A DECOMPOSITION THEOREM FOR UNITARY GROUP REPRESENTATIONS ON KAPLANSKY–HILBERT MODULES AND THE FURSTENBERG–ZIMMER STRUCTURE THEOREM

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To Rainer Nagel in tribute to his tremendous achievements within and outside mathematics

ABSTRACT. In this paper, a decomposition theorem for (covariant) unitary group representations on Kaplansky–Hilbert modules over Stone algebras is established, which generalizes the well-known Hilbert space case (where it coincides with the decomposition of Jacobs, deLeeuw and Glicksberg).

The proof rests heavily on the operator theory on Kaplansky–Hilbert modules, in particular the spectral theorem for Hilbert–Schmidt homomorphisms on such modules.

As an application, a generalization of the celebrated Furstenberg–Zimmer structure theorem to the case of measure-preserving actions of arbitrary groups on arbitrary probability spaces is established.

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The Furstenberg–Zimmer structure theorem is one of the central results in the structure theory of measure-preserving systems. It was conceived by Zimmer in [Zim77] and, independently, by Furstenberg [Fur77] in his seminal work on an ergodic-theoretic proof of Szemerédi’s theorem.

The fundamental insight at the heart of the Furstenberg–Zimmer theorem is the following dichotomy: a $G$-system $X$ is either relatively weakly mixing with respect to a given factor $Y$, or there exists a non-trivial intermediate factor $Z$ which has relative discrete spectrum with respect to $Y$. (One can use compact extensions instead, but that does not make a big difference, see the notes to Part III on page 62) This dichotomy was reformulated heuristically by Tao in terms of structure (rel. discrete spectrum) and (pseudo)-randomness (rel. weak mixing).

\footnote{Here and everywhere else in this paper, $G$ is an arbitrary group.}
In order to understand this result (as well as our contribution in this paper), it is helpful to consider the “non-relative” situation first. Recall that a measure-preserving system $X$ has discrete spectrum if $L^2(X)$ is generated by the finite-dimensional invariant subspaces, while it is said to be weakly mixing, if the product system on $X \times X$ is ergodic, i.e., the fixed space of the corresponding Koopman representation on $L^2(X \times X)$ is trivial.

The link between elements of the fixed space of the product dynamics and finite-dimensional invariant subspaces consists in the following observation: if $e_1, \ldots, e_n$ is an orthonormal basis of an invariant subspace, then $\sum_{j=1}^n e_j \otimes \overline{e_j}$ is an element of this fixed space. And if $\sum_{j=1}^n x_j \otimes \overline{y_j}$ is a non-zero element of the product fixed space then $\text{span}\{x_1, \ldots, x_n\}$ contains a nontrivial invariant subspace.

The said dichotomy is hence a corollary of the following purely operator-theoretic “key lemma”.

**Key Lemma.** Let $T : G \to \mathcal{L}(H)$ be a unitary representation of a group $G$ on a Hilbert space $H$. Then

$$\left\{ \sum_{j=1}^n e_j \otimes \overline{e_j} \mid e_1, \ldots, e_n \in H \text{ orthonormal basis of a } T\text{-invariant subspace} \right\}$$

spans a dense subspace of the fixed space $\text{fix}(T \otimes \overline{T}) \subseteq H \otimes H^*$.

(Here, $\overline{T}$ is the contragredient representation on the dual Hilbert space $H^*$ and $\overline{x}(y) := (y|x)$ for $x, y \in H$. If $H = L^2(X)$ and $T$ comes from a measure-preserving action of $G$ on the probability space $X$, then—under the natural identification $H^* = L^2(X)$ and $H \otimes H^* = L^2(X \times X)$—the representation $T \otimes \overline{T}$ is simply the Koopmanization of the product dynamics.)

The proof of the key lemma consists in two observations. The first is the identity

(1) 
$$\text{fix}(T \otimes \overline{T}) = \mathcal{H}_T(H),$$

under the natural identification $\mathcal{H}_S(H) \equiv H \otimes H^*$ of the space of Hilbert–Schmidt operators on $H$ with the tensor product $H \otimes H^*$, where

$$\mathcal{H}_T(H) := \{ A \in \mathcal{H}(H) \mid T_t A = A T_t \text{ for all } t \in G \}$$

is the space of $T$-intertwinning Hilbert–Schmidt operators on $H$. The second is

(2) 
$$\mathcal{H}_T(H) = \overline{\text{span}\{ A \in \mathcal{H}_T(H) \mid A \text{ is of finite rank} \}},$$

a consequence of the spectral theorem for self-adjoint Hilbert–Schmidt operators.

Apart from the dichotomy result, the key lemma also accounts (with a simple proof) for the following decomposition of the Hilbert space into a “discrete-spectrum” part and a “weakly mixing” part.

**Corollary.** Let $T : G \to \mathcal{L}(H)$ be a unitary group representation on a Hilbert space $H$. Then $H = H_{ds} \oplus H_{wm}$ orthogonally with closed invariant subspaces

$$H_{ds} := \text{cl} \bigcup \{ M \subseteq H \mid M \text{ finitely-generated, } T\text{-invariant subspace} \},$$

and

$$H_{wm} := \{ x \in H \mid x \otimes \overline{x} \perp \text{fix}(T \otimes \overline{T}) \}.$$
This decomposition (in slightly different form) has been applied to various fields (see, e.g., [EFHN15] Chap. 20, [MRR19], [BGKKST12]) and can be seen as a special case of the famous Jacobs–de Leeuw–Glicksberg decomposition (see the notes to Part II on page 43 below).

Furstenberg in [Fur77] employed these ideas and modified them in order to cover a “relative” situation in the presence of an extension $X \to Y$ of dynamical systems, where elements of $L^\infty(Y)$ take the role of the scalar field and $L^2(X)$ becomes a module over it. However, the technical realization of his program lacks the simple and elegant structural division into a purely functional-analytic, representation-theoretic part and its application to dynamical systems. Rather, it exhibits a strong reliance on measure-theoretic tools: disintegration of measures, measurable Hilbert bundles, almost-everywhere arguments almost everywhere. (The same applies to Zimmer’s alternative approach to the dichotomy theorem in [Zim77] as well as to Glasner’s presentation in [Gla03].) As a consequence, the classical version of the Furstenberg–Zimmer theorem is restricted to actions of Borel groups on standard Lebesgue spaces. (We shall call this the separability restriction in the following.)

The goal of our paper is to develop a natural functional-analytic (measure-theory free) framework that allows to derive the Furstenberg–Zimmer theorem in a manner which is completely parallel to the non-relative case sketched above (and, as a byproduct, to free the theorem from the separability restriction). It is based, firstly, on the observation that the space

$$L^2(X|Y) := \{ f \in L^2(X) \mid \mathbb{B}_Y |f|^2 \in L^\infty(Y) \},$$

already used by Tao in [Tao09], is a so-called Kaplansky–Hilbert module (KH-module) over the Stone algebra $\mathcal{A} = L^\infty(Y)$. And, secondly, that these modules are the “correct” generalization of Hilbert spaces to modules in the sense that each result from Hilbert space theory has its natural analogue for Kaplansky–Hilbert-modules. (A theory called “Boolean–valued analysis” explains this phenomenon, see also the comments to Part I on page 33.)

Applied to our situation, the KH-module analogue of the Hilbert space results from above—our main result in a nutshell—reads as follows (see Theorem 6.9 for the “full version”).

**Theorem A.** Let $E$ be a Kaplansky–Hilbert module over a Stone algebra $\mathcal{A}$, let $S : G \to \text{Aut}(\mathcal{A})$ be a representation of a group $G$ as automorphisms on $\mathcal{A}$ and $T : G \to \text{End}(E)$ an $S$-covariant unitary representation of $G$ on $E$. Then

$$\left\{ \sum_{j=1}^n e_j \otimes \overline{e}_j \mid e_1, \ldots, e_n \in E \text{ suborthonormal basis of a } T\text{-invariant KH-submodule} \right\}$$

spans an order-dense $\text{fix}(S)$-submodule of $\text{fix}(T \otimes \overline{T})$. Moreover, $E$ decomposes orthogonally into $T$-invariant KH-submodules $E = E_{ds} \oplus E_{wm}$, where

$$E_{ds} = \text{ocl} \bigcup \{ M \subseteq E \mid M \text{ finitely-generated, } T\text{-invariant submodule} \}$$

and $E_{wm} = \{ x \in E \mid x \otimes \overline{x} \perp \text{fix}(T \otimes \overline{T}) \}$.
Here, ocl denotes “order-closure” and refers—as well as the term “order-dense”—to the concept of “order-convergence”, which is the right translation of norm convergence into the KH-module setting. (See Chapter 5 and the explanations of “Key Concepts” below.) In the case $\mathbb{A} = \mathbb{C}$, where the Hilbert module $E$ is just an ordinary Hilbert space, it coincides with ordinary norm convergence. Hence, Theorem A generalizes the Hilbert space results from above.

Theorem A will be proved (as Theorem 6.9) in Part II. The proof is completely parallel to the Hilbert space case $\mathbb{A} = \mathbb{C}$ sketched above. In particular, it rests on KH-analogues of (1) and (2), see Lemma 6.6 and Lemma 6.10.

In Part I, we provide the necessary background on Stone algebras and Kaplansky–Hilbert modules, in particular the spectral theorem for self-adjoint Hilbert–Schmidt homomorphisms (Theorem 4.1).

Part III contains the application to extensions of dynamical systems. Our exposition differs from conventional ones in that we, following the approach in [EFHN15], exclusively work in the functional-analytic category with the corresponding notion of (Markov) isomorphisms (cf. the introductory remarks in Chapter 7 and Definition 7.1).

The main link between the abstract KH-module results and the dynamical systems world is the isomorphism

$$L^2(X|Y) \otimes L^2(X|Y) \cong L^2(X \times_Y X|Y),$$

where $X \times_Y X$ is the relatively independent joining (Proposition 7.11). In this context we define couplings and joinings and prove their relation to intertwining Markov operators (Proposition 7.7 and Lemma 7.8). This happens in an elegant and brief functional-analytic way and is, possibly, of independent interest.

On this conceptual basis, the application to extensions of dynamical systems is then completely analogous to the “non-relative” case. It results in the definition of the relative Kronecker factor and the characterizations of the corresponding Kronecker subspace $\mathcal{E}(X|Y)$ (Proposition 8.5) as well as its orthogonal complement (Propositions 8.8 and 8.9). Again in a nutshell, the results can be summarized as follows (see Proposition 8.2, Proposition 8.5 and Proposition 8.8 for the full versions).

**Theorem B.** Let $J: (Y; S) \to (X; T)$ be an extension of measure-preserving $G$-systems. Then

$$\left\{ \sum_{j=1}^{n} e_j \otimes f_j \right\} e_1, \ldots, e_n \in L^2(X|Y) \text{ suborthonormal basis of an invariant KH-module}$$

spans an $L^2$-dense fix($S$)-submodule of fix($T \times_T T$). Moreover, $L^2(X)$ decomposes orthogonally into

$$L^2(X) = \mathcal{E}(X|Y) \oplus \mathcal{E}(X|Y)^\perp,$$

where

$$\mathcal{E}(X|Y) := \text{cl}_{L^2} \bigcup \{M \subseteq L^2(X) \mid M \text{ finitely-generated, } T\text{-invariant } L^\infty(Y)\text{-submodule} \},$$

and for $f \in L^2(X)$:

$$f \perp \mathcal{E}(X|Y) \iff \inf_{r \in G} \max_{g \in \mathcal{F}} \|E_Y((T_r f)g)\|_{L^2} = 0 \quad \text{for each finite } F \subseteq L^\infty(X).$$
Finally, $\mathcal{E}(X|Y)$ is a $T$-invariant closed unital sublattice of $L^2(X)$.

The Furstenberg–Zimmer dichotomy (Theorem 8.14) as well as the full Furstenberg–Zimmer structure theorem (Theorem 8.16) are then mere corollaries.

Our exposition is monographic in style. Each part has at its end an own chapter with notes and remarks, comprising commented references to the literature.

**Key Concepts.** In the following we shall discuss some of the key concepts in more detail and highlight some particular points.

A (pre-)Hilbert module is, roughly speaking, a space $E$ endowed with an inner product $(\cdot|\cdot)$ that takes values in a commutative unital $C^*$-algebra $A$. By the Gelfand–Naimark theorem, $A \cong C(\Omega)$ for some compact Hausdorff space $\Omega$, and $A$ carries a canonical lattice structure. As a consequence, one obtains a “lattice-valued norm” on $E$, defined by $|x| := \sqrt{(x|x)} \in A_+$, i.e., $E$ is naturally a “lattice-normed space” (a concept extensively studied by Kusraev and his school, see [Kus00b]). Along with the order structure on $A_+$ there come natural (non-topological) notions of “order-convergence” and “order-completeness” on $A$ as well as on $E$, in general coarser than their norm-analogues.

Whereas in the largest part of the literature on Hilbert modules these order-based notions play no role, here they feature prominently. The reason is that if $Y$ is a probability space then $L^\infty(Y)$ is order-complete, and a sequence $(f_n)_n$ in $L^\infty(Y)$ order-converges if and only if it is uniformly bounded and almost everywhere convergent (Lemma 7.5). Hence, order-convergence is a generalization of bounded a.e.-convergence of sequences to convergence of nets.

Now, order-completeness of $A = C(\Omega)$ for a compact space $\Omega$ means that $\Omega$ is Stonean, i.e., extremally disconnected (Proposition 1.9). Stonean spaces may appear exotic on a first encounter, but they are quite natural objects from a functional analyst’s point of view [DDLS16, GvR16]. The same applies to order-convergence, which may be a cryptic notion in the beginning. In $C(\Omega)$ it is related to pointwise convergence on the Stonean space $\Omega$ in a peculiar way, involving the notion of (topological) almost everywhere convergence (Lemma 1.12).

Order-completeness of a Hilbert module $E$ over a Stonean algebra $A$ means that $E$ is a Kaplansky–Hilbert module (KH-module). This concept was introduced—under the name “AW*-module” and with a different but equivalent definition—by Kaplansky in [Kap53]. Our definition follows Kusraev [Kus00b] and has the advantage that (1) one can avoid the concept of a “mixing” and (2) the parallelism with conventional Hilbert space theory becomes strikingly apparent. Since, as far as we can see, a coherent and sufficiently complete account of KH-module theory is missing in the literature, we took the opportunity to sketch this theory in Chapter 2 (with explicit proofs reduced to a minimum).

In contrast to the cursory treatment of the general theory, we included a quite detailed proof of the spectral theorem for self-adjoint Hilbert–Schmidt homomorphisms on KH-modules (Theorem 4.1), the key auxiliary result in the paper. This theorem can be derived from the spectral theorem for self-adjoint “cyclically compact” operators obtained by Gönülü in [Gön16], cf. Remark 4.5. However, cyclical compactness is a technically
involved concept, whence we decided to present an alternative proof, based on the Hilbert bundle representation.

A Hilbert module $E$ over $A = \mathbb{C}(\Omega)$ can be identified with the space $\Gamma(H)$ of continuous sections of a (canonically constructible) topological Hilbert bundle $H$ over $\Omega$. (This establishes a categorical equivalence reminiscent of the Serre–Swan duality, see Proposition 3.3). By virtue of such a Hilbert bundle representation, one can literally employ already known Hilbert space results to prove their KH-module analogues, and we apply this method at one decisive point in the proof of the spectral theorem (Section 4.3).

Note that our use of Hilbert bundles differs from the classical one, where only measurable Hilbert bundles (equivalently: direct integrals) were used (for example by Zimmer [Zim76] and Glasner [Gla03, Chap. 9]), of course under the usual separability restriction. In our topological approach we dispense with this restriction and, what is more, we have to say “almost everywhere” almost nowhere.

Related Approaches and Further Applications. In [KL16], Kerr and Li prove the Furstenberg–Zimmer structure theorem effectively without separability restriction. Following Tao [Tao09] they employ the rather technical concept of conditional compactness. Also, instead of developing a structural result like our Theorem 6.9 they work ad hoc with the Hilbert module $L^2(X|Y)$ which, however, is not the same as ours. (Theirs is norm-closed but not order closed.) As a result, they only obtain “approximate statements” and the parallels with the non-relative case are not very apparent.

The deeper roots of our approach lie in the conviction that the theory of measure-preserving systems often is better studied in certain categories of functional-analytic objects (like AL-spaces with a distinguished quasi-interior point) than in the category of classical ergodic theory (probability spaces and measurable point mappings). This conviction is old and goes back at least to the work of our teacher and mentor, Rainer Nagel, on ergodic theory in the 1970s and ‘80s (see, e.g. [DNP 87]). It is present in a for a long time neglected paper by Ellis [Ell87] and it pervades the book [EFHN15] which, however, does not make explicit use of the language of category theory (on purpose).

In a series of papers [JT22, JT23a, JT23b] Jamneshan and Tao develop an alternative point-free approach to ergodic theory based on abstract measure theory. Combining this perspective with conditional analysis, Jamneshan develops an uncountable Furstenberg–Zimmer structure theory in a parallel work [Jam23]. In a work of Jamneshan and Spaas this is extended even further to the setting of dynamical systems on von Neumann algebras (see [JS22]). See also the notes to Part I on page 33. We also mention that in a recent article of Eisner (see [Eis2023]) the decomposition presented in Theorem B is stated for $\mathbb{Z}$-actions and used to give a new ergodic theoretic proof of Szemerédi’s theorem.

The techniques developed here form the basis for several more recent works: In [EdKr23] a theorem of Lindenstrauss on the connection between topological measurably distal systems is generalized to $\mathbb{Z}$-actions on general probability spaces; and [Ede2022] contains a Furstenberg–Zimmer structure theorem for stationary random walks. An “uncountable version” of Austin’s Mackey–Zimmer type representation theorem for ergodic extensions with relative discrete spectrum is established in [EJK23]. Finally, a follow-up
article [HK23] on KH-dynamical systems is in preparation, see also the notes to Part III on page 62.

**Notation and Terminology.** The set of natural numbers, in our use, is \( \mathbb{N} := \{1, 2, 3, \ldots \} \), and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Generic probability spaces are denoted by \( X, Y, Z \) where, e.g., \( X = (X, \Sigma_X, \mu_X) \). Integration with respect to \( \mu_X \) is denoted by

\[
\int_X f := \int_X f \, d\mu_X \quad (f \in L^1(X)).
\]

Here, \( L^p(X) = L^p(X; \mathbb{C}) \) denotes the space of equivalence classes of \( \mathbb{C} \)-valued \( p \)-integrable functions.

If \( E, F \) are normed spaces, then \( \mathcal{L}(E; F) \) denotes the space of all bounded linear operators \( E \to F \). We abbreviate the linear span of some subset \( M \) of a vector space \( E \) with span\( (M) \). If \( E \) is also an \( \mathbb{A} \)-module, where \( \mathbb{A} \) is some \( C^* \)-algebra, then span\( _{\mathbb{A}}(M) \) denotes the \( \mathbb{A} \)-linear span of \( M \).

We abbreviate “compact Hausdorff topological space” by speaking simply of a “compact space”. If \( \Omega \) is compact, then \( C(\Omega) = C(\Omega; \mathbb{C}) \) is the space of \( \mathbb{C} \)-valued continuous functions on \( \Omega \). The group of homeomorphisms on \( \Omega \) is Homeo\( (\Omega) \). For \( f \in C(\Omega) \) we abbreviate \( [f \neq 0] := \{\omega \in \Omega \mid f(\omega) \neq 0\} \) and likewise for expressions as \( [f = 0] \) or \( [f \leq 0] \) and so on.

The generic unit in a commutative \( C^* \)-algebra is \( 1 \), and the same is used for the function constantly equal to 1 on whatever set is considered. More generally, a characteristic function (= indicator function) of a set \( \mathcal{A} \) is denoted by \( 1_{\mathcal{A}} \).

The closure of a set \( M \) (with respect to some given topology) is denoted by \( \text{cl}(M) \) or \( \overline{M} \). If necessary we specify the respective topology by indexing (like \( \text{cl}_{L^2} \)). Nets are indexed like \( (x_\alpha)_{\alpha} \) as long as the underlying directed index set is not needed explicitly.

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**Part I. Stone Algebras and Kaplansky–Hilbert Modules**

Our general goal is to provide a conceptual view on the dichotomy at the heart of the Furstenberg–Zimmer structure theory and the underlying decomposition. This part
therefore gathers all the relevant terminology and background knowledge about Hilbert modules that generalize concepts and results from regular Hilbert space theory. Since many Hilbert space results do not carry over to general Hilbert modules, we focus on Kaplansky–Hilbert modules which are, in many ways, a well-behaved generalization of Hilbert spaces. Since KH-modules are defined in terms of a notion of order-completeness, order structures play an essential role and replace measure-theoretic almost everywhere notions. As we will see in Part III, the conditional $L^2$-space $L^2(X|Y)$ is indeed a Kaplansky–Hilbert module over $L^\infty(Y)$ and the reader is encouraged to think of this example along the way.

1. Lattice-Normed Spaces and Stone Algebras

1.1. Hilbert Modules. A pre-Hilbert space is a vector space $H$ over the field $\mathbb{C}$ together with a mapping $(\cdot|\cdot): H \times H \to \mathbb{C}$ obeying certain rules. In order to arrive at the definition of a pre-Hilbert module, the field $\mathbb{C}$ is replaced by a unital commutative $C^*$-algebra $A$. Recall here that, by the Gelfand–Naimark representation theorem, any such $A$ may be thought of as $C(\Omega)$ for a compact space $\Omega$ (see [Dix77, Sec.s 1.4 and 1.5] or [EPHN15, Sec. 4.4]). In particular, concepts such as complex conjugation, the modulus and square roots are defined in any unital (commutative) $C^*$-algebra via the continuous functional calculus.

We now collect some basic notions of Hilbert module theory; for more detailed information we refer to [Lan95].

Definition 1.1. Let $A$ be a unital commutative $C^*$-algebra. A unital $A$-module $E$ equipped with a mapping $(\cdot|\cdot): E \times E \to A$ is called a pre-Hilbert module over $A$ if the following conditions are satisfied.

1) For $x \in E$ we have $(x|x) \geq 0$. Moreover, $(x|x) = 0$ if and only if $x = 0$.

2) The map $(\cdot|y): E \to A, x \mapsto (x|y)$ is $A$-linear for every $y \in E$.

3) $(x|y) = (y|x)$ for all $x, y \in E$.

In a pre-Hilbert module $E$ the Cauchy–Schwarz inequality

$$|(x|y)| \leq \sqrt{(x|x) \cdot (y|y)}$$

holds for all $x, y \in E$. As a consequence, by

$$\|x\| := \|((x|x)^{\frac{1}{2}} A = \|((x|x)^{\frac{1}{2}} \text{ for } x \in E$$

a norm $\| \cdot \|$ is defined on $E$. The pre-Hilbert module $E$ is called a Hilbert module, if it is complete with respect to this norm. Note that a (pre-)Hilbert module over $A = \mathbb{C}$ is nothing but a usual (pre-)Hilbert space.

\footnote{This means that $1 \cdot x = x$ for every $x \in E$.}

\footnote{Complex conjugation is given by the involution in $A$.}

\footnote{This is due to the fact that $A$ is commutative and can be proved as in [DG83, page 49] by using the Cauchy–Schwarz inequality for scalar-valued positive-sesquilinear forms; in general one only has a weaker inequality, cf. [Lan95 Prop. 1.1.].}
One says that $x, y \in E$ are **orthogonal** if $(x|y) = 0$, and for a subset $M \subseteq E$ we define the **orthogonal complement** $M^\perp$ as

$$M^\perp := \{ x \in E \mid (x|y) = 0 \text{ for every } y \in M \}.$$  

Given pre-Hilbert modules $E$ and $F$, an $A$-linear map $T: E \to F$ is called a **module homomorphism**. The space of bounded module-homomorphisms is

$$\text{Hom}(E; F)$$

with $\text{End}(E) := \text{Hom}(E; E)$. Obviously, $\text{Hom}(E; F)$ is a closed subspace of $\mathcal{L}(E; F)$ (even with respect to the weak operator topology) and, naturally, an $A$-module.

A module homomorphism $T: E \to F$ is called $A$-**isometric** if

$$(Tx|Ty) = (x|y) \quad \text{for all } x, y \in E.$$  

By polarization, this is equivalent to $(Tx|Tx) = (x|x)$ for every $x \in E$. Clearly, every $A$-isometric homomorphism is (norm)-isometric, and hence bounded and injective.

**Examples 1.2.**

1. Let $\Omega$ be a compact space and $H$ a Hilbert space. The space $C(\Omega; H)$ of continuous maps from $\Omega$ to $H$ equipped with the pointwise scalar product defines a Hilbert module over $C(\Omega)$.

2. Let $\Omega$ be a non-finite compact space and consider $C(\Omega)$ as a Hilbert module over itself. If $\omega \in \Omega$ is an accumulation point of $\Omega$, then

$$I_\omega := \{ f \in C(\Omega) \mid f(\omega) = 0 \}$$

is a closed submodule of $C(\Omega)$ with $I_\omega^\perp = \{0\}$.

The last example shows that a closed submodule of a Hilbert module need not be orthogonally complemented (see [Lan95, page 7]). This is just one of many parts of Hilbert space theory that, contrary to what one might think at first, do not have immediate generalizations to Hilbert modules in general. The situation is better when one considers the subclass of Kaplansky–Hilbert modules, to be introduced in Chapter 2 below.

### 1.2. Lattice-Normed Spaces.

Any Hilbert module $E$ over a unital commutative $C^*$-algebra $A$ admits a “vector valued norm”

$$| \cdot |: E \to A_+, \quad x \mapsto (x|x)^{1/2}$$

where $A_+$ denotes the cone of the positive elements of $A$. This turns $E$ into a so-called lattice-normed space, see [Kus00b, Chap. 2].

**Definition 1.3.** Let $A$ be a unital commutative $C^*$-algebra. A vector space $E$ equipped with a mapping

$$| \cdot |: E \to A_+, \quad x \mapsto |x|$$

is a **lattice-normed space** (over $A$) if the following conditions are satisfied for all $x, y \in E$ and $\lambda \in \mathbb{C}$.

1. $|x| = 0$ if and only if $x = 0$;
2. $|\lambda x| = |\lambda| \cdot |x|$;
3. $|x + y| \leq |x| + |y|$.
If, in addition, $E$ is a unital $\mathbb{A}$-module, and 2) holds for all $x \in E$ and all $\lambda \in \mathbb{A}$, then $E$ is called a **lattice-normed module**.

A lattice-normed space $E$, like a pre-Hilbert module, carries the natural norm

$$\|x\| := \|\|x\|\|_\mathbb{A} \quad (x \in E).$$

If $E$ is a lattice-normed module, one has $\|fx\| \leq \|f\|_\mathbb{A}\|x\|$ for $f \in \mathbb{A}$, $x \in E$, hence $E$ is a normed module.

Note that if $\mathbb{A} = \mathbb{C}$, a lattice-normed space is nothing else than a normed space.

**Examples 1.4.**

1. Let $\mathbb{A}$ be a unital commutative $\mathbb{C}^*$-algebra. As mentioned before, each pre-Hilbert module over $\mathbb{A}$ defines a lattice-normed module.

2. Every unital commutative $\mathbb{C}^*$-algebra $\mathbb{A}$ is a lattice-normed space, where the lattice-norm is given by the usual modulus map $\mathbb{A} \to \mathbb{A}$, $f \mapsto |f|$ defined via functional calculus.

3. Let $\Omega$ be a compact space and $H$ be a Hilbert space. Consider the Hilbert module $C(\Omega; H)$ of Examples 1.2, part (1). Then the vector-valued norm is given by

$$C(\Omega; H) \to C(\Omega), \quad F \mapsto (\omega \mapsto \|F(\omega)\|).$$

For vector lattices there is a well-established concept of order-convergence [Kus00b, Subsection 1.3.4] generalizing the dominated almost everywhere convergence of a sequence of integrable functions on a probability space [Kus00b, Subsection 1.4.11]. With the lattice-norm replacing the modulus, the notion of order-convergence can be extended to lattice-normed spaces as follows [Kus00b, Subsection 2.1.5].

**Definition 1.5.** Let $E$ be a lattice-normed space over a unital commutative $\mathbb{C}^*$-algebra $\mathbb{A}$. A net $(f_i)_{i \in I}$ in $\mathbb{A}$ **decreases to** 0, if

$$i \leq j \quad \Rightarrow \quad 0 \leq f_j \leq f_i \quad \text{and} \quad \inf\{f_i | i \in I\} = 0.$$

A net $(x_\alpha)_\alpha$ in $E$ **order-converges** (or: is **order-convergent**) to $x \in E$ (in symbols: $\lim_\alpha x_\alpha = x$), if there is a net $(f_i)_{i \in I}$ in $\mathbb{A}$ decreasing to zero and satisfying

$$\forall i \in I \ \exists \alpha_i: \ |x_\alpha - x| \leq f_i \quad (\alpha \geq \alpha_i).$$

A net $(x_\alpha)_{\alpha \in \mathbb{A}}$ in $E$ is **order-Cauchy**, if the net $(x_\alpha - x_\beta)_{(\alpha, \beta) \in \mathbb{A} \times \mathbb{A}}$ order-converges to zero.\(^3\)

A subset $M \subseteq E$ of $E$ is **order-bounded**, if there is $f \in \mathbb{A}_+$ such that $|x| \leq f$ for all $x \in M$. It is **order-closed** in $E$, if the order-limit of every order-convergent net in $M$ is also contained in $M$. The **order-closure** $\text{ocl}(M) = M^0$ of $M \subseteq E$ is the smallest order-closed subset of $E$ containing $M$. (In general, it is possible that not every element of $\text{ocl}(M)$ is the limit of an order-convergent net in $M$. However, in the situations interesting to us this is indeed the case, see [Lemma 1.13] below.) We say that $M$ is **order-dense** in $E$, if $\text{ocl}(M) = E$.

\(^3\)Here, $\mathbb{A} \times \mathbb{A}$ is equipped with the product direction, i.e., $(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2)$ for $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{A} \times \mathbb{A}$ precisely when $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. 

A mapping \( f : E \to F \) between lattice-normed spaces is **order-continuous** if \( \lim_{\alpha} x_\alpha = x \in E \) implies \( \lim_{\alpha} f(x_\alpha) = f(x) \) whenever \( (x_\alpha)_{\alpha} \) is a net in and \( x \) an element of \( E \).

**Remarks 1.6.**

1. Recall that in a normed space \( E \) a net \( (x_\alpha)_{\alpha} \) converges to an element \( x \in E \) if and only if for every \( n \in \mathbb{N} \) there is \( \alpha(n) \) such that \( \|x_\alpha - x\| \leq \frac{1}{n} \) for every \( \alpha \geq \alpha(n) \). Hence, [Definition 1.5] is obtained by replacing the scalar-valued norm by a vector-valued norm and the sequence \( (\frac{1}{n})_{n \in \mathbb{N}} \) by a net decreasing to zero.

2. The order-limit of a net is unique (if it exists). Each order-convergent net is order-Cauchy and each order-Cauchy net is eventually order-bounded.

3. In a lattice-normed space, the vector space operations as well as the lattice-norm are order-continuous. In a pre-Hilbert module, the inner product and the module product are order-continuous.

4. One has \( \lim_{\alpha} x_\alpha = x \) in a lattice-normed space \( E \) if and only if \( \lim_{\alpha} \|x - x_\alpha\| = 0 \) in \( A \), and a similar statement holds for order-Cauchy nets.

It is natural to ask and important to understand how order-convergence and norm-convergence in a pre-Hilbert module are related.

**Lemma 1.7.** Let \( E, F \) be lattice-normed modules over a unital, commutative \( C^* \)-algebra \( A \). Then the following assertions hold:

1. If \( (x_\alpha)_{\alpha} \) is a net in \( E \) and \( x \in E \), then \( \lim_{\alpha} x_\alpha = x \) implies \( \lim_{\alpha} x_\alpha = x \).
2. Order-closed subsets of \( E \) are norm-closed, norm-dense subsets are order-dense.
3. A subset of \( E \) is order-bounded if and only if it is norm-bounded.
4. A module homomorphism \( A : E \to F \) is bounded if and only if it is order-continuous if and only if \( \lim_{\alpha} Ax_\alpha = Ax \) is order-continuous if and only if it is order-bounded, i.e., there exists \( c > 0 \) such that \( |Ax| \leq c|x| \) for all \( x \in E \). In this case, the latter estimate is true with \( c = \|A\| \).
5. If \( A = \mathbb{C} \), then order-convergence is the same as norm-convergence.

**Proof.**

(i) Without loss of generality we may suppose that \( x = 0 \). By definition of the norm on \( E \), \( |x_\alpha| \leq \|x_\alpha\|1 \). It follows that \( \lim_{\alpha} \|x_\alpha\|1 = 0 \) in \( A \), and hence \( \lim_{\alpha} |x_\alpha| = 0 \) as well.

(ii) follows from (i) and (iii) and (v) are obvious.

(iv) Suppose first that \( A \) is continuous and let \( c := \|A\| \). Then \( |x| \leq 1 \) implies \( |Ax| \leq c \). We claim that \( |Ax| \leq c|x| \) for all \( x \in E \). To prove this, fix \( \varepsilon > 0 \) and define \( f := (|x| + \varepsilon)^{-1} \). Then \( |fx| \leq 1 \) and hence \( |Ax| \leq c(|x| + \varepsilon) \). Letting \( \varepsilon \to 0 \) proves the claim.

Next, suppose that there is \( c \geq 0 \) with \( |Ax| \leq c|x| \) for all \( x \in E \). Then \( \lim_{\alpha} Ax_\alpha = Ax \) implies \( \lim_{\alpha} Ax_\alpha = Ax \) since \( |Ax_\alpha - Ax| = |A(x_\alpha - x)| \leq c|x_\alpha - x| \). Hence, \( A \) is order-continuous.

Finally, suppose that \( A \) is order-continuous but not continuous. Then there is a sequence \( (x_n)_n \) in \( E \) such that \( \|x_n\| \to 0 \) but \( \|Ax_n\| \to \infty \). By (i), \( \lim_{n} x_n = 0 \) and hence \( \lim_{\alpha} Ax_\alpha = 0 \). But order-convergent nets are eventually order-bounded and hence, by (iii), norm-bounded. This is a contradiction. \( \square \)
1.3. **Stone Algebras and Stonean Spaces.** Every order-convergent net in a lattice-normed space is order-Cauchy, but the converse may fail. This leads to the notion of “order-completeness”, to be introduced next.

**Definition 1.8.** A lattice-normed space $E$ is *(order-)complete* if every order-Cauchy net in $E$ is order-convergent in $E$. A commutative unital $C^*$-algebra $A$ is a Stone algebra if it is order-complete (as a lattice-normed space over itself).

A compact space $\Omega$ is called a Stonean space if it is extremally disconnected, i.e., if the closure of every open subset is open. And a unital commutative $C^*$-algebra (or rather: a Banach lattice) is called Dedekind complete if each subset of real elements, bounded from above, has a supremum. In view of the Gelfand–Naimark representation theorem, the following result is a complete characterization of Stone algebras.

**Proposition 1.9.** For a compact space $\Omega$ the following assertions are equivalent.

(a) $C(\Omega)$ is a Stone algebra.
(b) $C(\Omega)$ is Dedekind complete.
(c) $\Omega$ is a Stonean space.

**Proof.** The equivalence of (b) and (c) is standard, see for instance [Sch74, Prop. 7.7] or [Kus00b, Thm. 1.5.9]. The implication (c)$\Rightarrow$(a) is (admittedly not very explicitly) [Kus00b, 2.4.8], applied with $\mathcal{X} = \Omega \times \mathbb{R}$, the trivial Banach bundle over $\Omega$. (One has to know that $\Omega$ is Stonean if and only if it is the Stone–Čech compactification of each of its dense subsets, see [GJ76, Problem 6M].)

For the implication (a)$\Rightarrow$(b) let $M \subseteq C(\Omega)$ be a set of real-valued functions which is bounded above by an element $g \in C(\Omega)$. We consider the set $\mathcal{P}_\text{fin}(M)$ of all finite subsets of $M$. Then

$$\mathcal{P}_\text{fin}(M) \rightarrow C(\Omega), \quad N \mapsto \sup(N)$$

is an increasing net in $C(\Omega)$ bounded by $g$. If it order-converges, then its order-limit is necessarily the supremum of $M$. In view of (a) it hence suffices to show that every increasing net $(e_\alpha)_\alpha$ in $C(\Omega; \mathbb{R})$ which is order-bounded from above is order-Cauchy. Let $(e_\alpha)_{\alpha \in A}$ be such a net and consider the non-empty set

$$B := \{ h \in C(\Omega; \mathbb{R}) \mid e_\alpha \leq h \text{ for every } \alpha \in A \}.$$

Then $B$ is directed when equipped with the reverse order. With the product order, we therefore obtain a directed set $I = A \times B$ and a decreasing net

$$f : A \times B \rightarrow C(\Omega), \quad (\alpha, h) \mapsto 2(h - e_\alpha).$$

We show that $\inf_{(\alpha, h)} f(\alpha, h) = 0$. It is clear that $f(\alpha, h) \geq 0$ for every $(\alpha, h) \in A \times B$. Now take a real-valued function $s \in C(\Omega)$ with $0 \leq s \leq f(\alpha, h)$ for all $(\alpha, h) \in A \times B$. Then $e_\alpha \leq h - \frac{s}{2}$ for all $\alpha \in A$ and therefore $h - \frac{s}{2} \in B$ for every $h \in B$. But then $h - k \frac{s}{2} \in B$ for every $k \in \mathbb{N}$, which implies $s = 0$.

---

6In lattice theory the term “Stone algebra” has a different meaning. Since the Stone algebras from lattice theory do not show up in this work, there is no danger of confusion.

7The term “Stonian space” is also in use.
To finish the proof, take \((\alpha, h) \in A \times B\) and observe that
\[
|e_{\alpha_1} - e_{\alpha_2}| \leq |e_{\alpha_1} - h| + |h - e_{\alpha_2}| \leq 2(h - e_{\alpha})
\]
for \(\alpha_1, \alpha_2 \geq \alpha\). \(\square\)

As a consequence of Proposition 1.9 we note the following simplified description of order-convergence when \(\mathcal{A}\) is a Stone algebra.

**Lemma 1.10.** Let \(E\) be a lattice-normed space over a Stone algebra \(\mathcal{A}\), let \((x_\alpha)\) be a net in \(E\) and \(x \in E\). Then the following assertions hold:

(i) \(o\)-\(\lim_{\alpha} x_\alpha = x\) if and only if there is an index \(\alpha_0\) and a net \((f_\alpha)_{\alpha \geq \alpha_0}\) in \(\mathcal{A}\) decreasing to 0 with
\[
|x - x_\alpha| \leq f_\alpha \quad (\alpha \geq \alpha_0).
\]

(ii) \((x_\alpha)\) is order-Cauchy if and only if there is \(\alpha_0\) and a net \((f_\alpha)_{\alpha \geq \alpha_0}\) in \(\mathcal{A}\) decreasing to 0 with
\[
|\gamma - x_\gamma| \leq f_\alpha \quad (\beta, \gamma \geq \alpha \geq \alpha_0).
\]

**Proof.** For the proof of (i), suppose that \(x = o\)-\(\lim_{\alpha} x_\alpha\). Then choose \(\alpha_0\) such that the net \((x_\alpha)_{\alpha \geq \alpha_0}\) is order-bounded and define \(f_\alpha := \sup_{\beta \geq \alpha} |x - x_\beta|\). It is easy to see that \(\inf_{\alpha \geq \alpha_0} f_\alpha = 0\). The converse implication is trivial. Assertion (ii) follows from (i). \(\square\)

How are order convergence in a Stone algebra \(C(\Omega)\) and pointwise convergence in the function space \(C(\Omega)\) related? The following is the basic observation, noted by Wright in [Wri69a, Lem. 1.1].

Recall that a subset \(A\) of a compact space \(\Omega\) is **residual** if it contains a countable intersection of dense open sets. Since compact spaces have the Baire property, residual subsets are dense. Similarly to the terminology in measure theory, one says that a statement about elements \(\omega\) of \(\Omega\) holds **almost everywhere**, when it holds for all elements of a residual subset of \(\Omega\).

**Lemma 1.11** (Wright). Let \((f_i)_i\) be a family of elements in the Stone algebra \(\mathcal{A} = C(\Omega; \mathbb{R})\), bounded from below, and \(f = \inf_{i \in I} f_i\) its order-infimum. Then
\[
f(\omega) = \inf_i f_i(\omega) \quad \text{almost everywhere}.
\]

Employing Lemma 1.11 we obtain the following result, which entails a characterization of order-convergence for countable nets, cf. [Gut93, Subsection 0.3.3].

**Lemma 1.12.** Let \(\Omega\) be a Stonian space. Consider the following statements for a net \((e_\alpha)_{\alpha \in A}\) in \(C(\Omega)\) and \(e \in C(\Omega)\).

(a) \((e_\alpha)\) order-converges to \(e\).

(b) \((e_\alpha)\) is eventually bounded and \(\lim_{\alpha} e_\alpha(\omega) = e(\omega)\) almost everywhere.

Then (a) \(\implies\) (b). If the index set \(A\) is countable, then (a) \(\iff\) (b).

**Proof.** By passing to \(e_\alpha - e\) we may suppose \(e = 0\). Suppose that (a) holds and pick by Lemma 1.10 an index \(\alpha_0\) and a net \((f_\alpha)_{\alpha \geq \alpha_0}\) decreasing to 0 with \(|e_\alpha| \leq f_\alpha\) for \(\alpha \geq \alpha_0\). Then \(\|e_\alpha\| \leq \|f_\alpha\|\) for \(\alpha \geq \alpha_0\). Moreover,
pick a residual subset $D \subseteq \Omega$ such that $\inf_{\alpha \geq \alpha_0} f_\alpha(\omega) = 0$ for all $\omega \in D$ (Lemma 1.11). Then $\limsup_{\alpha} |e_\alpha(\omega)| \leq \inf_{\alpha \geq \alpha_0} f_\alpha(\omega) = 0$ for all $\omega \in D$. This proves (b).

Conversely, suppose that $A$ is countable and that (b) holds. Let $\alpha_0 \in A$ with $\sup_{\alpha \geq \alpha_0} \|e_\alpha\| < \infty$. For every $\alpha \in A$ with $\alpha \geq \alpha_0$ we write $f_\alpha := \sup_{\beta \geq \alpha} |e_\beta|$. Then $(f_\alpha)_{\alpha}$ is decreasing and it suffices to show that $f := \inf_{\alpha \geq \alpha_0} f_\alpha = 0$.

By hypothesis, the set $D := \{ \omega \mid \lim_{\alpha} e_\alpha(\omega) = 0 \}$ is residual. Moreover, by Lemma 1.11 the sets

$D' := \left\{ \omega \mid \inf_{\alpha} f_\alpha(\omega) = f(\omega) \right\}$ and $D_\alpha := \left\{ \omega \mid \sup_{\beta \geq \alpha} |e_\beta(\omega)| = f_\alpha(\omega) \right\}$ ($\alpha \geq \alpha_0$)

are all residual. Obviously, $f$ vanishes on the set $D'' := D \cap D' \cap \bigcap_{\alpha \geq \alpha_0} D_\alpha$. But since $A$ is countable, $D''$ is residual and hence dense in $\Omega$. Since $f$ is continuous, $f = 0$ as desired. \hfill \Box

Lattice-normed modules over Stone algebras have particularly nice properties. The following is a first example.

**Lemma 1.13.** Let $E$ be a lattice-normed module over the Stone algebra $\mathbb{A}$, and let $M \subseteq E$ be a submodule. Then its order-closure satisfies

$$\text{ocl}(M) = \{ x \in E \mid \text{there is a net } (x_\alpha)_\alpha \text{ in } M \text{ with } \liminf \alpha x_\alpha = x \}.$$  

Moreover, $\text{ocl}(M)$ is an order-closed submodule of $E$.

**Proof.** The inclusion $\subseteq$ is trivial. For the converse fix $x \in \text{ocl}(M)$. We first claim that $\inf_{x \in M} |x - z| = 0$. To prove this claim, it suffices to show that $N := \{ x \in E \mid \inf_{x \in M} |x - z| = 0 \}$ is order-closed (as it obviously contains $M$).

Let $(y_\alpha)_\alpha$ be a net in $N$ and $y = \liminf \alpha y_\alpha$. Then, for each $z \in M$ and each index $\alpha$

$$|y - z| \leq |y - y_\alpha| + |y_\alpha - z|.$$  

This implies $\inf_{x \in M} |y - z| \leq |y - y_\alpha|$ for each $\alpha$, hence $y \in N$ as well. It follows that $N$ is order-closed, and hence $x \in N$.

Next, we claim that the module $M$ is directed with respect to

$$z \leq z' \iff |x - z'| \leq |x - z|.$$  

To see this, fix $z_1, z_2 \in M$ and denote $f_j := |x - z_j|$ for $j = 1, 2$. By representing $\mathbb{A} = C(\Omega)$ for a Stonean space $\Omega$, we find a clopen subset $A \subseteq \Omega$ with $A \subseteq \{ f_1 \leq f_2 \}$ and $A^c \subseteq \{ f_2 \leq f_1 \}$. Define $z := 1_A z_1 + 1_A^c z_2 \in M$. Then

$$|x - z| = 1_A |x - z_1| + 1_A^c |x - z_2| = f_1 \wedge f_2,$$

and hence $z \geq z_1, z_2$. Finally, observe that $x = \liminf_{x \in M} z$ with respect to the direction on $M$. This establishes the inclusion $\subseteq$. \hfill \Box

The proof of the remaining assertion is straightforward.
2. Kaplansky–Hilbert Modules

Having Stone algebras at our disposal, we can now introduce Kaplansky–Hilbert mod-
ules (see also [Fra95, Theorem 4.1] for further equivalent definitions).

**Definition 2.1.** An order-complete lattice-normed module over a Stone algebra \( A \) is called a **Kaplansky–Banach module** (over \( A \)). A **Kaplansky–Hilbert module** (in short: KH-module) is an order-complete pre-Hilbert module \( \mathcal{E} \) over a Stone algebra.

A submodule of a Kaplansky–Hilbert module \( \mathcal{E} \) is called a **Kaplansky–Hilbert submodule** (KH-submodule) of \( \mathcal{E} \) if it is order-closed in \( \mathcal{E} \).

By [Kus00b, Thms 2.2.3 and 7.1.2] a Kaplansky–Banach module is automatically complete with respect to the norm. We shall focus on Kaplansky–Hilbert modules. The only Kaplansky–Banach module interesting to us and not being a KH-module is the space \( \text{Hom}(\mathcal{E}; F) \) of bounded module-homomorphisms between KH-modules, see Section 2.2 below.

**Examples 2.2.**

1. Our standard example for a KH-module is \( L^2(X\mid Y) \), when \( X \rightarrow Y \) is an extension of probability spaces, see Definition 7.3 and Proposition 7.6 below. Other examples of KH-modules are described in [Kus00b, Subsection 7.4.8].

2. Let \( \Omega \) be a Stonean space and \( H \) a Hilbert space. Then the Hilbert module \( C(\Omega; H) \) over \( C(\Omega) \) (see Examples 1.2 part (1)) is a Kaplansky–Hilbert module if and only if \( \Omega \) is finite or \( H \) is finite-dimensional [Kus00b, Subsection 2.3.3].

In view of the preceding example it is useful to remark that every pre-Hilbert module over a Stone algebra admits an order-completion, see Section 2.7 below.

2.1. Support of Elements and Modules. Let \( A \) denote a fixed Stone algebra. When convenient, we suppose without loss of generality \( A = C(\Omega) \) for a fixed Stonean space \( \Omega \).

The elements of \( B := \{ p \in A \mid p^2 = p \} \) are called **idempotents**. Endowed with the lattice structure coming from \( A \), the set \( B \) is a complete Boolean algebra with complementation \( p^c = 1 - p \) and meet \( p \land q = pq \). An element \( x \in E \) is called **normalized** if \( |x| \in B \). The normalization of \( x \in E \) is

\[
\frac{x}{|x|} := \lim_{\varepsilon \downarrow 0} \frac{1}{|x| + \varepsilon} x
\]

and its support is

\[
\text{supp}(x) := p_x := \frac{x}{|x|} = \lim_{\varepsilon \downarrow 0} \frac{|x|}{|x| + \varepsilon}.
\]

The following lemma collects the basic properties of these concepts.

**Lemma 2.3.** Let \( E \) be a Kaplansky–Banach module over a Stone algebra \( A \). Then the normalization and the support of \( x \in E \) are well-defined and have the following properties:

(i) \( \frac{x}{|x|} \) is normalized and satisfies \( |x| \frac{x}{|x|} = x \).

(ii) \( \text{supp}(x) = \min \{ p \in B \mid px = x \} = \text{supp}(|x|) = \text{supp}(\frac{x}{|x|}) \).

(iii) \( x \) is normalized \( \iff |x|x = x \iff x = \frac{x}{|x|} \iff |x| = \text{supp}(x) \).
(iv) If \( E \) is a KH-module, then \( (x|\frac{\alpha}{|x|}) = |x| \).

**Proof.** Observe that \( \varepsilon \mapsto \frac{|x|}{|x|+\varepsilon} \) is bounded by 1 and increasing as \( \varepsilon \searrow 0 \). Hence, it order-converges to its supremum \( p_x \). Identifying \( A = C(\Omega) \) we see that \( p_x = 1_{\{|x|>0\}} \in B \) and \( |x| = p_x|x| \). Moreover, as

\[
\left| \frac{x}{|x|+\varepsilon} - \frac{x}{|x|+\delta} \right| = \left| \frac{|x|}{|x|+\varepsilon} - \frac{|x|}{|x|+\delta} \right| \quad (\delta, \varepsilon > 0),
\]

the net \( \left( \frac{x}{|x|+\varepsilon} \right)_\varepsilon \) is order-Cauchy and hence order-convergent.

(i), (ii) Note that \( |x - p_x x| = (1 - p_x)|x| = |x| - p_x|x| = 0 \) and hence \( p_x x = x \). It follows that \( \frac{|x|}{|x|+\varepsilon} \to 1 \), \( \frac{|x|}{|x|+\varepsilon} x = p_x x = x \). If \( q \in B \) with \( qx = x \), then \( |x| = qx \). This implies

\[
\frac{|x|}{|x|+\varepsilon} = q \frac{|x|}{|x|+\varepsilon}
\]

for all \( \varepsilon > 0 \) and hence \( p_x = qp_x \leq q \).

(iii) If \( x \) is normalized, then \( |x| = |x||\frac{x}{|x|} - |x||\frac{x}{|x|} = x \). If \( |x| = x \) then \( \frac{x}{|x|+\varepsilon} = \frac{|x|}{|x|+\varepsilon} x \) for each \( \varepsilon > 0 \) and hence \( \frac{x}{|x|} \to p_x x = x \). If \( x = \frac{x}{|x|} \), then \( |x| = \frac{x}{|x|} = p_x \). If \( |x| = p_x \) then \( x \) is normalized.

(iv) If \( E \) is a KH-module, then \( (x|\frac{\alpha}{|x|}) = (|x|\frac{\alpha}{|x|}|\frac{x}{|x|}) = |x|p_x = |x| \).

With the help of supports one can easily derive the following:

\[
|x| \wedge |y| = 0 \implies |x + y| = |x| + |y| \quad (x, y \in E).
\]

(The identity \( |x| \wedge |y| = 0 \) is equivalent to \( p_x p_y = 0 \), from which it follows that \( |x| + |y| = p_x|x + y| + p_y|x + y| \leq |x + y| \).

The idempotent

\[
\text{supp}(E) := p_E := \sup\{\sup\{p_x \mid x \in E\} = \inf\{q \in B \mid q^c E = \{0\}\}
\]

is called the **support** of \( E \). The following is [Kap53, Lem. 5], but due to our different definition of a Kaplansky–Banach module we give a different proof here.

**Lemma 2.4.** Let \( E \) be a Kaplansky–Banach module. Then there is a normalized element \( x \in E \) such that \( p_x = p_E \).

**Proof.** Define a partial order on \( \mathcal{N} := \{x \in E \mid \{x\} \in B\} \) by \( x \leq y \iff x = p_x y \). Let \( K \subseteq \mathcal{N} \) be any chain and \( q := \sup_{a \in K} |a| \in B \). Then, for any \( a, x, y \in K \) and \( x, y \geq a \)

\[
|y - x| \leq |y - a| + |x - a| = p^c_a(|x| + |y|) \leq 2p^c_a q.
\]

This shows that the increasing net \( (x)_{x \in K} \) is order-Cauchy. Hence it converges, and its limit is an upper bound for \( K \). Zorn’s Lemma yields a maximal element \( x \in \mathcal{N} \).

Let \( y \in E \) be arbitrary and define \( z := x + p^c_x \frac{y}{|y|} \). Then \( z \in \mathcal{N} \) and \( p_x z = x \), hence \( z = x \) by maximality. This implies \( p^c_y z = 0 \), i.e., \( p_y \leq p_x \). It follows that \( p_x = p_E \) as desired.

If \( A = C(\Omega) \), then \( \text{supp}(x) = 1_{\{|x|>0\}} \) for \( x \in E \). In particular, for \( x \in C(\Omega) \) one has \( x \neq 0 \) almost everywhere (see Section [L3] before Lemma [L11]) if and only if \( \text{supp}(x) = 1 \). Consequently, one can speak of statements holding almost everywhere even without explicit recourse to the representation \( A = C(\Omega) \). Here is an illustration.
Lemma 2.5. Let $f, g \in \mathbb{A}$ with $fg = 0$ and $f \neq 0$ almost everywhere. Then $g = 0$.

Proof. Since $fg = 0$, one has $\frac{f}{|f|}g = 0$ and hence $0 = pf|g| = |g|$. □

2.2. Bounded Module Homomorphisms. Let $E, F$ be lattice-normed modules over a Stone algebra $\mathbb{A}$. Recall from [Lemma 1.7] that a module homomorphism $T: E \to F$ is bounded if and only if it is order-continuous. The space $\text{Hom}(E; F)$ of all bounded module homomorphisms is a unital $\mathbb{A}$-module in a canonical way.

We now turn $\text{Hom}(E; F)$ into a lattice-normed module. To this end, define the operator \textbf{lattice-norm} of $T \in \text{Hom}(E; F)$ by

$$|T| := \sup_{|x| \leq 1} |Tx|.$$ 

Here the supremum is taken in $\mathbb{A}_+$, and this supremum exists since the set $\{ |Tx| \mid x \in E, |x| \leq 1 \}$ is bounded from above, e.g., by $\|T\|_1$. The following identities and inequalities are analogous to their well-known counterparts in Banach space theory.

Proposition 2.6. Let $E, F$ be lattice-normed modules over a Stone algebra $\mathbb{A}$. Then

$$|T| : \text{Hom}(E; F) \to \mathbb{A}_+, \quad T \mapsto |T| = \sup_{|x| \leq 1} |Tx|$$

turns $\text{Hom}(E; F)$ into a lattice-normed module. Moreover, $\|T\| = \|T\|_\mathbb{A}$ and

$$|Tx| \leq |T||x| \quad (T \in \text{Hom}(E; F), x \in E).$$

If $G$ is another lattice-normed module over $\mathbb{A}$ and $S \in \text{Hom}(F; G)$, then $|ST| \leq |S||T|$. If $F$ is order-complete, then so is $\text{Hom}(E; F)$.

Proof. It is straightforward to show that $E$ is a lattice-normed module. Let $T \in \text{Hom}(E; F)$ and $x \in E$. Fix $\varepsilon > 0$ and define $f := (|x| + \varepsilon 1)^{-1}$. Then $|f| \leq 1$ and hence, after multiplying with $1/f$, $|Tx| \leq |T|(|x| + \varepsilon 1)$. This implies $|Tx| \leq |T||x|$.

Next, since $|x| \leq 1$ is equivalent to $\|x\| \leq 1$, $|T| = \sup_{|x| \leq 1} \|Tx\|_\mathbb{A} = \sup_{|x| \leq 1} \|T||x||_\mathbb{A}$.

The converse inequality follows since $|T| \leq \|T\|_1$ (cf. the proof of Lemma [1.7] part (iv).

The inequality $\|ST\| \leq \|S||T|$ is now proved exactly as in the case $\mathbb{A} = \mathbb{C}$. The same applies to the proof that $\text{Hom}(E; F)$ is order-complete whenever $F$ is. (One has to replace sequences by nets and norm convergence by order convergence.) □

A module homomorphism $T: E \to F$ between lattice-normed spaces is called $\mathbb{A}$-\textbf{isometric} if $|Tx| = |x|$ for all $x \in E$.

Proposition 2.7. Let $E, F$ be lattice-normed modules over the Stone algebra $\mathbb{A}$ and let $E_0$ be an order-dense submodule of $E$. Furthermore, let $F$ be order-complete. Then each $T \in \text{Hom}(E_0; F)$ has a unique extension to an element $T^E \in \text{Hom}(E; F)$.

If $T$ is $\mathbb{A}$-isometric, then so is $T^E$. The mapping

$$\text{Hom}(E_0; F) \to \text{Hom}(E; F), \quad T \mapsto T^E$$

is an $\mathbb{A}$-isometric isomorphism of lattice-normed modules.
The proof is a straightforward adaptation of the proof for the case \( A = \mathbb{C} \). The same applies for the following result.

**Lemma 2.8.** Let \( E, F \) be lattice-normed spaces over a Stone algebra and \( (T_a)_a \) a uniformly bounded net in \( \text{Hom}(E; F) \). Then the set \( \{ x \in E \mid \lim_a T_a x = 0 \} \) is an order-closed submodule of \( E \).

Finally, we note the following description of the support of a module homomorphism.

**Lemma 2.9.** Let \( E, F \) be a Kaplansky–Banach modules and \( T \in \text{Hom}(E; F) \). Then \( \text{supp}(T) = \text{supp}([T]) = \text{supp}(\text{ocl ran}(T)) \).

**Proof.** For \( q \in \mathbb{B} \) one has \( q^c T = 0 \iff q^c \text{ ran}(T) = \{0\} \iff q^c(\text{ocl ran}(T)) = \{0\} \). Taking the minimum of these \( q \) yields the claim. \( \square \)

### 2.3. (Sub)Orthonormal Systems

Let \( E \) be a pre-Hilbert module over a unital commutative \( C^* \)-algebra \( A \). A family \( (x_i)_{i \in I} \) in \( E \) is called an **orthogonal system** if \( (x_i|x_j) = 0 \) whenever \( i \neq j \). In this case if \( I \) is finite, one has

\[ \left| \sum_{i \in I} x_i \right|^2 = \sum_{i \in I} |x_i|^2. \]

An orthogonal system \( (x_i)_{i \in I} \) in \( E \) is called a **suborthonormal** system if each \( x_i \) is normalized. In this case, the system is called **homogeneous** if \( |x_i| = |x_j| \) for all \( i, j \in I \).

In other words, a suborthonormal system is homogeneous if \( (x_i|x_j) = \delta_{ij} p \) for all \( i, j \in I \) and some fixed idempotent \( p \in \mathbb{B} \). If \( p = 1 \) here, then \( (x_i)_{i \in I} \) is called an **orthonormal** system.

A (sub)orthonormal system \( (x_i)_{i \in I} \) is called a **(sub)orthonormal basis** if \( \{x_i \mid i \in I\}^\perp = \{0\} \). A subset \( B \subseteq E \) is called a (sub)orthonormal subset (basis) if the family \( (x)_{x \in B} \) is a (sub)orthonormal system (basis).

**Lemma 2.10.** Let \( E \) be a pre-Hilbert module over a Stone algebra \( A \), and let \( B \subseteq E \) be a suborthonormal set in \( E \). Then for each \( x \in E \) the formal series (= net of finite partial sums)

\[ \sum_{y \in B} (x|y)y := \left( \sum_{y \in F} (x|y)y \right)_{F \subseteq B \text{ finite}} \]

is an order-Cauchy net in \( E \). If it converges, its order-limit \( z \) satisfies \( x - z \in B^\perp \), and

\[ |z|^2 = \sum_{y \in B} |(x|y)|^2 \leq |x|^2 \quad \text{(Parseval identity/Bessel inequality)}. \]

**Proof.** The proof (see [Kus00b, Subsection 7.4.9]) very much follows the same steps as in case of Hilbert spaces (see, e.g., [Con85, Paragraph I.4]). \( \square \)

Contrary to Hilbert spaces, Hilbert modules need not have orthonormal bases. The best one can say is that a Kaplansky–Hilbert module always has a suborthonormal basis. This follows from [Kus00b, Subsection 7.4.10], but we shall give a direct proof here.

**Proposition 2.11.** Let \( E \) be a Kaplansky–Hilbert module. If \( N \subseteq E \) is a suborthonormal subset, then there is a suborthonormal basis \( B \) of \( E \) with \( N \subseteq B \).
Proof. Consider $\mathcal{N} := \{ M \subseteq E \mid M \text{ suborthonormal with } N \subseteq M \}$, partially ordered by set inclusion. By Zorn’s lemma we find a maximal element $B$ of $\mathcal{N}$. Now combine the maximality with Lemma 2.4 to conclude that $B^\perp = \{0\}$. □

As a consequence, we obtain the following decomposition result, cf. [Kap53, Thm. 3].

**Proposition 2.12.** Let $E$ be a KH-module over a Stone algebra $\mathbb{A}$ and $M \subseteq E$ a KH-submodule. Then $E = M \oplus M^\perp$.

Moreover, a suborthonormal system $\mathcal{B} \subseteq M$ is a suborthonormal basis for $M$ if and only if $M = \text{ocl span}_\mathbb{A}(\mathcal{B})$, and in this case $x \mapsto Px = \sum_{y \in \mathcal{B}} (x|y)y$ is the projection of $E$ onto $M$ along $M^\perp$.

**Proof.** Employ Lemma 2.10 and prove the second assertion as for Hilbert spaces. Since by Proposition 2.11 one always finds a suborthonormal basis for $M$, also the first assertion holds. □

2.4. **Operator Theory on KH-Modules.** If $E$ is a pre-Hilbert module over a Stone algebra $\mathbb{A}$ and $y \in E$, then

$$\overline{y}: E \to \mathbb{A}, \quad x \mapsto (x|y)$$

is an element of the dual module $E^* := \text{Hom}(E; \mathbb{A})$. If $\mathbb{A}$ is a Stone algebra, one has the following strengthening, comprising a version of the Riesz–Fréchet theorem for Kaplansky–Hilbert modules.

**Theorem 2.13.** Let $E$ be a pre-Hilbert module over a Stone algebra $\mathbb{A}$. Then

$$|\overline{y}| = |y| \quad \text{for all } y \in E.$$

If $E$ is a Kaplansky–Hilbert module, then the mapping

$$\Theta: E \to \text{Hom}(E; \mathbb{A}), \quad y \mapsto \overline{y}$$

is bijective.

**Proof.** By the Cauchy–Schwarz inequality in $E$, $|\overline{y}| = \sup_{|x| \leq 1} (x|y) \leq |y|$. Putting $x := (|y| + \varepsilon 1)^{-1} y$ yields $|y|^2 \leq |\overline{y}|(|y| + \varepsilon 1)$ for arbitrary $\varepsilon > 0$, and hence $|y| = |\overline{y}|$. The second part follows from [Kap53, Thm. 5]. □

If $E$ is a KH-module, we equip the dual module $E^*$ with the structure of a Kaplansky–Hilbert module over $\mathbb{A}$ turning $\Theta$ into an $\mathbb{A}$-antilinear and $\mathbb{A}$-isometric bijection.

The **conjugate homomorphism** of $T \in \text{End}(E)$ is $\overline{T} \in \text{End}(E^*)$ defined by $\overline{T} \circ \Theta = \Theta \circ T$ or, equivalently, $\overline{T} x = \overline{T} x$ for all $x \in E$. This is not to be confused with the classical dual operator $T'$, which would satisfy $T' \overline{x} = \overline{T' x}$, where $T^*$ is the adjoint of $T$, to be introduced next.

**Corollary 2.14.** Let $E, F$ be Kaplansky–Hilbert modules. For every $T \in \text{Hom}(E; F)$ there is a unique module homomorphism $T^* \in \text{Hom}(F; E)$ with

$$(Tx|y) = (x|T^* y) \quad \text{for all } x \in E, \ y \in F.$$
Moreover, \((T^*)^* = T, \|T\| = |T^*|\) and \(\text{ran}(T)^\perp = \ker(T^*)\).

**Proof.** The existence of the adjoint is proved in [Kap53, Thm. 6]. The remaining assertions are proved exactly as the analogues for bounded operators on Hilbert spaces. \(\square\)

If \(E\) is a KH-module, then with the involution

\[
\text{End}(E) \rightarrow \text{End}(E), \quad T \mapsto T^*,
\]

the space \(\text{End}(E)\) is a C*-algebra (and even an AW*-algebra, see [Kap53]). In particular, one can speak of **normal**, **self-adjoint** or **unitary** module homomorphisms on \(E\).

A homomorphism \(T: E \rightarrow F\) between KH-modules \(E, F\) is a **contraction** if \(|T| \leq 1\) (equivalently: \(\|T\| \leq 1\)). Recall that it is an isometry if \(|Tx| = |x|\) for all \(x \in E\). The results about contractions, isometries and unitaries on Hilbert spaces listed in Appendix D.4 of [EFHN15] carry over to such homomorphisms on KH-modules, as the proofs can be repeated mutatis mutandis.

Similarly, the results of Appendix D.5 of [EFHN15] about orthogonal projections in and onto closed subspaces of Hilbert spaces, carry over to KH-Modules. In particular, for \(P \in \text{End}(E)\) one has

\[
|P| \leq 1, \quad P = P^2 \iff P^2 = P = P^* \iff \text{ran}(I - P) \perp \text{ran}(P).
\]

In this case, \(P\) is called an **orthogonal projection**.

As a consequence of Proposition 2.12 one obtains, for a given KH-submodule of a KH-module \(E\), a unique orthogonal projection \(P \in \text{End}(E)\) such that \(\text{ran}(P) = M\) and \(\ker(P) = M^\perp\).

Because of its importance, we note explicitly the following **mean ergodic theorem**:

**Proposition 2.15** ([Mean Ergodic Theorem]). Let \(E\) be a KH-module \(E\) and \(T \in \text{End}(E)\) a contraction. Then \(\text{fix}(T) = \text{fix}(T^*)\), \(E\) decomposes orthogonally as \(E = \text{fix}(T) \oplus \text{ocl ran}(I - T)\), and for each \(x \in E\) one has

\[
o\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j x = Px,
\]

where \(P\) is the projection onto \(\text{fix}(T)\) along \(\text{ocl ran}(I - T)\).

**Proof.** The identity \(\text{fix}(T) = \text{fix}(T^*)\) is proved exactly as [EFHN15, Lem. D.14]. Then the orthogonal decomposition \(E = \text{fix}(T) \oplus \text{ocl ran}(I - T)\) follows from the last assertion of [Corollary 2.14]. To prove the remaining statement, one notes first that \(\frac{1}{n} \sum_{j=0}^{n-1} T^j x \rightarrow 0\) for \(x \in \text{ran}(I - T)\) and then passes to the order closure by Lemma 2.8. \(\square\)

2.5. **Modules and Homomorphisms of Finite Rank.** A KH-module \(E\) over a Stone algebra \(A\) is of **finite rank** if it has a finite suborthonormal basis. Note that if \(e_1, \ldots, e_n\) is such a suborthonormal basis of \(E\), then by Proposition 2.12

\[
E = \text{span}_A \{e_1, \ldots, e_n\} = A e_1 \oplus \cdots \oplus A e_n,
\]

i.e., the algebraic \(A\)-span of the basis vectors is already order-closed.
Lemma 2.16. Let $E$ be a KH-module and $n \in \mathbb{N}_0$. The following assertions are equivalent:

(a) There are $x_1, \ldots, x_n \in E$ such that $E = \operatorname{ocl} \operatorname{span}_\mathbb{A}\{x_1, \ldots, x_n\}$.
(b) There is a suborthonormal basis $e_1, \ldots, e_n$ of $E$.
(c) $\sum_{y \in B} |y|^2 \leq n1$ for each suborthonormal set $B \subseteq E$.

In (b) one can achieve $|e_1| \geq |e_2| \geq \cdots \geq |e_n|$.

Proof. (b)⇒(a) is clear, and (a)⇒(b) follows from defining (Gram–Schmidt)

$$e_1 := \frac{x_1}{|x_1|}, \quad e_k := \frac{y_k}{|y_k|}, \quad y_k := x_k - \sum_{j=1}^{k-1} (x_k|e_k)e_k \quad (1 \leq k \leq n).$$

(b)⇒(c): Let $F \subseteq E$ be any finite suborthonormal system. Then (by Parseval and Bessel)

$$\sum_{y \in F} |y|^2 = \sum_{y \in F} \sum_{j=1}^n |(y|e_j)|^2 = \sum_{j=1}^n \sum_{y \in F} |(e_j|y)|^2 \leq \sum_{j=1}^n |e_j|^2 \leq n1.$$

(c)⇒(b): We construct a suborthonormal system $e_1, e_2, \ldots$ recursively by requiring that $e_k$ is a normalized element of $\{e_1, \ldots, e_{k-1}\}^\perp$ with maximal support (Lemma 2.14). Then obviously $p_E = |e_1| \geq |e_2| \geq \ldots$. Clearly, (c) implies $e_{n+1} = 0$, and hence $\{e_1, \ldots, e_n\}$ is a suborthonormal basis.

It follows from Lemma 2.16 that each KH-submodule of a KH-module of finite rank is again of finite rank. A KH-submodule is called homogeneous (of rank $k$) if it has a homogeneous suborthonormal basis (of length $k$). Each finite-rank KH-module decomposes into homogeneous submodules as follows.

The dimension of a KH-module $E$ of finite-rank is

$$\dim_E := \sup\left\{ \sum_{y \in F} |y|^2 \left| F \text{ finite suborthonormal system in } E \right. \right\} = \sum_{y \in B} |y|^2 \in \mathbb{A},$$

where $B$ is any finite suborthonormal basis of $E$. (The identity follows from the proof of Lemma 2.16. Actually, $B$ may be any suborthonormal basis, but we shall not use this fact.)

By identifying $\mathbb{A} = C(\Omega)$ we can interpret $\dim_E$ as a continuous, $\mathbb{N}_0$-valued function on $\Omega$. The number $N := \| \dim_E \|_{\infty}$ is called the maximal rank of $E$. Define the idempotents

$$q_k := 1_{[\dim_E=k]} \in \mathbb{B} \quad (0 \leq k \leq N).$$

Then $(q_k)_k$ is a partition of unity of $\mathbb{B}$ with $q_N \neq 0$. We obtain a decomposition of $E$ as

$$E = q_0E \oplus q_1E \oplus \cdots \oplus q_NE$$

into its so-called homogeneous components.

Lemma 2.17. In the situation just described, either $q_k = 0$ or $E_k = q_k E$ is homogeneous of rank $k$.

Proof. Suppose $q_k \neq 0$ and let $e_1, \ldots, e_n$ be a suborthonormal basis for $E$ with $|e_1| \geq |e_2| \geq \cdots \geq |e_n|$. By definition, $kq_k = q_k \dim_E = \sum_{j=1}^n q_k|e_j|$. It follows that $q_k|e_j| = q_k$
for $1 \leq j \leq k$ and $q_k|e_j| = 0$ for $j > k$. Obviously, the system $q_k e_1, \ldots, q_k e_k$ is a 
homogeneous suborthonormal basis for $E_k$ of length $k$. □

Let $E, F$ be Kaplansky–Hilbert modules over a Stone algebra $\mathcal{A}$. For $y \in E$ and $z \in F$ 
define $A_{y,z} \in \text{Hom}(E; F)$ by

$$A_{y,z}: E \to F, \quad x \mapsto (x|z)y.$$ 

Any finite sum of homomorphisms of the form $A_{y,z}$ is called a 
**homomorphism of $\mathcal{A}$-finite rank**. Such homomorphisms can be characterized as follows.

**Proposition 2.18.** Let $E, F$ be a Kaplansky–Hilbert modules over a Stone algebra $\mathcal{A}$. For 
$A \in \text{Hom}(E; F)$ the following assertions are equivalent.

(a) $A$ is of $\mathcal{A}$-finite rank.

(b) The $\mathcal{A}$-module $\text{ran}(A)$ is contained in a finitely-generated submodule.

(c) $\text{ocl ran}(A)$ is a KH-submodule of finite rank.

(d) There are $z_1, \ldots, z_n \in F$ and a suborthonormal system $y_1, \ldots, y_n \in E$ such that

$$A = \sum_{i=1}^{n} A_{y_i, z_i}.$$ 

**Proof.** (d)$\Rightarrow$(a)$\Rightarrow$(b) is clear. (b)$\Rightarrow$(c) follows from Lemma 2.16. (d) follows from (c) 
by letting $y_1, \ldots, y_n \in F$ be a suborthonormal basis for $M := \text{ocl ran}(A)$ and applying 
Proposition 2.12 and Theorem 2.13 to find $z_1, \ldots, z_n \in E$. □

2.6. **Hilbert–Schmidt Homomorphisms.** Let $E$ and $F$ be Kaplansky–Hilbert modules 
over a Stone algebra $\mathcal{A}$. Moreover, let $\mathcal{F}$ be the family of all finite suborthonormal subsets 
of $E$. A homomorphism $A \in \text{Hom}(E; F)$ is called a **Hilbert–Schmidt homomorphism** if

$$|A|_{\text{HS}} := \sup \left\{ \left( \sum_{x \in B} |Ax|^2 \right)^{1/2} \mid B \in \mathcal{F} \right\}$$ 

exists in $\mathcal{A}_e$. We write $\text{HS}(E; F)$ for the $\mathcal{A}$-module of all $\mathcal{A}$-Hilbert–Schmidt homomorphisms 
from $E$ to $F$ and $\text{HS}(E)$ if $F = E$.

**Proposition 2.19.** Let $B$ be a fixed suborthonormal basis of $E$. Then for $A \in \text{End}(E)$ the 
following assertions are equivalent.

(a) $A \in \text{HS}(E)$.

(b) $A^* \in \text{HS}(E)$.

(c) $\sum_{x \in B} |Ax|^2$ order-converges in $\mathcal{A}$.

(d) $\sum_{x \in B} |A^* x|^2$ order-converges in $\mathcal{A}$.

In this case, $|A|_{\text{HS}}^2 = \sum_{x \in B} |Ax|^2$ and for each $T \in \text{Hom}(E)$ one has $TA, AT \in \text{HS}(E)$ 
with $|AT|_{\text{HS}}, |TA|_{\text{HS}} \leq |T||A|_{\text{HS}}$.

**Proof.** The proof is analogous to that of the case $\mathcal{A} = \mathbb{C}$, see [Gönl14, Prop. 3.2]. □

The space $\text{HS}(E)$ carries a natural KH-module structure:
Proposition 2.20. Let $E$ be a Kaplansky–Hilbert module over a Stone algebra $\mathcal{A}$. Then the mapping
\[
\text{HS}(E) \times \text{HS}(E) \to \mathcal{A}, \quad (A, B) \mapsto (A|B) := \omega\text{-lim}_{B \in \mathcal{F}} \sum_{x \in B} (Ax|Bx)
\]
turns $\text{HS}(E)$ into a Kaplansky–Hilbert module over $\mathcal{A}$, and
\[
(A|B)_{\text{HS}} = \sum_{x \in B} (Ax|Bx) \quad (A, B \in \text{HS}(E))
\]
as an order-convergent series for each suborthonormal basis $B$ of $E$.

Proof. Again, the proof is analogous to that of the case $\mathcal{A} = \mathbb{C}$, see [Gön14, Prop. 3.3 and Thm. 3.4]. □

If $y, z \in E$, then the finite-rank homomorphism $A_{y,z}$ (cf. Section 2.5) is Hilbert–Schmidt with
\[
|A_{y,z}|_{\text{HS}} = |y| \cdot |z|.
\]
Indeed, if $B$ is any suborthonormal basis of $E$, then
\[
|A_{y,z}|_{\text{HS}}^2 = \sum_{e \in B} |(e|z)y|^2 = |y|^2 \sum_{e \in B} |(z|e)|^2 = |y|^2 |z|^2
\]
by Parseval.

Lemma 2.21. Let $E$ be a Kaplansky–Hilbert module over a Stone algebra $\mathcal{A}$. Then the space of $\mathcal{A}$-finite-rank homomorphisms is order-dense in $\text{HS}(E)$.

Proof. This is again analogous to the Hilbert space case. Let $B$ be any suborthonormal basis of $E$. Define, for each $G \subseteq B$ finite, $A_G := \sum_{e \in G} A_{Ae,e} = \sum_{e \in G} (|e)Ae$. Then
\[
|A - A_G|_{\text{HS}}^2 = \sum_{e \in \mathcal{B}\setminus G} |Ae|^2 \to 0 \quad \text{as } G \nearrow B.
\]

2.7. Order-Completion and Tensor Products. Just as each pre-Hilbert space has a Hilbert space completion, each pre-Hilbert module over a Stone algebra has an “order-completion” in the following sense.

Proposition 2.22. Let $E$ be a pre-Hilbert module over a Stone algebra $\mathcal{A}$. Then there is a Kaplansky–Hilbert module $E^\sim$ and an $\mathcal{A}$-isometric homomorphism $\iota : E \to E^\sim$ with order-dense range.

Proof. The double dual $E^{**}$ of $E$ is an order-complete lattice-normed module. The mapping
\[
\iota : E \to E^{**}, \quad x \mapsto \iota(x) := (x^* \mapsto x^*(x))
\]
is $\mathcal{A}$-isometric. Indeed, $|\iota(x)| = \sup_{|x^*| \leq 1} |x^*(x)| \leq |x|$ and, by Theorem 2.13,
\[
|\iota(x)| = \sup_{|x^*| \leq 1} |x^*(x)| \geq \sup_{|y| \leq 1} |\overline{\mathcal{F}}(x)| = \sup_{|y| \leq 1} |(x|y)| = |x|
\]
for each $x \in E$. Let $E^\sim$ be the order-closure of $\iota(E)$ within $E^{**}$. Then $\iota(E)$ is an order-dense submodule of $E^\sim$. Finally, transport the $\mathcal{A}$-valued inner product from $E$ to $\iota(E)$ and then extend it (e.g. by a twofold application of Proposition 2.7) to all of $E^\sim$. □
The order-completion of a pre-Hilbert module $E$ over a Stone algebra $A$ is unique up to a canonical $A$-isometric isomorphism. We will therefore speak of the order-completion of $E$ in the following.

**Example 2.23.** Let $\Omega$ be a Stone space and let $H$ be a Hilbert space. The order-completion of the Hilbert module $C(\Omega; H)$ (see [Examples 1.2 part (1)]) can be identified with

$$ C_\#(\Omega; H) := \ell_c^\infty(\Omega; H)/\ell_0^\infty(\Omega; H). $$

Here, $\ell_c^\infty(\Omega; H)$ is the space of all bounded functions $\Omega \to H$ which are continuous on a residual set, and $\ell_0^\infty(\Omega; H)$ is the subspace of functions that vanish almost everywhere. See [Kus00b 2.3.3].

Using the order-completion, one can construct the tensor product of Kaplansky–Hilbert modules.

**Definition 2.24.** Let $E$ and $F$ be Kaplansky–Hilbert modules over a Stone algebra $A$. The algebraic tensor product $E \otimes_{\text{alg}} F$ as $A$-modules is equipped with the $A$-valued inner product

$$(\cdot, \cdot): (E \otimes_{\text{alg}} F) \times (E \otimes_{\text{alg}} F) \to A$$

defined on elementary tensors by

$$(x \otimes u, y \otimes v) := (x|y) \cdot (u|v) \quad (x \otimes u, y \otimes v \in E \otimes_{\text{alg}} F),$$

see [Lan95] pages 40–41. Its order-completion is called the **tensor product** $E \otimes F$.

See [Lan95 Chap. 4] for the tensor product of general Hilbert modules, but note that in our construction the order-completion is used instead of the norm-completion.

As in the case of ordinary Hilbert–Schmidt operators, the space $\text{HS}(E)$ can be identified with a tensor product.

**Proposition 2.25.** Let $E$ be a Kaplansky–Hilbert module over a Stone algebra $A$. Then there is a unique $A$-isometric isomorphism

$$ V: E \otimes E^* \to \text{HS}(E) $$

with $V(y \otimes \overline{z}) = A_{y,z}$ for all $y, z \in E$.

**Proof.** The mapping

$$ E \times E^* \to \text{HS}(E), \quad (y, \overline{z}) \mapsto A_{y,z} $$

is $A$-bilinear, and therefore induces an $A$-linear map $W: E \otimes_{\text{alg}} E^* \to \text{HS}(E)$ on the algebraic tensor product $E \otimes_{\text{alg}} E^*$. Moreover, for $\sum_{i=1}^n y_i \otimes \overline{z}_i \in E \otimes_{\text{alg}} E^*$ and any suborthonormal basis $B \subseteq E$

$$ \sum_{x \in B} \left| \sum_{i=1}^n A_{y_i,z_i} x^i \right|^2 = \sum_{x \in B} \sum_{i=1}^n \sum_{j=1}^n (x|z_i) (x|z_j) (y_i|y_j) = \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{x \in B} (z_j|x) z_i (y_i|y_j) \right) = \sum_{i=1}^n \sum_{j=1}^n (z_j|z_i) (y_i|y_j) = \left| \sum_{i=1}^n y_i \otimes \overline{z}_i \right|^2. $$
By Proposition 2.11 this means
\[ \left| \sum_{i=1}^{n} y_i \otimes z_i \right| = \left| \sum_{i=1}^{n} A_{y_i, z_i} \right|_{\text{HS}} \]
and therefore \( W \) induces an \( \mathcal{A} \)-isometric map \( V : [E \otimes_{\text{alg}} E^*] \to \text{HS}(E) \). By Proposition 2.7, \( V \) extends uniquely to an \( \mathcal{A} \)-isometry \( V : E \otimes E^* \to \text{HS}(E) \). Since the range of \( V \) is order-dense by Lemma 2.21 this extension is surjective and hence an isomorphism. \( \square \)

It is common to identify \( y \otimes z \) with \( A_{y,z} \) via \( V \).

3. Hilbert Bundle Representation

Similar to the representation of a unital commutative \( C^* \)-algebra as the space \( C(\Omega) \) of continuous functions on a compact Hausdorff space \( \Omega \), there is a representation of Hilbert modules over such algebras as the space of continuous sections on a Hilbert bundle over \( \Omega \). This chapter is devoted to these notions and describing the correspondence.

**Definition 3.1.** Let \( \Omega \) be a Hausdorff topological space. A (continuous) Hilbert bundle over \( \Omega \) is a topological space \( H \) together with a continuous, open surjection \( p : H \to \Omega \) with the following properties.

1) Every fiber \( H_\omega := p^{-1}(\omega) \), \( \omega \in H \), is a Hilbert space.
2) The mappings
\[ + : H \times_{\Omega} H \to H, \quad (e, f) \mapsto e + f \]
\[ \cdot : \mathbb{C} \times H \to H, \quad (\lambda, e) \mapsto \lambda e \]
\[ (\cdot | \cdot) : H \times_{\Omega} H \to \mathbb{C}, \quad (e, f) \mapsto (e | f) \]
are continuous. Here, \( H \times_{\Omega} H := \{(e, f) \in H \times H \mid p(e) = p(f)\} \).
3) For each \( \omega \in \Omega \), the sets
\[ \left\{ e \in H \mid pe \in U, \|e\| < \varepsilon \right\} \quad (\omega \in U \subseteq \Omega \text{ open, } \varepsilon > 0) \]
constitute an open neighborhood base of \( 0_\omega \in H_\omega \).

A simple, but important example for a Hilbert bundle is the following.

**Example 3.2.** Let \( \Omega \) be a Hausdorff space and \( H \) be a Hilbert space. Then the product \( \Omega \times H \) equipped with the product topology and the projection onto the first component is a continuous Hilbert bundle over \( \Omega \) called the **trivial bundle with fiber** \( H \).

Let \( p : H \to \Omega \) be a Hilbert bundle. Each (continuous) mapping \( x : O \to H \) with \( p \circ x = \text{id}_O \) for some open subset \( O \subseteq \Omega \) is called is called a local (continuous) section of \( H \). The space of all local continuous sections of \( H \) on \( O \) is denoted by \( \Gamma(O; H) \). Local sections with \( O = \Omega \) are called global sections, and we write \( \Gamma(H) := \Gamma(\Omega; H) \). One can show that for every \( e \in H \) there always exists a global section \( x \in \Gamma(H) \) with \( x(p(e)) = e \) (see, e.g., [Gie82] Theorem 3.2).

It is easy to see that \( \Gamma(H) \) is a Hilbert \( C(\Omega) \)-Module. Conversely, the following proposition—a module version of the Gelfand–Naimark representation theorem—tells that each Hilbert module is isomorphic to the module of continuous sections in a (naturally constructed) Hilbert bundle. See [DG83 Chap. 2] for a proof.
Proposition 3.3. Let \( \Omega \) be a compact space and \( E \) be a Hilbert module over \( C(\Omega) \). Then the disjoint union \( H \) of the Hilbert spaces
\[
H_\omega := E/\{ x \in E \mid |x|(\omega) = 0 \} \quad (\omega \in \Omega)
\]
equipped with the base point map \( p : H \to \Omega \) and the topology generated by the sets
\[
V(x, U, \varepsilon) := \{ e \in p^{-1}(U) \mid \|e - [x]_p(\varepsilon)\| < \varepsilon \}
\]
for \( x \in E, U \subseteq \Omega \) open and \( \varepsilon > 0 \) is a Hilbert bundle over \( \Omega \). Moreover, the mapping
\[
E \to \Gamma(H), \quad x \mapsto [\omega \mapsto [x]_\omega]
\]
is a \( C(\Omega) \)-isometric isomorphism of Hilbert modules over \( C(\Omega) \).

Suppose that \( E, F \) are Hilbert modules over \( \mathcal{A} = C(\Omega) \) and \( T \in \text{Hom}(E; F) \). Interpreting \( E = \Gamma(H) \) and \( F = \Gamma(K) \) for Hilbert modules \( H, K \) one obtains an induced bundle map
\[
T^\wedge : H \to K,
\]
which is overall continuous and restricts to uniformly bounded linear fiber maps \( T_\omega \in \mathcal{L}(H_\omega; K_\omega) \) for all \( \omega \in \Omega \) by
\[
T^\wedge e = (Tx)(\omega) \quad \text{whenever} \quad x \in E, x(\omega) = e.
\]
Conversely, if \( S : H \to K \) is a bundle map, then by
\[
(S^\vee x)(\omega) := S(x(\omega)), \quad (x \in E, \omega \in \Omega)
\]
an element \( S^\vee \in \text{Hom}(E; F) \) is defined. The mappings \( T \mapsto T^\wedge \) and \( S \mapsto S^\vee \) are mutually inverse. (See [Gie82, Summary 10.18])

One can frame this in the language of category theory: The assignments \( H \mapsto \Gamma(H) \) and \( E \mapsto H_E \) establish an equivalence of the category of Hilbert bundles and bundle maps over \( \Omega \) on one side and of Hilbert modules and bounded module homomorphisms over \( C(\Omega) \) on the other (see [DG83, Chap. 2] for more details). Consequently, as every unital commutative C*-algebra is isomorphic to a space \( C(\Omega) \), considering Hilbert bundles over compact spaces is, at least from a categorial perspective, equivalent to considering Hilbert modules over unital commutative C*-algebras. In particular, we can reformulate properties of and theorems on bundles in terms of modules and vice versa.

4. Spectral Theory of Hilbert–Schmidt Homomorphisms on KH-Modules

In this chapter we prove the spectral theorem for self-adjoint Hilbert–Schmidt homomorphisms on KH-modules.

4.1. Review of the Spectral Theorem on Hilbert Spaces. In this section we review the classical spectral theorem for self-adjoint Hilbert–Schmidt operators on a Hilbert space \( H \). We formulate and prove it in a way that can be transferred, more or less straightforwardly, to the module setting.

Let \( H \) be a Hilbert space, \( \text{HS}(H) \) the space of Hilbert–Schmidt operators on \( H \), and \( A = A^* \in \text{HS}(H) \) a self-adjoint Hilbert–Schmidt operator on \( H \). The following procedure which is commonly used in the proof of the spectral theorem for compact, self-adjoint operators shall be called the “spectral algorithm” applied to \( A \).
Define $\lambda := \|A\|$ and

$$A^\#:  \begin{cases} 
A/\|A\| & (A \neq 0) \\
0 & (A = 0).
\end{cases}$$

Then $\lambda A^\# = A$. Next, let

$$P^+ := P_{\text{fix}(A^\#)}, \quad P^- := P_{\text{fix}(-A^\#)}$$

be the orthogonal projections onto the fixed spaces of $A^\#$ and $-A^\#$, respectively. Then

$$AP^+ = \lambda P^+, \quad AP^- = -\lambda P^-$$

and so one sees that $P^+$ and $P^-$ are also the projections onto the eigenspaces $\ker(\lambda - A)$ and $\ker(-\lambda - A)$ of $A$ corresponding to $\lambda$ and $-\lambda$, respectively. Since $A$ is self-adjoint, $P^+P^- + P^-P^+ = 0$ [EFHN15, Lemma D.25]. Set $B_1 := A$, $\lambda_1 := \lambda$, $P_1^\pm := P^\pm$ and $B_2 := B_1 - \lambda_1(P_1^+ - P_1^-)$.

Then $B_2$ is again self-adjoint and Hilbert–Schmidt, and satisfies $B_2 \perp A(P^+ + P^-)$. Now repeat the procedure with $B_2$ in place of $A$, and iterate.

This “spectral algorithm” yields a sequence of pairs of projections $(P_j^+, P_j^-)$, a sequence of operators $(B_j)_j$ and a scalar sequence $(\lambda_j)_j$ with the following properties.

(i) All the occurring projections $P_j^+, P_k^-$ are pairwise orthogonal.

(ii) $P_j^+ A = AP_j^+ = \pm \lambda_j P_j^+$ for all $j \geq 1$.

(iii) All operators $B_j$ are self-adjoint and Hilbert–Schmidt.

(iv) $\lambda_j = \|B_j\|$ for all $j \geq 1$ (with $B_1 = A$).

(v) $\lambda_1 \geq \lambda_2 \geq \ldots$

(vi) For all $j \geq 1$: $\lambda_j \neq 0$ $\Rightarrow$ $\lambda_j > \lambda_{j+1}$ and $P_j^+ + P_j^- \neq 0$.

(vii) $A = B_{n+1} + \sum_{j=1}^{n} A(P_j^+ + P_j^-) = B_{n+1} + \sum_{j=1}^{n} \lambda_j(P_j^+ - P_j^-)$ ($n \in \mathbb{N}$).

Observe that this sum is an orthogonal decomposition of $A$ within $\text{HS}(H)$. Hence,

$$|A|_{\text{HS}}^2 = |B_n|_{\text{HS}}^2 + \sum_{j=1}^{n} \lambda_j^2 (|P_j^+|_{\text{HS}}^2 + |P_j^-|_{\text{HS}}^2)$$

($n \in \mathbb{N}$).

It follows that $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$ and hence $\|B_n\| \to 0$. This, in turn, implies

$$A = \sum_{j=1}^{\infty} \lambda_j(P_j^+ - P_j^-),$$

the series being convergent within $\text{HS}(H)$.

Before we can turn to the module analogue, we need to examine more closely how the projections $P_j^+$ can be obtained from the respective operator $B_j$. We only consider $P^+$, the case of $P^-$ is analogous.

Suppose that $T \in \mathcal{L}(H)$ is a contraction. Hence, by the mean ergodic theorem, $P^+ = P_{\text{fix}(T)}$ is the strong limit of the ergodic averages $C_n := \frac{1}{n}(I + T + \cdots + T^{n-1})$. If $T$ is compact, then $C_nT = TC_n \to TP^+ = P^+$ in operator norm.
To sum up: If we let
\[ p_n(z) := \frac{z}{n}(1 + z + \cdots + z^{n-1}), \]
then \( p_n(T) \to P_{\text{fix}(T)} \) in \( \mathcal{L}(H) \) whenever \( T \in \mathcal{L}(H) \) is compact with \( \|T\| \leq 1 \). Applied to our situation from above we obtain
\[ P_j^+ = \lim_{n} p_n(B_j^+) \quad \text{and} \quad P_j^- = \lim_{n} p_n(-B_j^+), \]
and the convergence is in the operator norm.

4.2. Spectral Theorem on KH-Modules. We now pass to Kaplansky–Hilbert modules and obtain the following analogue. Recall the definition of Hilbert–Schmidt homomorphisms on KH-modules from Section 2.6

Theorem 4.1 (Spectral Theorem). Let \( E \) be a KH-module over a Stone algebra \( \mathbb{A} \), and let \( A \in \text{HS}(E) \) be a self-adjoint Hilbert–Schmidt homomorphism on \( E \). Then there is a sequence \( (\lambda_j)_j \) in \( \mathbb{A} \) and orthogonal projections \( P_j^+, P_j^- \) (\( j \in \mathbb{N} \)) in \( \text{End}(E) \) such that
\[ A = \sum_{j=1}^{\infty} \lambda_j (P_j^+ - P_j^-) \]
in \( \text{HS}(E) \). Moreover, for each \( j \geq 1 \) the following assertions hold.

(i) \( \lambda_j P_j^\pm \in \text{HS}(E) \).
(ii) \( \text{ran} P_j^+ \perp \text{ran} P_j^- \) and \( \text{ran} P_j^\pm \perp \text{ran} P_k^\pm \) for each \( k \neq j \).
(iii) \( P_j^\pm A = AP_j^\pm = \pm \lambda_j P_j^\pm \).
(iv) \( \lambda_j \geq \lambda_{j+1} \).
(v) \( \lambda_j > \lambda_{j+1} \) almost everywhere on \( \text{supp}(\lambda_j) \).
(vi) \( \text{supp}(\lambda_j) = \text{supp}(P_j^+ + P_j^-) \).

The proof is completely analogous to the Hilbert space case. We fix a KH-module \( E \) over the Stone algebra \( \mathbb{A} \) and a Hilbert–Schmidt homomorphism \( A \in \text{HS}(E) \) and perform the following “spectral algorithm”:

Define \( B_1 := A, \lambda_1 := |B_1|, \) and \( B_1^\pm := \frac{B_1}{|B_1|} \) (see Section 2.1). Next, let
\[ P_1^+ := P_{\text{fix}(B_1^+)} \quad \text{and} \quad P_1^- := P_{\text{fix}(B_1^-)} \]
be the orthogonal projections onto the respective fixed spaces, see Proposition 2.15. Then let
\[ B_2 := B_1 - \lambda_1 (P_1^+ - P_1^-) \]
and iterate the procedure. The resulting sequences \( (\lambda_j)_j \) and \( ((P_j^+, P_j^-))_j \) are called the spectral decomposition of \( A \).

We shall now establish the claimed properties. First, note that \( \lambda_n = |B_n| \) and
\[ A = B_n + \sum_{j=1}^{n-1} \lambda_j (P_j^+ - P_j^-) \quad (n \in \mathbb{N}). \]

---

8This means that \( \text{supp}(\lambda_j - \lambda_{j+1}) = \text{supp}(\lambda_j) \), cf. page 17.
Moreover, since the mean ergodic theorem also holds for contractions on KH-modules (Proposition 2.15),
\[
P_j^+ x = \lim_{n \to \infty} p_n(B_j^2)x, \quad P_j^- x = \lim_{n \to \infty} p_n(-B_j^2)x \quad (x \in E),
\]
where \( p_n(z) = \frac{1}{n}(z + z^2 + \cdots + z^n) \). It follows that \( \text{supp}(P_j^+) \subseteq \text{supp}(B_j) = \text{supp}(|B_j|) = \text{supp}(\lambda_j) \).

**Lemma 4.2.** All the operators \( B_j, P_j^+, P_j^-; j \in \mathbb{N} \), are self-adjoint homomorphisms and contained in the double commutator \( \{A\}' \)' of \( A \) in \( \text{End}(E) \). In particular, they all commute with each other and with \( A \).

**Proof.** Let \( T \in \text{End}(E) \) commute with \( A \). Then it commutes with \( B_1 \). Suppose \( T \) commutes with \( B_j, P_j^\pm \) for \( j < n \). Then \( T \) commutes with \( B_n \), hence with \( B_n^\# \), hence with \( P_n^\pm \).

It is obvious that \( \text{ran} \, P_j^+ \perp \text{ran} \, P_j^- \), and hence \( Q_j := P_j^+ + P_j^- \) is again an orthogonal projection for each \( j \geq 1 \). (Indeed, it is the projection onto \( \text{fix}((B_j^\pm)^2) \).) Note that
\[
B_{j+1} = B_j - \lambda_j(P_j^+ - P_j^-) = B_j(I - Q_j).
\]
This yields \( \lambda_{j+1} = |B_{j+1}| \leq |B_j| = \lambda_j \) and, by induction,
\[
B_n = A \prod_{j<n}(I - Q_j).
\]
Since all the operators on the right-hand side commute, we obtain
\[
\text{ran} \, B_n \perp \text{ran} \, (Q_j) \quad \text{for all } j < n.
\]
This implies \( \text{ran}(B_n^\prime) \perp \text{ran}(Q_j) \) and hence also \( \text{ran}(Q_n) \perp \text{ran}(Q_j) \) for all \( j < n \). It follows that all the involved projections \( P_j^\pm \) are pairwise orthogonal.

Next, observe that from the representation
\[
B_n = A - \sum_{j<n} \lambda_j(P_j^+ - P_j^-)
\]
and the orthogonality of the projections it follows that
\[
\pm \lambda_n P_n^\pm = \pm B_n P_n^\pm = \pm A P_n^\pm.
\]
Since \( A \) is Hilbert–Schmidt, also \( \pm \lambda_j P_j^\pm \) is Hilbert–Schmidt. Thus, properties (i)–(iv) are established. It remains to show (v) and (vi) and that \( |B_n|_{\text{HS}} \to 0 \) (in order).

**Lemma 4.3.** Let \( 0 \leq \lambda \in \mathbb{A} \) and \( Q \in \text{End}(E) \) a projection such that \( \lambda Q \in \text{HS}(E) \) and \( \text{supp}(\lambda) = \text{supp}(Q) \). Then \( \lambda \leq |\lambda Q|_{\text{HS}} \).

**Proof.** Since \( \text{ran}(Q) \) is order-closed, we have \( \text{supp}(Q) = \text{supp}(\text{ran}(Q)) \) by Lemma 2.9. Take a normalized element \( e \in \text{ran}(Q) \) of maximal support \( |e| = \text{supp}(Q) \) (Lemma 2.4).

Then \( |\lambda Q|_{\text{HS}} \geq |\lambda Q e| = \lambda |e| = \lambda \) as claimed. \( \square \)
Now, note that \( B_n \perp \sum_{j<n} \lambda_j (P^+_j - P^-_j) \) in \( \text{HS}(E) \). This implies

\[
|A|_{\text{HS}}^2 = |B_n|_{\text{HS}}^2 + \sum_{j<n} |\lambda_j (P^+_j - P^-_j)|_{\text{HS}}^2.
\]

Since \( \text{HS}(E) \) is complete, \( C := \sum_{j=1}^{\infty} \lambda_j (P^+_j - P^-_j) \) exists (in order) in \( \text{HS}(E) \) and \( \lim_{n \to \infty} |B_n|_{\text{HS}} = |A - C|_{\text{HS}} \). Furthermore, suppose that (vi) is already established. Then, by Lemma 4.3 we can conclude that

\[
|A|_{\text{HS}}^2 \geq \sum_{j=1}^{\infty} |\lambda_j (P^+_j - P^-_j)|_{\text{HS}}^2 = \sum_{j=1}^{\infty} |\lambda_j Q_j|_{\text{HS}}^2 \geq \sum_{j=1}^{\infty} \lambda_j^2.
\]

It follows that \((\lambda_j)_j\) decreases to 0. But this implies \( C = A \) and hence that \(|B_n|_{\text{HS}}\) decreases to 0 as desired.

Hence, only (v) and (vi) remain to be shown. This will be done in the next section, where we employ the “bundle view” to reduce the problem to the Hilbert space case.

4.3. The “Bundle View”. As described in Section 3 we may (and do henceforth) suppose that \( \mathcal{A} = C(\Omega) \) for some Stonean space \( \Omega \) and \( E = \Gamma(H) \) for some Hilbert bundle \( H \) over \( \Omega \). Each homomorphism \( T \in \text{End}(E) \) induces fiber maps \( T_\omega \in \mathcal{L}(H_\omega) \) for each \( \omega \in \Omega \).

We intend to show that if one starts with a self-adjoint Hilbert–Schmidt homomorphism \( A \) on \( E \) and performs the “spectral algorithm” to it yielding the projections \( P_j^+ \) and the functions \( \lambda_j \), then for almost every \( \omega \) the fibre operator \( A_\omega \) is Hilbert–Schmidt in \( H_\omega \) and

\[
A_\omega = \sum_{j=1}^{\infty} \lambda_j(\omega) ((P^+_j)_\omega - (P^-_j)_\omega)
\]

is precisely the spectral decomposition of \( A_\omega \) arising from the “spectral algorithm” applied to \( A_\omega \). (Recall the topological definition of “almost everywhere” from the remarks preceding Wright’s Lemma 1.11.) From this insight we shall then infer the yet remaining statements to conclude the proof of Theorem 4.1.

The following lemma will be of utmost use in understanding Hilbert–Schmidt homomorphisms on a Hilbert module \( E \cong \Gamma(H) \) in terms of a representing Hilbert bundle. The Hilbert bundles corresponding to KH-modules are so-called complete Hilbert bundles, see [Gut93 Section 2.1]. However, it is enough for us to know that \( H \) is complete if and only if \( \Gamma(H) \) is a KH-module and the reader may take this as a definition.

Lemma 4.4. Let \( E = \Gamma(H) \) for some complete Hilbert bundle \( H \) over a Stonean space \( \Omega \) and \( A \in \text{End}(E) \). Then the following assertions hold:

(i) \( (A_\omega)^* = (A^*)_\omega \) for every \( \omega \in \Omega \).

In particular, if \( A \) is self-adjoint, then so is each \( A_\omega \).

(ii) If \( A \in \text{End}(E) \) is an orthogonal projection, then so is each \( A_\omega \).

(iii) \( |A|_\omega = \|A_\omega\| \) almost everywhere.

(iv) If \( A \in \text{HS}(E) \), then \( A_\omega \in \text{HS}(H_\omega) \) and \( \|A_\omega\|_{\text{HS}} = |A_\omega|_{\text{HS}}(\omega) \) almost everywhere.
apply the Gram–Schmidt procedure and suppose without loss of generality that
\( u_1 \) Gramian matrix is a suborthonormal system. Then

\[
|A_{\omega}v_{\omega}| = |(Ax)(\omega)| = |Ax|(\omega) \leq |A|(\omega|x|(\omega) = |A|(\omega)|v_{\omega}|
\]

It follows that \( |A_{\omega}| \leq |A|(\omega) \) for every \( \omega \in \Omega \). On the other hand, as \( |A| = \sup_{|x| \leq 1} |Ax| \), by Wright’s Lemma [1.11] one has

\[
|A|(\omega) = \sup_{|x| \leq 1} |Ax|(\omega) = \sup_{|x| \leq 1} |A_{\omega}x(\omega)| \leq |A_{\omega}| \quad \text{almost everywhere.}
\]

Conversely, fix \( \omega \in D \) and let \( v_1, \ldots, v_n \in H_\omega \) be orthonormal. Then there are sections \( x_1, \ldots, x_n \) such that \( x_i(\omega) = v_i \) for \( i = 1, \ldots, n \) (see Section [3]). By continuity, the Gramian matrix \( \langle (x_i(\cdot)|x_j(\cdot)) \rangle_{i,j} \) is invertible in a neighborhood of \( \omega \). Hence, one can apply the Gram–Schmidt procedure and suppose without loss of generality that \( x_1, \ldots, x_n \) is a suborthonormal system. Then

\[
\sum_{i=1}^{n} \|A_{\omega}v_i\|^2 = \sum_{i=1}^{n} \|(Ax_i)(\omega)\|^2 \leq |A|_{\text{HS}}(\omega)^2.
\]

Therefore, \( A_{\omega} \in \text{HS}(H_\omega) \) and \( |A_{\omega}|_{\text{HS}} \leq |A|_{\text{HS}}(\omega) \). \( \square \)

Now let us start again with the Hilbert–Schmidt operator \( A \in \text{HS}(E) \) and see what the algorithm yields. By (iii) of the preceding lemma, with \( \lambda := |A| \) we have

\[
\lambda(\omega) = \|A_{\omega}\| \quad \text{and} \quad (A^\sharp)_{\omega} \left( \frac{A}{|A|} \right)_{\omega} = (A_{\omega})^\sharp
\]

(almost everywhere). Although \( A^\sharp \) may not be a Hilbert–Schmidt operator, \( (A_{\omega})^\sharp \) is Hilbert–Schmidt (a.e.). Then \( p_n(A^\sharp)_{\omega} = p_n(A^\sharp_{\omega}) \to P_{\text{fix}(A^\sharp_{\omega})} \) in operator norm (a.e.). By (iii) of Lemma [4.4] again and by Lemma [1.12], \( p_n(A^\sharp) \) is convergent in \( \text{End}(E) \), and since the limit must be \( P^+ \), it follows that

\[
(P^+)_{\omega} = P_{\text{fix}(A^\sharp_{\omega})} \quad \text{(a.e.)}
\]

Likewise, \( (P^-)_{\omega} = P_{\text{fix}(A^\sharp_{\omega})} \) and \( Q_{\omega} = P_{\text{fix}(A^\sharp_{\omega})} \) (a.e.).
Since the procedure only involves countably many steps and countable intersections of residual sets in $\Omega$ are still residual, we conclude what we aimed at: There is a residual set $D$ in $\Omega$ such that for each $\omega \in D$ one has

$$\lambda_n(\omega) = |(B_n)\omega|, \quad (B_n)\omega = A_\omega - \sum_{j<n} \lambda_j(\omega)((P_j^+)\omega - (P_j^-)\omega)$$

for all $n \in \mathbb{N}$ and

$$A_\omega = \sum_{j=1}^{\infty} \lambda_j(\omega)((P_j^+)\omega - (P_j^-)\omega)$$

is the spectral decomposition of the Hilbert–Schmidt operator $A_\omega$.

By what we have seen in Section 4.1, $\lambda_j(\omega) > \lambda_{j+1}(\omega)$ whenever $\lambda_j(\omega) \neq 0$ and $\omega \in D$. But this just means $\lambda_j > \lambda_{j+1}$ almost everywhere on supp($\lambda_j$), i.e., assertion (v) of Theorem 4.1. Finally, again by what we have seen in Section 4.1 if $\omega \in D$ and $\lambda_j(\omega) = \|(B_n)\omega\| > 0$, then $(Q_n)\omega \neq 0$. And this is precisely assertion (vi) of Theorem 4.1.

With this observation, the proof of Theorem 4.1 is complete. \hfill $\blacksquare$

**Remark 4.5.** Theorem 4.1 can be recognized to be a special case of the spectral theorem for self-adjoint cyclically compact homomorphisms as proved by Gönülü in [Gön16]. In fact, it follows from [Gön14, Thm. 2.2, (iii)$\Rightarrow$(i)] that Hilbert–Schmidt homomorphisms (in our sense) are cyclically compact. (Gönülü himself operates with a different definition of a Hilbert–Schmidt homomorphism that presupposes cyclic compactness and rests on the spectral theorem, cf. [Gön14, Def. 3.1].)

Our approach to Theorem 4.1 has the advantage that it completely avoids the notion of cyclical compactness. Moreover, having made explicit the steps of the “spectral algorithm” in the proof is useful for proving an “equivariant” version in Part II (Proposition 6.5).

Finally, it seems worthwhile noting that our proof can easily be modified to cover all self-adjoint cyclically compact homomorphisms. Indeed, one only needs to replace $|\cdot|_{HS}$ by the operator lattice-norm and to employ that if $A \in \text{End}(E)$ is cyclically compact then almost every fiber operator $A_\omega$ is compact. (The latter follows from the fact that each cyclically compact homomorphism is the order-limit of a net of finite-rank homomorphisms, see [Kus00b, Thm. 8.5.6]). As a consequence we find that a homomorphism $A \in \text{End}(E)$ is cyclically compact if and only if almost every fiber operator $A_\omega$ is compact—a quite satisfying result.

**Notes and Comments**

Stonean (sometimes: Stonian) spaces are as old as Stone’s famous representation theorem from [Sto37]. Their central role in the representation theory of vector lattices was recognized soon after, e.g. in representation theorems by Kakutani [Kak41a, Kak41b]. See also Schaefer’s monograph [Sch74, Chap. 2] and the more recent book [GvK16].

In [Kap51], Kaplansky introduces the concept of an (in general non-commutative) $AW^*$-algebra. For commutative algebras, this means that the Boolean algebra of idempotents is complete and generates it as a $C^*$-algebra. Representing $A$ as $C(\Omega)$ for some compact
space it is routine to check that $A$ is a $\text{AW}^\ast$-algebra in this sense if and only if $\Omega$ is Stonean, i.e., $A$ is a Stone algebra. (As far as we know, the term “Stone algebra” was introduced by Wright in [Wri69a]. It is also used by Kusraev in [Kus00b].)

In [Kap53] Kaplansky introduced the concepts of Hilbert $C^\ast$-modules and Hilbert $\text{AW}^\ast$-modules, nowadays also called Kaplansky–Hilbert modules, and proved several analogues of Hilbert space results for them. In [DP63] Deckard and Pearcy observed that matrices with entries in a Stone algebra $A$ can be $A$-unitarily triangularized. This and its sequel [DP64] appear to be the first results on the spectral theory of homomorphisms on KH-modules. Wright continued this work in [Wri69b] and coined their modern name.

Kusraev and Kutateladze incorporated the theory of KH-modules into their systematic investigation of ordered functional-analytic structures, in particular of lattice-normed spaces, and operators thereon. Chapter 7 of [Kus00b] contains a brief introduction to Kaplansky–Hilbert modules. The definition there is ours (with “bo-convergence” being our “order-convergence”).

During the 1970s, the known results about Kaplansky–Hilbert modules were recognized as being embeddable into a theory called “Boolean-valued analysis” (BVA), which is the application of Boolean-valued models of set theory to analysis. Originally conceived by Takeuti [Tak78, Tak79a, Tak79b] it was continued by Ozawa [Oza83, Oza90] and, among others, by Kusraev and Kutateladze, see [Kus00b, Chap. 8] and [KK99].

The upshot of this theory, which has a major model-theoretic thrust, is that Kaplansky–Hilbert modules are simply the Hilbert spaces in a Boolean-valued universe. As the Boolean-valued universe is a model of ZFC, Hilbert space results can be re-interpreted in this new model and yield in this way valid statements about KH-modules (“transfer principle”).

Another, but essentially equivalent approach, is more recent and comes under the name of conditional analysis or conditional set theory, see, e.g., [DJKK16, CKV15, FKV09]. It is model-theory free but works with slightly different objects, essentially completions with respect to “mixing”.

We acknowledge these approaches, but we also feel that for most mathematicians (including ourselves), the model-theoretic path remains arcane and—in our topological set-up—the conditional world approach would require working with rather unfamiliar objects. Unfortunately, nowhere in the literature could we find a coherent account of the KH-module theory that would collect all the facts we needed and with minimal background requirements. Therefore, we decided to give such an account here.

In particular, we present a new proof of the spectral theorem for self-adjoint Hilbert–Schmidt homomorphisms on KH-modules. This theorem can also be derived from the spectral theorem for self-adjoint cyclically compact operators, cf. Remark 4.5. Such operators were introduced by Kusraev in 1983, see [Kus86], and are called “stably compact” operators in conditional analysis, see [IZ17]. In [Kus00a], a singular-value representation was established by BVA-techniques. An explicit formulation and a model theory-free proof of the spectral theorem for self-adjoint cyclically compact operators were given by Gönülü [Gön16]. However, cyclical compactness is a technically involved concept which fully reveals its naturality only in the light of the Boolean/conditional version of
the notions of "natural number" and "sequence". Our alternative proof of the spectral theorem avoids cyclical compactness altogether and is instead based on the representation of a KH-module as the space of continuous sections of a Hilbert bundle.

The duality between modules and bundles goes back to the famous result of Serre–Swan, see [Swa62]. Godement (in [God51]) as well as Douady and Dixmier (in [DD63]) seem to be among the first to systematically study continuous Hilbert bundles, although similar ideas date back further. Such bundles then became increasingly important in representation theory during the 1970s. In [HK77], Hofmann and Keimel summarize

the connections and equivalences between the different approaches to “relative functional analysis” based on Banach bundles, modules, and sheaves, respectively. The monographs by Gierz [Gie82] and Dupré–Gillette [DG83] appear to be the best references on Banach bundles and their connection with modules, the latter stressing the categorial equivalence of both worlds. Gutman [Gut93] investigates this correspondence in the context of lattice-normed spaces and, in particular, characterizes those Banach bundles that correspond to Kaplansky–Banach modules [Gut93, Thm. 2.1.1]. Proposition 3.3 for KH-modules has been obtained by Takemoto in [Tak73].

It turns out that when dealing with KH-modules it is often easier to prove things directly than appealing to a representation as sections in a bundle. We use the latter only at one point in the proof of the spectral theorem (Section 4.3).

Besides topological Hilbert bundles, there are of course also measurable Hilbert bundles and a connection of modules with spaces of measurable sections. Such Hilbert bundles were used, e.g. by Zimmer [Zim76] and Glasner [Gla03] in the context of the Furstenberg–Zimmer theory. They also appear, as direct integrals of Hilbert spaces, in the representation theory of von Neumann algebras [Tak02, Chap. VI.8]. Typically, this happens under the separability restriction and for Borel measure spaces only. By working in a topological setting, we avoid this restriction.

Part II. Covariant Unitary Group Representations on KH-Modules

In this part, we define (covariant) unitary group representations on Kaplansky–Hilbert modules and then prove a version of the Jacobs–de Leeuw–Glicksberg decomposition for such representations. This is done by using the spectral theorem to establish a correspondence between finitely-generated invariant submodules and intertwining Hilbert–Schmidt homomorphisms.

As stated in the introduction, $G$ is an arbitrary, but fixed group in this (and the next) chapter.

5. Covariant Group Representations on Hilbert Modules

5.1. KH-Dynamical Systems. Let $A$ a fixed commutative unital $C^*$-algebra. A $G$-representation on $A$ is a unital representation $S: G \to \text{Aut}(A)$, $t \mapsto S_t$.
of \( G \) as unital \(*\)-automorphisms of \( \mathbb{A} \) (as a \( C^* \)-algebra). It then follows that

\[
|S_t f| = S_t|f| \quad \text{for each } f \in \mathbb{A} \text{ and } t \in G,
\]
i.e., each \( S_t \) is a lattice homomorphism. Indeed, if we represent \( \mathbb{A} = C(\Omega) \) for some compact space \( \Omega \), then there is a unique group homomorphism

\[
\varphi : G \to \text{Homeo}(\Omega), \quad t \mapsto \varphi_t
\]
such that \( S_t f = f \circ \varphi_t^{-1} \), see [EFHN15, Thm. 4.13]. It follows that \( S_t \) commutes with all pointwise defined operations, like taking square roots for instance.

Let \( S : G \to \text{Aut}(\mathbb{A}) \) be a \( G \)-representation on \( \mathbb{A} \), and let \( E \) be a Hilbert module over \( \mathbb{A} \). We want to consider \( S \)-covariant unital representations of \( G \) on \( E \). For this, we need the following concept.

**Definition 5.1.** Let \( E \) be a pre-Hilbert module over the commutative unital \( C^* \)-algebra \( \mathbb{A} \), and let \( S \in \text{Aut}(\mathbb{A}) \). A bounded linear operator \( T \in \mathcal{L}(E) \) is called an \textbf{\( S \)-homomorphism} if

\[
T(fx) = Sf \cdot Tx \quad (x \in E, \ f \in \mathbb{A}).
\]

An \( S \)-homomorphism \( T \) is called an \textbf{\( S \)-isometry} if

\[
(Tx|Ty) = S(x|y) \quad \text{for all } x, y \in E,
\]
and it is called \textbf{\( S \)-unitary} if it is \( S \)-isometric and invertible.

**Remark 5.2.** An \( S \)-homomorphism \( T \in \mathcal{L}(E) \) is simply a homomorphism from \( E \) to \( E_S \), where \( E_S \) is the pre-Hilbert module that arises from \( E \) by changing the module multiplication to

\[
\mathbb{A} \times E \to E, \quad (f, x) \mapsto f \cdot_S x := Sf \cdot x
\]
and the \( \mathbb{A} \)-valued inner product to

\[
E \times E \to \mathbb{A}, \quad (x, y) \mapsto (x|y)_S := S^{-1}(x|y).
\]

Likewise, an \( S \)-isometry \( T \) is an isometry from \( E \) to \( E_S \). This change of perspective allows to transfer statements about bounded \( \mathbb{A} \)-homomorphisms between pre-Hilbert modules to statements about \( S \)-homomorphisms.

We are now able to define covariant representations.

**Definition 5.3.** Let \( S : G \to \text{Aut}(\mathbb{A}) \) be a \( G \)-representation on \( \mathbb{A} \), and let \( E \) be a Hilbert module over \( \mathbb{A} \). Then a \textbf{(covariant) unitary \( G \)-representation} on \( E \) over \( S \) is a mapping

\[
T : G \to \mathcal{L}(E), \quad t \mapsto T_t
\]
with the following properties.

(i) \( T_e = 1 \) and \( T_s T_t = T_{st} \) for all \( s, t \in G \) \quad (group homomorphism).

(ii) \( T_t \) is an \( S_t \)-unitary \( S_t \)-homomorphism for each \( t \in G \) \quad (\( S \)-covariance).

If \( T \) is an \( S \)-covariant representation on \( E \), then the tuple \((\mathbb{A}, S; E; T)\) is called a \textbf{\( G \)-system}, for short. If, in addition, \( \mathbb{A} \) is a Stone algebra and \( E \) is a Kaplansky–Hilbert module over \( \mathbb{A} \), then the \( G \)-system is called a \textbf{Kaplansky–Hilbert \( G \)-system}. And if \( G \) is understood, a Kaplansky–Hilbert \( G \)-system is simply called a \textbf{KH-dynamical system}.
Remarks 5.4. (1) If \( S: G \to \text{Aut}(\mathcal{A}) \) is a \( G \)-representation on \( \mathcal{A} \), then the triple \((\mathcal{A}, G, S)\) is commonly called a C\(^*\)-dynamical system, see [Ped79, 7.4.1]. This is one reason for using the term “KH-dynamical system”. The second is the application to extensions of \( G \)-dynamical systems, see Chapter 8 below and confer also [KS21, Def’s 3.5 and 3.12].

(2) We recall from [Ped79, 7.4.8] that an \( S \)-covariant unital representation of a C\(^*\)-dynamical system \((\mathcal{A}, G, S)\) on a Hilbert space \( H \) is a unitary representation \( T \) of \( G \) on \( H \) together with a unital \(*\)-representation \( \pi: \mathcal{A} \to \mathcal{L}(H) \) such that

\[
\pi(S_t f) T_t = T_t \pi(f) \quad (t \in G, \ f \in \mathcal{A}).
\]

Then \( H \) is an \( \mathcal{A} \)-module via \( f \cdot x := \pi(f)x \). Using module notation, the identity above becomes

\[
S_t f \cdot T_t x = T_t (f \cdot x) \quad (t \in G, \ f \in \mathcal{A}, \ x \in H),
\]

i.e., \( T_t \) is an \( S_t \)-homomorphism. It is trivially \( S_t \)-isometric, as the inner product maps into \( \mathbb{C}1 \subseteq \mathcal{A} \). Hence our notion of covariant representation generalizes the classical one from Hilbert spaces to Hilbert modules.

Let us list some examples.

Examples 5.5.

(1) Let \( \varphi: \Omega \to \Omega \) be a homeomorphism on a compact space \( \Omega \), \( H \) a Hilbert space and \( \Phi \in \mathcal{L}(H) \). Then the Koopman operator \( S_\varphi \in \mathcal{L}(C(\Omega)) \) given by \( S_\varphi f := f \circ \varphi^{-1} \) for every \( f \in C(\Omega) \) is a \(*\)-automorphism and

\[
C(\Omega; H) \to C(\Omega; H), \quad f \mapsto \Phi \circ f \circ \varphi^{-1}
\]

is an \( S_\varphi \)-homomorphism.

(2) More generally, suppose that \( \varphi: G \to \text{Homeo}(\Omega) \) is a representation of a group \( G \) as homeomorphisms of \( \Omega \), and \( \Phi: G \to \mathcal{L}(H) \) is a unitary representation of \( G \) on \( H \). Then

\[
S: G \to \text{Aut}(\mathcal{A}), \quad S_t f := f \circ \varphi_t^{-1} = f \circ \varphi_t^{-1},
\]

is a representation of \( G \) on \( \mathcal{A} \) as \(*\)-automorphisms, and with

\[
T: G \to \mathcal{L}(C(\Omega; H)), \quad T_t f := \Phi_t \circ f \circ \varphi_t^{-1}
\]

we obtain a \( G \)-system \( (C(\Omega), S; C(\Omega; H), T) \).

(3) If \( \Omega \) is a Stone space, then from (2) one obtains a KH-dynamical system by passing to the completion (see below). Other examples of KH-dynamical systems will be studied in Part III, see in particular Section 7.2.

5.2. Standard Constructions. Suppose that \( S \in \mathcal{L}(\mathcal{A}) \) is a \(*\)-automorphism of a Stone algebra \( \mathcal{A} \) and \( T \in \mathcal{L}(E) \) an \( S \)-unitary on a pre-Hilbert module \( E \) over \( \mathcal{A} \). Then there is a unique \( S \)-unitary operator \( T^{\sim} \in \mathcal{L}(E^{\sim}) \) on the order-completion \( E^{\sim} \) with \( T^{\sim}|_E = T \).

More generally, let \((\mathcal{A}, S; E, T)\) be a \( G \)-system and \( \mathcal{A} \) be a Stone algebra. Then by extending each operator \( T_t \) to the order completion \( E^{\sim} \) of \( E \) we obtain a KH-dynamical system \((\mathcal{A}, S; E^{\sim}, T)\), called the (order-)completion of the original system.
Example 5.6. In the situation of \[\text{Examples 5.5(2)}\] suppose that \(\Omega\) is a Stonean space. Then the \(G\)-system \((C(\Omega), S; C(\Omega; H), T)\) determines a KH-dynamical system
\[(C(\Omega), S; C_G(\Omega; H), T),\]
(cf.\[\text{Example 2.23}\]) where \(T_t\) is again defined as \(T_t f := \Phi_t \circ f \circ \varphi_t^{-1}\) (on representatives \(f\)).

Let \((A, S; E, T)\) be a KH-dynamical system. A subset \(F \subseteq E\) is \(T\)-invariant if \(T_t(F) \subseteq F\) for every \(t \in G\). If \(F\) is a \(T\)-invariant KH-submodule of \(E\), then \((A, S; F, T|_F)\) with
\[T|_F : G \to \mathcal{L}(F), \quad t \mapsto T|_F\]
is a KH-dynamical system. It is called a subsystem of \((A, S; E, T)\).

Let \((A, S; E, T)\) be a KH-dynamical system. Then the system \((A, S; E^*, \overline{T})\) with
\[\overline{T}_t x := \overline{T_t x} \quad (x \in E^*, \ t \in G)\]
defined using the Riesz–Fréchet identification from \[\text{Theorem 2.13}\] is called the dual system\[\footnote{Also the term contragredient system is used.}\].

Let \((A, S; E, T)\) and \((A, S; F, R)\) be KH-dynamical systems. Then their tensor product is \((A, S; E \otimes F, T \otimes R)\), where
\[T \otimes R : G \to \mathcal{L}(E \otimes F), \quad t \mapsto T_t \otimes R_t,\]
with \(T_t \otimes R_t\) being the unique extension (\[\text{Proposition 2.7}\]) of the linear operator on \(E \otimes_{\text{alg}} F\) given by
\[(T_t \otimes R_t)(x \otimes y) = T_t x \otimes R_t y\]
for \(x \in E, y \in F\) and \(t \in G\).

6. The Decomposition Theorem

In this chapter, we prove the main decomposition theorem for \(G\)-systems. Central to the proof is a re-examination of the spectral theorem (\[\text{Theorem 4.1}\]) for Hilbert–Schmidt homomorphisms that intertwine the dynamics.

Before we go in medias res, let us state some properties of \(G\)-systems.

Lemma 6.1. Let \((A, S; E, T)\) be a Kaplansky–Hilbert \(G\)-system. Then the following assertions hold:

(i) \(S_t |x|^2 = |T_t x|^2\) and \(S_t |x| = |T_t x|\) for all \(t \in G\) and \(x \in E\).
(ii) If \(x \in E\) is normalized, then so is \(T_t x\) for all \(t \in G\).
(iii) Each \(T_t\) maps the unit ball \(\{x \in E \mid |x| \leq 1\}\) bijectively onto itself.
(iv) \(T_t M^\perp = (T_t M)^\perp\) for all \(t \in G\) and \(M \subseteq E\).
(v) If \(\mathcal{B}\) is a suborthonormal subset (basis) of \(E\), then so is \(T_t \mathcal{B}\) for each \(t \in G\).
(vi) If \(M \subseteq E\) is \(T\)-invariant, then so are \(M^\perp\) and \(\text{ocl span}_A(M)\).

Proof. The first assertion of (i) is clear and the second follows from the first since \(S_t |x|^2 = (S_t |x|)^2\). (ii) follows from (i) as \(S_t\) must map idempotents to idempotents. (iii) follows since, by (i), \(|T_t x| = S_t |x| \leq S_t 1 = 1\) whenever \(|x| \leq 1\). (iv) is obvious and (v) follows from (ii) and (iii). Finally, (v) follows from (iv) and \[\text{Lemma 1.13}\].

\[\square\]
6.1. **Equivariant Spectral Theory.** Let \((\mathcal{A}, S; E, T)\) be a Kaplansky–Hilbert \(G\)-system. The fixed algebra

\[
\text{fix}(S) := \text{fix}_A(S) := \bigcap_{t \in G} \text{fix}(S_t) = \{ f \in \mathcal{A} | S_t f = f \text{ for all } t \in G \}
\]

is an order-closed unital \(\ast\)-subalgebra of \(\mathcal{A}\), and hence a Stone algebra in its own right. Its complete Boolean algebra of idempotents is denoted by \(\mathbb{B}_S\). The suprema (infima) in \(\text{fix}(S)\) and \(\mathcal{A}\) of a bounded subset of \(\text{fix}(S)\) coincide (by [EFHN15, Thm. 7.23] and order-closedness).

On the other hand, the fixed module

\[
\text{fix}(T) := \text{fix}_E(T) := \bigcap_{t \in G} \text{fix}(T_t) = \{ x \in E | T_t x = x \text{ for all } t \in G \}
\]

is an order-closed \(\text{fix}(S)\)-submodule of \(E\). Even more, it is a KH-module over \(\text{fix}(S)\) with the induced inner product. (Indeed, if \(x, y \in \text{fix}(T)\), then \(S_t(x|y) = (T_t x|T_t y) = (x|y)\) and hence \((x|y) \in \text{fix}(S)\).) The notions of order-convergence in \(\text{fix}(T)\) and in \(E\) coincide.

**Definition 6.2.** Let \((\mathcal{A}, S; E, T)\) be a Kaplansky–Hilbert \(G\)-system. A homomorphism \(A \in \text{End}(E)\) is \(T\)-intertwining if \(T_t A = A T_t\) for all \(t \in G\).

The set of \(T\)-intertwining homomorphisms is denoted by \(\text{End}_T(E)\). It is a (even strongly) order-closed \(\ast\)-subalgebra and a \(\text{fix}(S)\)-submodule of \(\text{End}(E)\).

Actually, it is a Kaplansky–Banach module over \(\text{fix}(S)\) and the notions of order-convergence in \(\text{End}_T(E)\) and in \(\text{End}(E)\) coincide. This is a consequence of the following lemma.

**Lemma 6.3.** Let \((\mathcal{A}, S; E, T)\) be a Kaplansky–Hilbert \(G\)-system and \(A \in \text{End}_T(E)\). Then \(|A| \in \text{fix}(S)\).

**Proof.** Let \(A \in \text{End}_T(E)\) and \(t \in G\). Since, by Lemma 6.1, \(T_t\) maps the (order) unit ball \(\{ x \in E | |x| \leq 1 \} \) bijectively onto itself,

\[
S_t |A| = S_t \sup\{|Ax| | |x| \leq 1\} = \sup\{|S_t Ax| | |x| \leq 1\} = \sup\{|AT_t x| | |x| \leq 1\} = \sup\{|Ay| | |y| \leq 1\} = |A|
\]

and hence \(|A| \in \text{fix}(S)\). \(\square\)

Here is another interpretation of \(\text{End}_T(E)\). It is easily checked that \(T_t A T_{t^{-1}}\) is a bounded homomorphism on \(E\) whenever \(A\) is. Hence we obtain an implemented (covariant) representation

\[
\mathcal{T} : G \to \text{End}(\text{End}(E)), \quad \mathcal{T}_t(A) := T_t A T_{t^{-1}} \quad (t \in G).
\]

Clearly, \(\text{End}_T(E) = \text{fix}(\mathcal{T})\).

Recall that \(\text{HS}(E)\) is a KH-module in its own right. The implemented representation restricts to a covariant representation on \(\text{HS}(E)\), as the following lemma shows. Its proof is straightforward.

\[\]
Lemma 6.4. Let \((\mathcal{A}, S; E, T)\) be a Kaplansky–Hilbert \(G\)-system and let \(t \in G\). Then 
\[ T_t(A) = T_t AT_t^{-1} \in \text{HS}(E) \text{ and } (T_t(A)|T_t(B))_{\text{HS}} = S_t(A|B)_{\text{HS}} \text{ for all } A, B \in \text{HS}(E). \]

Note that the fixed space \((\text{HS}(E))\) of this implemented representation is simply 
\[ \text{fix}_{\text{HS}(E)}(T) = \text{HS}_T(E) \coloneqq \text{End}_T(E) \cap \text{HS}(E). \]

We now look at the “equivariant version” of the spectral theorem.

Proposition 6.5. Let \((\mathcal{A}, S; E, T)\) be a Kaplansky–Hilbert \(G\)-system and let \(A \in \text{HS}_T(E)\) be self-adjoint. Furthermore, let 
\[ A = \sum_{j=1}^{\infty} \lambda_j (P_j^+ - P_j^-) \]
be the spectral decomposition of \(A\) (resulting from the “spectral algorithm” applied to \(A\)). Then \(\lambda_j \in \text{fix}(S)\) and \(P_j^+ \in \text{End}_T(E)\) for each \(j \in \mathbb{N}\).

Proof. We use the notation of Section 4.2. Since \(B_1 = A \in \text{End}_T(E)\) we obtain \(\lambda_1 = |B_1| \in \text{fix}(S)\) by Lemma 6.3. But then also \(B_1^\varepsilon = |B_1^\varepsilon| \in \text{End}_T(E)\) since \(|B_1^\varepsilon| \in \text{End}_T(E)\) for each \(\varepsilon > 0\) and \(\text{End}_T(E)\) is order-closed in \(\text{End}(E)\). This shows that \(\text{fix}(B_1^\varepsilon)\) and \(\text{fix}(-B_1^\varepsilon)\) are \(T\)-invariant, and hence also the orthogonal complements \(\text{fix}(B_1^\varepsilon)^{\perp}\) and \(\text{fix}(-B_1^\varepsilon)^{\perp}\) are \(T\)-invariant, see Lemma 6.1 (vi). This shows \(P_1^+, P_1^- \in \text{End}_T(E)\). But then also \(B_2 = A - \lambda_1(P_1^+ - P_1^-) \in \text{End}_T(E)\), and the claim follows by induction. \(\square\)

6.2. Main Theorem. Let \((\mathcal{A}, S; E, T)\) be a KH-dynamical system. From this we build the dual system \((\mathcal{A}, S; E^*, T)\) and then the tensor product system \((\mathcal{A}, S; E \otimes E^*, T \otimes T)\) as described in Section 2.7.

The following lemma relates the tensor product system with the implemented system on \(\text{HS}(E)\).

Lemma 6.6. Let \((\mathcal{A}, S; E, T)\) be a KH-dynamical system. Then the tensor product system \((\mathcal{A}, S; E \otimes E^*, T \otimes T)\) and the implemented system \((\mathcal{A}, S; \text{HS}(E), T)\) are isomorphic via the unitary isomorphism 
\[ V: E \otimes E^* \to \text{HS}(E), \quad V(y \otimes z) = (x \mapsto (x|z)y) \]
described in Proposition 2.25. Via this isomorphism, \(\text{fix}(T \otimes T) \cong \text{HS}_T(E)\).

Proof. For \(t \in G\) and \(x, y, z \in E\) one has 
\[ T_t V(y \otimes z)x = T_t((x|z)y) = S_t(x|z) \cdot T_t y = (T_t x|T_t z) T_t y = V((T_t y \otimes T_t z) T_t x) = V((T \otimes T)(y \otimes z)) T_t x. \]
This shows \(T_t V(u) = V((T_t \otimes T_t) u) T_t\) for \(u = y \otimes z\). The claim then follows by order-density of \(E \otimes_{\text{alg}} E^*\) in \(E \otimes E^*\). \(\square\)

The next lemma tells that finitely-generated invariant KH-submodules are in one-to-one correspondence with \(T\)-intertwining orthogonal projections of finite rank.
Lemma 6.7. Let \((\mathcal{A}, S; E, T)\) be a KH-dynamical system, \(M = \text{ran}(P) \subseteq E\) a KH-submodule with associated orthogonal projection \(P\). Then \(M\) is \(T\)-invariant if and only if \(P\) is \(T\)-intertwining, and \(M\) is of finite rank if and only if \(P\) is a finite-rank homomorphism if and only if \(P \in \text{HS}(E)\).

Proof. Apply Lemma 6.1 for the first assertion. The first part of the second is obvious. For the final equivalence, suppose that \(P \in \text{HS}(E)\). Then, for every finite suborthonormal system \(\mathcal{B} \subseteq M\), \(\sum_{e \in \mathcal{B}} |e|^2 = \sum_{e \in \mathcal{B}} |Pe|^2 \leq |P|_{\text{HS}}^2\). By Lemma 2.16, \(M\) is of finite rank. □

Recall from Proposition 2.12 that if \(M\) is a finite-rank KH-submodule of \(E\) and \(\mathcal{B} = \{e_1, \ldots, e_n\}\) is a suborthonormal basis for \(M\), then the associated orthogonal projection is

\[P = \mathcal{V}u_\mathcal{B}, \quad \text{where} \quad u_\mathcal{B} := \sum_{j=1}^n e_j \otimes \overline{e_j}.\]

By the previous two lemmas, \(M\) is \(T\)-invariant if and only if \(u_\mathcal{B} \in \text{fix}(T \otimes \overline{T})\).

Recall from Section 2.5 that a suborthonormal system \(e_1, \ldots, e_n\) is homogeneous if \(|e_1| = \cdots = |e_n|\); and KH-submodule \(M\) of finite rank is called homogeneous if it has a homogeneous suborthonormal basis. Finally, recall that a KH-submodule \(M\) of finite rank has a canonical decomposition into homogeneous KH-submodules.

Lemma 6.8. Let \((\mathcal{A}, S; E, T)\) be a KH-dynamical system and \(M \subseteq E\) a \(T\)-invariant KH-submodule of finite maximal rank \(N \in \mathbb{N}\).

(i) If \(M\) is homogeneous with homogeneous suborthonormal basis \(e_1, \ldots, e_N\). Then \(|e_1| = \cdots = |e_N| = \text{supp}(M) \in \text{fix}(S)\).

(ii) Let \(M = q_0M \oplus q_1M \oplus \cdots \oplus q_NM\) be the canonical decomposition of \(M\) into homogeneous KH-submodules. Then each \(q_k \in \text{fix}(S)\) and each \(q_kM\) is \(T\)-invariant.

Proof. (i) As each \(T_i\) is bijective, \(\text{supp}(M) \in \text{fix}(S)\). But \(|e_1| = \cdots = |e_N| = \text{supp}(M)\) is clear. (ii) Recall from Section 2.5 the definition of the dimension of \(M\) as \(\dim_M := \sum_{y \in \mathcal{B}} |y|^2\), where \(\mathcal{B}\) is any finite suborthonormal basis for \(M\). As \(M\) is \(T\)-invariant and each \(T_i\) maps a suborthonormal basis onto another suborthonormal basis, \(\dim_M \in \text{fix}(S)\). It follows that each \(q_k \in \text{fix}(S)\) (cf. its definition in Section 2.5) and hence \(q_kM\) is \(T\)-invariant. □

We are now ready to state and prove the main result of this article. (Recall the definition of \(u_\mathcal{B}\) from above.)

Theorem 6.9 (Decomposition Theorem). Let \((\mathcal{A}, S; E, T)\) be a KH-dynamical system. Then

\[\text{fix}(T \otimes \overline{T}) = \text{ocl span}_{\text{fix}(S)}(U).\]

where \(U = \{u_\mathcal{B} | \mathcal{B} \subseteq E \text{ finite, homogeneous, suborthonormal}\} \cap \text{fix}(T \otimes \overline{T})\).
Furthermore, $E$ decomposes orthogonally as $E = E_{\text{ds}} \oplus E_{\text{wm}}$ into $T$-invariant KH-submodules, where

$$E_{\text{ds}} = \text{ocl} \bigcup \{ M \subseteq E \mid M \text{ finitely-generated, } T \text{-invariant submodule} \}$$

$$= \text{ocl} \sum \{ M \subseteq E \mid M \text{ homogeneous } T \text{-invariant KH-submodule of finite rank } \}$$

$$= \text{ocl} \sum \{ \text{ran}(A) \mid A \in \text{HS}_T(E) \} \quad \text{and}$$

$$E_{\text{wm}} = \left\{ x \in E \mid x \otimes \overline{T} \perp \text{fix}(T \otimes \overline{T}) \right\}.$$  

The spaces $E_{\text{ds}}$ and $E_{\text{wm}}$ are called the **discrete spectrum part** and **weakly mixing part**, respectively.

The first statement of Theorem 6.9 is the KH-analogue of the “Key Lemma” of the Introduction. By Lemma 6.8, it is equivalent to the following result.

**Lemma 6.10.** Let $(\mathcal{A}, S; E, T)$ be a KH-dynamical system. Then

$$\text{HS}_T(E) = \text{ocl span}_{\text{fix}(S)} \left\{ P \in \text{HS}_T(E) \mid P \text{ orthogonal projection onto } \right.$$ 

a homogeneous finite-rank submodule \}

**Proof.** Fix $A \in \text{HS}_T(E)$. We need to show that one can approximate $A$ in order by sums of operators of the form $\lambda P$ where $\lambda \in \text{fix}(S)$ and $P \in \text{HS}_T(E)$ is an orthogonal projection onto a $T$-invariant, homogeneous, finite-rank submodule. This is done in several reduction steps.

Firstly, by the decomposition $A = \frac{1}{2} (A + A^*) + i \frac{1}{2} (A - A^*)$ we may suppose that $A$ is self-adjoint. Secondly, by virtue of the spectral decomposition (Proposition 6.5), we may further suppose that $A = \lambda P$ where $P$ is a $T$-intertwining orthogonal projection and $0 \leq \lambda \in \text{fix}(S)$.

Next, abbreviate $p := \text{supp}(\lambda) \in \text{fix}(S)$ and define $p_\varepsilon := \text{supp}(((\lambda - \varepsilon I)^+)$. Then $p_\varepsilon \in \mathbb{B} \cap \text{fix}(S)$, $\varepsilon p_\varepsilon \leq \lambda$ and $p_\varepsilon \nearrow p$ as $\varepsilon \searrow 0$. It follows that $p_\varepsilon \lambda P \rightarrow \lambda P = A$ in order, hence we may suppose $A = q P$ and $\varepsilon q \leq \lambda$ for some $\varepsilon > 0$ and some $q \in \mathbb{B} \cap \text{fix}(S)$.

Define $Q := q P$. Then $A = \lambda Q$ and $Q$ is a $T$-intertwining orthogonal projection. Moreover, $Q$ is $\mathcal{A}$-Hilbert–Schmidt, as

$$\sum_{y \in \mathcal{B}} |Qy|^2 \leq \frac{1}{\varepsilon^2} \sum_{y \in \mathcal{B}} \lambda^2 |Qy|^2 \leq \frac{1}{\varepsilon^2} |A|^2_{\text{HS}}$$

for each finite suborthonormal set $\mathcal{B} \subseteq E$. By Lemma 6.7 $Q$ is of finite rank.

Finally, by applying Lemma 6.8 we may suppose that $\text{ran}(Q)$ is homogeneous. $\square$

**Proof of Theorem 6.9** Each finitely-generated $T$-invariant submodule of $E$ is contained in a $T$-invariant KH-submodule of finite rank (e.g., its order-closure), and each $T$-invariant KH-submodule of finite rank is contained in a sum of homogeneous $T$-invariant KH-submodules of finite rank (Lemma 6.8). Hence, the first two alternative definitions of $E_{\text{ds}}$ coincide. The third characterization follows directly from Lemma 6.10.
It remains to prove $E_{ds}^+ = E_{wm}$. To this end, let $x \in E$ and $\mathcal{B}$ be a suborthonormal set. Then

$$(x \otimes \overline{x}|u_B) = \sum_{e \in \mathcal{B}} (x \otimes \overline{x}|e \otimes \overline{e}) = \sum_{e \in \mathcal{B}} |(x|e)|^2,$$

hence $x \perp \mathcal{B}$ if and only if $x \otimes \overline{x} \perp u_B$. It follows that $x \perp E_{ds}$ if and only if $x \otimes \overline{x} \perp u_B$ for all $u_B \in U$. And, by Lemma 6.10, this is the case if and only if $x \otimes \overline{x} \perp \text{fix}(T \otimes \overline{T})$. □

**Remark 6.11.** Employing the fact that the spectral theorem (Theorem 4.1) also holds for self-adjoint cyclically compact operators (cf. Remark 4.5) one can prove without difficulty that

$$E_{ds} = \text{ocl} \sum \{\text{ran}(A) \mid A \in \text{End}_T(E) \text{ cyclically compact}\}.$$

### Notes and Comments

The main example of a “KH-dynamical system” comes from an extension $X \to Y$ of measure-preserving systems, where $T$ and $S$ are the induced Koopman representations on $L^2(X|Y)$ and $L^\infty(Y)$, respectively. Of course, this example motivated our abstract notion. On the other hand, as demonstrated in Remarks 5.4, the concept assorts well with the established theory of covariant representations of $C^*$-algebras.

In the whole theory, the case $A = C$ is the classical Hilbert space situation, discussed in the Introduction. Indeed, as on $C$ there is only a trivial dynamics, a “KH-dynamical system” in the case $A = C$ is nothing but a unitary representation $T: G \to \mathcal{L}(H)$ of a group $G$ on a Hilbert space $H$. Theorem 6.9 then reduces to the Hilbert space decomposition $H = H_{ds} \oplus H_{wm}$ mentioned in the corollary on page 3.

It turns out that this decomposition coincides with the so-called Jacobs–deLeeuw–Glicksberg decomposition $H = H_{rev} \oplus H_{aws}$, which arises whenever a semigroup acts contractively on a Hilbert space $H$. Here, on $H_{rev}$ the semigroup embeds into a strongly continuous action of a compact group, and on $H_{aws}$ the zero vector is contained in the weak closure of each orbit, see [EFHN15] Chap. 16.

In order to see that indeed $H_{rev} = H_{ds}$ (and hence also $H_{wm} = H_{aws}$) one needs an essential part of the Peter–Weyl theorem, namely that $H = H_{ds}$ whenever $G$ is compact and $T$ is strongly continuous, see [EFHN15] Thm. 16.31 and its proof.

On the other hand, that part of the Peter–Weyl theorem is actually a straightforward consequence of the decomposition result of Theorem 6.9 for $A = C$ (i.e., the corollary stated on page 3 of the introduction). Indeed, suppose that $G$ is a compact group and the unitary representation $T$ of $G$ on $H$ is strongly continuous. Then $P := \int_G T_t \otimes \overline{T}_t \, dt$ is the orthogonal projection onto $\text{fix}(T \otimes \overline{T})$. Hence, for $x \in H_{wm}$ one has

$$0 = (x \otimes \overline{x}|P(x \otimes \overline{x})) = \int_G |(T_t x|x)|^2 \, dt,$$

and this implies $x = 0$ since the Haar measure on $G$ has full support. It follows that $H = H_{ds}$ as claimed.

These findings lead to the conclusion that although the decomposition of Jacobs–deLeeuw–Glicksberg and the discrete spectrum/weakly mixing decomposition coincide
extensionally, they are \textit{intensionally} different and the link between them is furnished only by a corollary of the latter.

It can be expected that there is an analogue of the Jacobs–deLeeuw–Glicksberg theory for semigroups on KH-modules (simply because in the Boolean/conditional world there is an analogue for every classical result, see the notes to Part I on page 33), which then relates to the ds/wm-decomposition just as in the Hilbert space case. This is the object of future work.

Since the spectral theorem also holds for cyclically compact operators (see Remark 4.5 and the notes to Part I on page 33), one can extend the description of the discrete spectrum part of $E$ involving cyclically compact operators (see Remark 6.11).

Part III. The Furstenberg–Zimmer Structure Theorem

In this part we apply the Decomposition Theorem (Theorem 6.9) to extensions of measure-preserving systems in order to establish the Furstenberg–Zimmer structure theorem. Again, $G$ denotes an arbitrary, but fixed group.

7. Extensions of Measure-Preserving Systems

Classically, a \textit{measure-preserving $G$-system} is a pair $X = (X; \varphi)$ of a probability space $X = (X, \Sigma_X, \mu_X)$ and a family $(\varphi_t)_{t \in G}$ of measurable and measure-preserving maps $\varphi_t: X \to X$ for $t \in G$ such that $\varphi_{ts} = \varphi_t \circ \varphi_s$ holds almost everywhere for $t, s \in G$ and $\varphi_0 = \text{id}_X$.

Such a dynamical system always induces a group $(T_t)_{t \in G}$ of (“Koopman”) operators on the corresponding $L^2$-space via $T_t f := f \circ \varphi_t^{-1}$ for $f \in L^2(X)$. The operators $T_t$ are examples of \textbf{Markov embeddings}, i.e., isometries $T \in \mathcal{L}(L^2(X))$ satisfying

- $|Tf| = |f|$ for every $f \in L^2(X)$,
- $T1 = 1$.

(See [EFHN15, Chap. 13] for a detailed discussion of general Markov operators. Bijective Markov embeddings are also called \textit{Markov isomorphisms}.)

In the case of a standard probability space $X$, every unitary group representation $G \to \mathcal{L}(L^2(X))$ as Markov isomorphisms is the “Koopmanization” of an underlying representation by means of point transformations. This follows from an important result of von Neumann (see [EFHN15, Prop. 7.19 and Thm. 7.20]). If the probability space is not standard, however, such an operator group is induced only by transformations of the associated measure algebra (see [EFHN15, Thm. 12.10]), but not necessarily by underlying point transformations.

Giving up point transformations and passing to the functional analytic side amounts to a change of category. This change involves a reversing of arrows, to the effect that factors and factor maps in the classical framework are replaced by extensions and embeddings in the functional analytic framework. To wit, where in classical ergodic theory one would write $X \to Y$ for an extension of systems, with $Y$ being the factor and $X$ the extension, we write $J: (Y, S) \to (X, T)$, where $J$ is the Markov embedding of $L^2(Y)$ into $L^2(X)$ and $S$ and $T$ are the respective Koopman representations. Actually, taking the change of
category seriously amounts to requiring only the functional-analytic properties of such representations and dispensing with the requirement that they arise from underlying point transformations. However unfamiliar this step might be at first, it is quite natural and advantageous for structural considerations, e.g., because the new category is free from countability restrictions and better suited for universal constructions.

Whereas the book [EFHN15] can be seen as a bridge between the two worlds, here we do not consider point transformations at all and work exclusively on the functional-analytic side. Hence the following definitions.

**Definition 7.1.** A **measure-preserving** $G$-**system** is a pair $(X; T)$ with a probability space $X$ and a representation 

$$T : G \to \mathcal{L}(L^2(X)), \ t \mapsto T_t$$

of $G$ as Markov lattice isomorphisms on $L^2(X)$.

An **extension** (or **morphism**) $J : (Y; S) \to (X; T)$ of measure-preserving systems is a Markov embedding $J \in \mathcal{L}(L^2(Y), L^2(X))$ such that the diagram

$$
\begin{array}{ccc}
L^2(X) & \xrightarrow{T_t} & L^2(X) \\
J \downarrow & & \downarrow J \\
L^2(Y) & \xrightarrow{S_t} & L^2(Y)
\end{array}
$$

is commutative for every $t \in G$.

Two extensions $J_1 : (Y_1; S^1) \to (X; T)$ and $J_2 : (Y_2; S^2) \to (X; T)$ are **equivalent** if there is a Markov lattice isomorphism $V : L^2(Y_1) \to L^2(Y_1)$ intertwining the dynamics and such that the diagram

$$
\begin{array}{ccc}
(X; T) & \xrightarrow{J_1} & (Y_1; S^1) \\
& V & \downarrow J_2 \\
& & (Y_2; S^2)
\end{array}
$$

is commutative.

We start from an extension $J : (Y; S) \to (X; T)$ such that the range $\text{ran} J \subseteq L^2(X)$ is an invariant and closed **unital vector sublattice** of $L^2(X)$. (The latter means that $\text{ran} J$ is a closed subspace containing 1 and closed with respect to taking real and imaginary parts as well as the modulus.) Each unital closed vector sublattice is of the form $L^2(X, \Sigma', \mu_X)$ for some (usually non-unique) $\sigma$-subalgebra $\Sigma'$ of $\Sigma_X$ [EFHN15, Thm. 13.19]. Moreover, two extensions $J_1$ and $J_2$ as above are equivalent precisely when $\text{ran} J_1 = \text{ran} J_2$. It follows that—up to equivalence—extensions into $(X; T)$ are in one-to-one correspondence with the invariant and closed unital vector sublattices of $L^2(X)$, and each extension can be seen as a proper inclusion. (See [EFHN15] Thms. 13.19 and 13.20 for these assertions.)
7.1. **Inductive limits.** For later use, let us recall here the concept of inductive limits of measure-preserving systems (see [EFHN15, Sec. 13.5]).

**Definition 7.2.** An **inductive system** is a net \((X_\alpha; T_\alpha)_\alpha\) of measure-preserving systems together with an extension \(\nu^\beta_\alpha: (X_\alpha; T_\alpha) \to (X_\beta; T_\beta)\) whenever \(\alpha \leq \beta\) and with the following properties:

- \(\nu^\gamma_\beta \nu^\beta_\alpha = \nu^\gamma_\alpha\) for \(\alpha \leq \beta \leq \gamma\) and
- \(\nu^\alpha_\alpha = \text{Id}\) for each \(\alpha\).

An **inductive limit** of such an inductive system is a measure-preserving system \((X; T)\) together with extensions \(\nu_\alpha: (X_\alpha; T_\alpha) \to (X; T)\) such that for each \(\alpha\) the diagram

\[
\begin{array}{ccc}
(X_\alpha; T_\alpha) & \xrightarrow{\nu_\alpha} & (X; T) \\
\downarrow J_\alpha & & \downarrow J \\
(Y; S) & \xrightarrow{I_\alpha} & (X_\alpha; T_\alpha)
\end{array}
\]

is commutative.

In this case, we write

\[
(X; T) = \lim_{\alpha \to} (X_\alpha; T_\alpha).
\]

By [EFHN15, Thm. 13.38] (which readily extends to the case of arbitrary group actions), every inductive system has an inductive limit. Moreover, by the universal property, an inductive limit is unique up to a natural isomorphism.

7.2. **The Associated KH-Dynamical System.** Let \(J: L^2(Y) \to L^2(X)\) be a Markov embedding and abbreviate \(\mathcal{A} := L^\infty(Y)\). The adjoint \(J^\ast\) of \(J\) is a Markov operator and hence extends uniquely to a Markov operator

\[
\mathbb{E}_Y: L^1(X) \to L^1(Y),
\]

see [EFHN15 Prop. 13.6]. We think of \(\mathbb{E}_Y\) as a **conditional expectation.** (Actually, the true conditional expectation is the orthogonal projection \(J^\ast J\) onto \(\text{ran} J\), see [EFHN15, Sec. 13.3]).

Since \(J\) is a Markov embedding, one has

\[
J(f g) = J(f) \cdot J(g) \quad \text{for all } f, g \in \mathcal{A},
\]

see [EFHN15, Sec. 13.2]. Hence, the product

\[
\mathcal{A} \times L^1(X) \to L^1(X), \quad (f, g) \mapsto (Jf)g
\]
turns \(L^1(X)\) into an \(A\)-module. The conditional expectation \(\mathbb{E}_Y : L^1(X) \to L^1(Y)\) is a module homomorphism \([EFHN15, \text{Thm. 13.12}]\). Define

\[
(f|g)_Y := \mathbb{E}_Y(f \overline{g}) \quad \text{and} \quad |f|_Y := \sqrt{(f|f)_Y} = \sqrt{\mathbb{E}_Y|f|^2} \quad \text{for} \ f, g \in L^2(X).
\]

Then the mapping

\[
(\cdot|\cdot)_Y : L^2(X) \times L^2(X) \to L^1(Y)
\]

is \(A\)-sesquilinear, positive definite and bounded. Hence, one has the inequalities

\[
|\langle f|g \rangle_Y| \leq |f|_Y|g|_Y \quad \text{and} \quad |f + g|_Y \leq |f|_Y + |g|_Y
\]

for all \(f, g \in L^2(X)\). (The second follows from the first and the first needs only be proved for \(f, g \in L^\infty(X)\), where it follows from standard arguments.) Consequently,

\[
\|f|_Y - |g|_Y\|_2 \leq \|f - g\|_2 \quad (f, g \in L^2(X)).
\]

In order to obtain a Hilbert module, we need to consider a submodule of \(L^2(X)\).

**Definition 7.3.** Let \(X\) and \(Y\) be probability spaces and \(J : L^2(Y) \to L^2(X)\) a Markov embedding. The **conditional \(L^2\)-space** is

\[
L^2(X|Y) := \{ f \in L^2(X) \mid \mathbb{E}_Y|f|^2 \in L^\infty(Y) \},
\]

equipped with the \(L^\infty(Y)\)-valued inner product \((\cdot|\cdot)_Y\) and the \(L^\infty(Y)\)-valued norm \(|\cdot|_Y\).

By what we have seen above, \(L^2(X|Y)\) is a pre-Hilbert module over the unital commutative \(C^*\)-algebra \(L^\infty(Y)\). Note that our definition of \(L^2(X|Y)\) coincides with Tao’s \([Tao09, \text{Sec. 2.13}]\) but differs from the one given in \([KL16, \text{p.51}]\).

**Remark 7.4.** Apart from the structures just introduced, the pre-Hilbert module \(E = L^2(X|Y)\) is equipped also with the usual inner product and norm inherited from \(L^2(X)\), and the usual modulus. These are denoted by \((f|g), \|f\|_2, |f|\), respectively, in order to avoid confusion with \((f|g)_Y, \|f\|_Y, |f|_Y\).

Order-related notions like order-convergence, order-Cauchy property or order-closure will always refer to \(L^2(X|Y)\) as a lattice-normed module over \(L^\infty(Y)\), and not in the sense of a sublattice of \(L^2(X)\), unless otherwise noted.

Since \(Y\) is a probability space, \(A := L^\infty(Y)\) is order-complete, i.e. a Stone algebra, see \([EFHN15, \text{Cor. 7.8 and Rem. 7.11}]\). We intend to show that \(L^2(X|Y)\) is a Kaplansky–Hilbert module over \(A\). To this aim, and also for later use, we need to relate the different convergence notions.

**Lemma 7.5.** Let \(X\) and \(Y\) be probability spaces and \(J : L^2(Y) \to L^2(X)\) a Markov embedding.

(i) For a bounded sequence \((f_n)_n\) in \(L^\infty(Y)\) one has \(\text{o-lim}_{n \to \infty} f_n = 0\) if and only if \(f_n \to 0 \mu_Y\)-almost everywhere.

(ii) Each order-bounded subset of \(L^2(X|Y)\) is \(\| \cdot \|_2\)-bounded. Each order-convergent net in \(L^2(X|Y)\) is \(L^2\)-convergent (to the same limit).
(iii) If $M$ is an $L^\infty(Y)$-submodule of $L^2(X|Y)$, then within $L^2(X|Y)$ the $\|\cdot\|_2$-closure of $M$ coincides with its order-closure. Moreover, $f \in \text{ocl}(M)$ if and only if there is a sequence $(f_n)_n$ in $M$ such that for each $\varepsilon > 0$ there is a $Y$-measurable set $A$ with $\mu_Y(A^c) \leq \varepsilon$ and $\|\mathbf{1}_A f_n - f|_Y\|_{L^\infty} \to 0$.

**Proof.** (i) This follows from the fact, that if $(f_n)_n$ is a bounded sequence, then the superior limits $\limsup_{n \to \infty} |f_n|$ in the order-sense and in the a.e.-sense coincide.

(ii) This follows from $\|f\|_2^2 = \int_Y \mathbb{E}_Y |f|^2 = \int_Y |f|_Y^2$ for $f \in L^2(X|Y)$ and the fact that if $(u_\alpha)_\alpha$ decreases to 0 in $L^\infty(Y)$, then $\|u_\alpha\|_2 \to 0$, see [EPHN15] Thm. 7.6.

(iii) By (ii) and Lemma 11.13 the order-closure is contained in the $L^2$-closure. Conversely, suppose that $f_n \in M$, $f \in L^2(X|Y)$ and $\|f_n - f\|_2 \to 0$. Passing to a subsequence we may suppose $|f_n - f|_Y \to 0$ almost everywhere. Fix $\varepsilon > 0$. Then by Egoroff’s theorem there is a $Y$-measurable set $A$ such that $\mu_Y(A^c) \leq \varepsilon$ and $\|\mathbf{1}_A f_n - \mathbf{1}_A f|_Y\|_{L^\infty} \to 0$. It follows that $\mathbf{1}_A f \in \text{ocl}(M)$. By (i), $f \in \text{ocl}(M)$ as desired. □

**Proposition 7.6.** Let $X$ and $Y$ be probability spaces and $J: L^2(Y) \to L^2(X)$ a Markov embedding. Then $L^2(X)$ is a Kaplansky–Hilbert module over $L^\infty(Y)$.

**Proof.** Let $(f_\alpha)_\alpha$ be an order-Cauchy net in $L^2(X|Y)$. Then the net $(f_\alpha - f_\beta)_{(\alpha, \beta)}$ order-converges to 0. By Lemma 11.13 $\|f_\alpha - f_\beta\|_2 \to 0$ as $(\alpha, \beta) \to \infty$, i.e., $(f_\alpha)_\alpha$ is $\|\cdot\|_2$-Cauchy.

As $L^2(X)$ is complete, there is $f \in L^2(X)$ with $\|f - f_n\|_2 \to 0$.

Since $(f_\alpha)_\alpha$ is order-Cauchy, we find an index $\alpha_0$ and a net $(u_\alpha)_{\alpha \geq \alpha_0}$ decreasing to 0 in $L^\infty(Y)$ such that

$$|f_\alpha - f_\beta|_Y \leq u_\gamma \quad (\alpha, \beta \geq \gamma \geq \alpha_0).$$

(Use Lemma 11.10) Since the mapping $g \mapsto |f_\alpha - g|_Y$ is continuous with respect to $L^2$-norms, by letting $\beta \to \infty$ we obtain

$$|f_\alpha - f|_Y \leq u_\gamma \quad (\alpha \geq \gamma \geq \alpha_0).$$

And this just means that $\text{o-lim}_\alpha f_\alpha = f$. □

Let $J: (Y; S) \to (X; T)$ be any extension of dynamical systems and, as before, $\mathcal{A} := L^\infty(Y)$. Then for every $t \in G$ the operator $S_t$ restricts to an automorphism $S_t \in \text{Aut}(\mathcal{A})$ of the Stone algebra $\mathcal{A}$. Moreover, $T_t$ restricts to an $S_t$-unitary operator (again denoted by $T_t$) on the KH-module $L^2(X|Y)$. In this manner we obtain the associated KH-dynamical system $(\mathcal{A}, S; L^2(X|Y), T)$.

7.3. **The “Dual Extension”**. As is common, we identify the dual of $L^2(X)$ with $L^2(X)$ via the duality $\langle f, g \rangle := \int_X f g$. Then $\langle f, \mathcal{F} \rangle = \langle f | g \rangle$ for all $f, g \in L^2(X)$.

Likewise, if $J: L^2(Y) \to L^2(X)$ is a Markov embedding and $L^2(X|Y)$ the associated Kaplansky–Hilbert module over $\mathcal{A} = L^\infty(Y)$, we identify the dual module $L^2(X|Y)^*$ with $L^2(X|Y)$ via the duality

$$\langle f, g \rangle_Y := \mathbb{E}_Y(fg) \quad (f, g \in L^2(X|Y)).$$

With this identification, the formal conjugation mapping $\Theta: L^2(X|Y) \to L^2(X|Y)$ is just ordinary conjugation, i.e., $\langle f, \mathcal{F} \rangle_Y = \langle f | g \rangle_Y$ for all $f, g \in L^2(X|Y)$. 
Let $T : L^2(X|Y) \to L^2(X|Y)$ be any linear operator. Recall from Section 2.4 that the conjugate operator is defined through $T^* = \overline{T^T}$. Hence, if $T$ is real, i.e., if $T$ maps real functions to real functions, then $\overline{T} = T$. This applies, in particular, to Markov homomorphisms, and hence to measure-preserving systems.

Given an extension $J : (Y; S) \to (X; T)$, the “dual extension” is therefore just $J$ again, and the dual $G$-system of the associated system $(A, S; L^2(X|Y), T)$ is precisely this system again.

### 7.4. Relatively Independent Joining

Suppose

$$J_X : L^2(Y) \to L^2(X) \quad \text{and} \quad J_{X'} : L^2(Y) \to L^2(X')$$

are Markov embeddings. Then we can form the associated KH-modules $L^2(X|Y)$ and $L^2(X'|Y)$. We aim at showing that their Kaplansky–Hilbert tensor product can be written as

$$L^2(X|Y) \otimes L^2(X'|Y) = L^2(Z|Y)$$

for certain Markov embeddings $I_X : L^2(X) \to L^2(Z)$ and $I_{X'} : L^2(X') \to L^2(Z)$.

In general, a pair of Markov embeddings $I_X : L^2(X) \to L^2(Z)$ and $I_{X'} : L^2(X') \to L^2(Z)$ is called a coupling, if the ranges $\text{ran}(I_X)$ and $\text{ran}(I_{X'})$ together generate $L^2(Z)$ as a Banach lattice. This is equivalent to saying that the $*$-subalgebra

$$\text{span}\{I_Xf \cdot I_{X'}g \mid f \in L^\infty(X), \ g \in L^\infty(X')\}$$

is dense in $L^2(Z)$.

Given such a coupling, one can form the Markov operator

$$P(I_X, I_{X'}) := \mathbb{E}_{X'} \circ I_X : L^2(X) \to L^2(X').$$

This so-called coupling operator determines the coupling up to a canonical isomorphism. And even more is true:

**Proposition 7.7.** Let $P : L^2(X) \to L^2(X')$ be a Markov operator. Then there is a coupling $(Z, I_X, I_{X'})$ such that $P = P(I_X, I_{X'})$. This coupling is unique in the sense that any two couplings with this property are canonically isomorphic.

**Proof.** By passing to Stone topological models as in [EFHN15 Chap. 12], we may suppose that $L^\infty(X) = C(X)$ and $L^\infty(X') = C(X')$ with Stonean spaces $X, X'$. Then $P$ restricts to a Markov operator $P : C(X) \to C(X')$. Moreover, $Z \equiv X \times X'$ is also compact and we define the operator $Q : C(Z) \to C(X')$ by $Qh(x') := (Ph(x', x'))(x')$. A moment’s thought reveals that $Q$ is a Markov operator and

$$Q(f \otimes g) = Pf \cdot g \quad (f \in C(X), \ g \in C(X')).$$

Endow $Z$ with the probability measure $\mu_Z = Q'\mu_{X'}$, i.e., through

$$\int_Z h = \int_{X'} Qh \quad (h \in C(Z)).$$

---

A Markov operator between $C(K)$-spaces is a positive linear operator that maps $1$ to $1$. 

The mapping \( I_X : C(X) \to C(Z) \). \( I_X f = f \otimes 1 \) is the Koopman operator of the projection \((x, x') \mapsto x'\). It extends uniquely to a Markov embedding,

\[
I_X : L^2(X) \to L^2(Z).
\]

Analogously we obtain the Markov embedding \( I_{X'} : L^2(X') \to L^2(Z) \). As \( C(X) \otimes C(X') \) is dense in \( C(Z) \), it follows that \((I_X, I_{X'}, Z)\) is a coupling.

Next, it is routine to show that \( Q \), or rather its extension to a Markov operator \( L^2(Z) \to L^2(X') \), is precisely the adjoint of \( I_{X'} \). (It suffices to test on elements of \( C(X) \otimes C(X') \).) Given \( f \in C(X) \) we finally obtain

\[
Q(I_X f) = Q(f \otimes 1) = P f,
\]

hence \( P = P(I_X, I_{X'}) \). This concludes the existence proof. The uniqueness proof is technical but routine, hence we omit it. \( \square \)

Now suppose that, as before, we start with two Markov embeddings

\[
J_X : L^2(Y) \to L^2(X) \quad \text{and} \quad J_{X'} : L^2(Y) \to L^2(X').
\]

Define \( P : = J_X \circ \mathbb{E}_X \), where \( \mathbb{E}_X = J_X^* \) is the “conditional expectation” from \( L^2(X) \) onto \( L^2(Y) \). Then \( P : L^2(X) \to L^2(X') \) is a Markov operator. The corresponding coupling is denoted by \( Z = X \times_Y X' \) and called the relatively independent coupling (associated with the embeddings \( J_X, J_{X'} \)).

Note that Markov operators between \( L^2 \)-spaces uniquely extend to the \( L^1 \)-spaces [EFHN15 Prop. 13.6]. If \( f \in L^2(X) \) and \( g \in L^2(X') \) then

\[
(f \otimes g) = (I_X f) \cdot (I_{X'} g) \in L^1(X \times_Y X').
\]

For \( h \in L^\infty(Y), f \in L^\infty(X) \) and \( g \in L^\infty(X') \) we have

\[
(f \otimes g \mid J_X h \otimes 1) = \int_Z (f \mathbb{E}_X h) \otimes g = \int_{X'} P(f \mathbb{E}_X h) g = \int_Y \mathbb{E}_Y (f \mathbb{E}_X h) \cdot \mathbb{E}_{X'} g
\]

\[
= (\mathbb{E}_X f \cdot \mathbb{E}_{X'} g \mid h)_{L^2(Y)} = \cdots = (f \otimes g \mid 1 \otimes J_{X'} h).
\]

This means \( I_X \circ J_X = I_{X'} \circ J_{X'} \) or, when we suppress explicit reference to the embeddings \( J_X \) and \( J_{X'} \),

\[
h \otimes 1 = 1 \otimes h \quad \text{for each} \ h \in L^2(Y).
\]

So the relatively independent coupling is a coupling over \( Y \). Furthermore, the computation from above also yields

\[
\mathbb{E}_Y (f \otimes g) = \mathbb{E}_X f \cdot \mathbb{E}_{X'} g \quad (f \in L^2(X), \ g \in L^2(X')).
\]

This explains the name of the coupling. Observe that this property uniquely determines the coupling in the sense of Proposition [7,7].

Couplings in the presence of dynamics are called joinings, cf. [Gla03 Chap. 6]. The following lemma describes under which condition a coupling can be turned into a joining.

**Lemma 7.8.** Let \((Z, I_X, I_{X'})\) be a coupling with associated Markov operator \( P = P(I_X, I_{X'}) \), and let \( L, R \) be Markov embeddings on \( L^2(X) \) and \( L^2(X') \), respectively. Then there is a
Markov embedding $T$ on $L^2(Z)$ with $I_X L = T I_X$ and $I_{X'} R = T I_{X'}$ if and only if $R^* PL = P$. In this case, $T$ is unique and satisfies

\[(7.1) \quad T(f \otimes g) = L f \otimes R g \quad (f \in L^\infty(X), \ g \in L^\infty(X')).\]

**Proof.** Since $T$ is supposed to be a Markov embedding and hence multiplicative on $L^\infty$, the commutation conditions $I_X L = T I_X$ and $I_{X'} R = T I_{X'}$ are equivalent to (7.1). Since the products $f \otimes g$ span a dense subspace, uniqueness is clear. The computation

\[
\left\| \sum_{j=1}^n L f_j \otimes R g_j \right\|_{L^2(Z)}^2 = \sum_{j,k=1}^n \int Z (L f_j L f_k) \otimes (R g_j R g_k) \, \mu = \sum_{j,k=1}^n \int Z (L f_j f_k) \otimes R (g_j g_k) \, \mu
\]

shows that the condition $R^* PL = P$ is necessary for the existence of $T$. Conversely, suppose that this condition holds. Then, since $R, L$ are multiplicative and real,

\[
\left\| \sum_{j=1}^n L f_j \otimes R g_j \right\|_{L^2(Z)}^2 = \sum_{j,k=1}^n \int Z (L f_j L f_k) \otimes (R g_j R g_k) = \sum_{j,k=1}^n \int Z (L (f_j f_k) \otimes R (g_j g_k)) = \sum_{j,k=1}^n \int Z P(f_j f_k) \otimes g_j g_k = \cdots = \left\| \sum_{j=1}^n f_j \otimes g_j \right\|_{L^2(Z)}^2.
\]

This shows that through (7.1) an isometric operator $T$ on $L^2(Z)$ is defined. It is multiplicative on $L^\infty(X) \otimes L^\infty(X')$ and hence a Markov embedding. \hfill \Box

If, in the situation of the previous lemma, $R$ is a Markov isomorphism, then $P = R^* PL$ is equivalent with $PL = RP$. Hence, the following corollary holds.

**Corollary 7.9.** Let $(X; L)$ and $(X'; R)$ be measure-preserving $G$-systems. Let $(Z, I_X, I_{X'})$ be a coupling such that its coupling operator $P$ intertwines the dynamics, i.e., satisfies $PL_t = R_t P$ for all $t \in G$. Then there is a unique $G$-dynamics $T$ on $Z$ such that

\[I_X: (X; L) \to (Z; T) \quad \text{and} \quad I_{X'}: (X'; R) \to (Z; T)\]

are extensions of $G$-systems. The $G$-dynamics $T$ is characterized by

\[T_t(f \otimes g) = L_t f \otimes R_t g \quad (f \in L^\infty(X), \ g \in L^\infty(X'), \ t \in G).\]

The system $(Z; T)$ is called the **joining** of the systems $(X; L)$ and $(X'; R)$ via the coupling operator $P$.

Let $J_X: (Y; S) \to (X; L)$ and $J_{X'}: (Y; S) \to (X'; R)$ be extensions and $Z = X \times_Y X'$ the relatively independent coupling. Then the coupling operator is $P = J_X J_{X'}^*$ and hence intertwines the dynamics. It follows that there is a unique $G$-dynamics, denoted by $L \times_Y R$, on $X \times_Y X'$ such that the coupling becomes a joining of the original systems.

The system $(X \times_Y X'; L \times_Y R)$ is called the **relatively independent joining** of the extensions $(J_X, J_{X'})$.

**Examples 7.10.**

1. Let $(X; L)$ and $(X'; R)$ be measure-preserving systems. Then the relatively independent joining of the trivial extensions $(\{pt\}; \text{Id}) \to (X; L), (X'; R)$ is simply the product system $(X \times X'; L \times R)$ with the natural embeddings $(X; L), (X'; R) \to (X \times X'; L \times R)$. 
Finally, we realize that the KH-dynamical system associated with the extension \((Y; S) \to (X \times_Y X'; L \times_Y R)\) is precisely the tensor product system arising from the KH-dynamical systems associated with the original extensions.

**Proposition 7.11.** Let \(J_X: (Y; S) \to (X; L)\) and \(J_{X'}: (Y; S) \to (X'; R)\) be extensions. Then there is a unique isomorphism of KH-dynamical systems

\[
W: (L^2(X|Y) \otimes L^2(X'|Y); L \otimes R) \cong (L^2(X \times_Y X'|Y); L \times_Y R)
\]

with \(W(f \otimes g) = (I_{Xf})(I_{X'}g)\) for all \(f \in L^2(X|Y)\) and \(g \in L^2(X'|Y)\).

**Proof.** For the time being, we need to distinguish \(f \otimes g \in L^2(X|Y) \otimes L^2(X'|Y)\) from \(f \otimes_Y g := (I_{Xf})(I_{X'}g) \in L^2(X \times_Y X')\). Uniqueness of \(W\) is clear.

To show existence we start with the (obviously well-defined) operator

\[
W: L^\infty(X) \otimes_{alg} L^\infty(X') \to L^2(X \times_Y X'|Y), \quad W(f \otimes g) := f \otimes_Y g.
\]

By employing the definition of the relatively independent coupling, the identity

\[
(f \otimes g|u \otimes v) = f \otimes_Y g|u \otimes_Y v
\]

is easily established. Since \(L^\infty(X)\) is, by [Lemma 7.5](iii), order-dense in \(L^2(X|Y)\) (and the analogous statement is true for \(X'\)), \(W\) extends to an isometric homomorphism

\[
W: L^2(X|Y) \otimes L^2(X'|Y) \to L^2(X \times_Y X'|Y)
\]

of KH-modules.

It is clear that \(W\) intertwines the dynamics. To show that \(W\) is surjective, it suffices to prove that the range of \(W\) is order-dense in \(L^2(X_1 \times_Y X_2|Y)\). However, this is an immediate consequence of [Lemma 7.5](iii).

Finally, take \(f \in L^2(X|Y)\) and \(g \in L^2(X'|Y)\). We then find sequences \((f_n)_{n \in \mathbb{N}}\) in \(L^\infty(X)\) and \((g_n)_{n \in \mathbb{N}}\) in \(L^\infty(X')\) order-converging to \(f\) and \(g\), respectively. Then \(\sigma\lim_{n \to \infty} f_n \otimes g_n = f \otimes g\) in \(L^2(X|Y) \otimes L^2(X'|Y)\) and hence \((I_{Xf_n})(I_{X'}g_n))_{n \in \mathbb{N}}\) order-converges to \(h := W(f \otimes g) \in L^2(X \times_Y X'|Y)\). In particular, \(\lim_{n \to \infty} (I_{Xf_n})(I_{X'}g_n) = h \in L^1(X \times_Y X)\) by [Lemma 7.5](ii). On the other hand, since \(\lim_{n \to \infty} f_n = f\) in \(L^2(X)\) and \(\lim_{n \to \infty} g_n = g\) in \(L^2(X')\) again by [Lemma 7.5](ii), we have \(\lim_{n \to \infty} (I_{Xf_n})(I_{X'}g_n) = (I_X f)(I_{X'}g)\) in \(L^1(X \times_Y X')\). This shows \(W(f \otimes g) = h = (I_X f)(I_{X'} g)\).

\[\square\]

8. **Splitting and Dichotomy for Extensions of Measure-Preserving Systems**

In this chapter we apply the Decomposition Theorem 6.9 to extensions of measure-preserving \(G\)-systems. Unless otherwise stated, \(G\) is an arbitrary but fixed group, and \(J: (Y; S) \to (X; T)\) is a fixed extension of a measure-preserving \(G\)-systems.

We usually suppress reference to \(J\) and consider \(L^2(Y)\) as a closed subspace of \(L^2(X)\). (Nevertheless, we distinguish notionally the dynamics \(S\) on \(Y\) and \(T\) on \(X\).) Furthermore,
we use the prefix “Y-” whenever we use notions pertaining to the associated KH-dynamical system on $L^2(X|Y)$, so that, for instance, we speak of Y-suborthonormal sets or Y-homogeneous finite-rank submodules.

**Remark 8.1.** The following observations will be helpful when transferring results from $L^2(X|Y)$ to the whole of $L^2(X)$.

1. If $f \in L^2(X)$, then multiplying by $\rho_n := 1_{|f|_Y \leq n} \in L^\infty(Y)$ yields $\rho_n f \in L^2(X|Y)$. As $\rho_n \to 1$, we obtain $\rho_n f \to f$ in $L^2(X)$.

2. If $M \subseteq L^2(X|Y)$ is a $L^\infty(Y)$-submodule, then $cl_2 M \cap L^2(X|Y) = ocl(M)$ (by Lemma 7.5). In particular, $f \in cl_2(M)$ if and only if $\rho_n f \in ocl(M)$ for all $n \in \mathbb{N}$.

3. If $f, g \in L^2(X)$, then $f \perp_Y g$ is equivalent to $f \perp_{L^2} L^\infty(Y)g$.

8.1. **The Fixed Space in the Relatively Independent Joining.** Let $\mathcal{B} \subseteq L^2(X)$ be a finite set. Similar as in Chapter 5 we write

$$u_\mathcal{B} := \sum_{g \in \mathcal{B}} g \otimes \overline{g} \in L^1(\times_Y X).$$

In general, $u_\mathcal{B}$ may be just an element of $L^1$, but if $\mathcal{B} \subseteq L^2(X|Y)$, then $u_\mathcal{B} \in L^2(\times_Y X|Y)$ and if $\mathcal{B} \subseteq L^\infty(X)$, then $u_\mathcal{B} \in L^\infty(\times_Y X|Y)$.

Note that if $f \in \text{fix}(T \times_Y T)$, then $\mathbb{E}_Y |f| \in \text{fix}(S)$, and hence by Remark 8.1(1)

$$\text{fix}(T \times_Y T) = cl_{L^2} \left( L^2(\times_Y X|Y) \cap \text{fix}(T \times_Y T) \right).$$

Therefore, Theorem 6.9 together with Remark 8.1(2) imply that $\text{fix}(T \times_Y T)$ is generated (in the $L^2$-sense) by those of its elements that have the form $\lambda u_\mathcal{B}$, where $\lambda \in \text{fix}(S) \cap L^\infty(Y)$ and $\mathcal{B} \subseteq L^2(X|Y)$ is a finite, homogeneous, suborthonormal set generating a $T$-invariant $L^\infty(Y)$ submodule. Unfortunately, it seems impossible to add the requirement $\mathcal{B} \subseteq L^\infty(X)$ here. The best we can say is the following:

**Proposition 8.2.** Let $J : (Y, S) \to (X, T)$ be an extension of measure-preserving $G$-systems. Then, in $L^2(\times_Y X)$,

$$\text{fix}(T \times_Y T) = cl_{L^2} \text{span} \{ u_\mathcal{B} \mid \mathcal{B} \subseteq L^\infty(X) \text{ finite}, u_\mathcal{B} \in \text{fix}(T \times_Y T) \}.$$ 

In order to prove Proposition 8.2 we need the following lemma.

**Lemma 8.3.** Let $J : (Y, S) \to (X, T)$ be an extension of measure-preserving $G$-systems and let $e_1, \ldots, e_n$ be a Y-suborthonormal system in $L^2(X|Y)$ satisfying that $\text{span}_{L^\infty(Y)} \{ e_1, \ldots, e_n \}$ is $T$-invariant. Then $\sum_{j=1}^n |e_j|^2 \in \text{fix}(T)$.

**Proof.** Define $e := \sum_{j=1}^n e_j \otimes \overline{e_j} \in \text{fix}(T \times_Y T)$. Then, within $L^1(\times_Y X)$,

$$|e|^2 = \sum_{j, k=1}^n e_j \overline{e_k} \otimes \overline{e_j e_k} \in \text{fix}(T \times_Y T).$$

---

Recall that $\cdot$ denotes the usual modulus mapping on $L^2(X)$, see Remark 7.4.
Hence, also $\mathbb{E}_X|e|^2 \in \text{fix}(T)$, where $\mathbb{E}_X$ denotes the conditional expectation onto the second factor. It is easy to see that $\mathbb{E}_X(f \otimes g) = (\mathbb{E}_Y f)g$ in general, hence

$$\text{fix}(T) \ni \mathbb{E}_X|e|^2 = \sum_{j,k=1}^n \mathbb{E}_Y(e_j e_k^* e_j e_k) = \sum_{j=1}^n |e_j|^2 Y_{\gamma j}^2 Y_{\gamma j} = \sum_{j=1}^n |e_j|^2,$$

cf. Lemma\kern-.5ex\cite{2003} (iii).

\begin{proof}[Proof of Proposition 8.2] Let $\mathcal{B} = \{e_1, \ldots, e_d\}$ be a suborthonormal subset of $L^2(X|Y)$ such that $u_B \in \text{fix}(T \times_Y T)$ and let $\lambda \in \text{fix}(S) \cap L^\infty(Y)$. We may assume that $\lambda \geq 0$. Let $e := \sum_{j=1}^d |e_j|^2 \in \text{fix}(T)$ by Lemma\kern-.5ex\cite{Furstenberg} and define $\eta_n := \frac{\sqrt{\lambda} e_n}{\sqrt{\lambda} + n}$. Then $0 \leq \eta_n \leq 1$, $\eta_n \in \text{fix}(T)$ and $\eta_n e_j \to e_j$ in $L^2$ as $n \to \infty$ for each $j = 1, \ldots, d$. Define

$$\mathcal{B}_n := \sqrt{\lambda} \eta_n \mathcal{B} = \{\sqrt{\lambda} \eta_n e_1, \ldots, \sqrt{\lambda} \eta_n e_d\} \subseteq L^\infty(X).$$

Then $\mathcal{B}_n$ is finite, $u_{\mathcal{B}_n} \in \text{fix}(T \times_Y T)$ for each $n \in \mathbb{N}$ and $u_{\mathcal{B}_n} \to u_B$ in $L^2$-norm as $n \to \infty$.
\end{proof}

**Remark 8.4.** Lemma\kern-.5ex\cite{Furstenberg} is due to Furstenberg\kern-.5ex\cite{Furstenberg} Thm. 9.13. However, Glasner seems to claim that the scaled elements $\eta_n e_j$ still form a $Y$-suborthonormal system, which, as far as we can see, need not be the case\kern-.5ex\cite{Glasner}.

### 8.2. Kronecker Subspace.

Theorem\kern-.5ex\cite{KH} states that the associated KH-dynamical system on $L^2(X|Y)$ induces a $Y$-orthogonal decomposition into KH-submodules

$$L^2(X|Y) = L^2(X|Y)_{\text{ds}} \oplus L^2(X|Y)_{\text{wm}},$$

into the discrete spectrum part and the weakly mixing part. The space

$$\mathcal{E}(X|Y) := \text{cl}_{L^2} L^2(X|Y)_{\text{ds}}$$

is called the \textit{(relative) discrete spectrum part} or the \textit{Kronecker subspace} of the extension.

**Proposition 8.5.** Let $J: (Y; S) \to (X; T)$ be an extension of measure-preserving $G$-systems. Then $\mathcal{E}(X|Y)$ is the $L^2$-closure of each of the following sets:

(1) $\bigcup \{M \mid M \text{ finitely-generated, } T\text{-invariant KH-submodule of } L^2(X|Y) \}$;
(2) $\bigcup \{M \mid M \text{ finitely-generated, } T\text{-invariant } L^\infty(Y)\text{-submodule of } L^2(X) \}$;
(3) $\bigcup \{M \mid M \text{ finitely-generated, } T\text{-invariant } L^\infty(Y)\text{-submodule of } L^\infty(X) \}$;
(4) $\{\text{span}_{L^\infty(Y)} \mathcal{B} \mid \mathcal{B} \subseteq L^\infty(X) \text{ finite, } u_B \in \text{fix}(T \times_Y T)\}$.

Moreover, $\mathcal{E}(X|Y)$ is a $T$-invariant closed unital submodule of $L^2(X)$.

\begin{proof} Let us denote the sets listed in (1)–(4) by $E_1$–$E_4$. Then $\text{cl}_{L^2} E_1 = \mathcal{E}(X|Y)$ by definition of $\mathcal{E}(X|Y)$ and Remark\kern-.5ex\cite{8} (2). The inclusions $E_4 \subseteq E_3 \subseteq E_2$ are clear. We shall show $E_2 \subseteq \text{cl}_{L^2} E_1$ and $E_1 \subseteq \text{cl}_{L^2} E_4$.

\footnote{Glasner uses different scaling functions $\eta_n$, but that is inessential here.}
\end{proof}
For the proof of the first inclusion, suppose that $M$ is a $T$-invariant $L^\infty(Y)$-submodule of $L^2(X)$ generated by the functions $f_1, \ldots, f_n$. Let $\lambda := 1 + \sum_{j=1}^n |f_j|_Y \in L^2(Y)$ and $g_j := \frac{1}{\lambda} f_j \in L^2(X|Y)$ for $1, \ldots, n$. We claim that $N := \operatorname{span}_{L^\infty(Y)} \{g_1, \ldots, g_n\}$ is $T$-invariant. Fix $t \in G$ and write

$$T_t f_j = \sum_{k=1}^n a_{jk} f_k, \quad T_{t^{-1}} f_j = \sum_{k=1}^n b_{jk} f_k$$

for $L^\infty(Y)$-elements $a_{jk}, b_{jk}$. Then

$$T_t g_i = S_t(\frac{1}{\lambda}) \sum_{k} a_{jk} f_k = \sum_{k} a_{jk} \lambda S_t(\frac{1}{\lambda}) g_k = \sum_{k} a_{jk} S_t(\frac{S^{-1}_{t}}{\lambda}) g_k.$$  

Next, note that

$$S^{-1}_{t} \lambda = 1 + \sum_{j=1}^n |T_{t^{-1}} f_j|_Y \leq 1 + \sum_{j=1}^n \sum_{k=1}^n |b_{jk}| |f_k|_Y \leq c \lambda$$

for some real number $c > 0$. This proves the claim. Finally, note that for each $j = 1, \ldots, n$

$$f_j = \lambda g_j \in \operatorname{cl}_{L^2} \operatorname{span}_{L^\infty(Y)} \{g_1, \ldots, g_n\} = \operatorname{cl}_{L^2} \operatorname{oc}(N) \subseteq \operatorname{cl}_{L^2}(E_1).$$

Hence, $M \subseteq \operatorname{cl}_{L^2}(E_1)$ as desired.

For the second inclusion let $M \subseteq L^2(X|Y)$ be a $T$-invariant, finite-rank KH-submodule. Pick a $Y$-suborthonormal basis $B = \{e_1, \ldots, e_d\}$ of $M$ and find, as in the proof of Proposition 8.2, functions $\eta_n \in \operatorname{fix}(T)$ with $0 \leq \eta_n \leq 1$ and $B_n := \eta_n B \subseteq L^\infty(X)$ and $\eta_n e_j \to e_j$ in $L^2$-norm for each $j = 1, \ldots, d$. Then $u_{B_n} \in \operatorname{fix}(T \times_Y T)$ and

$$M \subseteq \operatorname{cl}_{L^2} \bigcup_{n \in \mathbb{N}} \operatorname{span}_{L^\infty(Y)} B_n \subseteq \operatorname{cl}_{L^2} E_4.$$  

Finally, let $M$ be the norm-closure in $L^\infty(X)$ of the union of all finitely-generated $T$-invariant $L^\infty(Y)$-submodules of $L^\infty(X)$. Then $M$ is a unital $C^*$-subalgebra of $L^\infty(X)$ and therefore a unital sublattice of $L^\infty(X)$ (see [EFHN15, Thm. 7.23]). This implies that its $L^2$-closure $\mathcal{E}(X|Y)$ is a unital Banach sublattice of $L^2(X)$. □

By Proposition 8.5 and general theory [EFHN15 Prop. 13.19], $\mathcal{E}(X|Y) \cong L^2(Z)$ for some probability space $Z$ (determined up to a natural isomorphism). Moreover, as $\mathcal{E}(X|Y)$ is $T$-invariant, by restriction we obtain a system $(Z; T)$.

We write $\operatorname{Kro}(X|Y) \equiv Z$ and call the system $(\operatorname{Kro}(X|Y); T)$ the relative Kronecker factor of the extension. The original extension then factors as

$$(Y; S) \to (\operatorname{Kro}(X|Y); T) \to (X; T)$$

which amounts to a sequence

$$L^2(Y) \to L^2(\operatorname{Kro}(X|Y)) \to L^2(X)$$

of intertwining Markov embeddings on the level of function spaces.

**Definition 8.6.** An extension $J: (Y; S) \to (X; T)$ of measure-preserving systems has relative discrete spectrum (or: is a discrete spectrum extension) if $\mathcal{E}(X|Y) = L^2(X)$. 
If \((X; T)\) is a measure-preserving system, then the trivial extension \(J: ([pt]; \text{Id}) \to (X; T)\) has relative discrete spectrum if and only if the system \((X; T)\) has discrete spectrum, i.e., the finite-dimensional invariant subspaces of \(L^\infty (X)\) are dense in \(L^2 (X)\).

**Example 8.7.** Consider the torus \(\mathbb{T} := \{y \in \mathbb{C} | |y| = 1\}\) equipped with the Haar measure. For fixed \(a \in \mathbb{T}\) we consider the measure-preserving system \((Y; S)\) of the group \(\mathbb{Z}\) induced by the homeomorphism \(T \to T, y \mapsto ay\). The product space \(X = Y \times Y\) equipped with the action induced by

\[
\mathbb{T}^2 \to \mathbb{T}^2, \quad (y, z) \mapsto (ay, yz)
\]

is called the **skew torus** \((X; T)\). The projection onto the first component induces an extension \(J: (Y; S) \to (X; T)\). For every \(k \in \mathbb{Z}\) the continuous function

\[
f_k: \mathbb{T}^2 \to \mathbb{C}, \quad (y, z) \mapsto z^k
\]

generates an invariant submodule \(L^\infty (Y)_k \subseteq L^\infty (X)\) and these submodules are total in \(L^2(X)\). Thus, \(J: (Y; S) \to (X; T)\) has relative discrete spectrum. This example can be generalized to compact group extensions (cf. \[Zim76, Fur81, Sec. 6.2, Ell87, Sec.s 4 and 5\] and \[LTW02\]).

### 8.3. Orthogonal Complement of the Relative Kronecker Subspace.

**Proposition 8.8.** Let \(J: (Y; S) \to (X; T)\) be an extension of measure-preserving \(G\)-systems. Then for \(f \in L^2 (X)\) the following assertions are equivalent.

(a) \(f \perp\perp_{L^2} X(Y)\).

(b) \(\mathbb{E}_Y ((f \otimes f^T) h) = 0\) for each \(h \in \text{fix}(T \times_Y T) \cap L^\infty\).

(c) \(f \otimes f^T \perp \text{fix}(T \times_Y T) \cap L^\infty\).

(d) \(0 \in \overline{\text{co}} \{ T_t f \otimes T_t f^T \mid t \in G \} \) (in \(L^1\)).

(e) \(\inf_{t \in G} \max_{g \in F} \| \mathbb{E}_Y ((T_t f) g) \|_{L^2} = 0\) for each finite \(F \subseteq L^\infty (X)\).

**Proof.** The conditions (a) and (b) are invariant under multiplication with \(L^\infty (Y)\) and \(L^2 (X)\)-closed. Hence, for the proof of “(a)⇒(b)” we may suppose that \(f \in L^2 (X|Y)\) (cf. Remark 8.1). In this case—still by Remark 8.1—(a) is equivalent to

(a') \(f \perp_{Y} L^2 (X|Y)\),

and (b) is equivalent to

(b') \(f \otimes f^T \perp_Y \text{fix}(T \times_Y T) \cap L^2 (X \times_Y X|Y)\);

apply Lemma 7.5. Since (a') and (b') are equivalent by Theorem 6.9 we have established the equivalence (a)⇒(b).

Clearly, (b) implies (c). To prove that (c) implies (d) we first suppose that \(f \otimes f^T \in L^2\). Then (c) is equivalent to \(P(f \otimes f) = 0\), where \(P\) is the orthogonal projection in \(L^2 (X \times_Y X)\) onto \(\text{fix}(T \times_Y T)\). By the mean ergodic theorem for contraction semigroups on Hilbert spaces (Birkhoff–Alaoglu theorem \[EHNT13, Thm. 8.32\])

\[
P(f \otimes f) = 0 \iff 0 \in \overline{\text{co}} \{ T_t f \otimes T_t f^T \mid t \in G \}.
\]

This shows that (c) implies (d) if \(f \otimes f^T \in L^2\).
To see that this still holds if \( f \otimes \overline{f} \in L^1 \) one needs to realize that \( P \) is a Markov operator and hence extends continuously to \( L^1 \), and this extension—again denoted by \( P \)—is the mean ergodic projection (in the sense of [EFHN15 Def. 8.31] of the group \((T_i \times Y T_i)_{i \in G}\) acting on \( L^1 \). From (c) then follows that \( P(f \otimes \overline{f}) \perp \text{fix}(T \times Y T) \cap L^\infty \), and since \( P(f \otimes \overline{f}) \in \text{fix}(T \times Y T) \) it follows that \( P(f \otimes \overline{f}) = 0 \), i.e., (d).

Next, observe that \( |\mathbb{E}_Y((T,f)g)|^2 = \mathbb{E}_Y\left(T_i(f \otimes \overline{f}) \cdot (g \otimes \overline{g})\right) \) for \( g \in L^\infty(X) \). Hence, (d) implies

\[
0 \in \mathbb{C}\left\{ \sum_{g \in F} \|\mathbb{E}_Y((T,f)g)\|_{L^2}^2 \mid t \in G \right\}
\]

for each finite set \( F \subseteq L^\infty(X) \), and this is equivalent to (e).

Finally, suppose that (e) holds. Let \( \mathcal{B} \subseteq L^\infty(X) \) be finite such that \( u_\mathcal{B} = \sum_{g \in \mathcal{B}} g \otimes \overline{g} \in \text{fix}(T \times Y T) \). Then for each \( t \in G \)

\[
\sum_{g \in \mathcal{B}} |\mathbb{E}_Y((T,f)g)|^2 = \sum_{g \in \mathcal{B}} \mathbb{E}_Y\left(T_i(f \otimes \overline{f}) \cdot (g \otimes \overline{g})\right) = \mathbb{E}_Y\left(T_i(f \otimes \overline{f}) \cdot u_\mathcal{B}\right) = \mathbb{E}_Y\left(T_i(f \otimes \overline{f}) \cdot T_i u_\mathcal{B}\right) = S_i \sum_{g \in \mathcal{B}} |\mathbb{E}_Y(f g)|^2.
\]

Integrating yields

\[
\sum_{g \in \mathcal{B}} \|\mathbb{E}_Y((T,f)g)\|_{L^2}^2 = \sum_{g \in \mathcal{B}} \|\mathbb{E}_Y(f g)\|_{L^2}^2
\]

for each \( t \in G \). Hence, (e) implies \( f \perp_Y \overline{g} \) for all \( g \in \mathcal{B} \). Replacing \( \mathcal{B} \) by \( \mathcal{B}' := \{ \overline{g} \mid g \in \mathcal{B} \} \) and applying the characterization of \( \mathbb{F}(X|Y) \) from [Proposition 8.5](#) yields (a). \( \square \)

For amenable groups one can add to (a)–(e) another equivalent statement in terms of an asymptotic condition on ergodic nets. Recall that a group \( G \) is (discretely) amenable if \( G \) has a (left) Følner net \((N_\alpha)_\alpha\), i.e., \( N_\alpha \subseteq G \) is a finite subset of \( G \) for every \( \alpha \) such that

\[
\lim_{\alpha} \frac{|t N_\alpha \Delta N_\alpha|}{|N_\alpha|} = 0 \text{ for every } t \in G.
\]

Every abelian group is amenable. In case of \( G = \mathbb{Z} \), we obtain a Følner net \((N_k)_{k \in \mathbb{N}}\) by setting \( N_k := \{0, \ldots, k - 1\} \) for \( k \in \mathbb{N} \). Given a representation of \( G \) on a Hilbert space, any Følner net \((N_\alpha)_\alpha\) defines an ergodic net converging to the mean ergodic projection (cf. [Sch13 Thm. 1.7]). This leads to the following extension of [Proposition 8.8](#).

**Proposition 8.9.** Let \( G \) be a group with Følner net \((N_\alpha)_\alpha\), and let \( J : (Y; S) \to (X; T) \) be an extension of measure-preserving \( G \)-systems. Then for \( f \in L^2(X) \) the following assertions are equivalent.

(a) \( f \perp_{L^2} \mathbb{F}(X|Y) \).

(b) \( \lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{r \in N_\alpha} |\mathbb{E}_Y((T_i f) g)|^2 = 0 \) in \( L^1(Y) \) for each \( g \in L^\infty(X) \).

(c) \( \lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{r \in N_\alpha} |\mathbb{E}_Y((T_i f) \overline{g})|^2 = 0 \) in \( L^1(Y) \).

If \( f \in L^2(X|Y) \), then one can add the following assertion to these equivalences.

(d) \( f \perp_{L^2} \mathbb{F}(X|Y) \).

(e) \( \lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{r \in N_\alpha} |\mathbb{E}_Y((T_i f) g)|^2 = 0 \) in \( L^1(Y) \) for each \( g \in L^\infty(X) \).

(f) \( \lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{r \in N_\alpha} |\mathbb{E}_Y((T_i f) \overline{g})|^2 = 0 \) in \( L^1(Y) \).
Now by Proposition 8.8, (a) is equivalent to $P(f \otimes \overline{f}) = 0$, i.e.,

(c') $\lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} (T_i f \otimes \overline{T_i f}) = 0$.  

Multiplying with $g \otimes \overline{g}$ and taking $\mathbb{E}_Y$ we obtain (f).

We show that (f) implies (g). By the preliminary remark, (f) yields in particular

$$\lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) g)| = 0 \quad \text{in } L^1(Y) \quad \text{for each } g \in L^\infty(X).$$

With $f_m := 1_{|f| \leq m} f \in L^\infty(X)$ for $m \in \mathbb{N}$ we obtain

$$\frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) g)| \leq \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) f_m)| + \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} \mathbb{E}_Y(|T_i f| |f - f_m|)$$

and consequently

$$\left\| \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) g)| \right\|_{L^1} \leq \left\| \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) f_m)| \right\|_{L^1} + \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} \|f\|_{L^2} \|f - f_m\|_{L^2}$$

$$= \left\| \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) f_m)| \right\|_{L^1} + \|f - f_m\|_{L^2}$$

for every $\alpha$. This shows (g).

We next show that (g) $\Rightarrow$ (h) if $f \in L^2(X|Y)$. In this case, we have

$$|\mathbb{E}_Y((T_i f) f)| \leq (\mathbb{E}_Y |T_i f|^2)^{\frac{1}{2}} (\mathbb{E}_Y |f|^2)^{\frac{1}{2}}$$

for every $t$ in $G$,

and therefore $|\mathbb{E}_Y((T_i f) f)| \leq \|\mathbb{E}_Y |f|^2\|_{L^\infty(Y)} \cdot 1$ for every $t \in G$. Consequently,

$$\frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) f)|^2 \leq \|\mathbb{E}_Y |f|^2\|_{L^\infty(Y)} \cdot \left( \frac{1}{|N_\alpha|} \sum_{t \in N_\alpha} |\mathbb{E}_Y((T_i f) f)| \right)^2$$

in $L^1(Y)$ for every $\alpha$, and this proves the claimed implication.
Again by the preliminary remark we obtain that (h) implies (g) for \( f \in L^2(X|Y) \). To finish the proof, we show that (g) implies (a) for arbitrary \( f \in L^2(X) \). For \( m \in \mathbb{N} \) we set \( f_m := 1_{E_Y} |f|^2 \in L^2(X|Y) \). Then
\[
\frac{1}{|N_a|} \sum_{t \in N_a} |\mathbb{E}_Y((T_t f_m) f_m)| \leq \frac{1}{|N_a|} \sum_{t \in N_a} |\mathbb{E}_Y((T_t f) f)|
\]
in \( L^1(Y) \) for every \( \alpha \), and hence
\[
\lim_{\alpha} \frac{1}{|N_a|} \sum_{t \in N_a} |\mathbb{E}_Y((T_t f_m) f_m)|^2 = 0 \text{ in } L^1(Y)
\]
for every \( m \in \mathbb{N} \). But this means
\[
(P(f_m \otimes f_m) | f_m \otimes f_m) = \lim_{\alpha} \left\| \frac{1}{|N_a|} \sum_{t \in N_a} |\mathbb{E}_Y((T_t f_m) f_m)|^2 \right\|_{L^1(Y)} = 0
\]
and thus \( P(f_m \otimes f_m) = 0 \) for each \( m \in \mathbb{N} \). We obtain from Proposition 8.8 that \( f_m \perp_{L^2} \mathcal{E}(X|Y) \) for all \( m \in \mathbb{N} \), and then also \( f \perp_{L^2} \mathcal{E}(X|Y) \), hence (a). \( \Box \)

8.4. Ergodic and Weakly Mixing Extensions. A measure-preserving system \((X; T)\) is **ergodic** if the fixed space \( \text{fix}(T) \) is one-dimensional. More generally, an extension \( J: (Y; S) \to (X; T) \) of measure-preserving systems is **ergodic** if \( J(\text{fix}(S)) = \text{fix}(T) \) (cf. [Fur81, Definition 6.1]).

**Examples 8.10.**
1. Let \((X; T)\) be a measure-preserving system. Then the trivial extension \( J: \{\{pt\}; \text{Id}\} \to (X; T) \) is ergodic if and only if \((X; T)\) is ergodic.
2. Suppose that \((X; T)\) is an ergodic system. Then each extension \( J: (Y; S) \to (X; T) \) is ergodic.
3. Consider the skew torus \((X; T)\) and the extension \( J: (Y; S) \to (X; T) \) from Example 8.7. Then \((X; T)\) is ergodic if and only if \( a \) is not a root of unity (see [EFHN15, Prop. 10.17]). However, the extension \( J \) is always ergodic, as shown (in particular) in [EFHN15, Example 17.4.4].

**Definition 8.11.** An extension \( J: (Y; S) \to (X; T) \) is **weakly mixing** if the extension \((Y; S) \to (X \times_Y X; T \times_Y T)\) is ergodic.

**Proposition 8.12.** For an extension \( J: (Y; S) \to (X; T) \) of measure-preserving systems the following assertions are equivalent.

(a) The extension \( J \) is weakly mixing.
(b) \( \mathcal{E}(X|Y) = L^2(Y) \).
(c) \( P_{\text{fix}(T \times_Y T)}(f \otimes f) = 0 \) for all \( f \in L^2(X) \) with \( \mathbb{E}_Y f = 0 \).

(Here, \( P_{\text{fix}(T \times_Y T)} \) is the Markov projection onto the fixed space in \( L^1 \).) If \( G \) is amenable and \((N_a)_{a} \) is any Følner net in \( G \), (a)–(c) are equivalent to the following assertions.
(d) For all \( f \in L^2(X) \) and \( h \in L^\infty(X) \) one has
\[
\lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{i \in N_\alpha} |\mathbb{E}_Y(T_i f \cdot h) - (S_i \mathbb{E}_Y f) \cdot (\mathbb{E}_Y h)|^2 = 0 \quad \text{in } L^1(Y).
\]

(e) For all \( f \in L^2(X) \) with \( \mathbb{E}_Y f = 0 \) and \( h \in L^\infty(X) \) one has
\[
\lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{i \in N_\alpha} |\mathbb{E}_Y(T_i f \cdot h)|^2 = 0 \quad \text{in } L^1(Y).
\]

(f) For all \( f \in L^2(X) \) one has
\[
\lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{i \in N_\alpha} |\mathbb{E}_Y(T_i f \cdot \bar{f}) - (S_i \mathbb{E}_Y f) \cdot (\mathbb{E}_Y \bar{f})| = 0 \quad \text{in } L^1(Y).
\]

(g) For all \( f \in L^2(X) \) with \( \mathbb{E}_Y f = 0 \) one has
\[
\lim_{\alpha} \frac{1}{|N_\alpha|} \sum_{i \in N_\alpha} |\mathbb{E}_Y(T_i f \cdot \bar{f})| = 0 \quad \text{in } L^1(Y).
\]

**Proof.** (a)⇒(c): Suppose (a) and let \( f \in L^2(X) \) with \( \mathbb{E}_Y f = 0 \). Then, for \( h \in \text{fix}(T \times_Y T) \cap L^\infty \) one has \( h \in L^\infty(Y) \) and thus
\[
(f \otimes \bar{f})(h)_{L^2} = \int_{X \times_Y X} h \mathbb{E}_Y(f \otimes \bar{f}) = \int_{X \times_Y X} h |\mathbb{E}_Y f|^2 = 0.
\]

This yields (c).

(c)⇒(b): This follows directly from the implication (c)⇒(a) from Proposition 8.8.

(b)⇒(a): Suppose (b) and note that we need to show \( \text{fix}(T \times_Y T) \subseteq L^2(Y) \). Recall the characterization of \( \text{fix}(T \times_Y T) \) in Theorem 6.9. Let \( \mathcal{B} = \{e_1, \ldots, e_n\} \) be a suborthonormal system in \( L^2(X|Y) \) such that \( u_B = \sum_{j=1}^n e_j \otimes \overline{e_j} \in \text{fix}(T \times_Y T) \). Then \( \mathcal{B} \subseteq \mathcal{E}(X|Y) \subseteq L^2(Y) \) and hence \( \mathcal{B} \subseteq L^\infty(Y) \). It follows that \( u_B \in L^\infty(Y) \). As these elements generate \( \text{fix}(T \times_Y T) \) as a fix(\( S \))-module, we are done.

Finally, assume that \( G \) is amenable and let \((N_\alpha)_{\alpha}\) be any Følner net in \( G \). Observe that “(d) ⇔ (e)” and “(f) ⇔ (g)” (replace \( f \) with \( f - \mathbb{E}_Y f \)). By Proposition 8.9(e) and (g) are both equivalent to (b), and hence the proof is complete.

We restate the result in the important case of \( G = \mathbb{Z} \).

**Corollary 8.13.** Let \( J: (Y; S) \rightarrow (X; T) \) an extension of measure-preserving \( \mathbb{Z} \)-systems and set \( S := S_1 \) and \( T := T_1 \). Then the following assertions are equivalent.

(a) The extension \( J \) is weakly mixing.

(b) For all \( f \in L^2(X) \) and \( h \in L^\infty(X) \) one has
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}_Y(T^n f \cdot h) - (S^n \mathbb{E}_Y f) \cdot (\mathbb{E}_Y h)|^2 = 0 \quad \text{in } L^1(Y).
\]
(c) For all \( f \in L^2(X) \) with \( \mathbb{E}_Y f = 0 \) and \( h \in L^\infty(X) \) one has
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mathbb{E}_Y (T^n f \cdot h)|^2 = 0 \quad \text{in} \quad L^1(Y).
\]

(d) For all \( f \in L^2(X) \) one has
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mathbb{E}_Y (T^n f \cdot \overline{f}) - (S^n\mathbb{E}_Y f) \cdot (\mathbb{E}_Y f)|^2 = 0 \quad \text{in} \quad L^1(Y).
\]

(e) For all \( f \in L^2(X) \) with \( \mathbb{E}_Y f = 0 \) one has
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mathbb{E}_Y (T^n f \cdot \overline{f})|^2 = 0 \quad \text{in} \quad L^1(Y).
\]

8.5. **Furstenberg–Zimmer Structure Theorem.** We are now ready to state and prove the structure theorem of Furstenberg and Zimmer for measure-preserving systems, comprised of two parts. The first is the following dichotomy result.

**Theorem 8.14 (Furstenberg–Zimmer (dichotomy)).** Let \( J : (Y; S) \to (X; T) \) be an extension of measure-preserving systems. Then exactly one of the following statements is true.

(a) The extension \( J \) is weakly mixing.

(b) There is a non-trivial extension \( J_1 : (Y; S) \to (Z; R) \) with relative discrete spectrum and an extension \( J_2 : (Z; R) \to (X; T) \) such that the diagram

\[
\begin{array}{ccc}
(Y; S) & \xrightarrow{J} & (X; T) \\
\downarrow{J_1} && \downarrow{J_2} \\
(Z; R) & &
\end{array}
\]

commutes.

**Proof.** As we have seen in Section 8.3, one can always factor \( J \) through the relative Kronecker factor
\[
(Y; S) \to \text{Kro}(X|Y); T) \to (X; T).
\]

The first extension has, by construction, relative discrete spectrum. The result is therefore a direct consequence of Proposition 8.12. \( \square \)

In order to formulate the second part of the structure theorem we recall the following concept from [Fur77, Definition 8.3].

**Definition 8.15.** An extension \( J : (Y; S) \to (X; T) \) is **distal** if there is an ordinal \( \eta_0 \) and an inductive system \(((X_\eta; T_\eta))_{\eta < \eta_0}, (J_\eta^\mu)_{\eta < \sigma})\) such that

- \( J_1^\eta = J \),
- \( J_\eta^{\eta+1} \) has relatively discrete spectrum for every \( \eta < \eta_0 \),
- \( (X_\eta; T_\eta) = \lim_{\mu < \eta} (X_\mu; T_\mu) \) for every limit ordinal \( \mu \leq \eta_0 \).
An inductive system as above is called a **Furstenberg tower**.

**Theorem 8.16.** Let \( J: (\mathbb{Z}; R) \to (X; T) \) be an extension of measure-preserving systems. Then there is a distal extension \( J_1: (\mathbb{Z}; R) \to (Y; S) \) and a weakly mixing extension \( J_2: (Y; S) \to (X; T) \) such that the diagram

\[
\begin{array}{ccc}
(\mathbb{Z}; R) & \xrightarrow{J} & (X; T) \\
\downarrow{J_1} & & \downarrow{J_2} \\
(Y; S) & & \\
\end{array}
\]

commutes.

**Proof.** As described in Chapter 7, the extensions into \( L^2(X) \) can (and now will) be identified with the invariant unital Banach sublattices of \( L^2(X) \). In particular, we regard \( E_1 := L^2(\mathbb{Z}) \) as a sublattice of \( L^2(X) \). The intermediate extensions then correspond to invariant sublattices \( E \) containing \( E_1 \), and \( E \) will be called a distal sublattice if the corresponding extension \( E_1 \to E \) is a distal extension.

Starting with \( E_1 \) we shall construct by transfinite induction a Furstenberg tower. Suppose \( \eta \) is an ordinal and we have already constructed a Furstenberg tower \( (E_\sigma)_{\sigma < \eta} \).

If \( \eta = \eta' + 1 \) is not a limit ordinal, then to the sublattice \( E_\eta' \) we find an associated extension \( J_2: (Y; S) \to (X; T) \) and let \( E_\eta := \mathcal{E}(X|Y) \) be the Kronecker space of the extension.

If \( \eta = \lim_{\sigma < \eta} \sigma \) is a limit ordinal, we let \( E_\eta \) be the closure of the union of all \( E_\sigma \) for \( \sigma < \eta \). Then any system \( (Y; S) \) associated with \( E_\eta \) is the inductive limit of systems associated with \( E_\sigma \) for \( \sigma < \eta \).

In either case \( (E_\sigma)_{\sigma < \eta} \) is again a Furstenberg tower. Now, for reasons of cardinality, there must be an ordinal \( \eta \) with \( E_\eta = E_{\eta+1} \). Let \( J_2: (Y; S) \to (X; T) \) be an associated extension. Then, by the dichotomy theorem, \( J_2 \) is weakly mixing and the proof is complete. \( \square \)

**Notes and Comments**

The Furstenberg–Zimmer structure theorem was first proved, under separability and ergodicity assumptions, by Zimmer in [Zim77] and, independently, by Furstenberg in [Fur77]. In [FK78], Furstenberg and Katzenelson generalized Furstenberg’s result from \( \mathbb{Z} \)-actions to \( \mathbb{Z}^d \)-actions. An alternative presentation was given in [FKO82] and in Furstenberg’s book [Fur81], where the ergodicity assumption is dropped. Versions of the result can also be found in modern textbooks on ergodic theory, e.g., in [Gla03, Chapter 9], [EW11, Section 7.8] and [KL16, Section 3.3]. Of course, our exposition is strongly influenced by these works. However, we emphasize once more that here the result is derived as a mere corollary of the more general structure theorem on unitary group representations on Kaplansky–Hilbert modules proved in Part II of this article. This shows that the Furstenberg–Zimmer structure theorem can be viewed, in essence, as a result of “relative operator theory”.

As is well-known, there are several alternative notions of “structure” leading to the Furstenberg–Zimmer theorem. Here we follow Furstenberg’s original approach using
“(relative) discrete spectrum”, but likewise one can use “isometric” extensions or “compact” extensions. It is well-known that for a measure-preserving system \((X;\sigma)\) the following assertions are equivalent:

(a) The space \(L^2(X)\) is the closed union of all finite dimensional invariant subspaces.
(b) The orbit \(\{T_t f \mid t \in G\}\) is totally bounded in \(L^2(X)\) for every \(f \in L^2(X)\).

(This equivalence, by the way, tells that the decomposition into the “discrete spectrum” part and the “weakly mixing” part coincides with the Jacobs–deLeeuw–Glisckberg decomposition, cf. the Notes to Part II on page 63.)

Most approaches to the FZ-theorem take either (a) or (b) and transfer it to the relative setting. For example, Furstenberg and Zimmer introduce structured extensions based on (a) in their original articles, while in [FKO82] the description (b) is employed. It is of course a natural question how these different approaches are related. In his book, Furstenberg shows the equivalence of several different approaches to structured extensions (see [Fur81, Theorem 6.13]). His results have been extended recently by Jamneshan in [Jam23] showing, in particular, that the “algebraic approach” of (a) and the “topological approach” of (b) are still equivalent for extensions (if generalized suitably). This is even true in the context of dynamical systems of von Neumann algebras as shown by Jamneshan and Spaas in [JS22].

Although we chose (a) for our approach, also (b) can be relativized through our KH-module setting. Namely, we call a subset \(M\) of a KH-module \(E\) over a Stone algebra \(A\) totally order-bounded if the net

\[
F \mapsto \inf_{y \in F} |x - y| \quad (F \subseteq E \text{ finite})
\]

decreases to 0 uniformly in \(x \in M\). Employing Proposition 8.8 one can then show the following characterization of extensions with relative discrete spectrum.

**Theorem.** For an extension \(J : (Y, S) \to (X, T)\) the following assertions are equivalent.

(a) \(J\) has relative discrete spectrum.
(b) \(L^2(X|Y) = \text{ocl}\{f \in L^2(X|Y) \mid \{T_t f \mid t \in G\}\}\) is totally order bounded.

A proof of this result as well as a detailed examination of notions of “order compactness”, their relations to conditional Boolean valued analysis (“cyclical compactness”) and conditional set theory, and their applications to ergodic theory, will be the content of future work.

Finally, let us mention that Furstenberg’s “main result on fibered products” [Fur77, Thm. 7.1], [Cla03, Thm. 9.21], i.e., the identity

\[
\mathcal{G}(X \times_Y Z|Y) = \mathcal{G}(X|Y) \otimes_Y \mathcal{G}(Z|Y),
\]

can be reduced to the identity \((E \otimes F)_{ds} = E_{ds} \otimes F_{ds}\) being valid for KH-dynamical systems. This will be explained and proved in the forthcoming paper [HK23].

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