Certain and Uncertain Inference with Trivalent Conditionals*

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Abstract

Research on indicative conditionals usually aims either at determining their truth conditions, or at explaining how we should reason with them and when we can assert them. This paper integrates these semantic and epistemological projects by means of articulating trivalent, truth-functional truth conditions for indicative conditionals. Based on this framework, we provide a (non-classical) account of the probability of conditionals, and two logics of conditional reasoning: (i) a logic C of inference from certain premises that generalizes deductive reasoning; and (ii) a logic U of inference from uncertain premises that generalizes defeasible reasoning. Both logics are highly attractive in their domain. They provide a unified framework for conditional reasoning, generalize existing theories (e.g., Adams’s logic of “reasonable inference”) and yield an insightful analysis of the controversies about the validity of Modus Ponens, Import-Export and other principles of conditional logic.

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1 Introduction and Overview

Two research projects have dominated the literature on indicative conditionals (henceforth “conditionals”): the semantic project of determining their truth conditions, and the epistemological and pragmatic project of explaining how we should reason with them and when to assert them. The two projects are related: according to Jackson (1979, p. 589) and various other theorists, “the best approach to meaning and analysis is via truth conditions, and we should hope for a theory which explains the assertion conditions in terms of the truth conditions”. Specifically, probability plays a central role in connecting the semantic, epistemological and pragmatic dimension.

For example, David Lewis (1976, p. 297) claims that “assertability goes by subjective probability”, and similar positions are articulated by Adams (1965, pp. 173-174), Jackson (1979, p. 565) and, more recently, Leitgeb (2017, p. 278). Even if one rejects this identification (e.g., based on the arguments in Douven 2016; Carter forthcoming), a probabilistic treatment of indicative conditionals remains of the highest relevance for explaining how we reason with them, which conditional inferences are justified, and which role conditionals play in natural language. The question is whether we can ground an account of the probability of conditionals in their truth conditions, in order to provide a unified theory of truth conditions, probability and reasoning.

This paper develops such a theory. The basic idea is to understand an indicative conditional as a conditional assertion of the consequent, a view entertained by Quine (1953) and Belnap (1970, 1973), and pioneered by Reichenbach (1935) and de Finetti (1936a). Conditional assertions are evaluated as true if the antecedent and the consequent both hold, false when the antecedent holds, but the consequent not, and neither true nor false (=the third truth value) when the antecedent does not hold, since the commitment to the consequent is retracted.

This simple approach allows for a general semantics of conditionals and arbitrary compounds of conditionals. The probability of a proposition $X$ in this semantics is defined analogously to standard bivalent probability: the weight of possible worlds where $X$ is true, divided by the weight of the possible worlds where $X$ has classical truth value.\(^1\)

This definition yields an attractive logic $\mathcal{C}$ for certain inference in a propositional language $\mathcal{L}^\rightarrow$ endowed with an indicative conditional, and an equally attractive probabilistic logic $U$ for uncertain inference, where justified assertion (rather than truth) is preserved from the premises to the

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\(^1\)The reference to possible worlds does not make our semantics modal: it remains truth-functional. Only the probability of conditionals is defined via quantification over truth-functional valuations (here called “possible worlds”) of the appropriate kind. More on this in Section 6.
conclusion, and explicated as high subjective probability (pace Lewis, Jackson and Leitgeb). Both logics can be characterized purely probabilistically, as preservation of maximal/high probability in inference, but also in terms of consequence relations within trivalent logic (Proposition 1, 3 and 4). C and U turn out to be natural generalizations of deductive and defeasible reasoning to a language with a conditional; indeed, classical laws are theorems of C and U. Moreover, the difference between the principles validated by C and U explains to a great deal the controversies about principles such as Modus Ponens and Import-Export (e.g., McGee 1985; Sinnott-Armstrong, Moor, and Fogelin 1986).

These promises may look too good to be true. After all, there are well-known reasons to believe that strictly truth-functional accounts of indicative conditionals are doomed. Especially, we need to reply to the famous triviality results regarding the probability of indicative conditionals (Lewis 1976), and the collapse of the indicative to the material conditional (Gibbard 1980). To this end, we will note that the assumptions behind the triviality results cease to apply in a trivalent setting (see Section 5 and 6). Moreover, anybody skeptical about truth-functional accounts can read the paper as developing two attractive probabilistic logics of conditional reasoning, and consider the equivalent trivalent, truth-functional characterizations of these logics merely as a convenient calculation device.

Our paper is divided into three parts. The first part focuses on the semantic foundations, building on the technical work in Égré, Rossi, and Sprenger (2021a), but with more attention to normative questions. Section 2 points out how the standard accounts struggle to unify the truth conditions and the probability of indicative conditionals. Section 3 motivates the trivalent treatment of conditionals and Section 4 defines the trivalent truth tables for Boolean connectives and the indicative conditional, while Section 5 defends a specific relation of logical consequence for deductive inference with conditionals. The resulting logic C preserves non-falsity and interprets the connectives by means of Cooper’s (1968) truth tables for the trivalent conditional and negation, conjunction and disjunction.

The second part of the paper focuses on epistemology and reasoning. Section 6 defines the (non-classical) probability of trivalent propositions in analogy with standard bivalent probability. Section 7, the centerpiece of our paper, develops a logic U of uncertain reasoning where probability is preserved in passing from the premises to the conclusion. These sections also present the equivalent characterizations of C and U in terms of probabilistic and trivalent consequence relations. Section 8 examines which principles of conditional logic are valid in C and U (and which are violated, and why).
The third part contains applications, comparisons and evaluations: Section 9 discusses nested conditionals and McGee’s objection to Modus Ponens from the vantage point of our logic. Section 10 draws comparisons with other theories. Section 11 draws the balance and highlights the strengths and limits of our account. The appendices provide proof details (Appendix A and B), elaborate on trivalent probability (Appendix C) and explore an extension of our account to counterfactuals (Appendix D).

2 Truth Conditions vs. Probability

The subjective probability of a proposition \( X \) is typically interpreted as the degree of belief that \( X \) is true. Ideally, then, an account of the probability of conditionals would follow from a truth-conditional analysis of conditionals. Probability is then expressed as the expectation of semantic value, taken over all possible worlds. In other words, the probability of a conditional “if \( A \), then \( C \)”, written as \( A \rightarrow C \), would correspond to the total weight of the possible worlds where \( A \rightarrow C \) is true. (We use uppercase Latin letters to denote sentential variables.) If such an analysis were possible, it would unify the semantic and epistemological treatment of conditionals. Moreover, it would provide the tools for distinguishing two logics of conditional reasoning: a logic of deductive reasoning with certain premises, and a logic of defeasible reasoning with uncertain premises. Since the analogous distinction between classical logic for deductive inference and probabilistic logics for inductive inference is universally recognized as sensible and fruitful, we will also make it in conditional reasoning (see also Ciardelli 2020; Iacona forthcoming).

However, we first need to define the probability of a conditional \( A \rightarrow C \). The standard view holds that its probability \( A \rightarrow C \) should go by the conditional probability \( p(C|A) \) (e.g., Stalnaker 1970; Adams 1975; Égré and Cozic 2011). For conditional-free sentences \( A \) and \( C \), the equation

\[
p(A \rightarrow C) = p(C|A)
\]

(Adams’s Thesis)

figures as “Adams’s Thesis” in the literature. Whenever the antecedent is clearly relevant for the consequent, the principle enjoys also strong empirical support (e.g., Over, Hadjichristidis, et al. 2007; Skovgaard-Olsen, Singmann, and Klauer 2016). Because of its significance and intuitive appeal, Adams’s Thesis is also called “The Equation” in the psychology of reasoning, while the extension of Adams’s Thesis to arbitrary sentences \( A \) and \( C \) is known as “Stalnaker’s Thesis”. For more details on the terminological distinctions, see Égré and Rott (2021).
Unfortunately, David Lewis’s well-known triviality result complicates the picture. Lewis (1976) shows that, if (i) the probability of a sentence depends on its truth conditions in the above sense, and (ii) the probability function is closed under conditionalization, Adams’s Thesis implies $p(A \rightarrow C) = p(C)$ (whenever $A$ is compatible with both $C$ and its negation). This is clearly a reductio ad absurdum. Similar triviality results have been proven by Hájek (1989), Bradley (2000), and Milne (2003).

The triviality results limit the available strategies for a unified treatment of truth conditions and probability. Modal analyses of conditionals provide intuitive semantic models of the truth conditions of conditionals, and powerful formal systems for reasoning with them (e.g., Stalnaker 1968; Lewis 1973b). Moreover, they can integrate the treatment of “if” with other modal operators (e.g., Kratzer 1986). This has made modal semantics the dominant approach in current-day epistemology, but at a substantial price. First, most modal accounts do not integrate truth conditions with the probability of conditionals (e.g., Kratzer 2002; Khoo 2015; Mandelkern 2019), omitting an important part of the epistemological dimension of conditionals. Second, the meaning of conditionals may become at least partially belief-dependent (e.g., van Fraassen 1976; Bradley 2012). While we agree that the analysis of conditionals requires a subjective component, we do not think that the truth conditions, which have the function of settling disputes about the truth value of a sentence, are the right place for them. Third, some modal accounts propose the conditional probability $p(A|C)$ as a possible semantic value for the conditional $A \rightarrow C$ (e.g., McGee 1989; Stalnaker and Jeffrey 1994). However, this analysis reverses the traditional direction of the dependency between the probability and the truth conditions of a sentence.

On the other end of the spectrum, theorists such as Ernest W. Adams (1965, 1975, 1996a), Alan Gibbard (1980), Dorothy Edgington (1986, 1995, 2009) and David Over (e.g., Evans et al. 2007; Cruz et al. 2016; Over and Baratgin 2017) deny that indicative conditionals express propositions: at most they have partial truth conditions.

[...] the term ‘true’ has no clear ordinary sense as applied to conditionals, particularly to those whose antecedents prove to be false [...] In view of the foregoing remarks, it seems to us to be a mistake to analyze the logical properties of conditional statements in terms of their truth conditions. (Adams 1965, pp. 169-170)

By interpreting the meaning of a conditional $A \rightarrow C$ simply as expressing the subjective probability $p(C|A)$, these scholars avoid Lewis’s triviality results, but at the same time, they give up the hopes for a principled account unifying semantics and epistemology, where the probability of a proposition corresponds to its expected semantic value. It is generally recognized
that the non-propositional approach makes excellent predictions on how we reason with conditionals, and Adams’s (1975) probabilistic logic for indicative conditionals is still a central point of reference for any new approach. On the other hand, the focus on preserving probability in uncertain reasoning arguably narrows down the theoretical picture. Specifically, due to the lack of truth conditions, the approach does not clarify how one can argue and disagree about conditional sentences in a similar way as we do for normal, non-conditional propositions (Bradley 2012, p. 547).²

We claim that trivalent semantics provides the basis for unifying the truth conditions and the probability of conditionals, and resolving the above problems. The next section presents our basic idea.

3 The Basic Idea

It is controversial whether indicative conditionals have factual truth conditions and can be treated as propositions (e.g., see the dialogue in Jeffrey and Edgington 1991). However, even a defender of a non-truth-conditional view such as Adams (1965, p. 187) admits that we feel compelled to say that a conditional “if \( A \), then \( C \)” has been verified if we observe both \( A \) and \( C \), and falsified if we observe \( A \) and \( \neg C \). For example, take the sentence “if it rains, the match will be cancelled”; it seems to be true if it rains and the match is in fact cancelled, and false if the match takes place in spite of rain. Indeed, what else could be required for determining the truth or falsity of the sentence?

This “hindsight problem” (e.g., Khoo 2015) is a prima facie reason for treating conditionals as propositions, and assigning them factual truth conditions. Defenders of non-propositional accounts need to explain why observations in our actual world are sufficient for the truth or falsity of “if \( A \), then \( C \)”, and why \( A \rightarrow C \) behaves so differently when \( A \) is false.

Treating the indicative conditional “if \( A \), then \( C \)” as a conditional assertion—i.e., as an assertion about \( C \) upon the supposition that \( A \) is true—explains the asymmetry in truth conditions. On this account, when the

² Two other approaches deserve mention. First, there is Jackson’s view that the truth conditions of the indicative and the material conditional agree, and that perceived differences are due to pragmatic, not to semantic factors (Jackson 1979; Grice 1989). This account needs to explain the well-known paradoxes of material implication, and why the probability of an indicative conditional often differs from the probability of the corresponding material conditional (Adams 1975; Rott 2022, pp. 3-4). Second, inferentialist views on the meaning of conditionals focus on relations of relevance and evidential support between antecedent and consequent (e.g., Douven 2008, 2016; van Rooij and Schulz 2019). However, with the exception of the modal and probabilistic accounts by Crupi and Iacona (2021a,b), there is no integrated analysis of truth conditions, probability and reasoning in this paradigm (see also Iacona forthcoming).
antecedent is false, the speaker is committed to neither truth nor falsity of the consequent. This intuition has been voiced perhaps most prominently by Quine (1950, p. 12, our emphasis):

An affirmation of the form “if \( p \) then \( q \)” is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. If on the other hand the antecedent turns out to have been false, our conditional affirmation is as if it had never been made.

In other words, asserting a conditional makes an epistemic commitment only in case the antecedent turns out to be true. If it turns out to be false, the assertion is retracted: there is no factual basis for evaluating it (see also Belnap 1970, 1973). Therefore it is classified as neither true nor false. The “gappy” or “defective” truth table of Table 1 interprets this view as a partial assignment of truth values to conditionals (e.g., Reichenbach 1935; Adams 1975; Baratgin, Over, and Politzer 2013; Over and Baratgin 2017). Sometimes the probability \( p(C|A) \) also figures as a surrogate truth value in the lower row of the table (e.g., in the random variable proposal by Stalnaker and Jeffrey 1994; or in Sanfilippo et al. 2020).

$$\begin{array}{c|cc}
\text{Truth value of } A \rightarrow C & v(C) = 1 & v(C) = 0 \\
v(A) = 1 & 1 & 0 \\
v(A) = 0 & \text{(neither)} & \text{(neither)}
\end{array}$$

Table 1: ‘Gappy’ or ‘defective’ truth table for a conditional \( A \rightarrow C \) for a (partial) valuation function \( v : \mathcal{L}^\rightarrow \rightarrow \{0, 1\} \) in a language with conditional.

However, without a full truth-conditional treatment, such an account is limited: it neither evaluates nested conditionals, nor Boolean compounds of conditionals. If we could complete Table 1 and provide full truth conditions in a satisfactory way, this would greatly increase the scope and descriptive power of conditional reasoning, and facilitate the identification of theorems and valid inferences.

The obvious candidate for such truth conditions is a trivalent truth table, where the absence of commitment to the consequent \( C \) is represented by a third truth value. Instead of using partial valuations, we assign a third semantic value, \( 1/2 \) or “indeterminate”, when the antecedent is false (See Table 2). This is a recurring idea in the literature, defended, among others, by Jeffrey (1963), Cooper (1968), Belnap (1970, 1973), Manor (1975), Farrell (1986), McDermott (1996), Olkhovikov (2002/2016), Cantwell (2008), Rothschild (2014), and Égré, Rossi, and Sprenger (2021a,b).
Truth value of \( A \rightarrow C \)
\[
\begin{array}{cc}
\text{\( v(A) = 1 \)} & \text{\( v(C) = 1 \)} \\
1 & 0 \\
\text{\( v(A) = 0 \)} & \text{\( v(C) = 1 \)} \\
1/2 & 1/2 \\
\end{array}
\]

Table 2: Partial trivalent truth table for a conditional \( A \rightarrow C \) for a partial valuation function \( v : \mathcal{L}^n \rightarrow \{0, 1/2, 1\} \) in a language with conditional.

This basic idea has to be developed in various directions. Firstly, we need to decide how to extend the truth table of Table 1 to a fully trivalent truth table for \( A \rightarrow C \) where \( A \) and \( C \) can also take the value \( 1/2 \) (=neither true nor false, indeterminate). Secondly, we need to decide how to interpret the standard Boolean connectives \( \land, \lor, \neg \) in the context of propositions which can take three different truth values. Doing so will allow us to deal with nested conditionals, and more generally, with arbitrary compounds of atomic sentences connected by the standard connectives and \( \rightarrow \). Thirdly, we need to decide on a trivalent logical consequence relation for two kinds of reasoning: for deductive reasoning where premises are certain, and for defeasible reasoning (e.g., Adams’s “reasonable inference”) where premises are uncertain. The latter logic will be our analogue of System \( P \) (Adams 1975; Kraus, Lehmann, and Magidor 1990), Hawthorne and Makinson’s System \( O \) (Hawthorne 1996; Hawthorne and Makinson 2007), and other probabilistic logics for uncertain reasoning. But first, let us set up the truth tables for the conditional and the Boolean connectives.

4 Trivalent Truth Tables

We start by extending the basic idea of Table 2 to a full trivalent truth table for \( A \rightarrow C \). The two main options are shown in Table 3 and have been proposed by Bruno de Finetti (1936a) and William Cooper (1968), respectively. We abbreviate the two connectives with “DF” and “CC” (the latter after Cooper-Cantwell).\(^3\) In both of them the value \( 1/2 \) can be interpreted as “neither true nor false”, “void”, or “indeterminate”. There is moreover a systematic duality between those tables: whereas de Finetti treats indeterminate antecedents like false antecedents, Cooper treats them like true ones. Thus, in de Finetti’s table the second row copies the third, whereas in Cooper’s table it copies the first.

In principle, both options can be pursued fruitfully, and the choice between them also depends on the results which they yield. We will see later

\(^3\)Belnap (1973), Olkhovikov (2002/2016) and Cantwell (2008) rediscovered Cooper’s truth table independently.
that the Cooper table interacts more naturally with various notions of logical consequence (a detailed analysis is given in Égré, Rossi, and Sprenger 2021a).

The second choice concerns the definition of the standard logical connectives. A natural option is given by the familiar Łukasiewicz/Strong Kleene truth tables, displayed in Table 4. Conjunction corresponds to the “minimum” of the two values, disjunction to the “maximum”, and negation to inversion of the semantic value. In particular, the trivalent analysis admits, next to the indicative conditional \( A \rightarrow C \), a Strong Kleene “material” conditional \( A \supset C \), definable as \( \neg(\neg A \wedge \neg C) \), or equivalently, \( \neg A \lor C \).

\[
\begin{array}{c|ccc}
\neg & 1 & 1/2 & 0 \\
1 & 1 & 1/2 & 0 \\
1/2 & 1/2 & 1/2 & 1/2 \\
0 & 1/2 & 1/2 & 1/2
\end{array}
\begin{array}{c|ccc}
\wedge & 1 & 1/2 & 0 \\
1 & 1 & 1/2 & 0 \\
1/2 & 1/2 & 1/2 & 1/2 \\
0 & 1/2 & 1/2 & 1/2
\end{array}
\begin{array}{c|ccc}
\lor & 1 & 1/2 & 0 \\
1 & 1 & 1 & 1 \\
1/2 & 1/2 & 1/2 & 1/2 \\
0 & 1 & 1/2 & 0
\end{array}
\]

Table 3: Truth tables for the de Finetti conditional (left) and the Cooper conditional (right).

Table 4: Strong Kleene truth tables for negation, conjunction, and disjunction.

The Strong Kleene truth table for negation is uncontroversial and also yields the consequence that the conditional commutes with negation (for either the DF- or the CC-conditional): \( \neg(\neg A \rightarrow C) \) has the same truth table as \( A \rightarrow \neg C \). This is a very natural choice for interpreting conditional assertions: when we argue about \( A \rightarrow C \), both sides presuppose the antecedent \( A \) and argue about whether we should be committed to \( C \) or rather to \( \neg C \), given \( A \) (see also Ramsey 1929/1990, p. 247).

That said, the Strong Kleene truth tables for conjunction and disjunction have a very annoying consequence: “partitioning sentences” such as \( (A \rightarrow B) \land (\neg A \rightarrow C) \) will always be indeterminate or false (Belnap 1973; Bradley 2002, pp. 368-370). However, a sentence such as:

If the sun shines tomorrow, John goes to the beach; and if it rains, he goes to the museum.
seems to be true (with hindsight) if the sun shines tomorrow and John goes indeed to the beach. This intuition is completely lost in Strong Kleene semantics. Even worse, “obvious truths” such as \((A \rightarrow A) \land (\neg A \rightarrow \neg A)\) are always classified as indeterminate.\(^4\)

For this reason, we endorse alternative truth tables for conjunction and disjunction, advocated by Cooper (1968) and Belnap (1973), where true and false sentences do not change truth value when conjoined or disjoined with an indeterminate one, as in Table 5. In other words, indeterminate sentences are “truth-value neutral” in Boolean operations. This can be motivated by observing that they do not add determinate content as empirical statements do. The truth tables of Table 5 retain the usual properties of Boolean connectives (associativity, commutativity, the de Morgan laws, etc.), solve the problem of partitioning sentences, and have no substantial disadvantages with respect to Strong Kleene truth tables in conditional logic.\(^5\) Moreover, they have two non-trivial benefits.

| \(f\) | \(f'\) | \(1\) | \(1/2\) | \(0\) |
|------|------|-----|-----|-----|
| 1    | 0    | 1   | 1   | 0   |
| 1/2  | 1/2  | 1/2 | 1/2 | 0   |
| 0    | 1    | 0   | 0   | 0   |
| \(f''\) | 1 | 1/2 | 0 |

Table 5: Truth tables for Strong Kleene negation, paired with quasi-conjunction and quasi-disjunction as defined by Cooper (1968).

First, quasi-disjunction avoids the Linearity principle that \((A \rightarrow B) \lor (B \rightarrow A)\) cannot be false. This schema was famously criticized by MacColl (1908), who pointed out that neither “if John is red-haired, then John is a doctor”, nor “if John is a doctor, then he is red-haired”, nor their disjunction seems acceptable in ordinary reasoning. A semantics that qualifies such expressions as either true or indeterminate might thus be considered inadequate. Using quasi-conjunction and quasi-disjunction instead, \((A \rightarrow B) \lor (B \rightarrow A)\) is false when \(A\) is true and \(B\) is false (or vice versa).

\(^4\)An example due to P. Santorio.

\(^5\)In a logic that preserves non-falsity, quasi-disjunction makes Disjunction Introduction \((A \models A \lor B)\) invalid, e.g. when \(A\) has value \(1/2\) and \(B\) value \(0\). Similarly, in a logic that preserves (strict) truth, Conjunction Elimination \((A \land B \models B)\) can fail, e.g. when \(A\) has value \(1\) and \(B\) value \(1/2\). Having to choose between these options, it looks more reasonable to preserve the second type of inference than the first. (The combination of quasi-connectives and non-falsity preservation fails, e.g., in the inference from “if \(2 + 2 = 5\), then grass is green” to “if \(2 + 2 = 5\), then grass is green, or the Moon is made of Swiss cheese”, but this is a reasonable loss.) This is one of the reasons why quasi-connectives interact well with non-falsity-preserving logics.
Besides, unlike Strong Kleene disjunction, quasi-disjunction validates the rule of Disjunctive Syllogism when paired with a non-falsity-preserving definition of validity ($\neg A, A \lor B \models B$). As anticipated above, this suggests that the superiority of the Cooper truth tables is not independent from the choice of the consequence relation. That is, they offer the most promising combination of Boolean connectives and a relation of logical consequence for modeling conditional inference.

There is also a principled reason for adopting quasi-conjunction and quasi-disjunction, based on the connection between conditional bets and conditional assertions. How should we evaluate the conjunction of conditional assertions like $(A \rightarrow B) \land (C \rightarrow D)$? The interesting case occurs when $A$ is false, but $C$ and $D$ are true. McGee (1989, 496-501, in particular Theorem 1) generalizes from our attitudes toward non-conditional bets and shows by a Dutch Book argument that in this case, a bet on $(A \rightarrow B) \land (C \rightarrow D)$ should yield a strictly positive partial return. Also Sanfilippo et al. (2020, p. 156) argue that we should classify the compound bet as winning. Indeed, to the extent that the sentence $(A \rightarrow B) \land (C \rightarrow D)$ is testable, it has been verified when $A$ is false, but $C$ and $D$ are true. All this suggests to treat the assertion $(A \rightarrow B) \land (C \rightarrow D)$ as true rather than indeterminate. Unlike Strong Kleene conjunction, quasi-conjunction follows this line of reasoning in this case.

For all these reasons, we adopt quasi-conjunction, quasi-disjunction, Strong Kleene negation and the Cooper-Cantwell truth table for the conditional in the remainder. Note that all combinations of conditional and Boolean connectives surveyed in this section validate Import-Export: $(A \land B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$ are extensionally equivalent formulas.

5 Certain Inference

Next, we need to choose a trivalent relation of logical consequence for certain inference with conditionals. By certain inference, we mean inference either from known premises, or from hypothetical premises held with maximum probability.

The options that have been pursued most frequently in the literature are preservation of value 1 (SS-consequence), preservation of value greater or equal than $1/2$ (TT-consequence), and the combination of them (SS\cap TT-consequence), which demands that the truth value of the conclusion be at least as high as the truth value of the premises.\footnote{Mixed consequence relations such as “if the premise takes value 1, the conclusion needs to take value at least $1/2$” (aka. ST) are discussed, and discarded as inadequate for the Cooper-Cantwell conditional, in Égré, Rossi, and Sprenger (2021a,b).}
For evaluating these consequence relations in the context of deductive reasoning, we propose three basic constraints:

**Modus Ponens** From $A \rightarrow B$ and $A$ we should be able to infer $B$, that is: $A \rightarrow B, A \vdash B$.

**Law of Identity** The conditional “if $A$, then $A$”, should be a validity of the logic: $\vdash A \rightarrow A$.

**No Inference to the Converse** From the conditional “if $A$, then $B$”, we should not be able to infer the converse, that is: $A \rightarrow B \nvdash B \rightarrow A$.

We consider these constraints as eminently reasonable for a logic of conditional reasoning from certain premises: Modus Ponens is a standard desideratum for deductive logics, the Law of Identity is self-evident and the two conditionals $A \rightarrow B$ and $B \rightarrow A$ clearly have different meanings.\(^7\)

These consequence relations can be paired with the de Finetti or the Cooper truth table for the conditional, yielding various logics which we call DF/SS, DF/TT, CC/SS, and so on. Table 6 shows how our candidate logics fare with regard to the above adequacy conditions. The results are independent of the interpretation of conjunction and disjunction.

The only logics that satisfy all three adequacy conditions are of the CC/TT type—i.e., the Cooper truth table for the conditional paired with a TT-consequence relation. CC/TT logics satisfy various technically important principles such as Conditional Introduction, a full Deduction Theorem, and Conditional Excluded Middle, without validating problematic principles or

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\(^7\)Note that McGee’s famous counterexample to Modus Ponens is based on the premises being uncertain—see Section 9 for further discussion.
falling prey to triviality results (Égré, Rossi, and Sprenger 2021a).\(^8\) Hence the case for a logic of type CC/TT is very strong. In line with our arguments from the last section, we adopt the logic QCC/TT where quasi-conjunction and -disjunction (see Table 5) replace the Strong Kleene connectives. This logic will for brevity be called \(\mathbb{C}\) and it is our preferred logic for deductive reasoning with conditionals.

**Definition 1.** \(A_1, \ldots, A_n \models_{\mathbb{C}} B\) if and only if \(A_1, \ldots, A_n \models_{\text{QCC/TT}} B\)

\(\mathbb{C}\) is a paraconsistent logic almost equivalent to Cooper’s—his propositional logic of Ordinary Discourse—except that we do not restrict \(\mathbb{C}\) to bivalent valuations. Specifically, we give up Disjunction Introduction \((A \models A \lor B)\), but retain Disjunctive Syllogism \((A \lor B, \neg A \models B\), see footnote 5).

We now turn to the second part of the paper, where we explicate the probability of sentences of \(\mathcal{L}^-\), and identify an appropriate logic of uncertain inference with conditionals.

## 6 Probability for Trivalent Propositions

Epistemologists capture the standing of a proposition \(A\) by the *probability* of \(A\), which is supposed to reflect all the evidence the agent has for or against \(A\). When we identify propositions with sets of possible worlds, the probability of a proposition \(A\) is simply the cumulative credence assigned to all possible worlds where \(A\) is true.

Trivalent semantics for conditionals implements the same approach using a slight twist. As with bivalent probability, we start with a set of possible worlds \(W\) with associated algebra \(\mathcal{A}\), and a weight or credence function \(c: \mathcal{A} \to \mathbb{R}\) defined on the measurable space \((W, \mathcal{A})\), which represents the subjective plausibility of a particular element of the algebra, i.e., set of possible worlds (possibly a singleton). As anticipated in footnote 1, our use of possible worlds is only instrumental to define credence functions, as it is customary in probabilistic semantics: possible worlds are for us just complete trivalent valuation functions, i.e., assignment of values \(\{1, 1/2, 0\}\) to sentences of \(\mathcal{L}^-\) that respect the Cooper truth tables. Notably, this does not make the interpretation of the conditional modal or non-truth-functional: at each world \(w\), the truth-value of \(A \rightarrow C\) is given by a Cooper valuation function. Moreover, we assume that any algebra \(\mathcal{A}\) includes the singletons

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\(^8\)Regarding Gibbard’s (1980) triviality result, note that \((Q)\text{CC/TT}\) is not supraclassical since it invalidates explosion, i.e., \(A \land \neg A \not\models B\). Second, the material conditional of \((Q)\text{CC/TT}\) does not satisfy classical laws, and so we can’t infer \(A \land B\) from \((A \supset B) \land A\), as Gibbard does. Third, the conditional of QCC/TT is strictly weaker than its material counterpart. For details, see Égré, Rossi, and Sprenger (2022).
of worlds, i.e., for every $w \in W$, $\{w\} \in \mathcal{A}$. Finally, we assume that the credence function $c$ is finitely additive, i.e., for all singleton worlds $\{w\} \in \mathcal{A}$, we have $c(\{w\}) \geq 0$, whereas $c(\emptyset) = 0$, and $c(W) = 1$.

We now identify propositions with sentences of $L \to L$ and define a (non-classical) probability function $p : L \to [0, 1]$ on the language $L \to L$, taking into account that sentences of $L \to L$ can receive three values: true, false, or indeterminate.\footnote{If you do not like to use the term “probability” in a non-classical framework, because you prefer to reserve it for standard bivalent probability, just replace it by “degree of assertability” or a similar term. This is the choice of McDer mott (1996), whose definition is identical to ours.}

For convenience, define

$A_T = \{w \in W \mid v_w(A) = 1\}$

$A_I = \{w \in W \mid v_w(A) = 1/2\}$

$A_F = \{w \in W \mid v_w(A) = 0\}$

as the sets of possible worlds where $A$ is valued as true, false or indeterminate, relative to (Cooper) valuation functions $v_w : L \to \{0, 1/2, 1\}$, indexed by the possible worlds they represent.

In analogy to bivalent probability, we keep the isomorphism between probability and betting odds and define the probability of a (possibly conditional) proposition $A$ as derived from the conditional betting odds on $A$: how much more likely is a bet on $A$ to be won than to be lost? For this comparison, two quantities are relevant: (1) the cumulative weight of the worlds where $A$ is true (i.e., $c(A_T)$), and (2) the cumulative weight of the worlds where $A$ is false, i.e., $c(A_F)$). The decimal odds on $A$ are $O(A) = (c(A_T) + c(A_F)) / c(A_T)$, indicating the factor by which the bettor’s stake is multiplied in case $A$ occurs and she wins the bet. Then we calculate the probability of $A$ from the decimal odds on $A$ by the familiar formula $p(A) = 1/O(A)$, yielding

$$p(A) := \frac{c(A_T)}{c(A_T) + c(A_F)} \quad \text{if max}(c(A_T), c(A_F)) > 0.$$  

(Probability)

Hence, the probability of a sentence corresponds to its expected semantic value, restricted to the worlds where the sentence takes classical truth value. An identical definition to trivalent probability has been proposed by Cantwell (2006) on the basis of different truth conditions. Additionally, we let $p(A) = 1$ whenever $c(A_T) + c(A_F) = 0$, i.e., if it is certain that $A$ takes the value $1/2$ (e.g., when $A$ is $\bot \to \top$).

In other words, the trivalent probability of $A$ is the ratio between the credence assigned to the worlds where $A$ is true, and the credence assigned to the worlds where $A$ has classical truth value. Worlds where $A$ takes
indeterminate truth value are neglected for calculating the probability of \( A \), except when they take up the whole space. For conditional-free sentences \( A \) and their Boolean compounds, this corresponds to the classical picture since \( W = A_T \cup A_F \), or equivalently, \( A_I = \emptyset \).

The idea behind (Probability) is the same that motivates classical operational definitions of probability: a proposition is assertable, or probable, to the degree that we can rationally bet on it, i.e., to the degree that betting on this proposition will, in the long run, provide us with gains rather than losses (e.g., Sprenger and Hartmann 2019). This is a good reason for calling the object defined by equation (Probability) a “probability”, or a measure of the plausibility of a proposition.

Let us now look at the structural properties of \( p : \mathcal{L} \to [0,1] \) and compare them to standard axioms of probability:

1. \( p(\top) = 1 \) and \( p(\bot) = 0 \).
2. \( p(A) = 1 - p(\neg A) \).
3. \( p(A \lor B) \leq p(A) + p(B) \). The equality \( p(A \lor B) = p(A) + p(B) \) holds if and only if \( \models C (A \land B) \) and \( A_I = B_I \).

Just like standard probability, our trivalent probability is not additive, but subadditive. Equality holds here exactly when \( A \) and \( B \) are \( C \)-incompatible (i.e., \( \neg (A \land B) \) is a \( C \)-theorem) and they take classical truth values in the same set of worlds.

The third property is, however, a substantial weakening of the requirement \( p(A \lor B) + p(A \land B) = p(A) + p(B) \) typically imposed on probability functions. When \( A \) and \( B \) are false and \( C \) is true, the bet on \( (A \to B) \land C \) yields a positive return, while the bet on \( A \to B \) is called off. So we should not expect that in all circumstances \( p((A \to B) \land C) \leq p(A \to B) \), in notable difference to bivalent probability, and some non-classical probability functions (for a survey, see Williams 2016). Of course, \( p(A \land B) \leq p(A) \) will hold as long as \( A \) and \( B \) are conditional-free sentences.

Another standard requirement on probability functions is

4. If \( A \models B \), then \( p(A) \leq p(B) \), for a suitable relation of logical consequence.

Our logic of certain inference \( C \) does not support (4), but in the next section, we will show that (4) holds for the logic \( QCC/(SS\cap TT) \), and we will defend it as a logic of uncertain inference.

The “only if” direction presupposes the non-trivial assumption that \( p(A) > 0 \) and \( p(B) > 0 \).
What about conditional probability? Unfortunately, when \( A \) and \( C \) are not conditional-free, we cannot define a fully general conditional probability function that satisfies the ratio analysis \( p(C|A) = \frac{p(A \land C)}{p(A)} \). The reason is that we can have \( p(A \land C) > p(A) \) and this would allow for \( p(C|A) > 1 \). However, we can define a surrogate notion for conditional probability, allowing us to model rational belief updates similarly to Bayesian conditionalization. We propose that upon learning a conditional-free sentence \( A \in \mathcal{L} \) (i.e., \( A_I = \emptyset \)) with \( p(A) > 0 \), our probabilities should shift from \( p \) to the function \( p_A \) defined by

\[
p_A(C) := p(A \rightarrow C) = \frac{c(A_T \land C_T)}{c(A_T \land C_T) + c(A_T \land C_F)} = \frac{c(A_T \land C_T)}{c(A_T)} \tag{Update}
\]

for any sentence \( C \in \mathcal{L} \rightarrow \) (compare McGee 1989, p. 504). The above equation means that when we calculate the new probability of \( C \) according to (Probability), we evaluate \( C \) only in the worlds that make \( A \) true, without changing the relative weight of these worlds. This procedure corresponds exactly to a formal characterization of Bayesian Conditionalization in the bivalent framework. Moreover, it is straightforward to show that \( p_A \) satisfies the properties (1)–(3) listed above. The function \( f_A : p \mapsto p_A \) can thus rightfully be called a probabilistic update function on \( A \) and provides an adequate surrogate notion for conditionalizing on \( A \) (i.e., \( p'(C) = p(C|A) \)). See Appendix C for a more detailed discussion of probabilistic updates, including updates on conditionals, and for a comparison with McGee’s (1989) structurally similar account.

Several important corollaries follow. First, for conditional-free sentences \( A, C \in \mathcal{L} \), we obtain from the definitions that

\[
p(A \rightarrow C) = \frac{c(A_T \land C_T)}{c(A_T)} = \frac{p(A \land C)}{p(A)} = p(C|A) \tag{Adams’s Thesis}
\]

because for conditional-free sentences, \( p(X) = c(X_T) \). That is, instead of postulating Adams’s Thesis as a desideratum on the probability of a conditional, as in Stalnaker (1970) and Adams (1975, p. 3), we obtain it immediately from the semantics of trivalent conditionals, and the definition of probability as the inverse of rational betting odds.\(^{11}\) The well-known triviality results by Lewis (1976) and others are blocked since they depend on an application of the (bivalent) Law of Total Probability, which does not hold for trivalent, non-classical probability functions (Lassiter 2020).\(^{12}\)

\(^{11}\)For more discussion of Adams’s Thesis, including experimental evidence for and against, see Stalnaker 1968; Adams 1975; Dubois and Prade 1994; Douven and Verbrugge 2010, 2013; Evans et al. 2007; Over, Hadjichristidis, et al. 2007; Egré and Cozic 2011; Over 2016; Skovgaard-Olsen, Singmann, and Klauer 2016.

\(^{12}\)Bradley (2000) proposes a different triviality result: arguably we want indicative conditionals to satisfy the Preservation Condition—if \( p(A) > 0 \) and \( p(C) = 0 \), then
Second, the logic of certain inference with conditionals C can be represented as the preservation of maximal probability:

**Proposition 1.** \( A \models_C B \) if and only if for all probability functions \( p : \mathcal{L} \rightarrow [0, 1] \): if \( p(A) = 1 \), then also \( p(B) = 1 \).

More generally, \( A_1, A_2, \ldots, A_n \models_C B \) if and only if for all probability functions \( p : \mathcal{L} \rightarrow [0, 1] \): if \( p(\bigwedge A_i) = 1 \), then also \( p(B) = 1 \).

This provides an independent motivation for choosing C as a paraconsistent generalization of monotonic deductive inference to a language with a conditional. Indeed, the laws of classical logic in the conditional-free language \( \mathcal{L} \) (the Boolean fragment of \( \mathcal{L}^- \)) are also theorems of C, if we restrict ourselves to bivalent valuations. In Section 4, we used adequacy criteria inspired from classical logic, such as Modus Ponens, in order to select C as the best logical consequence relation for certain inference with a trivalent conditional. Proposition 1 provides an alternative justification: adopting a standard approach to defining the probability of trivalent sentences, we obtain C as the logic that generates certain conclusions from certain premises.

Third, the consequence relation of the Cooper SS-logics in the previous section can be represented as preserving strictly positive probability:

**Proposition 2.** If \( A \models_{\text{QCC/SS}} B \), then for all credence functions \( c : \mathcal{A} \rightarrow \mathbb{R} \) with \( c(A_i) < 1 \) and associated probability function \( p : \mathcal{L} \rightarrow [0, 1] \): if \( p(A) > 0 \), then \( p(B) > 0 \). The converse holds if we assume that B is no theorem of C.

In other words, A QCC/SS-implies B if and only if B is a real possibility (i.e., \( p(B) > 0 \)) in all probability functions that make A a real possibility. While QCC/TT preserves certainties, QCC/SS preserves possibilities (see also Adams 1996b).

The next step in our analysis is to provide a probabilistic notion of logical consequence that imposes an order on the probabilities (i.e., \( A \models B \) if and only if \( p(A) \leq p(B) \) for all probability functions) and explicates how we should reason with conditionals under uncertainty. This is a task that Adams (1975) and other champions of the probabilistic research program have approached with great skill (e.g., Sanfilippo et al. 2020). We will set

\[ p(A \rightarrow C) = 0, \]

but for this to hold in full generality, we need to posit strong logical dependencies between a conditional and its components, thus trivializing the conditional. This is indeed so for bivalent accounts, but our trivalent account implies the Preservation Condition as a theorem without having a vicious dependency between the truth values of \( A, C \) and \( A \rightarrow C \).

13The restriction to bivalent valuations is necessary. For instance, \( A \land \neg A \triangleright B \), which is by definition equivalent to \( \neg(A \land \neg A) \lor B \), has value 0 when A has value 1/2 and B has value 0.
up an attractive rival system generalizing Adams’s logic to complex conditional sentences.

7 Uncertain Inference

Certain inference with conditionals is arguably monotonic: in particular, when we already know \( B \) for certain, we also know that \( B \) is the case under the condition that \( A \). This is an instance of the property of structural monotonicity \((A, B \vdash A)\), paired with the fact that the conditional satisfies conditional introduction. Indeed, the logic \( C \) satisfies both properties, and it validates this inference, called True Consequent: \( B \vdash C \ A \rightarrow B \).\(^{14}\)

When we move to defeasible reasoning where instead of certainty, only high probability or degree of assertability is preserved, True Consequent should become invalid. We may accept, assert, or find plausible \( B \), but reject \( B \) under the condition that \( A \). For example, the conditional “if Juventus faces Toro in their next match, then Juventus will win” sounds highly plausible, whereas “if half of the Juventus squad is sick and Juventus faces Toro in their next match, then Juventus will win” seems much less plausible. A logic of inference with uncertain premises \( U \) should therefore, unlike the logic \( C \), be non-monotonic, i.e., we cannot infer from \( A \rightarrow C \) that \( A \land B \rightarrow C \) for any \( A, B \) and \( C \in \mathcal{L} \rightarrow \).\(^{15}\)

The canonical definition of validity for single-premise inference in a logic of uncertain inference preserves probability, as a proxy for rational acceptance or assertability. In other words, the probability of the premise \( A \) must never exceed the probability of the conclusion \( B \). Almost all logics of uncertain reasoning agree on this criterion for single-premise inference, which is the natural analogue of truth preservation in certain reasoning. We therefore adopt it as our definition of single-premise logical consequence:

**Definition 2** (Single Premise inference in \( U \)). \( A \vdash_U B \) if and only if \( p(A) \leq p(B) \) for all probability functions \( p : \mathcal{L} \rightarrow [0, 1] \) based on credence functions \( c : \mathcal{A} \rightarrow \mathbb{R}^{\geq 0} \).

**Corollary 1.** \( \vdash_U B \) if and only if \( p(B) = 1 \) for all probability functions \( p : \mathcal{L} \rightarrow [0, 1] \) based on credence functions \( c : \mathcal{A} \rightarrow \mathbb{R}^{\geq 0} \).

**Corollary 2.** All theorems of \( C \) are also theorems of \( U \).

It is easy to show that this inference criterion has the following characterization in trivalent logic:

\(^{14}\)Alternatively, we can rephrase this example as a meta-inference from \( \Gamma \vdash B \) to \( \Gamma, A \vdash B \), which is the same as \( \Gamma \vdash A \rightarrow B \) since \( C \) satisfies Conditional Proof.

\(^{15}\)The structural rule of Weakening (that is, inferring \( A, B \vdash C \) from \( A \vdash C \)) will remain valid in our logic \( U \).
Proposition 3 (Equivalences for Valid Single-Premise Inference in U).

The following are equivalent:

1. \( A \models_{U} B \)
2. \( A \models_{C} B \text{ and } \neg B \models_{C} \neg A, \text{ or } \models_{C} B \)
3. \( A \models_{QCC/SS\cap TT} B, \text{ or } \models_{C} B \).

Condition (3) expresses that the truth value of the conclusion (1, 1/2, or 0) is at least as high as the truth value of the premises in all possible valuations. The proposition states that this is equivalent to demanding that the conclusion be at least as probable as the premise for all probability functions. The validity of an inference \( A \models_{U} B \) can thus either be defined via quantifying over probability functions and weight assignments to possible models, or via the validity of \( A \models_{U} B \) in both QCC/SS and QCC/TT (=QCC/SS\cap TT).

Extending this criterion to multi-premise inference \( A_1, \ldots A_n \models_{U} B \) is non-trivial since many different criteria can be formulated. Should the probability of \( B \) not fall below the minimum probability of the premises? Should it follow Adams’s uncertainty preservation criterion (Adams 1975, 1996b)? Should \( B \) be at least as plausible as the (quasi-) conjunction of the premises?

A good intuitive case can be made for either of these criteria. Therefore, rather than deciding the matter based on first principles, we suggest to evaluate the resulting logics by their properties. We propose that \( A_1, \ldots A_n \models_{U} B \) if and only if for all credence functions \( c : W \mapsto [0,1] \) that induce a probability function \( p : \mathcal{L}^{-} \mapsto [0,1] \), the probability of the quasi-conjunction of the premises does not exceed the probability of the conclusion. This gives us a definition of validity for uncertain inference based on multiple premises:

Definition 3 (Multi-premise valid inference in U). \( A_1, \ldots A_n \models_{U} B \) iff for all probability functions \( p : \mathcal{L}^{-} \mapsto [0,1] \), \( p(A_1 \land \ldots \land A_n) \leq p(B) \).

There are three principled reasons for adopting this definition. First of all, like the definition for single-premise inference, Definition 3 allows for an equivalent characterization in trivalent logic:

Proposition 4 (Equivalences for Valid Multi-Premise Inference in U).

The following are equivalent:

1. \( A_1, \ldots A_n \models_{U} B \)
2. \( A_1 \land \ldots \land A_n \models_{C} B \text{ and } \neg B \models_{C} \neg (A_1 \land \ldots \land \neg A_n), \text{ or } \models_{C} B \).
From a computational point of view, the equivalence of (1) with (2) and (3) is extremely attractive. Note also that the logic QCC/SS∩TT that we have rejected in Section 5 for inference with certain premises, e.g. because it does not satisfy Modus Ponens, turns out to be an attractive candidate for inference with uncertain premises (where violating Modus Ponens may be an asset rather than a drawback: McGee 1985).

Secondly, Proposition 4 also provides sound and complete calculi for the logic U for free. For instance, since Cooper (1968) has a sound and complete Hilbert-style calculus for C, this automatically translates, thanks to Proposition 4, into a sound and complete calculus for U. Alternatively, still using Proposition 4, tableau- and sequent-style sound and complete axiomatizations of U can be extracted from Égré, Rossi, and Sprenger (2021b).

Thirdly and finally, defining multi-premise inference in this way yields an attractive logic of uncertain inference with conditionals, as we will see in the next two sections.

8 Properties of U

We now evaluate the logic U in terms of the inference schemes it validates, using the principles in Table 7, taken from the survey article by Égré and Rott (2021).17 The principles above the first horizontal line are generally considered to be desirable, or at least not harmful, in defeasible reasoning with conditionals. The principles between the lines—Modus Ponens, Modus Tollens, Supraclassicality for Inferences, Or-To-If, Import-Export, and Simplification of Disjunctive Antecedents—are typically a bone of contention between theorists. We also include some tautologies that are distinctive for connexive logics. The principles at the bottom—Contraposition, Monotonicity and Transitivity—are characteristic of most monotonic logics, and logics of deductive inference in particular, but should not be satisfied by a non-monotonic logic of uncertain reasoning (for compelling counterexamples, see Adams 1965). So we should expect that these principles are satisfied by C, but not by U.

Table 7 evaluates, in the rightmost columns, C and U with respect to all these principles. We cannot discuss each of them in detail, but we make some general observations. Many desirable or non-harmful principles are

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16For the TT-logics, but not for the SS-logics, \( A_1 \land \ldots \land A_n \models_{QCC/SS} B \) is the same as \( A_1, \ldots, A_n \models_{C} B \). In particular, \( A, B \models_{QCC/SS} B \) holds, while \( A \land B \models_{QCC/SS} B \) does not, due to the choice of quasi-conjunction instead of Strong Kleene conjunction (\( v(A) = 1, v(B) = 1/2 \)).

17We use C as an appropriate generalization of classical deductive logic in formulating principles like Left Logical Equivalence or Right Weakening.
| Constitutive and Generally Desirable Principles in Defeasible Reasoning | C | U |
|---------------------------------------------------------------|---|---|
| Logical Truth                                                | ✓ | ✓ |
| Law of Identity                                               | ✓ | ✓ |
| Supraclassicality (Laws) (for A without →) if |=_{CL} A, then |= A | ✓ | ✓ |
| Left Logical Equivalence if A |=_{C} B, B |=_{C} A, then A → C |= B → C | ✓ | ✓ |
| Stronger-Than-Material                                        | ✓ | ✓ |
| Conjunctive Sufficiency                                        | ✓ | ✓ |
| AND                                                          | ✓ | ✓ |
| OR                                                           | ✓ | ✓ |
| Cautious Transitivity                                         | ✓ | ✓ |
| Cautious Monotonocity                                         | ✓ | ✓ |
| Rational Monotonocity                                         | ✓ | ✓ |
| Reciprocity                                                   | ✓ | ✓ |
| Right Weakening                                               | ✓ | ✓ |
| Conditional Excluded Middle                                  | ✓ | ✓ |
| Rule of Conditional K                                         | ✓ | ✓ |

| Optional and Disputed Principles                              |   |   |
|---------------------------------------------------------------|---|---|
| Supraclassicality (Inferences) if Γ |=_{CL} B then Γ |= B     | X | X |
| Modus Ponens                                                 | ✓ | ✓ |
| Modus Tollens                                                | ✓ | ✓ |
| Simplifying Disjunctive Antecedents (A ∨ B) → C |= (A → C) ∧ (B → C) | ✓ | ✓ |
| Import-Export                                                | ✓ | ✓ |
| Or-to-If                                                     | ✓ | X |

| Connexive Principles (optional)                              |   |   |
|---------------------------------------------------------------|---|---|
| Aristotle's Thesis                                           | ✓ | ✓ |
| Boethius' Thesis                                              | ✓ | ✓ |
| Undesirable Principles                                        |   |   |
| Contraposition                                                | ✓ | X |
| Monotonicity                                                  | ✓ | X |
| Transitivity                                                  | ✓ | X |

Table 7: Overview of Inference Principles involving conditionals in defeasible reasoning. In the rightmost columns, it is shown whether C and U validate the principle generally (✓), only for bivalent valuations of the sentential variables (✓ in parentheses), or not at all (X). satisfied by U without restriction, whereas some of them only hold for bivalent (“atom-classical”) valuations of at least one sentential variable. This means that when all sentences are conditional-free, the inference is valid; only when one of the sentences contains a conditional connective (so that it can take the third truth value), it is possible that the inference fails. When we compare U to classical conditional logics (i.e., logics where all valuations are bivalent, such as Stalnaker-Lewis logics), we can consider the principles valid since making a comparison presupposes bivalent valuations. Specifically, U recovers all valid inferences of System P, a classical benchmark.
for conditional logics (Adams 1975; Kraus, Lehmann, and Magidor 1990).\footnote{Adams (1975) characterized his logic of probability preservation (or more precisely, of non-increase in uncertainty) syntactically by the principles Law of Identity, AND, OR, Cautious Monotonicity, Left Logical Equivalence, and Right Weakening.}

Moreover, both $C$ and $U$ validate connexive principles such as Aristotle’s Thesis ($\neg(\neg A \rightarrow A)$) and Boethius’s Thesis ($(A \rightarrow C) \rightarrow \neg(A \rightarrow \neg C)$).

Principles that are typically considered problematic—Monotonicity, Contraposition, Transitivity—are indeed not valid in $U$ (Egré and Rott 2021). These principles do not even hold when we restrict $U$ to bivalent valuations of sentential variables. However, they do (mainly) hold in our logic of certain inference $C$, in line with our view of $C$ as a generalization of classical deductive logic to a language with a conditional.

Most interesting are the six principles in the middle. Supraclassicality for inferences in $U$ fails because $C$ does not respect explosion, e.g., while $A \land \neg A \models_C B$ holds for any two sentences $A$ and $B$, it is not the case that $A \land \neg A \models_C B$. However, all classical laws are theorems of both $C$ and $U$ (when restricted to bivalent valuations). Modus Ponens and Modus Tollens hold for conditional-free sentences, but break down for nested conditionals—in line with McGee’s 1985 famous objections (see the next section for a detailed analysis). Also Simplification of Disjunctive Antecedent is preserved for bivalent valuations only.

Import-Export holds unrestrictedly, since $A \rightarrow (B \rightarrow C)$ and $(A \land B) \rightarrow C$ are congruent: they have exactly the same truth conditions.\footnote{This is a general feature of trivalent logics and holds regardless of how conjunction and the conditional are interpreted.} The principle is intuitively plausible: “it appears to be a fact of English usage, confirmed by numerous examples, that we assert, deny, or profess ignorance of a compound conditional $A \rightarrow (B \rightarrow C)$ under precisely the circumstances under which we assert, deny, or profess ignorance of $(A \land B) \rightarrow C$” (McGee 1989, p. 489). Experimental evidence seems to confirm this attitude (van Wijnbergen-Huitink, Elqayam, and Over 2015). Indeed, the main motivation for giving up Import-Export—e.g., in Stalnaker-Lewis semantics, but also in the probabilistic semantics of Sanfilippo et al. (2020)—is not its implausibility, but the pressure from Gibbard’s and Lewis’s triviality results, where Import-Export is an important premise. Some accounts therefore restrict the validity of Import-Export to simple conditionals and set up an error theory of why we infer from there to the general validity of the principle (e.g., Mandelkern 2020). By contrast, both $C$ and $U$ can just accept Import-Export since the triviality results do not apply to these logics (Egré, Rossi, and Sprenger 2022). This strategy is arguably preferable to ad hoc solutions and also yields benefits in the analysis of Modus Ponens.
Or-to-If fails in $U$: that $\neg A \lor B$ is highly plausible does not establish that $B$ is highly plausible, given $A$ (i.e., possibly $p(B|A) < p(\neg A \lor B)$). However, Or-To-If is highly plausible for reasoning from certain premises: if we know that $\neg A$ or $B$ is the case, then, if $A$, it must be the case that $B$. We can also look at this from a purely logical perspective: $C$ satisfies Or-To-If since it satisfies Disjunctive Syllogism and the Deduction Theorem. On the other hand, $U$ satisfies Disjunctive Syllogism, but it does not have a Deduction Theorem. The fact that Or-To-If is highly plausible in certain reasoning, but clearly invalid in uncertain reasoning, motivates that we need two distinct logics of conditional reasoning, in analogy to deductive and inductive, ampliative logics for conditional-free languages. Seen from this perspective, the failure of Stronger-Than-Material in $C$ is simply one way of avoiding the identification of the indicative and the material conditional.

9 Modus Ponens, Tollens, and Import-Export

Modus Ponens appears invariably valid in inference from certain premises, but there is a famous counterexample by McGee (1985) regarding its validity in inference from uncertain premises. It concerns the 1980 U.S. presidential elections.

If a Republican wins the election, then, if Reagan does not win, Anderson will win.
A Republican will win the election.

Therefore, if Reagan does not win the election, Anderson will.

At some point before the elections, the two premises were highly plausible: Ronald Reagan was predicted to win the election, and Anderson was the runner-up behind Reagan in the Republicans’ primary race. By Modus Ponens we infer that if Reagan did not win, Anderson did. The logical form of that inference is: from $A \rightarrow (B \rightarrow C)$ and $A$, infer, by Modus Ponens, $B \rightarrow C$. However, in the polls Anderson was actually trailing both Reagan and Carter, the democrat incumbent. Therefore, if Reagan was not elected president, the best explanation would be that Carter has been elected, contradicting the conclusion.

20 The Deduction Theorem states that $\Gamma, A \vdash B$ if and only if $\Gamma \vdash A \rightarrow B$. Both directions fail in $U$. A counterexample to the left-to-right direction is exactly provided by Or-To-If and Disjunctive Syllogism: while $A, \neg A \lor B \vdash_U B$ holds, a valuation $v$ s.t. $v(A) = 0$ and $v(B) = \frac{1}{2}$ shows that $\neg A \lor B \not\vdash_U A \rightarrow B$. A counterexample to the right-to-left direction, instead, is the following: $\vdash_U A \wedge B \rightarrow B$ (the latter is a theorem of $C$), but in general, as we have seen, $A \wedge B \not\vdash_U B$. 

23
McGee’s counterexample is usually considered forceful and convincing. Recently, Stern and Hartmann (2018) have pointed out that its intuitive appeal depends crucially on the use of nested conditionals. When the major premise of Modus Ponens is a nested conditional, the probability loss in inferring to the conclusion can be much higher than when we apply Modus Ponens to non-nested premises. For bivalent propositions $A$ and $B$, the term

$$p(B) = p(B|A)p(A) + p(B|\neg A)(1 - p(A))$$

is, by the Law of Total Probability, well controlled by the values of $p(A)$ and $p(B|A)$—the values that represent the probability of the two premises of Modus Ponens. For example, if both values exceed .9, then $p(B) \geq .81$, i.e., the product of the two probabilities and still a reasonably high value.

However, in the case of right-nested conditionals, the probability of the conclusion of Modus Ponens is poorly controlled:

$$p(C|B) = p(C|A \land B)p(A|B) + p(C|\neg A \land B))(1 - p(A|B))$$

Suppose that premises are highly plausible, e.g. $p(A) \geq .9$ and $p(C|A \land B) \geq .9$, where the latter probability has been calculated by applying Import-Export and Adams’s Thesis to $A \rightarrow (B \rightarrow C)$. Then you can still assign extremely low values to three of the four probabilities on the right hand side of equation (2), and derive a very low value of $p(C|B)$. Therefore the probability loss is more pronounced in McGee’s example than when we apply Modus Ponens to simple conditionals.

Our logics mirrors this diagnosis: Modus Ponens is valid in $C$, i.e., in certain inference, and valid in $U$ for bivalent valuations, i.e., when all involved propositional constants are classical. However, $U$ does not validate the unrestricted form of Modus Ponens, and in fact, the only countermodel to the schema $A \rightarrow B, A \models B$ is $\nu(A) = 1$ and $\nu(B) = 1/2$ (i.e., $B$ is a conditional with false antecedent).

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21 Sinnott-Armstrong, Moor, and Fogelin (1986) respond that the conclusion should be evaluated as a material conditional—which would be a plausible proposition—and argue that the burden is on McGee to show that this interpretation of the conditional is inadequate. But this defensive strategy is threatened by the strong theoretical and empirical arguments against the material conditional view, in particular the paradoxes of material implication, and the fact that judgments on the probability or assertability of $A \rightarrow C$ align with $p(C|A)$, not with $p(\neg A \lor C)$ (e.g., Over, Hadjichristidis, et al. 2007).

22 Note that the inference is both QCC/TT- and QCC/SS-valid. The fact that McGee’s argument is analyzed as valid in $C$ and as invalid in $U$ is also in accordance with the ambivalence generally felt regarding whether the argument is valid or not. Similar distinctions between the validity of Modus Ponens in certain and uncertain inference are, in different frameworks, defended by Neth (2019) and Santorio (2022).
The same kind of analysis can be applied to Modus Tollens, i.e., the schema \( A \rightarrow B, \neg B \models \neg A \). Consider the argument

If a Republican wins the election, then, if Reagan does not win, Anderson will win.
If Reagan does not win the election, then Anderson won’t either.

Therefore, no Republican will win the election.

This inference is supported by Modus Tollens, applied to a conditional-free sentence \( A \) and a conditional sentence \( B \). The premises agree with our knowledge about the 1980 US presidential elections, but the conclusion is unreasonable: odds were on a Republican victory. Indeed, \( U \) invalidates Modus Tollens for nested conditionals (the same valuations as before show it). However, for non-nested conditionals, Modus Tollens remains valid—as in System \( P \) and Stalnaker-Lewis semantics. This agrees with results from the psychology of reasoning, where Modus Tollens is a strongly endorsed inference, yet less strong than Modus Ponens (e.g., Johnson-Laird and Byrne 1991, 2002; Oaksford and Chater 2007).

By contrast, Denying the Antecedent and Affirming the Consequent are commonly seen as logical fallacies in uncertain reasoning. This is precisely what the validity criterion of \( U \) tells us: these schemes are not even valid for simple conditionals. That is, their failure is not a specific feature of inference with nested conditionals (as it is for Modus Ponens and Modus Tollens), but already present in the flat fragment of \( L \) where all sentences receive bivalent valuations.

Since Import-Export features crucially in McGee’s counterexample (e.g., in Stern and Hartmann’s probabilistic reconstruction), philosophers and logicians have often faced a choice between both principles. For example, Stalnaker (1968) and Lewis (1973b) give up Import-Export, but retain Modus Ponens. So does Mandelkern (2020), who restricts the validity of Import-Export.\(^{23}\) Our trivalent framework makes the opposite and arguably more natural choice: like McGee (1989), we let Import-Export be unrestrictedly valid and restrict the validity of Modus Ponens. This account does not only give a convincing analysis of McGee-style examples, which are typically recognized as a problem for Modus Ponens in uncertain reasoning, but also agrees with psychological evidence in favor of Import-Export and simple Modus Ponens.

\(^{23}\)More precisely, Mandelkern shows that a conditional satisfying Conditional Introduction (i.e., the meta-inference from \( \Gamma, A \models B \) to \( \Gamma \models A \rightarrow B \)) and both Modus Ponens and Import-Export is equivalent to the material conditional. After evaluating some examples where Import-Export leads to undesirable consequences when paired with reasonable principles, Mandelkern infers that the scope of Import-Export should be restricted to cases where the “middle proposition” \( B \) in \( A \rightarrow (B \rightarrow C) \) does not contain a conditional.
10 Comparisons

The trivalent treatment of indicative conditionals is first sketched in Reichenbach (1935) and de Finetti (1936a,b). A more detailed motivation of this approach, including an overview of the main consequence relations of interest, is given by Belnap (1970, 1973), but none of these authors provides a fully worked out account of the logic and epistemology of conditionals. The first complete trivalent account of a logic of conditionals is due to Cooper (1968), who originally created system $C$. However, Cooper restricts it to bivalent valuations of the sentential variables and does not develop an account of uncertain reasoning concerned with the probability of conditionals. Cantwell (2008) investigates the CC/TT-logic with Strong Kleene connectives for certain inference, but his treatment of “non-bivalent probability” ends up with an altogether different probabilistic logic (Cantwell 2006).

Most similar to our approach, both in spirit and content, is Michael McDermott’s (1996) proposal. Like McDermott, we explicate the truth conditions of conditionals within trivalent semantics, keep Import-Export unrestrictedly valid, define probability via conditional betting odds, and develop a non-classical logic of uncertain reasoning based on a probability preservation criterion. However, McDermott adopts de Finetti’s truth table for the conditional (as opposed to our Cooper conditional, compare Table 3), and he interprets conjunction and disjunction in natural language as ambiguous between strong Kleene connectives and Cooper’s quasi-connectives. McDermott’s criterion for multi-premise inference, according to which $A_1, \ldots, A_n \models C$ holds if for all probability functions $p : \mathcal{L} \rightarrow [0, 1]$: $p(A_1 \land \ldots \land A_n) \leq p(C)$,

is structurally identical to ours, but his conjunction follows the Strong Kleene tables. Therefore, the above criterion for valid uncertain inference can be reformulated as the conjunction of (1) $A_1 \land \ldots \land A_n \models_{\text{DF/SS}} C$ and (2) $A_1 \land \ldots \land A_n \models_{\text{DF/TT}} C$, yielding the logic $\text{MD}$. See Table 8 for a detailed comparison of $U$ and $\text{MD}$.

At the foundational level, the main differences between McDermott’s and our account are his use of the de Finetti truth table for the conditional, and the use of Strong Kleene conjunction in the definition of probabilistic validity. On the level of inferences, many features are similar, but $\text{MD}$ validates Transitivity ($A \rightarrow B, B \rightarrow C$, therefore $A \rightarrow C$). While this is acceptable and even desirable in the framework of certain inference, it is arguably problematic when reasoning from uncertain premises since the probability of $p(C|A)$ is in no way controlled by $p(C|B)$ and $p(B|A)$; in fact, it can be arbitrarily low. Suppose that you live in a very sunny, dry place.
Consider the sentences $A = \text{“it will rain tomorrow”}$, $B = \text{“I will work from home”}$, $C = \text{“I will work on the balcony”}$. Clearly, both $A \rightarrow B$ and $B \rightarrow C$ are highly plausible, but $A \rightarrow C$ isn’t. This structural feature offers, in our view, a decisive reason to prefer our model to McDermott’s.

Another close relative of our account is Dubois and Prade’s 1994 trivalent treatment of conditional objects, who use the same validity criterion as $U$ (i.e., the truth value of the conclusion must not be less than the truth value of the quasi-conjunction of the premises). Their system differs from $U$ by adopting de Finetti’s truth table for conditionals, and from $U$ and $MD$ by admitting only formulas from the flat fragment of $L^{-}$, i.e., Boolean sentences and conditionals only between Boolean sentences. Moreover, their tautologies are exactly the (classical) validities of the Boolean fragment $L$, and their semantics is sound and complete for System $P$.

All trivalent accounts of the probability of conditionals, including our own, can be regarded as extensions and developments of Adams’s seminal account of uncertain reasoning developed in his 1975 monograph *The Logic of Conditionals*. The point of departure of that book is to interpret the probability of a conditional $A \rightarrow C$ as the conditional probability $p(C | A)$. From there, Adams develops a probabilistic logic of defeasible reasoning in $L^{-}$, based on the $\varepsilon$-$\delta$ criterion. The descriptive accuracy of the predictions of Adams’s logic is widely acknowledged (e.g., McGee 1989, pp. 487-493).

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24Besides, according to $MD$, failure of Modus Ponens requires a left-nested conditional (counterexample: $v(A) = 1/2$, $v(B) = 0$—the TT-direction fails). However, $MD$ validates all McGee-type instances of Modus Ponens with right-nested conditionals, failing to explain why they look problematic.
488), but due to the lack of general truth conditions, it has limited scope. In particular, Adams cannot deal with nested conditionals and Boolean compounds of conditionals, which he has to paraphrase as rather complicated quasi-conjunctions and -disjunctions (Adams 1975, pp. 46-47). Our account recovers all the inferences in Adams’s logic of reasonable inference without suffering from these restrictions.

Specifically, Adams’s Thesis $p(A \rightarrow C) = p(C\mid A)$, for conditional-free sentences $A$ and $C$, emerges as a corollary of defining trivalent probability via the conditional betting odds of rational agents. Adams, by contrast, needs to postulate this equation as a basic fact. Our logic $U$ also goes beyond Adams by articulating Import-Export as a semantic fact (instead of postulating it as definition of the meaning of a nested conditional), accepting Rational Monotonicity, and giving a fine-grained analysis of Modus Ponens and Modus Tollens, including McGee-style counterexamples (see Table 8).

Also, the flat fragments (i.e., excluding nested conditionals, etc.) of Lewis’s logic $VC$ and Stalnaker’s logic $C2$ correspond to Adams’s System $P$. However, these logics give up on Import-Export and classify Modus Ponens and Modus Tollens as unrestrictedly valid. In the light of McGee-style examples, they are thus overshooting the target, whereas our account explains both why Modus Ponens and Modus Tollens appear plausible, and why they sometimes fail in uncertain reasoning.

We conclude with a note on modal accounts. Their basic idea consists in treating conditionals as (bivalent) propositions, and defining truth conditions by means of possible worlds: a conditional $A \rightarrow C$ is true if $C$ is true in a suitably restricted set of $A$-worlds (e.g., as defined by Stalnaker’s selection function or Lewisian spheres: Stalnaker 1968, 1975; Lewis 1973b,a; McGee 1989; Mandelkern 2019; see also Kratzer 1986). This approach is similar to the conditional assertion view: if $A$ is true in the actual world, the truth value of the conditional corresponds to the truth value of the consequent. The fundamental difference to our approach emerges when $A$ is false: while we assign a third truth value to the conditional, modal theorists assign a classical truth value, essentially based on epistemic considerations (“is $C$ the case in a plausible (set of) world(s) where $A$ is the case?”). In other words, modal semantics creates a disparity between the case where $A$ is true, where truth conditions are factual, and the case where $A$ is false, where truth conditions depend on considerations of plausibility and normality. This disparity stands in need of an explanation, and the trivalent approach presents a more unified semantic and epistemological package.

Moreover, probabilistic reasoning is less developed in modal accounts (though see McGee 1989). This has structural reasons: sticking to classical truth values and classical probability functions, modal accounts are
exposed to Lewis’ (1976) triviality result, and similar challenges raised by Hájek (1989) and Bradley (2000). Something plausible has to go: treating conditionals as propositions, the standard probability axioms, the closure of probability functions under conditionalization, or Import-Export. Modal accounts typically opt for giving up Import-Export, but this choice limits the scope and applicability of the probability calculus to complex conditionals, without a strong independent justification.

11 Conclusions

The philosophical upshot of the paper is simple: there is no gap between the truth conditions of conditionals and their probabilistic treatment. Trivalent semantics explicates the truth conditions and the probability of indicative conditionals and yield two attractive logics for reasoning with them: C explicates conditional reasoning with certain premises, U explicates conditional reasoning with uncertain premises. Although C is a paraconsistent logic, all theorems of classical logic are also theorems of C (restricted to bivalent valuations). The combination of C and U avoids Gibbard’s and Lewis’s triviality results and moreover, it provides a unified framework for conditional reasoning.

In conclusion, we structure the main features and results of our approach according to topics:

**Semantics** The indicative conditional expresses a conditional commitment to the consequent, retracted if the antecedent turns out false. This interpretation motivates a fully truth-functional trivalent analysis of the conditional along de Finetti’s lines. We interpret Boolean connectives according to Cooper’s truth tables for quasi-conjunction and -disjunction, and the conditional according to the Cooper-Cantwell table, grouping indeterminate antecedents with true ones.

**Probability** The probability of a conditional \( A \rightarrow C \) is given by the corresponding betting odds for a conditional assertion. These betting odds correspond to the ratio of the weight of possible worlds where \( A \rightarrow C \) is true, divided by the weight of possible worlds where \( A \rightarrow C \) is either true or false. Adams’s Thesis \( p(A \rightarrow C) = p(C|A) \) follows as a corollary for conditional-free sentences A and C, and need not be postulated as an axiom. The probability function \( p_A(C) := p(A \rightarrow C) \) describes updating on a conditional-free proposition A and serves as an adequate surrogate notion for conditional probability.

**Certain Inference** Conditional reasoning with certain premises (i.e., deductive inference in \( L^- \)) is captured by the logic C, which can be
characterized probabilistically, as preservation of maximal probability, and in terms of trivalent logic (Proposition 1). As a trivalent logic, $\mathcal{C}$ is based on the above semantics, and its consequence relation preserves non-falsity.

**Uncertain Inference** Conditional reasoning with *uncertain* premises (i.e., defeasible reasoning in $\mathcal{L}^-$) is captured by the logic $\mathcal{U}$, which preserves probability between the quasi-conjunction of the premises and the conclusion. Equivalently, $\mathcal{U}$ preserves truth and non-falsity for all trivalent valuations of the premises and the conclusion (Proposition 3 and 4).

Combining these semantic, logical and epistemological elements delivers a coherent, attractive and fruitful package on the level of inference and reasoning. Specifically, we can use it to analyze and to explain the controversy about the validity of Modus Ponens and other important inference principles. Comparisons with competitors are arguably favorable and the practical benefits of a truth-functional account (e.g., automated theorem-proving) are evident. We are eager to hear from critics and proponents of competing views.
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A Proofs of the Propositions

Given a model, consisting of a nonempty set of worlds $W$ and a valuation function $v$, recall that $A_T, A_i, A_F \subseteq W$ denote the set of possible worlds where $A$ is true, indeterminate, and false, respectively. Here and in the remainder, we identify possible worlds with complete valuation functions to all sentences in the language $\mathcal{L}$.

**Proposition 1.** $A \models_{\mathcal{C}} B$ if and only if for all probability functions $p : \mathcal{L} \rightarrow [0, 1]$: if $p(A) = 1$, then also $p(B) = 1$.

More generally, $A_1, A_2, \ldots, A_n \models_{\mathcal{C}} B$ if and only if for all probability functions $p : \mathcal{L} \rightarrow [0, 1]$: if $p(\land A_i) = 1$, then also $p(B) = 1$.

**Proof.** “⇒”. Suppose $A \models_{\mathcal{C}} B$. This means that for every model, $B_F \subseteq A_F$. Suppose now that $p(A) = 1$ for some probability function $p$: by (Probability), this requires $c(A_F) = 0$. But since $B_F \subseteq A_F$, and the measure properties of $c$, also $c(B_F) \leq c(A_F) = 0$ and hence $p(B) = 1$.

“⇐”. Suppose that for any $p$ with $p(A) = 1$, also $p(B) = 1$. Suppose further that $A \not\models_{\mathcal{C}} B$, i.e., there is a model and a world $w \in B_F$ with $w \not\in A_F$. Choose $c$ such that $c(w) = 1$, i.e., $w$ has maximal credence, and in particular, $c(w') = 0$, $\forall w' \neq w$. Then $c(A_F) = c(B_T) = 0$, and

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \frac{c(A_T)}{c(A_T) + 0} = 1,$$

$$p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1} = 0,$$

contradicting what we have assumed. Hence it must be the case that $A \models_{\mathcal{C}} B$.

The generalization to more than one premise is straightforward since $A_1, \ldots, A_n \models_{\mathcal{C}} B$ if and only if $\land A_i \models_{\mathcal{C}} B$. $\square$

**Proposition 2.** If $A \models_{\mathcal{QCC/SS}} B$, then for all credence functions $c : \mathcal{A} \rightarrow \mathbb{R}$ with $c(A_i) < 1$ and associated probability function $p : \mathcal{L} \rightarrow [0, 1]$: if $p(A) > 0$, then $p(B) > 0$. The converse holds if we assume that $B$ is no theorem of $\mathcal{C}$.

**Proof.** “⇒”. Suppose $A \models_{\mathcal{QCC/SS}} B$. This means that for every model, $A_T \subseteq B_T$. Suppose now that $p(A) > 0$; since we have excluded the case $c(A_i) = 1$ we have strictly positive credence that $A$ is true. In other words, $c(A_T) > 0$. Since $A_T \subseteq B_T$, it follows that $c(B_T) \geq c(A_T) > 0$, and hence $p(B) > 0$.

“⇐”. Suppose $A_T \neq \emptyset$ (otherwise the proof is trivial). We suppose further that $A \not\models_{\mathcal{QCC/SS}} B$, i.e., there is a world $w \in A_T$ with $w \not\in B_T$. Moreover, by assumption ($\models B$ is no theorem of $\mathcal{C}$) there must be a world $w' \in B_T$. Then
we choose \( c(w) = c(w') = \frac{1}{2} \) (for the case \( w = w' \), choose \( c(w) = 1 \)) and infer

\[
p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \left(1 + \frac{c(A_F)}{c(A_T)}\right)^{-1} > 0,
\]

and moreover, since \( w, w' \notin B_T \),

\[
p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1/2} = 0,
\]

contradicting what we assumed. Hence \( A \models_{\text{QCC/SS}} B \).

**Proposition 3** (Equivalences for Valid Single-Premise Inference in \( U \)).

The following are equivalent:

1. \( A \models_U B \)
2. \( A \models_C B \) and \( \neg B \models_C \neg A \), or \( \models_C B \);
3. \( A \models_{\text{QCC/SS\cap TT}} B \), or \( \models_C B \).

**Proof.** We reason by cases and begin with the case \( \models_C B \). In this case, \( p(B) = 1 \) and hence, (1), (2) and (3) are all true. In the remainder, we can therefore neglect this case and assume that there is at least a world \( w \in B_F \).

We simplify and unify notation and write "\( \models_S \) instead of "\( \models_{\text{QCC/SS}} \)”, and "\( \models_{TT} \) instead of "\( \models_{\text{QCC/TT}} \)" or "\( \models_C \). First, we show the equivalence of (2) and (3).

\[(2) \Rightarrow (3): \text{By assumption, we already have } A \models_{TT} B. \text{ Suppose } \neg B \models_C \neg A; \text{ this means that } (\neg A)_F \subseteq (\neg B)_F, \text{ or equivalently, } A_T \subseteq B_T. \text{ But the latter is the same as } A \models_{SS} B. \text{ So both the SS- and the TT-entailment holds between } A \text{ and } B.\]

\[(3) \Rightarrow (2): \text{Suppose } A \models_{\text{QCC/SS\cap TT}} B. \text{ This implies } A \models_{TT} B \text{ trivially; we still have to show } \neg B \models_{TT} \neg A. \text{ But since } A \models_{\text{QCC/SS}} B, \text{ we have } A_T \subseteq B_T \text{ and hence } (\neg A)_F \subseteq (\neg B)_F. \text{ The latter is equivalent to } \neg B \models_{TT} \neg A.\]

\[(3) \Rightarrow (1): \text{By assumption, } A_T \subseteq B_T \text{ and } B_F \subseteq A_F. \text{ Hence, } c(A_T) \leq c(B_T) \text{ and } c(A_F) \geq c(B_F). \text{ Thus, for all probability functions } p : L \rightarrow [0, 1],
\]

\[
p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \left(1 + \frac{c(A_F)}{c(A_T)}\right)^{-1} \leq \left(1 + \frac{c(B_F)}{c(B_T)}\right)^{-1} = p(B).
\]

\[(1) \Rightarrow (3): \text{Let us first deal with the case } A_T = \emptyset. \text{ In that case, } A \models_{SS} B \text{ is trivially satisfied. The only way for } (3) \text{ to be false is if there is a } w \in A_I \cap B_F,\]
such that $A \models_{TT} B$ fails. However, in that case, we can assign $c(w) = 1$, obtaining $p(A) = 1$ and $p(B) = 0$. So (1) would fail, too. For this reason, we can presuppose in the remainder that $A_T \neq \emptyset$.

We now prove the converse, i.e., $\neg(3) \Rightarrow \neg(1)$. Assume first that $A \not\models_{QCC/SS} B$, i.e., $A_T \cap (B_F \cup B_I) \neq \emptyset$.

Case 1: $A_T \cap B_F \neq \emptyset$. Choose a $w \in A_T \cap B_F$ and a probability distribution with $c(w) = 1$, yielding $p(A) = 1$ and $p(B) = 0$. So $\neg(1)$ holds.

Case 2: $A_T \cap B_F = \emptyset$. Choose a $w \in A_T \cap B_I$. However, since $B$ is by assumption no theorem of QCC/TT, we know that there is a $w' \in B_F$. Assign the credences $c(w) = c(w') = 1/2$. Then we obtain the following counterexample to (1):

\[
p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} \geq \frac{1/2}{1/2 + c(A_F)} \geq 1/2
\]

\[
p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1/2} = 0.
\]

Now assume that $A \not\models_{TT} B$, i.e., $B_F \cap (A_T \cup A_I) \neq \emptyset$. If there is a $w \in B_F \cap A_T$, we are done: simply assign maximal credence to this world, and we obtain that $p(A) > p(B)$. If there is only a $w \in B_F \cap A_I$, by contrast, we assign $c(w) = 1/2$, and moreover, we choose an arbitrary $w' \in A_T \cap (B_T \cup B_I)$ with $c(w') = 1/2$. Such a $w'$ must exist since we have assumed $A_T \neq \emptyset$. Then, we construct a counterexample to (1) as follows:

\[
p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \frac{1/2}{1/2 + 0} = 1
\]

\[
p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} \leq \frac{1/2}{1/2 + 1/2} = 1/2
\]

\[\square\]

The proof of Proposition 4 proceeds exactly as the proof of Proposition 3, with the (quasi-)conjunction $A_1 \land \ldots \land A_n$ taking the role of $A$. Since there are no structural differences, we omit it.

## B Proofs and Countermodels for Properties of $U$

This appendix contains the countermodels for the properties failed by $C$ and $U$, using the third criterion of Proposition 4: both the QCC/SS- and the QCC/TT-entailment ($=C$-entailment) needs to hold between the conjunction of the premises and the conclusion. Whenever that criterion is not directly applicable, we provide more details.
Logical Truth (\(\models_U A \rightarrow T\)): We know from the truth tables of the conditional that \(A \rightarrow T\) cannot be false. Hence \((A \rightarrow T)_T = \emptyset\). This implies that for all credence functions \(c\) and associated probability functions \(p\), \(p(A \rightarrow T) = 1\). Therefore, \(\models_U A \rightarrow T\).

Law of Identity (\(\models_U A \rightarrow A\)): We use the same reasoning as above to infer that \(p(A \rightarrow A) = 1\) and hence \(\models_U A \rightarrow A\).

Supraclassicality for Laws (for \(A \rightarrow\), and atom-classical valuations \(\models_{CL} A \Rightarrow \models_{U} A\)): Since \(A\) is a classical validity of the \(\rightarrow\)-free language, it is also a \(C\) (and \(U\)) validity (since the quasi-connectives output classical values on classical inputs), and therefore \(A_T = W\). Hence \(p(A) = 1\) for all probability functions \(p\).

Left Logical Equivalence \((A \models_C B, B \models_C AB \Rightarrow A \models U B \rightarrow C)\): Suppose it is not the case that \(A \models C B \models_{U} B \rightarrow C\). Then either SS- or the TT-entailment must fail. Suppose the TT-entailment fails: any countermodel has the form \(v(A) = 0\), \(v(B) \geq 1/2\), \(v(C) = 0\). But then \(B \not\models_C A\), in contradiction with our initial assumptions. Conversely, suppose the SS-entailment fails: any countermodel has the form \(v(A) \geq 1/2\), \(v(B) = 0\), \(v(C) = 1\). But then \(A \not\models_C B\). Hence \(A \rightarrow C \models_U B \rightarrow C\).

Stronger-Than-Material \((A \rightarrow B \models_U A \supset B)\): Valid in QCC/SS but valid in QCC/TT only when restricted to atom-classical valuations. The countermodel is \(v(A) = 1\), \(v(B) = 1/2\).

Conjunctive Sufficiency \((A, B \models_U A \rightarrow B)\): Valid in QCC/TT, but valid in QCC/SS only when restricted to atom-classical valuations. The countermodel is \(v(A) = 1\), \(v(B) = 1/2\). Not for the last time, the choice of quasi-conjunction turns out to be crucial for invalidating a principle of QCC/SS that would be valid using Strong Kleene connectives.

AND \((A \rightarrow B, A \rightarrow C \models_U A \rightarrow (B \land C))\): Valid in both QCC/TT and QCC/SS (proof is mechanical using the truth-tables method).

OR \((A \rightarrow C, B \rightarrow C \models_U A \lor B \rightarrow C)\): Valid in QCC/TT, but valid in QCC/SS only when restricted to atom-classical valuations. The countermodel is \(v(A) = 1/2\), \(v(B) = 0\), \(v(C) = 1\).

Cautious Transitivity \((A \rightarrow B, (A \land B) \rightarrow C \models_U A \rightarrow C)\): Valid in QCC/TT, valid in QCC/SS only when restricted to atom-classical valuations. The countermodel is \(v(A) = 1\), \(v(B) = 1\), \(v(C) = 1/2\).

Cautious Monotonicity \((A \rightarrow B, A \rightarrow C \models_U A \land C \rightarrow B)\): Valid in QCC/TT, valid in QCC/SS only when restricted to atom-classical valuations. The countermodel is \(v(A) = 1\), \(v(B) = 1/2\), \(v(C) = 1\).

Rational Monotonicity \((A \rightarrow B, \neg(A \rightarrow \neg C) \models_U A \land C \rightarrow B)\): This principle is in our semantics equivalent to Cautious Monotonicity since \(\neg(A \rightarrow \neg C)\) has the same truth table as \(A \rightarrow C\).
Reciprocity \( (A \rightarrow B, B \rightarrow A \models_U (A \rightarrow C) \equiv (B \rightarrow C)) \): Valid in QCC/TT, valid in QCC/SS only when restricted to atom-classical valuations. The countermodel is \( v(A) = 1, v(B) = 1/2 \) (or vice versa), and \( v(C) = 1/2 \).

Right Weakening \( (B \models_C C \Rightarrow A \rightarrow B \models_A A \rightarrow C) \): Valid in QCC/TT, but valid in QCC/SS only when restricted to atom-classical valuations. The countermodel is \( v(A) = 1, v(B) = 1, v(C) = 1/2 \).

Conditional Excluded Middle \( (\models_U (A \rightarrow B) \vee (A \rightarrow \neg B)) \): Valid in \( U \) because it is a theorem of \( C \).

Rule of Conditional K \( (\text{if the } A_i \text{ entail } C, \text{ then the } (B \rightarrow A_i) \text{ entail } (B \rightarrow C)) \): Valid in QCC/TT, but valid in QCC/SS only when restricted to atom-classical valuations. A countermodel is \( v(A_i) = 1 \) for all \( i \leq n \), \( v(B) = 1 \), \( v(C) = 1/2 \).

Modus Ponens \( (A \rightarrow B, A \models_U B) \): Valid in QCC/TT, but valid in QCC/SS only when restricted to atom-classical valuations. The countermodel is \( v(A) = 1, v(B) = 1/2 \).

Modus Tollens \( (A \rightarrow B, \neg B \models_U \neg A) \): Valid in QCC/SS (vacuously, there is no Cooper valuation \( v \) such that \( v((A \rightarrow B) \land \neg B) = 1 \)), but valid in QCC/TT only when restricted to atom-classical valuations. The countermodel is \( v(A) = 1, v(B) = 1/2 \) (i.e., identical to MP).

Supra-classicality for Inferences \( (\Gamma \models_{cL} B \Rightarrow \Gamma \models_U B) \): Fails even for single-premise inference, with \( \Gamma = \{A\} \) being of the form \( p \land \neg p \Rightarrow v(A) = 1/2 \). Inference from this premise would imply anything in classical logic, but not in \( C \) or \( U \) (e.g., set \( v(B) = 0 \)).

Or-to-If \( (\neg A \lor B \models_U A \rightarrow B) \): Valid in QCC/TT, but invalid in QCC/SS. Any valuation with \( v(A) = 0 \) is a countermodel.

Import-Export \( (A \rightarrow (B \rightarrow C) \text{ if and only if } (A \land B) \rightarrow C) \): This is a semantic validity for the trivalent conditional, combined with either Strong Kleene conjunction or quasi-conjunction. Both expressions have the same truth tables. A fortiori, both inferences are valid in \( C \) and \( U \).

Simplification of Disjunctive Antecedents \( ((A \lor B) \rightarrow \rightarrow C \models_U (A \rightarrow C) \land (B \rightarrow C)) \): Valid in QCC/SS but valid in QCC/TT only when restricted to atom-classical valuations. Countermodel: \( v(A) = 0, v(B) = 1/2 \) (or vice versa), and \( v(C) = 0 \).

Aristotle’s Thesis \( (\models_U \neg (\neg A \rightarrow A)) \): Valid in \( U \) because it is a theorem of \( C \).

Boethius’s Thesis \( (\models_U (A \rightarrow C) \rightarrow \neg (A \rightarrow \neg C)) \): Valid in \( U \) because it is a theorem of \( C \).

Contraposition \( (A \rightarrow C \models_U \neg C \rightarrow \neg A) \): Valid for atom-classical valuations in QCC/TT, but invalid in QCC/SS. Countermodel: \( v(A) = 1, v(C) = 1 \), \( Monotonicity (A \rightarrow C \models_U (A \land B) \rightarrow C) \): Valid in QCC/TT, but invalid in QCC/SS. Countermodel: \( v(A) = 1, v(B) = 0, v(C) = 1 \).
Transitivity \((A \rightarrow B, B \rightarrow C \models_{U} A \rightarrow C)\): Valid in QCC/TT, but invalid in QCC/SS. Countermodel: \(v(A) = 0, v(B) = 1, v(C) = 1\).

C Updating Probabilities in Trivalent Models

In this appendix, we study more features of the trivalent probability function defined in Section 6. First, we reconstruct Bayes’ Theorem for conditional-free sentences \(A, B \in \mathcal{L}\) as

\[
p(A \rightarrow B) = \frac{p(B \rightarrow A) p(B)}{p(A)} \quad \text{(Trivalent Bayes’ Theorem)}
\]

This is because for conditional-free sentences \(A\) and \(B\), the probabilities of the conditional can be written as

\[
p(A \rightarrow B) = \frac{p(A \land B)}{p(A)} \quad p(B \rightarrow A) = \frac{p(A \land B)}{p(B)}
\]

The probability \(p(A \rightarrow B)\) is thus an adequate surrogate notion for evaluating the probability of \(B\), conditional on \(A\). The above trivalent version of Bayes’ Theorem relates this value to the unconditional probability of \(A\) and \(B\), and the conditional probability of \(A\) given \(B\).

Second, in computer science, knowledge bases play an important role for updating beliefs and degrees of belief (e.g., Lehmann and Magidor 1992). The elements of these bases can be Boolean sentences, but also conditionals. The surrogate notion of Bayesian conditionalization that we developed in Section 6 can be extended to such sentences, too. Suppose we learn the sentence \(A \rightarrow B\) (for conditional-free \(A\) and \(B\)). Then we have

\[
p_{A \rightarrow B}(C) = p((A \rightarrow B) \rightarrow C)
\]

\[
= \frac{c([\neg A \lor B]_T \cap C_T)}{c([\neg A \lor B]_T \cap C_T) + c([\neg A \lor B]_F \cap C_F)}
\]

\[
= p((\neg A \lor B) \rightarrow C)
\]

\[
= p_{A \supset B}(C)
\]

\[
= p(C \mid A \supset B)
\]

In other words, learning a simple conditional is equivalent to conditionalizing on the corresponding material conditional in the bivalent fragment of the language.

At first sight, this looks like an unwelcome reduction of indicative to material conditionals, but in the context of learning and belief revision,
this feature is actually desirable. Any reasonable rule for “learning” the indicative conditional $A \rightarrow B$ should imply the constraint $p'(B|A) = 1$ in the posterior distribution $p'$. Otherwise one can hardly say that the conditional has been added to one’s beliefs.

Sprenger and Hartmann (2019, ch. 4) show in their Theorem 4.3 that conditioning on the material conditional $A \supset B$ is the unique way of minimizing the divergence between prior distribution $p$ and posterior distribution $p'$, subject to the constraint $p'(B|A) = 1$. Hence, the identification of learning (simple) indicative conditionals with updating on material conditionals is welcome and not worrying.

Third, the use of $p(A \rightarrow B)$ as a surrogate notion for $p(B|A)$ has also been suggested by McGee (1989) in order to extend standard probability on $\mathcal{L}$ to sentences in $\mathcal{L}^{-}$ involving a conditional. However, McGee does not provide a full truth-functional semantics for the probability function. So he cannot interpret $p(X)$, for all $X$, directly as the credence in the worlds where $X$ is true, or the credence ratio between worlds where $X$ is true or false, etc. Instead, McGee (1989, p. 504) gives an axiomatic characterization of the function $p(A \rightarrow B)$, and how it interacts with the probability of conditional-free sentences. The main pillar in his edifice is the

**(Simple) Independence Principle** (McGee 1989, p. 499). For conditional-free sentences $A, B, C \in \mathcal{L}$, and assuming that $A$ and $C$ are logically incompatible and $p(A) > 0$, then

$$p(C \land (A \rightarrow B)) = p(C) \cdot p(A \rightarrow B).$$

This principle actually amounts to a definition. After all, McGee does not have a truth-functional account that relates the probability of sentences containing Boolean and/or conditional operators to their truth conditions. So he needs to stipulate equation 3. But when we look at the truth conditions of $C \land (A \rightarrow B)$ according to our account, the Simple Independence Principle should not be valid: it is immediate from $C \models_{\text{CL}} \neg A$ that the truth value of $C \land (A \rightarrow B)$ is, for all bivalent valuations, identical to the truth value of $C$. Hence $p(C \land (A \rightarrow B)) = p(C)$, which is normally larger than $p(C) \cdot p(A \rightarrow B)$. Notably, in spite of this divergence, our probability function satisfies the adequacy conditions C2–C8 that McGee imposes, together with the Independence Principle, as necessary and sufficient conditions for a probability function on a language with a conditional.

McGee needs the Independence Principle in order to model the behavior of the conjunction of conditionals. For example, the betting odds on the conjunction of conditional bets $B|A$ and $D|C$ after settling the bet $B|A$ should equal the betting odds for $D$ give $C$ (assuming that $A, B, C, D$ are all conditional-free). This is an important adequacy condition on probabilistic
theories of conditionals, discussed in McGee (1989, pp. 499-503), but also in recent formal work such as Flaminio, Godo, and Hosni (2020, Section 6–7) and Sanfilippo et al. (2020). We obtain this result directly from the probabilistic semantics, without having to assume the Independence Principle as an axiom. When we observe, for instance, $A \land B$, the updated probability of $(A \rightarrow B) \land (C \rightarrow D)$ becomes

\[
p_{A,B}((A \rightarrow B) \land (C \rightarrow D)) = p((A \land B) \rightarrow ((A \rightarrow B) \land (C \rightarrow D)))
\]

\[
= \frac{c(A_T \cap B_T \cap C_T \cap D_T)}{c(A_T \cap B_T \cap C_T \cap D_T) + c(A_T \cap B_T \cap C_T \cap D_T^c)}
\]

\[
= \frac{p(A,B,C,D)}{p(A,B,C,D) + p(A,B,C,\neg D)}
\]

which conforms, in the Boolean fragment of $\mathcal{L}_{\rightarrow}$, to the usual conditional probability $p(D|A,B,C)$. This will, in turn, be equal to $p(D|C)$ if $D$ is independent of $A$ and $B$, given $C$. The update on $A \land B$ is implemented using the introduction of an outer conditional and yields, as predicted, the conditional probability of $D$ given $C$. In other words, the betting odds on the conjunction are changed in order to reflect that only one of the two component bets is still live. Where McGee needs a controversial axiom to obtain this result, we simply apply the trivalent truth tables and our definition of probabilistic updating.

Of course, all this stays at the level of a “proof of concept” and needs to be elaborated in a different paper. In particular, it seems promising to study in more detail how our probabilistic account relates to McGee’s, and whether it can be represented using Stalnaker-Lewis selection functions (as McGee shows for his approach in the last part of his paper). In this appendix, we just want to show that our conceptual framework recovers essential aspects of probabilistic reasoning with conditionals.

\section{Counterfactuals}

Counterfactuals are conditionals in the subjunctive mood where the antecedent is, in the context of discourse, assumed to be false. Since the syntactic structure of counterfactuals is close to indicatives, we would like
to explore whether our trivalent account says something interesting about counterfactuals (or subjunctive conditionals in general).

Our semantics looks at first useless since any conditional with a false antecedent receives the semantic value $\frac{1}{2}$, that is, “intedeterminate”. But this is only part of the story: we can define the probability of a counterfactual, or its degree of assertability, in the same way we did for indicatives, namely according to equation (Probability). Moreover, it is not obvious that U fails for counterfactuals, since no clear separating principles have been established for logics of indicatives and counterfactuals. There are moot points, such as whether the validity of Conditional Excluded Middle (CEM) and Import-Export differs between indicatives and counterfactuals (Lewis 1973b; Stalnaker 1980; Williams 2010; Briggs 2012), but none of the arguments has been decisive. That said, a satisfactory account of counterfactuals might wish to explain why these principles are discussed in different ways for indicatives and for counterfactuals.

The differences are more pronounced on the probabilistic level: while the probability of a (simple) indicative $A \rightarrow C$ should, according to most accounts, respect Adams’s Thesis, and be equal to the conditional probability $p(C|A)$, the probability of the corresponding counterfactual $A \rightarrow C$ is usually evaluated differently, for example by imaging the probability distribution on the antecedent (Lewis 1976; Günther 2017, 2022). An alternative interpretation, which can be represented as a specific form of imaging, is proposed by Causal Modeling Semantics (e.g., Galles and Pearl 1998; Pearl 2000, pp. 72–73; Briggs 2012): we determine $p(A \rightarrow C)$ as $p(C|do(A))$, i.e., we replace observing the antecedent $A$ by intervening on $A (=do(A))$, relative to a graph that represents causal dependencies between the sentential variables. Our account thus needs to be amended with a causal, and more generally, a hyperintensional dimension in order to spell out truth conditions, probability and logic of counterfactuals in a satisfactory way. This project, however, goes beyond the scope of this paper.