Bounds of Adj-TVaR Prediction for Aggregate Risk

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Abstract

In financial and insurance industries, risks may come from several sources. It is therefore important to predict future risk by using the concept of aggregate risk. Risk measure prediction plays important role in allocating capital as well as in controlling (and avoiding) worse risk. In this paper, we consider several risk measures such as Value-at-Risk (VaR), Tail VaR (TVaR) and its extension namely Adjusted TVaR (Adj-TVaR). Specifically, we perform an upper bound for such risk measure applied for aggregate risk models. The concept and property of comonotonicity and convex order are utilized to obtain such upper bound.

Keywords: coherent property; comonotonic rv; convex order; tail property; Value-at-Risk (VaR).

1. INTRODUCTION

In financial and insurance industries, risks may come from several sources. It is therefore important to predict future risk by using the concept of aggregate risk. Suppose that \( S_N \) represents aggregate risk of collection of random losses \( \{X_i : i = 1, 2, \ldots, N\} \) given by

\[
S_N = X_1 + X_2 + \cdots + X_N,
\]

where the random losses are not necessarily independent, e.g. McNeil et al. [1], Tse [2]. Note that the random variable \( N \) is usually assumed to follow a discrete distribution whilst \( X_i \) is continuous random loss.

For a single random loss, \( X \), with probability distribution determined by parameter vector \( \theta \), the risk measure Value-at-Risk (VaR) is defined as maximum tolerated risk at a given level of significance, e.g. McNeil et al. [1], Nieto and Ruiz [3]. VaR is very important in allocating capital as well as in controlling (and avoiding) worse risk. Basically, VaR is calculated via its inverse of distribution function i.e.
The risk measure TVaR may be interpreted as the second tolerated risk after VaR. TVaR is expected to occur for less than $1 - a$. We may differentiate the value of TVaR for class of distributions, such as normal, heavy-tailed and extreme. The heavier of tail distribution the greater value of TVaR.

**Theorem 2.1.** (Jadhav et al. [8]) For a random loss $X$ and some $\alpha \in (0,1)$ and $c \in [0,0.1]$, $Adj - TVaR_{(a,c)}(X)$ is defined as the mean loss in the interval between $VaR_{a}(X)$ and $VaR_{a+(1-a)c}(X)$ i.e.

$$Adj - TVaR_{(a,c)}(X) = E[X | VaR_{a}(X) \leq X \leq VaR_{a+(1-a)c}(X)]$$

where its confidence level is $\alpha + (1 - \alpha)^{1+c}$.

Let $VaR_{a}(X) = a$ and $VaR_{a+(1-a)c}(X) = b$. From (2) we obtain

$$Adj - TVaR_{(a,c)}(X) = \frac{1}{P(a \leq X \leq b)} \int_{a}^{b} xf_{X} (x) dx = \frac{1}{(1 - \alpha)^{1+c}} \int_{a}^{b} xf_{X} (x) dx.$$
By substituting $F_X(x) = \mu, x = F^{-1}(\mu)$, and $f(x)dx = d\mu$,

$$Adj-\text{TVaR}_{(\alpha,c)}(X) = \frac{1}{(1-\alpha)^{\text{c}}} \int_{\alpha}^{\alpha + (1-\alpha)^{\text{c}}} F_X^{-1}(\mu)d\mu.$$ 

For the case of an aggregate risk, $S_N$, the Adj-TVaR$(\alpha,c)(S_N)$ is given by

$$Adj-\text{TVaR}_{(\alpha,c)}(S_N) = \frac{1}{(1-\alpha)^{\text{c}}} \int_{\alpha}^{\alpha + (1-\alpha)^{\text{c}}} F_{S_N}^{-1}(\mu)d\mu.$$

**Property-2.1.** The Adj-TVaR is a coherent risk measure.

To verify that Adj-TVaR is coherent risk measure, it will be shown that it fulfill subadditivity properties [5]. Consider $X_1$ and $X_2$ be individual risk and $S_2$ aggregate of $X_1$ and $X_2$. Thus,

$$(1-\alpha)^{\text{c}}(\text{Adj-TVaR}_{(\alpha,c)}(X_1) + \text{Adj-TVaR}_{(\alpha,c)}(X_2) - \text{Adj-TVaR}_{(\alpha,c)}(S_2))$$

$$= (1-\alpha)^{\text{c}} \left( E \left[ X_1 | F_{X_1}^{-1}(\alpha) \leq X_1 \leq F_{X_1}^{-1}(\alpha + (1-\alpha)^{\text{c}}) \right] + \right.$$

$$E \left[ X_2 | F_{X_2}^{-1}(\alpha) \leq X_2 \leq F_{X_2}^{-1}(\alpha + (1-\alpha)^{\text{c}}) \right]$$

$$- (1-\alpha)^{\text{c}} \left. \left( E \left[ S_2 | F_{S_2}^{-1}(\alpha) \leq S_2 \leq F_{S_2}^{-1}(\alpha + (1-\alpha)^{\text{c}}) \right] \right) \right)$$

$$= E \left[ X_1 \mathbf{1}_{F_{X_1}^{-1}(\alpha) \leq X_1 \leq F_{X_1}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right] + E \left[ X_2 \mathbf{1}_{F_{X_2}^{-1}(\alpha) \leq X_2 \leq F_{X_2}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right] -$$

$$E \left[ S_2 \mathbf{1}_{F_{S_2}^{-1}(\alpha) \leq S_2 \leq F_{S_2}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right]$$

$$= E \left[ X_1 \left( \mathbf{1}_{F_{X_1}^{-1}(\alpha) \leq X_1 \leq F_{X_1}^{-1}(\alpha + (1-\alpha)^{\text{c}})} - \mathbf{1}_{F_{X_2}^{-1}(\alpha) \leq X_2 \leq F_{X_2}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right) \right] +$$

$$E \left[ X_2 \left( \mathbf{1}_{F_{X_2}^{-1}(\alpha) \leq X_2 \leq F_{X_2}^{-1}(\alpha + (1-\alpha)^{\text{c}})} - \mathbf{1}_{F_{S_2}^{-1}(\alpha) \leq S_2 \leq F_{S_2}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right) \right]$$

and we obtain

$$(1-\alpha)^{\text{c}}(\text{Adj-TVaR}_{(\alpha,c)}(X_1) + \text{Adj-TVaR}_{(\alpha,c)}(X_2) - \text{Adj-TVaR}_{(\alpha,c)}(S_2))$$

$$\geq F_{X_1}^{-1}(\alpha + (1-\alpha)^{\text{c}}) E \left[ X_1 \mathbf{1}_{F_{X_1}^{-1}(\alpha) \leq X_1 \leq F_{X_1}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right] -$$

$$F_{X_1}^{-1}(\alpha) E \left[ \mathbf{1}_{F_{X_1}^{-1}(\alpha) \leq X_1 \leq F_{X_1}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right] + F_{X_2}^{-1}(\alpha) E \left[ \mathbf{1}_{F_{X_2}^{-1}(\alpha) \leq X_2 \leq F_{X_2}^{-1}(\alpha + (1-\alpha)^{\text{c}})} \right]$$

$$= F_{X_1}^{-1}(\alpha + (1-\alpha)^{\text{c}}) \left( (\alpha + (1-\alpha)^{\text{c}}) - (\alpha + (1-\alpha)^{\text{c}} - \alpha) \right) +$$

$$F_{X_2}^{-1}(\alpha + (1-\alpha)^{\text{c}}) \left( (\alpha + (1-\alpha)^{\text{c}} - \alpha) - (\alpha + (1-\alpha)^{\text{c}} - \alpha) \right) = 0$$

Thus, for $S_2 = X_1 + X_2$, we have

$$\text{Adj-TVaR}_{(\alpha,c)}(S_2) \leq \text{Adj-TVaR}_{(\alpha,c)}(X_1) + \text{Adj-TVaR}_{(\alpha,c)}(X_2)$$

don that proves subadditivity property.

The following figures illustrate the value of Adj-TVaR for several distribution on the probability
function curve, in comparison to its VaR.

Figure 1: Adj-TVaR\(_\alpha(X)\) on probability function curves: (a) Normal, (b) Uniform, (c) Pareto, (d) Weibull

**Property-2.2.** The value Adj-TVaR, in the same level of confidence, has lower value than its corresponding TVaR.

To prove the above property, we will show that at \(a + (1 - a)^{1+c}\) significance level, TVaR\(_{\alpha+(1-\alpha)^c}\)\((X)\) ≥ Adj-TVaR\(_{(a,c)}\)(X). Suppose that on right-tailed there is two intervals, 

\[I_1 = [\alpha, 1 - \alpha^{1+c}]\] and \[I_2 = [\alpha + (1 - \alpha)^{1+c}, 1]\]. Thus, for every \(p \in I_1\) and \(q \in I_2\), \(F^{-1}_X(p) = F^{-1}_X(q)\), such that

Let \(1 - (1 - a)^{1+c} = a\) and \(a + (1 - a)^{1+c} = b\), we have

\[
\int_a^b F_X^{-1}_r dr + \int_a^b F_X^{-1}_p dp \geq \int_a^b F_X^{-1}_q dq + \int_a^b F_X^{-1}_r dr.
\]

\[
\int_a^b F_X^{-1}_p dp \geq \int_a^b F_X^{-1}_q dq
\]

from which we can conclude that TVaR\(_{\alpha+(1-\alpha)^c}\)\((X)\) ≥ Adj-TVaR\(_{(a,c)}\)(X).

**3. UPPER BOUND OF ADJ-TVaR FOR AGGREGATE RISK**

When a risk measure is applied to aggregate risk model, it may not be easy to the equality between the risk measure of aggregate risk and the aggregate of risk measure of individual risk. In other words, we may only seek whether subadditivity property applies for such risk measure.

As before, consider an aggregate risk of \(S_N\). Let \(X = (X_1, X_2, ..., X_N)\) be random vector and
$F_{X_1}, F_{X_2}, F_{X_3}, ..., F_{X_N}$ are their corresponding marginal distribution function. Now, for any random vector $X$, not necessarily comonotonic, its comonotonic counterpart is defined as any random vector with the same marginal distributions and with the comonotonic dependency structure. It can be proven that a random vector is comonotonic if and only if all its components are non-decreasing (or non-increasing) functions of the same random losses (see McNeil et al., 2005).

The comonotonic counterpart of $X$ will be denoted by $X^c = (X_1^c, X_2^c, ..., X_N^c)$ and

$$(X_1^c, X_2^c, ..., X_N^c) = (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_N}^{-1}(U)).$$

A random vector is comonotonic if and only if its marginal distribution function are non-decreasing function of the same random variable. Suppose that $S$ be the sum of every component of $X$. $S$ can be expressed as

$$S_N^c = X_1^c + X_2^c + ... + X_N^c.$$

**Definition 3.1.** Let $X_1$ and $X_2$ be two random variables.

(i) $X_1$ is said to be smaller than $X_2$ in convex order ($X_1 \preceq_{co} X_2$) if

$$E[g(X_1)] \leq E[g(X_2)]$$

for any convex function $g$ such that the expectation exist.

(ii) $X_1$ is said to be smaller than $X_2$ in stop loss order ($X_1 \preceq_{sl} X_2$) if

$$E[(X_1 - K)_+] \leq E[(X_2 - K)_+] \text{ and } E[X_1] = E[X_2],$$

for every $K \in R$.

**Property-3.1.** If $X_1$ precedes $X_2$ in convex order sense i.e if $X_1 \preceq_{co} X_2$, then $E[X_1] = E[X_2]$ and $\text{Var}[X_1] \leq \text{Var}[X_2]$.

**Property-3.2.** $X_1 \preceq_{co} X_2$ if only if $X_1 \preceq_{sl} X_2$.

**Theorem 3.1.** For any random vector $X = (X_1, X_2, ..., X_N)$ we have that [9]

$$S_N \preceq_{co} S_N^c$$

In the following proposition, we argue that risk measure of Adj-TVaR for aggregate risk is lower than the corresponding Adj-TVaR for its comonotonic counterpart.

**Proposition 3.1.** For any aggregate random variable $S_N$ and $S_N^c$ counterpart of it, we have that if $S_N \preceq_{sl} S_N^c$ then their respective Adj-TVaR are ordered:

$$S_N \preceq_{sl} S_N^c \Rightarrow Adj - TVaR_{(a,c)}(S_N) \leq Adj - TVaR_{(a,c)}(S_N^c)$$ (4)
**Proof:** First, we assume \( S_N \leq_d S_N^c \). According to Dhaene et al. [10] if \( S_N \leq_d S_N^c \) then

\[
\text{Var}_{\alpha}(S_N) \leq \text{Var}_{\alpha}(S_N^c).
\]

That inequality causes

\[
\alpha \text{Var}_{\alpha}(S_N) \leq \alpha \text{Var}_{\alpha}(S_N^c),
\]

The formula for \( \text{Adj-TVaR}_{\alpha,c}(S_N^c) \) is defined as

\[
\text{Adj-TVaR}_{\alpha,c}(S_N^c) = E\left[ S_N^c \mid \text{Var}_{\alpha}(S_N^c) \leq S_N \leq \text{Var}_{\alpha+(1-\alpha)^{1+c}}(S_N^c) \right]
\]

Let \( \text{Var}_{\alpha}(S_N^c) = a \) and \( \text{Var}_{\alpha+(1-\alpha)^{1+c}}(S_N^c) = b \). Thus

\[
\text{Adj-TVaR}_{\alpha,c}(S_N^c) = \frac{1}{P(a \leq S_N^c \leq b)} \int_a^b s f_{S_N^c}(s)ds
\]

\[
= \frac{1}{(1-\alpha)^{1+c}} \int_a^b s f_{S_N^c}(s)ds.
\]

By substituting \( F_{S_N^c}(s) = \mu, s = F_{S_N^c}^{-1}(\mu) \) and \( f_{S_N^c}(s)ds = d\mu \), we obtain

\[
\text{Adj-TVaR}_{\alpha,c}(S_N^c) = \frac{1}{(1-\alpha)^{1+c}} \int_a^{\alpha+(1-\alpha)^{1+c}} F_{S_N^c}^{-1}(\mu)d\mu.
\]

Suppose that \( \alpha + (1-\alpha)^{1+c} = b \), we have the following

\[
\text{Adj-TVaR}_{\alpha,c}(S_N^c) = \frac{1}{(1-\alpha)^{1+c}} \left( \int_a^b F_{X_N}^{-1}(\mu)d\mu + \cdots + \int_a^b F_{X_N}^{-1}(\mu)d\mu \right)
\]

\[
= \text{Adj-TVaR}_{(\alpha,c)}(X_1) + \cdots + \text{Adj-TVaR}_{(\alpha,c)}(X_N),
\]

and this proves our proposition.

4. **CONCLUDING REMARK**

The risk measure of Adj-TVaR for aggregate risk and its comonotonic counterpart may be applied to the Copula TVaR of Brahim et al. [11]. In practice, many random losses really depend on other losses that are not necessarily its component of aggregate risk.
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ACKNOWLEDGMENT

This research is partially funded by Riset Inovasi KK ITB (2019).

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