Maximal violation of Bell’s inequality in the case of real experiments

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Abstract

Einstein’s locality is invoked to derive a correlation inequality. In the case of ideal experiments, this inequality is equivalent to Bell’s original inequality of 1965 which, as is well known, is violated by a maximum factor of 1.5. The crucial point is that even in the case of real experiments where polarizers and detectors are non-ideal, the present inequality is violated by a factor of 1.5, whereas previous inequalities such as Clauser-Horne-Shimony-Holt inequality of 1969 and Clauser-Horne inequality of 1974 are violated by a factor of $\sqrt{2}$. The larger magnitude of violation can be of importance for the experimental test of locality. Moreover, the supplementary assumption used to derive this inequality is weaker than Garuccio-Rapisarda assumption. Thus an experiment based on this inequality refutes a larger family of hidden variable theories than an experiment based on Garuccio-Rapisarda inequality.

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I. Introduction

Local realism is a philosophical view which holds that external reality exists and has local properties. Quantum mechanics vehemently denies that such a world view has any meaning for physical systems because local realism assigns simultaneous values to non-commuting observables. In 1965 Bell [1] showed that the assumption of local realism, as postulated by Einstein, Podolsky, and Rosen (EPR) [2], leads to some constraints on the statistics of two spatially separated particles. These constraints, which are collectively known as Bell inequalities, are sometimes grossly violated by quantum mechanics. The violation of Bell inequalities therefore indicate that local realism is not only philosophically but also numerically incompatible with quantum mechanics. Bell’s theorem is of paramount importance for understanding the foundations of quantum mechanics because it rigorously formulates EPR’s assumption of locality and shows that all realistic interpretations of quantum mechanics must be nonlocal.

Bell’s original argument, however, can not be experimentally tested because it relies on perfect correlation of the spin of the two particles [3]. Faced with this problem, Clauser-Horne-Shimony-Holt (CHSH) [4], Freedman-Clauser (FC) [5], and Clauser-Horne (CH) [6] derived correlation inequalities for systems which do not achieve 100% correlation, but which do achieve a necessary minimum correlation. An Experiment based on CHSH, or FC, or CH inequality utilizes one-channel polarizers in which the dichotomic choice is between the detection of the photon and its lack of detection. A better experiment is one in which a truly binary choice is made between the ordinary and the extraordinary rays [7-10]. In this paper, we derive a correlation inequality for two-channel polarizer systems and we show that quantum mechanics violates this inequality by a factor of 1.5, whereas it violates the previous inequalities [4-10] by a factor of $\sqrt{2}$. Thus the magnitude of violation of the inequality derived in this paper is approximately 20.7% larger than the magnitude of violation of previous inequalities [4-10]. Moreover, we show that the present inequality requires the measurement of only three detection probabilities, whereas CH (or CHSH) inequality requires the measurements of five detection probabilities. Thus the present inequality can be used to test locality more simply than CH (or CHSH) inequality.

II. Experiments with pairs of atomic photons

We start by considering Bohm’s [11] version of EPR experiment in which an unstable source emits pairs of photons in a cascade from state $J = 1$ to $J = 0$ (see Fig. 1 of Ref. 10). The source is viewed by two apparatuses. The first (second) apparatus consists of a polarizer $P_1 (P_2)$ set at angle $m (n)$,
and two detectors $D_1^\pm (D_2^\pm)$ put along the ordinary and the extraordinary beams. During a period of time $T$ while the polarizers are set along axes $m$ and $n$, the source emits, say, $N$ pairs of photons.

Let $N^{\pm \pm}(m,n)$ be the number of simultaneous counts from detectors $D_1^\pm$ and $D_2^\pm$, $N^{\pm 0}(m,n)$ the number of counts when detectors $D_1^\pm$ are triggered but detectors $D_2^\pm$ are not triggered, $N^{0\pm}(m,n)$ the number of counts when detectors $D_2^\pm$ are triggered but $D_1^\pm$ are not triggered, and finally $N^{00}(m,n)$ the number of photons that are emitted by the source but not detected by either $D_1^\pm$ or $D_2^\pm$. If the time $T$ is sufficiently long, then the ensemble probabilities are defined as

$$p^{\pm \pm}(m,n) = \frac{N^{\pm \pm}(m,n)}{N}, \quad p^{\pm 0}(m,n) = \frac{N^{\pm 0}(m,n)}{N},$$
$$p^{0\pm}(m,n) = \frac{N^{0\pm}(m,n)}{N}, \quad p^{00}(m,n) = \frac{N^{00}(m,n)}{N}. \quad (1)$$

Similarly, we let $N^\pm(m) [N^\pm(n)]$ be the number of counts from detectors $D_1^\pm [D_2^\pm]$, and $N^0(m) [N^0(n)]$ the number of photons that are emitted by the source but not detected by $D_1^\pm [D_2^\pm]$. Again if the time $T$ is sufficiently long, then the ensemble probabilities are defined as

$$p^\pm(m) = \frac{N^\pm(m)}{N}, \quad p^0(m) = \frac{N^0(m)}{N},$$
$$p^\pm(n) = \frac{N^\pm(n)}{N}, \quad p^0(n) = \frac{N^0(n)}{N}. \quad (2)$$

It is important to emphasize that in real experiments, due to imperfection of polarizers and detectors, $p^{\pm 0}(m,n)$, $p^{0\pm}(m,n)$, and $p^{00}(m,n)$ are non-zero; in fact in experiments which are feasible with present technology, these probabilities are much larger than $p^{\pm \pm}(m,n)$ (similarly $p^0(m)$ [$p^0(n)$] are much larger than $p^\pm(m)$ [$p^\pm(n)$]). Since $p^{\pm 0}(m,n)$, $p^{0\pm}(m,n)$, $p^{00}(m,n)$, $p^0(m)$, and $p^0(n)$ can not be measured in actual experiments, it is crucial that they do not appear in any correlation inequality that is used to test locality.

We now consider a particular pair of photons and specify its state with a parameter $\lambda$. Following Bell, we do not impose any restriction on the complexity of $\lambda$. “It is a matter of indifference in the following whether $\lambda$ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous [1].”

The ensemble probabilities in Eqs. (1) and (2) are defined as

$$p^{\pm \pm}(a,b) = \int p(\lambda)p^\pm(a,\lambda)p^\pm(b,\lambda,\lambda),$$
$$p^\pm(a) = \int p(\lambda)p^\pm(a,\lambda),$$
$$p^\pm(b) = \int p(\lambda)p^\pm(b,\lambda). \quad (3)$$
Equations (3) may be stated in physical terms: The ensemble probability for detection of photons by detectors $D^\pm_1$ and $D^\pm_2$ [that is $p^\pm(a, b)$] is equal to the sum or integral of the probability that the emission is in the state $\lambda$ [that is $p(\lambda)$], times the conditional probability that if the emission is in the state $\lambda$, then a count is triggered by the first detector $D^\pm_1$ [that is $p^\pm(a | \lambda)$], times the conditional probability that if the emission is in the state $\lambda$ and if the first polarizer is set along axis $a$, then a count is triggered from the second detector $D^\pm_2$ [that is $p^\pm(b | \lambda, a)$]. Similarly the ensemble probability for detection of photons by detector $D^\pm_1 \left( D^\pm_2 \right)$ [that is $p^\pm(a) \left[ p^\pm(b) \right]$] is equal to the sum or integral of the probability that the photon is in the state $\lambda$ [that is $p(\lambda)$], times the conditional probability that if the photon is in the state $\lambda$, then a count is triggered by detector $D^\pm_1 \left( D^\pm_2 \right)$ [that is $p^\pm(a | \lambda) \left[ p^\pm(b | \lambda) \right]$]. Note that Eqs. (1), (2), and (3) are quite general and follow from the standard rules of probability theory. No assumption has yet been made that is not satisfied by quantum mechanics.

Hereafter, we focus our attention only on those theories that satisfy EPR criterion of locality: “Since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to first system [2]”. EPR’s criterion of locality can be translated into the following mathematical equation:

$$p^\pm(b | \lambda, a) = p^\pm(b | \lambda). \quad (4)$$

Equation (4) is the hallmark of local realism. It is the most general form of locality that accounts for correlations subject only to the requirement that a count triggered by the second detector does not depend on the orientation of the first polarizer. The assumption of locality, i.e., Eq. (4), is quite natural since the two photons are spatially separated so that the orientation of the first polarizer should not influence the measurement carried out on the second photon.

### III. Bell’s inequality

In the following we show that equation (4) leads to validity of an equality that is sometimes grossly violated by the quantum mechanical predictions in the case of real experiments. First we need to prove the following algebraic theorem.

**Theorem:** Given ten non-negative real numbers $x^+_1, x^-_1, x^+_2, x^-_2, y^+_1, y^-_1, y^+_2, y^-_2, U$ and $V$ such that $x^+_1, x^-_1, x^+_2, x^-_2 \leq U$, and $y^+_1, y^-_1, y^+_2, y^-_2 \leq V$, then the following inequality always holds:

$$Z = x^+_1 y^+_1 + x^-_1 y^-_1 - x^+_1 y^-_1 - x^-_1 y^+_1 + y^+_2 x^+_1 + y^-_2 x^-_1 - y^+_2 x^-_1 - y^-_2 x^+_1 + y^+_1 x^+_2 + y^-_1 x^-_2 + y^-_1 x^+_2 + 2x^+_2 y^+_2 + 2x^-_2 y^-_2 - V x^+_2 - V x^-_2 - U y^+_2 - U y^-_2 - U V \leq 0. \quad (5)$$


Proof: Calling $A = y_1^+ - y_1^-$, we write the function $Z$ as
\[
Z = x_2^+ (2y_2^+ - A - V) + x_2^- (2y_2^+ + A - V) + \left( x_1^+ - x_1^- \right) \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV. \tag{6}
\]

We consider the following eight cases:

1. First assume \[
\begin{align*}
2y_2^+ - A - V &\leq 0, \\
2y_2^- + A - V &\leq 0, \\
A + y_2^+ - y_2^- &\leq 0.
\end{align*}
\]

The function $Z$ is maximized if $x_2^+ = 0, x_2^- = 0, \quad x_1^+ - x_1^- = -U$. Thus
\[
Z \leq -U \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV = -U \left( A + 2y_2^+ + V \right). \tag{7}
\]

Since $V \geq A$ and $y_2^+ \geq 0$, $Z \leq 0$.

2. Next assume \[
\begin{align*}
2y_2^+ - A - V &> 0, \\
2y_2^- + A - V &\leq 0, \\
A + y_2^+ - y_2^- &\leq 0.
\end{align*}
\]

The function $Z$ is maximized if $x_2^+ = U, x_2^- = 0, \quad x_1^+ - x_1^- = -U$. Thus
\[
Z \leq U \left( 2y_2^+ - A - V \right) - U \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV = -2U \left( V + A \right). \tag{8}
\]

Since $V \geq A$, $Z \leq 0$.

3. Next assume \[
\begin{align*}
2y_2^+ - A - V &\leq 0, \\
2y_2^- + A - V &> 0, \\
A + y_2^+ - y_2^- &\leq 0.
\end{align*}
\]

The function $Z$ is maximized if $x_2^+ = 0, x_2^- = U, \quad x_1^+ - x_1^- = -U$. Thus
\[
Z \leq U \left( 2y_2^- + A - V \right) - U \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV = -2U \left( V - y_2^- + y_2^+ \right). \tag{9}
\]

Since $V \geq y_2^-$, and $y_2^+ \geq 0$, $Z \leq 0$.

4. Next assume \[
\begin{align*}
2y_2^+ - A - V &\leq 0, \\
2y_2^- + A - V &\leq 0, \\
A + y_2^+ - y_2^- &> 0.
\end{align*}
\]
The function $Z$ is maximized if $x_2^+ = 0$, $x_2^- = 0$, and $x_1^+ - x_1^- = U$. Thus

$$Z \leq U \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV$$

$$= -U \left( -A + 2y_2^+ + V \right). \quad (10)$$

Since $V \geq -A$ and $y_2^- \geq 0$, $Z \leq 0$.

(5) Next assume

$$\begin{cases} 
2y_2^+ - A - V > 0, \\
2y_2^- + A - V > 0, \\
A + y_2^+ - y_2^- \leq 0.
\end{cases}$$

The function $Z$ is maximized if $x_2^+ = U$, $x_2^- = U$, and $x_1^+ - x_1^- = -U$. Thus

$$Z \leq U \left( 2y_2^+ - A - V \right) + U \left( 2y_2^- + A - V \right) - U \left( A + y_2^+ - y_2^- \right)$$

$$- Uy_2^+ - Uy_2^- - UV$$

$$= -U \left( -2y_2^+ + A + 3V \right). \quad (11)$$

Since $V \geq A$ and $V \geq y_2^-$, $Z \leq 0$.

(6) Next assume

$$\begin{cases} 
2y_2^+ - A - V > 0, \\
2y_2^- + A - V \leq 0, \\
A + y_2^+ - y_2^- > 0.
\end{cases}$$

The function $Z$ is maximized if $x_2^+ = U$, $x_2^- = 0$, and $x_1^+ - x_1^- = U$. Thus

$$Z \leq U \left( 2y_2^+ - A - V \right) + U \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV$$

$$= -2U \left( -y_2^+ + y_2^- + V \right). \quad (12)$$

Since $V \geq y_2^+$, and $y_2^- \geq 0$ $Z \leq 0$.

(7) Next assume

$$\begin{cases} 
2y_2^+ - A - V \leq 0, \\
2y_2^- + A - V > 0, \\
A + y_2^+ - y_2^- > 0.
\end{cases}$$

The function $Z$ is maximized if $x_2^+ = 0$, $x_2^- = U$, and $x_1^+ - x_1^- = U$. Thus

$$Z \leq U \left( 2y_2^- + A - V \right) + U \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV$$

$$= -2U \left( -A + V \right). \quad (13)$$

Since $V \geq -A$, $Z \leq 0$. 

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(8) Finally assume \[
\begin{aligned}
-2y_2^+ + A + V > 0, \\
-2y_2^- - A + V > 0, \\
A + y_2^+ - y_2^- > 0.
\end{aligned}
\]
The function $Z$ is maximized if $x_2^+ = U$, $x_2^- = U$, and $x_1^+ - x_1^- = U$. Thus
\[
Z \leq U \left( 2y_2^+ - A - V \right) + U \left( 2y_2^- + A - V \right) + U \left( A + y_2^+ - y_2^- \right) - Uy_2^+ - Uy_2^- - UV = -U \left( -2y_2^+ - A + 3V \right).
\]
Since $V \geq -A$ and $V \geq y_2^+$, $Z \leq 0$, and the theorem is proved.

Now let $a$ ($b$) and $a'$ ($b'$) be two arbitrary orientation of the first (second) polarizer, and let
\[
\begin{aligned}
x_1^+ &= p^+(a | \lambda), & x_2^+ &= p^+(a' | \lambda), \\
y_1^+ &= p^+(b | \lambda), & y_2^+ &= p^+(b' | \lambda).
\end{aligned}
\]
Obviously for each value of $\lambda$, we have
\[
\begin{aligned}
p^\pm(a | \lambda) \leq 1, & \quad p^\pm(a' | \lambda) \leq 1, \\
p^\pm(b | \lambda) \leq 1, & \quad p^\pm(b' | \lambda) \leq 1.
\end{aligned}
\]
Inequalities (5) and (16) yield
\[
\begin{aligned}
p^+(a | \lambda)p^+(b | \lambda) + p^-(a | \lambda)p^-(b | \lambda) - p^+(a | \lambda)p^-(b | \lambda) \\
p^+(b | \lambda)p^+(a | \lambda) + p^-(b | \lambda)p^-(a | \lambda) - p^+(b | \lambda)p^-(a | \lambda) \\
p^+(b' | \lambda)p^-(a | \lambda) - p^-(b' | \lambda)p^+(a | \lambda) \\
p^+(b' | \lambda)p^-(a' | \lambda) + p^-(b' | \lambda)p^+(a' | \lambda) \\
p^+(a' | \lambda)p^-(b' | \lambda) + 2p^+(a' | \lambda)p^-(b' | \lambda) - p^-(a' | \lambda) \\
p^+(a' | \lambda) - p^-(b' | \lambda) - p^-(b' | \lambda) \leq 1.
\end{aligned}
\]
Multiplying both sides of (17) by $p(\lambda)$, integrating over $\lambda$ and using Eqs. (3), we obtain
\[
\begin{aligned}
p^{++}(a, b) + p^{--}(a, b) - p^{-+}(a, b) - p^{+-}(a, b) + p^{++}(b', a) + \\
p^{-+}(b', a) - p^{+-}(b', a) - p^{-+}(b', a) - p^{++}(a', b) - \\
p^{-+}(a', b) + p^{+-}(a', b) + p^{--}(a', b) + 2p^{++}(a', b') + \\
2p^{-+}(a', b') - p^{+-}(a') - p^{-+}(a') - p^{++}(b') - p^{--}(b') \leq 1.
\end{aligned}
\]
We now note that the expected value of detection probabilities while polarizers are set along orientations $m$ and $n$, i.e., $E(m, n)$ is defined as
\[
\begin{aligned}
E(m, n) &= p^{++}(m, n) - p^{-+}(m, n) \\
&\quad - p^{+-}(m, n) + p^{--}(m, n).
\end{aligned}
\]
Using (19), inequality (18) may be written as

\[
E(a, b) + E(b', a) - E(a', b) + 2p^{++}(a', b') + 2p^{--}(a', b') - p^+(a') - p^-(a') - p^+(b') - p^-(b') \leq 1.
\]

(20)

All local realistic theories must satisfy inequality (18) or (20).

IV. Violation of Bell’s inequality in the case of ideal experiments

First we consider an atomic cascade experiment in which polarizers and detectors are ideal. Assuming polarizers are set along axes \( m \) and \( n \) where \( \theta = |m - n| \), the expected values, the single and joint detection probabilities for a pair of photons in a cascade from state \( J = 1 \) to \( J = 0 \) are given by

\[
E(m, n) = E(\theta) = \cos 2\theta, \quad p^+(a') = p^-(a') = p^+(b') = p^-(b') = \frac{1}{2},
\]

\[
p^{++}(m, n) = p^{++}(\theta) = \frac{\cos^2 \theta}{2}, \quad p^{--}(m, n) = p^{--}(\theta) = \frac{\cos^2 \theta}{2}.
\]

(21)

Now if we choose the following orientation \( (a, b) = (b', a) = 30^\circ, (a', b) = 60^\circ \) and \( (a', b') = 0^\circ \) inequality (20) becomes

\[
2E(30^\circ) - E(60^\circ) + 2p^{++}(0^\circ) + 2p^{--}(0^\circ) - p^+(a') - p^-(a') - p^+(b') - p^-(b') \leq 1.
\]

(22)

Using (21), we obtain

\[
2 \cos(60^\circ) - \cos(120^\circ) + 2 \frac{\cos^2(0^\circ)}{2} + 2 \frac{\cos^2(0^\circ)}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 2 \times 0.5 - (-0.5) + 2 \times \frac{1}{2} + 2 \times \frac{1}{2} - 2 \leq 1,
\]

(23)

or

\[
1.5 \leq 1,
\]

(24)

which violates inequality (20) by a factor of 1.5 in the case of ideal experiments.

We now show that for ideal polarizers and detectors, inequality (20) is equivalent to CHSH inequality. In an ideal experiment, all emitted photons are analyzed by the detectors and the probability that a photon is not collected is zero, i.e.,

\[
p^{00}(m, n) = p^{00}(m, n) = p^{00}(m) = p^{00}(n) = 0
\]

(25)
Thus in an ideal experiment,
\[
p^+ (a') = p^{++} (a', b') + p^{+-} (a', b'),
\]
\[
p^- (a') = p^{--} (a', b') + p^{-+} (a', b').
\] (26)

Substituting (26) in (20), we obtain
\[
E(a, b) + E(b', a) - E(a', b) + p^+(b') - p^- (b') + p^+ (b') + p^- (b') \leq 1 \quad (27)
\]

Since we have assumed detectors and polarizers are ideal, we have
\[
p^+ (b') + p^- (b') = 1.
\]

Thus
\[
E(a, b) + E(b', a) - E(a', b) + E(a', b') \leq 2 \quad (28)
\]

which is the same as CHSH inequality. Thus for ideal polarizers and detectors, the inequality derived in this paper is equivalent to CHSH inequality.

It is important to emphasize that in the case of ideal experiments, neither the present inequality nor CHSH are necessary: they both immediately reduce to Bell’s original inequality of 1965 [1]. First we show that in ideal experiments, CHSH reduces to Bell’s original inequality. If we assume \(a'\) and \(b'\) are along the same direction, using Eq. (21), we have
\[
E(a', b') = 1.
\]

CHSH inequality (28) therefore becomes
\[
E(a, b) + E(b', a) - E(a', b) \leq 1 \quad (29)
\]

which is the same as Bell’s original inequality of 1965 [1]. If we choose the following orientations: \((a, b) = (b', a) = 30^\circ, (a', b) = 60^\circ\), Bell’s inequality (29) is violated by a maximum factor of 1.5.

We now show that inequality (20) also reduces to Bell’s original inequality [1] in an ideal experiment. Again if we assume \(a'\) and \(b'\) are along the same direction, using (21), we have \(p^{++}(a', b') = p^{--}(a', b') = \frac{1}{2}, p^{+}(a') = p^{-}(a') = p^{+}(b') = p^{-}(b') = \frac{1}{2}\). Inequality (20) therefore becomes
\[
E(a, b) + E(b', a) - E(a', b) \leq 1 \quad (30)
\]

which is the same as Bell’s original inequality of 1965 [1].

We have thus shown that for ideal polarizers and detectors Bell’s original inequality (29) is sufficient and there is no need for inequality (20) or CHSH inequality (28). Moreover, we have shown that for the case ideal experiments (see 25), i.e., for the case where \(p^{++}(m, n) + p^{+-}(m, n) + p^{-+}(m, n) + p^{--}(m, n) = 1\), inequality (20) is equivalent to CHSH inequality and to Bell’s original inequality. However, for the case of real experiments where \(p^{0+}(m, n), p^{0-}(m, n),\) and \(p^{00}(m, n)\) are non-zero, i.e., for the case where \(p^{++}(m, n) + p^{+-}(m, n) + p^{-+}(m, n) + p^{--}(m, n) < 1\), inequality (20) is a distinct and new inequality and is not equivalent to any of the previous inequalities.
V. Violation of Bell’s inequality in the case of real experiments

We now consider a real experiment in which polarizers and detectors are non-ideal. In the atomic cascade experiment, an atom emits two photons in a cascade from state $J = 1$ to $J = 0$. Since the pair of photons have zero angular momentum, they propagate in the form of spherical wave. Thus the probability $p(d_1, d_2)$ of both photons being simultaneously detected by two detectors in the directions $d_1$ and $d_2$ is \[ p(d_1, d_2) = \eta^2 \left( \frac{\Omega}{4\pi} \right)^2 g(\theta, \phi), \] (31)

where $\eta$ is the quantum efficiency of the detectors, $\Omega$ is the solid angle of the detector, $\cos \theta = d_1 . d_1$, and angle $\phi$ is related to $\Omega$ by

$$\Omega = 2\pi (1 - \cos \phi).$$

Finally the function $g(\theta, \phi)$ is the angular correlation function and in the special case is given by \[ g(\pi, \phi) = 1 + \frac{1}{8} \cos^2 \phi (1 + \cos \phi)^2. \] (33)

If we insert polarizers in front of the detectors, then the quantum mechanical predictions for joint detection probabilities are \[ p^+(a) = p^-(a) = \eta \left( \frac{\Omega}{8\pi} \right), \quad p^+(b) = p^-(b) = \eta \left( \frac{\Omega}{8\pi} \right), \]

$$p^{++}(a, b) = \eta^2 \left( \frac{\Omega}{8\pi} \right)^2 g(\theta, \phi) \left[ T^1_+ T^2_+ + T^1_- T^2_- F(\theta) \cos 2(a - b) \right],$$

$$p^{--}(a, b) = \eta^2 \left( \frac{\Omega}{8\pi} \right)^2 g(\theta, \phi) \left[ R^1_+ R^2_+ + R^1_- R^2_- F(\theta) \cos 2(a - b) \right],$$

$$p^{+-}(a, b) = \eta^2 \left( \frac{\Omega}{8\pi} \right)^2 g(\theta, \phi) \left[ T^1_+ R^2_+ - T^1_- R^2_- F(\theta) \cos 2(a - b) \right],$$

$$p^{-+}(a, b) = \eta^2 \left( \frac{\Omega}{8\pi} \right)^2 g(\theta, \phi) \left[ R^1_+ T^2_+ - R^1_- T^2_- F(\theta) \cos 2(a - b) \right],$$ (34)

where

$$T^i_+ = T^i_+ + T^i_-, \quad T^i_- = T^i_+ - T^i_-$$

$$R^i_+ = R^i_+ + R^i_-, \quad R^i_- = R^i_+ - R^i_-$$

for $i = 1, 2$, where $T^i_+(T^i_-)$ represents the prism transmittance along the transmitted path for incoming light polarized parallel (perpendicular) to the
transmitted-channel, and $R_\parallel (R_\perp)$ represents the prism transmittance along the reflected path for incoming light polarized parallel (perpendicular) to the reflected-channel. The function $F(\theta, \phi)$ is the so-called depolarization factor and for the special case $\theta = \pi$ and small $\phi$ is given by

$$F(\pi, \phi) \approx 1 - \frac{2}{3} (1 - \cos\phi)^2.$$  

The function $F(\theta, \phi)$, in general, is very close to 1 (the detailed expression for $F(\theta, \phi)$ is given in [4]).

In the atomic cascade experiments which are feasible with present technology [5,12], because $\frac{\Omega}{4\pi} \ll 1$, only a very small fraction of photons are detected. Thus inequality (18) can not be used to test the violation of Bell’s inequality. It is important to emphasize that a supplementary assumption is required primarily because the solid angle covered by the aperture of the apparatus, $\Omega$, is much less than $4\pi$ and not because the efficiency of the detectors, $\eta$, is much smaller than 1. In fact in the previous experiments (Ref. 12), the efficiency of detectors were larger than 90%. However, because $\frac{\Omega}{4\pi} \ll 1$, all previous experiments needed supplementary assumptions to test locality.

It is worth nothing that CHSH [4] and CH [6] combined the solid angle covered by the aperture of the apparatus $\frac{\Omega}{4\pi}$ and the efficiency of the detectors $\eta$ into one term and wrote $\eta_{\text{CHSH}} = \eta \frac{\Omega}{4\pi}$. They then referred to $\eta_{\text{CHSH}}$ as efficiency of the detectors (this terminology however, is not usually used in optics. In optics, the efficiency of detector refers to the probability of detection of a photon; it does not refer to the product of the solid angle covered by the detector and the probability of detection of a photon). CHSH and CH then pointed that since $\eta_{\text{CHSH}} \ll 1$, a supplementary assumption is required. To clarify CHSH and CH argument, it should be emphasized that a supplementary assumption is needed mainly because $\frac{\Omega}{4\pi} \ll 1$, not because $\eta \ll 1$.

We now state a supplementary assumption and we show that this assumption is sufficient to make experiments where $\frac{\Omega}{4\pi} \ll 1$ applicable as a test of local theories. The supplementary assumption is: For every emission $\lambda$, the detection probability by detector $D^+$ (or $D^-$) is less than or equal to the sum of detection probabilities by detectors $D^+$ and $D^-$ when the polarizer is set along any arbitrary axis. If we let $\mathbf{r}$ be an arbitrary direction of the first or second polarizer, then the above supplementary assumption may be translated into the following inequalities

$$p^+(\mathbf{a} | \lambda) \leq p^+(\mathbf{r} | \lambda) + p^-(\mathbf{r} | \lambda), \quad p^-(\mathbf{a} | \lambda) \leq p^+(\mathbf{r} | \lambda) + p^-(\mathbf{r} | \lambda),$$

$$p^+(\mathbf{a'} | \lambda) \leq p^+(\mathbf{r} | \lambda) + p^-(\mathbf{r} | \lambda), \quad p^-(\mathbf{a'} | \lambda) \leq p^+(\mathbf{r} | \lambda) + p^-(\mathbf{r} | \lambda),$$
\[ p^+(b \mid \lambda) \leq p^+(r \mid \lambda) + p^-(r \mid \lambda), \quad p^-(b \mid \lambda) \leq p^+(r \mid \lambda) + p^-(r \mid \lambda), \]
\[ p^+(b' \mid \lambda) \leq p^+(r \mid \lambda) + p^-(r \mid \lambda), \quad p^-(b' \mid \lambda) \leq p^+(r \mid \lambda) + p^-(r \mid \lambda). \]

(37)

This supplementary assumption is obviously valid for an ensemble of photons. The sum of detection probability by detector \( D^+ \) and \( D^- \) for an ensemble of photons when the polarizer is set along any arbitrary axis \( \mathbf{v} \)

\[ p^+(\mathbf{v}) + p^-(\mathbf{v}) = \eta \left( \frac{\Omega}{4\pi} \right), \]

whereas

\[ p^+(\mathbf{v}) = p^-(\mathbf{v}) = \eta \left( \frac{\Omega}{8\pi} \right). \]

(38)

(39)

The supplementary assumption requires that the corresponding probabilities be valid for each \( \lambda \).

It is worth noting that the present supplementary assumption is weaker than Garuccio-Rapisarda (GR) assumption \([8]\), that is, an experiment based on the present supplementary assumption refutes a larger family of hidden variable theories than an experiment based on GR assumption. The GR assumption is

\[ p^+(a \mid \lambda) + p^-(a \mid \lambda) = p^+(r \mid \lambda) + p^-(r \mid \lambda) \]

(40)

We now show that GR assumption implies the assumption of this paper. We first note that the following inequalities always hold

\[ p^+(a \mid \lambda) \leq p^+(a \mid \lambda) + p^-(a \mid \lambda), \quad p^-(a \mid \lambda) \leq p^+(a \mid \lambda) + p^-(a \mid \lambda). \]

(41)

Now using GR assumption (40), we can immediately conclude that

\[ p^+(a \mid \lambda) \leq p^+(r \mid \lambda) + p^-(r \mid \lambda), \quad p^-(a \mid \lambda) \leq p^+(r \mid \lambda) + p^-(r \mid \lambda), \]

(42)

which are the same as (37). Thus an experiment which refutes the hidden variable theories which are consistent with GR assumption also refutes the hidden variable theories which are consistent with the present assumption. The reverse however is not true. An experiment based on the present supplementary assumption refutes a larger family of hidden variable theories than an experiment based on GR assumption.
Now using relations (5), (15) and (37), and applying the same argument that led to inequality (18), we obtain the following inequality

\[
\begin{align*}
&\left[ p^{+-}(a, b) + p^{-+}(a, b) - p^{+}(a, b) - p^{-}(a, b) + p^{++}(b', a) + p^{-}(b', a) \\
&- p^{+}(b', a) - p^{-}(b', a) - p^{+}(a, b') - p^{-}(a, b') + p^{+}(a', b) \\
&+ 2p^{+}(a', b') + 2p^{-}(a', b') - p^{+}(a', r) - p^{-}(a', r) \\
&- p^{+}(a', r) - p^{-}(a', r) - p^{+}(r, b') - p^{-}(r, b') \\
&- p^{+}(r, b') \right] / \left[ p^{+}(r, r) + p^{-}(r, r) + p^{+}(r, r) + p^{-}(r, r) \right] \leq 1. \tag{43}
\end{align*}
\]

Note that inequality (43) contains only double-detection probabilities and the number of emissions \( N \) from the source is eliminated from the ratio. Quantum mechanics violates this inequality in the case of real experiments where the solid angle covered by the aperture of the apparatus, \( \Omega \), is much less than \( 4\pi \). In particular, the magnitude of violation is maximized if the following orientations are chosen \( (a, b) = (a', a) = 30^\circ, (a', b) = 60^\circ \) and \( (a', b') = (a', r) = (r, b') = 0^\circ \).

Inequality (43) may be considerably simplified if we invoke some of the symmetries that are exhibited in atomic-cascade photon experiments. For a pair of photons in cascade from state \( J = 1 \) to \( J = 0 \), the quantum mechanical detection probabilities \( p_{QM}^{\pm \pm} \) and expected value \( E_{QM} \) exhibit the following symmetry

\[
p_{QM}^{\pm \pm} (m, n) = p_{QM}^{\pm \pm} (| m - n |), \quad E_{QM} (m, n) = E_{QM} (| m - n |). \tag{44}
\]

We assume that the local theories also exhibit the same symmetry

\[
p^{\pm \pm} (m, n) = p^{\pm \pm} (| m - n |), \quad E (m, n) = E (| m - n |). \tag{45}
\]

Note that there is no harm in assuming Eqs. (45) since they are subject to experimental test (CHSH \( \square \), FC \( \square \), and CH \( \square \) made the same assumptions). Using the above symmetry, inequality (43) is simplified to

\[
\begin{align*}
&\left[ E (| a - b |) + E (| b' - a |) - E (| a' - b |) + 2p^{+}(| a' - b |) + 2p^{-}(| a' - b |) \\
&- p^{+}(| a' - r |) - p^{+}(| a' - r |) - p^{-}(| a' - r |) - p^{-}(| a' - r |) \\
&- p^{+}(| r - b' |) - p^{+}(| r - b' |) - p^{-}(| r - b' |) - p^{-}(| r - b' |) \right] / \\
&\left[ p^{+}(0^\circ) + p^{-}(0^\circ) + p^{+}(0^\circ) + p^{-}(0^\circ) \right] \leq 1. \tag{46}
\end{align*}
\]

We now take \( a' \) and \( b' \) to be along \( r \), and we take \( a, b, \) and \( a' \) to be three coplanar axes with the following orientations: \( | a - b | = | b' - a | = 30^\circ, \).
\[ |\mathbf{a}' - \mathbf{b}| = 60^\circ \text{ and } |\mathbf{a}' - \mathbf{b}'| = |\mathbf{a}' - \mathbf{r}| = |\mathbf{r} - \mathbf{b}'| = 0^\circ. \] Furthermore if we define \( K \) as

\[ K = p^{++}(0^\circ) + p^{+-}(0^\circ) + p^{-+}(0^\circ) + p^{- -}(0^\circ) \] (47)

then inequality (46) is simplified to

\[ \frac{2E(30^\circ) - E(60^\circ) - 2p^{+-}(0^\circ) - 2p^{-+}(0^\circ)}{K} \leq 1. \] (48)

Using the quantum mechanical probabilities [i.e., Eqs. (34)], inequality (48) becomes \( 1.5 \leq 1 \) in the case of real experiments (here for simplicity, we have assumed \( F(\theta, \phi) = 1 \); this is a good approximation even in the case of real experiments. In actual experiments where the solid angle of detectors \( \phi \) is usually less than \( \pi/6 \), from (36) it can be seen that \( F(\theta, \pi/6) \approx 0.99 \). Moreover, we have assumed \( T_i^i = R_i^i = 1 \) and \( T_i^\perp = R_i^\perp = 0 \), where \( i = 1, 2 \); this is also a good approximation, see for example [3] or the experiments by Aspect et. al. [12]).

Inequality (48) can be used to test locality more simply than CH or CHSH inequality. CH inequality may be written as

\[ \frac{3p(\phi) - p(3\phi) - p(\mathbf{a}', \infty) - p(\infty, \mathbf{b})}{p(\infty, \infty)} \leq 0. \] (49)

The above inequality requires the measurements of five detection probabilities:

1. The measurement of detection probability with both polarizers set along the \( 22.5^\circ \) axis [that is \( p(22.5^\circ) \)].
2. The measurement of detection probability with both polarizers set along the \( 67.5^\circ \) axis [that is \( p(67.5^\circ) \)].
3. The measurement of detection probability with the first polarizer set along \( \mathbf{a}' \) axis and the second polarizer being removed [that is \( p(\mathbf{a}', \infty) \)].
4. The measurement of detection probability with the first polarizer removed and the second polarizer set along \( \mathbf{b} \) axis [that is \( p(\infty, \mathbf{b}) \)].
5. The measurement of detection probability with both polarizers removed [that is \( p(\infty, \infty) \)].

In contrast, the inequality derived in this paper [i.e., inequality (48)] requires the measurements of only three detection probabilities:

1. The measurement of detection probability with both polarizers set along the \( 0^\circ \) axis [that is \( p^{\pm \pm}(0^\circ) \)].
2. The measurement of detection probability with both polarizers set along the \( 30^\circ \) axis [that is \( p^{\pm \pm}(30^\circ) \)].
3. The measurement of detection probability with both polarizers set along the \( 60^\circ \) axis [that is \( p^{\pm \pm}(60^\circ) \)].
VI. Violation of Bell’s inequality in phase-momentum and in high-energy experiments

It should be noted that the analysis that led to inequality (48) is not limited to atomic-cascade experiments and can easily be extended to experiments which use phase-momentum [13] or use high energy polarized protons or γ photons [14-15] to test Bell’s limit. For example in the experiment by Rarity and Tapster [13], instead of inequality (2) of their paper, the following inequality (i.e., inequality (48) using their notations) may be used to test locality:

\[
\frac{2E(30^\circ) - E(60^\circ) - 2\overline{C}_{a_3 b_4}(0^\circ) - 2\overline{C}_{a_4 b_3}(0^\circ)}{K} \leq 1, \tag{50}
\]

where \(C_{a_i b_j}(\phi_a, \phi_b)\) for \(i = 3, 4; j = 3, 4\) is the counting rate between detectors \(D_{ai}\) and \(D_{bj}\) with phase angles being set to \(\phi_a, \phi_b\) (See Fig. 1 of [13]). The following set of orientations \((\phi_a, \phi_b) = (\phi_a', \phi_b) = 30^\circ, (\phi_a, \phi_b) = 60^\circ,\) and \((\phi_a', \phi_b') = 0^\circ\) leads to the largest violation. Similarly, in high-energy experiments and spin correlation proton-proton scattering experiments [15], inequality (48) can be used to test locality.

VII. Summary

We have invoked Einstein’s locality (Eq. 4) to derive a correlation inequality [inequality (20)] that can be used to test locality. In the case of ideal experiments, this inequality is equivalent to Bell’s original inequality of 1965 [1] or CHSH inequality [4]. However, in the case of real experiments where polarizers and detectors are non-ideal, inequality (20) is a new and distinct inequality and is not equivalent to any of previous inequalities.

We have also demonstrated that the conjunction of Einstein’s locality [Eq. (4)] with a supplementary assumption [inequality (37)] leads to validity of inequality (48) that is sometimes grossly violated by quantum mechanics in the case of experiments where \(\Omega \ll 1\). Inequality (48), which may be called strong inequality [16], defines an experiment which can actually be performed with present technology. Quantum mechanics violates inequality (48) by a factor 1.5, whereas it violates CHSH or CH inequality by a factor of \(\sqrt{2}\). Thus the magnitude of violation of the inequality derived in this paper is approximately 20.7% larger than the magnitude of violation of previous inequalities [4-10]. The larger magnitude of violation can be useful for experimental test of locality.
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