On representation matrices of boundary conditions in \( SU(n) \) gauge theories compactified on two-dimensional orbifolds

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Abstract

We study the existence of diagonal representatives in each equivalence class of representation matrices of boundary conditions in \( SU(n) \) or \( U(n) \) gauge theories compactified on the orbifolds \( T^2/\mathbb{Z}_N \) (\( N = 2, 3, 4, 6 \)). We suppose that the theory has a global \( G' = U(n) \) symmetry. Using constraints, unitary transformations and gauge transformations, we examine whether the representation matrices can simultaneously become diagonal or not. We show that at least one diagonal representative necessarily exists in each equivalence class on \( T^2/\mathbb{Z}_2 \) and \( T^2/\mathbb{Z}_3 \), but the representation matrices on \( T^2/\mathbb{Z}_4 \) and \( T^2/\mathbb{Z}_6 \) can contain not only diagonal matrices but also non-diagonal \( 2 \times 2 \) ones and non-diagonal \( 3 \times 3 \) and \( 2 \times 2 \) ones, respectively, as members of block-diagonal submatrices. These non-diagonal matrices have discrete parameters, which means that the rank-reducing symmetry breaking can be caused by the discrete Wilson line phases.

1 Introduction

The standard model of particle physics has been established as an effective theory around the weak scale, but it possesses several riddles. The origin of the gauge bosons and the Higgs boson

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has been a big mystery. The standard model looks complicated at first glance, and it seems to suggest a simple and beautiful theory beyond it. Theories defined on a higher-dimensional space-time are possible candidates. The gauge bosons and the Higgs boson can be unified as a higher-dimensional gauge multiplet \cite{1}. To realize the chiral fermions in the higher-dimensional theories, an orbifold is considered as an extra space. Such models are phenomenologically attractive, because the Higgs mass splitting between the doublet components and the triplet ones in the grand unified theories (GUTs) \cite{2,3,4} can be elegantly realized by orbifolding \cite{5,6,7}. In addition, the idea to unify the gauge and Higgs bosons can be applied to the electroweak symmetry breaking \cite{8,9,10}, since the doublet components become able to be extracted from the original adjoint representation as zero modes. In this scenario, the effective potential of the Higgs field is finite without supersymmetry (SUSY) \cite{11,12}, since the Higgs field is a pseudo Nambu-Goldstone mode \cite{13}, with respect to spatially separated symmetry breakings. The finiteness is thanks to the non-locality of the breakings, and there are local divergences in subdiagrams \cite{14,15,16}. These divergences are removed by the lower-loop counter terms with no need to introduce additional counterterms \cite{14,15}, which means that the effective potential is free from the divergences when it is written with the finite renormalized couplings. This idea is also considered in GUT contexts to break the electroweak symmetry \cite{17,18,19} as above, or to reduce the rank of the GUT symmetry \cite{20}. It is also applied to the breaking of the unified symmetry to show that the boundary conditions for the Higgs mass splitting assumed in refs. \cite{5,6,7} can be naturally obtained as the minimum of the effective potential \cite{21,22,23}, and its phenomenologies are studied \cite{24,25}. The standard model particles are supposed to be composed of zero modes from bulk fields on a higher-dimensional space-time and 4-dimensional fields localized on boundaries called brane fields. Physical symmetries are determined in cooperation of boundary conditions of fields and the dynamics of the Wilson line phases, by the Hosotani mechanism \cite{26,27,28}. Hence, the study of the boundary conditions of fields as well as the dynamics is important. Boundary conditions on orbifolds are specified by representation matrices and classified by equivalence relations of gauge symmetries \cite{29}. We refer to the representation matrices as twist matrices. In ref. \cite{29}, the classification of twist matrices has been carried out on $S^1/Z_2$, and it is shown that each equivalence class has at least one diagonal representative. Relying on this generality, a general form of the effective potential was derived \cite{30}. For the orbifolds $T^2/Z_N$ ($N = 2, 3, 4, 6$), it has been done, in a limited way, for a class with diagonal representatives \cite{31,32,33}, and there is no definite answer whether every equivalence class has at least one diagonal representative on the orbifolds.\footnote{In refs. \cite{34,32}, the classification of the twist matrices has been examined in an $SU(2)$ gauge theory compactified on $T^2/Z_2$. In ref. \cite{35}, an attempt has been carried out to arrive at an answer by using matrix exponential representations, but it remains incomplete because the conditions are not fully considered.}

In this paper, we study the existence of diagonal representatives in each equivalence class of the twist matrices in $SU(n)$ or $U(n)$ gauge theories compactified on the orbifolds $T^2/Z_N$ ($N = 2, 3, 4, 6$). We suppose that the theory has a global $G' = U(n)$ symmetry. Using constraints, unitary transformations and gauge transformations, we examine whether the twist matrices can simultaneously become diagonal or not. We show that at least one diagonal representative necessarily exists in each equivalence class on $T^2/Z_2$ and $T^2/Z_3$, but the twist
matrices on $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$ can contain not only diagonal matrices but also non-diagonal $2 \times 2$ ones and non-diagonal $3 \times 3$ and $2 \times 2$ ones, respectively, as members of block-diagonal submatrices.

The outline of this paper is as follows. In the next section, we give a proof that there exists at least one diagonal representative in all equivalence classes of the twist matrices on $S^1/\mathbb{Z}_2$, using a notation which is useful to analyze twist matrices on $T^2/\mathbb{Z}_N$. We explain basic properties of $T^2/\mathbb{Z}_N$ in section 3. Using constraints and unitary transformations, we show that any twist matrices can become block-diagonal forms with specific types of submatrices on $T^2/\mathbb{Z}_2$, $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$ in section 4, 5, 6 and 7, respectively. Performing gauge transformations on $T^2/\mathbb{Z}_N$ for the twist matrices with the block-diagonal forms, we examine whether the block-diagonal ones can become diagonal or not in section 8. We explore physical implications on our results in section 9. In the last section, we give conclusions and discussions. We present the details of block-diagonalization of twist matrices on $T^2/\mathbb{Z}_2$, $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$ in appendix A, B, C and D, respectively. We explain possible forms of a matrix under a certain condition in appendix E and derivations of useful relations in our analyses in appendix F and G.

2 $S^1/\mathbb{Z}_2$

Before working on two-dimensional (2D) orbifolds, we examine the $S^1/\mathbb{Z}_2$ orbifold first, in a way slightly different from the literature [29]. Let $x^\mu$ ($\mu = 0, 1, 2, 3$) and $y$ be the coordinates on the Minkowski space-time $M^4$ and the orbifold $S^1/\mathbb{Z}_2$, respectively. The orbifold $S^1/\mathbb{Z}_2$ is described by $y$ that satisfies the identifications $y \sim y + 2\pi R$ and $y \sim -y$, where $R$ is the radius of $S^1$. Since the size of the extra dimension is irrelevant to the following discussions, we take $2\pi R = 1$ for simplicity. Related to the identifications, we define the operations $T : y \rightarrow y + 1$ and $P_0 : y \rightarrow -y$. Besides them, we can also define a parity operation around $y = \pi R = 1/2$ as $P_1 = TP_0$. Note that $P_{0}^{2} = P_{1}^{2} = I$ holds, where $I$ is the identity operation.

Let us consider a gauge theory on $M^4 \times S^1/\mathbb{Z}_2$, whose bulk gauge group is $G = SU(n)$ or $U(n)$, and suppose that the Lagrangian of the theory is invariant under global $G' = U(n)$ transformations for fields for simplicity. In general, the translation $T$ and the parity operations $P_0$ and $P_1$ accompany non-trivial transformations for fields in representation spaces of $G'$, which are described by boundary conditions. We define that the transformations corresponding to $T$ and $P_0$ in the fundamental representation of $G'$ are given by constant unitary matrices $T$ and $P_0$, respectively. We also denote the matrix representation of $P_1$ by $P_1 = TP_0$. We refer to them as twist matrices, which must satisfy the following constraints:

$$\begin{align*}
(P_0)^2 &= (TP_0)^2 = I, \\
(2.1)
\end{align*}$$

where $I$ is the unit matrix, in theories with a field belonging to the fundamental representation. In the following, we only examine the twist matrices on the fundamental representation.

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6More precisely, actions of $(P_0)^2$ and $(TP_0)^2$ in representation spaces on any fields contained in a theory must leave the fields unchanged for consistency. For instance, if only the adjoint representation exists in a
of \( G' \). Our study can be straightforwardly generalized to the other representation matrices, which are obtained via tensor products of the fundamental (and anti-fundamental) representations.

We can simplify the forms of the twist matrices by changing the basis in a representation space of \( G' \), which induces unitary transformations on the twist matrices as \( P_0 \to WP_0W^\dagger \) and \( T \to WTW^\dagger \) with \( WW^\dagger = I \). Since \( P_0 \) is a unitary matrix, eq. (2.1) implies \( P_0 = P_0^{-1} = P_0^\dagger \). Thus, by taking a suitable choice of the basis, \( P_0 \) is diagonalized as

\[
P_0 = \begin{pmatrix} -I_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix} = \begin{pmatrix} (P_0)(11) & (P_0)(12) \\ (P_0)(21) & (P_0)(22) \end{pmatrix}, \quad T = \begin{pmatrix} (T)(11) & (T)(12) \\ (T)(21) & (T)(22) \end{pmatrix}, \tag{2.2}
\]

where \( I_{n_k} \) are the \( n_k \times n_k \) unit matrices, and we have introduced \( n_k \times n_l \) matrices \( (P_0)(kl) \) and \( (T)(kl) \). Note that the \( n_k \) are non-negative integers and satisfy \( n_1 + n_2 = \text{rank}(P_0) = n \). From eq. (2.2), we can write \( (P_0)(kl) = (-1)^k \delta_{kl} I_{n_k} \). In the following, we introduce a notation as \( (T)(kl) = M_{k \bar{l}}^{[k-l]} \), where the superscript \( k - l = q \) is the \( \mathbb{Z}_2 \) charge defined by \( (P_0TP_0^{-1})(k_{k-q}) = (-1)^q M_{k_k-k}^q \). We also use the notation of \( M_{k \bar{l}}^{[k-l]} = M_{k \bar{l}}^{[k' \bar{l}']} = M_{k' \bar{l}'}^{[k-l]} \) with \( k' = k \) (mod 2) and \( l' = l \) (mod 2). This notation is more convenient for the later discussions in \( \mathbb{Z}_N \) \((N = 3, 4, 6)\) orbifold models, while it seems a little lengthy in \( \mathbb{Z}_2 \) orbifold models.

Keeping the diagonal form of \( P_0 \), we further simplify \( T \) by unitary transformations. Since \( TP_0 \) is unitary, eq. (2.1) implies \( (TP_0)^\dagger = TP_0 \). Thus, by using \( (T')(kl) = M_{k \bar{l}}^{[l-k]} \), we find

\[
M_{k_k-k}^q = (-1)^{-q} M_{k_k-k}^{-q}. \tag{2.3}
\]

From the condition \( TT^\dagger = I \), it follows that

\[
\delta_{q0} I_{n_k} = (TT^\dagger)(k_{k-q}) = \sum_{q'} M_{k_k-k+q''}^{-[q-q']} M_{k_k-k+q''}^{-[q-q']} = \sum_{q'} (-1)^{q'+q} M_{k_k-k+q''}^{-[q-q']} M_{k_k-k+q''}^{-[q'+q]}, \tag{2.4}
\]

where the summation over \( q' \) can be taken for any successive two integers, e.g., \( q' = 0, 1 \) or \( q' = 1, 2 \). The above equation implies \( M_{k_k-k}^{[0]} M_{k_k-k}^{[1]} = M_{k_k-k}^{[1]} M_{k_k-k}^{[0]} \), which is written in a more general form as

\[
M_{k_k-k}^{[0]} M_{k_k-k}^{[q]} = M_{k_k-k}^{[q]} M_{k_k-k}^{[0]}, \tag{2.5}
\]

where \( q = 0 \) gives the trivial relation.

From eq. (2.3), we find \( M_{k_k-k}^{[0]} = M_{k_k-k}^{[0]} \). Thus, \( M_{k_k-k}^{[0]} \) is diagonalized by a unitary matrix, and we can choose a basis where an \((i,j)\) element of \( M_{k_k-k}^{[0]} \) is given by \( (M_{k_k-k}^{[0]})_{ij} = a_k^i \delta_{ij} \) \((a_k^i \in \mathbb{R})\). In this basis, eq. (2.5) gives

\[
(a_k^i + a_k^j)(M_{k_k-k}^{[0]}(i,j) = 0. \tag{2.6}
\]

theory, \((P_0)^2\) and \((TP_0)^2\) can be chosen as nontrivial elements of the center subgroup \( \mathbb{Z}_n \) of \( SU(n) \). In some theories with more than two compact extra dimensions, a similar consideration to the above for the actions of twist matrices shows that there exist nontrivial degrees of freedom related to holonomy, so-called the ‘t Hooft flux [30, 67], which we do not discuss in this paper.
Thus, if \( a^i_k \neq a^j_{k-q} \) is satisfied, \((M^{[q]}_{k-k-q})_{ij} = 0\) holds. Therefore, by using a unitary transformation, we can take a basis where \( P_0 \) and \( T \) are written as the following block-diagonal forms:

\[
P_0 = \begin{pmatrix}
P_0^{(0)} & P_0^{(1)} & \cdots & P_0^{(M)} \\
\end{pmatrix}, \quad T = \begin{pmatrix}
T^{(0)} & T^{(1)} & \cdots & T^{(M)} \\
\end{pmatrix},
\]

where \( P_0^{(\lambda)} \) and \( T^{(\lambda)} \) (\( \lambda = 0, 1, \ldots, M \)) are

\[
P_0^{(\lambda)} = \begin{pmatrix}
(P_0^{(\lambda)})(11) & (P_0^{(\lambda)})(12) \\
(P_0^{(\lambda)})(21) & (P_0^{(\lambda)})(22) \\
\end{pmatrix}, \quad T^{(\lambda)} = \begin{pmatrix}
M_{11}^{(\lambda)[0]} & M_{12}^{(\lambda)[1]}[\lambda-1] \\
M_{21}^{(\lambda)[1]} & M_{22}^{(\lambda)[0]} \\
\end{pmatrix} = \begin{pmatrix}
a^{(\lambda)}I_{n_1^{(\lambda)}} & M_{12}^{(\lambda)[1]}[\lambda-1] \\
M_{21}^{(\lambda)[1]} & a^{(\lambda)}I_{n_2^{(\lambda)}} \\
\end{pmatrix}.
\]

In the above equations, \( n_1^{(\lambda)} \) and \( n_2^{(\lambda)} \) are non-negative integers, and \( a^{(\lambda)} \) are real parameters that satisfy \( a^{(\lambda)} \neq a^{(\lambda')} \) for \( \lambda \neq \lambda' \). Note that a similar discussion as the derivation of eq. \([2.10]\) gives the relation \( M^{(\lambda)[q] \dagger}_{k-k-q} = (-1)^q M^{(\lambda)[q]}_{k-k-q} \).

Since \( T \) is a unitary matrix, \( T^{(\lambda)\dagger} = I_{n_1^{(\lambda)}+n_2^{(\lambda)}} \) also holds and gives \( \sum_q M^{(\lambda)[q]}_{k-k+q} M^{(\lambda)[q] \dagger}_{k-k+q} = \delta_{q0} I^{(\lambda)}_{n_k^{(\lambda)}} \). Thus, we obtain

\[
(1 - a^{(\lambda)2}) I_{n_k^{(\lambda)}} = M^{(\lambda)[1]}_{k+k+1} M^{(\lambda)[1] \dagger}_{k+k+1},
\]

which implies that \( a^{(\lambda)2} \leq 1 \) is required as the diagonal elements of the right-hand side of the above is non-negative. Note that \( M^{(\lambda)[1]}_{k+k+1} = 0 \) holds for \( a^{(\lambda)2} = 1 \), which means that \( T^{(\lambda)} \) is already diagonal. We treat \( a^{(\lambda)2} = 1 \) case separately and take a basis such that \( a^{(0)2} = 1 \) and \( a^{(m)2} < 1 \) for \( m = 1, \ldots, M \) are satisfied. Then, \( P_0^{(0)} \) and \( T^{(0)} \) are diagonal matrices whose eigenvalues are 1 or -1.

We focus on the diagonalization of \( T^{(m)} \) (\( m = 1, \ldots, M \)). Here, we recall theorems of the linear algebra that the rank of the product of two matrices is not larger than the ranks of the two matrices and that the rank of \( m \times m' \) matrix is not larger than \( m \) nor \( m' \). These tell us that \( n_k^{(m)} = \text{rank}(M^{(m)[1]}_{k+k+1} M^{(m)[1] \dagger}_{k+k+1}) \leq \text{rank}(M^{(m)[1]}_{k+k+1}) \leq n_k^{(m)} \). By substituting \( k+1 \) for \( k \), we also find \( n_k^{(m)} = \text{rank}(M^{(m)[1]}_{k+k+1} M^{(m)[1] \dagger}_{k+k+1}) \leq \text{rank}(M^{(m)[1]}_{k+k+1}) \leq n_k^{(m)} \). Therefore, we conclude \( n_1^{(m)} = n_2^{(m)} = r^{(m)} \), which means \( M^{(m)[1]}_{k+k+1} \) is an \( r^{(m)} \times r^{(m)} \) square matrix. Then, by using a unitary matrix that satisfies \( U^{(m)}_{k+k+1} U^{(m) \dagger}_{k+k+1} = I_{r^{(m)}} \), we can rewrite \( M^{(m)[1]}_{k+k+1} \) as

\[
M^{(m)[1]}_{k+k+1} = \sqrt{1 - a^{(m)2}} U^{(m)}_{k+k+1}.
\]

Also, we find \( M^{(m)[1]}_{k+k+1} = -M^{(m)[1] \dagger}_{k+k+1} = -\sqrt{1 - a^{(m)2}} U^{(m) \dagger}_{k+k+1} \).
From the above discussions, we can denote $P_0^{(m)}$ and $T^{(m)}$ by

$$P_0^{(m)} = \begin{pmatrix} -I_r^{(m)} & 0 \\ 0 & I_r^{(m)} \end{pmatrix}, \quad T^{(m)} = \begin{pmatrix} \cos \theta^{(m)} I_r^{(m)} & \sin \theta^{(m)} U^{(m)}_1 \\ -\sin \theta^{(m)} U^{(m)}_1 & \cos \theta^{(m)} I_r^{(m)} \end{pmatrix},$$

where we have defined

$$\cos \theta^{(m)} = a^{(m)}, \quad \sin \theta^{(m)} = \sqrt{1 - a^{(m)2}}.$$  \hspace{1cm} (2.13)

The angle $\theta^{(m)}$ can be chosen to satisfy $0 < \theta^{(m)} < \pi$. Note that $\theta^{(m)} \neq \theta^{(m')}$ holds for $m \neq m'$ as $a^{(m)} \neq a^{(m')}$ does. By using the unitary matrix:

$$V^{(m)} = \begin{pmatrix} iU^{(m)}_1 & 0 \\ 0 & I_r^{(m)} \end{pmatrix},$$

we can change the basis as

$$P_0^{(m)} \rightarrow V^{(m)} P_0^{(m)} V^{(m)\dagger} = P_0^{(m)} = p_0 \otimes I_r^{(m)}, \quad T^{(m)} \rightarrow V^{(m)} T^{(m)} V^{(m)\dagger} = t^{(m)}_1 \otimes I_r^{(m)},$$

where $p_0$ and $t^{(m)}_1$ are defined by using the Pauli matrices $\sigma_i$ as

$$p_0 = -\sigma_3, \quad t^{(m)}_1 = i \sin \theta^{(m)} \sigma_1 + \cos \theta^{(m)} I_2 = e^{i\theta^{(m)} \sigma_1}. \hspace{1cm} (2.16)$$

As the final step to diagonalize the twist matrices, we consider a $y$–dependent gauge transformation, under which $p_0$ and $t^{(m)}_1$ transform as

$$p_0 \rightarrow p_0' = \Omega^{(m)}(-y)p_0\Omega^{(m)\dagger}(y) = p_0, \hspace{1cm} (2.17)$$

$$t^{(m)}_1 \rightarrow t^{(m)}_1' = \Omega^{(m)}(y + 1)t^{(m)}_1\Omega^{(m)\dagger}(y) = (-1)^{l^{(m)}} I_2,$$

where

$$\Omega^{(m)}(y) = \exp \left[ i \left( -\theta^{(m)} + \frac{l^{(m)}}{\pi} \right) y \sigma_1 \right], \quad l^{(m)} \in \mathbb{Z}. \hspace{1cm} (2.19)$$

Thus, we can choose a basis such that $P_0$ and $T$ are simultaneously diagonalized, where the submatrices in eq. (2.7) are given by

$$P_0^{(m)} = -\sigma_3 \otimes I_r^{(m)}, \quad T^{(m)} = (-1)^{l^{(m)}} I_2 \otimes I_r^{(m)}, \quad \text{for } m = 1, \ldots, M,$$

in addition to the already diagonal ones $P_0^{(0)}$ and $T^{(0)}$. It should be mentioned that there are several diagonal forms of twist matrices depending on $l^{(m)}$ in an equivalence class.
3 Two-dimensional orbifolds

In the following sections, we discuss gauge theories on $M^4 \times T^2 / \mathbb{Z}_N$ ($N = 2, 3, 4, 6$). We here summarize the basic properties of the 2D orbifolds. We particularly pay attention to relations satisfied by translation and rotation operations on $T^2 / \mathbb{Z}_N$ since these relations give rise to restriction of the forms of twist matrices corresponding to the operations.

A compactification on a 2D torus $T^2$ is obtained from the 2D Euclidean space $\mathbb{R}^2$, which we call the universal covering space, by modding out a 2D lattice $\Lambda$ as $T^2 = \mathbb{R}^2 / \Lambda$. We denote the 2D lattice by $\Lambda = \{ n_1 \lambda_1 + n_2 \lambda_2 | n_1, n_2 \in \mathbb{Z} \}$, where $\lambda_i$ ($i = 1, 2$) are linearly independent basis vectors. Coordinate vectors $y$ and $y'$ on $T^2$ are identified if $y' - y \in \Lambda$ is satisfied. In other words, any $y$ on $T^2$ satisfies the following identification:

$$y \sim y + n_1 \lambda_1 + n_2 \lambda_2.$$  \hfill (3.1)

It is convenient to use the complex coordinate system to deal with $T^2$ and 2D orbifolds. Let $z$ be a complex coordinate $z = y^1 + iy^2$, where $y^1$ and $y^2$ are Cartesian coordinates on $T^2$. Then, the identification in eq. (3.1) is expressed as

$$z \sim z + n_1 + n_2 \tau,$$  \hfill (3.2)

where we have taken $|\lambda_1| = 1$ for simplicity of the notation without loss of generality. The geometry of $T^2$ is encoded in the complex parameter $\tau$, which must satisfy $\text{Im}\tau \neq 0$ to span $T^2$. A fundamental region of $T^2$, which is an independent region of the covering space under the identification in eq. (3.2), is now given by $\{ p + q\tau | p, q \in [0, 1) \}$.

It is natural to define the translations $T_1$ and $T_2$ as

$$T_1 : z \rightarrow z + 1, \quad T_2 : z \rightarrow z + \tau.$$  \hfill (3.3)

Then, the identification in eq. (3.2) is expressed as $z \sim T_1^{n_1} T_2^{n_2} z$. Note that $[T_1, T_2] = 0$ holds, as expected from the 2D translational invariance.

The orbifold $T^2 / \mathbb{Z}_N$ is obtained from $T^2$ by further modding out an Abelian discrete group $\mathbb{Z}_N$ ($N = 2, 3, 4, 6$), whose elements are generated by the $N$–th root of unity $e^{2\pi i/N}$ on the complex coordinate system. We define the $\mathbb{Z}_N$ rotation $R_0$ as

$$R_0 : z \rightarrow e^{2\pi i/N} z.$$  \hfill (3.4)

Then, the orbifold $T^2 / \mathbb{Z}_N$ is given by imposing the identification under the rotation $R_0$ on the complex coordinate $z$ on the $T^2$ torus: $z \sim R_0 z$. For $N = 2$, $\tau$ is arbitrarily chosen as long as $\text{Im}\tau \neq 0$ is satisfied. On the other hand, $\tau$ is restricted by consistency in the $N = 3, 4, 6$ cases. We hereafter take $\tau = e^{2\pi i/N}$ for $N = 3, 4, 6$. From eq. (3.4), we see $(R_0)^N = I$, which is the identity operation.

\footnote{The 2D translational invariance may be broken by, e.g., non-trivial field configurations that give magnetic flux \cite{Note1}. While these possibilities are interesting, we do not consider them here.}
For the $N = 3, 4, 6$ cases, $\mathcal{T}_2$ can be rewritten as $\mathcal{T}_2 = \mathcal{R}_0 \mathcal{T}_1 \mathcal{R}_0^{-1}$. More generally, we can define the translations along $\tau^{m-1} = e^{2\pi i (m-1)/N}$ ($m \in \{1, \ldots, N\}$) direction as

$$\mathcal{T}_m = \mathcal{R}_0^{m-1} \mathcal{T}_1 \mathcal{R}_0^{-m}, \quad \mathcal{T}_m : z \rightarrow z + \tau^{m-1}, \quad \text{for } N = 3, 4, 6. \quad (3.5)$$

Since $\mathcal{T}_m$ are translations, they commute with each other and obey $[\mathcal{T}_m, \mathcal{T}_{m'}] = 0$ for any pairs of $m$ and $m'$. In addition, as $\tau$ is the $N$-th root of unity for the $N = 3, 4, 6$ cases, these translations satisfy the relations shown in Table 1. These relations are summarized as $(\mathcal{T}_m \mathcal{R}_0^p)_N^{p/p} = \mathcal{I}$ with integers $p$ and $N/p$ [39]. We can choose $\mathcal{T}_1$ and $\mathcal{T}_2$ as the basis of the 2D translations. Using the above mentioned relations, any $\mathcal{T}_m$ can be expressed by $\mathcal{T}_1$ and $\mathcal{T}_2$ and, thus, be expressed by $\mathcal{T}_1$ and $\mathcal{R}_0$ for the $N = 3, 4, 6$ cases. On the other hand, for the $N = 2$ case, $\mathcal{T}_2$ cannot be written by $\mathcal{T}_1$ and $\mathcal{R}_0$ for arbitrary $\tau$ with $\text{Im}\tau \neq 0$.

On the universal covering space of an orbifold $T^2/\mathbb{Z}_N$, there exist invariant points under $\mathcal{R}_0$ up to the translations in eq. (3.3), called fixed points. One can also define the rotations around the fixed points on the orbifold as shown below. If $z_F$ is a fixed point, it follows that

$$z_F = e^{2\pi i/N} z_F + n_1 + n_2 \tau, \quad n_1, n_2 \in \mathbb{Z}. \quad (3.6)$$

Let $z_{F,n_1,n_2}^{(n_1,n_2)}$ be the solutions of the above equation. We obtain the following solutions depending on $N$:

$$z_{F,2}^{(n_1,n_2)} = \frac{n_1 + n_2 \tau}{2} \quad (\text{Im}\tau \neq 0), \quad z_{F,3}^{(n_1,n_2)} = \frac{2n_1 - n_2 + (n_1 + n_2)e^{2\pi i/3}}{3}, \quad (3.7)$$

$$z_{F,4}^{(n_1,n_2)} = \frac{n_1 - n_2 + (n_1 + n_2)e^{2\pi i/4}}{2}, \quad z_{F,6}^{(n_1,n_2)} = -n_2 + (n_1 + n_2)e^{2\pi i/6}. \quad (3.8)$$

Notice that any operation $\mathcal{T}_1^{n_1} \mathcal{T}_2^{n_2} \mathcal{R}_0$ ($n_1, n_2 \in \mathbb{Z}$) gives a $\mathbb{Z}_N$ rotation around a fixed point $z_{F,n_1,n_2}^{(n_1,n_2)}$. This is understood because the solutions of the equation $\mathcal{T}_1^{n_1} \mathcal{T}_2^{n_2} \mathcal{R}_0 z = u = e^{2\pi i/N} (z - u)$ for $u$ are nothing but the fixed points defined in eq. (3.6). Thus, the relation $(\mathcal{T}_1^{n_1} \mathcal{T}_2^{n_2} \mathcal{R}_0)^N = \mathcal{I}$ is satisfied. Note that there is a $\mathbb{Z}_2$ subgroup in $\mathbb{Z}_4$. Also, there are $\mathbb{Z}_3$ and $\mathbb{Z}_2$ subgroups in $\mathbb{Z}_6$. Thus, $\mathcal{Z}_2$ operations in the $T^2/\mathbb{Z}_4$ case and $\mathbb{Z}_3$ and $\mathcal{Z}_2$ operations in the $T^2/\mathbb{Z}_6$ case are naturally defined. We denote the fixed points under the subgroup $\mathcal{Z}_2$ in the $N = 4$ case by $z_{F,4}^{(n_1,n_2)} = (n_1 + in_2)/2$. The $\pi$ rotation around $z_{F,4}^{(n_1,n_2)}$ is given by $\mathcal{T}_1^{n_1} \mathcal{T}_2^{n_2} \mathcal{R}_0^2$ in the $N = 4$ case, which satisfies $(\mathcal{T}_1^{n_1} \mathcal{T}_2^{n_2} \mathcal{R}_0^2)^2 = \mathcal{I}$. Also, the fixed points under the subgroup $\mathcal{Z}_2$ and $\mathbb{Z}_3$ in the $N = 6$ case are denoted by $z_{F,6}^{(n_1,n_2)} = (n_1 + n_2 e^{2\pi i/6})/2$ and

| $T^2/\mathbb{Z}_N$ | relations among translations | $\tau$ |
|-------------------|-----------------------------|-------|
| $T^2/\mathbb{Z}_3$ | $\prod_{m=1}^{3} \mathcal{T}_m = \mathcal{I}$ | $e^{2\pi i/3}$ |
| $T^2/\mathbb{Z}_4$ | $\prod_{m=1}^{4} \mathcal{T}_m = \mathcal{I}$, $\mathcal{T}_1 \mathcal{T}_3 = \mathcal{T}_2 \mathcal{T}_4 = \mathcal{I}$ | $e^{2\pi i/4}$ |
| $T^2/\mathbb{Z}_6$ | $\prod_{m=1}^{6} \mathcal{T}_m = \mathcal{I}$, $\mathcal{T}_1 \mathcal{T}_4 = \mathcal{T}_2 \mathcal{T}_5 = \mathcal{T}_3 \mathcal{T}_6 = \mathcal{I}$, $\mathcal{T}_1 \mathcal{T}_3 \mathcal{T}_5 = \mathcal{T}_2 \mathcal{T}_4 \mathcal{T}_6 = \mathcal{I}$ | $e^{2\pi i/6}$ |

Table 1: Relations among translations
\( z_{F,6,3}^{(n_1,n_2)} = [(n_1 - n_2) + (n_1 + 2n_2) e^{2\pi i/6}]/3 \), respectively. The \( \pi \) rotation around \( z_{F,6,2}^{(n_1,n_2)} \) and the \( 2\pi/3 \) rotation around \( z_{F,6,3}^{(n_1,n_2)} \) are given by \( T_1^{n_1} T_2^{n_2} R_0^3 \) and \( T_1^{n_1} T_2^{n_2} R_0^2 \) in the \( N = 6 \) case, where \((T_1^{n_1} T_2^{n_2} R_0^3)^2 = I \) and \((T_1^{n_1} T_2^{n_2} R_0^2)^3 = I \) are satisfied.

As the basis of the translations and rotations defined above, we can choose \( \{T_1, T_2, R_0\} \) for the \( N = 2 \) case and \( \{T_1, R_0\} \) for the \( N = 3, 4, 6 \) cases. Any operations are expressed by these basis operations. The translations \( T_m \) and the rotations around the fixed points satisfy several relations shown above. These relations are redundant but give sufficient ones that are satisfied by the basis operations.

For the \( N = 3, 4, 6 \) cases, except for \( R_0^N = I \), we can derive the relations for the rotations discussed above using the ones for the translations. For example, let us consider \( \mathbb{Z}_N \) (\( N = 3, 4, 6 \)) rotations around arbitrary fixed points, which are given by \( T_1^{n_1} T_2^{n_2} R_0^a \) and satisfy \((T_1^{n_1} T_2^{n_2} R_0)^N = I \) as discussed. This relation is derived by using the relations for the translations as follows. First, notice that we get the relation \( R_0^k T_1^{a_1} T_2^{a_2} = T_1^{a_1+k} T_2^{a_2+k} R_0^k \), where \( T_{m+N} = T_m \), from the definition of \( T_m \) given by eq. (3.5). Using this relation, we can gather \( R_0 \) in the following equation and obtain

\[
(T_1^{n_1} T_2^{n_2} R_0^N) = T_1^{n_1} T_2^{n_1+n_2} \cdots T_N^{n_1+n_2} T_1^{n_2} R_0^N = (T_1^{n_1} \cdots T_N^{n_2})^{n_1+n_2} = I^{n_1+n_2} = I,
\]  

where we have also used \( R_0^N = I, [T_m, T_{m''}] = 0 \) and \( \prod_{m=1}^N T_m = I \). As another example, we consider the \( \pi \) rotation \( T_1^{n_1} T_2^{n_2} R_0^3 \) in the \( N = 6 \) case. We see \((T_1^{n_1} T_2^{n_2} R_0^3)^2 = T_1^{n_1} T_2^{n_2} T_4^{n_1} T_5^{n_2} R_0^6 = (T_1 T_4)^{n_1} (T_2 T_5)^{n_2} = I \), where we have used \( R_0^6 = I, [T_m, T_{m''}] = 0 \) and \( T_3 T_4 = T_2 T_5 = I \). Thus, the relation \((T_1^{n_1} T_2^{n_2} R_0^3)^2 = I \) is derived. As in these cases, the other relations for the rotations discussed above are also derived using the relations for the translations. In addition, we can also consider \( N \) successive rotations around different fixed points, which are always expressed by the translations as \( \prod_{a=1}^N (T_1^{n_1(a)} T_2^{n_2(a)} R_0) = T_1^{n_1(a)} T_2^{n_2(a)} \), where \( n_1(a), n_2(a), n_1' \) and \( n_2' \) are integers. Such relations are also derived from the relations for the translations. Thus, it is sufficient to take care with \( [T_m, T_{m''}] = 0 \) \((m, m'' \in \{1, \ldots, N\})\), the relations in Table I and \( R_0^N = I \) as independent ones for the \( N = 3, 4, 6 \) cases.

In the following sections, we consider gauge theories on \( M^4 \times T^2/\mathbb{Z}_N \). We denote coordinates on \( M^4 \) and \( T^2/\mathbb{Z}_N \) by \( x^\mu (\mu = 0, 1, 2, 3) \) and \( z = x^5 + i x^6 \), respectively. As in the \( S^1/\mathbb{Z}_2 \) case, the translations and the \( \mathbb{Z}_N \) rotations accompany non-trivial twists in the representation space \( G' = U(n) \), under which the Lagrangian is supposed to be invariant. We denote the twist matrices corresponding to these operations by the Italic character symbol, e.g., \( T_m \) for \( T_m \), which are unitary matrices belonging to the fundamental representation. These matrices are constrained by the relations satisfied by the corresponding translations and rotations. For example, the relation \( [T_m, T_{m''}] = 0 \) gives the constraint \( [T_m, T_{m''}] = 0 \) for the twist matrices. As shown in the following sections, without loss of generality, the twist matrices are simplified by taking a suitable basis and gauge with the help of the constraints.
In this section, we discuss the twist matrices in the $T^2/\mathbb{Z}_2$ case. There are independent twist matrices $T_1$, $T_2$ and $R_0$ satisfying the constraints:

$[T_1, T_2] = 0, \quad R_0^2 = (T_1 R_0)^2 = (T_2 R_0)^2 = (T_1 T_2 R_0)^2 = I,$

(4.1)

where $I$ is the unit matrix. With a similar discussion as in the $S^1/\mathbb{Z}_2$ case given in section 2 to get eqs. (2.15) and (2.16), corresponding to eq. (2.7) for $P_0$ and $T_1$, we take block-diagonal forms of $R_0$ and $T_1$. Then, the twist matrices are given by

$$R_0 = \begin{pmatrix}
R_0^{(0)} & R_0^{(1)} \\
& \ddots \\
& & R_0^{(M)}
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
T_1^{(0)} & & \\
& \ddots & \\
& & T_1^{(M)}
\end{pmatrix},$$

(4.2)

$$T_2 = \begin{pmatrix}
T_2^{(00)} & T_2^{(10)} & \cdots & T_2^{(0M)} \\
T_2^{(10)} & T_2^{(11)} & \cdots & T_2^{(1M)} \\
& \ddots & \ddots & \ddots \\
T_2^{(M0)} & T_2^{(MM)}
\end{pmatrix},$$

(4.3)

where we have denoted submatrices of $R_0$, $T_1$ and $T_2$ by $R_0^{(\lambda)}$, $T_1^{(\lambda)}$ and $T_2^{(\lambda\lambda)}$ ($\lambda = 0, 1, \ldots, M$), respectively. As in section 2, we take $R_0^{(0)}$ and $T_1^{(0)}$ as $r^{(0)} \times r^{(0)}$ diagonal matrices whose eigenvalues are 1 or $-1$. Parameters such as $r^{(0)}$ which represent the size of matrices are non-negative integers. The same applies to the following ones. The other submatrices $R_0^{(m)}$ and $T_1^{(m)}$ ($m = 1, \ldots, M$) are defined to be $2r^{(m)} \times 2r^{(m)}$ ones and are given by

$$R_0^{(m)} = \begin{pmatrix}
-I_{r^{(m)}} & 0 \\
0 & I_{r^{(m)}}
\end{pmatrix} = -\sigma_3 \otimes I_{r^{(m)}},$$

(4.4)

$$T_1^{(m)} = \begin{pmatrix}
\cos \theta^{(m)} I_{r^{(m)}} & i \sin \theta^{(m)} I_{r^{(m)}} \\
i \sin \theta^{(m)} I_{r^{(m)}} & \cos \theta^{(m)} I_{r^{(m)}}
\end{pmatrix} = e^{i \theta^{(m)} \sigma_1} \otimes I_{r^{(m)}},$$

(4.5)

where $0 < \theta^{(m)} < \pi$ and $\theta^{(m)} \neq \theta^{(m')}$ for $m \neq m'$.

Without disturbing the above structure of $R_0$ and $T_1$, we can simplify the form of $T_2$ by using the constraints in eq. (4.1) and unitary transformations. As shown in appendix A.1 one finds that the submatrices $T_2^{(m0)}$ and $T_2^{(0m)}$ vanish. Then, as discussed in appendix A.2 we can simplify $T_2^{(00)}$, keeping diagonal forms of $R_0^{(0)}$ and $T_1^{(0)}$. The submatrices $R_0^{(0)}$, $T_0^{(0)}$ and $T_2^{(00)}$ in eqs. (4.2) and (4.3) are written as

$$R_0^{(0)} = \begin{pmatrix}
R_0^{(0),1} & 0 \\
0 & R_0^{(0),2}
\end{pmatrix}, \quad T_1^{(0)} = \begin{pmatrix}
T_1^{(0),1} & 0 \\
0 & T_1^{(0),2}
\end{pmatrix} = \begin{pmatrix}
-I_{r_1^{(0)}} & 0 \\
0 & I_{r_2^{(0)}}
\end{pmatrix},$$

(4.6)

$$T_2^{(00)} = \begin{pmatrix}
T_2^{(00),1} & 0 \\
0 & T_2^{(00),2}
\end{pmatrix},$$

(4.7)
where \( r_1^{(0)} + r_2^{(0)} = r^{(0)} \), and \( R_0^{(0),a}, T_1^{(0),a} \) and \( T_2^{(00),a} \) \( (a = 1, 2) \) are \( r_a^{(0)} \times r_a^{(0)} \) matrices. The submatrices \( R_0^{(0),a} \) \( (a = 1, 2) \) are diagonal and have eigenvalues 1 or \(-1\). The submatrices \( R_0^{(0),a}, T_1^{(0),a} \) and \( T_2^{(00),a} \) are further rearranged as

\[
R_0^{(0),a} = \begin{pmatrix}
R_0^{(0),a,0} & & \\
& R_0^{(0),a,1} & \\
& & \ddots & \\
& & & R_0^{(0),a,M_0^{(a)}}
\end{pmatrix},
\]

(4.8)

\[
T_1^{(0),a} = \begin{pmatrix}
T_1^{(0),a,0} & & \\
& T_1^{(0),a,1} & \\
& & \ddots & \\
& & & T_1^{(0),a,M_0^{(a)}}
\end{pmatrix},
\]

(4.9)

\[
T_2^{(00),a} = \begin{pmatrix}
T_2^{(00),a,0} & & \\
& T_2^{(00),a,1} & \\
& & \ddots & \\
& & & T_2^{(00),a,M_0^{(a)}}
\end{pmatrix},
\]

(4.10)

where \( R_0^{(0),a,0}, T_1^{(0),a,0}, \) and \( T_2^{(00),a,0} \) are \( r_a^{(0)} \times r_a^{(0)} \) diagonal matrices whose eigenvalues are 1 or \(-1\). In addition, for \( m = 1, \ldots, M_0^{(a)} \), the submatrices are given by

\[
R_0^{(0),a,m} = -\sigma_3 \otimes I_{r_a^{(0)},m}, \quad T_1^{(0),a,m} = (-1)^a I_2 \otimes I_{r_a^{(0)},m}, \quad T_2^{(00),a,m} = e^{i\phi^{(0),a,m}\sigma_1} \otimes I_{r_a^{(0)},m},
\]

(4.11)

which are defined to be \( 2r_a^{(0),m} \times 2r_a^{(0),m} \) matrices. The real parameters \( \phi^{(0),a,m} \) satisfy \( 0 < \phi^{(0),a,m} < \pi \) and \( \phi^{(0),a,m} \neq \phi^{(0),a,m'} \) for \( m \neq m' \).

Let us start to discuss \( T_2^{(mm')} \). As shown in appendix \[A.3\], there exist a basis where \( T_2^{(mm')} = 0 \) for \( m \neq m' \) is satisfied. Then, the twist matrix \( T_2 \) becomes a diagonal form as

\[
T_2 = \begin{pmatrix}
T_2^{(00)} & & \\
& T_2^{(1)} & \\
& & \ddots & \\
& & & T_2^{(M)}
\end{pmatrix},
\]

(4.12)

where \( T_2^{(00)} \) and \( T_2^{(m)} \) \( (m = 1, \ldots, M) \) are \( r^{(0)} \times r^{(0)} \) and \( 2r^{(m)} \times 2r^{(m)} \) submatrices, respectively.

The submatrices \( R_0^{(m)} \) and \( T_1^{(m)} \) in eq. (4.2) and \( T_2^{(m)} \) in eq. (4.12) are simplified by unitary transformations. As shown in appendix \[A.4\], they are finally decomposed into \( 2 \times 2 \) submatrices.
The submatrices $R_0^{(m)}$, $T_1^{(m)}$ and $T_2^{(m)}$ are written as

$$
R_0^{(m)} = \begin{pmatrix}
R_0^{(m),1} & R_0^{(m),2} & \cdots & R_0^{(m),M(m)} \\
R_0^{(m),1} & R_0^{(m),2} & \cdots & R_0^{(m),M(m)} \\
\vdots & \vdots & \ddots & \vdots \\
R_0^{(m),1} & R_0^{(m),2} & \cdots & R_0^{(m),M(m)}
\end{pmatrix},
$$

(4.13)

$$
T_1^{(m)} = \begin{pmatrix}
T_1^{(m),1} & T_1^{(m),2} & \cdots & T_1^{(m),M(m)} \\
T_1^{(m),1} & T_1^{(m),2} & \cdots & T_1^{(m),M(m)} \\
\vdots & \vdots & \ddots & \vdots \\
T_1^{(m),1} & T_1^{(m),2} & \cdots & T_1^{(m),M(m)}
\end{pmatrix},
$$

(4.14)

$$
T_2^{(m)} = \begin{pmatrix}
T_2^{(m),1} & T_2^{(m),2} & \cdots & T_2^{(m),M(m)} \\
T_2^{(m),1} & T_2^{(m),2} & \cdots & T_2^{(m),M(m)} \\
\vdots & \vdots & \ddots & \vdots \\
T_2^{(m),1} & T_2^{(m),2} & \cdots & T_2^{(m),M(m)}
\end{pmatrix},
$$

(4.15)

where $R_0^{(m),m'}$, $T_1^{(m),m'}$ and $T_2^{(m),m'}$ ($m = 1, \ldots, M(m)$) are $2r^{(m),m'} \times 2r^{(m),m'}$ submatrices, and $\sum_{m'=1}^{M(m)} r^{(m),m'} = r^{(m)}$. These submatrices are given by

$$
T_0^{(m),m'} = -\sigma_3 \otimes I_{r^{(m),m'}}, \quad T_1^{(m),m'} = e^{i\theta^{(m),m}} \sigma_1 \otimes I_{r^{(m),m'}},
$$

(4.16)

$$
T_2^{(m),m'} = \begin{pmatrix}
\cos \phi^{(m),m'} I_{r^{(m),m'}} & i \sin \phi^{(m),m'} I_{r^{(m),m'}} \\
i \sin \phi^{(m),m'} I_{r^{(m),m'}} & \cos \phi^{(m),m'} I_{r^{(m),m'}}
\end{pmatrix} = e^{i\phi^{(m),m'}} \sigma_1 \otimes I_{r^{(m),m'}}.
$$

(4.17)

The real parameters $\phi^{(m),m'}$ satisfy $\phi^{(m),m'} \neq \phi^{(m),m''}$ for $m' \neq m''$.

From the above discussions, the twist matrices are simultaneously rearranged to be block-diagonal with each diagonal blocks being a $2 \times 2$ matrix. For $R_0^{(0),a,m}$, $T_1^{(0),a,m}$ and $T_2^{(00),a,m}$ in eq. (4.14), the $2 \times 2$ matrices are given by

$$
r_0 = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}, \quad t_1 = \pm \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad t_2 = \begin{pmatrix}
b_2 & b_1 \\
b_1 & b_2
\end{pmatrix},
$$

(4.18)

where $b_1$ is a pure imaginary number with $\text{Im} b_1 > 0$ and $b_2$ is a real number which satisfy $|b_1|^2 + b_2^2 = 1$. For $R_0^{(0),m'}$, $T_1^{(0),m'}$ and $T_2^{(00),m'}$ in eqs. (4.10) and (4.17), we can rearrange them to be block-diagonal with each diagonal blocks being a $2 \times 2$ matrix in the following form:

$$
r_0 = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}, \quad t_1 = \begin{pmatrix}
a_2 & a_1 \\
a_1 & a_2
\end{pmatrix}, \quad t_2 = \begin{pmatrix}
b_2 & b_1 \\
b_1 & b_2
\end{pmatrix},
$$

(4.19)

where $a_1$ and $b_1$ are pure imaginary numbers with $\text{Im} a_1 > 0$ and $a_2$ and $b_2$ are real numbers which satisfy $|a_1|^2 + a_2^2 = 1$ and $|b_1|^2 + b_2^2 = 1$. As discussed in section 8, these matrices are simultaneously diagonalized by a gauge transformation.

The remaining $R_0^{(0),a,0}$, $T_1^{(0),a,0}$ and $T_2^{(00),a,0}$ are already diagonal and can be treated as block-diagonal matrices with each diagonal blocks being a $1 \times 1$ matrix.
Next we examine the $T^2/\mathbb{Z}_3$ orbifold. The independent twist matrices are $R_0$ and $T_1$, by which we can define the other ones as

$$T_2 = R_0 T_1 R_0^{-1}, \quad T_3 = R_0^{-1} T_1 R_0, \quad R_1 = T_1 R_0, \quad R_2 = T_1 T_2 R_0 = R_0^{-1} T_1^{-1} R_0^{-1}. \quad (5.1)$$

The twist matrices $R_1$ and $R_2$ correspond to $2\pi/3$ rotations around $z_{F,3}^{(1,0)}$ and $z_{F,3}^{(1,1)}$ defined in eq. (5.7), respectively. These satisfy the following relations to represent the corresponding operations,

$$R_0^3 = R_1^3 = R_2^3 = R_0 R_1 R_2 = R_1 R_2 R_0 = R_2 R_0 R_1 = I, \quad (5.2)$$
$$T_m T_m = T_{m+N}, \quad T_1 T_2 T_3 = I, \quad T_{m+1} R_0 = R_0 T_m, \quad (5.3)$$

where $I$ denotes the unit matrix and $T_{m+N} = T_m$ as before.

Without loss of generality, the twist matrix $R_0$ can be diagonalized, with three possible eigenvalues: $\omega^k$ ($k = 1, 2, 3$), where $\omega = e^{2\pi i/3}$. It is convenient to divide the twist matrices $R_0$ and $T_1$ into $3 \times 3$ blocks as

$$R_0 = \begin{pmatrix} \omega I_{n_1} & \omega^2 I_{n_2} \\ & I_{n_3} \end{pmatrix} = \begin{pmatrix} (R_0)(11) & (R_0)(12) & (R_0)(13) \\ (R_0)(21) & (R_0)(22) & (R_0)(23) \\ (R_0)(31) & (R_0)(32) & (R_0)(33) \end{pmatrix}, \quad (R_0)_{(kl)} = \omega^k \delta_{kl} I_{n_k}, \quad (5.4)$$

where $n_k$ is a non-negative integer and $I_{n_k}$ denotes the $n_k \times n_k$ unit matrix, and

$$T_1 = \begin{pmatrix} (T_1)(11) & (T_1)(12) & (T_1)(13) \\ (T_1)(21) & (T_1)(22) & (T_1)(23) \\ (T_1)(31) & (T_1)(32) & (T_1)(33) \end{pmatrix}, \quad (T_1)_{(kl)} = M_{kl}^{[k-l]} . \quad (5.5)$$

As before, we have introduced $n_k \times n_l$ matrices $M_{kl}^{[k-l]}$, and use a notation of $M_{kl}^{[k-l]} = M_{kl}^{[k'-l']} = M_{k'l'}^{[k-l]}$ with $k' = k \mod 3$ and $l' = l \mod 3$. The upper index $k-l = q$ represents the charge of the $\mathbb{Z}_3$ symmetry generated by $R_0$: $(R_0 R_1 R_0^{-1})_{(k,k-q)} = \omega^q M_{kk}^{[q]}$. 

As shown in appendix B.1, the conditions $R_0^3 = I$ ($a = 0, 1, 2$), with the help of the relations in eq. (5.1) and $T_1^\dagger = T_1^{-1}$, lead to those among $M_{kl}^{[k-l]}$ as

$$M_{kk}^{[q]} M_{kk}^{[-q]} = M_{kk}^{[-q]} M_{kk}^{[q]}, \quad M_{kk}^{[0]} M_{kk}^{[q]} = M_{kk}^{[q]} M_{kk}^{[0]}, \quad (5.6)$$

which are further summarized in eq. (F.3). From the second relations in eq. (5.6), we find that each $M_{kk}^{[0]}$ commutes with its dagger to be a normal matrix. Though $M_{kk}^{[q]}$ are not hermitian generally in this $T^2/\mathbb{Z}_3$ case in contrast to the other $T_2/\mathbb{Z}_N$ cases with even $N$, then they can be diagonalized by a unitary transformation.

In appendix B.2 we show that in the basis where $M_{kk}^{[0]}$ are diagonalized as $(M_{kk}^{[0]})_{ij} = a_{ij}^k \delta_{ij}$, $(M_{kk}^{[q]})_{ij}$ vanish if $a_{ij}^k \neq a_{ij}^{k-q}$, and thus the twist matrices $R_0$ and $T_1$ can be block-diagonalized
as

\[
R_0 = \begin{pmatrix} R_0^{(1)} & \cdots & R_0^{(m)} \\ \vdots & \ddots & \vdots \\ R_0^{(M)} & \cdots & R_0^{(M)} \end{pmatrix}, \quad R_0^{(m)} = \begin{pmatrix} \omega I_{n_1^{(m)}} & \omega^2 I_{n_2^{(m)}} & I_{n_3^{(m)}} \end{pmatrix},
\]

(5.7)

\[
T_1 = \begin{pmatrix} T_1^{(1)} & \cdots & T_1^{(m)} \\ \vdots & \ddots & \vdots \\ T_1^{(M)} & \cdots & T_1^{(M)} \end{pmatrix}, \quad T_1^{(m)} = \begin{pmatrix} a^{(m)} I_{n_1^{(m)}} & M_{12}^{(m)}[1] & M_{13}^{(m)}[1] \\ M_{21}^{(m)}[1] & a^{(m)} I_{n_2^{(m)}} & M_{23}^{(m)}[1] \\ M_{31}^{(m)}[1] & M_{32}^{(m)}[1] & a^{(m)} I_{n_3^{(m)}} \end{pmatrix},
\]

(5.8)

where \(n_k^{(m)} (m = 1, 2, \ldots, M)\) are non-negative integers and complex parameters \(a^{(m)}\) satisfy \(a^{(m)} \neq a^{(m')}\) for \(m \neq m'\).

The unitarity conditions \(T_1 T_1^\dagger = T_1^\dagger T_1 = I\) are examined in appendix B.3. The condition \(T_1 T_1^\dagger = I\) derives

\[
M_{k k-1}^{(m)[1]} M_{k k-1}^{(m)[1] \dagger} + M_{k k+1}^{(m)[1] - 1} M_{k k+1}^{(m)[1] - 1 \dagger} = (1 - |a^{(m)}|^2) I_{n_k^{(m)}},
\]

(5.9)

to show that, when \(|a^{(m)}| = 1, M_{k k-q}^{(m)[q]} = 0\) for \(q = \pm 1\) and \(T_1^{(m)}\) is already diagonal.

The cases with \(0 < |a^{(m)}| < 1\) and \(|a^{(m)}| = 0\) are studied respectively in appendix B.3.1 and B.3.2. In both cases, after some discussions, it is shown that \(M_{k k-q}^{(m)[q]}\) with \(q = \pm 1\) can be diagonalized. Then, we can rearrange \(T_1^{(m)}\) further to be block-diagonal, where \(R_0^{(m)}\) is still diagonal, with each diagonal blocks being a 3 × 3 matrix in the following form:

\[
r_0 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} a_3 & a_2 & a_1 \\ a_1 & a_3 & a_2 \\ a_2 & a_1 & a_3 \end{pmatrix},
\]

(5.10)

where \(a_1, a_2\) and \(a_3\) are complex numbers which satisfy \(a_1^2 + a_2^2 + a_3^2 = 3a_1 a_2 a_3 = 1\), as seen from the combination of eqs. B.13, B.14 and B.16 derived from \(T_1 T_2 T_3 = T_1 T_3 T_2 = I\) and \(T_1 T_1^\dagger = I\), and \(|a_1|^2 + |a_2|^2 + |a_3|^2 = 1\) and \(\bar{a_1} a_3 + \bar{a_3} a_2 + \bar{a_2} a_1 = 0\), as seen from eqs. B.16 and B.17, respectively, derived from \(T_1 T_1^\dagger = I\). In section 8 it is shown that 3 × 3 submatrices of this form are diagonalized by a suitable gauge transformation.

In the case with \(|a^{(m)}| = 1, T_1^{(m)}\) is diagonal to be treated as a block-diagonal matrix with each diagonal blocks being a 1 × 1 matrix.

6 \(T^2/\mathbb{Z}_4\)

We consider the \(T^2/\mathbb{Z}_4\) orbifold. As before, we choose the twist matrices \(R_0\) and \(T_1\) as independent ones, and show that they can become block-diagonal forms containing 4 × 4, 2 × 2
and $1 \times 1$ matrices, using constraints and unitary transformations. The details of a derivation are given in appendix C. Here, we present the outline.

The constraints restricting the forms of $R_0$ and $T_1$ are given by

$$T_1^\dagger = T_1^{-1}, \quad T_mT_m = T_mT_m', \quad T_1T_3 = I,$$

(6.1)

where $T_m = R_0^{m-1}T_1R_0^{1-m}$ as explained in section 3.

Starting from the following $R_0$ and $T_1$,

$$R_0 = \begin{pmatrix} 
i I_{n_1} & -I_{n_2} & & \\ & -I_{n_3} & & \\ & & \ddots & \\ & & & I_{n_4} \end{pmatrix}, \quad T_1 = \begin{pmatrix} (T_1)_{(11)} & (T_1)_{(12)} & (T_1)_{(13)} & (T_1)_{(14)} \\ (T_1)_{(21)} & (T_1)_{(22)} & (T_1)_{(23)} & (T_1)_{(24)} \\ (T_1)_{(31)} & (T_1)_{(32)} & (T_1)_{(33)} & (T_1)_{(34)} \\ (T_1)_{(41)} & (T_1)_{(42)} & (T_1)_{(43)} & (T_1)_{(44)} \end{pmatrix},$$

(6.2)

with the $n_k \times n_k$ unit matrices $I_{n_k}$ and $n_k \times n_l$ submatrices $(T_1)_{(kl)}$, and using eq. (C.4) derived from $T_1^\dagger = T_3 = R_0^2T_1R_0^{-2}$ and $T_m'T_m = T_mT_m'$, we can rearrange $R_0$ and $T_1$ to be block-diagonal forms in a similar way to the $S^1/Z_2$ case in section 2 such that

$$R_0 = \begin{pmatrix} R_0^{(1)} & & & \\ & R_0^{(2)} & & \\ & & \ddots & \\ & & & R_0^{(M)} \end{pmatrix}, \quad T_1 = \begin{pmatrix} T_1^{(1)} & & & \\ & T_1^{(2)} & & \\ & & \ddots & \\ & & & T_1^{(M)} \end{pmatrix},$$

(6.3)

where $R_0^{(m)}$ and $T_1^{(m)}$ ($m = 1, 2, \ldots, M$) are $n^{(m)} \times n^{(m)}$ matrices given by

$$R_0^{(m)} = \begin{pmatrix} iI_{n_1^{(m)}} & & & \\ & -I_{n_2^{(m)}} & & \\ & & -iI_{n_3^{(m)}} & \\ & & & I_{n_4^{(m)}} \end{pmatrix},$$

(6.4)

and

$$T_1^{(m)} = \begin{pmatrix} a^{(m)}I_{n_1^{(m)}}^{(m)} & M_1^{(m)} & M_1^{(m)[-1]} & M_1^{(m)[-2]} & M_1^{(m)[-1]} \\ M_2^{(m)[1]} & a^{(m)}I_{n_2^{(m)}}^{(m)} & M_2^{(m)[-1]} & M_2^{(m)[-2]} & M_2^{(m)[-1]} \\ M_3^{(m)[2]} & M_3^{(m)[-1]} & a^{(m)}I_{n_3^{(m)}}^{(m)} & M_3^{(m)[-1]} & M_3^{(m)[-1]} \\ M_4^{(m)[-1]} & M_4^{(m)[2]} & M_4^{(m)[-1]} & a^{(m)}I_{n_4^{(m)}}^{(m)} \end{pmatrix},$$

(6.5)

respectively. Here, $n^{(m)} = \sum_{k=1}^4 N_k^{(m)}$ and submatrices in $T_1^{(m)}$ are denoted as $(T_1^{(m)})_{(kl)} = M_{k,l}^{(m)[k-l]}$ for $k \neq l$ and $(T_1^{(m)})_{(kk)} = M_{k,k}^{(m)[0]} = a^{(m)}I_{n_k^{(m)}}$ with real parameters $a^{(m)}$ satisfying $a^{(m)} \neq a^{(m')}$ for $m \neq m'$ and restricted to $-1 \leq a^{(m)} \leq 1$. We use a notation of $M_{k,l}^{(m)[k-l]} = M_{k,l}^{(m)[k'-l]} = M_{k',l'}^{(m)[k-l]}$ with $k' = k \pmod{4}$ and $l' = l \pmod{4}$. Parameters such as $n_k$ and $N_k^{(m)}$ which represent the size of matrices are non-negative integers. The same applies to the
following ones. Because $T_1^{(m)}$ is already diagonal for $a^{(m)} = \pm 1$, we focus on $-1 < a^{(m)} < 1$ hereafter.

Next, using eqs. (C.8)–(C.11) derived from the constraints in eq. (6.1), we can choose the basis such that $M_{k,l}^{(m)[k-l]}$ have the forms of eqs. (C.22), (C.23) and (C.26), and rearrange $R_0^{(m)}$ and $T_1^{(m)}$ to be the forms of block-diagonal ones such as $R_0^{(m)} = R_0^{(m)'} \oplus R_0^{(m)''}$ and $T_1^{(m)} = T_1^{(m)'} \oplus T_1^{(m)''}$ given by

$$R_0^{(m)} = \begin{pmatrix} R_0^{(m)'} & \vspace{1mm} \vspace{1mm} \end{pmatrix} = R_0^{(m)'} \oplus R_0^{(m)''}, \quad (6.6)$$

$$R_0^{(m)'} = \begin{pmatrix} iI_{r(m)} & \vspace{1mm} & -I_{r(m)} \vspace{1mm} \end{pmatrix}, \quad (6.7)$$

$$R_0^{(m)''} = \begin{pmatrix} iI_{n_1^{(m)'} \vspace{1mm}} & \vspace{1mm} \vspace{1mm} \end{pmatrix} = \begin{pmatrix} -I_{n_2^{(m)'} \vspace{1mm}} \vspace{1mm} \end{pmatrix}, \quad (6.8)$$

and

$$T_1^{(m)'} = \begin{pmatrix} T_1^{(m)'} & \vspace{1mm} \vspace{1mm} \end{pmatrix} = T_1^{(m)'} \oplus T_1^{(m)''}, \quad (6.9)$$

$$T_1^{(m)'} = \begin{pmatrix} a^{(m)}I_{r(m)} & \vspace{1mm} \vspace{1mm} \end{pmatrix} = \begin{pmatrix} \vspace{1mm} & \vspace{1mm} \vspace{1mm} \end{pmatrix}, \quad (6.10)$$

respectively. Here, $M^{(m)[1]}$ are $r(m \times r(m)$ diagonal matrices with positive elements, i.e., $(M^{(m)[1]}_{\ell \ell}) > 0$, $M^{(m)[2]}$ are $r(m \times r(m)$ diagonal matrices with non-negative elements, i.e., $(M^{(m)[2]}_{\ell \ell}) \geq 0$, $U^{(m)}_{k,l}$ are $r(m \times r(m)$ unitary matrices and $U^{(m)}_{k,k-2}$ and $U^{(m)}_{k-k-2}$ are $n^{(m)'}_{k} \times n^{(m)'}_{k}$ unitary matrices. $R_0^{(m)''}$ and $T_1^{(m)''}$ can appear only in the case of $a^{(m)} = 0$.

We perform an appropriate unitary transformation that make all submatrices in $T_1^{(m)'}$ and $T_1^{(m)''}$ diagonalized simultaneously such as eqs. (C.42) and (C.43), keeping $R_0^{(m)}$ diagonal ones, and rearrange the rows and columns in $R_0^{(m)}$ and $T_1^{(m)}$. Then, we find that $R_0^{(m)}$ and $T_1^{(m)}$ can take simplified block-diagonal forms containing $4 \times 4$, $2 \times 2$ and $1 \times 1$ matrices as the diagonal blocks.
From eqs. (C.35) and (C.42), the $4 \times 4$ submatrices of $R_0^{(m)}$ and $T_1^{(m)}$ have the forms of $r_0$ and $t_1$ presented as

$$r_0 = \begin{pmatrix} i & -1 \\ -i & 1 \end{pmatrix}, \quad t_1 = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 \\ a_1 & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_2 & a_1 & a_4 \end{pmatrix} \tag{6.12}$$

respectively, where $a_1$ and $a_3$ are complex numbers with $a_3 = -\bar{a}_1$, $a_2$ is a non-negative number and $a_4$ is a real number, and they satisfy $2|a_1|^2 + a_2^2 + a_4^2 = 1$ and $2a_2a_4 = a_1^2 + a_2^2$. In section 8, it is shown that $4 \times 4$ submatrices of this form are diagonalized by a suitable gauge transformation.

In case with $a^{(m)} = 0$, from eqs. (C.37) and (C.43), $R_0^{(m)}$ and $T_1^{(m)}$ can contain the following type of $2 \times 2$ matrices as block-diagonal elements,

$$r_0' = i^{n'} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.13}$$

where $n'$ is an integer. Combining $r_0'$ with an odd $n'$ and that with an even $n'$, we can make the $4 \times 4$ matrices $r_0$ and $t_1$ with $a_2 = 1$ and $a_1 = a_3 = a_4 = 0$ in eq. (6.12), which are diagonalized simultaneously by a suitable gauge transformation. Hence, non-paired $r_0'$’s with $n' = \text{odd}$ alone or $n' = \text{even}$ alone remain independent from $r_0$ and $t_1$, and the number of diagonal-blocks composed of such $r_0'$ and $t_1'$ is $|n_1^{(m)'} - n_2^{(m)'|}$.

In case with $a^{(m)} = \pm 1$, $R_0^{(m)}$ and $T_1^{(m)}$ can contain $1 \times 1$ matrices as block-diagonal elements.

7 $T^2/\mathbb{Z}_6$

We consider the $T^2/\mathbb{Z}_6$ case. As before, we choose the twist matrices $R_0$ and $T_1$ as independent ones, and show that they can become block-diagonal forms containing $6 \times 6$, $3 \times 3$, $2 \times 2$ and $1 \times 1$ matrices, using constraints and unitary transformations. The details of derivations are given in appendix D. Here, we present the outline.

The constraints restricting the forms of $R_0$ and $T_1$ are given by

$$T_1^T = T_1^{-1}, \quad T_m T_m' = T_m T_m', \quad T_1 T_4 = I, \quad T_1 T_3 T_0 = I, \tag{7.1}$$

where $T_m = R_0^{m-1} T_1 R_1^{1-m}$ as explained in section 8.

Starting from the following $R_0$ and $T_1$,

$$R_0 = \begin{pmatrix} \eta I_{n_1} & \eta^2 I_{n_2} & -I_{n_3} \\ \eta I_{n_4} & -\eta^2 I_{n_5} \end{pmatrix} \tag{7.2}$$

$$T_1 = \begin{pmatrix} \eta^2 I_{n_3} & -I_{n_4} \\ I_{n_6} \end{pmatrix} \tag{7.3}$$

...
and

\[
T_1 = \begin{pmatrix}
(T_1)_{(11)} & (T_1)_{(12)} & (T_1)_{(13)} & (T_1)_{(14)} & (T_1)_{(15)} & (T_1)_{(16)} \\
(T_1)_{(21)} & (T_1)_{(22)} & (T_1)_{(23)} & (T_1)_{(24)} & (T_1)_{(25)} & (T_1)_{(26)} \\
(T_1)_{(31)} & (T_1)_{(32)} & (T_1)_{(33)} & (T_1)_{(34)} & (T_1)_{(35)} & (T_1)_{(36)} \\
(T_1)_{(41)} & (T_1)_{(42)} & (T_1)_{(43)} & (T_1)_{(44)} & (T_1)_{(45)} & (T_1)_{(46)} \\
(T_1)_{(51)} & (T_1)_{(52)} & (T_1)_{(53)} & (T_1)_{(54)} & (T_1)_{(55)} & (T_1)_{(56)} \\
(T_1)_{(61)} & (T_1)_{(62)} & (T_1)_{(63)} & (T_1)_{(64)} & (T_1)_{(65)} & (T_1)_{(66)} 
\end{pmatrix},
\]

(7.3)

with \( \eta = e^{2\pi i/6} \), the \( n_k \times n_k \) unit matrices \( I_{n_k} \) and \( n_k \times n_l \) submatrices \( (T_1)_{(kl)} \), and using eq. (D.5) derived from \( T_1^4 = T_1^{-1} = T_4 = R_0^0 T_1 R_0^{-3} \) and \( T_m T_m = T_n T_m' \), we can rearrange \( R_0 \) and \( T_1 \) to be block-diagonal forms in a similar way to the \( S^1/\mathbb{Z}_2 \) case in section 2 such that

\[
R_0 = \begin{pmatrix}
R_0^{(1)} & & \\
& \ddots & \\
& & R_0^{(M)}
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
T_1^{(1)} & & \\
& \ddots & \\
& & T_1^{(M)}
\end{pmatrix},
\]

(7.4)

where \( R_0^{(m)} \) and \( T_1^{(m)} \) \((m = 1, 2, \ldots, M)\) are \( n^{(m)} \times n^{(m)} \) matrices given by

\[
R_0^{(m)} = \begin{pmatrix}
\eta I_{n_1^{(m)}} & \eta^2 I_{n_2^{(m)}} & -I_{n_3^{(m)}} & -\eta I_{n_4^{(m)}} & -\eta^2 I_{n_5^{(m)}} & I_{n_6^{(m)}}
\end{pmatrix},
\]

(7.5)

and

\[
T_1^{(m)} = \begin{pmatrix}
a^{(m)} I_{n_1^{(m)}} & M_{12}^{(m)[-1]} & M_{13}^{(m)[-2]} & M_{14}^{(m)[-3]} & M_{15}^{(m)[2]} & M_{16}^{(m)[1]} \\
M_{21}^{(m)[1]} & a^{(m)} I_{n_2^{(m)}} & M_{23}^{(m)[-1]} & M_{24}^{(m)[-2]} & M_{25}^{(m)[-3]} & M_{26}^{(m)[2]} \\
M_{31}^{(m)[2]} & M_{32}^{(m)[1]} & a^{(m)} I_{n_3^{(m)}} & M_{34}^{(m)[-1]} & M_{35}^{(m)[-2]} & M_{36}^{(m)[-3]} \\
M_{41}^{(m)[3]} & M_{42}^{(m)[2]} & M_{43}^{(m)[1]} & a^{(m)} I_{n_4^{(m)}} & M_{45}^{(m)[-1]} & M_{46}^{(m)[-2]} \\
M_{51}^{(m)[-2]} & M_{52}^{(m)[3]} & M_{53}^{(m)[2]} & M_{54}^{(m)[1]} & a^{(m)} I_{n_5^{(m)}} & M_{56}^{(m)[-1]} \\
M_{61}^{(m)[-1]} & M_{62}^{(m)[-2]} & M_{63}^{(m)[3]} & M_{64}^{(m)[2]} & M_{65}^{(m)[1]} & a^{(m)} I_{n_6^{(m)}}
\end{pmatrix},
\]

(7.6)

respectively. Here, \( n^{(m)} = \sum_{k=1}^{6} n_k^{(m)} \) and submatrices in \( T_1^{(m)} \) are denoted as \( (T_1^{(m)})_{(kl)} = M_{kl}^{(m)[k-l]} \) for \( k \neq l \) and \( (T_1^{(m)})_{(kk)} = M_{kk}^{(m)[0]} = a^{(m)} I_{n_k^{(m)}} \) with real parameters \( a^{(m)} \) satisfying \( a^{(m)} \neq a^{(m')} \) for \( m \neq m' \) and restricted to \( -1 \leq a^{(m)} \leq 1 \). We use a notation of \( M_{kl}^{(m)[k-l]} = M_{k'l'}^{(m)[k'-l']} \) with \( k' = k \pmod{6} \) and \( l' = l \pmod{6} \). Parameters such as \( n_k \) and \( n_k^{(m)} \) which represent the size of matrices are non-negative integers. The same applies to the
following ones. Because \( T_1^{(m)} \) is already diagonal for \( a^{(m)} = \pm 1 \), we focus on \(-1 < a^{(m)} < 1\) hereafter.

Next, using eqs. (D.59)–(D.12) derived from the constraints in eq. (7.1), we can choose the basis such that \( M_k^{(m)[k−l]} \) have the forms of eqs. (D.20), (D.21) and (D.26)–(D.28), and rearrange \( R_0^{(m)} \) and \( T_1^{(m)} \) to be the forms of block-diagonal ones such as \( R_0^{(m)} = R_0^{(m)\prime} \oplus R_0^{(m)\prime\prime} \) and \( T_1^{(m)} = T_1^{(m)\prime} \oplus T_1^{(m)\prime\prime} \oplus T_1^{(m)\prime\prime\prime} \), where \( R_0^{(m)\prime} \), \( T_1^{(m)\prime} \), \( R_0^{(m)\prime\prime} \), \( T_1^{(m)\prime\prime} \), \( R_0^{(m)\prime\prime\prime} \) and \( T_1^{(m)\prime\prime\prime} \) are also block-diagonal matrices whose submatrices are given by

\[
\begin{align*}
(R_0^{(m)\prime})_{(kk-q)} &= \eta^2 \delta_{kk-q} I_{r^{(m)}}, \\
(T_1^{(m)\prime})_{(kk-q)} &= \tilde{M}^{(m)[q]} U_{r^{(m)}}, \quad (7.7) \\
(R_0^{(m)\prime\prime})_{(kk-q)} &= \eta^2 \delta_{kk-q} I_{n^{(m)\prime}}, \\
(T_1^{(m)\prime\prime})_{(kk-q)} &= -\frac{1}{3} I_{n^{(m)\prime}} \delta_{q0} + \frac{2}{3} \tilde{U}_{r^{(m)}} \delta_{q\pm 2}, \quad (7.8) \\
(R_0^{(m)\prime\prime\prime})_{(kk-q)} &= \eta^2 \delta_{kk-q} I_{n^{(m)\prime\prime}}, \\
(T_1^{(m)\prime\prime\prime})_{(kk-q)} &= -\frac{1}{2} I_{n^{(m)\prime\prime}} \delta_{q0} \pm \frac{\sqrt{3}}{2} \tilde{U}_{r^{(m)}} \delta_{q3}. \quad (7.9)
\end{align*}
\]

where \( \tilde{M}^{(m)[1]} \) are \( r^{(m)} \times r^{(m)} \) diagonal matrices with positive elements, i.e., \( (\tilde{M}^{(m)[1]})_{ii} > 0 \), \( \tilde{M}^{(m)[2]} \) and \( \tilde{M}^{(m)[3]} \) are \( r^{(m)} \times r^{(m)} \) diagonal matrices with non-negative elements, i.e., \( (\tilde{M}^{(m)[2]})_{ii} \geq 0 \) and \( (\tilde{M}^{(m)[3]})_{ii} \geq 0 \), \( \tilde{M}^{(m)[−q]} = \tilde{M}^{(m)[q]} \), \( U_{r^{(m)}} \) are \( r^{(m)} \times r^{(m)} \) unitary matrices and \( \tilde{U}_{r^{(m)}} \) and \( \tilde{U}_{r^{(m)}} \) are unitary matrices, respectively. \( R_0^{(m)\prime} \) and \( T_1^{(m)\prime} \) can appear only in the case of \( a^{(m)} = −1/3 \), and \( R_0^{(m)\prime\prime} \) and \( T_1^{(m)\prime\prime} \) can appear only in the case of \( a^{(m)} = −1/2 \).

We perform an appropriate unitary transformation that make all submatrices in \( T_1^{(m)\prime} \), \( T_1^{(m)\prime\prime} \) and \( T_1^{(m)\prime\prime\prime} \) diagonalized simultaneously such as eqs. (D.59)–(D.61), keeping \( R_0^{(m)} \) diagonal ones, and rearrange the rows and columns in \( R_0^{(m)} \) and \( T_1^{(m)} \). Then, we find that \( R_0^{(m)} \) and \( T_1^{(m)} \) can take simplified block-diagonal forms containing \( 6 \times 6, 3 \times 3, 2 \times 2 \) and \( 1 \times 1 \) matrices as the diagonal blocks.

From eqs. (D.59) and (D.59), the \( 6 \times 6 \) matrices of \( R_0^{(m)} \) and \( T_1^{(m)} \) have the form:

\[
\begin{pmatrix}
\eta & 0 \\
0 & \eta^2
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\eta & 0 \\
0 & \eta^2
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\eta & 0 \\
0 & \eta^2
\end{pmatrix}
\end{pmatrix}

\begin{pmatrix}
a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\
a_1 & a_6 & a_5 & a_4 & a_3 & a_2 \\
a_2 & a_1 & a_6 & a_5 & a_4 & a_3 \\
a_3 & a_2 & a_1 & a_6 & a_5 & a_4 \\
a_4 & a_3 & a_2 & a_1 & a_6 & a_5 \\
a_5 & a_4 & a_3 & a_2 & a_1 & a_6
\end{pmatrix}
\begin{pmatrix}
a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\
a_1 & a_6 & a_5 & a_4 & a_3 & a_2 \\
a_2 & a_1 & a_6 & a_5 & a_4 & a_3 \\
a_3 & a_2 & a_1 & a_6 & a_5 & a_4 \\
a_4 & a_3 & a_2 & a_1 & a_6 & a_5 \\
a_5 & a_4 & a_3 & a_2 & a_1 & a_6
\end{pmatrix} \quad (7.10)
\]

where \( a_1, a_2, a_4 \) and \( a_5 \) are complex numbers with \( a_4 = \overline{a_2}, a_5 = \overline{a_1}, a_3 \) is a pure imaginary number, \( a_6 \) is a real number, and they satisfy \( 2|a_1|^2 + 2|a_2|^2 + |a_3|^2 + a_6^2 = 1 \), \( |a_1|^2 + |a_2|^2 + |a_3|^2 + a_6^2 = 1 \), \( a_6^2 = a_6, 2a_2a_6 + \overline{a_2}^2 = a_2^2 - 2\overline{a_1}a_3, a_1a_6 + a_3a_2 + 2a_2a_1 = a_1, -a_2a_6 + a_2^2 + a_1a_3 + a_2^2 = a_2 \) and \( -2a_3a_6 = a_1a_2 - \overline{a_1a_2} = a_3 \). We note that these relations are not independent of each other and the number of independent parameters is two as seen from the parametrization in eq. (G.8). In the next section, it is shown that \( 6 \times 6 \) submatrices of this form are diagonalized by a suitable gauge transformation.
In the case with \( a^{(m)} = -1/3 \), from eqs. (D.46) and (D.60), \( R_0^{(m)} \) and \( T_1^{(m)} \) can contain following type of 3 \( \times \) 3 matrices as block-diagonal elements,

\[
\begin{align*}
    r'_0 &= e^{2\pi n'i/6} \begin{pmatrix}
        \omega & 0 & 0 \\
        0 & \omega^2 & 0 \\
        0 & 0 & 1
    \end{pmatrix}, \\
    t'_1 &= \begin{pmatrix}
        -1 & 2 & 2 \\
        2 & -1 & 2 \\
        2 & 2 & -1
    \end{pmatrix},
\end{align*}
\]  

(7.11)

where \( n' \) is an integer.

In the case with \( a^{(m)} = -1/2 \), from eqs. (D.47) and (D.61), \( R_0^{(m)} \) and \( T_1^{(m)} \) can contain following type of 2 \( \times \) 2 matrices as block-diagonal elements,

\[
\begin{align*}
    r''_0 &= e^{2\pi n''i/6} \begin{pmatrix}
        -1 & 0 \\
        0 & 1
    \end{pmatrix}, \\
    t''_1 &= \begin{pmatrix}
        \pm 1/2 & \pm \sqrt{3}/2i \\
        \pm \sqrt{3}/2i & -1/2
    \end{pmatrix}, \quad \text{(double sign in the same order)},
\end{align*}
\]  

(7.12)

where \( n'' \) is an integer.

In a similar way as the case of \( T^2/\mathbb{Z}_4 \), there can be block-diagonal elements which are diagonalized by a gauge transformation, with a suitable combination of 3 \( \times \) 3 matrices listed in eq. (7.11) with different \( n' \)'s or 2 \( \times \) 2 matrices listed in eq. (7.12) with different \( n'' \)'s.

In the case with \( a^{(m)} = \pm 1 \), \( R_0^{(m)} \) and \( T_1^{(m)} \) can contain 1 \( \times \) 1 matrices as block-diagonal elements.

### 8 Gauge Transformation

We show that 2 \( \times \) 2 submatrices \( t_1 \) and \( t_2 \) on \( T^2/\mathbb{Z}_2 \) and \( N \times N \) submatrix \( t_1 \) on \( T^2/\mathbb{Z}_N \) \( (N = 3, 4, 6) \) are diagonalized by suitable gauge transformations, keeping \( r_0 \) the diagonal one.

In section 4 we have found that \( R_0, T_1 \) and \( T_2 \) on \( T^2/\mathbb{Z}_2 \) contain following 2 \( \times \) 2 unitary matrices as members of block-diagonal ones,

\[
\begin{align*}
    r_0 &= X, \\
    t_1 &= a_1 Y + a_2 I, \\
    t_2 &= b_1 Y + b_2 I,
\end{align*}
\]  

(8.1)

where \( X \) and \( Y \) are defined by

\[
\begin{align*}
    X &\equiv \begin{pmatrix}
        -1 & 0 \\
        0 & 1
    \end{pmatrix}, \\
    Y &\equiv \begin{pmatrix}
        0 & 1 \\
        1 & 0
    \end{pmatrix},
\end{align*}
\]  

(8.2)

respectively (see eqs. (4.18) and (4.19)). The coefficients \( a_1, a_2, b_1 \) and \( b_2 \) are numbers. The matrices \( t_1 \) and \( t_2 \) are also written by

\[
\begin{align*}
    t_1 &= e^{i\theta_{(1)} Y}, \\
    t_2 &= e^{i\theta_{(2)} Y},
\end{align*}
\]  

(8.3)

where \( \theta_{(1)} \) and \( \theta_{(2)} \) are real numbers. Using \( \theta_{(1)} \) and \( \theta_{(2)} \), \( a_1, a_2, b_1 \) and \( b_2 \) are given by

\[
\begin{align*}
    a_1 &= i \sin \theta_{(1)}, \\
    a_2 &= \cos \theta_{(1)}, \\
    b_1 &= i \sin \theta_{(2)}, \\
    b_2 &= \cos \theta_{(2)}.
\end{align*}
\]  

(8.4)
Then, $r_0$, $t_1$ and $t_2$ are transformed into the diagonal ones:

\[
\begin{align*}
\tilde{r}_0 &= \Omega(-z, -\bar{z}) r_0 \Omega^\dagger(z, \bar{z}) = r_0, \\
\tilde{t}_1 &= \Omega(z + 1, \bar{z} + 1) t_1 \Omega^\dagger(z, \bar{z}) = (-1)^{l(1)} I, \\
\tilde{t}_2 &= \Omega(z + \tau, \bar{z} + \bar{\tau}) t_2 \Omega^\dagger(z, \bar{z}) = (-1)^{l(2)} I,
\end{align*}
\]

(8.5)

under the gauge transformation whose transformation function is given by $\Omega(z, \bar{z}) = e^{i(\beta z + \bar{\beta} \bar{z})}$ with $\beta = -\theta(1) + l(1) \pi \left(1 + \frac{\text{Re} \tau}{\text{Im} \tau}\right) - \frac{-\theta(2) + l(2) \pi}{\text{Im} \tau} (l(1) \text{ and } l(2); \text{integers}, \text{Im} \tau \neq 0)$.

In the same way, $R_0$ and $T_1$ on $T^2/\mathbb{Z}_N$ ($N = 3, 4, 6$) contain following $N \times N$ unitary matrices as members of block-diagonal ones,

\[
r_0 = X, \quad t_1 = \sum_{p=1}^{N} a_p Y^p,
\]

(8.6)

where $X$ and $Y$ are defined by

\[
X \equiv \begin{pmatrix}
\tau & \tau^2 & \cdots & \tau^{N-1} \\
& \tau & \cdots & \tau^{N-1} \\
& & \ddots & \vdots \\
& & & \tau^{N-1}
\end{pmatrix}, \quad Y \equiv \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix},
\]

(8.7)

respectively (see eqs. (5.10), (6.12) and (7.10)). Here, $\tau = e^{2\pi i/N}$, and $a_p$ ($p = 1, 2, \ldots, N$) are numbers which satisfy specific relations, e.g., $a_1^3 + a_2^3 + a_3^3 - 3a_1a_2a_3 = 1$, $|a_1|^2 + |a_2|^2 + |a_3|^2 = 1$ and $\bar{\alpha}_1 a_3 + \bar{\alpha}_3 a_2 + \bar{\alpha}_2 a_1 = 0$ for $N = 3$. The specific relations among $a_p$ are rewritten compactly as

\[
|a_j|^2 = 1, \quad (j = 1, \ldots, N),
\]

for $N = 3, 4, 6$, (8.8)

\[
a_1 a_2 a_3 = 1, \quad a_1 a_3 = a_2 a_4 = 1,
\]

for $N = 3, 4$, (8.9)

\[
a_1 a_4 = a_2 a_5 = a_3 a_6 = 1, \quad a_1 a_3 a_5 = a_2 a_4 a_6 = 1,
\]

for $N = 6$, (8.10)

where $\alpha_j \equiv \sum_{p=1}^{N} a_p \tau^{jp}$. From them, $t_1$ is expressed as

\[
t_1 = \sum_{p=1}^{N} a_p Y^p = e^{i(\beta Y + \bar{\beta} \bar{Y}^{N-1})}.
\]

(8.12)

The derivation of eqs. (8.8) - (8.12) is given in appendix C.

Based on eq. (8.12), we find that $r_0$ and $t_1$ are transformed as

\[
\begin{align*}
\tilde{r}_0 &= \Omega(\tau z, \tau \bar{z}) r_0 \Omega^\dagger(z, \bar{z}), \\
\tilde{t}_1 &= \Omega(z + 1, \bar{z} + 1) t_1 \Omega^\dagger(z, \bar{z}),
\end{align*}
\]

(8.13)

under a gauge transformation, and $\tilde{r}_0$ and $\tilde{t}_1$ become diagonal ones, using the gauge transformation function:

\[
\Omega(z, \bar{z}) = e^{i(\beta z Y + \bar{\beta} \bar{z} \bar{Y}^{N-1})}
\]

(8.14)
Table 2: Peculiar constraints and gauge transformed matrices

| N | peculiar constraints                          | $\beta$               | $\bar{r}_0$ | $\bar{t}_1$ |
|---|---------------------------------------------|-----------------------|-------------|-------------|
| 3 | $t_1t_2t_3 = I$                             | $-\theta - \frac{2}{3}\pi \tilde{l}$ | $r_0$       | $\omega \tilde{l}I$ |
| 4 | $t_1t_2t_3t_4 = I, \ t_1t_3 = t_2t_4 = I$  | $-\theta + \frac{1}{2}\pi \tilde{l}$ | $r_0$       | $(-1)^{\tilde{l}}I$ |
| 6 | $t_1t_2t_3t_4t_5t_6 = I, \ t_1t_4 = t_2t_5 = t_3t_6 = I, \ t_1t_3t_5 = t_2t_4t_6 = I$ | $-\theta$             | $r_0$       | $I$          |

with a suitable value $\beta$ including an integer $\tilde{l}$, as shown in Table 2.

In contrast, $R_0$ and $T_1$ on $T^2/\mathbb{Z}_4$ can contain $2 \times 2$ matrices:

$$r'_0 = i^{n'} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t'_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (n' : \text{integer}), \quad (8.15)$$

as block-diagonal elements, and $t'_1$ cannot become diagonal ones, keeping $r'_0$ diagonal ones, by the use of a gauge transformation. In the same way, $R_0$ and $T_1$ on $T^2/\mathbb{Z}_6$ can contain $3 \times 3$ matrices:

$$r'_0 = e^{2\pi n'i/6} \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t'_1 = \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}, \quad (n' : \text{integer}), \quad (8.16)$$

and $2 \times 2$ matrices:

$$r''_0 = e^{2\pi n''i/6} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t''_1 = \begin{pmatrix} -\frac{1}{2} & \pm \frac{\sqrt{3}}{2}i \\ \pm \frac{\sqrt{3}}{2}i & -\frac{1}{2} \end{pmatrix}, \quad (n'' : \text{integer}, \quad \text{double sign in the same order}), \quad (8.17)$$

as block-diagonal elements, and $t'_1$ and $t''_1$ cannot become diagonal ones, keeping $r'_0$ and $r''_0$ diagonal ones, by the use of a gauge transformation. Note that the matrices given in eq. (8.15) and those in eqs. (8.16) and (8.17) satisfy every relation required from transformation properties on $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$, respectively.

The absence of gauge transformations that make $t'_1$ and/or $t''_1$ diagonal ones is understood from the fact that there are no continuous Wilson line phases relating to such block-diagonal elements. The continuous Wilson line phases are non-integrable phases of, for instance, $W_1 = e^{i\sigma(A_2 + \langle A_2 \rangle)}T_1$ and $W_2 = e^{i\sigma(\tau(A_2) + \tau(A_3))}T_2$ with the vacuum expectation values (VEVs) of zero modes in $A_2 = (A_5 - iA_6)/2$ and $A_3 = \langle A_2 \rangle = (A_5 + iA_6)/2$. Here $A_3$ and $A_6$ are the gauge fields along the coordinates $y^1$ and $y^2$, respectively. We remind that $|\lambda_1| = 1$ is taken. The VEV $\langle A_2 \rangle$ should change as

$$R_0 \langle A_2 \rangle R_0^{-1} = \tau \langle A_2 \rangle, \quad (8.18)$$

under $\mathbb{Z}_N$ rotation. Actually, there are no non-vanishing VEVs that satisfy the above relation for $r'_0$ in eqs. (8.15) and (8.16) and for $r''_0$ in eq. (8.17).
9 Physical implications

9.1 Rank reduction

Here, we discuss a rank reduction of gauge group. The physical symmetries are determined in cooperation of boundary conditions of fields and the dynamics of the Wilson line phases, by the Hosotani mechanism [26, 27, 28]. In concrete terms, starting from boundary conditions specified by the twist matrices \((R_0, T_1, T_2)\), we calculate the effective potential \(V_{\text{eff}}\) for the Wilson line phases, which is flat at the tree level and generated by the quantum corrections. And then, minimization of \(V_{\text{eff}}\) tells us the VEV, \(\langle A_z \rangle\). After performing the gauge transformation with \(\Omega(z, \bar{z}) = e^{ig(\langle A_z \rangle z + \langle A_z \rangle \bar{z})}\), \(\langle A_z \rangle\) is shifted to \(\langle A_z' \rangle = 0\) and the twist matrices are transformed as

\[
(R_0, T_1, T_2) \to (R_0^{\text{sym}}, T_1^{\text{sym}}, T_2^{\text{sym}})
= (\Omega(-z, -\bar{z})R_0\Omega^\dagger(z, \bar{z}), \Omega(z + 1, \bar{z} + 1)T_1\Omega^\dagger(z, \bar{z}), \Omega(z + \tau, \bar{z} + \bar{\tau})T_2\Omega^\dagger(z, \bar{z})).
\] (9.1)

We note that, based on the definition of continuous Wilson line phases as eigenvalues of \(-i \ln W_1\) and \(-i \ln W_2\), they are gauge invariant because a change of \(\langle A_z \rangle\) is canceled out by that of \(T_1\) and \(T_2\). The physical gauge symmetry \(\mathcal{H}^{\text{sym}}\) is spanned by the generators \(T^a\) that commute with \((R_0^{\text{sym}}, T_1^{\text{sym}}, T_2^{\text{sym}})\):

\[
\mathcal{H}^{\text{sym}} = \{T^a; [T^a, R_0^{\text{sym}}] = [T^a, T_1^{\text{sym}}] = [T^a, T_2^{\text{sym}}] = 0\}.
\] (9.2)

Then the rank of unbroken gauge group agrees with the number of Cartan subalgebras that commute with \(R_0^{\text{sym}}\) and \(T_m^{\text{sym}}\), where \(m = 1, 2\) for \(T^2/\mathbb{Z}_2\) and \(m = 1\) for \(T^2/\mathbb{Z}_N\) \((N = 3, 4, 6)\). Thus the reduction of rank does not occur if \(R_0^{\text{sym}}\) and \(T_m^{\text{sym}}\) are diagonal. In other words, when diagonal \(R_0\) and \(T_m\) belong to the equivalence class, there are symmetry-enhanced points in the parameter space of the VEVs of the Wilson line phases, while the rank is reduced on a generic point. In contrast, the reduction always occurs in the presence of \(t_1^l\) on \(T^2/\mathbb{Z}_4\) and \(t_1^l\) and/or \(t_1^r\) on \(T^2/\mathbb{Z}_6\), because they take discrete values and cannot be diagonalized. Note that the \(2 \times 2\) matrices (\(3 \times 3\) matrix), \(t_1^l\) in eq. (8.15) and \(t_1^r\) in eq. (9.17) \((t_1^l\) in eq. (8.16)), are contained in the expression of the \(t_1^l\) in eq. (8.3) \((t_1^l\) in eq. (8.6) with \(N = 3\)). While the matrix \(t_1\) in the \(\mathbb{Z}_2\) \((\mathbb{Z}_3)\) orbifold can be diagonalized, keeping \(r_0\) a diagonal form, thanks to a singular gauge transformation that shifts the zero modes of \(A_z\), these degrees are projected out by the additional elements of the enlarged group in the \(\mathbb{Z}_4\) and \(\mathbb{Z}_6\) orbifolds, as examined in eq. (8.18). This observation indicates that the rank reduction by the \(t_1^l\) and/or \(t_1^r\), or the discrete Wilson line phase, can be realized only for \(\mathbb{Z}_N\) orbifolds with \(N\) being a nonprime number.

In refs. [39, 41], the rank reduction of gauge group has been studied in orbifold construction. It is pointed that the rank is not reduced by the discrete Wilson lines. In these references,
the rank reduction is discussed mainly focusing on the $\mathbb{Z}_2$ orbifolds with $N = 2$ being a prime number, and the possibility of the discrete Wilson lines that do not commute with the rotation $R_0$ has not been studied. Our results show counter examples to the claim.

### 9.2 Examples

Finally, we show some examples of the boundary conditions for illustration purpose, in the $T^2/\mathbb{Z}_4$ case.

The first one is an $SU(6)$ model with

$$R_0 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix},$$

by which the $SU(6)$ symmetry is broken down to $SU(3) \times SU(2) \times U(1)^2$. We note that, when these boundary conditions are chosen in an $S^1/\mathbb{Z}_2$ case, the continuous Wilson line generically induces further symmetry breaking $SU(2) \times U(1) \to U(1)$, which may be applied to the electroweak symmetry breaking \cite{17, 18}. In contrast, in the $T^2/\mathbb{Z}_4$ case, there are no degrees of the continuous Wilson line phases and the rank is not reduced.

To realize the degrees of the continuous Wilson line in the $T^2/\mathbb{Z}_4$ case, we modify the first example to $SU(8)$ model with

$$R_0 = \begin{pmatrix} i & -1 & -i \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix},$$

by which the $SU(8)$ symmetry is broken down to $SU(3) \times SU(2) \times U(1)^4$. Due to the degrees of the Wilson line phases, in the basis where $\langle A_z \rangle$ is gauged away, the upper left $4 \times 4$ submatrix in $T_1$ is written in general as the $t_1$ given in eq. \cite{6,12}. Generically, then, the further symmetry breaking $SU(2) \times U(1)^3 \to U(1)$ occurs, which might be identified with the electroweak symmetry breaking. In this second example, the rank can be reduced with the continuous Wilson line phases.

\footnote{In the case where the $(1,1)$ component of $R_0$, $-1$, is replaced by $i$, there appears a zero mode which can not be gauged away, discussed in the previous footnote. When it acquires a nonvanishing VEV, the rank is reduced \cite{40}.}
The third one is for the rank reduction without the continuous Wilson line phases: $SU(7)$ model with

\[
R_0 = \begin{pmatrix}
-1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & -1
\end{pmatrix}
\]

by which the $SU(7)$ symmetry is broken down to $SU(3) \times SU(2) \times U(1)^2$. In this example, the rank is reduced without the degrees of the continuous Wilson line phases.

### 10 Conclusions and discussions

We have studied the existence of diagonal representatives in each equivalence class of the twist matrices in $SU(n)$ or $U(n)$ gauge theories compactified on the orbifolds $T^2/Z_N$ ($N = 2, 3, 4, 6$), supposing that the theory has a global $G' = U(n)$ symmetry. Using constraints, unitary transformations and gauge transformations, we have examined whether the twist matrices can simultaneously become diagonal or not. We have shown that at least one diagonal representative necessarily exists in each equivalence class on $T^2/Z_2$ and $T^2/Z_3$, but the twist matrices on $T^2/Z_4$ and $T^2/Z_6$ can contain not only diagonal matrices but also non-diagonal $2 \times 2$ ones ($t'_1$ in eq. (8.15)) and non-diagonal $3 \times 3$ and $2 \times 2$ ones ($t''_1$ in eq. (8.16) and $t''_1$ in eq. (8.17)), respectively, as members of block-diagonal submatrices.

The absence of gauge transformations that make $t'_1$ and/or $t''_1$ diagonal ones is related to the absence of Wilson line phases with respect to such block-diagonal ones. When $R_0$ and $T_m$ contain diagonal elements alone, there are symmetry-enhanced points in the parameter space of the continuous Wilson line phases, while the rank is reduced on a generic point. In contrast, the reduction occurs in the presence of $t'_1$ on $T^2/Z_4$ and $t'_1$ and/or $t''_1$ on $T^2/Z_6$. We note that these matrices have discrete parameters, and thus our results show examples that the rank-reducing symmetry breaking is caused by the discrete Wilson line phases in $Z_N$ orbifolds with $N$ being a nonprime number.

In this article, we restrict our gauge group into $SU(n)$ or $U(n)$ to ensure the unitary transformations that diagonalize submatrices in our calculation are a part of the symmetry transformations. The extension to more general gauge group is an important issue to be addressed in a future work. Furthermore, there remains the arbitrariness problem of which type of boundary conditions should be chosen without relying on phenomenological information, and it would be challenging to find a mechanism or principle that determine boundary conditions of fields.
A Derivation of block-diagonal forms on $T^2/\mathbb{Z}_2$

In this section, we derive the results shown in section 4. We take the twist matrices $R_0$ and $T_1$ as shown in eqs. (4.2), (4.4) and (4.5). The matrix $T_2$ is now expressed by the submatrices $T_2^{(\lambda \lambda')}$ ($\lambda, \lambda' = 0, \ldots, M$) as

$$T_2 = \begin{pmatrix}
T_2^{(00)} & T_2^{(01)} & \cdots \\
T_2^{(10)} & T_2^{(11)} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}. \quad (A.1)
$$

A.1 Derivation of $T_2^{(0m)} = T_2^{(m0)} = 0$

Let us start to discuss how $T_2$ is simplified through unitary transformations that keep the forms of $R_0$ and $T_1$. First, we examine the condition $[T_1, T_2] = 0$ in eq. (4.1) to restrict the forms of $T_2$. From the above definitions, we obtain the following expressions:

$$T_1 T_2 = \begin{pmatrix}
T_1^{(0)} & T_1^{(0)T_2^{(00)}} & T_1^{(0)}T_2^{(01)} & \cdots \\
T_1^{(1)} & T_1^{(1)T_2^{(10)}} & T_1^{(1)T_2^{(11)}} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad T_2 T_1 = \begin{pmatrix}
T_2^{(00)}T_1^{(0)} & T_2^{(01)}T_1^{(1)} & \cdots \\
T_2^{(10)}T_1^{(0)} & T_2^{(11)}T_1^{(1)} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}, \quad (A.2)
$$

which imply the relations $T_1^{(0)}T_2^{(00)} = T_2^{(00)}T_1^{(0)}$, $T_1^{(0)}T_2^{(0m)} = T_2^{(0m)}T_1^{(m)}$, $T_1^{(m)}T_2^{(m0)} = T_2^{(m0)}T_1^{(0)}$ and $T_1^{(m)}T_2^{(mm')} = T_2^{(mm')}T_1^{(m')}$ for $m, m' = 1, \ldots, M$. From the second and third relations, we can derive that $T_2^{(0m)}$ and $T_2^{(m0)}$ vanish. To see this, let us examine $T_1^{(0)}T_2^{(0m)} = T_2^{(0m)}T_1^{(m)}$, which implies

$$\sum_{k=1}^{r^{(0)}} (T_1^{(0)})_{ik} (T_2^{(0m)})_{kj} = \sum_{k=1}^{2r^{(m)}} (T_2^{(0m)})_{ik} (T_1^{(m)})_{kj}, \quad (A.3)
$$

where $(i, j)$ elements of a matrix $A$ are denoted by $(A)_{ij}$. Since the elements of $T_1^{(0)}$ take $(T_1^{(0)})_{ik} = (-1)^{l_i} \delta_{ik}$ with $l_i \in \mathbb{Z}$, the above equation is rearranged as

$$\sum_{k=1}^{2r^{(m)}} (T_2^{(0m)})_{ik} \left[ (T_1^{(m)})_{kj} - (-1)^{l_i} \delta_{kj} \right] = 0. \quad (A.4)
$$

Let us define the $2r^{(m)} \times 2r^{(m)}$ matrix $\tilde{T}_1^{(m)}$ as $(\tilde{T}_1^{(m)})_{kj} = (T_1^{(m)})_{kj} - (-1)^{l_i} \delta_{kj}$, which is also expressed as follows:

$$\tilde{T}_1^{(m)} = \begin{pmatrix}
\cos \theta^{(m)} - (-1)^{l_i} & i \sin \theta^{(m)} \\
i \sin \theta^{(m)} & \cos \theta^{(m)} - (-1)^{l_i}
\end{pmatrix} \otimes I_{r^{(m)}}. \quad (A.5)
$$

Parameters such as $2r^{(m)}$ which represent the size of matrices are non-negative integers. The same applies to the following ones. Since $\cos \theta^{(m)} \neq \pm 1$, $\det \tilde{T}_1^{(m)} \neq 0$ holds. Thus, there exists the inverse matrix of $\tilde{T}_1^{(m)}$. Therefore, from eq. (A.4), we conclude $T_2^{(0m)} = 0$. One also finds $T_2^{(m0)} = 0$ through a similar examination with the relation $T_1^{(m)}T_2^{(m0)} = T_2^{(m0)}T_1^{(0)}$. 

26
A.2 Decomposition of $T_2^{(00)}$ into $2 \times 2$ submatrices

Next, let us focus on the relation $T_1^{(0)}T_2^{(00)} = T_2^{(00)}T_1^{(0)}$. We can choose a basis such that $R_0^{(0)}$ and $T_1^{(0)}$ are given by

$$R_0^{(0)} = \begin{pmatrix} R_0^{(0),1} & 0 \\ 0 & R_0^{(0),2} \end{pmatrix}, \quad T_1^{(0)} = \begin{pmatrix} T_1^{(0),1} & 0 \\ 0 & T_1^{(0),2} \end{pmatrix} = \begin{pmatrix} -I_r^{(0)} & 0 \\ 0 & I_r^{(0)} \end{pmatrix}, \quad (A.6)$$

where we have introduced $r_1^{(0)}$ and $r_2^{(0)}$ ($r_1^{(0)} + r_2^{(0)} = r^{(0)}$). The submatrix $R_0^{(0),a}$ ($a = 1, 2$) is diagonal and has eigenvalues 1 or $-1$. Then, from $T_1^{(0)}T_2^{(00)} = T_2^{(00)}T_1^{(0)}$, one sees that $T_2^{(00)}$ takes the block-diagonal form as

$$T_2^{(00)} = \begin{pmatrix} T_2^{(00),1} & 0 \\ 0 & T_2^{(00),2} \end{pmatrix}. \quad (A.7)$$

From the constraints $R_0^2 = (T_2R_0)^2 = I$ in eq. (A.11), we see that

$$(R_0^{(0),a})^2 = (T_2^{(00),a}R_0^{(0),a})^2 = I_{r_a}, \quad a = 1, 2. \quad (A.8)$$

If we make $R_0^{(0),a}$ and $T_2^{(00),a}$ correspond to $P_0$ and $T$ appeared in section 2, eq. (A.8) corresponds to eq. (2.1). In addition, since $T_1^{(0),a}$ are proportional to the unit matrices, if we take unitary transformations to simplify the form of $T_2^{(00),a}, T_1^{(0),a}$ is unchanged. Thus, as we have done in section 2 we can simplify the form of $T_2^{(00),a}$ keeping the diagonal form of $R_0^{(0),a}$. As a result, we obtain the simplified forms of the twist matrices $R_0^{(0),a}, T_1^{(0),a}$ and $T_2^{(00),a}$ as follows:

$$R_0^{(0),a} = \begin{pmatrix} R_0^{(0),a,0} & & \\ & R_0^{(0),a,1} & \\ & & \ddots & \\ & & & R_0^{(0),a,M_a} \end{pmatrix}, \quad R_0^{(0),a,m} = -\sigma_3 \otimes I_{r_a}, \quad (A.9)$$

$$T_1^{(0),a} = \begin{pmatrix} T_1^{(0),a,0} & & \\ & T_1^{(0),a,1} & \\ & & \ddots & \\ & & & T_1^{(0),a,M_a} \end{pmatrix}, \quad T_1^{(0),a,m} = (-1)^a I_2 \otimes I_{r_a}, \quad (A.10)$$

$$T_2^{(00),a} = \begin{pmatrix} T_2^{(00),a,0} & & \\ & T_2^{(00),a,1} & \\ & & \ddots & \\ & & & T_2^{(00),a,M_a} \end{pmatrix}, \quad T_2^{(00),a,m} = e^{i\phi^{(0),a,m} \sigma_1} \otimes I_{r_a}, \quad (A.11)$$

where $R_0^{(0),a,0}, T_1^{(0),a,0}$ and $T_2^{(00),a,0}$ are $r_a^{(0),0} \times r_a^{(0),0}$ diagonal matrices whose eigenvalues are 1 or $-1$. The real parameters $\phi^{(0),a,m}$ satisfy $0 < \phi^{(0),a,m} < \pi$ and $\phi^{(0),a,m} \neq \phi^{(0),a,m'}$ for $m \neq m'$. 

27
A.3 Block-diagonal form of $T_2$

Let us start to discuss $T_2^{(mm')}$ $(m,m' = 1,\ldots,M)$. First, we define

$$T_2^{(mm')} = \begin{pmatrix} (T_2^{(mm')})_{(11)} & (T_2^{(mm')})_{(12)} \\ (T_2^{(mm')})_{(21)} & (T_2^{(mm')})_{(22)} \end{pmatrix},$$

(A.12)

where $(T_2^{(mm')})_{(kl)}$ is an $r^{(m)} \times r^{(m')}$ matrix. Also, we can rewrite $T_2^{(mm')}$ by introducing $r^{(m)} \times r^{(m')}$ matrices $T_{2,\mu}^{(mm')}$ $(\mu = 0,1,2,3)$ as follows:

$$T_2^{(mm')} = \sum_{\mu=0}^{3} \sigma_\mu \otimes T_{2,\mu}^{(mm')},$$

(A.13)

where $\sigma_\mu = (\sigma_0,\sigma_i)$ and $\sigma_0 = I_2$. Since $T_1$ and $T_2$ commute with each other, it follows that

$$\left( e^{i\theta^{(m)}\sigma_1} - e^{-i\theta^{(m')\sigma_1}} \right) (\sigma_0 \otimes T_{2,0}^{(mm')} + \sigma_1 \otimes T_{2,1}^{(mm')})$$

$$+ \left( e^{i\theta^{(m)}\sigma_1} - e^{-i\theta^{(m')\sigma_1}} \right) (\sigma_2 \otimes T_{2,2}^{(mm')} + \sigma_3 \otimes T_{2,3}^{(mm')}) = 0.$$  

(A.14)

This equation implies $T_{2,\mu}^{(mm')} = 0$ for $m \neq m'$ and $T_{2,2}^{(mm')} = T_{2,3}^{(mm')} = 0$. Thus, $T_2$ takes the block-diagonal form. For convenience, we introduce the notation as $T_{2,0}^{(mm')} = A_2^{(m)}$ and $T_{2,1}^{(mm')} = B_2^{(m)}$. Then, $T_2$ is given by

$$T_2 = \begin{pmatrix} T_2^{(00)} \\ T_2^{(1)} \\ \cdots \\ T_2^{(M)} \end{pmatrix},$$

where

$$T_2^{(m)} = \sigma_0 \otimes A_2^{(m)} + \sigma_1 \otimes B_2^{(m)}.$$  

(A.15)

A.4 Decomposition of $T_2^{(m)}$ into $2 \times 2$ submatrices

From $(T_2 R_0)^2 = I$ in eq. (1.1), it follows that $(T_2 R_0)^\dagger = T_2 R_0$, which implies $(T_2^{(m)} R_0^{(m)})^\dagger = T_2^{(m)} R_0^{(m)}$. Using $R_0^{(m)} = -\sigma_3 \otimes I_{r^{(m)}}$ in eq. (1.4), we find $A_2^{(m)} = A_2^{(m)}$ and $B_2^{(m)} = -B_2^{(m)}$. Thus, by using a unitary transformation that keeps the forms of $R_0^{(m)}$ and $T_1^{(m)}$, we can diagonalize $A_2^{(m)}$ as $A_2^{(m)} \rightarrow \hat{A}_2^{(m)}$, where the element of $\hat{A}_2^{(m)}$ is given by $(\hat{A}_2^{(m)})_{ij} = \hat{a}_2^{(m)i} \delta_{ij}$ ($\hat{a}_2^{(m)i} \in \mathbb{R}$). In this new basis, let us redefine $B_2^{(m)}$ as $T_2^{(m)} = \sigma_0 \otimes \hat{A}_2^{(m)} + \sigma_1 \otimes B_2^{(m)}$. Then, from the condition $T_2 T_2^\dagger = I$, we get

$$\sigma_0 \otimes (I_{r^{(m)}} - \hat{A}_2^{(m)} \hat{A}_2^{(m)} + B_2^{(m)} B_2^{(m)}) + \sigma_1 \otimes (\hat{A}_2^{(m)} B_2^{(m)} - B_2^{(m)} \hat{A}_2^{(m)}) = 0.$$  

(A.16)

The term proportional to $\sigma_1$ in the above implies $(\hat{a}_2^{(m)i} - \bar{a}_2^{(m)j})(B_2^{(m)})_{ij} = 0$. Thus, using a unitary transformation that trivially acts on $\sigma_0$ and $\sigma_1$ in eq. (A.15), we can move to a basis
such that $R_0^{(m)}$, $T_1^{(m)}$, and $T_2^{(m)}$ take the block-diagonal forms as

\[
R_0^{(m)} = \begin{pmatrix}
R_0^{(m),1} & & \\
& R_0^{(m),2} & \\
& & \ddots & \\
& & & R_0^{(m),M(m)}
\end{pmatrix}, \quad R_0^{(m),m'} = -\sigma_3 \otimes I_{r(m),m'}, \quad (A.17)
\]

\[
T_1^{(m)} = \begin{pmatrix}
T_1^{(m),1} & & \\
& T_1^{(m),2} & \\
& & \ddots & \\
& & & T_1^{(m),M(m)}
\end{pmatrix}, \quad T_1^{(m),m'} = e^{i\phi(m)\sigma_1} \otimes I_{r(m),m'}, \quad (A.18)
\]

\[
T_2^{(m)} = \begin{pmatrix}
T_2^{(m),1} & & \\
& T_2^{(m),2} & \\
& & \ddots & \\
& & & T_2^{(m),M(m)}
\end{pmatrix}. \quad (A.19)
\]

The submatrices of $T_2^{(m)}$ in eq. (A.19) can be written more explicitly as

\[
T_2^{(m),m'} = a_2^{(m),m'} \sigma_0 \otimes I_{r(m),m'} + \sigma_1 \otimes B_2^{(m),m'}, \quad (A.20)
\]

where $a_2^{(m),m'} \in \mathbb{R}$ and $B_2^{(m),m'}$ is an $r(m) \times r(m)$ matrix, which satisfies $B_2^{(m),m'} = -B_2^{(m),m'}$. Now, we again use the condition $T_2 T_2^\dagger = I$ to get $(1 - a_2^{(m),m'}^2) I_{r(m),m'} = B_2^{(m),m'} B_2^{(m),m'}^\dagger$, which implies $a_2^{(m),m'}^2 \leq 1$. From the above, $B_2^{(m),m'}$ is rewritten by using a unitary hermitian matrix $U^{(m),m'}$, where $U^{(m),m'} U^{(m),m'}^\dagger = U^{(m),m'} U^{(m),m'} = I_{r(m),m'}$, as $B_2^{(m),m'} = i \sqrt{1 - a_2^{(m),m'}^2} U^{(m),m'}$. Keeping the forms of $R_0$ and $T_1$ in eqs. (A.17) and (A.18), we can diagonalize $U^{(m),m'}$ by the basis change as $T_2^{(m),m'} \rightarrow (\sigma_0 \otimes W^{(m),m'}) T_2^{(m),m'} (\sigma_0 \otimes W^{(m),m'})^\dagger$, where $W^{(m),m'} U^{(m),m'} W^{(m),m'}$ is chosen to be a diagonal matrix. Note that the possible eigenvalues of $U^{(m),m'}$ are $\pm 1$. Thus, by introducing a parameter $0 \leq \phi(m),m' < 2\pi$, $T_2^{(m),m'}$ is written as

\[
T_2^{(m),m'} = \begin{pmatrix}
\cos \phi(m),m' I_{r(m),m'} & i \sin \phi(m),m' I_{r(m),m'} \\
i \sin \phi(m),m' I_{r(m),m'} & \cos \phi(m),m' I_{r(m),m'}
\end{pmatrix} = e^{i\phi(m),m' \sigma_1} \otimes I_{r(m),m'}. \quad (A.21)
\]

**B  Derivation of block-diagonal forms on $T^2/\mathbb{Z}_3$**

Here, we show the details of the calculation for the $T^2/\mathbb{Z}_3$ orbifold. As discussed at the beginning of section 3, we choose $R_0$ and $T_1$ as the independent twist matrices, and work on a basis where these are written as

\[
R_0 = \begin{pmatrix}
\omega I_{n_1} \\
\omega^2 I_{n_2} \\
I_{n_3}
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
(T_1)_{(11)} & (T_1)_{(12)} & (T_1)_{(13)} \\
(T_1)_{(21)} & (T_1)_{(22)} & (T_1)_{(23)} \\
(T_1)_{(31)} & (T_1)_{(32)} & (T_1)_{(33)}
\end{pmatrix}, \quad (T_1)_{(kl)} = M_{k}^{l}. \quad (B.1)
\]
Here, \( n_k \) (\( k = 1, 2, 3 \)) is a non-negative integer, \( M_{kl}^{[k-l]} \) are \( n_k \times n_l \) matrices, and we adopt a notation of \( M_{kl}^{[k-l]} = M_{kl}^{[k'-l']} = M_{k' l'}^{[k-l]} \) with \( k' = k \) (mod 3) and \( l' = l \) (mod 3).

### B.1 \( R^3_a = I \)

Now, we investigate the conditions \( R^3_a = I \) (\( a = 0, 1, 2 \)). It is trivial for \( a = 0 \). For \( a = 1, 2 \), it is convenient to examine \( R^1_1 = R^2_2 \) and \( R_2 = R_2^{12} \) as \( R^3_a \) is rather complicated. With the help of the relations \( R_1 = T_1 R_{0} \) and \( R_2 = R_{0}^{-1} T_1^{-1} R_{0}^{-1} \) shown in eq. (5.1), these conditions lead to expressions of \( T^\dagger_1 \): \( T^\dagger_1 = R_0 T_1 R_0 T_1 R_{0} \), which is derived also from \( T_1 T_2 T_3 = T_1 T_3 T_2 = I \). Since \( (T^\dagger_1)_{(k-k-q)} = (T_1)_{(k-q-k)}^\dagger \), we find

\[
M_{k-k-q}^{[-q]^\dagger} = \sum_{q'} \omega^k M_{k-k+q^\prime}^{[-q]} \omega^{k+q^\prime} M_{k-k+q}^{[-q]} \omega^{k-q} = \sum_{q'} \omega^{q^\prime-q} M_{k-k+q^\prime}^{[-q]} M_{k-k+q}^{[q+q']} \omega^{k-q} = \sum_{q'} \omega^{q^\prime-q} M_{k-k+q^\prime}^{[-q]} M_{k-k+q}^{[q+q']},
\]

where the summation over \( q' \) can be taken for any successive three integers. The equality \( \sum_{q'} \omega^{q^\prime-q} M_{k-k+q^\prime}^{[-q]} M_{k-k+q}^{[q+q']} = \sum_{q'} \omega^{q^\prime-q} M_{k-k+q^\prime}^{[-q]} M_{k-k+q}^{[q+q']} \) requests that, with \( \omega = \omega^2 \)

\[
(\omega - \bar{\omega}) M_{k-k+1}^{[-1]} M_{k-k}^{[1]} + (\bar{\omega} - \omega) M_{k-k+1}^{[1]} M_{k-k}^{[-1]} = 0, \quad \text{for } q = 0, \tag{B.3}
\]
\[
(\omega - \bar{\omega}) M_{k-k+1}^{[0]} M_{k-k}^{[0]} + (\bar{\omega} - \omega) M_{k-k+1}^{[0]} M_{k-k}^{[0]} = 0, \quad \text{for } q = 1, \tag{B.4}
\]
\[
(\omega - \bar{\omega}) M_{k-k+1}^{[-1]} M_{k-k}^{[-1]} + (\bar{\omega} - \omega) M_{k-k+1}^{[-1]} M_{k-k}^{[-1]} = 0, \quad \text{for } q = -1. \tag{B.5}
\]

These are summarized as

\[
M_{k-k-q}^{[-q]} M_{k-k-q}^{[q]} = M_{k-k+q}^{[-q]} M_{k-k+q}^{[q]}, \tag{B.6}
\]
\[
M_{k-k-q}^{[q]} M_{k-k-q}^{[-q]} = M_{k-k+q}^{[q]} M_{k-k+q}^{[-q]}, \tag{B.7}
\]

which happen to hold also for \( q = 0 \) (trivially), and are further summarized in eq. (E.3).

Note that the second relations indicate that \( M_{k-k}^{[0]} \) commutes with a product \( M_{k-k+q}^{[-q]} M_{k-k+q}^{[q]} \), which appears in the expression of \( M_{k-k}^{[0]} \) in eq. (B.2) for \( q = 0 \). This means that each \( M_{k-k}^{[0]} \) commutes with its dagger, \( [M^{[0]}_{k-k}, M^{[0]}_{k-k}] = 0 \), to be a normal matrix. Then \( M_{k-k}^{[0]} \) can be diagonalized by a unitary transformation.

### B.2 Block-diagonal form

Since the unitary transformation that diagonalizes \( M^{[0]}_{k-k} \) does not modify \( R_0 \), we may move to a basis with \( (M^{[0]}_{k-k})_{ij} = a^i_k \delta_{ij} \). In this basis, the requirement in eq. (B.7) is written as

\[
(a^i_k - a^i_{k-q})(M^{[q]}_{k-k-q})_{ij} = 0, \tag{B.8}
\]
leading to \((M[\hat{q}])_{ij} = 0\) if \(a_k^i \neq a_k^j\). Thus, we see that the unitary matrices can be block-diagonalized as,

\[
R_0 = \begin{pmatrix}
R_0^{(1)} & \cdots & R_0^{(M)} \\
\end{pmatrix}, \quad R_0^{(m)} = \begin{pmatrix}
(R_0^{(0)})^{(11)} & (R_0^{(0)})^{(12)} & (R_0^{(0)})^{(13)} \\
(R_0^{(0)})^{(21)} & (R_0^{(0)})^{(22)} & (R_0^{(0)})^{(23)} \\
(R_0^{(0)})^{(31)} & (R_0^{(0)})^{(32)} & (R_0^{(0)})^{(33)} \\
\end{pmatrix},
\]

\[
T_1 = \begin{pmatrix}
T_1^{(1)} & \cdots & T_1^{(M)} \\
\end{pmatrix}, \quad T_1^{(m)} = \begin{pmatrix}
(T_1^{(0)})^{(11)} & (T_1^{(0)})^{(12)} & (T_1^{(0)})^{(13)} \\
(T_1^{(0)})^{(21)} & (T_1^{(0)})^{(22)} & (T_1^{(0)})^{(23)} \\
(T_1^{(0)})^{(31)} & (T_1^{(0)})^{(32)} & (T_1^{(0)})^{(33)} \\
\end{pmatrix},
\]

with

\[
(R_0^{(m)})_{kl} = \omega^k \delta_{kl} I_{n_k^{(m)}}, \quad (T_1^{(m)})_{(k-k)} = M_{k-k}^{[m]} \quad M_{k-k}^{(m)} = a^{(m)} I_{n_k^{(m)}}, \quad (B.9)
\]

where \(n_k^{(m)} \quad (m = 1, 2, \ldots, M)\) are non-negative integers, as before. To be more concrete,

\[
R_0^{(m)} = \begin{pmatrix}
\omega I_{n_k^{(m)}} & \omega^2 I_{n_k^{(m)}} \\
I_{n_k^{(m)}} & I_{n_k^{(m)}} \\
\end{pmatrix}, \quad T_1^{(m)} = \begin{pmatrix}
a^{(m)} I_{n_k^{(m)}} & M_{12}^{[1]} \\
M_{21}^{[1]} & a^{(m)} I_{n_k^{(m)}} \\
M_{31}^{[1]} & M_{32}^{[1]} & a^{(m)} I_{n_k^{(m)}} \\
\end{pmatrix},
\]

where parameters \(a^{(m)}\) satisfy \(a^{(m)} \neq a^{(m')}\) for \(m \neq m'\).

For later convenience, we rewrite the relation in eq. \[(B.2)\] as

\[
\overline{a^{(m)}} I_{n_k^{(m)}} = a^{(m)2} I_{n_k^{(m)}} - M_{k-k}^{[m][1]} M_{k-k}^{[1]}, \quad \text{for } q = 0, \quad (B.13)
\]

\[
M_{k-k}^{(m)[q][q']} = -a^{(m)} M_{k-k}^{(m)[q]} + M_{k-k}^{(m)[q]} M_{k-k}^{(m)[q']}, \quad \text{for } q = \pm 1. \quad (B.14)
\]

### B.3 \(T_1 T_1^\dagger = T_1^\dagger T_1 = I\)

Next, we examine a unitarity condition \(T_1 T_1^\dagger = I\), which implies for each block as

\[
(T_1^{(m)} T_1^{(m)^\dagger})_{(k-k)} = \sum_{q'} M_{k-k+q}^{(m)[q]} M_{k-k}^{(m)[q]} = \delta_{q0} I_{n_k^{(m)}}, \quad \text{for } q = 0, \quad (B.15)
\]

To be more concrete,

\[
M_{k-k}^{(m)[1]} M_{k-k}^{(m)[1]} + M_{k-k}^{(m)[1]} M_{k-k}^{(m)[1]} = (1 - |a^{(m)}|^2) I_{n_k^{(m)}}, \quad \text{for } q = 0, \quad (B.16)
\]

\[
a^{(m)} M_{k-k}^{(m)[q]} + \overline{a^{(m)}} M_{k-k}^{(m)[q]} + M_{k-k}^{(m)[q]} M_{k-k}^{(m)[q']} = 0, \quad \text{for } q = \pm 1. \quad (B.17)
\]

The condition for \(q = 0\) in eq. \[(B.16)\] shows that \(M_{k-k}^{(m)[1]} \) (double sign in the same order) vanishes when \(|a^{(m)}| = 1\). In this case, \(T_1^{(m)}\) is already diagonal, i.e., \((T_1^{(m)})_{kl} = a^{(m)} \delta_{kl} I_{n_k^{(m)}}\), \((|a^{(m)}| = 1)\), and thus we work on the case \(|a^{(m)}| < 1\) below.

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B.3.1 \(0 < |a^{(m)}| < 1\)

In the case with \(0 < |a^{(m)}| < 1\), the relation in eq. (B.13) tells us that \(M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger}\) is proportional to \(I_{n_1^{(m)}}\) with a non-vanishing proportional constant. This indicates that 
\[
n_2^{(m)} = \text{rank}(M_{12}^{(m)[1]} M_{31}^{(m)[1]^\dagger}) \leq \text{rank}(M_{31}^{(m)[1]^\dagger}) \leq n_3^{(m)} = \text{rank}(M_{12}^{(m)[1]} M_{23}^{(m)[1]^\dagger}) \leq n_1^{(m)} = \text{rank}(M_{12}^{(m)[1]} M_{23}^{(m)[1]^\dagger}),
\]
and thus that \(n_1^{(m)} = n_2^{(m)} = n_3^{(m)} (\equiv r^{(m)})\). Then, \(M_{k+q}^{(m)[q]}\) is an \(r^{(m)} \times r^{(m)}\) matrix of rank \(r^{(m)}\) and thus is invertible, and the above proportional relation shows \(M_{k+1}^{(m)} \propto (M_{k+1}^{(m)[1]}\)^{-1}.

Here, we note that a unitary transformation that diagonalizes a hermitian matrix \(M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger}\) does not modify \(R_0^{(m)}\) nor \(M_{k+1}^{(m)[0]}\) and thus we can operate it freely. After the transformation, the relation in eq. (B.16) tells us that \(M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger}\) is also diagonalized simultaneously. Since \(M_{k+1}^{(m)[1]} \propto (M_{k+1}^{(m)[1]}\)^{-1}, we can express each \(M_{k+1}^{(m)[1]}\) by a product of a diagonal matrix \(\hat{M}_k^{(m)[1]}\) with positive diagonal elements and a possible unitary matrix \(U^{(m)}\) that commutes with \(\hat{M}_k^{(m)[1]}\), in this basis. Note that \(U^{(m)}\) is a diagonal phase matrix if \(\hat{M}_k^{(m)[1]}\) has no degeneracy. We express \(M_{k+1}^{(m)[1]} = \hat{M}_k^{(m)[1]} U^{(m)}\) with \(\hat{M}_k^{(m)[1]} \propto (\hat{M}_k^{(m)[1]}\)^{-1}.

In order to find constraints on \(\hat{M}_k^{(m)[1]}\) and \(U^{(m)}\), let us examine \(M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger}\) and \(M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger}\). The relations in eq. (B.14) show that they are written as
\[
M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger} = M_{k+1}^{(m)[1]} (-a^{(m)} M_{k+1}^{(m)[1]^\dagger} + M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]}), \\
M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger} = (-a^{(m)} M_{k+1}^{(m)[1]} + M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]}),
\]
whose last terms in the two right-hand sides are common apparently. In addition, the first terms in the right-hand sides are also common due to the relation in eq. (B.6), and thus we find \(\hat{M}_k^{(m)[1]} = \hat{M}_k^{(m)[1]}\). Since this holds for all \(k\), \(\hat{M}_k^{(m)[1]}\) are common for all \(k\), and thus the possible unitary matrix \(U^{(m)}\) commutes with \(\hat{M}_k^{(m)[1]}\).

Note that \(M_{k+1}^{(m)[1]} U^{(m)} = M_{k+1}^{(m)[1]}\), in addition to \(M_{k+1}^{(m)[1]} U^{(m)} = M_{k+1}^{(m)[1]}\), is diagonal. Then, the relation in eq. (B.14) requires that \(M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger} U^{(m)} = M_{k+1}^{(m)[1]} M_{k+1}^{(m)[1]^\dagger} U^{(m)}\) is also diagonal, and so is the combination \(U^{(m)} U^{(m)} U^{(m)}\). In particular, we set \(U^{(m)} U^{(m)} U^{(m)} = \Theta^{(m)}\) where \(\Theta^{(m)}\) is a diagonal phase matrix. Using this observation, we see that the following unitary transformation of \(T_1^{(m)}\), which does not change \(R_0^{(m)}\), diagonalizes \(M_{k+q}^{(m)[q]}\):
\[
T_1^{(m)} = \begin{pmatrix}
a^{(m)} I_{r_1^{(m)}} \\
M_{1}^{(m)[1]} U_{1}^{(m)} \\
M_{1}^{(m)[1]} U_{1}^{(m)} \\
\end{pmatrix}, \quad
V^{(m)} T_1^{(m)} V^{(m)} = \begin{pmatrix}
a^{(m)} I_{r_1^{(m)}} \\
M_{1}^{(m)[1]} \Theta^{(m)} \\
M_{1}^{(m)[1]} \Theta^{(m)} \\
\end{pmatrix}.
\]
with \( V^{(m)} = \left( \hat{\Theta}^{(m)} U_2^{(m)} \right) \),

\[
\text{B.21}
\]

where \( k \) in \( M^{[m][\geq 1]}_k \) is omitted in \( V^{(m)} T_1^{(m)} V^{(m)\dagger} \) because \( M^{[m][\geq 1]}_k \) are independent of \( k \).

#### B.3.2 \( a^{(m)} = 0 \)

In the case with \( a^{(m)} = 0 \), the conditions in eqs. \( \text{(B.16)} \) and \( \text{(B.17)} \) are simplified as

\[
M^{(m)[1]}_{k-1} M^{(m)[1]\dagger} + M^{(m)[-1]}_{k+1} M^{(m)[-1]\dagger} = I_{n_k^{(m)}}, \quad M^{(m)[-q]}_{k+q} M^{(m)[q]\dagger} = 0,
\]

where \( q = \pm 1 \). Similarly, the conditions from \( T_1^\dagger T_1 = I \) are given as

\[
M^{(m)[1]\dagger} M^{(m)[1]} + M^{(m)[-1]\dagger} M^{(m)[-1]} = I_{n_k^{(m)}}, \quad M^{(m)[-q]\dagger} M^{(m)[q]} = 0.
\]

In addition, those in eqs. \( \text{(B.13)} \) and \( \text{(B.14)} \) are

\[
M^{(m)[1]}_{k-1} M^{(m)[-1]}_{k-1} = 0, \quad M^{(m)[-q]}_{k-q} M^{(m)[q]}_{k+q-1} = 0.
\]

Multiplying \( M^{(m)[1]}_{k-1} \) from the right to the first condition in eq. \( \text{(B.22)} \), we get

\[
M^{(m)[1]\dagger} M^{(m)[1]}_{k-1} M^{(m)[1]}_{k-1} = M^{(m)[1]}_{k-1},
\]

as \( M^{(m)[-1]\dagger} M^{(m)[1]}_{k-1} = 0 \). This indicates that, in the basis where \( M^{(m)[1]}_{k-1} M^{(m)[1]\dagger}_{k-1} \) are diagonal, which can be chosen without loss of generality, they can be written as

\[
M^{(m)[1]}_{k-1} M^{(m)[1]\dagger}_{k-1} = \begin{pmatrix}
I_{r_k^{(m)}} & 0 \\
0 & 0
\end{pmatrix},
\]

and then

\[
M^{(m)[-1]}_{k+1} M^{(m)[-1]\dagger}_{k+1} = \begin{pmatrix}
0 & I_{n_k^{(m)} - r_k^{(m)}} \\
I_{n_k^{(m)} - r_k^{(m)}} & 0
\end{pmatrix},
\]

where \( r_k^{(m)} \) are the rank of \( M^{(m)[1]}_{k-1} \). We note that \( \text{rank}(M^{(m)[-1]}_{k+1}) = n_k^{(m)} - r_k^{(m)} \).

The linear algebra tells us that the first conditions in eq. \( \text{(B.24)} \) imply that \( \text{rank}(M^{(m)[1]}_{k-1}) + \text{rank}(M^{(m)[-1]}_{k+1}) - n_{k-1}^{(m)} < \text{rank}(0) = 0 \), and thus \( 0 > r_k^{(m)} + n_k^{(m)} - r_{k-1}^{(m)} = r_k^{(m)} - r_{k-1}^{(m)} \). This means \( r_k^{(m)} < r_{k-1}^{(m)} < r_{k-2}^{(m)} = r_{k+1}^{(m)} < r_k^{(m)} \) and thus \( r_k^{(m)} \) are independent of \( k \).

Since the above discussion also holds after flip the sign of \( q \) in \( M^{(m)[q]}_{k+q} \), we see that \( \text{rank}(M^{(m)[-1]}_{k+1}) = n_k^{(m)} - r_k^{(m)} \), and thus \( n_k^{(m)} \) are also independent of \( k \).

Now, we discuss \( M^{(m)[-q]}_{k+q} M^{(m)[-q]}_{k+q} \). As in the previous \( 0 < |a^{(m)}| < 1 \) case, the second conditions in eq. \( \text{(B.24)} \) derive the conditions in eqs. \( \text{(B.18)} \) and \( \text{(B.19)} \) with \( a^{(m)} = 0 \). In this
case, the right hand sides of these equations are common and we see that \( M_{k+1,k}^{(m)[1]} \) and \( M_{k+1,k}^{(m)[1]} \) are diagonal from eq. (B.26), and thus from the first conditions in eq. (B.23), \( M_{k-1,k}^{(m)[-1]} \) and \( M_{k-1,k}^{(m)[-1]} \) are also diagonal in the present basis. This means that we can write as

\[
M_{k,k-1}^{(m)[1]} = \begin{pmatrix} U_k^{(m)} \\ 0 \end{pmatrix}, \quad M_{k-1,k}^{(m)[-1]} = \begin{pmatrix} 0 \\ U_k^{(m)[1]} \end{pmatrix},
\]

where \( U_k^{(m)} \) and \( U_k^{(m)[1]} \) are \( r_k \times r_k \) and \( (n_k^{(m)} - r_k) \times (n_k^{(m)} - r_k) \) unitary matrices, respectively.

Here, we can rearrange \( T_1^{(m)} \) (and of course also \( R_0^{(m)} \)) into two block-diagonal parts where both the diagonal blocks are written in a similar form as in eq. (B.12), but one with \( a^{(m)} = 0 \) and \( M_{k-1,k}^{(m)[-1]} = 0 \) and the other one with \( a^{(m)} = 0 \) and \( M_{k,k-1}^{(m)[1]} = 0 \).

The degrees of freedom of the unitary matrices \( U_k^{(m)} \) and \( U_k^{(m)[1]} \) are treated in a similar way as in the previous case discussed above eq. (B.20). At the end, we get the same form as in the last matrix in eq. (B.20) but with constraints of \( a^{(m)} = 0 \), \( \hat{M}^{(m)[q]} = I_{r(m)} \) and \( \hat{M}^{(m)[-q]} = 0 \) where \( q = \pm 1 \).

## C Derivation of block-diagonal forms on \( T^2/\mathbb{Z}_4 \)

In this appendix, we explain the details of block-diagonalization of twist matrices on \( T^2/\mathbb{Z}_4 \) given in section 3.

We can start with the following \( R_0 \) and \( T_1 \), without loss of generality,

\[
R_0 = \begin{pmatrix}
i I_{n_1} & -I_{n_2} \\
-I_{n_2} & I_{n_3} \
                    & I_{n_4}
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
(T_1)_{(11)} & (T_1)_{(12)} & (T_1)_{(13)} & (T_1)_{(14)} \\
(T_1)_{(21)} & (T_1)_{(22)} & (T_1)_{(23)} & (T_1)_{(24)} \\
(T_1)_{(31)} & (T_1)_{(32)} & (T_1)_{(33)} & (T_1)_{(34)} \\
(T_1)_{(41)} & (T_1)_{(42)} & (T_1)_{(43)} & (T_1)_{(44)}
\end{pmatrix},
\]

where \( I_{n_k} \) are \( n_k \times n_k \) unit matrices and \( (T_1)_{(kl)} \) are \( n_k \times n_l \) submatrices. We denote the submatrices as \( (T_1)_{(kl)} = M_{k,l}^{[k-l]} \) and use a notation of \( M_{k,l}^{[k-l]} = M_{k,l}^{[k'-l']} = M_{k',l'}^{[k-l]} \) with \( k = k' \) (mod 4) and \( l = l' \) (mod 4). The upper index \( k-l = q \) represents the charge of the \( \mathbb{Z}_4 \) symmetry generated by \( R_0: (R_0T_1R_0^{-1})_{(k,l)} = i^q M_{k,k-l}^{[q]} \). Parameters such as \( n_k \) which represent the size of matrices are non-negative integers. The same applies to the following ones.

### C.1 Block-diagonalization of \( T_1 \) and relations on submatrices

First, let us restrict the form of \( T_1 \) and derive relations on submatrices of \( T_1 \), using constraints \( T_1^\dagger = T_1^{-1}, T_m^\dagger T_m = T_m T_m^\dagger \) and \( T_1 T_3 = I \) where \( T_m = R_0^{m-1} T_1 R_0^{l} \) as explained in section 3.

From \( T_1^\dagger = T_1^{-1} = T_3 = R_0^2 T_1 R_0^{-2} \), we derive the relation:

\[
M_{k-k,q}^{[-q]} = (-1)^q M_{k,k-q}^{[q]}.
\]
Taking $q = 0$ in eq. (C.2), we find that $M^{[0]}_{k,k} = M^{[0]}_{k,k}$, and $M^{[0]}_{k,k}$ is diagonalized as $(M^{[0]}_{k,k})_{ij} = a^i_k \delta_{ij}$ ($a^i_k \in \mathbb{R}$) by a unitary transformation, without modifying $R_0$, where $(M^{[0]}_{k,k})_{ij}$ are $(i,j)$ elements of $M^{[0]}_{k,k}$.

As shown in appendix F from $T_m' T_m = T_m T_m'$, i.e., $T_1 R_0^{m-m'} T_1 = R_0^{m-m'} T_1 R_0^{m-m'} T_1 R_0^{m-m'}$, we derive the relation:

$$M[q'] M[q-q'] = M[q-q'] M[q']$$

(C.3)

Setting $q' = 0$ in eq. (C.3) and using $(M^{[0]}_{k,k})_{ij} = a^i_k \delta_{ij}$, we obtain the relation:

$$(a^i_k - a^i_{k-q})(M^{[q]}_{k,k})_{ij} = 0,$$

(C.4)

and rearrange $T_1$ to be a block-diagonal matrix, while keeping $R_0$ a diagonal one:

$$R_0 = \begin{pmatrix} R_0^{(1)} & 0 & \cdots & 0 \\ 0 & R_0^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_0^{(M)} \end{pmatrix} , \quad T_1 = \begin{pmatrix} T_1^{(1)} & 0 & \cdots & 0 \\ 0 & T_1^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_1^{(M)} \end{pmatrix} ,$$

(C.5)

where $R_0^{(m)}$ and $T_1^{(m)}$ $(m = 1, 2, \cdots, M)$ are $n^{(m)} \times n^{(m)}$ matrices given by

$$R_0^{(m)} = \begin{pmatrix} i I^{(m)}_{n_1} & 0 & \cdots & 0 \\ 0 & -I^{(m)}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I^{(m)}_{n_4} \end{pmatrix} ,$$

(C.6)

and

$$T_1^{(m)} = \begin{pmatrix} a^{(m)} I^{(m)}_{n_1} & M^{(m)}_{12} & M^{(m)}_{13} & M^{(m)}_{14} \\ M^{(m)}_{21} & a^{(m)} I^{(m)}_{n_2} & M^{(m)}_{23} & M^{(m)}_{24} \\ M^{(m)}_{31} & M^{(m)}_{32} & a^{(m)} I^{(m)}_{n_3} & M^{(m)}_{34} \\ M^{(m)}_{41} & M^{(m)}_{42} & M^{(m)}_{43} & a^{(m)} I^{(m)}_{n_4} \end{pmatrix} ,$$

(C.7)

respectively. Here, $n^{(m)} = \sum_{k=1}^{4} n_k^{(m)}$ and submatrices in $T_1^{(m)}$ are denoted as $(T_1^{(m)})_{kl} = M^{(m)}_{k,l}$ for $k \neq l$ and $(T_1^{(m)})_{kk} = M^{(m)}_{k,k} = a^{(m)} I_{n_k^{(m)}}$. Real parameters $a^{(m)}$ satisfy $a^{(m)} \neq a^{(m')}$ for $m \neq m'$. Submatrices obey the following relations that come from $T_1^{(1)} = T_3$ and $T_m' T_m = T_m T_m'$,

$$M^{(m)}_{k-k} = (-1)^q M^{(m)}_{k-k} ,$$

(C.8)

$$M^{(m)}_{k-k} M^{(m)}_{k-q} = M^{(m)}_{k-k} M^{(m)}_{k-q} .$$

(C.9)
From $T_1 T_3 = T_1 R_0^2 T_1 R_0^{-2} = I$, we derive the relation:

$$
\sum_{q'} (-1)^{q-q'} M_{k,k-q}^{(m)[q']} M_{k-q,k-q}^{(m)[q-q']} = \delta_{k,k} I_{n_k^{(m)}},
$$

(C.10)

where the summation over $q'$ can be taken for any successive four integers. Setting $q = 0$ and $q = 2$ in eq. (C.10) and using eqs. (C.8) and (C.9), we obtain the relations:

$$
2M_{k,k-1}^{(m)[1]} M_{k,k-1}^{(m)[1]\dagger} + M_{k,k-2}^{(m)[2]} M_{k,k-2}^{(m)[2]\dagger} = (1 - a^{(m)^2}) I_{n_k^{(m)}}, \quad \text{for } q = 0, \quad (C.11)
$$

$$
2a^{(m)} M_{k,k-2}^{(m)[2]} = M_{k,k-1}^{(m)[1]} M_{k-1,k-2}^{(m)[1]} + M_{k,k+1}^{(m)[1]} M_{k+1,k-2}^{(m)[1]}, \quad \text{for } q = 2. \quad (C.12)
$$

The relations obtained by setting $q = 1$ and $q = 3$ in eq. (C.10) are not independent of eq. (C.9), because they are also derived from linear combinations of eq. (C.9). From eq. (C.11), we find that $0 \leq a^{(m)^2} \leq 1$. Note that $T_1^{(m)}$ is already diagonal for $a^{(m)} = \pm 1$, and then $R_0$ and $T_1$ contain $n^{(m)} \times n^{(m)}$ diagonal matrices whose submatrices are given by $(R_0^{(m)})_{(kl)} = i^k \delta_{kl} I_{n_k^{(m)}}$ and $(T_1^{(m)})_{(kl)} = \pm \delta_{kl} I_{n_k^{(m)}}$ for $a^{(m)} = \pm 1$ (double sign in the same order) as members of block-diagonal matrices. Hereafter, we focus on $-1 < a^{(m)} < 1$.

### C.2 Restriction of $M_{k,l}^{(m)[k-l]}$

Next, let us restrict the form of $M_{k,l}^{(m)[k-l]}$, using relations obtained in the previous subsection.

For an arbitrary complex matrix $A$ with a rank $r$, $AA^\dagger$ is a hermitian matrix, and is rearranged to be the block-diagonal form of $(AA^\dagger)_{D}^{\oplus 0}$ where $(AA^\dagger)_{D}$ is an $r \times r$ diagonal matrix with positive elements and $0$ is a submatrix whose elements are zero, after a diagonalization by a unitary transformation. Then $(AA^\dagger)_{D} = (A^\dagger A)_{D}$ holds, and $A$ is written as $A = \hat{A}_r U \oplus 0$, where $\hat{A}_r$ is a diagonal $r \times r$ matrix with non-vanishing real positive elements and $U$ is an $r \times r$ unitary matrix which commutes with $\hat{A}_r$, i.e., $[\hat{A}_r, U] = 0$, as seen from appendix E.

Using eq. (C.8), eq. (C.9) with $q = 0$ is rewritten as

$$
M_{k,k-q}^{(m)[q']} M_{k,k-q}^{(m)[q']\dagger} = M_{k+q,k}^{(m)[q']} M_{k+q,k}^{(m)[q']\dagger}, \quad (C.13)
$$

and the following relation on the rank of submatrices is obtained

$$
\begin{align*}
\text{rank}(M_{k,k-q}^{(m)[q']}) &= \text{rank}(M_{k,k-q}^{(m)[q']\dagger}) = \text{rank}(M_{k,k-q}^{(m)[q']} M_{k,k-q}^{(m)[q']\dagger}) \\
&= \text{rank}(M_{k+q,k}^{(m)[q']} M_{k+q,k}^{(m)[q']\dagger}) = \text{rank}(M_{k+q,k}^{(m)[q']}).
\end{align*}
$$

(C.14)

As seen from eq. (C.11), $M_{k,k-1}^{(m)[1]} M_{k,k-1}^{(m)[1]\dagger}$ and $M_{k,k-2}^{(m)[2]} M_{k,k-2}^{(m)[2]\dagger}$ are diagonalized simultaneously by some unitary matrices $W_{k}^{(m)}$ such that

$$
\left( W_{k}^{(m)} M_{k,k-q}^{(m)[q']} M_{k,k-q}^{(m)[q']\dagger} W_{k}^{(m)} \right)_{R} = \left( M_{k,k-q}^{(m)[q']} M_{k,k-q}^{(m)[q']\dagger} \right)_{D} \oplus 0, \quad (q' = 1, 2),
$$

(C.15)
where \((\cdots)_R\) means that a suitable rearrangement by interchanges of rows and/or those of columns has been done. From eq. (C.13), we find that \(M^{(m)[1]}_{k+1 k} M^{(m)[1]}_{k+1 k}\) and \(M^{(m)[2]}_{k+2 k} M^{(m)[2]}_{k+2 k}\) are also diagonalized simultaneously by \(W^{(m)}_k\) such that

\[
(W^{(m)}_k M^{(m)[q']\dagger}_{k+q' k} M^{(m)[q']}_{k+q' k} W^{(m)}_k)^R = (M^{(m)[q']\dagger}_{k+q' k} M^{(m)[q']}_{k+q' k})'_D \oplus 0, \quad (q' = 1, 2),
\]

and obtain the relation:

\[
(M^{(m)[q']}_{k k-q'} M^{(m)[q']}_{k k-q'})'_D = (M^{(m)[q']\dagger}_{k+q' k} M^{(m)[q']}_{k+q' k})'_D \quad (q' = 1, 2).
\]

Replacing \(k\) with \(k + q'\) in eq. (C.15) and \(k\) with \(k - q'\) in eq. (C.16), we derive the relations:

\[
(W^{(m)}_{k+q'} M^{(m)[q']}_{k+q' k} M^{(m)[q']}_{k+q' k} W^{(m)}_{k+q'})^R = (M^{(m)[q']}_{k+q' k} M^{(m)[q']}_{k+q' k})'_D \oplus 0, \quad (q' = 1, 2),
\]

and

\[
(W^{(m)}_{k-q'} M^{(m)[q']}_{k-q' k} M^{(m)[q']}_{k-q' k} W^{(m)}_{k-q'})^R = (M^{(m)[q']\dagger}_{k-k-q'} M^{(m)[q']}_{k-k-q'})'_D \oplus 0, \quad (q' = 1, 2),
\]

respectively. Here \((M^{(m)[q']}_{k-k-q'} M^{(m)[q']}_{k-k-q'})'_D = (M^{(m)[q']\dagger}_{k-k-q'} M^{(m)[q']}_{k-k-q'})'_D = (M^{(m)[q']}_{k-k-q'} M^{(m)[q']}_{k-k-q'} M^{(m)[q']}_{k-k-q'})'_D\) hold on and, using eq. (C.17), we obtain the relation:

\[
(M^{(m)[q']}_{k-k-q'} M^{(m)[q']}_{k-k-q'})'_D = (M^{(m)[q']\dagger}_{k+q' k} M^{(m)[q']}_{k+q' k})'_D \quad (q' = 1, 2).
\]

Note that the rearrangement for \(q' = 1\) is generally different from that for \(q' = 2\) in eq. (C.15), but the same rearrangement has been done in eqs. (C.16), (C.18) and (C.19) as that in eq. (C.15) for each \(q'\). From eqs. (C.15) and (C.19), we see that both \(M^{(m)[q']}_{k-k-q'} M^{(m)[q']}_{k-k-q'}\) and \(M^{(m)[q']\dagger}_{k-k-q'} M^{(m)[q']}_{k-k-q'} (q' = 1, 2)\) become diagonal by the biunitary transformation \(W^{(m)}_k\) without modifying \(R^{(m)}_0\) and \(M^{(m)[0]}_k\), and hereafter we take a basis where they are already diagonal.

Using eq. (C.20) with \(q' = 1\) iteratively, we derive the relation:

\[
(M^{(m)[1]}_{k-k-1} M^{(m)[1]}_{k-k-1})'_D = (M^{(m)[1]}_{k+1 k} M^{(m)[1]}_{k+1 k})'_D = (M^{(m)[1]}_{k+2 k+1} M^{(m)[1]}_{k+2 k+1})'_D
= (M^{(m)[1]}_{k+3 k+2} M^{(m)[1]}_{k+3 k+2})'_D.
\]

From eq. (C.21), we find that \((M^{(m)[1]}_{k-k-1} M^{(m)[1]}_{k-k-1})'_D\) are independent of \(k\) and \(M^{(m)[1]}_{k-k-1}\) have a common rank \(r^{(m)}(= \text{rank}(M^{(m)[1]}_{k-k-1}))\). Then, based on the argument in appendix E, we choose the basis such that \(M^{(m)[1]}_{k-k-1}\) have the form:

\[
M^{(m)[1]}_{k-k-1} = \begin{pmatrix}
M^{(m)[1]}_{k-k-1} & U^{(m)}_{k-k-1} \\
0 & 0
\end{pmatrix},
\]

(3.22)
where $\hat{M}^{(m)[1]}$ are $r^{(m)} \times r^{(m)}$ diagonal matrices with positive elements, i.e., $(\hat{M}^{(m)[1]})_{ii} > 0$, and $U^{(m)}_{kk-1}$ are $r^{(m)} \times r^{(m)}$ unitary matrices that take a block-diagonal form such as $U$ in eq. (C.25). From eqs. (C.8) and (C.22), $M^{(m)[1]}_{k-1k}$ are given by

$$M^{(m)[1]}_{k-1k} = -M^{(m)[1]T}_{kk-1} = \begin{pmatrix} \hat{M}^{(m)[1]}U^{(m)}_{k-1k} & 0 \\ 0 & 0 \end{pmatrix}, \quad (C.23)$$

where $-U^{(m)T}_{kk-1}$ are denoted as $U^{(m)}_{k-1k}$.

From eqs. (C.14) and (C.20) with $q' = 2$, we derive the relations:

$$\text{rank}(M^{(m)[2]}_{kk-2}) = \text{rank}(M^{(m)[2]}_{k+2k}), \quad (C.24)$$

$$\left( M^{(m)[2]}_{kk-2} M^{(m)[2]T}_{kk-2} \right)'_D = \left( M^{(m)[2]}_{k+2k} M^{(m)[2]T}_{k+2k} \right)'_D. \quad (C.25)$$

Using eq. (C.11) and the fact that $\left( M^{(m)[1]}_{kk-1} M^{(m)[1]T}_{kk-1} \right)'_D$ are independent of $k$ and $M^{(m)[1]}_{kk-1} M^{(m)[1]T}_{kk-1}$ are diagonal in the present basis, i.e., $M^{(m)[1]}_{kk-1} M^{(m)[1]T}_{kk-1} = (\hat{M}^{(m)[1]})^2 \oplus 0$, we find that the $r^{(m)} \times r^{(m)}$ submatrices in $\left( M^{(m)[2]}_{kk-2} M^{(m)[2]T}_{kk-2} \right)'_D$ corresponding to $(\hat{M}^{(m)[1]})^2$ are also independent of $k$ and $M^{(m)[2]}_{kk-2} M^{(m)[2]T}_{kk-2}$ are also diagonal, and then the form of $M^{(m)[2]}_{kk-2}$ can be written as

$$M^{(m)[2]}_{kk-2} = \begin{pmatrix} \hat{M}^{(m)[2]} U^{(m)}_{kk-2} & 0 \\ 0 & \sqrt{1-a^{(m)^2}} \tilde{U}^{(m)}_{kk-2} \end{pmatrix}, \quad (C.26)$$

where $\hat{M}^{(m)[2]}$ are $r^{(m)} \times r^{(m)}$ diagonal matrices with non-negative elements, i.e., $(\hat{M}^{(m)[2]})_{ii} \geq 0$, and $U^{(m)}_{kk-2}$ are $r^{(m)} \times r^{(m)}$ unitary matrices, i.e., $U^{(m)}_{kk-2} U^{(m)T}_{kk-2} = U^{(m)}_{k-2k} U^{(m)T}_{k-2k} = I^{(m)}$. Here $U^{(m)}_{kk-2}$ are denoted as $U^{(m)}_{k-2k}$. We also use the notation $\tilde{U}^{(m)}_{kk-2} = \tilde{U}^{(m)T}_{kk-2}$ in the following. Using eqs. (C.11) and (C.26), we derive the relations $\tilde{U}^{(m)}_{kk-2} \tilde{U}^{(m)T}_{kk-2} = I_{n^{(m)y}_{k-2k}}$ where $n^{(m)y}_{k} = n^{(m)}_{k} - r^{(m)}$. Replacing $k$ with $k+2$ for $\tilde{U}^{(m)}_{k+2k} \tilde{U}^{(m)T}_{k+2k} = I_{n^{(m)y}_{k+2k}}$, we obtain $\tilde{U}^{(m)}_{k+2k} \tilde{U}^{(m)T}_{k+2k} = I_{n^{(m)y}_{k+2k}}$ where $n^{(m)y}_{k+2k} = n^{(m)y}_{k} + r^{(m)}$. Then, we obtain the relations:

$$\text{rank}(M^{(m)[2]}_{kk-2}) = \text{rank}(\hat{M}^{(m)[2]}_{kk-2} + n^{(m)y}_{k}), \quad \text{rank}(M^{(m)[2]}_{k+2k}) = \text{rank}(\hat{M}^{(m)[2]}_{k+2k} + n^{(m)y}_{k+2k}). \quad (C.27)$$

From eqs. (C.24) and (C.27), we find that $n^{(m)y}_{k} = n^{(m)y}_{k+2}$, i.e., $n^{(m)y}_{1} = n^{(m)y}_{2}$ and $n^{(m)y}_{2} = n^{(m)y}_{3}$, and then $\tilde{U}^{(m)}_{kk-2}$ are $n^{(m)y}_{k} \times n^{(m)y}_{k}$ unitary matrices. Thus, $M^{(m)[2]}_{kk-2}$ are $n^{(m)}_{k} \times n^{(m)}_{k}$ matrices where $n^{(m)}_{1} = n^{(m)}_{3}$ and $n^{(m)}_{2} = n^{(m)}_{4}$. The $(2,2)$ block in the right-hand side of eq. (C.26) can appear for $a^{(m)} = 0$ as will be seen from eq. (C.34).
Inserting eqs. (C.22), (C.23) and (C.26) into eq. (C.11), we obtain the relation:

\[
2(\hat{M}^{(m)[1]})^2 + (\hat{M}^{(m)[2]})^2 = (1 - a^{(m)^2}) I_{r(m)}. \tag{C.28}
\]

From eq. (C.28), the submatrices of \(\hat{M}^{(m)[q]}U^{(m)}_{k k-q} (q = 1, 2)\) are described as \(\hat{M}^{(m)[q]}U^{(m)}_{k k-q} (k') = m^{(m)[q]} u^{(m,k')}_{k k-q} \delta_{k'q}\) with non-negative numbers \(m^{(m)[q]}\) and unitary matrices \(u^{(m,k')}_{k k-q}\), i.e., \(u^{(m,k')}_{k k-q}\) can have off-diagonal elements in a block-diagonal matrix whose corresponding block-diagonal one in \(\hat{M}^{(m)[q]}\) is proportional to the unit matrix. Then we find that the commutation relation \([\hat{M}^{(m)[q]}, U^{(m)}_{k k-q}] = 0\) holds for any integers \(q\) and \(q'\), with \(\hat{M}^{(m)[0]} = a^{(m)} I_{r(m)}\), \(U^{(m)}_{k k} = I_{r(m)}\), \(\hat{M}^{(m)[q]} = M^{(m)[q]}\), and \(U^{(m)}_{k k-q} = (-1)^q U^{(m)^\dagger}_{k k-q}\).

Inserting eqs. (C.22), (C.23) and (C.26) into eq. (C.9), we obtain the relation \(U^{(m)}_{k k-q} U^{(m)}_{k q-q' k-q} = U^{(m)}_{k k-q+q} U^{(m)}_{k k-q+q' k-q}\) and write down the relations:

\[
U^{(m)}_{k k-q-q} U^{(m)}_{k k-q' k} = U^{(m)}_{k k-q+q} U^{(m)}_{k k-q' k}\quad \text{and} \quad U^{(m)}_{k k-q} U^{(m)}_{k q-q' k-q} = U^{(m)}_{k k-q+q} U^{(m)}_{k k-q' k+q}, \tag{C.29}
\]

where \(k\) and \(q\) are replaced with \(k - q\) and \(-q\) in the first relation, and \(q\) is replaced with \(-q\) in the second one. Exactly speaking, the relations (C.29) are obtained except for the parts involving \(m^{(m)[2]}\). Because unitary submatrices for \(m^{(m)[2]} = 0\), e.g., \(u^{(m,k')}_{k k-2}\), are arbitrary and they can be taken to make the relations also hold for the parts relating to \(m^{(m)[2]} = 0\), we choose them in this way and write down following relations in the same fashion that unitary submatrices for \(m^{(m)[2]} = 0\) can be contained as in eq. (C.29). Multiplying \(U^{(m)}_{k k-q}\) and \(U^{(m)}_{k + q k}\) by the above relations, respectively, we obtain the relations:

\[
U^{(m)}_{k k-q} U^{(m)}_{k k-q-q} U^{(m)}_{k k-q-q k-q} = U^{(m)}_{k k-q} U^{(m)}_{k k-q+q} U^{(m)}_{k k+q k-q}, \tag{C.30}
\]

\[
U^{(m)}_{k k-q} U^{(m)}_{k k-q+q} U^{(m)}_{k k+q q} = U^{(m)}_{k k-q} U^{(m)}_{k k+q+q} U^{(m)}_{k k+q+q}, \tag{C.31}
\]

Using eqs. (C.30) and (C.31), we derive the relation:

\[
J^{(m)}_{k k} \equiv U^{(m)}_{k k-1} (U^{(m)}_{k k-2} U^{(m)}_{k k-1} U^{(m)}_{k k-1}) U^{(m)}_{k k+1} U^{(m)}_{k k+1} = U^{(m)}_{k k+2} U^{(m)}_{k k+1} U^{(m)}_{k k+1} \tag{C.32}
\]

Inserting eqs. (C.22), (C.23) and (C.26) into eq. (C.12), we obtain the relations:

\[
2a^{(m)} \hat{M}^{(m)[2]} = (\hat{M}^{(m)[1]})^2 (J^{(m)}_{k k} + J^{(m)}_{k k}), \tag{C.33}
\]

\[
2a^{(m)} \sqrt{1 - a^{(m)^2}} \tilde{U}^{(m)}_{k k-2} = 0, \tag{C.34}
\]

and find \(n^{(m)} = 0\) for \(-1 < a^{(m)} < 1\) from eq. (C.34) because \(\tilde{U}^{(m)}_{k k-2}\) are unitary matrices.

In this way, the form of \(\hat{M}^{(m)[k-l]}\) is restricted as eqs. (C.22), (C.23) and (C.26) with \(\hat{M}^{(m)[1]}\) and \(\hat{M}^{(m)[2]}\) satisfying eqs. (C.28), (C.33), \(U^{(m)}_{k l}\) satisfying eqs. (C.29) and (C.33), and \(\tilde{U}^{(m)}_{k k-2}\) satisfying eq. (C.34).
C.3 Diagonalization of submatrices in $T_1^{(m)}$ and rearrangement

Based on eqs. (C.22), (C.23) and (C.26), we can rearrange $R_0^{(m)}$ and $T_1^{(m)}$ to be the form of block-diagonal ones such as $R_0^{(m)} = R_0^{(m)'} \oplus R_0^{(m)''}$ and $T_1^{(m)} = T_1^{(m)'} \oplus T_1^{(m)''}$, where $R_0^{(m)'}$, $T_1^{(m)'}$, $R_0^{(m)''}$ and $T_1^{(m)''}$ are given by

$$R_0^{(m)'} = \begin{pmatrix} iI_{r(m)} & -I_{r(m)} \\ -I_{r(m)} & I_{r(m)} \end{pmatrix},$$

$$T_1^{(m)'} = \begin{pmatrix} a^{(m)}I_{r(m)} & \hat{M}^{(m)}[1]U_{12}^{(m)} & \hat{M}^{(m)}[2]U_{13}^{(m)} & \hat{M}^{(m)}[1]U_{14}^{(m)} \\ \hat{M}^{(m)}[1]U_{21}^{(m)} & a^{(m)}I_{r(m)} & \hat{M}^{(m)}[1]U_{23}^{(m)} & \hat{M}^{(m)}[2]U_{24}^{(m)} \\ \hat{M}^{(m)}[1]U_{31}^{(m)} & \hat{M}^{(m)}[2]U_{32}^{(m)} & a^{(m)}I_{r(m)} & \hat{M}^{(m)}[1]U_{34}^{(m)} \\ \hat{M}^{(m)}[1]U_{41}^{(m)} & \hat{M}^{(m)}[2]U_{42}^{(m)} & \hat{M}^{(m)}[1]U_{43}^{(m)} & a^{(m)}I_{r(m)} \end{pmatrix},$$

$$R_0^{(m)''} = \begin{pmatrix} iI_{n_1^{(m)'}} & -I_{n_2^{(m)'}}, \\ -I_{n_1^{(m)'}}, & -I_{n_2^{(m)'}}, \\ I_{n_1^{(m)'}}, & I_{n_2^{(m)'}}, \end{pmatrix},$$

and

$$T_1^{(m)''} = \begin{pmatrix} 0 & 0 & \tilde{U}_{13}^{(m)} & 0 \\ 0 & 0 & 0 & \tilde{U}_{14}^{(m)} \\ \tilde{U}_{24}^{(m)} & 0 & 0 & 0 \\ 0 & \tilde{U}_{34}^{(m)} & 0 & 0 \end{pmatrix},$$

respectively.

For the convenience of a later calculation, we write down the following relation obtained by setting $k = 2$ in eq. (C.32),

$$J_{22}^{(m)} = U_{12}^{(m)}U_{14}^{(m)}U_{42}^{(m)} = U_{12}^{(m)}U_{13}^{(m)}U_{32}^{(m)} = U_{24}^{(m)}U_{43}^{(m)}U_{32}^{(m)}.$$

Let us perform a unitary transformation $V^{(m)}T_1^{(m)}V^{(m)\dagger}$ with $V^{(m)} = V^{(m)'} \oplus V^{(m)''}$, where $V^{(m)'}$ and $V^{(m)''}$ are given by

$$V^{(m)'} = \begin{pmatrix} \hat{\Theta}^{(m)[1]}U^{(m)}U_{21}^{(m)} \\ U^{(m)} \\ -\hat{\Theta}^{(m)[1]}U^{(m)}U_{23}^{(m)} \\ U^{(m)}U_{24}^{(m)} \end{pmatrix},$$

(40)
and

\[ V^{(m)\prime\prime} = \begin{pmatrix} \tilde{U}_{31}^{(m)} & \tilde{U}_{42}^{(m)} \\ I_{n_{1}^{(m)\prime}} & I_{n_{2}^{(m)\prime}} \end{pmatrix}, \quad (C.41) \]

respectively. Here \( U^{(m)} \) are unitary matrices that make \( j_{22}^{(m)} \) diagonal ones \( \hat{j}_{22}^{(m)} = U^{(m)} \hat{j}_{22}^{(m)} U^{(m)\dagger} \) and \( \hat{\Theta}^{(m)[1]} \) are diagonal matrices whose squares agree with \( \hat{j}_{22}^{(m)} \). It is understood that \( U^{(m)} \) as well as \( \hat{M}^{(m)[1]} \) and \( \hat{M}^{(m)[2]} \) from the fact that \( j_{22}^{(m)} \) can have off-diagonal elements only in a block-diagonal matrix whose corresponding block-diagonal one in \( \hat{M}^{(m)[1]} \) and \( \hat{M}^{(m)[2]} \) is proportional to the unit matrix. Then, we find that \( R_{Q}^{(m)\prime\prime} \) and \( R_{Q}^{(m)\prime} \) remain unchanged and, using \( U^{(m)}_{k-k-q} = (-1)^{q} U^{(m)\dagger}_{k-k-q} \), \( \hat{U}^{(m)}_{k-2-k} = \hat{U}^{(m)\dagger}_{k-2-k} \) and eq. \( (C.32) \), \( T_{1}^{(m)\prime} \) and \( T_{1}^{(m)\prime\prime} \) are transformed as

\[ V^{(m)\prime} T_{1}^{(m)\prime} V^{(m)\dagger} = \begin{pmatrix} a^{(m)} I_{r(m)} & \hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]\dagger} & -\hat{M}^{(m)[2]} & \hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]\dagger} \\ \hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]} & a^{(m)} I_{r(m)} & -\hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]\dagger} & \hat{M}^{(m)[2]} \\ -\hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]\dagger} & \hat{M}^{(m)[2]} & a^{(m)} I_{r(m)} & -\hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]\dagger} \\ \hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]} & -\hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]\dagger} & a^{(m)} I_{r(m)} & \hat{M}^{(m)[2]} \end{pmatrix}, \quad (C.42) \]

and

\[ V^{(m)\prime\prime} T_{1}^{(m)\prime\prime} V^{(m)\dagger} = \begin{pmatrix} 0 & 0 & I_{n_{1}^{(m)\prime}} & 0 \\ 0 & 0 & 0 & I_{n_{2}^{(m)\prime}} \\ I_{n_{1}^{(m)\prime}} & 0 & 0 & 0 \\ 0 & I_{n_{2}^{(m)\prime}} & 0 & 0 \end{pmatrix}, \quad (C.43) \]

respectively. Using \( \hat{\Theta}^{(m)[1]} \) and \( \hat{\Theta}^{(m)[1]\dagger} \), eq. \( (C.33) \) is rewritten as

\[ 2a^{(m)} \hat{M}^{(m)[2]} = (\hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]} \dagger)^{2} + (\hat{M}^{(m)[1]} \hat{\Theta}^{(m)[1]\dagger})^{2}. \quad (C.44) \]

### D Derivation of block-diagonal forms on \( T^2/\mathbb{Z}_6 \)

In this appendix, we explain the details of block-diagonalization of twist matrices on \( T^2/\mathbb{Z}_6 \) given in section 7.

We can start with the following \( R_{0} \) and \( T_{1} \), without loss of generality,

\[ R_{0} = \begin{pmatrix} \eta I_{n_{1}} & \eta^{2} I_{n_{2}} \\ \eta^{2} I_{n_{2}} & -I_{n_{3}} \end{pmatrix}, \quad (D.1) \]
and

\[ T_1 = \begin{pmatrix}
(T_1)_{(11)} & (T_1)_{(12)} & (T_1)_{(13)} & (T_1)_{(14)} & (T_1)_{(15)} & (T_1)_{(16)} \\
(T_1)_{(21)} & (T_1)_{(22)} & (T_1)_{(23)} & (T_1)_{(24)} & (T_1)_{(25)} & (T_1)_{(26)} \\
(T_1)_{(31)} & (T_1)_{(32)} & (T_1)_{(33)} & (T_1)_{(34)} & (T_1)_{(35)} & (T_1)_{(36)} \\
(T_1)_{(41)} & (T_1)_{(42)} & (T_1)_{(43)} & (T_1)_{(44)} & (T_1)_{(45)} & (T_1)_{(46)} \\
(T_1)_{(51)} & (T_1)_{(52)} & (T_1)_{(53)} & (T_1)_{(54)} & (T_1)_{(55)} & (T_1)_{(56)} \\
(T_1)_{(61)} & (T_1)_{(62)} & (T_1)_{(63)} & (T_1)_{(64)} & (T_1)_{(65)} & (T_1)_{(66)}
\end{pmatrix}, \quad (D.2)

where \( \eta = e^{2\pi i/6} \), \( I_{n_k} \) are \( n_k \times n_k \) unit matrices and \( (T_1)_{(kl)} \) are \( n_k \times n_l \) submatrices. We denote the submatrices as \( (T_1)_{(kl)} = M_{k,l}^{[k-l]} \) and use a notation of \( M_{k,l}^{[k-l]} = M_{l,k}^{[k-l]} = M_{k,l}^{[l-k]} \) with \( k = k \) (mod 6) and \( l = l \) (mod 6). The upper index \( k-l \) represents the charge of the \( \mathbb{Z}_6 \) symmetry generated by \( R_0 \): \( (R_0 T_1 R_0^{-1})_{(k-l)} = \eta^q M_{k-l}^{[k-l]} \). Parameters such as \( n_k \) which represent the size of matrices are non-negative integers. The same applies to the following ones.

### D.1 Block-diagonalization of \( T_1 \) and relations on submatrices

First, let us restrict the form of \( T_1 \) and derive relations on submatrices of \( T_1 \), using constraints \( T_1^\dagger = T_1^{-1}, T_m T_m = T_m T_m^\prime, T_1 T_4 = I \) and \( T_1 T_3 T_5 = I \) (or \( T_1 T_3 = T_2 \)) where \( T_m = R_0^{-1} T_1 R_0^{-m} \) as explained in section 3.

From \( T_1^\dagger = T_1^{-1} = T_4 = R_0^3 T_1 R_0^{-3} \), we derive the relation:

\[ M_{-q}^{[q]} T_k = (-1)^q M_{k-k-q}^{[q]} \quad (D.3)\]

Taking \( q = 0 \) in eq. (D.3), we find that \( M_{-q}^{[q]} = M_{0}^{[0]} \) and \( M_{0}^{[0]} \) is diagonalized as \( (M_{k,k}^{[0]})_{ij} = a_k^i \delta_{ij} \) by a unitary transformation, without modifying \( R_0 \), where \( (M_{k,k}^{[0]})_{ij} \) are \( (i, j) \) elements of \( M_{k,k}^{[0]} \).

As shown in appendix E, from \( T_m T_m = T_m T_m^\prime \), i.e., \( T_1 R_0^{m-m'} T_1 R_0^{m-m'} = T_1 R_0^{m-m'} T_1 R_0^{m-m'} \), we derive the relation:

\[ M_{k,k-q}^{[q]} M_{k,k-q}^{[q]} = M_{k,k-q+q}^{[q]} M_{k,k-q+q}^{[q]} (D.4)\]

Setting \( q = 0 \) in eq. (D.4) and using \( (M_{k,k}^{[0]})_{ij} = a_k^i \delta_{ij} \), we obtain the relation:

\[ (a_k^i - a_k^{i-q}) (M_{k,k-q}^{[q]})_{ij} = 0, \quad (D.5)\]

and rearrange \( T_1 \) to be a block-diagonal matrix, while keeping \( R_0 \) a diagonal one:

\[ R_0 = \begin{pmatrix} R_0^{(1)} & & \\
& R_0^{(2)} & \\
& & \ddots \\
& & & R_0^{(M)} \end{pmatrix}, \quad T_1 = \begin{pmatrix} T_1^{(1)} & & \\
& T_1^{(2)} & \\
& & \ddots \\
& & & T_1^{(M)} \end{pmatrix}, \quad (D.6)\]
where $R^{(m)}_0$ and $T^{(m)}_1$ $(m = 1, 2, \cdots, M)$ are $n^{(m)} \times n^{(m)}$ matrices given by

$$
R^{(m)}_0 = \begin{pmatrix}
\eta I_{n_1^{(m)}} & \eta^2 I_{n_2^{(m)}} & -I_{n_3^{(m)}} & -\eta I_{n_4^{(m)}} & -\eta^2 I_{n_5^{(m)}} & I_{n_6^{(m)}}
\end{pmatrix}, \quad (D.7)
$$

and

$$
T^{(m)}_1 = \begin{pmatrix}
(a^{(m)} I_{n_1^{(m)}})_{12} & M^{(m)[-2]}_{13} & M^{(m)[-3]}_{14} & M^{(m)[2]}_{15} & M^{(m)[1]}_{16} \\
M^{(m)[1]}_{21} & (a^{(m)} I_{n_3^{(m)}})_{23} & M^{(m)[-2]}_{24} & M^{(m)[-3]}_{25} & M^{(m)[2]}_{26} \\
M^{(m)[2]}_{31} & M^{(m)[1]}_{32} & (a^{(m)} I_{n_1^{(m)}})_{34} & M^{(m)[-1]}_{35} & M^{(m)[-2]}_{36} \\
M^{(m)[3]}_{41} & M^{(m)[2]}_{42} & M^{(m)[1]}_{43} & (a^{(m)} I_{n_2^{(m)}})_{44} & M^{(m)[-1]}_{45} \\
M^{(m)[-2]}_{51} & M^{(m)[3]}_{52} & M^{(m)[2]}_{53} & M^{(m)[1]}_{54} & M^{(m)[2]}_{56} \\
M^{(m)[-1]}_{61} & M^{(m)[-2]}_{62} & M^{(m)[3]}_{63} & M^{(m)[2]}_{64} & M^{(m)[1]}_{65} & (a^{(m)} I_{n_6^{(m)}})
\end{pmatrix}, \quad (D.8)
$$

respectively. Here, $n^{(m)} = \sum_{k=1}^6 n_k^{(m)}$ and submatrices in $T^{(m)}_1$ are denoted as $(T^{(m)}_1)_{(kl)} = M^{(m)[k-l]}_{k-l}$ for $k \neq l$ and $(T^{(m)}_1)_{(kk)} = M^{(m)[0]}_{k-k} = a^{(m)} I_{n_k^{(m)}}$. Real parameters $a^{(m)}$ satisfy $a^{(m)} \neq a^{(m')}$. Submatrices obey the following relations that come from $T^{(m)}_1 = T_4$ and $T_m T_m = T_m T_m'$,

$$
M^{(m)[q]}_{k-k} = (-1)^q M^{(m)[q]}_{k-k}, \quad (D.9)
$$

$$
M^{(m)[q]}_{k-k} M^{(m)[q']}_{k-k} = M^{(m)[q+q']}_{k-k} M^{(m)[q]}_{k-k} \quad (D.10)
$$

From $T_1 T_4 = T_1 R_0^3 T_1 R_0^{-3} = I$ and $T_2 = T_1 T_3$, i.e., $T_4 = R_0^{-1} T_4 R_0^2 T_1 R_0^{-1}$, we derive the relations:

$$
\sum_{q'} (-1)^{q-q'} M^{(m)[q']}_{k-k} M^{(m)[q]}_{k-k} = \delta_{k-k} I_{n_k^{(m)}}, \quad (D.11)
$$

and

$$
M^{(m)[q]}_{k-k} = \sum_{q'} \eta^{q-2q'} M^{(m)[q']}_{k-k} M^{(m)[q]}_{k-k} \quad (D.12)
$$

respectively. Here the summation over $q'$ can be taken for any successive six integers. Setting $q = 0, 2$ in eq. (D.11) and $q = 0, 1, 2, 3$ in eq. (D.12) and using eqs. (D.9) and (D.10), we obtain the relations:

$$
2 M^{(m)[1]}_{k-k-1} M^{(m)[1]}_{k-k-1} + 2 M^{(m)[2]}_{k-k-2} M^{(m)[2]}_{k-k-2} + M^{(m)[3]}_{k-k-3} M^{(m)[3]}_{k-k-3} = (1 - a^{(m)^2}) I_{n_k^{(m)}}, \quad (D.13)
$$

for $q = 0$, and
\[
2a^{(m)}M^{(m)[2]}_{k,k-2} + M^{(m)[-2]}_{k,k+2}M^{(m)[2]}_{k+2,k-2} = M^{(m)[1]}_{k,k-1}M^{(m)[1]}_{k-1,k-2} + 2M^{(m)[-1]}_{k,k+1}M^{(m)[3]}_{k+1,k-2}, \quad \text{for } q = 2,
\]
\[
\text{(D.14)}
\]
and
\[
M^{(m)[1]}_{k,k-1}M^{(m)[1]}_{k,k-1} = M^{(m)[2]}_{k,k-2}M^{(m)[2]}_{k,k-2} - M^{(m)[3]}_{k,k-3}M^{(m)[3]}_{k,k-3} = (a^{(m)} - a^{(m)2})I_{n_{k}^{(m)}}, \quad \text{for } q = 0,
\]
\[
\text{(D.15)}
\]
\[
(1 - a^{(m)})M^{(m)[1]}_{k,k-1} = M^{(m)[3]}_{k,k-3}M^{(m)[2]}_{k,k-2} - 2M^{(m)[2]}_{k,k-2}M^{(m)[-1]}_{k,k-1}, \quad \text{for } q = 1,
\]
\[
\text{(D.16)}
\]
\[
(1 + a^{(m)})M^{(m)[2]}_{k,k-2} = M^{(m)[1]}_{k,k-1}M^{(m)[1]}_{k-1,k-2} - M^{(m)[1]}_{k,k+1}M^{(m)[1]}_{k+1,k-2}
+ M^{(m)[2]}_{k,k+2}M^{(m)[2]}_{k+2,k-2}, \quad \text{for } q = 2,
\]
\[
\text{(D.17)}
\]
\[
(1 + 2a^{(m)})M^{(m)[3]}_{k,k-3} = M^{(m)[1]}_{k,k-1}M^{(m)[2]}_{k-1,k-3} + M^{(m)[-2]}_{k,k+2}M^{(m)[-1]}_{k+2,k-3}, \quad \text{for } q = 3.
\]
\[
\text{(D.18)}
\]

The relations obtained by setting \( q = 1, q = 3 \) and \( q = 5 \) in eq. \((D.11)\) are not independent of eq. \((D.10)\), because they are also derived from linear combinations of eq. \((D.10)\). The relation obtained by setting \( q = 4 \) in eq. \((D.11)\) is the Hermitian conjugate of eq. \((D.14)\). The relations obtained by setting \( q = 4 \) and \( q = 5 \) in eq. \((D.12)\) are the Hermitian conjugates of eqs. \((D.17)\) and \((D.16)\), respectively. From eq. \((D.13)\), we find that \( 0 \leq a^{(m)2} \leq 1 \). Note that \( T^{(m)}_{1} \) is already diagonal for \( a^{(m)} = \pm 1 \), and then \( R_{0} \) and \( T_{1} \) contain \( n_{m}^{(m)} \times n_{m}^{(m)} \) diagonal matrices whose submatrices are given by \((R_{0}^{(m)})_{{k,l}} = \eta^{k}\delta_{k}I_{n_{k}^{(m)}} \) and \((T_{1}^{(m)})_{{k,l}} = \pm\delta_{kl}I_{n_{k}^{(m)}} \) for \( a^{(m)} = \pm 1 \) (double sign in the same order) as members of block-diagonal matrices. Hereafter, we focus on \(-1 < a^{(m)} < 1\).

**D.2 Restriction of \( M^{(m)[k-l]}_{k,l} \)**

Next, let us restrict the form of \( M^{(m)[k-l]}_{k,l} \), using relations obtained in the previous subsection.

In the same way as on \( T^{2}/\mathbb{Z}_{4} \), we obtain the counterpart of eq. \((C.20)\), i.e., \((M^{(m)[q']_{k-k-q'}}_{k-k-q'}M^{(m)[q']_{k-k-q'}}_{k-k-q'})' = (M^{(m)[q']_{k-q'}k^{q'}k'})' \) \( (q' = 1, 2, 3) \), and using them with \( q' = 1 \) iteratively, we derive the relation:

\[
(M^{(m)[1]}_{k,k-1}M^{(m)[1]}_{k,k-1})' \rightarrow (M^{(m)[1]}_{k+1,k}M^{(m)[1]}_{k+1,k})' \rightarrow (M^{(m)[1]}_{k+2,k+1}M^{(m)[1]}_{k+2,k+1})' \rightarrow
(M^{(m)[1]}_{k+3,k+2}M^{(m)[1]}_{k+3,k+2})' \rightarrow (M^{(m)[1]}_{k+4,k+3}M^{(m)[1]}_{k+4,k+3})' \rightarrow
(M^{(m)[1]}_{k+5,k+4}M^{(m)[1]}_{k+5,k+4})' \rightarrow \cdots
\]
\[
\text{(D.19)}
\]

From eq. \((D.19)\), we find that \((M^{(m)[1]}_{k,k-1}M^{(m)[1]}_{k,k-1})' \) are independent of \( k \) and \( M^{(m)[1]}_{k,k-1} \) have a common rank \( r^{(m)}(= \text{rank}(M^{(m)[1]}_{k,k-1})) \). Then, based on the argument in appendix \( E \), we choose
the basis such that $M_{k,k-1}^{(m)[1]}$ have the form:

$$M_{k,k-1}^{(m)[1]} = \begin{pmatrix}
\hat{M}^{(m)[1]} U_{k,k-1}^{(m)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (D.20)$$

where $\hat{M}^{(m)[1]}$ are $r^{(m)} \times r^{(m)}$ diagonal matrices with positive elements, i.e., $(\hat{M}^{(m)[1]})_{ii} > 0$, and $U_{k,k-1}^{(m)}$ are $r^{(m)} \times r^{(m)}$ unitary matrices. From eqs. (D.9) and (D.20), $M_{k,k-1}^{(m)[1]}$ are given by

$$M_{k,k-1}^{(m)[1]} = -M_{k,k-1}^{(m)[1]\dagger} = \begin{pmatrix}
\hat{M}^{(m)[1]} U_{k,k-1}^{(m)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (D.21)$$

where $-U_{k,k-1}^{(m)\dagger}$ are denoted as $U_{k,k-1}^{(m)}$.

From the counterparts of eqs. (C.14) and (C.20) with $q’ = 2$ and $q’ = 3$, we derive the relations:

$$\text{rank}(M_{k,k-1}^{(m)[2]}) = \text{rank}(M_{k,k-1}^{(m)[1]}) \quad (D.22)$$

$$\left( M_{k,k-2}^{(m)[2]} M_{k,k-2}^{(m)[2]\dagger} \right)_D = \left( M_{k,k-2}^{(m)[2]} M_{k,k-2}^{(m)[2]\dagger} \right)_D \quad (D.23)$$

$$\text{rank}(M_{k,k-3}^{(m)[3]}) = \text{rank}(M_{k,k-3}^{(m)[3]}) \quad (D.24)$$

$$\left( M_{k,k-3}^{(m)[3]} M_{k,k-3}^{(m)[3]\dagger} \right)_D = \left( M_{k,k-3}^{(m)[3]} M_{k,k-3}^{(m)[3]\dagger} \right)_D \quad (D.25)$$

Using eqs. (D.13) and (D.15) and the fact that $\left( M_{k,k-1}^{(m)[1]} M_{k,k-1}^{(m)[1]\dagger} \right)_D$ are independent of $k$ and $M_{k,k-1}^{(m)[1]} M_{k,k-1}^{(m)[1]\dagger}$ are diagonal in the present basis, i.e., $M_{k,k-1}^{(m)[1]} M_{k,k-1}^{(m)[1]\dagger} = (\hat{M}^{(m)[1]})^2 \oplus 0$, we find that the $r^{(m)} \times r^{(m)}$ submatrices in $\left( M_{k,k-2}^{(m)[2]} M_{k,k-2}^{(m)[2]\dagger} \right)_D$ and $\left( M_{k,k-3}^{(m)[3]} M_{k,k-3}^{(m)[3]\dagger} \right)_D$ corresponding to $(\hat{M}^{(m)[1]})^2$ are also independent of $k$ and $M_{k,k-2}^{(m)[2]} M_{k,k-2}^{(m)[2]\dagger}$ and $M_{k,k-3}^{(m)[3]} M_{k,k-3}^{(m)[3]\dagger}$ are also diagonal. Then, the forms of $M_{k,k-2}^{(m)[2]}$ and $M_{k,k-3}^{(m)[3]}$ can be written as $M_{k,k-2}^{(m)[2]} = \hat{M}^{(m)[2]} U_{k,k-2}^{(m)} \oplus \hat{M}^{(m)}_{k,k-2}$ and $M_{k,k-3}^{(m)[3]} = \hat{M}^{(m)[3]} U_{k,k-3}^{(m)} \oplus \hat{M}^{(m)}_{k,k-3}$ where both $\hat{M}^{(m)[2]}$ and $\hat{M}^{(m)[3]}$ are $r^{(m)} \times r^{(m)}$ diagonal matrices with non-negative elements $(\hat{M}^{(m)[2]})_{ii} \geq 0$ and $(\hat{M}^{(m)[3]})_{ii} \geq 0$, and $U_{k,k-2}$ and $U_{k,k-3}$
are $r^{(m)} \times r^{(m)}$ unitary matrices. Using eqs. (D.13) - (D.18), $M^{(m)[2]}_{kk-2}$ and $M^{(m)[3]}_{kk-3}$ can be written as

\[
M^{(m)[2]}_{kk-2} = \begin{pmatrix}
\hat{M}^{(m)[2]} U^{(m)}_{kk-2} & 0 & 0 \\
0 & \frac{2}{3} \tilde{U}^{(m)}_{kk-2} & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

and

\[
M^{(m)[3]}_{kk-3} = \begin{pmatrix}
\hat{M}^{(m)[3]} U^{(m)}_{kk-3} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \pm \sqrt{\frac{3}{2}} \tilde{U}^{(m)}_{kk-3}
\end{pmatrix},
\]

where $\tilde{U}^{(m)}_{kk-2}$ and $\tilde{U}^{(m)}_{kk-3}$ satisfy the relations $\tilde{U}^{(m)}_{kk-2} U^{(m)}_{kk-2} = I_{n_k}^{(m)'}$ and $\tilde{U}^{(m)}_{kk-3} U^{(m)}_{kk-3} = I_{n_k}^{(m)''}$ with $n_k^{(m)} = r^{(m)} + n_k^{(m)'} + n_k^{(m)''}$. Here, $n_k^{(m)'} \neq 0$ and $n_k^{(m)''} \neq 0$ for $a^{(m)} = -1/3$ and $a^{(m)} = -1/2$, respectively. From eqs. (D.9) and (D.26), $M^{(m)[2]}_{k-2k}$ are given by

\[
M^{(m)[-2]}_{k-2k} = M^{(m)[2]^\dagger}_{kk-2} = \begin{pmatrix}
\hat{M}^{(m)[2]} U^{(m)}_{k-2k} & 0 & 0 \\
0 & \frac{2}{3} \tilde{U}^{(m)}_{k-2k} & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

where $U^{(m)}_{k-2k}$ and $\tilde{U}^{(m)}_{k-2k}$ are denoted as $U^{(m)}_{k-2k}$ and $\tilde{U}^{(m)}_{k-2k}$, respectively. In the same way, we often denote $-U^{(m)}_{kk-3}$ and $-\tilde{U}^{(m)}_{kk-3}$ as $U^{(m)}_{kk-3}$ and $\tilde{U}^{(m)}_{kk-3}$, respectively. Replacing $k$ with $k+2$ for $\tilde{U}^{(m)}_{kk-2} U^{(m)}_{kk-2} = I_{n_k}^{(m)'}$ and with $k+3$ for $\tilde{U}^{(m)}_{kk-3} U^{(m)}_{kk-3} = I_{n_k}^{(m)''}$, we obtain $\tilde{U}^{(m)}_{kk-2} U^{(m)}_{kk-2} = I_{n_k}^{(m)'}$ and $\tilde{U}^{(m)}_{kk-3} U^{(m)}_{kk-3} = I_{n_k}^{(m)''}$, respectively, where $n_k^{(m)} = r^{(m)} + n_k^{(m)'} + n_k^{(m)''}$ and $n_k^{(m)} = r^{(m)} + n_k^{(m)'} + n_k^{(m)''}$. Then, we obtain the relations:

\[
\begin{align*}
\text{rank}(M^{(m)[2]}_{kk-2}) &= \text{rank}(\hat{M}^{(m)[2]}_{kk-2}) + n_k^{(m)'} \quad \text{rank}(M^{(m)[2]}_{k+2k}) = \text{rank}(\hat{M}^{(m)[2]}_{k+2k}) + n_k^{(m)'} \quad \text{rank}(M^{(m)[3]}_{kk-3}) = \text{rank}(\hat{M}^{(m)[3]}_{kk-3}) + n_k^{(m)''} \quad \text{rank}(M^{(m)[3]}_{k+3k}) = \text{rank}(\hat{M}^{(m)[3]}_{k+3k}) + n_k^{(m)''}. 
\end{align*}
\]
From eqs. (D.22), (D.24), (D.29) and (D.30), we find that 
\[ n_k^{(m)'} = n_{k+2}^{(m)'} \text{, i.e.}, n_1^{(m)'} = n_3^{(m)'} = \]
\[ n_5^{(m)'} \text{ and } n_2^{(m)'} = n_4^{(m)'} = n_6^{(m)'} \text{, and } n_k^{(m)''} = n_{k+3}^{(m)''} \text{, i.e., } n_1^{(m)''} = n_4^{(m)''}, \]
\[ n_2^{(m)''} = n_5^{(m)''} \text{ and } n_3^{(m)''} = n_6^{(m)''}. \]
Thus \( U_{k+1}^{(m)} \) and \( U_{k+3}^{(m)} \) are \( n_k^{(m)'} \times n_k^{(m)''} \) and \( n_k^{(m)''} \times n_k^{(m)''} \) unitary matrices, respectively.

Inserting eqs. (D.20), (D.26) and (D.27) into eqs. (D.13) and (D.15), we obtain the relations:
\[
2(\tilde{M}^{(m)[1]} \tilde{q}^{k} + 2(\tilde{M}^{(m)[2]} q^{k} + (\tilde{M}^{(m)[3]} q^{k})^{2} = (1 - \tilde{a}^{(m)2}) I_{r(m)}, \quad (D.31)
\]
\[2(\tilde{M}^{(m)[1]} q^{k} - 2(\tilde{M}^{(m)[2]} q^{k} = (\tilde{a}^{(m)} - \tilde{a}^{(m)2}) I_{r(m)} \quad (D.32)
\]
In a similar way as on \( T^{2}/\mathbb{Z}_4 \), the submatrices of \( \tilde{M}^{(m)[q]} U_{k-q}^{(m)} \) \( (q = 1, 2, 3) \) are described as
\[
(\tilde{M}^{(m)[q]} U_{k-q}^{(m)})_{(k'q')} = M_{k'q'}^{(m)}, U_{k-q}^{(m)} q^{(m, k')} \text{ with non-negative numbers } M_{k'q'}^{(m)} \text{ and unitary matrices}
\]
\( U_{k-q}^{(m)} \), and the commutation relation \( [\tilde{M}^{(m)[q]} U_{k-q}^{(m)}, U_{k-q}^{(m)}] = 0 \) holds for any integers \( q \) and \( q' \), with
\[ M^{(m)[0]} = a^{(m)} I_{r(m)}, U_{k-q}^{(m)} = I_{r(m)} \text{, } M^{(m)[-q]} = M^{(m)} q^{k} \text{ and } U_{k-q}^{(m)} = (-1)^{q} U_{k-q}^{(m)}.
\]

Inserting eqs. (D.20), (D.21) and (D.26) – (D.28) into eq. (10), we obtain the relation
\[ U_{k-q}^{(m)} U_{k-q}^{(m)} = U_{k-q}^{(m)} U_{k-q}^{(m)} + q^{k} \text{ and write down the relation:}
\]
\[
U_{k-q}^{(m)} U_{k-q}^{(m)} = U_{k-q}^{(m)} U_{k-q}^{(m)} r^{(m)}
\]
where \( k \) is replaced with \( k - r \). In particular, from eq. (33), we obtain the relations:
\[
U_{k-q}^{(m)} U_{k-q}^{(m)} = U_{k-q}^{(m)} U_{k-q}^{(m)} + q^{k} \text{ and write down the following relations in the same fashion that unitary submatrices for } M_{k'}^{(m)[2]} = 0 \text{ and } M_{k'}^{(m)[3]} = 0 \text{ can be contained as in eq. (33). Using eq. (34), we derive the relations:}
\]
\[ J_{kk}^{(m)} = U_{k-k-1}^{(m)} U_{k-k-2}^{(m)} = U_{k-1-k}^{(m)} U_{k-2-k}^{(m)} = U_{k-1-k}^{(m)} U_{k-2-k}^{(m)} U_{k-1-k}^{(m)} = U_{k-2-k}^{(m)} U_{k-1-k}^{(m)} U_{k-2-k}^{(m)} \quad (D.35)
\]
\[ K_{kk}^{(m)} = U_{k-k+1}^{(m)} U_{k-k+2}^{(m)} = U_{k+1-k}^{(m)} U_{k+2-k}^{(m)} = U_{k+1-k}^{(m)} U_{k+2-k}^{(m)} U_{k+1-k}^{(m)} = U_{k+2-k}^{(m)} U_{k+1-k}^{(m)} U_{k+2-k}^{(m)} \quad (D.36)
\]
Using eqs. (34) – (36), it is shown that \( J_{kk}^{(m)} \) and \( K_{kk}^{(m)} \) commute with each other as follows:
\[
J_{kk}^{(m)} K_{kk}^{(m)} = U_{k-k+1}^{(m)} U_{k-k+2}^{(m)} U_{k-k+1}^{(m)} U_{k-k+2}^{(m)} \quad (D.37)
\]
Then, $J_{kk}^{(m)}$ and $K_{kk}^{(m)}$ are diagonalized simultaneously by a suitable unitary transformation. Furthermore, inserting $-U_{k,k-2}^{(m)} U_{k,k-3}^{(m)} U_{k,3}^{(m)} U_{k,k-2}^{(m)} U_{k,k-3}^{(m)} U_{k,3}^{(m)} U_{k,k-2}^{(m)} U_{k,k-3}^{(m)} U_{k,3}^{(m)} U_{k,k-2}^{(m)} = I_{n_{k}}^{(m)}$ into $U_{k,k+2}^{(m)} U_{k,k+2}^{(m)} - U_{k,k-2}^{(m)}$ and using eqs. (D.33)–(D.36), $U_{k,k+2}^{(m)} U_{k,k+2}^{(m)} U_{k,k-2}^{(m)} = -K_{kk}^{(m)} K_{kk}^{(m)} J_{kk}^{(m)}$ is derived as follows,

$$U_{k,k+2}^{(m)} U_{k,k+2}^{(m)} = -U_{k,k+2}^{(m)} U_{k,k+2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} U_{k,k-2}^{(m)} = K_{kk}^{(m)} K_{kk}^{(m)} J_{kk}^{(m)}$$

(D.38)

Inserting eqs. (D.20), (D.21) and (D.26)–(D.28) into eqs. (D.14), we obtain the relations:

$$2a^{(m)} M^{(m)[2]} - (\hat{M}^{(m)[1]} K_{kk}^{(m)} J_{kk}^{(m)} = (\hat{M}^{(m)[1]} K_{kk}^{(m)} J_{kk}^{(m)} + 2M^{(m)[1]} M^{(m)[3]} K_{kk}^{(m)}$$

(D.39)

$$U_{k,k-2}^{(m)} U_{k,k+2}^{(m)} U_{k,k+2}^{(m)} = 0.$$  

(D.40)

In the same way, inserting eqs. (D.20), (D.21) and (D.26)–(D.28) into eqs. (D.16)–(D.18), we obtain the relations:

$$(1 - a^{(m)}) \hat{M}^{(m)[1]} = -\hat{M}^{(m)[3]} \hat{M}^{(m)[2]} K_{kk}^{(m)} + 2M^{(m)[1]} M^{(m)[1]} J_{kk}^{(m)}$$

(D.41)

$$(1 + a^{(m)}) \hat{M}^{(m)[2]} = (\hat{M}^{(m)[1]} K_{kk}^{(m)} + \hat{M}^{(m)[3]} \hat{M}^{(m)[2]} K_{kk}^{(m)} - (\hat{M}^{(m)[2]} K_{kk}^{(m)} J_{kk}^{(m)}$$

(D.42)

$$(1 + 2a^{(m)}) \hat{M}^{(m)[3]} = -\hat{M}^{(m)[1]} M^{(m)[2]} (K_{kk}^{(m)} + K_{kk}^{(m)}$$

(D.43)

respectively, besides eq. (D.40).

In this way, the form of $M_{k,l}^{(m)[k-1]}$ is restricted as eqs. (D.20), (D.21) and (D.26)–(D.28) with $M^{(m)[1]}$, $M^{(m)[2]}$ and $M^{(m)[3]}$ satisfying eqs. (D.31), (D.32), (D.38) and (D.41)–(D.43), $U_{k,l}^{(m)}$ satisfying eqs. (D.33), (D.39) and (D.41)–(D.43), and $U_{k,k-2}^{(m)}$ satisfying eq. (D.40).

### D.3 Diagonalization of submatrices in $T_{1}^{(m)}$ and rearrangement

Based on eqs. (D.20), (D.21) and (D.26)–(D.28), we can rearrange $R_{0}^{(m)}$ and $T_{1}^{(m)}$ to be the form of block-diagonal ones such as

$$R_{0}^{(m)} = \left( \begin{array}{cc} \tilde{R}_{0}^{(m)[1]} & \tilde{R}_{0}^{(m)[2]} \\ \tilde{R}_{0}^{(m)[2]} & \tilde{R}_{0}^{(m)[3]} \end{array} \right) = \tilde{R}_{0}^{(m)[1]} \oplus \tilde{R}_{0}^{(m)[2]} \oplus \tilde{R}_{0}^{(m)[3]}$$

(D.44)

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\[
R^{(m)'}_0 = \begin{pmatrix}
\eta I_{r(m)} & \eta^2 I_{r(m)} & -I_{r(m)} & -\eta I_{r(m)} & -\eta^2 I_{r(m)} \\
-I_{r(m)} & -\eta I_{r(m)} & -I_{r(m)} & -\eta I_{r(m)} & -\eta^2 I_{r(m)} \\
-\eta I_{n_1(m)} & -\eta^2 I_{n_1(m)} & -I_{n_1(m)} & -\eta I_{n_1(m)} & -\eta^2 I_{n_1(m)} \\
-\eta^2 I_{n_2(m)} & -I_{n_2(m)} & -\eta I_{n_2(m)} & -\eta^2 I_{n_2(m)} & -I_{n_2(m)} \\
I_{n_3(m)} & I_{n_3(m)} & I_{n_3(m)} & I_{n_3(m)} & I_{n_3(m)}
\end{pmatrix},
\]

(D.45)

\[
R^{(m)''}_0 = \begin{pmatrix}
\eta I_{n_1(m)'} & \eta^2 I_{n_1(m)'} & -I_{n_1(m)'} & -\eta I_{n_1(m)'} & -\eta^2 I_{n_1(m)'} \\
-I_{n_1(m)'} & -\eta I_{n_1(m)'} & -I_{n_1(m)'} & -\eta I_{n_1(m)'} & -\eta^2 I_{n_1(m)'} \\
-\eta I_{n_2(m)'} & -\eta^2 I_{n_2(m)'} & -I_{n_2(m)'} & -\eta I_{n_2(m)'} & -\eta^2 I_{n_2(m)'} \\
-\eta^2 I_{n_3(m)'} & -I_{n_3(m)'} & -\eta I_{n_3(m)'} & -\eta^2 I_{n_3(m)'} & -I_{n_3(m)'} \\
I_{n_3(m)'} & I_{n_3(m)'} & I_{n_3(m)'} & I_{n_3(m)'} & I_{n_3(m)'}
\end{pmatrix},
\]

(D.46)

\[
R^{(m)'''}_0 = \begin{pmatrix}
\eta I_{n_1(m)''} & \eta^2 I_{n_1(m)''} & -I_{n_1(m)''} & -\eta I_{n_1(m)''} & -\eta^2 I_{n_1(m)''} \\
-I_{n_1(m)''} & -\eta I_{n_1(m)''} & -I_{n_1(m)''} & -\eta I_{n_1(m)''} & -\eta^2 I_{n_1(m)''} \\
-\eta I_{n_2(m)''} & -\eta^2 I_{n_2(m)''} & -I_{n_2(m)''} & -\eta I_{n_2(m)''} & -\eta^2 I_{n_2(m)''} \\
-\eta^2 I_{n_3(m)''} & -I_{n_3(m)''} & -\eta I_{n_3(m)''} & -\eta^2 I_{n_3(m)''} & -I_{n_3(m)''} \\
I_{n_3(m)''} & I_{n_3(m)''} & I_{n_3(m)''} & I_{n_3(m)''} & I_{n_3(m)''}
\end{pmatrix},
\]

(D.47)

and

\[
T^{(m)'}_1 = \begin{pmatrix}
T^{(m)'}_1 \\
T^{(m)''}_1 \\
T^{(m)'''}_1
\end{pmatrix} = T^{(m)'}_1 \oplus T^{(m)''}_1 \oplus T^{(m)'''}_1,
\]

(D.48)

\[
T^{(m)}_1 = \begin{pmatrix}
\hat{M}^{(m)}[1] U_{r(m)}^{(m)} & \hat{M}^{(m)}[2] U_{r(m)}^{(m)} & \hat{M}^{(m)}[3] U_{r(m)}^{(m)} & \hat{M}^{(m)}[4] U_{r(m)}^{(m)} & \hat{M}^{(m)}[5] U_{r(m)}^{(m)} & \hat{M}^{(m)}[6] U_{r(m)}^{(m)} \\
\hat{M}^{(m)}[1] I_{n_1(m)'} & \hat{M}^{(m)}[2] I_{n_1(m)'} & \hat{M}^{(m)}[3] I_{n_1(m)'} & \hat{M}^{(m)}[4] I_{n_1(m)'} & \hat{M}^{(m)}[5] I_{n_1(m)'} & \hat{M}^{(m)}[6] I_{n_1(m)'} \\
\hat{M}^{(m)}[1] I_{n_2(m)'} & \hat{M}^{(m)}[2] I_{n_2(m)'} & \hat{M}^{(m)}[3] I_{n_2(m)'} & \hat{M}^{(m)}[4] I_{n_2(m)'} & \hat{M}^{(m)}[5] I_{n_2(m)'} & \hat{M}^{(m)}[6] I_{n_2(m)'} \\
\hat{M}^{(m)}[1] I_{n_3(m)'} & \hat{M}^{(m)}[2] I_{n_3(m)'} & \hat{M}^{(m)}[3] I_{n_3(m)'} & \hat{M}^{(m)}[4] I_{n_3(m)'} & \hat{M}^{(m)}[5] I_{n_3(m)'} & \hat{M}^{(m)}[6] I_{n_3(m)'} \\
\hat{M}^{(m)}[1] I_{n_4(m)'} & \hat{M}^{(m)}[2] I_{n_4(m)'} & \hat{M}^{(m)}[3] I_{n_4(m)'} & \hat{M}^{(m)}[4] I_{n_4(m)'} & \hat{M}^{(m)}[5] I_{n_4(m)'} & \hat{M}^{(m)}[6] I_{n_4(m)'} \\
\hat{M}^{(m)}[1] I_{n_5(m)'} & \hat{M}^{(m)}[2] I_{n_5(m)'} & \hat{M}^{(m)}[3] I_{n_5(m)'} & \hat{M}^{(m)}[4] I_{n_5(m)'} & \hat{M}^{(m)}[5] I_{n_5(m)'} & \hat{M}^{(m)}[6] I_{n_5(m)'}
\end{pmatrix},
\]

(D.49)

\[
T^{(m)''}_1 = \begin{pmatrix}
-\frac{1}{3} I_{n_1(m)'} & 0 & \frac{2}{3} \bar{U}_{13}^{(m)} & 0 & \frac{2}{3} \bar{U}_{15}^{(m)} & 0 \\
0 & -\frac{1}{3} \bar{I}_{n_2(m)'} & 0 & \frac{2}{3} \bar{U}_{24}^{(m)} & 0 & \frac{2}{3} \bar{U}_{26}^{(m)} \\
\frac{2}{3} \bar{U}_{31}^{(m)} & 0 & -\frac{1}{3} \bar{I}_{n_1(m)'} & 0 & \frac{2}{3} \bar{U}_{35}^{(m)} & 0 \\
0 & \frac{2}{3} \bar{U}_{42}^{(m)} & 0 & -\frac{1}{3} \bar{I}_{n_2(m)'} & 0 & \frac{2}{3} \bar{U}_{46}^{(m)} \\
\frac{2}{3} \bar{U}_{51}^{(m)} & 0 & \frac{2}{3} \bar{U}_{53}^{(m)} & 0 & -\frac{1}{3} \bar{I}_{n_1(m)'} & 0 \\
0 & \frac{2}{3} \bar{U}_{62}^{(m)} & 0 & \frac{2}{3} \bar{U}_{64}^{(m)} & 0 & -\frac{1}{3} \bar{I}_{n_2(m)'}
\end{pmatrix},
\]

(D.50)

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\[ T_1^{(m)''} = \begin{pmatrix} -\frac{1}{2} I^{(m)''}_{n_1} & 0 & 0 & \pm \frac{\sqrt{2}}{2} U_4^{(m)} & 0 & 0 \\ 0 & -\frac{1}{2} I^{(m)''}_{n_2} & 0 & 0 & \pm \frac{\sqrt{2}}{2} U_5^{(m)} & 0 \\ 0 & 0 & -\frac{1}{2} I^{(m)''}_{n_3} & 0 & 0 & \pm \frac{\sqrt{2}}{2} U_6^{(m)} \\ \pm \frac{\sqrt{2}}{2} U_4^{(m)} & 0 & 0 & -\frac{1}{2} I^{(m)''}_{n_1} & 0 & 0 \\ 0 & \pm \frac{\sqrt{2}}{2} U_5^{(m)} & 0 & 0 & -\frac{1}{2} I^{(m)''}_{n_2} & 0 \\ 0 & 0 & \pm \frac{\sqrt{2}}{2} U_6^{(m)} & 0 & 0 & -\frac{1}{2} I^{(m)''}_{n_3} \end{pmatrix}, \]  

(double sign in the same order),

\[ (D.51) \]

respectively.

For the convenience of a later calculation, we write down the following relations obtained by setting \( k = 2 \) in eqs. \[ (D.35) \], \[ (D.36) \] and \[ (D.38) \],

\[ J_{22}^{(m)} = U_{21}^{(m)} U_{16}^{(m)} U_{62}^{(m)} = U_{21}^{(m)} U_{13}^{(m)} U_{32}^{(m)} = U_{24}^{(m)} U_{43}^{(m)} U_{32}^{(m)}, \]  

\[ (D.52) \]

\[ K_{22}^{(m)} = U_{23}^{(m)} U_{36}^{(m)} U_{62}^{(m)} = U_{23}^{(m)} U_{35}^{(m)} U_{52}^{(m)} = U_{24}^{(m)} U_{45}^{(m)} U_{52}^{(m)} = U_{24}^{(m)} U_{41}^{(m)} U_{12}^{(m)} = U_{25}^{(m)} U_{51}^{(m)} U_{12}^{(m)} = U_{25}^{(m)} U_{56}^{(m)} U_{62}^{(m)}, \]  

\[ (D.53) \]

\[ U_{24}^{(m)} U_{46}^{(m)} U_{62}^{(m)} = -K_{22}^{(m)} J_{22}^{(m)}. \]  

\[ (D.54) \]

From eq. \[ (D.40) \], we obtain the relations:

\[ \tilde{U}_{35}^{(m)} \tilde{U}_{51}^{(m)} \tilde{U}_{13}^{(m)} = I^{(m)''}_{n_1}, \quad \tilde{U}_{46}^{(m)} \tilde{U}_{62}^{(m)} \tilde{U}_{24}^{(m)} = I^{(m)''}_{n_2}. \]  

\[ (D.55) \]

Let us perform a unitary transformation \( V^{(m)} T_1^{(m)} V^{(m)\dagger} \) with \( V^{(m)} = V^{(m)'} \oplus V^{(m)''} \oplus V^{(m)'''} \), where \( V^{(m)'} \), \( V^{(m)''} \) and \( V^{(m)'''} \) are given by

\[ V^{(m)'} = \begin{pmatrix} \hat{\Theta}^{(m)[1]} U^{(m)} U_2^{(m)} \\ -\hat{\Theta}^{(m)[1]} U^{(m)} U_2^{(m)} \\ \hat{\Theta}^{(m)[2]} U^{(m)} U_2^{(m)} \\ iU^{(m)} U_2^{(m)} \end{pmatrix}, \]  

\[ (D.56) \]

\[ V^{(m)''} = \begin{pmatrix} \tilde{U}_{31}^{(m)} & \hat{U}_{42}^{(m)} & I^{(m)''}_{n_1} \\ \hat{U}_{31}^{(m)} & \tilde{U}_{42}^{(m)} & I^{(m)''}_{n_2} \\ \tilde{U}_{35}^{(m)} & \hat{U}_{46}^{(m)} \end{pmatrix}, \]  

\[ (D.57) \]
and

\[
V^{(m)'''} = \begin{pmatrix}
-i\tilde{U}_{11}^{(m)} & -i\tilde{U}_{52}^{(m)} & -i\tilde{U}_{63}^{(m)} & I_{n_1}^{(m)'''} \\
& & & I_{n_2}^{(m)'''} \\
& & & I_{n_3}^{(m)'''}
\end{pmatrix}.
\]

(D.58)

Here \(U^{(m)}\) are unitary matrices that make \(J_{22}^{(m)}\) and \(K_{22}^{(m)}\) diagonal ones. \(\tilde{J}_{22}^{(m)}\) and \(\tilde{K}_{22}^{(m)}\) simultaneously and commute with \(\tilde{M}^{(m)}[q]\), \(\tilde{\Theta}^{(m)[1]}\) are diagonal unitary matrices whose cubes agree with \(i\tilde{K}_{22}^{(m)\dagger}J_{22}^{(m)}\) and \(\tilde{\Theta}^{(m)[2]}\) are diagonal unitary matrices whose cubes agree with \(-\tilde{J}_{22}^{(m)\dagger}K_{22}^{(m)\dagger}\). Then, we find that \(R_0^{(m)'}\), \(R_0^{(m)''}\) and \(R_0^{(m)'''}\) remain unchanged and, using \(U_{k-k}^{(m)} = (-1)^{q}U_{k-k}^{(m)\dagger}\), \(\tilde{U}_{k-k}^{(m)} = (-1)^{q}\tilde{U}_{k-k}^{(m)\dagger}\) and eqs. (D.52) - (D.55), \(T_1^{(m)'}\), \(T_1^{(m)''}\) and \(T_1^{(m)'''}\) are transformed into matrices

\[
T_1^{(m)'} = \begin{pmatrix}
\alpha^{(m)}_{p \to (m)} & -\tilde{M}^{(m)[1]}\tilde{\Theta}^{[1]'} & \tilde{M}^{(m)[2]}\tilde{\Theta}^{[2]'} & -i\tilde{M}^{(m)[3]}I_{r^{(m)}} \\
\tilde{M}^{(m)[1]}\tilde{\Theta}^{[1]'} & \alpha^{(m)}_{p \to (m)} & -\tilde{M}^{(m)[2]}\tilde{\Theta}^{[2]'} & -i\tilde{M}^{(m)[3]}I_{r^{(m)}} \\
-\tilde{M}^{(m)[3]}I_{r^{(m)}} & \tilde{M}^{(m)[2]}\tilde{\Theta}^{[2]'} & \tilde{M}^{(m)[1]}\tilde{\Theta}^{[1]'} & \alpha^{(m)}_{p \to (m)} \\
-\tilde{M}^{(m)[1]}\tilde{\Theta}^{[1]'} & -\tilde{M}^{(m)[3]}I_{r^{(m)}} & \tilde{M}^{(m)[2]}\tilde{\Theta}^{[2]'} & \tilde{M}^{(m)[1]}\tilde{\Theta}^{[1]'}
\end{pmatrix}.
\]

(D.59)

\[
T_1^{(m)''} = \begin{pmatrix}
-\frac{1}{3}I_{n_1^{(m)''}} & 0 & \frac{2}{3}I_{n_1^{(m)''}} & 0 & \frac{2}{3}I_{n_1^{(m)''}} & 0 \\
0 & -\frac{1}{3}I_{n_2^{(m)''}} & 0 & \frac{2}{3}I_{n_2^{(m)''}} & 0 & \frac{2}{3}I_{n_2^{(m)''}} \\
\frac{2}{3}I_{n_1^{(m)''}} & 0 & -\frac{1}{3}I_{n_1^{(m)''}} & 0 & \frac{2}{3}I_{n_1^{(m)''}} & 0 \\
0 & \frac{2}{3}I_{n_2^{(m)''}} & 0 & -\frac{1}{3}I_{n_2^{(m)''}} & 0 & \frac{2}{3}I_{n_2^{(m)''}} \\
\frac{2}{3}I_{n_2^{(m)''}} & 0 & \frac{2}{3}I_{n_1^{(m)''}} & 0 & -\frac{1}{3}I_{n_1^{(m)''}} & 0 \\
0 & \frac{2}{3}I_{n_2^{(m)''}} & 0 & \frac{2}{3}I_{n_1^{(m)''}} & 0 & -\frac{1}{3}I_{n_2^{(m)''}}
\end{pmatrix},
\]

(D.60)

\[
T_1^{(m)'''} = \begin{pmatrix}
-\frac{1}{3}I_{n_1^{(m)''}} & 0 & 0 & \pm \frac{\sqrt{3}}{2}I_{n_1^{(m)''}} & 0 & 0 \\
0 & -\frac{1}{3}I_{n_2^{(m)''}} & 0 & 0 & \pm \frac{\sqrt{3}}{2}I_{n_2^{(m)''}} & 0 \\
0 & 0 & -\frac{1}{3}I_{n_1^{(m)''}} & 0 & 0 & \pm \frac{\sqrt{3}}{2}I_{n_3^{(m)''}} \\
\pm \frac{\sqrt{3}}{2}I_{n_1^{(m)'''}} & 0 & 0 & -\frac{1}{3}I_{n_1^{(m)'''}} & 0 & 0 \\
0 & \pm \frac{\sqrt{3}}{2}I_{n_2^{(m)'''}} & 0 & 0 & \pm \frac{\sqrt{3}}{2}I_{n_2^{(m)'''}} & 0 \\
0 & 0 & \pm \frac{\sqrt{3}}{2}I_{n_3^{(m)'''}} & 0 & 0 & -\frac{1}{3}I_{n_3^{(m)'''}}
\end{pmatrix},
\]

(double sign in the same order),

(D.61)
respectively. Those submatrices are compactly expressed as

\[(T'_1)^{(m')}_{(k-k-\varphi)} = \hat{M}^{(m')}\hat{\Theta}^{(m')\varphi}I_{r(m)}, \quad (T''_1)^{(m'')}_{(k-k-\varphi)} = \left(-\frac{1}{3}\delta_{q0} + \frac{2}{3}\delta_{q\pm2}\right)I_{n(k'')} , \quad (T'''_1)^{(m''')}_{(k-k-\varphi)} = \left(-\frac{1}{2}\delta_{q0} \pm \frac{\sqrt{3}}{2}i\delta_{q3}\right)I_{n(k''')} , \]

where \(\hat{\Theta}^{(m)[-q]} = (-1)^q\hat{\Theta}^{(m)[q]}\hat{\Theta}^{(m)[0]} = I_{r(m)}, \hat{\Theta}^{(m)[3]} = -iI_{r(m)}, n_{k'} = n_{k+2} \) and \(n_{k''} = n_{k+3}\). Using \(\hat{\Theta}^{(m)[1]}, \hat{\Theta}^{(m)[2]}\) and those hermitian conjugates, eqs. (D.39) and (D.41) – (D.43) are rewritten as

\[
2a^{(m)}\hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]} + (\hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]}\dagger)^2 = (\hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]})^2 - 2\hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]}\hat{M}^{(m)[3]}\hat{\Theta}^{(m)[3]},
\]

\[
(1 - a^{(m)})\hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]} = \hat{M}^{(m)[3]}\hat{\Theta}^{(m)[3]}\hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]} + 2\hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]}\hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]}\dagger ,
\]

\[
(1 + a^{(m)})\hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]} = (\hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]})^2 + \hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]}\hat{M}^{(m)[3]}\hat{\Theta}^{(m)[3]} + (\hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]}\dagger)^2 ,
\]

\[
(1 + 2a^{(m)})\hat{M}^{(m)[3]}\hat{\Theta}^{(m)[3]} = \hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]}\hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]} - \hat{M}^{(m)[2]}\hat{\Theta}^{(m)[2]}\hat{M}^{(m)[1]}\hat{\Theta}^{(m)[1]}\dagger ,
\]

respectively.

### E Possible forms of a matrix \(A\) under the condition \(AA^\dagger\) and \(A^\dagger A\) are diagonal

Let \(A\) be an \(l_1 \times l_2\) matrix with rank \(r\). In general, \(A\) can be expressed by using \(l_1 \times l_4\) unitary matrices \(U_i\) (\(i = 1, 2\)) and a matrix \(\hat{A}\) as \(A = U_1^\dagger \hat{A}U_2\), where \(\hat{A}\) is an \(l_1 \times l_2\) matrix and written by using a diagonal matrix \(\hat{A}_r\) with rank \(r\) as

\[
\hat{A} = \begin{pmatrix} \hat{A}_r & 0 \\ 0 & 0 \end{pmatrix} . \tag{E.1}
\]

We can take \(\hat{A}_r\) as

\[
\hat{A}_r = \begin{pmatrix} a_1I_{n_1} & & \\ & a_2I_{n_2} & \\ & & \ddots \\ & & & a_qI_{n_q} \end{pmatrix} , \quad a_k > 0 \quad \text{for} \quad k = 1, \ldots, q , \tag{E.2}
\]

where \(a_k \neq a_{k'}\) for \(k \neq k'\), \(n_k\) is a positive integer satisfying \(\sum_{k=1}^q n_k = r\), and \(I_{n_k}\) is the \(n_k \times n_k\) unit matrix. Then, we find

\[
U_1AA^\dagger U_1^\dagger = \hat{A}\hat{A}^\dagger = \begin{pmatrix} \hat{A}_r^2 & 0 \\ 0 & 0 \end{pmatrix} , \quad U_2A^\dagger AU_2^\dagger = \hat{A}^\dagger \hat{A} = \begin{pmatrix} \hat{A}_r^2 & 0 \\ 0 & 0 \end{pmatrix} . \tag{E.3}
\]
Note that the former (latter) is an \( l_1 \times l_1 \) \((l_2 \times l_2)\) matrix.

Let us take a basis where \( AA^\dagger \) and \( A^\dagger A \) are already diagonal and satisfy \( AA^\dagger = \hat{A} \hat{A}^\dagger \) and \( A^\dagger A = \hat{A}^\dagger \hat{A} \). Then, we find \( U_1 AA^\dagger U_1^\dagger = AA^\dagger \) and \( U_2 A^\dagger A U_2^\dagger = A^\dagger A \), which implies \([U_1, AA^\dagger] = 0\) and \([U_2, A^\dagger A] = 0\). In this case, a possible form of \( U_i \) is restricted as

\[
U_i = \begin{pmatrix}
    u_i^{(1)} & u_i^{(2)} & \cdots & u_i^{(q)} \\
    & & & \\
    & & & u_i^{(1)}
\end{pmatrix}, \tag{E.4}
\]

where \( u_i^{(k)} \) and \( \bar{u}_i \) are \( n_k \times n_k \) and \((l_i - r) \times (l_i - r)\) unitary matrices, respectively. Therefore, we find that a general form of \( A \) is given by

\[
A = U_1^\dagger \hat{A} U_2 = \begin{pmatrix}
    \hat{A}_r & 0 \\
    0 & 0
\end{pmatrix}, \quad \hat{A}_r = \begin{pmatrix}
    a_1 \bar{u}_r^{(1)} & a_2 \bar{u}_r^{(2)} & \cdots & a_q \bar{u}_r^{(q)} \\
    & & & \\
    & & & a_1 \bar{u}_r^{(1)}
\end{pmatrix} = \hat{A}_r U, \tag{E.5}
\]

where \( \hat{A}_r \) is a diagonal matrix with submatrices \((\hat{A}_r)_{(kl)} = a_k I_{n_k} \delta_{kl} (k, l = 1, \cdots, q)\) as seen from eq. (E.2), and \( U \) is a unitary matrix with a diagonal submatrices \((U)_{(kl)} = \bar{u}_r^{(k)} \delta_{kl}\). Here \( \bar{u}_r^{(k)} \) is an \( n_k \times n_k \) unitary matrix defined by \( \bar{u}_r^{(k)} = u_1^{(k)} u_2^{(k)} \). Then, it is easy to see that \( U \) commutes with \( A_r \).

### F Derivation of eqs. (C.3) and (D.4)

The unitary matrices \( R_0 \) and \( T_1 \) on \( T^2/\mathbb{Z}_N \) \((N = 3, 4, 6)\) contain submatrices denoted as \((R_0)_{(kl)} = \tau^k \delta_{kl} I_{n_k}\) with \( \tau = e^{2\pi i/N} \) and \((T_1)_{(kl)} = M_{kl}^{[k-l]}\). The upper index \( k - l = q \) represents the charge of the \( \mathbb{Z}_N \) symmetry generated by \( R_0 \): \((R_0 T_1 R_0^{-1})_{(k-k)} = \tau^q M_{k-k}^{[q]} \). We use a notation of \( M_{k-l}^{[k-l]} = M_{k-l}^{[k-l']} \) and \( k' = k \) \((\text{mod } N)\) and \( l' = l \) \((\text{mod } N)\).

Eqs. (C.3) and (D.4), i.e., \( M_{k-k}^{[q]} M_{k-k}^{[q-q']} = M_{k-q-k}^{[q-q']} \) \( M_{k-k}^{[q-q']} \), are derived as follows. We define \( T_m \) as \( T_m = R_0^{m-1} T_1 R_0^{-1-m} \). Then the relation \( T_{m'} T_m = T_m T_{m'} \) leads to \( T_1 R_0^{m-m'} T_1 = R_0^{m-m'} T_1 R_0^{m-m'} T_1 R_0^{m-m'} \) and, using it, we derive the relation:

\[
\sum_{q'} \tau^{(l-k-q')} M_{k-k}^{[q']} M_{k-k}^{[q-q']} = \sum_{q'} \tau^{(l-k+q'-q)} M_{k-k}^{[q']} M_{k-k}^{[q-q]}, \tag{F.1}
\]

where \( l = m - m' \) takes an integer and the summation over \( q' \) can be taken for any successive \( N \) integers. Changing \( q' \) into \( q - q' \) in the right-hand side of eq. (F.1), we obtain the relation:

\[
\sum_{q'} \tau^{(l-k-q')} M_{k-k}^{[q]} M_{k-k}^{[q-q']} = \sum_{q'} \tau^{(l-k-q)} M_{k-k}^{[q-q']} M_{k-k}^{[q-q]}, \tag{F.2}
\]
Multiplying $\frac{1}{N} \sum_{l=1}^{N} \tau^{l(q^\prime-q)}$ by eq. (E.2) and using the relation $\frac{1}{N} \sum_{l=1}^{N} \tau^{l(q^\prime-q)} = \delta_{q^\prime,q}$, we obtain the relation:

$$M[q']_k M[q'']_k = M[q']_k M[q'']_k M[q']_k M[q'']_k$$

where $q''$ is replaced with $q'$. For reference, using eq. (F.3), we obtain the relation:

$$M[q']_k M[q']_k M[q']_k M[q']_k = M[q']_k M[q']_k M[q']_k M[q']_k$$

Taking $q = 0$ and $q'' = 0$ in eq. (F.4), we obtain the relation:

$$[M[q']_k M[q']_k M[q']_k] = 0.$$  \hspace{1cm} (F.5)

## G Derivation of eqs. (8.8) - (8.12)

The specific relations for $N = 3$ are given by

$$a_1^2 + a_2^2 + a_3^2 - 3a_1a_2a_3 = 1, \quad |a_1|^2 + |a_2|^2 + |a_3|^2 = 1, \quad a_1a_3 + a_2a_2 + a_2a_1 = 0.$$  \hspace{1cm} (G.1)

From the first relation and the combination of the second and third ones in eq. (G.1), we obtain the relations:

$$\begin{align*}
(\omega a_1 + \omega^2 a_2 + a_3)(\omega^2 a_1 + \omega a_2 + a_3)(a_1 + a_2 + a_3) &= 1, \\
|\omega a_1 + \omega^2 a_2 + a_3|^2 &= |\omega^2 a_1 + \omega a_2 + a_3|^2 = |a_1 + a_2 + a_3|^2 = 1,
\end{align*}$$

respectively. Using $a_j = \sum_{p=1}^{3} a_j\omega^{jp}$, eqs. (G.2) and (G.3) are rewritten compactly as

$$\begin{align*}
\alpha_1\alpha_2\alpha_3 &= 1, \\
|\alpha_j|^2 &= 1, \quad (j = 1, 2, 3),
\end{align*}$$

respectively.

In the same way, eqs. (8.8), (8.10) and (8.11) are obtained, using the specific relations such that $2|a_1|^2 + a_2^2 + a_3^2 = 1$ and $2a_2a_4 = a_2^2 + \overline{a_1}^2$ with $a_2 = \overline{a_2}, a_3 = -\overline{a_3}$ and $a_4 = \overline{a_4}$ for $N = 4$ and $2|a_1|^2 + 2|a_2|^2 + |a_3|^2 + |a_5|^2 = 1, |a_1|^2 - |a_2|^2 - |a_3|^2 + |a_5|^2 = a_6, 2a_2a_4 + \overline{a_2}^2 = a_7 - 2\overline{a_1}a_3, a_1a_6 + a_3a_2 + 2a_2\overline{a_1} = a_1, -a_2a_6 + a_3 + \overline{a_1}a_3 + 2\overline{a_2} = a_2$ and $-2a_3a_6 + a_1a_2 - a_1a_2 = a_3$ with $a_3 = -\overline{a_3}, a_4 = \overline{a_4}, a_5 = -\overline{a_5}$ and $a_6 = \overline{a_6}$ for $N = 6$.

Here, we explain the reason why the relations (8.8) - (8.11) hold. From $t_1 = \sum_{p=1}^{N} a_p Y^p$ and $t_m = \sum_{p=1}^{N} a_p Y^p$, we obtain the relation:

$$t_m = X^{m-1}(\sum_{p=1}^{N} a_p Y^p)X^{1-m} = \sum_{p=1}^{N} a_p \tau^{(m-1)p} Y^p.$$  \hspace{1cm} (G.5)

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using the relation \(X^m Y^{m'} = \tau^{m m'} Y^{m'} X^m\). The unitary matrices \(t_m\) are diagonalized simultaneously by the unitary transformation:

\[
Ut_m U^\dagger = \sum_{p=1}^{N} a_p \tau^{(m-1)p} (UYU^\dagger)^p = \sum_{p=1}^{N} a_p \tau^{(m-1)p} X^p,
\]

(G.6)

where the \((j, j')\) elements of \(X, Y\) and \(U\) are given by

\[
(X)_{jj'} = \tau^j \delta_{jj'}, \quad (Y)_{jj'} = \delta_{jj'+1}, \quad (U)_{jj'} = \frac{1}{\sqrt{N}} \tau^j (U^\dagger)^{j'+1},
\]

(G.7)

respectively. Because \(Ut_m U^\dagger\) obey the same constraints as \(t_m\), those eigenvalues should do. The \((1, 1)\) element of \(Ut_m U^\dagger\) is given by \((Ut_m U^\dagger)_{11} = \sum_{p=1}^{N} a_p \tau^{mp}\), and we denote it as \(\alpha_m = \sum_{p=1}^{N} a_p \tau^{mp}\). Thus, we find that \(\alpha_m\) satisfy the relations (G.8) – (G.11) corresponding to \(t_m^* t_m = I\) and peculiar constraints in Table 2. The same relations are obtained for other diagonal elements of \(Ut_m U^\dagger\).

Finally, we derive eq. (G.12), i.e., \(t_1 = \sum_{p=1}^{N} a_p Y^p = e^{i(\theta Y + \bar{\theta} Y^{N-1})}\). From eqs. (G.8) – (G.11), \(\alpha_j = \sum_{p=1}^{N} a_p \tau^{j p}\) are parametrized by a complex number \(\theta\) as follows,

\[
\alpha_j = \sum_{p=1}^{N} a_p \tau^{j p} = e^{i(\theta j^i + \bar{\theta} j^i)},
\]

(G.8)

because the number of independent parameters on \(\alpha_j\) is two for the \(N = 3, 4, 6\) cases. Here, we derive eq. (G.8) for \(N = 6\). From eq. (G.8), \(\alpha_j\) are written by \(\alpha_j = e^{i \phi_j}\) using real parameters \(\phi_j\). From eq. (G.11), \(\phi_j\) obey \(\phi_1 + \phi_4 = \phi_2 + \phi_5 = \phi_3 + \phi_6 = \phi_1 + \phi_3 + \phi_5 = \phi_2 + \phi_4 + \phi_6 = 0\) (mod \(2\pi\)). Taking \(\phi_2\) and \(\phi_6\) as independent ones, others are determined as \(\phi_1 = \phi_2 + \phi_6, \quad \phi_3 = -\phi_6, \quad \phi_4 = -\phi_2 - \phi_6\) and \(\phi_5 = -\phi_2\) (mod \(2\pi\)). Then, using a complex parameter \(\theta = \frac{1}{2} \varphi_6 - \frac{1}{2\sqrt{2}} (2\varphi_2 + \varphi_6)\) made of \(\varphi_2\) and \(\varphi_6\), \(\varphi_j\) are parametrized as \(\varphi_j = \theta j^i + \bar{\theta} j^i\) with \(\eta = e^{2\pi i / 6}\). In the same way, \(\alpha_j\) are expressed as eq. (G.8) for \(N = 3, 4\). Using eq. (G.8) and \((X)_{jj'} = \tau^j \delta_{jj'}\), we obtain the relation:

\[
\sum_{p=1}^{N} a_p X^p = e^{i(\theta X + \bar{\theta} X)} = e^{i(\theta X + \bar{\theta} X^{N-1})}.
\]

(G.9)

Performing the unitary transformation such as \(U^\dagger X U = Y\) for eq. (G.9), we arrive at eq. (G.12):

\[
t_1 = \sum_{p=1}^{N} a_p Y^p = e^{i(\theta Y + \bar{\theta} Y^{N-1})}.
\]

(G.10)

For reference, multiplying \(\frac{1}{N} \sum_{j=1}^{N} \tau^{-jp}\) by eq. (G.8) and using \(\frac{1}{N} \sum_{j=1}^{N} \tau^{jp} = \delta_{pp'}\), we obtain the relation:

\[
ap = \frac{1}{N} \sum_{j=1}^{N} \alpha_j \tau^{-jp} = \frac{1}{N} \sum_{j=1}^{N} \tau^{jp} e^{i(\theta j^i + \bar{\theta} j^i)},
\]

(G.11)

where \(p'\) is replaced with \(p\).
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References

[1] N. Manton, A new six-dimensional approach to the Weinberg-Salam model, Nucl. Phys. B 158 (1979), 141.

[2] H. Georgi and S. L. Glashow, Unity of All Elementary Particle Forces, Phys. Rev. Lett. 32 (1974) 438.

[3] S. Dimopoulos and H. Georgi, Softly broken supersymmetry and SU(5), Nucl. Phys. B 193 (1981) 150.

[4] N. Sakai, Naturalness in supersymmetric GUTS, Z. Phys. C 11 (1981) 153.

[5] Y. Kawamura, Gauge Symmetry Reduction from the Extra Space $S^1/Z_2$, Prog. Theor. Phys. 103 (2000) 613 [hep-ph/9902423].

[6] Y. Kawamura, Triplet-doublet Splitting, Proton Stability and an Extra Dimension, Prog. Theor. Phys. 105 (2001) 999 [hep-ph/0012125].

[7] L. Hall and Y. Nomura, Gauge Unification in Higher Dimensions, Phys. Rev. D 64 (2001) 055003 [hep-ph/0103125].

[8] M. Kubo, C. S. Lim and H. Yamashita, The Hosotani mechanism in bulk gauge theories with an orbifold extra space $S^1/Z_2$, Mod. Phys. Lett. A 17 (2002) 2249 [hep-ph/0111327].

[9] C. Csaki, C. Grojean and H. Murayama, Standard model Higgs from higher dimensional gauge fields, Phys. Rev. D 67 (2003) 085012 [hep-ph/0210133].

[10] C. A. Scrucca, M. Serone and L. Silvestrini, Electroweak symmetry breaking and fermion masses from extra dimensions, Nucl. Phys. B 669 (2003) 128 [hep-ph/0304220].

[11] N. V. Krasnikov, Ultraviolet Fixed Point Behavior Of The Five-Dimensional Yang-Mills Theory, The Gauge Hierarchy Problem And A Possible New Dimension At The Tev Scale, Phys. Lett. B 273 246 (1991).

[12] H. Hatanaka, T. Inami and C. S. Lim, The gauge hierarchy problem and higher dimensional gauge theories, Mod. Phys. Lett. A 13 2601 (1998) [hep-th/9805067].

[13] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Electroweak symmetry breaking from dimensional deconstruction, Phys. Lett. B 513 (2001) 232 [hep-ph/0105239].
[14] N. Maru and T. Yamashita, Two-loop calculation of Higgs mass in gauge-Higgs unification: 5D massless QED compactified on $S^1$, Nucl. Phys. B 754 127 (2006) [hep-ph/0603237].

[15] Y. Hosotani, N. Maru, K. Takenaga and T. Yamashita, Two loop finiteness of Higgs mass and potential in the gauge-Higgs unification, Prog. Theor. Phys. 118 1053 (2007) [arXiv:0709.2844 [hep-ph]].

[16] J. Hisano, Y. Shoji and A. Yamada, To be, or not to be finite? The Higgs potential in Gauge Higgs Unification, JHEP 02 (2020), 193 [arXiv:1908.09158 [hep-ph]].

[17] N. Haba, Y. Hosotani, Y. Kawamura and T. Yamashita, Dynamical symmetry breaking in gauge Higgs unification on orbifold, Phys. Rev. D 70 (2004) 015010 [hep-ph/0401183].

[18] C. S. Lim and N. Maru, Towards a realistic grand gauge-Higgs unification, Phys. Lett. B 653 (2007) 320-324 [arXiv:0706.1397 [hep-ph]].

[19] Y. Hosotani and N. Yamatsu, Gauge–Higgs grand unification, Prog. Theor. Exp. Phys. 2015 (2015) 111B01 [arXiv:1504.03817 [hep-ph]].

[20] K. Kojima, K. Takenaga and T. Yamashita, The Standard Model Gauge Symmetry from Higher-Rank Unified Groups in Grand Gauge-Higgs Unification Models, JHEP 06 (2017) 018 [arXiv:1704.04840 [hep-ph]].

[21] K. Kojima, K. Takenaga and T. Yamashita, Grand Gauge-Higgs Unification, Phys. Rev. D 84 (2011) 051701 [arXiv:1103.1234 [hep-ph]].

[22] K. Kojima, K. Takenaga and T. Yamashita, Gauge symmetry breaking patterns in an SU(5) grand gauge-Higgs unification model, Phys. Rev. D 95 (2017) 015021 [arXiv:1608.05496 [hep-ph]].

[23] T. Yamashita, Doublet-Triplet Splitting in an SU(5) Grand Unification, Phys. Rev. D 84 (2011) 115016 [arXiv:1106.3229 [hep-ph]].

[24] M. Kakizaki, S. Kanemura, H. Taniguchi and T. Yamashita, Higgs sector as a probe of supersymmetric grand unification with the Hosotani mechanism, Phys. Rev. D 89 (2014) 075013 [arXiv:1312.7575 [hep-ph]].

[25] H. Nakano, M. Sato, O. Seto and T. Yamashita, Dirac gaugino from grand gauge-Higgs unification, Prog. Theor. Exp. Phys. 2022 (2022) 033B06 [arXiv:2201.04428 [hep-ph]].

[26] Y. Hosotani, Dynamical mass generation by compact extra dimensions, Phys. Lett. B 126 (1983) 309.

[27] Y. Hosotani, Dynamics of Nonintegrable Phases and Gauge Symmetry Breaking, Ann. of Phys 190 (1989) 233.
[28] N. Haba, M. Harada, Y. Hosotani and Y. Kawamura, *Dynamical rearrangement of gauge symmetry on the Orbifold $S^1/Z_2$*, Nucl. Phys. B 657 (2003) 169 [Errata ibid B 669 (2003) 381] [hep-ph/0212035].

[29] N. Haba, Y. Hosotani and Y. Kawamura, *Classification and Dynamics of Equivalence Classes in SU(N) gauge theory on the orbifold $S^1/Z_2$*, Prog. Theor. Phys. 111 (2004) 265 [hep-ph/0309088].

[30] N. Haba and T. Yamashita, *A General formula of the effective potential in 5-D SU(N) gauge theory on orbifold*, JHEP 02 (2004) 059 [hep-ph/0401185].

[31] Y. Kawamura, T. Kinami and T. Miura, *Equivalence Classes of Boundary Conditions in Gauge Theory on $Z_3$ Orbifold*, Prog. Theor. Phys. 120 (2008) 815 [arXiv:0808.2333].

[32] Y. Kawamura and T. Miura, *Equivalence Classes of Boundary Conditions in SU(N) Gauge Theory on 2-Dimensional Orbifolds*, Prog. Theor. Phys. 122 (2009) 847 [arXiv:0905.4123].

[33] Y. Goto and Y. Kawamura, *Orbifold family unification using vectorlike representation on six dimensions*, Phys. Rev. D 98 (2018) 035039 [arXiv:1712.06444].

[34] Y. Hosotani, S. Noda and K. Takenaga, *Dynamical gauge symmetry breaking and mass generation on the Orbifold $T^2/Z_2$*, Phys. Rev. D 69 (2004) 125014 [hep-ph/0403106].

[35] Y. Kawamura and Y. Nishikawa, *On diagonal representatives in boundary condition matrices on orbifolds*, Int. J. Mod. Phys. A 35 (2020) 2050206 [arXiv:2009.10958].

[36] G. 't Hooft, *A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories*, Nucl. Phys. B 153 (1979) 141.

[37] G. von Gersdorff, *A New Class of Rank Breaking Orbifolds*, Nucl. Phys. B 793 (2008) 192 [arXiv:0705.2410].

[38] C. Bachas, *A way to break supersymmetry*, [hep-ph/9503030].

[39] C. A. Scrucca and M. Serone, *Anomalies in field theories with extra dimensions*, Int. J. Mod. Phys. A 19 (2004) 2579 [hep-th/0403163].

[40] C. A. Scrucca, M. Serone, L. Silvestrini and A. Wulzer, *Gauge Higgs unification in orbifold models*, JHEP 02 (2004) 049 [hep-th/0312267].

[41] S. Förste, H. P. Nilles and A. Wingerter, *Geometry of Rank Reduction*, Phys. Rev. D 72 (2005) 026001 [hep-ph/0504117].