AN ITÔ FORMULA FOR ROUGH PARTIAL DIFFERENTIAL EQUATIONS. APPLICATION TO THE MAXIMUM PRINCIPLE

ANTOINE HOCQUET AND TORSTEIN NILSSEN

Abstract. We investigate existence, uniqueness and regularity for solutions of rough parabolic equations with transport noise of the form \( \partial_t u - A_t u - f = X_t \cdot \nabla u \), and then prove an Itô Formula (in the sense “chain rule”) for Nemytskii operations of the form \( u \mapsto F(u) \), where \( F \) is \( C^2 \). Our method is based on energy estimates, and a generalization of the Moser Iteration argument to prove boundedness of the solution. In particular, we do not make use of any flow transformation. We also address independently the case when \( F(u) = |u|^p \) with \( p \geq 2 \), which applies for continuous solutions of (non-necessarily parabolic) equations in \( L^p \). As an application of these results, we provide existence, uniqueness and a weak maximum principle for a relatively wide class of parabolic equations with homogeneous Dirichlet boundary conditions.

Keywords— rough paths, rough PDEs, energy method, weak solutions, Itô formula, maximum principle

Mathematics Subject Classification — 60H15, 35A15, 35B50, 35D30

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Date: June 28, 2018.

Financial support by the DFG via Research Unit FOR 2402 is gratefully acknowledged.
1. Introduction

In his seminal paper on differential equations, Lyons [Lyo98] noted: “we hope that by solving the one dimensional differential equation without using predictability, our ideas might produce a few pointers to the correct way to treat PDE’s driven by spatial noise”. Years later, based precisely on the insight that the variable in rough path theory could represent “space” rather than “time”, Hairer’s gave a solution to the so-called KPZ equation [Hai13] (see also [HW13]), hence showing that Lyon’s intuition was, in essence, right. Regarding e.g. [GT10, Hai13, HW13, Hai14, GIP15, OW16, BB16], the understanding of singular equations with space-time white noise $\xi(t, x)$ of the form

$$\partial_t u - \Delta u = F_0(u, \nabla u) + F_1(u) \xi$$

has now reached a certain level of maturity.

Our intention in this work is to take a direction which is in some sense, complementary to the aforementioned works. Namely, we are interested in perturbations that are irregular in time only, but for which an effect is observed on the space regularity of the solution. This situation is typically achieved in the case of a parabolic equation with transport noise of the form

$$\partial_t u - Au = b^i(t, x) \frac{dX}{dt} \partial_i u, \quad \text{on } (0, T] \times \mathbb{R}^d, \tag{1.1}$$

(with implicit summation over repeated indices) where $X$ denotes a Brownian motion, defined on some probability space and here the product with the noise term is subject to different possible meanings (Stratonovitch or Itô for instance). Note that rough PDEs such as (1.1) – that is with roughness in time only – have been investigated in [CF09, FO11, CFO11, FO14] where flow transformations, coupled with deterministic viscosity theory are used. More recently, an approach using a Feynmann-Kac representation has been proposed in [DFS17]. Here we will use a different method based on energy estimates and Sobolev spaces, following in some sense Krylov’s analytic approach [Kry99], but also using ideas from transport theory [DL89, Amb04] (see also [De 07] for a nice overview of the topic). We note that some interesting results on “rough” transport equations have been obtained e.g. in [DD16, CC+18, Cat16] (see also [FGP10], in a stochastic context). It should be observed that we do not use any probabilistic argument in our proofs, though stochastic analysis constitutes the main motivation for studying (1.2).

In this paper, we investigate the problem (1.1) under more general assumptions, by looking at rough parabolic equations of the form

$$\left\{ \begin{array}{lcl}
\text{du}_t - (A_t u + f_t(x))dt &=& dBu_t, \quad \text{on } (0, T] \times \mathbb{R}^d \\
u_0 \text{ given in } L^p(\mathbb{R}^d),
\end{array} \right.$$  

\tag{1.2}
where the unknown \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is identified with a path with values in the Lebesgue space \( L^p(\mathbb{R}^d) \), for some \( p \in [1, \infty] \). Here \((B_t)_{t \in [0,T]}\) should be a path with values in the space of differential operators of order one, and as usual, we adopt the differential notation “d” to say that (1.2) holds in the sense of increments (which means that we have a suitable “rough integration” map \( I_B \) in order to interpret the above right hand side, see section 2). We will assume throughout the paper that \( A_t : W^{1,p} \to W^{-1,p} \) is strictly elliptic for each \( t \in [0, T] \), and that \( t \mapsto B_t \) is continuous with finite \( 1/\alpha \)-variation, \( \alpha \) being in the interval \( (1/3, 1/2] \). Moreover, we are given a free term matching the expected regularity of \( Au \), that is \( f \in L^2(0,T;W^{-1,p}) \). We choose to work in the spaces of continuous paths of finite \( 1/\alpha \)-variation (denoted by \( V^\alpha(0,T; E) \)) instead of the Hölder spaces \( C^\alpha(0,T; E) \). Our main motivation in this work is to obtain an Itô Formula – in the sense “chain rule” – for (1.2), using the theory of unbounded rough drivers (as introduced in [BG17]). Note that similar results in the stochastic context (1.1) have been obtained for instance in [Kry11, Kry13], see also [ZdM06]. Our results are in some sense, closely related to the so-called “renormalization property” – in the sense of Di Perna and Lions in [DL89] – observed for transport equations in Sobolev spaces. Namely: for a solution \( u \) (in a suitable rough/distributional meaning), we obtain a similar equation on \( F(u) = F(u_t(x)) \), when \( F \) is in \( C^2(\mathbb{R}, \mathbb{R}) \). In particular, we will apply this result in order to obtain a weak maximum principle for some class of data \((u_0, A, f)\). We point out that, because of the lack of space regularity in the expansion (1.4) below, the solution \( u \) does not satisfy the hypotheses of [FH14, Proposition 7.6] in general. This is easily seen by observing that the integral “\( \int^t_s DF(u_r)[dBu_r] \)” is ill-defined a priori for the Nemytskii operation \( u \mapsto F(u) \), even if \( F \) is smooth.

Before presenting our main results, we shall make a few comments on rough paths analysis.

**Operator-valued rough paths; unbounded rough drivers.** Maybe the most common appeal of all the aforementioned works on singular equations (finite or infinite-dimensional) is the essential role played by algebra, which is illustrated by the following toy example. Let \( T > 0 \) be a fixed time-horizon and consider a one-dimensional, linear differential equation of the form

\[
\frac{du}{dt} = \frac{db}{dt} u \quad \text{on} \quad [0, T], \quad u_0 := \xi \in \mathbb{R}, \tag{1.3}
\]

where we consider continuous paths \( u, b : [0, T] \to \mathbb{R} \), while \( db/dt \) denotes a time-derivative. Although stochastic analysis is the main motivation behind rough path theory (think of \( b \) as a Brownian motion), an advantage of the latter is that it makes possible to treat equations of the form (1.3) deterministically, provided one adds extra information to the driving signal \( b \). The nature of the information needed
is related to the value of the iterated integrals

\[ B_{st}^1 := \int_{s<r_1<t} db_{r_1}, \quad B_{st}^2 := \int_{s<r_2<r_1<t} db_{r_1} db_{r_2}, \quad \ldots \]

seen as a collection \( B \equiv (B^1, B^2, \ldots) \) of 2-parameter mappings subject to the analytic condition \( |B_{st}^n| \lessapprox (t-s)^{\alpha n} \) for \( 0 \leq s \leq t \leq T \) and \( n \geq 0 \), where here \( \alpha > 0 \) denotes the Hölder regularity of \( b \) (for simplicity we exceptionally make use of the Hölder spaces here, however we point out that the results of this paragraph translate \( \text{mutatis mutandis} \) to the \( \frac{1}{\alpha} \)-variation setting). To illustrate why the value of \( B_{st}^1 \) is important in this context, let us integrate formally (1.3) in time. This gives

\[ u_t - u_s = \int_s^t db_r u_r = \left( \int_s^t db_r \right) u_s + \int_s^t db_r (u_r - u_s), \]

where in the sequel we adopt the differential notation \( db_r = \frac{db_r}{dt} \, dt \). We can now use (1.3) again to expand the term \( u_{st}^{1,1} := \int_s^t db_r (u_r - u_s) \) so that

\[ u_t - u_s = \left( \int_s^t db_{r_1} \right) u_s + \left( \int_{s<r_2<r_1<t} db_{r_1} db_{r_2} \right) u_s + \int_{s<r_3<r_2<r_1<t} db_{r_1} db_{r_2} db_{r_3} (u_{r_3} - u_s) \]

\[ \equiv: B_{st}^1 u_s + B_{st}^2 u_s + u_{st}^{2,2}, \quad \text{for every} \ 0 \leq s \leq t \leq T. \]  

(1.4)

(1.5)

Of course, the formula (1.4) makes sense only if one assumes suitable time-regularity for \( b \). For instance, a Hölder regularity of the form

\[ b \in C^\alpha(0,T;\mathbb{R}), \quad \text{with} \ \alpha > 1/2, \]  

(1.6)

will do since one can make sense of all the integrals by Young Theorem [You36]. In particular the regularity \( \alpha > \frac{1}{2} \) barely fails to include the brownian case, for the latter belongs to \( C^{1/2-\epsilon} \) for any \( \epsilon > 0 \). While it is clear that any “reasonable” notion of integral should lead to the equality \( B_{st}^1 \equiv \int_s^t dB_r = b_t - b_s \), on the other hand \( B^2 \) has no canonical meaning. If there exists such an object however, we expect it to satisfy the so-called Chen’s relation

\[ B_{st}^2 = B_{st}^2 + B_{st}^2 + B_{st}^2 B_{st}^1, \quad \text{for any} \ 0 \leq s \leq \theta \leq t \leq T, \]  

(1.7)

which reflects the fact that the integral should be linear with respect to its integrand, together with the additivity property “\( \int_{I_1 \cup I_2} = \int_{I_1} + \int_{I_2} \)”. The essential insight of rough paths theory is that, assuming we have an “offline interpretation” for \( B_{st}^2 \), then one can simply define the solution \( u \) to (1.3) by the expansion (1.5), to which we add the remainder term \( u^{2,2} \). Now, this remainder term is canonical, and it is obtained by projecting (in a suitable sense) the truncated series

\[ (s,t) \mapsto v_{st} := B_{st}^1 u_s + B_{st}^2 u_s \]

(1.8)

onto the space of increments \( \{(v_{st}), \ v_{st} = u_t - u_s \ \text{for some} \ u\}. \) Without going too much into details (this will be done in Section 2 below), this projection is well-defined thanks to the “compensated nature” of the local expansion (1.8), which itself stems from the fact that the algebraic relation (1.7) holds. This result is a particular case of what is called the “Sewing Lemma” (alternatively “Lyons Extension Theorem”, depending on the adopted point of view). We refer the reader to the basic references
on the topic [Lyo98, Gub04]. For the reader’s convenience, we shall recall a version of the Sewing Lemma in a Fréchet space, see Section 2 below.

More generally, one can ask whether there exists a “non-commutative” analogue of rough paths theory, namely where one substitutes numbers with operators. Assume now that \( u : [0, T] \to E \) for some topological vector space \( E \) and let \( b : [0, T] \to \mathcal{A} \), where \( \mathcal{A} \) is an algebra of operators in \( E \). In the case \( \text{dim}E < \infty \), using coordinates it can be shown that the above is just a particular case of a finite-dimensional rough differential equation. If one assumes however that \( E \) is a genuinely infinite dimensional Banach space, and \( \mathcal{A} \) an algebra of bounded linear operators, then the latter generalization becomes non trivial. However, as shown e.g. in [CL14] and [BG17, Section 2] (see also [FdLPM08]), the usual rough paths theory generalizes well to this situation.

Motivated by (1.1), one wishes now to look at the case where \( E \) is the Lebesgue space \( L^p(\mathbb{R}^d) \), for some \( p \in [1, \infty] \), and \( b \) takes values in the space \( \mathbb{D}_k \) consisting of differential operators of order \( k \in \mathbb{N} \). The composition of \( b \) with itself is now expected to be an element of \( \mathbb{D}_{2k} \), and so is for the second component of \( B_\ast \). In the transport case (1.1) this means that \( B_{\ast st} u_s \) is for a.e. \( 0 \leq s \leq t \leq T \) an element of the negative Sobolev space \( W^{-1,p} \), while \( B_{\ast st} u_s \) should be in \( W^{-2,p} \), but with a better time-regularity. Similarly, the remainder \( d^{st} \) is expected to be an element in \( W^{-3,p} \), with a time-regularity which is sufficiently large so that the expansion (1.4) fully characterizes the solution. The above loss of space-regularity (in exchange however of better time-regularity) is a crucial observation to make in the context of equations of the form (1.1). This phenomenon is at the origin of the difficulties encountered while trying to apply non-linear operations to solutions of (1.1). We can now take a step back and, at the cost of a little effort of abstraction, define an unbounded rough driver (URD in short) as a two parameter object \( B_{\ast st} = (B_{\ast st}^1, B_{\ast st}^2) \) such that for a suitable set of indices \( k \in S \subset \mathbb{Z} \), and \( i = 1, 2 \), it holds

\[
|B_{\ast st}^i|_{\mathcal{S}(W^k,p,W^{k-1},p)} \lesssim (t-s)^{1/\alpha}
\]

(here the left hand side denotes the operator norm). This point of view, which was introduced in [BG17], will be developed further here, by defining a notion of “geometric” URD, but also that of a “bracket” (by analogy with the bracket \( 2X_{\ast st} - X_{\ast st} \otimes X_{\ast st} \) of a 2-step rough path \( X \)). In particular, following the usual rough path approach, it will be assumed throughout the paper that \( B \) is given (although we shall provide some concrete examples for the reader’s convenience).

The ansatz. We consider the parabolic rough partial differential equation (1.2), where the principal part is an elliptic operator on divergence form

\[
A_i u(x) = \partial_i (a^{ij}(t, x) \partial_j u(x)),
\]

with coefficients \( a^{ij} \) being possibly discontinuous but bounded above and below (see assumption 1.2), and \( B = (B^1, B^2) \) is an unbounded rough driver. The free term \( f \) is an element of the Sobolev space \( L^2(0, T; H^{-1}) \).

We are essentially interested in the case where the rough driver \( B \) is “transport-like”, that is, we assume that \( B \) takes its values in the space of first order differential operators (see Definition 1.5 below). Our results in particular apply to the case
where \((B^1, B^2)\) is the unbounded rough driver, given by

\[
B^1_{st} := X^k_{st} \sigma^k \partial_i, \quad B^2_{st} := X^k_{st} \sigma^k \partial_i (\sigma^l \partial_j),
\]

where \((X^k, X^k_l)_{1 \leq k, l \leq m}\) is a geometric, continuous 2-step rough path, while the coefficients \(\sigma\) are in \(W^{3,\infty}\). The geometricity assumption is essential to avoid any problem related to the so-called “strong parabolicity” requirement (this is discussed more precisely in the next paragraph).

It should be possible to deal with general boundary problems, such as periodic and non-homogeneous Dirichlet or Neumann on a bounded open set \(D \subset \mathbb{R}^d\). However, in order to keep this work at a reasonable size, we refrain from doing so, and focus on the whole space scenario. We will show however how our results apply to the homogeneous Dirichlet problem on a smooth, bounded domain. This case is of particular interest as it displays a weak maximum principle (under suitable assumptions on the coefficients). Moreover, the approach used here generalizes well to more general operators \(A\) (for instance adding a perturbation \(b_i(t, x) \partial_i u + c(t, x) u\) with integrability conditions on \(b, c\)), but we choose to restrict to the above situation for the sake of simplicity.

**Geometricity and strong parabolicity.** We will provide an intrinsic notion of geometric unbounded rough driver, i.e. without referring to any finite-dimensional geometricity. Geometric URDs are, roughly speaking, those that can be well-approximated (in a suitable URD metric) by smooth paths \(b(n) : [0, T] \rightarrow T_1\), where \(T_1\) is the space of bounded linear maps from \(W^{k,p}\) to \(W^{k-1,p}\) for \(k \in \mathcal{S} \subset \mathbb{Z}\). Such URDs exhibit interesting properties, as for instance, if one assumes that \(B^1\) takes values in the space of differential operators \(\mathbb{D}_1 \subset T_1\), then its “bracket” – this term is used in the rough path sense namely: the difference between \(B^2_{st} - B^1_{st} B^1_{st}\), see [FH14, Chapter 5] and Definition 1.2 below – is also of order one, which is reminiscent to the fact that the Lie bracket of a vector field is itself a vector field, and not a second order differential operator. Interestingly, this gives a rough path interpretation of the usual “Stochastic Parabolicity” assumption (we refer e.g. to [KR77] and references therein). To wit, assume for instance that \(B^1 := \sigma \cdot \nabla(W_t - W_s)\) where \(W : \Omega \times [0, T] \rightarrow \mathbb{R}\) is a Brownian motion, and define \(B^2\) as the Itô enhancement, that is

\[
B^2 := \sigma \cdot \nabla(\sigma \cdot \nabla) W^\text{Itô}_{st} := \sigma \cdot \nabla(\sigma \cdot \nabla)[\frac{1}{2}(W_t - W_s)^2 - (t - s)].
\]

In this case one sees that the bracket \(L \equiv B^2 - \frac{1}{2} B^1 B^1\) is given by \(L_{st} := -(t - s) \sum_{ij} \sigma^i \sigma^j \partial_{ij}\), which has positive sign in the sense of linear operators, hence having a “de-regularizing effect”. If \(\sigma\) is taken bigger than \(1/2\) then, roughly speaking, one has to solve a parabolic equation backwards, leading to an impossibility. On the other hand, this problem disappears by assuming geometricity of \(B\) (which in the stochastic context boils down to Stratonovitch integration), in which case one observes that \(L = 0\) (in the setting of \(W : [0, T] \rightarrow \mathbb{R}^m\) with \(m > 1\), the bracket \(L\) is in general different from zero, however it is lower order in the sense of differential operators).
Organization of the Paper. The sequel of the present section will be devoted to
the definition and settings, as well as the statement of our main result, namely
the Itô Formula (Theorem 1). We also give an application to the weak maximum
principle. We complete our result by providing the chain rule for the case of interest
$F(u) := |u|^p$, $p \geq 2$, where we allow for broader conditions on $u$ (especially we do
not assume strong parabolicity here). This holds however, provided the solutions are
continuous (in time and space). In Section 2, we make some recalls, about controlled
rough paths theory mainly, but also concerning some interpolation properties in
Sobolev spaces. We also give the definition of the controlled path space $D_t^{\alpha,p}$, in the
case where the driving signal is an unbounded Rough Driver $B$. We state and prove
the a priori estimates in the latter space (alternatively called “remainder estimates”).
In Section 3, we define a suitable functional setting for rough parabolic equations,
and we state the essential argument of the paper, that is the “product formula”
(Proposition 3.1). In Section 4, we use this result to solve a class of rough
and we distinguish the case

where the driving signal is an unbounded Rough Driver $B$. We will first prove an Itô Formula

Section 6 we show, using a Moser Iteration, that a large class of solutions to rough parabolic problems of the
form (1.2) are bounded. Using this fact, together with a density argument, we will
conclude. The main interest in obtaining this formula is that it holds even if the
equation has no parabolic structure. In addition, the argument is local, namely it
does not rely on global integrability assumptions on $u$. In Section 6 we show, using
a Moser Iteration, that a large class of solutions to rough parabolic problems of the
form (1.2) are bounded. Using this fact, together with a density argument, we will
be able in Section 7 to prove Theorem 1. Section 8 is devoted to the proof of the
maximum principle, namely Theorem 3. Finally, in Appendix A, we detail some
needed results on the so-called “renormalization property” for unbounded rough
drivers.

1.1. Notation. Throughout the paper, the notation $K \subset \subset \mathbb{R}^d$ stands for “$K$ is a
compact set in $\mathbb{R}^d$.

By $\mathbb{N}$, we denote the set of natural integers $1, 2, \ldots$, and we let $N_0 := \mathbb{N} \cup \{0\}$, while $\mathbb{Z} := N_0 \cup \{-N\}$. Rationals are denoted by $\mathbb{Q}$, real numbers by $\mathbb{R}$ and complex
numbers by $\mathbb{C}$. We will moreover denote by $\mathbb{R}_+ := [0, \infty)$.

For an open smooth domain $U \subset \mathbb{R}^d$, we will consider the usual Lebesgue and
Sobolev spaces in the space-like variable: $L^p(U), W^{k,p}(U)$, for $k \in N_0$, and $p \in [1, \infty]$,
and we distinguish the case $p = 2$ by writing $H^k(U) := W^{2,k}(U)$. We denote their
respective norms by $\| \cdot \|_{L^p(U)}, \| \cdot \|_{W^{k,p}(U)}, \| \cdot \|_{H^k(U)}$. With the exception of Section 2.2,
the notations $L^p, W^{k,p}$ and $H^k$ refer to the whole space scenario $U = \mathbb{R}^d$. When $k$
is negative, we define $W^{k,p}$ to be the range of the linear mapping

$$T: (L^p)^{N_k} \ni h \mapsto Th := (D^\gamma h)_{|\gamma| \leq |k|} \quad \text{(weak sense)},$$

where $|\gamma| := \gamma_1 + \cdots + \gamma_d$ and $N_k := \sum_{|\gamma| = |k|} |\gamma|$. In this case, the space $W^{k,p}$ is endowed with the norm

$$\|f\|_{W^{k,p}} := \inf_{h \in (L^p)^{N_k}} \sum_{|\alpha| \leq |k|} \|h^\alpha\|_{L^p}.$$
Any elements \( h \in (L^p)_N \) such that \( Th = f \) is called a \textit{primitive} associated with \( f \). For \( p \neq 1 \), the space \( W^{k,p} \) is isomorphic to the dual space \((W_0^{k,\frac{1}{p}})^*\), where throughout the paper we use the notation \( W_0^{k,p} \) for the “boundary spaces”. Namely if \( n \) denotes the outward unit vector associated to \( \partial U \), and if \( p \in [0, \infty) \), we let
\[
W_0^{k,p}(U) \overset{\text{def}}{=} \{ f \in W^{k,p} \text{ s.t. } (n \cdot \nabla)^j f = 0 \text{ for } j \in \mathbb{N}_0, \ j < k - 1/p \}.
\]
As is well known, for \( p \neq \infty \) we have \( W_0^{k,p}(U) = \overline{C^\infty(U)W^{k,p}} \), where \( C^\infty(U) \) denotes the space of smooth and compactly supported functions. Note that for \( p = 1 \) we only have
\[
W^{-k,1} \equiv \left\{ \sum_{|\gamma| \leq k} D^\gamma h^\gamma, \ (h^\gamma)_{|\gamma| \leq k} \in (L^1)_N \right\} \subset (W_0^{k,\infty})^*, \quad (1.10)
\]
with strict embedding.

Throughout the paper we shall consider a finite, fixed time horizon \( T > 0 \).

Let \([s, t] \subset [0, T]\) be a subinterval. For space-time elements \( f_r(x) \) we use the notation
\[
\|f\|_{L^r(s,t;L^q)}
\]
to denote their norm in the space \( \|f\|_{L^r([s,t];L^q)} \), namely
\[
\|f\|_{L^r(s,t;L^q)} := \left( \int_s^t \left( \int_{\mathbb{R}^d} |f_r(x)|^q dx \right)^{r/q} dr \right)^{1/r}
\]
(the time variable will always be written with a subscript). If the domain of time integrability is clear from the context, we will write
\[
\|f\|_{L^r([s,t];L^q)}
\]
instead of \( \|f\|_{L^r(0,T;L^q)} \). We will also write \( C(0,T;E) \) for the space of continuous function with values in some Banach space \( E \), endowed with the norm \( \|f\|_{C(0,T;E)} := \sup_{t \in [0,T]} |f_t|_E \).

Given Banach spaces \( X, Y \), we will denote by \( \mathcal{L}(X,Y) \) the space of linear, continuous maps from \( X \) to \( Y \), endowed with the operator norm. For \( f \) in \( X^* := \mathcal{L}(X,\mathbb{R}) \), we denote the dual pairing by
\[
\langle X^*, \langle f, g \rangle \rangle_X
\]
(i.e. the evaluation of \( f \) at \( g \in X \)). When they are clear from the context, we will simply omit the underlying spaces and write \( \langle f, g \rangle \) instead.

In the sequel, we call a \textit{scale} any (possibly finite) sequence \( (E_k, | \cdot |_k)_{k \in S} \) of graded topological vector spaces, and such that \( E_{k+1} \) is continuously embedded into \( E_k \), for each \( k \in \mathbb{Z} \). In the context of (1.2), we do not really need the whole scale, but only a finite sequence. The relevant indices in this paper correspond to
\[
S := \{-3,-2,-1,0,1,2,3\}, \quad (1.11)
\]
but in general we could allow that \( S = \mathbb{Z} \). In some sense, our framework is similar as that of a Gelfand triple by using \( H^0 \equiv L^2 \) as a pivot space, a technical issue being that in \( \mathbb{R}^d \), there is of course no order between \( L^p \) and \( L^q \) for different values of \( p,q \). We can however circumvent this by considering local versions of the latter spaces.

In what follows, we will always have an embedding of \( E_k, k \in \mathbb{Z} \), in the space \( \mathcal{D}' \) of distributions in \( \mathbb{R}^d \). Following e.g. the terminology used in [Hör04] all the scales
considered in this paper admit a smallest local extension \((E_k, \text{loc})\). The space \(E_{k, \text{loc}}\) is 
the smallest Fréchet space containing \(E_k\), such that \(v \in E_{k, \text{loc}}\) if and only if \(\phi v \in E_k\) for every \(\phi \in C_c(\mathbb{R}^d; \mathbb{R})\). Otherwise said, \(E_{k, \text{loc}}\) is the completion of \(E_k\) for the families of seminorms 
\[ p_\phi(v) := |\phi v|_{E_k}, \quad \phi \in C_c(\mathbb{R}^d) \]
which, considering an appropriate countable family \(\{\phi_n\}\), makes \(E_{k, \text{loc}}\) a Fréchet space.

**Remark 1.1.** Elements of \(W^{k,p}_0(U)\) for \(0 \leq k \leq 3\) and \(p \in [1, \infty]\) are naturally identified in \(W^{k,p}(\mathbb{R}^d)\) through the embedding map 
\[ \iota_U : W^{3,p}_0(U) \hookrightarrow W^{3,p}(\mathbb{R}^d), \]
where for any \(\phi \in W^{3,p}_0(U)\), we define 
\[ \iota_U \phi(x) := \begin{cases} \phi(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases} \]
This operation is of course linear and continuous. In particular, by duality, for every distribution \(g \in W^{-3,1}_p(\mathbb{R}^d)\), the restriction \(g|_U \equiv \iota_U g\) to a smooth domain \(U\) is well defined. This fact will be used later on, where we will abuse notation and omit to write “\(\iota_U\)”.

We now introduce some notation related to rough paths theory. We will denote by \(\Delta, \Delta_2\) the simplices 
\[ \Delta := \{(s, t) \in [0, T]^2, \ s \leq t\}, \quad \Delta_2 := \{(s, \theta, t) \in [0, T]^3, \ s \leq \theta \leq t\}. \quad (1.12) \]
If \(E\) is a vector space and \(g : [0, T] \to E\), we define a two-parameter element \(\delta g\) as 
\[ \delta g_{st} := g_t - g_s, \quad \text{for all } s, t \in [0, T]. \]
Similarly, we define another operation \(\delta'\) by letting, for any \(g : \Delta \to E\), \(\delta'g\) be the quantity 
\[ \delta' g_{s\theta t} := g_{st} - g_{s\theta} - g_{\theta t}, \]
and we recall that \(\ker \delta' = \text{im} \delta\). As usual in the framework of controlled paths, we will omit the ‘\(\)’ on the second operation, and abusively write \(\delta\) instead of \(\delta'\).

We call control on \(I\) any superadditive map \(\omega : \Delta \to \mathbb{R}_+\), that is, for all \((s, \theta, t) \in \Delta_2\) there holds 
\[ \omega(s, \theta) + \omega(\theta, t) \leq \omega(s, t). \quad (1.13) \]
(Note that the property (1.13) implies in particular that \(\omega(t, t) = 0\) for any \(t \in [0, T]\).) We will call \(\omega\) regular if in addition \(\omega\) is continuous. All the controls considered in this paper satisfy this property.

If \(E\) is equipped with a family of semi-norms \((p_\gamma)_{\gamma \in \Gamma}, \) and \(\alpha > 0\), we denote by \(\mathcal{V}_1^\alpha(0, T; E)\) the set of paths \(g : [0, T] \to E\) admitting left and right limits with respect to each of the variables, and such that for each \(\gamma \in \Gamma\), there exist a regular control \(\omega_\gamma : \Delta \to \mathbb{R}_+\) with 
\[ p_\gamma(\delta g_{st}) \leq \omega_\gamma(s, t)^\alpha, \quad (1.14) \]
for every \((s, t) \in \Delta\). Similarly, we denote by \(\mathcal{V}_2^\alpha(0, T; E)\) the set of 2-index maps \(g : \Delta \to E\) such that \(g_{st} = 0\) for every \(t \in [0, T]\) and

\[
p_\gamma(g_{st}) \leq \omega_\gamma(s, t)\alpha,
\]

for all \((s, t) \in \Delta\), and some family of regular controls \(\omega_\gamma\). Note that \(g \in \mathcal{V}_2^\alpha(0, T; E)\) if and only if \(\delta g \in \mathcal{V}_2^\alpha(0, T; E)\).

If \((E, | \cdot |_E)\) is a Banach space (so that \((p_\gamma)_{\gamma \in \Gamma} = p \equiv | \cdot |_E\)), one defines a semi-norm \(\| \|_{\mathcal{V}_2^\alpha(0, T; E)}\) on \(\mathcal{V}_2^\alpha(0, T; E)\) by taking the infimum of every \(\omega(0, T)\alpha\) over every possible control \(\omega\) such that \((1.15)\) holds. Alternatively, it is equivalently defined as the \(q\)-variation of \(g\) with \(q := \frac{1}{\alpha}\). Namely, for \(0 \leq s \leq t \leq T\), denoting by \(C_{s,t}(g) := \{\omega : \Delta_{s,t} \to \mathbb{R}_+\text{ control s.t.} (1.15)\text{ holds}\}\), we have the property

\[
\|g\|_{\mathcal{V}_2^\alpha(s,t;E)} \overset{\text{def}}{=} \inf \{\omega(s, t)\alpha, \ \omega \in C_{s,t}(g)\} = \|g\|_{\mathcal{V}_2^\alpha(s,t;E)} \overset{\text{def}}{=} \left( \sup_{\|p\|=1} \sum_{(p)} g_{t_i,t_{i+1}}^q \right)^{\frac{1}{q}} ,
\]

where for we define \(\mathcal{P}_{s,t}\) as the set of partitions of \([s, t]\), that is

\[
\mathcal{P}_{s,t} := \{p \subset [s, t] : \exists \ell \geq 2, p = \{t_1 = s < t_1 < \cdots < t_\ell = t\}\},
\]

and where, throughout the paper, we use the notational convention:

\[
\sum_{(p)} h_{t_i,t_{i+1}} \overset{\text{def}}{=} \sum_{i=1}^{\#p-1} h_{t_i,t_{i+1}}
\]

for any 2-index element \(h\). The equality \([\cdot]_{\mathcal{V}_2^\alpha} = \| \cdot \|_{\mathcal{V}_2^\alpha}\) has been investigated in [HH18].

By \(\mathcal{V}_{2,\text{loc}}^\alpha(0, T; E)\) we denote the space of maps \(g : \Delta \to E\) such that there exists a countable covering \(\{I_k\}_k\) of \(I\) satisfying \(g \in \mathcal{V}_2^\alpha(I_k; E)\) for any \(k\). We also define the set \(\mathcal{V}_{2,+}^\alpha(0, T; E)\) of “negligible remainders” as

\[
\mathcal{V}_{2,+}^\alpha(0, T; E) := \bigcup_{\alpha > 1} \mathcal{V}_{2}^\alpha(0, T; E),
\]

and similarly for \(\mathcal{V}_{2,\text{loc}}^\alpha(0, T; E)\).

### 1.2. Unbounded rough drivers

Before we go to the formal definition of an unbounded rough driver, let us recall some basic elements of rough path theory.

Given \(\alpha \in (1/3, 1/2]\) and \(m \in \mathbb{N}_0\), recall that a continuous \((m\text{-dimensional})\ \alpha\text{-rough path is a pair}

\[
X = (X^k, X^{k\ell})_{1 \leq k, \ell \leq m} \in \mathcal{V}_2^{2\alpha}(0, T; \mathbb{R}^m) \times \mathcal{V}_2^{2\alpha}(0, T; \mathbb{R}^{m \times m}),
\]

such that Chen’s relations hold, namely:

\[
\delta X^k_{st \theta t} = 0, \quad \delta X^{k\ell}_{st \theta t} = X^k_{st \theta} X^\ell_{t}, \quad \text{for} \ (s, \theta, t) \in \Delta_2, \quad 1 \leq k, \ell \leq m.
\]

We refer the reader to the monographs [FV10, FH14] for a thorough introduction to the rough path theory. We will denote by \(\mathcal{C}^{\alpha}(0, T; \mathbb{R}^m)\) the set of all continuous rough paths as above. It is endowed with the metric \(d_{\mathcal{C}^{\alpha}}\) defined by

\[
d_{\mathcal{C}^{\alpha}}(X, Y) := \|X_0 - Y_0\|_{L^\infty(0, T)} + \|X - Y\|_{\mathcal{V}_2^{\alpha}} + \|X - Y\|_{\mathcal{V}_2^{\alpha}},
\]
for which it is complete. Note that, although \( d_{\varphi} \) is a function of the difference \( X - Y \), it is not a norm, because the space \( C^{\alpha}(0, T; \mathbb{R}^m) \) is not linear. For any element \( x \in \mathcal{V}^1(0, T; \mathbb{R}^m) \), there is a canonical lift \( S_2(x) \equiv (X, X) \) in \( C(0, T; \mathbb{R}^m) \) defined as

\[
X := \delta x, \quad \text{and for } k, \ell \in \{1, \ldots, m\} : \quad X_{st}^{k\ell} := \int_{\Delta} \delta x_r^{k} \delta x_r^{\ell}, \quad (s, t) \in \Delta.
\]

We shall denote by \( C^{\alpha}_g(0, T; \mathbb{R}^m) \subset C^{\alpha}(0, T; \mathbb{R}^m) \) the subset consisting of geometric rough paths. By definition, \( C^{\alpha}_g(0, T; \mathbb{R}) \) corresponds to the closure of the canonical lifts \( S_2(x) \), where \( x \in \mathcal{V}^1(0, T; \mathbb{R}) \), with respect to the rough path metric (1.20).

A useful notion is that of an unbounded rough driver (URD). It was first introduced in [BG17], in the context of transport equations.

**Definition 1.1** (unbounded rough driver). Let \( \alpha \in (1/3, 1/2] \) and fix \( p \in [1, \infty] \). A pair of 2-index maps \( B \equiv (B^1, B^2) \) is called an \( \alpha \)-unbounded rough driver in \( L^p \), if and only if

1. \( B^1_{st} \in \mathcal{L}(W^{k,p}, W^{k-1,p}) \) for \( i = 1, 2 \), and \(-3 \leq k - i \leq k \leq 3\), and there exists a regular control \( \omega_B : \Delta \to \mathbb{R}_+ \) such that

\[
|B^i_{st}|_{\mathcal{L}(W^{k,p}, W^{k-i,p})} \leq \omega_B(s, t)^{\alpha},
\]

for every \((s, t) \in \Delta \) and each \( k, i \) as above.

2. Chen’s relations hold true, namely, for every \((s, \theta, t) \in \Delta_2 \), we have

\[
\delta B^1_{s\theta t} = 0, \quad \delta B^2_{s\theta t} = B^1_{\theta t} B^1_{s\theta},
\]

as linear operators on the scale \((W^{k,p})_{k \in \mathbb{S}} \), where we recall notation (1.11).

We will say that an URD is differential if its components \((B^1, B^2)\) take their values in \( \mathbb{D}_1 \times \mathbb{D}_2 \), where \( \mathbb{D}_n, n \in \mathbb{N} \) consists of the space of linear differential operators of order \( n \). We will also say that \( B \) is differential if it operates only on \((W^{k,p})_{k \in \mathbb{S}} \), for a restricted set of indices, see (1.11), and hope that no confusion may arise from this. In the sequel, we shall restrict our attention to differential URDs (such property is implied by the transport property of Definition (1.5)).

**Example 1.1.** Let \( m \in \mathbb{N} \) and consider a continuous rough path \((X^k, X^{kl})_{1 \leq k, l \leq m} \) with values in \( \mathbb{R}^m \), of finite \( 1/\alpha \)-variation, \( \alpha \in (1/3, 1/2] \). Fix space-dependent coefficients \( \sigma \in W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^{m \times d}) \) and \( \nu \in W^{2,\infty}(\mathbb{R}^d, \mathbb{R}^m) \). For \((s, t) \in \Delta, i = 1, 2 \), define \( B^i_{st} : H^k \to H^{k-i} \), where \( i \leq k \leq 3 \) via the operation

\[
B^1_{st} \varphi := \sum_{1 \leq \beta \leq m} \sum_{1 \leq \beta \leq d} X^q_{st}(\sigma^{q\beta}_\beta \varphi + \nu^q \varphi), \quad B^2_{st} := \sum_{1 \leq \beta \leq m} \sum_{1 \leq \beta \leq d} X^{qr}_{st}(\sigma^{q\beta}_\beta \varphi + \nu^q \varphi),
\]

for \( \varphi \in H^k \). The latter is easily seen to define an URD with respect to the scale \((W^{k,p})_{k \in \mathbb{S}} \) for any \( p \in [1, \infty] \).

For a smooth path \( x \), the symmetric part of the tensor \( (\int \delta x^q \delta x^r)_{1 \leq q, r \leq m} \) is easily seen to be continuous with respect to the path \( x \) for the latter metric. In particular,
for any geometric rough path \((X^k, X^{kl})_{1 \leq k, l \leq m}\) as in the above example, considering an approximating sequence \(S_2(x(n)) \xrightarrow{\mathcal{C}} X\), one obtains at the limit:

\[
\text{sym}X^{kl}_{st} \equiv \frac{X^{kl}_{st} + X^{lk}_{st}}{2} = \frac{X^l_t X^k_s - X^k_t X^l_s}{2}, \quad \text{for all } 1 \leq k, l \leq m \quad \text{and all } (s, t) \in \Delta. \tag{1.23}
\]

Imposing in turn (1.23) yields the possibility of approximating \((X, X)\) by a sequence of absolutely continuous paths, so that a geometric rough path is alternatively defined as a couple \((X^k, A^{kl})_{1 \leq k, l \leq m}\) where the last component is called the \textit{Levy area} and corresponds to the antisymmetric tensor:

\[
A^{kl}_{st} := \frac{X^{kl}_{st} - X^{lk}_{st}}{2}, \quad 1 \leq k, l \leq m, \quad (s, t) \in \Delta.
\]

Consider \((B^1, B^2)\) as in Example 1.1, and for each \(k = 1, \ldots, m\), denote by \(V^k\) the first order differential operator corresponding to \(\sigma^k \cdot \nabla + \nu^k\). We see that formally, for any \((s, t) \in \Delta:\)

\[
B^2_{st} = \sum_{k,l} \frac{X^k_{st} X^l_{st}}{2} V^k V^l + \sum_{k,l} A^{kl}_{st} V^k V^l = \frac{1}{2} B^1_{st} B^1_{st} + \frac{1}{2} \sum_{k,l} A^{kl}_{st}[V^k, V^l], \tag{1.24}
\]

where interestingly one recognizes the “Lie Bracket” \([V^k, V^l] = V^k V^l - V^l V^k\). In the case where the \(V^k\)'s are vector fields we see in particular that the second sum is also a vector field, and hence an operator of first order only (in contrast with \(B^2\)).

Note that, if \(b : [0, T] \to \mathcal{T}_1 := \cap_{k=1}^3 \mathcal{L}(W^{k,p}, W^{k-1,p})\) is given and has finite variation, one can always define the \textit{canonical lift} \(B \equiv (B^1, B^2) := S_2(b)\) as the unbounded rough driver given by

\[
B^1_{st} := \delta b_{st} \in \mathcal{T}_1
\]

and

\[
B^2_{st} := \int_s^t (b_r - b_s) \, db_r \in \mathcal{T}_2,
\]

where \(\mathcal{T}_2 := \cap_{k=1}^3 \mathcal{L}(W^{k,p}, W^{k-2,p})\). The latter integral is well-defined in the sense of Riemann-Stieltjes, in the space \(\mathcal{T}_2\).

This discussion motivates the following.

**Definition 1.2** (Geometric URD). Let \(p \in [1, \infty]\). Given a continuous, \(\alpha\)-URD \(B\) on the scale \((W^{k,p})_{k \in \mathbb{S}}\) with \(\alpha \in (1/3, 1/2]\), we will say that \(B\) is geometric if there exists a sequence of paths \(b(n) \in C^1([0, T]; \mathcal{T}_1), n \geq 0\), such that letting \(B(n) := S_2(b(n))\) it holds for each \(n \geq 0\) and \(1 \leq i \leq k \leq 2:\)

\[
B^i(n) \in \mathcal{V}^i_2(0, T; \mathcal{T}_i), \tag{1.25}
\]

and moreover

\[
\rho_\alpha(B(n), B) \overset{\text{def}}{=} \sum_{k=-2}^3 \|b(n) - B^1_0\|_{L^\infty(0, T; \mathcal{L}(W^{k,p}, W^{k-1,p})}) + \sum_{i=1}^2 \sum_{k=-2+i}^3 \|B^i(n) - B^i\|_{\mathcal{V}^i_2(0, T; \mathcal{L}(W^{k,p}, W^{k-1,p}))} \to 0. \tag{1.26}
\]

\[
(1.27)
\]
We will denote by $L \equiv (L^1, L^2)$ the 2-index map
\[
(s, t) \in \Delta \mapsto \left( L^1_{st} := B^1_{st}, \ L^2_{st} := B^2_{st} - \frac{1}{2} B^1_{st} B^1_{st} \right)
\] (1.28)
and shall refer to it as the reduced form of the geometric URD $B$.

We shall also call $L^2$ the “bracket of $B$”, in analogy with the bracket $[X] \equiv X \otimes X - 2\text{sym}X$ of a given 2-step rough path $X \equiv (X, X)$. (Contrary to what is encountered in the usual rough path theory, note that this needs not be zero for a geometric URD.)

It is important to note that $L$ is not an unbounded rough driver, for it fails to fulfill Chen’s relations (1.22). Nevertheless, it will be convenient in the sequel to refer to this 2-parameter object rather than $B$ itself, because it displays remarkable properties when $B$ is “transport-like”, see Definition 1.5 and the comments below.

1.3. Notion of solution. Our ansatz is an equation of the form
\[
dv = f dt + dBv
\]
on $[0, T] \times \mathbb{R}^d$, where $f$ is $p$-integrable as a mapping with values in $W^{-1,p}$ for some given $p \in [1, \infty)$, and $v$ will always be assumed to be bounded as a path with values in $L^p$. We are ultimately interested in the case where the drift depends itself on the solution, that is when $f_t = \mathcal{D}_t[v_t]$, for some “reasonable” mapping $\mathcal{D} : [0, T] \times W^{1,p} \rightarrow W^{-1,p}$, but for now it will be convenient to see $f$ as a free term.

Before we proceed to the definition of a solution, we shall make some assumptions on $B$ that will be valid throughout the whole paper. The following property is easily seen to be fulfilled by the unbounded rough driver defined in Example 1.1.

Assumption 1.1. We are given $\alpha \in (1/3, 1/2]$ and a geometric, $\alpha$-unbounded rough driver $B$ in the scale $(H^k)_{k \in \mathbb{S}}$, with a regular control $\omega_B$ such that
\[
| (\nabla^{k-i} B^i_{st} - B^i_{st} \nabla^{k-i}) \phi(x) | \leq \omega_B(s, t)^{i\alpha} | \nabla^k \phi(x) |,
\]
for each $0 \leq k - i \leq k \leq 3$, $(s, t) \in \Delta$, $\phi \in C_c^\infty(\mathbb{R}^d)$, and a.e. $x \in \mathbb{R}^d$.

It is immediate that an URD $B$ as above admits a unique closed extension to every Sobolev scale $(W^{k,p})_{k \in \mathbb{N}_0}$ for $1 \leq p \leq \infty$. Namely: for each $-3 \leq k - i \leq k \leq 3$ and $(s, t) \in \Delta$ there exists an URD $B^i_{st} : W^{k,p} \rightarrow W^{k-i,p}$ such that $\varphi_n \rightarrow \varphi \in W^{k,p}$ and $B^i_{st} \varphi_n \rightarrow \psi \in W^{k-i,p}$ implies $\psi = B^i_{st} \varphi$. In the following, we will abuse notation and write systematically $B^i$ instead of $B^i_{st}$.

We can now proceed to the definition of a weak solution. The following has been first introduced in [BG17], see also [DGHT16].

Definition 1.3 (Weak solution). Let $T > 0$, $\alpha \in (1/3, 1/2]$ and fix $q \in [1, \infty]$. Assume that we are given a distribution $f \in L^1(0, T; (W^{1,q})^*)$. A path $v : [0, T] \rightarrow (L^q)^*$ is called a weak solution to the rough PDE (1.29) with respect to the scale of test functions $(W^{k,q})_{k \in \mathbb{N}}$, if $v$ is bounded as a mapping from $[0, T]$ to $(L^q)^*$ and such that moreover, for every $\phi \in W^{3,q}$, and every $(s, t) \in \Delta$, it holds
\[
\langle \delta v_{st}, \phi \rangle = \int_s^t \langle f_r, \phi \rangle dr + \langle (B^1_{st} + B^2_{st}) v_{st}, \phi \rangle + \langle v^3_{st}, \phi \rangle,
\]
for some $v^3 \in \mathcal{V}^{1+}_{2, loc}(0, T; (W^{3,q})^*)$. 

Though the notion of weak solution fulfills the minimal requirement under which remainder estimates (see Proposition 2.4) are possible, it is somehow too general for the scope of this paper. Due to the parabolic nature of the problem we are addressing in the present work, it is indeed expected that solutions live in a “better space” than just $L^\infty(0, T; (L^p)^*)$. This motivates the introduction of the following.

**Definition 1.4** ($L^p$-solution). Letting $p \in [1, \infty]$, we will say that $v$ is an $L^p$-solution of (1.29) if it is a weak solution with respect to the scale of test functions $(W^{k,p})_{k \in \mathbb{N}_0}$ (with the convention that $1/0 = \infty$ and $\infty/\infty = 1$), and such that moreover

$$v \in L^\infty(0, T; L^p) \cap L^p(0, T; W^{1,p}).$$  \hspace{1cm} (1.32)

Similarly, we will say that $v$ is an $L^p_{\text{loc}}$-solution (or, letting $U \subset \mathbb{R}^d$, an “$L^p(U)$ solution”) if it fulfills the above properties, where each occurrence of the Sobolev spaces in the space-like variable is replaced by its local counterpart.

We will now make some further assumptions on $B$.

### 1.4. Transport-like drivers.

We will assume throughout the paper that the geometric URD $B$ behaves algebraically as a derivation. More precisely, we will suppose the following.

**Definition 1.5.** Let $B$ be a geometric $\alpha$-unbounded rough driver, $\alpha \in (1/3, 1/2]$. We will say that $B$ is transport-like if for every $(s, t) \in \Delta$, and each $f, g \in C^\infty_c(\mathbb{R}^d)$, it holds a.e.

$$B_{st}^1(fg) = (B_{st}^1f)g + f(B_{st}^1g) \quad \text{and}$$

$$B_{st}^2(fg) = (B_{st}^2f)g + (B_{st}^1f)(B_{st}^1g) + f(B^2g).$$  \hspace{1cm} (1.33) \hspace{1cm} (1.34)

If $B$ is transport-like, denoting by $L^2$ its bracket (see Definition 1.2), then elementary algebraic computations using (1.33) and (1.34) yield

$$L^2_{st}(fg) = (B_{st}^2f)g + (B_{st}^1f)(B_{st}^1g) + f(B^2_{st}g)$$

$$- \frac{1}{2}(B^1_{st}B^1_{st}f)g - (B_{st}^1f)(B_{st}^1g)g - f\frac{1}{2}(B^1_{st}B^1_{st}g)$$

$$= (L^2_{st}f)g + f(L^2_{st}g),$$  \hspace{1cm} (1.35)

for any $f, g \in C^\infty_c$, and $(s, t) \in \Delta$. Hence we have a first order Leibniz rule for $L^2$, which means that $L^2$ is necessarily a first order differential operator (more specifically we have that for each $(s, t) \in \Delta$, $L^2_{st}$ is a derivation). Conversely, the property (1.35) ensures that (1.34) holds if one defines $B^2$ as the operator $\frac{1}{2}B^1B^1 + L^2$.

**Remark 1.2.** Let $B$ be transport-like and, in order to simplify the discussion, let us assume that each $B^i, i = 1, 2$, maps $C^\infty$ into itself. The transport property (1.33)-(1.34) asserts in particular that $B^i, i = 1, 2$, takes its values in $\mathbb{D}_i$, the space of linear operators of degree less than or equal to $i$. In fact, we have $B^1 : \Delta \to \mathbb{D}_i$, where we denote by $\mathbb{D}_1 \subset \mathbb{D}_i$ the space of derivations, that is the space of linear mappings $V : C^\infty \to C^\infty$ such that

$$V(fg) = (Vf)g + f(Vg), \quad \text{for all } f, g \in C^\infty.$$  \hspace{1cm} (1.36)
In particular, for \( i = 1, 2 \), there is a unique \( \sigma^i \in \mathcal{V}_2^{2\alpha}(0, T; C^\infty(\mathbb{R}^d; \mathbb{R}^d)) \), as well as a unique \( \beta \in \mathcal{V}_2^{2\alpha}(0, T; C^\infty(\mathbb{R}^d; \mathbb{R}^d)) \) such that \( B \) has the representation
\[
B_{st}^1 = \sigma_{st}^{1,i} \partial_i, \quad B_{st}^2 = \sigma_{st}^{2,jk} \partial_j \partial_k + \beta_{st}^i \partial_i. \tag{1.37}
\]

For such \( B \) we can define a formal adjoint \( B^\dagger \) by
\[
\langle B_{st}^i f, \phi \rangle = \langle f, B_{st}^{i\dagger} \phi \rangle, \quad \text{for every } f, \phi \in C^\infty, \tag{1.38}
\]
which is of course explicitly given by the formula
\[
B_{st}^{1\dagger} = -B_{st}^1 - \text{div} \sigma_{st}^1, \quad B_{st}^{2\dagger} = \partial_j \partial_k (\sigma_{st}^{2,jk} \cdot) - \beta_{st}^i \partial_j - \text{div} \beta_{st}. \tag{1.39}
\]

Note that \( B^\dagger \) is in general, neither an URD (because it fails to satisfy (1.22)), nor transport-like. It however fulfills the “backward” Chen’s relations
\[
\delta B_{s\theta t}^{2\dagger} = B_{s\theta}^1 B_{\theta t}^{1\dagger}, \quad \text{for } (s, \theta, t) \in \Delta_2. \tag{1.40}
\]

In the non-smooth case, it is not difficult to convince oneself that similar representations as that of (1.37) and (1.39) hold, the only difference being that \( \sigma^i \in W^{3,\infty} \) and \( \beta \in W^{2,\infty}. \)

Example 1.2. The geometric unbounded rough driver defined in Example 1.1 is transport-like if \( \nu = 0 \). Moreover, for any \( f \) and \( g \) in \( C^\infty \), it holds the modified Leibnitz rule
\[
B^{1\dagger}(fg) = (B^{1\dagger} f)g + f(B^{1\dagger} g) - Hfg \tag{1.41}
\]
where \( H : \Delta \to \mathbb{D}_0 \) is the operation
\[
Hf(x) = X^k \text{div} \sigma^k(x) f(x).
\]

We can now proceed further by being more explicit about the “drift term” \( f \) in (1.2).

1.5. **Assumptions on the coefficients and main results.** The ansatz considered in this paper involves a drift whose linear part is given by \( A_t v(t) \) where \( A \) is the following elliptic operator on divergence form:
\[
A_t = \partial_i (a^{ij}(t, \cdot) \partial_j \cdot) \tag{1.42}
\]
and where we will assume the following.

**Assumption 1.2.** We have \( a = (a^{ij})_{1 \leq i, j \leq d} \in L^\infty([0, T] \times \mathbb{R}^d, \mathbb{R}^{d \times d}) \), and there exists a coercivity constant \( \vartheta > 0 \) such that for a.e. \( (t, x) \in [0, T] \times \mathbb{R}^d : \)
\[
\vartheta \sum_{i=1}^d \xi_i^2 \leq \sum_{1 \leq i, j \leq d} a^{ij}(t, x) \xi_i \xi_j, \quad \xi \in \mathbb{R}^d. \tag{1.43}
\]

Before we state our main result, we shall first define a set of “admissible” functions \( F : \mathbb{R} \to \mathbb{R} \), namely such that there is an Itô Formula for \( F(u) \). We let
\[
C^2_{\text{adm}} := \{ F \in C^2(\mathbb{R}; \mathbb{R}) \text{, s.t. } F(0) = F'(0) = 0 \text{ and } |F''|_{L^\infty} < \infty \}. \tag{1.44}
\]

With this definition, we have the following,
This allows us to investigate the following homogeneous Dirichlet problem on \( \mathbb{B} \):

\[
\begin{aligned}
&\text{\text{Theorem 1 (Itô Formula). Let } A \text{ and } B \text{ such that the assumptions 1.1, 1.5 and 1.2 hold, and assume in addition that } B \text{ is transport-like, see Definition 1.5. Let } u \text{ be an } L^2\text{-solution of (1.2). For every } F \in C^2_{\text{adm}} \text{ it holds the chain rule}
\end{aligned}
\]

\[
dF(u) = F'(u)(Au + f)dt + dB(u),
\]

in the sense that the path \([0, T] \to L^2, t \mapsto F(u_t)\) is an \(L^1\)-solution to the above equation. More explicitly, we have for any \(\phi \in W^{3,\infty} \) :

\[
\langle \delta F(u), \phi \rangle = \int_{\mathbb{S}} (F'(u)A, u_r, \phi)dr + \langle \left(B_{st}^1 + B_{st}^2\right)(F(u)), \phi \rangle + \langle F_{st}, \phi \rangle
\]

(1.46)

for some unique remainder term \(F_{st} \in Y^{1,1+}_{2,\text{loc}}(0, T; W^{-3,1})\).

If in addition \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) is compactly supported, then (1.46) holds for any \(F \in C^2\).

For

\[
p \geq 2,
\]

note that \(C^2_{\text{adm}}\) contains the function \(x \mapsto |x|^p\) hence if \(u\) is a solution of (1.2), it holds in particular

\[
du = p|u|^{p-1}(Au + f)dt + dB(|u|^p)
\]

(1.47)

\((L^1\text{-sense}).\)

More generally, let \(u \in L^\infty(L^p) \cap L^1(W^{1,p})\), such that \(u\) is an \(L^p\)-solution of an abstract equation

\[
du = f dt + dBu, \quad u_0 \in L^p,
\]

(1.48)

with a given \(f \in L^1(0, T; W^{-1,p})\) (possibly depending on \(u\) itself). The question whether an Itô Formula of the form (1.47) holds is particularly relevant for applications. Such a statement is possible, even though no strong ellipticity condition enters into consideration here. There is however a "price to pay", for one needs to assume that \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) is a continuous mapping.

\textbf{Theorem 2. Fix } p \geq 2. \text{ Let } u \in C([0, T] \times \mathbb{R}^d), \text{ be a continuous, } L^p \text{ solution of (1.48), for some } f \in L^1(W^{-1,p}) \text{ and } u_0 \in L^p.

Then, we have in the } L^1\text{-sense:}

\[
d(|u|^p) = p|u|^{p-1}f dt + dB(|u|^p).
\]

(1.49)

To be more concrete we now give some by-products of our results. Consider \(B \equiv (B^1, B^2)\) as in Example 1.1 with \(\nu = 0\) and where \((X^k, X^{r, kl})_{1 \leq k, l \leq m}\) is geometric. It is easily seen that \(B\) is a geometric URD with respect to the scale \((W^k_{0, \infty}(D))_{k \in \mathbb{S}}\). This allows us to investigate the following homogeneous Dirichlet problem on \(D\):

\[
\begin{aligned}
&\left\{
\begin{array}{l}
du - A(t, x)u dt = dBu, \quad \text{on } \mathbb{R}^+ \times D, \\
u(0) = u_0, \\
u |_{\partial D} = 0 \quad (\text{trace sense}), \quad \text{for all } t \geq 0.
\end{array}
\right.
\end{aligned}
\]

(1.50)

But first, we need a suitable notion of solution for (1.50).
Definition 1.6. We will say that \( u \) solves the Dirichlet problem (1.50), if and only if \( u \) is an \( L^2(D) \)-solution to the equation \( du = Audt + dBu \), in the sense of Definition 1.4, and such that in addition
\[
 u \in L^2(0, T; W^{1,2}_0(D)). 
\] (1.51)

We have the following.

Theorem 3 (weak maximum principle for (1.50)). Assume that \( D \subset \mathbb{R}^d \) is open, bounded, and has a smooth boundary \( \partial D \). Let \( A, B^1, B^2 \), be subject to the above conditions. Assume moreover that \( \sigma \in W^{3,\infty}_0(D; \mathbb{R}^{m \times d}) \), (1.52) and \( a \in L^1(0, T; W^{1,\infty}_0(D)) \).

There exists a unique solution \( u \in C(0, T; L^2) \cap L^2(0, T; W^{1,2}_0(D)) \) of the Dirichlet problem (1.50). Moreover, it holds \( u \in L^\infty([0, T] \times D) \) and we have the following maximum principle for \( u \):
\[
\min (0, \text{ess inf}_D u_0) \leq u(t, x) \leq \max (0, \text{ess sup}_D u_0) \quad \text{a.e. for } (t, x) \in [0, T] \times D.
\]

2. Preliminaries

2.1. Some useful results. We recall first some elements of controlled path theory (see the basic reference [Gub04]). The main problem addressed by this theory is, roughly speaking, to give a meaning to incremental equations of the form
\[
u_t - u_s = \int_s^t dH, \quad \text{for } (s, t) \in \Delta,
\] (2.1)
where \( \Delta \ni (s, t) \mapsto H_{st} \) is a kind of “jet” associated to the quantity one wishes to integrate. Concrete examples are given by the integral
\[
\int_s^t dH \equiv \int_s^t f_s \, dX_r,
\]
where \( f \) and \( X \) are \( \alpha \)-Hölder, with \( \alpha > 1/2 \), the associated jet of which is for instance given by \( H_{st} := f_s \delta X_{st} \). If on the other hand \( 1/3 < \alpha \leq 1/2 \), one is instead led to consider a compensated riemann approximation of the form
\[
H_{st} := f_s \delta X_{st} + f'_s \left( \int_s^t \delta X_{sr} \, dX_r \right) = f_s X^1_{st} + f'_s X^2_{st},
\] (2.2)
where the quotation marks " " indicate that \( \int_s^t \delta X_{sr} \, dX_r \) is simply defined as the known quantity \( X^2_{st} \) (and not the other way around), and \( f' \) is the Gubinelli derivative of \( f \) with respect to \( X \) and needs to be determined. In fact, the meaning which will be given to equation (2.1) is really
\[
u_s - u_s = H_{st} + u^\circ_{st}
\]
where \( u^\circ_{st} \) is a (uniquely determined) remainder in \( V^{1+}_2(0, T) \). It depends upon the 3-parameter quantity \( h_{s\theta t} := \delta H_{s\theta t} \equiv H_{st} - H_{s\theta} - H_{\theta t} \), but not on \( H \) such that \( \delta H = h \).
To be more precise, we recall the following result, which is of fundamental importance in this paper.

**Proposition 2.1** (Sewing Lemma). Let \((E, (p_\gamma)_{\gamma \in \Gamma})\) be a Fréchet space. Define \(Z^1_+ (0,T;E)\) as the set of 3-index maps \(h : \Delta_2 \to E\) such that

- there exists a continuous \(H : \Delta \to E\) with \(h = \delta H\);
- for each \(\gamma \in \Gamma\), there is a regular control \(\omega_{h,\gamma} : \Delta \to \mathbb{R}_+\) and \(a_\gamma > 1\), such that
  \[ p_\gamma (h_{s\theta t}) \leq \omega_{h,\gamma}(s,t)^{a_\gamma}, \]  \((2.3)\)

uniformly as \((s,\theta,t) \in \Delta_2\).

Then, there exists a linear map \(\Lambda : Z^1_+ (0,T;E) \to V^1_+ (0,T;E)\), continuous in the sense that for every \(\gamma \in \Gamma\) and \(h \in Z^1_+ (0,T;E)\) there holds

\[ p_\gamma (\Lambda h_{s\theta t}) \leq C_{a_\gamma} \omega_{h,\gamma}(s,t)^{a_\gamma}, \]  \((2.4)\)

where the above constant only depends on the value of \(a_\gamma > 1\). In addition, \(\Lambda\) is a right inverse for \(\delta\), namely

\[ \delta \Lambda = \text{id} |_{Z^1_+}, \]  \((2.5)\)

and it is unique in the class of linear mappings fulfilling the properties (2.4)-(2.5).

Finally, for any \((s,t) \in \Delta\), we have the explicit formula:

\[ \Lambda_{st} h = \lim_{|p| \to 0} \left( H_{st} - \sum_{(p)} H_{it_{t_{i+1}}} \right), \]  \((2.6)\)

where we use the summation convention (1.17).

**Proof.** A proof of the Sewing Lemma in a Banach space can be found e.g. in [GT10]. The extension to Fréchet spaces is straightforward (a proof can be found in [HH18, Appendix A2]).

The conclusion of the Sewing Lemma can be alternatively formulated in terms of the existence (and uniqueness) of a “nice” integral mapping

\[ \mathcal{I} : \text{Dom}\mathcal{I} \subset V^\alpha_2 \to V^\alpha_2, \]

where the domain \(\text{Dom}\mathcal{I}\) corresponds to the space of admissible jets \(H\) such that \(\delta H \in Z_+^1\). Admissible jets are provided for instance by Taylor expansions such as \((2.2)\), in the case where \(f\) is a controlled path. We refer to Section 3 for precise definitions in the context of unbounded rough drivers. We will mostly use this result under the form given in Proposition 2.1, nevertheless we recall a few basic consequences of it, expressed in the (perhaps more familiar) language of integrals.

Given \(\alpha \in (0,1]\), let \(\text{Dom}\mathcal{I} := \{ H \in V^\alpha_2 (0,T;E) \text{ s.t. } \delta H \in Z^1_+ \}\) One can define a 2-index map as follows: for \((s,t) \in \Delta\), we let

\[ \mathcal{I}_{st} (H) := H_{st} - \Lambda_{st} \delta H \in V^\alpha_2 (0,T;E). \]  \((2.7)\)

The linear map \(\mathcal{I} : \text{Dom}\mathcal{I} \to V^\alpha_2 (0,T;E), H \mapsto \mathcal{I}(H)\) is called rough integral and fulfills the following properties:

- for every \((s,\theta,t) \in \Delta_2\) it holds \(\mathcal{I}_{st} = \mathcal{I}_{s\theta} + \mathcal{I}_{\theta t}\);
- if \(I \in V^\alpha_2 (I;E)\) is another 2-index map such that \(\delta I = 0\) and \(I - H \in V^1_+ (0,T;E)\), then \(I = \mathcal{I}(H)\);
for any $H$ as above, $\mathcal{I}(H)$ is given by the compensated Riemann sum approximation

$$I_{st}H = \lim_{|p| \to 0} \sum_{(p)} H_{t_i, t_{i+1}};$$

(2.8)

The rough integral extends the Riemann-Stieltjes integral in the following sense: assume that $E$ is a reflexive Banach space, and let $f : [0, T] \to \mathcal{L}(E, F)$, $g : [0, T] \to E$ be continuous maps. Assume furthermore that $g$ has a weak derivative $\dot{g} \in L^1(0, T; E)$ (in particular $g$ is absolutely continuous). Then, we have

$$\int f_{\dot{g}} dr < \infty$$

(2.9)

where $f_{\delta g}$ is to be understood as the jet $H : (s, t) \in \Delta \mapsto H_{st} := f_s \delta g_{st}$.

One of the core arguments that we shall use in this paper is a Gronwall-type argument, well-adapted to incremental equations of the form (2.1). We will extensively make use of the following result.

**Lemma 2.1** (Rough Gronwall). Let $G : [0, T] \to \mathbb{R}_+$ be a path such that there exist constants $\kappa, L > 0$, a regular control $\omega$, and a superadditive map $\varphi$ with:

$$\delta G_{st} \leq \left( \sup_{s \leq r \leq t} G_r \right) \omega(s, t) + \varphi(s, t),$$

(2.10)

for every $(s, t) \in \Delta$ under the smallness condition $\omega(s, t) \leq L$.

Then, there exists a constant $\tau_{\kappa, L} > 0$ such that

$$\sup_{0 \leq t \leq T} G_t \leq \exp \left( \frac{\omega(0, T)}{\tau_{\kappa, L}} \right) \left[ G_0 + \sup_{0 \leq t \leq T} |\varphi(0, t)| \right].$$

(2.11)

**Proof.** This result was first proven in [DGHT16]. For the reader’s convenience, and since our notations and setting are slightly different, we give a proof of it.

Let $\tau := L \wedge (2e^2)^{-\kappa}$, Since the control $\omega$ is regular, there exists an integer $K \geq 2$ and a sequence $t_0 \equiv 0 < t_1 < \cdots < t_{K-1} < t_K \equiv T$ such that for each $k$ in \{1, \ldots, K - 1\},

$$\omega(0, t_k) = k\tau,$$

(2.12)

while for $k = K$ it holds $\omega(0, t_K) \equiv \omega(0, T) \leq K\tau$. For $k \in \{0, \ldots, K - 1\}$, using superadditivity we obtain the property:

$$\omega(t_k, t_{k+1}) \leq \tau.$$  

(2.13)

Next, for $t \in [0, T]$, we let:

$$G_{\leq t} := \sup_{0 \leq r \leq t} G_r, \quad H_t := G_{\leq t} \exp \left( -\frac{\omega(0, t)}{\tau} \right), \quad H_{\leq t} := \sup_{0 \leq r \leq t} H_r.$$

Fix $t \in [t_{k-1}, t_k]$ for some $k \in \{1, \ldots, K\}$. Note that since $\tau \leq L$, we may apply the estimate (2.10) on each subinterval $[t_i, t_{i+1}]$. Hence, using (2.10), (2.13) and the
superadditivity of \( \varphi \), it holds
\[
G_t = G_0 + \sum_{i=0}^{k-2} \delta G_{t_i, t_{i+1}} + \delta G_{t_{k-1}, t}
\]
\[
\leq G_0 + \tau^{1/\kappa} \left( \sum_{i=0}^{k-2} G_{t_i, t_{i+1}} + G_{t_k, t} \right) + \sum_{i=0}^{k-2} \varphi(t_i, t_{i+1}) + \varphi(t_{k-1}, t)
\]
\[
\leq G_0 + \tau^{1/\kappa} \sum_{i=0}^{k-1} H_{t_i, t_{i+1}} \exp \left( \frac{\omega(t_i, t_{i+1})}{\tau} \right) + \varphi(0, t)
\]
which, according to (2.12) and the properties of the exponential map, is bounded above by
\[
G_0 + \tau^{1/\kappa} H_{\leq T} \exp(k+1) + \varphi(0, t).
\]
By the fact that \( \omega(0, t) \geq \omega(0, t_{k-1}) \), we deduce the following estimate on \( H \):
\[
H_t \leq \left\{ G_0 + |\varphi(0, t)| + \tau^{1/\kappa} \exp(k+1) H_{\leq t} \right\} \exp \left( \frac{-\omega(0, t)}{\tau} \right)
\]
\[
\leq G_0 + \sup_{t \leq T} \left\{ |\varphi(0, t)| \exp \left( \frac{-\omega(0, t)}{\tau} \right) \right\} + \tau^{1/\kappa} e^{2H_{\leq T}},
\]
According to our definition of \( \tau \), this yields the bound:
\[
H_{\leq T} \leq \frac{1}{1 - e^{2\tau^{1/\kappa}}} \left( G_0 + \sup_{t \leq T} \left\{ |\varphi(0, t)| \exp \left( \frac{-\omega(0, t)}{\tau} \right) \right\} \right),
\]
from which (2.11) follows. \( \blacksquare \)

We now recall some useful results of PDE theory. The first is a classical interpolation inequality, the proof of which can be found in [LSU68].

**Proposition 2.2.** For every \( f \) in the space \( L^\infty(0, T; L^2) \cap L^2(0, T; W^{1, 2}) \), then \( f \) belongs to \( L^\rho(0, T; L^\sigma) \) for every \( \rho, \sigma \) such that
\[
\frac{1}{\rho} + \frac{d}{2\sigma} \geq \frac{d}{4} \quad \text{and} \quad \begin{cases} 
\rho \in [2, \infty], & \sigma \in [2, \frac{2d}{d-2}] \quad \text{for } d > 2 \\
\rho \in (2, \infty], & \sigma \in [2, \infty) \quad \text{for } d = 2 \\
\rho \in [4, \infty], & \sigma \in [2, \infty) \quad \text{for } d = 1.
\end{cases}
\]

In addition, there exists a constant \( C_{2.2} > 0 \) (not depending on \( f \) in the above space) such that
\[
\|f\|_{L^\rho(0, T; L^\sigma)} \leq C_{2.2} \|f\|_{L^\infty(0, T; L^2) \cap L^2(0, T; H^1)} \equiv C_{2.2} \left( \|\nabla f\|_{L^2(0, T; L^2)} + \esssup_{r \in I} |f_r|_{L^2} \right).
\]
(2.15)

As a consequence, it can be checked that whenever \( r, q \in [1, \infty] \) are numbers satisfying
\[
\frac{1}{r} + \frac{d}{2q} \leq 1,
\]
then it holds the inequality
\[
\|u\|_{L^{\frac{2q}{d-2}}(L^\infty(0, T; L^2))} \leq C_{2.2} \|u\|_{L^\infty(0, T; L^2)}.
\]
(2.17)

Because \( L^1 \) does not identify to the dual of \( L^\infty \), we will need a specific weak compactness criterion. We recall the following classical result, see [AK16].
Proposition 2.3. Suppose that \( \mathcal{F} \) is a bounded set of \( L^1(\Omega, \Sigma, \mu) \), where \( \Omega \) denotes a measurable space such that \( \mu(\Omega) < \infty \). There is an equivalence between the two properties:

(i) \( \mathcal{F} \) is relatively weakly compact;
(ii) \( \mathcal{F} \) is equi-integrable, namely: given \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \) so that for every \( E \in \Sigma \) with \( \mu(E) < \delta(\epsilon) \), we have

\[
\sup_{f \in \mathcal{F}} \int_E |f| d\mu < \epsilon.
\]

As a consequence, we have the following weak compactness criterion in the scale of spaces \( W^{k,1}_{\text{loc}} \) when \( k \in \mathbb{Z} \).

Corollary 2.1. Let \( k \in \mathbb{Z} \). A sequence \( \{f_n, n \in \mathbb{N}\} \subset W^{k,1}_{\text{loc}}(\mathbb{R}^d) \) is weakly relatively compact if and only if the following holds:

(i) The sequence \( \{f_n\}_{n \geq 0} \) is bounded, namely for every \( K \subset \subset \mathbb{R}^d \), there exists a constant \( C_K > 0 \) such that

\[
\sup_{n \geq 0} |f_n|_{W^{k,1}(K)} \leq C_K. \tag{2.18}
\]

(ii) For every \( \epsilon > 0 \), there exists a \( \delta(\epsilon) > 0 \) such that whenever \( |K| \leq \delta(\epsilon) \), it holds

\[
\lim_{n \to \infty} |f_n|_{W^{k,1}(K)} \leq \epsilon.
\]

Proof. The case \( k \geq 0 \) is clear.

Assume that \( k = -m \) for some \( m \in \mathbb{N} \), and consider the covering \( \{B_l\}_{l \in \mathbb{N}_0} := (B(x_l, \rho_l))_{l \in \mathbb{N}_0} \), where \( \{(x_l, \rho_l), l \in \mathbb{N}_0\} = \mathbb{Q}^d \times \mathbb{Q}_+ \). For every \( n, l \geq 0 \) we let \( f^a_{n,l} \) be a primitive element for \( f_n \) i.e. such that \( f_n = \sum_{|\alpha| \leq m} D_{\alpha}f^a_{n,l} \) on \( B_l \) and with the property that

\[
|f^a_{n,l}|_{L^1(B_l)} \leq |f_n|_{W^{-m,1}(B_l)} + |B_l|.
\]

Fix \( \epsilon > 0 \). If \( K \) is a ball of finite volume, by our choice of cover \( \{B_l\} \), there exists an \( l = l(K) \) so that \( K \subset B_l \) and \( |K_l| \leq 2|K| \). Hence, if \( |K| < \frac{\epsilon}{\delta(\epsilon)} \) \( \land \), we have

\[
|f^a_{n,l}|_{L^1(K)} \leq |f^a_{n,l}|_{L^1(B_l)} \leq |f_n|_{L^1(B_l)} + |B_l| \leq 2\epsilon,
\]

showing equi-integrability of \( \{f^a_{n,l}\}_{n \in \mathbb{N}_0} \). The conclusion follows now from Proposition 2.3. \( \square \)

2.2. The space of controlled paths. In this paragraph we consider a smooth domain \( U \subset \mathbb{R}^d \) and we let

\[
p \in [1, \infty], \quad \text{while} \quad p' := \frac{p}{p - 1}.
\]

For notational simplicity, we will omit the domain of integrability and denote by \( L^p = L^p(U) \), \( W^{k,p} = W^{k,p}(U) \), and so on.

Given \( g \in L^\infty(0, T; L^p) \cap \mathcal{V}^a(0, T; W^{-1,p}) \) we will say that \( g \) is controlled by \( B \), if there exists \( g' \in L^\infty(0, T; L^p) \cap \mathcal{V}^a(0, T; W^{-1,p}) \) such that the element \( R^g \) of \( \mathcal{V}^2_2(0, T; W^{-1,p}) \) defined as

\[
R^g_{st} := \delta g_{st} - B^1_{st}g_s, \quad \text{for every} \ (s, t) \in \Delta,
\]

(2.19)
has better time regularity than each of its summands. More precisely we require that
\[ ||R^p||_{V^2_2([0,T];W^{-2,p})} < \infty \] (2.20)
(notice the loss of a space-derivative in the above). Abusively, we call \( g' \) “the Gubinelli derivative” of \( g \), although \( g' \) could be non-unique in principle (at least without any further assumption on \( B \), see Remark 3.1 below). Whenever \( k \geq 0 \), and \( y \in \mathcal{V}_2^\alpha(0,T;W^{-k,p}) \), we shall use the notations
\[ \gamma[y]_{-k}^{[\alpha]}(s,t) := ||y||_{\mathcal{V}_2^\alpha(s,t;W^{-k,p})}, \text{ for } (s,t) \in \Delta, \]
and
\[ \gamma[y]_{-k}^{[\alpha]} := \gamma[y]_{-k}^{[\alpha]}(0,T). \]

Having Proposition 2.4 in mind, the following definition provides a “natural” Banach space framework that comes with (1.2).

**Definition 2.1.** Given \( \alpha \in (1/3,1/2] \), we define the space of controlled paths \( \mathcal{D}_{B}^{\alpha,p}(U) \) as those couples \((g,g') \in \mathcal{V}_2^\alpha(0,T;L^p)^2 \) such that \( g \) is controlled by \( B \) with Gubinelli derivative \( g' \). Moreover, for any \((g,g')\) as above we denote by
\[ [g,g']_{\mathcal{D}_{B}^{\alpha,p}} := [R^\alpha]_{L^p} + [\delta g']_{L^p}, \]
which defines a seminorm.

The linear space \( \mathcal{D}_{B}^{\alpha,p}([0,T] \times U) \) is endowed with the norm
\[ ||g,g'||_{\mathcal{D}_{B}^{\alpha,p}([0,T] \times U)} := ||g||_{L^\infty(0,T;L^p(U))} + ||g'||_{L^\infty(0,T;L^p(U))} + [g,g']_{\mathcal{D}_{B}^{\alpha,p}}, \] (2.21)
and we shall omit the domain \([0,T] \times U\) when it is clear from the context.

Given a drift \( f \in L^p(0,T;W^{-1,p}) \), and \((g,g') \in \mathcal{D}_{B}^{\alpha,p} \), there exists a unique \( u \) in \( C(0,T;W^{-3,p}) \) solution to
\[ du = f dt + dBg, \] (2.22)
i.e. such that
\[ u^\delta_{st} := \delta u_{st} - \int_s^t f dr - B^1_{st}g_s - B^2_{st}g'_s \]
belongs to \( \mathcal{V}^1_{2,\text{loc}}(0,T;W^{-3,p}) \). Indeed, denoting by \( H_{st} \) the above right hand side, the assumption that \((g,g') \in \mathcal{D}_{B}^{\alpha,p} \) together with the properties of \( B \) guarantee the applicability of the sewning map \( \Lambda \) on \( g \) (see Proposition 2.1). Namely, we have \( \delta H \in \mathcal{Z}^{1+}(W^{-3,p}) \). Using the uniqueness property, we see that \( u^\delta \) is given by
\[ u^\delta := \Lambda[\delta H] \equiv \Lambda[B^1R^p + B^2\delta g'], \] (2.23)
We shall from now on denote by
\[ \pi(f;g,g') := u \] (2.24)
the unique solution \( u \) to (2.22), and by \( \pi(f;g,g')^\delta \) (or simply by \( u^\delta \)) its associated remainder term given by (2.23). The set of all bounded paths \( u : [0,T] \to L^p \) admitting a representation of the form (2.22) will be denoted by \( \Pi \), i.e.
\[ \Pi := \pi[L^p(0,T;W^{-1,p}(U)) \times \mathcal{D}_{B}^{\alpha,p}([0,T] \times U)] \cap L^\infty(0,T;L^p(U)). \] (2.25)
It defines a Banach space for the norm \( \| u \|_\Pi \) corresponding to the infimum over all possible representatives \((f; g, g')\)

\[
\| u \|_\Pi := \inf_{\pi(f; g, g') \in \mathcal{V}_{0,1}^{(J,p)}(W^{-3,p})} \left\{ \| f \|_{L^p(W^{-1,p})} + \| g, g' \|_{\mathcal{D}^{\alpha,p}} \right\}.
\]  

(2.26)

**Proof.** The homogeneity and separation axioms are trivial, let us show the triangle inequality. For \( u \equiv (f_u; g_u, g_u') \), and \( v \equiv (f_v; g_v, g_v') \) in \( \Pi \), we have by linearity: \( u + v = \pi(f_u + f_v; g_u + g_v, g_u' + g_v') \), whence

\[
\| u + v \|_\Pi \leq \| f_u + f_v \|_{L^p(W^{-1,p})} + \| g_u + g_v, g_u' + g_v' \|_{\mathcal{D}^{\alpha,p}}
\leq \| f_u \|_{L^p(W^{-1,p})} + \| g_u, g_u' \|_{\mathcal{D}^{\alpha,p}} + \| f_v \|_{L^p(W^{-1,p})} + \| g_v, g_v' \|_{\mathcal{D}^{\alpha,p}}.
\]

Since the above representatives are arbitrary, the triangle inequality follows.

The completeness of \( \| \cdot \|_\Pi \) follows the lines of [FV10, Section 1.3].

Notice from (2.23) that the Sewing Lemma yields the a priori estimate

\[
\| \pi(f; g, g') \|_{\mathcal{V}_{0,1}^{(J,p)}(s,t)} \leq C \left( \omega_B(s, t)^a [R^q]_{-2} (s, t) + \omega_B(s, t)^2 \omega_B(s, t)^2 \right),
\]

(2.27)

so that the drift term \( f \) does not seem to play a role a priori on the above remainder estimate. If on the other hand \( g, g' \) are functions of \( u \) itself, then any estimate of \( (g, g') \in \mathcal{D}^{\alpha,p}_B \) will involve \( f \) crucially, because in that case \( g \) and \( g' \) will be themselves solutions of a similar equation.

### 2.3. Remainder estimates.

Let \( U \subseteq \mathbb{R}^d \) be a (non-necessarily bounded) smooth domain and fix \( p \in [1, \infty] \). There exists a family \( (J_\eta)_{\eta \in (0,1)} \) of bounded linear maps \( J_\eta \in \mathcal{L}(W^{k,p}, W^{k,p}) \), \( \eta \in (0,1) \), \( k \in \mathbb{Z} \) being arbitrary, such that the following holds:

- \( J_\eta \) maps \( W^{k,p} \) into \( W^{k+2,p} \), for every \( \eta \in (0,1) \).

(2.28)

For some constant \( C_J > 0 \), for any \( \ell \in \mathbb{N}_0 \) with \( |k - \ell| \leq 2 \) : if \( 0 \leq k \leq \ell \leq 3 \), then

- \( \| J_\eta \|_{\mathcal{L}(W^{k,p}, W^{\ell,p})} \leq \frac{C_J}{\eta^{k-\ell}} \), for all \( \eta \in (0,1) \).

(2.29)

Finally, if \( 0 \leq \ell \leq k \leq 3 \), then

- \( \| \text{id} - J_\eta \|_{\mathcal{L}(W^{k,p}, W^{\ell,p})} \leq C_J \eta^{k-\ell} \), for all \( \eta \in (0,1) \).

(2.30)

Indeed, in the case when \( U \equiv \mathbb{R}^d \) and \( p \in [1, \infty] \) it suffices to consider for instance \( J_\eta f := \eta^{-d} \rho(\frac{x}{\eta}) * f \), where \( \rho \) is a radially symmetric, smooth function integrating to one.

**Remark 2.1.** If \( U \) is a smooth, compactly supported domain and \( k \geq 0 \), we can as well define \( J_\eta : W^{k,p}_0(U) \to W^{k,p}_0(U) \) by the same convolution operation, composed first with multiplication by a cut-off function \( \Theta_\eta(x) \). The cutoff is needed to circumvent the fact that convolution with \( \rho(\cdot / \eta) \eta^{-d} \) necessarily increases the support of an element. When \( k < 0 \) we can always identify \( W^{k,p} \) with a subspace of \( (W_0^{-k,p})^* \) and obtain similar estimates by duality.

For further details about this procedure, we refer the reader to the appendix in [HH18].
Remark 2.2. When \( p \in (1, \infty) \), it is possible to work with fractional powers and obtain similar estimates on the scale \( (W^{s,p})_{s \in \mathbb{R}} \). It is indeed sufficient in this case to define \( J_\eta \) as a Fourier cut-off. If \( U \) is a smooth bounded domain, one can alternatively let
\[
J_\eta = e^{\eta^2 \Delta},
\]
where \( \Delta \) denotes the Dirichlet Laplacian on \( U \).

From now on, we shall refer to \( (J_\eta)_{\eta \in (0,1)} \) as a family of smoothing operators.

We have the following observation.

Lemma 2.2. Any \( v \in \Pi \) with \( v = \pi(f;g,g') \) is controlled by \( B \), with Gubinelli derivative \( v' = g \) (so that in particular \( g' = v'' \triangleq (v')' \)). Moreover, the following estimate holds
\[
\|R^v\|_{-2}^{[2\alpha]}(s,t) \leq C \left( \int_s^t |f_r|_{W^{-2,p}} dr + \omega_B(s,t)^{2\alpha} \|v, v', v''\|_{L^\infty(s,t;L^\alpha)} \right)
\]
\[
+ \frac{1}{2} \left( \|R^v\|_{-2}^{[2\alpha]}(s,t) + \omega_B(s,t)^{\alpha} \|\delta v''\|_{-1}^{[1\alpha]}(s,t) \right).
\]

Proof. Note that
\[
R^v_{st} := \delta v_{st} - B_{st}^1 g_s \equiv \int_s^t f_r dr + B_{st}^2 g_s + v_{st}^3,
\]
where \( v^3 := \pi^3(f;g,g') \). Next, using (2.29)-(2.30), one can write
\[
|R^v_{st}|_{W^{-2,p}} \leq |J_\eta(\int_s^t f_r dr + B_{st}^2 g_s + v_{st}^3)|_{W^{-2,p}} + |(\text{id} - J_\eta)(\delta v_{st} - B_{st}^1 g_s)|_{W^{-2,p}}
\]
\[
\lesssim |\int_s^t f_r dr|_{W^{-2,p}} + |B^2 g'|_{W^{-2,p}} + \frac{|v_{st}^3|_{W^{-3,p}}}{\eta}
\]
\[
+ \eta^{2\alpha} \|v\|_{L^\infty(L^\alpha)} + \eta \omega_B(s,t)^{\alpha} \|g\|_{L^\infty(L^\alpha)}.
\]

Since the latter holds for arbitrary \( \eta \in (0,1) \), we can choose \( \eta := \lambda \omega_B(s,t)^{\alpha} \) for some \( \lambda > 0 \) big enough, depending on \( C \) in (2.27), we obtain:
\[
|R^v_{st}|_{W^{-2,p}} \leq \left( \int_s^t |f|_{W^{-2,p}} dr \right) + \omega_B(s,t)^{2\alpha} \|v, g, g'\|_{L^\infty(s,t;L^\alpha)} + \frac{\|v^3\|_{[3\alpha]}(s,t)}{\lambda \omega_B(s,t)^{\alpha}}
\]
\[
\leq \left( \int_s^t |f|_{W^{-2,p}} dr \right) + \omega_B(s,t)^{2\alpha} \|v, g, g'\|_{L^\infty(s,t;L^\alpha)}
\]
\[
+ \frac{1}{2} \left( \|R^v\|_{-2}^{[2\alpha]}(s,t) + \omega_B(s,t)^{\alpha} \|\delta g''\|_{-1}^{[1\alpha]}(s,t) \right),
\]
provided that \( \omega_B(s,t) \leq L \) for some absolute constant \( L \) depending on \( J \) only. This shows the claimed property. \( \blacksquare \)

The above lemma shows in particular that any element \( v \in \Pi \) of the form
\[
dv = f dt + dB(v', v'')
\]
(2.32)
is necessarily controlled by $B$ with Gubinelli derivative $v'$, whereas $v''$ is by definition the derivative of $v'$ itself, justifying a posteriori the notation. For such equations we are especially interested in the “sharpest control” majorizing $\omega(f; v', v'')^2$, that is

$$\omega_{v'}(s, t) := \inf_{(f,v',v'') \in L^p(W^{-1,p}) \times D^p_B} \|\pi(f; v', v'')\|_{\mathcal{V}(s, t; W^{-3,p})},$$

It is easily seen that $\omega_{v'}$ is a control. Using a similar approach as in [DGHT16, HH18], is should be possible to estimate $\omega_{v''}$ for a general $v \in \Pi$, provided that $[v, v']D^p_B(J \times U) \leq C_0[v', v'']D^p_B(J \times U)$ for some constant $C_0$, independent of $J \subset [0, T]$. The main scope of this paper consists however in linear equations of the form

$$dv = f dt + dBv. \quad (2.33)$$

By Lemma 2.2, in this case $v$ is controlled by $B$ with $v' = v$, so that we have in fact $v = v' = v''$. In order not to unnecessarily complicate the proof of the next statement, we will therefore establish the remainder estimates in this particular setting and refrain from stating the most general possible result.

**Proposition 2.4 (Remainder estimates).** Let $p \in [1, \infty]$ and fix $v \in \Pi$, given by (2.33) for some $f \in L^p(0, T; W^{-2,p})$.

Then, $v^2 \equiv \pi(f, v, v)$ has finite $\frac{1}{3}\alpha$-variation. Moreover, there are constants $C, L > 0$ depending only on $\alpha$ and $C_J$, such that for each $(s, t) \in \Delta$ subject to the smallness assumption

$$\omega_B(s, t) \leq L,$$

$$[v^2]_{2\alpha(s, t; W^{-3,p})} \leq C \left( \omega_B(s, t)^{3\alpha} \|v\|_{L^{\infty}(s, t; L^p)} + \omega_B(s, t)^{\alpha} \int_s^t |f_r|_{W^{-2,p}} dr \right). \quad (2.34)$$

**Proof.** The main idea goes back to [DGHT16], where a similar though slightly different statement was shown. See also [HH18, Proposition 3.1].

By definition of the remainder term $v^2 \equiv \pi^2(f; v, v)$, there exists some $z \in (1, 3\alpha]$ such that

$$\omega_z(s, t) \equiv \|v^2\|_{z(s, t; W^{-3,p})}^{1/z} < \infty.$$

Furthermore, recall that

$$\omega_z(s, t) = \inf \{\omega(s, t), \omega : \Delta_{[s, t]} \rightarrow \mathbb{R}_+, \text{control s.t. } (\omega)^z \geq \|v^2\|_{W^{-3,p}} \}. \quad (2.35)$$

Next, denote the integrated drift term by $\mathcal{D}_t := \int_0^t f_s dW$. For arbitrary $(s, \theta, t) \in \Delta_2$, the distribution $\delta v^2_{s \theta t}$ is given by (2.23), where $v = g = g'$, so that estimates on $v^2$ will follow by estimates on the latter, using Proposition 2.1. Using the smoothing operators, we write

$$\delta v^2_{s \theta t} = B_{1t}^0 R^v_{s \theta} + B_{2t}^0 \delta v$$

$$= B_{1t}^0 (\text{id} - J_{\theta}) (\delta v_s - B_{1s}^0 v_s) + B_{1t}^0 J_{\theta} (\delta \mathcal{D}_s + B_{1s}^0 v_s + v^2_{s \theta})$$

$$+ B_{2t}^0 (\text{id} - J_{\theta}) \delta v_s + B_{2t}^0 J_{\theta} (\delta \mathcal{D}_s + B_{2s}^0 v_s + B_{2s}^2 v_s + v^2_{s \theta})$$

$$=: \mathcal{T}^1 + \mathcal{T}^2 + \mathcal{T}^3 + \mathcal{T}^4.$$
The estimate on the “non-smooth parts” $T^1$ and $T^3$ works as follows:

$$
|T^1|_{W^{-3,p}} \leq \omega_B(\theta, t)^\alpha \left( |(\mathrm{id} - J_\eta)\delta v_{s\theta}|_{W^{2,p}} + |(\mathrm{id} - J_\eta)B^{1}_{s\theta}v_s|_{W^{2,p}} \right)
$$

$$
\leq C_J \left(2\omega_B(s, \theta)\alpha^2\eta^2 + \omega_B(\theta, t)2^\alpha\eta \|v\|_{L^\infty(L^p)}\right)
$$

(2.36)

where we have made use of the property (2.30). Similarly, we have

$$
|T^3|_{W^{-3,p}} \leq 2C_J\omega_B(\theta, t)2^\alpha\eta\|v\|_{L^\infty(L^p)}.
$$

(2.37)

Concerning the smooth parts $T^2$ and $T^4$, we have

$$
|T^2|_{W^{-3,p}} \leq \omega_B(\theta, t)^\alpha \left( |J_\eta\delta \varphi|_{W^{-1,p}} + |J_\eta B^2v_s|_{W^{-1,2}} + |J_\eta g^5|_{W^{-1,2}} \right)
$$

$$
\leq C_J \omega_B(\theta, t)^\alpha \left( \omega_\varphi(s, \theta) + \omega_B(s, \theta)2^\alpha\eta\|v\|_{L^\infty(L^p)} + \frac{\omega_z(s, \theta)^2}{\eta} \right),
$$

(2.38)

by definition of the control $\omega_z$.

Similarly we have for the fourth term:

$$
|T^4|_{W^{-3,p}} \leq \omega_B(\theta, t)^{2\alpha} \left( |J_\eta\delta \varphi|_{W^{-1,p}} + |J_\eta B^2v_s|_{W^{-1,2}} + |J_\eta g^5|_{W^{-1,2}} \right)
$$

$$
\leq C_J \omega_B(\theta, t)^{2\alpha} \left( \omega_\varphi(s, \theta) + \omega_B(s, \theta)^{2\alpha}\eta\|v\|_{L^\infty(L^p)} + \frac{\omega_z(s, \theta)^2}{\eta^2} \right).
$$

(2.39)

Choosing

$$
\eta := \lambda \omega_B(s, t)^\alpha,
$$

for some parameter $\lambda > 0$ (to be fixed later), we obtain the inequality

$$
|\delta v^5_{s\theta}|_{W^{-3,p}} \leq C_J \omega_B(s, t)^{3\alpha}\|v\|_{L^\infty(L^p)}(2\lambda^2 + 3\lambda + 2\lambda^{-1}) + C_J J_\omega_B(s, t)^\alpha\omega_\varphi(s, t)(1 + \lambda^{-1})
$$

$$
+ C_J \omega_z(s, t)^2(\lambda^{-1} + \lambda^{-2}).
$$

(2.40)

Next, choose $\lambda > 0$ sufficiently large so that

$$
|\Lambda| C_J(\lambda^{-1} + \lambda^{-2}) \leq \frac{1}{2}
$$

where $|\Lambda|$ denotes the norm of the Sewing Map, and consider $(s, t)$ sufficiently close to the diagonal so that

$$
\eta \equiv \lambda \omega_B(s, t)^\alpha < 1.
$$

Applying Proposition 2.1, it comes

$$
|v^5_{s\theta}|_{W^{-3,p}} \leq C \left( \omega_B(s, t)^{3\alpha}\|v\|_{L^\infty(L^p)} + \omega_B(s, t)^\alpha\omega_\varphi(s, t) \right) + \frac{1}{2}\omega_z(s, t)^2
$$

where the constant $C$ depends only on the quantities $C_J, \omega_B(0, T), \alpha$ and $|\Lambda|$. By the inequality $(a + b)^\epsilon \leq a^\epsilon + b^\epsilon$ for $a, b \geq 0$ and $\epsilon \in [0, 1]$, it holds

$$
|v^5_{s\theta}|_{W^{-3,p}} \leq C^{1/\epsilon} \left( \omega_B(s, t)^{3\alpha}\|v\|_{L^\infty(L^p)} + \omega_B(s, t)^{\alpha\epsilon}\omega_\varphi(s, t)^{1/\epsilon} \right) + \frac{1}{2\epsilon}\omega_z(s, t)
$$

By [FV10, p.22], the above right hand side is a control, hence we infer from the property (2.35) that

$$
\omega_z(s, t) \leq C^{1/\epsilon} \left( \omega_B(s, t)^{3\alpha}\|v\|_{L^\infty(L^p)} + \omega_B(s, t)^{\alpha\epsilon}\omega_\varphi(s, t)^{1/\epsilon} \right) + \frac{1}{2\epsilon}\omega_z(s, t),
$$
which shows that for any \( z \in (1, 3\alpha] \)
\[
\| v_{st}^z \|_{L^{3-\alpha}} \leq \omega_z(s, t) \leq \left( 1 - \frac{1}{2^3z} \right)^{-1} C^{1/z} \left( \omega_B(s, t)^{3\alpha / z} \| v \|_{L^{\infty}(s, t; L^p)}^{1/z} + \omega_B(s, t)^{\alpha / z} \omega_B(s, t)^{1/z} \right). \tag{2.41}
\]

Letting now \( z = 3\alpha \) yields the inequality (2.34). \( \blacksquare \)

From the latter proposition we deduce the following.

**Corollary 2.2.** Let \( v \) be as in Proposition 2.4. Then, there is a universal constant \( L > 0 \) such that
\[
\left\| R^v \right\|_{-2}^{[2\alpha]}(s, t) \leq C \left( \int_s^t \| f_r \|_{W^{-1, p}} dr + \| v \|_{L^{\infty}(s, t; L^p)} \omega_B(s, t)^{2\alpha} \right) \tag{2.42}
\]
for \( \omega_B(s, t) \leq L \), and
\[
\left\| \delta v \right\|_{-1}^{[\alpha]}(s, t) \leq C(1 + \| v \|_{L^{\infty}(s, t; L^p)}) \left( \left( \int_s^t \| f_r \|_{W^{-1, p}} dr \right)^{\alpha} + \omega_B(s, t)^{\alpha} \right) \tag{2.43}
\]
for \( \omega_B(s, t) + \int_s^t \| f_r \|_{W^{-1, p}} dr \leq L \).

**Proof.** Writing as before \( R^v = J_\eta R^v + (\text{id} - J_\eta) R^v \), then using that by the equation \( R^v = \delta v - B^1 v \equiv \int f dr + B^2 v + v^\varepsilon \), we have
\[
\left\| R^v \right\|_{-2}^{[\alpha]}(s, t) \leq \frac{C}{\eta} \left\| R^v \right\|_{-2}^{[\alpha]}(s, t) + C \| v \|_{L^{\infty}(s, t; L^p)}(\eta + \omega_B(s, t)^{\alpha}). \tag{2.44}
\]

This yields in particular
\[
\left\| \delta v \right\|_{-1}^{[\alpha]}(s, t) \leq \left\| R^v \right\|_{-1}^{[\alpha]}(s, t) + \left\| B^1 v \right\|_{-1}^{[\alpha]}(s, t)
\]
\[
\leq C \left( \frac{1}{\eta} \left\| R^v \right\|_{-2}^{[\alpha]}(s, t) + \| v \|_{L^{\infty}(s, t; L^p)}(\eta + \omega_B(s, t)^{\alpha}) + \| v \|_{L^{\infty}(s, t; L^p)} \omega_B(s, t)^{\alpha} \right). \tag{2.45}
\]

Now, using Lemma 2.2, with \( v = v' = v'' \), we can absorb the term \( \left\| R^{v''} \right\|_{-2}^{[\alpha]} \) in the left hand side of (2.31), yielding
\[
\left\| R^v \right\|_{-2}^{[\alpha]} \leq C \left( \int_s^t \| f_r \|_{W^{-1, p}} dr + \omega_B(s, t)^{\alpha} \left\| \delta v \right\|_{-1}^{[\alpha]}(s, t) \right)
\]
\[
\leq C \left( \int_s^t \| f_r \|_{W^{-1, p}} dr + \omega_B(s, t)^{\alpha} \left[ \frac{1}{\eta} \left\| R^v \right\|_{-2}^{[\alpha]}(s, t) + \| v \|_{L^{\infty}(s, t; L^p)}(\eta + \omega_B(s, t)^{\alpha}) \right] \right),
\]
by (2.45). Choosing \( \eta := \lambda \omega_B(s, t)^{\alpha} \) with \( \lambda > 0 \) big enough (depending on \( C \) in the above estimate) we obtain (2.42).

Using (2.45) but with \( \eta := \omega_B(s, t)^{\alpha} + \left( \int_s^t \| f_r \|_{W^{-1, p}} dr \right)^{\alpha} \), we obtain that \( \left\| \delta v \right\|_{-1}^{[\alpha]}(s, t) \) is bounded above by a constant times
\[
\left( \int_s^t \| f_r \|_{W^{-1, p}} dr \right)^{2\alpha} + \| v \|_{L^{\infty}(s, t; L^p)}(\int_s^t \| f_r \|_{W^{-1, p}} dr)^{\alpha} + \omega_B(s, t)^{\alpha},
\]
yielding (2.43). \( \blacksquare \)
3. The spaces "$\mathcal{H}^{\alpha,p}_B$" and "$\mathcal{H}^{\alpha,p}_{B,\text{loc}}"

3.1. A natural Banach space setting. Let $p \in [1, \infty)$, fix a domain $U \subset \mathbb{R}^d$, and consider an $\alpha$-URD $B$ which is transport-like (as before we assume $\alpha \in (1/3, 1/2)$). We define a space $\mathcal{H}^{\alpha,p}_B([0, T] \times U)$ as follows:

$$\mathcal{H}^{\alpha,p}_B([0, T] \times U) := \left\{ u \in C(L^p), \exists (f; g, g') \in L^p(W^{1,p}(U)) \times \mathcal{D}^{\alpha,p}_B, \text{ s.t. } u = \pi(f; g, g'), \right\}$$

where $\|u\|_{L^\infty(L^p(U))} + \|\nabla u\|_{L^p(L^p(U))} + \|f\|_{L^p(W^{1,p}(U))} + \|g, g'\|_{\mathcal{D}^{\alpha,p}_B([0, T] \times U)} < \infty$.

(3.1)

where in the sequel we use the notation (2.24).

The norm $\|u\|_{\mathcal{H}^{\alpha,p}_B}$ is defined as the smallest of the quantities appearing in the last line, among all the possible representatives $(f; g, g')$ of $u$. That is

$$\|u\|_{\mathcal{H}^{\alpha,p}_B([0, T] \times U)} := \|u\|_{L^\infty(L^p(U))} + \|\nabla u\|_{L^p(L^p(U))}$$

$$+ \inf_{u = \pi(f; g, g')} \left\{ \|f\|_{L^p(W^{1,p}(U))} + \|g, g'\|_{\mathcal{D}^{\alpha,p}_B([0, T] \times U)} \right\}.$$  
(3.2)

Similarly as for $\| \cdot \|_{\mathcal{H}^{\alpha,p}_B([0, T] \times U)}$ the quantity $\| \cdot \|_{\mathcal{H}^{\alpha,p}_B([0, T] \times U)}$ is a well-defined norm and $(\mathcal{H}^{\alpha,p}_B, \| \cdot \|_{\mathcal{H}^{\alpha,p}_B})$ is a Banach space (the proof of this fact is left to the reader). As before, we will omit to write the domain $[0, T] \times U$ when $U = \mathbb{R}^d$. As an important consequence of Corollary 2.2, the norm (3.2) is equivalent to the quantity

$$\|u\|_{L^\infty(L^p(U))} + \|\nabla u\|_{L^p(L^p(U))}$$

$$+ \inf_{\delta u - B^1 u - B^2 u - \int f \, dr \in \nabla^{1+1}(W^{1,3}(U))} \left\{ \|f\|_{L^p(W^{1,3}(U))} + \|\pi(f; u, u')\|_{\mathcal{D}^{\alpha,p}_B([0, T] \times U)} \right\}.$$  

for every $u$ lying in the subspace $\mathcal{L}^{\alpha,p}_B \subset \mathcal{H}^{\alpha,p}_B$, defined as

$$\mathcal{L}^{\alpha,p}_B := \{(u, u') \in \mathcal{H}^{\alpha,p}_B, \text{ s.t. } u' = u \}.$$  

Similarly, we define the local spaces $\mathcal{H}^{\alpha,p}_{B,\text{loc}} = \mathcal{H}^{\alpha,p}_{B,\text{loc}}([0, T] \times \mathbb{R}^d)$ by the property

$$u \in \mathcal{H}^{\alpha,p}_{B,\text{loc}} \iff u|_{[0, T] \times K} \in \mathcal{H}^{\alpha,p}_B([0, T] \times K) \text{ for every } K \subset \subset \mathbb{R}^d,$$

which, considering a sequence of compacts $K_n \uparrow \mathbb{R}^d$ and the family of seminorms $\|u\|_{\mathcal{H}^{\alpha,p}_B([0, T] \times K_n)}$, makes $\mathcal{H}^{\alpha,p}_{B,\text{loc}}$ a Fréchet space.

A useful result is the following.

**Lemma 3.1** (Stability). Let $p \in [1, \infty]$ and let $\mathcal{F} \subset \mathcal{H}^{\alpha,p}_{B,\text{loc}}$ be a bounded family. Then, $\mathcal{F}$ is relatively compact with respect to the topology of $C(0, T; H^{-1}_{\text{loc}})$.

Before we proceed to the proof, let us recall the following infinite dimensional version of Ascoli Theorem (see [Kel75, Chap.7]): Let $E$ be a Fréchet space. A family $\mathcal{F}$ of elements of $C(0, T; E)$ (endowed with the uniform convergence topology) is compact if and only if:

(a) $\mathcal{F}$ is closed in $C(0, T; E)$;
(b) $\mathcal{F}(t)$ has a compact closure for each $t \in [0, T]$, where $\mathcal{F}(t)$ denotes the set

$$\{ f_t, f \in \mathcal{F} \};$$
(c) the family $\mathcal{F}$ is equicontinuous.


Proof. Fix $K \subset \subset \mathbb{R}^d$, and consider a sequence $u(n) \in \mathcal{F}, n \geq 0$, as well as $f(n) \in L^p(W^{-1,p}(K))$ and $(g(n), g'(n)) \in D_B^{0,p}([0,T] \times K)$ such that $u(n) = \pi(f(n); g(n), g'(n))$ and

$$\|f(n)\|_{L^p(W^{-1,p}(K))} + \|g(n), g'(n)\|_{D_B^{0,p}([0,T] \times K)} \leq \inf_{u(n) = \pi(f(n); g(n), g'(n))} \left\{ \|\tilde{f}(n)\|_{L^p(W^{-1,p}(K))} + \|\tilde{g}(n), \tilde{g}'(n)\|_{D_B^{0,p}([0,T] \times K)} \right\} + 1.$$

By assumption, there exists a constant $C_K > 0$ such that for any $n \geq 0$:

$$\|u(n)\|_{L^\infty(L^p(K))} \leq \|u(n)\|_{H_B^{0,p}([0,T] \times K)} \leq C_K. \quad (3.3)$$

Denoting by $R^u(n) := \delta u(n) - B^1 g(n)$, we have by Lemma 2.2:

$$|\delta u(n)|_{W^{-1,p}(K)} \leq |R^u(n)|_{W^{-1,p}(K)} + |B^1 g(n)|_{W^{-1,p}(K)} \leq \left[ R^u(n) \right]_{1,0}^{loc}(s, t) + \omega_B(s, t)^\alpha \|g(n)\|_{L^\infty(L^p(K))} \leq C(\alpha, \|g\|_{L^\infty(L^p(K))}) \omega_1(s, t)^\alpha \quad (3.4)$$

where $\omega_1(s, t)$ denotes the control $\left( \left[ R^u(n) \right]_{1,0}^{loc}(s, t) \right) + \omega_B(s, t)$. From (3.4), one infers that the sequence $u(n) : [0, T] \rightarrow W_{loc}^{-1,p}, n \geq 0$, is uniformly equicontinuous. Since on the other hand the embedding $L^p \hookrightarrow W_{loc}^{-1,p}$ is compact, using (3.3) we can apply Ascoli Theorem, yielding the conclusion. \qed

Remark 3.1. It is natural to ask under which condition one can have uniqueness of a representation $u = \pi(f; g, g')$, which relates the Doob-Meyer decomposition Theorem for semi-martingales.

This is not true in general because our definition of an URD could accomodate that of a smooth driver $B := \dot{X} \partial_x$, where $X \in C^{\infty}(0, T; \mathbb{R})$ and for simplicity we let $d = 1$. Indeed, in this case one can arbitrarily choose $g' = 0$ for any $g$ and alternatively represent $u = \pi(f; g, 0)$ via $\pi(f + \dot{X} \partial_x g; 0, 0)$.

In the finite-dimensional case however (for instance replacing $B$ by a rough path of $1/\alpha$-finite variation with values in $\mathbb{R}$), there is indeed uniqueness of $g'$ provided $X$ is truly rough i.e. there exists a dense set of times $t \in [0, T]$ such that

$$\limsup_{s \rightarrow t} \frac{|X_{st}|}{\omega_X(s, t)^{2\alpha}} = \infty. \quad (3.5)$$

The situation here is different in the sense that assuming $B = X\sigma \cdot \nabla$ with $X$ as in (3.5) does not guarantee uniqueness of $g'$ for $(g, g') \in D_B^{0,p}$. Indeed, assume that $d = 2$, and let $B$ as above with $\sigma = (0, 1)$, for the Hilbertian scale $(H^k) \equiv (W^{k,2})_{k \geq 0}$. If $g'$ is a Gubinelli drivative for $g$, then it is immediate that any path of the form $t \mapsto g'_t(x, y) + f_t(x)$ where $f \in \mathcal{V}^\alpha(0, T; L^2(\mathbb{R}))$ is a function of the first variable only, is also a Gubinelli derivative for $t \mapsto g_t$. In this counterexample, one sees that the space-like variable plays an important role in the discussion, and that if one really wants to have some kind of Doob-Meyer decomposition, then some “non-degeneracy” assumptions on the differential operator $\sigma \cdot \nabla$ are in order. Let us now formulate a natural sufficient condition under which uniqueness of the Gubinelli derivative holds.
For technical reasons, assume this time that we are given an URD $B$ on the scale $(H^s_0(D))_{k \geq 0}$, $D$ being an open bounded set in $\mathbb{R}^d$. Assume that there exists a number

$$\gamma \in [2\alpha, 3\alpha),$$

such that it holds the following $H^2_0(D)$-coercivity assumption:

$$\left(B_{st}^1 \varphi, B_{st}^1 \varphi \right)_{H^1, H^1} \equiv (B_{st}^1 \varphi, B_{st}^1 \varphi)_{L^2} + (\nabla B_{st}^1 \varphi, \nabla B_{st}^1 \varphi)_{L^2} \geq \lambda \omega_B(s, t)^\gamma |\varphi|^2_{H^2} \quad (3.6)$$

for some $\lambda > 0$, independently of $\varphi$ in $H^2_0(D)$, and for each $(s, t) \in \Delta \cap I^2$, where we are given some dense subset $I$ of $[0, T]$.

Denote by $B_{st}^{1,*}$ the adjoint of $B_{st}^1$ with respect to the $H^1$-inner product. Then, by the Lax-Milgram Theorem, the operator $B_{st}^{1,*}B_{st}^1 : H^2 \to L^2$ is invertible, with a bounded inverse

$$T_{st} = (B_{st}^{1,*}B_{st}^1)^{-1} : H^2_0(D) \to H^2_0(D).$$

The operator norm of $\Lambda_{st}$ is estimated above as

$$|T_{st}|_{\mathcal{F}(H^2_0, H^2_0)} \leq \lambda^{-1} \omega_B(s, t)^{2\alpha - \gamma},$$

and by symmetry one obtains a similar bound on $\Lambda_{st}$, seen this time as an operator from $H^{-2}$ into itself.

If $g'_s$ is a candidate for the Gubinelli derivative of $g$ and if $R^g \in \mathcal{V}^2_2(0, T; H^{-1}) \cap \mathcal{V}_2(0, T; H^{-2})$ denotes the first order remainder $\delta g - Bg$, one infers from the above discussion the equality:

$$g'_s = T_{st}B_{st}^*\delta g_{st} - T_{st}B_{st}^*R^g_{st} =: T^1 + T^2. $$

But it holds, for any $s, t \in \Delta \cap I^2$,

$$|T^2|_{H^{-2}} \leq \lambda^{-1} \omega_B(s, t)^{2\alpha - \gamma} |B_{st}^{1,*}R^g_{st}|_{H^{-2}} \leq \lambda^{-1} \omega_B(s, t)^{3\alpha - \gamma} |R^g_{st}|_{H^{-1}}.$$

Assuming for simplicity that all the above controls are proportional to $t - s$ one sees by letting $t_n \downarrow s$, $t_n \in I_0$, that

$$T^2 \leq C(B, \lambda)(t_n - s)^\alpha \|R^g\|_{C^0(0, T; H^{-1})} \to 0 \quad \text{in} \quad H^{-1}.$$

Hence, $g'_s$ is uniquely determined by the relation

$$g'_t = \lim_{s \to t, s \in I} T_{st}B_{st}^{1,*}\delta g_{st} \quad \text{in} \quad H^{-1}.$$ 

**Example 3.1.** Let $d = 1$, and consider a 1-dimensional, $\alpha$-Hölder rough path $(X, \mathbb{X}) \in \mathcal{C}^\alpha(0, T; \mathbb{R})$ such that for some $I$ as above it holds

$$|X_{st}| \geq c(t - s)^\gamma, \quad \text{for every} \quad (s, t) \in \Delta_I,$$

where we are given some constant $\gamma \in [\alpha, 2\alpha)$ (this implies in particular true roughness for $X$, in the sense of (3.5)). Moreover, let $\sigma \in W^{3, \infty}(\mathbb{R}; \mathbb{R})$ be bounded below, namely such that there exists a constant $\sigma > 0$, with the property that $\sigma(x) \geq \sigma$ for almost every $x \in \mathbb{R}^d$.

Then, it is easily seen that (3.6) holds, with a coercivity constant $\lambda > 0$ depending on $c$ and $\sigma$. 
3.2. Main result: product formula. Given $u$ and $v$ in $\mathcal{H}^\alpha_{B,\text{loc}}$ such that $u' = u$ and $v' = v$, it seems natural to expect that the product $w := uv \in C(L^1)$ belongs to $\mathcal{H}^\alpha_{B,\text{loc}}$. The main result of this section gives a justification of this intuition, by showing that an “integration by parts formula” (or simply product formula) for $w$ is satisfied. Reiterating the product formula will ultimately allow us to prove the chain rule for polynomials of elements $u \in \mathcal{H}^\alpha_{B,\text{loc}}$ that are locally bounded, and then to conclude by a density argument (this last step is however not trivial, see Section 7).

Note that this approach is somewhat similar to the proof of the finite-dimensional Itô formula given for instance in [RY13].

**Proposition 3.1** (Product formula). Let $p \in [1, \infty]$, $u \in \mathcal{H}^{\alpha,p}_{B,\text{loc}}, v \in \mathcal{H}^{\alpha,p'}_{B,\text{loc}}$ where $p' := \frac{p}{p-1}$ (with the usual conventions “1/0 = $\infty$” and “$\infty/\infty = 1$”). Assume that $u$ is an $L^p_{\text{loc}}$-solution of
$$du = fdt + dBu$$
on $[0,T] \times \mathbb{R}^d$, in the sense of Definition 1.4, while $v$ is an $L^{p'}_{\text{loc}}$-solution of
$$dv = gdt + dBv.$$Then, we have $uv \in \mathcal{H}^{\alpha,1}_{B,\text{loc}}$ and $uv$ is an $L^1_{\text{loc}}$-solution of
$$d(uv) = (ug + fv)dt + dB(uv). \quad (3.7)$$

For convenience, we chose to state a local version of the product formula, but it should be observed that a similar “global version” holds, by adapting the arguments below in an obvious way. Namely if one assumes that $u \in \mathcal{H}^{\alpha,p}_{B,\text{loc}}$ and $v \in \mathcal{H}^{\alpha,p'}_{B,\text{loc}}$, then it holds true that $uv \in \mathcal{H}^{\alpha,1}_{B,\text{loc}}$, and moreover (3.7) is valid in the $L^1$-sense.

Before we proceed to the proof of Proposition 3.1, we need to introduce some additional notations. In what follows, we fix
$$K \subset \subset \mathbb{R}^d.$$

**Notation 3.1.** For $\epsilon \in (0, 1]$ we will denote by $K_\epsilon$ the $\epsilon$-fattening of $K$, namely
$$K_\epsilon := \{x + h \in \mathbb{R}^d, x \in K \text{ and } h \in B_1\}.$$

**Notation 3.2.** For $\epsilon \in (0, 1]$, one defines a set $\Omega_\epsilon \subset \mathbb{R}^d \times \mathbb{R}^d$ as follows:
$$\Omega_\epsilon^K := \left\{(x, y) \in K_1 \times K_1, \frac{x + y}{2} \in K, \frac{x - y}{2} \in B_\epsilon\right\}. \quad (3.8)$$

**Notation 3.3.** For each $k \geq 0$ and $p \in [1, \infty]$, and $\epsilon \in (0, 1]$, we define a linear, one-to-one mapping $T_\epsilon : W^{k,p}_0(\Omega_1^K) \to W^{k,p}_0(\Omega_\epsilon^K)$ by the formula
$$T_\epsilon \Phi(x, y) := \frac{1}{(2\epsilon)^d} \Phi \left(\frac{x + y}{2} + \frac{x - y}{2\epsilon}, \frac{x + y}{2}, \frac{x - y}{2\epsilon}\right), \quad (3.9)$$

for all $\Phi \in W^{k,p}_0(\Omega_1^K)$. The adjoint of $T_\epsilon^* : W^{-k,\frac{p}{p-1}}(\Omega_1^K) \to W^{-k,\frac{p}{p-1}}(\Omega_\epsilon^K)$ is well-defined by duality.

**Notation 3.4.** It will be convenient in the sequel to use the new system of coordinates $\chi : \Omega^K_1 \to K \times B_1$ defined by
$$(x_+, x_-) = \chi(x, y) := \left(\frac{x+y}{2}, \frac{x-y}{2}\right), \quad \text{for } (x, y) \in \Omega^K_1. \quad (3.10)$$
Note that $|\det D\chi| = 2^{-d}$ and that $\sqrt{2} \chi$ is a rotation.

Before we make sense of the product $u(x)v(x)$ as an element of $\mathcal{H}^{\alpha,1}_B([0,T] \times K)$, an intermediate step is to form first their tensor product $w(x,y) := u(x)v(y)$, and then to show that $w$ belongs to some similar space, for some unbounded rough driver $\Gamma(B)$ operating on both variables.

**Proposition 3.2.** Consider $u,v$ as in Proposition 3.1. Define $w := u \otimes v$ in the sense that

$$w(x,y) := u(x)v(y), \quad \text{for every } (x,y) \in \Omega^K_1. \quad (3.11)$$

Then, we have

$$w \in \mathcal{H}^{\alpha,1}_{\Gamma(B)}(\Omega^K_1) \quad (3.12)$$

where we denote by $\Gamma(B)$ the URD given for $(s,t) \in \Delta$ by

$$\Gamma(B)_{st} := B_{st}^1 \otimes \text{id} + \text{id} \otimes B_{st}^2, \quad \Gamma(B)_{st}^2 := B_{st}^2 \otimes \text{id} + B_{st}^1 \otimes B_{st}^1 + \text{id} \otimes B_{st}^2. \quad (3.13)$$

Moreover the mapping $r \mapsto f_r \otimes v_r + u_r \otimes g_r$, is Bochner integrable in the space $W^{-1,1}(\Omega^K_1)$, and $w$ is an $L^1_{\text{loc}}$-solution of

$$\frac{dw}{dt} = (f \otimes v + u \otimes g) dt + d\Gamma(B)w.$$ 

Denote by $\Gamma(B)$ is the symmetric driver defined in (3.13). If $B$ is transport-like, then the family of URD’s

$$\Gamma(B)^{i,\epsilon} := (T^*_r \Gamma(B)^1(T_r^*)^{-1}, T^*_r \Gamma(B)^2(T_r^*)^{-1})$$

has the remarkable property of being uniformly bounded, in the sense that there exists a control $\tilde{\omega}_B$ such that for $i = 1, 2$, and $-3 \leq k - i \leq k \leq 0$ it holds uniformly in $\epsilon > 0$:

$$|\Gamma(B)^{i,\epsilon}|_{L^\infty(W^{-k,1}(K),W^{-k-i,1}(K))} \leq \tilde{\omega}_B(s,t)^{i,\alpha} \quad (3.14)$$

(the above estimate turns out to be also independent of $K$). According to the terminology used in [DGHT16], we shall also say that the family $\Gamma(B)^{i,\epsilon}$ is renormalizable.

This is to be related to the notion of “renormalizable solution” in transport theory, where functions $F(u(t,x))$ of a weak solution $u$ to the PDE $\partial_t u = b(t,x) \cdot \nabla u$ are themselves solutions of a similar equation. As shown in the proof of Proposition 3.1, such property is also true for the ansatz (1.29), provided that the URD $B$ satisfies the estimates (3.14). Informally speaking, (3.14) mean that a commutator lemma à la Di Perna Lions can be performed, see [DL89]. We will investigate this question in detail in Appendix A.

**Proof of Proposition 3.2.** The proof is similar to that of [HH18]. It is based on the algebraic identity

$$\pi_{\Gamma(B)}(f \otimes v + u \otimes g; w, w)^{i,\epsilon}_{st} = u^{s}_{st} \otimes v_{st} + u_{st} \otimes v^{s}_{st} + \Lambda u \Xi + (\delta u_{st} - B^1_{st} u_{s}) \otimes \delta v_{st} + B^1_{st} u_{s} \otimes (\delta v_{st} - B^1_{st} u_{s}), \quad (3.15)$$

where $\Lambda$ denotes the Sewing Map, and $\Xi$ denotes the 3-parameter mapping $\Xi : \Delta_2 \to W^{-1,1}(\Omega^K_1), (s, \theta, t) \mapsto \Xi_{st} : \delta u_{st} \otimes \int_{\theta}^{s} g_{s}ds + \int_{s}^{t} f ds \otimes \delta v_{st}$.

It is also readily checked that $w$ defines an element of $\mathcal{H}^{\alpha,1}_{\Gamma(B)}(\Omega^K_1)$ (see the details given in [HH18, Section 5]).
Before we proceed to the proof of the main result, some comments about the product mapping
\[ \mu : W_{-k,p'} \otimes W_{k,p} \to (W_{k,\infty})^*, \quad k \in \mathbb{N}_0, \quad p \in [1, \infty), \]
are in order.

Fix \( k \in \mathbb{N}_0, \ p \in [1, \infty), \) denote by \( p' := \frac{p}{p-1} \in (1, \infty) \) and (with a slight abuse of notation) define \( f \otimes g(x, y) := f(x)g(y) \) where \( f \in W_{-k,p'} \) and \( g \in W_{k,p} \). We can make sense of the product \( \mu(f \otimes g)(x) \equiv "fg(x)" \) by the formula
\[ \langle \mu(f \otimes g), \phi \rangle \overset{\text{def}}{=} \int_{\mathbb{R}^d} (-1)^{|\gamma|} f_\gamma(x)D^\gamma(g\phi)(x)dx, \quad \tag{3.16} \]
for every \( \phi \in W_{k,\infty} \), where \( f = (D^\gamma f_\gamma)_{|\gamma| \leq k} \) in the distributional sense. Such a tuple exists (see e.g. \[Bre10,\ p.219\]) but is not unique in general. It is however a simple exercise to show that, given another \((\tilde{f}_\gamma)_{|\gamma| \leq k}\) as above, the corresponding distributions in (3.16) coincide for every \( g \) in \( W_{k,p} \), hence removing any ambiguity in the definition of \( \mu(f \otimes g) \). Using Leibniz formula, it is immediately seen that \( \mu : W_{-k,p'} \otimes W_{k,p} \to (W_{k,\infty})^* \) is bounded, namely
\[ \sup_{|\phi|_{W_{k,\infty}} \leq 1} |\langle \mu(f \otimes g), \phi \rangle| \leq |f|_{W_{-k,p'}}|g|_{W_{k,p}}, \quad \tag{3.17} \]
and because every factor of the form \( f^\gamma(D^\gamma g) \) in (3.16) is in \( L^1 \), the range of \( \mu \) is actually in the space \( W_{-k,1} \), see (1.10). Furthermore, one sees from (3.17) that \( \mu \) is nothing but the trace onto the diagonal \( \Upsilon := \{ x, y \in \mathbb{R}^{2d} : x = y \} \), namely it corresponds to what is obtained by first approximating \( f \) and \( g \) by sequences of compactly supported smooth functions, and then taking the limit in the corresponding products. (The fact that smooth functions are not dense in \( L^\infty \) is not a problem, because the other factor belongs to \( L^1 \equiv \overline{C^\infty}_{\infty}L^1 \), and it is enough to approximate only one of the factors.)

Consider now \( v = v(x, y) \in W_{-k,1}(\Omega^K) \) such that \( \mu(v) \) is well defined as an element of \( W_{-k,1}(\Upsilon) \), and identify \( \Upsilon \) with \( \mathbb{R}^d \). The adjoint of \( T_\varepsilon \) is formally given on \( v \) by
\[ T^*_\varepsilon v(x, y) = 2^{-d} \left[ (\tau_{-\varepsilon^2/2} \otimes \tau_{\varepsilon^2/2}) v \right] \left( \frac{x + y}{2}, \frac{x + y}{2} \right), \quad (x, y) \in \Omega^K, \quad \tag{3.18} \]
which, testing against \( \Phi \in F_k \), and letting \( (x_+, x_-) := \chi(x, y) \) yields the representation
\[ \langle T^*_\varepsilon v, \Phi \rangle = \int_{B_1} W_{k,\infty} \langle \mu \left[ (\tau_{-\varepsilon\cdot} \otimes \tau_{\varepsilon\cdot}) v \right] , \Phi(\cdot, x_-) \rangle_{W_{k,\infty} \mathbb{R}^d} dx_- ; \quad \tag{3.19} \]
where \( \Phi(x_+, x_-) := \Phi \circ \chi^{-1} \).

Having (3.19) at hand, we can now proceed to the proof of the main result.

**Proof of Proposition 3.1.** First note that, in Proposition 3.2, the domain \( \Omega^K_\varepsilon \) can be replaced by \( \Omega^K_\varepsilon \) for any \( \varepsilon > 0 \), yielding a similar conclusion. In particular, for any \( \varepsilon \in (0, 1) \), the restriction of \( w = u \otimes v \) to \( \Omega^K_\varepsilon \) belongs to the space \( \mathcal{H}^1_{\Gamma_\varepsilon}(\Omega^K_\varepsilon) \) and is an \( L^1 \)-solution to
\[ dw = (f \otimes v + u \otimes g)dt + d\Gamma(B)w, \quad \tag{3.20} \]
Similarly, the Bochner integral which defines the drift term
\[ \mathcal{D}_t := \int_0^t (f_r \otimes v_r + u_r \otimes g_r) \, dr, \quad t \in [0, T], \]
is to be understood in \( W^{-1,1}(\Omega^K) \).

Hence, testing (3.20) against \( \Phi := T_s \Psi \), where \( \Psi \) is a generic element in \( W^{3,\infty}_0(\Omega^K_1) \), and letting from now on
\[ w^\epsilon := T^*_\epsilon w, \quad \mathcal{D}^\epsilon := T^*_\epsilon \mathcal{D}, \quad \Gamma(B)^{1,\epsilon} := T^*_\epsilon \Gamma(B)^1(T^*_\epsilon)^{-1}, \quad \Gamma(B)^{2,\epsilon} := T^*_\epsilon \Gamma(B)^2(T^*_\epsilon)^{-1} \]
and \( w^{\epsilon,\hat{\kappa}} := T^*_\epsilon w^{\hat{\kappa}}, \quad (3.21) \)
we end up with the following equation
\[ dw^\epsilon = \mathcal{D}^\epsilon dt + d\Gamma(B)^\epsilon w^\epsilon \quad (3.22) \]
(\( L^1_{\text{loc}} \)-sense). Moreover, using the continuity of \( T_\epsilon \), it is clear that \( w^\epsilon \) belongs to \( \mathcal{H}^{\alpha,1}_{T(B),\text{loc}}(\Omega^K) \).

**First step: uniform bound on the drift.** Let \( \Phi \in W^{1,\infty}_0(\Omega^K_1) \). By definition, we have
\[ \langle \delta \mathcal{D}^\epsilon_{st}, \Phi \rangle = \int_s^t \langle u_r \otimes g_r + f_r \otimes v_r, T_r \Phi \rangle \, dr. \]
Fix \( r \in [s,t] \) such that \( u \equiv u_r \) belongs to \( W^{1,p} \), and let \( \Phi(x_+, x_-) := \Phi \circ \chi^{-1}(x_+, x_-) = \Phi(x_+ + x_-, x_+ - x_-) \). Making use of (3.19), there holds for the first term
\[ \langle u \otimes g, T_r \Phi \rangle = \int_{B_1} W_{-1,\tau(K)} \langle \mu [(\tau_{-\epsilon} - u)(\tau_{-\epsilon} - g)], \Phi(\cdot, x_-) \rangle_{W^{1,\infty}_0(K)} \, dx_- \]
\[ = \int_{B_1} W_{-1,\tau'}(K) \langle \tau_{-\epsilon} - g, (\tau_{-\epsilon} - u) \Phi(\cdot, x_-) \rangle_{W^{1,p}(K)} \, dx_- \]
\[ = \sum_{|\gamma| \leq 1} (-1)^{|\gamma|} \int_{B_1 \times K} \tau_{-\epsilon} g_{\gamma}(x_+) \Phi(s, x_-) \tau_{-\epsilon} u(x_+) \Phi(\cdot, x_-) \, dx_+ \, dx_- \]
where \( (g_{\gamma}) \in (L^p)^d \) denotes a primitive of \( g \), and \( D_+ \) denotes derivation with respect to the first variable. Hence we have
\[ \langle u \otimes g, T_r \Phi \rangle \leq \int_{B_1} |\tau_{-\epsilon} g|_{L^p(K)} |\tau_{-\epsilon} u|_{L^p(K)} |\Phi(\cdot, x_-)|_{L^\infty(K)} \, dx_- \]
\[ + \int_{B_1} |\tau_{-\epsilon} g|_{L^p(K)} |\tau_{-\epsilon} u|_{L^p(K)} |\nabla_+ \Phi(\cdot, x_-)|_{L^\infty(K)} \, dx_- \]
\[ \leq |B_1| |u|_{W^{1,p}(K)} |g|_{L^p(K)} |\Phi|_{W^{1,\infty}(\Omega^K)}, \]
where \( |B_1| \) denotes the Lebesgue measure of \( B_1 \).

Doing the same computations for the second term, and integrating in time, we end up with the estimate
\[ |\delta \mathcal{D}^*|_{W^{-1,1}(\Omega^K)} \leq |B_1| \left( |u|_{L^p(s,t;W^{1,p}(K))} |g|_{L^p(s,t;W^{1,p}(K))} \right) \]
\[ \leq |B_1| |u|_{L^p(s,t;W^{1,p}(K))} |\Phi|_{W^{1,\infty}(\Omega^K)}, \quad (3.23) \]
which bounded above by \( \omega_{\mathcal{D},K_1}(s,t) \), uniformly with respect to \( \epsilon \in (0,1] \).
Step 3: passage to the diagonal. Noticing that for a.e. \( r \in [0, T] \),

\[
|w_{s,t}^{\epsilon}|_{L^1(\Omega^K)} \leq |K| |u_r|_{L^p(K_n)} |v_r|_{L^{p'}(K_n)},
\]

and using the estimates (3.14) (see Appendix A), one can apply Proposition 2.4 to obtain the bound:

\[
|w_{s,t}^{\epsilon,2}|_{W^{-3,1}(\Omega^K)} \leq C \left( |K| |u_r|_{L^\infty(L^p(K_n))} |v_r|_{L^\infty(L^{p'}(K_n))} \omega_B(s,t)^{3\alpha} + \omega_{\beta,K}(s,t) \omega_B(s,t)^{\alpha} \right),
\]

(3.24)

for every \( (s, t) \in \Delta \) such that \( \omega(s, t) \leq L \) for some \( L > 0 \).

Fix \( (s, t) \) with \( \omega_B(s, t) \leq L \), and let

\[
\psi \in W_0^{3,\infty}(B_1), \quad \text{with} \quad \int_{B_1} \psi(x_-)dx_- = 1.
\]

(3.25)

Denote by \( \ell^\epsilon_{\psi,K} \) the element of \( W^{-3,1}(K) \) defined by

\[
\ell^\epsilon_{\psi,K}(\phi) := \langle w_{s,t}^{\epsilon,2}, (\phi \otimes \psi) \circ \chi \rangle, \quad \text{for} \ \phi \in W_0^{3,\infty}(K),
\]

which in particular satisfies the estimate

\[
|\ell^\epsilon_{\psi,K}|_{W^{-3,1}(K)} \leq |\psi|_{W^{3,\infty}(B_1)} |w_{s,t}^{\epsilon,2}|_{W^{-3,1}(\Omega^K)}.
\]

(3.26)

From (3.24), one deduces that for some constant \( C_K > 0 \) (depending on \( K \), but also on \( \|u\|_{H^{\alpha,p}([0,T] \times K_1)}, \|u\|_{H^{\alpha,p'}([0,T] \times K_1)} \) it holds, uniformly in \( \epsilon \in (0, 1] \):

\[
|\ell^\epsilon_{\psi,K}|_{W^{-3,1}(K)} \leq C_K
\]

(3.26)

Denote by \( \ell^\epsilon_{\psi} \in W_{\text{loc}}^{-3,1} \) the element obtained by “gluing together” all the \( \ell^\epsilon_{\psi,K} \)'s when \( K \subset \subset \mathbb{R}^d \). By (3.26), we see that \( \ell^\epsilon_{\psi} \) is uniformly bounded in \( W_{\text{loc}}^{-3,1} \). Recall that every closed and bounded set of \( W_{\text{loc}}^{-3,1} \), endowed with the weak topology \( \sigma(W_{\text{loc}}^{-3,1}, W_{\text{loc}}^{3,\infty}) \), is compact. Therefore, we infer the existence of a sequence \( \epsilon_n \to 0 \) and \( \ell \in (W_{\text{loc}}^{3,\infty})^* \) such that \( \ell^\epsilon_{\psi,K}(\phi) \to \ell(\phi) \), for every \( K \subset \subset \mathbb{R}^d \) and \( \phi \in W^{3,\infty}(K) \).

To show that \( \ell \) belongs to \( W_{\text{loc}}^{-3,1} \), we will apply the Dunford-Pettis Theorem (Corollary 2.1). If \( K \subset \subset \mathbb{R}^d \) has smooth boundary, the previous analysis shows that

\[
|\ell^\epsilon_{\psi}|_{W^{-3,1}(K)} \leq C_{\psi,|B_1|}(\|u\|_{L^\infty(L^p(K_n))} |v\|_{L^\infty(L^{p'}(K_n))} \omega_B(s,t)^{3\alpha} + \|u\|_{L^p(s,t;W^{1,\phi}(K_n))} |v|_{L^{p'}(s,t;W^{-1,p'}(K_n))} + \|v\|_{L^{p'}(s,t;W^{-1,p'}(K_n))} \|f\|_{L^p(s,t;W^{-1,\phi}(K_n))} \omega_B(s,t)^{\alpha}),
\]

(3.27)

by which one deduces boundedness of \( (\ell_{\psi,K}^\epsilon)_{n \geq 0} \). From the above inequality and the fact that \( K_n \downarrow K \), we also infer that

\[
\limsup_{n \to \infty} |\ell^\epsilon_{\psi,K}|_{W^{-3,1}(K)} \to 0,
\]

showing equi-integrability. Therefore the weak limit \( \ell \) belongs to \( W_{\text{loc}}^{-3,1} \).
It remains to take the limit in the other terms of the equation. Let \( K \subset \subset \mathbb{R}^d \), fix \( \phi \in W^{2,\infty}_0(K) \), and test all the terms in (3.20) against \( \Phi_\varepsilon(x,y) := (T_\varepsilon(\phi \otimes \psi)) \circ \chi(x,y) \equiv \phi \left( \frac{x+y}{2} \right) \psi \left( \frac{x-y}{2\varepsilon} \right) (2\varepsilon)^{-d} \), for \( (x,y) \in \Omega^K_\varepsilon \).

Using that \( \psi(\frac{r}{2\varepsilon})(2\varepsilon)^{-d} \) approximates the identity together with the \( L^\infty \)-renormalizability of \( B \), then it is seen by dominated convergence that

\[
W^{-1,1}(\Omega^K) \left\langle \delta D^\varepsilon_{st}, \Phi \right\rangle_{W^{1,\infty}_0(\Omega^K)} \rightarrow \int_s^t W^{-1,1}(K) \left\langle u_r g_r + f_r v_r, \phi \right\rangle_{W^{1,\infty}_0(K)} \, dr
\]

and also

\[
W^{-1,1}(\Omega^K) \left\langle (B^4_{st})^\varepsilon, \Phi \right\rangle_{W^{1,\infty}_0(\Omega^K)} \rightarrow \left\langle (B^4_{st} u_s) v_s + v_s B^4_{st} u_s, \phi \right\rangle_{W^{1,\infty}_0(\Omega^K)} \equiv \left\langle B^4_{st} (uv), \phi \right\rangle
\]

\[
W^{-2,1}(\Omega^K) \left\langle (B^2_{st})^\varepsilon, \Phi \right\rangle_{W^{2,\infty}(\Omega^K)} \rightarrow \left\langle (B^2_{st} u_s) v_s + B^2_{st} u_s B^2_{st} v_s + u_s B^2_{st} v_s, \phi \right\rangle_{W^{2,\infty}(\Omega^K)} \equiv \left\langle B^2_{st} (uv), \phi \right\rangle,
\]

where here we made use of the fact that \( B \) is transport-like, see Definition 1.5. Similarly, it is easily shown that \( \left\langle \delta w^\varepsilon_{st}, \Phi \right\rangle \rightarrow \left\langle \delta (uv)_{st}, \phi \right\rangle \). These convergences hold no matter which subsequence of \( \varepsilon_n \) we use, and hence the same is true for \( \ell^\varepsilon \). Therefore, taking the limit in (3.24) and using the lower semi-continuity of the norm \( | \cdot |_{W^{-3,1}} \) with respect to weak convergence, one sees that there exists a unique element \( (uv)^2 \) in \( W^{1,\infty}_{loc}(0,T; W^{2,1}_{loc}) \) such that

\[
\left\langle (uv)_{st}, \phi \right\rangle = \int_s^t \left( u g + f v, \phi \right) \, dr + \left\langle (B^1 + B^2)_{st} (uv), \phi \right\rangle + \left\langle (uv)^2_{st}, \phi \right\rangle,
\]

for every \( (s,t) \in \text{such that} \omega_B(s,t) \leq L \). This proves the proposition. \( \square \)

**Remark 3.2.** In the above proof we have used the fact that the trace \( \mu \) onto the diagonal \( \Upsilon \) is a well defined operator if it is endowed with the domain \( W^{-1,p} \otimes_{\text{alg}} W^{1,p'} \) consisting of linear combinations of elements of the form \( g_1 \otimes g_2 \), and where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

This result is not optimal, for the trace onto the diagonal can be extended to larger classes of couples \((g_1,g_2)\). For instance, consider the case where \( g \in W^{-1,1} \) is given by \( fu \) where \( u \in H^1 \cap L^\infty \), \( f \in H^{-1} \), and let \( v \in H^1 \cap L^\infty \). Then, although \( v \) does not belong to \( W^{1,\infty} \) a priori, it is still possible to define \( \mu (g,v) \in W^{-1,1} \) without ambiguity as the element

\[
\phi \in W^{1,\infty} \longmapsto L^1 \left( f \nabla u, v \phi \right)_{L^\infty} + L^1 \left( f \nabla v, u \phi \right)_{L^\infty} + L^1 \left( f u v, \nabla \phi \right)_{L^\infty},
\]

(3.28)

where \( f \equiv (f_t) \) is an arbitrary primitive of \( f \in H^{-1} \).

It is not difficult to convince oneself that in this case, the proof of Proposition 3.1 works similarly (this fact is left to the reader). In particular, denoting by \( A \) the set of all linear combinations \( u f \in W^{-1,1}_{loc} \) as above, let \( v, \tilde{v} \in \mathcal{H}^{1,2}_{B,loc} \cap L^\infty \) be solutions of

\[
dv = adt + dBv \quad \text{and} \quad d\tilde{v} = a\tilde{dt} + dB\tilde{v} \quad \text{for some} \ a, \tilde{a} \in A,
\]

in the \( L^1_{loc} \)-sense. Then, it is still the case that \( v \tilde{v} \in \mathcal{H}^{1,1}_{B,loc} \) and moreover it is an \( L^1_{loc} \)-solution of

\[
d(v\tilde{v}) = (v \tilde{a} + a\tilde{v}) dt + dB(v\tilde{v}).
\]
4. Parabolic equations with free terms

Here we deal with existence and uniqueness for parabolic, rough partial differential equations of the form

\[ du = (Au + f)dt + dBu, \quad \text{on } [0, T] \times \mathbb{R}^d, \]

where \( f \) belongs to the space \( L^2(0, T; H^{-1}) \), hence generalizing the case treated in [HH18]. The interest in doing so is that solutions of such equations need not be bounded. Since on the other hand we will show that an Itô formula still holds in that case, this will illustrate the fact that boundedness is not necessary. We have the following.

**Proposition 4.1** (Solvability in the energy space). Let \( f \in L^2(0, T; H^{-1}) \) and fix \( u_0 \in L^2 \). There exists a unique \( L^2 \)-solution \( u = u(u_0, f) \) to (4.1), and it belongs to the space \( H^{\alpha, 2}_B(\mathbb{R}^d) \).

The solution depends continuously on the indicated quantities, as a mapping \( L^2(0, T; L^2) \cap L^2(0, T; H^1) \rightarrow L^\infty(0, T; L^2) \), with respect to the strong topologies.

**Proof of Proposition 4.1.** The proof is quite similar to that of [HH18] but since our assumptions on \( B \) are somewhat more general, we provide a complete proof.

**Step 1: Bounds on the drift term.** Consider an \( L^2 \)-solution \( u \in H^{\alpha, 2}_B \) of the equation (4.1). Applying Proposition 3.1 with \( u = v \), we have that \( u^2 \in H^{\alpha, 1}_B \) and it is an \( L^1 \)-solution of

\[ du^2 = 2u(Au + f)dt + dB(u^2). \] (4.2)

We want to test against \( \phi = 1 \), and then apply Rough Gronwall, but for this we need first an estimate on \( u^{2\gamma} \), which itself follows from Proposition 2.4, together with the estimate on the drift. The analysis of the linear part of the drift leads to the estimate:

\[ \left| \int_s^t (uAu)dr \right|_{W^{-1, 1}} \leq \|a\|_\infty(\|\nabla u\|_{L^2(s, t; L^2)}^2 + \|u\nabla u\|_{L^1(s, t; L^1)}) \] (4.3)

(this expression defines a control, according to [FV10, Exercise 1.9]), whereas for the free term, considering primitive elements of \( f \), we find

\[ \int_s^t |uf|_{W^{-1, 1}}dr \leq (\|u\|_{L^2(s, t; L^2)} + \|\nabla u\|_{L^2(s, t; L^2)}) \|f\|_{L^2(s, t; H^{-1})}. \] (4.4)

**Step 2: Energy inequality and application to uniqueness.** Letting \( \omega_B(s, t) \) be the sum of the right hand sides in (4.3) and (4.4), one can then apply Proposition 2.4 to obtain

\[ |u^{2\gamma}_s|_{W^{-3, 1}} \leq C \left( \omega_B(s, t)^\alpha \omega_B(s, t) + \|u\|_{L^\infty(s, t; L^2)}^2 \omega_B(s, t)^{3\alpha} \right). \] (4.5)

for every \( (s, t) \in \Delta \) with \( \omega_B(s, t) \leq L \) for some absolute constant \( L > 0 \).
Next, one can take \( \phi = 1 \in W^{3,\infty} \) in (4.2), so that by Assumption 1.2 it holds for every \( s, t \) as above:

\[
\delta E_{st} := \delta (|u|_{L^2}^2)_{st} + \int_s^t |\nabla u_r|^2_{L^2} dr \leq \delta_0 \int_{[s,t]} (|u_r|^2_{L^2})_{x} dx dr + \langle u_s^2, (B_{st}^1 + B_{st}^2) \rangle + \langle u_s^2, 1 \rangle
\]

\[
\leq \delta_0 \int (|f|_{L^2(s,t;L^2)}^2) \|\nabla u\|_{L^2(s,t;L^2)}^2 + |u_s|_{L^2}^2 (\omega_B(s,t)^\alpha + \omega_B(s,t)^{2\alpha}) + |u_s^2|_{W^{-3,1}}^2,
\]

\[
\leq \delta_0 \|f\|_{L^2(s,t;L^2)}^2 \|\nabla u\|_{L^2(s,t;L^2)}^2 + (\omega_B(s,t)^\alpha + \omega_B(s,t)^{2\alpha} + \omega_B(s,t)^{3\alpha})(\sup_{r \in [s,t]} E_r) + \omega_B(s,t)^\alpha \|f\|_{L^2(s,t;H^{-1})}^2 (\|\nabla u\|_{L^2(s,t;L^2)} + \|u\|_{L^2(s,t;L^2)})
\]

Making use of Young Inequality \( \|f\|_{L^2(s,t;L^2)} \|\nabla u\|_{L^2(s,t;L^2)} \leq \frac{1}{2} \|f\|_{L^2(s,t;L^2)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(s,t;L^2)}^2 \) for a sufficiently small \( \epsilon(\delta) > 0 \), the first term in the right hand side can be absorbed to the left. Hence, taking \( L \) smaller if necessary, we infer that for any \( (s, t) \in \Delta \) with \( \omega_B(s,t) \leq L \), it holds the incremental inequality

\[
\delta E_{st} \leq \omega_B(s,t)^\alpha (\sup_{r \in [s,t]} E_r) + \|f\|_{L^2(s,t;H^{-1})}^2.
\]

By Lemma 2.1, we deduce the estimate

\[
(\sup_{t \in [0,T]} E_t) \equiv \|u\|_{L^\infty(0,T;L^2)}^2 + \|\nabla u\|_{L^2(0,T;L^2)}^2
\]

\[\leq C(\delta, \|a\|_{L^\infty}) \exp \left( \frac{\omega_B(0,T)}{\tau_{\alpha,L}} \right) \left[ \|u_0\|_{L^2}^2 + \|f\|_{L^2(0,T;H^{-1})}^2 \right]. \tag{4.6}
\]

The uniqueness is now straightforward, because the difference \( v \equiv u_1 - u_2 \) of two \( L^2 \)-solutions to (4.1) must be an \( L^2 \)-solution of (4.1), with \( f = 0 \) and \( v_0 = 0 \), hence yielding from (4.6) that \( v = 0 \).

**Step 3: Existence.** Existence and continuity are more delicate and rely on the stability results. It should be noted here that, the assumption that \( B \) be geometric is essential.

Consider a sequence \( B(n) \to B \) as in Definition 1.2. By Remark 1.2, we can assume without loss of generality that each \( B(n) \) is transport-like (it is indeed sufficient to approximate the coefficients). By standard results on parabolic equations, there exists a unique \( u(n) \) in the energy space \( L^\infty(L^2) \cap L^2(H^1) \), solving (4.1) in the sense of distributions.

Using moreover the fact that \( B(n) = S_2(b(n)) \), it is easily deduced from (4.1) that \( u(n) \) is an \( L^2 \)-solution of (4.1), in the sense of Definition 1.4. Consequently, the previous analysis shows that we have a uniform bound

\[
\|u(n)\|_{L^\infty(0,T;L^2)}^2 + \|\nabla u(n)\|_{L^2(0,T;H^1)}^2 \leq C(\delta, \|a\|_{L^\infty}, \|f\|_{L^2(0,T;H^{-1})}, \|u_0\|_{L^2}, T).
\]

As a straightforward consequence of this, we obtain as well the uniform estimate

\[
\|u(n)\|_{H_{B}^{\alpha,2}} \leq C',
\]

for another such constant \( C' \). By Lemma 3.1 we see that \( \{u(n), n \in \mathbb{N}\} \) is precompact in \( C(0,T;H_{loc}^{-1}) \), so that there exists a (possibly non-unique) limit point \( u \in H_{B}^{\alpha,2} \).

Due to Definition 1.5 and Appendix A, it is immediately checked that each of the terms in the equation on \( u(n) \) converge, up to a subsequence \( u(n_k), k \geq 0 \), to the
expected quantities associated to the limit $u$, the remainder $u^2(n_k)$ being uniformly bounded in $V^3(0, T; H^{-3})$. This shows the claimed existence.

Now, note that from the uniqueness part, such limit point $u$ has to be unique, hence every subsequence of $u(n)$ converges to the same limit. This shows in particular the convergence of the full sequence, with respect to the topology of $C(0, T; H^{-1}) \cap L^2_w(0, T; H^1_w)$.

**Step 4: Stability.** To show the stability, note first that if $u$ and $v$ are $L^2$-solutions of
\[
\begin{align*}
\frac{du}{dt} &= (Au + f)dt + dB_u, \quad u_0 = \xi \\
\frac{dv}{dt} &= (Av + g)dt + dB_v, \quad v_0 = \eta,
\end{align*}
\]
where $\xi, \eta \in L^2$, $f, g \in L^2(H^{-1})$, then $w := u - v$ solves the problem
\[
\frac{dw}{dt} = (Aw + f - g)dt + dB_w, \quad w_0 = \xi - \eta.
\]
Therefore, the strong continuity of the solution map $\mathcal{S} : L^2 \times L^2(0, T; H^{-1}) \to L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ follows from the estimate (4.6).

5. **Algebras of $\mathcal{H}^\alpha_{B, loc}$: proof of Theorem 2**

In Section 3, we have seen that given $u \in \mathcal{H}^\alpha_{B, loc} \cap L^\infty$ satisfying (1.48), for any $m \in \mathbb{N}$ we have $u^m$ in $\mathcal{H}^\alpha_{B, loc}$, and moreover $du^m = mu^{m-1}f dt + dB(u^m)$ ($L^1_{loc}$-sense). By linearity we obtain the chain rule for polynomials, that is:
\[
\frac{dP(u)}{dt} = P'(u)f dt + dB(P(u)), \quad \text{where} \quad P = \sum_{m \leq n} a_mX^m.
\]
Moreover we immediately see that
\[
|P(u)|_{W^{-3,1}} \leq \sum_{m \leq n} |a_m||u^{m-1}|_{W^{-3,1}},
\]
where for a polynomial $Q$ we denote by $Q(u)^2$ the remainder associated to the equation on $Q(u)$, that is $Q(u)^2 := \pi(Q'(u) f; Q(u), Q(u))^2$, see notation (2.24).

In this section we are more generally interested in the algebra generated by one element $u \in \mathcal{H}^\alpha_{B, loc} \cap L^\infty$, that is, roughly speaking, the smallest Banach space $\mathcal{A}[u] \hookrightarrow \mathcal{H}^\alpha_{B, loc}$ containing the polynomials $\sum a_m u^m$. The first step in the proof of Theorem 2 is to show that an Itô Formula for analytic functions of the solution (a property that is not fulfilled by the mapping $\mathbb{R} \to \mathbb{R}$, $F(z) := |z|^p$ when $p \not\in 2\mathbb{N}_0$).

We need first to introduce some notation.

**Notation 5.1.** Given $u \in \mathcal{H}^\alpha_{B, loc} \cap L^\infty$, we denote by
\[
\mathcal{P}[u] := \left\{ \sum_{0 \leq m \leq n} a_m u^m, \quad \text{for some} \ n \geq 0 \ \text{and} \ (a_m) \in \mathbb{C}^n \right\}.
\]
From Leibniz formula, it is clear that $\mathcal{P}[u]$ is included in $C(L^1_{loc}) \cap L^1(W^1_{loc}) \cap L^\infty$. 


Lemma 5.1. Given \( u \in \mathcal{H}_{B, \text{loc}}^{\alpha, 2} \cap L^\infty, v \in \mathcal{P}[u] \), we define
\[
\|v\| = \inf_{P:P(v)=v} \{|P| + |P'| + |P''|)(\|u\|_{L^\infty})\},
\]
where the infimum is taken over every possible polynomial with complex coefficients such that \( P(u) = v \), and from now on, given \( P = \sum_{0\leq m \leq n} a_m X^m \), we adopt the notation
\[
|P|(X) = \sum_{0\leq m \leq n} |a_m| X^m.
\]

With these notations at hand, we have the following Lemma.

**Lemma 5.1.** Let \( u \in \mathcal{H}_{B}^{\alpha, 2}([0, T] \times \mathbb{R}^d) \) be an \( L^2_{\text{loc}} \)-solution of (1.48). Assume furthermore that \( u \) is globally bounded, that is \( u \in L^\infty([0, T] \times \mathbb{R}^d) \). The following holds.

(P1) \( (\mathcal{P}, \|\cdot\|) \) is a complex normed algebra.

(P2) \( (\mathcal{P}, \|\cdot\|) \rightarrow \mathcal{H}_{B, \text{loc}}^{\alpha, 1} \) algebraically and topologically.

(P3) The completion of \( (\mathcal{P}, \|\cdot\|) \) is the Banach algebra
\[
\mathcal{A} := \{F(u), F \in \text{Hol}(0, \|u\|_{L^\infty})\},
\]
where \( \text{Hol}(0, \|u\|_{L^\infty}) \) is the space of holomorphic functions on some open set containing the ball \( \tilde{B}(0, \|u\|_{L^\infty}) \subset \mathbb{C} \). The latter space is endowed with the norm
\[
\|v\| := \inf_{v=F(u)} \sum_{m \geq 0} (m!)^{-1} |F^{(m)}(0)||\|u\|_{L^\infty}^m + m \|u\|_{L^\infty}^{m-1} + m(m-1)\|u\|_{L^\infty}^{m-2}).
\]

(P4) Any element \( F(u) \in \mathcal{A} \) is an \( L^1_{\text{loc}} \)-solution to
\[
dF(u) = F'(u)dt + dB(F(u)),
\]

in the sense of Definition 1.4.

**Proof of Lemma 5.1.** Proof of (P1). Clearly, the mapping \( \|\cdot\| \) is homogeneous, and it satisfies the triangle inequality. It is also point-separating, because \( \|v\| = 0 \) implies for every \( \epsilon > 0 \) the existence of \( P_\epsilon = \sum_{m=0}^{N_\epsilon} a_m X^m \) such that \( v = P_\epsilon(u) \) and \( \sum_{m=0}^{N_\epsilon} |a_m| \|u\|_{L^\infty}^m \leq \epsilon \). But then it holds \( \|v\|_{L^\infty} \leq \epsilon \), and since \( \epsilon > 0 \) is arbitrary, the claim follows.

From (5.6) we see in particular that a product \( PQ(u) \) in \( \mathcal{P}[u] \) has a norm estimated as
\[
\|PQ(u)\| \leq (|PQ| + |P'|Q + PQ'| + |P''Q + 2P'Q' + PQ''|)(M)
\]
\[
\leq (|P||Q| + |P'||Q| + |P||Q'| + |P''||Q| + 2|P'||Q'| + |P||Q''|)(M)
\]
\[
\leq \|P(u)\| \|Q(u)\|.
\]

Hence, \( \|\cdot\| \) is submultiplicative, which makes \( P[u] \) a complex normed algebra.

Proof of (P2). We now want to evaluate the \( \mathcal{H}_{B}^{\alpha, 1}(K) \)-norm of \( P(u) \), where \( K \subset \mathbb{R}^d \) is arbitrary, and compare it with \( \|P(u)\| \). Elementary computations yield for the drift
\[
\int_s^t |P'(u_r)|f_r|_{W^{-1,1}(K)}dr \leq |P''|(M)E_{s,t}^{1/2}w_f(s,t)^{1/2} + |P'|(M)|K|^{1/2}(t-s)^{1/2}w_f(s,t)^{1/2}
\]
where $M := \|u\|_{L^\infty}$, and we shall also denote by $E_{s,t} := \|\nabla u\|_{L^2(s,t;L^2)}^2$, and by $\omega_f(s,t) := \|f\|_{L^2(s,t;H^{-1})}$. Proposition 2.4 then yields

$$\|P^t(u)\|_{Y^\alpha(W^{-3,1})} \leq C \left[ \omega_B(s,t)^a |P''|(M)|E_{0,T}^{1/2}\omega_f(s,t)^{1/2}+|P'|(M)|K|^{1/2}T^{1/2}\omega_f(s,t)^{1/2} \right]$$

(5.5)

On the other hand, we have from the Leibniz rule:

$$\|P(u)\|_{L^1(W^{1,1}(K))} \leq T^{1/2}|K|^{1/2}|P'|(M)\|\nabla u\|_{L^2} + T|K||P|(M).$$

Adding all the above contributions, and using Lemma 2.2, we see that there exists a constant $C(T,|K|,|f|_{L^2(H^{-1})},\|\nabla u\|_{L^2(L^2)}) > 0$, such that for every $P \in C[X]$, it holds

$$\|P(u)\|_{H^\alpha_{m}(K)} \leq C(|P| + |P'| + |P''|)(M),$$

(5.6)

which yields the claimed embedding.

**Proof of (P3).** The embedding $\mathcal{A}[u] \mapsto (\mathcal{D}[u],\|\cdot\|)$ is easy and left to the reader. To prove that $(\mathcal{D}[u],\|\cdot\|) \rightarrow \mathcal{A}[u]$ holds, consider a Cauchy sequence $Q_n \equiv \sum a^{(n)}_m M^m$ in $\mathcal{D}[u]$. In particular it is bounded, so that there exists $C$ such that for every $n \geq 0$, it holds

$$\sum_{m \in \mathbb{N}_0} |a^{(n)}_m| M^m \leq C,$$

hence the sequence $\{b^{(n)}, n \geq 0\} \subset \ell_1$ defined for each $n \geq 0$ as $b^{(n)} := (a^{(n)}_m M^m)_{m \geq 0}$ is bounded. Since $\ell_1$ is the dual space of $c_0$, there is a weak-* limit $b \in \ell_1$ for some subsequence $(b^{(k)}_{\nu_k})_{k \geq 0}$ implying in particular the convergence of each coordinate. Namely, for every $m_0 \geq 0$, we have $a^{(n)}_{m_0} \rightarrow a_{m_0} \equiv b_{m_0}/M^{m_0}$ as $n \rightarrow \infty$, and hence Fatou Theorem implies:

$$\sum_{m \in \mathbb{N}_0} |b_m| \equiv \sum_{m \in \mathbb{N}_0} |a_m| M^m \leq C.$$

Defining now $F(z) := \sum_{m \in \mathbb{N}_0} a_m z^m$, we have $F(u) \in \mathcal{A}[u]$ and dominated convergence implies that $\|Q_n - F\| \equiv \sum |a^{(n)}_m - a_m| M^m \rightarrow 0$. This shows the claimed embedding.

**Proof of (P4).** The proof that the chain rule holds is just a consequence of the above steps. 

We can now show an Itô Formula for analytic functions $F(u)$ of continuous solutions.

**Corollary 5.1.** Let $u \in H_{B,loc}^{n_0,2} \cap C([0,T] \times \mathbb{R}^d)$, be an $L^2_{loc}$-solution of (1.48), for some $f \in L^2(H^{-1})$ and some $u_0 \in L^2_{loc}$.

For every analytic function $F : \mathbb{U} \rightarrow C$ where $\mathbb{U}$ is an open set of the complex plane containing $[-\|u\|_{L^\infty},\|u\|_{L^\infty}]$, we have $F(u) \in H_{B,loc}^{n_1,1}$ and moreover

$$dF(u) = F'(u) f dt + dB(F(u))$$

(5.7)

in the $L^1_{loc}$-sense.
Proof. We will proceed by a localization argument. Fix a parameter \( \rho \) such that
\[
0 < \rho < \text{dist}([-\|u\|_{L^\infty}, \|u\|_{L^\infty}], \mathbb{C} \setminus U),
\]
and let \( K \subset \mathbb{R}^d \).

Next, denote by \( \tau \) the modulus of continuity of \( u : [0, T] \times K \rightarrow \mathbb{R} \) evaluated at \( \rho/2 \), in the sense that \( \max(|t-s|, |x-y|) \leq \tau \) implies \( |u(t, x) - u(s, y)| \leq \rho/2 \). Let \( I_\rho := [\tau^2] \) and let \( \{x_j, j = 1, \ldots, J_\rho\} \subset K \) be a finite set such that \( \bigcup_{j=1}^{J_\rho} B(x_j, \tau) \supset K \), each point \( x_j \) being distinct from the others. Consider a family of smooth, compactly supported functions \( \chi_j, j = 1, \ldots, J_\rho \) such that
\[
1_{B(x_j, \tau)} \leq \chi_j \leq 1_{B(x_j, \frac{4}{3}\tau)}.
\]

Furthermore, for \( i, j \) as above, denote by \( u^{ij} := u(i\tau, x_j) \).

Fix now \( i \in \{0, \ldots, I_\rho\} \). For each \( n \geq 0 \), there exists a polynomial \( P_n^i(X) \) with degree less than or equal to \( nJ_\rho \), such that
\[
\frac{d^k P_n^i(x)}{dx^k} \bigg|_{x = u^{ij}} = \frac{d^k F(x)}{dx^k} \bigg|_{x = u^{ij}}, \quad \text{for every (} k, j \text{)} \in \{0, \ldots, n\} \times \{1, \ldots, J_\rho\}. \quad (5.9)
\]
(to prove the existence of \( P_n^i \), it suffices to invert a linear system, which yields a unique solution if the values \( (u^{ij}, j \leq J_\rho)^\prime \)'s are distinct.)

Having this definition at hand, we let for any \( 1 \leq j \leq J_\rho \):
\[
v^{ij} \overset{\text{def}}{=} u - u^{ij} \quad \text{and} \quad \begin{cases} Q_n^{ij}(X) \overset{\text{def}}{=} P_n^i(X + u^{ij}) \\ F_n^{ij}(X) \overset{\text{def}}{=} F_n(X + u^{ij}) \end{cases}.
\]

Then, for any \( \phi \in W_0^{3, \infty}(K) \) and \( (s, t) \in \Delta_{(i\tau, (i+1)\tau)} \), we see that it holds
\[
\langle \delta P_n^i(u)_{st}, \phi \rangle = \sum_{1 \leq j \leq J_\rho} \langle \delta [Q_n^{ij}(v^{ij})]_{st}, \chi_j \phi \rangle
\]
\[
= \sum_{1 \leq j \leq J_\rho} \left[ \int_s^t \left( \langle Q_n^{ij}(v^{ij}) f, \chi_j \phi \rangle \right) dr + \left( \langle B_{st}^1 + B_{st}^2 \rangle [Q_n^{ij}(v^{ij})], \chi_j \phi \rangle + \left( \langle Q_n^{ij}(v^{ij}) \rangle_{st}^2, \chi_j \phi \rangle \right) \right]. \quad (5.10)
\]

Clearly, we have
\[
\|P_n^i(u) - F(u)\|_{H_B^{m,1}((i\tau, (i+1)\tau) \times K)} \leq \sum_{j=1}^{J_\rho} \|\chi_j\|_{W^{3, \infty}} \|Q_n^{ij}(v^{ij}) - F_n^{ij}(v^{ij})\|_{H_B^{m,1}((i\tau, (i+1)\tau) \times B(x_j, \tau))},
\]
hence from Lemma 5.1 together with (5.9), we infer that
\[
\|P_n^i(u) - F(u)\|_{H_B^{m,1}((i\tau, (i+1)\tau) \times K)} \leq C \sum_{j=1}^{J_\rho} \|\chi_j\|_{W^{3, \infty}} \|Q_n^{ij}(v^{ij}) - F_n^{ij}(v^{ij})\|
\]
\[
\leq C \sum_{j=1}^{J_\rho} \sum_{l \geq n+1} \frac{F(l)(u^{ij})}{l!} \|u - u^{ij}\|_{L^\infty((i\tau, (i+1)\tau) \times K)} \rightarrow 0. \quad (5.11)
\]

This shows in particular that (5.7) holds with respect to the scale \( W^{k, \infty}([i\tau, (i+1)\tau) \times K; \mathbb{C}), k \geq 0 \), for each \( K \subset \mathbb{R}^d \), and each \( i \in \{0, \ldots, I_\rho\} \). Since our notion of solution is also local in time, the claim follows. \( \blacksquare \).
Proof of Theorem 2. We appeal to an argument used in complex interpolation theory (see e.g. [Lun09]).

Note first that the mapping $F : z \mapsto |z|^q/(q(q-1))$ is not holomorphic on the real line, except when $q$ is an even integer. This leads us to define, for $q \in \mathbb{C}$ with $\Re q > 1$, and $\epsilon \geq 0$:

$$F_\epsilon(z, q) := \frac{(\epsilon^2 + z^2)^{q/2}}{q(q-1)},$$

which for $\epsilon \neq 0$ is holomorphic with respect to $z$, inside the domain $U_\epsilon := \{z \in \mathbb{C}, \ |3z| < \epsilon \}$. In particular, using Corollary 5.1, we see that the remainder

$$F_\epsilon(u, q)^{(\epsilon)} := \delta \dot{F}_\epsilon(u, q) - \int_s^t \frac{\partial F_\epsilon}{\partial z}(u_r, q) f_r \, dr - (B_{st}^1 + B_{st}^2) \left[ F_\epsilon(u_s, q) \right]$$

lies in the space $\mathcal{V}^{1+}_{2, \text{loc}}(0,T;W^{-3,1}(K))$, for each $q$ such that $\Re q > 1$, and where from now on $K \subset \subset \mathbb{R}^d$ is fixed.

Let $n \in 2\mathbb{N}$ be the unique even integer such that $p \in \{n, n + 2\}$.

Applying Proposition 3.1 recursively, we have in the $L^1(K)$-sense:

$$d(u^n/n(n-1)) = \frac{u^{n-1}}{n-1} f dt + dB(u^n/n(n-1)),$$

and

$$d(u^{n+2}/(n+2)(n+1)) = \frac{u^{n+1}}{n+1} f dt + dB(u^n/(n+2)(n+1)).$$

For $m \in \{n, n + 2\}$, we have the estimates $|\frac{\partial F_\epsilon}{\partial z}(u, m+i\tau)| \leq C|z|^{m-1}$ and $|\frac{\partial^2 F_\epsilon}{\partial z^2}(u, m+i\tau)| \leq C|z|^{m-2}$, uniformly with respect to $\tau$ and $\epsilon > 0$. Consequently, letting $q = m + i\tau$ for some fixed $\tau \in \mathbb{R}$, the drift term

$$\mathcal{D}^{(q, \tau, \epsilon)}_{st} := \phi \in W_{0}^{1,q}(K) \mapsto \int_{[s,t] \times K} \left[ -\frac{\partial^2 F_\epsilon}{\partial z^2}(u, q) \phi \right] \, dx \, dr$$

is uniformly bounded above in $W^{-1,p}(K)$ by some control $\omega_{\mathcal{D}}(s, t)$ depending upon the quantities $q, \|f\|_{L^2(W^{-1,p})}, \|u\|_{L^\infty(L^p)} \cap L^2(W^{1,p})$, and $K$, but not on $\epsilon > 0$. Thanks to the a priori estimate (2.34), we see that the remainder is bounded above independently of $\epsilon > 0$. In particular, we can take the limit as $\epsilon \to 0$ in all the terms in (5.12), yielding the Itō Formula for $\phi \mapsto F_\epsilon(u_t(x), m + i\tau) \equiv |u_t(x)|^{m+i\tau}$, for fixed $m \in \{n, n + 2\}$ and $\tau \in \mathbb{R}$. Otherwise said, there exists a control $\omega_{\mathcal{D}}$ so that for each $\phi \in W_{0}^{3,\infty}(K)$:

$$\langle F_\epsilon(u, m + i\tau), \phi \rangle \leq \omega_{\mathcal{D}}(s, t)^{3\alpha} |\phi|_{W^{3,\infty}},$$

for each $(s, t) \in \Delta$, such that $\omega_{\mathcal{D}}(s, t) \leq L$ for some universal constant $L$. Now, the exponentiation of a positive term by an imaginary number is a unit, hence it is easily observed that the latter control can be defined independently of $\tau \in \mathbb{R}$.

To conclude we write

$$p = (1 - \theta)n + \theta(n + 2),$$
yields, after some computations: speaking, by approximation). For \( \kappa \) (this seemingly circular approach is not a problem because one can conclude then and then show that its \( L \) Moser Iteration.

6.1. case of solutions for RPDEs of the form

\[
(\kappa_{n+1}(s,t)) \leq \omega_n(s,t)^{3\alpha(1-\theta)} \omega_n(s,t)^{3\alpha\theta},
\]

for \( \omega_B(s,t) \leq L \). This proves that (1.49) holds in \( L^p(K) \), for each \( K \subset \subset \mathbb{R}^d \). To show that the equation holds in \( L^p \), it suffices to consider a sequence \( \phi_n \in W_0^{3,\infty}(\mathbb{R}^d) \) and \( \phi \in W^{3,\infty} \) such that \( \phi_n \to \phi \). Testing the equation against \( \phi_n \), and then using that \( u \in L^\infty(L^p) \cap L^2(L^p) \) and \( f \in L^2(W^{-1,p}) \), we can eventually take the limit as \( n \to \infty \). This finishes the proof of Theorem 2.

6. Boundedness of solutions

In this section, we go back to our parabolic setting and investigate boundedness of solutions for RPDEs of the form

\[
du = (Au + f)dt + dBu, \quad \text{in } [0,T] \times \mathbb{R}^d,
\]

where the free term \( f \) will be subject to additional conditions, see Assumption 6.1, and \( A \) fulfills Assumption 1.2.

6.1. Moser Iteration. Recall the basic idea of Moser’s iteration principle (in the case \( A = \Delta \) and \( f = 0 \)). We shall assume first that \( u \) is in \( C([0,T] \times K) \cap H^{\alpha,2}_B \), and then show that its \( L^\infty \)-norm is bounded above in terms of known quantities (this seemingly circular approach is not a problem because one can conclude then by approximation). For \( \kappa > 0 \) define the map \( v := u^{\kappa/2} \in A_B[u] \). Then, roughly speaking, \( v \) is the solution to a similar equation as \( u \). More precisely, Corollary 5.1 yields, after some computations:

\[
d(v^2) - dB(v^2) = \left( v \Delta v - \frac{\kappa - 2}{\kappa} |\nabla v|^2 \right) dt =: \mathcal{G}^{(\kappa)} dt
\]

(\( L^1 \) sense). In particular, it is easily seen that the drift term \( \mathcal{G}^{(\kappa)} \) is uniformly bounded as \( \kappa \) becomes arbitrarily large: we have in fact

\[
|\frac{\delta \mathcal{G}^{(\kappa)}}{\kappa} |_{L^{1,1}(K)} \leq \omega_{\kappa}(s,t) \lesssim \| \nabla v \|_{L^2(s,t;L^2)}^2 + (t-s) \| v \|_{L^\infty(s,t;L^2)}^2.
\]

Next, testing (6.2) against \( \phi = 1 \in W_0^{3,\infty}(\mathbb{R}^d) \), then estimating the remainder \( \langle v^{2\kappa}, 1 \rangle \) by the \( W^{-3,1} \)-norm as in Proposition 2.4, it follows thanks to the Rough Gronwall Lemma that

\[
\sup_{0 \leq r \leq T} |v_r|^2_{L^2} \equiv \sup_{0 \leq r \leq T} |u_r|^2_{L^\infty} \leq C |u_0|^2_{L^\infty}
\]

for a constant \( C \) depending on \( \alpha, \omega_B, T \), but not on the parameter \( \kappa \geq 2 \). Now, a classical result states that

\[
|f|_{L^\infty(X,M,\mu)} \to |f|_{L^\infty(X,M,\mu)},
\]
for any $\sigma$-finite measure space $(X, \mathcal{M}, \mu)$ and every $f \in L^\infty$ such that $f \in L^q$ for some $q \in [1, \infty)$. Therefore, taking the $\varpi$-th root of (6.3) and sending $\varpi$ to infinity yields then the claimed estimate.

The general case follows basically the same ideas, namely for any $\varpi \geq 2$ it is possible find a bound for $v := u^{\varpi/2}$ in the energy space, depending on known quantities. Thanks to (2.15), we will then obtain a recursive relation between different moments of $u$. The conclusion will be obtained by the following iteration lemma, whose proof is immediate by induction and therefore omitted.

**Lemma 6.1** (Recursive argument). Assume that we are given a sequence of numbers $\Phi_n, n \geq 0$, and positive constants $\epsilon, \gamma$, and $\tau \geq 1$, such that for all $n \geq 1$:

$$\Phi_n \leq \gamma \tau^{n-1} \Phi_{n+\epsilon}.$$  

Then, the following estimate holds: for any $n \geq 0$ we have

$$\Phi_n \leq \gamma (1+\epsilon)^{n-1} \tau^{n-1} \Phi_{0} (1+\epsilon)^n.$$  

Using this basic fact, together with an approximation argument, we will show that a sufficiently large class of non-degenerate parabolic equations have solutions $u \in \mathcal{H}_{B}^{\alpha,2}$ that are bounded.

We need now to consider a more restrictive class of free terms.

**Assumption 6.1.** We assume that

$$f \in \mathcal{M} := L^r(0,T;W^{-1,q}) \cap L^{2r}(0,T;W^{-1,2q}) \cap L^1(0,T;W^{-1,1}) \cap L^{2}(0,T;H^{-1}),$$

where the exponents $r \in (1, \infty]$ and $q \in (1 \vee \frac{d}{2}, \infty)$ are subject to the conditions

$$\frac{1}{r} + \frac{d}{2q} < 1.$$  

**Remark 6.1.** For instance, using Sobolev embeddings, it is easily checked that Assumption 6.1 is fulfilled for $f \in L^r(0,T;L^q)$, where $r$ and $q$ are subject to (6.6).

We have the following.

**Proposition 6.1.** Let Assumption 6.1 hold, suppose that $u_0 \in L^2 \cap L^{\infty}$, and assume that $u$ is the solution of (6.1) given by Proposition 4.1. Then, $u$ is bounded. Furthermore, we have the estimate

$$\|u\|_{L^\infty} \leq C(\|u_0\|_{L^\infty}, \vartheta, \|a\|_{L^\infty}, \|f\|_{\mathcal{M}}, \omega_B, \alpha, r, q),$$

where the above constant depends on the indicated quantities, but not on $u$ in the space $\mathcal{H}_{B}^{\alpha,2}$.

**6.2. Preliminary discussion.** Consider $u \in \mathcal{H}_{B}^{\alpha,2} \cap L^{\infty} \cap L^2$-solution of (1.2). Our purpose in the present paragraph is to set up the iteration argument, in order to apply Lemma 6.1. Let $\varpi \geq 2$, and assume moreover that $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is continuous. We have by Theorem 2:

$$\delta|u|^\varpi_{st} = \int_s^t \varpi|u|^\varpi-1(Au + f)dr + (B^1_{st} + B^2_{st})|u_s|^\varpi + v_s^\varpi$$  

in the $L^1$-sense.
Hence, assume from now on that (6.7) holds, and define

\[ v := |u|^{\frac{\nu}{2}}. \]

we have the following identities:

\[ v \partial_i v = \frac{\nu}{2} (\partial_i u) |u|^{\nu-1} \quad \partial_i v \partial_j v = \frac{\nu^2}{4} (\partial_i u)(\partial_j u) |u|^{\nu-2}. \]

Hence denoting by \((f)\) any primitive element of \(f\), and by \(v^{2,\frac{\nu}{2}} := u^{2,\frac{\nu}{2}}\), it holds for every \(\phi \in W^{3,\infty}\) that

\[
\langle \delta(v^2)_{st}, \phi \rangle - \langle (B_{st}^1 + B_{st}^2)(v^2), \phi \rangle - \langle v^{2,\frac{\nu}{2}}, \phi \rangle, \\
= \int_{[s,t] \times \mathbb{R}^d} \left[ -4 (\frac{\nu-1}{\nu}) a^{ij} (\partial_i v)(\partial_j v) - 2a^{ij} v(\partial_i v)(\partial_j \phi) \right] dx dr \\
+ \int_{[s,t] \times \mathbb{R}^d} \left[ -2(\nu - 1)f^i(\partial_i v)v^{1-\frac{\nu}{2}} - \nu f^i v^{2-\frac{\nu}{2}} \partial_i \phi \right] dx dr =: \langle \delta \mathscr{D}^{(\nu)}_{st}, \phi \rangle. \quad (6.8)
\]

Testing the drift against \(\phi \in W^{1,\infty}\) with \(|\phi|_1 \leq 1\), and making use of the estimates

\[ v^{1-2/\nu} \leq 1 + v, \quad v^{2-2/\nu} \leq 1 + v^2, \]

valid for \(\nu \geq 2\), it is immediately seen that

\[
|\delta \mathscr{D}^{(\nu)}_{st}|_{W^{-1,1}} \leq 4 (\frac{\nu-1}{\nu}) \|a\|_{L^\infty} \|\nabla v\|^2_{L^2(s,t;L^2)} + 2 \|a\|_{L^\infty} \|v \nabla v\|_{L^1(s,t;L^1)} \\
+ 2(\nu - 1) \left[ \|f\|_{L^{2\nu}(s,t;L^{2\nu})} \|\nabla v\|_{L^2(s,t;L^2)} \|v\|_{L^{2\nu}(s,t;L^{2\nu})} + \|f\|_{L^2(s,t;L^2)} \|\nabla v\|_{L^2(s,t;L^2)} \right] \\
+ \nu \left[ \|f\|_{L^{2\nu}(s,t;L^2)} \|v^2\|_{L^{2\nu}(s,t;L^{2\nu})} + \|f\|_{L^1(s,t;L^1)} \right].
\]

Whence, letting \(\rho_0 := 2r/(r - 1)\) and \(\sigma_0 := 2q/(q - 1)\), we infer that

\[
|\delta \mathscr{D}^{(\nu)}_{st}|_{W^{-1,1}} \lesssim \|\nabla v\|^2_{L^2(s,t;L^2)} + \|v \nabla v\|_{L^1(s,t;L^1)} \\
+ \nu \left( \|f\|_{L^{2\nu}(s,t;L^{2\nu})} \|\nabla v\|_{L^2(s,t;L^2)} \|v\|_{L^{2\nu}(s,t;L^{2\nu})} + \|f\|_{L^2(s,t;L^2)} \|v\|_{L^2(s,t;L^2)} \right) \\
+ \|f\|_{L^2(s,t;L^2)} \|\nabla v\|_{L^2(s,t;L^2)} + \|f\|_{L^1(s,t;L^1)}. \quad (6.9)
\]

We now let \(\phi = 1\) in (6.8) and transfer to the left hand side the negative term. For \(|t - s|\) small enough, this yields

\[
\delta(|v|^2)_{st} + \int_t^s |\nabla v|^2_{L^2} dr \lesssim \nu \left( \|f\|_{L^{2\nu}(s,t;L^{2\nu})} \|\nabla v\|_{L^2(s,t;L^2)} \|v\|_{L^{2\nu}(s,t;L^{2\nu})} \\
+ \|f\|_{L^2(s,t;L^2)} \|\nabla v\|_{L^2(s,t;L^2)} \right) + |v_s|_{L^2} \omega_B(s, t)^{\alpha} + |v_{st}^{2,2}|_{W^{-3,1}},
\]
where the above implicit constant depends on \( \vartheta \). Therefore, by Proposition 2.4 and (6.9), we get
\[
\delta([v]_t^2)_{sl} + \int_s^t |\nabla v|_{L^2}^2|dr \lesssim \chi\left(\|f\|_{L^{2r}(s,t;L^{2q})}\|\nabla v\|_{L^2(s,t;L^2)}\|v\|_{L^p_0(s,t;L^{p_0})}\right)
+ \|f\|_{L^r(s,t;L^q)}\|v\|_{L^p_0(s,t;L^{p_0})} + \|f\|_{L^2(s,t;L^2)}\|\nabla v\|_{L^2(s,t;L^2)} + \|f\|_{L^1(s,t;L^1)}
+ \|v\|_{L^\infty(s,t;L^2)}^2 \omega_B(s,t)^\alpha + \omega_B(s,t)^\alpha \left(\|\nabla v\|_{L^2(s,t;L^2)}^2 + \|v\nabla v\|_{L^1(s,t;L^1)}\right)
\] (6.10)

Using the interpolation inequality (2.15), the first term in the above right hand side is estimated as follows
\[
\chi\|f\|_{L^{2r}(s,t;L^{2q})}\|\nabla v\|_{L^2(s,t;L^2)}\|v\|_{L^p_0(s,t;L^{p_0})} \leq \frac{\epsilon}{2}\|\nabla v\|_{L^2(s,t;L^2)}^2 + \frac{\chi^2}{2\epsilon}\|f\|_{L^{2r}(s,t;L^{2q})}^2\|v\|_{L^p_0(s,t;L^{p_0})}^2,
\]
where \( \epsilon > 0 \) is arbitrary (note that the above right hand side is a control function). We proceed similarly for the third term, writing
\[
\chi\|f\|_{L^2(s,t;L^2)}\|\nabla v\|_{L^2(s,t;L^2)} \leq \frac{\epsilon}{2}\|\nabla v\|_{L^2(s,t;L^2)}^2 + \frac{\chi^2}{2\epsilon}\|f\|_{L^2(s,t;L^2)}^2.
\]

For \( t \in [0,T] \), define \( G_t := \|v\|_{L^\infty(0,t;L^2)}^2 + \int_0^t |\nabla v|_{L^2}^2|dr \), and fix \( \epsilon > 0 \) sufficiently small (depending on \( r, q \) only). Absorbing to the left in (6.10), we end up with the estimate
\[
\delta G_s \lesssim_{r,q,\vartheta} (\omega_B(s,t)^\alpha + (t-s)^{1/2}) \sup_{r \in [s,t]} G_r
+ \chi^2\left(\|f\|_{L^2(s,t;L^2)} + \|f\|_{L^1(s,t;L^1)} + \|f\|_{L^{2r}(s,t;L^{2q})}\|v\|_{L^p_0(s,t;L^{p_0})}^2 + \|f\|_{L^r(s,t;L^q)}\|v\|_{L^p_0(s,t;L^{p_0})}^2\right),
\]
for any \( |t-s| \) small enough.

Applying Lemma 2.1, we obtain the bound
\[
\left(\sup_{r \in I} G_r\right) \equiv \|v\|_{L^\infty(0,T;L^2)}^2 + \|v\|_{L^2(0,T;H^1)}^2 \leq \gamma \chi^2\left(1 + \|v\|_{L^p_0(0,T;L^{p_0})}^2\right),
\] (6.11)
where the above constant \( \gamma > 0 \) depends on the quantities \( r, q, \vartheta, T, \|f\|_{\mathcal{A}}, \omega_B \) and \( \alpha \) but not on \( \chi \), neither on \( v \) in \( C(L^2) \cap L^2(H^1) \).

We now want to apply Lemma 6.1. To this end, let \( \epsilon > 0 \) be such that
\[
\frac{1}{r} + \frac{d(1 + eq)}{2q} = 1
\] (note that such a number exists from (6.6)). For \( \epsilon \) as above, observe that
\[
\frac{1}{\rho_0(1 + \epsilon)} + \frac{d}{2(1 + \epsilon)\sigma_0} = \frac{d}{4},
\]
which means in particular that the exponents
\[
\rho := \rho_0(1 + \epsilon), \quad \sigma := \sigma_0(1 + \epsilon)
\]
satisfy the condition (2.14).
Let $n \geq 1$. In (6.11), making the substitution $\varphi := \varphi_n = 2(1 + \epsilon)^n$, we obtain

$$
\| u \|_{L^p(L^q)} \lesssim \| u \|_{L^\infty(L^2) \cap L^2(H^1)} \lesssim (1 + \epsilon)^n \left( 1 + \| u \|_{L^\infty(L^2)} \right)
$$

where to obtain the first inequality we have used the interpolation inequality (2.17) on $\| u \|^{1+\epsilon}$. Otherwise stated, if one defines the sequence

$$
\Phi_n := \| u \|_{L^p(\omega \cap L^q)}^{1+\epsilon} + 1, \quad n \geq 0,
$$

then the estimate (6.13) shows that for any $n \geq 1$:

$$
\Phi_n \leq \gamma (1 + \epsilon)^{n-1} \Phi_{n-1}^{1+\epsilon},
$$

where the constant $\gamma > 0$ depends only upon the quantities

$$
\vartheta, r, q, \| \alpha \|_{L^\infty}, \| f \|_{\mathcal{M}}, \omega_B, \quad \text{and } \alpha.
$$

Applying now (6.5), this yields for every $n \in \mathbb{N}_0$:

$$
\Phi_n \leq \gamma \frac{(1+\epsilon)^n - 1}{\epsilon} \left( 1 + \epsilon \right)^{n-1} \| u \|_{L^\infty(L^2)}^{1+\epsilon},
$$

from which

$$
\| u \|_{L^\infty} \equiv \lim_{n \to \infty} (\Phi_n)^{(1+\epsilon)^{-n}} \leq C(\| u \|_{C(L^2) \cap L^2(H^1)} + 1),
$$

for another constant $C > 0$ as above. Having this at hand, we can now proceed to the proof of Proposition 6.1.

### 6.3. Proof of Proposition 6.1.

Consider an approximating sequence $B(n) = S_2(b(n))$ as in Definition 1.2. By the classical PDE theory, if we denote by $u(n)$ the corresponding weak solution (i.e. in the sense of distributions) of

$$
\frac{\partial u(n)}{\partial t} - Au(n) = f + \hat{b}(n)u(n),
$$

then $u(n)$ is well defined and unique in the class $L^\infty(L^2) \cap L^2(H^1)$. It is easily seen that in fact, $u(n) \in \mathcal{H}^{\alpha^2}$ and is an $L^2$-solution of

$$
du(n) = (Au(n) + f)dt + dB(n)u(n)
$$

Indeed, writing the integral form of (6.16), then adding and subtracting $u(n)_s$ in the integrand, we have

$$
u(n)_t - u(n)_s - \int_s^t (Au(n) + f)dr = B^1(n)_{st}u(n)_s + \int_s^t dB_r(n)(u(n)_r - u(n)_s)
$$

$$
= B^1(n)_{st}u(n)_s + B^2(n)_{st}u(n)_s + \int_s^t dB_r(n)R_{ur}u(n),
$$

and the claim follows by direct evaluation of $R_{ur}u(n)$ in $\mathcal{V}_2^\alpha(0,T;H^{-2})$ (details are left to the reader).

Moreover, for $f$ as in (6.1), it is known that $u(n)$ is continuous, as a mapping from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}$ (it is even $\gamma$-Hölder for some $\gamma > 0$ depending on the data, see for instance [LSU68, Chapter 3]). As a consequence, using Corollary 5.1 the map
Consequently, the analysis made in the above paragraph ensures that

\[ \|u(n)\|_{L^\infty([0,T] \times \mathbb{R}^d)} \leq C, \]

for a constant depending on known quantities but not on \( n \geq 0 \). Using Banach Alaoglu Theorem, the weak-* lower-semicontinuity of the \( L^\infty \) norm, and uniqueness of the limit in \( C(L^2) \cap L^2(H^1) \) this implies that \( u \) satisfies the same estimate. This proves the proposition. ■

**Remark 6.2.** The above proof works for a more general URD with a multiplicative term, such as \( B = (\sigma \cdot \nabla + \nu)X \) where \( \nu \in W^{2,\infty} \).

### 7. Proof of Theorem 1

In order to prove Theorem 1, we first show that the Itô Formula holds when \( u \) is bounded. We will then proceed to the general proof, using an approximation argument.

Let \( u \) be an \( L^2 \)-solution of

\[ du = (Au + f)dt + dBu \]

\[ u_0 \in L^2 \cap L^\infty, \tag{7.1} \]

where \( f \) belongs to \( L^2(H^{-1}) \), and such that moreover \( u \in \mathcal{H}^{\alpha,2}_{L^2} \cap L^\infty \). By Lemma 5.1 we have \( P(u) \in \mathcal{H}^{\alpha,1}_{L^2} \) and moreover, Leibniz formula yields immediately

\[ \|P(u)\|_{L^\infty(L^1) \cap L^2(W^{1,1})} \leq C_K \|u\|_{L^\infty(L^2) \cap L^2(H^1)} (|P|_{L^\infty(-M,M)} + |P'|_{L^\infty([-M,M])}), \tag{7.2} \]

where \( M := \|u\|_{L^\infty} \).

Similarly, the drift \( \mathcal{D} := \int_0^t P'(u)(Au + f)dr \) is estimated as

\[ |\delta \mathcal{D}|_{W^{-1,1}} := \left| \int_s^t (A[P(u)] - a^{ij} \partial_i u \partial_j u P'(u)) dr \right|_{W^{-1,1}} \leq C \left( K, \|a\|_{L^\infty}, \|u\|_{L^\infty(L^2) \cap L^2(H^1)}, \|f\|_{L^2(H^{-1})} \right) \times (|P'|_{L^\infty(-M,M)} + |P''|_{L^\infty(-M,M)}), \tag{7.3} \]

so that from Proposition 2.4 and Corollary 2.2, we obtain as well the estimate

\[ \|P(u)\|_{\mathcal{D}^{\alpha,1}} \leq C(K, \|a\|_{L^\infty}, \|u\|_{L^\infty(L^2) \cap L^2(H^1)}) \times (|P|_{L^\infty(-M,M)} + |P'|_{L^\infty(-M,M)} + |P''|_{L^\infty(-M,M)}). \tag{7.4} \]

This being true for every \( K \subset \subset \mathbb{R}^d \), we have constructed a bounded linear mapping

\[ \varphi_u : \mathcal{P} \to \mathcal{H}^{\alpha,1}_{L^2}, \quad P \mapsto \varphi_u(P) := P(u) \]

where \( \mathcal{P} \) denotes the set of polynomial functions on \([-M,M]\), endowed with the norm of \( C^\alpha([-M,M]) \). The space \( \mathcal{P} \) is dense in \( C^\alpha([-M,M]) \) hence, from the so-called B.L.T. Theorem (we refer to [RS80, p. 12]), \( \varphi_u \) can be uniquely extended to a mapping

\[ u^* : C^\alpha([M,M]) \to \mathcal{H}^{\alpha,1}_{L^2}, \]

\[ v(n) := |u(n)|^{\nu/2} \text{ is an } L^1\text{-solution to (6.7) where } B \text{ has been replaced by } B(n). \]
(with same operator norm as \( \varphi_u \)). Finally, the stability results (Lemma 3.1) imply that \( F(u) \equiv u^*(F) \) is an \( L^1 \)-solution of

\[
d(F(u)) = F'(u)(Au + f)dt + dB(F(u)).
\]

We can now proceed to the proof of Theorem 1.

**Proof of Theorem 1.** Using Proposition 6.1 together with a density argument, one can consider sequences \( (f(n)) \) and \( (u_0(n)) \) with the property that for every \( n \in \mathbb{N}_0 \):

- Assumption 6.1 holds for \( f(n) \),
- \( u_0(n) \) is bounded;

and such that as \( n \to \infty \):

\[
\begin{align*}
  u_0(n) &\to u_0 \quad \text{strongly in } L^2, \\
  f(n) &\to f \quad \text{strongly in } L^2(H^{-1})
\end{align*}
\]

and thanks to Proposition 4.1, one can assume that the corresponding solution \( u(n) \in \mathcal{H}_{H^2}^{0,2} \cap L^\infty \) verifies

\[
u(n) \to u \equiv \text{the solution to the limiting equation, strongly in } L^2(H^1). \quad (7.7)
\]

From (7.7), there exists a subsequence (still denoted by \( u(n) \)) such that

\[
u(n) \to u \quad \text{almost everywhere on } [0,T] \times \mathbb{R}^d. \quad (7.8)
\]

For every \( F \in C^2 \) with \( F'(0) = 0 \), the above discussion shows that for \( \phi \in W^{3,\infty} \):

\[
\langle \delta F(u(n)), \phi \rangle - \langle F(u(n)), (B_{s,t}^{1,*} + B_{s,t}^{2,*})\phi \rangle - \langle F(u(n))_{s,t}, \phi \rangle
\]

\[
= \langle \delta \mathcal{D}(n), \phi \rangle \overset{\text{def}}{=} \iint_{[s,t] \times \mathbb{R}^d} \left[ -a^{ij} F'(u(n))\partial_j u(n)\partial_i \phi - a^{ij} F''(u(n))\partial_j u(n)\partial_i u(n)\phi 
- f_i(n)\partial_i u(n)F'(u(n))\phi - f_i(n)F'(u(n))\partial_i \phi \right] dx \, dr. \quad (7.9)
\]

As a consequence of our assumptions on \( F \), it holds \( |F'(X)| \leq |F''|_{L^\infty(\mathbb{R})}|X| \), so that

\[
|\delta \mathcal{D}_{st}|_{W^{-1,1}} \leq \|a\|_{L^\infty} \iint_{[s,t] \times \mathbb{R}^d} \left[ |F'(u(n))||\nabla u(n)| + |F''(u(n))||\nabla u(n)|^2 
+ |f_i(n)||F'(u(n))||\nabla u(n)| + |F'(u(n))| \right] \, dr.
\]

\[
\lesssim \|a\|_{L^\infty} \|F''|_{L^\infty} (t-s)^{1/2}\|u(n)\|_{L^\infty(s,t;L^2)} \|\nabla u(n)\|_{L^2(s,t;L^2)} + \|\nabla u(n)\|_{L^2(s,t;L^2)}^2

+ \|f(n)\|_{L^2(s,t;H^{-1})} \left( \|\nabla u(n)\|_{L^2(s,t;L^2)} + (t-s)^{1/2}\|u(n)\|_{L^\infty(s,t;L^2)} \right).
\]

Hence, from (7.7), we deduce the following uniform bound

\[
\|F(u(n))\|_{\mathcal{H}_{H^1}^{0,1}} \leq C. \quad (7.10)
\]

Next, by Lemma 3.1, we infer that (up to some new extraction)

\[
F(u(n)) \to F(u) \quad \text{strongly in } C(0,T;W^{-1,1}). \quad (7.11)
\]
Now, fixing \( \phi \in W^{3,\infty}(\mathbb{R}^d) \), \((s, t) \in \Delta\), it suffices to check that each term in (7.9) converges to what is expected. But using (7.8), (7.11) and dominated convergence, it holds:

\[
\int_{[s,t] \times \mathbb{R}^d} -a^{ij} F''(u(n)) \partial_j u(n) \partial_i u(n) \phi \to \int_{[s,t]} -a^{ij} F''(u) \partial_j u \partial_i u \phi.
\]

Similarly, using (7.6) we have

\[
\int_{[s,t] \times \mathbb{R}^d} -f_i(n) (F''(u(n)) \partial_i u(n) \phi + F'(u(n)) \partial_i \phi) dx dr \to \int_{[s,t]} -f_i(F''(u) \partial_i u \phi + F'(u) \partial_i \phi) dx dr = \int_s^t \langle f, F'(u) \phi \rangle dr.
\]

Finally, using (7.10), Banach Alaoglu Theorem and (7.11), we see that the left hand side in (7.9) converges to

\[
\langle \delta F(u), \phi \rangle - \langle F(u), (B^{1*}_{st} + B^{2*}_{st}) \phi \rangle - \langle R_{st}, \phi \rangle
\]

for some remainder \( R \in (W^{3,\infty})^* \). But proceeding as in Section 3, it is easily seen that \( R \) belongs to \( W^{-3,1} \).

This finishes the proof of Theorem 1. \( \blacksquare \)

8. PROOF OF THEOREM 3

**Proof of the solvability.** Identify the test functions \( W_0^{k,p}(D) \) as elements of \( W^{k,p}(\mathbb{R}^d) \) as in Remark 1.1, and then define

\[
\tilde{\sigma} := \iota_D(\sigma), \quad (\tilde{B}^1, \tilde{B}^2) := (\mathcal{X} \tilde{\sigma} \cdot \nabla, \mathcal{X} (\tilde{\sigma} \cdot \nabla)^2).
\]

Moreover, let \( \tilde{u}_0 := \iota_D(u_0) \). Concerning the principal part, we define

\[
\tilde{a}^{ij}(t,x) := \begin{cases} a^{ij}(t,x) & \text{if } (t, x) \in [0, T] \times D \\ 1 & \text{otherwise,} \end{cases}
\]

and for \( v \in L^2(H^1) \) : we define \( \tilde{A} v \in L^2(H^{-1}) \) such that

\[
\int_s^t \langle \tilde{A} v, v_r, \phi \rangle dr := -\int_s^t \langle \tilde{a}^{ij} \partial_j v, \partial_i \phi \rangle dr, \quad \text{for all } (s,t) \in \Delta, \ \phi \in H^1.
\]

With these definitions, \( \tilde{A}, \tilde{B} \), fulfill the hypotheses of Theorem 4.1 so that there exists a unique solution \( u \in \mathcal{H}_B^{a,2}([0, T] \times \mathbb{R}^d) \) to

\[
du = \tilde{A} u dt + d\tilde{B} u, \quad [0, T] \times \mathbb{R}^d.
\]  

(8.1)

The element \( v := \iota_D^*(u) \) is the natural candidate to solve the Dirichlet problem (1.50). In order to check this, let us remark that \( w := \iota_D^* \in L^2(H^1_0(D)) \) is a classical solution to

\[
\partial_t w = \Delta w \quad \text{on } [0, T] \times (\mathbb{R}^d \setminus D), \quad w_0 = 0,
\]

and hence \( w = 0 \). This shows that \( u \) is supported in \([0, T] \times D\). Since on the other hand \( u \) belongs to \( L^2(H^1(\mathbb{R}^d)) \), this implies that its trace onto \([0, T] \times \partial D\) is well defined, so that \( v \equiv \iota_D^*(u) \in L^2(H^1_0(D)) \). This shows that \( v \) solves the Dirichlet problem (1.50).
Proof of the maximum principle. The proof uses the so-called Stampacchia truncatures approach. Namely, let us fix a map $G \in C^1(\mathbb{R})$ such that the following properties are satisfied:

\[
|G'|_{L^\infty(\mathbb{R})} < \infty, \quad (8.2)
\]
\[
G \text{ is increasing on } (0, \infty), \quad (8.3)
\]
\[
G(x) = 0 \text{ whenever } x \leq 0. \quad (8.4)
\]

Let $F \in C^2(\mathbb{R})$ be defined by

\[
F(x) := \int_0^{x-M} G(y) dy, \quad x \in \mathbb{R}.
\]

where we denote by

\[
M = \max(0, \text{ess sup}_D u_0) < \infty.
\]

By the Itô Formula applied to $F$, the following equation holds:

\[
\langle \delta F(u), \phi \rangle = \int_s^t \langle G(u_r - M)Au_r, \phi \rangle dr + \langle (B^1 + B^2)F(u_s), \phi \rangle + \langle F^\delta_{st}, \phi \rangle,
\]

for some remainder $F^\delta \in V^{1+}(0, T; H^{-3})$. Next, we arrange the drift term as follows:

\[
\langle G(u - M)Au, \phi \rangle + \langle a^{ij}G'(u - M)\partial_i u \partial_j u, \phi \rangle = \langle -a^{ij}G(u - M)\partial_i u \partial_j \phi \rangle
\]
\[
= \langle F(u), \partial_j(a^{ij} \partial_i \phi) \rangle
\]

Hence, denoting by $\mathcal{D} := \int_0^t G(u_r - M)Au_r dr$, we have for each $(s, t) \in \Delta$:

\[
|\delta \mathcal{D}_{st}|_{W^{-2,1}} \leq \|a\|_{L^\infty} \iint_{[s,t] \times D} G'(u - M)|\nabla u|^2 dx dr + \|a\|_{L^1(s,t; W^{1,\infty})} \|F(u)\|_{L^\infty(s,t; L^1)}
\]

Hence, taking $\phi = 1$ and using Assumption 1.2 gives

\[
\delta|F|_{L^1_{st}} + \hat{\theta} \iint_{[s,t] \times D} G'(u - M)|\nabla u|^2 dx dr
\]
\[
\leq 3\|F\|_{L^\infty(s,t; L^1)} \omega_B(s, t)^\alpha + \|F\|_{L^\infty(s,t; L^1)} \omega_B(s, t)^\alpha \|a\|_{L^1(s,t; W^{1,\infty})}. \quad (8.5)
\]

for any $(s, t)$ such that $\omega_B(s, t) \leq \hat{\theta} \equiv \min(1, \tilde{\theta}, \|a\|_{L^\infty}^{-1})/2$. Applying Lemma 2.1, we obtain that

\[
\|F\|_{L^\infty(L^1)} \leq C (\hat{\theta}, \|a\|_{L^\infty}, \|a\|_{L^1(W^{1,\infty})}, \omega_B, \alpha) |F(u_0)|_{L^1} \equiv 0,
\]

from which we conclude that $u \leq M$ a.e. The proof of the estimate below is similar. Theorem 3 is now proved.

Appendix A. Renormalizability versus transport property

We want to prove that the transport property (1.33)-(1.34) imply uniform bounds for the drivers $\Gamma(B)^\epsilon \equiv (\Gamma(B)^1, \Gamma(B)^2)$ defined in (3.21). It should be noted that the above statements are made under the assumption that $B$ is a geometric, differential URD, which corresponds to a larger class as that of transport-like drivers. The proofs could certainly be adapted to the case where $B$ is symmetric and closed, see [BG17, Definition 5.3 & Definition 5.7], but we refrain from stating (and proving) such result.
In what follows, we make use of the notation
\[ T_i(\mathbb{R}^d) := \bigcap_{-3 \leq k-i \leq k, p \in [1, \infty]} \mathcal{L}(W^{k, p}(\mathbb{R}^d), W^{k-i, p}(\mathbb{R}^d)) \quad \text{for } i = 1, 2. \]

For a mapping
\[ U \equiv (U^1, U^2) : \Delta \to T_1(\mathbb{R}^d) \times T_2(\mathbb{R}^d) \]
we shall denote by \( \Gamma(U) : \Delta \to T_1(\mathbb{R}^d) \times T_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \) the element given by
\[
\Gamma(U)^1 := U^1 \otimes \text{id} + \text{id} \otimes U^1 \\
\Gamma(U)^2 := U^2 \otimes \text{id} + U^1 \otimes U^1 + \text{id} \otimes U^2.
\]

**Theorem 4.** Let \( B \) be a differential, geometric URD, and denote by \( \omega_B(s, t) \) the control \([B^1]_{\mathcal{V}^s(\mathbb{R}^d, \mathbb{R}^d)} + [B^2]_{\mathcal{V}^s(\mathbb{R}^d, \mathbb{R}^d)}\).

It holds the uniform bound, for \(-3 \leq k - i \leq k \leq 0\):
\[
|T_e^{-1} \Gamma(B)^i T_e|_{\mathcal{L}(W^{1, k}, W^{1, k-i})} \leq C \omega_B(s, t)^{i\alpha}
\]
where \( C > 0 \) denotes some absolute constant.

We start by a definition.

**Definition A.1.** For \( \beta > 0 \), we say that a mapping \( \mathbb{L} \in \mathcal{V}^\beta(0, T; T_1) \) has the renormalization property if there is a constant \( C > 0 \) such that
\[
\begin{align*}
|\nabla_{x,y} (\mathbb{L}_{x,s,t} + \mathbb{L}_{y,s,t})[\phi(x + y)]| &\leq |\nabla \phi(x + y)||[\mathbb{L}]_{\mathcal{V}^\beta(\mathbb{R}^d)}| \quad \text{for } \alpha = 1, 2, \\
|\nabla_{x,y} (\mathbb{L}_{x,s,t} + \mathbb{L}_{y,s,t})[\phi(x - y)]| &\leq |x - y||\nabla \phi(x - y)||[\mathbb{L}]_{\mathcal{V}^\beta(\mathbb{R}^d)}|
\end{align*}
\]
for \( i = 1, 2 \) and every \( \phi \in C_c^\infty \).

Theorem 4 is obtained as a consequence of the following Proposition, together with the observation that the bracket \( L_{st}^2 := B^2 - \frac{1}{2} B^1 B^1 \) of a differential, geometric URD \( B \), is of first order (see (1.35)).

**Proposition A.1.** Let \( U \equiv (U^1, U^2) : \Delta \to \mathcal{D}_1 \times \mathcal{D}_2 \) be a pair of two index maps. Assume that both \( \mathbb{L}^1 := U^1 \) and \( \mathbb{L}^2 := U^2 - \frac{1}{2} U^1 U^1 \) have the renormalization property with constant \( \beta := \alpha \) and (respectively) \( \beta = 2\alpha \). Then, the family \( \Gamma(U)^i \) satisfies the uniform bounds:
\[
|T_e^{-1} \Gamma(U)^i T_e|_{\mathcal{L}(W^{1, k}, W^{1, k-i})} \leq C [\mathbb{L}^i]_{\mathcal{V}^\alpha(\mathbb{R}^d)}
\]
for every \( \Phi \in U^\infty_c(\mathbb{R}^d) \), \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d \), and each \(-3 \leq k - i \leq k \leq 3\).

**Proof of Proposition A.1.** Let us first introduce some notation. Having Notation 3.4 in mind, we will make use of the gradient with respect to new variable \( x_+ \equiv \frac{x+y}{2} \) (respectively \( x_- \equiv \frac{x-y}{2} \)), and we will denote it by \( \nabla_+ \) (respectively \( \nabla_- \)). Note that since \( \sqrt{2}\chi \) is a rotation, we have the equivalence between the norms
\[
|\Phi|_{W^{k, \infty}} \quad \text{and} \quad \sum_{0 \leq i \leq k} |(\nabla_+)^i(\Phi \circ \chi^{-1})|_{L^\infty} + |(\nabla_-)^i(\Phi \circ \chi^{-1})|_{L^\infty}.
\]

Fix \( i \in \{1, 2\} \), and to avoid lengthy notations write \( \mathbb{L} := \mathbb{L}^i \), as well as \([\mathbb{L}]_{1}^{\alpha}(s, t) := [\mathbb{L}]_{\mathcal{V}^\beta(\mathbb{R}^d)}\). First note that by assumption it holds the modified Leibniz rule
\[
\mathbb{L}(fg) = (\mathbb{L}f)g + f(\mathbb{L}g) - Hfg
\]
where $H \equiv H_d(x) := U1(x)$ is some element in $W^{2,\infty}$. Moreover, for $\theta \in C_c^\infty(\mathbb{R}^d)$, we have the estimates
\[
|\langle \nabla^{k-1}\mathbb{L} - \mathbb{L}\nabla^{k-1}\rangle \theta(x)| \leq \llbracket \mathbb{L} \rrbracket_1^{[\alpha]} \langle \nabla^{k-1}\theta(x) \rangle \tag{A.4}
\]
uniformly with respect to $x \in \mathbb{R}^d$, $(s, t) \in \Delta$, and for $1 \leq k \leq 3$.

Using a density argument, it will be enough to show the claimed bounds on test functions of the form $\Phi(x) = \phi(x)\psi(x)$, with $\phi \in W^{k,\infty}$, $\psi \in W^{k,\infty}$, where $k \in \{1, 2, 3\}$ and $\psi$ is compactly supported in $B_1 = \{x_+ \in \mathbb{R}^d : |x_+| \leq 1\}$. For such $\Phi$, we have
\[
(\mathbb{L}_x + \mathbb{L}_y)T_\epsilon \Phi = \epsilon^{-d}(\mathbb{L}_x + \mathbb{L}_y)\left[\phi\left(\frac{x + y}{2}\right)\psi\left(\frac{x - y}{2\epsilon}\right)\right] + \epsilon^{-d}\phi\left(\frac{x + y}{2}\right)(\mathbb{L}_x + \mathbb{L}_y)\left[\psi\left(\frac{x - y}{2\epsilon}\right)\right] + \epsilon^{-d}(\mathbb{H}_x + \mathbb{H}_y)\phi\left(\frac{x + y}{2}\right)\psi\left(\frac{x - y}{2\epsilon}\right) =: I^+_\epsilon + I^-_\epsilon + I_0^\epsilon,
\]
where we recall that $T_\epsilon$ is the “blow-up transformation” defined in (3.9).

The evaluation of the first term $I^+_\epsilon$ is a direct consequence of the renormalization property, together with the commutator identities
\[
\nabla_+ T_\epsilon = T_\epsilon \nabla_+, \quad \text{and} \quad \nabla_- T_\epsilon = \epsilon^{-1}T_\epsilon^{-1}\nabla_-.
\tag{A.5}
\]
Making the change of variables $(x_+, x_-) := \chi(x, y)$ (see (3.10)) we see that for $k = 0, 1, 2$ :
\[
|\nabla^{k-1} T_\epsilon^{-1}(I^+_\epsilon)|_{L^\infty} = |T_\epsilon^{-1}(\nabla^{k-1} I^+_\epsilon)|_{L^\infty} = \text{ess sup}_{x_+, x_-} \left\{ |\psi(x_-)| \left| \nabla^{k-1}_+ \left( (\mathbb{L}_x + \mathbb{L}_y) \left[ \phi\left(\frac{x + y}{2}\right)\right] \right) \circ \chi^{-1}(x_+ - x_-) \right| \right\}.  \tag{A.6}
\]
Using (A.9), we have, uniformly in $\epsilon > 0$ :
\[
|\nabla^{k-1} T_\epsilon^{-1}(I^+_\epsilon)|_{L^\infty} \leq \llbracket \mathbb{L} \rrbracket_1^{[\alpha]}(s, t) |\psi|_{L^\infty} |\nabla \phi|_{L^\infty} \leq C \llbracket \mathbb{L} \rrbracket_1^{[\alpha]}(s, t) |\Phi|_{W^{1,\infty}}.
\]

Then, we write that
\[
|\nabla^{k-1} T_\epsilon^{-1}(I^-_\epsilon)|_{L^\infty} = \epsilon^k |T_\epsilon^{-1}\nabla^{k-i}(I^+_\epsilon)|_{L^\infty} = \text{ess sup}_{x_+, x_-} \left\{ |\psi(x_-)| \left| \nabla^{k-1}_- \left( (\mathbb{L}_x + \mathbb{L}_y) \left[ \phi\left(\frac{x + y}{2}\right)\right] \right) \circ \chi^{-1}(x_+ - x_-) \right| \right\}.
\]
This proves the claimed bound on $I^+_\epsilon$.

Concerning the other term, we have
\[
|T_\epsilon^{-1}(I^-_\epsilon)|_{L^\infty} = \text{ess sup}_{x_+, x_-} |\psi(x_-)| |\nabla \circ \chi^{-1}(x_+ - x_-)|,
\]
where we denote by $\Psi_\epsilon(x, y) := (\mathbb{L}_x + \mathbb{L}_y)(\psi\left(\frac{x - y}{\epsilon}\right))$. Note that $\Psi_\epsilon$ is supported inside the strip $\Omega^K_\epsilon$ (see (3.8)), because of the locality of $\mathbb{L}$.

Using now (A.10), it holds
\[
|\Psi_\epsilon \circ \chi^{-1}(x_+, \epsilon x_-)| \leq \llbracket \mathbb{L} \rrbracket_1^{[\alpha]}(s, t) |\epsilon x_-| |\nabla \left[ \psi\left(\frac{x_+}{\epsilon}\right)\right]| \leq \llbracket \mathbb{L} \rrbracket_1^{[\alpha]}(s, t) |\nabla \psi\left(\frac{x_+}{\epsilon}\right)|.
\]
Lemma A.1. Let $\beta > 0$. This finishes the proof. ■

Proof. where we denote by $k$ for some constant $K > 0$. From which we infer that

$$T\Gamma(U)^1T\Gamma(U)^1T = \frac{1}{2}\Gamma(U)^1\Gamma(U)^1 + L^c$$

(A.7)

from which we infer that

$$T\Gamma(U)^2T = \frac{1}{2}\Gamma(U)^1\Gamma(U)^1 + L^c$$

(A.8)

where we denote by

$$L^c = T\Gamma(U)^2T = T\Gamma(U)^1\Gamma(U)^1 + L^c$$

Now, for $k = 2, 3$, we have for every $\Phi \in W^{k,1}(\mathbb{R}^d \times \mathbb{R}^d)$:

$$|\Gamma(U)^2\Phi|_{W^{k,1}} \leq \frac{1}{2}[L^2]_{-1}^{[1]}(s,t)\Gamma(U)^1\Phi|_{W^{k-1,1}} + [L^2]_{-1}^{[1]}(s,t)|\Phi|_{W^{k-2,1}}$$

This finishes the proof.

To conclude, it suffices to show the following.

**Lemma A.1.** Let $L : \Delta \to \mathbb{D}_1$ be a continuous 2-index map with finite $1/\beta$ variation for some $\beta > 0$. Then $L$ has the renormalization property.

Proof. These are straightforward consequences of the representation (3.19), see Remark 1.2.

In fact, the time variable plays no particular role here. To wit, let $V : W^{3,\infty} \to W^{2,\infty}$ be a differential operator of order less than or equal to 1, and assume that for some constant $K > 0$, $|\nabla V - V\nabla k\theta| \leq K|\nabla k\theta|$, uniformly in $x \in \mathbb{R}^d$, and for $k \in \{0, 1, 2\}$.

Then, for every $\psi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$, $i = 1, 2$ and for any $k$ such that $0 \leq k-i \leq k \leq 3$, we have

$$|\nabla^{k-i}(V_x + V_y)(\phi(\frac{x+y}{2}))| \leq C(V|_{z(W^{3,\infty}, W^{2,\infty}), K})|\nabla^k \phi(\frac{x+y}{2})|$$

(A.9)
for a.e. \( x, y \in \mathbb{R}^d \), and similarly
\[
|\nabla^{k-1}(V_x + V_y)(\psi(\frac{x-y}{2}))| \leq C(|V|_{L^\infty(W^{3,\infty},W^{2,\infty})}, K)\left| \frac{x-y}{2} \right| |\nabla^k \psi(\frac{x-y}{2})|,
\]
(A.10)
for a.e. \( x, y \in \mathbb{R}^d \).

To obtain (A.10)-(A.10), it is sufficient to write \( V \) using coefficients, and then to use Taylor Formula
\[
\sigma^k(x) - \sigma^k(y) = (x-y) \cdot \int_0^1 \nabla \sigma^k(x + (y-x)\tau) \, d\tau.
\]
(A.11)

Details are left to the reader.

We can now proceed to the proof of the main result of this section.

**Proof of Theorem 4.** For \( i = 1, 2 \), let \( U^i := B^{i\dagger} \). Since \( B \) is a differential unbounded rough driver, it fulfills the hypotheses of Lemma A.1. Hence, the reduced adjoint \( L^i \equiv (L^{i,1},L^{i,2}) := (B^{i,1},B^{i,2} - \frac{1}{2}B^{i,1}B^{i,1}) \) satisfies the renormalization property.

By Lemma A.1 together with (1.35), the couple \((U^1,U^2 - \frac{1}{2}U^1U^1)\) has the renormalization property.

Hence, from Proposition A.1 we obtain the uniform estimate
\[
|\nabla^{k-i} T_{-\epsilon}^{-1} \Gamma(B)_{st} \nabla L^i|_{L^\infty(W^\infty,\infty,W^\infty,\infty)} \leq C(\nabla^k L^i)_{st} \leq C' \omega_B(s,t)^\alpha.
\]
(A.12)

The claimed estimates follow by duality, observing that \( \Gamma(B)^\dagger = \Gamma(B^\dagger) \).

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