Formation of Trapped Surfaces from Past Null Infinity

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Abstract

In this paper, we extend the results of Klainerman and Rodnianski in [K-R:Trapped], which were obtained for a finite region, by showing similar results from past null infinity. This allows us to recover and extend the results from past null infinity in the work of Christodoulou [Chr].

1 Introduction.

1.1 Main Goals.

A recent breakthrough in General Relativity is Christodoulou’s remarkable 589-page paper *The Formation of Black Holes in General Relativity*. In his paper, he has identified an open set of regular initial conditions on a finite outgoing null hypersurface. These initial conditions lead to the formation of a trapped surface in the corresponding vacuum spacetime, which is the future of the initial outgoing hypersurface and another incoming null hypersurface with prescribed Minkowskian data.

His paper is based on an initial data ansatz called *short pulse method*. In order to form a trapped surface, the initial data is required to be large in terms of a small parameter, \( \delta > 0 \). Remarkably, Christodoulou shows that due to the structure of the vacuum Einstein’s Equation, some components of curvature and Ricci coefficients remain small in terms of \( \delta \). This allowed him to carry out a careful continuity argument to prove all the desired estimates.

In order to construct a spacetime such that a trapped surface is formed by the focusing of gravitational waves from past null infinity Christodoulou also needs to carefully estimate the decay rate of all curvature components and Ricci coefficients towards null infinity.

In a subsequent paper [K-R:Trapped], Klainerman and Rodnianski simplify and extend Christodoulou’s result in a finite region. Based on a different scaling with respect to the small “short pulse” parameter \( \delta \), they introduce appropriate scale invariant norms and use them to derive estimates in a systematic fashion.

The scale invariant norms associate specific behaviors in powers of \( \delta \) to various geometric quantities, based on their specific signatures. This procedure allows them to show that all nonlinear terms, generated during the construction of the spacetime, are small and proportional to \( \delta^4 \), with the exception of a relatively small number of cases, which they trace down to what they call anomalies. The anomalies correspond to simple violations of the scale invariant norms.

In our paper we extend the results in [K-R:Trapped], which is done for a finite region, by showing a similar result from past null infinity. In the spirit of that paper we complement their \( \delta^- \) scale invariant norms with a new scaling corresponding to powers of \( u \), which describe the decay rates near past null infinity. We will thus see two different hierarchies corresponding to \( \delta \) and \( u \) respectively. To propagate these two hierarchies, we will meet some

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anomalous terms. In addition to the anomalies in [K-R:Trapped], we encounter additional anomalies, corresponding to the decay rate, which lead to “borderline terms”. In our paper, we will give a systematic way to treat these borderline terms. Moreover, based on this systematic way and a key observation on Einstein’s equation, we give a more direct and intuitive approach to do energy estimates in an infinite region by integrating the null Bianchi equations. We do not use the Bel-Robinson tensor or Lie derivatives.

Given that the estimates in [K-R:Trapped] are more systematic and thus easier to implement, it is natural to ask whether the results of [K-R:Trapped] can be used, as a stepping stone, to derive those of [Chr]. In our paper we show that this is indeed the case. More precisely, the results of [K-R:Trapped] hold true for a larger class of data than that of [Chr]. The spacetime estimates which are derived from these data are however weaker. We show, nevertheless, that starting with Christodoulou’s data the estimates derived in [K-R:Trapped] can be indeed improved consistent with Christodoulou’s results in [Chr].

Furthermore, along the methods developed in this paper, we can extend [Chr] by proving the formation of scarred surfaces from past null infinity and the formation of trapped surfaces from past null infinity without Minkowskian initial data condition. The details of these results will be given in future papers.

1.2 Structure of the Paper

The structure of this paper is as following:

- In Section 2, we give the basic setup and main results.
- In Section 3, we give the main equations and preliminaries.
- In Section 4 - Section 6, we give the estimates for the Ricci coefficient.
- In Section 7 - Section 8, we give the energy estimates.
- In Section 9, we give a heuristic argument for the formation of trapped surfaces from past null infinity.
- In Section 10, we prove the formation of trapped surfaces from past null infinity.
- In Section 11, we prove, by the given initial data in [Chr], we can get consistent estimates in [Chr].

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2 Basic Setup and Main Results

2.1 Definitions.

Double Null Foliations.
We consider a region $\mathcal{D} = \mathcal{D}(u_*, u_\infty)$ of a vacuum spacetime $(M, g)$ spanned by a double null foliation generated by the optical functions $(u, u_\infty)\leq u \leq 0$ and $0 \leq u \leq \delta$. We denote by $H_u$ the outgoing null hypersurfaces generated by the level surfaces of $u$ and by $H_{u\infty}$ the incoming null hypersurfaces generated by the level hypersurfaces of $u_\infty$. We write $S(u, u_\infty) = H_u \cap H_{u\infty}$ which is clearly a 2-sphere. We denote $H_{u1}(u_\infty)$ and $H_{u2}(u_\infty)$ to be the regions of these null hypersurfaces defined by $u_1 \leq u \leq u_2$ respectively. Let $L,L$ be the geodesic vectorfields associated to the two foliations and define,

$$\frac{1}{2} \Omega^2 = -g(L,L)^{-1}$$

Observe that for Minkowski Space the value $\Omega$ of $\Omega$ is 1. As is well known, our space-time slab $\mathcal{D}(u_*, u_\infty)$ is completely determined by data along the null, characteristic, hypersurfaces $H_u, H_{u\infty}$ corresponding to $u = u_\infty, u = 0$. Following [Chr] we assume that our data is trivial along $H_{u\infty}$, i.e. assume that $H_{u\infty}$ extends to $u < 0$ and the spacetime $(M, g)$ is Minkowskian for $u < 0$ and all values of $u < 0$. Moreover we can construct our double null foliation such that $\Omega = 1$ along $H_{u\infty}$, i.e.,

$$\Omega(u_\infty, u) = 1, \quad 0 \leq u \leq \delta.$$

Throughout this paper we work with the normalized null pair $(e_3, e_4)$,

$$e_3 = \Omega L, \quad e_4 = \Omega L, \quad g(e_3, e_4) = -2.$$

**Definition of coefficients.**

Given a 2-sphere $S(u, u_\infty)$ and $(e_a)$ $a = 1, 2$ an arbitrary frame tangent to it, we define the Ricci coefficients,

$$\Gamma_{(\lambda)(\mu)(\nu)} = g(e_{(\lambda)}, D_{e_{(\mu)}} e_{(\nu)}), \quad \lambda, \mu, \nu = 1, 2, 3, 4$$

These coefficients are completely determined by the following components,

\begin{align}
\chi_{ab} &= g(D_a e_4, e_b), \quad \chi_{ab} = g(D_a e_3, e_b), \\
\eta_a &= \frac{1}{2} g(D_a e_3, e_4), \quad \eta_a = -\frac{1}{2} g(D_a e_3, e_3) \\
\omega &= -\frac{1}{4} g(D_4 e_3, e_4), \quad \omega = -\frac{1}{4} g(D_3 e_4, e_3), \\
\zeta_a &= \frac{1}{2} g(D_a e_4, e_3)
\end{align}

(2.1)

Note that our normalization for $\Omega$ is the same as of [K-R:Trapped] and differs from [K-Ni].
where \( D_a = D_e a \). Moreover, we separate the trace and traceless part of \( \chi \) and \( \chi \). Let \( \hat{\chi} \) and \( \tilde{\chi} \) be the traceless parts of \( \chi \) and \( \chi \), respectively.

**Definition of curvatures.**

We also introduce the null curvature components.

\[
\alpha(R)_{ab} = R(e_a, e_4, e_b, e_4), \quad \alpha(R)_{ab} = R(e_a, e_3, e_b, e_4), \\
\beta(R)_a = \frac{1}{2} R(e_a, e_4, e_3, e_4), \quad \beta(R)_a = \frac{1}{2} R(e_a, e_3, e_4, e_4), \\
\rho(R) = \frac{1}{4} R(e_4, e_3, e_3, e_4), \quad \sigma(R) = \frac{1}{4} R(e_4, e_3, e_4, e_3)
\]  

(2.2)

Here *R denotes the Hodge dual of R. \( \nabla \) denotes the induced covariant derivative operator on \( S(u, \omega) \). \( \nabla_3 \) and \( \nabla_4 \) denote the projections to \( S(u, \omega) \) of the covariant derivatives \( D_3 \) and \( D_4 \), respectively. (see precise definitions in [K-Ni].) Observe that,

\[
\omega = -\frac{1}{2} \nabla_4 (\log \Omega), \quad \omega = -\frac{1}{2} \nabla_3 (\log \Omega), \\
\eta_a = \zeta_a + \nabla_a (\log \Omega), \quad \eta_a = -\zeta_a + \nabla_a (\log \Omega).
\]

**2.2 Signatures.**

To capture the structure of Einstein’s equation, we need the following definitions given below.

**2.2.1. Definitions of signatures.**

To every null curvature component \( \alpha, \beta, \rho, \sigma, \beta, \alpha, \chi, \chi, \zeta, \eta, \eta, \omega, \omega, \gamma \), null Ricci coefficient components \( \chi, \chi, \zeta, \eta, \eta, \omega, \omega, \gamma \), and metric \( \gamma \), we assign signatures according to the following rule:

\[
s(\phi) = (s_1(\phi), s_2(\phi)),
\]

(2.3)

\[
s_1(\phi) = 1 \cdot N_4(\phi) + \frac{1}{2} \cdot N_a(\phi) + 0 \cdot N_3(\phi) - 1,
\]

(2.4)

\[
s_2(\phi) = 0 \cdot N_4(\phi) + \frac{1}{2} \cdot N_a(\phi) + 1 \cdot N_3(\phi) - 1.
\]

(2.5)

where \( \phi \in \{ \alpha, \beta, \rho, \sigma, \beta, \alpha, \chi, \chi, \zeta, \eta, \eta, \omega, \omega, \gamma \} \) and \( N_4(\phi) \) is the number of times \( e_4 \) appears in the definition of \( \phi \) found in [2.1] and [2.2]. Similarly we define \( N_3(\phi) \) and \( N_a(\phi) \) where \( a = 1, 2 \).

**2.2.2. Signature tables.**

By the definition above, we have.

| \( s_1 \) | \( \alpha \) | \( \beta \) | \( \rho \) | \( \sigma \) | \( \beta \) | \( \alpha \) | \( \chi \) | \( \omega \) | \( \zeta \) | \( \eta \) | \( \eta \) | \( \text{tr}X \) | \( \chi \) | \( \omega \) | \( \gamma \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.5 | 1 | 1 | 0.5 | 0 | 1 | 1 | 0.5 | 0.5 | 0.5 | 0 | 0 | 0 | 0 |
| 0 | 0.5 | 1 | 1 | 1.5 | 2 | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 | 1 | 1 | 0 |

**2.2.3. Properties of signatures.**

Consistent with the definition, we have for any given null component \( \phi \),

\[
s_1(\nabla_4 \phi) = s_1(\phi) + 1, s_2(\nabla_4 \phi) = s_2(\phi),
\]

(2.6)
We also define the scale invariant norms.

2.3 Scale Invariant Norms.

2.3.1 Definitions of scale invariant norms.

We define the following scale invariant norms on the 2-sphere $S = S(u, \Omega)$. For any horizontal tensor-field $\psi$ with signature $s(\psi) = (s_1(\psi), s_2(\psi))$, we define,

$$
\|\psi\|_{L^2_0(S)} = \delta^{s_1(\psi)} \frac{1}{u} |\psi|_{L^2(\Omega)}^{2s_2(\psi)+1} |\psi|_{L^2_0(S)},
$$

(2.12)

$$
\|\psi\|_{L^4_0(S)} = \delta^{s_1(\psi)} \frac{1}{u} |\psi|_{L^4(\Omega)}^{2s_2(\psi)+\frac{1}{2}} |\psi|_{L^4_0(S)},
$$

(2.13)

$$
\|\psi\|_{L^4(\Omega)} = \delta^{s_1(\psi)-1} |u|^{2s_2(\psi)} |\psi|_{L^4_0(S)}.
$$

(2.14)

More generally, for $1 \leq p \leq \infty$, we define,

$$
\|\psi\|_{L^p_0(S)} = \delta^{s_1(\psi)-\frac{1}{2}} \frac{1}{u} |u|^{2s_2(\psi)+1-\frac{2}{p}} |\psi|_{L^p_0(S)}.
$$

(2.15)

2.3.2 Hölder’s inequalities for scale invariant norms.

The reason we introduce the signature and scale invariant norm above is to get the following Hölder’s Inequalities.

$$
\|\psi_1 \cdot \psi_2\|_{L^2_0(S)} \leq \frac{\delta^{\frac{1}{2}}}{|u|} \|\psi_1\|_{L^2_0(S)} \|\psi_2\|_{L^2_0(S)},
$$

(2.16)

$$
\|\psi_1 \cdot \psi_2\|_{L^2_0(S)} \leq \frac{\delta^{\frac{1}{2}}}{|u|} \|\psi_1\|_{L^2_0(S)} \|\psi_2\|_{L^2_0(S)},
$$

(2.17)

$$
\|\psi_1 \cdot \psi_2\|_{L^p_0(S)} \leq \frac{\delta^{\frac{1}{p}}}{|u|} \|\psi_1\|_{L^p_0(S)} \|\psi_2\|_{L^p_0(S)}.
$$

(2.18)

2.3.3 Scale invariant norms along a hypersurface.

For convenience, we also define the following scale invariant norms along the null hypersurfaces $H_u^{(0, \delta)}$ and $H_u^{(a, \omega)}$.

$$
\|\psi\|_{L^2_0(H_u^{(a, \omega)})}^2 = \delta^{-1} \int_0^{\delta} \|\psi\|_{L^2_0(S(u, \omega'))}^2 du'.
$$

(2.19)

\[ s_1(\nabla \phi) = s_1(\phi) + \frac{1}{2}, s_2(\nabla \phi) = s_2(\phi) + \frac{1}{2}, \]

(2.7)

\[ s_1(\nabla_3 \phi) = s_1(\phi), s_2(\nabla_3 \phi) = s_2(\phi) + 1, \]

(2.8)

2.2.4 Conservation of signatures. Using the definitions above, we have,

$$
s_1(\phi_1 \cdot \phi_2) = s_1(\phi_1) + s_1(\phi_2),
$$

(2.9)

$$
s_2(\phi_1 \cdot \phi_2) = s_2(\phi_1) + s_1(\phi_2),
$$

(2.10)

\[ s(\phi_1 \cdot \phi_2) = (s_1(\phi_1), s_2(\phi_1)) = (s_1(\phi_2), s_2(\phi_2)) = s(\phi_1) + s(\phi_2). \]

(2.11)

Remark: $s_1$ is the same signature used in [K-R:Trapped] and $s_2$ is introduced to study the decay rate near past null infinity.
\[ \|\psi\|^2_{L^2(S'(u, \infty))} \triangleq \int_{u, \infty}^u \frac{1}{|u'|^2} \|\psi\|^2_{L^2(S(u, \infty))} du'. \] (2.20)

In what follows, we also write \( H = H_u = H_u^{(0, \delta)} \) and \( \underline{H} = \underline{H}_u = H_u^{(u, \infty, \mu)} \).

### 2.4 Bootstrap Assumptions.

**Initial data assumptions.**

We define the initial data quantity,

\[ \mathcal{I}^{(0)} = \sup_{0 \leq \mu \leq \delta} \mathcal{I}^{(0)}(u) \]

where,

\[ \mathcal{I}^{(0)}(u) = \delta^\frac{1}{2} |u|_\infty \|\tilde{\chi}_0\|_{L^\infty(S(u, \infty))} + \sum_{0 \leq k \leq 2} \delta^\frac{1}{2} \|\tilde{\chi} \|_{L^2(S(u, \infty))} + \sum_{0 \leq k \leq 1} 1 \leq m \leq 4 \|\tilde{\chi} \|_{L^2(S(u, \infty))} \] (2.21)

Here \( \tilde{\chi}_0 \) denotes \( \tilde{\chi} \) along \( H_u^{(0, \delta)} \).

**Bootstrap assumptions in scale invariant norms.**

To give a precise formulation of our result, we need to introduce the following norms.

**Ricci coefficient norms:**

For any \( S(u, \mu) \), we introduce norms \((S)O_{s,p}(u, \mu)\).

\[(S)O_{0,\infty}(u, \mu) = \|\omega\|_{L^\infty(S)} + \|\tilde{\chi}\|_{L^\infty(S)} + \delta^\frac{1}{2} \|\tilde{\chi} \|_{L^2(S)} + \|\eta\|_{L^2(S)} + \|\hat{\omega}\|_{L^\infty(S)} \]
+ \frac{1}{|u|} \|\tilde{\chi}\|_{L^2(S)} + \frac{1}{|u|^2} \|\tilde{\chi} \|_{L^2(S)} + \|\hat{\omega}\|_{L^\infty(S)}, \] (2.22)

\[(S)O_{0,4}(u, \mu) = \|\omega\|_{L^\infty(S)} + \delta^\frac{1}{2} \|\tilde{\chi}\|_{L^2(S)} + \|\tilde{\chi} \|_{L^2(S)} + \|\eta\|_{L^2(S)} + \|\hat{\omega}\|_{L^\infty(S)} \]
+ \frac{\delta^\frac{1}{2}}{|u|} \|\tilde{\chi}\|_{L^2(S)} + \frac{\delta^\frac{1}{2}}{|u|^2} \|\tilde{\chi} \|_{L^2(S)} + \|\hat{\omega}\|_{L^\infty(S)}, \] (2.23)

\[(S)O_{0,2}(u, \mu) = \|\nabla \omega\|_{L^2(S)} + \delta^\frac{1}{2} \|\tilde{\chi}\|_{L^2(S)} + \|\tilde{\chi} \|_{L^2(S)} + \|\eta\|_{L^2(S)} + \|\hat{\omega}\|_{L^\infty(S)} \]
+ \frac{\delta^\frac{1}{2}}{|u|} \|\tilde{\chi}\|_{L^2(S)} + \|\tilde{\chi} \|_{L^2(S)} + \|\hat{\omega}\|_{L^\infty(S)}, \] (2.24)

\[(S)O_{1,2}(u, \mu) = \|\nabla \omega\|_{L^2(S)} + \|\nabla \tilde{\chi}\|_{L^2(S)} + \|\nabla \tilde{\chi} \|_{L^2(S)} + \|\nabla \eta\|_{L^2(S)} + \|\nabla \hat{\omega}\|_{L^\infty(S)} \]
+ \frac{1}{|u|} \|\tilde{\chi}\|_{L^2(S)} + \|\nabla \tilde{\chi} \|_{L^2(S)} + \|\nabla \hat{\omega}\|_{L^\infty(S)}, \] (2.25)

\[(S)O_{1,4}(u, \mu) = \|\nabla \omega\|_{L^2(S)} + \|\nabla \tilde{\chi}\|_{L^2(S)} + \|\nabla \tilde{\chi} \|_{L^2(S)} + \|\nabla \eta\|_{L^2(S)} + \|\nabla \hat{\omega}\|_{L^\infty(S)} \]
+ \frac{1}{|u|} \|\tilde{\chi}\|_{L^2(S)} + \|\nabla \tilde{\chi} \|_{L^2(S)} + \|\nabla \hat{\omega}\|_{L^\infty(S)}. \] (2.26)
We define the norms\(^{(S)}\mathcal{O}_{0.2}, (S)\mathcal{O}_{0.4}, (S)\mathcal{O}_{0,\infty}, (S)\mathcal{O}_{1.2}\) and \((S)\mathcal{O}_{1,4}\) to be the supremum of the corresponding norms over all values of \(u, u\) in our slab. Finally, we define the total Ricci norm \((S)\mathcal{O}\)
\[
(S)\mathcal{O} = (S)\mathcal{O}_{0.2} + (S)\mathcal{O}_{0.4} + (S)\mathcal{O}_{0,\infty} + (S)\mathcal{O}_{1.2} + (S)\mathcal{O}_{1,4}
\]
and let \((S)\mathcal{O}^{(0)}\) be the corresponding norm of the initial hypersurface \(H_0\).

**Curvature norms:**
Along the null hypersurfaces \(H = H^{(u,\alpha)}_{0}\) and \(H = H^{(u,\beta)}_{0}\), we introduce,
\[
\mathcal{R}_0(u, u) = \delta^s \|\alpha\|_{L^2_s(H)} + \|\beta\|_{L^2_s(H)} + \|\rho\|_{L^2_s(H)} + \|\sigma\|_{L^2_s(H)} + \|\beta\|_{L^2_s(H)}
\]
\[
(2.27)
\]
\[
\mathcal{R}_0(u, u) = \delta^s \|\beta\|_{L^2_s(H)} + \|\rho\|_{L^2_s(H)} + \|\sigma\|_{L^2_s(H)} + \|\beta\|_{L^2_s(H)} + \|u\|_{L^2_s(H)}
\]
\[
(2.28)
\]
\[
\mathcal{R}_1(u, u) = \|\nabla\alpha\|_{L^2_s(H)} + \|\nabla\beta\|_{L^2_s(H)} + \|\nabla\rho\|_{L^2_s(H)} + \|\nabla\sigma\|_{L^2_s(H)} + \|\nabla\beta\|_{L^2_s(H)}
\]
\[
(2.29)
\]
\[
\mathcal{R}_1(u, u) = \|\nabla\beta\|_{L^2_s(H)} + \|\nabla\rho\|_{L^2_s(H)} + \|\nabla\sigma\|_{L^2_s(H)} + \|\nabla\beta\|_{L^2_s(H)} + \|\nabla\beta\|_{L^2_s(H)}
\]
\[
(2.30)
\]
We set \(\mathcal{R}_0, \mathcal{R}_1\) to be the supremum over \(u, u\) in our spacetime slab of \(\mathcal{R}_0(u, u)\) and \(\mathcal{R}_1(u, u)\) respectively and similarly for the norm \(\mathcal{R}_0\) and \(\mathcal{R}_1\). Also, we write \(\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1\) and \(\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1\). \(\mathcal{R}^{(0)}\) denotes the initial value for the norm \(\mathcal{R}\) i.e.,
\[
\mathcal{R}^{(0)} = \sup_{0 \leq u, \delta} (\mathcal{R}_0(0, u) + \mathcal{R}_1(0, u)).
\]

**Remark:** Most of these quantities are scale invariant except for a small number of anomalous terms.

**Remark:** For \(\delta\), the anomalous terms are \(\alpha, tr\chi, L^2_s(S_s)\), and \(L^2_{sc}(H)\) norm for \(\beta\). Later, we will always try to use the \(L^\infty(S)\) norm for \(\hat{\chi}, \hat{\chi}\), and \(L^2_{sc}(H)\) norm for \(\beta\), which are normal. The genuine anomalous terms for \(\delta\) are \(\alpha\) and \(tr\chi\).

**Remark:** For \(u\), the anomalous terms are \(\hat{\chi}, \hat{\chi}\), and \(tr\chi\). Furthermore, \(tr\chi\) is more anomalous than \(\hat{\chi}\). Later we will see, that, the borderline terms for the decay rate of \(\alpha\) contain \(tr\chi\), most of the time. Usually, when we have \(\hat{\chi}\), we have \(tr\chi\) at the same time. This makes \(\hat{\chi}\), usually, a harmless anomalous term for the decay rate of \(u\).

To rectify the anomaly of \(\alpha\), we introduce an additional scale invariant norm
\[
\mathcal{R}^{(0)}_\delta[\alpha] := \sup_{H \subset H} \|\alpha\|_{L^2_s(\delta H)},
\]
where \(\delta H\) is a piece of the hypersurface \(H = H^{(u,\delta)}_{0}\) obtained by evolving a disc \(S^\delta \subset S(u, 0)\) of radius \(\delta^s\) along the integral curves of the vectorfield \(e_4\).

**Bootstrap Assumptions in the standard \(L^p\) norms.**
For convenience, we also list the bootstrap assumptions in the standard \(L^p\) norms.
\[
(2.31)
\]

\[
(2.31)
\]
We are now ready to state our main theorem:

**Theorem 1 (Main Theorem):** Assume that \( \mathcal{R}^{(0)} \leq \mathcal{I}^{(0)} \) for an arbitrary constant \( \mathcal{I}^{(0)} \).

Then, there exists a sufficiently small \( \delta > 0 \), such that,

\[
\mathcal{R} + \mathcal{R}_1 + \mathcal{O} \leq \mathcal{I}^{(0)}.
\]

**Strategy of the Proof.**

We divide proof of the main theorem in two parts. In the first part we derive estimates for the norm \( \mathcal{O} \) for the Ricci coefficients in terms of the initial data \( \mathcal{I}^{(0)} \) and the curvature norm \( \mathcal{R} \). More precisely we prove:

**Theorem 1A:** Assume that \( \mathcal{O} < \infty \) and \( \mathcal{R} < \infty \). There exists a constant \( C \) depending only on \( \mathcal{O}^{(0)}, \mathcal{R} \), and \( \mathcal{R}_1 \) such that,

\[
\mathcal{O} \leq C(\mathcal{O}^{(0)}, \mathcal{R}, \mathcal{R}_1).
\]  

We prove the theorem by a bootstrap argument. We start by assuming that there exists a sufficiently large constant \( \Delta_0 \) such that,

\[
(\mathcal{S}) \mathcal{O}_{0, \infty} \leq \Delta_0.
\]
Based on this assumption we show that, if $\delta$ is sufficiently small, estimate (2.39) also holds. This allows us to derive a better estimate than (2.40).

In the second part we need to use the estimates of Theorem 1A to derive estimates for the curvature norms $R$. This ends the proof of the main theorem.

**Theorem 1B**: The following estimate holds for a constant $C = C(I(0), R(0))$, and $\delta$ sufficiently small,

$$R + R \leq C(I(0), R(0)).$$

In the Main Theorem, we obtain a semi-global existence result. In Section 9 and Section 10, we will use these results to prove:

**Theorem 2** Given Minkowskian initial data on $H_{u,\infty}$ and initial data on $H_{u,0}^{(\delta)}$, a trapped surface must form in the slab $D(u \approx 1, \delta)$.

In Section 11 we will use the semi-global existence result above along with Christodoulou’s initial data in [Chr] to prove:

**Theorem 3** If given Minkowskian initial data on $H_{u,\infty}$ and [Chr] initial data on $H_{u,0}^{(\delta)}$, we can prove the consistent estimates in [Chr].

### 3 Main equations and preliminaries

#### 3.1 Null structure equations.

We recall the null structure equations: (see [K-Ni] or [K-R:Trapped].)

\[
\nabla_4 tr\chi + \frac{1}{2} (tr\chi)^2 = -|\hat{\chi}|^2 - 2\omega tr\chi, \quad (3.1)
\]

\[
\nabla_3 tr\chi + \frac{1}{2} (tr\chi)^2 = -|\hat{\chi}|^2 - 2\omega tr\chi, \quad (3.2)
\]

\[
\nabla_4 \hat{\chi} + tr\chi \hat{\chi} = -2\omega \hat{\chi} - \alpha, \quad (3.3)
\]

\[
\nabla_3 \hat{\chi} + tr\chi \hat{\chi} = -2\omega \hat{\chi} - \alpha, \quad (3.4)
\]

\[
\nabla_4 \eta = -\chi \cdot (\eta - \hat{\eta}) - \beta, \quad (3.5)
\]

\[
\nabla_3 \eta = -\chi \cdot (\eta - \eta) + \beta, \quad (3.6)
\]

\[
\nabla_4 \omega = 2\omega \omega + \frac{3}{4} |\eta - \hat{\eta}|^2 - \frac{1}{4} (\eta - \hat{\eta}) \cdot (\eta + \hat{\eta}) - \frac{1}{8} |\eta + \hat{\eta}|^2 + \frac{1}{2} \rho, \quad (3.7)
\]

\[
\nabla_3 \omega = 2\omega \omega + \frac{3}{4} |\eta - \hat{\eta}|^2 + \frac{1}{4} (\eta - \hat{\eta}) \cdot (\eta + \hat{\eta}) - \frac{1}{8} |\eta + \hat{\eta}|^2 + \frac{1}{2} \rho, \quad (3.8)
\]

\[
\nabla_4 tr\chi + \frac{1}{2} tr\chi tr\chi = 2\omega tr\chi + 2div\eta + 2|\eta|^2 + 2\rho \cdot \hat{\chi} \cdot \hat{\chi}, \quad (3.9)
\]

\[
\nabla_3 tr\chi + \frac{1}{2} tr\chi tr\chi = 2\omega tr\chi + 2div\eta + 2|\eta|^2 + 2\rho \cdot \hat{\chi} \cdot \hat{\chi}, \quad (3.10)
\]

\[
\nabla_4 \hat{\chi} + \frac{1}{2} tr\chi \hat{\chi} = \nabla \hat{\eta} + 2\omega \hat{\chi} - \frac{1}{2} tr\chi \hat{\chi} + \nabla \hat{\eta}, \quad (3.11)
\]
\[ \nabla_3 \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} = \nabla \hat{\chi} + 2 \omega \hat{\chi} - \frac{1}{2} \text{tr} \chi \hat{\chi} + \eta \hat{\chi}, \quad (3.12) \]

And the constraint equations:

\[ \text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \text{tr} \chi \delta_{ab}) - \beta, \quad (3.13) \]

\[ \text{curl} \eta = \hat{\chi} \wedge \hat{\chi} + \sigma \epsilon_{ab}, \quad (3.15) \]

\[ K = -\frac{1}{4} \text{tr} \chi \text{tr} \chi + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \rho, \quad (3.17) \]

with \( K \) the Gauss curvature of the surfaces \( S \).

### 3.2 Null Bianchi equations.

We recall the Null Bianchi equations in [K-Ni] or [K-R:Trapped]:

\[ \nabla_3 \alpha + \frac{1}{2} \text{tr} \chi \alpha = \nabla \hat{\chi} \beta + 4 \omega \alpha - 3 (\hat{\chi} \rho + \hat{\chi} \sigma) + (\zeta + 4 \eta) \hat{\chi} \beta, \quad (3.18) \]

\[ \nabla_4 \beta + 2 \text{tr} \chi \beta = \text{div} \alpha - 2 \omega \beta + \eta \cdot \alpha, \quad (3.19) \]

\[ \nabla_3 \beta + \text{tr} \chi \beta = \nabla \rho + \nabla \sigma + 2 \omega \beta + 2 \hat{\chi} \cdot \beta + 3 (\eta \rho + \eta \sigma), \quad (3.20) \]

\[ \nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma = -\text{div} \beta + \frac{1}{2} \hat{\chi} \cdot \alpha - \zeta \cdot \beta - 2 \eta \cdot \beta, \quad (3.21) \]

\[ \nabla_3 \rho + \frac{3}{2} \text{tr} \chi \rho = -\nabla \sigma - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2 \eta \cdot \beta, \quad (3.23) \]

\[ \nabla_4 \rho + \frac{3}{2} \text{tr} \chi \rho = -\text{div} \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2 \eta \cdot \beta, \quad (3.24) \]

\[ \nabla_4 \beta + \text{tr} \chi \beta = -\nabla \rho + \nabla \sigma + 2 \omega \beta + 2 \hat{\chi} \cdot \beta - 3 (\eta \rho - \eta \sigma), \quad (3.25) \]

Remark: All terms in a given null structure or null Bianchi equation have the same signature. For example, for \( \nabla_3 \alpha + \frac{1}{2} \text{tr} \chi \alpha = \nabla \hat{\chi} \beta + 4 \omega \alpha - 3 (\hat{\chi} \rho + \hat{\chi} \sigma) + (\zeta + 4 \eta) \hat{\chi} \beta \), each term in this equation has signature \((2,1)\).
3.3 Commutator Lemma.

We have the following Commutator Lemmas. (see [K-Ni])

Lemma 3.1. For a scalar function $f$,

$$\left[\nabla_3, \nabla_4\right]f = -2\omega\nabla_3 f + 2\omega\nabla_4 f + 4\zeta \cdot \nabla f$$

Lemma 3.2. For a scalar function $f$,

$$\left[\nabla_4, \nabla\right]f = \frac{1}{2} (\eta + \eta) D_4 f - \chi \cdot \nabla f$$

$$\left[\nabla_3, \nabla\right]f = \frac{1}{2} (\eta + \eta) D_3 f - \chi \cdot \nabla f$$

Lemma 3.3. For a 1-form tangent to $S$,

$$\left[D_4, \nabla\right]U_b = -\chi_{ac} \nabla_c U_b + \epsilon_{ac}^* \beta_b U_c + \frac{1}{2} (\eta_a + \eta_a) D_4 U_b - \chi_{ac} \eta_b U_c + \chi_{ab} \eta \cdot U$$

$$\left[D_3, \nabla\right]U_b = -\chi_{ac} \nabla_c U_b + \epsilon_{ac}^* \beta_a U_c + \frac{1}{2} (\eta_a + \eta_a) D_3 U_b - \chi_{ac} \eta_b U_c + \chi_{ab} \eta \cdot U$$

Lemma 3.4. For a 2-form tangent to $s$,

$$\left[D_4, \nabla\right]V_{bc} = \frac{1}{2} (\eta_a + \eta_a) D_4 V_{bc} - \eta_b V_{dc} \chi_{ad} - \frac{1}{2} \chi_{ac} \beta_b V_{dc} - \epsilon_{bd}^* \beta_a V_{bc}$$

$$\left[D_3, \nabla\right]V_{bc} = \frac{1}{2} (\eta_a + \eta_a) D_3 V_{bc} - \eta_b V_{dc} \chi_{ad} + \frac{1}{2} \chi_{ac} \beta_b V_{bc} - \epsilon_{bd}^* \beta_a V_{bc}$$

3.4 Evolution Lemma.

Given a scalar function $f$, we have, (see [K-Ni] or [K-R:Trapped] )

$$\frac{d}{du}\int_{S(u, \omega)} f = \int_{S(u, \omega)} \left( \frac{df}{du} + \Omega \chi f \right) = \int_{S(u, \omega)} \Omega (\nabla_4 f + tr \chi f),$$

(3.30)

$$\frac{d}{du}\int_{S(u, \omega)} f = \int_{S(u, \omega)} \left( \frac{df}{du} + \Omega \chi f \right) = \int_{S(u, \omega)} \Omega (\nabla_3 f + tr \chi f),$$

(3.31)

With these, we can establish the following Evolution Lemma.

Lemma 3.5. Evolution Lemma.

Assume $\delta > 0$ is sufficiently small, $c > 0$ is arbitrarily small, and $|\Omega \chi f + \frac{\omega}{m} | \leq \frac{\delta c}{m}.$

Let $U$ and $F$ be $k$-covariant $S$-tangent tensor fields satisfying the incoming evolutionary equation:

$$\frac{dU_{a_1 \ldots a_k}}{du} + \lambda_0 \Omega \chi U_{a_1 \ldots a_k} = F_{a_1 \ldots a_k},$$

(3.32)

Denoting $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$, we have along $F^{(u, \infty, a)}_{a_1 \ldots a_k}$,
Lemma 3.6. Integral Formulas.

For any horizontal tensor-field $\psi$, we have,

$$\|\psi\|_{L^2(S(u, \mathbf{u}))} \leq \|\psi\|_{L^2(S(0, \mathbf{u}))} + \int_{u_0}^{u} \|\nabla_4 \psi\|_{L^2(S(u, \mathbf{u}'))} du'. \quad (3.38)$$

$$\|\psi\|_{L^2(S(\mathbf{u}))} \leq \|\psi\|_{L^2(S(\mathbf{u}))} + \int_{u_0}^{u} \|\nabla_3 \psi\|_{L^2(S(\mathbf{u}'))} du'. \quad (3.39)$$

$$\|\psi\|_{L^4(S(u, \mathbf{u}))} \leq \|\psi\|_{L^4(S(0, \mathbf{u}))} + \int_{u_0}^{u} \|\nabla_4 \psi\|_{L^4(S(u, \mathbf{u}'))} du'. \quad (3.40)$$

We can rewrite the above inequalities using scale invariant norms,
\[ \|\psi\|_{L^2(S(u,w))} \leq \left\| \psi \right\|_{L^2(S(0,0))} + \int_0^u \delta^{-1} \left\| \nabla_4 \psi \right\|_{L^2(S(u',w))} du'. \] (3.42)

\[ \|\psi\|_{L^4(S(u,w))} \leq \left\| \psi \right\|_{L^4(S(u,\infty))} + \int_0^u \frac{1}{|u'|^5} \left\| \nabla_3 \psi \right\|_{L^4(S(u',w))} du'. \] (3.43)

\[ \|\psi\|_{L^6(S(u,w))} \leq \left\| \psi \right\|_{L^6(S(u,\infty))} + \int_0^u \delta^{-1} \left\| \nabla_4 \psi \right\|_{L^6(S(u',w))} du'. \] (3.44)

\[ \|\psi\|_{L^8(S(u,w))} \leq \left\| \psi \right\|_{L^8(S(u,\infty))} + \int_0^u \frac{1}{|u'|^4} \left\| \nabla_3 \psi \right\|_{L^8(S(u',w))} du'. \] (3.45)

### 3.5 Calculus inequalities
(see [K-R:Trapped])

**Lemma 3.7.** For any horizontal tensor \( F \), we have,
\[
\left\| F \right\|_{L^4(S(u,w))} \leq \left\| F \right\|_{L^4(S(u,\infty))} \left\| \nabla F \right\|_{L^2(S(u,w))} + \frac{1}{|u|^4} \left\| F \right\|_{L^2(S(u,w))},
\] (3.46)

Using scale invariant norms,
\[
\left\| F \right\|_{L^4(S(u,w))} \leq \left\| F \right\|_{L^4(S(u,\infty))} \left\| \nabla F \right\|_{L^4(S(u,w))} + \delta \left\| F \right\|_{L^4(S(u,w))},
\] (3.47)

**Lemma 3.8.** For any horizontal tensor \( F \), we have,
\[
\left\| F \right\|_{L^\infty(S(u,w))} \leq \left\| F \right\|_{L^\infty(S(u,\infty))} \left\| \nabla F \right\|_{L^4(S(u,w))} + \frac{1}{|u|^2} \left\| F \right\|_{L^4(S(u,w))},
\] (3.48)

Using scale invariant norms,
\[
\left\| F \right\|_{L^\infty(S(u,w))} \leq \left\| F \right\|_{L^\infty(S(u,\infty))} \left\| \nabla F \right\|_{L^\infty(S(u,w))} + \delta \left\| F \right\|_{L^\infty(S(u,w))},
\] (3.49)

### 3.6 Estimates for Hodge Systems
(see [K-R:Trapped])

We will use the following three types of Hodge Systems.

1. The operator \( D_1 \) takes any 1-form \( \psi \) to a pair of functions \((\text{div} \psi, \text{curl} \psi)\).
2. The operator \( D_2 \) takes any \( S \) tangent symmetric, traceless tensor \( \psi \) into the \( S \) tangent 1-form \( \text{div} \psi \).
3. The operator \( ^\ast D_1 \) takes the pair of scalar functions \((\psi, \psi^\dagger)\) to the \( S \) tangent one form \(-\nabla \psi + \ast \nabla \psi^\dagger\).

**Proposition 3.9.** Let \((S, \gamma)\) be a compact manifold with Gaussian curvature \( K \).

1. The following identity holds for vectorfields \( \psi \) on \( S \):
\[
\int_S (|\nabla \psi|^2 + K|\psi|^2) = \int_S (|\text{div} \psi|^2 + |\text{curl} \psi|^2) = \int_S |D_1 \psi|^2
\] (3.50)

2. The following identity holds for symmetric, traceless, 2-tensor field \( \psi \) on \( S \):
\[
\int_S (|\nabla \psi|^2 + 2K|\psi|^2) = 2 \int_S |\text{div} \psi|^2 = 2 \int_S |D_2 \psi|^2
\] (3.51)
Lemma 3.10. Hodge Estimates.

1. For a 1-form \( \psi \) on \( S \), we have,
\[
\| \nabla \psi \|_{L^2(S(u,u))} \leq \| \text{div} \psi \|_{L^2(S(u,u))} + \| \text{curl} \psi \|_{L^2(S(u,u))} + \frac{\delta^+}{\|u\|^2} \| K \|_{L^2(S(u,u))} \| \psi \|_{L^2(S(u,u))},
\]
(3.53)

2. For a symmetric, traceless, 2-tensor fields \( \psi \) on \( S \), we have,
\[
\| \nabla \psi \|_{L^2(S(u,u))} \leq \| \text{div} \psi \|_{L^2(S(u,u))} + \frac{\delta^+}{\|u\|^2} \| K \|_{L^2(S(u,u))} \| \psi \|_{L^2(S(u,u))},
\]
(3.54)

3. For a pair of functions \((\psi, \psi')\) on \( S \), we have,
\[
\| \nabla \psi \|_{L^2(S(u,u))} + \| \nabla \psi' \|_{L^2(S(u,u))} \leq \|^* D_1(\psi, \psi') \|_{L^2(S(u,u))},
\]
(3.55)

4. \((S)O_{0,2}(u,u)\) and \((S)O_{0,4}(u,u)\) Estimates.

4.1 Coefficient Estimates in Scale Invariant Norms.

For null Ricci coefficients \( \psi \) with signature \((s, s')\), it satisfies either transport equation
\[
\nabla_4 \psi^{(s,s')} = \sum_{s_1 + s_2 = s + 1} \psi^{(s_1, s_1')} \cdot \psi^{(s_2, s_2')} + \psi^{(s + 1, s')},
\]
(4.1)

or
\[
\nabla_3 \psi^{(s,s')} = \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \psi^{(s_1, s_1')} \cdot \psi^{(s_2, s_2')} + \psi^{(s, s' + 1)}.
\]
(4.2)

Here \( \psi^{(s,s')} \) and \( \Psi^{(s,s')} \) are \( S \)-tangent tensor fields with signatures \((s, s')\). In this Section, we consider \( \psi^{(s,s')} \) and \( \Psi^{(s,s')} \) as an arbitrary Ricci coefficient component with signature \((s, s')\) and a null curvature component with signature \((s, s')\), respectively.

Let’s establish several useful lemmas.

Lemma 4.1. For equation 4.1, we have,
\[
\| \psi^{(s,s')} \|_{L^2(S(u,u))} \leq \| \psi^{(s,s')} \|_{L^2(S(u,u))} + \sup_{\omega \in [0, \delta]} \sum_{s_1 + s_2 = s + 1, s_1' + s_2' = s'} \frac{\delta^+}{\|u\|^2} \| \psi^{(s_1, s_1')} \|_{L^2(S(u,u))} \| \psi^{(s_2, s_2')} \|_{L^2(S(u,u))} \nonumber
\]
\[
+ \| \psi^{(s + 1, s')} \|_{L^2(S(\mathbb{H}^n_d, u))},
\]
(4.3)
\[
\| \psi(s,s') \|_{L^2_c(S(u, \bar{u}))} \leq \| \psi(s,s') \|_{L^2_c(S(u, 0))} + \sup_{u' \in [0, \delta]} \sum_{s_1' + s_2' + s'' = s'} \frac{\delta^s}{|u'|} \| \psi(s_1, s_1') \|_{L^2_c(S(u, u'))} \| \psi(s_2, s_2') \|_{L^2_c(S(u, u'))} \\
+ \| \nabla \psi(s+1,s') \|_{L^2_c(H^w_0, \bar{u}')} + \frac{\delta^s}{|u'|} \| \psi(s+1,s') \|_{L^2_c(H^w_0, \bar{u}')} \]
\]

\begin{align}
\text{Proof.} \quad & \leq \| \psi(s,s') \|_{L^2_c(S(u, 0))} + \int_0^\delta \frac{\delta^{-1}}{|u'|} \| \nabla \psi(s,s') \|_{L^2_c(S(u, u'))} d\bar{u}' \\
& \leq \| \psi(s,s') \|_{L^2_c(S(u, 0))} + \int_0^\delta \frac{\delta^{-1}}{s_1' + s_2' + s'' = s'} \| \psi(s_1, s_1') \cdot \psi(s_2, s_2') \|_{L^2_c(S(u, u'))} d\bar{u}' \\
& + \int_0^\delta \frac{\delta^{-1}}{s_1' + s_2' + s'' = s'} \| \psi(s+1,s') \|_{L^2_c(S(u, u'))} d\bar{u}' \star \\
& \leq \| \psi(s,s') \|_{L^2_c(S(u, 0))} + \sup_{u' \in [0, \delta]} \sum_{s_1' + s_2' + s'' = s'} \frac{\delta^s}{|u'|} \| \psi(s_1, s_1') \|_{L^2_c(S(u, u'))} \| \psi(s_2, s_2') \|_{L^2_c(S(u, u'))} \\
+ \| \psi(s+1,s') \|_{L^2_c(H^w_0, \bar{u}')} \]
\end{align}

For *, we use Hölder’s inequality for scale invariant norms and the definition:
\[
\| \psi \|_{L^2_c(H^w_0, \bar{u}')}^2 \leq \int_0^\delta \| \psi \|_{L^2_c(S(u, u'))}^2 d\bar{u}' \quad \text{.} \quad (4.6)
\]
Similarly, we have,
\[
\|\psi(s,s')\|_{L^2_c(S(u,\omega))} \leq \|\psi(s,s')\|_{L^2_c(S(u,\omega))} + \int_0^u \delta^{-1} \|\nabla_4 \psi(s,s')\|_{L^2_c(S(u,\omega'))} du' \\
\leq \|\psi(s,s')\|_{L^2_c(S(u,\omega))} + \int_0^u \delta^{-1} \sum_{s_1 + s_2 = s+1, s_1' + s_2' = s'} \|\psi(s_1,s_1')\|_{L^2_c(S(u,\omega'))} \|\psi(s_2,s_2')\|_{L^2_c(S(u,\omega'))} du' \\
+ \int_0^u \delta^{-1} \|\psi(s+1,s')\|_{L^2_c(S(u,\omega))} du' \\
\leq \|\psi(s,s')\|_{L^2_c(S(u,\omega))} + \int_0^u \delta^{-1} \sum_{s_1 + s_2 = s+1, s_1' + s_2' = s'} \frac{\delta^{-1}}{|u|} \|\psi(s_1,s_1')\|_{L^2_c(S(u,\omega'))} \|\psi(s_2,s_2')\|_{L^2_c(S(u,\omega'))} du' \\
+ \|\psi(s+1,s')\|_{L^2_c(\Omega^{(n)}_u)} \|\nabla \psi(s+1,s')\|_{L^2_c(\Omega^{(n)}_u)} + \delta^{-1} \|\psi(s+1,s')\|_{L^2_c(\Omega^{(n)}_u)} \\
\leq \|\psi(s,s')\|_{L^2_c(S(u,\omega))} + \sup_{u' \in [0,\delta]} \sum_{s_1 + s_2 = s+1, s_1' + s_2' = s'} \frac{\delta^{-1}}{|u|} \|\psi(s_1,s_1')\|_{L^2_c(S(u,\omega'))} \|\psi(s_2,s_2')\|_{L^2_c(S(u,\omega'))} \\
+ \|\psi(s+1,s')\|_{L^2_c(\Omega^{(n)}_u)} \|\nabla \psi(s+1,s')\|_{L^2_c(\Omega^{(n)}_u)} + \delta^{-1} \|\psi(s+1,s')\|_{L^2_c(\Omega^{(n)}_u)} \\
(4.7)
\]

Lemma 4.2. For equation 4.2 if we do not have anomalies, then we have,
\[
\|\psi(s,s')\|_{L^2_c(S(u,\omega))} \leq \|\psi(s,s')\|_{L^2_c(S(u,\omega))} + \int_0^u \frac{\delta^{-1}}{|u'|} \sum_{s_1 + s_2 = s+1, s_1' + s_2' = s'} \|\psi(s_1,s_1')\|_{L^2_c(S(u,\omega'))} \|\psi(s_2,s_2')\|_{L^2_c(S(u,\omega'))} du' \\
+ \frac{1}{|u|^2} \|\psi(s,s'+1)\|_{L^2_c(\Omega^{(n)}_u)} \\
(4.8)
\]
\[
\|\psi(s,s')\|_{L^2_c(S(u,\omega))} \leq \|\psi(s,s')\|_{L^2_c(S(u,\omega))} + \int_0^u \frac{\delta^{-1}}{|u'|} \sum_{s_1 + s_2 = s+1, s_1' + s_2' = s'} \|\psi(s_1,s_1')\|_{L^2_c(S(u,\omega'))} \|\psi(s_2,s_2')\|_{L^2_c(S(u,\omega'))} du' \\
+ \frac{1}{|u|^2} \|\psi(s,s'+1)\|_{L^2_c(\Omega^{(n)}_u)} \|\nabla \psi(s,s'+1)\|_{L^2_c(\Omega^{(n)}_u)} + \frac{\delta^{-1}}{|u|^2} \|\psi(s,s'+1)\|_{L^2_c(\Omega^{(n)}_u)} \\
(4.9)
\]

Proof:
\[ \|\psi(s,s')\|_{L^2_{w}(S_w)} \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\nabla_3 \psi(s,s')\|_{L^2_{w}(S_w)} \, du' \]

\[ \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \sum_{s_1' + s_2' = s'} \|\psi(s_1,s_1') \cdot \psi(s_2,s_2')\|_{L^2_{w}(S_w)} \, du' \]

\[ + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi(s,s'+1)\|_{L^2_{w}(S_w)} \, du' \]

\[ \|\psi(s,s')\|_{L^2_{w}(S_w)} \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \sum_{s_1' + s_2' = s'} \|\psi(s_1,s_1') \cdot \psi(s_2,s_2')\|_{L^2_{w}(S_w)} \, du' \]

\[ + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi(s,s'+1)\|_{L^2_{w}(S_w)} \, du' \]

For **, we use Hölder’s inequality for scale invariant norms and the definition:

\[ \|\Psi\|_{L^2_{w}(U^{(u_{\infty},u)})}^2 \triangleq \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi\|_{L^2_{w}(S_w)}^2 \, du'. \]  

Similarly,

\[ \|\psi(s,s')\|_{L^2_{w}(S_w)} \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\nabla_3 \psi(s,s')\|_{L^2_{w}(S_w)} \, du' \]

\[ \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \sum_{s_1' + s_2' = s'} \|\psi(s_1,s_1') \cdot \psi(s_2,s_2')\|_{L^2_{w}(S_w)} \, du' \]

\[ + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi(s,s'+1)\|_{L^2_{w}(S_w)} \, du' \]

\[ \|\psi(s,s')\|_{L^2_{w}(S_w)} \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \sum_{s_1' + s_2' = s'} \|\psi(s_1,s_1') \cdot \psi(s_2,s_2')\|_{L^2_{w}(S_w)} \, du' \]

\[ + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi(s,s'+1)\|_{L^2_{w}(S_w)} \, du' \]

\[ \|\psi(s,s')\|_{L^2_{w}(S_w)} \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \sum_{s_1' + s_2' = s'} \|\psi(s_1,s_1') \cdot \psi(s_2,s_2')\|_{L^2_{w}(S_w)} \, du' \]

\[ + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi(s,s'+1)\|_{L^2_{w}(S_w)} \, du' \]

\[ \|\psi(s,s')\|_{L^2_{w}(S_w)} \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \sum_{s_1' + s_2' = s'} \|\psi(s_1,s_1') \cdot \psi(s_2,s_2')\|_{L^2_{w}(S_w)} \, du' \]

\[ + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi(s,s'+1)\|_{L^2_{w}(S_w)} \, du' \]

\[ \|\psi(s,s')\|_{L^2_{w}(S_w)} \leq \|\psi(s,s')\|_{L^2_{w}(S_w)} + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \sum_{s_1' + s_2' = s'} \|\psi(s_1,s_1') \cdot \psi(s_2,s_2')\|_{L^2_{w}(S_w)} \, du' \]

\[ + \int_{u_{\infty}}^{u} \frac{1}{|u'|^2} \|\Psi(s,s'+1)\|_{L^2_{w}(S_w)} \, du' \]

\[ \textbf{Remark:} \text{ If we don’t have anomalies in 4.1 from Lemma 4.1 we can actually prove,} \]
\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + \frac{\delta^+}{|u|} \| \| \| \psi(s,s') \|_{L^2(S(u,u))} + \delta^+ \mathcal{O}_{0,2} + R_0. \] (4.13)

\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + \mathcal{O}_{0,2} + R_0 + l.o.t. \] (4.14)

For convenience, we use lower order terms (l.o.t) to stand for terms which are a \( \delta^c \) smaller than the leading terms, where \( c > 0 \). For example, we can rewrite the inequalities in the above remark as,

\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + R_0 + l.o.t, \]

\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + R_0^\frac{1}{2} R_1^\frac{1}{2} + l.o.t, \]

**Remark:** If we don’t have anomalies in (4.12) then by Lemma 4.2 we can actually prove,

\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + \frac{\delta^+}{|u|^2} \mathcal{O}_{0,\infty} \mathcal{O}_{0,2} + \frac{1}{|u|^2} R_0. \] (4.15)

\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + \frac{\delta^+}{|u|^2} \mathcal{O}_{0,\infty} \mathcal{O}_{0,4} + \frac{1}{|u|^2} R_0^\frac{1}{2} R_1^\frac{1}{2} + \frac{\delta^+}{|u|^2} R_0. \] (4.16)

For convenience, we will rewrite these inequalities as

\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + \frac{1}{|u|^2} R_0 + l.o.t, \]

\[ \| \psi(s,s') \|_{L^2_c(S(u,u))} \leq \| \psi(s,s') \|_{L^2_c(S(u,0))} + \frac{1}{|u|^2} R_0^\frac{1}{2} R_1^\frac{1}{2} + l.o.t. \]

Thus if we don’t have anomalies, the nonlinear terms are actually lower order terms. Sometimes, for convenience, we will write

\[ \nabla_4 \psi(s,s') = \Psi(s+1,s') + l.o.t \]
\[ \nabla_3 \psi(s,s') = \Psi(s+1,s') + l.o.t. \]

For future reference, we will encounter some transport equations of following form,

\[ \nabla_3 \psi(s,s') = -\lambda_0 tr \chi \psi(s,s') + \sum_{s_1+\cdots+s_7 = s} \psi(s_1,s'_1) \psi(s_2,s'_2) + \psi(s,s+1). \] (4.17)

Here \( \lambda_0 \) is a positive constant that depends on the transport equation being considered.

For \( \mathcal{O}_{0,2}(u,u) \) estimates, if we use Lemma 4.2 we will have,
\[ \| \psi(s', s) \|_{L^2(S(u, \omega))} \leq \| \psi(s', s) \|_{L^2(S(u_\infty, \omega))} + \int_{u_\infty}^{u} \frac{\lambda^0 \delta^+}{|u'|^3} \| \nabla \psi \|_{L^2(S(u', \omega))} \| \psi(s', s') \|_{L^2(S(u', \omega))} du' \]
\[ + \int_{u_\infty}^{u} \frac{\delta^+}{|u'|^3} \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \| \psi(s_1, s_1') \|_{L^2(S(u', \omega))} \| \psi(s_2, s_2') \|_{L^2(S(u', \omega))} du' \]
\[ + \frac{1}{|u|^2} \| \psi(s, s'+1) \|_{L^2(S(u_\infty, \omega))} \]

(4.18)

From bootstrap assumptions in Section 2, we know \( \nabla \psi \) is the most anomalous term. We only can hope
\[ \frac{\delta^+}{|u|^2} \| \nabla \psi \|_{L^2(S(u, \omega))} \leq (S) O_{0, \infty}. \]

Thus, it’s very likely that the term \[ \int_{u_\infty}^{u} \frac{\lambda^0 \delta^+}{|u'|^3} \| \nabla \psi \|_{L^2(S(u', \omega))} \| \psi(s', s') \|_{L^2(S(u', \omega))} du' \]

will give us a \( \log |u_\infty| \) term, which will destroy our proof. Similar situations will happen for \( (S)O_{0, \infty}(u, \omega) \) estimates. Thus, in equation
\[ \nabla \psi(s', s') = -\lambda^0 \nabla \psi(s', s') + \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \psi(s_1, s_1'), \psi(s_2, s_2') + \psi(s, s'+1) \]

\( \lambda^0 \nabla \psi(s', s') \) is not a lower order term any more. It is actually a borderline term. To deal with these terms, we will use the Evolution Lemma and the following lemma.

**Lemma 4.3.** For equation 4.17, we have,
\[ |u|^2 \lambda^{0-1} \| \psi(s', s') \|_{L^2(S(u, \omega))} \leq |u_\infty|^2 \lambda^{0-1} \| \psi(s', s') \|_{L^2(S(u_\infty, \omega))} + \int_{u_\infty}^{u} |u'|^2 \lambda^{0-1} \| \psi(s, s'+1) \|_{L^2(S(u', \omega))} du' \]
\[ + \int_{u_\infty}^{u} |u'|^{2 \lambda^{0-1}} \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \| \psi(s_1, s_1'), \psi(s_2, s_2') \|_{L^2(S(u', \omega))} du' \]

(4.19)

\[ |u|^2 \lambda^{0-\frac{1}{2}} \| \psi(s', s') \|_{L^4(S(u, \omega))} \leq |u_\infty|^2 \lambda^{0-\frac{1}{2}} \| \psi(s', s') \|_{L^4(S(u_\infty, \omega))} + \int_{u_\infty}^{u} |u'|^2 \lambda^{0-\frac{1}{2}} \| \psi(s, s'+1) \|_{L^4(S(u', \omega))} du' \]
\[ + \int_{u_\infty}^{u} |u'|^{2 \lambda^{0-\frac{1}{2}}} \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \| \psi(s_1, s_1'), \psi(s_2, s_2') \|_{L^4(S(u', \omega))} du' \]

(4.20)

**Proof.**
Using the Evolution Lemma, we have,
\[ p = 2, \lambda_1 = 2 \lambda_0 - 1. \]
\[ |u|^{2\lambda_0 - 1} \| \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} \leq |u|^{2\lambda_0 - 1} \| \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} + \int_{u, \mathfrak{u}} |u'|^{2\lambda_0 - 1} \| \nabla_3 \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} \| du' \]
\[ \leq |u|^{2\lambda_0 - 1} \| \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} + \int_{u, \mathfrak{u}} |u'|^{2\lambda_0 - 1} \| \psi(s, s' + 1) \|_{L^2(S(u, \mathfrak{u}))} \| du' \]
\[ + \int_{u, \mathfrak{u}} |u'|^{2\lambda_0 - 1} \sum_{s' + 1 + 2 \equiv s_1 + s_2} \| \psi(s_1, s_1') \cdot \psi(s_2, s_2') \|_{L^2(S(u, \mathfrak{u}))} \| du' \]

(4.21)

Similarly, we have,
\[ p = 4, \lambda_1 = 2\lambda_0 - \frac{1}{2} \]
\[ |u|^{2\lambda_0 - \frac{1}{2}} \| \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} \leq |u|^{2\lambda_0 - \frac{1}{2}} \| \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} + \int_{u, \mathfrak{u}} |u'|^{2\lambda_0 - \frac{1}{2}} \| \nabla_3 \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} \| du' \]
\[ \leq |u|^{2\lambda_0 - \frac{1}{2}} \| \psi(s, s') \|_{L^2(S(u, \mathfrak{u}))} + \int_{u, \mathfrak{u}} |u'|^{2\lambda_0 - \frac{1}{2}} \| \psi(s, s' + 1) \|_{L^2(S(u, \mathfrak{u}))} \| du' \]
\[ + \int_{u, \mathfrak{u}} |u'|^{2\lambda_0 - \frac{1}{2}} \sum_{s' + 1 + 2 \equiv s_1 + s_2} \| \psi(s_1, s_1') \cdot \psi(s_2, s_2') \|_{L^2(S(u, \mathfrak{u}))} \| du' \]

(4.22)

In such a way, we can deal with the borderline terms and avoid the dangerous term \( \log |u| \).

**4.2 \((S)O_{0,2}(u, \mathfrak{u})\) Estimates.**

We will prove our results using a bootstrap argument. We start by assuming that there exists a sufficiently large constant \( \Delta_0 \) such that,

\[ \text{(S)}O_{0,\infty} \leq \Delta_0. \]

Let’s start with the \((S)O_{0,2}(u, \mathfrak{u})\) estimates.

**4.2.1. Estimates for \( \chi, \eta, \mathfrak{u} \).**

The null Ricci coefficients \( \chi, \eta \) and \( \mathfrak{u} \) satisfy transport equations of the form,

\[ \nabla_4 \psi = \psi \cdot \psi + \Psi \]

Here \( \psi \) denotes an arbitrary Ricci coefficient component while \( \Psi \) denotes a null curvature component.

More precisely, we have Null Structure equations which are

\[ \nabla_4 \eta = -\chi \cdot (\eta - \mathfrak{u}) - \beta, \]

(4.23)

\[ \nabla_4 \mathfrak{u} = 2\omega \mathfrak{u} + \frac{1}{2} \rho + (\eta, \eta)(\eta, \eta), \]

(4.24)

\[ \nabla_4 \chi + tr \chi \chi = -2\omega \chi - \alpha, \]

(4.25)

\[ \nabla_4 tr \chi = \frac{1}{2} (tr \chi)^2 - |\chi|^2 - 2\omega tr \chi. \]

(4.26)

Using Lemma 4.1 and the bootstrap assumption, we can prove,

\[ \| \eta \|_{L^2(S(u, \mathfrak{u}))} \leq \| \eta \|_{L^2(S(u, 0))} + R_0[\beta] + \delta \mathcal{O}_{0,\infty} \text{O}_{0,2} \]

(4.27)
\[ \|\omega\|_{L^2_r(S(\mu, \omega))} \leq \|\omega\|_{L^2_r(S(\mu, 0))} + \mathcal{R}_0[\rho] + \delta^2 \alpha_{0,\infty} \mathcal{O}_{0,\infty}, \]  
(4.28)

\[ \delta^2 \|\hat{\chi}\|_{L^2_r(S(\mu, \omega))} \leq \delta^2 \|\hat{\chi}\|_{L^2_r(S(\mu, 0))} + \mathcal{R}_0[\alpha] + \delta^2 \alpha_{0,\infty} \mathcal{O}_{0,\infty}, \]  
(4.29)

\[ \delta^2 \|\operatorname{tr}\chi\|_{L^2_r(S(\mu, \omega))} \leq \delta^2 \|\operatorname{tr}\chi\|_{L^2_r(S(\mu, 0))} + \delta^2 \alpha_{0,\infty} \mathcal{O}_{0,\infty}. \]  
(4.30)

### 4.2.2. Estimates for $\omega, \eta, \chi$

The Ricci coefficients $\omega, \eta$ and $\hat{\chi}$ satisfy equations of the form,

\[ \nabla_3 \psi = \psi \cdot \psi + \Psi. \]

For $\omega$, we have,

\[ \nabla_3 \omega = 2\omega \omega + \frac{1}{2} \rho + (\eta, \eta)(\eta, \eta). \]  
(4.31)

Using Lemma 4.2 and the bootstrap assumptions, we have,

\[ \|\omega\|_{L^2_r(S(\mu, \omega))} \leq \|\omega\|_{L^2_r(S(\mu, \omega))} + \mathcal{R}_0[\rho] + \delta^2 \alpha_{0,\infty} \mathcal{O}_{0,\infty}. \]  
(4.32)

For $\eta$, we have,

\[ \nabla_3 \eta = -\hat{\chi} \cdot (\eta - \eta) + \beta. \]  
(4.33)

This is equivalent to

\[ \nabla_3 \eta = -\frac{1}{2} \operatorname{tr}\chi \eta + \frac{1}{2} \operatorname{tr}\chi - \hat{\chi} \cdot (\eta - \eta) + \beta. \]  
(4.34)

Using Lemma 4.3, we have

\[ \|\eta\|_{L^2_r(S(\mu, \omega))} \leq \|\eta\|_{L^2_r(S(\mu, \omega))} + \int_{u_\infty}^u \frac{1}{2} \operatorname{tr}\chi \eta + \beta - \hat{\chi} \cdot (\eta - \eta) \|L^2_r(S(u, \omega)) \rho' \| \, du' \]

\[ \leq \|\eta\|_{L^2_r(S(\mu, \omega))} + \int_{u_\infty}^u \frac{1}{2} \|\operatorname{tr}\chi\|_{L^2_r(S(u, \omega))} \|\eta\|_{L^2_r(S(u, \omega))} \rho' \| \, du' + \frac{1}{|u|^2} \|u' \|^{2} \| \beta \|_{L^2_r(H^{-\infty, \infty})} + \text{l.o.t} \]

\[ \leq \|\eta\|_{L^2_r(S(\mu, \omega))} + \int_{u_\infty}^u \frac{1}{|u|^2} \|\operatorname{tr}\chi\|_{L^2_r(S(u, \omega))} \rho' \| \, du' + \frac{1}{|u|^2} \|u' \|^{2} \| \beta \|_{L^2_r(H^{-\infty, \infty})} + \text{l.o.t} \]

\[ \leq \|\eta\|_{L^2_r(S(\mu, \omega))} + \frac{c_0}{|u|^2} \|\Delta(\eta)\| + \frac{1}{|u|^2} \|u' \|^{2} \| \beta \|_{L^2_r(H^{-\infty, \infty})} + \text{l.o.t}. \]  
(4.35)

Since we have already estimated $\eta, \Delta(\eta)$ is now a constant depending only on $\mathcal{O}^{(0)}$ and $\mathcal{R}$. Thus, we obtain,

\[ \delta^{-\frac{1}{2}} \|u\|_{L^2_r(S(\mu, \omega))} \leq \frac{|u|}{|u_\infty|} \delta^{-\frac{1}{2}} \|u_\infty\| (\|\eta\|_{L^2_r(S(\mu, \omega))} + \Delta(\eta)) + \delta^{-\frac{1}{2}} \frac{|u|}{|u|^2} \|u' \|^{2} \| \beta \|_{L^2_r(H^{-\infty, \infty})} + \text{l.o.t}. \]  
(4.36)

\[ \delta^{-\frac{1}{2}} \|u\|_{L^2_r(S(\mu, \omega))} \leq \delta^{-\frac{1}{2}} \|u_\infty\| (\|\eta\|_{L^2_r(S(\mu, \omega))} + \Delta(\eta)) + \delta^{-\frac{1}{2}} \frac{|u|}{|u|^2} \|u' \|^{2} \| \beta \|_{L^2_r(H^{-\infty, \infty})} + \text{l.o.t}. \]  
(4.37)
which is equivalent to
\[
\|\eta\|_{L^4_x(S(u, \omega))} \leq \|\eta\|_{L^4_x(S(u, \omega))} + \Delta_1 \leq \frac{l.o.t.}{\|u\|^2} \|\beta\| + l.o.t.
\] (4.38)

Similarly, from
\[
\nabla_3 \hat{\chi} + \nabla_4 \chi = -2\omega \hat{\chi} - \alpha.
\] (4.39)
\[
\nabla_3 \nabla_4 \chi + \frac{1}{2} \nabla_4 \chi = -2\omega \nabla_3 \hat{\chi} - |\hat{\chi}|^2;
\] (4.40)
we can prove,
\[
\frac{1}{\|u\|^2} \|\hat{\chi}\|_{L^4_x(S(u, \omega))} \leq \frac{1}{\|u\|^2} \|\hat{\chi}\|_{L^4_x(S(u, \omega))} + \frac{\alpha}{\|u\|^2} + l.o.t.
\] (4.41)
\[
\frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, \omega))} \leq \frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, \omega))} + l.o.t.
\] (4.42)

We summarize the results of this subsection in the following proposition,

**Proposition 4.4.** Assuming that \(\delta \Delta_0\) is sufficiently small, there exists a constant \(C\) depending only on \(O(0)\), \(R_0\), and \(B_0\), such that,
\[
(S)\ O_{0.2} \leq C
\]

4.3 \(S)\ O_{0.4}(u, \omega)\ Estimates.

4.3.1 Estimates for \(\chi, \eta, \omega\).

Similarly, as in the \(S)\ O_{0.2}(u, \omega)\ estimates, using Lemma 4.3 and the bootstrap assumption, we can prove,
\[
\|\eta\|_{L^4_x(S(u, \omega))} \leq \|\eta\|_{L^4_x(S(u, 0))} + \|\nabla_4 \chi\|_{L^4_x(S(u, 0))} + \|\nabla_4 \beta\|_{L^4_x(S(u, 0))} + \Delta_1 \leq \|\eta\|_{L^4_x(S(u, 0))} + R_1 + \frac{\alpha}{\|u\|^2} + \frac{\alpha}{\|u\|^2} + l.o.t.
\] (4.43)
\[
\frac{\delta}{\|u\|^2} \|\hat{\chi}\|_{L^4_x(S(u, 0))} \leq \frac{\delta}{\|u\|^2} \|\hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \beta\|_{L^4_x(S(u, 0))} + \Delta_1 \leq \frac{\delta}{\|u\|^2} \|\hat{\chi}\|_{L^4_x(S(u, 0))} + \frac{\alpha}{\|u\|^2} + \frac{\alpha}{\|u\|^2} + l.o.t.
\] (4.44)
\[
\frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} \leq \frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \beta\|_{L^4_x(S(u, 0))} + \Delta_1 \leq \frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \frac{\alpha}{\|u\|^2} + \frac{\alpha}{\|u\|^2} + l.o.t.
\] (4.45)
\[
\frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} \leq \frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \beta\|_{L^4_x(S(u, 0))} + \Delta_1 \leq \frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \frac{\alpha}{\|u\|^2} + \frac{\alpha}{\|u\|^2} + l.o.t.
\] (4.46)
\[
\frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} \leq \frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \|\nabla_4 \beta\|_{L^4_x(S(u, 0))} + \Delta_1 \leq \frac{\delta}{\|u\|^2} \|\nabla_4 \hat{\chi}\|_{L^4_x(S(u, 0))} + \frac{\alpha}{\|u\|^2} + \frac{\alpha}{\|u\|^2} + l.o.t.
\] (4.47)
where $c$ is a fixed positive constant.

**4.3.2. Estimates for $\omega, \eta, \chi$.**

For $\omega$, using Lemma 4.3 and the bootstrap assumption, we can prove,

$$\|\omega\|_{L^2_t(S(u, w))} \leq \|\omega\|_{L^2_t(S(u, w))} + \|\nabla \rho\|_{L^2_t(H^{2(\infty, \infty), w})} + \|\rho\|_{L^2_t(H^{2(\infty, \infty), w})} + \delta O_0, 0.4, 0.4,$$

(4.48)

For $\eta, \chi$ and $tr \chi$, as in the $(S)O_{0.2}(u, w)$ estimates, we can use Lemma 4.3 and the bootstrap assumption to prove,

$$\|\eta\|_{L^2_t(S(u, w))} \leq \|\eta\|_{L^2_t(S(u, w))} + \|\nabla \beta\|_{L^2_t(H^{2(\infty, \infty), w})} + \|\beta\|_{L^2_t(H^{2(\infty, \infty), w})} + \delta \|\beta\|_{L^2_t(H^{2(\infty, \infty), w})} + \delta^c O_{0.0, 0.4},$$

(4.49)

$$\frac{\delta^4}{|u|^2} \|\chi\|_{L^2_t(S(u, w))} \leq \frac{\delta^4}{|u|^2} \|\chi\|_{L^2_t(S(u, w))} + \frac{\delta^4}{|u|^2} \|\alpha\|_{L^2_t(H^{2(\infty, \infty), w})} + \delta \|\alpha\|_{L^2_t(H^{2(\infty, \infty), w})} + \delta^c O_{0.0, 0.4},$$

(4.50)

where $c$ is a fixed positive constant.

We summarize the results of this subsection in the following proposition,

**Proposition 4.5.** Assuming that $\delta^c \Delta_0$ is sufficiently small, there exists a constant $C$ depending only on $\mathcal{O}^{(0)}, \mathcal{R}_0$, and $\mathcal{R}_0$, such that,

$$(S)O_{0.4} \leq C.$$

## 5 $(S)O_{1.2}(u, w)$ Estimates.

### 5.1 $L^2_{sc}(S)$ Estimates for $\alpha, \beta, \rho, \sigma, \beta, \alpha$

In this subsection, we prove $L^2_{sc}(S)$ estimates for the curvature components.

For $\beta, \rho, \sigma, \alpha$, we use,

$$\nabla_4 \beta + 2 tr \chi \beta = div \alpha - 2 \omega \beta + \eta \cdot \alpha,$$

(5.1)

$$\nabla_4 \sigma + \frac{3}{2} tr \chi \sigma = -div^* \beta + \frac{1}{2} \chi \cdot \alpha - \zeta \cdot \beta - 2 \eta \cdot \beta,$$

(5.2)

$$\nabla_4 \rho + \frac{3}{2} tr \chi \rho = div \beta - \frac{1}{2} \chi \cdot \alpha + \zeta \cdot \beta + 2 \eta \cdot \beta,$$

(5.3)

$$\nabla_4 \beta + tr \chi \beta = -\nabla \rho + ^* \nabla \sigma + 2 \omega \beta + 2 \chi \cdot \beta - 3 (\eta \rho - ^* \eta \sigma),$$

(5.4)
\[
\n\nabla_4 \alpha + \frac{1}{2} \operatorname{tr} \chi \alpha = -\nabla \hat{\otimes} \beta + 4 \omega \alpha - 3(\hat{\chi} \rho + \hat{\chi} \sigma) + (\zeta - 4 \eta) \hat{\otimes} \beta. \tag{5.5}
\]

Using the integral formula, if \( \delta \hat{\chi} \Delta_0 \) is sufficiently small then there exists a constant \( C \) depending on \( O(0) \), \( R \), and \( R \) such that,

\[
\| \beta \|_{L_2^2(S(u, \omega))] \leq C, \tag{5.6}
\]
\[
\| \rho \|_{L_2^2(S(u, \omega))] \leq C, \tag{5.7}
\]
\[
\| \sigma \|_{L_2^2(S(u, \omega))] \leq C, \tag{5.8}
\]
\[
\| \beta \|_{L_2^2(S(u, \omega))] \leq C, \tag{5.9}
\]
\[
\| \alpha \|_{L_2^2(S(u, \omega))] \leq C. \tag{5.10}
\]

For \( \alpha \) we use,

\[
\nabla_4 \alpha + \frac{1}{2} \operatorname{tr} \chi \alpha = \nabla \hat{\otimes} \beta + 4 \omega \alpha - 3(\hat{\chi} \rho + \hat{\chi} \sigma) + (\zeta + 4 \eta) \hat{\otimes} \beta, \tag{5.11}
\]

Using the Evolution Lemma, we get,

\[
\delta \hat{\chi} \| \alpha \|_{L_2^2(S(u, \omega))] \leq C. \tag{5.12}
\]

Later on, we will need to use the estimates for Hodge systems. To this end recall,

\[
K = -\rho + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \frac{1}{4} \operatorname{tr} \chi \operatorname{tr} \chi. \tag{5.13}
\]

Hence, we have,

\[
\frac{1}{|u|} \| K \|_{L_2^2(S(u, \omega))] \leq C. \tag{5.14}
\]

We summarize the results of this subsection in the following proposition:

**Proposition 5.1.** Assuming that \( \delta \hat{\chi} \Delta_0 \) is sufficiently small, there exist a constant \( C \), depending only on \( O(0) \), \( R \), and \( R \) such that,

\[
\| \beta, \rho, \sigma, \beta, \alpha \|_{L_2^2(S(u, \omega))] \leq C, \quad \delta \hat{\chi} \| \alpha \|_{L_2^2(S(u, \omega))] \leq C, \quad \frac{1}{|u|} \| K \|_{L_2^2(S(u, \omega))] \leq C.
\]

### 5.2 Estimates for \( \operatorname{tr} \chi, \hat{\chi} \)

For \( \operatorname{tr} \chi \) and \( \hat{\chi} \), we use

\[
\nabla_4 \operatorname{tr} \chi = -\frac{1}{2} (\operatorname{tr} \chi)^2 - |\hat{\chi}|^2 - 2 \omega \operatorname{tr} \chi, \tag{5.15}
\]
\[
[\nabla_4, \nabla] f = \frac{1}{2} (\eta + \omega) \nabla_4 f - \chi \cdot \nabla f, \tag{5.16}
\]

to get,
\( \nabla_4 \nabla \text{tr} \chi = -\nabla \text{tr} \chi \nabla \omega - 2\omega \nabla \text{tr} \chi - 2\nabla \chi \cdot \text{tr} \chi + \frac{1}{2} (\eta + \eta)(-\frac{1}{2} (\text{tr} \chi)^2 - |\hat{\chi}|^2 - 2\omega \text{tr} \chi) - \chi \nabla \text{tr} \chi. \)  
(5.17)

Note that there aren’t any anomalous terms for the \( u \) weight on the right hand side.

Using Lemma 4.1, we get,

\[
|u| \| \nabla \text{tr} \chi \|_{L^2_c(S(u, \omega))} \leq \mathcal{O}^{(0)} + \delta^c \mathcal{O}_{1, 2} \mathcal{O}_{0, \infty} + \delta^c \mathcal{O}_{1, 2} \mathcal{O}_{0, \infty} \mathcal{O}_{0, \infty}. \quad (5.18)
\]

Using \( \text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi + \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \text{tr} \chi) - \beta, \)  
(5.19)

and using the estimates for Hodge systems we have,

\[
\| \nabla \chi \|_{L^2_c(S(u, \omega))} \leq \mathcal{R}_1 + \mathcal{R}_0 + \mathcal{O}^{(0)} + \delta^c \mathcal{O}_{1, 2} \mathcal{O}_{0, \infty} + \delta^c \mathcal{O}_{1, 2} \mathcal{O}_{0, \infty} \mathcal{O}_{0, \infty}. \quad (5.20)
\]

### 5.3 Estimates for \( \text{tr} \chi \), \( \hat{\chi} \)

For \( \text{tr} \chi \) and \( \hat{\chi} \), similarly, we derive,

\[
\nabla_4 \nabla \text{tr} \chi + \frac{3}{2} \text{tr} \chi \nabla \text{tr} \chi = -\hat{\chi} \cdot \nabla \text{tr} \chi - (\nabla + \frac{1}{2} (\eta + \eta)) (2\omega \text{tr} \chi + |\hat{\chi}|^2) - \frac{1}{4} (\eta + \eta)(\text{tr} \chi)^2. \quad (5.21)
\]

From Lemma 4.3, we get,

\[
\| \nabla \text{tr} \chi \|_{L^2_c(S(u, \omega))} \leq \| \nabla \omega \|_{L^2_c(S(u, \omega))} + \mathcal{O}^{(0)} + \mathcal{R} + \text{l.o.t.} \quad (5.22)
\]

**Remark:** The most anomalous nonlinear term on the right hand side is \( (\eta + \eta)(\text{tr} \chi)^2 \).

To deal with this term, we notice that \( \eta + \eta = 2 \nabla (\log \Omega) \). To get an estimate for \( \nabla (\log \Omega) \), we use \( \omega = -\frac{1}{4} \nabla (\log \Omega) \). From this we get

\[
\nabla_4 \nabla \log \Omega = -2\nabla \omega + \frac{1}{2} (\eta + \eta) \nabla \log \Omega - \hat{\chi} \nabla \log \Omega.
\]

That’s the reason why we have \( \nabla \omega \) on the right hand side when we try to estimate \( \nabla \text{tr} \chi \).

Using,

\[
\text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi + \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \text{tr} \chi) + \beta, \quad (5.23)
\]

and using the estimates for Hodge systems we have,

\[
\frac{1}{|u|} \| \nabla \chi \|_{L^2_c(S(u, \omega))} \leq \| \nabla \omega \|_{L^2_c(S(u, \omega))} + \mathcal{O}^{(0)} + \mathcal{R} + \text{l.o.t.} \quad (5.24)
\]

### 5.4 Estimates for \( \eta \)

For \( \eta \), we define \( \mu \) as,

\[
\mu = -\text{div} \eta - \rho. \quad (5.25)
\]

We derive, (see [K-R:Trapped].)

\[
\nabla_4 \mu + \text{tr} \chi \mu = -\frac{1}{2} \text{tr} \chi \text{div} \eta + (\eta - \eta) \nabla \text{tr} \chi + \hat{\chi} \cdot \nabla (2\eta - \eta) + \frac{1}{2} \hat{\chi} \cdot \alpha - (\eta - 3\eta) \cdot \beta + \frac{1}{2} \text{tr} \chi \rho + \frac{1}{2} \text{tr} \chi (|\eta|^2 - \eta \cdot \eta) + \frac{1}{2} (\eta + \eta) \cdot \hat{\chi} \cdot (\eta - \eta). \quad (5.26)
\]
From Lemma 4.1, we have,
\[
\|\mu\|_{L^2_{sc}(S(u,u))} \leq \delta R_1[\alpha] + O^{(0)} + \text{l.o.t.} \tag{5.27}
\]
Applying the estimates for Hodge systems to
\[
\text{div}\eta = -\mu - \rho, \quad \text{curl}\eta = \sigma + \frac{1}{2} \hat{\chi} \wedge \hat{\chi}, \tag{5.28}
\]
we get,
\[
\|\nabla\eta\|_{L^2_{sc}(S(u,u))} \leq R_0[\rho,\sigma] + \delta \frac{1}{4} R_1[\alpha] + O^{(0)} + \text{l.o.t.} \tag{5.29}
\]

5.5 Estimates for $\eta$
For $\eta$, we define $\mu$ as,
\[
\mu = -\text{div}\eta - \rho. \tag{5.30}
\]
We derive,
\[
\nabla^3\mu + \text{tr}\chi\mu = -\frac{1}{2} \text{tr}\chi\text{div}\eta + \frac{1}{2} \text{tr}\chi\rho + (\eta - \hat{\eta})\nabla\text{tr}\chi + \frac{1}{2} \nabla(2\eta - \eta)
\]
\[
+ \frac{1}{2} \hat{\chi} \cdot \hat{\omega} - (\eta - 3\eta) \cdot \beta + \frac{1}{2} \text{tr}\chi(|\eta|^2 - \eta \cdot \eta) + \frac{1}{2} (\eta + \eta) \cdot \hat{\chi} \cdot (\eta - \eta). \tag{5.31}
\]
For equation 5.31, the terms having the worst decay rate are $-\frac{1}{2} \text{tr}\chi\text{div}\eta$ and $\frac{1}{2} \text{tr}\chi\rho$.
Thus, the equation can be rewritten as
\[
\nabla^3\mu + \text{tr}\chi\mu = -\frac{1}{2} \text{tr}\chi\text{div}\eta + \frac{1}{2} \text{tr}\chi\rho + \text{l.o.t.}
\]
We already have an estimate for $\nabla\eta$. Using Lemma 4.3, we have,
\[
\|\nabla\eta\|_{L^2_{sc}(S(u,u))} \leq R_1 + O^{(0)} + \text{l.o.t.} \tag{5.32}
\]
Applying the estimates for Hodge systems to
\[
\text{div}\eta = -\mu - \rho, \quad \text{curl}\eta = -\sigma + \frac{1}{2} \hat{\chi} \wedge \hat{\chi}, \tag{5.33}
\]
we get,
\[
\|\nabla\eta\|_{L^2_{sc}(S(u,u))} \leq R_1 + O^{(0)} + \text{l.o.t.} \tag{5.34}
\]

5.6 Estimates for $\omega$
We introduce the auxiliary quantities $\omega^\dagger$ and $\omega$ as follows,
\[
\nabla_4 \omega^\dagger = \frac{1}{2} \sigma, \quad \nabla_3 \omega = \frac{1}{2} \sigma
\]
with zero boundary conditions along $H_1$ and $H_0$, respectively.
We introduce the scalar pairs $\langle \omega \rangle = (\omega, \omega^\dagger)$ and $\langle \omega \rangle = (-\omega, \omega^\dagger)$.
Applying the Hodge operator $^*D_1$, we have,
\[
^*D_1 \langle \omega \rangle = -\nabla \omega + ^* \nabla \omega^\dagger, \quad ^*D_1 \langle \omega \rangle = \nabla \omega + ^* \nabla \omega^\dagger.
\]
For $\omega$, we have,
and we define $\kappa$ as,

$$\kappa = * D_1 < \omega > - \frac{1}{2} \beta.$$  

(5.36)

From this, we derive,

$$\nabla_3 \kappa + \text{tr} \chi \kappa = - \omega \beta - \hat{\chi} \cdot \beta + \frac{3}{2} (\eta \rho - * \eta \rho) - \frac{1}{2} (\eta + \eta) \rho$$

$$+ \frac{1}{2}(\eta^* + \eta^*) \sigma + \hat{\chi} \cdot (\nabla \omega + * \nabla \omega^\dagger) - \nabla F - \frac{1}{2} (\eta + \eta) F$$

(5.37)

where $F = 2 \omega \omega + (\eta, \eta)(\eta, \eta)$.

The only anomalous term on the right hand side is $\hat{\chi}$. Hence, we have

$$\nabla_3 \kappa = - \hat{\chi} \cdot \beta + \text{l.o.t.}$$

Using Lemma 4.1, we know,

$$\| \kappa \|_{L^2_{sc}(S(u, \omega))} \leq R_1 + O(0) + \text{l.o.t.}$$  

(5.38)

Using the estimates for Hodge systems, we have,

$$\| \nabla \omega \|_{L^2_{sc}(S(u, \omega))} \leq R_1 + O(0) + \text{l.o.t.}$$  

(5.39)

### 5.7 Estimates for $\omega$

For $\omega$, we have,

$$* D_1 < \omega > = \nabla \omega + * \nabla \omega^\dagger,$$  

(5.40)

we define $\kappa$ as,

$$\kappa = * D_1 < \omega > - \frac{1}{2} \beta.$$  

(5.41)

This allows us to derive the transport equation for $\kappa$ which is

$$\nabla_3 \kappa + \text{tr} \chi \kappa = \omega \beta - \hat{\chi} \cdot \beta + \frac{3}{2} (\eta \rho + * \eta \rho) - \frac{1}{2} (\eta + \eta) \rho$$

$$+ \frac{1}{2}(\eta^* + \eta^*) \sigma + \hat{\chi} \cdot (\nabla \omega + * \nabla \omega^\dagger) + \nabla F + \frac{1}{2} (\eta + \eta) F,$$  

(5.42)

where, $F = 2 \omega \omega + (\eta, \eta)(\eta, \eta)$.

For equation 5.42, the only borderline term is $\text{tr} \chi \kappa$. Thus, all the terms on the right hand side can be treated as lower order terms.

Thus, we have

$$\nabla_3 \kappa + \text{tr} \chi \kappa = \text{l.o.t.}$$

Using Lemma 4.3, we have,

$$\| \kappa \|_{L^2_{sc}(S(u, \omega))} \leq R_0 + O(0) + \text{l.o.t.}$$  

(5.43)

Using the estimates for Hodge systems, we have,

$$\| \nabla \omega \|_{L^2_{sc}(S(u, \omega))} \leq R_0 + R_1 + O(0) + \text{l.o.t.}$$  

(5.44)

We summarize the results of this subsection in the following proposition:
Proposition 5.2. Assuming $\delta \Delta_0$ is sufficiently small, there exists a constant $C$, depending only on $O^{(0)}$, $R$, and $\mathcal{R}$, such that
\[(S)O_{1,2} \leq C.\]

6 Second Angular Derivative Estimates for the Ricci Coefficients and $(S)O_{1,4}(u, u)$, $(S)O_{0,\infty}(u, u)$ Estimates.

6.1 Second Angular Derivative Estimates for the Ricci Coefficients.

In this subsection, we first derive the evolution equation for $\nabla^2 tr\chi$, $\nabla^2 tr\chi$, $\nabla \mu$, $\nabla \kappa$, and $\nabla \kappa$. Then, similar to how we proceeded in the previous section, we will obtain estimates for these quantities. Considering Hölder’s inequality for the scale invariant norm, and the anomalous terms for both the $\delta$ and $u$ weights, we only list the potentially worst terms. The rest can be treated as lower order terms.

We define:
\[\epsilon' = \frac{1}{4}.\] (6.1)

For $\nabla^2 tr\chi$, we have (see also [K-R:Trapped].)
\[\nabla_3 \nabla^2 tr\chi + 2 tr\chi \nabla^2 tr\chi = \nabla^2 \omega tr\chi + \nabla^2 \hat{\chi} + l.o.t.\] (6.2)

Later, we will show that if $\delta$ is sufficient small,
\[\|\nabla^2 \omega\|_{L^2(S(u,u))} \leq \frac{C}{|u|^4} + \|\nabla \beta\|_{L^2(S(u,u))}.\] (6.3)

Remark: In this subsection, the constant $C$ depends only on $O^{(0)}$, $R$, and $\mathcal{R}$.

From the Hodge system, for $\hat{\chi}$ in 5.23, we know that,
\[\nabla^2 \hat{\chi} = \nabla^2 tr\chi + \nabla (\eta, \eta) tr\chi + \nabla \beta + l.o.t.\]

Using Lemma 4.3, we get,
\[\|\nabla^2 tr\chi\|_{L^2(S(u,u))} \leq \frac{C}{|u|^{4-2\epsilon'}} = \frac{C}{|u|^2}.\] (6.4)

Similarly, for $\nabla^2 tr\chi$, $\nabla \mu$, $\nabla \kappa$, and $\nabla \kappa$, we have,
\[\|\nabla^2 tr\chi\|_{L^2(S(u,u))} \leq \frac{\delta^{-1}C}{|u|^3}.\] (6.5)
\[\|\nabla \mu\|_{L^2(S(u,u))} \leq \frac{\delta^{-\frac{1}{2}}C}{|u|^{\frac{3}{2}}}.\] (6.6)
\[\|\nabla \kappa\|_{L^2(S(u,u))} \leq \frac{\delta^{-1}C}{|u|^4}.\] (6.7)
\[\|\nabla \kappa\|_{L^2(S(u,u))} \leq \frac{C}{|u|^2}.\] (6.8)

For $\nabla^2 \omega, \nabla^2 \hat{\chi}, \nabla^2 \hat{\eta}, \nabla^2 \eta, \nabla^2 \hat{\chi}$, using the estimates for Hodge systems, we obtain,
To get estimates for (6.2) Codimension 1 Trace Formulas.

**Lemma 6.1. Codimension 1 Trace Formulas.**

\[
\|\nabla^2u\|_{L^2(S(u,\varpi))} \lesssim \frac{C}{|u|^4} + \|\nabla\beta\|_{L^2(S(u,\varpi))}, \quad (6.10)
\]

\[
\|\nabla^2\omega\|_{L^2(S(u,\varpi))} \lesssim \frac{\delta^{-1}C}{|u|^2} + \|\nabla\beta\|_{L^2(S(u,\varpi))}, \quad (6.11)
\]

\[
\|\nabla^2\chi\|_{L^2(S(u,\varpi))} \lesssim \frac{\delta^{-1}C}{|u|^2} + \|\nabla\beta\|_{L^2(S(u,\varpi))}, \quad (6.12)
\]

\[
\|\nabla^2\|_{L^2(S(u,\varpi))} \lesssim \frac{\delta^{-\frac{5}{2}}C}{|u|^3-2\epsilon} + \|\nabla\rho\|_{L^2(S(u,\varpi))} + \|\nabla\sigma\|_{L^2(S(u,\varpi))}, \quad (6.13)
\]

\[
\|\nabla^2\eta\|_{L^2(S(u,\varpi))} \lesssim \frac{\delta^{-\frac{5}{2}}C}{|u|^3} + \|\nabla\rho\|_{L^2(S(u,\varpi))} + \|\nabla\sigma\|_{L^2(S(u,\varpi))}, \quad (6.14)
\]

\[
\|\nabla^2\chi\|_{L^2(S(u,\varpi))} \lesssim \frac{C}{|u|^4} + \|\nabla\beta\|_{L^2(S(u,\varpi))}, \quad (6.15)
\]

**Remark:** For \(\|\nabla^2\chi\|_{L^2(S(u,\varpi))}\), \(\frac{C}{|u|^4}\) is the main term; for \(\|\nabla^2\|_{L^2(S(u,\varpi))}\), both \(\frac{C}{|u|^3-2\epsilon}\) and \(\|\nabla\rho\|_{L^2(S(u,\varpi))} + \|\nabla\sigma\|_{L^2(S(u,\varpi))}\) are the main terms; for the other terms, the derivatives of curvatures are the main terms.

### 6.2 Codimension 1 Trace Formulas.

To get estimates for \((S)\Omega_{0,4}\), we need the following lemma:

**Lemma 6.1. Codimension 1 Trace Formulas.**

\[
\|f\|_{L^4(S(u,\varpi))} \lesssim \|f - \bar{f}\|_{L^4(S(u,\varpi))} + (\|\nabla_3 f\|_{L^2(H)} + \|\frac{1}{|u|} f\|_{L^2(H)})\|\nabla f\|_{L^2(H)} + \frac{1}{|u|^2} \|f\|_{L^2(S)}, \quad (6.16)
\]

\[
\|f\|_{L^4(S(u,\varpi))} \lesssim \|f - \bar{f}\|_{L^4(S(u,0))} + (\|\nabla_4 f\|_{L^2(H)} + \|\frac{1}{|u|} f\|_{L^2(H)})\|\nabla f\|_{L^2(H)} + \frac{1}{|u|^2} \|f\|_{L^2(S)}, \quad (6.17)
\]

**Proof:**

Let’s prove \(6.16\) first.

We have,

\[
\sup_u \|f - \bar{f}\|_{L^4(S(u,\varpi))} \lesssim \int \int \left| \nabla_3 (f - \bar{f}) \right|^4 + \int \int |tr\chi(f - \bar{f})|^4 + \|f - \bar{f}\|_{L^4(S(u,\varpi))}^4, \quad (6.18)
\]

\[
\lesssim (\|\nabla_3 f\|_{L^2(H)} + \|\nabla_3 f\|_{L^2(H)})\|f - \bar{f}\|_{L^2(H)}^2 + (|tr\chi(f - \bar{f})|_{L^2(H)} + |tr\chi f|_{L^2(H)})\|f - \bar{f}\|_{L^2(H)}^2 + \|f - \bar{f}\|_{L^2(S)}^4, \quad (6.19)
\]

while, for \(\|f - \bar{f}\|_{L^2(H)}^2\), we have the estimate

\[
\|f - \bar{f}\|_{L^2(S)}^6 = \|f - \bar{f}\|_{L^2(S)}^2 \lesssim \|\nabla(f - \bar{f})\|_{L^2(S)}^2 \lesssim \left( \int_S (\nabla f)(f - \bar{f})^2 \right)^2 \leq \left( \int_S |\nabla f|^2 \right) \|f - \bar{f}\|_{L^2(S)}^4, \quad (6.20)
\]
Thus,
\[ \|f - \bar{f}\|_{L^4_u(S_u, u)}^4 \leq \left( \int \int_H |\nabla f|^2 \right)^{1/2} \sup_u \|f - \bar{f}\|_{L^4_u(S_u, u)} \]  \quad (6.21)

For \( \|\nabla_3 \bar{f}\|_{L^2(H)} \), we have the estimate,
\[ \|\nabla_3 \bar{f}\|_{L^2(H)} \approx \|\nabla_3 (\frac{1}{|u|^2} \int_S f)\|_{L^2(H)} \approx \| \frac{1}{|u|^2} \int_S f \|_{L^2(H)} + \| \frac{1}{|u|^2} \frac{d}{du} \int_S f \|_{L^2(H)} \]  \quad (6.22)

\[ \lesssim \| \frac{1}{|u|^2} \int_S f \|_{L^2(H)} + \| \frac{1}{|u|^2} \int_S (\nabla_3 f + tr X f) \|_{L^2(H)} \lesssim \|\nabla_3 f\|_{L^2(H)} + \| \frac{1}{|u|^2} f \|_{L^2(H)} \]  \quad (6.23)

Thus,
\[ \sup_u \|f - \bar{f}\|_{L^4_u(S_u, u)} \lesssim (\|\nabla_3 f\|_{L^2(H)} + \frac{1}{|u|} f) \|\nabla f\|_{L^2(H)} \sup_u \|f - \bar{f}\|_{L^4_u(S_u, u)} + \|f - \bar{f}\|_{L^4_u(S_u, u)} \]  \quad (6.24)

Therefore,
\[ \sup_u \|f - \bar{f}\|_{L^4_u(S_u, u)} \lesssim (\|\nabla_3 f\|_{L^2(H)} + \frac{1}{|u|} f) \|\nabla f\|_{L^2(H)} + \|f - \bar{f}\|_{L^4_u(S_u, u)} \] \quad (6.25)

Also,
\[ \|\bar{f}\|_{L^4(S)} \lesssim (\int S \bar{f}^4)^{1/4} = |\bar{f}| |u|^{1/4} \leq \frac{1}{|u|^2} \int_S |f| ds \cdot |u|^2 \leq \frac{\|f\|_{L^2(S)}}{|u|^2} \]  \quad (6.26)

Thus, we get,
\[ \|f\|_{L^4_u(S_u, u)} \lesssim \|f - \bar{f}\|_{L^4_u(S_u, u)} + (\|\nabla_3 f\|_{L^2(H)} + \frac{1}{|u|} f) \|\nabla f\|_{L^2(H)} + \|f - \bar{f}\|_{L^4_u(S_u, u)} \]  \quad (6.27)

Similarly, we obtain,
\[ \|f\|_{L^4_u(S_u, u)} \lesssim \|f - \bar{f}\|_{L^4_u(S_u, u)} + (\|\nabla_3 f\|_{L^2(H)} + \frac{1}{|u|} f) \|\nabla f\|_{L^2(H)} + \|f - \bar{f}\|_{L^4_u(S_u, u)} \]  \quad (6.28)

6.3 \((S)\mathcal{C}_{1,4}(u, u)\) Estimates and \((S)\mathcal{C}_{0,\infty}(u, u)\) Estimates.

Recall,
\[ \epsilon' = \frac{1}{4}. \]  \quad (6.29)

For \( \hat{\chi} \), we have,
\[ \|\nabla_4 \nabla \hat{\chi}\|_{L^2(H, u)} \leq \|\nabla \alpha\|_{L^2(H, u)} + l.o.t \leq \frac{\delta^{-1} C}{|u|}, \]  \quad (6.30)

\[ \|\nabla \nabla \hat{\chi}\|_{L^2(H, u)} \leq \|\nabla \beta\|_{L^2(H, u)} \leq \frac{\delta^{-\frac{1}{2}} C}{|u|^2}. \]  \quad (6.31)

From Lemma 5.1 ( Codimension 1 Trace Lemma ) and Lemma 2.4
\( \| F \|_{L^\infty(S(u,\omega))} \leq \| F \|_{L^1(S(u,\omega))}^{\frac{1}{2}} \| \nabla F \|_{L^4(S(u,\omega))}^{\frac{1}{2}} + \frac{1}{|u|^{\frac{1}{2}}} \| F \|_{L^4(S(u,\omega))}, \) \(6.32\)

Recall,
\[ \| \nabla \hat{\chi} \|_{L^4(S(u,\omega))} \leq \frac{\delta^{-\frac{1}{4}} C}{|u|^{\frac{1}{2}}}, \] \(6.33\)

and
\[ \| \hat{\chi} \|_{L^4(S(u,\omega))} \leq \frac{\delta^{-\frac{1}{4}} C}{|u|^{\frac{1}{2}}}, \] \(6.34\)

and
\[ \| \hat{\chi} \|_{L^4(S_{\delta}(u,\omega))} \leq \frac{\delta^{-\frac{1}{4}} C}{|u|^{\frac{1}{2}}}. \] \(6.35\)

Using, \(\text{see \cite{K-R:Trapped}}\),
\[ \| \psi \|_{L^\infty(S)} \leq \sup_{\delta S \subset S} (\delta^{\frac{1}{2}} |u|^{\frac{1}{2}} \| \nabla \psi \|_{L^4(S_{2\delta})} + \frac{\delta^{-\frac{1}{2}}}{|u|^{\frac{1}{2}}} \| \psi \|_{L^4(S_{2\delta})}), \]

we obtain,
\[ \| \hat{\chi} \|_{L^\infty(S(u,\omega))} \leq \frac{\delta^{-\frac{1}{2}} C}{|u|}, \]

which equals,
\[ \| \hat{\chi} \|_{L^\infty(S(u,\omega))} \leq C. \] \(6.36\)

For \(tr\chi\), we have,
\[ \| \nabla_4 \nabla tr\chi \|_{L^2(H^0(\omega),u)} \leq \frac{\delta^{-\frac{1}{2}} C}{|u|^2}, \] \(6.37\)

From Lemma 5.1 \(\text{ (Codimension 1 Trace Lemma)}\) and Lemma 2.4, we obtain,
\[ \| \nabla tr\chi \|_{L^4(S(u,\omega))} \leq \frac{\delta^{-\frac{1}{2}} C}{|u|^{\frac{1}{2}}}, \] \(6.38\)

\[ \| tr\chi \|_{L^\infty(S(u,\omega))} \leq \frac{\delta^{-\frac{1}{2}} C}{|u|}, \] \(6.39\)

Once this is proved, using \( \nabla_4 tr\chi + \frac{1}{2} (tr\chi)^2 = -|\hat{\chi}|^2 - 2\omega tr\chi \), we can prove:
\[ \| tr\chi \|_{L^\infty(S(u,\omega))} \leq \frac{C}{|u|}, \] \(6.40\)

which equals,
\[ \delta^{-\frac{1}{2}} \| tr\chi \|_{L^\infty(S(u,\omega))} \leq C. \] \(6.41\)

For \(tr\chi\), we have,
\[ \| \nabla_3 \nabla tr\chi \|_{L^2(H^{(\eta,\omega)},u)} \leq \frac{\delta^{\frac{1}{2}} C}{|u|^{\frac{1}{2}}}. \] \(6.42\)
\[
\|\nabla \nabla \nabla \nabla \chi\|_{L^2(H^{(u_{\infty},u)})} \leq \frac{C}{|u|^{\frac{1}{2} - 2\epsilon}}. 
\] (6.43)

From Lemma 5.1 (Codimension 1 Trace Lemma) and Lemma 2.4, we obtain,

\[
\|\nabla \chi\|_{L^4(S(u,u))} \leq \frac{\delta + C}{|u|^{\frac{1}{2} - \epsilon}} = \frac{\delta + C}{|u|^{\frac{1}{2}}}, 
\] (6.44)

\[
\|\nabla \chi\|_{L^\infty(S(u,u))} \leq \frac{C}{|u|}, 
\] (6.45)

\[
\frac{\delta}{|u|^2} \|\nabla \chi\|_{L^\infty(S(u,u))} \leq C. 
\] (6.46)

For \( \hat{\chi} \), we have,

\[
\|\nabla^4 \nabla \nabla \hat{\chi}\|_{L^2(H^{(u_{\infty},u)})} \leq \frac{C}{|u|^2}, 
\] (6.47)

\[
\|\nabla \nabla \nabla \nabla \hat{\chi}\|_{L^2(H^{(u_{\infty},u)})} \leq \frac{\delta + C}{|u|^2}. 
\] (6.48)

From Lemma 5.1 (Codimension 1 Trace Lemma) and Lemma 2.4, we obtain,

\[
\|\nabla \hat{\chi}\|_{L^4(S(u,u))} \leq \frac{\delta + C}{|u|^\frac{1}{2}}, 
\] (6.49)

\[
\|\hat{\chi}\|_{L^\infty(S(u,u))} \leq \frac{\delta + C}{|u|^2}, 
\] (6.50)

\[
\frac{1}{|u|} \|\hat{\chi}\|_{L^\infty(S(u,u))} \leq C. 
\] (6.51)

For \( \eta \), we have,

\[
\|\nabla^3 \nabla \eta\|_{L^2(H^{(u_{\infty},u)})} \leq \frac{\delta + C}{|u|^2}, 
\] (6.52)

\[
\|\nabla \nabla \eta\|_{L^2(H^{(u_{\infty},u)})} \leq \frac{C}{|u|^2}. 
\] (6.53)

From Lemma 5.1 (Codimension 1 Trace Lemma) and Lemma 2.4, we obtain,

\[
\|\nabla \eta\|_{L^4(S(u,u))} \leq \frac{\delta + C}{|u|^\frac{1}{2}}, 
\] (6.54)

\[
\|\eta\|_{L^\infty(S(u,u))} \leq \frac{C}{|u|^2}, 
\] (6.55)

\[
\|\eta\|_{L^\infty(S(u,u))} \leq C. 
\] (6.56)

For \( \bar{\eta} \), we have,

\[
\|\nabla \nabla \bar{\eta}\|_{L^2(H^{(u_{\infty},u)})} \leq \frac{C}{|u|^\frac{1}{2}}, 
\] (6.57)

\[
\|\nabla \bar{\eta}\|_{L^2(H^{(u_{\infty},u)})} \leq \|\nabla \rho\|_{L^2(H^{(u_{\infty},u)})} + l.o.t \leq \frac{\delta + C}{|u|^\frac{1}{2} - 2\epsilon}. 
\] (6.58)
From Lemma 5.1 (Codimension 1 Trace Lemma) and Lemma 2.4, we obtain,

\[
\|\nabla \eta\|_{L^4(S(u, u))} \leq \frac{\delta^{-1} C}{|u|^{\frac{5}{2}-\epsilon}}, \tag{6.59}
\]

\[
\|\eta\|_{L^\infty(S(u, u))} \leq C \frac{|u|}{|u|^{\frac{5}{2}-\epsilon}}, \tag{6.60}
\]

\[
C \frac{|u|}{|u|^{\frac{5}{2}-\epsilon}} \|\eta\|_{L^\infty(S(u, u))} \leq C. \tag{6.61}
\]

For \(\omega\), we have,

\[
\|\nabla^4 \omega\|_{L^2(H^0(u, u))} \leq C \frac{|\omega|}{|u|^{\frac{5}{2}}}, \tag{6.62}
\]

\[
\|\nabla \omega\|_{L^2(H^0(u, u))} \leq C \frac{|\omega|}{|u|^{\frac{7}{2}}}. \tag{6.63}
\]

From Lemma 5.1 (Codimension 1 Trace Lemma) and Lemma 2.4, we obtain,

\[
\|\nabla \omega\|_{L^4(S(u, u))} \leq C \frac{|\omega|}{|u|^{\frac{7}{2}-\epsilon}}, \tag{6.64}
\]

\[
\|\omega\|_{L^\infty(S(u, u))} \leq C \frac{|\omega|}{|u|^{\frac{9}{2}}}, \tag{6.65}
\]

\[
\|\omega\|_{L^\infty sc(S(u, u))} \leq C. \tag{6.66}
\]

For \(\omega\) we have,

\[
\|\nabla^3 \omega\|_{L^2(H^0(u, u))} \leq C \frac{|\omega|}{|u|^{\frac{7}{2}}}, \tag{6.67}
\]

\[
\|\nabla \omega\|_{L^2(H^0(u, u))} \leq C \frac{|\omega|}{|u|^{\frac{9}{2}}}; \tag{6.68}
\]

From Lemma 5.1 (Codimension 1 Trace Lemma) and Lemma 2.4, we obtain,

\[
\|\nabla \omega\|_{L^4(S(u, u))} \leq \frac{\delta^{-1} C}{|u|^{\frac{5}{2}-\epsilon}}, \tag{6.69}
\]

\[
\|\omega\|_{L^\infty(S(u, u))} \leq \frac{\delta^{-1} C}{|u|^{\frac{7}{2}}}. \tag{6.70}
\]

The last inequality implies,

\[
\|\omega\|_{L^\infty(S(u, u))} \leq C. \tag{6.71}
\]

To summarize, we should pay careful attention to these 4 terms which are slightly worse than expected:

\[
\frac{1}{|u|^{\frac{7}{2}}-\epsilon} \|\nabla tr \chi\|_{L^2(S(u, u))} \leq C, \tag{6.72}
\]

\[
\frac{1}{|u|^{\frac{9}{2}}-\epsilon} \|\nabla \eta\|_{L^2(S(u, u))} \leq C. \tag{6.73}
\]
\[
\frac{1}{|u|^2} \| \eta \|_{L^\infty(S(u, \omega))} \leq C, \quad (6.74)
\]
\[
\frac{1}{|u|'} \| \nabla \omega \|_{L^1(S(u, \omega))} \leq C. \quad (6.75)
\]

**Remark:** All the decay rates lost related to \( \epsilon \) are because we hope to close the bootstrap argument with only 1st derivatives of the curvatures using the Evolution Lemma. However, if we carefully go through the proof of the Evolution Lemma, we can prove sharper estimates to get rid of this \( \epsilon' \) still using only 1st derivatives of the curvatures.

### 6.4 Sharp Estimates without \( \epsilon' \) Loss

In this subsection, we will see that the \( \epsilon' \) loss found in the decay rate can be removed by using a Gagliardo-Nirenberg type inequality.

For \( \| \nabla \eta \|_{L^4(S(u, \omega))} \), we go through the proof of the Evolution Lemma.

\[
\frac{d}{du} \int_{S(u, \omega)} |u|^6 |\nabla \eta|^4 |dS = \int_{S(u, \omega)} |u|^6 \left( \frac{d}{du} |\nabla \eta|^4 + 4 \Omega \text{tr} \chi |\nabla \eta|^4 \right) + \text{l.o.t}
\]
\[
= \int_{S(u, \omega)} |u|^6 \left( \frac{1}{2} \Omega \text{tr} \chi \nabla \eta \cdot \nabla |\nabla \eta|^2 + 4 |\nabla \eta|^3 + 4 |\nabla \eta|^3 |\nabla \beta| \right) + \text{l.o.t}
\]
\[
\leq 4 |u|^5 \int_{S(u, \omega)} (\nabla \eta) (\nabla \eta)^3 + 4 \int_{S(u, \omega)} |u|^6 (\nabla \eta)^3 |\nabla \beta| + \text{l.o.t}
\]
\[
\leq 4 |u|^5 \| \nabla \eta \|_{L^4(S(u, \omega))} \| \nabla \eta \|_{L^2(S(u, \omega))}^3 + 4 \left( \int_{S(u, \omega)} |u|^6 |\nabla \beta|^2 \right)^{\frac{3}{2}} \left( \int_{S(u, \omega)} |u|^6 |\nabla \eta|^6 \right)^{\frac{1}{2}} + \text{l.o.t}
\]
\[
\leq 4 |u|^5 \| \nabla \eta \|_{L^4(S(u, \omega))} \| \nabla \eta \|_{L^2(S(u, \omega))}^3 + \| \nabla \beta \|_{L^2(S(u, \omega))} \| \nabla \eta \|_{L^2(S(u, \omega))} \| \nabla \sigma \|_{L^2(S(u, \omega))} + \text{l.o.t}
\]

(6.76)

For \( * \leq \), we use the Gagliardo-Nirenberg inequality,

\[
\| \nabla \eta \|_{L^2(S(u, \omega))} \leq \| \nabla \eta \|_{L^4(S(u, \omega))} \| \nabla \eta \|_{L^2(S(u, \omega))}^2,
\]

(6.77)

and

\[
\| \nabla \eta \|_{L^2(S(u, \omega))} \leq \frac{\delta^{-\frac{3}{2}} C}{|u|^2} + \| \nabla \eta \|_{L^2(S(u, \omega))} + \| \nabla \sigma \|_{L^2(S(u, \omega))},
\]

(6.78)

to get

\[
\| |u|^\frac{3}{2} \nabla \eta \|_{L^2(S(u, \omega))} \leq \| u \|_{L^4(S(u, \omega))} \| \nabla \eta \|_{L^2(S(u, \omega))}^2 + \int_{u=\infty}^u |u|^5 \| \nabla \eta \|_{L^4(S(u', \omega))} \| \nabla \eta \|_{L^4(S(u', \omega))} \| \nabla \eta \|_{L^2(S(u', \omega))} \| \nabla \eta \|_{L^2(S(u', \omega))} \| \nabla \sigma \|_{L^2(S(u', \omega))} + \text{l.o.t}
\]

(6.79)

This implies,
Then, we have,

\[
M_1 = |u|^4 \int_{u_{\infty}}^{u} |u'|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))} \| \nabla \eta \|_{L^4(S(u, u_{\infty}))} \\
\leq |u|^4 \sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))}^3 \int_{u_{\infty}}^{u} |u'|^{-5} \delta^{-\frac{1}{2}} C \\
\leq (\sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))})^3 \delta^{-\frac{1}{2}} C.
\]

\[
M_2 = |u|^4 \int_{u_{\infty}}^{u} \|u'|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))} \| \nabla \rho \|_{L^2(S(u, u_{\infty}))} \| \nabla \sigma \|_{L^2(S(u, u_{\infty}))} \\
\leq (\sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))})^2 \|u|^5 \| \nabla \rho \|_{L^2(S(u, u_{\infty}))} \| \nabla \sigma \|_{L^2(S(u, u_{\infty}))} \\
\leq (\sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))})^2 \|u|^5 \| \nabla \rho \|_{L^2(S(u, u_{\infty}))} |u'|^{-\frac{1}{2}} \delta^{-\frac{1}{2}} C \\
\leq (\sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))})^2 \|u|^5 \| \nabla \rho \|_{L^2(S(u, u_{\infty}))} |u'|^{\frac{1}{2}} \delta^{-\frac{1}{2}} C \\
\leq (\sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))})^2 \|u|^5 \| \nabla \rho \|_{L^2(S(u, u_{\infty}))} \delta^{-\frac{1}{2}} C.
\]

Denote, \( A := \sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))} \). Using the above, we get,

\[
A^4 \leq A^3 \delta^{-\frac{1}{2}} C + A^2 \|u|^5 \| \nabla \rho \|_{L^2(S(u, u_{\infty}))} \| \nabla \sigma \|_{L^2(S(u, u_{\infty}))} + A^2 \|u|^5 \| \nabla \beta \|_{L^2(S(u, u_{\infty}))} \delta^{-\frac{1}{2}} C.
\]

Hence, we have

\[
(\delta^{\frac{1}{2}} A)^4 \leq (\delta^{\frac{1}{2}} A)^3 C + (\delta^{\frac{1}{2}} A)^2 C.
\]

This implies,

\[
(\delta^{\frac{1}{2}} A)^4 \leq C,
\]

which implies,

\[
\delta^{\frac{1}{2}} \sup_u \|u|^5 \| \nabla \eta \|_{L^4(S(u, u_{\infty}))} \leq C.
\]
and this is equivalent to,
\[ \| \nabla \eta \|_{L^4_{sc}(S(u, u))} \leq C. \]  \hfill (6.87)

**Remark:** we can obtain similar improvements for \( \| \nabla \tr \chi \|_{L^4_{sc}(S(u, u))} \) and \( \| \nabla \omega \|_{L^4_{sc}(S(u, u))} \).

To summarize, we can prove that
\[ \| \nabla \eta \|_{L^4_{sc}(S(u, u))} \leq C. \]  \hfill (6.88)
\[ \| \nabla \tr \chi \|_{L^4_{sc}(S(u, u))} \leq C. \]  \hfill (6.89)
\[ \| \nabla \omega \|_{L^4_{sc}(S(u, u))} \leq C. \]  \hfill (6.90)

Furthermore, with the improved estimate for \( \| \nabla \eta \|_{L^4_{sc}(S(u, u))} \), we can get the improved estimate for \( \| \eta \|_{L^\infty_{sc}(S(u, u))} \) which is,
\[ \| \eta \|_{L^\infty_{sc}(S(u, u))} \leq C. \]  \hfill (6.91)

### 6.5 Estimates for \( \| \varOmega \tr \chi + \frac{2}{u^2} \|_{L^\infty(S)} \)

To prove the Evolution Lemma, we need the bootstrap assumption
\[ |\varOmega \tr \chi + \frac{2}{u^2}| \leq \frac{\delta c}{|u^2|}, \] where \( \delta > 0 \) is sufficiently small and \( c \) is an arbitrary small positive number.

Here we run a small bootstrap argument to verify this assumption.

From the null structure equation for \( \tr \chi \), we know,
\[ \nabla_3 \tr \chi + \frac{1}{2} \tr \chi \tr \chi = -2 \omega \tr \chi - |\hat{\chi}|^2. \]

From this equation, we get,
\[ \frac{d}{du} \varOmega \tr \chi + \frac{1}{2} \varOmega \tr \chi \varOmega \tr \chi = -2 \omega \tr \chi - \varOmega^2 |\hat{\chi}|^2 + l.o.t. \]

Also we have,
\[ \frac{d}{du} \frac{2}{u} + \frac{2}{u^2} = 0. \]

Thus, contracting these two equations, we have,
\[ \frac{d}{du} \left( \varOmega \tr \chi - \frac{2}{u} \right) + \frac{1}{2} \left( \varOmega \tr \chi + \frac{2}{u} \right) \left( \varOmega \tr \chi - \frac{2}{u} \right) = -2 \omega \tr \chi - \varOmega^2 |\hat{\chi}|^2 + l.o.t. \]

From the bootstrap assumption, we have,
\[ \frac{d}{du} \left( \varOmega \tr \chi - \frac{2}{u} \right) + \frac{2}{u} \left( \varOmega \tr \chi - \frac{2}{u} \right) + \frac{\delta c}{|u^2|} = -2 \omega \tr \chi - \varOmega^2 |\hat{\chi}|^2 + l.o.t. \]

Using the estimate results for coefficients in this section, we have,
\[ \frac{d}{du} \left( u^2 (\varOmega \tr \chi - \frac{2}{u}) \right) + \frac{\delta c}{|u^2|} \left( u^2 (\varOmega \tr \chi - \frac{2}{u}) \right) = \frac{\delta c}{|u^2|} + l.o.t. \]

From Gronwall’s inequality, we have,
\[ \| u^2 (\varOmega \tr \chi - \frac{2}{u}) \|_{L^\infty(S(u, u))} \leq C \left( |u^2_{\infty} (\varOmega \tr \chi - \frac{2}{u_{\infty}}) \|_{L^\infty(S(u_{\infty}, u))} + \frac{\delta c}{|u|} \right). \]  \hfill (6.92)
From the initial data in [Chr], we know,
\[ \| u_\infty^2 (\Omega \text{tr} \chi - \frac{2}{u}) \|_{L^\infty(S(u_\infty, u))} \leq C \delta^{\frac{1}{2}}. \]
Thus, we have,
\[ \| u^2 (\Omega \text{tr} \chi - \frac{2}{u}) \|_{L^\infty(S(u, u))} \leq C \delta^{\frac{1}{2}}. \]

Hence, we have improved the arbitrary small constant \( c \) to \( \frac{1}{2} \).

6.6 Estimates for \( \| \Omega - 1 \|_{L^\infty(S)} \)

Recall that \( \omega = -\frac{1}{2} \nabla_3 \log \Omega = \frac{1}{2} \nabla_3 (\Omega^{-1}) = \frac{1}{2} \frac{d}{du} (\Omega^{-1}). \) Since \( \Omega^{-1} = 1 \) on \( H_{u_\infty} \), we have,
\[ \| \Omega^{-1} - 1 \|_{L^\infty(S(u, u'))} \leq \int_{u_\infty}^{u} \| \omega \|_{L^\infty(S(u', u'))} du' \leq C \frac{\delta^{\frac{1}{2}}}{|u|^{\frac{3}{2}}}. \]
Thus, we have,
\[ \| \Omega - 1 \|_{L^\infty(S(u, u'))} \leq \int_{u_\infty}^{u} \| \omega \|_{L^\infty(S(u', u'))} du' \leq C \frac{\delta^{\frac{1}{2}}}{|u|^{\frac{3}{2}}}. \]

Thus, we close the small bootstrap argument and verify the assumption in the proof of the Evolution Lemma.

We summarize the results from subsection 6.1 to subsection 6.6 in the following proposition:

**Proposition 6.2.** Assuming \( \delta^\Delta_0 \) is sufficiently small, there exists a constant \( C \), depending only on \( O(0) \) and \( R, R \), such that
\[ (S) O_{1,4} \leq C \quad \text{and} \quad (S) O_{0,\infty} \leq C. \]
Collecting the results from Section 4 to Section 6, we have proved the following theorem:

**Theorem 1A:** Assume that \( O < \infty \) and \( R < \infty \). Then there exists a constant \( C \) depending only on \( O(0), R, R \), such that,
\[ O \leq C(O(0), R, R). \]

7 Curvature Estimates I,II.

In the next two sections, we give a direct and intuitive approach for proving energy estimates in an infinite region by integrating the null Bianchi equation.

In a finite region, similar work has been done in [L-R:Interaction]. Some of the ideas have originated in [Hol]. For an infinite region, we make a key observation which enables us to obtain such a new approach.

**A Key Observation.**

The key observation is on the coefficients in front of \( tr\chi \) and \( tr\chi \) in the following transport equations:

\[ \text{[Equation]} \]

\[ \text{[Equation]} \]

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Both curvature components $\Psi$ with signature $s(\Psi) = (s, s')$, and the $k$th-order angular derivatives of curvature components $\Psi$ with signature $s(\nabla^k \Psi) = (s, s')$, satisfy the following transport equations:

$$\nabla_3 \Psi^{(s,s')} + \left(\frac{1}{2} + s'\right) tr_N \Psi^{(s,s')} = \nabla \Psi^{(s-\frac{1}{2}, s'+\frac{1}{2})} + \sum_{s_1 + s_2 = s, \ s_1' + s_2' = s' + 1} \psi^{(s_1,s_1')} \cdot \psi^{(s_2,s_2')} \quad (7.1)$$

$$\nabla_4 \Psi^{(s,s')} + \left(\frac{1}{2} + s\right) tr_N \Psi^{(s,s')} = \nabla \Psi^{(s-\frac{1}{2}, s'+\frac{1}{2})} + \sum_{s_1 + s_2 = s, \ s_1' + s_2' = s' + 1} \psi^{(s_1,s_1')} \cdot \psi^{(s_2,s_2')} \quad (7.2)$$

where $\Psi^{(s,s')}$ and $\psi^{(s,s')}$ stand for 2 S-tangent tensor fields with signatures $(s, s')$.

Proof:
For $k = 0$, we can verify the equations above by simply checking the Null Bianchi Equations.

For $k = 1$, we can verify the equations above by taking 1st order angular derivatives of the Null Bianchi Equations and using the Commutator Lemma 3.3 once.

For general $k$, where $k$ is a positive integer, we can verify the equations above by taking the $k$th order angular derivatives of the Null Bianchi Equations and using the Commutator Lemma 3.3 $k$ times.

Remark: Making good use of the coefficients in front of $tr_N$ will lead us to the desired curvature estimates.

### 7.1 Curvature Estimates in the Scale Invariant Norms

Let’s first divide the curvature components into several pairs and prove several useful lemmas.

**Lemma 7.1.** Assuming that for a pair of S-tangent tensor fields $\Psi^{(s,s')}$ and $\Psi^{(s-\frac{1}{2},s'+\frac{1}{2})}$, we have the equations

$$\nabla_3 \Psi^{(s,s')} + \left(\frac{1}{2} + s'\right) tr_N \Psi^{(s,s')} = \nabla \Psi^{(s-\frac{1}{2}, s'+\frac{1}{2})} + \sum_{s_1 + s_2 = s, \ s_1' + s_2' = s' + 1} \psi^{(s_1,s_1')} \cdot \psi^{(s_2,s_2')} \quad (7.3)$$

$$\nabla_4 \Psi^{(s-\frac{1}{2},s'+\frac{1}{2})} + str_N \Psi^{(s-\frac{1}{2}, s'+\frac{1}{2})} = \nabla \Psi^{(s,s')} + \sum_{s_1 + s_2 = s, \ s_1' + s_2' = s' + 1} \psi^{(s_1,s_1')} \cdot \psi^{(s_2,s_2')} \quad (7.4)$$

Then we have,

$$\frac{d}{du} \left(\int_{S(u,u)} |\eta|^{4s'} (\Psi^{(s,s')} \eta)^2 \right) + \frac{d}{du} \left(\int_{S(u,u)} |u|^{4s'} (\Psi^{(s-\frac{1}{2}, s'+\frac{1}{2})} \eta)^2 \right)$$

$$= \int_{S(u,u)} -2 s' \delta u |u|^{4s'-2} (\Psi^{(s,s')})^2 - \int_{S(u,u)} \frac{\delta u}{\delta u} |u|^{4s'} (\eta + \frac{\eta}{u}) \cdot \Psi^{(s-\frac{1}{2}, s'+\frac{1}{2})} \cdot \Psi^{(s,s')}$$

$$+ \int_{S(u,u)} 2 \Omega |u|^{4s'} \Psi^{(s,s')} \cdot \sum_{s_1 + s_2 = s, \ s_1' + s_2' = s' + 1} \psi^{(s_1,s_1')} \cdot \psi^{(s_2,s_2')} + \int_{S(u,u)} 2 \Omega |u|^{4s'} \Psi^{(s-\frac{1}{2}, s'+\frac{1}{2})} \cdot \sum_{s_1 + s_2 = s, \ s_1' + s_2' = s' + 1} \psi^{(s_1,s_1')} \cdot \psi^{(s_2,s_2')} \quad (7.5)$$
where $c$ is a small constant number.

**Remark:** There isn’t a $\int_{S(u,w)} |u|^{4\varepsilon} \Omega r \chi (\Psi(s,s'))^2$ term on the right side, which might be a dangerous borderline term.

**Remark:** For convenience, we can rewrite the identity above as:

$$
\frac{d}{du} \left( \int_{S(u,w)} |u|^{4\varepsilon'} (\Psi(s,s'))^2 \right) + \frac{d}{du} \left( \int_{S(u,w)} |u|^{4\varepsilon'} (\Psi(s^{-\frac{1}{2}},s'^{-\frac{1}{2}} + \frac{1}{2}))^2 \right) =
\int_{S(u,w)} -2s\delta u^{4\varepsilon} (\Psi(s,s'))^2 - \int_{S(u,w)} 2|u|^{4\varepsilon'} \sum_{s_1, s_2 = 2s} \psi^{(s_1,s_1')}, \psi^{(s_2,s_2')}, \psi^{(s_3,s_3')}
$$

(7.6)

where, $\sum_{s_1 + s_2 + s_3 = 2s, s_1' + s_2' + s_3' = 2s + 1} \psi^{(s_1,s_1')}, \psi^{(s_2,s_2')}, \psi^{(s_3,s_3')}$ stands for,

$$
-2(\bar{\eta} + \bar{\mu})(s^{-\frac{1}{2}},s'^{-\frac{1}{2}} + \frac{1}{2}), \psi^{(s,s')} + \sum_{s_1' + s_2' + s_3' = 2s + 1} \psi^{(s_1,s_1')}, \psi^{(s_2,s_2')}, \psi^{(s_3,s_3')}
$$

Proof of Lemma 7.1

From (7.3) we have,

$$
\frac{d}{du} \left( \int_{S(u,w)} |u|^{4\varepsilon'} (\Psi(s,s'))^2 \right)
$$

$$
= \int_{S(u,w)} 4u^{4\varepsilon} (\Psi(s,s'))^2 + \int_{S(u,w)} 2u^{4\varepsilon'} (\Psi(s,s')) \frac{d}{du} (\Psi(s,s')) + \int_{S(u,w)} \Omega r \chi (\Psi(s,s'))^2
$$

$$
= \int_{S(u,w)} 4u^{4\varepsilon} (\Psi(s,s'))^2 + \int_{S(u,w)} 2u^{4\varepsilon'} (\Psi(s,s')) \cdot -\frac{1}{2} + s') \Omega r \chi (\Psi(s,s'))
$$

$$
+ \int_{S(u,w)} 2u^{4\varepsilon'} \Psi^{(s,s')} (\nabla \Psi^{(s^{-\frac{1}{2}},s'^{-\frac{1}{2}} + \frac{1}{2}}) + \sum_{s_1' + s_2' + s_3' = 2s' + 1} \psi^{(s_1,s_1')}) \psi^{(s,s')}) + \int_{S(u,w)} \Omega r \chi u^{4\varepsilon'} (\Psi(s,s'))^2
$$

$$
= \int_{S(u,w)} (4s^{s'} u^{4\varepsilon} - 2s^{s'} u^{4\varepsilon'} \Omega r \chi (\Psi(s,s'))^2 + \int_{S(u,w)} 2u^{4\varepsilon'} \Psi^{(s,s')} (\nabla \Psi^{(s^{-\frac{1}{2}},s'^{-\frac{1}{2}} + \frac{1}{2}}) + \sum_{s_1' + s_2' + s_3' = 2s' + 1} \psi^{(s_1,s_1')}) \psi^{(s,s')})
$$

(7.7)

For $S^0$, we use $\Omega r \chi = \frac{c}{|u|^2}$, where $c$ is a small positive constant.

From (7.4) we have,

$$
\frac{d}{du} \left( \int_{S(u,w)} u^{4\varepsilon'} ((s^{-\frac{1}{2}},s'^{-\frac{1}{2}} + \frac{1}{2}))^2 \right) = \int_{S(u,w)} 2u^{4\varepsilon'} (\Psi(s^{-\frac{1}{2}},s'^{-\frac{1}{2}} + \frac{1}{2})) \cdot \frac{d}{du} (\Psi(s^{-\frac{1}{2}},s'^{-\frac{1}{2}} + \frac{1}{2})
$$

$$
= \int_{S(u,w)} 2u^{4\varepsilon'} \Psi^{(s, s')} (\nabla \Psi^{(s,s')}) + \sum_{s_1' + s_2' + s_3' = 2s' + 1} \psi^{(s_1,s_1')}, \psi^{(s_2,s_2')}
$$

(7.8)

Hence,
\[ \frac{d}{du} \left( \int_{S(u, \omega)} |u|^{4s'} \left( \psi(s, s') \right)^2 \right) + \frac{d}{du} \left( \int_{S(u, \omega)} u^{4s'} \left( \psi(s - \frac{1}{2}s', \frac{1}{2}) \right)^2 \right) \]

\[ *1 = \int_{S(u, \omega)} -2s' \delta \cdot u^{-2} \left( \psi(s, s') \right)^2 - \int_{S(u, \omega)} 4u^{4s'} \nabla \Omega \psi(s - \frac{1}{2}s', \frac{1}{2}) \psi(s, s') \]

\[ + \int_{S(u, \omega)} 2\Omega u^{4s'} \psi(s, s') \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \psi(s_1, s_1') \psi(s_2, s_2') + \int_{S(u, \omega)} 2\Omega u^{4s'} \psi(s - \frac{1}{2}s', \frac{1}{2}) \sum_{s_1 + s_2 = s + \frac{1}{2}, s_1' + s_2' = s' + \frac{1}{2}} \psi(s_1, s_1') \psi(s_2, s_2') \]

\[ *2 = \int_{S(u, \omega)} -2s' \delta \cdot u^{-2} \left( \psi(s, s') \right)^2 - \int_{S(u, \omega)} 4\Omega u^{4s'} (\eta + \tilde{\eta}) \psi(s - \frac{1}{2}s', \frac{1}{2}) \psi(s, s') \]

\[ + \int_{S(u, \omega)} 2\Omega u^{4s'} \psi(s, s') \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \psi(s_1, s_1') \psi(s_2, s_2') + \int_{S(u, \omega)} 2\Omega u^{4s'} \psi(s - \frac{1}{2}s', \frac{1}{2}) \sum_{s_1 + s_2 = s + \frac{1}{2}, s_1' + s_2' = s' + \frac{1}{2}} \psi(s_1, s_1') \psi(s_2, s_2') \]

\[ (7.9) \]

For \(*1\), we use Stokes theorem on \(S(u, \omega)\). (See more details in \([\text{L-R-Interaction}]\).

For \(*2\), we use \(\eta + \tilde{\eta} = 2\nabla \log \Omega = \frac{1}{\Omega} \nabla \Omega\) and \(\nabla \Omega = \Omega (\eta + \tilde{\eta})\).

This finishes the proof for Lemma \(7.4\).

**Lemma 7.2.** Assuming a pair of \(S\)-tangent tensor fields \(\psi(s, s')\) and \(\psi(s - \frac{1}{2}s', \frac{1}{2})\) satisfy \(7.3\) and \(7.4\), then we have,

\[ \| \psi(s, s') \|^2_{L^2(H^0_\infty, \omega)} + \| \psi(s - \frac{1}{2}s', \frac{1}{2}) \|^2_{L^2(H^0_\infty, u)} \]

\[ \leq \| \psi(s, s') \|^2_{L^2(H^0_\infty, \omega)} + \| \psi(s - \frac{1}{2}s', \frac{1}{2}) \|^2_{L^2(H^0_\infty, u)} + \delta^c \sup_{u} \| \psi(s, s') \|^2_{L^2(H^0_\infty, \omega)} \]

\[ + \int_{u} \int_{u} \frac{\delta^{-\frac{1}{2}}}{|u'|} \left( (\eta + \tilde{\eta}) \cdot \psi(s, s'), \psi(s - \frac{1}{2}s', \frac{1}{2}) \right) \|L^1(S(u, u')) dg' du' \]

\[ + \int_{u} \int_{u} \frac{\delta^{-\frac{1}{2}}}{|u'|} \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \| \psi(s_1, s_1'), \psi(s_2, s_2') \|_{L^1(S(u, u'))} d' du' \]

\[ + \int_{u} \int_{u} \frac{\delta^{-\frac{1}{2}}}{|u'|} \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \| \psi(s_1, s_1'), \psi(s_2, s_2') \|_{L^1(S(u, u'))} d' du' \]

(7.10)

where \(c\) is a small constant.

**Remark:** For convenience, we will rewrite the inequality above as

\[ \| \psi(s, s') \|^2_{L^2(H^0_\infty, \omega)} + \| \psi(s - \frac{1}{2}s', \frac{1}{2}) \|^2_{L^2(H^0_\infty, u)} \]

\[ \leq \| \psi(s, s') \|^2_{L^2(H^0_\infty, \omega)} + \| \psi(s - \frac{1}{2}s', \frac{1}{2}) \|^2_{L^2(H^0_\infty, u)} + \delta^c \sup_{u} \| \psi(s, s') \|^2_{L^2(H^0_\infty, \omega)} \]

\[ + \int_{u} \int_{u} \delta^{-\frac{1}{2}} \sum_{s_1 + s_2 = s, s_1' + s_2' = s' + 1} \| \psi(s_1, s_1'), \psi(s_2, s_2') \|_{L^1(S(u, u'))} d' du' \]

(7.11)

**Proof of Lemma 7.2**

Multiplying \(\delta^{2s-\frac{1}{2}}\) on both sides of the conclusion of Lemma \(7.4\) we have,
\[
\frac{d}{du} \left( \int_{S(u,u')} \delta^{2s-3} u^{4s} (\Psi(s,s'))^2 \right) + \frac{d}{du} \left( \int_{S(u,u')} \delta^{2s-3} u^{4s} (\Psi(s-\frac{1}{2},s')^2) \right)
\]

\[
= \int_{S(u,u')} -2s' \delta^{2s-3} c_u u^{4s'-2} (\Psi(s,s'))^2 - \int_{S(u,u')} 4\Omega \delta^{2s-3} u^{4s'} (\eta + \eta) \Psi(s-\frac{1}{2},s') \Psi(s,s') \\
+ \int_{S(u,u')} 2\Omega \delta^{2s-3} u^{4s'} \sum_{s_1 + s_2 = s', \delta_1' + \delta_2' = s' + 1} \Psi(s_1) \Psi(s_2) \\
+ \int_{S(u,u')} 2\Omega \delta^{2s-3} u^{4s'} \Psi(s-\frac{1}{2},s' + \frac{1}{2}) \sum_{s_1 + s_2 = s', \delta_1' + \delta_2' = s' + \frac{1}{2}} \Psi(s_1) \Psi(s_2) 
\]

Using the definitions

\[
\|\Psi\|_{L^2_c(S(u,u'))} = \delta^{s_1} (\psi - \frac{1}{2}) u^{2s_1 (\psi + 1 - \frac{1}{2})} \|\psi\|_{L^2(S(u,u'))},
\]

and,

\[
\|\Psi(2s,2s'+1)\|_{L^2_c(S(u,u'))} = \delta^{2s-\frac{s'}{2}} u^{4s'+1} \|\Psi(2s,2s'+1)\|_{L^2(S(u,u'))}
\]

then by (7.12) we have,

\[
\frac{d}{du} \delta^{-1} \|\Psi(s,s')\|_{L^2_c(S(u,u'))}^2 + \frac{d}{du} \frac{1}{|u|^2} \|\psi(s',s') + \frac{1}{2}\|_{L^2_c(S(u,u'))}^2 \\
\leq \frac{\delta^c}{|u|^2} \delta^{-1} \|\Psi(s,s')\|_{L^2_c(S(u,u'))}^2 + \delta^{\frac{s'}{2}} \|\psi(s',s') \Psi(s-\frac{1}{2},s') \|_{L^2_c(S(u,u'))}^2 \\
+ \delta^{-\frac{s'}{2}} \sum_{s_1 + s_2 = s', \delta_1 + \delta_2 = s' + 1} \|\Psi(s_1) \Psi(s_2)\|_{L^2_c(S(u,u'))} \\
+ \delta^{-\frac{s'}{2}} \sum_{s_1 + s_2 = s', \delta_1 + \delta_2 = s' + 1} \|\Psi(s_1') \Psi(s_2')\|_{L^2_c(S(u,u'))}.
\]

Using the definitions

\[
\|\Psi\|_{L^2_c(U^{(0)},u')}^2 = \delta^{-1} \int_{u'} \|\Psi\|_{L^2_c(S(u,u'))}^2 du'.
\]

\[
\|\Psi\|_{L^2_c(U^{(\infty)},u')}^2 = \int_{u'} \frac{1}{|u'|} \|\Psi\|_{L^2_c(S(u,u'))}^2 du'.
\]

and integrating (7.15) over \(D(u_*, u_*')\), we get
\[ \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 + \| \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^0_\omega)}^2 \leq \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 + \| \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^0_\omega)}^2 + \delta^c \sup_{u'} \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 \]
\[ + \int_0^u \int_{u'_s} \delta^{-\frac{1}{2}} \frac{1}{|u'|} \| (\eta + \eta') \cdot \Psi(s,s'), \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^\infty_\omega)} \, du' \, du' \]
\[ + \int_0^u \int_{u'_s} \delta^{-\frac{1}{2}} \frac{1}{|u'|} \sum_{s,s',s''} \| \psi(s_1,s_1'), \psi(s_2,s_2'), \psi(s,s') \|_{L^2_t(L^\infty_\omega)} \, du' \, du' \]
\[ + \int_0^u \int_{u'_s} \delta^{-\frac{1}{2}} \frac{1}{|u'|} \sum_{s,s',s''} \| \tilde{\psi}(s_1,s_1'), \tilde{\psi}(s_2,s_2'), \psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^\infty_\omega)} \, du' \, du'. \]

(7.18)

This finishes the prove of Lemma 7.2.

From Lemma 7.2, we can easily get the following corollary.

**Corollary 7.3.** Assuming a pair of S-tangent tensor fields \( \Psi(s,s') \) and \( \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \) satisfy 7.3 and 7.4. Then we have,
\[ \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 + \| \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^0_\omega)}^2 \leq \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 + \| \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^0_\omega)}^2 + \delta^c \sup_{u'} \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 \]
\[ + \frac{\delta^{\frac{1}{2}}}{|u|} \sum_{s,s',s''} \sup_{u'_s} \frac{1}{|u'|} \| \psi(s_1,s_1') \|_{L^2_t(L^\infty_\omega)} \| \psi(s_2,s_2') \|_{L^2_t(L^\infty_\omega)} \| \psi(s,s') \|_{L^2_t(L^\infty_\omega)}, \]

(7.19)

or
\[ \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 + \| \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^0_\omega)}^2 \leq \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 + \| \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \|_{L^2_t(L^0_\omega)}^2 + \delta^c \sup_{u'} \| \Psi(s,s') \|_{L^2_t(L^0_\omega)}^2 \]
\[ + \frac{\delta^{\frac{1}{2}}}{|u|} \sum_{s,s',s''} \sup_{u'_s} \frac{1}{|u'|} \| \psi(s_1,s_1') \|_{L^2_t(L^\infty_\omega)} \| \psi(s_2,s_2') \|_{L^2_t(L^\infty_\omega)} \| \psi(s,s') \|_{L^2_t(L^\infty_\omega)}, \]

(7.20)

where \( c \) is a small constant.

We also obtain the following corollary:

**Corollary 7.4.** Assuming a pair of S-tangent tensor fields \( \Psi(s,s') \) and \( \Psi(s-\frac{1}{2},s'+\frac{1}{2}) \) satisfy 7.3 and 7.4. Then we have,
whenever \( \delta \) is sufficiently small, or

where \( c \) is a small constant.

Now we are ready to state the main theorem in this Section:

**Theorem 1B:** The following estimate holds for a constant \( C = C(\mathcal{I}(0), \mathcal{R}(0)) \), and \( \delta \) sufficiently small,

\[
\mathcal{R} + \mathcal{R} \leq C(\mathcal{I}(0), \mathcal{R}(0)).
\]

In the next two subsections, we will prove **Theorem 1B** by introducing a bootstrap assumption:

\[
\mathcal{R} + \mathcal{R} \leq \Delta.
\]

To this end we will prove the following proposition:

**Proposition 7.5.** There exists \( \delta_0 = \delta_0(\mathcal{I}(0), \mathcal{R}(0), \Delta) \), such that whenever \( \delta \leq \delta_0 \),

\[
\mathcal{R}_0 + \mathcal{R}_0 \leq C(\mathcal{I}(0), \mathcal{R}(0)),
\]

\[
\mathcal{R}_1 + \mathcal{R}_1 \leq C(\mathcal{I}(0), \mathcal{R}(0)).
\]

More precisely, we hope to prove, that there exists \( \delta_0 = \delta_0(\mathcal{I}(0), \mathcal{R}(0), \Delta) \), such that whenever \( \delta \leq \delta_0 \),

\[
\delta^\frac{1}{2} \| \alpha \|_{L^2_{\infty}(H)} + \| \beta \|_{L^2_{\infty}(H)} + \| \rho \|_{L^2_{\infty}(H)} + \| \sigma \|_{L^2_{\infty}(H)} + \| \beta \|_{L^2_{\infty}(H)} \leq C(\mathcal{I}(0), \mathcal{R}(0)),
\]

\[
\delta^\frac{1}{2} \| \beta \|_{L^2_{\infty}(\mathcal{U})} + \| \rho \|_{L^2_{\infty}(\mathcal{U})} + \| \sigma \|_{L^2_{\infty}(\mathcal{U})} + \| \alpha \|_{L^2_{\infty}(\mathcal{U})} \leq C(\mathcal{I}(0), \mathcal{R}(0)),
\]

\[
\| \nabla \alpha \|_{L^2_{\infty}(H)} + \| \nabla \beta \|_{L^2_{\infty}(H)} + \| \nabla \rho \|_{L^2_{\infty}(H)} + \| \nabla \sigma \|_{L^2_{\infty}(H)} + \| \nabla \beta \|_{L^2_{\infty}(H)} \leq C(\mathcal{I}(0), \mathcal{R}(0)),
\]

\[
\| \nabla \beta \|_{L^2_{\infty}(\mathcal{U})} + \| \nabla \rho \|_{L^2_{\infty}(\mathcal{U})} + \| \nabla \sigma \|_{L^2_{\infty}(\mathcal{U})} + \| \nabla \beta \|_{L^2_{\infty}(\mathcal{U})} + \| \nabla \alpha \|_{L^2_{\infty}(\mathcal{U})} \leq C(\mathcal{I}(0), \mathcal{R}(0)).
\]

**Remark:** From Corollary 7.3, we know that if we don’t have anomalies in 7.3 and 7.4 then we can easily prove Proposition 7.3 and Theorem 1B. Thus, we only need to care about the anomalies.
7.2 Curvature Estimates I.

Estimate for $\alpha$.

From the Null Bianchi Equations,

\[
\nabla_4 \alpha + \frac{1}{2} \text{tr} \chi \alpha = \nabla \otimes \beta + 4 \omega \alpha - 3(\chi \rho + \chi \sigma) + (\zeta + 4 \eta) \otimes \beta,
\]

and by the bootstrap assumption in the scale invariant norms, the anomalies are $\alpha$ and $\beta$ along $H$.

From Corollary 7.4 and Theorem 1A, we easily get

\[
\delta \|\alpha\|^2_{L^2_t(H^0_u \omega)} + \delta \|\beta\|^2_{L^2_t(H^0_u \omega)} \leq \delta \|\alpha\|^2_{L^2_t(H^0_u \omega)} + \delta c \|\alpha\|^2_{L^2_t(H^0_u \omega)} + \delta \frac{4}{3} C(I^0, \Delta) \Delta^2.
\]

where $c$ is a small positive number.

**Remark:** The borderline term and potentially dangerous term is $\Omega \text{tr} \chi \alpha^2$, but this term is killed off by our previous lemmas.

Estimate for $\beta$.

From the Null Bianchi Equations,

\[
\nabla_4 \beta + 2 \text{tr} \chi \beta = \nabla \rho + \nabla \sigma + 2 \omega \beta + 2 \chi \cdot \beta + 3(\eta \rho + \eta \sigma),
\]

and by the bootstrap assumption in the scale invariant norms, the anomalies are $\alpha$ and $\hat{\chi}$.

Using Lemma 7.2 we have,

\[
\|\beta\|^2_{L^2_t(H^0_u \omega)} + \|\rho, \sigma\|^2_{L^2_t(H^0_u \omega)} \\
\leq \|\beta\|^2_{L^2_t(H^0_u \omega)} + \delta c \|\beta\|^2_{L^2_t(H^0_u \omega)} + \delta \frac{4}{3} C(I^0, \Delta) \Delta^2 + \int_{u_0}^u \int \|\hat{\chi}\|_{L^1(S)} \|\alpha\|_{L^2(S)} \|\rho\|_{L^2(S)} \\
+ \int_{u_0}^u \int \|\hat{\chi}\|_{L^1(S)} \|\nabla \alpha\|_{L^2(S)} \|\rho\|_{L^2(S)} + \int_{u_0}^u \int \|\hat{\chi}\|_{L^1(S)} \|\rho\|_{L^2(S)} + \int_{u_0}^u \int \|\hat{\chi}\|_{L^1(S)} \|\rho\|_{L^2(S)}
\]

For $\hat{\chi}$ we have the estimate,

\[
\hat{\chi} \leq \hat{\chi} \times L^2_t(S(u, w)) \leq \delta \hat{\chi} \times L^2_t(S(u, w)) + \delta \frac{4}{3} R_1(a) \frac{1}{p} R_2(a) \frac{1}{p} + \delta \frac{4}{3} R_2(a) + \delta \frac{4}{3} C_{0, \infty} C_{0, \infty} (7.34)
\]
where \( c \) is a small positive number.

Later, for \( \alpha \), we will show,
\[
\mathcal{R}_0[\alpha] + \mathcal{R}_1[\alpha] \leq \mathcal{I}^{(0)}(\alpha) + \delta^\frac{1}{2}C,
\]
where \( C = C(\mathcal{I}^{(0)}, \mathcal{R}, \mathcal{R}) \).

Thus, for \( \delta \) sufficiently small, we will have,
\[
\frac{\delta^\frac{1}{2}}{|u|} \| \hat{\chi} \|_{L^2(S(\omega, u))} \leq C(\mathcal{I}^{(0)}),
\]
which is,
\[
\| \hat{\chi} \|_{L^2(S)} \leq \frac{\delta^\frac{1}{2}}{|u|}^2 C(\mathcal{I}^{(0)}).
\]

Returning to (7.33) we have,
\[
\begin{align*}
\| \beta \|_{L^2(H_0^0)}^2 + c & \leq \sup_{u'} \| \beta \|_{L^2(H_0^0)}^2 + \delta^\frac{1}{2}C(\mathcal{I}^{(0)}, \Delta)\Delta^2 &
+ \int_0^u \int_{u|\omega|} C(\mathcal{I}^{(0)}) \delta^\frac{1}{2} \| \nabla \alpha \|_{L^2(\omega)} \| \alpha \|_{L^2(\omega)} \| \rho \|_{L^2(\omega)} &
+ C(\mathcal{I}^{(0)}) \delta^\frac{1}{2} \| \alpha \|_{L^2(\omega)} \| \rho \|_{L^2(\omega)} &
\leq \| \beta \|_{L^2(u^0)}^2 + \frac{\delta^c}{|u|} \sup_{u'} \| \beta \|_{L^2(u_0^0)}^2 + \delta^\frac{1}{2}C(\mathcal{I}^{(0)}, \Delta)\Delta^2 + C(\mathcal{I}^{(0)}) \delta^\frac{1}{2} \| \alpha \|_{L^2(\omega)} \| \rho \|_{L^2(\omega)} &
+ \frac{C(\mathcal{I}^{(0)})}{|u|} \mathcal{R}_0[\alpha] \mathcal{R}_1[\alpha] \mathcal{R}_0[\rho] + \frac{C(\mathcal{I}^{(0)})}{|u|} \mathcal{R}_0[\alpha] \mathcal{R}_0[\rho] &
(7.35)
\end{align*}
\]

Later, we will see that for \( \delta \) sufficiently small, we will have,
\[
\mathcal{R}_0[\rho] + \mathcal{R}_0[\alpha] + \mathcal{R}_1[\alpha] \leq C(\mathcal{I}^{(0)}).
\]

Thus, we can prove,
\[
\begin{align*}
\| \beta \|_{L^2(H_0^0)}^2 + c & \leq \sup_{u'} \| \beta \|_{L^2(H_0^0)}^2 + \delta^\frac{1}{2}C(\mathcal{I}^{(0)}, \Delta)\Delta^2 + \delta^c\Delta^2 + C(\mathcal{I}^{(0)}) &
(7.36)
\end{align*}
\]

where \( c \) is a small positive constant.

Estimates for \( \rho, \sigma, \beta \).

We have the Null Bianchi Equations
\[
\nabla_3 \sigma + \frac{3}{2} \tr \chi \sigma = -\div \beta + \frac{1}{2} \hat{\chi} \cdot \alpha - \zeta \cdot \beta - 2\eta \cdot \beta, \tag{7.37}
\]
\[
\nabla_3 \rho + \frac{3}{2} \tr \chi \rho = -\div \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta - 2\eta \cdot \beta, \tag{7.38}
\]

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\[ \nabla_4 \beta + tr \chi \beta = -\nabla \rho + \nabla \sigma + 2 \omega \beta + 2 \bar{\chi} \cdot \beta - 3(\eta \rho - \eta \sigma), \quad (7.39) \]
\[ \nabla_3 \beta + 2tr \chi \beta = -div \alpha - 2\omega \beta + \eta \cdot \alpha, \quad (7.40) \]
\[ \nabla_4 \alpha + \frac{1}{2} tr \chi \alpha = -\nabla \hat{\beta} + 4 \omega \alpha - 3(\bar{\chi} \rho + \bar{\chi} \sigma) + (\zeta + 4\eta) \hat{\beta}, \quad (7.41) \]

**Remark:** \( \alpha \) doesn’t appear in the Null Bianchi equations for \( \rho, \sigma, \beta, \alpha \).

Using Corollary 7.3, we get,

\[
\| \rho, \sigma \|^2_{L^2_s(H^0_u \omega)} + \| \beta \|^2_{L^2_s(H^0_u \omega)} \\
\leq \| \rho, \sigma \|^2_{L^2_s(H^0_u \omega)} + \frac{\delta \epsilon}{|u|} \sup \| \rho, \sigma \|^2_{L^2_s(H^0_u \omega)} + \delta \hat{\chi} C(T^{(0)}, \Delta) \Delta^2 \]
\[
\leq \| \rho, \sigma \|^2_{L^2_s(H^0_u \omega)} + \frac{\delta \epsilon}{|u|} \Delta^2 + \delta \hat{\chi} C(T^{(0)}, \Delta) \Delta^2 \]
\[
\| \beta \|^2_{L^2_s(H^0_u \omega)} + \| \alpha \|^2_{L^2_s(H^0_u \omega)} \\
\leq \| \beta \|^2_{L^2_s(H^0_u \omega)} + \frac{\delta \epsilon}{|u|} \sup \| \beta \|^2_{L^2_s(H^0_u \omega)} + \delta \hat{\chi} C(T^{(0)}, \Delta) \Delta^2 \]
\[
\leq \| \beta \|^2_{L^2_s(H^0_u \omega)} + \frac{\delta \epsilon}{|u|} \Delta^2 + \delta \hat{\chi} C(T^{(0)}, \Delta) \Delta^2 \quad (7.42) \]

We summarize the above results above in the following proposition:

**Proposition 7.6.** The following estimate holds for a constant \( C = C(T^{(0)}, \mathcal{R}, \mathcal{R}), \) and \( \delta \) sufficiently small,

\[
\mathcal{R}_0 + \mathcal{R}_0 \leq T^{(0)} + \delta \hat{\chi} C. \]

**Estimate for \( \mathcal{R}_0^0[\alpha] \).**

Using the transported coordinates of the previous subsection we now derive estimates for the \( \mathcal{R}_0^0[\alpha] \) norm of the anomalous curvature component \( \alpha \).

**Proposition 7.7.** \( \mathcal{R}_0^0[\alpha](u) \leq \mathcal{R}_0^0[\alpha](u_{\infty}) + \mathcal{R} \)

**Proof:** Recall that, \( \mathcal{R}_0^0[\alpha] := \sup_{H \subseteq H_u} \| \alpha \|_{L^2_s(H^0_u \omega)}, \) where \( \delta H \) is the subset of \( H_u \) generated by transporting a disk \( \delta S \) of radius \( \delta \hat{\omega} \), embedded in the sphere \( S(u, 0) \), along the integral curves of the vector field \( e_3 \). We denote \( \delta S_\omega \) to be the intersection between \( \delta H \) and the level hypersurfaces of \( \hat{\omega} \) and \( \delta S(u', \hat{\omega}) \) to be the set obtained by transporting \( \delta S_\omega \) along the integral curves of \( e_3 \). According to the remark after the Evolution Lemma, since

\[
\nabla_3 \alpha + \frac{1}{2} tr \chi \alpha = \nabla \hat{\beta} + 4 \omega \alpha - 3(\bar{\chi} \rho + \bar{\chi} \sigma) + (\zeta + 4\eta) \hat{\beta}, \]

we have,

\[
\| \alpha \|_{L^2_s(S(u, \omega))} \leq \| \alpha \|_{L^2_s(S(u_{\infty}, \omega))} + \int_{u_{\infty}}^u \| \nabla \hat{\beta} + 4 \omega \alpha - 3(\bar{\chi} \rho + \bar{\chi} \sigma) + (\zeta + 4\eta) \hat{\beta} \|_{L^2_s(S(u', \omega))} \]
\[
\leq \| \alpha \|_{L^2_s(S(u_{\infty}, \omega))} + \int_{u_{\infty}}^u \nabla \hat{\beta} + 4 \omega \alpha - 3(\bar{\chi} \rho + \bar{\chi} \sigma) + (\zeta + 4\eta) \hat{\beta} \|_{L^2_s(S(u', \omega))} \]
\[
\leq \mathcal{R}_0^0[\alpha](u_{\infty}) + \mathcal{R}. \quad (7.44) \]
7.3 Curvature Estimates II.

Estimate for $\nabla \alpha$.

For $\alpha$, we use the Null Bianchi Equation

$$\nabla^3 \alpha + \frac{1}{2} \text{tr} \nabla \alpha = \nabla \hat{\nabla} \beta + 4 \omega \alpha - 3(\hat{\chi} \rho + \hat{\chi} \sigma) + (\zeta + 4 \eta) \hat{\beta}.$$  \hfill (7.45)

to get,

$$\nabla \nabla \nabla^3 \alpha + \frac{1}{2} \text{tr} \nabla \nabla \alpha + \frac{1}{2} \nabla \text{tr} \nabla \alpha = \nabla \nabla \hat{\nabla} \beta + 4 \nabla \omega \alpha + 4 \nabla \alpha - 3(\nabla \hat{\chi} \rho + \nabla \hat{\chi} \sigma) - 3(\hat{\chi} \nabla \rho + \hat{\chi} \nabla \sigma) + \nabla (\eta, \eta) \beta + (\eta, \eta) \nabla \beta.$$  \hfill (7.46)

Also, we obtain,

$$\nabla^3 \nabla \alpha - \nabla \nabla^3 \alpha = (\eta, \eta) \nabla \beta \alpha - \eta \nabla \alpha + \beta \nabla \alpha - \frac{1}{2} \text{tr} \nabla \nabla \alpha$$  \hfill (7.47)

Hence, we derive,

$$\nabla^3 \nabla \alpha + \text{tr} \nabla \nabla \alpha = \nabla \nabla \hat{\nabla} \beta - \frac{1}{2} \nabla \text{tr} \nabla \alpha + 4 \nabla \omega \alpha + 4 \nabla \alpha - 3 \nabla \hat{\chi} (\rho, \sigma) + \nabla (\eta, \eta) \beta + (\eta, \eta) \nabla \beta$$

$$+ (\eta, \eta) \nabla^3 \alpha - \eta \nabla \alpha + \beta \nabla \alpha - \hat{\chi} \nabla \alpha$$  \hfill (7.48)

Similarly, we derive,

$$\nabla^3 \nabla \beta + 2 \text{tr} \nabla \nabla \beta = \nabla \text{div} \alpha - 2 \nabla \text{tr} \nabla \beta - 2 \nabla \chi \beta - 2 \omega \nabla \beta + \nabla \eta \alpha + \eta \nabla \alpha$$

$$+ \chi \nabla \beta + \beta \nabla \alpha + (\eta, \eta) \nabla \beta(\text{div} - 2 \nabla \chi \beta - 2 \omega \beta + \eta \alpha) + \chi \eta \beta.$$  \hfill (7.49)

From Corollary 7.3 and Theorem 1A, we easily can get,

$$\|\nabla \alpha\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \|\nabla \beta\|^2_{L^2_{\delta}(H^{(0,\omega)})} \leq \|\nabla \alpha\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \delta^\bullet C(I^{(0)}, \Delta) \Delta^2.$$  \hfill (7.50)

Estimates for $\nabla \beta, \nabla \rho, \nabla \sigma, \nabla \beta$.

Using a similar method as in the estimate for $\nabla \alpha$, we obtain,

$$\|\nabla \beta\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \|\nabla \rho, \nabla \sigma\|^2_{L^2_{\delta}(H^{(0,\omega)})} \leq \|\nabla \beta\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \delta^\bullet C(I^{(0)}, \Delta) \Delta^2.$$  \hfill (7.51)

$$\|\nabla \rho, \nabla \sigma\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \|\nabla \beta\|^2_{L^2_{\delta}(H^{(0,\omega)})} \leq \|\nabla \rho, \nabla \sigma\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \delta^\bullet C(I^{(0)}, \Delta) \Delta^2.$$  \hfill (7.52)

$$\|\nabla \beta\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \|\nabla \alpha\|^2_{L^2_{\delta}(H^{(0,\omega)})} \leq \|\nabla \beta\|^2_{L^2_{\delta}(H^{(0,\omega)})} + \delta^\bullet C(I^{(0)}, \Delta) \Delta^2.$$  \hfill (7.53)

We summarize the results of this section in the following theorem:

**Theorem 1B**: The following estimate holds for a constant $C = C(I^{(0)}, R, R)$, and $\delta$ sufficiently small,

$$R_1 + R_1 \leq I^{(0)} + \delta^\bullet C.$$
Combining the results in Section 6 and Section 7, we prove:

**Theorem 1 (Main Theorem):** Assume that $R^{(0)} \leq \mathcal{I}^{(0)}$ for an arbitrary constant $\mathcal{I}^{(0)}$. Then, there exists a sufficiently small $\delta > 0$, such that,

$$R + \mathcal{R} + O \leq \mathcal{I}^{(0)}.$$

**8 Curvature Estimates III.**

For future applications, we will obtain more curvature estimates.

Let's first introduce the **Curvature norms**.

$$R_2(u, \mathcal{W}) = \|\nabla^2 \alpha\|_{L^2_u(H)} + \|\nabla^2 \beta\|_{L^2_u(H)} + \|\nabla^2 \rho\|_{L^2_u(H)} + \|\nabla^2 \sigma\|_{L^2_u(H)} + \|\nabla^2 \omega\|_{L^2_u(H)}.$$ (8.1)

$$R_3(u, \mathcal{W}) = \|\nabla^2 \beta\|_{L^2_u(H)} + \|\nabla^2 \rho\|_{L^2_u(H)} + \|\nabla^2 \sigma\|_{L^2_u(H)} + \|\nabla^2 \omega\|_{L^2_u(H)}.$$ (8.2)

**Estimate for $\nabla^2 \alpha$.**

For $\nabla \alpha$, we have,

$$\nabla \nabla \alpha + tr \chi \nabla \alpha = \nabla \nabla \alpha - \frac{1}{2} \nabla tr \chi \alpha + 4 \nabla \omega \alpha + 4 \nabla \nabla \alpha - 3 \nabla \chi (\rho, \sigma) + \nabla (\eta, \eta) \beta + (\eta, \eta) \nabla \beta$$

$$+ (\eta, \eta) \nabla \alpha - \eta \nabla \alpha + \beta \alpha - \nabla \alpha.$$ (8.3)

Thus, we have,

$$\nabla \nabla \nabla \alpha + tr \chi \nabla \alpha = -\nabla tr \chi \nabla \alpha + \nabla \nabla \alpha - \frac{1}{2} \nabla tr \chi \alpha - \frac{1}{2} \nabla \nabla \alpha + 4 \nabla \omega \alpha + 4 \nabla \nabla \alpha$$

$$+ \nabla (\eta, \eta) \nabla \alpha + (\eta, \eta) \nabla \alpha + \nabla \beta \alpha + \nabla \alpha.$$ (8.4)

Using a Commutator Lemma, we derive,

$$\nabla \nabla \nabla \alpha - \nabla \nabla \alpha = (\eta, \eta) \nabla \nabla \alpha - \eta \nabla \alpha \chi + \beta \nabla \chi + \nabla \alpha \eta - \nabla \nabla \alpha - \frac{1}{2} tr \chi \nabla \nabla \alpha.$$ (8.5)

Adding (8.3) to (8.5) we get

$$\nabla \nabla \nabla \alpha + \frac{3}{2} tr \chi \nabla \alpha = -\nabla tr \chi \nabla \alpha + \nabla \nabla \alpha - \frac{1}{2} \nabla tr \chi \alpha - \frac{1}{2} \nabla \nabla \alpha + 4 \nabla \omega \alpha + 4 \nabla \nabla \alpha$$

$$+ \nabla (\eta, \eta) \nabla \alpha + (\eta, \eta) \nabla \beta \alpha + \beta \nabla \alpha + (\eta, \eta) \nabla \nabla \alpha - \eta \nabla \alpha \chi$$

$$+ \beta \nabla \alpha + \nabla \alpha \eta - \nabla \nabla \alpha.$$ (8.6)

Similarly, we can calculate $\nabla \nabla \nabla \beta$. From Corollary (8.4) we easily can get,

$$\|\nabla^2 \alpha\|_{L^2_u(H^{[0,\omega]})}^2 + \|\nabla^2 \beta\|_{L^2_u(H^{[0,\omega]})}^2 \leq \|\nabla^2 \alpha\|_{L^2_u(H^{[0,\omega]})}^2 + \delta \chi C(\mathcal{I}^{(0)}, R_2, R_2)(R_2, R_2)^2.$$ (8.7)
Estimate for $\nabla^2 \beta, \nabla^2 \rho, \nabla^2 \sigma, \nabla^2 \beta$.

Using a similar method as in the estimate for $\nabla^2 \alpha$, we have,

$$\|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 + \|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2_u(H^0_v)}^2 \leq \|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 + \|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2_u(H^0_v)}^2 + \frac{\delta}{\epsilon} C(T(0), R_2, R_3) (R_2, R_3)^2. \quad (8.8)$$

$$\|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2_u(H^0_v)}^2 + \|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 \leq \|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 + \|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2_u(H^0_v)}^2 + \delta C(T(0), R_2, R_3) (R_2, R_3)^2. \quad (8.9)$$

$$\|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 + \|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2_u(H^0_v)}^2 \leq \|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 + \delta C(T(0), R_2, R_3) (R_2, R_3)^2. \quad (8.10)$$

Thus, as in [K-R:Trapped], let $0 < \delta \ll \epsilon \ll 1$. Then, if

$$\|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 + \|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2_u(H^0_v)}^2 \leq \epsilon^2 \quad (8.11)$$

holds, then we can prove:

$$\|\nabla^2 \beta\|_{L^2_u(H^0_v)}^2 + \|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2_u(H^0_v)}^2 \leq \epsilon \quad (8.12)$$

We rewrite (8.12) using the standard $L^p$ norms:

$$\delta \int_0^u \int_{S(u, \omega)} u^6 (\nabla^2 \beta)^2 + \delta \int_{S(u, \omega)} u^6 ((\nabla^2 \rho)^2 + (\nabla^2 \sigma)^2) \leq \epsilon^2. \quad (8.13)$$

$$\delta \int_0^u \int_{S(u, \omega)} u^8 ((\nabla^2 \rho)^2 + (\nabla^2 \sigma)^2) + \delta \int_{S(u, \omega)} u^8 (\nabla^2 \beta)^2 \leq \epsilon^2. \quad (8.14)$$

$$\delta \int_0^u \int_{S(u, \omega)} u^{10} (\nabla^2 \beta)^2 + \int_{S(u, \omega)} u^{10} (\nabla^2 \omega)^2 \leq \epsilon^2. \quad (8.15)$$

So far, we have energy estimates for $\nabla^2 \beta, \nabla^2 \rho, \nabla^2 \sigma, \nabla^2 \beta$.

9 Heuristic Argument

In order to get the trapped surfaces, we need to prove, that both $t r \chi < 0$ and $t r \chi < 0$ hold, pointwise, on a 2-sphere $S(u, \omega)$. Recalling, that along $H^0_s(v, u)$ ($v = 0$) we prescribe Minkowskian initial data. So, on $S(u, 0)$, we have $\Omega t r \chi = -\frac{1}{|\omega|}$ and $\Omega t r \chi = \frac{2}{|\omega|}$. To make $t r \chi < 0$, we need to use the following two Null Structure equations:

$$\nabla_4 t r \chi + \frac{1}{2} (t r \chi)^2 = -|\hat{\chi}|^2 - 2 \omega t r \chi,$$

$$\nabla_3 \hat{\chi} + \frac{1}{2} t r \chi \hat{\chi} = \hat{\nabla} \hat{\eta} + 2 \omega \hat{\chi} - \frac{1}{2} t r \chi \hat{\chi} + \eta \hat{\eta},$$

Using the 1st Null Structure equation from above, we have that,

$$\frac{d}{du} t r \chi \leq -|\hat{\chi}|^2.$$
Hence, we obtain,

\[ \text{tr} \chi(u, u) \lesssim \text{tr} \chi(u, 0) - \int_0^u |\hat{\chi}|^2(u, u') du' = \frac{2}{|u|} - \int_0^u |\hat{\chi}|^2(u, u') du' \]

Using the 2nd Null Structure Equation from above, we have that,

\[ \frac{d}{du} |\hat{\chi}|^2 + \Omega \text{tr} \chi |\hat{\chi}|^2 = \Omega \hat{\chi} \cdot E, \]

where \( E = \nabla \hat{\chi} + 2u \hat{\chi} - \frac{1}{2} \text{tr} \chi \hat{\chi} + \eta \hat{\chi} \).

Hence, we get,

\[ \frac{d}{du} (|u|^2 |\hat{\chi}|^2) = 2u |\hat{\chi}|^2 + |u|^2 (-\Omega \text{tr} \chi) |\hat{\chi}|^2 + \Omega |u|^2 \hat{\chi} \cdot E, \]

From the results obtained in Section 6, we know that

\[ \|u^2 (\Omega \text{tr} \chi - \frac{2}{u})\|_{L^\infty(S(u, u))} \leq C \delta^\frac{1}{2} \]

Thus, we have,

\[ \frac{d}{du} (|u|^2 |\hat{\chi}|^2) = \Omega |u|^2 \hat{\chi} \cdot E + C \delta^\frac{1}{2} |\hat{\chi}|^2. \]

If the \( L^\infty(S) \) norm for \( E \) is sufficiently small, then we can show that \( |u|^2 |\hat{\chi}|^2(u, u) \lesssim |u_\infty|^2 |\hat{\chi}|^2(u_\infty, u) \)

Thus,

\[ \text{tr} \chi(u, u) \lesssim \text{tr} \chi(u, 0) - \int_0^u |\hat{\chi}|^2(u, u') du' = \frac{2}{|u|} - \frac{|u_\infty|^2}{|u|^2} \int_0^u |\hat{\chi}|^2(u_\infty, u') du' \]

Hence, in order to get the trapped surfaces in the slab \( D(u \approx 1, \delta) \), we need,

\[ \frac{2}{|u_\infty|^2} < \int_0^u |\hat{\chi}|^2(u_\infty, u') du' \]

Similarly, to avoid trapped surfaces in the initial hypersurface \( H^{(0, u)}_{u_\infty} \), we need,

\[ \int_0^u |\hat{\chi}|^2(u_\infty, u') du' < \frac{2}{|u_\infty|} \]

In summary, we want the initial data on \( H^{(0, u)}_{u_\infty} \) to satisfy

\[ \frac{2}{|u_\infty|^2} < \int_0^u |\hat{\chi}|^2(u_\infty, u') du' < \frac{2}{|u_\infty|} \]

For the initial data in \( \text{[Crit]} \) and \( \text{[K-R:Trapped]} \), we have \( \|\hat{\chi}\|_{L^\infty(S(u_\infty, u))} \approx \frac{1}{|u_\infty|^2} \), and so this satisfies the required conditions.

Also, using the results obtained in Section 4 to Section 6, it’s easy to see that the \( L^\infty(S) \) norm for all the other terms in \( E \) are sufficiently small, except for \( \nabla \eta \). In the following section, we will focus on \( \nabla \eta \).
10 Formation of Trapped Surfaces.

In order to prove the formation of trapped surfaces, we hope that our heuristic argument works. Using the results obtained in previous Sections, we can easily obtain all the smallness we hope holds, except for $\int_0^u \int_{u_\infty}^u u^2 \hat{\chi} \nabla \eta$. To prove that $\int_0^u \int_{u_\infty}^u u^2 \hat{\chi} \nabla \eta$ is a low order term compared with 1, we only need to show that $\|\nabla \eta\|_{L^\infty(S)} \lesssim \frac{\delta + \epsilon}{|u|^2}$.

It is because, once we have this, and recalling that $\|\hat{\chi}\|_{L^\infty(S)} \lesssim \frac{\delta + \epsilon}{|u|^2}$, we will get

$$\left\| \int_0^u \int_{u_\infty}^u u^2 \hat{\chi} \nabla \eta \right\|_{L^\infty(S)} \leq \int_0^u \int_{u_\infty}^u u^2 \|\hat{\chi}\|_{L^\infty(S)} \|\nabla \eta\|_{L^\infty(S)} \lesssim \int_0^u \int_{u_\infty}^u u^2 \frac{\delta + \epsilon}{|u|^2} \lesssim \frac{\epsilon}{|u|^2},$$

and this will finish our proof.

10.1 $L^\infty(S)$ Estimates for $\nabla \eta$.

From Lemma 3.8, we know,

$$\|\nabla \eta\|_{L^\infty(S(u, \omega))} \lesssim \|\nabla \eta\|_{L^1(S(u, \omega))} \|\nabla^2 \eta\|_{L^1(S(u, \omega))} + \frac{1}{|u|} \|\nabla \eta\|_{L^1(S(u, \omega))}. \quad (10.2)$$

From Lemma 6.1, we know,

$$\|\nabla^2 \eta\|_{L^1(H)} \lesssim \|\nabla^4 \eta\|_{L^1(H)} \|\nabla^3 \eta\|_{L^1(H)}. \quad (10.3)$$

We also know,

$$\nabla_4 \nabla^2 \eta = \nabla^2 \beta + l.o.t., \quad \nabla^3 \eta = \nabla^2 \mu + \nabla^2(\rho, \sigma) + l.o.t. \quad (10.4)$$

Hence,

$$\|\nabla^2 \eta\|_{L^1(S)} \lesssim \|\nabla^4 \eta\|_{L^1(H)} \|\nabla^3 \eta\|_{L^1(H)} \lesssim \|\nabla^4 \eta\|_{L^3(H)} \|\nabla^3 \eta\|_{L^3(H)} + \|\nabla^2 \beta\|_{L^3(H)} + \|\nabla^3 \eta\|_{L^3(H)} \|\nabla^2 \mu\|_{L^3(H)} + \|\nabla^3 \eta\|_{L^3(H)} \|\nabla^2 (\rho, \sigma)\|_{L^3(H)}. \quad (10.5)$$

Next, we try to estimate $\|\nabla^2 \mu\|_{L^2(S(u, \omega))}$.

Recalling,

$$\nabla_4 \mu + \nabla_4 \mu = -\frac{1}{2} tr \chi \div \eta \nabla \eta + \chi \cdot \nabla (2\eta - \eta) + \frac{1}{2} \chi \cdot \alpha - (\eta - 2\eta) \cdot \beta + \frac{1}{2} tr \chi \rho + \frac{1}{2} tr \chi (\eta^2 - \eta \cdot \eta) + \frac{1}{2} (\eta + \eta) \cdot \hat{\chi} \cdot (\eta - \eta), \quad (10.6)$$
we obtain,
\[ \nabla_4 \Delta \mu = \frac{1}{2} \Delta \text{div} \chi + \frac{1}{2} \Delta \nabla (\eta + \chi) - 2 \text{tr} \chi \Delta \mu + \frac{1}{2} \nabla^2 \Delta \alpha + \frac{1}{2} \nabla \chi \cdot \nabla \alpha + \frac{1}{2} \nabla^2 \chi \cdot \nabla \alpha \]
\[ + (\eta - 3 \eta) \Delta \nabla \chi + (\eta - 3 \eta) \cdot \Delta \beta + \frac{1}{2} \text{tr} \chi \Delta \rho + \text{l.o.t.} \quad (10.7) \]

From,
\[ \alpha = -\nabla_4 \chi + \text{l.o.t}, \]
\[ \beta = -\nabla_4 \eta + \text{l.o.t}, \]
\[ \rho = 2 \nabla_4 \omega + \text{l.o.t}, \]

we have,
\[ \Delta \alpha = -\nabla_4 \Delta \chi + \text{l.o.t}, \]
\[ \Delta \beta = -\nabla_4 \Delta \eta + \text{l.o.t}, \]
\[ \Delta \rho = 2 \nabla_4 \Delta \omega + \text{l.o.t.} \]

Recalling,
\[ \nabla \Delta \eta = \nabla_4 \nabla^2 (\omega, \omega^T) + \nabla^2 \mu + \text{l.o.t}, \quad (10.14) \]

and defining,
\[ \dot{\eta} = \Delta \mu - \chi \cdot \nabla^2 (\omega, \omega^T) - (\eta - 3 \eta) \nabla^2 \eta + \frac{1}{2} \text{tr} \chi \nabla^2 \omega + \frac{1}{2} \nabla \chi \cdot \nabla^2 \chi, \]
\[ \quad (10.15) \]

then we have,
\[ \nabla_4 \dot{\eta} = \nabla^2 (\eta + \chi) \cdot \nabla \chi + \frac{1}{2} \nabla^2 \chi \cdot \alpha + \frac{1}{2} \nabla \chi \cdot \nabla \alpha + \text{l.o.t} \]
\[ = (\nabla \rho, \nabla \sigma) \nabla \chi + \frac{1}{2} \nabla^2 \chi \cdot \alpha + \frac{1}{2} \nabla \chi \cdot \nabla \alpha + \text{l.o.t.} \quad (10.16) \]

Thus, we have derived,
\[ \|\dot{\eta}\|_{L^2(S)} \leq \int_0^\infty \|\nabla \beta\|_{L^2(S)} \|\alpha\|_{L^2(S)} + \int_0^\infty \|\nabla \rho, \nabla \sigma\|_{L^2(S)} \|\nabla \chi\|_{L^2(S)} + \int_0^\infty \|\nabla \alpha\|_{L^2(S)} \|\nabla \chi\|_{L^2(S)} + \text{l.o.t} \]
\[ \leq \int_0^\infty \|\nabla \beta\|_{L^2(S)} \|\nabla \alpha\|_{L^2(S)} + \int_0^\infty \|\nabla \rho, \nabla \sigma\|_{L^2(S)} \|\nabla \chi\|_{L^2(S)} + \frac{\delta}{u \tau} \int_0^\infty \|\nabla \alpha\|_{L^2(S)} \|\nabla \chi\|_{L^2(S)} + \text{l.o.t.} \]
\[ \leq \frac{1}{u \tau} \|\nabla \beta\|_{L^2(H_0^0, \omega)} \|\nabla \alpha\|_{L^2(H_0^0, \omega)} + \frac{1}{u \tau} \|\nabla \rho, \nabla \sigma\|_{L^2(H_0^0, \omega)} \|\nabla \chi\|_{L^2(H_0^0, \omega)} + \frac{1}{u \tau} \|\nabla \chi\|_{L^2(H_0^0, \omega)} + \text{l.o.t.} \]
\[ + \frac{\delta}{u \tau} \|\nabla \rho, \nabla \sigma\|_{L^2(H_0^0, \omega)} \|\nabla \alpha\|_{L^2(H_0^0, \omega)} + \frac{1}{u \tau} \|\nabla^2 \alpha\|_{L^2(H_0^0, \omega)} + \text{l.o.t.} \]
\[ \leq \frac{\delta}{u \tau} + \frac{\delta}{u \tau} \leq \frac{\delta}{u \tau} \quad (10.17) \]
From this, we obtain,

\[ \| \nabla^2 \mu \|_{L^2(S(u, \omega))} \leq \delta^{-\frac{3}{4}} |u|^{-\frac{3}{4}}, \quad (10.18) \]

and

\[ \| \nabla^2 \mu \|_{L^2(H)} \leq \delta^{-\frac{1}{4}} |u|^{-\frac{1}{4}}. \quad (10.19) \]

Recalling the results from the Section Curvature Estimates III, we have

\[ \| \nabla^2 \beta \|_{L^2(H)} \leq \delta^{-\frac{1}{2}} |u|^{-\frac{1}{2}} \epsilon, \quad (10.20) \]

\[ \| \nabla^2 (\rho, \sigma) \|_{L^2(H)} \leq \delta^{-\frac{1}{4}} |u|^{-\frac{1}{4}} \epsilon. \quad (10.21) \]

Thus, we can prove,

\[ \| \nabla^2 \eta \|_{L^4(S(u, \omega))} \leq \delta^{-\frac{3}{4}} |u|^{-\frac{3}{4}} \epsilon. \quad (10.22) \]

We also already know that,

\[ \| \nabla \eta \|_{L^4(S(u, \omega))} \leq \delta^{-\frac{1}{4}} |u|^{-\frac{1}{4}}. \]

From,

\[ \| \nabla \eta \|_{L^\infty(S(u, \omega))} \leq \| \nabla \eta \|_{L^4(S(u, \omega))} \| \nabla^2 \eta \|_{L^2(S(u, \omega))} + 1 \| \nabla \eta \|_{L^4(S(u, \omega))}, \quad (10.23) \]

it follows that,

\[ \| \nabla \eta \|_{L^\infty(S(u, \omega))} \leq \delta^{-\frac{3}{4}} \epsilon^{-\frac{1}{4}}. \quad (10.24) \]

Recall that the initial data quantity is given by,

\[ \mathcal{I}^{(0)} = \sup_{0 \leq u \leq u_0} \mathcal{I}^{(0)}(u) \]

where,

\[ \mathcal{I}^{(0)}(u) = \delta^{-\frac{3}{2}} |u_\infty| \| \tilde{\chi}_0 \|_{L^\infty(S(u_\infty, \omega))} + \sum_{0 \leq k \leq 2} \delta^{-\frac{3}{2}} \| (\delta^2 |u_\infty| \nabla)^{k-1} (\delta^2 \nabla \tilde{\chi}_0) \|_{L^2(S(u_\infty, \omega))} \]

\[ + \sum_{0 \leq k \leq 1} \sum_{1 \leq m \leq 4} \delta^{-\frac{3}{2}} \| (\delta^2 |u_\infty| \nabla)^{k+m-1} (\delta^2 \nabla \tilde{\chi}_0) \|_{L^2(S(u_\infty, \omega))} \quad (10.25) \]

Thus, we have proved the following theorem:

**Theorem 2** Given Minkowskian initial data on \( H^{(u_\infty, \omega)} \), on \( H^{(0, \delta)} \) assume that, in addition to \( \mathcal{I}^{(0)} \), we also have for \( 2 \leq k \leq 4 \)

\[ \delta^{-\frac{3}{2}} \| (\delta^2 |u_\infty| \nabla)^{k} \tilde{\chi}_0 \|_{L^2(S(u_\infty, \omega))} \leq \epsilon \]

for a sufficiently small parameter \( \epsilon \) such that \( 0 < \delta \ll \epsilon \). Assume also that \( \tilde{\chi}_0 \) verifies

\[ \frac{2}{|u_\infty|^2} < \int_0^{u_\infty} |\tilde{\chi}_0|^2(u_\infty, u') du' < \frac{2}{|u_\infty|}. \]

Then, for \( \delta > 0 \) sufficiently small, a trapped surface must form in the slab \( \mathcal{D}(u \approx 1, \delta) \).
11 In the pursuit of Christodoulou’s results

In this section we will prove using the above methods, that given the same initial data as in [Chr], we can get the same results as in [Chr].

Above we have proved a semi-global existence result in weaker norms. As we explained in the introduction, here with initial data in [Chr], we hope to promote the results in weaker norms to stronger norms.

11.1 Initial data.

From Chapter 2 in [Chr], the initial data along $H_{u_0}$ he describes satisfies the conditions below:

$$\omega = 0, \quad (\|\omega\|_{L^\infty(S(u_0, \Omega))} \leq \frac{1}{|u_0|^2})$$

$$\|\hat{x}\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta^\frac{1}{2}}{|u_0|}, \quad \|\Omega r\chi - \frac{2}{|u_0|}\|_{L^\infty(S(u_0, \Omega))} \leq \frac{1}{|u_0|^2}, \quad \|\eta, \delta\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta^\frac{1}{2}}{|u_0|^2}$$

$$\|\chi\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta^\frac{1}{2}}{|u_0|^2}, \quad \|\Omega r\chi + \frac{2}{|u_0|}\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta}{|u_0|^2}, \quad \|\omega\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta}{|u_0|^2}$$

$$\|\beta - \frac{1}{|u_0|^2}\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta^\frac{1}{2}}{|u_0|^2}, \quad \|\beta\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta}{|u_0|^2}, \quad \|\alpha\|_{L^\infty(S(u_0, \Omega))} \leq \frac{\delta^\frac{1}{2}}{|u_0|^2}$$

where $|S(u_0, \Omega)| \approx |u_0|^2$. We will call the initial data along $H_{u_0}$ given in [Chr] Christodoulou initial data.

Remark: We also know, along $H_{u_0}$, for all the components above, once we have an extra $\nabla$, the weight for $\delta$ stays the same and the decay rate for $u$ is $\frac{1}{|u_0|^2}$ better.

11.2 Coefficients Estimates.

We hope to prove that all the inequalities for the coefficients in 11.1 still hold for $S = S(u, \Omega)$.

Before this subsection, we already have a semi-global existence result. For $(S)_{0, \infty}$, comparing these results with [Chr], we only need to improve $L^\infty(S)$ estimates for $\omega, \omega, \eta, \eta$. From [Chr], we hope to prove $\|\omega\|_{L^\infty(S)} \leq \frac{1}{|u_0|^2}$, $\|\omega\|_{L^\infty(S)} \leq \frac{\delta}{|u_0|^2}$, $\|\eta\|_{L^\infty(S)} \leq \frac{1}{|u_0|^2}$, and $\|\eta\|_{L^\infty(S)} \leq \frac{\delta}{|u_0|^2}$. We have already shown that, $\|\omega\|_{L^\infty(S)} \leq \frac{1}{|u_0|^2}$, $\|\omega\|_{L^\infty(S)} \leq \frac{1}{|u_0|^2}$, and $\|\eta\|_{L^\infty(S)} \leq \frac{1}{|u_0|^2}$.

Assume that we have the following bootstrap estimates for curvature:

$$\delta^\frac{5}{2} |u| \|\alpha\|_{L^\infty(S)} + \delta^\frac{7}{2} |u| |\beta| \|\alpha\|_{L^\infty(S)} + |u|^3 \|\rho, \sigma\|_{L^\infty(S)} + \delta^{-1} |u|^4 \|\beta\|_{L^\infty(S)} + \delta^{-\frac{1}{2}} |u|^5 \|\alpha\|_{L^\infty(S)} \leq C$$

$$\delta^\frac{5}{2} |u^2 \nabla \alpha|_{L^\infty(S)} + \delta^\frac{7}{2} |u^2 |\nabla \beta| \|\alpha\|_{L^\infty(S)} + |u^2 |\nabla \rho, \nabla \sigma\|_{L^\infty(S)} + \delta^{-1} |u^2 |\nabla \beta| \|\alpha\|_{L^\infty(S)} + \delta^{-\frac{1}{2}} |u^2 |\nabla \alpha\|_{L^\infty(S)} \leq C$$

$$\delta |u| \|\alpha\|_{L^2(H)} + |u| |\beta| \|\alpha\|_{L^2(H)} + \delta^{-\frac{1}{2}} |u| \|\rho, \sigma\|_{L^2(H)} + \delta^{-\frac{1}{2}} |u|^3 \|\beta\|_{L^2(H)} \leq C$$

$$\delta |u| \|\nabla \alpha\|_{L^2(H)} + |u|^2 \|\nabla \beta\|_{L^2(H)} + \delta^{-\frac{1}{2}} |u|^3 \|\nabla \rho, \nabla \sigma\|_{L^2(H)} + \delta^{-\frac{1}{2}} |u|^4 \|\nabla \beta\|_{L^2(H)} \leq C$$

$$\delta |u^2 |\nabla^2 \alpha\|_{L^2(H)} + |u|^3 \|\nabla^2 \beta\|_{L^2(H)} + \delta^{-\frac{1}{2}} |u|^4 \|\nabla^2 \rho, \nabla^2 \sigma\|_{L^2(H)} + \delta^{-\frac{1}{2}} |u|^5 \|\nabla^2 \alpha\|_{L^2(H)} \leq C$$

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\[ \delta^{-\frac{2}{p}} \|u^i\|_{L^2(S)} \leq C, \quad \delta^{-\frac{2}{p}} \|u^i\|_{L^2(S)} \leq C, \quad \delta^{-\frac{2}{p}} \|u^i\|_{L^2(S)} \leq C. \]

Using Christodoulou’s initial data and using the methods used to obtain the Ricci coefficient estimates before this subsection, we have:

For \( \omega \), using \( \nabla_3 \omega = 2\omega + \eta, \eta \) (\( \| \eta \| \leq 1 \)), we can prove \( \| \omega \|_{L^\infty(S)} \leq \frac{\delta}{|u|^2} \).

For \( \omega \), using \( \nabla_4 \omega = 2\omega + \eta, \eta \) (\( \| \eta \| \leq 1 \)), we can prove \( \| \omega \|_{L^\infty(S)} \leq \frac{\delta}{|u|^2} \).

For \( \eta \), using \( \nabla_4 \eta = -\chi \cdot (\eta - \eta) - \beta \), we can prove \( \| \eta \|_{L^\infty(S)} \leq \frac{\delta}{|u|^2} \).

Once we have the desired \( \mathcal{O}_{0,\infty} \) estimates, we can easily get the desired \( \mathcal{O}_{0,2} \) estimates and \( \mathcal{O}_{0,4} \) estimates on \( S(u, \bar{u}) \) with Hölder’s inequality.

For \( \mathcal{O}_{1,4} \), comparing our semi-global existence results with [Chr], we need to improve the results for every component. The main reason is that, in [Chr], we have shown that \( \| \omega \|_{L^2(S)} \leq \frac{\delta}{|u|^2} \).

Using Christodoulou’s initial data and using the methods used to obtain the Ricci coefficient estimates before this subsection, we have:

For \( \nabla \omega \), using the equation for \( \nabla_3 \nabla \omega \), we can show that \( \| \nabla \omega \|_{L^1(S)} \leq \frac{\delta}{|u|^2} \).

For \( \nabla \omega \), using the equation for \( \nabla_4 \nabla \omega \), we can show that \( \| \nabla \omega \|_{L^1(S)} \leq \frac{\delta}{|u|^2} \).

For \( \nabla \eta \), using the equation for \( \nabla_4 \nabla \eta \), we can show that \( \| \nabla \eta \|_{L^1(S)} \leq \frac{\delta}{|u|^2} \).

Once we have the desired \( \mathcal{O}_{1,4} \) estimates, we can easily get the desired \( \mathcal{O}_{1,2} \) estimates on \( S(u, \bar{u}) \) using Hölder’s inequality.

### 11.3 Energy Estimates.

For energy estimates, using the methods outlined in Section 7 and Section 8 with the new coefficients estimates in subsection 11.2, we need to close a small bootstrap arguments.

For the energy estimates for a curvature component \( \Psi \), in the Sections Curvature Estimates I, Curvature Estimates II and Curvature Estimates III, the only borderline terms are \( tr \chi (\Psi)^2 \), \( tr \chi (\nabla \Psi)^2 \), and \( tr \chi (\nabla^2 \Psi)^2 \), respectively.

We recall that the key point of the new type of energy estimates obtained in Section 7 and Section 8 is that it kills the \( tr \chi (\Psi)^2 \), \( tr \chi (\nabla \Psi)^2 \), and \( tr \chi (\nabla^2 \Psi)^2 \) terms by adding the \( u \) weight in the beginning. That is to say, all the borderline terms are cancelled. Thus we can
easily close our new bootstrap argument to get the desired energy estimates.

**Remark:** With the new type of energy estimates we have cancelled the borderline terms, which are the worst terms. The other terms are easy to deal with. Here, we will see an example for how to deal with the second worst terms.

Considering the energy estimates for \( \beta \) and \( \alpha \), we use

\[
\nabla \beta + 2 \text{tr} \chi \beta = -\text{div} \alpha - 2 \omega \beta + \eta \cdot \alpha. \tag{11.1}
\]

\[
\nabla \alpha + \frac{1}{2} \text{tr} \chi \alpha = -\nabla \beta + 4 \omega \alpha - 3 (\hat{\chi} \rho - \hat{\chi} \sigma) + (\zeta - 4 \eta) \hat{\beta}. \tag{11.2}
\]

When we do energy estimates using a weight \( u \), we cancel the borderline term \( \text{tr} \chi \beta^2 \).

The second worst term \( \hat{\chi} \alpha(\rho, \sigma) \) is left. From a heuristic view of point, \( \| \text{tr} \chi \beta^2 \|_{L^\infty(S)} \approx \frac{\| \rho \|^2}{\| u \|^2} \).

This term is the borderline term which cannot be dealt with by using Gronwall’s inequality.

While \( \hat{\chi} \alpha(\rho, \sigma) \) is better because it has a better decay rate. That is, \( \| \hat{\chi} \alpha(\rho, \sigma) \|_{L^\infty(S)} \leq \frac{\| \rho \|^2}{\| u \|^2} \).

So, because this term is better than the borderline term in terms of \( u \) weight, it suggests that this term can be dealt with by using Gronwall’s inequality. This heuristic argument is correct.

Moreover, we can prove the desired result:

\[
\int_{S(u, \omega)} |u|^6 \beta^2 + \int_{u_\infty}^u \int_{S(u, \omega)} |u|^6 \alpha^2 = \int_{0}^u \int_{u_\infty}^u \int_{S(u, \omega)} |u|^6 \hat{\chi}(\rho, \sigma) \alpha + \text{l.o.t.}
\]

Thus, by Gronwall’s inequality, we can prove the desired result:

\[
\int_{S(u, \omega)} |u|^6 \beta^2 + \int_{u_\infty}^u \int_{S(u, \omega)} |u|^6 \alpha^2 \leq \frac{\delta^3}{\| u \|} \leq \delta^3.
\]

We will have the same types of second worst term \( \hat{\chi} \alpha \rho \) in the energy estimates for \( \beta \) and \( \rho \). \( \hat{\chi} \alpha \rho \) can be dealt with by following the methods in the example above. All the other terms are truly lower order terms in terms of \( \delta \), and we can deal with these lower order terms very easily.

**In summary,** with Christodoulou initial data, and by using a new strategy, which is stated in this Section, we can easily prove the same results for all curvature components and coefficient coefficients as in [Chr].

**Theorem 3** If given Minkowskian initial data on \( H^{(u_\infty, u)} \) and Christodoulou initial data on \( H^{(0, \delta)} \), we can prove estimates that are consistent with [Chr].
References

[Chr] D. Christodoulou, *The Formation of Black Holes in General Relativity*, Monographs in Mathematics, European Mathematical Soc. 2009.

[Chr-Kl] D. Christodoulou, S. Klainerman, *The global nonlinear stability of he Minkowski space*, Princeton mathematical series 41, 1993.

[Hol] G. Holzegel, *Ultimately Schwarzschildian spacetimes and the black hole stability problem*, preprint, 2010, arXiv:1010.3216

[K-Ni] S. Klainerman, F. Nicolo, *The evolution problem in General Relativity*, Progress in Mathematical Physics, Birkhäuser.

[K-R:causal] S. Klainerman, I. Rodnianski, *Causal geometry of Einstein-Vacuum spacetimes with finite curvature flux*, Inventiones Math., 159, 437-529 (2005).

[K-R:LP] S. Klainerman, I. Rodnianski, *A geometric approach to the Littlewood-Paley theory*, GAFA, 16, no. 1, 126-163.

[K-R:Scared] S. Klainerman, I. Rodnianski, *On emerging scarred surfaces for the Einstein vacuum equations*, Discrete Contin. Dyn. Syst., 28 (2010), no. 3, 1007-1031.

[K-R:Trapped] S. Klainerman, I. Rodnianski, *On the the formation of trapped surfaces*, preprint, 2011, arXiv:0912.5097

[Luk] J. Luk, *On the Local Existence for the Characteristic Initial Value Problem in General Relativity*, preprint, 2011, arXiv:1107.0898

[L-R:Propagation] J. Luk, I. Rodnianski, *Local propagation of impulsive gravitational waves*, preprint, 2011

[L-R:Interaction] J. Luk, I. Rodnianski, *Nonlinear interactions of impulsive gravitational waves for the vacuum Einstein equations*, preprint, 2011