Wiener measure for Heisenberg group

Heping Liu\textsuperscript{1} and Yingzhan Wang\textsuperscript{2}

\textsuperscript{1} LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China
\textsuperscript{2} School of Sciences, South China University of Technology, Guangzhou 510641, P. R. China

\textsuperscript{2} Corresponding Author

E-mail: \textsuperscript{1}hpliu@pku.edu.cn, \textsuperscript{2}wyzde@pku.edu.cn

Abstract. In this paper, we build Wiener measure for the path space on the Heisenberg group by using of the heat kernel corresponding to the sub-Laplacian and give the definition of the Wiener integral. Then we give the Feynman-Kac formula.

Keywords: Heisenberg group, C-C distance, sub-Laplacian operator, Wiener measure, Feynman-Kac formula.

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1 Introduction

Wiener measure was first built by using of the heat kernel corresponding to Laplacian $\triangle$ by Wiener in 1920s. It gives a perfect explanation to some physics problem. (see [4]). The connection between Wiener measure and Schrödinger equation has been known for a long time. Kac in 1949 first gave the explicit solution expression of the following equation

$$\begin{align*}
\left\{ \begin{array}{l}
\partial_t u(t, \xi) = \frac{1}{2} \partial_{\xi}^2 u(t, \xi) - V(\xi) u(t, \xi) \\
u(0, \xi) = f(\xi)
\end{array} \right.
\end{align*}$$

through Wiener integral. And the upper Wiener integral solution is the famous Feynman-Kac formula. Over the recent years, there are still much work on Wiener measure. See [5],[6],[7]. In this paper, we study the Wiener space on Heisenberg group.

We suppose $\mathbb{H}$ Heisenberg group with underlining manifold $\mathbb{R}^{2n+1} \cong \mathbb{C}^n \times \mathbb{R}$. The multiplication is given by

$$(z, u)(z', u') = (z + z', u + u' + 2\text{Im}(zz')).$$

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There is a homogeneous norm $| \cdot |$ satisfying the subadditivity inequality:

$$|\xi| = (|z|^4 + u^2)^{\frac{1}{4}}, \text{ for } \xi = (z, u) \in \mathbb{H}.$$ 

Obviously it satisfies: $|\xi \eta| \leq |\xi| + |\eta|$, and $|\xi| = |\xi^{-1}|$ for $\xi, \eta \in \mathbb{H}$. Thus we can define the distance $N$ on $\mathbb{H}$.

$$N(x, y) = |xy^{-1}|, x, y \in \mathbb{H}.$$ 

Let $C_o[0, 1]$ denote the set of continuous functions $x(t)$ in the unit interval $[0,1]$ valued in $\mathbb{H}$ with $x(0) = o$, where $o$ is the unit element of $\mathbb{H}$. Then $C_o[0, 1]$ is a Banach space with the norm $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$. And $C_o[0, 1]$ is called the Wiener space in Heisenberg group.

If $X_i, Y_i, U (1 \leq i \leq n)$ denote the left invariant vector fields on $\mathbb{H}$ whose values at $o$ are given by $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial u}$ respectively, then

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial u}, Y_i = \frac{\partial}{\partial y_i}, U = \frac{\partial}{\partial u}.$$ 

Let $\mathcal{L}$ be the Kohn-laplacian operator sometimes called sub-Laplacian operator.

$$\mathcal{L} = -\sum_{i=1}^{n} (X_i^2 + Y_i^2).$$ 

For $\mathcal{L}$, the heat kernel’s explicit expression was first given by A. Hulanicki and B. Gaveau.

$$p_t(x, y, u) = (2\pi)^{-1} (4\pi)^{-n} \int_{\mathbb{R}} \left( \frac{|\lambda|}{\sinh |\lambda|} \right)^n \exp\left\{ -\frac{|\lambda||z|^2}{4\coth |\lambda|s - i\lambda \cdot u} \right\} d\lambda.$$ 

(see [1], [2].) Then we can use it to build the wiener measure in Heisenberg group.

This paper is organized in three sections. In the second section, we build the Wiener measure for the Wiener space in Heisenberg group. Then we define the Wiener integral. In the last section, we use it to give the Feynman-Kac formula corresponding to $\mathcal{L}$. Throughout the whole paper, constant $c$ may not be the same at every appearance.

### 2 Wiener measure and Wiener integral

As described in the introduction, Wiener space is the following set,

$$C_o[0, 1] = \{ x : [0, 1] \to \mathbb{H}, \text{continuous, } x(0) = o. \}$$

It is a Banach space with the norm $\| \cdot \|$. The corresponding Borel field is denoted by $\mathcal{B}$. To build the Wiener measure, first we will build a measure on one of its generating algebras. Next we extend the measure to the whole field.
For $E$ a Borel set in $\mathbb{R}^{2n+1}$, we define the cylinder set in $C_0[0,1]$:

$$I = \{ x \in C_0[0,1] : (x_{t_1}, x_{t_2}, ..., x_{t_m}) \in E \}$$

where $0 < t_1 < t_2 < t_3 < ... < t_m \leq 1$. And denote the set of all the cylinder sets by $\mathcal{B}$. Obviously $\mathcal{B}$ is a subfield of $\mathcal{B}$. In fact, we have the following lemma.

**Lemma 1.** $\sigma(\mathcal{B}) = \mathcal{B}$

**Proof.** Since $\mathcal{B}$ is a subfield of $\mathcal{B}$, so

$$\sigma(\mathcal{B}) \subseteq \mathcal{B}.$$  

And we can see obviously

$$\{ x \in C_0[0,1] \mid \|x\| \leq 1 \} = \bigcap_{m=1}^{\infty} \{ x \in C_0[0,1] \mid |x(\frac{k}{2^m})| \leq 1, k = 1, 2, ..., 2^m \}.$$  

So the closed unit ball is in $\sigma(\mathcal{B})$. Then

$$\sigma(\mathcal{B}) \supseteq \mathcal{B},$$

And we proved the $\sigma$-field generated by $\mathcal{B}$ is just $\mathcal{B}$.  

For $I, E, p_t(\cdot)$ described above, Denote

$$W(I) = \int_E \prod_{j=1}^{m} p_{t_j - t_{j-1}}(u_{j-1}^{-1}u_j) \prod_{j=1}^{m} du_j.$$  

(2)

Then we give our main theorem of this paper.

**Theorem 1.** $W$ defined above is a measure on $\mathcal{B}$. Furthermore $W$ is $\sigma$-additive on $\mathcal{B}$. Then it can be extended to a measure on the whole $\sigma$-field $\mathcal{B}$.

**Proof.** Denote by

$$I_1 = \left\{ x \in C_0[0,1] : (x_{t_1}, x_{t_2}, ..., x_{t_p}) \in B_1 \times B_2 \times ... \times B_p \right\},$$

$$I_2 = \left\{ x \in C_0[0,1] : (x_{t_2}, x_{t_3}, ..., x_{t_q}) \in C_1 \times C_2 \times ... \times C_q \right\},$$

where $B_i, C_j, i = 1, 2, ...p ; j = 1, 2, ..., q$, are Borel sets in $\mathbb{R}^{2n+1}$. To prove $W$ is a measure on $\mathcal{B}$, we need only prove when $I_1 \cap I_2 = \emptyset$, we have

$$W(I_1 \cup I_2) = W(I_1) + W(I_2).$$
We make time take the same values in the upper two sets. To do this, we need only add some $t_k$ and $B_k$ (or $C_k$), where $B_k$ (or $C_k$) is a set of the form $R^{2n+1}$. So that we get:

$$I'_1 = \{ x \in C_o[0,1] : (x_{t_1}, x_{t_2}, \ldots, x_{t_m}) \in B_1 \times B_2 \times \ldots \times B_m \},$$

$$I'_2 = \{ x \in C_o[0,1] : (x_{t_1}, x_{t_2}, \ldots, x_{t_m}) \in C_1 \times C_2 \times \ldots \times C_m \},$$

Obviously $I_1 = I'_1, I_2 = I'_2$. Since $I_1 \cap I_2 = \emptyset$, there must be some $j$ such that $B_j \cap C_j = \emptyset$. Then by expression (2),

$$W(I_1 \cup I_2) = W(I'_1 \cup I'_2) = W(I'_1) + W(I'_2).$$

So we need only prove this kind of equality $W(I) = W(I')$, where

$$I = \{ x \in C_o[0,1] : (x_{t_1}, x_{t_2}) \in B_1 \times B_2 : t_1 < t_2 \},$$

$$I' = \{ x \in C_o[0,1] : (x_{t_1}, x_{t_2}) \in B_1 \times R^{2n+1} \times B_2 : t_1 < t < t_2 \}.$$

The other cases can be proved analogously.

Through the definition of $W$, and using the properties of the heat kernel, we can get

$$W(I') = \int_{B_1} \int_{R^{2n+1}} \int_{B_2} p_{t_1}(u_1)p_{t_2-t_1}(u_1^{-1}u_2)p_{t_2-t}(u_2^{-1}u_3)du_1du_2du_3$$

$$= \int_{B_1} \int_{B_2} p_{t_1}(u_1)p_{t_2-t_1}(u_1^{-1}u_2)du_1du_2$$

$$= W(I).$$

Now we finished the proof that $W$ is a measure on $R$. If we can prove $W$ is $\sigma$-additive on $R$, then it has an unique extension to $R$. We need only prove the following claim:

If $\{I_m\}$ ($m = 1, 2, \ldots$) is a sequence of cylinder sets in $R$, $I_j \subset I_{j+1}$, and $\cap_{m=1}^{\infty} I_m = \emptyset$, then $\lim_{m \to \infty} W(I_m) = 0$.

Let

$$I_m = (t_{1}^{(m)}, t_{2}^{(m)}, \ldots, t_{s_m}^{(m)}, E_m)$$

$$\equiv \{ x \in C_o(0,1) : (x_{t_1}^{(m)}, x_{t_2}^{(m)}, \ldots, x_{t_{s_m}^{(m)}}) \in E_m \subset R^{s_m(2n+1)} \}.$$  

For $\forall \varepsilon > 0$, we choose a closed set $G_m \subset E_m$ such that

$$W(I_m \setminus K_m) < \frac{\varepsilon}{2m+1},$$

where

$$K_m = (t_{1}^{(m)}, t_{2}^{(m)}, \ldots, t_{s_m}^{(m)}; G_m).$$
Let \( L_m = \bigcap_{j=1}^{m} K_j \in \mathcal{R} \).
Then
\[
L_m \subset K_m \subset I_m.
\]

Hence
\[
W(I_m) = W(I_m \setminus L_m) + W(L_m).
\]

Scince
\[
I_m \setminus L_m = I_m \setminus \bigcap_{j=1}^{m} K_j = \bigcup_{j=1}^{m} (I_m \setminus K_j) \subset \bigcup_{j=1}^{m} (I_j \setminus K_j),
\]
Then
\[
W(I_m \setminus L_m) \leq \sum_{j=1}^{m} \frac{\varepsilon}{2^{m+1}} \leq \frac{\varepsilon}{2}.
\]

So we need prove there exists \( m_0 \) such that \( W(L_m) < \frac{\varepsilon}{2} \) for any \( m > m_0 \). Here we give the following lemma that we leave its proof behind:

**Lemma 2.** Let \( a > 0 \) and \( 0 < r < \frac{1}{2} \). Denote
\[
H^r(a) = \{ x \in C_0[0,1] \mid |x(t_2)^{-1}x(t_1)| \leq \frac{2a}{1-2^{-r}}|t_1 - t_2|^r, \forall t_1, t_2 \in [0,1] \}.
\]

Then for \( I \in \mathcal{R} \), if \( I \subset H^r(a)^c \), we have
\[
\lim_{a \to \infty} W(I) = 0.
\]

Now we suppose Lemma 2 is right. If we can prove when \( a \) is large enough, \( L_m \subseteq H^r(a)^c \), then for \( m \) large enough, \( W(L_m) < \frac{\varepsilon}{2} \) is obvious. So the only thing we should prove is there exists \( m_0 \) such that \( M_m = L_m \cap H^r(a) = \emptyset \) for any \( m > m_0 \). Note \( \{M_m\}_{m \geq 1} \) is a decreasing sequence, and \( \bigcap_{m=1}^{\infty} M_m = \emptyset \). If \( M_m \neq \emptyset \) for any \( m \), we choose \( x_m \in M_m \). Then \( \{x_m\}_{m \geq 1} \) is equi-continuous and uniformly bounded. Therefore by Ascoli-Arzelas theorem, we can choose a convergent sequence still denoted by \( \{x_m\}_{m \geq 1} \).

Suppose \( \lim_{m \to \infty} x_m = x_0 \). Then \( x_0 \in H^r(a) \). Note that \( M_m \) is a compact set, and \( x_m \in M_m, m \geq m_0 \). So \( x_0 \in M_m, \forall m \geq 1 \), which is a contrary to \( \bigcap_{m=1}^{\infty} M_m = \emptyset \). And theorem 1 is proved. So what left is to prove Lemma 2.

**Proof of Lemma 2.** Because \( N(g,h) = |g^{-1}h| \) is a distance on \( \mathbb{H} \), it is subadditive. Then, for \( x \in C_0[0,1] \), if there exist \( a > 0, r > 0 \) such that
\[
|x\left(\frac{k}{2^{m}}\right)x\left(\frac{k-1}{2^{m}}\right)^{-1}| \leq a\left(\frac{1}{2^{m}}\right)^r, \forall k = 1, 2, ..., 2^m, \forall m > 0,
\]
Then
\[
|x(t_1)x(t_2)^{-1}| \leq \frac{2a}{1-2^{-r}}|t_1 - t_2|^r,
\]
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where $t_1, t_2 \in [0, 1]$, and each of them can be written as $\frac{k}{2^n}, k$ is odd.

In fact, for $t_1 \leq t_2$, we can choose $t = \frac{q}{2^p} q$ is odd, and $p$ the smallest such that $t_1 \leq t \leq t_2$. If $t \neq t_1$, then we can write

$$t - t_1 = \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + ... + \frac{1}{2^{m_j}}, m_1 < m_2 < ... < m_j$$

and if $t \neq t_2$,

$$t_2 - t = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + ... + \frac{1}{2^{n_k}}, n_1 < n_2 < ... < n_k.$$

Consider the intervals,

$$[t_1, t_1 + \frac{1}{2^{m_j}}], [t_1 + \frac{1}{2^{m_j}}, t_1 + \frac{1}{2^{m_j}} + \frac{1}{2^{m_{j-1}}}], ... [t - \frac{1}{2^{m_1}}, t]$$

and

$$[t, t + \frac{1}{2^{n_1}}], [t + \frac{1}{2^{n_1}}, t + \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}}], ... [t_2 - \frac{1}{2^{n_k}}, t_2].$$

Let $l = \min\{m_1, n_1\}, s = \max\{m_j, n_k\}$, then we can see

$$|x(t_1)^{-1}x(t_2)| \leq 2a \sum_{k=l}^{s} \frac{1}{2^k}r \leq \frac{2a}{1-2^{-r}}|t_1 - t_2|^r.$$

Let

$$I_{a,k,m}^r = \left\{ x \in C_0(0, 1) \mid |x(\frac{k}{2^m})x(\frac{k-1}{2^m})^{-1}| > a(\frac{1}{2^m})^r \right\}, k = 1, 2, ..., 2^m.$$

Then we have

$$H^r(a) \supset \bigcap_{m=0}^{\infty} \bigcap_{k=1}^{2^m} (I_{a,k,m}^r)^c.$$  

So

$$I \subset (H^r(a))^c \subset \bigcup_{m=0}^{\infty} \bigcup_{k=1}^{2^m} I_{a,k,m}^r.$$

It follows that

$$W(I) \leq \sum_{m=0}^{\infty} \sum_{k=1}^{2^m} W(I_{a,k,m}^r). \quad (3)$$

Denote

$$E = \left\{ (u_1, u_2) \in \mathbb{H} : |u_2u_1^{-1}| > a(\frac{1}{2^m})^r \right\},$$
then
\[
W(I_{a,k,m}) = \int_E p_{k-1}(u_1)p_{\frac{k-1}{2m}}(u_2 u_1^{-1}) du_1 du_2
\]
\[
= \int_{|u_3|>a(\frac{1}{2m})^r} \int_H p_{k-1}(u_1)p_{\frac{k-1}{2m}}(u_3) du_1 du_3
\]
\[
= \int_{|u_3|>a(\frac{1}{2m})^r} p_{\frac{k-1}{2m}}(u_3) du_3. \tag{4}
\]

Now we need an estimate of the heat kernel. Denote \(d(x, y)\) the Carnot-Carathéodory distance associated to \(\mathcal{L}\) and \(B(x, r) = \{y \in M : d(x, y) < r\}\) the corresponding Balls. We refer the reader to [8] for those definitions and the following important estimate.

\[
p_t(x^{-1}y) \leq \frac{C_1}{\mu(B(x, t^{\frac{n}{2}}))} \exp\left(-\frac{C_1 d(x, y)^2}{t}\right), \forall x, y \in \mathbb{H}.
\]

From [3], we can get \(d(x, y) \geq N(x, y)\). Then there exists a constant \(M\), such that

\[
p_t(\xi) \leq cM t^{-n-1} \exp\left(-\frac{M^{-1} |\xi|^2}{t}\right), \forall \xi \in N. \tag{5}
\]

So

\[
(4) \leq cM \int_{|u_3|>a(\frac{1}{2m})^r} \exp\left(-M^{-1} 2^m |u_3|^2\right) \sum_{m=0}^{\infty} 2^m \rho^{m+1} d\rho
\]
\[
\leq c \int_{\rho>a(\frac{1}{2m})^r} \exp\left(-M^{-1} 2^m \rho^2\right) \rho^{2n+1} d\rho
\]
\[
\leq c 2^{\frac{m}{2}} \int_{\rho>a(\frac{1}{2m})^r} \exp\left(-M^{-1} 2^{m-1} \rho^2\right) d\rho
\]
\[
\leq c 2^{\frac{m}{2}} \int_{\rho>a(\frac{1}{2m})^r} \exp\left(-M^{-1} 2^{m-1} \rho^2\right) \frac{\rho}{a(\frac{1}{2m})^r} d\rho
\]
\[
= \frac{c}{a} 2^{m(r-\frac{1}{2})} e^{-(2M)^{-1} 2^{2m(1-2r)} a^2}.
\]

Then by (3),

\[
W(I) \leq \sum_{m=0}^{\infty} \sum_{k=1}^{2^m} \frac{c}{a} 2^{m(r-\frac{1}{2})} e^{-(2M)^{-1} 2^{2m(1-2r)} a^2}
\]
\[
\leq \sum_{m=0}^{\infty} \frac{c}{a} 2^{m(r+\frac{1}{2})} e^{-(4M)^{-1} m(1-2r) a^2}
\]
\[
= \frac{c}{a} \left(1 - 2^{r+\frac{1}{2}} e^{-(4M)^{-1} a^2(1-2r)}\right)^{-1}
\]
Then
\[ \lim_{a \to \infty} W(I) = 0. \]

The proof of the lemma 2 is finished. \( \square \)

So \( W \) on \( \mathcal{B} \) has an unique extension to \( \mathcal{B} \) still denoted by \( W \). We call \( W \) the Wiener measure on \( C_0[0,1] \). The correspondent integral is called Wiener integral. For \( f \in C_0[0,1] \), we define its Wiener integral by
\[ E^W[f] = \int_{C_0[0,1]} f(x)W(dx). \]

\( W \) can also be seen as the Wiener measure on the whole function space \( C[0,1] \), through \( \hat{W}(A) := W(A \cap C_0[0,1]) \), for any Borel set \( A \in C[0,1] \). For \( \xi \in \mathbb{H} \), we denote the translation by \( (T_{\xi}x)(t) := \xi x(t), \forall \ x \in C[0,1] \). Then we can define Wiener measure \( W_{\xi} \) on \( C[0,1], W_{\xi}(A) := W(T_{\xi}^{-1}A) \), for any Borel set \( A \in C[0,1] \). The corresponding Wiener integral can be defined similarly.

### 3 Feynman-Kac formula

In this section, we will use Wiener integral to give an explicit solution expression to the following type equation.
\[
\begin{aligned}
(\partial_t - \Delta_z - 4|z|^2 \partial^2_t + 4 \sum_{j=1}^n (x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}))u(t, \xi) &= -V(\xi)u(t, \xi) \\
u(0, \xi) &= f(\xi)
\end{aligned}
\]  
(6)

In the upper equation, let
\[
X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial u}, Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial u},
\]
which are just the left invariant vector fields of Heisenberg group. Then equation (6), turns into
\[
\begin{aligned}
(\partial_t + \mathcal{L})u(t, \xi) &= -V(\xi)u(t, \xi) \\
u(0, \xi) &= f(\xi)
\end{aligned}
\]  
(7)

Here we follow the classical method.

For almost every \( a \in \mathbb{H} \), we denote \( \delta_a(d\xi) \) the dot measure on \( \mathbb{H} \). That is, for any measurable function, it holds that
\[
\int_{\mathbb{H}} f(\xi)\delta_a(d\xi) = f(a).
\]
Note
\[ E^W[\delta_{x(t)}(B)] = \int_B p_t(u)du, \quad \text{for any Borel set } B \text{ in } \mathbb{H}. \]

So
\[ \frac{dE^W[\delta_{x(t)}(\cdot)]}{d\xi} = p_t(\xi). \]

We denote by \( \delta(\xi^{-1}a) \) the density function of \( \delta_a(d\xi) \). Denote \( \delta_{t,\xi}(x) = \delta(\xi^{-1}x(t)) \). Let \( E^W[\delta_{t,\xi}(x)] = \frac{d}{d\xi} E^W[\delta_{x(t)}(\cdot)] \). Then \( \delta_{t,\xi}(x) \) has its meaning.

Now we will give the next two lemmas.

**Lemma 3.** Let \( G(x) \) be a Wiener-integrable function on \( C[0,1] \), then \( E^W[G(x)\delta_{x(t)}(d\xi)] \), \((0 < t \leq 1)\) is a finite measure and absolutely continuous to the Haar measure on \( \mathbb{H} \). Furthermore for any measurable function on \( \mathbb{H} \), we have
\[ \int_{\mathbb{H}} f(\xi)E^W[G(x)\delta_{x(t)}(d\xi)] = E^W[f(x(t))G(x)] \quad (8) \]

**Proof.** Let \( B \) be a Borel set in \( \mathbb{H} \), and \( \delta_{x(t)}(B) \) a character function of cylinder \( I = \{x|x(t) \in B\} \). Obviously
\[ E^W[|G(x)\delta_{x(t)}(B)|] \leq E^W[|G(x)|]. \]

And if \( \mu(B) = 0 \), then from the definition of \( W \), we know \( W(I) = 0 \). It follows that
\[ E^W[G(x)\delta_{x(t)}(B)] = \int G(x)\delta_{x(t)}(B)W(dx) = \int I G(x)W(dx) = 0 \]

So \( E^W[G(x)\delta_{x(t)}(d\xi)] \) is absolutely continuous to the Haar measure on \( \mathbb{H} \).

For the proof of formula (8), we can see obviously it holds for simple function. For common functions, we can always make a sequence of simple function convergent to it. Then we can get the desired result. \( \square \)

Denote \( E^W[G(x)\delta_{t,\xi}(x)] = \frac{d}{d\xi} E^W[G(x)\delta_{x(t)}(\cdot)] \). And we have the following lemma.

**Lemma 4.** Let \( 0 < s < t \leq 1 \), \( G(x) \) a Wiener-integrable function and \( G(x) \) only depends on the value of \( x \) on \( [0,s] \). Then
\[ E^W[G(x)\delta_{t,\xi}(x)] = \int_{\mathbb{H}} E^W[\delta_{t-s,\eta^{-1}_1}(x)]E^W[G(x)\delta_{s,\eta}(x)]d\eta \quad (9) \]
Proof. By definition, we only need prove for any Borel set \( B \) in \( \mathbb{H} \), it holds that

\[
E^W[G(x)\delta_{x(t)}(B)] = \int_B \int_{\mathbb{H}} E^W[\delta_{t-s,\eta^{-1}\xi}(x)]E^W[G(x)\delta_{s,\eta}(x)]d\eta d\xi. \tag{10}
\]

Note first that

\[
\frac{d}{d\xi} E^W[\delta_{(x(s)^{-1}x(t))}(\cdot)] = E^W[\delta_{t-s,\xi}(x)].
\]

In fact for any Borel set \( B \),

\[
E^W[\delta_{(x(s)^{-1}x(t))}(B)] = \int_{(u_1^{-1}u_2)\in B} p_s(u_1)p_{t-s}(u_1^{-1}u_2)du_1du_2
\]
\[
= \int_{\mathbb{H}} \int_B p_s(u_1)p_{t-s}(u_3)du_1du_3
\]
\[
= \int_B p_{t-s}(u_3)du_3
\]
\[
= \int_B E^W[\delta_{t-s,\xi}(x)]d\xi
\]

Then using lemma 3,

\[
(right \ of \ (10)) = \int_{\mathbb{H}} \int_B E^W[\delta_{t-s,\eta^{-1}\xi}(x)]E^W[G(x)\delta_{s,\eta}(x)]d\xi d\eta
\]
\[
= \int_{\mathbb{H}} E^W[\chi_B(\eta x(s)^{-1}x(t))]E^W[G(x)\delta_{s,\eta}(x)]d\eta
\]
\[
= E^W[G(x)\delta_{x(t)}(B)]
\]
\[
= (left \ of \ (10)).
\]

Now we are ready to give the main theorem in this section, Feynman-Kac formula.

**Theorem 2.** Let \( V(\xi) \) be a lower bounded integrable function on the Heisenberg group \( \mathbb{H} \), and \( f(\xi) \) a bounded measurable function on \( \mathbb{H} \). Then

\[
u(t, \xi) = E^W[e^{\int_0^t V(x(s))ds}]
\]

is the solution of the differential equation (6) or (7).

To begin the proof of the theorem, we should first suppose the following conclusions hold.
Theorem 3. Let
\[ u(t, \xi) = E^W[\delta_{t, \xi}e^{-\int_0^t V(x(s))ds}] \]
Then \( u(t, \xi) \) is the solution of the following equation
\[
\begin{aligned}
(\partial_t + \mathcal{L})u(t, \xi) &= -V(\xi)u(t, \xi) \\
u(0, \xi) &= \delta_\xi \\
\lim_{\xi \to \infty} u(t, \xi) &= 0
\end{aligned}
\]
where \( \delta_\xi = 1 \), if \( \xi = 0 \); and 0, otherwise.

And through this theorem, we can get the following corollary,

Corollary 1. In upper partial differential equation, if \( V(\cdot) \) is replaced by \( V(\eta \cdot) \), Then
\[ p(t, \eta, \xi) = u(t, \eta^{-1}\xi) = E^W[\delta_{t, \eta^{-1}\xi}(x)e^{-\int_0^t V(\eta x(s))ds}] \]
is the solution of the following equation
\[
\begin{aligned}
(\partial_t + \mathcal{L})p(t, \eta, \xi) &= -V(\xi)p(t, \eta, \xi) \\
p(0, \eta, \xi) &= \delta_{\eta^{-1}\xi} \\
\lim_{\xi \to \infty} p(t, \eta, \xi) &= 0
\end{aligned}
\]

Lemma 5. \( p(t, \xi, \eta) = p(t, \eta, \xi) \).

Proof of the Theorem 2. From the definition of \( W_\xi \), for any function \( F \) on \( C[0,1] \), the following equality holds,
\[ E^{W_\xi}[F(x)] = E^W[F(\xi x)]. \]
at the meaning that one side of integral on the equal sign exists.
So
\[
\begin{aligned}
u(t, \xi) &= E^W[f(\xi x(t))e^{-\int_0^t V(\xi x(s))ds}] \\
&= \int_H f(\eta)p(t, \xi, \eta)d\eta \\
&= \int_H f(\eta)p(t, \eta, \xi)d\eta.
\end{aligned}
\]
Obviously \( u(t, \xi) \) satisfies equation (6). \( \square \)

Proof of Theorem 3. Since
\[ e^{-\int_0^t V(x(s))ds} = 1 - \int_0^t V(x(\tau))e^{-\int_0^\tau V(x(s))ds}d\tau. \]
so
\[ u(t, \xi) = E^W[\delta_t,\xi(x)] - \int_0^t E^W[V(x(\tau))e^{-\int_0^\tau V(x(s))ds}\delta_t,\xi(x)]d\tau. \]

by (8), (9), we get
\[
E^W[V(x(\tau))e^{-\int_0^\tau V(x(s))ds}\delta_t,\xi(x)]
\]
\[
= \int \mathbb{H} E^W[\delta_{t-\tau,\eta^{-1}\xi}(x)]E^W[\tau,\eta]e^{-\int_0^\tau V(x(s))ds}\delta_{\tau,\eta}(x)d\eta
\]
\[
= \int \mathbb{H} p_{t-\tau}(\eta^{-1}\xi)E^W[V(x(\tau))]e^{-\int_0^\tau V(x(s))ds}\delta_{\tau,\eta}(x)d\eta
\]
\[
= E^W[\int \mathbb{H} p_{t-\tau}(\eta^{-1}\xi)V(\eta)e^{-\int_0^\tau V(x(s))ds}\delta_{\tau,\eta}(x)d\eta]
\]
\[
= \int \mathbb{H} V(\eta)p_t(\xi,\eta)p_{t-\tau}(\eta^{-1}\xi)d\eta.
\]

So
\[ u(t, \xi) = p_t(\xi) - \int_0^t \int \mathbb{H} V(\eta)p_t(\xi,\eta)p_{t-\tau}(\eta^{-1}\xi)d\eta d\tau. \]

And (11) follows. \(\square\)

**Proof of lemma 5.** First we will see
\[ p(t, \eta, \xi) = E^W[\delta_{t,\eta^{-1}\xi}(x)e^{-\int_0^t V(\xi x(t)^{-1}x(t-\tau))d\tau}]. \] (13)

In fact for any Borel measurable function \( f \),
\[
\int \mathbb{H} f(\eta)p(t, \eta, \xi)d\eta
\]
\[
= \int \mathbb{H} f(\eta)E^W[\delta_{t,\eta^{-1}\xi}(x)e^{-\int_0^\tau V(\eta x(s))ds}]d\eta,
\]
\[
= \int \mathbb{H} f(\xi\eta^{-1})E^W[\delta_{t,\eta}(x)e^{-\int_0^\tau V(\xi\eta^{-1}x(s))ds}]d\eta
\]
\[
= E^W[\int \mathbb{H} f(\xi\eta^{-1})\delta_{t,\eta}(x)e^{-\int_0^\tau V(\xi\eta^{-1}x(s))ds}d\eta]
\]
\[
= E^W[f(\xi x(t)^{-1})e^{-\int_0^t V(\xi x(t)^{-1}x(s))ds}]
\]
\[
= \int \mathbb{H} f(\eta)E^W[\delta_{t,\eta^{-1}\xi}(x)e^{-\int_0^t V(\xi x(t)^{-1}x(s))ds}]d\eta
\]
\[
= \int \mathbb{H} f(\eta)E^W[\delta_{t,\eta^{-1}\xi}(x)e^{-\int_0^t V(\xi x(t)^{-1}x(t-\tau))d\tau}]d\eta.
\]
Next we only need prove
\[
E^W[\delta_{t_0\eta_1}(x)e^{-\int_0^t V(\xi_x(t)^{-1}x(t-r))dr}] = E^W[\delta_{t_0\eta_1}(x)e^{-\int_0^t V(\xi_x(t))dr}].
\] (14)

Define map \( T : C_0[0, t] \rightarrow C_0[0, t] \), \( Tx(s) = x(t)^{-1}x(t-s) \). Then the left of (14) is equal to \( E^W[\delta_{t_0\eta_1}(T x)e^{-\int_0^t V(\xi_x(T x)(t-r))dr}] \). So if we can prove Wiener measure is invariant under transform \( T \), formula (14) will holds.

In \( C_0[0, t] \), we need only consider the cylinder sets of the following forms
\[
I = \{ x \in C_0[0, t]; (x(t_1), x(t_2), ..., x(t_n)) \in (B_1, B_2, ..., B_n) \},
\]
where \( 0 < t_1 < t_2 < ... < t_n = t \), and \( B_1, B_2, ..., B_n \) are the Borel sets in \( \mathbb{R}^{2n+1} \).

Then
\[
TI = \{ (y(t - t_1), y(t - t_2), ..., y(t)^{-1}) \in (x(t)^{-1}B_1, x(t)^{-1}B_2, ..., B_n) \} = \{ (y(t - t_n - 1), y(t - t_n - 2), ..., y(t)) \in (y(t)B_{n-1}, y(t)B_{n-2}, ..., B_{n-1}^{-1}) \}.
\]

We need only prove \( W(I) = W(TI) \). In fact, by definition
\[
W(TI) = \int_{B_n^{-1}} \int_{B_{n-1}} ... \int_{u_nB_1} p_{t-t_n-1}(u_1)p_{t_n-1-t_n-2}(u_1^{-1}u_2)...p_{t_1}(u_1^{-1}u_n)du_1du_2...du_n
\]
\[
= \int_{B_n^{-1}} \int_{B_{n-1}} ... \int_{B_1} p_{t-t_n-1}(u_nv_1)p_{t_n-1-t_n-2}(v_1^{-1}v_2)...p_{t_1}(v_1^{-1})dv_1dv_2...du_n
\]
\[
= \int_{B_n} \int_{B_{n-1}} ... \int_{B_1} p_{t-t_n-1}(v_1^{-1}v_n)p_{t_n-1-t_n-2}(v_1^{-1}v_n)...p_{t_1}(v_1)dv_1dv_2...dv_n
\]
\[
= \int_{B_n} \int_{B_{n-1}} ... \int_{B_1} p_{t-t_n-1}(v_1^{-1}v_n)p_{t_n-1-t_n-2}(v_n^{-1}v_{n-1})...p_{t_1}(v_1)dv_1dv_2...dv_n
\]
\[
= W(I).
\]

Now the proof is finished. \( \square \)

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