Finite Size Effects and Conformal Symmetry of $O(N)$ Nonlinear Model in Three Dimensions

Akira FUJII and Takeo INAMI

Yukawa Institute for Theoretical Physics,
Kyoto University,
Kyoto 606-01, Japan.

January 31, 2022

Abstract

We study the $O(N)$ nonlinear model on a three-dimensional compact space $S^1 \times S^2$ (of radii $L$ and $R$ respectively) by means of large $N$ expansion, focusing on the finite size effects and conformal symmetry of this model at the critical point. We evaluate the correlation length and the Casimir energy of this model and study their dependence on $L$ and $R$. We examine the modular transformation properties of the partition function, and study the dependence of the specific heat on the mass gap in view of possible extension of the $C$ theorem to three dimensions.

E-mail address: fujii@jpnyitp.bitnet
1 Introduction

In two-dimensional field theories and statistical systems the conformal invariance provides a powerful tool which allows us to compute exactly physical quantities such as correlation functions, critical exponents and field contents [1]. This is due to the fact that the conformal symmetry is infinite-dimensional in two dimensions. Furthermore, we can classify conformal field theory (CFT) models and find their field contents by studying the finite-size effects and the modular transformation properties of such theories on the torus. Above two dimensions the conformal symmetry is infinite-dimensional, hence this symmetry appears less useful in analyzing critical systems. Despite this difficulty, extension of some of the notions of the two-dimensional CFT to higher-dimensional theories has been pursued by a few authors [2]. In particular, Cardy has studied modular transformation properties of the partition function of conformal invariant free field theories on a compact space of three and higher dimensions $D$ [3,4,5]. He has shown that the partition function of such theories on $S^1 \times S^{D-1}$ of radii $L$ and $R$ respectively and a moduli $= L=R$, or its derivative with respect to $\lambda$, is invariant under $\lambda \rightarrow 1$. It is important to examine whether interacting field theories have the same modular transformation properties at their fixed points where they are conformally invariant.

In this paper we will consider the $O(N)$ nonlinear (NL) model on a three-dimensional compact space as an example of models including interactions. Of a variety of three-dimensional compact spaces, $S^1 \times S^2$ allows us to study the properties with respect to conformal invariance of the field theory defined on it [5]. In $R^D$ with $D$ less than four $O(N)$ NL model is known to be renormalizable in the $1/N$ expansion and permits us a nonperturbative analysis. Furthermore, the model possesses order (symmetry breaking)-disorder phase transition in dimensions $2 < D < 4$ [6] and hence it serves our purpose of studying the modular properties at the critical value of the coupling constant. In condensed matter physics, the $(2+1)$-dimensional $O(3)$ NL model has been studied as
an effective field theory of the long wavelength behavior of 2-dimensional quantum antiferromagnet. Recently, from the analyses of the large $O(N)$ NL model on the semi-compact space $S^1 \times R^2$, interesting results have been obtained on the low temperature properties of the quantum antiferromagnet. Curiously the specific heat of $O(N)$ NL model is represented by Roger's polylogarithmic functions; this is analogous to the specific heat of certain two-dimensional integrable systems being represented by Roger's dilogarithmic function.

2 Saddle Point Method and Functions

The $O(N)$ NL model in the $D$-dimensional Euclidean space $M$ is defined by the action

$$S = \frac{1}{2g} \int_M d^n x \quad \square n;$$

(1)

where $n$ is an $N$-component vector obeying the constraint $n^2 = 1$. By introducing an auxiliary field, the partition function is expressed as

$$Z = \int_M d^n x \quad \exp \left( -\frac{1}{2g} \int_M d^n x \quad \square n \quad \square n + (n^2 - 1) \right);$$

(2)

By carrying out the path integration of $n$, the partition function is written in terms of the effective action as follows.

$$Z = \int_M d^n x \quad \frac{1}{g} \exp \left( (N = 2)S_e \right);$$

(3)

$$S_e = -\frac{1}{g} \ln \left( + \right);$$

(4)

where we have set $g = gN$.

We consider the limit $N \to 1$ keeping $g$ fixed. In this limit we can evaluate the partition function by means of the saddle point method. The gap equation for the saddle point value of is

$$\frac{1}{g} = \text{Tr} \left\{ \frac{1}{1 + } \right\};$$

(5)
It is known that the $O(\mathbb{N}) NL$ model on $R^3$ has an infrared (IR) xed point at $g = 0$ and an ultraviolet (UV) xed point at $g = g_\text{c}$ for $2 < D < 4$. We will consider the model on $M = S^1 \times S^2$ (of radii $L$ and $R$ respectively) and solve the gap equation \([3]\) at the UV xed point at $g = g_\text{c}$, at which the model is transform ed by a conformal mapping into a theory which possesses conformal invariance in $R^3$.

We recapitulate the renormalization group (RG) transformation of the $O(\mathbb{N}) NL$ model \([7,9]\). In accordance with the saddle point approximation, we calculate the $\sim$-function of the $O(\mathbb{N}) NL$ model in the large $N$ limit to obtain the xed point. Since the UV divergence behavior is independent of the global property of the space \([12]\), we may consider the model on $S^1 \times R^2$ (of radius $L$) instead of $S^1 \times S^2$. It is useful to introduce the weak constant magnetic eld $\tilde{\eta}$ for the purpose of computing the wave function renormalization constant. We choose the the rst component of $\tilde{\eta}$ in the direction of $\tilde{\eta} = (h; 0)$ and set $\tilde{\eta}(x) = (\sim x; \sim x)$. The partition function is now given, after performing the path integral for $(x)$, by

\[
Z = \frac{Z}{2g} \int_{\sim x}^{1} D \sim x \exp \left( \frac{1}{2g} \int_{0}^{L} d^2x \left( \frac{1}{2} \sim x^2 + \frac{1}{4} (\sim \sim x)^2 \right) + \int_{0}^{L} d^2x \left( \frac{1}{2} \sim g : \right) \right)
\]

In parallel with eld theories at nite temperature, UV divergences of the present model can be handled by introducing two bare coupling constants $b$ \([12]\),

\[
g = g_N; \quad t = g_{= L}; \quad (7)
\]

The RG transformation in the theory defined by \([4]\) amounts to changing the momentum cut-o in $R^2$ from $a^{-1}$ to $a^{1}$. $a^{-1}$ plays the role of the lattice spacing in the lattice version. The $\sim$-functions for $g$ and $t$ can be calculated by carrying out this RG transformation in the momentum space. The result is:

\[
g = \frac{\partial g}{\partial L} = g \frac{g^2}{4} \coth \frac{L}{2}; \quad (8)
\]
\[ t = \frac{dt}{dl} = \frac{qt}{4} \coth \frac{L}{2}; \quad \text{(9)} \]

where we have put \( h = 0 \). We end that the RG transformation has two fixed points,

\[ (g_c, t) = (0; \text{any value}) \quad \text{and} \quad (4 = \tilde{g}; 0); \quad \text{(10)} \]

where we have set \( \coth (L = 2) = 1 \) by considering the limit \( L = L = a \), \( L \). The fixed point \( g_c = 0 \) is IR stable and \( g_c = 4 \) \( \tilde{g} \) is UV stable. We note that the critical coupling constants \( g_c \) are the same as those on \( R^3 \) as they should be. The \( \tilde{g} \) -functions (8) and (9) can also be obtained in the saddle point method [13]. The \( t \) function vanishes only at \( t = 0 \).

This reflects the fact that no phase transition can occur at finite \( L \) in the spacetime \( S^1 \times R^2 \); it is a special case of the Mermin-Wagner-Cooperman's theorem [14].

The above result was derived by adopting the periodic boundary condition (PBC) in the \( S^1 \) coordinate. The \( \tilde{g} \) -functions for the model with the antiperiodic boundary condition (APBC) can be derived by repeating the same RG transformation as above. The result differs from eqs. (8) and (9) slightly:

\[ \tilde{g} = g \frac{g^2}{4} \tanh \frac{L}{2}; \quad \text{(11)} \]

\[ t = \frac{qt}{4} \tanh \frac{L}{2}; \quad \text{(12)} \]

The UV stable critical value of \( \tilde{g} \) is \( g_c = 4 \) and coincides with that in the case of PBC, as it should.

### 3 O(N) Nonlinear Model on \( S^1 \times S^2 \)

We are now ready to investigate the O(N) NL model on \( S^1 \times S^2 \). We consider this model with the value of \( \tilde{g} \) fixed at its UV critical value \( g_c \) while the value of \( t \) being left arbitrary. First we write the gap equation explicit. The eigenvalues of the Laplacian on \( S^1 \times S^2 \) are \( \lambda_n^2 + 1(l+1)R^2 \), where \( \lambda_n = 2(n+2)=L \) with \( n \) being 0 or 1.
depending on whether we take the PBC or APBC in the direction of $S^1$ respectively. $l$ takes non-negative integers, and $n$ integers. Write $l(l+1)=R^2 + \frac{1}{4} = (l+1)^2=R^2 + \frac{1}{4}$, where $\frac{1}{4}=1=4R^2$. The gap equation \( \text{(5)} \) takes the form

$$\frac{1}{g} = \frac{1}{32} \sum_{n=0}^{\infty} \frac{x^4}{(n+\frac{1}{2})^2+4R^2} = \frac{21+1}{4R^2}.$$  

We solve this equation \( \text{(13)} \) by keeping $g$ fixed at its critical value $g_c$ and find the critical value $c$ of $\text{or}_c$. 

### 3.1 Case of periodic boundary condition (PBC)

We solve the gap equation \( \text{(13)} \) in the three limiting cases $R = 1$, $L = 1$ and a $L \gg R$. 

\[ \text{(1a)} \quad R = 1 \quad (S^1 \quad R^2). \]

The gap equation \( \text{(13)} \) is further reduced to

$$\frac{1}{g} = \frac{1}{4L} \ln \sum_{n=1}^{\infty} \frac{(n^2 + (L=2))^2}{(n^2 + (L=2))^2} = \frac{1}{g_c} \quad \frac{1}{2L} \ln \left( \frac{2L}{\text{sinh} \left( \frac{L}{2} \right)} \right);$$  

On substituting the critical value $g = g_c = 4 = \text{for } g$ on the left side of \( \text{(14)} \), we get

$$\text{sinh} \left( \frac{L}{2} \right) = 4 = \text{for } g$$

and hence

$$c = \frac{2}{L} \text{sinh} \left( \frac{1}{2} \right) = \frac{2}{L} \ln \left( \frac{1+\frac{P}{5}}{2} \right) = 0.9624 = L;$$  

Regarding the length of $S^1$ as the inverse of the temperature, we have the $O(N)$ NL - model at finite temperature. Then $\frac{1}{c}$ is the correlation length at temperature $T = 1=L$. The specific heat of this model, $C = \frac{\partial}{\partial T} \ln Z$, has been calculated in \[10\]

$$\frac{C}{N} = (3)^{\frac{1}{2}} \text{Li}_3(e^{-L}) + (L_c) \text{Li}_2(e^{-L_c}) + \frac{(L_c)^3}{6} = \frac{4}{5};$$  

where $\text{Li}_p(x)$ and $\text{Li}_3(x)$ are Roger's dilogarithmic and trilogarithmic functions respectively, $\text{Li}_p(x) = \sum_{n=1}^{\infty} x^n/n^p$. It is curious to see that a rational number emerges from polylogarithms. It has been observed that Roger's dilogarithmic functions appear in the computation of the specific heat of two-dimensional integrable models and that a simple rational number comes out by use of the addition formula [7].
\[ L = 1 \quad (R \quad S^2). \]

The saddle point equation \((13)\) can be reduced to

\[
\frac{Z}{2} \sum_{l=0}^{\infty} \frac{1}{4R^2} \frac{x^4}{l^2 + (l+1)^2} = 0: \quad (17)
\]

We obtain to the order of \(l=R\)

\[
\frac{Z}{2} \sum_{l=0}^{\infty} \frac{1}{4R^2} \frac{x^4}{l^2 + (l+1)^2} = 0: \quad (17)
\]

The anomalous dimension \(d\) of \(n\) can be read from the \(R\) dependence of the correlation length, \(P_c = 1 = d=R\), which follows from \(R\)-size scaling \((3)\). We get \(d = 1=2\), which agrees with the usual computation in the large-\(N\) limit.

\((1c)\) General case.

We consider the case, \(L \quad R\). In eq.\((13)\), the summation over \(l\) is divergent, and is regularized by introducing the cut \(l=L\). The summation can be performed by means of the Euler-Maclaurin formula

\[
\frac{1}{N} \sum_{n=n_1}^{n=N} f(n) = \int_{n_1}^{n=N} f(x) \, dx + \frac{1}{2} \left[ f(n) + f(n+1) \right] + O \left( \frac{1}{N^3} \right): \quad (19)
\]

The result is

\[
\frac{1}{R} \sum_{l=0}^{\infty} \frac{1}{4R^2} \frac{x^4}{l^2 + (l+1)^2} = 0: \quad (17)
\]

\[
= \ln \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \right) + O \left( \frac{1}{N^3} \right): \quad (19)
\]

Using this approximation formula, we evaluate the saddle point equation \((13)\), and

\[
c = 2 \frac{L}{\sinh \frac{1}{2} + \frac{1}{12} \sinh \left( \frac{1}{2} R \right)} + O \left( \frac{L^3}{R^4} \right)
\]

\[
= \frac{0.9624 \pi}{2} + 2.684 \frac{L}{R^2} + O \left( \frac{L^3}{R^4} \right): \quad (21)
\]
From the above equation, we see the dependence of the correlation length on the temperature $T = 1=L$ and on the size $R$.

$$1 = q \frac{L}{c} + 1=4R^2;$$

$$= 0.9224=L + 2.713 \frac{L}{R^2} + O (L^3=R^4):$$ (22)

## 3.2 Case of antiperiodic boundary condition (APBC)

In the gap equation (13), the eigenvalues of the $S^2$ part of take the squares of halfodd integers times $R^2$. Those of the $S^1$ part of also take the square of halfodd integers if we impose the APBC in the direction of $S^1$. We solve the gap equation (13) in three cases depending on $R = 1, L = 1$ and a $L; R$. As pointed out in [13], there arise a few subtle problems in the study of quantum effects in scalar field theories with APBC. First, one cannot make use of the naive effective potential in the computation of the expectation value of $\langle x \rangle$, since a constant expectation value obviously contradicts APBC. One can circumvent this difficulty by considering instead linear modes. The saddle point equation for $\langle x \rangle$, then takes the same form as that in the PBC case, $\langle x \rangle = 0$. Second, the tachyonic modes will generally arise after taking account of the quantum effect of twisted scalar field on untwisted scalar field. We will see that the same type of tachyonic modes appear in the solution of the gap equation in the APBC case.

(2a) $R = 1$ ($S^1 \quad R^2$).

The gap equation can be further reduced to

$$\frac{1}{g} = \frac{1}{g_c} \frac{1}{2} \frac{L}{L} \ln \left(2 \cosh \frac{L}{2}\right):$$ (23)

At $g = g_c$, this equation has a solution $\langle x \rangle = i \hat{c}$, with

$$\hat{c} = (2=L) \cos^2 (1=2) = 2 = 3L:$$ (24)

This solution is tachyonic, $\hat{c} = 2 < 0$, and it should be interpreted that the correct value is $\langle x \rangle = 0$. We have not fully understood the mechanism of dealing with these
tachyonic modes. While we consider the unstable saddle point where we solve the gap equation of this model.

(2b) \( L = 1 \) \((S^2 \ R)\).

The boundary condition becomes immaterial in the limit \( L = 1 \). Hence we have the same result as (1b).

(2c) General case.

We consider the limit of \( R \); \( L \to 1 \) with \( L = R = xed \). In the case of \( L = 1 \), the gap equation can be evaluated by means of the Euler-Maclaurin formula \( (19) \). We have to the order of \( \frac{1}{4} \) as

\[
\frac{1}{g} = \frac{1}{g_c} \frac{1}{16} + O \left( \frac{1}{4} \right)
\]

By setting \( g = g_c \), we obtain \( c = 0 + O \left( \frac{1}{4} \right) \).

In the case of \( L = 1 \), we can also make use of \( (19) \) and find the critical value as

\[
c = c_{\frac{1}{4}=-1} + \left( \frac{1}{4}=12 \right) \quad (=L)
\]

formally. By the same argument as made in the case (2a), the definition should lead \( c = 0 \) under the condition of at least \( \frac{1}{4} \). (This upper limit can be seen if we consider a small \( \), which makes \( c \) equal to just 0 not to a pure imaginary number.)

4 Modulard Properties and Behavior of the Specific Heat

With the above result, let us discuss the physical aspects of the \( O(N) \) \( NL \) models on (semi)compact manifolds. First we discuss the modular invariance of this model. The simplest example model with the modular invariance is the two dimensional free scalar model on a torus with moduli. Its free energy \( F = T \frac{1}{\varphi_0} \frac{1}{n = 0} \ln (\varphi^2 + n^2) \) is invariant under \( L = 1 \) if we ignore the regularization of the divergence. We can calculate the free energy of \( O(N) \) \( NL \) model on \( S^1 \times S^2 \) with the APBC by means of the saddle point
If we choose $g = g_c$, the arguments in (27) enable us to calculate free energy as

$$F = T \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} (n+1)(n+2) f (l+1)(l+2) + (n+1)(n+2) (l=1) g$$

(27)

to the order $2$ in the case of $1$ and in the region $< \frac{1}{4}$ in the case of $1$. Obviously it is not invariant under $1=1$. Hence we deform the definition of the free energy (27) as follows according to [3],

$$F^0 = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} (n+1)(n+2) f (l+1)(l+2) + (n+1)(n+2) (l=1) ;$$

(28)

where $f$ is a some function. If we exchange the two radii of $S^2$ and $S^1$, i.e., $1=1$, the deformed free energy defined above is invariant. Because the every derivative of the free energy (27) by leads the factor $(n+1)(n+2)$, the deformed free energy (28) can be regarded as the half derivative by of the free energy (27).

Second we discuss the attempt of the expansion of Zamolodchikov’s C-theorem in dimensions higher than three. There exist an very important considerations about the C-theorem [16] in higher-dimensional field theory. The c-functions defined by the finite size correction and that introduced by the energy momentum tensor do not coincide in higher dimensions in general. We expect it is useful to analyze the partition functions to understand the C-theorem. We can calculate the specific heat $C (g) = dF / dT |_{T=T_0}$. Let $1=L$ of the large N O (N) NL model (PBC) with arbitrary $g$ by means of the saddle point method. If we obtain the mass gap $m = \hbar - \text{from the gap equation (3)}$ for the coupling constant $g$, the graph of the specific heat calculated as

$$C(mL) = (3)^1 L_{1} (e^{-mL}) + (mL) L_{1} (e^{-mL}) + \frac{(mL)^3}{6}$$

(29)

is presented as

Figure 1: Graph of $mL - C = N$. 

10
We see from the above graph that the specific heat has the local minimum at $m' = c (1 = \omega L = 1.039)$, i.e.: $g = g_c$. Therefore, although we consider the model on $S^1 \times S^2$ or $S^1 \times R^2$, which does not allow the phase transition, this behavior of the specific heat seems to justify to consider the model on $S^1 \times S^2$ or $S^1 \times R^2$ with the coupling constant $g_c$, which is the critical value on $R^3$. We expect some interpretation in terms of the RG.

In APBC case, we do not find such a local minimum of the specific heat. Therefore we cannot give a simple justification to $g_c$ on $S^1 \times S^2$ or $S^1 \times R^2$.

We have a variety of the expansion. Especially, we expect the analysis of the four-fermion model or supersymmetric NL model are important because they permit the nonperturbative analysis. Furthermore it is reported that the latter has zero $\beta$-function independent of the coupling constant [17]. And the combination of the compactification and $\beta$-expansion will be reported elsewhere [18].

References

[1] A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl.Phys.B 241 (1984) 333.

[2] A.M.Polyakov, JETP 28 (1968) 533, JETP Lett.12 (1970) 538; A.M.Igdal, JETP 28 (1969) 1036.

[3] J.L.Cardy, J.Phys.A 18 (1985) L757.

[4] J.L.Cardy, Nucl.Phys.B 290 (1987) 355.

[5] J.L.Cardy, Nucl.Phys.B 366 (1991) 403.

[6] J.L.Cardy, Nucl.Phys.B 270 (1986) 186.

[7] E.Brezin and J.Zinn-Justin, Phys.Rev.B 14 (1976) 3110; I.Ya.Aref'eva, Ann. Phys.117 (1979) 393; and references therein.
[8] F.D.M. Haldane, Phys. Lett. A 93 (1983) 464.

[9] D.R. Nelson and R.A. Pelcovits, Phys. Rev. B 16 (1977) 2191; S. Chakravarty, B.I. Halperin and D.R. Nelson, Phys. Rev. B 39 (1989) 2344; T. Yanagisawa, Phys. Rev. Lett. 68 (1992) 1026.

[10] S. Sachdev, Phys. Lett. B 309 (1993) 285.

[11] C.N. Yang and C.P. Yang, J. Math. Phys. 10 (1969) 1115; A I B. Zamolodchikov, Nucl. Phys. B 342 (1990) 695.

[12] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Second Edition) Oxford 1993.

[13] A. H. Castro Neto and E. Fradkin, Nucl. Phys. B 400 (1993) 525.

[14] N. D. Mermin and H. Wagner, Phys. Rev. 17 (1966) 1133; S. Coleman, Comm. Math. Phys. 31 (1973) 259.

[15] L. H. Ford, Phys. Rev. D 22 (1980) 3003; D. J. Toms, Ann. Phys. 129 (1980) 334.

[16] A. Cappelli, J. I. Latorre, Nucl. Phys. B 340 (1990) 659; A. Cappelli, D. Friedan, J. I. Latorre, Nucl. Phys. B 352 (1991) 616; A. Cappelli, J. I. Latorre and X. Vilasis-Cadona, Nucl. Phys. B 376 (1992) 510.

[17] V.G. Koures and K.T.M. Ahanthappa, Phys. Rev. D 43 (1991) 3428.

[18] In preparation.

Figure Caption

Figure 1. The graph of the behavior of the specific heat $C(mL) = N$ in terms of a dimensionless variable $1 = mL$. 

12
Figure 1.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9405189v1