Exotic Fluids Made of Open Strings

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Abstract

We compute the high energy entropy and the equation of state of a gas of open superstrings in the infinite volume limit focusing on the calculation of the number of strings as a function of energy and volume. We do it in the fixed temperature and fixed energy pictures to explicitly proof their equivalence. We find that, at high energy, an effective two dimensional behavior appears for the number of strings. Looking at the equation of state from a ten dimensional point of view, we show that the Hagedorn behavior can be seen as correcting the Zeldovich equation of state ($\rho = p$) that can be found from the two dimensional part of the entropy of the system. By the way, we show that, near the Hagedorn temperature, the equilibrium state obtained by sharing the total energy among open (super)strings of different length is stable.

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1 Introduction

String theory (with its extensions) hopefully will provide the primordial matter for the Universe in such a way that, after evolving from a probably non singular state at very high energy, we find a universe with the characteristics of the one we observe today. However, almost every important question is still waiting for a solution from String Theory. Meanwhile, it has been very clear from the very beginning that String Theory could provide us with a form of matter special enough so as to settle what seems to be the main question which is the one about the resolution of the initial singularity.

It seems that, to achieve this, we need a kind of reduction of degrees of freedom at high energy, at least with respect to the contribution coming from the ultraviolet (high momentum) region. When only perturbative strings were known, thermal duality showed itself as a mechanism to truncate ten dimensional string theories (Bosonic, Heterotic, Type II) at high energy to a kind of effective two dimensional (massless) field theories with polarizations given by the cosmological constant of a ten dimensional string theory \[1, 2\]. The problem was that only for Heterotic Superstrings a finite, but vanishing, high temperature free energy was found. For other string gases, like the ones made of Type II Closed Superstrings, thermal duality implies a two dimensional divergent free energy because the two-dimensional polarization degrees of freedom are measured by the cosmological constant of the ten dimensional Type-0 Bosonic String. In other cases, like heterotic compactifications, the dual phase at high energy is nonsense since it has positive free energy. Only in two dimensions, for generic heterotic non supersymmetric strings, we know of a meaningful high temperature dual phase \[2\], but we are already in a two dimensional space-time at low energy (low temperature). Anyhow, the high temperature dual phase is very peculiar because it has the cosmological constant of the theory as the coefficient of $\beta^{-2}$ and still might serve as a high temperature candidate to solve the Hagedorn behavior through a phase transition.

What we are going to show here is that the gas of open superstrings in a big nine dimensional container presents, at high energy, a two dimensional behavior that is corrected by the characteristic dominant Hagedorn comportment to produce an exotic equation of state of the form $p = \rho/(1 + k \sqrt{\rho})$ that can finally be approximated by the better known $p \propto \sqrt{\rho}$ equation of state. The entropic fundamental relation is given as the sum of the entropy of a gas of massless particles in two dimensions plus $\beta_H E$ which is the standard dominant Hagedorn behavior at high energy for the entropy of any gas made of strings (open or closed). The two dimensional contribution to the entropy alone would produce a Zeldovich’s fluid (also known as stiff fluid) equation of state $\rho = p$ that has appeared many times, for example in \[3\], as a good candidate to be the primordial cosmological fluid.
The two dimensional behavior of the system is stressed by the fact that the number of strings at high energy is the same function of $E$ as in a massless two dimensional ordinary gas.

2 The gas of open (super)strings

It is well known that, perturbatively, the critical behavior of a gas of free open strings at the Hagedorn temperature is such that the Helmholtz free energy diverges when approaching $T_H$ from below. In macrocanonical terms, this means that $T_H$ is a maximum temperature in the sense that any open string gas in equilibrium that, at vanishing chemical potential, has a finite internal energy $U(T,V)$ is necessarily kept at a temperature below $T_H$. On the other hand, in a description at given energy and null chemical potential, it is very natural to think about a maximum temperature because it would be a maximum for the temperature as a mathematical function of energy and volume.

2.1 The macrocanonical description of the gas of open superstrings

The black body of free open superstrings in the macrocanonical ensemble description has a free energy, $-PV = F(\beta,V)$, given by (see, e.g., [4])

$$F(\beta) = -\frac{V}{8(2\alpha')^5} \int_0^{+\infty} dt t^{-6} \left[ \theta_3 \left( 0, \frac{i\beta^2}{4\alpha' t} \right) - \theta_4 \left( 0, \frac{i\beta^2}{4\alpha' t} \right) \right] \theta_4^{-8} \left( 0, \frac{it}{\pi^2} \right) .$$

Let us remind some things to the reader. The exponential growth of the number of states with the mass is here expressed thorough the behavior of $\theta_4^{-8} \left( 0, \frac{it}{\pi^2} \right)$ that certainly diverges when $t$ approaches zero from the right. The behavior of the fourth Jacobi theta function is encoded in the relation $\theta_4 (0, i/t) = t^{1/2} \theta_3 (0, i t)$ that can be obtained by Poisson resummation of the series representing the Jacobi theta function.

From all this, one gets that, by using an ultraviolet cutoff, it is convenient to write the free energy as

$$F(\beta) \approx -\frac{V}{8(2\alpha')^5} \int_0^{+\infty} dt t^{-6} \left[ \theta_3 \left( 0, \frac{i\beta^2}{4\alpha' t} \right) - \theta_4 \left( 0, \frac{i\beta^2}{4\alpha' t} \right) \right] \theta_4^{-8} \left( 0, \frac{it}{\pi^2} \right)$$

$$- \frac{V}{2^9 \pi^8 (2\alpha')^5} \int_0^\epsilon dt t^{-2} e^{(8\alpha' \pi^2 - i\beta^2 \pi)/(4\alpha' t)} ,$$

(2.2)
where $\epsilon$ is a dimensionless cutoff that is taken small enough so that the second term on the right hand side of eq. (2.2) is a good approximation to the contribution to the free energy coming from the ultraviolet degrees of freedom\(^1\). Only the Maxwell-Boltzmann contribution has survived, so we are in the classical statistics approximation. This cut-off can be seen as separating the infrared from the ultraviolet degrees of freedom. We are going to be interested on the contribution to the high temperature free energy that precisely comes from the ultraviolet degrees of freedom (see [1]).

From here, it is very easy to read the Hagedorn temperature for open superstrings (treated without a gauge group contribution) as \(1/\beta_H = T_H = 1/(\pi\sqrt{8\alpha'})\)

The integral in the second term in eq. (2.2) can be evaluated to give \([5]\)

\[
\int_0^\epsilon dt \, t^{-2} e^{(8\alpha'\pi^3 - \beta^2 \pi)/(4\alpha')} = \frac{\beta_H^2}{2\pi^3 (\beta_H^2 - \beta^2_H)} e^{-2\pi^3(\beta^2_H - \beta^2)/(\epsilon\beta_H^3)}, \tag{2.3}
\]

that is valid when $\beta > \beta_H$. It is now very clear that the free energy diverges as $\beta$ approaches $\beta_H$ and eq. (2.3) tells us exactly how. For a fixed cut-off $\epsilon$, the contribution to the free energy coming from the ultraviolet region exponentially falls off when $(\beta - \beta_H)$ grows. If we then make the approximation $(\beta^2_H - \beta^2_H) = (\beta - \beta_H)^2 + 2\beta_H (\beta - \beta_H) \approx 2\beta_H (\beta - \beta_H)$, the contribution to $\beta PV = -\beta F(\beta, V) = \overline{N}(\beta, V)$ from the ultraviolet part can be approximated by\(^2\)

\[
-\beta F^h(\beta) = \frac{V}{2\pi \beta_H^8} \int_0^{+\infty} dE \, e^{-\beta E} e^{\beta_H E} \theta \left( E - \frac{4\pi^3}{\epsilon\beta_H} \right) \\
= \frac{V}{2\pi \beta^8_H} \frac{e^{-(\beta - \beta_H)4\pi^3/(\epsilon\beta_H)}}{\beta - \beta_H}. \tag{2.4}
\]

This integral representation tells us several things. First of all it is easy to read the high energy behavior of the single open string density of states given by (see [5, 7])

\[
\Omega^h_1(V, E) = \frac{V}{2\pi \beta^8_H} e^{\beta_H E} \theta \left( E - \frac{4\pi^3}{\epsilon\beta_H} \right), \tag{2.5}
\]

\(^1\)In a quantum field theory, finite temperature provides an ultraviolet exponential regulator. In String Theory at finite temperature as described by the analog model, the Hagedorn behavior appears as an ultraviolet divergence. In the analog model, the infrared and ultraviolet regions are defined for every quantum field in the infinite collection of them as vibrational modes of the string. Open string T-duality exchanges both regimes, but with different objects: a short distance between D-branes corresponds to a long distance propagation of open strings.

\(^2\)That $-\beta F(\beta, V) = \overline{N}(\beta, V)$ is the result of implementing $\mu = 0$ in the macrocanonical description when Maxwell-Boltzmann statistics is a good approximation [6].
where the energy cutoff has been directly derived from the dimensionless $\epsilon$ parameter.

It is also very clear that the contribution to the free energy near $\beta_H$ is dominated by $F^h$ that grows unbounded as long as the first integral in (2.2) gives a finite value at $\beta_H$.

The internal energy $U(\beta, V)$ can easily be computed as $U(\beta, V) = \frac{\partial}{\partial \beta} \beta F(\beta)$. Near the Hagedorn temperature, we have that

$$U^h(\beta, V) \approx V \left[ \frac{1}{(\beta - \beta_H)^2} - \frac{8\pi^6}{\epsilon^2 \beta_H^2} \right],$$

(2.6)

where we have presented it to show the only divergent term. Let us notice that $0 < (\beta - \beta_H) < \epsilon \beta_H / (\sqrt{8\pi^3})$.

The calculation of fluctuations is an important piece in the study of equivalence between different ensemble descriptions. Fluctuations computed for the energy $U$ are given around $\beta_h$ by

$$\left[ \frac{T^2 C_V(T, V)}{U} \right]^{1/2} \approx 2 \sqrt{\frac{\pi}{V}} \beta_H^4 (\beta - \beta_H)^{1/2} e^{(\beta - \beta_H) 2\pi^3 / (\epsilon \beta_H)} = \sqrt{2} \left( \frac{\beta H}{N} \right)^{-1/2},$$

(2.7)

that certainly vanishes as $\beta$ approaches $\beta_H$. In fact, fluctuations are $O(N^{-1/2})$ as in a typical closed isothermal system. This means that equivalence between the fixed temperature and the fixed energy descriptions for the system of open strings with $\mu = 0$ must occur in the sense that the states at given energy certainly would correspond to states with well defined averaged energy at a given temperature.

To compare with the fixed energy (or micro) calculation, it is useful to get the thermodynamical fundamental relation giving the entropy $S$ as a function of the canonical averaged energy $U$ and the volume $V$ for $\beta$ bigger and near $\beta_h$. One gets

$$S^h(U, V) = 2 \sqrt{\frac{U V}{2\pi \beta_H^8}} + \beta_H U,$$

(2.8)

where we have assumed that $U \gg 8 \pi^5 V / (\epsilon^2 \beta_H^{10})$.

Finally, it is worth to remark that the first term in (2.2) gives the contribution from the string modes for long propagation times. The massive modes are exponentially suppressed so the relevant contribution comes from the massless degrees of freedom $[8]$. 

5
2.2 The fixed energy description of the gas of open superstrings

The big number of open dimensions tells us that quantum statistical corrections are negligible for the system \([6]\). Then, the label \(N\) in \(\Omega_N\) certainly refers to the number of strings in the gas because Maxwell-Boltzmann statistics is applicable even when the system energy is not very high.

\(\Omega_1 (E, V)\) is the key ingredient to get \(\Omega_N (E, V)\) using Laplace convolutions \([5]\). The problem here is that such integrals cannot be performed analytically. However, we are interested in getting the high energy behavior of \(\Omega_N\) (for high \(N\)). For open strings, this can be obtained from the convolution of the high energy part of \(\Omega_n\) to get \(\Omega_{2n}^h\), \(^3\). The reason is that the pure exponential growth of \(\Omega_1^h\) implies that the convolution between the high and low energy parts of the single density of states gives a negligible contribution to \(\Omega_{2n}^h\) (this is not the case for closed strings when windings are absent \([6]\)). In Fig. 1 a numerical computation of \(\omega_2 (E, t) = \Omega_1 (E - t) \Omega_1 (t)\) at \(E = 5\) (\(\alpha' = 1\)) is showed together with the straight line which represents \(\omega_2^h\) as obtained from \(\Omega_1^h (E, V) \sim e^{\beta H E}\). In it, we can observe the two energy cutoffs for the validity of this approximation that, in general, will be placed, for \(\omega_2\), at \(\lambda\) and \(E - \lambda; \lambda = 4\pi^3 / (\epsilon \beta H)\). The plateau in the interval \([\lambda, 5 - \lambda]\) signals the feature that energy will be distributed with equal probability among open strings of different length, i.e., different energy. This is the way the equilibrium state shows the most probable decay of a highly excited open string (see \([9]\)). Looking at this plateau from the point of view of equipartition and the equivalence of ensembles, one might erroneously guess that the fixed temperature and preserved energy descriptions would not be equivalent. Instead, one would expect as the most probable configuration that each open string shared half of the total energy. However, the computation of energy fluctuations tells us that this is not the case. We are actually showing that the equilibrium got by shearing the total energy among open strings with different lengths is stable. Indeed this is a worthy feature of the behavior, around the Hagedorn temperature, of our system.

With this approximation one can get

\[
\Omega_N^h (E, V) = \frac{1}{N! (N - 1)!} \left( \frac{V}{2\pi \beta_H^8} \right)^N e^{\beta_H E} (E - N\lambda)^{N-1} \theta (E - N\lambda)
\approx \frac{1}{N!^2} \left( \frac{V}{2\pi \beta_H^8} \right)^N e^{\beta_H E} (E - N\lambda)^{N} \theta (E - N\lambda),
\]

where in the second equation we have assumed that \(N\) is very big. This high energy density of states for the gas of \(N\) strings holds as long as \(E\), the energy of the gas, is bigger than \(N\lambda\).

\(^3\)In fact, \(\Omega_{2n} (E) = \frac{\alpha'^2}{(2\pi)^{2n}} \int_0^E dt \Omega_n (E - t) \Omega_n (t)\) \([5]\).
Figure 1: $\omega_2(5, t)$ computed numerically. The straight line represents $\omega_2$ as given by using $\Omega^h_1$ only. $\alpha' = 1$.

The vanishing of the chemical potential $\mu$ implies that the high energy entropy is $S^h = \ln \sum_{N=0}^{\infty} \Omega^h_N$. In fact, the sum can be approximated by a single term $\Omega^h_{N^*}$ for $N^*$ such that $\Omega^h_{N=N^*}$ is a maximum of the density of states as a function of the number of particles. We then compute $\frac{\partial}{\partial N} \ln \Omega^h_N$ to get that, for $E \gg \lambda N^*$,

$$N^h(E, V) = N^* = \sqrt{\frac{E V}{2\pi \beta_H^8}}. \quad (2.10)$$

It is a very notorious fact that the number of open superstrings at high energy depends on energy and volume as in a gas of massless particles in two space-time dimensions described using Maxwell-Boltzmann statistics\(^4\).

The entropy is then

$$S^h(E, V) \approx 2N^h(E, V) + \beta_H E = 2 \sqrt{\frac{E V}{2\pi \beta_H^8}} + \beta_H E, \quad (2.11)$$

for $E \gg 8\pi^5 V / (e^2 \beta_H^10)$. It exactly coincides with $S^h(U, V)$ as computed in the canonical description, as we expected from the computation of the energy fluctuations in the fixed temperature picture. This expression of the entropy as a function of energy and volume was found in \([10], [11]\) (see also \([12]\)), although no physical interpretation of the term which scales with $E^{1/2}$ in terms of the number

\(^4\)For the gas of massless particles in $d > 2$ space-time dimensions with Bose-Einstein statistics, one finds: $N(E, V) = \frac{\zeta(d+1)}{\zeta(d)} N^*(E, V)$ where $N^*$ is the number of particles with $\mu = 0$ using Maxwell-Boltzmann statistics \([12]\). In two space-time dimensions, one gets the finite result: $N(E, L) = \frac{\sum_{r=1}^{\infty} \theta (ER-r)/r}{\zeta(2)} N^*(E, L)$, where $R$ is the radius of the compact space ($L = 2\pi R$) that, as the energy, is supposed to become big as one takes the thermodynamic limit. This expression at dimension two results from computing the free energy in the thermodynamic limit by converting the sum over discrete momenta to an integral together with the separation of the zero momentum contribution that represents the one of the the vacuum state (no particle state) that must not be second quantized.
of strings was given\(^5\). It is also worth to mention that this entropy is an extensive function of energy and volume that can be written in terms of the number of strings as

\[
S^h = 2N + 2\pi \beta_H^9 N^2 / V
\]  

(2.12)

This behavior in terms of the number of objects can be compared with a "regular" system for which entropy is \(O(N)\). In (2.11) and (2.12), the first term is very easily identified as the entropy for a two dimensional gas of massless particles that can be written as a function of the squared root of the energy or, exactly, as twice the average number of objects\(^6\). We also see the nine dimensional volume divided by the adequate power of the length scale of our problem which is the squared root of \(\alpha'\) (included in \(\beta_H\)). This is the way an effective one dimensional volume appears. This contribution to the entropy is corrected by the universal volume independent, and then pressureless, contribution coming from the Hagedorn behavior that is \(O(N^2)\). It is important to notice that the term \(\beta_H E\) shows up in the entropic fundamental relation for any critical string gas in the macrocanonical and the microcanonical computations. This term gives, for any gas of fundamental strings, a divergent contribution to \(C_V(E)\) and the effective two dimensional part finally makes \(C_V(E)\) finite and positive for the gas of open superstrings. The two terms in the entropy can be seen as the contribution of two different types of degrees of freedom at high energy that finally make \(T_H\) a maximum temperature for the system.

3 The equation of state

It is very easy to get the equation of state which relates the density of energy \(\rho\) and the pressure of the gas. We get

\[
P = \frac{\rho}{1 + \beta_H \sqrt{2\pi \beta_H^9 \rho}}.
\]  

(3.1)

Here, what makes the denominator different from the unity, and so the equation of state different from Zeldovich’s one, is precisely the term \(\beta_H E\) in the entropy.

\(^5\)The equivalence of ensembles was also not treated. It would be a mistake to think that it is trivial because the specific heat is positive and it would be cynic to say that equivalence has been implicitly assumed since the computation of energy fluctuations is so easy that does not deserve any mention. After all, we go further because we show here that the equilibrium got shearing the total energy with equal probability among a set of open strings with different lengths, i.e. different energies, is compatible with the stability of the system. This is more than stating that equivalence holds, it is a proof of the thermodynamic stability of the stringy matter that comes from the preferred decay mode of any highly excited open string \[9\].

\(^6\)The massless character of the particles does not seem to be something special because the high temperature limit for the free energy of a massive field in \(d\) space-time dimensions is dominated by the contribution to the free energy of a massless field in \(d\) dimensions.
This is dominant at high energy, because what we mean by high energy to get (2.8) and (2.11) implies the condition $\sqrt{2\pi\beta_H^{10}\rho} \gg 1$.

With this approximation the equation of state at very high energy looks

$$P \approx \sqrt{\frac{\rho}{2\pi\beta_H^{10}}}. $$

This fluid is causal because the sound would propagate in it with a speed given by

$$v_s^2 = \frac{\partial P}{\partial \rho} \approx \frac{1}{2\sqrt{2\pi\beta_H^{10}\rho}} \ll 1.$$

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