Regularized mean curvature flow for invariant hypersurfaces in a Hilbert space and its application to gauge theory

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Abstract

In this paper, we investigate a regularized mean curvature flow starting from an invariant hypersurface in a Hilbert space equipped with an isometric and almost free action of a Hilbert Lie group whose orbits are minimal regularizable submanifolds. We prove that, if the initial invariant hypersurface satisfies a certain kind of horizontally convexity condition and some additional conditions, then it collapses to an orbit of the Hilbert Lie group action along the regularized mean curvature flow. In the final section, we state a vision for applying the study of the regularized mean curvature flow to the gauge theory.

1 Introduction

C. L. Terng ([Te]) defined the notion of a proper Fredholm submanifold in a (separable) Hilbert space as a submanifold of finite codimension satisfying certain conditions for the normal exponential map. Note that the shape operators of a proper Fredholm submanifold are compact operator. By using this fact, C. King and C. L. Terng ([KiTe]) defined the regularized trace of the shape operator for each unit normal vector of a proper Fredholm submanifold. Later, E. Heintze, C. Olmos and X. Liu ([HLO]) defined another regularized trace of the shape operator for each unit normal vector of a proper Fredholm submanifold, which differs from one defined in [KiTe]. They called the regularized trace defined in [KiTe] $\zeta$-regularized trace. The regularized trace in [HLO] is easier to handle than one in [KiTe]. In almost all relevant cases, these regularized traces coincide. In this paper, we adopt the regularized trace defined in [HLO]. Let $M$ be a proper Fredholm submanifold in $V$ immersed by $f$. If, for each normal vector $\xi$ of $f$, the regularized trace $\text{Tr}_r A_\xi$ of the shape operator $A_\xi$ of $f$ and the trace $\text{Tr} A_\xi^2$ of $A_\xi^2$ exist, then $M$ (or $f$) is said to be regularizable. See Section 2 about the definition of the regularized trace $\text{Tr}_r A_\xi$. Let $M$ be a Hilbert manifold and $\{f_t\}_{t \in [0, T]}$ be a $C^\infty$-family of regularizable immersions.
of codimension one of $M$ into $V$ which admit a unit normal vector field $\xi_t$. The regularized mean curvature vector $H_t$ is defined by $H_t := -\text{Tr}_r((A_t - \xi_t) \cdot \xi_t)$, where $A_t$ denotes the shape tensor of $f_t$. Define a map $F : M \times [0, T) \to V$ by $F(x, t) := f_t(x)$ ($(x, t) \in M \times [0, T)$). We call $\{f_t\}_{t \in [0, T)}$ the regularized mean curvature flow if the following evolution equation holds:

\[
\frac{\partial F}{\partial t} = H_t.
\]

This notion was introduced in [Koi2].

R. S. Hamilton ([Ha]) proved the existence and the uniqueness of (in short time) of solutions of a weakly parabolic equation for sections of a finite dimensional vector bundle. The evolution equation (1.1) is regarded as the evolution equation for sections of the infinite dimensional trivial vector bundle $M \times V$ over $M$. Since $M$ is an infinite dimensional Hilbert manifold, we cannot apply the Hamilton’s result to this evolution equation (1.1). Also, since $M$ is of infinite dimension, $M$ is not locally compact. Thus, we cannot show the existence and the uniqueness of solutions of (1.1) starting from $f$ for a general regularizable $C^\infty$-immersion $f$ of codimension one. Hence, at least, we must impose the finiteness of the cohomogeneity of $f(M)$ to the initial data $f$ in order to use the compactness. Here “the finiteness of the cohomogeneity of $f(M)$” means that there exists a closed subgroup $G$ of the (full) isometry group $O(V) \ltimes V$ (which is not a Banach Lie group) of $V$ such that $f(M)$ is $G$-invariant and that $f(M)/G$ is a finite dimensional manifold with singularity. Furthermore, in order to use the compactness, we must impose $f$ the condition that $f(M)/G$ is compact.

So, we ([Koi2]) considered the following special case. Assume that an action of a Hilbert Lie group $G$ on a Hilbert space $V$ satisfies the following conditions:

(I) The action $G \curvearrowright V$ is isometric and almost free, where “almost free” means that the isotropy group of the action at each point is finite;

(II) All $G$-orbits are minimal regularizable submanifolds, that is, they are regularizable submanifold and their regularized mean curvature vectors vanish.

Note that $V/G$ is a finite dimensional orbifold by the condition (II). Denote by $N$ the orbit space $V/G$. Give $N$ the Riemannian orbimetric $g_N$ such that the orbit map $\phi : V \to N$ is a Riemannian orbisubmersion. Let $M(\subset V)$ be a $G$-invariant hypersurface in $V$. Furthermore, we assume the following condition:

(III) $\overline{M} := \phi(M)$ is compact.

Note that $\overline{M}$ is a compact hypersurface in $N$. Denote by $f$ the inclusion maps of $M$ into $V$ and $\overline{f}$ that of $\overline{M}$ into $N$. We ([Koi2]) showed that the regularized mean curvature flow starting from $M$ exists uniquely in short time. However there
were gaps in the statement and the proof. In this paper, we close the gaps (see the statement and the proof of Theorem 4.1 (also those of Theorem 3.1)). Here we note that the uniqueness of the flow is assured under the $G$-invariance of the flow.

In [Koi2], we mainly proved that the horizontally strongly convexity is preserved along the ($G$-invariant) regularized mean curvature flow in the case where $M$ is a $G$-invariant hypersurface in $V$ (see Theorem 6.1 in [Koi2]), where “$G$-invariance” of a regularized mean curvature flow means that the regularized mean curvature flow consists of $G$-invariant regularizable hypersurfaces. In the statement of Theorem 6.1 of [Koi2], it is not specified that the regularized mean curvature flow is $G$-invariant but the assumption of the $G$-invariance of the flow is needed in the statement.

In this paper, we prove that the $G$-invariant regularized mean curvature flow starting from a horizontally strongly convex $G$-invariant hypersurface in $V$ collapses to a $G$-orbit in finite time under some additional conditions (see Theorem A). We shall state this collapsing theorem in detail. Let $M$ be a hypersurface admitting a (global) unit normal vector field $\xi$. Denote by $\mathcal{K}$ the maximal sectional curvature of $(N, g_N)$, which is nonnegative because $V$ is flat. Set $b := \sqrt{\mathcal{K}}$. Let $\Sigma$ be the singular set of $(N, g_N)$ and $\{\Sigma_1, \cdots, \Sigma_k\}$ be the set of all connected components of $\Sigma$. Set $B^T_r(x) := \{v \in T_x N \mid \|v\| \leq r\}$. For $x \in N$ and $r > 0$, denote by $B_r(x)$ the geodesic ball of radius $r$ centered at $x$. Here we note that, even if $x \in \Sigma$, the exponential map $\exp_x : T_x N \rightarrow N$ is defined in the same manner as the Riemannian manifold-case and $B_r(x)$ is defined by $B_r(x) := \exp_x(B^T_r(x))$. We assume the following:

$\star_1$ $\overline{M}$ is included by $B_{\overline{\tau}}(x_0)$ for some $x_0 \in N$ and $\exp_{x_0} | B^T_{\overline{\tau}}(x_0)$ is injective.

Furthermore, we assume the following:

$\star_2$ $b^2(1 - \alpha)^{-2/n}(\omega_n^{-1} \cdot \text{Vol}_g(M))^{2/n} \leq 1,$

where $\alpha$ is a positive constant smaller than one and $\overline{g}$ denotes the induced metric on $\overline{M}$ and $\omega_n$ denotes the volume of the unit ball in the Euclidean space $\mathbb{R}^n$. Denote by $f$ the inclusion map of $M$ to $V$ and $\overline{f}$ that of $\overline{M}$ into $N$. Let $\{f_t\}_{t \in [0, T)}$ be the $G$-invariant regularized mean curvature flow starting from $f$. Denote by $H_t$ the regularized mean curvature vector of $f_t$ and set $H^\xi_t := -\langle H_t, \xi_t \rangle(=\text{Tr}_r((A_t - \xi_t)))$, where $\xi_t$ is a unit normal vector field of $f_t$ such that $\xi_0 = \xi$ and $t \mapsto \xi_t$ is continuous. Set $(H^\xi_t)_{\min} := \min_M H^\xi_t$ and $(H^\xi_t)_{\max} := \max_M H^\xi_t$. Define a constant $L$ by

$\star$ $L := \sup_{u \in V(\{x_1, \cdots, x_5\} \in (\tilde{H}_1)^\circ_u)} \max \left| \langle A^\phi_{X_1utely} (\tilde{\nabla}_{X_2} A^\phi_{X_3}) X_4, X_5 \rangle \right|,$

where $\tilde{H}$ denotes the horizontal distribution of $\phi$, $(\tilde{H}_1)^\circ_u$ denotes the set of all unit horizontal vectors of $\phi$ at $u$ and $A^\phi$ denotes one of the O'Neill’s tensors defined in
(see Section 4 about the definition of \(A^\phi\)). Note that the restriction \(A^\phi|_{\tilde{H} \times \tilde{H}}\) of \(A^\phi\) to \(\tilde{H} \times \tilde{H}\) is the tensor indicating the obstruction of the integrability of \(\tilde{H}\).

In this paper, we prove the following collapsing theorem.

**Theorem A.** Assume that the initial \(G\)-invariant hypersurface \(f : M \rightarrow V\) satisfies the above conditions \((\ast_1), (\ast_2)\) and \((H_0^s)^2(h_H)_0 > 2n^2L(g_H)_0\), where \((g_H)_0\) (resp. \((h_H)_0\)) denotes the horizontal component of the induced metric (resp. the scalar-valued second fundamental form) of \(f\). Then, for a \(G\)-invariant regularized mean curvature flow \(\{f_t\}_{t \in [0,T]}\) \((T : \text{the maximal time})\) starting from \(f\), the following statements (i) and (ii) hold:

(i) \(T < \infty\) and \(f_t(M)\) collapses to a \(G\)-orbit as \(t \rightarrow T\).

(ii) \(\lim_{t \rightarrow T} \frac{(H_t^s)_{\max}}{(H_t^s)_{\min}} = 1\) holds.

**Remark 1.1.** “\(\lim_{t \rightarrow T} \frac{(H_t^s)_{\max}}{(H_t^s)_{\min}} = 1\)” implies that \(f_t(M)\) converges to an infinitesimal constant tube over some \(G\)-orbit as \(t \rightarrow T\) (or equivalently, \(\phi(f_t(M))\) converges to a round point (= an infinitesimal round sphere) as \(t \rightarrow T\) (see Figure 1.2).

In Section 2, we recall the definition of the the regularized mean curvature flow and, in Section 3, we discuss the existence and the uniqueness of mean curvature...
flows starting from a compact orbifold in a Riemannian orbifold. In the first-half part of Section 4, we give a new proof of the existence and the uniqueness of a $G$-invariant regularized mean curvature flow starting from a $G$-invariant regularizable hypersurface satisfying the condition (III) in a Hilbert space $V$ equipped with a Hilbert Lie group action $G \curvearrowright V$ satisfying the conditions (I) and (II). In the second-half part of the section, we prepare the evolution equations for some basic geometric quantities along the $G$-invariant regularized mean curvature flow. In Section 5, we prove the Sobolev inequality for Riemannian suborbifolds. Sections 6-8 is devoted to prove Theorem A. In Section 9, we state a vision for applying the study of the regularized mean curvature flow to the gauge theory. In more detail, we state a vision for find and study interesting flows of hypersurfaces in the Yang-Milles (or self-dual) moduli space from regularized mean curvature flows in a Hilbert space.

![Diagram of figure 1.2: Collapse to a $G$-orbit]

2 The regularized mean curvature flow

Let $f$ be an immersion of an (infinite dimensional) Hilbert manifold $M$ into a Hilbert space $V$ and $A$ the shape tensor of $f$. If codim $M < \infty$, if the differential of the normal exponential map $\exp^\perp$ of $f$ at each point of $M$ is a Fredholm operator and if the restriction $\exp^\perp$ to the unit normal ball bundle of $f$ is proper, then $M$ is called a proper Fredholm submanifold. In this paper, we then call $f$ a proper Fredholm immersion. Then the shape operator $A_v$ is a compact operator for each normal vector $v$ of $M$. Furthermore, if, for each normal vector $v$ of $M$, the regularized trace $\text{Tr}_r A_v$ and $\text{Tr}_r A_v^2$ exist, then $M$ is called regularizable submanifold, where $\text{Tr}_r A_v$ is defined by $\text{Tr}_r A_v := \sum_{i=1}^{\infty} (\mu_i^+ + \mu_i^-) \left( \mu_1^- \leq \mu_2^- \leq \cdots \leq 0 \leq \cdots \leq \mu_2^+ \leq \mu_1^+ : \text{the spectrum of } A_v \right)$. In this paper, we then call $f$ regularizable immersion. If $\text{Tr}_r A_v = 0$
holds for any \( v \in T^\perp u M \), then \( f \) is said to be minimal. If \( f \) is a regulizable immersion and if \( \rho_u : v \mapsto \text{Tr}_r (A_u)_v (v \in T^\perp u M) \) is linear for any \( u \in M \), then the regularized mean curvature vector \( H \) of \( f \) is defined as the normal vector field satisfying \( \langle H_u, v \rangle = \text{Tr}_r (A_u)_v (\forall v \in T^\perp u M) (u \in M) \), where \( \langle , \rangle \) denotes the inner product of \( V \) and \( T^\perp u M \) denotes the normal space of \( f \) at \( u \).

**Example 2.1.** We consider the case where \( f \) is isoparametric. Then First we recall the notion of an isoparametric submanifold in a Hilbert space. If the normal connection of \( f \) is flat and if the principal curvatures of \( f \) for \( v \) are constant for any parallel normal vector field \( v \), then it is called an isoparametric submanifold. Then, by analyzing the focal structure of \( f \), we can show that the set \( \Lambda \) of all the principal curvatures of \( f \) is given by

\[
\Lambda = \bigcup_{a=1}^{k} \left\{ \frac{\lambda_a}{1 + b_a j} \mid j \in \mathbb{Z} \right\},
\]

where \( \lambda_a \)'s are parallel sections of the normal bundle \( T^\perp M \) and \( b_a \)'s are positive constants greater than one. See the first-half part of the proof of Theorem A in [Koi1] about the proof of this fact. Note that, even if Theorem A in [Koi1] is a result for isoparametric submanifolds in a Hilbert space arising from equifocal submanifolds in symmetric space of compact type, the first-half part of the proof is discussed for general isoparametric submanifolds in a Hilbert space. Hence the spectrum \( \text{Spec}(A_u)_v \) of the shape operator \( (A_u)_v \) for each normal vector \( v \) of \( f \) at \( u \in M \) is given by

\[
\text{Spec}(A_u)_v = \bigcup_{a=1}^{k} \left\{ \frac{(\lambda_a)_u(v)}{1 + b_a j} \mid j \in \mathbb{Z} \right\}.
\]

Hence the regularized trace \( \text{Tr}_r (A_u)_v \) is given by

\[
\text{Tr}_r (A_u)_v = \sum_{a=1}^{k} \sum_{j=1}^{\infty} \frac{(\lambda_a)_u(v)}{1 + b_a j}.
\]

From this fact, it directly follows that \( \rho_u \) is linear.

**Example 2.2.** We consider the case where \( f \) is a hypersurface. Then, since the normal space of \( M \) is of dimension one, \( \rho_u \) is linear for each point \( u \in M \).

We consider the case where \( f \) is a hypersurface and it admits a global unit normal vector field. Fix a global unit normal vector field \( \xi \). Then we call \( \text{Tr}_r A_{-\xi} (= -\langle H, \xi \rangle) \)
the regularized mean curvature of \( f \) and denote it by \( H^s \). Also, we call \( -A_\xi \) the shape operator and denote it by the same symbol \( A \).

**Remark 2.1.** In the research of the mean curvature flow starting from strictly convex hypersurfaces, it is general to take the outward unit normal vector field as the unit normal vector field \( \xi \) and \( -A_\xi \) as the shape operator and \( -\langle H, \xi \rangle \) as the mean curvature. Hence we take the shape operator \( A \) and the regularized mean curvature \( H^s \) as above.

Let \( \{ f_t \}_{t \in [0,T]} \) be a \( C^\infty \)-family of regularizable immersions of \( M \) into \( V \). Denote by \( H_t \) the regularized mean curvature vector of \( f_t \). Define a map \( F : M \times [0,T) \to V \) by \( F(x,t) := f_t(x) \ ((x,t) \in M \times [0,T)) \). If \( \frac{DF}{dt} = H_t \) holds, then we call \( \{ f_t \}_{t \in [0,T]} \) the regularized mean curvature flow. We cannot show that there uniquely exists a regularized mean curvature flow starting from \( f \) for any \( C^\infty \)-regularizable immersion \( f : M \to V \) because \( M \) is not compact. However, for a \( G \)-invariant regularizable immersion \( f : M \to V \) with the compact quotient \( f(M)/G \) in a Hilbert space \( V \) equipped with a special Hilbert Lie group action \( G \curvearrowright V \), it is shown that there uniquely exists a \( G \)-invariant regularized mean curvature flow starting from \( f \) (see Theorem 4.1).

3 The mean curvature flow in Riemannian orbifolds

The basic notions for a Riemannian orbifold and a suborbifold were defined in \([AK, BB, GKP, Sa, Sh, Th]\). In \([Koi2]\), we introduced the notion of the mean curvature flow starting from a suborbifold in a Riemannian orbifold. We shall recall this notion shortly. Let \( M \) be a paracompact Hausdorff space and \( \mathcal{O} := \{(U_\lambda, \varphi_\lambda, \tilde{U}_\lambda/\Gamma_\lambda) \mid \lambda \in \Lambda \} \) an \( n \)-dimensional \( C^k \)-orbifold atlas of \( M \), that is, a family satisfying the following condition (i)-(iv):

(i) \( \{ U_\lambda \mid \lambda \in \Lambda \} \) is an open covering of \( M \),
(ii) \( \tilde{U}_\lambda \) is an open set of \( \mathbb{R}^n \) and \( \Gamma_\lambda \) is a finite subgroup of the \( C^k \)-diffeomorphism group \( \text{Diff}^k(\tilde{U}_\lambda) \) of \( \tilde{U}_\lambda \),
(iii) \( \varphi_\lambda \) is a homeomorphism of \( U_\lambda \) onto \( \tilde{U}_\lambda/\Gamma_\lambda \),
(iv) for \( \lambda, \mu \in \Lambda \) with \( U_\lambda \cap U_\mu \neq \emptyset \), there exists \( (U_\nu, \phi_\nu, \tilde{U}_\nu/\Gamma_\nu) \in \mathcal{O} \) such that \( C^k \)-embeddings \( \rho_\lambda : \tilde{U}_\nu \hookrightarrow \tilde{U}_\lambda \) and \( \rho_\mu : \tilde{U}_\nu \hookrightarrow \tilde{U}_\mu \) satisfying \( \varphi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} \circ \rho_\lambda = \phi_\nu^{-1} \circ \pi_{\Gamma_\nu} \) and \( \varphi_\mu^{-1} \circ \pi_{\Gamma_\mu} \circ \rho_\mu = \phi_\nu^{-1} \circ \pi_{\Gamma_\nu} \), where \( \pi_{\Gamma_\lambda}, \pi_{\Gamma_\mu} \) and \( \pi_{\Gamma_\nu} \) are the orbit maps of \( \Gamma_\lambda, \Gamma_\mu \) and \( \Gamma_\nu \), respectively.

The pair \( (M, \mathcal{O}) \) is called an \( n \)-dimensional \( C^k \)-orbifold. and each \( (U_\lambda, \varphi_\lambda, \tilde{U}_\lambda/\Gamma_\lambda) \) is called an orbifold chart. Let \( (U_\lambda, \varphi_\lambda, \tilde{U}_\lambda/\Gamma_\lambda) \) be an orbifold chart around \( x \in M \).
Then the group \((\Gamma_{\lambda}\hat{\epsilon}) := \{ b \in \Gamma_{\lambda} \mid b(\hat{x}) = \hat{x} \}\) is unique for \(x\) up to the conjugation, where \(\hat{x}\) is a point of \(\hat{U}_{\lambda}\) with \((\varphi_{\lambda}^{-1} \circ \pi_{\Gamma_{\lambda}})(\hat{x}) = x\). Denote by \((\Gamma_{\lambda})_{\hat{x}}\) the conjugate class of this group \((\Gamma_{\lambda}\hat{\epsilon})\). This conjugate class is called the local group at \(x\). If the local group at \(x\) is not trivial, then \(x\) is called a singular point of \((M,\mathcal{O})\). Denote by Sing\((M,\mathcal{O})\) (or Sing\((M)\)) the set of all singular points of \((M,\mathcal{O})\). This set Sing\((M,\mathcal{O})\) is called the singular set of \((M,\mathcal{O})\).

Let \((M,\mathcal{O}_{M})\) and \((N,\mathcal{O}_{N})\) be orbifolds, and \(f\) a map from \(M\) to \(N\). If, for each \(x \in M\) and each pair of an orbifold chart \((U_{\lambda},\varphi_{\lambda},\hat{U}_{\lambda}/\Gamma_{\lambda})\) of \((M,\mathcal{O}_{M})\) around \(x\) and an orbifold chart \((V_{\mu},\psi_{\mu},\hat{V}_{\mu}/\Gamma_{\mu})\) of \((N,\mathcal{O}_{N})\) around \(f(x)\) \((f(U_{\lambda}) \subset V_{\mu})\), there exists a \(C^{k}\)-map \(\hat{f}_{\lambda\mu} : \hat{U}_{\lambda} \to \hat{V}_{\mu}\) with \(f \circ \varphi_{\lambda}^{-1} \circ \pi_{\Gamma_{\lambda}} = \psi_{\mu}^{-1} \circ \pi_{\Gamma_{\mu}} \circ \hat{f}_{\lambda\mu}\), then \(f\) is called a \(C^{k}\)-orbimorphism (or simply a \(C^{k}\)-map). Also \(\hat{f}_{\lambda\mu}\) is called a local lift of \(f\) with respect to \((U_{\lambda},\varphi_{\lambda},\hat{U}_{\lambda}/\Gamma_{\lambda})\) and \((V_{\mu},\psi_{\mu},\hat{V}_{\mu}/\Gamma_{\mu})\). Furthermore, if each local lift \(\hat{f}_{\lambda\mu}\) is an immersion, then \(f\) is called a \(C^{k}\)-orbiimmersion (or simply a \(C^{k}\)-immersion) and \((M,\mathcal{O}_{M})\) is called a \(C^{k}\)-(immersed) suborbifold in \((N,\mathcal{O}_{N})\). In the sequel, we assume that \(r = \infty\). If a \((0,2)\)-orbitensor field \(g\) of class \(C^{k}\) on \((M,\mathcal{O}_{M})\) is positive definite and symmetric, then we call \(g\) a \(C^{k}\)-Riemannian orbibetic and \((M,\mathcal{O}_{M},g)\) a \(C^{k}\)-Riemannian orbifold. See Section 3 of [Koi2] about the definition of \((0,2)\)-orbitensor field of class \(C^{k}\).

Let \(\{f_{t}\}_{t \in [0,T]}\) be a \(C^{\infty}\)-family of \(C^{\infty}\)-orbiimmersions of a \(C^{\infty}\)-orbifold \((M,\mathcal{O}_{M})\) into a \(C^{\infty}\)-Riemannian orbibetic \((N,\mathcal{O}_{N},\tilde{g})\). Assume that, for each \((x_{0},t_{0}) \in M \times [0,T]\) and each pair of an orbifold chart \((U_{\lambda},\varphi_{\lambda},\hat{U}_{\lambda}/\Gamma_{\lambda})\) of \((M,\mathcal{O}_{M})\) around \(x_{0}\) and an orbifold chart \((V_{\mu},\varphi_{\mu},\hat{V}_{\mu}/\Gamma_{\mu})\) of \((N,\mathcal{O}_{N})\) around \(f_{t_{0}}(x_{0})\) such that \(f_{t}(U_{\lambda}) \subset V_{\mu}\) for any \(t \in [t_{0},t_{0} + \varepsilon] \) \((\varepsilon : a \text{ sufficiently small positive number})\), there exists local lifts \((\hat{f}_{t})_{\lambda\mu} : \hat{U}_{\lambda} \to \hat{V}_{\mu}\) of \(f_{t}\) \((t \in [t_{0},t_{0} + \varepsilon])\) such that they give the mean curvature flow in \((\hat{V}_{\mu},\tilde{g}_{\mu}^{\lambda})\), where \(\tilde{g}_{\mu}^{\lambda}\) is the local lift of \(g\) to \(\hat{V}_{\mu}\). Then we call \(f_{t}\) \((0 \leq t < T)\) the mean curvature flow in \((N,\mathcal{O}_{N},\tilde{g})\).

In [Koi2], we proved the existence and the uniqueness theorem of a mean curvature flow starting from a \(C^{\infty}\)-orbiimmersion \(f\) of a compact \(C^{\infty}\)-orbifold into a \(C^{\infty}\)-Riemannian orbibetic (see Theorem 3.1 in [Koi2]). However, there was a gap in the proof. Hence we shall close the gap. The mean curvature flow equation is the same kind of partial differential equation as the Ricci flow equation. For the existence and the uniqueness of solutions of the Ricci flow equation in a compact orbibetic, the following fact is known. According to Subsection 5.2 of [KL], it is shown that, for any \(C^{\infty}\)-orbibetic \(g\) on a compact orbibetic \(M\), there uniquely exists a Ricci flow starting from \(g\) in short time. The method of the proof is as follows. The existence and the uniqueness of solutions of the Ricci flow equation in short time is reduced to those of a standard quasi-linear parabolic partial differential equation called the Ricci-de Turck equation by the de Turck trick. Since the Ricci-de Turck equation is a standard quasi-linear partial differential equation, it is shown that, in
the case where $M$ is a compact manifold (without boundary), there uniquely exists a solution of the Ricci-de Turck equation having $g$ as the initial data in short time for any $C^\infty$-Riemannian metric $g$. Hence, in this case, it is shown that, for any $C^\infty$-Riemannian metric $g$, there uniquely exists a Ricci flow starting from $g$ in short time. As stated in Subsection 5.2 of [KL], even if $M$ is a compact orbifold (not manifold), it is shown similarly that, for any $C^\infty$-Riemannian orbimetric $g$ on $M$, there uniquely exists a Ricci flow starting from $g$ in short time.

We prove the following statement by applying this method of the proof to the case of the mean curvature flow starting from a $C^\infty$-orbiimmersion of a compact orbifold $M$ into a Riemannian orbifold $(N, \tilde{g})$.

**Theorem 3.1.** Let $(M, O_M)$ be a compact orbifold and $(N, O_N, \tilde{g})$ a Riemannian orbifold. For any $C^\infty$-orbiimmersion $f$ of $M$ into $N$, there uniquely exists a mean curvature flow starting $f$ in short time.

**Proof.** First we consider the case where $M$ and $N$ are manifolds. Then the existence and the uniqueness of solutions of the mean curvature flow equation on a compact manifold (without boundary) in short time is reduced to those of a standard quasi-linear parabolic partial differential equation called the mean curvature-de Turck equation by the de Turck trick (see [Z], [CY] and [Koi3] etc. for example). Let’s recall the definition of the mean curvature-de Turck equation. Let $\{f_t\}_{t \in [0,T)}$ be a $C^\infty$-family of immersions of $M$ into $(N, \tilde{g})$. Set $g_t := f_t^* \tilde{g}$ and denote by $\nabla_t$ the Riemannian connection of $g_t$. Fix a torsion-free connection $\tilde{\nabla}$ on $M$. Define a $(1,2)$-tensor field $S_t$ on $M$ by $S_t := \nabla_t - \tilde{\nabla}$ and a vector field $V(f_t)$ on $M$ by $V(f_t) := \text{Tr}_{g_t} S_t$, where $\text{Tr}_{g_t} S_t$ is the trace of $S_t$ with respect to $g_t$. The mean curvature-de Turck equation is defined by

$$\frac{\partial f_t}{\partial t} = H_t + df_t(V(f_t)).$$

We give the local expressions of the mean curvature flow equation and the mean curvature-de Turck equation. Let $n := \dim M$ and $n + r := \dim N$. Take a local coordinate $(U, (x_1, \cdots, x_n))$ of $M$ and a local coordinate $(W, (y_1, \cdots, y_{n+r}))$ of $N$ with $f(U) \subset W$. The local expression of the mean curvature flow equation is given by

$$\frac{\partial (f_t)^\gamma}{\partial t} = \sum_{i,j=1}^n (g_t)^{ij} \left( \partial_t \partial_j (f_t)^\gamma + \sum_{\alpha,\beta=1}^{n+r} \partial_t (f_t)^\alpha \partial_j (f_t)^\beta (\tilde{g}_{\alpha\beta} \circ f_t) \right) - \sum_{k=1}^n \partial_k (f_t)^\gamma (\Gamma_t)^k_{ij},$$

for $t \in [0,T)$. The local expression of the mean curvature-de Turck equation is given by

$$\frac{\partial (f_t)^\gamma}{\partial t} = H_t + df_t(V(f_t)).$$

We give the local expressions of the mean curvature flow equation and the mean curvature-de Turck equation. Let $n := \dim M$ and $n + r := \dim N$. Take a local coordinate $(U, (x_1, \cdots, x_n))$ of $M$ and a local coordinate $(W, (y_1, \cdots, y_{n+r}))$ of $N$ with $f(U) \subset W$. The local expression of the mean curvature flow equation is given by

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for $t \in [0,T)$. The local expression of the mean curvature-de Turck equation is given by

$$\frac{\partial (f_t)^\gamma}{\partial t} = H_t + df_t(V(f_t)).$$
where \((f_t)^\gamma\)'s are the components of \(f_t\) with respect to \((U, (x_1, \ldots, x_n))\) and \((W, (y_1, \ldots, y_{n+r}))\), \(((g_t)^{ij})\) is the inverse matrix of the matrix \(((g_t)^{ij})\) consisting of the components \((g_t)^{ij}\)'s of the induced metric \(g_t\) with respect to \((U, (x_1, \ldots, x_n))\), \(\tilde{\Gamma}_{\alpha\beta}^\gamma\)'s are the Christoffel’s symbols of \(\bar{g}\) with respect to \((W, (y_1, \ldots, y_{n+r}))\) and \((\Gamma_t)^k_{ij}\) is the Christoffel’s symbols of \(g_t\) with respect to \((U, (x_1, \ldots, x_n))\). Here we note that \((g_t)^{ij}\) is given by

\[
(g_t)^{ij} = \sum_{\alpha, \beta = 1}^{n+r} (\partial_i(f_t)^\alpha)(\partial_j(f_t)^\beta)(\bar{g}_{\alpha\beta} \circ f_t)
\]

and \((\Gamma_t)^k_{ij}\) is given by

\[
(\Gamma_t)^k_{ij} = \sum_{l=1}^{n} \frac{(g_t)^{kl}}{2} (\partial_l(g_t)^{ij} + \partial_j(g_t)^{il} - \partial_i(g_t)^{lj}) .
\]

By the existence of the final term of the right-hand side of (3.2), the equation (3.2) is a quasi-linear parabolic partial differential equation but not strongly parabolic. On the other hand, the local expression of the mean curvature-de Tureck equation is given by

\[
\frac{\partial (f_t)^\gamma}{\partial t} = \sum_{i,j=1}^{n} (g_t)^{ij} \left( \partial_t \partial_j(f_t)^\gamma + \sum_{\alpha, \beta = 1}^{n+r} \partial_i(f_t)^\alpha \partial_j(f_t)^\beta \tilde{\Gamma}_{\alpha\beta}^\gamma \circ f_t \right) \\
- \sum_{k=1}^{n} \partial_k(f_t)^\gamma \tilde{\Gamma}_{ij}^k
\]

where \((f_t)^\gamma\)'s, \(((g_t)^{ij})\) and \(\tilde{\Gamma}_{\alpha\beta}^\gamma\) are as above, and \(\tilde{\Gamma}_{ij}^k\)'s are the Christoffel’s symbol of a fixed connection \(\tilde{\nabla}\) of \(M\). Since the final term of the right-hand side of (3.2) is changed by \(\sum_{i,j,k=1}^{n} (g_t)^{ij} \partial_k(f_t)^\gamma \tilde{\Gamma}_{ij}^k\), the equation (3.3) (hence (3.1)) is a strongly parabolic quasi-linear parabolic partial differential equation. Hence, if \(M\) is a compact manifold (without boundary), then it is shown that there uniquely exists a solution \(\{f_t\}_{t \in [0,T]}\) of (3.1) with \(f_0 = f\) in short time. As B. Kleiner and J. Lott state in Subsection 5.2 of [KL] in the case of the Ricci flow on a compact orbifold, even if \(M\) is a compact orbifold (not manifold), it is shown similarly that there uniquely exists a solution \(\{f_t\}_{t \in [0,T]}\) of (3.1) with \(f_0 = f\) in short time. For the solution \(\{f_t\}_{t \in [0,T]}\), we consider the ordinary differential equation \(\frac{\partial \psi_t}{\partial t} = -V(f_t) \circ \psi_t\). Let \(\{\psi_t\}_{t \in [0,T]}\) \((T' < T)\) be the solution of this ordinary differential equation with the initial condition \(\psi_0 = id_M\), where we note that each \(\psi_t\) is a \(C^\infty\)-diffeomorphism of \(M\) onto oneself. Then it is shown that \(\{f_t \circ \psi_t\}_{t \in [0,T']}\) is a mean curvature flow.
starting from $f$. Conversely, it is easy to show that any mean curvature flow starting from $f$ is given like this. This completes the proof. 

**Remark 3.1.** Let $f$ be a $C^\infty$-immersion of a manifold $M$ into a complete Riemannian manifold $(N, \tilde{g})$ and $D$ a relative compact domain of $M$. The existence of a mean curvature flow starting from $f|_D$ is shown but its uniqueness is not shown. In fact, it is shown that there exist infinitely many mean curvature flows starting from $f|_D$ as follows. Let $\tilde{f}|_D : M \to N$ be an immersion satisfying the following conditions:

(i) $(\tilde{f}|_D)|_D = f|_D$;

(ii) $\tilde{f}|_D(M)$ is an immersed complete Riemannian submanifold of bounded second fundamental form in $(N, \tilde{g})$, where we note that the condition of “bounded second fundamental form” controls the behavior of the submanifold $\tilde{f}|_D(M)$ near the infinity.

It is clear that there exists infinitely many complete extensions $f|_D$ satisfying the condition (ii). Assume that the norms of the curvature tensor of $(N, \tilde{g})$, its first derivative and its second derivative are bounded. Then there uniquely exists a mean curvature flow $\{(\tilde{f}|_D)_t\}_{t \in [0, T)}$ of bounded second fundamental form starting from $\tilde{f}|_D$ by the uniqueness theorem in [CY]. It is clear that $\{(\tilde{f}|_D)_t|_D\}_{t \in [0, T)}$ gives a mean curvature flow starting from $f|_D$ and that this flow depends on the choice of the complete extension $\tilde{f}|_D$. Thus we see that there exist infinitely many mean curvature flows starting from $f|_D$.

### 4 Evolution equations

Let $\mathcal{G} \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group $\mathcal{G}$ on a Hilbert space $V$ equipped with an inner product $\langle \cdot, \cdot \rangle$. The orbit space $V/\mathcal{G}$ is a (finite dimensional) $C^\infty$-orbifold. Let $\phi : V \to V/\mathcal{G}$ be the orbit map and set $N := V/\mathcal{G}$. Here we give an example of such an isometric almost free action of a Hilbert Lie group.

**Example 4.1.** Let $G$ be a compact semi-simple Lie group, $K$ a closed subgroup of $G$ and $\Gamma$ a discrete subgroup of $G$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively. Assume that a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ exists. Let $B$ be the Killing form of $\mathfrak{g}$ and $g$ the bi-invariant metric of $G$ induced from $-B$. Also, let $H^0([0, 1], \mathfrak{g})$ be the Hilbert space of all paths in the Lie algebra $\mathfrak{g}$ of $G$ which are $L^2$-integrable with respect to $-B$, and $H^1([0, 1], G)$ the Hilbert Lie group of all paths in $G$ which are of class $H^1$ with respect to $g$. This group $H^1([0, 1], G)$ acts on
$H^0([0, 1], g)$ isometrically and transitively as a gauge action:

$$(g \cdot u)(s) = \text{Ad}_G(g(s))(u(s)) - (R_{g(s)})^{-1}_s(g'(s)) \quad (s \in [0, 1])$$

$$(g \in H^1([0, 1], G), u \in H^0([0, 1], g)).$$

Set $P(G, \Gamma \times K) := \{g \in H^1([0, 1], G) \mid (g(0), g(1)) \in \Gamma \times K\}$. The group $P(G, \Gamma \times K)$ acts on $H^0([0, 1], g)$ almost freely and isometrically, and the orbit space of this action is diffeomorphic to the orbifold $\Gamma \setminus G / K$. Furthermore, each orbit of this action is regularizable and minimal (see [HLO], [PiTh], [Te1], [Te2], [TeTh], [Koi2]). In particular, in the case of $K = \Gamma = \{e\}$, $\phi$ is a Riemannian submersion of $H^0([0, 1], g)$ onto $(G, g)$ and is called the parallel transport map for $G$.

Let $g_N$ be the Riemannian orbimetric on $N$ such that $\phi$ is a Riemannian orsubmersion of $(V, \langle \ , \ \rangle)$ onto $(N, g_N)$. By using Theorem 3.1, we prove the following unique existence theorem for a $G$-invariant regularized mean curvature flow starting from a $G$-invariant regularizable hypersurface with compact quotient.

**Theorem 4.1.** Let $f : M \hookrightarrow V$ be an immersion such that $f(M)$ is a $G$-invariant regularizable hype-surface and that $(\phi \circ f)(M)$ is compact. Then there uniquely exists a $G$-invariant regularized mean curvature flow $\{f_t\}_{t \in [0, T]}$ starting from $f$.

**Proof.** Since $f(M)$ is $G$-invariant and $\phi(f(M))$ is compact, we can take an orbisubmersion $\overline{f}$ of a compact orbifold $M'$ into $N$ and an orbifold submersion $\phi_M : M \rightarrow M'$ with $\phi \circ f = \overline{f} \circ \phi_M$. Let $H$ be the regularized mean curvature vector of $f$ and $\overline{H}$ the mean curvature vector of $\overline{f}$. Then, since the fibres of $\phi$ are minimal regularizable submanifolds, we have $\phi_\ast(H) = \overline{H}$. In more detail, $H$ is the horizontal lift of $\overline{H}$ (with respect to the Riemannian orsubmersion $\phi$). According to Theorem 3.1, there uniquely exists the mean curvature flow $\overline{\{f_t\}}_{t \in [0, T]}$ starting from $\overline{f}$. Define a curve $c_x : [0, T) \rightarrow N$ by $c_x(t) := \overline{f}_t(x)$ and let $\overline{c}^a_x(t) := [0, T) \rightarrow V$ be the horizontal lift of $c_x$ to $f(u)$. Define an immersion $f_t : M \hookrightarrow V$ by $f_t(u) = \overline{c}^a_x(t)$ ($u \in \overline{M}$). Let $H_t$ be the regularized mean curvature vector of $f_t$ and $\overline{H}_t$ the mean curvature of $\overline{f}_t$. Then, it is clear that $f_t(M) = \phi^{-1}(\overline{f}_t(M'))$ holds. Hence, since the fibres of $\phi$ are minimal regularizable submanifolds, $H_t$ is the horizontal lift of $\overline{H}_t$. On the other hand, it follows from the construction of $f_t$ that $\frac{\partial f_t}{\partial t}$ is the horizontal lift of $\frac{\partial \overline{f}_t}{\partial t}$.

Also, since $\{\overline{f}_t\}_{t \in [0, T)}$ is the mean curvature flow, we have $\frac{\partial \overline{f}_t}{\partial t} = \overline{H}_t$. Hence $\{f_t\}_{t \in [0, T)}$ is a regularized mean curvature flow starting from $f$. It is clear that $f_t(M)$’s ($0 \leq t < T$) are $G$-invariant. Thus the existence of a $G$-invariant regularized mean curvature flow starting from $f$ is shown.
Next we shall show the uniqueness of a $\mathcal{G}$-invariant regularized mean curvature flow \( \{f_t\}_{t \in [0,T)} \) starting from \( f \). Let \( \{f_t\}_{t \in [0,T)} \) be such a flow. Then, since \( f_t(M) \) is $\mathcal{G}$-invariant, we can take an orbimmersion \( \mathcal{J}_t \) of a compact orbifold \( M' \) into \( N \) satisfying \( \phi \circ f_t = \mathcal{J}_t \circ \phi_M \). Then we have
\[
\frac{\partial \mathcal{J}_t}{\partial t} = \phi_*(\frac{\partial f_t}{\partial t}) = \phi_*(H_t) = \overline{\Pi}_t.
\]
Hence \( \{\mathcal{J}_t\}_{t \in [0,T)} \) is the mean curvature flow starting from \( \mathcal{J} \). Therefore \( \{f_t\}_{t \in [0,T)} \) is the flow constructed as above. Thus the uniqueness of a $\mathcal{G}$-invariant regularized mean curvature flow starting from \( f \) also is shown.

\begin{remark}
We cannot conclude whether there uniquely exists a (not necessarily $\mathcal{G}$-invariant) regularized mean curvature flow starting from a $\mathcal{G}$-invariant regularized immersion \( f \) as in the statement of Theorem 4.1.
\end{remark}

Let \( f : M \hookrightarrow V \) be an immersion such that \( f(M) \) is a $\mathcal{G}$-invariant regularizable hypersurface and that \( (\phi \circ f)(M) \) is compact and \( \{f_t\}_{t \in [0,T)} \) the $\mathcal{G}$-invariant regularized mean curvature flow starting from \( f \). Define a map \( \mathcal{F} : M \times [0,T) \to V \) by \( \mathcal{F}(u,t) := f_t(u) \) \((u,t) \in M \times [0,T)\) and a map \( \mathcal{F} : N \times [0,T) \to N \) by \( \mathcal{F}(x,t) := \mathcal{J}_t(x) \) \((x,t) \in N \times [0,T)\). Denote by \( H_t \) the regularized mean curvature vector of \( f_t \) and \( \overline{\Pi}_t \) the mean curvature vector of \( \mathcal{J}_t \). Since \( \phi \) has minimal regularizable fibres, \( H_t \) is the horizontal lift of \( \overline{\Pi}_t \), we can show that \( \phi \circ f_t = \mathcal{J}_t \circ \phi_M \) holds for all \( t \in [0,T) \). In the sequel, we consider the case where the codimension of \( M \) is equal to one. Denote by \( \hat{\mathcal{H}} \) (resp. \( \hat{\mathcal{V}} \)) the horizontal (resp. vertical) distribution of \( \phi \). Denote by \( \text{pr}_{\hat{\mathcal{H}}} \) (resp. \( \text{pr}_{\hat{\mathcal{V}}} \)) the orthogonal projection of \( TV \) onto \( \hat{\mathcal{H}} \) (resp. \( \hat{\mathcal{V}} \)). For simplicity, for \( X \in TV \), we denote \( \text{pr}_{\hat{\mathcal{H}}}(X) \) (resp. \( \text{pr}_{\hat{\mathcal{V}}}(X) \)) by \( X_{\hat{\mathcal{H}}} \) (resp. \( X_{\hat{\mathcal{V}}} \)). Define a distribution \( \mathcal{H}_t \) on \( M \) by \( f_t^*(\mathcal{H}(u)) = f_t^*(T_M(u)) \cap \mathcal{H}_t(u) \) \((u \in M)\) and a distribution \( \mathcal{V}_t \) on \( M \) by \( f_t^*(\mathcal{V}(u)) = \mathcal{V}_t(u) \) \((u \in M)\). Note that \( \mathcal{V}_t \) is independent of the choice of \( t \in [0,T) \). Fix a unit normal vector field \( \xi_t \) of \( f_t \). Denote by \( g_t, h_t, A_t, H_t \) and \( H^*_t \) the induced metric, the second fundamental form (for \( -\xi_t \)), the shape operator (for \( -\xi_t \)) and the regularized mean curvature vector and the regularized mean curvature (for \( -\xi_t \)), respectively. The group \( \mathcal{G} \) acts on \( M \) through \( f_t \). Since \( \phi : V \to V/\mathcal{G} \) is a $\mathcal{G}$-orbibundle and \( \hat{\mathcal{H}} \) is a connection of this orbibundle, it follows from Proposition 4.1 in [Koi2] that this action \( \mathcal{G} \acts M \) is independent of the choice of \( t \in [0,T) \). It is clear that quantities \( g_t, h_t, A_t, H_t \) and \( H^*_t \) are \( \mathcal{G} \)-invariant. Also, let \( \nabla^t \) be the Riemannian connection of \( g_t \). Let \( \pi_M \) be the projection of \( M \times [0,T) \) onto \( M \). For a vector bundle \( E \) over \( M \), denote by \( \pi^*_M \) the induced bundle of \( E \) by \( \pi_M \). Also denote by \( \Gamma(E) \) the space of all sections of \( E \). Define a section \( g \) of \( \pi^*_M(T^{(0,2)}M) \) by \( g(u,t) = (g_t)_{\cdot \cdot} \) \((u,t) \in M \times [0,T)\),
where $T^{0,2}M$ is the $(0,2)$-tensor bundle of $M$. Similarly, we define a section $h$ of $\pi^*_M(T^{0,2}M)$, a section $A$ of $\pi^*_M(T^{1,1}M)$, a section $H$ of $F^{*}TV$ and a section $\xi$ of $F^{*}TV$. We regard $H$ and $\xi$ as $V$-valued functions over $M \times [0, T)$ under the identification of $T_uV$’s ($u \in V$) and $V$. Define a subbundle $\mathcal{H}$ (resp. $\mathcal{V}$) of $\pi^*_M TM$ by $\mathcal{H}(u,t) := (H_t)_u$ (resp. $\mathcal{V}(u,t) := (V_t)_u$). Denote by $pr_{\mathcal{H}}$ (resp. $pr_{\mathcal{V}}$) the orthogonal projection of $\pi^*_M(TM)$ onto $\mathcal{H}$ (resp. $\mathcal{V}$). For simplicity, for $X \in \pi^*_M(TM)$, we denote $pr_{\mathcal{H}}(X)$ (resp. $pr_{\mathcal{V}}(X)$) by $X_\mathcal{H}$ (resp. $X_\mathcal{V}$). For a section $B$ of $\pi^*_M(T^{(r,s)}M)$, we define $\frac{\partial B}{\partial t}$ by $(\frac{\partial B}{\partial t})_{(u,t)} := \frac{dB_{(u,t)}}{dt}$, where the right-hand side of this relation is the derivative of the vector-valued function $t \mapsto B_{(u,t)} \in T^{(r,s)}_uM$. Also, we define a section $B_\mathcal{H}$ of $\pi^*_M(T^{(r,s)}M)$ by

$$B_\mathcal{H} = (pr_{\mathcal{H}} \otimes \cdots \otimes pr_{\mathcal{H}}) \circ B \circ (pr_{\mathcal{H}} \otimes \cdots \otimes pr_{\mathcal{H}}).$$

The restriction of $B_\mathcal{H}$ to $\mathcal{H} \times \cdots \times \mathcal{H}$ (s-times) is regarded as a section of the $(r,s)$-tensor bundle $\mathcal{H}^{(r,s)}$ of $\mathcal{H}$. This restriction also is denoted by the same symbol $B_\mathcal{H}$. Let $D_M$ (resp. $D_{[0,T]}$) be the subbundle of $T(M \times [0, T))$ defined by $(D_M)(u,t) := T_{(u,t)}(M \times \{t\})$ (resp. $(D_{[0,T]})(u,t) := T_{(u,t)}(\{u\} \times [0, T))$ for each $(u,t) \in M \times [0, T)$. Denote by $v^L_{u,t}$ the horozontal lift of $v \in T_uM$ to $(u,t)$ with respect to $\pi_M$ (i.e., $v^L_{u,t}$ is the element of $(D_M)_{(u,t)}$ with $(\pi_M)_{(u,t)}(v^L_{u,t}) = v$). Under the identification of $((u,t), v) = (v) \in (\pi^*_M T\mathcal{H})_{(u,t)}$ with $v^L_{u,t} \in (D_{[0,T]})(u,t)$, we identify $\pi^*_M TM$ with $D_M$. For a tangent vector field $X$ on $M$ (or an open set $U$ of $M$), we define $\overline{X} \in \Gamma(\pi^*_M TM) = \Gamma(D_M)$ (or $\Gamma((\pi^*_M TM)|_U) = \Gamma((D_M)|_U)$) by $\overline{X}_{(u,t)} := ((u,t), X_u) := (X_u|_{(u,t)}) ((u,t) \in U \times [0, T))$. Denote by $\overline{\nabla}$ the Riemannian connection of $\mathcal{V}$. Let $\nabla$ be the connection of $\pi^*_M TM$ defined by

$$(\nabla XY)_{(u,t)} := \nabla^t_{X_{(u,t)}} Y_{(t)} \text{ and } (\nabla_{\frac{\partial}{\partial t}} Y)_{(u,t)} := \frac{dY_{(u,.)}}{dt}$$

for $X \in \Gamma(D_M)$ and $Y \in \Gamma(\pi^*_M TM)$, where $X_{(u,t)}$ is identified with $(\pi_M)_*(u,t)$ $(X_{(u,t)}) \in T_uM$, $Y_{(t)}$ is identified with $(\pi_M)_*(Y_{(t)}) \in \Gamma(TM)$ and $Y_{(u,.)}$ is identified with $(\pi_M)_*(Y_{(u,.)}) \in C^\infty([0, T), T_uM)$. Note that $\nabla_{\frac{\partial}{\partial t}} \overline{X} = 0$. Denote by the same symbol $\nabla$ the connection of $\pi^*_M T^{(r,s)}M$ defined in terms of $\nabla^t$’s similarly. Define a connection $\nabla^\mathcal{H}$ of $\mathcal{H}$ by $\nabla^\mathcal{H} X Y := (\nabla_X Y)_\mathcal{H}$ for $X \in \Gamma(M \times [0, T))$ and $Y \in \Gamma(\mathcal{H})$. Similarly, define a connection $\nabla^\mathcal{V}$ of $\mathcal{V}$ by $\nabla^\mathcal{V} X Y := (\nabla_X Y)_\mathcal{V}$ for $X \in \Gamma(M \times [0, T))$ and $Y \in \Gamma(\mathcal{V})$.

Now we shall recall the evolution equations for some geometric quantities given in [Koi2]. By the same calculation as the proof of Lemma 4.2 of [Koi2] (where we replace $H = \|H\|\xi$ in the proof to $H = \langle H, \xi \rangle \xi = -H^s \xi$), we can derive the following evolution equation.
Lemma 4.2. The sections \((g_\mathcal{H})_t\)'s of \(\pi^*_M(T^{(0,2)}M)\) satisfy the following evolution equation:

\[
\frac{\partial g_\mathcal{H}}{\partial t} = -2H^s h_\mathcal{H}.
\]

According to the proof of Lemma 4.3 in [Koi2], we obtain the following evolution equation.

Lemma 4.3. The unit normal vector fields \(\xi_t\)'s satisfy the following evolution equation:

\[
\frac{\partial \xi}{\partial t} = -F_\ast (\text{grad}_{g^\mathcal{H}} H^s),
\]

where \(\text{grad}_{g^\mathcal{H}} (H^s)\) is the element of \(\pi^*_M(TM)\) such that \(dH^s(X) = g(\text{grad}_{g^\mathcal{H}} H^s, X)\) for any \(X \in \pi^*_M(TM)\).

Let \(S_t\) \((0 \leq t < T)\) be a \(C^\infty\)-family of a \((r,s)\)-tensor fields on \(M\) and \(S\) a section of \(\pi^*_M(T^{(r,s)}M)\) defined by \(S_{(u,t)} := (S_t)_u\). We define a section \(\Delta_\mathcal{H} S\) of \(\pi^*_M(T^{(r,s)}M)\) by

\[
(\Delta_\mathcal{H} S)_{(u,t)} := \sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} S,
\]

where \(\nabla\) is the connection of \(\pi^*_M(T^{(r,s)}M)\) (or \(\pi^*_M(T^{(r,s+1)}M)\)) induced from \(\nabla\) and \((e_1, \cdots, e_n)\) is an orthonormal base of \(\mathcal{H}_{(u,t)}\) with respect to \((g_\mathcal{H})_{(u,t)}\). Also, we define a section \(\tilde{\Delta}_\mathcal{H} S_\mathcal{H}\) of \(\mathcal{H}^{(r,s)}\) by

\[
(\tilde{\Delta}_\mathcal{H} S_\mathcal{H})_{(u,t)} := \sum_{i=1}^{n} \nabla^\mathcal{H}_{e_i} \nabla^\mathcal{H}_{e_i} S_\mathcal{H},
\]

where \(\nabla^\mathcal{H}\) is the connection of \(\mathcal{H}^{(r,s)}\) (or \(\mathcal{H}^{(r,s+1)}\)) induced from \(\nabla^\mathcal{H}\) and \((e_1, \cdots, e_n)\) is as above. Let \(A^\phi\) be the section of \(T^*V \otimes T^*V \otimes TV\) defined by

\[
A^\phi_X Y := (\tilde{\nabla}_X Y_{\mathcal{H}})_{\mathcal{V}} + (\tilde{\nabla}_X Y_{\mathcal{V}})_{\mathcal{H}} \quad (X, Y \in TV).
\]

Also, let \(T^\phi_X\) be the section of \(T^*V \otimes T^*V \otimes TV\) defined by

\[
T^\phi_X Y := (\tilde{\nabla}_X Y_{\mathcal{H}})_{\mathcal{V}} + (\tilde{\nabla}_X Y_{\mathcal{V}})_{\mathcal{H}} \quad (X, Y \in TV).
\]

Also, let \(A_t\) be the section of \(T^*M \otimes T^*M \otimes TM\) defined by

\[
(A_t)_X Y := (\tilde{\nabla}_X Y_{\mathcal{H}_t})_{\mathcal{V}_t} + (\tilde{\nabla}_X Y_{\mathcal{V}_t})_{\mathcal{H}_t} \quad (X, Y \in TM).
\]
Also let $A$ be the section of $\pi_M^*(T^*M \otimes T^*M \otimes TM)$ defined in terms of $A_i$’s ($t \in [0, T]$). Also, let $\mathcal{T}_t$ be the section of $T^*M \otimes T^*M \otimes TM$ defined by

$$(\mathcal{T}_t)_{XY} := (\nabla^\mathcal{T}_t X_{Y} Y_{X})_{\mathcal{H}_t} + (\nabla^\mathcal{T}_t Y_{Y} X_{X})_{\mathcal{V}_t} \quad (X, Y \in TM).$$

Also let $\mathcal{T}$ be the section of $\pi_M^*(T^*M \otimes T^*M \otimes TM)$ defined in terms of $\mathcal{T}_t$’s ($t \in [0, T]$). Clearly we have

$$F_*(A_X Y) = A_{\varphi X} F_* Y$$

for $X, Y \in \mathcal{H}$ and

$$F_*(T_W X) = T_{F_* W} F_* X$$

for $X \in \mathcal{H}$ and $W \in \mathcal{V}$. Let $E$ be a vector bundle over $M$. For a section $S$ of $\pi_M^*(T(0, r)M \otimes E)$, we define $\text{Tr}_{g_{\mathcal{H}}} \cdot S(\cdots, \bullet^j, \cdots, \bullet^k, \cdots)$ by

$$(\text{Tr}_{g_{\mathcal{H}}} S(\cdots, \bullet, \cdots, \bullet^k, \cdots))_{(u,t)} = \sum_{i=1}^n S_{(u,t)}(\cdots, \bullet^i, \cdots, \bullet^k, \cdots)$$

$((u, t) \in M \times [0, T])$, where $\{e_1, \cdots, e_n\}$ is an orthonormal base of $\mathcal{H}_{(u,t)}$ with respect to $(g_{\mathcal{H}})_{(u,t)}$, $S(\cdots, \bullet^i, \cdots, \bullet^k, \cdots)$ means that $\bullet$ is entered into the $j$-th component and the $k$-th component of $S$ and $S_{(u,t)}(\cdots, \bullet^i, \cdots, \bullet^k, \cdots)$ means that $e_i$ is entered into the $j$-th component and the $k$-th component of $S_{(u,t)}$.

In [Koi2], we derived the following relation.

**Lemma 4.4.** Let $S$ be a section of $\pi_M^*(T(0, 2)M)$ which is symmetric with respect to $g$. Then we have

$$(\triangle_{\mathcal{H}} S)_{\mathcal{H}}(X, Y) = (\triangle^\mathcal{H} S_{\mathcal{H}})(X, Y) - 2\text{Tr}_{g_{\mathcal{H}}}(\nabla \cdot S)(A X, Y) - 2\text{Tr}_{g_{\mathcal{H}}}(\nabla \cdot S)(A Y, X) - \text{Tr}_{g_{\mathcal{H}}}(\nabla \cdot S)(A, X, Y) - \text{Tr}_{g_{\mathcal{H}}}(\nabla \cdot S)(A, Y, X)$$

for $X, Y \in \mathcal{H}$, where $\nabla$ is the connection of $\pi_M^*(T(1, 2)M)$ induced from $\nabla$.

According to the proof of Lemma 4.5 in [Koi2], we obtain the following Simons-type identity.

**Lemma 4.5.** We have

$$\triangle_{\mathcal{H}} h = \nabla d h^s + H^s(A^2) h - (\text{Tr} (A^2)_{\mathcal{H}}) h.$$
where \((A^2)^\sharp\) is the element of \(\Gamma(\pi^*_M T^{(0,2)}M)\) defined by \((A^2)^\sharp(X,Y) := g(A^2X,Y)\) for \((X,Y) \in \pi^*_M TM\).

**Note.** In the sequel, we omit the notation \(F_\ast\) for simplicity.

Define a section \(\mathcal{R}\) of \(\pi^*_M(\mathcal{H}^{(0,2)})\) by

\[
\mathcal{R}(X,Y) := \text{Tr}_{g^H}(\mathcal{A}^\ast (\mathcal{A}^\ast X), Y) + \text{Tr}_{g^H} h(\mathcal{A}^\ast (\mathcal{A}^\ast Y), X) \\
+ \text{Tr}_{g^H}(\nabla^\ast \mathcal{A}^\ast X, Y) + \text{Tr}_{g^H} h(\nabla^\ast \mathcal{A}^\ast Y, X) \\
+ 2\text{Tr}_{g^H}(\nabla^\ast h)(\mathcal{A}^\ast X, Y) + 2\text{Tr}_{g^H}(\nabla^\ast h)(\mathcal{A}^\ast Y, X) \\
+ 2\text{Tr}_{g^H} h(\mathcal{A}^\ast X, \mathcal{A}^\ast Y) \\
(X,Y \in \mathcal{H}).
\]

According to Theorem 4.6 in [Koi2], we obtain the following evolution equation from Lemmas 4.3, 4.4 and 4.5.

**Lemma 4.6.** The sections \((h^H)_t\)’s of \(\pi^*_M(\mathcal{T}^{0,2})\) satisfies the following evolution equation:

\[
\frac{\partial h^H}{\partial t}(X,Y) = (\triangle^H h^H)(X,Y) - 2H^s((A^H)^2)^\sharp(X,Y) - 2H^s((A^\xi)^2)^\sharp(X,Y) \\
+ \text{Tr} \left((A^H)^2 - ((A^\xi)^2)^H\right) h^H(X,Y) - \mathcal{R}(X,Y)
\]

for \((X,Y) \in \mathcal{H}\).

According to the proof of Lemma 4.8 of [Koi2], we obtain the following relation from Lemma 4.2.

**Lemma 4.7.** Let \(X\) and \(Y\) be local sections of \(\mathcal{H}\) such that \(g(X,Y)\) is constant. Then we have \(g(\nabla^\mathcal{H} X,Y) + g(X,\nabla^\mathcal{H} Y) = 2H^s h(X,Y)\).

According to Lemmas 4.8 and 4.10 in [Koi2], we obtain the following relation.

**Lemma 4.8.** For \((X,Y) \in \mathcal{H}\), we have

\[
\mathcal{R}(X,Y) = 2\text{Tr}_{g^H}(\langle (A^\phi X, A^\phi (A_H Y)) \rangle + \langle (A^\phi Y, A^\phi (A_H X)) \rangle) \\
+ 2\text{Tr}_{g^H}(\langle (A^\phi X, A^\phi (A_H \bullet)) \rangle + \langle (A^\phi Y, A^\phi (A_H \bullet)) \rangle) \\
+ 2\text{Tr}_{g^H}(\langle (\nabla^\ast A^\phi \xi, A^\phi X) \rangle + \langle (\nabla^\ast A^\phi \xi, A^\phi Y) \rangle) \\
+ \text{Tr}_{g^H}(\langle (\nabla^\ast A^\phi \bullet X, A^\phi Y) \rangle + \langle (\nabla^\ast A^\phi \bullet Y, A^\phi X) \rangle) \\
+ 2\text{Tr}_{g^H}(\langle T^\phi A^\ast X \xi, A^\phi Y \rangle),
\]

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where we omit $F$. In particular, we have

\[
\mathcal{R}(X, X) = 4\text{Tr}_{g_H} \langle A^\phi_X, A^\phi_X (A_H X) \rangle + 4\text{Tr}_{g_H} \langle A^\phi_X, A^\phi_X (A_H \phi) \rangle \\
+ 3\text{Tr}_{g_H} \langle (\bar{\nabla} \cdot A^\phi)_{\xi} X, A^\phi X \rangle + 2\text{Tr}_{g_H} \langle (\bar{\nabla} \cdot A^\phi)_{\xi} X, A^\phi X \rangle \\
+ \text{Tr}_{g_H} \langle A^\phi_X, (\bar{\nabla}_X A^\phi)_{\xi} \phi \rangle
\]

and hence

\[
\text{Tr}_{g_H} \mathcal{R}(\cdot, \cdot) = 0.
\]

**Simple proof of the third relation.** We give a simple proof of $\text{Tr}_{g_H} \mathcal{R}(\cdot, \cdot) = 0$. Take any $(u, t) \in M \times [0, T)$ and an orthonormal base $(e_1, \ldots, e_n)$ of $H_{(u,t)}$ with respect to $g_{(u,t)}$. According to Lemma 4.4 and the definition of $\mathcal{R}$, we have

\[
(\text{Tr}_{g_H} \mathcal{R}(\cdot, \cdot))(u, t) = (\text{Tr}_{g_H} (\nabla^2 h)_{H}(\cdot, \cdot))(u, t) = -\sum_{i=1}^{n} (\nabla^2 h^s)(e_i, e_i) = 0,
\]

where we use $H^s = \sum_{i=1}^{n} h(e_i, e_i)$ (which holds because the fibres of $\phi$ is regularized minimal).

According to the proof of Corollary 4.11 in [Koi2], we obtain the following evolution equation.

**Lemma 4.9.** The norms $H^s_t$’s of $H_t$ satisfy the following evolution equation:

\[
\frac{\partial H^s_t}{\partial t} = \Delta_{H^s} H^s + H^s ||A_H||^2 - 3H^s \text{Tr}((A^\phi_{\xi})^2),
\]

According to the proof of Corollary 4.12 in [Koi2], we obtain the following evolution equation.
Lemma 4.10. The quantities $||(A_H)_t||^2$’s satisfy the following evolution equation:

$$\frac{\partial ||(A_H)_t||^2}{\partial t} = \Delta_H (||A_H||^2) - 2||\nabla^H A_H||^2 + 2||A_H||^2 (||A_H||^2 - \text{Tr}((A^2\xi)^2)_H) - 4H^s \text{Tr}(((A^2\xi)^2) \circ A_H) - 2\text{Tr}_{g_H}^\bullet \mathcal{R}(A_H \bullet, \bullet).$$

From Lemmas 4.9 and 4.10, we obtain the following evolution equation.

Lemma 4.11. The quantities $||(A_H)_t||^2 - \frac{(H^s)^2}{n}$’s satisfy the following evolution equation:

$$\frac{\partial (||(A_H)_t||^2 - \frac{(H^s)^2}{n})}{\partial t} = \Delta_H (||A_H||^2 - \frac{(H^s)^2}{n}) + \frac{2}{n} ||\text{grad}H^s||^2 + 2||A_H||^2 \times \left(||A_H||^2 - \frac{(H^s)^2}{n}\right) - 2||\nabla^H A_H||^2 - 2\text{Tr}((A^2\xi)^2) \times \left(||A_H||^2 - \frac{(H^s)^2}{n}\right) - 4H^s \text{Tr}((A^2\xi)^2 \circ (A_H - \frac{H^s}{n}\text{id})) - 2\text{Tr}_{g_H}^\bullet \mathcal{R}\left((A_H - \frac{H^s}{n}\text{id}) \bullet, \bullet\right),$$

where $\text{grad}H^s$ is the gradient vector field of $H^s$ with respect to $g$ and $||\text{grad}H^s||$ is the norm of $\text{grad}H^s$ with respect to $g$.

Set $n := \text{dim } H = \text{dim } \mathcal{M}$ and denote by $\bigwedge^n H^*$ the exterior product bundle of degree $n$ of $H^*$. Let $d\mu_{g_H}$ be the section of $\pi_M^*(\bigwedge^n H^*)$ such that $(d\mu_{g_H})(u,t)$ is the volume element of $(g_{H_t})_{(u,t)}$ for any $(u,t) \in M \times [0,T)$. Then we can derive the following evolution equation for $\{(d\mu_{g_H})(.,t)\}_{t \in [0,T)}$.

Lemma 4.12. The family $\{(d\mu_{g_H})(.,t)\}_{t \in [0,T)}$ satisfies

$$\frac{\partial d\mu_{g_H}}{\partial t} = -(H^s)^2 \cdot d\mu_{g_H}.$$ 

Proof. Let $(e_1, \cdots, e_n)$ be a local orthonormal base of $H_{(u_0,t_0)}$ with respect to $(g_{H})_{(u_0,t_0)}$ and $(E_1, \cdots, E_n)$ a local frame field of $H_{t_0}|_U$ ($U$ : an open set of $M$) with
where we used

\[ \nabla U = \frac{\partial}{\partial d\mu} \]

\[ = 1 \quad (4.2) \]

For simplicity, set \((g_H)_{ij} := g_H((\mathcal{E}_i)_H, (\mathcal{E}_j)_H)\) and \((h_H)_{ij} := h_H((\mathcal{E}_i)_H, (\mathcal{E}_j)_H)\). By using Lemma 4.2, we can show

\[
\frac{\partial (g_H)_{ij}}{\partial t} = \frac{\partial g_H}{\partial t}((\mathcal{E}_i)_H, (\mathcal{E}_j)_H) + g_H(\nabla \frac{\partial}{\partial t}((\mathcal{E}_i)_H), (\mathcal{E}_j)_H)
+ g_H((\mathcal{E}_i)_H, \nabla \frac{\partial}{\partial t}((\mathcal{E}_i)_H))
= \frac{\partial g_H}{\partial t}((\mathcal{E}_i)_H, (\mathcal{E}_j)_H) = -2H^*h_H((\mathcal{E}_i)_H, (\mathcal{E}_j)_H),
\]

where we used \(\nabla \frac{\partial}{\partial t}((\mathcal{E}_i)_H) \in V\) in the second equality. By using this relation, we can derive

\[
\frac{\partial d\mu_{g_H}}{\partial t} = \frac{\partial}{\partial t} \left( \sqrt{\det((g_H)_{ij})} \omega_1 \wedge \cdots \wedge \omega_n \right)
\]

\[
= \frac{1}{2 \sqrt{\det((g_H)_{ij})}} \sum_{j=1}^{n} \begin{vmatrix}
(g_H)_{11} & \cdots & \frac{\partial (g_H)_{1j}}{\partial t} & \cdots & (g_H)_{1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(g_H)_{n1} & \cdots & \frac{\partial (g_H)_{nj}}{\partial t} & \cdots & (g_H)_{nn}
\end{vmatrix} \omega_1 \wedge \cdots \wedge \omega_n
\]

\[
+ \sqrt{\det((g_H)_{ij})} \cdot \sum_{j=1}^{n} \omega_1 \wedge \cdots \wedge \frac{\partial \omega_j}{\partial t} \wedge \cdots \wedge \omega_n
\]

\[
= \frac{1}{2 \sqrt{\det((g_H)_{ij})}} \sum_{j=1}^{n} \begin{vmatrix}
(g_H)_{11} & \cdots & -2g_H((h_H)_{1j}, H) & \cdots & (g_H)_{1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(g_H)_{n1} & \cdots & -2g_H((h_H)_{nj}, H) & \cdots & (g_H)_{nn}
\end{vmatrix} \omega_1 \wedge \cdots \wedge \omega_n
\]

\[
+ \sqrt{\det((g_H)_{ij})} \cdot \sum_{j=1}^{n} \omega_1 \wedge \cdots \wedge \frac{\partial \omega_j}{\partial t} \wedge \cdots \wedge \omega_n
\]

\[
= - (H^n)^2 d\mu_{g_H} + \sqrt{\det((g_H)_{ij})} \cdot \sum_{j=1}^{n} \omega_1 \wedge \cdots \wedge \frac{\partial \omega_j}{\partial t} \wedge \cdots \wedge \omega_n.
\]
On the other hand, by using \( \nabla \frac{\partial}{\partial t} (E_i) \) again, we can show

\[
\frac{\partial \omega_j}{\partial t} ((E_i) \) = -\omega_1 (\nabla \frac{\partial}{\partial t} (E_i) ) = 0.
\]

Therefore we obtain the desired evolution equation.

5 Sobolev inequality for Riemannian suborbifolds

In this section, we prove the divergence theorem for a compact Riemannian orbifold and Sobolev inequality for a compact Riemannian suborbifold, which may have the boundary. Let \((\tilde{M}, \tilde{g})\) be an \( n \)-dimensional compact Riemannian orbifold, \( \Sigma \) the singular set of \((\tilde{M}, \tilde{g})\) and \( \{\Sigma_1, \cdots, \Sigma_k\} \) be the set of all connected components of \( \Sigma \). Set \( \tilde{M}' := \tilde{M} \setminus \Sigma \). For a function \( \rho \) over \( \tilde{M} \), we call \( \int_{\tilde{M}'} \rho dv_{\tilde{g}} \) the integral of \( \rho \) over \( \tilde{M} \) and denote it by \( \int_{\tilde{M}} \rho dv_{\tilde{g}} \). First we prove the divergence theorem for orbitangent vector fields on a compact Riemannian orbifold.

**Theorem 5.1.** For any \( C^1 \)-orbitangent vector field \( X \) on \((\tilde{M}, \tilde{g})\), the relation

\[
\int_{\tilde{M}} \text{div}_\tilde{g} X dv_{\tilde{g}} = 0
\]

holds.

**Proof.** Let \( U_i \) (\( i = 1, \cdots, k \)) be a sufficiently small tubular neighborhood of \( \Sigma_i \) with \( U_i \cap U_j = \emptyset \) (\( i \neq j \)). Set \( W := \tilde{M} \setminus \left( \bigcup_{i=1}^{k} U_i \right) \). Take families \( \{(U_{ij}, \varphi_{ij}, \tilde{U}_{ij}/\Gamma_{ij}) | j = 1, \cdots, m_i\} \) of orbifold charts of \( \tilde{M} \) such that the family \( \{\text{cl}(U_{ij})\}_{j=1}^{m_i} \) of the closure \( \text{cl}(U_{ij}) \) of \( U_{ij} \) gives a division of \( \text{cl}(U_i) \) (\( i = 1, \cdots, k \)). Denote by \( \pi_{ij} \) the projection \( \pi_{ij} : \tilde{U}_{ij} \to \tilde{U}_{ij}/\Gamma_{ij} \) and \( l_i \) the cardinal number of \( \Gamma_{ij} \), which depends only on \( i \). Let \( \xi_i \) be the outward unit normal vector field of \( \partial U_i \) and \( \iota_i \) is the inclusion map of \( \partial U_i \) into \( \tilde{M} \). Also, let \( \xi_{ij} \) be the outward unit normal vector field of \( \partial U_{ij} \) satisfying \( \xi_{ij} |_{\partial U_{ij}} = \xi_i |_{\partial U_{ij}} \). Also, let \( \tilde{\xi}_{ij} \) be the unit normal vector field of \( \partial \tilde{U}_{ij} \) satisfying \( (\varphi_{ij}^{-1} \circ \pi_{ij} \circ \tilde{\xi}_{ij}) = \xi_{ij} |_{\partial U_{ij}} \) and let \( \tilde{X}_{ij} \) be the vector field on \( \tilde{U}_{ij} \) satisfying \( (\varphi_{ij}^{-1} \circ \pi_{ij} \circ \tilde{X}_{ij}) = X |_{U_{ij}} \). Denote by \( \tau_{ij} \) the inclusion map of \( \partial \tilde{U}_{ij} \) into \( \mathbb{R}^n \) and \( \tilde{g}_{ij} \) the local lift of \( \tilde{g} \) with respect to \( (U_{ij}, \varphi_{ij}, \tilde{U}_{ij}/\Gamma_{ij}) \). Then, by using the
divergence theorem (for a compact Riemannian manifold with boundary), we have

\[
\int_{\mathcal{M}} \div g X \, dv_g = \int_W \div g X \, dv_g + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \int_{U_{ij}} \div g X \, dv_g
\]

\[
= - \sum_{i=1}^{k} \int_{\partial U_i} \gamma(X, \xi_i) \, dv_{\gamma} + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \int_{U_{ij}} \div g X \, dv_g.
\]

Also, by using the divergence theorem, we can show

\[
\int_{U_{ij}} \div g X \, dv_g = \frac{1}{l_i} \int_{\tilde{U}_{ij}} \div g \tilde{X}_{ij} \, dv_{\gamma}\]

\[
= \frac{1}{l_i} \int_{\partial \tilde{U}_{ij}} \tilde{\gamma}(\tilde{X}_{ij}, \tilde{\xi}_{ij}) \, dv_{\gamma}\]

\[
= \int_{\partial U_{ij}} \gamma(X, \xi_{ij}) \, dv_{\gamma}.
\]

and hence

\[
\sum_{j=1}^{m_i} \int_{U_{ij}} \div g X \, dv_g = \int_{\partial U_i} \gamma(X, \xi_i) \, dv_{\gamma}.
\]

From (5.1) and (5.2), we obtain \(\int_{\mathcal{M}} \div g X \, dv_g = 0\).

\[
\text{Figure 5.1: The relation of } \xi_i \text{ and } \xi_{ij}
\]

In 1973, J.H. Michael and L.M. Simon ([MS]) proved the Sobolev inequality for compact Riemannian submanifolds (which may have boundary) in a Euclidean space, where we note that the integrand must vanishes on the boundary of the submanifold.
In 1974, D. Hoffman and J. Spruck ([HoSp]) proved the same Sobolev inequality in a general Riemannian manifold, where we note that the integrand must vanishes on the boundary of the submanifold and furthermore, the volume of the support of the integrand must satisfy some estimate from above related to the curvature and the injective radius of the ambient space. We shall show the following Sobolev inequality for compact Riemannian suborbifolds.

**Theorem 5.2.** Let \((M, g)\) be a compact Riemannian suborbifold isometrically immersed into \((N, g_N)\) by \(f\). Assume that the sectional curvature \(K\) of a complete Riemannian orbifold \((N, g_N)\) satisfies \(K \leq b^2\) (\(b\) is a non-negative real number or the purely imaginary number), \(M\) satisfies the following condition \((\ast 1)\):

\((\ast 1)\) \(f(M)\) is included by \(B_{\pi b}(x_0)\) for some \(x_0 \in N\) and \(\exp_{x_0} |_{B_{\pi b}(x_0)} \) is injective.

Let \(\rho\) be any non-negative \(C^1\)-function on \(M\) satisfying

\((\ast 2)\)

\[ b^2(1 - \alpha)^{-2/n}(\omega_n^{-1} \cdot l \cdot \Vol_g(supp \rho))^{2/n} \leq 1, \]

where \(\overline{g}\) denotes the induced metric on \(M\), \(\omega_n\) denotes the volume of the unit ball in the Euclidean space \(\mathbb{R}^n\), \(l\) denotes the cardinality of the local group at \(x_0\) and \(\alpha\) is any fixed positive constant smaller than one. Then the following inequality for \(\rho\) holds:

\[(5.3) \left( \int_M \rho^{n-1} d\overline{g} \right)^\frac{-1}{n-1} \leq C(n, \alpha) \int_M \left( ||d\rho|| + \rho \cdot \|H\| \right) d\overline{g}, \]

where \(H\) denotes the mean curvature vector of \(f\) and \(C(n, \alpha)\) is the positive constant depending only on \(n\) and \(\alpha\).

**Remark 5.1.** In the case where \((M, \overline{g})\) is a compact Riemannian manifold, the statement of this theorem follows from the Sobolev’s inequality in [HoSp] because the condition \((\ast 1)\) assures the condition (2.3) in Theorem 2.1 of [HoSp].

We shall prepare some lemmas to prove this theorem. Let \(\Gamma\) be the local group at \(x_0\) and set \(\hat{x}_0 := (\exp_{x_0} \circ \pi)^{-1}(x_0)\). The orbitangent space \(T_{x_0}M\) is identified with the orbit space \(\mathbb{R}^{n+1}/\Gamma\). Let \(\pi : \mathbb{R}^{n+1} \to T_{x_0}N\) be the orbit map. Set \(\hat{B}_{\pi}(\hat{x}_0) := (\exp_{x_0} \circ \pi)^{-1}(B_{\pi}(x_0))\) and \(\hat{g}_N := (\exp_{x_0} \circ \pi)^*\hat{g}_N\). Also, set \(\hat{M} := \{(x, \hat{y}) \in \hat{M} \times \hat{B}_{\pi}(\hat{x}_0) \mid f(x) = (\exp_{x_0} \circ \pi)(\hat{y})\} \) and define a map \(\hat{f} : \hat{M} \to \hat{B}_{\pi}(\hat{x}_0)\) by \(\hat{f}(x, \hat{y}) = \hat{y} \quad ((x, \hat{y}) \in \hat{M})\). Also, define a map \(\pi_{\hat{M}} : \hat{M} \to \hat{M}\) by \(\pi_{\hat{M}}(x, \hat{y}) := x \quad ((x, \hat{y}) \in \hat{M})\). It is clear that \(\hat{M}\) is a \(C^\infty\)-manifold.
and that \( \hat{f} \) is a \( C^\infty \)-immersion. Also, it is clear that \( \pi_M \) is a \( C^\infty \)-orbisubmersion and that \( \exp_{x_0} \circ \pi \circ \hat{f} = f \circ \pi_M \) holds. Let \( \hat{g} \) be the Riemannian metric on \( \hat{M} \) such that \( \pi_M : (\hat{M}, \hat{g}) \rightarrow (M, g) \) is a Riemannian orbisubmersion. Let \( \nabla^N \) be the Riemannian connection of \( \hat{g}^N \) and \( \nabla \) that of \( \hat{g} \). Let \( X \in \Gamma^\infty(\hat{f}^*T(\hat{B}_{\hat{x}}(\hat{x}_0))) \). Let \( X^T \) (resp. \( X^\perp \)) be the tangential (resp. the normal) component of \( X \), that is,

\[
X_{\hat{x}} = f_{\hat{x}}(X^T_{\hat{x}}) + X^\perp_{\hat{x}} \quad (X^T_{\hat{x}} \in T_{\hat{x}}\hat{M}, \ X^\perp_{\hat{x}} \in T^\perp_{\hat{x}}\hat{M}).
\]

Define \( \text{div}_{\hat{f}}X \in C^\infty(\hat{M}) \) by

\[
(\text{div}_{\hat{f}}X)_{\hat{x}} := (\text{Tr}_{\hat{g}}((\nabla^N)\hat{f}X)^T)_{\hat{x}} = \sum_{i=1}^n \hat{g}((\nabla^N)_{e_i}X, \hat{f}_{\hat{x}}(e_i)) \quad (\hat{x} \in \hat{M}),
\]

where \( (e_1, \cdots, e_n) \) is an orthonormal base of \( T_{\hat{x}}\hat{M} \) with respect to \( \hat{g}_{\hat{x}} \) and \( (\nabla^N)\hat{f} \) denotes the induced connection of \( \nabla^N \) by \( \hat{f} \).

**Figure 5.2:** The lift of an orbisubmanifold in a singular geodesic ball (I)
First we prepare the following lemma.

**Lemma 5.3.** (i) \( \text{div}_{\hat{g}} X^T = \text{div}_{\hat{f}} X + \hat{\gamma}_N(X, H) \), where \( H \) denotes the mean curvature vector of \( \hat{f} \).

(ii) For \( \rho \in C^\infty(\hat{M}) \), we have
\[
\text{div}_{\hat{f}}(\rho X) = \rho \text{div}_{\hat{f}} X + \hat{\gamma}(X^T, \text{grad}_{\hat{g}} \rho),
\]
where \( \text{grad}_{\hat{g}}(\bullet) \) denotes the gradient vector field of \( \bullet \) with respect to \( \hat{g} \).

See the proof of Lemma 3.2 in [HoSp] about the proof of this lemma. Let \( r_{\hat{x}_0} : \hat{B}_{\hat{g}}(\hat{x}_0) \to [0, \infty) \) be the distance function from \( \hat{x}_0 \) with respect to \( \hat{g}_N \), that is, \( r_{\hat{x}_0}(\hat{x}) := d_{\hat{g}_N}(\hat{x}_0, \hat{x}) \) \( (\hat{x} \in \hat{B}_{\hat{g}}(\hat{x}_0)) \). Define a \( C^\infty \)-vector field \( \mathbb{P} \) on \( \hat{B}_{\hat{g}}(\hat{x}_0) \) by
\[
\mathbb{P}_{\hat{x}} := r_{\hat{x}_0}(\hat{x}) \cdot (\text{grad}_{\hat{g}_N} r_{\hat{x}_0})_{\hat{x}} \quad (\hat{x} \in \hat{B}_{\hat{g}}(\hat{x}_0)),
\]
where \( \text{grad}_{\hat{g}_N}(\bullet) \) denotes the gradient vector field of \( \bullet \) with respect to \( \hat{g}_N \). Also, define a \( C^\infty \)-vector field \( \tilde{\mathbb{P}} \) over the tangent space \( T_{\hat{x}_0} \mathbb{R}^{n+1} \) by
\[
\tilde{\mathbb{P}}_v := v \quad (v \in T_{\hat{x}_0} \mathbb{R}^{n+1}).
\]
By using the discussion in the proof Lemmas 3.5 and 3.6 in [HoSp], we can show the following fact.

**Lemma 5.4.** (i) For any unit vector $v$ of $\hat{B}_s(\hat{x}_0)$ at any $\hat{x} \in \hat{B}_s(\hat{x}_0)$, the following inequality holds:

$$\hat{g}_N(\nabla N, v) \geq b \cdot r_{\hat{x}_0}(\hat{x}) \cdot \cot(b \cdot r_{\hat{x}_0}(\hat{x})).$$

(ii) For any $(x, \hat{y}) \in \hat{M}$, the following inequality holds:

$$(\text{div}_{\hat{f}}(P \circ \hat{f}))(x, \hat{y}) \geq n \cdot b \cdot r_{\hat{x}_0}(\hat{y}) \cdot \cot(b \cdot r_{\hat{x}_0}(\hat{y})).$$

Figure 5.4 : The position vector field on the orbicovering of a geodesic ball

Let $\lambda$ be a $C^1$-function over $\mathbb{R}$ satisfying the following condition:

$$(5.4) \quad \lambda(t) = 0 \ (t \leq 0), \quad \lambda'(t) \geq 0 \ (t \geq 0).$$

Set $\hat{B}_{s}(\hat{x}_0) := \hat{f}^{-1}(B_s(\hat{x}_0))$, where $B_s(\hat{x}_0)$ denotes the geodesic ball of radius $s$ centered at $\hat{x}_0$. Let $\rho$ be a $C^1$-function as in the statement of Theorem 5.2 and set
Define functions $\Phi_{\hat{\rho}, \hat{x}_0, \lambda}$, $\eta_{\hat{\rho}, \hat{x}_0}$, $\Phi_{\hat{\rho}, \hat{x}_0}$, $\eta_{\hat{\rho}, \hat{x}_0}$ over $[0, \frac{\pi}{b})$ by

$$
\Phi_{\hat{\rho}, \hat{x}_0, \lambda}(s) := \int_{\hat{M}} \lambda (s - (r_{\hat{x}_0} \circ \hat{f})) \cdot \hat{\rho} \, dv_{\hat{g}},
$$

$$
\eta_{\hat{\rho}, \hat{x}_0, \lambda}(s) := \int_{\hat{B}_{\hat{M}}(\hat{x}_0)} \lambda (s - (r_{\hat{x}_0} \circ \hat{f})) \cdot (|\text{grad}_{\hat{g}} \rho| + \rho |H|) \, dv_{\hat{g}},
$$

According to the proof of Lemma 4.1 in [HoSp], we can derive the following fact by using Lemmas 5.3 and 5.4.

**Lemma 5.5.** For all $s \in [0, \frac{\pi}{b})$, the following inequality hold:

$$
\left\{
\begin{aligned}
& - \frac{d}{ds} ((\sin(bs))^{-n} \Phi_{\hat{\rho}, \hat{x}_0, \lambda}(s)) \leq (\sin(bs))^{-n} \eta_{\hat{\rho}, \hat{x}_0, \lambda}(s) \quad (b : \text{real}) \\
& - \frac{d}{ds} (s^{-n} \Phi_{\hat{\rho}, \hat{x}_0, \lambda}(s)) \leq s^{-n} \eta_{\hat{\rho}, \hat{x}_0, \lambda}(s) \quad (b : \text{purely imaginary}).
\end{aligned}
\right.
$$

According to the proof of Lemma 4.2 in [HoSp], we can show the following result by using Lemma 5.5.

**Lemma 5.6.** Let $\alpha$ and $\hat{\alpha}$ be constants with $0 < \alpha < 1 \leq \hat{\alpha}$, and $\{\lambda_{\varepsilon}\}_{\varepsilon > 0}$ be $C^1$-functions over $\mathbb{R}$ satisfying the above condition (5.4) and the following condition:

$$(5.5) \quad \lambda_{\varepsilon} \leq 1, \quad \lambda_{\varepsilon}^{-1}(1) = [\varepsilon, \infty).$$

Assume that the following conditions hold:

(i) $\hat{\rho}(\hat{x}_0) \geq 1$, (ii) $b^2 \left( \frac{1}{(1 - \alpha) \omega_n^2} \int_{\hat{M}} \hat{\rho} \, dv_{\hat{g}} \right)^{\frac{1}{n}} \leq 1$.

Set

$$
s_{\hat{\rho}} := \left\{
\begin{aligned}
& \frac{1}{b} \cdot \arcsin \left\{ b \left( \frac{1}{(1 - \alpha) \omega_n} \int_{\hat{M}} \hat{\rho} \, dv_{\hat{g}} \right)^{\frac{2}{n}} \right\} \quad (b : \text{real}) \\
& \left( \frac{1}{(1 - \alpha) \omega_n} \int_{\hat{M}} \hat{\rho} \, dv_{\hat{g}} \right)^{\frac{1}{n}} \quad (b : \text{purely imaginary}).
\end{aligned}
\right.$$

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which is defined by the above condition (ii). Furthermore, assume the following:

(iii) \( \hat{\alpha}_s \rho \leq \frac{\pi}{b} \).

Then there exists \( s_1 \in (0, s_\rho) \) satisfying

\[
\phi_{\hat{\rho}, \hat{x}_0}(\hat{\alpha} s_1) \leq \alpha^{-1} \cdot \hat{\alpha}^{n-2} \cdot s_\rho \cdot \lim_{\varepsilon \to 0} \eta_{\hat{\rho}, \hat{x}_0, \lambda_\varepsilon}(s_1).
\]

By using Lemma 5.6, we prove Theorem 5.2.

**Proof of Theorem 5.2.** We shall prove the statement in the case where \( b \) is real (similar also the case where \( b \) is purely imaginary). Let \( \alpha, \hat{\alpha} \) be constants with \( 0 < \alpha < 1 \leq \hat{\alpha} \) and \( \lambda_\varepsilon (\varepsilon > 0) \) be \( C^1 \)-function over \( \mathbb{R} \) satisfying (5.4) and (5.5). Define a function \( \overline{X}_\varepsilon \) (\( \varepsilon > 0 \)) over \( \mathbb{R} \) by \( \overline{X}_\varepsilon (s) := \lambda_\varepsilon (s + \varepsilon) \) and define a function \( \tilde{\rho}_{\varepsilon, t} \) (\( \varepsilon > 0 \)) over \( \hat{M} \) by \( \tilde{\rho}_{\varepsilon, t} := \overline{X}_\varepsilon (\hat{\rho} - t) \). Since \( \rho \) satisfies the condition \((\ast_2)\) in Theorem 5.2, \( \tilde{\rho}_{\varepsilon, t} \) satisfies the conditions (ii) and (iii) in Lemma 5.6. By using Lemma 5.6 and discussing as in the proof of Theorem 2.1 in [HoSp], we can derive

\[
\left( \int_{\hat{M}} \rho^{n-1} \, dv_{\overline{g}} \right)^{\frac{n-1}{n}} \leq \left( \int_{\hat{M}} \rho^{n-1} \, dv_{\overline{g}} \right)^{\frac{n-1}{n}} \leq \frac{n}{n-1} \cdot \frac{\pi}{2} \cdot \alpha^{-1} \cdot \hat{\alpha}^{n-2} \cdot (1 - \alpha)^{-\frac{1}{n}} \cdot \omega^{\frac{1}{n}} \cdot \left( \int_{\hat{M}} (\|\nabla_{\overline{g}} \hat{\rho}\| + \hat{\rho} \cdot \|H^s\|) \, dv_{\overline{g}} \right).
\]

Clearly we have

\[
\int_{\hat{M}} \rho^{n-1} \, dv_{\overline{g}} = l \cdot \int_{\hat{M}} \rho^{n-1} \, dv_{\overline{g}}
\]

and

\[
\int_{\hat{M}} (\|\nabla_{\overline{g}} \hat{\rho}\| + \hat{\rho} \cdot \|H^s\|) \, dv_{\overline{g}} = l \cdot \int_{\hat{M}} (\|\nabla_{\overline{g}} \rho\| + \rho \cdot \|H^s\|) \, dv_{\overline{g}}.
\]

Hence we obtain

\[
\left( \int_{\hat{M}} \rho^{n-1} \, dv_{\overline{g}} \right)^{\frac{n-1}{n}} \leq l^{\frac{1}{n}} \cdot \frac{n}{n-1} \cdot \frac{\pi}{2} \cdot \alpha^{-1} \cdot \hat{\alpha}^{n-2} \cdot (1 - \alpha)^{-\frac{1}{n}} \cdot \omega^{\frac{1}{n}} \cdot \left( \int_{\hat{M}} (\|\nabla_{\overline{g}} \rho\| + \rho \cdot \|H^s\|) \, dv_{\overline{g}} \right)
\]

\[\square\]
6 Approach to horizontally totally umbilicity

In this section, we recall the preservability of horizontally strongly convexity along the mean curvature flow. Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group $G$ on a Hilbert space $V$ equipped with an inner product $\langle \cdot, \cdot \rangle$ and $\phi : V \to N := V/G$ the orbit map. Denote by $\tilde{\nabla}$ the Riemannian connection of $V$. Set $n := \dim N - 1$. Let $M(\subset V)$ be a $G$-invariant hypersurface in $V$ such that $\phi(M)$ is compact. Let $f$ be an inclusion map of $M$ into $V$ and $f_t (0 \leq t < T)$ the $G$-invariant regularized mean curvature flow starting from $f$. We use the notations in Sections 4. In the sequel, we omit the notation $f_t^*$ for simplicity. As stated in Introduction, set

$$L := \sup_{u \in V} \max_{(X_1, \ldots, X_5) \in (\tilde{H}_1)_{\it u}^5} |\langle A_{h_{\it s}}^\phi((\tilde{\nabla} X_2) A_{h_{\it s}}^\phi X_3), X_5 \rangle|,$$

where $\tilde{H}_1 := \{ X \in \tilde{H} \mid ||X|| = 1 \}$. Assume that $L < \infty$. Note that $L < \infty$ in the case where $N$ is compact. In [Koi2], we proved the following horizontally strongly convexity preservability theorem by using evolution equations stated in Section 4 and the discussion in the proof of Theorem 5.1.

**Theorem 6.1 ([Koi2]).** If $M$ satisfies $(H^s_{\it h})^2(h_{\it h})_{(\cdot,0)} > 2n^2 L(g_{\it h})_{(\cdot,0)}$, then $T < \infty$ holds and $(H^s_{\it h})^2(h_{\it h})_{(\cdot,t)} > 2n^2 L(g_{\it h})_{(\cdot,t)}$ holds for all $t \in [0,T)$.

In this section, we shall prove the following result for the approach to the horizontally totally umbilicity of $f_t$ as $t \to T$.

**Proposition 6.2.** Under the hypothesis of Theorem 6.1, there exist positive constants $\delta$ and $C_0$ depending on only $f$, $L$, $K$ and the injective radius $i(N)$ of $N$ such that

$$||((A_{h_{\it h}})_{(\cdot,t)})^2 - \frac{(H^s_{\it h})^2}{n} < C_0(H^s_{\it h})^{2-\delta}$$

holds for all $t \in [0,T)$.

We prepare some lemmas to show this proposition. In the sequel, we denote the fibre metric of $\mathcal{H}^{(r,s)}$ induced from $g_{\mathcal{H}}$ by the same symbol $g_{\mathcal{H}}$, and set $||S|| := \sqrt{g_{\mathcal{H}}(S,S)}$ for $S \in \Gamma(\mathcal{H}^{(r,s)})$. Define a function $\psi_{\delta}$ over $M$ by

$$\psi_{\delta} := \frac{1}{(H^s)^{2-\delta}} \left( ||A_{h_{\mathcal{H}}}||^2 - \frac{(H^s)^2}{n} \right).$$
Lemma 6.2.1. Set $\alpha := 2 - \delta$. Then we have

\[
\frac{\partial \psi_\delta}{\partial t} = \nabla_t \psi_\delta + (2 - \alpha) \| A_H \|^2 \psi_\delta + \frac{(\alpha - 1)(\alpha - 2)}{(H^s)^2} \| dH^s \| \| \psi_\delta \|
\]

\[
+ \frac{2(\alpha - 1)}{H^s} g_H(dH^s, d\psi_\delta) - \frac{2}{(H^s)^{\alpha+2}} \| \nabla H \| \nabla^2 A_H - dH^s \otimes A_H \|^2
\]

\[
+ 3(\alpha - 2) \text{Tr}((A_H^\phi)^2) \psi_\delta - \frac{6}{(H^s)^{\alpha-1}} \text{Tr}((A_H^\phi)^2 \circ A_H)
\]

\[
+ \frac{6}{n(H^s)^{\alpha-2}} \text{Tr}((A_H^\phi)^2) - \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((\nabla \cdot A) \circ A_H) \cdot \cdot + \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((A \circ A) \cdot, A_H).
\]

Proof. By using Lemmas 4.9 and 4.11, we have

\[
\frac{\partial \psi_\delta}{\partial t} = (2 - \alpha) \| A_H \|^2 \psi_\delta + \frac{1}{(H^s)^{\alpha}} \nabla_t (\| A_H \|^2)
\]

\[
- \frac{1}{(H^s)^{\alpha+1}} \left( \alpha \| A_H \|^2 - \frac{(\alpha - 2)(H^s)^2}{n} \right) \nabla H \cdot \psi_\delta
\]

\[
- \frac{2}{(H^s)^{\alpha}} \| \nabla^2 A_H \|^2 + (3\alpha - 2) \text{Tr}((A_H^\phi)^2) \nabla \cdot \psi_\delta
\]

\[
- \frac{6}{(H^s)^{\alpha-1}} \text{Tr}((A_H^\phi)^2 \circ (A_H - \frac{H^s}{n} \text{id}))
\]

\[
- \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((A \circ A) \cdot, A_H) + \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((A \circ A) \cdot, A_H)
\]

\[
- \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((\nabla \cdot A) \circ A_H) \cdot \cdot.
\]

Also we have

\[
\nabla^2 \psi_\delta = \frac{1}{(H^s)^{\alpha}} \nabla^2 (\| A_H \|^2) - \frac{2\alpha}{(H^s)^{\alpha+1}} g_H(dH^s, d(\| A_H \|^2))
\]

\[
- \frac{1}{(H^s)^{\alpha+1}} \left( \alpha \| A_H \|^2 - \frac{(\alpha - 2)(H^s)^2}{n} \right) \nabla H \cdot \psi_\delta
\]

\[
+ \frac{1}{(H^s)^{\alpha+2}} \left( \alpha(\alpha + 1) \| A_H \|^2 - \frac{(\alpha - 1)(\alpha - 2)(H^s)^2}{n} \right) \| dH^s \|^2
\]

From (6.2) and (6.3), we obtain the desired relation. □

Then we have the following inequalities.
By using the Codazzi equation, we can derive the following relation.

**Lemma 6.2.2.** For any $X, Y, Z \in \mathcal{H}$, we have

$$
(\nabla^H_X h_{\mathcal{H}})(Y, Z) = (\nabla^H_Y h_{\mathcal{H}})(X, Z) + 2h(A_X Y, Z) - h(A_Y Z, X) + h(A_X Z, Y)
$$
or equivalently,

$$
(\nabla^H_X A_{\mathcal{H}})(Y) = (\nabla^H_Y A_{\mathcal{H}})(X) + 2(A \circ A_X)Y + (A_Y \circ A)(X) - (A_X \circ A)(Y).
$$

**Proof.** Let $(x, t)$ be the base point of $X, Y$ and $Z$ and extend these vectors to sections $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ of $\mathcal{H}_t$ with $(\nabla^H \tilde{X})(x, t) = (\nabla^H \tilde{Y})(x, t) = (\nabla^H \tilde{Z})(x, t) = 0$. Since $\nabla h$ is symmetric with respect to $g$ by the Codazzi equation and the flatness of $V$, we have

$$
(\nabla^H_X h_{\mathcal{H}})(Y, Z) = X(h(\tilde{Y}, \tilde{Z})) = \nabla_X h(Y, Z) + h(A_X Y, Z) + h(A_X Z, Y) = \nabla_Y h(X, Z) + h(A_X Y, Z) + h(A_X Z, Y) = Y(h(\tilde{X}, \tilde{Z})) - h(A_Y X, Z) - h(A_Y Z, X) + h(A_X Y, Z) + h(A_X Z, Y) = (\nabla^H_Y h)(X, Z) + 2h(A_X Y, Z) - h(A_Y Z, X) + h(A_X Z, Y).
$$

\[\square\]

Set

$$
K := \max_{(e_1, e_2) : \text{o.n.s. of } TV} ||A_{e_1}^\phi e_2||^2,
$$

where "o.n.s." means "orthonormal system". Assume that $K < \infty$. Note that $K < \infty$ if $N = V/G$ is compact. For a section $S$ of $\mathcal{H}^{(r, s)}$ and a permutation $\sigma$ of $s$-symbols, we define a section $S_\sigma$ of $\mathcal{H}^{(r, s)}$ by

$$
S_\sigma(X_1, \ldots, X_s) := S(X_{\sigma(1)}, \ldots, X_{\sigma(s)}) \quad (X_1, \ldots, X_s \in \mathcal{H})
$$

and $\text{Alt}(S)$ by

$$
\text{Alt}(S) := \frac{1}{s!} \sum_\sigma \text{sgn } \sigma S_\sigma,
$$

where $\sigma$ runs over the symmetric group of degree $s$. Also, denote by $(i, j)$ the transposition exchanging $i$ and $j$. Since $(H^s_t)^2(h_{\mathcal{H}})(\cdot, t) > n^2 L(g_{\mathcal{H}})(\cdot, t)$ ($t \in [0, T]$) and $\phi(M)$ is compact, there exists a positive constant $\varepsilon$ satisfying

$$
||H(\cdot, t)||^2(h_{\mathcal{H}})(\cdot, 0) \geq n^2 L(g_{\mathcal{H}})(\cdot, 0) + \varepsilon ||H(\cdot, 0)||^3(g_{\mathcal{H}})(\cdot, 0).
$$
Then we can show that
\[ |H(\cdot,t)|^2 (g_{\mathcal{H}})(\cdot,t) \geq n^2 L(g_{\mathcal{H}})(\cdot,t) + \varepsilon |H(\cdot,t)|^3 (g_{\mathcal{H}})(\cdot,t) \]
holds for all \( t \in [0, T) \). Without loss of generality, we may assume that \( \varepsilon \leq 1 \). Then we have the following inequalities.

**Lemma 6.2.3.** Let \( \varepsilon \) be as above. Then we have the following inequalities:

\[ (A_{\mathcal{H}} t)^3 - ||(A_{\mathcal{H}} t)||^4 \geq n^2(H^s)^2 \left( |(A_{\mathcal{H}} t)||^2 - \frac{(H^s)^2}{n} \right), \]

and

\[ \left\| H^s \nabla^H A_{\mathcal{H}} - dH^s \otimes A_{\mathcal{H}} \right\|^2 \geq -8 \varepsilon^{-2} K(H^s_{(u,t)})^2 + \frac{1}{8} \left\| (dH^s)_{(u,t)} \right\|^2 \varepsilon^2 (H^s_{(u,t)})^2. \]

**Proof.** First we shall show the inequality (6.4). Fix \( (u,t) \in \mathcal{M} \times [0, T) \). Take an orthonormal base \( \{e_1, \cdots, e_n\} \) of \( H_{(u,t)} \) with respect to \( g_{(u,t)} \) consisting of the eigenvectors of \( (A_{\mathcal{H}} t) \). Let \( (A_{\mathcal{H}} t)_{(u,t)}(e_i) = \lambda_i e_i \) (\( i = 1, \cdots, n \)). Note that \( \lambda_i > \varepsilon H^s(> 0) \) (\( i = 1, \cdots, n \)). Then we have

\[ H^s \text{Tr}_{\mathcal{H}} (A_{\mathcal{H}})^3 - ||(A_{\mathcal{H}} t)||^4 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 > \varepsilon^2 (H^s)^2 \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2. \]

On the other hand, we have

\[ ||(A_{\mathcal{H}} t)||^2 - \frac{(H^s)^2}{n} = \frac{1}{n} \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2. \]

From these inequalities, we can derive the inequality (6.4).

Next we shall show the inequality (6.5). By using Lemma 6.2.2, we can show

\[ \left\| H^s \nabla^H A_{\mathcal{H}} - dH^s \otimes A_{\mathcal{H}} \right\|^2 \geq \left\| \text{Alt} \left( H^s \nabla^H A_{\mathcal{H}} - dH^s \otimes A_{\mathcal{H}} \right) \right\|^2 \]

\[ \geq \left\| A \circ A - \frac{1}{2} A \circ (A \times \text{id}) - \frac{1}{2} A \circ (\text{id} \times A) - \text{Alt} \left( dH^s \otimes A_{\mathcal{H}} \right) \right\|^2. \]

For simplicity, we set

\[ S := A \circ A - \frac{1}{2} A \circ (A \times \text{id}) - \frac{1}{2} A \circ (\text{id} \times A). \]
It is clear that (6.5) holds at \((u, t)\) if \((dH^s)_{(u, t)} = 0\). Assume that \((dH^s)_{(u, t)} \neq 0\). Take an orthonormal base \((e_1, \cdots, e_n)\) of \(H_{(u, t)}\) with respect to \((g_H)_{(u, t)}\) with \(e_1 = \frac{(dH^s)_{(u, t)}}{||(dH^s)_{(u, t)}||}\). Then we have

\[
\left|\left| S_{(u, t)} - \text{Alt} (dH^s \otimes A_H)_{(u, t)} \right|\right|^2 \\
\geq ||S - \text{Alt} (dH^s \otimes A_H) (e_1, e_2)||^2 \\
\geq ||S(e_1, e_2)||^2 - ||(dH^s)_{(u, t)}||^2||g(S(e_1, e_2), A_He_2)|| + \frac{1}{4} \left|\left| (dH^s)_{(u, t)} \right|\right|^2 \cdot ||A_He_2||^2 \\
\geq ||S(e_1, e_2)||^2 - ||(dH^s)_{(u, t)}||^2||g(S(e_1, e_2), A_He_2)|| + \frac{1}{4} \left|\left| (dH^s)_{(u, t)} \right|\right|^2 \varepsilon^2 (H^s_{(u, t)})^2 \\
\geq (1 - 2\varepsilon^{-2})||S(e_1, e_2)||^2 + \left(\sqrt{2}\varepsilon^{-1}||S(e_1, e_2)|| - \frac{1}{2\sqrt{2}} \left|\left| (dH^s)_{(u, t)} \right|\right||\varepsilon H^s_{(u, t)}\right)^2 + \frac{1}{8} \left|\left| (dH^s)_{(u, t)} \right|\right|^2 \varepsilon^2 (H^s_{(u, t)})^2 \\
\geq -2\varepsilon^{-2}||S(e_1, e_2)||^2 + \frac{1}{8} \left|\left| (dH^s)_{(u, t)} \right|\right|^2 \varepsilon^2 (H^s_{(u, t)})^2 \\
\geq -8\varepsilon^{-2}K(H^s_{(u, t)})^2 + \frac{1}{8} \left|\left| (dH^s)_{(u, t)} \right|\right|^2 \varepsilon^2 (H^s_{(u, t)})^2,
\]

where we use \(||A_He|| \leq H^s\) holds for any unit vector \(e\) of \(H). Thus we see that (6.5) holds at \((u, t)\). This completes the proof. \(\square\)

From Lemma 6.2.1 and (6.5), we obtain the following lemma.

**Lemma 6.2.4.** Assume that \(\delta < 1\). Then we have the following inequality:

\[
\frac{\partial \psi_\delta}{\partial t} \leq \Delta_\mathcal{H} \psi_\delta + (2 - \alpha) ||A_H||^2 \psi_\delta + \frac{2(\alpha - 1)}{H^s} g_H (dH^s, d\psi_\delta) \\
- \frac{2}{(H^s)^{\alpha+2}} \left(\frac{1}{8} \left|\left| dH^s \right|\right|^2 \varepsilon^2 (H^s)^2 - \frac{1}{2} \frac{\partial}{\partial t} \left|\left| dH^s \right|\right|^2 \varepsilon^2 (H^s)^2 \right) \\
+ \frac{3(\alpha - 2)}{H^s} \text{Tr}((A^\mathcal{H}_\delta^2)\psi_\delta) - \frac{6}{(H^s)^{\alpha-2}} \text{Tr}(A^\mathcal{H}_\delta^2 \circ A_H) \\
+ \frac{6}{n(H^s)^{\alpha-2}} \text{Tr}((A^\mathcal{H}_\delta^2)\psi_\delta) - \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((\nabla_\bullet A)_{\bullet} \circ A_H, A_{\bullet}) \\
- \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((A_{\bullet} \circ A_H), A_{\bullet}) \\
+ \frac{4}{(H^s)^{\alpha}} \text{Tr}_{g_H} \text{Tr}_{g_H}^* h((A_{\bullet} \circ A_{\bullet}), A_{\bullet}).
\]

On the other hand, we can show the following fact for \(\psi_\delta\).
Lemma 6.2.5. We have

\[
\Delta_H \psi_{\delta} = \frac{2}{(H^s)_{\alpha+2}} \times \left\| H^s \cdot \nabla^H A_H - dH^s \cdot A_H \right\|^2 \\
+ \frac{2}{(H^s)_{\alpha-1}} \left( \text{Tr}((A_H)^3) - \text{Tr}((A^\phi_\xi)^2 \circ A_H) \right) \\
- \frac{2}{(H^s)_{\alpha}} \left( \text{Tr}((A_H)^2 - (A^\phi_\xi)^2 |_H) \right) \|A_H\|^2 \\
+ \frac{2}{(H^s)_{\alpha}} \text{Tr}_{g_H} \left( (\nabla^H dH^s) \left( \left( A_H - \frac{H^s}{n} \text{id} \right)(\bullet), \bullet \right) \right) \\
- \frac{\alpha}{H^s_{\alpha}} \psi_{\delta} \Delta_H H^s - \frac{(\alpha - 1)(\alpha - 2)}{(H^s)^2} \|dH^s\|^2 \psi_{\delta} \\
- \frac{2(\alpha - 1)}{H^s_{\alpha}} \text{Tr}_{g_H}(dH^s, d\psi_{\delta}) + \frac{2}{(H^s)_{\alpha}} \times \text{Tr}_{g_H} \mathcal{R}(A_H \bullet, \bullet).
\]

Proof. According to (4.16) in [Koi2], we have

\[
(6.6) \quad \text{Tr}_{g_H}((\Delta^H_H)H^s)(A_H \bullet, \bullet) = \frac{1}{2} \Delta_H \|A_H\|^2 - \|\nabla^H A_H\|^2.
\]

Also we have

\[
(6.7) \quad (A^2)_H = (A_H)^2 - (A^\phi_\xi)^2.
\]

By using Lemmas 4.4, 4.5 and these relations, we can derive

\[
(6.8) \quad \frac{1}{2} \Delta_H \|A_H\|^2 = \text{Tr}_{g_H}((\nabla^H dH^s)(A_H \bullet, \bullet) + H^s \text{Tr}((A_H)^3) \\
- H^s \text{Tr}((A^\phi_\xi)^2 \circ A_H) - \text{Tr}((A_H)^2 - (A^\phi_\xi)^2 |_H) \|A_H\|^2 \\
+ \text{Tr}_{g_H} \mathcal{R}(A_H \bullet, \bullet) + \|\nabla^H A_H\|^2.
\]

By substituting this relation into (6.3), we obtain

\[
\Delta_H \psi_{\delta} = \frac{2}{(H^s)_{\alpha}} \times \left\{ \text{Tr}_{g_H}((\nabla^H dH^s)(A_H \bullet, \bullet) + H^s \text{Tr}((A_H)^3) \\
- H^s \text{Tr}((A^\phi_\xi)^2 \circ A_H) - \text{Tr}((A_H)^2 - (A^\phi_\xi)^2 |_H) \|A_H\|^2 \\
+ \text{Tr}_{g_H} \mathcal{R}(A_H \bullet, \bullet) + \|\nabla^H A_H\|^2 \right\} \\
- \frac{2\alpha}{(H^s)_{\alpha+1}} \text{Tr}_{g_H}(dH^s, d(||A_H||^2)) \\
- \frac{1}{(H^s)_{\alpha+1}} \left( \alpha ||A_H||^2 - \frac{(\alpha - 2)(H^s)^2}{n} \right) \Delta_H H^s \\
+ \frac{1}{(H^s)_{\alpha+2}} \left( \alpha(\alpha + 1)||A_H||^2 - \frac{(\alpha - 1)(\alpha - 2)(H^s)^2}{n} \right) \|dH^s\|^2.
\]

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From this relation, we can derive the desired relation.

From this lemma, we can derive the following inequality for $\psi_\delta$ directly.

**Lemma 6.2.6.** We have

\[
\triangle_\mathcal{H} \psi_\delta \geq \frac{2}{(H^s)\alpha - 1} \left( \text{Tr}((A_\mathcal{H})^3) - \text{Tr}((A_\mathcal{H}^\phi)^2 \circ A_\mathcal{H}) \right) - \frac{2}{(H^s)\alpha} \left( ||A_\mathcal{H}||^2 - \text{Tr}(A_\mathcal{H}^\phi)^2 ||A_\mathcal{H}|| \text{Tr}(A_\mathcal{H})^2 \right) + \frac{2}{(H^s)\alpha} \text{Tr}_{g_\mathcal{H}} \left( \nabla_\mathcal{H}^2 dH^s \left( \left( A_\mathcal{H} - \frac{H^s}{n} \text{id} \right)(\bullet),\bullet \right) \right) - \frac{\alpha}{H^s} \psi_\delta \triangle_\mathcal{H} H^s - \frac{2(\alpha - 1)}{H^s} g_\mathcal{H}(dH^s, d\psi_\delta) + \frac{2}{(H^s)\alpha} \times \text{Tr}_{g_\mathcal{H}} \mathcal{R}(A_\mathcal{H} \bullet, \bullet).
\]

For a function $\rho$ over $M \times [0,T)$ such that $\rho(\cdot,t)$ ($t \in [0,T)$) are $G$-invariant, define a function $\rho_B$ over $\overline{M} \times [0,T)$ by $\rho_B = \rho_B \circ (\phi_M \times \text{id}_{[0,T)})$. We call this function the function over $M \times [0,T)$ associated with $\rho$. Denote by $g_N$ the Riemannian orbimetric of $N$ and set $\tilde{g}_t := \int_t^s g_N$. Also, denote by $d\tilde{v}_t$ the orbivolume element of $\tilde{g}_t$. Define a section $\bar{g}_t$ by $\pi_M(T(0,2\overline{M}))$ by $\bar{g}_t(x,t) = (\tilde{g}_t)_x ((x,t) \in \overline{M} \times [0,T))$ and a section $d\bar{v}_t$ of $\pi_M^*(\wedge^n T^* \overline{M})$ by $d\bar{v}_t(x,t) = (d\tilde{v}_t)_x ((x,t) \in \overline{M} \times [0,T))$, where $\pi_M$ is the natural projection of $M \times [0,T)$ onto $\overline{M}$ and $\pi_M^*(\bullet)$ denotes the induced bundle of $(\bullet)$ by $\pi_M$. Denote by $\nabla_t$ the Riemannian orbiconnection of $\tilde{g}_t$ and by $\Delta_t$ the Laplace operator of $\nabla_t$. Define an orbiconnection $\nabla$ of $\pi_M^*(T \overline{M})$ by using $\nabla_t$'s (see the definition of $\nabla$ in Section 4). Also, let $\nabla$ be the differential operator of $\pi_M^*(\overline{M} \times \mathbb{R})$ defined by using $\Delta_t$'s. Denote by $\int_M \rho_B d\bar{v}_t$ the function over $[0,T)$ defined by assigning $\int_M \rho_B d\bar{v}_t$ to each $t \in [0,T)$.

Let $\rho$ and $\rho_B$ be as above. According to Theorem 5.1, we have

\[
\int_M (\text{div}_\nabla \rho)_B d\bar{v} = \int_M \text{div}_\nabla (\rho_B) d\bar{v} = 0 \tag{6.9}
\]

and

\[
\int_M (\triangle_\mathcal{H} \rho)_B d\bar{v} = \int_M \overline{\Delta}(\rho_B) d\bar{v} = 0. \tag{6.10}
\]
From the inequality in Lemma 6.2.6 and (6.9), we can derive the following integral inequality.

**Lemma 6.2.7.** Assume that $0 \leq \delta \leq \frac{1}{2}$. Then, for any $\beta \geq 2$, we have

\[
\begin{align*}
&n \varepsilon^2 \int_M (H^s_B)^2 (\psi)_B^\beta d\bar{v} \\
&\leq \frac{3\beta \eta + 6}{2} \int_M (H^s_B)^{-\alpha}(\psi)_B^{\beta-1} || dH^s ||_B^2 d\bar{v} + \frac{3\beta}{2n} \int_M (\psi)_B^{\beta-2} || d\psi_v ||_B^2 d\bar{v} \\
&+ C_1 \int_M (H^s_B)^{-\alpha}(\psi)_B^{\beta-1} || A_H ||_B^2 d\bar{v} + C_2 \int_M (H^s_B)^{-\alpha}(\psi)_B^{\beta-1} d\bar{v},
\end{align*}
\]

where $C_i$ $(i = 1, 2)$ are positive constants depending only on $K$ and $L$ ($L$ is the constant defined in the previous section).

**Proof.** By using $\int_M \text{div}_{\nabla} \left( (H^s)^{-\alpha}(\psi)_B^{\beta-1} (A_H - (H^s/n) \text{id})(\text{grad} H^s) \right)_B d\bar{v} = 0$ and Lemma 6.2.2, we can show

\[
\begin{align*}
&\int_M (H^s_B)^{-\alpha}(\psi)_B^{\beta-1} \left( (\nabla H^s) ((A_H - (H^s/n) \text{id})(\bullet, \bullet)) \right)_B d\bar{v} \\
&= \alpha \int_M (H^s_B)^{-\alpha-1}(\psi)_B^{\beta-1} g_H ((dH^s \otimes dH^s, h_H - (H^s/n)g_H)_B d\bar{v} \\
&\quad - (\beta - 1) \int_M (H^s_B)^{-\alpha}(\psi)_B^{\beta-2} g_H ((dH^s \otimes d\psi_v, h_H - (H^s/n)g_H)_B d\bar{v} \\
&\quad + 3 \int_M (H^s_B)^{-\alpha}(\psi)_B^{\beta-1} \text{Tr}_{g_H} (A^\phi \circ A^\phi_{\text{grad} H^s})_B d\bar{v}.
\end{align*}
\]

Also, by using $\int_M (\Delta_H \psi^\beta)_B d\bar{v} = 0$, we can show

\[
\begin{align*}
&\int_M (\psi)_B^{\beta-1} (\Delta_H \psi)_B d\bar{v} = - (\beta - 1) \int_M (\psi)_B^{\beta-2} || d\psi_v ||_B^2 d\bar{v}
\end{align*}
\]

and hence

\[
\begin{align*}
&\int_M (H^s_B)^{-1}(\psi)_B^\beta (\Delta_H H^s)_B d\bar{v} \\
&= - 2\beta \int_M (H^s_B)^{-1}(\psi)_B^{\beta-1} g_H (dH^s, d\psi)_B d\bar{v} \\
&\quad + 2 \int_M (H^s_B)^{-2}(\psi)_B^\beta || dH^s ||_B^2 d\bar{v}.
\end{align*}
\]
By multiplying $\psi_\beta^{\beta-1}$ to both sides of the inequality in Lemma 6.2.6 and integrating the functions over $\mathcal{M}$ associated with both sides and using (6.11), (6.12) and (6.13), we can derive

$$\int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-1}_{B}\text{Tr}((A_\mathcal{H})^3)_{B}d\bar{v} - \int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-1}_{B}\|A_\mathcal{H}\|_{B}^4d\bar{v}$$

$$\leq \int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-1}_{B}((\text{Tr}((A_{\mathcal{H}}^\phi)^2)_{B} A_\mathcal{H}))_{B}d\bar{v}$$

$$-\int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-1}_{B}(\text{Tr}((A_{\mathcal{H}}^\phi)(\mathcal{H}))_{B}\|A_\mathcal{H}\|_{B}^2)_{B}d\bar{v}$$

$$-\frac{\beta-1}{2}\int_\mathcal{M} (\psi_\beta)^{\beta-2}_{B}\|d\psi_\beta\|_{B}^2d\bar{v}$$

$$-(\alpha \beta - \alpha + 1)\int_\mathcal{M} (H_B^s)^{-1}(\psi_\beta)^{\beta-1}_{B}g_\mathcal{H}(dH^s, d\psi_\beta)_{B}d\bar{v}$$

$$(6.14)$$

$$-\alpha\int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-1}_{B}g_\mathcal{H}((dH^s \otimes dH^s, h_\mathcal{H} - (H^s/n)g_\mathcal{H})_{B}d\bar{v}$$

$$+(\beta - 1)\int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-2}_{B}g_\mathcal{H}((dH^s \otimes d\psi_\beta, h_\mathcal{H} - (H^s/n)g_\mathcal{H})_{B}d\bar{v}$$

$$+(1 - 1/n)\int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-1}_{B}\|dH^s\|_{B}^2d\bar{v}$$

$$-3\int_\mathcal{M} (H_B^s)^{-\alpha}(\psi_\beta)^{\beta-1}_{B}\text{Tr}_{g_\mathcal{H}}(A_{\mathcal{H}}^\phi \circ A_{\mathcal{H}}^\phi)_{B}d\bar{v}$$

$$-\alpha\int_\mathcal{M} (H_B^s)^{-2}(\psi_\beta)^{\beta}_{B}\|dH^s\|_{B}^2d\bar{v}$$

$$-\int_\mathcal{M} (\psi_\beta)^{\beta-1}(H_B^s)^{-\alpha}(\text{Tr}_{g_\mathcal{H}}(A_{\mathcal{H}} \cdot \bullet))_{B}d\bar{v}.$$
calculations, we can derive

\[ \psi_2 \leq \alpha \int_M (H_B^s)^{-\alpha/2-1}(\psi_B)_{B-1}^2 ||dH^s||_B^2 d\bar{v} \]

\[ + (\alpha + \alpha - 1) \int_M (H_B^s)^{-1} \cdot (\psi_B)_{B-1}^2 \cdot ||dH^s||_B \cdot ||d\psi||_B d\bar{v} \]

\[
(6.16)
\]

\[ + (\beta - 1) \int_M (H_B^s)^{-\alpha/2}(\psi_B)_{B-3/2}^\beta \cdot ||dH^s||_B \cdot ||d\psi||_B d\bar{v} \]

\[ + \frac{n-1}{n} \int_M (H_B^s)^{-\alpha}(\psi_B)_{B-1}^\beta \cdot ||dH^s||_B^2 d\bar{v} , \]

where we use \( ||dH^s \otimes dH^s|| = ||dH^s||^2, ||dH^s \otimes d\psi|| = ||dH^s|| \cdot ||d\psi|| \) and \( ||h_H - \frac{H}{n} g_H||^2 = \psi_B(H^s)^2 \). By noticing \( ab \leq \frac{\alpha_2}{2\alpha} + \frac{\beta_2}{2\beta} \) for any \( a, b, \eta > 0 \) and \( \psi \leq (H^s)^\delta \) \( (0 < \delta < 1) \), we have

\[
(6.17) \quad \int_M (H_B^s)^{-1} \cdot (\psi_B)_{B-1}^\beta \cdot ||dH^s||_B \cdot ||d\psi||_B d\bar{v}
\]

\[
\leq \frac{\eta}{2} \int_M (H_B^s)^{-\alpha}(\psi_B)_{B-1}^\beta ||dH^s||_B^2 d\bar{v} + \frac{1}{2\eta} \int_M (\psi_B)_{B-2}^\alpha ||dH^s||_B^2 d\bar{v}
\]

and

\[
(6.18) \quad \int_M (H_B^s)^{-\alpha/2}(\psi_B)_{B-3/2}^\beta ||dH^s||_B \cdot ||d\psi||_B d\bar{v}
\]

\[
\leq \frac{\eta}{2} \int_M (H_B^s)^{-\alpha}(\psi_B)_{B-1}^\beta ||dH^s||_B^2 d\bar{v} + \frac{1}{2\eta} \int_M (\psi_B)_{B-2}^\alpha ||dH^s||_B^2 d\bar{v} .
\]

From (6.4) and (6.14) – (6.18), we can derive

\[
n_2^2 \int_M (H_B^s)^2(\psi_B)_{B}^\beta d\bar{v}
\]

\[
\leq \frac{\alpha + \alpha + \beta - 2}{2} \int_M (H_B^s)^{-\alpha}(\psi_B)_{B-1}^\beta ||dH^s||_B^2 d\bar{v}
\]

\[
+ \frac{\alpha + \alpha + \beta - 2}{2\eta} \int_M (\psi_B)_{B-2}^\beta ||d\psi||_B^2 d\bar{v}
\]

\[
+ \left( \alpha + \frac{n-1}{n} \right) \int_M (H_B^s)^{-\alpha}(\psi_B)_{B-1}^\beta ||dH^s||_B^2 d\bar{v}
\]

\[
+ 2\sqrt{\eta} \int_M (\psi_B)_{B-1}^\beta (H_B^s)^{-\alpha} \cdot ||(A^\phi)^2||_B \cdot ||A_H||_B d\bar{v}
\]

\[
+ 3\sqrt{\eta} \int_M (\psi_B)_{B-1}^\beta (H_B^s)^{-\alpha} \cdot ||A^\phi \circ A^\phi_{grad}\psi ||_B d\bar{v}
\]

\[
- \int_M (\psi_B)_{B-1}^\beta (H_B^s)^{-\alpha} Ty_{g_H}(A_H \bullet \bullet)_B d\bar{v} .
\]

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Since $0 \leq \delta \leq \frac{1}{2}$ (hence $\frac{2}{3} \leq \alpha \leq 2$), we can derive the desired inequality. \qed

Also, we can derive the following inequality.

**Lemma 6.2.8.** Assume that $0 \leq \delta \leq \frac{1}{2}$. Then, for any $\beta \geq 100\varepsilon^{-2}$, we have

$$
\frac{\partial}{\partial t} \int_M (\psi_b^\beta B d\bar{v}) + 2 \int_M (\psi_b^\beta B (H_b^\delta)^2 d\bar{v}) + \frac{\beta(\beta - 1)}{2} \int_M (\psi_b^\beta B - 2||d\psi_b||^2 B d\bar{v}) \\
+ \frac{\beta^2}{8} \int_M (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha} \cdot ||dH^\delta||^2 B d\bar{v}) \\
\leq \beta \delta \int_M (\psi_b^\beta B (H_b^\delta)^2 d\bar{v}) + 16\beta\varepsilon^{-2} K \int_M (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha} d\bar{v}) \\
- 3\beta \delta \int_M (\psi_b^\beta B \text{Tr}((A_b^\alpha)^2) B d\bar{v}) - 6\beta \int_M (\psi_b^\beta B - 1(H_b^\delta)^{\delta}\text{Tr}((A_b^\alpha)^2) B d\bar{v}) \\
+ 6\beta n \int_M (\psi_b^\beta B - 1(H_b^\delta)^{\delta}\text{Tr}((A_b^\alpha)^2) B d\bar{v}) \\
- 4\beta (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha}\text{Tr}_{gH} \text{Tr}_{gB} h(((\nabla \cdot A) \circ A) \cdot \cdot) B d\bar{v}) \\
- 4\beta (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha}\text{Tr}_{gH} \text{Tr}_{gB} h((A \circ A) \cdot \cdot) B d\bar{v}) \\
+ 4\beta (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha}\text{Tr}_{gH} \text{Tr}_{gB} h((A \circ A) \cdot \cdot) B d\bar{v}).
$$

**Proof.** By multiplying $\beta \psi_b^{\beta - 1}$ to both sides of the inequality in Lemma 6.2.4 and integrating over $M$, we obtain

$$
\int_M \left( \frac{\partial \psi_b^\beta}{\partial t} ight)_B d\bar{v} + \beta(\beta - 1) \int_M (\psi_b^\beta B - 2||d\psi_b||^2 B d\bar{v}) \\
+ \frac{\beta^2}{4} \int_M (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha} \cdot ||dH^\delta||^2 B d\bar{v}) \\
\leq \beta \delta \int_M (\psi_b^{\beta B}(H_b^\delta)^2 d\bar{v}) + 2\beta(\alpha - 1) \int_M (\psi_b^\beta B - 1(H_b^\delta)^{-1} \cdot ||dH^\delta|| B \cdot ||d\psi_b|| B d\bar{v}) \\
+ 16\beta\varepsilon^{-2} K \int_M (\psi_b^{\beta B}(H_b^\delta)^{-\alpha} d\bar{v}) + 6\beta \int_M (\psi_b^\beta B - 1(H_b^\delta)^{\delta}\text{Tr}((A_b^\alpha)^2) B d\bar{v}) \\
- 3\beta \delta \int_M (\psi_b^\beta B \text{Tr}((A_b^\alpha)^2) B d\bar{v}) - 6\beta \int_M (\psi_b^\beta B - 1(H_b^\delta)^{\delta}\text{Tr}((A_b^\alpha)^2) B d\bar{v}) \\
- 4\beta (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha}\text{Tr}_{gH} \text{Tr}_{gB} h(((\nabla \cdot A) \circ A) \cdot \cdot) B d\bar{v}) \\
- 4\beta (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha}\text{Tr}_{gH} \text{Tr}_{gB} h((A \circ A) \cdot \cdot) B d\bar{v}) \\
+ 4\beta (\psi_b^\beta B - 1(H_b^\delta)^{-\alpha}\text{Tr}_{gH} \text{Tr}_{gB} h((A \circ A) \cdot \cdot) B d\bar{v}),
$$

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where we use $\int_{\mathcal{M}} \triangle_{H}(\psi^\beta)_{B} d\tilde{v} = 0$ and $\|A_{H}\|^2 \leq (H^s)^2$. From this inequality, 
\[ \frac{\partial}{\partial t}(\psi^\beta)_{B} = -2(H^s)^2 d\tilde{v}, \quad \|dH^s\| \cdot \|\psi^\beta\| \leq \frac{\beta - 1}{\beta - 1} \|\psi^\beta\|^2 + \frac{(H^s)^{1-\alpha}}{\beta - 1} \|dH^s\|^2, \quad \alpha \leq 2, \]
$\|A_{H}\|^2 \leq (H^s)^2$, $\psi_{\delta} \leq (H^s)^{\delta}$ and $1 \leq 100\varepsilon^{-2} - 1 \leq 16\varepsilon^{-2}$ (which holds because of $\varepsilon \leq 1$), we can derive the desired inequality. \[ \square \]

For a function $\overline{g}$ over $\mathcal{M} \times [0, T)$, denote by $\|\overline{g}(\cdot, t)\|_{L^\beta, \overline{g}_t}$ the $L^\beta$-norm of with respect to $\overline{g}_t$ and $\|\overline{g}\|_{L^\beta, \overline{g}}$ the function over $[0, T)$ defined by assigning $\|\overline{g}(\cdot, t)\|_{L^\beta, \overline{g}_t}$ to each $t \in [0, T)$.

By using Lemmas 6.2.7 and 6.2.8, we can derive the fact.

**Lemma 6.2.9.** There exists a positive constant $C$ depending only on $K, L$ and $f$ such that, for any $\delta$ and $\beta$ satisfying

\[ 0 \leq \delta \leq \min \left\{ \frac{1}{2}, \frac{\varepsilon^2 \eta}{3}, \frac{\varepsilon^{-1}}{24(\eta + 1)} \right\} \quad \text{and} \quad \beta \geq \max \left\{ \frac{100\varepsilon^{-2}}{\varepsilon^{-2} \eta - 3\delta}, \frac{\varepsilon^2 \eta}{24(\eta + 1)} \right\}, \]

the following inequality holds:

\[ \sup_{t \in [0, T)} \|\psi^\beta(t)\|_{L^\beta, \overline{g}_t} < C. \]

**Proof.** Set

\[ C_1 := (\text{Vol}_{g_0}(M) + 1) \sup_{\delta \in [0, 1/2]} \max_{M} \psi^\beta(\cdot, 0). \]

Then we have $\|\psi^\beta(\cdot, 0)\|_{L^\beta, g_0} \leq C_1$. By using the inequalities in Lemmas 6.2.7 and 6.2.8, $\|A_{H}\|^2 \leq (H^s)^2$ and the Young’s inequality, we can show that

\[ \frac{\partial}{\partial t} \left( \|\psi^\beta\|_{L^\beta, \overline{g}} \right)^\beta \leq \beta ((3\delta - \varepsilon^2 \eta)\beta + \varepsilon^2 \eta) \] 
\[ + \frac{\beta (12\eta \varepsilon^2 \delta + 24\varepsilon^2 \delta - \varepsilon^4)}{8\varepsilon^2} \int_{\mathcal{M}} (\psi^\beta)_{B} d\psi^\beta d\tilde{v} \]
\[ + C_2 \|\psi^\beta\|_{L^\beta, \overline{g}} + C_3 \]
holds for some positive constants $C_2$ and $C_3$ depending only on $K$ and $L$. Hence we can derive

\[ \sup_{t \in [0, T)} \|\psi^\beta(t)\|_{L^\beta, \overline{g}_t} \]
\[ \leq \left( \left( \frac{C_3}{C_2} + \|\psi^\beta(\cdot, 0)\|_{L^\beta, g_0} \right) e^{C_2 T} - \frac{C_3}{C_2} \right)^{1/\beta} \]
\[ \leq \left( \left( \frac{C_3}{C_2} + C_1 \right) e^{C_2 T} - \frac{C_3}{C_2} \right)^{1/\beta}. \]

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By using this lemma, we can derive the following inequality.

**Lemma 6.2.10.** Take any positive constant \( k \). Assume that

\[
0 \leq \delta \leq \min \left\{ \frac{1}{2} - \frac{k}{\beta}, \frac{n \varepsilon^2 \eta}{3} - \frac{k}{\beta}, \frac{n \varepsilon^4}{24(\eta + 1)} - \frac{k}{\beta} \right\}
\]

and

\[
\beta \geq \max \left\{ 100 \varepsilon^{-2}, \frac{n \varepsilon^2 \eta}{n \varepsilon^2 \eta - 3 \delta} \right\}.
\]

Then the following inequality holds:

\[
\sup_{t \in [0,T)} \left( \int_{M} (H^s_t) B(\psi^\delta(\cdot, t)) B d\bar{v} \right)^{1/\beta} \leq C,
\]

where \( C \) is as in Lemma 6.2.9.

**Proof.** Set \( \delta' := \delta + \frac{k}{\beta} \). Clearly we have \((H^s_t) B(\psi^\delta(\cdot, t)) B = \psi^\delta \). From the assumption for \( \delta \) and \( \beta \), \( \delta' \) satisfies (6.20). Hence, from Lemma 6.2.9, we have

\[
\left( \int_{M} (H^s_t) B(\psi^\delta(\cdot, t)) B d\bar{v} \right)^{1/\beta} = \left( \int_{M} (\psi^\delta(\cdot, t)) B d\bar{v} \right)^{1/\beta} \leq C.
\]

By using Lemmas 6.2.9, 6.2.10 and Theorem 5.2, we shall prove the statement of Proposition 6.2.

**Proof of Proposition 6.2.** (Step I) First we shall show \( T < \infty \). According to Lemma 4.10, we have

\[
\frac{\partial H^s}{\partial t} \geq \triangle_H H^s + \frac{(H^s)^3}{n}.
\]

Let \( \rho \) be the solution of the ordinary differential equation \( \frac{dy}{dt} = \frac{1}{n} y^3 \) with the initial condition \( y(0) = \min_{M} H^s_0 \). This solution \( \rho \) is given by

\[
\rho(t) = \frac{\min_{M} H^s_0}{\sqrt{1 - (2/n) \min_{M} (H^s_0)^2 \cdot t}}.
\]
We regard $\rho$ as a function over $M \times [0, T)$. Then we have
\[
\frac{\partial (H^s - \rho)}{\partial t} \geq \Delta_M (H^s - \rho) + \frac{(H^s)^3 - \rho^3}{n}.
\]
Furthermore, by the maximum principle, we can derive that $H^s \geq \rho$ holds over $M \times [0, T)$. Therefore we obtain
\[
H^s \geq \min_M \frac{H^s_0}{\sqrt{1 - (2/n) \min_M (H^s_0)^2}}.
\]
This implies that $T \leq \frac{1}{(2/n) \min_M (H^s_0)^2}(< \infty)$.

(Step II) Take positive constants $\delta$ and $\beta$ satisfying (6.22) and (6.23). Define a function $\psi_{\delta, k}$ by $\psi_{\delta, k} := \max\{0, \psi_\delta(\cdot, t) - k\}$, where $k$ is any positive number with $k \geq \sup_M \psi_\delta(\cdot, 0)$. Set $A_t(k) := \{\phi(u) | \psi_\delta(u, t) \geq k\}$ and $\bar{A}(k) := \bigcup_{t \in [0, T)} (A_t(k) \times \{t\})$, which is finite because of $T < \infty$. For a function $\bar{\rho}$ over $M \times [0, T)$, denote by
\[
\int_{A_t(k)} \bar{\rho}(\cdot, t) d\bar{v}
\]
the function over $[0, T)$ defined by assigning $\int_{A_t(k)} \bar{\rho}(\cdot, t) d\bar{v}$ to each $t \in [0, T)$. By multiplying the inequality in Lemma 6.2.4 by $\beta \psi_{\delta, k}^{\beta - 1}$, we can show that the inequality in Lemma 6.2.8 holds for $\psi_{\delta, k}$ instead of $\psi_\delta$. From the inequality, the following inequality is derived directly:
\[
\frac{\partial}{\partial t} \int_{A_t(k)} (\psi_{\delta, k})^2_B d\bar{v} + \frac{\beta(\beta - 1)}{2} \int_{A_t(k)} (\psi_{\delta, k})^{\beta - 2}_B ||d\psi_{\delta, k}||_B^2 d\bar{v} \leq \beta \delta \int_{A_t(k)} (\psi_{\delta, k})^\beta_B (H^s_B)^2 d\bar{v}.
\]
Set $\hat{\psi} := \psi_{\delta, k}^{\beta/2}$. On $A_t(k)$, we have
\[
\frac{\beta(\beta - 1)}{2} (\psi_{\delta, k}^{\beta - 2})^{\beta - 2}_B (\cdot, t) ||d(\psi_{\delta, k})_B(\cdot, t)||_B^2 \geq ||d\hat{\psi}_B(\cdot, t)||_B^2
\]
and hence
\[
\frac{\partial}{\partial t} \int_{A_t(k)} \hat{\psi}_B^2 d\bar{v} + \int_{A_t(k)} ||d\hat{\psi}_B||_B^2 d\bar{v} \leq \beta \delta \int_{A_t(k)} \hat{\psi}_B^2 (H^s_B)^2 d\bar{v}.
\]
By integrating both sides of this inequality from 0 to any $t_0 \in [0, T)$, we have
\[
\int_{A_{t_0}(k)} \hat{\psi}_B^2(\cdot, t_0) d\bar{v}_{t_0} + \int_0^{t_0} \int_{A_t(k)} ||d\hat{\psi}_B||_B^2 d\bar{v} dt \leq \beta \delta \int_0^{t_0} \int_{A_t(k)} \hat{\psi}_B^2 (H^s_B)^2 d\bar{v} dt,
\]
where we use $k \geq \sup_M \psi_\delta(\cdot, 0)$. By the arbitrariness of $t_0$, we have
\[
\sup_{t \in [0, T)} \int_{A_t(k)} \hat{\psi}_B^2(\cdot, t) d\bar{v} + \int_0^T \int_{A_t(k)} ||d\hat{\psi}_B||_B^2 d\bar{v} dt \leq 2 \beta \delta \int_0^T \int_{A_t(k)} \hat{\psi}_B^2 (H^s_B)^2 d\bar{v} dt.
\]
From \( k \geq \sup_M \psi_\delta(\cdot, 0) \), we have \( A_0(k) = \emptyset \). Since \( f \) satisfies the conditions \((\ast_1)\) and \((\ast_2)\), so is also \( f_t (0 \leq t < T) \) because \( \text{Vol}_{\delta_t(M)} \) decreases with respect to \( t \) by Lemma 4.12. Hence we can apply the Sobolev's inequality in Theorem 5.2 to \( f_t \) \( (0 \leq t < T) \). By using the Sobolev's inequality in Theorem 5.2 and the Hölder's inequality, we can derive

\[
\left( \int_M \hat{\psi}^{\frac{2n}{n-2}}(\cdot, t) \, dv_t \right)^{\frac{n-1}{n}} \\
\leq C(n) \left( \int_M \|d(\hat{\psi})^{\frac{2(n-1)}{n-2}}(\cdot, t)\|_B \, dv_t + \int_M \hat{\psi}^{\frac{2(n-1)}{n-2}}(\cdot, t) \cdot (H_t^s)_B \, dv_t \right) \\
= C(n) \left\{ \frac{2(n-1)}{n-2} \int_M \hat{\psi}^{\frac{n}{n-2}}(\cdot, t) \cdot \|d\hat{\psi}(\cdot, t)\|_B \, dv_t + \int_M \hat{\psi}^{\frac{2(n-1)}{n-2}}(\cdot, t) \cdot (H_t^s)_B \, dv_t \right\} \\
\leq C(n) \left\{ \frac{2(n-1)}{n-2} \left( \int_M \hat{\psi}^{\frac{2n}{n-2}}(\cdot, t) \, dv_t \right)^{1/2} \cdot \left( \int_M \|d\hat{\psi}(\cdot, t)\|_B^2 \, dv_t \right)^{1/2} \\
+ \left( \int_M \hat{\psi}^{\frac{2n}{n-2}}(\cdot, t) \, dv_t \right)^{\frac{n-1}{n}} \cdot \left( \int_M (H_t^s)_B \, dv_t \right)^{1/2} \right\}.
\]

Also, since \( \psi_\delta(\cdot, t) \geq k \) on \( A_t(k) \), it follows from Lemma 6.2.10 that

\[
\left( \int_M (H_t^s)_B \, dv_t \right)^{1/2} \leq k^{-\beta/n} \left( \int_M (H_t^s)_B \psi_\delta^\beta \, dv_t \right)^{1/2} \leq k^{-\beta/n} \cdot C^{\beta/n},
\]

where \( C \) is as in Lemma 6.2.9. Hence we obtain

\[
\left( \int_M \hat{\psi}^{\frac{2n}{n-2}}(\cdot, t) \, dv_t \right)^{\frac{n-1}{n}} \\
\leq C(n) \left\{ \frac{2(n-1)}{n-2} \left( \int_M \hat{\psi}^{\frac{2n}{n-2}}(\cdot, t) \, dv_t \right)^{1/2} \cdot \left( \int_M \|d\hat{\psi}(\cdot, t)\|_B^2 \, dv_t \right)^{1/2} \\
+ \left( \int_M \hat{\psi}^{\frac{2n}{n-2}}(\cdot, t) \, dv_t \right)^{\frac{n-1}{n}} \cdot k^{-\beta/n} \cdot C^{\beta/n} \right\},
\]

that is,

\[
\left( \int_M \|d\hat{\psi}(\cdot, t)\|_B^2 \, dv_t \right)^{1/2} \\
\geq \frac{n-2}{2C(n)(n-1)} \left( \int_M \hat{\psi}^{\frac{2n}{n-2}}(\cdot, t) \, dv_t \right)^{\frac{n-2}{2n}} \left( 1 - C(n) \cdot \left( \frac{C}{k} \right)^{\beta/n} \right).
\]

Set

\[
k_1 := \max \left\{ \sup_M \psi_\delta(\cdot, 0), \ C(n)^{n/\beta} \cdot C \right\}.
\]
Assume that \( k \geq k_1 \). Then we have
\[
\int_M \|d\hat{\psi}(\cdot, t)\|_B^2 \, d\hat{v}_t
\geq \left( \frac{n - 2}{2C(n)(n - 1)} \right)^2 \left( \int_M \hat{\psi}_B^{2n} (\cdot, t) \, d\hat{v}_t \right)^{\frac{n-2}{n}} \left( 1 - C(n) \cdot \left( \frac{C}{k} \right)^{\beta/n} \right)^2.
\]

From (6.24) and (6.25), we obtain
\[
\sup_{t \in [0, T]} \int_M \hat{\psi}_B^2 (\cdot, t) \, d\hat{v}_t + \hat{C}(n, k) \int_0^T \left( \int_M \hat{\psi}_B^{2n} (\cdot, t) \, d\hat{v}_t \right)^{\frac{n-2}{n}} \, dt
\leq 2\beta \delta \int_0^T \left( \int_M \hat{\psi}_B^2 (H_t \delta t) \, d\hat{v}_t \right) \, dt,
\]
where \( \hat{C}(n, k) := \left( \frac{(n-2)(1-C(n))(C/k)^{\beta/n}}{2C(n)(n-1)} \right)^2 \). Set
\[
q := \begin{cases} 
\frac{n}{n-2} & (n \geq 3) \\
\text{any positive number} & (n = 2)
\end{cases}
\]
and \( q_0 := 2 - 1/q \) and
\[
||A_t(k)||_T := \int_0^T \left( \int_M d\hat{v}_t \right) \, dt.
\]

By using the interpolation inequality, we can derive
\[
\left( \int_M \hat{\psi}_B^{2q_0} \, d\hat{v}_t \right)^{\frac{1}{q_0}} \leq \left( \int_M \hat{\psi}_B^2 \, d\hat{v}_t \right)^{1-1/q_0} \cdot \left( \int_M \hat{\psi}_B^{2q} \, d\hat{v}_t \right)^{1/q_0}.
\]

By using this inequality and the Young inequality, we can derive
\[
\left( \int_0^T \left( \int_M \hat{\psi}_B^{2q_0} (\cdot, t) \, d\hat{v}_t \right) \, dt \right)^{\frac{1}{q_0}} \leq \left( \int_0^T \left( \int_M \hat{\psi}_B^2 (\cdot, t) \, d\hat{v}_t \right)^{q_0-1} \cdot \left( \int_M \hat{\psi}_B^{2q} (\cdot, t) \, d\hat{v}_t \right)^{1/q} \right)^{\frac{1}{q_0}} \cdot \left( \int_M \hat{\psi}_B^{2q} (\cdot, t) \, d\hat{v}_t \right)^{\frac{1}{q_0}}
\leq \sup_{t \in [0, T]} \int_M \hat{\psi}_B^2 (\cdot, t) \, d\hat{v}_t + \int_0^T \left( \int_M \hat{\psi}_B^{2q} (\cdot, t) \, d\hat{v}_t \right)^{1/q} \, dt.
\]

We may assume that $\hat{C}(n, k) < 1$ holds by replacing $C(n)$ to a bigger positive number and furthermore $k$ to a positive number bigger such that $1 - C(n) \cdot \left(\frac{\hat{C}}{T}\right)^{\beta/n} > 0$ holds for the replaced number $C(n)$. Then, from (6.26) and (6.27), we obtain

$$
\hat{C}(n, k) \left(\int_0^T \left(\int_M \hat{\psi}_B^{2q_0}(\cdot, t) \, d\bar{v}_t\right) \, dt \right)^{1/q_0} \\
\leq 2\beta\delta \int_0^T \left(\int_M \hat{\psi}_B^2(H^s_t)^2 \, d\bar{v}_t\right) \, dt.
$$

On the other hand, by using the Hölder’s inequality, we obtain

$$
\int_0^T \left(\int_M \hat{\psi}_B^2(H^s_t)^2 \, d\bar{v}_t\right) \, dt \leq ||A_t(k)||_{r^{-1}} \cdot \left(\int_0^T \left(\int_M \hat{\psi}_B^{2r}(H^s_t)^2 \, d\bar{v}_t\right) \, dt \right)^{1/r},
$$

where $r$ is any positive constant with $r > 1$. From (6.28) and this inequality, we obtain

$$
\left(\int_0^T \left(\int_M \hat{\psi}_B^{2q_0}(\cdot, t) \, d\bar{v}_t\right) \, dt \right)^{1/q_0} \\
\leq 2\hat{C}(n, k)^{-1}\beta\delta ||A_t(k)||_{r^{-1}} \cdot \left(\int_0^T \left(\int_M \hat{\psi}_B^{2r}(H^s_t)^2 \, d\bar{v}_t\right) \, dt \right)^{1/r}.
$$

On the other hand, according to Lemma 6.2.10, we have

$$
\int_M \hat{\psi}_B^2(H^s_t)^2 \, d\bar{v}_t \leq C^{2r}
$$

for some positive constant $C$ (depending only on $K, L$ and $f$) by replacing $r$ to a bigger positive number if necessary. Also, by using the Hölder inequality, we obtain

$$
\int_0^T \left(\int_M \hat{\psi}_B^2(\cdot, t) \, d\bar{v}_t\right) \, dt \\
\leq ||A_t(k)||_{q_0^{-1}} \cdot \left(\int_0^T \left(\int_M \hat{\psi}_B^{2q_0}(\cdot, t) \, d\bar{v}_t\right) \, dt \right)^{1/q_0}.
$$

From (6.29), (6.30) and this inequality, we obtain

$$
\int_0^T \left(\int_M \hat{\psi}_B^2(\cdot, t) \, d\bar{v}_t\right) \, dt \leq ||A_t(k)||_{T^{-1/q_0-1/r}} \cdot C^2 \cdot \hat{C}(n, k)^{-1} \cdot 2\beta\delta.
$$

We may assume that $2 - 1/q_0 - 1/r > 1$ holds by replacing $r$ to a bigger positive number if necessary. Take any positive constants $h$ and $k$ with $h > k \geq k_1$. Then
we have
\[
\int_0^T \left( \int_M (\psi^\beta_{\delta,k} - \psi^\beta_{\delta,h}) \, d\nu_t \right) dt \geq \int_0^T \left( \int_M |h - k|^\beta \, d\nu_t \right) dt = |h - k|^\beta \cdot ||A_t(h)||_T.
\]

From this inequality and (6.31), we obtain
\[
|h - k|^\beta \cdot ||A_t(h)||_T \leq ||A_t(k)||_T^{2^{-1/q_0 - 1/r}} \cdot C^2 \cdot \hat{C}(n, k) - 1 \cdot 2\beta\delta.
\]

Since \( \bullet \mapsto ||A_t(\bullet)||_T \) is a non-increasing and non-negative function and (6.32) holds for any \( h > k \geq k_1 \), it follows from the Stambaccha’s iteration lemma that \( ||A_t(k_1 + d)||_T = 0 \), where \( d \) is a positive constant depending only on \( \beta, \delta, q_0, r, C, \hat{C}(n, k) \) and \( ||A_t(k_1)||_T \). This implies that \( \sup_{t \in [0, T]} \max_M \psi_{\delta}(\cdot, t) \leq k_1 + d < \infty \). This completes the proof. \( \square \)

7 Estimate of the gradient of the mean curvature from above

In this section, we shall derive the following estimate of \( \text{grad} H^s \) from above by using Proposition 6.2.

**Proposition 7.1.** For any positive constant \( b \), there exists a constant \( C(b, f_0) \) (depending only on \( b \) and \( f_0 \)) satisfying
\[
||\text{grad} H^s||^2 \leq b \cdot (H^s)^4 + C(b, f_0) \quad \text{on} \quad M \times [0, T).
\]

We prepare some lemmas to prove this proposition.

**Lemma 7.1.1.** The family \( \{||\text{grad}_t H^s||^2\}_{t \in [0, T]} \) satisfies the following equation:
\[
\partial_t ||\text{grad}_t H^s||^2 = \Delta_H(||\text{grad}_t H^s||^2) - 2||\nabla^H \text{grad} H^s||^2 + 2||A_{Ht}||^2 \cdot ||\text{grad} H^s||^2 \\
+ 2H^s \cdot g_H(\text{grad}(||A_{Ht}||^2), \text{grad} H^s) \\
+ 2g_H(\langle A_{Ht}^\phi \rangle_{\hat{H}}, \text{grad} H^s) \\
- 6H^s \cdot g_H(\text{grad}((A_{\xi}^\phi)^2)_{\hat{H}}, \text{grad} H^s) \\
- 6\text{Tr}((A_{\xi}^\phi)^2)_{\hat{H}} \cdot ||\text{grad} H^s||^2.
\]
Hence we have the following inequality:

\[
\begin{align*}
\frac{\partial}{\partial t}||\text{grad } H^s||^2 & - \triangle_{\mathcal{H}}(||\text{grad } H^s||^2) \\
\leq & -2||\nabla_{\mathcal{H}}\text{grad } H^s||^2 + 4||A_{\mathcal{H}}||^2 \cdot ||\text{grad } H^s||^2 \\
& + 2H^s \cdot g_{\mathcal{H}}(||A_{\mathcal{H}}||^2, \text{grad } H^s) \\
& + 6H^s \cdot ||\text{grad}(\text{Tr}(\mathcal{A}_\xi^2))|| \cdot ||\text{grad } H^s|| \\
& - 6\text{Tr}(\mathcal{A}_\xi^2) \cdot ||\text{grad } H^s||^2.
\end{align*}
\]

(7.2)

**Proof.** By using Lemmas 4.2 and 4.9, we have

\[
\begin{align*}
\frac{\partial}{\partial t}||\text{grad } H^s||^2 &= \frac{\partial g_{\mathcal{H}}}{\partial t}(\text{grad } H^s, \text{grad } H^s) + 2g_{\mathcal{H}}\left(\text{grad} \left(\frac{\partial H^s}{\partial t}\right), \text{grad } H^s\right) \\
&= -2H^s \cdot h_{\mathcal{H}}(\text{grad } H^s, \text{grad } H^s) + 2g_{\mathcal{H}}(\triangle_{\mathcal{H}} H^s, \text{grad } H^s) \\
&+ 2g_{\mathcal{H}}(\text{grad}(H^s \cdot ||A_{\mathcal{H}}||^2), \text{grad } H^s) \\
&- 6g_{\mathcal{H}}(\text{grad}(H^s \cdot \text{Tr}(\mathcal{A}_\xi^2))_{\mathcal{H}}, \text{grad } H^s).
\end{align*}
\]

Also we have

\[
\triangle_{\mathcal{H}}(||\text{grad } H^s||^2) = 2g_{\mathcal{H}}(\triangle_{\mathcal{H}}^2(\text{grad } H^s), \text{grad } H^s) \\
+ 2g_{\mathcal{H}}(\nabla_{\mathcal{H}} \text{grad } H^s, \nabla_{\mathcal{H}} \text{grad } H^s)
\]

and

\[
\triangle_{\mathcal{H}}(\text{grad } H^s) = \text{grad}(\triangle_{\mathcal{H}} H^s) + H^s \cdot A_{\mathcal{H}}(\text{grad } H^s) - (A_{\mathcal{H}}^2)(\text{grad } H^s).
\]

By using these relations and noticing \(g_{\mathcal{H}}(A_{\mathcal{H}}(\bullet), \cdot) = -h_{\mathcal{H}}(\bullet, \cdot)\), we can derive the desired evolution equation (7.1). The inequality (7.2) is derived from (7.1) and

\[
g_{\mathcal{H}}((A_{\mathcal{H}}^2)(\text{grad } H^s), \text{grad } H^s) \leq ||A_{\mathcal{H}}||^2 \cdot ||\text{grad } H^s||^2.
\]

\[\square\]

**Lemma 7.1.2.** The family \(\left\{\frac{||\text{grad}_t H^s_t||^2}{H^s_t}\right\}_{t \in [0, T]}\) satisfies the following inequality:

\[
\begin{align*}
\frac{\partial}{\partial t} \left(\frac{||\text{grad } H^s||^2}{H^s}\right) - \triangle_{\mathcal{H}} \left(\frac{||\text{grad } H^s||^2}{H^s}\right) & \leq 3\frac{||\text{grad } H^s||^2}{H^s} \cdot ||A_{\mathcal{H}}||^2 + 2g_{\mathcal{H}}(\text{grad}(||A_{\mathcal{H}}||^2), \text{grad } H^s) \\
& + 6||\text{grad}(\text{Tr}(\mathcal{A}_\xi^2))|| \cdot ||\text{grad } H^s|| - 3\frac{H^s}{||\text{Tr}(\mathcal{A}_\xi^2)|| \cdot ||\text{grad } H^s||^2}
\end{align*}
\]

(7.3)
Proof. By a simple calculation, we have
\[
\frac{\partial}{\partial t} \left( \frac{||\text{grad } H^s||^2}{H^s} \right) = \nabla_H \left( \frac{||\text{grad } H^s||^2}{H^s} \right)
\]
\[
= \frac{1}{H^s} \left( \frac{\partial ||\text{grad } H^s||^2}{\partial t} - \nabla_H (||\text{grad } H^s||^2) \right) - \frac{||\text{grad } H^s||^2}{H^s} \left( \frac{\partial H^s}{\partial t} - \nabla_H H^s \right)
\]
\[
+ \frac{2}{(H^s)^2} g_H (\text{grad } H^s, \text{grad } ||\text{grad } H^s||^2).
\]
From this relation, Lemmas 4.9 and (7.2), we can derive the desired inequality.

From Lemma 4.10, we can derive the following evolution equation directly.

**Lemma 7.1.3.** The family \( \{ (H^s_t)^3 \} \in [0, T) \) satisfies the following evolution equation:
\[
\frac{\partial (H^s)^3}{\partial t} - \nabla_H ((H^s)^3) = 3(H^s)^3 \cdot ||A_H||^2 - 6H^s \cdot ||\text{grad } H^s||^2 - 9(H^s)^3 \cdot \text{Tr}((A^\phi)^2)_H.
\]

By using Lemmas 4.9, 4.11 and Proposition 6.2, we can derive the following evolution inequality.

**Lemma 7.1.4.** The family \( \{ \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \} \in [0, T) \) satisfies the following evolution inequality:
\[
\frac{\partial}{\partial t} \left( \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \right) - \nabla_H \left( \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \right) \leq -\frac{2(n - 1)}{3n} H^s \cdot \|\nabla^H A_H\|^2 + \tilde{C}(n, C_0, \delta) \cdot \|\nabla^H A_H\|^2
\]
\[
+ 3H^s \cdot ||A_H||^2 \cdot \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) - 2H^s \cdot \text{Tr}((A^\phi)^2)_H \cdot \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right)
\]
\[
- 4(H^s)^2 \cdot \text{Tr}((A^\phi)^2) \circ (A_H - \frac{H^s}{n} \cdot \text{id})
\]
\[
- 2H^s \cdot \text{Tr}_{g_H} \mathcal{R} \left( (A_H - \frac{H^s}{n} \cdot \text{id}) (\bullet, \bullet) \right)
\]
\[
- 3 \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \cdot \text{Tr}((A^\phi)^2)_H.
\]
Proof. By using Lemmas 4.9 and 4.11, we can derive

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \left( |A_H|^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \right) - \Delta_H \left( \left( |A_H|^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \right) &= \\
&= \frac{2H^s}{n} \cdot ||\text{grad } H^s||^2 + 2H^s \cdot ||A_H||^2 \cdot \left( \text{Tr}((A_H)^2) - \frac{(H^s)^2}{n} \right) \\
&\quad - 2H^s \cdot ||\nabla^H A_H||^2 + H^s \cdot ||A_H||^2 \cdot \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) \\
&\quad - g_H \left( \text{grad} \left( ||A_H||^2 - \frac{H^s}{n} \cdot \text{id} \right), \text{grad } H^s \right) \\
&\quad - 2H^s \cdot \text{Tr}((A_H^2)^2) \cdot \left( \text{Tr}((A_H)^2) - \frac{(H^s)^2}{n} \right) \\
&\quad - 4(H^s)^2 \cdot \text{Tr} \left( (A_H^2)^2 \circ (A_H - \frac{H^s}{n} \cdot \text{id}) \right) \\
&\quad - 2H^s \cdot \text{Tr}^\ast_H \mathcal{R} \left( \left( A_H - \frac{H^s}{n} \cdot \text{id} \right) (\bullet, \bullet) \right) \\
&\quad - 3 \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \cdot \text{Tr}((A_H^2)^2) \mathcal{H}.
\end{align*}
\]

On the other hand, by using \( ||A_H||^2 - \frac{(H^s)^2}{n} = ||A_H - \frac{H^s}{n} \cdot \text{id} || \), we can derive

\[
\begin{align*}
\left| g_H \left( \text{grad} \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right), \text{grad } H^s \right) \right| &= \\
&= d \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right) \left| \text{grad } H^s \right| \\
&= 2 \left| g_H \left( \nabla^\mathcal{H}_{\text{grad } H^s} \left( A_H - \frac{H^s}{n} \cdot \text{id} \right), A_H - \frac{H^s}{n} \cdot \text{id} \right) \right| \\
&\leq 2 ||\text{grad } H^s|| \cdot \left| \nabla^\mathcal{H} A_H \right| \cdot \left| A_H - \frac{H^s}{n} \cdot \text{id} \right| \\
&\leq 2n ||\nabla^\mathcal{H} A_H||^2 \cdot \left| A_H - \frac{H^s}{n} \cdot \text{id} \right|,
\end{align*}
\]

where we use \( \frac{1}{n} ||\text{grad } H^s||^2 \leq ||\nabla^\mathcal{H} A_H||^2 \). Also, according to Proposition 6.2, we have

\[
\left| A_H - \frac{H^s}{n} \cdot \text{id} \right| \leq \sqrt{C_0} \cdot (H^s)^{1-\delta/2}.
\]

Hence we have

\[
\begin{align*}
\left| g_H \left( \text{grad} \left( ||A_H||^2 - \frac{(H^s)^2}{n} \right), \text{grad } H^s \right) \right| &\leq 2n \sqrt{C_0} ||\nabla^\mathcal{H} A_H||^2 \cdot (H^s)^{1-\delta/2}.
\end{align*}
\]
Furthermore, according to the Young’s inequality:

\[(7.6) \quad ab \leq \varepsilon \cdot a^p + \varepsilon^{-1/(p-1)} \cdot b^q \quad (\forall \ a > 0, \ b > 0)\]

(where \(p\) and \(q\) are any positive constants with \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\varepsilon\) is any positive constant), we have

\[(7.7) \quad 2n\sqrt{C_0(H^s)^{1-\delta/2}} \leq \frac{2(n-1)}{3n} \cdot H^s + \tilde{C}(n, C_0, \delta),\]

where \(\tilde{C}(n, C_0, \delta)\) is a positive constant only on \(n, C_0\) and \(\delta\). Also, we have

\[||\nabla^H A_H||^2 \geq \frac{3}{n+2}||\nabla^H H||^2.\]

From (7.4) and these inequalities, we can derive the desired evolution inequality. \(\square\)

By using Lemmas 4.10, 7.1.2, 7.1.3 and 7.1.4, we shall prove Theorem Proposition 7.1.

**Proof of Proposition 7.1.** Define a function \(\rho\) over \(M \times [0, T)\) by

\[\rho := \frac{||\text{grad} H^s||^2}{H^s} + C_1 H^s \left(||A_H||^2 - \frac{(H^s)^2}{n}\right) + C_1 \cdot \tilde{C}(n, C_0, \delta)||A_H||^2 - b(H^s)^3,\]

where \(b\) is any positive constant and \(C_1\) is a positive constant which is sufficiently big compared to \(n\) and \(b\). By using Lemmas 4.10, 7.1.2, 7.1.3 and 7.1.4, we can derive

\[
\frac{\partial \rho}{\partial t} - \Delta_H \rho \\
\leq 3||\text{grad} H^s||^2 \cdot ||A_H||^2 + 2g_H(\text{grad}(||A_H||^2), \text{grad} H^s) \\
- \frac{2(n-1)}{3n} \cdot C_1 \cdot H^s \cdot ||\nabla^H A_H||^2 \\
+ 3C_1 \cdot H^s \cdot ||A_H||^2 \left(||A_H||^2 - \frac{(H^s)^2}{n}\right) \\
+ 2C_1 \cdot \tilde{C}(n, C_0, \delta) \cdot ||A_H||^4 - 3b(H^s)^3 \cdot ||A_H||^2 + 6bH^s \cdot ||\text{grad} H^s||^2 \\
+ 6||\text{grad} H^s|| \cdot ||\text{grad}(\text{Tr}(A^2_H))|| - \frac{3}{H^s} \cdot ||\text{grad} H^s||^2 \cdot \text{Tr}(A^2_H) \\
- 2C_1 H^s \cdot \text{Tr}(A^2_H) \cdot \left(||A_H||^2 - \frac{(H^s)^2}{n}\right).
\]
\[-4C_1(H^s)^2 \cdot \text{Tr} \left( (A_\xi^\phi)^2 \circ \left( A_H - \frac{H^s}{n} \cdot \text{id} \right) \right) \]
\[-2C_1 H^s \cdot \text{Tr}_{g_H}^* \mathcal{R} \left( \left( A_H - \frac{H^s}{n} \cdot \text{id} \right) (\bullet), (\bullet) \right) \]
\[-3C_1 \left( \|A_H\|^2 - \frac{(H^s)^2}{n} \right) \cdot H^s \cdot \text{Tr}((A_\xi^\phi)^2)_{\mathcal{H}} \]
\[-2C_1 \cdot \tilde{C}(n, C_0, \delta) \|A_H\|^2 \cdot \text{Tr}((A_\xi^\phi)^2)_{\mathcal{H}} \]
\[-4C_1 \cdot \tilde{C}(n, C_0, \delta) H^s \cdot \text{Tr} \left( ((A_\xi^\phi)^2)_{\mathcal{H}} \circ A_H \right) \]
\[-2C_1 \cdot \tilde{C}(n, C_0, \delta) \text{Tr}_{g_H}^* \mathcal{R}(A_H \bullet, \bullet) + 9b \cdot (H^s)^3 \cdot \text{Tr}((A_\xi^\phi)^2)_{\mathcal{H}}. \]

Also, in similar to (7.5), we obtain
\[|g_H(\text{grad}(\|A_H\|^2), \text{grad} H^s)| \leq 2n \sqrt{C_0 \|\nabla^H A_H\|^2 \cdot (H^s)^{1-\delta/2}}.\]

This implies together with (7.7) that
\[(7.9) \quad |g_H(\text{grad}(\|A_H\|^2), \text{grad} H^s)| \leq \left( \frac{2(n-1)}{3n} \cdot H^s + \tilde{C}(n, C_0, \delta) \right) \|\nabla^H A_H\|^2. \]

Denote by \(T^1V\) the unit tangent bundle of \(V\). Define a function \(\Psi\) over \(T^1V\) by
\[\Psi(X) := \|d(\text{Tr}(A_X^\phi)^2))_{\mathcal{H}}\| \quad (X \in T^1V).\]

It is clear that \(\Psi\) is continuous. Set \(\tilde{K}_1 := \sup_{t \in [0,T]} \max \|\nabla (\text{Tr}((A_\xi^\phi)^2)_{\mathcal{H}})\|\), which is finite because \(\Psi\) is continuous and the closure of \(\bigcup_{t \in [0,T]} \phi(f_t(M))\) is compact. Also, we have
\[(7.10) \quad \text{Tr}_{g_H}^* \mathcal{R} \left( \left( A_H - \frac{H^s}{n} \cdot \text{id} \right) (\bullet), (\bullet) \right) \leq \tilde{K}_2 \cdot \left\| A_H - \frac{H^s}{n} \cdot \text{id} \right\| \]

for some positive constant \(\tilde{K}_2\) because of the homogeneity of \(N\). By using (7.7), (7.9), (7.10),
$|A_H| \leq H^s$, $\frac{1}{n}||\text{grad} H^s||^2 \leq ||\nabla^H A_H||^2$ and Proposition 6.2, we can derive

$$\frac{\partial \rho}{\partial t} - \Delta \rho \leq \left(3n + \frac{4(n-1)}{3n} - \frac{2(n-1)C_1}{3n} + 6nb\right) H^s \cdot ||\nabla^H A_H||^2 + 2\hat{C}(n, C_0, \delta) \cdot ||\nabla^H A_H||^2 + 3C_0 \cdot C_1 (H^s)^5 - \delta + 2C_1 \cdot \hat{C}(n, C_0, \delta) (H^s)^4$$

(7.11)

Furthermore, by using the Young’s inequality (7.6) and the fact that $C_1$ is sufficiently big compared to $n$ and $b$, we can derive that

$$\frac{\partial \rho}{\partial t} - \Delta \rho \leq C_3(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2)$$

holds for some positive constant $C_3(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2)$ only on $n, C_0, C_1, b, \delta, \hat{K}_1$ and $\hat{K}_2$. This together with $T < \infty$ implies that

$$\max_M \rho_t \leq \max_M \rho_0 + C_3(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2) t$$

$$\leq \max_M \rho_0 + C_3(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2) \cdot T$$

$(0 \leq t < T)$. Therefore, we obtain

$$||\text{grad} H^s||^2 \leq b(H^s)^4 + \max_M \rho_0 \cdot H^s + C_3(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2) \cdot T \cdot H^s.$$  

Furthermore, by using the Young inequality (7.6), we obtain

$$||\text{grad} H^s||^2 \leq 2b(H^s)^4 + C_4(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2, T)$$

holds for some positive constant $C_4(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2, T)$ only on $n, C_0, C_1, b, \delta, \hat{K}_1$ $\hat{K}_2$ and $T$. Since $b$ is any positive constant and $C_4(n, C_0, C_1, b, \delta, \hat{K}_1, \hat{K}_2, T)$ essentially depends only on $n$ and $f_0$, we obtain the statement of Proposition 7.1. \hfill \Box

8 Proof of Theorem A.

In this section, we shall prove Theorem A. G. Huisken (Hui) obtained the evolution inequality for the squared norm of all iterated covariant derivatives of the shape...
operators of the mean curvature flow in a complete Riemannian manifold satisfying curvature-pinching conditions in Theorem 1.1 of [Hu]. See the proof of Lemma 7.2 (Page 478) of [Hu] about this evolution inequality. In similar to this evolution inequality, we obtain the following evolution inequality.

**Lemma 8.1.** For any positive integer $m$, the family $\{|||\nabla^m A_H|||_B^2\}_{t \in [0, T)}$ satisfies the following evolution inequality:

$$\frac{\partial}{\partial t}|||\nabla^m A_H|||_B^2 - \Delta_H|||\nabla^m A_H|||_B^2 \leq -2|||\nabla^m A_H|||_B^2 + C_4(n, m) \times \left( \sum_{i+j+k=m} |||\nabla^i A_H||| \cdot |||\nabla^j A_H||| \cdot |||\nabla^k A_H||| \cdot |||\nabla^m A_H||| \right) + C_5(m) \sum_{i \leq m} |||\nabla^i A_H||| \cdot |||\nabla^m A_H||| + C_6(m) |||\nabla^m A_H|||,$$

where $C_4(n, m)$ is a positive constant depending only on $n, m$ and $C_i(m)$ ($i = 5, 6$) are positive constants depending only on $m$.

In similar to Corollary 12.6 of [Ha], we can derive the following interpolation inequality.

**Lemma 8.2.** Let $S$ be an element of $\Gamma(\pi^*_M(T^{(1,1)} M))$ such that, for any $t \in [0, T)$, $S_t$ is a $G$-invariant $(1, 1)$-tensor field on $M$. For any positive integer $m$, the following inequality holds:

$$\int_M |||\nabla^i S_H|||_B^{2m/i} d\bar{v} \leq C(n, m) \cdot \max_M |||S_H|||_B^{2(m/i) - 1} \cdot \int_M |||\nabla^m S_H|||_B^2 d\bar{v},$$

where $C(n, m)$ is a positive constant depending only on $n$ and $m$.

From these lemmas, we can derive the following inequality.

**Lemma 8.3.** For any positive integer $m$, the following inequality holds:

$$\frac{d}{dt} \int_M |||\nabla^m A_H|||_B^2 d\bar{v} + 2 \int_M |||\nabla^{m+1} A_H|||_B^2 d\bar{v} \leq C_7(n, m, C_6(m), \text{Vol}(M_0)) \cdot \left( \max_M |||A_H|||_B^2 + 1 \right) \times \left( \int_M |||\nabla^m A_H|||_B^2 d\bar{v} + \left( \int_M |||\nabla^m A_H|||_B^2 d\bar{v} \right)^{1/2} \right),$$

(8.2)
where $C_7(n, m, C_0(m), \text{Vol}(M_0))$ is a positive constant depending only on $n, m, C_0(m)$ and the volume $\text{Vol}(M_0)$ of $M_0 = f_0(M)$.

**Proof.** By using (8.1) and the generalized Hölder inequality, we can derive

$$
\frac{d}{dt} \int_M \| (\nabla^H)^m A_H \|^2_B d\tilde{v} + 2 \int_M \| (\nabla^H)^{m+1} A_H \|^2_B d\tilde{v}
\leq C_4(n, m) \cdot \left( \sum_{i+j+k=m} \int_M \| (\nabla^H)^i A_H \|^\frac{2m}{2m-i} \| \tilde{v} \| d\tilde{v} \right)^\frac{1}{m} \cdot \left( \int_M \| (\nabla^H)^m A_H \|^\frac{2m}{2m-m} d\tilde{v} \right)^\frac{1}{m}
\times \left( \int_M \| (\nabla^H)^k A_H \|^\frac{2m}{2m-k} \| \tilde{v} \| d\tilde{v} \right)^\frac{1}{m} \cdot \left( \int_M \| (\nabla^H)^m A_H \|^\frac{2m}{2m-m} d\tilde{v} \right)^\frac{1}{m}
\times \frac{C(n, m)\tilde{C}(m) \sum_{i \leq m} \left( \int_M \| (\nabla^H)^i A_H \|^\frac{2m}{2m-i} \| \tilde{v} \| d\tilde{v} \right)^\frac{1}{m} \cdot \left( \int_M \| (\nabla^H)^m A_H \|^\frac{2m}{2m-m} d\tilde{v} \right)^\frac{1}{m}
\times \frac{C(n, m)\tilde{C}(m+1) \cdot \left( \int_M \| (\nabla^H)^m A_H \|^\frac{2m}{2m-m} d\tilde{v} \right)^\frac{1}{m} \cdot \left( \int_M d\tilde{v} \right)^\frac{1}{m}}{\frac{1}{m}}.
$$

From this inequality and Lemma 8.2, we can derive the desired inequality. \(\square\)

From this inequality and Lemma 8.2, we can derive the following statement.

**Proposition 8.4.** The family $\{\| A_H \|^2 \}_{t \in [0, T]}$ is not uniform bounded.

**Proof.** Suppose that $\sup_{t \in [0, T]} \max_M \| A_H \|^2 < \infty$. Denote by $C_A$ this supremum. Define a function $\Phi$ over $[0, T)$ by

$$
\Phi(t) := \int_M \| (\nabla^H)^m (A_H)_{i+} \|_B^2 d\tilde{v}_t \quad (t \in [0, T)).
$$

Then, according to (8.2), we have

$$
\frac{d\Phi}{dt} \leq C_7(n, m, C_0(m), \text{Vol}(M_0)) \cdot (C_A + 1) \cdot (\Phi + \Phi^{1/2}).
$$

Assume that $\sup_{t \in [0, T)} \Phi > 1$. Set $E := \{ t \in [0, T) \mid \Phi(t) > 1 \}$. Take any $t_0 \in E$. Then $\Phi \geq 1$ holds over $[t_0, t_0 + \varepsilon)$ for some a sufficiently small positive number $\varepsilon$. Hence we have

$$
\frac{d\Phi}{dt} \leq 2C_7(n, m, C_0(m), \text{Vol}(M_0)) \cdot (C_A + 1) \cdot \Phi
$$

on $[t_0, t_0 + \varepsilon)$. From this inequality, we can derive

$$
\Phi(t) \leq \Phi(t_0)e^{2C_7(n, m, C_0(m), \text{Vol}(M_0)) \cdot (C_A + 1)(t-t_0)} \quad (t \in [t_0, t_0 + \varepsilon))
$$

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and hence
\[ \Phi(t) \leq \Phi(t_0)e^{2C_\gamma(t,m,C_6(m),\text{Vol}(M_0)) \cdot (C_A+1)T} \quad (t \in [t_0, t_0 + \varepsilon]). \]
This fact together with the arbitrariness of \( t_0 \) implies that \( \Phi_t \) is uniform bounded. Thus, we see that
\[ \sup_{t \in [0,T]} \int_M \| (\nabla^H)^m (A_H)_t \|_B^2 \, d\nu_t < \infty \]
holds in general. Furthermore, since this inequality holds for any positive integer \( m \), it follows from Lemma 8.2 that
\[ \sup_{t \in [0,T]} \int_M \| (\nabla^H)^m (A_H)_t \|_B^l \, d\nu_t < \infty \]
holds for any positive integer \( m \) and any positive constant \( l \). Hence, by the Sobolev’s embedding theorem, we obtain
\[ \sup_{t \in [0,T]} \max_M \| (\nabla^H)^m (A_H)_t \| < \infty. \]
Since this fact holds for any positive integer \( m \), \( f_t \) converges to a \( C^\infty \)-embedding \( f_T \) as \( t \to T \) in \( C^\infty \)-topology. This implies that the mean curvature flow \( f_t \) extends after \( T \) because of the short time existence of the mean curvature flow starting from \( f_T \). This contradicts the definition of \( T \). Therefore we obtain
\[ \sup_{t \in [0,T]} \max_M \| A_H \| = \infty. \]

By imitating the proof of Theorem 4.1 of [A1, A2], we can show the following fact, where we note that more general curvature flows (including mean curvature flows as special case) is treated in [A1, A2].

**Lemma 8.5.** The following uniform boundedness holds:
\[ \inf_{t \in [0,T]} \max \{ \varepsilon > 0 \mid (A_H)_t \geq \varepsilon H^s_t \cdot \text{id on } M \} > 0 \]
and hence
\[ \sup_{(x,t) \in M \times [0,T]} \frac{\lambda_{\max}(x,t)}{\lambda_{\min}(x,t)} \leq \frac{1}{\varepsilon_0}, \]
where \( \lambda_{\max}(x,t) \) (resp. \( \lambda_{\min}(x,t) \)) denotes the maximum (resp. minimum) eigenvalue of \( (A_H)(x,t) \) and \( \varepsilon_0 \) denotes the above infimum.
Proof. Since
\[
\left( \frac{\partial h_H}{\partial t} - \Delta_H h_H \right)(X,Y) = -2H^s \cdot h_H(A_H(X),Y) + g_H \left( \left( \frac{\partial A_H}{\partial t} - \Delta_H A_H \right)(X),Y \right).
\]

From this relation, Lemmas 4.6 and 4.9, we can derive
\[
\frac{\partial A_H}{\partial t} - \Delta_H A_H = -2H^s \cdot h_H \left( (A_H)^2 - ((A^\phi)^2)_H \right) \cdot A_H - R^s.
\]
Furthermore, from this evolution equation and Lemma 4.10, we can derive
\[
\frac{\partial}{\partial t} \left( \frac{A_H}{H^s} \right) - \Delta_H \left( \frac{A_H}{H^s} \right) = \frac{1}{H^s} \nabla_{\text{grad} H^s} \left( \frac{A_H}{H^s} \right) + \left( \|\text{grad} H^s\|^3 \right) \cdot A_H - 2((A^\phi)^2)_H
\]
\[
+ \frac{2}{H^s} \cdot \text{tr}((A^\phi)^2)_H \cdot A_H - \frac{1}{H^s} R^s.
\]

For simplicity, we set
\[
S_H := g_H \left( \frac{1}{H^s} A_H(\bullet), \bullet \right)
\]
and
\[
P(S)_H := \frac{\|\text{grad} H^s\|^3}{(H^s)^3} \cdot h_H - 2((A^\phi)^2)_H \cdot h_H - 1
\]
\[
+ \frac{2}{H^s} \cdot \text{tr}((A^\phi)^2)_H \cdot h_H - \frac{1}{H^s} R,
\]
where \(((A^\phi)^2)_H)_b\) is defined by \(((A^\phi)^2)_H)_b(\bullet,\bullet) := g_H(((A^\phi)^2)_H(\bullet,\bullet)). Also, set
\[
\varepsilon_0 := \max\{\varepsilon > 0 \mid (S_H)_0 \geq \varepsilon g_H \}.
\]
Then, for any \((x,t) \in M \times [0,T)\), any \(\varepsilon > 0\) and any \(X \in \text{Ker}(S_H + \varepsilon g_H)_{(x,t)}\), we can show \(P(S_H + \varepsilon g_H)_{(x,t)}(X,X) \geq 0\). Hence, by the maximum principle (Theorem 5.1 of [Koi2]), we can derive that \((S_H)_t \geq \varepsilon_0 g_H\), that is, \((A_H)_t \geq \varepsilon_0 H^s g_H\) holds for all \(t \in [0,T)\). From this fact, it follows that \(\lambda_{\text{min}}(x,t) \geq \varepsilon_0 \|H(x,t)\|\) holds for all \((x,t) \in M \times [0,T)\). Hence we obtain
\[
\sup_{(x,t) \in \lambda_{\text{max}}(x,t) \leq \sup_{(x,t) \in (x,t) \in M \times [0,T)} \frac{\lambda_{\text{max}}(x,t)}{\varepsilon_0 \|H(x,t)\|} \leq \frac{1}{\varepsilon_0}.
\]
According to this lemma, we see that such a case as in Figure 8.1 does not happen.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure81.png}
\caption{The case where $\lim_{t \to T} \overline{M}_t$ is not a round point}
\end{figure}

By using Proposition 8.4 and Lemma 8.5, we shall prove the statement (i) of Theorem A.

Proof of (i) of Theorem A. According to Proposition 8.4 and Lemma 8.5, we have

$$\lim_{t \to T} \min_{x \in M} \lambda_{\min}(x, t) = \infty.$$ 

Set $\Lambda_{\min}(t) := \min_{x \in M} \lambda_{\min}(x, t)$. Let $x_{\min}(t)$ be a point of $\overline{M}$ with $\lambda_{\min}(x_{\min}(t), t) = \Lambda_{\min}(t)$ and set $\tilde{x}_{\min}(t) := \phi_{\overline{M}}(x_{\min}(t))$. Denote by $\gamma_{\overline{f}_t(\tilde{x}_{\min}(t))}$ the normal geodesic of $\overline{f}_t(\overline{M})$ starting from $\overline{f}_t(\tilde{x}_{\min}(t))$. Set $p_t := \gamma_{\overline{f}_t(\tilde{x}_{\min}(t))}(1/\Lambda_{\min}(t))$. Since $N$ is of non-negative curvature, the focal radii of $\overline{M}_t$ along any normal geodesic are smaller than or equal to $1/\Lambda_{\min}(t)$. This implies that $\overline{f}_t(\overline{M})$ is included by the geodesic sphere of radius $1/\Lambda_{\min}(t)$ centered at $p_t$ in $N$. Hence, since $\lim_{t \to T} 1/\Lambda_{\min}(t) = 0$, we see that, as $t \to T$, $\overline{M}_t$ collapses to a one-point set, that is, $M_t$ collapses to a $\mathcal{G}$-orbit.

Denote by $(\text{Ric}_{\overline{M}})_t$ the Ricci tensor of $\overline{g}_t$ and let $\text{Ric}_{\overline{M}}$ be the element of $\Gamma(\pi_{\overline{M}}^*(T(0,2)\overline{M}))$ defined by $(\text{Ric}_{\overline{M}})_t$'s. To show the statement (ii) of Theorem A, we prepare the following some lemmas.

**Lemma 8.6.** (i) For the section $\text{Ric}_{\overline{M}}$, the following relation holds:

\begin{equation}
(8.3) \quad \text{Ric}_{\overline{M}}(X, Y) = -3\text{Tr}(A_{X}^{\phi} \circ A_{Y}^{\phi})_{\mathcal{H}} - \overline{g}(A^2 X, Y) + ||\overline{H}|| \cdot \overline{g}(\overline{A} X, Y)
\end{equation}

$(X, Y \in \Gamma(\pi_{\overline{M}}^*(T\overline{M})))$, where $X^L$ (resp. $Y^L$) is the horizontal lift of $X$ (resp. $Y$) to $V$. 

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(ii) Let $\lambda_1$ be the smallest eigenvalue of $\overline{\mathbf{A}}(x,t)$. Then we have
\[
(8.4) \quad (\text{Ric}\overline{\mathbf{A}}(x,t))(v,v) \geq (n-1)\lambda_1^2 \overline{\mathbf{g}}(x,t)(v,v) \quad (v \in T_x\overline{\mathbf{M}}).
\]

Proof. Denote by $\overline{\text{Ric}}$ the Ricci tensor of $\overline{\mathbf{M}}$. By the Gauss equation, we have
\[
\overline{\text{Ric}}(X,Y) = \text{Ric}(X,Y) - \overline{\mathbf{g}}(\overline{\mathbf{A}}^2 X,Y) + ||\overline{\mathcal{P}}||\overline{\mathbf{g}}(\overline{\mathbf{A}}X,Y) - \overline{R}(\xi,X,Y,\xi)
\]
$$(X,Y \in T\overline{\mathbf{M}}).$$

Also, by a simple calculation, we have
\[
\overline{\text{Ric}}(X,Y) = -3\text{Tr}((\mathbf{A}_{\phi}^\phi \circ \mathbf{A}_{Y,L}^\phi)_{\mathcal{H}} + 3\mathbf{g}_{\mathcal{H}}((\mathbf{A}_{X,L}^\phi \circ \mathbf{A}_{Y,L}^\phi)(\xi), \xi)
\]
and
\[
\overline{R}(\xi,X,Y,\xi) = 3\mathbf{g}_{\mathcal{H}}((\mathbf{A}_{X,L}^\phi \circ \mathbf{A}_{Y,L}^\phi)(\xi), \xi)
\]
$$(X,Y \in \Gamma(\pi^\pm_\overline{\mathbf{M}}(T\overline{\mathbf{M}}))).$$ From these relations, we obtain the relation (8.3).

Next we show the inequality in the statement (ii). Since $\mathbf{A}_{\phi}^\phi$ is skew-symmetric, we have $\text{Tr}((\mathbf{A}_{\phi}^\phi)^2) \leq 0$. Also we have
\[
-\overline{\mathbf{g}}(x,t)(\overline{\mathbf{A}}^2(x,t))(v,v) + ||\overline{\mathcal{P}}(x,t)|| \cdot \overline{\mathbf{g}}(x,t)(\overline{\mathbf{A}}(x,t))(v,v) \geq (n-1)\lambda_1^2 \overline{\mathbf{g}}(x,t)(v,v).
\]

Hence, from the relation in (i), we can derive the inequality (8.4). \qed

By remarking the behavior of geodesic rays reaching the singular set of a compact Riemannian orbifold (see Figure 8.2) and using the discussion in the proof of Myers’s theorem ($[M]$), we can show the following Myers-type theorem for Riemannian orbifolds.

**Theorem 8.7.** Let $(N,g)$ be an $n$-dimensional compact (connected) Riemannian orbifold. If its Ricci curvature $\text{Ric}$ of $(N,g)$ satisfies $\text{Ric} \geq (n-1)K$ for some positive constant $K$, then the first conjugate radius along any geodesic in $(N,g)$ is smaller than or equal to $\frac{\pi}{\sqrt{K}}$ and hence so is also the diameter of $(N,g)$.

By using Propositions 7.1, 8.4, Lemmas 8.6 and Theorem 8.7, we prove the statement (ii) of Theorem A.

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The case where $g|_{U_i}$ is a flat metric

**Figure 8.2:** The behavior of geodesic rays around the singular set

**Proof of (ii) of Theorem A.** (Step I) According to Proposition 7.1, for any positive constant $b$, there exists a constant $C(b, f_0)$ (depending only on $b$ and $f_0$) satisfying

$$||\text{grad } H_s||^2 \leq b \cdot (H_s)^4 + C(b, f_0) \text{ on } M \times [0, T).$$

According to Proposition 8.4, we have

$$\lim_{t \to T}(H_s^t)_{\text{max}} = \infty.$$ 

Hence there exists a positive constant $t(b)$ with $(H_s^t)_{\text{max}} \geq \left( \frac{C(b, f_0)}{b} \right)^{1/4}$ for any $t \in [t(b), T)$. Then we have

$$||\text{grad } H_s^t|| \leq \sqrt{2b(H_s^t)_{\text{max}}}$$

for any $t \in [t(b), T)$. Fix $t_0 \in [t(b), T)$. Let $x_{t_0}$ be a maximal point of $||H_{t_0}||$. Take any geodesic $\gamma$ of length $\frac{1}{\sqrt{2||H_{t_0}||_{\text{max}}^b}^{1/4}}$ starting from $x_{t_0}$. According to (8.5), we
have
\[ ||H_{t_0}|| \geq (1 - b^{1/4})||H_{t_0}||_{\max} \]
along \( \gamma \). From the arbitrariness of \( t_0 \), this fact holds for any \( t \in [t(b), T) \).

(Step II) For any \( x \in \mathbb{M} \), denote by \( \gamma_{T_t(x)} \) the normal geodesic of \( \mathcal{T}_t(M) \) starting from \( \mathcal{T}_t(x) \). Set \( p_t := \gamma_{T_t(x)}(1/\lambda_{\min}(x,t)) \) and \( q_t(s) := \gamma_{T_t(x)}(s/\lambda_{\max}(x,t)) \). Since \( N \) is of non-negative curvature, the focal radii of \( \mathcal{T}_t(M) \) at \( x \) are smaller than or equal to \( 1/\lambda_{\min}(x,t) \). Denote by \( G_2(TN) \) the Grassmann bundle of \( N \) of 2-planes and \( \text{Sec} : G_2(TN) \to \mathbb{R} \) the function defined by assigning the sectional curvature of \( \Pi \) to each element \( \Pi \) of \( G_2(TN) \). Since \( \bigcup_{t \in [0,T)} \mathcal{T}_t(M) \) is compact, there exists the maximum of \( \text{Sec} \) over \( \bigcup_{t \in [0,T)} \mathcal{T}_t(M) \). Denote by \( \kappa_{\max} \) this maximum. It is easy to show that the focal radii of \( \mathcal{T}_t(M) \) at \( x \) are bigger than or equal to \( \hat{c}/\lambda_{\max}(x,t) \) for some positive constant \( \hat{c} \) depending only on \( \kappa_{\max} \). Hence a sufficiently small neighborhood of \( \mathcal{T}_t(x) \) in \( \mathcal{T}_t(M) \) is included by the closed domain surrounded by the geodesic spheres of radius \( 1/\lambda_{\min}(x,t) \) centered at \( p_t \) and that of radius \( \hat{c}/\lambda_{\max}(x,t) \) centered at \( q_t(\hat{c}) \). On the other hand, according to Lemma 8.5, we have
\[
\sup_{(x,t) \in \mathbb{M} \times [0,T)} \lambda_{\max}(x,t) \lambda_{\min}(x,t) < \infty.
\]
By using these facts, we can show
\[
\sup_{t \in [0,T)} \frac{(H^s_t)_{\max}}{(H^s_t)_{\min}} < \infty
\]
and
\[
\inf_{t \in [0,T)} \max \{ \varepsilon > 0 \mid (A_{\mathcal{T}})_t \geq \varepsilon H^s_t \cdot \text{id on } M \} > 0.
\]
Set
\[
C_0 := \sup_{t \in [0,T)} \frac{(H^s_t)_{\max}}{(H^s_t)_{\min}}
\]
and
\[
\varepsilon_0 := \inf_{t \in [0,T)} \max \{ \varepsilon > 0 \mid (A_{\mathcal{T}})_t \geq \varepsilon H^s_t \cdot \text{id on } M \}.
\]
Then, since \( A_{\mathcal{T}} \geq \varepsilon_0 H^s_{\min} \cdot \text{id on } M \times [0,T) \), it follows from (ii) of Lemma 8.6 that
\[
(Ric_{\mathcal{T}})(x,t)(v,v) \geq (n - 1)\varepsilon_0^2 \cdot (H^s_t)_{\min}^2 \cdot g(x,t)(v,v)
\]
for any \((x, t) \in M \times [0, T]\) and any \(v \in T_x \bar{M}\). Hence, according to Theorem 8.7, the first conjugate radius along any geodesic \(\gamma\) in \((\bar{M}, \bar{g}_t)\) is smaller than or equal to \(\frac{\pi}{\varepsilon_0 (H^s_t)_{\min}}\) for any \(t \in [0, T)\). This implies that

\[
\exp_{\bar{M}}(x) \left( B_{\bar{M}} \left( \frac{\pi}{\varepsilon_0 (H^s_t)_{\min}} \right) \right) = \bar{M}
\]

holds for any \(t \in [0, T)\), where \(\exp_{\bar{M}}(x)\) denotes the exponential map of \((\bar{M}, \bar{g}_t)\) at \(\bar{f}_t(x)\) and \(B_{\bar{M}} \left( \frac{\pi}{\varepsilon_0 (H^s_t)_{\min}} \right)\) denotes the closed ball of radius \(\frac{\pi}{\varepsilon_0 (H^s_t)_{\min}}\) in \(T_{\bar{f}_t(x)} \bar{M}\) centered at the zero vector \(0\). By the arbitrariness of \(b\) (in (Step I)), we may assume that \(b \leq \frac{\varepsilon_0}{4\pi^2 C_0}\). Then we have

\[
\frac{1}{\sqrt{2} (H^s_t)_{\max} \cdot b^{1/4}} \geq \frac{\pi}{\varepsilon_0 (H^s_t)_{\min}} \quad (t \in [0, T)).
\]

Let \(t_0\) be as in Step I. Then it follows from the above facts that

\[ ||H_{t_0}|| \geq (1 - b^{1/4}) ||H_{t_0}||_{\max} \]

holds on \(\bar{M}\). From the arbitrariness of \(t_0\), it follows that

\[ H^s \geq (1 - b^{1/4}) H^s_{\max} \]

holds on \(\bar{M} \times [t(b), T)\). In particular, we obtain

\[
\frac{H^s_{\max}}{H^s_{\min}} \leq \frac{1}{1 - b^{1/4}} \quad (t \in [0, T))
\]

on \(\bar{M} \times [t(b), T)\). Therefore, by approaching \(b\) to 0, we can derive

\[
\lim_{t \to T} \frac{(H^s_t)_{\max}}{(H^s_t)_{\min}} = 1.
\]

\[\square\]

9 Towards application to the Guage theory

In this section, we shall state the vision for applying the study of regularized mean curvature flows to the Gauge theory. In the future, we plan to find interesting
Riemannian submanifolds and interesting flows (of Riemannian submanifolds) in the Yang-Milles moduli space or the self-dual moduli space. To state the strategy of this plan, we first recall some basic notions in the theory of the connections of the principal bundles. Let \( \pi : P \to B \) be a principal bundle over a compact manifold \( B \) having a compact semi-simple Lie group \( G \) as the structure group. Fix an \( \text{Ad}(G) \)-invariant inner product \( \langle , \rangle_\theta \) (for example, the \((-1)\)-multiple of the Killing form) of the Lie algebra \( \mathfrak{g} \) of \( G \), where \( \text{Ad} \) denotes the adjoint representation of \( G \). Denote by \( g_G \) the bi-invariant metric of \( G \) induced from \( \langle , \rangle_\theta \).

Set

\[
\Omega^\infty_{T^1}(P, g) := \{ \hat{A} \in \Omega^\infty_1(P, g) \mid R^*_{g} \hat{A} = \text{Ad}(g^{-1}) \circ \hat{A} \ (\forall \ g \in G) \ \& \ \hat{A}|_\mathcal{V} = 0 \},
\]

where \( \mathcal{V} \) denotes the vertical distribution of the bundle \( P \). Each element of \( \Omega^\infty_{T^1}(P, g) \) is called a \( \mathfrak{g} \)-valued tensorial 1-form of class \( C^\infty \) on \( P \). Also, let \( \Omega^\infty_1(B, \text{Ad}(P)) \) (\( = \Gamma^\infty(T^*B \otimes \text{Ad}(P)) \)) be the space of all \( \text{Ad}(P) \)-valued 1-forms of class \( C^\infty \) over \( B \), where \( \text{Ad}(P) \) denotes the adjoint bundle \( P \times \text{Ad} \mathfrak{g} \). The space \( \mathcal{A}^\infty_\mathcal{V} \) is the affine space having \( \Omega^\infty_{T^1}(P, g) \) as the associated vector space. Furthermore, \( \Omega^\infty_{T^1}(P, g) \) is identified with \( \Omega^\infty_1(B, \text{Ad}(P)) \) under the correspondence \( \hat{A} \leftrightarrow A \) defined by \( u \cdot \hat{A}_u(X) = \hat{A}_{\pi(u)}(\pi_x X) \ (u \in P, \ X \in T_u P) \).

Denote by \( \mathcal{A}^\infty_{P^s} \) the space of all \( s \)-times weak differentiable connections of \( P \) and \( \Omega^\infty_{P^s}(B, \text{Ad}(P)) \) the space of all \( s \)-times weak differentiable \( \text{Ad}(P) \)-valued \( i \)-form on \( P \). Fix a \( C^\infty \)-connection \( \omega_0 \) of \( P \). Define an operator \( \Box_{\omega_0} : \Omega^\infty_{P^s}(B, \text{Ad}(P)) \to \Omega^\infty_{P^{s-2}}(B, \text{Ad}(P)) \) by

\[
\Box_{\omega_0} := \left\{ \begin{array}{ll}
  d_{\omega_0} \circ d_{\omega_0}^* + d_{\omega_0}^* \circ d_{\omega_0} + id & (i \geq 1) \\
  d_{\omega_0}^* \circ d_{\omega_0} + id & (i = 0) 
\end{array} \right.
\]

where \( d_{\omega_0} \) denotes the covariant exterior derivative with respect to \( \omega_0 \) and \( d_{\omega_0}^* \) denotes the adjoint operator of \( d_{\omega_0} \) with respect to the \( L^0 \)-inner products of \( \Omega^\infty_{P^j}(B, \text{Ad}(P)) \) \( (j \geq 0) \). The \( H^s \)-inner product \( \langle , \rangle_{s}^\omega \) of \( T_{\omega} \mathcal{A}^\infty_{P^s}(\approx \Omega^\infty_{P^s}(B, \text{Ad}(P)) \approx \Gamma^\infty_1(T^*B \otimes \text{Ad}(P))) \) is defined by

\[
\langle A_1, A_2 \rangle_{s}^\omega := \int_{x \in B} \langle (A_1)_x, (\Box_{\omega_0}^s(A_2))_x \rangle_{B, g} \ dv_B \\
(A_1, A_2 \in \Omega^\infty_{P^s}(B, \text{Ad}(P))),
\]

where \( \Box_{\omega_0}^s(A_2) \) denotes the element of \( \Omega^\infty_{P^0}(B, \text{Ad}(P)) \) corresponding to \( \Box_{\omega_0}^s(\hat{A}_2) \), \( \langle , \rangle_{B, g} \) denotes the fibre metric of \( T^*B \otimes \text{Ad}(P) \) defined by the the Riemannian metric of \( B \) and \( \langle , \rangle_\theta \) and \( dv_B \) denotes the volume element of the Riemannian metric of \( B \). Let \( \Omega^H_1(B, \text{Ad}(P)) \) be the completion of \( \Omega^\infty_1(B, \text{Ad}(P)) \) with respect to \( \langle , \rangle_{s}^\omega \), that is,

\[
\Omega^H_1(B, \text{Ad}(P)) := \{ A \in \Omega^\infty_{P^s}(B, \text{Ad}(P)) \mid \langle A, A \rangle_{s}^\omega < \infty \}
\]

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and $A^H_P$ the completion of $A^\infty_P$ with respect to $\langle \cdot, \cdot \rangle^\omega_0$, that is,

$$A^H_P := \{\omega_0 + A \mid A \in \Omega^H_1(B, \text{Ad}(P))\}.$$ 

Let $\Omega^H_{1,1}(P, g)$ be the completion of $\Omega^\infty_{1,1}(P, g)$ corresponding to $\Omega^H_1(B, \text{Ad}(P))$.

Let $\hat{\mathcal{G}}^\infty_P$ be the group of all $C^\infty$-gauge transformations $g$'s of $P$ with $\pi \circ g = \pi$. For each $g \in \hat{\mathcal{G}}^\infty_P$, $\hat{g} \in C^\infty(P, G)$ is defined by $g(u) = u\hat{g}(u)$ ($u \in P$). This element $\hat{g}$ satisfies

$$\hat{g}(ug) = \text{Ad}(g^{-1})(\hat{g}(u)) \quad (\forall u \in P, \forall g \in G),$$

where Ad denotes the homomorphism of $G$ to Aut$(G)$ defined by $\text{Ad}(g_1)(g_2) := g_1 \cdot g_2 \cdot g_1^{-1}$ ($g_1, g_2 \in G$). Under the correspondence $g \leftrightarrow \hat{g}$, $\mathcal{G}^\infty_P$ is identified with

$$\hat{\mathcal{G}}^\infty_P := \{\hat{g} \in C^\infty(P, G) \mid \hat{g}(ug) = \text{Ad}(g^{-1})(\hat{g}(u)) \quad (\forall u \in P, \forall g \in G)\}.$$ 

For $\hat{g} \in \hat{\mathcal{G}}^\infty_P$, the $C^\infty$-section $\hat{g}$ of the associated $G$-bundle $P \times_{\text{Ad}} G$ is defined by $\hat{g}(x) := u \cdot \hat{g}(u)$ ($x \in M$), where $u$ is any element of $\pi^{-1}(x)$. Under the correspondence $\hat{g} \leftrightarrow \hat{g}$, $\hat{\mathcal{G}}^\infty_P$ is isometrically identified with the space $\Gamma^\infty(P \times_{\text{Ad}} G)$ of all $C^\infty$-sections of $P \times_{\text{Ad}} G$. The $H^{s+1}$-completion of $\Gamma^\infty(P \times_{\text{Ad}} G)$ was defined by Groisser and Parker (see Section 1 (P668) of [GPI]). Denote by $\Gamma^{H^{s+1}}_P(P \times_{\text{Ad}} G)$ this completion. Also, denote by $\mathcal{G}^{H^{s+1}}_P$ (resp. $\hat{\mathcal{G}}^{H^{s+1}}_P$) the completion of $\mathcal{G}^\infty_P$ (resp. $\hat{\mathcal{G}}^\infty_P$) corresponding to $\Gamma^{H^{s+1}}_P(P \times_{\text{Ad}} G)$. Assume that $s > \frac{1}{2}$dim $M - 1$. Then, according to Lemma 1.2 of [U], this $H^{s+1}$-completion $\mathcal{G}^{H^{s+1}}_P$ is a $C^\infty$-Hilbert Lie group and the gauge action $\mathcal{G}^{H^{s+1}}_P \curvearrowright A^H_P$ is of class $C^\infty$. However, this action does not act isometrically on the Hilbert space $(A^H_P, \langle \cdot, \cdot \rangle^\omega_0)$. Define a Riemannian metric $g_s$ on $A^H_P$ by

$$(g_s)_\omega := \langle \cdot, \cdot \rangle^\omega_0 \quad (\omega \in A^H_P).$$

This Riemannian metric $g_s$ is non-flat and translation-invariant. The gauge action $\mathcal{G}^{H^{s+1}}_P \curvearrowright A^H_P$ acts isometrically on the Riemannian Hilbert manifold $(A^H_P, g_s)$. Note that the Hilbert space $(A^H_P, \langle \cdot, \cdot \rangle^\omega_0)$ is regarded as the tangent space of $(A^H_P, g_s)$ at $\omega_0$. Give the moduli space $M^H_P := A^H_P / \mathcal{G}^{H^{s+1}}_P$ the Riemannian orbimetric $\mathcal{F}$, such that the orbit map $\pi_{\mathcal{M}} : (A^H_P, g_s) \to (M^H_P, \mathcal{F})$ is a Riemannian orbisubmersion.

$$\mathcal{G}^{H^{s+1}}_P \curvearrowright A^H_P \curvearrowright \mathcal{M}^H_P \curvearrowright \mathcal{F}$$
In particular, denote by $\Lambda^H$ is of one-dimension. For closed subgroup $H$ is identified with bundle $G$. Example 9.1. Let $\phi: H^s([0,1],G)$ be a compact semi-simple Lie group and consider the trivial $G$-bundle $\Pi_o := S^1 \times G$ over $S^1$ by $\sigma$. Define an immersion $\iota_c$ of the induced bundle $c^*P$ into $P$ by $\iota_c(z(t),u) = u$ $((s(t),u) \in c^*P)$.  

**Definition 9.1.** Define a map $\text{hol}_c: A^H_{P} \to G$ by

$$\left(\mathcal{P}^{\omega}_{\sigma} \circ (\mathcal{P}^{\omega}_{\sigma})^{-1}\right)(u_0) = u_0 \cdot \text{hol}_c(\omega),$$

where $\mathcal{P}^{\omega}_{\sigma}$ (resp. $\mathcal{P}^{\omega}_{\sigma}$) denotes the parallel translation along $c \circ z$ with respect to $\omega$ (resp. $\omega_0$). We call this map $\text{hol}_c$ the holonomy map along $c$.

In particular, in the case where $P$ is the trivial $G$-bundle $\pi^o: S^1 \times G \to S^1$, $A^H_{P}$ is identified with the Hilbert space $H^s([0,1],g)$ of all $H^s$-curves in the Lie algebra $g$ of $G$ and the holonomy map $\text{hol}_c$ along $c(t) = t$ $((t \in [0,1])$ coincides with the parallel transport map $\phi: H^s([0,1],g) \to G$ for $G$ stated in Example 4.1, where $s$ may be any non-negative integer because $[0,1]$ is of one-dimension.

The based gauge group $(G^H_{P})_x$ at $x \in M$ is defined by

$$(G^H_{P})_x := \{ g \in G^H_{P} \mid e(g(\pi^{-1}(x))) = \{ e \},$$

where $e$ denotes the identity element of $G$. For a finite subgroup $\Gamma$ of $G$ and $u \in P$, we define a closed subgroup $(G^H_{P})_{u,\Gamma}$ by

$$(G^H_{P})_{u,\Gamma} := \{ g \in G^H_{P} \mid g(u) \in \Gamma \},$$

where $\tilde{g}$ is the element of $H^s+1(P,G)$ corresponding to $g$.

**Example 9.1.** Let $G$ be a compact semi-simple Lie group and consider the trivial $G$-bundle $P_o := S^1 \times G$ over $S^1$. Also let $s$ be any non-negative integer. Since $Ad(P_o)$ is identified with the trivial $g$-bundle $P_o := S^1 \times g$ over $S^1$, $A^{H_s}(\simeq \Omega^H_1([0,1],Ad(P_o)))$ is identified with $H^s([0,1],g)$. Also, $G^{H_s}_{P_o}$ is identified with the Hilbert Lie group $H^{s+1}([0,1],G)$ of all $H^s$-paths in $G$. Here we note that $H^{s+1}([0,1],G)$ is a $C^\infty$-Hilbert Lie group and $H^{s+1}([0,1],G) \simeq H^s([0,1],\mathbb{C})$ is a $C^\infty$-action because $[0,1]$ is of one-dimension. For closed subgroup $H$ of $G \times G$, we define a closed subgroup $P(G, H)$ of $H^s+1([0,1],G)$ by

$$P(G, H) := \{ g \in H^{s+1}([0,1],G) \mid (g(0),g(a)) \in H \}.$$
such that \((G, K)\) is a reductive pair, that is, there exists a subspace \(p\) of \(g\) satisfying \(g = \mathfrak{t} \oplus p\) and \([\mathfrak{t}, p] \subset p\), where \(g\) and \(\mathfrak{t}\) denote the Lie algebras of \(G\) and \(K\), respectively. Let \(B := A^H_{P_0}\) and \(G_1 := P(G, \Gamma \times K)\). Then the pair \((B, G_1)\) satisfies the conditions (I) and (II) stated in Introduction, where we note that the moduli space \(B/G_1\) is orbi-diffeomorphic to the orbifold \(\Gamma \setminus G / K\).

**Assumption.** Assume that \(s \geq \frac{1}{2} \dim B - 1\).

Denote by \(C^\infty([0, 1], M)\) the space of all \(C^\infty\)-paths in \(M\). Take \(c \in C^\infty([0, 1], M)\). Denote by \(\pi^c : c^*P \rightarrow [0, 1]\) the induced bundle of \(P\) by \(c\), which is isomorphic to the trivial \(G\)-bundle \(P_0([0, 1] \times G) \rightarrow [0, 1]\). Define an immersion \(\iota_c\) of the induced bundle \(c^*P\) into \(P\) by \(\iota_c(t, u) = u \ ((t, u) \in c^*P)\). Fix a base point \(\omega_0\) of \(A^P_{\infty}\). We shall define a map linking \(A^H_{P_0}\) to \(H^0([0, 1], g)\).

Let \(x_0, c, u_0\) be as above and \(\sigma\) the horizontal lift of \(c \circ z\) starting from \(u_0\) with respect to \(\omega_0\).

Then the pull-back bundle \(c^*P\) of \(P\) by \(c\) is identified with the trivial \(G\)-bundle \(S^1 \times G\) over \(S^1\) by \(\sigma\) as follows. Define a map \(\eta : S^1 \times G \rightarrow c^*P\) by

\[
\eta(z(t), g) := (z(t), \sigma(t)g) \quad \text{for} \quad ((t, g) \in [0, 1] \times G).
\]

It is clear that \(\eta\) is a bundle isomorphism. Throughout this bundle isomorphism \(\eta\), \(c^*P\) is identified with \(S^1 \times G\). Similarly, a bundle isomorphism \(\bar{\eta} : [0, 1] \times G \rightarrow (c \circ z)^*P\) is defined by

\[
\bar{\eta}(t, g) := (t, \sigma(t)g) \quad \text{for} \quad ((t, g) \in [0, 1] \times G).
\]

Throughout this bundle isomorphism \(\bar{\eta}\), \((c \circ z)^*P\) is identified with the trivial bundle \([0, 1] \times G\). The natural embedding \(\iota_\sigma : [0, 1] \times G \hookrightarrow P\) by

\[
\iota_\sigma(t, g) := \sigma(t)g \quad \text{for} \quad ((t, g) \in [0, 1] \times G).
\]

For \(\omega \in A^H_{P_0}\), the pull-back connection \(\iota_\sigma^*\omega\) of \((c \circ z)^*P\) is defined. Then we have

\[
(i_\sigma^*\omega)(t_0, g) \left( \frac{\partial}{\partial t} \right)_{(t_0, g)} := \omega_\sigma(t_0)(\sigma'(t_0)) \quad \text{for} \quad ((t, g) \in [0, 1] \times G).
\]

By the one-to-one correspondence

\[
i_\sigma^*\omega \quad \longleftrightarrow \quad t \mapsto \omega_\sigma(t)(\sigma'(t)) \quad \text{for} \quad (t \in [0, 1]),
\]

\(A^H_{(c \circ z)^*P}\) is identified with the Hilbert space \(H^s([0, 1], g)\) of all \(H^s\)-paths in \(g\).
Definition 9.2. Define a map \( \mu_c : \mathcal{A}^H_p \to H^s([0,1],g) \) by
\[
(\mu_c(\omega))(t) := -\omega_{\sigma(t)}(\sigma'(t)) \quad (t \in [0,1], \ \omega \in \mathcal{A}^H_p).
\]

As above, \( \omega_{\sigma(z(t_0))}(\sigma'(t_0)) \) is equal to \( (\sigma^* \omega)_{(t_0,g)} \left( \left( \frac{\partial}{\partial t} \right)_{(t_0,g)} \right) \). From this fact, we call this map \( \mu_c \) the pull-back connection map by \( c \). By the definitions of hol\(_c\) and \( \mu_c \), we can show the following relation.

Lemma 9.1. Among \( \text{hol}_c, \phi \) and \( \mu_c \), the relation \( \text{hol}_c = \phi \circ \mu_c \) holds.

Definition 9.3. Define a map \( \lambda_c : \mathcal{G}^{H^{s+1}}_p \to H^{s+1}([0,1],G) \) by
\[
\lambda_c(g)(t) := \hat{g}(\sigma(t)) \quad (t \in [0,1], \ g \in \mathcal{G}^{H^{s+1}}_p).
\]

Take a \( C^\infty \)-loop \( c : [0,1] \to M \) and \( \bar{c} \) the horizontal lift of \( c \) with respect to \( \omega_0 \). Easily we can show the following fact from the definitions of \( \mu_c \) and \( \lambda_c \).

Proposition 9.2. (i) Between \( \mu_c \) and \( \lambda_c \), the following relation holds:
\[
\mu_c(g \cdot \omega) = \lambda_c(g) \cdot \mu_c(\omega) \quad (g \in \mathcal{G}^{H^{s+1}}_p, \ \omega \in \mathcal{A}^H_p).
\]

(ii) \( \mu_c \) maps \( (\mathcal{G}^{H^{s+1}}_p)_{\bar{c}(0),\Gamma} \)-orbits in \( \mathcal{A}^H_p \) to \( P(G,\Gamma_{\omega_0}) \)-orbits in \( H^0([0,1],g) \) and hence there uniquely exists the map \( \overline{\pi}^\Gamma_c \) between the orbit spaces \( \mathcal{A}^H_p / (\mathcal{G}^{H^{s+1}}_p)_{\bar{c}(0),\Gamma} \) and \( H^0([0,1],g) / P(G,\Gamma_{\omega_0})(= G / (R_{\text{hol}_c(\omega_0)} \circ \text{Ad}(\Gamma) \circ R_{\text{hol}_c(\omega_0)}^{-1})) \) satisfying
\[
\overline{\pi}^\Gamma_c \circ \pi_{(\mathcal{G}^{H^{s+1}}_p)_{\bar{c}(0),\Gamma}} = \pi_{\Gamma_{\omega_0}} \circ \phi \circ \mu_c = \pi_{\Gamma} \circ \text{hol}_c,
\]
where \( \pi_{\Gamma} \) (resp. \( \pi_{(\mathcal{G}^{H^{s+1}}_p)_{\bar{c}(0),\Gamma}} \)) denotes the orbit map of the action \( \text{Ad}(\Gamma) \curvearrowright G \) (resp. \( (\mathcal{G}^{H^{s+1}}_p)_{\bar{c}(0),\Gamma} \curvearrowright \mathcal{A}^H_p \)).

Thus the study of \( P(G,\Gamma)-\)orbits in \( H^1([0,1],g) \) leads to that of \( (\mathcal{G}^{H^{s+1}}_p)_{\bar{c}(0),\Gamma} \)-orbits \( (u \in P) \) in \( \mathcal{A}^H_p \) through \( \mu_c \) (see Figure 9.1).
Remark 9.2. Since \( \mu_c \) is a bounded linear map of \((A^H_s, \langle , \rangle^\omega_s)\) (regarded as a Hilbert space) onto \( H^0([0, 1], g) \), the following facts hold:

(i) The inverse images \( \mu_c^{-1}(u) \)'s \((u \in H^0([0, 1], g))\) are closed affine subspaces of the affine space \( A^H_s \) and they are parallel. Hence, for any a \( C^\infty \)-orbsubmanifold \( S \) in \( H^0([0, 1], g)/P(G, \Gamma) \), \((\pi \circ \text{hol}_c)^{-1}(S)\) is a cylindrical \( C^\infty \)-orbsubmanifold of finite codimension in the Hilbert space \((A^H_s, \langle , \rangle^\omega_s)\);

(ii) The map \( \mu_c \) is a \( C^\infty \)-submersion as a map between the Riemannian Hilbert manifolds, where we regard \((A^H_s, \langle , \rangle^\omega_s)\) and \( H^0([0, 1], g) \) as Riemannian Hilbert manifolds.

Also, it is easy to show that the following facts holds:

(iii) The operator norm \( \| (d\mu_c)_\omega \|_{\text{ip}} \) of the differential \( (d\mu_c)_\omega \) of \( \mu_c \) at any point \( \omega \) is smaller than or equal to one.

Denote by \( \pi^H_{A^H_s} \) the orbit map of the action \( G_{H^s}^{H^{s+1}} \rtimes A^H_s \). Also, let \( \exp_{\omega_0} \) be the exponential map of the Riemannian Hilbert manifold \((A^H_s, g_s)\) at \( \omega_0 \). Then, for an orbsubmanifold \( S \) in \( H^0([0, 1], g)/P(G, \Gamma^{\omega_0}) \), we can construct an orbsubmanifolds \( \pi^H_{A^H_s}(\exp_{\omega_0}((\pi \circ \text{hol}_c)^{-1}(S))) \) in the moduli space \((M^H_s, \tilde{\pi})\) (see Figure 9.2).

Let \( \mathcal{M}_P^{H^s} \) be the Hilbert space of all Yang-Mills \( H^s \)-connections of the \( G \)-bundle \( \pi : P \rightarrow M \). Also, in the case of \( \dim M = 4 \), denote by \( SD_{H^s}^P \) the Hilbert space of all self-dual \( H^2 \)-connections of the \( G \)-bundle \( \pi : P \rightarrow M \), where we note that \( s \geq 2 \) in this case. Note that a self-dual connection means “instanton” because \( M \) is compact. The Yang-Mills moduli space \( \mathcal{M}_P^{H^s} := \mathcal{M}_P^{H^s}/G_{H^s}^{H^{s+1}} \) and the self-dual moduli space \( \mathcal{M}_{SD}^P := SD_{H^s}^P/G_{H^s}^{H^{s+1}} \) are finite dimensional manifolds with singularity in general. Give these moduli spaces the singular Riemannian metrics induced naturally from the (non-flat) Riemannian metric \( g_s \) on \( A^H_s \). Denote by \( \tilde{\pi} \) these singular Riemannian metrics.
Figure 9.2: Submanifolds in the moduli space defined by $\text{hol}_c$.

**Strategy**

(i) We plan to find a pair $(S, c)$ of an orbisubmanifold $S$ in the Riemannian orbifold $H^0([0, 1], g)/P(G, \Gamma)$ and a $C^\infty$-loop $c$ in $M$ such that

$$\pi^{H^c}_{\mathcal{M}_p} \left( \exp_\omega \left( (\pi_\Gamma \circ \text{hol}_c)^{-1}(S) \right) \right) \capYM_{H^c}$$

(resp. $\pi^{H^c}_{\mathcal{M}_p} \left( \exp_\omega \left( (\pi_\Gamma \circ \text{hol}_c)^{-1}(S) \right) \right) \cap SD_{H^c}$)

gives an interesting submanifold in $YM_{H^c}$ (resp. $SD_{H^c}$).

(ii) We plan to find a pair $\{S_t\}_{t \in [0, T]}$ of a mean curvature flow $\{S_t\}_{t \in [0, T]}$ in the Riemannian orbifold $H^0([0, 1], g)/P(G, \Gamma)$ and a $C^\infty$-loop $c$ in $M$ such that

$$\{\pi^{H^c}_{\mathcal{M}_p} \left( \exp_\omega \left( (\pi_\Gamma \circ \text{hol}_c)^{-1}(S_t) \right) \right) \capYM_{H^c}\}_{t \in [0, T]}$$

(resp. $\{\pi^{H^c}_{\mathcal{M}_p} \left( \exp_\omega \left( (\pi_\Gamma \circ \text{hol}_c)^{-1}(S_t) \right) \right) \cap SD_{H^c}\}_{t \in [0, T]}$)

gives an interesting flow in $YM_{H^c}$ (resp. $SD_{H^c}$) (for example, a good flow collapsing to a singular point of $YM_{H^c}$ (resp. $SD_{H^c}$)). We will use Theorem A to find a good flow collapsing to a singular point of $YM_{H^c}$ or $SD_{H^c}$.

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