Self-consistent theory of transport in superconducting wires

J. Sánchez-Cañizares and F. Sols

Departamento de Física Teórica de la Materia Condensada, C-V, and
Instituto Universitario de Ciencia de Materiales “Nicolás Cabrera”
Universidad Autónoma de Madrid, E-28049 Madrid, Spain

Abstract

We study superconducting transport in homogeneous wires in the cases of both equilibrium and nonequilibrium quasiparticle populations, using the quasiclassical Green’s function technique. We consider superconductors with arbitrary current densities and impurity concentrations ranging from the clean to the dirty limit. Local current conservation is guaranteed by ensuring that the order parameter satisfies the self-consistency equation at each point. For equilibrium transport, we compute the current, the order parameter amplitude, and the quasiparticle density of states as a function of the superfluid velocity, temperature, and disorder strength. Nonequilibrium is characterized by incoming quasiparticles with different chemical potentials at each end of the superconductor. We calculate the profiles of the electrostatic potential, order parameter, and effective quasiparticle gap. We find that a transport regime of current-induced gapless superconductivity can be achieved in clean superconductors, the stability of this state being enhanced by nonequilibrium. PACS numbers: 74.25.-q, 74.40.+k, 74.50.+r, 74.80.Fp

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I. INTRODUCTION

The quasiclassical theory of superconductivity was proposed by Eilenberger and Larkin and Ovchinnikov, and later developed by Usadel and Eliashberg. It is especially useful to study transport in systems with characteristic length scales much greater than the Fermi wave length \( \lambda_F \), which is effectively integrated out of the problem. In the seventies and eighties, the theory of nonequilibrium superconductivity was developed, and the response of the quasiparticle distribution function to external perturbations was investigated. Theoretical developments in Refs. were based on the quasiclassical Green's function (QCGF) technique. The calculation became possible of characteristic relaxation times due to phonons and paramagnetic impurities when an excess of quasiparticles is injected into the superconductor. In most of the cases, temperatures near the critical temperature and disorder strengths in the dirty limit were assumed, although the theory is potentially valid for all ranges of disorder and temperature. The effect of a moving condensate with a finite superfluid velocity \( v_s \) was studied in Refs. In particular, the quasiparticle relaxation time for energies between \( \Delta_- \) and \( \Delta_+ \), where \( \Delta_{\pm} \equiv |\Delta| \pm \hbar v_s p_F \), was calculated (\( \Delta_{\pm} \) are the direction-dependent effective quasiparticle energy gaps; \( \Delta \) is the conventional gap function playing the role of superconducting order parameter). However, the influence of \( v_s \) on \( |\Delta| \) through the self-consistency condition was not analysed.

During the nineties, a strong interest has developed on the physics of elastic, low current transport. Part of this work has been based on the resolution of the microscopic Bogoliubov – de Gennes (BdG) equations, whose equivalence with QCGF was proved in the clean limit by Beyer et al. The QCGF technique has allowed theorists to explain most of the experiments involving normal-superconductor (NS) interfaces. By permitting the study of spectral and spatial properties with inclusion of impurity averaged disorder, QCGF have become a most adequate tool to understand phenomena such as zero bias anomalies and re-entrance of the conductance, including its non-monotonic behavior as a function of temperature, voltage bias, or phase difference between external superconductors. These
effects are now understood as an interplay between proximity effect and disorder.\textsuperscript{18,19}

The study of equilibrium superconducting transport for arbitrary currents was already initiated in the sixties.\textsuperscript{20–22} Rogers\textsuperscript{20} and Bardeen\textsuperscript{21} developed a thermodynamic theory, and Maki\textsuperscript{22} gave a unified description of the clean and dirty limits using a Green’s function method. More recently, Bagwell\textsuperscript{23} has performed a similar study for clean superconductors within the BdG framework. The close connection between the implementation of self-consistency in the order parameter equation\textsuperscript{12} and current conservation has been noted by several authors.\textsuperscript{23–25} The implication is that a self-consistent description is essential in transport scenarios involving large supercurrents or superconductor lengths.\textsuperscript{25}

The combined effect of a moving condensate and a nonequilibrium distribution of quasiparticles has been addressed recently within a self-consistent scheme.\textsuperscript{26–31} These authors have solved the BdG equations for one-dimensional models of ballistic transport and have predicted physical features such as the existence of an Andreev-transmission dominated transport regime,\textsuperscript{26,28,29} the enhancement of excess current due to a finite $v_s$,\textsuperscript{29,31} and the possibility of current-induced gapless superconductivity (GS).\textsuperscript{26,27,29,30}

The primary purpose of this paper is to study the robustness of these effects against the inclusion of realistic physical factors such as scattering by impurities and the presence of many transverse channels. Specifically, we employ the QCGF technique to study both equilibrium and nonequilibrium transport in superconducting wires for arbitrary currents and applied voltages. Our equilibrium transport study complements work done by Maki\textsuperscript{22} in that we compute transport properties for different disorder strengths, ranging from the clean to the dirty limit. In particular, we calculate the critical current density for arbitrary disorder, thus going beyond the Usadel equations\textsuperscript{3} valid only in the dirty limit. We also assume that the length of the superconductor is much smaller than the inelastic scattering length.

In this work we have calculated stationary mean field solutions describing the response of the superconductor to an externally applied voltage. The obtained transport configurations are stable at sufficiently low temperatures and wire widths $W$ not much smaller than the
Meissner penetration length $\lambda$. In this regime (specifically for $W \gtrsim \lambda/25$, according to the estimate of Ref.13), both thermal and quantum phase-slips are energetically unfavored. On the other hand, we assume quasi–one-dimensional superconducting wires, for which the Meissner effect and transverse variations of the order parameter can be neglected. This gives us a window of parameters for which our theory is quantitatively valid. Despite these considerations, we wish to point out that a satisfactory understanding is still lacking of the crossover from stationary configurations to dynamic phase-slips in the response to externally applied voltages. A unified description of both phenomena should be the object of future theoretical study.

In section II, we give a brief, self-contained presentation of the QCGF technique (see Ref.17 for an updated review on this topic). Section III is devoted to equilibrium transport. The dependence of the order parameter, current density, and quasiparticle density of states on the superfluid velocity are computed exactly, and several critical magnitudes are calculated for different temperatures and disorder strengths. In section IV, we study nonequilibrium transport with elastic impurity scattering of arbitrary strength, and discuss the robustness of the GS regime. We solve the QCGF equation of motion with boundary conditions describing the injection of quasiparticles from normal reservoirs. The conclusions are presented in section V.

II. THE QUASICLASSICAL GREEN’S FUNCTION

We follow Refs.2,17 in the derivation of the equation for QCGF. The Dyson equation for the matrix Green’s function $\hat{G}$ in the non equilibrium Keldysh formalism2 is

$$[\hat{G}_0^{-1} - \hat{\Sigma}]\hat{G} = \hat{1},$$

(1)

where

$$\hat{G} \equiv \begin{bmatrix} \hat{G}^R & \hat{G} \\ 0 & \hat{G}^A \end{bmatrix},$$

(2)
$\hat{G}_0^{-1}$ being the usual, one-body, inverse matrix Green’s function, and $\hat{\Sigma}$ the self-energy matrix in Keldysh space. Following standard notation, the symbol $\wedge$ indicates $2 \times 2$ matrices in Nambu space. The retarded, advanced, and Keldysh Green’s functions are defined, respectively, as:

$$\hat{G}^R(1, 2) \equiv \theta(t_1 - t_2)[\hat{G}^<(1, 2) - \hat{G}^>(1, 2)] \tag{3}$$

$$\hat{G}^A(1, 2) \equiv -\theta(t_2 - t_1)[\hat{G}^<(1, 2) - \hat{G}^>(1, 2)] \tag{4}$$

$$\hat{G}(1, 2) \equiv [\hat{G}^<(1, 2) + \hat{G}^>(1, 2)] \tag{5}$$

where

$$i\hat{G}^<(1, 2) \equiv \begin{bmatrix} <\psi^\dagger(1)\psi^\dagger(2)> & <\psi^\dagger(1)\psi(2)> \\ -<\psi^\dagger(1)\psi^\dagger(2)> & -<\psi^\dagger(1)\psi(2)> \end{bmatrix} \tag{6}$$

$$i\hat{G}^>(1, 2) \equiv \begin{bmatrix} <\psi^\dagger(2)\psi^\dagger(1)> & <\psi^\dagger(2)\psi(1)> \\ -<\psi^\dagger(2)\psi^\dagger(1)> & -<\psi(2)\psi^\dagger(1)> \end{bmatrix} \tag{7}$$

In these expressions, we have used the standard abbreviations $1 \equiv (r_1, t_1)$ and $2 \equiv (r_2, t_2)$, which allow us to rewrite Eq. (1) as

$$\int d2(\hat{G}_0^{-1}(1, 2) - \hat{\Sigma}(1, 2))\hat{G}(2, 3) = \delta(1 - 3). \tag{8}$$

Subtracting from (1) its conjugate equation, we obtain

$$[\hat{G}_0^{-1} - \hat{\Sigma}, \hat{G}] = 0. \tag{9}$$

This equation may be simplified by going to the center-of-mass and relative coordinates, defined as $r_{1,2} \equiv R \pm r/2$, $t_{1,2} \equiv T \pm t/2$. One Fourier transforms now with respect to the relative variables $r$ and $t$, and introduces the QCGF defined by

$$\tilde{g}(R, T, \mathbf{p}, E) \equiv \frac{i}{\pi} \int_{-\infty}^{\infty} d\xi \tilde{G}(R, T, \mathbf{p}, E). \tag{10}$$
where $\xi \equiv p^2/2m - \mu$ is the free-electron energy measured from the Fermi level and $\hat{p} \equiv p/|p|$. The key assumption in the quasiclassical approximation is that the self-energy $\tilde{\Sigma}$ is almost $\xi$-independent. Then we can set $\xi = 0$ in $\Sigma$ and derive for the stationary case (neglecting the effect of magnetic fields),

$$\hbar v_F \hat{p} \cdot \nabla_R \tilde{g} = iE[\tilde{\tau}_3, \tilde{g}] - i[\tilde{\Sigma}, \tilde{g}].$$

(11)

$\tilde{\tau}_3$ is a block-diagonal matrix with block entries like in the third Pauli matrix $\hat{\tau}_3$. After the subtraction procedure, a normalization condition is needed for $\tilde{g}$. In Ref. 35, Shelankov studies the general, nonstationary case, and

$$\tilde{g} \tilde{g} = 1$$

(12)

is shown to be a useful choice.

Together with the normalization condition (12), Eq. (11) determines the QCGF. The remaining physical quantities can be expressed in terms of $\tilde{g}$. Specifically, the current density flowing through the system is

$$\mathbf{j} = -\frac{1}{4}eN(0)v_F \int dE \int \frac{d\hat{p}}{4\pi} \hat{p} \text{Tr}(\hat{\tau}_3 \tilde{g}),$$

(13)

$N(0)$ being the single-particle density of states per spin at the Fermi level.

Within the BCS approximation, the pairing effect is introduced in $\tilde{\Sigma}$ via the order parameter $\Delta$. The resulting BCS self-energy has retarded and advanced components

$$\tilde{\Sigma}_R^A_{BCS} = -i[\text{Re}(\Delta)\tilde{\tau}_1 - \text{Im}(\Delta)\tilde{\tau}_2].$$

(14)

In this language, the self-consistency equation for the order parameter reads

$$\Delta = -\frac{i}{8g} \int \frac{d\hat{p}}{4\pi} \int^{E_D}_{-E_D} dE \text{Tr}[(\tilde{\tau}_1 - i\tilde{\tau}_2)\tilde{g}],$$

(15)

with $g$ the electron-phonon coupling constant and $E_D$ the usual BCS cutoff energy. Eqs. (11), (12), (13), and (15) form the basic blocks for calculations in the following sections.
III. THE HOMOGENEOUS WIRE

We begin by studying transport in superconducting wires with arbitrary disorder and homogeneous on a scale much greater than the mean free path $l$. It is well known that, up to first order in perturbation theory, impurities are unable to lower $T_c$ in conventional s-wave superconductors, the reason being that non-magnetic impurities do not break time-reversal symmetry. However, the effect of impurities is expected to be important when a non negligible superfluid velocity is present in the system. For instance, it is known that the critical current decreases with impurity concentration. Moreover, the presence of a finite superflow introduces an intrinsic anisotropy in the system which one should expect to be sensitive to the presence of random scatterers. On the other hand, the study of transport in even nominally clean quasi–one-dimensional wires ($l \gg \xi_0 \gg W$, $\xi_0$ being the zero temperature coherence length) must include the effect of random disorder if scattering at the surface is diffusive. As mentioned above, the use of the QCGF technique permits an easy implementation of averages over impurity configurations. This allows for quantitative comparison with experiments and represents a notable advantage over techniques based on the resolution of the BdG equations. There have already been calculations exploring the extreme cases of ballistic and diffusive (dirty) superconductors in equilibrium. We wish to study the crossover between these two limits. In particular, we want to calculate how the critical current density decays with disorder and compare our results with the predictions based on macroscopic descriptions of the diffusive limit.

A. The model and its solution

We introduce disorder within the simplest approximation of incoherent multiple scattering by impurities. We average over impurity configurations compatible with a given degree of macroscopic disorder, retaining only the leading term in an expansion in powers of $(k_F l)^{-1}$. Under these rather standard assumptions, the contribution of disorder to the
self-energy may be written as
\[ \Sigma_{\text{imp}} = -\frac{i\hbar}{2\tau} \langle \hat{g} \rangle, \]  
(16)
where the brackets are meant to indicate angular average over the Fermi surface \[ \langle f \rangle \equiv \int \frac{d\hat{p}}{4\pi} f(\hat{p}) \]. The scattering time \( \tau \) coincides with the transport time \( \tau_{tr} \) when the impurity scattering is isotropic. Its precise definition is
\[ \frac{1}{\tau} = 2\pi n_{\text{imp}} N(0)|v(\hat{p}, \hat{p}')|^2, \]
(17)
where \( n_{\text{imp}} \) is the impurity concentration, and \( v(\hat{p}, \hat{p}') \) is the probability amplitude for an incoming electron with momentum direction \( \hat{p}' \) to be scattered into direction \( \hat{p} \) after collision with an impurity. Its square modulus is assumed to be independent of both the incoming and the outgoing direction. Finally, for a macroscopic description, impurity scattering is characterized by the disorder rate \( \hbar/\tau \). We will therefore use \( \tilde{\Sigma} = \tilde{\Sigma}_{\text{BCS}} + \tilde{\Sigma}_{\text{imp}} \) as the self-energy of our problem.

The inclusion of a finite superfluid velocity in our set of equations leads to the addition of phase factors \( e^{\pm i\mathbf{q}\cdot \mathbf{R}} \) in some physical quantities. Their effect is equivalent to that of shifting the energy variable by an amount \( \hbar v_F \hat{p} \cdot \mathbf{q} \) in Eq. (11) (\( \mathbf{q} \) is half the Cooper pair momentum, \( \mathbf{q} = mv_s/\hbar \)). The presence of \( \mathbf{q} \) leads to a nonzero supercurrent density and, through self-consistency equation (13), has a direct effect on the value of \( |\Delta| \). We wish then to solve the set of Eqs. (11), (12), (13), and (15) for different values of the superfluid velocity and \( \hbar/\tau \).

If quasiparticles are in equilibrium with themselves and with respect to the lattice, the Keldysh part of the Green’s function matrix (10) can be expressed in terms of the retarded and advanced elements as \( \hat{g} = \tanh(E/2k_B T)(\hat{g}^R - \hat{g}^A) \). These, in turn, may be written as
\[ \hat{g}^R = \alpha \hat{\tau}_3 + \beta \hat{\tau}_1 \]
\[ \hat{g}^A = -\alpha^* \hat{\tau}_3 + \beta^* \hat{\tau}_1, \]
(18)
where the scalar functions \( \alpha \equiv \alpha(\mathbf{R}, \hat{p}, E) \) and \( \beta \equiv \beta(\mathbf{R}, \hat{p}, E) \) are the generalized densities of states. The normalization condition (12) requires \( \alpha^2 + \beta^2 = 1 \). If one considers a bulk
superconducting wire with periodic boundary conditions, the problem becomes uniform and
the relevant equation of motion (11) yields
\[ (E - \hbar v_F u)\beta - i|\Delta|\alpha + \frac{i\hbar}{2\tau}(\beta <\alpha> - \alpha <\beta>) = 0, \] (19)
with \( u \equiv \hat{p} \cdot q / q \) for the angular variable. This equation is solved in Appendix A with full
inclusion of the \( u \)-dependence. Once the functions \( \alpha(u, E) \) and \( \beta(u, E) \) are obtained, we
compute the value of \( |\Delta| \) from Eq. (15). This new value is introduced in (19) to calculate
again the generalized densities of states, from which in turn we obtain a new order parameter.
The procedure is repeated until self-consistency in \( |\Delta| \) is achieved. After a self-consistent
pair potential is found for given superfluid velocity and disorder, the current density is
calculated with Eq. (13).

B. Discussion

In Fig. 1 the order parameter amplitude \( |\Delta| \) is plotted as a function of the superfluid
velocity for different temperatures, and dimensionless disorder strengths \( \Gamma \equiv \hbar / \tau \Delta_0 \). One
may note that Anderson’s theorem is satisfied at all temperatures, since \( |\Delta| \) for \( v_s = 0 \) is
independent of the disorder strength. For very small \( \Gamma \), one retrieves the expected result
that \( |\Delta| \rightarrow 0 \) when \( v_s p_F / |\Delta| \sim 1 \). Increasing the disorder seems to reinforce the pair
potential because of the much larger values of \( v_s \) needed to suppress \( \Delta \). This occurs at
\( v_s = v_d \), with \( v_d p_F / \Delta_0 \sim 3.1, 2.6, \) and \( 1.3 \) for \( T/T_c = 0.1, 0.5, \) and \( 0.9 \), when \( \Gamma = 10, \)
i.e., more than twice the ballistic (small \( \Gamma \)) value. This behavior may be interpreted as the
tendency of disorder to restore the spherical symmetry (and thus sustain the order parameter
amplitude), counteracting the anisotropy induced by the presence of superflow (see following
subsection).

This effect should not be viewed as an enhancement of superconductivity by disorder. Inspection of Fig. 2 shows that, as expected, the critical current density \( j_c \) (defined as the
maximum possible value of \( j \) with a nonzero \( \Delta \)) decays with increasing disorder. This is
compatible with the gap behavior discussed in the previous paragraph, because the density of superfluid electrons is also reduced.\textsuperscript{12} Fig. 2 also shows that, for a given macroscopic current density \(j\), there are two possible values of \(v_s\). The smaller \(v_s\) yields the more stable configuration. Finally, we note that, for \(T\) close to \(T_c\), we reproduce the smooth behavior expected from Ginzburg-Landau calculations.\textsuperscript{12}

As an illustration, we plot in Fig. 3 the values of several physical magnitudes at the point \(j = j_c\) as a function of disorder (left panels) and temperature (right panels). Despite the above mentioned different depairing behavior, both critical \(j\) and \(|\Delta|\) actually diminish with increasing disorder and temperature. However, the critical superfluid velocity is enhanced with \(\Gamma\) while it decreases with \(T\). This is a manifestation of the high anisotropy needed to break superconductivity when disorder is strong. It also shows that disorder is not intrinsically depairing, while temperature is.

Finally, in Fig. 4 we plot the decay of the critical current density with disorder at low temperatures \((T = 0.1T_c)\) for a wide range of \(\Gamma\) values. Inset (a) shows the linear behavior for small disorder \((\Gamma < 0.1)\). Inset (b) indicates that for \(\Gamma > 10\) the critical current follows a power-law \((j_c \propto \Gamma^{-\beta})\). Here we present a unified treatment encompassing the ballistic and the diffusive regimes. This requires going beyond the Usadel equations,\textsuperscript{3} valid only within the dirty limit. Our results may be compared with the predictions of a macroscopic theory based on energetic arguments. If we use a phenomenological formula for the density of superfluid electrons (valid for \(l \ll \xi_0\), see Ref.\textsuperscript{12}),

\[
\rho_s(l) = \frac{\rho_s(\infty)}{1 + \xi_0/l},
\]

and equate the kinetic energy to the constant condensation energy \(j_c^2/2\rho_s\), the law \(j_c(\Gamma) = j_c(0)/(1 + \Gamma)^{1/2}\) is obtained. This is the dotted line in Fig. 4. This simple law cannot reproduce the entire \(\Gamma\) dependence, and, in particular, it yields an exponent \(0.5\) slightly different from the exact \(\beta \approx 0.47\) which we obtain numerically.
C. Density of states

In Fig. 5 we plot the density of states (DOS) for different values of the superfluid velocity and the disorder strength. At low values of $v_s (\ll v_d)$ we recover the characteristic BCS density of states. The splitting of the gap $\Delta$ into $\Delta_+$ and $\Delta_-$ modifies the quasiparticle DOS. As $v_s$ raises from zero, the singularity at $E = \Delta_0$ evolves into two cusps at $\Delta_{\pm}$. The smoother character of the split singularity comes from the fact that the minimum quasiparticle energy depends on the momentum direction. Thus, the zero-velocity DOS singularity actually evolves into a distribution of singularities which, when integrated over the Fermi surface, yields two characteristic cusps.

For weak and moderate disorder ($\Gamma = 0.1$ and 1), the GS regime can be achieved with values of $v_s$ within the stable branch (see Fig. 2 for the corresponding temperature $T/T_c = 0.1$). This result is important, since it shows that the GS state occurs in a stable manner for equilibrium transport in relatively clean superconducting wires. GS can also be reached in dirty ($\Gamma = 10$) superconductors but only for unstable values of $v_s$.

Comparison of the DOS curves for $\Gamma = 1$ and $\Gamma = 10$ in Fig. 5 shows that, for some values of $v_s p_F/\Delta_0$ (such as 1 or 1.5) for which one would expect to have GS, the effect of disorder is that of restoring the gap. This effect may be understood as resulting from the directional randomization induced by multiple impurity scattering. The idea that GS should exist for $v_s > |\Delta|/p_F$ comes from a kinematic analysis that applies to plane waves or to quasiparticle states differing little from them. This is the case of weak disorder. By contrast, in strongly disordered superconductors, the exact quasiparticle states necessarily involve a strong mixture of plane waves pointing in many directions. Disorder helps to preserve the gap in the exact DOS because, at very low energies, semiclassical quasiparticle trajectories can only select plane waves from a narrow solid angle of gapless directions. Not being able to mix many momentum directions, quasiparticle states at such low energies cease to exist. This effect is analogous to the appearance of a minigap in the DOS of a diffusive normal metal in contact with a superconductor.
The last section of this paper is devoted to nonequilibrium transport in a disordered superconductor symmetrically connected to two normal reservoirs with different chemical potentials, as schematically depicted in Fig. 6. This boundary condition introduces a nontrivial spatial dependence of the physical magnitudes and makes the equation of motion (11) more difficult to solve. In particular, the Keldysh component of the Green’s function matrix \((\Pi)\) contains now all the relevant information on the quasiparticle local distribution function and on the momentum-relaxation processes taking place within the superconductor due to disorder. Of course, the main new physical ingredient involved in our calculation is the presence of a phase gradient in the order parameter, which has to be determined self-consistently at each point for every value of the chemical potentials at the reservoirs. Like in Ref. 30, the loss of translational invariance makes it harder to compute the pair potential, which now has to be determined both in its real and imaginary parts, as demanded by a locally self-consistent calculation.

We are interested in the behavior of the new transport regimes which were mentioned in the Introduction. In Refs. 28, 29 it was shown how the regime dominated by AT requires the presence of moderately reflecting barriers at the NS contacts in order to be realized. Unfortunately, the mathematical implementation of the appropriate boundary conditions in the context of the QCGF is very cumbersome, since it involves the solution of nonlinear equations. Due to this complication, we have chosen to perform the study for ideal NS contacts (where the boundary conditions reduce to continuity of the QCGF at the interfaces), leaving for the future a more systematic calculation for arbitrary barriers at the interfaces, in the spirit of Ref. 30. With this assumption, the AT regime cannot be studied, and we will focus instead on the sensitivity of the GS regime to the degree of disorder in the superconducting wire.
A. Solution of the kinetic problem

We assume a superconducting wire of length $L$, connected at its ends $z = \pm L/2$ with perfect normal leads (characterized by having $\Delta = 0$). These normal leads are connected to large normal reservoirs through ideal contacts, so that the chemical potentials characterizing the population of *incoming* electrons and holes are those of the reservoirs from which they were injected. Periodic boundary conditions are considered in the transverse directions, perpendicular to the transport direction $z$. For a given voltage $V = (\mu_L - \mu_R)/e$, we wish to solve again Eq. (11) with the normalization prescription (12) for a specific shape of the order parameter $\Delta(z) \equiv |\Delta(z)|e^{i\varphi(z)}$. Once this is done, the solution is used to compute a new $\Delta(z)$, and the procedure is repeated until self-consistency is achieved. Finally, the current density $j$ is calculated from Eq. (13).

In the stationary limit, and for an arbitrary shape of $\Delta$, it is possible to parametrize the retarded and advanced QCGF as

$$\hat{g}^R = \alpha \hat{\tau}_3 + \beta \hat{\tau}_1 + \gamma \hat{\tau}_2 \quad \hat{g}^A = -\alpha^* \hat{\tau}_3 + \beta^* \hat{\tau}_1 + \gamma^* \hat{\tau}_2,$$

(21)

where $\alpha$, $\beta$, and $\gamma$, are now scalar functions of the longitudinal coordinate $z$ (as well as of $u$ and $E$). When Eq. (21) is introduced into the equation of motion (11) a new set of equations is obtained:

$$huv_F \frac{\partial \alpha}{\partial z} = (i \frac{\hbar}{\tau} <\gamma> -2i\text{Im}\Delta)\beta - (i \frac{\hbar}{\tau} <\beta> +2i\text{Re}\Delta)\gamma$$

$$huv_F \frac{\partial \beta}{\partial z} = (-i \frac{\hbar}{\tau} <\gamma> +2i\text{Im}\Delta)\alpha + (i \frac{\hbar}{\tau} <\alpha> +2E)\gamma$$

$$huv_F \frac{\partial \gamma}{\partial z} = (i \frac{\hbar}{\tau} <\beta> +2i\text{Re}\Delta)\alpha - (i \frac{\hbar}{\tau} <\alpha> +2E)\beta,$$

(22)

where, as usual, the brackets stand for angular average. The normalization condition (12) implies $\alpha^2 + \beta^2 + \gamma^2 = 1$. In Appendix B we solve this set of equations with the appropriate boundary conditions.

While the retarded and advanced parts of the equation of motion still give us the generalized densities of states of the problem, the Keldysh part $\hat{g}$ of (10), which contains information
on actual occupations, is no longer trivial. Being in a nonequilibrium context, we need to write its corresponding equation of motion, which reads

\[
\hbar uv_F \frac{\partial \hat{g}}{\partial z} = iE(\hat{\tau}_3 \hat{g} - \hat{g} \hat{\tau}_3) - i(\hat{\Sigma}^R \hat{g} + \hat{\Sigma} \hat{g}^A - \hat{g}^R \hat{\Sigma} - \hat{g} \hat{\Sigma}^A).
\]  

(23)

One way of parametrising \( \hat{g} \) which automatically satisfies Eq. (12) is

\[
\hat{g} = \hat{g}^R \hat{h} - \hat{h} \hat{g}^A,
\]  

(24)

with \( \hat{h} \) an arbitrary distribution matrix. Schmid and Schön proposed a distribution matrix \( \hat{h} \) of the diagonal form \( \hat{h} = (1 - 2f_L) - 2f_T \hat{\tau}_3 \). This procedure is extremely useful in situations where the dirty limit is valid and the boundary conditions are independent of \( u \), but it does not permit a complete separation of the densities of states and the quasiparticle distribution function. An alternative parametrization was suggested by Shelankov,
The multiplicity of Eq. (23) by \( \hat{P}_1^R \hat{P}_1^A \) on the left, taking the trace, and using the cyclic properties as well as Eq. (26), one may find after some algebra,

\[
\hbar uv_F \text{Tr}(\hat{P}_1^A \hat{P}_1^R) \frac{\partial f_1}{\partial z} = -i \text{Tr}\{\hat{P}_1^A \hat{P}_1^R[(\hat{\Sigma}^R - \hat{\Sigma}^A) f_1 - \hat{\Sigma}]\}.
\]  

(27)

Eq. (27) is a closed expression that permits to obtain \( f_1(z, u, E) \) since, due to the symmetry of the problem, \( f_2(z, u, E) = -f_1(-z, u, E) \). Actually, \( (1 - f_1)/2 \) coincides with the usual Fermi distribution function in equilibrium, and with the quasiparticle distribution function within a semiconductor model. To fix the population of incoming electrons from each reservoir, it is natural to use the boundary conditions
\[ f_1(-L/2, u, E) = \tanh[(E − eV/2)/2k_BT], \quad \text{if } u > 0 \]
\[ f_1(L/2, u, E) = \tanh[(E + eV/2)/2k_BT], \quad \text{if } u < 0. \]  

(28)

The resolution of Eq. (27) with this boundary conditions also requires a self-consistent determination of \( \Sigma \). The contribution of \( \Sigma_{\text{imp}} \) has to be calculated at each energy from the solutions of Eqs. (22) and (27). The corresponding angular average of the whole \( \hat{g} \) is then performed and introduced in Eq. (16) until one achieves a self-consistent value of \( \Sigma_{\text{imp}} \). Once this is done at every energy \( E \), the self-consistent procedure for \( \Delta \) may be initiated.

Finally, when the self-consistent solutions for a given voltage are found, one may calculate the electrostatic potential in the structure as:
\[ e\phi(z) = -\frac{1}{4} \int dE < \text{Tr}\hat{g} >, \]  

(29)

which is directly related to the electronic density. Eqs. (29) and (13) will be used to compute the electrostatic potential profile and the current density which we discuss in the next subsection.

B. Discussion

In Fig. 7 we plot the variation of the current density with the applied bias \( V \) between the normal reservoirs for different values of the disorder. The variation of the order parameter amplitude at the center of the NSN structure \( z = 0 \) is represented in the lower figure. Throughout this subsection, we use the values of \( L = 3\xi_0 \) for the length of the superconductor and \( T = 0.01T_c \) for the temperature. One may see how when disorder is present (\( \Gamma \neq 0 \)), there exist two regimes with different current-voltage slopes. For low bias, \( dj/dV \) is always bigger than its corresponding normal value (which is that attained at higher voltages). This fact shows clearly the different effect of diffusive (dirty) regions and tunnel junctions on superconducting transport. In a purely ballistic conductor attached via tunnel barriers to the reservoirs, the conductance would always be bigger in the normal state. the main reason being the much smaller probability for the simultaneous tunneling of two electrons forming
a Cooper pair. However, if the contacts are good enough, and scattering is dominated by spatially distributed impurities, the superconducting state supports a greater amount of current due to the lack of normal reflection of quasiparticles decaying into Cooper pairs. In a perfectly clean sample, the superconducting and the normal states cannot be distinguished in what regards to transport (both display the same slope in the $\Gamma = 0$ case).

There is a correlation between the slope discontinuity for the current and the vanishing of $\Delta(z = 0)$. This is particularly clear for $\Gamma \gtrsim 1$ since then the effect is more marked and occurs at lower voltages ($eV \simeq 1.3\Delta_0$). This is the point at which the whole structure becomes normal and transport is governed by the Boltzmann equation. The case of weak disorder ($\Gamma = 0.2$) is a bit different. One may notice that the current density goes down slightly above $eV = 1.5\Delta_0$, while $\Delta(0)$ remains finite for still higher voltages. The structure is still superconducting (in the sense of having a nonzero order parameter) but the current is practically indistinguishable from that of the normal state. The superconductor is in the GS regime (see discussion below).

In Fig. 8 one may compare in more detail the differences between the situations with weak and strong disorder. We will concentrate on the different behavior of the GS regime. Several physical magnitudes have been represented as a function of position for different values of the applied voltage. The top panels represent the electrostatic potential along the superconductor S. Due to the interfaces with normal reservoirs and to the presence of an applied voltage, there is a penetration of the electric field within S, causing the quasiparticles to have a chemical potential different from that of the condensate (equal to zero in these graphs).

The relaxation of the electric field within a superconductor is one of the most heavily studied topics in the literature on nonequilibrium superconductivity. The physics we encounter here is somewhat different because we deal with a finite superconductor without inelastic scattering, connected symmetrically to normal reservoirs. Two different elastic mechanisms contribute to the spatial relaxation, after length $\Lambda$, of the excess charge density due to quasiparticles near the boundaries. One may have normal reflection of quasiparticles
returning to the reservoir. This translates into a law \( [1 + (z + L/2)/l]^{-1} \) for \( \phi(z) \) on the left boundary, characteristic of decay by incoherent multiple normal scattering. On the other hand, beyond a certain length, quasiparticles cannot propagate because their energy is insufficient to overcome the effective gap, since this increases as one gets deeper into the superconductor. This occurs at a characteristic length from the boundary \( \hbar v_F/\epsilon(V) \), where \( \epsilon(V) \) is an effective energy that diminishes with \( V \) because of the self-consistent decrease of \( |\Delta| \) in all the structure. This is the length scale at which quasiparticle conversion into Cooper pairs by Andreev reflection takes place. Andreev scattering generates a quasiparticle of opposite charge and thus tends to quickly suppress the charge excess. These considerations give a good account of the results for \( \phi(z) \) in Fig. 8, which shows a roughly constant \( \Lambda \) as a function of the applied voltage up to the transition to the normal state for \( \Gamma = 5 \) (dirty limit, \( l \) dominates), and a continuously increasing \( \Lambda \) when \( \Gamma = 0.2 \) (clean limit, \( \hbar v_F/\epsilon(V) \) dominates). When the structure becomes normal, the voltage profile is that which results from substracting the curves \( [1 + (L/2 \pm z)/l]^{-1} \). In the very clean limit (\( l \gg L \)) this results in an essentially straight line.

Center and bottom panels represent the \( |\Delta| \) and \( \Delta_- \) profiles, respectively. Since \( \varphi'(z) \) is no longer constant, \( \Delta_-(z) \equiv |\Delta(z)| - \hbar v_F\varphi'(z)/2 \). When \( \Delta_- < 0 \) at some point, the superconductor becomes locally gapless, at least for plane waves. In particular, near the interfaces, \( \Delta_- \) becomes negative very quickly as \( V \) increases, due to the smallness of \( |\Delta| \), while it remains positive in the bulk (near \( z = 0 \)). As the voltage increases, one may note a different evolution of the left and right graphs before the normal state is reached (this occurs at \( eV = 1.86\Delta_0 \) and \( eV = 1.35\Delta_0 \), respectively). *In the diffusive regime, the superconductor cannot be globally gapless.* All the \( \Delta_- \) curves are positive in the central region. Quasiparticles with energies below the effective gap penetrate the superconductor partially and become Cooper pairs. However, if \( \Delta_-(0) < 0 \), this transfer process becomes marginal. Normal scattering dominates and, due to the strong disorder, the quasiparticles can hardly be transmitted across the system. The directional randomization results in a sharp transition to the normal state. On the contrary, for \( \Gamma = 0.2 \), Fig. 8 shows that it
is still possible to have $\Delta_{-} < 0$ for all $z$ (for, e.g., $eV = 1.5\Delta_{0}$), while the system is still superconducting. The quasiparticle density of states is nonzero at all energies because the weak disorder is unable to randomize the quasiparticle nonequilibrium distribution, which results in the preservation of superconductivity. For sufficiently high voltages (depending on the wire length and on its effective dimensionality), the GS becomes unstable and the system goes normal.

Since realistically clean superconducting wires can be obtained nowadays, it could be possible to measure finite superflow effects passing a large current along a given direction of a superconducting sample, while probing the gap and/or the density of states with a weak tunnelling current.

V. CONCLUSIONS

We have used the technique of the quasiclassical Green’s function (QCGF) to study the effect of disorder and many transport channels in situations where superconductors present a non-negligible superfluid velocity. The QCGF technique has allowed us to include impurity scattering in a tractable manner, and to perform realistic calculations of disorder-averaged physical quantities.

The first part of this article has been devoted to the study of equilibrium transport in homogeneous wires. We have seen that disorder tends to restore the spherical symmetry and strengthen the order parameter amplitude, in contrast with the pair breaking effect of finite condensate flow. However, its net effect is that of diminishing the current. The critical current density has been calculated for disorder strengths ranging from the clean to the dirty limit. The self-consistent density of states for different values of the superfluid velocity and disorder has also been discussed. We find that, due to its direction mixing effect, sufficient disorder restores the quasiparticle gap in transport contexts where one would expect gapless superconductivity (GS).

The second part has focussed on nonequilibrium transport in normal-superconductor-
normal structures. Clean contacts between the superconductor and the normal reservoirs have been assumed, with boundary conditions that describe incoming quasiparticles from reservoirs with different chemical potentials. We have found that the current-induced GS regime is very sensitive to the presence of disorder. It is suppressed in the diffusive limit, but is stable in sufficiently clean superconductors. We have calculated the spatial profiles of the electrostatic potential, order parameter amplitude, and energy threshold for quasiparticle transmission. We have found that they show subtle differences in the ballistic and diffusive regimes in what regards to the current-induced phase transition to the normal state.

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APPENDIX A: IMPURITY-AVERAGED GENERALIZED DENSITIES OF STATES

Eq. (19) is actually an algebraic equation if one considers the averaged $c(E) \equiv <\alpha(u, E) >$ and $s(E) \equiv <\beta(u, E) >$ as known quantities. If it is so, one may define $t(u, E) \equiv \beta/\alpha$, whose formal solution is

$$ t(u, E) = \frac{i(|\Delta| + \hbar s(E)/2\tau)}{(E - \hbar v_F q u + i\hbar c(E)/2\tau)}. \tag{A1} $$

Now, $c(E)$ and $s(E)$ may be found if one notices the normalization prescription $\alpha^2 + \beta^2 = 1$, which makes $\alpha = (1 + t^2)^{-1/2}$ and $\beta = t(1 + t^2)^{-1/2}$. If we change variables in the angular integrals for $c(E)$ and $s(E)$:

$$ c(E) = \int_{t_+}^{t_-} \frac{i(|\Delta| + \hbar s(E)/2\tau)}{2\hbar v_F q} \frac{dt}{t^2(1 + t^2)^{1/2}} $$
\[ s(E) = \int_{t_+}^{t_-} \frac{i(\Delta| + \hbar s(E)/2\tau)}{2\hbar v_F q} \frac{dt}{t(1 + t^2)^{1/2}}, \]  
where \( t_\pm(E) \equiv i(|\Delta| + \hbar s(E)/2\tau)/(E \pm \hbar v_F q + \imath\hbar c(E)/2\tau) \). Integrals in Eq. (A2) give two coupled equations for \( c(E) \) and \( s(E) \)

\[
\begin{align*}
c &\equiv \frac{i(|\Delta| + \hbar s/2\tau)}{2\hbar v_F q} \left[ \left(1 + t_+^2\right)^{1/2} - \left(1 + t_-^2\right)^{1/2} \right] \\
s &\equiv \frac{i(|\Delta| + \hbar s/2\tau)}{2\hbar v_F q} \log \frac{t_-\left[1 + (1 + t_+^2)^{1/2}\right]}{t_+\left[1 + (1 + t_-^2)^{1/2}\right]},
\end{align*}
\]

where we have omitted the dependence on \( E \) for simplicity. These equations can be solved self-consistently at each energy using standard techniques.

**APPENDIX B: THE SCHOPHOL-MAKI TRANSFORMATION**

Eqs. (22) can be decoupled using the Schopohl-Maki transformation

\[
y_{1,2} = \frac{\beta \mp \imath\gamma}{1 + \alpha},
\]

which leads, with the help of Eq. (12), to the following Riccati differential equations:

\[
\begin{align*}
\hbar v_F \frac{\partial y_1}{\partial z} &\equiv 2i\bar{E}y_1 + \bar{\Delta}_2y_1^2 - \bar{\Delta}_1 = 0 \\
\hbar v_F \frac{\partial y_2}{\partial z} &\equiv 2i\bar{E}y_2 - \bar{\Delta}_1y_2^2 + \bar{\Delta}_2 = 0,
\end{align*}
\]

where \( \bar{E} \equiv E + \imath h < \alpha > /2\tau, \bar{\Delta}_1 \equiv \Delta + \hbar < \beta - \imath\gamma > /2\tau, \) and \( \bar{\Delta}_2 \equiv \Delta^* + \hbar < \beta + \imath\gamma > /2\tau. \)

When one solves Eq. (B2) in the normal leads (where \( \Delta = 0 \) and \( \Gamma = 0 \)), the solutions are

\[
\begin{align*}
y_1 &\equiv y_1(z_0)e^{2iE(z-z_0)/\hbar v_F} \\
y_2 &\equiv y_2(z_0)e^{-2iE(z-z_0)/\hbar v_F},
\end{align*}
\]

with \( z_0 \equiv \pm L/2. \) In the imaginary time representation, one has the freedom of choosing either the positive or the negative imaginary axis for the Matsubara energies involved in the problem. Considering the usual criterion of \( E = \imath \omega_n, \) with \( \omega_n > 0, \) one obtains as appropriate boundary conditions that ensure the finiteness of the solutions in the normal leads.
\begin{equation}
y_1(-L/2) = y_2(L/2) = 0 \quad \text{if } u > 0
\end{equation}
\begin{equation}
y_2(-L/2) = y_1(L/2) = 0 \quad \text{if } u < 0.
\end{equation}

On the other hand, the symmetry relation \( \Delta(z) = \Delta^*(-z) \) guarantees the following symmetries for the solutions of the generalized densities of states (regardless of the voltage difference between the normal reservoirs):

\begin{align*}
\alpha(z, u, E) &= \alpha(-z, u, E) \\
\beta(z, u, E) &= \beta(-z, u, E) \\
\gamma(z, u, E) &= -\gamma(-z, u, E),
\end{align*}

which translates into

\begin{equation}
y_1(z, u, E) = y_2(-z, u, E).
\end{equation}

Eq. (B2) reduces then to one effective equation for each \( y \), which can be solved numerically, and the generalized densities of states may finally be calculated from the relations

\begin{align*}
\alpha(z) &= \frac{1 - y_1(z)y_1(-z)}{1 + y_1(z)y_1(-z)} \\
\beta(z) &= \frac{y_1(z) + y_1(-z)}{1 + y_1(z)y_1(-z)} \\
\gamma(z) &= \frac{i[y_1(z) - y_1(-z)]}{1 + y_1(z)y_1(-z)},
\end{align*}

where we have omitted the \( u \) and \( E \) dependence. As explained at the end of subsection IV.A, a self-consistent procedure is also needed at each energy to obtain the right angular averages entering the equations of motion.
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FIG. 1. Impurity averaged order parameter amplitude as a function of the superfluid velocity, for different values of the normalized disorder rate $\Gamma = \frac{\hbar}{\tau \Delta_0}$ and temperature.
FIG. 2. Same as Fig. 1 for the current density.
FIG. 3. Impurity averaged *critical* magnitudes as a function of the disorder (left) and of the temperature (right).
FIG. 4. Impurity averaged critical current density as a function of the disorder for $T = 0.1T_c$. Insets: (a) Ballistic case ($\Gamma \ll 1$): Linear decay of the critical current with the disorder rate. (b) Diffusive case ($\Gamma \gg 1$): Power-law $j_c \propto \Gamma^{-\beta}$. We find $\beta \approx 0.47$, slightly different from the simple prediction of 0.5 (dotted line).
FIG. 5. Impurity averaged density of states for several superfluid velocities and disorder strengths. All results are given for $T = 0.1T_c$. 
FIG. 6. Schematic representation of a typical NSN structure, where S is a dirty superconductor. The normal leads are assumed to be perfect and connected to wide reservoirs through ideal contacts. Grey colour marks the superconducting zone of length \( L \). Small crosses signal possible impurity positions. The picture shows a specific realization of disorder within the superconductor.
FIG. 7. Top: Current density as a function of the applied voltage in the structure of Fig. 6, for different values of the disorder. Bottom: Order parameter amplitude at the center of the superconductor, for the same values of $\Gamma$. Here, $L = 3\xi_0$ and $T = 0.01T_c$. 
FIG. 8. Comparison between ballistic (left) and diffusive (right) limits in the structure of Fig. 7. Top panels: Profile of the electrostatic potential within the superconductor. Center panels: Order parameter amplitude. Bottom panels: Threshold $\Delta_- \equiv |\Delta| - \hbar v_F \varphi' / 2 = |\Delta| - m v_F v_s$. At positions where $\Delta_- < 0$ the superconductor is gapless. L and T as in Fig. 7.