Fréchet bounds of the 1-st kind for sets of half-rare events

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Abstract. Fréchet bounds of the 1-st kind for sets of events (s.e.) and its main properties are considered. The lemma on not more than two nonzero values of lower Fréchet-bounds of the 1-st kind for a set of half-rare events (s.h.r.e.) is proved with the corollary on the analogous assertion for s.e.’s with arbitrary event-probability distributions (e.p.d.’s.).

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In [1] it is shown that any s.e.’s are set-phenomena of some half-rare s.e. And its e.p.d.’s are characterized by the e.p.d.’s of the half-rare s.e.’s the set-phenomena of which they are. In view of this, a theory of Fréchet-bounds of the 1st kind of half-rare s.e.’s is being developed in this paper, i.e., the theory of such s.e.’s events of which happen with probabilities not greater than half. These “non-descript” restrictions are enough to build Fréchet-bounds for any s.e. from Fréchet-bounds of the half-rare s.e., relying on set-phenomenon transformations of e.p.d.’s and of its Fréchet-bounds of the 1st kind.

We will make one more purely technical assumption, which does not detract from the generality of our analysis. We will assume that events in each half-rare s.e. are ordered in descending of their probabilities and we introduce the following natural notation and abbreviations for half-rare N-s.e. \( \mathcal{X} = \{x, x \in \mathcal{X}\} = \{x_1, ..., x_N\} \) and its \( \mathcal{X} \)-set of probabilities of events \( \tilde{p} = \{p_x, x \in \mathcal{X}\} = \{p_{x_1}, ..., p_{x_N}\} = \{p_1, ..., p_N\} \) where

\[
1/2 \geq p_1 \geq p_2 \geq ... \geq p_N. \tag{*}
\]

1 Properties of Fréchet-bounds of the 1st kind of a half-rare s.e.

Consider the half-rare N-s.e. \( \mathcal{X} = \{x_1, ..., x_N\} \) with the \( \mathcal{X} \)-set of probabilities of events \( \tilde{p} = \{p_x, x \in \mathcal{X}\} = \{p_1, ..., p_N\} \) where, we recall, \( 1/2 \geq p_1 \geq ... \geq p_N \), and the e.p.d. of the 1st kind \( p(X/\mathcal{X}), X \subseteq \mathcal{X} \), and Fréchet-bounds of the 1st kind for \( X \subseteq \mathcal{X} \):

\[
p^{-}(X/\mathcal{X}) \leq p(X/\mathcal{X}) \leq p^{+}(X/\mathcal{X})
\]

where

\[
p^{-}(X/\mathcal{X}) = \max \left\{ 0, 1 - \sum_{x \in X} (1 - p_x) - \sum_{x \in X-X} p_x \right\}, \tag{1.1}
\]

\[
p^{+}(X/\mathcal{X}) = \min \left\{ \min_{x \in X} p_x, \min_{x \in X-X} (1 - p_x) \right\} \tag{1.2}
\]

are general formulas for the lower and the upper Fréchet-bounds of the 1st kind. They form the sets of Fréchet-bounds which are called the lower and the upper Fréchet-boundary distributions (F.-b.d.’s) of the 1st kind:

\[
\{p^{-}(X/\mathcal{X}), X \subseteq \mathcal{X}\}, \{p^{+}(X/\mathcal{X}), X \subseteq \mathcal{X}\}.
\]

The upper Fréchet-bound can be simplified by virtue of the characteristic property of a half-rare s.e. \((*)\) since then for non-empty \( \emptyset \neq X \subseteq \mathcal{X} \)

\[
\min_{x \in X} p_x \leq \min_{x \in X-X} (1 - p_x).
\]

We have the formula for non-empty \( \emptyset \neq X \subseteq \mathcal{X} \):

\[
p^{+}(X/\mathcal{X}) = \min_{x \in \mathcal{X}} p_x.
\]

If \( X = \emptyset \), then, as is not difficult to understand,

\[
p^{+}(\emptyset/\mathcal{X}) = \min_{x \in \mathcal{X}} (1 - p_x) = 1 - p_{x_1}.
\]

From this the formula for the upper Fréchet-bound of the 1st kind of a half-rare s.e. has the form:

\[
p^{+}(X/\mathcal{X}) = \begin{cases} 1 - p_{x_1}, & X = \emptyset, \\ \min_{x \in \mathcal{X}} p_x, & \emptyset \neq X \subseteq \mathcal{X}. \end{cases} \tag{1.3}
\]

and the upper F.-b.d. of the 1st kind has the form:

\[
\left\{ 1 - p_{x_1}, \min_{x \in \mathcal{X}} p_x, \emptyset \neq X \subseteq \mathcal{X} \right\}.
\]

With the lower Fréchet-bounds of the 1st kind of any half-rare s.e. \( \mathcal{X} \) the situation is not much more complicated.

Lemma 1 (on the lower Fréchet-bounds of the 1st kind of a half-rare s.e.). The lower Fréchet-bound of the 1st kind of the half-rare s.e. \( \mathcal{X} \) with maximum probability of
events $p_{\text{max}} = \max_{x \in \mathcal{X}} p_x = P(X_{\text{max}})$ can be nonzero for only two probabilities of the 1st kind: $p(\emptyset / \mathcal{X})$ and $P(X_{\text{max}}) / \mathcal{X}$.

**Proof.** From the definition of the lower Fréchet-bound of the 1st kind of the s.e. $\mathcal{X}$ (1.1) and due to the fact that $\mathcal{X}$ is the half-rare s.e., it is clear that for $X \subseteq \mathcal{X}$ such that $|X| > 1$, it is equal to zero, since in this case there are two such events $x, x' \in X$ that $1 - p_x > 1/2$ and $1 - p_{x'} > 1/2$. An therefore $1 - \sum_{x \in X} (1 - p_x) < 0$ and $p^-(X / \mathcal{X}) = 0$. Consequently, for the arbitrary half-rare s.e. $\mathcal{X}$ the lower Fréchet-bound of the 1st kind can differ from zero only for such $X \subseteq \mathcal{X}$ the power of which is not greater than unity $- |X| \leq 1$:

$$p^-(X / \mathcal{X}) = \begin{cases} 0 - \sum_{x \in \mathcal{X}} p_x, & X = \emptyset, \\ 0, & |X| > 1, X \subseteq \mathcal{X} \end{cases}$$

Since for the half-rare $N$-s.e. $\mathcal{X}$ there is only polynomial number, $N + 1$, of such subsets then the problem of constructing the lower Fréchet-bound of the 1st kind of a half-rare s.e. we can consider basically solved. After detailing the condition $|X| \leq 1$ the formula takes the following form:

$$p^-(X / \mathcal{X}) = \max \left\{ 0, 1 - \sum_{x \in \mathcal{X}} p_x \right\}, \quad X = \emptyset,$$

$$\max \left\{ 0, p_{\text{max}} - \sum_{z \in \mathcal{X} - \{x_{\text{max}}\}} p_z \right\}, \quad x \in \mathcal{X},$$

$$0, \quad |X| > 1, X \subseteq \mathcal{X}.$$

Now is just the time, once again take advantage of the fact that $\mathcal{X}$ is a half-rare s.e. Quite unexpectedly it turns out that for the half-rare s.e. $\mathcal{X}$ among $N$ conditions for $x \in \mathcal{X}$ only for the event $x_{\text{max}} = \max_{x \in \mathcal{X}} p_x$, the expression

$$p_{\text{max}} - \sum_{z \in \mathcal{X} - \{x_{\text{max}}\}} p_z$$

under the sign of maximum can be strictly greater than zero. For the rest events $z \neq x_{\text{max}}$ from $\mathcal{X}$ this expression can be only not greater than zero, since from its non-maximum probability $p_z$, since the maximum probability and all the rest ones will be subtracted from it:

$$p_z - p_{\text{max}} - \sum_{y \in \mathcal{X} - \{x_{\text{max}}, z\}} p_y \leq 0.$$

This implies the assertion of the lemma, since we obtain the following formula for the lower Fréchet-bound of the 1st kind of a half-rare s.e. for $X \subseteq \mathcal{X}$:

$$p^-(X / \mathcal{X}) = \max \left\{ 0, 1 - \sum_{x \in \mathcal{X}} p_x \right\}, \quad X = \emptyset,$$

$$\max \left\{ 0, p_{\text{max}} - \sum_{z \in \mathcal{X} - \{x_{\text{max}}\}} p_z \right\}, \quad X = \{x_{\text{max}}\},$$

and the lower F.-b.d. of the 1st kind has the form:

$$p^-(\emptyset / \mathcal{X}), p^- (\{x_{\text{max}}\} / \mathcal{X}), 0, ..., 0$$

where two possibly ono-zero the lower Fréchet-bounds are defined by the formula (1.4). The lemma is proved.

**Corollary 1** (on the lower Fréchet-bounds of the 1st kind of an arbitrary s.e.). The assertion of the lemma is valid not only for half-rare, but also for arbitrary s.e.'s.

Proof follows from the lemma on the characterization of an arbitrary s.e. by its half-rare projection [1], according to which the e.p.d. of the 1st kind of an arbitrary s.e., as well as its F.-b.d. of the 1st kind are set-phenomenal renumbering [1] of corresponding e.p.d.'s and F.-b.d.'s of its half-rare projection.

**Corollary 2** (on the lower Fréchet-bounds of the 1st kind of a half-rare doublet of events). Fréchet-bounds of the 1st kind of the half-rare doublet of events $\mathcal{X} = \{x, y\}$ with the $\{x, y\}$-set of marginal probabilities $\bar{p} = \{p_x, p_y\}$, such that $1/2 \geq p_x \geq p_y$, have the form:

$$1 - p_x - p_y = p^- (\emptyset) \leq p(\emptyset) \leq p^+ (\emptyset) = 1 - p_x,$$

$$p_x - p_y = p^- (x) \leq p(x) \leq p^+ (x) = p_x,$$

$$0 = p^- (y) \leq p(y) \leq p^+ (y) = p_y,$$

$$0 = p^- (xy) \leq p(xy) \leq p^+ (xy) = p_y.$$

**Proof.** The formulas (1.5) are an immediate consequence of formulas (1.3) and (1.4).

**Corollary 3** (on Fréchet-inequalities for a covariance of the 1st kind of a half-rare doublet of events). Fréchet-inequalities for a covariance of the 1st kind of the half-rare doublet of events $\mathcal{X} = \{x, y\}$ with the $\{x, y\}$-set of marginal probabilities $\bar{p} = \{p_x, p_y\}$, such that $1/2 \geq p_x \geq p_y$, have the form:

$$-p_x p_y \leq \text{Kov} (\emptyset) \leq (1 - p_x) p_y,$$

$$-(1 - p_x) p_y \leq \text{Kov} (x) \leq p_x p_y,$$

$$-(1 - p_x) p_y \leq \text{Kov} (y) \leq p_x p_y,$$

$$-p_x p_y \leq \text{Kov} (xy) \leq (1 - p_x) p_y.$$

Proof follows from the definition of covariance of the 1st kind and Corollary 2.

2 The pair illustrations

It is interesting to look at the graphs of F.-b.d.'s of the 1st kind, in order to “make sure with our own eyes” in the just proven of their amazing properties.
One of the concepts that appear on these graphs is the concept of the \textit{independent projection of an arbitrary s.e.} $X$, by which, recall, we understand a s.e., e.p.d. of the 1st kind of which is called \textit{independent e.p.d. (i.e.p.d.) of the 1st kind}; it is denoted by

$$\{p^*(X/\mathcal{X}), X \subseteq \mathcal{X}\}$$

and it is defined for $X \subseteq \mathcal{X}$ by formulas

$$p^*(X/\mathcal{X}) = \prod_{x \in X} p_x \prod_{x \notin \mathcal{X} - X} (1 - p_x)$$

where $p_x = P(x), x \in \mathcal{X}$, are probabilities of events from $\mathcal{X}$.

Each of the graphs illustrates the lower and the upper F.-b.d’s and i.e.p.d’s of independent projection of the half-rare s.e. $X$ of low power: $N = 2, 3, 4, 5, 6, 7$ for $X \subseteq \mathcal{X}$ and different values of the $X$-set of probabilities of events $\tilde{p} = \{p_x, x \in \mathcal{X}\}$. For $X \subseteq \mathcal{X}$ the interval

$$[p^*(X/\mathcal{X}), p^+(X/\mathcal{X})]$$

between values of the i.e.p.d. and the upper F.-b.d. is shown as \textcolor{blue}{blue}, and the interval

$$[p^-(X/\mathcal{X}), p^*(X/\mathcal{X})]$$

between values of the lower F.-b.d. and the i.e.p.d. is shown as \textcolor{red}{red}.

Subsets of events $X \subseteq \mathcal{X}$ is denoted by \textcolor{blue}{blue}-sets: $X \sim \{1_X(x), x \in \mathcal{X}\}$ of values of its indicators on $\mathcal{X}$. For example, the empty subset $\emptyset \subseteq \mathcal{X}$ is denoted by the $\mathcal{X}$-set from zeros: $\emptyset \sim \{0, ..., 0\}$, and the s.e. $\mathcal{X}$ by the $\mathcal{X}$-set from units: $\mathcal{X} \sim \{1, ..., 1\}$. The horizontal dotted line indicates the scale of the unit interval $[0, 1]$ along the vertical axis in $1/4$.

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