Wegner-type bounds for a two-particle lattice model with a generic "rough" quasi-periodic potential

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Abstract

In this paper, we consider a class of two-particle tight-binding Hamiltonians, describing pairs of interacting quantum particles on the lattice \( \mathbb{Z}^d, d \geq 1 \), subject to a common external potential \( V(x) \) which we assume quasi-periodic and depending on auxiliary parameters. Such parametric families of ergodic deterministic potentials ("grand ensembles") have been introduced earlier in [C07], in the framework of single-particle lattice systems, where it was proved that a non-uniform analog of the Wegner bound holds true for a class of quasi-periodic grand ensembles. Using the approach proposed in [CS08], we establish volume-dependent Wegner-type bounds for a class of quasi-periodic two-particle lattice systems with a non-random short-range interaction.

1 Introduction

\[
H(\omega; \theta) = \sum_{j=1}^{2} (\Delta_j + V(x_j; \omega; \theta)) + U(x_1, x_2),
\]

where
\[
(\Delta_1 f)(x_1, x_2) = f(x_1 - 1, x_2) + f(x_1 + 1, x_2),
\]
\[
(\Delta_2 f)(x_1, x_2) = f(x_1, x_2 - 1) + f(x_1, x_2 + 1).
\]

The function \( V : \mathbb{Z}^d \times \mathbb{T}^\nu \times \Theta \to \mathbb{R} \) is defined as follows. For every \( n \geq 1 \), consider the partition of the unit circle \( \mathbb{T}^1 = \mathbb{R}^1/\mathbb{Z}^1 \) into intervals \( I_{n,i} = [\frac{i-1}{2^n}, \frac{i}{2^n}) \) of size \( 2^{-n} \), \( i = 1, \ldots, 2^n \). Let \( \mathcal{C}_n = \{ \mathcal{C}_{n,k}, k = 1, \ldots, 2^{\nu n} \} \) be the family of all possible Cartesian products \( I_{n,i_1} \times \cdots \times I_{n,i_{\nu}}, \) i.e. a partition of the torus \( \mathbb{T}^\nu \) into a
family of cubes of sidelength $2^{-n}$. Further, let $\varphi_{n,k} = 1_{C_{n,k}}$ be the indicator functions of the cube $C_{n,k}$. Next, introduce a parametric family of functions on the torus

$$v(\omega; \theta) = \sum_{n=1}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega),$$

(1.2)

where $\{\theta_{n,k}, n \geq 1, 1 \leq k \leq K_n < \infty\}$ are IID random variables on some auxiliary probability space $\Theta$, with uniform distribution in $[0, 1]$. We will identify $\Theta$ with the set of all samples $\{\theta_{n,k}, n \geq 1, 1 \leq k \leq K_n\}$. Let $\{T^x, x \in \mathbb{Z}^d\}$ be an action of the additive group $\mathbb{Z}^d$ on the torus $\mathbb{T}^\nu$, which we assume to satisfy a Diophantine condition of the form

$$\forall x \in \mathbb{Z}^d \quad \text{dist}(\omega, T^x \omega) \geq \frac{\text{Const}}{\|x\|^B}, \quad 0 < B < \infty.$$  

(1.3)

Concerning the rate of decay of the coefficients $a_n$ in the expansion (1.2), we assume that $\forall n \geq 1$,

$$\frac{c''}{n^M} \leq |a_n| \leq \frac{c'}{n^\kappa},$$

(1.4)

for some $c', c'' \in (0, +\infty)$, $1 < \kappa \leq M < \infty$. Notice that the upper bound on $|a_n|$ guarantees the convergence of the above expansion, while the lower bound is required for our method.

Finally, we set

$$V(x; \omega; \theta) = v(T^x \omega; \theta), \quad x \in \mathbb{Z}^d, \omega \in \Omega \equiv \mathbb{T}^\nu, \theta \in \Theta.$$  

(1.5)

Following [C07], we will call such a parametric family of functions on $\Omega$ a grand ensemble of randedette type.

The expansions of the form (1.2) will be called randedette expansions.

## 2 Wegner-type bounds. Main results

In the spectral theory of random operators, e.g. Anderson tight-binding Hamiltonians of the form

$$H(\omega) = \Delta + V(x; \omega), \quad x \in \mathbb{Z}^d,$$

(2.1)

with an IID random potential $V(x; \omega)$, an important role is played by eigenvalue concentration bounds. The first fairly general result of such kind was obtained by F. Wegner [W81], so they are usually called Wegner-type bounds. Namely, consider the lattice Hamiltonian of the form (2.1), where the random variables $V(x; \cdot)$, identically distributed with a common probability cumulative distribution function (CDF, for short) $F(s) = F_V(s)$, admit a bounded probability density $p(s) = p_V(s)$, so that

$$\|p_V\|_\infty < \infty,$$

(2.2)
Given a finite lattice cube $\Lambda_L(u) \subset \mathbb{Z}^d$ of an arbitrary center $u \in \mathbb{Z}^d$ and of sidelength $2L + 1, L \geq 0$, consider the restriction of the Hamiltonian (???) on $\Lambda \equiv \Lambda_L(u)$ with Dirichlet boundary conditions. Further, let
\[
\Sigma(H_\Lambda) = \{ E_j^\Lambda, j = 1, \ldots, |\Lambda| = (2L + 1)^d \}
\]
be the (random) spectrum of the finite-volume Hamiltonian $H_\Lambda(\omega)$, i.e. the set of its (random) EVs counted with multiplicities. Then for all $E \in \mathbb{R}$ and all $\epsilon \in (0, 1)$, we have
\[
P \{ \text{dist} (\Sigma(H_\Lambda(\omega)), E) \leq \epsilon \} = P \{ \exists E_j^\Lambda \in [E - \epsilon, E + \epsilon] \} \leq |\Lambda| \cdot \| p_V \|_\infty \cdot \epsilon.
\]
(2.3)

The presence of the factor $|\Lambda| \equiv \text{card} \Lambda$ at the RHS of (2.3) allows to prove the absolute continuity of the so-called (limiting) density of states
\[
N(E) = \lim_{L \to \infty} \frac{1}{|\Lambda_L(0)|} \text{card} \{ j : E_j^\Lambda \leq E \},
\]
where the limit exists with probability one and is non-random (cf. [CyFKS87], [CarL90], [PF92] and references therein). In other words, the absolute continuity of the marginal CDF $F_V(s)$ implies that $N(E)$ also admits a density, called density of states (DoS) $\rho(E)$, so that
\[
N(E) = \int_{-\infty}^{E} \rho(E') dE'.
\]

Many generalizations of the Wegner bound (2.3) are well-known by now. In particular, analogs of the Wegner bound can be given in cases where the marginal distribution of the random potential field $V(x; \omega)$ does not admit a density, but the CDF $F_V(s)$ is Hölder-continuous; see, e.g., [St00, St01]. On the other hand, in the case of sufficiently regular marginal distributions optimal eigenvalue concentration bounds are known; see [CoHK07].

The main difficulty for an extension of the Wegner bound to deterministic (e.g., quasi-periodic) potentials is that all known ”probabilistic” methods applicable to the EV concentration problem require a greater freedom in varying individual values of the potential. For example, the conventional Wegner bound requires the RV $V(x; \cdot)$ to be independent. It is well-understood by now that the independence requirement can be replaced by that of asymptotic decay of dependence. In [CS08], even a greater degree of correlation was allowed. Still, an ensemble of quasi-periodic potentials
\[
V(x; \omega) = v(\omega + x\alpha), \quad \omega \in T^1 = \mathbb{R}/\mathbb{Z}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q},
\]
is too ”rigid” and does not allow a direct application of probabilistic methods to the EV concentration problem.

A reader interested in eigenvalue concentration bounds and, in particular, in the analysis of regularity of the density of states, can find an extensive bibliography in references [BCKP88], [CamK86], [CoHK07], [ConFS83], [St00, St01].
of two particles. Given a number $b$ where inequality holds (with $\theta$
respectively.

Theorem 2. Under the same assumptions as in Theorem 1, consider two lattices
$\Lambda$ cubes $\Theta$ by parameter $v$. Consider an ensemble of functions on the lattice
$V(x; \omega)$, $x \in \mathbb{Z}^d$, of the form (1.2) labeled by points of a specially constructed parameter space $\Theta$ is considered. While such an ensemble, (called "grand ensemble" in [C07]) is not ergodic, it turns out that some analog of Wegner-type bounds can be established for generic parameter values $\theta \in \Theta$.

A drawback of this method is that obtained EV concentration bounds are non-uniform, unlike the conventional Wegner-type bounds.

The main results of the recent paper are given in the following two statements. For notational simplicity, below we will denote by $\Sigma_{u,L}(\omega; \theta))$ the spectrum (i.e., the set of eigenvalues counted with multiplicities) of operator $H_{u,L}(\omega; \theta))$:

$$\Sigma_{u,L}(\omega; \theta)) = H_{u,L}(\omega; \theta)) = \left\{ E_j^{\Lambda_L(u)}, 1 \leq j \leq |\Lambda_L(u)| \right\}.$$ (2.4)

**Theorem 1.** Consider an ensemble of functions on the lattice $V_0(x; \omega) = v(T^x \omega; \theta)$, $x \in \mathbb{Z}^d$, of the form (1.2) labeled by points of the torus $T^\nu$ and by parameter $\theta \in \Theta$. Fix two numbers, $b > 0$ and $r > 1$. Given a lattice cube $\Lambda_L(u) \subset \mathbb{Z}^{2d}$, $\Lambda_L(u) \subset \Lambda_L(v)$, there exists a subset $\Theta_L \subset \Theta$ of measure $\mu(\Theta_L) \geq 1 - L^{-b}$, $b > 0$, such that for any $\theta \in \Theta_L$ the following inequality holds (with $\theta \in \Theta_L$ fixed):

$$P\{ \omega \in T^\nu : \text{dist}[\Sigma_u(\omega; \theta)), E] \leq \epsilon \} \leq \text{Const } L^{M+b+3d+r} \epsilon.$$

**Theorem 2.** Under the same assumptions as in Theorem 1, consider two lattice cubes $\Lambda_L(u')$, $\Lambda_L(u'') \subset \mathbb{Z}^{2d}$, such that

$$\|u' - u''\| \leq L^r, r > 1,$$

and

$$\min\{\|u - u'\|, \|u - S(u')\|\} > 8L$$ (2.5)

where $S : (u_1, u_2) \mapsto (u_2, u_1)$ is the symmetry in $\mathbb{Z}^{2d}$ exchanging the coordinates of two particles. Given a number $b \in (0, +\infty)$, there exists a subset $\Theta_L \subset \Theta$ of measure $\mu(\Theta_L) \geq 1 - L^{-b}$, $b > 0$, such that for any $\theta \in \Theta_L$ the following inequality holds (with $\theta \in \Theta_L$ fixed):

$$P\{ \omega \in T^\nu : \text{dist}[\Sigma_u(\omega; \theta)), \Sigma_{u''}(\omega; \theta)] \leq \epsilon \} \leq \text{Const } L^{M+b+r+5d} \epsilon.$$

The proofs of theorems 1 and 2 are given in Subsections 4.3 and 4.5 respectively.
Remark. To explain the role of condition (2.5), recall that the potential energy of a two-particle system is invariant under the symmetry $S: (u_1, u_2) \mapsto (u_2, u_1)$. Given a two-particle volume $\Lambda_L(u_1, u_2) = \Lambda_L(u_1) \times \Lambda_L(u_2)$, consider its "total shadow",

$$\Pi \Lambda_L(u_1, u_2) := \Lambda_L(u_1) \cup \Lambda_L(u_2).$$

Obviously, $\Pi \Lambda_L(u_1, u_2) = \Pi \Lambda_L(u_2, u_1)$, i.e.,

$$\Pi \Lambda_L(u) = \Pi \Lambda_L(S(u)) = S(\Lambda_L(u)),$$

so that the samples $\{V(x; \omega), x \in \Pi \Lambda_L(u)\}$ and $\{V(x; \omega), x \in \Pi \Lambda_L(S(u))\}$ are identical. As a consequence, Hamiltonians $H(\Lambda_L(u))$ and $H(\Lambda_L(S(u)))$ have identical spectra. Further discussion of this condition can be found in [CS08].

Naturally, the above upper bounds are useful only for sufficiently small $\epsilon$, when the RHS is smaller than 1.

3 Diagonally monotone operator families

Here we briefly recall some notions and results from [St00], [St01] and their extensions to two-particle systems proposed in [CS08].

Let $m \geq 1$ be a positive integer, and $J$ an abstract finite set with $|J| = m$. Consider the Euclidean space $\mathbb{R}^J \cong \mathbb{R}^m$ with standard basis $(e_1, \ldots, e_m)$, and its positive orthant

$$\mathbb{R}^+_J = \{ q \in \mathbb{R}^J : q_j \geq 0, j = 1, 2, \ldots, m \}.$$

For any measure $\mu$ on $\mathbb{R}$, we will denote by $\mu^m$ the product measure $\mu \times \cdots \times \mu$ on $\mathbb{R}^J$. Furthermore, for any probability measure $\mu$ and for any $\epsilon > 0$, define the following quantity:

$$s(\mu, \epsilon) = \sup_{a \in \mathbb{R}} \mu([a, a + \epsilon])$$

Furthermore, let $\mu^m_{m-1}$ be the marginal probability distribution induced by $\mu^m$ on $q' = (q_2, \ldots, q_m)$.

Definition 1. Let $J$ be a finite set with $|J| = m$. Consider a function $\Phi : \mathbb{R}^J \to \mathbb{R}$. It is called diagonally monotone (DM, for short) if it satisfies the following conditions:

1. for any $r \in \mathbb{R}^+_J$ and any $q \in \mathbb{R}^J$,

$$\Phi(q + r) \geq \Phi(q);$$

2. moreover, for $e = e_1 + \cdots + e_m \in \mathbb{R}^J$, for any $q \in \mathbb{R}^J$ and for any $t > 0$

$$\Phi(q + t \cdot e) - \Phi(q) \geq t.$$
It is convenient to introduce the notion of DM operators considered as quadratic forms. In the following definition, we use the same notations as above.

**Definition 2.** Let $\mathcal{H}$ be a Hilbert space. A family of self-adjoint operators $B(q): \mathcal{H} \to \mathcal{H}$, $q \in \mathbb{R}^J$, is called DM if,

$$\forall q \in \mathbb{R}^J \forall r \in \mathbb{R}^J \quad B(q + r) \geq B(q),$$

in the sense of quadratic forms, and for any vector $f \in \mathcal{H}$ with $\|f\| = 1$, the function $\Phi_f: \mathbb{R}^J \to \mathbb{R}$ defined by

$$\Phi_f(q) = (B(q)f, f)$$

is DM.

**Remark.** By virtue of the variational principle for self-adjoint operators, if an operator family $H(q)$ in a finite-dimensional Hilbert space $\mathcal{H}$ is DM, then each eigen-value $E_{B(q)}(k)$ of $B(q)$ is a DM function. In addition, it is readily seen that if $H(q)$, $q \in \mathbb{R}^J$, is a DM operator family in Hilbert space $\mathcal{H}$, and $H_0: \mathcal{H} \to \mathcal{H}$ is an arbitrary self-adjoint operator, then the family $H_0 + H(q)$ is also DM.

**Lemma 1** (Stollmann, [St00]). Let $J$ be a finite index set, $\mu$ be a probability measure on $\mathbb{R}$, and $\mu^J$ be the product measure on $\mathbb{R}^J$ with marginal measures $\mu$. If the function $\Phi: \mathbb{R}^J \to \mathbb{R}$ is DM, then for any open interval $I \subset \mathbb{R}$ we have

$$\mu^J \{ q : \Phi(q) \in I \} \leq |J| \cdot s(\mu, |I|).$$

**Remark.** It is not difficult to see that the identical distribution of the RV $V$ is not essential for the Stollmann’s bound. In a more general case, when the measure $\mu^J$ is a direct product of measures $\{\mu_j, j \in J\}$, the above bound can be replaced by

$$\mathbb{P} \{ q : \Phi(q) \in I \} \leq |J| \cdot |\Lambda| \cdot \max_j s(\mu_j, I).$$

In the particular case where $\mu_j$ admit a bounded probability densities $p_j(t)$, the Stollmann’s bound takes the form

$$\mathbb{P} \{ q : \Phi(q) \in [a, a + \epsilon] \} \leq |J| \cdot |\Lambda|^2 \cdot \max_j \|p_j\| \cdot \epsilon.$$

The proof of Stollmann’s lemma can be found in [St00], so we omit it here.

As was shown in [CS08], an ensemble of two-particle tight-binding Hamiltonians $H_A = H_{0,A} + U + (V(x_1; \omega) + V(x_1; \omega))$ in a finite cube $\Lambda = \Lambda^{(1)} \times \Lambda^{(2)} \subset \mathbb{Z}^{2d}$ with external potential $V(x; \omega)$ taking independent values is a diagonally monotone family, relative to the index set $J = \Pi A := \Lambda^{(1)} \cup \Lambda^{(2)}$. This allows a fairly straightforward application of Stollmann’s bound to two-particle systems. In our case, we achieve a similar goal, using a particular conditioning in the "grand ensemble" of quasi-periodic potentials $V(x; \omega; \theta)$; see Subsection 4.2.
4 Proof of the main result

4.1 Spacing along finite-volume trajectories

Owing to the Diophantine condition (1.3), for any cube \( \Lambda_L(u) \subset \mathbb{Z}^d \), \( L \geq 1 \), and any point \( \omega \in T^\nu \), we have

\[
\delta(\Lambda_L(u), \omega) := \min_{x,y \in \Lambda_L(u), x \neq y} \{ \text{dist}(T^x \omega, T^y \omega) \} \geq C L^{-B} \tag{4.1}
\]

for some \( C > 0 \). In fact, \( \delta(\omega) \) does not depend on \( \omega \), since the conventional distance on the torus is shift-invariant, so that below we drop the argument \( \omega \) in \( \delta(\Lambda_L(u)) = \delta(\Lambda_L(u), \omega) \). For the same reason, \( \delta(\Lambda_L(u)) \) does not depend upon \( u \), and we will use a simpler notation \( \delta_L \) for the quantity \( \delta(\Lambda_L(u)) \).

Given a positive integer \( n \), consider again the partition \( \mathcal{C}_n \) of the torus \( T^\nu \) into cubes \( C_{n,k} \), \( k = 1, \ldots, 2^{\nu n} \), of the form \( C_{n,k} = I_{n,i_1} \times \cdots \times I_{n,i_\nu} \), with \( I_{n,i} = \left[ \frac{i-1}{2^n}, \frac{i}{2^n} \right) \), as defined in Section 1. It is convenient for our purposes to use the max-norm in \( \mathbb{R}^\nu \),

\[
\|\omega\|_\infty := \max_{1 \leq i \leq \nu} |\omega_i|, \ \omega \in \mathbb{R}^\nu,
\]

and the distance induced by it on \( T^\nu = \mathbb{R}^\nu / \mathbb{Z}^\nu \). Below we always use this distance on the torus, unless otherwise specified. Then \( \text{diam} C_{n,k} = 2^{-n \nu} \), and we see that points of any finite-volume trajectory

\[
T(\omega, \Lambda_L(u)) = \{ T^x \omega, x \in \Lambda_L(u) \}
\]

are separated by elements of any partition \( \mathcal{C}_n \) with

\[
n \geq n_0(L) = \ln \delta_L / \ln 2 \geq C' \ln L \tag{4.2}
\]

for some \( 0 < C' < \infty \). As a consequence, given any point \( \omega \in T^\nu \), a positive integer \( n_0 \) and a lattice cube \( \Lambda_L(u) \), a finite family of RVs

\[
\{ \theta_{n_0,k} : \text{supp } \varphi_{n_0,k} \cap T(\omega, \Lambda_L(u)) \neq \emptyset \} \tag{4.3}
\]

(with \( n_0 \) fixed) is independent. Recall also that each of the above RVs \( \theta_{n_0,k} \) is uniformly distributed in \([0,1]\), so that its probability density is bounded by 1. Moreover, for any lattice cube \( \Lambda_L(u) \subset \mathbb{Z}^d \), so that \( \Pi \Lambda_L(u) \subset \Lambda_N(v) \), we see that points of any finite-volume trajectory

\[
T(\omega, \Pi \Lambda_L(u)) = \{ T^x \omega, x \in \Pi \Lambda_L(u) \}
\]

are separated by elements of any partition \( \mathcal{C}_n \) with

\[
n \geq n_0(N)
\]
4.2 Conditional independence of the potential values

Now we analyze the values of the function \( v : \mathbb{T}^v \times \Theta \rightarrow \mathbb{R} \) along the points of a given finite trajectory \( T(\omega, \Pi_{\Lambda_L}(u)) \). Relative to the product measure \( d\omega \times \mu \) on the product probability space \( \mathbb{T}^v \times \Theta \), these values are not independent. (Here, \( d\omega \) is the normalized Haar measure on the torus.) However, they are conditionally independent given the sigma-algebra \( \mathcal{B}(\Lambda_N(v)) \) generated by the RVs \( \omega_i, 1 \leq i \leq v \) and by all RVs \( \{\theta_{n,k} : n < n_0(N)\} \). Indeed, we can re-write the expansion (1.2) ("randelette" expansion) as follows:

\[
v(\omega; \theta) = \sum_{n=1}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega)
= \sum_{n < n_0(N)} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega) + \sum_{n \geq n_0(N)} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega)
\]

(4.3)

It is straightforward now that the first sum at the RHS becomes constant, given the sigma-algebra \( \mathcal{B}(\Lambda_N(v)) \). Fix two points \( \omega', \omega'' \in \mathbb{T}^v \) with \( \text{dist}(\omega', \omega'') \geq \delta_N \). For any \( n \geq n_0(\Lambda_L(u)) \), they are separated by the elements of partition \( \mathcal{C}_n \). Observe that, actually,

\[
a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega') = a_n \theta_{n,k(\omega')} \varphi_{n,k(\omega')}(\omega')
\]

where \( k(\omega') \) is uniquely defined by the condition

\[
\omega' \in \text{supp} \varphi_{n,k}
\]

(4.4)

Further, with \( k(\omega'') \) defined in a similar way for the point \( \omega'', k(\omega') \neq k(\omega'') \), since \( \omega' \) and \( \omega'' \) are separated by elements of \( \mathcal{C}_n \). This yields the representations

\[
v(\omega'; \theta) = \xi' + \sum_{n \geq n_0(N)} a_n \theta_{n,k(\omega')} \varphi_{n,k(\omega')}(\omega') = \xi' + \eta',
\]

\[
v(\omega''; \theta) = \xi'' + \sum_{n \geq n_0(N)} a_n \theta_{n,k(\omega'') \varphi_{n,k(\omega'')}(\omega'') = \xi'' + \eta'',
\]

(4.5)

where \( \xi', \xi'' \) are \( \mathcal{B}(\Lambda_N(v))-\text{measurable} \) and \( \eta', \eta'' \) are conditionally independent, given \( \mathcal{B}(\Lambda_N(v)) \). Actually, the entire family of the RVs \( v(T^x \omega; \theta), x \in \Pi_{\Lambda_L}(u) \), admits the decomposition

\[
v(T^x \omega; \theta) = \xi_x + \sum_{n \geq n_0(N)} a_n \theta_{n,k(T^x \omega)} \varphi_{n,k(T^x \omega)}(T^x \omega)
= \xi_x + \eta_x,
\]

(4.6)

where all \( \xi_x \) are \( \mathcal{B}(\Lambda_N(v))-\text{measurable} \) and the family of RVs \( \{\eta_x, x \in \Lambda_L(u)\} \) is independent. Respectively, the family \( \{\xi_x + \eta_x, x \in \Pi_{\Lambda_L}(u)\} \) is conditionally independent, given \( \mathcal{B}(\Lambda_N(v)) \), and so are the values \( \{v(T^x \omega, \eta), x \in \Pi_{\Lambda_L}(u)\} \).
Next, we re-write (4.6) as follows:

\[ v(T^x \omega; \theta) = \xi_x + a_{n_0(N)} \theta'_{n_0(N)} + \eta'_x, \]  

(4.7)

with

\[ \theta'_{n_0(N)} = \theta_{n_0(N), k(T^x \omega)} \varphi_{n_0(N), k(T^x \omega)}(T^x \omega) \]

and

\[ \eta'_x = \sum_{n > n_0(N)} a_n \theta_{n, k(T^x \omega)} \varphi_{n, k(T^x \omega)}(T^x \omega). \]

The RV \( \theta'_{n_0(N)} \) is uniformly distributed in \([-a_{n_0(N)}, a_{n_0(N)}]\), so its probability density \( p_{\theta'_{n_0(N)}} \) exists and is bounded by (2\( a_{n_0(N)} \))^{-1}. By virtue of (1.4) and (4.2), we have, therefore,

\[ a_{n_0(N)}^{-1} \leq \text{Const} n_0(N)^M \leq \text{Const} \ln M N. \]

The random variable \( \eta'_x \) admits some probability density \( p_{\eta'_x} \) (as a sum of a convergent series of RVs with uniform distributions). Since \( \theta'_{n_0(N)} \) and \( \eta'_x \) are independent, their sum \( \theta'_{n_0(N)} + \eta'_x \) admits a probability density given by a convolution \( p_{\theta'_{n_0(N)}} * p_{\eta'_x} \), which is bounded by the \( L^\infty \)-norm of any of them. Hence,

\[ \|p\|_\infty \leq (2a_{n_0(N)})^{-1} \leq \text{Const} \ln M L. \]

4.3 Two-particle Wegner-type bound for independent potentials

Here we recall the main results of ref. [CS08], with necessary notational adaptations. Their proofs are based on the observation that finite-volume two-particle Hamiltonians with an external random potential can be represented as diagonally monotone operator families, so that Stollmann’s method applies to such ensembles.

As before, \( F_V \) is the marginal CDF of the external random potential.

**Proposition 1** (Cf. Thm. 1 in [CS08]). Consider a two-particle Hamiltonian \( H = H_0 + (V(x_1; \omega) + V(x_2; \omega)) + U(x_1, x_2) \), where \( \{V(x; \omega), x \in \Pi \Lambda_L(u)\} \) is an IID random field relative to a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), with a marginal CDF \( F_V(t) \). Set

\[ s(\epsilon) = \sup_{a \in \mathbb{R}} \int_{a}^{a+\epsilon} dF_V(t), \quad 0 < \epsilon \leq 1. \]  

(4.7)

Then for all \( E \in \mathbb{R}, L \geq 1, u \in \mathbb{Z}^d \times \mathbb{Z}^d \) and \( \epsilon > 0 \),

\[ \mathbb{P} \{ \text{dist} \left[ \Sigma_{L,u} (\omega; \theta), E \right] \leq \epsilon \} \leq |\Lambda_L(u)|^{3/2} \cdot s(2\epsilon). \]  

(4.8)

**Proposition 2** (Cf. Thm. 2 in [CS08]). Under the same assumptions as in Theorem 4 consider a pair of two-particle cubes \( \Lambda_L(u), \Lambda_{L'}(u') \), \( L \geq L' \geq 1 \).
Let \( S : (u_1, u_2) \mapsto (u_2, u_1) \) be the symmetry in \( \mathbb{Z}^{2d} \) exchanging the coordinates of two particles and suppose that

\[
\min\{||u - u'||, ||u - S(u')||\} > 8L
\]  

(4.9)

Then for all \( E \in \mathbb{R} \) and \( \epsilon > 0 \),

\[
\mathbb{P} \{ \text{dist} [\Sigma_{u,L}(\omega; \theta), \Sigma_{u',L'}(\omega; \theta)] \leq \epsilon \} \leq |A_L(u)|^{3/2} \cdot |A_L(u')| \cdot s(2\epsilon).
\]  

(4.10)

### 4.4 One-volume Wegner-type bound for QP potentials

Now we return to the analysis of our two-particle Hamiltonian \( H_{A_L(u)} \) with quasi-periodic potential \( V(x; \omega; \theta) \) in a finite volume \( A_L(u) \subset \mathbb{Z}^{2d} \), where \( L \geq 1 \), \( u = (u_1, u_2) \in \mathbb{Z}^{2d} \), \( A_L(u) = A_L(u_1) \times A_L(u_2) \), and \( \Pi A_L(u) \subset A_N(v) \). Fix a real number \( E \). We have to estimate the probability

\[
\mathbb{P} \{ \text{dist} (\Sigma (H_{A_L(u)}), E) \leq \epsilon \} \leq \epsilon.
\]  

(4.7)

First, we can apply the identity

\[
\mathbb{P} \{ \text{dist} (\Sigma (H_{A_L(u)}), E) \leq \epsilon \} = \mathbb{E} [\mathbb{P} \{ \text{dist} (\Sigma (H_{A_L(u)}), E) \leq \epsilon \mid \mathcal{B}(A_N(v)) \} ],
\]  

(4.8)

so that it suffices to bound the inner, conditional probability. It is clear that the results of the Subsection 4.3 apply to the two-particle Hamiltonian \( H_{A_L(u)} \) under conditioning by the sigma-algebra \( \mathcal{B}(A_N(v)) \). Indeed, this conditioning makes the values of the external potential conditionally independent. Moreover, the conditional probability density for each value of the external potential \( V(x; \omega; \theta) \), \( x \in A_L(u_1) \cup A_L(u_2) \), admits a density bounded by \( a_{n_0(N)}^{-1} \) with \( n_0(N) \leq C' \ln N \).

By assumption (1.4), such a density is bounded by \( C'' M^N \), \( C'' M < \infty \). Therefore, owing to (1.4), the following inequality holds true:

\[
\mathbb{P} \{ \text{dist} (\Sigma (H_{A_L(u)}), E) \leq \epsilon \mid \mathcal{B}(A_N(v)) \} \leq C''' M^N \cdot L^{3d} \cdot \epsilon.
\]

Finally, we see that, by virtue of (4.8),

\[
\mathbb{P} \{ \text{dist} (\Sigma (H_{A_L(u)}), E) \leq \epsilon \} \leq C''' M^N \cdot L^{3d} \cdot \epsilon.
\]

This concludes the proof of Theorem 1. \( \square \)

### 4.5 Two-volume Wegner-type bound for QP potentials

We can argue as in the one-volume case, but apply now Proposition 2 instead of Proposition 1. We deal here with a pair of two-particle cubes \( A_L(u), A_L'(u') \), \( L \geq L' \geq 1 \). Suppose that \( \Pi (A_L(u') \cup A_L(u'')) \subset A_N(v) \subset \mathbb{Z}^d \) for some \( N \in \mathbb{N}, v \in \mathbb{Z}^d \) and condition (4.9) holds. We have to estimate the probability

\[
\mathbb{P} \{ \text{dist} [\Sigma_{u,L}(\omega; \theta), \Sigma_{u',L'}(\omega; \theta)] \leq \epsilon \}.
\]
First, we notice that
\[
\mathbb{P}\left\{ \text{dist} [\Sigma_{u,L}(\omega; \theta), \Sigma_{u',L'}(\omega; \theta)] \leq \epsilon \right\} = \mathbb{E}\left[ \mathbb{P}\left\{ \text{dist} [\Sigma_{u,L}(\omega; \theta), \Sigma_{u',L'}(\omega; \theta)] \leq \epsilon, |B(\Lambda_N(v))| \right\} \right].
\]
Therefore, one can apply the Proposition 2 of Subsection 4.3 to bound the conditional probability. Under this conditioning, the values of the external potential are independent, and the marginal conditional probability density for each value of the external potential \( V(x; \omega; \theta), x \in \Pi(\Lambda_L(u) \cup \Lambda_L(u')) \) is bounded by \( a_{n_0(N)}^{-1} \leq C'' \ln^M N \).

Therefore the following inequality holds true:
\[
\mathbb{P}\left\{ \text{dist} [\Sigma_{u,L}(\omega; \theta), \Sigma_{u',L'}(\omega; \theta)] \leq \epsilon, |B(\Lambda_N(v))| \right\} \leq C''' L^5 \ln^M N \cdot \epsilon.
\]
This concludes the proof of Theorem 2.

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