Quantum and Classical Binomial Distributions for the Charge Transmitted through Coherent Conductor.

G. B. Lesovik\textsuperscript{1)}, N. M. Chcthelkatchev\textsuperscript{1,‡}

\textsuperscript{1}L.D. Landau Institute for Theoretical Physics, Russian Academy of Sciences, 117940 Moscow, Russia.
\textsuperscript{‡}Institute for High Pressure Physics, Russian Academy of Sciences, Troitsk 142092, Moscow Region, Russia

Submitted

We discuss controversial results for the statistics of charge transport through coherent conductors. Two distribution functions for the charge transmitted was obtained previously, one actually coincides with classical binomial distribution the other is different and we call it here quantum binomial distribution. We show, that high order charge correlators, determined by the either distribution functions, can all be measured in different setups. The high order current correlators, starting the third order, reveal (missed in previous studies) special oscillating frequency dependence on the scale of the inverted time flight from the obstacle to the measuring point. Depending on setup, the oscillating terms give substantially different contributions.

PACS: 05.60.Gg

Last years has appeared new direction in quantum transport investigations — description of the statistics of a charge transmitted through a quantum conductor. Usually the distribution functions are investigated for a charge \(Q\), transmitted during a large interval of time \(t_0\) through a certain cross-section of the conductor, and, that is essential, all observables are calculated from the first principles. Despite of appreciable amount of articles (see the review \cite{Lesovik01}) and received results, some questions, in particular concerning the measurement theory, still remain unclear. One of the problems is that in a quantum case (in contrast to classical) arises the question how to define the observable that should be calculated. Technically this uncertainty is connected with noncommutativity of the current operators at various times. As it appears, it is possible to present several definitions for the distribution function (DF) and characteristic function (CF) which a) in the classical case coincide, b) satisfy some general principles (in particular, correlators \(\langle Q_n^m \rangle\) prove to be real), however lead to different answers in quantum case. It is possible to understand what definition is “correct” only having analyzed definite set-up for (at least gedanken) measurement.

In this paper we shall consider few variants of measurements and corresponding definitions of CF. We shall also comment the results received earlier. In the first paper devoted to microscopic description of the distribution function \cite{Lesovik02} the following definition was accepted:

\begin{equation}
\chi(\lambda) = \left\langle \exp \left( i\lambda \dot{Q}(t_0) \right) \right\rangle = \exp \left\{ i\lambda \int_0^{t_0} dt \dot{I}(t) \right\}
\end{equation}

(here \(\langle \ldots \rangle\) denotes ensemble averaging). This definition is the most direct generalization of the classical definition; it differs only in the replacement of the charge and current observables by the appropriate operators. Performing calculations we considered large intervals of time at which the correlators \(\langle \langle \dot{Q}(t_0)^n \rangle \rangle = \langle \langle \int_0^{t_0} dt_1 \ldots \int_0^{t_0} dt_n \dot{I}(t_1) \ldots \dot{I}(t_n) \rangle \rangle\) approximately equal to

\begin{equation}
\langle \langle \dot{Q}(t_0)^n \rangle \rangle = t_0 \langle \langle \dot{I}_0^n \rangle \rangle,
\end{equation}

where \(\langle \langle \dot{I}_0^n \rangle \rangle\) is the irreducible current correlator of \(n\)-th order at zero frequency limit. (The irreducible correlators satisfy the equation, \(\langle \exp \{i\lambda \dot{Q}(t)\} \rangle = \exp \{\sum_{n=1}^{\infty} (i\lambda)^n \langle \langle \dot{Q}(t)^n \rangle \rangle / n!\} \).

The method of calculation used in \cite{Lesovik01} can be generalized for the case of finite temperatures and (for normal single-channel conductors) we find

\begin{equation}
\chi(\lambda) = \exp \left\{ t_0 g \int \frac{d\epsilon}{2\pi \hbar} \ln \left\{ (1-n_L)(1-n_R) + n_Ln_R + n_L(1-n_R)\chi_e(\lambda) + n_R(1-n_L)\chi_e(-\lambda) \right\} \right\},
\end{equation}

where \(\chi_e(\lambda) = \cos \{\lambda e \sqrt{T(\epsilon)}\} + i \sqrt{T(\epsilon)} \sin \{\lambda e \sqrt{T(\epsilon)}\}\), \(g = 2\) is the factor taking into account spin degeneracy, \(T\) is the transparency, and \(n_L,R\) are the filling
factors in the left and right reservoirs correspondingly. This distribution (at $k_B T = 0, eV \neq 0$) we shall name quantum binomial distribution. In multi-channel case $\chi(\lambda)$ is a product $\Pi_n \chi_n(\lambda)$ of the characteristic functions $\chi_n(\lambda)$ corresponding to transmission eigenvalues $T_n$. The distribution function \(G\) formally describes fractional charge transport. At direct measurements of a charge in solid-state systems the fractional charge may in principle appear, for example, in the shot noise in conditions of fractional quantum Hall effect \[3\]. In our case the size of a "of a charge quantum" \(2e\sqrt{T}\) is determined by the eigen-value of the current operator provided that its action is considered on the subspace of one-particle excitations at the given energy \(e\sqrt{T} = \pm (\pm |I(t_0)|)\), where the normalized one-particle excitations \(|\pm\rangle\) satisfy the condition \(\langle +|I(t_0)|-\rangle = 0\), see also \[2\]. If we take into account the logarithmic on time \(t_0\) corrections to the irreducible correlators then the exact charge quantization which follows from the discrete distribution function is replaced apparently by small modulations in the continuous distribution function. If we limit ourselves to the corrections (occurring from the vacuum fluctuations) to the pair correlator then the distribution function becomes

\[
P(Q) = \sum_n P^{(0)}(ne\sqrt{T}) \left( \frac{2\pi}{Gh\ln(t_0\omega)} \right)^{1/2} \exp\left\{-(Q-ne\sqrt{T})^2/2Gh\ln(t_0\omega)\right\},
\]

where \(P^{(0)}(ne\sqrt{T})\) is the discrete distribution function with disregarded logarithmic corrections, \(G\) is the conductance, \(\omega\) is a characteristic frequency scale of the conductance dispersion.

Using Eq. \[5\] we find for the third order correlator:

\[
\langle \langle Q^3(t_0) \rangle \rangle = -t_0g \int \frac{d\epsilon}{2\pi\hbar} e^{3T^2(n_L-n_R)} \left\{ [3n_L(1-n_R) + n_R(1-n_L)] - 1 \right\} - 2T(n_L-n_R)^2 \bigg].
\]

At small voltage (and when the transparency \(T\) does not depend on energy) the correlator is proportional to \(V^3\) at large \(V\) we get \[2\]:

\[
\langle \langle Q^3(t_0) \rangle \rangle = -2e^3T^2(1-T)\frac{2eVt_0}{\hbar}.
\]

According to Eq. \[2\] the third order current correlator at zero frequencies is

\[
\langle \langle I^3(t) \rangle \rangle = -2e^3T^2(1-T)\frac{2eV}{\hbar}.
\]

Since it was not quite clear how to measure quantum binomial DF and correlators as in Eq. \[1\], it was suggested in Ref. \[1\] to use a spin located near to a wire as the counter of electrons passed through it. Thus it appears that the definition for CF in this case differs from \[1\] by the presence of the time ordering:

\[
\chi(\lambda) = \langle \hat{T}e^{\lambda/2} \int_0^{t_0} \int_{-t_0}^{t_0} e^{\lambda/2} e^{-\lambda/2} I(t) dt \rangle,
\]

in this (and only this) formula the symbol \(T\) means the usual time ordering, and \(\hat{T}\) means the ordering in the opposite direction.

It was found in \[14\] (see also references in \[1\]) for the third order correlator of the transmitted charge

\[
\langle \langle Q^3(t_0) \rangle \rangle = e^3T(1-T)(1-2T)\frac{2eVt_0}{\hbar}.
\]

As we see, the correlators of the third order \[4\] and \[7\] essentially differ. It would seem that there is nothing unexpected in such distinction as the definitions \[1\] and \[14\] differ. However, for example, the third order correlator of a charge according to both definitions \[1\] and \[14\] actually contains the correlator of currents at small frequencies; but such current correlator can be calculated with the help of the first definition \[1\] correctly at zero frequency limit \[14\] and it can be checked independently using the same machinery that was used in \[14\] for the calculation of the pair correlator \[7\]. It appears that the dispersion of the third order current correlator at small frequencies (along with the difference of the definitions) results in different answers for the third order correlators. Really, at frequencies \(\omega \ll eV/\hbar\) and \(x_{1,2,3} > 0\) we have

\[
\langle \langle I_{\omega_1}(x_1)I_{\omega_2}(x_2)I_{\omega_3}(x_3) \rangle \rangle = 2\pi\delta(\omega_1 + \omega_2 + \omega_3) \times T(1-T) e^{i\omega_1 x_1/V_F + i\omega_2 x_2/V_F + i\omega_3 x_3/V_F} \times [1 - 2T - e^{2i\omega_2 x_2/V_F}] eV \frac{2e^3}{\hbar}.
\]

When \(x_1 = x_2 = x_3 = x\)

\[
\langle \langle I_{\omega_1}(x)I_{\omega_2}(x)I_{\omega_3}(x) \rangle \rangle = 2\pi\delta(\omega_1 + \omega_2 + \omega_3) \times T(1-T)[1 - 2T - e^{2i\omega_2 x/V_F}] eV \frac{2e^3}{\hbar}.
\]

Formally assuming that such dependence on frequency is correct at all frequencies we find for the correlators in time representation:

\[
\langle \langle I(t_1,x)I(t_2,x)I(t_3,x) \rangle \rangle = eV \frac{2e^3}{\hbar} T(1-T) \times \text{Sym}
\]

\[
(1 - 2T)[\delta(t_1 - t_3) - \delta(t_1 - t_2) - \delta(t_2 - t_3) - \delta(t_1 - t_3)],
\]

where the symbol Sym means symmetrization on the indices \(i \neq j \neq k\); \(\tilde{t_2} \equiv t_2 + 2x/V_F\), \(\tilde{t_3} = t_1\), \(\tilde{t_3} = t_3\). (At the account of real frequency dependence instead
of \( \delta \)-functions there should stand functions which decay on characteristic times \( t \sim \tau_0 \), \(^2\) but for simplicity we shall describe the case with \( \delta \)-functions). Substituting this expression into the expression for the third order charge correlator which follows from (6) we get

\[
\langle \langle Q_T^3 \rangle \rangle = \frac{3}{4} \left[ \int_0^T dt_1 \int_0^T dt_2 \int_0^{t_2} dt_3 + \int_0^T dt_2 \int_0^{t_2} dt_1 \int_0^T dt_3 + \int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 + \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \right] \langle \langle I(t_1)I(t_2)I(t_3) \rangle \rangle \tag{11}
\]

and integrating over times we find at final \( x \) the answer \(^7\) proportional to \( T(1-T)(1-2T) \) that is typical for the classical binomial distribution (the same phenomenon takes place for other irreducible high-order correlators).

It occurs because terms in the Eq. (11) containing \( \delta \)-functions that depend from \( \tilde{t}_i \) do not give the contribution to the answer as they are not equal to zero only when simultaneously \( t_3 > t_2 \) and \( t_1 > t_2 \) and the integration volume in Eq. (11) does not cover such sector. Consider, e.g., the contribution to the correlator \( \langle \langle Q_T^3 \rangle \rangle \) from the term in Eq. (11) proportional to

\[
\delta(\tilde{t}_1 - \tilde{t}_3)\delta(\tilde{t}_3 - \tilde{t}_2) = \delta(t_1 - t_3)\delta(t_3 - t_2 - 2x/v_F). \tag{12}
\]

From Eq. (12) follows that the region where \( t_1 \approx t_3 \approx t_2 + 2x/v_F \) in Eq. (11) should give the leading contribution to the integrals. But from the requirement \( x > 0 \) follows that \( t_3 > t_2 \) and \( t_1 > t_2 \). The region defined by these inequalities does not overlap with the volume of the integration in Eq. (11): thus the contribution (12) to \( \langle \langle Q_T^3 \rangle \rangle \) is equal to zero. It is similarly possible to show that generally all terms in Eq. (11) proportional to \( \text{Sym}\delta(\tilde{t}_i - \tilde{t}_j)\delta(\tilde{t}_j - \tilde{t}_k) \) at \( x > 0 \) do not give the contribution to \( \langle \langle Q_T^3 \rangle \rangle \).

One could say that when the incident wave packet first fully passes the detector and only then, with a time delay, the detector goes back the part of the wave packet reflected from the barrier, the specific quantum interference disappears and the answer (11) is true. But if the distance to the detector is small then the incident wave packet interferes with the reflected one in the measurement region that leads eventually to the answer (11). So Eq. (7) is true when the time of electron flight from the scatterer to the spin of the detector that is situated at the distance \( d \) from the wire and \( L \) from the scatterer is larger than the decay time \( \tau_0 \) of the correlators. If these requirements are violated the answer will be different. In the calculations described in Ref. 4 though it was formally assumed that the distance \( L \) is equal to zero, in fact was considered the limit when \( L \) actually exceeded the wave packet size (which was also kept to be zero).

For the case when the spin-detector is located near to the scatterer \( x \ll v_F\tau_0 \) and it is close to the wire \( d \ll v_F\tau_0 \), the answer for \( \langle \langle Q_T^3 \rangle \rangle \) is proportional to \( -T^2(1-T) \) and it coincides with the answer (11) obtained from the quantum distribution. Really, using the general expression for the correlator (8) at \( x_{1,2,3} \ll v_F\tau_0 \) we get

\[
\langle \langle I_{\omega_1}(x_1)I_{\omega_2}(x_2)I_{\omega_3}(x_3) \rangle \rangle \simeq -2\pi\delta(\omega_1 + \omega_2 + \omega_3)2T^2(1-T)eV\frac{2e^3}{h}. \tag{13}
\]

Using this expression at \( \omega \ll eV/h \) and the definition (11) we obtain the expression for \( \langle \langle Q_T^3 \rangle \rangle \) proportional to \( -T^2(1-T) \) as well as in the calculation with the use of CF (11).

The measurement with the spin basically may be implemented in practice with the help of muons which can be trapped near to the conductor and then the measurement of the direction of their decay would give the angle of their spin rotation in a magnetic field. As an additional example of the measuring procedure which basically can be implemented practically, we have analyzed the measurement of the irreducible charge correlators with the help of a ammeter represented by a semiclassical system (for example, an oscillatory circuit) weakly interacting with the current in a quantum conductor. The state of the ammeter is characterized by the magnitude \( \phi \). Interaction of the ammeter with a quantum conductor is described by the interaction Hamiltonian \( H_i = \lambda\phi\hat{I}(t) \), where \( \lambda \) is the interaction constant, \( \hat{I}(t) \) is the current operator in the quantum conductor, \( \hat{I}(t) = \int \hat{I}(t,x)f(x)dx \) (we do not take into account effects of retardation), thus the area of the integration is determined by some kernel \( f(x) \). Correlators \( \phi \) are expressed through correlators of currents in a quantum conductor as follows:

\[
\langle \langle (\phi(t))^n \rangle \rangle = \left( -\frac{\lambda}{2} \right)^n \int d\tau_1 \ldots d\tau_n \kappa(|t - \tau_1|)\text{sign}(t - \tau_1) \ldots \kappa(|t - \tau_n|)\text{sign}(t - \tau_n) \times \langle \langle \langle T_{\tau_1}(I(\tau_1) \ldots I(\tau_n)) \rangle \rangle, \tag{14}
\]

where the integration is performed along the usual Keldysh contour; \( \kappa(\tau) \) is the susceptibility of the am-
meter. In that specific case when the ammeter represents an oscillator, the equation of motion for $\phi$ is:

$$\ddot{\phi} + \gamma \dot{\phi} + \Omega^2 \phi = \lambda(t) \exp(-\gamma t/2) \sin(\tilde{\Omega}t)/M\tilde{\Omega},$$

where $\tilde{\Omega} = \sqrt{\Omega^2 - \gamma^2}/4$. The case $\gamma \approx 2\Omega$, $\gamma \ll 1/\tau_0$ is the most interesting for us. Then

$$\langle \langle \phi^3(0) \rangle \rangle \approx \langle \langle I_0^3 \rangle \rangle \lambda^3 \int \frac{d\omega_1, \omega_2, \omega_3}{(2\pi)^3} \kappa(\omega_1) \kappa(\omega_2) \kappa(\omega_3) \times \delta(\omega_1 + \omega_2 + \omega_3) = 2 \frac{\langle \langle I_0^3 \rangle \rangle \lambda^3}{27M^4\Omega^4}, \quad (15)$$

$$\langle \langle \phi^n(0) \rangle \rangle \approx \frac{n!}{n+1} \frac{\langle \langle I_0^n \rangle \rangle \lambda^n}{M^n\Omega^{n+1}}, \quad (16)$$

where $\langle \langle I_0^n \rangle \rangle$ is the irreducible current correlator of the order of $n$, defined by the quantum binomial distribution [in (16) we neglected the contribution of the own thermal ammeter noise.]

Thus measuring irreducible correlators of coordinate $\phi$ of the ammeter it is possible to measure irreducible correlators of currents of high orders in a limit of zero frequencies (in particular $\langle \langle I_0^3 \rangle \rangle \propto -T^2(1-T)$). Such measurements are possible also under less restrictive requirements on frequencies of the ammeter if the kernel $f(x)$ defines the area of an integration so that $x \ll v_F\tau_0$. In the opposite limit it is natural to expect “classical” answers for the correlators (in particular $\langle \langle I_0^n \rangle \rangle \propto T(1-T)(1-2T)$).

We are grateful to M. Reznikov and especially to D. Ivanov for fruitful discussions. D.Ivanov paid our attention to special coordinate and frequency dependence of the third order correlators which appeared in essence important for reviewing various conditions of measuring. We also are grateful to M. Feigelman for reading manuscript and useful remarks.

Our work is supported by the Russian Science-Support Foundation, Russian Foundation for Basic Research (RFBR), the Russian ministry of science (the project “Physics of quantum computations”), SNF (Switzerland).

1. L.S. Levitov, The statistical theory of mesoscopic noise, cond-mat/0210284
2. L.S. Levitov, G.B. Lesovik, Pis’ma Zh. Eksp. Teor. Fiz. 55, 534 (1992) [JETP Lett. 55, 555 (1992)]
3. C.L. Kane, M.P.A. Fisher, Phys. Rev. Lett. 72, 724 (1994); L. Saminadayar et al., Phys. Rev. Lett. 79, 2526 (1997); de-Picciotto et al., Nature 389, 162 (1997).
4. L.S. Levitov, G.B. Lesovik, Quantum Measurement in Electric Circuit, cond-mat/ 9401004; L. S. Levitov, H.W. Lee, G.B. Lesovik, J. of Math. Phys. 37 4845-4866 (1996)

5. L.S. Levitov, G.B. Lesovik, Pis’ma Zh. Eksp. Teor. Fiz. 58, 225 (1993)[JETP Lett. 58, 230 (1993)]
6. G.B. Lesovik, Pis’ma Zh. Eksp. Teor. Fiz. 49, 513 (1989)[JETP Lett. 49, 592 (1989)]

7. Note, that quantum distribution function (3) may serve mathematically as a precise generating function for irreducible correlators of a current of any order at zero frequencies. The correspondence of the correlators of currents to the correlators of charges is given by the formula (2).