COMPUTATION OF ATOMIC FIBERS OF $\mathbb{Z}$-LINEAR MAPS

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Abstract. For given matrix $A \in \mathbb{Z}^{d \times n}$, the set $P_b = \{ z : Az = b, z \in \mathbb{Z}_+^n \}$ describes the preimage or fiber of $b \in \mathbb{Z}^d$ under the $\mathbb{Z}$-linear map $f_A : \mathbb{Z}_+^n \rightarrow \mathbb{Z}^d, x \mapsto Ax$. The fiber $P_b$ is called atomic, if $P_b = P_{b_1} + P_{b_2}$ implies $b = b_1$ or $b = b_2$. In this paper we present a novel algorithm to compute such atomic fibers. An algorithmic solution to appearing subproblems, computational examples and applications are included as well.

1. Introduction

Decomposition of rational polyhedra is at the heart of several interesting applications. However, there are different definitions of decomposability depending on the application. These definitions mainly differ in the treatment of the (integer) points of the polyhedron.

The simplest notion is that of linear decomposition of polyhedra. Two polyhedra $P, Q \subseteq \mathbb{R}^n$ are called homothetic if $P = \lambda Q + t$ for some $\lambda > 0$ and $t \in \mathbb{R}^n$. Here, a polyhedron $P$ is called indecomposable, if any decomposition $P = Q_1 + Q_2$ implies that both $Q_1$ and $Q_2$ are homothetic to $P$. It can be shown that there are only finitely many indecomposable rational polyhedra that are not homothetic to each other. For further details on this type of decomposition we refer the reader for example to Grünbaum (1967), Henk et al. (2003), Kannan et al. (1990), McMullen (1973), Meyer (1974), Smilanski (1987).

Let us now come to a bit more restrictive decomposition. Here we consider only polyhedra of the form $\{ x \in \mathbb{R}^n : Ax \leq b \}$ for a given matrix $A \in \mathbb{Z}^{d \times n}$ and varying $b \in \mathbb{Z}^d$. To emphasize that we only consider integer right-hand sides, we say that a polyhedron $P$ is integrally indecomposable, if any decomposition $P = Q_1 + Q_2$ (into polyhedra with integer right-hand sides) implies that both $Q_1$ and $Q_2$ are homothetic to $P$. This decomposition is more restrictive than the linear decomposition, since only such polyhedra $Q_1$ and $Q_2$ are allowed that have an integer right-hand side. Henk et al. (2003) showed finiteness of the system of integrally indecomposable polytopes. This result implies important applications: TDI-ness of each member of a family of systems $Ax \leq b, b \in \mathbb{Z}^d$, can be concluded from TDI-ness of the integrally indecomposable systems. Furthermore the finiteness of the system of integrally indecomposable polytopes enables us to compute a finite representation of a test set for a mixed-integer linear optimization problem.

Another important application of integral decomposability of polyhedra is that of factorizing a multivariate polynomial, see for example Abu Salem et al. (2004) and the references therein. Here, one considers only polyhedra of the form $\{ x \in \mathbb{R}^n : Ax \leq b \}$ for given matrix $A \in \mathbb{Z}^{d \times n}$ and varying $b \in \mathbb{Z}^d$, where each polyhedron is integer, that is, where each polyhedron has only integer vertices. Note that the notion of integral decomposability is restricted to integral polyhedra in this application whereas the definition of Henk et al. (2003) is valid for arbitrary rational polyhedra with integral right-hand side. The reason for this restriction is the simple observation that the so-called Newton polytope $\text{Newt}(f) := \text{conv}\{ \alpha_i \in \text{supp}(f) \}$ associated to a polynomial $f = \sum_{\alpha_i \in I} a_i x^{\alpha_i}$.

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with supp(f) = {αi: ai ≠ 0} is integer by definition. Moreover, the relation f = gh among three polynomials f, g, and h implies Newt(f) = Newt(g) + Newt(h), a theorem due to Ostrowski.

A direct generalization of the above notion of integral decomposition of integral polyhedra was introduced by Adams et al. (1999). They considered polytopes
\[ \tilde{P}_b := \text{conv}\{ z : Az = b, z \in \mathbb{Z}_+^n \} \]
called the fibers of b under the linear map \( f_A: \mathbb{Z}_+^n \rightarrow \mathbb{Z}^d, x \mapsto Ax \). A fiber \( \tilde{P}_b \) is called atomic if \( \tilde{P}_b = \tilde{P}_{b_1} + \tilde{P}_{b_2} \) implies \( b = b_1 \) or \( b = b_2 \). Note that \( \tilde{P}_b = \tilde{P}_{b_1} + \tilde{P}_{b_2} \) means that every vertex of \( \tilde{P}_b \) is the sum of a vertex of \( \tilde{P}_{b_1} \) and a vertex of \( \tilde{P}_{b_2} \) (and vice versa). Atomic fibers were used by Adams et al. (1999) to construct strong SAGBI bases for subalgebras of polynomial rings. They proved that the family of atomic fibers is finite and also gave an algorithm to compute atomic fibers via certain standard pairs. Via this algorithm, Adams et al. (1999) computed the atomic fibers of the twisted cubic, see Example 2.7.

In this paper, we consider a slight variation of the notion of atomic fibers that was introduced by Maclagan (2001). Instead of considering convex hulls \( \tilde{P}_b \) of the preimages
\[ P_b := \{ z : Az = b, z \in \mathbb{Z}_+^n \}, \]
of the map \( f_A: \mathbb{Z}_+^n \rightarrow \mathbb{Z}^d, x \mapsto Ax \), we consider the preimages \( P_b \) themselves. In Maclagan’s more general terminology, the sets \( P_b \) are called ((0), A)-fibers; we shall simply call them fibers in the remainder of this paper. We call a fiber \( P_b \) indecomposable or atomic, if \( P_b = P_{b_1} + P_{b_2} \) implies \( b = b_1 \) or \( b = b_2 \). Note that \( P_b = P_{b_1} + P_{b_2} \) means that every lattice point of \( P_b \) is the sum of a lattice point of \( P_{b_1} \) and a lattice point of \( P_{b_2} \) (and vice versa). This is indeed a very strong condition, but again it was shown that there are only finitely many (nonempty) atomic fibers for a given matrix \( A \) (Maclagan, 2001). Note that atomic fibers are not only minimal (with respect to decomposability) within the given family, but also generate every fiber \( P_b \) in this family as a Minkowski sum \( P_b = \sum_{i=1}^k \alpha_i P_{b_i}, \alpha_i \in \mathbb{Z}_+^n \), where \( \alpha_i P_{b_i} \) stands for iterated Minkowski-addition of \( P_{b_i} \) with itself. Atomic fibers (of this kind) were used in the computation of minimal vanishing sums of roots of unity (Steinberger, 2004).

Recently, the computation of atomic fibers also appeared as a subproblem in the capacitated design of telecommunication networks for a given communication demand under survivability conditions (Eisenschmidt et al., 2006). In this application, the right-hand side vectors \( b \) are taken from a sublattice or a submonoid of \( \mathbb{Z}^d \). Of course, restricting the set of “feasible” right-hand sides changes the notion of decomposability, thus the set of atomic fibers is changed.

Another related notion is that of extended atomic fibers. We call the set
\[ Q_b := \{ z : Az = b, z \in \mathbb{Z}^n \} \]
an extended fiber of the linear map of \( A \). We call it atomic, if \( (Q_b \cap O_j) = (Q_{b_1} \cap O_j) + (Q_{b_2} \cap O_j) \) holds for all the \( 2^n \) orthants \( O_j \) of \( \mathbb{R}^n \), then \( b = b_1 \) or \( b = b_2 \). Here, as well, it can be shown that there are only finitely many (nonempty) extended atomic fibers for a given matrix. Also this very strong notion of decomposability has an application: the set \( \mathcal{H}_\infty \) constructed in Hemmecke and Schultz (2003) for use in two-stage stochastic integer programming is in fact the set of extended atomic fibers of the family of extended fibers
\[ \{(x, y) : x = b, Tx + Wy = 0, x \in \mathbb{Z}^m, y \in \mathbb{Z}^n \} \]
where \( T \) and \( W \) are kept fixed and where \( b \) varies.

Outline. In this paper, we are mainly concerned about designing efficient algorithms for computing atomic and extended fibers. The outline of the paper is as follows. In section 2 we first define a hierarchy of partially extended fibers that interpolate between fibers and extended fibers. This hierarchy not only generalizes the notions of fibers and extended fibers, but also plays a significant
role in our algorithms. Motivated by our application in survivable network design, we define decomposability with respect to a given finitely generated monoid of feasible right-hand side vectors. We prove that, in this more general situation as well, there are only finitely many atomic fibers. We also present an algorithmic way to decompose a fiber into a Minkowski sum of indecomposable fibers.

In section 3 we present a first algorithm to compute the atomic extended fibers of a given matrix, following the pattern of a completion procedure. We present the algorithm in a simplified setting where the right-hand side vectors are restricted to a sublattice (rather than a submonoid) of \( \mathbb{Z}^d \). By restricting the atomic extended fibers to the positive orthants and performing a simple reduction step, the atomic fibers (or partially extended fibers) of a matrix can be easily obtained. However, this method is not a very efficient one for computing atomic fibers.

Therefore, we present a more efficient way to compute atomic fibers via a project-and-lift approach in section 4 and 5. We present the method in the general setting where a finitely generated monoid of right-hand sides is given by its generators.

Both our algorithms enable us to compute not only the atomic fibers \( P_b \) but also the atomic fibers \( \tilde{P}_b \) according to the definition in Adams et al. (1999). This will be shown at the end of section 3.

Finally, in section 6, we present first computational results of the project-and-lift algorithm.

2. (Partially Extended) Atomic Fibers

Let us now start our treatment with a formal definition of partially extended fibers.

**Definition 2.1.** Let \( A \in \mathbb{Z}^{d \times n} \) be a matrix, \( b \in \mathbb{Z}^d \) and \( 0 \leq k \leq n \).

(i) The set

\[
Q^{(k)}_b := \{ z : Az = b, z \in \mathbb{Z}^k_+ \times \mathbb{Z}^{n-k} \}
\]

is called an partially extended fiber of order \( k \) of the matrix \( A \). The set \( Q_b := Q^{(0)}_b \) is called an extended fiber, and \( P_b := Q^{(n)}_b \) is called a fiber of the matrix \( A \).

(ii) Let \( 0 \leq l \leq n \). For \( u, v \in \mathbb{R}^n \) we say that \( u \preceq_l v \) if \( u^{(i)} \geq 0 \) and \( |u^{(i)}| \leq |v^{(i)}| \) for all components \( i = 1, \ldots, l \). We will abbreviate \( \preceq_n \) by \( \preceq \). For \( U, V, W \subseteq \mathbb{R}^n \) we say that

\[
U = V \oplus W
\]

and call \( U \) the \( l \)-restricted Minkowski sum of \( V \) and \( W \), if for all \( u \in U \) there exist \( v \in V \), \( w \in W \) with \( v, w \preceq_l u \) and \( u = v + w \). Note that \( V \oplus W \) is just the ordinary Minkowski sum \( V + W \). We will abbreviate \( \oplus^{(n)} \) by \( \oplus \).

(iii) For \( 0 \leq m \leq n \) we will denote by \( \pi_m : \mathbb{R}^n \rightarrow \mathbb{R}^m \) with \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m) \) the projection onto the first \( m \) components.

Now we will go on defining atomic partially extended fibers w.r.t. a certain monoid \( M \subseteq \mathbb{Z}^d \). To accompany the hierarchy of partially extended fibers, we define a hierarchy of notions of decomposition that interpolates between ordinary Minkowski sums and orthant-wise Minkowski sums.

**Definition 2.2.** Let \( A \in \mathbb{Z}^{d \times n} \) be a matrix, \( b \in \mathbb{Z}^d \) and \( 0 \leq k, l \leq n \). Additionally, let \( M \subseteq \mathbb{Z}^d \) be a monoid.

(i) We call \( Q^{(k)}_b \) atomic w.r.t. \( \oplus^{(l)} \) and \( M \) if there is no decomposition

\[
Q^{(k)}_b = Q^{(k)}_{b_1} \oplus Q^{(k)}_{b_2}
\]

with \( b_1, b_2 \in M \) and \( \pi_l(Q^{(k)}_{b_1}), \pi_l(Q^{(k)}_{b_2}) \neq \pi_l(Q^{(k)}_0) \). By \( E^{(k)}_l(A, M) \) we denote the set of partially extended fibers of order \( k \) which are atomic w.r.t. \( \oplus^{(l)} \) and \( M \).
We denote by $E^{(k)}(A, M)$ the set $E^{(k)}_n(A, M)$ and call it the set of partially extended atomic fibers w.r.t. the monoid $M$. We denote by $F(A, M)$ the set $E^{(k)}(A, M)$ and call it the set of atomic fibers w.r.t. $M$.

Note that Definition 2.2 also applies to the special case where the monoid $M$ is a lattice. We will see later on that it is much easier to compute the atomic (partially extended) fibers of a matrix w.r.t. a lattice instead of an arbitrary monoid.

As our first step, we prove a generalization of the finiteness result for the family of atomic fibers.

**Lemma 2.3.** Let $0 \leq k \leq n$ be fixed. There are only finitely many partially extended fibers $Q_b^{(k)}$ which are atomic w.r.t. a finitely generated monoid $M$.

The proof of this lemma is based on the following nice theorem.

**Theorem 2.4 (Maclagan, 2001).** Let $k$ be a field. Let $I$ be an infinite family of monomial ideals in a polynomial ring $k[x_1, \ldots, x_n]$. Then there must exist ideals $I, J \in I$ with $I \subseteq J$.

To apply this theorem in our situation of partially extended fibers which are atomic w.r.t. a certain finitely generated monoid $M$, we introduce the following definition.

**Definition 2.5.** Let $A \in \mathbb{Z}^{d \times n}$ and $M = \langle m_1, \ldots, m_l \rangle \subseteq \mathbb{Z}^d$ a finitely generated monoid. Let $0 \leq k \leq n$ be fixed.

(i) Let $\alpha, \alpha \in \mathbb{Z}^l$ with

$$b := \sum_{i=1}^t \alpha_i m_i \quad \text{and} \quad \bar{b} := \sum_{i=1}^t \bar{\alpha}_i m_i.$$ 

We say that $(\alpha, Q_b^{(k)})$ reduces $(\alpha, Q_b^{(k)})$ and denote

$$(\alpha, Q_b^{(k)}) \leq (\alpha, Q_b^{(k)})$$ 

if $\alpha \subseteq \alpha$ and $Q_b^{(k)} = Q_b^{(k)} \oplus Q_b^{(k)}$. In particular, $b \bar{b} \in M$.

(ii) We call a pair $(\alpha, Q_b^{(k)})$ irreducible w.r.t. $\leq$ if there is no pair $(\alpha, Q_b^{(k)})$ different from $(\alpha, Q_b^{(k)})$ and $(0, Q_b^{(k)})$ with

$$(\alpha, Q_b^{(k)}) \leq (\alpha, Q_b^{(k)}).$$

**Lemma 2.6.** Let $0 \leq k \leq n$ be fixed. Let $A = \{ (\alpha^1, Q_{b_1}^{(k)}), (\alpha^2, Q_{b_2}^{(k)}), \ldots \} \subseteq \mathbb{A}$ be a set of pairs.

(i) Let $(\alpha^1, Q_{b_1}^{(k)}) \not\equiv (\alpha^2, Q_{b_2}^{(k)})$ for all $(\alpha^1, Q_{b_1}^{(k)}), (\alpha^2, Q_{b_2}^{(k)}) \in A$ with $i < j$. Then $A$ is finite.

(ii) There are only finitely many pairs $(\alpha, Q_{b}^{(k)})$ which are irreducible w.r.t. $\leq$.

**Proof.** (i) We associate with a pair $(\alpha^1, Q_{b_1}^{(k)})$ the monomial ideal

$$I_{\alpha^1} = \langle x_1^+, \ldots, z_k^+, \ldots z_{n-k}^+, \ldots, z_n^+, \alpha_1^{\alpha^1}, \ldots, \alpha_t^{\alpha^1} \rangle : A \mathbb{Z} = \sum_{i=1}^n \alpha_i^{\alpha^1} m_i = b_j, \ z \in \mathbb{Z}^+ \times \mathbb{Z}^{n-k} \subseteq Q[x_1, \ldots, x_{2n+t}]$$

where $z_k^+ = \max \{ 0, z_k \}$ and $z_n^+ = \max \{ 0, -z_i \}$. Then $(\alpha^1, Q_{b_1}^{(k)}) \not\equiv (\alpha^1, Q_{b_1}^{(k)})$ if $I_{\alpha^1}$ is not contained in $I_{\alpha^1}$. Consider the set $I = \{ I_{\alpha^1}, I_{\alpha^1}, \ldots \}$ of ideals associated to the elements in the set $A$. The set $I$ then is an antichain of ideals and is thus finite according to 2.4 (see Maclagan (2001)). The finiteness of $A$ follows from the finiteness of $I$.

(ii): A pair $(\alpha, Q_{b}^{(k)})$ is irreducible w.r.t. $\leq$ if and only if $(\alpha, Q_{b}^{(k)}) \not\equiv (\alpha, Q_{b}^{(k)})$ for any $\alpha \neq \alpha$. Let $A = \{ (\alpha^1, Q_{b_1}^{(k)}), (\alpha^2, Q_{b_2}^{(k)}), \ldots \}$ be the set of pairs which are irreducible w.r.t. $\leq$. Part (i) then yields that $A$ is finite.
We are now ready to prove Lemma 2.3.

**Proof of Lemma 2.3.** It is sufficient to show: for every \(Q^{(k)}_b\) atomic w.r.t. \(M\) there exists \(\alpha \in \mathbb{Z}_+^t\) with \(b = \sum_{i=1}^t \alpha_i m_i\) such that \((\alpha, Q^{(k)}_b)\) is irreducible w.r.t. \(\leq\). Then there is an injective mapping from the set of atomic extended fibers \(Q^{(k)}_b\) into the set of irreducible pairs \((\alpha, Q^{(k)}_b)\) and thus there are only finitely many extended atomic fibers w.r.t. \(M\).

Let \(b\) be fixed with \(Q^{(k)}_b\) an extended atomic fiber w.r.t. \(M\). Let \(\alpha \in \mathbb{Z}_+^t\) with \(b = \sum_{i=1}^t \alpha_i m_i\) be minimal w.r.t. \(\subseteq\), i.e., there is no \(\alpha \in \mathbb{Z}_+^t\) such that \(\alpha_i m_i \neq \alpha \) with \(\alpha \subseteq \alpha\) and \(b = \sum_{i=1}^t \alpha_i m_i\). We claim that the pair \((\alpha, Q^{(k)}_b)\) is irreducible w.r.t. \(\leq\). Suppose not. Then there is \((\bar{\alpha}, Q^{(k)}_b) \subseteq (\alpha, Q^{(k)}_b)\), i.e., \(\bar{\alpha} \subseteq \alpha\) and \(Q^{(k)}_b = Q^{(k)}_{\bar{b}} \oplus Q^{(k)}_{b-\bar{b}}\) implying \(b - \bar{b} \in M\). As \(Q^{(k)}_b\) is an extended atomic fiber we may w.l.o.g. assume that \(\bar{b} = b\) and \(b - \bar{b} = 0\). Therefore \(b = \sum_{i=1}^t \alpha_i m_i\) and as \(\alpha\) is minimally chosen w.r.t. \(\subseteq\) we have \(\bar{\alpha} = \alpha\). This proves our claim.

**Example 2.7.** In *Adams et al. (1999)*, it was shown how atomic fibers could be used to construct strong SAGBI bases for monomial subalgebra over principal ideal domains. As an example, they computed the atomic fibers of the matrix \(A = (\frac{1}{3} \frac{2}{1} \frac{1}{3})\) by hand via an approach different from the one we present below.

In the table below, we list the right-hand sides and all (finitely many) elements in these 18 atomic fibers.

\[
\begin{align*}
(0, 3) & \{ (0, 0, 0, 1) \} \\
(1, 2) & \{ (0, 1, 0, 0) \} \\
(2, 1) & \{ (0, 1, 0, 0) \} \\
(3, 0) & \{ (0, 0, 0, 0) \} \\
(2, 4) & \{ (0, 1, 0, 1), (0, 0, 2, 0) \} \\
(3, 3) & \{ (0, 0, 1, 0), (1, 1, 1, 0) \} \\
(4, 2) & \{ (0, 2, 0, 0), (1, 0, 1, 0) \} \\
(3, 6) & \{ (1, 0, 0, 2), (0, 1, 1, 1), (0, 0, 3, 0) \} \\
(4, 5) & \{ (0, 2, 0, 1), (0, 1, 2, 0), (1, 0, 1, 1) \} \\
(5, 4) & \{ (1, 1, 0, 1), (0, 2, 1, 0), (1, 0, 2, 0) \} \\
(6, 3) & \{ (2, 0, 1, 0), (1, 1, 1, 0), (0, 3, 0, 0) \} \\
(4, 8) & \{ (0, 2, 0, 2), (1, 0, 1, 2), (0, 1, 2, 1), (0, 0, 4, 0) \} \\
(6, 6) & \{ (2, 0, 0, 2), (0, 3, 0, 1), (1, 1, 1, 1), (1, 0, 3, 0), (0, 2, 2, 0) \} \\
(8, 4) & \{ (2, 1, 0, 1), (0, 4, 0, 0), (1, 2, 1, 0), (2, 0, 2, 0) \} \\
(6, 9) & \{ (2, 0, 3, 0, 2), (1, 1, 1, 2), (1, 0, 3, 1), (0, 2, 2, 1), (0, 1, 4, 0) \} \\
(9, 6) & \{ (3, 0, 0, 2), (1, 3, 0, 1), (2, 1, 1, 1), (2, 0, 3, 0), (1, 2, 2, 0), (0, 4, 1, 0) \} \\
(6, 12) & \{ (2, 0, 0, 4), (0, 3, 0, 3), (1, 1, 1, 3), (1, 0, 3, 2), (0, 2, 2, 2), (0, 1, 4, 1), (0, 0, 6, 0) \} \\
(12, 6) & \{ (4, 0, 0, 2), (2, 3, 0, 1), (3, 1, 1, 1), (3, 0, 3, 0), (2, 2, 2, 0), (0, 6, 0, 0), (1, 4, 1, 0) \}
\end{align*}
\]

Thus, for example, the fiber given by the right-hand side \((8, 7)\) is not atomic, since it can be decomposed into atomic fibers as

\[
P_{(\frac{8}{7})} = P_{(\frac{5}{4})} \oplus P_{(\frac{3}{3})}.
\]

This can be quickly verified by looking at the elements in these fibers:

\[
\{ (2, 1, 0, 2), (2, 0, 2, 1), (1, 1, 3, 0), (1, 2, 1, 1), (0, 4, 0, 1), (0, 3, 2, 0) \} = \{ (0, 1, 0, 1), (0, 0, 2, 0) \} \oplus \{ (2, 0, 0, 1), (1, 1, 1, 0), (0, 3, 0, 0) \}.
\]
Indeed, we have

\[
\begin{align*}
(2, 1, 0, 2) &= (0, 1, 0, 1) + (2, 0, 0, 1), \\
(2, 0, 2, 1) &= (0, 0, 2, 0) + (2, 0, 0, 1), \\
(1, 1, 3, 0) &= (0, 0, 2, 0) + (1, 1, 1, 0), \\
(1, 2, 1, 1) &= (0, 1, 0, 1) + (1, 1, 1, 0), \\
(0, 4, 0, 1) &= (0, 1, 0, 1) + (0, 3, 0, 0), \\
(0, 3, 2, 0) &= (0, 0, 2, 0) + (0, 3, 0, 0).
\end{align*}
\]

\[\square\]

In Example 2.7 above, it was easy to verify whether a given fiber is a summand in the decomposition of another fiber by simply checking the finitely many elements in the fiber for a decomposition. If the fibers are not bounded, however, this would not give a finite procedure. The following lemma tells us how to solve this problem via the (finitely many!) \(\subseteq\)-minimal elements in the given fibers.

**Definition 2.8.** Let \(A \in \mathbb{Z}^{d \times n}\) and \(b \in \mathbb{Z}^d\). Let \(0 \leq k \leq l \leq n\).

(i) An element \(v \in Q_b^{(k)}\) is called minimal w.r.t. \(\subseteq_l\) if there is no \(w \in Q_b^{(k)}\) with \(v \neq w\) and \(w \subseteq_l v\).

(ii) We define \(z, \tilde{z} \in Q_b^{(k)}\) to be equivalent if and only if \(\pi_l(z) = \pi_l(\tilde{z})\).

For \(l < n\) there are infinitely many \(\subseteq_l\)-minimal elements in general. Therefore we have to restrict ourselves to representatives of equivalence classes of \(\subseteq_l\)-minimal elements. Let \(R_{b,l}^{(k)}\) denote a set of representatives of the equivalence classes of the \(\subseteq_l\)-minimal elements in \(Q_b^{(k)}\). Let these representatives be chosen arbitrarily but fixed.

**Remark 2.9.** Let \(A \in \mathbb{Z}^{d \times n}\), \(b \in \mathbb{Z}^d\) and \(0 \leq k, l \leq n\). Then the set of representatives of \(\subseteq_l\)-minimal elements in \(Q_b^{(k)}\), \(R_{b,l}^{(k)}\), is finite by the Lemma of Gordan–Dickson (see for example Cox et al. (1992)).

**Lemma 2.10.** Let \(0 \leq k \leq l \leq n\) and let \(Q_{b_1}^{(k)} \neq \emptyset\), \(Q_{b_2}^{(k)} \neq \emptyset\). Then \(Q_{b_1+b_2}^{(k)} = Q_{b_1}^{(k)} \oplus (l) Q_{b_2}^{(k)}\) if and only if for every \(\subseteq_l\)-minimal vector \(v \in R_{b_1+b_2,l}^{(k)}\) there is a vector \(w \in Q_{b_1}^{(k)}\) with \(w \subseteq_l v\).

**Proof.** Let \(v \in Q_{b_1+b_2}^{(k)}\). Then there is \(\bar{v} \in R_{b_1+b_2,l}^{(k)}\) with \(\bar{v} \subseteq_l v\). Thus, by the assumption in the lemma, there is some \(\bar{w} \in Q_{b_1}^{(k)}\) such that \(\bar{w} \subseteq_l \bar{v} \subseteq_l v\). As \(k \leq l\) we have \(\bar{v} - \bar{w} \in \mathbb{Z}^k_+ \times \mathbb{Z}^{n-k}\) and thus \(\bar{v} - \bar{w} \subseteq_l \bar{v} \subseteq_l v\).

We now claim that \(v = (\bar{w} + v - \bar{v}) + (\bar{v} - \bar{w})\) with \(\bar{w} + v - \bar{v} \in Q_{b_1}^{(k)}, \bar{v} - \bar{w} \in Q_{b_2}^{(k)}, \bar{w} + v - \bar{v} \subseteq_l v,\) and \(\bar{v} - \bar{w} \subseteq_l v,\) is a desired representation of \(v\). The first two relations are trivial, if we keep in mind that \(A\bar{v} = A\bar{w} = b, A\bar{w} = b_1, b = b_1 + b_2\) and \(k \leq l\). We get the other two relations as follows:

\[
\begin{align*}
(\text{a}) \quad &\bar{w} + v - \bar{v} \subseteq_l \bar{v} + v - \bar{v} = v, \quad \text{since by construction } \pi_l(\bar{w}) \quad \text{and} \quad \pi_l(v - \bar{v}) \quad \text{lie in the same} \quad \text{orthant,} \\
(\text{b}) \quad &\bar{v} - \bar{w} \subseteq_l \bar{v} \subseteq_l v, \quad \text{since } \bar{w} \subseteq_l \bar{v}.
\end{align*}
\]

Thus, we have constructed for arbitrary \(v \in Q_{b_1+b_2}^{(k)}\) a valid representation of \(v\) as a sum of two elements from \(Q_{b_1}^{(k)}\) and \(Q_{b_2}^{(k)}\) whose projection onto the first \(l\) components lie in the same orthant as the projection of \(v\) onto its first \(l\) components. This concludes the proof. \[\square\]

Using this lemma repeatedly, we are now able to find, for a given right-hand side \(b \in M\), a decomposition \(Q_b^{(k)} = \bigoplus_{i=1}^s \alpha_i Q_{b_i}^{(k)}, \alpha_i \in \mathbb{Z}_+,\) that is, we can find a decomposition of a partially extended fiber into a sum of partially extended fibers which are atomic w.r.t. the monoid \(M\).
Algorithm 2.1 Algorithm to decompose extended fibers into sums of extended atomic fibers

Input: $A$, right-hand sides $\{b_1, \ldots, b_s\}$ of the set of extended atomic fibers $E^{(k)}(A, M)$
Output: $\alpha_1, \ldots, \alpha_s$ such that $Q_b^{(k)} = \bigoplus_{i=1}^s \alpha_i Q_{b_i}^{(k)}$

1: $\alpha_1 := \ldots := \alpha_s := 0$
2: for $i = 1$ to $s$ do
3: while $Q_b^{(k)} = Q_{b_i}^{(k)} \oplus Q_{b_i}^{(k)}$ and $b - b_i \in M$ do
4: \hspace{1em} $b := b - b_i$
5: \hspace{1em} $\alpha_i := \alpha_i + 1$
6: end while
7: end for
8: return: $\alpha_1, \ldots, \alpha_s$.

It remains to state an algorithm that computes the finitely many $\sqsubseteq$-minimal elements in $Q_b^{(k)}$ for fixed $k$. We will do this in the following paragraphs.

We have to find for some $l \in \{1, \ldots, n\}$ and some $k \in \{1, \ldots, l\}$ all $\sqsubseteq_l$-minimal elements in (projections of) fibers of the form

$$\pi_l(Q_b^{(k)}) = \{(x, y) \in \mathbb{Z}_+^k \times \mathbb{Z}^{(l-k)} : \exists z \in \mathbb{Z}^{(n-l)} \text{ with } A(x, y, z) = b\}.$$  

If $b = 0$, then 0 is the only $\sqsubseteq_l$-minimal element. If not, we reduce this problem to the problem of finding a Hilbert basis of a cone. It is not hard to show that all $\sqsubseteq_l$-minimal elements $(x, y, z)$ correspond to the elements $(x, y^+, y^-, z, 1)$ in a Hilbert basis of the cone

$$\{(x, y^+, y^-, z, u) \in \mathbb{Z}^n : A(x, y^+ - y^-, z) - bu = 0, x, y^+, y^-, u \geq 0\}.$$  

In general, this is not a pointed rational polyhedral cone (and thus need not have a unique inclusion-minimal Hilbert basis), since there can be linear relations among the (free) variables $z$. However, projected onto the space of the variables $x, y^+, y^-, u$, the nonnegativity constraints lead to a pointed rational polyhedral cone that possesses a unique inclusion-minimal Hilbert basis. Such a minimal Hilbert basis can be computed for example with 4ti2 (see 4ti2 team).

Note that the splitting of $y$ into $y^+$ and $y^-$ is only used for exposition here. In practice, one can directly use $y$ when computing the $\sqsubseteq_l$-minimal elements, see Hemmecke (2006, Section 2.6) for more details.

### 3. Computation of (Extended) Atomic Fibers

In the following we show how to compute the finitely many (extended) atomic fibers of a matrix $A \in \mathbb{Z}^{m \times n}$ w.r.t. a lattice $\Lambda$. In this section we will present a simple algorithm; we will give a more complex and much more efficient algorithm in the following sections. Both algorithms use the algorithmic pattern of a completion procedure.

We will denote the columns of matrix $A$ by $A_1, \ldots, A_n \in \mathbb{Z}^m$. Note that the function normal form$(s, G)$ in Algorithm 3.1 stems from Algorithm 3.2.

**Lemma 3.1.** Algorithm 3.1 terminates and computes a set $G$ such that $\{Q_{A,b}^i : b \in G\}$ contains all atomic fibers of $A$.

**Proof.** Associate with $b \in \Lambda$ the monomial ideal $I_{A,b} := \langle x^+z^-, z^+ : A z = b, z \in \mathbb{Z}^n \rangle \subseteq \mathbb{Q}[x_1, \ldots, x_{2n}]$, where $(z^+)^j = \max(0, z^j)$ and $(z^-)^j = \max(0, -z^j)$ for all components $j = 1, \ldots, n$. Algorithm 3.1 generates a sequence $\{f_1, f_2, \ldots\}$ in $G \setminus F$ such that $Q_{f_i} \neq Q_{f_i} \oplus Q_{f_j - f_i}$ whenever $i < j$. Thus, the corresponding sequence $\{I_{A,f_1}, I_{A,f_2}, \ldots\}$ of monomial ideals satisfies $I_{A,f_j} \nsubseteq I_{A,f_i}$.
Algorithm 3.1 Algorithm to compute extended atomic fibers

**Input:** $F := \{ \pm b_1, \ldots, \pm b_s \}$ with $\langle b_1, \ldots, b_s \rangle = \Lambda \cap AZ^n$

**Output:** a set $G$, such that $\{ Q_b : b \in G \}$ contains all extended fibers of $A$ which are atomic w.r.t. $\Lambda$

1: $G := F$
2: $C := \bigcup_{f,g \in G} \{ f + g \}$ (forming S-vectors)
3: while $C \neq \emptyset$ do
4: $s := \text{an element in } C$
5: $C := C \setminus \{ s \}$
6: $f := \text{normal form } (s, G)$
7: if $f \neq 0$ then
8: $G := G \cup \{ f \}$
9: $C := C \cup \bigcup_{g \in G} \{ f + g \}$ (adding S-vectors)
10: end if
11: end while
12: $G := G \cup \{ 0 \}$
13: return: $G$

Algorithm 3.2 Normal form algorithm

**Input:** $s, G$

**Output:** a normal form of $s$ with respect to $G$

1: while there is some $g \in G$ such that $Q_s = Q_g \oplus Q_{s-g}$ do
2: $s := s - g$
3: end while
4: return: $s$

whenever $i < j$. We conclude, by Theorem 2.4 given by Maclagan (2001), that this sequence of monomial ideals must be finite and thus, Algorithm 3.1 must terminate.

It remains to prove correctness. For this, let $G$ denote the set that is returned by Algorithm 3.1. Moreover, let $Q_b$ be an extended atomic fiber of $A$ with $b \neq 0$. We will show that $b \in G$.

Since $F \setminus \{ 0 \} \subseteq G \setminus \{ 0 \}$, we know that $Q_b = \sum Q_{b_j}$ for finitely many (not necessarily distinct) $b_j \in G \setminus \{ 0 \}$. This implies in particular, that every $z \in Q_b$ can be written as a sum $z = \sum v_j$ with $v_j \in Q_{b_j}$. We will show that we can find vectors $b_j \in G$ such that every $z \in Q_b$ can be written as a sum $z = \sum v_j$ with $v_j \in Q_{b_j}$ and $v_j \subseteq z$. This implies $Q_b = \bigoplus Q_{b_j}$. Since $Q_b$ is atomic, and thus indecomposable, this representation must be trivial, that is, it has to be $Q_b = Q_{b_0}$, and therefore we conclude $b \in G$.

With Lemma 2.10 it is sufficient to consider the $\sqsubseteq$-minimal elements in $Q_b$, $R_{b,n}^{(0)} = \{ z_1, \ldots, z_k \}$, to decide if it decomposes w.r.t. $\oplus$. From all representations $Q_b = \sum_{j \in J} Q_{b_j}$ with $b_j \in G \setminus \{ 0 \}$ choose a representation and elements $v_{i,j} \in Q_{b_j}$ with $z_i = \sum_{j \in J} v_{i,j}$ for $i = 1, \ldots, k$, such that the sum

$$\sum_{i=1}^{k} \sum_{j \in J} \| v_{i,j} \|_1$$

is minimal. By the triangle inequality we have that

$$\sum_{i=1}^{k} \sum_{j \in J} \| v_{i,j} \|_1 \geq \sum_{i=1}^{k} \| z_i \|_1.$$

(2)
Herein, equality holds if and only if all \( v_{i,j} \) have the same sign pattern as \( z_i \), \( i = 1, \ldots, k \), that is, if and only if we have \( v_{i,j} \subseteq z_i \) for all \( i \) and all \( j \). Thus, if we have equality in (2) for such a minimal representation \( Q_b = \sum_{j \in J} Q_{b_j} \), then \( v_{i,j} \in Q_{b_j} \) and \( v_{i,j} \subseteq z_i \) for all occurring \( v_{i,j} \), and we are done.

(It should be noted that we have required \( b_j \in G \setminus \{0\} \) for all appearing \( b_j \), that is in particular, \( b_j \neq 0 \). Those \( b_j \) will be sufficient to generate all \( \subseteq \)-minimal elements in the extended fiber \( Q_b \). We get the remaining elements in \( Q_b \) by adding elements from \( Q_0 \).

Therefore, let us assume that

\[
\sum_{i=1}^{k} \sum_{j \in J} \|v_{i,j}\|_1 > \sum_{i=1}^{k} \|z_i\|_1.
\]

(3)

In the following we construct a new representation \( Q'_b = \sum_{j' \in J'} Q_{b'_j} \) and elements \( v'_{i,j} \) whose corresponding sum (1) is smaller than the minimally chosen sum. This contradiction proves that we have indeed equality in (2) and our claim is proved.

From (3) we conclude that there are indices \( i_0, j_1, j_2 \) and a component \( m \in \{1, \ldots, n\} \) such that

\[
v_{i_0,j_1} = v_{i_0,j_2} < 0.
\]

As \( b_{j_1}, b_{j_2} \in G \), the sum \( b_{j_1} + b_{j_2} \) has been built and the extended fiber \( Q_{b_{j_1} + b_{j_2}} \) has either been reduced to \( Q_0 \) by sets \( Q_{b_{j''}}, j'' \in J'' \), during the Algorithm 3.2 or \( b_{j_1} + b_{j_2} \) has been added to \( G \). In the latter case we set \( J'' := \{j''\} \) with \( b_{j''} := b_{j_1} + b_{j_2} \). This gives representations

\[
v_{i_1,j_1} + v_{i_2,j_2} = \sum_{j'' \in J''} w_{i,j''} \text{ with } w_{i,j''} \in Q_{b_{j''}} \text{ and } w_{i,j''} \subseteq v_{i_1,j_1} + v_{i_2,j_2}
\]

for \( i = 1, \ldots, k \). As all \( w_{i,j''} \) lie in the same orthant of \( \mathbb{R}^n \) as \( v_{i_1,j_1} + v_{i_2,j_2} \), we get

\[
\left\| \sum_{j'' \in J''} w_{i,j''} \right\|_1 = \|v_{i_1,j_1} + v_{i_2,j_2}\|_1 \leq \|v_{i_1,j_1}\|_1 + \|v_{i_2,j_2}\|_1,
\]

with strict inequality for \( i = i_0 \).

Thus, replacing in \( Q_b = \sum_{j \in J} Q_{b_j} \) the term \( Q_{b_{j_1}} + Q_{b_{j_2}} \) by \( \sum_{j'' \in J''} Q_{b_{j''}} \), we arrive at a new representation \( Q_b = \sum_{j' \in J'} Q_{b_{j'}} \) whose corresponding sum (1) is at most

\[
\sum_{i=1}^{k} \sum_{j' \in J'} \|v_{i,j'}\|_1 < \sum_{i=1}^{k} \sum_{j \in J} \|v_{i,j}\|_1,
\]

contradicting the minimality of the representation \( Q_b = \sum_{j \in J} Q_{b_j} \). This concludes the proof. \( \square \)

**Remark 3.2.** One may of course use Algorithm 3.1 also for the problem of finding indecomposable extended fibers among the elements in the family of extended fibers \( Q_b = \{z : A z = b, z \in \mathbb{Z}^n\} \) where \( A \) is kept fixed and where \( b \) is allowed to vary on the lattice which is spanned by the columns of matrix \( A \). The input set then becomes \( F = \{\pm A_1, \ldots, \pm A_n\} \).

Having an algorithm available that computes all extended atomic fibers w.r.t. a given lattice \( \Lambda \), we can use it to compute partially extended atomic fibers w.r.t. \( \oplus \) and \( \Lambda \): If \( Q^{(k)}_b \) is atomic then so is \( Q_b \), as any decomposition of \( Q_b \), restricted to \( \mathbb{Z}^k_+ \times \mathbb{Z}^{n-k} \), would give a decomposition of \( Q^{(k)}_b \). This way of computing partially extended atomic fibers of a given matrix \( A \in \mathbb{Z}^{m \times n} \) is illustrated in Figure 1 and formalized in Algorithm 3.3.

The solid arrow from the bottom up in Figure 1 stands for the completion procedure which is given by Algorithm 3.1. The dashed arrow from the top to the bottom illustrates the procedure of intersecting the extended atomic fibers with \( \mathbb{Z}^k_+ \times \mathbb{Z}^{n-k} \) and dropping the reducible (or empty) fibers afterwards.
Algorithm 3.3 Computing partially extended atomic fibers

**Input:** \( F := \{ \pm b_1, \ldots, \pm b_s \} \) with \( \langle b_1, \ldots, b_s \rangle = \Lambda \cap A \mathbb{Z}^n, k \in \mathbb{Z}_+ \)

**Output:** A set \( G^* \) such that \( \{ Q^k_b : b \in G^* \} \) contains all partially extended fibers of order \( k \) which are atomic w.r.t. \( \oplus \) and \( \Lambda \)

1: Apply Algorithm 3.1 to the set \( F \). Let \( G \) denote the output.
2: \( G^* := \emptyset \).
3: for \( b \in G \) with \( Q^k_b \neq \emptyset \) do
   4: if \( Q^k_b \neq Q^k_g \oplus Q^k_{b-g} \) for all \( g \neq b \in G \) then
      5: \( G^* := G^* \cup \{ b \} \)
   6: end if
7: end for
8: return \( G^* \)

---

**Figure 1.** Computing (partially extended) atomic fibers via extended atomic fibers

Being given the atomic fibers \( P_b \) of a matrix \( A \) it is easy to compute the atomic fibers \( \tilde{P}_b \) which have been defined in Adams et al. (1999). Recall that \( \tilde{P}_b := \text{conv}\{ z : Az = b, z \in \mathbb{Z}^n_+ \} \) and that \( \tilde{P}_b \) is said to be atomic if each decomposition \( \tilde{P} = \tilde{P}_{b_1} + \tilde{P}_{b_2} \) implies either \( b = b_1 \) or \( b = b_2 \).

**Lemma 3.3.** If \( \tilde{P}_b \) is an atomic fiber of the matrix \( A \) then \( P_b \) is atomic, too.

**Proof.** Suppose \( P_b = P_{b_1} + P_{b_2} \) (and \( b_1, b_2 \neq 0 \)). Then we have: \( \tilde{P}_b = \text{conv}(P_b) = \text{conv}(P_{b_1} + P_{b_2}) = \text{conv}(P_{b_1}) + \text{conv}(P_{b_2}) = \tilde{P}_{b_1} + \tilde{P}_{b_2} \) which is a contradiction. \( \square \)

Lemma 3.3 enables us to compute the atomic fibers \( \tilde{P}_b \) via Algorithm 3.4.

### 4. Preliminaries of the project-and-lift algorithm

The way of computing atomic fibers presented in section 3, however, is pretty slow, since there are far more extended atomic fibers than atomic fibers. A similar behavior can be observed when one extracts the Hilbert basis of the cone \( \{ x : Ax = 0, x \in \mathbb{R}^n_+ \} \) from the Graver basis of \( A \), as the Graver basis is usually much bigger than the Hilbert basis one is interested in. Hemmecke (2002) showed that one can reduce this difference in sizes by a project-and-lift algorithm. With this algorithm, bigger Hilbert bases, even with more than 500,000 elements, can be computed nowadays.
In this section and in the following one, we will present a sim ilar algorithm to compute the atomic fibers of a given matrix $A \in \mathbb{Z}^{d \times n}$ which is significantly faster than Algorithm 3.3. This algorithm puts us in the position to compute not only the atomic fibers of a matrix but the atomic fibers w.r.t. an arbitrary (finitely generated monoid) $M$, i.e., the right-hand side $b$ is only allowed to vary in this monoid. During the algorithm we consider partially extended fibers $Q^{(k-1)}_b = \{ z \in \mathbb{Z}^{k-1} \times \mathbb{Z}^{n-k+1} : Az = b \}$ with varying $b \in M$ w.r.t. $k$-restricted Minkowski-sums.

Let $M = \langle m_1, \ldots, m_t \rangle \subseteq \mathbb{Z}^d$ be a finitely generated monoid and let $A \in \mathbb{Z}^{d \times n}$ be a matrix. We want to compute the atomic fibers of matrix $A$ w.r.t. the monoid $M$. The algorithm proceeds in $n$ individual steps. The $k$-th step is illustrated in Figure 2.

The $k$-th lifting step follows the arrows in the figure. It starts by performing a “preprocessing step” in which the input set is prepared for the main part of this lifting step. This process is illustrated by the dotted arrow and will be explained in more detail in section 5.3.

The $k$-th lifting step continues as follows: it performs a completion step similar to the one we presented in Algorithm 3.1, which is illustrated by the solid arrow going from the bottom up. This step will be explained in more detail in section 5.1.
The dashed arrow, finally, stands for a step where we drop all elements of the fibers having a negative \( k \)-th component. It might happen that an atomic partially extended fiber becomes empty or reducible when processing this last step. Therefore we have to perform another reducibility test. The details of this subroutine will be given in section 5.2.

Having performed the \( k \)-th lifting step we continue performing the \((k+1)\)-st lifting step. The whole project-and-lift algorithm is illustrated in Figure 3. After having performed \( n \) of these lifting steps we arrive at the finitely many fibers of the matrix \( A \) which are atomic w.r.t. \( M \).

| \( k \) | 0 | 1 | 2 | 3 | \( n \) |
|---|---|---|---|---|---|
| \( \leq k \) | \{\( Q_b^{(0)} \}_{b \in F_0} \} | \{\( Q_b^{(1)} \}_{b \in F_1} \} | \{\( Q_b^{(2)} \}_{b \in F_2} \} | \{\( Q_b^{(3)} \}_{b \in F_3} \} | \cdots | \{\( Q_b^{(n-1)} \}_{b \in G^{(n-1)}} \} |
| \( \leq k+1 \) | \{\( Q_b^{(0)} \}_{b \in F_0} \} | \{\( Q_b^{(1)} \}_{b \in F_1} \} | \{\( Q_b^{(2)} \}_{b \in F_2} \} | \{\( Q_b^{(3)} \}_{b \in F_3} \} | \cdots | \{\( Q_b^{(n)} \}_{b \in G^{*}} \} |

**Figure 3.** The scheme of the project-and-lift algorithm

**Dealing with infinitely many atomic fibers.** Let \( A \in \mathbb{Z}^{d \times n} \) be a matrix and \( M \subseteq \mathbb{Z}^{d} \) a monoid which is finitely generated. The project-and-lift algorithm will deal with partially extended fibers w.r.t. \( \oplus^{(l)} \) and the monoid \( M \) where \( l \leq n \). Recall from Definition 2.2 that \( Q_b^{(k)} \) is atomic w.r.t. \( \oplus^{(l)} \) and \( M \) if there is no decomposition

\[
Q_b^{(k)} = Q_{b_1}^{(l)} \oplus Q_{b_2}^{(l)}
\]

with \( b_1, b_2 \in M \) and \( \pi_l(Q_{b_1}^{(k)}), \pi_l(Q_{b_2}^{(k)}) \neq \pi_l(Q_0^{(k)}) \). Note that for \( l < n \) there are usually some \( \tilde{b} \in M \) with \( \pi_l(Q_{\tilde{b}}^{(k)}) = \pi_l(Q_0^{(k)}) \). Therefore, if \( Q_b^{(k)} \) is atomic w.r.t. \( \oplus^{(l)} \) and \( M \) then so is \( Q_{b+b'}^{(k)} \), \( Q_{b+2b}^{(k)} \), \ldots. This means that for \( l < n \) we usually have infinitely many partially extended fibers which are atomic w.r.t. \( \oplus^{(l)} \). It is clear that no terminating algorithm may compute the whole set of atomic partially extended fibers w.r.t. \( \oplus^{(l)} \) and \( M \). Therefore we introduce a preorder \( \preceq_l \) (i.e., a reflexive and transitive binary relation) on the set of right-hand side vectors \( b \in M \) with non-empty partially extended fiber \( Q_b^{(k)} \) and perform the \( l \)-th step of the project-and-lift algorithm w.r.t. the preorder \( \preceq_l \).

**Definition 4.1.** Let \( M^{(k)} \subseteq M \) be the submonoid of \( M \) with \( Q_b^{(k)} \neq \emptyset \) for \( b \in M^{(k)} \). Let \( A \in \mathbb{Z}^{d \times n} \), \( 0 \leq k \leq l \leq n \) and let \( b, \tilde{b} \in M^{(k)} \). We say that \( b \preceq_l \tilde{b} \) if \( \tilde{b} - b \in \tilde{S}^{(l)} \), where \( \tilde{S}^{(l)} = \{ \lambda_{l+1}A_{l+1} + \ldots + \lambda_{n}A_{n} : \lambda_{l} \in \mathbb{Z} \} \cap M \).

\( b \in M^{(k)} \) is called minimal w.r.t. \( \preceq_l \) if there is no \( b \neq \tilde{b} \in M^{(k)} \) with \( \tilde{b} \preceq_l b \).
Note that $\bar{b} \preceq_i b$ implies $\pi_i(Q^{(k)}_{b}) = \pi_i(Q^{(k)}_{\bar{b}})$ and $Q^{(k)}_{\bar{b}} = Q^{(k)}_{b} \oplus (Q^{(k)}_{b})^\perp$. The relation $\preceq_i$ defines a preorder on the set of right-hand sides $b \in M$ with non-empty partially extended fibers of order $k$. Additionally we have the following relation between the sets $S^{(l)}$: 
\[
\{0\} = \bar{S}^{(n)} \subseteq \bar{S}^{(n-1)} \subseteq \ldots \subseteq \bar{S}^{0} = M^{(0)}.
\] 

Lemma 4.2. Let $M = \langle m_1, \ldots, m_l \rangle$ be a monoid which is finitely generated.

(i) Let $0 \leq k \leq l \leq n$ and let $M \supseteq F = \{b_1, b_2, \ldots\}$ be a set of vectors with $b_i \not\preceq_i b_j$ for all $i < j$. Then $F$ is finite.

(ii) Let $0 \leq k \leq l \leq n$ and let $F = \{b_1, b_2, \ldots\}$ be a set of right hand sides satisfying $Q^{(k)}_{b_i}$ is atomic w.r.t. $\oplus^{(l)}$ and $M$ and $b_i \not\preceq_i b_j$ for all $b_i \neq b_j$. Then $F$ is finite.

Proof. (i): Let $b_i, b_j \in F$ with $i < j$ and let $\alpha^i, \alpha^j \in \mathbb{Z}^n_+$ with $b_i = \sum_{k=1}^l \alpha^i_k m_k$ and $b_j = \sum_{k=1}^l \alpha^j_k m_k$. Then $(\alpha^i, Q^{(k)}_{b_i}) \not\preceq (\alpha^j, Q^{(k)}_{b_j})$. Suppose not. Then we have $\alpha^i \subseteq \alpha^j$ and $Q^{(k)}_{b_i} = Q^{(k)}_{b_j} \oplus Q^{(k)}_{b_j-b_i}$ which implies that $Q^{(k)}_{b_j} = Q^{(k)}_{b_i} \oplus (Q^{(k)}_{b_j-b_i})$. But this last relation contradicts the fact that $b_i \not\preceq_i b_j$. Therefore $(\alpha^i, Q^{(k)}_{b_i}) \not\preceq (\alpha^j, Q^{(k)}_{b_j})$ for all $b_i, b_j \in F$ for $i < j$. Finiteness of $F$ follows with Lemma 2.6 (i).

(ii): This is a direct consequence of (i). \qed

Our algorithm will work with sets of vectors $F$ which have the property claimed in Lemma 4.2. Additionally they will admit the following property: if $b \in M$ is the right-hand side of a partially extended fiber $Q^{(k)}_b$ which is atomic w.r.t. $\oplus^{(l)}$ and $M$ then there is $b \in F$ with $\bar{b} \preceq_i b$. This means in particular: If $\bar{b} \in M$ is minimal w.r.t. $\preceq_i$ and $Q^{(k)}_{b}$ is atomic w.r.t. $\oplus^{(l)}$ and $M$ then $\bar{b} \in F$. Note, however, that the converse is not true in general: It is not guaranteed that for every $b \in M$ there is a $\bar{b} \preceq_i b$ that is minimal w.r.t. $\preceq_i$.

5. The $k$-th step of the project-and-lift algorithm

In the following subsections we will explain the individual steps the project-and-lift algorithm performs during one lifting step.

5.1. The completion procedure. In this subsection we will explain the so-called “completion procedure” in the $k$-th step of the algorithm. This part is illustrated in Figure 4.

Let $M$ be a monoid which is finitely generated. We denote by $M^{(k)} = \{b \in M : Q^{(k)}_b \neq \emptyset\}$ the submonoid of all right-hand sides $b \in M$ having non-empty partially extended fibers of order $k$. We have:
\[ M \supseteq M^{(0)} \supseteq M^{(1)} \supseteq \ldots \supseteq M^{(n)}. \]
As in the previous section, $\bar{S}^{(l)}$, $0 \leq l \leq n$, will denote the set $\{\lambda_{l+1} A_{l+1} + \ldots + \lambda_n A_n : \lambda_i \in \mathbb{Z}\} \cap M$.

Definition 5.1. We introduce a weight function $\omega_m$ for partially extended fibers $Q^{(k)}_b$ ($m \leq k \leq n$) by
\[ \omega_m(Q^{(k)}_b) = \min\{||\pi_m(v)||_1 : v \in Q^{(k)}_b\}. \]

Remark 5.2. Actually, it suffices to determine $||\pi_m(v)||_1$ for $\subseteq_m$-minimal elements $v$ in $Q^{(k)}_b$ to determine the value of $\omega_m(Q^{(k)}_b)$. To see this, suppose there is $w \in Q^{(k)}_b$ non-minimal w.r.t. $\subseteq_m$. Then there is $v \in Q^{(k)}_b$ with $v \subseteq_m w$ and thus $0 \leq v^j \leq w^j$ for $j = 1, \ldots, k$. Therefore $||\pi_m(v)||_1 \leq ||\pi_m(w)||_1$. 

Lemma 5.3. Algorithm 5.1 with input set $E_{k-1} := \{b_1, \ldots, b_s\}$ and monoid $M^{(k-1)}$ terminates and computes a set $G_{k-1} = G^{=0} \cup G^{>1} \cup \{0\} \subseteq M^{(k-1)}$ with properties (i) and (ii).
Figure 4. The completion procedure of the $k$-th lifting step

For the proof of Lemma 5.3, we have to introduce some more notation.

**Notation 5.4.** During the proof of Algorithm 5.1 we examine elements of partially extended fibers. These elements will be denoted as follows:

$$Q_{b}^{(k-1)} \ni z = (z_1, z_2, z_3) \in \mathbb{Z}_+^{k-1} \times \mathbb{Z} \times \mathbb{Z}^{n-k},$$

i.e., $z_1 \in \mathbb{Z}_+^{k-1}$ denotes the first $k-1$ components, $z_2 \in \mathbb{Z}$ the $k$-th component and $z_3 \in \mathbb{Z}^{n-k}$ denotes the last $n-k$ components.

We will use the following lemma in the proof of Lemma 5.3.

**Lemma 5.5.** Let $\tilde{F}_{k-1} \subseteq M^{(k-1)}$ be a set admitting the following property: for every right-hand side $b \in M^{(k-1)}$ of a partially extended fiber $Q_{b}^{(k-1)}$ which is atomic w.r.t. $\oplus^{(k-1)}$ and $M$ there exists $\tilde{b} \in \tilde{F}_{k-1}$ with $\tilde{b} \preceq_k b$. Let $\beta \in M^{(k-1)}$ be the right-hand side of an arbitrary partially extended fiber of order $(k-1)$. Then we find $\tilde{b}_i \in \tilde{F}_{k-1}$ such that for $M^{(k-1)} \ni \bar{\beta} := \sum_{i} \tilde{b}_i$ we have $\tilde{\beta} \preceq_k \beta$ and

$$Q^{(k-1)}_{\beta} = \bigoplus_{i \in I} Q^{(k-1)}_{\tilde{b}_i}. \quad (5)$$

*Proof.* Let $\beta \in M^{(k-1)} \setminus \{0\}$ be the right-hand side of a partially extended fiber of order $(k-1)$. Consider a decomposition of $Q^{(k-1)}_{\beta}$ into a sum of fibers of order $(k-1)$ which are atomic w.r.t. $\oplus^{(k-1)}$ and $M$:

$$Q^{(k-1)}_{\bar{\beta}} = \bigoplus_{i \in I} Q^{(k-1)}_{\tilde{b}_i}$$

As the partially extended fibers $Q^{(k-1)}_{b_i}$ are atomic w.r.t. $\oplus^{(k-1)}$ and $M$ there are $\tilde{b}_i \in \tilde{F}_{k-1}$ with $\tilde{b}_i \preceq_k b_i$ for all $i \in I$. Consider $M^{(k-1)} \ni \bar{\beta} := \sum_{i} \tilde{b}_i$. We have $\tilde{\beta} \preceq_k \beta$ because $b_i - \tilde{b}_i \in S^{(k)}$. 

Algorithm 5.1 The completion procedure to compute atomic partially extended fibers w.r.t. a monoid

Input: A set $\tilde{F}_{k-1} \subseteq M^{(k-1)}$ with the following properties:
   (i) For every right-hand side $b \in M^{(k-1)}$ of a partially extended fiber $Q_b^{(k-1)}$ which is atomic w.r.t. $\oplus^{(k-1)}$ and $M$ there exists $\hat{b} \in \tilde{F}_{k-1}$ with $\hat{b} \leq_k b$.
   (ii) $b_i \not\in k b_j$ for $b_i, b_j \in \tilde{F}_{k-1}$ with $b_i \neq b_j$.

Output: A set $G_{k-1} \subseteq M^{(k-1)}$ with the properties:
   (i) For every right-hand side $b \in M^{(k-1)}$ of a partially extended fiber $Q_b^{(k-1)}$ which is atomic w.r.t. $\oplus^{(k)}$ and $M$ there exists $\hat{b} \in G_{k-1}$ with $\hat{b} \leq_k b$.
   (ii) $b_i \not\in k b_j$ for $b_i \neq b_j \in G_{k-1}$.

1: $G^{\omega=0} := \{ f \in \tilde{F}_{k-1} : \omega_{k-1}(Q_f^{(k-1)}) = 0 \}
2: C^{\omega=0} := \bigcup_{f,g \in G^{\omega=0}} \{ f + g \}
3: \textbf{while } C^{\omega=0} \neq \emptyset \textbf{ do}
4: \hspace{1em} s := \text{an element in } C^{\omega=0}
5: \hspace{1em} C^{\omega=0} := C^{\omega=0} \setminus \{ s \}
6: \hspace{1em} f := \text{monoid-normal-form } (s, G^{\omega=0}, \emptyset, M^{(k-1)})
7: \hspace{1em} \textbf{if } f \notin S^{(k)} \textbf{ then}
8: \hspace{2em} G^{\omega=0} := G^{\omega=0} \cup \{ f \}
9: \hspace{2em} C^{\omega=0} := C^{\omega=0} \cup \bigcup_{g \in G^{\omega=0}} \{ f + g \}
10: \hspace{1em} \textbf{end if}
11: \textbf{end while}
12: G^{\omega=0} := \emptyset
13: \textbf{for all } b \in G^{\omega=0} \textbf{ do}
14: \hspace{1em} \textbf{if } Q_b^{(k-1)} \neq Q_g^{(k-1)} \oplus^{(k)} Q_{b-g}^{(k-1)} \textbf{ for all } b \neq g \in G^{\omega=0} \textbf{ with } b - g \in M \textbf{ then}
15: \hspace{2em} G^{\omega=1} := G^{\omega=0} \cup \{ b \}
16: \hspace{2em} \textbf{end if}
17: \hspace{1em} \textbf{end for}
18: G^{\omega=1} := \{ f \in \tilde{F}_{k-1} : \omega_{k-1}(Q_f^{(k-1)}) > 0 \}, G^{\omega=1} = \emptyset
19: \textbf{for all } g \in G^{\omega=1} \textbf{ do}
20: \hspace{1em} G^{\omega=1} := G^{\omega=1} \cup \text{monoid-normal-form } (g, G^{\omega=0}, \emptyset, M^{(k-1)})
21: \hspace{1em} \textbf{end for}
22: C^{\omega=1} := \bigcup_{f,g \in G^{\omega=1}} \{ f + g \}
23: \textbf{while } C^{\omega=1} \neq \emptyset \textbf{ do}
24: \hspace{1em} s := \text{an element in } C^{\omega=1} \text{ with smallest weight } \omega_{k-1}(Q_s^{(k-1)})
25: \hspace{1em} C^{\omega=1} := C^{\omega=1} \setminus \{ s \}
26: \hspace{1em} f := \text{monoid-normal-form } (s, G^{\omega=0}, G^{\omega=1}, M^{(k-1)})
27: \hspace{1em} \textbf{if } f \neq 0 \textbf{ then}
28: \hspace{2em} G^{\omega=1} := G^{\omega=1} \cup \{ f \}
29: \hspace{2em} C^{\omega=1} := C^{\omega=1} \cup \bigcup_{g \in G^{\omega=0} \cup G^{\omega=1}} \{ f + g \}
30: \hspace{1em} \textbf{end if}
31: \hspace{1em} \textbf{end for}
32: \hspace{1em} \textbf{end while}
33: \textbf{return } G_{k-1}

implies $\sum_{i \in I} b_i - \hat{b}_i = \beta - \hat{\beta} \in S^{(k)}$. Additionally

$$\pi_{k-1}(Q_{\beta}^{(k-1)}) = \pi_{k-1}(Q_{\hat{\beta}}^{(k-1)}) = \bigoplus_{i \in I} \pi_{k-1}(Q_{b_i}^{(k-1)}) = \bigoplus_{i \in I} \pi_{k-1}(Q_{\hat{b}_i}^{(k-1)}), \quad (6)$$
Algorithm 5.2: The monoid-normal-form algorithm

Input: $s, G^{\omega=0}, G^{\omega\geq 1}$, membership oracle for $M^{(k-1)}$
Output: a normal form of $s$ w.r.t. $G^{\omega=0} \cup G^{\omega\geq 1}$ and $M^{(k-1)}$

1: if $\exists g \in G^{\omega \geq 1}$ with $Q^Q_{s(g)} = Q^Q_{s(g)} \oplus (k) Q^Q_{s(g)-g}$ and $s - g \in M^{(k-1)}$ then
2: return 0
3: else
4: while $\exists g \in G^{\omega=0}$ with $Q^Q_{s(g)} = Q^Q_{s(g)} \oplus (k) Q^Q_{s(g)-g}$ and $s - g \in M^{(k-1)}$ do
5: $s := s - g$
6: end while
7: return $s$
8: end if

which together with $\beta = \sum_{i \in I} \tilde{b}_i$ implies that $Q^{(k-1)}_{\beta} = \bigoplus_{i \in I} Q^{(k-1)}_{b_i}$ and our claim is proved. $\square$

We are now in the position to prove Lemma 5.3.

Proof of Lemma 5.3. As the following proof will be slightly complex, consider the following outline of the proof first.

1. We show that $G_{k-1} \subseteq M^{(k-1)}$.
2. We show that the set $G^{\omega=0}$ is finite and that for all $b_i, b_j \in G^{\omega=0}$ with $b_i \neq b_j$ we have $b_i \not\leq_k b_j$.
3. We show that $G_{k-1}$ is finite. This implies that Algorithm 5.1 terminates. At the same time we show that the output set admits property (ii), i.e., $b_i \not\leq_k b_j$ for $b_i \neq b_j \in G_{k-1}$.
4. We show that if $Q^{(k-1)}_{b_k}$ is an atomic partially extended fiber w.r.t. $\oplus (k)$ and $M$ then there is $\tilde{b} \leq_k b$ with $\tilde{b} \in G_{k-1}$. This is property (i) of the output set.

Step 1.

It is clear that Algorithm 5.1 returns a set $G_{k-1} \subseteq M^{k-1}$. This is guaranteed by the monoid-normal-form algorithm, where we ensure that the elements added lie in $M^{(k-1)}$.

Step 2.

We will now prove that the set $G^{\omega=0}$ is finite. To this aim we show finiteness of $G^{\omega=0}$ first. Consider the sequence $\tilde{G}^{\omega=0} \setminus \{f \in \tilde{F}_{k-1} : \omega_{k-1}(Q^Q_f) = 0\} = \{f_1, f_2, \ldots\}$ produced in lines 1–11 of the algorithm. Clearly $f_i \in M^{(k-1)}$ for all $i$. Additionally $f_i \not\leq_k f_j$ for all $i < j$. Suppose not and let $f_i \preceq_k f_j$. Then $Q^Q_{f_j} = Q^Q_{f_i} \oplus (k) Q^Q_{f_j-f_i}$. As $f_j$ has been added to $G^{\omega=0}$, the second criterion of the monoid-normal-form algorithm is not satisfied, i.e., $f_j - f_i \notin M^{(k-1)}$. But $f_j - f_i \in S^{(k)}$ implies in particular that $f_j - f_i \in M$ and as $Q^Q_{f_j-f_i} \neq 0$ we have $f_j - f_i \in M^{(k-1)}$ which is a contradiction. Therefore $f_i \not\leq_k f_j$ for all $f_i, f_j \in \{f_1, f_2, \ldots\}$ with $i < j$. Finiteness of $G^{\omega=0}$ follows with Lemma 4.2. As $G^{\omega=0} \subseteq \tilde{G}^{\omega=0}$ it is clear now that $G^{\omega=0}$ is finite. Additionally lines 12–17 of Algorithm 5.1 guarantee that $b_i \not\leq_k b_j$ for all $b_i, b_j \in G^{\omega=0}$ with $b_i \neq b_j$.

Step 3.

Let $G^{\omega=\alpha} := \{b \in G_{k-1} : \omega_{k-1}(Q^Q_b) = \alpha\}$ for $\alpha \in \mathbb{Z}_+$. Furthermore let $G^{\omega\leq \alpha} := \{b \in G_{k-1} : \omega_{k-1}(Q^Q_b) \leq \alpha\}$ for $\alpha \in \mathbb{Z}_+$. We will show via induction that $b_i \not\leq_k b_j$ for $b_i, b_j \in G^{\omega \leq \alpha}$ with $b_i \neq b_j$. Lemma 4.2 then yields that $G^{\omega \leq \alpha}$ is finite. Clearly we have $G_{k-1} = \bigcup_{\alpha \in \mathbb{Z}_+} G^{\omega \leq \alpha}$. Let $b_i, b_j \in G_{k-1}$ with $b_i \neq b_j$. Then there is $\alpha \in \mathbb{Z}_+$ with $b_i, b_j \in G^{\omega \leq \alpha}$ yielding $b_i \not\leq_k b_j$. The
set $G_{k-1}$ admits property (ii) of the output set thus which together with Lemma 4.2 yields that $G_{k-1}$ is finite.

We will show via induction that $G^{\omega \leq \alpha}$ is finite. With step 2 of the proof we know that our claim is proved for $\alpha = 0$. Suppose that our assertions are true for all integers lower or equal than $\alpha$. We will prove our claim for $\alpha + 1$. Let $b_i, b_j \in G^{\omega \leq \alpha + 1}$ and suppose $b_i \preceq_k b_j$. There are several cases possible:

(i) $b_i, b_j \in \hat{F}_{k-1}$
   This contradicts input property (ii) of the input set $\hat{F}_{k-1}$.

(ii) $b_i \in \hat{F}_{k-1}, b_j \notin \hat{F}_{k-1}$
    This contradicts the if-clause of Algorithm 5.2 because $b_i$ then is an appropriate reducer of $b_j$.

(iii) $b_i \notin \hat{F}_{k-1}, b_j \in \hat{F}_{k-1}$
   As $\omega_{k-1}(b_{j-b_i}) = 0$ and as $G^{\omega=0}$ is completed before $G^{\omega \geq 1}$ we know that there is $\tilde{b} \in G^{\omega=0}$ with $\tilde{b} \preceq_k b_j - b_i \preceq_k b_j$. But this is a contradiction to lines 18-21 of Algorithm 5.1 because in this case $b_j$ would not have been added to $G^{\omega \geq 1}$ then.

(iv) $b_i, b_j \notin \hat{F}_{k-1}$
    Depending on whether either $b_i$ has been added to $G^{\omega = \alpha + 1}$ before $b_j$ was added or not we either have a contradiction to the if-clause of Algorithm 5.2 or to the else-clause of this algorithm.

We know via induction that $G^{\omega \leq \alpha}$ admits property (ii) of the output set. We will now show that this is also true for $G^{\omega \leq \alpha + 1}$. Let $b_i, b_j \in G^{\omega \leq \alpha + 1}$ and suppose $b_i \preceq_k b_j$. By induction, the previous discussion and monotonicity of the weight-function $\omega_k(\cdot)$: $b_i \in G^{\omega \leq \alpha}$ and $b_j \in G^{\omega = \alpha + 1}$. But this contradicts the if-clause of Algorithm 5.2. Therefore $G^{\omega \leq \alpha + 1}$ admits property (ii) of the output set which had to be proved.

Step 4.
Let $b \in M^{(k-1)}$ such that $Q_b^{(k-1)}$ is atomic with respect to $\bigoplus^{(k)}$ and $M$. With Lemma 5.5 we know that there is $\tilde{b} \in M$, $\tilde{b} \preceq_k b$, admitting a representation (5):

$$Q_b^{(k-1)} = \bigoplus_{i \in I} Q_{b_i}^{(k-1)},$$

where $b_i \in \hat{F}_{k-1}$. We will show that $\tilde{b} \in G_{k-1}$. The above representation implies in particular that every $z = (z_1, z_2, z_3) \in Q_{\tilde{b}}^{(k-1)}$ can be written as $(z_1, z_2, z_3) = \sum_{i \in I} (z_i^1, z_i^2, z_i^3)$ with $(z_i^1, z_i^2, z_i^3) \in Q_{b_i}^{(k-1)}$. In particular:

$$(z_i^1, z_i^2, z_i^3) \subseteq (z_1, z_2, z_3)$$

for all $i$. If $Q_{b_i}^{(k-1)} \ni (z_i^1, z_i^2, z_i^3)$ with $(z_i^1, z_i^2, z_i^3) \subseteq (z_1, z_2, z_3)$ was valid this then would imply: $Q_{\tilde{b}}^{(k-1)} = \bigoplus_{i \in I} Q_{b_i}^{(k-1)}$.

Let $R_{b,k}^{(k-1)} = \{(z_1^1, z_2^1, z_3^1), \ldots, (z_t^1, z_t^2, z_t^3)\}$ be the set of representatives of the $\subseteq_k$-minimal elements in $Q_{\tilde{b}}^{(k-1)}$ according to Definition 2.8. With Lemma 2.10 we know that it is sufficient to analyze the $\subseteq_k$-minimal elements in a partially extended fiber to decide decomposability w.r.t. $\bigoplus^{(k)}$.

From all representations $Q_{b_i}^{(k-1)} = \bigoplus_{j \in J} Q_{b_{j_i}}^{(k-1)}$ with $b_j \in G_{k-1}$ and $\pi_k(Q_{b_j}^{(k-1)}) \neq \pi_k(Q_{b_{j_i}}^{(k-1)})$ and where the $\subseteq_{k-1}$-minimal elements in $R_{b,k}^{(k-1)}$ are represented as $(z_1^i, z_2^i, z_3^i)$ with $(z_1^i, z_2^i, z_3^i) \in Q_{b_{j_i}}^{(k-1)}$ for $i = 1, \ldots, t$, choose a representation and
elements \((z_{1,j}^{i}, z_{2,j}^{i}, z_{3,j}^{i})\) such that the sum
\[
\sum_{i=1}^{t} \sum_{j \in J} \|z_{1,j}^{i}, z_{2,j}^{i}\|_1
\] is minimal.

By the triangle inequality we have
\[
\sum_{i=1}^{t} \sum_{j \in J} \|z_{1,j}^{i}, z_{2,j}^{i}\|_1 \geq \sum_{i=1}^{t} \|z_{1,j}^{i}\|_1
\] (8)

Herein equality holds if and only if all \((z_{1,j}^{i}, z_{2,j}^{i})\) have the same sign pattern as \((z_{1,j}^{i}, z_{2,j}^{i})\), \(i = 1, \ldots, t\), that is if and only if we have \((z_{1,j}^{i}, z_{2,j}^{i}, z_{3,j}^{i}) \subseteq_k (z_{1,j}^{i}, z_{2,j}^{i}, z_{3,j}^{i})\) for all \(j \in J\) and all \(i = 1, \ldots, t\).

Thus if we have equality in (9) for such a minimal representation \(Q_{b}^{(k-1)} = \bigoplus_{j \in J} Q_{b_{j}}^{(k-1)}\) then by Lemma 2.10 \(Q_{b_{j}}^{(k-1)} = \bigoplus_{j' \in J'} Q_{b_{j'}}^{(k-1)}\) and as \(\pi_k(Q_{b_{j}}^{(k-1)}) \neq \pi_k(Q_{b_{j'}}^{(k-1)})\) and as \(Q_{b_{j}}^{(k-1)}\) and thus \(Q_{b_{j}}^{(k-1)}\) is atomic w.r.t. \(\oplus(k)\) and \(M\) this representation must be trivial and we are done.

Therefore let us assume, that
\[
\sum_{i=1}^{t} \sum_{j \in J} \|z_{1,j}^{i}, z_{2,j}^{i}\|_1 > \sum_{i=1}^{t} \|z_{1,j}^{i}\|_1
\] (10)

In the following, we construct a new representation \(Q_{b'}^{(k-1)} = \bigoplus_{j' \in J'} Q_{b_{j'}}^{(k-1)}\) and elements \((z_{1,j}^{i'}, z_{2,j}^{i'}, z_{3,j}^{i'})\) whose corresponding sum (8) is smaller than the minimally chosen sum. This contradiction proves that we indeed have equality in (9) and our claim is proved.

From (10) and from (7), i.e., \(z_{1,j}^{i} \subseteq_k z_{2,j}^{i}\) for all \(i\) and \(j\), we conclude that there are indices \(i_0,j_1,j_2\) such that \(z_{1,j_1}^{i_0}, z_{2,j_1}^{i_0} < 0\). As \(b_{j_1}, b_{j_2} \in G^{\omega=0}\) the sum \(b_{j_1} + b_{j_2}\) was built during the algorithm. We have \(\omega_{k-1}(Q_{b_{j_1}+b_{j_2}}) = \omega_{k-1}(Q_{b_{j_1}^{(k-1)}}) + \omega_{k-1}(Q_{b_{j_2}^{(k-1)}}) = 0\) and thus there is no partially extended fiber with weight \(\omega_{k-1}\) greater than 0 which reduces \(Q_{b_{j_1}+b_{j_2}}\). Consequently the partially extended fiber \(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\) was either reduced to \(Q_{0}^{(k-1)}\) by sets \(Q_{b_{j'}}^{(k-1)}\), \(j' \in J''\), during the else-clause of the monoid-normal-form algorithm or the vector \(b_{j_1} + b_{j_2}\) has been added to the set \(\bigcup_i^n\) G then either \(b_{j_1} + b_{j_2} \in G^{\omega=0}\) or we find sets \(Q_{b_{j'}}^{(k-1)}\), \(j' \in J''\), with \(Q_{b_{j_1}+b_{j_2}}^{(k-1)} = \bigoplus_{j' \in J''} Q_{b_{j'}}^{(k-1)}\) with \(b_{j'} \in G^{\omega=0}\).

In the former case, set \(J'' := \{j''\}\) with \(b_{j''} := b_{j_1} + b_{j_2}\).

This gives representations
\[
(z_{1,j}^{i_1}, z_{2,j}^{i_1}, z_{3,j}^{i_1}) + (z_{1,j}^{i_2}, z_{2,j}^{i_2}, z_{3,j}^{i_2}) = \sum_{j'' \in J''} (z_{1,j''}^{i'}, z_{2,j''}^{i'}, z_{3,j''}^{i'}),
\]
for \(i = 1, \ldots, t\).

As all \((z_{1,j''}^{i'}, z_{2,j''}^{i'}, z_{3,j''}^{i'})\) lie in the same orthant as \((z_{1,j_1}^{i_0}, z_{2,j_1}^{i_0})\) we get:
\[
\left\| \sum_{j'' \in J''} (z_{1,j''}^{i'}, z_{2,j''}^{i'}) \right\|_1 = \|z_{1,j_1}^{i_0}, z_{2,j_1}^{i_0}\|_1 \leq \|z_{1,j_1}^{i_0} + z_{2,j_1}^{i_0}\|_1 + \|z_{1,j_1}^{i_0}, z_{2,j_1}^{i_0}\|_1
\]
with strict inequality for \(i = i_0\). Thus, by replacing in \(Q_{b}^{(k-1)} = \bigoplus_{j \in J} Q_{b_{j}}^{(k-1)}\) the term \(Q_{b_{j_1}+b_{j_2}}^{(k-1)}\) by \(\bigoplus_{j'' \in J''} Q_{b_{j''}}^{(k-1)}\) we arrive at a new representation \(Q_{b'}^{(k-1)} = \bigoplus_{j' \in J'} Q_{b_{j'}}^{(k-1)}\) whose corresponding sum (8) is at most
\[
\sum_{i=1}^{t} \sum_{j' \in J'} \|z_{1,j'}^{i'} + z_{2,j'}^{i'}\|_1 < \sum_{i=1}^{t} \sum_{j \in J} \|z_{1,j}^{i}, z_{2,j}^{i}\|_1
\]
contradicting the minimality of the representation $Q^{(k-1)}_b = \bigoplus_{j \in J} Q^{(k-1)}_{b_j}$. Therefore we have equality in (9) and thus $\tilde{b} \in G_{k-1}$ concluding our proof. □

5.2. Intersecting with the appropriate orthant and testing reducibility. In this subsection we want to illustrate the step of the project-and-lift algorithm which follows the completion procedure in each lifting step. This “intersection and reducibility test” is illustrated by the dashed arrow in Figure 5.

![Figure 5](image-url)

**Figure 5.** Intersecting with the appropriate orthant and dropping reducible partially extended fibers

### Algorithm 5.3 Intersecting and testing reducibility

**Input:** A set $G_{k-1} \subseteq M^{(k-1)}$ with the properties:

(i) For every right-hand side $b \in M^{(k-1)}$ of a partially extended fiber $Q^{(k-1)}_b$ which is atomic w.r.t. $\oplus^{(k)}$ and $M$ there exists $\tilde{b} \in G_{k-1}$ with $\tilde{b} \preceq_k b$.

(ii) $b_i \npreceq_k b_j$ for $b_i, b_j \in G_{k-1}$ with $b_i \neq b_j$.

**Output:** A set $F_k \subseteq M^{(k)}$ of right-hand sides with:

(i) For every right-hand side $b \in M^{(k)}$ of a partially extended fiber $Q^{(k)}_b$ which is atomic w.r.t. $\oplus^{(k)}$ and $M$ there exists $\tilde{b} \in F_k$ with $\tilde{b} \preceq_k b$.

(ii) $b_i \npreceq_k b_j$ for $b_i, b_j \in F_k$ with $b_i \neq b_j$.

1: $F_k := \emptyset$
2: for all $b \in G_{k-1}$ with $Q^{(k)}_b \neq \emptyset$ do
3: if $Q^{(k)}_b \neq Q^{(k)}_g \oplus^{(k)} Q^{(k)}_{b-g}$ for all $b \neq g \in G_{k-1}$ with $b - g \in M$ then
4: $F_k := F_k \cup \{b\}$
5: end if
6: end for
7: return $F_k$

**Lemma 5.6.** Algorithm 5.3 with input set $G_{k-1}$ terminates and computes a set $F_k \subseteq M^{(k)}$ with the properties (i) and (ii).
Proof. Termination of Algorithm 5.3 is clear. We have to show correctness of the algorithm. But this is easy as well: if \( b \in M^{(k-1)} \) and \( Q_b^{(k)} \neq \emptyset \) then \( b \in M^{(k)} \). Therefore \( F_k \subseteq M^{(k)} \). If \( Q_b^{(k)} \) is atomic w.r.t. \( \oplus^{(k)} \) and \( M \), then \( Q_b^{(k-1)} \) is atomic w.r.t. \( \oplus^{(k)} \) and \( M \) as well, because \( Q_b^{(k)} \subseteq Q_b^{(k-1)} \) and every decomposition of \( Q_b^{(k-1)} \) would give a decomposition of \( Q_b^{(k)} \). This characteristic immediately implies property (i) of the output set because we have property (i) of the input set.

To see property (ii) of the output set, suppose that there are \( b_1, b_2 \in F_k \) with \( b_2 \leq_k b_1 \). Then, \( Q_{b_2} = Q_{b_2} \oplus^{(k)} Q_{b_1-b_2} \) and \( b_1-b_2 \in \tilde{S}^{(k)} \). In particular, \( b_1-b_2 \in M \) which is a contradiction as \( b_1 \) would not have been added to \( F_k \) in this case. This yields that \( b_i \leq_k b_j \) for all \( b_i, b_j \in F_k \). Therefore Algorithm 5.3 is correct and terminates. \( \square \)

5.3. Refining the preorder. There is one more step to explain in the \( k \)-th lifting step of the project-and-lift algorithm. This step is illustrated by the dotted arrow in Figure 6; it is implemented in Algorithm 5.4.

![Figure 6](image-url)  
**Figure 6.** Refining the preorder to prepare the \( k+1 \)-st lifting step

**Lemma 5.7.** Algorithm 5.4 terminates and is correct.

Proof. Termination of the above algorithm is clear once we have shown that we can construct a finite set \( \mathcal{L} \) with the property that  
\[
\forall s \in \tilde{S}^{(k)} \exists s^{(j)} \in \mathcal{L} \text{ with } s^{(j)} \preceq_{k+1} s.
\]

For this, let us first construct generators for the monoid \( \tilde{S}^{(k+1)} \). These can be found by considering the homogeneous system of linear equations
\[
s = \sum_{j=k+2}^{n} \lambda_j A_j = \sum_{r=1}^{t} \alpha_r m_r,
\]
in the variables \( s \in \mathbb{Z}^d, \alpha \in \mathbb{Z}_+^t \), and in \( \lambda \in \mathbb{Z}^{n-k-1} \). If we extract the values of \( s \) for all finitely many minimal homogeneous solutions of this linear system, we obtain a generating set \( \{\bar{s}_1, \ldots, \bar{s}_p\} \) for the monoid \( \tilde{S}^{(k+1)} \).
Algorithm 5.4 Refining the preorder

Input: A set $F_k \subseteq M^{(k)}$ of right-hand sides with:

(i) For every right-hand side $b \in M^{(k)}$ of a partially extended fiber $Q_b^{(k)}$ which is atomic w.r.t. $\oplus^{(k)}$ and $M$ there exists $\hat{b} \in F_k$ with $\hat{b} \preceq_k b$. 
(ii) $b_i \not\preceq_k b_j$ for all $b_i, b_j \in F_k$.

Output: A set $\tilde{F}_k \subseteq M^{(k)}$ with the following properties:

(i) For every right-hand side $b \in M^{(k)}$ of a partially extended fiber $Q_b^{(k)}$ which is atomic w.r.t. $\oplus^{(k)}$ and $M$ there exists $\hat{b} \in \tilde{F}_k$ with $\hat{b} \preceq_{k+1} b$.
(ii) $b_i \not\preceq_{k+1} b_j$ for all $b_i, b_j \in \tilde{F}_k$ with $b_i \neq b_j$.

1. Compute a set $L = \{s^{(1)}, \ldots, s^{(r)}\} \subseteq S^{(k)}$ with:
   \[ \forall s \in S^{(k)} \exists s^{(j)} \in L \text{ with } s^{(j)} \preceq_{k+1} s \]

2. Set $\tilde{F}_k := \bigcup_{b \in F_k} \bigcup_{s \in L} \{b + s\}$.
3. for all $b \in \tilde{F}_k$ do
4. if $\exists b \in \tilde{F}_k$ with $\hat{b} \neq b$ and $\hat{b} \preceq_{k+1} b$ then
5. $\tilde{F}_k := \tilde{F}_k \setminus \{b\}$
6. end if
7. end for
8. return $\tilde{F}_k$

Now let us consider the finite set
\[ F = \left\{ \sum_{i=1}^p \lambda_i \bar{s}_i : 0 \leq \lambda_i < 1, i = 1, \ldots, p \right\} \cap \left\{ \lambda_{k+1} A_{k+1} + \ldots + \lambda_n A_n : \lambda_i \in \mathbb{Z} \right\}. \]

For each $f \in F$ we now consider the set $(f + \bar{S}^{(k+1)}) \cap \bar{S}^{(k)}$ and construct a finite set $L_f$ of vectors in $(f + \bar{S}^{(k+1)}) \cap \bar{S}^{(k)}$ such that
\[ \forall s \in (f + \bar{S}^{(k+1)}) \cap \bar{S}^{(k)} \exists s^{(j)} \in L_f \text{ with } s^{(j)} \preceq_{k+1} s. \]

Then $L = \bigcup_{f \in F} L_f \subseteq \bar{S}^{(k)}$ is finite and has the desired property.

In order to construct $L_f$, let us consider the inhomogeneous system of linear equations and inequalities
\[ f + \sum_{j=k+2}^n \lambda_j A_j = f + \sum_{r=1}^t \alpha_r m_r = s = \sum_{j=k+1}^n \mu_j A_j = \sum_{r=1}^t \beta_r m_r \]
in the variables $s \in \mathbb{Z}^d$, $\alpha, \beta \in \mathbb{Z}_+^n$, and in $\lambda \in \mathbb{Z}^{n-k-1}$, $\mu \in \mathbb{Z}^{n-k}$. The left-hand part states $s \in f + \bar{S}^{(k+1)}$ and the right-hand part encodes $s \in \bar{S}^{(k)}$. Then a suitable set $L_f$ can be found by computing the finitely many minimal inhomogeneous solutions to this linear system and by collecting the corresponding values of $s$. We have thus proved that we may construct a finite set $L$ admitting the claimed property. It remains to prove that the set $\tilde{F}_k$ constructed from $L$ admits the properties claimed in Algorithm 5.4.

It is clear that $\tilde{F}_k \subseteq M^{(k)}$ because $F_k \subseteq M^{(k)}$ and $L \subseteq \bar{S}^{(k)} \subseteq M$. Now let $b \in M^{(k)}$ be the right-hand side of an atomic partially extended fiber w.r.t. $\oplus^{(k)}$ and $M$. Because of property (i) of the input-set, we find $\bar{b} \in F_k$ with $\bar{b} \preceq_k b$, i.e., there is $\bar{s} \in \bar{S}^{(k)}$ with $\bar{b} = \bar{b} + \bar{s}$. As $\bar{s} \in \bar{S}^{(k)}$ there is $s \in \bar{L}$ with $s \preceq_{k+1} \bar{s}$ implying that there is $\bar{s} \in \bar{S}^{(k+1)}$ with $\bar{s} = s + \bar{s}$. Let $\bar{b} := \bar{b} + s$. Then clearly $b \preceq_{k+1} b$. Thus either $b \in \tilde{F}_k$ or there is $b' \in \tilde{F}_k$ with $b' \preceq_{k+1} \bar{b} \preceq_{k+1} b$. This concludes the proof. \qed
5.4. Initial input set and final output set. The previous subsections have shown how one step of the project-and-lift algorithm works. We have to perform $n$ of these steps to obtain a set $G^*$ which contains the right-hand sides of all atomic fibers w.r.t. the monoid $M = \langle m_1, \ldots, m_i \rangle$. We start with the initial input set $F_0 = \{\emptyset\}$. This is a valid input set because every non-empty extended fiber with right-hand side $b \in M$, i.e., $b \in M^{(0)}$, also lies in $S^{(0)}$.

5.5. Simplifications for the lattice case. As already mentioned at the beginning of section 4, the project-and-lift algorithm to compute the atomic fibers of a matrix with right-hand side $b$ varying on a lattice $\Lambda$ is much easier to treat than the case of general monoids.

The simplifications of the project-and-lift algorithm are based on the fact that the difference of two arbitrary lattice vectors $b_1, b_2 \in \Lambda$, $b_1 - b_2$, is again a lattice vector. This fact has implications for the preorder $\preceq_l$ on the right-hand side vectors $b \in \Lambda$. Let $A \in \mathbb{Z}^{d \times n}$, $\Lambda \subseteq \mathbb{Z}^d$ a lattice and $0 \leq l \leq n$. Consider the preorder introduced in section 4: $b_1 \preceq_l b_2$ if $b_2 - b_1 \in S^{(l)} = \{\lambda_{l+1}A_{l+1} + \cdots + \lambda_n A_n : \lambda_i \in \mathbb{Z}\} \cap \Lambda$. As $\Lambda$ is a lattice we obtain that $b_1 - b_2 \in S^{(l)}$ as well. This means: $b_1 \preceq_l b_2 \Leftrightarrow b_2 \preceq_l b_1$.

In other words, the preorder is in fact an equivalence relation.

Our aim in this subsection is to simplify the refining step in our project-and-lift algorithm. Recall from section 5.1 that the input set $F_{k-1}$ satisfies the following two properties which ensure finiteness and correctness of the algorithm:

(i) For every right-hand side $b \in M^{(k-1)}$ of a partially extended fiber $Q_b^{(k-1)}$ which is atomic w.r.t. $\oplus^{(k-1)}$ and $M$ there exists $\bar{b} \in F_{k-1}$ with $\bar{b} \preceq_k b$.

(ii) $b_i \not\preceq_k b_j$ for $b_i \neq b_j \in G_{k-1}$.

In this subsection, we will define a new input set $\bar{F}_{k-1}$ of Algorithm 5.1 which may be computed much easier than the set $F_{k-1}$. Having defined this new input set we will expose some properties of it. Finally we will show that the new input set $\bar{F}_{k-1}$ is sufficient to guarantee finiteness and correctness of the completion procedure, i.e., of Algorithm 5.1.

Let $F_{k-1}$ be the output set of Algorithm 5.3 and consider the following integer program:

\[
\begin{align*}
\min \lambda \\
\text{s.t.} \quad & \lambda_k A_k + \sum_{i \geq k+1} \lambda_i A_i = \sum_{j=1}^k \mu_j l_j \\
& \lambda_k - \lambda \leq 0 \\
& -\lambda_k - \lambda \leq 0 \\
& \lambda \geq 1 \\
& \lambda, \lambda_i, \mu_j \in \mathbb{Z}
\end{align*}
\]

There are two possible cases: either the integer program (11) is infeasible or it admits an optimal solution $\lambda^*, \lambda^*_i, \mu^*_j$. Consider the former case first and let $b_1, b_2 \in \Lambda$ with $b_1 \preceq_k b_2$. Then $b_1 \preceq_k b_2$.

This is the case because $b_2 - b_1 \in \Lambda$ and $b_2 - b_1 = \sum_{i \geq k} \lambda_i A_i$, $\lambda_i \in \mathbb{Z}$. If the absolute value of $\lambda_k$ was greater or equal than 1 the difference $b_2 - b_1$ would imply a feasible solution of the integer program (11). Thus $\lambda_k = 0$ and therefore we have $b_1 \preceq_k b_2$. In this case we set $\bar{F}_{k-1} := F_{k-1}$.

Now consider the latter case. Let $s := \sum_{i=k}^n \lambda^*_i A_i$. Again there are two possible cases: either $s \preceq_k -s$ or $s \not\preceq_k -s$. In the former case we set $\bar{F}_{k-1} := F_{k-1} \cup \{s\}$, in the latter case we set $\bar{F}_{k-1} := F_{k-1} \cup \{\pm s\}$.

Lemma 5.8. Let (11) admit an optimal solution $\lambda^*, \lambda^*_i, \mu^*_j$. We assume w.l.o.g. that $\lambda^*_k \in \mathbb{Z}_+$. Let $Q_b^{(k-1)}$ be atomic w.r.t. $\oplus^{(k-1)}$ and $\Lambda$. Then there is $\bar{b} \in F_{k-1}$ and $\lambda_0 \in \mathbb{Z}$ with $\bar{b} + \lambda_0 s \preceq_k b$.

Proof. Let $b \in \Lambda$ with $Q_b^{(k-1)}$ atomic w.r.t. $\oplus^{(k-1)}$ and $\Lambda$. Then there is $\bar{b} \in F_{k-1}$ with $\bar{b} \preceq_k b$, a consequence of $F_{k-1}$ being the output set of Algorithm 5.3. Let $\bar{s} := b - \bar{b} = \sum_{i \geq k} \lambda_i A_i$. Then
there is \( n \in \mathbb{Z} \) and \( 0 \leq \bar{r} < \lambda_k^* \) with \( \lambda_k = n \cdot \lambda_k^* + \bar{r} \). We will show that \( \bar{r} = 0 \). To this aim consider \( s_0 := \bar{s} - n \cdot s = \bar{r}A_k + \sum_{i \geq k+1} (\lambda_i - n \cdot \lambda_i^*)A_i. \) As \( s, \bar{s} \in \Lambda \) we have \( s_0 \in \Lambda \). If \( \bar{r} \neq 0 \) the feasible solution of (11) implied by \( s_0 \) admits an objective value lower than \( \lambda^* \), because \( \bar{r} < \lambda_k^* \). This contradicts the optimality of the solution \( \lambda^*, \lambda_i^*, \mu_i^* \). Therefore \( \bar{r} = 0 \) and thus \( \bar{b} + n \cdot s \not\leq_k b \). \( \square \)

**Lemma 5.9.** For \( b_i, b_j \in \hat{F}_{k-1} \) with \( b_i \neq b_j \) we have \( b_i \not\leq_k b_j \).

**Proof.** There are a few different cases to consider:

**Case 1:** \( b_k, b_j \in F_{k-1} \). Then \( b_i \not\leq_k b_j \) as \( F_{k-1} \) is the output set of Algorithm 5.3.

**Case 2:** \( b_i \in F_{k-1} \) and \( b_j = s \). We have to show that \( s \not\leq_k b_j \). Suppose not and consider \( b_j - s = \sum_{i \geq k+1} \lambda_i A_i \). But then \( b_j = \sum_{i 
 \geq k+1} \lambda_i A_i + \sum_{j \geq k} \lambda_j^* A_j \) and thus \( 0 \leq k-1 \) \( b_j \) which contradicts \( b_j \) being an element of \( F_{k-1} \).

**Case 3:** \( b_i = s \) and \( b_j = -s \). Here we have \( s \not\leq_k -s \) by our assumptions. \( \square \)

Lemma 5.9 implies that \( \hat{F}_{k-1} \) defined as above satisfies property (ii) of the input set of Algorithm 5.1. We continue giving another property of the set \( \hat{F}_{k-1} \).

**Lemma 5.10.** Let \( b \in \Lambda^{k-1} \). Then there is \( \bar{b} \leq_k b \) with
\[
Q_b^{(k-1)} = \bigoplus_i Q_{b_i}^{(k-1)} \quad \text{where} \quad b_i \in \hat{F}_{k-1}.
\]

**Proof.** As \( b \in \Lambda^{(k-1)} \) there is \( \bar{b} \leq_{k-1} b \) with
\[
Q_b^{(k-1)} = \bigoplus_j Q_{b_j}^{(k-1)} \quad \text{where} \quad b_j \in F_{k-1}.
\]

This representation is a consequence of \( \hat{F}_{k-1} \) being the output set of Algorithm 5.3. If the integer program (11) is infeasible then \( \bar{b} \leq_k b \) and our claim is proved. Therefore let (11) admit an optimal solution. W.l.o.g. we assume that \( \hat{F}_{k-1} = F_{k-1} \cup \{ \pm s \} \). Consider a decomposition of \( Q_b^{(k-1)} \) into a restricted Minkowski sum of partially extended fibers which are atomic w.r.t. this restricted Minkowski sum and the lattice \( \Lambda \):
\[
Q_b^{(k-1)} = \bigoplus_i Q_{b_i}^{(k-1)}.
\]

With Lemma 5.8 we know that for each \( b_i \) there is \( \bar{b}_i \in F_{k-1} \) and \( \lambda_{b_i} \in \mathbb{Z} \) such that \( \bar{b}_i + \lambda_{b_i} \cdot s \leq_k b_i \). We set \( \bar{b} := \sum_i \bar{b}_i + \lambda_{b_i} \cdot s \). Then we have \( \bar{b} \leq_k b \) and
\[
Q_b^{(k-1)} = \bigoplus_i (Q_{\bar{b}_i}^{(k-1)} \oplus \lambda_{b_i} Q_s^{(k-1)}) = \bigoplus_i Q_{\bar{b}_i}^{(k-1)} \oplus (\sum_i \lambda_{b_i}) Q_s^{(k-1)}.
\]

This proves our claim. \( \square \)

Now we want to show that the input set \( \hat{F}_{k-1} \) is sufficient to guarantee finiteness and correctness of Algorithm 5.1. An input set admitting properties (i) and (ii) is sufficient to do so. We have seen in Lemma 5.9 that our set \( \hat{F}_{k-1} \) admits property (ii). It admits property (i) as well if the integer program (11) is infeasible. But it does not admit this property in general if the integer program (11) is feasible. Note that in the proof of Algorithm 5.1 property (i) is only used to guarantee a representation (5) according to Lemma 5.5 with the projection of the summands satisfying \( \pi_k(Q_{b_i}^{(k-1)}) \neq \pi_k(Q_{b_i}^{(k-1)}) \). But with Lemma 5.10 this representation may be guaranteed as well. Furthermore \( \pi_k(Q_{b_i}^{(k-1)}) \neq \pi_k(Q_{b_i}^{(k-1)}) \). This is clear for \( b \in F_{k-1} \) because \( 0 \not\leq_{k-1} b \). As \( 0 \not\leq_k s \) we
Algorithm 5.5 Refining the preorder (equivalence relation) in the lattice case

**Input:** A lattice $\Lambda = \langle l_1, \ldots, l_t \rangle$ and a set $F_k \subseteq \Lambda^{(k)}$ of right-hand sides with:

(i) For every right-hand side $b \in \Lambda^{(k)}$ of a partially extended fiber $Q_b^{(k)}$ which is atomic w.r.t. $\oplus^{(k)}$ and $\Lambda$ there exists $\tilde{b} \in F_k$ with $\tilde{b} \preceq_k b$.

(ii) $b_i \nleq_k b_j$ for $b_i, b_j \in F_k$ with $b_i \neq b_j$.

**Output:** The set $\bar{F}_k \subseteq \Lambda^{(k)}$ defines as above.

1. Solve the following integer program:

   $\min \lambda$

   subject to:

   $\lambda_{k+1}A_{k+1} + \sum_{i \geq k+2} \lambda_i A_i = \sum_{j=1}^{k} \mu_j l_j$

   $\lambda_{k+1} - \lambda \leq 0$

   $-\lambda_{k+1} - \lambda \leq 0$

   $\lambda \geq 1$

   $\lambda, \lambda_i, \mu_j \in \mathbb{Z}$

2. if (11) is feasible then

3. Let $\lambda^*, \lambda_i^*, \mu_j^*$ be an optimal solution of (11).

4. Set $s := \sum_{i \geq k+1} \lambda_i^* A_i$.

5. if $s \leq_k -s$ then

6. return $\bar{F}_k := F_k \cup \{s\}$

7. else

8. return $\bar{F}_k := F_k \cup \{\pm s\}$

9. end if

10. else

11. return $\bar{F}_k := F_k$

12. end if

have an analogue result for $\pi_k(Q^{(k-1)}_s)$. This finally implies that Algorithm 5.1 terminates and is correct when given input set $F_{k-1}$.

Besides the modification of the input set, Algorithm 5.1 stays the same. Of course we may drop all tests if $b - g \in \Lambda$ during the normal-form algorithm because for $b, g \in \Lambda$ it is clear that the difference $b - g \in \Lambda$. The same is valid for Algorithm 5.3. It stays the same except for the dropping of tests whether $b - g \in \Lambda$.

Algorithm 6 is substituted by the above Algorithm 5.5. It does not compute the set $\tilde{F}_k$ but the set $\bar{F}_k$.

Lemma 5.11. Algorithm 5.5 terminates and is correct.

Proof. This is a direct consequence of the discussion in this subsection. □
We have created an implementation of the “project-and-lift” algorithm for the lattice case (section 5.5). The implementation is written in Allegro Common Lisp 8.0 and C. For the computation of the minimal elements of partially extended fibers, we use the library libzsolve, which is a part of 4ti2 (4ti2 team), version 1.3.1. In this section, we report on the computational experience with this code on several test problems. All computation times are given in CPU seconds on a Sun Fire V440 with UltraSPARC-IIIi processors running at 1.6 GHz.

6. First computational results

6.1. Results for number-partitioning problems. We first consider the problem of partitioning a natural number \( n \) into given parts (natural numbers) \( a_1, \ldots, a_k \) (with possible multiplicity). To this end, consider the set

\[
P_n = \left\{ (x_1, \ldots, x_k) \in \mathbb{Z}_+^k : n = \sum_{i=1}^{k} x_i \cdot a_i \right\}.
\]

We are interested in a minimal set \( \{n_1, \ldots, n_q\} \) of natural numbers such that the set \( P_n \) of partitions of every number \( n \) is the Minkowski sum of some of the sets \( P_{n_j} \). Thus we are interested in the atomic fibers corresponding to the matrix

\[
\begin{pmatrix}
a_1 & a_2 & a_3 & \cdots & a_k
\end{pmatrix}.
\]

(16)

We consider this problem for various sets of numbers \( a_1, \ldots, a_k \). The results are shown in Table 1.

Table 1. Results for number-partitioning problems

| Parts | Atomic fibers | Time (s) |
|-------|---------------|----------|
| 1     | 1             | 1        |
| 1 2   | 2             | 1        |
| 1 2 3 | 4             | 1        |
| 1 2 3 4 | 9          | 1        |
| 1 2 3 4 5 | 32       | 875      |
| 1 2 3 4 5 6 | 41        | >1000    |
| 2 3   | 3             | 1        |
| 2 3 5 | 14            | 1        |
| 2 3 5 7 | 72          | 149661   |
| 3 5   | 1             | 1        |
| 3 5 7 | 30            | 1        |

6.2. Results for homogeneous number-partitioning problems. Next we consider the problem of partitioning a natural number \( n \) into given natural numbers \( a_1, \ldots, a_k \) (with possible multiplicity), where we prescribe the number of summands. To this end, consider the set

\[
P^m_n = \left\{ (x_1, \ldots, x_k) \in \mathbb{Z}_+^k : n = \sum_{i=1}^{k} x_i \cdot a_i, \ m = \sum_{i=1}^{k} x_i \right\}.
\]

(17)

We are interested in a minimal set \( \{(m_1, n_1), \ldots, (m_q, n_q)\} \) of pairs \((m, n)\) such that the set \( P^m_n \) of partitions of every number \( n \) into \( m \) summands is the Minkowski sum of some of the sets \( P^m_{n_j} \). Thus we are interested in the atomic fibers corresponding to the matrix

\[
\begin{pmatrix}
1 & 1 & \cdots & 1
\end{pmatrix}.
\]

(18)

Again we consider the problem for various sets of numbers \( a_1, \ldots, a_k \). The results are shown in Table 2. We remark that the problem data \((1, 2, 3, 4)\) correspond to a problem equivalent to the one from Example 2.7.
Table 2. Results for homogeneous number-partitioning problems

| Parts | Atomic fibers | Time (s) |
|-------|---------------|----------|
| 1     | 1             | 1        |
| 1 2   | 2             | 1        |
| 1 2 3 | 4             | 1        |
| 1 2 3 4| 18            | 1        |
| 1 2 3 4 5| 79            | 19       |
| 1 2 3 5 | 12            | 1        |
| 1 2 3 6 | 35            | 2        |
| 1 2 3 7 | 19            | 1        |
| 1 2 3 8 | 58            | 30       |
| 1 2 3 9 | 28            | 2        |
| 1 2 3 10| 87            | 206      |
| 1 2 3 11| 39            | 6        |
| 1 2 3 12| 122           | 1620     |
| 1 2 3 13| 52            | 21       |
| 1 2 3 14| 163           | 5136     |
| 1 2 3 15| 67            | 72       |
| 1 2 3 17| 79            | 216      |
| 2 3   | 2             | 1        |
| 2 3 5 | 4             | 1        |
| 2 3 5 7| 26            | 1        |
| 2 3 5 7 11| 262         | 152792   |

6.3. Results for Steinberger’s sums of roots of unity. One example that appears and was solved in Steinberger (2004) is the computation of the atomic fibers of the matrix

\[
\begin{pmatrix}
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 & 0
\end{pmatrix}
\]

This matrix corresponds to a certain problem on 3 × 3 tables and has in fact 31 atomic fibers and 79 extended atomic fibers. The atomic fibers can be computed with our implementation in less than one CPU second.

The next higher problem on 4 × 4 tables leads to the matrix

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0
\end{pmatrix}
\]

Our implementation was able to compute the 12,675 atomic fibers for this matrix within 6.5 CPU days on a Sun Fire V440 with UltraSPARC-IIIi processors running at 1.6 GHz.
COMPUTATION OF ATOMIC FIBERS OF \( \mathbb{Z} \)-LINEAR MAPS

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