Long vs Short Time Scales: the Rough Dilemma and Beyond

Matthieu Garcin*          Martino Grasselli †

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Abstract

Using a large dataset on major stock indexes and FX rates, we test the robustness of the rough fractional volatility model over different time scales. We include the estimation error as well as the microstructure noise into the analysis. Our findings lead to new stylized facts regarding the volatility that are not described by models introduced so far: in the fractal analysis using the absolute moment approach, log-log plots are nonlinear and reveal very low perceived Hurst exponents at small scales, consistent with the rough framework, and higher perceived Hurst exponents for larger scales, along with stationarity of the volatility. These results, obtained for time series of realized volatilities are confirmed by another measure of volatility, namely Parkinson’s volatility, taking into account its specificities regarding measurement errors.

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1 Introduction

It is well known that a constant volatility is not consistent with time series data nor implied volatility surfaces. Several stochastic volatility models have been introduced in the last decades in order to reproduce the stylized facts of time series observed for both historical and implied volatility, see e.g. [Stein and Stein, 1994] [Heston, 1993] [Barndorff-Nielsen and Shephard, 2001] [Hagan et al., 2002], affine models like in [Duffie et al., 2003], ARCH, GARCH, their non-parametric extensions [Garcin and Goulet, 2019], and many others. In a different (implied volatility) perspective, [Dupire, 1994] assumed that the volatility becomes a deterministic function of time and of the current state of the asset, thus leading to the local volatility approach, which provides a theoretically perfect reproduction of the implied volatility surface.

All these approaches adopt the classic Brownian framework for the noise of both the underlying and its volatility. Motivated by an apparent presence of long memory in the volatility process, see e.g. [Lo, 1991], some researchers (see e.g. [Ding et al., 1993], [Baillie, 1996], [Bollerslev and Mikkelsen, 1996],

*Léonard de Vinci Pôle Universitaire, Research Center, 92916 Paris la Défense. Email: matthieu.garcin@devinci.fr.
†Department of Mathematics “Tullio Levi Civita”, University of Padova, via Trieste 63, 35121 Padova, Italy, and Léonard de Vinci Pôle Universitaire, Research Center, 92916 Paris la Défense. Email: grassell@math.unipd.it.
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[Andersen and Bollerslev, 1997], [Breidt et al., 1998]... modelled the log-volatility noise with a fractional Brownian motion, leading to the so-called fractional stochastic volatility (FSV) model[^1].

[Comte and Renault, 1998] suggested a model where the driving fractional Brownian motion has Hurst parameter $H > 0.5$, in order to take into account for the stylized fact suggesting that the volatility is a long memory process. Their estimation procedure for the Hurst parameter was based on the method of [Geweke and Porter-Hudak, 1983], involving the slow decay of the autocorrelation function (which is supposed to be of power law with exponent less than one) and it reveals to be problematic, as the asymptotic behaviour of the covariance function cannot be directly computed without assuming a specific functional form.

Recently, a new paradigm for the volatility process has been introduced by [Gatheral et al., 2018], who affirmed a universal phenomenon: volatility is rough and cannot be described by a SDE driven by a classic Brownian motion. In particular, they showed that the autocorrelation function of the volatility does not behave as a power law, at least at the time scales ranging from one day till 2 months considered in their observation, so that they disentangled the question about the long memory of the volatility from the asymptotic behaviour of the autocorrelation function. Then, [Gatheral et al., 2018] introduced a model, where the logarithm of the volatility is driven by a fractional Brownian motion (fBm) with a Hurst exponent that is empirically found to be very low, thus leading to rough trajectories for the volatility. For this reason, the approach is referred to as the rough fractional stochastic volatility model (RFSV). Using absolute moments estimation on a wide range of scales (from 1 day to approximately 50 days), the Hurst exponent is found to be close to 0.14 both for the log-volatility of S&P 500 and the NASDAQ, together with other major indexes. The series of volatilities used in the seminal paper [Gatheral et al., 2018] come from the realized variance estimates from the Oxford-Man Institute of Quantitative Finance Realized Library[^2] between January 3, 2000 and March 31, 2014. In their approach, the daily squared volatility is estimated using the quadratic variations of the log-prices at a five-minute frequency (about 96 observations per day, in total about 3500 days). Another fundamental result in [Gatheral et al., 2018] states that the estimation of the Hurst exponent $H$ is robust across time, scales and markets (equity indexes and FX).

One may wonder if the results of [Gatheral et al., 2018] depend on the particular estimation procedure adopted. There are indeed different ways to compute the volatility proxies (see e.g. the Fourier-based estimators for the realized variance in [Cuchiero and Teichmann, 2015] or the Parkinson estimator [Parkinson, 1980] that one can adopt when the realized variance is not available), or methods that even circumvent the absolute moment estimation procedure, like the Whittle-type estimation methods used in [Fukasawa et al., 2019] to find directly the Hurst exponent. However, as we will also recall in Subsection 2.3, all these methods lead to qualitatively similar results.

We thus observe two branches of the literature of fractional volatility leading to opposite conclusions:

- The traditional econometric approach, which studies the speed of the decay of the autocovariance function, concludes that there is long memory.
- The rough volatility approach, in which the Hurst exponent is estimated from scaling proper-

[^1]: A fractional Brownian motion (fBm) $B_H$ of Hurst exponent $H \in (0, 1)$ is a Gaussian process with a non-trivial covariance function, namely a non-Markovian process that allows for long or short memory, according to resp. $H > 0.5$ or $H < 0.5$. The case $H = 0.5$ corresponds to the classic (Markovian) Brownian motion.

[^2]: [http://realized.oxford-man.ox.ac.uk/data/download](http://realized.oxford-man.ox.ac.uk/data/download) The Oxford-Man Institute’s Realized Library contains a selection of daily non-parametric estimates of volatility of financial assets, including realized variance and realized kernel estimates.
ties of the series, uses thus estimators from literature of econophysics and of the statistics of stochastic processes. It concludes that there is no long memory.

In these two approaches, the tools are not the same, the range of scales analysed may differ, and the conclusions diverge.

A paper that somehow adopts both approaches is [Bennedsen et al., 2016], where the authors present a two-factor stochastic volatility model which is rough at the short time scales, but it presents stationarity at longer time scales, according to the traditional approach that considers the autocovariance function. This mixing effect leads to an effective Hurst parameter varying on different observation time scales. In particular, the authors find a Hurst parameter for the S&P 500 index ranging from few cents up to 0.2, as the observation time scale grows from one minute to few hundreds minutes, in line with our findings, where we get a similar result also for other equity indices. It is worth noticing that [Bennedsen et al., 2016] do consider just intra-day data: it would be interesting to see if their model is able to reproduce the stylized facts that we empirically observe also for longer time scales with daily data as we are going to describe.

Despite some technical issues arising from the fact that a volatility process driven by a fractional Brownian motion is not a semi-martingale (and the corresponding integrals require particular care), rough volatility models have been largely investigated in recent theoretical and empirical literature, to the point that today one can find more than one hundred papers on the subject[3]. Surprisingly, the empirical investigation basically relies on the dataset from the Oxford-Man Institute’s Realized Library, which is indeed very useful, but far to be the most complete.

In this paper, we extend previous empirical studies to a wider range of time scales, beyond the ones investigated in [Gatheral et al., 2018], in order to check whether data are consistent with the scaling properties predicted by fractional volatility models. We also include the mean reversion of the volatility in our investigation: when we consider large time scales, we expect to be able to observe the stationarity features that were precluded in the previous studies, since the time windows investigated so far were too small compared to the mean reversion frequency. In order to be able to observe such mean reversion effect, we obviously need long time series for the underlyings. This opens the door to an additional difficulty, because for equity indexes only daily data are available for time series starting before the 2000s, therefore we cannot use the realised volatility as a proxy of the unobserved daily volatility and we need another estimator. We suggest to adopt the Parkinson daily volatility estimator, see [Parkinson, 1980], which reveals to give qualitatively the same results of the realized volatility estimator (in terms of the estimation of the Hurst exponent), when tested on the same dataset where intraday data are available.

Endowed with a proxy for the daily volatility, we then repeat the empirical investigation of [Gatheral et al., 2018] and we confirm their findings for comparable time scales, while for larger scales our results are in contrast with the usual rough volatility paradigm, even by including the mean reversion effect into the analysis. In particular, by broadening the ranges of scales, we highlight new stylized facts about the scaling of volatility processes that cannot be reproduced by the usual rough volatility model driven by a fractional Brownian motion.

As volatility is unobserved, we include the estimation error into the analysis. In fact, what we observe is only noisy volatility. We model three types of noise, coming from microstructure, measurement and smoothing (arising from the fact that we replace the unobserved spot variance with the

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3 See e.g. the papers on the website https://sites.google.com/site/roughvol/home
We quantify the impact of all these noises in the estimation of the Hurst exponent and we filter out their effects. What remains after the filtering still deviates from the straight line behaviour predicted by the rough volatility model paradigm. In particular, our results show that the presence of only one fBm is not enough in order to meet all stylized facts empirically observed in our dataset.

The paper is organised as follows: in Section 2 we introduce the absolute moment estimation for the Hurst exponent and we compare alternative estimation procedures in the past literature. Section 3 shows the empirical results for the equity indexes, while in Section 4 we analyze the exchange rates market. In Section 5 we introduce and quantify the impact of the three sources of noise (namely, the measurement error, microstructure noise and the smoothing effect) that we identify in our estimation procedure. In Section 6 we filter out the three noises using the assumptions for the volatility process that are typically adopted in the literature. Section 8 concludes.

## 2 Estimation of the Hurst exponent

### 2.1 Absolute moments estimation of the Hurst exponent

Consider the dynamics of a price process $S$ that evolves according to the SDE

$$dS_t = (\cdot) dt + S_t \sigma_t dB_t, t \geq 0,$$

where $B$ is a standard Brownian motion defined in a probability space that satisfies the usual technical conditions. The process $S$ is assumed to be a semimartingale in order to avoid arbitrage opportunities.

The volatility price process $\sigma$ is not directly observable, one can only deduce indirectly its properties through some observable proxies like the realized variance process, defined as

$$\hat{\sigma}^2_{\delta, t} = \sum_{(t-1)\delta \leq u \leq t\delta} |\Delta \log S_u|^2,$$

where $S$ is a piecewise constant process which jumps at every sampling time of $S$ to the observed value of $S$ at the time. If there is no measurement error and the sampling frequency goes to infinity we have that

$$\hat{\sigma}^2_{\delta, t} \rightarrow \int_{(t-1)\delta}^{t\delta} \sigma^2 u \, du,$$

in probability, which justifies the choice of (2) for a proxy of the realized variance process, as well as its square root for the daily spot volatility, in the case where $\delta$ corresponds to the length of one day.

In [Gatheral et al., 2018], the authors performed a linear regression in order to fit the empirical absolute moment of order $k$ of the log-volatility, defined as

$$\frac{1}{n} \sum_{t=1}^{n} |\log \hat{\sigma}_{\delta, t+\tau} - \log \hat{\sigma}_{\delta, t}|^k.$$

They found a good fit, for different values of $k$, with the function

$$\log \frac{1}{n} \sum_{t=1}^{n} |\log \hat{\sigma}_{\delta, t+\tau} - \log \hat{\sigma}_{\delta, t}|^k \approx kH \log \tau + \eta_k,$$

for a very small value of $H \approx 0.1$. 

integrated variance estimator).
This special scaling property, together with some empirical stylized facts on the Gaussian nature of
the log-variance (see e.g. [Andersen et al., 2003a]), induced [Gatheral et al., 2018] to assume a
particular SDE for the log-volatility of the form
\[ \begin{align*}
d \log \sigma_t^2 &= \eta d B_t^H, \\
\end{align*} \]
where \( \eta \) is a constant and \( B_t^H \) is a fractional Brownian motion with Hurst parameter \( H \).

In fact, a fractional Brownian motion (fBm) of Hurst exponent \( H \in (0, 1) \) and scale parameter \( \eta^2 \)
has stationary increments that satisfy
\[ E \left[ \left| B^H_t - B^H_{t-\tau} \right|^k \right] = \frac{2^{k/2} \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \eta^k \tau^k, \quad \tau, k \geq 0, \]
where \( \Gamma(\cdot) \) denotes the Gamma function, see [Kolmogorov, 1940, Mandelbrot and van Ness, 1968].

Turning things around, we define the absolute empirical moment of order \( k \) of the increments of a
process \( \tilde{X} \) (playing the role of the log-volatility process), in a time interval \([0, N]\) for a given scale \( \tau \):
\[ M_{k, \tau, N}(\tilde{X}) = \frac{1}{\lfloor N/\tau \rfloor} \sum_{i=1}^{\lfloor N/\tau \rfloor} |X_{i\tau} - X_{(i-1)\tau}|^k. \]

Using equation (6), it follows that \( \ln(M_{k, \tau, N}(\tilde{X})) \) is proportional to \( H \) if \( \tilde{X} \) is a fBm as increments
are stationary. This is the basis for estimators of Hurst exponents, see e.g. [Benassi et al., 1998, Garcin, 2017]. In particular, we can compute such empirical absolute moments for a great number
of scales, and the estimator of \( H \) is then \( 1/k \) times the slope of the regression of \( \ln(M_{k, \tau, N}(\tilde{X})) \) on \( \ln(\tau) \) [Coeurjolly, 2005]. As a consequence, when plotting \( \ln(M_{k, \tau, N}(\tilde{X})) \) as a function of \( \ln(\tau) \) (also
called log-log plot), we should get a straight line if \( \tilde{X} \) is a fBm.

Note that the estimate \( H \approx 0.1 \) is in contrast with past literature on fractional volatility like
[Comte and Renault, 1998], where the long memory was associated to \( H > 0.5 \). In particular, \( H < 0.5 \)
means that the volatility path is rougher than semimartingales, and is consistent to a power law for the
term structure of implied volatility skew empirically observed in option markets, see [Gatheral et al., 2018]
and references therein.

### 2.2 Impact of the mean-reversion at large scales

An empirical stylized fact for the volatility is that it should be a stationary process, for both mathemat-
cal tractability and financial interpretability mostly at large times. In order to be consistent with a sta-
tionarity assumption, the rough volatility model in [Gatheral et al., 2018] extends (5) to the case where
the log-volatility follows a fractional Ornstein-Uhlenbeck process (fOU, see [Cheridito et al., 2003])
with a very long reversion time scale, so that the effect of this mean reversion is invisible at the scales of
the study (time scales considered are between one day and two months). However, when dealing with
scales longer than two months, we should expect a deviation from the linear behaviour in the log-log
plot for stationarized fBm processes (for example a fOU or the inverse Lamperti transform of a fBm,
see e.g. [Cheridito et al., 2003, Garcin, 2019, Šapina et al., 2017]). We illustrate the phenomenon in
the simple case where \( k = 2 \), namely the second absolute moment for the fBm, that can be rewritten
as follows:

\[ M_{2, \tau}(X) = E \left[ \left| X_t - X_{t-\tau} \right|^2 \right] = E \left[ |X_t|^2 \right] + E \left[ |X_{t-\tau}|^2 \right] - 2Cov \left( X_t, X_{t-\tau} \right) \]

\[ \text{4 In reality, we are using a version of } M_{k, \tau, N}(X) \text{ with overlapping increments. This allows us to slightly increase the convergence of this empirical absolute moment [Lo and MacKinlay, 1988].} \]
Now, for large scales, the first two terms become similar and independent of $\tau$. If the third term, the covariance, is a decreasing function wrt $\tau$ (decreasing to 0), then $M_{2,\tau}(X)$ becomes an increasing function that flattens for large $\tau$. As a consequence, the function $\log(\tau) \rightarrow \log(M_{2,\tau}(X))$ behaves as a straight line for small values of $\tau$, but as the time scale increases, the slope decreases gradually until reaching zero and we observe a concave behaviour for large scales. In conclusion, if we include the stationary volatility stylized fact into account, we should observe a potential decrease in the slope of the straight line in the log-log plot, or equivalently a smaller Hurst exponent for larger scales.

### 2.3 Spurious roughness arising from mean reversion and Whittle estimator

In a recent paper, [Fukasawa et al., 2019] confirm the findings of [Gatheral et al., 2018] on volatility roughness using another estimation methodology for the Hurst exponent. The authors argue that one should use the Whittle-type estimation for high-frequency self-similar Gaussian models developed in [Fukasawa and Takabatake, 2019], instead of the absolute moment method. The poor quality of the latter is underlined in a mean-reversion model for the log-volatility where the speed of reversion is so high that the fractal property of the series turns out to be invisible for this method, thus leading to a spurious effect of roughness. In particular, [Fukasawa et al., 2019] show that the absolute-moment method fails to estimate properly the Hurst exponent in the case of a standard Ornstein-Uhlenbeck model with a strong speed of reversion, $\theta = 10$:

$$dX_t = \theta(-3.2 - X_t)dt + 0.8dB_t,$$

where $B_t$ is a standard Brownian motion. In fact, this bias of the absolute-moment estimator in case of stationary time series is well documented e.g. in [Garcin, 2019]. The stationarity indeed tends to flatten the log-plot for high scales as emphasized in the previous subsection: the log-plot is thus nonlinear and the linear regression irrelevant. The larger the speed of reversion, the larger the range of scales with flat log-plot and the larger the bias.

However, in this subsection we are going to show that the Whittle-based estimator of [Fukasawa et al., 2019] also fails on such a pathological case. In order to fairly compare both Whittle and absolute moments estimators, we simulate 100 trajectories of 1,000 dates, with a time step of 0.1, of the Ornstein-Uhlenbeck process mentioned in Equation (8), for various values of $\theta$. We show that for $\theta = 10$, both estimators largely underestimate the Hurst exponent, thus both leading to a spurious roughness effect for the volatility. The average Hurst exponent estimated in this case by the Whittle approach is even irrelevant in Figure 1 because the implementation of the estimator, with the FGN library in R, floors this value at 0.05. Nevertheless, when decreasing progressively the speed of reversion, the bias decreases more rapidly for Whittle estimator than for the absolute-moment estimator. We could thus conclude that Whittle estimator is more accurate, but it is not so clear. Indeed, the output of the absolute-moment method depends on an arbitrary choice of scales for the regression. In particular, we considered 512 scales, from a time step of 0.1 to 51.2. When limiting to the two lowest scales, 0.1 and 0.2, we get another estimator, also based on the absolute-moment method, which is not as sensitive to stationarity and whose bias is lower than the one of Whittle estimator for similar speed of reversions, as one can see in Figure 1.

Besides, we find also some arguments in favour of the absolute-moment method. First, the computation time is much lower for this estimator, but, above all, the interpretability provided by this method is overriding. This estimator is based on the log-log plot which is linear in the case of the fBm. Every non-linearity of the log-log plot indicates some divergence from the fBm case and can be interpreted. For example, a flat plot for high scales suggests stationarity, as exposed above. A flat plot for low scales suggests the presence of white noise. On the opposite, the Whittle estimator is based on the power spectral density of the signal and after our observations, the empirical power spectral density is
Figure 1: Average of the estimated Hurst exponent for the model described in equation (8), for various values of $\log(\theta)$. The absolute-moment method is in black, for 512 scales (solid line) and 2 scales (dotted line). The Whittle estimator is in grey, but the value obtained for the highest speed of reversion is inconsistent with Whittle approach as the estimator used floors the estimate to 0.05.

often very erratic, so that it is much more difficult to understand if some additional effect, such as noise or stationarity, mitigates the fractional feature.

Our conclusion, regarding both estimators, is that there is no definitive reason to discard one or the other. We think that the estimation of the roughness of volatility is less a problem of choice of estimator than a problem of noise. Interestingly, [Fukasawa et al., 2019] adapted the Whittle estimator to take into account the measurement noise of the variance process, that is the difference between realized variance and integrated variance. This result is promising and unrelated to the comparison study we made between Whittle and absolute-moment estimators, because we did it on simulated series without the realized-variance layer. In fact, using the absolute-moment method, we are also able to filter out this measurement noise, as well as two other noises: the microstructure noise and the error between integrated variance and spot variance as we shall see in the next sections.

2.4 A useful proxy for daily volatility: Parkinson estimator

When intraday data are available, many proxies for the realized volatility have been introduced in literature, see e.g. the survey in [Gatheral and Oomen, 2010] or the estimator in [Cuchiero and Teichmann, 2015], based on previous works in Fourier analysis by [Mancino and Sanfelici, 2008]. These proxies are typically for the spot volatility when many intraday data are available, even tick-by-tick data, so that the measurement noise is small compared to the microstructure noise.

However, it is very useful to introduce a proxy for the daily volatility also when available data are few, which is the case for example when considering historical series for some equity indexes. For example, the dataset of realized variance of the Oxford-Man Institute begins in 2000 only. High-frequency data, which are used to compute a realized volatility, are also not easily available for previous dates. On the other extreme situation, it could be interesting to investigate the behaviour of the rough volatility model for higher resolution, say intraday, and once again if we cannot observe prices with a higher frequency we do not dispose of the realized volatility.

The limitation coming from the absence of intraday data compels us to use estimators of daily
volatility based on the range of prices reached during the day. Indeed, historical data including the highest, the lowest, the opening, and the closing prices (respectively \(H, L, O, C\)) each day are available, for example, since December 1988 for NASDAQ and April 1982 for S&P 500.

In the literature, mainly two proxies of daily volatility based on the highest and lowest prices of the day are proposed:

- Parkinson estimator \([Parkinson, 1980]\): \((h - l)P_1\),
- Rogers-Satchell estimator \([Rogers and Satchell, 1991]\): \((h[h - c] + l[l - c])R_1\),

where \(h = \log(H/O), l = \log(L/O), c = \log(C/O)\), and where \(P_1\) and \(R_1\) are scale parameters. In detail, \(1/P_1\) is the expected value of the high-low range, \(h - l\), for a series of volatility 1 and \(1/R_1\) is the expected value of \(h[h - c] + l[l - c]\) for the same series. Nevertheless, in what follows, we omit this scale parameter, because it neither affects the fractal properties of the series of volatility nor the estimation of the corresponding Hurst exponent.

These two methods have some limitations. First, they are based on observations of a continuous process in discrete time, the transaction times. The highest price is thus underestimated, the lowest price overestimated, and the volatility is underestimated. Adjustments of Rogers-Satchell formula exist, which take into account the time between two transactions \([Rogers et al., 1994]\). However, we are working here with massively traded indexes, so that we can consider that this bias is minor in our framework.

A second limitation is that we obtain each estimator of the time-varying volatility, at each instant of our high-frequency discretization, with only one observation of this range. In the standard estimators of Parkinson or of Rogers-Satchell, we should instead use an average of their statistic on higher frequencies, so as to define the daily volatility as this average. By doing so, we would benefit from strong convergence results. Nevertheless, calculating an average high-low range is infeasible in our framework, since we want to use long historical datasets, without available corresponding high-frequency prices. For this reason, using the isolated Parkinson or Rogers-Satchell statistic, instead of an average of these statistics on several time intervals, is not rare \([Alizadeh et al., 2002, Molnár, 2012]\). The high-low range solution is by far better than using an isolated daily quadratic return as a proxy of the variance. Indeed, the variance of this last proxy is much higher and if the closing price is equal to the opening price its value is zero, which is not the case of the high-low range statistic \([Alizadeh et al., 2002, Molnár, 2012]\).

In our work, we prefer Parkinson estimator to Rogers-Satchell estimator. Indeed, as soon as the opening price is the highest price and the closing price the lowest, which is a situation we encounter several times, Rogers-Satchell estimator is equal to zero. Rogers-Satchel method is intended to filter trends while estimating volatility. As the model we study, based on the fBm, does not include any significant trend, Parkinson estimator seems more pertinent.

### 3 Empirical results for major indexes

In this section we test the volatility roughness paradigm using the well-known dataset of realized volatility provided by the Oxford-Man Institute, between the 3rd January 2000 and the 21st November 2018. We thus have series of 4738 daily volatilities built on intraday prices sampled every 5 minutes. As we desire to work with longer time series of volatilities, we bypass the limitation of this dataset with Bloomberg data finishing on the 30th November 2018 and starting at various dates, depending on the stock index considered: the 21st April 1982 for S&P 500 (9553 days), the 15th December 1988 for NASDAQ (7817 days), the 1st April 1986 for FTSE (8520 days), the 2nd November 1988 for DAX (7834 days), the 1st July 1988 for CAC (7929 days), and the 14th May 1991 for SMI (7189 days).
For this longer dataset, we do not have access to intraday prices, but only to open, close, highest, and lowest prices. For this reason, we are unable to build series of realized volatilities starting so far in the past. We thus work with another estimate of the daily volatility known as Parkinson’s volatility.

3.1 Consistency test for the Parkinson estimator

We first start with a subset comparable with the sample used by [Gatheral et al., 2018], that is we take 5-minute price returns, between the 3rd January 2000 and the 21st November 2018 for both S&P 500 and NASDAQ indexes.

For this dataset both (square root of) realized variance and Parkinson estimator are available proxies for the daily volatility. The aim here is to repeat the procedure of [Gatheral et al., 2018] in a similar dataset and compare the estimation of the Hurst exponent using also the Parkinson estimator. We consider time scales till about 6 months (corresponding to $\log(\tau) \approx 5$), while the original empirical investigation in [Gatheral et al., 2018] allows for at most 2 months).

A comparison for both S&P 500 and NASDAQ indexes of daily Parkinson’s volatility and daily realized volatility built on 5-minute price returns, between the 3rd January 2000 and the 21st November 2018, shows that:

- For each index, both series of volatility, even though not being equal, are strongly correlated. The daily increments of the series of log-volatilities have a Pearson’s correlation of 63% for S&P and 57% for NASDAQ, whereas weekly increments have a correlation of 73% for S&P 500 and 70% for NASDAQ.

- For a range of scales between 1 day and 149 days, the linear regression of the log quadratic increments on their log time scale sounds relevant for both the log-volatility metrics, as one can see in Figure 2. Our estimator of the Hurst exponent is half the slope of the linear regression. For both indexes, we get a Hurst exponent slightly lower for Parkinson’s volatility than for realized volatility, always with a very high coefficient of determination, as reported in Table 1.

![Figure 2: $\log(\tau) \mapsto \log(M_{2,\tau,N}(X))$, where $N = 4738$ is the number of days between the 3rd January 2000 and the 21st November 2018, and $X$ the log of the volatility of respectively S&P index (on the left) and NASDAQ index (on the right). We consider a range of scales between 1 day and 149 days, corresponding to $\log(\tau) \approx 5$. Parkinson’s volatility is in black and realized volatility in grey.](image)

Depending on the volatility metric, we get different values for the Hurst exponent of the log-volatility series. However, the conclusion is the same concerning the model: a rough volatility model...
|                        | S&P index | NASDAQ index |
|------------------------|-----------|--------------|
| Realized volatility    | 0.125 (99.3%) | 0.115 (98.2%) |
| Parkinson’s volatility | 0.083 (98.8%) | 0.069 (98.8%) |

Table 1: Estimated Hurst exponent as half the slope of the linear regressions presented in Figure 2. In parenthesis, coefficient of determination of each linear regression.

sounds consistent with observations. Indeed, the straight lines in Figure 2 for both Parkinson’s log-volatility and realized log-volatility stresses the relevance of a fBm, and the slope of the lines shows that the Hurst exponent is very low. So, with Parkinson’s volatility, we get results which are similar to those obtained with realized volatility. In other words, the rough volatility model of [Gatheral et al., 2018] is validated for both realized volatility and Parkinson’s proxies for the volatility for time scales not only till 2 months as in the their paper, but even till 6 months.

In the next subsection, however, we are going to show that extending the range of resolution will lead to deviations from the straight line behaviour for the function $\ln(\tau) \mapsto \log(M_{2,\tau,N}(X))$.

### 3.2 Study at lower resolutions

Endowed with the Parkinson-based proxy for the daily volatility, which has been shown in the previous subsection to be consistent with the realized variance estimator, we now extend our study by considering the whole time series available on some indexes: for example, since December 1988 for NASDAQ and April 1982 for S&P 500. For these series, historical data including the highest, the lowest, the opening, and the closing prices each day are available, so that we can use the Parkinson estimator for the daily volatility, while the realized variance estimator cannot be implemented.

We see in Figure 3 the plot of the logarithm of the absolute moment as a function of the log-scale, for long historical series of proxies of log-volatility of various equity indexes. When we only consider scales below a certain threshold, which roughly corresponds to the abscissa 5 for most indexes (i.e. about 6 months), the linear approximation sounds correct, in line with the findings of the previous subsection. However, when we consider a larger range of scales, the linear approximation clearly fails. This is in contradiction with a fBm model. The shape is instead a convex increasing function.

**Remark 3.1.** As recalled in Subsection 2.2, the rough volatility model in [Gatheral et al., 2018] assumes that log-volatility follows a fBm, on which a mean reversion with a very weak strength is added so that the process is stationary. This mean reversion was not visible for short historical series as in Figure 2. Neither is it clearly visible for longer historical series and larger scales, at least for major indexes. Indeed, a mean reversion should flatten the curve of $\log(\tau) \mapsto \log(M_{2,\tau,N}(X))$ for large scales [Garcin, 2019]. In Figure 3, on the contrary, we observe an increased slope for time scales larger than 6 months.

With two linear regressions, one on small scales (below 6 months) and the other on larger scales (above 6 months), we were able to determine the Hurst exponent perceived at the corresponding scales, reported in Table 2. At larger scales, the perceived Hurst exponent is 1.6 to 2.7 times greater than what it is for small scales.

We also remark that above a given scale, it is difficult to describe the shape of the plot $\log(\tau) \mapsto \log(M_{2,\tau,N}(X))$. In Figure 3, we have displayed the plot for the corresponding scales in grey and we have limited our analysis of perceived Hurst exponents to smaller scales. This zone with erratic behaviour is roughly between abscissae 6.3 and 8.3, that is to say for scales between 545 and 4024 days. The shape of the plot in this zone depends on the equity index:
Figure 3: $\log(\tau) \mapsto \log(M_{2,\tau,N}(X))$, where $N$ is the number of days in the historical series (between 7189 for SMI index and 9553 for S&P 500 index), and $X$ the log of Parkinson’s volatility of various equity indexes. Straight lines are linear regressions either for small scales (below 6 months) or for large scales (above 6 months).
|                        | Small scales (≤ 6 months) | Large scales (> 6 months) |
|------------------------|---------------------------|---------------------------|
| S&P 500 index          | 0.047                     | 0.078                     |
| NASDAQ index           | 0.070                     | 0.189                     |
| FTSE index             | 0.059                     | 0.095                     |
| DAX index              | 0.057                     | 0.155                     |
| CAC index              | 0.064                     | 0.120                     |
| SMI index              | 0.064                     | 0.140                     |

Table 2: Perceived Hurst exponent at small (≤ 6 months) and large (> 6 months) scales, estimated as half the slope of the linear regressions presented in Figure 3.

- For NASDAQ index, we do not observe such a zone.
- For S&P 500, FTSE, and DAX indexes, we observe large fluctuations around a rising trend. These fluctuations evoke seasonality. Indeed, for a scale close to the quasi-period of this seasonality, variations of volatility are much narrower than for any other scale, and in particular than for smaller scales: this results in a negative scaling coefficient between these scales.
- For CAC and SMI indexes, in addition to the seasonality fluctuations, we observe an average slope close to zero. This null slope makes us think there is mean reversion for these indexes.

Remark 3.2. The observations concerning seasonality should be mitigated. The scales at which we observe this phenomenon are so high compared to the length of the datasets that it is not very significant, even though we built the empirical absolute moments with many overlapping increments. An abscissa of 8.3 corresponds more or less to half the length of the datasets, so that a seasonality for such a scale would simply consist in the observation of two periods. This is not enough to attest that seasonality exists in these series.

In conclusion, the stylized facts revealed by our analysis are the following:

1. The perceived Hurst exponent is gradually rising with the scale, starting from values close to 0.05 for one-day scales.
2. Mean reversion at very large scales may exist, as we can conjecture from observations of CAC and SMI indexes. But no strong argument nor confirmation for other indexes underpin this supposition. The rough volatility model of [Gatheral et al., 2018] also assumes a stationary volatility, consistently with the economic intuition.

3.3 Study at higher resolutions

We are now interested in the shape of $\log(\tau) \mapsto \log(M_{2,\tau,N}(X))$ for small values of $\tau$, in particular below one day. Since the intraday frequency of the dataset is one minute and we’re interested in time scales between 1 minute and 128 minutes, we cannot implement the realized variance proxy for the spot volatility. Again, we still use Parkinson’s volatility on one-minute time intervals. We work with a six-month high-frequency dataset for equity indexes, between the 17th May 2018 and the 29th November 2018. As market are closed during the night, we focus on shorter series of one trading day only, in which we have excluded the end of each daily series, during which the price is constant. We calculate then the empirical function $\log(\tau) \mapsto \log(M_{2,\tau,1}(X))$ at each day, for scales between 1 minute and 128 minutes, still with overlapping increments, and we average them for all the trading days observed.
Results are displayed in Figure 4. We observe shapes that are similar to those found for lower frequencies, except that we never observe a flattening of the curve for the greatest scales. The slope is very weak for small scales and it gradually increases. The cause of the absence of flattening may be twofold:

1. We may have restricted too much the range of scales. In particular, there is no continuity between the range of scales in this subsection and in the previous one. A global sight on scales between -7.3 and 8.3 is unfortunately not possible with our dataset.

2. The high-frequency volatility may not be stationary. Figure 5 shows the average log-volatility minute by minute of NASDAQ index and underpins this last hypothesis. Indeed, we observe that the high-frequency volatility in average decreases along the day, with an upsurge at the very end of the trading day. This is in contradiction with stationarity hypothesis.

The case of S&P 500 index is particular. For the smallest scales (one minute), we observe a half slope equal to 0.081, which suddenly changes at scale log(τ) = -5.5 (corresponding to 15 minutes) to become much lower, at 0.017. Then, the half slope gradually increases and asymptotically reaches 0.152. For all the other indexes, data seem to support a model involving a linear combination of at least two fBms driving the volatility process.

With the help of two linear regressions for small and large scales, we determine the Hurst exponent perceived at these scales, reported in Table 3. At large scales, the perceived Hurst exponent is 8.5 to 16.6 times greater than the value for small scales, which is much larger than the range we found with daily data. This difference is not surprising if we recall that high-frequency volatility is mainly explained by seasonality and microstructure phenomena that are peculiar to this intraday market.

| Small scales (1 minute) | Large scales (128 minutes) |
|------------------------|---------------------------|
| NASDAQ index           | 0.016                     | 0.184                     |
| DAX index              | 0.016                     | 0.265                     |
| CAC index              | 0.010                     | 0.089                     |
| SMI index              | 0.017                     | 0.144                     |

Table 3: Perceived Hurst exponent at small (1 minute) and large (128 minutes) scales for various indexes, estimated as half the slope of the linear regressions presented in Figure 4.

4 Empirical results for exchange rates

In this section we implement the empirical absolute moment regressions for volatilities of the most liquid exchange rates. We base our analysis on a real data set from Interactive Brokers. It gathers high-frequency rates between the 18th December 2006 and the 19th June 2019, sampled at a one-minute step, for a total of more than 4.6 millions data for each of the following pairs: EUR/USD, EUR/GBP, EUR/JPY, EUR/CAD, EUR/AUD, GBP/USD, GBP/JPY, USD/JPY, AUD/USD, and AUD/JPY. From these rates, we estimate daily volatilities, using the 1,440 observations in each day. We thus have for each series 3,206 consecutive observations of daily volatility that will be estimated in two different manners: a realized volatility, defined as the square root of the average quadratic one-minute log-variation, and Parkinson’s volatility, defined as above using the highest, the lowest and the opening price of each day. In the results, gathered in Figure 6 and Table 4, we see that, even though the estimated Hurst exponents vary between the realized volatility and Parkinson’s volatility, the shape of
Figure 4: Average of the map $\log(\tau) \mapsto \log(M_{2,\tau,1}(X))$, where $X$ is the log of Parkinson’s volatility of various equity indexes and for time scales ranging from 1 minute to 128 minutes. Straight lines are linear regressions either for small scales (1 minute) or for large scales (128 minutes).
the curve in the two cases is very close. This confirms the quality of the results obtained with a simple Parkinson statistic.

In Figure 5, we observe various regimes for the volatility process, depending on the time scale, as for equity indexes. But the scales at which regimes switch seem lower than for equity indexes. In general, for log-scales between 0 and 3 (that is to say between 1 day and 3 weeks), a linear regression of small slope holds for empirical absolute moments. Between 3 and a higher abscissa which depends on the rate considered (so between 3 weeks and a time horizon \( H \), which is located between 4.5 months and 12 months), the slope steepens. Above this long time horizon \( H \), the slope flattens and, at scales which depend again on the sample, aberrant oscillations appear, due to the limited size of the sample. Therefore, we observe a threefold behaviour of the dynamics: for small time scales (1 day till 3 weeks) we get a small Hurst exponent, at higher scales (3 weeks till \( H \)) we have a higher Hurst exponent, and for very large scales (above \( H \)) we observe stationarity. This is consistent with our findings about equity indexes, with the difference that it is much easier to observe stationarity for FX rates than for equity indexes, thanks to the flattening of the plot at very large scales.

We conclude this section with a robustness test for the realized variance estimator. It is well known that at a high frequency, the variations of prices reflect both the macroscopic dynamic led by realized volatility and the microstructure frictions. As a robustness check, instead of basing the estimation only on one-minute price returns, we plot the log-moment graph for a daily realized volatility estimated using thirty-minute returns (then 48 data for each day). By doing so, we use less returns in the estimation of the volatility, so that the convergence is not as good as we introduced some measurement noise. But, on the other hand, returns are less subject to microstructure noise, so that the realized volatility is less biased by these microstructure phenomena. In fact, a thirty-minute step seems to be a good tradeoff in order to deal with both noises for FX rates, according to the literature [Andersen et al., 2003b]. In Figure 7, we present the plot of the log-moments for realized volatilities computed on a one-minute and a thirty-minute basis. The conclusions are the same as before, regardless the duration of the increments in the computation of the realized volatility: we find a small Hurst exponent for small scales and larger perceived Hurst exponents for larger scales. This consistency makes us believe that the results are robust. This is not surprising, since we already obtained the (qualitatively) same results with an alternative volatility proxy, based on Parkinson, which does not depend on the choice of a step duration, as for the realized volatility.
Figure 6: Average of the map \( \log(\tau) \mapsto \log(M_{2,\tau,1}(X)) \), where \( X \) is the log of Parkinson’s volatility (black) or realized volatility (grey) of various FX rates between December 2006 and June 2019 sampled at a one-minute step, for a total of 4.6 millions data for each of the following pairs: EUR/USD, EUR/GBP, EUR/JPY, EUR/CAD, EUR/AUD, GBP/USD, GBP/JPY, USD/JPY, AUD/USD, and AUD/JPY. Time scales range between 1 day till 3 years.
| Type of volatility | Realized volatility | Parkinson’s volatility |
|-------------------|---------------------|-----------------------|
|                   | Small scales | Large scales | Small scales | Large scales |
| EUR/USD           | 0.075       | 0.244       | 0.020       | 0.158       |
| EUR/GBP           | 0.086       | 0.217       | 0.030       | 0.131       |
| EUR/JPY           | 0.117       | 0.190       | 0.048       | 0.102       |
| EUR/CAD           | 0.068       | 0.184       | 0.024       | 0.083       |
| EUR/AUD           | 0.106       | 0.192       | 0.041       | 0.090       |
| GBP/USD           | 0.093       | 0.157       | 0.028       | 0.096       |
| GBP/JPY           | 0.112       | 0.148       | 0.054       | 0.060       |
| USD/JPY           | 0.098       | 0.144       | 0.039       | 0.095       |
| AUD/USD           | 0.108       | 0.196       | 0.044       | 0.107       |
| AUD/JPY           | 0.135       | 0.152       | 0.077       | 0.082       |

Table 4: Perceived Hurst exponent, estimated as half the slope of linear regressions for the plots presented in Figure 6, restricted a time horizon lower than $H$. Small scales refer to the range from 1 day till 3 weeks, large scales refer to a range depending on the sample considered: from 2 months till 4.5 months for EUR/JPY, EUR/CAD, EUR/AUD, GBP/JPY, and AUD/JPY, from 2 till 5 months for AUD/USD, from 3 till 8 months for EUR/USD, from 5 till 9 months for USD/JPY, and from 5 till 12 months for EUR/GBP and GBP/USD.

Figure 7: Average of $\log(\tau) \mapsto \log(M_{2,\tau,1}(X))$, where $X$ is the log of realized volatility with 1-minute (grey) and 30-minute (black) returns of EUR/USD FX rates between December 2006 and June 2019.
5 Roughness and noisy volatility

In Subsection 2.3, we saw that it is possible to simulate a spurious effect of roughness in the volatility paths just by playing with the drift of the volatility process, in particular by introducing a strong mean reversion effect.

There is indeed another easy way to generate spurious roughness, namely by adding noise into the observations. In fact, what we measure is not volatility but just noisy volatility. In this section, we first provide a motivational example where we show that the presence of an additive noise may lead to a spurious roughness effect in a simulation study, where parameters are in line with empirical findings of the historical series considered above. We then quantify the impact of the noises in the estimation of the Hurst exponent in order to filter out the bias.

5.1 Spurious roughness in a simple additive model

We assume that the log-price \( \log(S_t) \) follows an Ito process with stochastic variance \( \sigma_t^2 \). Let \( \hat{\sigma}_t \) be the estimated volatility of day \( t \), obtained using for example the realized volatility or the Parkinson proxy. These estimators provide us with a noisy version of the true and unobserved volatility, \( \sigma_t \). We assume an additive model of measurement noise:

\[
\hat{\sigma}_t^2 = \sigma_t^2 + \varepsilon_t,
\]

where \( t \in \{1, ..., N\} \) and where \( \varepsilon_1, ..., \varepsilon_N \) are i.i.d. centered random variables. Moreover, we consider the simple case where the variance of the log-prices follows a geometric Brownian motion:

\[
\sigma_t^2 = \sigma_0^2 \exp \left( \xi B_t - \frac{1}{2} \xi^2 t \right).
\]

We need to estimate the two parameters of equation (10), \( \sigma_0 \) and \( \xi \), for the purpose of simulation.

First, the initial volatility \( \sigma_0 \) is fixed at the average realized volatility. For example, for EUR/USD we get the value \( \sigma_0 = 4.62 \times 10^{-3} \).

Second, the volatility of the volatility, \( \xi \), is obtained using the variance of the increments of the log-volatility on a one-day step, which is also the exponential of the ordinate at the origin of the log-log plot used in estimating Hurst exponent:

\[
\mathcal{V} := \mathbb{E} \left[ \left| \log(\sigma_t) - \log(\sigma_{t-1}) \right|^2 \right] = \frac{1}{7} \mathbb{E} \left[ \left| \log(\sigma_t^2) - \log(\sigma_{t-1}^2) \right|^2 \right] = \frac{1}{7} \mathbb{E} \left[ \xi(B_t - B_{t-1}) - \frac{1}{2} \xi^2 t \right]^2 = \frac{1}{7} \left( \xi^2 \mathbb{E}[(B_t - B_{t-1})^2] + \frac{1}{4} \xi^4 \right) = \frac{1}{7} \left( \xi^2 + \frac{1}{4} \xi^4 \right).
\]

Using Equation (11), we estimate \( \xi \) from an estimated \( \mathcal{V} \). A natural candidate for the estimation of \( \mathcal{V} \) is the empirical variance of the logarithm of the realized volatility:

\[
\hat{\mathcal{V}} = \frac{1}{N} \sum_{i=1}^{N} (\log(\hat{\sigma}_i) - \log(\hat{\sigma}_{i-1}))^2.
\]

However, the series of realized variance is noisy. To limit the impact of the noise, we compute instead an empirical variance of the same series, but on a larger time scale, \( 1 \leq \tau \ll N \), with a scaling term in

\footnote{Note that here we assume an additive model for the variance, not for the volatility. We can justify this choice by the additive nature of the variance, which makes it possible to apply the central limit theorem and to get an asymptotic distribution of the measurement error [Barndorff-Nielsen and Shephard, 2002], [Meddahi, 2002].}
front of the expression that it consistent with the geometric Brownian motion model of the variance of prices, as illustrated in Figure 8:

$$\hat{\nu}_\tau = \frac{1}{(N - \tau + 1)\tau} \sum_{i=\tau}^{N} (\log(\hat{\sigma}_i) - \log(\hat{\sigma}_{i-\tau}))^2.$$ 

From Equation (11), \(\xi^2\) is solution of the quadratic equation \(\frac{1}{4}x^2 + x - 4\nu = 0\), whose discriminant, \(1 + 4\nu\), is positive. From the two corresponding solutions, only one is positive and thus relevant: \(2(\sqrt{1 + 4\nu} - 1)\). As a consequence, we propose the following estimator for \(\xi\):

$$\hat{\xi}_\tau = \left[2 \left(\sqrt{1 + 4\hat{\nu}_\tau} - 1\right)\right]^{1/2}.$$ 

For our EUR/USD series, sampled every minute, after Figure 8, we estimate \(\xi\) with 3.8%, that is for a high \(\tau\), at which noise should have no impact. We note that this value is only relevant for the geometric Brownian motion model of volatility. In the case where the true volatility is stationary or follows a rough fBm, this value underestimates the true volatility of the variance process.

Endowed with the estimation of the parameters \(\sigma_0\) and \(\xi\) for (10), we simulate the model (9) for different values of the standard deviation of the noise. We shall see that for certain values of the noise variance, a spurious roughness effect appears naturally.

The standard deviation of the noise conditionally to the volatility process is noted \(\sqrt{\text{Var}[\varepsilon_t|\sigma_t]} = \alpha\sigma_t^2\), with \(\alpha \geq 0\). In Figure 9, we see that spurious roughness (\(H < 0.2\)) appears for \(\alpha\) above 8%. In our framework (FX sampled every minute), \(\alpha\) is only 3.7%, according to theoretical results exposed in the next Subsection 5.2 This factor \(\alpha\) moves to 6.2% for stocks sampled every minute, 13.9% for stocks sampled every 5 minutes, 24.1% for stocks sampled every 15 minutes, and 63.2% for Parkinson approach.

This result strongly depends on the input volatility of the volatility (and on the model too, which is here limited to the geometric Brownian motion for the variance process). In particular, we used the same vol of vol for all the simulations, which is the vol of vol relevant for EUR/USD. With higher vol of vol, the impact of the noise in the estimation of the Hurst exponent is decreased, as one can see in Figure 10.

In conclusion, a spurious roughness effect may appear in the presence of a noise. It will be therefore important to measure the standard deviation of the noise in order to filter out the bias from the estimation of the Hurst exponent. Figure 11 shows a typical spurious rough simulated path for a volatility satisfying (10) and a noise with \(\alpha = 25\%\).
Figure 9: Estimated Hurst exponent at small scales (lower than 21 days, in black) and high scales (between 54 and 149 days, in grey) for various noise amplitude ($\alpha$, which is the standard deviation of the noise divided by the variance of the log-price), obtained by simulation of a noisy volatility. The variance process of the price follows a geometric Brownian motion with the parameters estimated as for EUR/USD, i.e. $\sigma_0 = 4.62 \times 10^{-3}$ and $\xi = 3.8\%$.

Figure 10: Estimated Hurst exponent at small scales for various volatilities of the volatility, obtained by simulation of the prices and estimation of either the Parkinson volatility (in black) or the realized volatility (in grey). The variance process of the price follows a geometric Brownian motion with the volatility parameter estimated as for EUR/USD, i.e. $\sigma_0 = 4.62 \times 10^{-3}$, and $\xi$ ranging from 0.1% till 20%.
5.2 Theoretical noise of the volatility proxies

In this subsection we focus on the first source of bias in the estimates, namely the measurement error, coming from the fact that the volatility process is not observable. We will focus on the two estimators considered so far, i.e. the realized variance and the Parkinson estimators, for which we quantify the corresponding theoretical error.

5.2.1 Realized volatility approach

We can calculate the realized variance at a on-day scale for a given day, using $n$ log-returns:

$$RV_t(n) = \sum_{i=0}^{n-1} \left( \log \left( \frac{S_{t-i/n}}{S_{t-(i+1)/n}} \right) \right)^2.$$ 

The realized variance is an approximation of the integrated variance:

$$IV_t = \int_{t-1}^{t} \sigma_u^2 du.$$ 

If the log-price follows an Ito process, from [Barndorff-Nielsen and Shephard, 2002, Meddahi, 2002] we know the asymptotic distribution of the difference between the realized variance and the integrated variance, conditionally to the volatility process $(\sigma_u)_{u \in [t-1,t]}$:

$$\sqrt{n} \left( RV_t(n) - IV_t \right) \xrightarrow{n \to \infty} \mathcal{N} \left( 0, 2 \int_{t-1}^{t} \sigma_u^4 du \right).$$ 

That is, asymptotically, $RV_t(n) - IV_t$ is a centered Gaussian variable of variance $\frac{2}{n} \int_{t-1}^{t} \sigma_u^4 du$. By neglecting the variations of $\sigma_t$ in a day, we approximate the variance of the $\varepsilon_t$ in equation (9) by $2\sigma_t^4/n$. For FX rates, sampled every minute, we have $n = 1440$. In average, for EUR/USD, as $\sigma_t$ is about $4.62 \times 10^{-3}$ and $\sigma_t^2$ is $2.13 \times 10^{-5}$ in the geometric Brownian motion model, the standard deviation of $\varepsilon_t$ is in average $7.95 \times 10^{-7}$. 

Figure 11: Simulation of the noisy volatility model (9) with $\alpha = 25\%$, which is the standard deviation of the noise divided by the variance of the log-price. The variance process of the price follows a geometric Brownian motion with the parameters estimated as for EUR/USD, i.e. $\sigma_0 = 4.62 \times 10^{-3}$ and $\xi = 3.8\%$. 


Remark 5.1. When the time step used in the realized variance increases, the measurement noise increases as well. As a consequence, the slope of the log-log plot decreases (because of the greater impact of the noise in the estimation of the Hurst exponent), as one can see in Figure 12, but empirically the global shape of the log-log plot remains unchanged. In other words, the choice of the number of time steps in the computation of the realized variance has empirically a limited impact on the estimation of the Hurst exponent in our dataset.

Figure 12: Estimated Hurst exponent at small scales (lower than 21 days, in black) and high scales (between 54 and 149 days, in grey) for various time steps in the realized variance, obtained on EUR/USD.

5.2.2 Parkinson volatility approach

Parkinson’s volatility is based on the difference between the highest and the lowest log-prices in a given time interval. We thus focus on the statistic

\[
d_t = \log \left( \max_{s \in [t-1,t]} S_s \right) - \log \left( \min_{s \in [t-1,t]} S_s \right).
\]

(12)

If the price \( S_t \) follows a geometric Brownian motion, the high-low interval for one time unit, \( d_t \), is such that [Molnár, 2012; Parkinson, 1980]:

\[
\begin{align*}
E[d_t | \sigma_t] &= \sqrt{\frac{8}{\pi}} \sigma_t \\
E[d_t^2 | \sigma_t] &= 4 \log(2) \sigma_t^2,
\end{align*}
\]

(13)

and, for \( p > 2 \) (as well as for \( p = 1 \)):

\[
E\left[d_t^p | \sigma_t\right] = \frac{4}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \left(1 - \frac{4}{2p}\right) \zeta(p-1) \left(2\sigma_t^2\right)^{p/2},
\]

where \( \zeta \) is the Riemann zeta function [Molnár, 2012; Parkinson, 1980]. In particular, for \( p = 4 \):

\[
E\left[d_t^4 | \sigma_t\right] = 9 \zeta(3) \sigma_t^4.
\]
where $\zeta(3) \approx 1.202$ is the Apéry constant. The statistic $d_t^2/4 \log(2)$ is then an unbiased estimation of $\sigma_t^2$. The variance of the noise $\varepsilon_t$ in equation (9), in the Parkinson case, is thus:

$$\text{Var}[\varepsilon_t | \sigma_t] = \text{Var}[\hat{\sigma}_t^2 - \sigma_t^2 | \sigma_t] = E \left[ \left( \frac{d_t^2}{4 \log(2)} \right)^2 | \sigma_t \right] - E \left[ d_t^2/4 \log(2) | \sigma_t \right]^2 = \left( \frac{9 \zeta(3)}{16 \log(2)^2} - 1 \right) \sigma_t^4 \approx 0.4 \sigma_t^4. \tag{14}$$

The standard deviation of the measurement noise conditionally to the volatility process, $\sqrt{\text{Var}[\varepsilon_t | \sigma_t]} = \alpha \sigma_t^2$, is such that $\alpha = 63.2\%$ for Parkinson’s approach. This is much stronger than the standard deviation in the realized variance approach, which is $\sqrt{2/n} \sigma_t^2$ according to the theoretical noise amplitude exposed in Section 5.2.1. The measurement noise implied by Parkinson’s volatility is thus equivalent to the one of the realized volatility approach with only 5 price returns considered.

### 5.3 Microstructure noise

We now focus on the microstructure noise, coming from decimalization, absence of distinction between bid and ask, price formation and other issues typically faced in intraday data (see e.g. [Robert and Rosenbaum, 2011a, Robert and Rosenbaum, 2011b] and references therein). This microstructure noise affects the price at each observation in a roughly similar manner. It tends to overestimate the variance process. In order to limit the impact of the microstructure noise, we can consider higher time steps for estimating the integrated variance (as we did at the end of Section 4), or we can use the Parkinson volatility, which is less subject to microstructure noise. As already mentioned in the previous subsection, if one increases the time step, the number of observations will decrease and the estimator of the integrated variance will be less accurate, i.e. the measurement noise will increase. In order to keep the convergence of the variance estimator unchanged but to attenuate the impact of microstructure noise, we compute realized variances on a same amount of observations but with higher time steps. For example, the estimator with 5-minute time steps will now use 1440 observations, thus covering 5 full days. We observe that the shape of the log-log plot remains globally unchanged (even though we cannot observe what happens at the smallest scales) when increasing the time step, but with larger slopes and thus higher estimated Hurst exponents, as illustrated in Figure 13.

A spurious conclusion of this figure would be that the microstructure noise explains why estimated Hurst exponents are typically very low for volatility series. But, empirically, when we compute the log-log plot on a fBm, we get lower slopes than when we compute the log-log plot on a series of average values of fBm (for example, the value of the average process in $t$ is the average of the fBm in $4t$, $4t+1$, $4t+2$, $4t+3$, so that there is no overlapping). Such an averaging is concretely what is done when computing a realized variance.

Three conclusions arise:

- When averaging observations, as in the realized variance, a bias appears if we assume that the variance follows a fBm (this is related to the smoothing error exposed in the next subsection).

- The higher estimated Hurst exponent may result from the misspecification of the model. Therefore, the dynamics should behave as a fBm with low Hurst exponent for short scales and as a standard Brownian motion for long scales.

- The microstructure noise could be directly modeled and then estimated (e.g. as in [Robert and Rosenbaum, 2011a, Robert and Rosenbaum, 2011b]). However, in this case other problems arise, such as the convergence of our estimator towards the one-day integrated variance and of course the fact that any result would become model dependent.
Figure 13: Estimated Hurst exponent, as half the highest slope of the log-plot for various time steps in the realized variance (but always 1440 observations of prices), obtained on EUR/USD.

5.4 A third type of noise: the smoothing effect

In the previous subsections we considered the measurement and microstructure noises. Motivated by the inaccuracy of the Hurst estimation, we now focus on the third kind of noise, which comes from the fact that we try to estimate the Hurst exponent of the volatility process by applying estimation methods for the integrated volatility. Therefore, when we compute the variance of an increment of the realized volatility in the estimation of the Hurst exponent, for a given scale \( \tau \), we are in fact computing the integrated variance of the increment of the volatility process for a scale varying between \( \tau \) minus one day and \( \tau \) plus one day. This is referred to as the smoothing error, namely the bias introduced by taking the integral instead of the spot (unobservable) argument, see also Appendix 3 in [Gatheral et al., 2018], where methods and equations are similar to what we’re going to show in this subsection. In particular, in what follows, we show that this smoothing effect tends to overestimate the Hurst exponent. It may also explain why, in the Nasdaq log-log plots obtained by [Gatheral et al., 2018], the slope of straight line is slightly higher for the lowest log-scales, what is illustrated in Figure 14.

We assume that the variance process follows a fBm of Hurst exponent \( H \) and variance \( \xi^2 \). We observe the realized variance, which is an approximation of the integrated variance on \( d \) days (usually, \( d = 1 \)), which is the limit for \( N \to \infty \) of

\[
\nu_{t,N,d} = \frac{d}{N} \sum_{i=0}^{N-1} \sigma_{t-id/N}^2.
\]
Figure 14: Theoretical log second moment of the increments of $v_{t,N,d}$, with $d = 1$ and $N = 100$, for an input Hurst exponent equal to 0.15. The grey dotted lines are the tangents for scales 1 and 2 (log-scales 0 and 0.7), and scales 5 and 10 (log-scales 1.6 and 2.3), leading respectively to estimated Hurst exponents of 0.42 and 0.24.

Then, the variance of an increment of duration $\tau d$ of this integrated variance is:

\[
E \left[ (v_{t,N,d} - v_{t-\tau d,N,d})^2 \right] =
\]

\[
= \frac{d^2}{N^2} \sum_{i,j=0}^{N-1} \left( E \left[ \sigma_{t-iN}^2 \sigma_{t-jN}^2 \right] + E \left[ \sigma_{t-\tau d-iN}^2 \sigma_{t-\tau d-jN}^2 \right] - 2E \left[ \sigma_{t-iN}^2 \sigma_{t-\tau d-jN}^2 \right] \right)
\]

\[
= \frac{\xi^2 d^2}{N^2} \sum_{i,j=0}^{N-1} \left( |t - i d N|^{2H} + |t - j d N|^{2H} - |(j-i)d N|^{2H} \right)
\]

\[
+ \left( \left| t - \tau d - j d N \right|^{2H} + \left| t - \tau d - j d N \right|^{2H} - \left| (j-i) d N \right|^{2H} \right)
\]

\[
-2 \left( \left| t - j d N \right|^{2H} + \left| t - \tau d - j d N \right|^{2H} - \tau d + \left| (j-i) d N \right|^{2H} \right)
\]

\[
= \frac{\xi^2 d^2}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left( \left| \tau d + \frac{(j-i) d N}{N} \right|^{2H} - \left| \frac{(j-i) d N}{N} \right|^{2H} \right),
\]

using the fact, for the third equation, that $E \left[ \sigma_{t-iN}^2 \sigma_{t-jN}^2 \right] = \frac{\xi^2}{2} \left( |t_i|^{2H} + |t_j|^{2H} - |t_i - t_j|^{2H} \right)$, according to the fBm assumption, as well as the fact, for the fourth equation, that $\sum_{i,j=0}^{N-1} \left| t - \tau d - j d N \right|^{2H} = \sum_{i,j=0}^{N-1} \left| t - \tau d - j d N \right|^{2H}$ and that $\sum_{i,j=0}^{N-1} \left| t - i d N \right|^{2H} = \sum_{i,j=0}^{N-1} \left| t - j d N \right|^{2H}$.

As one can see in Figure 15, though the bias is limited for higher values of $H$, it is very significant for lower Hurst exponents.

Asymptotically, when $N \to +\infty$, we get a simple expression for $E \left[ (v_{t,N,d} - v_{t-\tau d,N,d})^2 \right]$. In particular, if we focus on $d = 1$, we get the following limit, which is consistent with the one provided.
Figure 15: Theoretically estimated Hurst exponent for \( v_{t,N,d} \), with \( d = 1 \) and \( N = 100 \), for various values of input Hurst exponents. The black line is the estimation on scales 1 and 2 (log-scales 0 and 0.7), the grey one on scales 5 and 10 (log-scales 1.6 and 2.3). The dotted line is the identity.

by [Gatheral et al., 2018]:

\[
\begin{align*}
\mathbb{E} \left[ (v_{t,N,1} - v_{t-\tau,N,1})^2 \right] &= \xi^2 \int_0^1 \int_0^1 \left( |\tau + v - u|^{2H} - |v - u|^{2H} \right) dudv \\
&= \xi^2 \tau^{2H} f(\tau, H),
\end{align*}
\]

(15)

where

\[
f(\tau, H) = \frac{\tau^2}{(2H + 1)(2H + 2)} \left( \left(1 + \frac{1}{\tau}\right)^{2H+2} - 2 - 2 \left( \frac{1}{\tau}\right)^{2H+2} + \left(1 - \frac{1}{\tau}\right)^{2H+2} \right). \tag{16}
\]

Similarly, Parkinson volatility is subject to two kinds of noise. The first one is the measurement noise and is not be neglected as this proxy is based on only two observations. On the contrary, this low number of observations makes the microstructure noise almost nonexistent. The second kind of noise is similar to the smoothing error: in fact, as the real volatility is not constant over one day, Parkinson volatility does not proxy a spot volatility. Depending on when the maximum and the minimum prices are reached in a day, the autocovariance structure of this proxy may differ from the one of the spot volatility. Unfortunately, we are not able to quantify accurately this noise. Simulations, presented in Section 6.1.2, suggest however that it has a lower amplitude than the measurement noise.

6 Filtering the noises

We have inventoried three sources of noise: a measurement noise, a microstructure noise, a smoothing error. A solution, in order to filter the microstructure noise, consists in using a higher time step in the computation of the realized variance. The estimator of the integrated variance is thus less accurate,
and as a consequence there is an underestimation of the slope in the log-log plot, i.e. a smaller Hurst exponent. This is visible in Figure [16] in which the time step for the realized variance was increased from 1 minute to 40 minutes in the historical series of EUR/USD.

In this section, we are only working with moments of order 2, with an increment duration $\tau \in \mathbb{N}$ such that $1 \leq \tau \ll N$:

$$M_{2,\tau}(X) = \frac{1}{(N - \tau + 1)} \sum_{i=\tau}^{N} |X_i - X_{i-\tau}|^2. \quad (17)$$

After having filtered the microstructure noise, we want to filter the measurement noise and the smoothing error. More precisely, for the measurement noise we simply correct the bias in the estimation of all the absolute moments in the log-log plot for the variance, and for the smoothing error we correct the multiplicative error in the moment appearing in equation (15). To put it simply, we are looking for the moments of increments of $SV_t$, but as we cannot observe $SV_t$ we work instead with $IV_t$, thus introducing the smoothing error. In addition, $IV_t$ is approximated by $RV_t$, whose difference with $IV_t$ is the measurement noise.

The way the noise is to be filtered strongly depends on the model one assumes for the volatility dynamic. In the next two subsections, we present two filtering methods based on two competing models. The first one is consistent with the RFSV approach. The second model uses the same underlying dynamic, a fBm, but applies it to the variance process instead of the volatility process. This approach is more consistent with dynamics inspired by the ARCH process, which are traditionally invoked in econometrics. It is worth noting that the filtered log-log plots still present some convexity. This confirms our findings regarding new stylized facts.

### 6.1 The log-volatility as a fBm

The first model we consider is the RFSV model [Gatheral et al., 2018]: $\sigma_t = \sigma \exp(\xi B^{H}_t)$, where $B^{H}_t$ is a fBm of Hurst exponent $H$ and $\sigma, \xi > 0$. With this assumption, the theoretical log-log plot may differ from a straight line because of measurement noise, microstructure noise and smoothing error. We present a method to filter these noises from the log-log plot. If the RFSV model depicted all the scaling features of the spot volatility series, the filtered plot should result in a straight line.

#### 6.1.1 Realized volatility approach

The log-plot of the log-volatility should be based on $M_{2,\tau}(\log(SV^{1/2}))$. However, we only observe $M_{2,\tau}(\log(RV^{1/2}))$. Thanks to the successive approximations explained below, we get the following relation between $M_{2,\tau}(\log(SV^{1/2}))$ and $M_{2,\tau}(\log(RV^{1/2}))$:

$$M_{2,\tau}(\log(SV^{1/2})) = \frac{1}{4} M_{2,\tau}(\log(SV))$$

$$\approx \frac{1}{4\sigma_0^2} M_{2,\tau}(SV)$$

$$\approx \frac{1}{4\sigma_0^2} f(\tau, H)^{-1} M_{2,\tau}(IV)$$

$$\approx \frac{1}{4} f(\tau, H)^{-1} M_{2,\tau}(\log(IV))$$

$$\approx \frac{1}{4} f(\tau, H)^{-1} \left[ M_{2,\tau}(\log(RV)) - \frac{4}{n} \right]$$

$$= f(\tau, H)^{-1} \left[ M_{2,\tau}(\log(RV^{1/2})) - \frac{4}{n} \right], \quad (18)$$

where $\sigma_0 \in \mathbb{R}$ and $f$ is defined in equation (16).

- The approximation between the first and the second line is based on a first-order Taylor expan-
sion of the logarithm, around an arbitrary value $\sigma_0^2$:

$$M_{2,\tau}(\log(SV)) = E[(\log(SV_{+\tau}) - \log(SV))^2]$$

$$\approx E\left[\left(\log(\sigma_0^2) + \frac{SV_{+\tau} - \sigma_0^2}{\sigma_0^2} - \log(\sigma_0^2) - \frac{SV - \sigma_0^2}{\sigma_0^2}\right)^2\right]$$

$$= \frac{1}{\sigma_0^4}E\left[(SV_{+\tau} - SV)^2\right]$$

$$= \frac{1}{\sigma_0^4}M_{2,\tau}(SV).$$

In addition to the error left by the Taylor expansion, another source of error may appear in the equations above and in all this section regarding the difference between the empirical moments, $M_{2,\tau}$, and the theoretical ones. However, for long time series as ours, the difference is small with respect to other approximations and is equal to zero in average.

- The approximation between the second and the third line is based on equation [15], that is $M_{2,\tau}(IV) = M_{2,\tau}(SV)f(\tau, H)$, and thus filters the smoothing error, assuming that $SV$ follows a fBm of Hurst exponent $H$. In fact, the assumption that $\log(SV^{1/2})$ follows a fBm, which is the assumption in line with the rough volatility model of [Gatheral et al., 2018], leads to the same approximation thanks to the first-order Taylor expansion of the logarithm introduced above, as $M_{2,\tau}(\log(SV^{1/2})) \approx \frac{1}{4\sigma_0^2}M_{2,\tau}(SV)$.

- The approximation between the third and the fourth line is based on the same kind of Taylor expansion than between the first and the second line.

- The approximation between the fourth and the fifth line exploits the measurement noise between $RV$ and $IV$. We have seen that if the log-price follows an Ito process, then $RV_t = IV_t + \varepsilon_t$ where $\varepsilon_t$ is asymptotically a centered Gaussian variable of variance $\frac{2}{n} \int_{t-1}^{t} SV_u^2 du \approx \frac{2}{n} IV_t^2$. Thus, as an approximation, we consider that $RV_t \approx IV_t(1 + \alpha_t)$, with $\alpha_t \sim N(0, 2/n)$. The $\alpha_t$ are independent variables, also independent of the $IV_t$, then:

$$M_{2,\tau}(\log(RV)) \approx E[(\log(IV_{+\tau}) - \log(IV) + \log(1 + \alpha_{+\tau}) - \log(1 + \alpha))^2]$$

$$= M_{2,\tau}(\log(IV)) + E[(\log(1 + \alpha_{+\tau}) - \log(1 + \alpha))^2]$$

$$= M_{2,\tau}(\log(IV)) + 2E[(\log(1 + \alpha))^2] - 2E[\log(1 + \alpha)]^2$$

$$\approx M_{2,\tau}(\log(IV)) + 2\text{Var}(\log(1 + \alpha)).$$

where we go from the first to the second line thanks to the fact that $\alpha_{+\tau}$ and $\alpha_t$ are identically distributed, then to the third line because they are iid and to the last line by a Taylor expansion for the second moment of the logarithm of a random variable.

In equation [18], we filter successively the measurement noise by translating each absolute moment by a value of $1/n$. Then, we filter the smoothing error by dividing the result by $f(\tau, H)$, with a properly chosen $H$, consistent with the rough framework.

We display the results for the EUR/USD time series in Figure [16] with the successive application of three filters. We observe that the filtering of the sole microstructure noise, by increasing the time step considered in the computation of the realized variance, leads to a flattening of the whole curve (from the black curve to the dark grey curve). This is counterintuitive, as the filtering of noise should lead to a higher estimated Hurst exponent, that is to a higher slope of the log-log plot. However, when we increased the time step to reduce the microstructure noise, other phenomena also come into play. For instance, the increase of the time step reduced the number of observations in a day and therefore

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6 For Parkinson volatility, this is where the filtering must differ.
we increased the measurement noise. The filtering of the measurement noise steepens back the log-log plot (light grey curve). For each of these curves, we observe convexity: higher estimated Hurst exponent for low-frequency increments than for high-frequency increments of volatility. The last filter is about the smoothing error. It accentuates the convexity (red curve). We conclude that the filtered log-log plot still shows convexity. The fBm assumption, for which we should have the same slope of the log-log plot for all scales, is not enough to explain the observations.

Figure 16: Filtering noise for EUR/USD. Microstructure noise has been filtered by increasing the time step for the realized variance from 1 minute (log-log plot in black) to 40 minutes (log-log plot in dark grey). Measurement noise has been filtered in addition by translating each absolute moment by a value of $1/n$ as in Equation (18) (log-log plot in light grey). Smoothing error has been additionally filtered by dividing the result by $f(\tau, H)$, with a properly chosen $H$, consistent with the rough framework (log-log plot in red). Time scales range between 1 day till 3 years.

The filtering relies on some approximations:

- Taylor expansions of the logarithm.
- Identification of the empirical moments to the theoretical ones.
- For the filtering of the smoothing error, we assume that the distortion of the log-plot is the one consistent with a model in which the variance follows a fBm. This is the most questionable approximation as the rough volatility model assumes instead that the log-volatility is a fBm. Nevertheless, the smoothing error tends to make the log-plot more concave because the increments of duration $\tau$ of the realized variance mitigates increments of the spot variance of duration $\tau$ with increments of shorter duration, for which the covariance is much stronger. However, we still observe that, even without this approximative filtering, the log-plot is convex in our empirical findings, see Figure [16].

We show on a theoretical example displayed in Figure [17] that, despite all the approximations cited above, the filtering method we propose is fairly accurate.
6.1.2 Parkinson volatility approach

In the Parkinson approach, the estimated volatility is \( PV_t = d_t \sqrt{\pi/8} \), where \( d_t \) is the high-low statistic defined by equation (12). According to equation (13), \( E[ PV_t | \sigma_t ] = \sigma_t \) and \( E[ PV_t^2 | \sigma_t ] = \pi \sigma_t^2 \log(2)/2 \), provided that \( \sigma_t \) is constant through the day \( t \). Therefore, we can write \( PV_t \approx \sigma_t (1 + \alpha_t^2) \), with \( \alpha_t^2 \) a centered variable of variance \( \pi \log(2)/2 - 1 \approx 0.0888 \). Therefore, similarly to equation (19), we have:

\[
M_{2,\tau}(\log(PV)) \approx M_{2,\tau}(\log(\sigma)) + 2 \text{Var}(\alpha_t^2) \\
= M_{2,\tau}(\log(\sigma)) + \pi \log(2) - 2.
\]

A filtering of the measurement noise of Parkinson’s volatility consists in removing the quantity \( \pi \log(2) - 2 \) from the absolute moments \( M_{2,\tau}(\log(PV)) \). This is an approximation because this is based on a first-order Taylor expansion of a logarithmic function as in the case of realized variance. Moreover, the filtering we propose does not take into account the equivalent of the smoothing error.

We examine by the mean of simulations to which extent this filtering method eliminates noise in the log-log plot. Using the Wood-Chan algorithm [Coeurjolly, 2000], which is both exact and rapid, we simulate trajectories of fBms of 4,6161,640 points, that is 3,206 days with 1,440 observations in a day. We then consider simulated volatility time series, based on the RFSV model, using the simulated fBms, with a vol of vol of 10\%. The Hurst exponent of the simulated fBms is 0.25. For each time series of volatilities, we simulate a series of prices, with 1,440 points each day. We then estimate the Parkinson volatility each day for each series. From these daily series of Parkinson volatilities, we draw series of volatilities, we simulate a series of prices, with 1,440 points each day. We then consider simulated volatility time series, based on the RFSV model, using the simulated fBms, with a vol of vol of 10\%.

The simulation first shows that the average unfiltered log-log plot is a convex function, with a slope far lower than the theoretical one, as one can see in Figure 17. Moreover, when we filter the log-log plot, we do not get the theoretical plot, in particular because the filter does not take into account the equivalent of the smoothing error. Nevertheless, the filtered version of the log-log plot is concave and, for log-scales above 3.5, is almost linear with a slope consistent with the theoretical log-log plot. According to these simulations, the filtering seems inefficient only for low scales. If we replace the quantity \( \pi \log(2) - 2 \approx 0.1776 \) removed from each absolute moment by the optimized quantity \( m^* = 0.1645 \), we get the red log-log plot in Figure 17, which fits the theoretical log-log plot. This suggests that the appropriate filter, incorporating the smoothing error correction, should be based on \( m^* \) instead of \( \pi \log(2) - 2 \). But this value is specific to a particular choice of parameters in the RFSV model, namely \( H \) and \( \xi \), and we cannot recommend a general use of it. For instance, with other simulations in which we have changed either \( \xi \) or \( H \), we obtain a value of \( m^* \) between 0.14 and 0.18.

We then apply the filtering to equity data. Indeed, for long time series of volatilities, we previously only used the Parkinson proxy and we want to check whether the observed stylized facts are significant.

We focus on the NASDAQ index. We observe that the shape of the filtered log-log plot for log-scales higher than 2 is similar to what it was before the filtering, with a higher slope. In particular, the convexity remains. We recall the observations made in Section 3 the unfiltered log-log plots of Parkinson volatilities had a lower slope than the ones of realized volatilities. For example, for NASDAQ index, on a short historical dataset, we estimated a Hurst exponent of 0.115 or 0.069, depending on the volatility proxy. For the extended dataset, we estimate in Figure 18 a Hurst exponent of 0.056 for log-scales between 2 and 5, whereas the estimated Hurst exponent reaches 0.104 for the same set of scales for the filtered log-log plot. This value is much more consistent with the value obtained on the restricted dataset for the realized volatility proxy. Table 5 displays similar results for other stock indices.

We have already stressed the fact that the filtering proposed for the Parkinson proxy was less accurate than the one used in the case of the realized volatility. However, simulations suggest that the true log-log plot should have a lower slope than the filtered one for scales lower than 3.5. A better
Figure 17: Average of \( \log(\tau) \mapsto \log(M_{2,\tau}(X)) \), where \( X \) is the log of either the realized volatility (left) or the Parkinson volatility (right) of simulated prices with a volatility following a RFSV model, with \( H = 0.25 \) and \( \xi = 10\% \). The black dots are the unfiltered observations and the straight line is the theoretical log-log plot for the RFSV model. For the realized volatility (left), the grey dots are the observations filtered by removing \( 1/n \) from each moment (filtering of the measurement noise), the red ones by filtering both the measurement noise and the smoothing error by the means of equation (18). For the Parkinson volatility (right), the grey (respectively red) dots are the observations filtered by removing \( \pi \log(2) - 2 \) (resp. \( m^* \)) from each moment.

Figure 18: \( \log(\tau) \mapsto \log(M_{2,\tau}(X)) \), where \( X \) is the log of Parkinson volatility of NASDAQ index between December 1988 and November 2018. The black dots are the unfiltered observations, the grey (respectively red) ones the observations filtered by removing \( \pi \log(2) - 2 \) (resp. \( m^* \)) from each moment, and the straight line are linear approximations for log-scales between 2 and 5.
|                      | Small scales (≤ 6 months) | Large scales (> 6 months) |
|----------------------|---------------------------|---------------------------|
| S&P 500 index        | 0.083 (+0.036)            | 0.104 (+0.026)            |
| NASDAQ index         | 0.164 (+0.094)            | 0.229 (+0.040)            |
| FTSE index           | 0.125 (+0.066)            | 0.133 (+0.038)            |
| DAX index            | 0.116 (+0.059)            | 0.204 (+0.049)            |
| CAC index            | 0.122 (+0.058)            | 0.168 (+0.048)            |
| SMI index            | 0.169 (+0.105)            | 0.198 (+0.058)            |

Table 5: Perceived Hurst exponent at small (≤ 6 months) and large (> 6 months) scales, and absolute variation in parenthesis with respect to the corresponding results displayed in Table 2, using the Parkinson volatility filtered by removing \( m^* \) from absolute moments.

filtering should thus increase the convexity phenomenon that we already observe on the filtered log-log plot for scales higher than 3.5, for which the filtering is appropriate according to simulations. The red log-log plot in Figure 18 which is based on the optimal filtering designed for the simulations displayed in Figure 17 confirms the convexity of the log-log plot\(^7\). As a consequence, we cannot assert that the convexity of the log-log plot is only a spurious effect implied by the choice of a particular volatility proxy. Again, the fBm assumption for the volatility is not consistent with the observations made for NASDAQ index and other stock indices, since we do not have the same slope of the filtered log-log plot at every scale.

### 6.2 The log-variance as a fBm

If, instead of the RFSV model assumed in the subsection above, the log-variance follows a fBm, the approximations are then transformed into the following, when using the realized variance proxy:

\[
M_{2,\tau}(\log(SV)) \approx \frac{1}{\sigma_0^2} M_{2,\tau}(SV) = \frac{1}{\sigma_0^2} f(\tau, H)^{-1} M_{2,\tau}(IV) \\
\approx f(\tau, H)^{-1} M_{2,\tau}(\log(IV)) \\
\approx f(\tau, H)^{-1} \left[ M_{2,\tau}(\log(RV)) - \frac{4}{\pi} \right].
\]

Two differences arise: a factor 4 in the filter of the measurement noise appears, and in this case the filter of the smoothing error is exact, not just an approximation.

In the Parkinson approach, the variance \( \sigma_t^2 \) is now estimated by \( d_t^2 / 4 \log(2) \), as exposed in Section 5.2.2, whose measurement noise has a conditional variance equal to \( V \sigma_t^4 \), with \( V = \left( \frac{9 \zeta(3)}{16 \log(2)} - 1 \right) \) according to equation (14). The extension of the filtering method exposed in Section 6.1.2 to the variance case thus consists in removing \( 2V \) from each absolute moment of the increments of the Parkinson variances series. Again, this method neglects the smoothing error.

We have conducted the same analysis as above on FX rates and equity indices, by replacing proxies of volatility by proxies of variance. We have determined the log-log plot for the Parkinson variance of equity indices as well as the realized variance of FX rates and observed again a convexity of the curve. The filtered version is very similar, with still a clear convexity: the slope of the log-log plot increases with the time scale. We have not displayed the log-log plots of empirical variance processes in this paper since they are very close to the ones we obtained for the volatility. It supplements our findings\(^7\) We have also tried the linear filtering of the moments with other values of \( m^* \) in the interval \([0.14, 0.18]\), but the convexity still holds.

\[7\]
of the previous subsection: neither the volatility nor the variance seems to follow a fBm, according to both raw and filtered log-log plots.

7 Conclusion

In this paper we performed an extensive empirical investigation on a large dataset including stock indexes and exchange rates. Our findings are twofold: first, we confirmed that the estimation of the Hurst exponent is indeed below 0.5 for time scales close to the ones considered in past literature, e.g. in \cite{Gatheral et al., 2018}. However, when larger time scales are considered (which is possible thanks to the size of our dataset), we face a violation of the stationarity assumption of the increments of the fBm driving the volatility process, in contrast with the rough volatility paradigm. In fact, in the log-log plot we observe a convexity effect that cannot be explained even by taking into account the different type of noises in the estimation procedures. As a consequence, we consider this convexity effect as a new stylized fact of the market that any advanced volatility model should be able to reproduce. A forthcoming paper will introduce a new volatility model that addresses these questions and that is able to reproduce the stylized facts.

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