Novel Scaling Behavior for the Multiplicity Distribution under Second-Order Quark-Hadron Phase Transition

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Deviation of the multiplicity distribution $P_q$ in small bin from its Poisson counterpart $p_q$ is studied within the Ginzburg-Landau description for second-order quark-hadron phase transition. Dynamical factor $d_q \equiv P_q/p_q$ for the distribution and ratio $D_q \equiv d_q/d_1$ are defined, and novel scaling behaviors between $D_q$ are found which can be used to detect the formation of quark-gluon plasma. The study of $d_q$ and $D_q$ is also very interesting for other multiparticle production processes without phase transition.

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Multiplicity distribution is one of the most important and most easily accessible experimental quantities in high energy leptonic and hadronic collisions. From the well-known KNO scaling and its violation\cite{1, 2} to the novel scaling form\cite{3} investigated very recently the distribution shows a lot about the dynamical features for the processes. Local multiplicity distributions have been studied for many years in terms of a variety of phase space variables\cite{4}, and substantial progress has been made recently in deriving analytical QCD predictions for those observables\cite{5}. Based on assuming the validity of the local parton hadron duality hypothesis, those analytical predictions for the parton level can be compared to experimental data. A global and local study of multiplicity fluctuations\cite{6} shows, however, that the theoretical predictions have significant deviation from experimental data. The significant deviation of theoretical predictions from experimental data indicates that we know only a little about multiparticle production processes since the hadronization process in soft QCD is far from being understood.

In this paper we try to investigate multiplicity distribution in some small two-dimensional kinetic region (which can be rapidity and transverse momentum or azimuthal angle, for example) in high energy heavy-ion collision processes. In such collisions a new matter form, quark-gluon plasma (QGP), may be formed which subsequently cools and decays into the experimentally observed hadrons, thus the system undergoes a quark-hadron phase transition. The hadrons produced in such processes may, in principle, carry some relic information about their parent state. Thus the investigation of the multiplicity distribution may be interesting and useful for probing the formation of QGP. In this paper we are limited to discussing multiplicity distribution under the assumption of a second-order phase transition, within the Ginzburg-Landau description for the phase transition. Within the same description for quark-hadron phase transition, the scaled factorial moments are studied by a lot of authors for second-order\cite{7, 8} and first-order\cite{9-12} phase transitions, and a universal scaling exponent $\nu \simeq 1.30$ is given in\cite{7, 8, 11, 12}.

It is useful to point out that the study of multiplicity fluctuations in photon production at the threshold of lasing, which shows a similar type of phase transition\cite{13}, is already in its mature age, although the theory and experiment for a quark-hadron phase transition are still in their infancy. As explained in\cite{7-12} the Ginzburg-Landau model, which has been used in describing superconducting transition and other macroscopic second-order phase transitions, can also be used to describe the multiplicity fluctuations in both second- and first-order phase transitions. In\cite{12} multiplicity distributions are studied for both first- and second-order phase transitions. The authors showed that for second-order phase transition the probability $P_q$ of finding $q$ hadrons in the small bin under investigation decreases monotonically with the increase of $q$ regardless of value of bin width and that for first-order phase transition $P_q$ is a decreasing function of $q$ for small bin width whereas the shape of the distribution changes with the increase of bin width. Thus the shape of the distribution was claimed a tool for telling the order of the phase transition.

In this paper we first show that the criterion in\cite{12} based on the shape of multiplicity distribution for the order of the phase transition is equivocal. This is easily seen once one considers the trivial case without dynamical fluctuations. For such a case, the multiplicity distribution $p_q$ is a Poisson one

$$ p_q(\bar{\sigma}) = \frac{\bar{\sigma}^q}{q!} \exp(-\bar{\sigma}) , \quad (1) $$

with $\bar{\sigma}$ the mean multiplicity. From this distribution one has

$$ \frac{p_{q+1}}{p_q} = \frac{\bar{\sigma}}{q+1} . \quad (2) $$

If $\bar{\sigma} < 2.0$ $p_q$ is a monotonically decreasing function of $q$ whereas $p_q$ changes its shape for $\bar{\sigma} > 2.0$. Using the same parameters as in\cite{12} $\bar{\sigma}$ is calculated and listed in Table 1.
Thus one can see that the shapes of multiplicity distributions given in [12] are similar to those of Poisson ones. In real experiments, one can always choose bin width to ensure the mean multiplicity larger than 2.0, then one cannot tell whether the distribution is shaped due to statistical fluctuations or due to the dynamics in the phase transitions. So one cannot give the order of quark-hadron phase transition just from the general shape of the distributions, and detailed information is needed. This result is not surprising, because the non-dominant dynamical fluctuations can only modify the shape of statistical distribution to some extent but cannot change its general behavior drastically.

Nevertheless, it should be pointed out that the study of multiplicity distribution is still very interesting and useful for processes with the onset of quark-hadron phase transitions. In Ginzburg-Landau theory, the multiplicity distribution turns out to be a Poisson one if the field is purely coherent. Conversely, the distribution turns into a negative binomial if the field is completely chaotic. In reality, one can assume multiplicity production arising from a mixture of chaotic and coherent fields, so the multiplicity distribution in real processes is not a Poisson one nor a negative binomial one, and the deviation of the distribution from a Poisson one is due to dynamical fluctuations. The real quantity concerned is the deviation of experimental \( P_q \) from its theoretical Poisson counterpart \( p_q \). Thus studying the deviation may reveal features of dynamical mechanism involved. Let the probability of having \( q \) hadrons in a certain bin is \( p_q \), the deviation of \( P_q \) from its Poisson counterpart \( p_q \) can be measured by a ratio \( d_q = P_q / p_q \). For the definition of \( d_q \) to make sense, it is necessary to let the mean multiplicity \( \bar{\sigma} \) for \( P_q \) and \( p_q \) be the same. Dynamical fluctuations are shown to be existent if the ratio is far from 1.0, either much larger or much smaller, for some \( q \). The ratio \( d_q \) can be called dynamical factor, since it is 1.0 unless there are dynamical fluctuations in the process. In the Ginzburg-Landau description for second-order phase transition \( P_q \) is given by [8]

\[
P_q(\delta) = Z^{-1} \int \mathcal{D}\phi p_q(\delta^2 | \phi |^2)e^{-F[\phi]}, \tag{3}
\]

where \( Z = \int \mathcal{D}\phi e^{-F[\phi]} \) the partition function, \( p_q(\bar{s}) \) the Poisson distribution with average \( \bar{s} \), and \( F[\phi] \) the free energy functional

\[
F[\phi] = \delta^2 [a | \phi |^2 + b | \phi |^4].
\]

It is instructive to note that a free energy functional with \( O(N) \) QCD order parameter was studied in [14]. The free energy functional used here is different from that in [14] because of the consideration that we are now only interested in the final state charged hadrons (most of them are \( \pi^\pm \)) so that a two-component order parameter is enough (which is written as a complex number) for our purpose. One can see that the functional used here can be derived from that in [14] by integrating out all other components and neglecting higher order powers of left components in the exponential. One more simplification used in present functional is that the derivative term is neglected since former studies (see the last two papers in [7] for details) find out that the term has little contribution to the universal exponent which is a measure of the fluctuations. Because of this simplification, the non-Gaussian functional integral can be treated as a normal integral and can be evaluated directly.

With the free energy functional above the system is in the plasma state for \( a > 0 \) (the order parameter \( | \phi_0 |^2 \) corresponding to the minimum of \( F[\phi] \) is zero) and in hadron phase for \( a < 0 \) (the order parameter \( | \phi_0 |^2 > 0 \)). In real experiments the temperature at which hadrons are emitted from the source is unknown and may be different from event to event. So we treat \( a \) as a free parameter and discuss only for \( a < 0 \) in the following since in the quark phase with \( a > 0 \) only a few hadrons can be produced through fluctuations. From the distribution of Eq. (3) one gets the mean multiplicity for \( a < 0 \)

\[
\bar{\sigma} = \frac{\int J_1(a | x |) d_q}{\int J_0(a | x |)}.
\]

with \( J_n(\alpha) = \int_0^\infty dy y^n \exp(-y^2 + \alpha y) \) representing the bin width \( \delta, a \propto T - T_c \) representing the temperature when the phase transition takes place. For small phase space bin the mean multiplicity in the bin is proportional to \( x \) thus can be very small. Under such circumstance the distribution must be concentrated around \( P_0 \), and both \( P_q \) and \( p_q \) for \( q > 1 \) must be very small, so a direct comparison between them could induce large uncertainty. This demands that the bin width in real experimental analysis should be large enough to ensure the mean multiplicity in the bin not too small (larger than 0.5, say). Of course, smaller bins can be used if a precise determination of both \( P_q \) and \( p_q \) can be obtained from high statistical experimental data. For zero bin width the relevant results are rather sensitive to the cascading production of particles through resonances, so an extremely small bin width should be avoided.

Because of the normalization of both \( P_q \) and \( p_q \), the dynamical factor \( d_q \) must be larger than 1.0 for some \( q \) and less than 1.0 for some other \( q \) if there exist dynamical fluctuations. One can easily derive

\[
d_q(x) = \frac{J_q(a | x |)}{J_0(a | x |)} \frac{d_q(x | a |)}{J_1(a | x |)} \exp(\bar{s}). \tag{5}
\]

The dependence of \( d_q \) on \( q \) for different \( x \) and \( | a | \) is shown in Fig. 1 by choosing \( | a | = \{1.0, \text{and } 2.0, -\ln x = \{-1.0, 0.0, 1.0, 2.0, 3.0\} \}. \) From this
ones can see that the general shapes of $d_q$ are similar for different choices of $|a|$ but depend strongly on the bin width $x$. In detail, for large $x$ (small $-\ln x$ or high mean multiplicity) $d_q(x)$ is quite large while $d_{q>1}$ are smaller than 1.0. For small $x$ (large $-\ln x$ or low mean multiplicity), however, $d_q(x)$ is smaller than but close to 1.0 while $d_{q>1}$ are larger than 1.0, indicating that two or more particles are more likely to be in the same small bin than for the pure statistical case. This phenomenon may be associated with the cluster effect or mini-jets in quark-hadron phase transition. For small $x$ the values for $d_q$ are independent of $|a|$. From Fig. 1 one can also see that the dependence of $d_q$ on $x$ is quite complicated. For some large $q$ $d_q$ is monotonically increasing with the decrease of $x$, but for small $q$, $d_q$ first decreases and then increases with the decrease of $x$. Complicated behaviors can be seen for $p_q(x)$, considering the fact that $s$ is an increasing function of $x$ while $p_q(s)$ changes its behavior at $s = q$. But, the ratio $p_q(x)/p_1(x)$ is an increasing function of $x$ for $q > 1$. Moreover there exists a scaling law between $p_q$

\[
\frac{p_q(s)}{p_1(s)} = \frac{2^{q-1}}{q! \left( \frac{p_q(s)}{p_1(s)} \right)^{q-1}}. \tag{6}
\]

Thus it may be more interesting to study the dependence on $x$ of

\[
D_q \equiv \frac{d_q}{d_1} = \frac{p_q/p_1}{p_q/p_1} \tag{7}
\]

instead of $d_q$, and one may expect some scaling behavior of $D_q$ when the resolution is changed. Now we turn to study $D_q$ for second-order quark-hadron phase transition. If there is no dynamical reason, $P_q = p_q$, one can see that $D_q$ for all $q$ can have only one value, 1.0, no matter how large or small the bin width is. So from the range of values $D_q$ takes, one can evaluate the strength of dynamical fluctuations. $D_q$ can be expressed in terms of $J_\nu(a)$ as

\[
\ln D_q = (q-1) \ln J_0(|a| x) + \ln J_q(|a| -1)) - J_1(|a| -1)). \tag{8}
\]

Besides $x$, there is in last expression another parameter $|a|$, which is a measure of how far from the critical temperature the hadronization process occurs and is unknown in current experiments. First let us fix $|a|$ to be 1.0. One can immediately see that, for any $x$, $D_q \propto D_\nu^{-1}$, a power law satisfied by Poisson, binomial and many other distributions in statistics. For any value of $|a|$, $D_q$ increases monotonically with the decrease of $x$. This shows that the dynamical influence can be observed more easily in high resolution analysis. This can be understood physically, since different dynamical fluctuations may offset each other and become less obviously observable in large bin analysis. The behaviors of $\ln D_q$ as functions of resolution $-\ln x$ are shown in Fig. 2 for $|a| = 1.0$ and 2.0 for $q = 2, 3, 4, 5, 6$. For small $-\ln x$ values of $D_q$ depend strongly on parameter $|a|$, but they approach parameter $|a|$ independent values for large $-\ln x$. The similarity in the shapes of $\ln D_q$ as functions of $-\ln x$ suggests a power law for other $|a|$ between $D_q$ and $D_2$ similar to the case we showed for $|a| = 1.0$. $\ln D_q$ is reshowm in Fig. 3 as functions of $\ln D_q$ with the same data as in Fig. 2. For both $|a| = 1.0$ and 2.0 perfect linear dependences of $\ln D_q$ on $\ln D_2$ can be seen. For other values of $|a|$ the similar linear dependence is checked to be true. Thus one has

\[
\ln D_q = A_q + B_q \ln D_2, \tag{9}
\]

with $A_q$ and $B_q$ depending on $|a|$. The fitted results of $A_q$ and $B_q$ from curves in Fig. 3 are shown in Fig. 4 as functions of $\ln(q - 1)$ for $|a| = 1.0$ and 2.0. It is obvious that both $\ln A_q$ and $\ln B_q$ have linear dependence on $\ln(q - 1)$ for fixed $|a|$. Especially, for the purpose of studying power law, we investigate $B_q$ and find that

\[
B_q = (q - 1)^\gamma, \tag{10}
\]

with $\gamma$ depending on $|a|$. For visualization, the linear fitting curves for $\ln B_q$ vs $\ln(q - 1)$ are shown also in Fig. 4 for $|a| = 1.0$ and 2.0. The slopes for $\ln A_q$ is about twice those for $\ln B_q$, and they increase with increasing $|a|$. When $|a|$ is zero, corresponding to the case in which hadrons are produced exactly at the critical point, numerical results show that $\gamma$ is 0.819. With the increase of $|a|$, $\gamma$ increases quickly. For sufficiently large $|a|$, when the difference between $|a| - 1$ and $|a|$ can be neglected, corresponding to the case in which hadrons are produced at temperature much below the critical point, one finds that $D_q(x) = F_q(|a|, x)$, with $F_q$ the scaled factorial moment which is given in [8] for second-order phase transition as

\[
F_q(x) = J_q(x) J_0(q-1) J_q^{|a|}(x). \tag{11}
\]

Similar relation between $D_q$ and $F_q$ is also true in the small $x$ limit. In these limiting cases, the scaling of $D_q$ is equivalent to that of the scaled factorial moments $F_q$, and one can get the exponent $\gamma = 1.3424$ for large $-\ln x$ [15] or large $|a|$. The dependence of $\gamma$ on $|a|$ is shown in Fig. 5. In real experiments, $|a|$ is not known for an event and may be increasing in the hadronization process. Thus one average over $|a|$ should be made. The smaller $|a|$, the less the number of produced particles. Thus the main contribution to $P_q$ comes from events with large $|a|$ or with high multiplicities. For those events, one should get $\gamma$ near 1.30, close to the universal exponent $\nu$ given in former studies of $F_q$ for second-order phase transition. For events with low multiplicity, one can get $\gamma > 0.819$. So the theoretical range for the exponent $\gamma$ is (0.819, 1.3424), corresponding to temperature range from $T = T_C$ to $T < T_C$. 

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In conclusion, two new quantities $d_q$ and $D_q$ are introduced to describe dynamical fluctuations in quark-hadron phase transitions. In the Ginzburg-Landau description for second-order quark-hadron phase transition, $d_q$ and $D_q$ are investigated analytically, and it is found that $D_q$ obeys a power law, $D_q \propto D_{Bq}^2$, with $B_q = (q - 1)^\gamma$. In experimental analysis, both $d_q$ and $D_q$ can be obtained quite easily. To get $P_q$ one only needs to count up the number of events with exact $q$ hadrons in the bin. $p_q$ is of Poisson type and can be calculated from the experimental $\bar{s}$. Simple algebras give $d_q$ and $D_q$. The existence of dynamical fluctuations can be confirmed if $d_q$ and $D_q$ can take values very different from 1.0. The scaling between $D_q$ and $D_{2q}$ is a possible signal for the formation of QGP, because up to now no other dynamical reason is known to induce such a scaling. The value of exponent $\gamma$ can be used to measure the deviation of the temperature, at which the hadronization occurs, from the critical point. The study of $D_q$ should be carried out in real experimental analysis in the future to see whether QGP has been formed in current high energy heavy-ion collisions. As a tool to study dynamical fluctuations $d_q$ and $D_q$ introduced in this paper may also be interesting in experimental analysis of leptonic and hadronic interactions without quark-hadron phase transitions. Study of $d_q$ and $D_q$ in first-order quark-hadron phase transition is in preparation.

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| $B=+1$ | 1     | 2     | 4     | 6     |
|--------|-------|-------|-------|-------|
| $x$    |       |       |       |       |
| 0.226  | 0.429 | 0.804 | 1.155 |
| 0.342  | 0.797 | 2.172 | 4.582 |

TABLE I. Mean multiplicities $\bar{s}$ for second-order ($B=+1$) and first-order ($B = -1$) phase transitions for different bin widths. $x$ is a parameter (different from the quantity used in this paper) associated with bin width $\delta$, parameter $s$ in [12]. The mean multiplicities are calculated using Eq. (13) in [12].
Figure Captions

Fig. 1 Dependence of dynamical factor $d_q$ on $q$ for $|a|=1.0$ and 2.0, for $-\ln x = -1.0, 0.0, 1.0, 2.0, 3.0$. 

Fig. 2 Dependences of $D_q$ on bin width $-\ln x$ for $|a|=1.0$ and 2.0 for $q=2, 3, 4, 5, 6$.  

Fig. 3 Scaling behaviors between $D_q$ and $D_2$ for $|a|=1.0$ and 2.0. The data are the same as in Fig. 2. From lower to upper are curves for $q=3, 4, 5, 6$, respectively. 

Fig. 4 Coefficients for the scaling between $D_q$ and $D_2$, $\ln D_q = A_q + B_q \ln D_2$, as functions of $\ln(q-1)$ for $|a|=1.0$ and 2.0. Linear fitting curves are shown for $B_q = (q-1)^\gamma$. 

Fig. 5 Dependence of exponent $\gamma$ on $|a|$. For large $|a|$, $\gamma$ is about 1.34.
