Abstract

Certain deformable families of vertex algebras acquire at a limit of the deformation parameter a large center, similar to affine Lie algebras at critical level. Then the vertex algebra and its representation category become a bundle over the variety defined by this large center. The zero-fibre becomes a vertex algebra, the other fibres become twisted modules over this vertex algebra.

We explore these ideas and a conjectural correspondence in a class of vertex algebras $A(p)[g,\kappa]$ associated to a choice of a finite-dimensional semisimple Lie algebra and an integer $(g,p)$ and a level $\kappa$, and some related algebras. These algebras were introduced under the name quantum geometric Langlands kernels and have an interpretation in 4-dimensional quantum field theory. In the limit $\kappa \to \infty$, they acquire a large central subalgebra identified with the ring of functions on the space of $g$-connections resp. $g$-opers. The zero-fibre is expected to be the Feigin-Tipunin algebra $W_p(g)$, whose category of representations is expected to be equivalent to the small quantum group $u_q(g)$, $q^{2p} = 1$ by the logarithmic Kazhdan Lusztig conjecture.

Our rigorous results focus on the cases $(g,1)$ and $(\mathfrak{sl}_2,2)$. In the first part of the paper, which might be of independent interest, we discuss the twisted modules (including twists by unipotent elements) of the affine Lie algebra and of the triplet algebra $W_2(\mathfrak{sl}_2)$, and introduce a method of twisted free field realization to obtain twisted modules for arbitrary $(g,p)$. We also match our findings to the respective quantum group with big center. In the second part of the paper we match the results to the vertex algebra bundle we compute from the limit of $A(p)[g,\kappa]$, which for $(g,1)$ and $(\mathfrak{sl}_2,2)$ have realizations as GKO coset resp. $N=4$ superconformal algebras.
## Contents

1 Introduction .................................................. 3
   1.1 Background ............................................. 3
   1.2 Goals .................................................. 4
   1.3 Content Part 1 .......................................... 7
   1.4 Content Part 2 .......................................... 10

2 Twisted modules ............................................ 13
   2.1 $G$-twisted modules of $\hat{g}_\kappa$ ................. 14
   2.2 $\text{SL}_2$-twisted modules of the triplet algebra ... 17
   2.3 Quantum groups ......................................... 24
   2.4 Twisted free field realization .......................... 26

3 Semiclassical limits and Sturm-Liouville operators ........ 30
   3.1 Limit of affine Lie algebras: Connections ............ 30
   3.2 The Virasoro algebra .................................... 31
   3.3 Limit of the Virasoro algebra ........................... 32
   3.4 Sturm-Liouville operators ............................... 33

4 Vertex algebras with big center ............................ 37
   4.1 Coupled vertex algebras .................................. 38
   4.2 Limits of these vertex algebras ......................... 38

5 The case of simply-laced $\mathfrak{g}$ at $p = 1$ ............ 39
   5.1 The affine coset .......................................... 39
   5.2 The quantum Hamiltonian reduction ..................... 41
   5.3 Representations and decomposition formulas ........... 41
   5.4 Bundle associated to the representations $\mathcal{V}_{\mu}^{\kappa-1} \otimes L^1_\nu$ ... 42
   5.5 Bundle associated to the spectral flow representations $\sigma'(\mathcal{V}_{\mu}^{\kappa-1}) \otimes L^1_\nu$ ... 44
   5.6 Bundle associated to the representations $T_{\mu,\mu}^{\kappa-1} \otimes L^1_\nu$ .......... 45
   5.7 Bundle associated to the representations $\mathcal{W}_{\mu,\mu}^{\kappa-1} \otimes L^1_\nu$, generic $\mu$ ... 45
   5.8 A second limit and the affine variant of the Feigin-Tipunin algebra ... 46

6 Explicit computations for $\mathfrak{sl}_2, p = 1$ ............ 47

7 Explicit computations for $\mathfrak{sl}_2, p = 2$ ............ 49
   7.1 The limit of the N=4 superconformal algebra ........... 49
   7.2 The limit of $\mathfrak{osp}(2|1)_\kappa$ ..................... 49
   7.3 The limit of the N=1 superconformal algebra and fermion ... 53
   7.4 The limit of the N=1 superconformal algebra and fermion II ... 55
1 Introduction

1.1 Background

A vertex algebra \( \mathcal{V} \) is, roughly speaking, a complex-analytic version of a commutative algebra, where the multiplication map depends on a formal complex variable \( \mathcal{Y} : \mathcal{V} \otimes \mathbb{C} \mathcal{V} \rightarrow \mathcal{V}[[z, z^{-1}]] \).

Here, \( \mathcal{V}[[z, z^{-1}]] \) denotes the Laurent series in a formal variable \( z \) with coefficients in \( \mathcal{V} \). The axioms of a vertex operator algebra include a version of commutativity or locality, which relates \( \mathcal{Y}(a, z_1) \mathcal{Y}(b, z_2) \) and \( \mathcal{Y}(b, z_2) \mathcal{Y}(a, z_1) \) up to formal delta-functions on the singularities \( z_1, z_2, z_1 - z_2 = 0 \). Another requirement is an action of the Virasoro algebra on \( \mathcal{V} \) compatible with conformal transformations of the variable \( z \). Standard mathematical textbooks on vertex algebras include [Kac98, FBZ04]. Vertex algebras have a strong relation to analysis, algebraic geometry and mathematical physics, in particular conformal field theory. A vertex algebra has a notion of representation, and under suitable conditions the category of representations is a braided tensor category [HLZ].

A fundamental example is the vertex algebra \( \mathcal{V}^\kappa(\hat{\mathfrak{g}}) \) associated to an affine Lie algebra \( \hat{\mathfrak{g}} \) at some level \( \kappa \), see Section 2.1. The conformal structure implies that certain analytic functions attached to the vertex algebra solve the Knizhnik-Zomolochikov differential equation. The solutions are multivalued around the singularities at \( z_i = z_j \), and the monodromy gives an action of the braid group on the space of solutions. The Drinfeld-Kohno theorem [Koh88, Drin90], see e.g. [Kas95] Chp. 19, states that this braid group action can be described by the braiding of the quantum group \( U_q(\mathfrak{g}) \), a deformation of the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) of a finite-dimensional semisimple Lie algebra \( \mathfrak{g} \), which has been defined by Drinfeld for this very purpose. The Kazhdan-Lusztig correspondence states that for generic \( \kappa \) the infinite semisimple tensor category of representations of \( \mathcal{V}^\kappa(\mathfrak{g}) \) is equivalent to those of the quantum group \( \mathcal{U}_q(\mathfrak{g}) \) for \( (\kappa + h^\vee)(\kappa^* + L h^\vee)r^\vee = 1 \), where \( L \) is the Langlands dual and \( r^\vee \) the lacing number. This result and its limit is of particular importance for the geometric Langlands conjecture, which again draws from conformal field theory, as discussed in Frenkel’s lecture [Fr05].

A closely related vertex algebra \( \mathcal{W}^\kappa(\mathfrak{g}) \) is obtained from \( \mathcal{V}^\kappa(\mathfrak{g}) \) by a quantum Hamiltonian reduction, see [FBZ04] Chapter 15. In the smallest case \( \mathfrak{g} = \mathfrak{sl}_2 \) the \( \mathcal{W} \)-algebra is the Virasoro algebra itself, see Section 3.2. For results on the representation theory, see [Ara07]. A fundamental result is Feigin-Frenkel duality [FF92] \( \mathcal{W}^\kappa(\mathfrak{g}) \cong \mathcal{W}^{\kappa^*}(\hat{\mathfrak{g}}) \) for \( (\kappa + h^\vee)(\kappa^* + L h^\vee)r^\vee = 1 \), where \( L \) is the Langlands dual and \( r^\vee \) the lacing number. This result and its limit is of particular importance for the geometric Langlands conjecture, which again draws from conformal field theory, as discussed in Frenkel’s lecture [Fr05].

A more recent example that has gathered considerable interest, is the Feigin-Tipunin algebra \( \mathcal{W}_p(\mathfrak{g}) \). It is defined as a kernel of screenings and, conjecturally, has a \( \mathfrak{g} \)-invariant subalgebra \( \mathcal{W}^{\kappa}(\mathfrak{g}) \). The conjectural logarithmic Kazhdan-Lusztig correspondence [FGST06, FT10, AM14, Len21, Sug21a, Sug21b] states that it has a finite non-semisimple category of representation, which is equivalent to the representations of the small quantum group \( U_q(\mathfrak{g}) \), \( q^{2p} = 1 \), a non-semisimple finite-dimensional Hopf algebra quotient of \( U_q(\mathfrak{g}) \) for \( q \) a root of unity defined by Lusztig [Lusz90], more precisely a quasi-Hopf algebra variant \( \tilde{U}_q(\mathfrak{sl}_2) \) [CGR20, GLO18, Ne21]. The smallest case \( \mathcal{W}_p(\mathfrak{sl}_2) \) has been studied previously under the name triplet algebra, and in this case the conjecture has been finally proven by the work of many authors [FGST06, AM08, NT09, TW13, 3]
The non-semisimplicity brings new challenges, see e.g. the surveys on logarithmic conformal field theory [CR] and the role of the coend [FS16].

Back to our example $V^\kappa(g)$, it is intriguing to study limits where the parameter $\kappa$ tends to degenerate values. This depends of course on choices, more precisely a choice of a $\mathbb{C}[\kappa^{-1}]$ integral form, which in most of our cases have been studied by [CL19] under the name deformable family. For a certain integral form of $V^\kappa(g)$ the limit $\kappa \to \infty$, which physically corresponds to the semiclassical limit, leads to a commutative algebra, which can be identified with the ring of functions on the space of regular $g$-valued connections, as we review in Section 3.1. The limit of critical level $\kappa \to -h^\vee$ acquires a large center, which is the ring of functions on the space of $g$-opers, see [FBZ04] Proposition 16.8.4. By [Ara11] this center is equal to the center of the reduction $W^\kappa(g)$ at critical level. By the Feigin-Frenkel duality mentioned above, this is equal to the center of the semiclassical limit of $W^\kappa(l^1g)$. For $l^1\mathfrak{sl}_2 = \mathfrak{sl}_2$ this means that the semiclassical limit of the Virasoro algebra is the ring of functions on the space of quadratic differentials resp. Sturm-Liouville operators, and many properties of the associated differential equation are reflected in the representation theory. We discuss this in more detail in Section 3.3 and 3.4.

### 1.2 Goals

The idea of the present article follows previous ideas in joint works of the first author in [FF92] and in [BBFLT13, BFL16, ACF22] and the line of work [CG17, CGL20, CDGG21, CN22]: Suppose we have a deformable family of vertex algebras $V^\kappa$, which in the semiclassical limit $\kappa \to \infty$ acquires a large central subalgebra $Z(V^\kappa)$, which is the ring of functions of some space $X$ of classical analytical objects with a composition law. In our cases this will be the space of $g$-connections resp. $g$-opers. Then the vertex algebra $V^\infty$ and its modules can be considered to be fiber bundles over $X$, where the fibre $V^\infty|_x$ at a point $x \in X$ is given by a quotient $V^\infty/Z(Z(x)) \oplus Z$. The zero-fiber $V^\infty|_0$ is again a vertex algebra, and all other fibers are modules over it.

However, it is an important observation that, in contrast to ring theory, the other fibres $V^\infty|_x$ are no proper modules over $V^\infty|_0$, but rather twisted modules, meaning that their vertex operator contains multivalued functions in the complex variable $z$, due to the nontrivial braiding present. Spoken more categorically, we would presume that the result is a $G$-graded tensor category of representation (with $G$-crossed braiding, see [ENOM09]), where $G$ in our case should be equivalence classes of $g$-connections resp. $g$-opers up to regular coordinate transformations. For example, regular singular connections up to regular coordinate transformations are parameterized by elements in the Lie group $G$.

A different, more familiar occurrence of such $G$-crossed tensor categories are orbifolds: Let $W$ be a vertex algebra with a finite group or Lie group $G$ acting as automorphisms. Then, again in general conjecturally, there is a $G$-crossed tensor category $\bigoplus_{g \in G} C_g$ with $C_0$ the usual braided tensor category of representations of $W$ and with $C_g$ the $C_0$-bimodule category of $g$-twisted representations, which are representations of the vertex algebra $W$ involving multivalued analytic functions with monodromy controlled by $g$, see the initial paragraphs of Section 2. The typical main application is, that all of these representations restrict to proper representations over the invariant subalgebra (or orbifold) $W^G$. 

CGR20, MY20, CLR21, GN21].
and in good cases the representation category of $\mathcal{W}^G$ consists precisely of twisted modules decomposed over this subalgebra. Categorically the decomposition is described by $G$-equivariantization and again returns a proper braided tensor category.

The goal of this article is to present two related classes of examples $\mathcal{V}^\kappa$ for each $(\mathfrak{g}, p)$, where we can study the correspondence between fibres in the limit, twisted representations of the zero-fibre, and corresponding representations of the quantum group with a large center. In these examples, the general conjecture is that the limit $\mathcal{V}^\infty$ acquires as center the ring of functions over $\mathfrak{g}$-connections resp. $\mathfrak{g}$-opers (for $\mathfrak{sl}_2$ Sturm-Liouville operators), and such that the zero fibre $\mathcal{W} = \mathcal{V}^\infty|_{0}$ is equal to the Feigin-Tipunin algebra $\mathcal{W}_p(\mathfrak{g})$ [CN22]. On the other hand, the logarithmic Kazhdan-Lusztig conjecture relates this representation theory to the small quantum group. Our ultimate goal would thus be four descriptions of the same tensor category, extending the logarithmic Kazhdan-Lusztig correspondence:

- Modules over the vertex algebra with big center $H^\kappa\mathcal{A}^{(p)}[\mathfrak{g}, \infty]$
- $g$-twisted modules over the Feigin-Tipunin vertex algebra $\mathcal{W}_p(\mathfrak{g})$
- Modules over the quantum group with big center $U_q(\mathfrak{g})$
- $G$-crossed extension of modules over the small quantum group $\tilde{u}_q(\mathfrak{g})$

In the specific cases $(\mathfrak{g}, 1)$, where the Feigin-Tipunin algebra is merely an affine Lie algebra, and $(\mathfrak{sl}_2, 2)$, where the Feigin-Tipunin algebra coincides with symplectic fermions, all of these questions are explicitly approachable: First, most work on the first and second deformable family is done in [ACL19, ACF22] and [CGL20] respectively, and we have explicit models for $\mathcal{V}^\kappa$. In particular, we can explicitly determine that the zero-fibre
is indeed $W_1(\mathfrak{g})$ and $W_2(\mathfrak{sl}_2)$ as conjectured and determine the fibres $\mathcal{C}_g$ as abelian categories. Second, using the explicit models for the Feigin-Tipunin algebra, we can completely determine the categories of $g$-twisted module. Third, as discussed the Kazhdan-Lusztig conjecture is proven for $(\mathfrak{sl}_2, p)$, and by construction it is true for $(\mathfrak{g}, 1)$.

Let us finally mention some further context: The role of the deformable family $\mathcal{V}^\kappa$ in this article is played by $\mathcal{A}^{[n]}[\mathfrak{g}, \kappa]$, which is an extension of $\tilde{\mathfrak{g}}_\kappa \times \tilde{\mathfrak{g}}_\kappa$ for $\frac{1}{\kappa + h^\vee} + \frac{1}{\kappa + h^\vee} = n$, $r\nu|n$, and by their quantum Hamiltonian reductions in either or both factors, see Section 4.1. These vertex algebras generalize the algebra of chiral differential operators on a Lie group $G$ [AG02], they play an important role for the geometric quantum Langlands correspondence and are called quantum geometric Langlands kernel. The geometric

Figure 1: Case $(\mathfrak{sl}_2, 2)$: We show the projective objects in three category fibres $\mathcal{C}_g$ over elements $(\begin{smallmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \end{smallmatrix})$, $(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix})$, $(\begin{smallmatrix} 1 & t & 0 \\ 0 & 1 & 0 \end{smallmatrix})$ representing the three types of conjugacy classes in $\text{SL}_2(\mathbb{C})$. 
Langlands duality is recovered in the limit $\kappa \to \infty$ and for $n = 0$. There is a 4D picture emerging from works by Kapustin and Witten [KW07], Creutzig and Gaiotto [CG17], Frenkel and Gaiotto [FG19], Costello, Dimofte, Linshaw and many others, for example [CL19, CDG20, CL22, CGL20, CDGG21]. Here, the vertex algebras $A^{(n)}[\mathfrak{g}, \kappa]$ appear as 2D junctions between 3D boundary conditions in a 4D topological twist of a gauge theory attached to $G, \kappa$. They label the 2D junction between 3D boundary conditions $D_{p',q'}$ and $D_{p,q}$ related by a modular transformation $(p' q') = \text{ST}_n S(p q)$, and their quantum Hamiltonian reductions in either factor label junctions between $N_{p,q}$ or $N_{p',q'}$. Here, the 3D boundary condition $D_{0,1}$ is labeled by the spin-ribbon category $D_{0,1} = \text{Rep}(D_\kappa)(\text{Gr}_G)$ of twisted $D$-modules over the affine Grassmannian and $N_{0,1}$ by the Kazhdan-Lusztig category associated to $G, \kappa$. For $\mathfrak{g} = \mathfrak{sl}_2, n = 2$ we review in Section 7 from [CGL20] the explicit realizations of $A^{(2)}[\mathfrak{sl}_2, \kappa]$ as $s\text{Vir}_{N=4}$ and of its one- and two-sided reductions as $\hat{\mathfrak{osp}}(2|1)$ and $s\text{Vir}_{N=1} \otimes \mathcal{F}$.

The article has two parts, according to the two sides of the correspondence:

1.3 Content Part 1

The first part, which might be of independent interest, addresses the following question:

**Question 1.1.** An affine Lie algebra $\hat{\mathfrak{g}}_\kappa$ and the associated vertex algebra $V^\kappa(\mathfrak{g})$ has an action by the corresponding Lie group $G$. Similarly, the Feigin-Tipunin algebra $W_{\mathfrak{g},p}$ has presumably an action by the corresponding dual Lie group $L^G$. What is the corresponding $G$-graded category of twisted modules $\bigoplus_{g \in G} C_g$, and how do the twisted modules and intertwining operators look explicitly? How do they decompose over the vertex subalgebra $W^G$? What are the relevant quantum group counterparts?

For the definition of $g$-twisted modules with respect to non-semisimple actions (and accordingly logarithmic singularities) we refer to the works of Huang [Hu10] and Bakalov [Bak16, BS16]. For most statements it seems more conceptual to talk about $x$-twisted modules for an infinitesimal automorphism (resp. derivation) $x$. Although we do not need it in our explicit treatment, we want to point out that we seem to miss a sufficient treatment of the categorical notions in the case of $G$ a Lie group. A notion of sheaves of categories has been proposed in [Gait13], a notion of an action of an algebraic group on a tensor category is found in [DEN17], and for vertex algebra orbifolds for compact Lie groups see [Mc20],

**Question 1.2.** For $G$ a Lie group acting on the Drinfeld center of a tensor category $C_0$ resp. a modular tensor category $C_0$, is there a version of the result in [ENOM09] that describe all $G$-graded resp. $G$-crossed extensions in terms of cohomological data? Such a result would be invaluable for notoriously hard proofs of tensor category equivalences.

We would also be very interesting a corresponding ”tangential” categorical notion, which would mean a first-order deformation of the abelian category, say $\mathfrak{g} \to \text{Ext}^1(1, L)$ for $L$ the coend, together with structural data data along the lines of [FGS22]. It is to be expected that the ENOM correspondence between an action of $\mathfrak{g}$ on a Drinfeld center of a tensor category resp. braided tensor category on the one hand and infinitesimal deformation on the other hand would be much easier and explicit in this setting.
We now discuss our concrete results:

In Section 2.1 we discuss the case of an affine Lie algebra, which also coincides with the Feigin-Tipunin algebra \( W_1(g) \) in the trivial case \( p = 1 \). The explicit \( g \)-twisted mode algebra was in this case defined in [BS16, Yang17], so one can explicitly determine the vertex algebra modules. It seems a common occurrence that, because \( g \) is an inner automorphism, there exists an isomorphism between the \( g \)-twisted mode algebra and the untwisted mode algebra, so as abelian categories all \( C_g \) are equivalent. However, the action of the Virasoro algebra is deformed by this isomorphism and the structure of the twisted module as a Virasoro module changes drastically. While a semisimple \( g \) shifts the conformal weight by some non-integer, which is the somewhat familiar case, we find for a unipotent \( g \) infinite towers of extensions and \( L_0 \) non-diagonalizable.

**Lemma (2.2).** Consider the inner automorphism \( g = e^{-2n(s + x)} \) of \( g \) with \( s, x \in g \) semisimple resp. nilpotent. The image of the twisted Virasoro action is as follows

\[
 f(L_n^{g, a}) = L_n^{g, a} - \frac{2\kappa}{2(\kappa + h^\vee)}(s + x)_n + \delta_n,0 \frac{\kappa^2}{2(\kappa + h^\vee)}(s, s)
\]

In Section 2.2 we turn to the Feigin-Tipunin algebra \( W_p(g) \) and explicitly treat the case \( (g, p) = (\mathfrak{sl}_2, 2) \), where it is equal to the even part of the symplectic fermions. The twisted mode algebra of symplectic fermions can be calculated explicitly, following [BS16]:

**Corollary (2.3).** The algebra of \( x \)-twisted mode operators for the infinitesimal automorphism (resp. derivation) \( x = \left( \begin{array}{cc} r_0 & 0 \\ 2t & -r \end{array} \right) \) is generated by \( \psi_n, n \in r + Z \) and \( \bar{\psi}_n, n \in -r + Z \) with the following relations

\[
\{ \psi^g_m, \psi^g_n \} = 2t\delta_{m, -n}
\]
\[
\{ \psi^g_m, \bar{\psi}^g_n \} = m\delta_{m, -n}
\]
\[
\{ \bar{\psi}_m^g, \bar{\psi}_n^g \} = 0
\]

From this we deduce the twisted representations of the symplectic fermions and of its even part, and then again the deformed Virasoro structure.

We find that for generic \( g \) there are two simple projective modules, comparable to the lattice vertex algebra modules (see below). The semisimple part of \( g \) causes as expected a shift in conformal weight. The unipotent part of \( g \) causes a screening operator to appear in the Virasoro action, which is not anymore compatible with the horizontal (i.e. lattice) grading. Instead of a direct sum of Fock/Wakimoto modules we get a massive tower of extensions and \( L_0 \)-Jordan blocks. For \( \pm g \) unipotent, we encounter additionally extensions between these modules. The projective modules look similar to the triplet algebra modules (which appears in the case \( g = 1 \)), but in general the two Verma modules appearing as composition factors cannot be further decomposed.

The \( g \)-twisted modules still possess an action of the centralizer of \( g \) in \( G \), and this is relevant if one finally wishes to decompose the \( g \)-twisted module as ordinary modules over \( W_G \), which is the Virasoro algebra. For the regular semisimple conjugacy class (the familiar case), this decomposes the direct sum of Virasoro modules with respect to the Cartan part into the single Virasoro modules with generic weights in some coset. For \( g \) unipotent, the action of the Borel part via the screening induces a very nontrivial filtration of the twisted module. In particular we find that the kernel and cokernel of the screening, acting on the deformed sum of Fock modules, are now Virasoro Verma and Coverma modules. We note however that the findings seem contrary to the expectation
we have for semisimple finite $G$, where the decomposition of $g$-twisted modules each produces new irreducible $W^G$-modules.

In Section 2.3 we match in the case $(\mathfrak{sl}_2, 2)$ our findings about the structure of the abelian categories $\mathcal{C}_g$ for the three types of conjugacy classes $g \in \text{SL}_2(\mathbb{C})$ with the infinite-dimensional quantum group of Kac-Procesi-DeConcini [DKP92a]. This Hopf algebra has a big center and its representation category correspondingly fibres over $\text{SL}_2(\mathbb{C})$. In a topological field theory context this algebra has been studied in [BGPR18] together with a so-called holonomy braiding, and in the present context it has been studied in [CDGG21]. From our perspective we would expect that the quantum group with big center has to be endowed with a nontrivial associator. It should extending the non-trivial associator on the small quantum group at even roots of unity $\tilde{u}_q(g)$ [CGR20, GLO18, Ne21], which turns the representation theory braided modular and which we expect to be the correct counterpart of the Feigin-Tipunin algebra. In the current paper we merely determine the structure of the abelian category from the known sets of irreducibles and match. For the principal block of $\tilde{u}_i(\mathfrak{sl}_2)$ and nilpotent $g$, the category equivalence is even directly visible as isomorphism to the zero-mode algebra generated by $\psi^g_0, \bar{\psi}^g_0$ and the parity operator $\pi$.

**Question 1.3.** We expect for general $(g, p)$ an equivalence of $G$-crossed tensor categories between fibres of the vertex algebra with big center, the categories of twisted modules, and the fibres of the quantum group with big center in some version.

If the ENOM uniqueness of $G$-crossed extensions were proven in our setting (Question 1.2), then this would follow from the logarithmic Kazhdan-Lusztig conjecture and Conjecture 4.3 concerning the zero-fibre (both of which we have in the example $(\mathfrak{sl}_2, 2)$), together with some mild calculations on the Cartan part to fix the choice in $\text{H}^3(G, \mathbb{C}^\times)$ for the associator.

In Section 2.4 we present an approach to construct $g$-twisted representation for general $W_p(g)$, which we would call twisted free field realization. The main idea is that, since any $g$ is conjugate to an element in the standard Borel subgroups, we have for any $g$ a free-field realization of $W_p(g)$ as subalgebra of a lattice vertex algebra (as kernel of short screenings), such that $g$ is also an automorphism of the lattice vertex algebra, which is now an inner automorphism. For $g$ unipotent, it is given by (exponentiated) long screening and all mode operators commute. We may hence apply the $\Delta$-deformation in [Li97, AM09] to long screening operators to obtain $g$-twisted modules. These modules can now be restricted to obtain twisted modules of $W_p(g)$. In the case $W_2(\mathfrak{sl}_2)$ we can explicitly observe, that at least the simple $g$-twisted modules can be extended to a module over the lattice vertex algebra (due to the trivial action of the opposite screening), and this procedure can be easily generalized. Conversely the twisted modules arise from decomposition. In general, we can only conjecture, that this produces all simple twisted modules. Technology to address this question has been developed in [Yang17].

**Question 1.4.** The appearance of short screening operators $\psi^g_0$ with nonzero-eigenvalue in the $g$-twisted module of $W_2(\mathfrak{sl}_2)$, and supposedly for general $g$, seems to suggest a connection to the representations of the $W_{-\mu}(g)$ [FF96], where the lattice is negative definite, causing the $p$-th power of the short screening to proportional to the long screening with arbitrary eigenvalue [Len21] Section 6.4.
1.4 Content Part 2

In the second part we turn our attention to the vertex algebras with big center. In Section 3.1 we briefly review how the semiclassical limit $\kappa \to \infty$ of an affine Lie algebra $\hat{\mathfrak{g}}$, $\kappa \to \infty$ is identified with the ring of functions on the space of $\mathfrak{g}$-valued connections, see [FBZ04] Section 16.

In Section 3.2 we briefly review the Virasoro algebra and its representations at generic central charge $V_h$ and $\varphi_{m,n}$. We also recall the formula from [BSA88] for the singular vectors of $\varphi_{1,n}$.

In Section 3.3 we discuss how the semiclassical limit of the Virasoro algebra can be identified with the ring of functions on the space of Sturm-Liouville operators $d^2/dz^2 + q(z)$, and how for regular Sturm-Liouville operators an additional action of $\mathfrak{sl}_2$ is visible. We also propose an interpretation of the limits of $\varphi_{1,n}$ as regular singular Sturm-Liouville operators with singular term $q(z) = -\frac{m^2-1}{4}z^{-2} + \cdots$ and with diagonalizable monodromy, and of $\bigoplus_m \varphi_{m,n}$ as ring of functions on such Sturm-Liouville operators with a fixed solution.

In Section 3.4 we briefly review some aspects of Sturm-Liouville differential equations, and discuss some examples. We are in particular interested in the monodromy of regular singular solution and in the nontrivial case of Fuchs’ theorem: If the leading exponents $z', z''$ of the two solutions have integer difference $s' - s'' = n$, which precisely correspond to the singular term above, then the monodromy can be a Jordan block, with one solution given by Frobenius’ method and another solution of ”second kind” involving logarithms. For example, the Bessel functions for integer parameter $\alpha$ exhibit such behavior, while the Bessel functions for half-integer parameter $\alpha$ still has diagonalizable monodromy. In both cases $s' - s''$ is an integer, but the singular vectors in $\varphi_{1,2\alpha}$ predicts the diagonalizability.

In Section 4.1 we presents the results [CG17, Mor21, CN22], which establish the existence of a remarkable deformable family of vertex algebras

**Theorem (4.1).** For each integer $p > 0$ divisible by the lacity $r^\vee$ there exists a deformable family of vertex algebras, which extends $V^\kappa(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$ and decomposes over it as follows

$$ \mathcal{A}^{(p)}[\mathfrak{g}, \kappa] = \bigoplus_{\lambda \in \mathbb{Q}^+} V^{\kappa}_\lambda(\mathfrak{g}) \otimes V^{\kappa^*}_\lambda(\mathfrak{g}) $$

for a pair of generic levels $\kappa, \kappa^*$ satisfying

$$ \frac{1}{\kappa + h^\vee} + \frac{1}{\kappa^* + h^\vee} = p $$

By performing a quantum Hamiltonian reduction on the right side we obtain an algebra $H^* \mathcal{A}^{(p)}[\mathfrak{g}, \kappa]$ extending $V^\kappa \otimes W^{\kappa^*}$ and performing a quantum Hamiltonian reduction on both sides we obtain an algebra $H^*H^* \mathcal{A}^{(p)}[\mathfrak{g}, \kappa]$, the Feigin-Tipunin algebra $W_p(\mathfrak{g})$. Note that reversing the roles of $\kappa, \kappa^*$ gives the fourth possibility extending $W^\kappa \otimes V^{\kappa^*}$.

In Section 4.2 we take the limit $\kappa \to \infty$ of this form, which follows straightforward from the previous results and the previously discussed limit of $V^{\kappa}_\lambda$ resp. $V^{\kappa^*}_\lambda$ as bundles over the space of $\mathfrak{g}$-connections resp. $\mathfrak{g}$-opers. We would expect that these two vertex algebra bundles are related by a classical Hamiltonian reduction. The remarkable conjecture raised in [CN22] is that the zero-fibre of $H^* \mathcal{A}^{(p)}[\mathfrak{g}, \infty]$ is the Feigin-Tipunin algebra $W_p(\mathfrak{g})$. On the other hand the limit of $\mathcal{A}^{(p)}[\mathfrak{g}, \kappa]$ and of the fourth possibility should have as zero-fibre an interesting vertex algebra, whose Hamiltonian reduction is $W_p(\mathfrak{g})$. In the example $(\mathfrak{sl}_2, 2)$ it is the small N=4 superconformal algebra [CGL20], see Section 7.1.
In Section 5.1 we turn to the case \( p = 1 \) and \( \mathfrak{g} \) simply-laced. There we have an explicit realization \( H^\kappa \mathcal{A}(1)[\mathfrak{g}, \kappa] = V^{\kappa-1}(\mathfrak{g}) \times V^1(\mathfrak{g}) \). The defining property is based on the diagonal embedding \( \hat{\mathfrak{g}}_{\kappa} \hookrightarrow \hat{\mathfrak{g}}_{\kappa-1} \times \hat{\mathfrak{g}}_1 \) and the GKO-type coset realization of \( W^{\kappa^*}(\mathfrak{g}) \) in [ACL19]

\[
V^{\kappa}(\mathfrak{g}) \otimes W^{\kappa^*}(\mathfrak{g}) \hookrightarrow V^{\kappa-1}(\mathfrak{g}) \times V^1(\mathfrak{g})
\]

The limit \( \kappa \to \infty \) of this algebra and its modules give bundles over the space of \( \mathfrak{g} \)-connections \( d + A(z) \) with \( A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \). For the algebra itself, the bundle is over regular \( \mathfrak{g} \)-connection, the zero-fibre is \( V^1(\mathfrak{g}) \), which is the lattice vertex algebra of the root lattice, and all other fibres are \( V^1(\mathfrak{g}) \) as vector spaces with a deformed Virasoro action

\[
L^{d+A}_n = L_n + \sum_{n' n'' = n} (A_{n''})_{n'} - \frac{1}{2} \sum_{n' + n'' = n} (A_{n'}, A_{n''})
\]

where \((A_{n''})_{n'}\) denotes the mode operator \( n' \) of the element \( A_{n''} \in \mathfrak{g} \). For arbitrary modules we encounter this deformed Virasoro for all types of singularity of \( d + A(z) \) at \( z = 0 \):

For regular connections, the additional terms are only a mild deformation by nilpotent degree-lowering elements. For regular singular connections \( A(z) = A_0 z^{-1} + \cdots \) we encounter degree-preserving elements, in particular we find \( L_0 \)-Jordan blocks according to the additional summand \( A_0 \). For irregular singularities, the effect becomes much wilder and we refrain from discussing them in the present article, see however Question 1.6 and the explicit Examples 5.15 and in particular Example 7.7. We expect our findings in the case \( p = 1 \) to be related to [ILT15, GIL20, GMS20].

In Section 5.2 and 5.3 we review the quantum Hamiltonian reduction of the coset model and review decomposition formulae for the restriction of certain modules to the subalgebra \( V^{\kappa}(\mathfrak{g}) \otimes W^{\kappa^*}(\mathfrak{g}) \), essentially following [ACF22].

In Section 5.4, 5.6 and 5.7 we discuss the limit and fibres of different types of modules and compare the results to the twisted modules. We also find limitations, since the integral form does not transport to the obvious integral form in the decomposition. Recall that for regular singular connections we have proven already established that the fibre is not a direct sum anymore.

In Section 6 we use the explicit realization of \( V^{\kappa} = H^\kappa H^\kappa \mathcal{A}(1)[\mathfrak{sl}_2, \kappa] \) by joint work of the first author [BFL16] as \( \text{Vir}^{b} \otimes V^{U \text{rod}}_{\sqrt{\mathbb{Z}}} \) with a modified Virasoro action called Urod algebra, and two explicit commuting Virasoro actions \( L^{(b_1)}_n, L^{(b_2)}_n \) giving

\[
\text{Vir}_{b_1} \otimes \text{Vir}_{b_2} \to \text{Vir}^{b} \otimes V^{U \text{rod}}_{\sqrt{\mathbb{Z}}}
\]

This explicit example \((\mathfrak{sl}_2, 1)\) was the initial motivation for the study of \( H^\kappa H^\kappa \mathcal{A}(1)[\mathfrak{g}, \kappa] \) in [ACF22] and also for our present article. We find by explicit calculation:

**Corollary (6.2).** The limit of \( V^{\kappa} \) fibres over the space of regular Sturm-Liouville operators \( \frac{d^2}{dz^2} + q(z) \) with \( q(z) = \sum_{n \in \mathbb{Z}} \ell_n z^{-2-n} \). The fibre \( V^{\kappa}_{|q(z)} \) is nonzero for \( \ell_n = 0, n \geq -1 \) and equal to the lattice vertex algebra \( V_{\sqrt{\mathbb{Z}}} \) with deformed Virasoro action

\[
L^{(b_1)}_n = \frac{1}{2\epsilon} f_{n+1} + L_n^V_{\sqrt{\mathbb{Z}}} - \ell_n - 2\epsilon \sum_{k \in \mathbb{Z}} \ell_k \epsilon_{n-k-1}
\]

In particular the zero-fibre \( V^{\kappa}_{|0} \) is isomorphic to \( V_{\sqrt{\mathbb{Z}}} \) deformed by a connection \( d + \frac{1}{2\epsilon} f z^0 \), which is what (in hindsight) we expect from classical Hamiltonian reduction.
Lemma 7.5. The limit of $H^\kappa A^{(2)}[\mathfrak{sl}_2, \kappa]$ and its modules fibres over the space of $\mathfrak{sl}_2$-connections, with fibre over a point $d + \sum_{n \in \mathbb{Z}} A_{-n-1} z^n$ given by the deformed mode algebra
\[
\{ \psi_m^{d+A}, \psi_n^{d+A} \} = - (A_{n+m}, f') \\
\{ \tilde{\psi}_m^{d+A}, \tilde{\psi}_n^{d+A} \} = (A_{n+m}, c') \\
\{ \psi_m^{d+A}, \tilde{\psi}_n^{d+A} \} = \frac{1}{2} (A_{n+m}, h') + m \delta_{m,-n}
\]
The Virasoro action on all fibres is given by the action of $\sum_{i+j} : \tilde{\psi}_i \psi_j :$ and we find in Lemma 7.5 that $[L_{-1}, -] = \frac{\text{d}}{\text{d}z} + A(z)$ if we interpret $A(z)$ as a $\mathfrak{sl}_2$-valued field acting on the vector-valued field with components $\psi(z), \tilde{\psi}(z).$ For example we hence find that indeed the zero-fibre indeed with symplectic fermions and the fibre over $d + A_0 z^{-1}$ corresponding to an (infinitesimal) automorphism $A_0$ indeed matches the twisted modules that we constructed in the first part.

We then repeat our computation explicitly in the Hamiltonian reduction originally given in the first authors joint work [BBFLT13] as the product of an $N=1$ superconformal algebra $sVir_{N=1}$ generated by $L_n, G_r$ and a free fermion $\mathcal{F}$ generated by $f,$ and with two explicit commuting Virasoro actions giving the embedding of
\[
Vir_{b_1} \otimes Vir_{b_2} \longrightarrow sVir_{N=1} \otimes \mathcal{F}
\]
As in the case $\mathfrak{sl}_2, p = 1$ the fibration in the limit looks quite natural, namely it corresponds in the limit to the Virasoso algebra contained in $sVir_{N=1},$ leaving the generators $G_{-3/2}/b$ and the free fermion. The zero-fibre is again a respective deformation by $d + f z^0$ of the algebra of symplectic fermions. Note that in comparison to [CGL20] we work with the Hamiltonian reduction by $e/\kappa - 1$ instead of $e - 1,$ which we feel has better limit behavior.

Question 1.5. The big center construction returns more than vertex algebra modules twisted by an (infinitesimal) automorphism $A_0 \in \mathfrak{g},$ but rather vertex algebra modules twisted by a $\mathfrak{g}$-connection $d + A(z)$ with $A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-1-n},$ where $d + A_0 z^{-1}$ corresponding to usual twisted modules. This more general notion should be compared with the notion of a twisted module in [FBZ04] Section 7 and 9.5 for $\tilde{\mathfrak{g}}_\kappa \rightarrow \mathcal{V},$ but one should work out an explicit definition of a $d + A(z)$-twisted module and from this a twisted commutator-formula with higher terms. Our explicit twisted modules in Section 7 and 6 as well as a generalized $\Delta$-deformation as in Section 2.4 should be examples for this notion.

Moreover, it should be clarified what the corresponding notion of a $G$-crossed tensor category is, in particular for a quantum group. The treatment in [CDGG21] suggests the higher terms can be understood as contributions in higher cohomology and higher categorical strucure, and they might be connected to the derived modular functor in joint work of the second author [LMSS20].
Question 1.6. It would be very interesting to continue our discussion to the case of irregular singularities. In this case, the differential equation exhibits angular sectors with different asymptotic behavior, the deformed Virasoro action in Theorem 5.3 has degree-raising terms and the modules we would have to discuss must be of Whittaker-type to accommodate $L_n/c \neq 0, n > 0$. We give instances of modules with irregular singularities in our cases $(g,p)$ in Example 5.15 using spectral flow, and in our case $(\mathfrak{sl}_2, 2)$ in Example 7.7, where we also compute the representations of an irregularly twisted mode algebra and briefly discuss associated differential equations. A theory of wild character varieties has been developed in [Boa14] and Versions of the Knishnik-Zomolodchikov equations with irregular singularities seem to appear in [FMTV00].

Question 1.7. All our discussion was aimed at vertex algebras, which, in the global picture, correspond to a disc. It would be interesting to compute $d + A$-twisted conformal blocks on a surface $\Sigma$ with respect to global $g$-connection on $\Sigma$.

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2 Twisted modules

A vertex operator algebra $V$ is, very roughly, a infinite-dimensional graded vector space with a multiplication map

$$\mathcal{Y}(-, z) : V \otimes V \to V[[z, z^{-1}]]$$

depending on a formal complex variable $z$ as a Laurent series, and which is endowed with an action of the Virasoro algebra spanned by $L_n$ for $\mathcal{Y}$ and the action of $L_n$ on the complex variable via $-z^{n+1, \frac{\partial}{\partial z}}$. Standard mathematical textbooks on the topic are [FBZ04, Kac98], to which we refer the reader for background.

A module over a vertex operator algebra is similarly a vector space $M$ with a map

$$\mathcal{Y}_M(-, z) : V \otimes M \to M[[z, z^{-1}]]$$

The Virasoro algebra acts also on $M$ via $\mathcal{Y}_M(L, z) = \sum_n L_n^M z^{-2-n}$. Let now $G$ be a group acting on $V$ as automorphisms. Then a $g$-twisted module $M$ of $V$ for $g \in G$ consists of a map with a more general dependency on $z$ as a multivalued functions (in more physical language, it is a non-local field), and where this multivaluedness is controlled by the action of $g$, namely

$$\mathcal{Y}_M(ga, z) = e^{2\pi i \frac{\theta}{\sigma}} \mathcal{Y}_M(a, z)$$

(1)

Note that $e^{2\pi i \frac{\theta}{\sigma}}$ acts as analytic continuation counterclockwise around $z = 0$, in particular we find explicitly from the exponential series

$$e^{2\pi i \frac{\theta}{\sigma}} \log(z) = \log(z) + 2\pi i$$
and thus \{1, \log(z)\} give a Jordan block. The previous definition is fairly standard when the automorphism acts semisimply, in which case \( \mathcal{Y}_M(a, z) \) for \( a \) a \( g \)-eigenvector with eigenvalue \( \lambda \) is a series of fractional powers \( z^m \) with \( \lambda = e^{2\pi i m} \). In the nonsemisimple case one allows \( \mathcal{Y}_M(a, -) \) to have a general regular singularity at \( z = 0 \), which means it is a power series in \( z^m \) and polynomially in \( \log(z) \). The respective theory has been rigorously developed by Huang [Hu10], similar to the tensor product theory in the nonsemisimple case using intertwining operators [HLZ]. A somewhat different view by Bakalov [Bak16] has the advantage of being quite explicit and handy, and several examples are discussed.

The main application of twisted modules is that they are ordinary modules over the vertex subalgebra \( \mathcal{V}^G \) fixed by \( G \), called the orbifold. The general principle, which can be proven under certain assumptions, is that the \( \mathcal{V}^G \)-modules all come from decomposing the restriction of \( g \)-twisted modules of \( \mathcal{V} \) for all \( g \in G \).

On the categorical side, if \( C \) is the braided tensor category of representations of \( \mathcal{V} \), then the category \( \mathcal{C}_g \) of \( g \)-twisted modules is a module category over \( C \) and conjecturally \( \bigoplus_{g \in G} \mathcal{C}_g \) has a \( G \)-action and a \( G \)-crossed braiding. The \( G \)-equivariantization is then a braided tensor category graded by \( G \)-conjugacy classes and should describe the \( \mathcal{V}^G \)-modules. For finite groups \( G \) it is shown in [ENOM09] that the \( G \)-crossed extensions are unique up to a choice in \( H^3(G, \mathbb{C}^\times) \). The respective theory for Lie groups is not fully available, to our knowledge, but the second author has recently studied the twisted modules of the orbifold of the 2-dimensional Heisenberg algebra under the action of \( \text{SO}_3 \).

We now discuss the description of twisted modules more explicitly, following essentially [BS16]. Assume we can write \( g = g_sg_n \) for an automorphism \( g_s \) acting semisimply and an automorphism \( g_n \) acting locally nilpotently. We find it convenient below to describe both in terms of derivations \( s, x \):

\[
g = e^{-2\pi i (s+x)}
\]

Then [BS16] Section 5 derives from his characterization the \( g \)-twisted commutator formula

\[
[a_m^g, b_n^g] = \sum_{j=0}^{\infty} \binom{m+x}{j} a_j b_{m+n-j}^g
\]

with \( m \in r + \mathbb{Z}, n \in r + \mathbb{Z} \) for \( s \)-eigenvectors \( a, b \) with eigenvalues \( r, t \), according to 1. Here, \( \binom{m+x}{j} \) is to be understood as a polynomial in the endomorphism \( x \) and the mode operators \( a_n \) define the vertex algebra via \( \mathcal{Y}(a, z) b =: \sum_{n \in \mathbb{Z}} (a_n b) z^{-n-1} \) and the twisted mode operators \( a_m^g \) define the vertex module via \( \mathcal{Y}_M(a, z) = \sum_{m \in r + \mathbb{Z}} a_m^g z^{-m} \) plus higher terms containing \( \log(x) \). Note that once these terms not containing logarithmic terms are known, the logarithmic terms follow with the action of \( x \) from 1.

### 2.1 \( G \)-twisted modules of \( \hat{\mathfrak{g}}_\kappa \)

The affine Lie algebra \( \hat{\mathfrak{g}}_\kappa \) at level \( \kappa \) has defining relations

\[
[a_m, b_n] = [a, b]_{m+n} + m\delta_{m,-n} \langle a, b \rangle \kappa
\]

with \( a, b \in \mathfrak{g} \) and with the Killing form \( \langle a, b \rangle \) in the normalization usual in vertex algebra theory, which is \( \frac{1}{\kappa \text{trace}} \) times the trace in the adjoint representation. Modules over the universal vertex algebra are given by (suitable) representations of this Lie algebra, where
the vertex operator for \( a \in g \) is
\[
\mathcal{Y}_M(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}
\]

By [Bak16] Section 6 we consider \( g = e^{-2\pi i(s+x)} \). The \( g \)-twisted modules of the universal affine vertex algebra is given by representations of the \( g \)-twisted mode algebra \( \hat{g}_{\kappa,g} \) with relations
\[
[a^g_n, b^g_n] = [a, b]^g_{m+n} + \delta_{m,-n}((m+x)a, b)\kappa
\]
and \( m, n \) in \( \mathbb{Z} \)-cosets of the \( s \)-eigenvalues \( r_a, r_b \) for the \( s \)-eigenvectors \( a, b \). The twisted Virasoro action is derived from the action of the familiar Sugawara element \( L \) on a twisted module, and we can express the the product by [Bak16] Section 6.2 in a twisted normally ordered product
\[
\hat{L}^{\hat{a},g}_{-1} = \frac{1}{2(\kappa + h^\vee)} \frac{1}{2}\mathcal{Y}_M(v^{(i)}_{(-1)}w^{(i)})_{z-n-2} = \frac{1}{2\kappa + h^\vee} \sum_{m, n} :v^{(i)g}_{m-n}w^{(i)g}_{n-m} - \delta(n+1)\kappa\delta_{n,0}d_n \frac{s_0}{2} 1
\]
where \( v^{(i)}, w^{(i)} \) is a dual pair of \( s \)-homogeneous bases, and where \( s_0 \) is defined by \( s_0a = r_0a \) with the choice of representative \( \Re(r_0) \in (-1,0] \) in the coset \( r + \mathbb{Z} \) of the \( s \)-eigenvalue \( r \).

**Example 2.1.** We assume for simplicity \( s = 0 \) and compute the commutator of a twisted translation operator with \( a^g_n \) for any \( a \in g \)
\[
[L^{\hat{a},g}_{-1}, a^g_n] = \frac{1}{2(\kappa + h^\vee)} \left( \sum_{m \geq 0} 2v^{(i)g}_{m-1}w^{(i)g}_{m+n} + 2v^{(i)g}_{m-1}a^g_n w^{(i)g}_{m-n} - [x.v^{(i)}, w^{(i)g}]_{-1}a^g_n \right)
\]
\[
= \frac{1}{2(\kappa + h^\vee)} \left( \sum_{m \geq 0} 2v^{(i)g}_{m-1}[w^{(i)}, a]_{m+n} + 2[v^{(i)}, a]_{-1-m+n}w^{(i)g}_{m-n} \right)
\]
\[
+ 2v^{(i)g}_{-1}((-n+x)w^{(i)}, a)\kappa - [[x.v^{(i)}, w^{(i)}], a]_{n-1} - \delta_{-1,-n}((-1) [[x.v^{(i)}, w^{(i)}], a)\kappa)
\]
Here, the term \( 2v^{(i)g}_{n-1}((-n+x)w^{(i)}, a)\kappa \) comes from the first or second commutator respectively for \( n \) positive or negative, and by the invariance of the form and then the defining property of the dual basis it evaluates to \( 2((-n-x)a)_{n-1}\kappa \). Assume for simplicity \( n < 0 \), then by invariance all terms \( 2v^{(i)g}_{m-1}[w^{(i)}, a]_{m+n} \) and \( 2[v^{(i)}, a]_{-1-m+n}w^{(i)g}_{m-n} \) for \( m' = m - (-n) \) cancel, leaving only the first term for \( m = 0, \ldots (-n)-1 \). By invariance and symmetry this can be rewritten as \( (-n)/2 \) identical commutators
\[
= \frac{1}{2(\kappa + h^\vee)} \left( \sum_{i} (-n)[v^{(i)}, [w^{(i)}, a]_{n-1} + 2((-n-x)a)_{n-1}\kappa - [[x.v^{(i)}, w^{(i)}], a]_{n-1}) \right)
\]
The expression \([v^{(i)}, [w^{(i)}, a]]\) is the action of the quadratic Casimir operator in the adjoint representation, which is the identity in the standard normalization and \( 2h^\vee \) in our normalization. Altogether
\[
[L^{\hat{a},g}_{-1}, a^g_n] = (-n)a^g_{n-1} - (x.a)^g_{n-1}
\]
We can interpret this result in the language of fields to make the connection visible
\[
[L^{\hat{a},g}_{-1}, a(z)] = \left( \frac{d}{dz} - A(z) \right) a(z)
\]
where \( a(z) = \sum_{n} a_n z^{-n-1} \) and in our case \( A(z) = xz^{-1} \). For an arbitrary connection \( d + A(z) \) we recover in Section 5.1 a similar formula.

The representations of the twisted mode algebra can be induced as modules from the twisted horizontal Lie algebra, spanned by \( a = a_0, a \in \mathfrak{g} \) with relations

\[
[a^g, b^g] = [a, b]^g + \langle x.a, b \rangle
\]

We are maybe a bit surprised at first to find that this has the same representation theory as \( \mathfrak{g} \). We now make explicit in our case the well-known statement that inner automorphisms do not change the category of representations (compare in the logarithmic case [BS16] Proposition 6.2 and [Yang17]) and compute how the Virasoro action is modified by this:

**Lemma 2.2.** Consider the inner automorphism \( g = e^{-2\pi i(s+x)} \) of \( \mathfrak{g} \) with \( s, x \in \mathfrak{g} \) semisimple resp. nilpotent.

a) We have a Lie algebra isomorphism \( f : \hat{\mathfrak{g}}_{s,g} \cong \hat{\mathfrak{g}}_s \) given explicitly by

\[
f : a_n^g \mapsto a_{n-r} - \eta(a)\delta_{n-r,0}
\]

for any eigenvector \( \text{ad}_s(a) = ra \), with the 1-cocycle \( \eta(a) = \langle s + x, a \rangle K \).

b) The image of the twisted Virasoro action is as follows

\[
f(L_n^{s,g}) = L_n^g - \frac{2\kappa}{2(\kappa + h^\vee)} (s + x)_n + \delta_{n,0} \frac{\kappa^2}{2(\kappa + h^\vee)} \langle s, s \rangle
\]

**Proof.** a) We check that we have an isomorphism

\[
[f(a_n^g), f(b_n^g)] = [a_{m-r} - \eta(a)\delta_{m-r,0}, b_{n-t} - \eta(b)\delta_{n-r,0}]
\]

\[
= [a_{m-r}, b_{n-t}]
\]

\[
= [a, b]_{m-n-(r+a+b)} + \delta_{m-r, -(n-r)}((m - r_a)a, b)\kappa
\]

\[
= [a, b]_{m-n-(r_a+r_b)} + \delta_{m-r, -(n-r)}(m\langle a, b \rangle \kappa - \langle [s, a], b \rangle \kappa)
\]

\[
f((a, b)^g_{m+n} + \delta_{m,-n}(m + x)a, b)\kappa)
\]

\[
= [a, b]_{m-n-(r+t)} - \eta([a, b])\delta_{m-n-(r_a+r_b)} + \delta_{m,-n}(m\langle a, b \rangle \kappa + [x, a], b)\kappa
\]

The first summands agree, the summands involving \( \langle a, b \rangle \kappa \) agree because the Killing forms vanishes unless \( r_a + r_b = 0 \), and the remaining terms agree with the definition \( \eta([a, b]) = \langle s + x, [a, b] \rangle \kappa = \langle [s + x, a], b \rangle \kappa \).

b) This isomorphism \( f : \hat{\mathfrak{g}}_{s,g} \to \hat{\mathfrak{g}}_s \) does not preserve the Sugawara vector giving the Virasoro action. We now compute the new Virasoro action on \( \hat{\mathfrak{g}}_s \):

We take a dual pair of \( s \)-homogeneous bases \( v^{(i)}, w^{(i)} \) and consider the action of the Sugawara element \( L \) on the twisted module, which [BS16] Section 6.2 can be expressed using the twisted normally ordered product

\[
L_n^{s,g} = \frac{1}{2(\kappa + h^\vee)} \mathcal{Y}_M(v^{(i)}_{(i)} w^{(i)}_{(i)})_{z^{-n-2}}
\]

\[
= \frac{1}{2(\kappa + h^\vee)} \sum_{m,i} v^{(i)}_m w^{(i)}_{n-m} : -[(s_0 + x)v^{(i)}_{m}, w^{(i)}_{n-m}] - \kappa \delta_{m,0} (\frac{s_0}{2}) \kappa
\]

16
Transporting this formula to \( \hat{\mathfrak{g}} \) using \( f \) we get

\[
\hat{f}(L_n^\tilde{\psi}_a) = \tilde{L}_n^\tilde{\psi}_a - \frac{2\kappa}{2(\kappa + h^\vee)} \eta(v^{(i)})\eta(w^{(i)}_n) + \delta_{n,0} \frac{\kappa^2}{2(\kappa + h^\vee)} \eta(v^{(i)})\eta(w^{(i)}_0)
\]

where \( f \) has no effect on the additional terms, so the additional terms can be reabsorbed into the definition of \( L_n^\tilde{\psi}_a \). Now: For \( \eta(a) = (s + x, a) \) we have by the defining property of the dual basis of the Killing form \( \eta(v^{(i)})w^{(i)}_n = (s + x)_n \) and \( \eta(v^{(i)})\eta(w^{(i)}) = \eta(s + x) = (s + x, s + x) = (s, s) \), since \( x \) acts nilpotently. This concludes the proof of the second assertion.

\[ \square \]

### 2.2 SL\(_2\)-twisted modules of the triplet algebra

We now turn to the triplet algebra \( \mathcal{W}_{\mathfrak{sl}_2,p} \) which is a \( C_2 \)-cofinite vertex algebra with non-semisimple category of modules, which is as a braided tensor category equivalent to the category of modules over a certain quasi-variant \( \hat{u}_q(\mathfrak{sl}_2) \) of the small quantum group \( u_q(\mathfrak{sl}_2) \) \cite{FGST06, CGR20, GLO18}. It has a conjectural action of SL\(_2\) in terms of long screening and divided power of short screening. The triplet algebra is a vertex subalgebra of the lattice vertex algebra \( \mathcal{V}_{\sqrt{2}\mathbb{Z}} \) as kernel of a short screening operator.

For \( p = 2 \) the triplet algebra is isomorphic to the even part of the vertex superalgebra of symplectic fermions, in this case there is an evident action of SL\(_2\) \cite{Kau00, GR17}. The symplectic fermions are defined by generating fields \( \psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-1-n} \) and \( \tilde{\psi}(z) = \sum_{n \in \mathbb{Z}} \tilde{\psi}_n z^{-1-n} \) with mode operators fulfilling

\[
\{\psi_m, \tilde{\psi}_n\} = \delta_{m,-n} m
\]

with \( \{\cdot, \cdot\} \) denoting the anticommutator. The twisted mode algebra for a nilpotent automorphism is discussed in \cite{BS16} Section 3 and 4, and for general \( g \) one can proceed similarly: Up to automorphisms resp. conjugation in SL\(_2\) we have \( g = e^{-2\pi i(s+x)} \) for

\[
s = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix},
\]

in the basis \( \psi, \tilde{\psi} \). From the twisted commutator formula 2 we get

**Corollary 2.3.** The algebra of twisted mode operators is generated by \( \psi, n \in r + \mathbb{Z} \) and \( \tilde{\psi}, n \in -r + \mathbb{Z} \) with the following relations

\[
\begin{align*}
\{\psi^g_m, \psi^g_n\} &= 2t\delta_{m,-n} \\
\{\psi^g_m, \tilde{\psi}^g_n\} &= m \delta_{m,-n} \\
\{\tilde{\psi}^g_m, \psi^g_n\} &= 0
\end{align*}
\]

The action of the Virasoro algebra is by \cite{BS16} Proposition 4.1 with shifted indices

\[
L_n = \begin{cases}
\sum_{j>0} \tilde{\psi}^g_{-j} \psi^g_j + \frac{1}{2}(\tilde{\psi}^g_0 \psi^g_0 - \psi^g_0 \tilde{\psi}^g_0) - \sum_{j>0} \psi^g_{-j} \tilde{\psi}^g_{-j} + \binom{s}{2}, & \text{for } n = 0 \\
\sum_{i+j=n} \tilde{\psi}^g_i \psi^g_j, & \text{for } n \neq 0
\end{cases}
\]

17
We now discuss the representation theory depending on \( g = g(\tau, t) \): We determine the modules induced by modules \( \mathcal{M}_0 \) of the zero-mode algebra \( \psi_0, \bar{\psi}_0 \) with trivial action of \( \psi_n, \bar{\psi}_n, n > 0 \), which also possess a compatible structure of a super vector space, given by an action of the parity operator \( \pi \) with \( \pi \psi_n^g = -\psi_{-n}^g \) and \( \pi \bar{\psi}_n^g = -\bar{\psi}_{-n}^g \). These induced modules correspond to twisted modules over the Vertex subalgebra of symplectic fermions. The action of the Virasoro algebra on the generating zero-mode module fulfills

\[
L_n|_{\mathcal{M}_0} = \begin{cases} 
0 & \text{for } n > 0 \\
\frac{1}{2}(\bar{\psi}_0^g \psi_0^g - \psi_0^g \bar{\psi}_0^g) + \left( \frac{s_0}{2} \right), & \text{for } n = 0 \\
\sum_{i+j=n, i,j<0} \bar{\psi}_i^g \psi_j^g & \text{for } n < 0 
\end{cases}
\]

To compute the twisted modules over the even vertex subalgebras, we have to determine also the \( \pi \)-twisted sectors, which correspond to the matrix \( s + \frac{1}{2} \). The subsequent decomposition over the even vertex subalgebra, which corresponds to taking the \( \mathbb{Z}_2 \)-equivariant objects, is simply a decomposition into the eigenspaces of \( \pi \).

**Case 1:** \( s + x = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \).

This is the case of untwisted modules and the result is well-known, but it is instructive to briefly repeat the computation: The zero-mode algebra is

\[
\psi_0^2 = 0, \quad \bar{\psi}_0^2 = 0, \quad \{\psi_0, \bar{\psi}_0\} = 0
\]

This superalgebra of dimension 4 has two irreducible modules, generated by \( v^\pm \) with \( \psi_0 v^\pm = \bar{\psi} v^\pm = 0 \) and parity \( \pi v^\pm = \pm v^\pm \), and projective covers

The Virasoro algebra acts on these zero-mode modules with \( L_0 1^\pm = 0 \) resp. on the projective covers with \( L_0 v^\pm = 1^\pm \).
To illustrate an example, we draw the module induced from the extension $1^+ \to \mathcal{M} \to 1^-$, which is usually called Wakimoto module or Verma module. It is in fact a lattice vertex superalgebra $\mathcal{V}_Z$ which serves as free field realization as discussed in the next section. In the diagram, where cones denote Virasoro modules, and as usual the Y-axis is the $L_0$-grading and the X-axis denotes the $\mathfrak{sl}_2$-grading resp. the $\mathbb{Z}$-lattice grading.
We now discuss the \( \pi \)-twisted sector with \( s + x = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \).

Due to the half-integer grade it has a trivial zero-mode algebra, so there are two irreducible modules generated by \( 1^\pm \) with \( \psi_0^\pi, \bar{\psi}_0^\pi \). The Virasoro algebra acts on these zero-mode modules with \( L_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) \( = -1/8 \) and the next nontrivial terms are \( L_0 \psi_{-1/2}^\pi = -1/8 \psi_{-1/2}^\pi \) and similarly \( L_0 \bar{\psi}_{-1/2}^\pi = -1/8 \bar{\psi}_{-1/2}^\pi \).

We also discuss the decomposition behaviour under restriction to the invariants \( V^G \), which is the Virasoro algebra, via the action of the centralizer \( \text{Cent}(g) = G = \text{SL}_2 \): The action of \( \mathfrak{sl}_2 \) preserves the (vertical) \( L_0 \)-grading and the irreducible module above apparently decompose into the \( \mathfrak{sl}_2 \)-representations of dimension \( 1, 3, 5, \ldots \) resp. \( 2, 4, 6, \ldots \) tensored with irreducible Virasoro representations \( \phi_{1,n}^\pi \).

**Case 2:** \( s + x = \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} \), \( t \neq 0 \).

The zero-mode algebra is

\[
(\psi_0^\pi)^2 = t, \quad (\bar{\psi}_0^\pi)^2 = 0, \quad \{\psi_0^\pi, \bar{\psi}_0^\pi\} = 0
\]

One-dimensional representations with \( \psi_0^\pi = \pm \sqrt{2t} \) do not possess a compatible structure of a supervectors space. Instead this only leaves a unique irreducible representation \( W(0, 0) \)

\[
(1^+, 1^-), \quad \psi_0^\pi 1^- = 1^+, \quad \psi_0^\pi 1^+ = t 1^-, \quad \bar{\psi}_0^\pi 1^+ = 0, \quad \pi 1^\pm = \pm 1^\pm
\]

\[
1^+ \iff \psi_0^\pi \iff 1^-
\]
and a unique indecomposable self-extension

\[
\begin{array}{c}
v_0^g \\ \rightarrow\\ \downarrow \\
v_0 \\
\psi_0 \\
v^- \rightarrow \psi^+_0 \rightarrow 1^+ \rightarrow 1^- \rightarrow \psi_0^g \\
\psi_0^g \\
\end{array}
\]

The structure of \( W_{(0,0;2t,0)} \) as an algebra over the Virasoro algebra is very rich. For illustration we now compute some instances, where we use green color to highlight the new terms in the twisted case. In particular we observe Jordan blocks for \( L_0 \) and new extensions between Virasoro modules.

\[
L_0 1^\pm = 0
\]

\[
L_0 \bar{\psi}^g_{-1} 1^\pm = \bar{\psi}^g_{-1} \psi^g_1 \psi^{-1}_{-1} 1^\pm = \bar{\psi}^g_{-1} 1^\pm
\]

\[
L_0 \psi^g_{-1} 1^\pm = (\bar{\psi}^g_{-1} \psi^g_1 - \psi^g_{-1} \bar{\psi}^g_1) \psi^g_{-1} 1^\pm = 2t \bar{\psi}^g_{-1} 1^\pm + \psi^g_{-1} 1^\pm
\]

\[
L_{-1} 1^\pm = \bar{\psi}^g_{-1} \psi^g_{0} 1^\pm = \begin{cases} t \bar{\psi}^g_{-1} 1^- \\ \bar{\psi}^g_{-1} 1^+ \end{cases}
\]

\[
L_1 \bar{\psi}^g_{-1} 1^\pm = \bar{\psi}^g_1 \psi^g_0 \bar{\psi}^g_{-1} 1^\pm = 0
\]

\[
L_1 \psi^g_{-1} 1^\pm = \psi^g_1 \psi^g_0 \psi^g_{-1} 1^\pm = \begin{cases} t 1^- \\ 1^+ \end{cases}
\]
We again draw these results in a diagram. Observe however, that the $L_0$-action is not semisimple and that the $\mathfrak{sl}_2$-grading is not preserved.
The $\pi$-twisted modules are again induced from the two irreducible representations of the trivial zero-mode algebra

We now discuss the decomposition behaviour under restriction to the invariants $\mathcal{V}^G$, which is the Virasoro algebra, via the action of the centralizer $\text{Cent}(g)$, which is the Borel part generated by $g$ itself, respectively by its Lie algebra generated by $x$.

**Lemma 2.4.** The kernel of $x$ on the $\pi$-untwisted module of parity $\pm 1$ are Verma modules generated by $1^\pm$. More precisely the singular vectors are

$1^\pm, \bar{\psi}^g_{-1}1^\mp, \bar{\psi}^g_{-2}\bar{\psi}^g_{-1}1^\mp, \ldots$

and the cokernel of $x$ are contragradient Verma modules with cosingular vectors

$1^\pm, \psi^g_{-1}1^\mp, \psi^g_{-2}\psi^g_{-1}1^\mp, \ldots$

**Proof.** It is clear that the kernel resp. cokernel are generated by the respective vectors. Be induction and the formulas for $L_0$ we see that

$$L_n \psi^g_{-n} \psi^g_{-(n-1)} \cdots \psi^g_{-1}1^\pm = (\bar{\psi}^g_{-n}\psi^g_{0}) \psi^g_{-n}\psi^g_{-(n-1)} \cdots \psi^g_{-1}1^\pm$$

$$+ \sum_{i=1}^{n-1} (\bar{\psi}^g_{-i}\psi^g_{-(n-i)}) \psi^g_{-n}\psi^g_{-(n-1)} \cdots \psi^g_{-1}1^\mp$$
The sum has nonzero contribution but these cancel each other out
\[
\sum_{i=1}^{n-1} (\psi_i^g \psi_{n-i}^g) \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^\pm = \\
= \sum_{i=n/2}^{n-1} (-i)(2t)(-1)^i(n-i) \psi_i^g \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-i}^g \psi_{-(n-i)}^g \cdots \psi_{-1}^g 1^+ \\
+ \sum_{i=1}^{n/2} (-i)(2t)(-1)^i(n-i) \psi_i^g \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-i}^g \psi_{-(n-i)}^g \cdots \psi_{-1}^g 1^+ = 0
\]
which leaves only the first term, which is depending on the sign
\[
L_n \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^- = (-n)(-1)^n \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^+ \\
L_n \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^+ = t(-n)(-1)^n \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^-
\]

We also note for later use that
\[
L_0 \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^+ = - \sum_{k=1}^{n} (\psi_k^g \psi_k^g) \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^+ \\
+ \sum_{k=1}^{n} (\psi_k^g \psi_k^g) \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^+ \\
= - \left( \sum_{k=1}^{n} (-1)^k \right) \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-1}^g 1^+ \\
+ t \left( \sum_{k=1}^{n} \psi_{-n}^g \psi_{-(n-1)}^g \cdots \psi_{-k}^g \cdots \psi_{-1}^g 1^+ 
\]

Case 3: If \( r \not\in \mathbb{Z} \), then the zero-mode algebra is trivial, so there are two induced modules, each generated by a vector \( v^\pm \) with fixed parity \( \pi v^\pm = \pm v \). Since the \( \pi \)-twisted sectors are irreducible?
For \( r \not\in \frac{1}{2} + \mathbb{Z} \) the \( \pi \)-twisted sectors have the same description, otherwise it is discussed in the previous bullet. Note that for \( r \not\in \frac{1}{2} \mathbb{Z} \) we have a conjugate with \( t = 0 \).

2.3 Quantum groups

There should be a quantum group setup that reproduces the categorical situation, more precisely a \( G \)-crossed extension of the category of \( u_q(g) \)-representations. The natural candidate would be the Kac-Procesi-DeConcini quantum group \( U_q^K(g) \) or unrestricted specialization with a big center, for which we refer to [DKP92a] and [CP94] Chapter 9.2 and Chapter 11.

For \( \text{ord}(g) = \ell \) odd and larger than all \( (\alpha, \alpha)/2 = d_\alpha \) there is a central subalgebra \( Z_0 \subset U_q^K(g) \), which contains in particular \( E_1^\ell, F_1^\ell, K_1^\ell \) and which is isomorphic to the ring of functions on the Lie group \( O(G) \) (including its Poisson structure). In particular, the representations of \( U_q^K(g) \) form a \( G \)-graded tensor category with zero-fibre representations of \( u_q(g) \). For \( \ell \) even, the Lie group has to be replaced by its Langlands dual \( L^G \), see [Beck92] for work in this direction.
**Question 2.5.** What is the abelian (module-) category associated to a conjugacy class $[g]$ of $G$ resp. $^lG$?

It is expected that this is connected to the geometric properties of the conjugacy class $[g]$, for example an influential conjecture by DeConcini, Kac. Procesi states that the dimensions of the irreducible representations are divisible by $\ell^{\dim([g])}$, where $\dim([g])$ is the dimension as a complex manifold, for example for regular elements of maximal dimension $|\Phi^+|$ it is true. Considerably less is known about the full structure as an abelian category, and to our knowledge there is no work in the case of even $\ell$ and the corresponding quantum group in [CGR20, GLO18], which is the one expected to correspond to the Feigin-Tipunin algebra.

As a consequence of standard algebra techniques it follows, see [CP94] Theorem 11.1.1 and Lemma 11.1.3

**Theorem 2.6.** There is a proper closed subvariety $\mathcal{D}$ of the spectrum of the (full) center, such that outside $\mathcal{D}$ there is a unique irreducible representation with prescribed central character, which has dimension $\ell^{\dim([\Phi^+])}$ with $\Phi^+$ the set of positive roots.

We now concentrate on the case $\mathfrak{sl}_0$. The full center consist of $Z_0$, generated by $E^\ell, F^\ell, K^\ell$ which we identify with the ring of functions on $G = \text{SL}_2$, together with Casimir element $C$. The conjugacy classes in $G$ consist of $\pm 1$, the regular semisimple elements and the regular unipotent element (possibly with sign). In this example [CP94] Example 11.1.7 discuss all irreducible representations: There are cyclic representations $V(a, b, \lambda)$ for generic points of the spectrum of the full center and the familiar representations $V_{k}^\pm, k \leq \ell$ for the $\ell - 1$ special points with central character $\chi$

$$\chi(E^\ell) = \chi(F^\ell) = 0, \quad \chi(K^\ell) = 1, \quad \chi(C) = \pm \frac{q^{r+1} - q^{-r-1}}{(q - q^{-1})^2}$$

We want to further derive the full abelian category structure of the fibre over any point $(e, f, k) \in \text{SL}_2$ in the spectrum of $Z_0$, which is the category of representation of

$$u_q(\mathfrak{sl}_2)^{(e, f, k)} := U_q^\ell(\mathfrak{sl}_2)/((E^\ell - e, F^\ell - f, K^\ell - k))$$

**Lemma 2.7.** The fibre $\text{Rep}(u_q(\mathfrak{sl}_2)^{(e, f, k)})$ over $(e, f, k) = (0, 0, \pm 1)$ is the familiar non-semisimple category of representations of $u_q(\mathfrak{g})$, the fibre over a regular semisimple element (conjugate to $(0, 0, k), k \neq \pm 1$) is a semisimple category with representations of dimension $\ell$, and the fibre over a regular unipotent element (conjugate to $(1, 0, \pm 1)$) is a nonsemisimple category with a unique simple representation of dimension $\ell$ and a projective cover with composition series of length $\ell$.

**Proof.** We use the results in [CP94] on irreducible modules. We first have to analyze the fibers of the map of parameters

$$V(a, b, \lambda) \mapsto (e, f, k) = \left( a \prod_{j=1}^{\ell-1} \left( ab + \frac{(\lambda q^{1-j} - \lambda^{-1} q^{-1+j})(q^j - q^{-j})}{(q - q^{-1})^2} \right), b, \lambda^\ell \right)$$

$$V_k^\pm \mapsto (e, f, k) = (0, 0, 1)$$

The preimage of $(e, f, k) \neq (0, 0, 1)$ consists generically of $\ell$ solutions $\lambda^\ell = k$ and for each $\ell$ solutions for the polynomial equation for $a$. If $f = 0$ and hence $b = 0$ this polynomial equation becomes a linear equation $e = aP$ with $P = \prod_{j=1}^{\ell-1} \frac{(\lambda q^{1-j} - \lambda^{-1} q^{-1+j})(q^j - q^{-j})}{(q - q^{-1})^2}$. The
product \( P \) is zero iff \( \lambda = \pm q^{j-1}, j = 1 \ldots \ell - 1 \), which is the case for all but one solution \( \lambda = q^{j-1} \) of \( \lambda^\ell = \pm 1 \). Hence for \( k \neq \pm 1 \) there are \( \ell \) preimages of \((e,0,k)\), and they are nonisomorphic, since there is a unique lowest-weight-vector. On the other hand for \( k = \pm 1 \) there is a unique preimage of \((e,0,\pm 1)\) of \( e \neq 0 \). All other \((e,f,k)\) are \( SL_2 \)-conjugate to these cases. In particular the points \((e,0,k), k \neq \pm 1 \) are conjugate to \((0,0,k)\), in which case the preimages are simply Verma-modules with generic highest weights \( \lambda \) with \( \lambda^\ell = k \).

Now since \( u_q(sl_2)^{(e,f,k)} \) is an algebra of dimension \( \ell^3 \) it is clear from standard algebra arguments (recall the decomposition of the regular bimodule into products of irreducibles and their projective cover) that if there are \( \ell \) irreducible modules of dimension \( \ell \), then the category of representations is semisimple, and if there is a unique irreducible module, then its projective cover has a composition series of length \( \ell \).

Since this result does not cover the case of even order root of unity and quantum group variant \( \tilde{u}_q(sl_2) \) relevant to this article, we also directly discuss the case \( q^4 = 1 \) and find it is in complete correspondence to our previous findings.

**Example 2.8 (Case \( sl_2, p = 2 \).** The quantum group \( \tilde{u}_q(sl_2) \) in [CGR20] is as an algebra isomorphic to \( u_q(sl_2) \) for \( \text{ord}(q) = 4 \). We consider the quantum group with a big center \( U^K_q(sl_2) \) for \( \text{ord}(q) = 4 \), which has central elements \( e := E^2, f := F^2, k = K^4 \), which correspond to the Langlands dual group, and define again

\[
u_q(sl_2)^{(e,f,k)} := U^K_q(sl_2)/(E^2 - e, F^2 - f, K^4 - k)
\]

Since \( K^2 \) is also central, the fiber \( \text{Rep}(u_q(sl_2)^{(e,f,k)}) \) decomposes into a direct sum (which corresponds to \( \pi \)-untwisted and \( \pi \)-twisted sectors in the previous section).

Over \((0,0,k)\) all modules are highest-weight modules and hence quotients of baby Verma modules, which are irreducible for \( k \neq 1 \) and for \( K^2 = -1 \). In particular the fibers over \((0,0,k), k \neq 1 \) are semisimple with two irreducible representations of dimension 2.

Over \((e,f,1)\) on the other hand we have \([E,F] = 0\) resp. \([E,FK] = 0\) and a fibre functor to super vector spaces via the action of \( K \). Hence the fibre of \((e,f,1)\) are representations of the following algebra in the category of super vector spaces

\[
\langle E, FK\rangle/(E^2 - e, F^2 - f)
\]

In particular over \((e,0,1)\) we have a 2-dimensional cyclic module with \( F \) acting identically zero and a \( 2+2 \)-dimensional self-extension with \( F \) acting nilpotently, which is the projective cover, just as the unipotent fibre in the previous section. Identifying \( E, FK \) in the previous algebra description with \( \psi_0^g, \bar{\psi}_0^g \) shows directly that it is isomorphic to the zero-mode algebra, and hence the equivalence of abelian categories.

We also note for the general case \( \tilde{u}_q(g) \) that the alternative approach in [Ne21] constructs this quasi-Hopf algebra as an equivariantization of \( U^K_q(sl_2) \), which would suggest to use this also for even order \( q \).

**Question 2.9.** Are these \( G \)-crossed categories for all \( \tilde{u}_q(g) \) in [GLO18]? Can the \( G \)-equivariantizations been described, for example by Lusztig-type quantum groups over the identity fibre and mixed quantum groups over the regular unipotent fibre?

### 2.4 Twisted free field realization

The twisted representations of \( \mathcal{W}_{sl_2,2} \) in the previous section, in particular the logarithmically twisted representations associated to nilpotent automorphisms, were found
by explicitly computing the twisted mode algebra, and they rely on the description as symplectic fermions. We now give a construction of twisted module, which relies on the definition of \( W_{g,p} \) as subalgebra of a lattice vertex algebra \( \mathcal{V}_{\sqrt{p}\Lambda^\vee} \) as kernel of a set of short screening operators. We cannot realize all twisted modules, in particular not the projectives, and we cannot prove that we can realize all irreducible twisted modules in this way, although we would conjecture so.

Consider the lattice vertex algebra \( \mathcal{V}_{\Lambda} \) for the rescaled coroot lattice \( \Lambda = \sqrt{p}\Lambda^\vee \). It has an action of the Borel subgroup \( ^L B \subset ^L G \) on the \( by \) exponentials of semisimple resp. nilpotent elements

\[
s_0^\alpha = \text{Res}(\mathcal{Y}(s^\alpha, z)), \quad x_0^\alpha = \text{Res}(\mathcal{Y}(x^\alpha, z))
\]

(the latter the so-called long screening operators) where \( \alpha^\vee \) runs over the basis of coroots associated to the choice of \( B \) and

\[
s^\alpha = \partial \phi_{\alpha^\vee}, \quad x^\alpha = e^{\sqrt{p}\alpha^\vee}
\]

an we have chosen a Virasoro action shifted by a background charge so that the \( x^\alpha \) have conformal dimension \( 1 \). If we assume \( g \) simply-laced, then this action is rigorously established in [FT10, Sug21a], but we expect the general proof to be similar.

Let \( g \in ^L G \). Up to conjugation, we can assume \( g \in ^L B \). We can now construct in general \( g \)-twisted modules of \( \mathcal{V}_{\sqrt{p}\Lambda^\vee} \), by using a method initiated in [DLM96, Li97] in the semisimple setting and developed in [FFHST02, AM09] for applications to short screening operators. To apply this method to the long screening operators, we collect the following properties:

- The vertex algebra infinitesimal automorphisms (or derivations) \( s_0^\alpha, x_0^\alpha \) are inner infinitesimal automorphism, meaning that we have obtained them as \( z^{-1} \)-mode of vertex algebra elements \( s^\alpha, x^\alpha \). Hence, suitable mode operators \( x_n, n \in \mathbb{Z} \) are available to us.

- \( s^\alpha, x^\alpha \) are primary of conformal dimension \( 1 \), meaning that

\[
L_n s^\alpha = 0, n > 0, \quad L_0 s^\alpha = s^\alpha
\]

\[
L_n x^\alpha = 0, n > 0, \quad L_0 x^\alpha = x^\alpha
\]

- The mode operators \( x_n^\alpha \) are self-commuting by the commutator formula

\[
[x_n^\alpha, x_m^\alpha] = \sum_{k \geq 0} \binom{n}{k} \mathcal{Y}(x_k^\alpha x_m^\alpha)_{m+n-k}
\]

because \( x_k^\alpha x_m^\alpha = 0 \) for \( -k - 1 < |\sqrt{p}\alpha^\vee| \). The mode operators \( s_n^\alpha \) span the Heisenberg algebra, so they are not self-commuting, but sufficiently controlled, see [DLM96, Li97].

Since twisted modules for semisimple elements are well-known, and on the other hand the last bullet requires modifications in this case, we now assume a unipotent element \( g \) resp. a nilpotent derivation \( x \), and study the logarithmically twisted modules as described. We now state the main result in [Hu10] Section 5:
Theorem 2.10. Assume that \( \mathcal{V} \) is a vertex operator algebra and \( x \) an element, which is a primary field of conformal dimension 1 and which has self-commuting mode operators, as discussed above. Let \( \mathcal{M} \) be a (weak) \( \mathcal{V} \)-module, then the following defines on the vector space \( \mathcal{M} \) the structure of an (weak) \( x_0 \)-twisted module \( \tilde{\mathcal{M}} \):

\[
\mathcal{Y}_{\tilde{\mathcal{M}}}(a, z) := \mathcal{Y}_\mathcal{M}(\Delta(x)a, z), \quad \Delta(x, z) = \exp \left( x_0 \log(z) + \sum_{n=1}^{\infty} \frac{x_n}{-n} (-z)^{-n} \right)
\]

Note that the source states this more generally for non-selfcommuting mode operators, on the other hand it is demanded in [AM09] Theorem 2.1. and in this case we find the exponential more difficult to handle. The defining property for \( g = e^{2\pi i x_0} \) holds by construction in the first term of \( \Delta \)

\[
e^{2\pi i z^0} \mathcal{Y}_{\tilde{\mathcal{M}}}(a, z) = \mathcal{Y}_\mathcal{M}(\exp \left( x_0 (\log(z) + 2\pi i) + \sum_{n=1}^{\infty} \frac{x_n}{-n} (-z)^{-n} \right) a, z)
\]

and we have

\[
\frac{\partial}{\partial z} \mathcal{Y}_{\tilde{\mathcal{M}}}(a, z) = \frac{\partial}{\partial z} \mathcal{Y}_\mathcal{M}(\exp \left( x_0 \log(z) + \sum_{n=1}^{\infty} \frac{x_n}{-n} (-z)^{-n} \right) a, z)
\]

+ \( \mathcal{Y}_\mathcal{M}( (x_0 z^{-1} + \sum_{n=1}^{\infty} x_n z^{-n-1}) \exp \left( x_0 \log(z) + \sum_{n=1}^{\infty} \frac{x_n}{-n} (-z)^{-n} \right) a, z) \)

= \( \mathcal{Y}_\mathcal{M}( (L_{-1} + \mathcal{Y}(x, z))(\exp \left( x_0 \log(z) + \sum_{n=1}^{\infty} \frac{x_n}{-n} (-z)^{-n} \right) a, z) \)

= \( \mathcal{Y}_\mathcal{M}(L_{-1}a, z) \)

Now we calculate the deformed Virasoro action in analogy to [Hu10] Section 5 or [AM09] Thm 2.2:

Example 2.11. Since \( L_0x = x \) and \( L_nx = 0 \) for \( n > 0 \) we have by skew-symmetry

\[
\mathcal{Y}(x, z)L = e^{zL_{-1}}\mathcal{Y}(L, -z)x = e^{zL_{-1}} \left( (-z)^{-2} x + (-z)^{-1} L_{-1} x + \sum_{n \leq -2} z^{-n-2} L_n x \right)
\]

in particular the singular terms are \( x_0L = -L_{-1}x + L_{-1}x = 0 \) and \( x_1L = x \), and further \( x_1x_1L = 0 \). Hence

\[
\mathcal{Y}_{\tilde{\mathcal{M}}}(L, z) = \mathcal{Y}_\mathcal{M}(\Delta(x, z)L, z) = \mathcal{Y}_\mathcal{M}(L + z^{-1} x, z)
\]

\[
\tilde{L}_n = L_n + x_n
\]

Next we consider a set of short screening operators \( y_0^{-\beta} = \text{Res}(\mathcal{Y}(y^{-\beta}, z)) \) with \( y^{-\beta} = e^{-a/\sqrt{p}} \) where \( -\beta \) runs over a set of simple roots of the opposite Borel. It is known that \( x_0^\delta \) and \( y_0^{-\beta} \) commute, due to the commutator formula: For \( \alpha \neq \beta \) this follows from \( \langle \alpha, -\beta \rangle \geq 0 \), whence the sum over \( k \) is empty, and for \( \alpha = \beta \) by explicit calculations of the summands \( k = 1, 2 \) and cancellation.
Corollary 2.12. The action of $^kB$ on $\mathcal{V}_\Lambda$ for $\Lambda = \sqrt{p}\Lambda'$ by vertex algebra automorphism descends to an action of $B$ on the vertex subalgebra $\mathcal{W}_{q,p}$ defined as intersection of the kernel of all short screenings $y^{-\beta}$.

Corollary 2.13. For $\lambda \in \Lambda'/\Lambda$ and $\mathcal{V}_\lambda$ the simple module of $\mathcal{V}_\Lambda$ generated by $e^\lambda$ and $x := x_0^{\alpha_0}$, we obtain an $x$-twisted module $\tilde{\mathcal{V}}_\lambda$ by Theorem 2.10 and by restriction an $x$-twisted module of $\mathcal{W}_{q,p}$, which we call the $x$-twisted Verma module for weight $\lambda$.

We now discuss this more explicitly in the case $\mathcal{W}_{sl_2,2}$ discussed in the previous subsection. Recall that for a derivation $x \psi = 2t \bar{\psi}$ we had $(\pi$-untwisted) $x$-twisted modules generated by for every representation of the zero-mode algebra

$$(\psi_0^\alpha)^2 = t, \quad (\bar{\psi}_0^\alpha)^2 = 0, \quad \{\psi_0^\alpha, \bar{\psi}_0^\alpha\} = 0$$

with compatible super vector space structure, more precisely a unique irreducible module with $\bar{\psi}_0^\alpha = 0$ and a unique self-extension, and unique $\pi$-twisted modules, each decomposed into $\pi$-eigenspaces. We now ask: Which of these modules can be extended to a representation of $\mathcal{V}_{\sqrt{2}\mathbb{Z}} \supset \mathcal{W}_{sl_2,2}$, where in this free-field realization $\psi = e^{-\alpha/\sqrt{2}}$ and $\bar{\psi} = L_{-1}e^{\alpha/\sqrt{2}}$. In fact $\mathcal{V}_{\sqrt{2}\mathbb{Z}}$ as a module over $\mathcal{W}_{sl_2,2}$ is an indecomposable extension with two composition factors generated by 1 and $e^{\alpha/\sqrt{2}}$, so the question is, whether the module can be extended by an action of the second generator. Now because

$$\frac{\partial}{\partial z} \mathcal{Y}(e^{\alpha/\sqrt{2}}, z) = \mathcal{Y}(L_{-1}e^{\alpha/\sqrt{2}}, z) = \sum_{n \in \mathbb{Z}} \bar{\psi}_n z^{-n-1}$$

this requires necessarily that $\bar{\psi}_0 = 0$, because the derivation cannot produce a term $z^{-1}$. Conversely, under this assumption we can define easily

$$\mathcal{Y}(e^{\alpha/\sqrt{2}}, z) := \sum_{n \in \mathbb{Z}} \bar{\psi}_n z^{-n}$$

This means that the unique irreducible $\pi$-untwisted and $\pi$-twisted $x$-twisted module of $\mathcal{W}_{sl_2,2}$ extends to a $x$-twisted module of $\mathcal{V}_\Lambda$, while the indecomposable $\pi$-untwisted $x$-twisted module does not extend.

We can compare this explicitly to the twisted free field realization using $\Delta$, with

$$x = e^{\alpha \sqrt{2}}, \quad \psi = e^{-\alpha/\sqrt{2}}, \quad \bar{\psi} = L_{-1}e^{\alpha/\sqrt{2}}$$

For example, since

$$x_0 e^{-\alpha/\sqrt{2}} = \partial_{e^{\alpha \sqrt{2}}} e^{-\alpha/\sqrt{2}} = 2L_{-1} e^{\alpha/\sqrt{2}}, \quad x_1 e^{-\alpha/\sqrt{2}} = e^{\alpha/\sqrt{2}}, \quad x_n e^{-\alpha/\sqrt{2}} = 0, n > 1$$

and all successive actions with $x_n$ are zero, we get upon expanding the exponential the following explicit expression for $\phi^\alpha(z)$

$$\mathcal{Y}_M(e^{-\alpha/\sqrt{2}}, z) = \mathcal{Y}_M(\Delta(x, z)e^{-\alpha/\sqrt{2}}, z)$$

$$= \mathcal{Y}_M((1 + x_0 \log(z) + x_1 z^{-1})e^{-\alpha/\sqrt{2}}, z)$$

$$= \mathcal{Y}_M(e^{-\alpha/\sqrt{2}}, z) + z^{-1} \mathcal{Y}_M(e^{\alpha/\sqrt{2}}, z) + 2 \log(z) \frac{\partial}{\partial z} \mathcal{Y}_M(e^{\alpha/\sqrt{2}}, z)$$

29
As intended, the logarithmic term has by construction the correct monodromy behaviour, if we recall again that in our case \((x_0)^2 = 0\). The non-logarithmic operator has the additional term \(z^{-1}Y_M(e^{-\alpha/\sqrt{2}}, z)\), which we claim corresponds to the additional (green) terms in the previous section. For example, we recover the representation of the zero-mode algebra on the groundstates:

\[
\psi_0^{1+} = \text{Res} \left( Y_M(e^{-\alpha/\sqrt{2}}, z) \right) 1 = \text{Res} \left( z^{-1}Y_M(e^{-\alpha/\sqrt{2}}, z) \right) 1 = e^{\alpha/\sqrt{2}} = 1^- \\
\psi_0^{1-} = \text{Res} \left( Y_M(-e^{-\alpha/\sqrt{2}}, z) \right) e^{\alpha/\sqrt{2}} = \text{Res} \left( Y_M(e^{-\alpha/\sqrt{2}}, z) \right) e^{\alpha/\sqrt{2}} = 1^+ \\
\bar{\psi}_0^{1+} = \text{Res} \left( Y_M(e^{\alpha/\sqrt{2}}, z) \right) 1 = \text{Res} \left( Y_M(e^{\alpha/\sqrt{2}}, z) \right) 1 = 0 \\
\bar{\psi}_0^{1-} = \text{Res} \left( Y_M(e^{\alpha/\sqrt{2}}, z) \right) e^{\alpha/\sqrt{2}} = \text{Res} \left( Y_M(e^{\alpha/\sqrt{2}}, z) \right) e^{\alpha/\sqrt{2}} = 0
\]

**Remark 2.14.** In [FT10, Sug21a] (again, simply-laced) the modules of \(W_{g,p}\) where constructed by the cohomology of \(V_\lambda \times_B G\), which is a vertex algebra bundle over \(kG/kB\) with fiber \(V_\Lambda\). The fact that we find a single \(g\)-twisted module of every untwisted module of \(V_\lambda\), in conjunction with the results of the second part, indicates some connection between these approaches.

**Question 2.15.** It would be nice to have explicit twisted intertwining operators for the tensor product.

## 3 Semiclassical limits and Sturm-Liouville operators

To speak about limits of vertex algebras for \(\epsilon \to 0\) (or respectively \(\epsilon \to \infty\)), we need to consider a vertex algebra \(V_\epsilon\) over the field of rational functions \(\mathbb{C}(\epsilon)\) and fix a choice of an integral form, which is \(\mathbb{C}([\epsilon])\)-submodule \(V_\epsilon^{C([\epsilon])}\) such that \(V_\epsilon^{C([\epsilon])} \otimes \mathbb{C}(\epsilon) \cong V_\epsilon\) and \(V_\epsilon^{C([\epsilon])}\) is closed under the vertex operator \(Y\). Differently spoken, we fix a \(\mathbb{C}(\epsilon)\)-basis such that the vertex operator on \(V_\epsilon^{C([\epsilon])}\) only consists of nonnegative (respectively nonpositive) powers of \(\epsilon\), so that they are well defined in the limit \(\epsilon \to 0\) resp. \(\epsilon \to \infty\) as the \(\epsilon^0\)-term. This procedure has been introduced in [CL15] under the name deformable family. The semiclassical limit of a vertex algebra acquires the additional structure of a Poisson vertex algebra, as discussed e.g. in [FBZ04] Sec. 16.

### 3.1 Limit of affine Lie algebras: Connections

We first briefly review the semiclassical limit of affine Lie algebras: Let \(\hat{\mathfrak{g}}_\kappa\) be an affine Lie algebra

\[
[a_m, b_n] = [a, b]_{m+n} + m\delta_{m,-n}\langle a, b \rangle \kappa
\]

with \(a, b \in \mathfrak{g}\) and Killing form \(\langle a, b \rangle\). Then with the choice of an integral form generated by \(a_n \kappa^{-1}\) we have the semiclassical limit \(\kappa \to \infty\), which due to

\[
[a_m \kappa^{-1}, b_n \kappa^{-1}] = \kappa^{-1} \cdot [a, b]_{m+n} \kappa^{-1} + \kappa^{-2} \cdot m\delta_{m,-n}\langle a, b \rangle \kappa
\]

becomes commutative and can be identified with the ring of functions on the space of \(\mathfrak{g}\)-valued connections

\[
d + A, \quad A = \sum_n A_n z^{-n-1}
\]
where $a_n^{\kappa+p}\kappa^{-1}$ corresponds to the function

$$d + A \mapsto \langle A_n, a \rangle$$

Clearly, the limit of the vacuum module (the Verma module for weight zero) can be identified with the ring of functions on the space of regular $g$-valued connections. As stated in [FBZ04] Section 16.3 this is in fact an isomorphism respecting the natural Poisson structure.

There are different integral forms and hence deformable families we can consider: Take as generators of the Verma module $a_0, a_m \kappa^{-1}, m > 0$ for $a \in g$. Then $a_0 \kappa^{-1}$ acts by zero, and there is still a nontrivial action of $g$. In this sense, the limit $\kappa \to \infty$ of $V_\lambda^\kappa(g)$, with $h$ not scaling with $b$ (so $\lambda b^{-2} \to 0$), is a bundle over the ring of functions on the space of regular $g$-connections, and the zero-fibre is the irreducible representation of $g$ with highest weight $\lambda$. The limit of the direct sum

$$\bigoplus_{\lambda=0}^\infty V_\lambda^\kappa \to \bigoplus_{\lambda=0}^\infty L_\lambda(sl_2)$$

can be identified with the ring of functions on the space of regular connections with a fixed solution, where $L_\lambda(sl_2)$ are the polynomials of degree $\lambda$ in some basis of two solutions under the natural $sl_2$-action.

**Question 3.1.** Presumably, under this identification the limit Poisson vertex algebra structure matches some natural Poisson structure on the bundle of solutions over the variety of regular connections.

### 3.2 The Virasoro algebra

We now discuss in more detail the semiclassical limit of $W^\kappa(sl_2)$, which is equal to the Virasoro algebra $\text{Vir}_b$ for $b = \sqrt{-\kappa - 2}$ defined by

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

where we parameterize central charge and conformal weights ($L_0$-eigenvalues) by

$$c = 1 + 6(b^{-1} + b)^2, \quad h_P = \frac{(b^{-1} + b)^2}{4} - P^2$$

The Verma module $V_{c,h}$ is the module with highest weight vector

$$L_0v = hv, \quad L_nv = 0, \quad n > 0$$

For generic $b^2 \notin \mathbb{Q}$ the Verma module is irreducible except for the following values $h_{m,n}$ for $m, n \in \mathbb{N}$, in which case there is a singular vector at level $mn$.

$$h_{m,n} = \frac{(b^{-1} + b)^2}{4} - \frac{(mb^{-1} + nb)^2}{4} = -b^{-2}m^2 - 1 + \frac{mn - 1}{2} - b^2n^2 - 1$$

In particular the vacuum module $h_{1,1} = 0$ has a singular vector $L_{-1}v = 0$. For later use we spell out

$$h_{m,1} = -b^{-2}m^2 - 1 - \frac{m - 1}{2}$$

$$h_{1,n} = -\frac{n - 1}{2} - b^2n^2 - 1$$

and note that in in the limit $b \to \infty$ we have $h_{m,1} \to -\frac{m-1}{2}$ and $h_{1,n}$ scales like $-b^2\frac{n^2-1}{4}$.

---

1. In physics literature, the former limit is called light, the latter heavy.
It is proven in [BSA88] that the singular vector for $h_{1,n}$, and after replacing $b \leftrightarrow b^{-1}$ also $h_{m,1}$, is given by

$$S_{1,n} = (b^{-2})^n \sum_{n_1 + \cdots + n_k = n} \frac{(n-1)!}{(\sum_{i=1}^k n_i)^n - \sum_{i=1}^k n_i} L_{-n_1} b^{-2} \cdots L_{-n_k} b^{-2}$$

For example $S_{1,2} b^4 = (L_{-1} b^{-2})^2 + L_{-2} b^{-2}$ and $S_{2,1} b^{-4} = (L_{-1} b^2)^2 + L_{-2} b^2$. We note that in the naive limit $b \to \infty$ we have for $h_{m,1}$ the singular vector $L_{-1}^m$ and for $h_{1,n}$ the singular vector $L_{-n}$. However, the result depends on the choice of integral form, see below.

The tensor structure on the representation category of the Virasoro vertex algebra has been obtained in [CJORY21].

### 3.3 Limit of the Virasoro algebra

There are several interesting choices of integral forms and hence deformable families. First, let us choose a system of generators $\ell_n = L_n b^{-2}$, then the limit of the Virasoro algebra $b \to \infty$ becomes commutative. It can be identified with the space of functions on the space of Sturm-Liouville operators

$$\frac{d^2}{dz^2} + q(z), \quad q(z) = \sum_{n \in \mathbb{Z}} q_n z^{-n-2}$$

where $\ell_n$ corresponds to the function

$$\frac{d^2}{dz^2} + q \mapsto q_n$$

There is also a different integral form generated by $L_1, L_0, L_{-1}$ and $\ell_n$ else, whence $\ell_1 = \ell_0 = \ell_{-1} = 0$. In this case $L_1, L_0, L_{-1}$ generate as usual the Lie algebra $\mathfrak{sl}_2$. The action on $\ell_{-n}, n \geq 2$ can be read off the Virasoro algebra relations

$$[L_1, \ell_n] = (-n+1)\ell_{n-1}[L_0, \ell_n] = (-n)\ell_n[L_{-1}, \ell_n] = (-n-1)\ell_{n+1}$$

so $\ell_n, n \leq -2$ resp. $n \geq 2$ span an $\mathfrak{sl}_2$ Verma module with lowest weight vector $\ell_{-2}$ resp. highest weight vector $\ell_2$.

**Remark 3.2.** Again, the limit acquires an additional Poisson structure, and the correspondence above relates this with a respective natural Poisson structure on the space of Sturm-Liouville operators. A similar principle relates the quantum Hamiltonian reduction of arbitrary $W_{q,k}$ to higher differential operators, which is related to the KdV-hierarchy and also puts more context on the formula for the singular vectors $S_{1,n}$, see the lecture [Zub91].

Some examples:

**Example 3.3.** The limit of the Verma module $\mathcal{V}_h$ at conformal weight $h = b^2 \ell_0$ is the ring of functions on the space of Sturm-Liouville operators with regular singularity and $\ell_0$ respectively.

**Example 3.4.** The limit of the vacuum representation $\varphi_{1,1} = \mathcal{V}_0/L_{-1}1$ is the ring $\mathbb{C}[\ell_{-2}, \ell_{-3}, \ldots]$ of functions on the space of Sturm-Liouville operators that are regular in $z = 0$.

More generally, the limit of $\varphi_{1,n}$ is the ring of functions on the space of Liouville operators with $\ell_0 = -\frac{n^2-1}{4}$ with diagonalizable monodromy. We discuss this in more depth in the next section.
Example 3.5. The limit of $\varphi_{m,1}$ is, in view of $\ell_0 \to 0$ and $S_{m,1} = b^4m\ell_{-1}^n + \cdots$ the truncated polynomial ring

$$\mathbb{C}[\ell_{-1}, \ell_{-2} \ldots]/(\ell_{-1}^n)$$

Note that now $L_0$ acts with a finite value, starting with $-m - 1$.

Let us now choose a different integral form on this module, generated by $L_{-1}, \ell_{-2}, \ell_{-3}, \ldots$.

Then the summands in the singular vector $S_{m,1}$ corresponding to $(n_1, \ldots, n_k)$, which we assume to be resorted such that $n_i = 1$ iff $i \neq l$, is

$$(L_{-n_1}b^2) \cdots (L_{-n_k}b^2) = b^{2l+4(k-l)}L_{-l}^1\ell_{-n_k-1}$$

and the highest scaling $b^{2m}$ is attained precisely for $(1, \ldots, 1, 2, \ldots, 2)$. Hence this limit of $\varphi_{m,1}$ is a ring

$$\mathbb{C}[L_{-1}, \ell_{-2} \ldots]/S_{m,1}$$

with $S_{m,1}$ a certain polynomial in $L_{-1}, \ell_{-2}$ starting with $L_{m}^m$. In particular, over any fibre with $\ell_{-2} = 0$ we have an $\mathfrak{sl}_2$-representation.

Question 3.6.

- Presumably, the $\mathfrak{sl}_2$ action is the action of the Möbius transformations on the space of Sturm-Liouville operators. Note that the trivial operator $\frac{d^2}{dz^2}$ where all $\ell_n = 0$ is invariant under $\mathfrak{sl}_2$, otherwise we have a respectively conjugated $\mathfrak{sl}_2$-action for regular $q(z)$.

- Presumably, the limit of the direct sum $\bigoplus_{m \geq 1} \varphi_{m,1}$ is the ring of functions on the space of regular Sturm-Liouville operators together with a fixed solution $f_1$. Presumably, the $\mathfrak{sl}_2$-action corresponds to the $\mathfrak{sl}_2$-action. Is there also a natural matching Poisson structure?

- What is the respective statement for $\varphi_{m,n}$? It would be reasonable to expect that the direct sum $\bigoplus_{m \geq 1} \varphi_{m,n}$ is the ring of functions on the space of Sturm-Liouville operators with $\ell_0 = -\frac{n^2}{4}$ and diagonal monodromy, as discussed below for $\varphi_{1,n}$, together with a fixed solution $f_1$.

- Similarly, one could presume that the semiclassical limit of the triplet algebra $\bigoplus \varphi_{m,1} \otimes \mathbb{C}^m$ is the ring of functions on the space of regular Sturm-Liouville operators together with a fixed basis of solutions $f_1, f_2$, and the according $\mathfrak{sl}_2$-action.

- We can also have $\ell_0, L_{-1}$ and $L_0, \ell_{-1}$ and then only action of part of $\mathfrak{sl}_2$. What does this correspond to?

3.4 Sturm-Liouville operators

The structure of the representations of the Virasoro algebra deeply reflects the solution properties of the associated differential equation, for example its monodromy. Let us briefly review some aspects of Sturm-Liouville differential equations:

$$\left(\frac{d^2}{dz^2} + q(z)\right)f(z) = 0$$

The Frobenius method makes a power series ansatz $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ around $z = 0$. If $q(z)$ is regular then $z = 0$ is called a regular point. If $q(z)z^2$ is regular, say $q(z) = \ell_0 + \cdots$, 

33
then \( z = 0 \) is called a regular singular point. In this case, we can start by solving the differential equation for the lowest term

\[
\left( \frac{d^2}{dz^2} + \frac{\ell_0}{z^2} \right) f_0(z) = 0, \quad f_0(z) = z^s, \quad s(s - 1) + \ell_0 = 0
\]

and obtain a recursive formula for the power series coefficients of a solution \( f(z) = \sum_{n=0}^{\infty} a_n z^{n+s} \). The equation \( s(s - 1) + \ell_0 = 0 \) is called indicial equation. Indeed, the theorem of Fuchs states, that the solutions of a Sturm-Liouville differential equation given by a convergent Frobenius series as above, or one of the solutions is of second kind:

\[ f_2 = f_1 \log(x) + \sum_{n=0}^{\infty} a_n z^{n+s} \]

where \( f_1 \) is a solution of the first kind.

To understand more precisely, when this can happen, we consider the monodromy of the 2-dimensional space of solutions around \( z = 0 \). Suppose the two solutions of the indicial equation \( s_1, s_2 \) differ not by an integer, then the monodromy is diagonalizable

\[
\begin{pmatrix}
e^{2\pi i s_1} & 0 \\
0 & e^{2\pi i s_2}
\end{pmatrix}
\]

in a basis of two distinguished solutions \( f_1, f_2 \) with Frobenius series for \( s_1, s_2 \). If \( s_1 - s_2 \in \mathbb{Z} \), it may happen that the monodromy matrix is a Jordan block

\[
\begin{pmatrix}
e^{2\pi i s_1} & 2\pi i \\
0 & e^{2\pi i s_2}
\end{pmatrix}
\]

with a solution \( f_1 \) with a Frobenius series for \( s_1 \) and a solution \( f_2 \) of the second type. Indeed, the classification theorem states that Sturm-Liouville operators with regular singular points are up to coordinate transformations classified by their monodromy operator, see e.g. [Ov01] Section 4.2.

From the perspective of the Virasoro algebra, \( \ell_0 \) is the rescaled conformal dimension, and the indicial equations \( s(s - 1) + \ell_0 = 0 \) with \( s_1 - s_2 = n \) precisely correspond to the values \( \ell_0 = -\frac{n^2 - 1}{4} \) of the series \( \varphi_{1,n} \). The striking observation is that the singular vector \( S_{1,n} \), as a polynomial equation in \( \ell_{-k} \), predicts precisely those coefficients leading to diagonalizable monodromy. In this sense, the limit of \( \varphi_{1,n} \) is identified with the ring of functions on the space of diagonalizable Sturm-Liouville operators with \( s_1 - s_2 = n \).

For Sturm-Liouville differential equations with irregular singularity, one observes the Stokes phenomenon that the vicinity of the singularity consists of angular sectors, called Stokes sectors, each with a different asymptotic expansion, see for example [Boa14]. It would be very interesting to continue the present discussion in this context, see Question 1.6.

**Example 3.7.** The Euler equation

\[
\left( \frac{d^2}{dz^2} + \frac{a}{z^2} + \frac{0}{z} \right) f(z) = 0
\]
is solved by \( z^s \) for the two solutions \( s(s-1) + a = 0 \), meaning \( s = \frac{1 \pm \sqrt{1-4a}}{2} \). For \( \phi_{1,2} \) we have \( \ell_0 = -\frac{3}{4} \) and thus \( s = \frac{3}{2},-\frac{3}{2} \). In particular the monodromy is

\[
\begin{pmatrix}
 e^{2\pi i(\frac{3}{2})} & 0 \\
 0 & e^{2\pi i(-\frac{3}{2})}
\end{pmatrix} = 
\begin{pmatrix}
 -1 & 0 \\
 0 & -1
\end{pmatrix}
\]

This diagonalizability is consistent with \( \ell_{-1} = 0 \) being the singular vector in \( \varphi_{1,1} \).

**Example 3.8.** The Bessel differential equation

\[
 \left( z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + (z^2 - \alpha^2) \right) F_\alpha(z) = 0
\]

becomes under substitution \( F_\alpha(z) = z^{-1/2} f_\alpha(z) \)

\[
 \left( \frac{d^2}{dz^2} + \frac{1 - 4\alpha^2}{4z^2} + \frac{0}{z} + 1 \right) F_\alpha(z)
\]

with a regular singularity at \( z = 0 \), solved by the Bessel function \( J_\alpha(z) \). Typically \( J_{\pm \alpha}(z) \) are two linearly independent solutions. However, if \( \alpha \) is an integer than some coefficient vanish and we have \( J_{-\alpha}(z) = (-1)^\alpha J_\alpha(z) \). Then, another linearly independent solution can be obtained by a limit procedure, called the Bessel function of the second kind

\[
 Y_n(z) = \frac{2}{\pi} J_n(z) \ln \frac{z}{2} - \frac{(\frac{z}{2})^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left( \frac{z^2}{4} \right)^k 
\]

\[
 - \frac{(\frac{z}{2})^n}{\pi} \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{(-\frac{z^2}{4})^k}{k!(n+k)!}
\]

We now discuss this from our perspective: The indicial equation predicts solutions with exponent \( s = \frac{1}{2} \pm \alpha \), which is consist with the behaviour \( z^{\pm \alpha} \) for the Bessel functions \( J_\alpha \) and \( J_{-\alpha} \) at the singularity. For \( \alpha \) half-integer, the exponents differ by an integer. Accordingly, the value \( \ell_0 = \frac{1-4\alpha^2}{4} \) matches \( \ell_0 = -\frac{n^2-1}{4} \) for \( n = 2\alpha \). Moreover

\[
 \ell_{-1} = 0, \quad \ell_{-2} = 1, \quad \ell_{-n} = 0, n > 2
\]

Thus the only contribution to \( S_{1,n} \) is \((n_1,n_2,...) = (2,2,...)\) and \( S_{1,n} = 0 \) iff \( n \) odd.

Accordingly the monodromies are for \( n = 2\alpha \) odd resp. even are in the basis \( J_\alpha \),

\[
 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 4i \\ 0 & -1 \end{pmatrix}
\]

According to their appearance in applications, the first type are called spherical Bessel functions and the second type cylindrical harmonics.

**Example 3.9.** The hypergeometric differential equation

\[
 \left( z(1-z) \frac{d^2}{dz^2} + (c - (a+b+1)z) \frac{d}{dz} - ab \right) F(z) = 0
\]

\[2\text{Wikipedia}\]
The case of positive integer one regular singular point.

For example, either one of the solutions is a rational function, or a hypergeometric function with an leading to solutions with exponents $s = c/2$ and $s = 1 - c/2$ at $z = 0$. For integer $c$, either one of the solutions is a rational function, or a hypergeometric function with an additional logarithms and a digamma function. This phenomenon is reviewed for example in [Vid07] and [DK17], which also nicely discusses this effect on a toy model $_0\Psi_1$ with one regular singular point.

The case of positive integer $c$ (resp. $2 - c$) corresponds by $c = n + 1$ we have $c(2 - c) = -\frac{n^2 - 1}{4}$ to $\phi_{1, n}$.

For example, $c = 0$ corresponds to $\phi_{1,1}$ and $z^{-2}$-term in $q(z)$ disappears. The relation $\ell_{-1}$ means that the $z^{-1}$-term in $q(z)$ also disappears and $b = 0$ resp. $a = 0$. In the hypergeometric equation in this case the third term vanishes and

$$
(z(1 - z) \frac{d^2}{dz^2} - (a + 1)z \frac{d}{dz}) F(z) = 0
$$

and after canceling $z$ and substituting $F'$ we find solutions

$$
F = 1, \quad (1 - z)^{a+2}, \quad \text{resp.} \quad F = 1, \quad \log(1 - z)
$$

with the respective diagonalizable monodromy with eigenvalues $+1$

On the other hand the hypergeometric function for $a + b = 1$ is related to the Legendre function $P^{1-c}_{-a}(1 - 2z)$. For integer $c$ the solutions are Legendre polynomials, together with the Legendre functions of the second kind, which involve logarithms.

We finally discuss the action by coordinate transformation and an $sl_2$-action:

If $f_1, f_2$ are solutions of the differential equation, then $f_1/f_2$ has Schwarzian derivative $2q(z)$. Here, the Schwarzian derivative $S$ has the following definition and defining property

$$
(S f)(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = \frac{f''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2
$$

$$
S(f \circ g) = (S(f) \circ g) \cdot (g')^2 + S(g)
$$

and is zero for Möbius transformations. We have the following action of precomposing with a diffeomorphism,

$$
\frac{d^2}{dz^2} + f'(z)^2 (q \circ f)(z) + \frac{1}{2} S(f)(z)
$$

and the following remarks:
• The space of Sturm-Liouville operators can be interpreted as the coadjoint representation of the Virasoro algebra, see [Ov01]. The Schwartz derivative appears here a 1-cocycle for the Virasoro algebra.

• From the analytic perspective, this transformation formula is the canonical transformation behaviour of a linear map between spaces of tensor densities, see [Ov01] Section 1.7. and [FBZ04] Section 8.2 and 16.5.

\[ \mathcal{F}_{-1/2} \to \mathcal{F}_{3/2} \]

• From a conformal field theory perspective, this transformation behaviour with Schwartzian derivative is precisely the transformation behaviour of the Virasoro field \( \mathcal{Y}(T, z) \) at nonzero or zero central charge.

• The functions \( f_1^2, f_1 f_2, f_2^2 \), which also appear as a basis of \( \phi_{3,1} \), are solutions to a third order differential equation, this should be viewed in the sense of the KdV hierarchy, see [Ov01, Zub91]

Suppose \( f_1, f_2 \) are solutions of a Sturm-Liouville problem. Then the transformation to the coordinate \( t = f_1 / f_2 \) brings the differential equation to the trivial form \( \frac{d^2}{dz^2} f(t) = 0 \).

One has for the trivial form a Lie algebra \( \mathfrak{sl}_2 \) via \( \frac{d}{dz}, z \frac{d}{dz}, z^2 \frac{d}{dz} \). Following Kirillov, we see that via the coordinate transformation, this leads to an action of \( \mathfrak{sl}_2 \) by vector fields

\[ f_1^2 \frac{d}{dz}, \quad f_1 f_2 \frac{d}{dz}, \quad f_2^2 \frac{d}{dz} \]

We check the commutator relations

\[ [f_1 f_2 \frac{d}{dz}, f_1^2 \frac{d}{dz}] = 2f_1^2 f_2 \frac{d}{dz} - f_1^2 f_2 \frac{d}{dz} - f_2^2 f_1 \frac{d}{dz} = W \cdot f_1^2 \frac{d}{dz} \]

\[ [f_1 f_2 \frac{d}{dz}, f_2^2 \frac{d}{dz}] = 2f_2^2 f_2 \frac{d}{dz} - f_2^2 f_2 \frac{d}{dz} - f_2^2 f_1 \frac{d}{dz} = -W \cdot f_1 \frac{d}{dz} \]

\[ [f_1^2 \frac{d}{dz}, f_2^2 \frac{d}{dz}] = 2f_1 f_2 f_2 \frac{d}{dz} - 2f_1 f_2 \frac{d}{dz} = 2W \cdot f_1 f_2 \frac{d}{dz} \]

where the Wronski determinant \( W = f_1' f_2 - f_1 f_2' \) is constant and nonzero. Since, at least for regular \( q(z) \), we may assume after a coordinate change a trivial Sturm-Liouville operator, it suffices to study \( \mathfrak{sl}_2 \)-action in this case

\[ \frac{d}{dz}, \quad z \frac{d}{dz}, \quad z^2 \frac{d}{dz} \]

For singular \( q(z) \) such a coordinate change is no longer holomorphic and we do not get an action of \( \mathfrak{sl}_2 \) by regular vector fields.

4  **Vertex algebras with big center**

Denote by \( V^\kappa(g) \) the universal affine vertex algebra at level \( \kappa \) and by \( M^\kappa_\lambda \) the induced module at weight \( \lambda \), by \( W^\kappa_\lambda \) the Wakimoto module, and by \( V^\kappa_\lambda \) their irreducible quotient for integral weight \( \lambda \). Denote by \( L^\kappa_\lambda(g) \) the irreducible affine vertex algebra for integral level \( \kappa^* \) and by \( L^\kappa_\lambda \) the irreducible quotients modules \( L^\kappa_\lambda \) for integral weight \( \lambda \). For \( g = \mathfrak{sl}_2, \kappa^* = 1 \) we have \( L^1(\mathfrak{sl}_2) = V_{\sqrt{2} \mathbb{Z}} \) a lattice vertex algebra and \( L_0^1 = V_0, \ L_{1/2}^1 = V_{1/2} \) their irreducible modules.
Denote by $W^\kappa(\mathfrak{g})$ the quantum Hamiltonian reduction by a principal nilpotent element $f$, and $V_{\lambda,f}$ and more generally $T^\kappa_{\lambda,\mu}$ the irreducible module obtained by reduction resp. $\mu$-twisted reduction of $V^\lambda_{\lambda}$. For $\mathfrak{g} = \mathfrak{sl}_2$ we have $V_{\lambda,f} = \varphi_{m,1}$ and $T^\kappa_{\lambda,\mu} = \varphi_{n,m}^{\mu-1}$ with Feigin-Frenkel duality $\varphi_{m,n} = \varphi_{n,m}^{\mu-1}$. Denote by $\pi^\kappa_{\lambda}$ the Fock module, which is the reduction of the Wakimoto module and for generic $\lambda$ coincides with the Verma module.

4.1 Coupled vertex algebras

Predecessors of the following general construction were the explicit constructions for $\mathfrak{sl}_2, p = 1$ in [BFL16] and $\mathfrak{sl}_2, p = 2$ in [BBFLT13] and $\mathfrak{g}, p = 1$ in [ACF22], whose conventions we use. On the other hand the construction builds for $p = 0$ on the algebra of chiral differential operators $D_{G,\kappa}$, which can be seen as a chiralization of the space of regular function on a finite group or Lie group.

The following construction first appeared in [CG17] from physical considerations, some form has been proven in [Mor21] using quantum groups and Kazhdan-Lusztig correspondence, and the form presented here including integral form and reduction is found in [CN22]

**Theorem 4.1.** For each integer $p > 0$ divisible by the lacity $\nu^\vee$ there exists a deformable family of vertex algebras, which extends $V^\kappa(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$ and decomposes over it as follows

$$\mathcal{A}^{(p)}[\mathfrak{g}, \kappa] = \bigoplus_{\lambda \in Q^+} V^\kappa_{\lambda}(\mathfrak{g}) \otimes V^{\kappa^*}_{\lambda}(\mathfrak{g})$$

for a pair of generic levels $\kappa, \kappa^*$ satisfying

$$\frac{1}{\kappa + \hbar^\vee} + \frac{1}{\kappa^* + \hbar^\vee} = p$$

**Corollary 4.2.** By quantum Hamiltonian reductions on either or both sides we obtain deformable families of vertex algebras extending $V^\kappa \otimes W^{\kappa^*}$

$$H^\vee \mathcal{A}^{(p)}[\mathfrak{g}, \kappa] = \bigoplus_{\lambda \in Q^+} V^\kappa_{\lambda}(\mathfrak{g}) \otimes T^\kappa_{\lambda,0}(\mathfrak{g})$$

(or sides reversed) and deformable families extending $W^\kappa \otimes W^{\kappa^*}$

$$H^\vee H^\vee \mathcal{A}^{(p)}[\mathfrak{g}, \kappa] = \bigoplus_{\lambda \in Q^+} T^\kappa_{\lambda,0}(\mathfrak{g}) \otimes T^{\kappa^*}_{\lambda,0}(\mathfrak{g})$$

4.2 Limits of these vertex algebras

The limits for $\kappa \to \infty$ or $\kappa^* \to \infty$ are, at least as modules, easily read off the defining decomposition formulas. They are by Section 3.1 resp. Section 3.3 bundles over the regular $\mathfrak{g}$-connections resp. regular $\mathfrak{g}$-opers with zero-fibres

$$\bigoplus_{\lambda \in Q^+} V^\kappa_{\lambda}(\mathfrak{g}) \otimes L^{\kappa^*}_{\lambda}(\mathfrak{g}), \quad \bigoplus_{\lambda \in Q^+} T^\kappa_{\lambda,0}(\mathfrak{g}) \otimes L^{\kappa^*}_{\lambda,0}(\mathfrak{g})$$

The quantum Hamiltonian reduction on the factor where the limit is taken are after the limit presumably related by a classical Hamiltonian reduction.

A conjecture communicated to us in the case $\mathfrak{sl}_2$ by T. Creutzig, and raised in this generality in his joint work [CN22] is
Conjecture 4.3. The zero-fibre $\bigoplus_{\lambda \in \mathbb{Q}^+} T^\kappa_{\lambda,0} \otimes \mathcal{L}^\bullet_{\lambda}(g)$ is the Feigin-Tipunin algebra $\mathcal{W}_p(g)$.

Note that the divisibility $r^\vee \mid p$ is precisely the condition we found in the non-simply-laced case in [FL18].

Question 4.4. The zero-fibre $\bigoplus_{\lambda \in \mathbb{Q}^+} T^\kappa_{\lambda,0} \otimes \mathcal{L}^\bullet_{\lambda}(g)$, whose quantum Hamiltonian reduction is presumably the Feigin-Tipunin algebra, seems to be a very interesting vertex algebra. What is known about it? Is there a free-field realization using nonlocal screening operators?

5 The case of simply-laced $g$ at $p = 1$

For $p = 1$ the algebra in the previous section has an explicit realization, which appeared in physics literature first for $\mathfrak{sl}_2$ under the name GKO-construction [GKO86] and was established in [ACL19] for $g$ simply-laced, see also [ACF22].

Theorem 5.1. Let $g$ be simply-laced, then we have the deformable family of vertex algebras extending $V^\kappa(g) \otimes W^\kappa_{\ast}(g)$

$$V^{\kappa-1}(g) \otimes L^1(g) = \bigoplus_{\lambda \in \mathbb{Q}^+} V^\kappa_{\lambda} \otimes T^\kappa_{\lambda,0}, \quad \frac{1}{\kappa + h^\vee} + \frac{1}{\kappa^* + h^\vee} = 1$$

More precisely, the full commutant of $V^\kappa(g)$ is $W^\kappa_{\ast}(g)$.

Remark 5.2. For non-simply-laced cases, we would have to consider as smallest case $p = r^\vee$ in order to have a well-defined Feigin-Tipunin algebra [FL18], corresponding to the largest $p$ for which the quantum group $u_q(g)$ collapses.

5.1 The affine coset

The explicit construction is based on the diagonal embedding of affine Lie algebras

$$\hat{\mathfrak{g}}_\kappa \to \hat{\mathfrak{g}}_{\kappa-1} \times \hat{\mathfrak{g}}_1$$

$$a_n^{(\kappa)} \to a_n^{(\kappa-1)} + a_n^{(1)}$$

and the commuting Virasoro embedding based on the three Sugawara elements

$$L_n^{\text{Coset}} = L_n^{(\kappa-1)} + L_n^{(1)} - L_n^{(\kappa)}$$

to which the $W^\kappa_{\ast}(g)$ reduces to in the case $\mathfrak{sl}_2$. The overall deformed action of $W^\kappa_{\ast}(g)$ could be calculated similarly.

Theorem 5.3. The limit $\kappa \to \infty$ of the affine Lie algebra $\hat{\mathfrak{g}}_{\kappa-1} \times \hat{\mathfrak{g}}_1$ is a bundle over the space of $g$-connections. The fibers over a point $d + A$, or the coinvariants with respect to $a_n^{(\kappa)}/\kappa - \langle A_n, a \rangle$ are isomorphic as a vector space to $\hat{\mathfrak{g}}_1$ and $L_n^{\text{Coset}}$ descends to a deformed Virasoro action

$$L_n^{d+A} = L_n + \sum_{n' + n'' = n} \langle A_{n''}, A_{n'} \rangle n' - \frac{1}{2} \sum_{n' + n'' = n} \langle A_{n'}, A_{n''} \rangle$$

where $(A_{n''})_{n'}$ denotes the mode operator $n'$ of the element $A_{n''} \in \mathfrak{g}$. 39
Proof. Now we take the limit $\kappa \to \infty$ and compute the coinvariants with respect to $a_n^{(\kappa)}/\kappa - \lambda_n$, where we express the eigenvalues in terms of the connections by $\lambda_n = \langle A_n, a \rangle$. This expression is by definition equal to the relation

$$a_n^{(\kappa-1)}/\kappa + a_n^{(1)}/\kappa \sim \langle A_n, a \rangle$$

Hence the coinvariants are given with the following isomorphism of vector spaces

$$(\hat{\mathfrak{g}}_{\kappa-1} \times \hat{\mathfrak{g}}_1)/\sim \sim \hat{\mathfrak{g}}_1$$

In particular, if $a_n^{(1)}$ acts on modules not scaling with $\kappa$, then taking coinvariants $a_n^{(\kappa)}/\kappa - \langle A_n, a \rangle \kappa$ gives the same as taking coinvariants $a_n^{(\kappa-1)}/\kappa - \langle A_n, a \rangle$. The Virasoro action of the coset, as defined, is explicitly

$$L^{\text{Coset}}_n = L^{(\kappa-1)}_n + L^{(1)}_n - L^{(\kappa)}_n$$

$$= -\frac{1}{2(\kappa - 1 + h^\vee)} \sum_{n' + n'' = n} :v^{(i)(\kappa-1)}_n u^{(i)\kappa(\kappa-1)}_n: - \frac{1}{2(1 + h^\vee)} \sum_{n' + n'' = n} :v^{(i)(1)}_n w^{(i)(1)}_n$$

$$+ \frac{1}{2(\kappa + h^\vee)} \sum_{n' + n'' = n} :v^{(i)(\kappa-1)}_n + v^{(i)(1)}_n(u^{(i)(\kappa-1)}_n + w^{(i)(1)}_n):$$

for $v^{(i)(\kappa)}, w^{(i)(\kappa)}$ a dual basis of $\mathfrak{g}$ corresponding to the level $\kappa$. Then the image under the previous isomorphism is

$$L^{d+A(z)}_n = L_n - \frac{1}{2(\kappa - 1 + h^\vee)} \sum_{n' + n'' = n} :(-v^{(i)(1)}_n) + \langle A_{n'}, v^{(i)(1)} \rangle \kappa(-w^{(i)(1)}_n) + \langle A_{n''}, w^{(i)(1)} \rangle \kappa:$$

$$+ \frac{1}{2(\kappa + h^\vee)} \sum_{n' + n'' = n} :A_{n'}, v^{(i)(1)} \kappa(A_{n''}, w^{(i)(1)} \kappa):$$

In the limit $\kappa \to \infty$ only terms involving at least one $A_n$ survive. Then taking normally ordered products is not necessary anymore. The quadratic term in $A_n$ survives with a prefactor

$$\left(-\frac{1}{2(\kappa - 1 + h^\vee)} + \frac{1}{2(\kappa + h^\vee)}\right) \kappa^2 \to -1/2$$

The two mixed terms are equal, because the dual basis is symmetric. With the property of the dual basis the

$$L^{d+A(z)}_n = L_n + \sum_{n' + n'' = n} \langle A_{n''}, A_{n'} \rangle - \frac{1}{2} \sum_{n' + n'' = n} \langle A_{n'}, A_{n''} \rangle$$

$\square$

Corollary 5.4. For $a \in \mathfrak{g}$ we find

$$[L^{d+A(z)}_n, a_k] = [L_n, a_k] + \sum_{n' + n'' = n} [(A_{n''})_{n'}, a_k]$$

$$= [L_n, a_k] + \sum_{n' + n'' = n} [A_{n''}, a]_{n' + k} + \sum_{n' + n'' = n} n' \delta_{n', -k} \langle A_{n''}, a \rangle \kappa$$

defines the quadratic terms in $L^{d+A}$ are central. In particular the modified Virasoro action has

$$L^{d+A(z)}_{n-1} = L_{n-1} + A(z) + \langle A(z)', - \rangle$$

where $A(z)$ acts as $\mathfrak{g}$-valued function with the $\mathfrak{g}$-adjoint action on $\mathfrak{g}$-valued functions $a(z)$ and $\langle , \rangle$ is evaluated degree-wise. Using an isomorphism like $f$ in Lemma 2.2 should remove this last term.

40
Example 5.5. We observe that for the connection $d + A = d + z^{-1}X$ with $X \in \mathfrak{g}$ we have

$$L_n^{d+A(z)} = L_n + X_n - \delta_{n,0} \frac{1}{2} \sum_{k \in \mathbb{Z}} \langle X, X \rangle$$

This coincides with the formula for the twisted Virasoro algebra in Lemma 2.2 with $\kappa = 1$.

5.2 The quantum Hamiltonian reduction

As suggested in [BBFLT13, BFL16] for $\mathfrak{sl}_2$ and conceptually established [ACF22] we now take the coset model and perform a quantum Hamiltonian reduction $H^0(-)$ with respect to the diagonally embedded $\hat{g}_\kappa \hookrightarrow \hat{g}_\kappa - p \times \hat{g}_p$.

A general principle [ACF22] Theorem 1 states that for any vertex algebra $V$ equipped with a homomorphism $V^k(\mathfrak{g}) \rightarrow V$ we have an isomorphism

$$H^\bullet(V \otimes L_p(\mathfrak{g})) \cong H^\bullet(V) \otimes L_p(\mathfrak{g})^{Urod}$$

where the first reduction is with respect to the diagonal embedding and the isomorphism only becomes an isomorphism of conformal vertex algebras if the Virasoro structure on $L_p(\mathfrak{g})$ is modified, which we denote $L_p(\mathfrak{g})^{Urod}$. This is called a generalization of the previously known $Urod$ algebra in the case $\mathfrak{sl}_2, p = 1$ [BFL16]. A similar statement holds of course for reductions of any module over $V \otimes L_p(\mathfrak{g})$. Precomposing this with the diagonal embedding $V^\kappa(\mathfrak{g}) \hookrightarrow V^{\kappa-p}(\mathfrak{g}) \otimes L_p(\mathfrak{g})$, we obtain a functor called $W$-algebra translation, which is proven to be exact

$$W^\kappa \otimes W^{\kappa'}(\mathfrak{g}) \hookrightarrow W^\kappa(\mathfrak{g}) \otimes L_p(\mathfrak{g})^{Urod} = A(\mathfrak{g}, \kappa, 1)$$

where

$$1 \frac{h^{\kappa'}}{\kappa + h^{\kappa'}} + 1 \frac{h^{\kappa}}{\kappa' + h^{\kappa}} = \rho_V, \quad \kappa + h^{\kappa'} \notin \mathbb{Q} \leq 0$$

The structure of the reduction $A(\mathfrak{g}, \kappa, 1)$ and also of the reduction of any module over $V^\kappa(\mathfrak{g}) \otimes L_1(\mathfrak{g})$ can be computed if one knows the decomposition of modules if restricted to $V^\kappa(\mathfrak{g}) \otimes W^{\kappa'}(\mathfrak{g})$. This is achieved with the decomposition formulas in the next section.

5.3 Representations and decomposition formulas

It is important to know the decomposition of a $V^\kappa(\mathfrak{g}) \otimes L_1(\mathfrak{g})^{Urod}$-module over the coset subalgebra $V^\kappa(\mathfrak{g}) \otimes W^{\kappa'}(\mathfrak{g})$. In particular this allows to compute the quantum Hamiltonian reduction with respect to $V^\kappa(\mathfrak{g})$ and obtain then similar decomposition of the corresponding $W^\kappa(\mathfrak{g}) \otimes L_1(\mathfrak{g})^{Urod}$-module over the coset subalgebra $\text{Vir}^{b_1} \otimes \text{Vir}^{b_2}$, which can be found in [BFL16] Theorem 2.1 and Theorem 3.3. and we will discuss more explicitly in the next section. For general $\mathfrak{g}$ and $p = 1$ this is in [ACF22] Section 8, which we now quote and we try to keep close to their notation.

---

3They introduce a parameter $\epsilon$ which interpolates between a reduction of the first and the second factor for $\epsilon \rightarrow 0, \infty$
Theorem 5.6. For generic level $\kappa$ we have the following decompositions of $V^\kappa(\mathfrak{g}) \otimes L_{\mathfrak{g}^1}\text{-module over the coset subalgebra } W^\kappa(\mathfrak{g}) \otimes W^\kappa(\mathfrak{g})$

1. We have for integral weight $\mu$

$$V^{\kappa-1}_\mu \otimes L^1_\nu = \bigoplus_{\lambda \in \mu + \nu + Q} V^\kappa_\lambda \otimes T^\kappa_\lambda$$

and reduction resp. reduction after spectral flow $\mu' \neq 0$

$$T^{\kappa-1}_{\mu,\mu'} \otimes L^1_\nu = \bigoplus_{\lambda \in \mu + \mu' + \nu + Q} T^\kappa_{\lambda,\mu'} \otimes T^\kappa_\lambda$$

2. We have for generic weight $\mu$

$$W^{\kappa-1}_\mu \otimes L^1_\nu = \bigoplus_{\lambda \in \mu + \nu + Q} W^\kappa_\lambda \otimes \pi^\kappa + h^\nu_\lambda$$

and reduction

$$\pi^{\kappa-1 + h^\nu}_\mu \otimes L^1_\nu = \bigoplus_{\lambda \in \mu + \nu + Q} \pi^\kappa + h^\nu_\lambda \otimes \pi^\kappa + h^\nu_{\lambda - (\kappa + h^\nu)}^\mu$$

Question 5.7. For modules from the Wakimoto module and non-generic weight $\mu$ (for example $\mu = 0$) we expect similar formulae but there can (and will) be nontrivial extensions between the summands. It would be very helpful to have results in this direction. In our context, these extensions are also visible for the twisted modules with nilpotent part, and connections with regular singular connections with nilpotent part.

Question 5.8. There are also decomposition formulae for admissible level. It would be very interesting to repeat out considerations in this case.

5.4 Bundle associated to the representations $V^{\kappa-1}_\mu \otimes L^1_\nu$

The affine Lie algebra $\hat{\mathfrak{g}}_\kappa$ has by Section 3.1 as limit $\kappa \to \infty$ the ring of functions over the space of $\mathfrak{g}$-connections on the punctured disc. Similarly, the affine vertex algebra $V^\kappa(\mathfrak{g})$ has as limit the ring of functions over the space of $\mathfrak{g}$-connections and its modules $V^\kappa_\mu$ with integral $\mu$ not scaled by $b$, have as limit a bundle over the space of regular connections with fibre the irreducible $\mathfrak{g}$-module $L^\mu_\mu$. This can be presumably identified with polynomials of degree $\mu$ in the solutions, such that the limit of $\bigoplus_\mu V^{\kappa-1}_\mu$ can be identified with the ring of functions over the space of regular connections together with a solution of the associated differential equation. Applying Theorem 5.3 directly yields

Corollary 5.9. For integral $\mu$, not scaled with $\kappa$, the limit of the $V^{\kappa-1}_\mu \otimes L^1_\nu$-module

$$V^{\kappa-1}_\mu \otimes L^1_\nu$$

is a bundle over the space of regular connections with zero-fibre $L^\mu_\mu \otimes L^1_\nu$ and modified Virasoro structure.
On the other hand, naively applying the limit to the decomposition formula in Theorem 5.6 yields the following matching result. A warning should be placed however, that the decomposition is not necessarily preserving the chosen integral form, as we will see. The direct expression for $p = 1$ and the decomposition formula give us two in general different integral forms to consider.

**Corollary 5.10.** The limit of the decomposition of $V_{\mu}^{\kappa-1} \otimes L_{\nu}^1$ as $V^\kappa \otimes W^\kappa^*$-module

$$
\bigoplus_{\lambda \in \mu + \nu + Q} \bigoplus_{\lambda \in P^+} V_{\lambda}^\kappa \otimes T_{\lambda,\mu}^* 
$$

is a bundle over the space of regular connections with zero-fibre $\bigoplus_{\lambda \in \mu + \nu + Q} \mathcal{L}_{\lambda} \otimes T_{\lambda,\mu}^*$

For $\mu = \nu = 0$ this is the vertex algebra $V^{\kappa-1}(g) \otimes L^1(g)$, which acquires as big center the ring of functions over regular connections. The vertex algebra with big center becomes a bundle over the space of regular connections with zero-fibre the vertex algebra

$$
L^1(g) = \bigoplus_{\lambda \in Q} \mathcal{L}_{\lambda} \otimes T_{\lambda,0}^*
$$

which it the Feigin-Tipunin algebra for simply-laced $g$ in the limiting case $p = 1$.

We now discuss the case $\mu \neq 0$. On the Grothendieck ring resp. on the level of characters, we have indeed an equality

$$
\mathcal{L}_{\mu}(g) \otimes L^1_{\nu}(g) = \bigoplus_{\lambda \in \mu + \nu + Q} \bigoplus_{\lambda \in P^+} \mathcal{L}_{\lambda} \otimes T_{\lambda,\mu}^*
$$

as we can see from computing with $g$ fusion rules

$$
\mathcal{L}_{\mu}(g) \otimes L^1_{\nu}(g) = \mathcal{L}_{\mu} \otimes \left( \bigoplus_{\eta \in \nu + Q} \bigoplus_{\eta \in P^+} \mathcal{L}_{\eta} \otimes T_{\eta,0}^* \right)
$$

$$
= \bigoplus_{\lambda \in \mu + \nu + Q} \bigoplus_{\lambda \in P^+} \mathcal{L}_{\lambda} \otimes \left( \bigoplus_{\eta \in P^+} \left( \frac{\mu \eta}{\lambda} \right) T_{\eta,0}^* \right)
$$

where we use the fusion rules coefficients $\left( \frac{\mu \eta}{\lambda} \right)$, which are nonzero only for $\mu + \eta \in \lambda + Q$, and comparing this to the formula in the limit $\kappa^* + h^* \to 1$

$$
T_{\lambda,\mu}^{\kappa^*} = \bigoplus_{\eta \in P^+} \left( \frac{\mu \eta}{\lambda} \right) T_{\eta,0}^{\kappa^*}
$$

Note however that the limit of $T_{\lambda,\mu}^{\kappa^*}$ depends on the choice of an integral form, that has to be compatible with the decomposition formula. We can read off from the previous calculation, that apparently this limit sends $T_{\lambda,\mu}^{\kappa^*}$ to a direct sum of irreducible $T_{\eta,0}^{\kappa^*}$, and not the respective indecomposable.
Example 5.11. For $\mathfrak{sl}_2$ we have $W^\kappa$ the Virasoro algebra at central charge 1 and $T_{\lambda,\mu} = \varphi_{2\lambda+1,2\mu+1}$. On the other the Clebsch-Gordan coefficients are $(^\lambda_\mu) = 1$ for $|\mu - \eta| \leq \lambda \leq \mu + \eta$, which is equivalent to $|\mu - \lambda| \leq \eta \leq \mu + \lambda$, and $\mu + \eta \in \lambda + 2\mathbb{Z}$. Altogether the previous formula reads

$$\varphi_{2\lambda+1,2\mu+1} = \bigoplus_{\substack{|\mu - \lambda| \leq \eta \leq \mu + \lambda \\ \mu + \eta \in \lambda + 2\mathbb{Z}}} \varphi_{2\eta+1,1}$$

Question 5.12. Which integral forms are compatible with the decomposition formula? Can we obtain similar results as the left-hand side for $p > 1$?

Question 5.13. We would like to study the case where $\mu/b^2$ is fixed in the limit. These correspond to regular singular connections.

5.5 Bundle associated to the spectral flow representations $\sigma^\ell(\mathbb{V}^{\kappa-1}_\mu) \otimes \mathbb{L}^1_\nu$

We also consider spectral flow $\sigma^\ell$, which is an automorphism of the affine Lie algebra associated to an element $\ell \in P$, viewed as subgroup of the affine Weyl group. As such it does not preserve the choice of Borel algebra used to define highest-weight modules.

Consider the previous decomposition

$$\mathbb{V}^{\kappa-1}_\mu(\mathfrak{sl}_2) \otimes \mathbb{L}^1_\nu(\mathfrak{sl}_2) = \bigoplus_{\lambda \in \mu+\nu+Q} \mathbb{V}^{\kappa}_\lambda \otimes T^{\kappa*}_{\lambda,\mu}$$

We apply $\sigma^\ell$ and use that coset is preserved and that $\sigma^\ell(\mathbb{L}^1_\nu) = \mathbb{L}^1_{\nu - \ell}$ and substitute $\nu$

$$\sigma^\ell(\mathbb{V}^{\kappa-1}_\mu) \otimes \mathbb{L}^1_\nu = \bigoplus_{\lambda \in \mu+\nu+\ell+Q} \sigma^\ell(\mathbb{V}^{\kappa}_\lambda) \otimes T^{\kappa*}_{\lambda,\mu}$$

For simplicity we now restrict ourselves to the case $\mathfrak{sl}_2$: The spectral flow automorphism is defined as follows, which we quote from [CG17] Section 9.1.1

$$\sigma^\ell(e_n) = e_{n-\ell}, \quad \sigma^\ell(f_n) = f_{n+\ell}, \quad \sigma^\ell(h_n) = h_n - \delta_{n,0}\ell K$$

$$\sigma^\ell(L_0) = L_0 - \frac{1}{2}\ell h_0 + \frac{1}{4}\ell^2 k$$

Example 5.14. The first spectral flow of the vacuum module $\sigma^1(\mathbb{V}^{\kappa-1}_0)$ is generated by a vector $v$ with

$$e_nv = 0, \quad f_nv = 0, \quad h_nv = -\delta_{n,0}\kappa, \quad n \geq 0$$

For the obvious integral form, the limit $\kappa \to \infty$ is hence the ring of functions on the space of regular singular $\mathfrak{sl}_2$-connections

$$d + A_0z^{-1} + A_{-1}z^0 + \sum_{n \leq -1} A_nz^{-1-n}, \quad A_0 \in \mathfrak{sl}_2^0, \quad A_{-1} \in \mathfrak{sl}_2^0$$

The first spectral flow of other modules with finite-dimensional groundstates have $e_1$ nonzero, but nilpotent, so this again corresponds to regular singular connections.
Example 5.15. The first spectral flow of a Verma module $M^\kappa_\mu$ corresponds to an irregular singular connection
\[ d + A_1 z^{-2} + \cdots, \quad A_1 \in \mathfrak{sl}_2^{>0} \]
Note that for non-generic weights the direct decomposition formula does presumably not hold. A second spectral flow on the vacuum module corresponds to an irregular singular connection
\[ d + A_1 z^{-2} + \cdots, \quad A_1 \in \mathfrak{sl}_2^{>0} \]
and the irreducible quotient corresponds to certain equations to be fulfilled by the $A_n$. It is to be expected that irreducible representations lead to connections with a particular easy behavior in contrast to the generic case, somewhat similar to Section 3.3. For example, $A_1 \in \mathfrak{sl}_2^{>0}$ nilpotent leads to a collapse of the essential singularity $\exp(-A_1 z^{-1})$.

5.6 Bundle associated to the representations $T^{\kappa-1}_{\mu,\mu'} \otimes \mathbb{L}^1_\nu$

The quantum Hamiltonian reduction resp. reduction after spectral flow $\mu' \neq 0$ of the decomposition in the previous section is by Theorem 5.6
\[ T^{\kappa-1}_{\mu,\mu'} \otimes \mathbb{L}^1_\nu = \bigoplus_{\lambda \in \mu+\mu'+\nu+Q} T^{\kappa}_\lambda \otimes T^{\kappa*}_{\lambda,\mu} \]

We have no analog of Theorem 5.3 for the reduction, but we can naively read off the limit of the right-hand side: It is a bundle over the $\mathfrak{g}$-opers with diagonalizable monodromy $\mu'$ and with zero-fibre
\[ \bigoplus_{\lambda \in \mu+\mu'+\nu+Q} \mathcal{L}_{\lambda} \otimes T^{\kappa*}_{\lambda,\mu} \]

We observe that this is the very same decomposition as for $\mu' = 0$ with $\nu + \mu'$ in order to account for the shift in the summation condition. The $\mu \neq 0$ has a similar effect as in the last section, so we can concentrate on the case $\mu = 0$.

Example 5.16. For $\mathfrak{sl}_2$ the limit of $\varphi_{2\lambda+1,2\nu+1}$ is for $\lambda = 0$ the ring of functions on the space of Sturm-Liouville operators $\frac{d^2}{dz^2} + q(z)$ with $q_0 = -\frac{n^2-1}{4}$ for $n = 2\mu' + 1$ and diagonal monodromy, which is then the monodromy of $z^s$ for $s = \frac{1+\nu}{2}$. More generally for $m = 2\lambda + 1$ it is a bundle over this set with zero-fibre $\mathbb{C}^m = \mathcal{L}_{\lambda}$ as $\mathfrak{sl}_2$-module. So for $\mu = 0$ the limit of the decomposition is, setting also $l = 2\nu + 1$,
\[ \bigoplus_{\lambda \in m+n+l \in 2\mathbb{Z}, m \geq 0} \mathbb{C}^m \otimes \varphi_{m,1} \]

Again, as a warning, it is not clear whether the limit of the decomposition agrees with the limit of the actual reduction (which we cannot compute) as modules or just on the level of Grothendieck rings.

5.7 Bundle associated to the representations $\mathbb{W}^{\kappa-1}_{\mu} \otimes \mathbb{L}^1_\nu$, generic $\mu$

For generic $\mu$ the Wakimoto module $\mathbb{W}^{\kappa-1}_{\mu}$ coincides with the Verma module
\[ M^{\kappa-1}_{\mu} = \hat{\mathfrak{sl}}_2 \otimes_{\langle h_0,e_0,f_1,h_1,e_1,\ldots \rangle} \mathbb{C} \]
with action $h_0v = \mu v$ and all others zero. Assume as first case that $\mu$ does not scale with $b^2$, and as second case that $\mu/b^2$ is kept constant. Then by Theorem 5.6 we have: The limit of $\mathcal{W}_{\mu}^{-1} \otimes \mathbb{L}_1^1$ is a bundle over the space of connection

$$d + A_0 z^{-1} + \sum_{k \geq 0} A_{-k-1} z^k, \quad A_0 = t \cdot e_0$$

for any $t \in \mathbb{C}$ and with fiber $\mathbb{L}_1^1 = \mathcal{V}_\sqrt{2}\mathbb{Z}$ with modified conformal structure

$$\tilde{L}_{n}^{\text{Coset}} = L_{n}^{\sqrt{2}\mathbb{Z}} + (t \cdot e_n + \mu/b^2 \cdot h_n) - \delta_{n,0} \frac{1}{2} \frac{\mu/b^2}{8} + \sum_{k \geq 0} 8 (A_{-k-1})_{n+k+1}$$

On the other hand we may employ the decomposition formula in Theorem 5.6

$$\mathcal{W}_{\mu}^{-1} \otimes \mathbb{L}_1^1 = \bigoplus_{\lambda \in \mu + \nu + Q} \mathcal{W}_\lambda \otimes \pi_{\lambda}^{\kappa^* + h^/}$$

In the limit the right-hand side is a bundle over the same space of connections and with fiber

$$\bigoplus_{\lambda \in \mu + \nu + Q} \pi_{\lambda}^{\kappa^* + h^/}$$

**Remark 5.17.** We could also choose a different integral form generated by $e_0$ and $e_n/\kappa, f_n/\kappa$ as before. Then instead of a regular singular connection we again get a regular connection, since $e_0/\kappa, f_0/\kappa \to 0$, but the bundle associated to $\mathcal{M}_\kappa^\lambda$ is suddenly the Verma module $\mathcal{M}_\lambda$ over $\mathfrak{g}$ and we get as limit

$$\bigoplus_{\lambda \in \mu + \nu + Q} \mathcal{M}_\lambda \otimes \pi_{\lambda}^{\kappa^* + h^/}$$

### 5.8 A second limit and the affine variant of the Feigin-Tipunin algebra

In Question 4.4 we asked for the affine variant of the Feigin-Tipunin algebra, which should extend $V^p(\mathfrak{g})$, should have a decomposition

$$\bigoplus_{\lambda \in Q^+} V^{-h^/ + p}_\lambda \otimes \mathcal{L}_\lambda^*,$$

have as Hamiltonian reduction the Feigin-Tipunin algebra, and should appear in the present work as the zero-fibre of the respective bundle in the limit of $H_{\kappa^*}^\lambda A^{(p)}[\mathfrak{g}, \kappa]$. In the case $p = 1$ we can calculate this in our explicit coset model as the second possible limit of

$$V^{\kappa - 1}(\mathfrak{g}) \otimes \mathbb{L}_1^1(\mathfrak{g}) = \bigoplus_{\lambda \in Q^+} V^{\lambda}_\lambda \otimes T^{\kappa, 0}_\lambda$$

$$\kappa \to -h^/ + 1, \quad \kappa^* \to \infty$$

The result is clearly $V^{\text{crit}}(\mathfrak{g}) \otimes \mathbb{L}_1^1(\mathfrak{g})$ being a bundle over the regular $\mathfrak{g}$-opers via the coset embedding of $W^*$ and extending $V^{\text{crit}}(\mathfrak{g})$ via the diagonal embedding. Note that
as discussed in the introduction, the center of $V^{\text{crit}}(g)$ is also the ring of functions on regular $g$-opers.

From the explicit formulas in the previous subsection we see that $L_n = \frac{1}{2} L_n^{\text{crit,rescaled}}$ plus terms of order $\ell^{-1}$ if all weights scale with order 1.

**Corollary 5.18.** The fibre of $V^{\text{crit}}(g) \otimes L^1(g)$ over a regular $g$-oper is, as a vertex algebra or twisted module, identified with the fibre of $V^{\text{crit}}$ tensored with $L^1(g)$.

As already stated, these fibres decompose over $V^{-h^\vee+1}$ as

$$\bigoplus_{\lambda \in Q^+} V_{\lambda}^{-h^\vee + 1} \otimes L_\lambda$$

**Question 5.19.** Compute the (twisted) representations of this algebra and compare it to the respective limits $\kappa \to \infty$ in the general setting of the previous section, and in particular for $p = 1$.

In the further case ($\mathfrak{sl}_2$, $p = 2$), the result should be the small N=4 superconformal algebra as discussed in the end of section 7.1.

### 6 Explicit computations for $\mathfrak{sl}_2$, $p = 1$

In this section we follow closely the explicit computations in [BFL16], but to be consistent with the general formulae in Section 4.1 we let $\kappa$ parametrizes $b_2$ instead of $b$, so our $\kappa$ corresponds to $\kappa + 1$ in [BFL16]. Note also that the sides and the roles of $b_1, b_2$ and are reversed and the factor with $b_1$ is dualized in comparison to the general formulae in Section 4.1.

The quantum Hamiltonian reduction of $V^\kappa(\mathfrak{sl}_2)$ is the Virasoro algebra with parameter (see Section 3.2)

$$b_2 = \sqrt{-\kappa - 2}$$

Accordingly, the coset and its reduction are

$$V^\kappa(\mathfrak{sl}_2) \otimes \text{Vir}^\kappa(\mathfrak{sl}_2) \hookrightarrow V^\kappa(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2)$$

$$\text{Vir}^{b_1} \otimes \text{Vir}^{b_2} \hookrightarrow \text{Vir}^b \otimes L_1(\mathfrak{sl}_2)$$

$$b_1^{-1} = \sqrt{-\kappa^* - 2} = \sqrt{-\kappa - 2}, \quad \kappa^* = -\frac{\kappa}{\kappa + 1}, \quad b = \sqrt{-\kappa - 1},$$

We check the following relations hold:

$$1 = \frac{1}{\kappa + 2} + \frac{1}{\kappa^* + 2}$$

$$1 = b_1^2 + b_2^2$$

$$b_1 = b/\sqrt{1 - b^2}$$

$$b_2 = \sqrt{b^2 - 1}$$

The Urod algebra is the vertex algebra $L_1(\mathfrak{sl}_2)$ with generators $f_{-1}, e_{-1}, h_{-1}$, or equivalently the vertex algebra $\mathcal{V}_{\sqrt{2}}$ with generators $e^{-\sqrt{2}}, e\sqrt{2}, \partial \phi \sqrt{2}$, with modified conformal element

$$T^{\text{Urod}} = \frac{1}{4}(\partial \phi \sqrt{2})^2 + \frac{1}{2} \partial^2 \phi \sqrt{2} + e\partial^2 e \sqrt{2 \phi}$$
[ACF22] state in Lemma 6.1 and Example 6.2 similar formulae for an Urod algebra \( L_1(\mathfrak{g}) \) with modified Virasoro action, corresponding to the case \( \mathfrak{g}, p = 1 \).

**Remark 6.1.** It might be interesting to try to understand the Urod algebra as affine Lie algebra at level 1 again deformed by a connection.

In [BFL16] Theorem 3.3 a) and Theorem 2.1 b) explicit instances of the defining decompositions in Section 4.1 are proven for this model

\[ \text{Vir}^b \otimes \mathcal{V}_{\mathbb{Z}} = \bigoplus_{n \geq 0} \varphi^{b_1}_{n,1} \otimes \varphi^{b_2}_{1,n} \]

In [BFL16] Remark 3.1 explicit conformal elements realizing the embedding of \( \text{Vir}^{b_1}, \text{Vir}^{b_1} \) in \( M^{b} \otimes \mathcal{V}_{\mathbb{Z}} \) are computed

\[ T^{(b_1)} = \frac{b + b^{-1}}{2(b - b^{-1})} e^{-2\phi} + \frac{b}{2(b - b^{-1})} (\partial \phi)^2 - \frac{b^{-1}}{\sqrt{2}(b - b^{-1})} \partial^2 \phi - \frac{(1 + 2b^{-2})}{b^2 - b^{-2}} (\partial \phi)^2 e^{2\phi} \]

\[ T^{(b_2)} = \frac{b - b^{-1}}{2(b - b^{-1})} e^{-2\phi} - \frac{b^{-1}}{2\sqrt{2}(b - b^{-1})} (\partial \phi)^2 + \frac{b}{\sqrt{2}(b - b^{-1})} \partial^2 \phi + \frac{1 + 2b^{-2}}{b^2 - b^{-2}} (\partial \phi)^2 e^{2\phi} \]

The limit \( b, b_2 \to \infty \) acquire a big central subalgebra generated by \( L_n^{b_1}/b^2 \). In the formula for \( T^{b_1}/b^2 \) all term except \( T/b \) vanish in the limit (assumed that we do not rescale the module with \( b \)), and we hence recover a similar result as in the proof of Theorem 5.3

\[ L_n^{(b_2)}/b_2^2 = L_n^{(b)}/b^2 \]

**Corollary 6.2.** The limit of \( \mathcal{V}^n \) fibres over the space of regular Sturm-Liouville operators \( \frac{d^2}{dz^2} + q(z) \) with \( q(z) = \sum_{n \in \mathbb{Z}} \ell_n z^{-2-n} \), the fibre \( \mathcal{V}^\infty \{q(z)} \) is nonzero for \( \ell_n = 0, n \geq -1 \) and equal to the lattice vertex algebra \( \mathcal{V}_{\mathbb{Z}} \) with deformed Virasoro action

\[ L_n^{(b_1)} = \frac{1}{2\epsilon} f_{n+1} + L_n^{\mathcal{V}_{\mathbb{Z}}} - \ell_n - 2\epsilon \sum_{k \in \mathbb{Z}} \ell_k \ell_{n-k-1} \]

Note that for regular connections all contributions are degree-lowering and hence not essentially change the structure as a Virasoro module.

In particular the zero-fibre \( \mathcal{V}^\infty|_0 \) is isomorphic to \( \mathcal{V}_{\mathbb{Z}} \) deformed by a connection \( d + \frac{1}{2\epsilon} f z^0 \), which is what (in hindsight) we expect from the classical Hamiltonian reduction.

We also have an \( \mathfrak{sl}_2 \)-action by \( L_n^{(b_2)} \), \( n = -1, 0, 1 \), which defined only for \( \ell_{-1}, \ell_0, \ell_1 = 0 \)

\[ L_n^{(b_2)} = -\frac{1}{2\epsilon} f_{n+1} + \frac{1}{\sqrt{2}} \partial^2 \varphi_n + (-n)(-n-1)e_{n+1} + L_n^{(b)} + 2\epsilon \sum_{k \in \mathbb{Z}} \ell_k \ell_{n-k-1} \]

48
Example 6.3. In particular the conformal weights resp. $L_0^{(b_1)}$-eigenvalues coincide with the standard choice of $V_{\sqrt{2}\mathbb{Z}}$, and the $sl_2$-weights of $L_0^{(b_2)}$ are according to the $\sqrt{2}\mathbb{Z}$-grading. $0, \frac{1}{4}, \frac{1}{2}, 1, 1, 1, \ldots$ simultaneous with $0, -\frac{1}{2}, +\frac{1}{2}, -1, 0, +1, \ldots$. The first eigenvectors are

$$v_{(1,1)\times(1,1)} := 1$$
$$v_{(2,1)\times(1,2)} := e^{-\frac{1}{\sqrt{2}}\varphi}$$
$$L_1^{(b_2)} v_{(2,1)\times(1,2)} := 2\epsilon e^{+\frac{1}{\sqrt{2}}\varphi}$$

and here $L_{-1}^{(b_1)} = -\frac{1}{2\epsilon} f_0$.

7 Explicit computations for $sll_2, p = 2$

7.1 The limit of the N=4 superconformal algebra

The large N=4 superconformal algebra $sVir_{N=4}$ as discussed in [CGL20] Section 2 is a quantum Hamiltonian reduction of $\hat{D}(2,1; a)_{k}$ and is generated by two commuting sets of $sl_2$ with generators $e, f, h$ and $e', f', h'$ at level

$$-\frac{a+1}{a} k - 1, \quad -(a+1)k - 1$$

and fermionic generators $G^{\pm\pm}$. There are two commuting Virasoro actions $L_{sl_2'}$ and $L_C := L - L_{sl_2'}$. All OPEs are explicitly given at the cited chapter.

In Section 2.4 the explicit case $k = 1/2$ is discussed

Example 7.1 ($k = 1/2$). The decomposition as $sl_2' \times sl_2$-module

$$sVir_{N=4}^{k,a} = \bigoplus_{\lambda \in \mathbb{Q}^+} V_{\lambda}^{-(a+3)/2} \otimes V_{\lambda}^{-(a-1+3)/2}$$

and from the proof of Theorem 2.5 following [Ad16] we find that the highest weight vectors are $X_n = (G^{++})(\partial G^{++}) \cdots (\partial^{n-1}G^{++})$.

The limit $a \to \infty$ of $sVir_{N=4}^{(k,a)}$ as deformable family defined by the generators $e'/\kappa, f'/\kappa, h'/\kappa$ and $L_C, G^{\pm\pm}$ is computed in [CGL20] Section 2.2. It acquires a large commutative subalgebra generated by $e'/\kappa, f'/\kappa, h'/\kappa$ isomorphic to the ring of functions on the space of $sl_2$-connections. The zero-fibre coincides for $k = 1/2$ with the so-called small N=4 superconformal algebra studied by Adamovic [Ad16] using free-field realization.

7.2 The limit of $osp(2|1)_{\kappa}$

The Hamiltonian reduction of $sVir_{N=4}^{1/2,a}$ with respect to $sl_2$ is calculated in [CGL20] Section 2.5:

Theorem 7.2. $H^\ast \mathcal{A}^{(2)}[sl_2, \kappa]$ with $\kappa = -(a + 3)/2$ is isomorphic to $V^{-(a+3)/2}(osp(1|2))$ with generators

$$e', f', h', x':= \frac{a+1}{\sqrt{2a}}G^{++}, y':= \frac{a+1}{\sqrt{2a}}G^{-+}$$

and the additional cohomology relation $e + 1 = 0$. 49
We now turn to the limit $\kappa \to \infty$ of
\[ H^\kappa A^{(2)}[\mathfrak{sl}_2, \kappa] \cong V^\kappa(\mathfrak{osp}(1|2)) \]
where $\kappa$ is related to the parameters in the previous section by $\kappa = -(a + 3)/2, k = 1/2$. It has generators $e'/\kappa$, $f'/\kappa$, $h'/\kappa$ forming $(\mathfrak{sl}_2)_\kappa$ and fermionic generators $x'/\sqrt{\kappa}, y'/\sqrt{\kappa}$. Their OPEs can be read off [CGL20] p. 16 to be
\[
(x'/\sqrt{\kappa})(z)(x'/\sqrt{\kappa})(w) \sim -(e'/\kappa)(w)(z - w)^{-1}
\]
\[
(y'/\sqrt{\kappa})(z)(y'/\sqrt{\kappa})(w) \sim (f'/\kappa)(w)(z - w)^{-1}
\]
\[
(x'/\sqrt{\kappa})(z)(y'/\sqrt{\kappa})(w) \sim (z - w)^{-2} + \frac{1}{2}(h'/\kappa)(w)(z - w)^{-1}
\]
An arbitrary fibre is defined by scalars $(e'/\kappa)_n, (f'/\kappa)_n, (h'/\kappa)_n$ and as in section 5.1 we define a $\mathfrak{sl}_2$-connection $d + A(z)$ with $A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}$ using the Killing form $(a/\kappa)_n = \langle A_n, a \rangle$, more explicitly for $\mathfrak{sl}_2$
\[ A_n = \frac{1}{4} f'(e'/\kappa)_n + \frac{1}{8} h'(h'/\kappa)_n + \frac{1}{4} e'(f'/\kappa)_n \]
We now introduce in the fibre $d + A$ the variables
\[ \psi^{d+A} := x'/\sqrt{\kappa}, \quad \bar{\psi}^{d+A} := y'/\sqrt{\kappa} \]
Then we find, by translating the singular part of OPE to (anti-)commutators
\[ [A_{-m-h_A}, B_{-n-h_B}] = \sum_{l \leq 0} \left( \frac{-m-1}{-l-1} \right) (A_{-l-h_A} B)_{-(n+m-l)-h_A-l-h_A} B \]

**Lemma 7.3.** The limit of $H^\kappa A^{(2)}[\mathfrak{sl}_2, \kappa]$ fibres over the space of regular $\mathfrak{sl}_2$-connections and the fibre over the point $d + \sum_{n \in \mathbb{Z}} A_{-n-1} z^n$ is given by the deformed mode algebra
\[
\{ \psi^{d+A}_m, \psi^{d+A}_n \} = -(A_{n+m}, f')
\]
\[
\{ \bar{\psi}^{d+A}_m, \bar{\psi}^{d+A}_n \} = (A_{n+m}, e')
\]
\[
\{ \psi^{d+A}_m, \bar{\psi}^{d+A}_n \} = \frac{1}{2} (A_{n+m}, h') + m \delta_{m,-n}
\]
We now study the operators $e'_0, f'_0, h'_0$. They are not part of the integral form, since $e'_0 = \kappa(e'_0/\kappa) \to \infty$, but they can be considered if $(e'_0/\kappa), (f'_0/\kappa), (h'_0/\kappa) = 0$, or respectively for a subset. From the remaining OPEs we read off easily that they form a well-defined Lie algebra $\mathfrak{sl}_2$ and that $x', y'$ transform as a duplett under this symmetry.

We now discuss Virasoro actions. The Virasoro action provided by the Sugawara construction of $\mathfrak{osp}(2|1)$ is
\[ (3 + 2\kappa)L^{\mathfrak{osp}(1|2)} = \frac{1}{2} : h' h' : + : e' f' : + : f' e' : - : x' y' : + : y' x' : \quad \kappa = -(a + 3)/2 \]
By construction we have the embedding
\[ (\mathfrak{sl}_2)_\kappa \times \text{Vir}_c \hookrightarrow \mathfrak{osp}(2|1)_\kappa \]
where $\text{Vir}_c$ is the reduction of $(\hat{\mathfrak{sl}}_2)_{-\frac{1}{2}k-1}$ for $k = 1/2$ and in the limit $(\hat{\mathfrak{sl}}_2)_{-\frac{1}{2}}$, so the central charge is $c = -2$. Explicitly, this embedding is given by the coset Virasoro action
\[
L_n^C = L_n^{\text{Vir}(1|2)} - L_n^\mathfrak{v} = \left(\frac{2(\kappa + 2)}{3 + 2\kappa} - 1\right) L_n^{\mathfrak{sl}_2} + \frac{2}{3 + 2\kappa} x y'z' : n
\]
In the limit $\kappa \to \infty$ the first term disappears which leaves the second term $\sum_{i+j=n} : \bar{\psi}_i \psi_j :$. Hence we precisely recover the Virasoro structure given by the energy-stress tensor of the symplectic fermions, for all fibres $d + A(z)$.

**Corollary 7.4.** The limit of $\text{H}^\ast \mathfrak{A}^{(2)}[\mathfrak{sl}_2, \kappa]$, fibres over the space of regular $\mathfrak{sl}_2$-connections, and the fibre over each connection $d + A(z)$ is isomorphic to a twist of symplectic fermions with the twisted mode algebra given above.

**Lemma 7.5.** We have
\[
\{ L_n^{d+A}, \psi_k^{d+A} \} = \sum_{i+j=n} \bar{\psi}_i^d A^{d+A} \bar{\psi}_j^{d+A} \psi_k^{d+A} - \psi_i^{d+A} \psi_k^{d+A} - \bar{\psi}_i^d A^{d+A} \psi_j^{d+A} - \bar{\psi}_i^d A^{d+A} \psi_j^{d+A} = 0
\]
and similarly for $\bar{\psi}$, where we can ignore the normally ordered product because the commutators are central. Differently spoken, the new Virasoro action $\text{L}_n^{d+A} = \frac{\partial}{\partial z} z^{n+1} + A(z) z^{n+1}$ acts on the vector-valued field with components $\psi(z)$, $\bar{\psi}(z)$. In particular $L_{-1}$ acts as the connection $d + A(z)$.

**Example 7.6.** The zero fibre coincides with symplectic fermions. The fibre over $d + A_0 z^{-1}$, assuming by conjugation that $A_0$ is contained in $\hat{\mathfrak{g}}^{>0}$, is isomorphic to the twisted sector in section 2.2.

**Example 7.7.** Let us consider the first type of irregular singularity
\[
A(z) = A_1 z^{-2} + A_0 z^{-1} + \cdots
\]
where we assume $A_1 \in \mathfrak{g}$ (called compatible framing in literature) and explicitly $A_1 = 4 \Xi h'$. As the easiest case let us assume $A_n = 0$, $n \leq 0$, then the twisted mode algebra is
\[
\{ \psi_m^{d+A}, \psi_n^{d+A} \} = 0
\]
\[
\{ \psi_m^{d+A}, \bar{\psi}_n^{d+A} \} = 0
\]
\[
\{ \psi_m^{d+A}, \bar{\psi}_n^{d+A} \} = \Xi \delta_{m+n,1} + m \delta_{m,n,0}
\]
In particular, in contrast to the regular singular case the $2^d$-dimensional subalgebra generated by $\psi_0, \bar{\psi}_0, \psi_1, \bar{\psi}_1$ is not any more commutative
\[
\{ \psi_0, \bar{\psi}_1 \} = \{ \bar{\psi}_0, \psi_1 \} = \Xi
\]
so that it admits no 1-dimensional representations, in particular no highest-weight vectors. Starting from the assumption of a vector \( v \) with \( \psi_1 v = \bar{\psi}_1 v = 0 \) we find for \( \Xi \neq 0 \) a 4-dimensional irreducible representation of this algebra (which is hence a full matrix algebra)

\[
\begin{pmatrix}
\bar{\psi}_1 & v & \psi_1 \\
\psi_0 & \bar{\psi}_0 & \psi_0 \\
\bar{\psi}_0 & \psi_0 & \bar{\psi}_0 \\
\psi_1 & \bar{\psi}_1 & \psi_1
\end{pmatrix}
\]

This can be trivially extended to all \( \psi_n, \bar{\psi}_n, n \geq 0 \), since all remaining generators are central, and induced up to full twisted representation. If we evaluate \( L_n, n \geq 0 \) on this top space we find

\[
L_n = 0, \quad n \geq 2
\]

\[
L_1 = \bar{\psi}_0 \psi_1 - \psi_0 \bar{\psi}_1
\]

\[
L_0 = \bar{\psi}_0 \psi_0 + (\bar{\psi}_{-1} \psi_1 - \psi_{-1} \bar{\psi}_1)
\]

Hence \( L_1 \) acts on \( v, \psi_0 \bar{\psi}_0 v \) by zero and on \( \psi_0 v, \bar{\psi}_0 v \), by eigenvalues \( \mp \Xi \) (a Whittaker-type behaviour), and \( L_0 \) acts on \( v, \psi_0 \bar{\psi}_0 v \) as a Jordan block (a logarithmic behaviour) and terms involving \( \psi_{-1}, \bar{\psi}_{-1} \), so the top space is not preserved, however these contributions are nilpotent, but they add higher correction terms on the true groundstates in a similar way the regular connection adds lower correction terms.

We also briefly describe the perspective of differential equations. Consider

\[
\left( \frac{d}{dz} - \Xi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) z^{-2} - \Lambda z^{-1} - A_{\text{reg}}(z) \left( \begin{array}{c} x(z) \\ y(z) \end{array} \right) = 0
\]

A substitution \( \Phi(z) z^\Lambda \exp \left( -\Xi z^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \) gives a power series approach for \( \Phi(z) \), with \( z^\Lambda \) causing monodromy around \( z = 0 \) prescribed by the regular singular term and \( \exp(\mp \Xi z^{-1}) \) causing an essential singularity at \( z = 0 \). In the easiest case \( \Lambda = 0, A_{\text{reg}}(z) = 0 \), corresponding to the twisted mode algebra above, and also for arbitrary diagonal \( \Lambda = (\lambda \ 0 \ \
lambda) \), where the involved matrices commute, we have explicit solutions

\[
\left( \begin{array}{c} x(z) \\ y(z) \end{array} \right) = \begin{pmatrix} \exp(-\Xi z^{-1}) z^\lambda & 0 \\ 0 & \exp(\Xi z^{-1}) z^{-\lambda} \end{pmatrix} \left( \begin{array}{c} x(z) \\ y(z) \end{array} \right)
\]

The next interesting and still explicit case is \( \Lambda = (\lambda \ b \ -\lambda) \), in which we can still quite explicitly determine, with substituting \( t = z^{-1} \)

\[
\left( \begin{array}{c} x(z) \\ y(z) \end{array} \right) = \begin{pmatrix} \exp(-\Xi z^{-1}) z^\lambda & \exp(-\Xi z^{-1}) z^{\lambda} (-b \int \exp(2\Xi t) t^{2\lambda-1} \, dt) \\ 0 & \exp(\Xi z^{-1}) z^{-\lambda} \end{pmatrix} \left( \begin{array}{c} x(z) \\ y(z) \end{array} \right)
\]

For \( \lambda = 1/2 \)

\[
\left( \begin{array}{c} x(z) \\ y(z) \end{array} \right) = \begin{pmatrix} \exp(-\Xi z^{-1}) z^{1/2} & \exp(\Xi z^{-1}) z^{1/2} (-b / 2 \Xi) \\ 0 & \exp(\Xi z^{-1}) z^{-1/2} \end{pmatrix} \left( \begin{array}{c} x(z) \\ y(z) \end{array} \right)
\]

With this example in mind, we now briefly review the Stokes phenomenon and the irregular Riemann Hilbert correspondence [Boa14] in our case \( \mathfrak{sl}_2 \) and an irregular singularity of
order $z^{-2}$: The anti-Stokes directions in $\mathbb{C}^\times$ (separating the sectors which the differential equation has a defined asymptotic $z \to 0$) are the positive and negative real axis and the associated subgroups of Stokes multipliers are the unipotent subgroups $U^\pm$, see [Boa02] Section 2. The irregular Riemann-Hilbert map in [Boa02] Theorem 2, further discussed in [Boa14] is a bijection of isomorphism classes of connections with these fixed singular terms and the group of Stokes factors $U^+ \times U^-$, which describe the transformation behaviour of the solution between adjacent sectors. We would hope that our approach now systematically explains the observed relation of the quantum Weyl group to the quantum behaviour of the solution between adjacent sectors. We would hope that our approach now systematically explains the observed relation of the quantum Weyl group to the quantum group with a big center and to the negative definite case in Question 1.4.

7.3 The limit of the $N=1$ superconformal algebra and fermion

The further Hamiltonian reduction of $sVir^{1/2,a}_{N=1}$ with respect to $\hat{\mathfrak{sl}}_2$ and then $\hat{\mathfrak{sl}}_2'$ is calculated in [CGL20] Section 2.5:

**Theorem 7.8.** $H^2H^\kappa A^{(2)}[\mathfrak{sl}_2,\kappa]$ with respect to the BRST differential $d' = b'(z)e'(z) + b'(z)$ is isomorphic to the $N=1$ superconformal algebra $sVir_{N=1}$ with central charge $c = 3/2 + 3(a + 2 + a^{-1})^2$ and generators the cohomology classes of

$$L' = L^{\text{up}(2|1)} + \frac{1}{2} \partial h' : b' \partial e' : = -\frac{1}{2} : (\partial x')x' :$$

$$G_{-3/2} = \sqrt{-1} \frac{1}{\sqrt{3} + 2\kappa} : h'x' : + 2 : e'y' : -(1 + 2\kappa)e'\partial x'$$

times a free fermion with OPE

$$x'(z)x'(w) \sim -e'(z - w)^{-1} = 1(z - w)^{-1}$$

and the additional relation $e' + 1 = 0$ up to coboundary.

It is not difficult to compute from this and the previous integral form the limit, the fibration and zero-fibre. We feel, however that this second reduction maybe not precisely the right one for our purposes. In the following we try to use a rescaled character to impose the relation $e'/\kappa + 1 = 0$, which makes not difference for finite $\kappa$, but in the limit. The deeper reason for this is probably that we wish to have a reduction compatible with the classical Hamiltonian reduction in then limit, and this requires for $\mathfrak{sl}_2$ a connection

$$d - \begin{pmatrix} 0 & q(z) \\ 1 & 0 \end{pmatrix} \quad \iff \quad \frac{\partial^2}{\partial z^2} - q(z)$$

which corresponds choosing a subset of fibres with $(h'/\kappa)_n = 0$ and $(e'/\kappa)_n = \delta_{n,-1}(-1)$, which translates into the cohomological condition $e' = 1$. Let us modify the previous result accordingly and add some further information we require

**Theorem 7.9.** $H^2H^\kappa A^{(2)}[\mathfrak{sl}_2,\kappa]$ with the modified BRST differential $d' = b'(z)(e'/\kappa)(z) + b'(z)$ is isomorphic to the $N=1$ superconformal algebra $sVir_{N=1}$ with central charge $c = 3/2 + 3(a + 2 + a^{-1})^2 = \frac{-3(1 + 2\kappa)(5 + 4\kappa)}{2(3 + 2\kappa)}$, with generators $G, L'$ given in the proof, times a free fermion with generator

$$(x'/\sqrt{\kappa})(z)(x'/\sqrt{\kappa})(w) \sim (-e'/\kappa)(z - w)^{-1} = 1(z - w)^{-1}$$

where up to coboundary $e'/\kappa + 1 = 0$. 
Proof. Let $\rho$ be the Lie algebra automorphism of $\mathfrak{osp}(1|2)_\kappa$ defined by
\[
\rho(e') = e'/\kappa, \quad \rho(h') = h', \quad \rho(f') = f'/\kappa, \quad \rho(x') = x'/\sqrt{\kappa}, \quad \rho(y') = y'/\sqrt{\kappa}
\]
which we see also preserves the three Virasoro elements $L^{\mathfrak{osp}(1|2)}$, $L^{\mathfrak{sl}_2}$, $L^C$. We extend $\rho$ trivial on $h', e'$ to the BRST complex $\mathbb{V}^n(\mathfrak{osp}(1|2)) \otimes \langle b', c' \rangle$, then $\rho$ sends the differential to the modified differential $\rho^d \rho^{-1} = \tilde{d}'$. In particular $\tilde{d}'(e') = \rho(e' + 1) = e'/\kappa + 1$ is a coboundary as intended. In this way we can apply $\rho$ to elements in the kernel of $d'$, or cochains, which are given in the proof of [CGL20] Lemma 2.11 and obtain cochains for $\tilde{d}'$ and their OPEs:

- $x'/\sqrt{\kappa}$ is a cocycle for $\tilde{d}'$, since because $e', x' = 0, n \geq 0$, and the cohomology class $[x'/\sqrt{\kappa}]$ has with itself the OPE of a free fermion:
\[
[x'/\sqrt{\kappa}](z)[x'/\sqrt{\kappa}](w) = [-e'/\kappa](w)(z - w)^{-1} = (z - w)^{-1}
\]

- The usual corrected Sugawara Virasoro elements
\[
\tilde{L}^{\mathfrak{sl}_2} := L^{\mathfrak{sl}_2} + \frac{1}{2} \partial h' - : b' \partial e' :
\]
\[
\tilde{L}^{\mathfrak{osp}(1|2)} := L^{\mathfrak{osp}(1|2)} + \frac{1}{2} \partial h' - : b' \partial e' :
\]
remain unchanged and cocycles, as does without correction the coset Virasoro element
\[
\tilde{L}^C = L^{\mathfrak{osp}(1|2)} - L^{\mathfrak{sl}_2} \rightarrow: (y'/\sqrt{\kappa})(x'/\sqrt{\kappa}) :
\]

- The Virasoro element for the free fermion changed accordingly to $\frac{1}{2} : (\partial x'/\sqrt{\kappa})(x'/\sqrt{\kappa}) :$, as a consequence we have a definition slightly different from [CGL20]
\[
L' := \tilde{L}^{\mathfrak{osp}(1|2)} - \frac{1}{2} : (\partial x'/\sqrt{\kappa})(x'/\sqrt{\kappa}) :
\]
The new $[L']$ again commutes with $[x]$ up to $d'$-coboundary.

- The following element is a $d'$-cocycle (the according modification of what is called $\psi$ in [CGL20])
\[
G := \frac{\sqrt{-1}}{\sqrt{3 + 2\kappa}} \left( : h' x' : / \sqrt{\kappa} + 2 : e' y' : / \sqrt{\kappa} - (1 + 2\kappa) : e' \partial x' : / \kappa \sqrt{\kappa} \right)
\]
\[
= \frac{\sqrt{-1}}{\sqrt{3 + 2\kappa}} \left( : h' x' : / \sqrt{\kappa} - 2y' \sqrt{\kappa} + \frac{1 + 2\kappa}{\kappa} \partial x' / \sqrt{\kappa} \right)
\]
and its class commutes with $[x]$ and generates with $[L']$ a copy of $s\text{Vir}_{N=1}$ with central charge $-\frac{3(1 + 2\kappa)(5 + 4\kappa)}{2(3 + 2\kappa)}$.

We have in the familiar way a Virasoro element cohomologous to $\tilde{L}^{\mathfrak{sl}_2}$ that is not unchanged under $\rho$ and now reads
\[
\tilde{L}^{\mathfrak{sl}_2} = \frac{1}{\kappa + 2} \sum_{i+j=n} \frac{1}{2} h_i^i + (bc)_i \left( \frac{1}{2} h_j^j + (bc)_j \right) + \left( 1 - \frac{1}{\kappa + 2} \right) (-1 - n) \left( \frac{1}{2} h_n^n + (bc)_n \right) + \frac{\kappa}{\kappa + 2} f_{n+1}'
\]
Assume now we are acting on monomials not containing $b', c'$ or $h'$ then $(bc)_i = 0$ unless $i < 0$. and $h'_i = 0$ unless $i \leq 0$

\[
\tilde{L}_{1}^{st} = \frac{\kappa}{\kappa + 2} f'_2 \\
\tilde{L}_{0}^{st} = \frac{1}{\kappa + 2} \left( \frac{1}{4} (h'_0)^2 + (1 - \frac{1}{\kappa + 2}) \frac{1}{2} (-1) h'_0 + \frac{\kappa}{\kappa + 2} f'_1 \right) \\
\tilde{L}_{-1}^{st} = \frac{1}{\kappa + 2} 2 \left( \frac{1}{2} h'_{-1} + (bc)_{-1} h'_0 + \frac{\kappa}{\kappa + 2} f'_0 \right)
\]

We now compute the limit $\kappa \to \infty$ in the integral form generated by $x'/\sqrt{\kappa}, y'/\sqrt{\kappa}$. Note that $d'(y'/\sqrt{\kappa}) = b_{-1} x'_{-1}/\kappa^{3/2} \to 0$ so up to asymptotically vanishing terms also $y'/\sqrt{\kappa}$ is a cocycle. Note also $L_{n}^{st}/\kappa - f_{n+1}/\kappa \to 0$. For $h'/\kappa = 0$ we have

\[
[x'/\sqrt{\kappa}(z)] x'/\sqrt{\kappa}(z) = (z-w)^{-1} \\
[y'/\sqrt{\kappa}(z)] y'/\sqrt{\kappa}(z) = (L_{n-1}^{st}/\kappa)(z-w)^{-1} \\
[x'/\sqrt{\kappa}(z)] y'/\sqrt{\kappa}(z) = (z-w)^{-2}
\]

which is consistent with $q^2 \frac{\partial}{\partial z} + q(z)$ with $q(z) = \sum_{n \in \mathbb{Z}} L_{m}^{st} z^{2-n}$. The zero-fibre or $L_{n-1}^{st}/\kappa = 0$ is symplectic fermions

\[
\tilde{L}_{0}^{st} x' = \frac{1}{2} x' \\
\tilde{L}_{0}^{st} y' = \frac{1}{2} y' \\
\tilde{L}_{-1}^{st} x' = y'
\]

To match this with $s\text{Vir}_{N=1} \times \mathcal{F}$, note that $x'/\sqrt{\kappa}$ is the fermion and

\[
y'/\sqrt{\kappa} = -\sqrt{-1/2} G/\sqrt{\kappa} - \partial(x'/\sqrt{\kappa})
\]

Note that the OPE cannot be preserved by the action of $L_{1}$, which is (in hindsight) what one should expect from a Hamiltonian reduction.

**Remark 7.10.** Note that we could also use the reduction $e' + 1 = 0$, at the price of loosing the classical Hamiltonian reduction picture including the remaining $\mathfrak{sl}_2$-action. On the other hand, then the fibre would again be symplectic fermions without any deformations, since $e'/\kappa \to 0$. We have no guess as to if this is interesting to consider.

### 7.4 The limit of the N=1 superconformal algebra and fermion II

We consider the same case as in the previous section but now follow closely the explicit computation by joint work of the first author in [BBFLT13] Section 3.2.1, which is related to the supersymmetric version of the GKO construction. We match these quite
Consider the vertex algebra $\mathcal{V} = s\text{Vir}_{N=1} \otimes \mathcal{F}$, where $s\text{Vir}_{N=1}$ is the Neveu-Schwartz sector of the super-Virasoro algebra with generators $L_n, G_r$, where $r \in \frac{1}{2} + \mathbb{Z}$ in the Neveu-Schwartz sector, and relations
\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{8}c(n^3 - n)\delta_{n+m}
\]
\[
\{G_r, G_s\} = 2L_{r+s} + \frac{1}{2}c(r^2 - \frac{1}{4})\delta_{r+s}
\]
\[
[L_n, G_r] = (\frac{1}{2} - n)G_{n+r}
\]
and $\mathcal{F}$ an additional fermion $(f_r, r \in \frac{1}{2} + \mathbb{Z})$
\[
\{f_r, f_s\} = \delta_{r+s}
\]
\[
[L_n, f_s] = 0
\]
\[
\{G_r, f_s\} = 0
\]
For the representation theory of $s\text{Vir}_{N=1}$ see [IK03]. We consider highest weight modules of $s\text{Vir}_{N=1}$, generated by a vector $|P\rangle$ with $L_0|P\rangle = \Delta_P|P\rangle$ for $\Delta_P = \frac{1}{2}(Q^2/4 - P^2)(Q/2 - P)$ and $L_n|P\rangle = 0, n > 0$, $G_r|P\rangle = 0, r > 0$, $f_n|P\rangle = 0, r > 0$.
Now, similar to the previous chapter, we have two explicit commuting actions of on $\text{Vir}^{b_1}, \text{Vir}^{b_2}$ on $s\text{Vir}_{N=1} \otimes \mathcal{F}$ with
\[
b_1 = \frac{2b}{\sqrt{2 - 2b^2}}, b_2^{-1} = \frac{2b^{-1}}{\sqrt{2 - 2b^{-2}}}
\]
fulfilling the relation $b_1^2 + b_2^{-2} = -2$.
\[
L_n^{(b_1)} = \frac{1}{1 - b^2} - \frac{1 + 2b^2}{2(1 - b^2)} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r : f_{n-r}f_r : + \frac{b}{1 - b^2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} : f_{n-r}G_r :
\]
\[
L_n^{(b_2)} = \frac{1}{1 - b^{-2}}L_n - \frac{1 + 2b^{-2}}{2(1 - b^{-2})} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r : f_{n-r}f_r : + \frac{b^{-1}}{1 - b^{-2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} : f_{n-r}G_r :
\]
We note $L_n^{(b_1)} + L_n^{(b_2)} = L_n^b + L_n^F$, where $L_n^F = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r : f_{n-r}f_r :$ is the usual Virasoro action for $\mathcal{F}$. 

56
Example 7.11. We compute $L^{(b_1)}_n$ on the first few elements, the respective expressions for $L^{(b_1)}_n$ are obtained by replacing $b \leftrightarrow b^{-1}$

$$L^{(b_1)}_0(P) = \frac{1}{1 - b^2} \Delta_P |P\rangle$$
$$L^{(b_1)}_0L^{-1}_{-1}|P\rangle = \frac{1}{1 - b^2} (\Delta_P + 1)L^{-1}_{-1}|P\rangle$$
$$L^{(b_1)}_0 f_{-\frac{1}{2}}|P\rangle = -\left(\frac{1 + 2b^2}{2(1 - b^2)} - \frac{1}{2}\right)f_{-\frac{1}{2}}|P\rangle + \left(\frac{b}{1 - b^2}\right)(-1)G_{-\frac{1}{2}}|P\rangle$$
$$L^{(b_1)}_0G_{-\frac{1}{2}}|P\rangle = \frac{1}{1 - b^2} (\Delta_P + \frac{1}{2})G_{-\frac{1}{2}}|P\rangle + \left(\frac{b}{1 - b^2}\right)(\frac{1}{2}c_0 + 2\Delta_P)f_{-\frac{1}{2}}|P\rangle$$

$$L^{(b_1)}_1|P\rangle = 0$$
$$L^{(b_1)}_1 f_{-\frac{1}{2}}|P\rangle = 0$$
$$L^{(b_1)}_1 f_{-\frac{3}{2}}|P\rangle = -\left(\frac{1 + 2b^2}{2(1 - b^2)} - \frac{3}{2}\right)f_{-\frac{3}{2}}|P\rangle + \left(\frac{b}{1 - b^2}\right)(1)G_{-\frac{3}{2}}|P\rangle$$
$$L^{(b_1)}_1G_{-\frac{3}{2}}|P\rangle = 0$$

$$L^{(b_1)}_{-1}|P\rangle = \frac{1}{1 - b^2} L_{-1}|P\rangle + \left(\frac{b}{1 - b^2}\right)G_{-\frac{1}{2}}L_{-\frac{1}{2}}|P\rangle$$
$$L^{(b_1)}_{-1} f_{-\frac{1}{2}}|P\rangle = \frac{1}{1 - b^2} f_{-\frac{1}{2}}L_{-1}|P\rangle - \left(\frac{1 + 2b^2}{2(1 - b^2)} - \frac{3}{2}\right)f_{-\frac{1}{2}}|P\rangle + \left(\frac{b}{1 - b^2}\right)(1)G_{-\frac{3}{2}}|P\rangle$$
$$L^{(b_1)}_{-1}G_{-\frac{1}{2}}|P\rangle = \frac{1}{1 - b^2} G_{-\frac{1}{2}}L_{-1}|P\rangle + \left(\frac{b}{1 - b^2}\right)(2f_{-\frac{1}{2}}L_{-1} + f_{-\frac{3}{2}}\Delta_P)|P\rangle$$

Example 7.12. For example, let $\Delta_P = 0$, then highest weight vectors of both $L^{(b_1)}, L^{(b_2)}$ are

$$|0\rangle, \quad G_{-\frac{1}{2}}|0\rangle, \quad f_{-\frac{1}{2}}|0\rangle + \frac{1}{b + b^{-1}}G_{-\frac{1}{2}}|0\rangle$$

with $L^{(b_1)}_0$-eigenvalues

$$0, \quad \frac{1/2}{1 - b^2}, \quad -\frac{1 + 2b^2}{2(1 - b^2)}$$
and correspondingly $L_0^{b_1} + L_0^{b_2}$-eigenvalues

\[
0, \quad \frac{1/2}{1 - b^2} + \frac{1/2}{1 - b^{-2}} = 1/2 \quad -\frac{1 + 2b^2}{2(1 - b^2)} - \frac{1 + 2b^{-2}}{2(1 - b^{-2})} = -\frac{1 + 2b^2}{2(1 - b^2)} + \frac{b^2 + 2}{2(1 - b^2)} = 1/2
\]

and they are obviously a basis of the $L_{(b_1)} + L_{(b_2)}$-eigenspace with eigenvalue 0 and 1/2. They are the highest weight vectors of the defining decomposition of a generic module. In the vacuum module we have $L_{-1}(0) = G_{-1/2}(0) = 0$.

Note for the following discussion that in the limit

\[
L_0^{(b_1)} : 0, 0, 1 \quad L_0^{(b_1)} : 0, \frac{1}{2}, -\frac{1}{2}
\]

We now want to consider the limit $b \sim 2b_2 \to \infty$. We choose the integral form generated by $f_r, G_r/b$. We first compute the elements in the large central subalgebra generated by $\ell_n = L_n^{(b_2)} / b_2$. As in the case $g, p = 1$ we find

\[
L_{(b_2)}^{b_2} / b_2 = 4L_n / b^2
\]

Hence the fibres $V^\infty|_{q(z)}$ for $q(z) = \sum_{n \in \mathbb{Z}} \ell_n z^{-n-2}$ are generated by $G_r, f_n$.

We observe that in the semiclassical limit of sVir$_{n=1}$ we have well-defined Lie brackets between $L_{-1}, L_0, L_1$ and $G_{-1/2}, G_{1/2}$ which form Lie superalgebra $\mathfrak{osp}(1|2)$. The $\mathfrak{sl}_2$-action on the generators, which is well-defined for $\ell_1 = \ell_0 = \ell_{-1} = 0$ can be obtained as limit of the calculations in Example 7.11 for the vacuum module, and we find a duplett

\[
L_{(b_2)}^{(b_2)} f_{-1/2} = (-\frac{1}{2}) f_{-1/2} \\
L_{(b_2)}^{(b_2)} f_{-1/2} = -f_{-3/2} + G_{-3/2}/b \\
L_0^{(b_2)}(-f_{-3/2} + G_{-3/2}/b) = (+\frac{1}{2})(-f_{-3/2} + G_{-3/2}/b)
\]

with conformal 1 with respect to the new Virasoro action $L_0^{(b_1)}$. Let us now abbreviate these doublet elements

\[
x = f_{-1/2}, \quad x(z) = \sum_{n \in \mathbb{Z}} f_{n+1/2} z^{-n-1} \\
y = -f_{-3/2} + G_{-3/2}/b, \quad y(z) = \sum_{n \in \mathbb{Z}} (n f_{n-1/2} + G_{n-1/2}/b) z^{-n-1}
\]

The relations of the mode algebra are

\[
\{x_m, x_n\} = \{f_{m+1/2}, f_{n+1/2}\} = \delta_{m+n, -1} \\
\{x_m, y_n\} = \{f_{m+1/2}, n f_{n-1/2}\} = \delta_{m-n, n} \\
\{y_m, y_n\} = \{m f_{m-1/2}, n f_{n-1/2}\} + \{G_{m-1/2}/b, G_{n-1/2}/b\} \\
= mn \delta_{m+n, 1} + \left(2L_{m+n-1} / b^2 + \frac{1}{2} c((m - \frac{1}{2})^2 - \frac{1}{4}) / b^2\right) \delta_{m+n, 1} = 2\ell_{m+n-1}
\]

with $c \sim 2b^2$. 

58
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