SERRE PRESENTATIONS OF LIE SUPERALGEBRAS

R.B. ZHANG

ABSTRACT. An analogue of Serre’s theorem is established for finite dimensional simple Lie superalgebras, which describes presentations in terms of Chevalley generators and Serre type relations relative to all possible choices of Borel subalgebras. The proof of the theorem is conceptually transparent; it also provides an alternative approach to Serre’s theorem for ordinary Lie algebras.

1. INTRODUCTION

1.0.1. A well known theorem of Serre gave presentations of finite dimensional semisimple Lie algebras in terms of Chevalley generators and Serre relations. It was generalised to Kac-Moody algebras with symmetrisable Cartan matrices by Gabber and Kac [9]. The theorem and its generalisation now provide the standard method to present simple Lie algebras and Kac-Moody algebras [14], as well as the associated quantised universal enveloping algebras [4, 12].

A natural question is how to present simple contragredient Lie superalgebras (i.e., Lie superalgebras with Cartan matrices) in a similar way. Surprisingly this was only seriously studied after quantised universal enveloping superalgebras [2] had become popular in the early 90s because of their applications in a variety of areas such as low dimensional topology [20, 29], statistical physics [2] and noncommutative geometry [22, 30, 31].

In the Lie superalgebra setting, unconventional higher order relations [19] are required beside the usual Serre relations, and their origin is somewhat mysterious. Since a Serre type presentation is always given relative to a chosen Borel subalgebra, the issue is further complicated by the fact [13, 14] that a simple contragredient Lie superalgebra admits classes of Borel subalgebras, which are not Weyl group conjugate.

1.0.2. At the present, investigation on Serre type presentations for Lie superalgebras is still rather incomplete even in the finite dimensional case. Presentations relative to many non-distinguished Borel subalgebras of such Lie superalgebras have never been constructed (see Remark 3.4). The crucial question on whether the Serre type relations obtained so far are complete (i.e., whether they are all the defining relations needed for...
for the Lie superalgebras under consideration) has not been answered satisfactorily. Therefore, there is the need of a systematic treatment of Serre presentations for the finite dimensional simple contragredient Lie superalgebras, and this paper aims to provide such a treatment.

1.0.3. It was Leites and Serganova [19] who first obtained the higher order Serre relations for $\mathfrak{sl}_{m|n}$ relative to the so-called distinguished Borel subalgebra (for which the simple roots are the easiest to describe). The corresponding quantum relations for $U_q(\mathfrak{sl}_{m|n})$ were constructed in [24, 5]. Yamane [26] wrote down higher order quantum Serre relations for quantised universal enveloping superalgebras of finite dimensional simple Lie superalgebras for the distinguished and some (but not all) non-distinguished Borel subalgebras. In the ensuing years, much further work was done to find Serre type relations for Lie superalgebras by Leites and collaborators [6, 7, 1] and by Yamane [27].

References [6, 7] and [26, 27] represent the current state of the problem of constructing Serre type presentations for the finite dimensional simple contragredient Lie superalgebras. [Reference [27] is largely on affine superalgebras.] However, the papers [26, 27] left out presentations of exceptional simple Lie superalgebras relative to non-distinguished Borel subalgebras. Reference [6] in principle treated all the Dynkin diagrams which could potentially require higher order Serre relations, but the relations in [6] and [26, 27] look very different and it is not clear at all whether they are equivalent.

1.0.4. The problem on whether the Serre type relations constructed were complete was only investigated by computer calculations. According to [6 §1], completeness of the relations of [6] was verified by computers for finite dimensional simple contragredient Lie superalgebras, but a conceptual proof is lacking. The problem is open for the Serre type relations given in [26, 27], and so is also in the infinite dimensional case.

We comment that in the cases considered in [26], completeness of the relations can in principle be deduced from the existence of a non-degenerate invariant bilinear form between the quantised universal enveloping superalgebras of the upper and low triangular Borel subalgebras, by using Geer’s result [10] that quantised universal enveloping superalgebras are trivial deformations. However, it is a highly complicated matter to establish the non-degeneracy of the bilinear form even in the case of ordinary quantised universal enveloping algebras (see, e.g., [21]). Many of the representation theoretical results required for proving the non-degeneracy are lacking for quantised universal enveloping superalgebras, rendering the super case much more difficult.

1.0.5. In this paper, we give a complete treatment of the Serre presentations of finite dimensional simple contragredient Lie superalgebras, proving an analogue of Serre’s theorem relative to all possible choices of Borel subalgebras. Comparing our results with those of [26] (in the $q \to 1$ limit), we have many more higher order Serre relations which are necessary, especially in the case of exceptional Lie superalgebras relative to
non-distinguished Borel subalgebras. Our method is also different from those in the literature. It in particular automatically shows the completeness of the relations which we construct.

1.0.6. Let us now describe more precisely the results of this paper. Given a realisation of the Cartan matrix $A = (a_{ij})$ of a simple contragredient Lie superalgebra with the set of simple roots $\Pi_0 = \{\alpha_1, \ldots, \alpha_r\}$, we introduce an auxiliary Lie superalgebra $\tilde{g}$, which is generated by Chevalley generators $\{e_i, f_i, h_i \mid i = 1, 2, \ldots, r\}$ subject to quadratic relations only (see Definition 3.1 where more informative notation is used). Let $\tau$ be the $\mathbb{Z}_2$-graded maximal ideal of $\tilde{g}$ that intersects trivially the Cartan subalgebra spanned by all $h_i$. Then $L := \tilde{g}/\tau$ is the simple Lie superalgebra which we started with in all cases except in type $A(n, n)$ where $L$ is $sl_{n+1}|n+1$ (see Theorem 3.3).

We introduce a $\mathbb{Z}_2$-graded ideal $s$ of the auxiliary Lie superalgebra, which is generated by explicitly given generators. A main result proved in Theorem 3.10 states that $s = \tau$, or equivalently, $g := \tilde{g}/s \cong L$. From this result, we deduce a super analogue of Serre’s theorem, Theorem 3.11 which gives presentations of the finite dimensional simple contragredient Lie superalgebras relative to all possible choices of Borel subalgebras. The completeness of the relations in Theorem 3.11 is guaranteed by Theorem 3.10.

1.0.7. The proof of Theorem 3.10 makes use of a $\mathbb{Z}$-grading of $\tilde{g}$, which descends to $L$ and $g$ to give $\mathbb{Z}$-gradings to these Lie superalgebras. Write $L = \bigoplus_k L_k$ and $g = \bigoplus_k g_k$ with respect the $\mathbb{Z}$-gradings. Lemma 3.8 states that $L_0 \cong g_0$ as Lie superalgebras and $L_k \cong g_k$ as $g_0$-modules for all $k \neq 0$. Then Theorem 3.10 follows from this lemma.

The unconventional Serre relations can now be understood as arising from two sources: the conditions for $g_{\pm 1}$ to be irreducible $g_0$-modules; and the requirement that $[g_{\pm 1}, g_{\pm 1}] = L_{\pm 2}$ and similar requirements at other degrees.

Recall that Yamane [27] used odd reflections [25] to find such relations. Leites and collaborators [19, 6] used homological algebra techniques and deduced relations from certain spectral sequences.

The approach developed here is quite different from the methods in [6, 7, 1] and in [26, 27] at both the conceptual and technical level. It has the advantage of automatically generating a complete set of relations that is minimal. Conceptually the approach is quite transparent in the sense that one can see how the defining relations arise. It also provides an alternative approach to Serre’s theorem for finite dimensional semi-simple Lie algebras, see Remark 5.2.

We also note that the proof in [9] of the generalised Serre theorem for Kac-Moody algebras with symmetrisable Cartan matrices relied on structural properties of Verma modules such as their embeddings, and also made use of the quadratic Casimir operator. The authors of both [27] and [6] commented on obstacles in generalising the proof to Lie superalgebras, especially difficulties related to the quadratic Casimir operator.
We may also add that one no longer has the properties of (generalised) Verma modules required by [9] in the context of Lie superalgebras, and this appears to be a more serious difficulty.

1.0.8. The organisation of the paper is as follows. Section 2 reviews Kac’s classification of finite dimensional simple classical Lie superalgebras [13], and also clarifies certain subtle points about Cartan matrices and Dynkin diagrams in this context. Section 3 contains the statements of the main results, Theorem 3.10 and Theorem 3.11, which give presentations of contragredient Lie superalgebras in arbitrary root systems. The proof of Theorem 3.10, which implies Theorem 3.11 as a corollary, is given by using the key lemma, Lemma 3.8. Sections 4 and 5 are devoted to the proof of the key lemma. An outline of the proof is given in Section 4.2 to explain its conceptual aspects. We end the paper with a discussion of possible generalisation of the method developed here to affine Kac-Moody superalgebras to construct Serre type presentations in Section 6.

Two appendices are also included. Appendix A gives the root systems and Dynkin diagrams of all simple contragredient Lie superalgebras [13, 8, 3]. The material is used throughout the paper, and is also necessary in order to make precise the description of Dynkin diagrams in non-distinguished root systems. Appendix B describes the structure of some generalised Verma modules of lowest weight type and their irreducible quotients, which enter the proof of Lemma 3.8.

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2. Finite dimensional simple Lie superalgebras

In this section, we present some background material, and clarify some tricky points about Cartan matrices and Dynkin diagrams of Lie superalgebras.

2.1. Finite dimensional simple Lie superalgebras. We work over the field \( \mathbb{C} \) of complex numbers throughout the paper.

2.1.1. Classification. A Lie superalgebra \( \mathfrak{g} \) is a \( \mathbb{Z}_2 \)-graded vector space \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) endowed with a bilinear map \([\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\), \((X,Y) \mapsto [X,Y]\), called the Lie superbracket, which is homogeneous of degree 0, graded skew-symmetric and satisfies the super Jacobian identity. The even subspace \( \mathfrak{g}_0 \) of a Lie superalgebra \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a Lie algebra in its own right, which is called the even subalgebra of \( \mathfrak{g} \). The odd subspace \( \mathfrak{g}_1 \) forms a \( \mathfrak{g}_0 \)-module under the restriction of the adjoint action defined by the Lie superbracket. If \( \mathfrak{g}_0 \) is a reductive Lie algebra and \( \mathfrak{g}_1 \) is a semi-simple \( \mathfrak{g}_0 \)-module, \( \mathfrak{g} \) is called classical [13, 23].

The classification of the finite dimensional simple Lie superalgebras was completed in the late 70s. The theorem below is taken from [13], which is still the best reference on Lie superalgebras. Historical information and further references on the classification can be found in [16, 17] (also see [23]).
The finite dimensional simple classical Lie superalgebras comprise of the simple contragredient Lie superalgebras

\[ A(m,n), \ B(0,n), \ B(m,n), \ m > 0, \ C(n), \ n > 2, \ D(m,n), \ m > 1, \ F(4), \ G(3), \ D(2, 1; \alpha), \ \alpha \in \mathbb{C}\backslash \{0, -1\}, \]

and simple strange Lie superalgebras \( P(n) \) and \( Q(n) \) (\( n \geq 1 \)).

The simple contragredient Lie superalgebras admit non-degenerate invariant bilinear forms, while the strange Lie superalgebras \( P(n) \) and \( Q(n) \) do not. In the remainder of the paper, we shall consider only contragredient simple Lie superalgebras.

The \( A, B, C \) and \( D \) series are essentially the special linear and orthosymplectic Lie superalgebras, which are familiar examples of Lie superalgebras. The exceptional Lie superalgebras \( F(4), G(3) \) and \( D(2, 1; \alpha) \) are less well-known, but one can understand their structures given the description of their roots in Appendix A.1.

Let \( g = g_{\bar{0}} \oplus g_{\bar{1}} \) be a simple contragredient Lie superalgebra, and choose a Cartan subalgebra \( h \) for \( g \), which by definition is just a Cartan subalgebra of \( g_{\bar{0}} \). Denote by \( g_{\alpha} \) the root space of the root \( \alpha \), and call \( \alpha \) even (resp. odd) if \( g_{\alpha} \subset g_{\bar{0}} \) (resp. \( g_{\alpha} \subset g_{\bar{1}} \)). Denote by \( \Delta_{0} \) and \( \Delta_{1} \) the sets of the even and odd roots respectively, and set \( \Delta = \Delta_{0} \cup \Delta_{1} \). Let \((\ , \ ) : h^{*} \times h^{*} \rightarrow \mathbb{C} \) denote the Weyl group invariant non-degenerate symmetric bilinear form on \( h^{*} \), where the Weyl group of \( g \) is by definition the Weyl group of \( g_{\bar{0}} \). A root \( \beta \) will be called isotropic if \((\beta, \beta) = 0 \). Note that all isotropic roots are odd.

A Borel subalgebra of \( g \) is a maximal soluble Lie super subalgebra containing a Borel subalgebra \( h \) for \( g \). A new feature in the present context is that Borel subalgebras are not always conjugate under the Weyl groups. All the conjugacy classes of Borel subalgebras were given in [13, pp. 51-52] [14, Proposition 1.2]. In particular, Kac described a particularly convenient Borel subalgebra, which he called distinguished, for each simple contragredient Lie superalgebra. We shall call a root system with the set of simple roots determined by this Borel subalgebra the distinguished root system. In this case, there exists only one odd simple root.

2.1.2. Cartan matrices and Dynkin diagrams. The precise forms of the Cartan matrices and Dynkin diagrams will be crucial in Section 3. However, there do not exist canonical definitions for them in the Lie superalgebra setting, thus we spell out the details of our definitions here.

Let \( \Pi_{\bar{0}} = \{ \alpha_{1}, \alpha_{2}, \ldots, \alpha_{r} \} \) be the set of simple roots of a simple contragredient Lie superalgebra \( g \) relative to a Borel subalgebra \( h \). The Cartan matrix and Dynkin diagram provide a convenient way to describe \( \Pi_{\bar{0}} \). We define a Cartan matrix in the following way. Denote by \( \Theta \subset \{1, 2, \ldots, r\} \) the subset such that \( \alpha_{t} \in \Delta_{1} \) for all \( t \in \Theta \). Let \( l_{m}^{2} \) be the minimum of \(|(\beta, \beta)|\) for all non-isotropic \( \beta \in \Delta \) if \( g \neq D(2, 1; \alpha) \). If \( g \) is \( D(2, 1; \alpha) \), let \( l_{m}^{2} \) be the minimum of all \(|(\beta, \beta)| > 0 \) (\( \beta \in \Delta \)), which are independent of the arbitrary
parameter \( \alpha \). Let
\[
\kappa = \begin{cases} 
0, & \text{if } g \text{ is of type } B, \\
1, & \text{otherwise}; 
\end{cases}
\]
d_i = \begin{cases} 
\frac{(\alpha_i, \alpha_i)}{2}, & \text{if } (\alpha_i, \alpha_i) \neq 0, \\
\frac{l^2}{2}, & \text{if } (\alpha_i, \alpha_i) = 0.
\end{cases}
\]

Introduce the matrices
\[
B = (b_{ij})_{i,j=1}^r, \quad b_{ij} = (\alpha_i, \alpha_j), \\
D = \text{diag}(d_1, \ldots, d_r),
\]
then the Cartan matrix \( A \) associated to the set of simple roots \( \Pi_b \) is defined by
\[
(2.1) \quad A = D^{-1}B.
\]

When it is necessary to indicate the dependence on \( \Theta \), we write \( (A, \Theta) \) for the Cartan matrix.

Note that if \( \alpha_i \) is non-isotropic, \( a_{ii} = \frac{2(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} \) is a non-positive integer for all \( t \). However, if \( \alpha_i \) is isotropic, then \( a_{ij} = \frac{2}{l^2}(\alpha_i, \alpha_j) \) can be an integer of any sign or zero (except in type \( D(2, 1; \alpha) \)). If \( b_{ij} \neq 0 \), we define
\[
(2.2) \quad \text{sgn}_{ij} = \text{sign of } b_{ij}.
\]

As we shall see in Section 2.2, these signs provide the additional information required to recover a Cartan matrix from its Dynkin diagram.

Remark 2.2. Our definition of the Cartan matrix differs from the usual one due to Kac [13]. In Kac’s definition, if \( b_{ss} = 0 \), then \( d_s = (\alpha_s, \alpha_s) \) for the smallest \( k \) such that \( d_s \neq 0 \). Note that in our definition, none of the signs \( \text{sgn}_{ij} \) is lost.

The Dynkin diagram associated with \( (A, \Theta) \) consists of \( r \) nodes, which are connected by lines. The \( i \)-th node is coloured white if \( i \notin \Theta \), black if \( i \in \Theta \) but \( \alpha_i \) is not isotropic, and grey if \( \alpha_i \) is isotropic.

If \( (A, \Theta) \) is of type \( D(2, 1; \alpha) \), the Dynkin diagram is obtained by simply connecting the \( i \)-th and \( j \)-th nodes by one line if \( a_{ij} \neq 0 \) and write \( b_{ij} \) at the line.

In all other cases, we join the \( i \)-th and \( j \)-th nodes by \( n_{ij} \) lines, where
\[
\begin{align*}
n_{ij} &= \max(|a_{ij}|, |a_{ji}|), & \text{if } a_{ii} + a_{jj} \geq 2; \\
n_{ij} &= |a_{ij}|, & \text{if } a_{ii} = a_{jj} = 0.
\end{align*}
\]

When the \( i \)-th and \( j \)-th nodes are not both grey, say, the \( i \)-th one is not grey, and connected by more than one lines, we draw an arrow pointing to the \( j \)-th node if \( -a_{ij} = 1 \) and pointing to the \( i \)-th node if \( -a_{ij} > 1 \).

The Dynkin diagrams of the simple contragredient Lie superalgebras are given in the tables in Appendix A.2.
2.2. Comments on Dynkin diagrams. From the Cartan matrices in our definition, one can recover the corresponding root systems. Dynkin diagrams also uniquely represent Cartan matrices, except in the cases of \( \mathfrak{osp}_{4|2} \) and \( \mathfrak{sl}_{2|2} \). The Dynkin diagrams of these superalgebras relative to the distinguished root systems are exactly the same, but the two Lie superalgebras are non-isomorphic.

This problem can be resolved by incorporating the signs \( sgn_{ij} \) into the Dynkin diagram, e.g., by placing \( sgn_{ij} \) at the line(s) connecting two grey nodes \( i \) and \( j \). Then the modified Dynkin diagram are respectively given by

\[
\mathfrak{sl}_{2|2}: \quad \begin{array}{c}
\bullet \quad \bullet
\end{array}, \quad \mathfrak{osp}_{4|2}: \quad \begin{array}{c}
\bullet \quad \bullet \quad \bullet
\end{array}.
\]

As we shall see, the signs enter the construction of higher order Serre relations.

In this paper we did not include the additional information of these signs in the definition of Dynkin diagrams, as they would make the diagrams look cumbersome. Also, there is no ambiguity about the signs in all the other Dynkin diagrams.

Similar signs were also discussed in [27].

Recall that if we remove a subset of vertices (i.e., nodes) and all the edges connected to these vertices from a Dynkin diagram of a semi-simple Lie algebra, we obtain the Dynkin diagram of another semi-simple Lie algebra of a smaller rank. This corresponds to taking regular subalgebras. In the context Lie superalgebras, the notion of regular subalgebras still exists, but some explanation is required at the level of Dynkin diagrams.

**Definition 2.3.** Call a sub-diagram \( \Gamma' \) of a Dynkin diagram \( \Gamma \) full if for any two nodes \( i \) and \( j \) in \( \Gamma' \), the edges between them in \( \Gamma \), the arrows on the edges, and also the \( b_{ij} \) labels of the edges when \( \Gamma \) is of type \( D(2,1;\alpha) \), are all present in \( \Gamma' \).

Consider for example the Dynkin diagram

\[
\begin{array}{c}
\bullet \quad \bullet
\end{array}
\]

of \( F(4) \), which has the following full sub-diagrams beside others:

\[
\begin{array}{c}
\bullet \quad \bullet
\end{array}, \quad \begin{array}{c}
\bullet \quad \bullet
\end{array}.
\]

Note that none of these appears in Tables 1 and 2.

The reason is that the sub-matrices in the Cartan matrix of \( F(4) \) associated with these full sub-diagrams are not Cartan matrices in the strict sense. The problem lies in the definition of \( a_{ij} \) when the node \( i \) is grey, which involves the number \( l_m \). The \( l_m \) for \( F(4) \) is not the correct ones for the full sub-diagrams. By properly renormalising the bilinear forms on the weight spaces associated with them, the full sub-diagrams can be cast into the form
which are respectively Dynkin diagrams for \(\mathfrak{sl}_{3|1}\) and \(\mathfrak{sl}_{2|1}\).

We call the Dynkin diagrams in Table 1 and Table 2 standard, and the ones like those in (2.4) non-standard.

We mention that if a Lie superalgebra \(\mathfrak{g}\) is contained as a regular subalgebra in another Lie superalgebra, defining relations of \(\mathfrak{g}\) can in principle be extracted from relations of the latter by considering sub-diagrams of Dynkin diagrams. However, this involves subtleties, as we have just discussed, and requires more care than hitherto exercised in the literature.

### 3. Presentations of Lie superalgebras

In this section, we generalise Serre’s theorem for semi-simple Lie algebras to contragredient Lie superalgebras, obtaining presentations for the Lie superalgebras in terms of Chevalley generators and defining relations.

#### 3.1. An auxiliary Lie superalgebra.

We start by defining an auxiliary Lie superalgebra following the strategy of [15]. Let \((A, \Theta)\) with \(A = (a_{ij})_{i,j=1}^{\infty}\) be the Cartan matrix of one of the simple contragredient Lie superalgebras relative to a given Borel subalgebra \(\mathfrak{b}\). Let \(\Pi_{\mathfrak{b}}\) be the set of simple roots relative to this Borel subalgebra.

**Definition 3.1.** Let \(\tilde{\mathfrak{g}}(A, \Theta)\) be the Lie superalgebra generated by homogeneous generators \(e_i, f_i, h_i (i = 1, 2, \ldots, r)\), where \(e_s, f_s\) for all \(s \in \Theta\) are odd while the rest are even, subject to the following relations

\[
\begin{align*}
[h_i, h_j] &= 0, \\
[h_i, e_j] &= a_{ij} e_j, \\
[h_i, f_j] &= -a_{ij} f_j, \\
[e_i, f_j] &= \delta_{ij} h_i, \quad \forall i, j.
\end{align*}
\]

Let \(\tilde{n}^+\) (resp. \(\tilde{n}^-\)) be the subalgebra generated by all \(e_i\) (resp. all \(f_i\)) subject to the relevant relations, and \(\mathfrak{h} = \bigoplus_{i=1}^{\infty} \mathfrak{c} h_i\), the Cartan subalgebra. Then it is well known and easy to prove (following the reasoning of [15, §1]) that \(\tilde{\mathfrak{g}}(A, \Theta) = \tilde{n}^+ \oplus \mathfrak{h} \oplus \tilde{n}^-\). The Lie superalgebra is graded \(\tilde{\mathfrak{g}}(A, \Theta) = \bigoplus_{\nu \in Q} \tilde{\mathfrak{g}}_{\nu}\) by \(Q = \mathbb{Z}\Pi_{\mathfrak{b}}\), with \(\tilde{\mathfrak{g}}_{0} = \mathfrak{h}\). Note that \(\tilde{n}^+\) (resp. \(\tilde{n}^-\)) is zero unless \(\nu \in Q_{\mathbb{N}}\), where \(\mathbb{N} = \{1, 2, \ldots\}\) and \(Q_{\mathbb{N}} = \mathbb{N}\Pi_{\mathfrak{b}}\), that is,

\[
\tilde{n}^+ = \bigoplus_{\nu \in Q_{\mathbb{N}}} \tilde{n}^+_{\nu}, \quad \tilde{n}^- = \bigoplus_{\nu \in Q_{\mathbb{N}}} \tilde{n}^-_{\nu}.
\]

Let \(\tau(\mathfrak{A}, \Theta)\) be the maximal \(\mathbb{Z}_2\)-graded ideal of \(\tilde{\mathfrak{g}}(A, \Theta)\) that intersects \(\mathfrak{h}\) trivially. Set \(\tau^\pm = \tau(\mathfrak{A}, \Theta) \cap \tilde{n}^\pm\). Then \(\tau(\mathfrak{A}, \Theta) = \tau^+ \oplus \tau^-\). The following fact follows from the maximality of \(\tau(\mathfrak{A}, \Theta)\).

**Lemma 3.2.** Let \(\Sigma = \Sigma^+ \cup \Sigma^-\) with \(\Sigma^\pm \subset \tilde{n}^\pm\) be a subset of \(\tilde{\mathfrak{g}}(A, \Theta)\) consisting of homogeneous elements. If \([f_i, \Sigma^+] \subset \mathbb{C}\Sigma^+\) and \([e_i, \Sigma^-] \subset \mathbb{C}\Sigma^-\) for all \(i\), then \(\Sigma \subset \tau(\mathfrak{A}, \Theta)\).
Proof. The given conditions on $\Sigma$ imply that the ideal generated by $\mathfrak{r}(A, \Theta) \cup \Sigma$ intersects $\mathfrak{h}$ trivially, hence must be equal to $\mathfrak{r}(A, \Theta)$ by the maximality of the latter. □

In particular, if $X^\pm \in \tilde{\mathfrak{n}}^\pm$ satisfy $[f_i, X^+] = 0$, and $[e_i, X^-] = 0$ for all $i$, then they belong to $\tilde{\mathfrak{n}}^\pm$ respectively.

Let us define the Lie superalgebra $L(A, \Theta) := \frac{\tilde{\mathfrak{g}}(A, \Theta)}{\mathfrak{r}(A, \Theta)}$.

We have the following result.

**Theorem 3.3.** Let $\mathfrak{g}$ be a finite dimensional simple contragredient Lie superalgebra, and let $(A, \Theta)$ be the Cartan matrix of $\mathfrak{g}$ relative to a given Borel subalgebra. Then $L(A, \Theta)$ is isomorphic to $\mathfrak{g}$ unless $\mathfrak{g} = \mathfrak{A}(n, n)$, and in the latter case $L(A, \Theta) \cong \mathfrak{sl}_{n+1} \oplus \mathfrak{sl}_{n+1}$.

Proof. This follows from Kac’s classification [13] of the simple contragredient Lie superalgebras (see Theorem 2.1) except in the case of $\mathfrak{A}(n, n)$. In the latter case, we have $\det A = 0$. Therefore, $L(A, \Theta)$ contains a 1-dimensional center, and the quotient of $L(A, \Theta)$ by the center is $\mathfrak{A}(n, n)$. Hence $L(A, \Theta)$ is isomorphic to $\mathfrak{sl}_{n+1} \oplus \mathfrak{sl}_{n+1}$.

3.2. **Main theorem.**

3.2.1. **Standard and higher order Serre elements.** Let us first define some elements of $\tilde{\mathfrak{g}}(A, \Theta)$, which will play a crucial role in studying the presentation of Lie superalgebras.

Call the following elements the standard Serre elements:

$(ad_{e_i})^{-a_{ij}}(e_j), \quad (ad_{f_i})^{-a_{ij}}(f_j), \quad \text{for } i \neq j, \text{ with } a_{ii} \neq 0 \text{ or } a_{ij} = 0;

[e_s, e_s], \quad [f_s, f_s], \quad \text{for } a_{ss} = 0.$

We also introduce higher order Serre elements if the Dynkin diagram of $(A, \Theta)$ contains full sub-diagrams of the following kind:

1. $\quad \begin{array}{c}
       \text{
       }
       \text{
       }
       \text{
       }
       \text{
       }$

   with $\text{sgn}_{ji} \text{sgn}_{rk} = -1$, the associated higher order Serre elements are

   $[e_t, [e_j, [e_t, e_k]]], \quad [f_t, [f_j, [f_t, f_k]]]$;

2. $\quad \begin{array}{c}
       \text{
       }
       \text{
       }
       \text{
       }$

   the associated higher order Serre elements are

   $[e_t, [e_j, [e_t, e_k]]], \quad [f_t, [f_j, [f_t, f_k]]]$;

3. $\quad \begin{array}{c}
       \text{
       }
       \text{
       }
       \text{
       }$

   the associated higher order Serre elements are

   $[e_t, [e_j, [e_t, e_k]]], \quad [f_t, [f_j, [f_t, f_k]]]$;
(4) \( \text{\includegraphics[width=0.8\textwidth]{diagram4}} \), the associated higher order Serre elements are
\[
[[e_j, e_l], [[e_j, e_l], [e_l, e_k]]],
[[f_j, f_l], [[f_j, f_l], [f_l, f_k]]];
\]

(5) \( \text{\includegraphics[width=0.8\textwidth]{diagram5}} \), the associated higher order Serre elements are
\[
[[e_i, [e_j, e_l]], [[e_i, e_l], [e_l, e_k]]],
[[f_i, [f_j, f_l]], [[f_i, f_l], [f_l, f_k]]];
\]

(6) \( \text{\includegraphics[width=0.8\textwidth]{diagram6}} \), the associated higher order Serre elements are
\[
[e_i, [e_s, e_l]] - [e_s, [e_l, e_i]],
[f_i, [f_s, f_l]] - [f_s, [f_l, f_i]];
\]

(7) \( \text{\includegraphics[width=0.8\textwidth]{diagram7}} \), which is a Dynkin diagram of \( F(4) \), the associated higher order Serre elements are
\[
[E, [E, [e_2, [e_3, e_4]]]],
[F, [F, [f_2, [f_3, f_4]]]],
\]
where \( E = [[e_1, e_2], [e_2, e_3]] \) and \( F = [[f_1, f_2], [f_2, f_3]] \);

(8) \( \text{\includegraphics[width=0.8\textwidth]{diagram8}} \), which is a Dynkin diagram of \( F(4) \), the associated higher order Serre elements are
\[
[[e_1, e_2], [[e_2, e_3], [e_3, e_4]] - [[e_2, e_3], [e_1, e_2], [e_3, e_4]],
[[f_1, f_2], [[f_2, f_3], [f_3, f_4]] - [[f_2, f_3], [f_1, f_2], [f_3, f_4]];
\]

(9) \( \text{\includegraphics[width=0.8\textwidth]{diagram9}} \), which only appears in Dynkin diagrams of \( F(4) \), the associated higher order Serre elements are
\[
[e_i, [e_j, [e_l, e_k]]],
[f_i, [f_j, [f_l, f_k]]];
\]

(10) \( \text{\includegraphics[width=0.8\textwidth]{diagram10}} \), which only appears in one of the Dynkin diagrams of \( F(4) \), the associated higher order Serre elements are
\[
2[e_i, [e_k, e_j]] + 3[e_j, [e_k, e_i]],
2[f_i, [f_k, f_j]] + 3[f_j, [f_k, f_i]];
(11) \[ \begin{array}{ccc}
1 & 2 & 3 \\
\hline
& \bullet & \circ \\
\end{array} \] which is one of the Dynkin diagrams of \( G(3) \), the associated higher order Serre elements are
\[\begin{align*}
[[e_1, e_2], [e_1, e_2], [e_1, e_2], [e_1, e_2]], \\
[[f_1, f_2], [f_1, f_2], [f_1, f_2], [f_1, f_2]],
\end{align*}\]

(12) \[ \begin{array}{ccc}
1 & 2 & 3 \\
\hline
& \bullet & \circ \\
\end{array} \] which is one of the Dynkin diagrams of \( G(3) \), the associated higher order Serre elements are
\[\begin{align*}
[[e_2, e_1], [e_3, e_2], [e_2, e_1], [e_2, e_1]], \\
[[f_2, f_1], [f_2, f_1], [f_2, f_1], [f_2, f_1]],
\end{align*}\]

(13) \[ \begin{array}{ccc}
1 & 2 & 3 \\
\hline
& \bullet & \circ \\
\end{array} \] which is one of the Dynkin diagrams of \( G(3) \), the associated higher order Serre elements are
\[\begin{align*}
[e_2, [e_3, e_1]] - 2[e_3, [e_2, e_1]], \\
[f_2, [f_3, f_1]] - 2[f_3, [f_2, f_1]],
\end{align*}\]

(14) \[ \begin{array}{ccc}
1 & \alpha & 2 \\
\hline
\end{array} \] which is one of the Dynkin diagram for \( D(2, 1; \alpha) \).

The higher order Serre elements are
\[\begin{align*}
\alpha[e_1, [e_2, e_3]] + (1 + \alpha)[e_1, e_3], \\
\alpha[f_1, [f_2, f_3]] + (1 + \alpha)[f_2, f_3],
\end{align*}\]

where we label the left, top and bottom nodes by 1, 2 and 3 respectively.

Remark 3.4. Cases (7) - (14) were not considered before in the literature.

Remark 3.5. The Dynkin diagrams of \( D(2, 1) \) and \( D(2, 1; \alpha) \) in their respective distinguished root systems are not among the full sub-diagrams listed above. Also, the diagram (9) above is a non-standard diagram of \( \mathfrak{sl}_3|_1 \) (see Section 2.2).

Denote by \( S^+(A, \Theta) \) (resp. \( S^-(A, \Theta) \)) the set of all the standard and higher order Serre elements (if defined) which involve generators \( e_k \) (resp. \( f_k \)) only. Set \( S(A, \Theta) = S^+(A, \Theta) \cup S^-(A, \Theta) \). We have the following result.

Lemma 3.6. The set \( S(A, \Theta) \) is contained in the maximal ideal \( \mathfrak{t}(A, \Theta) \) of \( \mathfrak{g} \).
Proof. Direct calculations show that
\[ [f_i, S^+ (A, \Theta)] \subset CS^+ (A, \Theta), \quad [e_i, S^- (A, \Theta)] \subset CS^- (A, \Theta), \quad \forall i. \]
Hence \( S(A, \Theta) \subset \tau(A, \Theta) \) by Lemma \[3.2\]. We leave out the details of the calculations. \( \square \)

Definition 3.7. Let \( s(A, \Theta) \) be the \( \mathbb{Z}_2 \)-graded ideal of \( \tilde{g}(A, \Theta) \) generated by the elements of \( S(A, \Theta) \).

Then \( s(A, \Theta) \subset \tau(A, \Theta) \) by Lemma \[3.6\]. Define the Lie superalgebra
\begin{equation} \tag{3.3} \begin{aligned} g(A, \Theta) := \frac{\tilde{g}(A, \Theta)}{s(A, \Theta)}. \end{aligned} \end{equation}
There exists a natural surjective Lie superalgebra map \( g(A, \Theta) \rightarrow L(A, \Theta) \). We shall show that it is in fact an isomorphism.

3.2.2. \( \mathbb{Z} \)-gradings. Let us discuss \( \mathbb{Z} \)-gradings for the Lie superalgebras \( g(A, \Theta) \) and \( L(A, \Theta) \). Fix a positive integer \( d \leq r \), where \( r \) is the size of \( A \). We assign degrees to the generators of \( \tilde{g}(A, \Theta) \) as follows:
\begin{equation} \tag{3.4} \begin{aligned} \deg(h_j) = 0, \quad \forall j, \\
\deg(e_i) = \deg(f_i) = 0, \quad \forall i \neq d, \\
\deg(e_d) = -\deg(f_d) = 1. \end{aligned} \end{equation}
This introduces a \( \mathbb{Z} \)-grading to the auxiliary Lie superalgebra \( \tilde{g}(A, \Theta) \), which is not required to be compatible with the \( \mathbb{Z}_2 \)-grading upon reduction modulo 2. In view of the \( Q \)-grading of \( \tilde{g}(A, \Theta) \) and \( \tau(A, \Theta) \), the maximal ideal \( \tau(A, \Theta) \) is \( \mathbb{Z} \)-graded. Since all elements in \( S(A, \Theta) \) are homogeneous with the \( \mathbb{Z} \)-grading, \( s(A, \Theta) \) is \( \mathbb{Z} \)-graded as well.

The Lie superalgebra \( L(A, \Theta) \) inherits a \( \mathbb{Z} \)-grading from \( \tilde{g}(A, \Theta) \) and \( \tau(A, \Theta) \). Write \( L(A, \Theta) = \bigoplus_{k \in \mathbb{Z}} L_k \). Since the roots of \( L(A, \Theta) \) are known, we have a detailed understanding of all \( L_k \) as \( L_0 \)-modules.

The Lie superalgebra \( g(A, \Theta) \) inherits a \( \mathbb{Z} \)-grading from \( \tilde{g}(A, \Theta) \) and \( s(A, \Theta) \). Write \( g(A, \Theta) = \bigoplus_{k \in \mathbb{Z}} g_k \), where \( g_k \) is the homogeneous component of degree \( k \). Note that \( g_1 \) (resp. \( g_{-1} \)) generates \( g_k \) (resp. \( g_{-k} \)) for all \( k > 0 \). Thus if \( g_p = 0 \) (resp. \( g_{-p} = 0 \)) for some \( p > 0 \), then \( g_q = 0 \) (resp. \( g_{-q} = 0 \)) for all \( q > p \). Also each \( g_k \) forms a \( g_0 \)-module in the obvious way.

We have the following result.

Lemma 3.8. There exist \( \mathbb{Z} \)-gradings for \( g(A, \Theta) \) and \( L(A, \Theta) \) determined by some \( d \) such that \( g_0 = L_0 \) as Lie superalgebras and \( g_k = L_k \) as \( g_0 \)-modules for all nonzero \( k \in \mathbb{Z} \).

This is the key lemma needed for establishing Theorem \[3.10\] below. Its proof is elementary but very lengthy, thus we relegate it to later sections. Here we consider some general properties of the Lie superalgebras \( g(A, \Theta) \) and \( L(A, \Theta) \), which will significantly simplify the proof of Lemma \[3.8\].
Recall that an anti-involution $\omega$ of a Lie superalgebra $\mathfrak{a}$ is a linear map on $\mathfrak{a}$ satisfying $\omega([X,Y]) = [\omega(Y),\omega(X)]$ for all $X,Y \in \mathfrak{a}$, and $\omega^2 = \text{id}_a$. The Lie superalgebra $\tilde{\mathfrak{g}}(A,\Theta)$ admits an anti-involution defined by

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(h_i) = h_i, \quad \forall i.$$ 

Note that $\omega(S^+) \subset -S^- \cup S^-$ and $\omega(S^-) \subset -S^+ \cup S^+$, where $S^\pm = S^\pm(A,\Theta)$ and $-S^\pm$ are respectively the sets consisting of the negatives of the elements of $S^\pm$. Therefore, $\omega$ descents to an anti-involution on $\mathfrak{g}(A,\Theta)$, which sends $\mathfrak{g}_k$ to $\mathfrak{g}_{-k}$ for all $k \in \mathbb{Z}$ and provides a $\mathfrak{g}_0$-module isomorphism between $\mathfrak{g}_{-k}$ and the dual space of $\mathfrak{g}_k$.

The anti-involution of $\tilde{\mathfrak{g}}(A,\Theta)$ also descends to an anti-involution of $L(A,\Theta)$, which maps $L_k$ to $L_{-k}$ for all $k \in \mathbb{Z}$, and provides an isomorphism between the $L_0$-module $L_{-k}$ and the dual $L_0$-module of $L_k$.

Therefore, if $\mathfrak{g}_0 = L_0$ and $\mathfrak{g}_k = L_k$ for all $k > 0$ as $\mathfrak{g}_0$-modules, the existence of the anti-involutions immediately implies that $\mathfrak{g}_{-k} = L_{-k}$ for all $k > 0$. Hence in order to prove Lemma 3.8 we only need to show that it holds for all $k > 0$.

The arguments above may be summarised as follows.

**Lemma 3.9.** If $\mathfrak{g}_0 = L_0$ as Lie superalgebras and $\mathfrak{g}_k = L_k$ for all $k > 0$ as $\mathfrak{g}_0$-modules, then Lemma 3.8 holds.

This result will play an essential role in the proof of Lemma 3.8.

#### 3.2.3. Main theorem

The following theorem is the main result of this paper.

**Theorem 3.10.** The Lie superalgebra $\mathfrak{g}(A,\Theta)$ coincides with $L(A,\Theta)$, or equivalently, the ideal $s(A,\Theta)$ of $\tilde{\mathfrak{g}}(A,\Theta)$ is equal to the maximal ideal $\tau(A,\Theta)$.

**Proof.** Note that Lemma 3.8 immediately implies the claim. Indeed, we have already shown in Lemma 3.7 that $s(A,\Theta) \subset \tau(A,\Theta)$, and this is an inclusion of $\mathbb{Z}$-graded ideals of $\tilde{\mathfrak{g}}(A,\Theta)$. If $s(A,\Theta) \neq \tau(A,\Theta)$, there would exist a surjective Lie superalgebra homomorphism $\mathfrak{g}(A,\Theta) \twoheadrightarrow L(A,\Theta)$ with a nonzero kernel. Thus for some $k$, the degree-$k$ homogeneous components of $L(A,\Theta)$ and $\mathfrak{g}(A,\Theta)$ are not equal. This contradicts Lemma 3.8.

---

#### 3.3. Presentations of Lie superalgebras

Since the generators of $s(A,\Theta)$ are known explicitly, Theorem 3.10 provides a presentation for each simple contragredient Lie superalgebra and $s_{n+1|n+1}$ in an arbitrary root system. We have the following result for the Lie superalgebra $L(A,\Theta)$.

**Theorem 3.11.** The Lie superalgebra $L(A,\Theta)$ is generated by the generators $e_i, f_i, h_i$ ($1 \leq i \leq r$), where $e_i$ and $f_i$ are odd if $i \in \Theta$, and even otherwise, subject to the quadratic relations

$$[h_i, h_j] = 0,$$

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,$$

$$[e_i, f_j] = \delta_{ij} h_i, \quad \forall i, j;$$

(3.5)
standard Serre relations
\[(ad_e)^{1-a_{ij}}(e_j) = 0,
\]
\[(3.6) \quad (ad_f)^{1-a_{ij}}(f_j) = 0, \quad \text{for } i \neq j, \text{ with } a_{ii} \neq 0 \text{ or } a_{ij} = 0;\]
\[\{e_1, e_i\} = 0, \quad \{f_1, f_i\} = 0, \quad \text{for } a_{ii} = 0;\]

and higher order Serre relations if the Dynkin diagram of \((A, \Theta)\) contains any of the following diagrams as full sub-diagrams:

(1) \begin{center}
\[\begin{array}{c}
\bullet \quad i \quad \bullet \quad j \quad \bullet \\
\end{array}\]
\end{center}
with \(\text{sgn}_i \text{sgn}_k = -1\), the associated higher order Serre relations are
\[\{e_t, [e_j, [e_t, e_i]]\} = 0, \quad \{f_t, [f_j, [f_t, f_i]]\} = 0;\]

(2) \begin{center}
\[\begin{array}{c}
\bullet \quad j \quad \bullet \quad i \quad \bullet \\
\end{array}\]
\end{center}
the associated higher order Serre relations are
\[\{e_t, [e_j, [e_t, e_i]]\} = 0, \quad \{f_t, [f_j, [f_t, f_i]]\} = 0;\]

(3) \begin{center}
\[\begin{array}{c}
\bullet \quad j \quad \bullet \quad i \quad \bullet \\
\end{array}\]
\end{center}
the associated higher order Serre relations are
\[\{e_t, [e_j, [e_t, e_i]]\} = 0, \quad \{f_t, [f_j, [f_t, f_i]]\} = 0;\]

(4) \begin{center}
\[\begin{array}{c}
\bullet \quad j \quad \bullet \quad i \quad \bullet \\
\end{array}\]
\end{center}
the associated higher order Serre relations are
\[\{[e_j, e_t], [e_j, e_i], [e_t, e_i]\} = 0,
\]
\[\{[f_j, f_t], [f_j, f_i], [f_t, f_i]\} = 0;\]

(5) \begin{center}
\[\begin{array}{c}
\bullet \quad j \quad \bullet \quad i \quad \bullet \\
\end{array}\]
\end{center}
the associated higher order Serre relations are
\[\{[e_i, [e_j, e_t]], [e_j, e_i], [e_t, e_i]\} = 0,
\]
\[\{[f_i, [f_j, f_t]], [f_j, f_i], [f_t, f_i]\} = 0;\]

(6) \begin{center}
\[\begin{array}{c}
\bullet \quad j \quad \bullet \\
\end{array}\]
\end{center}
the associated higher order Serre relations are
\[\{e_t, [e_s, e_i] - [e_s, [e_t, e_i]] = 0,
\]
\[\{f_t, [f_s, f_i] - [f_s, [f_t, f_i]] = 0;\]

(7) \begin{center}
\[\begin{array}{c}
\bullet \quad j \quad \bullet \quad i \quad \bullet \\
\end{array}\]
\end{center}
the associated higher order Serre relations are
\[\{E, [E, [e_2, [e_3, e_4]]] = 0,
\]
\[\{F, [F, [f_2, [f_3, f_4]]] = 0,
\]
where \(E = [[e_1, e_2], [e_2, e_3]]\) and \(F = [[f_1, f_2], [f_2, f_3]]\);
We have the following result.

When \((\Lambda, \Theta)\) is given in the distinguished root system, Theorem 3.11 simplifies considerably. We have the following result.
Theorem 3.12. Let \((A, \Theta)\) with \(\Theta = \{s\}\) be the Cartan matrix of a contragredient Lie superalgebra in the distinguished root system. Then \(L(A, \Theta)\) is generated by generators \(e_i, f_i, h_i (i = 1, 2, \ldots, r)\), where \(e_s\) and \(f_s\) are odd and the rest even, subject to the quadratic relations
\[ [h_i, h_j] = 0, \]
\[ [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \]
\[ [e_i, f_j] = \delta_{ij} h_i, \quad \forall i, j; \] 
standard Serre relations
\[ (ad_e)^{1-a_{ij}}(e_j) = 0, \]
\[ (ad_f)^{1-a_{ij}}(f_j) = 0, \quad \text{for } i \neq j, a_{ii} \neq 0; \]
and higher order Serre relations
\[ [e_s, [e_{s-1}, [e_s, e_{s+1}]]] = 0, \quad [f_s, [f_{s-1}, [f_s, f_{s+1}]]] = 0, \]
if the Dynkin diagram of \(A\) contains a full sub-diagram of the form
\[ \begin{array}{ccc}
  \bullet & - & \bullet \\
  \circ & - & \circ \\
  s-1 & s & s+1
\end{array} \quad \text{with } sgn_{s-1, s} sgn_{s, s+1} = -1, \quad \text{or} \quad \begin{array}{ccc}
  \bullet & - & \bullet \\
  \circ & - & \circ \\
  s-1 & s & s+1
\end{array}. \]

Remark 3.13. Note the importance of the signs \(sgn_{ij}\) in the above theorem. There are higher order Serre relations associated with the first Dynkin diagram in \((2.3)\), but none with the second. The Dynkin diagrams in \((2.3)\) are respectively those of \(\mathfrak{sl}_{2|2}\) and \(\mathfrak{osp}_{4|2}\) in their distinguished root systems. The Lie superalgebra \(D(2, 1; \alpha)\) in the distinguished root system has no higher order Serre relations either.

4. PROOF OF KEY LEMMA FOR DISTINGUISHED ROOT SYSTEMS

Throughout this section, we assume that the Cartan matrix \((A, \Theta)\) is associated with the distinguished root system of a simple Lie superalgebra. Thus \(\Theta\) contains only one element, which we denote by \(s\). To simplify notation, we write \(\tilde{\mathfrak{g}}(A)\) for \(\tilde{\mathfrak{g}}(A, \Theta)\), \(\mathfrak{g}(A)\) for \(\mathfrak{g}(A, \Theta)\), and \(L(A)\) for \(L(A, \Theta)\).

4.1. The proof. The proof of Lemma 3.8 will make essential use of Lemma 3.9. Define the \(\mathbb{Z}\)-gradings for \(\mathfrak{g}(A)\) and \(L(A)\) as in Section 3.2.2 by taking \(d = s\).

Lemma 4.1. As reductive Lie algebras, \(\mathfrak{g}_0 = L_0\).

Proof. In this case, both \(\mathfrak{g}_0\) and \(L_0\) are generated by purely even elements. Let \(\mathfrak{g}_0^\prime = [\mathfrak{g}_0, \mathfrak{g}_0]\) and \(L_0^\prime = [L_0, L_0]\) be the derived algebras. Then by Serre’s theorem for semi-simple Lie algebras \(\mathfrak{g}_0^\prime = L_0^\prime\). Now the claim immediately follows. \(\square\)

We now consider the \(\mathfrak{g}_0\)-modules \(\mathfrak{g}_1\) and \(L_1\).

Remark 4.2. For convenience, we continue to use \(e_i, h_i\) and \(f_i\) to denote the images of these elements in \(\mathfrak{g}(A)\).
Examine the following relations in \( g(A) \):

\[ (4.1) \quad [h_i, e_s] = a_{is} e_s, \quad [f_i, e_s] = 0, \quad (ad_{e_s})^{1-a_{is}} e_s = 0, \quad \forall i \neq s. \]

The first two relations imply that \( e_s \) is a lowest weight vector of the \( g_0 \)-module \( g_1 \), with weight \( \alpha_s \). Since \( a_{is} \) are non-positive integers for all \( i \neq s \), by [11, Theorem 21.4], the third relation implies that \( g_1 \) is an irreducible finite dimensional \( g_0 \)-module. The relations (4.1) also hold in \( L(A) \). This immediately shows the following result.

**Lemma 4.3.** Both \( g_1 \) and \( L_1 \) are irreducible \( g_0 \)-modules, and \( g_1 = L_1 \).

Note that \( g_2 \) is generated by \( g_1 \), that is \( g_2 = [g_1, g_1] \). By induction one can show that \( g_{k+1} = (ad_{g_1})^k (g_1) \) for all \( k \geq 1 \). If \( g_i = 0 \) for some \( i > 1 \), then \( g_j = 0 \) for all \( j \geq i \). We have the \( g_0 \)-module decomposition \( g_1 \otimes g_1 = S^2_2(g_1) \oplus \bigwedge^2_2(g_2) \), where \( S^2_2(g_1) \) denotes the second \( \mathbb{Z}_2 \)-graded symmetric power, and \( \bigwedge^2_2(g_1) \) the second \( \mathbb{Z}_2 \)-graded skew power, of \( g_1 \).

**Remark 4.4.** Throughout the paper, we use \( S^k_2(V) \) and \( \bigwedge^k_2(V) \) to denote the \( \mathbb{Z}_2 \)-graded symmetric and skew symmetric tensors of rank \( k \) in the \( \mathbb{Z}_2 \)-graded vector space \( V \), and \( S^k(V) \) and \( \bigwedge^k(V) \) to denote the usual symmetric and skew symmetric tensors of rank \( k \), ignoring the \( \mathbb{Z}_2 \)-grading of \( V \).

We have the following result:

**Lemma 4.5.** The Lie superbracket defines a surjective \( g_0 \)-map \( g_1 \otimes g_1 \rightarrow g_2, X \otimes Y \mapsto [X, Y] \). The \( g_0 \)-submodule \( S^2_2(g_1) \) is in the kernel of this map, and \( \bigwedge^2_2(g_1) \) is mapped surjectively onto \( g_2 \).

**Proof.** For any \( X, Y \in g_1 \), an element \( Z \in g_0 \) acts on \( X \otimes Y \) by

\[ Z \cdot (X \otimes Y) = [Z, X] \otimes Y + X \otimes [Z, Y]. \]

The Lie superbracket maps \( Z \cdot (X \otimes Y) \) to \( [[Z, X], Y] + [X, [Z, Y]] = [Z, [X, Y]] \). This proves the first claim. The second claim follows from the \( \mathbb{Z}_2 \)-graded skew symmetry of the Lie superbracket.

Therefore, the \( g_0 \)-map \( \Psi : \bigwedge^2_2(g_1) \rightarrow g_2 \) defined by the composition

\[ \bigwedge^2_2(g_1) \hookrightarrow g_1 \otimes g_1 \rightarrow g_2 \]

is also surjective, where the map on the left is the natural embedding. The structure of \( \bigwedge^2_2(g_1) \) as a \( g_0 \)-module can be understood; this enables us to understand the structure of \( g_2 \).

Recall that in the distinguished root systems, \( L_2 = 0 \) if \( L(A) \) is of type I, and \( L_2 \neq 0 \) but \( L_3 = 0 \) if \( L(A) \) is of type II. Thus in order to show that \( g_k = L_k \) for all \( k > 0 \), it remains to prove that \( g_2 = 0 \) if the Cartan matrix \( A \) is of type I, and \( g_2 = L_2 \) and \( g_3 = 0 \) if \( A \) is of type II. In view of Lemma 3.9 the proof of Lemma 3.8 is done once this is accomplished.
The rest of the proof will be based on a case by case study. Let us start with the type I Lie superalgebras.

4.1.1. The case of $\mathfrak{sl}_m|n$. If the Cartan matrix $A$ is that of $\mathfrak{sl}_m|n$, the Lie superalgebra $\mathfrak{g}(A)$ has $\mathfrak{g}_0 = \mathfrak{gl}_m \oplus \mathfrak{sl}_n$, and $\mathfrak{g}_1 \cong \mathbb{C}^m \otimes \mathbb{C}^n$ up to parity change, where $\mathbb{C}^m$ denotes the natural module for $\mathfrak{gl}_m$, and $\mathbb{C}^n$ denotes the dual of the natural module for $\mathfrak{sl}_n$. Assuming that both $m$ and $n$ are greater than 1. Then $\Lambda^2 \mathfrak{g}_1 = S^2(\mathbb{C}^m) \otimes S^2(\mathbb{C}^n) \oplus \Lambda^2(\mathbb{C}^m) \otimes \Lambda^2(\mathbb{C}^n)$.

The lowest weight vectors of the irreducible submodules are respectively given by

$$\nu(2) := e_s \otimes e_s;$$

$$\nu(1^2) := e_{s-1,s+2} \otimes e_{s+1} + e_{s,s+1} \otimes e_{s-1,s+2}$$

$$- (e_{s-1,s+1} \otimes e_{s,s+2} + e_{s,s+2} \otimes e_{s-1,s+1}),$$

where $s = m$, and

$$e_{s,s+1} = e_s, \quad e_{s+2,s+1} = [e_s, e_{s+1}],$$

$$e_{s-1,s+1} = [e_{s-1}, e_s], \quad e_{s-1,s+2} = [e_{s-1}, e_{s,s+2}].$$

We have $\Psi(\nu(2)) = [e_s, e_s] = 0$ by one of the standard Serre relations. It follows that the entire irreducible $\mathfrak{g}_0$-submodule $S^2(\mathbb{C}^m) \otimes S^2(\mathbb{C}^n)$ is mapped to zero. In particular, we have

$$[e_{s-1,s+2}, e_{s,s+1}] + [e_{s-1,s+1}, e_{s,s+2}] = 0. \tag{4.2}$$

The first term of (4.2) vanishes by the higher order Serre relation; this in turn forces the second term to vanish as well. Hence

$$\Psi(\nu(1^2)) = [e_{s-1,s+2}, e_{s,s+1}] - [e_{s-1,s+1}, e_{s,s+2}] = 0.$$

Therefore, $\nu(1^2)$ is in the kernel of $\Psi$, implying that the entire submodule $\Lambda^2(\mathbb{C}^m) \otimes \Lambda^2(\mathbb{C}^n)$ is mapped to zero by $\Psi$. This shows that $\mathfrak{g}_2 = 0$, and hence $\mathfrak{g}_k = 0$ for all $k \geq 2$.

Note that if $\min(m,n) = 1$, say, $n = 1$, $\Lambda^2 \mathfrak{g}_1$ is irreducible as $\mathfrak{g}_0$-module and is equal to $S^2(\mathbb{C}^m) \otimes \mathbb{C}$. The above proof obviously goes through but in a much simplified fashion.

Therefore, we have proved that $\mathfrak{g}_k = L_k$ for all $k \geq 2$ in the case $L(A) = \mathfrak{sl}_m|n$.

4.1.2. The case of $C(n+1)$ with $n > 1$. In this case, $\mathfrak{g}_0 = \mathfrak{sp}_{2n} \oplus \mathbb{C}$ and $\mathfrak{g}_1 = \mathbb{C}^{2n}$. The $\mathbb{Z}_2$-graded skew symmetric tensor $\Lambda^2 \mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module with the lowest weight vector $e_1 \otimes e_1$. Since $\Psi(e_1 \otimes e_1) = [e_1, e_1] = 0$ by the standard Serre relation, it immediately follows that $\mathfrak{g}_k = 0$ for all $k \geq 2$. 
4.1.3. The case of $D(m, n)$ with $m > 2$. In this case, $g_0 = gln_\mathbb{C} \oplus so_{2m}$, and $g_1$ is isomorphic to $C^n \otimes C^{2m}$ as $g_0$-module (up to parity) with $e_n$ being the lowest weight vector. Let us first assume that $n > 1$. Then we have

$$\bigwedge^2(g_1) = S^2(C^n) \otimes \left(\bigwedge^2(C^{2m}) \oplus \bigwedge^2(C^n) \otimes \bigwedge^2(C^{2m}) \oplus S^2(C^n) \otimes C\right).$$

Lowest weight vectors of the first two irreducible submodules can be explicitly constructed in exactly the same way as in the case of $sl_{m|n}$. The same arguments used there also show that the Lie superbracket maps both submodules to zero. Hence $g_2 \cong S^2(C^n) \otimes C$. Inspecting the roots of $D(m, n)$ given in Appendix A.1 we can see that $g_2 = L_2$.

Let us examine $g_2$ in more detail. We use notation from Appendix A.1 for roots of the Lie superalgebra $D(m, n)$. Let $X_{\delta_i \pm \epsilon_p}$, where $1 \leq i \leq n$ and $1 \leq p \leq m$, be a weight basis of $g_1$. Then in $g_2$, we have

$$[X_{\delta_i - \epsilon_p}, X_{\delta_j - \epsilon_q}] = [X_{\delta_i - \epsilon_q}, X_{\delta_j - \epsilon_p}] = 0, \quad \forall i, j, p, q,$$

and there exist scalars $c_{ij, pq}$ such that

$$[X_{\delta_i - \epsilon_p}, X_{\delta_j + \epsilon_p}] = c_{ij, pq}[X_{\delta_i - \epsilon_p}, X_{\delta_j - \epsilon_p}] \neq 0, \quad \forall i, j, p, q.$$ 

By multiplying the elements $X_{\delta_i \pm \epsilon_p}$ by appropriate scalars if necessary, we may assume

$$[X_{\delta_i - \epsilon_p}, X_{\delta_j + \epsilon_p}] = [X_{\delta_i - \epsilon_q}, X_{\delta_j + \epsilon_q}], \quad \forall i, j, p, q,$$

which we denote by $X_{\delta_i + \delta_j}$. Then the subset of $X_{\delta_i + \delta_j}$ with $1 \leq i \leq j \leq n$ forms a basis of $g_2$.

Now we consider $g_3$. It immediately follows from (4.3) that $[X_{\delta_i + \delta_j}, X_{\delta_k \pm \epsilon_p}] = 0$ for all $k, p$ and $i \leq j$, that is,

$$g_3 = [g_1, g_2] = 0.$$ 

Hence $g_k = 0$ for all $k \geq 3$.

When $n = 1$, the proof goes through much more simply. This completes the proof of Lemma 3.8 for the case of $D(m, n)$ with $m > 2$.

In contrast to the type I case, the complication here is that $g_3$ needs to be analysed separately as $g_2 \neq 0$.

4.1.4. The case of $D(2, n)$. In this case, $g_0 = gl_{\mathbb{C}} \oplus sl_2 \oplus sl_2$, and $g_1 = C^n \otimes C^2 \otimes C^2$. The $\mathbb{Z}_2$-graded skew-symmetric rank two tensor $\bigwedge^2(g_1)$ decomposes into the direct sum of four irreducible $g_0$-modules if $n > 1$:

$$\bigwedge^2(g_1) = L^n(2) \otimes L^2(2) \otimes L^1(2) \otimes L^0(2) \otimes L^n(1, 1) \otimes L^2(1, 1) \otimes L^0(1, 1) \otimes L^n(0) \otimes L^2(2) \otimes L^0(0) \otimes L^2(0).$$

If $n = 1$, then $L^n(1, 1) = 0$, the two modules in the middle are absent.
The lowest weight vectors of the first three submodules can be easily worked out. Below we give the explicit formulae for their images under the Lie superbracket. Let

\[ e_{s;5+1} = [e_s, e_{s+1}], \quad e_{s;5+2} = [e_s, e_{s+2}], \quad e_{s-1;5} = [e_{s-1}, e_s], \]

\[ e_{s-1;5+1} = [e_{s-1}, e_{s;5+1}], \quad e_{s-1;5+2} = [e_{s-1}, e_{s;5+2}]. \]

Then the images of the lowest weight vectors are given by

\[ (4.5) \quad [e_s, e_s], \quad [e_{s-1;5+1}, e_s] - [e_{s-1;5}, e_{s;5+1}], \quad [e_{s-1;5+2}, e_s] - [e_{s-1;5}, e_{s;5+2}]. \]

We have the Serre relation \([e_s, e_s] = 0\). This implies that the entire irreducible submodule \(L^n_{(2)} \otimes L^2_{(2)} \otimes L^2_{(2)}\) is mapped to zero by the Lie superbracket.

In the case \(n > 1\), this in particular implies

\[ [e_{s-1;5+1}, e_s] + [e_{s-1;5}, e_{s;5+1}] = 0, \quad [e_{s-1;5+2}, e_s] + [e_{s-1;5}, e_{s;5+2}] = 0. \]

Note that \([e_{s-1;5+1}, e_s] = 0\) and \([e_{s-1;5+2}, e_s] = 0\) are the two higher order Serre relations involving \(e_s\). Thus all the four terms on the left hand sides of the above equations should vanish separately. It then follows that the second and third elements in (4.5) are zero, that is, the lowest weight vectors of the irreducible submodules \(L^n_{(1,1)} \otimes L^2_{(2)} \otimes L^2_{(2)}\) and \(L^n_{(1,1)} \otimes L^2_{(0)} \otimes L^2_{(2)}\) are in the kernel of the Lie superbracket. Thus both irreducible submodules are mapped to zero by the Lie superbracket. The above analysis in vacuous if \(n = 1\).

Therefore, \(g_2 \cong L^n_{(2)} \otimes L^2_{(0)} \otimes L^2_{(0)}\), and this shows that \(g_2 \cong L_2\).

To analyse \(g_3\), we note that equation (4.4) still holds here as can be shown by adapting the arguments in the \(m > 2\) case. This completes the proof in this case.

4.1.5. The case of \(B(m,n)\). When \(m \geq 1\), the proof is much the same as in the case of \(D(m,n)\) with \(m > 2\). We omit the details.

If \(m = 0\), then \(g_0 = gl_n\), \(g_1 = \mathbb{C}^n\) and \(g_2 \cong \wedge^2(g_1) \cong L_2\). Every root vector in \(g_1\) is of the form \([X, e_s]\) for some positive root vector \(X \in g_0\), where \(s = n\). Thus it follows from the relation \((ad e_s)^3(e_{s-1}) = 0\) that \([g_1, [e_s, e_s]] = 0\). Since \([e_s, e_s]\) is a \(g_0\) lowest weight vector of \(g_2\), this implies \(g_3 = 0\).

Remark 4.6. The Lie superalgebra \(B(0,n)\) is essentially the same as the ordinary Lie algebra \(B_n\). As a matter of fact, the corresponding quantum supergroup is isomorphic to the smash product of \(U_q(B_n)\) with the group algebra of \(\mathbb{Z}_2^n [28] [18]\). The usual proof of Serre presentations for semi-simple Lie algebras (see, e.g., [11]) works for \(B(0,n)\). We gave the alternative proof here for the sake of uniformity.

4.1.6. The case of \(F(4)\). Let us order the nodes in the Dynkin diagram from the right to left:
We may express the simple roots as $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_3$ and $\alpha_4 = \frac{1}{2}\left(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3\right)$. The symmetric bilinear form on the weight space is defined in Appendix [A.1] where further details about roots of $F(4)$ are given.

The first three simple roots are the standard simple roots of $\mathfrak{so}_7$, thus $\mathfrak{g}_0 = \mathfrak{so}_7 \oplus \mathfrak{g}_1$. The subspace $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module, which has $e_4$ as a lowest weight vector, and restricts to the spinor module for $\mathfrak{so}_7$. Now $\wedge^2_2(\mathfrak{g}_1)$ decomposes into the direct sum of two irreducibles $\mathfrak{g}_0$-submodules, one of which is 1-dimensional, the other is 35-dimensional with lowest weight vector $e_4 \otimes e_4$.

The Serre relation $[e_4, e_4] = 0$ implies that the 35-dimensional submodule is in the kernel of the Lie superbracket, and hence $\mathfrak{g}_2$ is 1-dimensional. A basis element for $\mathfrak{g}_2$ is $E = [e_4, e_{\alpha_4 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3}]$.

For any weight $\beta$ of $\mathfrak{g}_1$, we use $e_{\beta} \in \mathfrak{g}_1$ to denote a basis vector of the associated weight space, and set $e_{\alpha_4} = e_4$. Then we have

$$[e_{\beta}, E] = 0, \quad \text{for all odd positive root } \beta.$$  

This is trivially true for $\beta = \alpha_4$ or $\alpha_4 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. For $\beta = \alpha_4 + \varepsilon_1$ or $\alpha_4 + \varepsilon_i + \varepsilon_j$ ($i \neq j$), we have $[e_{\beta}, E] = \left[[e_{\beta}, e_{\alpha_4}], e_{\alpha_4 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3}\right] = -\left[e_{\alpha_4}, [e_{\beta}, e_{\alpha_4 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3}]\right]$, where both terms vanish as they involve images in $\mathfrak{g}_2$ of elements in the 35-dimensional submodule of $\wedge^2_2(\mathfrak{g}_1)$. Therefore, $\mathfrak{g}_k = \{0\}$ for all $k \geq 3$.

4.1.7. The case of $G(3)$. In this case, $\mathfrak{g}_0$ is isomorphic to the reductive Lie algebra $G_2 \oplus \mathfrak{g}_1$, and $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module which restricts to the 7-dimensional irreducible $G_2$-module. The $\mathbb{Z}_2$-graded skew symmetric tensor $\wedge^2_2(\mathfrak{g}_1)$ decomposes into the direct sum $L(2\alpha_1) \oplus L(0)$ of two irreducible $\mathfrak{g}_0$-submodules. The submodule $L(2\alpha_1)$ has $e_1 \otimes e_1$ as lowest weight vector, thus its image under the Lie superbracket is zero by the Serre relation $[e_1, e_1] = 0$. The submodule $L(0)$ is 1-dimensional. Since the Lie superbracket maps $\wedge^2_2(\mathfrak{g}_1)$ surjectively to $\mathfrak{g}_2$, we immediately conclude that $\dim \mathfrak{g}_2 = 1$.

Let $X = e_{2\alpha_2 + \alpha_3}$ be the root vector of $G_2 \subset \mathfrak{g}$ associated with the positive root $2\alpha_2 + \alpha_3$. Then $e^+ := [X, [X, e_1]]$ is the highest weight vector of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module. Since $\mathfrak{g}_2$ is one-dimensional, it must be spanned by $E = [e_1, e^+]$.

If $e_{\beta} \in \mathfrak{g}_1$ is a weigh vector not proportional to $e_1$ or $e^+$, both $[e_{\beta}, e_1]$ and $[e_{\beta}, e^+]$ vanish since they lie in the image of $L(2\alpha_1) \subset \wedge^2_2(\mathfrak{g}_1)$ under the Lie superbracket. Hence $[e_{\beta}, E] = 0$. We also have $[e^+, e^+] = 0$, and the Serre relation $[e_1, e_1] = 0$. Thus $[e_1, E] = [e^+, E] = 0$. Therefore, $[\mathfrak{g}_1, E] = 0$, which implies $\mathfrak{g}_k = \{0\}$, for all $k \geq 3$.

4.1.8. The case of $D(2, 1; \alpha)$. We have $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{g}_1$, and $\mathfrak{g}_1 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. The tensor $\wedge^2_2(\mathfrak{g}_1)$ decomposes into the direct sum of two irreducible $\mathfrak{g}_0$-submodules,

$$\wedge^2_2(\mathfrak{g}_1) = L_{(2, 2)} \oplus L_{(1; 1, 1)}, \quad L_{(2, 2)} = L_{(2)} \otimes L_{(2)}, \quad L_{(1; 1, 1)} = L_{(1^2)} \otimes L_{(1^2)}.$$  

The notation here only reflects the $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$-module structure, as there is no need to specify the $\mathfrak{g}_1$-action explicitly (see Remark [4.7] below).
We have \( \dim L_{(2;2)} = 9 \) and \( \dim L_{(1^2;1^2)} = 1 \). The lowest weight vector for \( L_{(2;2)} \) is \( v(2) = e_1 \otimes e_1 \). Let

\[
e_- = e_1, \quad e_+ = [e_1, e_2], \quad e_{-+} = [e_1, e_3], \quad e_{++} = [e_-, e_3],
\]

\[
v(1^2) = e_- \otimes e_++ + e_+ \otimes e_- - e_- \otimes e_+ - e_+ \otimes e_-.
\]

Then the vector \( v(1^2) \) spans \( L_{(1^2;1^2)} \).

The Lie superbracket maps \( L_{(2;2)} \) to zero because \([e_1, e_1] = 0 \). Note that \([e_-, e_+] + [e_+, e_-] \) belongs to the image of \( L_{(2;2)} \), thus is zero. Hence \( g_2 \) is spanned by \( E = [e_-, e_+] \). Now it is easy to show that \([E, g_1] = 0 \).

**Remark 4.7.** This proof is essentially the same as that in the case of \( D(2, 1) \), except for that the \( g_1 \) subalgebra of \( g_0 \) acts on \( g_1 \) by different scalars in the two cases. However, this scalar is not important in the proof of Lemma 3.8, and that is the reason why we did not specify it explicitly.

### 4.2. Comments on the proof

Let us recapitulate the proof of Lemma 3.8 in the distinguished root systems.

1. By Lemma 3.9, the proof of Lemma 3.8 is reduced to showing that the parabolic subalgebras \( g(A)_{\geq 0} \) and \( L(A)_{\geq 0} \) are the same.
2. The elements \( \{ h_s \} \cup \{ h_i, f_i \mid i \neq s \} \) and those defining relations of \( g(A) \) obeyed by them give a Serre presentation for the reductive Lie algebra \( g_0 \).
3. Given item (2), it suffices to show that \( g(A)_{\geq 0} = \bigoplus_{k \geq 0} g_k \) and \( L(A)_{\geq 0} = \bigoplus_{k \geq 0} L_k \) are isomorphic as \( g_0 \)-modules.
4. Equation (4.1) gives the necessary and sufficient conditions for \( g_1 \) to be a finite dimensional irreducible \( g_0 \)-module with lowest weight \( \alpha_s \), hence \( g_1 = L_1 \) as \( g_0 \)-modules.
5. The standard and higher order Serre relations involving \( e_s \) are conditions imposed on \( g_0 \)-lowest weight vectors of \([g_1, g_1]\), which are the necessary and sufficient to guarantee that \( g_2 = L_2 \).
6. The fact that \( g_3 = 0 \) follows (trivially in the type I case) from the result on \( g_2 \) and graded skew symmetry of the Lie superbracket, thus no additional relations are required. The vanishing of \( g_3 \) implies that for all \( k \geq 3 \), \( g_k = 0 \), and hence \( g_k = L_k \).

In non-distinguished root systems, one can still prove Lemma 3.8 by following a similar strategy, as we shall see in the next section. However, there are important differences in several aspects.

There are many such \( \mathbb{Z} \)-gradings as defined in Section 3.2.2 for the Lie superalgebras \( g(A, \Theta) \) and \( L(A, \Theta) \). This works to our advantage.

Given any such \( \mathbb{Z} \)-grading \( g(A, \Theta) = \bigoplus_{k \in \mathbb{Z}} g_k \), the degree zero subspace \( g_0 \) forms a Lie superalgebra, which is not an ordinary Lie in general. Thus the requirement that
$g_1$ be an irreducible $g_0$-module is much more difficult to implement, and usually leads to unfamiliar higher order Serre relations.

In general $g_3 \neq 0$. In order for $g_k$ to be equal to $L_k$ for $k \geq 3$, higher order Serre relations are needed at degree $k \geq 3$.

5. Proof of Key Lemma for Non-Distinguished Root Systems

In this section we prove Lemma 3.8 in non-distinguished root systems by following a similar strategy as that in Section 4. In particular, Lemma 3.9 will be used in an essential way.

Assume that the Cartan matrix $A$ is of size $r \times r$. Fix a positive integer $d \leq r$, we consider the corresponding $\mathbb{Z}$-gradings for $g(A, \Theta)$ and $L(A, \Theta)$ defined in Section 3.2.2. We shall first establish that $g_0 = L_0$. Since the roots of $L(A, \Theta)$ are known explicitly (see Appendix A.1), we have a complete understanding of the $g_0$-module structure of every $L_k$. Thus once we have a description of the weight spaces of each $g_k$ as $g_0$-module for all $k > 0$, an easy comparison with the root spaces of $L_k$ will enable us to prove the key lemma.

Remark 5.1. In the proof of Lemma 3.8 given below, we shall only describe the weight spaces of $g_k$ ($k > 0$), and leave out the easy step of comparing them with those of $L_k$ in most cases.

For convenience, we introduce the parity map $p : \{1, 2, \ldots, r\} \rightarrow \{0, 1\}$ such that $p(i) = 1$ if $i \in \Theta$ and $p(i) = 0$ otherwise. Then $e_i$ and $f_i$ are odd if $p(i) = 1$, and even if $p(i) = 0$.

5.1. Proof in Type $A$. We use induction on the rank $r$ together with the help of Lemma 3.9 to prove Lemma 3.8 and Theorem 3.11.

If $r = 2$, the Dynkin diagram in the non-distinguished root system has two grey nodes. In this case, there exists no relation between $e_1$ and $e_2$, and $[e_1, e_2]$ is another positive root vector. Note that $[e_1, [e_1, e_2]] = 0$ and $[e_2, [e_1, e_2]] = 0$ by the graded skew symmetry of the Lie superbracket. Thus Lemma 3.8 is valid and $g(A, \Theta) = L(A, \Theta)$.

When $r > 2$, we take $d = r$. Then $g_0 = [g_0, g_0]$ is a special linear superalgebra of rank $r - 1$ by the induction hypothesis, and thus $g_0$ is a general linear superalgebra.

Define the following elements of $g_0$:

\[ X_{ij} = ad_{e_i} \cdots ad_{e_{i-2}}(e_{j-1}), \quad i < j \leq r, \]

where $X_{j,j+1} = e_j$. In view of the general linear superalgebra structure of $g_0$, we conclude that $g_1$ is isomorphic to the irreducible $g_0$-module with lowest weight $\alpha_r$ (which is in fact the natural module possibly upon a parity change) if and only if

\[ [X_{ik}, [X_{jr}, e_r]] = 0, \quad j \neq k. \]

By using the $g_0$-action, we can show that these conditions are equivalent to the relation

\[ [e_{r-1}, [e_{r-2}, e_{r-1}], e_r]] = 0 \]
and the relevant relations in (3.1). For \( p(r - 1) = 1 \), (5.2) is a higher order Serre relation associated with the sub-diagram with \( sgn_{r - 2, r - 1} = -sgn_{r - 1, r} \).

If \( p(r - 1) = 0 \), it can be derived from

\[
(5.3) \quad [e_{r - 1}, [e_{r - 1}, e_r]] = 0,
\]

which is a standard Serre relation.

Consider \( \wedge_5^2 g_1 \), which is an irreducible \( g_0 \)-module. The lowest weight vector is

\[
\begin{align*}
&\quad e_r \otimes e_r, \quad \text{if } p(r) = 1, \quad \text{or} \\
&\quad e_r \otimes [e_{r - 1}, e_r] - [e_{r - 1}, e_r] \otimes e_r, \quad \text{if } p(r) = 0.
\end{align*}
\]

Thus \( g_2 = 0 \) if and only if

\[
(5.4) \quad [e_r, e_r] = 0, \quad \text{if } p(r) = 1, \quad \text{or} \\
[ e_r, [e_r, e_{r - 1}] ] = 0, \quad \text{if } p(r) = 0,
\]

both of which are standard Serre relations. This proves that Lemma 3.8 and hence Theorem 3.10 are valid at rank \( r \).

**Remark 5.2.** The proof presented here includes an alternative proof for Serre’s theorem in the case of \( sl_n \). This can be generalised to all finite dimensional simple Lie algebras. In particular, the proof for the other classical Lie algebras can be extracted from the next two sections.

**5.2. Proof in type B.** Consider the first Dynkin diagram of type \( B \) in Table 2, where the last (that is, \( r \)-th) node is white, and take \( d = r \). In this case, \( g_0 \) is a general linear superalgebra, and we have already obtained a Serre presentation for it in Section 5.1.

We require \( g_1 \) be isomorphic to the irreducible \( g_0 \)-module with lowest weight \( \alpha_r \), which is in fact the natural module for \( g_0 \). This is achieved by relations formally the same as (5.2) or (5.3).

As \( g_0 \)-module, \( g_2 \) is isomorphic to \( \wedge_5^2 g_1 \), which is irreducible with the lowest weight vector \( E := [e_r, [e_r, e_{r - 1}]] \). Now \( g_3 = 0 \) if and only if \( [E, g_1] = 0 \). This in particular requires that

\[
(5.5) \quad (ad_{e_r})^3 (e_{r - 1}) = 0.
\]

We shall show that this in fact is the necessary and sufficient condition.

If \( p(r - 1) = 1 \), then \( [E, [e_{r - 1}, e_r]] = 0 \) trivially since \( [e_{r - 1}, e_r] = 0 \) in \( g_0 \). For \( K = [e_{r - 2}, [e_{r - 1}, e_r]] \), we also have \( [K, E] = 0 \). This follows from \( [K, e_{r - 1}] = 0 \), which is one of the higher order Serre relations associated with a sub-diagram of type \( A \). Applying \( ad_{e_r} \) to it twice and using (5.5), we obtain the desired relation. These relations imply that \( [X, E] = 0 \) for all \( X \in g_1 \) in this case. If \( p(r - 1) = 0 \), the fact that \( [X, E] = 0 \), for all \( X \in g_1 \), follows from

\[
[[e_{r - 1}, e_r], [[e_{r - 1}, e_r], e_r]] = 0,
\]

which can be derived from (5.5).
The other Dynkin diagram (where the last node is black) can be treated in essentially the same way. We omit the details.

5.3. **Proof in types C and D.** The Dynkin diagrams of type C formally have the same forms as two of the Dynkin diagrams of D. The only difference is in the numbers of grey nodes, see Remark [A.1](#). This enables us to treat both types of Lie superalgebras simultaneously.

5.3.1. **Case 1.** Consider the Dynkin diagram

![Dynkin diagram](image)

We label the nodes from left to right, thus the node is the one at the right end. Set \( d = r \), then \( g_0 \) is a general linear superalgebra.

As a \( g_0 \)-module, \( g_1 \) is generated by \( e_r \). We require it be isomorphic to the irreducible module \( \mathcal{T}_{\alpha_r} \) with lowest weight \( \alpha_r \). Appendix [B.2](#) describes the structure of the generalised Verma module \( \mathcal{V}_{\alpha_r} \) with lowest weight \( \alpha_r \) and the irreducible quotient \( \mathcal{L}_{\alpha_r} \). We immediately see that the relevant relations in (3.1) and the relations

\[
[X_{ir}, [X_{jr}, [X_{kr}, e_r]]] = 0, \quad \forall i \leq j \leq k \leq r - 1,
\]

(5.6)

are necessary and sufficient conditions to guarantee that \( g_1 \cong \mathcal{T}_{\alpha_r} \). Here \( X_{ir} \) are elements of \( g_0 \) defined by (5.1). The conditions (5.6) are equivalent to

\[
[e_{r-1}, [e_{r-2}, [e_{r-3}, e_r]]] = 0, \quad \text{if } e_{r-1} \text{ is even},
\]

\[
[X_{r-2,r}, [X_{r-2,r}, [e_{r-1}, e_r]]] = 0, \quad \text{if } e_{r-1}, e_{r-2} \text{ are both odd},
\]

\[
[X_{r-3,r}, [X_{r-2,r}, [e_{r-1}, e_r]]] = 0, \quad \text{if } e_{r-1} \text{ is odd, } e_{r-2} \text{ is even}
\]

(5.7)

because of the \( g_0 \)-action. Here Remark [B.1](#) is also in force.

Note that the different situations where the relations apply are mutually exclusive. The first relation is a standard Serre relation. The second and third are higher order Serre relations respectively associated with the sub-diagrams

![Sub-diagrams](image)

Recall that \( g_2 \) is the image of \( \wedge^2_2 g_1 \) under the Lie superbracket. As \( g_0 \)-module, \( \wedge^2_2 g_1 \) is irreducible with the lowest weight vector \( e_r \otimes [e_{r-1}, e_r] - [e_{r-1}, e_r] \otimes e_r \). Thus \( g_2 = 0 \) if and only if

\[
[e_r, [e_r, e_{r-1}]] = 0.
\]

(5.8)

This is again a standard Serre relation.

5.3.2. **Case 2.** Now we consider the case with the Dynkin diagram

![Dynkin diagram](image)
Let us first assume that \( r = 3 \). We have the Dynkin diagram of \( \mathfrak{osp}_{2|4} \) (resp. \( \mathfrak{osp}_{4|2} \)) if \( p(1) = 0 \) (resp. \( p(1) = 1 \)). Label by 1 the node marked by \( \times \), and take \( d = 1 \). The diagram obtained by deleting this node is
\[
\begin{array}{c}
\circ - - \circ \\
\end{array}
\]
This is a non-standard diagram of \( \mathfrak{osp}_{2|2} \cong sl_{2|1} \). Equation (3.11) by itself suffices to define this Lie superalgebra.

Now \( g_0 = \mathfrak{osp}_{2|2} \oplus \mathfrak{gl}_1 \) (isomorphic to \( gl_{2|1} \)). Let \( \bar{b}_0 \) be the Borel subalgebra of \( g_0 \) generated by \( f_2, f_3 \) and all \( h_i \), and define the lowest weight Verma module \( \overline{V}_{\alpha_1} := U(g_0) \otimes U(\bar{b}_0) \mathbb{C}_{\alpha_1} \) for \( g_0 \), where \( \mathbb{C}_{\alpha_1} \) is the irreducible \( \bar{b}_0 \)-module with lowest weight \( \alpha_1 \). Direct computations show that the maximal submodule \( M_{\alpha_1} \) is generated by the vector \((e_2e_3 - e_3e_2) \otimes 1\). The irreducible quotient \( \overline{L}_{\alpha_1} \) is four dimensional, with a basis consisting of the images of \( 1 \otimes 1, e_2 \otimes 1, e_3 \otimes 1, \) and \([e_2, e_3] \otimes 1\). Its restriction to \( \mathfrak{osp}_{2|2} \) is the natural module.

We need \( g_1 \cong \overline{L}_{\alpha_1} \), possibly up to a parity change depending on the parity of \( e_1 \).

From the description of \( \overline{V}_{\alpha_1} \) and \( M_{\alpha_1} \) above, we see that the necessary and sufficient conditions are the relevant quadratic relations involving \( e_1 \) in (3.11), and
\[
[e_2, [e_3, e_1]] - [e_3, [e_2, e_1]] = 0.
\]
Note that this is a higher order Serre relation associated with the sub-diagram \( \{6\} \) given in Theorem 3.11.

To proceed further, we need to specify the parity of \( e_1 \).

If \( e_1 \) is even, the Lie superalgebra \( L(A, \Theta) \) is \( \mathfrak{osp}_{2|4} \). Now \( \wedge^2 \mathfrak{g}_1 \) is the direct sum of a seven dimensional indecomposable \( g_0 \)-submodule and a one dimensional \( g_0 \)-submodule. The seven dimensional submodule is generated by the two lowest weight vectors
\[
e_1 \otimes [e_2, e_1] - [e_2, e_1] \otimes e_1, \quad e_1 \otimes [e_3, e_1] - [e_3, e_1] \otimes e_1,
\]
and the one dimensional submodule by
\[
[e_2, e_1] \otimes [e_3, e_1] + [e_3, e_1] \otimes [e_2, e_1] + e_1 \otimes [[e_2, e_3], e_1] - [[e_2, e_3], e_1] \otimes e_1.
\]
In this case, we need \( g_2 \) to be isomorphic to a one dimensional \( g_0 \)-module with weight \( 2\alpha_1 + \alpha_2 + \alpha_3 \). Thus the seven dimensional indecomposable submodule of \( \wedge^2 \mathfrak{g}_1 \) is sent to zero by the Lie superbracket, or equivalently,
\[
[e_1, [e_1, e_2]] = 0, \quad [e_1, [e_1, e_3]] = 0,
\]
which are standard Serre relations. The image of the one dimensional submodule is \( g_2 \), which is spanned by
\[
[[e_1, e_2], [e_1, e_3]] - [e_1, [e_1, [e_2, e_3]]] = -[[e_1, e_2], [e_1, e_3]],
\]
where (5.10) is used to obtain the identity. By using (5.9) and (5.10), one can easily show that \([g_2, g_1] = 0\), and hence \( g_3 = 0 \).
If \( e_1 \) is odd, the Lie superalgebra \( L(A, \Theta) \) is \( \mathfrak{osp}_{4|2} \). By dimension counting, we need \( \mathfrak{g}_2 = 0 \). Now \( \Lambda^2_{\alpha} \mathfrak{g}_1 \) is also a direct sum of a seven dimensional indecomposable \( \mathfrak{g}_0 \)-submodule and a one dimensional submodule. Given the condition \([e_1, e_1] = 0\), the seven dimensional submodule vanishes automatically under the Lie superbracket, and the image of the one dimensional submodule is spanned by \([e_1, e_2], [e_1, e_3]\). Taking the Lie superbracket of \( e_1 \) with both sides of (5.9), we obtain \([e_1, e_2], [e_1, e_3] = 0\). Hence \( \mathfrak{g}_2 = 0 \).

Now assume \( r \geq 4 \). We take \( d = r - 3 \), then \( \mathfrak{g}_0 \) is the direct sum of a general linear superalgebra and \( \mathfrak{osp}_{4|2} \) or \( \mathfrak{osp}_{2|4} \).

If \( e_{r-2} \) is even, the condition that \( \mathfrak{g}_1 \) is an irreducible \( \mathfrak{g}_0 \)-module of lowest weight \( \alpha_{r-3} \) is given by the relevant relations in (3.1),

\[
[e_{r-2}, [e_{r-2}, e_{r-3}]] = 0,
\]

and also

\[
[e_{r-4}, [e_{r-4}, e_{r-3}]] = 0, \quad \text{if } p(r - 4) = 0,
\]

\[
[e_{r-4}, [e_{r-5}, [e_{r-4}, e_{r-3}]]] = 0, \quad \text{if } p(r - 4) = 1.
\]

As \( \mathfrak{g}_0 \)-module, \( \Lambda^2_{\alpha} \mathfrak{g}_1 \) is the direct sum of three irreducibles. The \( \mathfrak{osp}_{2|4} \) subalgebra of \( \mathfrak{g}_1 \) acts trivially on one of the irreducible submodules, and \( \mathfrak{g}_2 \) is isomorphic to it. The necessary and sufficient condition for the Lie superbracket to annihilate the other two irreducible submodules is

\[
[e_{r-3}, [e_{r-3}, e_{r-2}]] = 0, \quad [e_{r-3}, [e_{r-3}, e_{r-4}]] = 0, \quad \text{if } p(r - 3) = 0,
\]

\[
[e_{r-3}, e_{r-3}] = 0, \quad [e_{r-3}, [e_{r-4}, [e_{r-3}, e_{r-2}]]] = 0, \quad \text{if } p(r - 3) = 1,
\]

as can be shown by examining lowest weight vectors of the submodules.

**Remark 5.3.** Let \( E = [e_{r-3}, e_{r-2}, e_{r-1}] \) and \( E' = [e_{r-3}, e_{r-2}, e_r] \). Then at least one of the vectors \([X, E]\) and \([X, E']\) vanishes for any \( X \in \mathfrak{g}_1 \).

Let \( v \) denote a lowest weight vector of \( \mathfrak{g}_2 \). We can take \( v = [E, E'] \) if \( e_{r-3} \) is even, and \( v = [e_{r-4}, E], E' \) if \( e_{r-3} \) is odd. Then by Remark 5.3 we have \([v, X] = 0\) for any \( X \in \mathfrak{g}_1 \). Hence \( \mathfrak{g}_3 = 0 \).

If \( e_{r-2} \) is odd, the condition that \( \mathfrak{g}_1 \) is an irreducible \( \mathfrak{g}_0 \)-module of lowest weight \( \alpha_{r-3} \) translates into the relations (5.11),

\[
[e_{r-2}, [e_{r-2}, e_{r-1}], e_{r-3}] = 0,
\]

\[
[e_{r-2}, [e_{r-2}, e_r], e_{r-3}] = 0,
\]

plus the relevant relations in (3.1). Here we have used some facts about generalised Verma modules for \( \mathfrak{osp}_{4|2} \).

As \( \mathfrak{g}_0 \)-module, \( \Lambda^2_{\alpha} \mathfrak{g}_1 \) is again a direct sum of three irreducibles. One of them restricts to a direct sum of one dimensional \( \mathfrak{osp}_{4|2} \)-modules, and \( \mathfrak{g}_2 \) is isomorphic to it. The other two irreducibles are both mapped to zero by the Lie superbracket. The necessary and sufficient condition for this to happen is still (5.12).
Note that Remark 5.3 remains valid in the present case if we define $E$ and $E'$ in the same way. Let $v = [E, E']$ if $e_{r-3}$ is odd, and $v = [e_{r-4}, E']$ if $e_{r-3}$ is even. Then $v$ is a nonzero lowest weight vector of $g_2$. It follows from Remark 5.3 that $[v, X] = 0$ for any $X \in g_1$. Hence $g_3 = 0$.

5.3.3. Case 3. Finally we consider the Dynkin diagram

\[
\begin{array}{c}
\circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

assuming that there are at least two grey nodes (as otherwise this would correspond to the distinguished root system of type $D$). This forces $r \geq 4$.

This case is quite easy, thus we shall be brief. We choose $d$ to be the largest integer such that $p(d) = 1$. Then $g_0$ is the direct sum of a general linear superalgebra and an even dimensional orthogonal Lie algebra.

From Section 5.1 we see that the necessary and sufficient conditions for $e_d$ (which must be odd) to generate an irreducible $g_0$-module are the relevant relations in (3.1) and the higher order Serre relation involving $e_d$ associated with the following sub-diagram

\[
\begin{array}{c}
\circ & \circ & \circ & \circ \\
\end{array}
\]

of the Dynkin diagram if $p(d - 1) = 1$. Note that if $d = 2$, this becomes vacuous.

As $g_0$-module, $g_1$ is the tensor product of the natural modules $V_A$ and $V_D$ respectively for the general linear superalgebra and orthogonal algebra contained in $g_0$. Here $V_D$ is purely even, and the grading of $V_A$ gives rise to the grading of $g_1$.

Now $\Lambda^2 g_1 \cong \Lambda^2(V_A) \otimes (S^2(V_D)/C) \oplus S^2(V_A) \otimes \Lambda^2 V_D \oplus \Lambda^2 V_A \otimes C$ as $g_0$-module. The images of the first two irreducibles under the Lie superbracket are set to zero by the relation $[e_d, e_d] = 0$ and the higher order Serre relation(s) associated with the sub-diagram(s) of the form

\[
\begin{array}{c}
\circ & \circ & \circ & \circ \\
\end{array}
\]

Note that if $d < r - 2$, there is only one such diagram, but there are two if $d = r - 2$, as the last node can be $(r - 1)$ or $r$. We have $g_2 \cong \Lambda^2(V_A) \otimes C$.

One can show that $[g_2, g_1] = 0$ by using the same arguments as those in Section 4.1.3 and Section 4.1.4 thus $g_3 = 0$.

5.4. Proof in type $F(4)$. Now we turn to $F(4)$, which is considerably more complicated than the other type of Lie superalgebras.

5.4.1. Case 1. Consider first the root system corresponding to the Dynkin diagram

\[
\begin{array}{c}
1 & 2 & 3 & 4 \\
\end{array}
\]

We take $d = 2$. Then $g_0 = sl_2 \oplus g_3$. The standard Serre relations plus the relevant relations in (3.1) are the necessary and sufficient conditions rendering the $g_0$-module $g_1$ irreducible. We have $g_1 \cong C^2 \otimes C^3$ up to a parity change.
As \( g_0 \)-module, \( \wedge^2 g_1 \) is a direct sum of two irreducibles. The condition \([e_2, e_2] = 0\) forces one of the irreducibles to be in the kernel of the map \( \wedge^2 g_1 \to g_2 \). Thus \( g_2 \) is an irreducible \( g_0 \)-module generated by the lowest weight vector \( E = \langle e_1, e_2, [e_2, e_3] \rangle \). We have \( g_2 = C \otimes \wedge^2(C^3) \).

Now \( g_3 = [g_2, g_1] \cong C^2 \otimes C \) with a basis consisting of vectors \( \langle E, [e_2, [e_3, e_4]] \rangle \) and \( \langle E, [e_2, [e_3, e_4]] \rangle \), where \( E = [e_1, e_2, e_3] \). One immediately sees that

\[
[g_3, e_2] = \langle E, [e_4] \rangle,
\]

which generates \( g_4 = C \otimes C^3 \).

To consider \( g_5 \), we only need to look at \([g_4, g_1]\). If \( X \in g_1 \) is any lowest weight vector for \( sl_2 \subset g_0 \), the higher order Serre relation associated with the Dynkin diagram (see diagram (7) in Theorem 3.11) renders \([g_4, X] = 0\). Since the \( sl_2 \) subalgebra of \( g_0 \) acts trivially on \( g_4 \), it follows that \([g_4, g_1] = 0\), that is, \( g_5 = 0 \).

### 5.4.2. Case 2

For the Dynkin diagram

```
1  2  3  4
```

we also take \( d = 2 \) as in the previous case. Then \( g_0 = gl_2 \oplus sp_4 \). The relevant relations in (3.11) and standard Serre relations guarantee that \( e_2 \) generates an irreducible \( g_0 \)-module, which is isomorphic to the tensor product \( C^2 \otimes C^4 \) of the natural modules for \( gl_2 \) and \( sp_4 \) up to a parity change.

Now \( \wedge^2 g_1 \) decomposes into the direct sum of three irreducible \( g_0 \)-modules, which are respectively isomorphic to \( S^2(C^2) \otimes S^2(C^4) \), \( \wedge^2(C^2) \otimes (\wedge^2(C^3)/C) \) and \( \wedge^2(C^2) \otimes C \). The necessary and sufficient conditions for the Lie superbracket to map the first and the third submodules to zero are \([e_2, e_2] = 0\) and the higher order Serre relation

\[
[[e_1, e_2], [e_2, e_2], [e_3, e_4]] - [[e_2, e_2], [e_1, e_2], [e_3, e_4]] = 0
\]

associated with the Dynkin diagram (see diagram (8) in Theorem 3.11). Now \( g_2 \) is isomorphic to \( \wedge^2(C^2) \otimes \wedge^2(C^2) \) with lowest weight vector

\[
E = \langle e_1, e_2, e_3 \rangle.
\]

Formally \([g_2, g_1]\) decomposes into the direct sum of two irreducibles, respectively having lowest weight vectors

\[
[e_1, e_2], \quad [e_2, [e_3, e_4]].
\]

The first vector vanishes by \([e_2, e_2] = 0\). The second vector is the supercommutator of \( e_2 \) with the left hand side of (5.13), thus is also zero. This shows that \( g_3 = 0 \).

### 5.4.3. Case 3

Consider the Dynkin diagram

```
    1  2  3  4
```

We take \( d = 4 \), and delete the 4-th node from the diagram to obtain

```
2  3  4  1
```
This is a non-standard diagram for \(\mathfrak{sl}_{1|3}\), where the double edges can be got rid of by a normalisation of the bilinear form on the weight space thus are immaterial. The presentation for \(\mathfrak{sl}_{1|3}\) involves no higher order Serre relation. We have \(\mathfrak{g}_0 = \mathfrak{gl}_{1|3}\).

Let \(\overline{p}\) be the lower triangular maximal parabolic subalgebra of \(\mathfrak{g}_0\) with Levi subalgebra \(l := \mathfrak{gl}_3 \oplus \mathfrak{gl}_1\). Let \(\mathcal{V}_{\alpha_4}^0 = \mathbb{C}v_0\) be the 1-dimensional \(\overline{p}\)-module with lowest weight \(\alpha_4\), which is assumed to be a purely odd superspace. Since \(\alpha_4\) is a typical \(\mathfrak{g}_0\) weight, the generalised Verma module \(\nabla_{\alpha_4} = U(\mathfrak{g}_0) \otimes_{U(\overline{p})} \mathcal{V}_{\alpha_4}^0\) is irreducible, i.e., \(\nabla_{\alpha_4} = \nabla_{\alpha_4}\). It is multiplicity free, and the set of weights is given by

\[
\Delta^+ \setminus \{\Delta^+(\mathfrak{g}_0) \cup \Delta_2^+\},
\]

where \(\Delta^+\) is the set of the positive roots of \(F(4)\) relative to the Borel subalgebra under consideration, \(\Delta^+(\mathfrak{g}_0)\) is the set of the positive roots of the subalgebra \(\mathfrak{g}_0\), and

\[
\Delta_2^+ = \left\{\frac{1}{2}(\delta + \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \varepsilon_i + \varepsilon_j, i \neq j \right\}.
\]

The \(\mathfrak{g}_0\)-module \(\wedge^2_{\overline{p}}\mathcal{V}_{\alpha_4}\) is not semi-simple. To avoid the laborious task of determining the indecomposable submodules, we simply examine the \(l\) lowest weight vectors in \(\wedge^2_{\overline{p}}\mathcal{V}_{\alpha_4}\). Of particular importance to us are the vectors

\[
\begin{align*}
z_1 & := v_0 \otimes v_0; \\
z_2 & := e_3 v_0 \otimes [e_2, e_3] e_3 v_0 - [e_2, e_3] e_3 v_0 \otimes e_3 v_0; \\
z_3 & := v_0 \otimes e_3 v_0 - e_3 v_0 \otimes v_0; \\
w_1 & := v_0 \otimes [e_2, e_3] e_3 v_0 + [e_2, e_3] e_3 v_0 \otimes v_0; \\
w_2 & := v_0 \otimes [e_1, [e_2, e_3]] [e_2, e_3] e_3 v_0 - [e_1, [e_2, e_3]] [e_2, e_3] e_3 v_0 \otimes v_0.
\end{align*}
\]

The space of \(l\)-lowest weight vectors of \(\wedge^2_{\overline{p}}\mathcal{V}_{\alpha_4}\) is spanned by \(w_1, w_2\) and the \(l\)-lowest weight vectors in the \(\mathfrak{g}_0\)-submodule \(M\) generated by \(z_1\) and \(z_2\). It is important to observe that \(w_1\) and \(w_2\) are not in \(M\), but \(w_1 \in U(\mathfrak{g}_0)w_2\). Furthermore, one can verify that \(\wedge^2_{\overline{p}}\mathcal{V}_{\alpha_4}/M\) is multiplicity free with the set of weights \(\Delta_2^+\).

Now we take \(v_0 = e_4\) and require \(\overline{p}\) act on it by the adjoint action. Then \(\mathfrak{g}_1 = \mathcal{V}_{\alpha_4}\).

We require that the Lie superbracket maps \(z_1\) and \(z_2\) to zero. This leads to the following relations:

\[
\begin{align*}
[e_4, e_4] & = 0; \\
[[e_3, e_4], [e_3, e_4], [e_2, e_3]] & = 0.
\end{align*}
\]

Under the first condition, the Lie superbracket automatically maps \(z_3\) to zero. Note that the second relation in equation (5.16) is the desired higher order Serre relation associated with the sub-diagram

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\]
The vectors $w_1$ and $w_2$ have non-zero images under the Lie superbracket, and we have $g_2 \cong \bigwedge_2 \mathfrak{T}_{\alpha_4}/M$. By considering the possible $l$-lowest weight vectors, we can show that $[g_1, g_2] = 0$, thus $g_3 = 0$.

Now the proof of Lemma 3.8 in this case is completed by comparing the weights in (5.14) and (5.15) with the roots in $L_1$ and $L_2$.

5.4.4. Case 4. Consider the Dynkin diagram

Take $d = 1$, then $g_0 = \mathfrak{osp}_{2|4} \oplus \mathfrak{gl}_1$. The presentation of $\mathfrak{osp}_{2|4}$ relative to the Dynkin diagram has been constructed, thus the defining relations among $e_i, f_i, h_i$ for $i > 1$ are all known. The parabolic subalgebra of $\mathfrak{osp}_{2|4}$ defined in Appendix B.1 together with the ideal $\mathfrak{gl}_1$ form a parabolic of $g_0$. Then $e_1$ spans a 1-dimensional module for this parabolic, which induces a generalised Verma module $\mathcal{V}_{\alpha_1}$ of lowest weight type for $g_0$. The structure of $\mathcal{V}_{\alpha_1}$ can be understood by using results of Section B.1. In particular, imposing the condition (B.1), which in the present case reads

(5.17) \[ [e_2, [[e_2, e_3], e_1]] = 0, \]

sends $\mathcal{V}_{\alpha_1}$ to the irreducible quotient, which is $g_1$. Note that (5.17) is a higher order Serre relation associated with diagram (9) in Theorem 3.11. It is a non-standard diagram of $\mathfrak{sl}_{1|3}$.

Now $g_1$ forms is 10-dimensional. A basis for it can be deduced from Section B.1. For every vector $b$ in this basis, we have $[b, e_1] = 0$. This holds trivially for most basis vectors, but for $b = [e_2, [[e_1, e_2], e_3], e_4]$, we have

\[ [e_1, b] = [[[e_1, e_2], e_3], [e_1, e_2], e_4]] = \frac{1}{2} (ad_{e_1})^2 [e_2, e_3, [e_2, e_4]]. \]

One can deduce from the defining relations for $\mathfrak{osp}_{2|4}$ that $[[e_2, e_3], [e_2, e_4]] = 0$, hence $[b, e_1] = 0$. This implies that the commutator of $e_1$ with all the remaining basis vectors are all zero. Therefore, $g_2 = [g_1, g_1] = 0$.

5.4.5. Case 5. In the case of the Dynkin diagram

we take $d = 4$ and delete the 4-th node to obtain the diagram
which is a non-standard diagram of $\mathfrak{sl}_1|3$. Thus we have a relation formally the same as (5.17).

Now $g_0 = \mathfrak{gl}_1|3$. The Verma module of lowest weight type for $g_0$ generated by $e_4$ contains the primitive vector $2[e_2, [e_3, e_4]] - 3[[e_2, e_3], e_4]$, which generates the maximal submodule. Thus the higher order Serre relation

\[(5.18) \quad 2[e_2, [e_3, e_4]] - 3[[e_3, e_2], e_4] = 0,\]

associated with diagram (10) in Theorem 3.11 is all that is needed to guarantee that $g_1$ is an irreducible $g_0$-module. This module is typical relative to the distinguished Borel subalgebra, and has dimension 8.

Restricted to a module for $\mathfrak{gl}_3 \subset g_0$, the even subspace of $g_1$ is the direct sum of the natural $\mathfrak{gl}_3$-module and a 1-dimensional module, while the odd subspace is the direct sum of the dual natural module (twisted by a scalar) and a 1-dimensional module.

Now consider $[g_1, g_1]$. We can easily work out its decomposition into irreducible $\mathfrak{gl}_3$-submodules. The corresponding $\mathfrak{gl}_3$ lowest weight vectors can be worked out, which include the following vectors:

\[e_2, e_2], \quad (ad_{e_2, e_4})^2 e_2, e_1], \quad [[e_2, e_3], [e_4], e_4].\]

It follows from the higher order Serre relation (5.18) that

\[[e_2, e_4], [e_3, e_4]] = 0.\]

Now we impose the relations

\[[e_2, e_2] = 0, \quad (ad_{e_2, e_4})^2 e_2, e_1] = 0,\]

where the first is a standard Serre relation, and the second is a higher order Serre relations associated with

\[1 \rightarrow 2 \rightarrow 4.\]

Under these conditions, all other $\mathfrak{gl}_3$ lowest vectors in $[g_1, g_1]$ vanish, except

\[(ad_{e_2, e_4})^2 e_1, \quad [[e_3, e_4], [e_1, e_2], [e_2, e_4]],\]

where the first one is actually a $g_0$ lowest weight vector. It generates an 4-dimensional irreducible $g_0$-module containing the second vector. This module is isomorphic to the dual of the natural $g_0$-module twisted by a scalar. This gives us $g_2 = [g_1, g_1]$. We can further show that $[g_2, e_4] = 0$, hence $g_3 = 0$.

5.5. Proof in type $G(3)$. 

5.5.1. **Case 1.** Consider the Dynkin diagram

```
1 = 2 --- 3
```

We take \( d = 3 \), then \( \mathfrak{g}_0 = \mathfrak{gl}_{2|1} \). Let \( \mathcal{V}_{\alpha_3} \) be the lowest weight Verma module for \( \mathfrak{g}_0 = \mathfrak{gl}_{2|1} \) with lowest weight \( \alpha_3 \). Denote by \( v_0 \) the lowest weight vector, which is assumed to be even. Then the maximal submodule of \( \mathcal{V}_{\alpha_3} \) is generated by \( e_1v_0 \) and \( e_2[e_1,e_2]^3v_0 \). The irreducible quotient \( \mathcal{L}_{\alpha_3} \) is multiplicity free and has weights

\[
\alpha_3 + k(\alpha_1 + \alpha_2), \quad k = 0, 1, 2, 3,
\]

\[
\alpha_3 + p(\alpha_1 + \alpha_2) + \alpha_2, \quad p = 0, 1, 2.
\]

In fact \( \mathcal{L}_{\alpha_3} \) is isomorphic to the third \( \mathbb{Z}_2 \)-graded symmetric power of the natural module for \( \mathfrak{g}_0 \) tensored with a 1-dimensional module. Thus \( \wedge_3^2 \mathcal{L}_{\alpha_3} \) is completely reducible; it is the direct sum of two irreducibles.

Now we take \( v_0 \) to be \( e_3 \), and let \( \mathfrak{g}_0 \) act on it by the adjoint action. Then the generators of the maximal submodule of \( \mathcal{V}_{\alpha_3} \) in this case are \( (ad_{[e_1,e_2]})^3 [e_2,e_3] \) and \( [e_1,e_3] \). Thus \([e_1,e_3] = 0 \) and the higher order Serre relation

(5.19) \((ad_{[e_1,e_2]})^3 [e_2,e_3] = 0\)

(associated with diagram \([11] \) in Theorem 3.11) render \( \mathfrak{g}_1 = \mathcal{L}_{\alpha_3} \).

One of the irreducible submodules of \( \wedge_3^2 \mathfrak{g}_1 \) has a lowest weight vector of the form \( e_3 \otimes [e_2,e_3] - [e_2,e_3] \otimes e_2 \). We require that this submodule be in the kernel of the Lie superbracket. This leads to the standard Serre relation \([e_3,[e_3,e_2]] = 0\).

The other irreducible submodule of \( \wedge_3^2 \mathfrak{g}_1 \) is mapped surjectively onto \( \mathfrak{g}_2 \). A lowest weight vector of \( \mathfrak{g}_2 \) is given by \( X := ad_{e_3} (ad_{[e_1,e_2]})^2 [e_2,e_3] \). This irreducible module is 4-dimensional and has weights

\[-(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2), \; \delta + \varepsilon_2 - \varepsilon_3, \; \delta + \varepsilon_1 - \varepsilon_3, \; 2\delta,\]

in the notation explained in Appendix A.1. It is easy to see that \([X,X] = 0\) for all \( X \in \mathfrak{g}_1 \). Thus \( \mathfrak{g}_3 = 0 \).

By examining the weights of \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \), we see that Lemma 3.8 holds.

5.5.2. **Case 2.** Consider the Dynkin diagram

```
1 = 2 --- 3
```

We take \( d = 1 \), and delete the first node from the Dynkin diagram to obtain

```
= = =
```

This is a nonstandard diagram for \( \mathfrak{sl}_{1|2} \), which can be cast into the usual Dynkin diagram of \( \mathfrak{sl}_{1|2} \) in the distinguished root system by normalising the bilinear form on the weight space. Note that no higher order Serre relations are required to present this Lie superalgebra. We have \( \mathfrak{g}_0 = \mathfrak{gl}_{1|2} \).
Now the \( g_0 \) Kac module of lowest weight type generated by \( e_1 \) is typical thus irreducible, hence \( g_1 \cong \mathcal{T}_{\alpha_1} \) with basis

\[
e_1, \quad [e_2, e_1], \quad [e_2, e_3], \quad [e_2, e_1], \quad [e_2, e_1].
\]

As \( g_0 \)-module \( \wedge^2 g_1 \) is the direct sum of two irreducible typical submodules, respectively generated by the lowest weight vectors \( e_1 \otimes e_1 \) and \( v - \frac{1}{2}v' \), where

\[
v = e_1 \otimes [e_2, e_3], \quad [e_2, e_1] + [e_2, e_3], \quad [e_2, e_1] \otimes e_1,
\]

\[
v' = [e_2, e_1] \otimes [e_3, e_2, e_1] - [e_3, [e_2, e_1]] \otimes [e_2, e_1].
\]

We require that \( v - \frac{1}{2}v' \) and thus the \( g_0 \)-submodule generated by it be mapped to zero by the Lie superbracket. This leads to

\[
[[e_2, e_1], [e_3, [e_2, e_1]] - [e_2, e_3], [[e_1, e_1], e_2]] = 0,
\]

which is one of the higher order Serre relations associated with the Dynkin diagram (see diagram (12) in Theorem 3.11). Therefore, \( g_2 \cong \mathcal{T}_{2\alpha_1} \) and has a basis

\[
[e_1, e_1], \quad [[e_1, e_1], e_2], \quad [e_3, [e_1, e_1], e_2], \quad [[e_2, e_3], [[e_1, e_1], e_2]].
\]

Now we consider \([g_2, g_1]\). One can easily see that \((ad_{e_1})^3 e_2\) is a \( g_0 \) lowest weight vector. We require that the \( g_0 \)-submodule generated by it be zero, hence we have the standard Serre relation

\[
(ad_{e_1})^3 e_2 = 0.
\]

This leaves \( g_3 = [g_2, g_1] \) to be an indecomposable \( g_0 \)-module cyclically generated by the lowest weight vector \([e_1, [e_2, e_3]], [[e_1, e_1], e_2]]\), which is 7-dimensional and multiplicity free. One can easily write down a basis for this module. We should remark that no \( g_0 \) lowest weight vector in \( g_3 \) is annihilated by all \( f_i \) for \( i = 1, 2, 3 \).

One can show by direct computations that \([e_1, g_3] = 0 \) and \([[e_1, e_1], g_2] = 0 \). Hence \( g_4 = 0 \).

An inspection of the weight spaces of \( g_i \) for \( 1 \leq i \leq 3 \) shows that they agree with those of \( L_i \) for \( 1 \leq i \leq 3 \). This completes the proof in this case.

5.5.3. Case 3. The final case of \( G(3) \) is the diagram

\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3
\end{array}
\]

We take \( d = 3 \), then \( g_0 = gl_2 | 1 \). The \( g_0 \) Kac module of lowest weight generated by \( e_3 \) is atypical. We set the primitive vector to zero to obtain

\[
2[[e_1, e_2], e_3] - [e_2, [e_1, e_3]] = 0,
\]
which is a higher order Serre relation in the present case. Then \( \mathfrak{g}_1 \) is an irreducible \( \mathfrak{g}_0 \)-module with lowest weight \( \alpha_3 \), which is isomorphic to the third \( \mathbb{Z}_2 \)-graded symmetric power of the natural module for \( \mathfrak{g}_0 \) twisted by a scalar. It has 3 odd and 4 even dimensions. A basis for \( \mathfrak{g}_1 \) is given by

\[
\begin{align*}
e_3, & \quad [e_1, e_3], \quad [e_1, [e_1, e_3]], \quad [e_2, e_3], \\
& \quad [[e_1, e_2], e_3], \quad [[e_1, e_2], [e_1, e_3]], \quad [[e_1, e_2], [e_1, [e_1, e_3]]].
\end{align*}
\]

The rest of the analysis is similar to Section 5.5.1. Now \( \wedge^2 \mathfrak{g}_1 \) is the direct sum of two irreducible \( \mathfrak{g}_0 \)-submodules. The images of theirs lowest weight vectors in \( \mathfrak{g}_1 \) are respectively \( [e_3, e_3] \) and \( E = [[e_1, e_3], [e_1, e_3]] \). Both generate typical \( \mathfrak{g}_0 \)-submodules, which respectively have dimensions 20 and 4. The standard Serre relation \( [e_3, e_3] = 0 \) removes the 20-dimensional submodule, thus \( \mathfrak{g}_2 \) is the 4-dimensional irreducible \( \mathfrak{g}_0 \)-module generated by \( E \).

We can also show that \( \mathfrak{g}_3 = 0 \) without imposing further relations. Inspecting the weights of \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \), we see that the claim of Lemma 3.8 indeed holds.

5.6. **Proof in type** \( \mathcal{D}(2, 1; \alpha) \). The Dynkin diagrams having only one grey node can be treated in exactly the same way as for the distinguished root system, thus we shall consider only the diagram with three gray nodes here. Set \( d = 3 \), then \( \mathfrak{g}_0 = \mathfrak{gl}_2 \). The \( \mathfrak{g}_0 \) Verma module of lowest weight type generated by \( e_3 \) contains the primitive vector

\[
\alpha[e_1, [e_2, e_3]] + (1 + \alpha)[e_2, [e_1, e_3]],
\]

which in fact generates the maximal submodule. The higher order Serre relation requires this vector to be zero. This is equivalent to taking the irreducible quotient of the Verma module, and we obtain \( \mathfrak{g}_1 \). A basis for \( \mathfrak{g}_1 \) is

\[
e_3, \quad [e_1, e_3], \quad [e_2, e_3], \quad [e_1, [e_2, e_3]].
\]

An easy computation using the higher order Serre relation shows that \( [\mathfrak{g}_1, e_3] = 0 \). Hence \( \mathfrak{g}_2 = 0 \). A quick inspection on the weights of \( \mathfrak{g}_1 \) shows that Lemma 3.8 indeed holds in this case.

6. **Remarks on affine Lie superalgebras**

We wish to mention that the generalisation of the method to affine Lie superalgebras is in principle straightforward conceptually. Consider, for example, the untwisted affine superalgebra \( \hat{\mathfrak{g}} \) of a contragredient Lie superalgebra \( \mathfrak{g} \). We want to present \( \hat{\mathfrak{g}} \) with the standard generators \( e_i, f_i, h_i \) with \( 0 \leq i \leq r \) and relations. Here the generators \( e_i, f_i, h_i \) with \( 1 \leq i \leq r \) are those for \( \mathfrak{g} \). By results of earlier sections, we may assume that all the Serre relations and higher order ones obeyed by \( e_i \) and \( f_i \) with \( 1 \leq i \leq r \) are given.

We introduce the standard \( \mathbb{Z} \)-grading of \( \hat{\mathfrak{g}} \) by decreeing that all \( h_j \) and \( e_i, f_i \) with \( 1 \leq i \leq r \) have degree 0, but \( e_0 \) and \( f_0 \) have degrees 1 and \(-1\) respectively. Then \( \hat{\mathfrak{g}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}_k \), with \( \mathfrak{h}_0 = \mathfrak{g} \bigoplus \mathfrak{g}_1 \). Now we require that as \( \mathfrak{g}_0 \)-modules, all \( \mathfrak{h}_k \) are isomorphic...
to \( g \). The (necessary and sufficient) conditions meeting this requirement give rise to the defining relations of \( \hat{g} \).

To illustrate how this may work, we consider the untwisted affine algebra \( \hat{g} = \hat{sl}_{r+1} \). The relations

\[
[e_1, [e_1, e_0]] = 0, \quad [e_r, [e_r, e_0]] = 0, \quad [e_i, e_0] = 0, \quad i \neq 1, r
\]

arise from the requirement that \( \hat{g}_1 \) be an irreducible \( g_0 \)-module. In \( [\hat{g}_1, \hat{g}_1] \), there are \( g_0 \) lowest weight vectors \( [[e_1, e_0], e_0] \) and \( [[e_r, e_0], e_0] \), which have weights different from any roots of \( g = sl_{r+1} \). Thus the condition that \( \hat{g}_2 \) is isomorphic to \( g \) as \( g_0 \)-module requires

\[
[[e_1, e_0], e_0] = 0, \quad [[e_r, e_0], e_0] = 0.
\]

Now we have derived at all the Serre relations needed for \( e_0 \), and those for \( f_0 \) can be similarly obtained. Together with relations defining \( g \), these relations define \( \hat{g} \).

We hope to treat the affine superalgebras on another occasion.

### Appendix A. Dynkin Diagrams

We describe the Dynkin diagrams for both the distinguished and non-distinguished root systems in this Appendix. The roots of all the simple contragredient Lie superalgebras will also be listed \([13, 14]\).

A.1. Roots. Let \( \epsilon_i \) \( (i = 1, 2, \ldots, k) \) and \( \delta_j \) \( (j = 1, 2, \ldots, l) \) be a basis of a real vector space \( E(k, l) \) equipped with a non-degenerate symmetric bilinear form. Then for each simple contragredient Lie superalgebra \( g \), the dual space \( h^* \) of the cartan subalgebra is either \( C \otimes_R E(k, l) \) for appropriate \( k, l \) or a subspace thereof, which inherits a non-degenerate bilinear form that is Weyl group invariant.

For the series \( A, B, C \) or \( D \), the bilinear form is defined by

\[
(\epsilon_i, \epsilon_{i'}) = \delta_{ii'}, \quad (\delta_j, \delta_{j'}) = -\delta_{jj'}, \quad (\epsilon_i, \delta_j) = 0, \quad \forall i, i', j, j'.
\]

The roots of the simple contragredient Lie superalgebras can be described as follows.

\( A(m|n) \):

\[
\Delta_0 = \{\epsilon_i - \epsilon_{i'} \mid i, i' \in [1, m+1], i \neq i'\} \cup \{\delta_j - \delta_{j'} \mid j, j' \in [1, n+1], j \neq j'\},
\]

\[
\Delta_1 = \{\pm(\epsilon_i - \delta_j) \mid i \in [1, m+1], j \in [1, n+1]\},
\]

where \( [1, N] \) denotes \( \{1, \ldots, N\} \) for any positive integer \( N \).

\( B(0,n) \):

\[
\Delta_0 = \{\pm\delta_j \pm \delta_{j'} \pm 2\delta_j \mid j, j' \in [1, n], j \neq j'\},
\]

\[
\Delta_1 = \{\pm\delta_j \mid j \in [1, n]\}.
\]
SERRE PRESENTATIONS OF LIE SUPERALGEBRAS

\( B(m, n), m > 1: \)

\[ \Delta_0 = \{ \pm \epsilon_i \pm \epsilon_{i'}, \pm \epsilon_i \mid i, i' \in [1, m], i \neq i' \} \]

\[ \cup \{ \pm \delta_j \pm \delta_{j'}, \pm 2\delta_j \mid j, j' \in [1, n], j \neq j' \}, \]

\[ \Delta_1 = \{ \pm \epsilon_i \pm \delta_j, \pm \delta_j \mid i \in [1, m], j \in [1, n] \}. \]

\( C(n+1): \)

\[ \Delta_0 = \{ \pm \delta_j \pm \delta_{j'}, \pm 2\delta_j \mid j, j' \in [1, n], j \neq j' \}, \]

\[ \Delta_1 = \{ \pm \epsilon_1 \pm \delta_j \mid j \in [1, n] \}. \]

\( D(m, n), m > 1: \)

\[ \Delta_0 = \{ \pm \epsilon_i \pm \epsilon_{i'} \mid i, i' \in [1, m], i \neq i' \} \]

\[ \cup \{ \pm \delta_j \pm \delta_{j'}, \pm 2\delta_j \mid j, j' \in [1, n], j \neq j' \}, \]

\[ \Delta_1 = \{ \pm \epsilon_i \pm \delta_j \mid i \in [1, m], j \in [1, n] \}. \]

\( F(4): \)

\[ \Delta_0 = \{ \pm \epsilon_i \pm \epsilon_j, \pm \epsilon_i \mid i, j = 1, 2, 3, i \neq j \} \cup \{ \pm \delta \}, \]

\[ \Delta_1 = \left\{ \frac{1}{2} ( \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta ) \right\}, \]

\[ (\delta, \delta) = -6, \quad (\epsilon_i, \epsilon_j) = 2\delta_{ij}, \quad (\epsilon_i, \delta) = 0, \quad \forall i, j = 1, 2, 3. \]

\( G(3): \)

\[ \Delta_0 = \{ \epsilon_i - \epsilon_j, \pm (2\epsilon_k - \epsilon_i - \epsilon_j) \mid 1 \leq i, j, k \leq 3, \text{pairwise distinct} \} \]

\[ \cup \{ \pm 2\delta \}, \]

\[ \Delta_1 = \{ \pm \delta + (\epsilon_i - \epsilon_j), \pm \delta \mid i \neq j \}, \]

\[ (\delta, \delta) = -2, \quad (\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\epsilon_i, \delta) = 0, \quad \forall i, j = 1, 2, 3. \]

\( D(2, 1; \alpha), \alpha \in \mathbb{C} \setminus \{0, -1\}: \)

\[ \Delta_0 = \{ \pm 2\epsilon_i \mid i = 1, 2 \} \cup \{ \pm 2\delta \}, \]

\[ \Delta_1 = \{ \pm \delta \pm \epsilon_1 \pm \epsilon_2 \}, \]

\[ (\epsilon_1, \epsilon_1) = 1, \quad (\epsilon_2, \epsilon_2) = \alpha, \quad (\delta, \delta) = -(1 + \alpha), \quad (\epsilon_i, \delta) = 0, \quad \forall i. \]
Denote by \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \) the set of simple roots of \( \mathfrak{g} \) relative to the distinguished Borel subalgebra. We have

\[
A(m|n) : \quad \Pi = \{ \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_m - \varepsilon_{m+1}, \varepsilon_{m+1} - \delta_1 - \delta_2, \ldots, \delta_n - \delta_{n+1} \};
\]

\[
B(0,n) : \quad \Pi = \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n \};
\]

\[
B(m,n), m > 1 : \quad \Pi = \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m \};
\]

\[
C(n+1) : \quad \Pi = \{ \varepsilon_1 - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n \};
\]

\[
D(m,n), m > 1 : \quad \Pi = \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m + \varepsilon_m \};
\]

\[
F(4) : \quad \Pi = \left\{ \frac{1}{2} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta), -\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3 \right\};
\]

\[
G(3) : \quad \Pi = \{ \delta - \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3 \};
\]

\[
D(2,1; \alpha), \alpha \in \mathbb{C} \setminus \{0,-1\} : \quad \Pi = \{ \delta - \varepsilon_1 - \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2 \}.
\]

Note that there is a unique simple root, which we denote by \( \alpha_s \), in each \( \Pi \). Thus \( \Theta = \{ s \} \).

The simple roots relative to other Borel subalgebras can be obtained by using odd reflections [25]. Let \( \Pi_b = \{ \alpha_1, \ldots, \alpha_r \} \) be the set of simple roots relative to a given Borel subalgebra \( \mathfrak{b} \subset \mathfrak{g} \). Take any isotropic odd simple root \( \alpha_t \in \Pi_b \), and define the odd reflection \( s_t \) by

\[
s_t(\alpha_t) = -\alpha_t,
\]

\[
s_t(\alpha_i) = \alpha_i + \alpha_t, \quad \text{if} \ i \neq t \text{ and } a_{it} \neq 0,
\]

\[
s_t(\alpha_i) = \alpha_i, \quad \text{if} \ i \neq t \text{ and } a_{it} = 0.
\]

Then \( s_t(\Pi_b) = \{ s_t(\alpha_1), \ldots, s_t(\alpha_r) \} \) is the set of simple roots relative to another Borel subalgebra, which is not Weyl group conjugate to \( \mathfrak{b} \). Further odd reflections can be defined with respect to isotropic roots in \( s_t(\Pi_b) \), which turn \( s_t(\Pi_b) \) into sets of simple roots relative to other Borel subalgebras. All the distinct sets obtained this way correspond bijectively to the conjugacy classes of Borel subalgebras.

### A.2. Dynkin diagrams.

#### A.2.1. Dynkin diagrams in distinguished root systems.

The Dynkin diagrams in the distinguished root systems are listed in Table 1 below, where \( r \) is the number of nodes and \( s \) is the element of \( \Theta \). Note that the form of Dynkin diagrams in the distinguished root systems is quite uniform in the literature. Table 1 is essentially the corresponding table in [13] with a slight modification in the Dynkin diagram for \( D(2,1; \alpha) \).

**Table 1. Dynkin diagrams in distinguished root systems**
### A.2.2. Dynkin diagrams in non-distinguished root systems

Table 2 gives the Dynkin diagrams of the non-distinguished root systems. A nice graphical explanation can be found in [3, §4] (see also [8]) on how to obtain the Dynkin diagrams in Table 2 by applying odd reflections to those in Table 1.

**Table 2. Dynkin diagrams in non-distinguished root systems**

| Lie superalgebra | Dynkin Diagram | r    | s    |
|------------------|----------------|------|------|
| A(m, n)          | ![Diagram](image) | m+n+1| m+1  |
| B(m, n), m>0     | ![Diagram](image) | m+n  | n    |
| B(0, n)          | ![Diagram](image) | n    | n    |
| C(n), n>2        | ![Diagram](image) | n    | 1    |
| D(m, n), m>1     | ![Diagram](image) | m+n  | n    |
| F(4)             | ![Diagram](image) | 4    | 1    |
| G(3)             | ![Diagram](image) | 3    | 1    |
| D(2, 1; α)       | ![Diagram](image) | 3    | 1    |

---

**Note:**
- The Dynkin diagrams are drawn using standard conventions: circles represent odd roots, and boxes represent even roots.
- The numbers r and s correspond to the number of nodes in the diagram for the Lie superalgebra.
- The diagrams are adapted to fit the text and are not to scale.
\[ D(m, n), m > 1 \]

\[ F(4) \]

\[ G(3) \]

\[ D(2, 1; \alpha) \]
In the diagrams in Table 2, a node marked with $\times$ can be white or grey. However, the precise rule for assigning colours requires the knowledge of the simple roots, which are described below.

**A(m,n).** An ordering $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{m+n+2})$ of $\epsilon_i$ and $\delta_j$ is called admissible if $\epsilon_i$ appears before $\epsilon_{i+1}$ for all $i$ and $\delta_j$ before $\delta_{j+1}$ for all $j$. Each admissible ordering corresponds to one Weyl group conjugate class of Borel subalgebras, with the associated simple roots given by $\epsilon_a - \epsilon_{a+1} (1 \leq a \leq m+n+1)$. In particular, the distinguished Borel corresponds to the admissible ordering such that all the $\epsilon_i$ appear before the $\delta_j$. Let us define $[\epsilon_a] (a = 1, 2, \ldots, m+n+2)$ by $[\epsilon_a] = 0$ (resp. $[\epsilon_a] = 1$) if $\epsilon_a$ is some $\epsilon_i$ (resp. $\delta_j$). The $a$-th node from the left in the Dynkin diagram is associated with the simple root $\epsilon_a - \epsilon_{a+1}$, which is white if $[\epsilon_a] = [\epsilon_{a+1}]$ and grey otherwise.

**B(m,n), m > 0.** Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{m+n})$ be an admissible ordering of $\epsilon_i (i = 1, \ldots, m)$ and $\delta_j (j = 1, \ldots, n)$. Then the corresponding simple roots are $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m+n-1} - \epsilon_{m+n}, \epsilon_{m+n}$. The first Dynkin diagram corresponds to the case $\epsilon_{m+n} = \epsilon_m$. The $a$-th node ($a < m + n$) from the left is associated with the simple root $\epsilon_a - \epsilon_{a+1}$, which is white if $[\epsilon_a] = [\epsilon_{a+1}]$ and grey otherwise. The second Dynkin diagram corresponds to the case $\epsilon_{m+n} = \delta_n$. The colours of the nodes marked $\times$ are assigned in the same way as in type $A$.

**C(n).** We have already specified the colours of the nodes in the Dynkin diagrams, but it is still useful to have an explicit description of the simple roots. Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be an admissible ordering of $\delta_j (j = 1, \ldots, n-1)$ and $\epsilon_1$. The first Dynkin diagram corresponds to the case with $\epsilon_n = \delta_{n-1}$, where simple roots are given by $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n$. The second Dynkin diagram corresponds to the case with $\epsilon_n = \epsilon_1$, where the simple roots are given by $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n$. The colours of the nodes marked with $\times$’s are assigned in the same way as in type $A$ and type $B$. 
Let \((E_1, E_2, \ldots, E_{m+n})\) be an admissible ordering of \(\varepsilon_i\) \((i = 1, \ldots, m)\) and \(\delta_j\) \((j = 1, \ldots, n)\). If \(E_{m+n-1} = \varepsilon_{m-1}\) and \(E_{m+n} = \varepsilon_m\), or \(E_{m+n-1} = \delta_n\) and \(E_{m+n} = \varepsilon_m\), the simple roots are given by
\[
E_1 - E_2, \ldots, E_{m+n-1} - E_{m+n}, E_{m+n-1} + E_{m+n}.
\]
The first Dynkin diagram corresponds to the former case, while the second Dynkin diagram corresponds to the latter. If \(E_{m+n-1} = \delta_{n-1}\) and \(E_{m+n} = \delta_n\), or \(E_{m+n-1} = \delta_n\) and \(E_{m+n} = \varepsilon_m\), the simple roots are given by
\[
E_1 - E_2, \ldots, E_{m+n-1} - E_{m+n}, 2E_{m+n}.
\]
The third Dynkin diagram corresponds to this case.

We assign colours to the nodes marked with \(\times\) in the same way as in the other cases.

**Remark A.1.** There are at least three grey nodes in the Dynkin diagrams of type \(D(m,n)\) in Table 2, but in each of the Dynkin diagrams of type \(C(n)\), there are only two grey nodes which are always next to each other.

## Appendix B. Presentations of Irreducible Modules

In general it is hard to give an explicit description of a finite dimensional irreducible module for a Lie superalgebra as the quotient of a (generalised) Verma module in a form similar to [11, Theorem 21.4] in the context of ordinary semi-simple Lie algebras. However, this is possible in some special cases, e.g., the natural module for \(\mathfrak{gl}_{m|n}\) in arbitrary root systems as discussed in Section 5.1. Here are two further cases, which are used in the proof of Lemma 3.8.

### B.1. An Irreducible \(osp_{2|4}\)-module.

Let \(\mathfrak{g}\) be the Lie superalgebra \(osp_{2|4}\) with the choice of Borel subalgebra corresponding to the Dynkin diagram

```
\[ \begin{array}{ccc}
1 & \rightarrow & 2 \\
\end{array} \]
```

We present \(\mathfrak{g}\) in the standard fashion using Chevalley generators \(e_i, f_i, h_i\) \((i = 1, 2, 3)\) and relations with the higher order Serre relations being those associated with diagram 6 in Theorem 3.11. To be specific, we denote by \(\alpha_i\) the simple roots and take
\[
(\alpha_1, \alpha_3) = (\alpha_2, \alpha_3) = -1, \quad (\alpha_1, \alpha_2) = 2, \quad (\alpha_3, \alpha_3) = 2.
\]

Let \(\overline{\mathfrak{g}}\) be the parabolic subalgebra generated by all the generators but \(e_1\). Then \(\overline{\mathfrak{g}} = l \oplus \overline{u}\) with \(l = \mathfrak{gl}_{2|1}\) and \(\overline{u}\) spanned by
\[
\begin{align*}
\zeta_1 & := e_1, \\
\zeta_2 & := [e_1, e_3], \\
x_1 & := [e_1, e_2], \\
x_2 & := [[e_1, e_2], e_3], \\
x_3 & := [[[e_1, e_2], e_3], e_3].
\end{align*}
\]

Given the irreducible \(\overline{\mathfrak{g}}\)-module \(\overline{\mathfrak{g}}^0_\lambda = \mathbb{C}v_0\) with lowest weight \(\lambda\) such that
\[
(\lambda, \alpha_2) = 0, \quad (\lambda, \alpha_3) = 0, \quad (\lambda, \alpha_1) = -2,
\]
we construct the generalised Verma module \( \nabla_{\lambda} = U(\mathfrak{g}) \otimes U(\bar{\mathfrak{p}}) T_{\lambda}^0 \). Then the maximal submodule \( M_{\lambda} \) of \( \nabla_{\lambda} \) is given by

\[
M_{\lambda} = U(\mathfrak{g}) \zeta_1 X_1 v_0.
\]

(B.1)

The irreducible quotient \( \mathcal{T}_{\lambda} = \nabla_{\lambda}/M_{\lambda} \) is 10-dimensional with a basis

\[
\begin{align*}
v_0, & \quad X_1 v_0, \quad X_2 v_0, \quad X_3 v_0, \quad X_1 X_3 v_0, \\
\zeta_1 v_0, & \quad \zeta_1 X_2 v_0, \quad \zeta_1 X_3 v_0, \quad \zeta_1 X_1 X_3 v_0, \quad \zeta_1 \zeta_2 v_0.
\end{align*}
\]

(B.2)

**Graded symmetric tensor for \( \mathfrak{gl}_{m|n} \).** Let \( \mathfrak{g} = \mathfrak{gl}_{m|n} \) and set \( r = m + n - 1 \). Choose an arbitrary homogeneous basis for the natural module \( \mathbb{C}^{m|n} \) with the last element being odd. We regard \( \mathfrak{g} \) as consisting of matrices relative to this basis. Take the subalgebra consisting of the upper triangular matrices as the Borel subalgebra, which corresponds to an admissible ordering \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m+n}) \) of \( \varepsilon_i \) \((1 \leq i \leq m)\) and \( \delta_j \) \((1 \leq j \leq n)\) with \( \varepsilon_{m+n} = \delta_n \). See Appendix [A.2] for more details.

Let \( l, u \) and \( \bar{u} \) be subalgebras respectively spanned by matrix units \( e_{r+1, r+1} \) and \( e_{ij} \) with \( 1 \leq i, j \leq r \), by \( e_{i, r+1} \) with \( 1 \leq r \), and by \( e_{r+1, i} \) with \( 1 \leq r \). Set \( \bar{p} = l \oplus \bar{u} \), which is a parabolic subalgebra, and \( \mathfrak{g} = \bar{p} \oplus u \).

For \( \lambda = 2\delta_n \), we consider the generalised Verma module \( \nabla_{\lambda} := U(\mathfrak{g}) \otimes U(\bar{\mathfrak{p}}) \mathbb{C}_{\lambda} \) of lowest weight type, where \( \mathbb{C}_{\lambda} \) denotes the irreducible \( \bar{p} \)-module with lowest weight \( \lambda \). Let \( v_0 \) denote a generator of \( \mathbb{C}_{\lambda} \), then

\[
\begin{align*}
f_r v_0 &= 0, \\
e_i v_0 &= 0, & f_i v_0 &= 0, & 1 \leq i \leq r - 1, \\
e_{ij} v_0 &= 2\delta_{j, r+1} v_0, & 1 \leq j \leq r + 1,
\end{align*}
\]

where \( e_i = e_{i, i+1} \) and \( f_i = e_{i+1, i} \).

Now \( \nabla_{\lambda} \cong U(u) \otimes \mathbb{C}_{\lambda} \) as \( l \)-module, where \( U(u) = S_*(u) \), the \( \mathbb{Z}_2 \)-graded symmetric algebra of \( u \). This superalgebra has a \( \mathbb{Z} \)-grading with \( u \) having degree 1. It induces a natural \( \mathbb{Z} \)-grading on \( \nabla_{\lambda} \). The unique maximal submodule \( M_{\lambda} \) of \( \nabla_{\lambda} \) is the direct sum of the homogeneous subspaces of degrees greater than or equal to 3, which is generated by \( U(u)_3 \otimes \mathbb{C}_{\lambda} \), the homogeneous subspace of degree 3. The irreducible quotient \( \mathcal{L}_{\lambda} \) of \( \nabla_{\lambda} \) is isomorphic to the \( \mathbb{Z}_2 \)-graded symmetric tensor of the natural \( \mathfrak{g} \)-module at rank 2. The natural \( l \) action on \( U(u) \) (obtained by generalising the adjoint action) respects the \( \mathbb{Z} \)-grading. In the present case, each homogeneous component is in fact an irreducible submodule. We are interested in \( U(u)_3 \). If \( u_3 \) is a nonzero lowest weight vector of \( U(u)_3 \), then \( M_{\lambda} \) is generated over \( \mathfrak{g} \) by \( u_3 \otimes \mathbb{C}_{\lambda} \). The form of \( u_3 \) depends on the ordering of the basis for \( \mathbb{C}^{m|n} \). Denote by \( E_{ij} \in U(\mathfrak{g}) \) the image of \( e_{ij} \in \mathfrak{g} \) under the natural embedding. The \( u_3 \) can be expressed as follows:

\[
\begin{align*}
u_3 &= E_{r+1, r+1}^3, & \text{if } & E_{r, r+1} \text{ is even;} \\
u_3 &= E_{r-1, r+1}^2 E_{r, r+1}, & \text{if both } & E_{r, r+1} \text{ and } E_{r-1, r} \text{ are odd;} \\
u_3 &= E_{r-2, r+1} E_{r-1, r+1} E_{r, r+1}, & \text{if } & E_{r, r+1} \text{ is odd but } E_{r-1, r} \text{ is even.}
\end{align*}
\]
Remark B.1. The third case becomes vacuous if $r = 2$; and both the second and third cases are vacuous if $r = 1$.

The irreducible quotient $\overline{L}_\lambda = \overline{V}_\lambda / M_\lambda$ is isomorphic to the graded skew symmetric rank two tensor $\wedge^2(C^{m/n})$ of the natural $g$-module.

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School of Mathematics and Statistics, University of Sydney, Sydney, Australia
E-mail address: ruibin.zhang@sydney.edu.au