MEAN VALUE RESULTS AND Ω-RESULTS FOR THE HYPERBOLIC LATTICE
POINT PROBLEM IN CONJUGACY CLASSES

DIMITRIOS CHATZAKOS

Abstract. For Γ a cofinite Fuchsian group, we study the lattice point problem in conjugacy classes on the Riemann surface \( \Gamma \setminus \mathbb{H} \). Let \( \mathcal{H} \) be a hyperbolic conjugacy class in \( \Gamma \) and \( \ell \) the \( \mathcal{H} \)-invariant closed geodesic on the surface. The main asymptotic for the counting function of the orbit \( \mathcal{H} \cdot z \) inside a circle of radius \( t \) centered at \( z \) grows like \( c_{\mathcal{H}} e^{t/2} \). This problem is also related with counting distances of the orbit of \( z \) from the geodesic \( \ell \). For \( X \sim e^{t/2} \) we study mean value and \( \Omega \)-results for the error term \( e(\mathcal{H}, X; z) \) of the counting function. We prove that a normalized version of the error \( e(\mathcal{H}, X; z) \) has finite mean value in the parameter \( t \). Further, we prove that if \( \Gamma \) is cocompact then

\[
\int_{\ell} e(\mathcal{H}, X; z) ds(z) = \Omega \left( X^{1/2} \log \log \log X \right).
\]

For \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) we prove the same \( \Omega \)-result, using a subconvexity bound for the Epstein zeta function associated to an indefinite quadratic form in two variables. We also study pointwise \( \Omega \) results for the error term. Our results extend the work of Phillips and Rudnick for the classical lattice problem to the conjugacy class problem.

1. Introduction

1.1. Mean value and \( \Omega \)-results for the classical hyperbolic lattice point problem. Let \( \mathbb{H} \) be the hyperbolic plane, \( z, w \) two fixed points in \( \mathbb{H} \) and \( \rho(z, w) \) their hyperbolic distance. For \( \Gamma \) a cocompact or cofinite Fuchsian group, the classical hyperbolic lattice point problem asks to estimate the quantity

\[
N(X; z, w) = \# \left\{ \gamma \in \Gamma : \rho(z, \gamma w) \leq \cosh^{-1} \left( \frac{X}{2} \right) \right\},
\]

as \( X \to \infty \). This problem has been extensively studied by many authors [1, 5, 6, 8, 10, 11, 21, 22, 24]. One of the main methods to understand this problem is using the spectral theory of automorphic forms. For this reason, let \( \Delta \) be the Laplacian of the hyperbolic surface \( \Gamma \setminus \mathbb{H} \) and let \( \{u_j\}_{j=0}^\infty \) be the \( L^2 \)-normalized eigenfunctions (Maass forms) of \( -\Delta \) with eigenvalues \( \{\lambda_j\}_{j=0}^\infty \). We also write \( \lambda_j = s_j(1-s_j) = 1/4 + t_j^2 \). Selberg [24], Günther [10], Good [8] et. al. proved that

\[
N(X; z, w) = \sum_{1/2 < s_j \leq 1} \sqrt{\frac{\Gamma(s_j - 1/2)}{\Gamma(s_j + 1)}} u_j(z) \overline{u_j(w)} X^{s_j} + E(X; z, w),
\]

where the error term \( E(X; z, w) \) satisfies the bound

\[
E(X; z, w) = O(X^{2/3}).
\]

Conjecturally, the optimal upper bound for the error term \( E(X; z, w) \) is

\[
E(X; z, w) = O_\epsilon(X^{1/2+\epsilon})
\]

for every \( \epsilon > 0 \) (see [21], [22]). This error term has a spectral expansion over all \( \lambda_j \geq 1/4 \). The contribution of \( \lambda_j = 1/4 \) is well understood. We subtract it from \( E(X; z, w) \) and we define the refined
error term $e(X; z, w)$ to be the difference

$$e(X; z, w) = E(X; z, w) - h(0) \sum_{t_j=0} u_j(z) u_j(w) = E(X; z, w) + O(X^{1/2} \log X),$$

where $h(t)$ is the Selberg/Harish-Chandra transform of the characteristic function $\chi_{[0, (X-2)/4]}$ (see [2, p. 2] for the details). Thus, bound (1.2) is equivalent with the bound

(1.3) $$e(X; z, w) = O(X^{1/2+\epsilon})$$

for every $\epsilon > 0$. For $z = w$, Phillips and Rudnick proved mean value results and $\Omega$-results (i.e. lower bounds for the $\limsup |e(X; z, z)|$) that support conjecture (1.3). For $\Gamma$ cofinite but not cocompact, let $E_a(z, s)$ be the nonholomorphic Eisenstein series corresponding to the cusp $a$. Phillips and Rudnick [22] proved the following theorems.

**Theorem 1.1** (Phillips-Rudnick [22]). (a) Let $\Gamma$ be a cocompact group. Then:

(1.4) $$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{e(2 \cosh r; z, z)}{e^{r/2}} dr = 0.$$

(b) Let $\Gamma$ be a cofinite but not cocompact group. Then:

(1.5) $$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{e(2 \cosh r; z, z)}{e^{r/2}} dr = \sum_a |E_a(z, 1/2)|^2.$$

**Theorem 1.2** (Phillips-Rudnick [22]). (a) If $\Gamma$ is cocompact or a subgroup of finite index in $PSL_2(\mathbb{Z})$, then for all $\delta > 0$,

$$e(X; z, z) = \Omega \left( X^{1/2} (\log \log X)^{1/4-\delta} \right).$$

(b) If $\Gamma$ is cofinite but not cocompact, and either has some eigenvalues $\lambda_j > 1/4$ or some cusp $a$ with $E_a(z, 1/2) \neq 0$, then,

$$e(X; z, z) = \Omega \left( X^{1/2} \right).$$

(c) If any other cofinite case, for all $\delta > 0$,

$$e(X; z, z) = \Omega \left( X^{1/2-\delta} \right).$$

In the proof of Theorem 1.2, the assumption $z = w$ is essential. In [2], we studied $\Omega$-results for the average

(1.6) $$M(X; z, w) = \frac{1}{X} \int_2^X \frac{e(x; z, w)}{x^{1/2}} dx$$

for two different points $z, w$. We proved that, if $\lambda_1 > 2.7823...$ and $z, w$ are sufficiently close to each other, the limit of $M(X; z, w)$ as $X \to \infty$ does not exist. In many cases, these results imply pointwise $\Omega$-results for $e(X; z, w)$ with $z \neq w$ as immediate corollaries.

There are specific groups $\Gamma$ for which we can provide refined $\Omega$-results. In [2], we proved that if $\Gamma$ is a cofinite group with sufficiently many cusp forms at the point $z$ in the sense that the series

$$\sum_{|t_j| < T} |u_j(z)|^2 \gg T^2$$

and satisfies $E_a(z, 1/2) \neq 0$ for some cusp $a$ then

$$e(X; z, w) = \Omega_{\pm} \left( X^{1/2} \right)$$

for $z$ fixed and $w$ sufficiently close to $z$ (see Corollary 1.9 in [2]).
1.2. The conjugacy class problem. In this paper we are interested in studying mean value results and $\Omega$-results for the hyperbolic lattice point problem in conjugacy classes. In this problem we restrict the action of $\Gamma$ in a hyperbolic conjugacy class $\mathcal{H} \subset \Gamma$; that means $\mathcal{H}$ is the conjugacy class of a hyperbolic element of $\Gamma$. Let $z \in \mathbb{H}$ be a fixed point. The problem asks to estimate the asymptotic behavior of the quantity

$$N_z(t) = \# \{ \gamma \in \mathcal{H} : \rho(z, \gamma z) \leq t \},$$

as $t \to \infty$. This problem was first studied by Huber in [12, 13]. The main reason we are interested in this problem is because it is related with counting distances of points in the orbit of the fixed point $z$ from a closed geodesic. This geometric interpretation was first explained by Huber in [12] and later in [13]. Assume $\mathcal{H}$ is the conjugacy class of the hyperbolic element $g^\nu$ with $g$ primitive and $\nu \in \mathbb{N}$. Let also $\ell$ be the invariant closed geodesic of $g$. Then $N_z(t)$ counts the number of $\gamma \in \langle g \rangle \backslash \Gamma$ such that $\rho(\gamma z, \ell) \leq t$. Equivalently, assume that $\ell$ lie on $\{ y_i, y > 0 \}$ (after conjugation). Let $\mu = \mu(\ell)$ be the length of $\ell$ and let $X$ be given by the change of variable

$$(1.7) \quad X = \frac{\sinh(t/2)}{\sinh(\mu/2)} \sim c_{\mathcal{H}} \cdot e^{t/2}.$$

Huber's interpretation shows that $N_z(t)$ actually counts $\gamma \in \langle g \rangle \backslash \Gamma$ such that $\cos \nu \geq X^{-1}$, where $\nu$ is the angle defined by the ray from $0$ to $\gamma z$ and the geodesic $\{ y_i, y > 0 \}$.

Under parametrization (1.7) denote $N_z(t)$ by $N(\mathcal{H}, X; z)$. Thus we have

$$N(\mathcal{H}, X; z) = \# \left\{ \gamma \in \mathcal{H} : \frac{\sinh(\rho(z, \gamma z)/2)}{\sinh(\mu/2)} \leq X \right\}.$$

The conjugacy class problem holds also a main formula similar to formula (1.1), which can be proved using the spectral theorem for $L^2(\Gamma \backslash \mathbb{H})$. This formula was first derived by Good in [8]; it can also be written in the following explicit form, see [4].

**Theorem 1.3 (Good [8], Chatzakos-Petridis [4]).** Let $\Gamma$ be a cofinite Fuchsian group and $\mathcal{H}$ a hyperbolic conjugacy class of $\Gamma$. Then:

$$N(\mathcal{H}, X; z) = \sum_{1/2 < s_j \leq 1} A(s_j) \tilde{u}_j u_j(z) X^{s_j} + E(\mathcal{H}, X; z),$$

where $A(s)$ is the product:

$$(1.8) \quad A(s) = 2^s \cos \left( \frac{\pi(s - 1)}{2} \right) \frac{\Gamma \left( \frac{s + 1}{2} \right)}{\Gamma \left( \frac{1 - s}{2} \right)} \frac{\Gamma \left( s - \frac{1}{2} \right)}{\pi \Gamma(s + 1)},$$

$$(1.9) \quad \tilde{u}_j = \int_{\sigma} \pi_j(z) ds(z)$$

is the period integral of $\pi_j$ along a segment $\sigma$ of the invariant closed geodesic of $\mathcal{H}$ with length $\int_{\sigma} ds(z) = \mu/\nu$ and $E(\mathcal{H}, X; z) = O(X^{2/3})$.

Notice that Theorem 1.3 implies the main asymptotic of $N(\mathcal{H}, X; z)$ is

$$N(\mathcal{H}, X; z) \sim \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})\nu} \mu X.$$

Once again we are interested in the growth of the error term. The similarities that arise between the two problems suggest that we should expect the bound

$$(1.10) \quad E(\mathcal{H}, X; z) = O_e(X^{1/2+\epsilon}).$$

(see [4, Conjecture 5.7]). As in the classical problem, the error term $E(\mathcal{H}, X; z)$ has a 'spectral expansion' over the eigenvalues $\lambda_j \geq 1/4$. We subtract the contribution of the eigenvalue $\lambda_j = 1/4$ and we denote
the expansion over the eigenvalues $\lambda_j > 1/4$ by $\epsilon(\mathcal{H}, X; z)$ (eq. (2.12)). In section 2 we prove that the bound (1.10) is equivalent with the bound

$$(1.11) \quad \epsilon(\mathcal{H}, X; z) = O_{\epsilon}(X^{1/2+\epsilon}).$$

In order to state our first result, we will need the following definition.

**Definition 1.4.** The Eisenstein period associated to the hyperbolic conjugacy class $\mathcal{H}$ is the period integral

$$(1.12) \quad \hat{E}_a(1/2 + it) = \int_\sigma E_a(z, 1/2 - it) ds(z),$$

across a segment $\sigma$ of the invariant geodesic $\ell$ with length $\int_\sigma ds(z) = \mu/\nu$.

In section 3 we prove that the error term $\epsilon(\mathcal{H}, X; z)$ has finite mean value in the radial parameter $t$.

**Theorem 1.5.** Let $\Gamma$ be a cofinite Fuchsian group.

(a) If $\Gamma$ is cocompact, then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\epsilon(\mathcal{H}, e^r; z)}{e^{r/2}} dr = 0.$$ 

(b) If $\Gamma$ is cofinite but not cocompact, then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\epsilon(\mathcal{H}, e^r; z)}{e^{r/2}} dr = \frac{\left|\Gamma(3/4)\right|^2}{\pi^{3/2}} \sum_a \hat{E}_a(1/2) E_a(z, 1/2).$$

**Remark 1.6.** Using the change of variables (1.7) we see that Theorem 1.5 is indeed a mean value result in the radial parameter $t$.

For the conjugacy class problem, proving pointwise $\Omega$-results is a more subtle problem comparing to the classical one, due to the appearance of the period integrals in the spectral expansion of $\epsilon(\mathcal{H}, X; z)$. In the proof of Theorem 1.2, Phillips and Rudnick choose $z = w$ so that the series expansion of the error term $\epsilon(X; z, w)$ contains the expressions $|u_j(z)|^2$ which are nonnegative. In this setting, the natural choice is to average over the $\mathcal{H}$-invariant geodesic $\ell$. For this reason, we will need the following result of Good and Tsuzuki which describes the exact asymptotic behaviour of the period integrals.

**Theorem 1.7** (Good [8], Tsuzuki [26]). The period integrals $\hat{u}_j$ of Maass forms and $\hat{E}_a(1/2 + it)$ of Eisenstein series satisfy the asymptotic

$$\sum_{|t_j| < T} |\hat{u}_j|^2 + \sum_a \frac{1}{4\pi} \int_{-T}^T |\hat{E}_a(1/2 + it)|^2 dt \sim \frac{\mu(\ell)}{\pi} \cdot T,$$ 

where $\mu(\ell)$ denotes the length of the invariant closed geodesic $\ell$.

We refer to [19, p. 3-4] for a detailed history of this result. We also give the following definition which is related to Theorem 1.7.

**Definition 1.8.** Fix $\mathcal{H}$ be a hyperbolic class of a cofinite but not cocompact group $\Gamma$. We say that the group $\Gamma$ has sufficiently small Eisenstein periods associated to $\mathcal{H}$ if for all cusps $a$ we have

$$\int_{-T}^T |\hat{E}_a(1/2 + it)|^2 dt \ll \frac{T}{(\log T)^{1+\delta}}$$

for a fixed $\delta > 0$. 

For the rest of this paper we write $\int_{\mathcal{H}} ds$ to indicate that we average over a segment of the invariant geodesic $\ell$ of length $\mu/\nu$. When $\mathcal{H}$ is the class of a primitive element we get $\nu = 1$, hence $\int_{\mathcal{H}} ds = \int_{\ell} ds$.

We distinguish the two cases of $\Omega$-results: if $g(X)$ is a positive function, we write $e(X; z, w) = \Omega_+(g(X))$ if

$$\limsup \frac{e(X; z, w)}{g(X)} > 0,$$

and $e(X; z, w) = \Omega_-(g(X))$ if

$$\liminf \frac{e(X; z, w)}{g(X)} < 0.$$

In section 4 we prove the following theorem, which is an average $\Omega$-result on the closed geodesic of $\mathcal{H}$.

**Theorem 1.9.** (a) If $\Gamma$ is either (i) cocompact or (ii) cofinite but not cocompact and has sufficiently small Eisenstein periods associated to $\mathcal{H}$ according to Definition 1.8, then

$$\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds(z) = \Omega_+(X^{1/2} \log \log X).$$

(b) If $\Gamma$ is cofinite but not cocompact and either (i) $\hat{u}_j \neq 0$ for at least one $\lambda_j > 1/4$ or (ii) $\hat{E}_a(1/2) \neq 0$ for a cusp $a$ then

$$\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds(z) = \Omega_+(X^{1/2}).$$

**Remark 1.10.** In subsection 4.3 we will see that the modular group $\Gamma = \text{PSL}_2(\mathbb{Z})$ has sufficiently small Eisenstein periods associated to a fixed conjugacy class $\mathcal{H} \subset \Gamma$. This follows from a subconvexity bound on the critical line for an Epstein zeta function associated to $\mathcal{H}$.

The asymptotic behaviour for the sums of period integrals in Theorem 1.7 is $cT$, where in local Weyl’s law (Theorem 2.6) we get an asymptotic $cT^2$. If $\Gamma$ is cocompact or cofinite but it has sufficiently small Eisenstein periods associated to $\mathcal{H}$ then

$$\sum_{|t_j| < T} |\hat{u}_j|^2 \sim \frac{\mu(\ell)}{\pi} T,$$

and summation by parts implies

$$\sum_{|t_j| < T} \frac{|\hat{u}_j|^2}{t_j} \gg \log T.$$  

In case (a) of Theorem 1.9 the triple logarithm should be compared with the extra factor $(\log \log X)^{1/4-\delta}$ in case (a) of Theorem 1.2. The first is a consequence of the asymptotic behaviour of period integrals in Theorem 1.7, and the second is a consequence of the local Weyl’s law.

To prove pointwise $\Omega$-results for $e(\mathcal{H}, X; z)$ we would like to have a fixed pair $(z, \mathcal{H})$ with $e(\mathcal{H}, X; z)$ large, i.e. a pair $(z, \mathcal{H})$ with a uniform ‘fixed sign’ property of all $\hat{u}_j u_j(z)$. That would allow us to prove a pointwise $\Omega$-result of the form

$$\limsup_{X} \frac{|e(\mathcal{H}, X; z)|}{X^{1/2}} = \infty.$$

However, Maass forms have complicated behaviour on the surfaces $\Gamma \backslash \mathbb{H}$; for instance, the nodal domains have very complicated shapes. For this reason we have not been able to determine any such specific pair $(z, \mathcal{H})$ with the desired fixed sign property. To overcome this problem we notice that the period integral is the limit of Riemann sums. Starting with a fixed conjugacy class $\mathcal{H}$, a discrete average allows us to prove the existence of at least one point $z = z_{\mathcal{H}}$ for which the error $e(\mathcal{H}, X; z_{\mathcal{H}})$ cannot be small.

We first prove the following proposition for discrete averages.
Proposition 1.11. Let $\mathcal{H}$ be a fixed hyperbolic class in $\Gamma$. If $\Gamma$ is either (i) cocompact or (ii) if $\Gamma$ is as in part (b) of Theorem 1.9, then there exist an integer $K = K_{\mathcal{H}}$ depending only on $\mathcal{H}$ and $z_1, z_2, \ldots, z_K$ points on $\ell$ such that:

$$\frac{1}{K} \sum_{m=1}^{K} e(\mathcal{H}, X; z_m) = \Omega_+(X^{1/2}).$$

In comparison with our results in [2], in order to prove $\Omega_-$-results for the error $e(\mathcal{H}, X; z)$ we are lead to investigate the behaviour of a modification of the average error term

$$\frac{1}{X} \int_{1}^{X} \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dx$$

on the geodesic $\ell$.

Proposition 1.12. Let $\Gamma$ be either (i) cocompact or (ii) cofinite but not cocompact, $\hat{a}_j \neq 0$ for at least one $\lambda_j > 1/4$ and $\hat{E}_a(1/2) = 0$ for all cusps $a$. Then there exist an integer $K = K_{\mathcal{H}}$ and $z_1, z_2, \ldots, z_K$ points in $\ell$ such that, as $X \to \infty$:

$$\frac{1}{K} \sum_{m=1}^{K} \frac{1}{X} \int_{1}^{X} \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dx = \Omega_-(1).$$

We deduce the following theorem on pointwise $\Omega$-results for the error term $e(\mathcal{H}, X; z)$ as an immediate corollary of Theorem 1.5 and Propositions 1.11, 1.12.

Theorem 1.13. Let $\Gamma$ be a Fuchsian group, $\mathcal{H}$ a hyperbolic conjugacy class of $\Gamma$ and $\ell$ the invariant closed geodesic of $\mathcal{H}$.

(a) If $\Gamma$ is as in Proposition 1.11, then there exist at least one point $z_\mathcal{H} \in \ell$ such that:

$$e(\mathcal{H}, X; z_\mathcal{H}) = \Omega_+(X^{1/2}).$$

(b) If $\Gamma$ is as in Proposition 1.12, then there exists at least one point $z_\mathcal{H} \in \ell$ such that:

$$e(\mathcal{H}, X; z_\mathcal{H}) = \Omega_-(X^{1/2}).$$

(c) If $\Gamma$ is not cocompact and the sum $\sum_a \hat{E}_a(1/2)E_a(z, 1/2)$ does not vanish then:

$$e(\mathcal{H}, X; z) = \Omega(X^{1/2}).$$

Finally, at the last section, as an application of Theorem 1.7 we obtain upper bounds for the error terms of both the classical problem and the conjugacy class problem on geodesics.

Remark 1.14. For the proof of Theorem 1.9 and Propositions 1.11, 1.12 we will crucially need some ‘fixed-sign’ properties of the $\Gamma$-function stated in Lemma 2.5. We emphasize that the differences in the signs in the two cases of Lemma 2.5 cause the different signs of our $\Omega$-results.

Remark 1.15. It follows from Theorem 1.13 that in order to prove a pointwise result $e(\mathcal{H}, X; z) = \Omega(X^{1/2})$ for one point $z$, we must only assume the nonvanishing of one period $\hat{a}_j$. In this case, the sign of our $\Omega$-result can be determined by the vanishing or not of the Eisenstein period integrals. If $\Gamma$ is cocompact or all Eisenstein periods vanish then there exists at least two points $z, w \in \ell$ such that:

$$e(\mathcal{H}, X; z) = \Omega_+(X^{1/2}),$$

$$e(\mathcal{H}, X; w) = \Omega_-(X^{1/2}).$$

(1.17)

These Eisenstein periods are of particular arithmetic interest; in fact $\hat{E}_a(1/2)$ is the constant term of the hyperbolic Fourier expansion of $E_a(z, s)$ (see [7, section 3.2]). In the arithmetic case, these periods are associated to special values of Epstein zeta functions (see subsection 4.3). We notice that, in principle, it is easier to check the nonvanishing of one period $\hat{E}_a(1/2)$ than the nonvanishing of the sum $\sum_a \hat{E}_a(1/2)E_a(z, 1/2)$.
Remark 1.16. Phillips and Rudnick in [22] generalized Theorem 1.1 and case c) of Theorem 1.2 in the case of the n-dimensional hyperbolic space $\mathbb{H}^n$ [22, p. 106].

Recently, Paarkonen and Paulin [20] studied the hyperbolic lattice point problem in conjugacy classes for the $n$-th hyperbolic space $H^n$ [22, p. 106]. However, their geometric approach cannot be used to generalise our results in dimensions $n \geq 3$. To do this, we need an explicit expression for the Huber transform $d_n(f, t)$ in the $n$-th dimension. In dimension $n = 3$, $d_3(f, t)$ was recently studied explicitly by Laaksonen in [18], where he obtained upper bounds for the second moments of the error term, generalising previous work by the author and Petridis [4].

1.3. Acknowledgments. I would like to thank my supervisor Y. Petridis for his helpful guidance and encouragement. I would also want to thank V. Blomer for bringing to my knowledge the subconvexity bound for the Epstein zeta function of an indefinite quadratic form. Finally, I would like to thank the two anonymous referees for many corrections and their valuable and helpful comments.

2. Spectral theory and counting

2.1. The Huber transform. We briefly state the basic results from the spectral theory of automorphic forms for the conjugacy class problem (see [4, section 2] for the details). Let $C_0^\infty[1, \infty)$ denote the space of real functions of compact support that are bounded in $[1, \infty)$ and have at most finitely many discontinuities.

Definition 2.1. Let $f \in C_0^\infty[1, \infty)$. The Huber transform $d(f, t)$ of $f$ at the spectral parameter $t$ is defined as

\begin{equation}
    d(f, t) = \int_0^{\pi/2} f \left( \frac{1}{\cos^2 v} \right) \frac{\xi_\lambda(v)}{\cos^2 v} dv,
\end{equation}

with $\lambda = 1/4 + t^2$, and $\xi_\lambda$ is the solution of the differential equation

\begin{equation}
    \xi_\lambda''(v) + \frac{\lambda}{\cos^2 v} \xi_\lambda(v) = 0, \quad v \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right),
\end{equation}

with $\xi_\lambda(0) = 1$, $\xi_\lambda'(0) = 0$.

The Huber transform plays a role analogous to that of the Selberg/Harish-Chandra transform in the classical counting (see [4, [13]). For this reason we work with $d(f, t)$ for an appropriate test function $f = f_X$.

2.2. The test function and counting. Assume first that $\Gamma \backslash \mathbb{H}$ is compact. For an $f \in C_0^\infty[1, \infty)$ we define the $\Gamma$-automorphic function

\begin{equation}
    A(f)(z) = \sum_{\gamma \in \mathcal{H}} f \left( \frac{\cosh \rho(z, \gamma z) - 1}{\cosh \mu - 1} \right).
\end{equation}

The following proposition gives the Fourier expansion of the counting function $A(f)(z)$ (see [4, p. 984], [13, p. 17]).

Proposition 2.2. The function $A(f)(z)$ has a Fourier expansion of the form

\begin{equation}
    A(f)(z) = \sum_j 2d(f, t_j) \hat{u}_j u_j(z),
\end{equation}

where $d(f, t)$ is the Huber transform of $f$.

The quantity $N(\mathcal{H}, X; z)$ can be interpreted as

\begin{equation}
    A(f_X)(z) = N(\mathcal{H}, X; z),
\end{equation}

for $f_X = \chi_{[1, X^2]}$, the characteristic function of the interval $[1, X^2]$. We have the following lemma.
Lemma 2.3. Let \( s = 1/2 + it \). Let also \( U = \sqrt{X^2 - 1} \), \( R = \log(X + U) \) and \( r = \log(x + \sqrt{x^2 - 1}) \) (thus \( X = 2 \cosh R \) and \( x = 2 \cosh r \)) and define the function

\[
G(t) = \frac{2\sqrt{2}}{\pi} \frac{\Gamma(3/4 + it/2)^2}{\Gamma(3/2 + it)} \cos(it/2 - \pi/4).
\]

Then, for the Huber transform of \( f_X \) we have the following estimates.

(a) If \( s \in (1/2, 1] \) then

\[
2d(f_X, t) = A(s)X^s + O\left((s - 1/2)^{-1}X^{1-s}\right),
\]

where \( A(s) \) is the \( \Gamma \)-product defined in (1.8).

(b) For \( t \in \mathbb{R} - \{0\} \) we have

\[
2d(f_X, t) = \Re \left(G(t)\Gamma(it)e^{itR}\right) X^{1/2} + \Re \left(V(R, t)e^{itR}\right),
\]

with \( V(R, t) = O\left((1 + |t|)^{-2}X^{-3/2}\right) \).

(c) For \( t = 0 \) we have

\[
d(f_X, 0) = O(X^{1/2} \log X).
\]

Remark 2.4. Stirling’s formula implies that, as \( |t| \to \infty \),

\[
|G(t)\Gamma(it)| \asymp (1 + |t|)^{-1}.
\]

We can now give the proof of the Lemma.

Proof. (a) Using the integral representation for \( d(f_X, t) \) in [4, p. 5] we get

\[
d(f_X, t) = (2\sqrt{\pi})^{-1} \Gamma\left(s + 1/2\right) \Gamma\left(1 - s/2\right) \int_0^U \left(P_{0}^{0} - P_{s-1}^{0}\right) dv.
\]

Using [9, p. 968, eq. (8.752.3)], this takes the form

\[
d(f_X, t) = (2\sqrt{\pi})^{-1} \Gamma\left(s + 1/2\right) \Gamma\left(1 - s/2\right) X \left(P_{s-1}^{1}(iU) - P_{s-1}^{1}(-iU)\right).
\]

Using formula [9, p. 971, eq. (8.776)], the statement follows.

(b) We use [9, p. 971, eq. (8.774)], so that equation (2.9) gives

\[
2d(f_X, t) = \Re \left(G(t)\Gamma(it)e^{itR}F\left(-1/2, 3/2; 1 + it; \frac{e^{-R}}{2X}\right)\right) X^{1/2},
\]

where \( F(a, b; c; z) \) denotes the Gauss’ hypergeometric function. As \( X \to \infty \), the definition of the hypergeometric function [9, p. 1005, eq. (9.100)] implies

\[
F\left(-1/2, 3/2; 1 + it; \frac{e^{-R}}{2X}\right) = 1 + O\left((1 + |t|)^{-1}X^{-2}\right).
\]

The statement of part (b) now follows.

(c) Plugging \( t = 0 \), i.e. \( s = 1/2 \), in eq. (2.9) and using formula [9, p. 961, eq. (8.713.2)], we calculate

\[
P_{-1/2}^{-1}(iU) - P_{-1/2}^{-1}(-iU) \ll X^{5/2} \int_0^\infty \left(\cosh^2 t + U^2\right)^{-3/2} dt
\]

\[
\ll X^{-1/2} \int_0^\infty \left(1 + \frac{\cosh t}{U}\right)^{-3/2} dt.
\]
Setting $x = \cosh t/U$ we get
\[
\int_0^\infty \left( \left( \frac{\cosh t}{U} \right)^2 + 1 \right)^{-3/2} dt = \int_{1/U}^\infty \frac{U}{(U^2x^2 - 1)^{1/2}} dx = \int_{1/U}^1 \frac{U}{(U^2x^2 - 1)^{1/2}} dx + \int_1^\infty \frac{U}{(U^2x^2 - 1)^{1/2}} dx.
\]

For $U \geq 2$ we get
\[
\int_1^\infty (x^2 + 1)^{-3/2} \frac{U}{(U^2x^2 - 1)^{1/2}} dx \ll \int_1^\infty (x^2 + 1)^{-3/2} dx \ll 1
\]
and, after setting $u = xU$,
\[
\int_{1/U}^1 (x^2 + 1)^{-3/2} \frac{U}{(U^2x^2 - 1)^{1/2}} dx = \int_1^U \frac{U^2}{(u^2 + U^2)^{3/2}} \frac{U}{(u^2 - 1)^{1/2}} du \leq \int_1^U \frac{1}{u - 1} du \ll \log U \ll \log X.
\]
Combining these estimates we get
\[
P_{-1/2}^{-1}(iU) + P_{-1/2}^{-1}(-iU) \ll X^{-1/2} \log X.
\]

If we ignore for a while any issue of convergence, then using (a) of Lemma 2.3 and Proposition 2.2 we obtain that, in the compact case, the error term $E(\mathcal{H}, X; z)$ has a formal ‘spectral expansion’ of the form
\[
E(\mathcal{H}, X; z) = \sum_{t_j \in \mathbb{R}} 2d(f_X, t_j) \tilde{u}_j u_j(z) + O \left( \sum_{1/2 < s_j \leq 1} (s_j - 1/2)^{-1} X^{1-s_j} \right).
\]
The $s_j$’s are discrete, thus we can find a constant $\sigma = \sigma_\mathcal{H} \in (0, 1/2]$ such that $s_j - 1/2 \geq \sigma$ for all $s_j \in (1/2, 1]$. This implies that the above $O$-term is $O(X^{1/2-\sigma})$. Using (c) of Lemma 2.3 and the finiteness of the eigenspace for the eigenvalue $t_j = 0$ we get the bound
\[
d(f_X, 0) \sum_{t_j = 0} \tilde{u}_j u_j(z) = O(X^{1/2} \log X).
\]
Since the contribution of the eigenvalue $\lambda_j = 1/4$ is well understood and does not affect the square root cancellation conjecture for the error term, we subtract this quantity from $E(\mathcal{H}, X; z)$ and we define the modified error term $e(\mathcal{H}, X; z)$ to be the difference
\[
e(\mathcal{H}, X; z) = E(\mathcal{H}, X; z) - d(f_X, 0) \sum_{t_j = 0} \tilde{u}_j u_j(z).
\]
Thus, if we ignore issues of convergence, for $\Gamma$ cocompact we conclude the principal series of the error $e(\mathcal{H}, X; z)$ takes the form:
\[
e(\mathcal{H}, X; z) = \sum_{t_j > 0} 2d(f_X, t_j) \tilde{u}_j u_j(z) + O(X^{1/2-\sigma}).
\]
2.3. Some more auxiliary lemmas. One of the key ingredients in the proofs of our results is the following lemma.

Lemma 2.5. For every $t \in \mathbb{R} - \{0\}$, we have:

a)\[
\Re (G(t) \Gamma(it)) > 0,
\]

b)\[
\Re \left( \frac{G(t) \Gamma(it)}{1 + it} \right) < 0.
\]

Proof. (of Lemma 2.5) a) Obviously, the first inequality is equivalent with

\[
\Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \cos(it/2 - \pi/4) \right) > 0.
\]

Since $\Gamma(x) = \Gamma(x)$, it suffices to prove the lemma for $t > 0$. Notice that

\[
\cos(it/2 - \pi/4) = \frac{\cosh \left( \frac{\pi t}{\sqrt{2}} \right)}{\sqrt{2}} + i \sinh \left( \frac{\pi t}{\sqrt{2}} \right).
\]

Using [9, p. 909, eq. (8.384.1)] we get

\[
\frac{\Gamma(it)}{\Gamma(3/2 + it)} = \frac{2}{\sqrt{\pi}} B(it, 3/2),
\]

where $B(x, y)$ is the Beta function. By the definition of Beta function [9, p. 908, eq. (8.380.1)] and the formula

\[
B(x + 1, y) = B(x, y) \frac{x}{x + y}
\]

we see that inequality (2.14) is equivalent with

\[
\Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \cosh \left( \frac{\pi t}{2} \right) - 3 \frac{\Gamma(it)}{\Gamma(3/2 + it)} \sinh \left( \frac{\pi t}{2} \right) \right) > 0,
\]

which is equivalent with $Q(t) > 0$, where $Q(t)$ is the function defined by

\[
Q(t) := \left( \int_{1}^{1} \cos(t \log s)(1 - s)^{1/2} ds \right) \left( 2t + 3 \tanh \left( \frac{\pi t}{2} \right) \right) + \left( \int_{0}^{1} \sin(t \log s)(1 - s)^{1/2} ds \right) \left( 3 - 2t \tanh \left( \frac{\pi t}{2} \right) \right).
\]

From [2, Lemma 2.2] it follows that if $f : (-\infty, 0) \to \mathbb{R}$ is a continuous and strictly decreasing real valued function such that $f(x) \sin(x)$ is integrable in $(-\infty, 0)$, then

\[
\int_{-\infty}^{0} f(x) \sin(x) dx > 0.
\]

To prove (2.16) we integrate by parts, we set $s = e^{x/t}$ and we apply (2.17) for

\[
f_{t}(x) = \frac{1}{t} \left( 1 - e^{x/t} \right)^{1/2} e^{x/t} \left( 2t + 3 \tanh \left( \frac{\pi t}{2} \right) \right) - \left( 1 - e^{x/t} \right)^{1/2} e^{x/t} \left( 2 - \frac{4t}{3} \tanh \left( \frac{\pi t}{2} \right) \right),
\]

which can be easily checked to be decreasing for $t \geq 2/\pi$. For $t \leq 2/\pi$, notice that

\[
\Re (G(t) \Gamma(it)) = \frac{4}{\pi^{3/2}} \left| \Gamma \left( \frac{3}{4} + \frac{it}{2} \right) \right|^{2} \cosh(\pi t/2) Q(t) t.
\]

Taking $t \to 0$ we get $\lim_{t \to 0} Q(t)/t > 0$, hence $\lim_{t \to 0} \Re (G(t) \Gamma(it)) > 0$ and the lemma holds for $t$ sufficiently small. Taking derivatives, we write $Q'(t)$ in the form $Q'(t) = \int_{-\infty}^{0} g_{t}(x) \sin(tx) dx$. Applying [2, Lemma 2.2] to $Q'(t)$ we conclude that $Q(t)$ is increasing. Hence, part a) follows. Part (b) can be proved along exactly the same lines, using [2, Lemma 2.2].
We will finally need the following estimate for the Maass forms and the Eisenstein series which is a local version of Weyl’s law for $L^2(\Gamma \backslash \mathbb{H})$.

**Theorem 2.6** (Local Weyl’s law). For every $z$, as $T \to \infty$,

$$
\sum_{|t_j| < T} |u_j(z)|^2 + \sum_a \frac{1}{4\pi} \int_{-T}^T |E_a(z, 1/2 + it)|^2 dt \sim cT^2,
$$

where $c = c(z)$ depends only on the number of elements of $\Gamma$ fixing $z$.

See [22, p. 86, lemma 2.3] for a proof of this result. We emphasize that if $z$ remains in a compact set of $\mathbb{H}$ the constant $c(z)$ remains uniformly bounded.

3. The mean value result

3.1. **Proof of Theorem 1.5** for $\Gamma \backslash \mathbb{H}$ compact. We first prove the error term $e(\mathcal{H}, X;z)$ has zero mean value for $\Gamma$ cocompact.

**Proof.** In this case $\Gamma$ has only discrete spectrum. The characteristic function $f_X$ is not smooth; thus when we apply the spectral theorem for $L^2(\Gamma \backslash \mathbb{H})$ [14, p. 69, Theorem 4.7 and p. 103, Theorem 7.3] directly to $A(f_X)$, we deduce the spectral expansion (2.13). This principal series is not absolutely convergent. To avoid convergence issues, for $x = 2 \cosh r \sim e^r$ we use the identity

$$
\frac{d}{dT} \left( \frac{1}{T} \int_0^T \frac{f_x dr}{e^{r/2}} \right) = \frac{1}{T} \int_0^T \frac{d(f_x, t)}{e^{r/2}} dr,
$$

i.e. the Huber transform commutes with multiplication of $f_x$ by a function that depends only on the radial variable $x$, and it commutes with integration over $r$. Hence, if we define the integrated error

$$
M_\mathcal{H}(T) = \frac{1}{T} \int_0^T e(\mathcal{H}, x;z) \frac{dr}{x^{1/2}},
$$

this has the spectral expansion

$$
M_\mathcal{H}(T) = \sum_{t_j > 0} 2\hat{u}_j u_j(z) \frac{1}{T} \int_0^T \frac{d(f_x, t)}{x^{1/2}} dr + O(T^{-\sigma}).
$$

Using part (b) of Lemma 2.3 we conclude

$$
M_\mathcal{H}(T) = \sum_{t_j > 0} \Re \left( G(t_j) \Gamma(it_j) \frac{1}{T} \int_0^T e^{it_j r} dr \right) \hat{u}_j u_j(z) + O \left( \sum_{t_j > 0} \frac{|\hat{u}_j||u_j(z)|}{T} \left| \int_0^T V(r, t_j) \frac{dr}{x^{1/2}} e^{it_j r} dr \right| + \frac{1}{T} \int_0^T e^{-r\sigma} dr \right).
$$

Using Theorems 1.7, 2.6 and Stirling’s formula (estimate (2.7)) we bound the main term by $O(T^{-1})$. For the first summand in the $O$-term we use integration by parts. Using that $V(R, t)$ is given by the formula

$$
V(r, t) = G(t) \Gamma(it) \left( F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{e^{-r}}{2x} \right) - 1 \right) \frac{1}{x^{1/2}}
$$

and using trivial estimates for the derivative of the hypergeometric function we obtain

$$
\int_0^T \frac{V(r, t_j)}{x^{1/2}} e^{it_j r} dr = O(t_j^{-2}).
$$

Hence the $O$-terms are also bounded by $T^{-1}$, and the statement follows. $\square$
3.2. **Proof of Theorem 1.5 for \( \Gamma \) for cofinite.** In this case the hyperbolic Laplacian \(-\Delta\) has also continuous spectrum which is spanned by the Eisenstein series \( E_\alpha(z, 1/2 + it) \) (see [14, chapters 3, 6 and 7]). To prove case (b) of Theorem 1.5 we have to consider the contribution of the continuous spectrum in \( M^\Gamma(T) \), which is given in terms of the Eisenstein series \( E_\alpha(z, 1/2 + it) \) and the period integrals \( \hat{E}_\alpha(1/2 + it) \). More specifically, using [4, eq. (4.1)] and [4, Lemma 4.2] we get that the contribution of the continuous spectrum is given by

\[
\sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{E}_\alpha(1/2 + it) E_\alpha(z, 1/2 + it) \left( \frac{1}{T} \int_{-\infty}^{T} \frac{2d(f_x, t)}{x^{1/2}} \, dr \right) \, dt.
\]

To justify this, as in the discrete spectrum we notice it is well-defined as coming from the spectral expansion of the integrated error (3.1). Hence, to complete the proof of Theorem 1.5, we need to prove that the expansion in (3.3) converges to

\[
\left( \frac{\Gamma(3/4)}{\pi^{3/2}} \right)^2 \sum_a \hat{E}_\alpha(1/2) E_\alpha(z, 1/2)
\]
as \( T \to \infty \). To deal with this expansion, we need the following lemma for the Huber transform.

**Lemma 3.1.** As \( T \to \infty \) we have

\[
\lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{1}{T} \int_{-\infty}^{T} \frac{2d(f_x, t)}{x^{1/2}} \, dr \, dt = \frac{4}{\sqrt{\pi}} |\Gamma(3/4)|^2.
\]

**Proof.** Using expression (2.10) we write

\[
\int_{-\infty}^{\infty} \frac{1}{T} \int_{0}^{T} \frac{2d(f_x, t)}{x^{1/2}} \, dr \, dt = \Re \left( \int_{-\infty}^{\infty} \frac{1}{T} \int_{0}^{T} G(t) \Gamma(it) e^{irt} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2\pi r} + 1} \right) \, dr \right).
\]

The convergence of the above integral can be justified as above, using that the Huber transform commutes with convolution in the \( x \) variable. Let \( \varepsilon > 0 \) be a fixed small number and \( M > 0 \) be a fixed large number. We consider the path integral

\[
\int_{\gamma} G(z) \Gamma(i\varepsilon) \frac{1}{T} \int_{0}^{T} e^{irz} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2\pi r} + 1} \right) \, dr \, dz,
\]

where \( \gamma \) is the contour \( \gamma = \bigcup_{i=1}^{6} C_i \) with

\[
\begin{align*}
C_1 &= [\varepsilon, M], \\
C_2 &= [M + iv, v \in [0, 1/2]], \\
C_3 &= [-M + i/2, M + i/2], \\
C_4 &= [-M + iv, v \in [0, 1/2]], \\
C_5 &= [-M, -\varepsilon], \\
C_6 &= \{ \varepsilon e^{i\theta}, \theta \in [0, \pi] \},
\end{align*}
\]

traversed counterclockwise. To calculate (3.5) we write \( G(z) \) as

\[
G(z) = \frac{\sqrt{\pi} \Gamma \left( \frac{3}{2} + iz \right) \Gamma \left( \frac{3}{2} - iz \right)}{\Gamma(3/2 + iz)} \left( e^{-\frac{i\pi}{4} - \frac{iz}{2}} + e^{\frac{i\pi}{4} + \frac{iz}{2}} \right),
\]

hence we see that the integrand is holomorphic inside the contour. The simple pole at \( z = 0 \) is coming from \( \Gamma(i\varepsilon) \). We note that \( \text{Res}_{z=0} \Gamma(i\varepsilon) = -i \). Applying Stirling’s formula and the asymptotics of the hypergeometric function (2.11) we deduce

\[
\begin{align*}
\int_{C_2 + C_4} G(z) \Gamma(i\varepsilon) \frac{1}{T} \int_{0}^{T} e^{irz} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2\pi r} + 1} \right) \, dr \, dz &= O \left( M^{-2} T^{-1} \right), \\
\int_{C_3} G(z) \Gamma(i\varepsilon) \frac{1}{T} \int_{0}^{T} e^{irz} F \left( -\frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2\pi r} + 1} \right) \, dr \, dz &= O \left( T^{-1} \right).
\end{align*}
\]
Further, as $\varepsilon \to 0$ we see that the term

$$\int_{C_{\varepsilon}} G(z)\Gamma(iz)\frac{1}{T} \int_0^T e^{itz} F\left(\frac{1}{2} \frac{3}{2}; 1 + iz; \frac{1}{e^{2T} + 1}\right) drdz$$

converges to

$$-i\pi G(0)\frac{1}{T} \int_0^T F\left(\frac{1}{2} \frac{3}{2}; 1; \frac{1}{e^{2T} + 1}\right) dr \text{Res}_{z=0} \Gamma(iz) = -\pi G(0)(1 + O(T^{-1})).$$

From Cauchy’s Theorem we conclude

$$\int_{-M}^{M} G(t)\Gamma(it)\frac{1}{T} \int_0^T e^{itF}\left(-\frac{1}{2} \frac{3}{2}; 1 + it; \frac{1}{e^{2T} + 1}\right) drdt = \pi G(0)(1 + O(T^{-1})) + O(M^{-2}T^{-1} + T^{-1}).$$

As $M \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} drdt = \frac{2\Gamma(3/4)^2}{\Gamma(3/2)} + O(T^{-1}),$$

and for $T \to \infty$ the statement follows.

We let $\phi_{\mathcal{H},a}(t)$ denote the function

$$\phi_{\mathcal{H},a}(t) = \hat{E}_a(1/2 + it)E_a(z,1/2 + it) - \hat{E}_a(1/2)E_a(z,1/2).$$

Thus, the contribution of the cusp $a$ in eq. (3.3) can be written in the form

$$\frac{1}{4\pi} \hat{E}_a(1/2)E_a(z,1/2) \int_{-\infty}^{\infty} \left(\frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} dr\right) dt + \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{\mathcal{H},a}(t) \left(\frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} dr\right) dt.$$  

(3.7)

The second term of (3.7) can be handled using Lemma 2.3. We calculate:

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{\mathcal{H},a}(t) \left(\frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} dr\right) dt = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_{\mathcal{H},a}(t)G(t)\Gamma(it)\frac{e^{itT} - 1}{it} dt + O\left(\frac{1}{T} \int_{-\infty}^{\infty} \phi_{\mathcal{H},a}(t) \frac{G(t)\Gamma(it)}{(1 + |t|)(2 + |t|)} dt\right).$$

Since $\phi_{\mathcal{H},a}(0) = 0$, applying Theorems 1.7 and 2.6 we conclude the bound

$$\int_{-\infty}^{\infty} \phi_{\mathcal{H},a}(t) \left(\frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} dr\right) dt = O(T^{-1}).$$

Hence, as $T \to \infty$ the contribution of the continuous spectrum converges to

$$\pi^{-3/2} \Gamma(3/4)^2 \sum_a \hat{E}_a(1/2)E_a(z,1/2).$$

This completes the proof of Theorem 1.5.

4. $\Omega$-RESULTS FOR THE AVERAGE ERROR TERM ON GEODESICS

In this section we give the proof of Theorem 1.9. For this reason, we mollify the average of the error term on the geodesic $\ell$. Let $\psi \geq 0$ be a smooth even function compactly supported in $[-1,1]$, such that $\hat{\psi} \geq 0$ and $\int_{-\infty}^{\infty} \psi(x)dx = 1$. For every $\epsilon > 0$ we also define the family of functions $\psi_{\epsilon}(x) = \epsilon^{-1}\psi(x/\epsilon)$. We have $0 \leq \psi_{\epsilon}(x) \leq 1$ and $\hat{\psi}_{\epsilon}(0) = 1$. As before, we study separately the contributions of the discrete and the continuous spectrum.
4.1. **The contribution of the discrete spectrum.** Let us denote by \( e(\mathcal{H}, R) \) the average of the normalized error term on the geodesic, evaluated at the parameter \( R = \log(X + U) \), i.e.

\[
e(\mathcal{H}, R) =: \int_{\mathcal{H}} \frac{e(\mathcal{H}, X; z)}{X^{1/2}} ds(z),
\]

and we consider the convolution

\[
(e(\mathcal{H}, \cdot) * \psi_{\epsilon}) (R) =: \int_{-\infty}^{+\infty} \psi_{\epsilon}(R - Y) e(\mathcal{H}, Y) dY.
\]

Notice that if \( |e(\mathcal{H}, Y)| \leq M \) for \( Y \) close to \( R \), then \( |e(\mathcal{H}, \cdot) * \psi_{\epsilon} (R)| \leq M \), by the properties of \( \psi(x) \). It follows that, in order to prove an \( \Omega \)-result for the average \( \int_{\mathcal{H}} e(\mathcal{H}, X; z) ds \), it suffices to prove an \( \Omega \)-result for the convolution \( e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R) \). Further, using Lemma 2.3, Stirling’s asymptotic (2.7), Theorem 1.7 and the properties of \( \psi \) we calculate the contribution of the discrete spectrum in \( (e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R) \) is given by

\[
\sum_{t_j > 0} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) \int_{-\infty}^{+\infty} \psi_{\epsilon}(Y - R) e^{it_j Y} dY \right) \nonumber \\
+ O \left( \sum_{t_j > 0} |\hat{u}_j|^2 \left| \int_{-\infty}^{+\infty} e^{-Y/2} \psi_{\epsilon}(Y - R) V(Y) e^{it_j Y} dY \right| + e^{-\sigma R} \right) \nonumber \\
= \sum_{t_j > 0} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) e^{it_j R} \right) \psi_{\epsilon}(t_j) + O \left( e^{-R/2} + e^{-\sigma R} \right),
\]

where the last estimate follows immediately from the properties of \( V \). Let \( A > 1 \). We split the sum of the above main term for \( t_j \geq A \) and \( t_j < A \). Using the bound

\[
\hat{\psi}_{\epsilon}(t_j) = O_k((|t_j|)^{-k})
\]

for every \( k \geq 1 \), for \( t_j \geq A \) we get

\[
\sum_{t_j \geq A} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) e^{it_j R} \right) \psi_{\epsilon}(t_j) = O_k(e^{-k} A^{-k}).
\]

For the partial sum part of the series we use the following lemma:

**Lemma 4.1** (Dirichlet’s box principle [22]). Let \( r_1, r_2, ..., r_n \) be \( n \) distinct real numbers and \( M > 0 \), \( T > 1 \). Then, there is an \( R \) satisfying \( M \leq R \leq MT^n \), such that

\[
|e^{i r_j R} - 1| < \frac{1}{T}
\]

for all \( j = 1, ..., n \).

We apply Lemma 4.1 to the sequence \( e^{i t_j R} \) and Lemma 1.7. Given \( T \) large we find an \( R \) such that \( M \ll R \ll MT^n \ll MT^2 \). The contribution of the discrete spectrum in the convoluted error term \( (e(\mathcal{H}, \cdot) * \psi_{\epsilon})(R) \) takes the form

\[
\sum_{t_j < A} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) \right) \psi_{\epsilon}(t_j) + O_k \left( T^{-1} \log A + e^{-k} A^{-k} + e^{-\sigma R} \right).
\]

The balance \( A \log A = T \), \( \log M \sim e^{-1} \), \( e^{-2} = A \) implies \( \log \log R \sim \log(e^{-1}) \) and for \( \epsilon \leq 1 \) we get:

\[
T^{-1} \log A + e^{-k} A^{-k} + e^{-\sigma R} = O(\epsilon + e^{-\sigma R}).
\]
From part a) of Lemma 2.5 we conclude the sum in (4.2) is positive. On the other hand there exists one \( \tau \in (0, 1) \) such that \( \hat{\psi}(x) \geq 1/2 \) for \( |x| \leq \tau \). Since \( \hat{\psi}(t_j) = \hat{\psi}(et_j) \), we get

\[
\sum_{t_j < \Lambda} \Re (G(t_j) \Gamma(it_j)) \hat{\psi}(t_j)|\hat{u}_{t_j}|^2 \gg \sum_{t_j < \tau/\epsilon} \Re (G(t_j) \Gamma(it_j)) |\hat{u}_{t_j}|^2 \\
\gg \sum_{t_j < \tau/\epsilon} t_j^{-1}|\hat{u}_{t_j}|^2.
\]

When \( \Gamma \) is cocompact or has sufficiently small Eisenstein periods in the sense of Definition 1.8, we have

\[
\sum_{t_j < \tau/\epsilon} t_j^{-1}|\hat{u}_{t_j}|^2 \gg \log(\epsilon^{-1}) \gg \log \log R.
\]

We conclude that the contribution of the discrete spectrum in \( e(\mathcal{H}, R) \) is \( \Omega_+(\log \log R) \). This implies that if \( \Gamma \) is cocompact or has sufficiently small Eisenstein periods, the contribution of the discrete spectrum in \( \int_{\mathcal{H}} e(\mathcal{H}, X; z) ds \) is \( \Omega_+(X^{1/2} \log \log X) \). In particular, this completes the proof of Theorem 1.9 for \( \Gamma \) cocompact.

### 4.2. The contribution of the continuous spectrum.

The contribution of the continuous spectrum in \( (e(\mathcal{H}, \cdot) \ast \psi_c)(R) \) is given by the quantity

\[
\sum_{\alpha} \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{E}_\alpha(1/2 + it)^2 \Re \left( G(t) \Gamma(it) e^{iRt} F \left( \frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \right) \hat{\psi}(t) dt.
\]

The convergence of the integral is justified as in section 3. Let \( \chi_{\mathcal{H}, \alpha}(t) \) denote the function \( \chi_{\mathcal{H}, \alpha}(t) = |\hat{E}_\alpha(1/2 + it)|^2 - |\hat{E}_\alpha(1/2)|^2 \). Thus the contribution of cusp \( \alpha \) in \( (e(\mathcal{H}, \cdot) \ast \psi_c)(R) \) splits in

\[
\frac{|\hat{E}_\alpha(1/2)|^2}{4\pi} \int_{-\infty}^{\infty} \Re \left( G(t) \Gamma(it) e^{iRt} F \left( \frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \right) \hat{\psi}(t) dt \\
+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \chi_{\mathcal{H}, \alpha}(t) \Re \left( G(t) \Gamma(it) e^{iRt} F \left( \frac{1}{2}, \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \hat{\psi}(t) dt.
\]

Let \( \gamma \) be the contour \( \gamma = \bigcup_{i=1}^{b} C_t \) defined in the proof of Lemma 3.1. The function \( \psi_c(x) \) is compactly supported in the interval \( [-\epsilon, \epsilon] \). Applying the Paley-Wiener Theorem \[17, \text{Theorem 7.4} \] we deduce that the holomorphic Fourier transform of \( \psi_c(x) \):

\[
\hat{\psi}_c(z) = \int_{-\infty}^{\infty} \psi_c(x)e^{-ixz} dx
\]

is an entire function of type \( \epsilon \), i.e. \( |\hat{\psi}_c(z)| \ll e^{\epsilon|z|} \), and it is square-integrable over horizontal lines:

\[
\int_{-\infty}^{\infty} |\hat{\psi}_c(v + iu)|^2 dv < \infty.
\]

For fixed \( \epsilon > 0 \) we have

\[
\int_{-\infty}^{\infty} |\hat{\psi}_c(v + iu)|^2 dv = \epsilon^{-1} \int_{-\infty}^{\infty} |\hat{\psi}(v + i\epsilon u)|^2 dv
\]

and since \( \int_{-\infty}^{\infty} |\hat{\psi}(v + i\epsilon u)|^2 dv \) converges uniformly to \( \int_{-\infty}^{\infty} |\hat{\psi}(v)|^2 dv \) as \( \epsilon \to 0 \) we get

\[
\int_{-\infty}^{\infty} |\hat{\psi}_c(v + i/2)|^2 dv \ll \epsilon^{-1}.
\]

Consider the integral

\[
\int_{\gamma} G(z) \Gamma(iz) e^{iRz} F \left( \frac{1}{2}, \frac{3}{2}; 1 + iz; \frac{1}{e^{2R} + 1} \right) \hat{\psi}_c(z) dz.
\]
The integrand is holomorphic inside the contour. Working as in the proof of Lemma 3.1 and applying Cauchy-Schwarz inequality and bound (4.4) for the integral over $C_3$ we deduce
\[
\int_{-\infty}^{\infty} G(t) \Gamma(it) e^{itT} \left( -\frac{1}{2} \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \check{\psi}_c(t) dt = \pi G(0) \check{\psi}_c(0) \left( 1 + O(e^{-2R}) \right)
\]
\[+O(e^{-1}e^{-R/2}).\]

To finish the proof of part (a) of Theorem 1.9, we notice that if
\[
\int_{-T}^{T} |\hat{E}_a(1/2 + it)|^2 dt \ll \frac{T}{(\log T)^{1+\delta}},
\]
then the function
\[
H_1(t) = \chi_{\mathcal{H},a}(t) G(t) \Gamma(it) F \left( -\frac{1}{2} \frac{3}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \check{\psi}_c(t)
\]
is in $L^1(\mathbb{R})$ independently of $\epsilon$ and $R$. To obtain this we notice that $\chi_{\mathcal{H},a}(t) G(t)$ remains bounded close to $t = 0$, we use the trivial bound $\check{\psi}_c(t) \leq 1$, Lemma 2.3 and we estimate
\[
\int_{-\infty}^{\infty} |H_1(t)| dt \ll \int_{-1}^{1} |H_1(t)| dt + \sum_{n=0}^{\infty} 2^{2n+1} \int_{2^n}^{2^{n+1}} |t|^{-1} |\hat{E}_a(1/2 + it)|^2 dt
\]
\[
\ll \int_{-1}^{1} |H_1(t)| dt + \sum_{n=0}^{\infty} 2^{-n} \int_{2^n}^{2^{n+1}} |\hat{E}_a(1/2 + it)|^2 dt
\]
\[
\ll \int_{-1}^{1} |H_1(t)| dt + \sum_{n=0}^{\infty} \frac{1}{(n+1)^{1+\delta}} \ll 1.
\]

Applying the Riemann–Lebesgue Lemma we conclude that
\[
\lim_{R \to \infty} \int_{-\infty}^{\infty} H_1(t) e^{iRT} dt = 0.
\]

Since $\check{\psi}_c(0) = 1$ and $\pi G(0) = 4\pi^{-1/2} |\Gamma(3/4)|^2$, the contribution of the continuous spectrum in $(\epsilon(\mathcal{H}, \cdot) \ast \psi_c)(R)$ takes the form
\[
\frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_a |\hat{E}_a(1/2)|^2 + O(e^{-1}e^{-R/2}) + o(1).
\]

As in the discrete spectrum (see the balance after expansion (4.2)) we choose the balance $\epsilon^{-1} \ll \log R \ll \log \log X$. Hence (4.8) takes the form
\[
\frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_a |\hat{E}_a(1/2)|^2 + O(X^{-1/2} \log \log X) + o(1).
\]

In particular, this completes the proof of part (a) of Theorem 1.9.

To prove part (b), we first notice that the contribution from the discrete spectrum is $c(R) + O_k \left( T^{-1} \log A + \epsilon^{-k} A^{-k} + e^{-\sigma R} \right)$, where $c(R) = \Omega_{+}(1)$ if there exists one $\hat{u}_j \neq 0$ and $c(R)$ vanishes otherwise. In this case, the contribution of the continuous spectrum takes the form
\[
\frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_a |\hat{E}_a(1/2)|^2 + O(e^{-1}e^{-R/2}) + \epsilon^{-1} \int_{-\infty}^{\infty} H_2(t) e^{iRT} dt
\]

where, using Theorem 1.7 and estimate (4.1), we deduce that the function $H_2(t) := \epsilon H_2(t)$ is in $L^1(\mathbb{R})$ independently of $\epsilon$ and $R$. Applying the Riemann–Lebesgue Lemma, the contribution of the continuous spectrum becomes
\[
\pi^{-3/2} |\Gamma(3/4)|^2 \sum_a |\hat{E}_a(1/2)|^2 + O(e^{-1}e^{-R/2}) + \epsilon^{-1} Q(R),
\]
Remark 4.2. For part (a) of Theorem 1.9, even if \( \Gamma \) has not sufficiently small Eisenstein periods associated to \( \mathcal{H} \) but has sufficiently many cusp forms in the sense that
\[
\sum_{0 < \ell, j < T} |\hat{a}_j|^2 \gg T,
\]
we can derive the \( \Omega_+(1) \) bound if we have a polynomial bound for the derivatives of the Eisenstein series on the critical line (see [3, Chapter 4] for details).

4.3. An arithmetic case: the modular group. In this subsection we concentrate to \( \Gamma = \text{PSL}_2(\mathbb{Z}) \). The set of primitive indefinite quadratic forms \( Q(x, y) = ax^2 + bxy + cy^2 \) in two variables (that means \((a, b, c) = 1 \) and \( b^2 - 4ac = d > 0 \) is not a square) is in one-to-one correspondence with the set of primitive hyperbolic elements of \( \Gamma \) (see [23, p. 232]). Here we briefly describe this correspondence.

The automorphs of \( \mathcal{Q} \) is the cyclic group \( \text{Aut}(\mathcal{Q}) \subset \text{SL}_2(\mathbb{Z}) \) which fixes \( \mathcal{Q} \), under the action
\[
\begin{pmatrix} a & b/2 \\ c & \end{pmatrix} \gamma = \gamma. 
\]
Let \( M_Q \) be a generator of \( \text{Aut}(\mathcal{Q}) \). Then the correspondence \( \mathcal{Q} \to M_Q \) is bijective between indefinite integral quadratic forms in two variables and primitive hyperbolic elements of the modular group.

Denote by \( \mathcal{H}_Q \) the conjugacy class of \( M_Q \) and by \( \ell_Q \) the \( M_Q \)-invariant geodesic. Define
\[
r(Q, n) = \#\{ (x, y) \in \mathbb{Z}^2 : Q(x, y) = n \}/\text{Aut}(\mathcal{Q}),
\]
and let \( \zeta(Q, s) \) denote the Epstein zeta function
\[
\zeta(Q, s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s},
\]
which is absolutely convergent in \( \Re(s) > 1 \). Hecke proved that the Eisenstein period \( \tilde{E}_a(s) \) along a normalized segment of \( \ell_Q \) satisfies
\[
\tilde{E}_a(s) = \frac{d^{s/2}\Gamma(2s)\zeta(2s)}{\Gamma(s)\zeta(2s)} \zeta(Q, s)
\]
(see [26, eq. (9.5)]). The functional equation of the Eisenstein series implies the functional equation of the Epstein zeta function:
\[
d^{(1-s)/2}\Gamma^2 \left( \frac{1-s}{2} \right) \pi^{s-1} \zeta(Q, 1-s) = d^{s/2}\Gamma^2 \left( \frac{s}{2} \right) \pi^{-s} \zeta(Q, s).
\]
The functional equation and the Phragmén-Lindelöf principle imply the convexity bound on the critical line:
\[
\zeta(Q, 1/2 + it) \ll_e (1 + |t|)^{1/2+\epsilon}, \quad t \in \mathbb{R}.
\]
Further, for the Epstein zeta function \( \zeta(Q, 1/2 + it) \) the following subconvexity bound holds:
\[
\zeta(Q, 1/2 + it) \ll_e (1 + |t|)^{1/3+\epsilon}
\]
To prove this, write the Epstein zeta function \( \zeta(Q, s) \) as a linear combination of zeta functions \( \zeta(s, \chi) \), where \( \chi \) runs through the class group characters of the number field \( Q(\sqrt{d}) \) [15, Ch. 12, p. 216]. The bound (4.14) now follows from the \( GL(1) \)-subconvexity bound over a number field and the subconvexity bound of Söhne [25] for Hecke zeta functions with Grössencharacters.
In this case we deduce that $\Gamma$ has sufficiently small Eisenstein periods; in fact

\[(4.15) \quad \int_{-T}^{T} |\hat{E}_a(1/2 + it)|^2 dt \ll T^{2/3+\epsilon}\]

for every $\epsilon > 0$. To prove this, we use the bound $|\zeta(1+2it)|^{-1} \ll (\log |t|)^{2/3}(\log \log |t|)^{1/3}$ as $|t| \to \infty$ [16, Th. 8.29] and Stirling’s formula, which imply

\[
\frac{|\Gamma(1/4 + it/2)|}{|\Gamma(1/2 + it)|} \ll (1 + |t|)^{-1/2}.
\]

Thus

\[
\hat{E}_a(1/2 + it) \ll (1 + |t|)^{-1/2}(\log |t|)^{2/3}(\log \log |t|)^{1/3}\zeta(1/2 + it) \ll_{\epsilon} (1 + |t|)^{-1/6+\epsilon}
\]

for every $\epsilon > 0$, and the bound (4.15) follows. In particular, the subconvexity bound (4.14) implies

\[
\int_{t_Q} e(\mathcal{H}_Q, X; z)ds(z) = \Omega_+(X^{1/2}\log \log X).
\]

5. Pointwise $\Omega$-results for the error term

In this section we prove Propositions 1.11, 1.12, and hence Theorem 1.13, where we consider pointwise $\Omega$-results for the error term $e(\mathcal{H}, X; z)$. We start with the discrete average. The arguments of the proofs follow the ideas from sections 3 and 4 (see [3, Chapter 4] for detailed proofs).

5.1. Proof of Proposition 1.11: The discrete spectrum. For $K > 0$ an integer we pick equally spaced $z_1, z_2, ..., z_K$ points on the invariant closed geodesic $\mathcal{H}$ with $\rho(z_{i+1}, z_i) = \delta$. Hence $\delta = \mu(\ell)/K$. For $R = \log(X + U)$ we define the quantity

\[
N_K(\mathcal{H}, R) = \frac{1}{K} \sum_{n=1}^{K} \frac{e(\mathcal{H}, X; z_m)}{X^{1/2}}
\]

and we consider the convolution

\[
(\psi_\epsilon * N_K(\mathcal{H}, \cdot))(R) = \int_{-\infty}^{\infty} \psi_\epsilon(R - Y)N_K(\mathcal{H}, Y)dY.
\]

Using Lemma 2.3, the properties of $\psi_\epsilon$, Theorem 2.6 and Theorem 1.7 we conclude

\[
(\psi_\epsilon * N(\mathcal{H}, \cdot)_K)(R) = \sum_{t_j > 0} \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^{K} u_j(z_m) \right) \Re \left( G(t_j) \Gamma(it_j)e^{it_j} \right) \hat{\psi}_\epsilon(t_j) + O(\epsilon^{-\sigma R}).
\]

For $A > 1$, using Stirling’s formula, Theorem 2.6, Theorem 1.7 and estimate 4.1 for $k \geq 1$ we estimate the tail of the series for $t_j > A$ is $O_k(\epsilon^{-k}A^{1/2-k})$. The partial sum of the series for $t_j \leq A$ can be handled as follows: by the definition of the period integral $\hat{u}_j$, as $K \to \infty$ we get

\[
\frac{\mu(\ell)}{K} \sum_{m=1}^{K} u_j(z_m) = \sum_{m=1}^{K} u_j(z_m) \delta \to \overline{u_j}
\]

uniformly, for every $j = 1, ..., n$ (where $n$ is such that $t_n \leq A < t_{n+1}$, hence $n \sim A^2$). That means for every small $\epsilon_1 > 0$ there exists a $K_0 = K_0(\epsilon_1) \geq 1$ such that

\[
(5.1) \quad \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^{K} u_j(z_m) \right) = \frac{\hat{u}_j^2}{\mu(\ell)} + O(\epsilon_1 \hat{u}_j)
\]
for every $K \geq K_0$. We get

\[
(\psi \ast N(\mathcal{H}, \cdot)_{\mathcal{K}})(R) = \frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re (G(t_j) \Gamma(\ell t_j) e^{it_j R}) \hat{\psi}(t_j)
\]

\[+ O_k \left( \epsilon_1 \sum_{t_j \leq A} \hat{u}_j \Re (G(t_j) \Gamma(\ell t_j) e^{it_j R}) \hat{\psi}(t_j) + \epsilon^{-k} A^{1/2-k} + e^{-\sigma R} \right).\]

Using Theorem 1.7 the $O$-term is bounded by $O(\epsilon_1 A^{1/2})$. For the main term, apply Dirichlet’s principle (Lemma 4.1) to the exponentials $e^{it_j R}$. For every $M$ and $T$ we find $M \ll R \ll M T^A$ such that

\[
(\psi \ast N(\mathcal{H}, \cdot)_{\mathcal{K}})(R) = \frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re (G(t_j) \Gamma(\ell t_j) \hat{\psi}(t_j))
\]

\[+ O_k (\epsilon^{-k} A^{1/2-k} + T^{-1} \log A + \epsilon_1 A^{1/2} + e^{-\sigma R}).\]

The balance $\epsilon^{-1} = A^{1-3/(2k+2)}$, $\epsilon_1 = A^{-1/2}$ implies the $O$-term is $O(T^{-1} \log A + \epsilon + e^{-\sigma R})$. By Lemma 2.5, the coefficients of the above sum are all positive. For the function $\psi$ we pick $\tau \in (0,1)$ such that $\hat{\psi}(x) \geq 1/2$ for $|x| \leq \tau$. It follows that if $\Gamma$ is cocompact or has sufficiently small Eisenstein periods we bound the above sum from below by

\[
\frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re (G(t_j) \Gamma(\ell t_j)) \hat{\psi}(t_j) \gg \log(\epsilon^{-1}).
\]

We deduce that for every $\epsilon > 0$ we can find a sufficiently large $K = K(\epsilon)$ such that

\[
(\psi \ast N_{\mathcal{K}}(\mathcal{H}, \cdot))(R) = k(\epsilon) + O(\epsilon + e^{-\sigma R}).
\]

with $k(\epsilon) = \Omega_+(\log(\epsilon^{-1}))$. If $\Gamma$ is cocompact, choosing $\epsilon = \epsilon_0$ sufficiently small and $K = K(\epsilon_0)$ sufficiently large, for $R,T \to \infty$ we conclude Proposition 1.11 for $\Gamma$ cocompact.

5.2. The continuous spectrum. The contribution of the continuous spectrum in the convolution $(\psi \ast N_{\mathcal{K}}(\mathcal{H}, \cdot))(R)$ is given by

\[
\sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{E}_a(1/2 + it) \left( \frac{1}{K} \sum_{m=1}^{K} E_a(z_m, 1/2 + it) \right)
\]

\[\times \Re \left( G(t) \Gamma(it) F \left( \frac{-1}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) e^{it R} \right) \hat{\psi}(t) dt.
\]

For $A > 0$, by Theorem 1.7, asymptotic (2.7) and estimate (4.1) it follows that the contribution of $|t| > A$ in the above integral is $O(\epsilon^{-k} A^{1/2-k})$. For $|t| \leq A$ and for any small $\epsilon_2 > 0$ we approximate the Eisenstein period integral as

\[
\frac{1}{K} \sum_{m=1}^{K} E_a(z_m, 1/2 + it) = \hat{E}_a(1/2 - it) + O(\epsilon_2)
\]

for every $K \geq K_0$ with $K_0 = K_0(\epsilon_2)$ sufficiently large. The contribution of the continuous spectrum (5.2) takes the form

\[
\sum_a \frac{1}{4\pi} \int_{|t| \leq A} |\hat{E}_a(1/2 + it)|^2 \Re \left( G(t) \Gamma(it) e^{it R} F \left( \frac{-1}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \right) \hat{\psi}(t) dt
\]

\[+ O_k \left( \epsilon_2 \sum_a \int_{|t| \leq A} \hat{E}_a(1/2 + it) G(t) \Gamma(it) e^{it R} F \left( \frac{-1}{2}; 1 + it; \frac{1}{e^{2R} + 1} \right) \hat{\psi}(t) dt + \epsilon^{-k} A^{1/2-k} \right).
\]

By subsection 4.2 and Theorem 1.7, the first summand of (5.4) takes the form

\[
\frac{1}{\pi^{3/2}} |\Gamma(3/4)|^2 \sum_a |\hat{E}_a(1/2)|^2 + O(\epsilon^{-1} Q_1(R) + \epsilon^{-k} A^{-k}),
\]
with \( Q_1(R) \to 0 \) as \( R \to \infty \). For the second summand of (5.4), we set \( \theta_{H,\ell}(t) = \hat{E}_\ell(1/2 + it) - \hat{E}_\ell(1/2) \) and we use the contour integral method to deduce that the contribution of the continuous spectrum in \((\psi \ast N_K(H, \cdot))(R)\) is
\[
\frac{1}{\pi^{1/2}} |\Gamma(3/4)|^2 \sum_{a} |\hat{E}_\ell(1/2)|^2 + O_k \left( \epsilon^{-1}Q_1(R) + \epsilon^{-1}A^{-k+1/2} + \epsilon_2 + \epsilon_2 \epsilon^{-1}R/2 + \epsilon_2 \log A \right).
\]
Choosing \( \epsilon_2 = \epsilon^2 \) and \( \epsilon^{-1} = A^{1-3/(2k+2)} \) as before we conclude the \( O \)-term is \( O(\epsilon^{-1}Q_1(R) + \epsilon) \). If \( \Gamma \) has at least one nonzero Eisenstein period integral then for fixed and sufficiently small \( \epsilon \) the contribution of the discrete spectrum in \((\psi \ast N_K(H, \cdot))(R)\) is \( \Omega_+(1) \). If \( \Gamma \) has at least one nonzero Eisenstein period integral then for fixed and sufficiently small \( \epsilon \) we get that the contribution of the continuous spectrum in \((\psi \ast N_K(H, \cdot))(R)\) is also \( \Omega_+(1) \). This completes the proof of Proposition 1.11.

5.3. Proof of Proposition 1.12. In this subsection we prove Proposition 1.12, where we study the average of a normalized error term on the geodesic \( \ell \). As we have already mentioned, this completes the proof of Theorem 1.13. In particular, to simplify the estimates we will prove Proposition 1.12 for the average
\[
M_{H, \ell}(X) = \frac{1}{Y} \int_{1}^{Y} \frac{e(H, x; z)}{x^{1/2}} \, dy,
\]
where we define \( Y \) and \( y \) be given by \( Y = X + \sqrt{X^2 - 1} \) and \( y = x + \sqrt{x^2 - 1} \). We will need the following lemma for the Huber transform.

Lemma 5.1. For \( y = x + \sqrt{x^2 - 1} \) we have
\[
\lim_{y \to \infty} \int_{-\infty}^{y} \frac{1}{Y} \int_{1}^{y} \frac{2d(f_x, t)}{x^{1/2}} \, dy \, dt = \frac{4}{\sqrt{\pi}} |\Gamma(3/4)|^2.
\]

The proof of Lemma follows similarly with that of Lemma 3.1. We can now prove Proposition 1.12.

Proof. (of Proposition 1.12). Assume first that \( \Gamma \) is cocompact. We pick \( z_1, z_2, \ldots, z_K \) equally spaced points on the invariant closed geodesic \( \ell \) with \( \rho(z_{i+1}, z_i) = \delta \). Using Lemma 2.3, Theorem 2.6 and Theorem 1.7 we conclude
\[
(5.6) \quad \frac{1}{K} \sum_{m=1}^{K} M_{H, z_m}(X) = \sum_{l_j > 0} \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^{K} u_j(z_m) \right) \Re \left( G(t_j) \Gamma(it_j) \frac{1}{Y} \int_{1}^{Y} e^{it_j r} \, dy \right) + O(Y^{-\sigma}),
\]
For \( A > 1 \), we use Theorem 1.7 and we apply the estimate (4.1) to bound the tail of the series in (5.6) for \( t_j \geq A \) by \( O(A^{-1/2}) \). For the partial sum of the series, we approximate the period integral \( \hat{u}_j \) uniformly, for every \( j = 1, \ldots, n \) (where \( n \asymp A^2 \)). For any \( \epsilon_1 > 0 \) we find a \( K_0 = K_0(\epsilon_1) \geq 1 \) such that for every \( K \geq K_0 \):
\[
(5.7) \quad \hat{u}_j \left( \frac{1}{K} \sum_{m=1}^{K} u_j(z_m) \right) = \frac{\hat{u}_j^2}{\mu(\ell)} + O \left( \epsilon_1 \hat{u}_j \right).
\]
We get
\[
\frac{1}{K} \sum_{m=1}^{K} M_{H, z_m}(X) = \frac{1}{\mu(\ell)} \sum_{l_j < A} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) \frac{Y \hat{u}_j}{1 + it_j} \right) + O(Y^{-1} + \epsilon_1 + A^{-1/2} + Y^{-\sigma}).
\]
For the main term, apply Dirichlet’s principle (Lemma 4.1) to the exponentials \( e^{it_j r} = Y^{it_j} \). For each \( T \) we can find \( R \ll T^{A^2} \) such that
\[
\frac{1}{K} \sum_{m=1}^{K} M_{H, z_m}(X) = \frac{1}{\mu(\ell)} \sum_{l_j < A} |\hat{u}_j|^2 \Re \left( G(t_j) \Gamma(it_j) \frac{Y \hat{u}_j}{1 + it_j} \right) + O(T^{-1} + \epsilon_1 + A^{-1/2} + Y^{-\sigma}).
\]
By Theorem 1.7, as \( A \to \infty \) the sum remains bounded and, for \( \Gamma \) cocompact, there exist infinitely many \( j \)'s such that \( \hat{u}_j \neq 0 \). By Lemma 2.5, all the nonzero terms are negative. Hence, there exists an \( A_0 \) such that for every \( A \geq A_0 \):

\[
\sum_{|t_j| < A} |\hat{u}_j|^2 \Re \left( \frac{G(t_j) \Gamma(it_j)}{1 + it_j} \right) \gg 1.
\]

For \( T, Y \) and \( A \) fixed and sufficiently large and \( \epsilon_1 \) fixed and sufficiently small, we find a \( K = K_0 \) fixed such that

\[
\frac{1}{K} \sum_{m=1}^{K} M_{\mathcal{H}, z_m}(X) = \Omega_-(1).
\]

Notice that the lower bound (5.8) holds if and only if there exists at least one nonzero \( \hat{u}_j \) with \( \lambda_j > \frac{1}{4} \).

Assume now that \( \Gamma \) is not cocompact. In this case, the contribution of the discrete spectrum in

\[
\frac{1}{K} \sum_{m=1}^{K} M_{\mathcal{H}, z_m}(X)
\]

is given by

\[
\frac{1}{K} \sum_{a \in \mathcal{A}} \int_{-\infty}^{\infty} \int_{0}^{Y} \frac{2d(f_x, t)}{\pi^{1/2}} dy \hat{E}_a(1/2 + it) E_a(z_m, 1/2 + it) dt.
\]

We cut the integral for \(|t| \leq A \) and \(|t| > A \). In the interval \(|t| \leq A \) we approximate the Eisenstein period \( \epsilon_2 \)-close. Applying Lemma 5.1 and following a standard calculation, expansion (5.9) takes the form

\[
\frac{\Gamma(3/4)^2}{\pi^{3/2}} \sum_{a} |\hat{E}_a(1/2)|^2 + \Re \left( \sum_{a} \frac{1}{4\pi} \int_{-\infty}^{\infty} \chi_{\mathcal{H}, a}(t) \frac{G(t) \Gamma(it)}{1 + it} Y it dt \right)
\]

\[
+ O(A^{-1/2} + \epsilon_2 + Y^{-1}),
\]

with \( K = K(\epsilon_2, A) \). Since for \( \Gamma \) all the Eisenstein periods \( \hat{E}_a(1/2) \) vanish, applying Riemann–Lebesgue Lemma for the second term the proposition follows for \( A, Y \) sufficiently large and \( \epsilon_2 \) sufficiently small.

\[ \square \]

6. Upper bounds on geodesics

In this section, we apply the key observation arising in the spectral theory of the conjugacy problem, that is the slower divergence for the sums of period integrals of Theorem 1.7, to the error terms of both the classical problem (described in subsection 1.1) and the conjugacy class problem. In particular, for the error \( e(X; z, w) \) we prove the following average result.

**Theorem 6.1.** Let \( \ell_0 \) be a closed geodesic of \( \Gamma \setminus \mathbb{H} \) and \( e(X; z, w) \) be the error term of the classical counting problem. Then

\[
\int_{\ell_0} e(X; z, w) ds(w) = O_{\ell_0}(X^{1/2} \log X).
\]

The proof of this result is similar to the proof for the classical pointwise bound \( O(X^{2/3}) \). The standard idea here is again to approximate the kernel defined \( k(u) = \chi_{[0, (x - 2)/4]} \) by appropriate step functions \( k_{\pm}(u) \) and use the observation

\[
\sum_{|t_j| < T} \frac{u_j(z) \hat{u}_j}{t_j^{3/2}} \ll \log T.
\]
Similarly, for the error term \( e(\mathcal{H}, X; z) \) of the conjugacy class problem we can deduce the upper bound
\[
\int_{t_0} e(\mathcal{H}, X; z) ds(z) = O_{t_0}(X^{1/2} \log X).
\]
Since the proof of this bound is similar with that of Theorem 6.1, it is omitted.

**Proof.** (of Theorem 6.1) The proof follows the steps of the proof for the classical pointwise bound \( O(X^{2/3}) \), sketched in [14, Ch. 12, p. 173]. Assume first the cocompact case. Define the functions \( k_-(u) \leq k(u) \leq k_+(u) \) by
\[
k_+(u) = \begin{cases} 
1, & \text{for } u \leq \frac{X-Y}{4}, \\
-4u + \frac{X+Y-2}{Y}, & \text{for } \frac{X-Y}{4} \leq u \leq \frac{X+Y-2}{Y}, \\
0, & \text{for } \frac{X+Y-2}{Y} \leq u,
\end{cases}
\]
and
\[
k_-(u) = \begin{cases} 
1, & \text{for } u \leq \frac{X-Y}{4}, \\
-4u + \frac{X-2}{Y}, & \text{for } \frac{X-Y}{4} \leq u \leq \frac{X-2}{4}, \\
0, & \text{for } \frac{X-2}{4} \leq u.
\end{cases}
\]
We denote their Selberg/Harish-Chandra transform by \( h_\pm(t) \). Using equations [2, p. 2, eq.(1.2)] we get
\[
e(X; z, w) \ll \sum_{t_j \in \mathbb{R} - \{0\}} h_\pm(t_j) u_j(z) u_j(w) + O(Y + X^{1/2}).
\]
Hence, using estimates [14, p. 173, eq. (12.9)] we conclude
\[
\int_{t_0} e(X; z, w) ds(w) \ll \sum_{t_j \in \mathbb{R} - \{0\}} h_\pm(t_j) u_j(z) u_j(w) + O(Y + X^{1/2})
\]
\[
\ll X^{1/2} \sum_{t_j} |t_j|^{-5/2} \min\{|t_j|, XY^{-1}\} |u_j(z)||u_j| + O(Y + X^{1/2}).
\]
Applying Cauchy-Schwarz inequality, local Weyl’s laws for the Maass forms \( u_j(z) \) and Theorem 1.7 for the periods \( \hat{u}_j \), we deduce that (6.3) is bounded by
\[
X^{1/2} \sum_{t_j \leq X/Y} |t_j|^{-3/2} |u_j(z)||\hat{u}_j| + X^{1/2} \sum_{t_j > X/Y} |t_j|^{-5/2} \frac{X}{Y} |u_j(z)||\hat{u}_j| \ll X^{1/2} \log(X/Y) + X^{1/2}.
\]
We conclude
\[
\int_{t_0} e(X; z, w) ds(w) \ll X^{1/2} \log(X/Y) + Y + X^{1/2}
\]
and the statement follows for \( Y = X^{1/2} \). For the cofinite case, the result follows similarly, using the relevant bounds for the Eisenstein series and their period integrals.

**References**

[1] F. Chamizo. *Some applications of large sieve in Riemann surfaces*. Acta Arith. 77, no. 4, 315–337, 1996.
[2] D. Chatzakos. *Ω-results for the hyperbolic lattice point problem*. Proc. of AMS, vol. 145 (2017), no. 4, 1421–1437.
[3] D. Chatzakos. *Lattice point problems in the hyperbolic plane*. Doctoral Thesis, University College of London, 2016.
[4] D. Chatzakos and Y. Petridis. *The hyperbolic lattice point problem in conjugacy classes*. Forum Math. 28, no. 5, 2371–2485, 2016.
[5] G. Cherubini and M. Risager. *On the variance of the error term in the hyperbolic circle problem*. (to appear in Revista Matemática Iberoamericana), arXiv:1512.04179 (2015)
[6] J. Delsarte. *Sur le gitter fuchsien*. (French); C. R. Acad. Sci. Paris, 214, 147–179, 1942.
[7] D. Goldfeld. *Automorphic forms and L-functions for the group GL(n, ℝ)*. Cambridge Studies in Advanced Mathematics, 99. Cambridge University Press, Cambridge, 2006. xiv+493 pp.
[8] A. Good. *Local analysis of Selberg’s trace formula*. Lecture Notes in Mathematics, 1040. Springer-Verlag, Berlin, 1983. i+128 pp.
[9] I. S. Gradshteyn and I. M. Ryzhik. Table of integrals, series, and products. Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007. xlviii+1171 pp.

[10] P. Günther. Gitterpunktprobleme in symmetrischen Riemannschen Rumen vom Rang 1. Math. Nachr. 94, 5–27, 1980.

[11] R. Hill and L. Parnovskii. The variance of the hyperbolic lattice point counting function. Russ. J. Math. Phys. 12 (2005), no. 4, 472–482.

[12] H. Huber. Über eine neue Klasse automorpher Funktionen und ein Gitterpunktproblem in der hyperbolischen Ebene. I. Comment. Math. Helv. 30, 20–62, 1956.

[13] H. Huber. Ein Gitterpunktproblem in der hyperbolischen Ebene. J. Reine Angew. Math. 496, 15–53, 1998.

[14] H. Iwaniec. Spectral methods of automorphic forms. Second edition. Graduate Studies in Mathematics, 53. American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002. xii+220 pp.

[15] H. Iwaniec. Topics in classical automorphic forms. Graduate Studies in Mathematics, 17. American Mathematical Society, Providence, RI, 1997. xii+259 pp.

[16] H. Iwaniec and E. Kowalski. Analytic Number Theory. American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004. xii+615 pp.

[17] Y. Katznelson. An introduction to harmonic analysis. Third edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2004. xviii+434 pp. ISBN: 0-521-83829-0; 0-521-54359-2

[18] N. Laaksonen. Lattice point counting in sectors of hyperbolic 3-space. Q. J. Math. 68 (2017), no. 3, 891–922.

[19] K. Martin, M. McKee and E. Wambach. A relative trace formula for a compact Riemann surface. Int. J. Number Theory 7, no. 2, 389–429, 2011.

[20] J. Parkkonen and F. Paulin. On the hyperbolic orbital counting problem in conjugacy classes. Math. Z. 279, no. 3-4, 1175–1196, 2015.

[21] S. J. Patterson. A lattice-point problem in hyperbolic space. Mathematika 22, no. 1, 81–88, 1975.

[22] R. Phillips and Z. Rudnick. The circle problem in the hyperbolic plane. J. Funct. Anal. 121, no. 1, 78–116, 1994.

[23] P. Sarnak. Class numbers of indefinite binary quadratic forms. J. Number Theory 15 (1982), no. 2, 229–247.

[24] A. Selberg. Equidistribution in discrete groups and the spectral theory of automorphic forms. http://publications.ias.edu/selberg/section/2491

[25] P. Sönhe. An upper bound for Hecke zeta-functions with Grössencharacters. J. Number Theory 66 (1997), no. 2, 225-250.

[26] M. Tsuzuki. Spectral square means for period integrals of wave functions on real hyperbolic spaces. J. Number Theory 129, no. 10, 2387–2438, 2009.

Université de Lille 1 Sciences et Technologies and Centre Européen pour les Mathématiques, la Physique et leurs Interactions (CEMPI), Cité Scientifique, 59655 Villeneuve d’Ascq Cédex, France

E-mail address: Dimitrios.Chatzakos@math.univ-lille1.fr