INTERMITTENCY PROPERTIES FOR A LARGE CLASS OF
STOCHASTIC PDES DRIVEN BY FRACTIONAL SPACE-TIME
NOISES

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ABSTRACT. In this paper, we study intermittency properties for various stochastic PDEs with varieties of space-time Gaussian noises via matching upper and lower moment bounds of the solution. Due to the absence of the powerful Feynman-Kac formula, the lower moment bounds have been missing for many interesting equations except for the stochastic heat equation. This work introduces and explores the Feynman diagram formula for the moments of the solution and the small ball nondegeneracy for the Green’s function to obtain the lower bounds for all moments which match the upper moment bounds. Our upper and lower moments are valid for various interesting equations, including stochastic heat equations, stochastic wave equations, stochastic heat equations with fractional Laplacians, and stochastic diffusions which are both fractional in time and in space.

1. INTRODUCTION

In this article, we consider the following stochastic partial differential equation in the whole $d$-dimensional Euclidean space $\mathbb{R}^d$:

$$\mathcal{L}u(t, x) = u(t, x)\dot{W}(t, x), \quad t > 0, \ x \in \mathbb{R}^d$$

(1.1)

with some given initial condition(s). Here $\mathcal{L}$ denotes a general (including fractional order) partial differential operator and $\dot{W}(t, x) = \sum_{i=1}^{d+1} \partial_i^2 W(t, x)$ is the Gaussian noise. Our approach can be applied to a large class of operator $\mathcal{L}$. For this reason as in [18, 31], instead of giving the concrete form of $\mathcal{L}$, we shall impose conditions satisfied by the Green’s function associated with $\mathcal{L}$.

Let us briefly recall the concept of Green’s function. Suppose $f(t, x), t \geq 0, x \in \mathbb{R}^d$ is a nice (smooth with compact support) function and consider the corresponding deterministic equation

$$\mathcal{L}u(t, x) = f(t, x), \quad t > 0, \ x \in \mathbb{R}^d.$$  \hspace{1cm} (1.2)

with the same initial condition(s) as in (1.1). The Green’s function associated with $\mathcal{L}$ is a (possibly generalized) function $G_{t,s}(x, y), 0 \leq s < t < \infty, x, y \in \mathbb{R}^d$ or a measure $G_{t,s}(x, y)dy := G_{t,s}(x, dy)$ (we omit the explicit dependence of $G$ on $\mathcal{L}$)

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such that the solution to (1.2) is given explicitly by

\[ u(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t,s}(x, y)f(s, y)dyds, \tag{1.3} \]

where the term \( I_0(t, x) \) depends on the initial data and the Green’s function.

If we formally replace \( f(s, y) \) in (1.3) by \( u(s, y)\dot{W}(s, y) \) and replace \( \dot{W}(s, y)dsdy \) by the Skorohod type stochastic integral \( W(ds, dy) \), then the solution to (1.1) satisfies

\[ u(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t,s}(x, y)u(s, y)W(ds, dy), \tag{1.4} \]

where the stochastic integral is interpreted in the Skorohod sense. However, Unlike the previous identity (1.3) the expression (1.4) is still an equation on \( u \). It is impossible to make sense for each of the terms \( \mathcal{L}u(t, x) \) and \( u(t, x)\dot{W}(t, x) \) in a straightforward way so it is impossible to find a solution satisfies (1.1) literally. But it is possible to find \( u(t, x) \) satisfies (1.4). A random field \( u(t, x) \) satisfying (1.4) will be called a mild solution (or random field solution) to (1.1). The existence and uniqueness of the solution of (1.4) have been well-studied together with the properties of the solution for different operators \( \mathcal{L} \) and for different noise structures of \( \dot{W} \). For a recent survey on stochastic heat equation we refer to [15] and the references therein.

One of the the most studied properties of solution is the intermittency property arose from the physics. This property is related the moment bounds of the solution. When (1.1) is parabolic Anderson model, namely, when \( \mathcal{L} \) is a heat operator or fractional heat operator, then the sharp (both lower and upper) moment bounds are known, see [2, 6, 5, 7, 8, 17, 19, 25], and we also refer to [15] and references therein. However, when \( \mathcal{L} \) is wave operators (namely the hyperbolic Anderson model) or when \( \mathcal{L} \) is (temporal) fractional differential operators, the situation is different and as far as we know here are the progress achieved.

(i) Similar to the stochastic heat equation (parabolic Anderson model) by using the chaos expansion and the hypercontractivity inequality one can obtain the (which we believe to be sharp) upper bounds. It is also possible to obtain the lower bound for the second moment. There are many contributions to this topic and among them we mention only a few [1, 2, 3, 4, 34] and the references therein. However, it is hard to obtain the sharp lower bound for any \( p \) moment which matches the upper bounds in terms of the growth of \( p \). On the other hand, there are works on the sharp second moment bound (ie \( p = 2 \)).

(ii) Until now the success to obtain a sharp lower moment bounds largely relies on the clever application of the Feynman-Kac formula. However, there is no effective corresponding analogous formula for other equations. The only work that we know is [10, Theorem 4.1], where the authors use an analogous Feynman-Kac formula for stochastic wave equation obtained in [11] to obtain a nice lower bound for all moments when the Gaussian noise is white in time and “smooth” in space. Let us mention that after the completion of this work, we learned the announcement of a work [33], where the Gaussian noise is what they called Dobrić-Ojeda one, namely, noise is still white in time but with a weight and the equation is one dimensional.
stochastic wave equation. The idea is still to make more careful use of the Feynman-Kac like formula obtained in [11].

The objective of this paper is to obtain the sharp lower bounds for all moments when the operator $\mathcal{L}$ in (1.1) is a wave operator or an operator which is fractional both in time and in space. The Gaussian noise $\dot{W}$ can also be general. It does not need to be white or fractional in time.

The approach that we use is a generalization of the Feynman diagram formula. This formula allows us to keep track the terms in the expectation of the product of several multiple Wiener-Itô integrals which is very sophisticated. It is in some sense a brutal force method. We fully explore the positivity of the Green’s function. This property enables us to throw away some complicated terms and keep the main terms so that the remaining ones are possible to manage although still very sophisticated. After the (fortunately successful) isolation of the leading terms there also remains an extremely challenging problem of how to bound them from below. This is also a very serious issue. We find that many Green’s functions satisfy what we call the small ball nondegeneracy property and this property can be used to ensure the sharp lower moment bounds.

The Feynman diagram type formula that we obtain is essentially analogous to the Feynman-Kac type formula obtained in [11]. However, the former one seems to be more convenient for us to manage. Since we only use the positivity and the small ball nondegeneracy properties, our approach is valid for a very large class of equations and for a large class of noise structure. The equations include stochastic wave equation (SWE, $\mathcal{L} = \partial_t^2 - \Delta$), stochastic heat equation which is inhomogeneous and fractional in space ($(\alpha,A)$-SHE, $\mathcal{L} = \partial_t - (\nabla(A(x))\nabla)^{\alpha/2}$), where $A$ is a positive definite symmetric matrix, stochastic partial differential equations which is both fractional in time and in space (SFDE, $\mathcal{L} = \partial_t^\beta - \frac{1}{2}\frac{1}{2}(\Delta)^{\alpha/2}$) (e.g. [1, 2, 4, 7, 9, 10] and references therein).

We can also allow the noise structure to be very general. We don’t need it to be white or fractional in time or in space. We follow the idea of [19] to assume that the covariance is bounded by some singular power functions.

Here is the organization of the paper. In Section 2 we give the noise structure and introduce the stochastic integral, mild solution, and chaos expansion. Section 3 proposes the general conditions satisfied by the Green’s function associated with $\mathcal{L}$ and state our main results. Sections 5 is devoted to prove the upper moment bounds for the solution. This is done by using the chaos expansion and the hypercontractivity inequality. Our main tool to prove the lower moment bounds is the generalization of the Feynman diagram formula for the expected value of product of several multiple Wiener-Itô integrals. This formula is presented in Section 4. After this preparation in Sections 5 we prove the lower moment bounds of the solution. To verify that some famous operators $\mathcal{L}$ satisfies the positivity and small ball nondegeneracy conditions so that our results can be applied to cover a large class of interesting stochastic partial equations, we verify these conditions for various interesting operators $\mathcal{L}$ in Section 7.

Throughout the entire paper, we shall use the notations $\lesssim$, $\gtrsim$, and $\simeq$ extensively. The meaning are conventional. This is, $A \lesssim B$ (or $A \gtrsim B$) means that there are constants $C \in (0,\infty)$ such that $A \leq CB$ (or $B \leq CA$, respectively). The notation $A \simeq B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold true.
2. Noise covariance structure, mild solution and chaos expansion

In this section, we give the conditions satisfied by the covariance of the noise $\dot{W}$ in (1.1). For this Gaussian noise we also define the (Skorohod type) integral, the mild solution, and the chaos expansion of the solution candidate. These concepts are known, so we recall them very quickly to fix the notation throughout the paper. We refer to [4, 14, 16, 17, 19] and the references therein for more details. The existence and uniqueness of the solution in our new situation will be a consequence of the upper moment bounds.

2.1. Noise covariance structure. We assume that the noise $\dot{W}(t,x) = \frac{\partial^d+1}{\partial t\partial x^1\cdots\partial x^d}W(t,x)$ is mean zero Gaussian with the following covariance structure:

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \gamma(t-s)\Lambda(x-y).$$

The restriction that the covariance of the noise has this product form of a function of time variables and a function of space variables is convenient. The reason that the time function is of the form $\gamma(t-s)$ means that the noise is stationary (or the original process $W$ has stationary increment). The space function is of the form $\Lambda(x-y)$ means that the noise is homogeneous.

In order to simplify our presentation and in order to cover the typical examples we make the following assumptions. For $\gamma$ we assume

(H1) There is a $\gamma \in (0, 1)$ such that

$$c|t|^{-\gamma} \leq \gamma(t) \leq C|t|^{-\gamma}, \quad \forall t \in \mathbb{R}_+$$

for some positive constants $c$, $C$. For convenience, when $\gamma = 1$ we mean $\gamma(t) = \delta(t)$.

For $\Lambda(\cdot)$ we assume that it satisfies one of the following three conditions:

(H2) There is a $\lambda \in (0, d)$ such that

$$c|x|^{-\lambda} \leq \Lambda(x) \leq C|x|^{-\lambda}, \quad \forall x \in \mathbb{R}^d.$$

(H3) There are constants $\lambda_j \in (0, 1), j = 1, \cdots, d$ such that

$$c \prod_{j=1}^d |x_j|^{-\lambda_j} \leq \Lambda(x) \leq C \prod_{j=1}^d |x_j|^{-\lambda_j}, \quad \forall x \in \mathbb{R}^d.$$

In this case we denote $\lambda = \sum_{i=1}^d \lambda_i$.

(H4) When $d = 1$ and $\gamma = 1$, we assume $\Lambda(x) = \delta(x)$.

2.2. Stochastic integral. We follow the approach of [2, 14, 19, 20, 30] to define stochastic integral. First, let us recall the Fourier transform with respect to the spatial variables. Denote by $D(\mathbb{R}^d)$ the space of real-valued infinitely differentiable functions with compact support on $\mathbb{R}^d$ (We can also introduce $D(\mathbb{R}_+ \times \mathbb{R}^d)$ in a similar way). The Fourier transform is defined as

$$\hat{f}(\xi) = \mathcal{F}[f(\cdot)](\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx,$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-d} \mathcal{F}[f(\cdot)](-x).$$
Let $H$ be the Hilbert space defined as the completion of $\mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ equipped with the inner product given by

$$
\langle \phi, \psi \rangle_H = \int_{(\mathbb{R} \times \mathbb{R}^d)^2} \phi(t,x)\psi(s,y)\gamma(t-s)\Lambda(x-y)dtdxdsy \tag{2.2}
$$

$$
= \frac{1}{(2\pi)^d} \int_{(\mathbb{R} \times \mathbb{R}^d)^2} \gamma(t-s)\hat{\phi}(t)\cdot\hat{\psi}(s)\mu(d\xi), \tag{2.3}
$$

where $\gamma : \mathbb{R} \to \mathbb{R}_+$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ are non-negative definite functions and satisfy (H1) and one of the conditions (H2)-(H4) introduced at the beginning of this section. Note that the space $H$ contains generalized functions.

The noise $\dot{W}$ can be described by an isonormal family of mean zero Gaussian random variables $\{W(\phi) : \phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)\}$ with the covariance $\mathbb{E}[W(\phi)W(\psi)] = \langle \phi, \psi \rangle_H$ for all $\phi$ and $\psi$ in $\mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$. This isometry can be extended to $H$ and is denoted by

$$
W(\phi) = \int_{\mathbb{R} \times \mathbb{R}^d} \phi(t,x)W(dt, dx), \quad \text{for all } \phi \in H.
$$

Let $\mathcal{P}$ be the set of smooth and cylindrical random variables of the form

$$
F = f(W(\phi_1), \ldots, W(\phi_n)),
$$

with $\phi_i \in H$, $f \in C^\infty_n(\mathbb{R}^n)$ (i.e. $f$ and all its partial derivatives have polynomial growth). For $F \in \mathcal{P}$ of the above form we define $DF$ as the $H$-valued random variable by the following expression

$$
DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \ldots, W(\phi_n))\phi_j.
$$

The operator $D$ is closable from $L^2(\Omega)$ into $L^2(\Omega; H)$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of $\mathcal{P}$ under the norm

$$
\|DF\|_{1,2} = \sqrt{\mathbb{E}[F^2] + \mathbb{E}[\|DF\|_H^2]}.
$$

Given any element $u \in L^2(\Omega; H)$ if there is a $v \in L^2(\Omega)$ such that

$$
\mathbb{E}(vF) = \mathbb{E}(\langle DF, u \rangle_H) \quad \text{for any } F \in \mathbb{D}^{1,2} \tag{2.4}
$$

then we say that $u$ is in the domain of $\delta$ and we call it the Skorohod integral of $u$, denoted by

$$
v = \delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x)W(dt, dx).
$$

Obviously, when such $v$ (satisfying (2.4)) exists, it is unique. We refer to [14, 19, 30] for more details. Now with the Skorohod integral introduced, we give the concept of mild solution as follows.

**Definition 2.1.** An adapted random field $\{u(t,x) : t \geq 0, x \in \mathbb{R}^d\}$ so that $\mathbb{E}[|u(t,x)|^2] < \infty$ for all $t \geq 0$ and $x \in \mathbb{R}^d$ is called a mild solution to equation (1.1) if for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ the process

$$
\{G_{t-s}(x,y)u(s,y)1_{[0,\delta]}(s) : s \geq 0, y \in \mathbb{R}^d\}
$$
is Skorohod integrable, and \( u(t, x) \) satisfies
\[
    u(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x, y) u(s, y) W(ds, dy) ,
\]
where \( G_t(x, y) \) is the Green’s function associated with \( \mathcal{L} \) and \( I_0(t, x) \) is from the initial condition(s) and the Green’s function.

If \( u \) is a mild solution to (1.1), namely if \( u \) satisfies (2.5), then \( u(s, y) = I_0(s, y) + \int_0^t \int_{\mathbb{R}^d} G_{s-r}(y, z) u(r, z) W(dr, dz) \). Substituting this expression into (2.5) we obtain
\[
    u(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x, y) I_0(s, y) W(ds, dy) + \int_0^t \int_0^r \int_{\mathbb{R}^d} G_{t-s}(x, y) G_{s-r}(y, z) u(r, z) W(dr, dz) W(ds, dy) .
\]
Repeating this procedure we obtain a solution candidate for the equation (2.5):
\[
    u(t, x) = I_0(t, x) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)) .
\]
Here
\[
    f_n(\cdot, t, x) := f_n(t_1, x_1, \ldots , t_n, x_n, t, x) \quad (2.7)
\]
\[
= \frac{1}{n!} \sum_{\sigma \in S_n} G_{t-t_{\sigma(1)}}(x, x_{\sigma(1)}) G_{t_{\sigma(1)}-t_{\sigma(2)}}(x_{\sigma(1)}, x_{\sigma(2)}) \cdots 
\times G_{t_{\sigma(n)}-t_{\sigma(n-1)}}(x_{\sigma(n)}, x_{\sigma(n-1)}) I_0(t_{\sigma(1)}, x_{\sigma(1)}) \mathbf{1}_{\{0 < t_{\sigma(i)} < \cdots < t_{\sigma(n)} < t\}}
\]
is the symmetrization of
\[
    G_{t-t_n}(x, x_n) G_{t_n-t_{n-1}}(x_n, x_{n-1}) \cdots 
\times G_{t_{n-1}-t_1}(x_2, x_1) I_0(t_1, x_1) \mathbf{1}_{\{0 < t_1 < \cdots < t_{n-1} < t\}} ,
\]
where \( S_n \) denotes the set of all permutations of \( \{1, \ldots , n\} \); and \( I_n(f_n(\cdot, t, x)) \) is the multiple Wiener-Itô integral (e.g. [14, 30]). The expression (2.6) is called the Wiener chaos expansion (or simply chaos expansion) of the solution. It is known that if (2.6) is convergent in \( L^2(\Omega) \), then (1.1) has a unique mild solution.

3. Small ball nondegeneracy and main results

3.1. Small ball nondegeneracy for Green’s function. Our main aim of this paper is to study the lower and upper asymptotics of the moments of mild solution defined in (1.4), which match with each other. What we need is the following assumptions on the Green’s function associated with the operator \( \mathcal{L} \). The following assumptions are made in order to derive the sharp lower asymptotics:

\textbf{(G1) [Positivity]}: \( G_t(\cdot, \cdot) \) is a positive function, measure, or generalized function.

\textbf{(G2) [Small ball nondegeneracy]}: \( G_t(\cdot, \cdot) \) satisfies the small ball nondegeneracy \((B(a,b))\). This is, there exist real numbers \( a \) and \( b \) (depending on the Green’s function) satisfying
\[
a > -1 , \quad b > 0 , \quad \text{and} \quad b(2a + 1) - \lambda > 0 , \quad (3.1)
\]
and there is a constant $C > 0$ such that
\[
\inf_{y \in B_r(x)} \int_{B_r(x)} G_t(y, z)dz \geq C \cdot t^a, \tag{3.2}
\]
for all $0 < t \leq \varepsilon^b \leq 1$ and $x \in \mathbb{R}^d$, where $B_r(x)$ is the ball of center $x$ with radius $\varepsilon$.

To obtain the upper bound for moments what we need is the following hypothesis for the Green’s function.

**Remark 3.1.** The task to verify the assumptions (G1)-(G3), in particular to find the sharp indices $a, b, \bar{h}$ in (3.2)-(3.4) is not trivial. We shall dedicate one section (Section 7) to verify these conditions for various partial differential operators $L$ that are currently interested by researchers. For different operators $L$, we shall obtain the best indices $a, b, \bar{h}$ in the sense that our upper and lower $p$-moment will match each other as $p$ or $t$ tends to infinity.

**Remark 3.2.** The hypothesis (G3) is quite standard for the upper moment bounds. When $G_t$ is a function (rather than a measure), then we can easily apply Hardy-Littlewood-Sobolev inequality ([24, Theorem 4.3]) to obtain
\[
\sup_{x, x' \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x, y)\Lambda(y - y')G_t(x', y')dydy' \leq C \cdot t^\bar{h}. \tag{3.4}
\]

**Remark 3.3.** In this remark, we give some intuitive connections between (G2) and (G3). If we assume $G_t(x, y) = G_t(x - y)$ satisfies what we shall call the total weighted mass property $\tilde{M}(\mu, \nu)$: there exist real numbers $\mu$ and $\nu$ (depending on the Green’s function) satisfying
\[
\mu > -1, \quad \nu \in \mathbb{R}, \quad \text{and} \quad \mu + \nu > -1, \tag{3.5}
\]
and there are two positive constants $C_1$ and $C_2$ such that
\[
\begin{align*}
\int_{\mathbb{R}^d} G_t(y)dy & \leq C_1 \cdot t^\mu, \\
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x - y)\Lambda(y)dy & \leq C_2 \cdot t^\nu. \tag{3.6}
\end{align*}
\]
Then we can easily see (3.4) holds with $\bar{h} = \mu + \nu > -1$. Furthermore, we notice that $\mu = a$, where $a$ is the same parameter in (G2).
Let us discuss the SHE and SWE in one dimension as examples. It is easy to see from Hardy-Littlewood rearrangement inequality (see [24, Theorem 3.4]) that
\[
\sup_{y \in \mathbb{R}} \int_{\mathbb{R}} G^h_t(x - y) \Lambda(x) dx \leq \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} |x|^{-\lambda} dx \leq C \cdot t^{-\frac{\lambda}{2}};
\]
\[
\sup_{y \in \mathbb{R}} \int_{\mathbb{R}} G^w_t(x - y) \Lambda(x) dx \leq \int_{-t}^t |x|^{-\lambda} dx \leq C \cdot t^{1-\lambda},
\]
where \(G^h_t\) and \(G^w_t\) are heat kernel and wave kernel, respectively. In addition, we know that
\[
\int_{\mathbb{R}} G^h_t(x) dx = 1, \quad \int_{\mathbb{R}} G^w_t(x) dx = t.
\]
Thus, (3.4) holds for SHE and SWE. Moreover, it will be shown in Section 7 that SHE and SWE satisfy small ball nondegeneracy with \(a = 0\) and \(a = 1\), respectively.

3.2. Main results. In this subsection we present our main results. This is, we give the upper moment estimates in Theorem 3.4 and lower moment in Theorem 3.6. In fact, with \(\gamma(\cdot), \Lambda(\cdot)\) and \(G(\cdot)\) satisfying conditions stated before, we also give the relation among the indices \(a, b\) and \(h\) so that the exponents in \(t\) and \(p\) in the lower and upper moments match with each others (see the Table 1).

First we state the result for the upper moment bounds.

Theorem 3.4. Assume \(\gamma(\cdot)\) satisfies (the upper inequality in) (H1) and \(\Lambda(\cdot)\) satisfies (the upper inequality in) one of (H2)-(H4). Let the Green function \(G(\cdot)\) satisfy (G3). Assume that the initial condition term \(I_0(t, x)\) is bounded, namely, there is a positive constant \(C\) such that \(\sup_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d} |I_0(t, x)| \leq C\). Then there is a unique mild solution \(u(t, x)\) satisfying (1.4). Moreover, there are some constants \(C_1\) and \(C_2\) do not depend on \(t, p\) and \(x\) such that
\[
\mathbb{E} [|u(t, x)|^p] \leq C_1 \exp \left( C_2 \cdot t^{1 + \frac{1}{\lambda + b}} \cdot p^{1 + \frac{1}{\lambda + b}} \right).
\]

The proof of this result will be given in next section (Section 4) by using the hypercontractivity inequality.

Remark 3.5. This result is new in the sense that it holds now for general operator \(\mathcal{L}\) satisfying (G3). When \(\mathcal{L}\) is the heat operator, wave operator, fractional \(\alpha\)-diffusion operator, or partial differential operator both fractional in time and space but homogeneous in space, the result is known (e.g. [2, 4, 15, 34], references therein and other references).

Our main contribution of this work is the following lower moment bounds for a general partial differential operator \(\mathcal{L}\).

Theorem 3.6. Assume \(\gamma(\cdot)\) satisfies (the lower inequality in) (H1) and \(\Lambda(\cdot)\) satisfies (the lower inequality in) one of (H2)-(H4). Let the Green function \(G(\cdot)\) satisfy (the lower inequality in) (G1) and (G2). If the initial condition satisfies \(\inf_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d} I_0(t, x) \geq c_0\) for some constant \(c_0 > 0\), then there are some positive constants \(c_1\) and \(c_2\) independent of \(t, p\) and \(x\) such that we have
\[
\mathbb{E} [|u(t, x)|^p] \geq c_1 \exp \left( c_2 \cdot t^{1 + \frac{b(\frac{1}{\lambda} - 1)}{b(\frac{1}{\lambda} - 1) - a}} \cdot p^{1 + \frac{b(\frac{1}{\lambda} - 1)}{b(\frac{1}{\lambda} - 1) - a}} \right).
\]

The proof of this theorem relies on the Feynman diagram formula for the moments of a chaos expansion. This formula will be presented in Section 5 and will be used in Section 6 to prove the above theorem.
Consequently, combing Theorem 3.4 and 3.6 we obtain the following theorem about the matching upper and lower moment bounds.

**Theorem 3.7.** Assume \( \gamma(\cdot) \) satisfy \((H1)\) and \( \Lambda(\cdot) \) satisfy one of \((H2)-(H4)\). Assume the Green function \( G_t(\cdot) \) satisfy \((G1)-(G3)\) with

\[ h := 2a - \frac{\lambda}{b} > -1 \]  

(3.9)

If the initial condition satisfies \( c_0 \leq I_0(t,x) \leq C_0 \) for some positive constants \( 0 < c_0 < C_0 < \infty \), then the mild solution \( u(t,x) \) to \((1.1)\) satisfies

\[
\begin{align*}
&c_1 \exp \left( c_2 \cdot t^{1+ \frac{b(1-\gamma)}{b(2\alpha+1)-\beta}} \cdot p^{1+ \frac{b}{b(2\alpha+1)-\beta}} \right) \\
&\leq \mathbb{E} \left[ |u(t,x)|^p \right] \leq c_1 \exp \left( c_2 \cdot t^{1+ \frac{b(1-\gamma)}{b(2\alpha+1)-\beta}} \cdot p^{1+ \frac{b}{b(2\alpha+1)-\beta}} \right)
\end{align*}
\]  

(3.10)

for all \( t \geq 0 \), \( x \in \mathbb{R}^d \), \( p \in \mathbb{Z}_+ \), where \( c_1, c_2, C_1, C_2 \) are some positive constants, independent of \( t, x, p \).

**Proof.** It is obvious that under the conditions of this theorem both the conditions of Theorems 3.4 and 3.6 hold. Thus both \((3.8)\) and \((3.7)\) hold true. Replacing \( h \) by \((3.9)\) we see that \((3.7)\) becomes the second inequality in \((3.10)\). The theorem is then proved.

We shall demonstrate that \((3.9)\) holds true for the Green’s function of various partial differential operators: SHE, \( \alpha \)-SHE, SWE and SFD (see \((7.1), (7.13), (7.22)\) and \((7.33)\) respectively). We summarize the results of that section here in following table. Notice that Table 1 only includes the exponent parts of \((3.10)\).

**Table 1. Matching Lower and Upper Moments**

| SPDEs | \((a,b)\) | \( h \) | Moment | When \( \gamma = 2 - 2H \) |
|-------|-----------|-------|--------|----------------|
| SHE   | \((0,2)\) | \(- \frac{\lambda}{2} \) | \( t^{1+ \frac{2(1-\gamma)}{2(2\alpha+1)-\beta}} \cdot p^{\frac{\beta}{2\alpha+1}} \) | \( t^{\frac{-2H-\lambda}{2\alpha+1}} \cdot p^{\frac{\beta}{2\alpha+1}} \) |
| \( \alpha \)-SHE | \((0,\alpha)\) | \(- \frac{\lambda}{\alpha} \) | \( t^{1+ \frac{\alpha(1-\gamma)}{2\alpha}} \cdot p^{\frac{\beta}{2\alpha}} \) | \( t^{\frac{-2H-\lambda}{2\alpha}} \cdot p^{\frac{\beta}{2\alpha}} \) |
| SWE   | \((1,1)\) | \( 2 - \lambda \) | \( t^{1+ \frac{2}{2\alpha+1}} \cdot p^{\frac{2}{2\alpha+1}} \) | \( t^{\frac{-2H-2\lambda}{2\alpha+1}} \cdot p^{\frac{2}{2\alpha+1}} \) |
| SFD   | \((\beta - 1,\frac{\beta}{\beta})\) | \( 2(\beta - 1) - \frac{\lambda\beta}{\alpha} \) | \( t^{1+ \frac{\alpha(1-\gamma)}{2\alpha}} \cdot p^{\frac{\beta(2\alpha+\lambda)}{2\alpha+2\lambda}} \) | \( t^{\frac{-2H+2\beta+2\lambda-2\alpha}{2\alpha+2\lambda}} \cdot p^{\frac{\beta(2\alpha+\lambda)}{2\alpha+2\lambda}} \) |

4. **Upper moment bounds**

Our goal of this section is to prove the upper moment bounds assuming that \((G1), (G3)\) hold for the Green’s function \( G \) associated with the operator \( \mathcal{L} \) and assuming that \((H1)\) and one of \((H2)-(H4)\) or one of \((H2’)-(H3’)\) hold true for the noise covariance structure.

Sometimes it is convenient to use Fourier transformation to represent the covariance function in spatial variables. Assume \( \Lambda(x) \geq 0 \) for all \( x \in \mathbb{R}^d \) and assume that
there is a measure $\mu$ on $\mathbb{R}^d$ such that
\[ \Lambda(x) = \int_{\mathbb{R}^d} e^{ix\xi} \mu(d\xi). \] (4.1)

We now assume some conditions on the Fourier mode that are similar to (H2)-(H3) and (G3).

(H2) There is a $\hat{\Lambda} : \mathbb{R}^d \to \mathbb{R}$ such that $\mu(d\xi) = \hat{\Lambda}(\xi)d\xi$ and there are constants $\lambda_j \in (0, 1), j = 1, \ldots, d$ and $C > 0$ such that
\[ |\hat{\Lambda}(\xi)| \leq C \prod_{j=1}^{d} |\xi|^\lambda_j - 1, \quad \forall \xi \in \mathbb{R}^d. \]

In this case we denote $\lambda = \lambda_1 + \cdots + \lambda_d$.

(H3′) There is a $\hat{\Lambda} : \mathbb{R}^d \to \mathbb{R}$ such that $\mu(d\xi) = \hat{\Lambda}(\xi)d\xi$ and there are constants $\lambda \in (0, d)$ and $C > 0$ such that
\[ |\hat{\Lambda}(\xi)| \leq C|\xi|^{\lambda - d}, \quad \forall \xi \in \mathbb{R}^d. \]

(G3′) [Majorized property] $G_t(\cdot)$ satisfies the Majorized property $(M(h))$. This is, there exists a positive function or measure $Q_t$ such that $G_t(x, y) \leq Q_t(x - y)$ for any $t > 0$ and $x, y \in \mathbb{R}$, and there exist a real number $h > -1$ (the same ones in (G3)) and a constant $C > 0$ such that
\[ \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{Q}_t(\xi - \eta)|^2 |\mu|(d\xi) \leq C \cdot t^h, \] (4.2)

where $|\mu|(\xi) = |\hat{\Lambda}(\xi)|d\xi$ with (H2′) or (H3′) holds.

**Theorem 4.1.** Let the Green function $G_t(\cdot)$ satisfy (G3′). Assume $\gamma(\cdot)$ satisfies (the upper inequality in) (H1) and $\Lambda(\cdot)$ satisfies one of (H2′)-(H3′). Assume that the initial condition term $I_0(t, x)$ is bounded, namely, there is a positive constant $C$ such that $\sup_{t, x \in \mathbb{R}^d} |I_0(t, x)| \leq C$. Then there is a unique mild solution $u(t, x)$ is satisfying (1.4). Moreover, there are some constants $C_1$ and $C_2$ do not depend on $t$, $p$ and $x$ such that (3.7) holds.

**Proof of Theorem 3.4 and Theorem 4.1.** As indicated in [15], there are mainly three approaches to obtain the upper moments, effective in different situations. In our case, we choose to use the approach of combining chaos expansion and hypercontractivity inequality.

**Step 1:** We shall show the upper bound under assumptions (G3), (H1) and one of (H2)-(H4). In the following, we shall only prove the case (H2). The cases (H3) and (H4) can be done similarly. Recall the Wiener-Itô chaos expansion (2.6) for the mild solution to (2.5)

\[ u(t, x) = I_0(t, x) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)), \]

where $f_n(\cdot, t, x)$ is given by (2.7). Denote $u_n(t, x) = I_n(f_n(\cdot, t, x))$. Then it follows from the Itô isometry for the multiple Wiener-Itô integral (e.g. [14]) that
\[ \|u_n(t, x)\|_{L^2}^2 = \mathbb{E} |I_n(f_n(\cdot, t, x))|^2 = n! \|f_n(\cdot, t, x) \|^2_{\mathbb{H}^\otimes n}. \]
Littlewood-Sobolev inequality: By the Cauchy-Schwarz inequality \( \psi \) (or Stirling’s formula for Gamma function.

Thus, we have

\[
\|u_n(t, x)\|_{L^2}^2 = n!\|f_n(\cdot; t, x)\|_{\mathbb{H}^n}^2
= \frac{1}{n!}\Phi_n(t) := \frac{c_H^n}{n!} \int_{[0,t]^{2n}} \gamma(t_j - s_j)\Psi_n(t, \tilde{s})d\tilde{s}.
\]

By the Cauchy-Schwarz inequality \( \Psi_n(t, \tilde{s}) \leq \left[ \Psi_n(t, \tilde{t})\Psi_n(s, \tilde{s}) \right]^{1/2} \) and Hardy-Littlewood-Sobolev inequality [20, Inequality (2.4)], we obtain with \( \gamma = 2 - 2H \) (or \( H = 1 - \frac{1}{2} \)) from (4.3)

\[
\Phi_n(t) \lesssim \int_{[0,t]^{2n}} \gamma(t_j - s_j) \left[ \Psi_n(t, \tilde{t})\Psi_n(s, \tilde{s}) \right]^{1/2}d\tilde{s}
\leq \left( \int_{[0,t]^{2n}} |\Psi_n(s, \tilde{s})|^{1/H}d\tilde{s} \right)^{2H}.
\]

Now we need to resort to the key assumption (G3), i.e. \textit{Hardy-Littlewood-Sobolev type mass property} \( M(h) \) to obtain the bound for \( \Psi_n \). Repeatedly using (G3) (namely, (3.4)), we have

\[
\Psi_n(s, \tilde{s}) \leq \int_{\mathbb{R}^{2n}} f_n(s, \tilde{s}; t, x) \prod_{j=1}^{n} \Lambda(x_j - y_j)f_n(s, \tilde{s}; y, t, x)d\tilde{s}
\leq \left( \prod_{j=1}^{n} |s_{\sigma(j+1)} - s_{\sigma(j)}|^{h}1_{0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t} \right),
\]

where \( h > -1 \) and \( 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < s_{\sigma(n+1)} = t \). Denote the simplex \( \prod_n(t) = \{ (s_1, \cdots, s_n); 0 < s_1 < \cdots < s_n < t \} \). Then, using the bound we just obtained for \( \Psi_n \), we obtain the upper bound for \( \Phi_n(t) \):

\[
\Phi_n(t) \leq C_H^n \left( n! \int_{\prod_n(t)} \prod_{j=1}^{n} [s_{j+1} - s_j]^{h/2H}ds \right)^{2H}
\leq C_H^n \left( \frac{n^n h^{+2H} + n}{\Gamma(nh/2H + n + 1)} \right)^{2H} \simeq C_H^n \frac{n^n (h+2H)}{(n!)^h}
\]

by Stirling’s formula for Gamma function.

As a result, the second moment can be estimated as

\[
\|u_n(t, x)\|_{L^2}^2 = \frac{1}{n!}\Phi_n(t) \leq C_H^n \frac{n^n (h+2H)}{(n!)^h+1}.
\]
It is now easy to bound the \( p \)-th moment from the above second moment bound by using the hypercontractivity inequality (e.g., [14, p.54, Theorem 3.20])

\[
\|u_n(t, x)\|_{L^p} \leq (p - 1)^{n/2} \|u_n(t, x)\|_{L^2} \leq C_H^n (p - 1)^{n/2} \left[ \frac{t^{n(h+2H)}}{(n!)^{h+1}} \right]^{1/2}
\]

Thus

\[
\|u(t, x)\|_p \leq C + \sum_{n=1}^\infty \|u_n(t, x)\|_p \leq C + \sum_{n=1}^\infty C_H^n (p - 1)^{n/2} \left[ \frac{t^{n(h+2H)}}{(n!)^{h+1}} \right]^{1/2} \leq C \exp \left( C \cdot t^{\frac{h+2H}{p-1}} \right).
\]

This means \( \mathbb{E}[\|u(t, x)\|^p] \leq C_1 \exp \left( C_2 \cdot t^{\frac{h+2H}{p-1}} \right) \) for some positive constants \( C_1 \) and \( C_2 \) and hence we conclude the proof of Theorem 3.4.

**Step 2:** We shall show the upper bound under assumptions (G3′), (H1) and one of (H2′)-(H3′). Denote

\[
f_n^Q(\cdot, t, x) := f_n(t_1, x_1, \cdots, t_n, x_n, t, x) = \frac{1}{n!} \sum_{\sigma \in S_n} Q_{t-t_\sigma(n)}(x - x_\sigma(n)) Q_{t_\sigma(n) - t_{\sigma(n-1)}}(x_\sigma(n) - x_{\sigma(n-1)}) \cdots \times Q_{t_{\sigma(2)} - t_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) I_0(t_{\sigma(1)}, x_{\sigma(1)}).
\]

Namely, we replace \( G \) in the expression of \( f_n(\cdot, t, x) \) by \( Q \). Then by the positivity of \( G, \Lambda \), and the fact that \( G \leq Q \), we have

\[
\Psi_n(\bar{s}, \bar{s}) \leq \int_{\mathbb{R}^{2n d}} f_n(\bar{s}, \bar{x}; t, x) \prod_{j=1}^n \Lambda(x_j - y_j) f_n(\bar{s}, \bar{y}; t, x) d\bar{x} d\bar{y}
\]

\[
\leq \int_{\mathbb{R}^{2n d}} f_n^Q(\bar{s}, \bar{x}; t, x) \prod_{j=1}^n \Lambda(x_j - y_j) f_n^Q(\bar{s}, \bar{y}; t, x) d\bar{x} d\bar{y}
\]

\[
\leq \int_{\mathbb{R}^{2n d}} |F[f_n(\bar{s}, \cdot; t, x)](\xi)|^2 |\mu|(d\xi) \leq \prod_{j=1}^n |s_{\sigma(j+1)} - s_{\sigma(j)}|^H.
\]

Thus, we get \( \mathbb{E}[\|u(t, x)\|^p] \leq C_1 \exp \left( C_2 \cdot t^{\frac{1}{p-1}} \right) \) for some positive constants \( C_1 \) and \( C_2 \).

\[ \square \]

5. **Feynman diagram formula**

Now we turn to the proof of Theorem 3.6, i.e., the lower bounds for the moments. The main difficulty is the lack of the Feynman-Kac formula for general partial differential operators. To get around of this difficulty our strategy is a brutal force one. We try to handle the \( p \)-th moment of \( u(t, x) \) directly, where \( p \) is an arbitrary positive integer and \( u \) is the mild solution to (1.1), given by its chaos expansion (2.6). Since the solution is an infinite sum of multiple Wiener-Itô integrals so we
need first to use the product formulas of the multiple Wiener-Itô integrals (with respect to Gaussian noise). This is called the Feynman diagram formulas and they can be found in Theorem 5.7 and Theorem 5.8 in [14] (for general Gaussian noise case), Theorem 10.2 in [27] or Theorem 5.3 in [28] (for White noise cases). In this section we shall present this formula “graphically” so that we can keep track the terms.

Recall that the Gaussian space $\mathcal{H}$ in our situation is the Hilbert space obtained by the completion of $D(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to the scalar product defined by (2.2). Since the work of [27] or [28] are for the “White noise” case, we will follow the product formula of [14, Theorem 5.7]. Since we are only interested in the expectation of the product of multiple integrals and since $E[I_k(f)] = 0$ for all $k \geq 1$ we only need to sum the terms with $|\gamma| = 0$ in [14, Theorem 5.7] when we take the expectation of the left hand side of [14, Equation 5.3.5] (The notation $\gamma$ used in [14] is different than the one used in this paper).

To visualize these summation terms graphically, we recall the concept of diagram associated with only these terms. A Feynman diagram $D$ is a set of some vertices and some edges connecting them so that the vertices are arranged into some finite rows and each row contains some finite many vertices. The set of vertices of the diagram $D$ can then be represented by $\mathcal{V}(D) = \{(k, r) : 1 \leq k \leq m, 1 \leq r \leq n_m\}$. We use $\mathcal{E}(D) = \{[(\vec{k}, \tau), (\vec{k}, \tau^*)] : \vec{k} < \vec{k}^*\}$ denote the set of all edges of a diagram $D$, where $\vec{k} < \vec{k}^*$ means $(\vec{k}, \tau)$ is the upper (row) and $(\vec{k}, \tau^*)$ is the lower (row) end point of an edge. The strict inequality is important here since two vertices in the same row are not allowed to form an edge. For an edge $[(\vec{k}, \tau), (\vec{k}, \tau^*)] \in \mathcal{E}(D)$, we call $(\vec{k}, \tau)$ the upper vertex and $(\vec{k}, \tau^*)$ the lower vertex of the edge and we call a vertex associates with an edge if it is either upper or lower vertex of the edge. We use $\mathcal{V}(D)$ and $\mathcal{E}(D)$ to the sets of all upper and lower vertices, respectively. We require that one vertex associates with at most one edge. Thus we have $\mathcal{V}(D) \cap \mathcal{V}(D) = \emptyset$.

After taking the expectation of [14, Equation 5.3.5], the terms will be significantly reduced. To account the remaining terms we only need to consider the following special diagrams.

**Definition 5.1.** A diagram $D = (\mathcal{V}(D), \mathcal{E}(D))$ is called admissible if every vertex is associated with one and only one edge. The set of all admissible diagrams associated with the vertices $\{(k, r), 1 \leq k \leq m, 1 \leq r \leq n_k\}$ is denoted by $\mathbb{D}(n_1, \ldots, n_m)$.

It is clear that if a diagram $D$ is admissible then $n_1 + \cdots + n_m = 2|\mathcal{E}(D)|$, in particular, $n_1 + \cdots + n_m$ is an even integer.

Let $f_k : (\mathbb{R}_+ \times \mathbb{R}^d)^{n_k} \to \mathbb{R}$, $k = 1, \ldots, m$ be some given measurable functions. Associated with these functions we have naturally the set of Feynman diagrams $\mathbb{D}(f_1, \ldots, f_m)$. The correspondence is described as follows. Each Feynman diagram $\tilde{D} \in \mathbb{D}(f_1, \ldots, f_m)$ contains $m$ rows, corresponding to $f_1, \ldots, f_m$, and the $k$-th row of $\mathcal{D}$ contains $n_k$ vertices, which is the number of independent variables of the function $f_k$. We use $\mathbb{D}(f_1, \ldots, f_m)$ to denote the set of all admissible Feynman diagrams associated with $f_1, \ldots, f_m$.

For the sake of convenience we consider $(t,x)$ as one vector independent variable, where $t \geq 0$, $x \in \mathbb{R}^d$. So we shall say that $f_k : (\mathbb{R}_+ \times \mathbb{R}^d)^{n_k} \to \mathbb{R}$ has $n_k$ independent (vector) variables. From the functions $f_k : (\mathbb{R}_+ \times \mathbb{R}^d)^{n_k} \to \mathbb{R}$, $k = 1, \ldots, m$, we define their concatenation $f_1 \circ \cdots \circ f_m$ as a function of $n_1 + \cdots + n_m$ independent vector variables. We name the $n_k$ independent variables of the function $f_k$ by
(t_{(k,1)}, x_{(k,1)}), \ldots, (t_{(k,n_k)}, x_{(k,n_k)}), \) associated with the \( k \)-th row vertices. Thus for an admissible Feynman diagram \( \mathcal{D} \in \mathbb{D}(f_1, \ldots, f_m) \), the concatenation \( f_1 \circ \cdots \circ f_m \) is a function of \( n_1 + \cdots + n_m \) vector variables and we write it as
\[
   f_1 \circ \cdots \circ f_m((t_{(1)}, x_{(1)}), (t_{(2)}, x_{(2)}), \ldots, (t_{(m)}, x_{(m)})).
\]
The edges in \( \mathcal{D} \in \mathbb{D}(f_1, \ldots, f_m) \) are used to form the (tensor) scalar product in the Gaussian space \( \mathcal{H} \) of the above concatenation. Here is the detail of this construction. If \( [(\vec{r}, \vec{s}), (\vec{t}, \vec{v})] \) is an edge of the diagram \( \mathcal{D} \), then we form a factor
\[
   \gamma(t(\vec{r}, \vec{s}) - t(\vec{t}, \vec{v})) \Lambda(x(\vec{r}) - x(\vec{v})).
\]
For the set of \( \mathcal{E}(\mathcal{D}) \), we denote the product of all above factors as
\[
   \gamma(t(\vec{r}, \vec{s}) - t(\vec{t}, \vec{v})) \Lambda(x(\vec{r}) - x(\vec{v})) = \prod_{[(\vec{r}, \vec{s}), (\vec{t}, \vec{v})] \in \mathcal{E}(\mathcal{D})} \gamma(t(\vec{r}, \vec{s}) - t(\vec{t}, \vec{v})) \Lambda(x(\vec{r}) - x(\vec{v})). \tag{5.1}
\]
With these notations, we define finally a real number associated with \( f_1, \ldots, f_m \) and associated with an admissible diagram \( \mathcal{D} \) as follows:
\[
   F_{\mathcal{D}}(f_1, \cdots, f_m) = \int f_1 \circ \cdots \circ f_m((t_{(1)}, x_{(1)}), (t_{(2)}, x_{(2)}), \ldots, (t_{(m)}, x_{(m)})) \gamma(t(\vec{r}, \vec{s}) - t(\vec{t}, \vec{v})) \Lambda(x(\vec{r}) - x(\vec{v})) dt_{\vec{r}} dt_{\vec{v}} dx_{\vec{r}} dx_{\vec{v}}. \tag{5.2}
\]
To illustrate the above notation, we give one example.

**Example 5.2.** Given three functions of four independent of variables. Let us take the following admissible diagram \( \mathcal{D} \in \mathbb{D}(4,4,4) \) in Figure 1 as an example. The upper vertices are colored in red and the lower vertices are colored in blue. The upper and lower variables are gives as follows:
\[
\begin{align*}
   x(\vec{r}) &\in \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,4), \} \\
x(\vec{v}) &\in \{(2,2), (2,3), (3,1), (3,2), (3,3), (3,4) \}
\end{align*}
\]
The corresponding set of edges of this diagram is
\[
\mathcal{E}(\mathcal{D}) = \{ [(1,1), (3,1)], [(1,2), (2,3)], [(1,3), (3,4)], [(1,4), (2,2)], [(2,1), (3,2)], [(2,4), (3,3)] \}.
\]
In this case, \( n_1 = n_2 = n_3 = 4 \). It is easy to see that \( |\mathcal{E}(\mathcal{D})| = 6 = (n_1 + n_2 + n_3)/2 \) and
\[
F_{\mathcal{D}}(f_1, f_2, f_3) = \int f_1((t(1,1), x(1,1)), (t(1,2), x(1,2)), (t(1,3), x(1,3)), (t(1,4), x(1,4))) \cdot f_2((t(2,1), x(2,1)), (t(2,2), x(2,2)), (t(2,3), x(2,3)), (t(2,4), x(2,4))) \cdot f_3((t(3,1), x(3,1)), (t(3,2), x(3,2)), (t(3,3), x(3,3)), (t(3,4), x(3,4))) \\
\cdot \gamma(t(1,1) - t(3,1)) \gamma(t(1,2) - t(2,3)) \gamma(t(1,3) - t(3,4)) \\
\cdot \gamma(t(1,4) - t(2,2)) \gamma(t(1,4) - t(2,3)) \gamma(t(2,4) - t(3,3)) \\
\cdot \Lambda(x(1,1) - x(3,1)) \Lambda(x(1,2) - x(2,3)) \Lambda(x(1,3) - x(3,4)) \\
\cdot \Lambda(x(1,4) - x(2,2)) \Lambda(x(2,1) - x(3,3)) \Lambda(x(2,4) - x(3,3)) dt dx,
\]
where \( dt = dt_{(1,1)} \cdots dt_{(3,4)} \) and similar notation for \( dx \). Notice that the Feynman diagrams are used to track the terms and to provide guidance for the variables inside \( \gamma \) and \( \Lambda \).

Let us also notice that the operation \( F_D \) can also be defined for any \( f_k \in \mathcal{H}^{\otimes n_k}, \) \( k = 1, \cdots, m \), which may contain measures or generalized functions.

\[ \begin{array}{c}
\text{Figure 1. An example the admissible diagram}
\end{array} \]
Proof. We only need to prove the second equality in (5.4). We may only consider the time variable without loss of generality (i.e. $d = 0$). Namely, we reduce the symmetric function $f_n(t_1, x_1, \cdots, t_n, x_n; t, x)$ to the symmetric function $f_n(t_1, \cdots, t_n; t)$. Then what we need to show is the following equality for any $n_1, \ldots, n_m$ and for any corresponding admissible diagrams $\mathcal{D}$, (5.4) holds true. We shall prove (5.4) recursively on $n$. Denote the function of (2.8) by $f_n(t_1, \cdots, t_n; t)$ and its symmetrization by $\tilde{f}_n(t_1, \cdots, t_n; t)$. Then

$$
\sum_{D \in \mathcal{D}(f_{n_1}, \cdots, f_{n_m})} F_D \left( \tilde{f}_{n_1}(\cdot,t), \cdots, \tilde{f}_{n_m}(\cdot,t) \right) = \sum_{D \in \mathcal{D}(f_{n_1}, \cdots, f_{n_m})} \prod_{j=1}^m \tilde{f}_{n_j}(t(j,1), \cdots, t(j,n_j); t) \times \gamma(t_{\mathcal{F}(D)} - t_{\mathcal{L}(D)}) dt_D
$$

$$
\sum_{D \in \mathcal{D}(f_{n_1}, \cdots, f_{n_m})} \frac{1}{n_1!} \sum_{\sigma \in S_{n_1}} \prod_{j=2}^m \tilde{f}_{n_j}(t(j,1), \cdots, t(j,n_j); t) \times \gamma(t_{\mathcal{F}(D)} - t_{\mathcal{L}(D)}) dt_D
$$

$$
\sum_{D \in \mathcal{D}(f_{n_1}, \cdots, f_{n_m})} \frac{1}{n_1!} \sum_{\sigma \in S_{n_1}} I_{\sigma(1), \cdots, \sigma(n_1), D},
$$

where $S_{n_1}$ denotes the set of all permutations of $\{1, 2, \cdots, n_1\}$ and $I_{\sigma(1), \cdots, \sigma(n_1), D}$ denotes the above integral. Suppose that $\mathcal{D} \in \mathcal{D}(f_{n_1}, \cdots, f_{n_m})$ is a Feynman diagram. Then there there are $(j_1, r_1), \cdots, (j_{n_1}, r_{n_1})$ such that the following edges

$$[(1, 1), (j_1, r_1)], \cdots, [(1, n_1), (j_{n_1}, r_{n_1})]$$

are in $\mathcal{D}$. In this diagram $\mathcal{D}$, we replace the above edges by the following ones

$$[(1, \sigma(1)), (j_1, r_1)], \cdots, [(1, \sigma(n_1)), (j_{n_1}, r_{n_1})]$$

and retain all other edges unchanged. Then we obtain another diagram $\mathcal{D}_\sigma$. See Figure 2 for an illustrating example. This transformation $\mathcal{D} \rightarrow \mathcal{D}_\sigma$ has the following

(i) If $\mathcal{D}$ is an admissible diagram, so is $\mathcal{D}_\sigma$.

(ii) For any fixed permutation $\sigma$, the mapping $\mathcal{D} \rightarrow \mathcal{D}_\sigma$ is a bijection from $\mathcal{D}(f_{n_1}, \cdots, f_{n_m})$ to itself.
(iii) \( \gamma \left( t_{\mathcal{F}(D)} - t_{\mathcal{F}(D')} \right) \) remains unchanged:

\[
\gamma \left( t_{\mathcal{F}(D)} - t_{\mathcal{F}(D)} \right) = \gamma \left( t_{\mathcal{F}(D)} - t_{\mathcal{F}(D)} \right).
\]

These properties imply

\[
\sum_{D \in \mathcal{B}(n_1, \ldots, n_m)} I_{\sigma(1), \ldots, \sigma(n_1), \sigma(n_1)} = \sum_{D \in \mathcal{B}(n_1, \ldots, n_m)} I_{1, \ldots, n_1, D}.
\]

Substituting it to (5.6) we have

\[
F_D \left( f_{n_1}(\cdot), \ldots, f_{n_m}(\cdot) \right) = \sum_{D \in \mathcal{B}(n_1, \ldots, n_m)} \int f_{n_1}(t_{(1,1)}, \ldots, t_{(1,n_1)}; t) \cdot \prod_{j=2}^{m} f_{n_j}(t_{(j,1)}, \ldots, t_{(j,n_j)}; t) \times \gamma \left( t_{\mathcal{F}(D)} - t_{\mathcal{F}(D)} \right) dt_D.
\]

This can be used to prove the theorem by induction.

\[\square\]

**Example 5.5.** The above formula (5.4) can be used to compute all moments of a chaos expansion. This will be done in the next section when we prove the lower moment bounds. As an example, it is interesting to consider the second moment. By orthogonality of multiple Wiener-Itô chaos expansion, we have

\[
\mathbb{E}[|u(t,x)|^2] = 1 + \sum_{n=1}^{\infty} \mathbb{E} \left[ |I_n(f_n)|^2 \right].
\]

Then by (5.4) in Theorem 5.4, one finds

\[
\mathbb{E} \left[ |I_n(f_n)|^2 \right] = \sum_{D \in \mathcal{B}(n,n)} \int \prod_{j=1}^{n} \prod_{r=1}^{n} G_{t_{(j),r}} \left( x_{r+1}^{(j)} - x_{r}^{(j)} \right) 1_{\{0<t_{(j),r}^1<\cdots<t_{(j),r}^n<t\}} \times \gamma \left( t_{\mathcal{F}(D)} - t_{\mathcal{F}(D)} \right) \Lambda \left( x_{\mathcal{F}(D)} - x_{\mathcal{F}(D)} \right) dt_D dx_D.
\]

An example of admissible diagram \( D \in \mathcal{D}(4,4) \) can be illustrated in the Figure 3. In this diagram, we have \( T_{\mathcal{F}(D)} := (t_{\mathcal{F}(D)}, x_{\mathcal{F}(D)}) = \{ T_j^{(2)} : 1 \leq j \leq 4 \} \) colored in red, \( T_{\mathcal{F}(D)} := (t_{\mathcal{F}(D)}, x_{\mathcal{F}(D)}) = \{ T_j^{(1)} : 1 \leq j \leq 4 \} \) colored in blue. Moreover,

\[
\gamma \left( t_{\mathcal{F}(D)} - t_{\mathcal{F}(D)} \right) := \gamma \left( t_1^{(2)} - t_1^{(1)} \right) \gamma \left( t_2^{(2)} - t_2^{(1)} \right) \gamma \left( t_3^{(2)} - t_3^{(1)} \right) \gamma \left( t_4^{(2)} - t_4^{(1)} \right),
\]

\[
\Lambda \left( x_{\mathcal{F}(D)} - x_{\mathcal{F}(D)} \right) := \Lambda \left( x_1^{(2)} - x_1^{(1)} \right) \Lambda \left( x_2^{(2)} - x_2^{(1)} \right) \Lambda \left( x_3^{(2)} - x_3^{(1)} \right) \Lambda \left( x_4^{(2)} - x_4^{(1)} \right).
\]

and \( \Lambda \left( x_{\mathcal{F}(D)} - x_{\mathcal{F}(D)} \right) \) is also expressed in the same way. Obviously, there are 4! such diagram.
If $\gamma(\cdot) = \delta(\cdot)$, then (5.7) reduced to

$$
E\left[\left|I_n^W(f_n)\right|^2\right] = \int \prod_{r=1}^{n} G_{t_{r+1} - t_r}(x_{r+1} - x_r) \Lambda(x_r - y_r)
	imes G_{t_{r+1} - t_r}(y_{r+1} - y_r) \cdot 1_{\{0 < t_1 < \cdots < t_n < t\}} dt dx dy .
$$

(5.8)

This is because the only admissible admissible diagram is the ‘trivial’ one shown in Figure 4 in this case. Otherwise, in some non-trivial admissible diagrams (e.g. the one in Figure 3) the indicate function $1_{\{0 < t_1 < \cdots < t_n < t\}}$ is not compatible with $1_{\{0 < t_1^{(2)} < \cdots < t_n^{(2)} < t\}}$.

6. LOWER MOMENT BOUNDS

In this section we use the formula (5.4) to obtain the lower moment bounds for the mild solution of (1.1). In the remaining part of the paper we shall use the index $(t_l^j, x_l^j)$ to represent the independent variable $(t_{(l,j)}, x_{(l,j)})$ associated with the vertice $(l, j)$: the superscript indicates the row that variable corresponds to and the subscript indicates the column that variable corresponds to. Again, in the following, we shall only prove the case (H2). The cases (H3) and (H4) can be done similarly.
Proof of Theorem 3.6. Let \( u(t,x) \) be the mild solution given by (2.6)-(2.7). Let \( p \) be an even positive integer. Applying Theorem 5.4, we have

\[
E \left[ \prod_{j=1}^{p} u(t,x_j) \right] = E \left[ \prod_{j=1}^{p} \sum_{n_j=0}^{\infty} I_{n_j} (f_{n_j} (\cdot, t, x_j)) \right] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} E \left[ I_{n_1} (f_{n_1} (\cdot, t, x_1)) \cdots I_{n_p} (f_{n_p} (\cdot, t, x_p)) \right] = \sum_{m=0}^{\infty} \sum_{n_1+n_2+\cdots+n_p=2m} F_D (f_{n_1}, \cdots, f_{n_p}).
\]

(6.1)

Notice that the last equality follows from the fact that the number of all vertices of an admissible diagram \( D \) must be even.

Our next strategy is to find the suitable lower bounds for the term

\[
\sum_{n_1+\cdots+n_p=2m} \sum_{D \in D} F_D
\]

in (6.1) when \( p \) and \( m \) are sufficiently large. We shall divide our proof into three steps.

**Step 1:** By the assumption \((G1)\), namely, all the kernels \( f_{n_k} \) are nonnegative, to obtain the lower bounds, we can discard any terms we wish. As in [10] we shall keep only those terms such that \( n_1 = \cdots = n_p \) (see the Figure 5 for a graphical illustration). To be more precise, among all the admissible diagrams \( D \in D (n_1, \cdots, n_p) \) such that \( n_1 + \cdots + n_p = 2m \), we take into account only those diagrams satisfying the following conditions:

**(D.1)** We consider only the diagram so that the number of vertices in each row are the same. This is, we set

\[
n_1 = \cdots = n_p = \frac{2m}{p} = m_p.
\]

(6.2)
We set the first \( \frac{p}{2} \) rows to be the upper vertices \( T_{\mathcal{F}(D)} := (t_{\mathcal{F}(D)}, x_{\mathcal{F}(D)}) \) (which are colored in red in the Figure 5), and the remaining rows to be the lower vertices \( T_{\mathcal{X}(D)} := (t_{\mathcal{X}(D)}, x_{\mathcal{X}(D)}) \) (which are colored in blue in the Figure 5).

**Remark 6.1.** Fix the set of upper vertices. Any permutation of the lower vertices corresponds to an admissible diagram in one to one manner. Then there are \( m! \) such admissible diagrams satisfying the conditions (D.1) and (D.2).

Let \( t \in \mathbb{R}_+ \). Denote \( L = \frac{t}{2(m_p + 1)} \), \( t_j = \frac{jt}{2(m_p + 1)} \) and \( I_j = [a_j, b_j] \) for \( j = 1, \ldots, m_p \), where \( a_j = t_j - L/4 \) and \( b_j = t_j + L/4 \). We assure \( t_j^l \) is in \( I_j \) for \( 1 \leq l \leq p \) and \( 1 \leq j \leq m_p \). And we put some restriction on these points such that \( t_{j+1}^l - t_j^l \) are smaller than \( \varepsilon^b \) (the same one in \( B(a, b) \)) for any \( \varepsilon(\leq 1) \), then

\[
\frac{t}{4m_p} \simeq \frac{t}{4(m_p + 1)} \leq t_{j+1}^l - t_j^l \leq \frac{t}{m_p + 1} \simeq \frac{t}{m_p} \leq \varepsilon^b. \tag{6.3}
\]

Moreover, combining (6.2) in (D.1) and (6.3), we must have \( m_p = \frac{2m}{p} \geq \frac{t}{\varepsilon^b} \), which is equivalent to the following condition:

\[
m \geq \frac{p \cdot t}{2 \varepsilon^b}. \tag{6.4}
\]

**Step 2:** Since \( f_n(\cdot, t, x) \) is defined by (2.7) we can use (5.4) in Theorem 5.4 to bound \( F_D \) in (6.1).

We only consider particular scenario specified in **Step 1**. We denote the set of all admissible diagrams satisfying satisfying the conditions (D.1) and (D.2) by \( \mathcal{D} := \mathcal{D}(f_{n_1}, \ldots, f_{n_m}) \). When \( \mathcal{D} \in \mathcal{D} \), we have

\[
F_D(f_{n_1}, \ldots, f_{n_p}) = \int \prod_{i=1}^{p} \prod_{l=1}^{m_p} G_{t_{j+1}^l - t_j^l} (x_{j+1}^l, x_j^l) 1_{I_j}(t_j^l)1_{(0 < t_j < \cdots < t_{m_p} < t)} \tag{6.5}
\]

\[
\times \gamma \left( (t_{\mathcal{F}(D)} - t_{\mathcal{X}(D)}), \Lambda \left( x_{\mathcal{F}(D)} - x_{\mathcal{X}(D)} \right) \right) dt_{\mathcal{D}} dx_{\mathcal{D}},
\]

with the convention that \( x_{m_p} = x \) and \( t_{m_p} = t \) for all \( 1 \leq l \leq p \).

It seems very difficult to compute the multiple integral in (6.5). We need to find a suitable lower bounds of the integral that are the main parts and that are relatively easier to handle. Since \( \Lambda(x) \rightarrow \infty \) when \( x \rightarrow 0 \), we shall first bound the above integral with respect to the spatial variables \( dx_{\mathcal{D}} \) from below by the integration over small balls \( B_\varepsilon(x) \) centered at \( x = x_1 = \cdots = x_p \) with radius \( \varepsilon \). By the assumption (H2) or (H3), it is easy to see \( \Lambda \left( x_{\mathcal{F}(D)} - x_{\mathcal{X}(D)} \right) \gtrsim \varepsilon^{-m\gamma} \) since \( \#(\mathcal{F}(D)) = \#(\mathcal{X}(D)) = m \) and since we always have \( |x_i - x_j| \leq 2\varepsilon \) for any \( i \in \mathcal{F}(D) \) and \( j \in \mathcal{X}(D) \). Similarly, it is obvious to control \( \gamma \left( t_{\mathcal{F}(D)} - t_{\mathcal{X}(D)} \right) \gtrsim t^{-m\gamma} \) because \( |t_i - t_j| \leq t \) for any \( i \in \mathcal{F}(D) \) and \( j \in \mathcal{X}(D) \).

On each space-time line, there are \( m_p \) space-time points. By (6.3) we have

\[
t_{j+1}^l - t_j^l \leq \varepsilon^b \quad \text{for} \quad 1 \leq l \leq p \quad \text{and} \quad 1 \leq j \leq m_p.
\]

The small ball nondegeneracy property \( B(a, b) \) implies

\[
\int_{B_\varepsilon(x)} G_{t_{j+1}^l - t_j^l} (x_{j+1}^l, x_j^l) dx_j^l \geq C \cdot |t_{j+1}^l - t_j^l|^a
\]

for some
if \( x^l_j \) belong to \( B_\varepsilon(x) \) for all \( l \) and \( j \). Thus, on the domain
\[
\Omega_\varepsilon := \cap_{j=1}^p \cap_{j=1}^{m_p} \{ x^l_j \in B_\varepsilon(x) \}
\]
we have from the simple fact \( \Lambda \left( x_{\overline{f}}(D) - x_{\underline{f}}(D) \right) \gtrsim \varepsilon^{-m_\lambda} \):
\[
\int \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t^l_{j+1} - t^l_j} \left( x^l_{j+1}, x^l_j \right) \Lambda \left( x_{\overline{f}}(D) - x_{\underline{f}}(D) \right) dx_D
\]
\[
\gtrsim \int_{\Omega_\varepsilon} \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t^l_{j+1} - t^l_j} \left( x^l_{j+1}, x^l_j \right) \Lambda \left( x_{\overline{f}}(D) - x_{\underline{f}}(D) \right) dx_D
\]
\[
\gtrsim \varepsilon^{-m_\lambda} \int_{B_\varepsilon(x)^{2m}} \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t^l_{j+1} - t^l_j} \left( x^l_{j+1}, x^l_j \right) dx_D
\]
\[
= \varepsilon^{-m_\lambda} \int_{B_\varepsilon(x)^{2m-1}} \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t^l_{j+1} - t^l_j} \left( x^l_{j+1}, x^l_j \right) dx_1
\]
\[
\times \prod_{l=1,j,j \neq 1} G_{t^l_{j+1} - t^l_j} \left( x^l_{j+1}, x^l_j \right) dx_D \setminus x_1^l,
\]
where we used (6.3) and \( dx_D \setminus x_1^l \) means that \( dx_1^l \) is removed from \( dx_D \). We integrate the spatial variables iteratively to find
\[
\int \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t^l_{j+1} - t^l_j} \left( x^l_{j+1}, x^l_j \right) \Lambda \left( x_{\overline{f}}(D) - x_{\underline{f}}(D) \right) dx_D
\]
\[
\gtrsim \varepsilon^{-m_\lambda} \prod_{l=1}^p \prod_{j=1}^{m_p} \left[ t^l_{j+1} - t^l_j \right]^2 (6.6)
\]
for all
\[
t_D \in \tilde{\Omega}_\varepsilon := \cap_{j=1}^p \cap_{j=1}^{m_p} \{ t^l_j \in I_j \}.
\]
From this inequality, Remark 6.1, and (6.3) we can bound (6.5) from below by
\[
F_D(f_1, \ldots, f_{m_p})
\]
\[
\gtrsim \varepsilon^{-m_\lambda t^{-m_\gamma}} \int \prod_{l=1}^p \prod_{j=1}^{m_p} \left[ t^l_{j+1} - t^l_j \right]^\alpha I_{l_j} \left( t^l_j \right) 1_{\{ 0 < t^l_1 < \cdots < t^l_{m_p} < t \}} dt_D
\]
\[
\gtrsim \varepsilon^{-m_\lambda t^{-m_\gamma}} \left( \frac{t}{4m_p} \right)^{2m_a} \int \prod_{l=1}^p \prod_{j=1}^{m_p} I_{l_j} \left( t^l_j \right) 1_{\{ 0 < t^l_1 < \cdots < t^l_{m_p} < t \}} dt_D
\]
\[
= \varepsilon^{-m_\lambda t^{-m_\gamma}} \left( \frac{t}{4m_p} \right)^{2m_a} I_{\varepsilon,p,m}, \quad (6.7)
\]
where $I_{\varepsilon,p,m}$ denotes the above multiple integral with respect to $dt_D$. Now let us deal with this integral $I_{\varepsilon,p,m}$. It is easy to see

$$I_{\varepsilon,p,m} = \left[ \prod_{j=1}^{m_p} 1_{I_j(t_j)} dt_1 \cdots dt_{m_p} \right]^p = \left( \frac{L}{2} \right)^{m_p \times p} \simeq \left( \frac{t}{m_p} \right)^{2m}.$$  

Let $\mathbb{D}(m_p)$ denote $\mathbb{D}(f_{m_p}, \cdots, f_{m_p})$. Substituting this bound into (6.7) we have for $D \in \mathbb{D}(m_p)$,

$$F_D(f_{m_p}, \cdots, f_{m_p}) \gtrsim \varepsilon^{-m\lambda t - m \gamma} \cdot \left( \frac{t}{4m_p} \right)^{2m} \int \prod_{l=1}^{p} \prod_{j=1}^{m_p} 1_{I_j(t_j)} 1_{0 < t_1 < \cdots < t_{m_p} < t_j} dt_D$$

$$\gtrsim \varepsilon^{-m\lambda t - m \gamma} \cdot \left( \frac{t}{m} \right)^{2m(a+1)}.$$  

Since there are $m!$ elements in $\mathbb{D}(m_p)$, we have

$$\sum_{D \in \mathbb{D}(m_p)} F_D(f_{m_p}, \cdots, f_{m_p}) \gtrsim m! \varepsilon^{-m\lambda t - m \gamma} \cdot \left( \frac{t}{m} \right)^{2m(a+1)}. \quad (6.8)$$  

**Step 3:** In this step, we obtain the asymptotic behaviors of the term appearing in (6.8) when $m$ is sufficiently large. According to Stirling’s formula $m! \simeq \sqrt{2\pi m \cdot \left( \frac{m}{e} \right)^m}$, we arrive at

$$\sum_{D \in \mathbb{D}(m_p)} F_D(f_{m_p}, \cdots, f_{m_p}) \gtrsim \varepsilon^{-m\lambda t - m \gamma} \cdot \left( \frac{t}{m} \right)^{2m(a+1)} \simeq \left( -\lambda \times \frac{t^{2(a+1) - \gamma} \cdot p^{2(a+1)}}{m^{2a+1}} \right)^m. \quad (6.9)$$  

Let us recall that to obtain the above inequality we assumed that $t, x$ are sufficiently large and $b$ is sufficiently small. Consequently, $m$ is also large enough since it satisfies (6.4). Now in (6.9), we can take the value

$$m_0(\varepsilon) = \left[ C \varepsilon^{-\lambda} t^{2(a+1) - \gamma} p^{2(a+1)} \right]^{\frac{1}{2a+1}} = C \cdot \varepsilon^{-\frac{\lambda}{2a+1}} t^{1 + \frac{1}{2a+1}} p^{1 + \frac{1}{2a+1}}.$$  

With this choice of $m = m_0(\varepsilon)$, the condition (6.4) i.e. $m \geq \frac{2t}{2a+1}$ together with (3.1) (i.e. $b(2a+1) - \lambda > 0$) imply that

$$\varepsilon^{\frac{b(2a+1) - \lambda}{2a+1}} \gtrsim t^{\frac{1 - \gamma}{2a+1}} p^{-\frac{1}{2a+1}}$$

$$\iff \varepsilon \gtrsim t^{\frac{1 - \gamma}{b(2a+1) - \lambda}} p^{-\frac{1}{b(2a+1) - \lambda}} =: \varepsilon_{t,p}.$$  

Thus, putting $\varepsilon = \varepsilon_{t,p}$ and $m = m_0(\varepsilon_{t,p})$ into (6.9) we obtain

$$\sum_{D \in \mathbb{D}(m_p)} F_D(f_{m_p}, \cdots, f_{m_p}) \gtrsim \exp \left( C \cdot \varepsilon_{t,p}^{\frac{1 - \gamma}{2a+1}} t^{1 + \frac{1 - \gamma}{2a+1}} p^{1 + \frac{1}{2a+1}} \right)$$

$$= \exp \left( C \cdot t^{\frac{1 - \gamma}{b(2a+1) - \lambda}} p^{\frac{1 - \gamma}{b(2a+1) - \lambda}} \lambda \times t^{1 + \frac{1 - \gamma}{2a+1}} p^{1 + \frac{1}{2a+1}} \right),$$

where

$$1 + \frac{1 - \gamma}{2a+1} + \frac{1 - \gamma}{b(2a+1) - \lambda} = \frac{\lambda}{2a+1} = 1 + \frac{b(1 - \gamma)}{b(2a+1) - \lambda}.$$
Intermittency properties

\[
1 + \frac{1}{2a + 1} + \frac{1}{b(2a + 1) - \lambda} = 1 + \frac{b}{b(2a + 1) - \lambda}.
\]

This is

\[
\sum_{D \in \mathbb{D}(m_p)} F_D(f_{m_p}, \ldots, f_{m_p}) \gtrsim \exp \left( C \cdot t^{1 + \frac{b(1-\gamma)}{2(2a+1)}} \cdot p^{1 + \frac{b}{2(2a+1)-\lambda}} \right). \tag{6.10}
\]

As a result, from (6.1), (6.8) and (6.10) we obtain that

\[
\mathbb{E} \left[ \prod_{j=1}^p u(t, x_j) \right] = \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_p=2m} \sum_{D \in \mathbb{D}(f_{m_1}, \ldots, f_{m_p})} F_D(f_{m_1}, \ldots, f_{m_p})
\]

\[
\gtrsim \sum_{p \cdot m_p=2m_0} \sum_{D \in \mathbb{D}(m_p)} F_D(f_{m_p}, \ldots, f_{m_p})
\]

\[
\gtrsim \exp \left( t^{1 + \frac{b(1-\gamma)}{2(2a+1)}} \cdot p^{1 + \frac{b}{2(2a+1)-\lambda}} \right).
\]

We have completed the proof of Theorem 3.6. \(\square\)

7. Some important SPDEs

In this section, we shall explain the positivity property (G1), the small ball nondegeneracy property \((B(\alpha, \beta))\) (G2) and the HLS total weighted mass property (G3) for some important stochastic PDEs: SHE, \(\alpha\)-SHE, SWE and SFD.

7.1. Stochastic heat equation (SHE). Firstly, we consider the well known stochastic heat equation that has been extensively studied in literature, see [15] and the references therein. The equation has the following form.

\[
(SHE) \quad \begin{cases}
\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x) + u(t, x) \dot{W}(t, x), & t > 0, \quad x \in \mathbb{R}^d, \\
u(0, x) = u_0(x).
\end{cases}
\tag{7.1}
\]

In this case the partial differential operator in the setting of equation (1.1) is

\[
\mathcal{L} u(t, x) = \frac{\partial u(t, x)}{\partial t} - \frac{1}{2} \Delta u(t, x).
\]

There is only one initial condition \(u(0, x) = u_0(x)\). The Green’s function and its Fourier transform in spatial variable are respectively:

\[
G^h_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp \left( -\frac{|x|^2}{2t} \right) \quad \text{and} \quad \mathcal{F}[G^h_t(\cdot)](\xi) = \exp \left( -\frac{t|\xi|^2}{2} \right). \tag{7.2}
\]

It is clear that \(G^h_t(x) \geq 0\) is a positive kernel. So, the assumption (G1) is obviously satisfied. We shall show the small ball nondegeneracy property \((B(\alpha, \beta))\) (G2) and the HLS mass property \(M(\mu, \nu)\) (G3) in the following proposition 7.1 and proposition 7.2 respectively.

Proposition 7.1 (Small Ball Nondegeneracy Property and Lower Moments for SHE). For the heat kernel \(G^h_t(x)\), the small ball nondegeneracy \(B(0,2)\) holds. In fact we have the following statements:

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(i) For all $d \in \mathbb{N}$, there exist some strict positive constants $C_1$ and $C_2$ independent of $t$, $x$, and $\varepsilon$ such that
\[
\inf_{y \in B_1(x)} \int_{B_1(x)} G^d_t(y - z)dz \geq C_1 \exp \left(-C_2 \frac{t}{\varepsilon^d} \right). \quad (7.3)
\]
(ii) Consequently, $B(0,2)$ holds for $G^d_t$, i.e. there exist a strict positive constant $C$ independent of $t$, $x$, and $\varepsilon$ so that
\[
\inf_{y \in B_1(x)} \int_{B_1(x)} G^d_t(y - z)dz \geq C, \quad (7.4)
\]
for $0 < t \leq \varepsilon^2$.

As a result, assuming $\gamma(\cdot)$ (with $\gamma = 2 - 2H$) and $\Lambda(\cdot)$ satisfy the same conditions of Theorem 3.6, there are some positive constants $c_1$ and $c_2$ independent of $t$, $p$, and $x$ such that
\[
\mathbb{E}[|u|^p(t,x)|^p] \geq c_1 \exp \left(c_2 \cdot t^{\frac{1+\gamma}{2-\gamma}} \right). \quad (7.5)
\]

Proof. We only need to prove (7.3), which is related to what is known as small ball property of Brownian motion. The readers can find the related result in immense literatures, for example (5.6.20) in [14] for one dimension. We divide the proof into two steps.

**Step 1:** Clearly, we may assume $x = (0, \cdots, 0)$. It may be possible to work on the integral directly. However, we feel easier to use the spherical coordinate for the computation of the integral. We employ the following $d$-dimensional spherical coordinate $(z_1, \cdots, z_d) = \Phi(r, \theta, \phi_1, \cdots, \phi_{d-2})$:
\[
\begin{align*}
z_1 &= r \cdot \cos(\phi_1) \\
z_2 &= r \cdot \sin(\phi_1) \cos(\phi_2) \\
\vdots \\
z_{d-2} &= r \cdot \sin(\phi_1) \cdots \sin(\phi_{d-3}) \cos(\phi_{d-2}) \\
z_{d-1} &= r \cdot \sin(\phi_1) \cdots \sin(\phi_{d-2}) \cos(\theta) \\
z_d &= r \cdot \sin(\phi_1) \cdots \sin(\phi_{d-2}) \sin(\theta),
\end{align*}
\]
where $0 \leq \phi_n < \pi$, $n = 1, \cdots, d-2$, $0 \leq \theta \leq 2\pi$. The Jacobian determinant of this transformation is
\[
J_d = r^{d-1} \prod_{k=1}^{d-2} \sin^{d-1-k}(\phi_k).
\]
Since $G^d_t(\cdot)$ is rotation invariant as a function in $\mathbb{R}^d$ we only need to consider $y = (r_0, 0, \cdots, 0)$ for some fixed $r_0 \in (0, \varepsilon)$. Set $B_\varepsilon(r_0) := B_\varepsilon(y)$, therefore
\[
\int_{B_\varepsilon(x)} G^d_t(y - z)dz \geq \int_{B_\varepsilon(r_0) \cap B_\varepsilon(0)} \frac{1}{(2\pi t)^{d/2}} \exp \left(-\frac{|z|^2}{2t} \right) dz \\
\simeq \int_0^\varepsilon \int_{[0,\pi]^{d-2}} \int_0^{2\pi} \frac{1}{(2\pi t)^{d/2}} \exp \left(-\frac{r^2}{2t} \right) \times 1_{B_\varepsilon(r_0)}(\Phi(r, \theta, \phi)) \cdot J_d d\theta d\phi dr. \quad (7.6)
\]
Notice that the identity
\[
1_{B_\varepsilon(r_0)}(\Phi(r, \theta, \phi)) = 1_{B_\varepsilon(0)}((r_0, 0, \cdots, 0) - \Phi(r, \theta, \phi))
\]
can be expressed as
\[
\{ (r, \theta, \phi) \in [0, \varepsilon] \times [0, 2\pi] \times [0, \pi)^{d-2} : r^2 \sin^2(\phi_1) + [r \cos(\phi_1) - r_0]^2 \leq \varepsilon^2 \} \\
= \{ (r, \theta, \phi) \in [0, \varepsilon] \times [0, 2\pi] \times [0, \pi)^{d-2} : r^2 + r_0^2 - 2r \cdot r_0 \cos(\phi_1) \leq \varepsilon^2 \}. 
\]

(7.7)

In order to estimate the lower bound of (7.6), we need the following particular subset of \( \{ (r, \theta, \phi) \in [0, \varepsilon] \times [0, 2\pi] \times [0, \pi)^{d-2} : \Psi(r, \theta, \phi) \in \mathcal{B}_c(r_0) \} : 
\[
S_{\varepsilon}(r, \theta, \phi) := \{ (r, \theta, \phi) \in [0, \varepsilon] \times [0, 2\pi] \times [0, \pi/2)^{d-3} : r^2 + r_0^2 - 2r \cdot r_0 \cos(\phi_1) \leq \varepsilon^2 \}. 
\]

(7.8)

Because for \( \phi_1 \in [0, \pi/3) \), we always have
\[
r^2 + r_0^2 - 2r r_0 \cos(\phi_1) \leq r^2 + r_0^2 - r r_0 \leq \varepsilon^2 ,
\]
if \( 0 \leq r, r_0 \leq \varepsilon \). On the domain \( S_{\varepsilon}(r, \theta, \phi) \), the indicate function \( 1_{S_{\varepsilon}}(r, \theta, \phi) := 1_{S_{\varepsilon}(r, \theta, \phi)}(r, \theta, \phi) = 1 \). Then we have from (7.6)
\[
\int_{B_c(0) \cap \{ x \in \mathbb{R}^d : r \leq \varepsilon \}} C_{d}^b(y - z) \, dz \\
\geq \int_0^\varepsilon \int_0^{2\pi} \int_0^{\delta} \exp \left( -\frac{r^2}{2t} \right) \times 1_{S_{\varepsilon}}(r, \theta, \phi) \cdot |J_d| \, d\theta \, d\phi \, dr \\
\geq \int_0^\varepsilon \frac{1}{(2\pi t)^{d/2}} \exp \left( -\frac{r^2}{2t} \right) \, r^{d-1} \, dr \\
\geq \int_0^\delta e^{-\frac{r^2}{2}} \, dr \geq c_{d}^{-1} \cdot e^{-\frac{\delta^2}{2}} .
\]

(7.9)

where we have used the change of variable \( r \to \tilde{r} = r/\sqrt{t} \) in the last line.

**Step 2:** We shall prove (7.9) is greater than \( C_1 \exp \left( -\frac{\delta^2}{4d^2} \right) \) by showing the following claim. For fixed \( \nu > 0 \), one can find a constant \( c_{\nu} \) such that \( c_{\nu} \cdot \int_0^\infty \exp \left( -\frac{1}{2} r^\nu \right) \, dr = 1 \). We claim that there exists a constant \( \bar{c} > \frac{(\nu+1)^2}{4\nu} \) such that \( \forall \delta := \left( \frac{\delta}{\sqrt{\nu}} \right)^d > 0, \)
\[
\int_0^\delta e^{-\frac{r^2}{2}} \, dr \geq c_{\nu}^{-1} \cdot e^{-\bar{c} \delta^2}.
\]

(7.10)

This is equivalent to prove
\[
c_{\nu} \cdot \int_0^\infty \exp \left( -\frac{1}{2} r^\nu \right) \, dr + e^{-\bar{c} \delta^2} \leq 1 .
\]

Let
\[
g(\delta) = c_{\nu} \cdot \int_\delta^\infty e^{-\frac{r^\nu}{2}} \, dr + e^{-\bar{c} \delta^2}.
\]

It is easy to see that \( g(\delta) \) is continuous and \( g(0) = g(\infty) = 1 \). So in order to prove \( g(\delta) \leq 1 \) for all \( \delta > 0 \), it suffices to show that if \( c > \frac{(\nu+1)^2}{4\nu} \), then
\[
g'(\delta) = \frac{\nu \cdot c}{\delta^{\nu+1}} e^{-\bar{c} \delta^2} - c_{\nu} \cdot e^{-\bar{c} \delta^2} = 0 .
\]
has exactly one root. It is clear that this is equivalent to
\[
\frac{\nu \cdot c}{c} e^{\frac{\nu}{c}} = \nu^{\nu+1} e^{\frac{\nu+1}{c}} \iff \exp\left(\frac{c}{\nu} + (\nu + 1) \ln(\delta) - \frac{\nu^2}{2} - \ln\left(\frac{\nu \cdot c}{c}\right)\right) = 1
\]
\[
\iff h(\delta) = \frac{c}{\nu} + (\nu + 1) \ln(\delta) - \frac{\nu^2}{2} - \ln\left(\frac{\nu \cdot c}{c}\right) = 0
\]
has exactly one root. One can notice that \(h(0^+) = +\infty\) and \(h(+\infty) = -\infty\). Then \(h(\varepsilon)\) has at least one root. Next, we shall show it has at most one root, which suffices to argue that
\[
h'(\delta) = -\frac{1}{\nu + 1} \left[\left(\frac{\nu}{\delta}\right)^2 - \varepsilon - \frac{(\nu + 1)^2}{4}\right] = 0
\]
has no root for \(\delta > 0\). But this is verified when \(\varepsilon > \frac{(\nu+1)^2}{4\nu}\). Lastly, the fact \(g'(\delta) = 0\) has only one root and the intermediate value theorem imply that the claim (7.10) holds.

Letting \(\nu = 2/d\) and \(\delta = (\frac{c}{\nu})^d\) in (7.10), we get (7.9) is greater than \(C_1 \exp\left(-\frac{C_2 \cdot c}{\delta}\right)\) for some constant \(C_1\) and \(C_2\). Thus, we have completed the proof of (7.3). □

**Proposition 7.2 (HLS mass Property and Upper Moments for SHE).**

Assume \(\gamma(\cdot)\) (with \(\gamma = 2 - 2H\)) and \(A(\cdot)\) with \(\lambda < 2\) satisfy the same conditions as in Theorem 3.4 or Theorem 4.1. Then for the heat kernel \(G_t^h(x - y)\), we have (G3) or (G3') with \(M(-\frac{\lambda}{2})\) hold. In other words, for all \(d \in \mathbb{N}\), there exist some strict positive constants \(C_1, C_2\) and \(C_3\) do not depend on \(t\) and \(x\) such that

\[
\sup_{x, x' \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t^h(x - y) A(y - y') G_t^h(x' - y') dy dy' \leq C \cdot t^{-\frac{\lambda}{2}}, \quad (7.11)
\]

or denoting \(\mu(d\xi) = \hat{V}(\xi) d\xi\)

\[
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |G_t^h(\xi - \eta)|^2 \mu(d\xi) \leq C_3 \cdot t^{-\frac{\lambda}{2}}. \quad (7.12)
\]

As a result, we have the upper p-th (\(p \geq 2\)) moments for \(u_t^h(t, x)\) for any \(d \geq 1\). More precisely, for some constants \(C_1\) and \(C_2\) that are independent of \(t\), \(p\) and \(x\) we can get

\[
\mathbb{E}[|u_t^h(t, x)|^p] \leq C_1 \cdot \exp\left(C_2 \cdot t^{-\frac{\lambda}{2}} p^{\frac{\lambda}{2 - \frac{H}{2}} - \frac{\lambda}{2}}\right).
\]

**Proof.** We only need to prove (3.6) with \(M(0, -\frac{\lambda}{2})\). This is,

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t^h(x - y) dy = \int_{\mathbb{R}^d} G_t^h(y) dy = 1,
\]
\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t^h(x - y) A(y) dy \lesssim \sup_{x \in \mathbb{R}^d} \mathbb{E} |\sqrt{t}X - x|^{-\lambda} \leq C \cdot t^{-\frac{\lambda}{2}},
\]

where \(X\) is a standard normal random variable and the above last inequality follows from [21, Lemma A.1].
For the (7.12), it is easy to
\[
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{G}_t^h(\xi - \eta)|^2 \mu(d\xi) = \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-t|\xi - \eta|^2} \mu(d\xi) \\
\leq t^{-\frac{d}{2}} \cdot \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\xi|^\lambda - d \cdot |\xi|^2 \mu(d\xi) \leq C \cdot t^h .
\]
So, we obtain the upper moment bound. \(\square\)

7.2. Fractional spatial equations: Space nonhomogeneous case. The next model is the generalized \(d (\geq 1)\)-spatial dimensional fractional stochastic \(\alpha\)-heat equation (\(\alpha\)-SHE) that has been considered in [1, 2, 9]:

\[(\alpha\text{-SHE}) \left\{ \begin{array}{l}
\frac{\partial u(t,x)}{\partial t} = -(-\nabla (a(x)\nabla))^{\alpha/2} u(t,x) + u(t,x)W(t,x), \quad t > 0, \quad x \in \mathbb{R}^d, \\
u(0,x) = u_0(x) ,
\end{array} \right.\]

(7.13)

where \(0 < \alpha < 2, a(\cdot) : \mathbb{R}^d \to \mathbb{R}^d\) is a matrix valued function whose entries are Hölder continuous, and there exists a constant \(c \geq 1\) such that \(c^{-1} \cdot Id \leq a(x) \leq c \cdot Id\). The operator \(\mathcal{L}\) is

\[\mathcal{L}u(t,x) = \frac{\partial u(t,x)}{\partial t} + (-\nabla (a(x)\nabla))^{\alpha/2} u(t,x)\]

and the corresponding Green’s function \(G_{t}^{h,\alpha}(x)\) satisfies the following Nash’s Hölder estimates (see e.g. [9] for more details):

\[\frac{1}{C} \left( t^{-\frac{d}{2}} \wedge \frac{t}{|x - y|^{d + \alpha}} \right) \leq G_{t}^{h,\alpha}(x,y) \leq C \left( t^{-\frac{d}{2}} \wedge \frac{t}{|x - y|^{d + \alpha}} \right) ,\]

(7.14)

and \(I_0(t,x) = G_{t}^{h,\alpha} \ast u_0(x)\). Clearly, (7.14) ensures the positivity of \(G_{t}^{h,\alpha}(x)\) when \(\alpha \in (0, 2)\). We still need to take care of the small ball nondegeneracy property \((\text{G2})\) with \(B(\alpha, \beta)\) and the HLS mass property \((\text{G3})\) with \(M(0, -\frac{d}{\alpha})\).

Proposition 7.3 (Small Ball Nondegeneracy Property and Lower Moments for \(\alpha\)-SHE). For the heat kernel \(G_{t}^{h,\alpha}(x)\), we have \(B(0,\alpha)\) holds:

(i) For \(\alpha \in (0, 2)\) and \(d \in \mathbb{N}\), there exist some strict positive constants \(C_1\) and \(C_2\) do not depend on \(t\) and \(\varepsilon\) such that

\[\inf_{y \in B_\varepsilon(x)} \int_{B(x)} G_{t}^{h,\alpha}(y,z)dz \geq C_1 \exp \left( -C_2 \frac{t}{\varepsilon^\alpha} \right) .\]

(7.15)

(ii) Consequently, \(B(0,\alpha)\) holds for \(G_{t}^{h,\alpha}\), i.e. there exist a strict positive constant \(C\) independent of \(t\) and \(\varepsilon\) so that

\[\inf_{y \in B_\varepsilon(x)} \int_{B(x)} G_{t}^{h,\alpha}(y,z)dz \geq C ,\]

(7.16)

for \(0 < t \leq \varepsilon^\alpha\).

As a result, assuming \(\gamma(\cdot)\) (with \(\gamma = 2 - 2H\)) and \(\Lambda(\cdot)\) satisfy the same conditions of Theorem 3.6, we have the lower \(p\)-th \((p \geq 2)\) moment bound: there are constants \(c_1\) and \(c_2\) independent of \(t, p\) and \(x\) such that

\[\mathbb{E}[|u^{h,\alpha}(t,x)|^p] \geq c_1 \exp \left( c_2 \cdot t^{\frac{2\alpha H - \lambda}{\alpha - \lambda}} \cdot \frac{2\alpha - \lambda}{\alpha - \alpha} \right) .\]
Proof. The proof is similar to the SHE case except now we have the Nash’s Hölder estimates (7.14) instead of the the precise form of $G^h_{t}(x,y)$.

By lower bound in the Nash’s inequality (7.14), we have

$$G^h_{t}(x,y) \gtrsim t^{-\frac{d}{2}} \exp \left( -C_{\alpha,d} \cdot \frac{|x-y|^\alpha}{t} \right), \quad (7.17)$$

since $1 \land |x|^{-1} \gtrsim C_{1,\alpha} \cdot \exp \left( -C_{2,d} \cdot |x|^\alpha \right)$ for $\alpha > 0$. Thus (7.16) can be proved the same way as that of (7.15). \hfill \Box

**Proposition 7.4 (HLS mass Property and Upper Moments for $\alpha$-SHE).**

Assume $\gamma(\cdot)$ (with $\gamma = 2 - 2H$) and $\Lambda(\cdot)$ with $\lambda < \alpha$ satisfy the same conditions of Theorem 3.4 or Theorem 4.1. Then for the heat kernel $G^h_{t}(x,y)$, we have \((G3)\) \ and \((G3')\) \ with $M\left(-\frac{2}{\alpha}\right)$ hold. In other words, for all $d \in \mathbb{N}$, there exist some strict positive constants $C_1$ and $C_2$ independent of $t$ and $x$ such that

$$\sup_{x,x' \in \mathbb{R}^d} \int_{\mathbb{R}^d} G^h_{t}(x,y)\Lambda(y-y')G^h_{t}(x',y')dydy' \leq C \cdot t^{-\frac{d}{2}}. \quad (7.18)$$

Furthermore, there is a positive kernel $Q_t(x-y)$ such that $G^h_{t}(x,y) \leq Q_t(x-y)$ and

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{Q}_t(\xi - \eta)|2|\mu|(d\xi) \leq C_3 \cdot t^{-\frac{d}{2}} \quad (7.19)$$

with $|\mu|(d\xi) = |\hat{V}(\xi)|d\xi$.

Consequently, we have the upper $p$-th ($p \geq 2$) moment bounds. This is, for some constants $C_1$ and $C_2$ that are independent of $t$, $p$ and $x$ we have

$$\mathbb{E} |u^h(t,x)|^p \leq C_1 \cdot \exp \left( C_2 \cdot t^{\frac{2\alpha H - \lambda}{\alpha - \lambda}} \right).$$

Proof. Presumably, we may use (7.14) to obtain the desired bounds. However, we will use Pollard’s formula in [9] to prove this proposition.

$$e^{-u^{\frac{d}{2}}} = \int_0^\infty e^{-us}g(\alpha/2,s)ds, \quad u \geq 0, \quad (7.20)$$

where $g(\alpha,s)$ is a probability density function of $s \geq 0$ and defined in (1.2) in [9].

By Proposition 2.2 there, we have

$$G^h_{t}(x,y) = \int_0^\infty p(t^{\frac{2}{\alpha}}s,x,y)g(\alpha/2,s)ds$$

$$\leq C \int_0^\infty t^{-\frac{d}{2}}s^{-\frac{d}{2}} \exp \left( -\frac{|x-y|^2}{Ct^{2/\alpha} s} \right) g(\alpha/2,s)ds =: Q_t(x-y). \quad (7.21)$$

Therefore, it is sufficient to show the assumption \((G3)\) can be archived with $M(0,-\frac{2}{\alpha})$ (i.e. the estimates (3.6)) for $Q_t(x-y)$. It is not hard to derive that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} Q_t(x-y)dy \lesssim \int_0^\infty g(\alpha/2,s)ds < \infty,$$
and
\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} Q_t(x - y) \Lambda(y) dy \\
\lesssim \int_0^\infty t^{- \frac{d}{2}} s^{- \frac{d}{2}} \left[ \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \exp \left( - \frac{|x - y|^2}{C t^{2/\alpha} s} \right) \Lambda(y) dy \right] g(\alpha/2, s) ds \\
\lesssim t^{- \frac{d}{2}} \int_0^\infty s^{- \frac{d}{2}} g(\alpha/2, s) ds \leq C_2 \cdot t^{- \frac{d}{2}} ,
\]
where we have applied rearrangement inequality and [9, Proposition 2.1].

Moreover, for the Fourier transform of \( Q_t(x) \) with respect to \( x \), we have
\[
\mathcal{F}[Q_t(\cdot)](\xi) \simeq \int_0^\infty \exp \left( -Cs \cdot t^{2/\alpha} |\xi|^2 \right) g(\alpha/2, s) ds \\
\simeq \exp \left( -\left[ C t^{2/\alpha} |\xi|^2 \right] ^{n/2} \right) = e^{-C_\alpha t |\xi|^n} .
\]

Finally, it is relatively easy to see that the assumption (7.19) can be archived. Then the upper moment bound follows. \( \square \)

7.3. Stochastic wave equations. the lower moment bounds for \( d \)-dimensional stochastic wave equation (SWE) is one of the SPDEs that motivated this study. This type of equations has been well-studied in literature. There are several works on the upper bounds for any moments. But the lower bounds are only known for the second moments except in a few cases. (see e.g. [2, 10]). We give a more complete results for all moments. This equation has the following form (we consider only \( d = 1, 2, 3 \)):
\[
\text{(SWE)} \begin{cases}
\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) \dot{W}(t,x), & t > 0, \ x \in \mathbb{R}^d, \\
u(0,x) = u_0(x), & \frac{\partial}{\partial t} u(0,x) = v_0(x) .
\end{cases}
\]

The operator \( \mathcal{L} \) has the form
\[
\mathcal{L} u(t,x) = \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} .
\]

The associated Green’s function has different forms for different dimensions. More precisely, it is given by
\[
\begin{cases}
G_t^w(x) = \frac{1}{4\pi} \mathbf{1}_{\{|x| < t\}} , & d = 1 , \\
G_t^w(x) = \frac{1}{4\sigma} \sqrt{t^2 - |x|^2} \mathbf{1}_{\{|x| < t\}} , & d = 2 , \\
G_t^w(dx) = \frac{1}{4\sigma} \mathbf{1}_{\{|x| < t\}} , & d = 3 ,
\end{cases}
\]

where \( \sigma_t(dx) \) is a surface measure on the sphere \( \partial B_t(0) \subseteq \mathbb{R}^3 \) with center at 0 and radius \( t \), with total mass \( 4\pi t^2 \) and \( G_t^w(\mathbb{R}^3) = t \). It is well known that \( G_t^w(\cdot) \) may not be positive when \( d \geq 4 \). On the other hand for any dimension \( d \), the Fourier transform of \( G_t^w(\cdot) \) has the same form given by
\[
\mathcal{F}[G_t^w(\cdot)](\xi) = \frac{\sin(t|\xi|)}{|\xi|} , \quad \xi \in \mathbb{R}^d .
\]

In this case we also have \( I_0^w(t,x) := \frac{\partial}{\partial t} G_t^w * u_0(x) + G_t^w * v_0(x) \).

When \( d = 1, 2 \), \( G_t^w(x) \) are positive functions and when \( d = 3 \) it is a positive measure. Thus, the assumption (G1) is satisfied for wave kernel \( G_t^w(dx) \). The next two propositions are devoted to (G2) and (G3).
Proposition 7.5 (Small Ball Nondegeneracy Property and Lower Moments for SWE). For the wave kernel $G_t^w(x)$ defined by (7.23), we have $B(1,1)$ holds:

(i) When $d = 1$ and $d = 2$, there exist strict positive constants $C_1$ and $C_2$, independent of $t$, $\varepsilon$ and $y$ such that

$$\inf_{y \in B_{\varepsilon}(x)} \int_{B_{\varepsilon}(x)} G_t^w(y - z)dz \geq C_1 \cdot t \exp\left(-C_2 \frac{t}{\varepsilon}\right).$$  \hfill (7.24)

Consequently, there exist a strict positive constant $C$ independent of $t$, $\varepsilon$ and $y$ so that

$$\inf_{y \in B_{\varepsilon}(x)} \int_{B_{\varepsilon}(x)} G_t^w(y - z)dz \geq C \cdot t,$$  \hfill (7.25)

for $0 < t \leq \varepsilon$.

(ii) When $d = 3$, there exists a strict positive constant $C$ independent of $t$, $\varepsilon$ and $y$ such that

$$\inf_{y \in B_{\varepsilon}(x)} \int_{B_{\varepsilon}(x)} G_t^w(y - dz) \geq C \cdot t,$$  \hfill (7.26)

for $0 < t \leq \varepsilon$.

As a consequence, assuming $\gamma(\cdot)$ (with $\gamma = 2 - 2H$) and $\Lambda(\cdot)$ satisfy the same conditions of Theorem 3.6, we have the following lower moment bounds for the solution:

$$E[|u^w(t,x)|^p] \geq c_1 \exp\left(c_2 \cdot t^{\frac{2H + 2 - \lambda}{4 - \lambda}} \cdot p^\frac{4 - \lambda}{4 - \lambda}\right)$$

for some constants $c_1$ and $c_2$ independent of $t$, $p$ and $x$.

Remark 7.6. The small ball nondegeneracy property of wave kernel $G_t^w$ is motivated by the following fact when $d = 1$. Let us illustrate it with $x = y = 0$. Then the left hand of (7.24) can be evaluated exactly as

$$\int_{-\varepsilon}^{\varepsilon} G_t^w(z)dz = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2}1_{\{|z| \leq t\}}dz = t \wedge \varepsilon.$$

And then it is not hard to see

$$t \wedge \varepsilon = \varepsilon \cdot \left(\frac{t}{\varepsilon} \wedge 1\right) \geq \varepsilon \cdot \left(C_1 \cdot \frac{t}{\varepsilon} \exp\left(-C_2 \frac{t}{\varepsilon}\right)\right) = C_1 \cdot t \exp\left(-C_2 \frac{t}{\varepsilon}\right),$$

which is the right hand of (7.24).

Proof. We shall give the proof of Proposition 7.5 for $d = 1, 2, 3$ in three steps separately.

Step 1 ($d = 1$): It is clear that we only need to show (7.24). Without loss of generality, we may assume $x = 0$. Let us consider $d = 1$ at first. Because
Intermittency properties

\[ G_t^w(y - z) = \frac{1}{t} 1_{\{|y - z| < t\}}, \]  
then (7.25) becomes

\[
\int_{\mathbb{R}} G_t^w(y - z) G_w^w(z) \, dz \\
\simeq \int_{\mathbb{R}} \mathcal{F}[G_t^w(y - \cdot)](\xi) \mathcal{F}[G_w^w(\cdot)](\xi) \, d\xi \\
\simeq \int_{\mathbb{R}} e^{-y^2 \xi^2} \frac{\sin(t|\xi|)}{|\xi|} \, d\xi \\
\simeq \int_{\mathbb{R}} e^{-y^2 |\xi|^2} \left[ \sin^2 \left( \frac{1}{2} t + \varepsilon |\xi| \right) - \sin^2 \left( \frac{1}{2} t - \varepsilon |\xi| \right) \right] \, d\xi \\
\simeq (|t + \varepsilon| - y) 1_{\{|y| < |t + \varepsilon|\}} - (|t - \varepsilon| - y) 1_{\{|y| < |t - \varepsilon|\}} , \tag{7.27}
\]

where in the last line we have applied the Fourier transform (e.g. 17.34(21) in [13])

\[
\mathcal{F}[x^{-2} \sin^2(ax)](\xi) = \mathcal{F}_c[x^{-2} \sin^2(ax)](\xi) = \frac{\pi}{2} (a - \xi/2) 1_{\{\xi < 2a\}} .
\]

The rest is routine. We split (7.27) into two cases: \( t > \varepsilon \) and \( t \leq \varepsilon \). Noticing \(|y| \leq \varepsilon\), when \( t > \varepsilon \) we can bound (7.27) below by

\[
((t + \varepsilon) - y) - ((t - \varepsilon) - y) 1_{\{|y| < |t - \varepsilon|\}} \geq 2\varepsilon 1_{\{|y| < |t - \varepsilon|\}} + t 1_{\{|y| \geq |t - \varepsilon|\}} \geq \varepsilon .
\]

The case \( t \leq \varepsilon \) can be done similarly, so we omit the details. Therefore, we obtain

\[
\int_{B_\varepsilon(x)} G_t^w(y - z) \, dz \geq t \wedge \varepsilon \geq C_1 \cdot t \exp \left( -C_2 \frac{t}{\varepsilon} \right) .
\]

We have completed the proof of (7.25) when \( d = 1 \).

**Step 2** \((d = 2)\): Recall that \( G_t^w(y - z) = \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |y - z|^2}} 1_{\{|y - z| < t\}} \). Then

\[
\int_{\mathbb{R}^2} G_t^w(y - z) 1_{B_\varepsilon(z)}(z) \, dz \\
\geq \int_{\mathbb{R}^2} \frac{1}{t} 1_{\{|y - z| < t\}} 1_{\{|z| < \varepsilon\}} \, dz \\
\simeq \frac{1}{t} \int_{\mathbb{R}^2} 1_{\{|y_1 - z_1| < t\}} 1_{\{|y_2 - z_2| < t\}} 1_{\{|z_1| < \varepsilon\}} 1_{\{|z_2| < \varepsilon\}} \, dz \\
\simeq \frac{1}{t} \left( \int_{\mathbb{R}} 1_{\{|y - z| < t\}} 1_{\{|z| < \varepsilon\}} \, dz \right)^2 \\
\geq \frac{1}{t} \left( C_1 \cdot t \exp \left( -C_2 \frac{t}{\varepsilon} \right) \right)^2 = C_1 \cdot t \exp \left( -C_2 \frac{t}{\varepsilon} \right) , \tag{7.28}
\]

where we have applied the result in \( d = 1 \) to derive the inequality last line in (7.28). Thus, the proof of (7.25) when \( d = 2 \) has been completed.

**Step 3** \((d = 3)\): Let us recall that now \( G_t^w dz) = \frac{1}{4\pi} \frac{\sigma_t( dz)}{t} \) where \( \sigma_t( dz) \) is the surface measure on \( \partial B_t(0) \). We may assume \( x = 0 \) and simplify \( B_\varepsilon(0) \) as \( B_\varepsilon \). Then
(7.26) becomes
\[
\int_{\mathbb{R}^3} 1_{B_r}(z)G^w_t(y - dz) = \frac{1}{4\pi t} \int_{\partial B_r} 1_{B_r}(y - z)\sigma_t(dz) = \frac{1}{4\pi t} \int_0^{2\pi} \int_0^\pi 1_{B_r}(y - \Psi(\theta, \phi)) \left| \frac{\partial \Psi}{\partial \theta} \times \frac{\partial \Psi}{\partial \phi} \right| d\theta d\phi
\]
\[
= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi 1_{B_r}(y - \Psi(\theta, \phi))|\sin(\phi)|d\phi d\theta,
\]
where the parametrization is the three dimensional spherical coordinate (i.e. \( d = 3 \) in (7.5)):
\[
\Psi(\theta, \phi) = (z_1(\theta, \phi), z_2(\theta, \phi), z_3(\theta, \phi)) = (t \sin(\phi) \cos(\theta), t \sin(\phi) \sin(\theta), t \cos(\phi)).
\]
Similarly, we can select the particular subset as in (7.8) so that we can bound (7.29) below as
\[
\frac{t}{4\pi} \int_0^{2\pi} \int_0^\pi 1_{B_r}(y - \Psi(\theta, \phi))|\sin(\phi)|d\phi d\theta \geq \frac{t}{4\pi} \int_0^{2\pi/3} \int_0^{\pi/3} |\sin(\phi)|d\phi d\theta = t/4.
\]
As a result, we have completed the proof of (7.26). \( \square \)

**Proposition 7.7 (HLS mass Property and Upper Moments for SWE).** Assume \( d = 1, 2, 3, \gamma(\cdot) \) (with \( \gamma = 2 - 2H \)) and \( \Lambda(\cdot) \) with \( \lambda < 2 \) and \( d \) satisfy the same conditions of Theorem 3.4 or Theorem 4.1. Then for the wave kernel \( G^w_t(x) \), we have \((G3)\) with \( M(2 - \lambda) \) or \((G3')\) with \( M(2 - \lambda) \) hold. In other words, for \( d = 1, 2, 3 \), there exists some strict positive constants \( C \) independent of \( t \) and \( x \) such that
\[
\sup_{x, x' \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} G^w_t(x - y)\Lambda(y - y')G^w_t(x' - y')dydy' \leq C \cdot t^{2-\lambda}, \quad (7.30)
\]
Denoting \( \mu(d\xi) = \hat{V}(\xi)d\xi \)
\[
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{G}^w_t(\xi - \eta)|^2 \mu(d\xi) \leq C \cdot t^{2-\lambda}. \quad (7.31)
\]
Consequently, we have the desired upper \( p \)-th (\( p \geq 2 \)) moment bounds for the solution \( u^w(t, x) \) when \( d = 1, 2, 3 \). This is, we can find constants \( C_1 \) and \( C_2 \) that are independent of \( t \), \( p \) and \( x \) such that
\[
\mathbb{E}[|u^w(t, x)|^p] \leq C_1 \cdot \exp \left( C_2 \cdot t^{\frac{2H+2-\lambda}{2-\lambda}} \cdot p^{\frac{4-\lambda}{2-\lambda}} \right).
\]
**Proof.** It is clear we only need to show \((G3)\) holds for \( G^w_t(x) \) with \( M(2 - \lambda) \), i.e. the estimates (7.30).
When \( d = 1, 2 \), we can easily apply Hardy-Littlewood-Sobolev inequality ([24, Theorem 4.3]) for \( \lambda < d \) to bound

\[
\sup_{x, x' \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t^w(x - y) \Lambda(y - y') G_t^w(x' - y') dy dy'
\leq \sup_{x, x' \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x - y) |y - y'|^{-\lambda} G_t(x' - y') dy dy'
\leq \left[ \int_{\mathbb{R}^d} |G_t^w(y)|^{\frac{2d}{2d - \lambda}} dy \right]^{\frac{2d - \lambda}{2d}}.
\]

For \( d = 1 \), we have

\[
\left[ \int_{\mathbb{R}^d} |G_t^w(y)|^{\frac{2d}{2d - \lambda}} dy \right]^{\frac{2d - \lambda}{2d}} \simeq \left[ \int_{-t}^{t} |1/2|^{\frac{2d}{2d - \lambda}} dt \right]^{\frac{2d - \lambda}{2d}} \leq C \cdot t^{2 - \lambda}.
\]

For \( d = 2 \), we have

\[
\left[ \int_{\mathbb{R}^d} |G_t^w(y)|^{\frac{2d}{2d - \lambda}} dy \right]^{\frac{2d - \lambda}{2d}} \simeq \left[ \int_{\mathbb{R}^2} |t^2 - x^2|^{-\frac{\lambda}{4\pi t}} \left| 1_{|x| < t} \right| dx \right]^{\frac{2d - \lambda}{2d}}
\approx t^{2 - \lambda} \cdot \left[ \int_{\mathbb{R}^2} |1 - x^2|^{-\frac{\lambda}{4\pi t}} \left| 1_{|x| < t} \right| dx \right]^{\frac{2d - \lambda}{2d}}
= C \cdot t^{2 - \lambda},
\]

where the integral is finite if \( \lambda < 2 \).

Now we shall apply the HLS inequality on sphere (see e.g. [24, Theorem 4.5]) to show (7.30) for \( d = 3 \) and \( \lambda < 3 \). Denote by \( S^3 \) the unit sphere in \( \mathbb{R}^3 \). We have

\[
\sup_{x, x' \in \mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} G_t^w(x - y) \Lambda(y - y') G_t^w(x' - y') dy dy'
\leq \sup_{x, x' \in \mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |y - y'|^{-\lambda} \sigma_t(x - dy) \sigma_t(x' - dy')
\leq t^{2 - \lambda} \cdot \sup_{x, x' \in \mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} 1_{x + S^3}(y) |\sigma_t(x' - dy')| = C \cdot t^{2 - \lambda},
\]

where we have made use of the scaling property of the surface measure \( \sigma_t(dy) = t^2 \sigma_1(dy) \) with \( y = t\tilde{y} \) in the third line and the HLS inequality [24, Theorem 4.5] on sphere in the last line. This proves (7.30).

In regard to the bound (7.31), it is easy to see that if \( \lambda < 2 \wedge d \)

\[
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{G}_t^w(\xi)|^2 \mu(d\xi - \eta) = \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| \frac{\sin(t|\xi|)}{\xi} \right|^2 \mu(d\xi - \eta)
\leq t^{2 - \lambda} \cdot \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\xi|^{\lambda - d}}{1 + |\xi + t\eta|^2} d\xi \leq C \cdot t^{2 - \lambda}.
\]

Thus, we complete the proof of Proposition 7.7. \( \square \)
Remark 7.8. The properties we obtained in Proposition 7.1 (i) and Proposition 7.5 (i) can be also rewritten as the following small ball property (B(a,b,c)): if $y \in B_\varepsilon(x)$, then
\[
\int_{B_\varepsilon(x)} G_t(y-z)dz \geq C_1 \cdot t^a \exp \left( -C_2 \frac{t^b}{\varepsilon^c} \right),
\]
where $a, b$ and $c$ are parameters depending on the kernel. Obviously, $B(a,b,c)$ is stronger than $B(a,b)$ because (7.32) holds for all $t > 0$ other than $0 < t \leq \varepsilon^3$.

For example, we have proved that $(a,b,c)=(0,1,2)$ for the heat kernel, $(a,b,c) = (0,1,1)$ for the $\alpha$-heat kernel and $(a,b,c)=(1,1,1)$ for the wave kernel.

Our effort to take into account $B(a,b)$ rather than $B(a,b,c)$ is mainly stimulated by Proposition 7.5 (ii). One should note that when $d = 3$, the wave kernel cannot satisfy the $B(a,b,c)$. Because the three dimensional wave kernel is a surface measure on the sphere $\partial B_1(0)$, there might be no intersection between the surface measure $G_t^e(y-z)$ and the ball $B_\varepsilon(x)$ if $t \gg \varepsilon$. Then the lower bound in (7.26) might be 0.

7.4. Fractional temporal and fractional spatial equations: space homogeneous case. In this section we consider the following $d$-spatial dimensional stochastic partial differential equation of fractional orders both in time and space variables, which will be called the stochastic fractional diffusion (SFD). The existence, uniqueness, upper moment bounds have been obtained earlier (e.g. [4] and references therein). But the sharp lower bounds for any moment has not been known. We shall apply Theorem 3.6 to obtain a sharp lower moment bounds for this equation.

This type of equations takes the following form:

\[
\begin{aligned}
&\mathcal{L}u(t,x) = \partial_t^\beta u(t,x) + \frac{1}{2}(-\Delta)^{\alpha/2}u(t,x) + u(t,x)W(t,\cdot), \quad t > 0, \quad x \in \mathbb{R}^d, \\
&\partial_t^k u(t,x)|_{t=0} = u_k(x), \quad 0 \leq k \leq \lfloor \beta \rfloor - 1.
\end{aligned}
\]

As in [4, 29], we shall assume that $\beta \in (1/2, 2)$ and $\alpha \in (0, 2]$. We refer to [23] for the precise meaning of the fractional derivative in time and the fractional Laplacian. Notice that the SWE coincide with the case $(\alpha, \beta) = (2, 2)$ in (7.33) formally.

In this case, the operator $\mathcal{L}$ is given by
\[
\mathcal{L}u(t,x) = \partial_t^\beta u(t,x) + \frac{1}{2}(-\Delta)^{\alpha/2}u(t,x).
\]

The associated Green's function can be represented by the Fox $H$-function.
\[
G_t^Y(x) := G_t^{Y;\alpha,\beta,d}(x) = \frac{t^{[\beta]-1}}{\pi^{d/2}|x|^d} H_{2,3}^{2,1} \left( \frac{|x|^\alpha}{2^{\alpha-1}t^\beta}, (1,1,\frac{1}{\beta},\frac{1}{\beta},1,1,1) \right),
\]
where $H$ is a Fox H-function (e.g. [22]). When $\beta > 1$ we also need another Green function
\[
G_t^Z(x) := G_t^{Z;\alpha,\beta,d}(x) = \frac{t^{[\beta]-1}}{\pi^{d/2}|x|^d} H_{2,3}^{2,1} \left( \frac{|x|^\alpha}{2^{\alpha-1}t^\beta}, (1,1,\lfloor \beta \rfloor,\beta,\frac{1}{\beta},1,1,1) \right)
\]
to represent $I_0(t,x)$, namely,
\[
I_0^Y(t,x) = \sum_{k=0}^{[\beta]-1} \int_{\mathbb{R}^d} u_{[\beta]-1-k}(y) \partial_t^k G_t^Y(x-y)dy.
\]
The Fourier transforms of \( G^Y_t(x) \) and \( G^Z_t(x) \) are given by the following:

\[
\mathcal{F}[G^Z_t(\cdot)](\xi) = t^{[\beta] - 1} E_{\beta, [\beta]} \left( -\frac{t^\beta |\xi|^\alpha}{2} \right), \\
\mathcal{F}[G^Y_t(\cdot)](\xi) = t^{\beta - 1} E_{\beta, \beta} \left( -\frac{t^\beta |\xi|^\alpha}{2} \right),
\]

(7.37)

where \( E_{\beta, \gamma} \) is the Mittag-Leffler function (e.g. [23]).

As before, we may assume \( u_0 = 1 \) and \( u_k = 0 \) for \( k \geq 1 \) to simplify the form of moments without loss of generality (also see Remark 3.6 in [4]). We have \( l_0(t, x) = 1 \) by our particular initial conditions. Whence, we can prove Theorem 3.4 with the notations introduced before.

Positivity of \( G^Y_t(x) \) (as well as \( G^Z_t(x) \)) have been obtained in the following three cases in [4, Theorem 3.1]:

\[
\begin{cases}
  d = 1, \beta \in (1, 2) \text{ and } \alpha \in [\beta, 2] ; \\
  d = 2, 3, \beta \in (1, 2) \text{ and } \alpha = 2 ; \\
  d \in \mathbb{N}, \beta \in (0, 1] \text{ and } \alpha \in (0, 2] .
\end{cases}
\]

Notice that although \( \beta \) is allowed to be smaller than \( \frac{1}{2} \), the existence and uniqueness of solutions to (7.33) can be proved only under the conditions \( \beta \in \left( \frac{1}{2}, 2 \right) \) and \( \alpha \in (0, 2] \). Therefore, we will replace last condition by

\[
d \in \mathbb{N}, \beta \in \left( \frac{1}{2}, 1 \right], \text{ and } \alpha \in (0, 2] .
\]

This means that we will assume that \((\alpha, \beta, d)\) satisfies one of the following three conditions:

\[
\begin{cases}
  (a) \beta \in \left( \frac{1}{2}, 1 \right], \alpha \in (0, 2], \ d \in \mathbb{N} ; \\
  (b) \beta \in (1, 2), \alpha \in (0, 2], \ d = 2, 3 ; \\
  (c) \beta \in (1, 2), \alpha \in [\beta, 2], \ d = 1 .
\end{cases}
\]

(7.38)

As we indicated above the assumption (G1) is met under the above parameter range of (7.38). In the remaining part of this subsection, we shall prove (G2) and (G3) for the Green’s function \( G^Y_t \).

**Proposition 7.9 (Small Ball Nondegeneracy Property and Lower Moments for SFD).** For the kernel \( G^Y_t(x) \) defined in (7.34), the small ball nondegeneracy property \( B(\beta - 1, \frac{\beta}{2}) \) holds for the parameter ranges given in (7.38). More precisely, there exist a strictly positive constant \( C \) independent of \( t, \varepsilon \) and \( y \) such that

\[
\inf_{y \in B_{\varepsilon}(x)} \int_{B_{\varepsilon}(x)} G^Y_t(y - z) dz \geq C \cdot t^{\beta - 1}
\]

(7.39)

for any \( 0 < t \leq \varepsilon^{\frac{\beta}{2}} \).

As a result, if \( \gamma(\cdot) \) (with \( \gamma = 2 - 2H \)) and \( \Lambda(\cdot) \) satisfy the same conditions as in Theorem 3.6, the lower \( p \)-th \((p \geq 2)\) moment bounds hold

\[
\mathbb{E}[|u^f(t, x)|^p] \geq c_1 \exp \left( c_2 \cdot \frac{\alpha (2\alpha + 2H - 2) - \beta \lambda}{2(\alpha - \beta) - \beta \lambda} \cdot p^{\frac{\beta(2\alpha - \beta - \lambda)}{2(\alpha - \beta) - \beta \lambda}} \right)
\]

for some constants \( c_1 \) and \( c_2 \) independent of \( t, p \) and \( x \).
Proof. We divide the proof into three steps to deal with three cases in (7.38) separately.

**Step 1: case (a).** The special case \( \beta = 1 \) was treated in (7.16), so we can assume \( \beta \in (1/2, 1) \). By the convolution property of [4], we get a subordination law for the Green’s function:

\[
G_t^Y (x) = \frac{t^{\beta-1}}{\pi^{d/2} |x|^d} H_{2,3}^{1,1} \left( \frac{|x|^{\alpha}}{2\alpha-1}, t \beta, \alpha \right) H_{1,2}^{1,1} \left( \frac{|x|^{\alpha}}{2\alpha-1}, t \beta, \alpha \right) \]

When \( y, z \in B_\varepsilon(x), t \leq \varepsilon^2 \) and when \( \varepsilon \) is small enough we have

\[
\int_{B_\varepsilon(x)} G_t^Y (y - z) dz \approx \int_{B_\varepsilon(x)} \frac{\beta t^{\beta-1}}{|y - z|^d} \int_0^\infty H_{1,2}^{1,1} \left( \frac{|y - z|^{\alpha}}{2\alpha-1}, t \beta, \alpha \right) \times H_{1,1}^{1,0} \left( (ts)^{\beta} \right) \frac{ds}{s} dz \]

\[
\approx \int_{B_\varepsilon(x)} \frac{\beta t^{\beta-1}}{|y - z|^d} \int_0^\infty H_{1,2}^{1,1} \left( \frac{|y - z|^{\alpha}}{2\alpha-1}, t \beta, \alpha \right) \times H_{1,1}^{1,0} \left( \frac{1}{s} \right) \frac{ds}{s} dz . \]

Notice that the second \( H \)-function is nonnegative by Lemma 4.5 in[4]. Moreover, recall that the characteristic function and the density of a centered, \( d \)-dimensional spherically symmetric \( \alpha \)-stable random variable are given, respectively, by

\[
f_{\alpha,d}(\xi) = \exp(-|\xi|^\alpha), \quad \xi \in \mathbb{R}^d , \tag{7.42}
\]

and

\[
\rho_{\alpha,d}(x) = \frac{1}{(\sqrt{\pi})^d |x|^d} H_{1,2}^{1,1} \left( \frac{|x|^{\alpha}}{2\alpha-1}, \frac{1}{(1, 1)} \right), \quad x \in \mathbb{R}^d . \tag{7.43}
\]

This means that the first Fox H-function is related to the spherically symmetric \( \alpha \)-stable distribution (see also [4, Theorem 3.3] for more details). Therefore, one can apply the Pollard’s formula in [9] together with (7.42) and (7.43) to find

\[
\frac{1}{|y - z|^d} H_{1,2}^{1,1} \left( \frac{|y - z|^{\alpha}}{2\alpha-1}, t \beta, \alpha \right) \approx G_{\alpha,d}^{h,\alpha}(y - z) \]

\[
\approx \left( \frac{t}{s} \right)^{-\frac{d}{\alpha}} \wedge \left( \frac{(t/s)^{\beta}}{|y - z|^{d+\alpha}} \right) \]

\[
\approx \left( \frac{t}{s} \right)^{-\frac{d}{\alpha}} \exp \left( -C_{\alpha,d} \frac{|y - z|^\alpha}{(t/s)^{\alpha}} \right) ,
\]

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where $G_t^{h,\alpha}(x)$ is the $\alpha$-heat kernel associated to (7.13). Whence, by Proposition 7.1 (i), and (7.41) we get

$$
\int_{B_t(x)} G_t^Y(y-z)dz 
\ge t^{\beta-1} \int_0^\infty \exp \left( -c \cdot \frac{(t/s)^\beta}{\varepsilon^\alpha} \right) \times H_{1,1}^{1,0} \left( s^{-\beta} (1,1) \right) \frac{ds}{s} 
\ge t^{\beta-1} \int_0^\infty \exp (-c \cdot s) \times H_{1,1}^{1,0} \left( s^{-\beta} (1,1) \right) \frac{ds}{s},
$$

(7.44)

if $y, z \in B_t(x)$ and $t < \varepsilon^{\alpha/\beta}$.

Next, we need to analyze $H_{1,1}^{1,0} \left( s^{-\beta} (1,1) \right)$. We only need to consider its asymptotics for $s$ near to 0 and near $\infty$. We shall use the results in the Appendix of [4] replacing the notations there by $\Delta = \beta_1 - \alpha_1 = 1 - \beta, \alpha^* = \beta_1 - \alpha_1 = 1 - \beta, \delta = \beta^\beta$ and $\mu = 1 - \beta$. Let us recall the asymptotic expansion for the Fox H-function (e.g. [4, (A10)]):

$$
H^{[m,n]}_{p,q} \left( s^{-\beta_1} (a_1, \ldots, a_p \beta_1, \ldots, \beta_q) \right) = \sum_{j=1}^m \sum_{l=0}^\infty h_j^* \cdot s^{l+1}. \quad (7.45)
$$

Thus, when $s \to 0$ we have

$$
H_{1,1}^{1,0} \left( s^{-\beta_1} (1,1) \right) = \sum_{l=0}^\infty h_l^* \cdot s^{l+1},
$$

(7.46)

since $m = 1$ and $(b_1, \beta_1) = (1,1), h_l^*$ is given by (e.g. [4, (A12)])

$$
h_l^* = \frac{(-1)^l}{l! \beta_1} \cdot \frac{1}{\Gamma \left( a_1 - [b_1 + l] \beta_1 \right)} = \frac{(-1)^l}{l!} \cdot \frac{1}{\Gamma (-l \beta_1)}.
$$

Therefore, one can easily see that $h_0^* = 0, h_1^* = -1/\Gamma(-\beta) > 0$, and

$$
H_{1,1}^{1,0} \left( s^{-\beta_1} (1,1) \right) = \sum_{l=0}^\infty h_l^* \cdot s^{l+1} \simeq h_1^* \cdot s, \quad |s| \simeq 0.
$$

When $s$ goes to infinity, by [22, Corollary 1.10.2], we have the following asymptotic:

$$
H_{1,1}^{1,0} \left( s^{-\beta_1} (1,1) \right) = O \left( s^{[\beta/2] - \beta/(1-\beta)} \exp \left[ -C_\beta s^{1/(1-\beta)} \right] \right), \quad s \to \infty,
$$

(7.47)

where $C_\beta = (1 - \beta)\beta^{(1-\beta)}$. Whence we can observe that the integral in (7.44) is finite. So, we have for some constant $C_\beta > 0$

$$
\int_{B_t(x)} G_t^Y(y-z)dz \simeq C_\beta \cdot t^{\beta-1}.
$$

As a result, we have proved the small ball nondegeneracy property $B(\beta - 1, \delta^\alpha)$ for the case (a).

**Step 2: case (b).** In this case $d = 2$ or $d = 3, \beta \in (1,2)$ and $\alpha = 2$. By equations (43) and (85) in [32], we have for $\beta \in [1,2]$

$$
G_t^Y(x) = \Gamma_{\beta,d}(t,x),
$$

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where
\[ \Gamma_{\beta,2}(t, x) = \frac{C \cdot t^{-1}}{\Gamma(1/2)} \int_1^{\infty} \phi(-\beta/2, 0, -|x|t^{-\beta}\tau)(\tau^2 - 1)^{-1/2}d\tau, \tag{7.48} \]
\[ \Gamma_{\beta,3}(t, x) = Ct^{-\beta/2-1} \int_1^{\infty} \phi(-\beta/2, -\beta/2; -|x|t^{-\beta/2})d\tau. \tag{7.49} \]

Here \( \phi(a, b, c) \) is the Wright function.

Let us check the small ball nondegeneracy property \( B(\beta - 1, \frac{2}{\beta}) \) for \( d = 2 \) first. If
\( y, z \in B_\varepsilon(x) \) and \( t \leq \varepsilon^{2/\beta} \), by the representation (7.48)
\[ \int_{B_\varepsilon(x)} G^Y_t(y - z)dz = \int_{B_\varepsilon(x)} \Gamma_{\beta,2}(t, y - z)dz \]
\[ \simeq t^{-1} \int_{B_\varepsilon(x)} \int_1^{\infty} \phi(-\beta/2, 0, -|y - z|t^{-\beta}\tau)(\tau^2 - 1)^{-1/2}d\tau dz \]
\[ \simeq t^{-1} \int_{B_\varepsilon(x)} \int_0^{\infty} \phi(-\beta/2, 0, -\tau)(\frac{t\tau}{\sqrt{t^2\tau^2 - |y - z|^2}}) \frac{t\tau}{\sqrt{t^2\tau^2 - |y - z|^2}} d\tau dz \]
\[ \simeq t^{-1} \int_0^{\infty} \phi(-\beta/2, 0, -\tau) \cdot t^\beta \tau \cdot \exp\left(-\frac{t^\beta/2\tau}{\varepsilon}\right)d\tau, \]
where the last inequality is derived analogously to the argument used in (7.28) for the wave kernel when \( d = 2 \) and the fact that \( \phi(-\beta/2, 0, -\tau) \) is positive (see [32, Section 2]). Then since \( t^\beta/2 \leq \varepsilon \) and \( \exp(-t^\beta/2\tau/\varepsilon) \geq \exp(-\tau) \), we obtain by the relation between the Wright function and the Fox \( H \)-function
\[ \int_{B_\varepsilon(x)} G^Y_t(y - z)dz = \int_{B_\varepsilon(x)} \Gamma_{\beta,2}(t, y - z)dz \]
\[ \simeq t^\beta \int_0^{\infty} \phi(-\beta/2, 0, -\tau) \cdot \tau \exp(-\tau)d\tau \]
\[ \simeq t^\beta \int_0^{\infty} H^1_{1,0}(\tau) \cdot \exp(-\tau)d\tau \simeq C_\beta \cdot t^\beta - 1, \tag{7.50} \]
where the integral in the last equality of (7.50) is finite by the similar asymptotic analysis of \( H^1_{1,0} \) as in case (a). Thus, we proved \( B(\beta - 1, \frac{2}{\beta}) \) for \( d = 2 \).

Next, let us check the small ball nondegeneracy property \( B(\beta - 1, \frac{2}{\beta}) \) for \( d = 3 \). We have by the equation (7.49)
\[ \int_{B_\varepsilon(x)} G^Y_t(y - z)dz = \int_{B_\varepsilon(x)} \Gamma_{\beta,3}(t, y - z)dz \]
\[ \simeq \int_{B_\varepsilon(x)} t^{-\beta/2-1} \int_1^{\infty} \phi(-\beta/2, -\beta/2; -|y - z|t^{-\beta/2})d\tau dz \]
\[ \simeq t^{-\beta-1} \int_{B_\varepsilon(x)} \frac{1}{|y - z|} \int_0^{\infty} \phi(-\beta/2, -\beta/2; -\tau) \chi_{|y - z| \leq t^{\beta/2}\tau}d\tau dz. \tag{7.51} \]
Now we can apply the same three dimensional spherical coordinate transformation as in the proof of Proposition 7.5 (now for \( d = 3 \)). Assuming \( x = 0 \), the integral
with respect to $z$ in (7.51) becomes

\[
\int_{B_x(0)} \frac{1}{|y-z|} 1_{(|y-z| \leq e^{\beta/2} \tau)} dz \simeq \int_{B_{e^{\beta/2} \tau}(0)} \frac{1_{B_x(0)}(y-z)}{|z|} dz
\]

\[
\simeq \tau^2 \int_0^\tau \int_0^{2\pi} \int_0^{\pi/2} r \cdot 1_{B_x}(y - \Psi(\theta, \phi)) |\sin(\phi)| d\phi d\theta dr
\]

\[
\gtrsim \tau^4 t^{2\beta} \cdot \int_0^{2\pi} \int_0^{\pi/2} |\sin(\phi)| d\phi d\theta \simeq \tau^4 t^{2\beta}. 
\]

Thus, plugging it back to (7.51), we get

\[
\int_{B_x(x)} G_t^Y (y-z) dz \gtrsim t^{\beta-1} \int_0^\infty \phi(-\beta/2, -\beta/2; -\tau) \cdot \tau^4 d\tau \simeq t^{\beta-1},
\]

where the last equality follows from the asymptotic behavior of the Wright function. Hence we complete the proof of the proposition in case (b).

**Step 3: case (c).** We have $d = 1$, $\beta \in (1, 2)$ and $\alpha \in [\beta, 2]$. By Remark 3.2 (3) and convolution property Theorem 1.8 in [4], the Fox H-function admits an alternative representation:

\[
G_t^Y (x) = \frac{t^{\beta-1}}{|x|} H_{2,1}^{1,1} \left( \frac{|x|^\alpha}{t^\beta}; \left| \begin{array}{c} (1,1), (\beta,\beta), (1/2) \\ (1,1), (1,1), (1,2) \end{array} \right| \right)
\]

\[
= \frac{\beta t^{\beta-1}}{|x|} \int_0^\infty H_{2,2}^{1,1} \left( \frac{|x|^\alpha s^\beta}{t^\beta}; \left| \begin{array}{c} (1,1), (\beta,\beta), (1/2) \\ (1,1), (1,1), (1,2) \end{array} \right| \right) H_{1,1}^{1,0} \left( \left| \begin{array}{c} (s)/(1,\alpha) \end{array} \right| \right) ds.
\] (7.52)

(The representation is well defined since $\Delta_1 = \sum_{j=1}^2 \beta_j - \sum_{j=1}^2 \alpha_j = 0$, $a_1^\alpha = \alpha - \alpha_2 + 1 - \beta_2$ $= 0$, $\delta_2 = \frac{\alpha}{2} - \alpha = 1$, $s = 2 - 2 = 0$; $\Delta_2 = \beta - 1 - \alpha = -\beta$, $a_2^\alpha = \beta - 1 - \alpha = -\beta$, $\delta_2 = \delta - \beta$ and $\mu_2 = 1 \beta$.) Note that the second Fox H-function is nonnegative combining [22, Property 2.4] with [4, Lemma 4.5]. By [26, (4.38)], the first Fox H-function can be identified as the Green function of neutral-fractional diffusion, namely,

\[
\frac{1}{|x|} H_{2,1}^{1,1} \left( |x|^{\alpha}; \left| \begin{array}{c} (1,1), (1/2) \\ (1,1), (1,2) \end{array} \right| \right) = N_0^\alpha (|x|) = K_0^\alpha,\alpha (|x|)
\]

\[
= \frac{1}{\pi} \frac{|x|^{\alpha-1}\sin(\alpha\pi/2)}{1 + 2|x|^{\alpha}\cos(\alpha\pi/2) + |x|^{2\alpha}}.
\]

From (7.52) it then follows

\[
G_t^Y (x) = \beta t^{\beta-1} \int_0^\infty \left( \frac{s}{t} \right)^{\beta/\alpha} N_0^\alpha \left( |x|(s/t)^{\beta/\alpha} \right) H_{1,1}^{1,0} \left( s^{-\beta}; \left| \begin{array}{c} (\beta,\beta), (1,\alpha) \end{array} \right| \right) ds.
\]
Thus, we have (without loss of generality we can set \( x = 0 \) in the following),

\[
\int_{B_r(x)} G_t^Y(y - z) \, dz = \int_{B_r(0)} \beta^\beta \int_0^\infty \left( \frac{t}{s} \right)^{\beta/\alpha} \mathcal{N}_0 \left( |y - z| (s/t)^{\beta/\alpha} \right) H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) \, \frac{ds}{s} \, dz \\
\geq \sin \left( \frac{\alpha \pi}{2} \right) \beta^{\beta - 1} \int_0^\infty \int_{B_r(y)} \left( \frac{s}{t} \right)^{\beta/\alpha} \frac{|\mathcal{H}(s/t)^{\beta/\alpha}|^{\alpha - 1}}{|\mathcal{H}(s/t)^{\beta/\alpha}|^{2\alpha} + 1} \, dz \cdot H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) \, \frac{ds}{s} \\
\geq \sin \left( \frac{\alpha \pi}{2} \right) \beta^{\beta - 1} \int_0^\infty \int_{B_r(y)} \left( \frac{s}{t} \right)^{\beta/\alpha} \frac{|z| (s/t)^{\beta/\alpha}|^{\alpha - 1}}{|z| (s/t)^{\beta/\alpha}|^{2\alpha} + 1} \, dz \cdot H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) \, \frac{ds}{s} \\
\approx \sin \left( \frac{\alpha \pi}{2} \right) \beta^{\beta - 1} \int_0^\infty \frac{\arctan \left( \frac{s^{\beta - \alpha}}{b^\beta} \right)}{b^\beta} \cdot H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) \, \frac{ds}{s} \\
\geq \sin \left( \frac{\alpha \pi}{2} \right) \beta^{\beta - 1} \int_0^\infty \frac{\arctan \left( \frac{s^{\beta - \alpha}}{b^\beta} \right)}{b^\beta} \cdot H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) \, \frac{ds}{s} \\
\quad \text{(7.53)}
\]

for \( y, z \in B_r(x) \), and \( t \leq \varepsilon^\beta \).

Next, we need to take care of the asymptotics of \( H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) \) (with the notations \( \Delta = \beta_1 - \alpha_1 = \alpha - \beta, \ a^* = \beta_1 - \alpha_1 = \alpha - \beta, \ s^* = \beta - \mu = 1 - \beta \) when \( s \) goes to infinity. Similar to (7.46) in case (a), we find that

\[
H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) = \sum_{l=0}^\infty h_t^* \cdot s^{-\beta(l+1)/\alpha} \approx h_t^* s^{-2\beta/\alpha} \quad \text{as} \; s \to \infty,
\]

with \( h_t^* = \frac{-1}{\alpha \pi} \cdot \frac{1}{|\mathcal{H}(s/t)|} \) and \( h_t^* > 0 \). When \( s \to 0 \), similar to (7.47), we have the following asymptotic estimate

\[
H_{1,1}^{1,0} \left( s^{-\beta} \left| \cdots \right| \right) = O \left( s^{-\beta[3/2-\beta]/(\beta-\beta)} \exp \left[ -C_{\alpha, \beta} \cdot s^{-\beta/\alpha} \right] \right), \quad s \to 0
\]

for some constant \( C_{\alpha, \beta} > 0 \).

Finally, we obtain from (7.53) and the asymptotics

\[
\int_{B_r(x)} G_t^Y(y - z) \, dz \geq \sin \left( \frac{\alpha \pi}{2} \right) \beta^{\beta - 1} \int_0^\infty \frac{\arctan \left( \frac{s^{\beta - \alpha}}{b^\beta} \right)}{b^\beta} \cdot H_{1,1}^{1,0} \left( s^{-\beta} \left| \left( \frac{\beta, \beta}{1, \alpha} \right) \right| \right) \, \frac{ds}{s} \geq C_{\alpha, \beta} \cdot t^{\beta - 1},
\]

for some constant \( C_{\alpha, \beta} > 0 \). Thus, we complete the proof of the small ball nondegeneracy property \( B(\beta - 1, \beta) \) for case (c).

\[ \Box \]

**Proposition 7.10 (HLS mass Property and Upper Moments for SFD).**
Assume that \( \gamma(\cdot) \) (with \( \gamma = 2 - 2H \)) and \( \Lambda(\cdot) \) satisfy the same conditions of Theorem 3.4 (under the condition \( \lambda < \min(2\alpha - \alpha/\beta, d) \)) or Theorem 4.1 (under the condition \( \lambda < \min(\alpha, d) \)). When the parameters are in the range given by (7.38) the Green’s function \( G_t^Y(x) \) satisfies the \((G3)\) or \((G3')\) with \( M(2(\beta - 1) - \frac{\beta}{\alpha}) \). In other words, there exist strict positive constants \( C_1 \) and \( C_2 \) independent of \( t \) and \( x \) such that

\[
\sup_{x, x' \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} G_t^Y(x - y) \Lambda(y - y') G_t^Y(x' - y') \, dy \, dy' \leq C \cdot t^{2(\beta - 1) - \frac{\beta}{\alpha}}, \quad (7.54)
\]
and furthermore, denoting \( \mu(d\xi) = \check{V}(\xi)d\xi \)

\[
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\check{G}_t^Y(\xi - \eta)|^2|\mu|(d\xi) \leq C_3 \cdot t^{2(\beta-1)-\frac{\alpha}{\alpha}} . \tag{7.55}
\]

Consequently, we have the upper p-th \((p \geq 2)\) moment bounds for the solution \(u^f(t,x)\). Namely, there are positive constants \(C_1\) and \(C_2\) independent of \(t, p \) and \( x \) satisfying

\[
E[|u^f(t,x)|^p] \leq C_1 \cdot \exp \left( C_2 \cdot t^{\frac{\alpha(2\beta+2\beta-2)}{2(\alpha-\beta)\lambda}} \cdot p^{\frac{\beta(2\alpha-\lambda)}{2(\alpha-\beta)\lambda}} \right) .
\]

**Proof.** We need to show \( M(2(\beta - 1) - \frac{\beta \lambda}{\alpha}) \) under conditions \((7.38)\) and \( \lambda < \min(2\alpha - \alpha/\beta, d) \), i.e. the estimates \((7.54)\). This gives the upper bound accordingly. This has been proved in [4, Theorem 3.14 and Lemma 7.3]. For the sake of completeness, we give some details here. Applying Hardy-Littlewood-Sobolev inequality ([24, Theorem 4.3]), we can find

\[
\sup_{x,x' \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} G_t^Y(x - y)\Lambda(y - y') G_t^Y(x' - y') dy dy' \\
\leq \sup_{x,x' \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} G_t^Y(x - y)|y - y'|^{-\lambda} G_t^Y(x' - y') dy dy' \\
\leq \left[ \int_{\mathbb{R}^d} |G_t^Y(y)|^2 \ dy \right]^\frac{2d-\lambda}{d} \cdot \left[ \int_{\mathbb{R}^d} \frac{t^{\beta - 1}}{|y|^d} H_{2,3}^{2,1} \left( \frac{|y|^\alpha}{2(\alpha - 1)\beta} \right) \right]^\frac{2d}{2d-\lambda} \leq C \cdot t^{2(\beta - 1) - \frac{\alpha}{\alpha}} ,
\]

where we have employed change of variable \( y \rightarrow t^{\beta/\alpha} \cdot y \) and the estimate of H-function \( H_{2,3}^{2,1}(y) \) obtained in [4, Lemma 7.1].

Next, we need to prove the inequality \((7.55)\) under conditions \((7.38)\) and \( \lambda < \min(\alpha, d) \). Let us recall some useful estimates for the Mittag-Leffler function \( E_{\beta, \alpha}(-|z|) \). Using the equation \((7.37)\) and the assumptions on \( \Lambda(\cdot) \), we have

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t^Y(x - y)\Lambda(y) dy \lesssim \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_t^Y(x - y)|y|^{-\lambda} dy \\
\lesssim \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \check{G}_t^Y(\xi) \cdot e^{1-x \cdot \xi} |\xi|^{\lambda - d} d\xi \\
\lesssim \psi^{\beta - 1} \cdot \int_{\mathbb{R}^d} E_{\beta, \alpha} \left( -\frac{t^\beta \xi^\alpha}{2} \right) \cdot |\xi|^{\lambda - d} d\xi \\
\lesssim t^{(\beta - 1) - \frac{\alpha}{d}} \cdot \int_{\mathbb{R}^d} E_{\beta, \alpha} \left( -|\xi|^\alpha \right) \cdot |\xi|^{\lambda - d} d\xi .
\]

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And the integral is well defined since
\[
\int_{\mathbb{R}^d} |E_{\beta,\beta} (-|\xi|^\alpha) \cdot |\xi|^{\lambda-d} d\xi \lesssim \int_{\mathbb{R}^d} [1 + |\xi|^\alpha]^{-1} \cdot |\xi|^{\lambda-d} d\xi < \infty,
\]
under the assumption \( \lambda < \min(\alpha, d) \). Thus, we complete the proof. \( \Box \)

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