On the Breakdown of Regular Solutions with Finite Energy for 3D Degenerate Compressible Navier–Stokes Equations

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Abstract. In this paper, the three-dimensional (3D) isentropic compressible Navier–Stokes equations with degenerate viscosities (ICND) is considered in both the whole space and the periodic domain. First, for the corresponding Cauchy problem, when shear and bulk viscosity coefficients are both given as a constant multiple of the density’s power ($\rho^\delta$ with $0 < \delta < 1$), based on some elaborate analysis of this system’s intrinsic singular structures, we show that the $L^\infty$ norm of the deformation tensor $D(u)$ and the $L^6$ norm of $\nabla \rho^{\delta-1}$ control the possible breakdown of regular solutions with far field vacuum. This conclusion means that if a solution with far field vacuum of the ICND system is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of $D(u)$ or $\nabla \rho^{\delta-1}$ as the critical time approaches. Second, when $0 < \delta \leq 1$, under the additional assumption that the shear and second viscosities (respectively $\mu(\rho)$ and $\lambda(\rho)$) satisfy the BD relation $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$, if we consider the corresponding problem in some periodic domain and the initial density is away from the vacuum, it can be proved that the possible breakdown of classical solutions can be controlled only by the $L^\infty$ norm of $D(u)$. It is worth pointing out that, except the conclusions mentioned above, another purpose of the current paper is to show how to understand the intrinsic singular structures of the fluid system considered now, and then how to develop the corresponding nonlinear energy estimates in the specially designed energy space with singular weights for the unique regular solution with finite energy.

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1. Introduction

The time evolution of the mass density $\rho \geq 0$ and the velocity $u = (u^{(1)}, u^{(2)}, u^{(3)})^\top \in \mathbb{R}^3$ of a general viscous isentropic compressible fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following isentropic compressible Navier–Stokes equations:

$$\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}T.
\end{aligned}$$

(1.1)

Here, $x = (x_1, x_2, x_3) \in \Omega$, $t \geq 0$ are the space and time variables, respectively. For the polytropic gases, the constitutive relation is given by

$$P = A\rho^n, \quad A > 0, \quad \gamma > 1,$$

(1.2)

where $A$ is an entropy constant and $\gamma$ is the adiabatic exponent. $T$ denotes the viscous stress tensor with the form

$$T = 2\mu(\rho)D(u) + \lambda(\rho)\text{div}uI_3,$$

(1.3)

for some constant $\delta \geq 0$, $\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho) + \frac{2}{3}\mu(\rho)$ is the bulk viscosity coefficient, $\alpha$ and $\beta$ are both constants satisfying

$$\alpha > 0 \quad \text{and} \quad \alpha + \beta \geq 0.$$  

(1.5)

In the current paper, assuming $0 < \delta < 1$, we consider the unique smooth solution $(\rho, u)$ with finite energy to the following two types of problems:

- Cauchy problem ($\Omega = \mathbb{R}^3$) for (1.1)–(1.5) with the following initial data and far field behavior:

  $$(\rho, u)|_{t=0} = (\rho_0(x) > 0, u_0(x)) \quad \text{for} \quad x \in \Omega,$$

  (1.6)

  $$(\rho, u)(t, x) \to (0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0.$$  

(1.7)

- Periodic problem ($\Omega = \mathbb{T}^3$) for (1.1)–(1.5) with the initial data (1.6), where $\mathbb{T}^3$ is the three-dimensional torus.

In the theory of gas dynamics, the CNS can be derived from the Boltzmann equations through the Chapman–Enskog expansion, cf. Chapman–Cowling [9] and Li–Qin [29]. Under some proper physical assumptions, the viscosity coefficients and heat conductivity coefficient $\kappa$ are not constants but functions of the absolute temperature $\theta$ such as:

$$\mu(\theta) = a_1\theta^\frac{2}{3} F(\theta), \quad \lambda(\theta) = a_2\theta^\frac{1}{2} F(\theta), \quad \kappa(\theta) = a_3\theta^\frac{1}{2} F(\theta)$$

(1.8)

for some constants $a_i$ ($i = 1, 2, 3$) (see [9]). Actually for the cut-off inverse power force models, if the intermolecular potential varies as $r^{-a}$, where $r$ is intermolecular distance, then in (1.8):

$$F(\theta) = \theta^b \quad \text{with} \quad b = \frac{2}{a} \in [0, +\infty).$$

In particular (see §10 of [9]), for ionized gas,

$$a = 1 \quad \text{and} \quad b = 2;$$

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for Maxwellian molecules,
\[ a = 4 \quad \text{and} \quad b = \frac{1}{2}; \]
while for rigid elastic spherical molecules,
\[ a = \infty \quad \text{and} \quad b = 0. \]

According to Liu–Xin–Yang [33], for isentropic and polytropic fluids, such a dependence is inherited through the laws of Boyle and Gay–Lussac:
\[ P = R\rho \theta = A\rho^\gamma, \quad \text{for constant} \quad R > 0, \]
i.e., \( \theta = AR^{-1}\rho^{\gamma-1} \), and one finds that the viscosity coefficients are functions of the density of the form (1.4). Generally, for most of the physical processes, \( \gamma \in (1, 3) \), which implies that for rigid elastic spherical molecules, \( \delta \in (0, 1) \), which is exactly the case that we are going to study. Actually, the similar assumption that viscosity coefficients depend on the density can be seen in a lot of fluid models, such as Korteweg system, shallow water equations, lake equations and quantum Navier–Stokes system (see [5–7, 16, 17, 23, 25, 32, 35, 42]).

Throughout this paper, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces:
\[
\begin{align*}
\|f\|_s &= \|f\|_{H^s(\Omega)}, \quad |f|_p = \|f\|_{L^p(\Omega)}, \quad \|f\|_{m,p} = \|f\|_{W^{m,p}(\Omega)}, \\
|f|_{C^k} &= \|f\|_{C^k(\Omega)}, \quad \|f\|_{X(Y(t))} = \|f\|_{X([0,t];Y(\Omega))}, \\
D^{k,r} &= \{f \in L^1_{loc}(\Omega) : |f|_{D^{k,r}} = |\nabla^k f|_r < +\infty\}, \quad D^k = D^{k,2}, \\
D^1 &= \{f \in L^1(\Omega) : |f|_{D^1} = |\nabla f|_2 < \infty\}, \quad |f|_{D^1} = \|f\|_{D^1(\Omega)}, \\
\|f\|_{X_1 \cap \gamma X_2} &= \|f\|_{X_1} + \|f\|_{X_2}, \quad \int_{\Omega} f \, dx = \int_{\Omega} f, \\
X([0, T]; Y(\Omega)) &= X([0, T]; Y), \quad \|(f, g, h)\|_X = \|f\|_X + \|g\|_X + \|h\|_X.
\end{align*}
\]

We will clearly indicate that \( \Omega = \mathbb{R}^d \) or \( \mathbb{T}^d \) where the above notations are used. A detailed study of homogeneous Sobolev spaces can be found in [14].

Because the momentum equations (1.1) is a double degenerate system when the density loses its positive lower bound, i.e.,
\[
\begin{align*}
\rho(u_t + u \cdot \nabla u) + \nabla P = \operatorname{div}(2\mu(\rho)D(u) + \lambda(\rho)\text{div}uI_3), \\
\end{align*}
\]
\text{Degenerate time evolution operator}

\[
\begin{align*}
\end{align*}
\]
\text{Degenerate elliptic operator}

usually, it is very hard to control the behavior of the fluids velocity \( u \) near the vacuum. Moreover, it should be pointed out here that unlike the case of constant viscosities, the elliptic operator \( \text{div} \mathbb{T} \) not only loses strong regularizing effect on solutions, but also cause some troubles in the high order regularity estimates for the velocity. For example, in order to establish some uniform a priori estimates independent of the lower bound of density in \( H^3 \) space, we need to handle the extra nonlinear terms such as

\[ \text{div}(\nabla^k \rho^\delta Q(u)) \quad \text{for} \quad Q(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta \text{div}uI_3, \]

where \( k = 1, 2, 3 \). Therefore, many attentions need to be paid in order to control these strong nonlinearities, especially for considering the related problems with vacuum state for arbitrarily large time.

For the cases \( \delta \in (0, \infty) \), if \( \rho > 0 \), (1.1) can be formally rewritten as
\[
\begin{align*}
\begin{align*}
\end{align*}
\end{align*}
\]
\text{(1.9)}

\[
\begin{align*}
\end{align*}
\]
where the quantities $\psi$ and $Lu$ are given by

$$
\psi \triangleq \nabla \log \rho \quad \text{when} \quad \delta = 1;
$$

$$
\psi \triangleq \frac{\delta}{\delta - 1} \nabla \rho^{\delta - 1} \quad \text{when} \quad \delta \in (0, 1) \cup (1, \infty);
$$

$$
Lu \triangleq -\alpha \triangle u - (\alpha + \beta) \nabla \text{div} u.
$$

(1.10)

When $\delta = 1$, from (1.9)–(1.10), the degeneracies of the time evolution and viscosities on $u$ caused by the vacuum have been transferred to the possible singularity of the term $\nabla \log \rho$, which actually can be controlled by a symmetric hyperbolic system with a source term $\nabla \text{div} u$ in Li–Pan–Zhu [30]. Then via establishing a uniform a priori estimates in $L^6 \cap D^1 \cap D^2$ for $\nabla \log \rho$, the existence of two-dimensional (2D) local classical solution with far field vacuum to (1.1) has been obtained in [30], which also applies to the 2D shallow water equations. When $\delta > 1$, (1.9)–(1.10) imply that actually the velocity $u$ can be governed by a nonlinear degenerate parabolic system without singularity near the vacuum region. Based on this observation, by using some hyperbolic approach which bridges the parabolic system (1.9) when $\rho > 0$ and the hyperbolic one

$$
u_t + u \cdot \nabla u = 0 \quad \text{when} \quad \rho = 0,$$

the existence of 3D local classical solutions with vacuum to (1.1) was established in Li–Pan–Zhu [31]. The corresponding global well-posedness in some homogeneous Sobolev spaces has been established by Xin–Zhu [44] under some initial smallness assumptions. Moreover, under the well-known B-D relation for viscosities:

$$
\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)),
$$

(1.11)

which is introduced by Bresch–Desjardins and their collaborators in [3–6], recently the global existences of the multi-dimensional weak solutions with finite energy for some kinds of $\mu(\rho)$ have been given by Bresch–Vasseur–Yu [8], Li–Xin [28] and Vasseur–Yu [43].

However, the approaches used in [30,31,44,49] fail to apply to the case $\delta \in (0, 1)$. Indeed, when vacuum appears only at far fields, the velocity field $u$ is still governed by the quasi-linear parabolic system (1.9)–(1.10). Yet, some new essential difficulties arise compared with the case $\delta \geq 1$:

(1) first, the source term contains a stronger singularity as:

$$
\nabla \rho^{\delta - 1} = (\delta - 1)\rho^{\delta - 1} \nabla \log \rho,
$$

whose behavior will become more singular than that of $\nabla \log \rho$ in [30] due to $\delta - 1 < 0$ when the density $\rho \to 0$;

(2) second, the coefficient $\rho^{\delta - 1}$ in front of the Lamé operator $L$ will tend to $\infty$ as $\rho \to 0$ in the far filed instead of equaling to 1 in [30] or tending to 0 in [31,44]. Then it is necessary to show that the term $\rho^{\delta - 1} Lu$ is well defined.

Recently, via introducing an elaborate (linear) elliptic approach on the operators $L(\rho^{\delta - 1} u)$ and some initial compatibility conditions, Xin–Zhu [45] identifies one class of initial data admitting one unique 3D local regular solution with far field vacuum and finite energy to the corresponding Cauchy problem of (1.1) in some inhomogeneous Sobolev spaces. Some other interesting results on the degenerate compressible Navier–Stokes equations can be found in [12,13,15,18–20,22,27,36,41,46,47].

In the current paper, we will do some study on the breakdown of the regular solution of 3D degenerate compressible Navier–Stokes equations obtained in [45] for the case $\delta \in (0, 1)$ (see Theorem 1.1). In order to state our main results clearly, we divide the rest of the introduction into two subsections.

### 1.1. Cauchy Problem with Far Field Vacuum

Let $\Omega = \mathbb{R}^3$. We first introduce a proper class of solutions called regular solutions to the Cauchy problem (1.1)–(1.7).
**Definition 1.1.** Let $T > 0$ be a finite constant. A solution $(\rho, u)$ to the Cauchy problem (1.1)–(1.7) is called a regular solution in $[0, T] \times \mathbb{R}^3$ if $(\rho, u)$ satisfies this problem in the sense of distribution and:

(A) $\rho > 0$, $\rho^{\gamma - 1} \in C([0, T]; H^3)$, $\nabla \rho^{\delta - 1} \in L^\infty([0, T]; L^\infty \cap D^2)$;

(B) $u \in C([0, T]; H^3) \cap L^2([0, T]; H^1)$, $u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2)$,

\[
\begin{align*}
\rho^{\frac{\delta - 1}{2}} \nabla u & \in C([0, T]; L^2), \\
\rho^{\frac{\delta - 1}{2}} \nabla u_t & \in L^\infty([0, T]; L^2), \\
\rho^{\delta - 1} \nabla u & \in L^\infty([0, T]; D^1), \\
\rho^{\delta - 1} \nabla^2 u & \in C([0, T]; H^1) \cap L^2([0, T]; D^2).
\end{align*}
\]

**Remark 1.1.** It follows from the regularity (A) shown above and the Gagliardo–Nirenberg inequality that $\nabla \rho^{\delta - 1} \in L^\infty$, which means that the vacuum occurs if and only in the far field. Moreover, it should be pointed out that the definition of regular solutions above is essentially based on the careful analysis on the intrinsic singular structures of (1.1) for finite energy solutions (1 $\leq \gamma \leq 2$), which can be seen in Sect. 2 of the current paper, or on pages 8 to 11 of [45].

The local-in-time well-posedness of the regular solution obtained by Xin–Zhu [45] can be given as follows:

**Theorem 1.1.** [45] Let parameters $(\gamma, \delta, \alpha, \beta)$ satisfy

\[
\gamma > 1, \quad 0 < \delta < 1, \quad \alpha > 0, \quad \alpha + \beta \geq 0.
\]

If the initial data $(\rho_0, u_0)$ satisfies

\[
\rho_0 > 0, \quad (\rho_0^{\gamma - 1}, u_0) \in H^3, \quad \nabla \rho_0^{\delta - 1} \in D^1 \cap D^2, \quad \nabla \rho_0^{\frac{\delta - 1}{2}} \in L^4,
\]

and the initial compatibility conditions:

\[
\nabla u_0 = \rho_0^{\frac{\delta - 1}{2}} g_1, \quad Lu_0 = \rho_0^{\frac{\delta - 1}{2}} g_2, \quad \nabla \left( \rho_0^{\delta - 1} L u_0 \right) = \rho_0^{\frac{\delta - 1}{2}} g_3,
\]

for some $(g_1, g_2, g_3) \in L^2$, then there exist a time $T_* > 0$ and a unique regular solution $(\rho, u)$ in $[0, T_*) \times \mathbb{R}^3$ to the Cauchy problem (1.1)–(1.7) satisfying:

\[
\begin{align*}
t^\frac{\delta}{2} u & \in L^\infty([0, T_*]; D^\delta), \\
t^\frac{\delta}{2} u_t & \in L^\infty([0, T_*]; D^2) \cap L^2([0, T_*]; D^3), \\
u_{tt} & \in L^2([0, T_*]; L^2), \\
t^\frac{\delta}{2} u_{tt} & \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^1), \\
\rho^{1 - \delta} & \in L^\infty([0, T_*]; L^\infty \cap D^{1, 6} \cap D^{2, 3} \cap D^3), \\
\nabla \rho^{\delta - 1} & \in C([0, T_*]; D^1 \cap D^2), \\
\nabla \log \rho & \in L^\infty([0, T_*]; L^\infty \cap L^6 \cap D^{1, 3} \cap D^2).
\end{align*}
\]

Moreover, if $1 < \gamma \leq 2$, $(\rho, u)$ is a classical solution to (1.1)–(1.7) in $(0, T_*] \times \mathbb{R}^3$.

**Remark 1.2.** For $(\alpha, \beta)$, it is required that $\alpha > 0$ and $2\alpha + 3\beta \geq 0$ in [45], which, via the same arguments used in their paper, can be easily replaced by (1.5).

**Remark 1.3.** The conditions (1.13)–(1.14) in Theorem 1.1 identify a class of admissible initial data that makes the problem (1.1)–(1.7) solvable, which are satisfied by, for example,

\[
\rho_0(x) = \frac{1}{1 + |x|^{2a}}, \quad u_0(x) \in C^\infty_0(\mathbb{R}^3), \quad \text{and} \quad \frac{3}{4(\gamma - 1)} < a < \frac{1}{4(1 - \delta)}.
\]

Particularly, when $\nabla u_0$ is compactly supported, the compatibility conditions (1.14) are satisfied automatically.

Naturally we will consider that the local regular solutions obtained above to the Cauchy problem (1.1)–(1.7) may cease to exist globally, or what is the key estimate to make sure that this solution could be extended to be a global one? The similar question has been studied for the 3D incompressible Euler equation by Beale–Kato–Majda (BKM) in their pioneering work [1], in which they showed: if $0 < T < +\infty$
is the maximum existence time for the smooth solution, then the \( L^\infty \)-bound of vorticity must blow up, i.e.,
\[
\limsup_{T \to T^*} \int_0^T \| \nabla \times u(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} dt = \infty.
\]
(1.16)

Later, Ponce [38] rephrased the above criterion in terms of the deformation tensor \( D(u) \), which has been applied to the strong solution with vacuum (see [10]) for some 3D compressible viscous flow systems [21,48] for the case \( \delta = 0 \) in (1.4), which degenerate only at the time evolution: \( \rho(u_t + u \cdot \nabla u)'' \).

Remark 1.4. First, actually, if we assume that for any time \( T^* > 0 \),
\[
\lim_{T \to T^*} \left( \sup_{0 \leq t \leq T} \| \nabla \rho^{\delta-1}(t, \cdot) \|_{L^6(\mathbb{R}^3)} + \int_0^T \| D(u)(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} dt \right) < \infty,
\]
or
\[
\lim_{T \to T^*} \left( \sup_{0 \leq t \leq T} \| \nabla \rho^{\delta-1}(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} + \int_0^T \| D(u)(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} dt \right) < \infty,
\]
then for \( \gamma \in (1,2] \), the regular solution \( (\rho, u) \) obtained in Theorem 1.1 satisfies the Cauchy problem (1.1)–(1.7) in the classical sense in \( (0, T^*) \times \mathbb{R}^3 \).

Second, when \( \gamma \in (1,2] \), the solution obtained in Theorem 1.1 has finite energy.

Moreover, by replacing \( \nabla \rho^{\delta-1} \) with \( \nabla \log \rho \), the criterion (1.17) also holds for the case \( \delta = 1 \) and the shallow water system, which can be found in [30].

At last, the condition on \( L^\infty L^6_t \) (or \( L^\infty_t L^\infty_x \)) norm of \( \nabla \rho^{\delta-1} \) is motivated from the well-posedness theory developed in Theorem 1.1, which essentially requires that the density can not decay very fast in the far field due to \( 0 < \delta < 1 \). Actually, the initial condition \( \nabla \rho_0^{\delta-1} \in D^1 \cap D^2 \) in (1.13) implies that \( \nabla \rho_0^{\delta-1} \in L^6 \cap L^\infty \).

1.2. Periodic Problem Away from the Vacuum

Let \( \Omega = \mathbb{T}^3 \). Now we state that, under the additional assumption that the shear and second viscosities (respectively \( \mu(\rho) \) and \( \lambda(\rho) \)) satisfy the B-D relation (1.19) (see [3–6,8]), if we consider the corresponding problem in some periodic domain and the initial density is away from the vacuum, the possible breakdown of classical solutions can be controlled only by the \( L^\infty \) norm of \( D(u) \). This conclusion can be stated precisely as follows.
Theorem 1.3. Let (1.2), (1.5) hold, $\delta \in (0, 1]$ and
\[\lambda = 2(\mu'(\rho)\rho - \mu(\rho)) = 2\alpha(\delta - 1)\rho^\delta.\] (1.19)
Assume the initial data $(\rho_0, u_0)$ satisfies
\[\rho_0 > 0, \quad (\rho_0, u_0) \in H^3,\] (1.20)
and $(\rho(t, x), u(t, x))$ is the corresponding unique classical solution in $[0, T] \times T^3$ for some positive time $T > 0$ to the periodic problem (1.1)–(1.6) which satisfies
\[\rho \in C([0, T]; H^3), \quad u \in C([0, T]; H^3) \cap L^2([0, T]; H^4).\]
If the maximal existence time $T$ of this solution is finite, then
\[\lim_{T \to T} \int_0^T \|D(u)(t, \cdot)\|_{L^\infty(T^3)}^2 \, dt = \infty.\] (1.21)

Remark 1.5. Actually, the conclusion obtained above can also be applied to some initial boundary value problems of the system (1.1) with $\delta \in (0, 1]$ in (1.4) under proper boundary conditions in smooth and bounded domains. For simplicity, here we only consider the periodic problem. Moreover, it should be pointed out that according to [7, 35], for two-dimensional shallow water system, the viscous stress tensor has the form $T = \text{div}(2\rho D(u) + 2\rho \text{div}u I^2)$, which does not satisfy (1.19).

The rest of this paper will be divided into five sections. In Sect. 2, we introduce two intrinsic singular structures (2.2) and (2.5) of the system (1.1), and then show the main strategy of our proof. In Sect. 3, we show some new elliptic approaches that are related to singular structures (2.2) and (2.5). Moreover, in order to make sure that we can continue to apply the local existence theory obtained in Theorem 1.1 at any positive time within the solution’s life span, we also verify all the compatibility conditions and initial conditions for our solution at any positive time before the singularity appears. Based on the analysis on the mathematical structure of system (1.1), we give the proof for Theorem 1.2 in Sect. 4. Section 5 is devoted to proving Theorem 1.3. Finally, we will give an appendix to list some basic lemmas that were used frequently in our proof. It is worth pointing out that, except the conclusions mentioned above, another purpose of the current paper is to show how to understand the intrinsic singular structures of the fluid system considered now, and then how to develop the corresponding nonlinear energy estimates for the regular solution with finite energy.

2. Reformulation and Main Strategy

In this section, we always assume that (1.12) holds and $\Omega = \mathbb{R}^3$. We first introduce two intrinsic singular structures of system (1.1), and then show the main strategy of our proof. For simplicity, in the rest of this paper, we always denote
\[a = \left(\frac{A\gamma}{\gamma - 1}\right)^{\frac{\gamma - 1}{\gamma - 2}}, \quad \text{and} \quad e = \frac{\delta - 1}{2(\gamma - 1)} < 0.\] (2.1)

Moreover, we use $\text{DTE}$ to denote the degenerate time evolution operator, $\text{WSS}$ to denote the weak singular source term, $\text{SSS}$ to denote the strong singular source term, and $\text{SSE}$ to denote the strong singular elliptic operator.

Generally, because the momentum equations (1.1)$_2$ is a double degenerate system when the density loses its positive lower bound, i.e.,
\[\rho(u_t + u \cdot \nabla u) + \nabla P = \text{DTE} (\text{Degenerate elliptic operator} \left(\text{div}(2\rho D(u) + \lambda(\rho)\text{div}u I^2)\right)),\]
usually, it is very hard to obtain higher order regularities of the fluids velocity $u$ near the vacuum. Then it is necessary that we need to find some intrinsic structures of this system to make some effective analysis on $u$. Due to $\delta \in (0, 1)$, formally, we have two choices. The first one is the “Degenerate”–“Weak-Singular”
structure shown in (2.2), which has a degeneracy in the time evolution, but provides one uniform elliptic operator $L \rho$. However, this structure still has one WSS: $\nabla \log \rho \cdot Q(u)$. The other one is the strong singular structure shown in (2.5), which has a nice time evolution operator, but also has one SSE: $\rho^{\delta-1}L \rho$ and one SSS: $\nabla \rho^{\delta-1} \cdot Q(u)$.

### 2.1. “Degenerate”–“Weak-Singular” Structure

In terms of variables

$$\varphi = a \rho^{1-\delta}, \quad g = aA \rho^{1-\delta}, \quad f = a \delta \nabla \log \rho = (f^{(1)}, f^{(2)}, f^{(3)}),$$

and $u$, for $\delta \in (0, 1]$, the system (1.1) can be rewritten as

$$\begin{cases}
\varphi_t + u \cdot \nabla \varphi + (1 - \delta) \varphi \nabla u = 0, \\
g_t + u \cdot \nabla g + (\gamma - \delta) g \nabla u = 0, \\
\varphi(u_t + u \cdot \nabla u) + \nabla g + aL u = f \cdot Q(u), \\
\varphi_t + \sum_{i=1}^{3} A_i \partial_i f + B^* f + a \delta \nabla \delta u = 0,
\end{cases}$$

(2.2)

where $A_i = (a_{ij}^l)_{3 \times 3}$ for $i, j, l = 1, 2, 3$, are symmetric with $a_{ij}^l = u^{(l)}$ for $i = j$; otherwise $a_{ij}^l = 0$, and $B^* = (\nabla u)^\top$. Actually, when $\delta = 1$, the momentum equations can be reformulated into

$$u_t + u \cdot \nabla u + A_1 \nabla \rho^{-1} + Lu = \nabla \log \rho \cdot Q(u).$$

(2.3)

### 2.2. Strong Singular Structure

In terms of variables

$$\phi = \frac{A_1 \rho^{-1}}{\gamma - 1}, \quad \psi = \frac{\delta}{\delta - 1} \nabla \rho^{\delta-1} = (\psi^{(1)}, \psi^{(2)}, \psi^{(3)}),$$

(2.4)

and $u$, for $\delta \in (0, 1]$, the system (1.1) can be rewritten as

$$\begin{cases}
\phi_t + u \cdot \nabla \phi + (\gamma - 1) \phi \nabla u = 0, \\
\psi_t + u \cdot \nabla \psi + \phi \nabla \phi + a \phi^{2e} L u = \psi \cdot Q(u), \\
\psi_t + \sum_{i=1}^{3} A_i \partial_i \psi + B \psi + \delta a \phi^{2e} \nabla \delta u = 0,
\end{cases}$$

(2.5)

where $B = (\nabla u)^\top + (\delta - 1) \nabla \delta u$. In Sects. 3 and 4, we will consider the Cauchy problem (2.5) with the initial data

$$(\phi, u, \psi)|_{t=0} = (\phi_0, u_0, \psi_0) = \left( \frac{A_1 \rho^{-1}}{\gamma - 1} (x), u_0(x), \frac{\delta}{\delta - 1} \nabla \rho^{\delta-1}(x) \right), \quad x \in \mathbb{R}^3,$$

(2.6)

and the far field behavior:

$$(\phi, u, \psi) \to (0, 0, 0), \quad \text{as} \quad |x| \to +\infty, \quad t \geq 0.$$

(2.7)

For the simplicity of the proof in Sects. 3 and 4, we also give some relations between the new variables:

$$\varphi = \phi^{-2e}, \quad f = \psi \varphi, \quad g = \frac{\gamma - 1}{\gamma - \delta} \phi \varphi, \quad \psi = \frac{a \delta}{\delta - 1} \nabla \phi^{2e} = \frac{a \delta}{\delta - 1} \nabla \varphi^{-1}.$$
2.3. Main Strategy

Now, based on the two intrinsic singular structures of the system (1.1) mentioned above, we show our main strategy of our proof for Theorem 1.2.

2.3.1. Necessity of the Strong Singular Structure (2.5). It is well known that the single degenerate structure
\[
\rho(u + u \cdot \nabla u) + \nabla P + Lu = 0,
\]
has been widely used for the well-posed or singularity formation theory of smooth solutions with vacuum to compressible fluid systems for the case \( \delta = 0 \) ([10, 21, 48]). For such kind of structures, we can make sure that the velocity belongs to \( D^1 \cap D^3 \) and also \( \sqrt{\rho} \in L^2 \), if the initial data is smooth enough and satisfy some necessary compatibility conditions, which implies that the velocity itself only belongs \( L^6 \cap L^\infty \) in the whole space.

However, for our “Degenerate”–“Weak-singular” structure (2.2), if the velocity \( u \) only belongs to \( D^1 \cap D^3 \) and \( \sqrt{\rho} \in L^2 \), it is not good enough to close the desired energy estimates. For example, considering the basic weighted \( L^2 \) norm of \( u \): \( |\varphi u|_2 \), it follows from the equations (2.2)_3 that
\[
\frac{1}{2} \frac{d}{dt} \int \varphi u^2 + a \int (\alpha |\nabla u|^2 + (\alpha + \beta)|\text{div} u|^2)
= \frac{1}{2} \int \varphi u^2 - \int \varphi (u \cdot \nabla) u \cdot u - \int \nabla g \cdot u + \int f \cdot Q(u) \cdot u.
\]
For the last term
\[ I^* = \int f \cdot Q(u) \cdot u, \]
one has \( Q(u) \in L^2 \cap L^\infty \) and \( u \in L^6 \cap L^\infty \). Then in order to make sure that
\[ f \cdot Q(u) \cdot u \in L^1, \]
at least we need \( f \in L^p \) for some \( p \in [1, 3] \). However, even \( \rho > 0 \) only decays to zero in the far field. It is still very hard to find the initial data such that
\[ f(0, x) = a \delta \nabla \log \rho_0 \in L^p \quad \text{for some} \quad p \in [1, 3]. \]
Generally, for some initial density decays to zero in the form of polynomials:
\[ \rho_0(x) = \frac{1}{1 + |x|^{2q}} \quad \text{for some proper} \quad q, \]
it is easy to see that, actually,
\[ f(0, x) = a \delta \nabla \log \rho_0 \in L^p \quad \text{for any} \quad p > 3. \]
We know that \( f \) should still keep in \( L^p \) within the life span due to the standard theory of the symmetric hyperbolic system (2.2)_4. Then according to the Holder’s inequality, in order to make sure that \( f \cdot Q(u) \cdot u \in L^1 \), at least we need \( u \in L^1 \), where \( l = \frac{2p}{p - 2} \in [2, 6] \). Formally we know that the “Degenerate”–“Weak-Singular” structure (2.2) at most provides the information \( u \in L^6 \cap L^\infty \) due to the degeneracy in the time evolution. Then it is obvious that the “Degenerate”–“Weak-Singular” structure (2.2) can not give enough information for closing the desired nonlinear energy estimates. Then we need to introduce another structure (2.5) to give the \( L^2 \) integrability of \( u \).

Next we formally show how to use the system (2.5) to give one close energy estimates for the nonlinear problems. First, for the behavior of the possible singular term \( \nabla \rho^{\delta - 1} \), it follows from (2.5)_3 that it could be controlled by a symmetric hyperbolic system with a possible singular higher order term \( \delta a \phi^{2c} \nabla \text{div} u \). Due to the fact that \( \phi^{2c} \) has an uniformly positive lower bound in the whole space, then for this special strong singular system, one can find formally that, even though the coefficients \( a \phi^{2c} \) in front of Lamé
operator $L$ will tend to $\infty$ as $\rho \to 0$ in the far filed, yet this structure could give a better a priori estimate on $u$ in weighted-$H^3$ than those of [10,30,31,44]. In order to close the estimates, we need to control $\phi^{2e}\nabla \psi \cdot u$ in $D^1 \cap D^2$, which can be obtained by regarding the momentum equations as the following inhomogeneous Lamé equations:

$$a L(\phi^{2e} \partial_ku) = a \phi^{2e} L \partial_ku - a G(\nabla \phi^{2e}, \partial_ku)$$

with

$$G(v, u) = \alpha v \cdot \nabla u + \alpha \text{div}(u \otimes v) + (\alpha + \beta)(\text{div}u + v \cdot (\nabla u) + u \cdot \nabla v).$$

Actually, one has

$$|\phi^{2e}\nabla^2 u|_{D^1} \leq C(|\phi^{2e}\nabla u|_{D^2} + |\psi|_{\infty} |\nabla \phi|_{L^2}^2 + |\nabla u|_{\infty} |\nabla \psi|_{D^1}).$$

Similar calculations can be done for $|\phi^{2e}\nabla^2 u|_{D^2}$ and $|\phi^{2e}\nabla^2 u|_{2}$. It should be pointed out that we have used the following only way to estimate

$$\psi \cdot \nabla u_i \in L^\infty([0, T]; L^6), \quad \phi^{2e}\nabla^2 u \in L^\infty([0, T]; L^2), \quad \phi^{2e}\nabla u_i \in L^\infty([0, T]; L^2).$$

Thus, it seems that at least, the reformulated system (2.5) can provide a closed energy estimates for the regular solution we defined in the introduction.

### 2.3.2. Advantage of the “Degenerate”—“Weak-Singular” Structure (2.2)

From (2.5), the assumption on the boundedness of $\psi$ and $D(u)$ in (4.1) can only ensure the boundedness of $\nabla u$ and $\phi^{2e}Lu$. For the higher order estimates on $\nabla^k u$ ($k = 3, 4$) with singular weights, we need the estimates on $\nabla^l \psi$ ($l = 1, 2$). Actually, the thing will become very complex if we want to do these estimates directly from (2.5). The introduction of (2.2) will make the estimates look clearer.

Taking the estimate on $\|u\|_{L^\infty([0, T]; D^3)}$ for example, first we need to consider

$$\frac{1}{2} \frac{d}{dt} \left( a \alpha |\phi^{2e}\nabla u_t|^2 + a(\alpha + \beta)|\phi^{2e}\text{div}u_t|^2 \right) + |u_t|^2 = \int \left( -u \cdot \nabla u_t - \nabla \phi_t - a \phi^{2e} L u - a \nabla \phi^{2e} \cdot \psi \right) \cdot u_t$$

$$+ \int \left( a \frac{1}{2} \phi^{2e} (\alpha |\nabla u_t|^2 + (\alpha + \beta)|\text{div}u_t|^2) + (\psi \cdot Q(u_t))_t \right) u_t.$$  \hspace{1cm} (2.11)

For the one component of the last term $J^*$

$$\int \psi \cdot Q(u_t) \cdot u_t,$$

it can only be controlled by $|\psi \cdot Q(u)|_{2}^2 |u_t|^2$. Here, we may use $|\psi|_\infty |Q(u)|_{2}$ or $|\psi|_6 |Q(u)|_{3}$ to control the norm $|\psi \cdot Q(u)|_{2}$. However, in order to control $|\psi|_{\infty}$, we need $\|\psi\|_{D^1 \cap D^2}$, and then $|\phi^{2e}\nabla^2 u|_{L^1([0, T]; D^1 \cap D^2)}$. This means that at least we need the estimate on $\int_0^t |\nabla^2 u_t|^2 ds$, which is also required in $|Q(u)|_{3}$.

If we continue to use the structure (2.5) for obtaining the estimate on $\int_0^t |\nabla^2 u_t|^2 ds$, formally one has

$$a L(\phi^{2e} u_t) = -\phi^{2e} (\phi^{-2e} (u_t + u \cdot \nabla u + \nabla \phi - \psi \cdot Q(u)))_t = \frac{1}{\delta} G(\psi, u_t) = Y.$$

On the one hand, the following way only to estimate $|\nabla^2 u_t|_2$ might not hold:

$$|\phi^{2e} u_t|_{D^2} \leq C |Y|_{2},$$

for some constant $C > 0$. Because we do not have $\phi^{2e} u_t \in L^p$ for some $p > 0$, or in some weak or strong sense,

$$\phi^{2e} u_t \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty.$$
by $|\nabla f|_2 + ||\nabla^2 \phi||_1$. Actually, at the current step, what we need is only the estimate on $|\nabla^2 u_t|_2$, not on $|\phi^2 \nabla^2 u_t|_2$. Then we should ask help from the structure (2.2), where one has

$$aLu_t = -\varphi u_{tt} - \varphi (u \cdot \nabla u)_t - \varphi_t (u_t + u \cdot \nabla u) - \nabla g_t + (f \cdot Q(u))_t. \quad (2.13)$$

It follows from Lemma 3.1 that

$$|u_t|_{D^2} \leq C(|-\varphi u_{tt} - \varphi (u \cdot \nabla u)_t - \varphi_t (u_t + u \cdot \nabla u) - \nabla g_t + (f \cdot Q(u))_t|_2). \quad (2.14)$$

for some constant $C > 0$, which only depends on $|\nabla f|_2$ but not on $|\nabla \psi|_2$. The estimate on $|\nabla f|_2$ is much easier than that of $|\nabla \psi|_2$, which makes the estimates clearer compared with the one in (2.12). Once obtaining the estimate on $\int^t_0 |\nabla^2 u_t|_2^2 ds$, then we can come back to the structure (2.5) again for further estimates, which is good enough for extending our solution beyond the time $T$ (see Sect. 4.2).

3. Elliptic Argument and Compatibility Conditions

In this section, we always assume that (1.12) holds and $\Omega = \mathbb{R}^3$. First, we will show some elliptic approaches that are related to singular structures (2.2) and (2.5). Second, in order to make sure that we can continue to apply the local existence theory obtained in Theorem 1.1 at any positive time, we will verify the compatibility conditions in (1.14) for any positive time within the regular solution’s life span. At last, we will show some additional regularities of the regular solution that are not shown in Theorem 1.1.

3.1. Standard Elliptic Approach

The following regularity estimate for the Lamé operator is standard in harmonic analysis.

**Lemma 3.1.** [10,40] Let $1 < q < +\infty$, $k \in \mathbb{Z}$ and $Z \in D^{k,q}$. If $u \in D^{1,q}_0(\mathbb{R}^3)$ is a weak solution to the following elliptic problem

$$Lu = Z, \quad (3.1)$$

then $u \in D^{k+2,q}$, and it holds that

$$|u|_{D^{k+2,q}} \leq C|Z|_{D^{k,q}},$$

where the constant $C > 0$ depends only on $\alpha$, $\beta$ and $q$.

3.2. Density Involved Elliptic Approach

Let $T > 0$ be some time. In the rest of this section, let $(\rho, u)$ in $[0, T] \times \mathbb{R}^3$ be the unique regular solution to the Cauchy problem (1.1)–(1.7) obtained in Theorem 1.1. For simplicity, we denote

$$G(v, u) = \alpha v \cdot \nabla u + \text{adiv}(u \otimes v) + (\alpha + \beta)(\text{div} v + v \cdot (\nabla u) + u \cdot \nabla v),$$

$$H(u, \phi, \psi) = H = u_t + u \cdot \nabla u + \nabla \phi - \psi \cdot Q(u).$$

**Lemma 3.2.** For $k = 1, 2, 3$,

$$|\phi^c \partial_k u|_{D^2} \leq C(\phi^-_{\infty} |\partial_k H|_2 + |\partial_k \phi^{-\epsilon} H|_2 + |G(\phi^{-\epsilon} \psi, \partial_k u)|_2),$$

where the constant $C > 0$ depends only on $\alpha$, $\beta$, $A$, $\gamma$ and $\delta$. 
Proof. According to the equations (2.5)\textsubscript{2}, for \( k = 1, 2, 3 \), one has
\[
aL(\phi^e \partial_k u) = a \phi^e L \partial_k u - aG(\nabla \phi^e, \partial_k u)
= -\phi^e \partial_k (\phi^{-2e} H) - \frac{\delta - 1}{2\delta} G(\phi^{-e} \psi, \partial_k u)
= -\phi^{-e} \partial_k H - 2\partial_k \phi^{-e} H - \frac{\delta - 1}{2\delta} G(\phi^{-e} \psi, \partial_k u).
\]
(3.2)

Lemma 3.3. For \( k = 1, 2, 3 \),
\[
|\phi^{2e} \partial_k u|_{D^2} \leq C(|H|_{D^1} + |f H|_2 + |G(\psi, \partial_k u)|_2),
\]
where the constant \( C > 0 \) depends only on \( \alpha, \beta, A, \gamma \) and \( \delta \).

Proof. According to the equations (2.5)\textsubscript{2}, for \( k = 1, 2, 3 \), one has
\[
aL(\phi^{2e} \partial_k u) = a \phi^{2e} L \partial_k u - aG(\nabla \phi^{2e}, \partial_k u)
= -\phi^{2e} \partial_k (\phi^{-2e} H) - \frac{\delta - 1}{\delta} G(\psi, \partial_k u)
= -\partial_k H - \frac{1 - \delta}{a\delta} f H - \frac{\delta - 1}{\delta} G(\psi, \partial_k u).
\]
(3.3)

Lemma 3.4.
\[
|\phi^{2e} \nabla^\xi u|_{D^2} \leq C(|\nabla^\xi H|_2 + |f \cdot \nabla H|_2 + ||f||^2 H|_2 + |\nabla f \cdot H|_2 + |G(\nabla \phi^{2e}, \nabla^\xi u)|_2),
\]
where the constant \( C > 0 \) depends only on \( (\alpha, \beta, A, \gamma, \delta) \), and \( \xi \in \mathbb{R}^3 \) with \(|\xi| = 2\) is any multi-index whose components are all non-negative integers.

Proof. According to the equations (2.5)\textsubscript{2}, for multi-index \( \xi \in \mathbb{R}^3 \), one has
\[
aL(\phi^{2e} \nabla^\xi u) = a \phi^{2e} \nabla^\xi Lu - aG(\nabla \phi^{2e}, \nabla^\xi u)
= -\phi^{2e} \nabla^\xi (\phi^{-2e} H) - \frac{\delta - 1}{\delta} G(\psi, \nabla^\xi u).
\]
(3.4)

Remark 3.1. In the above three lemmas, one has used the facts: for \( k = 1, 2 \),
\[
\phi^e \nabla u \in H^1(\mathbb{R}^3), \quad \phi^{2e} \nabla^k u \in L^6(\mathbb{R}^3).
\]

3.3. Verification of Compatibility Conditions

According to Theorem 1.1 and the initial assumption (1.13)–(1.14), now we have the unique regular solution \((p, u)\) to the Cauchy problem (1.1)–(1.7) in \([0, T] \times \mathbb{R}^3 \). In order to extend this local solution from \([0, T]\) to the time interval \([0, T_1]\) with \( T_1 > T \), we need to apply Theorem 1.1 at \( t = T \) again. Thus we need to make sure that the compatibility condition (1.14) still holds at time \( t = T \). For this purpose, we first need the following lemma:

Lemma 3.5.
\[
\begin{align*}
\phi^e \nabla u \in C([0, T]; L^2), \quad & \phi^{2e} Lu \in C([0, T]; L^2), \\
\phi^e \nabla u_t \in L^\infty([0, T]; L^2), \quad & \phi^{2e} \nabla^2 u_t \in L^2([0, T]; L^2), \\
t^{\frac{1}{2}} \phi^e \nabla u_{tt} \in L^2([0, T]; L^2), \quad & t\phi^e \nabla (\phi^{2e} Lu) \in C([0, T]; L^2).
\end{align*}
\]
(3.5)
Proof. First, the first line and the first property in the second line of (3.5) can be obtained directly from Theorem 1.1.

Second, the second property in the second line and the first one in the third line of (3.5), actually have been proven in [45], but is not indicated clearly in Theorem 1.1 and Definition 1.1, which can be implied by the estimates (3.74) and (3.80), convergences (3.85), (3.127) and (3.141) of [45].

Next, for the second property in the third line of (3.5), it follows from equations (2.5), Definition 1.1, the proved properties of (3.5) and Theorem 1.1 that

\[ a\phi^e_3 \partial_k Lu = -a\partial_k \phi^e_3 Lu - \partial_k \phi^e u_t - \phi^e \partial_k u_t - \phi^e \partial_k (u \cdot \nabla u) \]
\[ - \partial_k \phi^e u \cdot \nabla u - \partial_k (\phi^e \nabla \phi) + \partial_k \phi^e \psi \cdot Q(u) \]
\[ + \phi^e \partial_k (\psi \cdot Q(u)) \in L^\infty([0, T]; L^2), \tag{3.6} \]

for \( k = 1, 2, 3 \).

It follows from the following relations
\[
\begin{align*}
\phi_t^e &= -u \cdot \nabla \phi^e - \frac{\delta - 1}{2} \phi^e \text{div} u, \\
\psi_t &= -\sum_{l=1}^3 A_l \partial_l \psi - B \psi - \delta a \phi^e \nabla \text{div} u, \tag{3.7}
\end{align*}
\]

Definition 1.1 and Theorem 1.1 that
\[ \phi_t^e, \ \nabla \phi_t^e, \ \psi_t, \ \nabla \psi_t \in L^\infty([0, T]; L^2), \tag{3.8} \]
which, along with the relation (3.6), implies that
\[ a_t (\phi^e_3 \partial_k Lu)_t \in L^2([0, T]; L^2). \tag{3.9} \]

Then it follows from (3.6), (3.9) and the classical Sobolev imbedding theorem that
\[ a_t \phi^e_3 \partial_k Lu \in C([0, T]; L^2). \tag{3.10} \]

Similarly, one can also show that
\[ a \phi^e \partial_k \phi^e_2 Lu \in C([0, T]; L^2), \tag{3.11} \]
which, together with (3.10), implies the desired conclusion.

The above lemma implies that, if \((\rho, u)\) is the unique regular solution in \([0, T] \times \mathbb{R}^3\) to the Cauchy problem (1.1)–(1.7), then one can obtain that
\[ (\phi^e \nabla u)(T, x), \ (\phi^e_2 Lu)(T, x), \ (\phi^e \nabla (\phi^e_2 Lu))(T, x) \in L^2(\mathbb{R}^3). \tag{3.12} \]

This means that the compatibility conditions in (1.14) are still available at \( t = T \). \qed

### 3.4. Verification of Some Special Initial Condition

In the initial assumption (1.13) of Theorem 1.1, the authors require that
\[ \nabla \rho_0^{\frac{\gamma-1}{2-\gamma}} \in L^4, \tag{3.13} \]
which, actually, is only used in the initial data’s approximation process from the non-vacuum flow to the flow with the far field vacuum, and does not appear in the corresponding energy estimates process in the proof for the local existence of the regular solution shown in [45]. Therefore, in Theorem 1.1, the regularity of the quantity \( \nabla \rho^{\frac{\gamma-1}{2-\gamma}} \) has not been mentioned.

However, whether the fact \( \nabla \rho_0^{\frac{\gamma-1}{2-\gamma}} \in C([0, T]; L^4) \) holds or not is very crucial for our following proof, not only for the application of the local existence theorem at some positive time, but also for the estimate on \( \phi^e \nabla u_t \) (see Lemma 4.11). Now we will verify this desired information in this subsection.
Denote $\omega = \nabla \rho^{\frac{\delta - 1}{2}}$, it follows from the equation (1.1) that $\omega$ satisfies the following equations

$$\omega_t + \sum_{i=1}^{3} A_i \partial_i \omega + A^*(u) \omega + \frac{\delta - 1}{2} \sqrt{a\phi^s} \nabla \text{div} u = 0,$$

(3.14)

where $A^*(u) = (\nabla u)^{\top} + \frac{\delta - 1}{2} \text{div} u$.

**Lemma 3.6.**

$$\omega \in C([0, T]; L^6 \cap L^\infty \cap D^{1, 3} \cap D^2), \quad \omega_t \in L^\infty([0, T]; H^1).$$

The proof of the above lemma can be easily got from the regularity of the solution $(\rho, u)$ obtained in Theorem 1.1. Here we omit it.

**Lemma 3.7.**

$$\omega \in C([0, T]; L^4) \quad \text{and} \quad \nabla f \in L^\infty([0, T]; L^2).$$

**Proof.** According to Lemma 3.6 and the Sobolev imbedding theorem, for the time continuity of $\omega$, we only need to show that

$$\omega \in L^\infty([0, T]; L^4).$$

(3.15)

In fact, let $f : \mathbb{R}^+ \to \mathbb{R}$ satisfy

$$f(s) = \begin{cases} 1, & s \in [0, \frac{1}{2}] \\ \text{non-negative polynomial}, & s \in \left[\frac{1}{2}, 1\right] \\ e^{-s}, & s \in [1, \infty) \end{cases}$$

such that $f \in C^2$. Then there exists a generic constant $C_\ast > 0$ such that $|f'(s)| \leq C_\ast f(s)$. Define, for any $R > 0$, $f_R(x) = f(\frac{|x|}{R})$.

First, according to Lemma 3.6, it is easy to see that for any given $R > 0$,

$$\int \left( \sum_{i=1}^{3} A_i \partial_i \omega + A^*(u) \omega + \frac{\delta - 1}{2} \sqrt{a\phi^s} \nabla \text{div} u \right) \cdot \omega |\omega|^2 < \infty,$$

$$\int \left( \sum_{i=1}^{3} \partial_i A_i + \partial_i A_i f_R(x) + A_i \partial_i f_R(x) \right) |\omega|^4 < \infty,$$

(3.16)

$$\int \left( |\omega|^4 f_R(x) + \omega_t \cdot |\omega|^2 + \omega_t \cdot |\omega|^2 f_R(x) \right) < \infty.$$

Second, since the equations (3.14) holds almost everywhere, one can multiply (3.14) by $4|\omega|^2 f_R(x)$ on its both sides and integrate with respect to $x$ over $\mathbb{R}^3$, then one has

$$\frac{d}{dt} \int |\omega|^4 f_R(x) = \int \left( \sum_{i=1}^{3} \partial_i A_i f_R(x) + A_i \partial_i f_R(x) \right) |\omega|^4$$

$$- \int 4f_R(x) \left( A^*(u) \omega + \frac{\delta - 1}{2} \sqrt{a\phi^s} \nabla \text{div} u \right) \cdot \omega |\omega|^2$$

$$\leq C|\nabla u|_\infty \int |\omega|^4 f_R(x)$$

$$+ C((1 + R^{-1})|\omega|_\infty |u|_2 + |\phi^s \nabla \text{div} u|_2) |\omega|^3$$

$$\leq C|\nabla u|_\infty \int |\omega|^4 f_R(x) + C(1 + R^{-1}),$$

(3.17)

for some constant $C > 0$ depending only on $(\rho_0, u_0, \alpha, \beta, A, \gamma, \delta, T)$ but not on $R$, which, along with Gronwall’s inequality, implies that

$$\int |\omega|^4 f_R(x) \leq C(1 + R^{-1}) \quad \text{for} \quad t \in [0, T].$$

(3.18)
Note that $|\omega|^4 f_R(x) \to |\omega|^4$ almost everywhere as $R \to \infty$, then it follows from the Fatou lemma (see Lemma 5.15) that

$$\int |\omega|^4 \leq \liminf_{R \to \infty} \int |\omega|^4 f_R(x) \leq C \quad \text{for} \quad t \in [0, T].$$

Then (3.15) has been obtained and the desired conclusion follows quickly.

At last, it follows from the direct calculations that

$$\nabla f = \frac{a\delta}{\delta - 1} \nabla \left( \rho^{1-\delta} \nabla \rho^{\delta-1} \right) = \frac{a\delta}{\delta - 1} \left( \rho^{1-\delta} \nabla \rho^{\delta-1} - \rho^{2-2\delta} |\nabla \rho^{\delta-1}|^2 \right) \in L^\infty([0, T]; L^2),$$

where one has used the fact that $\nabla \rho^{\delta-1} \in L^\infty([0, T]; L^4)$.

\[3.5. \text{Further Explanation on the Initial Assumptions}\]

Finally, we show some additional information that is implied by (1.13)–(1.14).

**Lemma 3.8.**

$$\rho_0^{\delta-1} \nabla u_0 \in D^1(\mathbb{R}^3).$$

**Proof.** First, it follows from the argument used in the proof for Lemma 4.13 that

$$\phi_0^{2\varepsilon} \nabla^2 u_0 \in L^2.$$  \hspace{1cm} (3.20)

Second, for any positive constant $\eta > 0$, one has

$$(\phi_0 + \eta)^{2\varepsilon} \nabla u_0 \in L^6 \quad \text{and} \quad (\phi_0 + \eta)^{2\varepsilon} \nabla^2 u_0 \in L^2.$$  \hspace{1cm} (3.21)

Then it follows from the Sobolev imbedding theorem that

$$| (\phi_0 + \eta)^{2\varepsilon} \nabla u_0 |_6 \leq C ( | (\phi_0 + \eta)^{2\varepsilon} \nabla^2 u_0 |_2 + | \nabla (\phi_0 + \eta)^{2\varepsilon} |_{L^6} |\nabla u_0|_2 )$$

$$\leq C ( |\phi_0^{2\varepsilon} \nabla^2 u_0|_2 + |\nabla \phi_0^{2\varepsilon}|_{L^6} |\nabla u_0|_2 ) \leq C,$$

for some constant $C > 0$ that is independent of $\eta > 0$.

Moreover, it is very easy to see that, for every $x \in \mathbb{R}^3$,

$$(\phi_0 + \eta)^{2\varepsilon} \nabla u_0 \to \phi_0^{2\varepsilon} \nabla u_0 \quad \text{a.e. as} \quad \eta \to 0,$$

which, along with Fatou’s lemma (see Lemma 5.15), implies that

$$\int |\phi_0^{2\varepsilon} \nabla u_0|^6 \leq \liminf_{\eta \to 0} \int |(\phi_0 + \eta)^{2\varepsilon} \nabla u_0|^6 < \infty.$$  \hspace{1cm} (4.1)

Then the proof of this lemma is completed. \hfill \square

4. **Cauchy Problem with Far Field Vacuum**

This section will be devoted to proving Theorem 1.2, and we always assume that (1.12) holds and $\Omega = \mathbb{R}^3$. In this section, we only give the proof for (1.17), and the proof for (1.18) is similar. In order to prove (1.17), we use a contradiction argument. Let $(\rho, u)$ be the unique regular solution to the Cauchy problem (1.1)–(1.7) with the life span $\bar{T}$ obtained in Theorem 1.1. It is worth pointing out that this solution also satisfies the properties stated in Lemmas 3.2–3.8. We assume that $\bar{T} < +\infty$ and

$$\lim_{T \to \bar{T}} \left( \sup_{0 \leq t \leq T} \| \nabla \rho^{\delta-1}(t, \cdot) \|_{L^6(\mathbb{R}^3)} + \int_0^T \| D(u)(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} \, dt \right) = C_0 < \infty,$$  \hspace{1cm} (4.1)
for some constant $C_0 > 0$. We will show that under the assumption \( (4.1) \), $T$ is actually not the maximal existence time for the regular solution.

By assumptions \( (4.1) \) and the system \( (2.5) \), we first show that $\rho$ is uniformly bounded.

**Lemma 4.1.**

\[
C^{-1} \leq \|\rho\|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq C \quad \text{and} \quad \|\phi\|_{L^\infty([0,T];L^q(\mathbb{R}^3))} \leq C \quad \text{for} \quad 0 \leq T < T^*,
\]

where the constant $C > 0$ only depends on \((\rho_0, u_0), C_0, A, \gamma, \) the constant $q \in [2, +\infty]$ and $T^*$.

**Proof.** First, it is obvious that $\phi$ can be represented by

\[
\phi(t, x) = \phi_0(W(0, t, x)) \exp \left( -\gamma - 1 \int_0^t \text{div}(s, W(s, t, x)) ds \right),
\]

where $W \in C^1([0, T] \times [0, T] \times \mathbb{R}^3)$ is the solution to the initial value problem

\[
\begin{aligned}
\frac{d}{ds} W(s, t, x) &= u(s, W(s, t, x)), & 0 \leq s \leq T, \\
W(t, t, x) &= x, & 0 \leq s \leq T, & x \in \mathbb{R}^3.
\end{aligned}
\]

Then it follows from \( (4.1) - (4.2) \) that

\[
|\phi_0|_\infty \exp(-CC_0) \leq \|\phi\|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq |\phi_0|_\infty \exp(CC_0) \quad \text{for} \quad 0 \leq T < T^*,
\]

where one has used the fact that $|\text{div}u|_\infty \leq |D(u)|_\infty$.

Second, it follows from \( (2.5) \) that

\[
\frac{d}{dt} |\phi|_2 \leq C|\text{div}u|_\infty |\phi|_2,
\]

which, along with Gronwall’s inequality and \( (4.4) \), implies the desired conclusions. \( \square \)

### 4.1. Lower Order Estimates from the Strong Singular Structure \( (2.5) \)

Now we give the basic energy estimate.

**Lemma 4.2.**

\[
\sup_{0 \leq t \leq T} |u(t)|_2^2 + \int_0^T |\phi^c \nabla u|_2^2 dt \leq C \quad \text{for} \quad 0 \leq T < T^*,
\]

where the constant $C > 0$ only depends on \((\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta \) and $T^*$.

**Proof.** Multiplying \( (2.5) \) by $2u$ and integrating over $\mathbb{R}^3$, it follows from Lemma 4.1 that

\[
\frac{d}{dt} |u|_2^2 + 2a\alpha |\phi^c \nabla u|_2^2 + 2a(\alpha + \beta)|\phi^c \text{div}u|_2^2 \\
= \int 2 \left( -u \cdot \nabla u - \nabla \phi + \delta^{-1}\psi \cdot Q(u) \right) \cdot u \\
\leq C \left( |\text{div}u|_\infty |u|_2^2 + |\phi|_2 |\text{div}u|_2 + |\psi|_6 |\phi^c \nabla u|_2 |u|_3 |\phi|_\infty^e \right) \\
\leq \frac{a\alpha}{2} |\phi^c \nabla u|_2^2 + C (|\text{div}u|_\infty |u|_2^2 + |u|_2^2 + 1),
\]

which, along with Gronwall’s inequality and \( (4.1) \), implies that

\[
|u(t)|_2^2 + \int_0^t |\phi^c \nabla u(s)|_2^2 ds \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]

\( \square \)

The next lemma provides a key estimate on $\nabla \phi$ and $\nabla u$. 
Lemma 4.3.

\[ \sup_{0 \leq t \leq T} \left( |\phi^c \nabla u|_2^2 + |\nabla \phi|_2^2 \right)(t) + \int_0^T (|\phi^{2c} L u|_2^2 + |\nabla^2 u|_2^2 + |u_t|_2^2) dt \leq C \]

for \( 0 \leq T < T \), where the constant \( C > 0 \) only depends on \( (\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta \) and \( T \).

**Proof.** Multiplying (2.5) by \((-a\phi^{2c} L u - \nabla \phi)\) and integrating over \( \mathbb{R}^3 \), one has

\[
\frac{d}{dt} \int (a|\phi^c \nabla u|_2^2 + (\alpha + \beta)|\phi^c \text{div} u|_2^2) + \int (a\phi^{2c} L u + \nabla \phi)^2
\]

\[
= -aa \int \phi^{2c} (u \cdot \nabla) u \cdot \nabla \times \text{curl} u + a(2\alpha + \beta) \int \phi^{2c} (u \cdot \nabla) u \cdot \nabla \text{div} u
\]

\[
- \int (u \cdot \nabla) u \cdot \nabla \phi - \int u_t \cdot \nabla \phi + \frac{1-\delta}{\delta} \int \psi \cdot Q(u) \cdot u_t
\]

\[
+ \frac{a}{2} \int \alpha (\phi^{2c}) t |\nabla u|^2 + (\alpha + \beta) \phi^{2c} t |\text{div} u|^2
\]

\[
+ \int \psi \cdot Q(u) \cdot \nabla \phi + a \int (\psi \cdot Q(u)) \cdot \phi^{2c} L u \equiv \sum_{i=1}^{8} L_i,
\]

where one has used the fact that \(-\Delta u + \nabla \text{div} u = \nabla \times \text{curl} u\).

First, it follows from the standard elliptic estimate shown in Lemma 3.1 that

\[
|\nabla^2 u|_2^2 - C|\nabla \phi|_2^2 \leq C - aLu|_2^2 - C|\nabla \phi|_2^2
\]

\[
\leq C|a\phi^{2c} L u + \nabla \phi|_2^2 - C|\nabla \phi|_2^2 \leq C|a\phi^{2c} L u + \nabla \phi|_2^2.
\]

(4.9)

According to momentum equations (2.5)2, one can also obtain that

\[
|u_t|_2 \leq C((a\phi^{2c} L u + \nabla \phi|_2 + |u_3| |\nabla u|_6 + |\psi|_6 |\nabla u|_3)
\]

\[
\leq C((a\phi^{2c} L u + \nabla \phi|_2 + |\nabla u|_2^\frac{1}{3} |\nabla^2 u|_2 + |\nabla u|_2^\frac{1}{2} |\nabla^2 u|_2^\frac{1}{2}).
\]

(4.10)

Second, we start to estimate the right-hand side of (4.8) term by term. Due to

\[
\begin{align*}
|L_1| &= a\alpha \int \phi^{2c} (u \cdot \nabla) u \cdot \nabla \times \text{curl} u \\
&= a\alpha \int \text{curl} u \cdot \nabla \times \phi^{2c} (u \cdot \nabla) u \\
&= a\alpha \int \left( \phi^{2c} \text{curl} u \cdot \nabla \times ((u \cdot \nabla) u) + \text{curl} u \cdot \tilde{g} \right) \\
&= a\alpha \int \left( \phi^{2c} \text{curl} u \cdot \nabla \times (u \times \text{curl} u) - \text{curl} u \cdot \tilde{g} \right) \\
&= a\alpha \int \phi^{2c} \left( \frac{1}{2} |\text{curl} u|^2 \text{div} u - \text{curl} u \cdot D(u) \cdot \text{curl} u \right) \\
&\quad + a\alpha \int \left( -\frac{1}{2} \nabla \phi^{2c} u |\text{curl} u|^2 - \text{curl} u \cdot \tilde{g} \right) \\
&\leq C|D(u)|_\infty |\phi^c \nabla u|_2^2 + C|\psi|_6 |u_3| |\nabla u|_3 |\nabla u|_6 \\
&\leq C|D(u)|_\infty |\phi^c \nabla u|_2^2 + C|\phi^c \nabla u|_2^2 + C(\varepsilon)|\phi^c \nabla u|_2^2.
\end{align*}
\]

(4.11)
where one has used (4.1) and Lemma 4.2, and $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ is given by

\[\tilde{g}_1 = \frac{\partial \phi^{2e}}{\partial x_2} (u \cdot \nabla)u^3 - \frac{\partial \phi^{2e}}{\partial x_3} (u \cdot \nabla)u^2,\]
\[\tilde{g}_2 = \frac{\partial \phi^{2e}}{\partial x_3} (u \cdot \nabla)u^1 - \frac{\partial \phi^{2e}}{\partial x_2} (u \cdot \nabla)u^3,\]
\[\tilde{g}_3 = \frac{\partial \phi^{2e}}{\partial x_1} (u \cdot \nabla)u^2 - \frac{\partial \phi^{2e}}{\partial x_2} (u \cdot \nabla)u^1.\]

Similarly,

\[|L_2| = a(2\alpha + \beta)\left|\int \phi^{2e} (u \cdot \nabla)u \cdot \nabla \text{div}u\right| \leq C|\text{div}u|_{\infty} |\phi^{2e} \nabla u|_2^2 + C|\psi|_{6}|u|_{3}\nabla u|_3|\nabla u|_6 \leq C|D(u)|_{\infty} |\phi^{2e} \nabla u|_2^2 + \epsilon |\nabla^2 u|_2^2 + C(\epsilon) |\phi^{2e} \nabla u|_2^4,\]

\[|L_3| = \left|\int (u \cdot \nabla)u \cdot \nabla \phi\right| \leq C|\nabla u|_{\infty} |\phi^{2e} \nabla u|_2^2 + \epsilon |\nabla^2 u|_2^2 + C(\epsilon) |\phi^{2e} \nabla u|_2^4,\]

\[L_4 = -\int u_t \cdot \nabla \phi + \frac{d}{dt} \int \phi \text{div}u - \int \phi_t \text{div}u = \frac{d}{dt} \int \phi \text{div}u + \int u \cdot \nabla \phi \text{div}u + (\gamma - 1) \int \phi (\text{div}u)^2 \leq \frac{d}{dt} \int \phi \text{div}u + C|\text{div}u|_{\infty} |u|_2 |\nabla \phi|_2 + C |\phi|_{-2} |\phi^{2e} \nabla u|_2^2,\]

\[L_5 = \frac{1}{\delta} \int \psi \cdot Q(u) \cdot u_t \leq C |\phi|_{-2} |u|_2 |\phi^{2e} \nabla u|_2^2 |\nabla^2 u|_2^2 |\psi|_6,\]

\[L_6 = \frac{a}{2} \int \alpha((\phi^{2e})_t |\nabla u|_2^2 + (\alpha + \beta)(\phi^{2e})_t |\text{div}u|_2^2) \leq \frac{a}{2} \int (u_t \cdot \nabla \phi^{2e} - (\delta - 1) \phi^{2e} \text{div}u)(\alpha |\nabla u|_2^2 + (\alpha + \beta) |\text{div}u|_2^2) \leq C |\psi|_{6} |\nabla u|_{3} |\nabla u|_{6} + C |D(u)|_{\infty} |\phi^{2e} \nabla u|_2^2 \leq C |\psi|_{6} |\nabla u|_{3} |\nabla u|_{6} + C |\phi^{2e} \nabla u|_2^2 + C(\epsilon) |\phi^{2e} \nabla u|_2^4,\]

\[L_7 = \int \psi \cdot Q(u) \cdot \nabla \phi \leq C |\phi|_{-2} |\nabla \phi|_2 |\phi^{2e} \nabla u|_2^3 |\nabla^2 u|_2^2 |\psi|_6 \leq C |\phi^{2e} \nabla u|_2^3 + C(\epsilon) (|\phi^{2e} \nabla u|_2^3 + |\nabla \phi|_2^3),\]

\[L_8 = \int \psi \cdot Q(u) \cdot \phi^{2e} Lu \leq C |\psi|_{-2} |\phi^{2e} Lu|_2 |\phi^{2e} \nabla u|_2^3 |\nabla^2 u|_2^2 \leq C(\epsilon) |\phi^{2e} \nabla u|_2^3 + C(\epsilon) (|\phi^{2e} \nabla u|_2^3 + |\phi^{2e} Lu|_2^3),\]

where $\epsilon > 0$ is a sufficiently small constant.
It follows from (4.8)–(4.12) that
\[
\frac{d}{dt} \int \left( \frac{1}{2} \alpha \phi \nabla u^2 + \frac{1}{2} (\alpha + \beta) |\phi|^2 \text{div} u^2 - \phi \text{div} u \right) + C |\nabla u|^2 + \frac{a}{2} |\phi^2 L u|^2 \\
\leq C (|\phi \nabla u|^2 + |\nabla \phi|^2 + |D(u) + |D(u)|_\infty + |\phi^2 \nabla u|^2).
\tag{4.13}
\]

Second, applying \( \nabla \) to (2.5) and multiplying by \( (\nabla \phi)^T \), one has
\[
(\langle \nabla \phi \rangle^2_t + \text{div}(\langle \nabla \phi \rangle^2 u + (\gamma - 2)|\nabla \phi |^2 \text{div} u \\
= -2(\nabla \phi)^T \nabla u(\nabla \phi) - (\gamma - 1)\phi \nabla \phi \cdot \nabla \text{div} u \\
= -2(\nabla \phi)^T D(u)(\nabla \phi) - (\gamma - 1)\phi \nabla \phi \cdot \nabla \text{div} u.
\tag{4.14}
\]

Integrating (4.14) over \( \mathbb{R}^3 \), it follows from Lemma 4.1 that
\[
\frac{d}{dt} |\nabla \phi|^2 \leq C(\epsilon(|D(u)|_\infty + 1)|\nabla \phi|^2 + \epsilon |\nabla^2 u|^2.
\tag{4.15}
\]

Adding (4.15) to (4.13), it follows from Hölder’s inequality, Young’s inequality and Gronwall’s inequality that
\[
|\phi \nabla u(t)|^2 + |\nabla \phi(t)|^2 + \int_0^t (|\nabla u|^2 + |\phi^2 L u|^2)ds \leq C \quad \text{for} \quad 0 \leq t \leq T,
\]
which, together with (4.10), implies that
\[
\int_0^t |u_t|^2 ds \leq C \int_0^t (|\phi^2 L u|^2 + |\nabla u|^2 + |\nabla \phi|^2 + |\nabla u|^2 + |\nabla |^2)ds \leq C.
\]

\[\square\]

**Lemma 4.4.**
\[
\sup_{0 \leq t \leq T} \left( |u_t|^2 + |u|^2 + |\phi^2 L u|^2 \right)(t) + \int_0^T (|\phi \nabla u_t|^2 + |u|^2)dt \leq C,
\]
for \( 0 \leq T < \infty \), where the constant \( C > 0 \) only depends on \( (\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta \) and \( T \).

**Proof.** It follows from Lemma 3.1, equations (2.2)_3 and (2.5)_2 that
\[
|u|_D^2 + |\phi^2 L u|^2 \leq C \left( |u_t|^2 + |u|^2 |\nabla u|^2 + |\nabla \phi|^2 + |\nabla \phi|^2 \right),
\]
which, along with Lemmas 4.1–4.3, implies that
\[
|u|_D^2 + |\phi^2 L u|^2 \leq C(1 + |u_t|^2).
\tag{4.16}
\]

Next, differentiating (2.5)_3 with respect to \( t \), it reads
\[
u_{tt} + a\phi^{2e} L u_t = -(u \cdot \nabla u)_t - \nabla \phi_t - a\phi^{2e} L u + (\psi \cdot Q(u))_t.
\tag{4.17}
\]

Multiplying (4.17) by \( u_t \) and integrating over \( \mathbb{R}^3 \), one has
\[
\frac{1}{2} \frac{d}{dt} |u_t|^2 + a|\phi \nabla u_t|^2 + (\alpha + \beta) |\phi \text{div} u_t|^2 \\
= -\int \left( (u \cdot \nabla u + \nabla \phi)_t + a\phi^{2e} L u - \psi \cdot Q(u) - \frac{1}{\beta} \psi \cdot Q(u)_t \right) \cdot u_t \equiv \sum_{i=9}^{13} L_i.
\tag{4.18}
\]
Based on (4.1), Lemmas 4.1–4.3 and (4.16), one can obtain that

\[ L_9 = -\int (u \cdot \nabla u)_t \cdot u_t = -\int ((u_t \cdot \nabla)u + (u \cdot \nabla)u_t) \cdot u_t \]
\[ = -\int (u_t \cdot D(u) - u_t - \frac{1}{2}(u_t)^2 \text{div} u) \leq C|D(u)|_\infty |u_t|^2, \]
\[ L_{10} = -\int \nabla \phi_t \cdot u_t = \int \phi_t \text{div} u_t \]
\[ = -\frac{(\gamma - 1)}{2} \frac{d}{dt} \int \phi (\text{div} u)^2 - \frac{(\gamma - 1)}{2} \int u \cdot \nabla \phi (\text{div} u)^2 \]
\[ - \frac{(\gamma - 1)^2}{2} \int \phi (\text{div} u)^3 - \int u \cdot \nabla \phi \text{div} u_t \]
\[ \leq -\frac{(\gamma - 1)}{2} \frac{d}{dt} \int \phi (\text{div} u)^2 + C|u|_6 |\nabla \phi|_6 |\nabla u|_2 |\nabla u|_6 \]
\[ + C|\phi|_\infty D(u)|_\infty |\nabla u|^2_2 + C|\nabla \phi|_6 |\nabla u|_2 \]
\[ \leq -\frac{(\gamma - 1)}{2} \frac{d}{dt} \int \phi (\text{div} u)^2 + C(|D(u)|_\infty + |u_t|^2) + \frac{a\alpha}{4} |\phi|^2 \nabla u_t^2, \]

\[ L_{11} = -\int a \phi_t^2 e^{Lu} \cdot u_t = \int a (u \cdot \nabla \phi^2 + (\delta - 1) \phi^2 \text{div} u)Lu \cdot u_t \]
\[ \leq C|u|_6 |\psi|_6 |Lu|_2 |u_t|^2 + C|\text{div} u|_\infty |\phi^2 Lu|_2 |u_t|_2 \]
\[ \leq C(1 + |D(u)|_\infty) (|u_t|_2^2 + 1) + \frac{a\alpha}{4} |\phi|^2 \nabla u_t^2, \]
\[ L_{12} + L_{13} = \int \left( \psi_t \cdot Q(u) + \frac{1}{\delta} \psi \cdot Q(u)_t \right) \cdot u_t \]
\[ = \int \frac{1}{\delta} \psi \cdot Q(u)_t \cdot u_t - a\delta \int \phi^2 \nabla \text{div} u \cdot Q(u) \cdot u_t \]
\[ + \int \psi \cdot u \text{div} (Q(u) \cdot u_t) - (\delta - 1) \int \text{div} u \psi \cdot Q(u) \cdot u_t \]
\[ \leq C |\psi|_6 (|\nabla u|_2 |u_t|^2 + |u_t|_2^2 + |u|_6 |\nabla^2 u|_2 |u_t|_6 + |u|_6 |Q(u)|_6 |\nabla u_t|_2) \]
\[ + C |\psi|_6 |u_t|_2 |Q(u)|_6 |\nabla u|_6 - a\delta \int \phi^2 \nabla \text{div} u \cdot Q(u) \cdot u_t \]
\[ \leq C(|u_t|^2_2 + 1) + \frac{a\alpha}{4} |\phi|^2 \nabla u_t^2 + L^*, \]

where, via integration by parts, the last term \( L^* \) in \( L_{12} + L_{13} \) can be estimated as follows:

\[ L^* = -a\delta \int \phi^2 \nabla \text{div} u \cdot Q(u) \cdot u_t \]
\[ = a\delta \int (\phi^2 \text{div} u Lu \cdot u_t + \phi^2 \text{div} Q(u) : \nabla u_t + \text{div} u \phi^2 \cdot Q(u) \cdot u_t) \]
\[ \leq C |\phi^2 Lu|_2 |\text{div} u|_\infty |u_t|^2 + C |\text{div} u|_\infty |\phi^2 \nabla u_t|^2 + |\phi^2 \nabla u|^2 \]
\[ + C |\psi|_6 |u_t|_2 |\text{div} u|_6 |\nabla u|_6 \]
\[ \leq C(|u_t|^4_2 + 1) + \frac{a\alpha}{4} |\phi|^2 \nabla u_t^2 + C |\text{div} u|^2_\infty. \]

It follows from Lemma 3.1 and equations (2.2) that

\[ |\nabla^2 u|_6 \leq C |\varphi|_\infty (|u_t|^2_6 + |u \cdot \nabla u|_6 + |\nabla \phi|_6 + |\psi \cdot Q(u)|_6) \]
\[ \leq C(1 + |\nabla u_t|^2 + |\nabla \phi|^2 + |\nabla u_t|^2 |\nabla^2 u|_6), \]

(4.21)
where one has used the fact that $|\nabla u|_\infty \leq C |\nabla u|_0^2 |\nabla^2 u|_0^2$.

Then, according to the Young’s inequality, one has

$$|\nabla^2 u|_0 \leq C (1 + |\nabla u|_2 + |u_t|_2),$$

which, along with (4.20), implies that

$$|\text{div} u|_\infty^2 \leq C (1 + |u_t|_2^2) + \frac{a\alpha}{4} |\phi^\varepsilon \nabla u|_2^2,$$

$$L^* \leq C (|u_t|_2^2 + 1) + \frac{1}{4} a\alpha |\phi^\varepsilon \nabla u|_2^2.$$

Lemma 4.5.

Letting $\text{div} u|_\infty^2 \leq C (1 + |D(u)|_\infty + |u_t|_2^2)(|u_t|_2^2 + 1)$.

Integrating (4.24) over $(\tau, t)$ $(\tau \in (0, t))$, one has

$$|u_t|_2^2 + |\text{div} u(t)|_2^2 + \int_\tau^t |\phi^\varepsilon \nabla u_t(s)|_2^2 ds$$

$$\leq |u_t(\tau)|_2^2 + C \int_\tau^t (1 + |D(u)|_\infty + |u_t|_2^2)(|u_t|_2^2 + 1)ds.$$

It follows from momentum equations (2.5)2 that

$$|u_t(\tau)|_2 \leq C (|u|_\infty |\nabla u|_2 + |\nabla \phi|_2 + |\phi^2 L u|_2 + |\psi|_\infty |\nabla u|_2) (\tau),$$

which, along with the definition of the regular solution and the assumption (1.14), implies that

$$\limsup_{\tau \to 0} |u_t(\tau)|_2 \leq C (|u_0|_\infty |\nabla u_0|_2 + |\nabla \phi_0|_2 + |g_2|_2 + |\psi_0|_\infty |\nabla u_0|_2) \leq C.$$

Letting $\tau \to 0$ in (4.25), applying Gronwall’s inequality and (4.1), we arrive at

$$|u_t(t)|_2^2 + |\phi^2 L u(t)|_2^2 + |u(t)|_2^2 + \int_0^t (|\phi^\varepsilon \nabla u_t|^2_2 + |u|^2_{L^2(0, t)}) ds \leq C \quad \text{for} \quad 0 \leq t \leq T.$$

This completes the proof of this lemma.

Lemma 4.5.

$$\sup_{0 \leq t \leq T} \left( \|g\|_{H^1 \cap D^{1,6}} + \|\phi_t, g_t, \phi_t\|_{L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)} \right)(t) \leq C \quad \text{for} \quad 0 \leq T < T,$$

where the constant $C > 0$ only depends on $(\rho_0, u_0)$, $C_0$, $\alpha$, $\beta$, $A$, $\gamma$, $\delta$ and $T$.

The proof of this lemma can be directly obtained from the equations (2.2)1, (2.2)2 and (2.5)1, and the conclusions of Lemmas 4.1–4.4. Here we omit its details.

4.2. Higher Order Estimates from the “Degenerate”–“Weak-Singular” Structure (2.2)

Lemma 4.4 implies that

$$\int_0^t |\nabla u(\cdot, s)|_\infty^2 ds \leq C \quad \text{for} \quad 0 \leq t < T,$$

where $C > 0$ is some finite constant. Noting that (2.5) is essentially a hyperbolic-singular parabolic system, it is very hard to derive other higher order estimates for the regularity of the regular solutions via using this structure directly. Indeed, we need to ask for help from the so-called ”Degenerate”–”Weak-Singular” structure (2.2), which will be shown in the following 3 lemmas.
Lemma 4.6.
\[
\sup_{0 \leq t \leq T} \left( \| (g, \phi) \|^2_{D^2} + |f|^2_{D^1} + \| (g_t, \phi_t) \|^2_1 + |f_t|^2_{D^2} \right) (t) \\
+ \int_0^T \left( |u|^2_{D^3} + |\phi|^2_{D^2} + |g_t|^2_{D^2} \right) dt \leq C \quad \text{for} \quad 0 \leq T < T,
\]
where the constant \( C > 0 \) only depends on \((\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta \) and \( T \).

Proof. First, it follows from equations (2.2)_3, Lemmas 3.1 and 4.1–4.5 that
\[
|u|_{D^3} \leq C(|\varphi u_t|_{D^1} + |\varphi u \cdot \nabla u|_{D^1} + |\nabla g|_{D^1} + |f \cdot Q(u)|_{D^1}) \\
\leq C(1 + |u_t|_{D^1} + |\phi|_{D^2} + |\nabla^2 u|_{3} + |\nabla f|_{2} |\nabla u|_{\infty}). \tag{4.29}
\]
where one has used the following relation
\[
\nabla^2 g = C_1 \rho^{1-\delta} \nabla^2 \phi + C_2 \nabla \phi \otimes \nabla \phi, \tag{4.30}
\]
for two constants \( C_1 > 0 \) and \( C_2 > 0 \). Then it follows from Young’s inequality that
\[
|u|_{D^3} \leq C(1 + |u_t|_{D^1} + |\phi|_{D^2} + |\nabla f|_{2}). \tag{4.31}
\]

Next, applying \( \nabla \) to (2.2)_4, multiplying the resulting equations by \( 2\nabla f \) and integrating over \( \mathbb{R}^3 \), then according to (4.31), one has
\[
\frac{d}{dt} |\nabla f|^2_{D^1} \leq C|\nabla u|_{\infty}|\nabla f|^2_{D^1} + C|\nabla^3 u|_{2} |\nabla f|_{2} + |\nabla^2 u|_{3} |f|_{6} |\nabla f|_{2} \tag{4.32}
\]
\[
\leq C(1 + |\nabla u|_{\infty})|f|^2_{D^1} + C(1 + |\phi|^2_{D^2} + |u_t|^2_{D^1}).
\]

On the other hand, applying \( \nabla^2 \) to (2.5)_1, multiplying the resulting equations by \( 2\nabla^2 \phi \) and integrating over \( \mathbb{R}^3 \), it follows from Lemmas 4.1–4.4 and (4.31) that
\[
\frac{d}{dt} |\phi|^2_{D^2} \leq C|\nabla u|_{\infty}|\phi|^2_{D^2} + C|\nabla |_{6} |\phi|_{D^2} |\nabla^2 u|_{3} + C|\nabla |_{\infty} |\phi|_{D^2} |\nabla^3 u|_{2} \tag{4.33}
\]
\[
\leq C(1 + |\nabla u|_{\infty})(|\phi|^2_{D^2} + |f|^2_{D^1}) + C(1 + |\nabla u_t|^2_{D^1}).
\]
which, together with (4.32), gives that
\[
\frac{d}{dt} (|\phi|^2_{D^2} + |f|^2_{D^1}) \leq C(1 + |\nabla u|_{\infty})(|\phi|^2_{D^2} + |f|^2_{D^1}) + C(1 + |\nabla u_t|^2_{D^1}). \tag{4.34}
\]
Then it follows from Gronwall’s inequality, (4.34) and (4.28) that
\[
|\phi(t)|^2_{D^2} + |f(t)|^2_{D^1} + \int_0^t |u(s)|^2_{D^3} ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \tag{4.35}
\]

Finally, according to (4.30) and the following relations
\[
\begin{align*}
\phi_t &= -\nabla(u \cdot f) - \nabla \div u, \\
g_t &= -u \cdot \nabla g - (\gamma - \delta) g \div u, \\
\phi_{tt} &= -(u \cdot \nabla) \phi_t - (\gamma - \delta)(\phi \div u)_t, \\
g_{tt} &= -(u \cdot \nabla) g_t - (\gamma - \delta)(g \div u)_t,
\end{align*}
\]
we conclude the proof of this lemma. \( \square \)

In order to obtain higher order regularity, we need the following improved estimate.

Lemma 4.7.
\[
\sup_{0 \leq t \leq T} ([u_t]^2_{D^1} + |u|^2_{D^3}) (t) + \int_0^T (|\nabla u_t|^2_{D^3} + |u_t|^2_{D^2}) dt \leq C \quad \text{for} \quad 0 \leq T < T,
\]
where the constant \( C > 0 \) only depends on \((\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta \) and \( T \).
Proof. First,
\[ aL u_t = -\varphi u_{tt} - \varphi (u \cdot \nabla u)_t - \varphi_t (u_t + u \cdot \nabla u) - \nabla g_t + (f \cdot Q(u))_t, \]  
(4.37)

Lemmas 3.1 and 4.5 yield
\[ |u_t|_{D^2} \leq C(|\varphi u_{tt}|_2 + |\varphi (u \cdot \nabla u)_t|_2 + |\varphi_t u_t|_2 + |\varphi_t u \cdot \nabla u|_2) + |\nabla g_t|_2 + |(f \cdot Q(u))_t|_2 \]
\[ \leq C(1 + |\sqrt{\varphi} u_{tt}|_2 + |\nabla u_t|_2 + |\nabla u_t|_3 + |\nabla u|_\infty), \]
(4.38)

which implies, with the help of Young’s inequality, that
\[ |u_t|_{D^2} \leq C(1 + |\sqrt{\varphi} u_{tt}|_2 + |\nabla u_t|_2 + |u|_{D^2,a}). \]
(4.39)

Now, multiplying (4.17) by \(u_{tt}\) and integrating over \(\mathbb{R}^3\), one has
\[ \frac{a}{2} \frac{d}{dt} \left( |\nabla u_t|^2 + (\alpha + \beta) |\text{div} u_t|^2 \right) + |\sqrt{\varphi} u_{tt}|^2 \]
\[ = \int \left( -\varphi (u \cdot \nabla u)_t - \varphi_t (u_t + u \cdot \nabla u) - \nabla g_t + (f \cdot Q(u))_t \right) u_{tt} \equiv \sum_{i=14}^{18} L_i. \]
(4.40)

For the terms \(L_{14} - L_{18}\), it follows from Lemmas 4.1–4.6 and (4.39) that
\[ L_{14} = -\int \varphi (u \cdot \nabla u)_t \cdot u_{tt} \]
\[ \leq C |\varphi|_\infty^2 (|u_t|_6 |\nabla u|_3 + |u|_\infty |\nabla u_t|_2) |\sqrt{\varphi} u_{tt}|_2 \]
\[ \leq C |\nabla u_t|^2 + \frac{1}{10} |\sqrt{\varphi} u_{tt}|^2, \]

\[ L_{15} = -\int \varphi u_t \cdot u_{tt} = \int (u \cdot \nabla \varphi + (1 - \delta) \varphi \text{div} u) u_t \cdot u_{tt} \]
\[ \leq C (|\varphi|^2 |u|_6 |\psi|_6 + |\varphi|^2 |\nabla u|_6) |u_t|_6 |\sqrt{\varphi} u_{tt}|_2 \]
\[ \leq C |\nabla u_t|^2 + \frac{1}{10} |\sqrt{\varphi} u_{tt}|^2, \]

\[ L_{16} = -\int \varphi (u \cdot \nabla) u \cdot u_{tt} \]
\[ = \int (u \cdot \nabla \varphi + (1 - \delta) \varphi \text{div} u)(u \cdot \nabla) u \cdot u_{tt} \]
\[ \leq C (|\varphi|^2 |u|_\infty |\psi|_6 + |\varphi|^2 |\nabla u|_6) |u_t|_6 |\sqrt{\varphi} u_{tt}|_2 \]
\[ \leq C |\nabla u_t|^2 + \frac{1}{10} |\sqrt{\varphi} u_{tt}|^2, \]

\[ L_{17} = -\int \nabla g_t \cdot u_{tt} = \frac{d}{dt} \int g_t \text{div} u_t - \int g_t \text{div} u_t \]
\[ \leq \frac{d}{dt} \int g_t \text{div} u_t + C |\nabla u_t|^2 |g_t|_2 \leq \frac{d}{dt} \int g_t \text{div} u_t + C (|\nabla u_t|^2 + |g_t|^2), \]

\[ L_{18} = \int (f \cdot Q(u))_t \cdot u_{tt} = \int f \cdot Q(u)_t \cdot u_{tt} + \int f_t \cdot Q(u) \cdot u_{tt} \]
\[ \leq C |\varphi|^2 |\psi|_6 |\nabla u_t|_3 |\sqrt{\varphi} u_{tt}|_2 + \frac{d}{dt} \int f_t \cdot Q(u) \cdot u_t \]
\[ - \int f_t \cdot Q(u) \cdot u_t - \int f_t \cdot Q(u_t) \cdot u_t \]
\[ \leq C |\varphi|^2 |\psi|_6 |\nabla u_t|_3 |\sqrt{\varphi} u_{tt}|_2 + C |f_t|_2 |\nabla u_t|_6 |u_t|_3 + \frac{d}{dt} \int f_t \cdot Q(u) \cdot u_t \]
\[ - \int (u \cdot f)_t \text{div}(Q(u) \cdot u_t) + \int \nabla \text{div} u_t \cdot Q(u) \cdot u_t \]
where the constant which, along with Lemma 4.1 and (4.35), implies that
\[
\leq \frac{d}{dt} \int f_t \cdot Q(u) \cdot u_t + C(\|\varphi_t\|_{D}^2 + |f|_{D^2}^2 + |\varphi|_{D^2}^2 + |\varphi_t|_{D}^2 + |\phi_t|_{D}^2 + |\phi|_{D}^2) dt
\]
\[
+ C|u_t|_{D}^2 + |\varphi|_{D^2}^2 + |\varphi_t|_{D}^2 + |\phi|_{D}^2 + |f|_{D^2}^2
\]
\[
\leq C|u_t|_{D}^2 + C(1 + |\varphi|_{D}^2 + |\varphi_t|_{D}^2 + |\phi|_{D}^2 + |f|_{D^2}^2).
\]
(4.41)

Therefore, (4.40)–(4.41) imply that
\[
\frac{d}{dt} \left( |\varphi_t|_{D}^2 + \int |\nabla u|_{D}^2 dt \right)
\leq C(1 + |\varphi|_{D}^2 + |\varphi_t|_{D}^2 + |\phi_t|_{D}^2),
\]
which, upon integrating over \((\tau, t)\), along with Lemma 4.6, yields
\[
|\nabla u_t(t)|_{D}^2 + \int_{\tau}^{t} |\nabla u|_{D}^2 ds \leq C + |\nabla u(\tau)|_{D}^2, \quad 0 \leq t \leq T,
\]
(4.43)
where one has used the fact that for any \(\epsilon > 0\),
\[
\int (g_t \cdot \nabla u + f_t \cdot Q(u) \cdot u_t) \leq \epsilon |\nabla u_t|_{D}^2 + C.
\]
(4.44)

On the other hand, it follows from the momentum equations (2.5) that
\[
|\nabla u_t(\tau)|_{D}^2 \leq (|\nabla (u \cdot \nabla u + \nabla \phi + \psi \cdot Q(u))|_{D})(\tau).
\]
(4.45)

Then by the assumption (1.14), one has
\[
\limsup \frac{|\nabla u_t(\tau)|_{D}^2}{\tau^2} \leq C(|\nabla (u_0 \cdot \nabla u_0)|_{D}^2 + |\nabla \phi_0|_{D}^2)
\]
\[
+ C|\nabla (\psi_0 \cdot Q(u_0))|_{D}^2 + |\rho_0 \cdot g_3|_{D}^2 \leq C.
\]
(4.46)

Letting \(\tau \to 0\) in (4.43), we finally prove that
\[
|\nabla u_t(t)|_{D}^2 + \int_{0}^{t} |\nabla u|_{D}^2 ds \leq C, \quad 0 \leq t \leq T.
\]
(4.47)

The rest of desired estimates follows quickly from (4.31), (4.39) and (4.47). □

It remains to prove the following lemma for the required regularity estimate.

**Lemma 4.8.**
\[
\sup_{0 \leq t \leq T} (|\varphi|_{D}^2 + |f|_{D^2}^2 + |\varphi|_{D^2}^2 + |\varphi_t|_{D}^2 + |\phi_t|_{D}^2)(t)
\]
\[
+ \sup_{0 \leq t \leq T} |f_t(t)|_{D^2}^2 + \int_{0}^{T} (|u|_{D^4}^2 + |\varphi_t|_{D}^2) dt \leq C \quad \text{for} \quad 0 \leq T < T,
\]
where the constant \(C > 0\) only depends on \((\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta, \) and \(T\).

**Proof.** First, for the \(D^2\) estimate of \(\varphi\), it follows from direct calculations that
\[
\nabla^2 \varphi = -2e \varphi^2 \nabla \log \phi = 4e^2 \varphi^2 \nabla \log \phi \otimes \nabla \log \phi - 2e \varphi^2 \nabla^2 \log \phi,
\]
which, along with Lemma 4.1 and (4.35), implies that
\[
|\nabla^2 \varphi|_{3} \leq C(|f|_{D^2}^2 + \varphi^2 \nabla \log \phi)_{3} \leq C(1 + |f|_{D^2}^2) \quad \text{for} \quad 0 \leq t \leq T.
\]

It follows from equations (2.2) and Lemma 3.1 and the relation (4.37) that
\[
|u|_{D^4} \leq C(|\varphi u_t|_{D^2} + |\varphi u \cdot \nabla u|_{D^2} + |\nabla g|_{D^2} + |f \cdot Q(u)|_{D^2})
\]
\[
\leq C(1 + |u|_{D^2} + |\varphi|_{D^2} + |\nabla^2 f|_{2})
\]
(4.48)
Next, applying $\nabla^2$ to (2.2)$_4$, multiplying the resulting equations by $2\nabla^2 f$ and then integrating over $\mathbb{R}^3$, one has
\begin{equation}
\frac{d}{dt} |\nabla^2 f|^2 \leq C (|f|_{D^2}^2 + |\nabla^4 u|_2 |f|_{D^2} + 1)
\leq C (|f|_{D^2}^2 + |\phi|_{D^3} + |u|_{D^2}^2 + 1). \tag{4.49}
\end{equation}

On the other hand, applying $\nabla^3$ to (2.5)$_1$, multiplying the resulting equations by $2\nabla^3 \phi$ and then integrating over $\mathbb{R}^3$, one has
\begin{equation}
\frac{d}{dt} |\phi|^2_{D^3} \leq C (|\phi|^2_{D^3} + |\nabla^4 u|_2 |\phi|_{D^3} + 1)
\leq C (|\phi|^2_{D^3} + |u|_{D^2}^2 + |f|_{D^2}^2 + 1), \tag{4.50}
\end{equation}
which, together with (4.48)–(4.49), gives that
\begin{equation}
|\phi(t)|_{D^3}^2 + |f(t)|_{D^2}^2 + \int_0^t |u|_{D^2}^2 ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \tag{4.51}
\end{equation}

Finally, the rest of the estimates follows from the relation (4.36) and
\[ \varphi_{tt} = -u_t \cdot \nabla \varphi - u \cdot \nabla \varphi_t - (1 - \delta)\varphi_t \text{div} u - (1 - \delta) \varphi \text{div} u_t. \]

\section*{4.3. Higher Order Estimates from the Strong Singular Structure (2.5)}

\textbf{Lemma 4.9.}
\[ \text{ess sup}_{0 \leq t \leq T} (|\psi|^2_{D^1 \cap D^2} + |\psi_t|^2_{D^1}) + |\phi^{2e} \nabla^2 u|_{D^1}^2 (t) + \int_0^T |\phi^{2e} \nabla^2 u|_{D^2}^2 dt \leq C \]
for $0 \leq T < \bar{T}$, where the constant $C > 0$ only depends on $(\rho_0, u_0)$, $C_0$, $\alpha$, $\beta$, $A$, $\gamma$, $\delta$ and $\bar{T}$.

\textbf{Proof.} First, set $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3)^T$ ($1 \leq |\varsigma| \leq 2$ and $\varsigma_i = 0, 1, 2$). Applying $\partial^2_x$ to (2.5)$_3$, multiplying by $2\partial^2_x \psi$ and then integrating over $\mathbb{R}^3$, one can get
\begin{equation}
\frac{d}{dt} |\partial^2_x \psi|^2 \leq \left( \sum_{i=1}^{3} |\partial_i A_i|_{\infty} + |B|_{\infty} \right) |\partial^2_x \psi|^2 + |\Theta|_2 |\partial^2_x \psi|_2, \tag{4.52}
\end{equation}
where
\[ \Theta = \partial^2_x (B \psi) - B \partial^2_x \psi + \sum_{i=1}^{3} (\partial^2_x (A_i \partial_i \psi) - A_i \partial_i \partial^2_x \psi) + a \delta \partial^2_x (\phi^{2e} \nabla \text{div} u). \]

For $|\varsigma| = 1$, it is easy to obtain
\begin{equation}
|\Theta|_2 \leq C (|\nabla^2 u|_2 |\psi|_{\infty} + |\nabla u|_{\infty} |\nabla \psi|_2 + |\phi^{2e} \nabla^2 u|_{D^1}). \tag{4.53}
\end{equation}

Similarly, for $|\varsigma| = 2$, one has
\begin{equation}
|\Theta|_2 \leq C (|\nabla u|_{\infty} |\nabla^2 \psi|_2 + |\nabla^2 u|_3 |\nabla \psi|_6 + |\nabla^3 u|_2 |\psi|_{\infty} + |\phi^{2e} \nabla^2 u|_{D^2}). \tag{4.54}
\end{equation}

It follows from (4.52)–(4.54) and the Gagliardo–Nirenberg inequality that
\begin{equation}
\frac{d}{dt} \|\psi(t)\|_{D^1 \cap D^2} \leq C \|\psi(t)\|_{D^1 \cap D^2} + C \|\phi^{2e} \nabla^2 u\|_{D^1 \cap D^2}, \tag{4.55}
\end{equation}
where $C > 0$ only depends on $(\rho_0, u_0)$, $C_0$, $\alpha$, $\beta$, $A$, $\gamma$, $\delta$ and $\bar{T}$.\[ \square \]
Second, for the estimates of $\|\phi^{\alpha}\nabla^2 u\|_{D^1 \cap D^2}$, it follows from direct calculations, Lemmas 3.3–3.4 and 4.1–4.8 that
\[
|\phi^{\alpha} \nabla^2 u|_{D^1} \leq C(1 + |\phi^{\alpha} \nabla^2 u|_{D^2} + |\nabla \phi|_{D^2} + |\nabla^2 \phi|_{D^2} + |\nabla^3 \phi|_{D^2})
\]
\[
\leq C(1 + |\nabla \phi|_{D^2} + |f|_{D^2} + |G(\phi, \partial_k u)|_{D^2})
\]
\[
\leq C(1 + |\nabla \phi|_{D^2} + |\psi(t)|_{D^1 \cap D^2}),
\]
(4.56)
\[
|\phi^{\alpha} \nabla^2 u|_{D^2} \leq C(|\nabla^2 H|_{D^2} + |f \cdot H|_{D^2} + |f|^2 |H|_{D^2} + |G(\phi^{\alpha}, \nabla^2 u)|_{D^2})
\]
\[
\leq C(1 + |u_t|_{D^2} + |\psi(t)|_{D^1 \cap D^2}).
\]

According to (4.39), one has
\[
|u_t|_{D^2} \leq C(1 + |\nabla u_t|_{D^2} + |\nabla u_t|_{D^2}) \leq C(1 + |\nabla u_t|_{D^2}),
\]
(4.57)
which, together with (4.55)–(4.56), implies that
\[
\frac{d}{dt}|\psi(t)|_{D^1 \cap D^2} \leq C|\psi(t)|_{D^1 \cap D^2} + C(1 + |\nabla u_t|_{D^2}).
\]

Then according to Gronwall’s inequality, one has
\[
|\psi(t)|_{D^1 \cap D^2} \leq C\left(1 + \int_0^t |\nabla u_t|_{D^2} \right) \exp(Ct) \leq C(T).
\]
(4.58)

Second, according to equations (2.5)3, for $0 \leq t \leq T$, it holds that
\[
|\nabla \psi(t)|_{D^2} \leq C(|u|_{D^2} + |\nabla u|_{D^2} + |\nabla^2 u|_{D^2} + |\nabla^3 u|_{D^2} + |\phi^{\alpha} \nabla^2 u|_{D^1}) \leq C.
\]
(4.59)

Lemma 4.10.
\[
\sup_{0 \leq t \leq T} (|\phi^{\alpha} \nabla^2 u|_{2} + |\omega|_{4})(t) \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]

where the constant $C > 0$ only depends on $(\rho_0, u_0)$, $C_0$, $\alpha$, $\beta$, $A$, $\gamma$, $\delta$ and $T$.

Proof. First, according to Lemmas 3.2 and 4.7–4.9, one has
\[
|\phi^{\alpha} \partial_k u|_{D^2} \leq C(|\phi^{\alpha} \nabla^2 u|_{D^2} + |\nabla \phi|_{D^2} + |\phi^{\alpha} \partial_k H|_{D^2} + |G(\phi^{\alpha}, \partial_k u)|_{D^2}) \leq C,
\]

Then it follow from Lemmas 4.3 and 5.12 that
\[
|\phi^{\alpha} \nabla u|_{D^1} + |\phi^{\alpha} \nabla^2 u|_{2} \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]
(4.60)
Second, since the equations (3.14) holds almost everywhere, one can multiply (3.14) by $4\omega |\omega|^2$ on its both sides and integrate with respect to $x$ over $\mathbb{R}^3$, then one has
\[
\frac{d}{dt} \int |\omega|^2 = \int \sum_{i=1}^3 A_i A_i |\omega|^4 - \int 4 \left( A^*(u) \omega + \frac{\delta - 1}{2} \sqrt{a} |\phi^{\alpha} \nabla \text{div} u| \right) \cdot |\omega|^2 \leq C(\nabla u|_{D^2}^2 + C\omega^3 |\phi^{\alpha} \nabla \text{div} u|_{D^2}^2 \leq C(\omega^4 + 1),
\]
(4.61)
which, along with Gronwall’s inequality, implies the desired estimate.

Lemma 4.11.
\[
\text{ess sup}_{0 \leq t \leq T} (|\phi^{\alpha} \nabla u|_{2}^2 + |\phi^{\alpha} \nabla (\phi^{\alpha} L u)|_{2}^2)(t) + \int_0^T |u_t|_{2}^2 + |\phi^{\alpha} L u|_{2}^2 \, dt \leq C,
\]
for $0 \leq T < T$, where the constant $C > 0$ only depends on $(\rho_0, u_0)$, $C_0$, $\alpha$, $\beta$, $A$, $\gamma$, $\delta$ and $T$. 

Proof. Multiplying (4.17) by $u_{tt}$ and integrating over $\mathbb{R}^3$ give

$$\frac{1}{2} \frac{d}{dt} \left( a\alpha |\phi^c \nabla u_t|^2 + a(a + \beta)|\phi^c \text{div} u_t|^2 \right) + |u_{tt}|^2 = \int \left( -(a \cdot \nabla u)_t - \nabla \phi_t - a\phi^c \nabla Lu - a\nabla \phi^c \cdot Q(u) \right) \cdot u_{tt}$$

$$+ \int \left( \frac{a}{2} \phi^{2c} (a|\nabla u_t|^2 + (a + \beta)|\text{div} u_t|^2) + (\psi \cdot Q(u))_t \cdot u_{tt} \right) = \sum_{i=19}^{24} L_i. \tag{4.62}$$

For the terms $L_{19}$–$L_{24}$, it follows from Lemmas 4.1–4.10 that

$L_{19} = - \int (u \cdot \nabla u)_t \cdot u_{tt}$

$$\leq C(|u||\nabla u| + |u||\nabla u|)u_{tt} \leq C(|\phi^c \nabla u_t|^2 + 1) + \frac{1}{10} |u_{tt}|^2,$$

$L_{20} = - \int \nabla \phi_t \cdot u_{tt} \leq C|\nabla \phi_t|u_{tt} \leq C|u_{tt}|^2,$

$L_{21} = - \int a\phi^2 \nabla u \cdot u_{tt} = a \int (u \cdot \nabla \phi^2 + (\delta - 1) \phi^2 \text{div} u)Lu \cdot u_{tt}$

$$\leq C(|u||\nabla u|L u + C|\phi^2 \nabla u||\nabla u|)u_{tt} \leq C + \frac{1}{10} |u_{tt}|^2,$$

$L_{22} = - \int a\nabla \phi^2 \cdot Q(u) \cdot u_{tt}$

$$\leq C|\nabla u_t|^2 |\nabla u_t|u_{tt} \leq C|\phi^c \nabla u_t|^2 + \frac{1}{10} |u_{tt}|^2,$$ \tag{4.63}

$L_{23} = \int \frac{a}{2} \phi^{2c} (a|\nabla u_t|^2 + (a + \beta)|\text{div} u_t|^2)$

$$= - \int \frac{a}{2} (u \cdot \nabla \phi^2 + (\delta - 1) \phi^2 \text{div} u)(a|\nabla u_t|^2 + (a + \beta)|\text{div} u_t|^2)$

$$\leq C|u||\nabla u_t|^2 + C|\nabla u| \phi^c \nabla u_t|^2,$$

$L_{24} = \int (\psi \cdot Q(u))_t \cdot u_{tt} = \int \psi \cdot Q(u) \cdot u_{tt} dx + \int \psi_t \cdot Q(u) \cdot u_{tt}$

$$\leq C|\psi||\nabla u_t|^2 u_{tt} - \int (u \cdot \nabla \psi + \delta \text{div} u + a\delta \phi^2 \nabla \text{div} u) \cdot Q(u) \cdot u_{tt}$

$$\leq C(|\psi||\nabla u_t|^2 + |\nabla u|^2|\nabla u| + |\psi| \nabla u_6 + |\phi^c \nabla u^2|)u_{tt} \leq C|\phi^c \nabla u_t|^2 + \frac{1}{10} |u_{tt}|^2 + C,$$

which, along with (4.62), implies that

$$\frac{1}{2} \frac{d}{dt} \left( a\alpha |\phi^c \nabla u_t|^2 + a(a + \beta)|\phi^c \text{div} u_t|^2 \right) + |u_{tt}|^2 \leq C|\phi^c \nabla u_t|^2 + C. \tag{4.64}$$

Integrating (4.62) over $(\tau, t)$ shows that for $0 \leq t \leq T$,

$$|\phi^c \nabla u_t(t)|^2 + \int_\tau^t |u_{tt}|^2 ds \leq C|\phi^c \nabla u_t(\tau)|^2 + C. \tag{4.65}$$

On the other hand, it follows from the momentum equations (2.5)2 that

$$|\phi^c \nabla u_t(\tau)| \leq |(\phi^c \nabla (u \cdot \nabla u + \nabla \phi + a\phi^2 Lu - \psi \cdot Q(u))| |(\tau). \tag{4.66}$$
Then according to the assumptions (1.13)–(1.14) and Lemma 3.8, one has
\[
\limsup_{\tau \to 0} |\phi^\tau \nabla u_t(\tau)|_2 \leq C \left( |\phi_0^\tau \nabla (u_0 \cdot \nabla u_0)|_2 + |\phi_0^\tau \nabla^2 \phi_0|_2 \right. \\
+ |\phi_0^\tau \nabla (\psi_0 \cdot Q(u_0))|_2 + |g_3|_2 \\
\left. \leq C \left( |\phi_0^\tau \nabla^2 u_0|_2 |u_0|_\infty + |\nabla u_0|_\infty |\phi_0^\tau \nabla u_0|_2 + |\phi_0^\tau \nabla^2 \phi_0|_2 \right) \\
+ C \left( |\phi_0^\tau \nabla^2 u_0|_2 |\psi_0|_\infty + |\nabla \psi_0|_3 |\phi_0^\tau \nabla u_0|_6 + 1 \right) \leq C.
\]
(4.67)

Actually, for the estimates of $|\phi^\tau \nabla^2 \phi|_2$, due to
\[
\phi^\tau \nabla^2 \phi = C_3(\rho^{\frac{\delta-1}{2}} \nabla^2 \rho^{\gamma-1}) \\
= C_3 \rho^{\frac{\delta-1}{2}} ((\gamma - 1)(\gamma - 2)\rho^{\gamma-3} \nabla \rho \otimes \nabla \rho + (\gamma - 1)\rho^{\gamma-2} \nabla^2 \rho) \\
= C_4 \rho^{\gamma + \frac{1-3\delta}{3}} \nabla \psi + C_5 \rho^{\gamma + \frac{\delta-5}{3}} \nabla^2 \rho,
\]
(4.68)
for some constants $C_i$ ($i = 3, \ldots, 9$), one can obtain that
\[
|\rho^{\delta-2} \nabla^2 \rho|_2 + |\rho^{\frac{\delta-1}{2}} \nabla^2 \rho|_2 + |\phi^\tau \nabla^2 \phi|_2 \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]
(4.69)

Letting $\tau \to 0$ in (4.65), one can obtain that
\[
|\phi^\tau \nabla u_t|_2^2 + |\nabla u_t|_2^2 + \int_0^t |u_{tt}|_2^2 ds \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]
(4.70)

It follows from (2.5) that
\[
a \phi^{2e} Lu_t = -u_{tt} - (u \cdot \nabla u)_t - \nabla \phi_t - a \phi^{2e} Lu + (\psi \cdot Q(u))_t,
\]
(4.71)
which implies that
\[
|\phi^\tau \nabla u_t(t)|_2 \leq C \left| (u_t + (u \cdot \nabla u)_t + \nabla \phi_t - (\psi \cdot Q(u))_t + a \phi^{2e} Lu) \right|_2 \\
\leq C \left| (u_t|_2 + |u_t|_3 \nabla u|_3 + |u|_\infty \nabla u_t|_2 + |\nabla \phi_t|_2 + |\psi|_\infty \nabla u_t|_2) \\
+ C \left| (\psi|_6 \nabla u_3 + |\psi|_\infty u|_\infty \nabla^2 u_2 + |\phi^{2e} Lu|_6 \div u_3) \right| \\
\leq C(1 + |u_{tt}|_2). \tag{4.72}
\]

Using the equations (2.5)_2, for multi-index $\xi \in \mathbb{R}^3$ with $|\xi| = 2$, one has
\[
|a \phi^\tau \nabla(\phi^{2e} Lu)|_2 = | -a \phi^\tau \nabla (u_t + u \cdot \nabla u + \nabla \phi - \psi \cdot Q(u))|_2 \\
\leq C \left| (\phi^\tau \nabla u)(t)|_2 + |\nabla u|_\infty |\phi^\tau \nabla u|_2 + |u|_\infty |\phi^\tau \nabla^2 u|_2 \right) \\
+ C \left| (\phi^\tau \nabla^2 \phi)|_2 + |\nabla \psi|_3 |\phi^\tau \nabla u|_6 + |\psi|_\infty |\phi^\tau \nabla^2 u|_2 \right| \\
\leq C(1 + |\phi^\tau \nabla^2 \phi|_2 + |\phi^\tau \nabla^2 u|_2).
\]
(4.73)

Then according to (4.72), (4.73), (4.69) and (4.60), one finally gets
\[
|\phi^\tau \nabla(\phi^{2e} Lu)|_2^2 + \int_0^t |\phi^{2e} Lu_t|_2^2 ds \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]
□

**Lemma 4.12.**

\[
\sup_{0 \leq t \leq T} |t| u_{tt}(t)|_2^2 + \int_0^T t|\phi^\tau u_{tt}|_2^2 dt \leq C \quad \text{for} \quad 0 \leq T < T,
\]
where the constant $C > 0$ only depends on $(\rho_0, u_0)$, $C_0$, $\alpha$, $\beta$, $A$, $\gamma$, $\delta$ and $T$. 

Proof. Now applying \( \partial_t \) to (4.17) yields

\[
\begin{align*}
{u}_{ttt} + a \phi^2 e^{Lu_{tt}} &= -(u \cdot \nabla u)_{tt} - \nabla \phi_{tt} - a \phi^2 e^{Lu_t} - 2a \phi^2 e^{Lu_t} \\
&\quad + 2\psi_t \cdot Q(u_t) + \psi_t \cdot Q(u) + \psi \cdot Q(u_{tt}). \tag{4.74}
\end{align*}
\]

Multiplying (4.74) by \( u_{tt} \) and integrating over \( \mathbb{R}^3 \) give

\[
\frac{1}{2} \frac{d}{dt} |u_{tt}|^2 + a \alpha |\phi^e \nabla u_{tt}|^2 + a(\alpha + \beta) |\phi^e \text{div} u_{tt}|^2
\]

\[
= \int \left( - (u \cdot \nabla u)_{tt} - \nabla \phi_{tt} - a \nabla \phi^2 e^{Q(u) - a \phi^2 e^{Lu_t}} \right) \cdot u_{tt}
\]

\[
+ \int \left( - 2a \phi^2 e^{Lu_t} + 2\psi_t \cdot Q(u_t) + \psi_t \cdot Q(u) + \psi \cdot Q(u_{tt}) \right) \cdot u_{tt} = \sum_{i=25}^{32} L_i. \tag{4.75}
\]

For the terms \( L_{25} - L_{32} \), it follows from Lemmas 4.1–4.11 that

\[
L_{25} = - \int (u \cdot \nabla u)_{tt} \cdot u_{tt}
\]

\[
\leq C(|\nabla u|_6|u_t|_3 + |\nabla u|_\infty|u_{tt}|_2 + |u|_\infty|\nabla u_{tt}|_2)|u_{tt}|_2
\]

\[
\leq C(1 + |u_{tt}|_2^2 + |\nabla^2 u_{tt}|_2^2) + \frac{a \alpha}{10} |\phi^e \nabla u_{tt}|_2^2;
\]

\[
L_{26} = - \int \nabla \phi_{tt} \cdot u_{tt}
\]

\[
\leq C(|\phi_{tt}|_2|\phi^e \nabla u_{tt}|_2|\phi|_3^2 \leq C|\phi_{tt}|_2^2 + \frac{a \alpha}{10} |\phi^e \nabla u_{tt}|_2^2;
\]

\[
L_{27} = - \int a \nabla \phi^2 e^{Q(u_{tt})} \cdot u_{tt}
\]

\[
\leq C(|\psi|_\infty|u_{tt}|_2|\phi^e \nabla u_{tt}|_2|\psi|_3^2 \leq C|u_{tt}|_2^2 + \frac{a \alpha}{10} |\phi^e \nabla u_{tt}|_2^2,
\]

and

\[
L_{28} = - \int a \phi^2 e^{Lu} \cdot u_{tt} = a \int u \cdot \nabla \phi^2 e^{Lu} \cdot u_{tt}
\]

\[
- a \int ((\delta - 1) \text{div} uu \cdot \nabla \phi^2 e^{(\delta - 1)^2 \phi^2 e(\text{div} u)^2} Lu \cdot u_{tt}
\]

\[
- a \int (u \cdot \nabla \psi + (\delta - 1) \phi^2 e \text{div} u_{tt}) Lu \cdot u_{tt}
\]

\[
\leq C(|\nabla^2 u|_6|\psi|_6|u|_6 + |u|_\infty|\nabla u|_\infty|\nabla |\nabla^2 u|_2 + |\phi^2 e Lu|_2|\text{div} u^2_\infty|)|u_{tt}|_2
\]

\[
+ C(|u_{tt}|_6|\psi|_6|u|_t|_2 + |\phi^2 e Lu|_6|\nabla u_t|_2|u_{tt}|_3) \leq C||u_{tt}||_1,
\]

\[
L_{29} = - \int 2a \phi^2 e^{Lu_t} \cdot u_{tt}
\]

\[
= - \int 2a(u \cdot \nabla \phi^2 e + (\delta - 1) \phi^2 e \text{div} u)Lu_t \cdot u_{tt}
\]

\[
\leq C(|\psi|_\infty|u|_\infty|Lu|_t|_2 + |\phi^2 e Lu|_t|_2|\text{div} u|_\infty)|u_{tt}|_2
\]

\[
\leq C(1 + |u_{tt}|_2^2 + |\nabla^2 u_{tt}|_2^2),
\]

\[
L_{30} = \int 2\psi_t \cdot Q(u_t) \cdot u_{tt}
\]

\[
\leq C|\psi|_t|\nabla u_t|_2|u_{tt}|_3 \leq C(1 + |u_{tt}|_2^2 + \frac{a \alpha}{10} |\phi^e \nabla u_{tt}|_2^2,
\]

\[
L_{31} = \int \psi_t \cdot Q(u) \cdot u_{tt}
\]
Lemma 4.13. Finally, we show that for some constant $\tau$

\[ L_{32} = \int \psi \cdot Q(u_{tt}) \cdot u_{tt} \]

\[ \leq C|\psi|_\infty|\nabla u_{tt}|_2 \leq C|u_{tt}|_2 + \frac{aa}{10} |\phi^e \nabla u_{tt}|_2 \]

which, along with (4.72) and (4.75)–(4.76), implies that

\[ \frac{1}{2} \frac{d}{dt} |u_{tt}|_2^2 + a\alpha |\phi^e \nabla u_{tt}|_2^2 + a(\alpha + \beta) |\phi^e \text{div} u_{tt}|_2^2 \leq C|u_{tt}|_2^2 + |\nabla^2 u_{tt}|_2^2 + |\phi_{tt}|_2^2 + 1. \]  

(4.78)

Multiplying both sides of (4.78) by $t$ and integrating over $(\tau, t)$, one can get

\[ t|u_{tt}(t)|_2^2 + \frac{aa}{2} \int_\tau^t s|\phi^e \nabla u_{tt}|_2^2 ds \leq \tau|u_{tt}(\tau)|_2^2 + C. \]  

(4.79)

It follows from (4.70) and Lemma 5.11 that there exists a sequence $s_k$ such that

\[ s_k \to 0, \quad \text{and} \quad s_k|u_{tt}(s_k, x)|_2 \to 0, \quad \text{as} \quad k \to +\infty. \]

The desired estimates follows from taking $\tau = s_k$ and letting $k \to +\infty$ in (4.79).

4.4. Second Order Estimates on the Velocity with Singular Weight

Finally, we show that

Lemma 4.13.

\[ \sup_{0 \leq t \leq T} |\phi^{2e} \nabla^2 u(t)|_2 + \int_0^T |\phi^{2e} \nabla^2 u_{tt}|_2^2 dt \leq C \quad \text{for} \quad 0 \leq T < T, \]

where the constant $C > 0$ only depends on $(\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta$ and $T$.

Proof. First, it follows from the equations (2.5) that for any constant $\eta > 0$,

\[ aL((\phi + \eta)^{2e} u) = a(\phi + \eta)^{2e} Lu - aG(\nabla (\phi + \eta)^{2e}, u) = \Lambda_\eta. \]  

(4.80)

First, due to

\[ \nabla (\phi + \eta)^{2e} = \frac{\phi^{2e+1}}{(\phi + \eta)^{-2e+1}} \nabla \phi^{2e}, \]

\[ \nabla^2 (\phi + \eta)^{2e} = \nabla \left( \frac{\phi^{2e+1}}{(\phi + \eta)^{-2e+1}} \nabla \phi^{2e} \right) = \frac{\phi^{2e+1}}{(\phi + \eta)^{-2e+1}} \nabla^2 \phi^{2e} - \frac{2e - 1}{e(\phi + \eta)^{-2e+2}} \nabla \phi^e \otimes \nabla \phi^e, \]  

(4.81)

one quickly obtains that

\[ ||\nabla (\phi + \eta)^{2e}||_{L^\infty L^\infty \cap D^1} + |\Lambda_\eta|_2 \leq C \quad \text{for} \quad 0 \leq t \leq T, \]  

(4.82)

for some constant $C > 0$ that is independent of $\eta$.

Second, due to $\eta > 0$ and (4.82), one has $(\phi + \eta)^{2e} u \in L^\infty([0, T]; H^2)$. Then according to Lemma 3.1 and (4.81), one has

\[ |(\phi + \eta)^{2e} u|_{D^2} \leq C|\Lambda_\eta|_2 \leq C \quad \text{for} \quad 0 \leq t \leq T, \]  

(4.83)

for some constant $C > 0$ that is independent of $\eta$. 

\[ aL((\phi + \eta)^{2e} u) = a(\phi + \eta)^{2e} Lu - aG(\nabla (\phi + \eta)^{2e}, u) = \Lambda_\eta. \]  

(4.80)
Thus one has
\[(\phi + \eta)^{2e}\nabla^2 u|_2 \leq C(1 + |\nabla (\phi + \eta^{2e})|_3 + |\nabla^2 (\phi + \eta)^{2e}|_3) \leq C, \quad (4.84)\]
for \(0 \leq t \leq T\), where the constant \(C > 0\) is also independent of \(\eta\).

It is very easy to see that, for every \((t, x) \in [0, T] \times \mathbb{R}^3\),
\[(\phi + \eta)^{2e}\nabla^2 u \to \phi^{2e}\nabla^2 u \quad \text{a.e. as} \quad \eta \to 0,
\]
which, along with Fatou’s lemma (see Lemma 5.15), implies that
\[\int |\phi^{2e}\nabla^2 u|^2 \leq \liminf_{\eta \to 0} \int |(\phi + \eta)^{2e}\nabla^2 u|^2 \leq C \quad \text{for} \quad 0 \leq t \leq T.\]

Finally, the estimate on \(\phi^{2e}\nabla^2 u_t\) follows from the following approximate scheme
\[aL((\phi + \eta)^{2e} u_t) = a(\phi + \eta)^{2e} Lu_t - aG(\nabla (\phi + \eta)^{2e}, u_t),\]
and the similar argument used above for the estimate on \(\phi^{2e}\nabla^2 u\).

\section*{4.5. Contradiction Argument}

Now we know that if the regular solution \((\rho, u)(t, x)\) exists up to the time \(T > 0\), with the maximal time \(T < +\infty\) such that the assumption (4.1) holds, then Lemmas 4.1–4.10 hold.

Then first, according to Lemmas 4.1–4.10 and the standard weak compactness theory, one has, for any sequence \(\{t_k\}_{k=1}^\infty\) satisfying \(0 < t_k < T\) and
\[t_k \to T \quad \text{as} \quad k \to \infty,\]
there exists one subsequence \(\{t_{k_l}\}_{l=1}^\infty\) and functions \((\phi(T, x), u(T, x), \psi(T, x), \omega(T, x))\) such that
\[
\begin{align*}
\phi(t_{k_l}, x) & \to \phi(T, x) \quad \text{in} \quad H^3 \quad \text{as} \quad k \to \infty, \\
u(t_{k_l}, x) & \to u(T, x) \quad \text{in} \quad H^3 \quad \text{as} \quad k \to \infty, \\
\psi(t_{k_l}, x) & \to \psi(T, x) \quad \text{in} \quad D^1 \cap D^2 \quad \text{as} \quad k \to \infty, \\
\omega(t_{k_l}, x) & \to \omega(T, x) \quad \text{in} \quad L^4 \quad \text{as} \quad k \to \infty.
\end{align*}
(4.85)
\]

Second, we want to show that functions \((\phi(T, x), u(T, x), \psi(T, x), \omega(T, x))\) satisfy all the initial assumptions shown in Theorem 1.1, which include (1.13)–(1.14) and the relations between \(\phi\) and \((\psi, \omega)\):
\[
\psi = \frac{a\delta}{\delta - 1} \nabla \phi^{2e} \quad \text{and} \quad \omega = a^{\frac{1}{2}} \nabla \phi^{e}.
(4.86)
\]
(1.13) can be obtained immediately from the weak convergences in (4.85).

Next we consider the relations in (4.86). For this purpose, we consider the following equation
\[\phi_t + u \cdot \nabla \phi + (\gamma - 1)\phi \text{div} u = 0, \quad (4.87)\]
which holds in \([0, T] \times \mathbb{R}^3\) in the classical sense. Actually, if we regard \((\phi(T, x), u(T, x), \psi(T, x), \omega(T, x))\) and
\[
\phi_t(T, x) = -u(T, x) \cdot \nabla \phi(T, x) - (\gamma - 1)\phi(T, x) \text{div} u(T, x)
\]
as the extended definitions of \((\phi(t, x), u(t, x), \psi(t, x), \omega(t, x), \phi_t(t, x))\) at the time \(t = T\), then one has
\[\text{ess sup}_{0 \leq t \leq T} (||\phi||_3^2 + ||\phi_t||_2^2) \leq C < \infty,\]
which, along with the classical Sobolev embedding theorem, implies that
\[\phi \in C([0, T]; H^2), \quad (4.88)\]
It follows from the first line in (4.85) and the consistency of the weak convergence and the strong convergence in $H^2$ space, one has 
\[
\phi(t_k, x) \to \phi(T, x) \quad \text{in} \quad H^2 \quad \text{as} \quad k \to \infty.
\] (4.89)

Notice that for any $0 < R < \infty$, there exists one finite constant $C(R, T)$ (see Lemma 3.9 of [45]) such that
\[
\phi(t, x) \geq C(R, T) > 0 \quad \text{for any} \quad (t, x) \in [0, T] \times B_R,
\] (4.90)

which, along with the last two lines in (4.85) and (4.89), yields that (4.86) holds.

Moreover, it follows from Lemma 3.5, Lemmas 4.1–4.13, (4.85)–(4.86) and (4.88)–(4.90) that,
\[
\sup_{\tau \leq t \leq T} \left( |\phi^c \nabla u(t)|_2^2 + |\phi^{2c} Lu(t)|_2^2 + |\phi^c \nabla (\phi^{2c} Lu(t))|_2^2 \right) \leq C,
\]
for any $\tau \in (0, T)$, which means that (1.14) still holds at $t = T$. Then $(\phi(T, x), u(T, x), \psi(T, x), \omega(T, x))$ satisfy all the conditions on the initial data of Theorem 1.1.

Finally, if we solve the system (1.1) with the initial time $T$, then Theorem 1.1 ensures that for some constant $T_0 > 0$, $(\rho, u)(t, x)$ is the unique regular solution in $[T, T + T_0] \times \mathbb{R}^3$ to this problem. It follows from the boundedness of all required norms of the solution $(\rho, u)(t, x)$ in $[0, T + T_0] \times \mathbb{R}^3$ and the standard arguments for proving the time continuity of the regular solution (see page 45 of [45] and Subsection 3.4) that, $(\rho, u)(t, x)$ is actually the unique regular solution in $[0, T + T_0] \times \mathbb{R}^3$ to the Cauchy problem (1.1)–(1.7), which contradicts to the fact that $0 < T < \infty$ is the maximal existence time.

Until now, we have completed the proof of Theorem 1.2.

Remark 4.1. According to the proof in this subsection, the definitions of $(\phi(T, x), u(T, x), \psi(T, x), \omega(T, x))$ do not depend on the choice of the time sequences $\{t_k\}_{k=1}^\infty$.

Finally, we give a detail explanation for Remark 1.4.

4.6. The Classical Solution to the Cauchy Problem (1.1)–(1.7)

Now we show that, if $\gamma \in (1, 2]$, the regular solution obtained in the above step is indeed a classical one in $[0, T] \times \mathbb{R}^3$. First, due to $1 < \gamma \leq 2$, one has
\[
(\rho, \nabla \rho, \rho_t, u, \nabla u) \in C([0, T] \times \mathbb{R}^3).
\]

Second, it follows from the classical Sobolev embedding theorem:
\[
L^2([0, T]; H^1) \cap W^{1,2}([0, T]; H^{-1}) \hookrightarrow C([0, T]; L^2),
\] (4.91)

and the regularity (1.15) that for any $\tau \in (0, T)$,
\[
tu_t \in C([0, T]; H^2), \quad \text{and} \quad u_t \in C([\tau, T] \times \mathbb{R}^3).
\] (4.92)

Finally, it remains to show that $Lu \in C([\tau, T] \times \mathbb{R}^3)$. According to the following elliptic system
\[
aLu = -\phi^{-2c} (u_t + u \cdot \nabla u + \nabla \phi - \psi \cdot Q(u)) = \phi^{-2c} H,
\] (4.93)

the regularity (1.15) implies that
\[
t\phi^{-2c} H \in L^\infty([0, T]; H^2).
\] (4.94)

Note that
\[
(t\phi^{-2c} H)_t = \phi^{-2c} H + t\phi^{-2c} H + t\phi^{-2c} H_e \in L^2([0, T]; L^2).
\] (4.95)

So it follows from the classical Sobolev imbedding theorem:
\[
L^\infty([0, T]; H^1) \cap W^{1,2}([0, T]; H^{-1}) \hookrightarrow C([0, T]; L^q),
\] (4.96)

for any $q \in [2, 6]$, and (4.93)–(4.95) that
\[
t\phi^{-2c} H \in C([0, T]; W^{1,4}), \quad tLu \in C([0, T]; W^{1,4}).
\]

Again the Sobolev embedding theorem implies that $Lu \in C((0, T] \times \mathbb{R}^3)$. 

5. Periodic Problem Away from the Vacuum

This section will be devoted to proving Theorem 1.3, and we always assume that (1.2), (1.5), \( \delta \in (0,1] \)
and (1.19) hold, and \( \Omega = \mathbb{T}^3 \). When \( \inf_x \rho_0(x) > 0 \), the local existence of the unique classical solution to the periodic problem (1.1)-(1.6) stated in Theorem 1.3 follows from the standard theory of the symmetric hyperbolic-parabolic structure which satisfies the well-known Kawashima’s condition, c.f. [24,37].

In order to prove (1.21), we use a contradiction argument. Let \( (\rho, u) \) be the unique classical solution stated in Theorem 1.3 to the periodic problem (1.1)-(1.6) with the life span \( T \). We assume that \( T < +\infty \) and

\[
\lim_{T \to -T} \int_0^T \| D(u)(t, \cdot) \|_{L^\infty(\mathbb{T}^3)}^2 \, dt = C_0 < \infty,
\]

for some constant \( C_0 > 0 \). We will show that under the assumption (5.1), \( T \) is actually not the maximal existence time for this classical solution.

First, it follows from the assumption (5.1) and (1.1) that \( \rho \) is uniformly bounded.

**Lemma 5.1.**

\[
C^{-1} \leq \rho(t,x) \leq C \quad \text{for} \quad 0 \leq t < T,
\]

where the constant \( C > 0 \) only depends on \( (\rho_0, u_0), C_0 \) and \( T \).

The proof is similar to that of Lemma 4.1. Here we omit it.

Second, we give the well-known B-D entropy estimate.

**Lemma 5.2.** [6]

\[
\int \left( \frac{1}{2} \rho |u + \Phi|^2 + \frac{P}{\gamma - 1} \right)(t, \cdot) + \int_0^t \int_{\mathbb{T}^3} \left( \frac{2P'(\rho)}{\rho} |\nabla \rho|^2 + \frac{1}{2} \mu(\rho) |\nabla u - \nabla u^\top|^2 \right) dx \, ds \leq C,
\]

for \( 0 \leq t < T \), where \( \Phi = \frac{2ad}{\delta - 1} \nabla \rho^{\delta - 1} \) when \( \delta \in (0,1) \), \( \Phi = \alpha \nabla \log \rho \) when \( \delta = 1 \), and the constant \( C > 0 \) only depends on \( (\rho_0, u_0), A, \gamma, \alpha \) and \( \delta \).

The proof of this lemma can be found in [6]. Here we omit it. Now we give the basic energy estimate.

**Lemma 5.3.** [6]

\[
\int \left( \frac{1}{2} \rho |u|^2 + \frac{P}{\gamma - 1} \right)(t, \cdot) + \int_0^t \int_{\mathbb{T}^3} \left( \frac{1}{2} \mu(\rho) |\nabla u + \nabla u^\top|^2 + \lambda(\rho)(\operatorname{div} u)^2 \right) dx \, ds \leq C,
\]

for \( 0 \leq t < T \), where the constant \( C > 0 \) only depends on \( (\rho_0, u_0), A \) and \( \gamma \).

The conclusion obtained above is classical. Here we omit its proof.

It follows quickly from Lemmas 5.1–5.3 that

**Lemma 5.4.**

\[
\sup_{0 \leq t \leq T} \left( |u|^2 + |\nabla \rho|^2 \right)(t) + \int_0^T |\nabla u|^2 dt \leq C \quad \text{for} \quad 0 \leq T < T,
\]

where the constant \( C > 0 \) only depends on \( (\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta \) and \( T \).

Now we improve the energy estimate obtained in Lemma 5.4.

**Lemma 5.5.**

\[
\frac{d}{dt} \int \phi |u|^4(t, \cdot) + \int |u|^2 |\nabla u|^2(t, \cdot) \leq C(1 + |\operatorname{div} u|_{\infty} |\sqrt{\rho} u|^2_{L^2} + |D(u)|_{L^2} |u|^2_{L^2}) \quad \text{for} \quad 0 \leq t < T,
\]

where the constant \( C > 0 \) only depends on \( (\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta \) and \( T \).
Proof. First, multiplying \((2.2)_3\) by \(r|u|^{-2}u\) \((r \geq 3)\) and integrating the resulting equation over \(T^3\) by parts, then one has

\[
\frac{d}{dt} \int \varphi |u|^r + a \int \mathcal{H}_r = -ar(r-2)(\alpha + \beta) \int \operatorname{div} |u|^{r-3}u \cdot \nabla |u| \\
+ \int \left( \delta \varphi \operatorname{div} |u|^r + rg \operatorname{div} (|u|^{-2}u) + rf \cdot Q(u) \cdot |u|^{-2}u \right),
\]

where

\[
H_r = r|u|^{-2}(\alpha |\nabla u|^2 + (\alpha + \beta)|\operatorname{div} u|^2 + \alpha(r-2)|\nabla |u||^2),
\]

and specially, \(\varphi = 1\) when \(\delta = 1\).

For any given \(\varepsilon_1 \in (0, 1)\), we define a nonnegative function which will be determined in Step 2 as follows

\[
\Phi(\varepsilon_0, \varepsilon_1, r) = \begin{cases} \frac{\alpha \varepsilon_1 (r-1)}{3 - \frac{\alpha (4-\varepsilon_0)}{3} - \beta}, & \text{if } \frac{r^2(\alpha + \beta)}{4(r-1)} - \frac{\alpha(4-\varepsilon_0)}{3} - \beta > 0, \\
0, & \text{otherwise}. \end{cases}
\]

**Step 1:** We assume that

\[
\int_{T^3 \cap \{ |u| > 0 \}} |u|^r \nabla \left( \frac{u}{|u|} \right)^2 \, dx > \Phi(\varepsilon_0, \varepsilon_1, r) \int_{T^3 \cap \{ |u| > 0 \}} |u|^{-2} |\nabla |u||^2 \, dx.
\]

A direct calculation gives for \(|u| > 0\) that the following formula holds.

\[
|\nabla u|^2 = |u|^2 \left( \frac{u}{|u|} \right)^2 + |\nabla |u||^2.
\]

According to \((5.3)\) and the Cauchy’s inequality, one has

\[
\frac{d}{dt} \int \varphi |u|^r + a \int_{T^3 \cap \{ |u| > 0 \}} \mathcal{H}_r \, dx
= -ar(r-2)(\alpha + \beta) \int_{T^3 \cap \{ |u| > 0 \}} \operatorname{div} |u|^{r-3}u |^{r-4}u \cdot \nabla |u| \, dx
+ \int \left( \delta \varphi \operatorname{div} |u|^r + rg \operatorname{div} (|u|^{-2}u) + rf \cdot Q(u) \cdot |u|^{-2}u \right) \leq ar(\alpha + \beta) \int_{T^3 \cap \{ |u| > 0 \}} |u|^{-2} \left( |\operatorname{div} u|^2 + \frac{(r-2)^2}{4} |\nabla |u||^2 \right) \, dx
+ \int \left( \delta \varphi \operatorname{div} |u|^r + rg \operatorname{div} (|u|^{-2}u) + rf \cdot Q(u) \cdot |u|^{-2}u \right).
\]

It follows from Lemmas 5.1–5.4, Gagliardo–Nirenberg inequality, Hölder’s inequality and Young’s inequality that

\[
M_1 = \int \delta \varphi \operatorname{div} |u|^{r} \leq C |\operatorname{div} u|_\infty \sqrt{\varphi} |u|^\frac{2}{2},
M_2 = r \int g |u|^{-2} |\nabla u| \leq C \left( \int |u|^{-2} |\nabla u|^2 \right)^\frac{1}{2} \left( \int |u|^{-2} g^2 \right)^\frac{1}{2}
\leq C \left( \int |u|^{-2} |\nabla u|^2 \right)^\frac{1}{2} ||u||_6^{1-\frac{1}{2}} g_{\frac{n+2}{2r+2}} \leq \frac{1}{4} a ar \varepsilon_0 \int |u|^{-2} |\nabla u|^2 \, dx + C(a, \varepsilon_0),
M_3 = \int rf \cdot Q(u) \cdot |u|^{-2} u \leq C |\nabla \rho|_2 D(u)|_\infty ||u|^{r-1}|_2,
\]

where \(a = \frac{4}{3} - \frac{\alpha(4-\varepsilon_0)}{3} - \beta\).
where \( \epsilon_0 \in (0, \frac{1}{2}) \) is independent of \( r \). Then combining (5.4)–(5.7), one easily has
\[
\frac{d}{dt} \int \varphi |u|^r + ar \Psi(\epsilon_0, \epsilon_1, \epsilon_2, r) \int_{\Omega \cap \{|u| > 0\}} |u|^{-2} |\nabla u|^2 dx \\
+ \int_{\Omega \cap \{|u| > 0\}} a \epsilon_0 (1 - \epsilon_0) \epsilon_2 |u|^{r-2} |\nabla (\frac{u}{|u|})|^2 dx \\
\leq C(a, \alpha, r, \epsilon_0) + C |\text{div} u|_{\infty} |\sqrt{\varphi} |u|^2|_2 + C |D(u)|_{\infty} ||u|^{-1}|_2;
\]
where
\[
\Psi(\epsilon_0, \epsilon_1, \epsilon_2, r) = \alpha(1 - \epsilon_0)(1 - \epsilon_2) \Phi(\epsilon_0, \epsilon_1, r) + \alpha(r - 1 - \epsilon_0) - \frac{(r - 2)^2 (\alpha + \beta)}{4}.
\]
Via choosing \( \epsilon_0 < 2(1 - \delta) \) small enough, then \( \beta < -\epsilon_0 \alpha \), i.e.,
\[
4 \not\in \left\{ r \left| \frac{r^2 (\alpha + \beta)}{4(r - 1)} - \frac{(4 - \epsilon_0)\alpha}{3} - \beta > 0 \right\}.
\]
In this case, for \( r = 4 \) and \( 0 < \epsilon_0 < \min\{2(1 - \delta), \frac{1}{4}\} \), it is easy to get
\[
r \left[ \alpha(1 - \epsilon_0)(1 - \epsilon_2) \Phi(\epsilon_0, \epsilon_1, r) + \alpha(r - 1 - \epsilon_0) - \frac{(r - 2)^2 (\alpha + \beta)}{4} \right] \\
> 4 \left( \frac{11}{4} \alpha - (\alpha + \beta) \right) = 4 \left( \frac{7\alpha}{4} - \beta \right) \geq 4 \left( \frac{7\alpha}{4} + \epsilon_0 \alpha \right) > 7\alpha,
\]
which, together with (5.8), implies that
\[
\frac{d}{dt} \int \varphi |u|^4 + C \int_{\Omega \cap \{|u| > 0\}} |u|^2 |\nabla u|^2 dx \\
\leq C(a, \alpha, r, \epsilon_0) + C |\text{div} u|_{\infty} |\sqrt{\varphi} |u|^2|_2 + C |D(u)|_{\infty} ||u|^3|_2,
\]

**Step 2** : we assume that
\[
\int_{\Omega \cap \{|u| > 0\}} |u|^{r-2} |\nabla (\frac{u}{|u|})|^2 dx \leq \Phi(\epsilon_0, \epsilon_1, r) \int_{\Omega \cap \{|u| > 0\}} |u|^{-2} |\nabla u|^2 dx.
\]
A direct calculation gives for \( |u| > 0 \),
\[
\text{div} u = |u| \text{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|}.
\]
Then combining (5.12) and (5.6)–(5.7), one quickly has
\[
\frac{d}{dt} \int \varphi |u|^r + a \int_{\Omega \cap \{|u| > 0\}} \alpha r (1 - \epsilon_0) |u|^{r-2} |\nabla u|^2 dx \\
+ a \int_{\Omega \cap \{|u| > 0\}} (r(\alpha + \beta)|u|^{r-2} |\text{div} u|^2 + \alpha r (r - 2) |u|^{r-2} |\nabla |u| |^2) dx \\
= -ar(r - 2)(\alpha + \beta) \int_{\Omega \cap \{|u| > 0\}} |u|^{r-2} u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) dx \\
- ar(r - 2)(\alpha + \beta) \int_{\Omega \cap \{|u| > 0\}} |u|^{r-4} u \cdot \nabla |u|^2 dx \\
+ \int \left( \delta \varphi \text{div} u |u|^r + rf \cdot Q(u) \cdot |u|^{r-2} u \right).
\]
This gives
\[
\frac{d}{dt} \int \varphi |u|^r + \int_{\Omega \cap \{|u| > 0\}} a r |u|^{r-4} \Gamma dx \\
\leq C(a, \alpha, r, \epsilon_0) + C |\text{div} u|_{\infty} |\sqrt{\varphi} |u|^2|_2 + C |D(u)|_{\infty} ||u|^{-1}|_2,
\]
where
\[ \Gamma = \alpha(1 - \epsilon_0)|u|^2|\nabla u|^2 + (\alpha + \beta)|u|^2|\text{div} u|^2 + \alpha(r - 2)|u|^2|\nabla|u|^2 \]
\[ + (r - 2)(\alpha + \beta)|u|^2 u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + (r - 2)(\alpha + \beta)|u| \cdot \nabla |u|^2. \]
\tag{5.15} \]

Now we consider how to make sure that $\Gamma \geq 0$.
\[ \Gamma = \alpha(1 - \epsilon_0)|u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u|^2 | \]
\[ + (\alpha + \beta)|u|^2 \left( |u| \text{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|} \right)^2 + \alpha(r - 2)|u|^2|\nabla |u|^2 | \]
\[ + (r - 2)(\alpha + \beta)|u|^2 u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + (r - 2)(\alpha + \beta)|u| \cdot \nabla |u|^2 | \]
\[ = \alpha(1 - \epsilon_0)|u|^4 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + \alpha(r - 1 - \epsilon_0)|u|^2|\nabla |u|^2 | \]
\[ + (r - 1)(\alpha + \beta) \left( u \cdot \nabla |u| + \frac{r}{2(r - 1)}|u|^2 \left( \text{div} \frac{u}{|u|} \right) \right)^2 | \]
\[ + (\alpha + \beta)|u|^4 \left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 - \frac{r^2(\alpha + \beta)}{4(r - 1)}|u|^4 \left( \text{div} \left( \frac{u}{|u|} \right) \right)^2, \]

which, combining with the fact $|\text{div} \left( \frac{u}{|u|} \right)| \leq 3 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2$, implies that
\[ \Gamma \geq \alpha(r - 1 - \epsilon_0)|u|^2|\nabla |u|^2 | + \left( \frac{4 - \epsilon_0}{3} \alpha + \beta - \frac{r^2(\alpha + \beta)}{4(r - 1)} \right)|u|^4 \left( \text{div} \left( \frac{u}{|u|} \right) \right)^2. \]

Thus
\[ \int_{T^3 \cap \{|u| > 0\}} r|u|^{-4} \Gamma dx \geq \alpha r(r - 1 - \epsilon_0) \int_{T^3 \cap \{|u| > 0\}} |u|^{-2} |\nabla |u|^2 |^2 dx \]
\[ + r \left( \frac{4 - \epsilon_0}{3} \alpha + \beta - \frac{r^2(\alpha + \beta)}{4(r - 1)} \right) \int_{T^3 \cap \{|u| > 0\}} |u|^r \left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 dx \]
\tag{5.16} \]
\[ \geq \Lambda(\epsilon_0, \epsilon_1, r) \int_{T^3 \cap \{|u| > 0\}} |u|^{-2} |\nabla |u|^2 |^2 dx, \]

where
\[ \Lambda(\epsilon_0, \epsilon_1, r) = 3r \left( \frac{4 - \epsilon_0}{3} \alpha + \beta - \frac{r^2(\alpha + \beta)}{4(r - 1)} \right) \Phi(\epsilon_0, \epsilon_1, r) + \alpha r(r - 1 - \epsilon_0). \tag{5.17} \]

Here we need that $\epsilon_0$ is sufficiently small such that
\[ 0 < \epsilon_0 < \min\{2(1 - \delta), 1/4, (r - 1)(1 - \epsilon_1)\}. \]

Then combining (5.5), (5.14) and (5.16)-(5.17), when $r = 4$, one quickly has
\[ \frac{d}{dt} \int \varphi|u|^4(t, \cdot) + \int |u|^2|\nabla u|^2(t, \cdot) \]
\[ \leq C(1 + |\text{div} u|_\infty)\sqrt{\varphi}|u|^2 + |D(u)|_\infty||u|^2|_{L^2} \text{ for } 0 \leq t < T. \tag{5.18} \]

Finally, (5.2) follows from (5.10) and (5.18). \qed

The next lemma provides a key estimate on $\nabla u$.

**Lemma 5.6.**
\[ \sup_{0 \leq t \leq T} \left( |\nabla u|^2 + |u|^4 \right)(t) + \int_0^T (|\nabla^2 u|^2 + |u|^2 + ||u||\nabla u|^2)dt \leq C, \]

for $0 \leq T < T'$, where the constant $C > 0$ only depends on $(\rho_0, u_0)$, $C_0$, $\alpha$, $\beta$, $A$, $\gamma$, $\delta$ and $T$. \hfill \square
Proof. First, it follows from the proof of Lemma 4.3 that
\[
\frac{a}{2} \frac{d}{dt} \left( \alpha|\phi^e \nabla u|^2 + (\alpha + \beta)|\phi^e \text{div} u|^2 \right) + \int (a\phi^{2e}Lu + \nabla \phi)^2 \equiv \sum_{i=1}^{8} L_i, \tag{5.19}
\]
where, specially, \( e = 0 \) and \( \psi = \nabla \log \rho \) when \( \delta = 1 \).

The definitions of \( L_i \) \((i = 1, \ldots, 8)\) are same as those of \( L_i \) \((i = 1, \ldots, 8)\) in Lemma 4.3 with \( \mathbb{R}^3 \) replaced by \( T^3 \).

Second, for the terms \( L_1, L_3 \) and \( L_4 \), similarly one still has
\[
\begin{align*}
|L_1| &\leq C|D(u)_{\infty}| |\phi^e \nabla u|^2 + \epsilon|\nabla^2 u|^2 + C(\epsilon)|\phi^e \text{div} u|^2, \\
|L_3| &\leq C|\phi^{1-2e}| |\phi^e \nabla u|^2 + C|\text{div} u|_{\infty} |u|_{2} |\nabla \phi|_2, \\
L_4 &\leq \frac{d}{dt} \int \phi \text{div} u + C|\text{div} u|_{\infty} |u|_{2} |\nabla \phi|_2 + C|\phi^e |_{\infty} |\phi^e \nabla u|^2.
\end{align*}
\tag{5.20}
\]

For other terms, according to Hölder’s inequality, Gagliardo–Nirenberg inequality and Young’s inequality, one gets
\[
|L_2| = a(2\alpha + \beta) \int \phi^{2e} (-\nabla u : \nabla u \nabla \text{div} u + \frac{1}{2}(\text{div} u)^3)
+ a(2\alpha + \beta) \int (u \cdot \nabla u) \cdot \nabla \phi^{2e} \text{div} u
\leq C|\text{div} u|_{\infty} |\phi^e \nabla u|^2 + C|\psi_{2}| |\nabla u|_{6} |\text{div} u|_{\infty}
\leq C(1 + |D(u)|^2_{\infty})(1 + |\phi^e \nabla u|^2) + \epsilon|\nabla^2 u|^2,
\]
\[
L_5 = \frac{1 - \delta}{\delta} \int \psi \cdot Q(u) \cdot u \leq C|D(u)|_{\infty} |\psi_{2}| |u|_{2},
\]
\[
L_6 = \frac{a}{2} \int (-\text{div}(u\phi^{2e}) - (\delta - 2)\phi^{2e} \text{div} u)(\alpha|\nabla u|^2 + (\alpha + \beta)|\text{div} u|^2)
\leq C||u||\nabla u|_{2} |\nabla^2 u|_{2} |\phi^{2e}|_{\infty} + C|D(u)|_{\infty} |\phi^e \nabla u|_2^2
\leq C|D(u)|_{\infty} |\phi^e \nabla u|_2^2 + \epsilon|\nabla^2 u|^2 + C(\epsilon)||u||\nabla u|_{2}^2,
\]
\[
L_7 = \int \psi \cdot Q(u) \cdot \nabla \phi \leq C|D(u)|_{\infty} |\psi_{2}| |\nabla \phi|_2,
\]
\[
L_8 = \int \psi \cdot Q(u) \cdot \phi^{2e} Lu \leq C|D(u)|_{\infty} |\psi_{2}| |\phi^{2e} Lu|_2
\leq C(\epsilon)|D(u)|^2_{\infty} + \epsilon|\phi^{2e} Lu|_2^2,
\]
where \( \epsilon > 0 \) is a sufficiently small constant.

It follows from (5.19)–(5.21) that
\[
\frac{d}{dt} \int \left( \frac{a}{2} \alpha|\phi^e \nabla u|^2 + \frac{a}{2}(\alpha + \beta)|\phi^e \text{div} u|^2 - \phi \text{div} u \right) dx
+ C|\nabla^2 u|^2_{2} + \frac{a}{2}|\phi^{2e} Lu|^2_{2}
\leq C(1 + |D(u)|^2_{\infty} + |\phi^e \nabla u|_2^2)(1 + |\phi^e \nabla u|_2^2) + C(\epsilon)||u||\nabla u|_{2}^2.
\tag{5.22}
\]

Let \( \eta > 0 \) be a sufficiently small constant. We add \( \eta(5.22) \) to (5.2), and it follows from Gronwall’s inequality that
\[
|\phi^e \nabla u|^2_{2} + |\sqrt{\epsilon} u|^2_{2} + \int_0^t \left(|\nabla^2 u|^2_{2} + |\phi^{2e} Lu|^2_{2} + ||u||\nabla u|_{2}^2\right) ds \leq C.
\]
for \(0 \leq t \leq T\), which, together with (4.10), implies that
\[
\int_0^t |u_t|^2 ds \leq C \int_0^t (|\phi^{2e} Lu|^2 + |\nabla u|^2 u_t^2 + |\nabla \phi|^2 + |D(u)|_\infty^2 |\psi|^2) ds \leq C.
\]
\[\square\]

**Lemma 5.7.**
\[
\sup_{0 \leq t \leq T} \left( |u_t|^2 + |u|^2_{D^2} + |\nabla \rho|^2 \right) (t) + \int_0^T (|\nabla u_t|^2 + |u|^2_{D^2, \rho}) dt \leq C
\]
for \(0 \leq T < T\), where the constant \(C > 0\) only depends on \((\rho_0, u_0), C_0, \alpha, \beta, A, \gamma, \delta\) and \(T\).

**Proof.** First, it follows from Lemma 3.1, equations (2.2) and Young’s inequality that
\[
|u|_{D^2} \leq C(1 + |u_t|^2 + |\nabla \rho|^2) \quad \text{or} \quad |u|_{D^2} \leq C(1 + |u_t|^2 + |D(u)|_\infty).
\]
which, along with Lemmas 5.1–5.6, implies that
\[
|u|_{D^2} \leq C(1 + |u_t|^2 + |\nabla \rho|^2) \quad \text{or} \quad |u|_{D^2} \leq C(1 + |u_t|^2 + |D(u)|_\infty). \tag{5.24}
\]

Second, differentiating \((1.1)_2\) with respect to \(t\), it reads
\[
\rho u_{tt} - \text{div}(2\mu(\rho)D(u_t) + \lambda(\rho)\text{div}u_t \mathbb{I}_3)
= -\rho_t u_t - \rho u \cdot \nabla u_t - \rho u_t \cdot \nabla u - \nabla P_t
+ \text{div}(2\mu(\rho)_tD(u) + \lambda(\rho)_t\text{div}u_t \mathbb{I}_3). \tag{5.25}
\]

Multiplying (5.25) by \(u_t\) and integrating over \(\mathbb{T}^3\), one has
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 + \int \left( \frac{1}{2} \mu(\rho) |\nabla u_t + \nabla u_t^\top|^2 + \lambda(\rho)(\text{div}u_t)^2 \right)
= -\int \rho u \cdot \nabla |u_t|^2 - \int \rho u \nabla (u \cdot \nabla u_t) - \int \rho u_t \cdot \nabla u \cdot u_t
+ \int P_t \text{div}u_t - \int (2\mu(\rho)_tD(u) : D(u_t) + \lambda(\rho)_t\text{div} \text{div}u_t)
\equiv: \sum_{i=9}^{13} \tilde{L}_i. \tag{5.26}
\]

According to Lemmas 5.1–5.6, Hölder’s inequality, Gagliardo–Nirenberg inequality and Young’s inequality, we deduce that
\[ \tilde{L}_9 = -\int \rho u \cdot \nabla |u_t|^2 \leq C|\rho|_{\infty}|u|_{\infty}|u_t|_2|\nabla u_t|_2 \leq C\|\nabla u\|_1^2|u_t|_2^2 + \epsilon|\nabla u_t|_2^2, \]

\[ \tilde{L}_{10} = -\int \rho u \cdot \nabla (u \cdot \nabla u_t) \leq C(|u|_6^2|\nabla u|^2 + |u|_6 + |u|^2|\nabla^2 u|_2) \leq C(|\nabla u|_2^2|\nabla u_t| + |\nabla u_t|_1|\nabla u|_2) \leq C\|\nabla u\|_1|\nabla u_t|_2 \leq \epsilon|\nabla u_t|_2^2 + C(\epsilon)|\nabla u|_1^2, \]

\[ \tilde{L}_{11} = -\int \rho u_t \cdot \nabla u \cdot u_t \leq C|\rho|_{\infty}|u_t|_2|\nabla u|_3 \leq \epsilon|\nabla u_t|_2^2 + C(\epsilon)|u_t|_2^2\|\nabla u\|_1^2, \tag{5.27} \]

\[ \tilde{L}_{12} = \int P_t \text{div}u_t \leq \int \Omega |u_t| \nabla P + \gamma P \text{div} |\nabla u_t| \leq C(|u|_{\infty}|\nabla P|_2 + |P|_{\infty}|\text{div} u|_2)|\nabla u_t|_2 \leq \epsilon|\nabla u_t|_2^2 + C(\epsilon)|\nabla u|_1^2, \]

\[ \tilde{L}_{13} = \int \left( 2\mu(\rho)D(u) : D(u_t) + \lambda(\rho)\text{div} u \right) \leq C|D(u)|_{\infty}|\nabla u_t|_2(|\rho|_{\infty}|\nabla u|_2 + |u|_3|\nabla \rho|_6) \leq \epsilon|\nabla u_t|_2^2 + C(\epsilon)|D(u)|_{\infty}^2(1 + |\nabla \rho|_6). \]

Then it follows from (5.26)–(5.27) and Lemmas 5.6 and 5.14 that

\[ \frac{1}{2} \frac{d}{dt} \int \rho|u_t|^2 + \int |\nabla u_t|^2 \leq C(\|\nabla \rho\|_6^2 + |u_t|_2^2 + 1)(\|\nabla u\|_1^2 + |D(u)|_{\infty}^2 + 1). \tag{5.28} \]

Next, applying \( \nabla \) to (1.1) and multiplying by \( 6|\nabla \rho|^4 \nabla \rho \), one has

\[ (\|\nabla \rho\|_6^6)_t + \text{div}(\nabla \rho)^6 u + 5|\nabla \rho|^6 \text{div} u = -6|\nabla \rho|^4 (\nabla \rho)^7 D(u)(\nabla \rho) - 6\rho|\nabla \rho|^4 \nabla \rho \cdot \nabla \text{div} u. \tag{5.29} \]

It follows from Lemma 3.1 and equations (2.2) that

\[ |\nabla^2 u|_6 \leq C(1 + |\nabla u_t|_2 + |D(u)|_{\infty}(1 + |\nabla \rho|_6)). \tag{5.30} \]

Then integrating (5.29) over \( \Omega^3 \) and noticing (5.30), one immediately obtains

\[ \frac{d}{dt}|\nabla \rho|_6 \leq C|D(u)|_{\infty}(|\nabla \rho|_6 + 1) + C(\epsilon) + C|\nabla u_t|_2^2, \tag{5.31} \]

which, together with (5.28) and Gronwall’s inequality, implies (5.23).

This completes the proof of this lemma.

Until now, we have obtained that

\[ \lim_{T \to \infty} \left( \sup_{0 \leq t \leq T} \|\nabla \rho(t, \cdot)\|_{L^6(\Omega^3)} + \int_0^T \|D(u)(t, \cdot)\|_{L^\infty(\Omega^3)} \, dt \right) = C_0 < \infty, \tag{5.32} \]

for some constant \( C_0 > 0 \), then the rest of the proof can be obtained by the completely same argument used in the proof for Lemmas 4.6–4.8.
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Declarations

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Appendix: Some Basic Lemmas

In this section, we list some basic lemmas to be used later. The first one is the well-known Gagliardo–Nirenberg inequality.

Lemma 5.8. [26] For $p \in [2, 6]$, $q \in (1, \infty)$, and $r \in (3, \infty)$, there exists some generic constant $C > 0$ that may depend on $q$ and $r$ such that for

$$f \in H^1(\mathbb{R}^3), \quad \text{and} \quad g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3),$$

it holds that

$$|f|^p_p \leq C|f|^2_2^{(6-p)/2} |\nabla f|^2_2^{(3p-6)/2}, \quad |g|_\infty \leq C|g|^q(r-3)/(3r+q(r-3)) |\nabla g|^{3r/(3r+q(r-3))}.$$  \hspace{1cm} (5.33)

Some special cases of this inequality are

$$|u|_6 \leq C|u|_{D^1}, \quad |u|_\infty \leq C|u|_6 \frac{1}{2} |\nabla u|_6^{\frac{1}{2}}, \quad |u|_\infty \leq C\|u\|_{W^{1,r}}.$$  \hspace{1cm} (5.34)

The second lemma gives some compactness results obtained via the Aubin-Lions Lemma.

Lemma 5.9. [39] Let $X_0 \subset X \subset X_1$ be three Banach spaces. Suppose that $X_0$ is compactly embedded in $X$ and $X$ is continuously embedded in $X_1$. Then the following statements hold.

i) If $J$ is bounded in $L^p([0,T]; X_0)$ for $1 \leq p < +\infty$, and $\partial J/\partial t$ is bounded in $L^1([0,T]; X_1)$, then $J$ is relatively compact in $L^p([0,T]; X)$;

ii) If $J$ is bounded in $L^\infty([0,T]; X_0)$ and $\partial J/\partial t$ is bounded in $L^p([0,T]; X_1)$ for $p > 1$, then $J$ is relatively compact in $C([0,T]; X)$.

The third one can be found in Majda [34].

Lemma 5.10. [34] Let $r$, $a$ and $b$ be constants such that

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}, \quad \text{and} \quad 1 \leq a, \ b, \ r \leq \infty.$$

$\forall s \geq 1$, if $f, g \in W^{s,a} \cap W^{s,b}(\mathbb{R}^3)$, then it holds that

$$|\nabla^s(fg) - f\nabla^s g|_r \leq C_s((|\nabla f|_a |\nabla^{s-1} g|_b + |\nabla^s f|_a |g|_b),$$  \hspace{1cm} (5.35)

$$|\nabla^s(fg) - f\nabla^s g|_r \leq C_s((|\nabla f|_a |\nabla^{s-1} g|_b + |\nabla^s f|_a |g|_b),$$  \hspace{1cm} (5.36)

where $C_s > 0$ is a constant depending only on $s$, and $\nabla^s f$ $(s \geq 1)$ is the set of all $\partial^s f$ with $|\zeta| = s$. Here $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$ is a multi-index.

The following lemma is important in the derivation of the a priori estimates in Sect. 3, which can be found in Remark 1 of [2].
Lemma 5.11. [2] If $f(t,x) \in L^2([0,T];L^2)$, then there exists a sequence $s_k$ such that

$$s_k \to 0, \quad \text{and} \quad s_k |f(s_k,x)|_2^2 \to 0, \quad \text{as} \quad k \to +\infty.$$ 

Next we give one Sobolev inequalities on the interpolation estimate in the following lemma.

Lemma 5.12. [34] Let $u \in H^s$, then for any $s' \in [0,s]$, there exists a constant $C_s$ only depending on $s$ such that

$$\|u\|_s \leq C_s \|u\|_0^{1-\frac{s'}{s}} \|u\|_{s'}^{\frac{s'}{s}}.$$ 

In order to improve a weak convergence to the strong convergence, we give the following lemma.

Lemma 5.13. [34] If the function sequence $\{w_n\}_{n=1}^{\infty}$ converges weakly to $w$ in a Hilbert space $X$, then it converges strongly to $w$ in $X$ if and only if

$$\|w\|_X \geq \limsup_{n \to \infty} \|w_n\|_X.$$ 

The next lemma is used to give the estimate on $\nabla u_\epsilon$ in the periodic problem away from the vacuum.

Lemma 5.14. [11] Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be an open, connected domain. Then there is a constant $C > 0$, known as the Korn constant of $\Omega$, such that, for all vector fields $v = (v^1, \ldots, v^n) \in H^1(\Omega)$,

$$\|v\|_{H^1(\Omega)}^2 \leq C \int_{\Omega} (|v|^2 + |D(v)|^2) \, dx.$$ 

Finally, we give the well-known Fatou’s lemma.

Lemma 5.15. Given a measure space $(V, \mathcal{F}, \nu)$ and a set $X \in \mathcal{F}$, let $\{f_n\}$ be a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$-measurable non-negative functions $f_n : X \to [0, \infty]$. Define the function $f : X \to [0, \infty]$ by setting

$$f(x) = \liminf_{n \to \infty} f_n(x),$$

for every $x \in X$. Then $f$ is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\geq 0}})$-measurable, and

$$\int_X f(x) \, d\nu \leq \liminf_{n \to \infty} \int_X f_n(x) \, d\nu.$$ 

References

[1] Beale, T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equation. Commun. Math. Phys. 94, 61–66 (1984)

[2] Boldrini, J.L., Rojas-Medar, M.A., Fernández-Cara, E.: Semi-Galerkin approximation and regular solutions to the equations of the nonhomogeneous asymmetric fluids. J. Math. Pures Appl. 82, 1499–1525 (2003)

[3] Bresch, D., Desjardins, B.: Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. Commun. Math. Phys. 238, 211–223 (2003)

[4] Bresch, D., Desjardins, B.: On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids. J. Math. Pures Appl. 87, 57–90 (2007)

[5] Bresch, D., Desjardins, B., Lin, C.: On some compressible fluid models: Korteweg, Lubrication, and Shallow water systems. Commun. Part. Differ. Equ. 28, 843–868 (2003)

[6] Bresch, D., Desjardins, B., Métivier, G.: Recent mathematical results and open problems about shallow water equations. In: Analysis and Simulation of Fluid Dynamics. Adv. Math. Fluid Mech., pp. 15–31. Birkhäuser, Basel (2007)

[7] Bresch, D., Noble, P.: Mathematical derivation of viscous shallow-water equations with zero surface tension. Indiana Univ. Math. J. 60, 1137–1169 (2011)

[8] Bresch, D., Vasseur, A., Yu, C.: Global existence of entropy-weak solutions to the compressible Navier–Stokes equations with non-linear density dependent viscosities (2019). arXiv:1906.02701

[9] Chapman, S., Cowling, T.: The Mathematical Theory of Non-Uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases. Cambridge University Press, Cambridge (1990)

[10] Cho, Y., Choe, H.J., Kim, H.: Unique solvability of the initial boundary value problems for compressible viscous fluids. J. Math. Pure Appl. 83, 243–275 (2004)

[11] Ciarlet, P.G.: On Korn’s inequality. Chin. Ann. Math. Ser. (B) 31, 607–618 (2010)

[12] Constantin, P., Drivas, T., Nguyen, H., Pasqualotto, F.: Compressible fluids and active potentials. Ann. Inst. Henri Poincaré Anal. Nonlinéaire 37, 145–180 (2020)
Gent, P.: The energetically consistent shallow water equations. J. Atmos. Sci. 50, 1323–1325 (1993)

Gerbeau, J., Perthame, B.: Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. Discrete Contin. Dyn. Syst. (B) 1, 89–102 (2001)

Germain, P., LeFloch, P.: Finite energy method for compressible fluids: The Navier–Stokes–Korteweg model. Commun. Pures Appl. Math. LXIX, 3–61 (2016)

Guo, Z., Li, H., Xin, Z.: Lagrange structure and dynamics for solutions to the spherically symmetric compressible Navier–Stokes equations. Commun. Math. Phys. 309, 371–412 (2012)

Haspot, B.: Global $bmo^{-1}(\mathbb{R}^N)$ radially symmetric solution for compressible Navier–Stokes equations with initial density in $L^p(\mathbb{R}^N)$, arXiv:1901.03143v1 (2019)

Huang, X., Li, J., Xin, Z.: Blow-up criterion for the compressible flows with vacuum states. Commun. Math. Phys. 301, 23–35 (2010)

Jia, Q., Wang, Y., Xin, Z.: Global well-posedness of 2D compressible Navier–Stokes equations with large data and vacuum. J. Math. Fluid Mech. 16, 483–521 (2014)

Jüngel, A.: Global weak solutions to compressible Navier–Stokes equations for quantum fluids. SIAM. J. Math. Anal. 42, 1025–1045 (2010)

Kawashima, S.: Systems of A Hyperbolic-Parabolic Composite Type, with Applications to The Equations of Magnetohydrodynamics, Ph.D. thesis, Kyoto University, https://doi.org/10.14989/doctor.k3193 (1983)

Kloeeden, P.E.: Global existence of classical solutions in the dissipative shallow water equations. SIAM. J. Math. Anal. 16, 301–315 (1985)

Ladyzenskaja, O.A., Ural’ceva, N.N.: Linear and Quasilinear Equations of Parabolic Type. American Mathematical Society, Providence, RI (1968)

Li, H., Li, J., Xin, Z.: Vanishing of vacuum states and blow-up phenomena of the compressible Navier–Stokes equations. Commun. Math. Phys. 281, 401–444 (2008)

Li, J., Xin, Z.: Global existence of weak solutions to the barotropic compressible Navier–Stokes flows with degenerate viscosities, preprint (2016) arXiv:1504.06826

Li, T., Qin, T.: Physics and Partial Differential Equations. SIAM, Philadelphia, Higher Education Press, Beijing (2014)

Li, Y., Pan, R., Zhu, S.: On classical solutions to 2D shallow water equations with degenerate viscosities. J. Math. Fluid Mech. 19, 151–190 (2017)

Li, Y., Pan, R., Zhu, S.: On classical solutions for viscous polytropic fluids with degenerate viscosities and vacuum. Arch. Ration. Mech. Anal. 234, 1281–1334 (2019)

Lions, P.L.: Mathematical Topics in Fluid Mechanics: Compressible Models, vol. 2. Oxford University Press, New York (1998)

Liu, T., Yin, Z., Yang, T.: Vacuum states for compressible flow. Discrete Contin. Dyn. Syst. 4, 1–32 (1998)

Majda, A.: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Applied Mathematical Science 53. Spinger Berlin Heidelberg, New York (1986)

Marche, F.: Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. Eur. J. Mech. B/Fluids 26, 49–63 (2007)

Mellet, A., Vasseur, A.: On the barotropic compressible Navier–Stokes equations. Commun. Part. Differ. Equ. 32, 431–452 (2007)

Nash, J.: Le probleme de Cauchy pour les equations differentielles dun fluide general. Bull. Soc. Math. France 90, 197–217 (1962)

Ponce, G.: Remarks on a paper: remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. 98, 349–353 (1985)

Simon, J.: Compact sets in $L^p(0,T;B)$. Ann. Mat. Pura. Appl. 146, 65–96 (1987)

Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, Princeton (1970)

Sunbye, L.: Global existence for the Cauchy problem for the viscous shallow water equations. Rocky Mt. J. Math. 28, 1135–1152 (1998)

Vasseur, A., Yu, C.: Global weak solutions to compressible Navier–Stokes equations with damping. SIAM J. Math. Anal. 48, 1489–1511 (2016)

Vasseur, A., Yu, C.: Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations. Invent. Math. 206, 935–974 (2016)

Xin, Z., Zhu, S.: Global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier–Stokes equations with degenerate viscosities and vacuum, arXiv:1806.02383 (2019, submitted)

Xin, Z., Zhu, S.: Well-posedness of three-dimensional isentropic compressible Navier–Stokes equations with degenerate viscosities and far field vacuum. J. Math. Pures Appl. (2021, to appear). arXiv:1811.04744v2

Yang, T., Zhao, H.: A vacuum problem for the one-dimensional compressible Navier–Stokes equations with density-dependent viscosity. J. Differ. Equ. 184, 163–184 (2002)

Yang, T., Zhu, C.: Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum. Commun. Math. Phys. 230, 329–363 (2002)
[48] Zhu, S.: On classical solutions of the compressible Magnetohydrodynamic equations with vacuum. SIAM J. Appl. Math. 425, 928–953 (2015)
[49] Zhu, S.: Well-Posedness and Singularity Formation of Isentropic Compressible Navier–Stokes Equations, Ph.D. Thesis, Shanghai Jiao Tong University (2015)

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