A flexible family of distributions on the cylinder

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Abstract

We propose a flexible family of distribution, generalized $t$-distribution, on the cylinder which is obtained as a conditional distribution of a trivariate normal distribution. The new distribution has bimodality and asymmetry depending on the values of parameters and flexibly fits the cylindrical data. The circular marginal of this distribution is distributed as a generalized $t$-distribution on the circle. Some other properties are also investigated. The proposed distribution is applied to the real cylindrical data.

Key words and phrases: circular-linear correlation, circular-linear regression, generalized von Mises distribution, Johnson–Wehrly model, Mardia–Sutton model

1 Introduction

The directional or circular data often appear in a variety of scientific fields and various stochastic models have been proposed and investigated for analyzing such data. For univariate circular data, there have been many distributions investigated in terms of both tractability and applicability, see Jones and Pewsey (2005), Kato and Jones (2010) and Kato and Jones (2014). However, we sometimes encounter situations which involve both circular and linear variables, namely cylindrical data such as the pair of wind direction and temperature (Mardia and Sutton, 1978) or directions and distances of animal movements (Fisher, 1993). It is clear that the univariate circular distribution is not enough to analyze such cylindrical data. Thus, the distribution on the cylinder is needed, but there are not so many distributions available compared to univariate circular distributions. We give a brief review below for several cylindrical distributions known in the literature. Johnson and Wehrly (1978) gave a distribution based on the principle of maximum entropy subject to constraints on certain moments, and Mardia and Sutton (1978) provided another distribution as a conditional distribution of a trivariate normal distribution or a maximum entropy distribution. An extension of the distribution by Mardia and Sutton (1978) was studied by Kato and Shimizu (2008), which can also be derived as a maximum entropy distribution or a conditional of a trivariate normal.

In this paper, we propose the generalized $t$-distribution on the cylinder, which is a natural extension of the member of the exponential family given by Kato and Shimizu (2008). The proposed distribution is also regarded as a cylindrical extension of the generalized $t$-distribution on the circle proposed by Siew, Kato and Shimizu (2008). In fact, the circular marginal

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distribution is the generalized $t$-distribution on the circle. The proposed distribution can be obtained as a conditional distribution of a trivariate normal distribution and characterized as the maximum $\beta$-entropy distribution. This is a quite flexible distribution which can express both bimodality and unimodality in terms of parameter values. We investigate some properties such as marginal and conditional distributions, modality, moments, circular-linear correlation and skewness. We briefly discuss a circular-linear regression model derived from the conditional distribution.

Subsequent sections are organized as follows. Section 2 provides a derivation of the distribution. In Section 3 some properties of the new distribution are studied. Section 4 deals with a method for estimating parameters in the new distribution and empirical application is discussed as well.

2 Derivation

Suppose that a trivariate random vector $W$ given $U = u > 0$ is distributed as a normal distribution $N_3(\mu, \Sigma/u)$ with mean vector $\mu = (\eta_1, \eta_2, \eta_3)' \in \mathbb{R}^3$ and variance-covariance matrix $\Sigma/u$, where

$$
\Sigma = \begin{pmatrix}
\sigma_1^2 & \rho_{12} \sigma_1 \sigma_2 & \rho_{13} \sigma_1 \sigma_3 \\
\rho_{12} \sigma_1 \sigma_2 & \sigma_2^2 & \rho_{23} \sigma_2 \sigma_3 \\
\rho_{13} \sigma_1 \sigma_3 & \rho_{23} \sigma_2 \sigma_3 & \sigma_3^2 
\end{pmatrix}
$$

for $\sigma_j > 0$ ($j = 1, 2, 3$), $-1 < \rho_{12} < 1$ and $1 + 2 \rho_{12} \rho_{13} \rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 > 0$. The conditional density function of $W | (U = u)$ is given by

$$
f(w|u) = \frac{u^{3/2}}{(2\pi)^{3/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{u}{2} (w - \eta)' \Sigma^{-1} (w - \eta) \right\}, \quad w \in \mathbb{R}^3.
$$

Let $U$ follow a gamma distribution with density

$$
g(u) = \frac{\alpha^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\alpha u}, \quad u > 0
$$

for $\alpha > 0$. Then the unconditional distribution of $W$ has density

$$
f(w) = \int_0^\infty f(w|u)g(u)du
$$

$$
= \frac{\Gamma(\alpha + 3/2)}{(2\pi)^{3/2} \alpha^{3/2} |\Sigma|^{1/2} \Gamma(\alpha)} \left\{ 1 + \frac{(w - \eta)' \Sigma^{-1} (w - \eta)}{2\alpha} \right\}^{-(\alpha+3/2)}, \quad w \in \mathbb{R}^3.
$$

We use a cylindrical coordinate $W = (X, X_1, X_2)'$, where $X_1 = R \cos \Theta$ and $X_2 = R \sin \Theta$ with $R > 0$ and $0 \leq \Theta < 2\pi$, and consider the conditional distribution of $(X, \Theta)'$ given $R = r$, which provides a distribution on the cylinder. Define new parameters as

$$
\mu(\theta) = \mu + \lambda \cos(\theta - \nu), \quad \tau^2 = \frac{\sigma_1^2 \rho_2^2}{1 - \rho_{23}^2},
$$

$$
\kappa_1^* \cos \mu_1 = r (b_1 \eta_2 - b_2 \eta_3), \quad \kappa_1^* \sin \mu_1 = r (b_3 \eta_3 - b_2 \eta_2),
$$

$$
\kappa_2^* \cos 2\mu_2 = \frac{r^2 (b_3 - b_1)}{4}, \quad \kappa_2^* \sin 2\mu_2 = \frac{r^2 b_2}{2}
$$
with \(-\infty < \mu < \infty, \lambda, \kappa_1^*, \kappa_2^* \geq 0, 0 \leq \nu, \mu_1 < 2\pi\) and \(0 \leq \mu_2 < \pi\), where
\[
\lambda \cos \nu = -a_1 r, \quad \lambda \sin \nu = -a_2 r.
\]

Then we have
\[
\frac{1}{2}(w - \eta)\Sigma^{-1}(w - \eta) = \frac{1}{2\tau^2}\{x - \mu(\theta)\}^2 - \kappa_1^* \cos(\theta - \mu_1) - \kappa_2^* \cos 2(\theta - \mu_2) + d,
\]
where \(d = (b_1 \eta_2^2 + b_3 \eta_3^2 - 2b_2 \eta_2 \eta_3)/2 \geq 0\). The conditional probability density function \(f(x, \theta|R)\) of \((X, \Theta)|R = r\) is represented as
\[
f(x, \theta|R) = C^{-1} \left[ 1 + \frac{1}{2\sigma^2} \left\{x - \mu(\theta)\right\}^2 - \kappa_1 \cos(\theta - \mu_1) - \kappa_2 \cos 2(\theta - \mu_2) \right]^{-(\alpha+3)/2},
\]
where \(\kappa_1 = \kappa_1^*/\gamma, \kappa_2 = \kappa_2^*/\gamma, \sigma^2 = \gamma \tau^2, \gamma = \alpha + d > 0\) and the normalizing constant is
\[
C = 2\sqrt{2\pi} B(1/2, \alpha + 1) \sigma \left\{ F_4 \left( \frac{\alpha}{2} + \frac{1}{2}, \frac{\alpha}{2} + 1, 1, 1; \kappa_1^2, \kappa_2^2 \right) + 2 \sum_{j=1}^{\infty} \frac{(\alpha + 1)_{3j}}{(2j)!j!} \left( \frac{\kappa_1}{2} \right)^{2j} \left( \frac{\kappa_2}{2} \right)^j \cos 2j(\mu_2 - \mu_1) \times F_4 \left( \frac{\alpha + 3j}{2} + \frac{1}{2}, \frac{\alpha + 3j}{2} + 1, 2j + 1, j + 1; \kappa_1^2, \kappa_2^2 \right) \right\}.
\]

Here \(F_4\) denotes Appell’s double hypergeometric function (Gradshteyn and Ryzhik, 2007, 9.180.4) defined by
\[
F_4(\alpha_1, \alpha_2, \beta_1, \beta_2; z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha_1)_{i+j}(\alpha_2)_{i+j} z_1^i z_2^j}{(\beta_1)_i(\beta_2)_j i! j!}, \quad \sqrt{z_1} + \sqrt{z_2} < 1
\]
with Pochhammer’s symbol
\[
(c)_j = \begin{cases} 
c(c+1) \cdots (c+j-1), & j \geq 1, \\
1, & j = 0,
\end{cases}
\]
and \(B\) the beta function. The distribution with density function \((1)\) has nine parameters \(\alpha, \sigma > 0, -\infty < \mu < \infty, \kappa_1, \kappa_2, \lambda \geq 0, 0 \leq \nu, \mu_1 < 2\pi\) and \(0 \leq \mu_2 < \pi\) with restriction \(\kappa_1 + \kappa_2 < 1\). The resulting distribution should be called the generalized \(t\)-distribution on the cylinder. Relationships among the new and original parameters are:

(a) \(\eta_2 = \eta_3 = 0 \iff \kappa_1 = 0\).
\( \kappa_2 = 0 \iff \rho_{23} = 0, \sigma_2 = \sigma_3. \)

\( \lambda = 0 \iff \rho_{12} = \rho_{13} = 0. \)

As being introduced in (1), the parameter \( \alpha \) was assumed positive. However, the density is still valid when the parameter space of \( \alpha \) is extended to \( \alpha \geq -1/2. \) Note that \( \alpha + 3/2 \) corresponds to the degree of freedom in the generalized \( t \)-distribution on the cylinder (1) and the parameter \( \alpha \) determines the degree of concentration around the mode. To see this, we give contour plots of the proposed density (1) in Figure 1, where \( \alpha = -1/2, 1, 3 \) and the other parameters are specified as \( \lambda, \mu, \mu_1, \mu_2 = 0, \nu = \pi/3, \kappa_1 = 0.1, \kappa_2 = 0.5 \) and \( \sigma = 1. \)

Under the assumption \( \gamma = \alpha + 3/2, \) if we use reparametrization \( 1/\psi = -(\alpha + 3/2), -\kappa_1 = \tanh(\kappa_1^* \psi), -\kappa_2 = \tanh(\kappa_2^* \psi), \) then the density (1) has a similar expression to the family of distributions introduced by Jones and Pewsey (2005):

\[
 f(x, \theta | r) \propto \left[ 1 - \frac{\psi}{2\tau^2} (x - \mu(\theta))^2 + \tanh(\kappa_1^* \psi) \cos(\theta - \mu_1) + \tanh(\kappa_2^* \psi) \cos 2(\theta - \mu_2) \right]^{1/\psi}.
\]

The restriction \( \alpha \geq -1/2 \) in (1) is changed into \( -1 \leq \psi < 0. \)

\[\text{Figure 1: Contour plots of the proposed density (1) when (a) } \alpha = -1/2, \text{ (b) } \alpha = 1 \text{ and (c) } \alpha = 3. \text{ The other parameters are set as } \lambda, \mu, \mu_1, \mu_2 = 0, \nu = \pi/3, \kappa_1 = 0.1, \kappa_2 = 0.5 \text{ and } \sigma = 1. \]

### 3 Properties

#### 3.1 Special cases

1. Under the assumption \( \gamma = \alpha \) in (1), letting \( \gamma (= \alpha) \rightarrow \infty, \) we have an extension of the distribution by Mardia and Sutton (1978). Its joint probability density function (Kato and Shimizu, 2008) is

\[
f(x, \theta) = C_1^{-1} \exp \left[ -\frac{(x - \mu(\theta))^2}{2\tau^2} + \kappa_1^* \cos(\theta - \mu_1) + \kappa_2^* \cos 2(\theta - \mu_2) \right]
\]

with the normalizing constant

\[
 C_1 = (2\pi)^{3/2} \tau \left[ I_0(\kappa_1^*)I_0(\kappa_2^*) + 2 \sum_{j=1}^{\infty} I_j(\kappa_1^*)I_{2j}(\kappa_2^*) \cos 2j(\mu_1 - \mu_2) \right].
\]

\[(a) \quad \text{Figure 1: Contour plots of the proposed density (1) when (a) } \alpha = -1/2, \text{ (b) } \alpha = 1 \text{ and (c) } \alpha = 3. \text{ The other parameters are set as } \lambda, \mu, \mu_1, \mu_2 = 0, \nu = \pi/3, \kappa_1 = 0.1, \kappa_2 = 0.5 \text{ and } \sigma = 1. \]
Here \( I_j \) denotes the modified Bessel function of the first kind and order \( j \) given by
\[
I_j(z) = \frac{1}{2\pi} \int_0^{2\pi} \cos(j\theta)e^{z\cos\theta}d\theta = \sum_{r=0}^{\infty} \frac{1}{\Gamma(r+j+1)r!} \left( \frac{z}{2} \right)^{2r+j}, \quad z \in \mathbb{C}.
\]

2. When \( \kappa_2 = 0 \), (1) reduces to
\[
f(x, \theta) = C_1^{-1} \left[ 1 + \frac{1}{2\sigma^2} \{ x - \mu(\theta) \}^2 - \kappa_1 \cos(\theta - \mu_1) \right]^{-(\alpha+3/2)}.
\]

The normalizing constant is represented as
\[
C_2 = 2\sqrt{2\pi\sigma B(1/2, \alpha + 1)} F_1 \left( \frac{\alpha}{2} + \frac{1}{2}, \frac{\alpha}{2} + 1, 1; \kappa_1^2 \right)
\]
using the Gauss hypergeometric function \( _2F_1 \). If we replace \( \kappa_1 = \kappa_1^* / \gamma, \sigma^2 = \gamma \tau^2 \) and let \( \gamma = \alpha \to \infty \), we have the distribution proposed by Mardia and Sutton (1978) and the constant \( C_2 \) tends to \((2\pi)^{3/2} \tau I_0(\kappa_1^*)\). This agrees with the fact that \( C_1 = (2\pi)^{3/2} \tau I_0(\kappa_1^*) \) when \( \kappa_2^* = 0 \) in (4).

### 3.2 Marginal and conditional distributions

The marginal distribution of \( \Theta \) is
\[
f_\Theta(\theta) = C_\Theta \{ 1 - \kappa_1 \cos(\theta - \mu_1) - \kappa_2 \cos 2(\theta - \mu_2) \}^{-(\alpha+1)}, (6)
\]
where
\[
C_\Theta = 2\pi \left\{ F_4 \left( \frac{\alpha}{2} + \frac{1}{2}, \frac{\alpha}{2} + 1, 1, 1; \kappa_1^2, \kappa_2^2 \right) 
+ 2 \sum_{j=1}^{\infty} \frac{(\alpha + 1)_{2j}}{(2j)!j!} \left( \frac{\kappa_1}{2} \right)^{2j} \left( \frac{\kappa_2}{2} \right)^{2j} \cos 2j(\mu_2 - \mu_1) \right.
\times F_4 \left( \frac{\alpha + 3j}{2} + 1, 1, 2j + 1, 1; \kappa_1^2, \kappa_2^2 \right) \right\}.
\]

The distribution with density (3) is a member of the generalized \( t \)-distributions on the circle proposed by Siew, Kato and Shimizu (2008), and is possibly bimodal and asymmetric. Cosine and sine moments of the generalized \( t \)-distributions are given in their paper. The generalized \( t \)-distributions include the generalized von Mises distribution (cf. Yfantis and Borgman, 1982) as a special case. As another special case when \( \kappa_2 = 0 \) in (1), the marginal distribution of \( \Theta \) belongs to the family of symmetric distributions by Jones and Pewsey (2005). Note that the marginal density (6) is independent of \( \lambda, \mu, \sigma \) and \( \nu \) which are the parameters of the proposed density (1). On the other hand, the marginal distribution of \( X \) does not have a closed form in general. When \( \lambda = 0 \), we can obtain the marginal distribution of \( X \) in a closed form given by
\[
f_X(x) = C^{-1} D(x) \left\{ 1 + \frac{1}{2\sigma^2} (x - \mu)^2 \right\}^{-(\alpha+3/2)},
\]
where
\[
D(x) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{2\sigma^2}} dy.
\]

Similarly, the marginal distribution of \( \Theta \) can be obtained in a closed form.
where $C$ is defined by (2) and $D(x)$ is obtained by replacing $\alpha$ and $\kappa_i$ ($i = 1, 2$) with $\alpha + 1/2$ and $\kappa_i/(1 + (x - \mu)^2/(2\sigma^2))$, respectively, in $C_\Theta$ defined in (7). Note that this density is symmetric about $\mu$.

In (11), the conditional distribution of $X$ given $\Theta = \theta$ has the generalized $t$-density function provided by

$$f_{X|\Theta}(x|\theta) = C_{X|\Theta}^{-1} \left(1 + \frac{(x - \mu(\theta))^2}{2\sigma^2 [1 - \{\kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos(2(\theta - \mu_2))\}]}\right)^{-(\alpha+3/2)} \tag{8}$$

with the normalizing constant

$$C_{X|\Theta} = \sqrt{2\sigma B \left(\frac{1}{2}, \alpha + 1\right)} \left[1 - \{\kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos(2(\theta - \mu_2))\}\right]^{1/2}.$$

The conditional distribution of $\Theta$ given $X = x$ is a member of the generalized $t$-distributions on the circle, and is obtained by replacing $\alpha$ and $\kappa_i$ ($i = 1, 2$) with $\alpha + 1/2$ and $\kappa_i/(1 + (x - \mu)^2/(2\sigma^2))$, respectively, in (6).

### 3.3 Modality

We consider the modality of the distribution with density (11). The mode $x^*$ of (11), whenever the value of $\theta$ is specified, is $x^* = \mu(\theta)$. Similar to Siew, Kato and Shimizu (2008), we discuss maximization of the function $m(\theta) = \kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos(2(\theta - \mu_2))$ with respect to $\theta$. The solution $\theta^*$ of an equation

$$\kappa_1 \sin(\theta - \mu_1) + 2\kappa_2 \sin(2(\theta - \mu_2)) = 0 \tag{9}$$

is a value which maximizes $m(\theta)$ if the sign of $h(\theta^*)$ is positive, where

$$h(\theta) = \kappa_1 \cos(\theta - \mu_1) + 4\kappa_2 \cos(2(\theta - \mu_2)).$$

Equation (9) can be solved numerically for any combinations of $\mu_1$ and $\mu_2$. Without loss of generality, we let $\mu_1 = 0$. Then (9) has closed form solutions when $\mu_2 = 0, \pi/2, \pi/4$ and $3\pi/4$. The results are given in Table 1, where $\theta_0 = \arccos(\kappa_1/(4\kappa_2))$, $\theta_1 = \arcsin\left(-\kappa_1 + \sqrt{\kappa_1^2 + 32\kappa_2^2}/(8\kappa_2)\right)$ and $\theta_2 = \arcsin\left(\kappa_1 + \sqrt{\kappa_1^2 + 32\kappa_2^2}/(8\kappa_2)\right)$. See Yfantis and Borgman (1982) for more discussion as to the solutions of (9).

Contour plots and marginal density plots of the proposed density (11) are given in Figures 2 and 3. Figure 2 shows the case when $\sigma = 1, \mu = 0, \lambda = 0, \nu = \pi/3, \kappa_1 = 0.5, \kappa_2 = 0.1, \mu_1 = \mu_2 = 0$ and $\alpha = 3$, and Figure 3 when $\sigma = 1, \mu = 0, \lambda = 1, \nu = \pi/3, \kappa_1 = 0.2, \kappa_2 = 0.3, \mu_1 = \mu_2 = 0$ and $\alpha = 3$. In Figure 2, the joint density is unimodal and the marginal densities of $X$ and $\Theta$ are symmetric. On the other hand, in Figure 3, the joint density is bimodal and the marginal density of $X$ is asymmetric. In fact, the marginal density of $X$ is possibly skew depending on the values of the parameters, as will be seen in Section 3.5. The marginal density of $\Theta$ in Figure 3 is bimodal.
Table 1: Summary of the modality of the proposed distribution for some values of $\mu_2$ when $\mu_1 = 0.$

| $\mu_2$ | Condition | Modes $(x, \theta)$ |
|---------|-----------|---------------------|
| 0       | $4\kappa_2 > \kappa_1$ | $(\mu + \lambda \cos \nu, 0), (\mu + \lambda \cos \nu, \pi)$ |
|         | $4\kappa_2 < \kappa_1$ | $(\mu + \lambda \cos \nu, 0)$ |
| $\pi/2$ | $4\kappa_2 > \kappa_1$ | $(\mu + \lambda \cos(\theta_0 - \nu), \theta_0), (\mu + \lambda \cos(\theta_0 + \nu), 2\pi - \theta_0)$ |
|         | $4\kappa_2 < \kappa_1$ | $(\mu + \lambda \cos \nu, \theta_0)$ |
| $\pi/4$ | $2\kappa_2 > \kappa_1$ | $(\mu + \lambda \cos(\theta_1 - \nu), \theta_1), (\mu + \lambda \cos(\theta_2 - \nu), \pi + \theta_2)$ |
|         | $2\kappa_2 < \kappa_1$ | $(\mu + \lambda \cos(\theta_1 - \nu), \theta_1)$ |
| $3\pi/4$ | $2\kappa_2 > \kappa_1$ | $(\mu + \lambda \cos(\theta_2 + \nu), \pi - \theta), (\mu + \lambda \cos(\theta_1 + \nu), 2\pi - \theta_1)$ |
|         | $2\kappa_2 < \kappa_1$ | $(\mu + \lambda \cos(\theta_1 + \nu), 2\pi - \theta_1)$ |

### 3.4 Moments and circular-linear correlation

We consider moments of the proposed distribution. Let $\alpha_{m,k} = E(\cos m\theta)$ and $\beta_{m,k} = E(\sin k\theta)$ be the trigonometric moments of (6) under replacement $\alpha$ with $\alpha - k$, which are obtainable using the results by Siew, Kato and Shimizu (2008). Moments of a random vector $(X, \Theta)'$ having (1) are given by

$$E[(X - \mu(\Theta))^2 \cos m\Theta] = C_{\Theta,k} C^{-1/2} \sigma^{2k+1} B \left( k + \frac{1}{2}, \alpha - k + 1 \right) \alpha_{m,k}$$  \hspace{1cm} (10)

and

$$E[(X - \mu(\Theta))^2 \sin m\Theta] = C_{\Theta,k} C^{-1/2} \sigma^{2k+1} B \left( k + \frac{1}{2}, \alpha - k + 1 \right) \beta_{m,k},$$  \hspace{1cm} (11)

where $C_{\Theta,k}$ is obtained by replacing $\alpha$ with $\alpha - k$ in $C_{\Theta}$ in (7).

Moreover, we derive another type of moments of a random vector $(X, \Theta)'$ having (5) given by putting $\kappa_2 = 0$ in (11). After some calculations, the moments for nonnegative integers $k (< \alpha + 1)$ and $m$ turn out to be

$$E[(X - \mu(\Theta))^2 \cos m(\Theta - \mu_1)]$$

$$= \frac{2^k \sigma^{2k} B \left( k + \frac{1}{2}, \alpha - k + 1 \right) (-1)^{m+\alpha-k+1} P_{\alpha-k}^{m} \left( \frac{1}{\sqrt{1-\kappa_1^2}} \right) (1 - \kappa_1^2)^{-(\alpha-k+1)/2}}{B \left( \frac{1}{2}, \alpha + 1 \right) F_1 \left( \frac{1}{2} + \frac{1}{2}, \frac{1}{2} + 1, 1; \kappa_1^2 \right) (\alpha - k - m + 1)_m},$$  \hspace{1cm} (12)

where $P$ denotes the associated Legendre function (Gradshteyn and Ryzhik, 2007, 8.711.2) defined by

$$P_{\nu}^{m}(z) = \frac{\nu + 1)(\nu + 2) \cdots (\nu + m)}{\pi} \int_{0}^{\pi} \left( z + \sqrt{z^2 - 1} \cos \varphi \right)^{\nu} \cos m\varphi \, d\varphi.$$
Figure 2: (a) Contour plot of log-density, and marginal density plots of (b) the linear random variable $X$ and (c) the circular random variable $\Theta$ for the proposed density (1) when $\sigma = 1$, $\mu = 0$, $\lambda = 0$, $\nu = \pi/3$, $\kappa_1 = 0.5$, $\kappa_2 = 0.1$, $\mu_1 = \mu_2 = 0$, $\alpha = 3$. The marginal distribution of $X$ is symmetric and the marginal distribution of $\Theta$ is unimodal.

Next, we study the circular-linear correlation $R_{x\theta}$ between $X$ and $\Theta$ (cf. Mardia and Jupp, 2000, p. 245) which is defined as

$$R_{x\theta}^2 = \frac{r_{xs}^2 + r_{xc}^2 - 2r_{cs}r_{xs}r_{xc}}{1 - r_{cs}^2},$$

where $r_{xs} = \text{Corr}(X, \cos \Theta), r_{xc} = \text{Corr}(X, \sin \Theta)$ and $r_{cs} = \text{Corr}(\cos \Theta, \sin \Theta)$ are Pearson’s correlation coefficients. We only consider the case where $(X, \Theta)'$ has density (5) for simplicity because the circular-linear correlation of $(X, \Theta)$ with density (1) is not feasible to compute. A straightforward calculation shows that

$$R_{x\theta}^2 = \frac{\lambda^2 U}{q + \lambda^2 U},$$

where

$$U = \frac{1}{2} (1 - p_2) \sin^2(\mu_1 - \nu) + \left\{ \frac{1}{2} (1 + p_2) - p_1^2 \right\} \cos^2(\mu_1 - \nu),$$
Figure 3: (a) Contour plot of log-density, and marginal density plots of (b) the linear random variable $X$ and (c) the circular random variable $\Theta$ for the proposed density (1) when $\sigma = 1$, $\mu = 0$, $\lambda = 1$, $\nu = \pi/3$, $\kappa_1 = 0.2$, $\kappa_2 = 0.3$, $\mu_1 = \mu_2 = 0$, $\alpha = 3$. The marginal distribution of $X$ is asymmetric and the marginal distribution of $\Theta$ is bimodal.

and $p_1, p_2$ and $q$ denote $p_m = E\{\cos m(\Theta - \mu_1)\}, m = 1, 2$, and $q = E[\{X - \mu(\Theta)\}^2]$, calculable from (12). Note that $U > 0$ since $1 - p_2 > 0$ and $(1 + p_2)/2 - p_1^2 = \text{Var}\{\cos(\Theta - \mu_1)\} > 0$. We can observe that $R^2_{x\theta}$ is an increasing function of $\lambda$, and $R^2_{x\theta} = 0$ if and only if $\lambda = 0$. Furthermore, letting $\gamma = \alpha \to \infty$, it is seen that (13) reduces to the circular-linear correlation of the distribution proposed by Mardia and Sutton (1978) because (5) goes to the Mardia–Sutton model. This is confirmed from the fact that $q \to \tau^2$ and $p_m \to I_m(\kappa^*)/I_0(\kappa^*), m = 1, 2$ as $\gamma = \alpha \to \infty$.

### 3.5 Skewness of marginal distribution on the real line

For the proposed density (1), we derive the skewness of the marginal density of $X$ defined as

$$sk = \frac{E\left[\{X - E(X)\}^3\right]}{(E\left[\{X - E(X)\}^2\right])^{3/2}}.$$
Straightforward calculation shows that

\[ sk = \frac{3v_2\lambda + v_3\lambda^3}{(v_1\lambda^2 + q)^{3/2}}, \tag{14} \]

where \( v_1 = \text{Var}\{\cos(\Theta - \nu)\} \), \( v_2 = \text{Cov}\{\{X - \mu(\Theta)\}^2, \cos(\Theta - \nu)\} \) and \( v_3 = E([\cos(\Theta - \nu) - E\{\cos(\Theta - \nu)\}]^3) \), which are calculable from (10) and (11). We can easily obtain from (14) that \( sk \to v_3/v_1^{3/2} \) as \( \lambda \to \infty \) and \( sk = 0 \) when \( \lambda = 0 \). Figure 4 shows that a graph of skewness as a function of \( \lambda \). We see from Figure 4 that the marginal distribution of \( X \) of the proposed model can be left and right skewed according to the values of the parameters.

![Figure 4](image_url)

Figure 4: Skewness of the marginal distribution of \( X \) as a function of \( \lambda \) for (a) \( \sigma = 3, \mu = 0, \kappa_1 = 0.5, \kappa_2 = 0, \mu_1 = \mu_2 = 0, \alpha = 9/2 \) with \( \nu = \pi \) (solid), \( \nu = \pi/2 \) (dashed) and \( \nu = 0 \) (dotted), and (b) \( \sigma = 3, \mu = 0, \nu = 0, \kappa_1 = 0.5, \mu_1 = \mu_2 = 0, \alpha = 9/2 \) with \( \kappa_2 = 0 \) (solid), \( \kappa_2 = 0.1 \) (dashed) and \( \kappa_2 = 0.5 \) (dotted).

### 3.6 Regression

We can derive a circular-linear regression model from the conditional distribution with density \( \Phi \). In fact, the conditional mean of \( X \) given \( \Theta = \theta \) is

\[ E(X|\Theta = \theta) = \mu(\theta) = \mu + \lambda \cos(\theta - \nu) \]

and the conditional variance of \( X \) given \( \Theta = \theta \) is

\[ \text{Var}(X|\Theta = \theta) = \frac{\sigma^2}{\alpha} \{1 - \kappa_1 \cos(\theta - \mu_1) - \kappa_2 \cos 2(\theta - \mu_2)\}. \]

Note that the conditional variance is dependent on \( \theta \), i.e. the regression model possibly has heterogeneity. As a reduced model, if we let \( \gamma = \alpha \to \infty \), we have \( \text{Var}(X|\Theta = \theta) = \tau^2 \), which is independent of \( \theta \). Moreover if \( \kappa_1 = 0 \) and \( \kappa_2 = 0 \) in \( \Phi \), we have \( \text{Var}(X|\Theta = \theta) = \sigma^2/\alpha \) and, in this case, we obtain a regression model

\[ x_i = \mu + \lambda \cos(\theta_i - \nu) + \varepsilon_i, \quad i = 1, \ldots, n, \]

with random errors \( \varepsilon_i \) which are independent and identically distributed according to the generalized t-distribution.
3.7 Maximizing $\beta$-entropy

The related distribution proposed by Mardia and Sutton (1978) and Kato and Shimizu (2008) can be characterized as the maximum entropy distribution under certain moment conditions. Also a maximum entropy distribution under certain moment conditions relates to the proposed distribution with density (1). We consider the $\beta$-entropy (see Eguchi, 2009, Section 13.2.4) defined as

$$E(f) = \left\{ \int f^{\beta+1}(x, \theta) dx \theta - \beta - 1 \right\} / \beta(\beta + 1).$$

Then the maximum entropy distribution subject to constraints on the moments

$$E(X^2), E(X \cos \Theta), E(X \sin \Theta), E(\cos p \Theta), E(\sin p \Theta), p = 1, 2,$$

is the distribution with density

$$f(x, \theta) \propto \left( 1 + \beta \left[ -\frac{(x - \mu(\theta))^2}{2\tau^2} + \kappa_1^* \cos(\theta - \mu_1) + \kappa_2^* \cos 2(\theta - \mu_2) \right] \right)^{1/\beta}.$$ (15)

If we take $\beta = -\tau^{-1}$, (15) gives a density related to (1).

4 Parameter estimation and empirical application

4.1 Parameter estimation

We provide a method for calculating the maximum likelihood estimates of the parameters in generalized $t$-distribution with density (1). When we observe $(x_i, \theta_i), i = 1, \ldots, n$, the log-likelihood function is given by

$$L(\psi) = -n \log B \left( \frac{1}{2}, \alpha + 1 \right) - n \log C_{\theta} - \frac{n}{2} \log 2 - n \log \sigma$$

$$- \left( \alpha + \frac{3}{2} \right) \sum_{i=1}^{n} \log \left[ 1 + \frac{1}{2\sigma^2} \{x_i - \mu - \lambda \cos(\theta_i - \nu)\}^2 - \kappa_1 \cos(\theta_i - \mu_1) - \kappa_2 \cos 2(\theta_i - \mu_2) \right],$$

where $\psi = (\sigma, \mu, \lambda, \nu, \kappa_1, \mu_1, \kappa_2, \mu_2, \alpha)'$. For obtaining the maximizer of $L(\psi)$, we propose the conditional maximization algorithm. We first divide the parameter $\psi$ into $\psi = (\sigma, \psi_1', \psi_2')'$, where $\psi_1 = (\mu, \lambda, \nu)'$ and $\psi_2 = (\kappa_1, \mu_1, \kappa_2, \mu_2, \alpha)'$. Given the value of $\sigma$ and $\psi_2$, maximizing $L(\psi)$ is equivalent to maximizing

$$- \left( \alpha + \frac{3}{2} \right) \sum_{i=1}^{n} \log \left[ C_i + \frac{1}{2\sigma^2} \{x_i - \mu - \lambda \cos(\theta_i - \nu)\}^2 \right],$$

with respect to $\psi_1$, where $C_i = 1 - \kappa_1 \cos(\theta_i - \mu_1) - \kappa_2 \cos 2(\theta_i - \mu_2)$. Let $x = (x_1, \ldots, x_n)'$, $T = (t_1', \ldots, t_n')'$ for $t_i = (1, \cos \theta_i, \sin \theta_i)'$, $\beta = (\mu, \lambda \cos \nu, \lambda \sin \nu)'$ and $W = \text{diag}(w_1, \ldots, w_n)$ for

$$w_i = \frac{\sigma^2}{\sigma^2 + \{x_i - \mu - \lambda \cos(\theta_i - \nu)\}^2 / 2}.$$
Using the theory of weighted regression (see Andrews, 1974), the maximizer of \( \psi_1 \) given \( \sigma \) and \( \psi_2 \) can be obtained as
\[
\hat{\beta} = (T'WT)^{-1}T'Wx, \tag{16}
\]
which deduces the maximizer \( \hat{\psi}_1 \). Since \( W \) depends on \( \psi_1 \), we calculate \( W \) based on the current values in each iteration.

Given \( \psi_1 \) and \( \psi_2 \), maximizing \( L(\psi) \) with respect to \( \sigma^2 \) is equivalent to solving the following equation:
\[
n \left( \alpha + \frac{3}{2} \right)^{-1} = \sum_{i=1}^{n} \frac{2 \{ x_i - \mu - \lambda \cos(\theta_i - \nu) \}^2}{\{ x_i - \mu - \lambda \cos(\theta_i - \nu) \}^2 + C_i \sigma^2}, \tag{17}
\]
which deduces the maximizer \( \hat{\sigma} \).

Finally for maximizing \( \psi_2 \) under given \( \sigma \) and \( \psi_1 \), we maximize
\[
- n \log B \left( \frac{1}{2}, \alpha + 1 \right) - n \log C_\theta - \left( \alpha + \frac{3}{2} \right) \sum_{i=1}^{n} \log \{ D_i - \kappa_1 \cos(\theta_i - \mu_1) - \kappa_2 \cos(2(\theta_i - \mu_2)) \}, \tag{18}
\]
with respect to \( \psi_2 \), where \( D_i = 1 + (2\sigma^2)^{-1} \{ x_i - \mu - \lambda \cos(\theta_i - \nu) \}^2 \). This maximization problem is quite similar to obtaining the maximum likelihood estimates of the generalized t-distribution on the circle (Siew, Kato and Shimizu, 2008), so that we can obtain the maximizer \( \hat{\psi}_2 \) given \( \sigma \) and \( \hat{\psi}_1 \).

Therefore, the proposed estimation method is described in the following.

**Estimation Algorithm**

1. Determine the initial values \( \psi^{(0)} \).

2. For \( k = 0, \ldots, R \) (for suitable iteration number \( R \)), we repeat the following.
   - Calculate \( W \) based on \( \sigma^{(k)} \) and \( \psi_1^{(k)} \), then obtain \( \psi_1^{(k+1)} \) based on (16).
   - Obtain \( \sigma^{(k+1)} \) based on (17) with \( \psi_1 = \psi_1^{(k+1)} \) and \( \psi_2 = \psi_2^{(k)} \).
   - Obtain \( \psi_2^{(k+1)} \) by maximizing (18) with \( \psi_1 = \psi_1^{(k+1)} \) and \( \sigma = \sigma^{(k+1)} \).

3. Return \( \hat{\psi} = (\sigma^{(R+1)}, \psi_1^{(R+1)}, \psi_2^{(R+1)}) \) as the estimated values of \( \psi \).

4.2 Empirical application

For an illustrative example, we consider a cylindrical dataset treated in Kato and Shimizu (2008) on the January surface wind direction and temperature at 12h GMT at Kew for the years from 1956 to 1960, given in Mardia and Sutton (1978). We fitted the proposed generalized t-distribution using the estimation method given in the previous section. We also fitted the member of the exponential family given by Kato and Shimizu (2008) with density (3) for comparison. Table 2 provides the maximum likelihood estimates of the parameters, maximum log-likelihood and AIC values of the two models. Judging from AIC, we see that the generalized t-distribution gives a better fit than the Kato and Shimizu distribution. Figure 5 shows a scatter plot of the data and a contour plot of the fitted proposed density.
Table 2: Maximum likelihood estimates of the parameters, the maximum log-likelihood (MLL), and AIC values of model [1] (GT) and the Kato and Shimizu model [3] (KS) fitted to the data from Mardia and Sutton (1978).

| Model | σ  | μ  | λ  | ν  | κ₁ | μ₁  | μ₂  | α   | MLL | AIC |
|-------|----|----|----|----|----|-----|-----|-----|-----|-----|
| GT    | 3.44 | 42.3 | 4.88 | 3.44 | 0.00 | 0.00 | 0.227 | 3.14 | 0.07 | -109.8 | 237.5 |

| Model | τ  | μ  | λ  | ν  | κ₁ | μ₁  | μ₂  | MLL | AIC |
|-------|----|----|----|----|----|-----|-----|-----|-----|
| KS    | 4.86 | 42.1 | 5.01 | 3.50 | 1.02 | 4.23 | 0.53 | 0.481 | -126.7 | 269.4 |

Figure 5: Sample plot and contour plot of the fitted proposed density.

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