How to best exploit patchy resources? This long-standing question belongs to the extensively studied class of explore/exploit problems that arise in a wide range of situations, from animal foraging, to robotic exploration, and to human decision processes. Despite its broad relevance, the issue of optimal exploitation has previously only been tackled through two paradigmatic limiting models—patch-use and random search—that do not account for the interplay between searcher motion within patches and resource depletion. Here, we bridge this gap by introducing a minimal patch exploitation model that incorporates this coupling: the searcher depletes the resources along its random-walk trajectory within a patch and travels to a new patch after it takes $S$ consecutive steps without finding resources. We compute the distribution of the amount of resources $F_t$ consumed by time $t$ for this non-Markovian random walker and show that exploring multiple patches is beneficial. In one dimension, we analytically derive the optimal strategy to maximize $F_t$. We show that this strategy is robust with respect to the distribution of resources within patches and the criterion for leaving a given patch. We also show that $F_t$ can be optimized in the ecologically-relevant case of two-dimensional patchy environments.

Optimal foraging theory focuses on how to best exploit food resources. Patch-use [1] and random search [2, 3] represent two paradigms for this resource search that have attracted much attention in the physics, mathematical and ecological literatures. In patch-use models, a forager consumes resources within a patch until a specified depletion level (and concomitant decrease in resource intake rate) is reached before the forager moves to another virgin patch. In his pioneering work [4], Charnov predicted optimal strategies to maximize the total resource intake. However, models of this genre do not account for the motion of the organism within a patch, and the depletion rate of resources within a patch is given a priori [5–7] (Fig. 1(a)).

Random search represents a different perspective in which the motion of the searcher—typically a simple or a generalized random walk—is concretely modeled. The search efficiency is quantified by the time to reach targets (see Fig. 1(b)). Diverse classes of searches, including Lévy strategies [8], intermittent strategies [9–12] and persistent random walks [13], have been shown to minimize this search time under general conditions. However, in most of these models, the depletion of targets is not even considered.

An important but unaddressed issue in both of the models outlined above is the interplay between searcher motion within patches and resource depletion. In this Letter, we introduce a minimal patch exploitation model that incorporates this interplay and thus reconciles patch-use and random search models (Fig. 1(c)). Each patch is modeled as a lattice (taken as infinite for simplicity), in which every site initially contains one unit of resource (termed food in the following). A searcher undergoes a simple random walk within a patch and food at a site is completely consumed whenever the site is visited.
ited for the first time. Thus resources within a patch are gradually depleted and eventually it is advantageous for the searcher to move to another unexploited patch. A crucial aspect of the resource landscape is that it is generated by the trajectory of the searcher and is not given a priori, as in patch-use models.

We assume that the searcher moves to a virgin patch when resources become too scarce. We define the scarcity criterion by the condition that the searcher leaves its current patch after traveling $S$ consecutive steps without finding food. This widely used criterion, with $S$ known as the “give-up time” (see [6, 7, 14]), is simple to implement, even for a searcher with limited intelligence, and ensures that the searcher does not experience excessively long periods of scarcity. The searcher thus spends a random time $T_i$ and consumes $N_i$ food units in patch $i$. We also assume that the searcher requires a time $Z$ to traverse between patches, which we take as deterministic for simplicity. We define $\tau_i \equiv T_i + Z$ the duration of the $i$th phase, which consists of the exploitation of patch $i$ and the time to travel to patch $i + 1$.

We quantify the efficiency of the exploitation of this patchy environment by the amount of consumed food $F_i$ up to time $t$. Note that $F_i$ is also the number of distinct sites visited at time $t$ by the random walker, for which numerous results have been established in the case of Markovian random walks [15, 16]. For our model, implementing the scarcity criterion requires keeping track of all previously visited sites, which renders the dynamics non-Markovian.

Our model also belongs to the class of composite search strategies that combines two temporal phases: (i) intensive search and (ii) fast displacement [17–19], but we extend these approaches to explicitly account for resource depletion. Beyond its ecological interest, this explore/exploit duality underlies a wide class of problems that are controlled by the tradeoff between remaining in the current depleting patch and the time or energetic cost to move to a new patch. Examples of this phenomenon include economic growth [20], evolutionary dynamics, and materials design [21].

We begin by giving a simple argument to show that the temporal behavior of $F_i$ must be governed by a non-trivial optimization. If the random-walk searcher remains in a single patch forever (pure exploitation limit, $S \to \infty$), the amount of food consumed $F_i$, which coincides with the number of distinct sites visited, grows sublinearly in time for spatial dimension $d \leq 2$. On the contrary, if the searcher leaves a patch as soon as it fails to find food (pure exploration limit, $S = 1$), $F_i$ clearly grows linearly in time, albeit with a small amplitude $1/Z$. Thus, at long times, exploring multiple patches is beneficial in low spatial dimensions. Since the walker wastes considerable time traveling between patches when $S$ is small, the total amount of consumed food is optimized at an intermediate value of $S$. In this Letter, we derive the temporal behavior of $F_i$, determine the optimal search strategy, and test its robustness with respect to the exploitation conditions.

Let $M_t$ be the (random) number of patches visited by time $t$. Then $F_i$ can be written as

$$F_i = N_1 + \ldots + N_{M_t},$$

while the phase durations $\{\tau_i\}$ satisfy the sum rule (Fig. 1(d))

$$\tau_1 + \ldots + \tau_{M_t} = t.$$

The time $\tau_i$ is the sum of the residence time $T_i$ in the $i$th patch and the inter-patch transit time $Z$. Note that the amount of food $N_i$ eaten and the time $T_i$ spent in the $i$th patch are correlated. Moreover, because of the sum rule (2), the $\{\tau_i\}$ and thus the $\{T_i\}$ are not independent, so that $F_i$ is a sum of correlated random variables. To determine $F_i$, we show that its distribution can be expressed in terms of the joint distribution of $(\tau_i, N_i)$, which we determine in $d = 1$. Note that the coupled variables $(\tau_i, N_i)$ are identically distributed except for $(\tau_{M_t}, N_{M_t})$. In the long-time limit, the quantity of food $N_{M_t}$ consumed in the last patch can be neglected in Eq. (1).

To determine the joint distribution of $(\tau_i, N_i)$, we extend the approach developed in [22] for standard renewal processes to our situation where the variables $N_i$ and $T_i$ are coupled. We define $t_i$ as the time at which the searcher arrives at patch $i + 1$ and $\tau_i = t_i - t_{i-1}$ as the time interval between successive patch visits. The Laplace transform $\langle e^{-pF_i} \rangle$ of the amount of consumed food $F_i$ can be written as

$$\langle e^{-pF_i} \rangle = \sum_{m=0}^{\infty} \int_{\mathbb{R}^m} dy_1 \ldots dy_m \sum_{n_1, \ldots, n_m} e^{-p(n_1 + \ldots + n_m)} \times \Pr\{\{\tau_i = y_i, \{N_i = n_i\}, M_t = m + 1\}.$$

We rewrite the joint probability in Eq. (3) as

$$\Pr\{\{\tau_i = y_i, \{N_i = n_i\}, M_t = m + 1\} = \langle I(t_m < t < t_{m+1}) \prod_{i=1}^{m} \delta(\tau_i - y_i)\delta_{N,i} \rangle,$$

with the indicator function $I(z) = 1$ if the logical variable $z$ is true, and $I(z) = 0$ otherwise. The joint Laplace transform of this probability with respect to all the variables $y_i$ and $t$ is obtained in the Supplementary Material [23] by rewriting $t_i$ in terms of $\tau_i$ and then using the independence of the times $\tau_i$. In turn, this expression can be inverted with respect to the conjugate Laplace variables of all the $y_i$, after which the sum over the number of patches $m$ of Eq. (3) can be performed. Following these steps, the temporal Laplace transform of $\langle e^{-pF_i} \rangle$ is

$$\int_0^\infty dt e^{-st} \langle e^{-pF_i} \rangle = \frac{1}{s} \frac{1}{1 - (e^{-s\tau} - pN)},$$
where $\langle e^{-sT-pN} \rangle$ is the double Laplace transform of the joint distribution of $(\tau, N)$. Equation (5) is general and applies for any spatial dimension and for any search process or distribution of food within patches. The expression in (5) can be made explicit in $d = 1$ by mapping our model of exploitation with memory in a single patch onto the survival of the starving random walk [24] (see Methods), whose average lifetime $(T)$ and distribution of food eaten $N$ are known. In the Supplementary Material, we determine the full distribution of the pair $(T, N)$, from which we finally extract the quantity $\langle e^{-sT-pN} \rangle$ that appears in Eq. (5):

$$\langle e^{-sT-pN} \rangle = \int_0^\infty d\theta P(\theta) e^{-p\pi \theta \sqrt{S/2}} e^{-s(Z+S)} \times \exp \left[ 4 \int_0^{\theta} \frac{du}{u} \sum_{j=0}^{\infty} q_j \right],$$

(6)

where

$$q_j = \frac{1 - e^{-(s+2j+1)^2/u^2}}{1 + su^2 S/(2j+1)^2} - \left( 1 - e^{-(2j+1)^2/u^2} \right),$$

$$P(\theta) = \frac{4}{\theta} \sum_{j=0}^{\infty} e^{-(2j+1)^2/\theta^2} \exp \left[ -2 \sum_{k=0}^{\infty} E_1 \left( (2k+1)^2/\theta^2 \right) \right],$$

(7)

and $E_1(x) = \int_x^\infty dt e^{-xt}/t$ is the exponential integral.

The key step in the derivation of Eq. (6) consists of mapping the exploitation of a single patch onto the survival probability of the starving random walk [24]. In this latter model, a random walk has a metabolic capacity $S$, which is the number of steps it can take without food before starving. The medium is an infinite $d$-dimensional lattice, with one unit of food initially at each site. Upon encountering a food-containing site, the walker instantaneously and completely consumes the food and can again travel $S$ additional steps without eating before starving. Upon encountering an empty site, the walker comes one time unit closer to starvation. The fundamental point is that the statistics of $(T, N)$ for a random walker within a patch that leaves the patch after $S$ steps coincides with the lifetime $T$ and number of distinct sites $N$ visited by a starving random walk with metabolic capacity $S$ at the instant of starvation. Detailed calculations based on this insight are given in the Supplementary Material.

We now focus on the first two moments of $F_t$, whose Laplace transforms are obtained from the small-$p$ expansion of Eq. (5). By analyzing this expansion in the small-$s$ limit, the long-time behavior of these moments are

$$\langle F_t \rangle \sim \frac{\langle N \rangle}{(T) + Z},$$

$$\frac{\text{Var}(F_t)}{t} \sim \frac{\langle N \rangle^2 \text{Var}(T)}{(T + Z)^3} + \frac{\text{Var}(N) \text{Var}(T)}{(T + Z)^2} + \frac{\text{Cov}(N, T)}{(T + Z)^2},$$

(8)

where $\text{Var}(X) = \langle X^2 \rangle - \langle X \rangle^2$ and $\text{Cov}(X, Y) = \langle XY \rangle - \langle X \rangle \langle Y \rangle$. From the small-$p$ and $s$ behavior of Eqs. (6) and (7), the limiting behavior of the moments for $S \gg 1$ are (see the Supplementary Material for details):

$$\frac{\langle F_t \rangle}{t} \sim \frac{K_1 \sqrt{S}}{K_2 S + Z},$$

$$\frac{\text{Var}(F_t)}{t} \sim \left[ \frac{K_3 S^3}{(K_2 S + Z)^3} + \frac{K_4 S}{K_2 S + Z} - \frac{K_5 S^2}{(K_2 S + Z)^2} \right],$$

(9)

where the $K_i$’s are constants given in the Supplementary Material.

The salient feature of these results is that $\langle F_t \rangle$ reaches a maximum for $S = Z/K_2$, corresponding to $(T) = Z$ (Fig. 2). That is, the optimal strategy to maximize the amount of food consumed involves spending on average the same time in exploiting each patch and in moving between patches. At optimality, we use Eqs. (9) to find that the normalized variance is given by

$$\frac{\text{Var}(F_t)}{\langle F_t \rangle^2} \approx 0.203 \ldots Z.$$

(10)

Thus relative fluctuations are not small and increase with $Z$—stochasticity is an essential feature of food consumption in a patchy environment. It is worth noting that we can reproduce the first of Eqs. (8) by neglecting correlations between $N$ and $T$. In this case $\langle F_t \rangle$ is simply the average amount of food $\langle N \rangle$ consumed in one patch multiplied by the mean number $t/(T + Z)$ of patches explored at large time $t$. However, this approach cannot give the variance of $F_t$, which involves the covariance between $N$ and $T$ (second of Eqs. (8)). The covariance term reduces fluctuations in food consumption by a factor three with respect to the case where correlations are neglected.

The optimal strategy outlined above turns out to be robust with respect to: (i) the distribution of food within
a patch and (ii) searcher volatility. For the former attribute, consider, for example, a Poissonian distribution of food, where each site in a patch initially contains food with probability \( \rho \). Using again the mapping onto starving random walks, it can be shown that the mean starvation time is not modified when \( S \gg 1 \). However, the mean number of distinct sites visited at starvation is multiplied by an overall factor \( \rho \) [25]. Thus for \( S \gg 1 \),

\[
\frac{\langle F_i \rangle}{t} \approx \frac{\rho K_1 \sqrt{S}}{K_2 S + Z},
\]

so that the amount of food consumed is optimized for the same conditions as in the case where each site initially contains food (Fig. 3(a)).

For the second attribute, suppose that the searcher has a fixed probability to leave the patch at each step, independent of the current resource density. Equation (5), and thus Eqs. (8), still hold for any distribution of times spent in each patch. For example, for an exponential distribution of residence times with mean \( \lambda^{-1} \), we use the recently-obtained expression for the mean number of distinct sites visited by an exponential one-dimensional evanescent random walk [26] in the first of Eqs. (8) to obtain

\[
\frac{\langle F_i \rangle}{t} \approx \frac{\sqrt{\coth \lambda/2}}{Z + \lambda^{-1}}.
\]

For this example, \( \langle F_i \rangle \) is maximized for \( 1/\lambda \approx Z \) in the \( Z \gg 1 \) limit. Again, the optimal strategy is to spend the same amount of time on average in exploiting a patch and in moving between patches (Fig. 3(a)).

For the ecologically-relevant case of two-dimensional resource patches, we also find that the average amount of food consumed is governed by a similar optimization as in \( d = 1 \) (Fig. 3(b)). However, the optimal strategy consists in spending more time exploiting a single patch rather than traveling between patches. This preference arises because patch exploitation—whose efficiency is quantified by the mean number of distinct sites visited by a given time—is relatively more profitable in two than in one dimension [15, 16].

To summarize, we introduced a minimal explore/exploit model that quantifies the impact of the coupling between searcher motion within patches and resource depletion on the amount of resource consumed. At the theoretical level, this model can also be viewed as a resetting process, in which a random walker stochastically resets to a new position inside a virgin patch. In contrast to existing studies [27–31], the times between resets are not given a priori but determined by the walk itself. This modification may open a new perspective in the burgeoning field of resetting processes.

Financial support for this research was provided in part by starting grant FPTOpt-277998 (OB), and by grants DMR-1623243 from the National Science Foundation and from the John Templeton Foundation (SR).

![Figure 3](image-url)
[21] J. D. Cohen, S. M. McClure, and A. J. Yu, Phil. Trans. R. Soc. B 362, 933 (2007).
[22] C. Godreche and J. Luck, J. Stat. Phys. 104, 489 (2001).
[23] M. Chupeau, O. Bénichou, S. Redner, Supplementary Material.
[24] O. Bénichou and S. Redner, Phys. Rev. Lett. 113, 238101 (2014).
[25] M. Chupeau, O. Bénichou, S. Redner (unpublished).
[26] S. B. Yuste, E. Abad, and K. Lindenberg, Phys. Rev. Lett. 110, 220603 (2013).
[27] M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. 106, 160601 (2011).
[28] M. R. Evans and S. N. Majumdar, J. Phys. A 44, 435001 (2011).
[29] A. Pal, A. Kundu, and M. R. Evans, J. Phys. A 49, 225001 (2016).
[30] A. Nagar and S. Gupta, arXiv preprint arXiv:1512.02092 (2015).
[31] S. Eule and J. J. Metzger, New J. Phys. 18, 033006 (2016).
Supplementary Information to
Random Search with Memory in Patchy Media: Exploration-Exploitation Tradeoff
by M. Chupeau, O. Bénichou O., and S. Redner

We first derive the double Laplace transform of the joint probability for the time $T$ that a searcher spends in a patch and the corresponding amount of food $N$ consumed in this patch, in the limit of large give-up time $S$. This double Laplace transform can be written as

$$\langle e^{-sT-pN} \rangle = \sum_{n=0}^{\infty} P(N = n)e^{-pn} \int_{0}^{\infty} dt e^{-st} P(T = t|N = n).$$  \hspace{1cm} (1)$$

To evaluate the above quantity, we need to characterize the trajectory of a one-dimensional starving random walk that has visited $n$ distinct sites at the instant of starvation. This trajectory consists of $n - 1$ returns to food at either end of a food-free interval, within a time $R_k < S$ for the $k^{th}$ return, and a final lethal excursion of $S$ steps without encountering food. The lifetime of this walk is therefore $T = R_1 + \ldots + R_n + S$. The integral in Eq. (1), which we denote as $\langle e^{-sT}|N = n \rangle$, can thus be written as

$$\langle e^{-sT}|N = n \rangle = \int_{0}^{\infty} dt e^{-st} P(T = t|N = n) = e^{-sS} \prod_{k=1}^{n} (e^{-sR_k}),$$ \hspace{1cm} (2)$$

with

$$\langle e^{-sR_k} \rangle = \int_{0}^{S} dt e^{-st} F_k(t).$$ \hspace{1cm} (3)$$

Here $F_k(t)$ denotes the first-passage probability that the walk first exits an interval of length $(k+1)a$ at time $t$ when the walk starts at a distance $a$ from one end of the interval. This probability is given by [1]

$$F_k(t) = \frac{4\pi D}{(ka)^2} \sum_{j=0}^{\infty} (2j+1) \sin \left(\frac{(2j+1)\pi}{k}\right) \exp \left\{- \left[\frac{(2j+1)\pi}{ka}\right]^2Dt\right\}.$$ \hspace{1cm} (4)$$

Note that the denominator in Eq. (3) gives the probability that the walk reaches either end of the interval within $S$ steps. When $S$ is large, this probability equals 1 up to an exponentially small correction. In this limit, we thus have

$$\langle e^{-sR_k} \rangle \sim \int_{0}^{S} dt e^{-st} F_k(t).$$ \hspace{1cm} (5)$$

Moreover, when the give-up time $S$ is large, the amount of food consumed in a single patch is large, as this quantity scales as $\sqrt{S}$ [2]. It also corresponds to the mean number of returns to the ends of the food-free interval. The elapsed time $R_k$ between two successive encounters with food is then negligible compared to the total time $T$ spent in the patch for any $k$. The range of the Laplace variable $s$ where $\langle e^{-sT} \rangle$ is not small thus satisfies the constraint $sS \ll 1$. We therefore have $\langle e^{-sR_k} \rangle \simeq 1$ in the limit of large $S$.

We use this latter fact to determine the quantity

$$U_n = \prod_{k=1}^{n} \langle e^{-sR_k} \rangle,$$ \hspace{1cm} (6)$$

which is the product of a large number of terms that are close to 1. Thus it is more convenient to first compute $\ln U_n$ and then re-exponentiate:

$$\ln U_n = \sum_{k=1}^{n} \ln \langle e^{-sR_k} \rangle = \sum_{k=1}^{n} \ln (1 + \langle e^{-sR_k} \rangle - 1),$$

$$= \sum_{k=1}^{n} \ln \left[1 + \frac{4}{ku^2S} \sum_{j=0}^{\infty} (2j+1)^2 \left(\frac{1 - e^{-(sS+(2j+1)/u^2)}}{s + (2j+1)^2/(u^2S)} - \frac{1 - e^{-(2j+1)^2/u^2}}{1/(u^2S)}\right)\right],$$ \hspace{1cm} (7)$$

where \( u \equiv ak/(\pi \sqrt{DS}) \) and where the second line follows by explicitly evaluating the integral in Eq. (5). Since the argument of the logarithm is close to 1, we expand to lowest order to give

\[
\ln U_n = \sum_{k=0}^{n} \frac{4}{ku^2S} \sum_{j=0}^{\infty} (2j + 1)^2 \left( \frac{1 - e^{-sS + (2j+1)^2/u^2}}{s + (2j + 1)^2/(u^2S)} - \frac{1 - e^{-(2j+1)^2/u^2}}{1/(u^2S)} \right).
\] (8)

We now introduce \( \theta \equiv an/(\pi \sqrt{DS}) \) and take the continuum limit of Eq. (8), using again \( u = ak/(\pi \sqrt{DS}) \), to give

\[
\ln U(\theta) \simeq 4 \int_0^\theta \frac{du}{u} \sum_{j=0}^{\infty} \left\{ \frac{1 - e^{-sS + (2j+1)^2/u^2}}{1 + su^2S/(2j + 1)^2} - \left[ 1 - e^{-(2j+1)^2/u^2} \right] \right\}. \] (9)

Furthermore, the distribution \( P(N = n) \) that appears in Eq. (1) was determined in the continuum limit in Ref. [2] in terms of the rescaled variable \( \theta \):

\[
P(\theta) = \frac{4}{\theta} \sum_{k=0}^{\infty} e^{-(2j+1)^2/\theta^2} \exp \left\{ -2 \sum_{k=0}^{\infty} E_1 \left[ (2k+1)^2/\theta^2 \right] \right\}, \] (10)

where \( E_1(x) \equiv \int_1^\infty dt e^{-xt}/t \) is the exponential integral function. The double Laplace transform for \((T, N)\) is then

\[
\langle e^{-sT-pN} \rangle = \int_0^\infty d\theta P(\theta) e^{-p\theta \sqrt{DS}/a} \exp \left\{ 4 \int_0^\theta \frac{du}{u} \sum_{j=0}^{\infty} \left[ \frac{1 - e^{-sS + (2j+1)^2/u^2}}{1 + sSu^2/(2j + 1)^2} - \left( 1 - e^{-(2j+1)^2/u^2} \right) \right] \right\}, \] (11)

and using the relation \( \tau = T + Z \), we obtain Eqs. (6) and (7) of the main text.

We now extract from Eq. (11) the asymptotic expression of the moments of \( T \) and \( N \) that appear in Eq. (8) of the main text. The moments of \( N \) can be readily obtained from the marginal distribution \( P(\theta) \)

\[
\langle N \rangle = \frac{\pi}{a} \sqrt{DS} \int_0^\infty d\theta P(\theta) \equiv K_1 \sqrt{S},
\]

\[
\langle N^2 \rangle = \frac{\pi^2 DS}{a^2} \int_0^\infty d\theta \theta^2 P(\theta).
\] (12)

Hence

\[
\text{Var}(N) = \frac{\pi^2 DS}{a^2} \left[ \int_0^\infty d\theta \theta^2 P(\theta) - \left( \int_0^\infty d\theta P(\theta) \right)^2 \right] \equiv K_4 S. \] (13)

The values of \( \langle T \rangle, \text{Var}(T) \) and \( \langle NT \rangle \) are obtained by taking the small-\( p \) and small-\( s \) limits of the Laplace transform:

\[
\langle e^{-sT-pN} \rangle \underset{s,p \to 0}{\sim} 1 - s\langle T \rangle - p\langle N \rangle + sp\langle TN \rangle + \frac{s^2}{2} \langle T^2 \rangle. \] (14)

In the limit of \( s \to 0 \), we have

\[
\frac{1 - e^{-sS + (2j+1)^2/u^2}}{1 + u^2 sS/(2j + 1)^2} - \left( 1 - e^{-(2j+1)^2/u^2} \right) = sS \left[ \left( 1 + \frac{u^2}{(2j+1)^2} \right) e^{-(2j+1)^2/u^2} - \frac{u^2}{(2j+1)^2} \right] + s^2 S^2 \left[ \frac{u^4}{(2j+1)^4} - \left( \frac{1}{2} + \frac{u^2}{(2j+1)^2} + \frac{u^4}{(2j+1)^4} \right) e^{-(2j+1)^2/u^2} \right]
\]

so that

\[
\langle e^{-sT-pN} \rangle \underset{s,p \to 0}{\sim} \left( 1 - sS + \frac{s^2}{2} S^2 \right) \int_0^{+\infty} d\theta P(\theta) \left( 1 - p \frac{\pi \sqrt{DS} \theta}{a} \right) \left( 1 + sS A(\theta) + s^2 S^2 \frac{2B(\theta) + A^2(\theta)}{2} \right), \] (15)

and

\[
\langle e^{-sT-pN} \rangle \underset{s,p \to 0}{\sim} \left( 1 - sS + \frac{s^2}{2} S^2 \right) \int_0^{+\infty} d\theta P(\theta) \left( 1 - p \frac{\pi \sqrt{DS} \theta}{a} \right) \left( 1 + sS A(\theta) + s^2 S^2 \frac{2B(\theta) + A^2(\theta)}{2} \right), \] (16)
with
\[
A(\theta) \equiv \sum_{j=0}^{\infty} \int_0^{\theta} \frac{du}{u} \left( 1 + \frac{u^2}{(2j+1)^2} \right) e^{-\frac{(2j+1)^2}{u^2}} - \frac{u^2}{(2j+1)^2},
\]
\[
B(\theta) \equiv \sum_{j=0}^{\infty} \int_0^{\theta} \frac{du}{u} \left[ \frac{u^4}{(2j+1)^4} - \frac{u^2}{(2j+1)^2} + \frac{u^4}{(2j+1)^4} e^{-\frac{(2j+1)^2}{u^2}} \right].
\]

Identifying (16) with (14), we obtain
\[
\langle T \rangle = \left[ 1 - \int_0^\infty d\theta P(\theta) A(\theta) \right] S \equiv K_2 S,
\]
\[
\langle T^2 \rangle = \left[ 1 + 2 \int_0^\infty d\theta P(\theta) \left( B(\theta) + \frac{1}{2} A^2(\theta) - A(\theta) \right) \right] S^2.
\]

Hence
\[
\text{Var}(T) = \left[ \int_0^\infty d\theta P(\theta) \left( 2B(\theta) + A^2(\theta) \right) - \left( \int_0^\infty d\theta P(\theta) A(\theta) \right)^2 \right] S^2.
\]

Equation (14) also yields
\[
\langle NT \rangle = \frac{\pi}{a} \sqrt{DS^3} \int_0^\infty d\theta P(\theta) \theta (1 - A(\theta)),
\]
so that
\[
\text{Cov}(N, T) \equiv \langle NT \rangle - \langle N \rangle \langle T \rangle = \frac{\pi}{a} \sqrt{DS^3} \int_0^\infty d\theta P(\theta) \theta \left[ \int_0^\infty d\varphi P(\varphi) A(\varphi) - A(\theta) \right].
\]

Finally, we substitute the asymptotic expressions of the moments of \( T \) and \( N \) in Eq. (8) of the main text and obtain the constants \( K_1 \) to \( K_5 \) that appear in Eq. (9) of the main text
\[
K_1 \equiv \frac{\pi}{a} \sqrt{D} \int_0^\infty \theta P(\theta) d\theta \approx 2.90 \ldots,
\]
\[
K_2 \equiv 1 - \int_0^\infty \theta P(\theta) A(\theta) d\theta \approx 3.27 \ldots,
\]
\[
K_3 \equiv \frac{\pi^2 D}{a^2} \left[ \int_0^\infty \theta P(\theta) d\theta \right]^2 \left[ \int_0^\infty \theta P(\theta) (2B(\theta) + A^2(\theta)) - \left( \int_0^\infty \theta P(\theta) A(\theta) \right)^2 \right] \approx 16.1 \ldots,
\]
\[
K_4 \equiv \frac{\pi^2 D}{a^2} \left[ \int_0^\infty \theta P(\theta) d\theta \right]^2 \left[ \int_0^\infty \theta P(\theta) (2B(\theta) + A^2(\theta)) - \left( \int_0^\infty \theta P(\theta) A(\theta) \right)^2 \right] \approx 178 \ldots,
\]
\[
K_5 \equiv \frac{2\pi^2 D}{a^2} \int_0^\infty \theta P(\theta) \int_0^\infty \theta P(\theta) d\theta \left[ \int_0^\infty \theta P(\theta) A(\theta) \right] \approx 851 \ldots.
\]

[1] S. Redner, A Guide to First-Passage Processes (Cambridge University Press, 2001).
[2] O. Bénichou and S. Redner, Phys. Rev. Lett. 113, 238101 (2014).