Diophantine approximation by Piatetski-Shapiro primes

S. I. Dimitrov

1 Introduction and statement of the result

Let \([ \cdot ]\) be the floor function. In this paper we show that whenever \(\eta\) is real, the constants \(\lambda_i\) satisfy some necessary conditions, then for any fixed \(1 < c < 38/37\) there exist infinitely many prime triples \(p_1, p_2, p_3\) satisfying the inequality

\[|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{37c - 38} (\log \max p_j)^{10}\]

and such that \(p_i = [n_i^c], i = 1, 2, 3\).

Keywords Diophantine approximation · Piatetski-Shapiro primes

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1 Introduction and statement of the result

Let \(\mathbb{P}\) denotes the set of all prime numbers. In 1953 Piatetski-Shapiro [6] showed that for any fixed \(\gamma \in (11/12, 1)\) the set

\[\mathbb{P}_\gamma = \{ p \in \mathbb{P} \mid p = [n^{1/\gamma}] \text{ for some } n \in \mathbb{N}\}\]

is infinite. The prime numbers of the form \(p = [n^{1/\gamma}]\) are called Piatetski-Shapiro primes of type \(\gamma\). Subsequently the interval for \(\gamma\) was sharpened many times and the best result up to now belongs to Rivat and Wu [7] for \(\gamma \in (205/243, 1)\).

Twenty years later Vaughan [9] proved that whenever \(\delta > 0, \eta\) is real and constants \(\lambda_i\) satisfy some conditions, there are infinitely many prime triples \(p_1, p_2, p_3\) such that

\[|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi + \delta}\]

for \(\xi = 1/10\). Latter the upper bound for \(\xi\) was improved several times and the best result up to now is due to K. Matomäki [5] with \(\xi = 2/9\). In relation to solvability of inequality (1) with prime numbers of a special form we find papers by the author and Todorova [3] and the author [1]. In order to establish our result we solve the inequality (1) with Piatetski-Shapiro primes. Thus we prove the following theorem.
Theorem 1 Suppose that $\lambda_1, \lambda_2, \lambda_3$ are non-zero real numbers, not all of the same sign, $\eta$ is real, $\lambda_1/\lambda_2$ is irrational and $\gamma$ be fixed with $37/38 < \gamma < 1$. Then there exist infinitely many ordered triples of Piatetski-Shapiro primes $p_1, p_2, p_3$ of type $\gamma$ such that

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{37/38 \gamma} (\log \max p_j)^{10}.$$ 

2 Notations

The letter $p$ will always denote prime number. As usual $[t]$ and $\{t\}$ denote the integer part of $t$ and the fractional part of $t$. Moreover $\psi(t) = \{t\} - 1/2$. Let $\gamma$ be a real constant such that $37/38 < \gamma < 1$. Since $\lambda_1/\lambda_2$ is irrational, there are infinitely many different convergents $a_0/q_0$ to its continued fraction, with

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}, \quad (a_0, q_0) = 1, \quad a_0 \neq 0$$

and $q_0$ is arbitrary large. Denote

$$X = q_0^{13/6};$$
$$\Delta = X^{12/38} \log X;$$
$$\epsilon = X^{37/38 \gamma} \log^{10} X;$$
$$H = \frac{\log^2 X}{\epsilon};$$
$$S(\alpha, X) = \sum_{\lambda_0 X < p \leq X \atop p \in \mathbb{P}} p^{1-\gamma} e(\alpha p) \log p, \quad 0 < \lambda_0 < 1;$$
$$\Sigma(\alpha, X) = \gamma \sum_{\lambda_0 X < p \leq X} e(\alpha p) \log p;$$
$$\Omega(\alpha, X) = \sum_{\lambda_0 X < p \leq X} X p^{1-\gamma} (\psi(-p+1) - \psi(-p')) e(\alpha p) \log p;$$
$$I(\alpha, X) = \gamma \int_{\lambda_0 X}^X e(\alpha y) dy.$$ 

3 Preliminary lemmas

Lemma 1 Let $\epsilon > 0$ and $k \in \mathbb{N}$. There exists a function $\theta(y)$ which is $k$ times continuously differentiable and such that

$$\theta(y) = 1 \quad \text{for} \quad |y| \leq 3\epsilon/4;$$
$$0 < \theta(y) < 1 \quad \text{for} \quad 3\epsilon/4 < |y| < \epsilon;$$
$$\theta(y) = 0 \quad \text{for} \quad |y| \geq \epsilon.$$ 

and its Fourier transform

$$\Theta(x) = \int_{-\infty}^{\infty} \theta(y) e(-xy) dy$$

satisfies the inequality

$$|\Theta(x)| \leq \min \left( \frac{7\epsilon}{4}, \frac{1}{\pi |x|}, \frac{1}{\pi |x|} \left( \frac{k}{2\pi |x| \epsilon/8} \right)^k \right).$$
Proof. See (Lemma 1, [8]).

**Lemma 2.** Let $|\alpha| \leq \Delta$. Then for the sum denoted by (8) and the integral denoted by (10) the asymptotic formula

$$\Sigma(\alpha, X) = I(\alpha, X) + O\left(\frac{X}{e^{(\log X)^{1/5}}}\right)$$

holds.

Proof. This lemma is very similar to result of Tolev [8]. Inspecting the arguments presented in ([8], Lemma 14), the reader will easily see that the proof of Lemma 2 can be obtained by the same way.

**Lemma 3.** For the sum denoted by (9) the upper bound

$$\Omega(\alpha, X) \ll X^{\frac{17 - 12\gamma}{26}} \log^5 X.$$  

holds.

Proof. It follows by the same argument used in ([2], (36)).

**Lemma 4.** Suppose that $\alpha \in \mathbb{R}, a \in \mathbb{Z}, q \in \mathbb{N}$, $|\alpha - \frac{a}{q}| \leq \frac{1}{q^\gamma}, (a, q) = 1$.

Let

$$\Psi(X) = \sum_{p \leq X} e(\alpha p) \log p.$$  

Then

$$\Psi(X) \ll \left(Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2}\right) \log^4 X.$$  

Proof. See ([4], Theorem 13.6).

### 4 Outline of the proof

Consider the sum

$$\Gamma(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X, \frac{1}{q} \in \mathbb{Z}, \gamma \in \mathbb{R}, i = 1, 2, 3} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)p_1^{-\gamma} p_2^{-\gamma} p_3^{-\gamma} \log p_1 \log p_2 \log p_3.$$  

Using the inverse Fourier transform for the function $\theta(x)$ we get

$$\Gamma(X) = \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X, \frac{1}{q} \in \mathbb{Z}, \gamma \in \mathbb{R}, i = 1, 2, 3} \int_{-\infty}^{\infty} \Theta(t)e(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t) \, dt = \int_{-\infty}^{\infty} \Theta(t)S(\lambda_1 t, X)S(\lambda_2 t, X)S(\lambda_3 t, X)e(\eta t) \, dt.$$  

We decompose $\Gamma(X)$ as follows

$$\Gamma(X) = \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X),$$  

where

$$\Gamma_1(X) = \int_{|\gamma| \leq \Delta} \Theta(t)S(\lambda_1 t, X)S(\lambda_2 t, X)S(\lambda_3 t, X)e(\eta t) \, dt.$$
\[ \Gamma_2(X) = \int_{|t| \leq H} \Theta(t) S(\lambda_1 t, X) S(\lambda_2 t, X) S(\lambda_3 t, X) e(\eta t) \, dt, \quad (14) \]
\[ \Gamma_3(X) = \int_{|t| > H} \Theta(t) S(\lambda_1 t, X) S(\lambda_2 t, X) S(\lambda_3 t, X) e(\eta t) \, dt. \quad (15) \]

We shall estimate \( \Gamma_1(X), \Gamma_2(X) \) and \( \Gamma_3(X) \), respectively, in the sections 5, 6 and 7. In section 8 we shall complete the proof of Theorem 1.

5 Lower bound of \( \Gamma_1(X) \)

In order to find the lower bound of \( \Gamma_1(X) \) we need to prove the following two lemmas.

**Lemma 5** For the sum denoted by (7) and the integral denoted by (10) the asymptotic formula
\[ S(\alpha, X) = I(\alpha, X) + \mathcal{O}\left(\frac{X}{(\log X)^{1/2}}\right) \quad (16) \]
holds.

**Proof** From (7)–(9) we have
\[ S(\alpha, X) = \sum_{\lambda_0 X < p \leq X} p^{1-\gamma} ([1-p\gamma] - [(p+1)\gamma]) e(\alpha p) \log p \]
\[ = \sum_{\lambda_0 X < p \leq X} p^{1-\gamma} ((p+1)\gamma - p\gamma) e(\alpha p) \log p \]
\[ + \sum_{\lambda_0 X < p \leq X} p^{1-\gamma} (\psi(-(p+1)\gamma) - \psi(-p\gamma)) e(\alpha p) \log p \]
\[ = \Sigma(\alpha, X) + \Omega(\alpha, X) + \mathcal{O}(\log X). \quad (17) \]

Bearing in mind (17), Lemmas 2 and 3 we obtain the asymptotic formula (16). □

**Lemma 6** Let \( \lambda \neq 0 \). Then for the sum denoted by (7) and the integral denoted by (10) we have
\[ (i) \quad \int_{-\Delta}^{\Delta} |S(\lambda \alpha, X)|^2 \, d\alpha \ll X \log^3 X, \]
\[ (ii) \quad \int_{-\Delta}^{\Delta} |I(\lambda \alpha)|^2 \, d\alpha \ll X \log X, \]
\[ (iii) \quad \int_{0}^{1} |S(\alpha, X)|^2 \, d\alpha \ll X^{2-\gamma} \log^2 X. \]

**Proof** We only prove (i). The inequalities (ii) and (iii) can be proved likewise.

Using (4), (7) and Lagrange’s mean value theorem we obtain
\[ \int_{-\Delta}^{\Delta} |S(\lambda \alpha, X)|^2 \, d\alpha = \sum_{\lambda_0 X < p_1, p_2 \leq X, p_1 \equiv p_2 \bmod 1, 2} (p_1 p_2)^{1-\gamma} \log p_1 \log p_2 \int_{-\Delta}^{\Delta} e(\lambda(p_1 - p_2)\alpha) \, d\alpha \]
\[ \ll X^{2-2\gamma} (\log X)^2 \sum_{\lambda_0 X < n_1, n_2 \leq X, n_1 \equiv n_2 \bmod 1, 2} \min\left(\Delta, \frac{1}{|n_1 - n_2|}\right) \]
We use the identity

\begin{equation}
\sum_{(\lambda_0 X) \leq m_1, m_2 \leq X} \frac{1}{m_1 - m_1^{1/\gamma}} - \sum_{(\lambda_0 X) \leq m_1, m_2 \leq X} \frac{1}{m_2 - m_1^{1/\gamma}} \ll \varepsilon.
\end{equation}

\begin{equation}
\sum_{(\lambda_0 X) \leq m_1, m_2 \leq X} \frac{1}{m_2 - m_1} \ll \varepsilon.
\end{equation}

\begin{equation}
\sum_{(\lambda_0 X) \leq m_1, m_2 \leq X} \frac{1}{m_2 - m_1} \ll X \log^3 X.
\end{equation}

The lemma is proved. \hfill \Box

Put

\begin{align*}
S_1 &= S(t, X), \\
I_i &= I(t, X).
\end{align*}

We use the identity

\begin{equation}
S_1 S_2 S_3 = I_1 I_2 I_3 + (S_1 - I_1) I_2 I_3 + S_1 (S_2 - I_2) I_3 + S_1 S_2 (S_3 - I_3).
\end{equation}

Replace

\begin{equation}
J(X) = \int_{|t| < \Delta} \Theta(t) I(t, X) I(t, X) I(t, X) e(\eta t) \, dt.
\end{equation}

Now from (13), (18), (19), Lemmas 1, 5 and 6 it follows

\begin{equation}
\Gamma_1(X) - J(X) = \int_{|t| < \Delta} \Theta(t) \left( S(t, X) - I(t, X) \right) I(t, X) I(t, X) e(\eta t) \, dt
\end{equation}

\begin{align*}
&+ \int_{|t| < \Delta} \Theta(t) S(t, X) \left( S(t, X) - I(t, X) \right) I(t, X) e(\eta t) \, dt \\
&+ \int_{|t| < \Delta} \Theta(t) S(t, X) S(t, X) \left( S(t, X) - I(t, X) \right) e(\eta t) \, dt
\end{align*}

\begin{align*}
&\ll \varepsilon \frac{X}{e(\log X)^{1/5}} \left( \int_{|t| < \Delta} |I(t, X) I(t, X)| \, dt \\
&+ \int_{|t| < \Delta} |S(t, X) I(t, X)| \, dt + \int_{|t| < \Delta} |S(t, X) S(t, X)| \, dt \right) \\
&\ll \varepsilon \frac{X}{e(\log X)^{1/5}} \left( \int_{|t| < \Delta} |I(t, X)|^2 \, dt + \int_{|t| < \Delta} |I(t, X)|^2 \, dt \\
&+ \int_{|t| < \Delta} |S(t, X)|^2 \, dt + \int_{|t| < \Delta} |S(t, X)|^2 \, dt \right)
\end{align*}

\begin{equation}
\ll \varepsilon \frac{X^2}{e(\log X)^{1/6}}.
\end{equation}
On the other hand for the integral defined by (19) we write

\[ J(X) = B(X) + \Phi, \]

where

\[ B(X) = \gamma^3 \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \int_{\lambda_0 X}^{X} \theta(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta) \, dy_1 \, dy_2 \, dy_3 \]

and

\[ \Phi \ll \int_{\Delta} |\Theta(t)| |I(\lambda_1 t, X)I(\lambda_2 t, X)I(\lambda_3 t, X)| \, dt. \]  

According to ([3], Lemma 4) we have

\[ B(X) \gg \varepsilon X^2. \]  

By (10) we get

\[ I(\alpha, X) \ll \frac{1}{|\alpha|}. \]  

Using (22), (24) and Lemma 1 we deduce

\[ \Phi \ll \frac{e}{\Delta^2}. \]  

Bearing in mind (4), (20), (21), (23) and (25) we obtain

\[ \Gamma_1(X) \gg \varepsilon X^2. \]  

\section*{6 Upper bound of $\Gamma_2(X)$}

Suppose that

\[ \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}, \quad (a, q) = 1 \]

with

\[ q \in \left[ X^{\frac{1}{12}}, X^{\frac{13}{12}} \right]. \]

Then (8), (27), (28) and Lemma 4 yield

\[ \Sigma(\alpha, X) \ll X^{\frac{25}{26}} \log^4 X. \]

Now (17), (29) and Lemma 3 give us

\[ S(\alpha, X) \ll X^{\frac{37}{36} - \frac{12}{26}} \log^5 X. \]

Let

\[ \mathcal{S}(t, X) = \min \{ |S(\lambda_1 t, X)|, |S(\lambda_2 t, X)| \}. \]

We shall prove the following lemma.
Lemma 7 Let $t$, $X$, $\lambda_1$, $\lambda_2 \in \mathbb{R}$, 

$$\Delta \leq |t| \leq H,$$  \hspace{1cm} (32)

where $\Delta$ and $H$ are denoted by (4) and (6), $\lambda_1/\lambda_2 \in \mathbb{R}\setminus \mathbb{Q}$ and $\mathcal{S}(t, X)$ is defined by (31). Then there exists a sequence of real numbers $X_1, X_2, \ldots \to \infty$ such that

$$\mathcal{S}(t, X_j) \ll X_j^{37/12} \log^5 X_j, \quad j = 1, 2, \ldots.$$  \hspace{1cm} (33)

Proof Our aim is to prove that there exists a sequence $X_1, X_2, \ldots \to \infty$ such that for each $j = 1, 2, \ldots$ at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ with $t$, subject to (32) can be approximated by rational numbers with denominators, satisfying (28). Then the proof follows from (30) and (31).

Since $\lambda_1, \lambda_2, \lambda_3$ are not all of the same sign one can assume that $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$. Let us notice that there exist $a_1, q_1 \in \mathbb{Z}$, such that

$$|\lambda_1 t - \frac{a_1}{q_1}| < \frac{1}{q_1 q_0^2}, \quad (a_1, q_1) = 1, \quad 1 \leq q_1 \leq q_0^2, \quad a_1 \neq 0.$$  \hspace{1cm} (34)

From Dirichlet’s approximation theorem it follows the existence of integers $a_1$ and $q_1$, satisfying the first three conditions. If $a_1 = 0$ then

$$|\lambda_1 t| < \frac{1}{q_1 q_0^2}$$

and (32) gives us

$$\lambda_1 \Delta < \lambda_1 |t| < \frac{1}{q_0^2}, \quad q_0^2 < \frac{1}{\lambda_1 \Delta}.$$  \hspace{1cm} (35)

The last inequality, (3) and (4) yield

$$X_{i+2}^\frac{12}{13} < \frac{X_{i+2}^{\frac{12}{13}}}{\lambda_1 \log X},$$

which is impossible for large $X$. Therefore $a_1 \neq 0$. By analogy there exist $a_2, q_2 \in \mathbb{Z}$, such that

$$|\lambda_2 t - \frac{a_2}{q_2}| < \frac{1}{q_2 q_0^2}, \quad (a_2, q_2) = 1, \quad 1 \leq q_2 \leq q_0^2, \quad a_2 \neq 0.$$  \hspace{1cm} (36)

If $q_i \in \left[ X_i^{\frac{12}{13}}, X_{i+1}^{\frac{12}{13}} \right]$ for $i = 1$ or $i = 2$, then the proof is completed. By (3), (33) and (34) we deduce

$$q_i \leq X_{i+1}^{\frac{12}{13}} = q_0^2, \quad i = 1, 2.$$  \hspace{1cm} (37)

It remains to show that the case $q_i < X_i^{\frac{12}{13}}, i = 1, 2$ is impossible. Assume that

$$q_i < X_i^{\frac{12}{13}}, \quad i = 1, 2.$$  \hspace{1cm} (38)

From (5), (6), (32)–(35) it follows

$$1 \leq |a_i| < \frac{1}{q_0^2} + q_i |t| < \frac{1}{q_0^2} + q_i \lambda_i H,$$

$$1 \leq |a_i| < \frac{1}{q_0^2} + \lambda_i X^{\frac{38p-35}{50}} (\log X)^{-8}, \quad i = 1, 2.$$  \hspace{1cm} (39)

We have

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 t}{\lambda_2 t} = \frac{\frac{a_1}{q_1}}{\frac{a_2}{q_2}} + \left( \frac{\lambda_1 t - \frac{a_1}{q_1}}{\frac{a_2}{q_2}} \right) = \frac{a_1 q_2 + \lambda_1 t - \frac{a_1}{q_1}}{a_2 q_1}, \quad 1 + X_i,$$  \hspace{1cm} (40)

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where
\[ x_i = \frac{q_i}{a_i} \left( \lambda_i t - \frac{a_i}{q_i} \right), \quad i = 1, 2. \]  

(38)

Bearing in mind (33), (34), (37) and (38) we get
\[ |x_i| < \frac{|q_i|}{|a_i|} \cdot \frac{1}{q_i q_0^2} \leq \frac{1}{q_i q_0}, \quad i = 1, 2, \]
\[ \frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + O \left( \frac{1}{q_i q_0} \right)}{1 + O \left( \frac{1}{q_i q_0} \right)} = \frac{a_1 q_2}{a_2 q_1} \left( 1 + O \left( \frac{1}{q_i q_0} \right) \right). \]

Thus
\[ \frac{a_1 q_2}{a_2 q_1} = O(1) \]

and
\[ \frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + O \left( \frac{1}{q_i q_0} \right). \]  

(39)

Therefore, both fractions \( \frac{a_0}{q_0} \) and \( \frac{a_1 q_2}{a_2 q_1} \) approximate \( \frac{\lambda_1}{\lambda_2} \). Using (3), (33), (35) and inequality (36) with \( i = 2 \) we obtain
\[ |a_2| q_1 < 1 + \lambda_2 X^{2 - \frac{36}{26}} (\log X)^{-8} \leq \frac{q_0}{\log X}. \]  

(40)

Consequently \( |a_2| q_1 \neq q_0 \) and \( \frac{a_0}{q_0} \neq \frac{a_1 q_2}{a_2 q_1} \). Now (40) implies
\[ \left| \frac{a_0}{q_0} - \frac{a_1 q_2}{a_2 q_1} \right| = \left| \frac{a_0 a_2 q_1 - a_1 q_2 q_0}{|a_2| q_1 q_0} \right| \geq \frac{1}{|a_2| q_1 q_0} > \frac{\log X}{q_0^2}. \]  

(41)

On the other hand, from (2) and (39) we deduce
\[ \left| \frac{a_0}{q_0} - \frac{a_1 q_2}{a_2 q_1} \right| \leq \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_2} \leq \frac{1}{q_0^2}, \]

which contradicts (41). This rejects the assumption (35). Let \( q_0^{(1)}, q_0^{(2)}, \ldots \) be an infinite sequence of values of \( q_0 \), satisfying (2). Then using (3) one gets an infinite sequence \( X_1, X_2, \ldots \) of values of \( X \), such that at least one of the numbers \( \lambda_1 t \) and \( \lambda_2 t \) can be approximated by rational numbers with denominators, satisfying (28). Hence, the proof is completed. \( \square \)

Taking into account (14), (31), Lemmas 1 and 7 we deduce
\[ \Gamma_2(X_j) \ll \varepsilon \int_{\Delta \leq |t| \leq H} \mathfrak{S}(t, X_j) \left( \left| S(\lambda_1 t, X_j) S(\lambda_3 t, X_j) \right| + \left| S(\lambda_2 t, X_j) S(\lambda_3 t, X_j) \right| \right) dt 
\ll \varepsilon \int_{\Delta \leq |t| \leq H} \mathfrak{S}(t, X_j) \left( \left| S(\lambda_1 t, X_j) \right|^2 + \left| S(\lambda_2 t, X_j) \right|^2 + \left| S(\lambda_3 t, X_j) \right|^2 \right) dt 
\ll \varepsilon X_j^{3 - \frac{12}{26}} (\log X_j)^5 T_k. \]  

(42)
where
\[ T_k = \int_{\Delta} H \left| S(\lambda_k t, X_j) \right|^2 dt. \]

Using Lemma 6 (iii) and working as in ([3], pp. 17–18) we obtain
\[ T_k \ll H X_j^{2-\gamma} \log^2 X_j. \]

From (5), (6), (42), (43) we get
\[ \Gamma_2(X_j) \ll X_j^{\frac{37-12\gamma}{36}} X_j^{2-\gamma} \log^9 X_j \ll X_j^{\frac{89-38\gamma}{36}} \log^9 X_j \ll \frac{\varepsilon X_j^2}{\log X_j}. \]

7 Upper bound of \( \Gamma_3(X) \)

By (7), (15) and Lemma 1 it follows
\[ \Gamma_3(X) \ll X^{3-3\gamma} \int_{\frac{1}{H}}^{\infty} \left( \frac{k}{2\pi t \varepsilon/8} \right)^k \varepsilon dt = \frac{X^{3-3\gamma}}{k} \left( \frac{4k}{\pi \varepsilon H} \right)^k. \]

Choosing \( k = [\log X] \) from (6) and (45) we obtain
\[ \Gamma_3(X) \ll 1. \]

8 Proof of the Theorem

Summarizing (5), (12), (26), (44) and (46) we deduce
\[ \Gamma(X_j) \gg \varepsilon X_j^2 = X_j^{\frac{89-38\gamma}{26}} \log^{10} X_j. \]

The last estimation implies
\[ \Gamma(X_j) \rightarrow \infty \quad \text{as} \quad X_j \rightarrow \infty. \]

Bearing in mind (11) and (47) we establish Theorem 1.

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