CUSPIDAL DISCRETE SERIES
FOR PROJECTIVE HYPERBOLIC SPACES

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Dedicated to Sigurður Helgason on the occasion of his 85th birthday

Abstract. We have in [1] proposed a definition of cusp forms on semisimple symmetric spaces \( G/H \), involving the notion of a Radon transform and a related Abel transform. For the real non-Riemannian hyperbolic spaces, we showed that there exists an infinite number of cuspidal discrete series, and at most finitely many non-cuspidal discrete series, including in particular the spherical discrete series. For the projective spaces, the spherical discrete series are the only non-cuspidal discrete series. Below, we extend these results to the other hyperbolic spaces, and we also study the question of when the Abel transform of a Schwartz function is again a Schwartz function.

1. Introduction

We initiated, in joint work with Henrik Schlichtkrull, in [1] a generalization of Harish-Chandra’s notion of cusp forms for real semisimple Lie groups \( G \) to semisimple symmetric spaces \( G/H \). In the group case, all the discrete series are cuspidal, and this plays an important role in Harish-Chandra’s work on the Plancherel formula. However, in the established generalizations to \( G/H \), cuspidality plays no role and, in fact, was hitherto not defined at all.

The notion of cuspidality relates to the integral geometry on the symmetric space by using integration over a certain unipotent subgroup \( N^* \subset G \), whose definition is given in [1]. The map \( f \mapsto \int_{N^*} f(\cdot nH) \, dn \), which maps functions on \( G/H \) to functions on \( G/N^* \), is a kind of a Radon transform for \( G/H \). A discrete series is said to be cuspidal if it is annihilated by this transform.

Let \( p,q \) denote positive integers. The Radon transform, and the question of cuspidality, on the real hyperbolic spaces \( SO(p,q+1)_e/\SO(p,q)_e \), was treated in detail in [1]. We showed that there is at most a finite number of non-cuspidal discrete series, including in particular all the spherical discrete series, but also some non-spherical discrete series. The non-spherical non-cuspidal discrete series are given by odd functions on the real hyperbolic space, which means that they do not descend to functions on the real projective hyperbolic space.

In the present paper, we consider the projective hyperbolic spaces over the classical fields \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H} \),

\[
G/H = \text{O}(p+1,q+1)/(\text{O}(p+1,q) \times \text{O}(1)), \ U(p+1,q+1)/(\text{U}(p+1,q) \times \text{U}(1)), \\
\text{Sp}(p+1,q+1)/(\text{Sp}(p+1,q) \times \text{Sp}(1)),
\]

for \( p \geq 0, q \geq 1 \). Notice the change of indices from \( p \) to \( p+1 \), to simplify formulae and calculations.

2010 Mathematics Subject Classification. Primary 43A85; Secondary 22E30.
Our main result, Theorem 6.1, states that the non-cuspidal discrete series for the projective hyperbolic spaces precisely consist of the spherical discrete series. The Radon transform of the generating functions is also given explicitly. Finally, we show that the Abel transform maps (a dense subspace of) the Schwartz functions on \(G/H\) perpendicular to the non-cuspidal discrete series into Schwartz functions. The latter result also holds for the non-projective real case, and is a new result for all cases.

Our calculations and main results are also valid, with \(p = 0, q = 1\) and \(d = 8\), for the Cayley numbers \(\mathbb{O}\), corresponding to the exceptional symmetric space \(F_{4(-20)}/\text{Spin}(1,8)\). Although the model for this space, and the group action on it, is more complicated, this space can for our purposes be viewed as

\[
F_{4(-20)}/\text{Spin}(1,8) = "U(1, 2; \mathbb{O})/U(1, 1; \mathbb{O}) \times U(1; \mathbb{O})".
\]

We state our results in full generality, but only give complete proofs for the non-projective real case, and only for the exceptional symmetric case in split rank one, using general theory.

We would like to thank Henrik Schlichtkrull for input and fruitful discussions, which in the real case lead to the explicit formulae involving the Hypergeometric function. We also want to thank Job Kuit for discussions of part (vi) of Theorem 6.1, which in the real case lead to the explicit formulae involving the Hypergeometric function. We also thank Job Kuit for discussions of part (vi) of Theorem 6.1, explaining how to prove a similar result in split rank one, using general theory.

Part of this work was outlined by the first author at the Special Session ‘Radon Transforms and Geometric Analysis in Honor of Sigurdur Helgason’s 85th Birthday’, at the 2012 AMS National Meeting in Boston. He is grateful to the organizers Jens Christensen, Fulton Gonzalez, and Todd Quinto, for their invitation to speak, and the hospitality at the meeting, and the subsequent Workshop on Geometric Analysis on Euclidean and Homogeneous Spaces.

2. Model and structure

Let \(\mathbb{F}\) be one of the classical fields \(\mathbb{R}\), \(\mathbb{C}\) or \(\mathbb{H}\), and let \(x \mapsto \overline{x}\) be the standard (anti-)involution of \(\mathbb{F}\). We make the standard identifications between \(\mathbb{C}\) and \(\mathbb{R}^2\), and between \(\mathbb{H}\) and \(\mathbb{R}^4\). Let \(p \geq 0, q \geq 1\) be integers, and consider the Hermitian form \([\cdot, \cdot]\) on \(\mathbb{F}^{p+q+2}\) given by

\[
[x, y] = x_1\overline{y}_1 + \cdots + x_{p+1}\overline{y}_{p+1} - x_{p+2}\overline{y}_{p+2} - \cdots - x_{p+1+q+1}\overline{y}_{p+1+q+1}, \quad (x, y \in \mathbb{F}^{p+q+2}).
\]

Let \(G = U(p+1, q+1; \mathbb{F})\) denote the group of \((p+q+2) \times (p+q+2)\) matrices over \(\mathbb{F}\) preserving \([\cdot, \cdot]\). Thus \(U(p+1, q+1; \mathbb{R}) = O(p+1, q+1)\), \(U(p+1, q+1; \mathbb{C}) = U(p+1, q+1)\) and \(U(p+1, q+1; \mathbb{H}) = \text{Sp}(p+1, q+1)\) in standard notation. Put \(U(p; \mathbb{F}) = U(p, 0; \mathbb{F})\).

Let \(x_0 = (0, \ldots, 0, 1)^T\), where superscript \(T\) indicates transpose. Let \(H = U(p+1, 1; \mathbb{F}) \times U(1; \mathbb{F})\) be the subgroup of \(G\) stabilizing the line \(\mathbb{F} \cdot x_0\) in \(\mathbb{F}^{p+q+2}\). An involution \(\sigma\) of \(G\) fixing \(H\) is given by \(\sigma(g) = JgJ\), where \(J\) is the diagonal matrix with entries \((1, \ldots, 1, -1)\). The reductive symmetric space \(G/H\) (of rank 1) can be identified with the projective hyperbolic space \(\mathbb{X} = \mathbb{X}(p+1, q+1; \mathbb{F})\):

\[
\mathbb{X} = \{z \in \mathbb{F}^{p+q+2} : [z, z] = -1\} / \sim,
\]

where \(\sim\) is the equivalence relation \(z \sim zu, u \in \mathbb{F}^*\).

The Lie algebra \(\mathfrak{g}\) of \(G\) consists of \((p+q+2) \times (p+q+2)\) matrices

\[
\mathfrak{g} = \left( \begin{array}{cc} A & B \\ B^* & C \end{array} \right),
\]
where $A$ is a skew Hermitian $(p + 1) \times (p + 1)$ matrix, $C$ is a skew Hermitian $(q + 1) \times (q + 1)$ matrix, and $B$ is an arbitrary $(p + 1) \times (q + 1)$ matrix. Here $B^*$ denotes the conjugated transpose of $B$.

Let $K = K_1 \times K_2 = U(p + 1; \mathbb{F}) \times U(q + 1; \mathbb{F})$ be the maximal compact subgroup of $G$ consisting of elements fixed by the classical Cartan involution on $G$, $\theta(g) = (g^*)^{-1}, g \in G$. Here $g^*$ denotes the conjugated transpose of $g$. The Cartan involution on $g$ is given by: $\theta(X) = -X^*$. Let $g = k \oplus p$ be the decomposition of $g$ into the $\pm 1$-eigenspaces of $\theta$, where $k = \{X \in g : \theta(X) = X\}$ and $p = \{X \in g : \theta(X) = -X\}$. Similarly, let $g = h \oplus q$ be the decomposition of $g$ into the $\pm 1$-eigenspaces of $\sigma(X) = JXJ$, where $h = \{X \in g : \sigma(X) = X\}$ and $q = \{X \in g : \sigma(X) = -X\}$.

We choose a maximal abelian subalgebra $a_q \subset p \cap q$ as

$$a_q = \left\{ X_{t_1} = \begin{pmatrix} 0 & 0 & t_1 \\ 0 & 0_{p,q} & 0 \\ t_1 & 0 & 0 \end{pmatrix} : t_1 \in \mathbb{R} \right\},$$

where $0_{p,q}$ is the $(p+q) \times (p+q)$ null matrix. The exponential of $X_{t_1}$, $a_{t_1} = \exp(X_{t_1})$, is given by

$$a_{t_1} = \exp(X_{t_1}) = \begin{pmatrix} \cosh t_1 & 0 & \sinh t_1 \\ 0 & I_{p,q} & 0 \\ \sinh t_1 & 0 & \cosh t_1 \end{pmatrix},$$

where $I_{p,q}$ is the $(p + q) \times (p + q)$ identity matrix. Also define $A_q = \exp(a_q)$.

Let $A_q^+ = \{a_{t_1} : t_1 > 0\}$. Let $a(x) = a(kah) = a$ denote the projection onto the $A_q^+$ component in the Cartan decomposition $G = K \bar{A}_q H$ of $G$. Let $M$ be the centralizer of $X_1 \in a_q$ (i.e., when $t_1 = 1$) in $K \cap H$. Then $M$ is the stabilizer of the line $F(1, 0, \ldots, 0, 1)$, and the homogeneous space $K/M$ can be identified with the product of unit spheres $S^p \times S^q$:

$$\mathbb{Y} = \{y \in F^{p+q+2} : |y_1|^2 + \cdots + |y_{p+1}|^2 = |y_{p+2}|^2 + \cdots + |y_{p+q+2}|^2 = 1\} / \sim.$$

The image of the set $\{z \in F^{p+q+2} : |z_1| = -1, (z_1, \ldots, z_{p+1}) \neq 0\}$ in $X$ is an open dense subset, which we will denote by $X'$. The map

$$K/M \times \mathbb{R}^+ \to \mathbb{X}, (kM, t_1) \mapsto ka_{t_1}H,$$

is a diffeomorphism onto $X'$.

We introduce spherical coordinates on $X$ as the pull back of the map:

$$x(t_1, y) = (u \sinh t_1; v \cosh t_1), \quad t_1 \in \mathbb{R}^+, \quad y = (u; v) \in S^p \times S^q.$$

We define a ($K$-invariant) ‘distance’ from $x \in X$ to the origin as $|x| = |x(t_1, y)| = |t_1|$. Then $X' = \{x \in X : |x| > 0\}$. We note that $\cosh^2(|x|) = |x_{p+2}|^2 + \cdots + |x_{p+q+2}|^2$.

For $g \in G$, we define $|g| = |gH|$.

Let $r = \min\{p, q\}$, and let $X_t$ be the $(r + 1) \times (r + 1)$ anti-diagonal matrix with entries $t = (t_1, \ldots, t_{r+1}) \in \mathbb{R}^{r+1}$, starting from the upper right corner. We extend $a_q$ (viz. as $t_2 = \cdots = t_{r+1} = 0$) to a maximal subalgebra $a \subset p$ as

$$a = \left\{ X_t = \begin{pmatrix} 0 & 0 & X_t \\ 0 & 0 & 0 \\ X_t^* & 0 & 0 \end{pmatrix} \right\}.$$

We will also consider the sub-algebra $a_h = a \cap h = \{X_t \in a : t_1 = 0\}$.
Let (considered as row vectors) 
\[ u = (u_1, \ldots, u_p) \in \mathbb{F}^p \quad \text{and} \quad v = (v_q, \ldots, v_1) \in \mathbb{F}^q. \]
It turns out to be convenient to number the entries of \( v \) from right to left as indicated. Let furthermore \( w \in \text{Im} \mathbb{F} \) (i.e., \( w = 0 \) for \( \mathbb{F} = \mathbb{R} \)). Now define \( N_{u,v,w} \in \mathfrak{g} \) as the matrix given by
\[
N_{u,v,w} = \begin{pmatrix}
-w & u & v & w \\
-\pi^T & 0 & 0 & \pi^T \\
\pi^T & 0 & 0 & -\pi^T \\
-w & u & v & w \\
\end{pmatrix}.
\]
Then \( \exp(N_{u,v,w}) = I + N_{u,v,w} + 1/2N_{u,v,w}^2 \), and
\[
(2.1) \quad \exp(N_{u,v,w}) \cdot x_0 = (1/2(|u|^2 - |v|^2) + w, \pi^T; -\pi^T, 1 + 1/2(|u|^2 - |v|^2) + w)^T.
\]
A small calculation also yields that
\[
(2.2) \quad a_{t_1} \exp(N_{u,v,w}) \cdot x_0 = (\sinh t_1 + 1/2e^{t_1}(|u|^2 - |v|^2) + e^{t_1}w, \pi^T; -\pi^T, \cosh t_1 + 1/2e^{t_1}(|u|^2 - |v|^2) + e^{t_1}w)^T,
\]
for any \( t_1 \in \mathbb{R} \).

We note that \([X_{t_1}, N_{u,v,0}] = t_1 N_{u,v,0}\), and \([X_{t_1}, N_{0,0,w}] = 2t_1 N_{0,0,w}\). Let \( \gamma(X_{t_1}) = t_1 \). Then the root system \( \Sigma_q \) for \( \mathfrak{n}_q \) is given by \( \Sigma_q = \{ \pm \gamma \} \), for \( \mathbb{F} = \mathbb{R} \), and \( \Sigma_q = \{ \pm \gamma \} \cup \{ \pm 2\gamma \} \), for \( \mathbb{F} = \mathbb{C}, \mathbb{H} \). The associated nilpotent subalgebra \( \mathfrak{n}_q \) is given by \( \mathfrak{n}_q = \mathfrak{g}^\gamma = \{ N_{u,v,0} : u \in \mathbb{F}^p, v \in \mathbb{F}^q \} \), when \( \mathbb{F} = \mathbb{R} \), and \( \mathfrak{n}_q = \mathfrak{g}^\gamma + \mathfrak{g}^{2\gamma} = \{ N_{u,v,w} : u \in \mathbb{F}^p, v \in \mathbb{F}^q, w \in \text{Im} \mathbb{F} \} \), when \( \mathbb{F} = \mathbb{C}, \mathbb{H} \). Half the sum of the positive roots, \( \rho_q = \frac{1}{2} \sum_{\alpha \in \Sigma_q^+} m_\alpha \alpha \), where \( m_\alpha \) is the multiplicity of the root \( \alpha \), is thus
\[
(\rho_q, X_{t_1}) = \frac{1}{2}(dp + dq + 2(d-1))t_1,
\]
where \( d = \text{dim}_\mathbb{R} \mathbb{F} \). Using the identification \( A_q \sim \mathbb{R} \), we will also sometimes use the definition \( \rho_q = \frac{1}{2}(dp + dq + 2(dp - dq)) \in \mathbb{R} \).

The (restricted) \( \Sigma \) for \( \mathfrak{a} \) is given by \( \{ \pm t_i \pm t_j \}, i \neq j, i, j \in \{ 1, \ldots, r + 1 \}, \{ \pm t_i \}, i \in \{ 1, \ldots, r + 1 \}, \) if \( p \neq q \), and \( \{ \pm 2t_i \}, i \in \{ 1, \ldots, r + 1 \} \), if \( d \geq 2 \). Let \( \alpha_{i,j}(X_i) = t_i + t_j, i < j; \beta_{i,j}(X_i) = t_i - t_j, i < j \), and \( \gamma_{i,j}(X_i) = t_i \).

We choose two sets of positive roots
\[
\Sigma^+ = \{ \alpha_{i,j}, \beta_{i,j}, \gamma_i, 2\gamma_i \},
\]
which corresponds to the (standard) ordering \( t_1 > t_2 > \cdots > t_{r+1} \), and
\[
\Sigma^+_1 = \{ \alpha_{i,j}, \gamma_i, 2\gamma_i \} \cup \{ \beta_{i,j} : i \neq 1 \} \cup \{ -\beta_{i,j} : i = 1 \},
\]
which corresponds to the ordering \( t_2 > t_3 > \cdots > t_{r+1} > t_1 \). The double roots \( \{ \pm 2\gamma_i \} \) are not present for \( \mathbb{F} = \mathbb{R} \), the single roots \( \{ \pm \gamma_i \} \) are not present when \( p = q \). The associated nilpotent subalgebras are denoted by \( \mathfrak{n} \) and \( \mathfrak{n}_1 \) respectively. The half sum of positive roots \( \rho_1 \) with regards to \( \Sigma^+_1 \) is given by (restricted to \( A_q \))
\[
(\rho_1, X_{t_1}) = \frac{1}{2}((dp - dq) + 2(dp - dq))t_1.
\]
As before, we will sometimes use the definition \( \rho_1 = \frac{1}{2}((dp - dq) + 2(dp - dq)) \in \mathbb{R} \).

We note that \( \gamma \in \Sigma^+_q \) is the restriction of the roots \( \{ \alpha_{1,1+j}, \gamma_1, \beta_{1,1+j} \} \), with \( j \in \{ 1, \ldots, r \} \), where
\[
\mathfrak{g}^{n_{1,i+1}} = \{ N_{u,v,0} : u_j = -\pi_j, u_i = v_i = 0, i \neq j \},
\]
\[ g^{\beta_{i,j}} = \{ N_{u,v,0} : u_j = \nu_j, \, u_i = v_i = 0, \, i \neq j \}, \]
and, for \( p > q \),
\[ \mathfrak{g}^\gamma = \{ N_{u,v,0} : u = (0, \ldots, 0, u_{q+1}, \ldots, u_p), \, v = 0 \}, \]
which for \( p < q \) becomes \( u = 0, \, v = (v_q, v_{q-1}, \ldots, v_{p+1}, 0, \ldots, 0) \).
Define \( \mathfrak{n}^+ = \mathfrak{n}_1 \cap \mathfrak{n}_q \) as the subalgebra associated to the roots \( \{ \alpha_{1+j}, \gamma_1, 2\gamma_1 \} \).
Then, for \( p \geq q \)
\[ \mathfrak{n}^+ = \{ N_{u,v,w} : u = (-w^r, u'), \, v \in \mathbb{F}^q, \, u' \in \mathbb{F}^{q-p} \}, \]
and, for \( p < q \)
\[ \mathfrak{n}^* = \{ N_{u,v,w} : v = (-w^r, v'), \, u \in \mathbb{F}^p, \, v' \in \mathbb{F}^{q-p} \}, \]
where \( u', v' \) means that the order of the indices is reversed. In the following we shall by abuse of notation leave out the \( r \).

**Remark 2.1.** We have the identity \( \Sigma^+_q = \{ \alpha \in \Sigma^+: \alpha|_{\mathfrak{n}_q} > 0 \} \). We then have the disjoint union \( \Sigma_q^+ = \Sigma^+ \cup \Sigma^0 \cup \Sigma^- \), where the second sign refers to \( \alpha|_{\mathfrak{n}_q} \). The choice of the nilpotent subalgebra \( \mathfrak{n}^+ \) can thus be described by the correspondence \( \mathfrak{n}^* \sim \Sigma^+ + \Sigma^0 \).

### 3. The discrete series

From [3, Section 8] and [4, Table 2], we have the following parametrization of the discrete series for the projective hyperbolic spaces, with an exception for \( q = d = 1 \):
\[ \{ T_\lambda \mid \lambda = \frac{1}{2} (dq - dp) - 1 + \mu_\lambda > 0, \mu_\lambda \in 2\mathbb{Z} \}. \]
The spherical discrete series are given by the parameters \( \lambda \) for which \( \mu_\lambda \leq 0 \), including the ‘exceptional’ discrete series corresponding to the (finitely many) parameters \( \lambda > 0 \) for which \( \mu_\lambda < 0 \). We notice that spherical discrete series exists if, and only if, \( d(q-p) > 2 \). For \( q = d = 1 \), the discrete series is parameterized by \( \lambda \in \mathbb{R} \setminus \{ 0 \} \) such that \( |\lambda| + \rho_q \in 2\mathbb{Z} \), and there are no spherical discrete series.

The parameter \( \lambda \) is, via the formula \( \Delta f = (\lambda^2 - p_q^2) f \), related to the eigenvalue of the Laplace-Beltrami operator \( \Delta \) of \( G/H \) on functions \( f \) in the corresponding representation space in \( L^2(G/H) \) (with suitable normalization of \( \Delta \)). Using [4, Theorem 5.1] (see [1, Proposition 3.2] for more details), we can explicitly describe the discrete series by generating functions \( \psi_\lambda \) as follows. Let \( s = s_1 \in \mathbb{R} \) describe the elements \( a_s = a_{s_1} \in \mathfrak{a}_q \). Let \( \lambda \) be a discrete series parameter. For \( \mu_\lambda \geq 0 \), we have
\[ \psi_\lambda(ka_s H) = \psi_\lambda(x(s, y)) = \phi_{\mu_\lambda}(k)(\cosh s)^{-\lambda - \rho_q}, \]
where \( \phi_{\mu_\lambda} \) is a \( K \cap H \)-invariant zonal spherical function, in particular \( \phi_0 = 1 \). For \( \mu_\lambda = -2m \leq 0 \), we have
\[ \psi_\lambda(ka_s H) = P_\lambda(\cosh^2 s)(\cosh s)^{-\lambda - \rho_q - 2m}, \]
where \( P_\lambda \) is a polynomial of degree \( m \). For \( q = d = 1 \), consider the one-parameter subgroup \( T = \{ k_\theta \} \subset K_2 \) defined by
\[ k_\theta = \begin{pmatrix} f_{p+1} & \theta & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}, \]
where $I_j$ denotes the identity matrix of size $j$, then $\psi_\lambda(k_o a_s H) = e^{im\theta}(\cosh \lambda)^{-|\lambda|-\rho_q}$, with $m = \lambda \pm \rho_q$, and the sign determined by the sign of $\lambda$. See [1] Section 3 for further details.

4. SCHWARTZ FUNCTIONS

In this section we recall some results from [2, Chapter 17] regarding $L^2$-Schwartz functions on $G/H$. Let $\Xi$ denote Harish-Chandra’s bi-$K$-invariant elementary spherical function $\varphi_0$ on $G$, and define the real analytic function $\Theta : G/H \to \mathbb{R}^+$ by

$$\Theta(x) = \sqrt{\Xi(x\sigma(x)^{-1})} \quad (x \in G).$$

We notice that there exists a positive constant $C$, and a positive integer $m$, such that

$$a^{-\rho_s} \leq \Theta(a) \leq Ca^{-\rho_s}(1 + |a|)^m, \quad (a \in A^+_q).$$

Here we use the definition $a^\lambda = e^{(\lambda, \log a)}$, for $a \in A^+_q$, $\lambda \in a^*_C$.

The space $C^2(G/H)$ of $L^2$-Schwartz functions on $G/H$ can be defined as the space of all smooth functions on $G/H$ satisfying

$$\mu_{n,D}(f) = \sup_{x \in G/H} \Theta^{-1}(x)(1 + |x|)^n|f(D, x)| < \infty,$$

for all $n \in \mathbb{N} \cup \{0\}$ and $D \in U(g)$.

Let $f \in C^2(G/H)$. Let $S \subset G$ be a compact set. Then, for any $n \in \mathbb{N} \cup \{0\}$, there exists a positive constant $C$, such that

$$|f(g \cdot x)| \leq C \Theta(a(x))(1 + |x|)^{-n} \quad (g \in S, x \in G/H).$$

5. A RADON TRANSFORM AND AN ABEL TRANSFORM

Let $N^* = \exp(n^*)$ and $N_1 = \exp(n_1)$ denote the two nilpotent subgroups generated by $n^*$ and $n_1$ respectively. For functions on $G/H$ we define, assuming convergence,

$$Rf(g) = \int_{N^*} f(g n^* H) \, dn^* \quad (g \in G).$$

Let $H^A$ denote the centralizer of $A$ in $H$. Then $Rf(gm) = Rf(g)$, $m \in H^A$, and

**Theorem 5.1.** Let $f \in C^2(G/H)$.

(i) The integral defining the Radon transform $R$ converges uniformly on compact sets.

(ii) $Rf \in C^\infty(G/H^A N_1)$.

(iii) The Radon transform is $G$- and $g$-equivariant.

**Proof.** We first assume $p \geq q$. Let $f \in C^2(G/H)$, and fix a compact set $S \subset G$. Let $n \in \mathbb{N}$. Then

$$\int_{N^*} |f(g n^* H)|dn^* \leq C \int_{N^*} a(n^*)^{-\rho_s}(1 + |n^*|)^{-n+m}dn^* \quad (g \in S),$$

for the constants $C$ and $m$ given by (4.1) and (4.2). From (2.1) and (2.3), we have

$$\cosh^2(\exp(N_{(v,w),v,w})) = (1 + 1/2|u|^2)^2 + |v|^2 + |w|^2.$$
Using that \( \log s \leq \arccosh s \leq \log s + \log 2 \), when \( s \geq 1 \), we see that the last integral in (5.2) is bounded by

\[
C \int_{\mathbb{R}^{dp-dq} \times \mathbb{R}^{dq} \times \mathbb{R}^{d-1}} \left( (1 + 1/2|u'|^2)^2 + |v|^2 + |w|^2 \right)^{-\frac{dp}{2} + dq + 2(d-1)} \times (1 + \log((1 + 1/2|u'|^2)^2 + |v|^2 + |w|^2)^{-n+m} du' dv dw,
\]

where \( C \) is a positive constant.

Consider the integral \( (x \in \mathbb{R}^k, y \in \mathbb{R}^l) \), with \( n > 2 \),

\[
\int_{\mathbb{R}^k \times \mathbb{R}^l} (1 + |x|^4 + |y|^2)^{-a} (1 + \log(1 + |x|^4 + |y|^2))^{-n} dx dy.
\]

With the substitution \( y = \sqrt{1 + |x|^4} z \in \mathbb{R}^l \), we get

\[
\int_{\mathbb{R}^k \times \mathbb{R}^l} (1 + |x|^4)^{-a + \frac{1}{2}} (1 + |z|^2)^{-a} (1 + \log(1 + |x|^4) + \log(1 + |z|^2))^{-n} dz dx \leq \int_{\mathbb{R}^k} (1 + |x|^4)^{-a + \frac{1}{2}} (1 + \log(1 + |x|^4))^{-\frac{n}{2}} dx \int_{\mathbb{R}^l} (1 + |z|^2)^{-a} (1 + \log(1 + |z|^2))^{-\frac{n}{2}} dz,
\]

which is finite if, and only if, \( k \leq 4a - 2l \) and \( l \leq 2a \).

We have \( k = dp - dq, l = dq + d - 1 \) and \( a = (dp + dq + 2(d-1))/4 \), whence \( k = 4a - 2l \) and \( l \leq 2a \), and the integral (5.1) converges uniformly on compact sets.

In the \( p < q \) case, we see from (2.1) and (2.4), that \( \cosh^2(|\exp(N_{u,v})|) = |v'|^2 + |u|^2 + (1 - 1/2|v'|^2)^2 + |w|^2 = 1 + |u|^2 + 1/4|v'|^4 + |w|^2 \), and we proceed as before, reversing the roles of \( u \) and \( v \). \( \square \)

We define the Abel transform \( A \) by \( Af(a) = a^\rho Rf(a) \), for \( a \in A_q \).

**Theorem 5.2.** Let \( g \in G \) and \( f \in \mathcal{C}^2(G/H) \). Let \( \Delta \) denote the Laplace–Beltrami operator on \( G/H \) and let \( \Delta_{A_q} \) denote the Euclidean Laplacian on \( A_q \). Then

\[
\mathcal{A}(\Delta f) = (\Delta_{A_q} - \rho_\lambda^2) Af \quad (a \in A_q).
\]

**Proof.** See [3] Lemma 2.4], and the discussion before and after this lemma. \( \square \)

Let \( \psi_\lambda \) belong to the discrete series with parameter \( \lambda \). Since \( \Delta \psi_\lambda = (\lambda^2 - \rho_\lambda^2) \psi_\lambda \), we see that \( \mathcal{A} \psi_\lambda \) is an eigenfunction for the Euclidean Laplacian \( \Delta_{A_q} \) on \( A_q \) with the eigenvalue \( \lambda^2 \). This implies in particular that \( s \mapsto R\psi_\lambda(a_s) \) is a linear combination of \( e^{(\lambda-\rho_1)s} \) and \( e^{(-\lambda-\rho_1)s} \).

### 6. The main result

Here we state the main theorem, to be proven in the following sections. We will in particular be interested in the values of \( Rf \) on the elements \( a_s \in A_q \), so for simplicity we write \( Rf(a_s) = Rf(a_s) \), and, similarly, \( Af(s) = Af(a_s) \).

Let \( R > 0 \), and let \( \mathcal{C}_R^\infty(G/H) \) denote the subspace of smooth functions on \( G/H \) with support inside the \( (K\text{-invariant}) \) ‘ball’ of radius \( R \). Let similarly \( \mathcal{C}_R^\infty(\mathbb{R}) \) denote the subspace of smooth functions on \( \mathbb{R} \) with support inside \([-R, R]\). Finally, let \( \mathcal{S}(\mathbb{R}) \) denote the Schwartz functions on \( \mathbb{R} \).

**Theorem 6.1.** Let \( G/H \) be a projective hyperbolic space over \( \mathbb{R}, \mathbb{C}, \mathbb{H} \), with \( p \geq 0, q \geq 1 \), or over \( \mathbb{O} \), with \( p = 0, q = 1 \).

(i) If \( d(q-p) \leq 2 \), then all discrete series are cuspidal.
(ii) If \( d(q-p) > 2 \), then non-cuspidal discrete series exists, given by the parameters \( \lambda > 0 \) with \( \mu_\lambda \leq 0 \). More precisely, if \( 0 \neq f \in C^2(G/H) \) belongs to \( T_\lambda \), then \( \mathcal{A}f(s) = C e^{\lambda s} \), with \( C \neq 0 \).

(iii) \( T_\lambda \) is non-cuspidal if and only if \( T_\lambda \) is spherical.

(iv) If \( p \geq q \), and \( f \in C^\infty_R(G/H) \), for \( R > 0 \), then \( \mathcal{A}f \in C^\infty_R(\mathbb{R}) \).

(v) If \( d(q-p) \leq 1 \), and \( f \in C^2(G/H) \), then \( \mathcal{A}f \in S(\mathbb{R}) \).

(vi) Assume \( d(q-p) > 1 \). Let \( D \) be the \( G \)-invariant differential operator \( \Delta_p(\Delta_2 - \lambda_1^2) \ldots (\Delta_2 - \lambda_r^2) \), where \( \lambda_1, \ldots, \lambda_r \) are the parameters of the non-cuspidal discrete series, and \( \Delta_2 = \Delta + \rho^2 \). Then \( \mathcal{A}(Df) \in S(\mathbb{R}) \), for \( f \) in a dense subspace of \( C^2(G/H) \).

Remark 6.2. The theorem also holds for the non-projective spaces \( SO(p+1,q+1) \), except for item (iii), due to the existence of non-cuspidal non-spherical discrete series, corresponding to the parameters \( \lambda > 0 \), with \( \mu_\lambda \in 2\mathbb{Z} + 1 \) and \( \mu_\lambda < 0 \).

Remark 6.3. The conditions in item (vi) essentially state that \( \mathcal{A}f \) is a Schwartz function if \( f \) is perpendicular to all non-cuspidal discrete series. The factor \( \Delta_2 \), however, cannot be avoided, except in the cases \( d = 1 \) and \( q-p \) odd.

Remark 6.4. For the exceptional case, only (ii), (iii) and (vi) are relevant. The spherical discrete series corresponds to \( \lambda = 3 (\mu_\lambda = 0) \) and \( \lambda = 1 (\mu_\lambda = -2) \).

7. Proof of the Main Theorem for \( p \geq q \)

Proposition 7.1. Let \( p \geq q \).

(i) Let \( f \in C^\infty_R(G/H) \), for \( R > 0 \). Then \( \mathcal{A}f \in C^\infty_R(\mathbb{R}) \).

(ii) Let \( f \in C^2(G/H) \). Then \( \mathcal{A}f \in S(\mathbb{R}) \).

Proof. Let \( f \in C^\infty_R(G/H) \), for \( R > 0 \). By (2.2) and (2.3), we have

\[
\cosh^2(|a_s \exp(N(v,w),v,w)|) = (\cosh s + 1/2e^s|u'|^2)^2 + |v|^2 + |e^w|^2 \geq \cosh^2 s,
\]

and thus \( Rf(s) = 0 \), for \( |s| > R \), which shows (i).

For (ii), let \( f \in C^2(G/H) \). As before we have, for \( n \in \mathbb{N} \),

\[
\tag{7.1} \int_{N^*} |f(a_sn^*H)|dn^* \leq C \int_{N^*} a(a_sn*)^{-\rho_s}(1 + |a_sn^*|)^{-n}dn^*,
\]

where \( C \) is a positive constant.

The integral in (7.1) is bounded by

\[
\int_{\mathbb{R}^{dp+dq} \times \mathbb{R}^{dq} \times \mathbb{R}^{(d-1)}} ((\cosh s + 1/2e^s|u'|^2)^2 + |v|^2 + |e^w|^2)^{\frac{dp+dq+2(d-1)}{2}} \times (1 + \log((\cosh s + 1/2e^s|u'|^2)^2 + |v|^2 + |e^w|^2(1/2))^{-n}du'dv'dw
\]

\[
\leq (\cosh s)^{-\frac{dp+dq+2(d-1)}{2}} \int_{\mathbb{R}^{dp+dq} \times \mathbb{R}^{dq} \times \mathbb{R}^{(d-1)}} (1 + (1/\sqrt{2}(\cosh s)^{-\frac{1}{2}}e^{s/2}|u'|)^2)^2 \times (1 + \log((1 + (1/\sqrt{2}(\cosh s)^{-\frac{1}{2}}e^{s/2}|u'|)^2)^2 + ((\cosh s)^{-1}|v|^2 + ((\cosh s)^{-1}|e^w|^2)^{\frac{dp+dq+2(d-1)}{2}}) \times (1 + \log((1 + (1/\sqrt{2}(\cosh s)^{-\frac{1}{2}}e^{s/2}|u'|)^2)^2 + ((\cosh s)^{-1}|v|^2 + ((\cosh s)^{-1}|e^w|^2)^{-n}du'dv'dw,
\]

since \( \log \cosh s \geq 0 \).
Consider the substitutions \( \bar{\pi} = 1/\sqrt{2}(cosh s)^{-1/2}e^{s/2}u', \) \( \bar{\nu} = (cosh s)^{-1}v \) and \( \bar{\pi} = (cosh s)^{-1}e^{s}w. \) Then \( du' = (\sqrt{2}(cosh s)^{-1/2})dp - dq d\bar{\pi}, \) \( dv = (cosh s)^{d}d\bar{\nu}, \) and \( dw = ((cosh s)e^{-s})^{d-1}d\bar{\pi}, \) and the above integral becomes

\[
\sqrt{2}e^{-dp - dq} \int e^{-dp - dq + 2(d - 1)}((1 + |\bar{\pi}|^2 + |\bar{\nu}|^2 + |\bar{\mu}|^2) - \frac{dp + dq + 2(d - 1)}{4}) \times (1 + \log((1 + |\bar{\pi}|^2 + |\bar{\nu}|^2 + |\bar{\mu}|^2)) - n dudw \\
\leq C_{p,q}d_{s}^{-\rho_{\bar{\nu}}},
\]

where \( C_{p,q} \) is a constant only depending on \( p \) and \( q. \) The proposition follows using the \( U(\mathfrak{g}) \)-equivariance of the Radon transform from Theorem 5.1 (iii).

Let \( C^{2}(G/H)_{d} = L^{2}(G/H)_{d} \cap C^{2}(G/H) \) denote the span of the discrete series in \( C^{2}(G/H). \)

**Proposition 7.2.** Let \( p \geq q. \) Then \( Rf = 0, \) for \( f \in C^{2}(G/H)_{d}. \)

**Proof.** Let \( f \in C^{2}(G/H)_{d}. \) Then \( Af \) belongs to \( S(A_{q}) \) by Theorem 7.1 but at the same time \( Af \) is also an eigenfunction of \( \Delta_{A_{q}} \) on \( A_{q}. \) We conclude that \( Af = 0, \) and thus \( Rf = 0. \)

\(\)
Using the substitution \(\tilde{u} = (1 + |\tilde{v}|^4 - 2(tanh s)|\tilde{v}'|^2)^{-1/2}\), and likewise for \(u\), the integral becomes

\[
\int_{R^d\times R^d\times R^{d-1}} \left(1 + |\tilde{v}|^4 - 2(tanh s)|\tilde{v}'|^2\right)^{-\frac{\lambda + sp}{2} + \frac{d-1}{2}} \times (1 + |\tilde{u}|^2 + |\tilde{v}|^2)^{-\frac{\lambda + sp}{2} - \frac{d-1}{2}} d\tilde{u}' d\tilde{v} d\tilde{u} = C \int_0^\infty (1 + \xi^4 - 2(tanh s)\xi^2)^{-\frac{\lambda + sp}{2} - \frac{d-1}{2}} \xi^d q dq d\xi
\]

using polar coordinates, where \(C\) is the positive constant given by

\[
C = \int_0^\infty \int_0^\infty (1 + \eta^2 + \sigma^2)^{-\frac{\lambda + sp}{2} - \eta^d \sigma d^2} d\eta d\sigma < \infty.
\]

From \([5, 3.252(10)]\), we get

\[
\int_0^\infty (1 + x^2 + 2(cost)x)^{-\nu} x^{\mu-1} dx = 2^{\nu - \frac{1}{2}} (sin t)^{-\nu} \Gamma(\nu + \frac{1}{2}) B(\mu, 2\nu - \mu) P_{\mu - \nu - \frac{1}{2}}^\frac{1}{2} (cost).
\]

We also have

\[
P_{\mu - \nu - \frac{1}{2}}^\frac{1}{2}(y) = \Gamma(\nu + \frac{1}{2}) \left(1 + \frac{1 + y}{1 - y}\right)^\frac{1}{2} 2F_1 \left(-\mu + \nu + \frac{1}{2}, \mu - \nu + \frac{1}{2}; \nu + \frac{1}{2}; 1 - \frac{1}{2} y\right).
\]

With \(y = cos t = -tanh s\), for \(0 < t < \pi\), we get \(sin t = 1/ cosh s\) and \(1 - y = e^s / cosh s\) and \(1 + y = e^{-s} / cosh s\). Putting this together, we get

\[
\int_0^\infty (1 + x^2 - 2(tanh s)x)^{-\nu} x^{\mu-1} dx = B(\mu, 2\nu - \mu)/2(cos s)^{\nu - \frac{1}{2}} \times 2F_1 \left(-\mu + \nu + \frac{1}{2}, \mu - \nu + \frac{1}{2}; \nu + \frac{1}{2}; 1 - e^{-2s}\right).
\]

With \(\mu = \frac{dq - dp}{2}\) and \(\nu = \frac{\lambda + sp - dp - (d-1)}{2}\), we get

\[
R\psi_\lambda(a_s) = C_\lambda e^{-ds}(1 + e^{-2s})^{-\frac{\lambda}{2}} 2F_1 \left(\frac{\mu_\lambda}{2}, 1 - \frac{\mu_\lambda}{2}; \frac{\mu_\lambda + dq - dp}{2}; 1 + e^{-2s}\right),
\]

where \(C_\lambda\) is a positive constant depending on \(p, q\) and \(\lambda\).

The hypergeometric function \(z \mapsto 2F_1(\mu\lambda/2, 1 - \mu\lambda/2; (\mu\lambda + dq - dp)/2; z)\) is a polynomial of degree \(-\mu\lambda/2\) for \(\mu\lambda \leq 0\), and degree \(-\mu\lambda/2 - 1\) for \(\mu\lambda > 0\). We thus immediately get (8.1) for \(\mu_\lambda = 0\).

Now let \(\mu_\lambda = -2m < 0\). We can write \(\psi_\lambda(a_s)\) as the sum

\[
\psi_\lambda(a_s) = (cosh s)^{-\lambda - p\lambda} + \sum_{j=1}^m C_j (cosh s)^{-\lambda + 2j - p\lambda};
\]

or

\[
\psi_\lambda = \tilde{\psi}_\lambda + \sum_{j=1}^m C_j \tilde{\psi}_{\lambda + 2j}.
\]
It follows that $R\psi_\lambda(a_s)$ can be written as a sum

$$R\psi_\lambda(a_s) = C_0 e^{-ds}(1 + e^{-2s})^m + \sum_{j=0}^{m-1} C_j e^{-ds}(1 + e^{-2s})^j,$$

where $C_0$ is a non-zero constant corresponding to the factor $\tilde{\psi}_\lambda(a_s) = (\cosh s)^{-\lambda - \rho_s}$. Thus, since we know that $R\psi_\lambda(a_s)$ is a linear combination of $e^{(\lambda - \rho_1)s}$ and $e^{(-\lambda - \rho_1)s}$, we get $R\psi_\lambda(a_s) = Ce^{(\mu_\lambda - d)s}$, for a non-zero constant $C$.

Let finally $\mu_\lambda > 0$. Then $|\psi_\lambda| \leq \|\phi_{\mu_\lambda}\|_\infty \tilde{\psi}_\lambda$, and $|R\psi_\lambda| \leq \|\phi_{\mu_\lambda}\|_\infty R\tilde{\psi}_\lambda$. We have

$$|R\psi_\lambda(a_s)| \leq C_1 R\tilde{\psi}_\lambda(a_s) \leq C_2 e^{-ds} \quad \text{for} \quad s \to \infty,$$

and

$$|R\psi_\lambda(a_s)| \leq C_1 R\tilde{\psi}_\lambda(a_s) \leq C_2 e^{(\mu_\lambda - d)s} \quad \text{for} \quad s \to -\infty,$$

for positive constants $C_i$. Since $s \mapsto R\psi_\lambda(a_s)$ again is a linear combination of $e^{(\mu_\lambda - d)s}$ and $e^{(-\lambda - \rho_1)s}$, we see from (8.3) and (8.4) that $R\tilde{\psi}_\lambda = 0$. \qed

Consider the cases where $p < q$ and $d(q - p) \leq 2$, i.e., the cases $(d, p, q) = (1, q - 1, q), (d, p, q) = (1, q - 2, q)$ and $(d, p, q) = (2, q - 1, q)$. In the first case $\mu_\lambda = \lambda + 1/2$, and $\mu_\lambda = \lambda$ in the last two cases. This means that $\mu_\lambda > 0$ and (i) follows from Proposition \[8.1\].

For the proof of (v), we need to consider the cases where $p < q$ and $d(q - p) \leq 1$, i.e., the cases $(d, p, q) = (1, q - 1, q)$. From \[1\] Theorem 5.1 (iii)(a) (recall that in that paper $p := p - 1$), we see that the Schwartz condition in the real case is also satisfied for $p = q - 1$.

9. Reduction to the real case ($d = 1$)

Some of our results above for the projective hyperbolic spaces could also be established from \[1\] Theorem 5.2] via the remark below. However, we feel that the new and different presentation, and in particular the new proof of Proposition \[8.1\] merits the space given.

Let $F = \mathbb{C}, \mathbb{H}$, with $p \geq 0, q \geq 1$, and $d = \dim_{\mathbb{R}} F$. There is a natural projection

$$\mathcal{X}(dp, dq, \mathbb{R}) \to \mathcal{X}(p, q, F),$$

with a natural action of $U(1; F)$ on $\mathcal{X}(dp, dq; \mathbb{R})$. Let

$$\mathcal{E}_\lambda(p, q, F) = \{ f \in C^\infty(\mathcal{X}(p, q, F)) \mid \Delta f = (\lambda^2 - \rho_s^2)f \},$$

then there is a $G$-homomorphism,

$$\mathcal{E}_\lambda(p, q, F) \to \mathcal{E}_\lambda(dp, dq, \mathbb{R}),$$

which is an isomorphism onto the $U(1; F)$ invariant functions in $\mathcal{E}_\lambda(dp, dq, \mathbb{R})$. We refer to \[7\] for more details.

Let $p' + 1 = d(p + 1)$ and $q' + 1 = d(q + 1)$. We note that $\rho_q = \frac{1}{2}(dp + dq + 2(d - 1)) = \frac{1}{2}(p' + q') = \rho_q'$. Let $\tilde{\psi}_\lambda$ be as in the proof of Theorem \[8.1\] and let $R_{p,q}^d$ and $R_{p',q'}^1$ denote the Radon transforms corresponding to the spaces $\mathcal{X}(p + 1, q + 1, F)$ and $\mathcal{X}(p' + 1, q' + 1, \mathbb{R})$ respectively. Using the substitution $w' = e^s w$ in \[8.2\], we get the identity:

$$R_{p,q}^d \tilde{\psi}_\lambda(a_s) = e^{-(d-1)s} R_{p',q'}^1 \tilde{\psi}_\lambda(a_s), \quad (s \in \mathbb{R}),$$

(9.1)
which shows that some of our results for the projective hyperbolic spaces follow from the real \((d = 1)\) case. Notice though, that the elements \(a_s \in A_q\) on the left-hand and the right-hand side of (9.1) belong to different groups, and that the reduction only works for the Abelian part. Similarly, we get
\[
\mathcal{A}_{p,q}^d \tilde{\psi}_\lambda = A_{p',q'}^1 \tilde{\psi}_\lambda .
\]

10. Proof of (vi) of the main theorem - A Closer study of \(Rf\) near \(+\infty\)

We want to prove that \(\mathcal{A}(Df) \in \mathcal{S}(\mathbb{R})\), for \(f \in \mathcal{C}^2(G/H)\) and \(d(q-p) > 1\). Although we believe this to be true in general, our proof, near \(+\infty\), is only valid for the dense \(G\)-invariant subspace generated by the \(K\)-finite and \((K \cap H)\)-invariant functions.

For \(d = 1\), Theorem 5.1 (iii)(b)] yields that the Schwartz decay conditions are satisfied near \(-\infty\) for \(\mathcal{A}(f)\), and thus also for \(\mathcal{A}(Df)\). For \(d > 1\), the proof from [1] is easily adapted in the same way as formula (9.1), which leaves us to study \(Rf\) near \(+\infty\). We will concentrate on the proof for \(d = 2, 4\) below, with some comments on the \(d = 1\) case, and further remarks in Section 11.

Consider the subgroup \(T\) given by
\[
k_\theta = \begin{pmatrix}
I_{p+1} & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
0 & 0 & I_{q-1} & 0 \\
0 & -\sin \theta & 0 & \cos \theta
\end{pmatrix}.
\]

Then
\[
k_\theta a_t \cdot x_0 = (\sinh t, 0, \ldots, 0; \sin \theta \cosh t, 0, \ldots, 0, \cos \theta \cosh t).
\]

We see that \(H \supseteq K_1\), with \(K_1\) normalizing \(T\), and \(K_2 = (K_2 \cap H)^T(K_2 \cap H)\), where \(K_2 \cap H = U(q, \mathbb{F}) \times U(1, \mathbb{F})\). Furthermore \(U(q, \mathbb{F})\) centralizes \(A_q\), and as is easily seen, \((K \cap H)k_\theta a_t H = (K \cap H)k_\theta a_t H\), for \(w \in U(1, \mathbb{F})\). From this we deduce that
\[
(K \cap H)k_\theta a_t H = (K \cap H)A_s n_u, v', w H, \quad \text{and} \quad G = (K \cap H)T A H,
\]
where \((K \cap H)^T\) and \((K \cap H)^{A_s}\) denote the centralizers of \(T\) and \(A_q\) in \(K \cap H\) respectively. It follows that a \(K \cap H\)-invariant function is uniquely determined by the values \(f(k_\theta a_t H)\), for \((\theta, t) \in [0, \pi] \times \mathbb{R}^+\).

From the equation \((K \cap H)k_\theta a_t H = (K \cap H)A_s n_u, v', w H\), we get
\[
(cosh t)^2 = (cosh s - 1/2e^s|v'|^2)^2 + |v'|^2 + |u|^2 + |e^s w|^2,
\]
and
\[
(cosh \theta \cosh t)^2 = (cosh s - 1/2e^s|v'|^2)^2 + |e^s w|^2.
\]

Let \(x = |u|, y = |v'|\) and \(z = e^s |w|\). Let \(v = -\sinh s + 1/2e^s y^2\), then \(y^2 = 1 + 2e^{-s} v - e^{-2s}\), and
\[
(cosh t)^2 = 1 + x^2 + v^2 + z^2, \quad \text{and} \quad (cosh \theta \cosh t)^2 = (v - e^{-s})^2 + z^2.
\]

For \(p = 0\), the variable \(x = 0\) and the integration over \(x\) disappears, and for \(d = 1\), the integration over \(z\) disappears. Furthermore, the equations (10.2) and (10.3) are slightly different in these cases, see Section 11.

Consider a \(K \cap H\)-invariant function \(f\) of irreducible \(K\)-type. Then the function \(k \mapsto f(k_\theta a_t)\) is a zonal spherical function on \(K/(K \cap H)\), a Jacobi polynomial in
Applying Lebesgue’s theorem, we get

\[ h(k_0 a_i) = (\cos \theta \cosh t)^m h(a_i), \]

where for each \( N \in \mathbb{N} \),

\[ |H(1 + x^2 + v^2 + z^2)| < C(1 + x^2 + v^2 + z^2)^{-\frac{d - \beta}{2} m} (1 + \log(1 + x^2 + v^2 + z^2))^{-N}. \]

With the above substitutions, we find

\[ R_\theta(s) = e^{-ds} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty H(1 + x^2 + v^2 + z^2)((v - e^{-s})^2 + z^2)^{\frac{s-1}{2}} x^\alpha z^{d-2} dv \, dx \, dz, \]

where \( \alpha = dp - 1, \beta = d(q-p) - 1 \geq 0 \), i.e., \( \beta \) is a positive integer.

We have the following upper bound, for \( s \geq 0 \), since \( \beta \geq 1 \):

\[ |R_\theta(s)| \leq C e^{-ds} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \frac{(1 + x^2 + v^2 + z^2)^{-\frac{d - \beta}{2} m}}{(1 + \log(1 + x^2 + v^2 + z^2))^{N}} (1 + v^2 + z^2)^{\frac{s-1}{2}} x^\alpha z^{d-2} dv \, dx \, dz < +\infty. \]

Applying Lebesgue’s theorem, we get

\[ \lim_{s \to \infty} e^{ds} R_\theta(s) = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty H(1 + x^2 + v^2 + z^2)(v^2 + z^2)^{\frac{s-1}{2}} x^\alpha z^{d-2} dv \, dx \, dz. \]

For convenience, we replace \( z \) by \( u \). We can define \( R_\theta(s) \) as a function of the variable \( z = e^{-s} \) near \( z = 0 \), for \( z > 0 \). Let \( F(z) = e^{ds} R_\theta(s) \), then

\[ F(z) = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty H(1 + x^2 + v^2 + u^2)((v - z)^2 + u^2)^{\frac{s-1}{2}} x^\alpha u^{d-2} dv \, dx \, du. \]

Let \( k_0 \) be the largest integer such that \( k_0 < (\beta - 1)/2 + 1 \), and \( 0 \leq k < k_0 \). The derivatives \( d^k/dz^k \) of the integrand are zero at \( v = -\sinh s = \frac{1}{2}(z - z^{-1}) \), whence the integrand is at least \( k_0 \) times differentiable near \( z = 0 \), and we can compute the derivatives \( d^k/dz^k F(z) \). For \( k_0 > 0 \), we will use Taylor’s formula to express \( F(z) \) as a polynomial of degree \( k_0 - 1 \), plus a remainder term involving \( d^{k_0}/dz^{k_0} F(\xi) \), for some \( 0 < \xi(z) < z \).

**Lemma 10.1.** Fix \( v, u \in \mathbb{R}, m \in 2\mathbb{Z}^+ \) and \( \delta \in \frac{1}{2} \mathbb{Z}^+ \), and define

\[ S(z) = S_{v,u,m,\delta}(z) = ((v - z)^2 + u^2)^{\frac{s-1}{2}} (1 + 2zv - z^2)^{\delta}. \]

For \( 0 \leq j < \delta + 1 \), \( d^j/dz^j S(z) \) is a polynomial in \( (v - z)^2 + u^2 \), \( (v - z) \), and \( (1 + 2zv - z^2) \), of degree at most \( m + \delta \) in \( v \), \( m \) in \( u \), \( m + 2\delta - j \) in \( z \). For \( z = 0 \), the degree is at most \( m + j \) in \( v \), and \( m \) in \( u \). When \( j \) is odd, \( d^j/dz^j S \) is an odd function of \( v \) at \( z = 0 \).

**Proof.** Straightforward, using that \( d/dz(1 + 2zv - z^2) = -d/dz((v - z)^2 + u^2) = 2(v - z). \)
Note, that for \( d(q - p) \) odd, that is, \( d = 1 \) and \( q - p \) odd, the term \( (\beta - 1)/2 = ((q - p) - 2)/2 \) is a half-integer, and the statements in Lemma 10.1 have to be changed accordingly.

Using Taylors formula, we get

\[
F(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_{k_0-1} z^{k_0-1} + R_{k_0}(\xi) z^{k_0},
\]

where \( 0 < \xi < z \), and

\[
c_j = \frac{1}{j!} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty H(1 + x^2 + v^2 + u^2) \frac{d^j}{dz^j} S_{v,u,m,(\beta-1)/2}(0) x^\alpha u^{d-2} \, dv \, dx \, du,
\]

for \( j \in \{0, \ldots, k_0 - 1\} \). By Lemma 10.1, \( c_j = 0 \), for \( j \) odd. The remainder term \( R_{k_0}(\xi) \) is given by:

\[
\frac{1}{k_0!} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty H(1 + x^2 + v^2 + u^2) \frac{d^{k_0}}{dz^{k_0}} S_{v,u,m,(\beta-1)/2}(\xi) x^\alpha u^{d-2} \, dv \, dx \, du.
\]

Consider \( A_h(s) = e^{\rho_1 s} R_h(s) = z^{-(\rho_1 - d)} F(z) \), which is equal to

\[
c_0 z^{-(\rho_1 - d)} + c_2 z^{-(\rho_1 - d - 2)} + \cdots + c_{k_0-2} z^{-2} + c_{k_0-1} z^{-1} + R_{k_0}(\xi).
\]

The exponents \( \rho_1 - d - 2j = d(q - p)/2 - 1 - 2j \), for \( j \in \{0, \ldots, k_0 - 1\} \), correspond to the parameters \( \lambda_1, \ldots, \lambda_r \) of the non-cuspidal discrete series. From (6.3), and the definition of the differential operator \( D \) in Theorem 6.1 (vi), \( A(Dh) \) thus only has a positive contribution from the remainder term, and, due to the term \( d^2/ds^2 \), no constant term at \( \infty \).

Note, that for \( d(q - p) \) odd, the last two terms are: \( c_{k_0-1} z^{-\frac{1}{2}} + z^{\frac{1}{2}} R_{k_0}(\xi) \), where the last term is rapidly decreasing. For the other cases, the constant term \( C_{R_{k_0}} = \lim_{s \to \infty} R_{k_0}(e^{-s}) \) could be non-zero, but we will prove that \( R_{k_0}(\xi) - C_{R_{k_0}} \) is rapidly decreasing at \( +\infty \), where \( \xi = \xi(s) \), with \( 0 < \xi < e^{-s} \). We also consider the case \( k_0 = 0 \), with \( \xi = e^{-s} \).

Let \( G(v,u,z) = 1/k_0! d^{k_0}/dz^{k_0} S_{v,u,m,(\beta-1)/2}(z) \). Then \( G(v,u,z) - G(v,u,0) = z P(v,u,z) \), where \( P \) is a polynomial of degree less than \( m + (\beta - 1)/2 \) in \( v \) and \( u \), where \( \xi = \xi(s) \), with \( 0 < \xi < e^{-s} \). The constant \( P \) and \( G \) denote the polynomials defined from \( P \) and \( G \) by taking absolute values in all coefficients.

Let \( C \) denote (possibly different) positive constants. With \( 0 < \xi < e^{-s} \), we get the following estimates

\[
|R_{k_0}(\xi) - C_{R_{k_0}}| \leq e^{-s} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty |H(1 + x^2 + v^2 + u^2)| |P|(v,u,1) x^\alpha u^{d-2} \, dv \, dx \, du \]

\[
+ \int_0^\infty \int_0^\infty \int_{-\infty}^{-\sinh s} |H(1 + x^2 + v^2 + u^2)| |G|(v,u,0) x^\alpha u^{d-2} \, dv \, dx \, du.
\]

The first integral is bounded by \( C e^{-s} \), since the double integral is convergent. The second integral is bounded near infinity by \( C s^{-N} \), for all \( N \), which is seen as follows. For \( s \) large, the integrand is for every \( N \in \mathbb{N} \) bounded by \( C x^2 + v^2 + u^2)^{-(\rho + m)/2} |v|^{m+k_0} \log(x^2 + v^2 + u^2)^{-N} \). Substituting \( v = -v \), \( x = x'v \) and \( u = u'v \),
we have the estimates
\[
\leq C \int_0^\infty \int_0^\infty \int_{\sinh s}^\infty (1 + x'^2 + u'^2)^{-\frac{\alpha + m}{2}} v^{-\rho s - m + m_0 + \alpha + 1 + d - 1} \\
\times (\log(v^2) + \log(1 + x'^2 + u'^2))^{-N} x'^{d-2} dv \, dx' \, du'
\leq C \int_{\sinh s}^\infty v^{-\rho s + k_0 + \alpha + d} (\log(v))^{-N} \, dv.
\]
Inserting the values \( \rho_q = d(p + q)/2 + d - 1 \), \( k_0 = (d(q - p) - 2)/2 \), and \( \alpha = dp - 1 \), we end up with
\[
C \int_{\sinh s}^\infty v^{-1} (\log(v))^{-N} \, dv = C(N - 1)^{-1} (\log(\sinh s))^{-N + 1} \leq Cs^{-N + 1}.
\]
It follows that \( R_{k_0}(\xi) - C_{R_{k_0}} \) is rapidly decreasing at \( +\infty \), whence \( A(Dh) \) is rapidly decreasing at \( +\infty \), since the constant term is not present, which finishes the proof of Theorem 6.1 (vi) for \( K \)-irreducible \( (K \cap H) \)-invariant functions.

Finally, consider the \( G \)-invariant subspace \( V \) of \( C^2(G/H) \) generated by the \( K \)-irreducible \( (K \cap H) \)-invariant functions. The conclusion in (vi) is clearly satisfied for \( f \in V \). We need to show that \( V \) is dense in \( C^2(G/H) \). Let \( 0 \neq f \in L^2(G/H) \) be perpendicular to \( V \). Let \( \mathcal{U} \) be the closed \( G \)-invariant subspace of \( L^2(G/H) \) generated by \( f \). Then \( \mathcal{U} \) contains a non-zero \( C^\infty \)-vector \( f_1 \in C^2(G/H) \), and after a translation, we may assume that \( f_1(eH) \neq 0 \). The function \( f_2 \) defined by \( 0 \neq f_2(gH) = \int_{K \cap H} f_1(kgH) \, dk \) is then a \( (K \cap H) \)-invariant element in \( \mathcal{U} \), belonging to the closure of \( V \), which is a contradiction.

11. Final Remarks - The Remaining Cases

Theorem 6.1 also holds for the real non-projective space \( G/H = SO(p + 1, q + 1)_{c}/SO(p + 1, q)_{c} \), except for item (iii). The statements (i), (ii), (iv) and (v) are proved in [4]. For the proof of (vi), the last equations in (10.2) and (10.3) should be replaced by
\[
\cos \theta \cosh t = \cosh s - 1/2e^s |v'|^2, \text{ and}
\]
\[
\cos \theta \cosh t = (v - e^{-s}).
\]
Then \( (\theta, t) \in [0, 2\pi] \times \mathbb{R}^+ \), and \( m \) could be odd. For \( p = 0 \), the first equations in (10.2) and (10.3) should be replaced by
\[
\sinh t = \sinh s - 1/2e^sv'^2, \text{ and}
\]
\[
\sinh t = -v,
\]
with \( (\theta, t) \in [0, 2\pi] \times \mathbb{R}^+ \), \( H \) defined by \( H(-\sinh t) = (\cosh t)^{-m} h(a_t) \), and
\[
|H(v)| < C(1 + v^2)^{-\frac{\alpha + m}{2}} (1 + \log(1 + v^2))^{-N}.
\]
With these remarks it is not difficult to modify Lemma 10.1 and complete the proof. Notice, that a priori all constants \( c_j \) in the Taylor expansion could be non-zero.

Finally, we consider the exceptional case, with \( F = \mathbb{O} \) (and \( p = 0, q = 1, d = 8 \)). We will show that the formulas (10.1) and (10.2) are meaningful and true for this case as well. The formula (10.1) was already shown to be true and used in [4]. We give a brief outline of the proof of (10.2).
According to [6] and [8], the exceptional group $G$ can be defined by the automorphisms of a 27 dimensional Jordan Algebra $J_{1,2}$ parameterized by $\xi, u \in \mathbb{R}^3 \times \mathbb{O}^3$, with basis $E_1, E_2, E_3, F_1, F_2, F_3$. We denote an element in $J_{1,2}$ by $X(\xi, u)$.

The subgroups $H$ and $K$ are in fact equal to the stabilizers of $E_3$, respectively of $E_1$. The subgroup $N$, which in this case equals $N^*$, is defined in [6] as $u(y, z), y \in \text{Im}(\mathbb{O})$ and $z \in \mathbb{O}$. The subgroups $A = \{a_i\}$ and $T = \{k_\theta\}$ are also defined there. In [6] the expression $a_xu(y, z)E_3$ is calculated, and in [8] the expression $k_\theta X(\xi, u)$ is calculated; combining these two calculations $k_\theta a_xE_3 = k_\theta a_xu(0, 0)E_3$ can be calculated.

Recall that $\xi_1, \xi_3$ are invariant under $K \cap H$. To derive the first formula in (10.12), we only need to compare the first coordinates of $k_\theta a_xE_3$ and $k_\theta a_xu(y, z)E_3$, $\xi_1(s, y, z) = \xi_1(t, \theta)$; to derive the second formula in (10.12), we only need to compare the third coordinates of $k_\theta a_xE_3$ and $k_\theta a_xu(y, z)E_3$, $\xi_3(s, y, z) = \xi_3(t, \theta)$. We have

\[
\begin{align*}
\xi_1(s, y, z) &= -\cosh(2t) - 1)/2, \\
\xi_1(s, y, z) &= -\cosh(2s)((1 - z)/2 + |z|^2/4 + |y|^2) \\
&\quad + \sinh(2s)(1/2|z|^2(1 - |z|^2/2) - |y|^2) + (1 - |z|^2)/2, \\
\xi_3(s, y, z) &= \cosh(2s)(((1 - z)/2 + |z|^2/4 + |y|^2) \\
&\quad - \sinh(2s)(1/2|z|^2(1 - |z|^2/2) - |y|^2) + (1 - |z|^2)/2.
\end{align*}
\]

A tedious, but straightforward calculation, leads to the formulas (10.12), with $v'$ replaced by $z$, and $w$ replaced by $y$.

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