Feynman’s Path Integrals as Evolutionary Semigroups

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Abstract
We show that, for a class of systems described by a Lagrangian

\[ L(x, \dot{x}, t) = \frac{1}{2} \dot{x}^2 - V(x, t) \]

the propagator

\[ K\left(x'', t''; x', t'\right) = \int e^{\frac{i}{\hbar} \int_{t'}^{t''} dt L(x, \dot{x}, t)} D[x(t)] \]

can be reduced via Noether’s Theorem to a standard path integral multiplied by a phase factor. Using Henstock’s integration technique, this path integral is given a firm mathematical basis. Finally, we recast the propagator as an evolutionary semigroup.

1 Noether Invariants and Path Integrals

Here we deal with systems described by a Lagrangian of the form \((m \equiv 1)\)

\[ L(x, \dot{x}, t) = \frac{1}{2} \dot{x}^2 - V(x, t) \] (1)

The propagator for this system is given by the path integral

\[ K\left(x'', t''; x', t'\right) = \int e^{\frac{i}{\hbar} \int_{t'}^{t''} dt L(x, \dot{x}, t)} D[x(t)] \] (2)
We will use Noether invariants to help simplify (2). Our goal is to remove the time-dependence from the integral in the exponential. The way this will be accomplished is through global time and scale transformations.

Suppose the transformations (assumed to be sufficiently smooth in what follows)

\[ x \rightarrow \hat{x} = x + \epsilon \chi(x, t) \]  
\[ t \rightarrow \hat{t} = t + \epsilon \tau(x, t) \]

leave the action

\[ S[x] = \int_{t'}^{t''} L dt \]

invariant (for particular functions \( \chi \) and \( \tau \)). That is, there exists a function \( \Omega(x, t) \) such that, for every \( x(t) \) we have (see for example [19])

\[ \int_{t'}^{t''} \hat{L}(\hat{x}, \frac{d\hat{x}}{dt}, \hat{t}) d\hat{t} = \int_{t'}^{t''} L(x, \dot{x}, t) dt + \epsilon \int_{t'}^{t''} \frac{d\Omega}{dt}(x, t) dt \]  

(5)

where we have ignored terms of \( O(\epsilon^2) \) and higher. Note that the integral involving \( \Omega \) adds only a constant term since \( x(t') \) and \( x(t'') \) are fixed. Also, if \( \hat{L} \) has the same form as \( L \) then \( \Omega \equiv 0 \). We also assume that (3) and (4) are invertible in the sense that there are functions \( \hat{\chi} \) and \( \hat{\tau} \) (also assumed to be sufficiently smooth) such that

\[ x = \hat{x} + \epsilon \hat{\chi}(\hat{x}, \hat{t}) \]  
\[ t = \hat{t} + \epsilon \hat{\tau}(\hat{x}, \hat{t}) \]

(6)  
(7)

For (8) to hold it is necessary and sufficient to have [12]

\[ \hat{L}(\hat{x}, \frac{d\hat{x}}{dt}, \hat{t}) \frac{d\hat{t}}{dt} = L(x, \dot{x}, t) + \epsilon \frac{d\Omega}{dt}(x, t) \]  

(8)

Now, (8) must hold for all infinitesimal values of \( \epsilon \) in (3) and (4). So, if we differentiate both sides of (8) with respect to \( \epsilon \), remember that \( L \) does not depend on \( \epsilon \), and set \( \epsilon = 0 \) we have

\[ \left( \frac{d}{d\epsilon} L \right) \frac{d\hat{t}}{dt} \bigg|_{\epsilon=0} + \hat{L} \left( \frac{d}{d\epsilon} \frac{d\hat{t}}{dt} \right) \bigg|_{\epsilon=0} = \frac{d\Omega}{dt} \]

(9)
At $\epsilon = 0$ we have $\hat{x} = x$, $d\hat{x}/d\hat{t} = \dot{x}$, $\hat{t} = t$ and $\hat{L} = L$. Also,

$$\left.\frac{d\,d\hat{t}}{d\epsilon\,dt}\right|_{\epsilon=0} = \left.\frac{d}{d\epsilon}\left(1 + \epsilon \frac{\partial \tau}{\partial x} \dot{x} + \epsilon \frac{\partial \tau}{\partial t} \right)\right|_{\epsilon=0}$$

$$= \left.\left(\frac{\partial \tau}{\partial x} \dot{x} + \frac{\partial \tau}{\partial t} \right)\right|_{\epsilon=0}$$

$$= \frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x} \dot{x}$$

$$= \frac{d\tau}{dt}$$

So,

$$\hat{L} \left(\frac{d\,d\hat{t}}{d\epsilon\,dt}\right)\bigg|_{\epsilon=0} = L\dot{x}$$

Also notice from (10) that

$$\left.\frac{d\hat{t}}{dt}\right|_{\epsilon=0} = 1$$

(11)

Returning to (9) we have

$$\left.\frac{d\hat{L}}{d\epsilon}\right|_{\epsilon=0} = \left(\frac{\partial \hat{L}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial \epsilon} + \frac{\partial \hat{L}}{\partial (d\hat{x}/d\hat{t})} \frac{\partial (d\hat{x}/d\hat{t})}{\partial \epsilon} + \frac{\partial \hat{L}}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial \epsilon}\right)\bigg|_{\epsilon=0}$$

Now, from (8) and using (3)

$$\left.\frac{\partial \hat{L}}{\partial \hat{x}}\right|_{\epsilon=0} = \left.\frac{\partial L}{\partial x}\right|_{\epsilon=0} + \epsilon \frac{\partial \Omega}{\partial \hat{x}}\bigg|_{\epsilon=0}$$

$$= \left.\frac{\partial L}{\partial x}\right|_{\epsilon=0} + \frac{\partial L}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial \epsilon} \bigg|_{\epsilon=0} + \epsilon \frac{\partial \Omega}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial \epsilon} \bigg|_{\epsilon=0}$$

$$= \frac{\partial L}{\partial x}$$

Similarly, we have

$$\left.\frac{\partial \hat{L}}{\partial (d\hat{x}/d\hat{t})}\right|_{\epsilon=0} = \left.\frac{\partial L}{\partial \dot{x}}\right|_{\epsilon=0}$$

(13)
\[
\frac{\partial \hat{L}}{\partial t} \big|_{\epsilon=0} = \frac{\partial L}{\partial t}
\]  

Looking at (3) and (4) we see that

\[
\frac{\partial \hat{x}}{\partial \epsilon} \big|_{\epsilon=0} = \chi
\]  

\[
\frac{\partial \hat{t}}{\partial \epsilon} \big|_{\epsilon=0} = \tau
\]

So now using (11) to (16)

\[
\left( \frac{d}{d\epsilon} \hat{L} \right) \frac{d\hat{t}}{dt} \big|_{\epsilon=0} = \left( \frac{\partial L}{\partial x} \chi + \frac{\partial L}{\partial \hat{x}} \left( \frac{\partial (d\hat{x}/dt)}{\partial \epsilon} \right) \big|_{\epsilon=0} + \frac{\partial L}{\partial \hat{t}} \tau \right)
\]

We only need to find

\[
\frac{\partial (d\hat{x}/dt)}{\partial \epsilon} \big|_{\epsilon=0}
\]

Notice that

\[
\frac{d\hat{x}}{dt} \frac{d\hat{t}}{dt} = \frac{d\hat{x}}{dt}
\]

So

\[
\left( \frac{\partial d\hat{x}}{\partial \epsilon dt} \right) \frac{d\hat{t}}{dt} + \frac{d\hat{x}}{dt} \left( \frac{\partial d\hat{t}}{\partial \epsilon dt} \right) = \frac{\partial d\hat{x}}{\partial \epsilon dt}
\]

Evaluating (18) at \(\epsilon = 0\) and using (10) and (11) gives us

\[
\frac{\partial d\hat{x}}{\partial \epsilon dt} \big|_{\epsilon=0} + \frac{d\hat{x}}{dt} \big|_{\epsilon=0} \hat{\tau} = \frac{\partial d\hat{x}}{\partial \epsilon dt} \big|_{\epsilon=0}
\]

Using (3) in (17) we see that

\[
\frac{d\hat{x}}{dt} \big|_{\epsilon=0} = \hat{x}
\]
Also, from (3)
\[
\frac{d\hat{x}}{dt} = \dot{x} + \epsilon \frac{\partial \chi}{\partial x} \dot{x} + \epsilon \frac{\partial \chi}{\partial t}
\]  
(21)

Using (20) and (21) in (19) gives us
\[
\frac{\partial}{\partial \epsilon} \left. \frac{d}{dt} \hat{L} \right|_{\epsilon=0} = \frac{\partial}{\partial \epsilon} \left( \dot{x} + \epsilon \frac{\partial \chi}{\partial x} \dot{x} + \epsilon \frac{\partial \chi}{\partial t} \right) \bigg|_{\epsilon=0} - \dot{x} \dot{\tau}
\]
\[
= \frac{\partial \chi}{\partial x} \dot{x} + \frac{\partial \chi}{\partial t} - \dot{x} \dot{\tau}
\]
\[
= \frac{d\chi}{dt} - \dot{x} \dot{\tau}
\]
\[
= \dot{\chi} - \dot{x} \dot{\tau}
\]

So now we have
\[
\left( \frac{d}{d\epsilon} \hat{L} \right) \frac{dt}{dt} \bigg|_{\epsilon=0} = \frac{\partial L}{\partial \chi} \chi + \frac{\partial L}{\partial \dot{x}} (\dot{\chi} - \dot{x} \dot{\tau}) + \frac{\partial L}{\partial \tau} \dot{\tau}
\]

Then (9) becomes
\[
\frac{\partial L}{\partial \chi} \chi + \frac{\partial L}{\partial \dot{x}} (\dot{\chi} - \dot{x} \dot{\tau}) + \frac{\partial L}{\partial \tau} \dot{\tau} = \frac{d\Omega}{dt}
\]  
(22)

Now
\[
\frac{\partial L}{\partial \dot{x}} \chi = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \chi \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \chi
\]

Then (22) becomes
\[
\frac{\partial L}{\partial x} \chi + \dot{L} + \frac{d \dot{L}}{dt} \tau - \frac{\partial L}{\partial \dot{x}} \dot{x} \tau + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \chi \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \chi
\]
\[
- \left( \frac{\partial L}{\partial \dot{x}} \dot{\dot{x}} \tau + \frac{\partial L}{\partial \dot{x}} \dot{\tau} \right) = \frac{d\Omega}{dt}
\]

Using
\[
\frac{\partial L}{\partial \dot{x}} \dot{x} \dot{\tau} + \frac{\partial L}{\partial \dot{x}} \dot{\tau} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \dot{x} \tau \right) - \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \dot{\tau}
\]

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gives
\[ \frac{\partial L}{\partial x} \chi - \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \chi + L \ddot{\tau} + \frac{dL}{dt} \tau - \frac{\partial L}{\partial x} \ddot{\tau} + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \chi \right) \]
\[ - \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{x}} \right) + \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \ddot{\tau} = \frac{d\Omega}{dt} \]

Rearranging some we have
\[ \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \right] (\chi - \dot{x} \tau) + \frac{d}{dt} \left[ L \tau + \frac{\partial L}{\partial \dot{x}} (\chi - \dot{x} \tau) \right] = \frac{d\Omega}{dt} \] (23)

So, this gives us
\[ (\chi - \dot{x} \tau) \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \right] = \frac{d}{dt} I(x, \dot{x}, t) \] (24)

where
\[ I(x, \dot{x}, t) = -L \tau - \frac{\partial L}{\partial \dot{x}} (\chi - \dot{x} \tau) + \Omega \] (25)

Hamilton’s principle states that the actual path followed will be an extreme value of the action integral. That is
\[ \delta S[x] = \delta \int_{t'}^{t''} L dt = 0 \]

This yields the Euler-Lagrange equation of motion
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \] (26)

Using (26) in (24) we have

**Theorem 1.1 (Noether’s First Theorem (for one parameter))** Along a path \( x(t) \) satisfying (24), \( I(x, \dot{x}, t) \) as given in (23) is constant.

Now we put our Lagrangian (1) into (25) to get
\[ I(x, \dot{x}, t) = \frac{1}{2} \dot{x}^2 \tau - \dot{x} \chi + V \tau + \Omega \] (27)
So a Lagrangian of the form

\[ L(x, \dot{x}, t) = \frac{1}{2} \dot{x}^2 - V(x, t) \]

leads to an invariant quadratic in \( \dot{x} \). We ask, is there a restriction on the form of \( V \) that will allow an invariant \( I \)? The answer is yes.

In order to determine the form of \( V \) we follow the method developed by Lewis and Leach (see [9], [10], [11], [16] and [17]). Since we know our invariant is quadratic in \( \dot{x} \), we start with a general invariant

\[ I(x, \dot{x}, t) = f_2(x, t)\dot{x}^2 + f_1(x, t)\dot{x} + f_0(x, t) \tag{28} \]

where \( f_0, f_1 \) and \( f_2 \) are functions of \( x \) and \( t \) satisfying the condition

\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x} \dot{x} + \frac{\partial I}{\partial x} \ddot{x} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x} \dot{x} - \frac{\partial I}{\partial \dot{x}} \frac{\partial V}{\partial x} = 0 \]

We used (26) in going to the second equality above. For (28)

\[ \frac{\partial I}{\partial t} = f_2 \dot{x}^2 + f_1 \dot{x} + f_0 \]
\[ \frac{\partial I}{\partial x} \dot{x} = f_2 x \dot{x}^3 + f_1 x \dot{x}^2 + f_0 x \dot{x} \]
\[ -\frac{\partial I}{\partial \dot{x}} \frac{\partial V}{\partial x} = -2 f_2 \dot{x} \frac{\partial V}{\partial x} - f_1 \frac{\partial V}{\partial x} \]

where the additional subscripts denote partial differentiation with respect to \( t \) or \( x \).

So now we have

\[ (f_{2x}) \dot{x}^3 + (f_{2t} + f_{1x}) \dot{x}^2 + (-2 f_2 \frac{\partial V}{\partial x} + f_{1t} + f_{0x}) \dot{x} \]
\[ + (-f_1 \frac{\partial V}{\partial x} + f_{0t}) = 0 \tag{29} \]

Since \( f_0, f_1 \) and \( f_2 \) are independent of \( \dot{x} \), we require each of the coefficients in (29) to vanish separately. This gives us

\[ f_{2x} = 0 \]

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so
\[ f_2(t) = 2\alpha(t) \]
Then,
\[ f_{1x} = -2\dot{\alpha} \]
which leads to
\[ f_1(x, t) = -2\dot{\alpha}(t)x + \beta(t) \]
Then we have
\[
-4\alpha \frac{\partial V}{\partial x} - 2\ddot{\alpha}x + \dot{\beta} + f_{0x} = 0 \quad (30)
\]
\[
(2\dot{\alpha}x - \beta) \frac{\partial V}{\partial x} + f_0t = 0 \quad (31)
\]
Integrating (30) with respect to \( x \) gives us
\[
-4\alpha [V + \gamma_1(t)] - \ddot{\alpha}x^2 + \dot{\beta}x + [f_0 + \gamma_2(t)] = \gamma_3(t)
\]
or
\[
V(x, t) = \frac{1}{4\alpha}(f_0 - \ddot{\alpha}x^2 + \dot{\beta}x) + \gamma(t) \quad (32)
\]
Using \( V \) in (31) and solving the PDE for \( f_0 \) via the method of characteristics results in
\[
f_0(x, t) = G\left(\frac{x}{\alpha^{3/2}} + \frac{1}{4} \int^t \frac{\beta(t^*)}{\alpha^{3/2}(t^*)} dt^*\right) + \frac{1}{2\alpha}(\dot{\alpha}x - \frac{1}{2}\beta)^2
\]
where \( G(\cdot) \) is an arbitrary function of its argument.
If we define \( \rho(t) \) and \( a(t) \) by
\[
\rho(t) = 2\alpha(t)^{1/2} \quad \quad -\frac{a(t)}{\rho(t)} = \frac{1}{8} \int^t \frac{\beta(t^*)}{\alpha^{3/2}(t^*)} dt^*
\]
and use the solution for $f_0$ in (32), we have that the potential is

$$V(x, t) = \left( \frac{\ddot{a} - \dot{a}}{\rho} \right) x - \frac{1}{2} \frac{\ddot{\rho}}{\rho} x^2 + \frac{1}{\rho^2} F \left( \frac{x - a}{\rho} \right)$$

where $F(u) = G(2u)$. We have removed an irrelevant function of $t$ from $V$ since it only adds a constant to the value of the action $S[x]$. This potential has the associated Noether invariant

$$I(x, \dot{x}, t) = \frac{1}{2} \left[ \rho (\dot{x} - \dot{a}) - \dot{\rho} (x - a) \right] \dot{x} + \cdots$$

Finding the parts of $I$ that depend on $\dot{x}$, and $\dot{x}^2$

$$I = \frac{\rho^2}{2} \dot{x}^2 - \rho [\rho \dot{a} + \dot{\rho} (x - a)] \dot{x} + \cdots$$

and comparing with (27) leads to

$$\tau = \rho^2$$
$$\chi = \rho [\rho \dot{a} + \dot{\rho} (x - a)]$$

In summary, we can start with any two functions $a(t)$ and $\rho(t)$ that have continuous second derivatives on our time period of interest (since we will want $V$ to be continuous below). With these we define a Lagrangian of the form

$$L(x, \dot{x}, t) = \frac{1}{2} \dot{x}^2 - V(x, t)$$

where

$$V(x, t) = \left( \frac{\ddot{a} - \dot{a}}{\rho} \right) x - \frac{1}{2} \frac{\ddot{\rho}}{\rho} x^2 + \frac{1}{\rho^2} F \left( \frac{x - a}{\rho} \right)$$  \hspace{1cm} (33)$$

and $F(\cdot)$ is an arbitrary function of its argument. The action

$$S[x] = \int_{t'}^{t''} L dt$$

will be invariant under the transformations

$$x \rightarrow \hat{x}(x, t) = x + \epsilon \rho [\rho \dot{a} + \dot{\rho} (x - a)]$$
$$t \rightarrow \hat{t}(t) = t + \epsilon \rho^2(t)$$
Associated with this system will be the Noether invariant

\[
I(x, \dot{x}, t) = \frac{1}{2} \left[ \rho(\dot{x} - \dot{a}) - \dot{\rho}(x - a) \right]^2 + F \left( \frac{x - a}{\rho} \right)
\]  

(34)

Note that \( V \) can have an arbitrary additive function of \( t \)

\[ V \to V + g(t) \]

This only adds a constant to the action so we can set \( g(t) \equiv 0 \).

**Example 1.2 (The 1-D Time-Dependent Harmonic Oscillator)** Consider the case where

\[ C\rho(t) = a(t) \quad 0 \leq C < \infty \]

and \( F \equiv 0 \). Then we have

\[
V(x, t) = -\frac{1}{2} \ddot{\rho} \rho x^2
\]

\[ = \frac{1}{2} \omega(t)x^2 \]

where \( \omega(t) = -\ddot{\rho}/\rho \). This is the harmonic oscillator with a time-dependent frequency. Now, we see that the Lagrangian

\[
L(x, \dot{x}, t) = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega(t)x^2
\]

will be invariant under the transformations

\[
\dot{x} = x + \frac{\epsilon}{2} \left( \frac{d}{dt} \rho^2 \right)x
\]

\[
\dot{t} = t + \epsilon \rho^2
\]

The Noether invariant of this system is given by

\[
I(x, \dot{x}, t) = \frac{1}{2} \left[ \rho \ddot{x} - \dot{\rho} \dot{x} \right]^2
\]

The reader is invited to show what happens when we have \( C = 0 \) above and

\[
F(t) = \frac{1}{2} K \left( \frac{x}{\rho} \right)^2 \quad 0 < K < \infty
\]
Now that we have an expression for $V$, we put this into (1) to get (see [5])

$$L(x, \dot{x}, t) = \frac{d}{dt}X + L_0$$

where

$$L_0(x, \dot{x}, t) = \frac{\rho^2}{2} \left[ \frac{d}{dt} \left( \frac{x - a}{\rho} \right) \right]^2 - \frac{1}{\rho^2}F \left( \frac{x - a}{\rho} \right)$$

and

$$X(t) = \frac{\dot{\rho}}{2\rho}x^2 + \frac{W}{\rho}x - G$$

$$W(t) = \dot{a} \rho - a \dot{\rho}$$

$$G(t) = \int_{t'}^t \frac{W^2}{2\rho^2}ds$$

Letting

$$Q(t) = \frac{x - a}{\rho}$$

$$\dot{\hat{t}} = \int_{t'}^t \rho^{-2}ds$$

we have

$$\int_{t'}^{t''} L_0 dt = \int_{\hat{t}'}^{\hat{t}''} \bar{L}_0 d\hat{t}$$

where

$$\bar{L}_0 \left( \bar{Q}(\hat{t}), \dot{\bar{Q}}(\hat{t}) \right) = \frac{1}{2} \left[ \frac{d}{d\hat{t}} \bar{Q}(\hat{t}) \right]^2 - F \left( \bar{Q}(\hat{t}) \right)$$

Then our original propagator (2) is given by

$$K \left( x'', t'' ; x', t' \right) = \left[ \rho(t'') \rho(t') \right]^{-1/2} e^{\frac{i}{\hbar} \left[ X(t'') - X(t') \right]} \bar{K}_0 \left( Q'', \dot{\bar{Q}}'' ; \bar{Q}', \dot{\bar{Q}}' \right)$$

where

$$\bar{K}_0 \left( \bar{Q}', \dot{\bar{Q}}' ; \bar{Q}'', \dot{\bar{Q}}'' \right) = \int e^{\frac{i}{\hbar} \int_{\hat{t}'}^{\hat{t}''} d\hat{t} \bar{L}_0(\bar{Q}(\hat{t}), \dot{\bar{Q}}(\hat{t}))} \mathcal{D} \left[ \bar{Q}(\hat{t}) \right]$$

See [7] for the measure factor $\left[ \rho(t'') \rho(t') \right]^{-1/2}$ in (35). Notice that $\bar{K}_0$ is the "standard" path integral formula. We will now rigorously define this using Muldowney’s [13] application of the Henstock integral.
2 Henstock Integration

The Henstock (gauge or generalized Riemann) integral is a simple, though powerful generalization of the Riemann integral. Essentially it amounts to letting the interval length bounds depend on position. In the definition of the Riemann integral, the bound is the same over the whole domain of integration. Thus, the Henstock integral allows us to take into account the local variation of the functions we integrate. Also, since it is a non-absolute integral, it leads to a particularly simple and conceptually pleasing definition of the Feynman path integral. For more information on the Henstock integral in finite dimensions see DePree and Swartz [6]. For the application to functional integration see Muldowney [14]. The following is largely based on [13] and [15].

We begin with the Henstock integral in one dimension. We do this in \( \mathbb{R} \). As shown in Example (2.2) below, the generalization to \( E \subseteq \mathbb{R} \) is easy. Let \( I = [u, v) \), \( u, v \in \mathbb{R} \) \( \subseteq [-\infty, \infty] \) and \( |I| = v - u \). If either \( u = -\infty \) or \( v = \infty \) we let \( |I| = 0 \). Let \( \delta(x) > 0 \) for all \( x \in \mathbb{R} \). We call \( \delta \) a gauge. We say that \( I \) is attached to \( x \) if any of the following is true

\[
I = \begin{cases} (-\infty, v) \\ [u, v) \\ [u, \infty) \end{cases} \quad \text{and} \quad x = \begin{cases} -\infty \\ u \text{ or } v \\ \infty \end{cases}
\]

We call the attached pair \((x, I)\) \( \delta \)-fine if, respectively

\[
\begin{cases}
v < -\frac{1}{\delta(x)} \\
v - u < \delta(x) \\
u > \frac{1}{\delta(x)}
\end{cases}
\]

We say that

\[
\mathcal{E} = \{(x^{(i)}, I^{(i)})\}_{i=1}^{n}
\]

is a division of \( \mathbb{R} \) if \( I^{(i)} \cap I^{(j)} = \emptyset \) for \( i \neq j \) and

\[
\bigcup_{i=1}^{n} I^{(i)} = \mathbb{R}
\]

Further, \( \mathcal{E}_\delta \) is \( \delta \)-fine when every \((x^{(i)}, I^{(i)})\) is \( \delta \)-fine for \( i = 1, \ldots, n \). When we sum over the point-interval pairs in \( \mathcal{E}_\delta \), we denote this by \((\mathcal{E}_\delta) \Sigma\).
Definition 2.1 Let $h(x, I)$ be a function of point-interval pairs $(x, I)$ where $h(x, I) = 0$ when $x = \pm \infty$. We say $h(x, I)$ is Henstock integrable over $\mathbb{R}$, to the value $\alpha$, if for every $\epsilon > 0$ there exists a $\delta(x), x \in \mathbb{R}$ such that
\[
|(\mathcal{E}_\delta) \sum h(x, I) - \alpha| < \epsilon
\]
whenever the division $\mathcal{E}_\delta$ is $\delta$-fine. We write
\[
\int_{\mathbb{R}} h(x, I) = \alpha
\]

Example 2.2 Let
\[
f(x) = \begin{cases} 
0 & x \in [0, 1] \text{ and rational} \\
1 & x \in [0, 1] \text{ and irrational}
\end{cases}
\]
Here we'll let $h(x, I) = f(x) |I|$ where we'll define the gauge below. The gauge then puts an upper bound on the possible size of the $|I|$'s. Note that we do not actually form an explicit set $\mathcal{E}_\delta$. We have, with our domain of integration being $[0, 1]$
\[
| (\mathcal{E}_\delta) \sum f(x_j) |I_j| - 1 | = | (\mathcal{E}_\delta) \sum (f(x_j) - 1) |I_j|
\]
If $x_j$ is irrational, $f(x_j) - 1 = 0$. We can let $\delta(x) = 1$ for $x$ irrational. For $x_j = q_k$ rational, where $q_k \in Q = \{q_n : q_n \text{ is irrational} \}_{n=1}^{\infty}$, set $\delta(x_j) = \frac{\epsilon}{2k+1}$. Then,
\[
|f(q_k) - 1| |I_k| < \frac{\epsilon}{2k+1}
\]
Since $q_k$ is attached to at most two intervals $I$, we have (even if every rational number is attached to two intervals)
\[
| (\mathcal{E}_\delta) \sum (f(x_j) - 1) |I_j| \leq 2 \sum_{k=1}^{\infty} \frac{\epsilon}{2k+1}
\]
Hence,
\[
\int_{0}^{1} f(x) dx = 1
\]
We'll now extend the Henstock integral to multiple dimensions. For \( \mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2 \) let \( I \subset \mathbb{R}^2 \) be of the form

\[
I = I_1 \times I_2 = [u_1, v_1) \times [u_2, v_2)
\]

where \( u_i = -\infty \) and \( v_i = \infty \) are allowed. All the definitions are as in the one-dimensional case where now \( x \) is attached to \( I \) if \( x \) is at a corner of \( I \).

With \( \delta(x) > 0 \) for \( x \in \mathbb{R}^2 \) being our gauge, we say \( (x, I) \) is \( \delta \)-fine if \( (x_1, I_1) \) and \( (x_2, I_2) \) are both \( \delta \)-fine. Thus, the \( I \)'s will be rectangles in \( \mathbb{R}^2 \). Finally, \( h(x, I) \) is integrable on \( \mathbb{R}^2 \), written as

\[
\int_{\mathbb{R}^2} h(x, I) = \alpha
\]

if, for every \( \epsilon > 0 \) there exists a \( \delta(x) \) such that

\[
\| (E_\delta) \sum h(x, I) - \alpha \| < \epsilon
\]

whenever \( E_\delta \) is a \( \delta \)-fine division of \( \mathbb{R}^2 \). The generalization to \( \mathbb{R}^n \) is obvious.

We now define the Henstock integral in infinite dimensions. Note the definition of the gauge here. This is different than that defined in \([13]\) (see \([13]\)). Let \( B = [t', t''] \), \( x(t) \) be a function defined on \( B \) and, for \( N = \{t_i\}_{i=1}^{n-1} \subseteq B \) let

\[
I(N) = I_{t_1} \times \cdots \times I_{t_{n-1}} \subseteq \mathbb{R}^{n-1}
\]

where \( I_{t_i} = [u_i, v_i), (-\infty, v_i) \) or \( [u_i, \infty) \). We call \( N \) a dimension set. \( I(N) \) is attached to the vector \( x(N) \in \mathbb{R}^{n-1} \) when

\[
x(N) = (x_1, \ldots, x_{n-1})
\]

where \( x_i = x(t_i), t_i \in N \) and \( x_i = u_i \) or \( v_i, -\infty \) or \( \infty \), respectively. Define

\[
I[N] \doteq I(N) \times \mathbb{R}^{B \setminus N} = \left\{ x(t) : x \in \mathbb{R}^B, x_i \in I_{t_i} \text{ for } i = 1, \ldots, n-1 \right\}
\]

\[
\|I[N]\| \doteq |I(N)| = |I_{t_1}| \cdots |I_{t_{n-1}}| = \text{Volume}(I(N))
\]

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We say $x(t)$, $N$ and $I[N]$ are attached if $I(N)$ is attached to $x(N)$.

For $M = \{t_i\} \subseteq B$ suppose we have $t' = t_0 < t_1 < \cdots < t_{m-1} < t_m = t''$. Let

$$J[M] \doteq \{x(t) : x(t') = x', x(t'') = x'', x_i \in I'_{t_i} \text{ for } i = 1, \ldots, m-1\}$$

$J[M]$ is the set of functions starting (ending) at $x'$ ($x''$) and passing through the interval $I'_{t_i}$ at time $t_i$. $J[M] \subseteq I[N]$ when $N \subseteq M$ or when $I'_{t_i} \subseteq I_{t_i}$ for $t_i \in M \cap N$.

Let $A = \{t_i\}_{i=1}^{\infty} \subseteq B$. Define $L_A$ as the collection of all possible finite subsets of $A$

$$L_A \doteq \{\{t^k_j : t^k_j \in A \text{ for } j = 1, \ldots, l_k\}_{k=1}^{\infty}\}$$

Let

$$L_A(x(t)) : \mathbb{R}^B \to L_A$$

and

$$\delta_{L_A} \doteq \{\delta_N(x(N)) : L_A(x(t)) \subseteq N \text{ and } x(N) \in \mathbb{R}^{n-1}\}$$

for $x(t) \in \mathbb{R}^B$. Note that $\delta_N(x(N)) \in \delta_{L_A}$ is allowed to depend on both $x(N)$ and $N$.

A gauge is defined on $\mathbb{R}^B$ by

$$\gamma = (A, L_A, \delta_{L_A})$$

Let $x(t)$, $N$, and $I[N]$ be attached. Then $(x(t), N, I)$ is $\gamma$-fine if $L_A \subseteq N$ and $(x(N), I(N))$ is $\delta_N$-fine on the finite dimensional space $\mathbb{R}^N$. Here $\delta_N = \delta_N(x(N)) \in \delta_{L_A}$.

So, for each $x(t) \in \mathbb{R}^B$ we have to sample the function at the times $t_i \in L_A(x(t))$. These times are constrained by our choice of $A$. However, we can sample the $x(t)$’s on sets larger than the minimum, including $t_i$’s not in $A$. We call these larger sets $N \supseteq L_A(x(t))$. By allowing the size of $N$ to increase we can discriminate more among those functions that give the same minimal set $L_A$. The reason for this is that these functions may take on different values at $t_i$ that are not in $L_A$ even if they have the same values on $L_A$. Now that we have a set $N$, we choose a finite dimensional gauge $\delta_N$. This
gauge gives us the size of the interval associated with each \( x(N) \in \bar{\mathbb{R}}^{n-1} \). We can shrink this interval by choosing a finer gauge.

When \( \left( x^{(i)}(t), I^{(i)}[N_i] \right) \) is \( \gamma \)-fine for every \( i \) we call

\[
\mathcal{E}_\gamma = \left\{ \left( x^{(i)}(t), I^{(i)}[N_i] \right) \right\}_{i=1}^{n}
\]

\( \gamma \)-fine. \( \mathcal{E}_\gamma \) is a division of \( \mathbb{R}^B \) when

\[
I^{(i)}[N_i] \cap I^{(j)}[N_j] = \emptyset
\]

for \( i \neq j \) and

\[
\bigcup_{i=1}^{n} I^{(i)}[N_i] = \mathbb{R}^B
\]

**Definition 2.3** Let \( h(x, N, I) \) be a function of point-dimension set-interval triplets. Define \( h(x, N, I) \equiv 0 \) whenever \( x(t_i) = \pm \infty \) for \( t_i \in N \). We say that \( h(x, N, I) \) is **Henstock integrable over** \( \mathbb{R}^B \), to the value \( \alpha \), if for every \( \epsilon > 0 \) there exists a \( \gamma \) such that

\[
\left| \left( \mathcal{E}_\gamma \right) \sum h(x, N, I) - \alpha \right| < \epsilon
\]

whenever the division \( \mathcal{E}_\gamma \) is \( \gamma \)-fine. We write

\[
\int_{\mathbb{R}^B} h(x, N, I) = \alpha
\]

### 3 Feynman Path Integration

Our path integral is given by (36)

\[
\bar{K}_0 \left( \bar{Q}'', \hat{t}' ; \bar{Q}', \hat{t}' \right) = \int e^{\int_{\bar{t}'}^{\hat{t}''} d\bar{t}_0(\bar{Q}(\bar{t}), \bar{Q}(\hat{t}))} D[\bar{Q}(\hat{t})]
\]

We start with a finite dimensional approximation to this

\[
\bar{K}_0 \approx \int_{u_1}^{v_1} \cdots \int_{u_{n-1}}^{v_{n-1}} e^{i \sum_{j=1}^{n} \frac{(\bar{Q}_j - \bar{Q}_{j-1})^2}{2(t_j - t_{j-1})} - \hat{F} (\bar{Q}_{j-1}) (\hat{t}_j - \hat{t}_{j-1})} \times \prod_{j=1}^{n} \left[ 2h\pi i (\hat{t}_j - \hat{t}_{j-1}) \right]^{1/2} d\bar{Q}_1 \cdots d\bar{Q}_{n-1}
\]
where $\bar{Q}_j \doteq \bar{Q}(\hat{t}_j)$. Define

$$g_F(\bar{Q}, N) = \exp \left[ i \sum_{j=1}^{n} \left\{ \frac{(\bar{Q}_j - \bar{Q}_{j-1})^2}{2(\hat{t}_j - \hat{t}_{j-1})} - \bar{F}(\bar{Q}_{j-1})(\hat{t}_j - \hat{t}_{j-1}) \right\} \right] \times \prod_{j=1}^{n} \left[ 2\hbar \pi i(\hat{t}_j - \hat{t}_{j-1}) \right]^{-1/2}$$

where

$$g_0(\bar{Q}, N) = \exp \left[ i \sum_{j=1}^{n} \left\{ \frac{(\bar{Q}_j - \bar{Q}_{j-1})^2}{2(\hat{t}_j - \hat{t}_{j-1})} \right\} \right] \prod_{j=1}^{n} \left[ 2\hbar \pi i(\hat{t}_j - \hat{t}_{j-1}) \right]^{-1/2}$$

is the free particle case. Note that the above can be interpreted either as integrating over functions or over the finite dimensional space $\mathbb{R}^{n-1}$. If we wish to specify the latter case we will write $g_F(\bar{Q}(N))$ instead.

Now define

$$g_F(\bar{Q}, N, I) \doteq g_F(\bar{Q}, N) |I[N]|$$

$$g_F(\bar{Q}(N), I(N)) \doteq g_F(\bar{Q}(N)) |I(N)|$$

and, having chosen a $J[N]$ for a fixed $N$, let

$$G_F(J(N)) \doteq \int_{J(N)} g_F(\bar{Q}(N), I(N))$$

This is a finite dimensional integral over a hypercube in $\mathbb{R}^{n-1}$ given by

$$I'_{t_1} \times \cdots \times I'_{t_{n-1}}$$

where $\bar{Q}_i \in I'_{t_i}$ for $\bar{Q}(t) \in J[N]$. We then let $G_F(J[N]) = G_F(J(N))$. In particular we have

$$G_0(J) = \int_J g_0(\bar{Q}(N), I(N))$$

$$= \int_J g_0(\bar{Q}(N)) |I(N)|$$

$$= \int_J \exp \left[ i \sum_{j=1}^{n} \left\{ \frac{(\bar{Q}_j - \bar{Q}_{j-1})^2}{2(\hat{t}_j - \hat{t}_{j-1})} \right\} \right] \prod_{j=1}^{n} \left[ 2\hbar \pi i(\hat{t}_j - \hat{t}_{j-1}) \right]^{-1/2} |I(N)|$$
for the free particle case. Let

\[ \bar{f}(\bar{Q}, N) = \exp \left[ -\frac{i}{\hbar} \sum_{j=1}^{n} \bar{F}(\bar{Q}_{j-1}) (\hat{t}_j - \hat{t}_{j-1}) \right] \]

Then, if \( \bar{F} \) is continuous we have

\[ K_0(\bar{Q}^{\prime\prime}, \hat{t}^{\prime\prime}; \bar{Q}^{\prime}, \hat{t}^{\prime}) = \int_{\Re(t', t'')} \bar{f}(\bar{Q}, N) G_0(I) \] (37)

Letting \( h(\bar{Q}, N, I) = \bar{f}(\bar{Q}, N) G_0(I) \) this integral is defined in Definition (2.3).

We would like a better way to evaluate (37). To that end we let \( \tau_j = \hat{t}' + j(\hat{t}'' - \hat{t}')/m \) for \( j = 1, \ldots, m-1 \) and \( m = 2^q \). Then we have \( M = \{ \tau_j \}_{j=1}^{m-1} \) and, with \( y_j = y(\tau_j) \),

\[ y = (y_1, \ldots, y_{m-1}) \in \Re^{m-1} \]

where \( y_0 = \bar{Q}' \), \( y_m = \bar{Q}'' \). Letting

\[ g_F^{(m)}(y) = \exp \left[ \frac{i}{\hbar} \sum_{j=1}^{m} \left\{ \frac{(y_j - y_{j-1})^2}{2(\tau_j - \tau_{j-1})} - \bar{F}(y_{j-1}) (\tau_j - \tau_{j-1}) \right\} \right] \times \prod_{j=1}^{m} [2\hbar \pi i(\tau_j - \tau_{j-1})]^{-1/2} \]

we have that if

\[ \int_{\Re^{m-1}} g_F^{(m)}(y) dy = \Gamma_m \]

then

\[ \int_{\Re(t', t'')} \bar{f}(\bar{Q}, M) G_0(I) = \Gamma_m \]

We'll now show that Feynman's limit

\[ \lim_{m \to \infty} \int_{\Re^{m-1}} g_F^{(m)}(y) dy \]

is in fact the correct method for determining the propagator. First, we need the following definition.
Definition 3.1 Let $h_m(y^{(m)})$ be defined for $y^{(m)} \in \mathbb{R}^m$ and, for every $\epsilon > 0$ let there exist positive functions $\{\delta_0^{(m)}\}$ such that $\delta_0^{(m)}(y^{(m)}) = \delta_0^{(m+1)}(y^{(m+1)})$.

Then $h_m$ is uniformly integrable if for all $\delta_0^{(m)}$-fine divisions of $\mathbb{R}^m$ we have

$$\left| \left( \mathcal{E} \delta_0^{(m)} \right) \sum h_m(y^{(m)}) |I(M)| - H_m \right| < \epsilon$$

If $g_F^{(m)}(y)$ is uniformly integrable and $\Gamma_m \to \Gamma$ as $m \to \infty$ then

$$\lim_{m \to \infty} \int_{\mathbb{R}^{m-1}} g_F^{(m)}(y)dy = \tilde{K}_0 \left( \tilde{Q}'' , \tilde{t}'' ; \tilde{Q}' , \tilde{t}' \right)$$

4 Evolutionary Semigroups

In this section we develop some evolution semigroup theory. For a more detailed coverage of this see Chicone and Latushkin [3]. For the use of evolution semigroups in control theory see Curtain and Zwart [4], Clark, et. al [18] and Bensoussan, et. al [1, 2].

4.1 Evolutionary Families

Our goal is to be able to deal with the non-autonomous abstract Cauchy problem in a Hilbert space $\mathcal{H}$

$$\dot{u}(\theta) = A(\theta)u(\theta)$$

$$u(\tau) = x_\tau$$

where $\theta \geq \tau \in \mathbb{R}_+$ and $x_\tau \in D(A(\tau))$ is the initial condition that must lie in the domain of $A$, $D(A(\cdot))$, at time $\tau$. When we think about solutions to (38) the following concept proves useful.

Definition 4.1 A strongly continuous evolutionary family on $\mathbb{R}_+$ of bounded operators $U(\theta, \tau)$ on a Hilbert space $\mathcal{H}$ has the following properties

- $U(\theta, \theta) = I$ for $\theta \in \mathbb{R}_+$.
- $U(\theta, \tau) = U(\theta, r)U(r, \tau)$ for $\theta \geq r \geq \tau \in \mathbb{R}_+$.
- The map $(\theta, \tau) \to U(\theta, \tau)$ is strongly continuous for $\theta \geq \tau \in \mathbb{R}_+$. 

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For a given $U(\theta, \tau)$, its growth bound is given by

$$\omega_0(U) = \inf \{ \omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ where } \|U(\theta, \tau)\| \leq M_\omega e^{\omega(\theta - \tau)} \ \forall \ \theta \geq \tau \}$$

If $\omega_0(U) < 0$ we call $U(\theta, \tau)$ exponentially stable and if $\omega_0(U) < \infty$ we call $U(\theta, \tau)$ exponentially bounded. It is fairly easy to see that our propagator

$$K(x'', t''; x', t')$$

gives rise to an exponentially bounded, strongly continuous evolutionary family through the operator

$$U_K^{(x'', \theta)}(\tau, x') = \int dx' K(x'', \theta; x', \tau) \psi(\tau)$$

### 4.2 Howland Semigroups on $\mathbb{R}_+$

Now, if $U(\theta, \tau)$ acts on the Hilbert space $\mathcal{H}$, we will let $\mathcal{H}_+ = L_p(\mathbb{R}_+, \mathcal{H})$, $1 \leq p < \infty$ where for $h \in \mathcal{H}_+$ we have $h : \mathbb{R}_+ \rightarrow \mathcal{H}$. If, for example, $\mathcal{H}$ is $L_p(\mathbb{R}_+)$ then $\mathcal{H}_+ = L_p(\mathbb{R}_+ \times \mathbb{R}_+) = L_p(\mathbb{R}_+ \times \mathbb{R}_+)$. For $h \in \mathcal{H}_+$, $h : \mathbb{R}_+ \rightarrow L_p(\mathbb{R}_+) = L_p(\mathbb{R}_+ \times \mathbb{R}_+)$ or $h : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow L_p(\mathbb{R}_+) = L_p(\mathbb{R}_+ \times \mathbb{R}_+)$ where $h(x, y = a) = h(x)|_{y=a}$.

**Definition 4.2** If $U(\theta, \tau)$, $\theta \geq \tau \geq 0$, is a strongly continuous exponentially bounded evolution family on Hilbert space $\mathcal{H}$, the associated evolutionary semigroup $E_t^+$, $t \geq 0$, acting on $\mathcal{H}_+ = L_p(\mathbb{R}_+, \mathcal{H})$ is given by

$$(E_t^+ h)(\theta) = \begin{cases} U(\theta, \theta - t) h(\theta - t) & \theta \geq t \geq 0 \\ 0 & 0 \leq \theta \leq t \end{cases}$$

(39)

We also call (39) a *Howland semigroup on the half line* ($\mathbb{R}_+)$.

For our propagator, the associated evolutionary semigroup is given by

$$(E_t^+ \psi)(\theta) = \begin{cases} \int K(x'', \theta; x', \theta - t) \psi(x', \theta - t) dx' & \theta \geq t \geq 0 \\ 0 & 0 \leq \theta \leq t \end{cases}$$

We have the following.

**Proposition 4.3** $E_t^+$ as defined in (39) is a strongly continuous semigroup.

**Proof.** We first show that $E_t^+$ is a semigroup. We have

$$(E_t^+ h)(\theta) = U(\theta, \theta - t) h(\theta - t) = g(\theta)$$

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for $\theta \geq t \geq 0$. Then

$$(E^s_+ g)(\theta) = U(\theta, \theta - s)g(\theta - s) = U(\theta, \theta - s)U(\theta - s, \theta - s - t)h(\theta - s - t) = U(\theta, \theta - s - t)h(\theta - s - t) = (E^{s+t}_+ h)(\theta)$$

Also,

$$(E^0_+ h)(\theta) = U(\theta, \theta)h(\theta) = Ih(\theta)$$

Then, $E^t_+ E^s_+ = E^{s+t}_+$ and $E^0_+ = I$. So $E^t_+$ is a semigroup.

Now we show strong continuity of $E^t_+$. First, the set of compactly supported, continuous functions $C_c(\mathbb{R}, \mathcal{H})$ is dense in $L^p$. We have for $h \in C_c(\mathbb{R}, \mathcal{H})$

$$\lim_{t \to 0^+} E^t_+ h = h$$

Also, we can find a $\delta > 0$ and an $N \geq 1$ such that, for all $t \in [0, \delta]$

$$\|E^t_+\| \leq Me^{\omega t} \leq N$$

This follows from the exponential boundedness of $U(\theta, \tau)$ as given in Definition (4.2). These two conditions imply strong continuity [8, page 38]. Hence, $E^t_+$ is a strongly continuous semigroup on $L^p(\mathbb{R}_+, \mathcal{H})$ where $1 \leq p < \infty$ and $t \geq 0$. ☐

Consider the right translation semigroup on $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$

$$(T_r(t)f)(\theta) = \begin{cases} f(\theta - t), & \theta - t \geq 0 \\ 0, & \theta - t \leq 0 \end{cases}$$

We have that

$$A_rf = \lim_{h \to 0^+} \frac{T_r(h)f - f}{h} = \lim_{h \to 0^+} \frac{f(\theta - h) - f(\theta)}{h} = -\frac{d}{d\theta}f(\theta)$$
So, the generator of $T_r(t)$ is $A_r = -\frac{d}{d\theta}$. We have \[ D(A_r) = \left\{ f \in L_p(\mathbb{R}_+) : f \text{ is absolutely continuous and } f' \in L_p(\mathbb{R}_+) \right\} \]

when $\mathcal{H} = L_p(\mathbb{R}_+), \; 1 \leq p < \infty$. With $(T_r(t)h)(\theta) = h(\theta - t)$ and defining the weight $w(\theta; t) = U(\theta, \theta - t)$ we have that $E^t_+ = w(\theta; t)T_r(t)$ is a weighted translation operator.

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