Proof of some supercongruences concerning truncated hypergeometric series

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Abstract

In this paper, we prove some supercongruences concerning truncated hypergeometric series. For example, we show that for any prime \( p > 3 \) and positive integer \( r \),

\[
\sum_{k=0}^{p^r-1} (3k+1) \frac{\left(\frac{1}{3}\right)_k^3 4^k}{(1)_k^3} \equiv p^r + \frac{7}{6} p^{r+3} B_{p-3} \pmod{p^{r+4}}
\]

and

\[
\sum_{k=0}^{(p^r-1)/2} (4k+1) \frac{\left(\frac{1}{4}\right)_k^4}{(1)_k^4} \equiv p^r + \frac{7}{6} p^{r+3} B_{p-3} \pmod{p^{r+4}},
\]

where \( (x)_k = x(x+1) \cdots (x+k-1) \) is the Pochhammer symbol and \( B_0, B_1, B_2, \ldots \) are Bernoulli numbers. These two congruences confirm conjectures of Sun (Sci China Math 54:2509–2535, 2011) and Guo (Adv Appl Math 120:102078, 2020), respectively.

Keywords  Truncated hypergeometric series · Binomial coefficients · Supercongruences · WZ pairs

Mathematics Subject Classification  Primary 33C20 · 11A07; Secondary 11B65 · 05A10
1 Introduction

For \( m, n \in \mathbb{N} = \{0, 1, 2\ldots\} \), the truncated hypergeometric series \( m+1 F_m \) is defined by

\[
m+1 F_m \left[ \begin{array}{c} x_0 \ x_1 \ldots \ x_m \\ y_1 \ldots y_m \end{array} \right] = \sum_{k=0}^{n} \frac{(x_0)_k(x_1)_k\ldots(x_m)_k}{(y_1)_k\ldots(y_m)_k} \frac{z^k}{k!},
\]

where \((x)_k = x(x+1)\ldots(x+k-1)\) is the Pochhammer symbol. During the past few decades, supercongruences concerning truncated hypergeometric series have been widely studied (cf.\ [2, 9, 11, 16, 17, 19, 22–26]).

In 2011, Sun [16] proposed some conjectural supercongruences which relate truncated hypergeometric series to Euler numbers and Bernoulli numbers (see [16] for the definitions of Euler numbers and Bernoulli numbers). For example, he conjectured that, for any prime \( p > 3 \),

\[
\sum_{k=0}^{\frac{(p-1)/2}{2}} (3k+1) \left(\frac{1}{3}\right)_k^3 4^k \equiv p + 2(-1)^{\frac{(p-1)/2}{2}} p^3 E_{p-3} \quad (\text{mod } p^4), \tag{1.1}
\]

and for any \( r \in \mathbb{Z}^+ \),

\[
\sum_{k=0}^{p^r-1} (3k+1) \left(\frac{1}{3}\right)_k^3 4^k \equiv p^r + \frac{7}{6} p^{r+3} B_{p-3} \quad (\text{mod } p^{r+4}), \tag{1.2}
\]

where \( E_0, E_1, E_2, \ldots \) are Euler numbers and \( B_0, B_1, B_2, \ldots \) are Bernoulli numbers. Note that \( ak+1 = (1+1/a)_k/(1/a)_k \). Thus the sums in (1.1) and (1.2) are actually the truncated hypergeometric series. In 2012, using the WZ method (cf.\ [13]), Guillera and Zudilin [2] proved that

\[
\sum_{k=0}^{p-1} (3k+1) \left(\frac{1}{3}\right)_k^3 4^k \equiv \sum_{k=0}^{\frac{(p-1)/2}{2}} (3k+1) \left(\frac{1}{3}\right)_k^3 4^k \equiv p \quad (\text{mod } p^3), \tag{1.3}
\]

which is (1.2) modulo \( p^3 \) with \( r = 1 \). In 2019, Mao and Zhang [11] confirmed (1.1) via a WZ pair found by Guillera and Zudilin [2]. Recently, Guo [3] first obtained a \( q \)-analogue of (1.3) and later Guo and Schlosser [5] obtained another \( q \)-analogue of (1.3). It is worth noting that analogously to (1.3), Guo and Schlosser [5] conjectured that for any odd prime \( p \),

\[
\sum_{k=0}^{\frac{(p+1)/2}{2}} (3k-1) \left(\frac{1}{2}\right)_k^3 \left(\frac{1}{2}\right)_k^4 4^k \equiv p \quad (\text{mod } p^3),
\]

which has been confirmed by the first author [23] by extending it to the modulus \( p^4 \) case. The reader is referred to [18] for further conjectures involving the sums in (1.1) and (1.2).

Our first theorem confirms (1.2).

**Theorem 1.1** For any prime \( p > 3 \) and integer \( r \geq 1 \), we have

\[
\sum_{k=0}^{p^r-1} (3k+1) \left(\frac{1}{3}\right)_k^3 4^k \equiv p^r + \frac{7}{6} p^{r+3} B_{p-3} \quad (\text{mod } p^{r+4}). \tag{1.4}
\]
Using the same technique as the one used in the proof of Theorem 1.1 and using (1.1), we can also prove that for any prime \( p > 3 \) and positive integer \( r \),

\[
\sum_{k=0}^{(p'-1)/2} (3k + 1) \frac{\left(\frac{1}{3}\right)^3 4^k}{(1)_k^4} \equiv p^r + 2(-1)^{(p-1)/2} p^{r+2} E_{p-3} \pmod{p^{r+3}}.
\]

In 2011, as a refinement of the (C.2) supercongruence of Van Hamme [22], Long [9] proved that

\[
\sum_{k=0}^{(p-1)/2} (4k + 1) \frac{\left(\frac{1}{2}\right)^4}{(1)_k^4} \equiv p \pmod{p^4}.
\] (1.5)

Guo and Wang [6] obtained a generalization of (1.5). For any prime \( p > 3 \) and positive integer \( r \), they proved that

\[
\sum_{k=0}^{(p'-1)/2} (4k + 1) \frac{\left(\frac{1}{2}\right)^4}{(1)_k^4} \equiv p^r \pmod{p^{r+3}}.
\] (1.6)

Our next theorem confirms a conjecture of Guo [4, Conjecture 6.2] which extends (1.6) to the modulus \( p^{r+4} \) case.

**Theorem 1.2** Let \( p > 3 \) be a prime and \( r \) a positive integer. Then

\[
\sum_{k=0}^{(p'-1)/2} (4k + 1) \frac{\left(\frac{1}{2}\right)^4}{(1)_k^4} \equiv p^r + \frac{7}{6} p^{r+3} B_{p-3} \pmod{p^{r+4}}.
\] (1.7)

Note that Guo [4] proved that for any odd prime \( p \) and positive integer \( r \),

\[
\sum_{k=0}^{(p'-1)/2} (4k + 1) \frac{\left(\frac{1}{2}\right)^4}{(1)_k^4} \equiv \sum_{k=0}^{p'-1} (4k + 1) \frac{\left(\frac{1}{2}\right)^4}{(1)_k^4} \pmod{p^{r+4}}.
\]

Clearly, the two sums in (1.4) and (1.7) are the same modulo \( p^{r+4} \). Guo [4, Conjecture 6.3] conjectured that it is also true for \( p = 3 \).

**Theorem 1.3** Let \( p \) be an odd prime and \( r \) a positive integer. Then

\[
\sum_{k=0}^{p'-1} (3k + 1) \frac{\left(\frac{1}{3}\right)^3 4^k}{(1)_k^3} \equiv \sum_{k=0}^{(p'-1)/2} (4k + 1) \frac{\left(\frac{1}{2}\right)^4}{(1)_k^4} \pmod{p^{r+4}}.
\] (1.8)

Note that Guo [4, Conjecture 6.4] also conjectured a \( q \)-analogue of (1.8).

Our main strategy to prove Theorems 1.1–1.3 is using the WZ method (the reader is referred to [2, 13, 26] for further details and some well-known WZ pairs). In fact, the case \( r = 1 \) is easy to deal with since the dominators appearing in the WZ pairs are not divisible by \( p \). However, the cases \( r \geq 2 \) are very sophisticated. In these cases, we need to reduce the sums in (1.4) and (1.7) to the case \( r = 1 \) via some complicated calculation.

The paper is organized as follows. In both Sects. 2 and 3, we shall first establish preliminary results which connect the cases \( r \geq 2 \) with the case \( r = 1 \) and play important role in the proof of Theorem 1.3. Then we will use the preliminary results to prove Theorems 1.1 and 1.2. In the end of Sect. 3, we shall give the proof of Theorem 1.3.
2 Proof of Theorem 1.1

We first establish the following result.

**Theorem 2.1** For any odd prime \( p \) and positive integer \( r \), we have

\[
\frac{1}{p^r} \sum_{k=0}^{p-1} (3k + 1) \left(\frac{1}{k} \right) 4^k \equiv \frac{1}{p} \sum_{k=0}^{p-1} (3k + 1) \left(\frac{1}{k} \right) 4^k \pmod{p^4}.
\]

Define the multiple harmonic sum (cf. [21]) as follows:

\[
H_n(s_1, s_2, \ldots, s_r) = \sum_{1 \leq k_1 < k_2 < \cdots < k_r \leq n} \frac{1}{k_1^{s_1} k_2^{s_2} \cdots k_r^{s_r}},
\]

where \( n \geq r > 0 \) and each \( s_j \) is a positive integer. Multiple harmonic sums have many congruence properties. For example, for any prime \( p > s + 2 \), Sun [15] proved that

\[
H_{p-1}(s) \equiv \begin{cases} 
\frac{s(s + 1)}{2s + 4} p^2 B_{p-s-2} \pmod{p^3} & \text{if } 2 \nmid s, \\
\frac{s}{s + 1} p B_{p-s-1} \pmod{p^2} & \text{if } 2 \mid s;
\end{cases} \tag{2.1}
\]

for any \( p > 5 \), Hessami Pilehrood and Hessami Pilehrood [7, Lemma 3] proved that

\[
H_{p-1}(1, 2) \equiv -\frac{3H_{p-1}(1)}{p^2} - \frac{5H_{p-1}(3)}{12} \pmod{p^3}. \tag{2.2}
\]

**Lemma 2.1** For any odd prime \( p \) and positive integer \( r \), we have

\[
p^{2r} \sum_{n=1}^{p-1} \frac{1}{n^2} \binom{2n}{n} \equiv p^2 \sum_{n=1}^{p-1} \frac{1}{n^2} \binom{2n}{n} \pmod{p^4},
\]

\[
p^{2r} \sum_{n=1}^{p-1} \frac{H_{n-1}(1)}{n} \binom{2n}{n} \equiv p^2 \sum_{n=1}^{p-1} \frac{H_{n-1}(1)}{n} \binom{2n}{n} \pmod{p^4},
\]

\[
p^{3r} \sum_{n=1}^{p-1} \frac{H_{n-1}(1)^2}{n} \binom{2n}{n} \equiv p^3 \sum_{n=1}^{p-1} \frac{H_{n-1}(1)^2}{n} \binom{2n}{n} \pmod{p^4},
\]

\[
p^{3r} \sum_{n=1}^{p-1} \frac{H_{n-1}(2)}{n} \binom{2n}{n} \equiv p^3 \sum_{n=1}^{p-1} \frac{H_{n-1}(2)}{n} \binom{2n}{n} \pmod{p^4},
\]

\[
p^{3r} \sum_{n=1}^{p-1} \frac{H_{n-1}(1)}{n^2} \binom{2n}{n} \equiv p^3 \sum_{n=1}^{p-1} \frac{H_{n-1}(1)}{n^2} \binom{2n}{n} \pmod{p^4},
\]

\[
p^{2r} \sum_{k=1}^{(p^r-1)/2} \frac{1}{(2k - 1)^2} \equiv p^2 \sum_{k=1}^{(p^r-1)/2} \frac{1}{(2k - 1)^2} \pmod{p^4},
\]

\[
p^{3r} \sum_{k=1}^{(p^r-1)/2} \frac{H_{2k-2}(1)}{(2k - 1)^2} \equiv p^3 \sum_{k=1}^{(p^r-1)/2} \frac{H_{2k-2}(1)}{(2k - 1)^2} \pmod{p^4},
\]

\[
p^{3r} \sum_{k=1}^{(p^r-1)/2} \frac{H_{k-1}(1)}{(2k - 1)^2} \equiv p^3 \sum_{k=1}^{(p^r-1)/2} \frac{H_{k-1}(1)}{(2k - 1)^2} \pmod{p^4}.
\]
**Proof** We only prove the second congruence, since the other ones can be showed in a similar way. We shall finish the proof by induction on $r$.

Clearly, the second congruence holds for $r = 1$. Assume that it holds for $r = k > 1$. For any $p$-adic integer $x$, let $\text{ord}_p(x)$ denote its $p$-adic order (i.e., $\text{ord}_p(x) = \max\{n \in \mathbb{Z}: x/p^n \in \mathbb{Z}_p\}$). It is easy to see that $\text{ord}_p(H_{n-1}(1)) \geq -k$ for $1 \leq n \leq p^{k+1} - 1$. Therefore

$$p^{2k+2} \sum_{n=1}^{p^{k+1}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2n}{n} \right) = p^{2k+2} \sum_{n=1}^{p^{k+1}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2n}{n} \right) + p^{2k+2} \sum_{n=1}^{p^{k+1}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2n}{n} \right)$$

$$\equiv p^{2k+2} \sum_{n=1}^{p^{k+1}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2n}{n} \right)$$

$$\equiv p^{2k+1} \sum_{n=1}^{p^{k}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2pn}{pn} \right) \pmod{p^4}.$$ 

Note that

$$\text{ord}_p(p^{2k+1}/n) \geq k + 2 \geq 4 \text{ for } 1 \leq n \leq p^k - 1.$$ 

Hence, for $1 \leq n \leq p^k - 1$, we have

$$\frac{p^{2k+1}H_{pn-1}(1)}{n} \equiv \frac{p^{2k+1}}{n} \sum_{j=1 \atop p|j}^{p^n-1} 1 \equiv \frac{p^{2k}H_{n-1}(1)}{n} \pmod{p^4}$$

and

$$\frac{p^{2k}H_{n-1}(1)}{n} \equiv 0 \pmod{p^2}.$$ 

By the well-known Kazandzidis’ congruence (cf. [14, p. 380]), for any odd prime $p$, we have

$$\left( \frac{2pn}{pn} \right) \equiv \left( \frac{2n}{n} \right) \pmod{p^2}.$$ 

Combining the above and by the induction hypothesis, we arrive at

$$p^{2k+2} \sum_{n=1}^{p^{k+1}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2n}{n} \right) \equiv p^{2k} \sum_{n=1}^{p^{k}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2n}{n} \right)$$

$$\equiv p^{2} \sum_{n=1}^{p^{k}-1} \frac{H_{n-1}(1)}{n} \left( \frac{2n}{n} \right) \pmod{p^4}.$$ 

We are done. $\square$

**Proof of Theorem 2.1** As in [2], we shall use the following WZ pair

$$F(n, k) = (3n + 2k + 1) \frac{\left( \frac{1}{2} \right)_n \left( \frac{1}{2} + k \right)_n^2 4^n}{(1)_n^3}$$
and
\[ G(n, k) = -\frac{(\frac{1}{2})n(\frac{1}{2} + k)^2}{(1)^2_{n-1}} 4^n. \]

Then we have
\[ F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k) \]
and
\[ \sum_{k=0}^{p'-1} (3k + 1) \frac{(\frac{1}{2})^3 k^4}{(1)^3_k} = \sum_{n=0}^{p'-1} F(n, 0). \]

Clearly,
\[ \sum_{n=0}^{p'-1} F(n, 0) = \sum_{n=0}^{p'-1} (p'-1)/2 \left( F(n, k - 1) - F(n, k) \right) + \sum_{n=0}^{p'-1} F \left( n, \frac{p'-1}{2} \right) \]
\[ = \sum_{k=1}^{p'-1/2} \sum_{n=0}^{p'-1} \left( G(n + 1, k) - G(n, k) \right) + \sum_{n=0}^{p'-1} F \left( n, \frac{p'-1}{2} \right) \]
\[ = \sum_{k=1}^{p'-1/2} G(p'^r, k) + \sum_{n=0}^{p'-1} F \left( n, \frac{p'-1}{2} \right), \]
where the last step follows from the fact \( G(0, k) = 0 \). It suffices to show
\[ \frac{1}{p^r} \sum_{n=0}^{p'-1} F \left( n, \frac{p'^r - 1}{2} \right) \equiv \frac{1}{p} \sum_{n=0}^{p-1} F \left( n, \frac{p-1}{2} \right) \pmod{p^4} \] (2.3)
and
\[ \frac{1}{p^r} \sum_{k=1}^{(p'-1)/2} G(p'^r, k) \equiv \frac{1}{p} \sum_{k=1}^{(p-1)/2} G(p, k) \pmod{p^4}. \] (2.4)

We first consider (2.3).

\[ F \left( n, \frac{p'^r - 1}{2} \right) = \begin{cases} 
\frac{p^{2r}(3n + p'^r)}{4n^2} \cdot \left( \frac{2n}{n} \right) \left( \frac{1 + p'^r}{2} \right)_{n-1}^2, & \text{if } n \geq 1, \\
p^r, & \text{if } n = 0.
\end{cases} \]

Therefore,
\[ \frac{1}{p^r} \sum_{n=1}^{p'-1} F \left( n, \frac{p'^r - 1}{2} \right) = \frac{3p'^r}{4} \sum_{n=1}^{p'-1} \left( \frac{1 + p'^r}{2} \right)_{n-1}, \left( \frac{2n}{n} \right) \frac{(2n)}{n} \\
+ \frac{p^{2r}}{4} \sum_{n=1}^{p'-1} \left( \frac{1 + p'^r}{2} \right)_{n-1} \cdot \left( \frac{2n}{n} \right). \]

For \( 1 \leq n \leq p'^r - 1 \), it is clear that \( \text{ord}_p(n) \leq r - 1 \). Note that
\[ \frac{(1 + p'^r)}{(1)_{n-1}} = 1 + \frac{p^r}{2} H_{n-1}(1) + \frac{p^{2r}}{4} H_{n-1}(1, 1) + \cdots \]
Note that in (2.5) we actually obtain a result which is independent of $r$. Thus we have proved (2.3).

Now we consider (2.4). Clearly,

$$
\frac{1}{p^r} \sum_{k=1}^{\left(\frac{p^r-1}{2}\right)} G(p^r, k) = -\frac{4p^{2r}}{16p^r} \left(\frac{2p^r}{p^r}\right)^3 \sum_{k=1}^{\left(\frac{p^r-1}{2}\right)} \frac{\left(\frac{1}{2} + p^r\right)^2 k^{-1}}{(2k-1)^2 \left(\frac{1}{2}\right)^2 k^{-1}}.
$$

By a similar argument as above, $\text{ord}_p(2k - 1) \leq r - 1$ and

$$
\frac{\left(\frac{1}{2} + p^r\right)_{k-1}}{(\frac{1}{2})_{k-1}} \equiv 1 + 2p^r H_{2k-2}(1) - p^r H_{k-1}(1) \pmod{p^2}.
$$
Therefore, by Lemma 2.1 we have
\[
\frac{1}{pr} \sum_{k=1}^{(p-1)/2} G(p^r, k) \equiv -4p^2r \left( \frac{2p^r}{p^r} \right)^3 \sum_{k=1}^{(p-1)/2} \frac{1 + 4p^r H_{2k-2}(1) - 2p^r H_{k-1}(1)}{(2k-1)^2} \quad \text{(mod } p^4) \]

Note that
\[
\sum_{k=1}^{(p-1)/2} \frac{1}{(2k-1)^2} = H_{p-1}(2) - \frac{1}{4} H_{(p-1)/2}(2) \equiv 0 \quad \text{(mod } p) \]

Hence we have
\[
\sum_{k=1}^{(p-1)/2} \frac{1 + 4p^r H_{2k-2}(1) - 2p H_{k-1}(1)}{(2k-1)^2} \equiv 0 \quad \text{(mod } p) \]

Then, from Kazandzidis’ congruence and Fermat’s little theorem, we immediately obtain
\[
\frac{1}{pr} \sum_{k=1}^{(p-1)/2} G(p^r, k) \equiv -2p^2 \left( \frac{2p^r}{p^r} \right)^3 \sum_{k=1}^{(p-1)/2} \frac{1 + 4p H_{2k-2}(1) - 2p H_{k-1}(1)}{(2k-1)^2} \quad \text{(mod } p^4) \]

This proves (2.4) since the right-hand side of the above congruence is independent of \( r \).

The proof of Theorem 2.1 is now complete. \( \square \)

We are now in a position to prove Theorem 1.1. We need the following lemmas.

**Lemma 2.2** For any prime \( p > 3 \), we have
\[
\sum_{k=1}^{p-1} \frac{1}{k^3} \binom{2k}{k} \equiv -\frac{2H_{p-1}(1)}{p^2} \quad \text{(mod } p), \quad (2.8)
\]
\[
\sum_{k=1}^{p-1} \frac{H_k(2)}{k} \binom{2k}{k} \equiv \frac{2H_{p-1}(1)}{3p^2} \quad \text{(mod } p), \quad (2.9)
\]
\[
\sum_{k=1}^{p-1} \left( \frac{2}{k^2} - \frac{3H_k(1)}{k} \right) \binom{2k}{k} \equiv \frac{2H_{p-1}(1)}{p} \quad \text{(mod } p^2). \quad (2.10)
\]

**Proof** Note that the modulus \( p^4 \) case of (2.8) was originally conjectured by Sun [16, Conjecture 1.1]. Hessami Pilehrood and Hessami Pilehrood [7] first proved (2.8), and Tauraso [21] proved the modulus \( p^2 \) case. We can directly verify (2.9) and (2.10) for \( p = 5 \). By [21, Theorem 3] we know these two congruences hold for \( p > 5 \). \( \square \)

Sun [18, Conjecture 52] conjectured that for any prime \( p > 3 \),
\[
\sum_{k=1}^{p-1} \left( \frac{3H_k(1)^2}{k} - \frac{4H_k(1)}{k^2} \right) \binom{2k}{k} \equiv \frac{6H_{p-1}(1)}{p^2} + \frac{8}{5} p^2 B_{p-5} \quad \text{(mod } p^3). \]

The next lemma confirms the modulus \( p \) case of the above conjecture.
Lemma 2.3 For any prime \( p > 3 \) we have
\[
\sum_{k=1}^{p-1} \left( \frac{3H_k(1)^2}{k} - \frac{4H_k(1)}{k^2} \right) \left( \frac{2k}{k} \right) \equiv \frac{6H_{p-1}(1)}{p^2} \pmod{p}.
\]

Proof As in [21], consider the WZ pair
\[
F(n, k) = \frac{1}{k} \binom{n+k}{k} \quad \text{and} \quad G(n, k) = \frac{k}{(n+1)^2} \binom{n+k}{k}.
\]
Then, for any \( n, k \in \mathbb{N} \) we have
\[
F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k).
\]
Let \( S_n = \sum_{k=1}^{n} F(n, k)H_k(1)^2 \). Then
\[
S_{n+1} - S_n = \sum_{k=1}^{n+1} F(n+1, k)H_k(1)^2 - \sum_{k=1}^{n} F(n, k)H_k(1)^2
\]
\[
= F(n+1, n+1)H_{n+1}(1)^2 + \sum_{k=1}^{n} (F(n+1, k) - F(n, k))H_k(1)^2
\]
\[
= F(n+1, n+1)H_{n+1}(1)^2 + \sum_{k=1}^{n} (G(n, k+1) - G(n, k))H_k(1)^2
\]
\[
= F(n+1, n+1)H_{n+1}(1)^2 + \sum_{k=1}^{n} \left( G(n, k+1)H_k(1)^2 - G(n, k)H_{k-1}(1)^2 \right)
\]
\[
- \frac{2G(n, k)H_k(1)}{k} + \frac{G(n, k)}{k^2}
\]
\[
= \frac{(2n+2)H_{n+1}(1)^2}{n+1} + \frac{(2n+2)H_n(1)^2}{2n+2} - 2\sum_{k=1}^{n} \frac{G(n, k)H_k(1)}{k} + \sum_{k=1}^{n} \frac{G(n, k)}{k^2}.
\]
(2.11)

By [21, (16)] we have
\[
\sum_{k=1}^{n} \frac{G(n, k)}{k^2} = \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{k} \binom{n+k}{k} = \frac{1}{(n+1)^2} \left( \frac{3}{2} \sum_{k=1}^{n} \frac{(2k)}{k} - H_n(1) \right)
\]
(2.12)

From [1, (1.49)] we know that
\[
\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n}.
\]

Therefore,
\[
\sum_{k=1}^{n} \frac{G(n, k)H_k(1)}{k} = \frac{1}{(n+1)^2} \sum_{k=1}^{n} \binom{n+k}{k} \sum_{j=1}^{k} \frac{1}{j} = \frac{1}{(n+1)^2} \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} \binom{n+k}{k}
\]
\[
= \frac{1}{(n+1)^2} \sum_{j=1}^{n} \frac{1}{j} \left( \binom{2n+1}{n} - \binom{n+j}{j-1} \right)
\]
\[ \frac{1}{(n + 1)^2} \left( \frac{1}{2} \binom{2n + 2}{n + 1} H_n(1) - \frac{1}{n + 1} \sum_{j=1}^{n} \binom{n + j}{j} \right) \]

Substituting (2.12) and (2.13) into (2.11), we have

\[
S_{n+1} - S_n = \frac{3\binom{2n+2}{n+1} H_{p+1}(1)^2}{2n+2} - \frac{2\binom{2n+2}{n+1} H_n(1)}{(n+1)^2} + \frac{5\binom{2n+2}{n+1}}{2(n+1)^3} - \frac{H_{p+1}(1)}{(n+1)^2}
\]

Now summing both sides over \( n \) from 0 to \( p - 2 \) and noting that \( S_0 = 0 \), we have

\[
S_{p-1} = \frac{3}{2} \sum_{n=1}^{p-1} \frac{\binom{2n}{n} H_n(1)^2}{n} - \frac{2}{n+1} n^2 \sum_{n=1}^{p-1} \frac{\binom{2n}{n} H_n(1)}{n} + \frac{5}{2} \sum_{n=1}^{p-1} \frac{\binom{2n}{n}}{n^3} \]

\[
- \sum_{n=1}^{p-1} \frac{H_n(1)}{n^2} - H_{p-1}(3) + \frac{3}{2} \sum_{n=1}^{p-1} \frac{1}{n^2} \sum_{k=1}^{n-1} \binom{2k}{k} \]

Clearly, for \( 1 \leq k \leq p - 1 \),

\[
F(p - 1, k) = \frac{1}{k} \left( \frac{p - 1 + k}{k} \right) \equiv \frac{1}{k} \left( \frac{k - 1}{k} \right) \equiv 0 \pmod{p}.
\]

Thus we have

\[
S_{p-1} = \sum_{k=1}^{p-1} F(p - 1, k) H_k(1)^2 \equiv 0 \pmod{p}. \tag{2.15}
\]

In view of (2.1) and (2.2), we have

\[
\sum_{n=1}^{p-1} H_n(1) \equiv H_{p-1}(1, 2) \equiv - \frac{3 H_{p-1}(1)}{p^2} \pmod{p}. \tag{2.16}
\]

Note that

\[
\sum_{n=1}^{p-1} \frac{1}{n^2} \sum_{k=1}^{n-1} \binom{2k}{k} \equiv \sum_{k=1}^{p-1} \frac{1}{k} \sum_{n=k}^{p-1} \frac{\binom{2n}{n}}{n^2} - \sum_{n=1}^{p-1} \frac{\binom{2n}{n}}{n^3} \equiv \sum_{k=1}^{p-1} \frac{\binom{2k}{k} H_k(2)}{k} \pmod{p}. \tag{2.17}
\]

Substituting (2.15)–(2.17) into (2.14) and using (2.8) and (2.9), we immediately obtain the desired result.

\[
\text{Lemma 2.4} \quad \text{Let} \ p > 3 \ \text{be a prime. Then}
\]

\[
\sum_{k=1}^{(p-1)/2} \frac{1}{(2k - 1)^2} \equiv \frac{1}{12} p B_{p-3} \pmod{p^2}, \tag{2.18}
\]

\[
\sum_{k=1}^{(p-1)/2} \frac{H_{2k-2}(1)}{(2k - 1)^2} \equiv \frac{3}{8} B_{p-3} \pmod{p}. \tag{2.19}
\]
Proof. By [15, Corollaries 5.1 and 5.2] we have

\[ H_{p-1}(2) \equiv \frac{2}{3} p B_{p-3} \pmod{p^2} \quad \text{and} \quad H_{(p-1)/2}(2) \equiv \frac{7}{3} p B_{p-3} \pmod{p^2}. \]

It follows that

\[ \sum_{k=1}^{(p-1)/2} \frac{1}{(2k-1)^2} = H_{p-1}(2) - \frac{1}{4} H_{(p-1)/2}(2) \equiv \frac{1}{12} p B_{p-3} \pmod{p^2}. \]

This proves (2.18).

Clearly,

\[ \sum_{k=1}^{(p-1)/2} \frac{H_{2k-2}(1)}{(2k-1)^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{H_{2((p+1)/2-k)-2}(1)}{(2((p+1)/2-k)-1)^2} \equiv \frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{H_{2k}(1)}{k^2} \pmod{p}, \]

where we used the fact \( H_{p-1-k}(1) \equiv H_k(1) \pmod{p} \) for any \( 0 \leq k \leq p-1 \). In view of [10, Lemma 2.4], we have

\[ \sum_{k=1}^{(p-1)/2} \frac{H_{2k}(1)}{k^2} \equiv \frac{3}{2} B_{p-3} \pmod{p}. \]  

This proves (2.19).

Note that

\[ H_{(p-1)/2-k}(1) = H_{(p-1)/2}(1) - \sum_{j=1}^{k} \frac{1}{(p+1)/2 - j} \]

\[ \equiv H_{(p-1)/2}(1) + 2 H_{2k}(1) - H_{k}(1) \pmod{p}. \]

Therefore,

\[ \sum_{k=1}^{(p-1)/2} \frac{H_{k-1}(1)}{(2k-1)^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2-k}(1)}{(p-2k)^2} \]

\[ \equiv \frac{1}{4} \sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2}(1) + 2 H_{2k}(1) - H_{k}(1)}{k^2} \]

\[ \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{H_{2k}(1)}{k^2} - \frac{1}{4} H_{(p-1)/2}(3) - \frac{1}{4} H_{(p-1)/2}(1, 2) \pmod{p}. \]

It is known (cf. [8]) that

\[ H_{(p-1)/2}(3) \equiv -2 B_{p-3} \pmod{p} \quad \text{and} \quad H_{(p-1)/2}(1, 2) \equiv \frac{3}{2} B_{p-3} \pmod{p}. \]

Then (2.20) follows from the above two congruences and (2.21) immediately. \( \Box \)
Proof of Theorem 1.1. By (2.5) and (2.7) we have

\[
\frac{1}{p^r} \sum_{k=0}^{p^r-1} (3k + 1)(\frac{1}{3})_k 4^k \equiv 1 + \frac{1}{4} p^2 \left( \sum_{n=1}^{p-1} \frac{(2n)}{n} H_n(1) - 2 \sum_{n=1}^{p-1} \frac{(2n)}{n^2} \right) \\
+ \frac{1}{8} p^3 \left( 3 \sum_{n=1}^{p-1} \frac{(2n)}{n} H_n(1)^2 - 4 \sum_{n=1}^{p-1} \frac{(2n)}{n^2} H_n(1) \right) \\
+ \frac{3}{4} p \sum_{n=1}^{p-1} \frac{(2n)}{n} + \frac{5}{16} p^3 \sum_{n=1}^{p-1} \frac{(2n)}{n^3} - \frac{3}{16} p^3 \sum_{n=1}^{p-1} \frac{(2n)}{n} H_n(2) \sum_{n=1}^{p-1} \frac{(2n)}{n^2} H_n(1) - 2 p^2 \sum_{k=1}^{(p-1)/2} \frac{1}{(2k-1)^2} \\
- 8 p^3 \sum_{k=1}^{(p-1)/2} \frac{H_{2k-2}(1)}{(2k-1)^2} + 4 p^3 \sum_{k=1}^{(p-1)/2} \frac{H_{2k-1}(1)}{(2k-1)^2} \pmod{p^4}.
\]

Then, by (2.1), (2.6) and Lemmas 2.2–2.4, we have

\[
\frac{1}{p^r} \sum_{n=0}^{p^r-1} (3n + 1)(\frac{1}{3})_n 4^n \equiv 1 + \frac{7}{6} p^3 B_{p-3} \pmod{p^4}.
\]

The proof of Theorem 1.1 is now complete. \(\square\)

3 Proof of Theorem 1.2

Similarly as in Sect. 2, we first establish the following result.

Theorem 3.1 For any odd prime \(p\) and positive integer \(r\), we have

\[
\frac{1}{p^r} \sum_{k=0}^{(p^r-1)/2} (4k + 1)(\frac{1}{2})_k 4^k \equiv \frac{1}{p} \sum_{k=0}^{(p-1)/2} (4k + 1)(\frac{1}{2})_k 4^k \pmod{p^4}.
\]

Lemma 3.1 For any odd prime \(p\) and positive integer \(r\), we have

\[
p^r H_{p^r-1}(1) \equiv p H_{p-1}(1) \pmod{p^4}, \quad (3.1)
\]

\[
p^{2r} H_{p^r-1}(1, 1) \equiv p^2 H_{p-1}(1, 1) \pmod{p^4}, \quad (3.2)
\]

\[
p^{3r} H_{p^r-1}(1, 1, 1) \equiv p^3 H_{p-1}(1, 1, 1) \pmod{p^4} \quad (3.3)
\]

Proof We first prove (3.1). Clearly, it holds for \(r = 1\). Assume that it holds for \(r = k > 1\). Then, for \(r = k + 1\), we have

\[
p^{k+1} H_{p^{k+1}-1}(1) = p^{k+1} \sum_{j=1}^{p^{k+1}-1} \frac{1}{j} + p^{k+1} \sum_{j=1}^{p^{k+1}-1} \frac{1}{j} = p^k \sum_{j=1}^{p^{k-1}-1} \frac{1}{j} + p^{k+1} \sum_{l=0}^{p-1} \frac{1}{l p + j}
\]

\[
\equiv p^k \sum_{j=1}^{p^{k-1}-1} \frac{1}{j} + p^{k+1} \sum_{l=0}^{p^{k-1}-1} \left( \frac{1}{j} - \frac{1}{j^2} \right)
\]
we have

\[ p^k H_{p^k-1}(1) + \frac{p^{2k+2}(p^k - 1)}{2} H_{p-1}(2) - p^{2k+1} H_{p-1}(1) \]

\[ \equiv p^k H_{p^k-1}(1) \pmod{p^4}. \]

By the induction hypothesis we obtain (3.1).

(3.2) and (3.3) can be proved similarly by noting that

\[ H_{p^r-1}(1, 1) = \frac{H_{p^r-1}(1)^2 - H_{p^r-1}(2)}{2} \]

and

\[ H_{p^r-1}(1, 1, 1) = \frac{H_{p^r-1}(1)^3 - 3H_{p^r-1}(1)H_{p^r-1}(2) + 2H_{p^r-1}(3)}{6}. \]

\[ \square \]

**Lemma 3.2** For any odd prime \( p \) and positive integer \( r \), we have

\[ p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k + 1)^3(k^2)} \equiv p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k + 1)^3(k^2)} \pmod{p^4}. \]

**Proof** Clearly, the congruence holds for \( r = 1 \).

Now suppose that \( r \geq 2 \). In view of the proof of [12, Theorem 1.1], for \( 0 < k < p^r/2 \), we have

\[ \frac{p^r}{k(k)} \equiv 0 \pmod{p}. \] (3.4)

Thus we obtain

\[ p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k + 1)^3(k^2)} = p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k + 1)^3(k^2)} + p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k + 1)^3(k^2)} \]

\[ \equiv p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k + 1)^3(k^2)} \pmod{p^4}, \]

where we have used \( \text{ord}_p(p^{3r}/(2k)^2) \geq r+2 \geq 4 \) by (3.4). It is routine to check that

\[ p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k + 1)^3(k^2)} = p^{3r} \sum_{k=0}^{(p^r-1-3)/2} \frac{4(2k+1)p-1}{(2k+1)^3(2k+1)^2(2k+1)^2}. \]

Now

\[ \frac{(2k + 1)p - 1}{((2k + 1)p - 1)/2} = \frac{\Gamma((2k + 1)p)}{\Gamma((2k+1)p+1)/2} = -\frac{\Gamma_p((2k + 1)p)}{\Gamma_p((2k+1)p+1)}(2k), \]
where \( \Gamma_p(x) \) is the \( p \)-adic Gamma function (see [14, Chapter 7] for properties of this function). By (3.4), for \( 0 \leq k \leq (p^r - 1) / 2 \) we have
\[
\frac{p^{3r-3}}{(2k+1)^3(\frac{2k}{k})^2} = \frac{4p^{3r-3}}{(2k+1)(k+1)^2(\frac{2k}{k+1})^2} \equiv 0 \pmod{p^3}.
\]

By Fermat’s little theorem we have
\[
4^{(2k+1)p-1} = 16^k \cdot 4^{p-1} \equiv 16^k \pmod{p}.
\]

Furthermore,
\[
\frac{\Gamma_p((2k+1)p)^2}{\Gamma_p((2k+1)p+1)^4} = \frac{\Gamma_p(0)^2}{\Gamma_p(\frac{1}{2})^4} = 1 \pmod{p}.
\]

Combining the above we arrive at
\[
p^{3r} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k+1)^3(\frac{2k}{k})^2} = p^{3r-3} \sum_{k=0}^{(p^r-3)/2} \frac{16^k}{(2k+1)^3(\frac{2k}{k})^2} \pmod{p^4}.
\]

Then the desired result follows from induction on \( r \). \( \square \)

**Proof of Theorem 3.1.** We need the following pair which appeared in [25]
\[
F(n, k) = (-1)^k (4n + 1) \frac{(\frac{1}{2})_n^3(\frac{1}{2})_{n+k}}{(1)_n^3(1)_{n-k}(\frac{1}{2})_k^2},
\]
\[
G(n, k) = (-1)^{k-1} \frac{4(\frac{1}{2})_n^3(\frac{1}{2})_{n+k-2}}{(1)_n^3(1)_{n-k}(\frac{1}{2})_k^2}.
\]

One may easily check that for any \( n \in \mathbb{N} \) and \( k \in \mathbb{Z}^+ \),
\[
(2k - 1)F(n, k - 1) - 2kF(n, k) = G(n + 1, k) - G(n, k).
\]

Note that such pair is not a WZ pair. Nevertheless, it is also very useful as classical WZ pairs. For \( m \in \mathbb{N} \), by induction on \( m \) and noting that
\[
F(n, k) = 0 \quad \text{for } k > n
\]

and
\[
G(0, k) = 0 \quad \text{for } k > 0,
\]

we have
\[
\sum_{n=0}^{m} F(n, 0) = \frac{4^m}{(\frac{1}{2})^m} F(m, m) + \sum_{k=1}^{m} \frac{4^{k-1}G(m + 1, k)}{(2k - 1)(\frac{2k-2}{k-1})}.
\]

Taking \( m = (p^r - 1)/2 \) in (3.5), we have
\[
\sum_{k=0}^{(p^r - 1)/2} \frac{(4k + 1)(\frac{1}{2})_k^4}{(1)_k^2} = \sum_{k=0}^{(p^r - 1)/2} F(n, 0) = \frac{2^{p^r-1}}{(p^r - 1)/2} F \left( \frac{p^r - 1}{2}, \frac{p^r - 1}{2} \right)
\]
\[
+ \sum_{k=1}^{(p^r - 1)/2} \frac{4^{k-1}}{(2k - 1)(\frac{2k-2}{k-1})} G \left( \frac{p^r + 1}{2}, k \right).
\]
It suffices to show
\[ \frac{1}{p^r} \cdot \frac{2^{p^r-1}}{(p^r-1)/2} F \left( \frac{p^r - 1}{2}, \frac{p^r - 1}{2} \right) \equiv \frac{1}{p^r} \cdot \frac{2^{p-1}}{(p-1)/2} F \left( \frac{p - 1}{2}, \frac{p - 1}{2} \right) \pmod{p^4} \] (mod $p^4$) (3.6)

and
\[ \frac{1}{p^r} \cdot \sum_{k=1}^{(p^r-1)/2} \frac{4^{k-1}}{(2k-1)(2k-2)} G \left( \frac{p^r + 1}{2}, k \right) \equiv \frac{1}{p^r} \cdot \sum_{k=1}^{(p-1)/2} \frac{4^{k-1}}{(2k-1)(2k-2)} G \left( \frac{p + 1}{2}, k \right) \pmod{p^4}. \] (mod $p^4$) (3.7)

We first consider (3.6). Note that
\[ \frac{(1/2)_k}{(1)_k} = \frac{(2k)_{1/2}}{4k}. \]

Thus we have
\[ \frac{1}{p^r} \cdot \frac{2^{p^r-1}}{(p^r-1)/2} F \left( \frac{p^r - 1}{2}, \frac{p^r - 1}{2} \right) \equiv \frac{1}{p^r} \cdot \frac{(-1)^{(p^r-1)/2}2^{p^r-1}(2p^r - 1)}{(p^r-1)/2} \cdot \frac{\frac{1}{2} \left( p - 1 \right)^{(p^r-1)/2} \left( 2 \right)^{(p^r-1)/2}}{(1)^{(p^r-1)/2} \left( 2 \right)^{(p^r-1)/2}} \]
\[ = \left( \frac{1}{2} p^r - 1 \right) \left( 2p^r - 1 \right) \pmod{p^4}. \] (3.8)

For any $p$-adic integer $a$, it is easy to see that
\[ \binom{ap^r - 1}{p^r - 1} = \frac{(1 + (a - 1)p^r)_{p^r-1}}{(1)_{p^r-1}} \equiv 1 + (a - 1)p^r H_{p^r-1}(1) + (a - 1)^2 p^{2r} H_{p^r-1}(1, 1) + (a - 1)^3 p^{3r} H_{p^r-1}(1, 1, 1) \]
\[ \equiv 1 + (a - 1)p H_{p-1}(1) + (a - 1)^2 p^2 H_{p-1}(1, 1) + (a - 1)^3 p^3 H_{p-1}(1, 1, 1) \]
\[ \equiv \binom{ap - 1}{p - 1} \pmod{p^4}, \] (3.9)

where we have used Lemma 3.1. Therefore,
\[ \frac{1}{p^r} \cdot \frac{2^{p^r-1}}{(p^r-1)/2} F \left( \frac{p^r - 1}{2}, \frac{p^r - 1}{2} \right) \equiv \left( \frac{1}{2} p - 1 \right) \left( 2p - 1 \right) \pmod{p^4}. \] (3.9)

Then (3.6) follows by noting that the right-hand side of the above congruence is independent of $r$.

Below we consider (3.7). It is easy to see that
\[ \frac{1}{p^r} \cdot \sum_{k=1}^{(p^r-1)/2} \frac{4^{k-1}}{(2k-1)(2k-2)} G \left( \frac{p^r + 1}{2}, k \right) \]
\[ = \frac{1}{p^r} \cdot \sum_{k=1}^{(p-1)/2} \frac{4^{k-1}}{(2k-1)(2k-2)} \cdot \frac{4(-1)^{k-1} \left( 2 \right)^{(p^r+1)/2} \left( 2 \right)^{(p^r-1)/2+k}}{(1)^{(p^r-1)/2} \left( 2 \right)^{(p^r+1)/2+k} \left( 2k \right)^{(p^r-1)/2+k}}. \]

Note that
\[ \left( \frac{1}{2} \right)_{(p^r-1)/2+k} = \left( \frac{1}{2} \right)_{(p^r-1)/2} \left( \frac{p^r}{2} \right) = \frac{p^r}{2} \left( \frac{1}{2} \right)_{(p^r-1)/2} \left( 1 + \frac{p^r}{2} \right)_{k-1} \]

\[ \frac{1}{p^r} \cdot \sum_{k=1}^{(p^r-1)/2} \frac{4^{k-1}}{(2k-1)(2k-2)} G \left( \frac{p^r + 1}{2}, k \right) \equiv \frac{1}{p^r} \cdot \sum_{k=1}^{(p-1)/2} \frac{4^{k-1}}{(2k-1)(2k-2)} G \left( \frac{p + 1}{2}, k \right) \pmod{p^4}. \] (mod $p^4$)
and
\[(1)_{(p'+1)/2-k} = (-1)^{k-1} \frac{(1)_{(p'-1)/2}}{(\frac{1}{2} - \frac{p'}{2})_{k-1}}.\]
Therefore,
\[1 \cdot \frac{(p'-1)/2}{p'} \sum_{k=1}^{4k-1} G \left( \frac{p' + 1}{2}, k \right) \]
\[= \frac{p'^3}{16^{p-1}} \left( \frac{p' - 1}{p' - 1/2} \right)^4 \sum_{k=0}^{(p'-3)/2} \frac{164^k (1 + \frac{p'}{2}) (\frac{1}{2} - \frac{p'}{2})_k}{(2k + 1)^3 \left( \frac{2k}{k} \right)^3 (1)^2} \cdot\]
By (3.4) we have
\[\frac{p'^3}{(2k + 1)^3 \left( \frac{2k}{k} \right)^3} \equiv 0 \pmod{p^3}.\]
Thus, by Fermat’s little theorem, (3.8) and Lemma 3.2, we further obtain
\[1 \cdot \frac{(p'-1)/2}{p'} \sum_{k=1}^{4k-1} G \left( \frac{p' + 1}{2}, k \right) \equiv \frac{16^k}{(2k + 1)^3 \left( \frac{2k}{k} \right)^2} \pmod{p^4}.\]
(3.10)
This proves (3.7).
The proof of Theorem 3.1 is now complete.

**Proof of Theorem 1.2.** By (3.9) and (3.10) we arrive at
\[\frac{1}{p'} \sum_{k=0}^{(p'-1)/2} \left( \frac{4k + 1}{1} \right)_{1/4} \equiv \left( \frac{2p - 1}{p - 1} \right) \pmod{p^4}.\]
(3.11)
By (3.8),
\[\left( \frac{2p - 1}{p - 1} \right) \equiv 1 + \frac{1}{2} p H_{p-1}(1) + \frac{1}{4} p^2 H_{p-1}(1, 1) - \frac{1}{8} p^3 H_{p-1}(1, 1, 1) \pmod{p^4}\]
and
\[\left( \frac{2p - 1}{p - 1} \right) \equiv 1 + p H_{p-1}(1) + p^2 H_{p-1}(1, 1) + p^3 H_{p-1}(1, 1, 1) \pmod{p^4}.\]
It is known (cf. [8]) that
\[H_{p-1}(1, 1) \equiv -\frac{1}{3} p B_{p-3} \pmod{p^2}\]
and
\[H_{p-1}(1, 1, 1) \equiv 0 \pmod{p}.\]
Then, in view of (2.1), we get that
\[
\left(\frac{1}{2} p - 1\right)\left(\frac{2p - 1}{p - 1}\right) \equiv 1 - \frac{7}{12} p^3 B_{p-3} \pmod{p^4}.
\] (3.12)

Sun [17, Theorem 1.2] proved that
\[
\sum_{k=0}^{(p-3)/2} \frac{16^k}{(2k + 1)^3 (2k^2)} \equiv \frac{7}{4} B_{p-3} \pmod{p}.
\] (3.13)

Substituting (3.12) and (3.13) into (3.11), we immediately obtain Theorem 1.2. □

Now we can easily obtain Theorem 1.3.

**Proof of Theorem 1.3.** The case \( p > 3 \) is the immediate corollary of Theorems 1.1 and 1.2.

Now we consider the case \( p = 3 \). In view of Theorems 2.1 and 3.1, we only need to prove Theorem 1.3 for \( r = 1 \). In fact, if \( p = 3 \) and \( r = 1 \), one may check Theorem 1.3 directly. □

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