SHARP PHASE TRANSITIONS FOR THE ALMOST MATHIEU OPERATOR

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Abstract. It is known that the spectral type of the almost Mathieu operator depends in a fundamental way on both the strength of the coupling constant and the arithmetic properties of the frequency. We study the competition between these factors and locate the point where the phase transition from singular continuous spectrum to pure point spectrum takes place, which solves Jitomirskaya’s conjecture in [28, 30]. Together with [3], we give the sharp description of phase transitions for the almost Mathieu operator.

1. Main results

This paper concerns the spectral measure of the Almost Mathieu operator:

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(n\alpha + \theta)u_n,$$

where $\theta \in \mathbb{R}$ is the phase, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is the frequency and $\lambda \in \mathbb{R}$ is the coupling constant, which has been extensively studied because of its strong backgrounds in physics and also because it provides interesting examples in spectral theory [35]. We will find the exact transition point from singular continuous spectrum to purely point spectrum of Almost Mathieu operator, thus solve Jitomirskaya’s conjecture in 1995 [28] (see also Problem 8 in [30]). More precisely, let $\frac{p_n}{q_n}$ be the $n$–th convergent of $\alpha$, and define

$$(1.1) \quad \beta(\alpha) := \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n},$$

our main results are the following:

Theorem 1.1. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ with $0 < \beta(\alpha) < \infty$, then we have the following:

1. If $|\lambda| < 1$, then $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for all $\theta$.
2. If $1 \leq |\lambda| < e^\beta$, then $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for all $\theta$.
3. If $|\lambda| > e^\beta$, then $H_{\lambda,\alpha,\theta}$ has purely point spectrum with exponentially decaying eigenfunctions for a.e. $\theta$.

Remark 1.1. Part (1) is proved by Avila [3], we state here just for completeness.

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Remark 1.2. The cases $\beta = 0, \infty$ have been solved in previous works [3, 9]. Together with Theorem 1.1, one sees the sharp phase transition scenario of three types of the spectral measure. Moreover, the type of the spectral measure is clear for all $(\lambda, \beta)$ except the line $\lambda = e^\beta$. See Figure 1 and Figure 2 below.

![Figure 1. Phase transition diagram](image1)

![Figure 2. Phase transition for fixed $\alpha$.](image2)

Remark 1.3. Theorem 1.1(3), also called Anderson localization (AL), is optimal in the sense that the result can not be true for $G_\delta$ dense $\theta$ [25]. The arithmetic property of $\theta$ will influence the spectral measure.

Now we briefly recall the history of this problem. By symmetry, we just need to consider the case $\lambda > 0$. In 1980, Aubry-André [1] conjectured that the spectral measure of $H_{\lambda,\alpha,\theta}$ depends on $\lambda$ in the following way:

1. If $\lambda < 1$, then $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and all $\theta \in \mathbb{R}$.
2. If $\lambda > 1$, then $H_{\lambda,\alpha,\theta}$ has pure point spectrum for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and all $\theta \in \mathbb{R}$.

However, Aubry and André overlooked the role of the arithmetic property of $\alpha$. Avron-Simon [12] soon found that by Gordon’s lemma [21], $H_{\lambda,\alpha,\theta}$ has
no eigenvalues for any $\lambda \in \mathbb{R}$, $\theta \in \mathbb{R}$ if $\beta(\alpha) = \infty$. Since then, people pondered how the arithmetic property of $\alpha$ influences the spectral type and under which condition Aubry-André’s conjecture [1] is true.

When $\alpha$ is Diophantine (i.e. there exist $\gamma, \tau > 0$ such that $\|k\alpha\| > \frac{\gamma - 1}{|k|\tau}$, for all $0 \neq k \in \mathbb{Z}$), and $\lambda$ is large enough, the operator has pure point spectrum [17, 19, 39], and when $\lambda$ is small enough, the operator has absolutely continuous spectrum [14, 15, 16]. The common feature of the above results is that they both rely on KAM-type arguments, thus the largeness or smallness of $\lambda$ depend on Diophantine constant $\gamma, \tau$, we therefore call such results perturbative results. Non-perturbative approach to localization problem was developed by Jitomirskaya, based on partial advance [26, 27], she finally proved that if $\alpha$ is Diophantine, $H_{\lambda, \alpha, \theta}$ has AL for all $\lambda > 1$ and a.e. $\theta \in \mathbb{R}$. It follows from the strong version of Aubry duality [22], $H_{\lambda^{-1}, \alpha, \theta}$ has purely absolutely continuous spectrum for a.e. $\theta \in \mathbb{R}$. Therefore, Jitomirskaya [29] proved Aubry-André’s conjecture in the measure setting, i.e. the conjecture holds for almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\theta \in \mathbb{R}$.

Before Jitomirskaya’s result, Last [33], Gesztesy-Simon [20], Last-Simon [36] have already showed that $H_{\lambda, \alpha, \theta}$ has absolutely continuous components for every $\lambda < 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\theta \in \mathbb{R}$, so the conjecture in subcritical regime still has some hope to be true, which was also conjectured by Simon [38]. Recently, Avila-Jitomirskaya [10] showed that if $\alpha$ is Diophantine, then $H_{\lambda, \alpha, \theta}$ is purely absolutely continuous for every $\theta \in \mathbb{R}$. For $\beta > 0$, Avila-Damanik [7] proved that the conjecture (1) for almost every $\theta$. The complete answer of Aubry-André’s conjecture (1) was provided by Avila [3]. One thus sees that $\lambda = 1$ is the phase transition point from absolutely continuous spectrum to singular spectrum.

The remained issue is Aubry-André’s conjecture (2) when $\alpha$ is Liouvillean. People already knew that the spectral measure is pure point for Diophantine $\alpha$ and almost every phases, while it is purely singular continuous for $\beta(\alpha) = \infty$ and all phase. So there must be phase transition when $\beta(\alpha)$ goes from zero to infinity. In 1995, Jitomirskaya [28] modified the second part of the Aubry-André’s conjecture and conjectured the following

(1) If $1 < \lambda < e^{\beta}$, the spectrum is purely singular continuous for all $\theta$.

(2) If $\lambda > e^{\beta}$, the spectrum is pure point with exponential decaying eigenfunctions for a.e. $\theta$.

Thus $\lambda = e^{\beta}$ is conjectured to be the exact phase transition point from continuous spectrum to pure point spectrum. There are some partial results on Jitomirskaya’s conjecture. By Gordon’s lemma [21] and the exact formula of Lyapunov exponent [13], one can prove that $H_{\lambda, \alpha, \theta}$ has purely singular continuous spectrum for any $\theta \in \mathbb{R}$ if $1 < \lambda < e^{\frac{2}{\beta}}$, see also Remark 3.1 for more discussions. For the pure point part, Avila-Jitomirskaya [9] showed that if $\lambda > e^{\frac{8}{15}}$, then $H_{\lambda, \alpha, \theta}$ has AL for a.e. $\theta \in \mathbb{R}$. You-Zhou [40] proved that if
\[ \lambda > C e^\beta \text{ with } C \text{ large enough}, \]
then the eigenvalues of \( H_{\lambda, \alpha, \theta} \) with exponentially decaying eigenfunctions are dense in the spectrum. Readers can find more discussions on these two results in section 4. The main contribution of this paper is to give a full proof of Jitomirskaya’s conjecture.

We remark that the spectral type at the transition points \( \lambda = 1 \) and \( \lambda = e^\beta \) have not been completely understood so far. Partial results include the following: in case \( \lambda = 1 \), since the Lebesgue measure of the spectrum is zero for every \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) \([11, 34]\), by Aubry duality \([22]\), we know \( H_{\lambda, \alpha, \theta} \) is purely singular continuous for a.e. \( \theta \in \mathbb{R} \). In fact, Avila \([2]\) has proved more: if \( \theta \) is not rational w.r.t \( \alpha \), then \( H_{\lambda, \alpha, \theta} \) is purely singular continuous.

We remark that, by Gordon’s lemma \([21]\), if \( \beta > 0 \), then \( H_{\lambda, \alpha, \theta} \) is purely singular continuous for \( \lambda = 1 \) and every \( \theta \in \mathbb{R} \), we include this in Theorem 1.1(2). Excluding or proving the existence of point spectrum in case that \( \alpha \) is Diophantine is one of the major interesting problems for the critical almost Mathieu operator. For the second transition point \( \lambda = e^\beta \), one knows almost nothing but purely singular continuous spectrum for a \( G_\delta \) set of \( \theta \) \([25]\). The spectral type possibly depends on the finer properties of approximation of \( \alpha \), as conjectured by Jitomirskaya in \([30]\).

2. Preliminaries

For a bounded analytic (possibly matrix valued) function \( F \) defined on \( \{ \theta | |\Im\theta| < h \} \), let \( \| F \|_h = \sup_{|\Im\theta| < h} \| F(\theta) \| \) and denote by \( C^\omega_h(\mathbb{T}, *) \) the set of all these \(*\)-valued functions (\(*\) will usually denote \( \mathbb{R}, SL(2, \mathbb{R}) \)).

2.1. Continued Fraction Expansion. Let \( \alpha \in (0, 1) \) be irrational. Define \( a_0 = 0, a_0 = \alpha \), and inductively for \( k \geq 1 \),

\[
 a_k = [\alpha_{k-1}], \quad \alpha_k = \alpha_{k-1}^{-1} - a_k = G(\alpha_{k-1}) = \left\{ \frac{1}{\alpha_{k-1}} \right\},
\]

Let \( p_0 = 0, p_1 = 1, q_0 = 1, q_1 = a_1 \), then we define inductively \( p_k = a_k p_{k-1} + p_{k-2}, q_k = a_k q_{k-1} + q_{k-2} \). The sequence \( (q_n) \) is the denominators of best rational approximations of \( \alpha \) since we have

\[
(2.1) \quad \forall 1 \leq k < q_n, \quad \| k\alpha \|_\mathbb{T} \geq \| q_{n-1}\alpha \|_\mathbb{T},
\]
and

\[
(2.2) \quad \| q_n\alpha \|_\mathbb{T} \leq \frac{1}{q_{n+1}}.
\]

Note that (1.11) is equivalent to

\[
(2.3) \quad \limsup_{k \to \infty} \frac{1}{|k|} \ln \frac{1}{\| k\alpha \|_\mathbb{T}} = \beta.
\]

\(^1\)If one check carefully the proof, it already gives \( C = 1 \).
2.2. Cocycles. A cocycle \((\alpha, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{T}, SL(2, \mathbb{R}))\) is a linear skew product:

\[
(\alpha, A) : \quad \mathbb{T}^1 \times \mathbb{R}^2 \rightarrow \mathbb{T}^1 \times \mathbb{R}^2
\]

\[
(\theta, v) \mapsto (\theta + \alpha, A(\theta) \cdot v),
\]

for \(n \geq 1\), the products are defined as

\[
A_n(\theta) = A(\theta + (n - 1)\alpha) \cdots A(\theta),
\]

and \(A_{-n}(\theta) = A_n(\theta - n\alpha)^{-1}\). For this kind of cocycles, the Lyapunov exponent

\[
L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int \ln \|A_n(\theta)\| d\theta,
\]

is well defined.

Assume now \(A \in C^0(\mathbb{T}, SL(2, \mathbb{R}))\) is homotopic to the identity. Then there exists \(\psi : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}\) and \(u : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}^+\) such that

\[
A(x) \cdot \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = u(x, y) \begin{pmatrix} \cos 2\pi (y + \psi(x, y)) \\ \sin 2\pi (y + \psi(x, y)) \end{pmatrix}.
\]

The function \(\psi\) is called a lift of \(A\). Let \(\mu\) be any probability measure on \(\mathbb{T} \times \mathbb{T}\) which is invariant by the continuous map \(T : (x, y) \mapsto (x + \alpha, y + \psi(x, y))\), projecting over Lebesgue measure on the first coordinate (for instance, take \(\mu\) as any accumulation point of \(\frac{1}{n} \sum_{k=0}^{n-1} T^k \nu\) where \(\nu\) is Lebesgue measure on \(\mathbb{T} \times \mathbb{T}\)). Then the number

\[
\text{rot}_f(\alpha, A) = \int \psi d\mu \mod \mathbb{Z}
\]

does not depend on the choices of \(\psi\) and \(\mu\), and is called the fibered rotation number of \((\alpha, A)\), see \[31\] and \[23\].

Let

\[
R_\phi = \begin{pmatrix} \cos 2\pi \phi & -\sin 2\pi \phi \\ \sin 2\pi \phi & \cos 2\pi \phi \end{pmatrix},
\]

then any \(A \in C^0(\mathbb{T}, SL(2, \mathbb{R}))\) is homotopic to \(\theta \mapsto R_{n\theta}\) for some \(n \in \mathbb{Z}\), we call \(n\) the degree of \(A\), and denote \(\deg A = n\). The fibered rotation number is invariant under conjugation in the following sense: For cocycles \((\alpha, A_1)\) and \((\alpha, A_2)\), if there exists \(B \in C^0(\mathbb{T}, PSL(2, \mathbb{R}))\), such that \(B(\theta + \alpha)^{-1} A_1(\theta) B(\theta) = A_2(\theta)\), then we say \((\alpha, A_1)\) is conjugated to \((\alpha, A_1)\). If \(B\) has degree \(n\), then we have

\[
(2.4) \quad \text{rot}_f(\alpha, A_1) = \text{rot}_f(\alpha, A_2) + \frac{1}{2} n\alpha.
\]

If furthermore \(B \in C^0(\mathbb{T}, SL(2, \mathbb{R}))\) with \(\deg B = n \in \mathbb{Z}\), then we have

\[
(2.5) \quad \text{rot}_f(\alpha, A_1) = \text{rot}_f(\alpha, A_2) + n\alpha.
\]

The cocycle \((\alpha, A)\) is \(C^\omega\) reducible, if it can be \(C^\omega\) conjugated to a constant cocycle. The cocycle \((\alpha, A)\) is called \(C^\omega\) rotations reducible, if there exist \(B \in C^\omega(\mathbb{T}, SL(2, \mathbb{R}))\) such that \(B(\theta + \alpha)^{-1} A(\theta) B(\theta) \in SO(2, \mathbb{R})\). The crucial reducibility results for us is the following:
Theorem 2.1. [24] Let \((\alpha, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega_h(\mathbb{T}, SL(2, \mathbb{R}))\) with \(h > \tilde{h} > 0\), \(R \in SL(2, \mathbb{R})\), for every \(\tau > 1\), \(\gamma > 0\), if \(\text{rot}_f(\alpha, A) \in DC_\alpha(\tau, \gamma)\), where

\[
DC_\alpha(\tau, \gamma) = \{\phi \in \mathbb{R} | \parallel 2\phi - m\alpha \parallel_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|m| + 1)^\tau}, m \in \mathbb{Z}\}
\]

then there exist \(T = T(\tau), \kappa = \kappa(\tau)\), such that if

\[
\|A(\theta) - R\|_h < T(\tau)^\gamma \kappa(h - \tilde{h})^\kappa,
\]

then there exist \(B \in C^\omega(\mathbb{T}, SL(2, \mathbb{R})), \varphi \in C^\omega(\mathbb{T}, \mathbb{R})\), such that

\[
B(\theta + \alpha)A(\theta)B(\theta)^{-1} = R_{\varphi(\theta)},
\]

with estimates \(\|B - \text{id}\|_h \leq \|A(\theta) - R\|_h^\frac{2}{h}, \|\varphi(\theta) - \tilde{\varphi}(0)\|_h \leq 2\|A(\theta) - R\|_h\).

2.3. Almost Mathieu cocycle. Note that a sequence \((u_n)_{n \in \mathbb{Z}}\) is a formal solution of the eigenvalue equation \(H_{\lambda, \alpha, \theta} u = Eu\) if and only if it satisfied

\[
\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S^\lambda_E(\theta + n\alpha) \cdot \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},
\]

where

\[
S^\lambda_E(\theta) = \begin{pmatrix} E - 2\lambda \cos 2\pi(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{R}).
\]

We call \((\alpha, S^\lambda_E)\) an almost Mathieu cocycle.

Denote the spectrum of \(H_{\lambda, \alpha, \theta}\) by \(\Sigma_{\lambda, \alpha}\), which is independent of \(\theta\) when \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\). If \(E \in \Sigma_{\lambda, \alpha}\), then the Lyapunov exponent of almost Mathieu cocycle can be computed directly.

Theorem 2.2. [25] If \(\alpha \in \mathbb{R} \setminus \mathbb{Q}, E \in \Sigma_{\lambda, \alpha}\), then we have

\[
L(\alpha, S^\lambda_E) = \max\{0, \ln |\lambda|\}.
\]

2.4. Global theory of one frequency quasi-periodic \(SL(2, \mathbb{R})\) cocycle.

We make a short review of Avila’s global theory of one frequency quasi-periodic \(SL(2, \mathbb{R})\) cocycle [4]. Suppose that \(A \in C^\omega(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{R}))\) admits a holomorphic extension to \(|3\theta| < \delta\), then for \(|\varepsilon| < \delta\) we can define \(A_\varepsilon \in C^\omega(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))\) by \(A_\varepsilon(\theta) = A(\theta + i\varepsilon)\). The cocycles which are not uniformly hyperbolic are classified into three regimes: subcritical, critical, and supercritical. In particular, \((\alpha, A)\) is said to be subcritical, if there exists \(\delta > 0\), such that \(L(\alpha, A_\varepsilon) = 0\) for \(|\varepsilon| < \delta\).

The heart of Avila’s global theory is his “Almost Reducibility Conjecture” (ARC), which says that subcriticality implies almost reducibility. Recall the cocycle \((\alpha, A)\) is called almost reducible, if there exists \(h > 0\), and a sequence \(B_n \in C^\omega_{h_\varepsilon}(\mathbb{T}, PSL(2, \mathbb{R}))\) such that \(B_n(\theta + \alpha)^{-1} A(\theta)B_n(\theta)\) converges to constant uniformly in \(|3\theta| < h_\varepsilon\). For our purpose, we need this strong version of almost reducibility, and \(h_\varepsilon\) should be chosen to be \(\delta - \varepsilon\) with \(\varepsilon\) arbitrary small.

The full solution of ARC was recently given by Avila in [5, 6]. In the case \(\beta(\alpha) > 0\), it is the following:
**Theorem 2.3.** [5] Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( \beta(\alpha) > 0 \), \( h > 0 \), \( A \in C^\omega_h(\mathbb{T}, \mathbb{R}) \). If \((\alpha, A)\) is subcritical, then for any \( 0 < h_* < h \) there exists \( C > 0 \) such that if \( \delta > 0 \) is small enough, then there exist \( B \in C^\omega_h(T_*, \text{PSL}(2, \mathbb{R})) \) and \( R_* \in SO(2, \mathbb{R}) \) such that \( \|B\|_{h_*} \leq e^{C\delta q} \) and
\[
\|B(\theta + \alpha)^{-1} A(\theta)B(\theta) - R_*\|_{h_*} \leq e^{-\delta q}.
\]

2.5. **Aubry duality.** Suppose that the quasi-periodic Schrödinger operator
\[
(H_{V, \alpha, \theta} x)_n = x_{n+1} + x_{n-1} + V(n \alpha + \theta) x_n = Ex_n,
\]
has an analytic quasi-periodic Bloch wave \( x_n = e^{2\pi i n \varphi} (n \alpha + \varphi) \) for some \( \varphi \in C^\omega(T, \mathbb{C}) \) and \( \varphi \in [0, 1) \). It is easy to see the Fourier coefficients of \( \varphi(\theta) \) satisfy the following Long-range operator:
\[
(\hat{L}_{V, \alpha, \varphi} u)_n = \sum_{k \in \mathbb{Z}} V_k u_{n-k} + 2\cos 2\pi(\varphi + n \alpha) u_n = Eu_n,
\]

Almost Mathieu operator is the only operator which is invariant under Aubry duality, and the dual of \( H_{\lambda, \alpha, \theta} \) is \( H_{\lambda^{-1}, \alpha, \varphi} \).

Rigorous spectral Aubry duality was founded by Gordon-Jitomirskaya-Last-Simon in [22], where they proved that if \( H_{\lambda, \alpha, \theta} \) has pure point spectrum for a.e. \( \theta \in \mathbb{R} \), then \( H_{\lambda^{-1}, \alpha, \varphi} \) has purely absolutely continuous spectrum for a.e. \( \varphi \in \mathbb{R} \). Readers can find more discussions about dynamical Aubry duality in section 4.

3. **Singular continuous spectrum**

In this section, we prove Theorem 1.1 (2). We re-state it as in following

**Theorem 3.1.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( 0 < \beta(\alpha) \leq \infty \). If \( 1 \leq \lambda < e^\beta \), then \( H_{\lambda, \theta, \alpha} \) has purely singular continuous spectrum for any \( \theta \in \mathbb{T} \).

**Remark 3.1.** We stress again by classical Gordon’s argument [21], one can only obtain result in regime \( 1 \leq \lambda < e^{\frac{\beta}{2}} \). The reason why one can only obtain \( e^{\frac{\beta}{2}} \) is that, in the classical Gordon’s lemma, one has to approximate the solution by periodic ones along double periods.

**Proof.** If \( 1 < \lambda < e^\beta \), \( E \in \Sigma_{\lambda, \alpha} \), then by Theorem 2.2, one always has \( L(E, \alpha) = \ln \lambda > 0 \). By Kotani’s theory [32], the operator \( H_{\lambda, \theta, \alpha} \) doesn’t support any absolutely continuous spectrum, thus one only needs to exclude the point spectrum. In the case \( \lambda = 1 \), since Lebesgue measure of \( \Sigma_{1, \alpha} \) is zero for any \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) [11, 34], then \( H_{1, \theta, \alpha} \) also doesn’t support any absolutely continuous spectrum, thus to prove Theorem 3.1 it is also enough to exclude the point spectrum.

As in classical Gordon’s lemma, we approximate the quasi-periodic cocycles by periodic ones. Denote \( A(\theta) = S^\alpha_{E}(\theta) \) and
\[
A_m(\theta) = A(\theta + (m-1) \alpha) \cdots A(\theta + \alpha) A(\theta),
\]
\[
= A^m(\theta) \cdots A^2(\theta) A(\theta)
\]
\[ A_m(\theta) = A(\theta + m \frac{p_n}{q_n}) \cdots A(\theta + \frac{p_n}{q_n}) A(\theta), \]
\[ = \tilde{A}^m(\theta) \cdots \tilde{A}^2(\theta) \tilde{A}^1(\theta), \]

for \( m \geq 1 \). We also denote \( A_{-m}(\theta) = A_m(\theta - m\alpha)^{-1}, \tilde{A}_{-m}(\theta) = \tilde{A}_m(\theta - m\frac{p_n}{q_n})^{-1} \). Our proof is based on the following

**Proposition 3.1.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). If \( \lambda \geq 1 \) and \( E \in \Sigma_{\lambda, \alpha} \), then for any \( \epsilon > 0 \), there exists \( N = N(E, \lambda, \epsilon) > 0 \) such that if \( q_n > N \), then we have

\[
\sup_{\theta \in \mathbb{T}} \| \tilde{A}_{\pm q_n}(\theta) - A_{\pm q_n}(\theta) \| \leq \frac{1}{q_{n+1}} e^{(\ln \lambda + \epsilon) q_n},
\]

\[
\sup_{\theta \in \mathbb{T}} \| A_{q_n}(\theta + q_n \alpha) - A_{q_n}(\theta) \| \leq \frac{1}{q_{n+1}} e^{(\ln \lambda + \epsilon) q_n}.
\]

**Proof.** Furman’s result [18] gives

\[
\lim_{m \to \pm \infty} \sup_{\theta \in \mathbb{T}} \frac{1}{|m|} \log \| A_m(\theta) \| \leq L(\alpha, S_E^\lambda).
\]

Then by Theorem 2.2, we know for any \( \epsilon > 0 \), there exists \( K = K(E, \lambda, \epsilon) > 0 \), such that for any \( |m| \geq K \), we have

\[
\sup_{\theta \in \mathbb{T}} \| A_m(\theta) \| \leq e^{|m|(\ln \lambda + \epsilon/2)}.
\]

In the following, we only consider \( m \) is positive, the proof is similar for negative \( m \). In order to prove (3.3), we need the following:

**Lemma 3.1.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). If \( \lambda \geq 1 \) and \( E \in \Sigma_{\lambda, \alpha} \), then for any \( \epsilon > 0 \), there exists \( N_\epsilon = N_\epsilon(E, \lambda, \epsilon) > 2K \), such that

\[
\sup_{\theta \in \mathbb{T}} \| \tilde{A}_m(\theta) \| \leq e^{m(\ln \lambda + 2\epsilon/3)}
\]

for any \( q_n \geq N_\epsilon \), \( m \geq K \).

**Proof.** Clearly, for fixed \( m \in \mathbb{Z} \) and \( \delta > 0 \), if \( q_n \) is sufficiently large we have

\[
\sup_{\theta \in \mathbb{T}} \frac{1}{m} \ln \| \tilde{A}_m(\theta) \| - \frac{1}{m} \ln \| A_m(\theta) \| < \delta.
\]

Thus, there exists \( N_\epsilon = N_\epsilon(E, \lambda, \epsilon) > 0 \) such that if \( q_n \geq N_\epsilon \) then (3.7) holds for \( K \leq m \leq 2K - 1 \). Since any \( m \geq K \) can be written as a sum of integers \( m_i \) satisfying \( K \leq m_i \leq 2K - 1 \), this implies that (3.7) holds for all \( m \geq K \). \( \square \)
Once we have Lemma 3.1, we can prove directly by telescoping arguments. In fact, if \( q_n \geq N_\varepsilon \) we can write

\[
A_{q_n} - \tilde{A}_{q_n} = \sum_{i=1}^{q_n} A_{q_n} \cdots A_{i+1} \left( A_i ^{-1} \tilde{A}_i \right) \tilde{A}_{i-1} \cdots \tilde{A}_1
\]

\[
= \left( \sum_{i=1}^{K} \sum_{i=K+1}^{q_n} + \sum_{i=q_n-K}^{q_n} \right) A_{q_n} \cdots A_{i+1} \left( A_i ^{-1} \tilde{A}_i \right) \tilde{A}_{i-1} \cdots \tilde{A}_1
\]

\[
= (I) + (II) + (III),
\]

since for \( i \leq q_n \), we have \( \|A_i ^{-1} \tilde{A}_i\| \leq \frac{4\pi \lambda(i-1)}{q_n q_n + 1} \leq \frac{4\pi \lambda}{q_n + 1} \), then by (3.6) and Lemma 3.1, we can estimate

\[
(I) \leq \frac{4\pi \lambda}{q_n + 1} \sum_{i=1}^{K} (4\lambda + 3) i - 1 e^{(q_n - i)(\ln \lambda + 2\varepsilon / 3)},
\]

\[
(II) \leq \frac{4\pi \lambda}{q_n + 1} \sum_{i=K+1}^{q_n-K} e^{(q_n - 1)(\ln \lambda + 2\varepsilon / 3)},
\]

\[
(III) \leq \frac{4\pi \lambda}{q_n + 1} \sum_{i=q_n-K+1}^{q_n} (4\lambda + 3) q_n - i e^{(i-1)(\ln \lambda + 2\varepsilon / 3)}.
\]

If \( q_n \) is sufficiently large, then (3.3) follows directly. Using the similar argument as above, we can prove (3.4).

Now we finish the proof of Theorem 3.1 by contradiction. For any fixed \( \theta \), we suppose that \( E \) is an eigenvalue of \( H_{\lambda, \alpha, \theta} \), then there exists \( \pi = \left( \begin{array}{c} \nu_0 \\ \nu_{-1} \end{array} \right) \) with \( \|\pi\| = 1 \), and for any \( \varepsilon > 0 \), there exists \( N = N(E, \lambda, \varepsilon) \), such that if \( |m| > N(E, \lambda, \varepsilon) \), then \( \|A_m(\theta)\pi\| \leq \varepsilon \). In particular, for any \( 0 < 2\varepsilon < \ln \lambda - \beta \), we can select \( q_n > \max\{N(E, \lambda, \varepsilon), N(E, \lambda, \varepsilon)\} \), and \( q_n+1 > e^{(\beta - \varepsilon)q_n} \), such that

\[
\|A_{q_n}(\theta)\pi\| \leq \varepsilon, \quad \|A_{-q_n}(\theta)\pi\| \leq \varepsilon,
\]

where \( N(E, \lambda, \varepsilon) \) is defined in Proposition 3.1.

What’s important is the following observation:

**Lemma 3.2.** The following estimate holds:

\[
\|A_{q_n}(\theta + q_n\alpha) + A_{-q_n}(\theta + q_n\alpha)\| \leq 2\varepsilon + 10e^{-(\beta - \ln \lambda - 2\varepsilon)q_n}.
\]

**Proof.** By (3.3), it is sufficient for us to prove

\[
\|A_{q_n}(\theta + q_n\alpha) + A_{-q_n}(\theta + q_n\alpha)\| \leq 2\varepsilon + 8e^{-(\beta - \ln \lambda - 2\varepsilon)q_n}.
\]

By Hamilton-Clay Theorem, for any \( M \in SL(2, \mathbb{R}) \), one has

\[
M + M^{-1} = \text{tr} M \cdot Id,
\]
for every \( \theta' \in \mathbb{T} \). Take \( M = \tilde{A}_{q_n}(\theta') \), then

\begin{equation}
\tilde{A}_{q_n}(\theta') + \tilde{A}_{-q_n}(\theta') = \text{tr} \tilde{A}_{q_n}(\theta') .
\end{equation}

By assumptions (3.2) and (3.3), we have

\[
\| \text{tr} \tilde{A}_{q_n}(\theta) \| \\
\leq \| A_{q_n}(\theta) \| + \| \tilde{A}_{q_n}(\theta) - A_{q_n}(\theta) \| + \| \tilde{A}_{q_n}(\theta) \| - \| A_{-q_n}(\theta) \| \\
\leq 2 \varepsilon + 2 e^{-\beta \ln \lambda} q_n .
\]

As a result of (3.3) and (3.4), we have

\[
\| \text{tr} \tilde{A}_{q_n}(\theta + q_n \alpha) \| \\
\leq \| \text{tr} \tilde{A}_{q_n}(\theta) - \text{tr} A_{q_n}(\theta) \| + \| \text{tr} A_{q_n}(\theta + q_n \alpha) - \text{tr} A_{q_n}(\theta) \| \\
\leq 2 \varepsilon + 8 e^{-\beta \ln \lambda} q_n ,
\]

then (3.9) follows from (3.11). \( \square \)

However by Lemma 3.2 we have

\[
\| A_{2q_n}(\theta) \| = \| A_{q_n}(\theta + q_n \alpha) A_{q_n}(\theta) \| \\
\geq \| A_{-q_n}(\theta + q_n \alpha) A_{q_n}(\theta) \| - \| \tilde{A}_{q_n}(\theta + q_n \alpha) + \tilde{A}_{-q_n}(\theta + q_n \alpha) \| A_{q_n}(\theta) \| \\
\geq 1 - 2 \varepsilon^2 - 10 \varepsilon e^{-\beta \ln \lambda} q_n > \frac{1}{2} ,
\]

which contradicts with the assumption that \( E \) is an eigenvalue. \( \square \)

4. **Anderson localization**

In this section, we prove Theorem 1.1 (3). We re-state it as the following

**Theorem 4.1.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) be such that \( 0 < \beta(\alpha) < \infty \). If \( \lambda > e^\beta \), then the almost Mathieu operator \( H_{\lambda, \alpha, \phi} \) has Anderson Localization for a.e. \( \phi \).

Traditional method for Anderson Localization is to prove the exponentially decay of Green function [9, 26, 27, 29]. Due to the limitation of the method, Anderson Localization can be proved only for Liouvillean frequency with \( \lambda > e^{166} \) so far [9]. So there is still a gap between \( e^\beta \) and \( e^{166} \).

In this paper, we develop a new approach depending on the reducibility and Aubry duality. We will show that Theorem 4.1 can be obtained by dynamical Aubry duality and the following full measure reducibility result:

**Theorem 4.2.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( \beta(\alpha) > 0 \), if \( \lambda > e^\beta \), \( \text{rot}_f(\alpha, S_{E}^{\lambda-1}) \) is Diophantine w.r.t. \( \alpha \), then \( (\alpha, S_{E}^{\lambda-1}) \) is reducible.

The dynamical Aubry duality was established by Puig [37], who proved that Anderson localization of the Long range operator \( \tilde{L}_{\nu, \alpha, \phi} \) for almost
every $\varphi \in \mathbb{T}$ implies reducibility of $(\alpha, S^V_E)$ for almost every energies. Conversely, to deal with localization problem by reducibility was first realized by You-Zhou in [40]. However, in [40] they can only prove the eigenvalues of $\hat{L}_{V,\alpha,\varphi}$ with exponentially decaying eigenfunctions are dense in the spectrum. The main issue remained is to prove those eigenfunctions form a complete basis. The key point in this paper is that, we find that the quantitative estimates in the proof of Theorem 4.2 actually provides an asymptotical distribution of the eigenvalues and eigenfunctions, which ultimately implies pure point spectrum for almost every phases. Compared with tradition localization argument, the price we have to pay is that we lose precise arithmetic control on the localization phases. However, by this approach, one can indeed establish a kind of equivalence between quantitative full measure reducibility of Schrödinger operator (or Schrödinger cocycle) and Anderson localization of its dual Long-range operator.

**Proof Theorem 4.1** We need the following definition:

**Definition 4.1.** For any fixed $N \in \mathbb{N}, C > 0, \varepsilon > 0$, a normalized eigenfunction $u(n)$ is said to be $(N, C, \varepsilon)$-good, if $|u(n)| \leq e^{-C|n|}$, for $|n| \geq (1 - \varepsilon)N$.

We label the $(N, C, \varepsilon)$-good eigenfunctions of $H_{\lambda, \alpha, \varphi}$ by $u^\phi_j(n)$, denote the corresponding eigenvalue by $E^\phi_j$, also we denote

$$\mathcal{E}_{N,C,\varepsilon}^\phi = \{E^\phi_j | u^\phi_j(n) \text{ is a } (N, C, \varepsilon)-\text{good normalized eigenfunction}\}$$

and denote $\mathcal{E}(\phi) = \bigcup_{N>0} \mathcal{E}_{N,C,\varepsilon}^\phi$. Let $\mu_{\delta_0,\phi}$ be the spectral measure supported on $\mathcal{E}(\phi)$ with respect to $\delta_0$.

The following spectral analysis is completely new and will be crucial for our proof.

**Proposition 4.1.** Suppose that there exists $C > 0$, such that for any $\delta > 0$, there exists $\varepsilon > 0$, and for a.e. $\varphi$,

$$\#\{\text{linearly independent } (N, C, \varepsilon)-\text{good eigenfunctions}\} \geq (1 - \delta)2N,$$

for $N$ large enough, then for a.e. $\varphi$, we have $\mu_\phi = \mu_{\delta_0,\phi} = \mu_{\delta_0,\phi}^{pp}$.

**Proof.** Fix $\varphi \in \mathbb{T}$ such that (4.1) is satisfied. Denote

$$K^\phi_{N,C,\varepsilon} = \{j \in \mathbb{N} | u^\phi_j(n) \text{ is a } (N, C, \varepsilon)-\text{good eigenfunction}\}$$

Notice that for any fixed $N, C, \varepsilon$, $\#K^\phi_{N,C,\varepsilon}$ is finite, and also

$$\sum_{|n| \leq (1-\varepsilon)N} |u^\phi_j(n)|^2 > 1 - e^{-C\varepsilon N}$$

for $(N, C, \varepsilon)$-good eigenfunction $u^\phi_j(n)$. 


Let \( \tilde{\mu}_{\delta_n, \phi}^{pp} = \tilde{\mu}_{\delta_n, \phi}^{pp}(N, C, \varepsilon) \) be the truncated spectral measure supported on \( \mathcal{E}_{N,C,\varepsilon}^{\phi} \). Then by spectral theorem and (4.2), we have
\[
\frac{1}{2N} \sum_{|n| \leq N} \left| \mu_{\delta_n, \phi}^{pp} \right| > \frac{1}{2N} \sum_{|n| \leq N} \left| \tilde{\mu}_{\delta_n, \phi}^{pp} \right|
= \frac{1}{2N} \sum_{|n| \leq N} \langle P_{\mathcal{E}_{N,C,\varepsilon}^{\phi}} \delta_n, \delta_n \rangle
= \frac{1}{2N} \sum_{|n| \leq N} \sum_{j \in K_{N,C,\varepsilon}^{\phi}} \langle P_{E_j^{\phi} \delta_n}, \delta_n \rangle
> \frac{1}{2N} \sum_{|n| \leq (1-\varepsilon)N} \sum_{j \in K_{N,C,\varepsilon}^{\phi}} |u_j^{\phi}(n)|^2
> \frac{1}{2N} \sum_{|n| \leq (1-\varepsilon)N} \sum_{j \in K_{N,C,\varepsilon}^{\phi}} (1 - e^{-C\varepsilon N})
> (1 - \delta)(1 - e^{-C\varepsilon N}).
\]

Since \( \mathcal{E}(\phi) = \mathcal{E}(\phi + \alpha) \), we can rewrite the above inequalities as
\[
\frac{1}{2N} \sum_{|n| \leq N} \left| \mu_{\delta_0, \phi + n\alpha}^{pp} \right| > (1 - \delta)(1 - e^{-C\varepsilon N}),
\]

Let \( N \) go to \( \infty \), since \( \delta \) is arbitrary small, we have
\[
\int_{T^1} \chi_{\Theta_\gamma}(\phi) d\phi = 1,
\]
by Birkhoff’s ergodic theorem. Thus for a.e. \( \phi \in T^1 \), \( \mu_\phi = \mu_{\delta_0, \phi} = \mu_{\delta_0, \phi}^{pp} \). \( \square \)

Let \( \Theta_\gamma = \{ \phi | \phi \in DC_{\alpha}(\tau, \gamma) \} \). We have \( \bigcup_{\gamma > 0} \Theta_\gamma = 1 \), which implies that for any \( \delta > 0 \), there exists \( \bar{\epsilon} > 0 \), such that if \( |\gamma| < \bar{\epsilon} \), then \( |\Theta_\gamma| > 1 - \frac{\delta}{\bar{\epsilon}} \). By Birkhoff’s ergodic theorem again, we have
\[
\lim_{\tilde{N} \to \infty} \frac{1}{2\tilde{N}} \sum_{|k| \leq \tilde{N}} \chi_{\Theta_\gamma}(\phi + k\alpha) = \int_{T^1} \chi_{\Theta_\gamma}(\phi) d\phi.
\]
Thus for \( N \) large enough (we take \( \tilde{N} = N(1 - \frac{\delta}{\bar{\epsilon}}) \)), we have
\[
(4.3) \quad \# \{k | \phi + k\alpha \in \Theta_\gamma, |k| \leq 2N(1 - \frac{\delta}{\bar{\epsilon}}) \} \geq (1 - \delta)2N.
\]

For any \( \phi \in \Theta_\gamma \), we choose \( \tilde{N} \) sufficiently large such that (4.3) holds for \( N > \tilde{N} \). We will prove that \( H_{\lambda, \alpha, \phi} \) has at least \((1 - \delta)2N\) different eigenvalues \( E_k^{\phi} \) whose eigenfunctions \( u_k^{\phi}(n) \) are \((N, \ln \lambda - \beta - \varepsilon, \varepsilon)\)-good for any \( \varepsilon \). To prove this, we need the following quantitative version of Theorem 4.2:

Proposition 4.2. Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( \beta(\alpha) > 0 \) and \( \lambda > e^\beta \). Suppose that \( \text{rot}_f(\alpha, S_{\lambda - 1}^{\lambda - 1} E_k) = \phi + k\alpha \in DC_{\alpha}(\tau, \gamma) \). Then for any fixed \( \gamma > 0 \),
Let $\tau > 0$ and small enough $\epsilon > 0$, there exist $c_1(\lambda, \gamma, \tau, \epsilon, \alpha), c_2(\lambda, \gamma, \tau, \epsilon)$ and $B_k \in C_{\ln \lambda - \beta - \epsilon}(\mathbb{T}, SL(2, \mathbb{R}))$, such that

\begin{equation}
B_k(\theta + \alpha)^{-1}S_{\lambda - 1}^{\lambda - 1}E_k(\theta)B_k(\theta) = R_{\phi + k' \alpha},
\end{equation}

with estimates:

\begin{align}
\|B_k\|_{\ln \lambda - \beta - \epsilon} & \leq c_1(\lambda, \gamma, \tau, \epsilon, \alpha), \\
|k - k'| & \leq c_2(\lambda, \gamma, \tau, \epsilon).
\end{align}

**Proof.** If $\lambda > e^{\beta} > 1$, $\lambda^{-1}E_k \in \Sigma_{\lambda^{-1}, \alpha}$, then the almost Mathieu cocycle $(\alpha, S_{\lambda^{-1}}^{\lambda - 1}E_k)$ is subcritical in the regime $|3\theta| < \ln \lambda$. To prove Proposition 4.2, we need Theorem 2.2 and the following:

**Lemma 4.1.** If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda > 1$, $E \in \mathbb{R}$, then for $\epsilon \geq 0$,

$L(\alpha, (S_{\lambda}^{\lambda - 1})_\epsilon) = \max\{L(\alpha, S_{\lambda}^{\lambda - 1}), (\epsilon - \ln \lambda)\}$.

**Proof.** The proof can be found in Appendix A of [4].

Now by Theorem 2.3, for $0 < \epsilon < \ln \lambda - \beta$, there exists a sequence of $\tilde{B}_n \in C_{\ln \lambda - \epsilon/2}(\mathbb{T}, PSL(2, \mathbb{R}))$ such that

$\tilde{B}_n(\theta + \alpha)^{-1}S_{\lambda - 1}^{\lambda - 1}E_k(\theta)\tilde{B}_n(\theta) = R_{\phi_n} + F_n(\theta)$,

with estimate

\begin{equation}
\|\tilde{B}_n\|_{\ln \lambda - \epsilon/2} \leq e^{C\delta' q_n}, \\
\|F_n\|_{\ln \lambda - \epsilon/2} \leq e^{-\delta' q_n},
\end{equation}

which implies

\begin{equation}
|\deg \tilde{B}_n| \leq c(\lambda, \epsilon)q_n.
\end{equation}

One may consult footnote 5 of [5] in proving this.

If $\phi + k\alpha \in DC_\alpha(\tau, \gamma)$, we have

$$
\|2(\phi + k\alpha) - m\alpha - k'\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|m| + k'| + 1)^\tau} \geq \frac{(1 + |k'| - \tau \gamma)}{(|m| + 1)^\tau}.
$$

By (2.4), this formula implies that $\rot(\alpha, R_{\phi_n} + F_n(\theta)) \in DC_\alpha(\tau, (1 + |\deg \tilde{B}_n| - \tau \gamma))$. Let $q_s$ be the smallest denominator such that

$$
e^{-q_s \delta'} < T(\tau)(\frac{\gamma}{(1 + c(\lambda, \epsilon)q_s)^\gamma})^\kappa(\frac{\epsilon}{2})^\kappa,
$$

where $T = T(\tau)$, $\kappa = \kappa(\tau)$ are defined in Theorem 2.1. By Theorem 2.1, there exist $\overline{B}_k(\theta) \in C_{\ln \lambda - \epsilon}(\mathbb{T}, SL(2, \mathbb{R}))$, $\eta_k(\theta) \in C_{\ln \lambda - \epsilon}(\mathbb{T}, \mathbb{R})$, such that

$\overline{B}_k(\theta + \alpha)^{-1}(R_{\phi_n} + F_s(\theta))\overline{B}_k(\theta) = R_{\eta_k(\theta)}$.
with estimates \( \|\eta_k\|_{\ln \lambda - \epsilon} \leq e^{-q_0'} \) and
\[
(4.9) \quad \|\mathcal{B}_k - id\|_{\ln \lambda - \epsilon} \leq e^{-q_0'/2}.
\]
Let \( \psi_k(\theta) \) satisfy
\[
(4.10) \quad \psi_k(\theta + \alpha) - \psi_k(\theta) = \eta_k(\theta) - \hat{\eta}_k(0).
\]
since \( \ln \lambda > \beta \), by (2.3), we know that there exists \( c = c(\alpha, \epsilon) \) such that (4.10) has analytic solution \( \psi_k(\theta) \in C^\omega_{\ln \lambda - \beta - \epsilon}(\mathbb{T}, \mathbb{R}) \) with estimate
\[
(4.11) \quad \|\psi_k\|_{\ln \lambda - \beta - \epsilon} \leq c(\alpha, \epsilon) \|\eta_k\|_{\ln \lambda - \epsilon} \leq c(\alpha, \epsilon) e^{-q_0'}. 
\]
Let \( B_k(\theta) = \tilde{B}_k(\theta)\mathcal{B}_k(\theta) \) then there exists \( k' \in \mathbb{Z} \), such that
\[
B_k(\theta + \alpha)^{-1}S_{\lambda^{-1}E_k}(\theta)B_k(\theta) = R_{\hat{\eta}_k(0)} = R_{\phi + k'\alpha}.
\]
Since \( \text{rot}_f(\alpha, S_{\lambda^{-1}E_k}) \) is irrational w.r.t \( \alpha \), then \( B_k(\theta) \in C^\omega_{\ln \lambda - \beta - \epsilon}(\mathbb{T}, SL(2, \mathbb{R})) \), one can consult Remark 1.5 of [11] for this proof. Notice that \( \deg R_{\hat{\psi}_k(\theta)} = 0 \) and by (4.9), we have \( \deg \mathcal{B}_k = 0 \). Consequently by (2.3), we have
\[
(4.12) \quad k' = k - \deg \tilde{B}_k,
\]
(4.6) then follows from (4.8) and (4.12), and (4.5) follows from (4.7), (4.9) and (4.11).

Rewrite (4.4) as
\[
(4.13) \quad B_k(\theta + \alpha)^{-1}S_{\lambda^{-1}E_k}(\theta)B_k(\theta) = \begin{pmatrix} e^{2\pi i(\phi + k'\alpha)} & 0 \\ 0 & e^{-2\pi i(\phi + k'\alpha)} \end{pmatrix},
\]
and write \( B_k(\theta) = \begin{pmatrix} z_{11}(\theta) & z_{12}(\theta) \\ z_{21}(\theta) & z_{22}(\theta) \end{pmatrix} \), then we have
\[
(4.14) \quad (\lambda^{-1}E_k - 2\lambda^{-1}\cos(\theta))z_{11}(\theta) = z_{11}(\theta - \alpha)e^{-2\pi i(\phi + k'\alpha)} + z_{11}(\theta + \alpha)e^{2\pi i(\phi + k'\alpha)}.
\]
Taking the Fourier transformation for (4.14), we have
\[
\tilde{z}_{11}(n + 1) + \tilde{z}_{11}(n - 1) + 2\lambda \cos(\phi + k'\alpha + n\alpha)\tilde{z}_{11}(n) = E_k \tilde{z}_{11}(n),
\]
then \( \tilde{z}_{11}(n) \) is a eigenfunction, since \( z_{11} \in C^\omega_{\ln \lambda - \beta - \epsilon}(\mathbb{T}, \mathbb{C}) \). To normalize \( \tilde{z}_{11}(n) \), we need the following observation:

**Lemma 4.2.** We have the following:
\[
\|\tilde{z}_{11}\|_2 \geq (2\|B\|_{C^0})^{-1}.
\]

**Proof.** Write
\[
u = \begin{pmatrix} z_{11}(\theta) \\ z_{21}(\theta) \end{pmatrix}, \quad v = \begin{pmatrix} z_{12}(\theta) \\ z_{22}(\theta) \end{pmatrix},
\]
then \( \|u\|_{L^2}\|v\|_{L^2} > 1 \) since \( \det B_k(\theta) = 1 \). This implies that
\[
\|z_{11}\|_{L^2} + \|z_{21}\|_{L^2} = \|u\|_{L^2} > \|v\|_{L^2} > (\|B\|_{C^0})^{-1}.
\]
By (4.13), we have $z_{21}(\theta + \alpha) = e^{-2\pi i (\phi + k' \alpha)} z_{11}(\theta)$, therefore, we have
$$\|\hat{z}_{11}\|_2 = \|z_{11}\|_{L^2} \geq (2\|B\|_{C^0})^{-1}.$$ 

Normalizing $\hat{z}_{11}(n)$ by $u^\phi_k(n) = \frac{\hat{z}_{11}(n+k')}{\|\hat{z}_{11}\|_2}$. Now we prove it is in fact $(N, \ln \lambda - \beta - \epsilon, \epsilon)$-good. Let
$$2\epsilon < \frac{\delta}{3} - \frac{c_3(\lambda, \gamma, \tau, \epsilon, \alpha)}{N} - \frac{c_2(\lambda, \gamma, \tau, \epsilon)}{N},$$
where $c_3(\lambda, \gamma, \tau, \epsilon, \alpha) = \frac{\ln 2c_1(\lambda, \gamma, \tau, \epsilon, \alpha)}{\ln \lambda - \beta - \epsilon}$. Since $u^\phi_k(n) = u^{\phi+k'\alpha}(n-k')$, then by Proposition 4.2 and Lemma 4.2, we have
$$|u^\phi_k(n)| = |u^{\phi+k'\alpha}(n-k')| \leq \|B_k\|_{L^2} e^{-n-k'}(\ln \lambda - \beta - \epsilon)$$
$$\leq e^{(c_3(\lambda, \gamma, \tau, \epsilon, \alpha) + |k| + c_2(\lambda, \gamma, \tau, \epsilon))(\ln \lambda - \beta - \epsilon) - |n|(\ln \lambda - \beta - \epsilon)}$$
$$\leq e^{(N(1-k') + c_2(\lambda, \gamma, \tau, \epsilon) + c_3(\lambda, \gamma, \tau, \epsilon, \alpha))(\ln \lambda - \beta - \epsilon) - |n|(\ln \lambda - \beta - \epsilon)}$$
$$\leq e^{-|n|(\ln \lambda - \beta - \epsilon) \epsilon},$$
for $|n| \geq N(1-\epsilon)$, which means $(u^\phi_k(n))$ is $(N, \ln \lambda - \beta - \epsilon, \epsilon)$-good.

By Proposition 4.1 and the above estimate, we know for a.e. $\phi \in T^1$, $H_{\lambda, \alpha, \phi}$ has Anderson Localization.

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