On the modification of the Hamiltonians’ spectrum in gravitational quantum mechanics

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received 20 January 2010; accepted in final form 22 February 2010
published online 25 March 2010

PACS 04.60.-m – Quantum gravity

Abstract – Different candidates of quantum gravity such as string theory, doubly special relativity, loop quantum gravity and black-hole physics all predict the existence of a minimum observable length or a maximum observable momentum which modifies the Heisenberg uncertainty principle. This modified version is usually called the generalized (gravitational) uncertainty principle (GUP) and changes all Hamiltonians in quantum mechanics. In this letter, we use a recently proposed GUP which is consistent with string theory, doubly special relativity and black-hole physics and predicts both a minimum measurable length and a maximum measurable momentum. This form of GUP results in two additional terms in any quantum-mechanical Hamiltonian, proportional to $\alpha p^3$ and $\alpha^2 p^4$, respectively, where $\alpha \sim 1/M_{Pl}$ is the GUP parameter. By considering both terms as perturbations, we study two quantum-mechanical systems in the framework of the proposed GUP: a particle in a box and a simple harmonic oscillator. We demonstrate that, for the general polynomial potentials, the corrections to the highly excited eigenenergies are proportional to their square values. We show that this result is exact for the case of a particle in a box.

Introduction. – The existence of a minimum measurable length is one of the common properties of various quantum gravity theories such as string theory, loop quantum gravity and doubly special relativity. Moreover, some Gedanken experiments in black-hole physics show that a minimum length of the order of the Planck length arises naturally from any theory of quantum gravity. We can also realize a minimal measurable length in the context of spacetime non-commutativity. We should notice that the minimal observable length can be probed essentially: we can use a $D_0$ brane to probe the minimal length, but this needs a very long time. In fact, one can probe the Planck length on a $D_0$ brane if the proposed experiment lasts an infinite time.

On the other hand, the Heisenberg uncertainty principle does not exert any restriction on the measurement precision of the particles’ positions or momenta. So, in principle, there is no minimum measurable length in the usual Heisenberg picture. In the past few years, many papers have appeared in the literature to address the presence of a minimum measurable length by redefinition of the uncertainty principle in the context of the generalized uncertainty principle (GUP) [1]. This leads to modification of the commutation relations between position and momentum operators in the Hilbert space. The similar modified commutation relations have also appeared in doubly special relativity (DSR) theories [2,3]. In fact, DSR theories also indicate the presence of maximum measurable momenta.

The application of GUP in quantum mechanics would incorporate the effects of a minimum measurable length on the quantum-mechanical systems which results in the modification of the Hamiltonian (see [4] and references therein). Since the GUP-corrected Hamiltonian usually contains momentum polynomials of the order of greater than two, the resulting Schrödinger equation has a completely different differential structure. However, since the effect of quantum gravity is only considerable at the order of Planck energy, we can use the perturbation method to find the GUP-corrected spectrum of the system. Furthermore, when the corrected Hamiltonian is naturally perturbative, the perturbation method is more appropriate than other techniques to find the effect of the minimum observable length on the energy spectrum.

In this letter, we consider a recently proposed GUP which is consistent with string theory, doubly special relativity and black-hole physics and predicts both a minimum measurable length and a maximum measurable length.
momentum [5]. First, we obtain the GUP-corrected Hamiltonian to $O(\alpha^2)$ where $\alpha \sim 1/M_P c$ is the GUP parameter. Then, using the perturbation theory, we will obtain the corrected spectrum for two well-known and instructive cases up to the leading order. These cases consist of a particle in a box (PB) and a simple harmonic oscillator (SHO). We can consider these models as the limiting cases of the potential $V(x) = |a|x^{2(1+\epsilon)}$, where $j = 0$ denotes SHO and $j = \infty$ denotes PB.

A generalized uncertainty principle. – In a recent paper, Ali et al. have proposed a generalized uncertainty principle to address the discreteness of space with the following commutation relation [5]:

$$[x_i, p_j] = i\hbar \left[ \delta_{ij} - \alpha \left( \frac{p_i p_j}{p^2} \right) + \alpha^2 \left( p^2 \delta_{ij} + 3p_i p_j \right) \right],$$

where $p^2 = \sum_{j=1}^{3} p_j p_j$, $\alpha = \alpha_0/M_P c = \alpha_0 \ell_P / h$, $M_P \equiv$ Planck mass, $\ell_P \equiv$ Planck length $\approx 10^{-35}$ m, and $M_P c^2 \equiv$ Planck energy $\approx 10^{19}$ GeV. Moreover, the space of positions and momentums is separately assumed commutative, i.e. $[x_i, x_j] = [p_i, p_j] = 0$. In one dimension, the above commutation relations result in the following form of the uncertainty relation to $O(\alpha^2)$ [6]:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle \right]$$

$$\geq \frac{\hbar}{2} \left[ 1 + \left( \frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \right) \Delta p^2 + 4\alpha^2 \langle p^2 \rangle - 2\alpha \sqrt{\langle p^2 \rangle} \right].$$

Note that, the particular form of the above inequality implies both a minimum observable length and a maximum observable momentum at the same time [5]:

$$\Delta x \geq (\Delta x)_{min} \approx \alpha_0 \ell_P,$$

$$\Delta p \leq (\Delta p)_{max} \approx \frac{M_P c}{\alpha_0}.$$  

(3)

Now, let us define

$$x_i = x_{0i},$$

$$p_i = p_{0i} \left( 1 - \alpha_0 p_0^2 + 2\alpha_0^2 p_0^2 \right),$$

where $x_{0i}$ and $p_{0i}$ obey the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar \delta_{ij}$. It is easy to check that using eq. (4), eq. (1) is satisfied to $O(\alpha^2)$. Moreover, from the above equation we can interpret $p_{0i}$ as the momentum operator at low energies ($p_{0i} = -i\hbar \partial_{x_{0i}}$), $p_i$ as the momentum operator at high energies, and $p_0$ as the magnitude of the $p_{0i}$ vector ($p_0^2 = \sum_{j=1}^{3} p_{0j} p_{0j}$). It is usually assumed that $\alpha_0$ is of the order of unity. So, the $\alpha$-dependent terms are important only for high-energy (Planck energy) or high-momentum regime. Now, consider the following general form of the Hamiltonian:

$$H = \frac{p^2}{2m} + V(\vec{x}),$$

(5)

which by using eq. (4) can be written as

$$H = H_0 + \alpha H_1 + \alpha^2 H_2 + O(\alpha^3),$$

(6)

where $H_0 = \frac{p_0^2}{2m} + V(\vec{r})$ and

$$H_1 = -\frac{p_0^3}{m}, \quad H_2 = \frac{5p_0^4}{m}.$$  

(7)

In the next two sections, we are interested in studying the effect of $H_1$ and $H_2$ on two one-dimensional quantum-mechanical systems and generalize some results to general polynomial potential cases.

GUP and a particle in a box. – Here, we apply the GUP formalism to a particle in a box of length $L$. The boundaries of the box are located at $x = 0$ and $x = L$. The Hamiltonian of the unperturbed system $H_0 = \frac{p_0^2}{2m}$ results in the following Schrödinger equation:

$$H_0 \psi_n(x) = E_n^0 \psi_n(x),$$

(8)

where the wave functions should vanish at the boundaries ($\psi_n(0) = \psi_n(L) = 0$). So, the corresponding eigenvalues and normalized eigenfunctions for the unperturbed system are $E_n^0 = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$ and $\langle x|n\rangle \equiv \psi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$, respectively. Now, using the perturbation theory, we find the effect of $H_1$ on the energy eigenvalues to $O(\alpha)$:

$$E_n^1 = \alpha\langle n|H_1|n\rangle = \frac{2\alpha \text{m}^3 \pi^3 \hbar^3}{mL^2} \int_0^L \sin(n\pi x/L) \cos(n\pi x/L) \, dx = 0.$$  

(9)

This result shows that, for this case, GUP formalism has no contribution in energy spectrum to $O(\alpha)$. To proceed to the next order, we need to consider both $H_1$ and $H_2$ and denote their corresponding energy corrections by $E_n^{1,2}$ and $E_n^{2,2}$, respectively. To obtain $E_n^{2,1}$, we consider the second-order perturbation, namely

$$E_n^{2,1} = \alpha^2 \sum_{k \neq n} \frac{|\langle k|H_1|n\rangle|^2}{E_n^0 - E_k^0} = -\frac{3\alpha^2 \pi^2 \hbar^4 n^6}{mL^4} \times \sum_{k=\text{odd}}^{\infty} \frac{k^2}{(k^2 - n^2)^3}, \quad \text{for } n \text{ even},$$

$$\times \sum_{k=\text{even}}^{\infty} \frac{k^2}{(k^2 - n^2)^3}, \quad \text{for } n \text{ odd}.$$  

(10)

On the other hand, we have $\sum_{k=\text{odd}}^{\infty} \frac{k^2}{(k^2 - n^2)^3}$ for $n$ even $= \sum_{k=\text{even}}^{\infty} \frac{1}{k^2 - n^2}$ for $n$ odd $= \frac{\pi^2}{24n^2}$. So, we find $E_n^{2,1} = -\frac{\alpha^2 \pi^2 \hbar^4 n^6}{2mL^4}$. Since the contribution of $H_2$ at least
is second order in $\alpha$, after a straightforward calculation, one finds
\[
E_{n}^{2.2} = \alpha^{2}(n|H_{2}|n) = \frac{5\alpha^{2}}{m} (n|p_{0}^{2}|n) = \frac{5\alpha^{2}\pi^{4}h^{4}n^{4}}{mL^{4}}. \tag{11}
\]
Therefore, the total change in energy spectrum is $\Delta E_{n} = E_{n}^{2.1} + E_{n}^{2.2} = \frac{2\alpha^{2}\pi^{4}h^{4}n^{4}}{2mL^{4}}$ to $\mathcal{O}(\alpha^{2})$. Moreover, the above equations show that the contributions of $H_{1}$ and $H_{2}$ are in the same order which result in the modification of the previous results [7]. Note that, we can exactly write $\Delta E_{n}$ in terms of unperturbed energy eigenvalues $E_{n}^{0}$ as
\[
\Delta E_{n} = 18m\alpha^{2}E_{n}^{2}, \tag{12}
\]
or $\frac{\Delta E_{n}}{E_{n}^{0}} \propto E_{n}^{0}$. In other words, the relative change in each energy level is proportional to its unperturbed energy eigenvalue.

**GUP and simple harmonic oscillator.** Simple harmonic oscillator is one of the most important systems in quantum mechanics because an arbitrary potential can be approximated as a harmonic potential at the vicinity of a stable equilibrium point. The Hamiltonian for this system in the absence of GUP is given by
\[
H_{0} = \frac{p_{0}^{2}}{2m} + \frac{1}{2}m\omega^{2}x^{2}. \tag{13}
\]
In the quantum-mechanical picture ($H_{0}\psi_{n}(x) = E_{n}^{0}\psi_{n}(x)$), this model has well-known eigenvalues and eigenfunctions:
\[
\psi_{n}(x) = \left(\frac{\omega}{\pi}\right)^{1/4} H_{n}(\sqrt{\omega}x) e^{-\omega x^{2}/2}, \tag{14}
\]
\[
E_{n}^{0} = (n+1/2)\hbar\omega, \tag{15}
\]
where $H_{n}(x)$ is the Hermite polynomial and the orthonormality and completeness of the basis functions follow from those of the Hermite polynomials. We can also express the Hamiltonian in terms of non-Hermitian ladder operators $a = \sqrt{\frac{m}{\hbar\omega}}(x + \frac{1}{m\omega}p_{0})$ and $a^{\dagger} = \sqrt{\frac{m}{\hbar\omega}}(x - \frac{1}{m\omega}p_{0})$ as $H_{0} = \hbar\omega(a^{\dagger}a + 1/2)$. $a$ and $a^{\dagger}$ act on an eigenstate of energy $E$ to produce, up to a multiplicative constant, another eigenstate of energy $E \pm \hbar\omega$, respectively:
\[
|a|n\rangle = \sqrt{n}|n-1\rangle, \quad |a\rangle n\rangle = \sqrt{n+1}|n+1\rangle, \tag{16}
\]
with $a|0\rangle = 0$. Now, we have enough tools to find the effect of $H_{1}$ on energy eigenvalues. Similar to the case of a particle in a box and without any calculation, we can show that $E_{1}^{1} = \alpha^{2}|n|H_{1}|n\rangle$ vanishes also for this case. Note that, since $p_{0}$ is proportional to $a$ and $a^{\dagger}$ ($p_{0} = i\sqrt{\frac{\hbar\omega}{2}}(a^{\dagger} - a)$),
\[
H_{1} = -\frac{\hbar\omega}{m} \text{consists of odd number of } a \text{ and } a^{\dagger}. \text{ Thus, because of eq. (16), } E_{1}^{1} \text{ vanishes for all eigenstates. We can also conclude this result from reality of energy eigenvalues.}
\]
To obtain the second-order correction of $H_{1}$, we need to find the explicit form of $H_{1}$ in terms of ladder operators, namely
\[
H_{1} = -\frac{i}{m}(\frac{\hbar m\omega}{2})^{3/2} \left(a^{3} - 3a^{2}a - 3a^{1}a^{2} + 3a - a^{3}\right), \tag{17}
\]
where we have used $[a, a^{\dagger}] = 1$. So, we have
\[
\langle k|H_{1}|n\rangle = -\frac{i}{m}(\frac{\hbar m\omega}{2})^{3/2} \left\{ (\sqrt{n+1} + (n+3)\delta_{n+3,k} - 3(n+1)\delta_{n+1,k} + 3n\delta_{n-1,k} - \sqrt{n(n-1)}\delta_{n-3,k} \right\}, \tag{18}
\]
where $\delta$ is the Kronecker delta symbol. After some algebraic manipulations, we find the second-order correction of $H_{1}$ as
\[
E_{n}^{2.1} = \alpha^{2} \sum_{k \neq n} \frac{\langle k|H_{1}|n\rangle^{2}}{E_{n}^{0} - E_{k}^{0}} = -m\alpha^{2}h^{2}\omega^{2} \left( \frac{30n^{2} + 30n + 11}{8} \right), \tag{19}
\]
where we have used $E_{n}^{0} - E_{k}^{0} = (n-k)\hbar\omega$. Now, let us consider the contribution of $H_{2}$ on energy spectrum of SHO. Since $H_{2}$ is proportional to the forth power of the momentum, we need to express $p_{0}^{2}$ in terms of ladder operators
\[
p_{0}^{2} = \left(\frac{m\hbar\omega}{2}\right)^{2} |12a^{1}a + 6a^{2}a^{2} + 3
+(a^{4} - 4a^{3}a - 6a^{2} - 4a^{1}a^{3} - 6a^{2} + a^{4}|). \tag{20}
\]
Note that, in the above equation, the terms which are in parentheses have unequal number of $a$ and $a^{1}$ and consequently they do not contribute in $\langle n|H_{2}|n\rangle$. So, we can write the leading-order contribution of $H_{2}$ as
\[
E_{n}^{2.2} = \alpha^{2}(n|H_{2}|n) = \frac{5\alpha^{2}}{m} \left(\frac{m\hbar\omega}{2}\right)^{2}
× \langle n|12a^{1}a + 6a^{2}a^{2} + 3n\rangle
= m\alpha^{2}h^{2}\omega^{2} \left( \frac{30n^{2} + 30n + 15}{4} \right). \tag{21}
\]
Therefore, using eqs. (19) and (21), the total effect of the GUP on energy levels to $\mathcal{O}(\alpha^{2})$ is $\Delta E_{n} = E_{n}^{2.1} + E_{n}^{2.2} = \frac{1}{2}m\alpha^{2}h^{2}\omega^{2}(30n^{2} + 30n + 19)$. Moreover, eqs. (19) and (21) show that, also in this case, the corrections of $H_{1}$ and $H_{2}$ to energy levels are in the same order which modify the previous results for SHO [6]. For large values of $n$ ($n \gg 1$), we have $\Delta E_{n} \cong \frac{5}{4}m\alpha^{2}h^{2}\omega^{2}n^{2}$ and $E_{n}^{0} \cong n\hbar\omega$. Therefore, for large $n$, we can write
\[
\Delta E_{n} \cong \frac{15}{4}m\alpha^{2}E_{n}^{2}, \tag{22}
\]
or $\frac{\Delta E_{n}}{E_{n}^{0}} \propto E_{n}^{0}$. So, also in this case, the relative change in each energy level is proportional to its energy eigenvalue.
Since now, we have studied two limiting cases of \( V(x) = |a|x^{2(j+1)} \) potentials \((j \in \text{positive integers})\), where \( j = 0, \infty \) correspond to SHO and PB, respectively. Moreover, we found that the relation \( \frac{\Delta E_n}{\hbar^2} \propto E_n \) is exact for all energy levels of PB (12) and is valid for high energy levels of SHO (22). Since these two systems are limiting cases, we can generalize this result for all other values of \( 0 < j < \infty \).

To justify this generalization, we can use a general operational procedure called the factorization method [8]. In this method, the Hamiltonian of the system is written as the multiplication of two ladder operators plus a constant \( H = a^1a + E \). Then, these operators are used to obtain the Hamiltonian’s eigenfunctions. In general, in contrast to the case of SHO, one ladder operator is not enough to form all the Hamiltonian’s eigenfunctions and for each eigenfunction a ladder operator is needed.

The procedure of finding the ladder operators and the eigenfunctions consists of some steps: We find operators \( a_1, a_2, a_3, \ldots \) and real constants \( E_0^1, E_0^2, E_0^3, \ldots \) from the following recursion relations:

\[
a_1^\dagger a_1 + E_0^1 = H_0, \\
 a_2^\dagger a_2 + E_0^2 = a_1^\dagger a_1 + E_1^1, \\
 a_3^\dagger a_3 + E_0^3 = a_2^\dagger a_2 + E_2^1, \\
\]

or generally

\[
a_{n+1}^\dagger a_{n+1} + E_0^{n+1} = a_n^\dagger a_n + E_n^0, \quad n = 1, 2, \ldots ,
\]

where the real constants \( E_0^0 \) are the eigenvalues of the Hamiltonian and the operators \( a_n, a_n^\dagger \) are the ladder operators used to form the eigenfunctions. Also, assume that there exists a null eigenfunction (root function) \( |\xi_n\rangle \) with zero eigenvalue for each \( a_n \), namely

\[
a_n |\xi_n\rangle = 0.
\]

Hence, \( E_0^n \) is the \( n \)th eigenvalue of the Hamiltonian with the following corresponding eigenfunction:

\[
|n\rangle = a_1^\dagger a_2^\dagger \ldots a_{n-1}^\dagger |\xi_n\rangle.
\]

Because of recursion relations each of the annihilation operators \( (a_n) \) should contain a linear momentum term. Thus, \( a_n \) can be written as

\[
a_n = \frac{1}{\sqrt{2m}} (p_i + if_n(x)),
\]

where \( f_n(x) \) is a real function of \( x \) \( f_n(x) = -m\omega x \) for SHO and \( f_n(x) = \frac{n\pi \hbar}{L} \cot (\frac{nx}{L}) \) for PB. From the above equation, we have \( p_0 = \sqrt{2m(a_1 + a_1^\dagger)} \). So, the relevant terms of \( p_0^4 \) which have non-zero contribution in the expectation value of \( H_2 \) are

\[
(p_0^4)_{\text{relevant}} = 4m^2 \left[ (a_1^\dagger a_1)^2 + (a_1 a_1^\dagger)^2 + a_1^2 a_1^\dagger \right] \\
+ a_1^2 a_1^\dagger + a_1 a_1^\dagger a_1 + a_1^2 a_1^\dagger a_1^\dagger,
\]

where \( [a_1, a_1^\dagger] = -\frac{\hbar}{m} \frac{df_1(x)}{dx} \). Note that, for high-energy levels, we have \( \langle n|a_1 a_1^\dagger|n\rangle \approx \langle n|a_1^\dagger a_1|n\rangle \approx \langle n|H_0|n\rangle \). In other words, in this limit, the effect of \( \frac{\hbar}{m} \frac{df_1(x)}{dx} \) is negligible with respect to the Hamiltonian \( H_0 \). Now, let us verify this result for two studied cases. For PB we have \( \frac{\hbar}{m} \frac{df_1(x)}{dx} = -\frac{2E_0^0}{\sin^2(\pi x/L)} \) which results in \( \langle n|a_1 a_1^\dagger|n\rangle \approx \langle n|H_0|n\rangle + \frac{1}{2} \hbar \omega \) which result in

\[
\langle n|a_1 a_1^\dagger|n\rangle \approx \langle n|H_0|n\rangle + \frac{1}{2} \hbar \omega
\]

For the case of simple harmonic oscillator, we have \( \frac{\hbar}{m} \frac{df_1(x)}{dx} = -\hbar \omega \) and \( a_1 a_1^\dagger = H_0 + \frac{1}{2} \hbar \omega \) which in turn results in

\[
\langle n|a_1 a_1^\dagger|n\rangle \approx \langle n|H_0|n\rangle + \frac{1}{2} \hbar \omega
\]

Therefore, in the high-energy regime, to calculate the expectation value of \( H_2 \) all terms in eq. (28) act as the first term. So, we have

\[
\alpha^2 \langle n|H_2|n\rangle \approx O(1)ma^2 \langle n|H_0|n\rangle \approx O(1)ma^2 E_n^0.
\]

Moreover, because of the normalization condition of the energy eigenstates (26), for \( n \gg 1 \) we have

\[
a_1 |n\rangle \sim \sqrt{E_0^n} |n-1\rangle, \quad a_1^\dagger |n\rangle \sim \sqrt{E_0^n} |n+1\rangle,
\]

which also imply eq. (31) in a more straightforward manner. On the other hand, the Hermiticity of the Hamiltonian shows that, for a general polynomial potential, \( H_1 \) has no first-order contribution in the energy spectrum. Also, the explicit form of \( H_1 = \frac{p_0^2}{2m} \) in terms of ladder operators \( (a_1, a_1^\dagger) \) and eq. (32) show that \( \langle k|H_1|n\rangle \sim E_n^{3/2} \) and consequently the second-order perturbation correction \( E_n^{1/2} \) is also proportional to \( E_n^0 \).

On the other hand, for \( n \gg 1 \) the spectrum of \( H_0 = \frac{p_0^2}{2m} + |a|x^{2(j+1)} \) coincides with

\[
H_0 = \frac{p_0^2}{2m} + |a|x^{2(j+1)} + b x^{2(j+1)-1} + c x^{2(j+1)-2} + \ldots,
\]

which can be obtained from Sommerfeld-Wilson quantization rule

\[
\int p_x \, dx = nh, \quad n = 1, 2, \ldots ,
\]

in the high-energy limit, where \( p_x = \sqrt{2m(E - V(x))} \). In this energy limit, for the same value of the classical turning points \( (x_{TP} \gg 1) \), this integral for \( V_1(x) = |a|x^{2(j+1)} \) is approximately equal to \( V_2(x) = |a|x^{2(j+1)} + b x^{2(j+1)-1} + c x^{2(j+1)-2} + \ldots \). Because, the
dominance of $V_2$ over $V_1$ is around $x \approx 0$ which for $E \gg 1$
(hereafter we choose $\hbar^2 = 1$) does not alter considerably
the value of $E - V$ in the integrand. Moreover, since $|x_T L| \gg 1$, we have $E_{1,n} = V_1(L) \approx V_2(L) = E_{2,n}$. For instance, for $V_1(x) = x^4$, using eq. (34), we have $E_{1,n} = L^4 = \beta n^{4/3}$, where $\beta = \left( \frac{\sqrt{\Gamma(7/4)}}{\Gamma(5/4)} \right)^{4/3}$. For $V_2(x) = x^4 + x^2$ we have $\int_L - L \sqrt{L^4 + L^2 - x^4 - x^2} dx = \frac{L^5}{\sqrt{L^4 - x^4}} x + O(1) L = \frac{n \pi}{2}$ which results in $E_{2,n} = L^4 + L^2 = \beta n^{4/3} + O(n^{2/3})$. So, for $n \gg 1$ we have $E_{1,n} \approx E_{2,n}$.

Now, following eqs. (12) and (22), for the general form of the Hamiltonian (33), we can write the following relation:

$$\Delta E_n \approx O(1) n a^2 E_n^{\alpha^2},$$

which is an approximate relation for $n \gg 1$.

Conclusions. – In this letter, we have considered the consequence of a generalized (gravitational) uncertainty principle on the spectrum of some quantum-mechanical systems. This principle comes from the presence of a minimum observable length and modifies all Hamiltonians in quantum mechanics. Following the recently proposed GUP which is consistent with string theory, doubly special relativity, black-hole physics and also implies a maximum observable momentum, we found the energy eigenvalues of a particle in a box and a simple harmonic oscillator up to the second order of the minimum length ($\ell_{Pl}$). We showed that, for the case of a particle in a box, the corrections to the eigenenergies are exactly proportional to their square values. We also concluded that, for the general polynomial potentials in the form $V(x) = |a|x^{2(j+1)} + bx^{2(j+1)-1} + cx^{2(j+1)-2} + \ldots$, this result is approximately valid for highly excited eigenenergies.

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I would like to thank K. NOZARI for useful discussions and comments.

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