Abstract

Graphons are analytic objects representing convergent sequences of large graphs. A graphon is said to be finitely forcible if it is determined by finitely many subgraph densities, i.e., if the asymptotic structure of graphs represented by such a graphon depends only on finitely many density constraints. Such graphons appear in various scenarios, particularly in extremal combinatorics.

Lovász and Szegedy conjectured that all finitely forcible graphons possess a simple structure. This was disproved in a strong sense by Cooper, Král' and Martins, who showed that any graphon is a subgraphon of a finitely forcible graphon. We strengthen this result by showing for every $\varepsilon > 0$ that any graphon spans a $1 - \varepsilon$ proportion of a finitely forcible graphon.

1 Introduction

The theory of graph limits is an emerging area of combinatorics, which offers analytic tools to study large graphs. The range of applications of analytic methods offered by the theory of graph limits has been constantly expanding. The most prominent examples of such applications come from the closely related flag algebra method of Razborov [32], which changed the landscape of extremal graph combinatorics by providing progress on numerous important problems in the area, e.g. [1–5, 17, 19–22, 30–34]. Among other applications of the methods provided by the theory, we would like to highlight those from computer science related to property and parameter testing algorithms [29]. We refer the reader to the recent monograph by Lovász [25] for further results.
In this paper, we are interested in limits of sequences of dense graphs. An analytic object representing a sequence of dense graphs is called a graphon. Formally, a graphon is a measurable function \( W \) from the unit square \([0,1]^2\) to the unit interval \([0,1]\) that is symmetric; i.e., \( W(x,y) = W(y,x) \) for every \((x,y) \in [0,1]^2\). Given a graphon \( W \), we can define the density \( d(H,W) \) of a graph \( H \) in \( W \) (we give the definition in Section 2). Every graphon is uniquely determined, up to weak isomorphism, by the densities of all graphs. The main objects of our study are finitely forcible graphons, which are graphons that are uniquely determined by the densities of finitely many graphs. We refer the reader to Section 2 for further introduction to these concepts.

Results on finitely forcible graphons can be found in disguise in various settings in graph theory. For example, a classical result of Thomason [35] (see also Chung, Graham and Wilson [7]) on quasirandom graphs is equivalent to saying that the constant graphon is finitely forcible by the densities of 4-vertex graphs. Another source of motivation for studying finitely forcible graphons comes from extremal graph theory. For example, Proposition 3 given in Section 2 says that a graphon \( W \) is finitely forcible if and only if there exists some linear combination of subgraph densities such that \( W \) is its unique minimizer.

Lovász and Szegedy [27] initiated a systematic study of properties of finitely forcible graphons and conjectured, based on examples of finitely forcible graphons known at that time, that all finitely forcible graphons must possess a simple structure.

**Conjecture 1** (Lovász and Szegedy, [27, Conjecture 9]). *The space of typical vertices of every finitely forcible graphon is compact.*

**Conjecture 2** (Lovász and Szegedy, [27, Conjecture 10]). *The space of typical vertices of every finitely forcible graphon has finite dimension.*

Conjectures 1 and 2 were disproved by counterexample constructions in [16] and [15], respectively. A stronger counterexample to Conjecture 2 was given in [9]; if true, Conjecture 2 would imply that the minimum number of parts of a weak \( \varepsilon \)-regular partition of a finitely forcible graphon is bounded by a power of \( \varepsilon^{-1} \). For the finitely forcible graphon constructed in [9], any weak \( \varepsilon \)-regular partition must have a number of parts almost exponential in \( \varepsilon^{-2} \) for infinitely many \( \varepsilon > 0 \), which is close to the general lower bound from [8]. This line of research culminated with the following general result of Cooper, Martins and the first author [10].

**Theorem 1.** *For every graphon \( W_F \), there exists a finitely forcible graphon \( W_0 \) such that \( W_F \) is a subgraphon of \( W_0 \) induced by a 1/14 fraction of the vertices of \( W_0 \).*

Theorem 1 yields counterexamples to Conjectures 1 and 2 and provides a universal framework for constructing finitely forcible graphons with very complex structure. In view of Proposition 3, Theorem 1 says that problems on minimizing a linear combination of subgraph densities, which are among the problems of the simplest kind in extremal graph theory, may have unique optimal solutions with highly complex structure. Given the general nature of Theorem 1, it is surprising [10] that the family of graphs whose densities force \( W_0 \) in Theorem 1 can be chosen to be independent of \( W_F \).

It is natural to ask whether the fraction 1/14 in Theorem 1 can be replaced by a larger quantity. The proof techniques from [10] allows replacing the fraction by any number smaller than 1/2. The purpose of this paper is to show that the fraction can be replaced by any number smaller than 1.
Theorem 2. For every $\varepsilon > 0$ and every graphon $W_F$, there exists a finitely forcible graphon $W_0$ such that $W_F$ is a subgraphon of $W_0$ induced by a $1 - \varepsilon$ fraction of the vertices of $W_0$.

The proof of Theorem 2 is based on the method of decorated constraints, which was introduced in [15,16], and uses Theorem 1 as one of the main tools. Informally speaking, Theorem 1 is used to embed the graphon $W_F$ on a small part of $W_0$ and other auxiliary structure of $W_0$ is then used to magnify the graphon $W_F$ to the $1 - \varepsilon$ fraction of the vertices of $W_0$. We remark that, in contrast to the proof of Theorem 1, the family of graphs used to force $W_0$ in Theorem 2 depends on $\varepsilon$, and we show in Section 3 that this dependence is necessary.

2 Preliminaries

We now introduce the notation and terminology used in the paper; our notation mostly follows that used in [10] in relation to graph limits. We start with some general notation. For $k \in \mathbb{N}$, $[k]$ denotes the set of integers $\{1,2,\ldots,k\}$. If $\mathcal{F}$ is a family of sets, we use $\bigcup \mathcal{F}$ to denote the union of all sets $F \in \mathcal{F}$. Unless stated otherwise, we work with the Lebesgue measure on $[0,1]^d$ throughout the paper. If $X \subseteq \mathbb{R}^d$ is a measurable set, we write $|X|$ for its measure and for two measurable sets $X,Y \subseteq \mathbb{R}^d$, and we write $X \subseteq Y$ to mean $|X \setminus Y| = 0$.

2.1 Graphs and graphons

The order of a graph $G$, which is denoted by $|G|$, is its number of vertices. The density of a graph $H$ in $G$, which is denoted $d(H,G)$, is the probability that a uniformly randomly chosen set of $|H|$ vertices of $G$ induces a graph isomorphic to $H$. If $|H| > |G|$, then we set $d(H,G)$ to zero.

Recall that a graphon is a measurable function $W$ from the unit square $[0,1]^2$ to the unit interval $[0,1]$ that is symmetric; i.e., $W(x,y) = W(y,x)$ for every $(x,y) \in [0,1]^2$. Conceptually, a graphon $W$ can be thought of as an infinite weighted graph on the vertex set $[0,1]$ with the edge $(x,y) \in [0,1]^2$ having the weight $W(x,y)$. Following this intuition, we refer to the points of $[0,1]$ as vertices. To visualize the structure of a graphon, we shall use a figure that may be seen as a continuous version of the adjacency matrix. More precisely, in a figure depicting $W$, the domain of $W$ is represented by the unit square $[0,1]^2$ with the origin in the top left corner and the values of $W$ are represented by appropriate shades of gray, with 0 corresponding to white and 1 to black.

A graphon can be associated with a probability distribution on graphs of a fixed order. Formally, for a graphon $W$ and an integer $k \in \mathbb{N}$, a $W$-random graph of order $k$ is a graph $G$ obtained by the following two step procedure. Let $x_1,\ldots,x_k \in [0,1]$ be $k$ points chosen uniformly and independently at random. Form a graph $G$ with the vertex set $[k]$ such that the edge $ij$ is present with probability $W(x_i,x_j)$ for every pair of distinct vertices $i,j \in [k]$. The density of a graph $H$ in the graphon $W$ is the probability that a $W$-random graph of order $|H|$ is isomorphic to $H$; we denote this quantity by $d(H,W)$. Note that $d(H,W)$ is also the expected density of $H$ in a $W$-random graph $G$ of order $k$ for every $k \geq |H|$.

Consider a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ such that the orders $|G_n|$ tend to infinity. We say that the sequence $(G_n)_{n \in \mathbb{N}}$ is convergent if for every graph $H$, the sequence of the densities of $H$ in $G_n$, i.e., the sequence $(d(H,G_n))_{n \in \mathbb{N}}$, is convergent. We say that the sequence $(G_n)_{n \in \mathbb{N}}$ converges to a graphon $W$ if

$$\lim_{n \to \infty} d(H,G_n) = d(H,W)$$
for every graph $H$. Lovász and Szegedy [28] showed that every convergent sequence of graphs converges to a graphon. Conversely, they showed that for every graphon, there exists a sequence of graphs converging to it. Indeed, the sequence of $W$-random graphs of increasing orders converges to $W$ with probability one.

For a graphon $W$ and a vertex $x \in [0, 1]$, we define the degree of $x$ in $W$ as

$$\deg_W(x) = \int_{[0,1]} W(x, y) \, dy.$$ 

Note that the degree is well-defined for almost every vertex $x \in [0, 1]$. We also define the neighbourhood of a vertex $x \in [0, 1]$ as the set $\{y \in [0, 1] : W(x, y) > 0\}$ and denote it $N_W(x)$. Note that the set $N_W(x)$ is measurable for almost every $x \in [0, 1]$. When the graphon $W$ is clear from context, we omit the subscripts. Finally, we define a graphon parameter $t(C_4, W)$ as follows:

$$t(C_4, W) = \int_{[0,1]^4} W(x_1, x_2)W(x_2, x_3)W(x_3, x_4)W(x_4, x_1) \, dx_1 \, dx_2 \, dx_3 \, dx_4.$$ 

Note that $t(C_4, W) = d(C_4, W)/3 + d(K_4^-, W)/3 + d(K_4, W)$, where $K_4^-$ is the graph obtained from $K_4$ by removing an edge. In other words, $t(C_4, W)$ is the probability that a randomly chosen 4-tuple forms a cycle of length four, in a specific order, in a $W$-random graph.

We say that two graphons $W_1$ and $W_2$ are weakly isomorphic if $d(H, W_1) = d(H, W_2)$ for every graph $H$. Clearly, weakly isomorphic graphons are limits of the same sequences of graphs. It is natural to ask how weakly isomorphic graphons can differ in their structure; this was answered in [3]. A function $\varphi : [0, 1] \to [0, 1]$ is measure-preserving if it is measurable and $|\varphi^{-1}(X)| = |X|$ for every measurable subset $X \subseteq [0, 1]$. It is easy to check that if $\varphi : [0, 1] \to [0, 1]$ is a measure-preserving map, then the graphon $W^\varphi$ defined as $W^\varphi(x, y) = W(\varphi(x), \varphi(y))$ is weakly isomorphic to $W$. In [3], it was shown in particular that two graphons $W_1$ and $W_2$ are weakly isomorphic if and only if there exist measure-preserving maps $\varphi_1, \varphi_2 : [0, 1] \to [0, 1]$ such that $W^{\varphi_1} = W^{\varphi_2}$ almost everywhere.

Let $W_1$ and $W_2$ be two graphons and $X \subseteq [0, 1]$ a non-null measurable set. We say that $W_1$ is a subgraphon of $W_2$ induced by $X$ if there exist measure-preserving maps $\varphi_1 : X \to [0, |X|]$ and $\varphi_2 : X \to X$ such that

$$W_1 (|X|^{-1} \cdot \varphi_1(x), |X|^{-1} \cdot \varphi_1(y)) = W_2 (\varphi_2(x), \varphi_2(y))$$ 

for almost every $(x, y) \in X \times X$.

A graphon $W$ is finitely forcible if there exist graphs $H_1, \ldots, H_m$ such that any graphon $W'$ satisfying that $d(H_i, W') = d(H_i, W)$ for every $i \in [m]$ is weakly isomorphic to $W$. A family of graphs $H_1, \ldots, H_m$ whose densities determine the graphon $W$ up to weak isomorphism is called a forcing family. Examples of a finitely forcible graphons include constant graphons, step graphons [26] and the half-graphon [11,27]. We remind that a step graphon is a graphon $W$ such that there exists a partition of $[0, 1]$ into non-null measurable sets $U_1, \ldots, U_k$ such that $W$ is constant on $U_i \times U_j$ for every $i, j \in [k]$, and the half-graphon is the graphon $W_\Delta$ such that $W_\Delta(x, y) = 1$ if $x + y \geq 1$ and $W_\Delta(x, y) = 0$ otherwise. The following proposition provides a link between finitely forcible graphons and extremal graph theory.
Proposition 3. A graphon \( W \) is finitely forcible if and only if there exist graphs \( H_1, \ldots, H_m \) and reals \( \alpha_1, \ldots, \alpha_m \) such that for every graphon \( W' \),
\[
\sum_{i=1}^{m} \alpha_i d(H_i, W) \leq \sum_{i=1}^{m} \alpha_i d(H_i, W') ,
\]
and equality holds if and only if \( W \) and \( W' \) are weakly isomorphic.

On the other hand, it is not the case that every linear combination of subgraph densities has a unique minimizer (e.g. \( d(K_3, W) \) is minimized by any triangle-free graphon \( W \)). However, Lovász \([23–25, 27]\) conjectured that one can always add further density constraints to make the solution unique. Specifically, he conjectured the following. Let \( \omega \) and \( \alpha \) and reals \( \ell, \ldots, d \) graphs and \( \nu \) measures has a unique minimizer (e.g. \( d \)) for any measurable set \( A \). Since it holds that \( \lVert \cdot \rVert_1 \) \( \nu \) is defined as \( \nu(X) = \int_X g \). Note that \( \lVert g \rVert_1 = \lVert f \rVert_1 \). Furthermore, if a function \( g \) satisfies
\[
\int_{[0,1]^2} g(x) (1 - W^\varphi(x,y)) g(y) \, dx \, dy = 0 ,
\]
then the corresponding function \( f \) satisfies that
\[
\int_{[0,1]^2} f(x) (1 - W(x,y)) f(y) \, dx \, dy = 0 .
\]
This implies \( \omega(W) \geq \omega(W^\varphi) \), and we can conclude that \( \omega(W) = \omega(W^\varphi) \) as desired. \( \square \)
We close this subsection with a well-known measure-theoretic result which we will apply throughout the paper. It follows from the Monotone Reordering Theorem [25, Proposition A.19] and the fact that for measurable subsets from the fact that any standard probability space is isomorphic to the unit interval.

**Proposition 5.** Let $X$ be a measurable subset of $[0,1)$ and $h : X \to \mathbb{R}$ a measurable function. There exists a measure-preserving map $\varphi : X \to [0,|X|)$ and a non-decreasing function $f : [0,|X|) \to \mathbb{R}$ such that $h(x) = f(\varphi(x))$ for almost every $x \in X$.

### 2.2 Partitioned graphons and decorated constraints

The most direct way of showing that a graphon $W$ is finitely forcible is by explicitly providing the forcing family of graphs $H_1, \ldots, H_m$ and their densities $d_1, \ldots, d_m$ and analyzing all graphons $W'$ such that $d(H_i, W') = d_i$. However, this approach often becomes impractical when $m$ is very large and, even more so, when $H_1, \ldots, H_m$ depend on $\varepsilon$, as is required to prove Theorem 2. We now introduce the method of decorated constraints that was developed in [15, 16], which allows us to use more advanced constraints to establish that a graphon is finitely forcible.

A **density expression** is a formal polynomial in graphs; i.e., graphs and real numbers are density expressions, and if $D_1$ and $D_2$ are density expressions, then so are $D_1 + D_2$ and $D_1 \cdot D_2$. A density expression $D$ can be evaluated with respect to a graphon $W$ by replacing every graph $H$ in $D$ with $d(H, W)$. A **constraint** is an equality between two density expressions and it is satisfied by a graphon $W$ if both density expressions evaluated with respect to $W$ are equal. A simple example of a constraint is the equality $H = d$, which is satisfied by a graphon $W$ if and only if $d(H, W) = d$.

If $C$ is a finite set of constraints and $W$ is the unique graphon, up to weak isomorphism, that satisfies all of the constraints in $C$, then $W$ is finitely forcible. Indeed, $W$ is the unique graphon, up to weak isomorphism, with the density of $H$ equal to $d(H, W)$ for all graphs $H$ appearing in a constraint in $C$. We will often say that the constraints contained in $C$ force the graphon $W$.

A graphon $W$ is **partitioned** if there exist positive reals $a_1, \ldots, a_k$ summing to one and distinct reals $d_1, \ldots, d_k \in [0,1]$ such that the set $A_i$ of vertices of $W$ with degree $d_i$ has measure $a_i$ for all $i \in [k]$. The sets $A_i$ are called **parts** and the **degree** of a part $A_i$ is $d_i$. We will abuse the notation here and if $W$ and $W'$ are two partitioned graphons with parts of measures $a_i$ and degrees $d_i$, we will use the same letters to denote the corresponding parts of $W$ and $W'$. This is technically incorrect since the part $A_i$ can be a different subset of $[0,1]$ in $W$ and $W'$ but we will make sure that the graphon that we have in mind is always clear from the context. The property of being a partitioned graphon can be forced in the following sense; see [16, Lemma 2] for a proof.

**Lemma 6.** Let $a_1, \ldots, a_k$ be positive reals summing to one and $d_1, \ldots, d_k$ distinct reals from $[0,1]$. There exists a finite set of constraints $C$ such that a graphon $W$ satisfies all constraints in $C$ if and only if $W$ is a partitioned graphon with parts of measures $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$.

Consider a partitioned graphon $W$ and let $\mathcal{P}$ be the set of its parts. The **relative degree** of a vertex $x \in [0,1]$ in $W$ with respect to a set $\mathcal{X} \subseteq \mathcal{P}$ of parts is defined as

$$\deg_{W}^{\mathcal{X}}(x) = \left| \bigcup \mathcal{X} \right|^{-1} \cdot \int_{\bigcup \mathcal{X}} W(x, y) \, dy.$$
Similarly, the relative neighbourhood of \( x \in [0,1] \) with respect to \( \mathcal{X} \), which is denoted by \( N^X_W(x) \), is the set \( N_W(x) \cap \bigcup \mathcal{X} \). If \( \mathcal{X} = \{X\} \) for some part \( A \), then we simply write \( \deg^X_W(x) \) and \( N^X_W(x) \). As before, if the graphon \( W \) is clear from context, then we omit the subscripts. For two non-empty subsets \( \mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{P} \), the restriction of the graphon \( W \) to \( \bigcup \mathcal{X}_1 \times \bigcup \mathcal{X}_2 \) will be referred to as the tile \( \mathcal{X}_1 \times \mathcal{X}_2 \). If both \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are singular, we call the tile simple; otherwise, it is composite.

We now introduce a formally stronger (but technically equivalent) version of constraints, which we call decorated constraints. These are similar to decorated constraints used in \([9,10,15,16]\) except that we will allow vertices of graphs appearing in constraints to be assigned to multiple parts, as opposed to just a single part as in \([9,10,15,16,18]\). We discuss the difference in more detail further. We will always have a particular set of parts in mind when working with decorated constraints. So, fix a partition \( \mathcal{P} \). A decorated graph \( G \) is a graph with \( 0 \leq m \leq |G| \) distinguished vertices labelled from 1 to \( m \), which are called roots, and with every vertex \( v \) (including the roots) assigned a non-empty subset of \( \mathcal{P} \), which is called the decoration of \( v \). If the decoration of a vertex is a single element set, \( \{A\} \), we just write \( A \) as the decoration to simplify our notation. Two decorated graphs \( G_1 \) and \( G_2 \) are compatible if the subgraphs induced by their roots are isomorphic, respecting both the labels of roots and the decorations assigned to them. A decorated density expression is a formal polynomial in decorated graphs such that all graphs in the expression are mutually compatible and a decorated constraint is an equality between two decorated density expressions such that all graphs in the expression are mutually compatible.

Let \( W \) be a partitioned graphon with parts \( \mathcal{P} \) and \( C \) a decorated constraint of the form \( D = 0 \) where \( D \) is a decorated density expression. We now describe what we mean when we say that the graphon \( W \) satisfies \( C \). Let \( H_0 \) be the decorated graph induced by the roots \( v_1, \ldots, v_m \) of the decorated graphs in \( C \). Call an \( m \)-tuple \( (x_1, \ldots, x_m) \in [0,1]^m \) feasible if each \( x_i \) belongs to one of the parts that \( v_i \) is decorated with, \( W(x_i, x_j) > 0 \) for every edge \( v_iv_j \in E(H_0) \) and \( W(x_i, x_j) < 1 \) for every non-edge \( v_iv_j \notin E(H_0) \). Given a feasible \( m \)-tuple \( (x_1, \ldots, x_m) \in [0,1]^m \), the evaluation of \( D \) at the \( m \)-tuple is obtained by replacing each decorated graph \( H \) with the probability that a \( W \)-random graph of order \( |H| \) is the graph \( H \), conditioned on the event that the roots are chosen as the vertices \( x_1, \ldots, x_m \) and they induce the graph \( H_0 \), and that each non-root vertex is chosen from the union of the parts in its decoration. The graphon \( W \) satisfies the constraint \( C \) if for almost every feasible \( m \)-tuple, the evaluation of \( D \) is equal to zero. We say that \( W \) satisfies a decorated constraint of the form \( D = D' \) if it satisfies \( D - D' = 0 \).

We next describe a convention of depicting decorated constraints that we use in this paper, which is analogous to that used in \([9,10]\). The roots of decorated graphs will be represented by squares and the non-root vertices by circles. The decoration of every vertex will be depicted as a label inside the square or circle. If \( X \) is any letter such that \( X_1, X_2, \ldots \) are parts, then \( X_* \) will denote the label \( \{X_1, X_2, \ldots\} \). For example, if \( B_1, B_2 \) are all the parts with the letter \( B \), then \( B_* \) will refer to the label \( \{B_1, B_2\} \). The roots in all decorated graphs appearing in a constraint will be placed on the same mutual positions; i.e., the corresponding roots of different graphs in the constraint are on the same respective positions. Edges are represented as solid lines between vertices and non-edges are represented as dashed lines. The absence of any line between two root vertices indicates that the constraint should hold for both cases when the edge between the root vertices is present and when it is not present. Finally, the absence of a line between a non-root vertex and another vertex represent the sum of decorated graphs with this edge present and without this edge. Thus, if \( k \) such lines are absent in a
We now give an example, depicted in Figure 1. We consider the graphon $W$ depicted in the left part of the figure: the graphon $W$ has three parts $A$, $B_1$ and $B_2$, each of measure $1/3$. The densities between the parts are as given in the figure. In particular, the degree of $A$ is $2/3$, the degree of $B_1$ is $11/18$ and the degree of $B_2$ is $1$. We next consider the decorated graph $H$ depicted in the right part of Figure 1. The graph $H$ has two roots $v_1$ and $v_2$ that are adjacent and decorated with $B_1$ and $A$, respectively, and it has two non-root vertices $v_3$ and $v_4$ that are also adjacent and decorated with $\{B_1, B_2\}$ (denoted $B_*$) and $A$, respectively. The vertex $v_3$ is adjacent to both roots and $v_4$ is adjacent only to $v_1$. The probability described in the previous paragraph is independent of the choice of $x_1$ and $x_2$ in $B_1$ and $A$ and is equal to $13/96$. In particular, the graphon $W$ satisfies the decorated constraint $H = 13/96$ depicted in Figure 1.

In [16, Lemma 3], it was shown that decorated constraints where each vertex is decorated with a single element set are equivalent to (ordinary) constraints. Let us call such decorated constraints simple; i.e., a decorated constraint is simple if all decorations appearing in it are single element sets. However, each decorated constraint is equivalent to a set of simple decorated constraints: if a decorated graph $H$ contains a non-root vertex $v$ decorated by a set of parts $X$, we may replace $H$ with a convex combination of graphs $H$ decorated by elements of $X$, where the coefficients are proportional to the measures of the parts from $X$. If one of the roots, say $v$, appearing in a decorated constraint is decorated by a set $X$, we consider all decorated constraints with $v$ labelled by elements of $X$. In this way, we can convert any decorated constraint to an equivalent set of simple decorated constraints. Hence, we can conclude, using [16, Lemma 3], that the following holds.

**Proposition 7.** For every decorated constraint $C$, there exists a finite collection of constraints $C'$ such that a partitioned graphon $W$ satisfies $C$ if and only if it satisfies $C'$.

We next describe how decorated constraints can be used to embed a finitely forcible graphon inside another graphon. Suppose that $W_0$ is a finitely forcible graphon that is forced by constraints $H_i = d_i$ for $i \in [k]$, where $H_1, \ldots, H_k$ are graphs and $d_1, \ldots, d_k$ are their densities. If $W$ is a partitioned graphon and $A$ one of its parts, replacing each $H_i$ with the decorated graph where each vertex is decorated with $A$ results in a set of constraints that are satisfied if and only if the subgraphon of $W$ induced by $A$ is weakly isomorphic to $W_0$. The same holds if instead of a single part $A$ we consider a set of parts. We state this observation as a separate lemma.

**Lemma 8.** Let $W_0$ be a finitely forcible graphon, $\mathcal{P}$ a set of parts and $\mathcal{X} \subseteq \mathcal{P}$. There exists a finite set $\mathcal{C}$ of decorated constraints such that every partitioned graphon $W$ with parts $\mathcal{P}$
satisfies $C$ if and only if the subgraphon of $W$ induced by $\bigcup X$ is weakly isomorphic to $W_0$.

We conclude this subsection by stating a lemma, which appeared implicitly in [27] proof of Lemma 2.7 or Lemma 3.3 and was explicitly stated in [9] Lemma 8.

**Lemma 9.** Let $X, Z \subseteq \mathbb{R}$ be measurable non-null sets and let $F: X \times Z \to [0,1]$ be a measurable function. If there exists a constant $C \in \mathbb{R}$ such that

$$\int_Z F(x,z)F(x',z) \, dz = C$$

for almost every $(x, x') \in X^2$, then it holds that

$$\int_Z F(x,z)^2 \, dz = C$$

for almost every $x \in X$.

## 3 Main Proof

In this section, we prove Theorem 2. For technical reasons, it is easier to consider graphons as functions from $[0,1)^2$ rather than $[0,1]^2$, and we will do so throughout the section. Note that this change affects a graphon on a set of measure zero only. We will also work with real intervals of the type $[a,b)$, which we call *half-open*.

For the proof of Theorem 2, fix a graphon $W_F$ and $\varepsilon > 0$. We can assume that $\varepsilon = \frac{1}{2^r - 1}$ for an integer $r$, and that almost every vertex of $W_F$ has degree less than one. If either assumption does not hold, choose $\varepsilon' < \varepsilon$ such that $\frac{1}{2^r - 1}$ is a power of two and apply the theorem with the graphon $W_F'$ such that $W_F'(x,y) = W_F\left(\frac{1-\varepsilon'}{1-\varepsilon} \cdot x, \frac{1-\varepsilon'}{1-\varepsilon} \cdot y\right)$ for $(x,y) \in [0,\frac{1-\varepsilon}{1-\varepsilon'}]^2$ and $W_F'(x,y) = 0$ elsewhere.

Set $M = 4\left(\frac{1}{2^r} - 1\right)$ and $m = \log_2 M$. By applying the Monotone Reordering Theorem and, if needed, changing the graphon $W_F$ on a set of measure zero, we can assume that there exists a partition of $[0,1)$ into half open intervals $Q_1, \ldots, Q_M$ such that the degree of every vertex $x$ of $W_F$ contained in $Q_k$, $k \in [M]$, belongs to the interval $[(k-1)/M, k/M)$ and the subinterval $Q_k$ precedes $Q_{k+1}, \ldots, Q_M$ for every $k \in [M]$. Note that some of the subintervals $Q_k$ can be empty.

### 3.1 Overview of $W_0$

We next provide a description of the general structure of the graphon $W_0$ and present the detailed definition of individual tiles throughout this section together with the decorated constraints enforcing its structure. We also refer the reader to Figure 2 where the graphon $W_0$ is visualized, and to Table 1 which provides references to subsections where individual tiles are forced. The graphon $W_0$ is a partitioned graphon with

- $M$ parts $A_1, \ldots, A_M$,
- $M + 9$ parts $B_A, \ldots, B_F, B_{G,1}, \ldots, B_{G,M}, B_P, B_Q, B_R$,
- $m + 1$ parts $C_1, \ldots, C_m$ and $C_\infty$,
Table 1: References to the subsections where the corresponding tiles are analyzed.

|    | \(A_s\) | \(B_s\) | \(C_s\) | \(D_s\) | \(E_s\) | \(F_s\) | \(G_1\) | \(G_2\) |
|----|---------|---------|---------|---------|---------|---------|---------|---------|
| \(A_s\) | 3.8     | 3.9     | 3.7     | 3.7     | 3.5     | 3.4     | 3.9     | 3.10    |
| \(B_s\) | 3.9     | 3.3     | 3.7     | 3.7     | 3.5     | 3.4     | 3.9     | 3.10    |
| \(C_s\) | 3.7     | 3.7     | 3.6     | 3.6     | 3.5     | 3.4     | 3.9     | 3.10    |
| \(D_s\) | 3.7     | 3.7     | 3.6     | 3.6     | 3.5     | 3.4     | 3.9     | 3.10    |
| \(E_s\) | 3.5     | 3.5     | 3.5     | 3.5     | 3.5     | 3.4     | 3.9     | 3.10    |
| \(F_s\) | 3.4     | 3.4     | 3.4     | 3.4     | 3.4     | 3.4     | 3.9     | 3.10    |
| \(G_1\) | 3.9     | 3.9     | 3.9     | 3.9     | 3.9     | 3.9     | 3.9     | 3.10    |
| \(G_2\) | 3.10    | 3.10    | 3.10    | 3.10    | 3.10    | 3.10    | 3.10    | 3.10    |

- \(m + 1\) parts \(D_1, \ldots, D_m\) and \(D_\infty\),
- \(m\) parts \(E_1, \ldots, E_{m-1}\) and \(E_\infty\),
- \(M\) parts \(F_1, \ldots, F_M\), and
- two parts \(G_1\) and \(G_2\).

In total, \(W_0\) has \(3M + 3m + 13\) parts, and the set of the parts contained in each of the seven groups above is denoted \(A_s, B_s, \ldots, G_s\), respectively, following the notation that we have introduced for visualizing decorated constraints. The set of all \(3M + 3m + 13\) parts of \(W_0\) is denoted \(\mathcal{P}\), i.e., \(\mathcal{P} = A_s \cup B_s \cup \cdots \cup G_s\). We will also use \(B_{G_\ast}\) to denote the set containing the parts \(B_{G,1}, \ldots, B_{G,M}\), and \(|X_s|\) to denote the measure of the union of the parts contained in \(X_s\) for \(X \in \{A, B, \ldots, G\}\). For some arguments that we present, it may be convenient to think of parts contained in each of the groups as a single part. Indeed, the parts contained in the same group serve a similar purpose. However, note that in order to apply Lemma 6 all parts must have similar degrees. For this reason, we will have to divide them further so that we are able to perform the degree balancing in Subsection 3.9. Since we only have vertices of measure \(\varepsilon\) to balance degrees, the number of parts we will need depends on \(\varepsilon\).

We now describe the actual structure of the graphon \(W_0\). Each part \(X \in \mathcal{P}\) of the graphon is a half-open subinterval of \([0, 1)\) with measure given in Table 2 and these subintervals follow the order in which they were listed when we introduced the parts of \(W_0\). For the rest of this section, the part \(X \in \mathcal{P}\) of \(W_0\), i.e., the subinterval of \([0, 1)\) forming this part in \(W_0\), will be denoted by \(X^0\). This allows us to clearly distinguish the subintervals of \([0, 1)\) forming the parts of \(W_0\) from the parts with the same name in other graphons that we will consider. We will also use \(A_0^s, \ldots, G_0^s\) to denote the unions of the subintervals associated with the parts contained in \(A_s, \ldots, G_s\), respectively.

The graphon \(W_F\) is contained on the composite tile \(A_s \times A_s\) and we set \(W_0((1 - \varepsilon)x, (1 - \varepsilon)y) = W_F(x, y)\) for every \((x, y) \in [0, 1)^2\). In this way, the part \(Q_k\) of the graphon \(W_F\) corresponds to the part \(A_k\) of the graphon \(W_0\) for every \(k \in [M]\). The graphon \(W_0\) outside the composite tile \(A_s \times A_s\) will be defined in the following subsections, and we use the convention that when the value \(W_0(x, y)\) is defined, the definition also sets the value \(W_0(y, x)\).

The parts contained in \(B_s \cup \cdots \cup F_s\) of the graphon \(W_0\) are used to enforce its structure, and the parts \(G_1\) and \(G_2\) are used to balance the degrees inside the parts. For each part \(X \in \mathcal{P}\) besides \(G_1\) and \(G_2\), we define a real number \(\text{pre-deg}(X)\), which we call the pre-degree...
Figure 2: A sketch of the graphon $W_0$. The composite tile $A_s \times A_s$ contains the graphon $W_F$ and the composite tile $B_s \times B_s$ contains the graphon $\tilde{W}_0$ from Theorem 10. The details of the structure between the parts contained in $A_s$ and $B_{G,s}$ and the parts contained in $C_s, D_s, E_s$ and $F_s$ are given in Figure 3.

Figure 3: The structure of the graphon $W_0$ between the parts contained in $A_s$ and $B_{G,s}$ and the parts contained in $C_s, D_s, E_s$ and $F_s$. 
\begin{table}
| Part       | Measure | Pre-Degree |
|------------|---------|------------|
| $A_k$      | $\frac{1}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $B_{G,k}$  | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $B_{A,k}, B_{F,k}, B_{P,k}, B_{R,k}$ | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $C_k$      | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $D_k$      | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $E_k$      | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $F_k$      | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $G_1$      | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $G_2$      | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $G_1, D_\infty$ | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |
| $G_2$      | $\frac{\varepsilon}{2^k}$ | $\frac{\varepsilon}{4}$ |

Table 2: The sizes and the pre-degrees of the parts of the graphon $W_0$. 

of $X$. These numbers are given in Table 2. The definition of the graphon $W_0$ will ensure that

$$
\int_{[0,1) \setminus G_2^0} W_0(x, z) \, dz = \text{pre-deg}(X)
$$

(1)

for every $x \in X^0$; further details are given in Subsection 3.9. We next fix an irrational number $\delta_X \in (0, \varepsilon/4)$ for each part $X \in \mathcal{P}$ such that the numbers $\delta_X$, $X \in \mathcal{P}$, are rationally independent; in particular, all the numbers $\delta_X$, $X \in \mathcal{P}$, are mutually distinct. The part $G_2$ will be used to distinguish different parts of the graphons by guaranteeing that the degree of each part $X \in \mathcal{P} \setminus \{G_1, G_2\}$ is $\text{pre-deg}(X) + \delta_X$. The graphon $W_0$ is constant on each tile $X \times G_2$, $X \in \mathcal{P}$, and the sole purpose of these tiles is to guarantee that different parts have distinct degrees.

We conclude this subsection by defining a notation that will be convenient in our exposition. Let $X^0$ be a non-empty set of parts of $W_0$ such that $\bigcup X^0$ is a half-open interval. We define a mapping $\gamma_X : [0, 1) \to \bigcup X^0$ as

$$
\gamma_X(x) = x \cdot \left| \bigcup X^0 \right| + \min \bigcup X^0.
$$

(2)

For example, $W_0(\gamma_{A_0}(x), \gamma_{A_0}(y)) = W_F(x, y)$ for every $x, y \in [0, 1)$. If $X^0 = \{X^0\}$, we will just write $\gamma_X$ instead of $\gamma_{\{X\}}$.

### 3.2 Universal graphon

In this subsection, we revisit the construction of the graphon $W_0$ from Theorem 1 given in [10]. In the proof of Theorem 2, we apply Theorem 1 with the same graphon $W_F$ for which we are proving Theorem 2. To distinguish the graphons $W_0$ from Theorems 1 and 2, we will be using $\tilde{W}_0$ for the graphon from Theorem 1. The graphon $\tilde{W}_0$ obtained in this way is visualized in Figure 3 and we now review some of the properties of the graphon $\tilde{W}_0$ and the proof of its finite forcibility given in [10].

**Theorem 10.** The graphon $\tilde{W}_0$ is a partitioned graphon with 10 parts $\tilde{A}, \ldots, \tilde{G}$, $\tilde{P}, \tilde{Q}$ and $\tilde{R}$ that has the following properties in particular.
Figure 4: The graphon $\widetilde{W}_0$ constructed in [10].
(a) The parts $\tilde{A}, \ldots, \tilde{G}$, $\tilde{P}$ and $\tilde{R}$ are half-open intervals $[0/14, 1/14)$, $[6/14, 7/14)$, $[7/14, 8/14)$ and $[13/14, 14/14)$, respectively. In particular, each of these parts has measure $1/14$.

(b) The part $\tilde{Q}$ is $[8/14, 13/14)$, i.e., its measure is $5/14$.

(c) It holds that
\[
\tilde{W}_0 \left( \frac{6 + x}{14}, \frac{6 + y}{14} \right) = W_F(x, y)
\]
for every $(x, y) \in [0, 1)^2$.

(d) It holds that
\[
\tilde{W}_0 \left( \frac{6 + x}{14}, \frac{7 + y}{14} \right) = W_\Delta(x, y)
\]
for every $(x, y) \in [0, 1)^2$, where $W_\Delta$ is the half-graphon defined in Section 2.

(e) For every graphon $W$ that is weakly isomorphic to $\tilde{W}_0$, there exists a measure-preserving map $\tilde{g} : [0, 1) \to [0, 1)$ such that
\[
W(x, y) = \tilde{W}(\tilde{g}(x), \tilde{g}(y))
\]
for almost every $(x, y) \in [0, 1)^2$.

We use the graphon $\tilde{W}_0$ to define $W_0$ on the composite tile $B_* \times B_*$ by setting
\[
W_0(\gamma_{B_*}(x), \gamma_{B_*}(y)) = \tilde{W}_0(x, y)
\]
for every $[0, 1)^2$. In this way, the vertices of the parts $\tilde{A}, \ldots, \tilde{F}, \tilde{P}, \tilde{Q}$ and $\tilde{R}$ of the graphon $\tilde{W}_0$ correspond to the vertices of the parts $B^0_A, \ldots, B^0_F, B^0_P, B^0_Q$ and $B^0_R$ of the graphon $W_0$, respectively, and the vertices of the part $\tilde{G}$ to the union of the vertices of $B^0_G \cup \cdots \cup B^0_{G,M}$.

3.3 General structure of $W_0$

In this subsection, we provide an overview of the constraints that witness the finite forcibility of $W_0$, and use some of them to establish the general structure of any graphon satisfying them. The constraints that we use are the following:

(a) the constraints given in Lemma 6 such that any graphon satisfying them is partitioned graphon with parts $P$ that have the same degrees and measures as those of $W_0$,

(b) the decorated constraints from Lemma 8 applied to the graphon $\tilde{W}_0$ and with $X = B_*$, and

(c) the decorated constraints that we present in this and the following subsections of this section.

Suppose that $W$ is a graphon satisfying all these constraints. We will construct a particular measure-preserving map $g : [0, 1) \to [0, 1)$ and prove that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in [0, 1)^2$. 
for almost every \( x \) that \( P \) each part of The Monotone Reordering Theorem implies that for each part \( X \), there is a measure-preserving map \( \varphi_X : X \rightarrow [0,|X|) \) and a non-decreasing function \( f_X : [0,|X|) \rightarrow \mathbb{R} \) such that

\[
f_X(\varphi_X(x)) = \deg^F_{\tilde{W}_0}(x)
\]

for almost every \( x \in X \). Theorem \textbf{10} implies that there exists a measure-preserving map \( \tilde{g} : B_* \rightarrow [0,|B_*|) \) such that

\[
W(x, y) = \tilde{W}_0\left(\frac{\tilde{g}(x)}{|B_*|}, \frac{\tilde{g}(y)}{|B_*|}\right)
\]

for almost every \((x, y) \in B_* \times B_*\). We next define the mapping \( g : [0,1) \rightarrow [0,1) \) as follows.

\[
g(x) = \begin{cases} 
\gamma_{B_*}(\tilde{g}(x)/|B_*|) & \text{if } x \text{ belongs to a part contained in } B_* \text{, and} \\
\gamma_X(\varphi_X(x)/|X|) & \text{if } x \text{ belongs to a part } X \not\in B_*.
\end{cases}
\]

Recall that our goal is to show that \( W(x, y) = W_0(g(x), g(y)) \) for almost every \((x, y) \in [0,1)^2\). Note that the definition of \( g \) directly implies that \( W(x, y) = W_0(g(x), g(y)) \) for almost every \((x, y) \in B_* \times B_*\).

The definition of the function \( g \) implies that each part \( X \not\in B_* \) of \( W \) is mapped to the part \( X^0 \) of \( W_0 \) by \( g \), however, this property is not implied by the definition for the parts \( X \in B_* \). We now introduce decorated constraints that will guarantee this; these constraints are depicted in Figure \textbf{5}. Recall that for almost every \( x, y \in B_* \times B_* \), we \( W(x, y) = \tilde{W}_0\left(\frac{\tilde{g}(x)}{|B_*|}, \frac{\tilde{g}(y)}{|B_*|}\right) \). The first constraint in the figure implies that for every \( X \in \{A,\ldots,F, P, Q, R\} \), each vertex of the part \( B_X \) of the graphon \( W \) belongs to the part \( \tilde{X} \) of the graphon \( \tilde{W}_0 \), which is embedded in the composite tile \( B_* \times B_* \). Likewise, the second constraint implies that each vertex of one of the parts of \( B_{G,i} \) belongs to the part \( \tilde{G} \) of the graphon \( \tilde{W}_0 \). It follows that for each \( X \in \{A,\ldots,F, P, Q, R\} \), the part \( B_X \) of \( W \) is mapped by \( g \) to the part \( B^0_X \) of \( W_0 \), and that \( B_{G,i} \) is mapped to \( B^0_{G,i} \). Note that we have not yet proven that each \( B_{G,i} \) is mapped to \( B^0_{G,i} \); we will do so in the next section.

\[\text{Figure 5: Set-decorated constraints aligning the parts contained in } B_* \]. The value of \( X \) ranges among \( A,\ldots,F, P, Q, R \), the value of \( d_Z \) is equal to the degree of the part \( Z \) in the graphon \( \tilde{W}_0 \), \( Z \in \{A,\ldots,G, P, Q, R\} \).

3.4 Coordinate system

In this subsection, we introduce some structure of the graphon \( W_0 \) that allows us to define a coordinate system inside most of its parts, following similar arguments used in \textbf{9, 10, 15}. 

\[
\begin{align*}
B_* &= d_X \\
B_{G,i} &= d_G
\end{align*}
\]
By Lemma 8, there exist decorated constraints such that the composite tile $B$ almost every $(x,y)$ with $i,j \in [M]$ such that $i + j \leq M$, the second for all $i,j \in [M]$ such that $i + j \geq M + 2$, and the last for all $X \in \{B_A, \ldots , B_F, B_P, B_Q, B_R\}$.

Since the graphon on the composite tile $\gamma$ one if $X \in \{A, B_G\}$, with $i,j \in [M]$ and with $i,j \in [m-1] \cup \{\infty\}$ if $X = E$ (using the convention that $i < \infty$ for every $i \in \mathbb{N}$).

By Lemma 8, there exist decorated constraints such that the composite tile $F_* \times F_*$ is weakly isomorphic to the half-graphon, which we introduced in Section 2. In addition to the constraints given by Lemma 8, we also include the constraints depicted in Figure 6.

The first constraint in Figure 6 implies that each of the tiles $F_i \times F_j$ with $i,j \in [M]$ and $i + j \leq M$ is equal to zero almost everywhere and the second constraint implies that each of the tiles $F_i \times F_j$ with $i,j \in [M]$ and $i + j \geq M + 2$ is equal to one almost everywhere. Since the graphon on the composite tile $F_0 \times F_0$ is weakly isomorphic to the half-graphon by Lemma 8, the choice of $\varphi_X$ for $X \in F_*$ yields that $W(x,y) = W_0(g(x),g(y))$ for almost every $(x,y) \in F_* \times F_*$. The composite tiles $X^0 \times F_*$ of $W_0$ are equal to zero for $X \in \{B_A, \ldots , B_F, B_P, B_Q, B_B\}$ and this is enforced by the last constraint Figure 6. In particular, it holds that $W(x,y) = W_0(g(x),g(y))$ for almost every $(x,y) \in F_* \times F_*$ for $X \in \{B_A, \ldots , B_F, B_P, B_Q, B_B\}$.

Next fix $X \in \{A, B_G, C, D, E\}$. For $(x,y) \in X^0 \times F^0$, we define $W_0(x,y)$ to be equal to one if $\gamma_X^{-1}(x)/|X_*| + \gamma_X^{-1}(y)/|F_*| \geq 1$, and equal to zero otherwise. The first two constraints in Figure 7 enforce that the composite tile $X_* \times F_*$ is a scaled half-graphon. The last constraint guarantees that the degrees relative to $F_*$ of the vertices contained in the parts of $X_*$ are ordered in the same way in $W$ as in $W_0$. In particular, this implies that $g$ maps the part $B_{G,i}$ to the part $B_{G,i}^0$ for every $i \in [M]$. The choice of $\varphi_X$ for $X \in A_* \cup C_* \cup D_* \cup E_*$ yields that $W(x,y) = W_0(g(x),g(y))$ for almost every $(x,y) \in X \times F_*$. Finally, since $g$ maps the part $B_{G,i}$ to the part $B_{G,i}^0$ for every $i \in [M]$, we also obtain that $W(x,y) = W_0(g(x),g(y))$ for almost every $(x,y) \in B_{G,i} \times F_*$. 

Figure 6: Decorated constraints forcing the structure of some of the tiles involving parts from $F_*$. The first constraint should hold for all $i, j \in [M]$ such that $i + j \leq M$, the second for all $i, j \in [M]$ such that $i + j \geq M + 2$, and the last for all $X \in \{B_A, \ldots , B_F, B_P, B_Q, B_R\}$.

Figure 7: Decorated constraints forcing the structure of the rest of the tiles involving parts from $F_*$. The constraints should hold for $X \in \{A, B_G, C, D, E\}$. The last constraint should hold for all $i < j$ with $i, j \in [M]$ if $X \in \{A, B_G\}$, with $i,j \in [M] \cup \{\infty\}$ if $X \in \{C, D\}$ and with $i,j \in [m-1] \cup \{\infty\}$ if $X = E$ (using the convention that $i < \infty$ for every $i \in \mathbb{N}$).
3.5 Checker tiles

The checker graphon $W_C$ is the graphon defined as follows; the graphon is also depicted in Figure 8. Let $I_k$ denote the interval $[1 - 2^{-k+1}, 1 - 2^{-k})$ for $k \in \mathbb{N}$. Set $W_C(x, y) = 0$ if $x$ and $y$ belongs to the same interval $I_k$ for some $k \in \mathbb{N}$, and $W_C(x, y) = 0$ otherwise.

We now define the graphon $W_0$ on the tiles involving the parts from $E_*$. For $X \in \{A_1, \ldots, A_M, B_{G,1}, \ldots, B_{G,M}, C_*, D_*, E_*\}$, set $W_0(\gamma_{E_*}(x), \gamma_X(y)) = W_C(x, y)$ for all $(x, y) \in [0, 1)^2$. We also set $W(x, y) = 0$ for all $(x, y) \in E_* \times X$ where $X \in B_* \setminus B_{G*}$.

![Figure 8: The checker graphon $W_C$.](image)

![Figure 9: The decorated constraints forcing the structure of the composite tile $E_* \times E_*$.](image)

We next consider the constraints depicted in Figures 9 and 10. Since the arguments follow the lines of those presented in [10], we present them here on a general level and refer the reader for further details to [10, Section 3.2]. The first constraint on the first line in Figure 9 implies that there exists a collection $J$ of disjoint measurable subsets of $[0, 1)$ such that the following holds for almost every $(x, y) \in E_* \times E_*$: $W(x, y) = 1$ if and only if $\gamma_{E_*}(x)$ and $\gamma_X(y)$ belong to the same $J \in J$. The second constraint on the first line implies that each $J \in J$ differs from an interval on a set of measure zero. The first constraint on the second line yields that the length $|J|$ of each interval $J \in J$ is equal to $1 - \sup J$. Finally, the remaining constraint is equivalent to saying that

$$
\sum_{J \in J} |J|^2 = \frac{1}{3}.
$$
Together with the fact that $|J| = 1 - \sup J$, this implies that, up to changing each set contained in $J$ on a set of measure zero, $J$ contains exactly the sets $I_k$, $k \in \mathbb{N}$. It follows that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in E_\ast \times E_\ast$.

$$E_\ast = 0 \quad E_\ast = 0 \quad E_\ast = 0$$

$$E_\ast = 0 \quad E_\ast = 0$$

Figure 10: The decorated constraints forcing the structure of the composite tiles $E_\ast \times X$ for $X \in \{A_1, \ldots, A_M, B_{G,1}, \ldots, B_{G,M}, C_\ast, D_\ast, E_\ast\}$, and $E_\ast \times Z$ for $Z \in B_\ast \setminus B_{G_\ast}$.

Finally, we consider the constraints depicted in Figure 10. Fix $X$ to be one of $A_1, \ldots, A_M$, $B_{G,1}, \ldots, B_{G,M}$, $C_\ast$, $D_\ast$ and $E_\ast$. The first two constraints on the first line imply that there exist disjoint measurable subsets $K_J \subseteq [0, 1)$, $J \in \mathcal{J}$, such that the following holds for almost every $(x, y) \in E_\ast \times X$: $W(x, y) = 1$ iff there exists $J \in \mathcal{J}$ such that $\gamma_{E_\ast}^{-1}(g(x)) \in J$ and $\gamma_X^{-1}(g(y)) \in K_J$. The next two constraints yield that $|J| = |K_J|$ and that each $K_J$ differs from an interval on a set of measure zero, respectively. Finally, the first constraint on the second line implies that if $J \in \mathcal{J}$ precedes $J' \in \mathcal{J}'$, then $K_J$ precedes $K_{J'}$. Hence, we conclude that $J$ and $K_J$ differ on a set of measure zero for every $J \in \mathcal{J}$. It follows that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in E_\ast \times X$. The last constraint in Figure 10 implies that $W(x, y) = 0 = W_0(g(x), g(y))$ for almost every $(x, y) \in E_\ast \times Z$, $Z \in B_\ast \setminus B_{G_\ast}$.

### 3.6 Exponential checker tiles

We next define an iterated version of the checker graphon. Informally speaking, for each $r \in \mathbb{N}$, we form $W^r_C$ by splitting the parts of the checker graphon into $1, 2^{r-1}, 2^{2(r-1)}, 2^{3(r-1)}$, etc. parts. So, fix $r \in \mathbb{N}$ and define the graphon $W^r_C$ as follows: $W^r_C(x, y) = 1$ if and only if $x$
Figure 12: The decorated constraints forcing the tiles $C_x \times C_x$ and $D_x \times D_x$. The constraints on the first line should hold for $X = C_x$ and $X = D_x$.

and $y$ belongs to the same interval $I_k$, $k \in \mathbb{N}$, and

$$\left\lfloor \frac{x - \min I_k}{|I_k|} \cdot 2^{(k-1)(r-1)} \right\rfloor = \left\lfloor \frac{y - \min I_k}{|I_k|} \cdot 2^{(k-1)(r-1)} \right\rfloor,$$

and $W_C^r(x, y) = 0$ otherwise. The graphons $W^2_C$ and $W^3_C$ are depicted in Figure 11. Also note that the graphon $W^1_C$ is the checker graphon itself. Finally, we define $W_0(\gamma_{C*}(x), \gamma_{C*}(y)) = W_0(\gamma_{C*}(x), \gamma_{D*}(y)) = W_1^C(x, y)$ and $W_0(\gamma_{D*}(x), \gamma_{D*}(y)) = W^2_C(x, y)$ for $(x, y) \in [0, 1)^2$.

Consider now the decorated constraints given in Figure 12. The arguments are similar to those presented in Subsection 3.3, so we present them briefly. Fix $X$ to be $C_x$ or $D_x$. The constraints on the first line imply that there a family $J_X$ of disjoint subintervals $[0, 1)$ such that the following holds for almost every $(x, y) \in X \times X$: $W(x, y) = 1$ iff $\gamma_X^{-1}(g(x))$ and $\gamma_X^{-1}(g(y))$ belong to the same interval $J$, and $W(x, y) = 0$ otherwise. The first constraint on the second line implies that almost every $x \in C_x$ with $\gamma_{C*}^{-1}(g(x)) \in I_k$, $k \in \mathbb{N}$ satisfies that $\gamma_{C*}^{-1}(g(x))$ belongs to an interval $J \in J_{C*}$ such that $|J| = 2|I_k|^2 = 2^{-2k+1}$. It follows that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in C_x \times C_x$. Similarly, the second constraints imply that almost every $x \in D_x$ with $\gamma_{D*}^{-1}(g(x)) \in I_k$, $k \in \mathbb{N}$ satisfies that $\gamma_{D*}^{-1}(g(x))$ belongs to an interval $J \in J_{D*}$ such that $|J| = 4|I_k|^3 = 2^{-3k+2}$, which yields that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in D_x \times D_x$.

We next consider the constraints depicted in Figure 13. The two constraints on the first line imply that there exist disjoint measurable sets $K_J$, $J \in J_{C*}$, such that the following holds for almost every $(x, y) \in C_x \times D_x$: $W(x, y) = 1$ iff $\gamma_{C*}^{-1}(g(x))$ belongs to $J \in J_{C*}$ and $\gamma_{D*}^{-1}(g(y))$ belongs to $K_J$, and $W(x, y) = 0$ otherwise. The first constraint on the second line implies that each $K_J$ differs from an interval in a set of measure zero, the second that $|J| = |K_J|$ and the last that if $J$ precedes $J'$, $J, J' \in J_{C*}$, then $K_J$ precedes $K_{J'}$. We conclude that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in C_x \times D_x$. 

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\[ C \ast C \ast D \ast = 0 \]
\[ D \ast D \ast D \ast = 0 \]
\[ C \ast C \ast F \ast = 0 \]
\[ C \ast F \ast C \ast = 0 \]

Figure 13: The decorated constraints forcing the structure of the composite tile \( C \ast D \ast \).

3.7 Referencing dyadic parts

Fix \( X \in A \ast \cup B \ast \). For \((x, y) \in X \times C\ast \), we define \( W_0(x, y) \) to be equal to one if \( \gamma_{C\ast}(y) \in I_k \), \( k \in \mathbb{N} \), and
\[
\left\lfloor \frac{y - \text{min } I_k}{|I_k|} \cdot 2^{k-1} \right\rfloor = \left\lfloor x \cdot 2^{k-1} \right\rfloor ,
\]
and equal to zero otherwise. Similarly, for \((x, y) \in X \times D\ast \), we define \( W_0(x, y) \) to be equal to one if \( \gamma_{D\ast}(y) \in I_k \), \( k \in \mathbb{N} \), and
\[
\left\lfloor \frac{y - \text{min } I_k}{|I_k|} \cdot 2^{2(k-1)} \right\rfloor = \left\lfloor x \cdot 2^{k-1} \right\rfloor \pmod{2^{k-1}},
\]
and equal to zero otherwise. See Figure 14 for the illustration. Finally, we define \( W_0(x, y) = 0 \) for \((x, y) \in X \times (C\ast \cup D\ast)\) for \( X \in B \ast \setminus B \ast \).

Figure 14: The composite tiles \( X \times C\ast \) (in the left) and \( X \times D\ast \) (in the right) for \( X \in A \ast \cup B \ast \).

Fix \( X \in A \ast \cup B \ast \) and \((Y, Z)\) to be either \((C\ast, E\ast)\) or \((D\ast, C\ast)\), and consider the constraints depicted in Figure 15. To make our presentation less cumbersome, we will write \( J_{E\ast} \) for \( \{I_k, k \in \mathbb{N}\} \). The first two constraints on the first line imply that for every \( J \in J_Y \), there exists an interval \( K_J \subseteq [0, 1) \) such that the following holds for almost every \((x, y) \in X \times Y\): \( W(x, y) = 1 \) if \( \gamma_Y^{-1}(g(y)) \in J \), \( J \in J_Y \) and \( \gamma_X^{-1}(g(x)) \in K_J \), and \( W(x, y) = 0 \) otherwise. Note that the intervals \( K_J \) need not be disjoint. Suppose that \( J, J' \in J_Y \), \( J \neq J' \), are subintervals (up to measure zero) of the same interval contained in \( J_Z \). The third constraint on the first line implies that \( K_J \) and \( K_{J'} \) are disjoint, and the first constraint on the second line implies that if \( J \) precedes \( J' \), then \( K_J \) precedes \( K_{J'} \). The final constraint yields that if an interval...
\[X \times Y = 0\]  
\[X \times Y \times F = 0\]  
\[Z \times Y = 0\]

\[X \times F = 0\]  
\[Y \times F = 0\]  
\[X \times Y \times E = 2\]

Figure 15: The decorated constraints forcing the structure of the tiles \(X \times C_*\) and \(X \times D_*\) for \(X \in A_* \cup B_{G*}\). The constraints should hold for every such \(X\) with \((Y,Z) = (C_*,E_*)\) and \((Y,Z) = (D_*,C_*)\).

\(J \in \mathcal{J}_Y\) is a subset (up to measure zero) of \(I_k, k \in \mathbb{N}\), then \(|K_j| = 2^{-k+1}\). However, this can only be possible if for every interval \(J_0 \in \mathcal{J}_Z\), the intervals \(K_j\) for \(J \subseteq J_0, J \in \mathcal{J}_Y\), partition the interval \([0,1)\) up to a set of measure zero. We conclude that \(W(x,y) = W_0(g(x),g(y))\) for almost every \((x,y) \in X \times Y\). Finally, the two constraints depicted in Figure 16 imply that \(W(x,y) = 0 = W_0(g(x),g(y))\) for almost every \((x,y) \in X \times (C_* \cup D_*)\) where \(X \in B_* \setminus B_{G*}\).

\[C_* = 0\]  
\[D_* = 0\]

Figure 16: The decorated constraints forcing the structure of the tiles \(X \times C_*\) and \(X \times D_*\) for \(X \in B_* \setminus B_{G*}\).

### 3.8 Forcing the graphon \(W_F\)

We now define the most important part of the graphon \(W_0\), which is the composite tile \(A_* \times A_*\): \(W_0(\gamma_{A*}(x), \gamma_{A*}(y)) = W_F(x,y)\) for every \((x,y) \in [0,1)^2\).

Let \(I^i, i \in [M]\), be the half-open interval \(\gamma_{A*}^{-1}(A_i)\), and define \(I_{s,t}, s \in \mathbb{N}\) and \(t \in [2^{s-1}]\), to be the \(t\)-th half-open subinterval of \(I^i\) when the interval \(I^i\) is subdivided into \(2^{s-1}\) half-open subintervals of the same length. Consider now the first constraint in Figure 17. The constraint implies that the following holds for almost every \(x \in C_*\) and \(y \in D_*\) with \(W(x,y) > 0\). Suppose that \(\gamma_{D_*}^{-1}(g(y)) \in I_s\) and let \(t \in [2^s]\) and \(t' \in [2^s]\) be such that

\[
\left\lfloor \frac{\gamma_{D_*}^{-1}(g(y)) - \min I_s}{|I_s|} \cdot 2^{2(s-1)} \right\rfloor = (t - 1) \cdot 2^{s-1} + (t' - 1).
\]
Since $W(x, y) > 0$, it follows that
\[
\left\lfloor \frac{\gamma_{C_i}^{-1}(g(x)) - \min I_s}{|I_s|} \cdot 2^{s-1} \right\rfloor = t - 1.
\]
Hence, the left side of the constraint for a particular choice of $i, j \in [M]$ is equal to
\[
\frac{d_W(g^{-1}(\gamma_{A_i}(I_{s,t}^i)), g^{-1}(\gamma_{A_j}(I_{s',t'}^j)))}{|A_i||A_j|}.
\]
Similarly, the right side is equal to
\[
\frac{d_W(g^{-1}(\gamma_{B_i}(I_{s,t}^i)), g^{-1}(\gamma_{B_j}(I_{s',t'}^j)))}{|B_i||B_j|}.
\]
We conclude that the following holds for every $i, j \in [M]$, $s \in \mathbb{N}$ and $t, t' \in [2^s]$:
\[
\frac{d_W(g^{-1}(\gamma_{A_i}(I_{s,t}^i)), g^{-1}(\gamma_{A_j}(I_{s',t'}^j)))}{|A_i||A_j|} = \frac{1}{|B_i||B_j|} \int_{I_{s,t}^i \times I_{s',t'}} W_F(x, y) \, dx \, dy.
\]
Since $\gamma_{A_*}$ is a linear function and any half-open subinterval of $[0, 1)$ can be expressed a
countable union of intervals of the from $I_{s,t}^i$, we obtain that the following holds
\[
\frac{d_W(g^{-1}(\gamma_{A_*}(J)), g^{-1}(\gamma_{A_*}(J')))}{|A_*||A_*|} = \int_{J \times J'} W_F(x, y) \, dx \, dy
\] (3)
for any two two measurable subsets $J$ and $J'$ of $[0, 1)$.

Fix a measurable bijection $\psi: [0, 1) \to A$ such that $\psi^{-1}(X) = |X|/|A|$ for every measurable subset of $A$. Define a graphon $W_A$ as $W_A(x, y) = W(\psi(x), \psi(y))$ and a map $\tilde{g}: [0, 1) \to [0, 1)$ as $\tilde{g}(x) = \gamma_{A_*}^{-1}(g(\psi(x)))$. Note that $\tilde{g}$ is a measure-preserving map from $[0, 1]$ to $[0, 1]$. Since it holds that $W_0(\gamma_{A_*}(x), \gamma_{A_*}(y)) = W_F(x, y)$ for every $(x, y) \in [0, 1]^2$, it therefore suffices to show that $W_A(x, y) = W_F(\tilde{g}(x), \tilde{g}(y))$ for almost every $(x, y) \in [0, 1]^2$, which would imply that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in A_* \times A_*$. This can be done following the lines of [10] Section 5.4. However, we present a shorter proof, which is a special case of [12] Proof of Theorem 3.12 given by Doležal et al.

We start by summarizing some facts on the graphon $W_A$. Using (3), we obtain that
\[
d_{W_A}(\tilde{g}^{-1}(J), \tilde{g}^{-1}(J')) = d_{W_F}(J, J')
\] (4)
for any two measurable subsets $J$ and $J'$ of $[0,1)$. In addition, the second constraint in Figure 17 yields that
\[ t(C_4, W_A) = t(C_4, W_F). \]  
(5)

A graphon $W$ can be viewed an operator $T_W$ on $L^2([0,1))$, the $L^2$-space of functions from $[0,1)$, defined as
\[ T_W(h)(x) = \int_{[0,1]} W(x,y)h(y) \, dy \]
for $h \in L^2([0,1))$. The operator $T_W$ is self-adjoint and compact for any graphon $W$ [25 Section 7.5]. Moreover, it has a discrete spectrum, all its eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$ are real and it holds [25 Section 7.5] that
\[ t(C_4, W) = \sum_{i=1}^{\infty} \lambda_i^4. \]

The measure-preserving map $\tilde{g} : [0,1) \to [0,1)$ yields a pullback embedding $\tilde{g}^* : L^2([0,1)) \to L^2([0,1))$, i.e., $\tilde{g}^*(h)(x) := h(g(x))$ for $h \in L^2([0,1))$. Similarly, we define $\tilde{g}^*(W)(x,y) := W(\tilde{g}(x), \tilde{g}(y))$ for a graphon $W$. Let $K$ be the image of $L^2([0,1))$ under $\tilde{g}^*$; note that $K$ is a closed subspace of $L^2([0,1))$. The equation (4) implies that for any $h_1, h_2 \in K$, we have $\langle T_{\tilde{g}^*(W)}h_1, h_2 \rangle = \langle T_{W_A}h_1, h_2 \rangle$. In other words, the operator $T_{\tilde{g}^*(W)}$ is the composition of the restriction of $T_{W_A}$ to $K$ and the projection to $K$. This implies that if $(\lambda_i)_{i \in \mathbb{N}}$ are the eigenvalues of $T_{W_A}$ and $(\mu_i)_{i \in \mathbb{N}}$ the eigenvalues of $T_{\tilde{g}^*(W_F)}$, listed in the non-increasing order in terms of absolute value, then $|\mu_i| \leq |\lambda_i|$ for every $i \in \mathbb{N}$, in particular,
\[ \sum_{i \in \mathbb{N}} \mu_i^4 \leq \sum_{i \in \mathbb{N}} \lambda_i^4, \]
and equality holds if and only if $T_{W_A}$ is the extension of $T_{\tilde{g}^*(W_F)}$ by mapping each element of the orthogonal complement of $K$ to the zero function. Since $t(C_4, W_A) = t(C_4, \tilde{g}^*(W_F))$ by (5), it follows that $W_A$ and $\tilde{g}^*(W_F)$ are equal almost everywhere. We conclude that $W(x,y) = W_0(g(x), g(y))$ for almost every $(x,y) \in A_* \times A_*$.  

3.9 Degree balancing

We now define the graphon $W_0$ on the remaining tiles except for those involving the part $G_2$. First, set $W_0(x,y) = 0$ for all $(x,y) \in A_* \times B_*$. The graphon $W_0$ is now defined on all tiles except for those involving the part $G_1$ or $G_2$. For $x \in [0,1) \setminus (G_1 \cup G_2)$, we define
\[ h(x) = \int_{[0,1) \setminus (G_1 \cup G_2)} W_0(x,y) \, dy, \]
and set
\[ W_0(x,y) = \frac{2}{\varepsilon} \text{(pre-deg}(X) - h(x)) \]
for every $(x,y) \in X^0 \times G_1^0$, $X \in \mathcal{P} \setminus \{G_1, G_2\}$. Finally, let $\rho \in [0,1]$ be such that
\[ \frac{\rho \varepsilon}{2} + \sum_{X \in \mathcal{P} \setminus (G_1, G_2)} \frac{2}{\varepsilon} \int_X \text{(pre-deg}(X) - h(x)) \, dx \]
is a rational number, and set $W_0(x,y) = \rho$ for every $(x,y) \in G_1^0 \times G_1^0$.  

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At this point, it may not be clear that all the values of \( W_0 \) belong \([0, 1]\). So, we need to prove that \( \text{pre-deg}(X) - h(x) \in [0, \varepsilon/2] \) for every \( x \in X^0, X \in \mathcal{P} \setminus \{G_1, G_2\} \). Since the total measure of the parts \( B_*, C_*, D_*, E_*, \) and \( F_* \) is \( \varepsilon/4 \), it is enough to show that

\[
\int_{A^0} W_0(x, y) \, dy \in [\text{pre-deg}(X) - \varepsilon/2, \text{pre-deg}(X) - \varepsilon/4]
\tag{6}
\]

for every \( x \in X^0, X \in \mathcal{P} \setminus \{G_1, G_2\} \).

If \( X = A_0^k, k \in [M] \), the value of the integral in (6) belongs to \([ (1 - \varepsilon)\frac{k-1}{M}, (1 - \varepsilon)\frac{k}{M} ]\) for every \( x \in X^0 \) by the definition of \( Q_k \), which was given at the beginning of this section. Since \( M = 4\frac{1-\varepsilon}{\varepsilon} \), it follows that the value of the integral belongs to the interval \([ (k-1)\frac{1}{4}, k\frac{1}{4} ]\), which is the interval on the right side of (6). Similarly, if \( X = F_0^k \), the value of the integral in (6) belongs to \([ (1 - \varepsilon)\frac{k-1}{M}, (1 - \varepsilon)\frac{k}{M} ]\) for every \( x \in X^0 \) by the definition of \( W_0 \), and this interval again coincide with that on the right side of (6).

If \( X \in B_* \), then the integral in (6) is zero and (6) is also satisfied. If \( X = C_0^k \) or \( X = D_0^k \), \( k \in [m] \), then the integral in (6) is equal to \( \frac{1-\varepsilon}{2m} \) for every \( x \in X^0 \) and (6) is satisfied. If \( X = E_0^k \), \( k \in [m - 1] \), then the integral in (6) is equal to \( \frac{1-\varepsilon}{2m} \) for every \( x \in X^0 \) and (6) is again satisfied. Finally, if \( X \in \{C_0^\infty, D_0^\infty, E_0^\infty\} \), then the integral in (6) is at most \( \frac{1-\varepsilon}{2m} \), and so its value belongs to the interval on the right side of (6).

We now force the structure of the tiles that we have just defined. First, the constraint in Figure 18 implies that \( W(x, y) = W_0(g(x), g(y)) = 0 \) for almost every \((x, y) \in A \times B\). We
now analyze the constraints depicted in Figure 19. The two constraints on the first line imply that, for every $X \in P \setminus \{G_2\}$, the integral
\[ \int_X W(x, y)W(x, y') \, dx \] is the same for almost every $y, y' \in G_1$ (and is equal to the value of the corresponding integral in $W_0$). By Lemma 9, the integral
\[ \int_X W(x, y)^2 \, dx \] is the same for almost every $y \in G_1$ and its value is equal to (7). Therefore, for almost every $(y, y') \in G_1^2$, we have that
\[ \int_X (W(x, y) - W(x, y'))^2 \, dx = 0. \]
In particular, there exists a $y' \in G_1$ such that for almost every $y \in G_1$,
\[ \int_X (W(x, y) - W(x, y'))^2 \, dx = 0. \]
This is equivalent to saying that for almost every $y \in G_1, x \in X$, $W(x, y) = W(x, y')$. Thus, there exists a function $\tilde{h} : [0, 1) \setminus G_2 \to \mathbb{R}$ such that $W(x, y) = \tilde{h}(x)$ for almost every $(x, y) \in ([0, 1) \setminus G_2) \times G_1$. The two constraints on the second line in the figure imply that
\[ \int_{G_1} W(x, y) \, dy = \int_{G_1^2} W_0(x, g(g(y))) \, dy \] for almost every $x \in [0, 1) \setminus G_2$. We conclude that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in ([0, 1) \setminus G_2) \times G_1$.

3.10 Degree distinguishing

We now conclude the definition of the graphon $W_0$. Recall that we fixed irrational numbers $\delta_X \in (0, \varepsilon/4)$ for each part $X \in P$ such that the numbers $\delta_X, X \in P$, are rationally independent. For each $X \in P$, we set $W_0(x, y) = \delta_X/|G_2^0| = 4\delta_X/\varepsilon$ for all $(x, y) \in X^0 \times G_2^0$. Observe that the degree of each part $X \in P \setminus \{G_1, G_2\}$ is equal to pre-deg$(X) + \delta_X$, the degree of $G_1$ is $r + \delta_{G_1}$ where $r$ is a rational number (this follows from the choice of $\rho$), and the degree of $G_2$ is a rational combination of the values $\delta_X, X \in P$ (recall that $\varepsilon$ is rational). Since the values $\delta_X, X \in P$, are rationally independent, the degrees of all the parts are distinct.

Fix $X \in P$ and consider the constraints depicted in Figure 20. The first constraint yields that
\[ \frac{1}{|G_2|} \int_{G_2} W(x, y) \, dy = \delta_X \] for almost every $x \in X$, and the second constraint yields that
\[ \frac{1}{|G_2|} \int_{G_2} W(x, y)W(x', y) \, dy = \delta_X^2. \]
for almost every $x, x' \in X$. The latter implies by Lemma 9 that
\[ \frac{1}{|G_2|} \int_{G_2} W(x, y)^2 \, dy = \delta^2_X \]
for almost every $x \in X$. Hence, for almost every $x \in X$,
\[ \frac{1}{|G_2|} \int_{G_2} (W(x, y) - \delta_X)^2 \, dy = \frac{1}{|G_2|} \int_{G_2} W(x, y)^2 \, dy - \frac{2\delta_X}{|G_2|} \int_{G_2} W(x, y) \, dy + \delta^2_X = 0, \]
which implies that $W(x, y) = \delta_X = W_0(g(x), g(y))$ for almost every $(x, y) \in X \times G_2$. This concludes the argument that $W(x, y) = W_0(g(x), g(y))$ for almost every $(x, y) \in [0,1)^2$ and the proof of Theorem 2 is now finished.

4 Sizes of Forcing Families

In this section, we show that there is no finite family $G$ of graphs such that Theorem 2 would hold for all graphons $W_F$ and $\varepsilon > 0$ with $G$ being the forcing family. In particular, the main result of this section is the following theorem.

Theorem 11. For every positive integer $n$, there exists a graphon $W_F$ and a real number $\varepsilon > 0$ such that if $W$ is a finitely forcible graphon containing $W_F$ as a subgraphon on at least $1 - \varepsilon$ fraction of its vertices, then every forcing family for $W$ contains a graph of order greater than $n$.

To prove Theorem 11 we need a modification of a result of Erdős, Lovász and Spencer [13, Lemma 5]. The proof follows the lines of the proof in [13] but we include its sketch for completeness.

Lemma 12. Let $n$ be a positive integer and let $H_1, \ldots, H_m$ be all connected graphs on at most $n$ vertices. There exist graphons $W_1, \ldots, W_m$ such that the vectors
\[ (d(H_1, W_i), \ldots, d(H_m, W_i)), \ i \in [m], \]
are linearly independent in $\mathbb{R}^m$, and there is no index $i \in [m]$ and non-null set $A \subseteq [0,1]$ such that $W_i$ is positive almost everywhere on $A \times A$.

Proof. Fix an integer $n$ and the graphs $H_1, \ldots, H_m$. Let $k_i$ be the number of vertices of $H_i$, $i \in [m]$. For $i \in [m]$ and $\bar{s}_i \in [0,1]^{k_i}$ such that $\bar{s}_{i,1} + \cdots + \bar{s}_{i,k_i} \leq 1$, define $W_{i,\bar{s}_i}$ to be the following step graphon with $k_i + 1$ parts $S_{i,1}, \ldots, S_{i,k_i+1}$. The measure of the part $S_j$, $j \in [k_i]$,
Finally, define the graphon \( W \) equal to 0 otherwise. In other words, \( W \) is weakly isomorphic to \( W \). We claim that the span of \( S \) is \( \mathbb{R}^m \). Suppose not. Then we may choose \( (c_1, \ldots, c_m) \) to be a non-zero vector in the orthogonal complement of the span of \( S \). That is, \( c_1, \ldots, c_m \) are real numbers, not all zero, such that
\[
c_1 \cdot d(H_1, W_{i,\bar{s}_i}) + \cdots + c_m \cdot d(H_m, W_{i,\bar{s}_i}) = 0
\]
for every \( i \in [m] \) and \( \bar{s}_i \in [0, 1]^{k_i} \) such that \( \bar{s}_{i,1} + \cdots + \bar{s}_{i,k_i} \leq 1 \). Take \( i \) such that \( c_i \neq 0 \). The left side of (8) is a polynomial in \( \bar{s}_{i,1}, \ldots, \bar{s}_{i,k_i} \). Observe that the only term that contributes to the coefficient of the monomial \( \bar{s}_{i,1} \cdots \bar{s}_{i,k_i} \) is the term \( c_i d(H_i, W_{i,\bar{s}_i}) \). It follows that the left side of (8) is polynomial that is not identically zero, which implies that the equality (8) cannot hold for all choices of \( \bar{s}_i \in [0, 1]^{k_i} \). Thus, the span of \( S \) is \( \mathbb{R}^m \), and so we can choose \( m \) linearly independent vectors from \( S \). The corresponding graphons can be taken as \( W_i \).

We are now ready to prove Theorem 11

**Proof of Theorem 11.** Fix an integer \( n \). Note that the densities of induced connected subgraphs on at most \( n \) vertices determine the densities of all subgraphs on at most \( n \) vertices. Therefore, to prove the theorem, it is enough to show the following: there exists a graphon \( W_F \) and a real number \( \varepsilon > 0 \) such that no graphon \( W \) that contains \( W_F \) as a subgraph on at least \( 1 - \varepsilon \) fraction of its vertices is a finitely forcible graphon such that the set of all connected graphs with at most \( n \) vertices is a forcing family. Let \( H_1, \ldots, H_m \) be all connected graphs on at most \( n \) vertices, and let \( k_i \) be the number of vertices of \( H_i, \ i \in [m] \).

Let \( W_1, \ldots, W_m \) be the graphons from Lemma 12. In addition, let \( W_{m+1} \) be the graphon equal to one everywhere. We define \( W_F(x, y) \) to be equal to
\[
W_i((m+2)x-(i-1), (m+2)y-(i-1)) \quad \text{if } (x, y) \in \left[\frac{i-1}{m+2}, \frac{i}{m+2}\right]^2 \quad \text{for } i \in [m+1],
\]
equal to 0 otherwise. In other words, \( W_F \) contains each of the graphons \( W_1, \ldots, W_{m+1} \) on a \( \frac{1}{m+2} \) fraction of its vertices, and it is zero elsewhere.

Suppose that \( W \) is a graphon that contains \( W_F \) on a \( 1 - \varepsilon' \) fraction of its vertices for some \( \varepsilon' \leq \varepsilon \), where \( \varepsilon \) will be specified later. By applying a suitable measure preserving transformation, we can assume that the subgraphon of \( W \) on \( \left[\frac{i-1}{m+2}, 1 - \varepsilon'\right], \left[\frac{i}{m+2}, 1 - \varepsilon'\right] \) is weakly isomorphic to \( W_i \) for every \( i \in [m+1] \), and the graphon \( W \) is zero almost everywhere else on \([0, 1 - \varepsilon')^2 \). Consider a vector \( \bar{s} \in \left[0, \frac{1}{m+1}\right]^{m+1} \), and let \( t_0, \ldots, t_{m+2} \in [0, 1] \) be such that \( t_0 = 0, t_i = t_{i-1} + \bar{s}_i \) for \( i \in [m+1] \), and \( t_{m+2} = 1 \). Define a function \( \varphi : [0, 1] \to [0, 1] \) as follows:
\[
\varphi(x) = \begin{cases} 
\frac{(i-1)(1-\varepsilon')}{m+2} + \frac{x-t_{i-1}-(1-\varepsilon')}{m+2}(t_i-t_{i-1}) & \text{if } x \in [t_{i-1}(1-\varepsilon'), t_i(1-\varepsilon')] \text{ for } i \in [m+2], \text{ and} \\
\frac{x}{t_i} & \text{otherwise.}
\end{cases}
\]
Finally, define the graphon \( W_{\bar{s}_i} \) as \( W_{\bar{s}_i}(x, y) = W(\varphi(x), \varphi(y)) \). Informally speaking, the part of \( W \) containing \( W_i \) is stretched to size \( \bar{s}_i(1-\varepsilon') \) for every \( i \in [m+1] \). In particular, the graphons \( W \) and \( W_{\frac{1}{m+2}, \ldots, \frac{1}{m+2}} \) are the same.
We now analyze $d(H_i, W_{\bar{z}})$ as a function of $\bar{s}_1, \ldots, \bar{s}_{m+1}$. Each of the $k_i$ vertices of $H_i$ can be chosen either from one of the $m + 2$ intervals $[t_{i-1}(1 - \varepsilon'), t_i(1 - \varepsilon')]$, $i \in [m + 2]$, or from the interval $[1 - \varepsilon', 1]$. The choices where no vertex of $H_i$ is chosen to be in the interval $[1 - \varepsilon', 1]$ contribute to $d(H_i, W_{\bar{z}})$ by a total of

$$\sum_{j=1}^{m+1} ((1 - \varepsilon')^{k_i} d(H_i, W_{\bar{z}})).$$

Each of the other choices contributes by a term that is a product of a constant between 0 and $\varepsilon'$ (since at least one vertex is mapped to $[1 - \varepsilon', 1)$, which has measure $\varepsilon'$) and less than $k_i$ terms of the form $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{m+1}$, or $(1 - \bar{s}_1 - \bar{s}_2 - \ldots - \bar{s}_{m+1})$. Since there are at most $(m + 3)^{k_i}$ ways to assign vertices of $H_i$ to the $m + 3$ intervals described above and the derivative of a product of the above form with respect to $\bar{s}_j$ is at most $k_i$, we obtain that the derivative of $d(H_i, W)$ with respect to $\bar{s}_j$ is between

$$(1 - \varepsilon')^{k_i} k_i \bar{s}_j^{k_i-1} d(H_i, W_{\bar{z}}) - (m + 3)^{k_i} k_i \varepsilon'$$

and $(1 - \varepsilon')^{k_i} k_i \bar{s}_j^{k_i-1} d(H_i, W_{\bar{z}}) + (m + 3)^{k_i} k_i \varepsilon'$.

We next examine the derivatives of $d(H_i, W_{\bar{z}})$ at the point $\bar{s} = (\frac{1}{m+2}, \frac{1}{m+2}, \ldots, \frac{1}{m+2})$. Observe that the square matrix $A$ such that

$$A_{ij} = (1 - \varepsilon')^{k_i} k_i \left( \frac{1}{m+2} \right)^{k_i-1} d(H_i, W_j)$$

for $i, j \in [m]$, is invertible. Indeed, multiplying each row with a suitable coefficient yields the matrix that has $d(H_i, W_j)$ as the entry in the $i$-th row and $j$-th column. Hence, there exists $\varepsilon_0$ such that any matrix obtained from $A$ by perturbing each of its entries by at most $\varepsilon_0$ is invertible.

We now set $\varepsilon = \min \left\{ \frac{\varepsilon_0}{(m+3)^n \cdot \frac{1}{m+1}} \right\}$ (note that the definition of $\varepsilon$ depends on $n$ only) and assume that $\varepsilon' \leq \varepsilon$ in the rest of the proof.

Observe that there exists an open ball contained in $\left(0, \frac{1}{m+1}\right)^{m+1}$ that contains the point $\bar{s} = (\frac{1}{m+2}, \ldots, \frac{1}{m+2})$ such that each $d(H_i, W_{\bar{z}})$, $i \in [m]$, is well-defined on this ball. Also observe that the entries of the Jacobian matrix $\left( \frac{\partial d(H_i, W_{\bar{z}})}{\partial s_{ij}} \right)$ for $\bar{s} = (\frac{1}{m+2}, \ldots, \frac{1}{m+2})$ differ from the entries of $A$ by at most $(m + 3)^{k_i} k_i \varepsilon' \leq (m + 3)^n n \varepsilon \leq \varepsilon_0$. In particular, the Jacobian matrix is invertible. Hence, the Implicit Function Theorem implies that there exist $\delta \in \left(0, \frac{1}{m+1} - \frac{1}{m+2}\right)$ and a continuous function $g : \left(\frac{1}{m+2} - \delta, \frac{1}{m+2} + \delta\right) \rightarrow \left(0, \frac{1}{m+1}\right)^m$ such that $g\left(\frac{1}{m+2}\right) = \left(\frac{1}{m+2}, \ldots, \frac{1}{m+2}\right)$ and

$$d(H_i, W) = d(H_i, W_{g(z)_1, \ldots, g(z)_m, z})$$

for every $z \in \left(\frac{1}{m+2} - \delta, \frac{1}{m+2} + \delta\right)$. Fix $z \in \left(\frac{1}{m+2}, \frac{1}{m+2} + \delta\right) \subseteq \left(0, \frac{1}{m+1}\right)$. We set $W' = W_{g(z)_1, \ldots, g(z)_m, z}$. Observe that the densities of all graphs $H_1, \ldots, H_m$ are the same in $W$ and $W'$ by the choice of $g$ and $z$.

It remains to argue that the graphons $W$ and $W'$ are not weakly isomorphic. Note that $\omega(W') = \frac{1 - \varepsilon}{m+2}$, as this is certified by taking $A$ to be the interval $[\frac{m}{m+2}(1 - \varepsilon'), \frac{m+1}{m+2}(1 - \varepsilon')]$. Let $A$ be any measurable set $A \subseteq [0, 1]$ such that $W$ is equal to 1 almost everywhere on
Define \( A_1 = A \cap [0, 1 - \epsilon'] \) and \( A_2 = A \cap [1 - \epsilon', 1] \). Observe that \( A_1 \subseteq \left[ \frac{m}{m+2}(1-\epsilon'), \frac{m+1}{m+2}(1-\epsilon') \right) \) by the second assertion of Lemma \[12\] and that \( |A_2| \leq \epsilon' \leq \frac{1}{m+4} \) by the choice of \( \epsilon \). This implies that

\[
|A_1| \geq \frac{1 - \epsilon'}{m + 2} - \epsilon' = \frac{1 - (m + 3)\epsilon'}{(m + 2)} \geq \frac{1}{(m + 2)(m + 4)}.
\]

Since the interval \( \left[ \frac{m}{m+2}(1-\epsilon'), \frac{m+1}{m+2}(1-\epsilon') \right) \) is stretched to an interval of size \( z(1-\epsilon') > \frac{1}{m+2}(1-\epsilon') \) in \( W' \), we obtain that

\[
\omega(W') \geq z(m + 2)|A_1| + |A_2| = |A| + (z(m + 2) - 1)|A_1| \geq |A| + \frac{z(m + 2) - 1}{(m + 2)(m + 4)}.
\]

Since the choice of \( A \) was arbitrary, it follows that

\[
\omega(W') \geq \omega(W) + \frac{z(m + 2) - 1}{(m + 2)(m + 4)} > \omega(W)
\]

and we conclude that the graphon \( W \) and \( W' \) cannot be weakly isomorphic by Lemma \[4\].

We would like to remark that Theorem \[11\] excludes the existence of a finite family \( \mathcal{G} \) of graphs such that for every graphon \( W_F \) and every \( \epsilon > 0 \), there exists a finitely forcible graphon \( W_0 \) that contains \( W_F \) as subgraphon on a \( 1 - \epsilon \) fraction of its vertices, and \( \mathcal{G} \) is a forcing family for \( W_0 \). In other words, Theorem \[2\] cannot be proven with a universal forcing family (unlike Theorem \[1\]). However, we were not able to show that the number of graphs needed to force the structure of graphons containing \( W_F \) must grow with \( \epsilon^{-1} \), i.e., we do not know whether the following stronger statement is true: for every \( K \in \mathbb{N} \), there exist a graphon \( W_F \) and \( \epsilon > 0 \) such that if \( W_0 \) is a finitely forcible graphon that contains \( W_F \) as a subgraphon on a \( 1 - \epsilon \) fraction of its vertices, then every forcing family of \( W_0 \) contains at least \( K \) graphs.

**Acknowledgment**

We would like to thank Jan Hladký for useful discussions on the step forcing property of \( C_4 \), in particular, the arguments used at the end of Subsection 3.8. We would also like to thank the authors of \[12\] for sharing an early version of their manuscript.

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