The normalized Laplacian spectrum of $n$-polygon graphs and applications

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ABSTRACT
Given an arbitrary connected graph $G$, the $n$-polygon graph $\tau_n(G)$ is obtained by adding a path with length $n$ ($n \geq 2$) to each edge of graph $G$, and the iterated $n$-polygon graphs $\tau_g^n(G)$ ($g \geq 0$) are obtained from the iteration $\tau_g^n(G) = \tau_n(\tau_{g-1}^n(G))$, with the initial condition $\tau_0^n(G) = G$. In this paper, a method for calculating the eigenvalues of the normalized Laplacian matrix for graph $\tau_n(G)$ is first given. The normalized Laplacian spectra for the graph $\tau_n(G)$ and graphs $\tau_g^n(G)$ ($g \geq 0$) can also then be derived. Finally, as applications, we calculate the multiplicative degree-Kirchhoff index, Kemeny’s constant, and the number of spanning trees for the graph $\tau_n(G)$ and graphs $\tau_g^n(G)$ by exploring their connections with the normalized Laplacian spectrum, and obtain exact results for these quantities.

1. Introduction

The spectra of the adjacency matrices, Laplacian, and normalized Laplacian for a graph is of particular importance because many important structural and dynamical properties of the graph can be obtained from the eigenvalues and eigenvectors of these matrices [1,2]. For example, the spectra of these matrices provide information on the degree distribution, multiplicative degree-Kirchhoff index, community structure, local clustering, total number of links, and number of spanning trees, etc [2–4]. In addition, the first-passage properties of the networks [5,6], such as the mean first-passage time, mean hitting time, mixing time, and Kemeny’s constant, can be expressed in terms of the eigenvalues of the Laplacian and normalized Laplacian [2,7–9]. During the past several years, there has been particular interest in studying the spectra for the normalized Laplacian matrices of different graphs [10–17].

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A class of iterated graphs, obtained by replacing each edge of a ‘base graph’ with a given structure, have recently attracted significant attention [18,19] because they not only exhibit rich structural properties, such as self-similarity, fractal [19–21], scale-free, and small-world properties [18,22], they also show distinctive dynamic properties [20–24].

For an arbitrary connected ‘base graph’ $G$, if the normalized Laplacian spectrum of graph $G$ is known, calculating the normalized Laplacian spectrum and the related quantities of graph $\tau(G)$, which is obtained from $G$ by replacing each edge of $G$ with a given structure, becomes an interesting topic with wide applications, for which numerous results have been obtained. Examples include the graph $\tau(G)$ obtained by replacing every edge of $G$ with a geometry, such as triangles [25–27], a quadrilateral [28–30], or a pentagon [31]. However, for more general graphs $\tau_n(G)$ obtained by replacing every edge of $G$ by an $(n + 1)$-polygon ($n \geq 2$), the related topic is more difficult and general results for the normalized Laplacian spectrum $\tau_n(G)$ remain unknown.

To fill in this gap, we consider the case in which all edges of $G$ are replaced by an $(n + 1)$-polygon. We analyse the relation between the normalized Laplacian spectrum of the graph $\tau_n(G)$ and that of graph $G$, and present a detailed spectra of the graph $\tau_n(G)$ for any $n \geq 2$. Using our results recursively, we also obtain the normalized Laplacian spectra of the iterated graph $\tau^g_n(G)$ ($g \geq 0$), where $\tau^0_n(G) = G$, and $\tau^g_n(G) = \tau_n(\tau^{g-1}_n(G))$. As applications, we also analyse the multiplicative degree-Kirchhoff index, Kemeny’s constant, and the number of spanning trees of the graph $\tau^g_n(G)$. Exact results for these quantities are also presented.

This paper is organized as follows. First, in Section 2, we present the related definitions and notations used in the manuscript. In Section 3, we present some classical results, quotes, and some preliminary results used in this study. In Section 4, we present the main results on the normalized Laplacian spectrum of the graph $\tau_n(G)$, and in Section 5, we calculate the multiplicative degree-Kirchhoff index, Kemeny’s constant, and the number of spanning trees of the graphs $\tau_n(G)$ and $\tau^g_n(G)$. The detailed proofs of our results are presented in Sections 6–9 and Appendices A and B.

2. Definitions and notations

Let $G = (V(G), E(G))$ be a undirected graph with vertex set $V(G)$ and edge set $E(G)$, $N_0 = |V(G)|$ represent the total number of nodes of $G$, and $E_0 = |E(G)|$ be the total number of edges of graph $G$. For any two nodes $i, j \in V(G)$, if there is an edge between node $i$ and $j$ in $E(G)$, we state that $i$ is a neighbour of $j$, or that $i$ and $j$ are adjacent (i.e. $i \sim j$). If $e$ is an edge with end-vertices $i$ and $j$, we state that edge $e$ is incident to nodes $i$ and $j$, or that nodes $i$ and $j$ are incident with edge $e$. The degree of vertex $i$, referred to as $d_i$, is the number of edges incident to node $i$.

**Definition 2.1 ([2]):** For any undirected graph $G$, the Laplacian matrix of $G$ is defined as

$$L(G) = D(G) - A(G),$$

where $D(G)$ is the degree diagonal matrix of graph $G$, and $A(G) = (A_{ij})_{N_0 \times E_0}$ is the adjacency matrix of $G$, with

$$A_{ij} = \begin{cases} 1 & i \sim j, \\ 0 & \text{others}. \end{cases}$$

(1)
Definition 2.2 ([2,32]): For any undirected graph $G$, the normalized Laplacian of $G$ is defined as
\[
\mathcal{L}_G = D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}} = I - D(G)^{-\frac{1}{2}}A(G)D(G)^{-\frac{1}{2}}.
\] (2)

Let $\delta_{ij}$ be the Kronecker delta function. The $(i,j)$-th entry of the matrix $\mathcal{L}_G$ can then be written as
\[
\mathcal{L}_G(i,j) = \delta_{ij} - \frac{A_{ij}}{\sqrt{d_id_j}}.
\] (3)

Definition 2.3 ([2]): For any undirected graph $G$ with $N_0$ vertexes, the normalized Laplacian spectrum of graph $G$ is defined as
\[
\sigma(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_0}\},
\] (4)
where $\lambda_i$ $(i = 1, 2, \cdots, N_0)$ are the eigenvalues of $\mathcal{L}_G$.

Definition 2.4 ([3]): Let $G$ be a undirected graph, with vertex set $V(G) = \{1, 2, \ldots, N_0\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_{E_0}\}$. The incidence matrix of graph $G$ is then defined as
\[
B = (b_{ij})_{N_0 \times E_0},
\]
where
\[
b_{ij} = \begin{cases} 
1, & \text{if vertex } i \text{ and edge } e_j \text{ are incident}, \\
0, & \text{otherwise}. 
\end{cases}
\]

Definition 2.5 ([3,7]): Let $G'$ be a directed graph, with vertex set $V(G') = \{1, 2, \ldots, N_0\}$ and directed edge set $E(G') = \{e'_1, e'_2, \ldots, e'_{E_0}\}$. The incidence matrix of directed graph $G'$ is then defined as
\[
B' = (b'_{ij})_{N_0 \times E_0},
\]
where
\[
b'_{ij} = \begin{cases} 
-1, & \text{if vertex } i \text{ is the source vertex of edge } e'_j, \\
0, & \text{if vertex } i \text{ is not incident on edge } e'_j, \\
1, & \text{if vertex } i \text{ is the target vertex of edge } e'_j. 
\end{cases}
\]

Definition 2.6 ([3,7]): Let $G'$ be a directed graph, and is said weakly connected if its underlying undirected graph (i.e. a graph obtained by replacing each directed edge of $G'$ with an undirected edge) is connected.

Definition 2.7: The $n$-polygon $(n \geq 2)$ graph of $G$, denoted by $\tau_n (G)$, is the graph obtained by adding a path with length $n$ to each edge of $G$.

In other words, $\tau_n$ is an operator that turns each edge of a graph into an $(n + 1)$-polygon. Thus, $\tau_n (G)$ is a graph obtained by replacing each edge $ij$ of $G$ with an $(n + 1)$-polygon, as shown in Figure 1.
**Figure 1.** Construction method of $n$-polygon graph. The $n$-polygon graph of $G$, denoted by $\tau_n(G)$, is obtained from $G$ by replacing each edge $ij$ of $G$ with the $(n+1)$-gon on the right-hand side of the arrow.

**Definition 2.8:** For any $g > 0$, the iterated $n$-polygon graph with generation $g$ ($g \geq 1$), referred to as $\tau^g_n(G)$, is defined as the graph obtained through the iteration $\tau^g_n(G) = \tau_n(\tau^{g-1}_n(G))$, with the initial condition $\tau^0_n(G) = G$.

Let $N_g$ and $E_g$ be the total number of vertices and edges of graph $\tau^g_n(G)$, respectively. For any $g > 0$, we have

$$N_g = N_{g-1} + (n-1)E_{g-1}, \quad E_g = (n+1)E_{g-1}. \quad (5)$$

Therefore,

$$N_g = N_0 + (n-1)\frac{(n+1)^g - 1}{n}E_0, \quad (5)$$

and

$$E_g = (n+1)^gE_0. \quad (6)$$

### 3. Preliminaries

In this section, we present some classical and preliminary results used to derive our main results described in the present manuscript.

**Lemma 3.1 ([3]):** For any connected undirected graph $G$, let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N_0}$ be the eigenvalues of $L_G$. Then,

(i) $\lambda_1 = 0$, and for any $2 \leq i \leq N_0$, $0 < \lambda_i \leq 2$;
(ii) $\lambda_{N_0} = 2$ if and only if graph $G$ is bipartite;
(iii) graph $G$ is bipartite if and only if both $\lambda_i$ and $2 - \lambda_i$ are eigenvalues of $L_G$; and $m_{L_G}(\lambda_i) = m_{L_G}(2 - \lambda_i)$, where $m_{L_G}(\lambda_i)$ is the multiplicity of eigenvalue $\lambda_i$.

Therefore, all eigenvalues of $L_G$ are non-negative. Given the normalized Laplacian spectrum of graph $G$, the multiplicative degree-Kirchhoff index, Kemeny’s constant, and the number of spanning trees of graph $G$ can be expressed as follows.

**Lemma 3.2 ([2,3,7]):** Letting $G$ be a connected undirected graph with $N_0$ vertices and $E_0$ edges, $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{N_0}$ are the eigenvalues of the normalized Laplacian $L_G$. Then,
(i) the multiplicative degree-Kirchhoff index of $G$ can be written as

$$Kf'(G) = 2E_0 \sum_{i=2}^{N_0} \frac{1}{\lambda_i};$$

(ii) Kemeny’s constant of $G$ is

$$K(G) = \sum_{i=2}^{N_0} \frac{1}{\lambda_i};$$

and

(iii) the number of spanning trees $N_{st}(G)$ of $G$ is

$$N_{st}(G) = \frac{1}{2E_0} \left( \prod_{i=1}^{N_0} d_i \right) \prod_{i=2}^{N_0} \lambda_i.$$

**Lemma 3.3 ([33])**: Let $f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ ($b_n \neq 0$) be a polynomial of degree $n$ and $r_i$ ($i=1, 2, \cdots, n$) be the roots of $f(x) = 0$. Then, Vieta’s formula states that

$$\prod_{i=1}^{n} r_i = (-1)^n \frac{b_0}{b_n},$$

and

$$\sum_{i=1}^{n} \frac{1}{r_i} = -\frac{b_1}{b_0}.$$

**Lemma 3.4 ([2])**: Let $B$ be the incidence matrix of a connected graph $G$ with $N_0$ vertices. Then,

$$\text{rank } (B) = \begin{cases} N_0 - 1 & \text{if } G \text{ is bipartite,} \\ N_0 & \text{if } G \text{ is non-bipartite.} \end{cases}$$

**Lemma 3.5 ([3,7])**: Let $B'$ be the incidence matrix of weakly connected directed graph $G'$ with $N_0$ vertices. Then, $\text{rank } (B') = N_0 - 1$.

**Lemma 3.6**: Let $u$ be an arbitrary real number, and \{${a_n(\mu)}$\}_{n \geq -1} be a series defined by the recursive equation $a_n(\mu) = 2(1-\mu)a_{n-1}(\mu) - a_{n-2}(\mu)$, with the initial conditions $a_{-1}(\mu) = 0$ and $a_0(\mu) = 1$. Then,

(i) for any $n \geq 1$, $a_n(2-\mu) = (-1)^n a_n(\mu)$;

(ii) if $\mu = 0$,

$$a_n(0) = n + 1,$$

and if $\mu = 2$,

$$a_n(2) = (-1)^n (n + 1).$$
(iii) for any \( n \geq 0 \), \( a_n(\mu) \) is an \( n \)-th order polynomial in \( \mu \), which can be written as

\[
a_n(\mu) = \sum_{i=0}^{n} a_n^{(i)} \mu^i,
\]

where \( a_n^{(i)} \) is the coefficient of \( \mu^i \); furthermore,

\[
a_n^{(0)} = n + 1,
\]

\[
a_n^{(1)} = -\frac{n^3 + 3n^2 + 2n}{3},
\]

\[
a_n^{(n)} = (-1)^n \frac{2^n}{n}.
\]

(iv) for any \( n \geq 2 \), \( a_n(\mu) \) and \( 1 + a_n(\mu) \) can be expanded as

\[
a_n(\mu) = \begin{cases} 
\left( \frac{a_{n-1}^{n+1}(\mu) - a_{n-1}^{n-1}(\mu)}{2} \right) a_{n-1}^n(\mu), & \text{if } n \text{ is odd}, \\
\left( \frac{a_{n/2}^n(\mu) - a_{n/2-1}^n(\mu)}{2} \right) \left( a_{n/2}^n(\mu) + a_{n/2-1}^n(\mu) \right), & \text{if } n \text{ is even},
\end{cases}
\]

and

\[
1 + a_n(\mu) = \begin{cases} 
\left( \frac{a_{n+1}^{n+1}(\mu) - a_{n+1}^{n-1}(\mu)}{2} \right) \left( a_{n+1}^n(\mu) + a_{n+1}^n(\mu) \right), & \text{if } n \text{ is odd}, \\
\left( \frac{a_{n/2}^n(\mu) - a_{n/2-2}^n(\mu)}{2} \right) a_{n/2}^n(\mu), & \text{if } n \text{ is even}.
\end{cases}
\]

The proof of Lemma 3.6 is presented in Appendix A.

**Lemma 3.7:** If \( n \ (n \geq 3) \) is an odd number, and \( \{a_k(\mu)\}_{k \geq 0} \) is a series defined in Lemma 3.6, we have the following results.

(i) Let \( \mu \) be an arbitrary root of equation \( a_{n-1}^\mu(\mu) - a_{n-1}^{n-1}(\mu) = 0 \). Then,

\[
1 - \frac{a_{n-1}^{n+1}(\mu) - a_{n-1}^{n-1}(\mu)}{a_{n-1}^n(\mu) - a_{n-1}^{n-2}(\mu)} = 2, \quad a_{n-2}(\mu) = 1.
\]

(ii) Let \( \mu \) be an arbitrary root of equation \( a_{n-1}^\mu(\mu) + a_{n-1}^{n-1}(\mu) = 0 \). Then,

\[
1 - \frac{a_{n-1}^{n+1}(\mu) - a_{n-1}^{n-1}(\mu)}{a_{n-1}^n(\mu) - a_{n-1}^{n-2}(\mu)} = 0, \quad a_{n-2}(\mu) = -1.
\]

(iii) Let \( \mu \) be an arbitrary root of equation \( a_{n-1}^{n\mu}(\mu) = 0 \). Then,

\[
1 - \frac{a_{n-1}^{n+1}(\mu) - a_{n-1}^{n-1}(\mu)}{a_{n-1}^n(\mu) - a_{n-1}^{n-2}(\mu)} \in (-\infty, 0) \cup (2, +\infty), \quad a_{n-1}(\mu) = -1 \text{ and } a_{n-2}(\mu) = -2 (1 - \mu).
\]
The proof of the Lemma is presented in Appendix B.

Lemma 3.8: If \( n (n \geq 2) \) is an even number, and \( \{a_k(\mu)\}_{k \geq 0} \) is a series defined in Lemma 3.6, we have the following results.

(i) Let \( \mu \) be an arbitrary root of equation \( a_n^2(\mu) - a_{n-2}(\mu) = 0 \). Then,
\[
1 - \frac{a_n^2(\mu) - a_{n-1}^2(\mu)}{a_{n-1}^2(\mu) - a_{n-2}^2(\mu)} = 2, \tag{20}
\]
\[
a_{n-2}(\mu) = 1. \tag{21}
\]

(ii) Let \( \mu \) be an arbitrary root of equation \( a_{n-2}^2(\mu) = -1 \). Then,
\[
1 - \frac{a_n^2(\mu) - a_{n-1}^2(\mu)}{a_{n-1}^2(\mu) - a_{n-2}^2(\mu)} = 0, \tag{22}
\]
\[
a_{n-2}(\mu) = -1. \tag{23}
\]

(iii) Let \( \mu \) be an arbitrary root of equation \( a_n^2(\mu) + a_{n-1}^2(\mu) = 0 \). Then,
\[
1 - \frac{a_n^2(\mu) - a_{n-1}^2(\mu)}{a_{n-1}^2(\mu) - a_{n-2}^2(\mu)} \in (-\infty,0) \cup (2, +\infty), \tag{24}
\]
\[
a_{n-1}(\mu) = -1 \text{ and } a_{n-2}(\mu) = -2 (1 - \mu). \tag{25}
\]

The proof of Lemma 3.8 is similar to the proof of Lemma 3.7 and we omit the detailed derivation.

4. Main results

For any connected undirected graph \( G \) with \( N_0 \) vertices, if the normalized Laplacian spectrum \( \sigma_G = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_0}\} \) is known, we will present the normalized Laplacian spectra of the \( n \)-polygon graph \( \tau_n(G) \), which is defined by Definition 2.7. First, in Lemma 4.1, we present the general relation between the eigenvalues of \( \mathcal{L}_G \) and \( \mathcal{L}_{\tau_n(G)} \), where \( \mathcal{L}_G \) and \( \mathcal{L}_{\tau_n(G)} \) are normalized Laplacian matrices for graphs \( G \) and \( \tau_n(G) \), respectively. Two Remarks follow Lemma 4.1 to clarify the detailed formulas of the relations when \( n \) is odd or even. Then, in Theorems 4.2 and 4.3, we present a way to derive the eigenvalues of \( \mathcal{L}_{\tau_n(G)} \). Using Theorems 4.2 and 4.3 recursively, we can also derive the eigenvalues of graph \( \tau_n^g(G) \) \((n \geq 2, g \geq 1)\), defined in Definition 2.8, which is the iterated \( n \)-polygon graph of \( G \) with generation \( g \).

Lemma 4.1: For any integer \( n (n \geq 2) \), let \( \mu \) be a real number such that \( a_{n-1}(\mu) \neq 0 \) and \( a_{n-1}(\mu) + 1 \neq 0 \), where \( \{a_k(\mu)\} \) is a series defined in Lemma 3.6. Then, \( \mu \) is an eigenvalue of \( \mathcal{L}_{\tau_n(G)} \) with multiplicity \( k \) \((k > 0)\) if and only if \( \lambda \equiv 1 - \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} \) is an eigenvalue of \( \mathcal{L}_G \) with multiplicity \( k \) and \( \lambda \neq 0, \lambda \neq 2 \).

The proof of this Lemma is presented in Section 6.
Remark 4.1: If \( n \) is an odd number, replacing \( a_n(\mu) \) and \( 1 + a_{n-1}(\mu) \) from Equations (12) and (13), respectively, we obtain

\[
\lambda \equiv 1 - \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} = 1 - \frac{\left(\frac{a_{\frac{n-1}{2}+1}(\mu)}{\frac{n-1}{2}+1} - \frac{a_{\frac{n-1}{2}-1}(\mu)}{\frac{n-1}{2}-1}\right) a_{\frac{n-1}{2}}(\mu)}{\left(\frac{a_{\frac{n-1}{2}}(\mu)}{\frac{n-1}{2}} - \frac{a_{\frac{n-1}{2}-2}(\mu)}{\frac{n-1}{2}-2}\right) a_{\frac{n-1}{2}-1}(\mu)}
\]

and \( 1 + a_{n-1}(\mu) \neq 0 \) is equivalent to \( a_{\frac{n-1}{2}} - a_{\frac{n-1}{2}-2} \neq 0 \) and \( a_{\frac{n-1}{2}-1} \neq 0 \).

Remark 4.2: If \( n \) is an even number, replacing \( a_n(\mu) \) and \( 1 + a_{n-1}(\mu) \) from Equations (12) and (13), respectively, we obtain

\[
\lambda \equiv 1 - \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} = 1 - \frac{\left(\frac{a_{\frac{n}{2}}(\mu)}{\frac{n}{2}} - \frac{a_{\frac{n}{2}-1}(\mu)}{\frac{n}{2}-1}\right) \left(\frac{a_{\frac{n}{2}}(\mu)}{\frac{n}{2}} + \frac{a_{\frac{n}{2}-1}(\mu)}{\frac{n}{2}-1}\right)}{\left(\frac{a_{\frac{n}{2}-1}(\mu)}{\frac{n}{2}-1} - \frac{a_{\frac{n}{2}-2}(\mu)}{\frac{n}{2}-2}\right) \left(\frac{a_{\frac{n}{2}}(\mu)}{\frac{n}{2}} + \frac{a_{\frac{n}{2}-1}(\mu)}{\frac{n}{2}-1}\right)}
\]

and \( 1 + a_{n-1}(\mu) \neq 0 \) is equivalent to \( a_{\frac{n}{2}-1}(\mu) - a_{\frac{n}{2}-2}(\mu) \neq 0 \) and \( a_{\frac{n}{2}}(\mu) + a_{\frac{n}{2}-1}(\mu) \neq 0 \).

Theorem 4.2: Let \( G \) be a connected graph with \( N_0 \) vertices and \( E_0 \) edges, and \( \tau_n(G) \) be the \( n \)-polygon graph of \( G \), where \( n \geq 3 \) and \( n \) is odd. The eigenvalues for the normalized Laplacian \( \mathcal{L}_{\tau_n(G)} \) can be obtained in the following way.

(i) \( \mu = 0 \) is an eigenvalue of \( \mathcal{L}_{\tau_n(G)} \) with multiplicity 1. If \( G \) is a bipartite graph, \( \mu = 2 \) is an eigenvalue of \( \mathcal{L}_{\mathcal{G}(n)} \) with multiplicity 1.

(ii) Letting \( \mu \) be an arbitrary root of equation \( a_{\frac{n-1}{2}}(\mu) = 0 \), \( \mu \) is an eigenvalue of \( \mathcal{L}_{\tau_n(G)} \) with multiplicity \( N_0 \).

(iii) Letting \( \mu \) be an arbitrary root of equation \( a_{\frac{n-1}{2}}(\mu) + a_{\frac{n-1}{2}-1}(\mu) = 0 \), \( \mu \) is an eigenvalue of \( \mathcal{L}_{\tau_n(G)} \) with multiplicity \( E_0 - N_0 + 1 \).

(iv) When \( G \) is a non-bipartite graph, letting \( \mu \) be an arbitrary root of equation \( a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2}-1}(\mu) = 0 \), \( \mu \) is an eigenvalue of \( \mathcal{L}_{\tau_n(G)} \) with multiplicity \( E_0 - N_0 \).

(v) When \( G \) is a bipartite graph, letting \( \mu \) be an arbitrary root of equation \( a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2}-1}(\mu) = 0 \), \( \mu \) is an eigenvalue of \( \mathcal{L}_{\tau_n(G)} \) with multiplicity \( E_0 - N_0 + 1 \).

(vi) Let \( \lambda \) be an arbitrary eigenvalue of \( \mathcal{L}_G \) such that \( \lambda \neq 0 \), and \( \lambda \neq 2 \), \( \mu_i(\lambda), (i = 1, 2, \ldots, \frac{n+1}{2}) \) be the roots of the equation

\[
1 - \frac{a_{\frac{n+1}{2}+1}(x) - a_{\frac{n+1}{2}-1}(x)}{a_{\frac{n}{2}}(x) - a_{\frac{n}{2}-2}(x)} = \lambda.
\]

Then, \( \mu_i(\lambda), (i = 1, 2, \ldots, \frac{n+1}{2}) \) are eigenvalues of \( \mathcal{L}_{\tau_n(G)} \) with \( m_G(\lambda) = m_{\tau_n(G)}(\mu_i(\lambda)), i = 1, 2, \ldots, \frac{n+1}{2} \).
The proof of Theorem 4.2 is presented in Section 7.

**Theorem 4.3:** Let $G$ be a connected graph with $N_0$ vertices and $E_0$ edges, and $\tau_n(G)$ be the $n$-polygon graph of $G$, where $n \geq 2$ and $n$ is even. The eigenvalues for the normalized Laplacian $\mathcal{L}_{\tau_n(G)}$ can be obtained in the following way.

(i) $\mu = 0$ is an eigenvalue of $\mathcal{L}_{\tau_n(G)}$ with multiplicity 1.

(ii) Letting $\mu$ be an arbitrary root of equation $a_2(\mu) + a_2^{-1}(\mu) = 0$, $\mu$ is an eigenvalue of $\mathcal{L}_{\tau_n(G)}$ with multiplicity $N_0$.

(iii) Letting $\mu$ be an arbitrary root of equation $a_2^{-1}(\mu) = 0$, $\mu$ is an eigenvalue of $\mathcal{L}_{\tau_n(G)}$ with multiplicity $E_0 - N_0 + 1$.

(iv) When $G$ is a non-bipartite graph, letting $\mu$ be an arbitrary root of equation $a_2(\mu) - a_2^{-1}(\mu) = 0$, $\mu$ is an eigenvalue of $\mathcal{L}_{\tau_n(G)}$ with multiplicity $E_0 - N_0$.

(v) When $G$ is a bipartite graph, letting $\mu$ be an arbitrary root of equation $a_2^{-1}(\mu) = 0$, $\mu$ is an eigenvalue of $\mathcal{L}_{\tau_n(G)}$ with multiplicity $E_0 - N_0 + 1$.

(vi) When $G$ is a non-bipartite graph, letting $\mu$ be an arbitrary root of equation $a_2(\mu) - a_2^{-1}(\mu) = 0$, $\mu$ is an eigenvalue of $\mathcal{L}_{\tau_n(G)}$ with multiplicity $E_0 - N_0$.

Then, $\mu_i(\lambda)$, $(i = 1, 2, \ldots, \frac{n}{2})$ are eigenvalues of $\mathcal{L}_{\tau_n(G)}$ with $m_{\mathcal{L}_G}(\lambda) = m_{\mathcal{L}_{\tau_n(G)}}(\mu_i(\lambda))$, $i = 1, 2, \ldots, \frac{n}{2}$.

The proof of Theorem 4.3 is similar to the proof of Theorem 4.2. Owing to a length restriction, we omit the detailed derivation herein.

**Remark 4.3:** Using Theorems 4.2 and 4.3, we can obtain the complete normalized Laplacian spectrum $\sigma(\tau_n(G))$ of $n$-polygon graph $\tau_n(G)$. We can also derive the complete normalized Laplacian spectrum of graph $\tau^g_n(G)$ ($n \geq 2, g \geq 1$), defined in Definition 2.8, which is the iterated $n$-polygon graph of $G$ with generation $g$, by applying Theorems 4.2 and 4.3 recursively. In fact, letting $n = 2$ and $n = 4$ in Theorem 4.3, we can recover the results obtained in [25,31], respectively; in addition, letting $n = 3$ in Theorem 4.2, we can recover the results obtained in [30].

**5. Applications**

Recalling that the multiplicative degree-Kirchhoff index, Kemeny’s constant, and the number of spanning trees can be expressed as functions of the normalized Laplacian spectrum, as shown in Lemma 3.2, we obtain the following Theorems.

**Theorem 5.1:** Let $G$ be a connected undirected graph with $N_0$ vertices and $E_0$ edges, $\tau_n(G)$ ($n \geq 2$) be an $n$-polygon graph of $G$, and $\tau^g_n(G)$ ($n \geq 2, g \geq 1$), defined in Definition 2.8, be an iterated $n$-polygon graph of $G$ with generation $g$, and let $K^f(G)$, $K^f(\tau_n(G))$, and $K^f(\tau^g_n(G))$ represent the multiplicative degree-Kirchhoff indexes of graph $G$, $\tau_n(G)$, and
\[ \tau_n^g(G), \] respectively. Then, for any \( n \geq 2, \)
\[ Kf'(\tau_n(G)) = (n^2 + n) Kf'(G) + \frac{2}{3} (n + 1) (n^2 - 1) E_0 - \frac{2}{3} (n^2 - 1) E_0 N_0 - \frac{1}{6} (n^2 - 1) (n - 2) E_0, \]
and for \( n \geq 2 \) and \( g \geq 1, \)
\[ Kf'(\tau_n^g(G)) = (n^2 + n)^g Kf'(G) - \frac{1}{3} (n - 2) (n + 1)^g (n^g - 1) E_0 + \frac{2 (n - 1)}{3 n} (n + 1)^g \left((n + 1)^g (n^2 + 1) - n^{g+2} - 1\right) E_0^2 - \frac{2}{3} (n + 1)^g (n^g - 1) E_0 N_0. \] (31)

The proof of Theorem 5.1 is presented in Section 8.

**Theorem 5.2:** Let \( G \) be a connected undirected graph with \( N_0 \) vertices and \( E_0 \) edges, and \( K(G), K(\tau_n(G)), \) and \( K(\tau_n^g(G)) \) represent Kemeny’s constant of graph \( G, \tau_n(G), \) and \( \tau_n^g(G), \) respectively. Then, for any \( n \geq 2, \)
\[ K(\tau_n(G)) = n K(G) + \frac{1}{3} (n^2 - 1) E_0 - \frac{1}{3} (n - 1) N_0 - \frac{1}{12} (n - 1) (n - 2), \]
and for \( n \geq 2 \) and \( g \geq 1, \)
\[ K(\tau_n^g(G)) = n^g K(G) - \frac{1}{3} (n^g - 1) N_0 - \frac{1}{12} (n - 2) (n^g - 1) \left(\frac{(n - 1) (n^2 + 1)}{3 n} (n + 1)^g + \frac{1}{3} (-n^3 + n^2 + 1) n^{g-1} - \frac{1}{3}\right) E_0. \]

This Theorem can be obtained directly from Theorem 5.1 and Lemma 3.2.

**Theorem 5.3:** Let \( G \) be a connected undirected graph with \( N_0 \) vertices and \( E_0 \) edges, and let \( N_{st}(G), N_{st}(\tau_n(G)), \) and \( N_{st}(\tau_n^g(G)) \) represent the number of spanning trees of graph \( G, \tau_n(G), \) and \( \tau_n^g(G), \) respectively. Then, for \( n \geq 2, \)
\[ N_{st}(\tau_n(G)) = (n + 1)^{N_0 - 1} n E_0 - N_0 + 1 \times N_{st}(G), \] (32)
and for \( n \geq 2 \) and \( g \geq 1, \)
\[ N_{st}(\tau_n^g(G)) = (n + 1)^{\frac{(n-1)(n+1)^g-n^g-1}{n^g}} E_0 + g N_0 - g \times n^{\frac{(n+1)^g+(n-1)^g-1}{n^g}} E_0 - g N_0 + g \times N_{st}(G). \] (33)

The proof of Theorem 5.3 is presented in Section 9.
We will show the relation between the corresponding entries of \( -\text{polygon} \), whose nodes are labelled as \( n \).

Thus, for any vertex \( v \in V \),

\[
\text{denote the set of the new neighbours of vertex } v.
\]

First, we prove the ‘only if’ part of this Lemma.

Similarly, for any vertex \( v \in V \),

\[
\text{let } \vec{v} = (v_1, v_2, \ldots, v_{N1})^T \text{ be an eigenvector with respect to the eigenvalue } \mu \text{ of } \tau_n(G). \text{ Thus,}
\]

\[
\mathcal{L}_{\tau_n(G)} \vec{v} = \mu \vec{v}. \quad (34)
\]

Replacing every entry of \( \mathcal{L}_{\tau_n(G)} \) from Equation (3), we find that for any vertex \( q \) of graph \( \tau_n(G) \), the corresponding entry \( v_q \) in \( \vec{v} \) satisfies

\[
(1 - \mu) v_q = \sum_{t=1}^{N1} A_{\tau_n(G)}(q, t) \sqrt{d'_q d'_t} v_t, \quad (35)
\]

where \( A_{\tau_n(G)}(q, t) \) is the \((q, t)\) entry for the adjacency matrix of \( \tau_n(G) \), and \( d'_q \) and \( d'_t \) are the degrees for vertex \( q \) and \( t \) in graph \( \tau_n(G) \), respectively.

Let \( V(G) \) and \( V(\tau_n(G)) \) be the vertexes set of graph \( G \) and \( \tau_n(G) \), respectively, and \( V_N = \{ v \in V(\tau_n(G)), v \notin V(G) \} \). We have \( V(G) \subset V(\tau_n(G)) \) and \( V(\tau_n(G)) = V_N \cup V(G) \).

We will show the relation between the corresponding entries of \( \vec{v} \) for the nodes in \( V(G) \) and those for the nodes in \( V_N \).

For an arbitrary edge of graph \( G \), let \( i \) and \( j \) be the two ends of the edge. Recalling the construction of the graph \( \tau_n(G) \), as shown in Figure 1, the edge is replaced with an \((n + 1)\)-polygon, whose nodes are labelled as \( i, i'_1, i'_2, \ldots, i'_{n-1}, i'_{n-1}, \text{ and } j \).

Let \( N_O^i \subseteq V(G) \) denote the set of neighbours of vertex \( i \) in graph \( G \), and \( N_N^i \subseteq V_N \) denote the set of the new neighbours of vertex \( i \) in graph \( \tau_n(G) \). Note that \( d'_i = 2d_i \) and \( d'_j = 2d_j \). We can rewrite Equation (35) as

\[
(1 - \mu) v_i = \sum_{i'_1 \in N_N^i} \frac{1}{\sqrt{d'_i d'_1}} v_{i'_1} + \sum_{j \in N_O^i} \frac{1}{\sqrt{d'_i d'_j}} v_j
\]

\[
= \sum_{i'_1 \in N_N^i} \frac{1}{2\sqrt{d_i}} v_{i'_1} + \sum_{j \in N_O^i} \frac{1}{2\sqrt{d_i}} v_j. \quad (36)
\]

Similarly, for any vertex \( i'_1 \in N_N^i \),

\[
(1 - \mu) v_{i'_1} = \frac{1}{\sqrt{d_{i'_1} d_{i'_2}}} v_{i'_2} + \frac{1}{\sqrt{d_{i'_1} d_i}} v_i
\]

\[
= \frac{1}{2} v_{i'_2} + \frac{1}{2\sqrt{d_i}} v_i, \quad (37)
\]

for the vertex \( i'_{n-1} \in N_N^i \),

\[
(1 - \mu) v_{i'_{n-1}} = \frac{1}{2} v_{i'_{n-2}} + \frac{1}{2\sqrt{d_j}} v_j, \quad (38)
\]
and for the vertex \( i_k \) (2 ≤ \( k \) ≤ \( n - 2 \)),

\[
(1 - \mu) v_k' = \frac{1}{2} \left( v_{k-1}' + v_{k+1}' \right).
\]

(39)

Eliminating the variables \( v_2', v_3', \ldots, v_{n-2}' \) in Equations (37) and (39), we have

\[
a_{n-2}(\mu) v_1' = v_{n-1}' + a_{n-3}(\mu) \frac{v_i}{\sqrt{d_i}},
\]

(40)

where \( \{a_n(\mu)\} \) is just the series defined in Lemma 3.6.

Similarly, eliminating the variables \( v_2', v_3', \ldots, v_{n-1}' \) in Equations (37)–(39), we obtain

\[
a_{n-1}(\mu) v_1' = a_{n-2}(\mu) \frac{1}{\sqrt{d_i}} v_i + \frac{1}{\sqrt{d_j}} v_j.
\]

(41)

Therefore, when \( a_{n-1}(\mu) \neq 0 \), multiplying both sides of Equation (36) by a factor of \( a_{n-1}(\mu) \), and replacing \( a_{n-1}(\mu) v_i' \) from Equation (41), we have

\[
2 (1 - \mu) a_{n-1}(\mu) v_i = a_{n-2}(\mu) v_i + \sum_{j \in N_O^i} \frac{1 + a_{n-1}(\mu)}{\sqrt{d_i d_j}} v_j,
\]

(42)

which is equivalent to

\[
a_{n}(\mu) v_i = \sum_{j \in N_O^i} \frac{1 + a_{n-1}(\mu)}{\sqrt{d_i d_j}} v_j.
\]

(43)

In the case of \( a_{n-1}(\mu) \neq 0 \) and \( 1 + a_{n-1}(\mu) \neq 0 \), Equation (43) is also equal to

\[
\frac{a_n(\mu)}{1 + a_{n-1}(\mu)} v_i = \sum_{j \in N_O^i} \frac{1}{\sqrt{d_i d_j}} v_j = \sum_{j \in V(G)} \frac{A_{ij}}{\sqrt{d_i d_j}} v_j,
\]

(44)

which implies that \( \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} \) is simply an eigenvalue of the matrix \( D(G)^{-\frac{1}{2}} A(G) D(G)^{-\frac{1}{2}} \).

Therefore, when \( a_{n-1}(\mu) \neq 0 \) and \( 1 + a_{n-1}(\mu) \neq 0 \), \( \lambda \equiv 1 - \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} \) is an eigenvalue of the matrix \( L_G = I - D(G)^{-\frac{1}{2}} A(G) D(G)^{-\frac{1}{2}} \).

Furthermore, Equations (37)–(39) also show that the entries \( v_k' \) (\( k = 1, 2, \ldots, n-1 \)) of \( \vec{v} \) are completely determined by the entries \( v_i \) and \( v_j \). This is the number of dimensions of the solution space for the linear equation \( \mathcal{L}_G \vec{v} = \mu \vec{v} \) is the same as that for the linear equation \( \mathcal{L}_G \vec{v}' = \lambda \vec{v}' \). Therefore, \( \lambda \equiv 1 - \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} \) is an eigenvalue of the matrix \( \mathcal{L}_G \) with the same multiplicity as the eigenvalue \( \mu \) of \( \mathcal{L}_{\tau_n(G)} \).

As shown in Lemmas 3.7 and 3.8, if \( a_{n-1}(\mu) \neq 0 \) and \( 1 + a_{n-1}(\mu) \neq 0 \), \( \lambda \equiv 1 - \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} \neq 0 \) and \( \lambda \equiv 1 - \frac{a_n(\mu)}{1 + a_{n-1}(\mu)} \neq 2 \).

Therefore, we obtain the ‘only if’ part of this Lemma, and we then prove the ‘if’ part of this Lemma.
Let $\lambda$ be an arbitrary eigenvalue of $L_G$ with multiplicity $k$ ($k > 0$), and $\vec{v}' = (v_1, v_2, \ldots, v_N)^T$ be an eigenvector with respect to the eigenvalue $\lambda$ of $L_G$, i.e.,

$$L_G \vec{v}' = \lambda \vec{v'},$$

which can be rewritten as

$$(1 - \lambda) v_i = \sum_{j \in V(G)} \frac{A_{ij}}{d_i d_j} v_j = \sum_{j \in N^O_i} \frac{1}{d_i d_j} v_j,$$

for any node $i \in V(G)$. Therefore, for any $\mu$ that satisfies $\lambda = 1 - \frac{a_{n-1}(\mu)}{1 + a_{n-1}(\mu)}$, Equation (44) holds and $1 + a_{n-1}(\mu) \neq 0$.

Note that $\lambda \neq 0$ and $\lambda \neq 2$. As proved in Lemmas 3.7 and 3.8, we obtain $a_{n-1}(\mu) \neq 0$, and for any node $i \in V(G)$, Equation (42) holds.

For any edge with ends $i$ and $j$, let $v_i$ and $v_j$ be the two corresponding entries in $\vec{v}'$, and let

$$v_{i1}' = \frac{a_{n-2}(\mu)}{a_{n-1}(\mu)} \frac{1}{\sqrt{d_i}} v_i + \frac{1}{a_{n-1}(\mu) \sqrt{d_j}} v_j,$$

$$v_{i2}' = 2(1 - \mu) v_{i1}' - \frac{1}{\sqrt{d_i}} v_i.$$

In addition, for any $k$ ($2 \leq k \leq n - 2$),

$$v_{k+1}' = 2(1 - \mu) v_k' - v_{k-1}'.$$

In this way, we can obtain an $N_1$-dimensional vector $\vec{v}$, whose entries satisfy Equations (36)–(39). Then, for any node $q \in V(\tau_n(G))$, the entry $v_q$ in $\vec{v}$ satisfies Equation (35).

Therefore, the vector $\vec{v}$ satisfies Equation (34), and $\mu$ is an eigenvalue of $L_{\tau_n(G)}$. Note that the corresponding eigenvectors $\vec{v}$ of $\mu$ is in a one-to-one correspondence with the eigenvectors $\vec{v}'$ of $\lambda$. Thus, the eigenvalue $\mu$ of $L_{\tau_n(G)}$ has the same multiplicity as the eigenvalue $\lambda$ of $L_G$.

This ends the proof.

7. **Proof of Theorem 4.2**

(i) It can be obtained directly from Lemma 3.1.

(ii) For any root $\mu$ of equation $a_{n-1}(\mu) = 0$, we can obtain from Lemma 3.7 (see Equation (19)) that

$$a_{n-1}(\mu) = -1, \quad \text{and} \quad a_{n-2}(\mu) = -2(1 - \mu).$$

Inserting the two equations into Equation (41), we have

$$v_{i1}' = \frac{2(1 - \mu)}{\sqrt{d_i}} v_i - \frac{1}{\sqrt{d_j}} v_j.$$


Therefore, for any node \( i \in V(G) \),

\[
\sum_{i_1 \in N_N^i} \frac{1}{\sqrt{d_i' d_{i_1}'}} v_{i_1}' + \sum_{j \in V(G)} \frac{1}{\sqrt{d_j' d_j}} v_j
\]

\[
= \sum_{i_1 \in N_N^i} \frac{1}{2\sqrt{d_i}} \left[ \frac{2(1 - \mu)}{\sqrt{d_i}} v_i - \frac{1}{\sqrt{d_j}} v_j \right] + \sum_{j \in N_O^i} \frac{1}{2\sqrt{d_j d_j}} v_j
\]

\[
= \sum_{j \in N_O^i} \frac{(1 - \mu)}{d_j} v_j = (1 - \mu) v_i,
\]  

(51)

where \( N_N^i \) and \( N_O^i \) represent the neighbours of node \( i \) in \( V_N \) and \( V(G) \), respectively. Thus, Equation (36) holds regardless the value of \( v_i \).

Let \( \vec{v}' = (v_1, v_2, \ldots, v_{N_0})^T \) be an arbitrary vector. Equation (51) inform us that Equation (36) holds if \( \mu \) is a root of equation \( a_{n-1}^{\mu} = 0 \). For an arbitrary edge of graph \( G \), let nodes \( i \) and \( j \) be the two ends of the edge. Considering the \((n + 1)\)-polygon shown on the right-hand side of Figure 1, and calculating \( v_i', v_i^{'2}, \ldots, v_i^{n-1} \) using Equations (47)–(49), we obtain an \( N_1 \)-dimensional vector \( \vec{v} \), whose entries satisfy Equations (36)–(39). Then, for any node \( q \in V(\tau_n(G)) \), the entry \( v_q \) in \( \vec{v} \) satisfies Equation (35).

Therefore, the vector \( \vec{v} \) satisfies Equation (34), and \( \mu \) is an eigenvalue of \( L_{\tau_n(G)} \). Furthermore, there are strict one-to-one correspondences between the eigenvectors \( \vec{v} \) of \( \mu \) and \( \vec{v}' \), which is an arbitrary vector in an \( N_0 \)-dimensional space. Thus, the multiplicity of the eigenvalue \( \mu \) is \( N_0 \).

(iii) Let \( \mu \) be an arbitrary root of equation \( a_{n-1}^{\mu} + a_{n-2}^{\mu} = 0 \). Then,

\[
a_{n-1}^{\mu} = [a_{n-1}^{\mu} - a_{n-1}^{\mu-1}][a_{n-1}^{\mu} + a_{n-1}^{\mu-1}] = 0,
\]

and we can obtain from Lemma 3.7 (see Equation (17)) that

\[
a_{n-2}^{\mu} = -1.
\]

Therefore,

\[
a_{n-3}^{\mu} = 2(1 - \mu)a_{n-2}^{\mu} - a_{n-1}^{\mu} = -2(1 - \mu).
\]

Replacing \( a_{n-1} \) and \( a_{n-2} \) with 0 and \(-1\) in Equation (41), respectively, for any two nodes \( i \) and \( j \) of graph \( G \), if \( i \sim j \),

\[
\frac{v_i}{\sqrt{d_i}} = \frac{v_j}{\sqrt{d_j}}.
\]

(52)

Because \( G \) is a connected graph, Equation (52) shows that \( \frac{v_i}{\sqrt{d_i}} \) is a constant for any \( i \in V(G) \). Let \( \theta = \frac{v_i}{\sqrt{d_i}} \), for any \( i \in V(G) \). Equations (40) and (36) can then be rewritten
as
\[ v_{i'_1} + v_{i'_{n-1}} = 2 (1 - \mu) \theta, \quad i'_1 \in N^i_N, i'_{n-1} \in N^j_N, \] (53)
\[ \sum_{i'_1 \in N^i_N} v_{i'_1} = [2 (1 - \mu) - 1] \theta d_i, \quad i \in V(G). \] (54)

Calculating \( \sum_{i \in V(G)} \sum_{i'_1 \in N^i_N} v_{i'_1} \) using Equation (53), we have
\[ \sum_{i \in V(G)} \sum_{i'_1 \in N^i_N} v_{i'_1} = \frac{1}{2} \sum_{i \sim j, i'_1 \in N^i_N, i'_{n-1} \in N^j_N} (v_{i'_1} + v_{i'_{n-1}}) = 2 (1 - \mu) \theta E_0. \] (55)

By contrast, calculating \( \sum_{i \in V(G)} \sum_{i'_1 \in N^i_N} v_{i'_1} \) using Equation (54), we obtain
\[ \sum_{i \in V(G)} \sum_{i'_1 \in N^i_N} v_{i'_1} = [2 (1 - \mu) - 1] \theta \sum_{i \in V(G)} d_i = [4 (1 - \mu) - 2] \theta E_0. \] (56)

Therefore, \( [4 (1 - \mu) - 2] \theta = 2 (1 - \mu) \theta, \) which leads to \( \mu = 0 \) or \( \theta = 0. \)

If \( \mu = 0, \) replacing \( a_{n-1}^\mu \) and \( a_{n-2}^\mu \) from Equation (7), we have
\[ a_{n-1}^\mu + a_{n-2}^\mu = \frac{n-1}{2} + 1 + \frac{n-1}{2} - 1 + 1 = n \neq 0. \]

Thus, \( \mu \neq 0, \) and \( \theta = 0, \) which leads to
\[ v_i = 0, \] (57)
for any node \( i \) of graph \( G. \)

Substituting \( v_i \) with 0, Equation (36) can be rewritten as
\[ \sum_{i'_1 \in N^i_N} v_{i'_1} = 0, \] (58)
for any node \( i \in V(G). \)

Similarly, replacing \( v_i \) and \( a_{n-2}(\mu) \) with 0 and \( -1, \) respectively, in Equation (40), we obtain
\[ v_{i'_1} + v_{i'_{n-1}} = 0. \] (59)

Furthermore, if \( n \) is odd and \( v_i = 0, \) in any \((n + 1)\)-polygon of \( \tau_n(G), \) as shown in Figure 1, we find that Equations (37)–(39) are equivalent to
\[ v_{i'_t+1} + v_{i'_{n-t-1}} = 0, \quad t = 0, 1, 2, \ldots, \frac{n-1}{2} - 1, \] (60)
\[ a_t(\mu_i) v_{i'_t} = v_{i'_{t+1}}, \quad t = 1, 2, \ldots, \frac{n-1}{2} - 1. \] (61)

Therefore, Equation (35) holds if and only if Equations (57), (58), (60), and (61) hold for any \((n + 1)\)-polygon of graph \( \tau_n(G). \)
Note that the total number of \((n + 1)\)-polygons in graph \(\tau_n(G)\) is \(E_0\). For an arbitrary \((n + 1)\)-polygon \(P_r\) of graph \(\tau_n(G)\), let \(v_{t+1}^r = -v_{n-t}^r = x_r\), and let \(X = (x_1, x_2, \ldots, x_{E_0})^T\). We can see that Equations (58) and (59) are equivalent to \(B'X = 0\), where \(B'\) is the incident matrix of weakly connected directed graph \(G'\) (see Definitions 2.5 and 2.6).

Similarly, for an arbitrary \((n + 1)\)-polygon \(P_r\) of graph \(\tau_n(G)\), let

\[
v_{t+1}^r = -v_{n-t}^r = y_t^r, \quad t = 1, 2, \ldots, \frac{n-1}{2} - 1,
\]

and \(Y(t) = (y_1^t, y_2^t, \ldots, y_{E_0}^t)^T\), Equations (60) and (61) can be rewritten as

\[
Y(t) - a_t(\mu)X = 0, \quad t = 1, 2, \ldots, \frac{n-1}{2} - 1.
\]

Therefore, there are strict one-to-one correspondences between the roots of Equation (34) and the roots of the following equation:

\[
\begin{pmatrix}
I & 0 & \cdots & 0 & -a_1(\mu)I \\
0 & I & \cdots & 0 & -a_2(\mu)I \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I & -a_{n-1}(\mu)I \\
0 & 0 & \cdots & 0 & B'
\end{pmatrix}
\begin{pmatrix}
Y(1) \\
Y(2) \\
\vdots \\
Y(n-1) \\
X
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \tag{62}
\]

where \(I\) is an identity matrix of order \(E_0\), and \(B'\) is the incidence matrix of weakly connected directed graph \(G'\) whose underlying undirected graph (where all edges are replaced with undirected edges) is \(G\).

Because \(\text{rank}(B') = N_0 - 1\) (see Lemma 3.5), the rank for the coefficient matrix of linear Equation (62) is \(\frac{n}{2}E_0 + N_0\), and the number of dimensions for the vector space spanned by the roots of linear Equation (62) is \(E_0 - N_0 + 1\). Thus, the number of dimensions for the eigenspace of \(\mu\) is also \(E_0 - N_0 + 1\), and the multiplicity for eigenvalue \(\mu\) is \(E_0 - N_0 + 1\).

(iv) Let \(\mu\) be an arbitrary root of equation \(a_{n-1}(\mu) - a_{n-1}(\mu) = 0\). Then,

\[
a_{n-1}(\mu) = [a_{n-1}(\mu) - a_{n-1}(\mu)] [a_{n-1}(\mu) + a_{n-1}(\mu)] = 0,
\]

and we can obtain from Lemma 3.7 (see Equation (A15)) that

\[
a_{n-2}(\mu) = 1.
\]

Replacing \(a_{n-1}\) and \(a_{n-1}\) with 0 and 1 in Equation (41), for any two nodes \(i\) and \(j\) of graph \(G\), if \(i \sim j\),

\[
\frac{v_i}{\sqrt{d_i}} = -\frac{v_j}{\sqrt{d_j}}. \tag{63}
\]

Note that \(G\) is non-bipartite, and there is at least an odd cycle \(C\) in \(G[9,34]\). Let \(i_1, i_2, \ldots, i_k\) (where \(k\) is odd) be the nodes series of cycle \(C\). Equation (63) informs us
that

\[
\frac{v_i}{\sqrt{d_i}} = -\frac{v_j}{\sqrt{d_j}} = \frac{v_k}{\sqrt{d_k}} = \cdots = \frac{v_i}{\sqrt{d_i}}.
\]

Therefore, \( v_i = v_j = \cdots = v_k = 0 \). Because \( G \) is connected, Equation (63) informs us that

\[
v_i = 0.
\] (64)

for any node \( i \) of graph \( G \).

Substituting \( v_i \) with 0, Equation (36) can be rewritten as

\[
\sum_{i'_i \in N^i_i} v_{i'_i} = 0,
\] (65)

for any node \( i \) of graph \( G \).

Similarly, replacing \( v_i \) and \( a_{n-2}(\mu) \) with 0 and 1, respectively, in Equation (40), we obtain

\[
v_{i'_t} - v_{i'_{t-1}} = 0.
\] (66)

Furthermore, if \( n \) is odd and \( v_i = 0 \) for any node \( i \in V(G) \), we can see that Equations (37)–(39) are equivalent to

\[
v_{i_{t+1}} - v_{i_{t-1}} = 0, \quad t = 0, 1, 2, \ldots, \frac{n-1}{2} - 1,
\]

\[
a_{t}(\mu) v_{i'_t} = v_{i'_{t+1}}, \quad t = 1, 2, \ldots, \frac{n-1}{2} - 1.
\] (68)

Therefore, Equation (35) holds if and only if Equations (64), (65), (67), and (68) hold for any \((n + 1)\)-polygon of graph \( \tau_n(G) \).

Note that the total number of \((n + 1)\)-polygons in graph \( \tau_n(G) \) is \( E_0 \). For an arbitrary \((n + 1)\)-polygon \( P_r \) of graph \( \tau_n(G) \), let \( v_{i'_r} = v_{i'_{r-1}} = x_r \), and let \( X = (x_1, x_2, \ldots, x_{E_0})^T \). We can see that Equations (65) and (66) are equivalent to \( BX = 0 \), where \( B \) is the incident matrix of graph \( G \).

Similarly, for an arbitrary \((n + 1)\)-polygon, \( P_r \), of graph \( \tau_n(G) \), letting

\[
v_{i'_{t+1}} = v_{i'_{t-1}} = y_r^{(t)}, \quad t = 1, 2, \ldots, \frac{n-1}{2} - 1,
\]

and letting \( Y^{(t)} = (y_1^{(t)}, y_2^{(t)}, \ldots, y_{E_0}^{(t)})^T \), Equations (67) and (68) can be rewritten as

\[
Y^{(t)} - a_{t}(\mu)X = 0, \quad t = 1, 2, \ldots, \frac{n-1}{2} - 1.
\]

Therefore, there are strict one-to-one correspondences between the roots of Equation (34) and the roots of the equation

\[
\begin{pmatrix}
I & 0 & \cdots & 0 & -a_1(\mu)I & 0 & 0 & \cdots & 0 \\
0 & I & \cdots & 0 & -a_2(\mu)I & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I & -a_{n-1}(\mu)I & B & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\begin{pmatrix}
Y_1^{(1)} \\
Y_2^{(2)} \\
\vdots \\
Y_{E_0}^{(n-1)} \\
X \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
= 0.
\] (69)
where $B$ is the incident matrix of $G$, and $I$ is an identity matrix of order $E_0$.

Because $G$ is non-bipartite, $\text{rank}(B) = N_0$ (see Lemma 3.4). Therefore the rank for the coefficient matrix of linear Equation (69) is $\frac{n-3}{2}E_0 + N_0$, and the number of dimensions for the vector space spanned by the roots of linear Equation (69) is $E_0 - N_0$. Thus, the number of dimensions for the eigenspace of $\mu$ is also $E_0 - N_0$, and the multiplicity for eigenvalue $\mu$ is $E_0 - N_0$.

(v) Let $\mu$ be an arbitrary root of equation $a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2}-1}(\mu) = 0$. That is,

$$a_{\frac{n-1}{2}}(\mu) = a_{\frac{n-1}{2}-1}(\mu).$$

As shown in Lemma 3.6, for any $n \geq 1$, $a_n(2 - \mu) = (-1)^n a_n(\mu)$. Therefore,

$$a_{\frac{n-1}{2}}(2 - \mu) = (-1)^{\frac{n-1}{2}} a_{\frac{n-1}{2}}(\mu) = (-1)^{\frac{n-1}{2}} a_{\frac{n-1}{2}-1}(\mu) = (-1)^{\frac{n-1}{2}} \times (-1)^{\frac{n-1}{2}-1} a_{\frac{n-1}{2}-1}(2 - \mu) = (-1)^{n-2} a_{\frac{n-1}{2}-1}(2 - \mu) = (-1)a_{\frac{n-1}{2}-1}(2 - \mu).$$

Thus, $2 - \mu$ is a root of equation $a_{\frac{n-1}{2}}(\mu) + a_{\frac{n-1}{2}-1}(\mu) = 0$.

Therefore, $2 - \mu$ is an eigenvalue $\tau_n(G)$ with multiplicity $E_0 - N_0 + 1$ (see (iii) of this Theorem).

Note that $G$ is bipartite, and thus $\tau_n(G)$ is also bipartite. We can obtain from Lemma 3.6 that $\mu$ is also an eigenvalue $\tau_n(G)$ with multiplicity $E_0 - N_0 + 1$.

(vi) This is simply the result of Lemma 4.1.

The proof is completed.

8. Proof of Theorem 5.1

Proof: First, we prove that Equation (30) holds when $n$ is an odd number.

Let $\lambda$ ($\lambda \neq 0$ and $\lambda \neq 2$) be an arbitrary eigenvalue of $L_G$, and $\mu_i(\lambda)(i = 1, 2, \cdots, \frac{n+1}{2})$ be the roots of equation

$$1 - \frac{a_{\frac{n-1}{2}+1}(x) - a_{\frac{n-1}{2}-1}(x)}{a_{\frac{n-1}{2}}(x) - a_{\frac{n-1}{2}-2}(x)} = \lambda. \hspace{1cm} (71)$$

Because $a_{\frac{n-1}{2}}(x) - a_{\frac{n-1}{2}-2}(x) \neq 0$, Equation (71) can be rewritten as

$$f(x) = a_{\frac{n+1}{2}}(x) - (1-\lambda)a_{\frac{n-1}{2}}(x) - a_{\frac{n-3}{2}}(x) + (1-\lambda)a_{\frac{n-5}{2}}(x) = 0, \hspace{1cm} (72)$$

which is a polynomial equation of degree $\frac{n+1}{2}$. 


Let $b_i$ be the coefficient of $x^i$ ($i = 1, 2, \cdots, \frac{n+1}{2}$) for polynomial $f(x)$. According to Lemma 3.6(iv), we obtain
\[
b_0 = \alpha_{\frac{n+1}{2}}^{(0)}(x) - (1 - \lambda) \alpha_{\frac{n+1}{2}}^{(1)}(x) - \alpha_{\frac{n+1}{2}}^{(2)}(x) + (1 - \lambda) \alpha_{\frac{n+1}{2}}^{(3)}(x) = 2\lambda,
\]
\[
b_1 = \alpha_{\frac{n+1}{2}}^{(1)}(x) - (1 - \lambda) \alpha_{\frac{n+1}{2}}^{(2)}(x) - \alpha_{\frac{n+1}{2}}^{(3)}(x) + (1 - \lambda) \alpha_{\frac{n+1}{2}}^{(4)}(x) = -\frac{(n - 1)^2}{2}\lambda - 2n,
\]
and
\[
b_{\frac{n+1}{2}} = a_{\frac{n+1}{2}}^{(\frac{n+1}{2})}(x) = (-1)^{\frac{n+1}{2}} 2^{\frac{n+1}{2}}.
\]
Using Lemma 3.3, we have
\[
\sum_{i=1}^{\frac{n+1}{2}} \frac{1}{\tilde{\mu}_i(\lambda)} = -\frac{b_1}{b_0} = \frac{n}{\lambda} + \left(\frac{n - 1}{2}\right)^2.
\]
Similarly, let $\tilde{\mu}_i(i = 1, 2, \cdots, \frac{n-1}{2})$ be the roots of equation $a_{\frac{n-1}{2}}^{(n-1)} - a_{\frac{n-1}{2} - 1}^{(n-1)} = 0$, $\hat{\mu}_i(i = \frac{n-1}{2} + 1, \frac{n-1}{2} + 2, \ldots, n - 1)$ be the roots of equation $a_{\frac{n-1}{2}}^{(n-1)} + a_{\frac{n-1}{2} - 1}^{(n-1)} = 0$, and $\tilde{\mu}_i(i = 1, 2, \ldots, \frac{n-1}{2})$ be the roots of equation $a_{\frac{n-1}{2}}^{(n-1)} = 0$. Rewriting these equations as a polynomial and using Lemma 3.3, we have
\[
\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{\tilde{\mu}_i} = \frac{n^2 - 1}{4},
\]
\[
\sum_{i=\frac{n-1}{2} + 1}^{\frac{n-1}{2}} \frac{1}{\hat{\mu}_i} = \frac{n^2 - 1}{12},
\]
and
\[
\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{\tilde{\mu}_i} = \frac{n^2 + 2n - 3}{12}.
\]
Let $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{N_0}$ be the eigenvalues of the normalized Laplacian $L_G$. If $G$ is non-bipartite, Theorem 4.2 informs us that the normalized Laplacian spectrum of graph $\tau_n(G)$ can be written as follows:
\[
\sigma(\tau_n(G)) = \bigcup_{j=2}^{N_0} \left[ \bigcup_{i=1}^{\frac{n+1}{2}} \{ \mu_i(\lambda_j) \} \bigcup_{i=1}^{\frac{n-1}{2}} \{ \hat{\mu}_i, \tilde{\mu}_i, \ldots, \tilde{\mu}_i \} \bigcup_{i=1}^{\frac{n-1}{2}} \{ \hat{\mu}_i, \tilde{\mu}_i, \ldots, \tilde{\mu}_i \} \cup \{0\} \right]
\]
\[
\bigcup_{i=\frac{n-1}{2} + 1}^{n-1} \{ \hat{\mu}_i, \tilde{\mu}_i, \ldots, \tilde{\mu}_i \} \cup \{0\}.
\]
Therefore, when \( n \) is an odd number, and \( G \) is non-bipartite, using Lemma 3.2(i) and Equations (76)–(79), we have

\[
Kf' (\tau_n(G)) = 2E_1 \sum_{u \in \sigma(\tau_n(G)), \mu \neq 0} \frac{1}{\mu}
\]

\[
= 2E_1 \left[ \sum_{i=2}^{n_0} \sum_{j=1}^{n+1} \frac{1}{\mu_j (\lambda_i)} + \sum_{i=1}^{n-1} \frac{1}{\mu_i} N_0 + \sum_{i=1}^{n-1} \frac{1}{\mu_i} (E_0 - N_0) \right]
\]

\[
+ \sum_{i=\frac{n-1}{2}+1}^{n-1} \frac{1}{\mu_i} (E_0 - N_0 + 1) \right]
\]

\[
= 2(n+1)E_0 \left[ n \sum_{i=2}^{n_0} \frac{1}{\lambda_i} + \frac{n^2-1}{3} E_0 - \frac{n-1}{3} N_0 - \frac{(n-1)(n-2)}{6} \right]
\]

\[
= \left[ n^2 + n \right] Kf' (G) + \frac{2}{3} (n+1) \left( n^2 - 1 \right) E_0 - \frac{2}{3} \left( n^2 - 1 \right) E_0 N_0
\]

\[
- \frac{1}{6} \left( n^2 - 1 \right) (n-2) E_0.
\]

(81)

Thus, Equation (30) holds when \( n \) is an odd number and \( G \) is non-bipartite.

Similarly, we can also verify that Equation (30) holds when \( n \) is an odd number and \( G \) is bipartite, and Equation (30) holds when \( n \) is an even number.

Therefore, Equation (30) holds for any \( n (n \geq 2) \).

Next, we will prove that Equation (31) holds for any \( n \geq 2 \) and \( g \geq 1 \).

Recalling the definition of the iterated \( n \)-polygon graph with generation \( g \) (see Definition 2.8), for any \( g \geq 1 \), \( \tau_n^g(G) = \tau_n(\tau_n^{g-1}(G)) \). Therefore, we can obtain from Equation (30) that, for any \( g \geq 1 \),

\[
Kf' \left( \tau_n^g(G) \right) = \left( n^2 + n \right) Kf' \left( \tau_n^{g-1}(G) \right) + \frac{2}{3} (n+1) \left( n^2 - 1 \right) E_{g-1}
\]

\[
- \frac{2}{3} \left( n^2 - 1 \right) E_{g-1} N_{g-1} - \frac{1}{6} \left( n^2 - 1 \right) (n-2) E_{g-1}.
\]

(82)

Using Equation (82) recursively and replacing \( N_g \) from Equation (5), we obtain Equation (31).

\section{Proof of Theorem 5.3}

\textbf{Proof:} First, we prove that Equation (32) holds when \( n \) is an odd number.

Let \( \lambda (\lambda \neq 0 \text{ and } \lambda \neq 2) \) be an arbitrary eigenvalue of \( \mathcal{L}_G \), and \( \mu_i(\lambda)(i = 1, 2, \cdots, \frac{n}{2}) \) be the roots of equation

\[
1 - \frac{a_{\frac{n-1}{2}+1}(x)}{a_{\frac{n-1}{2}}(x)} - \frac{a_{\frac{n-1}{2}-1}(x)}{a_{\frac{n-1}{2}-2}(x)} = \lambda.
\]

(83)
Note that Equation (83) can be rewritten as Equation (72), which is a polynomial, with some coefficients shown as Equations (73) and (75). Using Lemma 3.3, we have

\[ \prod_{i=1}^{n+1} \mu_i(\lambda) = (-1)^n \frac{b_0}{b_{n+1}} = \frac{\lambda}{2^{n-1}}. \]  

(84)

Similarly, let \( \tilde{\mu}_i (i = 1, 2, \ldots, \frac{n-1}{2}) \) be the roots of equation \( a_{\frac{n-1}{2}} - a_{\frac{n-1}{2}+1} = 0 \), \( \tilde{\mu}_i (i = \frac{n-1}{2} + 1, \frac{n-1}{2} + 2, \ldots, n-1) \) be the roots of equation \( a_{\frac{n-1}{2}} + a_{\frac{n-1}{2}+1} = 0 \), and \( \hat{\mu}_i (i = 1, 2, \ldots, \frac{n-1}{2}) \) be the roots of equation \( a_{\frac{n-1}{2}} = 0 \). Rewriting these equations as a polynomial and using Lemma 3.3, we have

\[ \prod_{i=1}^{\frac{n-1}{2}} \tilde{\mu}_i = \frac{1}{2^{n-1}}, \]  

(85)

\[ \prod_{i=\frac{n-1}{2}+1}^{n-1} \tilde{\mu}_i = \frac{n}{2^{n-1}}, \]  

(86)

and

\[ \prod_{i=1}^{\frac{n-1}{2}} \hat{\mu}_i = \frac{n+1}{2^{n+1}}. \]  

(87)

Let \( 0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{N_0} \) be the eigenvalues of the normalized Laplacian \( L_G \). If \( G \) is non-bipartite (note that the normalized Laplacian spectrum \( \sigma(\tau_n(G)) \) of graph \( \tau_n(G) \) is shown in Equation (80)), we have

\[
\prod_{\mu \in \sigma(\tau_n(G)), \mu \neq 0} \mu = \left( \prod_{i=1}^{\frac{n-1}{2}} \tilde{\mu}_i \right)^{N_0} \times \left( \prod_{i=1}^{\frac{n-1}{2}} \tilde{\mu}_i \right)^{E_0-N_0} \times \left( \prod_{i=\frac{n-1}{2}+1}^{n-1} \tilde{\mu}_i \right)^{E_0-N_0+1} \times \prod_{i=2}^{N_0} \prod_{j=1}^{\frac{n-1}{2}} \mu_j(\lambda_i) \\
= \left( \frac{n+1}{2^{n+1}} \right)^{N_0} \times \left( \frac{1}{2^{n-1}} \right)^{E_0-N_0} \times \left( \frac{n}{2^{n-1}} \right)^{E_0-N_0+1} \times \prod_{i=2}^{N_0} \lambda_i \\
= \frac{n^{E_0-N_0+1}}{(2^{n-1})^{E_0}} \left( \frac{n+1}{2} \right)^{N_0} \times \prod_{i=2}^{N_0} \lambda_i. \]  

(88)

Therefore,

\[
\frac{N_{st}(G(n))}{N_{st}(G)} = \frac{2^{N_1} \prod_{\mu \in \sigma(\tau_n(G)), \mu \neq 0} \mu}{(n+1) \prod_{i=2}^{N_0} \lambda_i} = n^{E_0-N_0+1}(n+1)^{N_0-1}. \]  

(89)

Thus, Equation (32) holds when \( n \) is an odd number and \( G \) is non-bipartite.
Similarity, we can also verify that Equation (32) holds in the case \( n \) is an odd number and \( G \) non-bipartite, and Equation (32) holds in the case \( n \) is an even number.

Thus, Equation (32) holds for any \( n \geq 2 \).

Recalling the definition of the iterated \( n \)-polygon graph with generation \( g \) (see Definition 2.8), for any \( g \geq 1 \), \( \tau^g_n(G) = \tau_n(\tau^{g-1}_n(G)) \). Therefore, we can obtain from Equation (32) that, for any \( g \geq 1 \),

\[
N_{st}(\tau^g_n(G)) = (n + 1)^{N_{s-1}-1} n^{E_{s-1}-N_{s-1}+1} N_{st}(\tau^{g-1}_n(G)). \tag{90}
\]

Using Equation (90) recursively, we obtain Equation (33).

10. Conclusion

Given the normalized Laplacian spectrum of an arbitrary connected graph \( G \), we have get the normalized Laplacian spectrum of the graph \( \tau_n(G) \), which is obtained by replacing each edge of \( G \) with a \((n + 1)\)-polygon \((n \geq 2)\). The normalized Laplacian spectrum of the graphs \( \tau^g_n(G) \) \((g \geq 0)\) can also then be obtained. As applications, we also calculated the multiplicative degree-Kirchhoff index, Kemeny’s constant, and the number of spanning trees for graphs \( \tau^g_n(G) \) \((g \geq 0)\). Note that ‘edge replacing’ is a common graph operator that is widely used to construct different meaningful networks. The results obtained here will be helpful for network optimization.

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The proof of the Lemma 3.6

Here, we use ‘$a_k$’ to represent ‘$a_k(\mu)$’ to lighten the notations. Firstly, we solve the recursive relation which series $\{a_n\}$ satisfies and present the general formula of $\{a_n\}$. Let $\beta = 2(1 - \mu)$. Then the recursive relation of series $\{a_n(\mu)\}$ can be rewritten as

$$a_n = \beta a_{n-1} - a_{n-2}, \quad (A1)$$

with initial conditions $a_{-1} = 0$, $a_0 = 1$. Solving the recursive relation, we obtain

$$a_n(\mu) = c_1 \left( \frac{\beta + i\sqrt{4 - \beta^2}}{2} \right)^n + c_2 \left( \frac{\beta - i\sqrt{4 - \beta^2}}{2} \right)^n \equiv \alpha_n(\beta), \quad (A2)$$

where

$$\begin{align*}
c_1 &= \frac{1}{2} - \frac{i\beta}{2\sqrt{4 - \beta^2}}, \\
c_2 &= \frac{1}{2} + \frac{i\beta}{2\sqrt{4 - \beta^2}}. 
\end{align*} \quad (A3)$$

Then, we prove the results of Lemma 3.6 item-by-item.

(i) Note that $2[1 - (2 - \mu)] = 2(\mu - 1) = -\beta$. We have,

$$a_n(2 - \mu) = \alpha_n(-\beta) = (-1)^n \alpha_n(\beta) = (-1)^n a_n(\mu).$$

(ii) Note that $\beta = 2(1 - \mu)$. Then

$$\beta = \begin{cases} 2 & \text{if } \mu = 0, \\
-2 & \text{if } \mu = 2. \end{cases}$$

Replacing $\beta$ with 2 and $-2$ respectively in Equation (A2), we get Equations (7) and (8).

(iii) Firstly, we prove $a_n(\mu)$ is a polynomial of degree $n$ while $(n \geq 0)$ by mathematical induction.

Step 1: for $n = 0$ and $n = 1$,

$$a_0(\mu) = 1, \quad (A4)$$
$$a_1(\mu) = -2\mu + 2. \quad (A5)$$

It is obvious that $a_0(\mu)$ and $a_1(\mu)$ are polynomials of degree 0 and 1 respectively.

Step 2: for any $n \geq 2$, we assume $a_{n-1}(\mu)$ and $a_{n-2}(\mu)$ are polynomials of degree $n-1$ and $n-2$ respectively.

$$a_2(\mu) = 4\mu^2 - 8\mu + 3. \quad (A6)$$

Note that $a_n(\mu) = 2(1 - \mu)a_{n-1}(\mu) - a_{n-2}(\mu)$. Then, the degree of polynomial $a_n(\mu)$ is just 1 plus the degree of polynomial $a_{n-1}(\mu)$. Therefore $a_n(\mu)$ is a polynomial of degree $n$, which can be expressed as

$$a_n(\mu) = \sum_{i=0}^{n} a_n^{(i)} \mu^i, \quad (A7)$$

where $a_n^{(i)}$ is the coefficient of $\mu^i$. 
Replacing \(a_n(\mu), a_{n-1}(\mu)\) and \(a_{n-2}(\mu)\) from Equation (A7) in recursive equation \(a_n(\mu) = 2(1 - \mu)a_{n-1}(\mu) - a_{n-2}(\mu)\), we find \(\{a_n^{(0)}\}\) satisfies the following recurrence relation

\[a_n^{(0)} = 2a_{n-1}^{(0)} - a_{n-2}^{(0)},\]  

(A8)

with initial conditions \(a_0^{(0)} = 1\) and \(a_1^{(0)} = 2\).

By solving the recursive equation as shown in Equation (A8), we obtain Equation (9).

Similarly, we find \(\{a_n^{(1)}\}\) satisfies the following recurrence relation

\[a_n^{(1)} = -2a_{n-1}^{(1)} + 2a_{n-2}^{(1)} = -2n + 2a_{n-1}^{(1)} - a_{n-2}^{(1)},\]  

(A9)

with initial conditions \(a_1^{(1)} = -2, a_2^{(1)} = -8\); and \(\{a_n^{(n)}\}\) satisfies the following recurrence relation

\[a_n^{(n)} = -2a_{n-1}^{(n-1)},\]  

(A10)

with initial condition \(a_0^{(0)} = 1\). By solving the recurrence relation as shown in Equations (A9) and (A10), we obtain Equations (10) and (11).

(iv) Here, we prove Equation (12) by mathematical induction and Equation (13) can also be obtained similarly.

**Step 1:** It is easy to verify that Equation (12) is true for \(n = 2\) and \(n = 3\).

**Step 2:** assume Equation (12) is true for \(n = 2t\) and \(n = 2t + 1\) \((t\) is an arbitrary nonnegative integer), i.e.

\[a_{2t} = (a_t - a_{t-1}) (a_t + a_{t-1}) , \quad a_{2t+1} = (a_{t+1} - a_{t-1}) a_t.\]

Then,

\[a_{2t+2} = \beta a_{2t+1} - a_{2t} = \beta (a_{t+1} - a_{t-1}) a_t \]  

\[= \beta a_t + a_{t-1}) a_t - (a_t^2 - a_{t-1}^2)\]

\[= \beta a_{t+1} - a_t - a_{t-1} a_{t+1}\]

\[= a_{t+1}^2 - a_t^2\]

\[= (a_{t+1} + a_t)(a_{t+1} - a_t),\]

and

\[a_{2t+3} = \beta a_{2t+2} - a_{2t+1} = a_{t+1} (a_{t+2} - a_t).\]

Therefore Equation (12) is true for \(n = 2t + 2\) and \(n = 2t + 3\). Thus Equation (12) is true for any \(n \geq 2\).

Similarly, we can also prove Equation (13) by mathematical induction.

**Proof of Lemma 3.7**

(i) Let \(n\) \((n \geq 3)\) be an arbitrary odd number and \(\mu\) be an arbitrary root of equation \(a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 1}(\mu) = 0\). Therefore,

\[a_{\frac{n-1}{2} - 1}(\mu) = a_{\frac{n-1}{2}}(\mu) = 2 (1 - \mu) a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 2}(\mu).\]  

(A11)

Then

\[2 (1 - \mu) - 1] a_{\frac{n-1}{2} - 1}(\mu) = a_{\frac{n-1}{2} - 2}(\mu).\]  

(A12)
Replacing \( a_{\frac{n-1}{2} + 1}(\mu) \) and \( a_{\frac{n-1}{2}}(\mu) \) with \( 2(1 - \mu)a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 1}(\mu) \) and \( 2(1 - \mu)a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 2}(\mu) \) respectively, we obtain

\[
1 - \frac{a_{\frac{n-1}{2} + 1}(\mu) - a_{\frac{n-1}{2} - 1}(\mu)}{a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 2}(\mu)} = 1 - \frac{2(1 - \mu)a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 1}(\mu)}{2(1 - \mu)a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 2}(\mu) - a_{\frac{n-1}{2} - 2}(\mu)}
\]

\[
= 1 - \frac{[2(1 - \mu) - 2]a_{\frac{n-1}{2} - 1}(\mu)}{[2(1 - \mu) - 2][2(1 - \mu) - 1]} a_{\frac{n-1}{2} - 1}(\mu)
\]

\[
= 2.
\]

(A13)

One can easily obtain from Equation (12) that

\[
an_{n-1} = \left( a_{\frac{n-1}{2}} - a_{\frac{n-1}{2} - 1} \right) \left( a_{\frac{n-1}{2}} + a_{\frac{n-1}{2} - 1} \right) = 0.
\]

(A14)

By using Equations (12) and (13), we have

\[
a_{n-2}(\mu) = a_{\frac{n-2}{2} - 1}(\mu) \left( a_{\frac{n-2}{2} + 1}(\mu) - a_{\frac{n-2}{2} - 1}(\mu) \right)
\]

\[
= a_{\frac{n-1}{2} - 1}(\mu) \left( a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 2}(\mu) \right)
\]

\[
= a_{\frac{n-1}{2}}(\mu) \left( a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 2}(\mu) \right)
\]

\[
= 1 + a_{n-1}(\mu)
\]

\[
= 1.
\]

(A15)

(ii) Let \( \mu \) be an arbitrary root of equation \( a_{\frac{n-1}{2}}(\mu) + a_{\frac{n-1}{2} - 1}(\mu) = 0 \). We have

\[
- a_{\frac{n-1}{2} - 1}(\mu) = a_{\frac{n-1}{2}}(\mu) = 2(1 - \mu)a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 2}(\mu).
\]

(A16)

Thus

\[
[2(1 - \mu) + 1]a_{\frac{n-1}{2} - 1}(\mu) = a_{\frac{n-1}{2} - 2}(\mu).
\]

(A17)

Replacing \( a_{\frac{n-1}{2} + 1}(\mu) \) and \( a_{\frac{n-1}{2}}(\mu) \) with \( 2(1 - \mu)a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 1}(\mu) \) and \( 2(1 - \mu)a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 2}(\mu) \) respectively, we obtain

\[
1 - \frac{a_{\frac{n-1}{2} + 1}(\mu) - a_{\frac{n-1}{2} - 1}(\mu)}{a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 2}(\mu)} = 1 - \frac{2(1 - \mu)a_{\frac{n-1}{2}}(\mu) - a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 1}(\mu)}{2(1 - \mu)a_{\frac{n-1}{2} - 1}(\mu) - a_{\frac{n-1}{2} - 2}(\mu) - a_{\frac{n-1}{2} - 2}(\mu)}
\]

\[
= 1 - \frac{[2(1 - \mu) + 2]a_{\frac{n-1}{2} - 1}(\mu)}{[2(1 - \mu) - 2][2(1 - \mu) + 1]} a_{\frac{n-1}{2} - 1}(\mu)
\]

\[
= 0.
\]

(A18)
Note that \( a_{n-1} = (a_{n-1} - a_{n-1}^{-1})(a_{n-1} + a_{n-1}^{-1}) = 0 \). By using Equations (12) and (13), we have

\[
a_{n-2}(\mu) = a_{n-2}^{-1}(\mu) \left( a_{n-2}^{-1} + a_{n-2}^{-1}(\mu) \right)
= a_{n-1}^{-1}(\mu) \left( a_{n-1}^{-1} + a_{n-1}^{-1}(\mu) \right)
= -a_{n-1}^{-1}(\mu) \left( a_{n-1}^{-1} + a_{n-1}^{-1}(\mu) \right)
= -(1 + a_{n-1}(\mu))
= -1. \tag{A19}
\]

(iii) Let \( \mu \) be an arbitrary root of equation \( a_{n-1}^{-1}(\mu) = 0 \). Then

\[
0 = a_{n-1}^{-1}(\mu) = 2(1 - \mu) a_{n-1}^{-1}(\mu) - a_{n-2}^{-1}(\mu), \tag{A20}
\]

which yields

\[
2(1 - \mu) a_{n-1}^{-1}(\mu) = a_{n-2}^{-1}(\mu).
\]

Replacing \( a_{n-1}^{-1}(\mu) \) and \( a_{n-2}^{-1}(\mu) \) with \( 2(1 - \mu)a_{n-1}^{-1}(\mu) - a_{n-1}^{-1}(\mu) \) and \( 2(1 - \mu)a_{n-1}^{-1}(\mu) \) in \( 1 - \frac{a_{n-1}^{-1} - a_{n-2}^{-1}}{a_{n-1}^{-1} - a_{n-2}^{-1}} \) respectively, we obtain

\[
1 - \frac{a_{n-1}^{-1} + a_{n-2}^{-1}(\mu)}{a_{n-1}^{-1}(\mu) - a_{n-2}^{-1}(\mu)} = 1 - \frac{2(1 - \mu) a_{n-1}^{-1}(\mu) - a_{n-1}^{-1}(\mu) - a_{n-1}^{-1}(\mu)}{-2(1 - \mu)a_{n-1}^{-1}(\mu)}
= 1 - \frac{1}{1 - \mu}. \tag{A21}
\]

Since \( \mu \in (0, 2) \), we have \( 1 - \frac{1}{1 - \mu} \in (-\infty, 0) \cup (2, +\infty) \).

It is easy to verify from Equation (12) that

\[
a_n(\mu) = \left( a_{n+1}^{-1}(\mu) - a_{n-1}^{-1}(\mu) \right) a_{n-1}^{-1}(\mu) = 0.
\]

Thus

\[
1 = 1 + a_n(\mu)
= \left( a_{n+1}^{-1}(\mu) - a_{n-1}^{-2}(\mu) \right) \left( a_{n+1}^{-1}(\mu) + a_{n-1}^{-1}(\mu) \right)
= \left( a_{n+1}^{-1}(\mu) - a_{n-1}^{-1}(\mu) \right) \left[ 2(1 - \mu) a_{n+1}^{-1}(\mu) - a_{n-1}^{-1}(\mu) + a_{n+1}^{-1}(\mu) \right]
= \left( a_{n+1}^{-1}(\mu) \right)^2. \tag{A22}
\]

Therefore

\[
a_{n-1}(\mu) = \left( a_{n+1}^{-1}(\mu) - a_{n-1}^{-1}(\mu) \right) \left( a_{n+1}^{-1}(\mu) + a_{n-1}^{-1}(\mu) \right)
= -\left( a_{n+1}^{-1}(\mu) \right)^2
= -1. \tag{A23}
\]

Replacing \( a_n(\mu) \) and \( a_{n-1}(\mu) \) with 0 and \( -1 \) in \( a_n(\mu) = 2(1 - \mu)a_{n-1}(\mu) - a_{n-2}(\mu) \), we have

\[
a_{n-2}(\mu) = -2(1 - \mu). \tag{A24}
\]