The endpoint theorem

Susan M Scott\(^1,2\) and Ben E Whale\(^{1,**}\)

\(^1\) Centre for Gravitational Astrophysics, Department of Quantum Science, Research School of Physics, College of Science, The Australian National University, Australia
\(^2\) Australian Research Council Centre of Excellence for Gravitational Wave Discovery (OzGrav), Australia

E-mail: susan.scott@anu.edu.au and ben@benwhale.com

Received 1 July 2020, revised 24 January 2021
Accepted for publication 1 February 2021
Published 16 February 2021

Abstract

The endpoint theorem links the existence of a sequence (curve), without accumulation points, in a manifold to the existence of an open embedding of that manifold so that the image of the given sequence (curve) has a unique endpoint. It plays a fundamental role in the theory of the abstract boundary as it implies that there is always an abstract boundary point to represent the endpoint of such sequences and curves. The endpoint theorem will be of interest to researchers analysing specific spacetimes as it shows how to construct a chart in the original manifold which contains the sequence (curve). In particular, it has application to the study of singularities predicted by the singularity theorems.

Keywords: sequence, accumulation point, endpoint, embedding, envelopment, abstract boundary, singularity theorems

1. Introduction

This paper addresses a technical question about the ability to build an open embedding, \(\phi: M \rightarrow M_\psi\), for a manifold \(M\), in which the image of a given accumulation point free sequence \((x_i)_{i \in \mathbb{N}} \subset M\), has a unique limit point in the larger manifold. This question has significant physical interest if one wishes, as we do, to understand the physical consequences of the Penrose–Hawking singularity theorems using only the context of those theorems. It is apparent that both the motivation for doing this and the “rules of the game” are no longer well understood, which is disappointing, since a great deal of effort was spent in the 70s and 80s attempting to understand the implications of the singularity theorems in this way. As an example,
Christopher Clarke who worked on the original version of the endpoint theorem with Scott, produced a book [1], and numerous other publications, for example [2–11], which provide ample examples of what we mean by ‘the context for the singularity theorems’. Nevertheless, for the benefit of the reader, we provide here an explanation of these topics relevant for this paper. The reader familiar with applications of differential topology to relativity (e.g. [12]), or more broadly with global Lorentzian geometry (e.g. [13]), may wish to skip directly to section 2.

We begin with a discussion of the Penrose–Hawking singularity theorems. For more detail refer to [14, 15] for general overviews, [12, 16] for self-contained presentations by Penrose, and Hawking and Ellis, respectively, or [[13], chapter 12] for a more modern, mathematically focussed, presentation. Theorem 12.43 of [13] is an example of a modern singularity theorem that aligns with the spirit of the Penrose–Hawking singularity theorems:

**Theorem 1.1 ([[13], theorem 12.43]).** No Lorentzian manifold \((M, g)\) of dimension greater than 2 can satisfy all of the following three requirements together:

(a) The manifold \((M, g)\) has no closed timelike curves.

(b) There exists a future trapped or past trapped set in \((M, g)\).

(c) Every inextendible, nonspacelike geodesic in \((M, g)\) contains a pair of conjugate points.

The proof boils down to showing that the first and second conditions imply that there exists a geodesic without a pair of conjugate points. The third condition then gives a contradiction.

When one also assumes the weak energy condition, the Raychaudhuri equation can be used to show that there exists an inextendible, incomplete, causal geodesic. If the manifold is assumed to be maximally extended, then one can claim that the inextendible, incomplete, causal geodesic ‘approaches a singularity’. We write ‘approaches a singularity’ in quotes because the predicted causal geodesic has no accumulation points. It ‘approaches’ no points in the manifold. Moreover, beyond defining a singularity to be an incomplete, inextendible curve in a maximally extended manifold, it is not clear exactly what ‘singularity’ means. The singularity theorems do not predict any kind of physical experience for an observer on the incomplete, inextendible curve. Even today, more than 50 years after Penrose’s paper [17] (the basis for his receipt of the 2020 Nobel Prize in Physics), there is no accepted definition of what a singularity is in this context. There are accepted definitions within classes of Lorentzian manifolds, for example certain parameter dependent solutions of Einstein’s equations, but no accepted definition for the very general Lorentzian manifolds to which the singularity theorems apply. Interpretation and resolution of the Gordian knot of a phrase, ‘approaches a singularity’, sums up the goal of the program of research in which this paper lies.

Theorem 1.1 is beautifully simple. It assumes very little, and only assumes things that have strong physical justifications [[16], chapter 8]. Yet, with the additions of the weak energy condition and the assumption of maximal extension, it manages to show that the manifold is causal geodesically incomplete. This then has the interpretation that it is possible for a freely falling observer to exist for only a finite amount of time; a staggeringly profound statement about cosmology.

At the time when Penrose published the first singularity theorem [17], it was important that the theorem was very general. Two years earlier Lifshitz and Khalatnikov [18] had claimed that, ‘An attempt is made to provide an answer to one of the principal questions of modern cosmology: ‘does the general solution of the gravitational equations have a singularity?’ The authors give a negative answer to this question.’ Because Penrose’s result was very general, it demonstrated that this particular claim of Lifshitz and Khalatnikov could not be true. Six years later Belinskii et al published their famous paper claiming to have derived the properties...
of generic cosmological singularities [19]. This paper birthed the BKL conjecture. It was the
generality of Penrose’s result that made it physically relevant. It is this generality that we seek
to preserve in our study of gravitational singularities.

Some researchers assume that it is necessary to involve Einstein’s field equations to study
what might happen to an observer along an incomplete, inextendible geodesic. They wish to
phrase Einstein’s equations as an initial value (or initial boundary value) problem and study
the evolution of curvature via application of PDE techniques. This assumes that the manifold
is globally hyperbolic. Often a global splitting of the manifold, e.g. into a $3+1$ form, is also
used. We see this as an interesting subcase of the full problem.

Within the context of PDE techniques, ‘approaches a singularity’ has a clear meaning and
interpretation. For example, given a $3+1$ decomposition of the manifold, one has a canoni-
cal set of charts within which to perform computations. The properties and relations of these
charts are known. These charts play a special role in supporting the analysis of the manifold.
In particular, points on the boundary of the images of the charts in Euclidean space are taken
to play the role of the location of any singularity, points at infinity, or points through which
the manifold may be extended. We will call these boundary points. Thus the canonical charts
define what ‘approach’ means. The curvature, or tensors related to the connection, can also be
directly studied in the chart and ‘on approach’ to boundary points. The result is that one can
determine properties of the singularity.

In the more general context of the singularity theorems the manifold cannot be assumed to
be globally hyperbolic. Any $3+1$ decomposition can only be assumed to be a local decompo-
sition with respect to a particular chart. There is no reason why one can assume that some nice
initial boundary value problem for Einstein’s field equations can be given. There is no clear
meaning nor interpretation for what ‘approaches a singularity’ means. In particular, the lack
of global charts means that one must take additional care in the use of chart based techniques.
There is no a priori reason to select one chart’s boundary points over another.

The inappropriateness of assumptions related to the application of PDE techniques in our
work means that we must necessarily address questions that those assumptions usually render
solved. Thus we must find other ways of solving these questions. Theorem 2.1 is an example
of this. In a globally hyperbolic manifold every sequence without accumulation points, in the
manifold, has a set of boundary points to which it limits. This happens because of global hyper-
bolicity. Theorem 2.1 shows that the same is true for any manifold. For researchers unaware of
the field of differential topology in relativity, a result like theorem 2.1 is likely to seem esoteric
and unnecessary. We hope that the literature that we cite here can be used, by academics for
which our result is esoteric, as a way to gain more understanding of other approaches to the
open problems of general relativity.

The field of boundary constructions in general relativity consists of attempts to build math-
ematical structures via which the phrase ‘approaches a singularity’ can find valid, chart inde-
dependent, interpretation. Even though a Lorentzian manifold has a metric, that metric does not
induce a distance and thus the naïve approach of using a Cauchy completion fails; which dis-
tance on the manifold should you choose and why should you choose that distance? Sormani
and Vega have recently published a paper discussing this issue [20].

Modern examples of boundary constructions can be found in [20–24]. At a minimum, a
boundary construction should produce a topological space into which the manifold $\mathcal{M}$ can
be embedded. The topological space should involve as few choices as possible and should
be geometrically defined. Points in the topological space that are not in the image of $\mathcal{M}$
can be considered as boundary points. Boundary points make up the boundary of $\mathcal{M}$. Ide-
ally such points should be classifiable as a singularity, points at infinity, regular points, and so
on. A boundary construction is a generalisation of Penrose’s conformal compactification. The
modern boundaries are subtle and nuanced constructions. They can be categorised by the type of structure on $M$ that they use to perform the construction of the boundary. The $c$-boundary \cite{21} uses past and future sets. The isocausal boundary \cite{22} uses causality respecting embeddings. The null distance \cite{20} selects a time function which induces a distance and then builds the Cauchy completion. The boundary given in \cite{24} allows for any subsets of the atlas satisfying a particular compatibility condition. The abstract boundary \cite{23} uses the set of all open embeddings of the manifold. A good review of the field of boundary constructions, circa 2002, can be found in \cite{25}.

The abstract boundary takes its inspiration from the practice of studying global behaviour via charts. Examples of chart based global analysis can be found in every textbook on general relativity; our favourites are \cite{16, 26, 27}. The paper \cite{28} gives an example where a $3 + 1$ decomposition is used, and the two papers \cite{29, 30} provide an example where the $3 + 1$ decomposition is not used. In all three papers \cite{28–30} different charts are constructed based on an analysis of curvature in order to ensure that the boundary points of the new chart somehow provide a better representation of the limiting behaviour of tensors related to the curvature.

The construction of the maximal analytic extension of the Kerr solution through the ring singularity \cite[section 5.6]{16} provides another example. As presented in Boyer and Lindquist coordinates $(r, \theta, \phi, t)$, the set of points $\{(0, \theta, \phi, t) : \theta \in (0, \pi), \phi \in (0, 2\pi), t \in \mathbb{R}\}$ all represent the same chart boundary point; the origin of $\mathbb{R}^4$. Analysis of the behaviour of $R_{abcd}R^{abcd}$ shows that this scalar has path dependent limits to the set $\{(0, \theta, \phi, t) : \theta \in (0, \pi), \phi \in (0, 2\pi), t \in \mathbb{R}\}$. Changing to Kerr–Schild coordinates resolves the path dependent limits and reveals a ring singularity as well as points towards which the metric has regular limits. The ring singularity and the regular points are still chart boundary points. They are not in the image of the chart; they are on the topological boundary of the image of the chart in Euclidean space. One can analytically extend through the regular points by constructing a new chart. The chart boundary points defining the ring singularity remain chart boundary points.

In \cite{24} it is shown that any nice enough chart with boundary points induces an open embedding and vice versa. In this sense the abstract boundary uses the traditional chart based method of analysing global structure to build a non-chart dependent way of interpreting the word ‘approach’ in the phrase ‘approaches a singularity’.

Theorem 2.1 implies that the topological space generated by the abstract boundary \cite{24, 31} is actually a compactification of $M$. Without this result there could exist sequences without accumulation points, or equivalently incomplete, inextendible curves, which approach no boundary point. If a boundary construction does not have enough boundary points then it might be possible that ‘approach to a singularity’ could have no meaning for that boundary construction. This would demonstrate that more data than assumed in the singularity theorems is needed to understand their physical implications. Thus theorem 2.1 is a foundational part of the abstract boundary and absolutely necessary for its relevance within the context of the singularity theorems.

In the Kerr example the calculations using $R_{abcd}R^{abcd}$ are geometric but the claim that the singularity is a ring should really be worded, ‘with respect to Kerr–Schild coordinates the chart boundary points at which $R_{abcd}R^{abcd}$ diverges is represented as a ring’. There is an example \cite{28} of a manifold in which there are two tensors such that there is no chart which resolves the path dependent limits of each tensor at the same time. This shows that the ring singularity of the Kerr solution is specific to $R_{abcd}R^{abcd}$ and the chosen chart. The ring singularity is not a geometric quantity; it is a representation of the values of the path dependent limits of a geometric quantity in a particular chart. Because of the manner in which the abstract boundary constructs the larger
topological space, every chart boundary point corresponds to a point in the topological space. This provides the statement 'The Kerr black hole contains a ring singularity' with a non-chart dependent interpretation. It also, for example, can be used to give geometric interpretation to important chart dependent calculations like those in [32].

The abstract boundary is the only boundary construction with a geometric definition of a singularity that does not rely on the introduction of curvature quantities [[33], theorems 1.2 and 1.3]. Thus it aligns with the context of the Penrose–Hawking singularity theorems. The results contained in [33] rely on theorem 2.1 to prove that every incomplete, inextendible curve is represented by a boundary point that can be interpreted as an essential singularity. Theorems 1.2 and 1.3 of [33] show that incomplete, inextendible curves, within very general classes of curves, in very general spacetimes, must end at essential abstract boundary singularities; points through which the metric cannot be extended. It is known that abstract boundary singularities satisfy generic stability properties [10, 34, 35]. Thus they are geometrically defined, physically reasonable, and closely tied to the incomplete, inextendible curves predicted by the singularity theorems. Theorems 1.2 and 1.3 of [33] therefore provide an answer to the implicit ‘location’ aspect of the word ‘approach’ for exactly the type of singularity predicted by the singularity theorems. These theorems are vital steps towards solving the other main problem in the completion of the singularity theorems: ‘What are their physical properties?’ The endpoint theorem has, and will continue to facilitate further research to probe the nature of these singularities, hopefully one day leading to the full completion of the singularity theorems; that is, a statement regarding the physical behaviour of the predicted incomplete, inextendible curves, without assumptions beyond those used in the singularity theorems themselves. See [36] for further details about the intended program.

2. The endpoint theorem

Because the theorem below, in effect, attaches endpoints to sequences and curves we have dubbed it ‘The Endpoint Theorem’.

**Theorem 2.1 (The endpoint theorem).** Let \(M\) and \(M_\partial\) be smooth, connected, Hausdorff, paracompact manifolds of dimension \(n\). If \((x_i)_{i\in\mathbb{N}}\) is a sequence of points in \(M\) without an accumulation point, then there exists an open embedding \(\phi : M \to M_\partial\), such that \(\partial \phi(M)\) is diffeomorphic to the \((n - 1)\) dimensional unit ball and the sequence \((\phi(x_i))_{i\in\mathbb{N}}\) converges to some \(y \in \partial \phi(M)\).

The proof of this result proceeds in three steps. The first step is to construct a non-self-intersecting curve, \(\lambda\) in \(M\) so that \((x_i) \subset \lambda\). The second step is to construct a coordinate system about this curve. In the third step we use this coordinate system to construct an open embedding with the required properties.

**Proof** Let \(h\) be a complete Riemannian metric on \(M\). In order to construct a non-self-intersecting curve \(\lambda\) so that \((x_i) \subset \lambda\) we need to construct a cover of \(M\) by a collection \(V_i\) of connected open submanifolds of \(M\) with compact closure so that for all \(i > 0\),

\[
\nabla_i \subset V_i \quad \text{and} \quad x_i \in V_i \setminus \nabla_{i-1}.
\]

We shall do this inductively.

Before we give the induction we define a sequence \((r_i)_{i\in\mathbb{N}}\) in \(\mathbb{R}^+\). Given \(p \in M\) and \(r \in \mathbb{R}^+\) let \(B_{ds}(p)\) be the open ball of radius \(r\) centred on \(p\) with respect to the complete distance, \(d_{ds}\),
induced by \( h \). For each \( i \in \mathbb{N} \) define

\[
    r_i = \frac{1}{2} \min_{k > i} \{ d_k(x_k, x_0) \} > 0.
\]

Since \( h \) is complete and \( (x_i) \) has no accumulation points, \( \lim_{i \to \infty} r_i = \infty \). In particular, for each \( i \in \mathbb{N} \), \( k > i \) implies that \( x_k \notin B_{h,r_i}(x_0) \) and \( \bigcup B_{h,r_i}(x_0) = M \).

For the base case of the induction we define \( V_0, V_1 \subset M \) so that \( V_0 \subset V_1 \) and \( x_1 \in V_1 \setminus V_0 \). Figure 1 provides an illustration of the construction. Let \( V_0 = B_{h,x_0}(x_0) \). Note that for all \( k > 0 \), \( x_k \notin V_0 \). Let \( c_1 < \frac{1}{4} \min_{i>1} \{ d_i(x_i, V_0) \} \) and define \( V_1^{i} = \{ x \in M : d_h(x, V_0) < c_1 \} \). Note that since \( V_0 \) is connected and compact, and \( h \) is complete, \( V_1^{i} \setminus V_0 \) is path connected. By construction, for all \( k > 1 \), \( x_k \notin V_1^{i} \cup B_{h,r_i}(x_0) \). Choose a curve \( c_1 : [0,1] \to M \) so that \( c_1(0) \in V_1^{i} \setminus V_0 \), \( c_1(0) \neq x_1 \), \( c_1(1) = x_1 \) and \( c_1((0,1)) \cap \{ x_k : k > 1 \} \cup \{ V_0 \} = \emptyset \). Let

\[
    \zeta_1 = \frac{1}{2} \min \left\{ \min_{k>1} \{ d_k(x_k, c_1) \}, d(c_1, V_0) \right\}
\]

and

\[
    N(c_1, \zeta_1) = \{ x \in M : d_h(x, c_1) < \zeta_1 \}.
\]

Note that \( N(c_1, \zeta_1) \cap V_0 = \emptyset \). Define \( V_1 = V_1^{i} \cup B_{h,r_i}(x_0) \cup N(c_1, \zeta_1) \).

Since \( V_1 \) is the union of open subsets of \( M \) it is an open submanifold. By construction, each subset \( V_1^{i}, B_{h,r_i}(x_0) \) and \( N(c_1, \zeta_1) \) is connected. Since \( x_0 \in V_1^{i} \cap B_{h,r_i}(x_0) \), and \( c_1(0) \in N(c_1, \zeta_1) \cap V_1^{i} \), \( V_1 \) itself is connected. Since each of \( V_1^{i}, B_{h,r_i}(x_0) \) and \( N(c_1, \zeta_1) \) has compact closure, \( V_1 \) has compact closure. By definition \( \overline{V_0} \subset V_1^{i} \subset V_1 \). By construction \( x_1 = c_1(1) \in N(c_1, \zeta_1) \subset V_1 \) and by construction \( x_1 \notin V_0 \). Hence \( x_1 \in V_1 \setminus V_0 \). Let \( k > 1 \). By construction \( d_h(x_k, V_0) > \zeta_1 \) so that \( x_k \notin V_1^{i} \). Similarly, \( d_h(x_k, c_1) > \zeta_1 \) so that \( x_k \notin N(c_1, \zeta_1) \).

Since \( d_h(x_k, x_0) > r_1 \) it is also the case that \( x_k \notin B_{h,r_1}(x_0) \). Thus \( x_k \notin V_1^{i} \).

We now handle the inductive case. Let \( j \in \mathbb{N}, j \neq 0 \). The argument is the same as for the base case. Suppose that \( V_0, \ldots, V_{i-1} \) exist and are such that for all \( j = 1, \ldots, i-1 \) \( \overline{V_j} \subset V_{j-1} \subset V_j, x_j \in V_j \setminus \overline{V_{j-1}} \) and for all \( k > j \), \( x_k \notin \overline{V_j} \). Let \( \epsilon_j < \frac{1}{2} \min_{i > j} \{ d_i(x_i, V_{i-1}) \} \). Define \( V_i^{j} = \{ x \in M : d_h(x, V_{j-1}) < \epsilon_j \} \). Note that since \( V_{j-1} \) is connected and compact, and \( h \) is complete, \( V_i^{j} \setminus \overline{V_{j-1}} \) is path connected. By construction, for all \( k > i \), \( x_k \notin V_i^{j} \cup B_{h,r_i}(x_0) \). Choose a curve \( c_i : [0,1] \to M \) so that \( c_i(0) \in V_i^{j} \setminus V_{j-1}, c_i(0) \neq x_i, c_i(1) = x_i \) and \( c_i((0,1)) \cap \overline{V_{j-1}} \neq \emptyset \).
\{ x_k : k > i \} \cup \nabla_{i+1} = \emptyset. \) Let

\[ \zeta = \frac{1}{2} \min \left\{ \min_{k<i} \{ d_b(x_k, c_i) \}, \min_{k>i} \{ d_b(x_k, c_i) \} \right\} \]

and

\[ \mathcal{N}(c_i, \zeta) = \{ x \in \mathcal{M} : d_b(x, c_i) < \zeta \}. \]

Note that \( \mathcal{N}(c_i, \zeta) \cap \nabla_{i+1} = \emptyset. \) Define \( V_i = V_i^i \cup B_{h_{a+r}}(x_0) \cup \mathcal{N}(c_i, \zeta). \) To complete the proof we need to prove that \( V_i \) is a connected open submanifold of \( \mathcal{M} \) with compact closure so that \( \nabla_{i+1} \subset V_i, x_i \in V_i \setminus \nabla_{i+1} \) and that for all \( k > i, x_k \notin \nabla_i. \)

Since \( V_i \) is the union of open subsets of \( \mathcal{M} \) it is an open submanifold. By construction, each subset \( V_i^i, B_{h_{a+r}}(x_0) \) and \( \mathcal{N}(c_i, \zeta) \) is connected. Since \( x_0 \in V_i^i \cap B_{h_{a+r}}(x_0) \) and \( c_i(0) \in \mathcal{N}(c_i, \zeta) \cap V_i^i, V_i \) itself is connected. Since each of \( V_i^i, B_{h_{a+r}}(x_0) \) and \( \mathcal{N}(c_i, \zeta) \) has compact closure, \( V_i \) has compact closure. By definition \( \nabla_{i+1} \subset V_i^i \subset V_i. \) By construction \( x_i = c_i(1) \in \mathcal{N}(c_i, \zeta) \subset V_i \) and by assumption for all \( j = 0, \ldots, i - 1, x_i \notin \nabla_j. \) Hence \( x_i \in V_i \setminus \nabla_{i+1}. \) Let \( k > i. \) By construction \( d_b(x_k, \nabla_{i+1}) > \epsilon_i, \) so that \( x_k \notin \nabla_i. \) Similarly, \( d_b(x_k, c_i) > \zeta, \) so that \( x_k \notin \mathcal{N}(c_i, \zeta). \)

Since \( d_b(x_k, x_0) > r, \) it is also the case that \( x_k \notin B_{h_{a+r}}(x_0). \) Thus \( x_k \notin \nabla_i. \)

From above, \( \{ B_{h_{a+r}}(x_0) : i \in \mathbb{N} \} \) is a cover of \( \mathcal{M} \). Since, for each \( i \in \mathbb{N}, B_{h_{a+r}}(x_0) \subset V_i \) we know that \( \{ V_i : i \in \mathbb{N} \} \) is also a cover of \( \mathcal{M}. \)

Now, let \( \lambda_0 \) be a non-self-intersecting curve in \( V_1 \) which joins \( x_0 \) to \( x_1. \) For each \( i > 0, \)

\( x_i \in V_i \setminus \nabla_{i+1}. \) Thus, for each \( i > 0, x_i, x_{i+1} \in V_{i+1} \setminus \nabla_{i+1}. \) We seek to show that each \( V_{i+1} \setminus \nabla_{i+1} \) is connected as this will allow us to find a suitable curve from \( x_i \) to \( x_{i+1}. \)

We begin by showing that \( V_i \setminus \nabla_{i+1} \) is connected, where \( V_i \setminus \nabla_{i+1} = V_i^i \setminus \nabla_{i+1} \cup B_{h_{a+r}}(x_0) \setminus \nabla_{i+1} \cup \mathcal{N}(c_i, \zeta) \setminus \nabla_{i+1}. \) As mentioned above, \( V_i^i \setminus \nabla_{i+1} \) is connected. By construction, for each \( i > 0, N(c_i, \zeta) \setminus \nabla_{i+1} = \emptyset \) and \( c_i(0) \in V_i^i \setminus \nabla_{i+1} \cap N(c_i, \zeta). \) Thus, \( (V_i^i \cup N(c_i, \zeta)) \setminus \nabla_{i+1} \) is connected. If \( B_{h_{a+r}}(x_0) \subset V_{i+1}, \) then \( V_i \setminus \nabla_{i+1} = (V_i^i \cup N(c_i, \zeta)) \setminus \nabla_{i+1} \) which is connected. Otherwise, there exists \( y_1 \in B_{h_{a+r}}(x_0) \). Let \( \gamma_1 \) be a non-self-intersecting curve in \( B_{h_{a+r}}(x_0) \) which joins \( x_0 \) to \( y_1. \) There must exist a point \( p_1 \in \gamma_1 \setminus \nabla_{i+1} \cap \nabla_i \). Since \( y_1 \) is path connected to \( p_1 \) in \( B_{h_{a+r}}(x_0) \setminus \nabla_{i+1}, y_1 \) is path connected in \( V_i \setminus \nabla_{i+1} \) to any point in \( (V_i^i \cup N(c_i, \zeta)) \setminus \nabla_{i+1}. \) Let \( y_2 \in B_{h_{a+r}}(x_0) \setminus \nabla_{i+1} \) where \( y_1 \) and \( y_2 \) are distinct points. Let \( \gamma_2 \) be a non-self-intersecting curve in \( B_{h_{a+r}}(x_0) \) which joins \( x_0 \) to \( y_2, \) where \( \gamma_1 \cap \gamma_2 = \emptyset. \) There must exist a point \( p_2 \in \gamma_2 \setminus \nabla_{i+1} \cap \nabla_i. \) Since \( y_2 \) is path connected to \( p_2 \) in \( B_{h_{a+r}}(x_0) \setminus \nabla_{i+1}, y_2 \) is path connected in \( V_i \setminus \nabla_{i+1} \) to \( p_1 \) and \( y_1. \) Thus \( V_i \setminus \nabla_{i+1} \) is connected.

We now show that \( V_{i+1} \setminus \nabla_{i+1} \) is connected. The paragraph above implies that \( V_{i+1} \setminus \nabla_i \) and \( V_i \setminus \nabla_{i+1} \) are connected. Consider a point \( p \in V_{i+1} \setminus \nabla_i. \) Since \( V_{i+1} = \{ x \in \mathcal{M} : d_b(x, \nabla_i) < \epsilon_{i+1} \} \) and \( V_{i+1} \setminus \nabla_i \) is connected, there exists a path in \( V_{i+1} \setminus \nabla_i \) which connects \( p \) to a point \( q \in \partial V_{i+1} \setminus \nabla_i \) and \( \partial V_{i+1} \setminus \nabla_i \). Due to the connectedness of \( V_{i+1} \setminus \nabla_i, q \) and \( p \) are path connected in \( V_{i+1} \setminus \nabla_i \) to every point in \( V_i \setminus \nabla_i \) and \( \partial V_{i+1} \setminus \nabla_i \). This implies that \( V_{i+1} \setminus \nabla_{i+1} \) is connected.

We now introduce the required curve from \( x_i \) to \( x_{i+1}. \) Since \( V_{i+1} \setminus \nabla_{i+1} \) is connected there exists a non-self-intersecting curve, \( \lambda_i, \) in \( V_{i+1} \setminus \nabla_{i+1} \) that joins \( x_i \) and \( x_{i+1}. \) We may do this in such a way that \( \lambda_i \cap \lambda_{i-1} = x_i. \) By ‘smoothing’ at the joins we get a \( C^\infty \) curve \( \lambda : [0, 1] \to \mathcal{M} \) which is non-self-intersecting, such that \( x_i \subset \lambda. \) Note that \( \lambda \) may need to be rescaled so that \( \lambda'(t) \neq 0 \) for \( i \in [0, 1]. \) Let \( (t_i)_{i \in \mathbb{N}} \) be a sequence in \( [0, 1] \) so that \( t_0 = 0, \) and for all \( i, j \in \mathbb{N}, \lambda(t_i) = x_i, t_i < t_j \iff i < j, \) and \( t_i \to 1 \) as \( i \to \infty. \) By construction, for each \( i \in \mathbb{N}, \) \( \lambda \) eventuates, and never returns to \( \nabla_i. \) Since \( \{ \nabla_i : i \in \mathbb{N} \} \) is a compact exhaustion of \( \mathcal{M} \) we know that \( d_b(x_0, \lambda(t)) \to \infty \) as \( t \to 1. \) This implies that the non-self-intersecting curve \( \lambda \) has no accumulation points and therefore approaches the ‘edge’ of the manifold as \( t \to 1. \)
Let $N\lambda = \{v \in T_{(0)}\mathcal{M} : h(v, X(t)) = 0, t \in [0, 1)\}$ be the normal bundle of $\lambda$. It is a sub-bundle of $T\mathcal{M}$, hence we can restrict $h$ to $N\lambda$. The restriction of the metric $h$ induces a Levi–Civita connection on $N\lambda$. Let $\{E_i\}_{i=1,...,n}$ be a linearly independent frame at $N_{(0)}\lambda$. Parallelly propagate, with respect to the induced Levi–Civita connection, this frame along $\lambda$ in $N\lambda$ to get $n-1$ linearly independent vector fields $E_i : [0, 1) \to N\lambda$. The bundle $N\lambda$ is an inclusion into $TM$. In an abuse of notation denote the image of $E_i$ under this inclusion by $E_i$.

By construction for all $i = 1, \ldots, n-1$, $h(E_i, X') = 0$.

Let $B_{n-1}$ be the unit ball in $\mathbb{R}^{n-1}$. The existence of a normal neighbourhood of $\lambda$, [[37], proposition 7.26], implies that there exists a smooth function $f : (0, 1) \to \mathbb{R}^+$ so that the function $\mu : (0, 1) \times B_{n-1} \to \mathcal{M}$ defined by $\mu(t, p) = \exp_{(0)}(f(t)p'E_i(t))$, where $i$ sums over 1 to $n-1$, is a chart.

Now we use the chart $\mu$ to construct an open embedding of $\mathcal{M}$. Let $\mathcal{N}$ be the set $\mathcal{M} \cup \{(1, 2) \times B_{n-1}\}$. Give $\mathcal{N}$ the atlas generated by the set containing all the charts of $\mathcal{M}$ as well as a new chart $\psi : (0, 2) \times B_{n-1} \to \mathcal{N}$ given by

$$\psi(t, p) = \begin{cases} \exp_{(0)}(f(t)p'E_i(t)) & \text{if } t \in (0, 1) \\ (t, p) & \text{otherwise.} \end{cases} \tag{1}$$

The only non-trivial part is to show that $\mathcal{N}$ is a Hausdorff manifold. This reduces to showing that the points $\{1\} \times B_{n-1}$ are Hausdorff separated from $\mathcal{M}$. This must be true, however, since for any $p \in \mathcal{M}$ there exists $i$ so that $p \in V_i$, and $\mathcal{N} \setminus V_i$ is an open neighbourhood containing every point in $\{1\} \times B_{n-1}$. Thus $\mathcal{N}$ is a smooth, connected, Hausdorff, paracompact manifold of dimension $n$. By definition $\partial \mathcal{M}$ considered as a subset of $\mathcal{N}$ is $\{1\} \times B_{n-1}$ which is diffeomorphic to the $n-1$ dimensional unit ball.

Let $\phi : \mathcal{M} \to \mathcal{N}$ be the identity. Then $\phi$ is an open embedding of $\mathcal{M}$ and the sequence $(\phi(x_i))_{i \in \mathbb{N}}$ converges to the point $(1, 0) \in [1, 2) \times B_{n-1}$ of $\mathcal{N}$, as required. \qed

**Corollary 2.2.** Let $\mathcal{M}$ and $\mathcal{M}_3$ be smooth, connected, Hausdorff, paracompact manifolds of dimension $n$. Let $\gamma : [a, b) \to \mathcal{M}$ be a non-self-intersecting curve in $\mathcal{M}$ without limit points in $\mathcal{M}$. Then there exists an open embedding $\phi : \mathcal{M} \to \mathcal{M}_3$ so that the curve $\phi(\gamma)$ has an endpoint in $\partial \phi(\mathcal{M})$.

**Proof** Since $\gamma$ is non-self-intersecting, we may use it as the curve in the proof of the endpoint theorem. The construction of $\phi$ then implies that $\phi(\gamma) \to (1, 0) \in \mathcal{N}$. \qed

**Data availability statement**

No new data were created or analysed in this study.

**Acknowledgments**

The authors give their thanks to Christopher J S Clarke who worked with Susan M Scott on the original development of this result and sadly passed away on 16 April 2019. As a student, Scott was inspired by the work of Clarke on singularities, and they subsequently collaborated in this field a number of times, including for the production of the endpoint theorem presented here.
References

[1] Clarke C J S 1993 The Analysis of Space-Time Singularities (Cambridge Lecture Notes in Physics vol 1) (Cambridge: Cambridge University Press)
[2] Slupinski M J and Clarke C J S 1980 Singular points and projective limits in relativity Commun. Math. Phys. 71 289–97
[3] Tipler F J, Clarke C J S and Ellis G F R 1980 Singularities and horizons—a review article General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein vol 2 ed A Held and J L Anderson (New York: Plenum)
[4] Clarke C J S 1979 Boundary definitions Gen. Relativ. Gravit. 10 977–80
[5] Clarke C J S 1978 The singular holonomy group Commun. Math. Phys. 58 291–7
[6] Clarke C J S 1975 The classification of singularities Gen. Relativ. Gravit. 6 35–40
[7] Clarke C J S 1975 Singularities in globally hyperbolic space-time Commun. Math. Phys. 41 65–78
[8] Clarke C J S and Schmidt B G 1977 Singularities: the state of the art Gen. Relativ. Gravit. 8 129–37
[9] Clarke C J S 1983 The cardinality of manifold atlases Israel J. Math. 45 9–16
[10] Fama C J and Clarke C J S 1998 A rigidity result on the ideal boundary structure of smooth spacetimes Class. Quantum Grav. 15 2829–40
[11] Clarke C J S and Królak A 1985 Conditions for the occurrence of strong curvature singularities J. Geom. Phys. 2 127–43
[12] Penrose R 1987 Techniques of Differential Topology in Relativity (Conf. Board of the Mathematical Sciences Regional Conf. Series in Applied Mathematics vol 7) (Philadelphia, PA: SIAM)
[13] Beem J K, Ehrlich P E and Easley K L 1996 Global Lorentzian Geometry (Pure and Applied Mathematics: a Series of Monographs and Textbooks vol 202) (New York: Dekker)
[14] Senovilla J M M 1998 Singularity theorems and their consequences Gen. Relativ. Gravit. 30 701–848
[15] Senovilla J M M and Garfinkle D 2015 The 1965 Penrose singularity theorem Class. Quantum Grav. 32 124008
[16] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[17] Penrose R 1965 Gravitational collapse and space-time singularities Phys. Rev. Lett. 14 57–9
[18] Lifshitz E M and Khalatnikov I M 1963 Investigations in relativistic cosmology Adv. Phys. 12 185–249
[19] Belinskii V A, Lifshitz E M and Khalatnikov I M 1971 Oscillatory approach to a singular point in relativistic cosmology Sov. Phys. Usp. 13 745–65
[20] Sormani C and Vega C 2016 Null distance on a spacetime Class. Quantum Grav. 33 085001
[21] Flores J L, Herrera J and Sánchez M 2011 On the final definition of the causal boundary and its relation with the conformal boundary Adv. Theor. Math. Phys. 15 991–1057
[22] Garcia-Parrado A and Senovilla J M M 2003 Causal relationships: a new tool for the causal characterization of Lorentzian manifolds Class. Quantum Grav. 20 625
[23] Scott S M and Szekeres P 1994 The abstract boundary—a new approach to singularities of manifolds J. Geom. Phys. 13 223–53
[24] Whale B E 2014 The chart based approach to studying the global structure of a spacetime induces a coordinate invariant boundary Gen. Relativ. Gravit. 46 1624
[25] Ashley M J S L 2002 Singularity theorems and the abstract boundary construction PhD Thesis Department of Physics, Australian National University Located at http://hdl.handle.net/1885/46055
[26] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (Physics Series) (San Francisco, CA: Freeman)
[27] Plebanski J and Krasinski A 2006 An Introduction to General Relativity and Cosmology (Cambridge: Cambridge University Press)
[28] Beyer F and Henning J 2014 An exact smooth Gowdy-symmetric generalized Taub-NUT solution Class. Quantum Grav. 31 095010
[29] Scott S M and Szekeres P 1986 The Curzon singularity. I: spatial sections Gen. Relativ. Gravit. 18 557–70
[30] Scott S M and Szekeres P 1986 The Curzon singularity. II: global picture Gen. Relativ. Gravit. 18 571–83
[31] Barry R A and Scott S M 2011 The attached point topology of the abstract boundary for spacetime Class. Quantum Grav. 28 165003
[32] Klainerman S and Rodnianski I 2010 On the breakdown criterion in general relativity J. Am. Math. Soc. 23 345–82
[33] Whale B E, Ashley M J S L and Scott S M 2015 Generalizations of the abstract boundary singularity theorem Class. Quantum Grav. 32 135001
[34] Ashley M J S L 2002 The stability of abstract boundary essential singularities Gen. Relativ. Gravit. 34 1625–35
[35] Fama C J and Scott S M 1994 Invariance properties of boundary sets of open embeddings of manifolds and their application to the abstract boundary Differential Geometry and Mathematical Physics (Contemporary Mathematics vol 170 Vancouver, BC, 1993)) (Providence, RI: American Mathematical Society) pp 79–111
[36] Ashley M J S L and Scott S M 2003 Curvature singularities and abstract boundary singularity theorems for space-time Recent Advances in Riemannian and Lorentzian Geometries (Contemporary Mathematics vol 357) ed K L Duggal and R Sharma (Providence, RI: American Mathematical Society) pp 9–19
[37] O’Neill B 1983 Semi-Riemannian Geometry with Applications to Relativity (Pure and Applied Mathematics vol 103) (New York: Academic)