Learning with Smooth Hinge Losses

Junru Luo *, Hong Qiao †, and Bo Zhang ‡

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Abstract

Due to the non-smoothness of the Hinge loss in SVM, it is difficult to obtain a faster convergence rate with modern optimization algorithms. In this paper, we introduce two smooth Hinge losses \( \psi_G(\alpha; \sigma) \) and \( \psi_M(\alpha; \sigma) \) which are infinitely differentiable and converge to the Hinge loss uniformly in \( \alpha \) as \( \sigma \) tends to 0. By replacing the Hinge loss with these two smooth Hinge losses, we obtain two smooth support vector machines (SSVMs), respectively. Solving the SSVMs with the Trust Region Newton method (TRON) leads to two quadratically convergent algorithms. Experiments in text classification tasks show that the proposed SSVMs are effective in real-world applications. We also introduce a general smooth convex loss function to unify several commonly-used convex loss functions in machine learning. The general framework provides smooth approximation functions to non-smooth convex loss functions, which can be used to obtain smooth models that can be solved with faster convergent optimization algorithms.

1 Introduction

Consider binary classification problems. Suppose we have a training dataset \( \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \), which is assumed to be independent and identically distributed realizations of a random pair \( \mathcal{X} \times \mathcal{Y} \), where \( \mathcal{X} \subset \mathbb{R}^p \) and \( \mathcal{Y} = \{+1, -1\} \). Our purpose is to learn a linear classifier \( w \in \mathbb{R}^p \) to predict a new instance \( x \in \mathcal{X} \) correctly. The instance \( x \) will be assigned to be positive if \( w^T x > 0 \), and negative otherwise. Moreover, we use the 0/1 loss to evaluate the performance of the classifier \( w \), that is, if the classifier \( w \) makes a correct decision, then there is no loss and, otherwise, the loss is 1. To avoid overfitting, it is necessary to apply a regularization term to penalize the classifier.

*J. Luo is with the School of Computer Science and Artificial Intelligence & Aliyun School of Big Data, Changzhou University, Changzhou, Jiangsu province, China e-mail: (luojunru@cczu.edu.cn).
†H. Qiao is the Institute of Automation, Chinese Academy of Sciences, Beijing 100190, China and School of Artificial Intelligence, University of Chinese Academy of Sciences, Beijing 100049, China e-mail: (hong.qiao@ia.ac.cn).
‡B. Zhang is the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China e-mail: (b.zhang@amt.ac.cn).
$L_2$-regularization is the mostly used one in machine learning problems, which allows for the use of a kernel function as a way of embedding the original data in a higher dimension space \[4, 29\]. In certain practical applications such as text classification, one hopes to learn a sparse classifier through $L_1$-regularization \[12, 7\]. By minimizing the structural risk, the binary classification problem is equivalent to the optimization problem:

$$\arg \min_{w \in \mathbb{R}^p} \frac{1}{2} \lambda \|w\|^2 + \frac{1}{n} \sum_{i=1}^{n} I(y_i w^T x_i \leq 0),$$

where $I(\cdot)$ returns 1 if its argument is true and 0 otherwise, and $\frac{1}{2} \lambda \|w\|^2$ is the $L_2$-regularization term. Due to the non-differentiability and non-convexity of the 0/1 loss, it is an NP-hard problem to optimize (1) directly. To overcome this difficulty, it is common to replace the 0/1 loss with a convex surrogate loss function. And many efficient convex optimization methods can thus be applied to obtain a good solution, such as the gradient-based method and the coordinate descent method \[29, 15, 6, 42\], which are iteration methods based on taking the gradient or coordinate as a descent direction to decrease the objective function.

What kind of convex functions can be applied to replace the 0/1 loss has been studied \[22, 33, 43, 3\]. The weakest possible condition on the surrogate loss $\ell$ is that it is classification-calibrated, which is a pointwise form of the Fisher consistency for binary classification \[21\]. \[3\] obtained a necessary and sufficient condition for a convex surrogate loss $\ell$ to be classification-calibrated, as stated in the following theorem.

**Theorem 1** [3, Theorem 2] A convex function $\ell$ is classification-calibrated if and only if it is differentiable at 0 and $\ell'(0) < 0$.

A brief overview of surrogate loss functions frequently used in practice is given in \[21, 34\]. The mostly used surrogate loss functions include the Hinge loss for SVM, the logistic loss for logistical regression, and the exponential loss for AdaBoost. They are all classification-calibrated. Note that the logistic and exponential losses are smooth, but the Hinge loss is not. As a result, solving SVMs with gradient-based methods only gives a suboptimal convergence rate, and no second-order algorithm is available for solving SVMs. To address this issue, the squared Hinge loss was introduced, but this leads to a new model since the squared Hinge loss is not an approximation of the Hinge loss \[6, 17\].

In this paper, we propose two new convex surrogate losses $\psi_G(\alpha; \sigma)$ and $\psi_M(\alpha; \sigma)$ for binary classification, where $\sigma$ is a tunable hyper-parameter (see \[2\] below), which are called smooth Hinge losses due to the following reasons. First, $\psi_G(\alpha; \sigma)$ and $\psi_M(\alpha; \sigma)$ converge to the Hinge loss uniformly in $\alpha$ as $\sigma$ approaches to 0, so they can keep the advantage of the Hinge loss in SVMs. Secondly, $\psi_G(\alpha; \sigma)$ and $\psi_M(\alpha; \sigma)$ are infinitely differentiable. By replacing the Hinge loss with these two smooth Hinge losses, we obtain two smooth support vector machines (SSVMs) which can be solved with second-order methods. In particular, they can be solved by the inexact Newton method with a quadratic
convergence rate as conducted in [1, 20] for the logistic regression. Although first-order methods are often sufficient in machine learning, there will be a great improvement in training time experimentally on the large scale sparse learning problems by using second-order methods.

Motivated by the proposed smooth Hinge losses, we also propose a general smooth convex loss function \( \psi(\alpha) = \Phi_c(v)(\theta - \alpha) + \phi_c(v)\sigma \) with \( v = (\theta - \alpha)/\sigma \), where \( \Phi_c(v) \) and \( \phi_c(v) \) satisfy the conditions given in Theorem 3 below. This general smooth convex loss function \( \psi(\alpha) \) provides a smooth approximation to several surrogate loss functions usually used in machine learning, such as the non-differentiable absolute loss which is usually used as a regularization term, and the rectified linear unit (ReLU) activation function used in deep neural networks.

This paper is organized as follows. In Section 2, we first briefly review several SVMs with different convex loss functions and then introduce the smooth Hinge loss functions \( \psi_G(\alpha; \sigma) \), \( \psi_M(\alpha; \sigma) \). The general smooth convex loss function \( \psi(\alpha) \) is then presented and discussed in Section 3. In Section 4, we give the smooth support vector machine by replacing the Hinge loss with the smooth Hinge loss \( \psi_G \) or \( \psi_M \). The first-order and second-order algorithms for the proposed SSVMs are also presented and analyzed. Several empirical examples of text categorization with high dimensions and sparse features are implemented in Section 5; the results show that the smooth Hinge losses are efficient for binary classification. Some conclusions are given in Section 6.

2 Smooth Hinge Losses

The support vector machine (SVM) is a famous algorithm for binary classification and has now also been applied to many other machine learning problems such as the AUC learning, multi-task learning, multi-class classification and imbalanced classification problems [27, 18, 2, 14]. [5] is a recent survey work about the applications, challenges and trends of SVM.

The SVM model can be described as the following optimization problem

\[
\arg \min_{w \in \mathbb{R}^d} \frac{1}{2}\|w\|^2 + \frac{1}{n} \sum_{i=1}^{n} \ell(y_i w^T x_i),
\]

where the used surrogate loss is the Hinge loss \( \ell(\alpha) = \max\{0, 1 - \alpha\} \). The model \( \ell(\alpha) = \max\{0, 1 - \alpha\} \) is called L1-SVM.

Since the Hinge loss is not smooth, it is usually replaced with a smooth function. One is the squared Hinge loss \( \ell(\alpha) = \max\{0, 1 - \alpha\}^2 \), which is convex, piecewise quadratic and differentiable [6, 33]. The SVM model with the squared Hinge loss is called L2-SVM. [17] proposed to replace the squared Hinge loss by its smooth approximation \( \ell(\alpha) = (1 - \alpha) + \sigma \ln(1 + e^{-\frac{1}{\sigma} \alpha}) \). In order to make the objective of the Lagrangian dual problem of L1-SVM strongly convex, which is needed in developing an accurate optima estimation with dual methods, the
The following smoothed hinge loss is proposed in [30] to replace the hinge loss:

\[
\ell_{\gamma}(\alpha) = \begin{cases} 
0 & \text{if } \alpha \geq 1 \\
1 - \alpha - \frac{\gamma}{2} & \text{if } \alpha \leq 1 - \gamma \\
\frac{1}{2\gamma}(1 - \alpha)^2 & \text{otherwise.}
\end{cases}
\]

The stochastic dual coordinate ascent method is then applied to accelerate the training process [30, 31]. With the help of \(L_1\)-regularization and the smoothed hinge-loss \(\ell_{\gamma}(\cdot)\), a sparse and smooth support vector machine is obtained in [12]. By simultaneously identifying the inactive features and samples, a novel screening method was further developed in [12], which is able to reduce a large-scale problem to a small-scale problem.

Motivated by the smoothing technique in quantile regression, [39] presents the smooth approximation \(K_h(\alpha) = (1 - \alpha)H((1 - \alpha)/h)\) to the hinge loss, where \(h\) is a bandwidth and \(H(\cdot)\) is the smooth function defined by

\[
H(\alpha) = \begin{cases} 
0 & \text{if } \alpha \leq -1 \\
\frac{1}{2} + \frac{15}{16}(\alpha - \frac{2}{3}\alpha^3 + \frac{1}{5}\alpha^5) & \text{if } -1 < \alpha < 1 \\
1 & \text{otherwise.}
\end{cases}
\]

By replacing the hinge loss with its smooth approximation \(K_h(\alpha)\), a smooth SVM is obtained, and a linear-type estimator is constructed for the smooth SVM in a distributed setting in [39].

Although there have already been several smooth loss functions to replace the Hinge loss in practice, they may not be ideal choices due to the fact that they are either not approximations to the Hinge loss or at most twice differentiable, such as the squared Hinge loss and its approximations as well as \(\ell_{\gamma}\) and \(K_h\). In this section, we propose two (infinitely differentiable) smooth Hinge loss functions which overcome the above weakness and are given by

\[
\psi_G(\alpha; \sigma) = \Phi(v)(1 - \alpha) + \phi(v)\sigma, \\
\psi_M(\alpha; \sigma) = \Phi_M(v)(1 - \alpha) + \phi_M(v)\sigma,
\]

where \(\sigma > 0\) is a given parameter and \(v = (1 - \alpha)/\sigma\). Here, \(\Phi(\cdot)\) and \(\phi(\cdot)\) are the cumulative distribution function (CDF) and probability density function (PDF) of the standard normal distribution, respectively, \(\Phi_M(v) = (1 + v/\sqrt{1 + v^2})/2, \phi_M(v) = 1/(2\sqrt{1 + v^2})\). The following theorem gives the approximation property of \(\psi_G\) and \(\psi_M\).

**Theorem 2** \(\psi_G(\alpha; \sigma)\) and \(\psi_M(\alpha; \sigma)\) satisfy the estimates:

1) \(0 \leq \psi_G(\alpha; \sigma) - \max\{0, 1 - \alpha\} \leq \sigma/\sqrt{2\pi}\);

2) \(0 \leq \psi_M(\alpha; \sigma) - \max\{0, 1 - \alpha\} \leq \sigma/2\).

Thus \(\psi_G(\alpha; \sigma)\) and \(\psi_M(\alpha; \sigma)\) converge to the Hinge loss uniformly in \(\alpha\) as \(\sigma\) tends to 0.

**Proof.** Let \(\ell(\alpha) = \max\{0, 1 - \alpha\}\) and \(v = (1 - \alpha)/\sigma\). Taking the derivative of \(\psi_G(\alpha; \sigma)\), we have

\[
\psi_G'(\alpha; \sigma) = -(\Phi'(v)v + \phi'(v)) - \Phi(v).
\]
By the definition of $\phi(v)$ and $\Phi(v)$ we have
\[
\Phi'(v) = \phi(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}.
\]
It then follows that
\[
\Phi'(v)v + \phi'(v) = 0.
\]
Thus we have $\psi'(\alpha; \sigma) = -\Phi(v) \leq 0$, which means that $\psi_G$ is a monotonically decreasing function of $\alpha$. For $\alpha \geq 1$, we have $\ell(\alpha) = 0$, so
\[
0 = \lim_{\alpha \to \infty} \psi_G(\alpha; \sigma) \leq \psi_G(\alpha; \sigma) - \ell(\alpha) \leq \psi_G(1; \sigma).
\]
For $\alpha \leq 1$, $\ell(\alpha) = 1 - \alpha$ and $(\psi_G(\alpha; \sigma) - \ell(\alpha))' = 1 - \Phi(v) \geq 0$, implying that
\[
0 = \lim_{\alpha \to \infty} [\psi_G(\alpha; \sigma) - \ell(\alpha)] \leq \psi_G(\alpha; \sigma) - \ell(\alpha) \leq \psi_G(1; \sigma).
\]
Thus,
\[
0 \leq \psi_G(\alpha; \sigma) - \max\{0, 1 - \alpha\} \leq \psi_G(1; \sigma) = \frac{\sigma}{\sqrt{2\pi}}.
\]
For $\psi_M(\alpha; \sigma)$ we have
\[
\psi'_M(\alpha; \sigma) = -(\Phi'_M(v) + \phi'_M(v)) - \Phi_M(v).
\]
By the definition of $\Phi_M(v)$ and $\phi_M(v)$ it is easy to see that
\[
\Phi'_M(v)v + \phi'_M(v) = 0.
\]
Thus, $\psi'_M(\alpha; \sigma) = -\Phi_M(v) \leq 0$, implying that $\psi_M$ is monotonically decreasing. Similarly as for $\psi_G$, we can easily prove that
\[
0 \leq \psi_M(\alpha; \sigma) - \max\{0, 1 - \alpha\} \leq \psi_M(1; \sigma) = \sigma/2.
\]
The proof is thus complete. \(\square\)

3 A General Smooth Convex Loss

Motivated by (3) and (4) we propose a general smooth convex loss function as stated in the following theorem.

**Theorem 3** Given $\theta \in \mathbb{R}$ and $\sigma > 0$, define $\psi(\alpha) = \Phi_c(v)(\theta - \alpha) + \phi_c(v)\sigma$, where $v = (\theta - \alpha)/\sigma$, $\Phi_c(v)$ and $\phi_c(v)$ are differentiable and satisfy that $\Phi'_c(v)v + \phi'_c(v) = 0$ and $\Phi'_c(v) \geq 0$. Then we have
1) $\psi(\alpha)$ is twice differentiable with $\psi'(\alpha) = -\Phi_c(v)$ and $\psi''(\alpha) = \Phi'_c(v)/\sigma$;
2) $\psi(\alpha)$ is convex;
3) $\psi(\alpha)$ is $\gamma$-strongly convex if $\Phi'_c(v) \geq \gamma\sigma$ for all $v \in \mathbb{R}$;
4) $\psi(\alpha)$ is $\mu$-smooth convex if $\Phi'_c(v) \leq \mu\sigma$ for all $v \in \mathbb{R}$;
5) $\psi(\alpha)$ is classification-calibrated for binary classification if $\Phi_c(\theta/\sigma) > 0$;
6) the conjugate of $\psi(\alpha)$ is $\psi^*(\beta) = \beta\theta - \phi_c(\Phi^{-1}_c(-\beta))\sigma$, $\beta \in -R(\Phi_c)$, where $\Phi^{-1}_c$ is the inverse function of $\Phi_c$ and $R(\Phi_c)$ is the range of $\Phi_c$. 

Proof. 1) It is easy to obtain that
\[
\psi'(\alpha) = -(\Phi'_c(v) + \phi'_c(v)) - \Phi_c(v) = -\Phi_c(v),
\]
\[
\psi''(\alpha) = \Phi'_c(v)/\sigma.
\]
2) Since \( \psi''(\alpha) = \Phi'_c(v)/\sigma \geq 0, \) \( \psi \) is convex.
3) If \( \Phi'_c(v) \geq \gamma \sigma \) for all \( v \in \mathbb{R}, \) then \( \psi''(\alpha) \geq \gamma. \) Thus \( \psi(\alpha) \) is \( \gamma \)-strongly convex, that is,
\[
\psi(\alpha) - \psi(\beta) \geq \psi'(\beta)(\alpha - \beta) + \frac{\gamma}{2}|\alpha - \beta|^2, \quad \forall \alpha, \beta \in \mathbb{R}.
\]
4) If \( \Phi'_c(v) \leq \mu \sigma, \) then \( \psi''(\alpha) \leq \mu, \) so the convex function \( \psi(\alpha) \) is \( \mu \)-smooth, that is,
\[
|\psi'(\alpha) - \psi'(\beta)| \leq \mu|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}.
\]
5) Since \( \psi'(0) = -\Phi_c(\theta/\sigma) < 0, \) by Theorem 4, the convex function \( \psi(\alpha) \) is classification-calibrated.
6) The conjugate function of \( \psi \) is
\[
\psi^*(\beta) = \sup_{\alpha \in \mathbb{R}}(\beta \alpha - \psi(\alpha))
\]
\[
= \sup_{\alpha \in \mathbb{R}}(\beta \alpha - \Phi_c(v)(\theta - \alpha) - \phi_c(v)\sigma)
\]
\[
= \beta \theta - \inf_{v \in \mathbb{R}}(\beta v + \Phi_c(v)v + \phi_c(v))\sigma.
\]
Let \( L(v) = \beta v + \Phi_c(v)v + \phi_c(v) \) and \( v^* = \arg \inf_{v \in \mathbb{R}} L(v). \) Then \( L'(v) = \beta + \Phi_c(v). \) Thus, \( L(v) \) reaches its minimum at \( v^* = \Phi_c^{-1}(\beta), \) and so we have
\[
\psi^*(\beta) = \beta \theta - (\beta v^* + \Phi_c(v^*)v^* + \phi_c(v^*))
\]
\[
= \beta \theta - \phi_c(v^*) = \beta \theta - \phi_c(\Phi_c^{-1}(\beta)).
\]
\[\square\]

Note that if \( \Phi_c \) and \( \phi_c \) satisfy the conditions in Theorem 3, and \( \Phi_c(v) \geq 0, \) suppose \( v = v(\alpha) \) is convex and differentiable, then \( \psi_\alpha(v) = \Phi_c(v) + \phi_c(v) \) is also convex. The general smooth convex loss function \( \psi(\alpha) \) includes many surrogate loss functions mostly used in binary classification as special cases, as shown below. Figure 1 below presents several surrogate convex loss functions.

Example 1: Least Square Loss. Let \( \Phi_c(v) = \sigma v \) and \( \phi_c(v) = -\sigma v^2/2. \) Let \( \theta > 0. \) Then
\[
\psi(\alpha) = \Phi_c(v)(\theta - \alpha) + \phi_c(v)\sigma = (\theta - \alpha)^2/2,
\]
which is the least square loss. The parameter \( \theta \) satisfies \( \Phi_c(\theta/\sigma) = \theta > 0. \) The conditions in Theorem 3 are easy to verify, and it is easy to obtain that
\[
\psi'(\alpha) = -\Phi_c(v) = \alpha - \theta, \quad \psi''(\alpha) = \Phi'_c(v)/\sigma = 1,
\]
\[
\psi'^*(\beta) = \beta \theta - \phi_c(\Phi_c^{-1}(\beta)) = \beta \theta + \sigma(\beta/\sigma)^2/2
\]
\[
= \beta \theta + \beta^2/(2\sigma), \quad \beta \in \mathbb{R}.
\]
Example 2: Smooth Hinge Loss $\psi_G$. Let $\Phi_c(v) = \Phi(v)$ and $\phi_c(v) = \phi(v)$ with $\Phi$ and $\phi$ given in Section 2. Then

$$\psi(\alpha) = \Phi(v)(\theta - \alpha) + \phi(v)\sigma.$$ 

By (3), $\Phi'_c(v)v + \phi'_c(v) = 0$, so $\psi'(\alpha) = -\Phi(v) \leq 0$. Then $\psi(\alpha) \geq \lim_{\alpha \to \infty} \psi(\alpha) = 0$. Setting $\theta = 1$ gives the smooth Hinge loss $\psi_G$. Further, $\Phi'_c(v) = \phi(v) \geq 0$, $\forall v \in \mathbb{R}$, $\Phi_c(\theta/\sigma) = \Phi(\theta/\sigma) > 0$, $\forall \theta \in \mathbb{R}$, and $\psi(\alpha)$ tends to the Hinge loss as $\sigma \to 0$. It is also easy to see that

$$\psi''(\alpha) = \Phi'_c(v)/\sigma = \phi(v)/\sigma,$$

$$\psi^*(\beta) = \beta\theta - \phi_c(\Phi^{-1}_c(-\beta)) = \beta\theta - \phi(\Phi^{-1}(-\beta))$$

with $\beta \in (-1, 0)$.

Example 3: Smooth Hinge Loss $\psi_M$. Let $\Phi_c(v) = \Phi_M(v)$ and $\phi_c(v) = \phi_M(v)$. Then it follows that

$$\psi(\alpha) = \Phi_c(v)(\theta - \alpha) + \phi_c(v)\sigma$$

$$= \frac{1}{2}(\theta - \alpha) + \frac{1}{2}\sqrt{(\theta - \alpha)^2 + \sigma^2}.$$ 

By (4), $\Phi'_c(v)v + \phi'_c(v) = 0$, so $\psi'(\alpha) = -\Phi_M(v) \leq 0$. Then $\psi(\alpha) \geq \lim_{\alpha \to \infty} \psi(\alpha) = 0$. Moreover, $\Phi'_c(v) \geq 0$, $\forall \theta \in \mathbb{R}$, $\Phi'_M(v) = (1 + v^2)^{-3/2}/2 > 0$, $\forall v \in \mathbb{R}$, and $\psi(\alpha)$ tends to the Hinge loss as $\sigma \to 0$. Setting $\theta = 1$ gives the smooth Hinge loss $\psi_M$. It is easy to get

$$\psi''(\alpha) = \Phi'_M(v)/\sigma = (1 + v^2)^{-3/2}/(2\sigma).$$

In addition, we have

$$\psi^*(\beta) = \beta\theta - \phi_c(\Phi^{-1}_c(-\beta)) = \beta\theta - \frac{1}{2}\sqrt{1 - (2\beta + 1)^2}$$

with $\beta \in (-1, 0)$.

Example 4: Exponential Loss. Let $\Phi_c(v) = e^v$, $\phi_c(v) = (1 - v)e^v$. Then

$$\psi(\alpha) = \Phi_c(v)(\theta - \alpha) + \phi_c(v)\sigma = e^v \geq 0.$$ 

Further, $\Phi'_c(v) = e^v \geq 0$ and $\Phi_c(\theta/\sigma) = e^{\theta/\sigma} > 0$ for all $\theta \in \mathbb{R}$. Moreover, $\Phi'_c(v)v + \phi'_c(v) = e^v + (1 - v)e^v - e^v = 0$. Letting $\theta = 0$ and $\sigma = 1$ gives the exponential loss. It is easy to get that

$$\psi'(\alpha) = -\Phi_c(v) = -e^v, \quad \psi''(\alpha) = \Phi'_c(v)/\sigma = e^v/\sigma,$$

and

$$\psi^*(\beta) = \beta\theta - \phi_c(\Phi^{-1}_c(-\beta)) = \beta\theta + \beta(1 - \ln(-\beta))$$

$$= \beta(1 + \theta - \ln(-\beta)), \quad \beta \in (-\infty, 0).$$

Example 5: Logistic Loss. Let $\Phi_c(v) = e^v/(1 + e^v)$, $\phi_c(v) = \ln(1 + e^v) - ve^v/(1 + e^v)$. Then

$$\psi(\alpha) = \Phi_c(v)(\theta - \alpha) + \phi_c(v)\sigma = \sigma\ln(1 + e^v) \geq 0.$$
It is easy to get that \( \Phi'_c(v) = e^v(1 + e^v)^{-2} \), \( \Phi_c(\theta/\sigma) > 0 \) for all \( \theta \in \mathbb{R} \), and \( \Phi'_c(v) + \phi'_c(v) = 0 \). Letting \( \theta = 0 \) and \( \sigma = 1 \) gives the logistic loss. Further, we have
\[
\psi'(\alpha) = -\frac{e^\alpha}{1 + e^\alpha}, \quad \psi''(\alpha) = \frac{\Phi'_c(v)}{\sigma} = \frac{1}{\sigma} e^\alpha (1 + e^\alpha)^{-2}.
\]

For \( \beta \in (-1, 0) \), \( \Phi_c^{-1}(-\beta) = \ln(-\beta/(1 + \beta)) \), so
\[
\psi^*(\beta) = \beta \theta - \phi_c(\Phi_c^{-1}(-\beta)) = \beta(\theta - \ln(-\beta)) + (1 + \beta) \ln(1 + \beta).
\]

**Example 6: Smooth Absolute Loss.** Let \( \Phi_c(v) = \arctan(v) \), \( \phi_c(v) = -\frac{1}{2} \ln(1 + v^2) \). Then it follows that
\[
\psi(\alpha) = \arctan(v)(\theta - \alpha) - \frac{\sigma}{2} \ln(1 + v^2),
\]
\( \Phi'_c(v) + \phi'_c(v) = 0 \) and \( \psi'(\alpha) = -\arctan((\theta - \alpha)/\sigma) \), so \( \psi'(\alpha) < 0 \) for \( \alpha < \theta \) and \( \psi'(\alpha) \geq 0 \) for \( \alpha \geq \theta \). Thus, \( \psi(\alpha) \geq \lim_{\alpha \to \theta} \psi(\alpha) = 0 \). To make the condition \( \Phi_c(\theta/\sigma) = \arctan(\theta/\sigma) > 0 \) to hold, we must have \( \theta > 0 \). Further, we have \( \Phi'_c(v) = 1/(1 + v^2) \geq 0 \). It is also easy to derive that \( \lim_{\alpha \to 0} \psi(\alpha) = (\pi/2) |\theta - \alpha| \), making us to call \( \psi(\alpha) \) the smooth absolute loss function. A direct calculation gives
\[
\psi''(\alpha) = \frac{\Phi'_c(v)}{\sigma} = \frac{1}{\sigma(1 + v^2)}.
\]
Finally, for \( \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) we have \( \Phi_c^{-1}(-\beta) = \tan(-\beta) \) and
\[
\psi^*(\beta) = \beta \theta - \phi_c(\Phi_c^{-1}(-\beta)) = \beta \theta + \frac{1}{2} \ln(1 + \tan^2(-\beta)).
\]

**Example 7: Smooth ReLU.** ReLU is a famous non-smooth activation function in deep neural networks (DNN), which is defined as \( \psi_{ReLU}(\alpha) = \max(0, \alpha) \). Define the smooth ReLU (sReLU) function as
\[
\psi_{sReLU}(\alpha; \sigma) = \Phi(\alpha/\sigma) \alpha + \phi(\alpha/\sigma) \sigma
\]
with \( \Phi \) and \( \phi \) given in Section 2. Then \( \psi_{sReLU}(\alpha; \sigma) = \psi(\alpha) \), where \( \psi \) is defined as in Example 2 with \( \theta = 0 \). By Example 2 we know that \( \psi_{sReLU}(\alpha; \sigma) \) uniformly converge to ReLU as \( \sigma \) goes to 0.

By Theorem 3 we have the following remarks.

1) Any smooth convex function \( \psi(\alpha) \) can be rewritten in the form \( \psi(\alpha) = \Phi_c(v)(\theta - \alpha) + \phi_c(v) \sigma \), where \( \Phi_c(v) = -\psi'(\theta - \sigma v) \) and \( \phi_c(v) = \psi(\theta - \sigma v) - \Phi_c(v) v \).

2) Given a monotonically increasing, differentiable function \( \Phi_c(v) \), we are able to construct a convex, smooth surrogate loss which is classification-calibrated for binary classification. Moreover, there is no need to know the explicit expression of the loss function when learning with gradient-based methods.
3) There is a great interest to develop a smoothing technique to approximate a non-smooth convex function [25, 32]. For example, [25] improved the traditional bounds on the number of iterations of the gradient methods based on a special smoothing technique in non-smooth convex minimization problems. Theorem 3 provides a new smoothing technique by searching a monotonically increasing and differentiable function $\Phi_c(v)$ which approximates the sub-gradient of the non-smooth convex function.

4 Algorithms

There are many algorithms for SVMs. In the early time, the decomposition methods such as SMO and SVMlight, which overcome the memory requirement of the quadratic optimization methods, have been proposed to solve L1-SVM in its dual form [26, 15] (see [41] for convergence analysis of the decomposition methods including SMO and SVMlight). Later, several convex optimization methods have been introduced, such as the gradient-based methods [42], the bundle method [35, 11], the coordinate descent method [41, 35], the dual method [13, 30, 45] and online learning methods [29, 37, 10]. Based on the generalized Hessian matrix of a convex function with locally Lipschitz gradient proposed in [23], several second-order methods have been applied to solve L2-SVM [16, 20].

In this paper, we focus on the following smooth support vector machine

$$\arg \min_{w \in \mathbb{R}^p} L(w) := \frac{1}{2} \lambda \|w\|^2 + \frac{1}{n} \sum_{i=1}^n \psi(y_i w^T x_i; \sigma),$$

(5)

where $\psi$ is the smooth Hinge loss $\psi_M$ or $\psi_G$. We will analyze the first and second-order convex algorithms for the above SSVM.
4.1 First-Order Algorithms

The gradient descent (GD) method has taken the stage as the primary workhorse for convex optimization problems. It iteratively approaches the optimal solution. In each iteration, the standard full gradient descent (FGD) method takes the negative gradient as the descent direction, and the classifier updates as follows

\[
 w_{t+1} = w_t - \eta_t \left[ \lambda w_t - \frac{1}{n} \sum_{i=1}^{n} \Phi_c(v^t_i) y_i x_i \right],
\]

where \( \eta_t \) is a predefined step size and \( v^t_i = (1 - y_i w^T x_i) / \sigma \). The FGD method has \( O(1/t) \) convergence rate under standard assumptions [24]. Nesterov proposed a famous accelerated gradient (AGD) method in 1982 (see [24]). The AGD method achieves the optimal convergence rate of \( O(1/t^2) \) for smooth objective functions.

However, for large-scale problems it is computationally very expensive to compute the full gradient in each iteration. To accelerate the learning procedure, the linear classifier can be updated with a stochastic gradient (SG) step instead of a full gradient (FG) step, such as Pegasos [29]. Precisely, at the \( t \)-th iteration, \( w_t \) is updated based on a randomly chosen example \((x_{i_t}, y_{i_t})\):

\[
 w_{t+1} = w_t - \eta_t \left[ \lambda w_t - \Phi_c(v^t_{i_t}) y_{i_t} x_{i_t} \right],
\]

where \( \eta_t \) is a predefined step size which is required to satisfy that \( \sum_{t=1}^{\infty} \eta_t = \infty \) and \( \sum_{t=1}^{\infty} \eta_t^2 < \infty \) for convergence. \( \eta_t \Phi_c(v^t_{i_t}) \) can be seen as the learning rate of the chosen example \((x_{i_t}, y_{i_t})\). Noting that \( \eta_t \) is independent of \((x_{i_t}, y_{i_t})\) and \( \Phi_c(v^t_{i_t}) \) depends on the margin of the chosen example \((x_{i_t}, y_{i_t})\), we call \( \eta_t \) the exogenous learning rate and \( \Phi_c(v^t_{i_t}) \) the endogenous learning rate.

An advantage of the above stochastic gradient descent (SGD) algorithm is that its iteration cost is independent of the number of training examples. This property makes it suitable for large-scale problems. In the SGD algorithm, the full gradient \( -(1/n) \sum_{i=1}^{n} \Phi_c(v^t_i) y_i x_i \) is replaced by a stochastic gradient \( -\Phi_c(v^t_{i_t}) y_{i_t} x_{i_t} \) of a randomly chosen example in the \( t \)-th iteration. Though both gradients are equivalent in expectation, they are rarely the same. Thus, it is not a natural to use the norm of the stochastic gradient, \( \| \Phi_c(v^t_{i_t}) y_{i_t} x_{i_t} \| \), as a stopping criterion. The discrepancy between the FG and the SG also has a negative effect on the convergence rate of the SGD method. Since there is no guarantee that \( \| \Phi_c(v^t_{i_t}) y_{i_t} x_{i_t} \| \) will approach to zero, we need to employ a monotonically decreasing step size series \( \{ \eta_t \}_{t=0}^{\infty} \) with \( \eta_t \to 0 \) for convergence. The small step size leads to a sub-linear convergence of the SGD method even for the strongly convex objective function.

Many first-order algorithms have been proposed by combining the low computation cost of SGD and the faster convergence of the FGD method to provide variance reduction [8, 30]. These novel algorithms are known as stochastic variance reduction gradient (SVRG) algorithms, such as SAG [28], SAGA [8], SDCA [30], Finito [9], and SVRG [36]. Their convergence analysis can be found in the corresponding references.
**Algorithm 1** The Framework of TRON Algorithm

**Input:** Given $w_0 = 0$, training data $X$, label vector $Y$ and parameter $\lambda, \sigma$, $\Delta_0 = \| \nabla L(w_0) \|

1: for $t = 1, 2, \ldots$ do
2: Calculate $g_t = \nabla L(w_t)$ and stop if $\| \nabla L(w_t) \| \leq 0.0005$
3: Calculate the diagonal matrix $D$ with respect to the Hessian matrix $\nabla^2 L(w_t)$
4: Find an approximate solution $s_t$ of the trust region sub-problem with the conjugate gradient method in Algorithm 2
5: $\rho_t = \frac{L(w_t+s_t)-L(w_t)}{q_t(s_t)}$, where
   
   $q_t(s_t) = \nabla^T L(w_t)s_t + \frac{1}{2} s_t^T (\lambda E + \frac{1}{n} X^T D X) s_t$
   
   $= \nabla^T L(w_t)s_t + \frac{1}{2} \lambda \|s_t\|^2 + \frac{1}{2n} (X s_t)^T D (X s_t)$

6: Update the classifier $w_{t+1}$:
   
   $w_{t+1} = \begin{cases} w_t + s_t & \text{if } \rho_t > \eta_0 \\ w_t & \text{if } \rho_t \leq \eta_0 \end{cases}$

7: Update $\Delta_{t+1}$ according to the rule:
   
   $\Delta_{t+1} \in [\delta_1 \min(\|s_t\|, \Delta_t), \delta_2 \Delta t]$, if $\rho \leq \eta_1$;
   
   $\Delta_{t+1} \in [\delta_1 \Delta_t, \delta_2 \Delta t]$, if $\eta_1 < \rho < \eta_2$;
   
   $\Delta_{t+1} \in [\Delta_t, \delta_3 \Delta t]$, if $\rho \geq \eta_2$.

8: end for
Algorithm 2 Conjugate Gradient Algorithm

Input: $s_1^t \leftarrow 0$, $r_1^t \leftarrow -\nabla^2 L(w_t))s_1^t - \nabla L(w_t) = -\nabla L(w_t)$, $p_1^t = 0$, $\Delta_t$.

1: for $k = 1, 2, 3, \ldots$ do
2:   If $\|r_k^t\| \leq \xi \|g_t\|$, output $s_t = s_t^k$ and Terminate.
3:   $\alpha_t^k = \frac{\nabla^2 L(w_t)}{p_t^k} + \frac{1}{n}(\nabla^2 L(w_t))T(D(w_t))$
4:   If $\|s_t^k + \alpha_t^k p_t^k\| \leq \Delta_t$, set $s_t^{k+1} = s_t^k + \alpha_t^k p_t^k$ and continue with Step 5.
5:   Otherwise, compute $\tau > 0$ so that $\|s_t^k + \tau p_t^k\| = \Delta_t$. Set $s_t^{k+1} = s_t^k + \tau p_t^k$ and Terminate. Compute $\tau = \frac{-\langle s_t^k \rangle^T}{\|p_t^k\|^2}$.
6:   $\beta = \frac{(r_t^k)^T r_t^{k+1}}{\|r_t^k\|^2}$
7:   $p_t^{k+1} = r_t^{k+1} + \beta p_t^k$
8: end for

4.2 Second-Order Algorithms

Recently, inexact Newton methods without computing the inverse Hessian matrix have been proposed to obtain a superlinear convergence rate in machine learning, such as LiSSA [1] and TRON [19, 20]. LiSSA constructs a natural estimator of the inverse Hessian matrix by using the Taylor expansion, while TRON is a trust region Newton method introduced in [19] to deal with general bound-constrained optimization problems, which generates an approximate Newton direction by solving a trust-region subproblem. In TRON, the direction step should give as much reduction as the Cauchy step. [20] applied the TRON method to maximize the log-likelihood of the logistic regression model, in which a conjugate gradient method was used to solve the trust-region subproblem approximately. TRON was also extended to solve the L2-SVM model by introducing a general Hessian for convex objective functions having a Lipschitz continuous gradient [23].

We want to apply TRON to solve our SSVM problem [5]. By Theorem 2, we have

$$\nabla^2 \tilde{\psi}(y_i w^T x_i) = x_i \left( \frac{\Phi_c'(v_i)}{\sigma} \right) x_i^T = d_i x_i x_i^T,$$

where $v_i = (1 - y_i w^T x_i)/\sigma$ and $d_i = \Phi_c'(v_i)/\sigma$. Then the Hessian matrix of the total loss function $L(w)$ is given by

$$\nabla^2 L(w) = \lambda E + \frac{1}{n} \sum_{i=1}^n d_i x_i x_i^T = \lambda E + \frac{1}{n} X^T D X,$$

where $E$ is the $p \times p$ identity matrix, $D = \text{diag}(d_1, \cdots, d_n)$ is a diagonal matrix, and $X$ is the input feature matrix with its each row representing an instance.

The Newton step is given as $\nabla^2 L(w) \nabla L(w)$ which requires a huge computation cost for high dimensional machine learning problems. The trust-region
method is to provide an approximate Newton direction. For recent advances in the trust-region methods see [40]. Suppose \( w_t \) is the solution at the \( t \)-th iteration. Then the trust-region method generates a direction step \( s_t \) by solving the quadratic subproblem

\[
s_t = \arg \min_{s \in \mathbb{R}^p} \langle \nabla L(w_t), s \rangle + \frac{1}{2} s^T B s, \text{ s.t. } \|s\| \leq \Delta_t,
\]

(8)

where \( \Delta_t \) is the trust region and \( B \) is a positive semi-definite matrix. Here, we choose the matrix \( B \) to be the true Hessian \( \nabla^2 L(w_t) \). By solving (8) with the conjugate-gradient (CG) method, the objective function is an adequate approximation of the reduction of the total loss \( L(w_t + s_t) - L(w_t) \). Thus, if the trust region \( \Delta_t \) is large enough so that \( \| \nabla^2 L(w_t) \nabla L(w_t) \| \leq \Delta_t \), then we will get an inexact Newton step. The framework of TRON for the SSVM (5) is presented in Algorithm 1, and the CG method for the subproblem (8) is given in Algorithm 2. In updating \( \Delta_{t+1} \) from \( \Delta_t \), we use the same parameters \( \eta_0, \eta_1, \eta_2, \delta_1, \delta_2, \delta_3 \) as in [20].

Note that there is no matrix inversion or matrix-matrix multiplication but a matrix-vector multiplication in Algorithm 1. The structure of the Hessian matrix \( \nabla^2 L(w) \) makes it appropriate for \( \nabla^2 L(w) \) to be applied to the second-order algorithm, especially when the input feature matrix \( X \) is sparse. For any vector \( s \in \mathbb{R}^p \), the Hessian matrix-vector multiplication is given as

\[
\nabla^2 L(w)s = (\lambda E + \frac{1}{n}X^TDX)s = \lambda s + \frac{1}{n}X^T(D(Xs)),
\]

so there is no need to compute and store the dense and high-dimensional Hessian matrix \( \nabla^2 L(w) \). We can sequentially calculate \( s_1 = Xs, s_2 = Ds_1, s_3 = X^Ts_2 \), which involves only sparse matrix-vector multiplication. Thus, both the computation and storage costs can be controlled even for very high-dimensional problems. The following theorem establishes the convergence rate of the TRON algorithm (Algorithm 1), which is similar to that of the TRON algorithm developed for logistic regression in [20].

**Theorem 4** The sequence \( w_t \) generated by Algorithm 1 globally converges to the unique optimal solution \( w^* \) of the SSVM (5). If \( \xi_t < 1 \), then Algorithm 1 has a Q-linear convergence, that is, \( \lim_{t \to \infty} \frac{\|w_{t+1} - w^*\|}{\|w_t - w^*\|} < 1 \). If \( \xi_t \to 0 \) as \( t \to \infty \), then Algorithm 1 is of Q-superlinear convergence, that is, \( \lim_{t \to \infty} \frac{\|w_{t+1} - w^*\|}{\|w_t - w^*\|} = 0 \).

Moreover, if \( \xi_t = \kappa \|\nabla L(w_t)\| \) for a positive constant \( \kappa \), then \( w_t \) converges quadratically to \( w^* \), that is, \( \lim_{t \to \infty} \frac{\|w_{t+1} - w^*\|^2}{\|w_t - w^*\|^2} < 1 \).

According to the convex optimization theory, it is not surprising that the trust region Newton method achieves a quadratic convergence rate for a smooth convex objective function \( L(w) \) if \( \|\nabla^2 L(w)\| \leq \mu \) for some \( \mu \). This is also true for other second-order algorithms such as BFGS and LBFGS. Therefore, we can also apply other second-order algorithms such as BFGS and LBFGS to solve the SSVM (5) with similar convergence results as Theorem 4. Further, the second-order
stochastic optimization algorithm LiSSA proposed in [1] for logistical regression can also be extended to solve the SSVM [3]. Note that the TRON algorithm may be applied to L2-SVM with the help of the generalized Hessian matrix. However, the squared Hinge loss in L2-SVM is not twice differentiable, so there is no guarantee to achieve the quadratic convergence rate of TRON for L2-SVM.

5 Experiment

In this section, we conduct two experiments to study the first-order and second-order algorithms for our SSVM [5] with the smooth Hinge loss $\psi_G(\alpha; \sigma)$ or $\psi_M(\alpha; \sigma)$.

5.1 Datasets

In this paper, we focus on the linear SVM model. We consider three data sets: NEWS20, RCV1 and REAL-SIM, which come from document classification and can be download from LIBSVM website[1]. Table 1 lists the number of instances and features as well as the sparsity metric of these datasets which is the proportion of non-zero elements in the input feature. Details can be found in [20].

Table 1: The three sparse datasets used in the experiments.

| DATASET   | DATA SIZE | FEATURE | SPARSITY   |
|-----------|-----------|---------|------------|
| NEWS20    | 19996     | 1355191 | 0.034%     |
| RCV1      | 697641    | 47236   | 0.155%     |
| REAL-SIM  | 72309     | 20958   | 0.245%     |

5.2 The Smooth Parameter Sensitivity Study

We first study the sensitivity of the smooth parameter $\sigma$. As discussed in Section 2, the smooth Hinge loss functions $\psi_G(\alpha; \sigma)$ and $\psi_M(\alpha; \sigma)$ approach to the Hinge loss if $\sigma$ tends to 0. If $\sigma$ is too small, then SSVM is almost equivalent L1-SVM. If $\sigma$ is too large, then the margin $yw^T x$ has little effect on the smooth loss, which makes it impossible to learn a good classifier. We set $\lambda = 10^{-5}$, $\xi_i = 0.1$ and choose $\sigma$ to be $2^{-30}$, $2^{-25}$, $2^{-20}$, $2^{-15}$, $2^{-10}$, $2^{-9}$, $\ldots$, $2^{5}$. For each dataset, we randomly divide it into 5 parts, choose 4 parts for training, and the remaining one for testing. To reduce the variance in the results, for each data set, we generate 4 independent 5-part partitions, leading to a total 20 runs for each dataset. Then the training accuracy is obtained by averaging over these 20 runs. We present the training accuracy of the three datasets NEWS20, REAL-SIM, and RCV in Figure 2.

1 https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
Figure 2: The training accuracy of the two new models on the datasets with respect to the parameter $\sigma$: NEWS20 (top left), REAL-SIM (top right) and RCV1 (bottom). The vertical axis shows the average accuracy and the horizontal axis is $\log_2 \sigma$. The red and green lines represent $\psi_G$ and $\psi_M$, respectively.

Table 2: The test accuracy (%) of different models on the three datasets.

| METHODS | LOSSES                | NEWS20     | REALSIM    | RCV1       |
|---------|-----------------------|------------|------------|------------|
| TRON    | LOGISTIC LOSS         | 95.99 ± 0.21 | 97.20 ± 0.12 | 97.23 ± 0.04 |
| TRON    | SQUARED HINGE LOSS $\ell_2$ | 96.57 ± 0.23 | 97.40 ± 0.12 | 97.65 ± 0.04 |
| TRON    | SMOOTH HINGE LOSS $\psi_M$ | 96.72 ± 0.22 | 97.36 ± 0.13 | 97.60 ± 0.04 |
| TRON    | SMOOTH HINGE LOSS $\psi_C$ | 96.73 ± 0.21 | 97.41 ± 0.12 | 97.61 ± 0.04 |
| PEGASOS | HINGE LOSS $\ell_1$   | 94.12 ± 0.41 | 96.42 ± 0.16 | 97.50 ± 0.05 |
| SIFS    | SMOOTH HINGE LOSS $\ell_\gamma$ | 94.20 ± 0.40 | 96.77 ± 0.14 | 97.52 ± 0.04 |

Table 3: The running time(s) of different models on the three datasets.

| METHODS | LOSSES                | NEWS20 | REALSIM | RCV1 |
|---------|-----------------------|--------|---------|------|
| TRON    | LOGISTIC LOSS         | 1.82   | 0.46    | 7.43 |
| TRON    | SQUARED HINGE LOSS $\ell_2$ | 10.23  | 1.31    | 31.97 |
| TRON    | SMOOTH HINGE LOSS $\psi_M$ | 16.43  | 2.01    | 24.25 |
| TRON    | SMOOTH HINGE LOSS $\psi_G$ | 19.95  | 2.45    | 19.63 |
| PEGASOS | HINGE LOSS $\ell_1$   | 683.46 | 86.95   | 2486.50 |
| SIFS    | SMOOTH HINGE LOSS $\ell_\gamma$ | 238.19 | 39.06   | 430.18 |
5.3 Comparison with Other Algorithms

We now compare our SSVMs with several state-of-the-art binary classification models: Pegasos for L1-SVM, TRON for L2-SVM, and TRON for logistic regression [20, 29]. These experiments were conducted with MATLAB on a workstation with 8GB memory and 3.60GHZ CPU. We also compare our method with the new method SIFS proposed in 2019 [12] (the code of this method is obtained from https://github.com/jiewangustc/SIFS). We use $\sigma = 2^{-6}, 2^{-1}, 2^{-3}$ for the datasets NEWS20, REAL-SIM and RCV1, respectively. The averaged testing accuracy is presented in Table 2. The results show that the second-order method is better than the first-order methods in the large-scale sparse problems and our new model is effective for binary classification problems. By using the same stopping criterion $\|\nabla L(w_t)\| \leq 0.001$ for the same optimization algorithm TRON, we compare several different surrogate loss functions: the logistic loss, the squared Hinge loss, and the smooth Hinge losses. The logistic loss has minimum iteration steps, and the smooth Hinge losses achieve the highest accuracy.

6 Conclusions

In this paper, we proposed two smooth Hinge loss functions $\psi_G$ and $\psi_M$ to build two smooth support vector machines for binary classification problems. We have also discussed several modern first-order and second-order convex algorithms which can be applied to solve our SSVMs effectively. The second-order algorithms can achieve a quadratic convergence rate for the SSVMs, which is different from the traditional SVMs. For our SSVMs and several state-of-the-art binary classification models, experiments have also been carried out on three real-world data sets from document classification, and the experimental results illustrated that our SSVMs are useful for binary classification problems. Further, motivated by the smooth Hinge loss functions $\psi_G$ and $\psi_M$, we gave a general smooth convex loss function which unifies several commonly-used convex loss functions in machine learning, including the L1 regularization and the ReLU activation function in neural networks. The unified framework provides a tool to approximate a non-smooth convex function with a smooth one which can be used to build a smooth machine learning model to be solved with a faster convergent optimization algorithm.

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