Statistical and Scaling Properties of the $ac$ conductivity in Thin Metal-Dielectric Composites

L. Zekri$^{1,2}$, N. Zekri$^{1,2}$, R. Bouamrane$^{1,2}$, and F. Brouers$^{1,3}$

$^1$International Centre for Theoretical Physics, 34100 Trieste, Italy.

$^2$U.S.T.O., Departement de Physique, L.E.P.M., B.P.1505 El M’Naouar, Oran, Algeria.

$^3$Universite de Liege, Institut de Physique, Sart Tilman 4000, Liege Belgium.

August 23, 2019

Abstract

We study in this paper the scaling and statistical properties of the $ac$ conductivity of thin metal-dielectric films in different regions of the loss in metallic components and particularly in the limit of vanishing loss. We model the system by a 2D $RL - C$ network and calculate the effective conductivity by using a real space renormalization group method. It is found that the real conductivity strongly fluctuates for very small losses. The correlation length, which seems to be equivalent to the localization length, diverges for vanishing losses confirming our previous results for the decay of the real conductivity with the loss. We found also that the distribution of the real conductivity becomes log-normal below a certain critical loss $R_c$ which is size dependent for finite systems. For infinite systems this critical loss vanishes and corresponds to the phase transition between localized modes for finite losses and the extended ones at zero loss.

Keywords: Optical properties, Percolation, Disorder, Localization, Scaling, Statistical properties.

PACS Nos. 72.15.Gd;05.70.Jk;71.55.J
1 Introduction

In a previous work [1], we found that the real part of the conductivity of thin metal-dielectric composites vanishes when the loss in the metallic component vanishes. This behavior can easily be explained by the absence of dissipation in the medium. However, this result is in contradiction with the predictions of the effective medium theory where [2]

\[ \epsilon_{eff} = \sqrt{\epsilon_m \epsilon_d} \]  

(1)

where the indices \( m, d \) and \( eff \) stand respectively for the metal, dielectric and effective medium. Indeed, effective medium theory is valid only when the length scale is larger than the correlation length \[ 2, 3 \] which can eventually diverge when the loss vanishes.

Recently, Brouers et al. [4] have studied the scaling effects in such films at the percolation threshold and for various losses and frequencies. They showed the existence of two characteristic lengths for these systems: the coherence length \( L_\omega \) and the correlation length \( L_c \). The first characteristic length corresponds to a ballistic transport without a modification of the phase while the second one corresponds to the size above which the field fluctuations become negligible. For small frequencies the real part of the conductivity is power-law decreasing up to the coherence length and then increases before saturating at the correlation length. However, when the frequency increases the coherence length becomes smaller than the inter-grains distance and this decrease of the real conductivity disappears. On the other hand, they observed at the characteristic frequency \( \omega_{res} \) (where the conductivities of the two components have the same magnitude at vanishing loss) a mutual effect between giant field fluctuations and the dynamic characteristic length \( L_c \). Therefore, the scaling behavior of the conductivity allows us to measure qualitatively the correlation length particularly for vanishing loss where there is still a controversy as discussed above [4]. We note that the scaling study of Brouers et al. [4] has been restricted only for two values of the loss: \( R = 0.1 \) and \( R = 10^{-4} \). However, the above mentioned discrepancy with the effective medium theory has been observed for smaller losses [1]. It is then necessary to measure \( L_c \) for smaller losses in order to check weather it is divergent or not in the limit superconductor-dielectric.
On the other hand, a non monotonic behavior of the real conductivity has been observed in the previous work [1] in the region around $R = 10^{-6}$ most probably due to strong fluctuations of conductivity. This quantity, averaged over 100 samples could not behave 'well' statistically. Therefore, the competition between the scaling effect and the statistical properties can play an important role in such systems and in particular, the transition from localized modes below the correlation length to extended ones above this length [1] can be statistically characterized as for quantum systems. Indeed, in the quantum counterpart part the conductance fluctuations change from the insulating phase to the conducting one. In particular, in the metallic phase the conductance fluctuations are universal and of the order of $e^2/h$ with a Gaussian distribution, while in the insulating phase these fluctuations become large and the distribution is log-normal [5].

In this paper, we examine the scaling properties of this system for a wide range of losses including the limit superconductor-insulator and determine the behavior of the correlation length in this limit. We study also and characterize the different eigenmode phases in these systems by the statistical properties of the conductivity for various losses. We restrict ourselves to the concentration of the metal corresponding to the percolation threshold (which is the minimum concentration corresponding to the appearance of a continuous metallic path through the sample. In 2D $p_c = 0.5$). As in the previous work [1], this system is modelled by a two dimensional $RL-C$ network where the inductance $L$ stands for the metallic grains with a small loss (resistance) $R$ while the dielectric grains correspond to the capacitance $C$ which is assumed without dissipation. The major part of the calculations is done for the characteristic frequency $\omega_{res}$ (At this frequency the dielectric $ac$ conductivity has the same magnitude in modulus as the metallic one for small losses). We can use, without loss of generality the framework where $L=C=\omega_{res}=1$. Frequencies different from the characteristic one are normalized by $\omega_{res}$. We use for the calculation of the effective conductivity the Real Space Renormalization Group Method (RSRG) extensively studied during the two last decades [1, 4, 6, 7] which consists in a representation of the network in Wheatstone bridges transformed into two equivalent conductivities following the directions $x$ and $y$ (see Fig.1). This method has been shown be a good approximation for the calculation of the conductivity and
the critical exponents near the percolation threshold \cite{1,4}.

2 Scaling properties and characteristic lengths

Since, from the previous work \cite{1}, the composite metal-dielectric films showed localization properties, then it should exist in addition to the two characteristic lengths found previously ($L_\omega$ and $L_c$) \cite{4}, a third one which is the localization length $L_{loc}$. However, this length seems to be equivalent to $L_c$. Indeed, the localization length is defined as the mean size of the sample above which the local field strength (the equivalent to the wavefunction in Helmoltz equation) decays exponentially and becomes negligible. This means also that above this length the field fluctuations become small which is the definition of the correlation length $L_c$.

On the other hand, we found previously \cite{1}, by means the inverse participation ratio (IPR) that the eigenmodes are delocalized when the loss in the metallic grains tends to vanish without becoming purely extended (the exponent of the IPR does not reach the value $-2$). This probably means that the localization length increases but remains finite. We should then confirm the divergence of the correlation length for vanishing losses before discussing its relation to the localization length. We re-examine now the scaling properties of the real part of the effective conductivity as done by Brouers et al. \cite{4} but for a range of losses extended to vanishing ones and for two frequencies: the characteristic one ($\omega_{res} = 1$) and $\omega = 1/8$ (obviously we find the same results as Brouers et al. for the values of the loss studied by them, i.e. $R = 0.1$ and $R = 10^{-4}$). As they found, the real conductivity increases and saturates at the correlation length (see Figs.2). Furthermore, for small losses the effective conductivity is shown to fluctuate strongly in opposition to the case of large losses where it seems to behave 'well' statistically. We remark also (as expected) that the saturation takes place at larger sizes when the loss is small, and becomes much greater than 1024 (which is the maximum size we can reach with our computers) below $R = 10^{-6}$.

In Fig.3 we see that the correlation length increases rapidly when the loss decreases and is power-law diverging for vanishing losses (see the insert of Fig.3) with the power law exponent 0.41 (close to 0.5 expected by Brouers et al. \cite{4}). This correlation length is estimated from the
saturation of the effective conductivity shown in Fig. 2 and is not accurate because of the strong fluctuations of the conductivity particularly for small losses. But this divergence is clearly shown from the general behavior of the effective conductivity. Therefore, from the above discussion, this means that the localization length diverges also. Indeed, although the exponent of the IPR saturates at a value smaller than $-2$, this quantity itself (the IPR) does not saturate and continues decreasing with the length scale. Therefore, the localization and the correlation length have at least similar behaviors and confirm the decrease of the conductivity for vanishing losses.

3 statistical behavior of the conductivity

As we can see in Fig. 2, when the loss becomes small the conductivity (which is averaged over 100 samples) becomes strongly fluctuating. Therefore, as discussed before, the conductivity could not obey to the central limit theorem [8], and we should study its statistics before averaging it. On the other hand, for quantum systems, in the insulating phase, the conductance distribution is log-normal and then, its logarithm obeys to the central limit theorem [8]. The conductance fluctuations for classical systems have been studied for a long time [9] but have not characterized the electromagnetic eigenmodes.

In Figs. 4, we show the distribution of the conductivity for various losses. As we can see in these figures, in the region of large losses ($R = 10^{-1}$ and $10^{-3}$) the distribution of the real conductivity is Gaussian and becomes narrower when the length scale increases indicating that the real conductivity obeys to the central limit theorem [8]. We see also that when the loss decreases these distributions become broaden, and consequently the conductivity fluctuations increase (as clearly shown in Fig. 5). In the region of very small losses ($R = 10^{-6}$ and $10^{-9}$ in Figs. 4c and 4d respectively) this distribution becomes Poissonian and narrows for larger sample sizes for $R = 10^{-6}$ (Fig. 4c) while it seems to be less affected by the size for $R = 10^{-9}$ (Fig. 4d). These two different behaviors of the conductivity distribution appear clearly in Fig. 5 for the variance which decreases for $R$ smaller than $10^{-8}$ while the relative variance (called also normalized noise [9]) saturates in this region. Therefore, the averaged real conductivity seems to be not 'stable' statistically for very
small losses, while it seems to be so in the region around $R = 10^{-6}$.

In Fig.6 we show the distribution of $-\log(\sigma)$ in the region of the loss where $\sigma$ shows a Poissonian distribution in Figs.4 (i.e. $R = 10^{-6}$ and $10^{-9}$). We can see that the distribution for $R = 10^{-6}$ seems to become Gaussian and becomes narrower for larger system sizes while for $R = 10^{-9}$ the distribution has a long tail for large $\sigma$, but the narrowing seems to be slowly varying with the size. From these distributions we conclude that in this region of very small losses $\log(\sigma)$ is the relevant quantity for statistical averaging and not $\sigma$. Note that these distributions have been done only from 100 samples which is probably not sufficient for obtaining the Gaussian shape, but their narrowing for larger sizes it is clearly observed. We estimate the critical loss separating the two regions (Gaussian distribution and log-normal one) at $R_c = 10^{-5}$. Therefore, below this critical loss the real conductivity should be estimated from the average of its logarithm, i.e.

$$<\sigma> = \exp(<\log(\sigma)>)$$ (2)

In Fig.7 we compare the real conductivity calculated by direct averaging ($<\sigma>$) with the averaging of this quantity by Eq.(2) in the regions of the log-normal distributions. We see that the second method gives average effective conductivities for very small losses one magnitude smaller than in the first one. This implies that the correlation length should increase for vanishing losses more rapidly than in Fig.3.

Finally this statistical behavior of the real conductivity characterizes the two phases: localized modes for sizes above the correlation length and extended modes below this length (which diverges at zero loss). This transition seems to occur (for a sample size 1024x1024) at the critical loss $R_c = 10^{-5}$. The correlation length $L_c$ at this loss is about 1024. Therefore, there is a scaling effect for $R_c$ for any finite system. We can then avoid this scaling effect if we consider an infinite system size. In this case, there is a phase transition from localized modes to extended ones at the critical loss $R_c = 0$ and the conductivity distribution is log-normal only at this critical loss.

This statistical characterization of the classical eigenmodes is exactly the inverse of that
of the electronic states. Indeed, in electronic systems, we look for the statistical behavior of the conductance while in this case the analog of the electronic conductance is the optical transmission which should show a similar statistical behavior. Nevertheless, this characterization of the electromagnetic eigenmodes by the statistical properties of the \( ac \) conductivity is also related to that of the electronic states. Actually, we see for vanishing losses (corresponding to extended modes) that the distribution of the real conductivity seems to become independent of the system size (see Fig.6b) and its relative variance is constant (Fig.5). This behavior is also shown in electronic systems where in the metallic regime, the conductance fluctuations become independent of the length and magnetic field (called universal conductance fluctuations \( \mathbb{I} \)).

4 Conclusion

We have studied in this paper the characteristic lengths and the statistical properties of the real part of the effective conductivity in 2D metal-dielectric composites at the percolation threshold and for a characteristic frequency where the conductivities of the two components have the same magnitude for a vanishing loss. We found that the correlation length and the localization length (studied in the previous paper \( \mathbb{II} \)) have a similar behavior. These lengths seem to diverge for vanishing losses which leads to the limit of validity of the effective medium theory (since the size of the system is never greater than these lengths) and confirms the behavior of the real conductivity observed previously \( \mathbb{I} \) in this region of the loss.

We examined also the statistical properties of this effective conductivity and found two different distributions for small and large losses. For large losses the distribution of conductivity is Gaussian while for small losses it becomes log-normal with a long tail and strong fluctuations. For vanishing losses, the fluctuations seem to change very slowly with the size of the system. We found the critical loss separating these two statistical behaviors at \( R_c = 10^{-5} \). This critical loss separates also the localized modes at large losses from the extended modes at small ones \( \mathbb{II} \). However, this critical loss is size dependent and actually the correlation length at \( R_c \) is about 1024 which is the size of the system. Nevertheless, for infinite systems there is a phase transition from
localized modes for finite losses to extended modes at zero loss which is then the critical loss for this phase transition.

We found an analogy between the statistical characterization of the classical eigenmodes studied here and that of the electronic states. In particular, we found that in the limit of vanishing loss where the eigenmodes are extended, constant conductivity fluctuations (with the length scale) which corresponds to the metallic states in electronic systems. Therefore, these scaling and statistical behaviors of the real conductivity explain its anomalous decay with the loss and its non monotonic variation observed in the previous paper [1]. However, since the distributions observed here have long tails and are not really Gaussians, an extensive study in this way is needed by using the Lévy distributions [10] and examining the variation with the loss of the exponent of the power-law tail of these distributions. We need also an accurate determination of the correlation length in order to study its critical divergence exponent for vanishing loss. Indeed, the divergence of $L_c$ is power-law with the exponent 0.41 close to the estimation of Brouers et al. [4] (0.5) but no accurate determination has been done because of the strong fluctuations of the effective conductivity. Therefore an exact calculation of the conductivity (by Frank and Lobb method [11]) and its averaging by taking into account their statistical distributions are important for the accurate determination of $L_c$ by the above mentioned scaling behavior. These questions should be a subject of a forthcoming investigation.

ACKNOWLEDGEMENTS

We would like to acknowledge the support and the hospitality of ICTP during the progress of this work.
References

[1] L.Zekri, R.Bouamrane, N.Zekri and F.Brouers, ICTP Preprint IC/98/36, submitted to Physica A (1998)

[2] A.M. Dykhne, Zh.Eksp.Teor.Fiz. 59, 110 (1970) [Sov.Phys.JETP, 32, 348 (1971)].

[3] D.Staufer and A.Aharony, Percolation theory, 2nd Ed., Taylor and Francis,London, 1994; D.J.Bergman and D.Stroud, Solid State Phys. 46, 147 (1992).

[4] F.Brouers, A.K.Sarychev, S.Blacher and O.Lothaire, Physica A241, 146 (1997).

[5] For a review see C.W.J.Beenakker, Rev.Mod.Phys. 69, 731 (1997); T.Guhr, A.Mueller-Groeling and H.A. Weidenmueller, Phys.Rep. 299 189 (1998). See also K.Senouci, N.Zekri et R.Ouasti, Physica A234, 23 (1996); K.S.Chase and A.McKinnon, J.Phys. C20, 6189 (1987); P.Sheng and Z.Zhang, J.Phys.: Condens.Matt. 3, 4257 (1991); P.Markos and B.Kramer, Ann.Physik 2, 339 (1993).

[6] F.Brouers, S.Blacher and A.K.Sarychev in Fractals in the Natural and Applied Sciences, Ch and H, London (1995) p. 237.

[7] A.K.Sarychev, D.J.Bergman and Y.Yagil, Phys.Rev. B51, 5366 (1995).

[8] H.Ventel, Theory of probabilities, Ed. Mir Moscow, 1987.

[9] C.D.Essoh, These de doctorat, Marseille (France) 1990.

[10] M.F.Shlesinger, G.M.Zaslavski et U.Frish, Lévy flights and related topics in physics, Springer Berlin, 1994.

[11] D.J.Frank and C.J.Lobb, Phys.Rev. B37, 302 (1988).
Figure Captions

**Fig.1** The real space renormalization group for a square network.

**Fig.2** The real part of the conductivity versus the sample size for different losses $R$ and for a) $\omega = 1/8$ and b) $\omega = 1$.

**Fig.3** The correlation length as a function of the loss $R$. The insert is a log-log plot of this figure for the estimation of the power-law exponent.

**Fig.4** The distribution of the real part of the effective conductivity for a sample size $512 \times 512$ (thin curve) and $1024 \times 1024$ (thick curve) and for 4 values of the loss: a) $R = 10^{-1}$, b) $R = 10^{-3}$, c) $R = 10^{-6}$ and d) $R = 10^{-9}$.

**Fig.5** The variance of the real part of the effective conductivity (open circles) and the relation variance (filled triangle up).

**Fig.6** The distribution of $-Log(Re[\sigma])$ for sample sizes $512 \times 512$ (thin curve) and $1024 \times 1024$ (thick curve) and for: a) $R = 10^{-6}$ and b) $R = 10^{-9}$.

**Fig.7** Averaged real part of the conductivity versus $R$ by averaging $Re[\sigma]$ (filled squares) and $Log(Re[\sigma])$ (open squares).
FIGURE 2a

Real conductivity vs. sample size for different values of \( R \):
- \( (R=10^{-1}) \)
- \( (R=10^{-3}) \)
- \( (R=10^{-6}) \)
- \( (R=10^{-8}) \)
\[ L_c = \frac{1}{R^{0.41}} \]

**FIGURE 3**
FIGURE 4

Real conductivity

Occurrence
Variance of $\text{Re} \{ \sigma \}$

(Variance)

(Relative variance)

FIGURE 5
FIGURE 6
FIGURE 7

Direct averaging of $\text{Re}\{\sigma\}$

Averaging of $\text{Re}\{\sigma\}$ by its logarithm

$\langle \text{Re}\{\sigma\}\rangle$ vs. Loss $R$