Quantization of spinning particle in arbitrary background

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Abstract

The article is a natural continuation of the papers by Gavrilov and Gitman (Class.Quant.Grav. 17 (2000) L133; Int. J. Mod. Phys. A15 (2000) 4499) devoted to relativistic particle quantization. Here we generalize the problem, considering the quantization of a spinning particle in arbitrary gravitational background. The nontriviality of such a generalization is related to the necessity of solving complicated ordering problems. Similar to the flat space-time case, we show in the course of the canonical quantization how a consistent relativistic quantum mechanics of spinning particle in gravitational and electromagnetic backgrounds can be constructed.

I. INTRODUCTION

The problem of quantizing the classical (pseudoclassical) models of relativistic particles was discussed in numerous articles [1][2]. In our last publications [3], we presented a new solution of this problem, which shows how a consistent relativistic quantum mechanics can

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be obtained in the course of the canonical quantization of relativistic particle models. We stress that the new construction gives a solution to the old problem of how to construct a consistent quantum mechanics on the base of a relativistic wave equation. The quantization was performed for a spinless particle in arbitrary electromagnetic and gravitational backgrounds, as well as for a spinning particle, but only in an external electromagnetic background. The case of the spinning particle in curved space-time was not considered in the above publications. In the present article we discuss in detail the canonical quantization of spinning particle moving in arbitrary electromagnetic and gravitational backgrounds in 3+1 dimensions. Here we meet both technical and conceptual problems, in particular the ordering problem. It is enough to mention that quantization in the even simpler corresponding nonrelativistic particle case is an open problem. It attracts attention up to the present and several points of view have been brought up on its solution [4]. The relativistic case, which naturally absorbs all known difficulties of its nonrelativistic analog, is essentially richer and more complicated due to its gauge nature.

II. PSEUDOCLASSICAL MODEL OF A SPINNING PARTICLE IN CURVED SPACE-TIME

A pseudoclassical action of a spin one-half relativistic particle in 3+1 dimensions, with spinning degrees of freedom describing by anticommuting (Grassmann) variables, was discussed in [4]. In flat space-time (\(g_{ab} = \eta_{ab} = \text{diag}(1, -1, -1, -1), \ a, b = 0, 1, 2, 3\), Latin letters from the beginning of the alphabet are used for the Lorentz indices), and in the presence of an external electromagnetic field \(A_a(x)\), the action can be written in the following Lorentz invariant form:

\[
S = \int_0^1 Ld\tau, \quad L = -\frac{\eta_{ab}}{2e}(\dot{x}^a - i\xi^a\chi)(\dot{x}^b - i\xi^b\chi) - \frac{e}{2}m^2 - q\dot{x}^aA_a(x) + eF_{ab}(x)\xi^a\xi^b - i\xi^a\dot{\xi}^a + i\xi_4\dot{\xi}_4 + im\xi_4\chi, \quad (1)
\]
where the coordinates \( x^a \) of the particle and the variable \( e \) are Grassmann-even; the Lorentz vector \( \xi^a \), the (pseudo) scalar \( \xi_4 \), and the scalar \( \chi \) are Grassmann-odd. All the variables depend on the parameter \( \tau \in [0, 1] \), which plays here the role of time. Dots above the variables denote their derivatives with respect to \( \tau \). There are two types of gauge transformations under which the action (1) is invariant: the reparametrizations and the supertransformations

\[
\begin{align*}
\delta x^a &= \dot{x}^a \varepsilon, \quad \delta e = d(e \varepsilon) / d\tau, \quad \delta \xi^a = \dot{\xi}^a \varepsilon, \quad \delta \xi_4 = \dot{\xi}_4 \varepsilon, \quad \delta \chi = d(\chi \varepsilon) / d\tau; \\
\delta x^a &= i \xi^a \varepsilon, \quad \delta e = i \chi \varepsilon, \quad \delta \xi^a = (\dot{x}^a - i \xi^a \chi) / 2e, \quad \delta \xi_4 = -m / 2, \\
\delta x^a &= i \xi^a \varepsilon, \quad \delta e = i \chi \varepsilon, \quad \delta \xi^a = (\dot{x}^a - i \xi^a \chi) / 2e, \quad \delta \xi_4 = -m / 2, \\
\delta \xi^a &= \dot{\xi}^a \varepsilon, \quad \delta \xi^a = \dot{\xi}^a \varepsilon, \quad \delta \chi = \dot{\chi} / 2e, \quad \delta \xi_4 = -m / 2,
\end{align*}
\]

where \( \varepsilon(\tau) \) and \( \epsilon(\tau) \) are \( \tau \)-dependent gauge parameters, the first one is even and the second one is odd.

We generalize the action (1) to the curved space-time case (without torsion) with a metric tensor \( g_{\mu \nu} (x) \) as follows:

\[
S = \int_0^1 \! L d\tau, \quad L = -\frac{g_{\mu \nu}(x)}{2e} [\dot{x}^\mu - i \xi^\mu (x) \chi] [\dot{x}^\nu - i \xi^\nu (x) \chi] - \frac{e}{2m^2} q \dot{x}^\mu A_\mu(x) \\
+ i q e F_{\mu \nu}(x) \xi^\mu(x) \xi^\nu(x) - i \xi^\nu(x) D_\tau \xi^\mu(x) + i \xi_4 \dot{\xi}_4 + i m \xi_4 \chi.
\]

Here \( \xi^\mu(x) = e^\mu_a(x) \xi^a \) are world vectors (Greek letters are used for the world indices, e.g. \( \mu = 0, 1, 2, 3 \)), and \( \xi^a \) are Lorentz vectors, where \( e^\mu_a(x) \) is the vierbein field \( [3] \), and \( D_\tau \) is the covariant derivative with respect to \( \tau \),

\[
D_\tau \xi^\mu(x) = \xi^\mu_{\sigma \tau}(x) \dot{x}^\sigma + e^\mu_a(x) \dot{\xi}^a, \quad D_\tau \xi^a = \dot{\xi}^a + \xi^\nu_{b \tau}(x) \dot{x}^\nu = e^a_\mu(x) D_\tau \xi^\mu(x), \\
e^a_\mu(x) e_{\nu \tau}(x) = g_{\mu \nu}(x), \quad e^a_\mu(x) e_{\nu a}(x) = \eta_{ab}, \quad \xi^\mu_{\sigma \tau}(x) = \partial_\sigma \xi^\mu_\tau(x) + \Gamma^\mu_{\nu \sigma} \chi_\nu \tau(x).
\]

Here \( \omega^a_{\nu}(x) = [\partial_\nu e^a_\lambda(x) + e^{a \sigma}(x) \Gamma^\lambda_{\nu \sigma}(x)] e^b_\lambda(x) \) are spin connections for the torsion-free case, \( \omega^a_{\nu} = -\omega^a_{\nu} \), and \( \Gamma^\mu_{\nu \sigma} \) is the affine connection. This action (3) is invariant under general coordinate transformations and is invariant under the gauge transformations

\[1\text{Introduction of the interaction with a torsion field in the model was discussed in [3]}\]

\[2\text{A different (nonsupersymmetric) form of the spinning particle action in curved space time was considered in [7]. That action follows from (3) in a special gauge.}\]

3
\[ \delta x^\mu = \dot{x}^\mu \varepsilon, \ \delta e = d(e \varepsilon)/d\tau, \ \delta \xi_a = \dot{\xi}_a \varepsilon, \ \delta \xi_4 = \dot{\xi}_4 \varepsilon, \ \delta \chi = d(\chi e)/d\tau; \]
\[ \delta x^\mu = i\dot{\xi}^\mu e, \ \delta e = i\chi e, \ \delta \chi = \dot{\varepsilon}, \ \delta \xi^\mu = (\dot{\varepsilon} - i\xi^\mu \chi) e/2e, \ \delta \xi_4 = -m\varepsilon/2, \quad (5) \]

For the purpose of quantization, we select a reference frame which admits a time synchronization over all space. Such a reference frame corresponds to a special gauge \( g_{0i} = 0 \) for which \( g^{00} = g_{00}^{-1}, \ g^{ij}g_{kj} = \delta^i_j \). Such a reference frame always exists for any real space-time. Besides, we choose a special gauge for the vierbein, \( e^a_0(x) = \delta^a_0 \sqrt{g_{00}(x)}, \ e^0_i(x) = 0 \). Then, \( \omega^0_0 = 0, \ \xi_0 \chi \omega^0_k = 0 \). In our index conventions, barred Latin letters from the beginning of the alphabet denote spatial Minkowski vectors, \( a = (0, \bar{a}) \), \( \bar{a} = 1, 2, 3 \); Latin letters from the middle of the alphabet represent spatial world vectors, so that \( \mu = (0, i), \ i = 1, 2, 3 \).

### III. HAMILTONIAN STRUCTURE OF THE THEORY

The expressions for the canonical momenta have the form
\[ p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = \left( -e^{-1}g_{\mu\nu} \left( \dot{x}^\nu - i\xi^\nu \chi \right) - qA_\mu - i\xi_a \xi_b \omega_{ab}^\mu, \right); \quad p_e = \frac{\partial L}{\partial \dot{e}} = 0, \]
\[ \pi_a = \frac{\partial L}{\partial \dot{\xi}_a} = -i\xi_a, \quad \pi_4 = \frac{\partial L}{\partial \dot{\xi}_4} = -i\xi_4, \quad P_\chi = \frac{\partial L}{\partial \dot{\chi}} = 0. \quad (6) \]

They imply the following primary constraints \( \phi_B^{(1)} = 0, \ B = 1, 2, (3, n), \ n = 0, 1, 2, 3, 4 \).
\[ \phi_1 = P_\chi, \ \phi_2 = P_e, \ \phi_{3,n} = \pi_n + i\xi_n. \quad (7) \]

The total Hamiltonian \[ H^{(1)} \] has the form \[ H^{(1)} = H + \lambda^B \phi_B^{(1)}, \] where
\[ H = -\frac{e}{2} \left[ (p_\mu + qA_\mu + i\xi_a \xi_b \omega_{ba}^\mu) g^{\mu\nu} (p_\nu + qA_\nu + i\xi_c \xi_d \omega_{dc}^\nu) - m^2 + 2iqF_{\mu\nu} \xi^\mu \xi^\nu \right] + i \left[ (p_\mu + qA_\mu + i\xi_a \xi_b \omega_{ba}^\mu) \xi^\mu - m\xi_4 \right] \chi. \quad (8) \]

Using the consistency conditions \( \dot{\phi}^{(1)} = 0 \) for the primary constraints, we find the secondary constraints \( \phi^{(2)} = 0 \),
\[ \phi_1^{(2)} = (p_\mu + qA_\mu + i\xi_a \xi_b \omega_{ba}^\mu) \xi^\mu + m\xi^4, \]
\[ \phi_2^{(2)} = (p_\mu + qA_\mu + i\xi_a \xi_b \omega_{ba}^\mu) g^{\mu\nu} (p_\nu + qA_\nu + i\xi_c \xi_d \omega_{dc}^\nu) - m^2 + 2iqF_{\mu\nu} \xi^\mu \xi^\nu. \quad (9) \]
and determine the $\lambda$'s, which correspond to the primary constraints $\phi^{(1)}_{3,a}$. No more secondary constraints arise from the consistency conditions, and the $\lambda$'s that correspond to the constraints $\phi^{(1)}_1, \phi^{(1)}_2$ remain undetermined. The Hamiltonian $H$ is proportional to the constraints, $H = -\frac{e}{2} \phi^{(2)}_2 + i \phi^{(2)}_1 \chi$.

It is convenient to replace the initial set of constraints $\phi^{(1)}, \phi^{(2)}$ by an equivalent one, which we define below. To this end we define the principal value of the square root of an expression containing Grassmann variables as the one which is positive whenever all generating elements of the Grassmann algebra are set to zero. Suppose $r$,

$$r = \sqrt{g_{00}[m^2 - (p_k + A_k) g^{kl}(p_l + A_l) + 2q F_{\mu\nu} \xi^\mu \pi^\nu]}, \quad A_\mu = q A_\mu - \pi_a \xi_b \omega^{ba}_\mu,$$

is such a principal value of the expression indicated. Then we introduce a set of constraints $\phi^{(1)}, T$, equivalent to $\phi^{(1)}, \phi^{(2)}$, where

$$T_1 = \left( p_\mu + q A_\mu + i \xi_a \xi_b \omega^{ba}_\mu \right) (\pi^\mu - i \xi^\mu) - m (\pi_4 - i \xi_4) + 2i \xi_a (\pi_b + i \xi_b) \omega^{ba}_\mu \xi^\mu, \quad T_2 = p_0 + A_0 + \zeta r, \quad \zeta = \pm 1.$$  

To check that, it is useful to take into account the following relation

$$\phi^{(2)}_2 = (-2 \zeta r + T_2) g^{00} T_2 - \frac{i}{2} \phi^{(1)}_{3,a} \left\{ \phi^{(1)}_{3,b}, \phi^{(2)}_2 \right\} - \frac{i}{2} \phi^{(1)}_{3,a} 2i \xi_b \omega^{ba}_\mu g^{\mu\nu} \frac{i}{2} \phi^{(1)}_{3,c} 2i \xi_d \omega^{dc}_\nu.$$  

In fact the constraint $T_2 = 0$ is a linearized analog of the quadratic primary constraint $\phi^{(2)}_2 = 0$ (compare with the flat space-time case [3]). We can regard the discrete variable $\zeta = \pm 1$ as the sign of the Grassmann valued quantity $p_0 + A_0$,

$$\zeta = -\text{sign} [p_0 + A_0],$$

is an analog of the charge sign variable, well known in the flat-space case [3] and especially important for all further constructions. One can easily see from the equations (3) that, similar to scalar particle case, $\text{sign}(\dot{x}^0) = \zeta$. 

5
The new set of constraints \( \phi^{(1)}, T \) is explicitly divided in a subset of the first-class constraints \( \phi^{(1)}_1, \phi^{(1)}_2, T \), and in a subset of second-class constraints \( \phi^{(1)}_{3, n} \). Indeed,

\[
\{ \phi^{(1)}_\nu, \phi^{(1)} \} = \{ \phi^{(1)}_\nu, T \} = \{ T, \phi^{(1)}_{3, n} \} \bigg|_{\phi = T = 0} = \{ T, T \} \bigg|_{\phi = T = 0} = 0, \quad \nu = 1, 2.
\]

(13)

Similar to the flat space-time case we impose first two gauge conditions \( \phi^G = 0 \),

\[
\phi^G_1 = \pi^0 - i \xi^0 - \zeta (\pi_4 - i \xi_4), \quad \phi^G_2 = x^0 - \zeta \tau.
\]

(14)

From the consistency conditions \( \dot{\phi}^G_{1, 2} = 0 \), we find two additional constraints

\[
\dot{\phi}^G_3 = \chi + 2 \zeta \alpha \Delta^{-1} = 0, \quad \phi^G_4 = e - g_{00} \tilde{\omega}^{-1} \left[ 1 - \alpha \Delta^{-1} (\pi^a - i \xi^a) e^0_0 \right] = 0,
\]

(15)

where

\[
\alpha = \frac{i \zeta g_{00}}{r + \tilde{\omega}} \left[ 2 (\pi_b - i \xi_b) \omega_{k}^{b_0} g^{k l} \left( p_l + \tilde{A}_l \right) - 2 q F_{k \mu} e_0^\mu \left( \pi^k - i \xi^k \right) \right],
\]

\[
\Delta = 2 \left[ \zeta (\tilde{\omega}_0 + m) \right], \quad \tilde{A}_\mu = q A_\mu - \pi_b \xi_b \omega_{\mu}^{b_0} \tilde{\omega} = r \bigg|_{\pi_0 = \pi_0, \xi_0 = \pi_0} = \sqrt{g_{00} (\tilde{\omega}_0^2 + \tilde{\rho})},
\]

\[
\tilde{\omega}_0 = \sqrt{\left[ m^2 - (p_k + \tilde{A}_k) g^{k l} \left( p_l + \tilde{A}_l \right) + 2 q F_{k l} \xi^k n^l \right]},
\]

\[
\tilde{\rho} = 2 \left[ q F_{k \mu} e_0^\mu \left( \xi^k + i \pi^k \right) \pi^0 - \pi_0 (\xi_b + i \pi_b) \omega_{k}^{b_0} g^{k l} \left( p_l + \tilde{A}_l \right) \right],
\]

\[
\pi_0 = \Delta^{-1} \left[ (p_l + \tilde{A}_l) (\pi^l - i \xi^l) + 2 i \zeta_0 (\pi_b + i \xi_b) \omega_{j}^{b_0} \xi^j \right].
\]

(16)

Then, from the consistency conditions \( \dot{\phi}^G_{3, 4} = 0 \) we can find \( \lambda^{1, 2} \). All the constraints \((\phi^{(1)}, T, \phi^G)\) are already of second-class. As in flat-space case we pass from these constraints to an equivalent set of second-class constraints \( \Phi_a, a = 1, 2, ..., 13 \),

\[
\Phi_1 = t_1 T_1 + t_2 T_2 + f_1 \phi^G_1 + f_0 \phi^G_{3, 0} + f_{50} \phi^G_{3, 0} \phi^{(1)}_{3, 0} = p_0 + \tilde{A}_0 + \zeta \tilde{\omega}, \quad \Phi_2 = \phi^G_2, \quad \Phi_3 = \phi^{(1)}_{3, 1},
\]

\[
\Phi_4 = \phi^{(1)}_{3, 2}, \quad \Phi_5 = \phi^{(1)}_{3, 3}, \quad \Phi_6 = T_1 + b T_2 + c \phi^G_2, \quad \Phi_7 = \phi^G_1, \quad \Phi_8 = \phi^G_3 + d \phi^G + v \phi^G_1 + u \phi^{(1)}_2,
\]

\[
\Phi_9 = \phi^{(1)}_4, \quad \Phi_{10} = \phi^G_4 + w \phi^G_2 + z \Phi_7 + s \Phi_6, \quad \Phi_{11} = \phi^{(1)}_2, \quad \Phi_{12} = \phi^{(1)}_{3, 0}, \quad \Phi_{13} = \phi^{(1)}_{3, 4}
\]

(17)

where
\[ t_1 = -\alpha \Delta^{-1}, \quad t_2 = 1 + \alpha \Delta^{-1} (\pi^a - i \xi^a) e^0_a, \quad f_1 = \zeta m \alpha \Delta^{-1}, \quad f_0 = \alpha \zeta \left( r g_0^{1/2} + m \right), \]
\[ f_{a0} = \alpha \Delta^{-1} \xi_b \omega_0^{b0} g_{00}^{-1/2}, \quad b = -\frac{\phi^G_2, T_1}{\phi^G_2, T_1}, \quad c = \frac{T_1 + bT_2, \Phi_1}{\phi^G_2, \Phi_1}, \quad u = -\frac{\phi^G_3 + v \phi^G_4, \Phi_4}{\phi^G_2, \Phi_4}, \]
\[ v = -\frac{\phi^G_5, \Phi_6}{\Phi_7, \Phi_6}, \quad d = -\frac{\phi^G_3, \Phi_1}{\phi^G_2, \Phi_1}, \quad w = -\frac{\phi^G_4, \Phi_1}{\phi^G_2, \Phi_1}, \quad z = -\frac{\phi^G_4, \Phi_6}{\Phi_7, \Phi_6}, \quad s = -\frac{\phi^G_4, \Phi_7}{\Phi_6, \Phi_7}. \]

The matrix \( \{ \Phi_a, \Phi_b \} \) has now a simple quasi-diagonal form with the following nonzero elements

\[ \{ \Phi_2, \Phi_1 \} = -\{ \Phi_1, \Phi_2 \} = 1, \quad \{ \Phi_3, \Phi_3 \} = \{ \Phi_4, \Phi_4 \} = \{ \Phi_5, \Phi_5 \} = -2i, \]
\[ \{ \Phi_6, \Phi_7 \} = \{ \Phi_7, \Phi_6 \} = 2i \left[ \zeta (\bar{\omega}_0 + m) + \xi_b \omega_0^{b0} (\pi^i - i \xi^j) \right], \quad \{ \Phi_8, \Phi_9 \} = \{ \Phi_9, \Phi_8 \} = 1, \]
\[ \{ \Phi_{10}, \Phi_{11} \} = -\{ \Phi_{11}, \Phi_{10} \} = 1, \quad \{ \Phi_{12}, \Phi_{12} \} = -\{ \Phi_{13}, \Phi_{13} \} = 2i. \quad (18) \]

We call the variables \( \eta = (x^k, p_k, \zeta, \xi^a, \pi_0) \) the independent variables, since the remaining variables can be expressed via the independent ones by means of the constraints. Similar to the flat space-time case, we can prove that the Hamilton equations of motion and the corresponding constraints for the independent variables have the form

\[ \dot{\eta} = \{ \eta, H_{eff} \}_D(U), \quad U = \phi^{(1)}_{3,k} = 0, \quad k = 1, 2, 3. \quad (19) \]

The effective Hamiltonian \( H_{eff} \) reads:

\[ H_{eff} = [\zeta \bar{\omega}_0 + \omega]_{x^a = \zeta^a}, \quad \bar{\omega}_\mu = q A_\mu + i \xi_a \omega_\mu^{\bar{b}a}, \quad \omega = \bar{\omega}_{|\pi_a = -i \xi_a} = \sqrt{g_{00} (\omega^2_0 + \rho)}, \]
\[ \omega_0 = \sqrt{\left[ m^2 - (p_k + \bar{A}_k) g^{k1} (p_l + \bar{A}_l) - 2iq F_{kl} \xi^k \xi^l \right]}, \quad \rho = 4 \left[ q F_{k\mu} e^{\mu}_{0} \xi^k \bar{\pi}_0 + \bar{\pi}_0 \bar{\omega}_0 \omega_0^{b0} g^{k1} (p_l + \bar{A}_l) \right], \]
\[ \bar{\pi}_0 = -i \xi^k (p_k + \bar{A}_k) [\zeta (\omega_0 + m)]^{-1}, \quad (20) \]

and the only nonzero Dirac brackets between the independent variables are

\[ \{ x^k, p_l \}_D(U) = \{ x^k, p_l \} = \delta^k_l, \quad \{ \xi^b, \xi^\bar{a} \}_D(U) = \frac{i}{2} \eta^{\bar{a}b}. \quad (21) \]

**IV. QUANTIZATION**

Equal time commutation relations for the operators \( \hat{X}^k, \hat{P}_k, \zeta, \hat{\Xi}^a \), which correspond to the variables \( x^k, p_k, \zeta, \xi^a \), are defined according to their Dirac brackets. Thus, the nonzero
commutators (anticommutators) are
\[ [\hat{X}^k, \hat{P}_j] = i\hbar \delta^k_j, \quad [\hat{\Xi}^\alpha, \hat{\Xi}^\beta ] = -\frac{\hbar}{2} \eta^{\alpha\beta}, \quad k, j = 1, 2, 3; \quad \alpha, \beta = 1, 2, 3. \] (22)

We assume \( \hat{\zeta}^2 = 1 \) (see [3]), and realize the operator algebra in the state space \( \mathcal{R} \) whose elements \( \Psi \in \mathcal{R} \) are \( x \)-dependent eight-component columns \( \Psi = (\Psi_{+1}(x), \Psi_{-1}(x)) \), where \( \Psi_\zeta(x), \zeta = \pm 1 \) are four component columns. The inner product in \( \mathcal{R} \) is defined as follows:
\[ (\Psi, \Psi') = (\Psi_{+1}, \Psi'_{+1})_D + (\Psi'_{-1}, \Psi_{-1})_D. \] (23)

For the inner product between the four component columns we select the following equivalent expressions,
\[ (\Psi, \Psi')_D = \int \Psi^\dagger(x)\Psi(x)g_{00}^{-1/2}\sqrt{-g}dx = \int \Psi(x)e^0_a(x)\gamma^a\Psi(x)\sqrt{-g}dx \]
\[ = \int \Psi(x)\gamma^\mu(x)\Psi(x)d\sigma_\mu, \quad \Psi(x) = \Psi^\dagger(x)\gamma^0, \quad \gamma^\mu(x) = e^\mu_a(x)\gamma^a, \quad g = \det ||g_{\mu\nu}||. \] (24)

To obey the above operator algebra in the space \( \mathcal{R} \), we can choose the following realization:
\[ \hat{X}^k = x^k \mathbf{I}, \quad \hat{P}_k = \hat{p}_k \mathbf{I}, \quad \hat{p}_k = -i\hbar \partial_k, \]
\[ \hat{\Xi}^\alpha = \text{bdiag}(\hat{\xi}^\alpha, \hat{\xi}^\alpha), \quad \hat{\xi}^\alpha = \frac{i}{2}\hbar^{1/2}\gamma^\alpha, \quad \hat{\zeta} = \text{bdiag}(I, -I), \] (25)
where \( \mathbf{I} \) is 8 \( \times \) 8 unit matrix, \( I \) is 4 \( \times \) 4 unit matrix, and \( \gamma^\alpha, \bar{\alpha} = 1, 2, 3 \), are three usual \( \gamma \)-matrices in \((3 + 1)\)–dimensions, \( \left[ \gamma^\alpha, \gamma^\beta \right]_+ = 2\eta^{\bar{\alpha}\bar{\beta}} \). One can easily see that such defined operators are Hermitian with respect to the inner product (24).

The quantum Hamiltonian \( \hat{H}_\tau \) that defines the \( \tau \)-evolution of state vectors of the system has to be constructed as a quantum operator in the space \( \mathcal{R} \) on the base of the correspondence

3Here and in what follows we use the following notations
\[ \text{bdiag} (A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \]
where \( A \) and \( B \) are some matrices.
principle starting with its classical image, which is $\mathcal{H}_{eff}$ given by Eq. (20). There exist many quantum operators, which have the same classical image. That corresponds to the well-known ambiguity of the general quantization. We construct $\hat{H}_r$ as follows:

$$
\hat{H}_r = \zeta \hat{A}_0 + \hat{\Omega}, \quad \hat{A}_0 = q \hat{A}_0 + i \hat{\omega}_0 \hat{\omega}_k \hat{\lambda}_k, \quad \hat{\omega}_0 = \text{bdiag} \left( A_0 \big|_{x^0=\tau} I, \quad A_0 \big|_{x^0=-\tau} I \right),
$$

$$
\hat{\omega}_k = \text{bdiag} \left( \omega_0^{\bar{b}a} \big|_{x^0=\tau} I, \quad \omega_0^{\bar{b}a} \big|_{x^0=-\tau} I \right), \quad \hat{\Omega} = \text{bdiag} \left( \hat{\omega} \big|_{x^0=\tau}, \quad -\hat{\omega} \big|_{x^0=-\tau} \right),
$$

$$
\hat{\omega} = \epsilon_0^a \gamma_a \left[ m + \gamma^k (x) \left( \hat{p}_k + q \hat{A}_k - i \frac{\hbar}{4} \gamma^a \gamma_0 \omega_0^{\bar{b}a} \right) \right]. \tag{26}
$$

where we have used the Dirac matrix $\gamma^0, (\gamma^0)^2 = 1, \quad [\gamma^0, \gamma^a] = 0$. (One ought to remark that we can write $\epsilon_0^a \gamma_a = \sqrt{g_{00}} \gamma_0$). The first term in the expression (26) is a natural quantum image of the classical quantity $\zeta \hat{A}_0 \big|_{x^0=\xi_\tau}$. The term $\hat{\Omega}$ is a possible quantum image of the classical quantity $\omega \big|_{x^0=\xi_\tau}$. In fact, we have to justify the following symbolic relation

$$
\lim_{\text{classical}} \hat{\Omega} = \omega \big|_{x^0=\xi_\tau}. \tag{27}
$$

To be more rigorous, one has to work with operator symbols. However, we remain here in terms of the operators, hoping that our manipulations have a clear sense and do not need to be confirmed on the symbol language. First, we replace the operator $\hat{\Omega}$ under the sign of the limit by another one $\hat{\Omega}' = \hat{\Omega} + \hat{\Delta}$,

$$
\hat{\Delta} = \text{bdiag} \left( \hat{\delta} \big|_{x^0=\tau}, \quad -\hat{\delta} \big|_{x^0=-\tau} \right), \quad \hat{\delta} = \sqrt{g_{00}} \gamma_0 \hat{\lambda}_k + \frac{i \hbar}{4} g^{jk} \partial_0 g_{jk}, \quad \hat{\lambda}_k = -m \hbar^{-1/2} \left[ \gamma_j, \hat{\xi}_k \right],
$$

$$
\hat{\gamma}_k = \hat{\xi} \left[ q F_{kl} \epsilon_0^a \hat{\xi}^k \frac{1}{2} \left[ \hat{\xi}_0 \omega_k g^{kl}, \hat{p}_l + \hat{\tilde{A}}_l \right] + \frac{1}{(\hat{\omega}_0^2 - m^2)} \right] + \hat{\tilde{A}}_k = q \hat{A}_k + i \hat{\xi} \hat{\xi}_0 \hat{\omega}_0^{\bar{b}a},
$$

since the classical limit of $\hat{\Delta}$ is zero. Indeed, the leading contributions in $\hbar$ to the operator $\hat{\Delta}$ result from terms that contain $\left( \hat{\xi}^k \right)^2$. In the classical limit such terms turn out to be proportional to $\left( \xi^k \right)^2$, which is zero due to the Grassmann nature of $\xi$’s. The square of the operator $\hat{\Omega}'$ in the classical limit corresponds to the square of the classical quantity $\omega \big|_{x^0=\xi_\tau}$. Indeed, $\left( \hat{\Omega}' \right)^2 = \text{bdiag} \left( \hat{\omega}^2 \big|_{x^0=\tau, \xi_\tau = 1}, \quad \hat{\omega}^2 \big|_{x^0=-\tau, \xi_\tau = -1} \right)$, $\hat{\omega}^2 = g_{00} \left( \hat{\omega}_0^2 + \hat{\rho}_1 + \hat{\rho}_2 \right)$, where
\[ \hat{\omega}_0^2 = \left[ m^2 - \frac{1}{\sqrt{-\det g_{ij}}} \left( \hat{p}_i + \hat{A}_i \right) \right] \sqrt{-\det g_{ij} g^{kl} \left( \hat{p}_l + \hat{A}_l \right)} - \frac{\hbar^2}{4} \hat{R} \right] I - i q F_{ij}[\hat{\xi}^j, \hat{\xi}^i], \]

\[ \rho_1 = \frac{1}{2i} \left[ \hat{y}, \left[ \hat{\xi}^i \left( \hat{p}_i + \hat{A}_i \right), \hat{\omega}_0 - m \right]_+ \right], \]

\[ \rho_2 = \frac{m}{2i} \left\{ \left[ \hat{\xi}^i, \hat{\xi}^j \right] - \left[ g^{ij}, \hat{\xi}_k \right] \left( \hat{p}_l + \hat{A}_l \right) + g^{kl} \left[ \hat{p}_l + \hat{A}_l, \hat{\xi}_k \right] \hat{y} - g^{kl} \left[ \left( \hat{p}_l + \hat{A}_l \right) \hat{y}, \hat{\xi}_k \right] \right\} \]

\[ + 2i \hbar^{-1/2} \left\{ \frac{1}{2} \left[ \hat{\xi}^k, \hat{\xi}^i \right] \left[ \hat{\lambda}_k, \hat{p}_l + \hat{A}_l \right] + \xi^i \left[ \hat{\lambda}_k, \hat{\xi}^j \right] \left( \hat{p}_l + \hat{A}_l \right) + \hat{\xi}^j \left[ \hat{p}_l + \hat{A}_l, \hat{\xi}^i \right] \hat{\lambda}_k \right\} - \left( \hat{\xi}^k \hat{\lambda}_k \right)^2, \]

where \( \hat{R} \) is scalar curvature related to the stationary metric \( \bar{g}_{\mu \nu} = g_{\mu \nu}\big|_{x^0 = \text{const}} \), and the following expression for \( \hat{\omega} \) is used,

\[ \hat{\omega} = \sqrt{g_{00}} \left( \omega_0 + i \frac{\hbar}{2} \gamma^k(x) \gamma^0 \omega^0_k \right) = \sqrt{g_{00}} \omega_0 - i \frac{\hbar}{4} g^{jk} \partial_0 g_{jk}, \quad \hat{D}_k = \partial_k + \frac{1}{4} \gamma^a \gamma^0 \omega^b_k, \]

\[ \omega_0 = \gamma_0 \left( m + \gamma^k(x) \left( -i \hbar \hat{D}_k + q A_k \right) \right) = \gamma_0 \left( m - 2i \hbar^{-1/2} \xi^k \left( \hat{p}_k + \hat{A}_k \right) \right). \]

In the classical limit \( \hat{\omega}_0^2 \to \omega_0^2, \hat{\rho}_1 \to \rho, \hat{\rho}_2 \to 0, (\hat{\rho}_2 \text{ does not contain terms without } \hbar) \), Thus, the classical limit of \( \left( \hat{\Omega}^\tau \right)^2 \) as well, is the classical quantity \( \omega^2\big|_{x^0 = \zeta \tau} \).

The Hamiltonian \( \hat{H}_\tau \) can be written in the following block-diagonal form

\[ \hat{H}_\tau = \text{bdig} \left( \hat{h}(\tau), -\hat{h}(-\tau) \right), \quad \hat{h}(x^0) = q A_0 - i \frac{\hbar}{4} \gamma a \gamma_0 \omega^b_k + \hat{\omega}. \] (27)

The \( \tau \)-evolution of the state vectors is defined by the corresponding Schrödinger equation \( i \hbar \partial_\tau \Psi(\tau) = \hat{H}_\tau \Psi(\tau) \), where the state vectors now depend parametrically on \( \tau \), \( \Psi(\tau) = (\Psi_{+1}(\tau, x), \Psi_{-1}(\tau, x)) \). Similar to the flat space-time case \( \Psi \), we may reformulate the evolution in terms of the physical time \( x^0 = \zeta \tau \). The corresponding Schrödinger equation has the form

\[ i \hbar \partial_0 \Psi(x^0) = \hat{H}_{x^0} \Psi(x^0), \quad \hat{H}_{x^0} = \text{bdig} \left( \hat{h}(x^0), \hat{h}^\epsilon(x^0) \right), \]

\[ \hat{h}^\epsilon(x^0) = \gamma^2 \left( \hat{h}(x^0) \right)^* \gamma^2 = \hat{h}(x^0) \big|_{q \to -q}, \quad \Psi(x^0) = (\Psi(x), \Psi^\epsilon(x)), \] (28)

where \( \Psi(x) = \Psi_{+1}(x^0, x) \) and \( \Psi^\epsilon(x) = \gamma^2 \Psi_{-1}(-x^0, x) \) are Dirac bispinors. The inner product of two states vectors in such a representation reads

\[ (\Psi, \Psi') = (\Psi, \Psi')_D + (\Psi^\epsilon, \Psi^\epsilon')_D. \] (29)
In accordance with the classical interpretation (see [3]) we regard \( \zeta \) as the charge sign operator. Let \( \Psi_{\zeta} \) be states with a definite charge, thus, \( \hat{\zeta} \Psi_{\zeta} = \zeta \Psi_{\zeta}, \ \zeta = \pm 1 \). The states \( \Psi_{+1} \) have \( \Psi^c = 0 \). Then (28) is reduced to the Dirac equation in curved space-time for the spinor field \( \Psi(x) \) of the charge \( q \),

\[
[\gamma^\mu (i\hbar D_\mu - q A_\mu) - m] \Psi(x) = 0, \quad D_\mu = \partial_\mu + \frac{1}{4} \gamma_a \gamma^b \omega_{ba}^\mu .
\]  

(30)

States \( \Psi_{-1} \) have \( \Psi = 0 \). Then (28) is reduced to the Dirac equation in curved space-time for the spinor field \( \Psi^c(x) \) of the charge \( -q \),

\[
[\gamma^\mu (i\hbar D_\mu + q A_\mu) - m] \Psi^c(x) = 0.
\]  

(31)

The Hamiltonian \( \hat{h}(x^0) \) can be considered as a one-particle Dirac Hamiltonian in the case under consideration for the charge \( q \).

Let us restrict ourselves to those backgrounds that do not create particles from the vacuum. For such backgrounds the one-particle sector of the corresponding QFT can be consistently defined, see the corresponding discussion in [3] and some remarks at the end. Consider for simplicity, the eigenvalue problem \( \hat{h} \Psi(x) = \epsilon \Psi(x) \) for the Dirac Hamiltonian in a time-independent external background (in fact, \( \hat{h}(x^0) \) does not depend on \( x^0 \) in such a case, thus, \( \hat{h}(x^0) = \hat{h} \)). Presenting the spinor \( \Psi \) in the form

\[
\Psi(x) = \left[ g^{00}_{-1/2} \gamma^0 \left( \epsilon - q A_0 + \frac{i}{4} \gamma_a \gamma^b \omega_{ba}^0 \right) + \gamma^k (i\hbar D_k - q A_k) + m \right] \varphi(x) .
\]

we get for \( \varphi(x) \) the corresponding squared Dirac equation,

\[
\left[ g^{00} \left( \epsilon - q A_0 + \frac{i}{4} \gamma_a \gamma^b \omega_{ba}^0 \right)^2 - D \right] \varphi(x) = 0 ,
\]

\[
D = m^2 - \frac{\hbar^2}{4} R - \frac{1}{\sqrt{-g}} (i\hbar D_k - q A_k) \sqrt{-g} g^{kl} (i\hbar D_l - q A_l) + \frac{\hbar}{4} q F_{\mu\nu}[\gamma^\mu, \gamma^\nu] - ,
\]

(32)

where \( R \) is the scalar curvature. We can see that a pair \( (\varphi, \epsilon) \) is a solution of the equation (32) if it obeys either the equation \( \epsilon = q A_0 - \frac{i}{4} \gamma_a \gamma^b \omega_{ba}^0 + \sqrt{\varphi^{-1}} g^{00} D \varphi \), or the equation \( \epsilon = q A_0 - \frac{i}{4} \gamma_a \gamma^b \omega_{ba}^0 - \sqrt{\varphi^{-1}} g^{00} D \varphi \). Let us denote via \( (\varphi_{+n}, \epsilon_{+n}) \) the solutions of the first
equation, and via \((\varphi_{-,n}, \epsilon_{-,n})\) the solutions of the second equation, where \(n\) are quantum numbers. Thus, the eigenvalue problem has the solutions

\[
\epsilon_{+,n} = qA_0 - \frac{i}{4} \gamma_a \gamma_b \omega_{b0} - i \sqrt{\varphi_{+,n}^{-1} g_{00} D \varphi_{+,n}} \quad \varphi_{+,n}^{-1} g_{00} D \varphi_{+,n}, \\
\epsilon_{-,\alpha} = qA_0 - \frac{i}{4} \gamma_a \gamma_b \omega_{b0} + \sqrt{\varphi_{-,\alpha}^{-1} g_{00} D \varphi_{-,\alpha}},
\]

and

\[
\Psi_{+,n}(x) = \begin{bmatrix} g_{00}^{-1/2} \gamma^0 \left( \epsilon_{+,n} - qA_0 + \frac{i}{4} \gamma_a \gamma_b \omega_{b0} \right) + \gamma^k (i \hbar D_k - qA_k) + m \end{bmatrix} \varphi_{+,n}(x), \\
\Psi_{-,n}(x) = \begin{bmatrix} g_{00}^{-1/2} \gamma \left( \epsilon_{-,n} - qA_0 + \frac{i}{4} \gamma_a \gamma_b \omega_{b0} \right) + \gamma^k (i \hbar D_k - qA_k) + m \end{bmatrix} \varphi_{-,n}(x),
\]

(33)

Since \(\epsilon_{+,n} > \epsilon_{-,\alpha}\), we call \(\epsilon_{+,n}\) the upper branch and \(\epsilon_{-,\alpha}\) the lower branch of the energy spectrum. The square norm of the eigenvectors \(\Psi_{\kappa,n}\) is always positive, all the eigenvectors are mutually orthogonal and can thus be orthonormalized as follows,

\[
(\Psi_{\kappa,n}, \Psi_{\kappa',n'})_D = \delta_{\kappa,\kappa'} \delta_{n,n'}, \quad \kappa = \pm .
\]

(34)

A solution of the eigenvalue problem \(\hat{h} \epsilon \Psi = \epsilon \Psi\) for the charge conjugated Hamiltonian can be analyzed in a similar manner. Here we get the set \((\epsilon_{\kappa,n}, \Psi_{\kappa,n}^c)\),

\[
\Psi_{\kappa,n}^c = \gamma^2 \Psi_*^{-\kappa,n}, \quad \epsilon_{\kappa,n}^c = - \epsilon_{-\kappa,n}, \quad (\Psi_{\kappa,n}^c, \Psi_{\kappa',n'}^c) = \delta_{\kappa,\kappa'} \delta_{n,n'}, \quad \kappa = \pm .
\]

(35)

From this point on, we can repeat all the arguments from [3] and construct a consistent relativistic quantum mechanics in Hilbert space without an indefinite metric, which is a reduction of the space \(\mathcal{R}\) to its physical subspace. The latter can be defined as a linear envelope of vectors of the form

\[
\Psi_{+,n} = \begin{pmatrix} \psi_{+,n}(x) \\ 0 \end{pmatrix}, \quad \Psi_{+,\alpha}^c = \begin{pmatrix} 0 \\ \psi_{+,\alpha}^c(x) \end{pmatrix}.
\]

In such a Hilbert space the operator \(\hat{\Omega}\) has a positively defined spectrum and the Hamiltonian \(\hat{H}_{x^0}\) has the right spectrum of particle and antiparticle energies in the background under consideration, which coincides with the spectrum of particles and antiparticles in the one-particle sector of the corresponding QFT.
V. SOME REMARKS

Returning to our choice of the quantum Hamiltonian (in fact, of the operator $\hat{\Omega}$ from (26)), one has to stress that the classical theory gives enough information to resolve the ordering problem in an unique way. The operator ordering and the nonclassical parts of the operator $\hat{\Omega}$ were chosen to maintain the invariance of the quantum theory under general coordinate transformations and under $U(1)$ transformations of the electromagnetic background. In particular, such a choice provides the invariance of the inner product (24) under general coordinate transformations as well as under the choice of the space-like hypersurface where the inner product is defined. One can also see that $\hat{\Omega}$ is positive defined in the Hilbert space constructed. The positivity condition helps to fix an ambiguity in the definition of $\hat{\Omega}$ as well.

We recall that a one-particle sector of QFT (as well as any sector with a definite particle number) may be defined in an unique way for all time instants only in external backgrounds which do not create particles from the vacuum. Nonsingular time independent external backgrounds are important examples of the above backgrounds. That is why we have presented the detailed discussion for such kinds of backgrounds to simplify our analyses. A generalization to arbitrary backgrounds, in which the vacuum remains stable, may be done in a similar manner. In backgrounds that violate the vacuum stability, a more complicated multi-particle interpretation of the constructed quantum mechanics, which establishes a connection to the QFT, is also possible.

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