Metafluid dynamics and Hamilton-Jacobi formalism

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Abstract

Metafluid dynamics was investigated within Hamilton-Jacobi formalism and the existence of the hidden gauge symmetry was analyzed. The obtained results are in agreement with those of Faddeev-Jackiw approach.

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1 Introduction

The hydrodynamic turbulence represents an ancient important issue from theoretical and experimental point of view. Recently, this problem was subjected to an intense debate because there are many phenomena which are turbulent, for example in astrophysics, cosmology, biomechanics, meteorology [1, 2, 3]. Recently, an alternative description for the Clebsch decomposition of currents in fluid mechanics was proposed [4] and its non-abelian extensions were obtained [5]. The hidden gauge invariance was studied in Dirac and extended frame formalism [6]. A new approach for investigation of the fluid turbulence was proposed recently [7]. The method named the metafluid dynamics, is based on the use of the analogy between Maxwell electromagnetism and turbulent hydrodynamics and it describes the dynamical behavior of average flow quantities in incompressible fluids flows having high Reynolds numbers in a similar way as it was done to get the macroscopic electromagnetic fields [8]. The average procedure was obtained using the spatial filtering method indicated in [9]. A Lagrangian for the metafluid dynamics was proposed recently in [10]. The theory was analyzed for the first time as a constrained system from the symplectic point of view and a hidden gauge symmetry was reported [10]. Therefore, a new procedure describing the gauge symmetries of the constrained systems should be applied to the same Lagrangian. Hamilton-Jacobi (HJ) formalism, based on Carathéodory's equivalent Lagrangians method [11] is an alternative method for quantization of constrained...
The main aim of this paper is to investigate the metafluid dynamics within (HJ) formalism and to give the corresponding justification for the existence of its hidden symmetry.

The plan of the paper is as follows:

In Sec. 2 the (HJ) formalism is briefly introduced. In Sec. 3 the metafluid dynamics is presented and its treatment within symplectic formalism is briefly reviewed. In Sec. 4 the (HJ) formulation of metafluid dynamics was investigated. Sec. 5 is dedicated to our conclusions.

2 Hamilton-Jacobi formalism

The basic idea of this new approach [12] is to consider the constraints as "Hamiltonians" and to involve all of them in the process of finding the action. Let us assume that a given degenerate Lagrangian \( L \) admits the following primary "Hamiltonians"

\[
H'_{\alpha} = p_{\alpha} + H_\alpha(t, q_a, p_a)
\]

We define the canonical Hamiltonian \( H_0 \) as

\[
H_0 = -L(t, q_a, \dot{q}_a, \dot{q}_a = w_a) + p_a w_a + \dot{q}_a p_\mu; \nu = 0, n - r + 1, \cdots, n.
\]

The equations of motion are obtained as total differential equations in many variables as given below:

\[
dq_a = \frac{\partial H'_{\alpha}}{\partial p_a} dt_\alpha, dp_a = -\frac{\partial H'_{\alpha}}{\partial q_a} dt_\alpha, dp_\mu = -\frac{\partial H'_{\alpha}}{\partial t_\mu} dt_\alpha, \mu = 1, \cdots, r.
\]

and the (HJ) function is given by

\[
dz = (-H_\alpha + p_{\alpha} \frac{\partial H'_{\alpha}}{\partial p_a}) dt_\alpha
\]

The set of equations (3) is integrable if and only if [12]

\[
[H'_{\alpha}, H'_{\beta}] = 0, \forall \alpha, \beta.
\]

The method is straightforward for constrained systems [17] having finite degree of freedom [18] but it becomes, in some cases, quite difficult to be used for field theories. The main difficulty comes from the fact that some surface terms
may play an important role in closing the algebra of "Hamiltonians" but some of them have no physical meaning from the (HJ) point of view. Another problem is the treatment of the second-class constraints systems within (HJ) formalism. In this particular case the "Hamiltonians" are not in involution and it is not a unique way to solve this problem [19, 20, 21].

3 Metafluid dynamics

Based on the analogy between Maxwell electromagnetism and turbulent hydrodynamics Marmanis (see for more details Ref. [7] and the references therein) proposed an approximative theory such that the equations describing the dynamic variables are linear but the nonlinearities emerge as sources of turbulent motion. Marmanis constructed a system of equations containing the vorticity $\vec{\omega} = \nabla \times \vec{u}$ and the Lamb vector $\vec{l} = \vec{w} \times \vec{u}$ as follows

$$\nabla \cdot \vec{\omega} = 0, \frac{\partial \vec{\omega}}{\partial t} = -\nabla \times \vec{l} + v \nabla^2 \vec{\omega}, \nabla \cdot \vec{l} = n(\vec{x},t)$$

(6)

$$\frac{\partial \vec{l}}{\partial t} = u^2 \nabla \times \vec{w} - \vec{j}(\vec{x},t) + v \nabla n(\vec{x},t) - v \nabla^2 \vec{l},$$

(7)

where the turbulent current is given as

$$\vec{j}(\vec{x},t) = \vec{u} n + \nabla \times (\vec{u} \vec{w}) \vec{u} + \vec{w} \times \nabla (\Phi + \vec{u}^2) + 2(\vec{l} \cdot \nabla) \vec{u},$$

(8)

and the turbulent charge $n(\vec{x},t)$ has the following form

$$n(\vec{x},t) = -\nabla^2 \Phi(\vec{x},t).$$

(9)

In (9) the Bernoulli energy function $\Phi(\vec{x},t)$ has the expression

$$\Phi(\vec{x},t) = \frac{p}{\rho} + \frac{\vec{u}^2}{2},$$

(10)

where $p(\vec{x},t)$ is the pressure, $\rho$ is the density, $\vec{u}(\vec{x},t)$ represents the velocity field and $v$ is the kinematic viscosity.

Taking the averaging process of (6) and (7) we obtain (see for more details [7])

$$\nabla \cdot \vec{\omega} = 0, \frac{\partial \vec{\omega}}{\partial t} = -\nabla \times \vec{l} + v \nabla^2 \vec{\omega}, \nabla \cdot \vec{l} = n(\vec{x},t),$$

(11)

$$\frac{\partial \vec{l}}{\partial t} = c^2 \nabla \times \vec{\omega} - \vec{j}(\vec{x},t) + v \nabla n(\vec{x},t) - v \nabla^2 \vec{l},$$

(12)

where
\[ \vec{w} = \langle \vec{w} \rangle, \vec{T} = \langle \vec{T} \rangle, \vec{J} = \langle \vec{J} \rangle, \chi^2 = \langle u^2 \rangle, \vec{u} = \langle \vec{u} \rangle, \] (13)

and

\[ \phi = \langle \Phi \rangle, n(\vec{x}, t) = \langle n(\vec{x}, t) \rangle. \] (14)

Since the Lagrangian density of the classical electromagnetism [8] is given by the very well known expression

\[ L = \frac{1}{2}(E^2 - B^2), \] (15)

the analogy between electromagnetism and turbulence allow us to write the Lagrangian density turbulence [10] as

\[ L = \frac{1}{2}(\vec{t}^2 - \chi^2 \vec{w}^2). \] (16)

Inserting the expressions of \( \vec{t} \) and \( \vec{w} \) into (16) the form of the Lagrangian density becomes

\[ L = \frac{1}{2}(\nabla \phi - \frac{\partial \vec{u}}{\partial t} + v \nabla^2 \vec{u})^2 - \frac{1}{2}c^2(\nabla \times \vec{u})^2. \] (17)

In [10] the authors considered the case when sources are not zero. The interaction Lagrangian density

\[ L_{int} = \vec{J} \cdot \vec{u} - n \phi - v \vec{u} \cdot \nabla n. \] (18)

was added to (17) and the total Lagrangian density corresponding to the metafluid dynamics can be written as

\[ L = \frac{1}{2}(\nabla \phi - \frac{\partial \vec{u}}{\partial t} + v \nabla^2 \vec{u})^2 - \frac{1}{2}c^2(\nabla \times \vec{u})^2 + \vec{J} \cdot \vec{u} - n \phi - v \vec{u} \cdot \nabla n. \] (19)

Using the Faddeev-Jackiw analysis [22] the set of constraints corresponding to (19) was obtained [10] as

\[ \pi^0 = 0, \nabla \cdot \vec{w} + n = 0. \] (20)

Imposing \( \phi \) being a constant and using the condition of incompressible of fluid \( \nabla \cdot \vec{u} = 0 \), the Dirac’s brackets among the space space fields were calculated as

\[ \{ u_i(\vec{x}), \pi_i(\vec{y}) \}_{DB} = (\delta_{ij} - \frac{\partial x^i}{\partial \xi^j} \frac{\partial \xi^j}{\partial y^i}) \delta(\vec{x} - \vec{y}). \] (21)

Finally, it was proved that the Lagrangian (19) admits the following gauge symmetry [10]

\[ \delta u_i = \partial_i \epsilon, \delta \pi_i = 0, \delta \phi = -\dot{\epsilon}. \] (22)
4 Hamilton-Jacobi analysis

The starting point in (HJ) is the degenerate Lagrangian density given by (19). The canonical momentum conjugate to $\vec{u}$ has the form

$$\vec{\pi}(\vec{\pi}, t) = -\vec{\pi}(\vec{\pi}, t),$$

but

$$\pi^0 = 0.$$  \hspace{1cm} (24)

Here $\pi^0$ is the canonical momenta associated to $\phi$. Making use of (19), (23) and (24) the canonical Hamiltonian density is given by

$$H_c = \frac{1}{2} \vec{\pi}^2 - \vec{\pi} \cdot \nabla \phi + \frac{1}{2} c^2 (\nabla \times \vec{u})^2 + v \vec{\pi} \cdot \nabla^2 \vec{u} - \vec{\pi} \cdot \vec{J} + \phi n + v \vec{u} \cdot \nabla n.$$  \hspace{1cm} (25)

The Hamiltonians densities to start with are

$$H'_0 = p_0 + H_c, H'_1 = \pi^0.$$  \hspace{1cm} (26)

The equations of motion corresponding to (26) have the following expressions

$$d \vec{u} = (\vec{\pi} + v \nabla^2 \vec{u} - \nabla \phi) d\tau,$$

$$d \vec{\pi} = (\vec{J} - v \nabla n - c^2 \nabla \times (\nabla \times \vec{u}) - v \nabla^2 \vec{\pi}) d\tau.$$  \hspace{1cm} (28)

The next step is to impose the integrability conditions and to make the above system integrable. Imposing $dH'_1 = 0$ we obtain another ”Hamiltonian” as

$$H_2 = \nabla \cdot \vec{\pi} + n$$  \hspace{1cm} (29)

Using the fact that $n = -\nabla^2 \phi$ and making zero the variation of (29) obtain

$$H_3 = (\nabla \cdot \vec{J} + 2v \nabla^2 n + \hat{n}) d\tau.$$  \hspace{1cm} (30)

Analyzing the form of (30) we observed that the process of finding new ”Hamiltonians” finished. The gauge variable is determined from $H_3 = 0$ but the problem of imposing the gauge fixing $\nabla \cdot \vec{u} = 0$ inside of (HJ) formalism remains not yet justified. At the first sight, the set of ”Hamiltonian” densities

$$H_0'' = p_0 + \frac{1}{2} \vec{\pi}^2 + \frac{1}{2} c^2 (\nabla \times \vec{u})^2 + v \vec{\pi} \cdot \nabla^2 \vec{u} - \vec{\pi} \cdot \vec{J} + v \vec{u} \cdot \nabla n, H_1' = \pi^0, H_2 = \nabla \cdot \vec{\pi} + n,$$  \hspace{1cm} (31)

are in involution but $H_2$ is not in the form required in (1). We have two options at this stage: to use an extended phase-space or to make a canonical transformation such that $H_2$ becomes a momentum. Knowing that all Poisson brackets are canonical invariants [23] we conclude that it is possible to perform
a canonical transformation for metafluid theory. In this way the existence of the hidden gauge symmetry is justified inside of (HJ) formalism.

In addition, we observed that in the inertial range $\eta \ll \text{InertialRange} \ll L$ (see Ref. [2] for notations and more details) the constraints obtained in [10] by using the symplectic analysis are the same as those produced by (HJ) formalism. By inspection, we can easily check the "Hamiltonians" $H_0, H_1$ and $H_2$ produce the same Dirac’s brackets as obtained in the symplectic formalism [10] but in (HJ) approach we are working on the original phase space extended with the pair $(p_0, \tau)$.

5 Conclusion

In (HJ) one of the main problem is to put all "Hamiltonians" in the form given by (1). The analysis of the metafluid dynamics is one example of theory possessing a gauge symmetry but one "Hamiltonian" is not in the requested form, so it creates problems in managing the hidden gauge symmetry. We observed that it is possible to perform a canonical transformation such that $H_2$ becomes a momentum and the "Hamiltonians" remain in involution. In this manner we argue the existence of the gauge hidden symmetry of metafluid dynamics within (HJ) formalism. In [18] and very recently in [24] the important role of the canonical transformations for (HJ) analysis of second-class constrained systems was discussed. In this paper we claimed that the canonical transformations are needed and for some of the first class constrained systems. In addition, we found that the surface terms played an important role in finding the total differential equations (27) and (28). It was reported that (HJ) and Faddeev-Jackiw approaches gave the same set of constraints although the formalisms have completely different structures and different mechanisms of identified the gauge variables.

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