A topological model for inflation

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In this paper we will discuss a new model for inflation based on topological ideas. For that purpose we will consider the change of the topology of the spatial component seen as compact 3-manifold. We analyzed the topology change by using Morse theory and handle body decomposition of manifolds. For the general case of a topology change of a $n$–manifold, we are forced to introduce a scalar field with quadratic potential or double well potential. Unfortunately these cases are ruled out by the CMB results of the Planck mission. In case of 3-manifolds there is another possibility which uses deep results in differential topology of 4-manifolds. With the help of these results we will show that in case of a fixed homology of the 3-manifolds one will obtain a scalar field potential which is conformally equivalent to the Starobinsky model. The free parameter of the Starobinsky model can be expressed by the topological invariants of the 3-manifold. Furthermore we are able to express the number of e-folds as well as the energy and length scale by the Chern-Simons invariant of the final 3-manifold. We will apply these results to a specific model which was used by us to discuss the appearance of the cosmological constant with an experimentally confirmed value.

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I. INTRODUCTION

Because of the influx of observational data, recent years have witnessed enormous advances in our understanding of the early universe. To interpret the present data, it is sufficient to work in a regime in which spacetime can be taken to be a smooth continuum as in general relativity, setting aside fundamental questions involving the deep Planck regime. However, for a complete conceptual understanding as well as interpretation of the future, more refined data, these long-standing issues will have to be faced squarely. As an example one may ask, can one show from first principles that the smooth spacetime of general relativity is valid at the onset of inflation? At the same time, this approach has some problems like special initial conditions and free parameters like the amount of increase (number of e-folds). Furthermore, there are many possibilities for a specific model (chaotic or fractal inflation, Starobinsky model etc.). But the impressive results of the PLANCK mission excludes many models. Nevertheless, the main questions for inflation remain: what is the scalar field? what is the number of e-folds? which model is realistic? what is the energy scale? etc. In this paper we will focus mainly on the question about the origin of inflation. Today inflation is the main theoretical framework that describes the early Universe and that can account for the present observational data. In thirty years of existence, inflation has survived, in contrast with earlier competitors, the tremendous improvement of cosmological data. In particular, the fluctuations of the Cosmic Microwave Background (CMB) had not yet been measured when inflation was invented, whereas they give us today a remarkable picture of the cosmological perturbations in the early Universe. In nearly all known models, the inflation period is caused by one or more scalar field(s). But the question about the origin of this scalar field remains among other problems.

In this paper we will go a different way to explain inflation. Inspired by Wheelers idea of topology change at small scales in quantum gravity, we will consider a spatial topology change. The description of the change using the concept of a cobordism (representing the spacetime) will lead automatically to a scalar field. On general grounds, one can show that there are two kinds of changes: adding a submanifold or change/deform a submanifold. In the first case, we will get the quadratic potential of chaotic inflation whereas in the second case we will obtain the double well potential of topological inflation. But both models were ruled out by the PLANCK mission. Amazingly, only in four dimension there is another possibility of a topology change. This modification is an infinite process of submanifold deformations (arranged along a tree). At the first view, it seems hopeless to calculate something. But in contrast to the two cases above, we are able to determine everything in this model. The potential of the scalar field \( \phi \) is \( (1 - e^{-\phi})^2 \) which is conformally equivalent to the Starobinsky model. The number of e-folds is determined by a topological invariant of the spatial space. This invariant will be used to get expressions for \( \alpha \) (free parameter in the Starobinsky model) or the energy scale of inflation. Why is this miracle possible? Mostow-Prasad rigidity (see [C]) is the cause for this behavior. The infinite process of submanifold deformations implies a hyperbolic geometry for the underlying space. But any deformation of a hyperbolic space must be an isometry. Therefore geometric expressions like volume or curvature are topological invariants. It is the point where geometry and topology meet. In a previous paper [12] we discussed a concrete model of topology change in the evolution of the cosmos with two phases. In particular, we obtained a realistic value of the cosmological constant. Here we will use this model to calculate the values of the inflation parameters in the Starobinsky model, i.e. \( \alpha \) (coupling of the \( R^2 \) term), energy scale, number of e-folds \( N \), the spectral tilt \( n_s \) and the tensor-scalar ratio \( r \). We will also compare these values with the current measurements. One point remains, how does this model couples to matter (reheating)? In a geometric/topological theory of inflation one also needs a geometric model of matter to explain this coupling. Fortunately, this theory was partly developed in previous work [10, 14]. By using these ideas, we will explain the coupling between the scalar field and matter. The coupling constant is given by a topological invariant again.

II. THE MODEL

The main idea of our model can be summarized by a simple assumption: during the cosmic evolution (i.e. directly after the Big Bang) the spatial component (space) undergoes a topology change. This assumption is mainly motivated by all approaches to quantum gravity. Notable are first ideas by Wheeler [47]. But topology changes are able to...
produce singularities and causal discontinuities as shown in [5]. In many cases one can circumvent these problems as discussed in [24, 20]. Here, we will implicitly assume that the topology changes is causal continuous.

At first we have to discuss the description of a spatial topology change. Let \( \Sigma_1 \) and \( \Sigma_2 \) be two compact closed 3-manifolds so that \( \Sigma_1 \) is changed to \( \Sigma_2 \). Now there is a spacetime \( M \) with \( \partial M = \Sigma_1 \cup \Sigma_2 \) called a cobordism. For now we have to face the question to characterize the topology of the cobordism. It is obvious that two diffeomorphic 3-manifolds \( \Sigma_1 = \Sigma_2 \) will generate a trivial cobordism \( \Sigma_1 \times [0, 1] \). Interestingly, it is not true if there is a counterexample to the smooth Poincare conjecture in dimension 4. Then this cobordism between diffeomorphic 3-manifold can be also non-trivial (i.e. non-product). In general, the difference between two manifolds is expressed in a complicated topological structure of the interior. The prominent example is a cobordism between two disjoint circles and one circle, the so-called trouser. There, the non-triviality of the cobordism is given by the appearance of a ‘singular’ point, the crotch. Fortunately, this behavior can be generalized to all other cases too. To understand this solution we have to introduce Morse theory and handlebody decomposition of manifolds. By using these methods we will show that a topology change requires a scalar field including an interaction potential (related to the so-called Morse function).

### A. Morse theory and handles

In Morse theory one analyzed the (differential-)topology of a manifold \( M \) by using a (twice-)differentiable function \( f : M \to \mathbb{R} \). The main idea is the usage of this function to generate a diffeomorphism via the gradient equation

\[
\frac{d}{dt} \vec{x} = -\nabla f(\vec{x})
\]

in a coordinate system. Away from the fix point \( \nabla f = 0 \), the solution of this differential equation is the desired diffeomorphism. This behavior breaks down at the fix points. The fix points of this equation are the critical points \( f \). Now one has to assume that these critical points are isolated and that the matrix of second derivatives has maximal rank (non-degenerated critical points). This function \( f \) is called a Morse function. Then, the function \( f \) in a neighborhood of an isolated, non-degenerated point \( x^{(0)} \) (i.e. \( \nabla f|_{x^{(0)}} = 0 \), \( \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} |_{x^{(0)}} \right) \neq 0 \) looks generically like

\[
f(x) = f(x^{(0)}) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2
\]

where the number \( k \) is called the index of the critical point. In the physics point of view, the Morse function is a scalar field over the manifold. The Morse function \( 2 \) at the critical point is a quadratic form w.r.t the coordinates. This form is invariant by the action of the group \( SO(n-k,k) \). If one interpret this group as isometry group then one can determine the geometry in the neighborhood of the critical point. But the the Morse function reflects the topological properties of the underlying space, i.e. the analytic properties of \( f \) are connected with the topology of \( M \). For that purpose we will define the level set of \( f \), i.e.

\[
M(a) = f^{-1}((-\infty, a]) = \{ x \in M | f(x) \leq a \}
\]

Now consider two sets \( M(a) \) and \( M(b) \) for \( a < b \). If there is no critical point in the compact set \( f^{-1}((a,b]) \) then \( M(a) \) and \( M(b) \) are homotopy equivalent (and also topologically equivalent, at least in dimension smaller than 5). If the compact set \( f^{-1}((a,b]) \) contains a critical point of index \( k \) then \( M(a) \) and \( M(b) \) are related by the attachment of a \( k \)-handle \( D^k \times D^{n-k} \) \( (D^k = \{ x \in \mathbb{R}^{k+1} | ||x||^2 \leq 1 \} \) is the \( k \)-disc), i.e. \( M(b) = M(a) \cup D^k \times D^{n-k} \). Therefore, the topology of \( M \) is encoded in the critical values of the Morse function. But let us give two words of warning: firstly this approach gives only the number handles but not the detailed attachment of the handles and secondly the number of handles as induced by the Morse function can be larger as the minimal number of handles used to decompose the manifold. Both facts are expressed in the Morse relations: let \( n_k \) be the number of critical points of index \( k \) and \( b_k \) the \( k \)th Betti number (= the rank of the homology group with values in \( \mathbb{R} \)). Then, one has

\[
b_k \leq n_k \quad \sum_k (-1)^k n_k = \sum_k (-1)^k b_k = \chi(M)
\]

where \( \chi(M) \) is the Euler characteristics of \( M \). So, Morse theory extracted only the homological properties of the manifold but not the whole topological information.
Figure 1: a pant as cobordism

Figure 2: Killing a 0- and a 1-handle which can be described by the function $x^3 - t \cdot x$ from $t = 1$ over $t = 0$ to $t = -1$ (from left to right)

B. Cobordism, handles and Cerf theory

Now we will discuss the cobordism $W(\Sigma_1, \Sigma_2)$ between two different 3-manifolds $\Sigma_1$ and $\Sigma_2$. In case of 3-manifolds, the word 'different' means non-diffeomorphic which agrees with non-homeomorphic (see [13]). What is the structure of the cobordism $W(\Sigma_1, \Sigma_2)$ for different 3-manifolds? The answer can be simply expressed that there are one or more $k$-handles in the interior of the cobordism. A proof can be found in [42] and we will discuss a simple example now.

Let us consider a pant or trouser like above, i.e. a cobordism between two disjoint circles and one circle (see Fig. 1). A circle is the union of one 0-handle ($D^0 \times D^1$) and one 1-handle ($D^1 \times D^0$) glued together along $\partial D^1 \times D^0 = S^0 \times D^0$ and the boundary of the 0-handle $\partial (D^0 \times D^1) = S^0$, i.e. along the end-points of the two intervals $D^1$. Now lets go from two disjoint circles on one side of the cobordism to one circle of the other side of the cobordism. The two disjoint circles are build from two 0-handles and two 1-handles whereas the one circle is decomposed by one 0-handle and one 1-handle. Therefore the pair of one 0-handle and one 1-handle was destroyed. Each process of this kind will produce a handle in the interior of the cobordism. In this case it is a 1-handle (the crotch of the pant). The critical point of this handle represents the topology change. It is the critical point of the corresponding Morse function for the cobordism.

Of course the process can be reversed (cobordism classes are forming a group). Then a 0-/1-handle pair appears and one circles splits into two disjoint circles but the main observation is the same: A topology change produces an additional handle in the interior of the cobordism. For the discussion later, we have to make an important remark. The 1-handle in the interior of the cobordism is a saddle and the critical point is a saddle point. The corresponding Morse function is given by $f(x_1, x_2) = x_2^2 - x_1^2$. Now we are interested in the geometry of the cobordism. The group $SO(1,1)$ fixes the function $f(x_1, x_2)$ by the usual action. The group $SO(1,1)$ is the group of hyperbolic rotations preserving the area and orientation of a unit hyperbola. It is a subgroup $SO(1,1) \subset SL(2, \mathbb{R})$ of the Möbius group, the isometry group of the 2-dimensional hyperbolic space. Because of the saddle point, the interior of the cobordism is a saddle surface having a hyperbolic geometry (with negative curvature). This observation can be generalized to any saddle point of a cobordism (or to any $k$-handle for $0 < k < n$ with the dimension $n$ of the cobordism). In dimension four, one obtains hyperbolic geometries for 1- and 3-handles and the geometry $AdS_3 \times S^1$ for 2-handles [46] for the isometry group $SO(2,2)$ inside of the cobordism. Then two geodesics will be separated exponentially after passing the critical point of the handle. This behavior explains also the appearance of an inflationary phase after a topology change which will be discussed later.

The process of 'killing' the 0-/1-handle pair is visualized in Fig. 2. There, one can also find an analytic expression for this process. For completeness, we remark that the theory behind this description is called Cerf theory [23].
general, the modification of any handle structure can be simplified to one process of this kind. The idea of Cerf theory can be simply expressed by considering the function $W \to \mathbb{R}$, i.e. a one-parameter family of Morse functions at the boundary of the cobordism. Central point of Cerf theory is the existence of two generic singularities (with vanishing first derivatives). The first kind is the Morse singularity: $\pm x^2$ and the two other cases $\pm x^3$ and $\pm x^4$. For these two cases $x^3, x^4$, there are resolutions as one-parameter families: $x^3 - tx$ and $x^4 - tx^2$ (with the parameter $t \in \mathbb{R}$). Interestingly, the $x^4$-case can be reduced to the $x^3$-case. The one-parameter family $x^3 - tx$ is visualized in Fig. 2 for $t = 1, 0, -1$ (from left to right). Now the cancellation of a $k - (k+1)$-handle pair is described by the function

$$f(x) = f(x^{(0)}) - x_1^2 - x_2^2 - \cdots - x_k^2 + (x_{k+1}^3 - t \cdot x_{k+1}) + \cdots + x_n^2$$

and we will use this function to describe the change in topology as the effect of handle canceling.

One example is the simplification of a handle decomposition. As explained in the previous subsection, this handle decomposition of the cobordism can be non-uniquely given. An example is the following picture Fig. 3. A cobordism between two circles (usually the trivial cylinder $S^1 \times [0,1]$) is decomposed by an extra pair of one 1-handle and one 2-handle. But as the figure indicated, there is a flow (determined by the corresponding Morse function, see (1)) from one critical point (1-handle) to the other critical point (2-handle). This flow is a diffeomorphism which can be used to cancel both critical points, see [41] and [42] for the details. For the successful canceling, one needs an implicit assumption which will be explained later.

There is another example where canceling pairs appear, the so-called homology cobordism. This example will become important later. It is motivated by the following question: how does a simple-connected, 4-dimensional cobordism with trivial homology look like? In general one would expect that the corresponding 3-manifolds have to be simply connected too. But let us consider a disk $D^2$ with boundary $S^1$. $D^2$ is simply connected in contrast to the boundary. So, every non-contractable curve in a 3-manifold can be transformed to a contractable curve inside of the cobordism by attaching a disk or better a 2-handle $D^2 \times D^2$. The corresponding change of the cobordism can be changed by the attachment of a 3-handle which cancels the 2-handle. A simple argument using the Mayer-Vietoris sequence shows that a cobordism $W(\Sigma_1, \Sigma_2)$ which looks like $S^3 \times [0,1]$ is a cobordism between two homology 3-spheres $\Sigma_1, \Sigma_2$, i.e. 3-manifolds with the same homology like the 3-sphere. One call this cobordism, a homology cobordism. A sequence of these homology cobordism will look like $S^3 \times \mathbb{R}$ but inside of this cobordism you have an ongoing topology change. Later on we will construct the Starobinsky model from this cobordism.

C. The physics view on cobordism

In [49], Witten presented a physics view on Morse theory using supersymmetric quantum mechanics. This work was cited by many followers and it was the beginning of the field of topological quantum field theory. Part of this work will be used to describe the cobordism and its handle decomposition. As explained above, a cobordism $W(\Sigma_1, \Sigma_2)$ between different manifolds $\Sigma_1$ and $\Sigma_2$ contains at least one handle, say $k$–handle $H_k$. This handle $H_k$ is embedded
in the cobordism by using a map
\[ \Phi_H : H_k \hookrightarrow W(\Sigma_1, \Sigma_2) \]
to visualizing the attachment of the handle. It can be locally modeled by a map
\[ \Phi_{H,loc} : H_k \hookrightarrow U \subset \mathbb{R}^4 \]
where \( U \subset W(\Sigma_1, \Sigma_2) \) is a chart of the cobordism representing the attachment of the handle. In principle, this local description is enough to understand the adding of a handle to the interior of the cobordism (and representing the non-triviality of the cobordism). So, if we are choosing the map
\[ \Phi : W(\Sigma_1, \Sigma_2) \rightarrow \mathbb{R}^4, \quad supp(\Phi) = H_k \]
then we have an equivalent description given by a set of four scalar fields \( (\phi_0, \ldots, \phi_3) = \Phi \). Importantly, this description can be generalized to all dimensions expressing the topology change. In the special case of 3-manifolds we will later present another description using a \( SU(2) \)-valued scalar field.

But now we will consider the case of a \( n \)-manifold. It is not an accident that scalar fields are describing a topology change. Critical points of a scalar field (the Morse function) express the topology of the underlying space, at least partly. Here, in the case of a topology change we have to consider the change of a scalar field. This change leads to the appearance of one or more handles in the interior of the cobordism whose location is a tuple of four scalar fields to describe the embedding. A vector field is not suitable because there is no ‘direction’ in an embedding. But we are able to simplify this description. The embedding can be chosen in such a manner that the flow to the critical point of the handle \( H_k \) is normal to the boundary. The Morse function for the handle \( H_k \) is given by \( h(\Phi) = \sum_i(\pm \phi_i^2) \) in the coordinate system \( (\phi_1, \ldots, \phi_n) \). The normal direction in the cobordism is expressed by a coordinate \( t \) seen as a (locally) non-vanishing vector field (as section of the normal bundle of the boundary \( \partial W(\Sigma_1, \Sigma_2) \)). The case of a cobordism with more than one critical point (or handle) is more interesting. This case includes also the situation to simplify the cobordism (see Fig. 3). In this situation the Morse function \( h \) inside of the cobordism generates the ‘potential energy’ of the problem to be \( V(\Phi) = (dh)^2 \). Then the flow from one critical point to another critical point can be described as a tunneling path. These paths are the paths of steepest descent (leading from one critical point to another critical point) expressed as solutions of the equation \( (1) \) now written as
\[ \frac{d\phi_i}{dt} = g_{ij} \frac{\partial h}{\partial \phi_j} \]
with respect to a metric \( g_{ij} \) of \( W(\Sigma_1, \Sigma_2) \). Here, the variable \( t \) the path of steepest descent. But as shown in [42], one can order the handles so that the coordinate \( t \) of the cobordism can be identified with this parameter which will be done now. As Witten pointed out in the paper [49], the relevant action is given by
\[ S = \int dt \left[ \frac{1}{2} g_{ij} \frac{d\phi_i}{dt} \frac{d\phi_j}{dt} + \frac{1}{2} g_{ij} \frac{\partial h}{\partial \phi_i} \frac{\partial h}{\partial \phi_j} \right] \]
(4)
Now, we identify the coordinates \( (x^0, \ldots, x^{n-1}) \) of the handle \( H^k \) with one direction in the coordinate system of the cobordism, say \( x^0 = t \), and using a Lorentz transformation at the same time then no direction is preferred (a standard argument). Finally we obtain the action of the nonlinear sigma model
\[ S = \int \left( \frac{1}{2} g^{kl} \partial_k \phi^i \partial_l \phi^i + \frac{1}{2} g^{kl} \frac{\partial h}{\partial \phi_k} \frac{\partial h}{\partial \phi_l} \right) \]
and the path of steepest descent is given by a choice of a function \( t(t^i) \) (as embedding of the curve). This action can be also used to describe the canceling process of a \( k -/(k + 1) \)-handle pair. Both handles agreed in nearly all directions except one direction. It is the direction \( k + 1 \) with coordinate \( \phi_{k+1} \). The \( k \)-handle is given by the Morse function \( -\phi_1^2 - \cdots - \phi_k^2 + \phi_{k+1}^2 + \cdots + \phi_n^2 \) whereas the \( (k + 1) \)-handle is determined by the Morse function \( -\phi_1^2 - \cdots - \phi_{k+1}^2 + \phi_{k+2}^2 + \cdots + \phi_n^2 \). The difference between both Morse functions is concentrated at the \( (k + 1) \)-direction: the function \( +\phi_{k+1}^2 \) for the \( k \)-handle and \( -\phi_{k+1}^2 \) for the \( (k + 1) \)-handle. Both handles are connected along this direction and the canceling of both handles has its origin in this connection. In the above mentioned Cerf theory [23], this handle pair is described by one function
\[ -\phi_1^2 - \cdots - \phi_k^2 + (\phi_{k+1}^2 - T \cdot \phi_{k+1}) + \phi_{k+2}^2 + \cdots + \phi_n^2 \]
(5)
with one parameter $T$. The main result of Cerf theory states that this expression is unique (or better generic) up to diffeomorphisms. The canceling is described schematically in Fig. 2 for the parameter $T = 1$, $T = 0$ and $T = -1$. But then we need only one scalar field in the action and the function $h$ is given by the bracket term in the expression\footnote{In dimension three the canceling is always possible. But in dimension 4, there is a problem which is at the heart of all problems.}. Finally we obtain the action

$$ S = \int dt \left[ \frac{1}{2} \frac{d\phi}{dt} + 2 \frac{d\phi}{d\phi} + 1 \right]$$

for $h = \phi^3/3 - T \cdot \phi$. But as explained above, one has the freedom to embed the handle pair into the $x$–coordinate system of the cobordism. For the curve $x(t)$ connecting the two handles, one can rewrite the total derivative

$$ \frac{d}{dt} = \dot{x}^\mu \partial_\mu$$

and in a small neighborhood we can use the usual relation $\dot{x}^\mu \dot{x}_\mu = c^2$ between the four velocities. Instead to integrate only along the curve, we will consider a field $\phi$ of handle pairs on the cobordism $W(\Sigma_1, \Sigma_2)$. Then $\phi$ can be interpreted as a kind of density for handle pairs in $\Sigma_{1,2}$. Then we have to integrate over the whole cobordism to obtain the action

$$ S = \int_{W(\Sigma_1, \Sigma_2)} d^n x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + 1 \right]$$

where we set $c = 1$. The argumentation can be generalized to other cases like the appearance of a handle as described by the function $h = \pm \phi^2$ leading to the general action

$$ S = \int_{W(\Sigma_1, \Sigma_2)} d^n x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + 1 \right]$$

As remarked above, we looked for the generic cases like $h = \pm \phi^2$ and $h = \phi^3 - T \cdot \phi$ but higher powers are also possible. In combination with the Einstein-Hilbert action we obtain the two generic models

$$ S_{\text{chaotic}} = \int_{W(\Sigma_1, \Sigma_2)} d^n x \sqrt{g} \left[ R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + 1 \right]$$

$$ S_{\text{topological}} = \int_{W(\Sigma_1, \Sigma_2)} d^n x \sqrt{g} \left[ R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + 1 \right]$$

of chaotic inflation and topological inflation.

By using the model of a topology change, we are able to reproduce two known inflationary models. The advantage of this model is the natural appearance of the scalar field $\phi$ which is associated to the topology change. Unfortunately, both models were ruled out by recent results of the Planck mission\footnote{In dimension four a pair of handles like $1 - /2$–or $2 - /3$–handle pairs cannot be canceled smoothly but topologically using an infinite process. In the next section we will describe this process which will lead to the Starobinsky model which is one of the favored models of the Planck mission.}. Only potentials like $\phi^m$ with $1 < m < 2$ or long-tailed expressions like $(1 - e^{-\phi})^2$ are possible. But functions like $\phi^m$ cannot be generated by Morse functions and one cannot reproduce this model by topological methods. Interestingly, only in dimension four there is the possibility to obtain long-tailed expressions. There is a simple reason why this is possible: in dimension four a pair of handles like $1 - /2$–or $2 - /3$–handle pairs cannot be canceled smoothly but topologically using an infinite process. In the next section we will describe this process which will lead to the Starobinsky model which is one of the favored models of the Planck mission.

### III. INFLATION IN FOUR DIMENSIONS

In this section we will specialize to 4D spacetime. Here, the cancellation of a handle pair is described by an infinite process i.e. one needs a special handle known as Casson handle. Casson handles are parametrized by all trees. In principle, this fact is the reason for the exponential potential leading to Starobinsky inflation.

#### A. Cobordism between 3-manifolds and 4-manifold topology

In the previous section we described the general case of a topology change. Implicitly we assumed that the canceling of $k - /k + 1$–handles is always possible. But in dimension 4, there is a problem which is at the heart of all problems...
As mentioned above, the root of the tree is located at the center of the disk. Then the whole tree is located between the disk carries a hyperbolic metric. This metric can be simply transformed into

\[ x^2 + y^2 = 1, \]

and one 2-handle \( D^1 \times D^3 \). The 1-handle and 2-handle cancel each other if the attaching sphere \( \partial D^2 \times 0 = S^1 \) of the 2-handle meets the belt sphere \( 0 \times \partial D^3 = S^2 \) of the 1-handle transversally in one point. To understand the problem, we have to consider the attachment of handles. A \( k \)-handle \( D^k \times D^{n-k} \) is attached to \( D^n \) via the boundary by the map \( \partial D^k \times D^{n-k} \rightarrow \partial D^n \).

Then a 2-handle is attached to \( \partial D^4 = S^3 \) by an embedding of \( \partial D^2 \times D^2 = S^1 \times D^2 \), the solid torus. But this map is equivalent to \( S^1 \rightarrow S^3 \), i.e. the attaching of a 2-handle is determined by a knot. There is also an additional number, the framing, which describes how a parallel copy of the knots wind around the knot. The attachment of 1- and 3-handles are easier to describe. In case of a 1-handle, Akbulut [4] found another amazing description: a 1-handle is a removed 2-handle with fixed framing (see [36] section 5.4 for the details). An example of a non-canceling pair of one 1-handle and one 2-handle is visualized in Fig. 4. In this example, the attaching sphere of the 2-handle (the knot without the dot) meets the belt sphere of the 1-handle (the knot with the dot) twice. Usually, a curve meeting a second curve twice can be separated. This process is called the Whitney trick. For the realization of this process in a controlled manner, one needs an embedded disk. But it is known that the Whitney trick fails because the disk contains self-intersections (it is immersed in contrast to embedded), see [9]. Interestingly, it is possible to realize the Whitney trick topologically. But then one needs an infinite process as shown by Freedman [31]. Now we will describe this process to use it for the Starobinsky model.

B. Casson handles or the infinite process of handle-cancellations

In dimension 4, the process of handle canceling can be an infinite process. One can understand the reason simply. Two disks in a 4-manifold intersect in a point. At the same time, disks in a 4-manifold can admit self-intersections, i.e. we obtain an immersed disk (in contrast to an embedded disk with no self-intersections). As explained above, one can cancel the self-intersections but one needs another disk admitting self-intersections again. As Freedman [31] showed, one needs infinitely many disks (or stages) to cancel the self-intersection topologically. In this process, an immersed disk can admit more than one self-intersection and therefore needs more than one disk for its canceling. Thus, one obtains a tree \( T_{CH} \) of disks, called a Casson handle \( CH \) (see the appendix and [22, 30, 31, 36]). Here we have also a special situation: an infinite object, the Casson handle \( CH \), has to be embedded into a compact 4-manifold, the cobordism \( W(\Sigma_1, \Sigma_2) \). Furthermore, the tree can be exponentially large (see [19]). But then we have to find an embedding of this tree into a compact submanifold. For simplicity we can assume a disk \( D^2 \) of radius 1. The infinite tree has a root and a continuum of leaves at infinity. If we put the root of the tree in the middle of the disk then the leaves have to be at the boundary of the disk. The corresponding disk has to admit a special geometry to reflect this properties, it is the Poincare hyperbolic disk with metric

\[ ds^2_{\mathbb{H}} = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} \]

The boundary of the disk (i.e. \( x^2 + y^2 = 1 \)) represents the point at infinity. The (scalar) curvature is negative, i.e. the disk carries a hyperbolic metric. This metric can be simply transformed into

\[ ds^2 = g_{rr} dr^2 + g_{\xi \xi} d\xi^2 = \frac{dr^2 + r^2 d\xi^2}{(1 - r^2)^2} \quad (7) \]

by using \( x = r \cdot \cos \xi, y = r \cdot \sin \xi \). The tree of the Casson handle is embedded along a fixed angle \( \xi \), i.e. \( d\xi = 0 \).

As mentioned above, the root of the tree is located at the center of the disk. Then the whole tree is located between \( 0 < r < 1 \) where \( r = 1 \) is containing the leaves of the tree. By using \( d\xi = 0 \), we obtained for the metric

\[ ds^2_{\text{tree}} = \frac{dr^2}{(1 - r^2)^2} = \left( d \left( \ln \left( \frac{1 + r}{1 - r} \right) \right) \right)^2 = d\phi^2 \quad (8) \]
by choosing
\[ \phi = \ln \left( \frac{1 + r}{1 - r} \right), \quad r = \frac{e^\phi - 1}{e^\phi + 1}. \]  

A Morse function on the disk is given by
\[ h(r) = \pm r^2 \]
and a cancelling pair (by using Cerf theory) can be expressed by
\[ h(r) = r^3 + T \cdot r \]
with the deformation parameter \( T \). For \( T < 0 \) one has the canceling pair of two handles and for \( T > 0 \) both handles are disappeared. The cancellation point is given by \( T = 0 \) or by \( h(r) = r^3 \). For the following argumentation, we will start with a Morse function \( h(r) = r^2/2 \) and deform it to a pair of two handles by choosing
\[ h(r) = r^2 - \frac{r^3}{3}. \]  

One handle is at the middle of the disk \((r = 0)\) and the canceling handle is located at infinity (after adding the whole Casson handle), i.e. at the boundary of the disk \( r = 1 \). One can also construct the previous Morse function directly from this data. For that purpose, we have to choose the first derivative to be
\[ dh = r (1 - r) \, dr \]
to get critical points at \( r = 0 \) (minimum) and \( r = 1 \) (maximum). By a simple integration, one will get the Morse function (10) above. Then in the action (9), one has the potential
\[ V(r) = g_{rr} \left( \frac{\partial h}{\partial r} \right)^2 = \frac{(1 - r)^2 r^2}{(1 - r^2)^2} \]
with respect to the metric (7)
\[ g_{rr} = \frac{1}{(1 - r^2)^2}. \]

In our philosophy, we have to use the coordinate \( \phi \) instead of \( r \) which is equivalent to transform the problem back into the Euclidean space. Then we will obtain the scalar field \( \phi \) and the new potential \( V(\phi) \) in these coordinates
\[ V(\phi) = e^{-2\phi} \left( e^\phi - 1 \right)^2 = (1 - e^{-\phi})^2 \]
leading to the action
\[ S_{\text{Starobinsky}} = \int_{W(\Sigma_1, \Sigma_2)} d^4 x \sqrt{g} \left( R + \partial^\mu \phi \partial_\mu \phi - A \cdot (1 - e^{-\phi})^2 \right) \]
(12)
of the Starobinsky model written as scalar field action. Here we take the opportunity to scale the potential by the free parameter \( A \), i.e. by scaling the function \( V(\phi) \to A \cdot V(\phi) \). The classical Starobinsky model can be constructed after performing a conformal transformation
\[ g \to g' = e^\phi g \]
(13)
with
\[ e^\phi = 1 + 2\alpha \cdot R \]
(14)
with the scalar curvature \( R \) and \( \alpha = \frac{1}{8A} \). Then one obtains
\[ S_{\text{Starobinsky}} = \int_{W(\Sigma_1, \Sigma_2)} d^4 x \sqrt{g} \left( R + \alpha \cdot R^2 \right) \]
(15)
the usual Starobinsky model. It is one of the few models which agrees with the results of the Planck mission. But what is the meaning of this conformal transformation? Is it possible to determine the free parameter \( \alpha \) and what is its meaning? What is the real geometric background of this model? All these questions have to be addressed to get a full derivation of the model. Therefore we will start with the model (12) to obtain (15).
C. A geometric interpretation of the Starobinsky model

The infinite process of handle-cancellation in dimension four was used to construct the scalar field model (12), which is conformally equivalent to the Starobinsky model. Before we start we have to give an overview about the model leading to this action. We considered a topology change of a 3-manifold $\Sigma_1$ into another 3-manifold $\Sigma_2$ represented by a cobordism, i.e., by a 4-manifold with boundary $\Sigma_1 \cup \Sigma_2$. In this process, one changes the handle structure of $\Sigma_1$ into the handle structure of $\Sigma_2$. Some handles will be canceled and some other handles are created. For the special case of 3-manifolds, one can consider a scalar field $\phi$ where the variation of this field gives the topology change. Analytically one has to consider a one parameter family of functions $\phi^2 - T \cdot \phi$ for this creation/annihilation process. But in dimension four, there are problems to realize this process. One needs a complicated infinite tree-like structure (Casson handle) to manage this process which has to be embedded into the compact cobordism. The embedding can be realized by using the hyperbolic metric of the Poincare hyperbolic disk. At the end we obtained an analytic expression (see the potential (11)) for the corresponding handle structure of this Casson handle.

For that purpose, we have to consider the potential (11). We derived it for the Casson handle embedded in the Poincare hyperbolic disk. A quick look at the defining formula (9) for the scalar field $\phi$ will give us the defining area: $r$ is between $0 \leq r < 1$ leading to $0 \leq \phi < \infty$. But the final potential told us more: outside of the Poincare hyperbolic disk, the scalar field can admit negative values and the potential increased exponentially. At this point we have to remember on the interpretation of $\phi$: it is directly the deformation of the 3-manifold $\Sigma_1$ into $\Sigma_2$. Obviously, this deformation will lead to a deformation of the metric at the 3-manifold as well. Now we will consider the metric (7)

$$ds^2 = d\phi^2 + \sinh(\phi)^2 d\xi^2$$

by using the coordinates $(\phi, \xi)$. Along the $\phi$-coordinate we have the exponentially crowing tree and along $\xi$ we have also an exponential increase given by $\exp \phi$ or large positive $\phi$. For $\phi > 0$, one has an exponential increase $e^\phi$ of the metric (see 8) induced by the hyperbolic metric used to embed the tree of the Casson handle.

The embedding of the tree will mimic also the embedding of the whole Casson handle. The Casson handle (see Appendix) is homeomorphic to $D^2 \times \mathbb{R}^2$ (see [31]) and therefore we will need a four-dimensional version of the Poincare hyperbolic disk, the Poincare hyperbolic 4-ball with metric

$$ds^2_{4D} = \frac{dr^2 + r^2 d\Omega^2}{(1 - r^2)^2}$$

(16)

with the angle coordinates $\Omega$ (a tupel of 3 angles) and the radius $r$. Interestingly the calculation remained the same because the expression (10) qualitatively agreed with (8). The interesting part is independent of the dimension (see [21]). But there is an important difference: now we have a four-dimensional hyperbolic submanifold admitting Mostow rigidity or Mostow-Prasad rigidity. Mostow rigidity is a powerful property. As shown by Mostow [44], every hyperbolic $n$–manifold $n > 2$ with finite volume has this property: Every diffeomorphism (especially every conformal transformation) of a hyperbolic $n$–manifold with finite volume is induced by an isometry. See C for more information.

Therefore one cannot scale a hyperbolic 3- and 4-manifold with finite volume. The volume $\text{vol}()$ and the curvature (or the Chern-Simons invariant) are topological invariants. Now one may ask that the embedding of the Casson handle is rather artificial then generic. But there is a second argument. In the appendix we worked out how the canceling 1-/2-handle pair with a Casson handle attached looks like. Especially we will show that the corresponding sequence of 3-manifolds is a sequence of hyperbolic 3-manifolds of finite volume. Now we are able to argue similarly: one has

$$ds^2_{4D} = d\phi^2 + \sinh(\phi)^2 d\Omega^2$$

and one will get an exponential increase by $\exp \phi$ along all directions. This discussion showed that the scalar field $\phi$ can be interpreted as the deformation of the cobordism metric via a conformal transformation

$$g' = e^\phi g$$

for all positive values $\phi > 0$. Which geometrical expression determines this conformal transformation? The field $\phi$ is directly related to the radius of the hyperbolic disk via (19). In a very small neighborhood of $\phi = 0$, one has approximately an Euclidean metric (vanishing curvature $R = 0$). For large values $\phi > 0$, the curvature of the curves inside the disk increases (relative to the background metric). The negative values $\phi < 0$ correspond to the area outside of the hyperbolic disk. Here a curve passing the hyperbolic disk will be changed according to the negative curvature of the disk. Then we obtain the simple relation

$$e^\phi = 1 + f(R)$$
with the strictly increasing function $f$ (i.e. $f(R) < 0$ for $R < 0$ and $f(R) > 0$ for $R > 0$). The simplest function is the linear function, i.e.

$$e^\phi = 1 + \epsilon \cdot R$$

and the positivity of the exponential function implied

$$R \geq -\frac{1}{\epsilon}.$$ 

This special conformal transformation will transfer the action (12) to the action (15)

$$S_{\text{Starobinsky}} = \int_{W(\Sigma_1, \Sigma_2)} d^4x \sqrt{g} \left( R + \alpha \cdot R^2 \right)$$

with $\epsilon = 2 \cdot \alpha$. But with the discussion above, we can interpret the Starobinsky model geometrically. The potential $V(\phi)$ of the scalar action is given by

$$V(\phi) = \frac{1}{8\alpha} \left( 1 - e^{-\phi} \right)^2$$

or in terms of the curvature

$$\frac{1}{8\alpha} \left( 1 - \frac{1}{1 + 2\alpha \cdot R} \right)^2$$

This expression is flat for positive curvatures reaching slowly the value $1/8\alpha$. But for negative values ($-1/2\alpha < R < 0$), it grows rapidly. As discussed above, the positive value is related to the curvature of a curve in the interior of the hyperbolic disk. The limit values corresponds to the curvature of the whole disk which is needed to embeds the whole tree of the Casson handle (see above). But negative values (or the contraction of the metric) are leading to a strongly increasing potential or the contraction of the disk (containing the embedded tree) is impossible. This behavior goes over to the 4-dimensional case (the embedding of the whole Casson handle into the Poincare hyperbolic 4-ball). Then we can state:

The Starobinsky model is the simplest realization of Mostow rigidity, i.e. there is a 4-dimensional hyperbolic submanifold of curvature $-\frac{1}{2\alpha}$ which cannot be contracted.

This submanifold is also the cause for inflation: it is the reaction of the incompressibility of the submanifold. But Mostow rigidity has a great advantage: geometric expression are topological invariants. For this reason we should be able to determine the free parameter by the topological invariants of the cobordism.

**D. Determine the number of e-folds**

In the usual models of inflation, the number of e-folds $N$ is a free parameter which will be choose to be $50 < N < 60$. But topology in combination with Mostow rigidity should determine this value by purely topological methods. In [12], we described the way to get this value in principle. Here we will adapt the derivation of the formula to the case in this paper. For that purpose we will state two deep mathematical results, the details can be found in [12]. Above we introduced a hyperbolic disk (Poincare disk) to embed the infinite tree. This deep result can be expressed a different manner: there is no freedom or we have to choose the hyperbolic metric (which is up to isometries given by (8)). The second deep result is the representation of the topology change as an infinite chain of 3-manifolds $Y_1 \rightarrow \cdots \rightarrow Y_\infty$ where each spatial 3-manifold $Y_n$ admits a a homogenous metric of constant negative curvature. Equivalently this change is given by an infinite chain of cobordisms

$$W_\infty = W(Y_1, Y_2) \cup Y_2 W(Y_2, Y_3) \cup \cdots$$

representing the chain of changes. This chain of cobordisms $W_\infty$ is also embedded in the spacetime. As shown in [12], $W_\infty$ is a model for an end of an exotic $\mathbb{R}^4$ or a model of an exotic $S^3 \times \mathbb{R}$. As shown in this paper, the embedded $W_\infty$ admits a hyperbolic geometry. This hyperbolic geometry of the cobordism is best expressed by the metric

$$ds^2 = dt^2 - a(t)^2 h_{ik} dx^i dx^k$$ (17)
also called the Friedmann-Robertson-Walker metric (FRW metric) with the scaling function \( a(t) \) for the (spatial) 3-manifold (denoted as \( \Sigma \) in the following). As explained above, the spatial 3-manifold admits (at least for the pieces) a homogenous metric of constant curvature. Now we have the following situation: each spatial 3-manifold admits a hyperbolic metric and the whole process (given as infinite chain \( W_\infty \) of cobordisms) admits also a hyperbolic metric of constant negative scalar curvature \( R = -4\Lambda = \text{const.} < 0 \) which is realized by the equation \( \text{Ric} = -\Lambda g \), i.e. \( \Lambda \) is the so-called cosmological constant. Then one obtains the equation

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2}
\]  

(18)

having the solutions \( a(t) = a_0 \sqrt{3|k|/\Lambda} \sinh(t \sqrt{\Lambda/3}) \) for \( k < 0 \), \( a(t) = a_0 \exp(t \sqrt{\Lambda/3}) \) for \( k = 0 \) and \( a(t) = a_0 \sqrt{3|k|/\Lambda} \cosh(t \sqrt{\Lambda/3}) \) for \( k > 0 \) all with exponential behavior. At first we will consider this equation for constant topology, i.e. for the spacetime \( Y_n \times [0,1] \). But as explained above and see \([12]\), the embedding and every \( Y_n \) admits a hyperbolic structure. Now taking Mostow-Prasad rigidity seriously, the scaling function \( a(t) \) must be constant, or \( \dot{a} = 0 \). Therefore we will get

\[
\Lambda = \frac{3k}{a^2} \]

(19)

by using (18) for the parts of constant topology. Formula (19) can be now written in the form

\[
\Lambda = \frac{1}{a^2} = 3 \, R
\]

(20)

so that \( \text{CC} \) is related to the curvature of the 3D space. By using \( a_n = \sqrt[3]{\text{vol}(Y_n)} \), we are able to define a scaling parameter for every \( Y_n \). By Mostow-Prasad rigidity, \( a_n \) is also constant, \( a_n = \text{const.} \). But the change \( Y_n \rightarrow Y_{n+1} \) increases the volumes of \( Y_n \), \( \text{vol}(Y_{n+1}) > \text{vol}(Y_n) \), by adding specific 3-manifolds (i.e. complements of the Whitehead links). Therefore we have the strange situation that the spatial space changes by the addition of new (topologically non-trivial) spaces. To illustrate the amount of the change, we have to consider the embedding directly. It is given by the embedding of the Casson handle \( CH \) as represented by the corresponding infinite tree \( T_{CH} \). As explained above, this tree must be embedded into the hyperbolic space. For the tree, it is enough to use a 2D model, i.e. the hyperbolic space \( \mathbb{H}^2 \). There are many isometric models of \( \mathbb{H}^2 \) (see the appendix \([11]\) for two models). Above we used the Poincare disk model but now we will use the half-plane model with the hyperbolic metric

\[
ds^2 = \frac{dx^2 + dy^2}{y^2}
\]

(21)

to simplify the calculations. The infinite tree must be embedded along the \( y \)-axis and we set \( dx = 0 \). The tree \( T_{CH} \), as the representative for the Casson handle, can be seen as metric space instead of a simplicial tree. In case of a simplicial tree, one is only interested in the structure given by the number of levels and branches. The tree \( T_{CH} \) as a metric space (so-called \( \mathbb{R} \)-tree) has the property that any two points are joined by a unique arc isometric to an interval in \( \mathbb{R} \). Then the embedding of \( T_{CH} \) is given by the identification of the coordinate \( y \) with the coordinate of the tree \( a_T \) representing the distance from the root. This coordinate is a real number and we can build the new distance function after the embedding as

\[
ds_T^2 = \frac{da_T^2}{a_T^2} = \frac{dt^2}{L^2}
\]

But as discussed above, the tree \( T_{CH} \) grows with respect to a time parameter so that we need to introduce an independent time scale \( t \). From the physics point of view, the time scale describes the partition of the tree into slices. This main idea was used in \([12]\) to get the relation between the number of e-folds \( N \) and a topological invariant (Chern-Simons invariant) of the 3-manifold (as result of the change). In the following we will describe only the main points in the derivation of the formula (see \([12]\) for the details):

- The growing \( ds_T^2 \) of the tree with respect to the hyperbolic structure is given by

\[
ds_T^2 = \frac{da_T^2}{a_T^2} = \frac{dt^2}{L^2}
\]

This equation agrees with the Friedman equation for a (flat) deSitter space, i.e. the current model of our universe with a CC. This equation can be formally integrated yielding the expression

\[
a_T(t,L) = a_0 \cdot \exp \left( \frac{t}{L} \right)
\]

(22)
One important invariant of a cobordism is the signature $\sigma(W)$, i.e. the number of positive minus the number of negative eigenvalues of the intersection form. Using the Hirzebruch signature theorem, it is given by the first Pontryagin class

$$\sigma(W(\Sigma_1, \Sigma_2)) = \frac{1}{3} \int \limits_{W(\Sigma_1, \Sigma_2)} tr(R \wedge R)$$

with the curvature 2-form $R$ of the tangent bundle $TW$. By Stokes theorem, this expression is given by the difference

$$\sigma(W(\Sigma_1, \Sigma_2)) = \frac{1}{3} CS(\Sigma_2) - \frac{1}{3} CS(\Sigma_1)$$

of two boundary integrals where

$$\int \limits_\Sigma tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = 8\pi^2 CS(\Sigma)$$

is known as Chern-Simons invariant of a 3-manifold $\Sigma$.

Using ideas of Witten [50–52] we will interpret the connection $A$ as ISO(2, 1) connection. Note that ISO(2, 1) is the Lorentz group SO(3, 1) by Wigner-Inönü contraction or the isometry group of the hyperbolic geometry. For that purpose we choose

$$A_i = \frac{1}{\ell} e^a_i P_a + \omega^a_i J_a$$

with the length $\ell$ and 1-form $A = A_i dx^i$ with values in the Lie algebra ISO(2, 1) so that the generators $P_a, J_a$ fulfill the commutation relations

$$[J_a, J_b] = \epsilon_{abc} J^c \quad [P_a, P_b] = 0 \quad [J_a, P_b] = \epsilon_{abc} P^c$$

with pairings $\langle J_a, P_b \rangle = Tr(J_a P_b) = \delta_{ab}, \langle J_a, J_b \rangle = 0 = \langle P_a, P_b \rangle$. This choice was discussed in [48] in the context of Cartan geometry.

For vanishing torsion $T = 0$, we obtain

$$\int \limits_\Sigma tr(A \wedge F) = \frac{1}{7} \int \limits_\Sigma 3 R \sqrt{h} d^3 x$$

and finally the relation

$$8\pi^2 \cdot \ell \cdot CS(\Sigma) = \frac{3}{2} \int \limits_\Sigma 3 R \sqrt{h} d^3 x.$$  \hspace{1cm} (25)

From (23) it follows that

$$\sigma(W(\Sigma_1, \Sigma_k)) = \frac{1}{3} CS(\Sigma_k) - \frac{1}{3} CS(\Sigma_{k-1}) + \frac{1}{3} CS(\Sigma_{k-1}) - \ldots - \frac{1}{3} CS(\Sigma_1) = \frac{1}{3} CS(\Sigma_k) - \frac{1}{3} CS(\Sigma_1).$$

Then we identify $t = \ell$ with the time and using (25) we will obtain the expression

$$t \cdot CS(\Sigma_2) = \frac{3}{2} \int \limits_{\Sigma_2} 3 R_{\text{ren}} \sqrt{h} d^3 x$$

where the extra factor $8\pi^2$ (equals $4 \cdot vol(S^3)$) is the normalization of the curvature integral. This normalization of the curvature changes the absolute value of the curvature into

$$|3R_{\text{ren}}| = \frac{1}{8\pi^2 L^2}$$

and we choose the scaling factor by the relation to the volume $L = \sqrt[3]{vol(\Sigma_2)}/(8\pi^2)$. 
Then we will obtain formally

\[ \int_{\Sigma_2} |3R_{\text{ren}}| \sqrt{h} \, d^3 x = \int_{\Sigma_2} \frac{1}{8\pi^2 L^2} \sqrt{h} \, d^3 x = L^3 \cdot \frac{1}{L^2} = L \]  

by using

\[ L^3 = \frac{\text{vol}(\Sigma_2)}{8\pi^2} = \frac{1}{8\pi^2} \int_{\Sigma_2} \sqrt{h} \, d^3 x \]

in agreement with the normalization above. Let us note that Mostow-Prasad rigidity enforces us to choose a rescaled formula

\[ \text{vol}_{\text{hyp}}(\Sigma_2) \cdot L^3 = \frac{1}{8\pi^2} \int_{\Sigma_2} \sqrt{h} \, d^3 x, \]

with the hyperbolic volume (as a topological invariant). The volume of all other 3-manifolds can be arbitrarily scaled. In case of hyperbolic 3-manifolds, the scalar curvature \( 3R < 0 \) is negative but above we used the absolute value \( |3R| \) in the calculation. Therefore we have to modify \( 20 \), i.e., we have to use the absolute value of the curvature \( |3R| \) and of the Chern-Simons invariant \( |CS(\Sigma_2)| \). By \( 20 \) and \( 28 \) using

\[ \frac{t}{L} = \begin{cases} \frac{3}{2} \cdot \text{vol}_{\text{hyp}}(\Sigma_2) & \Sigma_2 \text{ non-hyperbolic 3-manifold} \\ \frac{3 \cdot \text{vol}_{\text{hyp}}(\Sigma_2)}{2 |CS(\Sigma_2)|} & \Sigma_2 \text{ hyperbolic 3-manifold} \end{cases} \]

a simple integration \( 22 \) gives the following exponential behavior

\[ a(t) = a_0 \cdot e^{t/L} = \begin{cases} a_0 \cdot \exp \left( \frac{3}{2} \cdot \text{vol}_{\text{hyp}}(\Sigma_2) \right) & \Sigma_2 \text{ non-hyperbolic 3-manifold} \\ a_0 \cdot \exp \left( \frac{3 \cdot |CS(\Sigma_2)|}{2 |CS(\Sigma_2)|} \right) & \Sigma_2 \text{ hyperbolic 3-manifold} \end{cases} \]

For the following, we will introduce the shortening

\[ \vartheta = \begin{cases} \frac{3}{2} \cdot \text{vol}_{\text{hyp}}(\Sigma_2) & \Sigma_2 \text{ non-hyperbolic 3-manifold} \\ \frac{3 \cdot |CS(\Sigma_2)|}{2 |CS(\Sigma_2)|} & \Sigma_2 \text{ hyperbolic 3-manifold} \end{cases} \]  

In principle, the value \( \vartheta \) is the number of e-folds but above \( 27 \) we used another normalization of the curvature. But the curvature is related to \( a(t) \) by \( 1/a^2 \). Therefore we have to correct the number of e-folds by the logarithm \( \ln(8\pi^2) \) of the normalization and we will obtain

\[ N = \vartheta + \ln(8\pi^2) \]  

or

\[ N = \begin{cases} \frac{3}{2} \cdot \text{vol}_{\text{hyp}}(\Sigma_2) + \ln(8\pi^2) & \Sigma_2 \text{ non-hyperbolic 3-manifold} \\ \frac{3 \cdot |CS(\Sigma_2)|}{2 |CS(\Sigma_2)|} + \ln(8\pi^2) & \Sigma_2 \text{ hyperbolic 3-manifold} \end{cases} \]

for the number e-folds. This result determines the number of e-folds and connect it with topological information of the final 3-manifold. Interestingly, this result is independent of the embedding and of the particular Casson handle. As shown in the following, using this result we are able to determine also the other parameters like the energy scale or the parameter \( \alpha \) in the Starobinsky model.

### E. Determine the Energy scale and the Parameter \( \alpha \)

Starting point is the formula

\[ a = a_0 \cdot \exp(\vartheta) \]
with the definition (29) of $\vartheta$. In [16], we also derived this formula by relating it to the levels of the tree representing the Casson handle. By using the shortening $\vartheta$, we obtain

$$a = a_0 \cdot \sum_{n=0}^{\infty} \frac{\vartheta^n}{n!}$$

or the $n$th level will contribute by $\frac{\vartheta^n}{n!}$. To calculate the energy scale, we need the argumentation that the energy scale as represented by an energy change $\Delta E$ is related to a time change by $\Delta t \sim h/\Delta E$. Therefore we are enforced to determine the shortest time change. But this change must agree with the number of levels in the tree of the Casson handle where the topology change appears. The Casson handle is designed to produce a (flat) disk with no self-intersections. As explained above, this disk will be used to cancel additional self-intersections and at the end it will lead to the topology change. Therefore we have to ask how many levels are necessary to get the first disk with no self-intersections. In [32] Freeman answered this question: three levels are needed! Now it seems natural that the shortest time scale will be assumed to be the Planck time $t_{Planck}$. Then we will get

$$\Delta t_{inflation} = \left(1 + \vartheta + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6}\right) t_{Planck}$$

for the shortest time interval of the topology change. Finally we will obtain for the energy scale

$$\Delta E_{inflation} = \frac{E_{Planck}}{1 + \vartheta + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6}}. \quad (32)$$

of the inflation. In subsection [HIC] we gave a geometrical interpretation of the parameter $\alpha$ as the radius of a non-contractable core. Following the argumentation above, then this core has to consist of at least three levels. Furthermore, $\alpha$ has to be expressed as energy in Planck units. But then using (32) we will obtain

$$\alpha \cdot M_P^2 = \frac{1}{\left(1 + \vartheta + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6}\right)}$$

and as we will see below, $\alpha$ will be of order $10^{-5}$ below the Planck energy. Finally, the spectral tilt $n_s$ and the tensor-scalar ratio $r$ can be determined to be

$$n_s = 1 - \frac{2}{\vartheta + \ln(8\pi^2)} \quad r = \frac{12}{(\vartheta + \ln(8\pi^2))^2}$$

with the topological invariant $\vartheta$.

F. Reheating and topology

Now we will discuss the Einstein-Hilbert action for the sequences of cobordism $W(Y_1, Y_2) \cup Y_2 W(Y_2, Y_3) \cup Y_3 \cdots$ following our work [10]. Let us start with the (Euclidean) Einstein-Hilbert action functional

$$S_{EH}(M) = \int_M R \sqrt{g} \, d^4x \quad (33)$$

of the 4-manifold $M$ and fix the Ricci-flat metric $g$ as solution of the vacuum field equations of the exotic 4-manifold. As discussed above, we consider a sequences of cobordism

$$W(Y_1, Y_2) \cup Y_2 W(Y_2, Y_3) \cup Y_3 \cdots$$

and one has to consider the Einstein-Hilbert action functional for every cobordism $W(Y_n, Y_{n+1})$. In general, for a manifold $M$ with boundary $\partial M = \Sigma$ one has the expression (see [34])

$$S_{EH}(M) = \int_M R \sqrt{g} \, d^4x + \int_{\Sigma} H \sqrt{h} \, d^3x$$
and for the cobordism, one obtains

$$S_{EH}(W(Y_n, Y_{n+1})) = \int_{W(Y_n, Y_{n+1})} R \sqrt{g} \, d^4 x + \int_{Y_{n+1}} H \sqrt{h} \, d^3 x - \int_{Y_n} H \sqrt{h} \, d^3 x$$

where \( H \) is the mean curvature of the boundary with metric \( h \). In the following we will discuss the boundary term, i.e. we reduce the problem to the discussion of the action

$$S_{EH}(\Sigma) = \int_{\Sigma} H \sqrt{h} \, d^3 x$$  \hspace{1cm} (34)

(see also [3, 6] for this boundary term) along the boundary \( \Sigma \) (a 3-manifold). Now we will show that the action (34) over a 3-manifold \( \Sigma \) is equivalent to the Dirac action of a spinor over \( \Sigma \). For completeness we present the discussion from [10]. At first let us consider the general case of an embedding of a 3-manifold into a 4-manifold. Let \( \iota: \Sigma \rightarrow M \) be an embedding of the 3-manifold \( \Sigma \) into the 4-manifold \( M \) with the normal vector \( \vec{N} \). A small neighborhood \( U_\varepsilon \) of \( \iota(\Sigma) \subset M \) looks like \( U_\varepsilon = \iota(\Sigma) \times [0, \varepsilon] \). Furthermore we identify \( \Sigma \) and \( \iota(\Sigma) \) (\( \iota \) is an embedding). Every 3-manifold admits a spin structure with a spin bundle (a spin bundle) as a lift of the frame bundle (principal \( SO(3) \) bundle associated to the tangent bundle). There is a (complex) vector bundle associated to the spin bundle (by a representation of the spin group), called spinor bundle \( S_\Sigma \). A section in the spinor bundle is called a spinor field (or a spinor). In case of a 4-manifold, we have to assume the existence of a spin structure. But for a manifold like \( M \), there is no restriction, i.e. there is always a spin structure and a spinor bundle \( S_M \). In general, the unitary representation of the spin group in \( D \) dimensions is \( 2^{[D/2]} \)-dimensional. From the representational point of view, a spinor in 4 dimensions is a pair of spinors in dimension 3. Therefore, the spinor bundle \( S_M \) of the 4-manifold splits into two sub-bundles \( S^+_M \) where one subbundle, say \( S^+_M \), can be related to the spinor bundle \( S_\Sigma \) of the 3-manifold. Then the spinor bundles are related by \( S_\Sigma = \iota^* S^+_M \) with the same relation \( \phi = \iota_* \Phi \) for the spinors \( \phi \in \Gamma(S_\Sigma) \) and \( \Phi \in \Gamma(S^+_M) \). Let \( \nabla^M_X, \nabla_X \) be the covariant derivatives in the spinor bundles along a vector field \( X \) as section of the bundle \( T\Sigma \). Then we have the formula

$$\nabla^M_X \Phi = \nabla_X \psi - \frac{1}{2} (\nabla_X \vec{N}) \cdot \vec{N} \cdot \psi$$  \hspace{1cm} (35)

with the embedding \( \phi \mapsto \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \Phi \) of the spinor spaces from the relation \( \phi = \iota_* \Phi \). Here we remark that of course there are two possible embeddings. For later use we will use the left-handed version. The expression \( \nabla_X \vec{N} \) is the second fundamental form of the embedding where the trace \( tr(\nabla_X \vec{N}) = 2H \) is related to the mean curvature \( H \). Then from (35) one obtains the following relation between the corresponding Dirac operators

$$D^M \Phi = D^\Sigma \psi - H \psi$$  \hspace{1cm} (36)

with the Dirac operator \( D^\Sigma \) on the 3-manifold \( \Sigma \). This relation (as well as (34)) is only true for the small neighborhood \( U_\varepsilon \) where the normal vector points is parallel to the vector defined by the coordinates of the interval \([0, \varepsilon]\) in \( U_\varepsilon \). In [14], we extend the spinor representation of an immersed surface into the 3-space to the immersion of a 3-manifold into a 4-manifold according to the work in [33]. Then the spinor \( \phi \) defines directly the embedding (via an integral representation) of the 3-manifold. Then the restricted spinor \( \Phi|_{\Sigma} = \phi \) is parallel transported along the normal vector and \( \Phi \) is constant along the normal direction (reflecting the product structure of \( U_\varepsilon \)). But then the spinor \( \Phi \) has to fulfill

$$D^M \Phi = 0$$  \hspace{1cm} (37)

in \( U_\varepsilon \) i.e. \( \Phi \) is a parallel spinor. Finally we get

$$D^\Sigma \psi = H \psi$$  \hspace{1cm} (38)

with the extra condition \( |\psi|^2 = \text{const.} \). (see [33] for the explicit construction of the spinor with \( |\psi|^2 = \text{const.} \) from the restriction of \( \Phi \)). Then we can express the action (34) by using (38) to obtain

$$\int_{\Sigma} H \sqrt{h} \, d^3 x = \int_{\Sigma} \bar{\psi} D^\Sigma \psi \sqrt{h} \, d^3 x$$  \hspace{1cm} (39)
using $|\psi|^2 = \text{const.}$

Above we obtained a relation (36) between a 3-dimensional spinor $\psi$ on the 3-manifold $\Sigma$ fulfilling a Dirac equation $D^\Sigma \psi = H \psi$ (determined by the embedding $\Sigma \to M$ into a 4-manifold $M$) and a 4-dimensional spinor $\Phi$ on a 4-manifold $M$ with fixed chirality ($\in \Gamma(S^3_M)$ or $\in \Gamma(S^3_M)$) fulfilling the Dirac equation $D^M \Phi = 0$ for the 4-dimensional spinor $\Phi$ by using the embedding

$$\Phi = \begin{pmatrix} 0 \\ \psi \end{pmatrix}.$$  (40)

In [10] we went a step further and discussed the topology of the 3-manifold leading to a fermion. On general grounds, one can show that a fermion is given by a knot complement admitting a hyperbolic structure. The connection between the knot and the particle properties is currently under investigation. But first calculations seem to imply that the particular knot is only important for the dynamical state (like the energy or momentum) but not for charges, flavors etc.

Now we will reverse the argumentation. Starting with the 4D Dirac action, the restriction to the boundary is given by

$$S_{\Sigma} = \int_{\Sigma} \bar{\psi} D^\Sigma \psi \sqrt{h} d^3x - \int_{\Sigma} H |\psi|^2 \sqrt{h} d^3x$$

(where we forget the condition $|\psi|^2 = \text{const.}$). The conformal transformation (13) will also influence the 3-metric $h$ to get $e^\phi h$ so that the last term is given by

$$\int_{\Sigma} H |\psi|^2 \sqrt{h} d^3x \to \int_{\Sigma} H e^\phi |\psi|^2 \sqrt{h} d^3x = S_{WW}$$

which is an interaction term of the inflaton field $\phi$ with the fermion field $\psi$. The coupling constant is the mean curvature $H$ of $\Sigma$ which is constant (by using the hyperbolic structure). In [10], the extension of this action to the spacetime $M$ was also discussed for the Dirac operator. If we fix $H$ as coupling parameter, the extension of $S_{WW}$ to 4D can be done by using the embedding (40). Then we will get the Lagrangian of the matter-scalar field coupling

$$\mathcal{L} = \bar{\Phi} D^M \Phi + |H| e^\phi |\Phi|^2$$

and we have to determine the coupling $|H|$ now. The argumentation follows from the calculation of $\vartheta$ in the previous subsections. The mean curvature is constant by the hyperbolic geometry. Furthermore, the fermion $\Phi$ couples geometrically to one level (1-level) of the Casson handle. Therefore using the scaling formula (31), we will get

$$|H| = \exp \left( -\frac{3}{CS(1 - \text{level})} \right)$$

by using the reference to the Planck scale. Now we have to discuss the value for the Chern-Simons invariant of 1-level. The corresponding 3-manifold to 1-level is given by complements of the Whitehead link $Wh$ with $CS(S^3 \setminus Wh) = \frac{1}{2}$ mod 1 (see the Fig. in [12]) so that

$$|H| = e^{-12} \approx 0.6 \cdot 10^{-5}$$  (41)

gives the coupling between the scalar field $\phi$ and fermion $\Phi$. In [28], they gave an upper bound of order $10^{-5}$ for the coupling together with an e-fold $N \approx 51.7$ to get the right reheating temperature. Later, we will present a particular model with this number of e-folds. Here, we remark that the coupling above does not depend on the topology change. It is a universal coupling between the scalar field (as substitute for the Casson handle) and the fermion field (as substitute for the knot complement). Therefore, we obtained a natural model for the coupling of matter to the inflaton field $\phi$ after inflation.

### IV. A REALISTIC EXAMPLE

The distinguished feature of differential topology of manifolds in dimension 4 is the existence of open 4-manifolds carrying a plenty of non-diffeomorphic smooth structures. In the cosmological model presented here, the special role is played by the topologically simplest 4-manifold, i.e. $\mathbb{R}^4$, which carries a continuum of infinitely many different
smoothness structures. Each of them except one, the standard \( \mathbb{R}^4 \), is called *exotic* \( \mathbb{R}^4 \). All exotic \( \mathbb{R}^4 \) are Riemannian smooth open 4- manifolds homeomorphic to \( \mathbb{R}^4 \) but non-diffeomorphic to the standard smooth \( \mathbb{R}^4 \). The standard smoothness is distinguished by the requirement that the topological product \( \mathbb{R} \times \mathbb{R}^3 \) is a smooth product. There exists only one (up to diffeomorphisms) smoothing, the standard \( \mathbb{R}^4 \), where the product above is smooth. There are two types of exotic \( \mathbb{R}^4 \): small exotic \( \mathbb{R}^4 \) can be embedded into the standard \( S^4 \) whereas large exotic \( \mathbb{R}^4 \) cannot. In the following, an exotic \( \mathbb{R}^4 \), presumably small if not stated differently, will be denoted as \( \mathbb{R}^4 \). In cosmology, one usually considers the topology \( S^3 \times \mathbb{R} \) for the spacetime. But by using the simple topological relations \( \mathbb{R}^4 \setminus \{0\} = S^3 \times \mathbb{R} \) or \( \mathbb{R}^4 \setminus \{0\} = S^3 \times \mathbb{R} \), one obtains also an exotic \( S^3 \times \mathbb{R} \) from every exotic \( \mathbb{R}^4 \). In the following we will denote the exotic \( S^3 \times \mathbb{R} \) by \( S^3 \times_\theta \mathbb{R} \) to indicate the important fact that there is no global splitting of \( S^3 \times_\theta \mathbb{R} \) or it is not globally hyperbolic. This fact has a tremendous impact on cosmology and therefore we will consider our main hypothesis:

**MainHypo:** The spacetime, seen as smooth four-dimensional manifold, admits an exotic smoothness structure.

### A. Introduction of the model

The main hypothesis above has the following consequences (see [12]):

- Any \( \mathbb{R}^4 \) has necessarily non-vanishing Riemann curvature. Also the \( S^3 \times_\theta \mathbb{R} \) has a non-vanishing curvature.
- Inside of \( \mathbb{R}^4 \), there is a compact 4-dimensional submanifold \( K \subset \mathbb{R}^4 \), which is not surrounded by a smoothly embedded 3-sphere. Then there is a chain of 3-submanifolds of \( \mathbb{R}^4 \) \( Y_1 \rightarrow \cdots \rightarrow Y_\infty \) and the corresponding infinite chain of cobordisms

  \[
  \text{End}(\mathbb{R}^4) = W(Y_1, Y_2) \cup_{Y_2} W(Y_2, Y_3) \cup \cdots
  \]

  where \( W(Y_k, Y_{k+1}) \) denotes the cobordism between \( Y_k \) and \( Y_{k+1} \) so that \( \mathbb{R}^4 = K \cup_{Y_1} \text{End}(\mathbb{R}^4) \) where \( \partial K = Y_1 \). Furthermore one has \( \text{End}(\mathbb{R}^4) \subset S^3 \times_\theta \mathbb{R} \).
- \( \mathbb{R}^4 \) and \( S^3 \times_\theta \mathbb{R} \) embeds into the standard \( \mathbb{R}^4 \) or \( \mathbb{S}^4 \) but also in some other complicated 4-manifolds. The construction of \( \mathbb{R}^4 \) gives us a natural smooth embedding into the compact 4-manifold \( E(2) \# \overline{\mathbb{CP}^2} \) (with the K3 surface \( E(2) \)) (see [20]).

But every subset \( K', K' \subset \mathbb{R}^4 \), is surrounded by a 3-sphere. This fact is the starting point of our model. Now we choose Planck-size 3-sphere \( S^3 \) inside of the compact subset \( K \subset \mathbb{R}^4 \). This is the initial point where our cosmos starts to evolve. By the construction of \( \mathbb{R}^4 \), as mentioned above, there exists the homology 3-sphere

\[
\Sigma(2, 5, 7) = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^5 + z^7 = 0, \ |x|^2 + |y|^2 + |z|^2 = 1\}
\]

inside of \( K \) which is the boundary of the Akbulut cork for \( E(2) \# \overline{\mathbb{CP}^2} \). (see chapter 9, [36]). If \( S^3 \) is the starting point of the cosmos as above, then \( S^3 \subset \Sigma(2, 5, 7) \). But then we will obtain the first topological transition

\[
S^3 \rightarrow \Sigma(2, 5, 7)
\]

inside \( \mathbb{R}^4 \). The construction of \( \mathbb{R}^4 \) was based on the topological structure of \( E(2) \) (the K3 surface). \( E(2) \) splits topologically into a 4-manifold \( |E_8 \oplus E_8| \) with intersection form \( E_8 \oplus E_8 \) (see [36]) and the sum of three copies of \( \mathbb{S}^2 \times \mathbb{S}^2 \). In this topological splitting

\[
|E_8 \oplus E_8| \times (S^2 \times S^2) \times (S^2 \times S^2) \times (S^2 \times S^2) \quad (42)
\]

the 4-manifold \( |E_8 \oplus E_8| \) has a boundary which is the sum of two Poincaré spheres \( P \# P \). Here we used the fact that a smooth 4-manifold with intersection form \( E_8 \) must have a boundary (which is the Poincaré sphere \( P \)), otherwise it would contradict the Donaldson’s theorem. Then any closed version of \( |E_8 \oplus E_8| \) does not exist and this fact is the reason for the existence of \( \mathbb{R}^4 \). To express it differently, the \( \mathbb{R}^4 \) lies between this 3-manifold \( \Sigma(2, 5, 7) \) and the sum of two Poincaré spheres \( P \# P \). We analyzed this spacetime in [11]. It is interesting to note that the number of \( \mathbb{S}^2 \times \mathbb{S}^2 \) components must be three or more otherwise the corresponding spacetime is not smooth!

Therefore we have two topological transitions resulting from the embedding into \( E(2) \# \overline{\mathbb{CP}^2} \)

\[
S^3 \xrightarrow{\text{cork}} \Sigma(2, 5, 7) \xrightarrow{\text{giving}} P \# P.
\]

These two topological transition are the main idea of our model.
B. Consequences for the inflation

In this section we will show how the change of the energy scale is driven by the topological transitions

\[ S^3 \xrightarrow{\text{cobordism}} \Sigma(2,5,7) \xrightarrow{\text{gluing}} P \# P. \]

Both transitions have different topological descriptions. The first transition \( S^3 \to \Sigma(2,5,7) \) can be realized by a smooth cobordism. To realize this cobordism, one has to add a 1-/2-handle pair together with one relation. It is a characteristic property of the 4-dimensional spacetime that this adding of a handle pair will also produce extra intersections between the handles. For that purpose we need a structure, which is an infinite tree of self-intersecting disks, also known as Casson handle. In [12] we described this situation extensively. If one assumes a Planck-size \( (L_P) \) 3-sphere at the Big Bang then the scale \( a \) of \( \Sigma(2,5,7) \) changes like

\[ a = L_P \cdot \exp \left( \frac{3}{2 \cdot CS(\Sigma(2,5,7))} \right) \]

with the Chern-Simons invariant and \( \vartheta \)

\[ CS(\Sigma(2,5,7)) = \frac{9}{4 \cdot (2 \cdot 5 \cdot 7)} = \frac{9}{280} \quad \vartheta = \frac{140}{3} \]

and the Planck scale of order \( 10^{-34} \) changes to \( 10^{-15} m \). Obviously, this transition has an exponential or inflationary behavior. Surprisingly, the number of e-folds can be explicitly calculated (see [13]) by the formula (30) to be

\[ N = \frac{3}{2 \cdot CS(\Sigma(2,5,7))} + \ln(8\pi^2) \approx 51 \]

and we also obtain the energy and time scale of this transition (see [13])

\[ E_{\text{GUT}} = \frac{E_P}{1 + \vartheta + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6}} \approx 10^{15} \text{GeV} \quad t = t_P \left( 1 + \vartheta + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6} \right) \approx 10^{-39} s \]

right at the conjectured GUT scale \( (E_P, t_P) \) Planck energy and time, respectively). Above, we showed that this transition is described. But then, the dimension-less free parameter \( \alpha \cdot M_P^{-2} \) as well spectral tilt \( n_s \) and the tensor-scalar ratio \( r \) can be determined to be

\[ \alpha \cdot M_P^{-2} = 1 + N + \frac{N^2}{2} + \frac{N^3}{6} \approx 10^{-5} \quad n_s \approx 0.961 \quad r \approx 0.0046 \]

using (43) which is in good agreement with current measurements. As discussed in the previous section, this number of e-folds is compatible with the coupling constant (41) in the reheating process.

V. CONCLUSION

In this paper we presented a complete topological picture to describe inflation. We started with a general formalism of topology change using the cobordism concept. This description lead naturally to the introduction of a scalar field \( \phi \). The potential \( V(\phi) \) is given by the squared derivative of the Morse function. By using Cerf theory, we obtained two possible models: chaotic inflation \( V(\phi) \sim \varphi^2 \) and topological inflation \( V(\phi) \sim (\varphi^2 - 1)^2 \). Both models were ruled out by the Planck mission. But in particular for 4-dimensional spacetime, there is another possibility which leads naturally to Starobinsky inflation. According to this model, an inflationary phase in the cosmic evolution is caused by the exotic smoothness structure of our spacetime. The exotic smoothness structure is constructed by a hyperbolic homology 3-sphere \( \Sigma \). The exponential expansion has its origin in the hyperbolic structure of the spacetime. This expansion is determined by a single parameter \( \vartheta \), the fraction of two topological invariants for the hyperbolic homology 3-sphere: the volume and the Chern-Simons invariant. Furthermore, we were able to calculate the number of e-folds which is by Mostow-Prasad rigidity a topological invariant. With the help of the invariant, we were also able to determine all parameters of the Starobinsky model like \( \alpha \), energy scale, e-folds and the coupling constant for the reheating process. One question remains: But what is about the inflation without quantum effects? Fortunately, there is growing evidence that the differential structures constructed above (i.e. exotic smoothness in dimension 4) is directly related to quantum gravitational effects [8, 13, 27]. Maybe we touch only the tip of the iceberg.
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Appendix A: Connected and boundary-connected sum of manifolds

Now we will define the connected sum # and the boundary connected sum ∂ of manifolds. Let M, N be two n-manifolds with boundaries ∂M, ∂N. The connected sum M#N is the procedure of cutting out a disk D^n from the interior int(M) \ D^n and int(N) \ D^n with the boundaries S^{n−1} ∪ ∂M and S^{n−1} ∪ ∂N, respectively, and gluing them together along the common boundary component S^{n−1}. The boundary ∂(M#N) = ∂M ∪ ∂N is the disjoint sum of the boundaries ∂M, ∂N. The boundary connected sum M♮N is the procedure of cutting out a disk D^{n−1} from the boundary ∂M \ D^{n−1} and ∂N \ D^{n−1} and gluing them together along S^{n−2} of the boundary. Then the boundary of this sum M♮N is the connected sum ∂(M♮N) = ∂M#∂N of the boundaries ∂M, ∂N.

Appendix B: Casson Handles

Let us start with the basic construction of the Casson handle CH. Let M be a smooth, compact, simple-connected 4-manifold and f : D^2 → M a (codimension-2) mapping. By using diffeomorphisms of D^2 and M, one can deform the mapping f to get an immersion (i.e. injective differential) generically with only double points (i.e. #||f^−1(f(x))|| = 2) as singularities. But to incorporate the generic location of the disk, one is rather interesting in the mapping of a 2-handle D^2 × D^2 induced by f × id : D^2 × D^2 → M from f. Then every double point (or self-intersection) of f(D^2) leads to self-plumbings of the 2-handle D^2 × D^2. A self-plumbing is an identification of D^2 × D^2 with D^2 × D^2 where D^2, D^2 ⊂ D^2 are disjoint sub-disks of the first factor disk. Consider the pair (D^2 × D^2, ∂D^2 × D^2) and produce finitely many self-plumbings away from the attaching region ∂D^2 × D^2 to get a kinky handle (k, ∂−k) where ∂−k denotes the attaching region of the kinky handle. A kinky handle (k, ∂−k) is a one-stage tower (T_1, ∂−T_1) and an (n + 1)-stage tower (T_{n+1}, ∂−T_{n+1}) is an n-stage tower union kinky handles U_{i=1}^n(T_ℓ, ∂−T_ℓ) where two towers are attached along ∂−T_ℓ. Let T_n be (interior T_n) ∪ ∂−T_n and the Casson handle

\[ CH = \bigcup_{\ell=0} T_\ell^- \]

is the union of towers (with direct limit topology induced from the inclusions T_n ↪ T_{n+1}).

The main idea of the construction above is very simple: an immersed disk (disk with self-intersections) can be deformed into an embedded disk (disk without self-intersections) by sliding one part of the disk along another (embedded) disk to kill the self-intersections. Unfortunately the other disk can be immersed only. But the immersion can be deformed to an embedding by a disk again etc. In the limit of this process one "shifts the self-intersections into infinity" and obtains the standard open 2-handle (D^2 × S^1, ∂D^2 × S^1).

A Casson handle is specified up to (orientation preserving) diffeomorphism (of pairs) by a labeled finitely-branching tree with base-point *, having all edge paths infinitely extendable away from *. Each edge should be given a label + or -. Here is the construction: tree → CH. Each vertex corresponds to a kinky handle; the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self plumbing. The whole process generates a tree with infinite many levels. In principle, every tree with a finite number of branches per level realizes a corresponding Casson handle. Technically speaking, each building block of a Casson handle, the "kinky" handle with n kinks, is diffeomorphic to the n−times boundary-connected sum S^1 × S^1 (see appendix A) with two attaching regions. One region is a tubular neighborhood of band sums of Whitehead links connected with the previous block. The other region is a disjoint union of the standard open subsets S^1 × S^1 in S^1 × S^2 = ∂(S^1 × D^3) (this is connected with the next block).

Appendix C: Hyperbolic 3-/4-Manifolds and Mostow-Prasad rigidity

In short, Mostow–Prasad rigidity theorem states that the geometry of a complete, finite-volume hyperbolic manifold of dimension greater than two is uniquely determined by the fundamental group. The corresponding theorem was proven by Mostow for closed manifolds and extended by Prasad for finite-volume manifolds with boundary. In dimension 3, there is also an extension for non-compact manifolds also called ending lamination theorem. It states
that hyperbolic 3-manifolds with finitely generated fundamental groups are determined by their topology together with invariants of the ends admitting a kind of foliation at surfaces in the end. The end of a 3-manifolds has always the form $S \times [0,1)$ with the compact surfaces $S$. Then a lamination on the surface $S$ is a closed subset of $S$ that is written as the disjoint union of geodesics of $S$.

A general formulation of the Mostow-Prasad rigidity theorem is:

Let $M, N$ be compact hyperbolic $n-$manifolds with $n \geq 3$. Assume that $M$ and $N$ have isomorphic fundamental groups. Then the isomorphism of fundamental groups is induced by a unique isometry.

An important corollary states that geometric invariants are topological invariants. The Mostow-Prasad rigidity theorem has special formulations for dimension 3 and 4. Both manifolds $M, N$ have to be homotopy-equivalent and every homotopy-equivalence induces an isometry. In dimension 3, the homotopy-equivalence of a 3-manifold of non-positive sectional curvature implies a homeomorphism (a direct consequence of the geometrization theorem, the exception are only the lens spaces) and a diffeomorphism (see Moise [43]). In dimension 4, compact homotopy-equivalent simply-connected 4-manifolds are homeomorphic (see Freedman [31]). This result can be extended to a large class of compact non-simply connected 4-manifolds (having a good fundamental group), see [29]. Therefore, if a 3- or 4-manifold admits a hyperbolic structure then this structure is unique up to isometry and all geometric invariants are topological invariants among them the volume and the curvature.

Then a hyperbolic 3-manifold $M^3$ is given by the quotient space $\mathbb{H}^3/\Gamma$ where $\Gamma \subset Isom(\mathbb{H}^3) = SO(3,1)$ is a discrete subgroup (Kleinian group) so that $\Gamma \simeq \pi_1(M^3)$. A hyperbolic structure is a homomorphism $\pi_1(M^3) \to SO(3,1)$ up to conjugacy (inducing the isometry). The analogous result holds for the hyperbolic 4-manifold which can be written as quotient $\mathbb{H}^4/\pi_1(M^4)$.

Let $X^4$ be a compact hyperbolic 4-manifold with metric $g_0$ and let $M^4$ be a compact manifold together with a smooth map $f : M \to X$. As shown in [17] or in the survey [18] (Main Theorem 1.1), the volumes of $X, M$ are related

$$Vol_r(M) \geq deg(f)Vol(X,g_0)$$

where $deg(f)$ denotes the degree of $f$. If equality holds, and if the infimum of the relation is achieved by some metric $g$, then $(M, g)$ is an isometric Riemannian covering of $(X, g_0)$ with covering map $M \to X$ homotopic to $f$. In particular, if $f$ is the identity map $X \to X$ (having degree $deg(f) = 1$) then it implies that $g_0$ is the only Einstein metric on $X$ up to rescalings and diffeomorphisms.

Appendix D: Models of Hyperbolic geometry

In the following we will describe two main models of hyperbolic geometry which were used in this paper. For simplicity we will concentrate on the two-dimensional versions.

The Poincare disk model also called the conformal disk model, is a model of 2-dimensional hyperbolic geometry in which the points of the geometry are inside the unit disk, and the straight lines consist of all segments of circles contained within that disk that are orthogonal to the boundary of the disk, plus all diameters of the disk. The metric in this model is given by

$$ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

which can be transformed to expression [16] by a radial coordinate transformation. In this model, the hyperbolic geometry is confined to the unit disk, where the boundary represents the 'sphere at infinity'.

The Poincare half-plane model is the upper half-plane, denoted by $\mathbb{H}^2 = \{ (x, y) \mid y > 0, x, y \in \mathbb{R} \}$, together with a metric, the Poincare metric,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

(see [21]) that makes it a model of two-dimensional hyperbolic geometry. Here the line $y = 0$ represents the infinity (so-called ideal points).

Both models are isometric to each other. A point $(x, y)$ in the disk model maps to the point

$$\left( \frac{2x}{x^2 + (1 - y)^2}, \frac{1 - x^2 - y^2}{x^2 + (1 - y)^2} \right)$$

in the half-plane model conversely a point $(x, y)$ in the half-plane model maps to the point

$$\left( \frac{2x}{x^2 + (1 + y)^2}, \frac{x^2 + y^2 - 1}{x^2 + (1 + y)^2} \right)$$
in the disk model. This transform is known as Cayley transform.
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[53] In particular, see the recent results of the Planck satellite in arXiv from 1303.5062 to 1303.5090.

[54] In complex coordinates the plumbing may be written as \((z, w) \mapsto (w, z)\) or \((z, w) \mapsto (\bar{w}, \bar{z})\) creating either a positive or negative (respectively) double point on the disk \(D^2 \times 0\) (the core).

[55] In the proof of Freedman [31], the main complications come from the lack of control about this process.

[56] The number of end-connected sums is exactly the number of self intersections of the immersed two handle.