A PARALLEL SPLITTING METHOD FOR WEAKLY COUPLED MONOTONE INCLUSIONS*

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Abstract

A parallel splitting method is proposed for solving systems of coupled monotone inclusions in Hilbert spaces. Convergence is established for a wide class of coupling schemes. Unlike classical alternating algorithms, which are limited to two variables and linear coupling, our parallel method can handle an arbitrary number of variables as well as nonlinear coupling schemes. The breadth and flexibility of the proposed framework is illustrated through applications in the areas of evolution inclusions, dynamical games, signal recovery, image decomposition, best approximation, network flows, and variational problems in Sobolev spaces.

Keywords: coupled systems, evolution inclusion, forward-backward algorithm, maximal monotone operator, operator splitting, parallel algorithm, Sobolev space, weak convergence.

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1 Problem statement

This paper is concerned with the numerical solution of systems of coupled monotone inclusions in Hilbert spaces. A simple instance of this problem is to find $x_1 \in H$, $x_2 \in H$ such that

$$\begin{align*}
0 &\in A_1 x_1 + x_1 - x_2 \\
0 &\in A_2 x_2 + x_2 - x_1,
\end{align*}$$

(1.1)

where $(H, \|\cdot\|)$ is a real Hilbert space, and where $A_1$ and $A_2$ are maximal monotone operators acting on $H$. This formulation arises in various areas of nonlinear analysis [20]. For example, if $A_1 = \partial f_1$ and $A_2 = \partial f_2$ are the subdifferentials of proper lower semicontinuous convex functions $f_1$ and $f_2$ from $H$ to $]-\infty, +\infty]$, (1.1) is equivalent to

$$\minimize_{x_1 \in H, x_2 \in H} f_1(x_1) + f_2(x_2) + \frac{1}{2} \|x_1 - x_2\|^2.$$

(1.2)

This joint minimization problem, which was first investigated in [1], models problems in disciplines such as the cognitive sciences [9], image processing [34], and signal processing [38] (see also the references therein for further applications in mechanics, filter design, and dynamical games). In particular, if $f_1$ and $f_2$ are the indicator functions of closed convex subsets $C_1$ and $C_2$ of $H$, (1.2) reverts to the classical best approximation pair problem [19, 28, 40]

$$\minimize_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\|.$$

(1.3)

On the numerical side, a simple algorithm is available to solve (1.1), namely,

$$x_{1,0} \in H \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases}
\frac{x_{2,n}}{x_{1,n+1}} = (\text{Id} + A_2)^{-1} x_{1,n} \\
\frac{x_{2,n}}{x_{1,n+1}} = (\text{Id} + A_1)^{-1} x_{2,n}.
\end{cases}$$

(1.4)

This alternating resolvent method produces sequences $(x_{1,n})_{n \in \mathbb{N}}$ and $(x_{2,n})_{n \in \mathbb{N}}$ that converge weakly to points $x_1$ and $x_2$, respectively, such that $(x_1, x_2)$ solves (1.1) if solutions exist [20, Theorem 3.3]. In [5], the variational formulation (1.2) was extended to

$$\minimize_{x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2} f_1(x_1) + f_2(x_2) + \frac{1}{2} \|L_1 x_1 - L_2 x_2\|^2_G,$$

(1.5)

where $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{G}$ are Hilbert spaces, $f_1 : \mathcal{H}_1 \to ]-\infty, +\infty]$ and $f_2 : \mathcal{H}_2 \to ]-\infty, +\infty]$ are proper lower semicontinuous convex functions, and $L_1 : \mathcal{H}_1 \to \mathcal{G}$ and $L_2 : \mathcal{H}_2 \to \mathcal{G}$ are linear and bounded. This problem was solved in [5] via an inertial alternating minimization procedure first proposed in [9] for the case of the strongly coupled problem (1.2).

The above problems and their solution algorithms are limited to two variables which, in addition, must be linearly coupled. These are serious restrictions since models featuring more than two variables and/or nonlinear coupling schemes arise naturally in applications. The purpose of this paper is to address simultaneously these restrictions by proposing a parallel algorithm for solving systems of monotone inclusions involving an arbitrary number of variables and nonlinear coupling. The breadth and flexibility of this framework will be illustrated through applications in the areas of evolution inclusions, dynamical games, signal recovery, image decomposition, best approximation, network flows, and decomposition methods in Sobolev spaces.

We now state our problem formulation and our standing assumptions.
Problem 1.1 Let $(\mathcal{H}_i)_{1 \leq i \leq m}$ be real Hilbert spaces, where $m \geq 2$. For every $i \in \{1, \ldots, m\}$, let $A_i: \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be maximal monotone and let $B_i: \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to \mathcal{H}_i$. It is assumed that there exists $\beta \in [0, +\infty]$ such that

$$\langle x_1, \ldots, x_m \rangle \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m, (\forall y_1, \ldots, y_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m$$

$$\sum_{i=1}^{m} \langle B_i(x_1, \ldots, x_m) - B_i(y_1, \ldots, y_m) \rangle \left| x_i - y_i \right| \geq \beta \sum_{i=1}^{m} \| B_i(x_1, \ldots, x_m) - B_i(y_1, \ldots, y_m) \|^2. \quad (1.6)$$

The problem is to

$$\text{find } x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m \text{ such that}$$

$$\begin{cases} 0 \in A_1 x_1 + B_1(x_1, \ldots, x_m) \\
\vdots \\
0 \in A_m x_m + B_m(x_1, \ldots, x_m), \quad (1.7) \end{cases}$$

under the assumption that such points exist.

In abstract terms, the system of inclusions in (1.7) models an equilibrium involving $m$ variables in different Hilbert spaces. The $i$th inclusion in this system is a perturbation of the basic inclusion $0 \in A_i x_i$ by addition of the coupling term $B_i(x_1, \ldots, x_m)$. This type of coupling will be referred to as weak in that it is not restricted to a simple linear combination of the variables as in (1.1). As will be seen in Section 4, our analysis captures various linear and nonlinear coupling schemes. For example, if

$$\langle \forall i \in \{1, \ldots, m\} \rangle \quad \mathcal{H}_i = \mathcal{H} \quad \text{and} \quad (\forall x \in \mathcal{H}) \quad B_i(x, \ldots, x) = 0, \quad (1.8)$$

then Problem 1.1 is a relaxation of the standard problem (1.2) of finding a common zero of the operators $(A_i)_{1 \leq i \leq m}$, i.e., of solving the inclusion $0 \in \bigcap_{i=1}^{m} A_i x$. In particular, if $m = 2$, $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $B_1 = -B_2: (x_1, x_2) \mapsto x_1 - x_2$, and $\beta = 1/2$, then Problem 1.1 reverts to (1.1). On the other hand, if $m = 2$, $A_1 = \partial f_1$, $A_2 = \partial f_2$, $B_1: (x_1, x_2) \mapsto L^* (L_1 x_1 - L_2 x_2)$, $B_2: (x_1, x_2) \mapsto -L^*_2 (L_1 x_1 - L_2 x_2)$, and $\beta = (\|L_1\|^2 + \|L_2\|^2)^{-1}$, then Problem 1.1 reverts to (1.5).

The paper is organized as follows. In Section 2, we present our algorithm for solving Problem 1.1 and prove its convergence to solutions of Problem 1.1. In Section 3, we describe various instances of (1.7) resulting from specific choices for the operators $(A_i)_{1 \leq i \leq m}$, e.g., minimization problems, variational inequalities, saddle-point problems, and evolution inclusions. In Section 4, we discuss examples of linear and nonlinear coupling schemes that can be obtained through specific choices of the operators $(B_i)_{1 \leq i \leq m}$ in Problem 1.1. Applications to systems of evolution inclusions are treated in Section 5. Section 6 is devoted to variational formulations deriving from Problem 1.1 and features various special cases. The applications treated in that section include dynamical games, signal recovery, image decomposition, best approximation, and network flows. Finally, Section 7 describes an application to decomposition methods in Sobolev spaces.

Notation. Throughout, $\mathcal{H}$ and $(\mathcal{H}_i)_{1 \leq i \leq m}$ are real Hilbert spaces. Their scalar products are denoted by $\langle \cdot, \cdot \rangle$ and the associated norms by $\| \cdot \|$. Moreover, Id denotes the identity operator on these spaces. The indicator function of a subset $C$ of $\mathcal{H}$ is

$$\iota_C: x \mapsto \begin{cases} 0, & \text{if } x \in C; \\
+\infty, & \text{if } x \notin C \end{cases} \quad (1.9)$$
and the distance from \(x \in \mathcal{H}\) to \(C\) is \(d_C(x) = \inf_{y \in C} \|x - y\|\); if \(C\) is nonempty closed and convex, the projection of \(x\) onto \(C\) is the unique point \(P_Cx\) in \(C\) such that \(\|x - P_Cx\| = d_C(x)\). We denote by \(\Gamma_0(\mathcal{H})\) the class of lower semicontinuous convex functions \(f: \mathcal{H} \to [-\infty, +\infty]\) which are proper in the sense that \(\text{dom } f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \emptyset\). The subdifferential of \(f \in \Gamma_0(\mathcal{H})\) is the maximal monotone operator

\[
\partial f: \mathcal{H} \to 2^{\mathcal{H}}: x \mapsto \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \langle y - x \mid u \rangle + f(x) \leq f(y)\}.
\]

(1.10)

If \(\mathcal{G}\) is a real Hilbert space, \(\mathcal{B}(\mathcal{H}, \mathcal{G})\) is the space of bounded linear operators from \(\mathcal{H}\) to \(\mathcal{G}\) and \(\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})\). We denote by \(\text{gra } A = \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}\) the graph of a set-valued operator \(A: \mathcal{H} \to 2^{\mathcal{H}}\), by \(\text{dom } A = \{x \in \mathcal{H} \mid Ax \neq \emptyset\}\) its domain, and by \(J_A = (\text{Id} + A)^{-1}\) its resolvent. If \(A\) is monotone, then \(J_A\) is single-valued and nonexpansive and, furthermore, if \(A\) is maximal monotone, then \(\text{dom } J_A = \mathcal{H}\). For complements and further background on convex analysis and monotone operator theory, see [11, 25, 62, 65, 66].

\section{Algorithm}

Let us start with a characterization of the solutions to Problem 1.1.

\textbf{Proposition 2.1} Let \((x_i)_{1 \leq i \leq m} \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m\), let \((\lambda_i)_{1 \leq i \leq m} \in [0, 1]^m\), and let \(\gamma \in ]0, +\infty[\). Then \((x_i)_{1 \leq i \leq m}\) solves Problem 1.1 if and only if

\[
(\forall i \in \{1, \ldots, m\}) \quad x_i = \lambda_i x_i + (1 - \lambda_i) J_{\gamma A_i} (x_i - \gamma B_i(x_1, \ldots, x_m)).
\]

(2.1)

\textbf{Proof.} Let \(i \in \{1, \ldots, m\}\). Then, since \(B_i\) is single-valued,

\[
0 \in A_i x_i + B_i(x_1, \ldots, x_m) \iff x_i - \gamma B_i(x_1, \ldots, x_m) \in x_i + \gamma A_i x_i
\]

\[
\iff x_i = J_{\gamma A_i} (x_i - \gamma B_i(x_1, \ldots, x_m))
\]

\[
\iff x_i = x_i + (1 - \lambda_i) (J_{\gamma A_i} (x_i - \gamma B_i(x_1, \ldots, x_m)) - x_i),
\]

and we obtain (2.1). \(\Box\)

The above characterization suggests the following algorithm, which constructs \(m\) sequences \(((x_{i,n})_{n \in \mathbb{N}})_{1 \leq i \leq m}\). Recall that \(\beta\) is the constant appearing in (1.6).

\textbf{Algorithm 2.2} Fix \(\varepsilon \in ]0, \min\{1, \beta\}[\), \((\gamma_n)_{n \in \mathbb{N}}\) in \([\varepsilon, 2\beta - \varepsilon], (\lambda_n)_{n \in \mathbb{N}}\) in \([0, 1 - \varepsilon]\), and \((x_{i,0})_{1 \leq i \leq m}\) \(\in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m\). Set, for every \(n \in \mathbb{N}\),

\[
\begin{cases}
    x_{1,n+1} = \lambda_{1,n} x_{1,n} + (1 - \lambda_{1,n}) (J_{\gamma_n A_{1,n}} (x_{1,n} - \gamma_n (B_{1,n}(x_{1,n}, \ldots, x_{m,n}) + b_{1,n})) + a_{1,n}) \\
    \vdots \\
    x_{m,n+1} = \lambda_{m,n} x_{m,n} + (1 - \lambda_{m,n}) (J_{\gamma_n A_{m,n}} (x_{m,n} - \gamma_n (B_{m,n}(x_{1,n}, \ldots, x_{m,n}) + b_{m,n})) + a_{m,n})
\end{cases}
\]

(2.3)

where, for every \(i \in \{1, \ldots, m\}\), the following hold.

(i) \((A_{i,n})_{n \in \mathbb{N}}\) are maximal monotone operators from \(\mathcal{H}_i\) to \(2^{\mathcal{H}_i}\) such that

\[
(\forall p \in ]0, +\infty[) \quad \sum_{n \in \mathbb{N}} \sup_{\|y\| \leq p} \|J_{\gamma_n A_{i,n}} y - J_{\gamma_n A_{i,n}} y\| < +\infty.
\]

(2.4)
Lemma 2.5

(ii) \((B_{i,n})_{n \in \mathbb{N}}\) are operators from \(\mathcal{H}_1 \times \cdots \times \mathcal{H}_m\) to \(\mathcal{H}_i\) such that

(a) the operators \((B_{i,n} - B_i)_{n \in \mathbb{N}}\) are Lipschitz-continuous with respective constants \((\kappa_{i,n})_{n \in \mathbb{N}}\) in \([0, +\infty]\) satisfying \(\sum_{n \in \mathbb{N}} \kappa_{i,n} < +\infty\); and

(b) there exists \(z \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m\), independent of \(i\), such that \((\forall n \in \mathbb{N}) B_{i,n}z = B_i z\).

(iii) \((a_{i,n})_{n \in \mathbb{N}}\) and \((b_{i,n})_{n \in \mathbb{N}}\) are sequences in \(\mathcal{H}_i\) such that \(\sum_{n \in \mathbb{N}} \|a_{i,n}\| < +\infty\) and \(\sum_{n \in \mathbb{N}} \|b_{i,n}\| < +\infty\).

(iv) \((\lambda_{i,n})_{n \in \mathbb{N}}\) is a sequence in \([0, 1]\) such that \(\sum_{n \in \mathbb{N}} |\lambda_{i,n} - \lambda_n| < +\infty\).

Conditions (i) and (ii) describe the types of approximations to the original operators \((A_i)_{1 \leq i \leq m}\) and \((B_i)_{1 \leq i \leq m}\) which can be utilized. Examples of approximations will be provided in Proposition 3.7 and Remark 4.7, respectively. Condition (iii) quantifies the tolerance which is allowed in the implementation of these approximations (see \([33, 42, 45]\) for specific examples), while (iv) quantifies the tolerance which is allowed in the agent-dependent departure from the global relaxation scheme. The parallel nature of Algorithm 2.2 stems from the fact that the \(m\) evaluations of the resolvent operators in (2.3) can be performed independently and, therefore, simultaneously on concurrent processors.

Our asymptotic analysis of Algorithm 2.2 requires the following result on the convergence of the forward-backward algorithm. This algorithm finds its roots in the projected gradient method \([48]\) and certain methods for solving variational inequalities \([15, 26, 49, 61]\) (see also the bibliography of \([31]\) for more recent developments). First, we need to define the notion of cocoercivity.

Definition 2.3 Let \(\chi \in [0, +\infty[.\) An operator \(B : \mathcal{H} \to \mathcal{H}\) is \(\chi\)-cocoercive if

\[
(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Bx - By \rangle \geq \chi \|Bx - By\|^2. \tag{2.5}
\]

If \(\chi = 1\) in (2.5), then \(B\) is firmly nonexpansive.

Lemma 2.4 \([31]\) Corollary 6.5] Let \((\mathcal{H}, \|\cdot\|)\) be a real Hilbert space, let \(\chi \in [0, +\infty[\), let \(A : \mathcal{H} \to 2^\mathcal{H}\) be a maximal monotone operator, and let \(B : \mathcal{H} \to \mathcal{H}\) be a \(\chi\)-cocoercive operator such that \((A + B)^{-1}(0) \neq \emptyset.\) Fix \(\varepsilon \in [0, \min\{1, \chi\}[,\) let \((\gamma_n)_{n \in \mathbb{N}}\) be a sequence in \([\varepsilon, 2\chi - \varepsilon]\), let \((\lambda_n)_{n \in \mathbb{N}}\) be a sequence in \([0, 1 - \varepsilon]\), and let \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) be sequences in \(\mathcal{H}\) such that \(\sum_{n \in \mathbb{N}} ||a_n|| < +\infty\) and \(\sum_{n \in \mathbb{N}} ||b_n|| < +\infty.\) Fix \(x_0 \in \mathcal{H}\) and, for every \(n \in \mathbb{N},\) set

\[
x_{n+1} = \lambda_n x_n + (1 - \lambda_n)(J_{\gamma_n}A(x_n - \gamma_n(Bx_n + b_n)) + a_n). \tag{2.6}
\]

Then \((x_n)_{n \in \mathbb{N}}\) converges weakly to a point in \((A + B)^{-1}(0)\).

We shall also use the following fact.

Lemma 2.5 \([31]\) Lemma 2.3] Let \((\mathcal{H}, ||\cdot||)\) be a real Hilbert space, let \(\chi \in [0, +\infty[\), let \(B : \mathcal{H} \to \mathcal{H}\) be a \(\chi\)-cocoercive operator, and let \(\gamma \in [0, 2\chi[.\) Then \(\text{Id} - \gamma B\) is nonexpansive.

The main result of this section is the following theorem.

Theorem 2.6 Let \((x_{i,n})_{n \in \mathbb{N}}\) be sequences generated by Algorithm 2.2. Then, for every \(i \in \{1, \ldots, m\}, (x_{i,n})_{n \in \mathbb{N}}\) converges weakly to a point \(x_i \in \mathcal{H}_i\), and \((x_i)_{1 \leq i \leq m}\) is a solution to Problem 1.1.
Proof. Throughout the proof, a generic element $x$ in the Cartesian product $H_1 \times \cdots \times H_m$ will be expressed in terms of its components as $x = (x_i)_{1 \leq i \leq m}$. We shall show our algorithmic setting reduces to the situation described in Lemma 2.4 in the real Hilbert space $H$ obtained by endowing $H_1 \times \cdots \times H_m$ with the scalar product

$$
\langle \langle \cdot | \cdot \rangle \rangle : (x, y) \mapsto \sum_{i=1}^{m} \langle x_i | y_i \rangle, \quad (2.7)
$$

with associated norm

$$
||| \cdot ||| : x \mapsto \sqrt{\sum_{i=1}^{m} \| x_i \|^2}. \quad (2.8)
$$

To this end, we shall show that the iterations (2.3) can be cast in the form of (2.6). First, define

$$
A : H \to 2^H : x \mapsto \times_{i=1}^{m} A_i x_i \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad A_n : H \to 2^H : x \mapsto \times_{i=1}^{m} A_{i,n} x_i. \quad (2.9)
$$

It follows from the maximal monotonicity of the operators $(A_i)_{1 \leq i \leq m}$, condition (i) in Algorithm 2.2, (2.7), and (2.9) that $A$ and $(A_n)_{n \in \mathbb{N}}$ are maximal monotone,

$$
(2.10)
$$

with resolvents

$$
J_A : H \to H : x \mapsto (J_{A_i} x_i)_{1 \leq i \leq m} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad J_{A_n} : H \to H : x \mapsto (J_{A_{i,n}} x_i)_{1 \leq i \leq m}. \quad (2.11)
$$

respectively. Moreover, for every $\rho \in ]0, +\infty[$, we derive from (2.8), (2.11), and condition (i) in Algorithm 2.2 that

$$
\sum_{n \in \mathbb{N}} \sup_{||| y ||| \leq \rho} ||| J_{A_n} y - J_{A} y ||| = \sum_{n \in \mathbb{N}} \sup_{||| y ||| \leq \rho} \sqrt{\sum_{i=1}^{m} ||| J_{A_{i,n}} y_i - J_{A_i} y_i |||^2} 
\leq \sum_{n \in \mathbb{N}} \sup_{||| y ||| \leq \rho} \sum_{i=1}^{m} ||| J_{A_{i,n}} y_i - J_{A_i} y_i |||
\leq \sum_{i=1}^{m} \sum_{n \in \mathbb{N}} \sup_{||| y_i ||| \leq \rho} ||| J_{A_{i,n}} y_i - J_{A_i} y_i ||| < +\infty. \quad (2.12)
$$

Now define

$$
B : H \to H : x \mapsto (B_i x)_{1 \leq i \leq m} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad B_n : H \to H : x \mapsto (B_{i,n} x)_{1 \leq i \leq m}. \quad (2.13)
$$

Then (1.7) is equivalent to

$$
\text{find} \quad x \in H \quad \text{such that} \quad 0 \in A x + B x. \quad (2.14)
$$

Moreover, in the light of (2.7), (2.8), and (2.13), (1.6) becomes

$$
(\forall x \in H)(\forall y \in H) \quad \langle \langle x - y | B x - B y \rangle \rangle \geq \beta ||| B x - B y |||^2. \quad (2.15)
$$
In other words, $B$ is $\beta$-cocoercive. Next, let $n \in \mathbb{N}$ and set

$$c_n = (a_{i,n})_{1 \leq i \leq m} \quad \text{and} \quad d_n = (b_{i,n})_{1 \leq i \leq m}. \quad (2.16)$$

We deduce from (2.8) and condition (iii) in Algorithm 2.2 that

$$\sum_{k \in \mathbb{N}} \|c_k\| \leq \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{m} \|a_{i,k}\|^2 \right) \leq m \sum_{i=1}^{m} \|a_{i,k}\| < +\infty \quad (2.17)$$

and, likewise, that

$$\sum_{k \in \mathbb{N}} \|d_k\| < +\infty. \quad (2.18)$$

Now set

$$x_n = (x_{i,n})_{1 \leq i \leq m} \quad \text{and} \quad \Lambda_n : \mathcal{H} \rightarrow \mathcal{H} : x \mapsto (\lambda_{i,n} x_i)_{1 \leq i \leq m}. \quad (2.19)$$

It follows from (2.8) and condition (iv) in Algorithm 2.2 that

$$\|\Lambda_n\| = \max_{1 \leq i \leq m} \lambda_{i,n} \leq 1 \quad \text{and} \quad \|\text{Id} - \Lambda_n\| = 1 - \min_{1 \leq i \leq m} \lambda_{i,n} \leq 1. \quad (2.20)$$

Hence,

$$\|\Lambda_n\| + \|\text{Id} - \Lambda_n\| = 1 + \max_{1 \leq i \leq m} (\lambda_{i,n} - \lambda_n) - \min_{1 \leq i \leq m} (\lambda_{i,n} - \lambda_n) \leq 1 + \tau_n, \quad (2.21)$$

where

$$\tau_n = 2 \max_{1 \leq i \leq m} |\lambda_{i,n} - \lambda_n|. \quad (2.22)$$

We observe that, by virtue of condition (iv) in Algorithm 2.2,

$$\sum_{k \in \mathbb{N}} \tau_k = 2 \sum_{k \in \mathbb{N}} \max_{1 \leq i \leq m} |\lambda_{i,k} - \lambda_k| \leq 2 \sum_{i=1}^{m} \sum_{k \in \mathbb{N}} |\lambda_{i,k} - \lambda_k| < +\infty. \quad (2.23)$$

Moreover, in view of (2.11), (2.13), (2.16), and (2.19), the iterations (2.3) are equivalent to

$$x_{n+1} = \Lambda_n x_n + (\text{Id} - \Lambda_n) \left( J_{\gamma_n} A_n (x_n - \gamma_n (B_n x_n + d_n)) + c_n \right). \quad (2.24)$$

Now define

$$D_n = B_n - B. \quad (2.25)$$

It follows from condition (ii)(a) in Algorithm 2.2, (2.8), and (2.13) that $D_n$ is Lipschitz continuous with constant $\kappa_n = \sqrt{\sum_{i=1}^{m} \kappa_{i,n}^2}$ and that

$$\sum_{k \in \mathbb{N}} \kappa_k = \sum_{k \in \mathbb{N}} \sqrt{\sum_{i=1}^{m} \kappa_{i,k}^2} \leq \sum_{i=1}^{m} \sum_{k \in \mathbb{N}} \kappa_{i,k} < +\infty. \quad (2.26)$$

Furthermore, set

$$b_n = D_n x_n + d_n \quad (2.27)$$

and let $x$ be a solution to Problem 1.1. Then

$$\|b_n\| \leq \|D_n x_n\| + \|d_n\| \leq \|D_n x_n - D_n x\| + \|D_n x - D_n z\| + \|d_n\| \leq \kappa_n (\|x_n - x\| + \|x - z\|) + \|d_n\|, \quad (2.28)$$

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where \( z \) is provided by assumption \[(ii)(b) \] in Algorithm 2.2. We now set
\[
T_n = \text{Id} - \gamma_n B 
\quad \text{and} \quad
e_n = J_{\gamma_n A_n}(T_n x) - x. \tag{2.29}
\]
On the one hand, the inequality \( \sup_{k \in \mathbb{N}} \|x_k\| \leq 2\beta \) yields
\[
\|T_n x\| \leq \rho, \quad \text{where} \quad \rho = \|x\| + 2\beta \|B x\|. \tag{2.30}
\]
On the other hand, Proposition 2.1 and (2.11) supply
\[
x = J_{\gamma_n A}(T_n x). \tag{2.31}
\]
Therefore, (2.29), (2.30), and (2.12) imply that
\[
\sum_{k \in \mathbb{N}} \|e_k\| = \sum_{k \in \mathbb{N}} \|J_{\gamma_k A_k}(T_k x) - x\| = \sum_{k \in \mathbb{N}} \|J_{\gamma_k A_k}(T_k x) - J_{\gamma_n A}(T_n x)\| < +\infty. \tag{2.32}
\]
In addition, (2.25), (2.27), and (2.29) yield
\[
J_{\gamma_n A_n}(x_n - \gamma_n (B_n x_n + d_n)) - x = J_{\gamma_n A_n}(T_n x_n - \gamma_n b_n) - J_{\gamma_n A_n}(T_n x) + e_n. \tag{2.33}
\]
Since \( J_{\gamma_n A} \) is nonexpansive as a resolvent (see [11, Proposition 3.5.3] or [25, Proposition 2.2.iii]) and \( T_n \) is nonexpansive by Lemma 2.5, we derive from (2.33) and (2.28) that
\[
\|J_{\gamma_n A_n}(x_n - \gamma_n (B_n x_n + d_n)) - x\| \leq \|J_{\gamma_n A_n}(T_n x_n - \gamma_n b_n) - J_{\gamma_n A_n}(T_n x)\| + \|e_n\|
\leq \|T_n x_n - \gamma_n b_n - T_n x\| + \|e_n\|
\leq \|x_n - x\| + \gamma_n \|b_n\| + \|e_n\|
\leq (1 + 2\beta \kappa_n) \|x_n - x\| + 2\beta \kappa_n \|x - z\|
\quad + 2\beta \|d_n\| + \|e_n\|. \tag{2.34}
\]
Thus, it results from (2.24), (2.31), (2.21), and (2.20) that
\[
\|x_{n+1} - x\| \leq \|A_n(x_n - x) + (\text{Id} - A_n)(J_{\gamma_n A_n}(x_n - \gamma_n (B_n x_n + d_n)) - x + c_n)\|
\leq \|A_n\| \|x_n - x\| + \|\text{Id} - A_n\| \|c_n\|
\leq \|A_n\| \|x_n - x\| + \|\text{Id} - A_n\| \|c_n\|
\leq (1 + \tau_n) \|x_n - x\| + \|c_n\| + 2\beta \kappa_n \|x_n - x\|
\quad + 2\beta \kappa_n \|x - z\| + \|d_n\| + \|e_n\|
\leq (1 + \alpha_n) \|x_n - x\| + \|c_n\| + 2\beta \kappa_n \|x_n - x\|
\quad + 2\beta \kappa_n \|x - z\| + \|d_n\| + \|e_n\|
\leq (1 + \alpha_n) \|x_n - x\| + \delta_n, \tag{2.35}
\]
where
\[
\alpha_n = \tau_n + 2\beta \kappa_n \quad \text{and} \quad \delta_n = \|c_n\| + 2\beta \kappa_n \|x - z\| + 2\beta \|d_n\| + \|e_n\|. \tag{2.36}
\]
In turn, it follows from (2.23), (2.26), (2.17), (2.18), and (2.32) that \( \sum_{k \in \mathbb{N}} \alpha_k < +\infty \) and \( \sum_{k \in \mathbb{N}} \delta_k < +\infty \). Thus, (2.35) and Lemma 2.2 yield

\[
\sup_{k \in \mathbb{N}} \| x_k - x \| < +\infty \tag{2.37}
\]

and, using (2.26) and (2.18), we derive from (2.28) that

\[
\sum_{k \in \mathbb{N}} \| b_k \| < +\infty. \tag{2.38}
\]

In view of (2.27) and (2.29), (2.24) is equivalent to

\[
x_{n+1} = \Lambda_n x_n + (\text{Id} - \Lambda_n) (J_{\gamma_n A}(T_n x_n - \gamma_n b_n) + h_n), \tag{2.39}
\]

where

\[
h_n = J_{\gamma_n A}(T_n x_n - \gamma_n b_n) - J_{\gamma_n A}(T_n x_n - \gamma_n b_n) + c_n. \tag{2.40}
\]

Now set \( \mu = \sup_{k \in \mathbb{N}} \| x_k - x \| + \rho + 2\beta \sup_{k \in \mathbb{N}} \| b_k \| \). Then it follows from (2.37), and (2.38) that \( \mu < +\infty \). Moreover, we deduce from the nonexpansivity of \( T_n \) and (2.30) that

\[
\| T_n x_n - \gamma_n b_n \| \leq \| T_n x_n - T_n x \| + \| T_n x \| + 2\beta \| b_n \|
\leq \| x_n - x \| + \rho + 2\beta \| b_n \|
\leq \mu. \tag{2.41}
\]

Hence, appealing to (2.12) and (2.17), we deduce from (2.40) that

\[
\sum_{k \in \mathbb{N}} \| h_k \| < +\infty. \tag{2.42}
\]

Note that, upon introducing

\[
a_n = h_n + \frac{1}{1 - \lambda_n} (\Lambda_n - \lambda_n \text{Id}) (x_n - J_{\gamma_n A}(T_n x_n - \gamma_n b_n) - h_n), \tag{2.43}
\]

we can rewrite (2.39) in the equivalent form (2.6), namely

\[
x_{n+1} = \lambda_n x_n + (1 - \lambda_n) (J_{\gamma_n A}(x_n - \gamma_n (B x_n + b_n)) + a_n). \tag{2.44}
\]

Using (2.31) and the nonexpansivity of \( J_{\gamma_n A} \) and \( T_n \), we get

\[
\| x_n - J_{\gamma_n A}(T_n x_n - \gamma_n b_n) - h_n \|
\leq \| x_n - x \| + \| J_{\gamma_n A}(T_n x) - J_{\gamma_n A}(T_n x_n - \gamma_n b_n) \|
+ \| h_n \|
\leq 2 \| x_n - x \| + 2\beta \| b_n \| + \| h_n \|. \tag{2.45}
\]

Therefore, we derive from (2.37), (2.38), and (2.42) that

\[
\nu = \sup_{k \in \mathbb{N}} \| x_k - J_{\gamma_k A}(T_k x_k - \gamma_k b_k) - h_k \| < +\infty, \tag{2.46}
\]

and hence, from (2.43) and the inequality \( \lambda_n \leq 1 - \varepsilon \), that

\[
\| a_n \|
\leq \| h_n \| + \frac{1}{1 - \lambda_n} \| \Lambda_n - \lambda_n \text{Id} \| \| x_n - J_{\gamma_n A}(T_n x_n - \gamma_n b_n) - h_n \|
\leq \| h_n \| + \frac{\nu}{\varepsilon} \max_{1 \leq i \leq m} \| \lambda_{i,n} - \lambda_n \|. \tag{2.47}
\]
Thus, using (2.42) and arguing as in (2.23), we get
\[\sum_{k \in \mathbb{N}} ||a_k|| < +\infty. \] (2.48)

However, Lemma 2.4 asserts that, under (2.10), (2.15), (2.38), (2.48), and the hypotheses on \((\gamma_n)_{n \in \mathbb{N}}\) and \((\lambda_n)_{n \in \mathbb{N}}\) in Algorithm 2.2, the sequence \((x_n)_{n \in \mathbb{N}}\) generated by (2.44) converges weakly to a solution to (2.14). Since (2.44) is equivalent to (2.3), and (2.14) is equivalent to (1.7), the proof is complete. \[\square\]

3 Specific scenarios

Problem 1.1 covers various scenarios, depending on the type of operators \((A_i)_{1 \leq i \leq m}\) utilized in (1.7). We now provide some specific examples which will serve as a basis for the concrete problems to be discussed in Sections 5–7.

**Example 3.1** Suppose that, for every \(i \in \{1, \ldots, m\}\), \(A_i = \partial f_i\) where \(f_i \in \Gamma_0(H_i)\). Then (1.7) reduces to the system of coupled variational inequalities
\[
\text{find } x_1 \in H_1, \ldots, x_m \in H_m \text{ such that } \quad (\forall i \in \{1, \ldots, m\})(\forall y \in H_i) \quad \langle x_i - y \mid B_i(x_1, \ldots, x_m) \rangle + f_i(x_i) \leq f_i(y). \quad (3.1)
\]
A particular case of this type of problem will be investigated in Section 6.

**Example 3.2** Suppose that, for every \(i \in \{1, \ldots, m\}\), \(A_i\) is the normal cone operator to a nonempty closed convex subset \(C_i\) of \(H_i\), that is
\[
A_i = N_{C_i} : H_i \to 2^{H_i} : x \mapsto \begin{cases} \{ u \in H \mid \sup_{y \in C_i} \langle y - x \mid u \rangle \leq 0 \}, & \text{if } x \in C_i; \\ \emptyset, & \text{otherwise}. \end{cases} \quad (3.2)
\]
Then (1.7) becomes a system of coupled variational inequalities of the form
\[
\text{find } x_1 \in C_1, \ldots, x_m \in C_m \text{ such that } \quad (\forall i \in \{1, \ldots, m\})(\forall y \in C_i) \quad \langle x_i - y \mid B_i(x_1, \ldots, x_m) \rangle \leq 0. \quad (3.3)
\]
Such formulations will be investigated in Example 6.6 and Example 6.9.

**Example 3.3** For every \(i \in \{1, \ldots, m\}\), let \(Y_i\) and \(Z_i\) are real Hilbert spaces, and suppose that
\[
A_i : Y_i \oplus Z_i \to 2^{Y_i \oplus Z_i} : (y, z) \mapsto \{(u, v) \in Y_i \oplus Z_i \mid u \in \partial (-F_i(\cdot, z))(y) \text{ and } v \in \partial (F_i(y, \cdot))(z)\}, \quad (3.4)
\]
where \(F_i : Y_i \oplus Z_i \to [-\infty, +\infty]\) satisfies
\begin{enumerate}
\item \((\exists (y_0, z_0) \in Y_i \oplus Z_i)(\forall (y, z) \in Y_i \oplus Z_i) \quad F_i(y_0, z) > -\infty \text{ and } F_i(y, z_0) < +\infty;\)
\item for every \((y, z) \in Y_i \oplus Z_i\), \(-F_i(\cdot, z)\) and \(F_i(y, \cdot)\) are lower semicontinuous and convex.
\end{enumerate}
Then, for every $i \in \{1, \ldots, m\}$, $A_i$ is a maximal monotone operator acting on $\mathcal{H}_i = \mathcal{Y}_i \oplus \mathcal{Z}_i$ [57] and, upon setting $B_i = (B_{i1}, B_{i2})$, where $B_{i1}: \mathcal{H}_i \to \mathcal{Y}_i$ and $B_{i2}: \mathcal{H}_i \to \mathcal{Z}_i$, (1.7) reduces to the system of coupled saddle-point problems

$$\text{find } x_1 = (y_1, z_1) \in \mathcal{H}_1, \ldots, x_m = (y_m, z_m) \in \mathcal{H}_m \text{ such that } (\forall i \in \{1, \ldots, m\})$$

$$\begin{cases}
    \sup_{y \in \mathcal{Y}_i} F_i(y, z_i) - \langle y | B_{i1}(x_1, \ldots, x_m) \rangle_{\mathcal{Y}_i} = F_i(y_1, z_i) - \langle y_1 | B_{i1}(x_1, \ldots, x_m) \rangle_{\mathcal{Y}_i} \\
    \inf_{z \in \mathcal{Z}_i} F_i(y, z) + \langle z | B_{i2}(x_1, \ldots, x_m) \rangle_{\mathcal{Z}_i} = F_i(y_1, z_i) + \langle z_1 | B_{i2}(x_1, \ldots, x_m) \rangle_{\mathcal{Z}_i}.
\end{cases} \tag{3.5}$$

Such formulations will arise in Example 4.11.

**Example 3.4** Let us recall some standard notation [25, 66]. Fix $T \in [0, +\infty[$ and $p \in [1, +\infty[$. Then $\mathcal{D}([0, T])$ is the set of infinitely differentiable functions from $[0, T]$ to $\mathbb{R}$ with compact support in $]0, T[$. Given a real Hilbert space $H$, $\mathcal{C}([0, T]; H)$ is the space of continuous functions from $[0, T]$ to $H$ and $L^p([0, T]; H)$ is the space of classes of equivalences of Borel measurable functions $x: [0, T] \to H$ such that $\int_0^T \|x(t)\|_H^p dt < +\infty$. $L^2([0, T]; H)$ is a Hilbert space with scalar product $(x, y) \mapsto \int_0^T \langle x(t) | y(t) \rangle_H dt$. Now take $x$ and $y$ in $L^1([0, T]; H)$. Then $y$ is the weak derivative of $x$ if $\int_0^T \phi(t)y(t)dt = -\int_0^T (\phi(t)/dt)x(t)dt$ for every $\phi \in \mathcal{D}([0, T])$, in which case we use the notation $y = x'$. Moreover,

$$W^{1,2}([0, T]; H) = \{ x \in L^2([0, T]; H) \mid x' \in L^2([0, T]; H) \}. \tag{3.6}$$

Now, for every $i \in \{1, \ldots, m\}$, let $\mathcal{H}_i$ be a real Hilbert space, let $A_i: \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be maximal monotone, let $B_i: \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to \mathcal{H}_i$, and let $\mathcal{H}_i = L^2([0, T]; \mathcal{H}_i)$. Then, under standard assumptions, the operator

$$A_i: \mathcal{H}_i \to 2^{\mathcal{H}_i}; x \mapsto \begin{cases} x' + A_i x, & \text{if } x \in W^{1,2}([0, T]; \mathcal{H}_i) \text{ and } x(0) = x(T); \\
\emptyset, & \text{otherwise}
\end{cases} \tag{3.7}$$

is maximal monotone (see [10], [25, Section 3.6], [62]). In this context, with a suitable construction of the operators $(B_i)_{1 \leq i \leq m}$ in terms of $(B_i)_{1 \leq i \leq m}$, (1.7) assumes the form of the system of coupled evolution inclusions

$$\text{find } x_1 \in W^{1,2}([0, T]; \mathcal{H}_1), \ldots, x_m \in W^{1,2}([0, T]; \mathcal{H}_m) \text{ such that } (\forall i \in \{1, \ldots, m\})$$

$$\begin{cases} 0 \in x_i'(t) + A_i (x_i(t)) + B_i(x_1(t), \ldots, x_m(t)) \text{ a.e. on } [0, T] \mid x_i(0) = x_i(T). \end{cases} \tag{3.8}$$

This type of problem will be revisited in Section 5.

In Algorithm 2.2 maximal monotone approximations $A_{i,n}$ [1 \leq i \leq m] to the original operators $A_i$ [1 \leq i \leq m] can be used at iteration $n$, as long as (2.4) is satisfied. In order to illustrate this condition, we need a couple of definitions and some technical facts.

**Definition 3.5** Let $A$ and $B$ be set-valued operators from $\mathcal{H}$ to $2^\mathcal{H}$ and let $\rho \in [0, +\infty[$ be such that $E_\rho \cap (\text{gra } A \cup \text{gra } B) \neq \emptyset$, where $E_\rho = \{ (x, y) \in \mathcal{H} \times \mathcal{H} \mid \max \{ \|x\|, \|y\| \} \leq \rho \}$. The $\rho$-Hausdorff distance between $A$ and $B$ is [7, Section 1.1]

$$\text{haus}_\rho(A, B) = \max \left\{ \sup_{z \in E_\rho \cap \text{gra } B} d_{\text{gra } A}(z), \sup_{z \in E_\rho \cap \text{gra } A} d_{\text{gra } B}(z) \right\}. \tag{3.9}$$
Moreover, the Yosida approximation of $A$ of index $\gamma \in [0, +\infty[$ is $\gamma A = (\text{Id} - J_{\gamma A})/\gamma$ [11] [25].

**Lemma 3.6** Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone, let $x \in \mathcal{H}$, let $\gamma \in [0, +\infty[$, and let $\mu \in ]0, +\infty[$. Then the following hold.

(i) $J_{\mu A}x = J_{\gamma A}(x + (1 - \gamma/\mu)(J_{\mu A}x - x))$.

(ii) $\gamma \leq \mu \Rightarrow \|J_{\gamma A}x - x\| \leq 2\|J_{\mu A}x - x\|$.

(iii) $J_{(\gamma A)}x = x + \gamma(J_{(\gamma + \mu A)}x - x)/(\gamma + \mu)$.

(iv) $\|J_{(\gamma A)}x - J_{\gamma A}x\| \leq 2\mu\|J_{\gamma A}x - x\|/(\gamma + \mu)$.

**Proof.** (i) See [25, Section II.4].

(ii) Set $\lambda = \gamma/\mu$ and observe that $\lambda \in [0, 1]$. It follows from the nonexpansivity of $J_{\gamma A}$ [11, Proposition 3.5.3] and (i) that $\|J_{\gamma A}x - x\| \leq \|J_{\gamma A}x - J_{\mu A}x\| + \|J_{\mu A}x - x\| \leq \|J_{\gamma A} - J_{\gamma A}(\lambda x + (1 - \lambda)J_{\mu A}x)\| + \|J_{\mu A}x - x\| \leq \|x - \lambda x - (1 - \lambda)J_{\mu A}x\| + \|J_{\mu A}x - x\| \leq 2\|J_{\mu A}x - x\|$. [11, 25].

(iii) This identity follows at once from the semigroup property $\gamma + \mu A = (\gamma A)$, which can be found in [25, Proposition 2.6(ii)].

(iv) It follows from [iii] that

$$\|J_{(\gamma A)}x - J_{\gamma A}x\| = \left\|x + \frac{\gamma}{\gamma + \mu}(J_{(\gamma + \mu A)}x - x) - J_{\gamma A}x\right\| = \left\|\frac{\gamma}{\gamma + \mu}(J_{(\gamma + \mu A)}x - x) - (J_{\gamma A}x - x)\right\| - \frac{\mu}{\gamma + \mu}(J_{\gamma A}x - x)\right\| \leq \frac{\gamma}{\gamma + \mu}\|J_{(\gamma + \mu A)}x - J_{\gamma A}x\| + \frac{\mu}{\gamma + \mu}\|J_{\gamma A}x - x\|. \quad (3.10)$$

On the other hand, it follows from (i) and the nonexpansivity of $J_{(\gamma + \mu A)}$ that

$$\|J_{(\gamma + \mu A)}x - J_{\gamma A}x\| = \left\|J_{(\gamma + \mu A)}x - J_{(\gamma + \mu A)}\left(x + \left(1 - \frac{\gamma + \mu}{\gamma}\right)(J_{\gamma A}x - x)\right)\right\| \leq \frac{\mu}{\gamma}\|J_{\gamma A}x - x\|. \quad (3.11)$$

which, combined with (3.10), yields the announced inequality. \(\square\)

**Proposition 3.7** Let $i \in \{1, \ldots, m\}$ and let $(\gamma_n)_{n \in \mathbb{N}}$ be as in Algorithm 2.2. Then condition (2.4) holds if one of the following is satisfied for every $n \in \mathbb{N}$.

(i) $A_{i,n} = (\gamma_{i,n}/\gamma_n)A_i$, where $(\gamma_{i,n})_{n \in \mathbb{N}}$ lies in $]0, 2\beta]$ and satisfies $\sum_{n \in \mathbb{N}} |\gamma_{i,n} - \gamma_n| < +\infty$.

(ii) $A_{i,n} = \mu_{i,n}A_i$, where $(\mu_{i,n})_{n \in \mathbb{N}}$ lies in $]0, +\infty[$ and satisfies $\sum_{n \in \mathbb{N}} \mu_{i,n} < +\infty$.

(iii) $\gamma_n = \gamma \in [\varepsilon, 2\beta - \varepsilon]$, and

$$\forall \theta \in \left[\|J_{\gamma A}\| \max\left\{1, 1/\gamma\right\}, +\infty \right[ \sum_{n \in \mathbb{N}} \text{haus}_\theta(A_i, A_{i,n}) < +\infty. \quad (3.12)$$
Proof. Let $\rho \in [0, +\infty[. Since $\sup_{n \in \mathbb{N}} \gamma_n \leq 2\beta$, we derive from Lemma 3.6(ii) and the nonexpansivity of $\text{Id} - J_{2\beta A_i} = J_{(2\beta A_i)^{-1}}$ that

$$
(\forall n \in \mathbb{N})(\forall y \in \mathcal{H}_i) \quad \|J_{\gamma_n A_i} y - y\| \leq 2\|J_{2\beta A_i} y - y\| \\
\leq 2\|\text{Id} - J_{2\beta A_i} y - (\text{Id} - J_{2\beta A_i})0\| + 2\|J_{2\beta A_i}0\| \\
\leq 2\|y\| + 2\|J_{2\beta A_i}0\|. 
$$

(3.13)

In addition, set $\mu = 2\rho + 2\|J_{2\beta A_i}0\|$. We now prove assertions (i)–(iii).

(i) It follows from Lemma 3.6(i) and the nonexpansivity of $J_{\gamma_i A_i}$ that

$$
(\forall n \in \mathbb{N})(\forall y \in \mathcal{H}_i) \quad \|J_{\gamma_n A_i} y - J_{\gamma_i A_i} y\| = \|J_{\gamma_n A_i} y - J_{\gamma_i A_i} (y + (1 - \gamma_i / \gamma_n)(J_{\gamma_n A_i} y - y))\| \\
\leq |1 - \gamma_i / \gamma_n| \|J_{\gamma_n A_i} y - y\|. 
$$

(3.14)

Hence, in view of (3.12), (3.13), and the inequality $\inf_{n \in \mathbb{N}} \gamma_n \geq \varepsilon$ we have

$$
\sum_{n \in \mathbb{N}} \sup_{\|y\| \leq \rho} \|J_{\gamma_n A_i} y - J_{\gamma_i A_i} y\| \leq \mu \sum_{n \in \mathbb{N}} \left|1 - \gamma_i / \gamma_n\right| \leq \frac{\mu}{\varepsilon} \sum_{n \in \mathbb{N}} \left|\gamma_n - \gamma_i\right| < +\infty, 
$$

(3.15)

which yields (2.4).

(ii) For every $y \in \mathcal{H}_i$ such that $\|y\| \leq \rho$ and every $n \in \mathbb{N}$, Lemma 3.6(iv) and (3.13) yield

$$
\|J_{\gamma_n (\mu_i A_i)} y - J_{\gamma_i A_i} y\| \leq \frac{2\mu_i}{\gamma_n + \mu_i} \|J_{\gamma_n A_i} y - y\| \leq \frac{2\mu_i}{\varepsilon} \mu. 
$$

(3.16)

Consequently, (2.4) holds.

(iii) Set $\varrho = \max\{\rho + \|J_{\gamma A_i}0\|, (\rho + \|J_{\gamma A_i}0\|) / \gamma\}$ and let $E_{\varrho} \subset \mathcal{H}_i \times \mathcal{H}_i$ be as in Definition 3.5. It follows from [7] Proposition 1.2 that $E_{\varrho} \cap \text{gra} A_i \neq \emptyset$ and that

$$
(\forall n \in \mathbb{N}) \sup_{\|y\| \leq \rho} \|J_{\gamma A_i n} y - J_{\gamma A_i} y\| \leq (2 + \gamma) \text{haus}_{\varrho}(A_{i,n}, A_i). 
$$

(3.17)

Since, in view of (3.12), $\sum_{n \in \mathbb{N}} \text{haus}_{\varrho}(A_{i,n}, A_i) < +\infty$, we conclude that (2.4) holds.

4 Coupling schemes

The coupling between the $m$ inclusions in Problem 1.1 is determined by the operators $(B_i)_{1 \leq i \leq m}$, which must satisfy (1.6). In this section, we describe various situations in which this property holds. In each case, the value of $\beta$ in (1.6) will be specified, as it is explicitly required in Algorithm 2.2. In this connection, the notion of cocoercivity (see Definition 2.3) will play an important role. Examples of cocoercive operators include firmly nonexpansive operators (e.g., resolvents of maximal monotone operators, proximity operators, and projection operators onto nonempty closed convex sets). In addition, the Yosida approximation of a maximal monotone operator of index $\chi$ is $\chi$-cocoercive [4] (further examples of cocoercive operators can be found in [67]). It is clear from (2.5) that if $T$ is $\chi$-cocoercive, then it is $\chi^{-1}$-Lipschitz continuous. The next lemma, which provides a converse implication, supplies us with another important instance of cocoercive operator (see also [37]).
Lemma 4.1 [16, Corollaire 10] Let \( \varphi: \mathcal{H} \rightarrow \mathbb{R} \) be a differentiable convex function and let \( \tau \in [0, +\infty[ \). Suppose that \( \nabla \varphi \) is \( \tau \)-Lipschitz continuous. Then \( \nabla \varphi \) is \( \tau^{-1} \)-cocoercive.

Lemma 4.2 Let \( L \in \mathcal{B}(\mathcal{H}) \) be a nonzero self-adjoint operator such that \( (\forall x \in \mathcal{H}) \langle Lx \mid x \rangle \geq 0 \). Then \( L \) is \( \|L\|^{-1} \)-cocoercive.

Proof. Set \( \varphi: x \mapsto \langle Lx \mid x \rangle/2 \). Then \( \varphi \) is convex and differentiable, and its gradient \( \nabla \varphi: x \mapsto Lx \) is \( \|L\| \)-Lipschitz continuous. Hence, the assertion follows from Lemma 4.1. \( \Box \)

4.1 Linear coupling

We examine the case in which the operators \( (B_i)_{1 \leq i \leq m} \) are linear, which reduces (1.6) to

\[
(\forall(x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m) \quad \sum_{i=1}^{m} \langle B_i(x_1, \ldots, x_m) \mid x_i \rangle \geq \beta \sum_{i=1}^{m} \|B_i(x_1, \ldots, x_m)\|^2. \tag{4.1}
\]

We assume that, for every \( i \) and \( j \) in \( \{1, \ldots, m\} \), there exists \( M_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i) \) such that

\[
(\forall i \in \{1, \ldots, m\}) \quad B_i: \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \rightarrow \mathcal{H}_i: (x_j)_{1 \leq j \leq m} \mapsto \sum_{j=1}^{m} M_{ij}x_j. \tag{4.2}
\]

Thus, (4.1) is equivalent to

\[
(\forall(x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m) \quad \sum_{i=1}^{m} \sum_{j=1}^{m} \langle M_{ij}x_j \mid x_i \rangle \geq \beta \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} M_{ij}x_j \right\|^2. \tag{4.3}
\]

Our objective is to determine tight values of \( \beta \) for which this inequality holds in various scenarios. As in the proof of Theorem 2.6, it will be convenient to let \( \mathcal{H} \) be the direct Hilbert sum of the spaces \( (\mathcal{H}_i)_{1 \leq i \leq m} \) with the notation (2.7) and (2.8), and to set

\[
B: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto (B_i x)_{1 \leq i \leq m} = \left( \sum_{j=1}^{m} M_{ij}x_j \right)_{1 \leq i \leq m}. \tag{4.4}
\]

Proposition 4.3 Suppose that the following hold.

(i) \( (\exists (i, j) \in \{1, \ldots, m\}^2) \) \( M_{ij} \neq 0 \).

(ii) \( (\forall(i, j) \in \{1, \ldots, m\}^2) \) \( M_{ji} = M_{ij}^* \).

(iii) \( (\forall(x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m) \sum_{i=1}^{m} \sum_{j=1}^{m} \langle M_{ij}x_j \mid x_i \rangle \geq 0 \).

Then (4.3) is satisfied with \( \beta = 1/\|B\| \) and, a fortiori, with

\[
\beta = \frac{1}{\sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} \|M_{ij}\|^2}}. \tag{4.5}
\]
Proof. It follows from (1) that \( B \neq 0 \) and from (ii) that \( B^* = B \). In addition, (2.7) and (iii) imply that \( (\forall x \in \mathcal{H}) \langle Bx \mid x \rangle \geq 0 \). Hence, we derive from Lemma 4.2 that \( B \) is \( \|B\|^{-1} \)-cocoercive which, in view of (4.4), (2.7), and (2.8), can be expressed as

\[
(\forall (x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m) \sum_{i=1}^{m} \sum_{j=1}^{m} \langle M_{ij} x_j \mid x_i \rangle \geq \frac{1}{\|B\|} \sum_{i=1}^{m} \sum_{j=1}^{m} \|M_{ij} x_j\|^2 . \tag{4.6}
\]

Hence, (4.3) holds with \( \beta = 1/\|B\| \). Now take \( x \in \mathcal{H} \) such that \( \|x\| \leq 1 \). Then, (4.4) and Cauchy-Schwarz yield

\[
\|Bx\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \|M_{ij} x_j\|^2 \\
\leq \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \|M_{ij}\| \|x_j\| \right)^2 \\
\leq \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \|M_{ij}\|^2 \right) \left( \sum_{j=1}^{m} \|x_j\|^2 \right) \\
\leq \sum_{i=1}^{m} \sum_{j=1}^{m} \|M_{ij}\|^2 . \tag{4.7}
\]

Thus, \( \|B\|^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{m} \|M_{ij}\|^2 \) and it follows from (4.6) that (4.3) holds with (4.5). \( \square \)

In practice, one is interested in obtaining the tightest bound in (4.3). If \( \|B\| \) is known, one will use \( \beta = 1/\|B\| \) in Algorithm 2.2. This is for instance the case in the next proposition. In many situations, however, \( \|B\| \) will be hard to compute and one can use the value supplied by (4.5), which requires only knowledge of the norms of the individual operators \( (M_{ij})_{1 \leq i, j \leq m} \).

**Proposition 4.4** Let \( \Xi = [\xi_{ij}] \) be a nonzero real \( m \times m \) positive semidefinite symmetric matrix with largest eigenvalue \( \lambda_{\text{max}} \). Set

\[
(\forall i \in \{1, \ldots, m\}) \quad \mathcal{H}_i = \mathcal{H} \quad \text{and} \quad B_i : \mathcal{H}^m \to \mathcal{H} : (x_j)_{1 \leq j \leq m} \mapsto \sum_{j=1}^{m} \xi_{ij} x_j . \tag{4.8}
\]

Then (4.3) holds with \( \beta = 1/\lambda_{\text{max}} \).

**Proof.** Let \( \Lambda \) be the diagonal matrix with the diagonal entries of which are the eigenvalues \( \lambda_i \) of \( \Xi \). There exists an \( m \times m \) orthogonal matrix \( \Pi = [\pi_{ij}] \) such that \( \Xi = \Pi \Lambda \Pi^t \). Now set \( D : \mathcal{H} \to \mathcal{H} : x \mapsto (\lambda_i x_i)_{1 \leq i \leq m} \) and \( U : \mathcal{H} \to \mathcal{H} : x \mapsto \left( \sum_{j=1}^{m} \pi_{ij} x_j \right)_{1 \leq i \leq m} \). Then \( U \) is unitary and \( \|B\|^2 = \|U D U^*\|^2 = \|D\|^2 = \sup_{\|x\| \leq 1} \sum_{i=1}^{m} \lambda_i^2 \|x_i\|^2 = \lambda_{\text{max}}^2 \). Hence, the assertion follows from Proposition 4.3. \( \square \)

As shown next, equality can be achieved in (4.3).

**Example 4.5** Set

\[
(\forall i \in \{1, \ldots, m\}) \quad \mathcal{H}_i = \mathcal{H} \quad \text{and} \quad B_i : \mathcal{H}^m \to \mathcal{H} : (x_j)_{1 \leq j \leq m} \mapsto x_i - \frac{1}{m} \sum_{j=1}^{m} x_j . \tag{4.9}
\]

Then equality is achieved in (4.3) with \( \beta = 1 \).
Proof. Let \((x_1, \ldots, x_m) \in \mathcal{H}^m\). Then
\[
\sum_{i=1}^m \left< x_i - \frac{1}{m} \sum_{j=1}^m x_j \right| x_i \right> = \frac{1}{m} \left< \sum_{j=1}^m x_j \right|^2 + \sum_{i=1}^m \left< x_i - \frac{2}{m} \sum_{j=1}^m x_j \right| x_i \right>
= \sum_{i=1}^m \left< x_i - \frac{1}{m} \sum_{j=1}^m x_j \right| ^2,
\]
(4.10)
which provides the announced identity. □

Our last example concerns a specific structure of the operators \((M_{ij})_{1 \leq i, j \leq m}\).

**Proposition 4.6** For every \(k \in \{1, \ldots, p\}\), let \(\mathcal{G}_k\) be a real Hilbert space and, for every \(i \in \{1, \ldots, m\}\), let \(L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k)\). Assume that \(\min_{1 \leq k \leq p} \sum_{i=1}^m \|L_{ki}\|^2 > 0\) and set
\[(\forall (i, j) \in \{1, \ldots, m\}^2) \quad M_{ij} = \sum_{k=1}^p L_{ki}^* L_{kj}\]
in (4.2). Then (4.3) holds with
\[
\beta = \frac{1}{\sum_{k=1}^p \sum_{i=1}^m \|L_{ki}\|^2}.\]

Proof. For every \(i\) and \(j\) in \(\{1, \ldots, m\}\), (4.11) and Cauchy-Schwarz yield
\[
\|M_{ij}\|^2 = \left\| \sum_{k=1}^p L_{ki}^* L_{kj} \right\|^2 \leq \left( \sum_{k=1}^p \|L_{ki}\| \|L_{kj}\| \right)^2 \leq \left( \sum_{k=1}^p \|L_{ki}\|^2 \right) \left( \sum_{k=1}^p \|L_{kj}\|^2 \right).
\]
Consequently,
\[
\sum_{i=1}^m \sum_{j=1}^m \|M_{ij}\|^2 \leq \left( \sum_{k=1}^p \sum_{i=1}^m \|L_{ki}\|^2 \right)^2 \left( \sum_{k=1}^p \sum_{j=1}^m \|L_{kj}\|^2 \right)^2 \leq \left( \sum_{k=1}^p \sum_{i=1}^m \|L_{ki}\|^2 \right)^2 \left( \sum_{k=1}^p \sum_{j=1}^m \|L_{kj}\|^2 \right)^2.
\]
(4.14)
On the other hand, it follows from (4.11) that conditions (i)–(iii) in Proposition 4.3 are satisfied. Therefore, we derive from Proposition 4.3 that (4.3) holds with \(\beta\) as defined in (4.12). □

**Remark 4.7** For every \(i \in \{1, \ldots, m\}\) and \(n \in \mathbb{N}\), suppose that \(B_{i,n} \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\) in Algorithm 2.2 say
\[
B_{i,n} : \mathcal{H} \rightarrow \mathcal{H}_i; (x_j)_{1 \leq j \leq m} \mapsto \sum_{j=1}^m M_{ij,n} x_j, \text{ where } (\forall j \in \{1, \ldots, m\}) \ M_{ij,n} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i).\]
(4.15)
Then assumption (ii)(b) in Algorithm 2.2 is satisfied with \(z = 0\). In addition, suppose that
\[
\max_{1 \leq i \leq m} \sum_{n \in \mathbb{N}} \left\| \sum_{j=1}^m \|M_{ij,n} - M_{ij}\|^2 < +\infty.\right\|
\]
(4.16)
Then assumption (ii)(a) in Algorithm 2.2 is satisfied. Indeed, let \(x \in \mathcal{H}, i \in \{1, \ldots, m\},\) and \(n \in \mathbb{N},\) and set \(\kappa_{i,n} = \sqrt{\sum_{j=1}^m \|M_{ij,n} - M_{ij}\|^2}.\) Then, by Cauchy-Schwarz,
\[
\|(B_{i,n} - B_i)x\| = \left\| \sum_{j=1}^m (M_{ij,n} - M_{ij}) x_j \right\| \leq \sum_{j=1}^m \|M_{ij,n} - M_{ij}\| \|x_j\| \leq \kappa_{i,n} \||x||,\]
(4.17)
where (4.16) yields \(\sum_{n \in \mathbb{N}} \kappa_{i,n} < +\infty,\) as desired.
4.2 Nonlinear coupling

In this section we turn our attention to the determination of the parameter \( \beta \) in (1.6) when the operators \((B_i)_{1 \leq i \leq m}\) are nonlinear. Our first model is a nonlinear version of Proposition 4.6.

**Proposition 4.8** For every \( k \in \{1, \ldots, p\} \), let \( \mathcal{G}_k \) be a real Hilbert space, let \( \beta_k \in ]0, +\infty[ \), let \( T_k: \mathcal{G}_k \to \mathcal{G}_k \) be \( \beta_k \)-cocoercive, and, for every \( i \in \{1, \ldots, m\} \), let \( L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k) \). Assume that \( \min_{1 \leq k \leq p} \sum_{i=1}^{m} \|L_{ki}\|^2 > 0 \) and set

\[
(\forall i \in \{1, \ldots, m\}) \quad B_i: \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to \mathcal{H}_i: (x_j)_{1 \leq j \leq m} \mapsto \sum_{k=1}^{p} L_{ki}^* T_k \left( \sum_{j=1}^{m} L_{kj} x_j \right).
\]

Then (1.6) holds with

\[
\beta = \frac{1}{p} \min_{1 \leq k \leq p} \frac{\beta_k}{\sum_{i=1}^{m} \|L_{ki}\|^2}.
\]

**Proof.** For every \( i \in \{1, \ldots, m\} \), let \( x_i \) and \( y_i \) be points in \( \mathcal{H}_i \). It follows from (4.18), (4.19), and the convexity of \( \|\cdot\|^2 \) that

\[
\sum_{i=1}^{m} \langle B_i(x_1, \ldots, x_m) - B_i(y_1, \ldots, y_m) \mid x_i - y_i \rangle \\
= \sum_{i=1}^{m} \sum_{k=1}^{p} \langle L_{ki}^* (T_k \left( \sum_{j=1}^{m} L_{kj} x_j \right) - T_k \left( \sum_{j=1}^{m} L_{kj} y_j \right)) \mid x_i - y_i \rangle \\
= \sum_{i=1}^{m} \sum_{k=1}^{p} \langle T_k \left( \sum_{j=1}^{m} L_{kj} x_j \right) - T_k \left( \sum_{j=1}^{m} L_{kj} y_j \right) \mid L_{ki} (x_i - y_i) \rangle \\
= \sum_{k=1}^{p} \beta_k \|T_k \left( \sum_{j=1}^{m} L_{kj} x_j \right) - T_k \left( \sum_{j=1}^{m} L_{kj} y_j \right)\|^2 \\
\geq \beta \sum_{k=1}^{p} \sum_{i=1}^{m} L_{ki}^* \left( T_k \left( \sum_{j=1}^{m} L_{kj} x_j \right) - T_k \left( \sum_{j=1}^{m} L_{kj} y_j \right) \right) \|^2,
\]

which establishes the inequality. \( \square \)

**Remark 4.9** Suppose that \( T_k \equiv \text{Id} \) in Proposition 4.8. Then the operators \((B_i)_{1 \leq i \leq m}\) of (4.18) are simply those resulting from Proposition 4.6. However, since \( \beta_k \equiv 1 \), the bound given in (4.12) is sharper than that given in (4.19).
Corollary 4.10 For every $k \in \{1, \ldots, p\}$, let $G_k$ be a real Hilbert space, let $\tau_k \in [0, +\infty[$, let $\varphi_k: G_k \to \mathbb{R}$ be a $\tau_k$-Lipschitz-differentiable convex function, and, for every $i \in \{1, \ldots, m\}$, let $L_{ki} \in \mathcal{B}(H_i, G_k)$. Assume that $\min_{1 \leq k \leq p} \sum_{i=1}^{m} \|L_{ki}\|^2 > 0$ and set

$$(\forall i \in \{1, \ldots, m\}) \quad B_i: \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to \mathcal{H}_i: (x_j)_{1 \leq j \leq m} \mapsto \sum_{k=1}^{p} L_{ki}^* \nabla \varphi_k \left( \sum_{j=1}^{m} L_{kj} x_j \right). \quad (4.21)$$

Then $(B_i)_{1 \leq i \leq m}$ satisfies (1.6) with

$$\beta = \frac{1}{\max_{1 \leq k \leq p} \tau_k \sum_{i=1}^{m} \|L_{ki}\|^2}. \quad (4.22)$$

Proof. Lemma 4.1 asserts that, for every $k \in \{1, \ldots, p\}$, $T_k = \nabla \varphi_k$ is $\tau_k^{-1}$-cocoercive. The result therefore follows from Proposition 4.3. $\square$

Example 4.11 (saddle point problems) For every $k \in \{1, \ldots, p\}$ and $l \in \{1, \ldots, q\}$, let $G_k$ and $\mathcal{K}_l$ be real Hilbert spaces, let $\tau_k \in [0, +\infty[$, let $\kappa_l \in [0, +\infty[$, let $\varphi_k: G_k \to \mathbb{R}$ be a $\tau_k$-Lipschitz-differentiable convex function, let $\psi_l: \mathcal{K}_l \to \mathbb{R}$ be a $\kappa_l$-Lipschitz-differentiable convex function. Furthermore, for every $i \in \{1, \ldots, m\}$, let $\mathcal{Y}_i$ and $\mathcal{Z}_i$ be real Hilbert spaces, let $F_i: \mathcal{Y}_i \oplus \mathcal{Z}_i \to [-\infty, +\infty]$ satisfy (1) and (ii) in Example 3.3, let $L_{ki} \in \mathcal{B}(\mathcal{Z}_i, G_k)$ and $M_{li} \in \mathcal{B}(\mathcal{Y}_i, \mathcal{K}_l)$. It is assumed that $\min_{1 \leq k \leq p} \sum_{i=1}^{m} \|L_{ki}\|^2 > 0$ and that $\min_{1 \leq l \leq q} \sum_{i=1}^{m} \|M_{li}\|^2 > 0$. Consider the problem

$$\max_{y_1, \ldots, y_m \in \mathcal{Y}_m} \min_{z_1, \ldots, z_m \in \mathcal{Z}_m} \sum_{i=1}^{m} F_i(y_i, z_i) - \sum_{l=1}^{q} \psi_l \left( \sum_{i=1}^{m} M_{li} y_i \right) + \sum_{k=1}^{p} \varphi_k \left( \sum_{i=1}^{m} L_{ki} z_i \right). \quad (4.23)$$

Now set

$$B_{i1}: \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m \to \mathcal{Y}_i: (y_j)_{1 \leq j \leq m} \mapsto \sum_{j=1}^{q} M_{lj}^* \psi_l \left( \sum_{l=1}^{m} M_{lj} y_j \right)$$

and

$$B_{i2}: \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_m \to \mathcal{Z}_i: (z_j)_{1 \leq j \leq m} \mapsto \sum_{k=1}^{p} L_{ki}^* \varphi_k \left( \sum_{j=1}^{m} L_{kj} z_j \right), \quad (4.24)$$

and

$$\beta_1 = \frac{1}{\max_{1 \leq l \leq q} \kappa_l \sum_{j=1}^{m} \|M_{lj}\|^2} \quad \text{and} \quad \beta_2 = \frac{1}{\max_{1 \leq k \leq p} \tau_k \sum_{j=1}^{m} \|L_{kj}\|^2}. \quad (4.25)$$

We derive from Corollary 4.10 that, for every $(y_1, \ldots, y_m)$ and $(\overline{y}_1, \ldots, \overline{y}_m)$ in $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$,

$$\sum_{i=1}^{m} \left\langle B_{i1}(y_1, \ldots, y_m) - B_{i1}(\overline{y}_1, \ldots, \overline{y}_m) \mid y_i - \overline{y}_i \right\rangle_{\mathcal{Y}_i} \geq \beta_1 \sum_{i=1}^{m} \left\| B_{i1}(y_1, \ldots, y_m) - B_{i1}(\overline{y}_1, \ldots, \overline{y}_m) \right\|^2_{\mathcal{Y}_i} \quad (4.26)$$

and that an analogous inequality is satisfied by $B_{i2}$ with $\beta_2$. On the other hand, using minimax theory [57], we can cast (4.23) in the form of (3.3) where, for every $i \in \{1, \ldots, m\}$, $H_i = \mathcal{Y}_i \oplus \mathcal{Z}_i$ and

$$B_i = (B_{i1}, B_{i2}): (y_j, z_j)_{1 \leq j \leq m} \mapsto (\overline{B}_{i1}(y_1, \ldots, y_m), \overline{B}_{i2}(z_1, \ldots, z_m)). \quad (4.27)$$

Altogether, it follows from Example 3.3 that (4.23) is a special case of Problem 1.1 in which $(B_i)_{1 \leq i \leq m}$ satisfies (1.6) with $\beta = \min\{\beta_1, \beta_2\}$. 

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5 Coupling evolution inclusions

Evolution inclusions arise in various fields of applied mathematics [41, 60]. In this section, we address the problem of solving systems of coupled evolution inclusions with periodicity conditions. The notation and definitions introduced in Example 3.4 will be used.

5.1 Problem formulation and algorithm

Problem 5.1 Let \((H_i)_{1 \leq i \leq m}\) be real Hilbert spaces and let \(T \in [0, +\infty]\). For every \(i \in \{1, \ldots, m\}\), set

\[
W_i = \{ x \in C([0, T]; H_i) \cap W^{1,2}([0, T]; H_i) \mid x(T) = x(0) \},
\]

let \(f_i \in \Gamma_0(H_i)\), and let \(B_i : H_1 \times \cdots \times H_m \to H_i\). It is assumed that there exists \(\beta \in ]0, +\infty[\) such that

\[
\forall (x_1, \ldots, x_m) \in H_1 \times \cdots \times H_m, (\forall (y_1, \ldots, y_m) \in H_1 \times \cdots \times H_m) \sum_{i=1}^m (B_i(x_1, \ldots, x_m) - B_i(y_1, \ldots, y_m)) \mid x_i - y_i \rvert_{H_i} \geq \beta \sum_{i=1}^m \|B_i(x_1, \ldots, x_m) - B_i(y_1, \ldots, y_m)\|^2_{H_i}.
\]

The problem is to find \(x_1, \ldots, x_m \in W_m\) such that

\[
\forall i \in \{1, \ldots, m\} \quad 0 \in x'_i(t) + \partial f_i(x_i(t)) + B_i(x_1(t), \ldots, x_m(t)) \text{ a.e. on } ]0, T[,
\]

under the assumption that such solutions exist.

Algorithm 5.2 Fix \(\varepsilon \in ]0, \min\{1, \beta\}\), \((\gamma_n)_{n \in \mathbb{N}}\) in \([\varepsilon, 2\beta - \varepsilon]\), and \((\lambda_n)_{n \in \mathbb{N}}\) in \([0, 1 - \varepsilon]\). Let, for every \(n \in \mathbb{N}\) and every \(i \in \{1, \ldots, m\}\), \(y_{i,n}\) be the unique solution in \(W_i\) to the inclusion

\[
\frac{x_{i,n}(t) - y_{i,n}(t)}{\gamma_n} - (B_i(x_{1,n}(t), \ldots, x_{m,n}(t)) + b_{i,n}(t))
\]

\[
\in y_{i,n}(t) + \partial f_i(y_{i,n}(t)) + e_{i,n}(t) \text{ a.e. on } ]0, T[, \quad \text{(5.4)}
\]

and set

\[
x_{i,n+1} = \lambda_{i,n} x_{i,n} + (1 - \lambda_{i,n}) y_{i,n} \quad \text{(5.5)}
\]

where, for every \(i \in \{1, \ldots, m\}\), the following hold.

(i) \(x_{i,0} \in W^{1,2}([0, T]; H_i)\).

(ii) \((b_{i,n})_{n \in \mathbb{N}}\) and \((e_{i,n})_{n \in \mathbb{N}}\) are sequences in \(L^2([0, T]; H_i)\) such that

\[
\sum_{n \in \mathbb{N}} \sqrt{\int_0^T \|b_{i,n}(t)\|^2_{H_i} dt} < +\infty \text{ and } \sum_{n \in \mathbb{N}} \sqrt{\int_0^T \|e_{i,n}(t)\|^2_{H_i} dt} < +\infty. \quad \text{(5.6)}
\]

(iii) \((\lambda_{i,n})_{n \in \mathbb{N}}\) is a sequence in \([0, 1]\) such that \(\sum_{n \in \mathbb{N}} |\lambda_{i,n} - \lambda_n| < +\infty\).

In (5.4), \(b_{i,n}(t)\) models the error tolerated in the computation of \(B_i(x_{1,n}(t), \ldots, x_{m,n}(t))\), while \(e_{i,n}(t)\) models the error tolerated in solving the inclusion with respect to \(\partial f_i(y_{i,n}(t))\).
5.2 Convergence

Theorem 5.3 Let \((x_{i,n})_{n \in \mathbb{N}}\) be sequences generated by Algorithm 5.2. Then, for every \(i \in \{1, \ldots, m\}\), \((x_{i,n})_{n \in \mathbb{N}}\) converges weakly in \(W^{1,2}([0,T];H_i)\) to a point \(x_i \in W_i\), and \((x_i)_{1 \leq i \leq m}\) is a solution to Problem 5.1.

Proof. For every \(i \in \{1, \ldots, m\}\), set \(H_i = L^2([0,T];H_i)\) and

\[
A_i : H_i \rightarrow 2^{H_i},
\]

\[
x \mapsto \begin{cases} u \in H_i | u(t) \in x'(t) + \partial f_i(x(t)) \text{ a.e. in } [0,T] \}, & \text{if } x \in W_i; \\
\varnothing, & \text{otherwise.}
\end{cases}
\]

Let us first show that the operators \((A_i)_{1 \leq i \leq m}\) are maximal monotone. For this purpose, let \(i \in \{1, \ldots, m\}\), \((x,u) \in \text{gra} A_i\), and \((y,v) \in \text{gra} A_i\). It follows from (5.7) that, almost everywhere on \([0,T], u(t) - x'(t) \in \partial f_i(x(t))\) and \(v(t) - y'(t) \in \partial f_i(y(t))\). Therefore, by monotonicity of \(\partial f_i\), we have

\[
\int_0^T \langle (u(t) - x'(t)) - (v(t) - y'(t)) \mid x(t) - y(t) \rangle_{H_i} dt \geq 0.
\]

Hence,

\[
\langle u - v \mid x - y \rangle = \int_0^T \langle u(t) - v(t) \mid x(t) - y(t) \rangle_{H_i} dt
\]

\[
= \int_0^T \langle (u(t) - x'(t)) - (v(t) - y'(t)) \mid x(t) - y(t) \rangle_{H_i} dt
\]

\[
+ \int_0^T \langle x'(t) - y'(t) \mid x(t) - y(t) \rangle_{H_i} dt
\]

\[
\geq \frac{1}{2} \int_0^T \frac{d\|x(t) - y(t)\|_{H_i}^2}{dt} dt
\]

\[
= \frac{1}{2} (\|x(T) - y(T)\|_{H_i}^2 - \|x(0) - y(0)\|_{H_i}^2)
\]

\[
= 0.
\]

Thus, \(A_i\) is monotone. To prove maximality, set \(g_i = (1/2)\| \cdot \|_{H_i}^2 + f_i\). Then \(g_i \in \Gamma_0(H_i)\) and \(\partial g_i = \text{Id} + \partial f_i\). Moreover, since \(f_i \in \Gamma_0(H_i)\), it is minorized by a continuous affine functional, say \(f_i \geq \langle \cdot \mid v \rangle_{H_i} + \eta\) for some \(v \in H_i\) and \(\eta \in \mathbb{R}\). Now, let \(y \in \text{dom} f_i = \text{dom} g_i\) and take \((x,u) \in \text{gra} \partial g_i\). Then (5.10) and Cauchy-Schwarz imply the coercivity property

\[
\langle x - y \mid u \rangle_{H_i} \geq \frac{g_i(x) - g_i(y)}{\|x\|_{H_i}}
\]

\[
= \frac{\|x\|_{H_i} f_i(x) - g_i(y)}{\|x\|_{H_i}}
\]

\[
\geq \frac{\|x\|_{H_i}}{2} - \|v\|_{H_i} + \frac{\eta - g_i(y)}{\|x\|_{H_i}}
\]

\[
\rightarrow +\infty \text{ as } \|x\|_{H_i} \rightarrow +\infty.
\]

(5.10)
Therefore, [25, Corollaire 3.4] asserts that for every \( w \in \mathcal{H}_i \) there exists \( z \in \mathcal{W}_i \) such that
\[
w(t) \in z'(t) + \partial g_i(z(t)) = z'(t) + z(t) + \partial f_i(z(t)) \quad \text{a.e. on } [0, T],
\]
i.e., by (5.7), such that \( w - z \in A_i \). This shows that the range of \( \text{Id} + A_i \) is \( \mathcal{H}_i \) and hence, by Minty’s theorem \[11, Theorem 3.5.8\], that \( A_i \) is maximal monotone.

Next, for every \( i \in \{1, \ldots, m\} \) and every \( (x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \), define almost everywhere
\[
B_i(x_1, \ldots, x_m) : [0, T] \to \mathcal{H}_i
\]
\[
t \mapsto B_i(x_1(t), \ldots, x_m(t)).
\]

Now let \( (x_1, \ldots, x_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \) and set \((\forall i \in \{1, \ldots, m\}) b_i = B_i(0, \ldots, 0)\). Then it follows from (5.2) and Cauchy-Schwarz that, almost everywhere on \([0, T]\),
\[
\beta \sum_{j=1}^m \| B_j(x_1(t), \ldots, x_m(t)) - b_j \|_{\mathcal{H}_j}^2 \leq \sum_{j=1}^m \langle B_j(x_1(t), \ldots, x_m(t)) - b_j, x_j(t) - 0 \rangle_{\mathcal{H}_j}
\]
\[
\leq \sum_{j=1}^m \| B_j(x_1(t), \ldots, x_m(t)) - b_j \|_{\mathcal{H}_j} \| x_j(t) \|_{\mathcal{H}_j}
\]
\[
\leq \sqrt{\sum_{j=1}^m \| B_j(x_1(t), \ldots, x_m(t)) - b_j \|_{\mathcal{H}_j}^2} \sqrt{\sum_{j=1}^m \| x_j(t) \|_{\mathcal{H}_j}^2},
\]
(5.13)

Therefore, for every \( i \in \{1, \ldots, m\} \),
\[
\| B_i(x_1, \ldots, x_m)(t) \|_{\mathcal{H}_i}^2 \leq 2\left( \| b_i \|_{\mathcal{H}_i}^2 + \| B_i(x_1, \ldots, x_m)(t) - b_i \|_{\mathcal{H}_i}^2 \right)
\]
\[
\leq 2\left( \| b_i \|_{\mathcal{H}_i}^2 + \sum_{j=1}^m \| B_j(x_1(t), \ldots, x_m(t)) - b_j \|_{\mathcal{H}_j}^2 \right)
\]
\[
\leq 2\left( \| b_i \|_{\mathcal{H}_i}^2 + \frac{1}{\beta^2} \sum_{j=1}^m \| x_j(t) \|_{\mathcal{H}_j}^2 \right) \quad \text{a.e. on } [0, T],
\]
(5.14)

which yields
\[
\int_0^T \| B_i(x_1, \ldots, x_m)(t) \|_{\mathcal{H}_i}^2 dt \leq 2T \| b_i \|_{\mathcal{H}_i}^2 + \frac{2}{\beta^2} \sum_{j=1}^m \| x_j \|_{\mathcal{H}_j}^2,
\]
(5.15)

so that we can now claim that \( B_i : \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to L^2([0, T]; \mathcal{H}_i) = \mathcal{H}_i \). In addition, upon integrating, we derive from (5.2) and (5.12) that, for every \( (y_1, \ldots, y_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \),
\[
\sum_{i=1}^m (B_i(x_1, \ldots, x_m) - B_i(y_1, \ldots, y_m)) | x_i - y_i | \geq \beta \sum_{i=1}^m \| B_i(x_1, \ldots, x_m) - B_i(y_1, \ldots, y_m) \|_2^2.
\]
(5.16)

We have thus established (1.6).

Let us now make the connection between Algorithm 5.2 and Algorithm 2.2. For every \( n \in \mathbb{N} \) and every \( i \in \{1, \ldots, m\} \), it follows from (5.4), (5.7), (5.12), and the maximal monotonicity of \( A_i \) that \( y_{i,n} \) is uniquely defined and can be expressed as
\[
y_{i,n} = J_{\gamma_n A_i} \left( x_{i,n} - \gamma_n (B_i(x_{1,n}, \ldots, x_{m,n}) + b_{i,n}) \right) + a_{i,n},
\]
(5.17)
As a result, we deduce from (5.18) that
\[ (5.19) \]
and we therefore derive from (5.4) and (5.5) that
\[ (5.19) \]
We observe that (5.19) derives from (2.3), where \( A_{i,n} \equiv A_i \) and \( B_{i,n} \equiv B_i \). On the other hand, for every \( i \in \{1, \ldots, m\} \), by nonexpansivity of the operators \((J_{\gamma_n} A_i)_{n \in \mathbb{N}}\), we deduce from (5.18) and (5.6) that
\[ (5.20) \]
As a result, all the hypotheses of Algorithm 2.2 are satisfied and hence Theorem 2.6 asserts that, for every \( i \in \{1, \ldots, m\} \), by nonexpansivity of the operators \((J_{\gamma_n} A_i)_{n \in \mathbb{N}}\), we deduce from (5.18) and (5.5) that
\[ (5.20) \]
and the (5.19) derives from (5.2) are satisfied and hence Theorem 2.6 asserts that, for every \( i \in \{1, \ldots, m\} \), (\( x_{i,n}\))\( n \in \mathbb{N} \) converges weakly in \( H_i = L^2([0, T]; H_i) \) to a point \( x_i \), and \( (x_i)_{1 \leq i \leq m} \) satisfies
\[ (5.21) \]
Accordingly,
\[ (5.22) \]
and (5.19) derives from (2.3), where \( A_{i,n} = A_i \) and \( B_{i,n} = B_i \). On the other hand, for every \( i \in \{1, \ldots, m\} \), by nonexpansivity of the operators \((J_{\gamma_n} A_i)_{n \in \mathbb{N}}\), we deduce from (5.18) and (5.6) that
\[ (5.20) \]
As a result, all the hypotheses of Algorithm 2.2 are satisfied and hence Theorem 2.6 asserts that, for every \( i \in \{1, \ldots, m\} \), (\( x_{i,n}\))\( n \in \mathbb{N} \) converges weakly to \( x_i \) in \( \mathcal{W}_i = L^1, 2([0, T]; H_i) \) to a point \( x_i \), and \( (x_i)_{1 \leq i \leq m} \) satisfies
\[ (5.21) \]
Accordingly,
\[ (5.22) \]
and the (5.19) derives from (5.2) are satisfied and hence Theorem 2.6 asserts that, for every \( i \in \{1, \ldots, m\} \), (\( x_{i,n}\))\( n \in \mathbb{N} \) converges weakly to \( x_i \) in \( \mathcal{W}_i = L^1, 2([0, T]; H_i) \) to a point \( x_i \), and \( (x_i)_{1 \leq i \leq m} \) satisfies
\[ (5.21) \]
Accordingly,
\[ (5.22) \]
and the (5.19) derives from (5.2) are satisfied and hence Theorem 2.6 asserts that, for every \( i \in \{1, \ldots, m\} \), (\( x_{i,n}\))\( n \in \mathbb{N} \) converges weakly to \( x_i \) in \( \mathcal{W}_i = L^1, 2([0, T]; H_i) \) to a point \( x_i \), and \( (x_i)_{1 \leq i \leq m} \) satisfies
\[ (5.21) \]
Accordingly,
\[ (5.22) \]
and the (5.19) derives from (5.2) are satisfied and hence Theorem 2.6 asserts that, for every \( i \in \{1, \ldots, m\} \), (\( x_{i,n}\))\( n \in \mathbb{N} \) converges weakly to \( x_i \) in \( \mathcal{W}_i = L^1, 2([0, T]; H_i) \) to a point \( x_i \), and \( (x_i)_{1 \leq i \leq m} \) satisfies
\[ (5.21) \]
and, furthermore,
\[
\int_0^T \langle y_{i,n}(t), y'_{i,n}(t) \rangle_{H_i} dt = \frac{1}{2} \int_0^T d\|y_{i,n}(t)\|^2_{H_i} dt = \frac{\|y_{i,n}(T)\|^2_{H_i} - \|y_{i,n}(0)\|^2_{H_i}}{2} = 0. \tag{5.28}
\]

We deduce from (5.27), (5.24), and (5.28) that
\[
0 = \int_0^T \langle x_{i,n}(t), y'_{i,n}(t) \rangle_{H_i} dt - \int_0^T \langle B_i(x_{1,n}(t), \ldots, x_{m,n}(t)), y'_{i,n}(t) \rangle_{H_i} dt \\
- \int_0^T \|y'_{i,n}(t)\|^2_{H_i} dt - \int_0^T \|e_{i,n}(t)\|_{H_i} dt. \tag{5.29}
\]

Thus, using Cauchy-Schwarz, the inequality \( \gamma_n \geq \varepsilon \), and (5.12), we obtain
\[
\|y'_{i,n}\|^2 \leq \left( \frac{1}{\varepsilon} \|x_{i,n}\| + \|B_i(x_{1,n}, \ldots, x_{m,n})\| + \|b_{i,n}\| + \|e_{i,n}\| \right) \|y'_{i,n}\|. \tag{5.30}
\]

In turn, it follows from (5.5) that
\[
\|x'_{i,n+1}\| \leq \lambda_i \|x'_{i,n}\| + (1 - \lambda_i) \left( \frac{1}{\varepsilon} \|x_{i,n}\| + \|B_i(x_{1,n}, \ldots, x_{m,n})\| + \|b_{i,n}\| + \|e_{i,n}\| \right). \tag{5.31}
\]

On the other hand, arguing as in (5.15), we derive from (5.22) that
\[
\|B_i(x_{1,n}, \ldots, x_{m,n})\| \leq \sqrt{2T} \|b_i\|^2_{H_i} + \frac{2m\sigma^2}{\beta^2} \leq \sqrt{2T} \|b_i\|_{H_i} + \sqrt{2m} \frac{\sigma}{\beta}. \tag{5.32}
\]

Hence, using (ii) in Algorithm 5.2 we derive by induction from (5.31) that
\[
\|x'_{i,n}\| \leq \max \left\{ \|x'_{i,0}\|, \frac{\sigma}{\varepsilon} + \sqrt{2T} \|b_i\|_{H_i} + \sqrt{2m} \frac{\sigma}{\beta} + \sup_{k \in \mathbb{N}} \left( \|b_{i,k}\| + \|e_{i,k}\| \right) \right\}. \tag{5.33}
\]

This shows the boundedness of \((x'_{i,n})_{n \in \mathbb{N}}\) in \(L^2([0, T]; H_i)\). Now let \(z\) be the weak limit in \(L^2([0, T]; H_i)\) of an arbitrary weakly convergent subsequence of \((x'_{i,n})_{n \in \mathbb{N}}\). Since \((x_{i,n})_{n \in \mathbb{N}}\) converges weakly in \(L^2([0, T]; H_i)\) to \(x_i\), it therefore follows from [66, Proposition 23.19] that \(z = x'\). In turn, this shows that \((x'_{i,n})_{n \in \mathbb{N}}\) converges weakly in \(L^2([0, T]; H_i)\) to \(x'\). \(\square\)

## 6 The variational case

In this section, we study a special case of Problem 1.1 which yields a variational formulation that extends (1.5). This framework can be regarded as a particular instance of Example 4.11.

### 6.1 Problem formulation and algorithm

Recall that, for every \(f \in \Gamma_0(\mathcal{H})\) and every \(x \in \mathcal{H}\), the function \(y \mapsto f(y) + \|x - y\|^2/2\) admits a unique minimizer, which is denoted by \(\text{prox}_f x\). The proximity operator thus defined can be expressed as \(\text{prox}_f = J_{\beta f} \) [51].
Problem 6.1 Let \( (\mathcal{H}_i)_{1 \leq i \leq m} \) and \( (\mathcal{G}_k)_{1 \leq k \leq p} \) be real Hilbert spaces. For every \( i \in \{1, \ldots, m\} \), let \( f_i \in \Gamma_0(\mathcal{H}_i) \) and, for every \( k \in \{1, \ldots, p\} \), let \( \tau_k \in [0, +\infty[ \), let \( \varphi_k : \mathcal{G}_k \to \mathbb{R} \) be a \( \tau_k \)-Lipschitz-differentiable convex function, and let \( L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k) \). It is assumed that \( \min_{1 \leq k \leq p} \sum_{i=1}^m \|L_{ki}\|^2 > 0 \). The problem is to

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^p \varphi_k \left( \sum_{i=1}^m L_{ki}x_i \right), \\
\text{subject to} & \quad x_i \in \mathcal{H}_i, \quad i = 1, \ldots, m.
\end{align*}
\]

under the assumption that solutions exist.

Algorithm 6.2 Set

\[
\beta = \frac{1}{\max_{1 \leq k \leq p} \tau_k \sum_{i=1}^m \|L_{ki}\|^2}.
\]

Fix \( \varepsilon \in ]0, \min\{1, \beta\}[ \), \( (\gamma_n)_{n \in \mathbb{N}} \) in \( [\varepsilon, 2\beta - \varepsilon[ \), \( (\lambda_n)_{n \in \mathbb{N}} \) in \( [0, 1 - \varepsilon[ \), and \( (x_i,0)_{1 \leq i \leq m} \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \). Set, for every \( n \in \mathbb{N} \),

\[
\begin{align*}
x_{1,n+1} &= \lambda_{1,n} x_{1,n} + (1 - \lambda_{1,n}) \left( \operatorname{prox}_{\gamma_n f_1,n} \left( x_{1,n} - \gamma_n \left( \sum_{k=1}^p L_{k1}^* \nabla \varphi_k \left( \sum_{j=1}^m L_{kj} x_{j,n} \right) + b_{1,n} \right) \right) + a_{1,n} \right), \\
&\vdots \\
x_{m,n+1} &= \lambda_{m,n} x_{m,n} + (1 - \lambda_{m,n}) \left( \operatorname{prox}_{\gamma_n f_m,n} \left( x_{m,n} - \gamma_n \left( \sum_{k=1}^p L_{km}^* \nabla \varphi_k \left( \sum_{j=1}^m L_{kj} x_{j,n} \right) + b_{m,n} \right) \right) + a_{m,n} \right),
\end{align*}
\]

where, for every \( i \in \{1, \ldots, m\} \), the following hold.

(i) \( (f_{i,n})_{n \in \mathbb{N}} \) are functions in \( \Gamma_0(\mathcal{H}_i) \) such that

\[
(\forall \rho \in ]0, +\infty[) \sum_{n \in \mathbb{N}} \sup_{\|y\| \leq \rho} \| \operatorname{prox}_{\gamma_n f_i,n} y - \operatorname{prox}_{\gamma_n f_i} y \| < +\infty. \tag{6.4}
\]

(ii) \( (a_{i,n})_{n \in \mathbb{N}} \) and \( (b_{i,n})_{n \in \mathbb{N}} \) are sequences in \( \mathcal{H}_i \) such that \( \sum_{n \in \mathbb{N}} \|a_{i,n}\| < +\infty \) and \( \sum_{n \in \mathbb{N}} \|b_{i,n}\| < +\infty \).

(iii) \( (\lambda_{i,n})_{n \in \mathbb{N}} \) is a sequence in \( [0, 1[ \) such that \( \sum_{n \in \mathbb{N}} |\lambda_{i,n} - \lambda_{i}| < +\infty \).

Theorem 6.3 Let \( ((x_{i,n})_{n \in \mathbb{N}})_{1 \leq i \leq m} \) be sequences generated by Algorithm 6.2 Then, for every \( i \in \{1, \ldots, m\} \), \( (x_{i,n})_{n \in \mathbb{N}} \) converges weakly to a point \( x_i \in \mathcal{H}_i \), and \( (x_{i})_{1 \leq i \leq m} \) is a solution to Problem 6.1.

Proof. Problem 6.1 is a special case of Problem 1.1 where, for every \( i \in \{1, \ldots, m\} \),

\[
A_i = \partial f_i \quad \text{and} \quad B_i : (x_j)_{1 \leq j \leq m} \mapsto \sum_{k=1}^p L_{ki}^* \nabla \varphi_k \left( \sum_{j=1}^m L_{kj} x_j \right). \tag{6.5}
\]
Indeed, define $H$ as in the proof of Theorem 2.6 and set

$$f: H \to ]-\infty, +\infty[: (x_i)_{1 \leq i \leq m} \mapsto \sum_{i=1}^{m} f_i(x_i) \quad (6.6)$$

and

$$g: H \to \mathbb{R}: (x_i)_{1 \leq i \leq m} \mapsto \sum_{k=1}^{p} \varphi_k \left( \sum_{i=1}^{m} L_{ki} x_i \right). \quad (6.7)$$

Then $f$ and $g$ are in $\Gamma_0(H)$ and it follows from Fermat’s rule and elementary subdifferential calculus that, for every $(x_1, \ldots, x_m) \in H$,

$$(x_1, \ldots, x_m) \text{ solves } (6.1) \iff (0, \ldots, 0) \in \partial (f + g)(x_1, \ldots, x_m)$$

$$\iff (0, \ldots, 0) \in \partial f(x_1, \ldots, x_m) + \nabla g(x_1, \ldots, x_m)$$

$$\iff (\forall i \in \{1, \ldots, m\}) \ 0 \in \partial f_i(x_i) + \sum_{k=1}^{p} L_{ki} \nabla \varphi_k \left( \sum_{j=1}^{m} L_{kj} x_j \right)$$

$$\iff (\forall i \in \{1, \ldots, m\}) \ 0 \in A_i x_i + B_i(x_1, \ldots, x_m). \quad (6.8)$$

In addition, Lemma 4.1 asserts that, for every $k \in \{1, \ldots, p\}$, $\nabla \varphi_k$ is $\tau_k^{-1}$-cocoercive. In turn, we derive from Corollary 4.10 that the family $(B_i)_{1 \leq i \leq m}$ in (6.5) satisfies (1.6) with $\beta$ as in (6.2). Setting

$$(\forall i \in \{1, \ldots, m\})(\forall n \in \mathbb{N}) \ A_{i,n} = \partial f_{i,n} \quad \text{and} \quad B_{i,n} = B_i, \quad (6.9)$$

we deduce from (6.4) that Algorithm 6.2 is a particular case of Algorithm 2.2. Altogether, Theorem 6.3 follows from Theorem 2.6.

### 6.2 Applications

Let us consider some applications of Theorem 6.3 starting with a game-theoretic interpretation of Problem 6.1.

**Example 6.4 (coordinated games)** Consider a game with $m$ players indexed by $i \in \{1, \ldots, m\}$. The strategy $x_i$ of the $i$th player lies in the real Hilbert space $\mathcal{H}_i$ and his individual utility is modeled by a proper upper semicontinuous concave function $h_i: \mathcal{H}_i \to ]-\infty, +\infty[$. In the absence of coordination, the goal of each player is to maximize his own payoff, which can be described by the variational problem

$$\max_{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m} \sum_{i=1}^{m} h_i(x_i). \quad (6.10)$$

A coordinator having a global vision of the common interest of the group of players (say, a benevolent dictator [52]) imposes that, instead of solving the individualistic problem (6.10), the players solve the joint equilibration problem

$$\max_{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m} \sum_{i=1}^{m} h_i(x_i) + g(x_1, \ldots, x_m), \quad (6.11)$$
where $g: \bigoplus_{i=1}^{m} \mathcal{H}_i \to \mathbb{R}$ is a Lipschitz-differentiable concave utility function that models the collective welfare of the group. A finer model consists in considering $p$ subgroups of players and writing $g = \sum_{k=1}^{p} g_k$, where the payoff $g_k$ of subgroup $k \in \{1, \ldots, p\}$ can be expressed as

$$g_k: (x_1, \ldots, x_m) \mapsto \psi_k \left( \sum_{i=1}^{m} L_{ki} x_i \right), \tag{6.12}$$

where $\psi_k$ is a Lipschitz-differentiable concave function on a real Hilbert space $\mathcal{G}_k$ and where, for every $i \in \{1, \ldots, m\}$, $L_{ki} \in \mathcal{B} (\mathcal{H}_i, \mathcal{G}_k)$. In this model, player $i$ is involved in the activity of subgroup $k$ if $L_{ki} \neq 0$. Upon setting $f_i = -h_i$ for every $i \in \{1, \ldots, m\}$ and $\phi_k = -\psi_k$ for every $k \in \{1, \ldots, p\}$, we recover precisely Problem 6.1. Let us notice that a solution $(x_1, \ldots, x_m)$ to (6.11)–(6.12) can be interpreted as a Nash equilibrium of the potential game [50] in which the payoff of player $i$ in terms of the strategies of the remaining $m - 1$ players is given by

$$x_i \mapsto h_i(x_i) + \sum_{k=1}^{p} \psi_k \left( \sum_{j=1}^{m} L_{kj} x_j \right). \tag{6.13}$$

In this framework, Theorem 6.3 provides a numerical construction of a Nash equilibrium of the game, and Algorithm 6.2 provides a dynamical model for the interaction between the players. At iteration $n$ of Algorithm 6.2, each player $i$ aims at maximizing the utility given in (6.13). This is carried out by the proximal step (6.3), which is a relaxed version of the exact proximal step

$$x_{i,n+1} = \text{prox}_{\gamma_n f_i} \left( x_{i,n} - \gamma_n \sum_{k=1}^{p} L_{ki}^* \nabla \phi_k \left( \sum_{j=1}^{m} L_{kj} x_{j,n} \right) \right), \tag{6.14}$$

in which the function $f_i$ is replaced by an approximation $f_{i,n}$, and some errors $a_{i,n}$ and $b_{i,n}$ are tolerated in the numerical implementation of $\text{prox}_{\gamma_n f_i,n}$ and $(\nabla \phi_k)_{1 \leq k \leq p}$, respectively. The last ingredient of this step concerns risk aversion and is modeled by the relaxation parameter $\lambda_i,n$. When $\lambda_i,n = 0$, player $i$ makes a full proximal step; otherwise, his step is more heavily anchored to his current position $x_{i,n}$ due, for instance, to uncertainty concerning the next performance of his payoff. Let us note that, in the absence of coordination ($\phi_k \equiv 0$) the dynamics of each player would just evolve independently through pure proximal iterations. The coordinator modifies the current strategy $x_{i,n}$ by adding to it a component in the direction of the gradient of the collective utility, namely $-\gamma_n \sum_{k=1}^{p} L_{ki}^* \nabla \phi_k (\sum_{j=1}^{m} L_{kj} x_{j,n})$. In this simultaneous dynamical game, the players choose strategies in a decentralized fashion and without knowledge of the strategies that are being chosen by other players.

**Example 6.5 (2-agent problem)** In Problem 6.1 set $m = 2$ and $p = 1$. Then (6.1) becomes

$$\text{minimize}_{x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2} f_1(x_1) + f_2(x_2) + \phi_1 (L_{11} x_1 + L_{12} x_2). \tag{6.15}$$

Now suppose that $\phi_1$ is the Moreau envelope of a function $\psi \in \Gamma_0 (\mathcal{G}_1)$, i.e.,

$$\phi_1: x \mapsto \inf_{y \in \mathcal{G}_1} \psi(y) + \frac{1}{2} \|x - y\|_{\mathcal{G}_1}^2. \tag{6.16}$$

Then $\nabla \phi_1 = \text{Id} - \text{prox}_\psi$ has Lipschitz constant $\tau_1 = 1$ [51]. Let us employ the simple form of (6.3) in which $\lambda_0 \equiv 0$, $\lambda_{1,n} \equiv 0$, $\lambda_{2,n} \equiv 0$, $a_{1,n} \equiv 0$, $a_{2,n} \equiv 0$, $f_{1,n} \equiv f_1$, $f_{2,n} \equiv f_2$, $b_{1,n} \equiv 0$, and $b_{2,n} \equiv 0$, namely

$$\begin{cases} 
  x_{1,n+1} = \text{prox}_{\gamma_n f_1} (x_{1,n} + \gamma_n L_{11} (\text{prox}_\psi - \text{Id}) (L_{11} x_{1,n} + L_{12} x_{2,n})) \\
  x_{2,n+1} = \text{prox}_{\gamma_n f_2} (x_{2,n} + \gamma_n L_{12} (\text{prox}_\psi - \text{Id}) (L_{11} x_{1,n} + L_{12} x_{2,n})).
\end{cases} \tag{6.17}$$

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Theorem 6.3 asserts that, if \((\gamma_n)_{n \in \mathbb{N}}\) lies in \([\varepsilon, 2(\|L_{11}\|^2 + \|L_{12}\|^2)^{-1} - \varepsilon]\) for some arbitrarily small \(\varepsilon \in [0, +\infty[\), then \(((x_{1,n}, x_{2,n}))_{n \in \mathbb{N}}\) converges weakly to a solution \((x_1, x_2)\) to (6.15). In particular, if \(\psi\) is the indicator function of a nonempty closed convex subset \(C\) of \(G_1\), then (6.15) and (6.17) become respectively

\[
\begin{align*}
\text{minimize} & \quad f_1(x_1) + f_2(x_2) + \frac{1}{2} L_{11}^2 (L_{11} x_1 + L_{12} x_2) \\
\text{subject to} & \quad x_1, x_2 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2,
\end{align*}
\]

and

\[
\begin{align*}
x_{1,n+1} &= \text{prox}_{\gamma_n f_1} \left( x_{1,n} + \gamma_n L_{11}^* (L_{11} x_{1,n} + L_{12} x_{2,n}) \right) \\
x_{2,n+1} &= \text{prox}_{\gamma_n f_2} \left( x_{2,n} + \gamma_n L_{12}^* (L_{11} x_{1,n} + L_{12} x_{2,n}) \right).
\end{align*}
\]

A further special case of interest is when \(C = \{0\}\), meaning that (6.18) reduces to (1.5), i.e.,

\[
\begin{align*}
\text{minimize} & \quad f_1(x_1) + f_2(x_2) + \frac{1}{2} \|L_{11} x_1 + L_{12} x_2\|^2_{G_1}, \\
\text{subject to} & \quad x_1, x_2 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2,
\end{align*}
\]

and that (6.17) assumes the form

\[
\begin{align*}
x_{1,n+1} &= \text{prox}_{\gamma_n f_1} \left( x_{1,n} - \gamma_n L_{11}^* (L_{11} x_{1,n} + L_{12} x_{2,n}) \right) \\
x_{2,n+1} &= \text{prox}_{\gamma_n f_2} \left( x_{2,n} - \gamma_n L_{12}^* (L_{11} x_{1,n} + L_{12} x_{2,n}) \right).
\end{align*}
\]

In [5], (6.20) was approached via an inertial alternating proximal algorithm. Finally, if we further specialize (6.20) by choosing \(\mathcal{H}_1 = \mathcal{H}_2 = G_1\) and \(L_{11} = \text{Id} = -L_{12}\), then (6.20) reduces to (1.2), which was first considered in [1]. In this case, upon setting \(\gamma_n \equiv 1/2\) in (6.21) we obtain the parallel proximal algorithm

\[
\begin{align*}
x_{1,n+1} &= \text{prox}_{f_1/2} \left( (x_{1,n} + x_{2,n})/2 \right) \\
x_{2,n+1} &= \text{prox}_{f_2/2} \left( (x_{1,n} + x_{2,n})/2 \right).
\end{align*}
\]

In view of the above analysis, the sequence \(((x_{1,n}, x_{2,n}))_{n \in \mathbb{N}}\) thus generated converges weakly to a solution to (1.2). In [1], the same conclusion was reached for the sequential algorithm (see also [9] for an alternative algorithm with costs-to-move)

\[
\begin{align*}
x_{1,n+1} &= \text{prox}_{f_1} x_{2,n} \\
x_{2,n+1} &= \text{prox}_{f_2} x_{1,n+1}.
\end{align*}
\]

**Example 6.6 (traffic theory)** Consider a network with \(M\) links indexed by \(j \in \{1, \ldots, M\}\) and \(N\) paths indexed by \(l \in \{1, \ldots, N\}\), linking a subset of \(Q\) origin-destination node pairs indexed by \(k \in \{1, \ldots, Q\}\). There are \(m\) types of users indexed by \(i \in \{1, \ldots, m\}\) transiting on the network. For every \(i \in \{1, \ldots, m\}\) and \(l \in \{1, \ldots, N\}\), let \(\xi_{il} \in \mathbb{R}\) be the flux of user \(i\) on path \(l\) and let \(x_i = (\xi_{il})_{1 \leq l \leq N}\) be the flow associated with user \(i\). A standard problem in traffic theory is to find a Wardrop equilibrium [64] of the network, i.e., flows \((x_i)_{1 \leq i \leq m}\) such that the costs in all paths actually used are equal and less than those a single user would face on any unused path. Such an equilibrium can be obtained by solving the variational problem [21] [54] [59]

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{M} \int_{0}^{h_j(x_1, \ldots, x_m)} \phi_j(h) dh, \\
\text{subject to} & \quad x_1, x_m \in C_1, \ldots, x_m \in C_m
\end{align*}
\]

where \(\phi_j : \mathbb{R} \to [0, +\infty[\) is a strictly increasing \(\tau\)-Lipschitz continuous function modeling the cost of transiting on link \(j\) and \(h_j(x_1, \ldots, x_m)\) is the total flow through link \(j\), which can be expressed
as $h_j(x_1, \ldots, x_m) = \sum_{i=1}^m (Lx_i)^\top e_j$, where $e_j$ is the $j$th canonical basis vector of $\mathbb{R}^M$ and $L$ is an $M \times N$ binary matrix with $j$th entry equal to 1 or 0, according as link $j$ belongs to path $l$ or not. Furthermore, each closed and convex constraint set $C_i$ in (6.24) is defined as $C_i = \{(y_i)_{1 \leq i \leq N} \in [0, +\infty]^N \mid (\forall k \in \{1, \ldots, Q\}) \sum_{l \in N_k} y_l = \delta_{ik}\}$, where $\emptyset \neq N_k \subset \{1, \ldots, N\}$ is the set of paths linking the pair $k$ and $\delta_{ik} \in [0, +\infty]$ is the flow of user $i$ that must transit from the origin to the destination of pair $k$ (for more details on network flows, see [58, 59]). Upon setting

$$\varphi_1: \mathbb{R}^M \rightarrow \mathbb{R}: (\nu_j)_{1 \leq j \leq M} \mapsto \sum_{j=1}^M \int_0^{\nu_j} \phi_j(h) dh,$$

(6.25)

problem (6.24) can be written as

$$\min_{x_1 \in \mathbb{R}^N, \ldots, x_m \in \mathbb{R}^N} \sum_{i=1}^m \iota C_i(x_i) + \varphi_1 \left( \sum_{i=1}^m Lx_i \right).$$

(6.26)

Since $\varphi_1$ is strictly convex and $\tau$-Lipschitz-differentiable, (6.26) is a particular instance of Problem 6.1 with $p = 1$, $G_1 = \mathbb{R}^M$ and $(\forall i \in \{1, \ldots, m\}) \mathcal{H}_i = \mathbb{R}^N$, $f_i = \iota C_i$, and $L_{1i} = L$. Accordingly, Theorem 6.3 asserts that (6.26) can be solved by Algorithm 6.2 which, with the choice of parameters $\gamma_n = \gamma \in [0, 2/\tau]$, $\lambda_{1n} \equiv 0$, $\lambda_n \equiv 0$, $a_{i, n} \equiv 0$, and $b_{i, n} \equiv 0$, yields

$$(\forall i \in \{1, \ldots, m\}) \quad x_{i, n+1} = P_{C_i} \left( x_{i, n} - \gamma L^\top \left( \phi_1 \left( \sum_{i=1}^m Lx_{i, n} \right), \ldots, \phi_m \left( \sum_{i=1}^m Lx_{i, n} \right) \right) \right).$$

(6.27)

In the special case when $m = 1$ the algorithm described in (6.24) is proposed in [22]. Let us note that, as an alternative to $\varphi_1$ in (6.25), we can consider the function

$$\varphi_1: \mathbb{R}^M \rightarrow \mathbb{R}: (\nu_j)_{1 \leq j \leq M} \mapsto \sum_{j=1}^M \nu_j \phi_j(\nu_j),$$

(6.28)

under suitable assumptions on $(\phi_j)_{1 \leq j \leq M}$. In this case, (6.26) reduces to the problem of finding the social optimum in the network [59], that is

$$\min_{x_1 \in C_1, \ldots, x_m \in C_m} \sum_{j=1}^M h_j(x_1, \ldots, x_m) \phi_j(h_j(x_1, \ldots, x_m)),$$

(6.29)

which can also be solved with Algorithm 6.2.

**Example 6.7 (source separation)** Consider the problem of recovering $m$ signals $(x_i)_{1 \leq i \leq m}$ lying in respective Hilbert spaces $(\mathcal{H}_i)_{1 \leq i \leq m}$ from $p$ observations $(z_k)_{1 \leq k \leq p}$ lying in respective Hilbert spaces $(\mathcal{G}_k)_{1 \leq k \leq p}$. The data formation model is

$$(\forall k \in \{1, \ldots, p\}) \quad z_k = \sum_{i=1}^m L_{ki} x_i + w_k,$$

(6.30)

where $L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k)$ and where $w_k \in \mathcal{G}_k$ models observation noise (see in particular [23, 43]). In other words, the objective is to recover the original signals $(x_i)_{1 \leq i \leq m}$ from the $p$ mixtures $(z_k)_{1 \leq k \leq p}$. This situation arises in particular in audio signal processing, when $p$ microphones record the superpositions $(z_k)_{1 \leq k \leq p}$ of $m$ sources $(x_i)_{1 \leq i \leq m}$ that have undergone linear distortions and noise.
corruption. Let us note that the same type of model arises in multicomponent signal deconvolution problems [34]. A variational formulation of the problem is

$$\min_{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m} \sum_{i=1}^{m} f_i(x_i) + \sum_{k=1}^{p} D_k \left( \sum_{i=1}^{m} L_{ki} x_i, z_k \right).$$

(6.31)

In this formulation, each function $f_i \in \Gamma_0(\mathcal{H}_i)$ models some prior knowledge about the signal $x_i$. On the other hand, each function $D_k : \mathcal{G}_k \times \mathcal{G}_k \to [0, +\infty]$ promotes data fitting: it vanishes only on the diagonal $\{ (z, z) \mid z \in \mathcal{G}_k \}$ and, for every $z \in \mathcal{G}_k$, $D_k(\cdot, z)$ is convex and Lipschitz-differentiable (for instance, $D_k$ can be a Bregman distance under suitable assumptions [18, 27], and in particular the standard quadratic fitting term $D_k : (y, z) \mapsto \|y - z\|_{\mathcal{G}_k}^2$). It is clear that (6.31) is a special realization of Problem 6.1 (with $\varphi_k = D_k(\cdot, z_k)$ for every $k \in \{1, \ldots, p\}$) and that it can therefore be solved via Algorithm 6.2.

Example 6.8 (image decomposition) A standard problem in image processing is to find the decomposition $(x_i)_{1 \leq i \leq m}$ of an image $x = \sum_{i=1}^{m} x_i$ in some Hilbert space $\mathcal{H}$, from some observation $z$. When $m = 2$, a common instance of this problem is the geometry/texture decomposition problem [12, 14]. The variational formulations studied in these papers are special instances of the problem

$$\min_{x_1 \in \mathcal{H}, \ldots, x_m \in \mathcal{H}} \sum_{i=1}^{m} f_i(x_i) + \frac{1}{4} \left\| z - \sum_{i=1}^{m} x_i \right\|^2,$$

(6.32)

where $(f_i)_{1 \leq i \leq m}$ are functions in $\Gamma_0(\mathcal{H})$. The first term in the objective is a separable function, the purpose of which is to promote certain known features of each component $x_i$, and the second is a least-squares data fitting term. As shown in [34], for $m = 2$, (6.32) can be solved by alternating proximal methods, which produce weakly convergent sequences. In [13], a finer 3-component model of the form (6.32) was investigated in $\mathcal{H} = \mathbb{R}^N$, and a coordinate descent algorithm was proposed to solve it. This algorithm, however, has modest convergence properties, and it was proved only that the cluster points of the sequence it generates are solutions of the particular finite dimensional problem considered there. By contrast, since (6.32) is a special case of Problem 6.1 (with $\mathcal{H}_i \equiv \mathcal{H}$, $k = 1$, $\varphi_1 = \|z - \cdot\|^2/4$, and $L_{ii} \equiv \text{Id}$), we can derive from Theorem 6.3 an iterative method the orbits of which are guaranteed to converge weakly to a solution to (6.32), under the sole assumption that solutions exist. For instance, for $m = 3$, (6.2) yields $\beta = 2/3$. Taking for simplicity $\gamma_n \equiv 1$, $\lambda_n \equiv 0$, and, for $i \in \{1, 2, 3\}$, $\lambda_{i,n} \equiv 0$, $a_{i,n} \equiv 0$, $f_{i,n} \equiv f_i$, and $b_{i,n} \equiv 0$, (6.3) becomes

$$\begin{cases}
    x_{1,n+1} = \text{prox}_{f_1} \left( \left( z + x_{1,n} - x_{2,n} - x_{3,n} \right)/2 \right) \\
    x_{2,n+1} = \text{prox}_{f_2} \left( \left( z - x_{1,n} + x_{2,n} - x_{3,n} \right)/2 \right) \\
    x_{3,n+1} = \text{prox}_{f_3} \left( \left( z - x_{1,n} - x_{2,n} + x_{3,n} \right)/2 \right).
\end{cases}$$

(6.33)

Let us note that, since Theorem 6.3 allows for more general coupling terms than that used in (6.32), more sophisticated image decomposition problems can be solved in our framework.

Example 6.9 (best approximation) The convex feasibility problem is to find a point in the intersection of closed convex subsets $(C_i)_{1 \leq i \leq m}$ of a real Hilbert space $\mathcal{H}$ [17, 29]. In many instances, this intersection may turn out to be empty and a relaxation of this problem in the presence of a hard constraint represented by $C_1$ is to

$$\min_{x_1 \in C_1} \frac{1}{2} \sum_{i=2}^{m} \omega_i d_{C_i}^2(x_1),$$

(6.34)
where \((\omega_i)_{2 \leq i \leq m}\) are strictly positive weights such that \(\max_{2 \leq i \leq m} \omega_i = 1\). Since, for every \(i \in \{2, \ldots, m\}\) and every \(x_1 \in C_1\), \(a_{C_1}^2(x_1) = \min_{x_i \in C_i} \|x_1 - x_i\|^2\), (6.34) can be reformulated as

\[
\minimize_{x_1 \in C_1, \ldots, x_m \in C_m} \frac{1}{2} \sum_{k=1}^{m-1} \omega_{k+1} \|x_1 - x_{k+1}\|^2.
\] (6.35)

This is a special instance of Problem 6.1 with \(p = m - 1\) and, for every \(i \in \{1, \ldots, m\}\), \(f_i = t_{C_i}\) and

\[
(\forall k \in \{1, \ldots, m-1\}) \quad \varphi_k = \frac{\omega_{k+1}}{2} \|\cdot\|^2 \quad \text{and} \quad L_{ki} = \begin{cases} 
\text{Id}, & \text{if } i = 1; \\
-\text{Id}, & \text{if } i = k + 1; \\
0, & \text{otherwise}.
\end{cases}
\] (6.36)

We can derive from Algorithm 6.2 an algorithm which, by Theorem 6.3, generates orbits that are guaranteed to converge weakly to a solution to (6.35). Indeed, in this case, (6.2) yields \(\beta = 1/(2(m-1))\). For example, upon setting \(\gamma_n = \gamma = 0, 1/(m-1)\), \(\lambda_n \equiv 0, \lambda_{i,n} \equiv 0, a_{i,n} \equiv 0, b_{i,n} \equiv 0\), and \(f_{i,n} = t_{C_i}\) for simplicity, Algorithm 6.2 becomes

\[
\begin{aligned}
&x_{1,n+1} = P_{C_1}((1 - \gamma \sum_{i=2}^m \omega_i)x_{1,n} + \gamma \sum_{i=2}^m \omega_i x_{i,n}) \\
&(\forall i \in \{2, \ldots, m\}) \quad x_{i,n+1} = P_{C_i}(\gamma \omega_i x_{1,n} + (1 - \gamma \omega_i)x_{i,n}).
\end{aligned}
\] (6.37)

In the particular case when \(m = 2\) and \(\gamma = 1/2\), then \(\omega_2 = 1\), (6.35) is equivalent to finding a best approximation pair relative to \((C_1, C_2)\) [19], and (6.37) reduces to

\[
\begin{aligned}
x_{1,n+1} &= P_{C_1}((x_{1,n} + x_{2,n})/2) \\
x_{2,n+1} &= P_{C_2}((x_{1,n} + x_{2,n})/2).
\end{aligned}
\] (6.38)

### 7 Variational problems over decomposed domains in Sobolev spaces

In this section, we consider a particular case of Problem 6.1 involving Sobolev trace operators in coupling terms modeling constraints or transmission conditions at the interfaces of subdomains.

#### 7.1 Notation and definitions

We set some notation and recall basic definitions. For details and complements, see [2, 36, 39, 53, 66].

We denote by \(\mathbb{R}^N\) the usual \(N\)-dimensional Euclidean space and by \(|\cdot|\) its norm, where \(N \geq 2\). Let \(\Omega\) be a nonempty open bounded subset of \(\mathbb{R}^N\) with Lipschitz boundary \(\text{bdry} \Omega\). The space \(H^1(\Omega) = \{x \in L^2(\Omega) \mid Dx \in (L^2(\Omega))^N\}\), where \(D\) denotes the weak gradient, is a Hilbert space with scalar product \((\cdot, \cdot)_{H^1(\Omega)} : (x, y) \mapsto \int_\Omega xy + \int_\Omega (Dx)^\top Dy\). We denote by \(S\) the surface measure on \(\text{bdry} \Omega\) [53, Section 1.1.3]. Now let \(\Upsilon\) be a nonempty open set in \(\text{bdry} \Omega\) and let \(L^2(\Upsilon)\) be the space of square \(S\)-integrable functions on \(\Upsilon\). Endowed with the scalar product

\[
(\cdot, \cdot)_{L^2(\Upsilon)} : (v, w) \mapsto \int_\Upsilon vw \, dS,
\] (7.1)
\( L^2(\Gamma) \) is a Hilbert space. The Sobolev trace operator associated with \( \Omega \) is the unique operator \( T \in \mathcal{B}(H^1(\Omega), L^2(\text{bdry} \Omega)) \) such that \((\forall x \in C^1(\overline{\Omega})) \) \( T x = x|_{\text{bdry} \Omega} \). Endowed with the scalar product
\[
\langle \cdot | \cdot \rangle: (x, y) \mapsto \int_\Omega (Dx)^\top Dy,
\]
the space \( H^1_{0, \Gamma}(\Omega) = \{ x \in H^1(\Omega) \mid T x = 0 \text{ on } \Gamma \} \) is a Hilbert space \([66, \text{Section 25.10}]\). Finally, for \( S \)-almost every \( \omega \in \text{bdry} \Omega \), there exists a unit outward normal vector \( \nu(\omega) \).

### 7.2 Problem formulation and algorithm

**Problem 7.1** Let \( \Omega \) be a nonempty open bounded subset of \( \mathbb{R}^N \) with Lipschitz boundary \( \text{bdry} \Omega \). Let \((\Omega_i)_{1 \leq i \leq m}\) be disjoint open subsets of \( \Omega \) (see Fig. 1) such that the boundaries \((\text{bdry} \Omega_i)_{1 \leq i \leq m}\) are Lipschitz, \( \overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega}_i \), and
\[
(\forall i \in \{1, \ldots, m\}) \quad \Upsilon_{i,i} = \text{int}_{\text{bdry} \Omega}(\text{bdry} \Omega_i \cap \text{bdry} \Omega) \neq \emptyset,
\]
where \( \text{int}_{\text{bdry} \Omega} \) denotes the interior relative to \( \text{bdry} \Omega \). For every \( i \in \{1, \ldots, m\} \), set
\[
(\forall j \in \{i + 1, \ldots, m\}) \quad \Upsilon_{i,j} = \Upsilon_{j,i} = \text{int}_{\text{bdry} \Omega_i}(\text{bdry} \Omega_i \cap \text{bdry} \Omega_j),
\]
let
\[
J(i) = \{ j \in \{1, \ldots, m\} \setminus \{i\} \mid \Upsilon_{i,j} \neq \emptyset \},
\]
be the set of indices of active interfaces, let \( T_i: H^1(\Omega_i) \to L^2(\text{bdry} \Omega_i) \) be the trace operator, let
\[
\mathcal{H}_i = H^1_{0, \Upsilon_{i,i}}(\Omega_i) = \{ x \in H^1(\Omega_i) \mid T_i x = 0 \text{ on } \Upsilon_{i,i} \},
\]
let \( f_i \in \Gamma_0(\mathcal{H}_i) \), and, for every \( j \in J(i) \), let \( \tau_{ij} \in [0, +\infty[ \), let \( \varphi_{ij}: L^2(\Upsilon_{i,j}) \to \mathbb{R} \) be convex and \( \tau_{ij} \)-Lipschitz-differentiable, and set \( T_{ij}: \mathcal{H}_i \to L^2(\Upsilon_{i,j}): x \mapsto (T_i x)|_{\Upsilon_{i,j}} \). The problem is to
\[
\minimize_{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m} \sum_{i=1}^m f_i(x_i) + \sum_{i=1}^m \sum_{j \in J(i)} \varphi_{ij}(T_{ij} x_i - T_{ji} x_j),
\]
under the assumption that solutions exist.
In the above formulation, each function \( x_i \) is defined on a subdomain \( \Omega_i \). The potential \( f_i \) models intrinsic properties of \( x_i \) while, for every \( j \in J(i) \), the potential \( \varphi_{ij} \) arising in the coupling term models the interaction with the \( j \)th subdomain as a function of the difference of the Sobolev traces of \( x_i \) and \( x_j \) on \( \Upsilon_{i,j} \), i.e., of the jump across the interface between \( \Omega_i \) and \( \Omega_j \). Such variational formulations arise in the modeling of transmission problems through thin layers, of Neumann’s sieve (transmission through a finely perforated surface), and of cracks in material \([4, 5, 6, 8]\). Note that, contrary to these approaches, our setting can handle \( m > 2 \) domains as well as nonquadratic functions \( \varphi_{ij} \). We also observe that, if each \( \varphi_{ij} : L^2(\Upsilon_{i,j}) \rightarrow [0, +\infty) \) and vanishes only at 0, \( (7.7) \) can be regarded as a relaxation of some domain decomposition problems, in which one typically imposes the “no-jump” conditions \( T_{ij} x_i = T_{ji} x_j \) across interfaces \([24, 56, 63] \). More generally, \( (7.7) \) can promote various properties of the jump. For instance, if \( \varphi_{ij} = d_{ij}^2 \), where \( C_{ij} \) is a closed convex subset of \( L^2(\Upsilon_{i,j}) \), the underlying constraint is \( T_{ij} x_i - T_{ji} x_j \in C_{ij} \). Unilateral conditions \([35, 40] \) can be modeled in this fashion.

**Algorithm 7.2**

Set

\[
\beta = \frac{1}{p \max_{(k,l) \in K} \tau_{kl}(\|T_{kl}\|^2 + \|T_{lk}\|^2)}, \tag{7.8}
\]

where \( p \) is the cardinality of \( K = \{(k,l) \mid 1 \leq k \leq m, \, l \in J(k)\} \), and fix \( \varepsilon \in [0, \min\{1, \beta\}] \), \( (\gamma_n)_{n \in \mathbb{N}} \) in \([\varepsilon, 2\beta - \varepsilon]\), and \( (\lambda_n)_{n \in \mathbb{N}} \) in \([0, 1 - \varepsilon]\). For every \( n \in \mathbb{N} \) and every \( i \in \{1, \ldots, m\} \), let \( y_{i,n} \) be the unique solution in \( \mathcal{H}_i \) to the problem

\[
\underset{y \in \mathcal{H}_i}{\text{minimize}} \quad \gamma_n f_i(y) + \frac{1}{2} \int_{\Omega_i} |Dy - Dz_{i,n} + \gamma_n D(z_{i,n} + b_{i,n})|^2, \tag{7.9}
\]

where \( z_{i,n} \) is the unique weak solution in \( H^1(\Omega_i) \) to the Dirichlet-Neumann boundary problem (\( v_i(\omega) \) is the unit outward normal vector at \( \omega \in \text{bdry} \Omega_i \))

\[
\begin{cases}
\Delta z_{i,n} = 0 & \text{on } \Omega_i \\
z_{i,n} = 0 & \text{on } \Upsilon_{i,i} \\
v_i^T Dz_{i,n} = \sum_{j \in J(i)} v_{ij,n} & \text{on } \bigcup_{j \in J(i)} \Upsilon_{i,j},
\end{cases} \tag{7.10}
\]

where, for every \( j \in J(i) \),

\[
\tilde{v}_{ij,n} = \begin{cases}
v_{ij,n} = \nabla \varphi_{ij}(T_{ij} x_i,n - T_{ji} x_j,n) - \nabla \varphi_{ji}(T_{ji} x_j,n - T_{ij} x_i,n) & \text{on } \Upsilon_{i,j} \\
0 & \text{on } \text{bdry} \Omega_i \setminus \Upsilon_{i,j},
\end{cases} \tag{7.11}
\]

and set

\[
x_{i,n+1} = \lambda_{i,n} x_{i,n} + (1 - \lambda_{i,n})(y_{i,n} + a_{i,n}), \tag{7.12}
\]

where, for every \( i \in \{1, \ldots, m\} \), the following hold.

(i) \( x_{i,0} \in \mathcal{H}_i \).

(ii) \( (a_{i,n})_{n \in \mathbb{N}} \) and \( (b_{i,n})_{n \in \mathbb{N}} \) are sequences in \( \mathcal{H}_i \) such that

\[
\sum_{n \in \mathbb{N}} \sqrt{\int_{\Omega_i} |Da_{i,n}|^2} < +\infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} \sqrt{\int_{\Omega_i} |Db_{i,n}|^2} < +\infty. \tag{7.13}
\]

(iii) \( (\lambda_{i,n})_{n \in \mathbb{N}} \) is a sequence in \([0, 1]\) such that \( \sum_{n \in \mathbb{N}} |\lambda_{i,n} - \lambda_n| < +\infty. \)

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7.3 Convergence

**Theorem 7.3** Let \( ((x_{i,n})_{n \in \mathbb{N}})_{1 \leq i \leq m} \) be sequences generated by Algorithm 7.2. Then, for every \( i \in \{1, \ldots, m\} \), \( (x_{i,n})_{n \in \mathbb{N}} \) converges weakly in \( \mathcal{H}_i \) to a point \( x_i \in \mathcal{H}_i \), and \( (x_i)_{1 \leq i \leq m} \) is a solution to Problem 7.1.

**Proof.** Given \( i \in \{1, \ldots, m\} \) and \( j \in J(i) \), we first observe that \( T_{ij} \in \mathcal{B}(\mathcal{H}_i, L^2(\mathcal{Y}_{i,j})) \). Indeed, since the embedding \( \mathcal{H}_i \hookrightarrow H^1(\Omega_i) \) is continuous [66, p. 1033] and \( T_i \in \mathcal{B}(H^1(\Omega_i), L^2(\text{bdry} \Omega_i)) \), the operator \( T_{ij}: \mathcal{H}_i \rightarrow L^2(\mathcal{Y}_{i,j}): x \mapsto (T_i x)|_{\mathcal{Y}_{i,j}} \) is indeed linear and continuous. Let us now show that Problem 7.1 is a special case of Problem 6.1. For every \( (k,l) \in \mathcal{K} \) and every \( i \in \{1, \ldots, m\} \), set

\[
G_{kl} = L^2(\mathcal{Y}_{k,l}) \quad \text{and} \quad \mathcal{L}_{kl} = \begin{cases} T_{kl}, & \text{if } i = k; \\ -T_{lk}, & \text{if } i = l; \\ 0, & \text{otherwise}, \end{cases}
\]

and note that \( \mathcal{L}_{kl} \in \mathcal{B}(\mathcal{H}_i, G_{kl}) \) since (7.4) entails \( L^2(\mathcal{Y}_{k,l}) = L^2(\mathcal{Y}_{l,k}) \). Thus, (7.7) can be written as

\[
\min_{x_i \in \mathcal{H}_i, \ldots, x_m \in \mathcal{H}_m} \sum_{i=1}^{m} f_i(x_i) + \sum_{(k,l) \in \mathcal{K}} \varphi_{kl} \left( \sum_{i=1}^{m} \mathcal{L}_{kl} x_i \right),
\]

which conforms to (6.1). Next, let us show that Algorithm 7.2 is a particular case of Algorithm 6.2. To this end, let \( i \in \{1, \ldots, m\} \) and \( n \in \mathbb{N} \). Since \( \text{bdry} \Omega_i = \bigcup_{i \in J(i)} \Omega_{i,j} \), we deduce from [66, Theorem 25.1] that (7.10) admits a unique weak solution \( z_{i,n} \in \mathcal{H}_i \). Accordingly [7.2], [66, Definition 25.31], (7.11), and (7.11) yield

\[
(\forall x \in \mathcal{H}_i) \quad \langle x \mid z_{i,n} \rangle = \int_{\Omega_i} (Dx)^\top Dz_{i,n} = \int_{\bigcup_{j \in J(i)} \mathcal{Y}_{i,j}} (T_i x) \left( \sum_{j \in J(i)} \bar{v}_{ij,n} \right) dS
\]

\[
= \sum_{j \in J(i)} \int_{\mathcal{Y}_{i,j}} (T_{ij} x) v_{ij,n} dS
\]

\[
= \sum_{j \in J(i)} \langle T_{ij} x \mid v_{ij,n} \rangle_{L^2(\mathcal{Y}_{i,j})}
\]

\[
= \left( x \right| \sum_{j \in J(i)} T^*_{ij} v_{ij,n} \right).
\]

Therefore \( z_{i,n} = \sum_{j \in J(i)} T^*_{ij} v_{ij,n} \) and hence (7.14) and (7.11) yield

\[
z_{i,n} = \sum_{k=1}^{m} \sum_{l \in J(k)} L^*_{kli} \nabla \varphi_{kl} \left( \sum_{j=1}^{m} L_{kli}^* x_{j,n} \right) = \sum_{(k,l) \in \mathcal{K}} L^*_{kli} \nabla \varphi_{kl} \left( \sum_{j=1}^{m} L_{kli}^* x_{j,n} \right).
\]

On the other hand, it follows from (7.6) and (7.2) that (7.9) is equivalent to

\[
\min_{y \in \mathcal{H}_i} \gamma_n f_i(y) + \frac{1}{2} \|y - (x_{i,n} - \gamma_n (z_{i,n} + b_{i,n}))\|^2,
\]

33
the unique solution of which is

$$y_{i,n} = \text{prox}_{\gamma_n f_i} \left( x_{i,n} - \gamma_n \left( \sum_{(k,l) \in K} L^*_{kl} \nabla \varphi_{kl} \left( \sum_{j=1}^{m} L_{klj} x_{j,n} \right) + b_{i,n} \right) \right).$$

(7.19)

Moreover, (6.2) is implied by (7.14) and (7.8). Hence, in view of (7.12) and (7.13), Algorithm 7.2 is a particular case of Algorithm 6.2. Altogether, Theorem 6.3 asserts that, for every $i \in \{1, \ldots, m\}$, the sequence $(x_{i,n})_{n \in \mathbb{N}}$ converges weakly in $H_i$ to a point $x_i \in H_i$, where $(x_i)_{1 \leq i \leq m}$ is a solution to Problem 7.1.

Example 7.4 Let $y \in L^2(\Omega)$. With the same notation and hypotheses as in Problem 7.1 let, for every $i \in \{1, \ldots, m\}$,

$$f_i: H_i \to \mathbb{R}: x \mapsto \frac{1}{2} \int_{\Omega_i} |Dx|^2 - \int_{\Omega_i} xy \quad \text{and} \quad (\forall j \in J(i)) \quad \varphi_{ij} = d^2_{C_{ij}},$$

(7.20)

where $C_{ij}$ is a nonempty closed convex subset of $L^2(\Upsilon_{i,j})$. For every $i \in \{1, \ldots, m\}$ the solution to the problem

$$\text{minimize} \ f_i(x)$$

(7.21)

is the weak solution to the Poisson equation with mixed Dirichlet-Neumann conditions [66, Theorem 25.1]

$$\begin{cases}
-\Delta x = y & \text{on } \Omega_i \\
x = 0 & \text{on } \Upsilon_{i,i} \\
\nu^\top_i Dx = 0 & \text{on } \bigcup_{j \in J(i)} \Upsilon_{i,j}.
\end{cases}$$

(7.22)

Problem 7.1 couples these Poisson problems by penalizing the violation of the constraints $T_{ij} x_i - T_{ji} x_j \in C_{ij}$.

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