ELEMENTARY FORMULAS FOR INTEGER PARTITIONS

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Abstract. In this note we will give various exact formulas for functions on integer partitions including the functions \( p(n) \) and \( p(n,k) \) of the number of partitions of \( n \) and the number of such partitions into exactly \( k \) parts respectively. For instance, we shall prove that

\[
p(n) = \sum_{d|n} \frac{d}{d/n} \left( \sum_{i_1=1}^{\frac{d-1}{2}} \cdots \sum_{i_k=1}^{\frac{n-10-12-\cdots-1k-4}{2k}} \frac{\mu(c)}{c \left( \frac{i_k - 1}{c} - \frac{i_{k-3} - 1}{c} \right)} \right),
\]

Our proofs are elementary.

1. Introduction

Among challenges that faced mathematicians who interested in integer partitions was the problem to find a formula to compute the number of partitions of any positive integer. Hardy and Ramanujan in [5] gave the following asymptotic formula for \( p(n) \),

\[
p(n) \sim \frac{e^{\pi \sqrt{2n/3}}}{4n\sqrt{3}},
\]

and Rademacher in [6] gave the following exact formula for \( p(n) \),

\[
p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k \geq 1} A_k(n) \sqrt{k} \left[ \frac{d}{dx} \sinh \left( \frac{\pi}{k} \sqrt{\frac{2}{3}(x - \frac{1}{24})} \right) \right]_{x=n},
\]

where

\[
A_k(n) = \sum_{h \mod k, (h,k)=1} \omega_{h,k} e^{-2\pi i n h / k}
\]

and \( \omega_{h,k} \) is a certain 24th root of unity. To find such deep formulas the authors used tools from complex analysis. A standard reference for more details about integer partitions is [2]. Our purpose in this work is to give exacts formulas involving only finite sums for functions on integer partitions.

A nonempty finite set \( A \) of positive integers is relatively prime if \( \gcd(A) = 1 \) and it is relatively prime to \( m \) if \( \gcd(A,m) = 1 \). Accordingly, a partition of \( n \) is called relatively prime if its parts form a relatively prime set and it is called relatively prime to \( m \) if its parts form a set which is relatively prime to \( m \). Throughout let
Let $k, l, m, n$ be positive integers, let $\lfloor x \rfloor$ be the floor of $x$, and let $\mu(n)$ be the Möbius function.

**Definition 1.** Let $p(n)$ be the number of (unrestricted) partitions of $n$, let $p_{\phi(m)}(n)$ be the number of partitions of $n$ which are relatively prime to $m$, and let $p_{\phi}(n)$ be the number of relatively prime partitions of $n$. Let $p(n, k)$ be the number of partitions of $n$ into exactly $k$ parts, let $p_{\phi(m)}(n, k)$ be the number of partitions of $n$ into exactly $k$ parts which are relatively prime to $m$, and let $p_{\phi}(n, k)$ be the number of relatively prime partitions of $n$ into exactly $k$ parts. Let $p(n, k, l)$ be the number of partitions of $n$ into exactly $k$ parts the smallest of which is $l$, let $p_{\phi(m)}(n, k, l)$ be the number of partitions of $n$ into exactly $k$ parts which are relatively prime to $m$, and let $p_{\phi}(n, k, l)$ be the number of relatively prime partitions of $n$ into exactly $k$ parts which are relatively prime to $m$ with smallest part $l$.

Theorem 1. We have

1. $p_{\phi(m)}(n) = p_{\phi}(n), p_{\phi(m)}(n, k) = p_{\phi}(n, k), \text{ and } p_{\phi(m)}(n, k, l) = p_{\phi}(n, k, l)$.
2. $p(n) = \sum_{d \mid n} p_{\phi}(d)$ or equivalently $p_{\phi}(n) = \sum_{d \mid n} \mu(d)p(n/d)$.
3. $p(n, k) = \sum_{d \mid n} p_{\phi}(d, k)$ or equivalently $p_{\phi}(n, k) = \sum_{d \mid n} \mu(d)p(n/d, k)$.
4. $p(n, k, l) = \sum_{d \mid n} p_{\phi}(n/d, k, l/d)$ or equivalently $p_{\phi}(n, k, l) = \sum_{d \mid n} \mu(d)p(n/d, k, l/d)$.
5. $p(n) = \sum_{k=1}^{n} p(n, k)$ and $p_{\phi(m)}(n) = \sum_{k=1}^{n} p_{\phi(m)}(n, k)$.
6. $p(n, k) = \sum_{d=1}^{[n/k]} p(n, k, l)$ and $p_{\phi(m)}(n, k) = \sum_{d=1}^{[n/k]} p_{\phi(m)}(n, k, l)$.
7. If $k > 1$, then $p_{\phi(m)}(n, k, l) = p_{\phi(m)}(n - l, k - 1, \geq l)$.
8. If $l \leq \lfloor n/k \rfloor$, then $p(n, k, \geq l) = \sum_{j=l}^{[n/k]} p(n, k, j)$ and $p_{\phi(m)}(n, k, \geq l) = \sum_{j=l}^{[n/k]} p_{\phi(m)}(n, k, j)$.

Note that the equivalence of the two identities in Theorem 1 (4) follows by the Möbius inversion formula for arithmetical functions of several variables, see [3, Theorem 2]. Further it is understood that

$$p(n, k) = p_{\phi(m)}(n, k) = 0, \text{ if } k > n$$

and

$$p(n, k, l) = p_{\phi(m)}(n, k, l) = p(n, k, \geq l) = p_{\phi(m)}(n, k, \geq l) = 0, \text{ if } k > n \text{ or } l > \lfloor n/k \rfloor.$$ 

The following result is crucial to our formulas.

**Theorem 2 (14).** Let $a$ and $b$ be positive integers such that $a \leq b$ and let

$$\Phi([a, b], n) = \#\{c \in \{a, a + 1, \ldots, b\} : \gcd(c, n) = 1\}.$$ 

Then

$$\Phi([a, b], n) = \sum_{d \mid n} \mu(d)(\lfloor b/d \rfloor - \lfloor (a - 1)/d \rfloor).$$

Note that this result generalizes the Euler phi function since

$$\Phi(n) = \#\{c \in [1, n] : \gcd(c, n) = 1\} = \Phi([1, n], n).$$
2. Formulas for $p_{\Psi(m)}(n, k, l)$ and $p_{\Psi}(n, k, l)$

**Theorem 3.** If $n \geq 2$ and $l \leq \lfloor n/2 \rfloor$, then

$$p_{\Psi(m)}(n, 2, \geq l) = \sum_{d|(n,m)} \mu(d) \left( \left\lfloor \frac{n}{2d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor \right).$$

**Proof.** We have

$$p_{\Psi(m)}(n, 2, \geq l) = \# \{ a \in [l, \lfloor n/2 \rfloor] : \gcd(a, n-a, m) = 1 \}$$
$$= \# \{ a \in [l, \lfloor n/2 \rfloor] : \gcd(a, (n, m)) = 1 \}$$
$$= \Phi([l, \lfloor n/2 \rfloor], \gcd(n, m))$$
$$= \sum_{d|(n,m)} \mu(d) \left( \left\lfloor \frac{n}{2d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor \right),$$

where the last identity follows by Theorem 2. \qed

**Theorem 4.** We have

(a) $p_{\Psi(m)}(n, k, i_0) = \sum_{i_1=i_0}^{\lfloor n-i_0 \rfloor} \sum_{i_2=i_1}^{\lfloor n-i_0-i_1 \rfloor} \cdots \sum_{i_{k-3}=i_{k-4}}^{\lfloor n-i_0-i_1-\cdots-i_{k-4} \rfloor} \sum_{d|(n,m,i_0,i_1,i_2,\ldots,i_{k-3})} \mu(d) \left( \left\lfloor \frac{n-i_0-i_1-\cdots-i_{k-3}}{2d} \right\rfloor - \left\lfloor \frac{i_{k-3}-1}{d} \right\rfloor \right).$

(b) $p_{\Psi}(n, k, i_0) = \sum_{i_1=i_0}^{\lfloor n-i_0 \rfloor} \sum_{i_2=i_1}^{\lfloor n-i_0-i_1 \rfloor} \cdots \sum_{i_{k-3}=i_{k-4}}^{\lfloor n-i_0-i_1-\cdots-i_{k-4} \rfloor} \sum_{d|(n,i_0,i_1,i_2,\ldots,i_{k-3})} \mu(d) \left( \left\lfloor \frac{n-i_0-i_1-\cdots-i_{k-3}}{2d} \right\rfloor - \left\lfloor \frac{i_{k-3}-1}{d} \right\rfloor \right).$
Theorem 1 (1).

(a) and Theorem 1 (1).

□

(b) This part follows directly from part (b) since the last identity follows by Theorem 3.

Proof. (a) Repeatedly application of Theorem 1 (7, 8) yields

\[ p_{\Phi(m)}(n, k, i_0) = p_{\Phi(m, i_0)}(n - i_0, k - 1, \geq i_0) \]

\[ = \sum_{i_1 = i_0}^{\left\lceil \frac{n-i_0}{k-1} \right\rceil} p_{\Phi(m, i_0)}(n - i_0, k - 1, i_1) \]

\[ = \sum_{i_1 = i_0}^{\left\lceil \frac{n-i_0}{k-1} \right\rceil} \sum_{i_2 = i_1}^{\left\lceil \frac{n-i_0-i_1}{k-2} \right\rceil} \ldots \sum_{i_{k-3} = i_{k-4}}^{\left\lceil \frac{n-i_0-i_1-\ldots-i_{k-4}}{2} \right\rceil} \]

\[ p_{\Phi(m, i_0, i_1, \ldots, i_{k-4})}(n - i_0 - i_1 - \ldots - i_{k-4}, 3, i_{k-3}) \]

\[ = \sum_{i_1 = i_0}^{\left\lceil \frac{n-i_0}{k-1} \right\rceil} \sum_{i_2 = i_1}^{\left\lceil \frac{n-i_0-i_1}{k-2} \right\rceil} \ldots \sum_{i_{k-3} = i_{k-4}}^{\left\lceil \frac{n-i_0-i_1-\ldots-i_{k-4}}{2} \right\rceil} \sum_{i_{k-3} = i_{k-4}}^{\left\lceil \frac{n-i_0-i_1-\ldots-i_{k-4}}{2} \right\rceil} \]

\[ \mu(d) \left( \left\lfloor \frac{n-i_0-i_1-i_2-\ldots-i_{k-3}}{2d} \right\rfloor - \left\lfloor \frac{i_{k-3}-1}{d} \right\rfloor \right), \]

where the last identity follows by Theorem 3.

(b) This part follows directly from part (b) since \( p_\Phi(n, k, i_0) = p_{\Phi(n)}(n, k, i_0) \) by Theorem 1 (1).

3. Formulas for \( p_{\Phi(m)}(n, k) \), \( p_\Phi(n, k) \), and \( p(n, k) \)

Theorem 5. We have

(a) \( p_{\Phi(m)}(n, k) = \sum_{i_0 = 1}^{\left\lceil \frac{n}{k} \right\rceil} \sum_{i_1 = i_0}^{\left\lceil \frac{n-i_0}{k-1} \right\rceil} \sum_{i_2 = i_1}^{\left\lceil \frac{n-i_0-i_1}{k-2} \right\rceil} \ldots \sum_{i_{k-3} = i_{k-4}}^{\left\lceil \frac{n-i_0-i_1-\ldots-i_{k-4}}{2} \right\rceil} \sum_{i_{k-3} = i_{k-4}}^{\left\lceil \frac{n-i_0-i_1-\ldots-i_{k-4}}{2} \right\rceil} \mu(d) \left( \left\lfloor \frac{n-i_0-i_1-i_2-\ldots-i_{k-3}}{2d} \right\rfloor - \left\lfloor \frac{i_{k-3}-1}{d} \right\rfloor \right). \]

(b) \( p_\Phi(n, k) = \sum_{i_0 = 1}^{\left\lceil \frac{n}{k} \right\rceil} \sum_{i_1 = i_0}^{\left\lceil \frac{n-i_0}{k-1} \right\rceil} \sum_{i_2 = i_1}^{\left\lceil \frac{n-i_0-i_1}{k-2} \right\rceil} \ldots \sum_{i_{k-3} = i_{k-4}}^{\left\lceil \frac{n-i_0-i_1-\ldots-i_{k-4}}{2} \right\rceil} \sum_{i_{k-3} = i_{k-4}}^{\left\lceil \frac{n-i_0-i_1-\ldots-i_{k-4}}{2} \right\rceil} \mu(d) \left( \left\lfloor \frac{n-i_0-i_1-i_2-\ldots-i_{k-3}}{2d} \right\rfloor - \left\lfloor \frac{i_{k-3}-1}{d} \right\rfloor \right). \)

Proof. Part (a) follows by Theorem 1 (6) and Theorem 4. Part (b) follows by part (a) and Theorem 1 (1). □
Theorem 6. We have
\[ p(n, k) = \sum_{d|n} \sum_{i_0=1}^{\lfloor n/d \rfloor} \sum_{i_1=1}^{\lfloor n/d - i_0 \rfloor} \ldots \sum_{i_{k-3}=1}^{\lfloor n/d - i_0 - i_1 - \ldots - i_{k-4} \rfloor} \sum_{c|(d,i_0,i_1,i_2,\ldots,i_{k-3})} \frac{c(d,i_0,i_1,i_2,\ldots,i_{k-3})}{\mu(c)} \left( \frac{d - i_0 - i_1 - i_2 - \ldots - i_{k-3}}{2c} - \left\lfloor \frac{i_{k-3} - 1}{c} \right\rfloor \right). \]

Proof. The is a consequence of Theorem 3 and Theorem 5. \(\square\)

4. Formulas for \(p_{\Psi(n)}(n)\), \(p_{\Psi}(n)\), and \(p(n)\)

Theorem 7. We have
(a) \[ p_{\Psi(m)}(n) = \sum_{k=1}^{n} \sum_{i_0=1}^{\lfloor n/k \rfloor} \sum_{i_1=1}^{\lfloor n/k - i_0 \rfloor} \ldots \sum_{i_{k-3}=1}^{\lfloor n/k - i_0 - i_1 - \ldots - i_{k-4} \rfloor} \sum_{d|(n,m,i_0,i_1,i_2,\ldots,i_{k-3})} \mu(d) \left( \frac{n - i_0 - i_1 - i_2 - \ldots - i_{k-3}}{2d} - \left\lfloor \frac{i_{k-3} - 1}{d} \right\rfloor \right). \]

(b) \[ p_{\Psi}(n) = \sum_{k=1}^{n} \sum_{i_0=1}^{\lfloor n/k \rfloor} \sum_{i_1=1}^{\lfloor n/k - i_0 \rfloor} \ldots \sum_{i_{k-3}=1}^{\lfloor n/k - i_0 - i_1 - \ldots - i_{k-4} \rfloor} \sum_{d|(n,i_0,i_1,i_2,\ldots,i_{k-3})} \mu(d) \left( \frac{n - i_0 - i_1 - i_2 - \ldots - i_{k-3}}{2d} - \left\lfloor \frac{i_{k-3} - 1}{d} \right\rfloor \right). \]

Proof. Combine Theorem 5(5) with Theorem 5 to obtain part (a). As to part (b) combine part (a) with Theorem 1(1). \(\square\)

Theorem 8. We have
\[ p(n) = \sum_{d|n} \sum_{k=1}^{d} \sum_{i_0=1}^{\lfloor d/k \rfloor} \sum_{i_1=1}^{\lfloor d/k - i_0 \rfloor} \ldots \sum_{i_{k-3}=1}^{\lfloor d/k - i_0 - i_1 - \ldots - i_{k-4} \rfloor} \sum_{c|(d,i_0,i_1,i_2,\ldots,i_{k-3})} \mu(c) \left( \frac{d - i_0 - i_1 - i_2 - \ldots - i_{k-3}}{2c} - \left\lfloor \frac{i_{k-3} - 1}{c} \right\rfloor \right). \]

Proof. Use Theorem 1(2) and Theorem 7. \(\square\)

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