Exhaustive Search for Low Autocorrelation Binary Sequences

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June 1996

Abstract: Binary sequences with low autocorrelations are important in communication engineering and in statistical mechanics as groundstates of the Bernasconi-model. Computer searches are the main tool to construct such sequences. Due to the exponential size of the configuration space, exhaustive searches are limited to short sequences. We discuss an exhaustive search algorithm with run time characteristic $O(2^N)$ and apply it to compile a table of exact groundstates of the Bernasconi-model up to $N = 48$. The data suggests $F > 9$ for the optimal merit factor in the limit $N \to \infty$.

1 Introduction

Binary sequences $S = \{s_1 = \pm 1, \ldots, s_N\}$ with low off-peak autocorrelations

$$C_k(S) = \sum_{i=1}^{N-k} s_is_{i+k}. \quad (1)$$

have applications in many communication engineering problems [1]. One exciting example has been their use in high precision interplanetary radar measurements to check out space-time-curvature [2].

Physicists prefer to consider binary sequences as one dimensional systems of Ising-spin. In this context, low autocorrelation binary sequences appear as minima of the energy

$$E(S) = \sum_{k=1}^{N-1} C_k^2(S). \quad (2)$$

This is the Bernasconi-model [3]. It has long-range 4-spin interactions and is completely deterministic, i.e. there is no explicit or quenched disorder like in spin-glasses. Nevertheless the ground states are highly disordered – quasi by definition. This self-induced disorder resembles very much the situation in real glasses. In fact, the Bernasconi-model exhibits features of a glass transition like a jump in the specific heat [3] and slow dynamics and aging [4].

A clever variation of the replica method allows an analytical treatment of the Bernasconi-model in the high-temperature regime [5, 6]. For the low-temperature regime, analytical results are rare – especially the ground states are not known. With periodic boundary conditions, i.e. with

$$C_k = \sum_{i=1}^{N} s_is_{(i+k-1)(\mod N) + 1}, \quad (3)$$

instead of eq. 1, the construction of ground states is possible for special values of $N$. Example: For $N = 4n + 3$ prime, the modified Legendre-sequence

$$s_j = \begin{cases} j^{(N-1)\mod N} & 1 \leq i < N \\ \pm 1 & i = N \quad (4) \end{cases}$$

yields $C_k^2 = 1$, the minimum possible value for odd $N$. Other ground states can be constructed from linear shift register sequences based on primitive polynomials over Galois fields. This construction requires $N = 2^p - 1$ with $p$ prime. See [1, 6] for details.

For the model with open boundary conditions (eq. 1) no construction of ground states is known, not even for special values of $N$. The Legendre-sequences are far from the true ground states [7]. The only exact results have been provided by exhaustive enumerations, limited however by the exponential complexity of the problem to systems smaller than $N = 32$ [8]. Partial enumerations allow larger values of $N$ but cannot guarantee to yield true ground states. Promising candidates for partial enumerations are the skew-symmetric sequences of odd length $N = 2n - 1$. These sequences satisfy

$$s_{n+l} = (-1)^l s_{n-l}, \quad l = 1, \ldots, n - 1 \quad (5)$$

from which it follows that all $C_k$ with $k$ odd vanish. The restriction to skew-symmetric sequences reduces the effective size of the problem by a factor of 2, but the true ground states are not skew-symmetric for several values of $N$, as we will see below.

Finding the ground states of the Bernasconi-model has turned out to be a hard mathematical problem. Golay [8, 3] has conjectured that the maximal merit factor

$$F = \frac{N^2}{2E} \quad (6)$$
should obey $F \lesssim 12.32$ for $N \gg 1$. However, heuristic searches among skew-symmetric sequences up to $N = 199$ suggest $F \approx 6$ for long sequences [9], a value consistent with results from simulated annealing [3]. This large discrepancy indicates that the groundstates, i.e. the sequences with high merit factors $6 < F \lesssim 12$, must be extremely isolated energy minima in configuration space. Stochastic search procedures including simulated annealing are not well suited to find these “golf-holes”. Exhaustive search seems to be the only approach at least for small systems. The complete configuration space has been exhaustively searched up to $N = 71$ [10, 11]. Fifty days of CPU-time on a special purpose computer have been used for an exhaustive search for binary sequences up to $N = 40$ that minimize $\max_k |C_k|$ [12].

In this contribution we discuss a fast algorithm for the exhaustive enumeration. It is fast enough to yield exact groundstates of the Bernasconi-model up to $N = 48$ and can easily be modified for partial enumerations. The data is used to estimate the optimal merit factor in the large $N$ limit.

## 2 The Algorithm

Any algorithm that performs an exhaustive search for the ground state of the Bernasconi-model has to cope with the enormous size $(2^N)$ of the configuration space. This exponential complexity limits the accessible values of $N$ very soon and calls for methods to restrict the search to smaller subspaces without missing the true ground state. Symmetries are an obvious device to cut out portions of the configuration space. We will see below, that the use of symmetries can reduce the size of the search space by a factor of about $1/8$. A method borrowed from combinatorial optimization - branch and bound - will prove useful to reduce the complexity from $O(2^N)$ to $O(b^N)$ with $b < 2$. We will further see, that the enumeration problem is suited almost perfectly for parallelization.

### 2.1 Symmetries

The correlations $C_k$ (eq. 1) are unchanged when the sequence is complemented or reversed. When alternate elements of the sequence are complemented, the even-indexed correlations are not affected, the odd-indexed correlations only change sign. Hence, with the exception of a small number of symmetric sequences, the $2^N$ sequences will come in classes of eight which are equivalent. The total number of nonequivalent sequences is slightly larger than $2^{N-3}$.

The $m$ left- and $m$ rightmost elements of the sequence can be used to parameterize the symmetry-classes. For $m = 3$ and $N$ odd, this gives 12 classes:

- $---\cdots---$  $---\cdots++-$  $---\cdots+++$  $---\cdots+++$
- $---\cdots+--$  $---\cdots+++-$  $---\cdots+++-$  $---\cdots+++-$
- $---\cdots+++$  $---\cdots+++$  $---\cdots+++$  $---\cdots+++$
- $---\cdots+++$  $---\cdots+++$  $---\cdots+++$  $---\cdots+++$

For $N$ even there are 10 classes. In general the number $c$ of symmetry-classes that can be distinguished by $m$ left- and $m$ right-border elements reads

$$c(m) = 2^{2m-3} + 2^{m-2+\left(N \mod 2\right)} \quad (7)$$

and the number of nonequivalent configurations reduces to a fraction

$$\frac{c(m)}{2^{2m}} = \frac{1}{8} + \frac{1}{2^{m+2-(N \mod 2)}} \quad (8)$$

The optimal value $\frac{1}{s}$ is approached with increasing $m$.

### 2.2 Branch and Bound

Branch and bound methods are commonly used in combinatorial optimization [13] and (less frequently) in statistical mechanics [14, 15]. They solve a discrete optimization problem by breaking up its feasible set into successively smaller subsets (branch), calculating bounds on the objective function value over each subset, and using them to discard certain subsets from further consideration (bound). The procedure ends when each subset has either produced a feasible solution, or has been shown to contain no better solution than the one already in hand. The best solution found during this procedure is a global optimum.

The idea is of course to discard many subsets as early as possible during the branching process, i.e. to discard most of the feasible solutions before actually evaluating them. The success of this approach depends on the branching rule and very much on the quality of the bounds, but it can be quite dramatic. Numerical investigations have shown, for example, that the $n$-city Traveling Salesman Problem can be solved exactly in time $O(n^\alpha)$ with $\alpha < 3$ using branch and bound methods [13]! This is no contradiction to the exponential complexity of the TSP since the latter is the guaranteed, i.e. worst case complexity, while the former refers to the typical case, averaged over many instances of the TSP.

In accordance with our symmetry-classes, we specify a set of feasible solutions by fixing the $m$ left- and $m$ rightmost elements of the sequence. The $N - 2m$ center elements are not specified, i.e. the set contains $2^{N-2m}$ feasible solutions. Given a feasible set specified by the $m$ border elements, 4 smaller sets are created by fixing the elements $s_{m+1}$ and $s_{N-m}$ to $\pm 1$. This is the branching rule. It is applied recursively until all elements have been fixed. The energy of the resulting sequence is compared
Algorithm 1 Procedure search \((S, m)\) – search for the minimum energy configuration \(S_{\text{opt}}\) within the subset \((S, m)\) of all configurations. 

1. \(n \leftarrow N - 2m\); \{number of free elements \(s_i\)\}
2. if \(n > 1\) then \{\(>\) 2 seq. in subset\}
3. if \(E_b(S, m) \geq E(S_{\text{opt}})\) then \{bound\}
4. return;
5. else \{branch\}
6. search \((S, m + 1)\);
7. \(s_{m+1} \leftarrow -s_{m+1}\); search \((S, m + 1)\);
8. \(s_{N-m} \leftarrow -s_{N-m}\); search \((S, m + 1)\);
9. \(s_{m+1} \leftarrow -s_{m+1}\); search \((S, m + 1)\);
10. end if
11. else if \(n = 1\) then \{2 seq. in subset\}
12. if \(E(S) < E(S_{\text{opt}})\) then
13. \(S_{\text{opt}} \leftarrow S\);
14. end if
15. \(s_{m+1} \leftarrow -s_{m+1}\);
16. if \(E(S) < E(S_{\text{opt}})\) then
17. \(S_{\text{opt}} \leftarrow S\);
18. end if
19. else \{1 seq. in subset\}
20. if \(E(S) < E(S_{\text{opt}})\) then
21. \(S_{\text{opt}} \leftarrow S\);
22. end if
23. end if

most by 2. Let \(f_k\) denote the number of terms \(s_is_{i+k}\) in \(C_k\) that contain at least one free element. This leads to

\[
E_b = \sum_{k=1}^{N-k} \min\{k, (|C_k| - 2f_k)^2\}
\]  

(11)

where \(b_k = (N - k) \mod 2 \in \{0, 1\}\) is the minimum value \(|C_k|\) can attain. The \(f_k\) are given by

\[
f_k = \begin{cases} 
0 & k \geq N - m \\
2(N - m - k) & N/2 \leq k < N - m \\
N - 2m & k < N/2 
\end{cases}
\]  

(12)

i.e. the long range correlations are not affected by our relaxation. \(E_b\) is not the strongest bound to \(E_{\text{min}}\), but its calculation is very fast.

Now we have gathered all ingredients to formulate the branch and bound procedure search (algorithm 1). This procedure is called with two parameters specifying the subset to search: a binary sequence \(S = \{s_1, \ldots, s_N\}\) and an integer \(m\). The subset consists of all \(2^{N-2m}\) sequences that can be generated from \(S\) by varying the \(N - 2m\) center elements. \(S_{\text{opt}}\) is a global variable that holds the sequence with the minimum energy found so far. On entry, the size of the subset is checked: If it contains more than 2 sequences, branch and bound (lines 3–10) is applied. Otherwise the sequences in the subset are evaluated (lines 11–23).

The procedure search is called from a driving procedure with \(c(m_0)\) subsets, each representing a different symmetry-class. In practice, we used \(m_0 = 6\) with 528 \((N\) even) resp. 544 \((N\) odd) symmetry-classes.

Figure 1 CPU time for exhaustive search algorithm vs. \(N\). Times are measured on a Sun UltraSparc I 170 workstation.
To measure the impact of branch and bound, we started two runs on the same machine: one “straight” enumeration (omitting lines 3–5) and the other with activated bound-mechanism. Figure 1 displays the effect of branch and bound on the CPU time. The straight enumeration shows the expected \( O(2^N) \) behavior. Branch and bound reduces the complexity to \( O(1.85^N) \). Albeit this is still exponential, the gain in speed is worth the little effort. The branch and bound enumeration for \( N = 44 \) took about 2 days on a Sun UltraSparc I 170 workstation. This compares well with the extrapolated 68 days for the straight enumeration!

2.3 Parallelization

The different symmetry classes can be searched independently. Hence the straight enumeration is perfectly parallelizable into \( c(m) \) threads of control. Branch and bound complicates the situation: Whenever a better sequence is found by one thread, it should be communicated immediately to all other threads to ensure that always the best \( E(S_{\text{opt}}) \) is used in the bounding test (line 3). But \( E(S_{\text{opt}}) \) is accessed very frequently, so the necessary synchronization would spoil the parallelization. Giving each thread its own local copy of \( S_{\text{opt}} \) preserves perfect parallelization but abandons most of the benefits of branch and bound.

A solution to this dilemma is provided by the workpile paradigm [16]: The symmetry classes to be searched for are put on a central workpile and a number of worker threads are launched together. Each worker thread requests an assignment of work from the workpile (i.e. a symmetry class), performs the search, and then asks for a new work assignment. This process repeats until all symmetry classes have been considered.

The access to the workpile has to be protected with a mutual exclusion lock, allowing only one thread at a time to read or modify data from the workpile. If each worker thread uses its own local copy of \( S_{\text{opt}} \), this is the only synchronization needed. To propagate the best \( S_{\text{opt}} \) as fast as possible among the workers, it is stored in the workpile. A worker that requests a new work assignment compares its own local \( S_{\text{opt}} \) with the global one and updates the one with the higher energy under the protection of the lock. This method limits the use of a suboptimal \( S_{\text{opt}} \) to the search within \( n - 1 \) symmetry classes, where \( n \) is the number of worker threads. The delay in propagation of the optimal \( S_{\text{opt}} \) is minimized by choosing \( c(m) \gg n \). In this case, the workpile paradigm has the additional advantage of evenly distributing the load among all worker threads. Due to branch and bound, the enumerations in some symmetry classes may take considerably less time than in others. A worker that encounters these “easy” classes simply gets more classes to search.

On a 4 processor Sun SPARCstation 20, the number of worker threads (\( \leq 4 \)) is much smaller than \( c(m) \) for \( m = 6 \), so the workpile paradigm should yield almost perfect parallelization. Figure 2 shows that this is indeed the case. The low speed-up factors for small \( N \) are due to the relative costs of thread-generation and synchronization compared to the actual enumeration.

3 Results

Using the multithreaded branch and bound algorithm and 313 hours of CPU time on a 4-processor Sun SPARCstation 20, the ground states of the Bernasconi-model have been found up to \( N = 48 \) (table 1). The enumeration for \( N = 32 \) (the previous peak value) took only 80 seconds, \( N = 39 \) was done in 1 hour. It is remarkable that from the 22 optimal skew-symmetric sequences in the range \( 5 \leq N \leq 47 \) [10] 7 have energies well above the true ground-state energy - a third. This should be kept in mind if one uses skew-symmetric sequences to estimate the groundstate energy in the limit \( N \to \infty \).

Figure 3 shows the groundstate energies \( E \) vs. \( N \). In contrast to the model with periodic boundary conditions there are no visible regular patterns for special values of \( N \) [6]. The energies seem to follow \( E \propto N^2 \) for all values of \( N \). A quadratic fit yields

\[
F = \lim_{N \to \infty} \frac{N^2}{2E} = 9.3
\]

and leads us to the tentative conclusion that

\[
F = \lim_{N \to \infty} \frac{N^2}{2E} > 9.
\]

This estimate is in agreement with Golay’s conjecture \( F \lesssim 12.32 \) and has to be compared to the value \( F \approx 6.0 \) found.
Table 1 Ground states of the Bernasconi-model for $3 \leq N \leq 48$. Sequences are written in run-length notation: Each figure indicates the number of consecutive equal signed elements.

| $N$ | $E_{\text{min}}$ | sequence |
|-----|-----------------|----------|
| 3   | 1               | 21       |
| 4   | 2               | 211      |
| 5   | 2               | 311      |
| 6   | 7               | 1111     |
| 7   | 3               | 1123     |
| 8   | 8               | 12113    |
| 9   | 12              | 42111    |
| 10  | 13              | 22114    |
| 11  | 5               | 112133   |
| 12  | 10              | 1221114  |
| 13  | 6               | 5221111 |
| 14  | 19              | 2221115  |
| 15  | 15              | 52221111 |
| 16  | 24              | 225111121|
| 17  | 32              | 252211121|
| 18  | 25              | 441112221|
| 19  | 29              | 4111142212|
| 20  | 26              | 5113112321|
| 21  | 26              | 272211111221|
| 22  | 39              | 5122111233|
| 23  | 47              | 212121111632|
| 24  | 36              | 2236111112121|
| 25  | 36              | 337111121121|
| 26  | 45              | 21212111116322|
| 27  | 37              | 343131312111211|
| 28  | 50              | 343131312111212|
| 29  | 62              | 212112131313431|
| 30  | 59              | 551212111113231|
| 31  | 67              | 733212121112111|
| 32  | 64              | 71112111113221221|
| 33  | 64              | 642121111111122221|
| 34  | 65              | 8421211111111122221|
| 35  | 73              | 712212111112111332|
| 36  | 82              | 3632311131211211121|
| 37  | 86              | 8642112111111122221|
| 38  | 87              | 8442112111111122221|
| 39  | 99              | 82121121234321111111|
| 40  | 108             | 4441211123112131313|
| 41  | 108             | 3431111111222211211|
| 42  | 101             | 313131314134312112121|
| 43  | 109             | 113243211111222121121|
| 44  | 122             | 52531311311222111211121|
| 45  | 118             | 8212112123123432111111|
| 46  | 131             | 8234312312112211111111|
| 47  | 135             | 9234312312112211111111|
| 48  | 140             | 3111111832143212221121121|

Figure 3 Groundstate energy of the Bernasconi-Hamiltonian vs. $N$.

by heuristic searches for long skew-symmetric sequences [9] and by simulated annealing [3]. This indicates once more that heuristic and probabilistic methods fail to find the groundstates of the Bernasconi-model. Every algorithm of such a kind should be judged by the percentage of values it finds from table 1.

Acknowledgements: Thanks are due to A. Engel and J. Richter for guiding the author’s attention to the wonderful world of branch and bound and to S. Kobe for providing helpful references.

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