SOLVING SYSTEM OF INTERVAL LINEAR EQUATIONS BY GAUSS JORDAN METHOD USING GENERALIZED INTERVAL ARITHMETIC

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Abstract. We introduce an interval version of Gauss Jordon method for solving system of interval linear equations involving generalized interval numbers by applying the generalized interval arithmetic operations. The given interval system is solved without converting to equivalent crisp linear systems. Also numerical example is given to support the proposed algorithm.

1. Introduction
In modelling and solving real-world scenarios, it is essential to overcome uncertainty and vagueness. Such uncertainty and vagueness can be due to several sources, such as not precisely taken measurements, physical model design, variations of system parameters, errors during estimation to name a few. In such a case situation, the exact values of the model parameters are not specified, but only the possible values of the intervals can be identified. Interval analysis is an efficient and accurate tool that allows one to effectively manage such uncertainty and vagueness. Systems of linear equations play a significant role in designing and solving real world problems. The problem can be formulated as a system of interval linear equations when the model parameters are uncertain and ambiguous. In literature, there are various techniques for computing the “smallest” interval vector containing the definite solution of the systems of interval linear equations.

In literature several authors referred in [5],[6],[12],[15],[16], [21] and many more have discussed solving system of interval linear equations. Esmaeil Siahlooei and Seyed Abolfazl Shahzadeh Fazeli[5] have discussed solving system by using two iterative methods. Nirmala and Ganesan[20] have studied solving interval system by interval form of Jacobi iterative method. Nirmala and Ganesan[19] have proposed direct method for solving interval linear system based on interval version of Gaussian Elimination method and also Cramer’s rule. Shohreh Abolmasoumi and Majid Alav [22] have introduced the algorithm based on gradient vector for finding solution of interval system. Millan Hladik[15] developed distinct feasible transformations and to enhance the findings to further broad categories of AE interval systems and linear parametric systems. Further, these transformations suffice to generalise the evidence of certain characterization theorems. Dymova [6] have introduced solving interval linear system by modified interval division.

Dehghani-Madiseh [12] has introduced to find the generalized solution set of the symmetric interval linear systems based on the Cholesky decomposition. Here, an algorithm is introduced for determining the system of interval linear equations by employing interval version of Gauss Jordon method without converting to crisp equivalent forms.
The paper is being organized as: Section 2 gives the basic concepts and arithmetic operations of generalized interval numbers. Section 3 gives the basic concepts of interval matrices and arithmetic operations on interval matrices. In section 4, an algorithm is proposed for solving system of interval linear equations by using interval version of Gauss Jordan method. In section 5, to demonstrate the efficacy of the proposed algorithm, numerical examples are also given.

2. Preliminaries

Let $\mathbb{IR} = \{ [x_1, x_2] : x_1 \leq x_2 \text{ and } x_1, x_2 \in \mathbb{R} \}$ be the set of all proper intervals and $\overline{\mathbb{IR}} = \{ [x_1, x_2] : x_1 > x_2 \text{ and } x_1, x_2 \in \mathbb{R} \}$ be the set of all improper intervals on the real line $\mathbb{R}$. If $x_1 = x_2 = x$, then $\bar{x} = [x, x] = x$ is a real number (or a degenerate interval). The mid-point and width (or half-width) of an interval number $[x, x]$ are defined as $m(x) = \frac{x_1 + x_2}{2}$ and $w(x) = \frac{x_2 - x_1}{2}$. We denote the set of generalized intervals (proper and improper) by $D = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{ [x_1, x_2] : x_1, x_2 \in \mathbb{R} \}$.

The “dual” is an essential monadic operator proposed by Kaucher [10] that reverses the end-points of the intervals expresses an element to element symmetry between proper and improper intervals in $D$. For $\bar{x} = [x_1, x_2] \in D$, its dual is defined by $\text{dual}(\bar{x}) = \text{dual}([x_1, x_2]) = [x_2, x_1]$. The opposite of an interval $\bar{x} = [x_1, x_2]$ is $\text{opp} \{ [x_1, x_2] \} = [-x_1, -x_2]$ which is the additive inverse of $[x_1, x_2]$ and $\left[ \frac{1}{x_1}, \frac{1}{x_2} \right]$ is the multiplicative inverse of $[x_1, x_2]$, provided $0 \notin [x_1, x_2]$. That is, $\bar{x} + (-\text{dual} \bar{x}) = \bar{x} - \text{dual} \bar{x} = [x_1, x_2] - \text{dual}([x_1, x_2]) = [x_1, x_2] - [x_2, x_1] = [x_1 - x_1, x_2 - x_2] = [0, 0]$ and $\bar{x} \times \left( \frac{1}{\text{dual}(\bar{x})} \right) = [x_1, x_2] \times \frac{1}{\text{dual}([x_1, x_2])} = [x_1, x_2] \times \frac{1}{[x_2, x_1]} = [x_1, x_2] \times \left[ \frac{1}{x_1}, \frac{1}{x_2} \right] = \left[ \frac{x_1}{x_1}, \frac{x_2}{x_2} \right] = [1, 1]$.

2.1. A new interval arithmetic

Ganesan and Veeramani [7] extend the arithmetic operations to the set of generalized interval numbers $D$ and incorporating the concept of dual. For $\bar{x} = [x_1, x_2]$, $\bar{y} = [y_1, y_2] \in D$ and for $* \in \{ \cdot, - \}$, we define $\bar{x} * \bar{y} = [m(\bar{x}) * m(\bar{y}) - k, m(\bar{x}) * m(\bar{y}) + k]$, where $k = \min \left\{ \left( (m(\bar{x}) * m(\bar{y})) - \alpha, (m(\bar{x}) * m(\bar{y})) - \beta \right) \right\}$, $\alpha$ and $\beta$ are the end points of the interval under the existing interval arithmetic. Consider $\bar{x} = [x_1, x_2]$, $\bar{y} = [y_1, y_2] \in D$,

(i) Addition

\[ \bar{x} + \bar{y} = [x_1, x_2] + [y_1, y_2] = [m(\bar{x}) + m(\bar{y}) - k, m(\bar{x}) + m(\bar{y}) + k] , \text{ where } k = \left( \frac{(y_2 + x_2) - (y_1 + x_1)}{2} \right) \]
(ii) Subtraction

\[ \tilde{x} - \tilde{y} = [x_1, x_2] - [y_1, y_2] = \{(m(\tilde{x}) - m(\tilde{y})) - k, (m(\tilde{x}) - m(\tilde{y})) + k\}, \text{ where } k = \frac{(y_2 + x_2) - (y_1 + x_1)}{2}. \]

Also if \( \tilde{x} = \tilde{y} \) i.e. \([x_1, x_2] = [y_1, y_2]\), then

\[ \tilde{x} - \tilde{y} = \tilde{x} - \text{dual}(\tilde{x}) = [x_1, x_2] - \text{dual}([x_1, x_2]) = [x_1, x_2] - [x_2, x_1] = [x_1 - x_2, x_2 - x_1] = [0, 0]. \]

(iii) Multiplication

\[ \tilde{x} \cdot \tilde{y} = [x_1, x_2] \cdot [y_1, y_2] = \{(m(x) - m(y)) - k, (m(x) - m(y)) + k\}, \text{ where } k = \min\{m(x) - m(y), (m(x) - m(y))\}. \]

\[ \alpha = \min(x_1y_1, x_1y_2, x_2y_1, x_2y_2) \quad \text{and} \quad \beta = \max(x_1y_1, x_1y_2, x_2y_1, x_2y_2) \]

(iv) Division:

\[ \frac{1}{\tilde{x}} = \frac{1}{[x_1, x_2]} = \left[ \frac{1}{m(\tilde{x})} \cdot k, \frac{1}{m(\tilde{x})} + k \right], \text{ where } k = \min\left\{ \frac{1}{x_2} \left( \frac{x_2 - x_1}{x_1 + x_2} \right), \frac{1}{x_1} \left( \frac{x_2 - x_1}{x_1 + x_2} \right) \right\} \]

and 0 \( \not\in [x_1, x_2] \).

Also if \( \tilde{x} = \tilde{y} \) i.e. \([x_1, x_2] = [y_1, y_2]\), then

\[ \tilde{x} = \tilde{y} = \tilde{x} \cdot \text{dual}(\tilde{x}) = [x_1, x_2] \cdot \text{dual}([x_1, x_2]) = [x_1, x_2] \cdot \text{dual}([x_1, x_2]) = [x_1 - x_2, x_2 - x_1] = [0, 0]. \]

3. Main results

An interval matrix \( \tilde{X} \) is a matrix whose elements are interval numbers. An interval matrix \( \tilde{X} \) will be written as 

\[ \tilde{A} = \left( \begin{array}{cccc} \tilde{x}_{11} & \cdots & \tilde{x}_{1n} \\ \cdots & \cdots & \cdots \\ \tilde{x}_{m1} & \cdots & \tilde{x}_{mn} \end{array} \right) = (\tilde{x}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}. \]

We use \( D^{m \times n} \) to denote the set of all \((m \times n)\) interval matrices. The midpoint of an interval matrix \( \tilde{X} \) is defined as 

\[ m(\tilde{X}) = \left( \begin{array}{cccc} m(\tilde{x}_{11}) & \cdots & m(\tilde{x}_{1n}) \\ \cdots & \cdots & \cdots \\ m(\tilde{x}_{m1}) & \cdots & m(\tilde{x}_{mn}) \end{array} \right). \]

The width of an interval matrix \( \tilde{X} \) is defined as 

\[ w(\tilde{X}) = \left( \begin{array}{cccc} w(\tilde{x}_{11}) & \cdots & w(\tilde{x}_{1n}) \\ \cdots & \cdots & \cdots \\ w(\tilde{x}_{m1}) & \cdots & w(\tilde{x}_{mn}) \end{array} \right) \]

which is always nonnegative. We use \( O \) to denote the null
matrix \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} and \( \tilde{O} \) to denote the null interval matrix \begin{pmatrix} \tilde{0} & \ldots & \tilde{0} \\ \vdots & \ddots & \vdots \\ \tilde{0} & \ldots & \tilde{0} \end{pmatrix}. Also we use \( I \) to denote the identity matrix \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} and \( \tilde{I} \) to denote the identity interval matrix \begin{pmatrix} \tilde{1} & \ldots & \tilde{0} \\ \vdots & \ddots & \vdots \\ \tilde{0} & \ldots & \tilde{1} \end{pmatrix}.

The interval matrices \( \tilde{X} \) and \( \tilde{Y} \) are said to be equivalent if \( m(\tilde{X}) = m(\tilde{Y}) \) and is denoted by \( \tilde{X} \approx \tilde{Y} \). In particular if \( m(\tilde{X}) = m(\tilde{Y}) = 0 \) and \( w(\tilde{X}) = w(\tilde{Y}) = 0 \), then \( \tilde{X} = \tilde{Y} \).

If \( m(\tilde{X}) = 0 \), then we say that \( \tilde{X} \) is a zero interval matrix and is denoted by \( \tilde{O} \). In particular if \( m(\tilde{X}) = 0 \) and \( w(\tilde{X}) = 0 \), then \( \tilde{X} = \begin{pmatrix} [0,0] & \ldots & [0,0] \\ \vdots & \ddots & \vdots \\ [0,0] & \ldots & [0,0] \end{pmatrix} \). Also if \( m(\tilde{X}) = 0 \) and \( w(\tilde{X}) \neq 0 \), then \( \tilde{X} \approx \tilde{O} \).

If \( m(\tilde{X}) = 1 \) then we say that \( \tilde{X} \) is an identity interval matrix and is denoted by \( \tilde{I} \). In particular if \( m(\tilde{X}) = 1 \) and \( w(\tilde{X}) = 0 \), then \( \tilde{X} = \begin{pmatrix} [1,1] & \ldots & [0,0] \\ \vdots & \ddots & \vdots \\ [0,0] & \ldots & [1,1] \end{pmatrix} \). Also, if \( m(\tilde{X}) = 1 \) and \( w(\tilde{X}) \neq 0 \), then \( \tilde{X} \approx \tilde{I} \).

### 3.1 Arithmetic Operations on Interval Matrices

We define arithmetic operations on interval matrices as follows: If \( \tilde{X}, \tilde{Y} \in \mathbb{D}^{m \times n}, \alpha \in \mathbb{D}^{n} \) and \( \tilde{\alpha} \in \mathbb{D} \), then

(i) \( \tilde{\alpha} \tilde{X} \approx (\tilde{\alpha}, \tilde{x}_{ij}) \) for \( 1 \leq i \leq m, \ 1 \leq j \leq n \)

(ii) \( (\tilde{X} + \tilde{Y}) \approx (\tilde{x}_{ij} + \tilde{y}_{ij}) \) for \( 1 \leq i \leq m, \ 1 \leq j \leq n \)

(iii) \( (\tilde{X} - \tilde{Y}) \approx \begin{cases} (\tilde{x}_{ij} - \tilde{y}_{ij}) & \text{if } \tilde{X} \neq \tilde{Y} \\
 & \text{if } \tilde{X} \approx \tilde{Y} \end{cases} \) for \( 1 \leq i \leq m, \ 1 \leq j \leq n \)

(iv) \( \tilde{X} \tilde{Y} \approx \sum_{k=1}^{n} \tilde{x}_{ik} \tilde{y}_{kj} \) for \( 1 \leq i \leq m, \ 1 \leq j \leq n \)

(v) \( \tilde{\alpha} \tilde{X} \approx \sum_{j=1}^{n} \tilde{a}_{j} \tilde{x}_{ij} \) for \( 1 \leq i \leq m \)
4. Algorithm for solving system of interval linear equations $\tilde{A}\tilde{x} \approx \tilde{b}$ by Gauss Jordon Method

Consider the system of interval linear equations $\tilde{A}\tilde{x} \approx \tilde{b}$, where

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{pmatrix}$$

**Step: 1** Write down the augmented interval matrix (i.e.) $(\tilde{A} | \tilde{b}) = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{b}_1 \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & \tilde{b}_2 \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & \tilde{b}_3 \end{pmatrix}$

**Step: 2** Fix the pivot row and identify the pivot element $\tilde{a}_{11}$

Modify $R_2$, using $\tilde{m}_{21} = -\frac{\tilde{a}_{21}}{\tilde{a}_{11}}$. The modified $R_2$ is $R_2 + \tilde{m}_{21} * R_1$

where new $\tilde{a}_{21} = \tilde{a}_{21} + \tilde{m}_{21} * \tilde{a}_{11} \approx 0$; $\tilde{a}_{22} = \tilde{a}_{22} + \tilde{m}_{21} * \tilde{a}_{12}$; $\tilde{a}_{23} = \tilde{a}_{23} + \tilde{m}_{21} * \tilde{a}_{13}$;

$\tilde{b}_2 = \tilde{b}_2 + \tilde{m}_{21} * \tilde{b}_1$.

Similarly, Modify $R_3$, using $\tilde{m}_{31} = -\frac{\tilde{a}_{31}}{\tilde{a}_{11}}$. The new $R_3$ is $R_3 + \tilde{m}_{31} * R_1$.

**Step: 3** In the reduced augmented matrix, fix the pivot row $R_2$ and identity the pivot element $\tilde{a}_{22}$.

Modify $R_1$, using $\tilde{m}_{12} = -\frac{\tilde{a}_{12}}{\tilde{a}_{22}}$. The new $R_1$ is $R_1 + \tilde{m}_{12} * R_2$.

Modify $R_3$, using $\tilde{m}_{32} = -\frac{\tilde{a}_{32}}{\tilde{a}_{22}}$. The new $R_3$, $R_3 + \tilde{m}_{32} * R_2$.

**Step: 4** Fix the pivot row and identify the pivot element $\tilde{a}_{33}$.

Modify $R_1$, using $\tilde{m}_{13} = -\frac{\tilde{a}_{13}}{\tilde{a}_{33}}$. The new $R_1$ is $R_1 + \tilde{m}_{13} * R_3$.

Modify $R_2$, using $\tilde{m}_{23} = -\frac{\tilde{a}_{23}}{\tilde{a}_{33}}$. The new $R_2$ is $R_2 + \tilde{m}_{23} * R_3$.

**Step: 5** The reduced augmented interval matrix is

$$(\tilde{A} | \tilde{b}) = \begin{pmatrix} \tilde{a}_{11} & 0 & 0 & \tilde{b}_1 \\ 0 & \tilde{a}_{22} & 0 & \tilde{b}_2 \\ 0 & 0 & \tilde{a}_{33} & \tilde{b}_3 \end{pmatrix}$$

and the row echelon form is

$$(\check{A} | \check{b}) = \begin{pmatrix} 1 & 0 & 0 & \check{c}_1 \\ 0 & 1 & 0 & \check{c}_2 \\ 0 & 0 & 1 & \check{c}_3 \end{pmatrix}$$

Finally, the solution set is $\check{x}_1 = \check{c}_1$; $\check{x}_2 = \check{c}_2$; $\check{x}_3 = \check{c}_3$
5. Numerical Example

Example 5.1:
In the example presented by Marzieh Dehghani-Madiseh [12]

\[
\tilde{A} = \begin{bmatrix}
[2, 4] & [-1, 1] \\
[-1, 1] & [2, 4]
\end{bmatrix}, \quad b = \begin{bmatrix}
[-3, 3] \\
0
\end{bmatrix}
\]

By using the proposed algorithm,

Step 1: The augmented interval matrix is
\[
(\tilde{A}|\tilde{b}) = \begin{bmatrix}
[2, 4] & [-1, 1] & [-3, 3] \\
[-1, 1] & [2, 4] & 0
\end{bmatrix}
\]

Step 2: The pivot element is \( \tilde{a}_{11} = [2, 4]; \tilde{m}_{21} = [-0.41667, 0.41667] \)
\( \tilde{a}_{21} = 0; \tilde{a}_{22} = [1.58333, 4.41667]; \tilde{b}_2 = [-1.25, 1.25] \)

Step 3: The pivot element is \( \tilde{a}_{22} = [1.58333, 4.41667]; \tilde{m}_{12} = [-0.4143, 0.4143] \)
\( \tilde{a}_{11} = [2, 4]; \tilde{a}_{12} = [0, 0]; \tilde{b}_1 = [-3.5179, 3.5179] \)

Step 4: The reduced augmented interval matrix is

\[
\begin{bmatrix}
[2, 4] & 0 & [-3.5179, 3.5179] \\
0 & [1.58333, 4.41667] & [-1.25, 1.25]
\end{bmatrix}
\]

Step 5: The row echelon form is
\[
\begin{bmatrix}
1 & 0 & [-1.4658, 1.4658] \\
0 & 1 & [-0.5179, 0.5179]
\end{bmatrix}
\]
\( R_1 \rightarrow R_1/[2, 4] \)
\( R_2 \rightarrow R_2/[1.58333, 4.41667] \)

Therefore, the solution set is \( \tilde{x}_1 = [-1.4658, 1.4658]; \tilde{x}_2 = [-0.5179, 0.5179] \)

6. Conclusion

In order to obtain the solution of a system of interval linear equations by means of Gauss Jordan method, we have recommended an algorithm, where we have made use of a new set of arithmetic operations on generalized interval numbers. A (3x3) interval matrix is considered to explain the proposed algorithm. It is applicable for any \((n \times n)\) interval matrix. It is to be observed that this method is used to produce an interval matrix inversion program that uses a minimum of storage.

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