The Langlands–Kottwitz approach for the modular curve

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We show how the Langlands–Kottwitz method can be used to determine the local factors of the Hasse–Weil zeta-function of the modular curve at places of bad reduction. On the way, we prove a conjecture of Haines and Kottwitz in this special case.

1 Introduction

The aim of this paper is to extend the method of Langlands [19] and Kottwitz [18] to determine the Hasse–Weil zeta function of some moduli schemes of elliptic curves with level-structure, at all places. Fix a prime \( p \) and an integer \( m \geq 3 \) prime to \( p \). Let \( \mathcal{M}_m/\mathbb{Z}[\frac{1}{m}] \) be the moduli space of elliptic curves with level-\( m \)-structure and let \( \pi_n: \mathcal{M}_{\Gamma(p^n),m} \to \mathcal{M}_m \) be the finite covering by the moduli space of elliptic curves with Drinfeld-level-\( p^n \)-structure and level-\( m \)-structure. Inverting \( p \), this gives a finite Galois cover \( \pi_{m_0}: \mathcal{M}_{\Gamma(p^n),m}[\frac{1}{p}] \cong \mathcal{M}_{p^{nm}} \to \mathcal{M}_m[\frac{1}{p}] \) with Galois group \( \text{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) \).

We make use of the concept of semisimple trace [21] (cf. also [11, Section 3.1]). Recall that in the case of a proper smooth variety \( X \) over \( \mathbb{Q} \) with good reduction at \( p \), the local factor of the Hasse–Weil zeta function is given by

\[
\log \zeta_p(X, s) = \sum_{r \geq 1} \left| \chi(\mathbb{F}_{p^r}) \right| \frac{p^{-rs}}{r}. \tag{1}
\]
for any proper smooth model $\mathfrak{X}$ over $\mathbb{Z}_{(p)}$ of $X$. This follows from the proper base change theorem for étale cohomology and the Lefschetz trace formula.

In general, for the semisimple local factor, $\zeta_{p}^{ss}$, and a proper smooth variety $X$ over $\mathbb{Q}$ with proper model $\mathfrak{X}$ over $\mathbb{Z}_{(p)}$, one has

$$\log \zeta_{p}^{ss}(X, s) = \sum_{r \geq 1} \sum_{x \in \mathfrak{X}(\mathbb{F}_{p^r})} \text{tr}^{ss}(\Phi_{p^r}|(R\psi \bar{\mathbb{Q}}_{\ell})_{x}) \frac{p^{-rs}}{r}.$$ 

Here $\Phi_{p^r}$ is a geometric Frobenius and $R\psi \bar{\mathbb{Q}}_{\ell}$ denotes the complex of nearby cycle sheaves. In the case that $\mathfrak{X}$ is smooth over $\mathbb{Z}_{(p)}$, this gives back (1) since then $\zeta_{p}^{ss} = \zeta_{p}$ and

$$\text{tr}^{ss}(\Phi_{p^r}|(R\psi \bar{\mathbb{Q}}_{\ell})_{x}) = 1.$$ 

Using the compatibility of the nearby cycles functor $R\psi$ with proper maps, we get in our situation

$$\log \zeta_{p}^{ss}(\mathcal{M}_{\Gamma(p^n), m}, s) = \sum_{r \geq 1} \sum_{x \in \mathcal{M}_{m}(\mathbb{F}_{p^r})} \text{tr}^{ss}(\Phi_{p^r}|(R\psi \mathcal{F}_n)_{x}) \frac{p^{-rs}}{r},$$

where $\mathcal{F}_n = \pi_{m|s} \bar{\mathbb{Q}}_{\ell}$. (For problems related to noncompactness of $\mathcal{M}_m$, see Theorem 7.9.) This essentially reduces the problem to that of computing the semisimple trace of Frobenius on the nearby cycle sheaves.

Our first result is a computation of the semisimple trace of Frobenius on the nearby cycles for certain regular schemes. Let $\mathcal{O}$ be the ring of integers in a local field $K$. Let $X/\mathcal{O}$ be regular and flat of relative dimension 1, with special fiber $X_s$. Let $X_{\text{ur}}$ be the base-change of $X$ to the maximal unramified extension $K_{\text{ur}}$ of $K$, let $X_{\mathcal{O}_{\text{ur}}}$ be the base-change to the ring of integers in $K_{\text{ur}}$ and let $X_{\bar{s}}$ be the geometric special fiber. Then, we have $\iota : X_{\bar{s}} \to X_{\mathcal{O}_{\text{ur}}}$ and $j : X_{\text{ur}} \to X_{\mathcal{O}_{\text{ur}}}$.

**Theorem A.** Assume that $X_{s}$ is globally the union of regular divisors. Let $x \in X_{s}(\mathbb{F}_{q})$ and let $D_1, \ldots, D_i$ be the divisors passing through $x$. Let $W_i$ be the $i$-dimensional $\bar{\mathbb{Q}}_{\ell}$-vector space with basis given by the $D_i$ and let $W_2$ be the kernel of the map $W_i \to \bar{\mathbb{Q}}_{\ell}$ sending all $D_i$ to 1. Then there are canonical isomorphisms

$$(\iota^* R^k j_* \bar{\mathbb{Q}}_{\ell})_x \cong \begin{cases} \bar{\mathbb{Q}}_{\ell}, & k = 0, \\ W_1(-1), & k = 1, \\ W_2(-2), & k = 2, \\ 0 & \text{else.} \end{cases}$$

□
The main ingredient in the proof of this lemma is Thomason’s purity theorem [24], a special case of Grothendieck’s purity conjecture. Together with some general remarks made in Section 7, this is enough to compute the semisimple trace of Frobenius. It is known that the assumptions of this lemma are fulfilled in the case of interest to us, as recalled in Section 6.

To state our second main result, we introduce some notation. For any integer \( n \geq 0 \), we define a function \( \phi_{p,n} : \text{GL}_2(\mathbb{Q}_p') \to \mathbb{Q} \). If \( n = 0 \), it is simply \( 1/p^r-1 \) times the characteristic function of the set 

\[
\text{GL}_2(\mathbb{Z}_p') \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) \text{GL}_2(\mathbb{Z}_p').
\]

If \( n \geq 0 \), we need further notation to state the definition. For \( g \in \text{GL}_2(\mathbb{Q}_p') \), let \( k(g) \) be the minimal integer such that \( p^kg \) has integral entries. Further, let \( \ell(g) = v_p(1-\text{tr } g + \det g) \). (For a more conceptual interpretation of these numbers, see Section 14.) Then

- \( \phi_{p,n}(g) = 0 \) except if \( v_p(\det g) = 1 \), \( v_p(\text{tr } g) \geq 0 \) and \( k(g) \leq n-1 \). Assume now that \( g \) has these properties.
- \( \phi_{p,n}(g) = -1 - p^r \) if \( v_p(\text{tr } g) = 1 \).
- \( \phi_{p,n}(g) = 1 - p^{2\ell(g)r} \) if \( v_p(\text{tr } g) = 0 \) and \( \ell(g) < n - k(g) \).
- \( \phi_{p,n}(g) = 1 + p^{2(n-k(g)-1)r} \) if \( v_p(\text{tr } g) = 0 \) and \( \ell(g) \geq n - k(g) \).

To any point \( x \in M_m(\mathbb{F}_p') \), there is an associated element \( \delta = \delta(x) \in \text{GL}_2(\mathbb{Q}_p') \), well defined up to \( \sigma \)-conjugation. Its construction is based on crystalline cohomology and is recalled in Section 5.

Finally, let

\[
\Gamma(p^n) = \ker(\text{GL}_n(\mathbb{Z}_p') \to \text{GL}_n(\mathbb{Z}_p'/p^n\mathbb{Z}_p')).
\]

We normalize the Haar measure on \( \text{GL}_2(\mathbb{Q}_p') \) by giving \( \text{GL}_2(\mathbb{Z}_p') \) volume \( p^r-1 \).

**Theorem B.** (i) The function \( \phi_{p,n} \) lies in the center of the Hecke algebra of compactly supported functions on \( \text{GL}_2(\mathbb{Q}_p') \) that are biinvariant under \( \Gamma(p^n) \).

(ii) For any point \( x \in M_m(\mathbb{F}_p') \) with associated \( \delta = \delta(x) \),

\[
\text{tr}^\text{ss}(\Phi | (R\psi F_n)_x) = TO_{\delta\sigma}(\phi_{p,n})(TO_{\delta\sigma}(\phi_{p,0}))^{-1}.
\]
(iii) For any irreducible admissible smooth representation $\pi$ of $GL_2(\mathbb{Q}_p)$ with $\pi^{\Gamma(p^n)\mathbb{Q}_p} \neq 0$, the function $\phi_{p,n}$ acts through the scalar

$$p^{2r} \text{tr}^{ss}(\Phi|_{\sigma_\pi})$$

on $\pi^{\Gamma(p^n)\mathbb{Q}_p}$, where $\sigma_\pi$ is the representation of the Weil–Deligne group of $\mathbb{Q}_p$ associated to $\pi$ by the local Langlands correspondence.

Part (ii) furnishes an explicit formula for the semisimple trace of Frobenius on the nearby cycles. Also note that the description of the Bernstein center implies that parts (i) and (iii) uniquely characterize the function $\phi_{p,n}$. In fact, we will use this as the definition and then verify that it agrees with the explicit function only at the end, in Section 14.

This proves a conjecture of Haines and Kottwitz in the special case at hand. The conjecture states roughly that the semisimple trace of a power of Frobenius on the $\ell$-adic cohomology of a Shimura variety can be written as a sum of products of a volume factor, an orbital integral away from $p$ and a twisted orbital integral of a function in the center of a certain Hecke algebra. This is provided by Corollary 10.1 in our case, upon summing over all isogeny classes.

In order to proceed further, one has to relate the twisted orbital integrals to usual orbital integrals. To accomplish this, we prove a base-change identity for central functions.

Let

$$\Gamma(p^n)\mathbb{Q}_p = \ker(GL_n(\mathbb{Z}_p) \to GL_n(\mathbb{Z}/p^n\mathbb{Z})).$$

Further, for $G = GL_n(\mathbb{Q}_p)$ or $G = GL_n(\mathbb{Q}_p')$, let $\mathcal{Z}(G)$ be the Bernstein center of $G$ (see Section 2). For any compact open subgroup $K$, we denote by $e_K$ its associated idempotent.

**Theorem C.** Assume that

$$f \in \mathcal{Z}(GL_2(\mathbb{Q}_p)), \quad \phi \in \mathcal{Z}(GL_2(\mathbb{Q}_p'))$$

are given such that for every tempered irreducible smooth representation $\pi$ of $GL_2(\mathbb{Q}_p)$ with base-change lift $\Pi$, the scalars $c_{f,\pi}$ resp. $c_{\phi,\Pi}$ through which $f$ resp. $\phi$ act on $\pi$ resp. $\Pi$, agree: $c_{f,\pi} = c_{\phi,\Pi}$.
Assume that \( h \in C_c^\infty(GL_2(\mathbb{Q}_p)) \) and \( h' \in C_c^\infty(GL_2(\mathbb{Q}_{p'})) \) are such that the twisted orbital integrals of \( h' \) match with the orbital integrals of \( h \) (cf. Definition 3.2). Then also \( f \ast h \) and \( \phi \ast h' \) have matching (twisted) orbital integrals.

Furthermore, \( e_{\Gamma(p^n)_{\mathbb{Q}_p}} \) and \( e_{\Gamma(p^n)_{\mathbb{Q}_{p'}}} \) have matching (twisted) orbital integrals. \( \square \)

This generalizes the corresponding result for a hyperspecial maximal compact subgroup, known as the base-change fundamental lemma. Versions of this result for general groups and parahoric subgroups have recently been obtained by Haines [10]. Together with the Arthur–Selberg trace formula and an analysis of the contribution of the “points at infinity”, Theorems B and C imply the following theorem. Recall that there is a smooth projective curve \( \tilde{\mathcal{M}}_m \) over \( \mathbb{Z}[\frac{1}{m}] \) containing \( \mathcal{M}_m \) as a fiberwise open dense subset.

**Theorem D.** Assume that \( m \) is the product of two coprime integers, both at least 3. Then the Hasse–Weil zeta-function of \( \tilde{\mathcal{M}}_m \) is given by

\[
\zeta(\tilde{\mathcal{M}}_m, s) = \prod_{\pi \in \Pi_{\text{disc}}(GL_2(\hat{\mathbb{A}^1}))} L\left( \pi, s - \frac{1}{2} \right)^{\frac{m(\pi) \chi(\pi_{\infty}) \dim \pi_f K_m}{2}}
\]

where \( \Pi_{\text{disc}}(GL_2(\hat{\mathbb{A}^1})) \) is the set of automorphic representations \( \pi = \pi_f \otimes \pi_{\infty} \) of \( GL_2(\hat{\mathbb{A}^1}) \) that occur discretely in \( L^2(GL_2(\mathbb{Q}) \mathbb{R}^\times \backslash GL_2(\hat{\mathbb{A}^1})) \) such that \( \pi_{\infty} \) has trivial central and infinitesimal character. Furthermore, \( m(\pi) \) is the multiplicity of \( \pi \) inside \( L^2(GL_2(\mathbb{Q}) \mathbb{R}^\times \backslash GL_2(\hat{\mathbb{A}^1})) \), \( \chi(\pi_{\infty}) = 2 \) if \( \pi_{\infty} \) is a character and \( \chi(\pi_{\infty}) = -2 \) otherwise, and \( K_m = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv 1 \mod m \} \). \( \square \)

**Remark.** Of course, multiplicity 1 for \( GL_2 \) tells us that \( m(\pi) = 1 \). \( \square \)

A much stronger version of this theorem has been proved by Carayol [5]. Decompose (the cuspidal part of) the \( \ell \)-adic cohomology of the modular curves according to automorphic representations \( \pi = \bigotimes \pi_p \) as

\[
\bigoplus \pi \otimes \sigma_{\pi},
\]

where \( \sigma_{\pi} \) is a two-dimensional representation of the absolute Galois group of \( \mathbb{Q} \). Then Carayol determines the restriction of \( \sigma_{\pi} \) to the absolute Galois group of \( \mathbb{Q}_p, p \neq \ell \), by
showing that it is paired (up to an explicit twist) with $\pi_p$ through the local Langlands correspondence. In particular, their $L$-functions agree up to shift, which gives our Theorem D upon taking the product over all automorphic representations $\pi$.

It would not be a serious problem to extend the methods used here to prove that all local $L$-factors of $\sigma_\pi$ and $\pi$ agree (up to shift), by allowing the action of arbitrary Hecke operators prime to $p$ in our considerations in order to “cut out” a single representation $\pi$ in the cohomology. If one could prove that the local $\epsilon$-factors of $\sigma_\pi$ and $\pi$ agree as well, this would give a new proof of Carayol’s result, but we do not see any way to check this.

It should be pointed out, however, that Carayol uses advanced methods, relying on the “local fundamental representation” constructed by Deligne [8], strong statements about nearby cycles, the consideration of more general Shimura curves and some instances of automorphic functoriality, notably the Jacquet–Langlands correspondence and base-change for $GL_2$.

By contrast, except for base-change for $GL_2$, all of these methods are avoided in this article. (In the form that our article is written, it makes use of (unramified) base-change for $GL_2$, but this is needed only for Theorem B and Theorem C and could be avoided if one is only interested in Theorem D. Only the spherical base-change identity is really needed, whose proof reduces to explicit combinatorics as in [20].) Our approach relies on the geometry of the modular curve itself, the main geometric ingredient being Theorem A which relies on Thomason’s purity theorem.

We now briefly describe the content of the different sections.

Sections 2–7 mainly recall results that will be needed later. Here, the first two sections are of a representation-theoretic nature, describing some results from local harmonic analysis, in particular, the base-change identity, Theorem C. The next sections are of a more algebro-geometric nature, describing results about the moduli space of elliptic curves with level-structure and particularly their bad reduction (Sections 4 and 6). We also recall the Langlands–Kottwitz method of counting points in Section 5 and the definition of the semisimple local factor in Section 7.

Sections 8 and 9 are technically the heart of this work. In Section 8, we prove our result on vanishing cycles, Theorem A, which allows us to compute the semisimple trace of Frobenius in the given situation. Then, in Section 9, we rewrite this result in terms of twisted orbital integrals of certain functions naturally defined through the local Langlands correspondence and prove Theorem B, modulo the explicit formula for $\phi_{p,n}$.

The rest of the article, Sections 10–13, employs the standard method of comparing the Lefschetz and the Arthur–Selberg trace formula to prove Theorem D.
Finally, Section 14 provides the explicit formula for the function $\phi_{p,n}$ and finishes the proof of Theorem B.

1.1 Notation

For any field $K$, we denote by $G_K = \text{Gal}(\overline{K}/K)$ its absolute Galois group.

2 The Bernstein Center

Let $G = \text{GL}_n(F)$, where $F$ is a local field. Let $\mathcal{H}(G, K)$ be the Hecke algebra of locally constant functions on $G$ with compact support and biinvariant under $K$, for a compact open subgroup $K$ of $G$.

We will recall the description of the center of the Hecke algebras $\mathcal{H}(G, K)$, where $K$ ranges through compact open subgroups of $G$ (cf. [3]). Denote the center by $\mathcal{Z}(G, K)$ and let $\mathcal{Z}(G) = \varprojlim \mathcal{Z}(G, K)$. Note that this is not a subset of the Hecke algebra $\mathcal{H}(G)$. Rather, it is a subset of $\hat{\mathcal{H}}(G) = \varprojlim \mathcal{H}(G, K) \supset \mathcal{H}(G)$ which can be identified (after choosing a Haar measure) with the space of distributions $T$ of $G$ such that $T * e_K$ is of compact support for all compact open subgroups $K$. Here $e_K$ is the idempotent associated to $K$, that is, the characteristic function of $K$ divided by its volume. Then $\hat{\mathcal{H}}(G)$ has an algebra structure through convolution, and its center is $\mathcal{Z}(G)$. In fact, $\mathcal{Z}(G)$ consists of the conjugation-invariant distributions in $\hat{\mathcal{H}}(G)$.

Let $\hat{G}$ be the set of irreducible smooth representations of $G$ over $\mathbb{C}$. By Schur’s lemma, we have a map $\phi : \mathcal{Z}(G) \to \text{Map}(\hat{G}, \mathbb{C}^\times)$. We will now explain how to describe the center explicitly using this map.

Let $P$ be a parabolic subgroup of $G$ with Levi subgroup $L \cong \prod_{i=1}^k \text{GL}_{n_i}$ and fix a supercuspidal representation $\sigma$ of $L$. Let $D = (\mathbb{G}_m)^k$. Then, we have a universal unramified character $\chi : L \to \Gamma(D, \mathcal{O}_D) \cong \mathbb{C}[T_1^{\pm 1}, \ldots, T_k^{\pm 1}]$ sending $(g_i)_{i=1,...,k}$ to $\prod_{i=1}^k T_i^{\psi(\det(g_i))}$. We get a corresponding family of representations $n\text{-}\text{Ind}_P^G(\sigma \chi)$ of $G$ parameterized by the scheme $D$ (here $n\text{-}\text{Ind}$ denotes the normalized induction). We will also write $D$ for the set of representations of $G$ one gets by specializing to a closed point of $D$.

Let $\text{Rep} G$ be the category of smooth admissible representations of $G$ and let $(\text{Rep} G)(L, D)$ be the full subcategory of $\text{Rep} G$ consisting of those representations that can be embedded into a direct sum of representations in the family $D$.

**Theorem 2.1.** $\text{Rep} G$ is the direct sum of the categories $(\text{Rep} G)(L, D)$ where $(L, D)$ are taken up to conjugation.
Proof. This is [3, Proposition 2.10].

Let \(W(L, D)\) be the subgroup of \(\text{Norm}_G(L)/L\) consisting of those \(n\) such that the set of representations \(D\) coincides with its conjugate via \(n\).

Theorem 2.2. Fix a supercuspidal representation \(\sigma\) of a Levi subgroup \(L\) as above. Let \(z \in \mathcal{Z}(G)\). Then \(z\) acts by a scalar on \(n\text{-Ind}_G^L(\sigma \chi_0)\) for any character \(\chi_0\). The corresponding function on \(D\) is a \(W(L, D)\)-invariant regular function. This induces an isomorphism of \(\mathcal{Z}(G)\) with the algebra of regular functions on \(\bigcup_{(L, D)} D / W(L, D)\).

Proof. This is [3, Theorem 2.13].

3 Base Change

We will establish a base change identity that will be used later. This also allows us to recall certain facts about base change of representations.

Let \(\sigma\) be the lift of Frobenius on \(\mathbb{Q}_{p'}\).

Definition 3.1. For an element \(\delta \in \text{GL}_2(\mathbb{Q}_{p'})\), we let \(N\delta = \delta \delta^\sigma \cdots \delta^{\sigma^{r-1}}\).

One easily sees that the conjugacy class of \(N\delta\) contains an element of \(\text{GL}_2(\mathbb{Q}_p)\). For \(\gamma \in \text{GL}_2(\mathbb{Q}_p)\), define the centralizer

\[ G_\gamma(R) = \{g \in \text{GL}_2(R) \mid g^{-1}\gamma g = \gamma\} \]

and for \(\delta \in \text{GL}_2(\mathbb{Q}_{p'})\), the twisted centralizer

\[ G_{\delta\sigma}(R) = \{h \in \text{GL}_2(R \otimes \mathbb{Q}_{p'}) \mid h^{-1}\delta h^\sigma = \delta\}. \]

It is known that \(G_{\delta\sigma}\) is an inner form of \(G_{N\delta}\). We choose associated Haar measures on their groups of \(\mathbb{Q}_p\)-valued points.

For any smooth function \(f\) with compact support on \(\text{GL}_2(\mathbb{Q}_p)\), put

\[ O_\gamma(f) = \int_{G_\gamma(\mathbb{Q}_p) \backslash \text{GL}_2(\mathbb{Q}_p)} f(g^{-1}\gamma g) \, dg \]
and for any smooth function $\phi$ with compact support on $GL_2(\mathbb{Q}_p)$, put

$$TO_{\delta \sigma}(\phi) = \int_{G_{\delta \sigma}(\mathbb{Q}_p) \backslash GL_2(\mathbb{Q}_p)} \phi(h^{-1} \delta h') \, dh.$$ 

**Definition 3.2.** We say that functions

$$\phi \in C_c^\infty(GL_2(\mathbb{Q}_p^r)), \quad f \in C_c^\infty(GL_2(\mathbb{Q}_p))$$

have matching (twisted) orbital integrals (sometimes we simply say that they are “associated”) if

$$O_\gamma(f) = \begin{cases} \pm TO_{\delta \sigma}(\phi) & \text{if } \gamma \text{ is conjugate to } N\delta \text{ for some } \delta, \\ 0 & \text{else.} \end{cases}$$

for all semisimple $\gamma \in GL_2(\mathbb{Q}_p)$. Here, the sign is plus except if $N\delta$ is a central element, but $\delta$ is not $\sigma$-conjugate to a central element, when it is minus. \hfill \Box

**Remark.** This definition depends on the choice of Haar measures on $GL_2(\mathbb{Q}_p)$ and $GL_2(\mathbb{Q}_p^r)$ that we will not yet fix; it does not depend on the choice of Haar measures on $G_{\delta \sigma}(\mathbb{Q}_p)$ and $G_{N\delta}(\mathbb{Q}_p)$ as long as they are chosen compatibly. \hfill \Box

**Proposition 3.3.** There is a unique bijection between $\sigma$-conjugacy classes in $GL_2(\mathbb{Z}_p^r/p^n\mathbb{Z}_p)$ and conjugacy classes in $GL_2(\mathbb{Z}/p^n\mathbb{Z})$ satisfying the requirement that if $\delta \in GL_2(\mathbb{Z}_p^r/p^n\mathbb{Z}_p)$ has the property that $N\delta = \delta \delta^{\sigma} \cdots \delta^{\sigma^{r-1}} \in GL_2(\mathbb{Z}/p^n\mathbb{Z})$, then the $\sigma$-conjugacy class of $\delta$ is mapped to the conjugacy class of $N\delta$. We denote this bijection by $\delta \mapsto N\delta$ in general.

Further, for all $\delta \in GL_2(\mathbb{Z}_p^r/p^n\mathbb{Z}_p)$,

$$G_{\delta \sigma}(\mathbb{Z}/p^n\mathbb{Z}) = \{ h \in GL_2(\mathbb{Z}_p^r/p^n\mathbb{Z}_p^r) \mid h^{-1} \delta h^\sigma = \delta \}$$

has as many elements as

$$G_{N\delta}(\mathbb{Z}/p^n\mathbb{Z}) = \{ g \in GL_2(\mathbb{Z}/p^n\mathbb{Z}) \mid g^{-1} N\delta g = N\delta \}. \hfill \Box$$

**Proof.** Let $\gamma \in GL_2(\mathbb{Z}/p^n\mathbb{Z})$. We get the commutative groups $Z_{\gamma,p} = (\mathbb{Z}/p^n\mathbb{Z}[\gamma])^\times$ and $Z_{\gamma,p^r} = (\mathbb{Z}_p/p^n\mathbb{Z}_p[\gamma])^\times$. The norm map defines a homomorphism $d_\delta : Z_{\gamma,p^r} \to Z_{\gamma,p}$. Also
define the homomorphism $d_1: \mathbb{Z}_\gamma, \text{pr} \to \mathbb{Z}_\gamma, \text{pr}$ by $d_1(x) = xx^\sigma$. By definition, we have $H^1(\text{Gal}(\mathbb{Q}^p'/\mathbb{Q}_p), \mathbb{Z}_\gamma, p) = \ker(d_2)/\text{im}(d_1)$.

Lemma 3.4. This cohomology group vanishes: $H^1(\text{Gal}(\mathbb{Q}_p'/\mathbb{Q}_p), \mathbb{Z}_\gamma, p) = 0$. Hence the following complex is exact

$$0 \to \mathbb{Z}_\gamma, p \to \mathbb{Z}_\gamma, p^{r} \xrightarrow{d_1} \mathbb{Z}_\gamma, p^{r} \xrightarrow{d_2} \mathbb{Z}_\gamma, p \to 0.$$

Proof. We have a $\text{Gal}(\mathbb{Q}_p'/\mathbb{Q}_p)$-invariant filtration on $\mathbb{Z}_\gamma, p$ given by $X_i = \ker(\mathbb{Z}_\gamma, p \to GL_2(\mathbb{Z}_p'/\mathbb{Z}_p))$ for $i = 0, \ldots, n$. By the long exact sequence for cohomology, it is enough to prove the vanishing of the cohomology for the successive quotients. But for $i \geq 1$, the quotient $X_i/X_{i+1}$ is a $\mathbb{F}_p$-subvectorspace of $\ker(GL_2(\mathbb{Z}_p'/p^{i+1}\mathbb{Z}_p') \to GL_2(\mathbb{Z}_p'/p^i\mathbb{Z}_p')) \cong \mathbb{F}_p^4$.

But by Lang’s lemma,

$$H^1(\text{Gal}(\mathbb{Q}_p'/\mathbb{Q}_p), \mathbb{F}_p) = 0.$$

For $i = 0$, Lang’s lemma works just as well, noting that the groups considered are connected.

The complex is clearly exact at the first two steps. We have just proved that it is exact at the third step. Hence the surjectivity of the last map follows by counting elements.

Given $\gamma \in GL_2(\mathbb{Z}/p^n\mathbb{Z})$, choose some $\delta \in \mathbb{Z}_\gamma, p$ with $N\delta = \gamma$. This exists by the last lemma. We claim that in this case, $G_d(\mathbb{Z}/p^n\mathbb{Z}) = G_{\gamma}(\mathbb{Z}/p^n\mathbb{Z})$ as sets.

Take $x \in G_d(\mathbb{Z}/p^n\mathbb{Z})$. Then $x^{-1}\delta x^\sigma = \delta$ and hence $x^{-1}\delta x^\sigma x^{i+1} = \delta x^{i+1}$ for all $i = 0, \ldots, r - 1$ and multiplying these equations gives $x^{-1}N\delta x = N\delta$, hence $x$ commutes with $\gamma = N\delta$. But then $x$ commutes with $\delta \in \mathbb{Z}_p'/p^n\mathbb{Z}_p'[\gamma]$ and therefore $x^{-1}\delta x^\sigma = \delta$ implies $x = x^\sigma$, whence $x \in G_{\gamma}(\mathbb{Z}/p^n\mathbb{Z})$.

The other inclusion $G_{\gamma}(\mathbb{Z}/p^n\mathbb{Z}) \subset G_d(\mathbb{Z}/p^n\mathbb{Z})$ follows directly from $\delta \in \mathbb{Z}_p'/p^n\mathbb{Z}_p'[\gamma]$. This proves the claim $G_d(\mathbb{Z}/p^n\mathbb{Z}) = G_{\gamma}(\mathbb{Z}/p^n\mathbb{Z})$ and hence the first part of the proposition in this case.

Now, for representatives $\gamma_1, \ldots, \gamma_t$ of the conjugacy classes in $GL_2(\mathbb{Z}/p^n\mathbb{Z})$, we have constructed elements $\delta_1, \ldots, \delta_t$ with $N\delta_i = \gamma_i$ for all $i$, whence representing different...
σ-conjugacy classes. We know that the size of their σ-conjugacy classes is

\[ \frac{|GL_2(\mathbb{Z}_{p'}/p^n\mathbb{Z}_{p'})|}{|G_{\delta_1}(\mathbb{Z}/p^n\mathbb{Z})|} = \frac{|GL_2(\mathbb{Z}_{p'}/p^n\mathbb{Z}_{p'})|}{|G_{\gamma_i}(\mathbb{Z}/p^n\mathbb{Z})|}. \]

The sum gives

\[ \frac{|GL_2(\mathbb{Z}_{p'}/p^n\mathbb{Z}_{p'})|}{|GL_2(\mathbb{Z}/p^n\mathbb{Z})|} \sum_{i=1}^{t} \frac{|GL_2(\mathbb{Z}/p^n\mathbb{Z})|}{|G_{\gamma_i}(\mathbb{Z}/p^n\mathbb{Z})|} = |GL_2(\mathbb{Z}_{p'}/p^n\mathbb{Z}_{p'})| = |GL_2(\mathbb{Z}_{p'}/p^n\mathbb{Z}_{p'})|. \]

Hence every element of GL₂(\mathbb{Z}_{p'}/p^n\mathbb{Z}_{p'}) is σ-conjugate to one of \( \delta_1, \ldots, \delta_t \), proving the rest of the proposition.

We use this proposition to prove the following identity. Define the principal congruence subgroups

\( \Gamma(p^n)_{\mathbb{Q}_p} = \{ g \in GL_2(\mathbb{Z}_p) \mid g \equiv 1 \mod p^k \} \), \( \Gamma(p^n)_{\mathbb{Q}_{p'}} = \{ g \in GL_2(\mathbb{Z}_{p'}) \mid g \equiv 1 \mod p^k \} \).

For any compact open subgroup \( K \) of GL₂(\mathbb{Q}_p) or GL₂(\mathbb{Q}_{p'}), let \( e_K \) be the idempotent which is the characteristic function of \( K \) divided by its volume.

**Corollary 3.5.** Let \( f \) be a conjugation-invariant locally integrable function on GL₂(\mathbb{Z}_p). Then the function \( \phi \) on GL₂(\mathbb{Z}_{p'}) defined by \( \phi(\delta) = f(N\delta) \) is locally integrable. Furthermore,

\( (e_{\Gamma(p^n)_{\mathbb{Q}_p}} * \phi)(\delta) = (e_{\Gamma(p^n)_{\mathbb{Q}_{p'}}} * f)(N\delta) \)

for all \( \delta \in GL_2(\mathbb{Z}_{p'}) \).

**Proof.** Assume first that \( f \) is locally constant, say invariant by \( \Gamma(p^n)_{\mathbb{Q}_p} \). Of course, \( \phi \) is then invariant under \( \Gamma(p^n)_{\mathbb{Q}_{p'}} \) and, in particular, locally integrable. The desired identity follows on combining Proposition 3.3 for the integers \( k \) and \( n \).

The corollary now follows by approximating \( f \) by locally constant functions. ■

Now we explain how to derive a base change fundamental lemma for elements in the center of Hecke algebras, once base change of representations is established.

Let tempered representations \( \pi, \text{ resp. } \Pi \), of GL₂(\mathbb{Q}_p), resp. GL₂(\mathbb{Q}_{p'}), be given.
Definition 3.6. In this situation, $\pi$ is called a base-change lift of $\pi$ if $\Pi$ is invariant under $\text{Gal}(\mathbb{Q}_{p'}/\mathbb{Q}_p)$ and for some extension of $\Pi$ to a representation of $\text{GL}_2(\mathbb{Q}_{p'}) \rtimes \text{Gal}(\mathbb{Q}_{p'}/\mathbb{Q}_p)$, the identity

$$\text{tr}(Ng|\pi) = \text{tr}((g, \sigma)|\Pi)$$

holds for all $g \in \text{GL}_2(\mathbb{Q}_{p'})$ such that the conjugacy class of $Ng$ is regular semi-simple. □

It is known that there exist unique base-change lifts (cf. [20]) or more generally [2].

Theorem 3.7. Assume that

$$f \in \mathcal{Z}(\text{GL}_2(\mathbb{Q}_p)) \quad \phi \in \mathcal{Z}(\text{GL}_2(\mathbb{Q}_{p'}))$$

are given such that for every tempered irreducible smooth representation $\pi$ of $\text{GL}_2(\mathbb{Q}_p)$ with base-change lift $\Pi$, the scalars $c_{f,\pi}$ resp. $c_{\phi,\Pi}$ through which $f$ resp. $\phi$ act on $\pi$ resp. $\Pi$, agree: $c_{f,\pi} = c_{\phi,\Pi}$.

Then for any associated $h \in C_c^\infty(\text{GL}_2(\mathbb{Q}_p))$ and $h' \in C_c^\infty(\text{GL}_2(\mathbb{Q}_{p'}))$, also $f \ast h$ and $\phi \ast h'$ have matching (twisted) orbital integrals.

Furthermore, $e_{\Gamma(p^n)\mathbb{Q}_p}$ and $e_{\Gamma(p^n)\mathbb{Q}_{p'}}$ are associated. □

Proof. Because $h$ and $h'$ are associated, we have $\text{tr}(h|\pi) = \text{tr}((h', \sigma)|\Pi)$ if $\Pi$ is a base-change lift of $\pi$, as follows from the Weyl integration formula (cf. [20, p. 99]) for the twisted version. We find

$$\text{tr}(f \ast h|\pi) = c_{f,\pi} \text{tr}(h|\pi) = c_{\phi,\Pi} \text{tr}((h', \sigma)|\Pi) = \text{tr}((\phi \ast h', \sigma)|\Pi).$$

We may find a function $f' \in \mathcal{H}(\text{GL}_2(\mathbb{Q}_p))$ that has matching (twisted) orbital integrals with $\phi \ast h'$ (cf. [20, Proposition 6.2]). This implies that $\text{tr}((\phi \ast h', \sigma)|\Pi) = \text{tr}(f'|\pi)$. Hence $\text{tr}((f \ast h - f')|\pi) = 0$ for all tempered irreducible smooth representations $\pi$ of $\text{GL}_2(\mathbb{Q}_p)$. By Kazhdan’s density theorem ([15, Theorem 1]) all regular semi-simple orbital integrals of $f \ast h - f'$ vanish. Hence $f \ast h$ and $\phi \ast h'$ have matching regular semi-simple (twisted) orbital integrals. By [7, Proposition 7.2], all semi-simple (twisted) orbital integrals of $f \ast h$ and $\phi \ast h'$ match.
To show the last statement, we first check that

\[ \text{tr}(e_{\Gamma(p^n)\otimes k}\pi) = \text{tr}((e_{\Gamma(p^n)\otimes k}, \sigma)|\Pi). \]

But this follows directly from Corollary 5.5 with \( f \) the character of \( \pi \) restricted to \( \text{GL}_2(\mathbb{Z}_p) \), \( k = n \) and \( \delta = 1 \), because characters are locally integrable. Now the rest of the argument is precisely as above.

\[ \square \]

4 The Moduli Space of Elliptic Curves with Level Structure: Case of Good Reduction

We will briefly recall some aspects of the theory of the moduli space of elliptic curves with level structure that we shall need. All of the material presented in this section is contained in [9].

**Definition 4.1.** A morphism \( p : E \to S \) of schemes with a section \( e : S \to E \) is said to be an elliptic curve over \( S \) if \( p \) is proper, flat, and all geometric fibers are elliptic curves (with zero section given by \( e \)). \( \square \)

We simply say that \( E/S \) is an elliptic curve, omitting the morphisms \( p \) and \( e \) in the notation. It is well known that an elliptic curve is canonically a commutative group scheme over \( S \), with \( e \) as unit section.

One might try to represent the functor

\[ \mathfrak{M} : \text{(schemes)} \to \text{(sets)} \]

\[ S \mapsto \{ E/S \text{ elliptic curve over } S \text{ up to isomorphism} \}, \]

but it is well known that this is not representable by a scheme. We need the following definition.

**Definition 4.2.** A level-\( m \)-structure on an elliptic curve \( E/S \) is an isomorphism of group schemes over \( S \)

\[ \alpha : (\mathbb{Z}/m\mathbb{Z})^2_S \to E[m], \]

where \( E[m] \) is the preimage of (the closed subscheme) \( e \) under multiplication by \( m : E \to E \). \( \square \)
This is motivated by the fact that for \( S = \text{Spec} \, k \) the spectrum of an algebraically closed field \( k \) of characteristic prime to \( m \), one always has (noncanonically) \( E[m] \cong (\mathbb{Z}/m\mathbb{Z})^2 \). However, for algebraically closed fields \( k \) the characteristic of which divides \( m \), there are no level-\( m \)-structures at all and it follows that if \((E/S, \alpha)\) is an elliptic curve with level-\( m \)-structure then \( m \) is invertible on \( S \). Consider now the following functor:

\[
\mathcal{M}_m : (\text{schemes}/\mathbb{Z}[m^{-1}]) \rightarrow (\text{sets})
\]

\[
S \mapsto \left\{ (E/S, \alpha) \text{ elliptic curve } E \text{ over } S \text{ with level-}m\text{-structure } \alpha, \text{ up to isomorphism} \right\}.
\]

**Theorem 4.3.** For \( m \geq 3 \), the functor \( \mathcal{M}_m \) is representable by a smooth affine curve \( \mathcal{M}_m \) over \( \text{Spec} \, \mathbb{Z}[1/m] \). There is a projective smooth curve \( \bar{\mathcal{M}}_m \) containing \( \mathcal{M}_m \) as an open dense subset such that the boundary \( \partial \mathcal{M}_m = \bar{\mathcal{M}}_m \setminus \mathcal{M}_m \) is étale over \( \text{Spec} \, \mathbb{Z}[1/m] \). □

Because the integer \( m \) plays a minor role in the following, we will write \( \mathcal{M} \) for \( \mathcal{M}_m \).

5 Counting Points: The Langlands–Kottwitz Approach

We will explain the method of Langlands–Kottwitz to count the number of points mod \( p \) of Shimura varieties with good reduction, in the case of the modular curve. This is based on some unpublished notes of Kottwitz [17].

Let \( p \) be a prime not dividing \( m \). Fix an elliptic curve \( E_0 \) over \( \mathbb{F}_{p^r} \), for some positive integer \( r \). Let \( \mathbb{A}_f^p \) be the ring of finite adèles of \( \mathbb{Q} \) with trivial \( p \)-component and \( \hat{\mathbb{Z}}^p \cong \prod_{\ell \neq p} \mathbb{Z}_\ell \) be the integral elements in \( \mathbb{A}_f^p \).

We wish to count the number of elements of

\[
\mathcal{M}(\mathbb{F}_{p^r})(E_0) := \{ x \in \mathcal{M}(\mathbb{F}_{p^r}) \mid E_x \text{ is } \mathbb{F}_{p^r}-\text{isogeneous to } E_0 \}.
\]

Define

\[
H^p = H^1_{et}(E_0, \mathbb{A}_f^p), \quad H_p = H^1_{cris}(E_0/\mathbb{Z}_{p^r}) \otimes_{\mathbb{Z}_{p^r}} \mathbb{Q}_{p^r}.
\]

Now take \( x \in \mathcal{M}(\mathbb{F}_{p^r})(E_0) \) to be arbitrary. Choosing an \( \mathbb{F}_{p^r} \)-isogeny \( f : E_0 \to E_x \), we get a \( G_{\mathbb{F}_{p^r}} = \text{Gal}(\mathbb{F}_{p^r}/\mathbb{F}_{p^r}) \)-invariant \( \hat{\mathbb{Z}}^p \)-lattice

\[
L = f^*(H^1_{et}(E_x, \hat{\mathbb{Z}}^p)) \subset H^p.
\]
an $F$, $V$-invariant $\mathbb{Z}_{p'}$-lattice

$$\Lambda = f^*(H^1_{\text{cris}}(E_0/\mathbb{Z}_{p'})) \subset H_p,$$

and a $G_{\mathbb{F}_{p'}}$-invariant isomorphism

$$\phi : (\mathbb{Z}/m\mathbb{Z})^2 \longrightarrow L \otimes \mathbb{Z}/m\mathbb{Z}$$

(where the left-hand side has the trivial $G_{\mathbb{F}_{p'}}$-action), corresponding to the level-$m$-structure. Let $Y^p$ be the set of such $(L, \phi)$ and $Y_p$ be the set of $\Lambda$ as above. Dividing by the choice of $f$, we get a map

$$\mathcal{M}(\mathbb{F}_{p'})(E_0) \longrightarrow \Gamma \backslash Y^p \times Y_p,$$

where $\Gamma = (\text{End}(E_0) \otimes \mathbb{Q})^\times$.

**Theorem 5.1.** This map is a bijection. \hfill \Box

**Proof.** Assume that $(E_1, \phi_1)$ and $(E_2, \phi_2)$ have the same image. Choose isogenies $f_1 : E_0 \rightarrow E_1$, $f_2 : E_0 \rightarrow E_2$. Then the corresponding elements of $Y^p \times Y_p$ differ by an element $h \in \Gamma$. Write $h = m^{-1}h_0$ where $m$ is an integer and $h_0$ is a self-isogeny of $E_0$. Changing $f_1$ to $f_1h$ and $f_2$ to $f_2m$, we may assume that the elements of $Y^p \times Y_p$ are the same. We wish to see that $f = f_1f_2^{-1}$, a priori an element of $\text{Hom}(E_2, E_1) \otimes \mathbb{Q}$, actually belongs to $\text{Hom}(E_2, E_1)$. Analogously, $f_2f_1^{-1}$ will be an actual morphism, so that they define inverse isomorphisms.

Now, let $M$ be an integer such that $Mf : E_2 \rightarrow E_1$ is an isogeny. Our knowledge of what happens on the cohomology implies by the theory of étale covers of $E_1$ and the theory of Dieudonné modules, that $Mf$ factors through multiplication by $M$. This is what we wished to show. Note that $\phi_1$ and $\phi_2$ are carried to each other by assumption.

For surjectivity, let $(L, \phi, \Lambda) \in Y^p \times Y_p$ be given. By changing by a scalar, we may assume that $L$ and $\Lambda$ are contained in the integral lattices $H^1_{\text{et}}(E_0, \mathbb{Z}_{p'}, \hat{\mathbb{Z}}_p)$, $H^1_{\text{cris}}(E_0/\mathbb{Z}_{p'})$. Then the theory of Dieudonné modules provides us with a subgroup of $p$-power order $G_p$ corresponding to $\Lambda$ and the theory of étale covers of $E_0$ provides us with a subgroup $G_p$ of order prime to $p$, corresponding to $L$. We then take $E_1 = E_0/G_pG_p$. It is easy to see that this gives the correct lattices. Of course, $\phi$ provides a level-$m$-structure. \hfill \blacksquare

From here, it is straightforward to deduce the following corollary. Let $\gamma \in \text{GL}_2(\mathbb{A}_f^p)$ be the endomorphism induced by $\Phi_{p'}$ on $H^p$ (after choosing a basis of $H^p$).
Similarly, let $\delta \in \text{GL}_2(\mathbb{Q}_p)$ be induced by the $p$-linear endomorphism $F$ on $H_p$ (after choosing a basis of $H_p$): If $\sigma$ is the $p$-linear isomorphism of $H_p$ preserving the chosen basis, define $\delta$ by $F = \delta \sigma$. Then we have the centralizer

$$G_\gamma(\mathbb{A}^P_f) = \{g \in \text{GL}_2(\mathbb{A}^P_f) \mid g^{-1} \gamma g = \gamma\}$$

of $\gamma$ in $\text{GL}_2(\mathbb{A}^P_f)$ and the twisted centralizer

$$G_{\delta \sigma}(\mathbb{Q}_p) = \{h \in \text{GL}_2(\mathbb{Q}_p) \mid h^{-1} \delta h^p = \delta\}$$

of $\delta$ in $\text{GL}_2(\mathbb{Q}_p)$. Let $f^p$ be the characteristic function of the set $K^p = \{g \in \text{GL}_2(\hat{\mathbb{Z}}^p) \mid g \equiv 1 \mod m\}$ divided by its volume and let $\phi_{p,0}$ be the characteristic function of the set

$$\text{GL}_2(\mathbb{Z}_p^r) \left( \begin{array}{cc} p & 0 \\
0 & 1 \end{array} \right) \text{GL}_2(\mathbb{Z}_p^r)$$

divided by the volume of $\text{GL}_2(\mathbb{Z}_p^r)$. For any smooth function with compact support $f$ on $\text{GL}_2(\mathbb{A}^P_f)$, put

$$O_\gamma(f) = \int_{G_\gamma(\mathbb{A}^P_f) \setminus \text{GL}_2(\mathbb{A}^P_f)} f(g^{-1} \gamma g) \, dg.$$

**Corollary 5.2.** The cardinality of $\mathcal{M}(\mathbb{F}_p^r)(E_0)$ is

$$\text{vol}(\Gamma \setminus G_\gamma(\mathbb{A}^P_f) \times G_{\delta \sigma}(\mathbb{Q}_p)) O_\gamma(f^p) T O_{\delta \sigma}(\phi_{p,0}),$$

where the Haar measure on $\Gamma$ gives point measure 1. \hfill \Box

**Proof.** Choose the integral cohomology of $E_0$ as a base point in $Y^p$ and $Y_p$. Then, we may identify the set $X^p$ of pairs $(L, \phi)$ as above, but without the Galois-invariance condition, with $\text{GL}_2(\mathbb{A}^P_f)/K^p$. Similarly, we may identify $X_p$, the set of all lattices $\Lambda$, with $\text{GL}_2(\mathbb{Q}_p)/K_p$, where $K_p = \text{GL}_2(\mathbb{Z}_p^r)$. The condition that an element $gK^p$ of $X^p$ lies in $Y^p$ is then expressed by saying that $\gamma gK^p = gK^p$, or equivalently $g^{-1} \gamma g \in K^p$. Similarly, the condition that an element $hK_p$ of $X_p$ lies in $Y_p$ is expressed by $FhK_p \subset hK_p$ and $VhK_p \subset hK_p$. Noting that $FV = p$, this is equivalent to $phK_p \subset FhK_p \subset hK_p$, that is, $pK_p \subset h^{-1} \delta h^p K_p \subset K_p$. The Weil pairing gives an isomorphism of the second exterior power of $H_p$ with $\mathbb{Q}_p^r(-1)$, so that $v_p(\det \delta) = 1$. In particular, the condition on $h$ can
be rewritten as
\[ h^{-1}\delta h' \in K_p \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} K_p. \]

This means that the cardinality of \( \Gamma \backslash Y^p \times Y_p \) is equal to
\[
\int_{\Gamma \backslash \text{GL}_2(\mathbb{A}_K) \times \text{GL}_2(\mathbb{Q}_p)} f^p(g^{-1} \gamma g)\phi_{p,0}(h^{-1}\delta h') \, dg \, dh.
\]

The formula of the corollary is a simple transcription. 

\[\blacksquare\]

**Remark.** In particular \( TO_{\delta_\sigma}(\phi_{p,0}) \neq 0 \) whenever \( \mathcal{M}(\mathbb{F}_{p^r})(E_0) \neq \emptyset \).

\[\square\]

6 The Moduli Space of Elliptic Curves with Level Structure: Case of Bad Reduction

We are interested in extending the moduli spaces \( \mathcal{M}_m \), defined over \( \text{Spec } \mathbb{Z}[\frac{1}{m}] \), to the remaining primes, where they have bad reduction. The material presented here is contained in [14]. Let us fix a prime \( p \) first and choose some integer \( m \geq 3 \) prime to \( p \). For any integer \( n \geq 0 \), we wish to extend the scheme \( \mathcal{M}_{p^nm} \) over \( \text{Spec } \mathbb{Z}[\frac{1}{m}] \), noting that we already have defined it over \( \text{Spec } \mathbb{Z}[\frac{1}{pm}] \).

**Definition 6.1.** A Drinfeld-level-\( p^n \)-structure on an elliptic curve \( E/S \) is a pair of sections \( P, Q : S \rightarrow E[p^n] \) such that there is an equality of relative Cartier divisors
\[
\sum_{i,j \in \mathbb{Z}/p^n\mathbb{Z}} [iP + jQ] = E[p^n].
\]

Since for \( p \) invertible on \( S \), the group scheme \( E[p^n] \) is étale over \( S \), a Drinfeld-level-\( p^n \)-structure coincides with an ordinary level-\( p^n \)-structure in this case. Hence the following gives an extension of the functor \( \mathcal{M}_{p^nm} \) to schemes over \( \text{Spec } \mathbb{Z}[\frac{1}{m}] \):

\[
\mathcal{M}_{\Gamma(p^n),m} : \text{(schemes}/\mathbb{Z}[m^{-1}]\text{)} \rightarrow \text{(Sets)}
\]

\[ S \mapsto \left\{ \begin{array}{l}
(E/S, (P, Q), \alpha) \text{ elliptic curve } E \text{ over } S \text{ with} \\
\text{Drinfeld-level-}p^n\text{-structure } (P, Q) \text{ and} \\
\text{level-}m\text{-structure } \alpha, \text{ up to isomorphism} \end{array} \right\}. \]
Theorem 6.2. The functor $\mathcal{M}_{\Gamma(p^n), m}$ is representable by a regular scheme $\mathcal{M}_{\Gamma(p^n), m}$ which is an affine curve over $\text{Spec } \mathbb{Z}[\frac{1}{m}]$. The canonical (forgetful) map

$$\pi_n : \mathcal{M}_{\Gamma(p^n), m} \to \mathcal{M}_m$$

is finite. Over $\text{Spec } \mathbb{Z}[\frac{1}{pm}]$, it is an étale cover with Galois group $\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$. □

Again, the integer $m$ plays a minor role, so we will suppress it from the notation and write $\mathcal{M}_{\Gamma(p^n)}$ for $\mathcal{M}_{\Gamma(p^n), m}$.

In this situation, the problem of compactification is slightly more difficult. Recall that the Weil pairing is a perfect pairing $E[p^n] \times S E[p^n] \to \mu_{p^n}$, $S$. It allows us to define a morphism

$$\mathcal{M}_{\Gamma(p^n)} \to \text{Spec } \mathbb{Z}[m^{-1}][\zeta_{p^n}],$$

where $\zeta_{p^n}$ is a primitive $p^n$th root of unity, by sending $\zeta_{p^n}$ to the image of the universal sections $(P, Q)$ under the Weil pairing.

Theorem 6.3. There is a smooth proper curve $\tilde{\mathcal{M}}_{\Gamma(p^n)/\mathbb{Z}[m^{-1}][\zeta_{p^n}]}$ with $\mathcal{M}_{\Gamma(p^n)}$ as an open subset such that the complement is étale over $\text{Spec } \mathbb{Z}[m^{-1}][\zeta_{p^n}]$ and has a smooth neighborhood. □

We end this section with a description of the special fiber in characteristic $p$ of $\mathcal{M}_{\Gamma(p^n)}$. For any direct summand $H \subset (\mathbb{Z}/p^n\mathbb{Z})^2$ of order $p^n$, write $\mathcal{M}^{H}_{\Gamma(p^n)}$ for the reduced subscheme of the closed subscheme of $\mathcal{M}_{\Gamma(p^n)}$ where

$$\sum_{(i, j) \in H \subset (\mathbb{Z}/p^n\mathbb{Z})^2} [iP + jQ] = p^n[e].$$

Theorem 6.4. For any $H$, the closed subscheme $\mathcal{M}^{H}_{\Gamma(p^n)}$ is a regular divisor on $\mathcal{M}_{\Gamma(p^n)}$ which is supported in $\mathcal{M}_{\Gamma(p^n)} \otimes_{\mathbb{Z}} \mathbb{F}_p$. Any two of them intersect exactly at the supersingular points of $\mathcal{M}_{\Gamma(p^n)} \otimes_{\mathbb{Z}} \mathbb{F}_p$, that is, those points such that the associated elliptic curve
is supersingular. Furthermore,

$$\mathcal{M}_{\Gamma(p^n)} \otimes_{\mathbb{Z}} \mathbb{F}_p = \bigcup_H \mathcal{M}^H_{\Gamma(p^n)}. \quad \square$$

7 The (semisimple) Local Factor

In this section, we wish to recall certain invariants attached to (the cohomology) of a variety $X$ over a local field $K$ with a residue field $\mathbb{F}_q$. Recall that we denoted $G_K = \text{Gal}(\bar{K}/K)$. Further, let $I_K \subset G_K$ be the inertia subgroup and let $\Phi_q$ be a geometric Frobenius element.

Let $\ell$ be a prime which does not divide $q$.

**Definition 7.1.** The Hasse–Weil local factor of $X$ is

$$\zeta(X, s) = \prod_{i=0}^{2 \dim X} \det(1 - \Phi_q q^{-s} | H^i_c(X \otimes_K \bar{K}, \bar{\mathbb{Q}}_{\ell})^{I_K})^{(-1)^{i+1}}. \quad \square$$

Here $H^i_c$ denotes étale cohomology with compact supports.

Note that this definition depends on $\ell$; it is however conjectured that it is independent of $\ell$, as follows from the monodromy conjecture. As we are working only with curves and the monodromy conjecture for curves is proved in [22], we get no problems.

It is rather hard to compute the local factors if $X$ has bad reduction. However, there is a slight variant which comes down to counting points “with multiplicity”. For this, we need to introduce the concept of semisimple trace, for which we also refer the reader to [11].

Let $V$ be a continuous representation of $G_K$ in a finite dimensional $\bar{\mathbb{Q}}_{\ell}$-vector space, where $\ell$ is prime to the residue characteristic of $K$. Furthermore, let $H$ be a finite group acting on $V$, commuting with the action of $G_K$.

**Lemma 7.2.** There is a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

into $G_K \times H$-invariant subspaces $V_i$ such that $I_K$ acts through a finite quotient on $\text{gr } V_i = \bigoplus_{i=1}^k V_i/V_{i-1}. \quad \square$
Proof. Note that this contains Grothendieck’s local monodromy theorem [25]. We will repeat the proof here. By induction, it suffices to find a nonzero $G_K \times H$-stable subspace $V_1$ on which $I_K$ acts through a finite quotient. In fact, it is enough to find a $I_K \times H$-stable subspace with this property, as the maximal $I_K \times H$-stable subspace on which $I_K$ acts through a finite quotient is automatically $G_K \times H$-stable, because $I_K \times H$ is normal in $G_K \times H$.

First, we check that the image of $I_K \times H$ is contained in $GL_n(E)$ for some finite extension $E$ of $\mathbb{Q}_\ell$. Denote $\rho : I_K \times H \to GL_n(\bar{\mathbb{Q}}_\ell)$. Since $I_K \times H$ is (locally) compact and Hausdorff, it is a Baire space, that is, the intersection of countably many dense open subsets is nonempty. Assume that there was no such extension $E$. For all $E$, $\rho^{-1}(GL_n(\bar{\mathbb{Q}}_\ell) \setminus GL_n(E))$ is an open subset of $I_K \times H$. Clearly, their intersection is empty and there are only countably many finite extensions $E$ of $\mathbb{Q}_\ell$ inside $\bar{\mathbb{Q}}_\ell$. Hence one of them is not dense. But then some subgroup of finite index maps to $GL_n(E)$, which easily implies the claim, after passing to a finite extension.

Since $I_K \times H$ is compact, the map $\rho : I_K \times H \to GL_n(E)$ factors through some maximal compact subgroup, which after conjugation may be assumed to be $GL_n(O)$, where $O$ is the ring of integral elements of $E$. Let $F$ be the residue field of $E$.

There is a surjection $t : I_K \to \mathbb{Z}_\ell$ the kernel $I_{\ell}^t$ of which is an inverse limit of groups of order prime to $\ell$. But the kernel of the map $GL_n(O) \to GL_n(F)$ is a pro-$\ell$-group and hence meets $I_{\ell}^t$ trivially. This means that $I_{\ell}^t$ acts through a finite quotient on $V$. Let $I_{\ell}^{t'}$ be the kernel of $I_{\ell}^t \to GL_n(O)$ and let $I_{\ell}^t = I_K / I_{\ell}^{t'}$ have center $Z$. Our considerations show that $t|_Z : Z \to \mathbb{Z}_\ell$ is nontrivial and has finite kernel.

Let $\lambda \in Z$ with $t(\lambda) \neq 0$. Recall that $\Phi^{-1}_q t(g) \Phi_q = qt(g)$ for all $g \in I_K$. In particular, there are positive integers $r$ and $s$ such that $\Phi^{-3s}_q \Phi^s_q = \lambda^{qs}$, so that the image $\rho(\lambda)^q$ in $GL_n(E)$ is conjugate to $\rho(\lambda)^{qs}$. This implies that all eigenvalues of $\rho(\lambda)$ are roots of unity, so that by replacing $\lambda$ by a power, we may assume that $W = V^{\lambda=1}$ is nontrivial. But since $\lambda \in Z$, $W$ is $I_K \times H$-stable. □

Definition 7.3. For $h \in H$, we define

$$tr^{ss}(\Phi^r_q h|V) = tr(\Phi^r_q h|(gr V_\bullet)^{I_K})$$

for any filtration $V_\bullet$ as in the previous lemma. □
Proposition 7.4. This definition is independent of the choice of the filtration. In particular, the semisimple trace is additive in short exact sequences.

Proof. Taking a common refinement of two filtrations, this reduces to the well-known statement that for any endomorphism \( \phi \) of a vector space \( V \) with \( \phi \)-invariant subspace \( W \), one has

\[
\text{tr}(\phi | V) = \text{tr}(\phi | W) + \text{tr}(\phi | V/W).
\]

This allows one to define the semisimple trace on the Grothendieck group, or on the derived category of finite-dimensional continuous \( \ell \)-adic representations of \( G_K \times H \).

Next, we explain a different point of view on the semisimple trace. Let us consider the bounded derived category \( D^b(\text{Rep} \mathbb{Q}_\ell(G_K \times H)) \) of continuous representations of \( G_K \times H \) in finite dimensional \( \mathbb{Q}_\ell \)-vector spaces.

Remark. Note that the correct version of the derived category of \( \ell \)-adic sheaves on a scheme \( X \) is defined as direct 2-limit over all finite extensions \( E \subset \mathbb{Q}_\ell \) of the inverse 2-limit of the derived categories of constructible \( \mathbb{Z}/\ell^n\mathbb{Z} \)-sheaves, tensored with \( E \). We use the same definition of \( D^b(\text{Rep} \mathbb{Q}_\ell(G_K \times H)) \) as the direct 2-limit of the inverse 2-limit of \( D^b(\text{Rep} \mathbb{Z}/\ell^n\mathbb{Z}(G_K \times H)) \), tensored with \( E \), here. See [16, Chapter 2], for a detailed discussion.

Consider the derived functor

\[
R_{I_K} : D^b(\text{Rep} \mathbb{Q}_\ell(G_K \times H)) \rightarrow D^b(\text{Rep} \mathbb{Q}_\ell(G_{\mathbb{F}_q} \times H))
\]

of taking invariants under \( I_K \).

Remark. Again, this is abuse of language as only with finite coefficients, this really is the derived functor.

The finiteness properties needed here are special cases of the finiteness theorems for étale cohomology: Consider

\[
\text{Spec } \mathbb{F}_q \rightarrow \text{Spec } \mathcal{O}_{\text{ur}} \leftarrow \text{Spec } K_{\text{ur}},
\]
where \( K^{ur} \) is the maximal unramified extension of \( K \), and \( \mathcal{O}^{ur} \) are its integral elements. Then \( R_{I_K} = \iota^* R_{I_S} \).

We have defined a map

\[
\text{tr}^{ss}(\Phi^r q h) : D^b(\text{Rep}_{\overline{Q}_\ell}(G_K \times H)) \to \overline{\mathbb{Q}}_\ell
\]

that is additive in distinguished triangles. There is a second map

\[
\text{tr}(\Phi^r q h) \circ R_{I_K} : D^b(\text{Rep}_{\overline{Q}_\ell}(G_K \times H)) \to \overline{\mathbb{Q}}_\ell.
\]

Again, it is additive in distinguished triangles.

**Lemma 7.5.** These two linear forms are related by

\[
\text{tr}(\Phi^r q h) \circ R_{I_K} = (1 - q^r) \text{tr}^{ss}(\Phi^r q h).
\]

**Proof.** Because of the additivity of both sides and the existence of filtrations as in Lemma 7.2, it suffices to check this for a complex

\[
\cdots \to 0 \to V_0 \to 0 \to \cdots
\]

concentrated in degree 0 and with \( I_K \) acting through a finite quotient on \( V_0 \). We can even assume that this quotient is cyclic, as taking invariants under the wild inertia subgroup is exact, and the tame inertia group is procyclic. Then \( \text{tr}^{ss}(\Phi^r q h|_{V_0}) = \text{tr}(\Phi^r q h|_{V_0^{I_K}}) \) and \( R_{I_K}(V_0) \) is represented by the complex

\[
\cdots \to 0 \to V_0^{I_K} \to V_0^{I_K}(-1) \to V_0^{I_K} \to \cdots
\]

The lemma is now obvious.

Let \( X \) be a variety over \( K \).

**Definition 7.6.** The semisimple local factor is defined by

\[
\log \zeta^{ss}(X, s) = \sum_{r \geq 1} \sum_{i=0}^{2 \dim X} (-1)^i \text{tr}^{ss}(\Phi^r q H^i_c(X \otimes \tilde{K}, \overline{\mathbb{Q}}_\ell)) \frac{q^{-r s}}{r}.
\]
Note that if $I_K$ acts through a finite quotient (e.g., if it acts trivially, as is the case when $X$ has good reduction), this agrees with the usual local factor.

Let $O \subset K$ be the ring of integers. For a scheme $X_O/O$ of finite type, we write $X_s, X_{\bar{s}}, X_{\bar{\eta}}$ resp. $X_{\bar{\eta}}$ for its special, geometric special, generic resp. geometric generic fiber. Let $X_{\bar{\phi}}$ denote the base change to the ring of integers in the algebraic closure of $K$. Then we have maps $\bar{i}: X_{\bar{s}} \to X_{\bar{\phi}}$ and $\bar{j}: X_{\bar{\eta}} \to X_{\bar{\phi}}$.

**Definition 7.7.** For a $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $X_{\bar{\eta}}$, the complex of nearby cycle sheaves is defined to be

$$R\psi \mathcal{F} = \bar{i}^* R\bar{j}_* \mathcal{F}_{\bar{\eta}},$$

where $\mathcal{F}_{\bar{\eta}}$ is the pullback of $\mathcal{F}$ to $X_{\bar{\eta}}$. This is an element of the (so-called) derived category of $\bar{\mathbb{Q}}_\ell$-sheaves on $X_{\bar{s}}$ with an action of $G_K$ that is compatible with its action on $X_{\bar{s}}$. □

**Theorem 7.8.** Assume that $X_O/O$ is a scheme of finite type such that there exists an open immersion $X_O \subset \tilde{X}_O$ where $\tilde{X}_O$ is proper over $O$, with complement $D$ a relative normal crossings divisor (i.e., there is an open neighborhood $U$ of $D$ in $\tilde{X}_O$ which is smooth over $O$, such that $D$ is a relative normal crossings divisor in $U$). Then there is a canonical $G_K$-equivariant isomorphism

$$H^i_c(X_{\bar{s}}, \bar{\mathbb{Q}}_\ell) \cong H^i_c(X_{\bar{s}}, R\psi \bar{\mathbb{Q}}_\ell)$$

and

$$\log \xi^{ss}(X_{\bar{s}}, s) = \sum_{r \geq 1} \sum_{x \in X_s(F_q^r)} \tr^{ss}(\Phi_q^r| (R\psi \bar{\mathbb{Q}}_\ell)_x) \frac{q^{-rs}}{r}.$$

**Proof.** The first statement follows from [26, XIII, Proposition 2.1.9]. For the second statement, it comes down to

$$\sum_{t=0}^{2 \dim X} (-1)^t \tr^{ss}(\Phi_q^t| H^i_c(X_{\bar{s}}, R\psi \bar{\mathbb{Q}}_\ell)) = \sum_{x \in X_s(F_q^r)} \tr^{ss}(\Phi_q^r| (R\psi \bar{\mathbb{Q}}_\ell)_x),$$

which follows from the version of the Lefschetz trace formula in [11, Proposition 10, part (2)].

□

With these preparations, we deduce the following result.
Theorem 7.9. There is a canonical \( G_{\mathbb{Q}_p} \)-equivariant isomorphism

\[
H^i_c(\mathcal{M}_{\Gamma(p^n), \eta}, \bar{Q}_\ell) \longrightarrow H^i_c(\mathcal{M}_{\Gamma(p^n), \bar{s}}, R\psi \bar{Q}_\ell).
\]

In particular, the formula for the semisimple local factor from Theorem 7.8 holds true. \( \square \)

Proof. We cannot apply Theorem 7.8 to the scheme \( X = \mathcal{M}_{\Gamma(p^n)} \) as its divisor at infinity is not étale over \( S = \text{Spec} \mathbb{Z}[m^{-1}] \). There is the following way to circumvent this difficulty.

We always have a canonical \( G_{\mathbb{Q}_p} \)-equivariant morphism

\[
H^i_c(X_{\bar{s}}, \bar{Q}_\ell) \longrightarrow H^i_c(X_{\bar{s}}, R\psi \bar{Q}_\ell).
\]

To check that it is an isomorphism, we can forget about the Galois action.

We may also consider \( X \) as a scheme over \( S = \text{Spec} \mathbb{Z}[m^{-1}]/(\zeta^{p^n}) \). Let \( X' \) be the normalization of \( X \times_S S' \). As \( X \) is normal (since regular) and on the generic fibers, \( (X \times_S S')_{\bar{s}} \) is a disjoint union of copies of \( X \), parameterized by the primitive \( p^n \)th roots of unity, it follows that \( X'/S' \) is a disjoint union of copies of \( X/S' \). With Theorem 6.3, it follows that we may use Theorem 7.8 for \( X' \). Hence

\[
H^i_c(X_{\bar{s}}, \bar{Q}_\ell) \cong H^i_c(X_{\bar{s}}', R\psi' \bar{Q}_\ell),
\]

where \( R\psi' \bar{Q}_\ell \) are the nearby cycles for \( X' \) and \( \bar{s}' \) is the geometric special point of \( S' \). Note that \( g : X_{\bar{s}}' \to X_{\bar{s}} \) is an infinitesimal thickening. Furthermore, the composite morphism \( f : X' \to X \times_S S' \to X \) is finite and hence \( g^* R\psi \bar{Q}_\ell \cong f_{\bar{s}}^* R\psi' \bar{Q}_\ell \). Therefore

\[
H^i_c(X_{\bar{s}}', R\psi' \bar{Q}_\ell) \cong H^i_c(X_{\bar{s}}', f_{\bar{s}}^* R\psi' \bar{Q}_\ell) \cong H^i_c(X_{\bar{s}}, R\psi \bar{Q}_\ell).
\]

8 Calculation of the Nearby Cycles

Again, let \( X_O/\mathcal{O} \) be a scheme of finite type. Let \( X_{O_{ur}} \) be the base-change of \( X_O \) to the maximal unramified extension \( K_{ur} \) of \( K \) and let \( X_{O_{ur}} \) be the base-change to the ring of integers in \( K_{ur} \). Then we have \( i : X_{\bar{s}} \to X_{O_{ur}} \) and \( j : X_{\bar{s}} \to X_{O_{ur}} \).

Lemma 8.1. In this setting \( R_{i*}(R\psi \mathcal{F}) = i^* R_{j*}(\mathcal{F})_{O_{ur}} \). \( \square \)
We now give a calculation in the case of interest to us.

**Theorem 8.2.** Let \( X/\mathcal{O} \) be regular and flat of relative dimension 1 and assume that \( X_s \) is globally the union of regular divisors. Let \( x \in X_s(\mathbb{F}_q) \) and let \( D_1, \ldots, D_i \) be the divisors passing through \( x \). Let \( W_1 \) be the \( i \)-dimensional \( \overline{\mathbb{Q}}_\ell \)-vector space with basis given by the \( D_t, t = 1, \ldots, i \), and let \( W_2 \) be the kernel of the map \( W_1 \to \overline{\mathbb{Q}}_\ell \) sending all \( D_t \) to 1. Then there are canonical isomorphisms

\[
(t^* R^k j_* \overline{\mathbb{Q}}_\ell)_x \cong \begin{cases} 
\overline{\mathbb{Q}}_\ell, & k = 0, \\
W_1(-1), & k = 1, \\
W_2(-2), & k = 2, \\
0, & \text{else.}
\end{cases}
\]

**Proof.** We use a method similar to the one employed in [22, pp. 36–38].

By passing to a suitable open subset, we may assume that the special fiber is the union of the divisors \( D_1, \ldots, D_i \) and \( x \) is the only possible intersection point of these divisors. Denote by \( b_i : D_i \to X \) and \( b : x \to X \), the closed embeddings. Let \( I^* \) be an injective resolution of \( \mathbb{Q}_\ell \) on \( X \).

**Remark.** Note that this is abuse of language, as in Remark 7. The proper meaning is to take a compatible system of injective resolutions of \( \mathbb{Z}/\ell^n \mathbb{Z} \).

Using diverse adjunction morphisms, we get a complex of sheaves on \( X_{\mathcal{O}^{ur}} \)

\[
\cdots \to 0 \to W_2 \otimes b_* b^! I^* \to \bigoplus_i b_{i*} b_i^! I^* \to t^* I^* \to t^* j_* j^* I^* \to 0.
\]

**Proposition 8.3.** The hypercohomology of this complex vanishes.

**Proof.** This is almost exactly [22, Lemma 2.5]. We repeat the argument here.

Let us begin with some general remarks. Recall that for a closed embedding \( i : Y \to Z \), there is a right-adjoint functor \( i^! \) to \( i_* = i_! \), given by \( i^! \mathcal{F} = \ker(\mathcal{F} \to j_* j^* \mathcal{F}) \), where
The inclusion of the complement. We get an exact sequence

\[ 0 \to i_! i^! F \to F \to j_* j^* F. \]

Being right-adjoint, \( i_! \) is left-exact. Furthermore, \( i_* \) has a left adjoint \( i^* \) and a right adjoint \( i^! \), hence is exact. Thus \( i^! \) has an exact left adjoint and hence preserves injectives. Similarly, \( i_* \) has the exact left adjoint \( i^* \) and thus preserves injectives. We see that if \( F \) is injective, then \( i_! i^! F = i_* i^* F \) is injective. Therefore \( F = i_! i^! F \oplus F' \) for some injective sheaf \( F' \). We get an injection \( F' \to j_* j^* F \). Since \( F' \) is injective, this is a split injection, with a cokernel supported on \( Y \). But

\[ \text{Hom}(i_* G, j_* j^* F) = \text{Hom}(j^* i_* G, j^* F) = 0 \]

for any sheaf \( G \) on \( Y \), hence the cokernel is trivial and \( F' = j_* j^* F \). This shows that

\[ F = i_* i^! F \oplus j_* j^* F \]

for any injective sheaf \( F \) on \( Y \), where \( i^! F \) and \( j^* F \) are injective, as \( j^* = j^! \) has the exact left adjoint \( j_* \).

We prove the proposition for any complex of injective sheaves \( I^* \). This reduces the problem of doing it for a single injective sheaf \( I \). Let \( U \) be the complement of \( X_s \) in \( X \) and let \( U_i \) be the complement of \( x \) in \( D_i \). In our situation, we get a decomposition of \( I \) as

\[ I = f_{U_s} I_U \oplus \bigoplus_i f_{U_i} I_{U_i} \oplus f_{x_s} I_x, \]

where \( I_T \) is an injective sheaf on \( T \) and \( f_T : T \to X \) is the locally closed embedding, for any \( T \) occurring as an index.

Now we check case by case. First, \( b^! f_{U_s} I_U = b^! f_{U_s} I_U = 0 \) and the complex reduces to \( \tau^* f_{U_s} I_U \cong \tau^* f_{U_s} I_U \). Second, \( b^! f_{U_i} I_{U_i} = b^! f_{U_i} I_{U_i} = 0 \) for \( j \neq i \), while \( b^! f_{U_i} I_{U_i} = I_{U_i} \). Hence the complex reduces to the isomorphism \( I_{U_i} \cong I_{U_i} \) in this case. In the last case, \( b^! f_{x_s} I_x = b^! f_{x_s} I_x = f_{x_s} I_x \) for all \( i \), and hence the complex reduces to

\[ \cdots \to 0 \to W_2 \otimes f_{x_s} I_x \to W_1 \otimes f_{x_s} I_x \to f_{x_s} I_x \to 0, \]

and this is exact by our definition of \( W_2 \).
Let us recall one important known special case of Grothendieck’s purity conjecture.

**Theorem 8.4.** Let $X$ be a regular separated Noetherian scheme of finite type over the ring of integers in a local field and let $f: Y \to X$ be a closed immersion of a regular scheme $Y$ that is of codimension $d$ at each point. Let $\ell$ be a prime that is invertible on $X$. Then there is an isomorphism in the derived category of constructible ${\bar \mathbb{Q}}_{\ell}$-sheaves

$$Rf^! {\bar \mathbb{Q}}_{\ell} \cong {\bar \mathbb{Q}}_{\ell}(-d)[-2d].$$

□

**Proof.** This is contained in [24, Corollary 3.9]. □

We use this to get isomorphisms $b_i^* b_i^! I^* \cong b_i^* {\bar \mathbb{Q}}_{\ell}(-1)[-2]$ and $b_i^* b_i^! I^* \cong b_i^* {\bar \mathbb{Q}}_{\ell}(-2)[-4]$ in the derived category. Hence, since the spectral sequence for hypercohomology of

$$\cdots \to 0 \to W_2 \otimes b_i^* b_i^! I^* \to \bigoplus_i b_i^* b_i^! I^* \to \iota^* I^*$$

is equivariant for the Galois action and its only nonzero terms are of the form ${\bar \mathbb{Q}}_{\ell}(-k)$ for different $k$, it degenerates and we get the desired isomorphism. □

As a corollary, we can compute the semisimple trace of Frobenius on the nearby cycles in our situation. Let $B$ denote the Borel subgroup of $GL_2$. Recall that we associated an element $\delta \in GL_2(\mathbb{Q}_{p^r})$ to any point $x \in \mathcal{M}(\mathbb{F}_{p^r})$ by looking at the action of $F$ on the crystalline cohomology. We have the covering $\pi_n : \mathcal{M}_{\Gamma(p^n)} \to \mathcal{M}$ and the sheaf $\mathcal{F}_n = \pi_{n!} {\bar \mathbb{Q}}_{\ell}$ on the generic fiber of $\mathcal{M}_{\Gamma(p^n)}$.

**Corollary 8.5.** Let $x \in \mathcal{M}(\mathbb{F}_{p^r})$ and let $g \in GL_2(\mathbb{Z}_p)$.

(i) If $x$ corresponds to an ordinary elliptic curve and $\alpha$ is the unique eigenvalue of $N\delta$ with valuation 0, then

$$\text{tr}^{ss}(\Phi_{p^r} g | (R\psi \mathcal{F}_n)_x) = \text{tr}(\Phi_{p^r} g | V_n),$$

where $\Phi_{p^r}$ is the Frobenius $p^r$-power.
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where $V_n$ is a $G_{\mathbb{F}_p} \times GL_2(\mathbb{Z}/p^n\mathbb{Z})$-representation isomorphic to

$$
\bigoplus_{\chi \in ((\mathbb{Z}/p^n\mathbb{Z})^*)^r} \text{Ind}_{B(\mathbb{Z}/p^n\mathbb{Z})}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes \chi
$$

as a $GL_2(\mathbb{Z}/p^n\mathbb{Z})$-representation. Here $\Phi^r$ acts as the scalar $\chi(a)^{-1}$ on $\text{Ind}_{B(\mathbb{Z}/p^n\mathbb{Z})}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes \chi$.

(ii) If $x$ corresponds to a supersingular elliptic curve, then

$$
\text{tr}_{ss}(\Phi^r g | (R\psi_f)_x) = 1 - \text{tr}(g | St^p),
$$

where

$$
St = \ker(\text{Ind}_{B(\mathbb{Z}/p^n\mathbb{Z})}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes 1 \rightarrow 1)
$$

is the Steinberg representation of $GL_2(\mathbb{Z}/p^n\mathbb{Z})$. □

**Proof.** Note first that we have

$$
R_{k, R}\psi_{M, F_f} = R_{k, R}\psi_{M, \pi_{\eta\varnothing}} \bar{Q}_\ell = \pi_{\eta\varnothing} R_{k, R}\psi_{M, \rho(p)\eta} \bar{Q}_\ell = \pi_{\eta\varnothing} R_{k, j} \bar{Q}_\ell
$$

because $\pi_n$ is finite. Here subscripts for $R\psi$ indicate with respect to which scheme the nearby cycles are taken, and $i$ and $j$ are as defined before Lemma 8.1, for the scheme $M_{\Gamma(p^n)}$. Note that we may apply Theorem 8.2 because of Theorem 6.4.

Let $x$ be any point above $x$ in $M_{\Gamma(p^n)}$. In case (i), we see that

$$
(t^* R^k j_s \bar{Q}_\ell)_x \cong \begin{cases} 
\bar{Q}_\ell(-k), & k = 0, 1, \\
0 & \text{else}.
\end{cases}
$$

It remains to understand the action of $GL_2(\mathbb{Z}/p^n\mathbb{Z}) \times G_{\mathbb{F}_p}$ on $\pi_n^{-1}(x)$. Let $E$ be the elliptic curve corresponding to $x$. Fix an identification $E[p^\infty] \cong \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$ and in particular $E[p^n] \cong \mu_{p^n} \times \mathbb{Z}/p^n\mathbb{Z}$. Then the Drinfeld-level-$p^n$-structures are parameterized by surjections

$$(\mathbb{Z}/p^n\mathbb{Z})^2 \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}.$$ 

The right action of $GL_2(\mathbb{Z}/p^n\mathbb{Z})$ is given by precomposition.

The identification $E[p^\infty] \cong \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$ gives $\delta$ the form $\left(\begin{smallmatrix} pb_0 & 0 \\
0 & aw_0
\end{smallmatrix}\right)$, because the crystalline cohomology of $E$ agrees with the contravariant Dieudonné module of $E[p^\infty]$. 

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Then
\[ \Phi_{pr} = N \delta = \begin{pmatrix} p'Nb_0 & 0 \\ 0 & Na_0 \end{pmatrix}. \]

Hence \( a = Na_0 \) and \( \Phi_{pr} \) acts through multiplication by \( a^{-1} \) on the factor \( \mathbb{Z}/p^n\mathbb{Z} \) of \( E[p^n] \). Hence it sends a Drinfeld-level-\( p^n \)-structure given by some surjection to the same surjection multiplied by \( a^{-1} \). In total, we get
\[
(\pi_n \bar{s} \ast \iota \ast R^k j \ast \bar{\Psi}_{\ell})_x \cong \begin{cases} V_n(-k), & k = 0, 1, \\ 0 & \text{else}, \end{cases}
\]
where \( V_n \cong \mathbb{Q}_\ell^M \) with \( M \) the set of surjections \( (\mathbb{Z}/p^n\mathbb{Z})^2 \to \mathbb{Z}/p^n\mathbb{Z} \). Then \( \text{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) \times G_{\bar{F}_{pr}} \) acts on those pairs and hence on \( V_n \), compatibly with the action on the left-hand side. The action of diagonal multiplication commutes with this action and gives rise to the decomposition
\[ V_n = \bigoplus_{\chi \in (\mathbb{Z}/p^n\mathbb{Z})^*} V_\chi. \]

Then one checks that \( V_\chi = \text{Ind}_{B(\mathbb{Z}/p^n\mathbb{Z})}^{\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes \chi \) and one easily arrives at the formula in case (i).

In case (ii), there is only one point \( \tilde{x} \) above \( x \) in \( \mathcal{M}_{\Gamma(p^n)} \) and \( p^n + p^{n-1} \) irreducible components meet at \( x \), parameterized by \( \mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z}) \). This parameterization is \( \text{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) \)-equivariant, so that
\[
W_1 \cong \text{Ind}_{B(\mathbb{Z}/p^n\mathbb{Z})}^{\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes 1
\]
and
\[
W_2 \cong \ker(\text{Ind}_{B(\mathbb{Z}/p^n\mathbb{Z})}^{\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes 1 \to 1) = \text{St}
\]
in the notation of Theorem 8.2. This yields the desired result.

Define for \( x \in \mathcal{M}(\mathbb{F}_{p^n}) \)
\[
(R\Psi \mathcal{F}_{\infty})_x = \lim_{\to}(R\Psi \mathcal{F}_n)_x.
\]

It carries a natural smooth action of \( \text{GL}_2(\mathbb{Z}_p) \) and a commuting continuous action of \( G_{\bar{Q}_{p^n}} \). Then we can define \( \text{tr}^{\text{ss}}(\Phi_{pr} h(R\Psi \mathcal{F}_{\infty})_x) \) for \( h \in C_\infty^c(\text{GL}_2(\mathbb{Z}_p)) \) in the following way:
Choose \( n \) such that \( h \) is \( \Gamma(p^n)_{Q_p} \)-biinvariant and then take invariants under \( \Gamma(p^n)_{Q_p} \) first:

\[
\text{tr}^{ss}(\Phi_{p'} h | (R\psi F_{\infty})_x) = \text{tr}^{ss}(\Phi_{p'} h | (R\psi F_n)_x).
\]

It is easily checked that this gives something well-defined.

We see that for \( h \in C_{\infty}(GL_2(Z_p)) \), the value of \( \text{tr}^{ss}(\Phi_{p'} h | (R\psi F_{\infty})_x) \) depends only on the element \( \gamma = N\delta \) associated to \( x \). This motivates the following definition.

**Definition 8.6.** For \( \gamma \in GL_2(Q_p) \) and \( h \in C_{\infty}(GL_2(Z_p)) \), define

\[
c_r(\gamma, h) = \sum_{\chi_0 \in (\mathbb{Z}/p^n\mathbb{Z})^*} \text{tr}(h|\text{Ind}_{E_{\mathbb{Z}/p^n\mathbb{Z}}}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \otimes \chi_0) (t_2)^{-1},
\]

where \( t_2 \) is the unique eigenvalue of \( \gamma \) with \( v_p(t_2) = 0 \). For \( v_p(\text{tr} \gamma) \geq 1 \), we take

\[
c_r(\gamma, h) = \text{tr}(h|1) - p^r \text{tr}(h|\text{St}).
\]

Since \( x \) is supersingular if and only if \( \text{tr} N\delta \equiv 0 \mod p \), we get

\[
\text{tr}^{ss}(\Phi_{p'} h | (R\psi F_{\infty})_x) = c_r(N\delta, h)
\]

whenever \( \delta \) is associated to \( x \in M(F_{p'}) \).

9 The Semisimple Trace of Frobenius as a Twisted Orbital Integral

First, we construct the function \( \phi_p \) which will turn out to have the correct twisted orbital integrals.

**Lemma 9.1.** There is a function \( \phi_p \) of the Bernstein center of \( GL_2(Q_{p'}) \) such that for all irreducible smooth representations \( \Pi \) of \( GL_2(Q_{p'}) \), \( \phi_p \) acts by the scalar

\[
p^{r_1} \text{tr}^{ss}(\Phi_{p'} | \sigma_\Pi).
\]

where \( \sigma_\Pi \) is the representation of the Weil group \( W_{Q_{p'}} \) of \( Q_{p'} \) with values in \( \bar{Q}_\ell \) associated to \( \Pi \) by the local Langlands correspondence.
Remark. Of course, the definition of the semisimple trace of Frobenius makes sense for representations of $W_{\mathbb{Q}_p}$. For a representation $\sigma$ of $W_{\mathbb{Q}_p}$, we write $\sigma^{ss}$ for the associated semisimplification.

Proof. By Theorem 2.2, we only need to check that this defines a regular function on $D/W(L, D)$ for all $L$ and $D$. First, note that the scalar agrees for a one-dimensional representation $\Pi$ and the corresponding twist of the Steinberg representation, because we are taking the semisimple trace. This shows that we get a well-defined function on $D/W(L, D)$. But if one fixes $L$ and $D$ and takes $\Pi$ in the corresponding component, then the semi-simplification $\sigma^{ss}_\Pi$ decomposes as $(\sigma_1 \otimes \chi_1 \circ \det) \oplus \cdots \oplus (\sigma_t \otimes \chi_t \circ \det)$ for certain fixed irreducible representations $\sigma_1, \ldots, \sigma_t$ and varying unramified characters $\chi_1, \ldots, \chi_t$ parameterized by $D$. In particular,

$$\text{tr}^{ss}(\Phi_{p^r} | \sigma_\Pi) = \sum_{i=1}^t \text{tr}^{ss}(\Phi_{p^r} | \sigma_i) \chi_i(p)$$

which is clearly a regular function on $D$ and necessarily $W(L, D)$-invariant, hence descends to a regular function on $D/W(L, D)$.

We also need the function $\phi_{p,0} = \phi_p * e_{GL_2(\mathbb{Z}_p)} \in \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$. This definition is compatible with our previous use of $\phi_{p,0}$:

Lemma 9.2. The function $\phi_{p,0}$ is the characteristic function of the set

$$GL_2(\mathbb{Z}_p) \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) GL_2(\mathbb{Z}_p)$$

divided by the volume of $GL_2(\mathbb{Z}_p)$.

Proof. Both functions are elements of the spherical Hecke algebra $\mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$. Since the Satake transform is an isomorphism, it suffices to check that the characteristic function of the given set divided by its volume acts through the scalars

$$p^{\frac{t}{2}} \text{tr}^{ss}(\Phi_{p^r} | \sigma_\Pi)$$
on unramified representations. In general, this is done in [18, Theorem 2.1.3]. Let us explain what it means here. By the Satake parameterization, an unramified representation $\Pi$ is given by two unramified characters $\chi_1$ and $\chi_2$. Then $\sigma_\Pi = \chi_1 \oplus \chi_2$ and

$$tr^{ss}(\Phi_{p^r}|\sigma_\Pi) = \chi_1(p) + \chi_2(p).$$

Hence this is just the usual formula for the trace of the classical Hecke operators, usually called $T_p$, in terms of the Satake parameters (at least for $r = 1$).

Theorem 9.3. Let $\delta \in GL_2(\mathbb{Q}_{p^r})$ with semisimple norm $\gamma \in GL_2(\mathbb{Q}_p)$. Let $h \in C^\infty_c(GL_2(\mathbb{Z}_p))$ and $h' \in C^\infty_c(GL_2(\mathbb{Z}_{p^r}))$ have matching (twisted) orbital integrals. Then

$$TO_{\delta\sigma}(\phi_p * h') = TO_{\delta\sigma}(\phi_{p,0})G_\tau(\gamma, h).$$

Proof. Let $f_1 = \phi_p * h'$ and let $f_2 = \phi_{p,0}$. Let $\Pi$ be the base-change lift of some tempered representation $\pi$ of $GL_2(\mathbb{Q}_p)$. Then, tracing through the definitions and taking $n$ so that $h$ and $h'$ are $\Gamma(p^n)_{\mathbb{Q}_p}$ resp. $\Gamma(p^n)_{\mathbb{Q}_{p^r}}$-biinvariant,

$$tr((f_1, \sigma)|\Pi) = p^{\frac{1}{2}r}tr((h', \sigma)|\Pi_{\Gamma(p^n)})tr^{ss}(\Phi_{p^r}|\sigma_\Pi) = p^{\frac{1}{2}r}tr(h|\pi_{\Gamma(p^n)})tr^{ss}(\Phi_{p^r}|\sigma_\Pi),$$

(because $h$ and $h'$ have matching (twisted) orbital integrals) and

$$tr((f_2, \sigma)|\Pi) = p^{\frac{1}{2}r} \dim \pi^{GL_2(\mathbb{Z}_p)}tr^{ss}(\Phi_{p^r}|\sigma_\Pi),$$

because $e_{\Gamma(1)_{\mathbb{Q}_p}}$ and $e_{\Gamma(1)_{\mathbb{Q}_{p^r}}}$ are associated by Theorem 3.7.

As a first step, we prove the theorem for special $\delta$.

Lemma 9.4. Assume that $\delta = \left( \begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix} \right)$, with $Nt_1 \neq Nt_2$. Then, the twisted orbital integrals $TO_{\delta\sigma}(\phi_p * h')$, $TO_{\delta\sigma}(\phi_{p,0})$ vanish except in the case where, up to exchanging $t_1$ and $t_2$, we have $v_p(t_1) = 1$ and $v_p(t_2) = 0$. In the latter case,

$$TO_{\delta\sigma}(\phi_p * h') = \text{vol}(T(\mathbb{Z}_p))^{-1} \sum_{\chi_0 \in (\mathbb{Z}/p^r\mathbb{Z})^\times} tr\left(h|\text{Ind}_{B(\mathbb{Z}/p^r\mathbb{Z})}^{GL_2(\mathbb{Z}/p^r\mathbb{Z})} 1 \boxtimes \chi_0\right) \chi_0(Nt_2)^{-1}$$

and $TO_{\delta\sigma}(\phi_{p,0}) = \text{vol}(T(\mathbb{Z}_p))^{-1}$. □
We remark that this implies the theorem in this case.

**Proof.** Let $B$ be the standard Borel subgroup consisting of upper triangular elements and let $\chi$ be a unitary character of $T(\mathbb{Q}_p)$ (and hence of $B(\mathbb{Q}_p)$). Take the normalized induction $\pi_\chi = n-\text{Ind}_{B(\mathbb{Q}_p)}^{G_2(\mathbb{Q}_p)} \chi$, an irreducible tempered representation of $G_2(\mathbb{Q}_p)$. Then, by [20, Lemma 7.2], the character $\Theta_{\pi_\chi}$, a locally integrable function, is supported on the elements conjugate to an element of $T(\mathbb{Q}_p)$ and for $t = (t_1, t_2) \in T(\mathbb{Q}_p)$ regular,

$$\Theta_{\pi_\chi}(t) = \frac{\chi(t_1, t_2) + \chi(t_2, t_1)}{|t_2|^{-2} + |t_1|^{1/2}}.$$

Let $\Pi_\chi$ be the base-change lift of $\pi_\chi$, with twisted character $\Theta_{\Pi_\chi, \sigma}$. For $t \in T(\mathbb{Q}_p)$, define

$$TO_t(f) = \begin{cases} TO_{\tilde{t}}(f), & t = N\tilde{t} \text{ for some } \tilde{t} \in T(\mathbb{Q}_p'), \\ 0 & \text{else.} \end{cases}$$

This definition is independent of the choice of $\tilde{t}$ as all choices are $\sigma$-conjugate. We get by the twisted version of Weyl's integration formula (cf. [20, p. 99]) for any $f \in C_c^\infty(\text{GL}_2(\mathbb{Q}_p'))$

$$\text{tr}((f, \sigma)|\Pi_\chi) = \int_{\text{GL}_2(\mathbb{Q}_p')} f(g) \Theta_{\Pi_\chi, \sigma}(g) \, dg$$

$$= \frac{1}{2} \int_{T(\mathbb{Q}_p)} \left| \frac{t_1}{t_2} - 2 + \frac{t_2}{t_1} \right| TO_t(f) \frac{\chi(t_1, t_2) + \chi(t_2, t_1)}{|t_2|^{-2} + |t_1|^{1/2}} \, dt$$

$$= \int_{T(\mathbb{Q}_p)} \left| \frac{t_1}{t_2} - 2 + \frac{t_2}{t_1} \right|^{1/2} TO_t(f) \chi(t) \, dt.$$

By Fourier inversion, we arrive at

$$TO_t(f) = \left| \frac{t_1}{t_2} - 2 + \frac{t_2}{t_1} \right|^{-1/2} \int_{T(\mathbb{Q}_p)_{u}} \text{tr}((f, \sigma)|\Pi_\chi) \chi(t)^{-1} \, d\chi,$$

where $T(\mathbb{Q}_p)_{u}$ denotes the set of unitary characters of $T(\mathbb{Q}_p)$. Measures need to be chosen so that $\text{vol}(T(\mathbb{Q}_p)_{u}) = \text{vol}(T(Z_p))^{-1}$, where $T(\mathbb{Q}_p)_{u}$ is the identity component of $T(\mathbb{Q}_p)_{u}$; it consists precisely of the unramified characters.

Note that $TO_t(f)$ is a locally constant function on the set of regular elements of $T(\mathbb{Q}_p)$ and hence this gives an identity of functions there. From here, it is immediate that $TO_t(f_2) = 0$ and $TO_t(f_1) = 0$ for all $t = (t_1, t_2)$ with $t_1 \neq t_2$, except in the case where (up to...
exchanging $t_1$ and $t_2$, $v_p(t_1) = r$ and $v_p(t_2) = 0$. In the latter case, $TO_t(f_2) = \text{vol}(T(Z_p))^{-1}$.

The calculation of $TO_t(f_1)$ is slightly more involved:

$$TO_t(f_1) = p^{-\frac{1}{2} r} \int_{T(Q)_{\text{u}}} \text{tr}((f_1, \sigma)|\Pi\chi) \chi(t)^{-1} d\chi$$

$$= \text{vol}(T(Z_p))^{-1} \sum_{\chi_0 \in ((Z/p^nZ)^*)^\vee} \text{tr}(h|\text{Ind}_{B(Z/p^nZ)}^{GL_2(Z/p^nZ)} 1 \otimes \chi_0) \chi_0(t)^{-1},$$

giving the desired result.

Next, we remark that if $\delta$ is not $\sigma$-conjugate to an element as in Lemma 9.4, then the eigenvalues of $N\delta$ have the same valuation. Let

$$f = f_1 + (H_1 p^r - H_2) f_2,$$

where we have set

$$H_1 = \text{tr}(h|\text{St}), \quad H_2 = \text{tr}(h|1),$$

with $\text{St}$ resp. 1 the Steinberg resp. trivial representation of $GL_2(Z/p^nZ)$. Then the next lemma finishes the proof of the theorem.

**Lemma 9.5.** Assume that the eigenvalues of $N\delta$ have the same valuation. Then the twisted orbital integral $TO_{\delta\sigma}(f)$ vanishes.

**Proof.** Let $V$ be the set of all $\delta \in GL_2(Q_{p^r})$ such that the eigenvalues of $N\delta$ have the same valuation. Note that $V$ is open. In particular, its characteristic function $\chi_V$ is locally constant and hence $\tilde{f}(g) = f(g)\chi_V(g)$ defines a function $\tilde{f} \in C_c^\infty(GL_2(Q_{p^r}))$. Then, obviously, the twisted orbital integrals of $\tilde{f}$ and $f$ agree on all elements $\delta$ such that the eigenvalues of $N\delta$ have the same valuation. We will prove that for all tempered irreducible smooth representations $\pi$ of $GL_2(Z_p)$ with base-change lift $\Pi$, we have $\text{tr}((\tilde{f}, \sigma)|\Pi) = 0$. By the usual arguments (cf. proof of Theorem 3.7), this implies that all twisted orbital integrals of $\tilde{f}$ for elements $\delta$ with $N\delta$ semisimple vanish. This then proves the lemma.

First, we find another expression for $\text{tr}((\tilde{f}, \sigma)|\Pi)$. Note that we have seen in Lemma 9.4 that the twisted orbital integrals of $f_i$ vanish on all elements of $\delta \in T$ with $N\delta$ having distinct eigenvalues of the same valuation, whence the same is true for $f$. 
In particular,

$$\text{tr}((\tilde{f}, \sigma)|\Pi) = \text{tr}((f, \sigma)|\Pi)_{\sigma} = \int_{\text{GL}_2(\mathbb{Q}_{p'} \sigma)_{\sigma-\text{ell}}} f(g) \Theta_{\Pi, \sigma}(g) \, dg,$$

where $\text{GL}_2(\mathbb{Q}_{p'})_{\sigma-\text{ell}}$ is the set of elements of $\delta \in \text{GL}_2(\mathbb{Q}_{p'})$ with $N\delta$ elliptic (since the character $\Theta_{\Pi, \sigma}$ is locally integrable, one could always restrict the integration to regular semisimple elements and hence non-semisimple elements need not be considered). This reduces us to proving that $\text{tr}((f, \sigma)|\Pi)_{\sigma} = 0$.

But for $\pi = \text{n-Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)} \chi$ the normalized induction of a unitary character, with base-change lift $\Pi$, we have $\text{tr}((f, \sigma)|\Pi)_{\sigma} = 0$ because the character $\Theta_{\pi}$ is supported in elements conjugate to an element of $T(\mathbb{Q}_p)$. For $\pi$ supercuspidal with base-change lift $\Pi$,

$$\text{tr}((f, \sigma)|\Pi)_{\sigma} = \text{tr}((f, \sigma)|\Pi) = 0.$$

Here the second equation follows from the definitions of $f$ and the way $(f, \sigma)$ acts on $\Pi$, whereas the first equation holds because the character $\Theta_{\pi}$ is supported in the elements the eigenvalues of which have the same valuation—this easily follows from the fact that $\pi$ is compactly induced from a representation of an open subgroup that is a compact modulo center, as proved in [4]. This leaves us with checking that $\text{tr}((f, \sigma)|\Pi)_{\sigma} = 0$ for any unitary twist of the Steinberg representation $\pi$ with base-change lift $\Pi$. However, restricted to the elliptic elements, the character of a twist of the Steinberg representation agrees up to sign with the character of the corresponding one-dimensional representation. Hence it is enough to check that $\text{tr}((f, \sigma)|\Pi)_{\sigma} = 0$ for any a one-dimensional representation $\pi = \chi \circ \det$ with base-change lift $\Pi = \chi \circ \text{Norm}_{\mathbb{Q}_{p'}/\mathbb{Q}_p} \circ \det$.

Then

$$\text{tr}((f, \sigma)|\Pi)_{\sigma-\text{ell}} = \text{tr}((f, \sigma)|\Pi) - \frac{1}{2} \int_{T(\mathbb{Q}_p)} \left| \frac{t_1}{t_2} - 2 + \frac{t_2}{t_1} \right| T O_1(f) \chi(t_1 t_2) \, dt.$$ 

Note that the function in the integral only takes nonzero values if $v_p(t_1) = r$ and $v_p(t_2) = 0$, or the other way around. Hence, we may rewrite the equality as

$$\text{tr}((f, \sigma)|\Pi)_{\sigma-\text{ell}} = \text{tr}((f, \sigma)|\Pi) - p^r \chi(p^r) \int_{T(\mathbb{Z}_p)} T O_1(p^{t_1}, t_2)(f) \chi(t_1 t_2) \, dt.$$
Now, if $\chi$ is ramified, then $\text{tr}((f, \sigma)|\Pi) = 0$, while the integral is zero as well, because $TO_{(p't_1, t_2)}(f)$ does not depend on $t_1$ and hence keeping $t_2$ fixed and integrating over $t_1$ gives zero.

On the other hand, if $\chi$ is unramified, then

$$\text{tr}((f, \sigma)|\Pi) = \text{tr}((f_1, \sigma)|\Pi) + (H_1 p^r - H_2)\text{tr}((f_2, \sigma)|\Pi) = (1 + p^r)\chi(p^r)H_2 + (H_1 p^r - H_2)(1 + p^r)\chi(p^r) = (1 + p^r)H_1\chi(p^r)$$

and the integral gives

$$\int_{T(Z_p)} TO_{(p't_1, t_2)}(f) \, dt = \int_{T(Z_p)} TO_{(p't_1, t_2)}(f_1) \, dt + (H_1 p^r - H_2)\int_{T(Z_p)} TO_{(p't_1, t_2)}(f_2) \, dt = (H_1 + H_2) + (H_1 p^r - H_2) = (1 + p^r)H_1.$$ 

Putting everything together, we get the conclusion.

---

We get the following corollary.

**Corollary 9.6.** Let $x \in \mathcal{M}(\mathbb{F}_{p^r})$ with associated $\delta$. Let $h \in C_c^\infty(\text{GL}_2(\mathbb{Z}_p))$ and $h' \in C_c^\infty(\text{GL}_2(\mathbb{Z}_{p^r}))$ have matching (twisted) orbital integrals. Then

$$TO_{\delta_\sigma}(\phi_p * h') = TO_{\delta_\sigma}(\phi_{p,0})\text{tr}^{\text{ss}}(\Phi_{p^r}^\sigma h)(R\psi_F)_x).$$

**Proof.** Combining Theorem 9.3 and Corollary 8.5, all we have to check is the following lemma.

**Lemma 9.7.** For any $\delta \in \text{GL}_2(\mathbb{Q}_{p^r})$ associated to an elliptic curve over $\mathbb{F}_{p^r}$, the norm $N\delta$ is semisimple.

**Proof.** As $N\delta$ is the endomorphism of crystalline cohomology associated to the geometric Frobenius $\Phi_{p^r}$ of $E_0$, it is enough to prove that any $\mathbb{F}_{p^r}$-self-isogeny $f : E \to E$ of an elliptic curve $E/\mathbb{F}_{p^r}$ gives rise to a semisimple endomorphism on the crystalline
cohomology. If not, we may find \( m, n \in \mathbb{Z} \) such that \( f' = mf - n \) is nilpotent on crystalline cohomology, but nonzero. But if \( f' \) is nonzero, then for the dual isogeny \( (f')^* \), the composition \( f'(f)^* \) is a scalar, and hence induces multiplication by a scalar on the crystalline cohomology. Hence \( f' \) induces an invertible endomorphism on the (rational) crystalline cohomology, contradiction.

We note that by Theorem 3.7, we can take \( h \) to be the idempotent \( e_{\Gamma(p^n)\bar{\mathbb{Q}}_p} \) and \( h' \) to be the idempotent \( e_{\Gamma(p^n)\bar{\mathbb{Q}}_{p'}} \) and get the following corollary, proving Theorem B for \( \phi_{p} \ast e_{\Gamma(p^n)\bar{\mathbb{Q}}_{p'}} \) instead of \( \phi_{p,n} \). For the comparison of these functions, we refer to Section 14.

**Corollary 9.8.** Let \( x \in \mathcal{M}(\mathbb{F}_{p'}) \) with associated \( \delta \). Then

\[
\text{tr}^{ss}(\Phi_{p'}|(R\psi F_n)_x) = TO_{\delta}(\phi_{p} \ast e_{\Gamma(p^n)\bar{\mathbb{Q}}_{p'}})(TO_{\delta}(\phi_{p,0}))^{-1}.
\]

**Proof.** Use the remark after Corollary 5.2 to see that the right-hand side is well defined.

**10 The Langlands–Kottwitz Approach: Case of Bad Reduction**

By Theorem 7.9, we get

\[
\log \zeta^{ss}(\mathcal{M}_{\Gamma(p^n)}, \bar{\mathbb{Q}}_l) = \sum_{r \geq 1} \sum_{x \in \mathcal{M}(\mathbb{F}_{p'})} \text{tr}^{ss}(\Phi_{p'}|(R\psi F_n)_x) \frac{D_{rs}^{-1}}{r}.
\]

Again, we may split the terms according to their \( \mathbb{F}_{p'} \)-isogeny class. This leads us to consider, for \( E_0 \) fixed,

\[
\sum_{x \in \mathcal{M}(\mathbb{F}_{p'})(E_0)} \text{tr}^{ss}(\Phi_{p'}|(R\psi F_n)_x).
\]

Now Corollary 9.8 tells us that

\[
\text{tr}^{ss}(\Phi_{p'}|(R\psi F_n)_x) = TO_{\delta}(\phi_{p} \ast e_{\Gamma(p^n)\bar{\mathbb{Q}}_{p'}})(TO_{\delta}(\phi_{p,0}))^{-1}.
\]
Corollary 10.1. The sum
\[
\sum_{x \in M(F_{\text{pr}})(E_0)} \text{trss}(\Phi_{\text{pr}}| (R\psi \mathcal{F}_n)_x)
\]
equals
\[
\text{vol}(\Gamma \backslash G_{\mathfrak{f}}(\mathbb{A}_f) \times G_{\delta \sigma} (\mathbb{Q}_p) ) O_{\delta \sigma} (\phi_p * e_{\Gamma(p^n)_{\mathcal{O}_p'}}).
\]
\[\square\]

Proof. This is obvious from what was already said and Theorem 5.2. \[\blacksquare\]

First, we eliminate the twisted orbital integral. Let \( f_{p,r} \) be the function of the Bernstein center for \( \text{GL}_2(\mathbb{Q}_p) \) such that for all irreducible smooth representations \( \pi \) of \( \text{GL}_2(\mathbb{Q}_p) \), \( f_{p,r} \) acts by the scalar \( p^{1/2} \text{trss}(\Phi^r_{\text{pr}}|\sigma_\pi) \), where, again, \( \sigma_\pi \) is the associated representation of the Weil group \( \mathcal{W}_{\mathbb{Q}_p} \) over \( \bar{\mathbb{Q}}_l \). The existence of \( f_{p,r} \) is proved in the same way as Lemma 9.1. By [12], we know that if \( \pi \) is tempered and \( \Pi \) is a base-change lift of \( \pi \), then \( \sigma_\Pi \) is the restriction of \( \sigma_\pi \). Perhaps, it is worth remarking that the statement on the semisimple trace of Frobenius that we need is much simpler.

Lemma 10.2. For any tempered irreducible smooth representation \( \pi \) of \( \text{GL}_2(\mathbb{Q}_p) \) with base-change lift \( \Pi \), we have
\[
\text{trss}(\Phi^r_{\text{pr}}|\sigma_\pi) = \text{trss}(\Phi^r_{\text{pr}}|\sigma_\Pi).
\]
\[\square\]

Proof. Assume first that \( \text{trss}(\Phi^r_{\text{pr}}|\sigma_\Pi) \neq 0 \). Then the semisimplification of \( \sigma_\Pi \) is a sum of two characters \( \chi_1 \) and \( \chi_2 \), one of which, say \( \chi_2 \), is unramified and in particular invariant under \( \text{Gal}(\mathbb{Q}_{p'}/\mathbb{Q}_p) \). Because \( \Pi \) is invariant under the Galois group \( \text{Gal}(\mathbb{Q}_{p'}/\mathbb{Q}_p) \), the character \( \chi_1 \) needs to factor over the norm map. We see that there is a principal series representation \( \pi' \) with base-change lift \( \Pi \). By the uniqueness properties of base-change (cf. [20]) \( \pi \) is also a principal series representation. The claim then follows from the explicit description of base-change for principal series representation.

Now assume \( \text{trss}(\Phi_{\text{pr}}|\sigma_\Pi) = 0 \). If \( \text{trss}(\Phi^r_{\text{pr}}|\sigma_\pi) \neq 0 \), then the semisimplification of \( \sigma_\pi \) is a sum of two characters (one of which is unramified), whence \( \pi \) is again a principal series representation. This yields the claim as before. \[\blacksquare\]

This shows that \( f_{p,r} * e_{\Gamma(p^n)_{\mathcal{O}_p}} \) and \( \phi_p * e_{\Gamma(p^n)_{\mathcal{O}_p}} \) satisfy the hypothesis of Theorem 3.7. Thus, by Lemma 9.7, we may rewrite the expression in Corollary 10.1 as follows.
Corollary 10.3. The sum
\[ \sum_{x \in M(F_{pr})(E_0)} \text{tr}^{ss}(\Phi_{pr}|(R\psi F_n)x) \]
equals \[ \pm \text{vol}(\Gamma \backslash G_{\gamma}(\mathbb{A}_f^P) \times G_{\delta_\sigma}(\mathbb{Q}_p)) O_{\gamma}(f^P) O_{N\delta}(f_{p,r} \ast e_{\Gamma(p^n)_{\mathbb{Q}_p}}). \] (2)

We need to recall certain facts from Honda–Tate theory to simplify our expression further.

Theorem 10.4. Fix a finite field \( \mathbb{F}_q \) of characteristic \( p \).

(a) For any elliptic curve \( E/\mathbb{F}_q \), the action of Frobenius on \( H^1_{\text{et}}(E, \mathbb{Q}_\ell) \) is semisimple with characteristic polynomial \( p_E \in \mathbb{Z}[T] \) independent of \( \ell \). Additionally, if \( F \) acts as \( \delta_\sigma \) on \( H^1_{\text{cris}}(E/\mathbb{Z}_q) \otimes \mathbb{Q}_q \), then \( N\delta \) is semisimple with characteristic polynomial \( p_E \).

Let \( \gamma_E \in \text{GL}_2(\mathbb{Q}) \) be semisimple with characteristic polynomial \( p_E \). Then

(b) The map \( E \mapsto \gamma_E \) gives a bijection between \( \mathbb{F}_q \)-isogeny classes of elliptic curves over \( \mathbb{F}_q \) and conjugacy classes of semisimple elements \( \gamma \in \text{GL}_2(\mathbb{Q}) \) with \( \text{det} \gamma = q \) and \( \text{tr} \gamma \in \mathbb{Z} \) which are elliptic in \( \text{GL}_2(\mathbb{R}) \).

(c) Let \( G_{\gamma_E} \) be the centralizer of \( \gamma_E \). Then \( \text{End}(E)^\times \) is an inner form of \( G_{\gamma_E}^\times \). In fact,

\[ (\text{End}(E) \otimes \mathbb{Q}_\ell)^\times \cong G_{\gamma_E} \otimes \mathbb{Q}_\ell \quad \text{for } \ell \neq p. \]

\[ (\text{End}(E) \otimes \mathbb{Q}_p)^\times \cong G_{\delta_\sigma}. \]

Furthermore, \((\text{End}(E) \otimes \mathbb{R})^\times \) is anisotropic modulo center. □

Proof. This combines the fixed point formulas in étale and crystalline cohomology, the Weil conjectures (here Weil’s theorem) for elliptic curves and the main theorems of [13, 23]. ■

Regarding our expression for one isogeny class, we first obtain that (2) equals

\[ \pm \text{vol}((\Gamma \backslash (\text{End}(E) \otimes \mathbb{A}_f)^\times) O_{\gamma}(f^P) O_{\gamma}(f_{p,r} \ast e_{\Gamma(p^n)_{\mathbb{Q}_p}}). \] (3)
writing \( \gamma = \gamma_E \in \text{GL}_2(\mathbb{Q}) \) as in the theorem and using that by part (a), this is compatible with our previous use. Define the function

\[
f = f^p(f_{p,r} \ast e_{\Gamma(p^n)_{\mathbb{Q}_p}}) \in C_c^\infty(\text{GL}_2(\mathbb{A}_f)).
\]

Recalling that \( \Gamma = (\text{End}(E) \otimes \mathbb{Q})^\times \), we see that (3) equals

\[
\pm \text{vol}((\text{End}(E) \otimes \mathbb{Q})^\times \backslash (\text{End}(E) \otimes \mathbb{A}_f)^\times) \int_{G_{\gamma}(\mathbb{A}_f) \backslash \text{GL}_2(\mathbb{A}_f)} f(g^{-1} \gamma g) \, dg. \tag{4}
\]

For any reductive group \( G \) over \( \mathbb{Q} \), let \( \tilde{G} \) be any inner form of \( G \) over \( \mathbb{Q} \) which is anisotropic modulo center over \( \mathbb{R} \), if existent. The terms where \( \tilde{G} \) occurs will not depend on the choice made because of the invariance of the Tamagawa number under inner twists. Collecting everything so far, we see that:

**Theorem 10.5.** The Lefschetz number

\[
\sum_{x \in M_{\Gamma(p^n)}(F_p)} \text{tr} \tilde{\Phi}_{p'} |(R\psi \mathcal{F}_n)_x|
\]

equals

\[
- \sum_{\gamma \in Z(\mathbb{Q})} \text{vol}(\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}_f)) f(\gamma)
+ \sum_{\gamma \in \text{GL}_2(\mathbb{Q}) \backslash Z(\mathbb{Q}) \text{ semisimple conj. class with } \gamma_\infty \text{ elliptic}} \text{vol}(\tilde{G}_\gamma(\mathbb{Q}) \backslash \tilde{G}_\gamma(\mathbb{A}_f)) \int_{G_{\gamma}(\mathbb{A}_f) \backslash \text{GL}_2(\mathbb{A}_f)} f(g^{-1} \gamma g) \, dg.
\]

**Remark.** If \( \gamma \in \text{GL}_2(\mathbb{Q}) \backslash Z(\mathbb{Q}) \) is semisimple with \( \gamma_\infty \) elliptic, then \( G_\gamma \) is already anisotropic modulo center over \( \mathbb{R} \), so that one may take \( \tilde{G}_\gamma = G_\gamma \) in this case. We will not need this fact. \( \square \)

**Proof.** We only need to check that the contributions of \( \gamma \) with \( \det \gamma \neq p^r \) or \( \text{tr} \gamma \notin \mathbb{Z} \) vanish. Assume that \( \det \gamma \neq p^r \). The orbital integrals of \( f^p \) vanish except if the determinant is a unit away from \( p \), so that \( \det \gamma \) is up to sign a power of \( p \). The orbital integrals of \( f_{p,r} \ast e_{\Gamma(p^n)_{\mathbb{Q}_p}} \) vanish except if \( v_p(\det \gamma) = r \), so that \( \det \gamma = \pm p^r \). But if \( \det \gamma = -p^r < 0 \), then \( \gamma \) is hyperbolic at \( \infty \), contradiction.
Assume now that $\gamma \notin \mathbb{Z}$. The orbital integrals of $f^p$ vanish as soon as a prime $\ell \neq p$ is in the denominator of $\text{tr} \gamma$. The orbital integrals of $f_{p, r} \ast e_{l(p^n) \mathcal{Q}_p}$ match with the twisted orbital integrals of $\phi_p \ast e_{l(p^n) \mathcal{Q}_{p'}}$, which were computed in Theorem 9.3. In particular, they are nonzero only if $v_p(\text{tr} \gamma) \geq 0$, so that $\text{tr} \gamma$ is necessarily integral. ■

It turns out that it is easier to apply the Arthur–Selberg trace formula for the cohomology of the compactification $\tilde{\mathcal{M}}_{p^n m}$ instead of the cohomology with compact supports of $\mathcal{M}_{p^n m}$. The corresponding modifications are done in the next section.

11 Contributions from Infinity

Recall that the smooth curve $\mathcal{M}_{p^n m}/\text{Spec } \mathbb{Z}[\frac{1}{p^n}]$ has a smooth projective compactification $j: \mathcal{M}_{p^n m} \to \tilde{\mathcal{M}}_{p^n m}$ with boundary $\partial \mathcal{M}_{p^n m}$. We use a subscript $\mathcal{Q}$ to denote base change to $\mathbb{Q}$. We are interested in the cohomology groups $H^i(\tilde{\mathcal{M}}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell)$. Let

$$H^*(\tilde{\mathcal{M}}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell) = \sum_{i=0}^2 (-1)^i H^i(\tilde{\mathcal{M}}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell), \quad H^*_c(\mathcal{M}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell) = \sum_{i=0}^2 (-1)^i H^i_c(\mathcal{M}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell)$$

in the Grothendieck group of representations of $G_{\mathcal{Q}} \times \text{GL}_2(\mathbb{Z}/p^n m \mathbb{Z})$. Then the long exact cohomology sequence for

$$0 \to j_i^! \mathcal{Q}_\ell \to \mathcal{Q}_\ell \to \bigoplus_{x \in \partial \mathcal{M}_{p^n m \mathcal{Q}}} \tilde{\mathcal{Q}}_{\ell, x} \to 0$$

implies that

$$H^*(\tilde{\mathcal{M}}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell) = H^*_c(\mathcal{M}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell) + H^0(\partial \mathcal{M}_{p^n m \mathcal{Q}}, \mathcal{Q}_\ell).$$

Lemma 11.1. There is a $G_{\mathcal{Q}} \times \text{GL}_2(\mathbb{Z}/p^n m \mathbb{Z})$-equivariant bijection

$$\partial \mathcal{M}_{p^n m \mathcal{Q}} \cong \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \backslash \text{GL}_2(\mathbb{Z}/p^n m \mathbb{Z}),$$

where $\text{GL}_2(\mathbb{Z}/p^n m \mathbb{Z})$ acts on the right-hand side by multiplication from the right, and $G_{\mathcal{Q}}$ acts on the right-hand side by multiplication from the left through the map

$$G_{\mathcal{Q}} \to \text{Gal}(\mathbb{Q}(\zeta_{p^n m})/\mathbb{Q}) \cong (\mathbb{Z}/p^n m \mathbb{Z})^\times \to \text{GL}_2(\mathbb{Z}/p^n m \mathbb{Z}),$$

the last map being given by $x \mapsto (x^{-1} \ 0 \ \ 0 \ \ 1)$. ■
Proof. This is contained in [14]. The point is that the points at infinity correspond to level-$p^m$-structures on the rational $p^m$-gon (cf. also [9]) and the automorphism group of the $p^m$-gon is isomorphic to $\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \}$. The Galois group acts on the $p^m$-torsion points of the rational $p^m$-gon only by its action on the $p^m$th roots of unity.

We get the following corollary.

Corollary 11.2. The semisimple trace of the Frobenius $\Phi^r_p$ on $H^0(\partial M_{p^m\widehat{Q}}, \widehat{Q}_\ell)$ is given by

$$\frac{1}{2} \int_{\text{GL}_2(\widehat{\mathbb{Z}})} \int_{A_f} f^p k^{-1} \begin{pmatrix} 1 & u \\ 0 & p^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} k \ du dk.$$

Remark. Here, for all $p'$ we use the Haar measure on $\mathbb{Q}_{p'}$ that gives $\mathbb{Z}_{p'}$ measure 1; in particular, the subgroup $\widehat{\mathbb{Z}}$ of $A_f$ gets measure 1.

Proof. Note that if

$$f^p k^{-1} \begin{pmatrix} 1 & u \\ 0 & p^r \end{pmatrix} k \neq 0,$$

then $p^r \equiv 1 \mod m$, so that the integral is identically zero if $p^r \not\equiv 1 \mod m$. In fact, in this case, $\Phi^r_p$ has no fixed points on $\partial M_{p^m\widehat{Q}}$. So assume now that $p^r \equiv 1 \mod m$. In that case, the inertia subgroup at $p$ groups the points of $\partial M_{p^m\widehat{Q}}$ into packets of size $p^{n-1}(p-1)$ on which $\Phi^r_p$ acts trivially. Therefore, the semisimple trace of $\Phi^r_p$ is

$$\frac{1}{p^{n-1}(p-1)} \# \left( \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right) \setminus \text{GL}_2(\mathbb{Z}/p^m\mathbb{Z}) \right).$$

But

$$f^p k^{-1} \begin{pmatrix} 1 & u \\ 0 & p^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} k = \# \text{GL}_2(\mathbb{Z}/m\mathbb{Z}) \text{vol}(\text{GL}_2(\widehat{\mathbb{Z}}_p))^{-1}$$

if $u \equiv 0 \mod m$ and is 0 otherwise, so that

$$\int_{\text{GL}_2(\widehat{\mathbb{Z}}_p)} \int_{A_f} f^p k^{-1} \begin{pmatrix} 1 & u \\ 0 & p^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} k \ du dk = \# \left( \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right) \setminus \text{GL}_2(\mathbb{Z}/m\mathbb{Z}) \right).$$
This reduces us to the statement

\[
\int_{GL_2(Z_p) \backslash Q_p} (f_{p,r} \ast e_{\Gamma(p^n)_{QP}}) \left(k^{-1} \begin{pmatrix} 1 & u \\ 0 & p^r \end{pmatrix} k\right)dudk = p^{2n} - p^{2n-2}.
\]

But note that the left-hand side is the orbital integral of \( f_{p,r} \ast e_{\Gamma(p^n)_{QP}} \) for \( \gamma = \begin{pmatrix} 1 & 0 \\ 0 & p^r \end{pmatrix} \), because more generally for \( \gamma_1 \neq \gamma_2 \) and any \( h \in C_c^\infty(GL_2(Q_p)) \)

\[
\int_{GL_2(Z_p) \backslash Q_p} h \left(k^{-1} \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k\right)dudk
\]

\[
= \left|1 - \frac{\gamma_2}{\gamma_1}\right|^{-1}_p \int_{GL_2(Z_p) \backslash Q_p} h \left(k^{-1} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k\right)dudk
\]

\[
= \left|1 - \frac{\gamma_2}{\gamma_1}\right|^{-1}_p \text{vol}(T(Z_p)) \int_{T(Q_p) \backslash GL_2(Q_p)} h \left(g^{-1} \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} g\right)dg,
\]

as \( GL_2(Q_p) = B(Q_p)GL_2(Z_p) \). Note that in our case, \( |1 - \frac{\gamma_2}{\gamma_1}|^{-1}_p = |1 - p^r|^{-1}_p = 1 \). But the orbital integral of \( f_{p,r} \ast e_{\Gamma(p^n)_{QP}} \) equals the corresponding twisted orbital integral of \( \phi_p \ast e_{\Gamma(p^n)_{QP}} \) which was calculated in Lemma 9.4. \( \blacksquare \)

12 The Arthur–Selberg Trace Formula

This section serves to give the special case of the Arthur–Selberg trace formula for \( GL_2 \) that will be needed. We simply specialize the formula in [1] for the trace of Hecke operators on the \( L^2 \)-cohomology of locally symmetric spaces.

Let

\[
H_{(2)} = \lim_{i \to \infty} H_{(2)}^i(M_m(\mathbb{C}), \mathbb{C})
\]

be the inverse limit of the \( L^2 \)-cohomologies of the spaces \( M_m(\mathbb{C}) \). It is a smooth admissible representation of \( GL_2(\mathbb{A}_f) \). Again, we define the element

\[
H_{(2)}^s = \sum_{i=0}^{2} (-1)^i H_{(2)}^i
\]

in the Grothendieck group of smooth admissible representations of \( GL_2(\mathbb{A}_f) \). Then let

\[
\mathcal{L}(h) = \text{tr}(h|H_{(2)}^s)
\]

for any \( h \in C_c^\infty(GL_2(\mathbb{A}_f)) \).
Let $Z \subset T \subset B \subset \text{GL}_2$ be the center, the diagonal torus and the upper triangular matrices. Recall that for any reductive group $G$ over $\mathbb{Q}$, $\hat{G}$ is an inner form of $G$ over $\mathbb{Q}$ that is anisotropic modulo center over $\mathbb{R}$. For $\gamma \in \text{GL}_2$, let $G_\gamma$ be its centralizer. Finally, let

$$T(\mathbb{Q})' = \left\{ \gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \mid \gamma_1 \gamma_2 > 0, |\gamma_1| < |\gamma_2| \right\}.$$

Here and in the following, absolute values always denote the real absolute value.

**Theorem 12.1.** For any $h \in C^\infty_c(\text{GL}_2(\mathbb{A}_f))$, we have

$$\frac{1}{2} \mathcal{L}(h) = \sum_{\gamma \in Z(\mathbb{Q})} \text{vol}(\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}_f)) h(\gamma) + \sum_{\gamma \in \text{GL}_2(\mathbb{Q}) \backslash Z(\mathbb{Q}) \text{ semisimple conj. class with } \gamma_\infty \text{ elliptic}} \text{vol}(\hat{G}_\gamma(\mathbb{Q}) \backslash \hat{G}_\gamma(\mathbb{A}_f)) \int_{G_\gamma(\mathbb{A}_f) \backslash \text{GL}_2(\mathbb{A}_f)} h(g^{-1} \gamma g) \, dg$$

$$+ \frac{1}{2} \sum_{\gamma \in T(\mathbb{Q})'} \int_{\text{GL}_2(\mathbb{Z}) \backslash \mathbb{A}_f} h(k^{-1} \gamma k^{-1}) \, dk \int_{\text{GL}_2(\mathbb{Z}) \backslash \mathbb{A}_f} h(k^{-1} \gamma k^{-1}) \, dk.$$

**Proof.** Specialize [1, Theorem 6.1] to this case. In this proof, we will use freely the notation from that article. As a preparation, we note the following formula for a discrete series character.

**Lemma 12.2.** Let $\pi$ be the admissible representation of $\text{GL}_2(\mathbb{R})$ given by the space of $O(2)$-finite functions on $\mathbb{P}^1(\mathbb{R})$ modulo the constant functions. Let $\Theta_\pi$ be its character. Then for regular elliptic $\gamma \in \text{GL}_2(\mathbb{R})$, $\Theta_\pi(\gamma) = -1$, for $\gamma \in T(\mathbb{Q})$ regular with diagonal entries $\gamma_1$ and $\gamma_2$, one has $\Theta_\pi(\gamma) = 0$ if $\gamma_1 \gamma_2 < 0$, whereas if $\gamma_1 \gamma_2 > 0$, then

$$\Theta_\pi(\gamma) = 2 \frac{\min(|\frac{\gamma_1}{\gamma_2}|, |\frac{\gamma_2}{\gamma_1}|) \frac{1}{2}}{|\frac{\gamma_1}{\gamma_2}|^2 - |\frac{\gamma_2}{\gamma_1}|^2}.$$

**Proof.** This directly follows from the formula for induced characters.
We remark that the representation $\pi$ from the lemma is the unique discrete series representation with trivial central and infinitesimal character.

Now we begin to analyze the formula of [1, Theorem 6.1]. We claim that the term $\frac{1}{2} L(h)$ is equal to $L_\mu(h)$ for $\mu = 1$ in Arthur’s notation. For this, we recall that the $\mathbb{C}$-valued points of $M_m$ are given by

$$M_m(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / (SO_2(\mathbb{R}) \times K_m),$$

where

$$K_m = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv 1 \text{ mod } m \}.$$  

Because $SO_2(\mathbb{R})$ has index 2 in a maximal compact subgroup, we get a factor of 2 by Remark 3 after [1, Theorem 6.1], giving $L(h) = 2 L_\mu(h)$ for $\mu = 1$, as claimed.

The outer sum in [1, Theorem 6.1] runs over $M = GL_2$ and $M = T$.

Consider first the summand for $M = GL_2$. The factor in front of the inner sum becomes 1. The inner sum runs over semisimple conjugacy classes in $\gamma \in GL_2(\mathbb{Q})$ which are elliptic at $\infty$. For the first factor $\chi(G_\gamma)$, we use Remark 2 after [1, Theorem 6.1], noting that the term called $|D(G, B)|$ in that formula is equal to 1, and the sign is $-1$ or $+1$ corresponding to (in this order) $\gamma$ being central or not. The second factor $|\iota^{GL_2}(\gamma)|$ equals 1 in all cases, because all centralizers $G_\gamma$ are connected (as algebraic groups over $\mathbb{Q}$) (cf. [1, Equation (6.1)]). Now, by Lemma 12.2 and the definition of $\Phi_M(\gamma, \mu)$ [1, Equation (4.4)] $\Phi_{GL_2}(\gamma, \mu) = 1$ for regular elliptic $\gamma$ and hence all elliptic $\gamma$, by the way that $\Phi_{GL_2}(\gamma, \mu)$ is extended, cf. [1, p. 275]. Finally, the term $h_{GL_2}(\gamma)$ is precisely the orbital integral $O_h(\gamma)$, by [1, Equation (6.2)]. This takes care of all the terms for $M = GL_2$, giving the first two summands in our formula.

Now, consider the case $M = T$. The factor in front of the inner sum becomes $-\frac{1}{2}$. The inner sum runs over the elements $\gamma \in T(\mathbb{Q})$. Let the diagonal entries of $\gamma$ be $\gamma_1$ and $\gamma_2$. To evaluate $\chi(T_\gamma) = \chi(T)$, we use Remark 2 after [1, Theorem 6.1] again, to get $\chi(T_\gamma) = \text{vol}(T(\mathbb{Q}) \backslash T(\mathbb{A}_f)) = \frac{1}{4}$. The term $|\iota^T(\gamma)|$ gives 1, by the same reasoning as above. We wish to evaluate the term $\Phi_M(\gamma, \mu) = \Phi_{M}(\gamma, 1)$. Consider first the case of regular $\gamma$.

By Lemma 12.2 and the definition (4.4) in [1], we get that $\Phi_T(\gamma, \mu) = 0$ if $\gamma_1 \gamma_2 < 0$, and otherwise equal to $-2 \min(|\gamma_1|, |\gamma_2|)$. The same reasoning as above shows that this result continues to hold for nonregular $\gamma$. If $|\gamma_1| \leq |\gamma_2|$, then the fourth factor $h_T(\gamma)$ appears in the form of [1, Equation (6.3)] in our formula, noting that $\delta_B(\gamma_{\text{fin}})^2 = |\frac{\gamma_1}{\gamma_2}|^\frac{1}{2}$. Finally, note that exchanging $\gamma_1$ and $\gamma_2$ does not change $h_T(\gamma)$, so that we may combine those terms. This gives the desired formula.
We shall also need the spectral expansion for $\mathcal{L}(h)$. Let $\Pi_{\text{disc}}(GL_2(\mathbb{A}), 1)$ denote the set of irreducible automorphic representations $\pi = \bigotimes_{p \leq \infty} \pi_p$ of $GL_2(\mathbb{A})$ with $\pi_\infty$ having trivial central and infinitesimal character, that occur discretely in

$$L^2(GL_2(\mathbb{Q})_{\mathbb{R} \geq 0} \backslash GL_2(\mathbb{A})) .$$

For $\pi \in \Pi_{\text{disc}}(GL_2(\mathbb{A}), 1)$, let $m(\pi)$ be the multiplicity of $\pi$ in $L^2(GL_2(\mathbb{Q})_{\mathbb{R} \geq 0} \backslash GL_2(\mathbb{A}))$. Using the relative Lie algebra cohomology groups, we have the following lemma.

**Lemma 12.3.** For any $i = 0, 1, 2$, there is a canonical $GL_2(\mathbb{A}_f)$-equivariant isomorphism

$$H^i_{(2)} \cong \bigoplus_{\pi \in \Pi_{\text{disc}}(GL_2(\mathbb{A}), 1)} m(\pi) H^i(gl_2, SO_2(\mathbb{R}), \pi_\infty) \pi_f .$$

There are the following possibilities for the representation $\pi_\infty$, which has trivial central and infinitesimal character:

(i) $\pi_\infty$ is the trivial representation or $\pi_\infty = \text{sgn det}$. Then

$$H^i(gl_2, SO_2(\mathbb{R}), \pi_\infty) = \begin{cases} \mathbb{C}, & i = 0 \\ 0, & i = 1, \\ \mathbb{C} & i = 2; \end{cases}$$

(ii) $\pi_\infty$ is the representation from Lemma 12.2. Then

$$H^i(gl_2, SO_2(\mathbb{R}), \pi_\infty) = \begin{cases} 0, & i = 0, \\ \mathbb{C} \oplus \mathbb{C}, & i = 1, \\ 0 & i = 2. \end{cases} \Box$$

**Proof.** The first part is taken from the discussion in [1, Section 2]. The second part is contained in [6].

Denote $\chi(\pi_\infty) = \sum_{i=0}^{2} (-1)^i \dim H^i(gl_2, SO_2(\mathbb{R}), \pi_\infty)$. We get the following corollary.

**Corollary 12.4.** For any $h \in C_c^\infty(GL_2(\mathbb{A}_f))$, we have $\mathcal{L}(h) = \sum_{\pi \in \Pi_{\text{disc}}(GL_2(\mathbb{A}), 1)} m(\pi) \chi(\pi_\infty) \text{tr}(h|\pi_f)$. \boxed{\square}
13 Comparison of the Lefschetz and Arthur–Selberg Trace Formula

We deduce the following theorem. For this, we need to fix an isomorphism \( \overline{\mathbb{Q}} \cong \mathbb{C} \).

**Theorem 13.1.** With \( f \) as above, \( 2 \text{tr}^{ss}(\Phi_p^r|H^*(\tilde{M}_{p^m\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^r})) = \mathcal{L}(f) \). □

**Proof.** We compare the formulas given by Theorem 10.5, Corollary 11.2 and Theorem 12.1. We are left to show that whenever

\[
\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \in T(\mathbb{Q})
\]

with \( \gamma_1 \gamma_2 > 0 \) and \( |\gamma_1| \leq |\gamma_2| \), then

\[
\int_{GL_2(\hat{\mathbb{Z}})} \int_{\mathbb{A}_f} f^{-1}(1) \gamma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k \, du \, dk = 0
\]

except for \( \gamma_1 = 1 \) and \( \gamma_2 = p^r \).

However, the integral factors into a product of local integrals and the integral for a prime \( \ell \neq p \) is only nonzero if \( \gamma \in GL_2(\mathbb{Z}_\ell) \). It follows that \( \gamma_1 \) and \( \gamma_2 \) are up to sign a power of \( p \).

Next, we claim that

\[
\int_{GL_2(\mathbb{Z}_p)} \int_{\mathbb{Q}_p} (f_{p,r} \ast e_{r}(p^n)_{\mathbb{Q}_p}) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k \, du \, dk \neq 0
\]

only if \( v_p(\gamma_1) = 0 \) and \( v_p(\gamma_2) = r \), or the other way around. Indeed, as long as \( \gamma_1 \neq \gamma_2 \), the term is, up to a constant, an orbital integral of \( f_{p,r} \ast e_{r}(p^n)_{\mathbb{Q}_p} \), cf. proof of Corollary 11.2, and we have computed those, by computing the twisted orbital integrals of the matching function \( \Phi_p \ast e_{r}(p^n)_{\mathbb{Q}_p} \). The case \( \gamma_1 = \gamma_2 \) follows by continuity of the integrals.

As \( \gamma_1 \) and \( \gamma_2 \) are up to sign powers of \( p \), we are left with either \( \gamma_1 = 1 \) and \( \gamma_2 = p^r \) or \( \gamma_1 = -1 \) and \( \gamma_2 = -p^r \). But the second case also gives 0, because no conjugate of \( \gamma \) will be \( \equiv 1 \mod m \).

This finally allows us to compute the zeta function of the varieties \( \tilde{M}_m \). Here, \( m \) is any integer which is the product of two coprime integers, both at least 3, and we do not consider any distinguished prime. Recall that the Hasse–Weil zeta-function of a
variety $X$ over a number field $K$ is defined as a product of the local factors,

$$
\zeta(X, s) = \prod_\lambda \zeta(X_{K_\lambda}, s),
$$

convergent for all complex numbers $s$ the real part of which is large enough. Here $\lambda$ runs through the finite places of $K$ and $X_{K_\lambda}$ denotes the base-change of $X$ to the local field $K_\lambda$.

**Theorem 13.2.** The Hasse–Weil zeta-function of $\tilde{\mathcal{M}}_m$ is given by

$$
\zeta(\tilde{\mathcal{M}}_m, s) = \prod_{\pi \in \Pi_{\text{disc}}(\text{GL}_2(\hat{\mathbb{Z}}), 1)} L \left( \pi, s - \frac{1}{2} \right)^{\frac{1}{2} m(\pi) \chi(\pi_\infty) \dim \pi_{K_m}^{f}},
$$

where $K_m = \{ g \in \text{GL}_2(\hat{\mathbb{Z}}) \mid g \equiv 1 \mod m \}$. □

**Proof.** We compute the semisimple local factors at all primes $p$. For this, write $m = p^n m'$, where $m'$ is not divisible by $p$. By assumption on $m$, we get $m' \geq 3$. Combining Theorem 13.1 and Corollary 12.4, one sees that

$$
\sum_{i=0}^{2} (-1)^i \text{tr}^s(\Phi_p^r|H^i(\tilde{\mathcal{M}}_m, \tilde{\mathbb{Q}}_p, \tilde{\mathbb{Q}}_\ell)) = \frac{1}{2} \text{tr}^s \sum_{\pi \in \Pi_{\text{disc}}(\text{GL}_2(\hat{\mathbb{Z}}), 1)} m(\pi) \chi(\pi_\infty) \text{tr}^s(\Phi_p^r|_\sigma \pi_p) \dim \pi_{K_m}^{f}. \quad (5)
$$

We check manually that also

$$
\sum_{i \in \{0, 2\}} (-1)^i \text{tr}^s(\Phi_p^r|H^i(\tilde{\mathcal{M}}_m, \tilde{\mathbb{Q}}_p, \tilde{\mathbb{Q}}_\ell)) = \frac{1}{2} \text{tr}^s \sum_{\pi \in \Pi_{\text{disc}}(\text{GL}_2(\hat{\mathbb{Z}}), 1)} m(\pi) \chi(\pi_\infty) \text{tr}^s(\Phi_p^r|_\sigma \pi_p) \dim \pi_{\infty} = 1. \quad (6)
$$

Indeed, the sum on the right-hand side gives nonzero terms only for one-dimensional representations $\pi$ which are trivial on $K_m$. Using $\chi(\pi_\infty) = 2$, $\dim \pi_{K_m}^{f} = 1$ and $m(\pi) = 1$, the statement then reduces to class field theory, as the geometric connected components of $\tilde{\mathcal{M}}_m$ are parameterized by the primitive $m$th roots of unity. Note that in (6) one may replace the semisimple trace by the usual trace on the $I_{\mathbb{Q}_p}$-invariants everywhere. This gives

$$
\prod_{i \in \{0, 2\}} \text{det}(1 - \Phi_p p^{-s}|H^i(\tilde{\mathcal{M}}_m, \tilde{\mathbb{Q}}_p, \tilde{\mathbb{Q}}_\ell) I_{\mathbb{Q}_p}) = \prod_{\pi \in \Pi_{\text{disc}}(\text{GL}_2(\hat{\mathbb{Z}}), 1)} L \left( \pi_p, s - \frac{1}{2} \right)^{\frac{1}{2} m(\pi) \chi(\pi_\infty) \dim \pi_{K_m}^{f}}. \quad (7)
$$
Subtracting (6) from (5), we see that

\[- \text{tr}^{ss}(\Phi^p_r|H^1(\tilde{M}_{m, \Q_p}, \tilde{Q}_r)) = \frac{1}{2} p^{\frac{1}{2} r} \sum_{\pi \in \Pi_{\text{disc}}(\text{GL}_2(\A_f), 1)} m(\pi) \chi(\pi_\infty) \dim \pi_{\infty} > 1 \dim \pi_f^K_m,\]

or equivalently

\[\det^{ss}(1 - \Phi_p P^{-s}|H^1(\tilde{M}_{m, \Q_p}, \tilde{Q}_r))^{-1} = L\left(\sigma^{ss}_{\pi_p}, s - \frac{1}{2}\right)^{\frac{1}{2} m(\pi) \chi(\pi_\infty) \dim \pi_f^K_m},\]

with the obvious definition for the semisimple determinant. All zeros of the left-hand side have imaginary part 0, 1/2, or 1: Indeed, if \(\tilde{M}_{m, \Q_p}\) had good reduction, the Weil conjectures would imply that all zeros have imaginary part 1/2. In general, the semistable reduction theorem for curves together with the Rapoport–Zink spectral sequence imply that all zeros have imaginary part 0, 1/2, or 1. Changing the semisimple determinant to the usual determinant on the invariants under \(I_{\Q_p}\) exactly eliminates the zeros of imaginary part 1, by the monodromy conjecture, proved in dimension 1 in [22].

We also see that all zeros of the right-hand side have imaginary part 0, 1/2, or 1. Assume \(\pi\) gives a nontrivial contribution to the right-hand side. Then \(\pi_p\) cannot be one-dimensional, because otherwise \(\pi\) and hence \(\pi_\infty\) would be one-dimensional. Hence \(\pi_p\) is generic. Being also unitary, the \(L\)-factor \(L(\pi_p, s - \frac{1}{2})\) of \(\pi_p\) cannot have poles with imaginary part greater than or equal to 1, so that replacing \(L(\sigma^{ss}_{\pi_p}, s - \frac{1}{2})\) by \(L(\pi_p, s - \frac{1}{2})\) consists again in removing all zeros of imaginary part 1.

We find that

\[\det(1 - \Phi_p P^{-s}|H^1(\tilde{M}_{m, \Q_p}, \tilde{Q}_r)^{I_{\Q_p}})^{-1} = \prod_{\pi \in \Pi_{\text{disc}}(\text{GL}_2(\A_f), 1)} L\left(\pi_p, s - \frac{1}{2}\right)^{\frac{1}{2} m(\pi) \chi(\pi_\infty) \dim \pi_f^K_m}.\]

Combining (7) and (10) yields the result.

\[\text{14 \ Explicit Determination of } \phi_p \ast e_{r/p^n}\]

In this section, we aim to determine the values of the function \(\phi_p \ast e_{r/p^n}\) for \(n \geq 1\). Set \(q = p^r\).
For any \( g \in \text{GL}_2(\mathbb{Q}_q) \), we let \( k(g) \) denote the minimal number \( k \) such that \( p^k g \) has integral entries. If additionally, \( v_p(\det g) \geq 1 \) and \( v_p(\tr g) = 0 \), then \( g \) has a unique eigenvalue \( x \in \mathbb{Q}_q \) with \( v_p(x) = 0 \); we define \( \ell(g) = v_p(x - 1) \) in this case. The choice of the maximal compact subgroup \( \text{GL}_2(\mathbb{Z}_q) \) gives a vertex \( v_0 \) in the building of \( \text{PGL}_2 \). We will need another characterization of \( k(g) \).

**Lemma 14.1.** For any \( g \in \text{GL}_2(\mathbb{Q}_q) \) which is conjugate to an integral matrix, consider the set \( V_g \) of all vertices \( v \) such that \( g(\Lambda_v) \subset \Lambda_v \), where \( \Lambda_v \) is the lattice corresponding to \( v \). Then the distance of \( v \) and \( v_0 \) is at least \( k(g) \) for all \( v \in V_g \) and there is a unique vertex \( v(g) \in V_g \) such that the distance of \( v(g) \) and \( v_0 \) is equal to \( k(g) \).

**Proof.** Note that if \( k(g) = 0 \), then this is trivial. So assume \( k(g) > 0 \).

It is technically more convenient to use norms (or equivalently valuations) instead of lattices. So let \( \text{val}_v : \mathbb{Z}_q^2 \to \mathbb{R} \) be the valuation associated to \( v \) in the building of \( \text{PGL}_2 \); it is well defined up to a constant. Then the distance of \( v \) and \( v_0 \) is

\[
\max(\text{val}_v(y) - \text{val}_v(x) + \text{val}_{v_0}(x) - \text{val}_{v_0}(y)).
\]

Now by definition of \( k(g) \), one has \( \text{val}_{v_0}(gx) \geq \text{val}_{v_0}(x) - k(g) \) for all \( x \in \mathbb{Z}_q^2 \), but there is some \( x \in \mathbb{Z}_q^2 \) with \( \text{val}_{v_0}(gx) = \text{val}_{v_0}(x) - k(g) \). Fix such an \( x \) and set \( y = gx \). Assuming that \( v \in V_g \), we have \( \text{val}_v(gx) \geq \text{val}_v(x) \), so that

\[
\text{dist}(v, v_0) \geq \text{val}_v(gx) - \text{val}_v(x) + \text{val}_{v_0}(x) - \text{val}_{v_0}(gx) \geq k(g),
\]

giving the first claim. But we may more generally set \( y = gx + ax \) for any \( a \in \mathbb{Z}_q \). Then we still have \( \text{val}_{v_0}(y) \leq \text{val}_{v_0}(x) - k(g) \) (since \( k(g) > 0 \)) and \( \text{val}_v(y) \geq \text{val}_v(x) \), giving

\[
\text{dist}(v, v_0) \geq \text{val}_v(y) - \text{val}_v(x) + \text{val}_{v_0}(x) - \text{val}_{v_0}(y) \geq k(g),
\]

as before. It follows that if \( \text{dist}(v, v_0) = k(g) \), then necessarily \( \text{val}_v(gx + ax) = \text{val}_v(x) \) for all \( a \in \mathbb{Z}_q \). But then it is clear that \( \Lambda_v \) is the lattice generated by \( gx \) and \( x \), so that we get uniqueness.

Finally, define

\[
\text{val}_g(x) = \min(\text{val}_{v_0}(x), \text{val}_{v_0}(gx)).
\]
Since \( g^2x = (\text{tr } g)gx - (\det g)x \) and both \( \text{tr } g \) and \( \det g \) are integral, one sees that \( \text{val}_g(gx) \geq \text{val}_g(x) \). Furthermore,

\[
\text{val}_g(y) - \text{val}_g(x) + \text{val}_v(x) - \text{val}_v(y) \leq \min(0, \text{val}_v(x) - \text{val}_v(gx)) \leq k(g),
\]

so that by what we have already proved, \( \text{val}_g(x) \) corresponds to a point \( v(g) \in V_g \) with distance \( k(g) \) to \( v_0 \).

**Remark.** This lemma is related to the fact that for all matrices \( g \in \text{GL}_2(\mathbb{Q}_q) \), the set \( V_g \) is convex.

For a fixed vertex \( v \), the set of all \( g \in \text{GL}_2(\mathbb{Q}_q) \), conjugate to some integral matrix, with \( v(g) = v \) is denoted \( G_v \). By \( \tilde{G}_v \), we mean the set of all \( g \) which map the lattice corresponding to \( v \) into itself.

It is clear that if \( v_p(\det g) \geq 1 \) and \( v_p(\text{tr } g) = 0 \) (so that \( \ell(g) \) is defined), then \( \ell(g) = v_p(1 - \text{tr } g + \det g) \). For \( n \geq 1 \), we define a function \( \phi_{p,n} : \text{GL}_2(\mathbb{Q}_q) \to \mathbb{C} \) by the following requirements:

- \( \phi_{p,n}(g) = 0 \) except if \( v_p(\det g) = 1 \), \( v_p(\text{tr } g) \geq 0 \) and \( k(g) \leq n - 1 \). Assume now that \( g \) has these properties.
- \( \phi_{p,n}(g) = -1 - q \) if \( v_p(\text{tr } g) \geq 1 \),
- \( \phi_{p,n}(g) = 1 - q^{2\ell(g)} \) if \( v_p(\text{tr } g) = 0 \) and \( \ell(g) < n - k(g) \),
- \( \phi_{p,n}(g) = 1 + q^{2(n-k(g))} - 1 \) if \( v_p(\text{tr } g) = 0 \) and \( \ell(g) \geq n - k(g) \).

**Theorem 14.2.** Choose the Haar measure on \( \text{GL}_2(\mathbb{Q}_q) \) such that a maximal compact subgroup has measure \( q - 1 \). Then \( \phi_{p,n} = \phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_q}} \). □

**Proof.** It is enough to check that \( \phi_{p,n} \) lies in the center of the Hecke algebra and that the semisimple orbital integrals of \( \phi_{p,n} \) and \( \phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_q}} \) agree.

In the case \( q = p \), we have computed the orbital integrals of \( \phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_q}} \) in Theorem 9.3. In fact, the calculation goes through for all powers \( q \) of \( p \). Recall that \( \phi_{p,0} \) is the characteristic function of \( \text{GL}_2(\mathbb{Z}_q)(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}) \text{GL}_2(\mathbb{Z}_q) \) divided by the volume of \( \text{GL}_2(\mathbb{Z}_q) \).

**Theorem 14.3.** Let \( \gamma \in \text{GL}_2(\mathbb{Q}_q) \) be semisimple. Then

\[
O_\gamma(\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_q}}) = O_\gamma(\phi_{p,0})c(\gamma).
\]
where
\[ c(\gamma) = \begin{cases} (1 + q)(1 - q^n), & v_p(\det \gamma) = 1, v_p(\tr \gamma) \geq 1, \\ q^{2n} - q^{2n-2}, & v_p(\det \gamma) = 1, v_p(\tr \gamma) = 0, \ell(\gamma) \geq n, \\ 0, & \text{else}. \end{cases} \]

**Proof.** Note that when \( q = p \), then \( c(\gamma) = c_1(\gamma, er_{(p^n)q_p}) \), so that this is Theorem 9.3. But note that we never used that \( p \) is a prime in the local harmonic analysis, so that replacing \( \mathbb{Q}_p \) by \( \mathbb{Q}_q \) everywhere gives the result for general \( q \). □

We now aim at proving the same formula for \( \phi_{p,n} \).

**Proposition 14.4.** Let \( \gamma \in \text{GL}_2(\mathbb{Q}_q) \) be semisimple. Then \( O_{\gamma}(\phi_{p,n}) = O_{\gamma}(\phi_{p,0}) c(\gamma) \). □

**Proof.** First of all, note that \( O_{\gamma}(\phi_{p,n}) \) can only be nonzero if \( \gamma \) is conjugate to an integral matrix and \( v_p(\det \gamma) = 1 \). Of course, the same holds for \( O_{\gamma}(\phi_{p,0}) \). Hence we only need to consider the case that \( \gamma \) is integral and \( v_p(\det \gamma) = 1 \).

For any vertex \( v \) of the building of \( \text{PGL}_2 \), let \( G_{v,\gamma} = \{ g \in \text{GL}_2(\mathbb{Q}_q) \mid v(g^{-1}g) = v \} \).

**Lemma 14.5.** For any \( v \neq v_0 \), we have
\[
\frac{\text{vol}(G_{v,\gamma}(\mathbb{Q}_q) \setminus G_{v,\gamma})}{\text{vol}(G_{v,\gamma}(\mathbb{Q}_q) \setminus G_{v_0,\gamma})} = \begin{cases} \frac{q}{q + 1}, & \tr \gamma \equiv 0 \text{ mod } p, \\ \frac{q - 1}{q + 1}, & \tr \gamma \not\equiv 0 \text{ mod } p. \end{cases}
\]

**Proof.** Let \( v' \) be the first vertex on the path from \( v \) to \( v_0 \). Then \( G_v = \tilde{G}_v \setminus \tilde{G}_{v'} \). Furthermore, \( \tilde{G}_v \) is conjugate to \( \tilde{G}_{v_0} \). Under this conjugation, \( v' \) is taken to some vertex \( v'_0 \) that is a neighbor of \( v_0 \). In fact, one may choose \( v'_0 \) arbitrarily. We see that \( G_v \) is conjugate to \( G_{v_0} \setminus \tilde{G}_{v'_0} \) for all neighbors \( v'_0 \) of \( v_0 \) (note that \( G_{v_0} = \tilde{G}_{v_0} \)). Hence \( G_{v,\gamma} \) is conjugate to the set
\[ \{ g \in \text{GL}_2(\mathbb{Q}_q) \mid g^{-1}g \in G_{v_0} \setminus \tilde{G}_{v'_0} \} \]

which is obviously a subset of \( G_{v_0,\gamma} \). We check that if \( \gamma \in G_{v_0} \) and \( \tr \gamma \equiv 0 \text{ mod } p \), then there are \( q \) (out of \( q + 1 \)) neighbors \( v'_0 \) of \( v_0 \) such that \( \gamma \notin \tilde{G}_{v'_0} \) and if \( \tr \gamma \not\equiv 0 \text{ mod } p \), then there are \( q - 1 \) neighbors with this property. In fact, \( \gamma \in \tilde{G}_{v'_0} \) if and only if \( \gamma \text{ mod } p \) stabilizes the line in \( \mathbb{F}_q^2 \) corresponding to \( v'_0 \). Now in the first case, \( \gamma \text{ mod } p \) has only eigenvalue 0, with geometric multiplicity 1, whereas in the second case, \( \gamma \text{ mod } p \) has two distinct eigenvalues 0 and \( \tr \gamma \).
Using this with $g^{-1} \gamma g$ in place of $\gamma$, we see that each element of $G_{v_0, \gamma}$ lies in precisely $q$ (resp. $q - 1$) of the $q + 1$ sets

$$\{ g \in \text{GL}_2(\mathbb{Q}_q) \mid g^{-1} \gamma g \in G_{v_0} \setminus \bar{G}_{v'_0} \}$$

indexed by $v'_0$. This gives the claim. ■

Note that we have (by our choice of Haar measure, giving a maximal compact subgroup measure $q - 1$)

$$O_{\gamma}(\phi_{p,0}) = \frac{1}{q - 1} \text{vol}(G_{\gamma}(\mathbb{Q}_q) \setminus G_{v_0, \gamma}).$$

Now we are reduced to a simple counting argument. Assume first that $\text{tr} \gamma \equiv 0 \mod p$. Then $\phi_{p,n}(g^{-1} \gamma g) = -1 - q$ as long as $k(g^{-1} \gamma g) \leq n - 1$; otherwise, it gives 0. There are $(q + 1)(1 + q + q^2 + \cdots + q^{n-2})$ vertices $v \neq v_0$ with distance at most $n - 1$. Hence, by the lemma,

$$O_{\gamma}(\phi_{p,n}) = -(1 + q) \text{vol}(G_{\gamma}(\mathbb{Q}_q) \setminus G_{v_0, \gamma}) - (1 + q)(q + 1)(1 + q + \cdots + q^{n-2})$$

$$\times \frac{q}{q + 1} \text{vol}(G_{\gamma}(\mathbb{Q}_q) \setminus G_{v_0, \gamma})$$

$$= -(1 + q)(1 + q + \cdots + q^{n-1}) \text{vol}(G_{\gamma}(\mathbb{Q}_q) \setminus G_{v_0, \gamma}).$$

Comparing, we get the claim.

Now assume that $\text{tr} \gamma \not\equiv 0 \mod p$, so that $\ell(\gamma)$ is defined. Assume that $\ell(\gamma) < n$. Then for $k(g^{-1} \gamma g) < n - \ell(g)$, we have

$$\phi_{p,n}(g^{-1} \gamma g) = 1 - q^{2\ell(g)},$$

for $n - \ell(g) \leq k(g^{-1} \gamma g) < n$, we have

$$\phi_{p,n}(g^{-1} \gamma g) = 1 + q^{2(n-k(g^{-1} \gamma g)) - 1},$$
and in all other cases we get 0. Therefore, again by the lemma,
\[
\frac{O_\gamma(\phi_{p,n})}{\text{vol}(G_\gamma(\mathbb{Q}_q) \backslash G_{\nu_0,\gamma})} = (1 - q^{2\ell(g)})\text{vol}(G_\gamma(\mathbb{Q}_q) \backslash G_{\nu_0,\gamma})
\]
\[
+ (1 - q^{2\ell(g)})(q + 1)(1 + q + \cdots + q^{n-\ell(g)-2})\frac{q - 1}{q + 1}
\]
\[
+ (1 + q^{2\ell(g)-1})(q + 1)q^{n-\ell(g)-1} + \cdots + (1 + q^3)(q + 1)q^{n-3}
\]
\[
+ (1 + q)(q + 1)q^{n-2}\frac{q - 1}{q + 1}
\]
\[
= (1 - q^{2\ell(g)})q^{n-\ell(g)-1} + (q - 1)(q^{n-\ell(g)-1}) + \cdots + q^{n-2}
\]
\[
+ q^{n-1} + \cdots + q^{n+\ell(g)-2}
\]
\[
= (1 - q^{2\ell(g)})q^{n-\ell(g)-1} + (q^{2\ell(g)} - 1)q^{n-\ell(g)-1} = 0,
\]
as claimed.

Finally, assume \(\ell(\gamma) = n\). Then \(\phi_{p,n}(g^{-1}\gamma g) = 1 + q^{2(n-k(g^{-1}\gamma g))} - 1\) if \(k(g^{-1}\gamma g) < n\) and vanishes otherwise. This shows that
\[
\frac{O_\gamma(\phi_{p,n})}{\text{vol}(G_\gamma(\mathbb{Q}_q) \backslash G_{\nu_0,\gamma})} = (1 + q^{2n-1}) + ((1 + q^{2n-3})(q + 1)q^0 + \cdots + (1 + q)(q + 1)q^{n-2})\frac{q - 1}{q + 1}
\]
\[
= (1 + q^{2n-1}) + (q - 1)(1 + q + \cdots + q^{n-2} + q^{n-1} + \cdots + q^{2n-3})
\]
\[
= q^{2n-1} + q^{2n-2}.
\]
Again, this is what we have asserted.

It remains to see that \(\phi_{p,n}\) lies in the center of \(\mathcal{H}(GL_2(\mathbb{Q}_q), \Gamma(p^n)_{\mathbb{Q}_q})\). Our argument will be slightly indirect, as the direct approach would run into some convergence issues. We consider the following deformation \(\phi_{p,n,t}\) of \(\phi_{p,n}\):

- \(\phi_{p,n,t}(g) = 0\) except if \(v_p(\det g) = 1\), \(v_p(\text{tr} g) \geq 0\) and \(k(g) \leq n - 1\). Assume now that \(g\) has these properties.
- \(\phi_{p,n,t}(g) = -q^{1-\ell^2} / q^{-\ell^2}\) if \(v_p(\text{tr} g) \geq 1\).
- \(\phi_{p,n,t}(g) = 1 - t^{2\ell(g)}\) if \(v_p(\text{tr} g) = 0\) and \(\ell(g) < n - k(g)\).
- \(\phi_{p,n,t}(g) = 1 - (q-1)^{2(n-k(g))} / q^{-\ell^2}\) if \(v_p(\text{tr} g) = 0\) and \(\ell(g) \geq n - k(g)\).
Then specializing to \( t = q \), we have \( \phi_{p,n,q} = \phi_{p,n} \). We claim that for all \( t \), \( \phi_{p,n,t} \) lies in the center of \( \mathcal{H}(\text{GL}_2(\mathbb{Q}_q), \Gamma(p^n)_{\mathbb{Q}_q}) \). Since as a function of \( t \), \( \phi_{p,n,t} \) is a rational function and hence any identity of the form \( \phi_{p,n,t} \cdot f = f \cdot \phi_{p,n,t} \) reduces to a polynomial identity in \( t \), it suffices to check this for \( t \) infinitesimally small. Hence consider \( \phi_{p,n,t}(g) \) as a function on \( \text{GL}_2(\mathbb{Q}_q) \) with values in \( \mathbb{C}[[t]] \). First we check that these functions are compatible for varying \( n \).

**Proposition 14.6.** We have \( \phi_{p,n,t} = \phi_{p,n+1,t} \cdot e_{\Gamma(p^n)\mathbb{Q}_q} \). \( \square \)

**Proof.** We compare function values at \( g \in \text{GL}_2(\mathbb{Q}_q) \). We may assume that \( v_p(\det g) = 1 \), as otherwise both functions vanish. Also if \( k(g) \geq n + 1 \), then so is \( k(gu) \geq n + 1 \) for all \( u \in \Gamma(p^n)_{\mathbb{Q}_q} \), so that both sides give 0. Hence we may assume \( k(g) \leq n \). Note that in all cases, \( k(gu) = k(g) \) for all \( u \in \Gamma(p^n)_{\mathbb{Q}_q} \). Also recall that if \( \ell(g) \) is defined, then \( \ell(g) = v_p(1 - \text{tr} g + \det g) \).

Consider the case \( k(g) = n \). We need to check that the right-hand side gives 0. Note that in this case, the value \( \phi_{p,n+1,t}(gu) \) depends only on \( \text{tr}(gu) \mod p \). It is easy to see that each value of \( \text{tr}(gu) \mod p \) is taken the same number of times. For \( \text{tr}(gu) = 0 \mod p \), we have

\[
\phi_{p,n+1,t}(gu) = -q \frac{1 - t^2}{q - t^2},
\]

for \( \text{tr}(gu) = 1 \mod p \), we have

\[
\phi_{p,n+1,t}(gu) = 1 - \frac{(q - 1)t^2}{q - t^2} = \frac{1 - t^2}{q - t^2},
\]

and for all other values of \( \text{tr}(gu) \mod p \), we have \( \phi_{p,n+1,t}(gu) = 0 \). This gives the result.

Now we can assume that \( k(g) \leq n - 1 \). Assume first that \( \text{tr} g \equiv 0 \mod p \). Then \( \text{tr}(gu) \equiv 0 \mod p \) and hence \( \phi_{p,n+1,t}(gu) = \phi_{p,n,t}(g) \) for all \( u \in \Gamma(p^n)_{\mathbb{Q}_q} \), giving the claim in this case.

We are left with \( \text{tr} g \not\equiv 0 \mod p \). If \( k(g) + \ell(g) < n \), then \( k(gu) = k(g) \) and \( \ell(gu) = \ell(g) \) for all \( u \in \Gamma(p^n)_{\mathbb{Q}_q} \) and in particular again \( k(gu) + \ell(gu) < n \), so that by definition \( \phi_{p,n+1,t}(gu) = \phi_{p,n,t}(g) \).

So, finally, we are in the case \( \text{tr} g \not\equiv 0 \mod p \), \( k(g) + \ell(g) \geq n \), but \( k(g) \leq n - 1 \). Then

\[
\phi_{p,n,t}(g) = 1 - \frac{(q - 1)t^{2(n-k(g))}}{q - t^2}.
\]
We know that \( \text{tr}(gu) \equiv 1 \mod p^{n-k(g)} \), but all values of \( \text{tr}(gu) \mod p^{n-k(g)+1} \) with this restriction are taken equally often. If

\[
\text{tr}(gu) \equiv 1 + \det g \mod p^{n+1-k(g)},
\]

then \( \ell(gu) \geq n + 1 - k(g) \) (since \( \det(gu) \equiv \det g \mod p^{n+1} \)), so that

\[
\phi_{p,n+1,t}(gu) = 1 - \frac{(q-1)t^{2(n+1-k(g))}}{q-t^2}.
\]

In all other cases, we have \( \ell(gu) = n - k(g) \), so that

\[
\phi_{p,n+1,t}(gu) = 1 - t^{2(n-k(g))}.
\]

Hence we get

\[
(\phi_{p,n+1,t} \ast e_{\Gamma(p^n)_{\mathbb{Q}_q}})(g) = \frac{1}{q} \left( 1 - \frac{(q-1)t^{2(n+1-k(g))}}{q-t^2} \right) + \frac{q-1}{q} (1 - t^{2(n-k(g))}) = 1 - \frac{(q-1)t^{2(n-k(g))}}{q-t^2} = \phi_{p,n,t}(g),
\]

as claimed.

Hence we may consider the system \( \phi_{p,t} = (\phi_{p,n,t})_n \) as a distribution with values in \( \mathbb{C}[[t]] \) on the compactly supported, locally constant functions on \( GL_2(\mathbb{Q}_q) \) with the property that \( \phi_{p,t} \ast e_K \) is compactly supported for all compact open subgroups \( K \). To check that \( \phi_{p,n,t} \) is central for all \( n \), it remains to see that \( \phi_{p,t} \) is conjugation-invariant. But note that \( \phi_{p,t} \mod t^m \) is represented by a locally constant function for all \( m \)—the important point here is that \( \phi_{p,t} \mod t^m \) becomes constant when one eigenvalue of \( g \) approaches 1. Here we need our deformation parameter \( t \). It is also clear that \( \phi_{p,t} \mod t^m \) is conjugation-invariant for all \( m \), which finishes the proof.

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