Non-solidity of high index Fano 3-folds

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Abstract

We give conditions for a $\mathbb{Q}$-factorial terminal Fano 3-fold $X$ with Fano index $\iota_X \geq 2$ to be non-solid. This is complemented in the index $\iota_X = 2$ case by an explicit analysis of birational links for codimension 4 Fano 3-folds. This study provides further evidence to a conjecture by Abban and Okada on the solidity of Fano 3-folds in high codimension. We also reformulate their conjecture for high index: if the codimension and index of $X$ are at least 2, then $X$ is non-rigid and if the codimension of $X$ is at least 3, then it is non-solid.

1 Introduction

The birational classification of Fano varieties can be divided into two classes: those that are birational to a strict fibration, and those that are not. The notion that encodes this nature is the one of solidity. A Fano variety $X$ is birationally solid if there is no birational map from $X$ to a Mori fibre space $\sigma : Y \to S$ with $\dim S > 0$ (cf \cite[Definition 1.4]{AO18}). Notice that a stronger form of birational solidity was introduced by Corti (cf \cite[Definition 1.3]{Cor00}): a Fano variety is birationally rigid if it is not birational to any non-isomorphic Mori fibre space. Both of these imply irrationality of $X$ in a strong sense. In this paper we partially answer a question of Abban and Okada on the rarity of solid Fano 3-folds.

In this paper we look at terminal $\mathbb{Q}$-factorial Fano 3-folds $X$ with Fano index $\iota_X \geq 2$, where

$$\iota_X := \max\{q \in \mathbb{Z}_{\geq 1} : -K_X \sim qA \text{ for some } A \in \text{Cl}(X)\}.$$ 

These come with an embedding into weighted projective space given by the ring of sections

$$\bigoplus_{m \geq 0} H^0(X, mA).$$ 

(1)

We establish sufficient conditions on the embedding of such varieties such that they are non-solid. We prove the following Main Theorem.

**Theorem 1.1.** Let $X$ be a $\mathbb{Q}$-Fano 3-fold with $\rho_X = 1$ and Fano index $\iota_X \geq 2$ embedded in a weighted projective space $X \subset \mathbb{P}(a_0, a_1, a_2, \ldots, a_N)$ such that $a_i \leq a_{i+1}$. Suppose that $l = \text{lcm}(a_0, a_1) < \iota_X$. Then, $X$ is birational to a Mori fibre space $Y \to S$ where $\dim S > 0$.

Together with Theorem 1.2 below, our Main Theorem partly answers, and extends to higher index, the question posed by Abban and Okada in \cite[Question 1.5]{AO18} regarding the occurrence of solid varieties in high codimension, and confirms the prediction that solid varieties cease to appear when codimension and Fano index increase.

We also formulate a version of this statement for index $\iota_X = 2$ in Corollary 2.1. Dwelling on the index 2 varieties, we take Fano 3-folds in codimension 4 as a case study. In Table I we list the 11 families of codimension 4 and index 2 Fano 3-folds that fall in the description of Corollary 2.1 identified by their Graded Ring Database ID (GRDB, \cite{BK}). There are 34 families of codimension 4 Fano 3-folds of index 2, each corresponding to a different Hilbert series in GRDB. We examine the remaining 23 using explicit methods; except for three families (which are birationally non-rigid), we show that they are all non-solid in the following theorem.
Theorem 1.2. Let $X \subset \mathbb{P}$ be a quasi-smooth codimension 4 and index 2 Fano 3-fold that is not in the families #39890, #39928, and #39660. Then, there is at least one deformation family of $X$ that is non-solid. Moreover, the excluded families are birationally non-rigid.

We prove Theorem 1.2 by studying the birational links initiated by blowing up a Type I centre (or a Type II centre for families #39569 and #39607) in each of the 23 families, as in Subsection 3.2 and Section 4. Such analysis of these birational links relies on the explicit description of index 2 Fano 3-folds in codimension 4. Several approaches to their explicit construction can be found in the literature. Prokhorov and Reid [PR16] build one family in codimension 4 via a divisorial extraction of a specific curve in $\mathbb{P}^4$ and running the Sarkisov Program starting with such extraction. In [Duc18], Ducat generalises their construction, recovering the family of Prokhorov-Reid and finding two new deformation families in codimension 4 having the same Hilbert series. Of the latter two families, we study only one in this paper; in particular, we retrieve the birational links of [PR16, Duc18] (see Subsection 4.2). To Coughlan and Ducat [CD20] it is due a different approach to constructing Fano 3-folds that relies on rank 2 cluster algebras. More recently, Campo [Cam21] has constructed a total of 52 families of codimension 4 Fano 3-folds of index 2 by means of equivariant unprojections, some of which correspond to the same Hilbert series, also in accordance to [Duc18, CD20]. We mostly refer to this approach in the rest of this paper. In Section 3 we give a brief overview of the construction in [Cam21].

The investigation of non-solid codimension 2 Fano 3-folds complete intersections with index $\iota \geq 2$ has been carried out by Guerreiro in [DG21, Theorem 1.0.1]. Concerning Fano hypersurfaces, in [ACP20] the authors prove that there is no Fano 3-fold hypersurface with index greater than 1 that is birationally non-rigid [ACP20, Theorem 1.1], and conjecture which are the only families of solid hypersurfaces [ACP20, Conjecture 1.4]. The families of codimension 3 Fano 3-folds with index 2 are only two: one is smooth (in [Isk78]), while the other is studied in [PR16, Example 6.3.3 and Section 6.4], and they are both rational.

These works, together with our paper, give evidence to the following conjecture.

Conjecture 1.3. Suppose $X$ is a Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities embedded via (1) in a weighted projective space with codimension and Fano index at least 2. Then $X$ is not birationally rigid. Moreover, if the codimension is at least 3, $X$ is non-solid.

Our case study also highlights an interesting phenomenon (end of Case II.a below), already occurring in [DG21, Section 4.4], that is: in one instance (#39660) the birational link terminates with a divisorial contraction to a 3-fold embedded in a fake weighted projective space ([Buc08, Kas09]). Moreover, we are able to compute the Picard rank of the two families #40671 and #40672 that behave à la Ducat (Section 4.3), which is equal to 2.

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2 Non-solidity of Fano 3-folds

The Main Theorem 1.1 determines the sufficient conditions for a terminal $\mathbb{Q}$-Fano 3-fold to be non-solid.

Proof of Main Theorem 1.1 The anticanonical divisor of $X$ is $-K_X = \iota_X H|_X$, where $H$ is a Weil divisor on $\mathbb{P} := \mathbb{P}(a_0, a_1, a_2, \ldots, a_N)$ whose restriction to $X$ generates $\text{Cl}(X)$. Since $-K_X$ is divisible in $\text{Cl}(X)$ it follows from [BS07, Theorem 1] that $H^0(X, -K_X) \neq 0$ and therefore $-K_X$ is effective. Consider the projection $\mathbb{P}(a_0, a_1, a_2, \ldots, a_N) \dasharrow \mathbb{P}^1$ to the first two coordinates $u_0, u_1,$
and its restriction $\pi: X \rightarrow \mathbb{P}^1$ to $X$. A generic fibre $\pi^{-1}(t)$ is a surface, $S_\lambda$, such that $S_\lambda \sim tH|_X$ since

$$S_\lambda: (x_0^{l/a_0} - \lambda x_1^{l/a_1} = 0, \lambda \in \mathbb{C}^*) \subset X.$$ 

We have that $K_X + S_\lambda$ is $\mathbb{Q}$-Cartier and since $X$ is terminal codim $\text{Sing}(X) \geq 3$. Therefore,

$$-K_{S_\lambda} = (-K_X - S_\lambda)|_{S_\lambda} + \text{Diff}_{S_\lambda}(0) = (\iota_X H|_X - lH|_X)|_{S_\lambda} + \text{Diff}_{S_\lambda}(0) = (\iota_X - l)H|_{S_\lambda} + \text{Diff}_{S_\lambda}(0)$$

and $-K_{S_\lambda}$ is big because $\iota_X - l > 0$ and $\text{Diff}_{S_\lambda}(0)$ is effective. Let $\psi: S' \rightarrow S_\lambda$ be the minimal resolution of $S_\lambda$. In particular, $-K_S'$ is big by [Laz04, Corollary 2.2.7]. Hence, $K_{S'}$ is not pseudoeffective and it follows by [BCHM10, Corollary 1.3.3] that $S_\lambda$ and $S'$ are uniruled. Resolving the indeterminacy of $\pi$, we get a commutative diagram,

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\varphi} & \mathbb{P}^1 \\
\psi \downarrow & & \downarrow \\
X & \longrightarrow & \mathbb{P}^1
\end{array}$$

where $\tilde{X}$ is a smooth threefold and $\varphi$ is onto. Let $\tilde{S}$ be the proper transform of $S_\lambda$ on $\tilde{X}$. Then, $\tilde{S}$ is uniruled and we can run a $K_{\tilde{X}}$-MMP on $\tilde{X}$ over $\mathbb{P}^1$. This is a (finite) sequence of divisorial contractions and small modifications $\chi$, fitting in one of the commutative diagrams,

$$\begin{array}{c}
\tilde{X} \\
\varphi \downarrow \\
\mathbb{P}^1
\end{array} \xrightarrow{\chi} \begin{array}{c}
\tilde{X} \rightleftharpoons \mathbb{P}^1 \\
\varphi' \downarrow \\
\mathbb{P}^1
\end{array} \xrightarrow{\chi} \begin{array}{c}
\tilde{X} \\
\varphi' \downarrow \\
\mathbb{P}^1
\end{array}$$

On the left hand side, $\varphi': Y \rightarrow \mathbb{P}^1$ is a del Pezzo fibration, while on the right hand side, the map $\varphi': Y \rightarrow B$ is a conic bundle with base a normal surface $B$, and $v$ is a surjective morphism with connected fibres.

In fact, specifying Theorem 1.1 to the index 2 case, we have the following corollary.

**Corollary 2.1.** Let $X$ be a $\mathbb{Q}$-Fano 3-fold with $\rho_X = 1$ and Fano index $\iota_X = 2$ embedded in a weighted projective space $X \subset \mathbb{P}(a_0, a_1, a_2, \ldots, a_N)$ such that $a_i \leq a_{i+1}$. Suppose that $a_0 = a_1 = 1$. Then, $X$ is birational to a Mori fibre space $Y \rightarrow S$ where $\dim S > 0$.

Corollary 2.1 does not give an explicit description of the birational map between $X$ and the strict Mori fibre space $Y \rightarrow S$. We recover it using the Sarkisov Program in the case of codimension 4 Fano 3-folds having index $\iota_X = 2$. As an immediate consequence of Corollary 2.1, we have the following corollary.

**Corollary 2.2.** Let $X$ be a family in Table 1. Then $X$ is not solid.

| ID     | $\dim |A|$ |
|--------|--------|
| #40360 | 2      |
| #40370 | 2      |
| #40397 | 2      |
| #40399 | 2      |
| #40400 | 2      |
| #40407 | 2      |
| #40633 | 3      |
| #40671 | 3      |

Table 1: Families for which $\dim |A| \geq 2$. 

3
3 Case Study: Fano 3-folds in Codimension 4 and index 2

3.1 Construction

We work over the field of complex numbers. Let $\bar{X} \subset \mathbb{P}^7$ be a quasi-smooth codimension 4 Q-Fano 3-fold and $\bar{Z} \subset \mathbb{P}^6$ a codimension 3 Q-Fano 3-fold, both of index $\iota = 1$, and suppose that $\bar{X}$ is obtained as Type I unprojection of $Z$ at a divisor $D \cong \mathbb{P}^2$ embedded as a complete intersection inside $Z$ (for a detailed study of Type I unprojections, see [KM83, PR04, Pap04, BKR12]). For $W$ a set of seven positive non-zero integers, call $x_0, x_1, x_2, y_1, y_2, y_3, y_4, s$ the coordinates of $\mathbb{P}^7 = \mathbb{P}^7(1, W)$, and consider $\gamma$ the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{P}^7$ that changes sign to the coordinate of $x_0$ of weight 1, that is,

$$
\gamma: (x_0, x_1, x_2, y_1, y_2, y_3, y_4, s) \mapsto (-x_0, x_1, x_2, y_1, y_2, y_3, y_4, s).
$$

(2)

Here, the divisor $D$ is defined by the ideal $I_D := \langle y_1, \ldots, y_4 \rangle$. Provided that the equations of $\bar{X}$ are invariant under $\gamma$, it is possible to perform the quotient of $\bar{X}$ by $\gamma$. The 3-fold $X := \bar{X}/\gamma$ obtained as such is Fano, has terminal singularities, is quasi-smooth, its ambient space is $\mathbb{P}^7(2, W)$, and has index $\iota = 2$ (cf [Cam21] Lemmas 3.1, 3.2, 3.3, 3.4).

This construction can be summarised by the following diagram.

$$
\begin{array}{ccc}
\iota & = & 1 \\
\bar{X} & \overset{\text{unprojection}}{\leftarrow} & - - - - - - \bar{Z} \\
\downarrow_{\mathbb{Z}/2\mathbb{Z}} & & \downarrow_{\gamma} \\
\iota & = & 2 \\
\bar{X} & \overset{\text{unprojection}}{\leftarrow} & - - - - - - \bar{Z} \\
\end{array}
$$

(3)

The key point is to find an appropriate index 1 double cover $\bar{X}$ for $X$ of index 2. The double cover is ramified on the half elephant $-\frac{1}{2}K_X$, and the equations of $X$ are inherited from the ones of $\bar{X}$ (see [Cam21] Theorem 1.1]). The Fano 3-folds $X$ of index 2 built in such a way are quasi-smooth and have terminal singularities.

Recall from [BKR12] that there are between two and four different deformation families for $X$ of index 1 sharing the same Hilbert series. They are derived from just as many so-called formats for the $5 \times 5$ antisymmetric graded matrix $M$ whose five maximal Pfaffians determine the equations of $Z$. These formats, defined by specific constraints on the polynomial entries of the matrix $M$ (cf [BKR12] Definition 2.2), are called Tom and Jerry formats. Accordingly, $X$ is said to be either of Tom type or of Jerry type (cf [Cam20] Definition 2.2]). Not all the formats are compatible with the double-cover construction (cf [Cam21] Theorems 4.3, 5.1]), but exactly one Tom format always is.

When a Type I unprojection is employed, we only focus on $\bar{X}$ of Tom type, which represent 32 families out of the 34 we study in this paper. In this context, $\bar{X}$ is a general member in its Tom family, provided the $\gamma$-invariance. Thus, $X$ is general under the above constraints.

A close analysis of the unprojection equations of $\bar{X}$, and therefore of $X$ gives crucial insights on the behaviour of the birational links from $X$. The unprojection equations of $\bar{X}$ are of the form $s y_i = g_i(x_0, x_1, x_2, y_1, \ldots, y_4)$ for $1 \leq i \leq 4$ (cf [Pap04] Theorem 4.3)). Consequently, four of the nine equations defining $X$ are of the form $s y_i = g_i(\xi, x_1, x_2, y_1, \ldots, y_4)$ for $1 \leq i \leq 4$; with a little abuse of notation, we call them unprojection equations as well. The point $p_s \in X$ is a Type I centre ([BKR12] Theorem 3.2] and [Cam21] Lemma 3.4]). Note that the unprojection variable $s$ appears linearly in the unprojection equations of $\bar{X}$ and $X$, hence the point $p_s \in X$ is called a linear cyclic quotient singularity in [DG21] Definition 2.6.1]; we will occasionally use this nomenclature in the following. From [Cam20] Lemma 3.5] we have that each unprojection equation of the index 1 double cover $\bar{X}$ contains at least one monomial only in the orbirates $x_0, x_1, x_2$. The following lemma is a consequence of the constraint of $\bar{X}$ invariant under the action $\gamma$.

Lemma 3.1. Consider the four unprojection equations $s y_i = g_i$ of $X$ for $1 \leq i \leq 4$, and suppose that $\text{wt}(x_1)$ is even. Then, there are at least three $g_i$ that are of the form $g_i = f_i(\xi, x_1) + h_i(\xi, x_1, x_2, y_1, \ldots, y_4)$.
Proof. We do not recall the construction and notation of unprojection equations necessary to this proof, but we refer to [Pap04, Section 5.3] and [Cam20, Appendix].

Since $\xi X = 2$, for terminality reasons the basket of singularities of $X$ consists only of cyclic quotient singularities with odd order (cf [Suz04, Lemma 1.2 (3)]). Consequently, the weight of the unprojection variable $s$ is always odd. By hypotheses, the weights of $\xi$ and $x_1$ are even, so the orbinate $x_2$ has odd weight. By direct observation, at least three of the coordinates $y_1, \ldots, y_4$ have odd weight. Note that if $s y_i$ for some $1 \leq i \leq 4$ has odd weight, the corresponding $g_i$ does not contain any pure monomial in $\xi, x_1$; that is, $w y_i$ must be odd for it to happen.

Suppose that the matrix $M$ is in $\text{Tom}_1$ format; the proof for the other $\text{Tom}$ formats is analogous. Such matrix is of the form

\[
M = \begin{pmatrix}
p_1 & p_2 & p_3 & p_4 \\
q_1 & q_2 & q_3 \\
q_4 & q_5 & q_6
\end{pmatrix}
\]

for $p_i \notin I_D$ and $q_i \in I_D$ homogeneous polynomials in the given degrees of $M$. Without loss of generality, we can fill the entries of $M$ with linear monomials when the $\text{Tom}$ constraints and the degree prescription on $M$ allow us to do so (for details see [BKR12, Section 6.2]). In this context, at least three of the $q_i$ can be filled with one of the $y_1, \ldots, y_4$. By homogeneity of the Pfaffians, at least two of the $p_i$ have even degree; thus, $\xi$ and $x_1$ can occupy those entries (not necessarily linearly).

Referring to the notation in [Pap04, Section 5.3], [Cam20, Appendix] such entries are multiplied by 1 at least once in the Pfaffians of the matrices $N_j$. Thus, at least two entries of each row of the $4 \times 4$ matrix $Q$ contain a monomial purely in $\xi, x_1$. Such entries get multiplied with one another when taking the determinant of the four $3 \times 3$ submatrices of $Q$. This is up to a $p_i$ factor, which disappears in the definition of $g_i$ thanks to [Pap04, Lemma 5.3].

At worst, the two entries of $Q$ containing pure monomials in $\xi, x_1$ are all concentrated in a $2 \times 3$ block (or two $1 \times 3$ blocks). This implies that only three of the $g_i$ have the desired monomials $f_i(\xi, x_1)$. Otherwise, all $g_i$ contain $f_i(\xi, x_1)$.

Also the Pfaffians equations of $\tilde{X}$ and $X$ play a significant role in how the birational links unfold. The following statements regard the presence of certain monomials in such equations.

To fix ideas, suppose that $M$ is in $\text{Tom}_1$ format. Recall from [Cam20, Section 4, (a), (b)] the following configurations of weights of $M$. If the entries $a_{24}, a_{25}, a_{34}, a_{35}$ in $I_D$ have the same weight $\pi$, we say that $M$ is in configuration "A". On the other hand, if the entries $a_{25}, a_{34}$ in $I_D$ have the same weight $\pi$, we say that $M$ is in configuration "B". For instance, in $\text{Tom}_1$ format, the weights of $M$ of the index 1 double cover #1405 of #39678 are in configuration A (matrix on the left), and the weights of $M$ of the index 1 double cover #569 of #39576 are in configuration B (matrix on the right).

\[
\text{wt} (M_{#1405}) = \begin{pmatrix}
3 & 4 & 4 & 4 \\
4 & 5 & 5 \\
5 & 5 \\
6
\end{pmatrix} \quad \text{wt} (M_{#569}) = \begin{pmatrix}
3 & 4 & 5 & 6 \\
5 & 6 & 7 \\
7 & 8 \\
9
\end{pmatrix}.
\]

The configurations when $M$ is in another $\text{Tom}$ format are analogous. We therefore have the following lemma.

Lemma 3.2. Suppose that the weights of $M$ are in configuration B.

(a) If there is only one coordinate $y \in I_D$ to fill the entries of weight $\pi$, then the pure power $y^2$ appears in one of the Pfaffian equations ($\text{Pf}_1$ for $M$ in $\text{Tom}_1$ format) of $X$.

(b) If there are two coordinates $y_1, y_2 \in I_D$ to fill the entries of weight $\pi$, then the monomial $y_1 y_2$ appears in one of the Pfaffian equations of $X$. 

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Suppose that the weights of $M$ are in configuration $A$.

(i) If there are two coordinates $y_1, y_2 \in I_D$ to fill the entries of weight $\pi$, then there is a non-degenerate quadratic form in $y_1, y_2$ defined on $\mathbb{P}^7(2, W)$.

(ii) If there are three coordinates $y_1, y_2, y_3 \in I_D$ to fill the entries of weight $\pi$, then the pure power $y_3^2$ appears in one of the Pfaffian equations of $X$, and there is a non-degenerate quadratic form in $y_1, y_2$ defined on $\mathbb{P}^7(2, W)$.

Proof. For case (a) and without loss of generality, the entries $a_{25}, a_{34}$ of $M$ can be occupied by the generator $y$ of the ideal $I_D$. The first Pfaffian of $M$ contains $y^2$.

Analogously for case (b) we can fill the entry $a_{25}$ with $y_1$ and $a_{34}$ with $y_2$. Therefore, $\text{Pf}_1$ contains the monomial $y_1 y_2$.

In case (i), the entries $a_{24}, a_{25}, a_{34}, a_{35}$ of $M$ are filled with (combinations of) the coordinates $y_1, y_2$. The restriction of $\text{Pf}_1(M)$ to $\mathbb{P}_1^{y_1, y_2}$ is a rank 2 quadratic form in $y_1, y_2$. Up to a change of basis, it is of the form $y_1^2 - y_1 y_2 + y_2^2$. Here $a_{24} = y_1$, $a_{35} = y_2$, $a_{25} = y_1 + y_2$, and $a_{34} = y_1 - y_2$. Case (ii) is analogous to (i) here the coordinate $y_3$ also appears, say, in the entries $a_{25}, a_{34}$. Therefore the only occurrence of $y_3$ in $\text{Pf}_1(M)$ is the pure power $y_3^2$.  

Note that for reasons of complexes’ resolutions (cf. [BK12]), it never occurs to have only one coordinate of weight $\text{wt} y = \pi$ in configuration $A$.

The purpose of Lemma 3.2 concerns the study of the flips in the birational links for $X$. We refer to the notation introduced in Lemma 3.1 and in Equation (4). Consider the Poisson structure in $\mathbb{C}$ and Equation (4). Consider the equations of $X$ in Equation (4) and evaluate them at $(y_i = x_2 = 0)$ for $1 \leq i \leq 4$. Hence we get the system of polynomial equations $f_i(\xi, x_1) = 0$ for $1 \leq i \leq 3$. Now suppose that the $f_i(\xi, x_1)$ are not algebraically independent: they therefore have a common solution, that is, a finite set of points in $X$. Such points are quotient singularities with even order. This is not possible as $X$ cannot have singularities with even order (cf. [Suz14] Lemma 1.2 (3)), as it would not be quasi-smooth. Thus, $f_i(\xi, x_1)$ for $1 \leq i \leq 3$ are algebraically independent. 

Remark 3.3. Not all matrices $M$ considered here fall into either configuration A or B. In fact, when $M$ is in neither one of these configurations we can conclude that none of the coordinates $y_i$ appear as a pure power in $\text{Pf}_1(M)$.

Remark 3.4. For the family with GRDB ID #40933, we have that two of the weight 1 coordinates in $I_D$ appear as pure powers in $\text{Pf}_3(M)$.

Remark 3.5. The orbinate $x_3$ does not appear as a pure power in the Pfaffian equations. This is a feature of Tom formats: all the entries $p_i \notin I_D$ of $M$ are multiplied by entries $q_i \in I_D$ in the Pfaffians.

Lemma 3.6. Suppose that $\text{wt}(y_1)$ is even. Then, the polynomials $f_i(\xi, x_1)$ for $1 \leq i \leq 3$ are algebraically independent.

Proof. We refer to the notation introduced in Lemma 3.1 and in Equation (4). Consider the equations of $X$ in Equation (4) and evaluate them at $(y_i = x_2 = 0)$ for $1 \leq i \leq 4$. Hence we get the system of polynomial equations $f_i(\xi, x_1) = 0$ for $1 \leq i \leq 3$. Now suppose that the $f_i(\xi, x_1)$ are not algebraically independent: they therefore have a common solution, that is, a finite set of points in $X$. Such points are quotient singularities with even order. This is not possible as $X$ cannot have singularities with even order (cf. [Suz14] Lemma 1.2 (3)), as it would not be quasi-smooth. Thus, $f_i(\xi, x_1)$ for $1 \leq i \leq 3$ are algebraically independent. 

The remaining two families (GRDB ID #39569 and #39607) that we investigate here are constructed in a similar fashion as in diagram (3): in these cases, their respective double covers $X$ only have Type $I_2$ centres, or worse. Hence, the Tom and Jerry formats are not applicable as $X$ is obtained via Type $I_2$ unprojections from a Fano hypersurface: see [Cam21], Section 7 for the construction of these double covers.

Among the Hilbert series listed in the GRDB there is also #41028 corresponding to the smooth Fano 3-fold of Iskovskih in [Isk77, Isk78 Case 13, Table (6.5)]. There are two distinguished deformation families associated to this Hilbert series: these are the classical examples of a divisor of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ (cf. [BK12] Section 2). These are rational, and therefore non-solid. This confirms the statement of Theorem 1.1 also in the case of #41028.
3.2 Lift under the Kawamata blowup

We want to initiate a birational link by blowing up a cyclic quotient singularity on $X$. In order to understand what the equations of the blowup $Y$ are we first explain how the sections $H^0(X, mA)$ lift to $Y$ under a Kawamata blowup of $p_s \in X$, for $m \geq 1$.

Recall that by [Pap04, Theorem 4.3], [BKR12, Theorem 3.2], and [Cam21, Lemma 3.4], $p_s$ is a linear cyclic quotient singularity of $X$ as in [DG21, Definition 2.6.1]: locally, $p_s \sim \frac{1}{a_s}(a_0, a_1, a_2)$. In the notation introduced in Subsection 3.1, $a_0, a_1, a_2$ are the weights of the orbinates $\xi, x_1, x_2$.

We can assume (cf [Suz04, Lemma 1.2 (3)]) that $a_0 = 2$ is equal to the Fano index of $X$. Moreover, since $p_s$ is terminal, $\gcd(a_s, a_0a_1a_2) = 1$ and, in particular, there is $k \in \mathbb{Z}$ such that $ka_0 \equiv 1 \pmod{a_s}$. Denote by $\bar{s} < a_s$ the unique remainder of $a$ mod $a_s$. Then,

$$p_s \sim \frac{1}{a_s}(1, ka_1, ka_2) \sim \frac{1}{a_s}(1, ka_1, a_s - ka_1) = \frac{1}{a_s}(1, \frac{a_1}{2}, a_s - \frac{a_1}{2}).$$

**Lemma 3.7.** Let $\varphi: Y \to X$ be the Kawamata blowup centred at $p_s$. Then

$s \in H^0(Y, -m_1K_Y + m_2E)$ for some $m_1, m_2 > 0$;

$\xi, x_1 \in H^0(Y, -mK_Y)$ for some $m > 0$ depending on the chosen variable;

$x_\mu \in H^0(Y, -m_1(\mu)K_Y - m_2(\mu)E)$ for some $m_1, m_2 > 0$, depending on the chosen variable.

**Proof.** Recall that for the Kawamata blowup $\varphi: Y \to X$ we have $K_Y = \varphi^*(K_X) + \frac{1}{r}E$, where $\frac{1}{r}$ is its discrepancy and $r$ is the index of the cyclic quotient singularity.

Let $\nu_i$ be the vanishing order of $x_i$ at $E$. Then,

$$x_i \in H^0 \left( Y, \frac{a_i}{2}\varphi^*(-K_X) - \nu_i E \right)$$

$$= H^0 \left( Y, \frac{a_i}{2}(-K_Y + \frac{1}{r}E) - \nu_i E \right)$$

$$= H^0 \left( Y, -\frac{a_i}{2}K_Y + \frac{a_i - 2a_s\nu_i}{2a_s}E \right).$$

By the description of $p_s$ it follows that $\xi$ vanishes at $E$ with order $\nu_0 = \frac{1}{a_s}$. Similarly, the vanishing order of $x_1$ and $x_2$ at $E$ is $\nu_1 = \frac{a_1}{2a_s}$ and $\nu_2 = \frac{2a_s - a_1}{2a_s}$. Hence,

$$\xi \in H^0(Y, -K_Y), \quad x_1 \in H^0 \left( Y, -\frac{a_1}{2}K_Y \right), \quad x_2 \in H^0 \left( Y, -\frac{a_2}{2}K_Y - \frac{1}{2}E \right).$$

On the other hand, $s \in H^0(Y, \frac{a_0}{2}\varphi^*(-K_X)) = H^0(Y, -\frac{a_0}{2}K_Y + \frac{1}{2}E)$. By lemma 3.1 the four unprojection equations of $X$ are of the form

$$\begin{cases} sy_i = f_i(\xi, x_1) + h_i(\xi, x_1, x_2, y_1, \ldots, y_4), & 1 \leq i \leq 3 \\ sy_4 = h_4(\xi, x_1, x_2, y_1, \ldots, y_4) \end{cases} \tag{4}$$

where $f_i$ is not identically zero by quasi-smoothness of $X$ and $\text{wt}(y_4)$ is even.

We compute the vanishing order of $y_i$, $1 \leq i \leq 3$ at $E$. The monomials of $f_i(\xi, x_1)$ are of the form $\xi^\alpha x_1^\beta$ and if $d_i$ is homogeneous degree of the equation $sy_i - g_i = 0$, we have $\alpha a_0 + \beta a_1 = d_i$.

On the other hand such monomials vanish at $E$ with order

$$\alpha \frac{1}{a_s} + \beta \frac{a_1}{2a_s} = \frac{d_i}{2a_s} < 1.$$

Hence, $\xi^\alpha x_1^\beta$ is pulled back by $\varphi$ to $\xi^\alpha x_1^\beta u^{d_i/2a_s}$ where $E := (u = 0)$ is the exceptional divisor. Since $\frac{d_i}{2a_s} < 1$, when saturating the ideal of $Y$ with respect to $u$, we find that $\xi^\alpha x_1^\beta u^{d_i/2a_s}$ becomes $\xi^\alpha x_1^\beta$.

Hence $sy_i - g_i = 0$ is a divisor in $|-\frac{d_i}{2}K_Y|$ and we conclude that $y_i \in H^0 \left( Y, -\frac{d_i}{2}K_Y - \frac{1}{2}E \right).$
For \( y_4 \), since \( sy_4 - h_4 = 0 \) contains no pure monomials in \((\xi, x_1)\), we can only say that \( y_4 \) vanishes at \( E \) with order at least \( \frac{\nu_4}{2a_s} \). That is, \( \nu_4 \geq \frac{\nu_4}{2a_s} \) and therefore \( \frac{\nu_4}{2a_s} \leq -\frac{1}{2} \) with equality when the vanishing order is exactly \( \frac{\nu_4}{2a_s} \). In other words,

\[
y_4 \in H^0 \left( Y, -\frac{\nu_4}{2}K_Y - m_4E \right), \quad m_4 \geq \frac{1}{2} > 0.
\]

The Lemma below follows from Lemma 3.7 and Lemma 3.1.

**Lemma 3.8.** Let \( X \subset \mathbb{P}^{\nu_4} \) be defined by the equations \( \text{Pf}_i = sy_j - g_j = 0, \ 1 \leq i \leq 5, \ 1 \leq j \leq 4 \), where \( \text{Pf}_i, g_j \in \mathbb{C}[\xi, x_1, x_2, y_1, \ldots, y_4] \). Then, the Kawamata blow up \( Y \) of \( X \) at \( p_s \in X \) is defined by equations \( \text{Pf}_i = sy_j - g_j = 0, \ 1 \leq i \leq 5, \ 1 \leq j \leq 4 \), with \( \text{Pf}_i, g_j \in \mathbb{C}[t, \xi, x_1, x_2, y_1, \ldots, y_4] \), where \( sy_j - g_j \in |-m_jK_Y| \) for exactly three values of \( j \), and \( \text{Pf}_i \in |-m_iK_Y - n_iE|, \ with \ n_i > 0 \).

## 4 Birational Links and Mori Dream Spaces

We make use of the technology of Mori Dream Spaces to study the birational links discussed later in this Section. We recall the definition of Mori Dream Space as in \[HK00\] and some of their properties within the context of the Sarkisov Program, as explained in the works \[AK16\] and \[BZ10\]. This section is devoted to give a theoretical background to show that we can perform the 2-ray game for codimension 4 index 2 Fano 3-folds.

**Definition 4.1 (\[HK00\] Definition 1.10).** A normal projective variety \( Z \) is a Mori Dream Space if the following hold

- \( Z \) is \( \mathbb{Q} \)-factorial and \( \text{Pic}(Z) \) is finitely generated;
- \( \text{Nef}(Z) \) is the affine hull of finitely many semi-ample line bundles;
- There exists a finite collection of small \( \mathbb{Q} \)-factorial modifications \( f_i: Z \dashrightarrow Z_i \) such that each \( Z_i \) satisfies the previous point and \( \text{Mov}(Z) = \bigcup f_i^*(\text{Nef}(Z_i)) \).

**Remark 4.2.** As pointed out in \[Oka16\] Remark 2.4, if we work over a field which is not the algebraic closure of a finite field, then the condition that \( \text{Pic}(Z) \) is finitely generated is equivalent to \( \text{Pic}(Z)_{\mathbb{R}} \simeq N^1(Z)_{\mathbb{R}} \).

In characteristic zero, it is known that any klt pair \((Z, \Delta)\) where \( Z \) is \( \mathbb{Q} \)-factorial and \(-K_Z + \Delta\) is ample is a Mori Dream Space (see \[BCHM10\] Corollary 1.3.2]). Examples include weak Fano varieties. We have the following lemma.

**Lemma 4.3 (\[HK00\] Proposition 1.11 (2))**. Let \( Z \) be a Mori Dream Space. Then, there are finitely many birational contractions \( g_i: Z \dashrightarrow Z_i \) where for each \( i, Z_i \) is a Mori Dream Space and

\[
\text{Eff}(Z) = \bigcup_i C_i, \quad C_i = g_i^*(\text{Nef}(Z_i)) + \mathbb{R}_+[E_1] + \cdots + \mathbb{R}_+[E_k],
\]

where \( E_1, \ldots, E_k \) are the prime divisors contracted by \( g_i \). If \( Z_i \) and \( Z_j \) are in adjacent chambers, then they are related by a small \( \mathbb{Q} \)-factorial modification.

We have the following result:

**Lemma 4.4 (\[AK16\] Lemma 2.9).** Let \( X \) be a \( \mathbb{Q} \)-Fano 3-fold and \( \varphi: Y \to X \) be a divisorial extraction. Then \( \varphi \) initiates a Sarkisov link if and only if the following hold:

1. \( Y \) is a Mori Dream Space;
2. If \( \tau: Y \to Y' \) is a small birational map and \( Y' \) is \( \mathbb{Q} \)-factorial, then \( Y' \) is terminal;
3. \([-K_Y] \in \text{Int}(\overline{\text{Mov}(Y)}).

It is not true that the blowup of a Mori Dream Space is a Mori Dream Space (see \cite{Cas18} for many examples). However, the Kawamata blowup \(\varphi: Y \to X\) of a \(\mathbb{Q}\)-Fano 3-fold centred at a linear cyclic quotient singularity is at worst a weak Fano 3-fold with \(\mathbb{Q}\)-factorial terminal singularities and \([-K_Y] \in \text{Int}(\overline{\text{Mov}(Y)}).

More than that, we prove that this corresponds to the first variation of GIT quotient.

In this case, the only small \(\mathbb{Q}\)-factorial modifications which are not isomorphisms are flips. This allows us to always remain in the Mori category since the discrepancies increase (see \cite[Lemma 3.38]{KM98}).

One can always play a 2-ray game on a Mori Dream Space \(Y\) with \(\rho(Y) = 2\) and this is uniquely defined by the first move.

**Birational links and Toric Varieties.** Let \(T\) be a rank 2 toric variety (up to isomorphism) for which the toric blowup \(\Phi: T \to \mathbb{P}\) restricts to the unique Kawamata blowup \(Y \to X\) centred at \(p_s \in X\). Then, \(\text{Cl}(T) = \mathbb{Z}[\Phi^*H] + \mathbb{Z}[E]\), where \(H\) is the generator of the Class Group of \(\mathbb{P}\) and \(E = \Phi^{-1}(p_s)\) is the exceptional divisor. Notice that \(p_s\) is not in the support of \(H\). Then, the Cox ring of \(T\) is

\[
\text{Cox}(T) = \bigoplus_{(m_1, m_2) \in \mathbb{Z}^2} H^0(T, m_1 \Phi^*H + m_2 E)
\]

Since \(T\) is toric, the Cox ring of \(T\) is isomorphic to a polynomial ring, in this case,

\[
\text{Cox}(T) \simeq \mathbb{C}[t, \xi, x_1, x_2, y_1, \ldots, y_4, s].
\]

Over \(N^1(T)_\mathbb{R} \simeq \mathbb{R}^2\), we can depict the rays generated by the sections of Lemma 3.7 as in Figure 1. The movable and effective cone of \(T\) in \(N^1(T)_\mathbb{R}\) are

\[
\text{Mov}(T) = \mathbb{R}_+[M_1] + \mathbb{R}_+[M_2] \subset \text{Eff}(T) = \mathbb{R}_+[E] + \mathbb{R}_+[E']
\]

According to Lemma 3.7, notice that the rays \(E, M_1\) and \(-K_Y\) cannot coincide. On the other hand, it can happen that some of the other rays do coincide.

![Figure 1: A representation of the Mori chamber decomposition of \(T\). The outermost rays generate the cone of pseudo-effective divisors of \(T\) and in red it is represented the subcone of movable divisors of \(T\).](image)

We run several 2-ray games on \(T\). We divide the construction of the elementary Sarkisov links in two main cases, depending roughly on the Mori chamber decomposition of \(T\) in Figure 1, namely the behaviour of its movable cone of divisors near the boundary of the effective cone of divisors of \(T\). We consider three cases:

1. **Fibration:** the class of \(M_2\) is linearly equivalent to a rational multiple of \(E'\).

2. **Divisorial Contraction:** the class of \(M_2\) is not linearly equivalent to a rational multiple of \(E'\). Moreover, we contract the divisor \(E'\):
(a) **to a point**: no generator class of the Cox ring of $T$ is linearly equivalent to a rational multiple of $M_2$, or

(b) **to a rational curve**: there is one (and only one) generator class of the Cox ring of $T$ which is linearly equivalent to a rational multiple of $M_2$.

The diagram below sets the notation used in the rest of the paper. Note that we only consider Fano 3-folds not in Table [1]. In each case we have a birational link at the level of toric varieties

$$
\begin{array}{cccccccc}
\Phi & T & \tau_0 & T_1 & \cdots & \tau_n & T' \\
\alpha_0 & \beta_0 & \alpha_1 & \cdots & \beta_{n-1} & \beta_0' \\
\mathbb{P} & F_0 & \cdots & F_n & \cdots & F' \\
\end{array}
$$

and dim $\mathcal{F}' \leq \dim T'$, with equality if and only if we are in the second case. We restrict the diagram above to the Kawamata blowup $\varphi: Y \subset T \to X \subset \mathbb{P}$ in order to obtain a birational link between 3-folds.

It follows from [HK00, Proposition 2.11], that the birational contractions of a Mori Dream Space are induced from Toric Geometry. By [HK00, Proposition 1.11], one can always carry out a classical Mori Program for any divisor on a Mori Dream space $Y$. In particular, when $\rho(Y) = 2$ it is called a 2-ray game. We refer the reader to [Cor00] for the precise definition of 2-ray game.

The next three lemmas describe the nature of the restriction of the maps $\tau, \tau_i$ to the birational link relative to $X \subset \mathbb{P}$.

**Lemma 4.5.** The map $\tau_0$ restricts to an isomorphism on $Y$.

**Proof.** In this proof, call $z_i$ all variables that are not $t, s, \xi, x_1$. The small modification $\tau_0: T \to T_1$ can be decomposed as

$$
\begin{array}{cccccccc}
T & \tau_0 & T_1 \\
\alpha_0 & \beta_0 & \\
F_0 & \cdots & F_n & \cdots & F' \\
\end{array}
$$

where $F_0 := \text{Proj} \bigoplus_{m \geq 1} H^0(T, mO_T(\frac{1}{2}))$, in coordinates the map $\alpha_0$ is,

$$
\alpha_0: T \to F_0 \\
(t, s, \xi, x_1, \ldots, z_i, \ldots) \mapsto (\xi, x_1, \ldots, u_j, \ldots)
$$

and $u_j$ is some monomial which is a multiple of $z_i$ and of either $t$ or $s$.

Notice that the irrelevant ideal of $T$ is $(t, s) \cap (\xi, x_1, \ldots, z_i, \ldots)$. Hence, $\alpha_0$ contracts the locus $(z_i = 0) \subset T$. This is indeed a small contraction since the ray $\mathbb{R}_+ \left( \frac{1}{2} \right) \subset \text{Mov}(T)$. Now we restrict this small contraction to $Y$. By Lemma 3.7 we know that $z_i$ are the sections in $H^0 \left( Y, -\frac{n_i}{2} K_Y - n_i E \right)$ with $n_i > 0$. On the other hand, by Lemma 3.8 the Pfaffian equations $\text{Pf}_j$ must vanish identically and the remaining equations are $f_j(\xi, x_1) = 0$ for $1 \leq j \leq 3$. However, this is empty by Lemma 3.6.

**Remark 4.6.** It is interesting to observe that the behaviour of the restriction of $\tau_0$ as in Lemma 4.5 is not a feature of the codimension of $X$ but rather of its Fano index. When the index is 1, the map $\tau$ restricts to a number of simultaneous Atiyah flops (see [Cam20, Theorem 4.1] and [Oka14]). On the other hand, for higher indices it is an isomorphism (cf [DG21, Theorem 2.5.6]).

**Lemma 4.7.** Suppose that the map $\tau_1: T_i \to T_{i+1}$ restricts to a small $\mathbb{Q}$-factorial modification over a point which is not an isomorphism. Then, it restricts to a flip.
Proof. By assumption there are curves $C_i \subset Y_i$ and $C_{i+1} \subset Y_{i+1}$ such that the diagram

$$
C_i \subset Y_i \dashrightarrow \tau_i \dashrightarrow Y_{i+1} \supset C_{i+1}
$$

is a small $\mathbb{Q}$-factorial modification. Clearly, we have $K_{Y_i} \cdot C_i < 0$. Indeed, by Lemma 3.8, $C_i$ intersects $-K_{Y_i}$ transversely and the claim follows. On the other hand, there exists a divisor

$$
L \sim -m_1 K_Y - m_2 E \in \mathbb{R}_+[A_i] + \mathbb{R}_+[E']
$$

(implying in particular $m_i > 0$) for which $L \cdot C_{i+1} > 0$. On the other hand,

$$
A_i \sim -n_1 K_Y - n_2 E, \ n_i > 0
$$

and $A_i \cdot C_{i+1} = 0$. Then, $m_2 A_i - n_2 L = (m_2 n_1 - m_1 n_2)(-K_{Y_{i+1}})$. Indeed, $m_2 n_1 - m_1 n_2 = 0$ if and only if $A_i \sim L$ which is not possible since they have different intersections with $C_{i+1}$. In fact, $m_2 n_1 - m_1 n_2 > 0$ since $L$ is in the cone $\mathbb{R}_+[A_i] + \mathbb{R}_+[E']$ but is not linearly equivalent to any non-zero rational multiple of $A_i$. In particular,

$$
-K_{Y_{i+1}} \cdot C_{i+1} = \frac{1}{m_2 n_1 - m_1 n_2}(m_2 A_i \cdot C_{i+1} - n_2 L \cdot C_{i+1}) < 0
$$

since $A_i$ and $L$ were chosen so that $A_i \cdot C_{i+1} = 0$ and $L \cdot C_{i+1} > 0$. \hfill \Box

Lemma 4.8 (DG[21] Lemma 2.5.7). Let $\sigma: X \dashrightarrow X'$ be an elementary birational link between $\mathbb{Q}$-Fano 3-folds initiated by a divisorial extraction $\varphi: E \subset Y \to X$. Then, there is a birational map $\Psi: Y \dashrightarrow X'$ which is the composition of small $\mathbb{Q}$-factorial modifications followed by a divisorial contraction $\varphi': E' \subset Y' \to X'$ with discrepancy

$$
a = \frac{m_2}{n_2 m_1 - n_1 m_2}
$$

where $m_i$ and $n_i$ are positive rational numbers such that $\varphi'^*(K_{X'}) = -m_1 K_Y - m_2 E$ and $E' \sim -n_1 K_Y - n_2 E$.

In practice, to successfully run this game we need to guarantee that each step $\tau_i: T_i \dashrightarrow T_{i+1}$ of the birational link contracts finitely many curves, with the exception of $\tau_i$ an isomorphism on $Y_i \subset T_i$. However, in each case it is possible to retrieve explicitly the loci contracted and extracted by the maps $\tau_i$.

In the rest of this section we will present several Tables containing information about the links for each family examined. We specify the Type I centre whose blowup initiates the link in cases where there is more than one Type I centre; for instance, we write "#39993 1/5" instead of just "#39993" if the link starts with the Kawamata blowup of a 1/5 singularity.

4.1 Case I: Fibrations

In each case in Figure[2] whose notation we refer to, the movable cone of $T$ is not strictly contained in the effective cone of $T$. Indeed, the rays generated by $z_4$ and $z_5$ are both in $\partial \text{Mov}(T)$ and $\partial \text{Eff}(T)$. Hence we have a diagram of toric varieties

$$
\begin{array}{ccc}
T & \xrightarrow{\Phi} & T' \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{\psi} & \mathcal{F}'
\end{array}
$$
where $\Phi$ is a divisorial contraction, $\Phi'$ is a fibration into $\mathcal{F}' \simeq \text{Proj} \mathbb{C}[z_4, z_5] \simeq \mathbb{P}^1$. The map $\tau$ is a small $\mathbb{Q}$-factorial modification. We restrict $\Phi: T \rightarrow \mathbb{P}$ to be the unique Kawamata blowup $\varphi: E \subset Y \rightarrow p_8 \in X$. By Lemmas 4.5 and 4.7, the map $\tau|_Y: Y \rightarrow Y'$ is an isomorphism followed by a finite sequence of isomorphisms or flips. Referring to the notation in Figure 2, by assumption the rays $\mathbb{R}_+[z_4]$ and $\mathbb{R}_+[z_5]$ are given by linearly dependent vectors $v_4$ and $v_5$ in $\mathbb{Z}^2$. Let $B$ be the matrix whose columns are the vectors $v_4$ and $v_5$. Then, there is $A \in \text{GL}(2, \mathbb{Z})$ such that

$$A \cdot B = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$ 

Then, performing a row operation on the grading of $T'$ via the matrix $A$, $T'$ is isomorphic to a toric variety with $\mathbb{C}^* \times \mathbb{C}^*$-action given by

$$T': \begin{pmatrix} \kappa_0 & \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 & \kappa_6 & a & b \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & 0 & 0 \end{pmatrix}$$

where every $2 \times 2$ minor is non-positive. The irrelevant ideal of $T'$ is $(t, s, \xi, x_1, z_1, z_2, z_3) \cap (z_4, z_5)$. The map $\Phi'$ can be realised as

$$\Phi': T' \rightarrow \mathcal{F}'$$

$$(t, s, \xi, x_1, z_1, z_2, z_3, z_4, z_5) \mapsto (z_4, z_5)$$

and the fibre of $\Phi'$ over each point is isomorphic to $\mathbb{P}(\lambda_0, \ldots, \lambda_6)$. The following Table 2 shows what the restrictions of $\Phi'$ to $Y'$ are.

**Example 4.9.** Let $X \subset \mathbb{P} := \mathbb{P}(1, 2, 3, 4, 5, 7, 5)$ with $x_2, \xi, y_4, y_1, x_1, y_2, y_3, s$ its homogeneous variables.

We take the toric blowup $\Phi: T \rightarrow \mathbb{P}$ centred at $p_8 = (0 : \cdots : 0 : 1) \in \mathbb{P}$ and restrict it to the unique Kawamata blowup $\varphi: Y \rightarrow X$ centred at $p_8$. The point $p_8$ is a cyclic quotient singularity of type $\frac{1}{d}(1, 2, 3)$ and local analytical variables $\xi, u, x_1$ called the orbinates. By Lemma 3.7 in order to restrict $\Phi$ to the the Kawamata blowup of $X$ at $p_8$ we need for $T$ to have a certain bi-grading: the one relative to #39961 is listed in Table 2. In particular,

$$\text{Mov}(T) = \langle (\frac{1}{7}), (\frac{1}{3}) \rangle \subset \langle (\frac{0}{1}), (\frac{1}{0}) \rangle = \text{Eff}(T) \subset \mathbb{R}^2.$$

Now we run the 2-ray game on $T$ following the movable cone in Case I.b of Figure 2 and then we restrict it to $Y$. After saturation with respect to the new variable $t$, the equations defining
Table 2: Elementary birational links to a fibration (cases in Table 1 excluded). The family 
\( \#39607 \) is embedded in the weighted projective space \( \mathbb{P}^7(2, 3, 3, 4, 5, 6, 7) \) with coordinates 
\( \xi, u, z, v, s_0, s_1, s_2 \).

\[
Y := \Phi_+^{-1}X \subset T \text{ are the following}
\]

\[
\begin{align*}
&x_2\xi y_1 - y_1^3 - y_4 x_1 + x_2 y_2 = 0 \\
&-x_2^7 \xi^3 - x_2^3 \xi^2 t - x_2^2 \xi^3 + x_2^2 \xi y_1 t + x_2\xi x_1 - x_2^2 y_2 t - y_1 x_1 + y_4 s = 0 \\
&\xi^3 y_4 - y_1^3 t^3 + x_2 \xi y_2 - y_1 y_2 + x_2 y_3 = 0 \\
x_2^6 y_1 t^3 - x_2^2 y_4^2 t^4 + \xi y_1 - y_1^3 y_1 t^3 - x_2^2 y_4^2 x_1 t^2 + x_2^2 y_3 t + x_1 y_2 = 0 \\
x_2^6 y_4 y_1 t^3 - y_1^3 t^4 - y_1^2 x_1 t^2 + x_2 y_4 y_3 t + y_2^3 - y_1 y_3 = 0 \\
x_2^7 y_4 t^4 - x_2^2 e_1 x_1^3 + x_2^2 e_4 t^3 - x_2^2 e_5 = x_2^6 y_4^2 x_1 t^2 \\
-x_2^2 y_4^2 x_1 t^3 - \xi^3 x_1 + y_1^3 x_1 t^3 - x_2 y_4 y_1 x_1 t^2 - x_2 y_1^3 y_2 t^3 + x_2 y_3 t^2 + y_2 = 0 \\
x_2^6 y_4 x_1 t^3 - x_2^6 y_2 t^3 - x_2^6 y_4^2 t^4 + y_4 y_1 t^4 - x_2^6 y_4^2 x_1 t^2 \\
+ x_2^4 y_1 x_1 t^3 - \xi y_2 + y_1^3 y_1 t^3 + x_2^4 y_3 t - x_2 y_1 y_3 t - x_1 y_3 = 0 \\
-x_2^6 y_3 t^3 + x_2^6 y_4^3 t^6 - x_2^6 x_1 t^3 + x_2^6 \xi^2 y_1 t^4 + x_2^6 y_2 x_1 t^3 - \xi^6 + \xi^3 y_1 t^3 - x_2^6 y_4^3 y_1 t^4 \\
+ x_2^6 y_4 x_1 t^2 - y_4 x_1 t^4 - x_2^6 y_4 y_1 x_1 t^2 - x_2^6 \xi^2 y_1 t^3 + x_2^6 y_1^3 y_2 t^4 \\
- y_2^2 x_1 t^2 + y_2^2 y_1 y_2 t^3 + x_2 y_4 x_1 y_2^2 t + x_2 x_1 y_3 t - y_3 s = 0
\end{align*}
\]
These consist of the five maximal Pfaffians Pf_i = 0 together with the four unprojection equations sy_j - g_j = 0. Also, Y is inside the rank 2 toric variety T whose irrelevant ideal is \((t, s) \cap (\xi, x_1, y_3, y_2, y_1, x_2, y_4)\). By Lemma 4.3, the map \(\tau_0: T \rightarrow T_1\) restricts to an isomorphism on \(Y\) and therefore we can assume \(Y \subset T_1\) where \(T_1\) has irrelevant ideal \(I_1 = (t, s, \xi, x_1) \cap (y_3, y_2, y_1, x_2, y_4)\) Crossing the \(y_3\)-wall contracts the locus \(\mathcal{V}(y_2, y_1, x_2, y_4) \subset T_1\). Its restriction to \(Y\) is \(\mathcal{V}(y_3, y_2, y_1, x_2, y_4) \subset \mathcal{I}(I_1)\), where \(I_1\) is the ideal defining \(Y_1 \subset T_1\). Hence, the contraction happens away from \(Y_1\), so \(\tau_1\) restricts to an isomorphism \(Y_1 \cong Y_2\). Just as before we just set \(Y_1 \cong Y_2 \subset T_2\) where \(I_2 = (t, s, \xi, x_1, y_3) \cap (y_2, y_1, x_2, y_4)\). Crossing the \(y_2\)-wall restricts to a contraction of \(C_2 := (y_1 = x_2 = y_4 = 0) | y_2 \cong \mathbb{P}(7, 1)\) and an extraction of \(C_3 := (t = s = \xi = x_1 = 0) | y_3 \cong \mathbb{P}(1, 5)\). Hence, the map \(\tau_2\) corresponds to a toric flip over a point, denoted by \((-7, -1, 1, 5)\) and \(Y_3 \subset T_3\) with irrelevant ideal \(I_3 = (t, s, \xi, x_1, y_3, y_2) \cap (y_1, x_2, y_4)\). The map \(\tau_3\) restricts to \((x_2 = y_4 = 0) | y_3 = (y_1 = x_2 = y_4 = 0) \subset \mathcal{V}(I_3)\) where \(I_3\) is the ideal defining \(Y_3 \subset T_3\). Therefore the small contraction \(\tau_3: T_3 \rightarrow T_4\) happens away from \(Y_3\). Finally, we have a map \(\varphi': Y_4 \rightarrow \mathcal{F'} = \text{Proj} \mathbb{C}[x_2, y_4]\). A generic fibre is a surface \(S\) given by \(\left(t^3 \xi + t^2 y_2 + t \xi y_2 - y_2^2 + t^2 \xi y_1 - \xi^3 y_1 - t y_2 y_1 - \xi y_1 y_2 - t^2 y_1^2 + y_2 y_1^2 = 0\right) \subset \mathbb{P}(1, 1, 1, 1, 2, 2)\).

Hence, \(S\) is a del Pezzo surface of degree 2. Therefore we have the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow \phi & & \downarrow \phi' \\
F & \xrightarrow{\alpha_2} & Y_2 \\
\downarrow \beta_2 & & \downarrow \beta_2 \\
\mathbb{P} & \xrightarrow{\tau} & \mathbb{P}(1, 2, 2) \\
\end{array}
\]

where \(\varphi': Y_4 \rightarrow \mathbb{P}(1, 2, 2)\) is a del Pezzo fibration of degree two.

4.2 Case II: Divisorial contractions

In each case in Figure 3 whose notation we refer to, the movable cone of \(T\) is strictly contained in the effective cone of \(T\). Hence we have a diagram of toric varieties

\[
\begin{array}{ccc}
T & \xrightarrow{\Phi} & T' \\
\downarrow \tau & & \downarrow \Phi' \\
\mathbb{P} & \xrightarrow{\tau'} & \mathbb{F}' \\
\end{array}
\]

where \(\Phi\) and \(\Phi'\) are divisorial contractions and \(\tau\) is a small \(\mathbb{Q}\)-factorial modification. As usual, we restrict \(\Phi: T \rightarrow \mathbb{P}\) to be the unique Kawamata blowup \(\varphi: E \subset Y \rightarrow p_0 \in X\). By Lemmas 4.4 and 4.7 the map \(\tau|_Y: Y \rightarrow Y'\) is an isomorphism followed by a finite sequence of isomorphisms or flips. Referring to the notation in Figure 3 by assumption the rays \(\mathbb{R}_+[z_4]\) and \(\mathbb{R}_+[z_5]\) are given by linearly independent vectors \(v_4\) and \(v_5\) in \(\mathbb{Z}^2\). Let \(-d \neq 0\) be the determinant of the matrix \(B\) whose columns are \(v_4\) and \(v_5\); without loss of generality we can assume that \(d > 0\) (cf. [Ahn17, Lemma 2.4]). Then, there is \(A \in \text{GL}(2, \mathbb{Z})\) such that

\[
A \cdot B = \begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix}.
\]

After a row operation on the grading of \(T'\) via the matrix \(A\), \(T'\) is isomorphic to a toric variety with \(\mathbb{C}^* \times \mathbb{C}^*\)-action given by

\[
T' : \begin{pmatrix} t & s & \xi & x_1 & z_2 & z_3 & z_4 & z_5 \\ \kappa_0 & \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 & \kappa_6 & 0 & -d \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & d & 0 \end{pmatrix}
\]
Assume \( \mathbb{P} \) weighted projective space. \( \Phi \) whenever where every \( 2 \times 2 \) minor is non-positive. The irrelevant ideal of \( T \) is \((t, s, \xi, x_1, z_1, z_2, z_3) \cap (z_4, z_5)\) in cases II.a and \((t, s, \xi, x_1, z_1, z_2) \cap (z_3, z_4, z_5)\) in cases II.b.

**Cases II.a. Divisorial contractions to a point.** We have \( \kappa_\xi \neq 0 \) by assumption. The map \( \Phi' \) is given by the sections multiples of \( D \) (in the notation of [BZ10, Section 4.1.7]) and is a divisorial contraction to a point in \( F' \). This can be realised as

\[
\Phi': T' \longrightarrow F'
\]

\[
(t, s, \xi, x_1, z_1, z_2, z_3, z_4, z_5) \mapsto (t \xi^{\kappa_0/d}, s \xi^{\kappa_1/d}, \xi \xi^{\kappa_2/d}, x_1 \xi^{\kappa_3/d}, z_1 \xi^{\kappa_4/d}, z_2 \xi^{\kappa_5/d}, z_3 \xi^{\kappa_6/d}, z_4).
\]

Assume \( X \) is not the family \#39660. Then, \( F' = \mathbb{P}(\lambda_0, \ldots, \lambda_6, d) \) and \( \Phi' \) contracts the divisor \( E': (z_5 = 0) \cong \mathbb{P}(\kappa_0, \ldots, \kappa_6) \) to the point \( p = (0 : \cdots : 0 : 1) \in F' \). In particular \( p \) is smooth whenever \( d = 1 \). The contraction \( \Phi' \) restricts to a weighted blowup \( \varphi': Y' \rightarrow X' \) as in Table 3.

The following explicit example illustrates a link that terminates with a 3-fold in a fake weighted projective space.

**Example 4.10.** Consider the deformation family with ID \#39660 in Tom format. This is \( X \subset \mathbb{P} := \mathbb{P}(2, 2, 3, 5, 7, 12, 17) \) with homogeneous coordinates \( \xi, y_1, y_2, y_3, y_4, x_1, s \). By Lemma 8.7 we know that the weighted blowup of \( \mathbb{P}_3 = (0 : \cdots : 0 : 1) \in \mathbb{P} \) restricts to the Kawamata blowup \( \varphi: Y \subset T \rightarrow X \subset \mathbb{P} \) provided that the bi-grading of \( T \) is

\[
\begin{array}{c|cccccccc}
  t & 3 & 10 & 1 & 6 & 2 & 1 & 1 & 0 \\
  s & 1 & 9 & 1 & 6 & 3 & 2 & 2 & 1
\end{array}
\]

up to multiplication by a matrix in \( \text{GL}(2, \mathbb{Z}) \). We have a sequence of small \( \mathbb{Q} \)-factorial modifications \( \tau: T \rightarrow T' \) where \( T' \) has the same Cox ring as \( T \) but whose irrelevant ideal is \((t, s, \xi, x_1, y_3, x_2, y_2) \cap (y_1, y_4)\). Let \( T'y_4 \) be the same rank 2 toric variety as \( T' \), except that \( y_1 \) has bi-degree \((-1,0)\), instead of \((-2,0)\). Then, there is a map \( q: T'y_4 \rightarrow T' \) given by \( y_1 \mapsto y'_4 \) and \( T'y_4 \)

Figure 3: The possible effective cones of \( T \) ending with a contraction of the divisor \( D_{z_5}: (z_5 = 0) \). In cases II.a.1, II.a.2, II.a.3 the divisor \( D_{z_5} \) is contracted to the point \( F' = \text{Proj} \mathbb{C}[z_4] \) and in cases \( 3c \) and \( 3d \) to the rational curve \( F' = \text{Proj} \mathbb{C}[z_3, z_4] \). The variables \( z_1, \ldots, z_5 \) are \( y_1, \ldots, y_4, x_2 \) up to permutation. The exceptional divisor of the Kawamata blowup \( \varphi: Y \rightarrow X \) is \( E: (t = 0) \).
Table 3: We list the restriction $\Phi'_{|T'} = \varphi'$ and the model to which $\varphi'$ contracts to. In each case $\varphi'$ is a weighted blowup with weights $\frac{1}{r}(a_1, \ldots, a_t)$ with $r \geq 1$. For case #39890 $\varphi'$ is a contraction to a hyperquotient singularity; in the other instances, $\varphi'$ is a contraction to a Gorenstein point. The family #39569 is embedded in the weighted projective space $\mathbb{P}^7(2, 3, 5, 6, 7, 8, 9)$ with coordinates $\xi, z, u, v, s, 0, s_1, s_2$.

| ID   | Centre | $T'$ | $\varphi'$ | $X' \subset T'$ |
|------|--------|------|------------|----------------|
| #39569 | 1/7 (I12) | $\begin{pmatrix} t & s_0 & \xi & y & s_1 & s_2 & v & u & z \end{pmatrix}$ | $\begin{pmatrix} 5 & 6 & 1 & 3 & 4 & 2 & 1 & 0 & -1 \end{pmatrix}$ | $(5, 1, 3, 1)$ | $X'_{0,6} \subset \mathbb{P}(1^2, 2, 3^3)$ |
| #39660 | 1/17 | $\begin{pmatrix} 3 & 10 & 1 & 6 & 2 & 1 & 1 & 0 & -2 \end{pmatrix}$ | $\frac{1}{2}(1, 1, 1)$ | $X'_{1,5} \subset \mathbb{P}(1^3, 2^2, 3)/\mu_2$ |
| #39890 | 1/11 | $\begin{pmatrix} 8 & 15 & 2 & 10 & 5 & 3 & 1 & 0 & -3 \end{pmatrix}$ | $\frac{1}{3}(2, 10, 5, 3, 1)$ | $X' \subset \mathbb{P}(1^2, 2, 3^2, 4, 5, 6)$ |
| #39906 | 1/7 | $\begin{pmatrix} 1 & 4 & 1 & 3 & 3 & 1 & 1 & 0 & -1 \end{pmatrix}$ | $(2, 5, 3, 1)$ | $X' \subset \mathbb{P}(1^4, 2, 3, 4)$ |
| #39912 | 1/11 | $\begin{pmatrix} 4 & 13 & 2 & 3 & 2 & 3 & 1 & 0 & -1 \end{pmatrix}$ | $(2, 3, 3, 1)$ | $X' \subset \mathbb{P}(1^4, 2^2, 3)$ |
| #39913 | 1/5 | $\begin{pmatrix} 4 & 9 & 3 & 6 & 2 & 3 & 2 & 1 & 1 \end{pmatrix}$ | $(2, 3, 3, 1)$ | $X' \subset \mathbb{P}(1^3, 2^2, 3, 4)$ |
| #39928 | 1/13 | $\begin{pmatrix} 1 & 7 & 1 & 2 & 4 & 2 & 1 & 1 & 0 \end{pmatrix}$ | $(2, 4, 7, 3, 1)$ | $X'_{3,4} \subset \mathbb{P}(1^4, 2^2)$ |
| #39929 | 1/5 | $\begin{pmatrix} 4 & 7 & 2 & 4 & 2 & 3 & 1 & 0 & -1 \end{pmatrix}$ | $(4, 2, 7, 3, 1)$ | $X' \subset \mathbb{P}(1^4, 2^2, 3)$ |
| #39934 | 1/5 | $\begin{pmatrix} 4 & 7 & 2 & 1 & 4 & 3 & 1 & 0 & -1 \end{pmatrix}$ | $(2, 3, 3, 1)$ | $X' \subset \mathbb{P}(1^4, 2^3, 3)$ |

can be seen as a double cover of $T'$. On the other hand, $T' = \text{Proj } \mathbb{C}[t, s, \xi, x_1, y_3, x_2, y_2, y_1, y_4]^{\mu_2}$, where $\mu_2$ is the multiplicative cyclic group of order 2 acting on $\mathbb{C}[t, s, \xi, x_1, y_3, x_2, y_2, y_1, y_4]$ via

$$
\varepsilon \cdot (t, s, \xi, x_1, y_3, x_2, y_2, y_1, y_4) = (t, s, \xi, x_1, y_3, x_2, y_2, y_1, \varepsilon y_4).
$$

Hence, $q$ is the quotient map of $T'^{y_4}$ under this group action. Consider the map

$$
\Phi^{y_4} : T'^{y_4} \longrightarrow \mathbb{P}(1, 9, 1, 6, 3, 2, 2, 1)
$$

$$(t, s, \xi, x_1, y_3, x_2, y_2, y_1, y_4) \mapsto (t y_3^2, s y_4^3, \xi y_4, x_1 y_4^6, y_3 y_4^2, x_2 y_4, y_2 y_4, y_1).$$

This is the map given by the sections multiples of $H^0(T'^{y_4}, \mathcal{O}_{T'^{y_4}}(0, 1))$ and it corresponds to a divisorial contraction to a point in $\mathbb{P}(1, 9, 1, 6, 3, 2, 2, 1)$. Since $\Phi^{y_4}$ is an isomorphism away from the locus $(y_4 = 0)$, the action on $T'^{y_4}$ is carried through to $\mathbb{P}(1, 9, 1, 6, 3, 2, 2, 1)$.

The action at each point of this weighted projective space is then given by

$$
\varepsilon \cdot (t : s : \xi : x_1 : y_3 : x_2 : y_2 : y_1 : y_4) = \left( \varepsilon t : \varepsilon s : \varepsilon \xi : x_1 : y_3 : \varepsilon x_2 : \varepsilon y_2 : y_1 \right) = \left( t : \varepsilon s : \xi : x_1 : y_3 : x_2 : y_2 : \varepsilon y_1 \right).
$$

Then, $\Phi^{y_4}$ descends to a divisorial contraction of quotient spaces

$$
\Phi' : T' = T'^{y_4}/\mu_2 \longrightarrow \mathbb{P}(1, 9, 1, 6, 3, 2, 2, 1)/\mu_2.
$$
The map \( \Phi' \) restricts to a divisorial contraction \( \varphi' \) to the point \( p_0 \) in the complete intersection of a quartic and a quintic \( Z \subset \mathbb{P}(1,1,1,2,2,1) \) with homogeneous variables \( t, \xi, x_2, y_2, y_1 \) which is given by the equations

\[
\begin{align*}
t^4 + t\xi^3 - y_3y_1 - y_2^2 &= 0 \\
tx_2^2 + ty_1^4 - \xi^5 - \xi^2x_2y_4 + y_3y_2 &= 0.
\end{align*}
\]

Note that \( Z \) is given by the equations \( a \) quartic and \( b \) quintic and its exceptional divisor is isomorphic to \( \mathbb{P}^2 \). Hence, \( \varphi' \) is the Kawamata blowup centred at \( p_{y_1} \in Z \).

**Cases II.b. Divisorial contractions to a rational curve.** By assumption, \( \alpha_6 = 0 \). The map \( \Phi' \) is given by the sections that are multiples of the divisor \( (z_4 = 0) \), and is a divisorial contraction to the rational curve \( \Gamma' := \text{Proj} \mathbb{C}[\lambda_6, d] \). By Lemma 3.8, \( \Phi' \) restricts to a divisorial contraction to a curve at the level of 3-folds. By terminality, it follows that \( \gcd(\lambda_6, d) = 1 \) since, otherwise, the curve \( \Gamma' \) would be a line of singularities. In fact, by looking at each case we can see that \( d = 1 \). The map \( \Phi' \) is then

\[
(\Phi, T', \mathcal{F}') \rightarrow (\mathcal{F}')
\]

Hence, the divisor \( (z_5 = 0) \subset T' \) which is isomorphic to \( \mathbb{P}(\kappa_0, \ldots, \kappa_5) \) is contracted to \( \Gamma' \subset \mathbb{P}(\lambda_0, \ldots, \lambda_6, 1) \). In Table 3.8 we summarise the details regarding all cases falling into II.b.

| ID     | Centre | \( T' \) | \( \Gamma' \subset \mathcal{X'} \subset \mathcal{F}' \) |
|--------|--------|----------|-------------------------------------------------|
| #39557 | 1/11   | \( \begin{pmatrix} t & 5 & 8 & 1 & 3 & 2 & 1 & 0 & 0 & -1 \\ s & 8 & 15 & 2 & 6 & 5 & 3 & 1 & 1 & 0 \end{pmatrix} \) | \( \mathbb{P}^1 \subset X'_1 \subset \mathbb{P}(1,1,2,3,5) \) |
| #39605 | 1/13   | \( \begin{pmatrix} t & 3 & 8 & 1 & 5 & 2 & 1 & 0 & 0 & -1 \\ s & 15 & 4 & 2 & 10 & 5 & 3 & 1 & 1 & 0 \end{pmatrix} \) | \( \mathbb{P}^1 \subset X'_{6,8} \subset \mathbb{P}(1,1,2,3,4,5) \) |
| #39675 | 1/9    | \( \begin{pmatrix} t & 3 & 6 & 1 & 1 & 2 & 1 & 0 & 0 & -1 \\ s & 11 & 4 & 2 & 5 & 3 & 1 & 1 & 0 \end{pmatrix} \) | \( \mathbb{P}^1 \subset X'_{6,9} \subset \mathbb{P}(1,1,2,3,5) \) |
| #39678 | 1/5    | \( \begin{pmatrix} t & 3 & 4 & 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ s & 7 & 4 & 2 & 2 & 3 & 3 & 1 & 1 & 0 \end{pmatrix} \) | \( \mathbb{P}^1 \subset X'_{4,6} \subset \mathbb{P}(1,1,2,3,3) \) |
| #39676 | 1/7    | \( \begin{pmatrix} t & 3 & 5 & 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ s & 9 & 2 & 1 & 2 & 3 & 3 & 1 & 1 & 0 \end{pmatrix} \) | \( \mathbb{P}^1 \subset X'_{4,6} \subset \mathbb{P}(1,1,2,3,3) \) |
| #39898 | 1/9    | \( \begin{pmatrix} t & 3 & 6 & 1 & 4 & 2 & 1 & 0 & 0 & -1 \\ s & 1 & 5 & 1 & 4 & 3 & 2 & 1 & 2 & 0 \end{pmatrix} \) | \( \mathbb{P}^1(1,2) \subset \mathcal{X}' \subset \mathbb{P}(1^3, 2^2, 3, 4, 5) \) |

Table 4: We list the restriction \( \Phi'|_{\mathcal{X}'} = \varphi' \) and the model to which \( \varphi' \) contracts to. In each case \( \varphi' \) is a contraction to a curve \( \Gamma' \) inside a Fano 3-fold \( X' \).

**Divisorial contractions to curves with non-rational components.** Here we look into more detail at the case in which the birational link terminates with a divisorial contraction to a non-complete intersection curve. The families falling into this description are #40663, #40671,
Notice that the families #40663 and #40993 have been treated in [Duc18, Table 1] and are referred to as A.3 and A.2 respectively. In that paper, Ducat constructs these two families via simple Sarkisov links initiated by blowing up a curve $\Gamma$ on a rational 3-fold.

It turns out that these are not the only codimension 4 and index 2 Fano 3-folds which can be obtained in this way. Here we rely on the construction of $X$ as in Section 3 and show that #40671 and #40672 are rational via a reversed procedure to the one in [Duc18]. These two examples are interesting also because the Sarkisov link at the toric level ends with a fibration while its restriction to the 3-folds ends with a divisorial contraction to a non-rational curve. Moreover, we compute the Picard rank of #40671 and #40672.

**#40672.** Let $X \subset \mathbb{P} := \mathbb{P}(1, 1, 2, 2, 2, 3, 3)$, with homogeneous coordinates $y_1, y_2, x_2, \xi, y_4, x_1, y_3, s$ be a quasi-smooth member of the family #40672 as in Section 3 and [Cam21].

After the Kawamata blowup of the point $p_s \sim \frac{1}{3}(1, 1, 2)$ and are referred to as A.3 and A.2 respectively. In that paper, Ducat constructs these two families via simple Sarkisov links initiated by blowing up a curve $\Gamma$ on a rational 3-fold.

The small $\mathbb{Q}$-factorial modification $\tau_1$ contracts $\mathbb{P}(3, 3, 1, 1) \subset T_1$ to a point and extracts $\mathbb{P}(1, 1, 1, 2) \subset T_2$ where $T_2$ has the same Cox ring as $T_1$ but irrelevant ideal $(t, s, \xi, x_1, y_3) \cap (y_1, y_2, x_2, y_4)$. This restricts to the flip

$$C_1 \subset Y_1 \dashrightarrow \cdots \dashrightarrow Y_2 \supset C_2$$

where $C_1: (t + s + 2x_1^3 = \xi - x_1 = 0) \subset \mathbb{P}(3, 3, 1, 1)$, that is, $C_1 \simeq \mathbb{P}(3t, 1, x_1)$ and $C_2: (y_1 + x_2 = y_2 + x_2 = 0) \subset \mathbb{P}(1, 1, 1, 2)$, that is $C_2 \simeq \mathbb{P}(1, x_2, 2y_4)$. Also $-KY_1 \sim (\xi = 0) \subset Y_1$ and $E \sim (t = 0) \subset Y_1$. It is clear that $-KY_1 \cdot C_1 = \frac{1}{3}$. On the other hand, we have $-KY_2 \sim D_{y_3} - D_{x_2}$.

Hence, $-KY_2 \cdot C_2 = -\frac{1}{2}$. Consider the map $\Phi'$

$$\Phi': T_2 \to \mathbb{P}(1, 1, 1, 2)$$

$$(t, s, \xi, x_1, y_3, y_1, y_2, x_2, y_4) \mapsto (y_1, y_2, x_2, y_4).$$

Then $\Phi'$ is a fibration whose fibres are isomorphic to $\mathbb{P}(1, 2, 1, 1, 1)$. Consider the two consecutive projections $X \dashrightarrow X' \dashrightarrow X''$ where $X \dashrightarrow X'$ is the projection away from $p_s \in X$ and $X' \dashrightarrow X''$ is the projection away from $p_{y_3} \in X'$. The equations of $X''$ are given explicitly by

$$
\begin{pmatrix}
-1 & y_2 & y_3 x_2 + y_1 x_2^2 - x_2 y_4 \\
y_1 - y_2 & y_2 - y_1 x_2 - y_2 x_2 & -y_4 - y_1 x_2^2 + y_2 y_4 + y_1 x_2 y_4 - y_4^2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\xi
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Let $\Gamma \subset \mathbb{P}(1, 1, 1, 2)$ be defined by the three $2 \times 2$ minors of the matrix above. The curve $\Gamma$ has two irreducible and reduced components: one is rational, and the other has genus 4. We have that $\Phi'|_{y_2}$ is a divisorial contraction to $\Gamma \subset \mathbb{P}(1, 1, 1, 2)$. Since $\Gamma$ has two irreducible components, $\rho_X = 2$.

**#40671.** This case is completely analogous to the previous one. We give the $2 \times 3$ matrix whose $2 \times 2$ minors define $\Gamma \subset \mathbb{P}(1, 1, 1, 2)$. This is

$$
\begin{pmatrix}
y_1 & -y_2 & -y_3 y_4 \\
y_2 y_3 & -y_2 y_3 + y_2 y_3 & -y_1 y_3^2 + y_2 y_3^2 + y_4^2
\end{pmatrix}.
$$

The curve $\Gamma$ has one rational irreducible component, and another irreducible component with genus 5. As before, $\rho_X = 2$. 

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4.2.1 Wrapping up: conclusion of non-solidity proof

We finalise the proof of non-solidity for the families whose link terminates with a divisorial contraction. We prove that all models $X'$ in Tables 3 and 4 admit a structure of a strict Mori fibre space with the possible exception of two.

**Lemma 4.11.** Let $X$ be a family in Table 4. Then, $X$ is non-solid.

**Proof.** We consider the projection $\pi: X' \to \mathbb{P}^1$ in each case. Then, the generic fibre $S \subset X'$ is a surface for which $-K_S \sim O_S(1)$ by adjunction since $X'$ has Fano index 2. The conclusion follows from Corollary 2.1.

We show next that the families #39890 and #39928 in Table 3 admit singular birational models in a family whose general member is a quasismooth complete intersection of a cubic and a quartic $X'_{3,4} \subset \mathbb{P}(1,1,1,1,2,2)$. It was shown by Corti and Mella that these general quasismooth members are bi-rigid. In this case, we were not able to show that these are non-solid: these are the non-birationally rigid families mentioned in Theorem 1.1.

**Lemma 4.12.** Let $X$ be a family #39890 or #39928 in Table 3. Then $X$ is birational to a non-quasismooth complete intersection of a cubic and a quartic $X'_{3,4} \subset \mathbb{P}(1,1,1,1,2,2)$.

**Example 4.13.** Consider the singular complete intersection

\[
\begin{align*}
-t\xi z_2 + tz_3^2 + t z_1 + \xi u + z_1 y + uy &= 0 \\
t^3 \xi - \xi^4 + ty^3 - t^3 z_2 - t \xi u - \xi z_1 y - t uy + tu z_2 - z_1^2 &= 0
\end{align*}
\]

inside $\mathbb{P}(1,1,1,1,2,2)$ with homogeneous variables $t, \xi, y, z_2, z_1, u$. Then, there is a weighted blowup from the point $p_{z_2}$ that initiates a birational link to a quasismooth family in Tom format with ID #39928 (see Table 3).

**Lemma 4.14.** Let $X$ be a member of a family in Table 3 not treated in Lemma 4.12. Then, the model $X'$ is birational to a del Pezzo fibration.

**Proof.** Notice that each model $X'$ in Table 3 has Fano index 1. General (quasi-smooth) members in each of these families have been treated in [BZ10], [Oka14] and [Cam20]. In particular it was shown that these are non-solid. The same results extend to terminal families provided that the key monomials are still present in the equations of $X'$, which is the case for each model.

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