Classification of Genus 3 Curves in Special Strata of the Moduli Space

Martine Girard, David R. Kohel

School of Mathematics and Statistics,
The University of Sydney
\{girard,kohel\}@maths.usyd.edu.au

Abstract. We describe the invariants of plane quartic curves — nonhyperelliptic genus 3 curves in their canonical model — as determined by Dixmier and Ohno, with application to the classification of curves with given structure. In particular, we determine modular equations for the strata in the moduli space \( M_3 \) of plane quartics which have at least seven hyperflexes, and obtain an computational characterization of curves in these strata.

1 Introduction

The classification of curves of genus 0, 1, and 2 is aided by use of various geometric and arithmetic invariants. In this work we consider nonhyperelliptic genus 3 curves, for which the canonical model is an embedding as a projective plane quartic. The work of Hess \[3,4\] gives generic algorithms for determining the locus of Weierstrass points and for finding whether two curves are isomorphic.

Such a generic approach to isomorphism testing works well for curves over finite fields, where a small degree splitting field for the Weierstrass places exists, and when one wants to test only two curves. In this work, we investigate the geometric invariants of nonhyperelliptic genus 3 curves, which are much more suited to classifying curves which are already given in terms of their canonical embeddings.

In particular, plane quartic curves admit explicit formulas for the Weierstrass locus, invariants of Dixmier and Ohno by which the curves may be classified up to isomorphism over an algebraically closed field, and moreover can be classified into strata following Vermeulen’s characterization in terms of the number and configuration of Weierstrass points of weight two.

In the generic case, Harris \[2\] proved that a generic curve of any genus over a field of characteristic zero, is expected to have generic Galois action on the Weierstrass points. Thus in order to establish an isomorphism between the sets of Weierstrass points one needs in general an excessively large degree extension to apply the algorithm of Hess. Thus it becomes essential to exploit any special structure of the Weierstrass points to facilitate this algorithm. In this article we focus on curves whose moduli lie in special strata of the moduli space of genus three curves. We use a classification by invariants to reduce to a trivial calculation of invariants on certain strata of Vermeulen of dimensions 0 and 1.
2 The Weierstrass Locus of Quartic Curves

A nonhyperelliptic curve \( C \) of genus 3 can be defined via the canonical embedding by a quartic equation \( F(X, Y, Z) = 0 \) in the projective plane. The Hessian \( H(X, Y, Z) \) of the form \( F(X, Y, Z) \) is defined by

\[
H = \begin{vmatrix}
\frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial X \partial Z} \\
\frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Y \partial Z} \\
\frac{\partial^2 F}{\partial X \partial Z} & \frac{\partial^2 F}{\partial Y \partial Z} & \frac{\partial^2 F}{\partial Z^2}
\end{vmatrix}.
\]

This form is a sextic, which meets the curve \( C \) in the 24 inflection points of the curve (counting multiplicities). These inflection are also the Weierstrass points, hence they may be determined in an elementary way. The inflection points which meet the Hessian with multiplicity 2 are those inflection points which meet their tangent line to multiplicity 4, and are called hyperflexes.

The hyperflexes are intrinsic points of the genus 3 curve, since they are Weierstrass of weight 2 – those which have a deficit of 2 in their gap sequences. Thus they are preserved by any isomorphism of curves, and reflect the underlying geometry of the curves (rather than solely of a choice of projective embedding).

We focus in this article on the classification of those curves which have an exceptional number of hyperflexes. The partitioning of the Weierstrass points into those of weight 1 and weight 2 can also be efficiently determined since it is the singular subscheme of the intersection \( F = H = 0 \), defined by the vanishing of the Jacobian minors:

\[
\frac{\partial F}{\partial X} \frac{\partial H}{\partial Y} - \frac{\partial F}{\partial Y} \frac{\partial H}{\partial X} = \frac{\partial F}{\partial X} \frac{\partial H}{\partial Z} - \frac{\partial F}{\partial Z} \frac{\partial H}{\partial X} = \frac{\partial F}{\partial Y} \frac{\partial H}{\partial Z} - \frac{\partial F}{\partial Z} \frac{\partial H}{\partial Y} = 0.
\]

The calculation of the hyperflex locus can be reduced to polynomial factorization, without the need for Gröbner basis calculations. Let \( R \) be the resultant \( \text{Res}(H, F, Z) \) of degree 24 and let set \( G(X, Y) = \text{GCD}(R, R_X, R_Y) \). Then \( G \) determines the \((X, Y)\)-coordinates of the hyperflex locus for which \( XYZ \neq 0 \).

Since plane quartics are canonical embeddings of a genus 3 curve, any isomorphism of such curves is induced by a linear isomorphism of their ambient projective planes. As a result, the problem of determining isomorphisms is reduced to the intersection of a linear algebra problem of finding such an isomorphism and a combinatorial one, of mapping a finite set of Weierstrass points to Weierstrass points. By combining classification of quartics by their moduli invariants into strata determined by the numbers and configurations of hyperflexes, we facilitate the problem of establishing isomorphisms between curves.
3 Quartic Invariants

The $j$-invariant of an elliptic curve or the Igusa invariants of a genus 2 curve provide invariants by which a curve of genus 1 or 2 can be classified up to isomorphism. Recall that the $j$-invariant of an elliptic curve is defined in terms of weighted projective invariants $E_4, E_6, \Delta$ such that

$$j = \frac{E_4}{\Delta} \quad \text{and} \quad E_4^3 - E_6^2 = 12 \Delta.$$

Similarly Igusa [5] defined weighted invariants $J_2, J_4, J_6, J_8, J_{10}$ with one relation $J_2 J_6 - J_4^2 = 4 J_8$, from which one can determine a set of absolute invariants (see also Mestre [7]).

For genus $g \geq 2$, the moduli space of curves of genus $g$ is a space of dimension $3g - 3$, thus the determination of generators for the ring of projective invariants becomes increasingly difficult. However, for genus 3, Dixmier [1] provided an explicit set of 7 weighted invariants and proves their algebraic independence over $\mathbb{C}$. The determination of these invariants builds on on the explicit 19th century methods of Salmon [11]). By comparison of the Poincaré series of this ring with that computed by Shioda [12], Dixmier finds that his invariants $I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}$ determine a subring over which the ring $\mathcal{A}$ of all invariants is free of rank 50.

Recently, Ohno [8] determined a complete set of generators and relations for the full ring of invariants of plane quartics. In particular he shows that there exist six additional invariants $J_9, J_{12}, J_{15}, J_{18}, I_{21}, J_{21}$, which generate $\mathcal{A}$ as a finite algebraic extension of $\mathbb{C}[I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}]$.

Definition of covariants and contravariants

In this section we recall the definitions of covariants and contravariants, and basic constructions appearing in Dixmier [1], Ohno [8], and Salmon [11]. We first introduce the definitions of covariant and contravariant, following the modern terminology used in Poonen, Schaefer, and Stoll [10, §7.1]. For other modern treatments of the subject, see Sturmfels [13] and Olver [9].

Let $V = \mathbb{C}^n$ be be equipped with the standard left action of $\text{GL}_n(\mathbb{C})$, which induces a right action on the algebra $\mathbb{C}[x_1, \ldots, x_n] = \text{Sym}(V^*)$. For $\gamma \in \text{GL}_n(\mathbb{C})$ and $F \in \mathbb{C}[x_1, \ldots, x_n]$ we define this action by $F^\gamma(x) = F(\gamma(x))$ for all $x \in V$.

We denote $\mathbb{C}[x_1, \ldots, x_n]_d$ the $d$-th graded components of polynomials homogeneous of degree $d$.

**Definition 1.** A covariant of degree $r$ and order $m$ is a $\mathbb{C}$-linear function

$$\psi : \mathbb{C}[x_1, \ldots, x_n]_d \rightarrow \mathbb{C}[x_1, \ldots, x_n]_m,$$

which satisfies

1. $\text{SL}_n(\mathbb{C})$-module homomorphism, i.e. $\psi(F^\gamma) = \psi(F)\gamma$ for all $\gamma \in \text{SL}_n(\mathbb{C})$, and
2. The coefficients of $\psi(F)$ depend polynomially in the coefficients of $x_1^{i_1} \cdots x_n^{i_n}$.
3. And $\psi(\lambda F) = \lambda^r \psi(F)$ for all $\lambda \in \mathbb{C}$.

We note in particular that the last two conditions imply that $\psi$ is homogeneous of degree $r$ in the coefficients of the degree $d$ form $F$. An invariant is a covariant of order 0.

N.B. One “usually” defines a covariant to satisfy

$$\psi(F^\gamma(x)) = \text{det}(\gamma)^k \psi(F(x))$$

where $\gamma(x) = x$, or equivalently, $\psi(F^\gamma) = \text{det}(\gamma)^k \psi(F)^\gamma$, for all $\gamma \in \text{GL}_n(V)$ and $x$ in $V$. One defines $k$ to be the weight (or index) of $\psi$. Clearly we then have the relation $2k = dr - m$. Applying a scalar matrix $\gamma$ to an invariant implies that $k \equiv 0 \mod n$.

The definition of Poonen, Schaefer, and Stoll admits the possibility of covariants with a multiplicative character. Following the classical definitions we include the stronger condition above in our definition of covariants.

In order to define a contravariant, we set $\mathbb{C}[[u_1, \ldots, u_n]] = \text{Sym}(V)$, where $\{u_1, \ldots, u_n\}$ is a basis for $V$ dual to the basis $\{x_1, \ldots, x_n\}$ of $V^\ast$. Then $\text{GL}_n(\mathbb{C})$ has the right contragradient action on polynomials in $\mathbb{C}[[u_1, \ldots, u_n]]$, which we denote $G^\ast$, where $\gamma_s$ is the inverse transpose of $\gamma$.

**Definition 2.** A contravariant of degree $r$ and order $m$ is a $\mathbb{C}$-linear function

$$\psi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[u_1, \ldots, u_n],$$

which satisfies

1. $\text{SL}_n(\mathbb{C})$-module homomorphism, i.e. $\psi(F^\gamma) = \psi(F)^\gamma$ for all $\gamma \in \text{SL}_n(\mathbb{C})$,
2. The coefficients of $\psi(F)$ depend polynomially in the coefficients of $x_1^{i_1} \cdots x_n^{i_n}$,
3. And $\psi(\lambda F) = \lambda^{-r} \psi(F)$ for all $\lambda \in \mathbb{C}$.

N.B. As noted in [10], we may formally identify $u_1, \ldots, u_n$ with $x_1, \ldots, x_n$, via the isomorphism $V \to V^\ast$ implied by the choice of basis for $V$. We nevertheless distinguish the $\text{GL}_n(\mathbb{C})$-modules structures by denoting the action by $G \mapsto G^\ast$ for $\gamma$ in $\text{GL}_n(\mathbb{C})$. In our mathematical exposition we preserve the notational distinction between $x_i$ and $u_i$.

**Covariant and contravariant operations**

We extend the linear pairing $V \times V^\ast \to \mathbb{C}$ given by $(u_i, x_j) \mapsto \delta_{ij}$ to a differential operation

$$D : \mathbb{C}[u_1, \ldots, u_n] \times \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n],$$

by identifying a monomial $u_1^{i_1} \cdots u_n^{i_n}$ of total degree $m$ with the operator

$$\frac{\partial^m}{\partial^{i_1} x_1 \cdots \partial^{i_n} x_n}.$$
We denote the $D(\psi, \varphi)$ by $D_\psi(\varphi)$. By symmetry, we define a differential operator

$$D : \mathbb{C}[x_1, \ldots, x_n] \times \mathbb{C}[u_1, \ldots, u_n] \rightarrow \mathbb{C}[u_1, \ldots, u_n],$$

and denote $D(\varphi, \psi)$ by $D_\varphi(\psi)$. (We resolve the notational ambiguity from the arguments.) We recall as a lemma a classical result.

**Lemma 3.** Let $\varphi$ be a covariant and $\psi$ be a contravariant on $\mathbb{C}[x_1, \ldots, x_n]_d$. Then $D_\varphi(\psi)$ is a contravariant of order $\text{ord}(\psi) - \text{ord}(\varphi)$ and and $D_\psi(\varphi)$ a covariant of order $\text{ord}(\psi) - \text{ord}(\varphi)$, both of degree $\text{deg}(\varphi) + \text{deg}(\psi)$.

When specializing to ternary forms, we denote $(x_1, x_2, x_3)$ by $(x, y, z)$. For ternary quadratic forms, Dixmier \[1\] used the additional operations. Let $\varphi$ be a ternary quadratic form in $x, y, z$ and

$$D(\varphi) = \frac{1}{2} \begin{bmatrix} \frac{\partial^2 \varphi}{\partial x^2} & \frac{\partial^2 \varphi}{\partial x \partial y} & \frac{\partial^2 \varphi}{\partial x \partial z} \\ \frac{\partial^2 \varphi}{\partial y \partial x} & \frac{\partial^2 \varphi}{\partial y^2} & \frac{\partial^2 \varphi}{\partial y \partial z} \\ \frac{\partial^2 \varphi}{\partial z \partial x} & \frac{\partial^2 \varphi}{\partial z \partial y} & \frac{\partial^2 \varphi}{\partial z^2} \end{bmatrix}$$

and let $D(\varphi)^*$ be its classical adjoint. Then for $\varphi$ and $\psi$ covariant and contravariant forms, respectively, we define

$$J_{11}(\varphi, \psi) = \langle D(\varphi), D(\psi) \rangle \text{ and } J_{22}(\varphi, \psi) = \langle D(\varphi)^*, D(\psi)^* \rangle,$$

where $\langle A, B \rangle$ is a matrix dot product, and

$$J_{30}(\varphi, \psi) = J_{30}(\varphi) = \det(D(\varphi)) \text{ and } J_{03}(\varphi, \psi) = J_{03}(\psi) = \det(D(\psi)).$$

The expressions $J_{ij}$ play a role in invariant theory of ternary quadratic forms, but more generally we have the following classical lemma.

**Lemma 4.** Let $\varphi$ be a covariant and $\psi$ be a contravariant on $\mathbb{C}[x, y, z]_d$, each of order 2. Then $J_{ij}(\varphi, \psi)$ is an invariant on $\mathbb{C}[x, y, z]_d$ of degree $i \deg(\varphi) + j \deg(\psi)$.

In particular we will apply this to describe the construction of the complete invariants of ternary quartics by Dixmier \[1\] and Ohno \[2\].

Finally, for two binary forms $F(x, y)$ and $G(x, y)$ of degrees $r$ and $s$, we define the $k$-th transvectant of is defined to be

$$(F, G)^k = \frac{(r-k)!(s-k)!}{r!s!} \left( \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2} \right)^k F(x_1, y_1)G(x_2, y_2) \bigg|_{(x_1, y_1) = (x, y)}$$

**Lemma 5.** Let $F(x, y) = a_{40}x^4 + 4a_{31}x^3y + 6a_{22}x^2y^2 + 4a_{13}xy^3 + a_{04}y^4$ be a binary quartic form, and set $G = (F, F)^2$. Then $F$ has invariants $\sigma$ and $\psi$ defined by

$$\sigma = \frac{1}{2}(F, F)^4 = a_{40}a_{04} - 4a_{31}a_{13} + 3a_{22}^2,$$

and

$$\psi = \frac{1}{6}(F, G)^4 = a_{40}a_{22}a_{04} - a_{40}a_{13}^2 - a_{31}^2 + 2a_{31}a_{22}a_{13} - a_{22}^3.$$

The invariant $\sigma^3 - 27\psi^2$ is the discriminant of the form $F(x, y)$ (up to a scalar).
Covariants and contavariants of quartics

In this section we use the above construction to define the invariants of Dixmier and Ohno classifying ternary quartics, i.e. genus 3 curves of general type. As noted above, we denote \((x_1, x_2, x_3)\) by \((x, y, z)\), and dual \((u_1, u_2, u_3)\) by \((u, v, w)\).

The polynomial rings \(k[x, y, z]\) and \(k[u, v, w]\) represent coordinate rings of the ambient projective space \(\mathbb{P}^2\) and the dual projective space \((\mathbb{P}^2)^*\), respectively, of the quartic \(F(x, y, z) = 0\). Where necessary, \(k[x_1, x_2, x_3, u_1, u_2, u_3]\) will be the bi-graded coordinate ring of \(\mathbb{P}^2 \times (\mathbb{P}^2)^*\).

N.B. In the spirit of the classical literature, we speak of covariants and contravariants of a quartic form \(F(x, y, z)\), though formally the covariant or contravariant is a function from \(C[x, y, z]_4\) to \(C[x, y, z]_m\) or \(C[u, v, w]_m\). Similarly, we may express a homogeneous form as

\[ F(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n} \frac{d!}{i_1! \cdots i_n!} a(i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n}, \]

where the sum is over all indices with \(i_1 + \cdots + i_n = d\). The calculation of invariants is thus normalized to be primitive with respect to such classically integral forms (as in Lemma 5). In the case of quartics, the constructions and expressions often require the primes 2 and 3 to be invertible, even if the final invariants can be made well-defined in these characteristics. In what follows we follow Dixmier and Ohno in normalizing the expressions to be primitive with respect to the coefficients \(\{a_{ijk}\}\), and only at the end provide the scalars by which the invariants must be normalized to be integral with respect to the coefficients \(a_{ijk}\) of an integral form

\[ F(x, y, z) = \sum_{i,j,k} a_{i,j,k} x^i y^j z^k. \]  

In the spirit of Igusa’s article on genus 2 curves \([5]\), the determination of a complete set of integral invariants over any ring, and their algorithmic construction, remains open.

The first covariants at our disposal for a form \(F\) is the form itself (i.e. the identity covariant) and the Hessian \(H\). We additionally require two contravariants \(\sigma\) and \(\psi\), from which the Dixmier and Ohno invariants are derived by the operations of the previous section. The contravariant \(\sigma\) appears in Salmon \([11]\) (§92 and §292), has degree 2 and order 4, and the construction of the degree 3 and order 6 contravariant \(\psi\) appears (in Salmon §92, p.78). Formally intersect \(ux + vy + wz = 0\) with the form \(F\), and setting \(w = 1\), eliminate \(z\) a binary quartic \(R(x, y) = F(x, y, -ux - vy)\). Then the invariants \(\sigma\) and \(\psi\) of Lemma \([5]\) rehomogenized with respect to \(w\), provide us with the covariants \(\sigma(u, v, w)\) and \(\psi(u, v, w)\).

We can now define a system of covariants and contravariants for ternary quartics, from the covariants \(F\) and \(H\) and contravariants \(\sigma\) and \(\psi\).
Covariants  Contravariants
\[ \tau = 12^{-1}D_\rho(F) \quad \rho = 144^{-1}D_F(\psi) \]
\[ \xi = 72^{-1}D_\sigma(H) \quad \eta = 12^{-1}D_\xi(\sigma) \]
\[ \nu = 8^{-1}D_\eta D_\rho(H) \quad \chi = 8^{-1}D_\nu^2(\psi) \]

Subsequently we can define the invariants of Dixmier:
\[ I_3 = 144^{-1}D_\sigma(F), \quad I_9 = J_{11}(\tau, \rho), \quad I_{15} = J_{30}(\tau), \]
\[ I_6 = 4608^{-1}(D_\psi(H) - 8I_3^2), \quad I_{12} = J_{03}(\rho), \quad I_{18} = J_{22}(\tau, \rho). \]

together with the discriminant \( I_{27} \). Dixmier [1] proved that these invariants are algebraically independent over \( \mathbb{C} \) and generate a subring of the ring \( \mathcal{A} \) of ternary quartic invariants of index 50. Ohno [8] proved computationally that the additional six invariants
\[ J_9 = J_{11}(\xi, \rho), \quad J_{15} = J_{30}(\xi), \quad I_{21} = J_{03}(\eta), \]
\[ J_{12} = J_{11}(\tau, \eta), \quad J_{18} = J_{22}(\xi, \rho), \quad J_{21} = J_{11}(\nu, \eta), \]
generate \( \mathcal{A} \); he moreover determined a complete set of algebraic relations for the ring \( \mathcal{A} = \mathbb{C}[I_k, J_l] \).

In the following table we summarise the covariant and contravariant degrees and orders, as can be determined from Lemma 3, beginning with the forms \( F, H, \sigma \) and \( \psi \).

| Covariants | Contravariants |
|------------|----------------|
| \( \tau, \xi \) | \( \rho \) |
| \( \nu \) | \( \chi \) |

As noted above, the natural normalization for the invariant to be integral depends whether one considers classically integral forms or integral forms. On integral forms one normalizes the Dixmier–Ohno invariants as follows:
\[ 2^{12}3^2I_5, \quad 2^{12}3^6I_6, \quad 2^{12}3^8I_9, \quad 2^{16}3^{12}I_{12}, \quad 2^{23}3^{15}I_{15}, \quad 2^{27}3^{17}I_{18}, \quad 2^{40}I_{27}, \]
\[ 2^{12}3^7J_9, \quad 2^{17}3^{10}J_{12}, \quad 2^{23}3^{12}J_{15}, \quad 2^{27}3^{15}J_{18}, \quad 2^{31}3^{18}J_{21}, \quad 2^{33}3^{16}J_{21}. \]
We refer to these normalizations of the Dixmier–Ohno invariants as the integral Dixmier–Ohno invariants (as opposed to the classically integral invariants). Hereafter we will make these normalizations and write $I_{3k}$ or $J_{3l}$ to denote the above integral invariants.

In what follows we invert $I_3$ in order to define six algebraically independent functions on the moduli space of quartic plane curves $(i_1, i_2, i_3, i_4, i_5, i_6) = (I_6 I_9 I_{12} I_{15} I_{18} I_{27})$, and those defining an algebraic extension of $\mathbb{C}(i_1, \ldots, i_6)$:

$(j_1, j_2, j_3, j_4, j_5, j_6) = (J_9 I_3^3, J_{12} I_3^3, J_{15} I_3^3, J_{18} I_3^3, J_{21} I_3^3, J_{27} I_3^3)$.

4 Vermeulen Stratification

In 1983, Vermeulen [14] constructed a stratification of the moduli space $\mathcal{M}_3$ of curves of genus 3 in terms of the number of hyperflexes and their geometric configuration. Similar results were obtained independently around the same time by Lugert [6]. The classification of curves by the structure of their Weierstrass points identifies more subtle structure of the curves than that provided by the automorphism group.

Let $\mathcal{M}_3^\circ$ be $\{[C] \in \mathcal{M}_3, C$ non-hyperelliptic $\}$, $M_s = \{[C] \in \mathcal{M}_3^\circ, C$ has at least $s$ hyperflexes $\}$, and $M_s^\circ = \{[C] \in \mathcal{M}_3^\circ, C$ has exactly $s$ hyperflexes $\}$. All strata but $\mathcal{M}_3^\circ$ are closed irreducible subvarieties of $\mathcal{M}_3$. Each $M_s^\circ$ is the union of the strata with $s$ hyperflexes. For instance, $M_{12}^\circ$ consists of the two moduli points corresponding to the Fermat curve and the curve $X^4 + Y^4 + Z^4 + 3(X^2Z^2 + X^2Y^2 + Y^2Z^2) = 0$ and $M_{11}^\circ$ and $M_{10}^\circ$ are both empty. In the diagram below, we summarize the relevant data from Vermeulen’s stratification. We denote by $s$ the number of hyperflexes.

**N.B.** The $X_i$ have dimension 3, the $Y_i$ have dimension 2, the $Z_i$ have dimension 1 and the strata with a Greek letter have dimension zero.

5 Special Strata of Plane Quartics

For the special strata $S$ of $\mathcal{M}_3$ with more than six hyperflexes we determine a parametrization of the stratum and a model for a generic curve $C/\tilde{S}$ for some finite cover $\tilde{S} \to S$. In each case the structure of $\tilde{S} \to S$ is a Galois cover over which the hyperflexes locus splits completely. The Dixmier–Ohno invariants are computed over $\tilde{S}$ by their sequences of covariants and contravariants, rather than evaluation of symbolic expressions.
Table 1. Vermeulen’s stratification of $\mathcal{M}_3$

| $\mathcal{X}$ | $s$ | dim | Substrata | $\mathcal{X}$ | $s$ | dim | Substrata |
|---------------|----|-----|-----------|---------------|----|-----|-----------|
| $\mathcal{M}_3^1$ | 0 | 6 | $M_1$ | $\mathcal{Z}_5$ | 5 | 1 | $\Theta, \Pi_i, \Sigma, \Psi$ |
| $M_1^1$ | 1 | 5 | $M_2^1$ | $\mathcal{Z}_2$ | 6 | 1 | $\Pi_i, \Omega_i, \Phi, \Psi$ |
| $M_2^2$ | 2 | 4 | $X_1, X_2, X_3$ | $\mathcal{Z}_3$ | 6 | 1 | $\Theta, \Pi_i, \Omega_i, \Psi$ |
| $X_2$ | 3 | 3 | $Y_1, Y_2, Y_3$ | $\mathcal{Z}_5$ | 6 | 1 | $\Sigma, \Phi, \Psi$ |
| $X_3$ | 3 | 3 | $Y_1, Y_3, Y_4, Y_5$ | $\mathcal{Z}_9$ | 6 | 1 | $\Omega_i, \Phi, \Psi$ |
| $X_1$ | 4 | 3 | $Y_1$ | $\mathcal{Z}_4$ | 7 | 1 | $\Omega_i, \Psi$ |
| $Y_2$ | 4 | 2 | $Z_1, Z_5$ | $\Theta$ | 7 | 0 | |
| $Y_3$ | 4 | 2 | $Z_i, 1 \leq i \leq 8$ | $\Pi_i$ | 7 | 0 | |
| $Y_4$ | 4 | 2 | $Z_i, i \neq 3, 6, 8$ | $Z_1$ | 8 | 1 | $\Phi, \Psi$ |
| $Y_5$ | 2 | 4 | $Z_2, Z_3, Z_6, Z_9$ | $\Sigma$ | 8 | 0 | |
| $Y_1$ | 5 | 2 | $Z_1, Z_2, Z_3, Z_4$ | $\Omega$ | 9 | 0 | |
| $Y_6$ | 5 | 1 | $\Theta, \Pi_i, \Omega_i, \Phi$ | $\Phi$ | 12 | 0 | |
| $Z_7$ | 5 | 1 | $\Pi_i, \Sigma, \Omega_i, \Psi$ | $\Psi$ | 12 | 0 | |

Stratum $\mathcal{Z}_4$.

The moduli space $\mathcal{Z}_4$ is a one-dimensional subspace of $\mathcal{M}_3$, for which we can find a generic curve defined over $\mathbb{Q}(i, t)$ of the form:

$$C : t(t+i)(X^2-YZ)^2-YZ(2X-Y-Z)((i-1)t^2+2t+(i+1))X-(it+1)(Y+Z)$$

where $i^2 = -1$ and $t$ is a parameter. Let $\tilde{Z}_4$ be the rational curve with function field $\mathbb{Q}(i, t)$. This model parametrizes the curve plus the triple of Weierstrass points

$$\{(0 : 0 : 1), (0 : 1 : 0), (1 : 1 : 1)\},$$

with tangent lines defined by $Y = 0$, $Z = 0$, and $2X - Y - Z = 0$, respectively, and moreover, the remaining hyperflexes of $C$ split over $\mathbb{Q}(i, t)$. Thus $\tilde{Z}_4$ defines a pointed moduli space parametrizing three hyperflexes, and should determine a Galois cover of the moduli space $\mathcal{Z}_4$ on which the hyperflex locus splits.

A computation of the Dixmier–Ohno invariants reveals an automorphism $t \mapsto \frac{(-i+1)t^3+3t^2-i}{(t^2+(-i+1)t-i)}$, which provides the Galois cover $\tilde{Z}_4 \to \mathcal{Z}_4$.

The first of the Dixmier–Ohno invariants $(i_1, i_2)$ can be expressed in terms of the invariant $z$ as:

$$z = \frac{(z-3)(5z-3)}{4z^2} \left( \frac{79z^5-1059z^4-2670z^3+12366z^2-13203z+4455}{16(z^2-12z+9)z^3} \right).$$

Reciprocally, the expression

$$z = \frac{-165(880i_1^3+1336i_1^2-558i_1-1047)}{(53680i_1^3-10560i_1^2+10120i_1^2-57750i_1+7920i_1-6105)}$$

(2)
gives $z$ as a rational function in $(i_1, i_2)$ so $z$ generates the function field of $Z_4$. We note in particular that $Z_4$ is defined over $\mathbb{Q}$.

Solving for the algebraic relations in $(i_1, i_2)$ and renormalizing, we find a weighted projective equation for $Z_4$ in terms of the first Dixmier–Ohno invariants:

$$
193600I_6^5 - 35776I_3^2I_6^4 + 86784I_4I_6^3I_9 - 961040I_3I_9^3 \\
- 2304I_6^2I_9 + 100608I_5I_6I_9 - 526608I_3^6I_6^2 - 65376I_3I_6I_9 \\
+ 72152I_5I_6^2I_9 + 1728I_2^6 - 78048I_3^2I_6 - 515889I_3^{10} = 0. \quad (3)
$$

Thus from the invariants $I_3, I_6,$ and $I_9$ we determine a necessary condition for a given quartic curve to be in the stratum $Z_4$. The remaining invariants have rational expressions in $z$, so can be readily computed from the rational expression (2). A comparison with the remaining Dixmier–Ohno invariants verifies or contradicts the hypothesis that a curve with invariants satisfying (3) lies in $Z_4$.

Given the invariant $z$ for a point in $Z_4$, we can determine a field of definition for a representative curve with model $C$ above, by solving for a root $t$ of the degree six polynomial

$$
2T^6 + 2(z - 3)T^5 + (z - 3)^2T^4 + 2(z^2 - 4z + 1)T^3 + 2z^2T^2 + 2z(z - 1)T + (z - 1)^2,
$$

which is reducible over any field containing a square root of $-1$.

**Stratum $Z_1$.**

The moduli space $Z_1$ is a one-dimensional subspace of $M_3$ which consists of moduli of curves with 8 hyperflexes and automorphism group $D_4$. There exists a generic curve over some cover $\tilde{Z}_1$ on which the of $Z_1$, defined by

$$
C : (t^2 + 1)(X^2 - YZ)^2 = YZ(2X - Y - Z)(2tX - Y - t^2Z).
$$

By computing the Dixmier–Ohno invariants, we find that there exists a cyclic degree 4 cover $\tilde{Z}_1 \to Z_1$. In particular the transformation $t \mapsto (1 + it)/(t + i)$ of order four maps $C$ to an isomorphic curve. As above, we find an invariant function $z = t + 1/t + u + 1/u - 1/2$, where $u = (1 + it)/(t + i)$, such that the absolute Dixmier–Ohno invariants can be expressed in $z$. Specifically, the first two are:

$$(i_1, i_2) = \left( \frac{(2z - 9)(2z + 9)}{4z^2}, \frac{(2z + 9)(8z^2 - 24z + 459)}{24z^3} \right),$$

Reciprocally, we find an expression for $z$ in terms of the first invariants

$$z = (-153i_1 - 171)/(26i_1 - 12i_2 + 38),$$
so that \( z \) generates the function field of \( Z_1 \). The remaining invariants can be expressed in terms of the invariant \( z \):

\[
(i_3, i_4, i_5, i_6) = \left( \frac{(2z + 9)(4z - 27)(8z + 9)^2}{2^7 z^4}, \frac{(z + 9)^2(2z - 45)(2z + 9)^2}{4z^5}, \frac{(z + 9)(2z + 9)^2(8z + 9)(8z^2 - 129z + 837)}{2^3 z^6}, \frac{(2z - 7)^3}{2^1 z^9} \right),
\]

including \((j_1, \ldots, j_6)\):

\[
\left( \frac{(2z + 9)(2z^2 - 11z - 9)}{4z^3}, \frac{(2z + 9)(4z^3 - 16z^2 - 99z - 1215)}{2^3 z^4}, \frac{(2z + 9)^2(z - 6)}{2z^4}, \frac{(2z + 9)^2(2z - 15)^2(2z^2 - 3z + 27)}{2^3 z^6}, \frac{(2z + 9)^2(56z^5 - 748z^4 + 1122z^3 + 20907z^2 - 38880z - 374706)}{16z^7} \right).
\]

As in the case of \( Z_4 \), the moduli space \( Z_1 \) is defined over \( \mathbb{Q} \). Given the invariant \( z \) for a point in \( Z_1 \), we can find a curve \( C \) which is defined in terms of a root \( t \) of the degree four polynomial:

\[
(2T^4 - T^3 + 12T^2 - T + 2) - 2z(T^3 + T),
\]

defining a cyclic cover of degree 4 over \( Z_1 \).

### 6 Zero Dimensional Strata

It remains to classify the strata of dimension zero in terms of moduli, which we denote \( \Theta, \Pi_1, \Pi_2, \Sigma, \Omega_1, \Omega_2, \Phi, \) and \( \Psi \). In the case of \( Z_1 \) and \( Z_4 \), the known models for curves in these families were not obviously definable over their field of moduli. With the exception of \( \Theta \) and \( \Sigma \) below, we will see that these exceptional strata have a curve which may be defined over its field of moduli.

**Stratum \( \Theta \).**

The moduli space \( \Theta \) is a zero-dimensional subspace of \( M_3 \) which is represented by the curve \( C_\Theta \) of the form

\[
637(X^2 - YZ)^2 = (2X - Y - Z)(aX + bY + cZ)YZ
\]

where

\[
\begin{align*}
a &= (-132i - 240)t^2 + (702i + 1068)t - 219i + 59 \\
b &= (-88i - 160)t^2 + (468i + 712)t - 146i - 810 \\
c &= (108i + 428)t^2 + (-806i - 2032)t + 295i + 415
\end{align*}
\]
with \( i^2 = -1 \) and \( t^3 = ((10 + i)t/2 - 1)(t - 1) \), represented by the Dixmier invariants

\[
(i_1, i_2, i_3, i_4, i_5, i_6) = \left( -\frac{3^3}{2^472^2}, \frac{3^2173}{2^472^2}, -\frac{3^5149}{2^772}, \frac{3^510223}{2^774}, \frac{3^55^23527}{2^974}, 11^413^2 \right).
\]

and the point in \( M_3 \) is completely determined by the \( i_k \) and the additional invariants of Ohno:

\[
(j_1, j_2, j_3, j_4, j_5, j_6) = \left( -\frac{3^311}{2^377}, \frac{3^24817}{2^773}, \frac{3^3}{2 \cdot 7^2}, \frac{3^4535}{2^573}, \frac{3^455291}{2^774}, -\frac{3^25486023}{2^975} \right).
\]

**Strata \( II_1, II_2 \).**

The strata \( II_i \) are zero-dimensional subspaces of \( M_3 \) which are represented by the curves \( C_{II_i} \) of the form

\[
49(X^2 - YZ)^2 = YZ(2X - Y - Z)(aX + b(Y + Z))
\]

where

\[
a = (-52i + 46)t^2 + (49i + 25)t - 82i - 114 \\
b = (5i - 37)t^2 + (-28i + 19)t + 83i - 6,
\]

with \( i^2 = -1 \) and \( t \) satisfying \( 2t^3 = -(i+1)^2t^2 + (i+1) \). The ideal of relations for the absolute Dixmier invariants \((i_1, i_2, i_3, i_4, i_5, i_6)\) for \( II = II_1 \cup II_2 \cup II_3 \), is the degree three ideal

\[
(614656i_1^2 + 21952i_2^2 - 231516i_1 - 62613,
-2^53^22297i_2 + 15135904i_1^2 - 681236i_1 + 418761,
-2^72297i_3 + 7519344i_1^2 - 7084828i_1 - 3230271,
2^22297i_4 + 14936160i_1^2 + 20448508i_1 + 5686083,
-2^77^22297i_5 + 12234260416i_1^2 + 10161115868i_1 + 1386276669,
-2^{21}3^37^32297i_6 + 87127555902240i_1^2 - 19953560617372i_1 - 29171717887351)
\]

We note, in addition, that the curves \( C_{II_i} \) admit an automorphism \( \sigma(X,Y,Z) = (X,Z,Y) \), which induces a quotient to a non-CM elliptic curve \( E_{II_i} \), whose \( j \)-invariant satisfies

\[
j^3 - \frac{3907322953}{3^77^4} j^2 + \frac{429408710168}{3^77^4} j - \frac{126488474356752}{7^4} = 0.
\]

**Stratum \( \Sigma \).**

The stratum \( \Sigma \) is represented by the curve

\[
X^4 + Y^4 + 6\sqrt{-7}X^2Y^2 - 3(-1 + \sqrt{-7})XYZ^2 = (7 + 3\sqrt{-7})/8Z^4.
\]
with absolute Dixmier invariants
\((i_1, i_2, i_3, i_4, i_5, i_6) = \left( \frac{3^3}{2^7}, \frac{1557}{2^7}, \frac{18225}{2^7}, \frac{28403}{2^7}, \frac{2419065}{2^7}, \frac{1}{21331877} \right)\),
and absolute Ohno invariants
\((j_1, j_2, j_3, j_4, j_5, j_6) = \left( -\frac{159}{56}, \frac{3249}{112}, 9, \frac{14445}{224}, \frac{166617}{896}, -\frac{2076561}{1568} \right)\).

**Strata \(\Omega_1, \Omega_2\).**

These strata consists of two curves
\[(X^2 - YZ)^2 = (3 \pm \sqrt{7})(2X - Y - Z)(X - Y - Z)YZ.\]

In terms of the absolute Dixmier invariants \((i_1, i_2, i_3, i_4, i_5, i_6)\), the ideal of relations for \(\Omega_1 \cup \Omega_2\) is the degree two ideal
\[(64 i_2^2 + 64 i_1 + 9, 1864 i_1 + 64 i_2 - 153, 512 i_3 + 66416 i_1 + 15435, 32 i_4 + 28504 i_1 + 16695, 64 i_5 + 383138 i_1 + 37737, 1624959306694656 i_6 + 34973684392 i_1 + 5920507885).\]

**Stratum \(\Phi\).**

The stratum \(\Phi\) is represented by the Fermat quartic
\[X^4 + Y^4 + Z^4 = 0,\]
with absolute Dixmier invariants
\[(i_1, i_2, i_3, i_4, i_5, i_6) = (0, 0, 0, 0, -2^{43}3^{18}),\]
and absolute Ohno invariants
\[(j_1, j_2, j_3, j_4, j_5, j_6) = (0, 0, 0, 0, 0).\]

**Stratum \(\Psi\).**

The stratum \(\Psi\) is represented by the quartic
\[X^4 + Y^4 + Z^4 + 3(X^2Z^2 + X^2Y^2 + Y^2Z^2)\]
with absolute Dixmier invariants
\[(i_1, i_2, i_3, i_4, i_5, i_6) = \left( \frac{9}{16}, \frac{3^372}{2^7}, \frac{3^473}{2^7}, \frac{3^573}{2^7}, \frac{3^674}{2^7}, -\frac{5^6}{2^{26}3^{18}} \right),\]
and absolute Ohno invariants
\[(j_1, j_2, j_3, j_4, j_5, j_6) = \left( \frac{63}{25}, \frac{2457}{2^7}, \frac{9}{2}, \frac{3969}{2^7}, \frac{177957}{2^{12}}, \frac{606879}{2^{10}} \right).\]
References

1. J. Dixmier, On the projective invariants of quartic plane curves, Adv. in Math., 64 (1987), no. 3, 279–304.
2. J. Harris, Galois groups of enumerative problems, Duke Math. J., 46 (1979), 685–724.
3. F. Hess, An algorithm for computing Weierstrass points, Algorithmic number theory (Sydney, 2002), 357–371, Lecture Notes in Comput. Sci., 2369, Springer, Berlin, 2002.
4. F. Hess, An algorithm for computing isomorphisms of algebraic function fields, Algorithmic number theory, 263–271, Lecture Notes in Comput. Sci., 3076, Springer, Berlin, 2004.
5. J. Igusa, Arithmetic variety of moduli for genus two, Ann. of Math. (2), 72 (1960), 612–649.
6. E. Lugert, Weierstrapunkte kompakter Riemannscher Flächen vom Geschlecht 3, Ph.D. thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 1981.
7. J.-F. Mestre, Construction de courbes de genre 2 à partir de leurs modules, Effective methods in algebraic geometry (Castiglioncello, 1990), 313–334, Progr. Math., 94, Birkhäuser Boston, 1991.
8. T. Ohno, Invariant subring of ternary quartics I – generators and relations, preprint.
9. P. J. Olver, Classical invariant theory, London Mathematical Society Student Texts, 44, Cambridge University Press, Cambridge, 1999.
10. B. Poonen, E. Schaefer, and M. Stoll, Twists of X(7) and primitive solutions to $x^2 + y^3 = z^7$, preprint, 2005.
11. G. Salmon, A treatise on the higher plane curves, 3rd ed., 1879; reprinted by Chelsea, New York, 1960.
12. T. Shioda, On the graded ring of invariants of binary octavics, Amer. J. Math., 89 (1967), 1022–1046.
13. B. Sturmfels, Algorithms in invariant theory, Texts and Monographs in Symbolic Computation, Springer-Verlag, Vienna, 1993.
14. A. M. Vermeulen. Weierstrass points of weight two on curves of genus three. PhD thesis, Universiteit van Amsterdam, 1983.