q-Witt Algebras, q-Virasoro algebra, q-Lie Algebras,
q-Holomorph Structure and Representations

Naihong Hu
Department of Mathematics, East China Normal University
Shanghai 200062, China
E-mail: nhhu@math.ecnu.edu.cn

Abstract. For $q$ generic or $q = \varepsilon$ a primitive $l$-th root of 1, $q$-Witt algebras are described by means of $q$-divided power algebras. The structure of the universal $q$-central extension of the $q$-Witt algebra, the $q$-Virasoro algebra $\text{Vir}^q$, is also determined. $q$-Lie algebras are investigated and the $q$-PBW theorem for the universal enveloping algebras of $q$-Lie algebras is proved. A realization of a class of representations of the $q$-Witt algebras is given. Based on it, the $q$-holomorph structure for the $q$-Witt algebras is constructed, which interprets the realization in the context of representation theory.

1991 Mathematics Subject Classification: 17B37
Keywords: $q$-divided power algebra, $q$-Witt algebra, $q$-Virasoro algebra, $q$-Lie algebra, $q$-holomorph structure, representation

1 Introduction and Notation

Much of the recent work in quantum group theory has focused on the investigation of the structure and representations of the quantum groups $R_q[G]$ for the semisimple Lie groups $G$ or the quantized universal enveloping algebras $U_q(g)$ for the classical semisimple (or affine) Lie algebras $g$. As well-known, especially for the primitive root of unity case, this leads to an interesting representation theory mirroring in a remarkable fashion the prime characteristic case for both algebraic groups and classical Lie algebras (cf.[5], etc.). In the discussion, $q$-integers and $q$-binomial numbers play important roles. From the viewpoint of Lie theory, however, the attention to quantum groups has been paid mainly on those Lie algebras associated to (generalized) Cartan matrices. The interest here is to wonder how certain “quantum phenomenon” might be related to those Lie algebras of Cartan type. We begin with such an attempt for a simple situation of Witt type.

As mentioned above, $q$-integers is of basic importance in quantum phenomenon. The Jackson differential operator as skew derivation was used by Kassel ([9]) and other authors. It actually makes a close connection between the Jackson differential operator and $q$-integers. As known in the modular Lie algebra theory, the divided power structure over a commutative algebra plays a crucial role in the structure and representation theory of Lie algebras of Cartan type ([14], [11–13]). So we hope that there exists a similar divided power structure reflecting the structure nature of the quantum version of the Witt algebras. Starting from Jackson operators, we introduce a $q$-divided power structure in terms of $q$-binomial numbers such that it is compatible

The work was supported in part by the National Natural Science Foundation of China (No: 19731004) and the Alexander von Humboldt Foundation.
with the characterization of skew derivations. Thereby, a natural development can be proceeded by following the approach to modular Lie algebras of Cartan type. As in the modular case, such a divided power algebra structure not only describes intrinsically the structure of \( q \)-Witt algebras, but plays a nice role in a realization of a class of representations as well. A further insight into the realization leads to the construction of the \( q \)-holomorph structure, which interprets the intrinsic characteristic of the realization in representation theory, so that more representations (not necessarily graded) can be obtained by it.

The paper is organized as follows. Section 2 is about a description of quantum version of the Witt algebras. A weighted \( q \)-Jacobi identity under a \( q \)-Lie product also holds for them. \( q \)-Holomorph structure for \( q \)-Witt algebras, which still obeys the weighted \( q \)-Jacobi identity, is constructed. These examples allow us to extract a general algebraic object, \( q \)-Lie algebras, in section 3. The \( q \)-PBW theorem for their universal enveloping algebras is also valid. As an application of the weighted \( q \)-Jacobi identity, the structure of the universal \( q \)-central extension of the \( q \)-Witt algebra, the \( q \)-Virasoro algebra \( \text{Vir}_q \), is also determined. In section 4, the behavior of the universal enveloping algebra of the \( \varepsilon \)-Witt algebra \( \mathcal{W}^\varepsilon(1, 1) \) over its central subalgebra is understood and, a conjecture of de Concini and Procesi is connected with our case. Section 5 concerns with a realization of a class of representations for \( \mathcal{W}^\varepsilon(1, 1) \), which is established over a tensor vector space \( \mathfrak{A}^\varepsilon(1, 1) \otimes_K V \). By virtue of it, the irreducible modules are easily described. An analogous realization for the \( q \)-Witt algebra \( \mathcal{W}^q(1) \) can be transferred automatically to a tensor vector space \( \mathfrak{A}^q(1) \otimes_K V \). A more general nature for such a realization is further revealed in section 6 by taking advantage of the \( q \)-holomorph structure \( \mathcal{H}^\varepsilon(1) \) of the \( \varepsilon \)-Witt algebra \( \mathcal{W}^\varepsilon(1, 1) \) with its associated quantum base \( l \)-space \( 1_K^l, 1 \).

**Notation.** Let \( K \) be an arbitrary field, \( \text{char}(K) \neq 2, 3 \), and \( q \in K, q \neq 0, 1 \). For \( n, r \in \mathbb{Z}^+ (n \geq r) \), set

\[
(n)_q = \frac{1 - q^n}{1 - q}, \quad (n)_q! = (1)_q(2)_q \cdots (n)_q, \quad \left(\frac{n}{r}\right)_q = \frac{(n)_q!}{(n-r)_q! r_q!}.
\]

Then \((-n)_q := -q^{-n}(n)_q\). We make the convention: \((0)_q! = 1\), \((n)_0! = 1\) and \((n)_r! = 0\) for \( n < r \). It is well known that the \( q \)-binomial number \( \left(\frac{n}{r}\right)_q \) is a Gaussian polynomial in the variable \( q \). Therefore, \( \left(\frac{n}{r}\right)_q \) is well defined for all \( q \in K \). Notice also that if \( q = \varepsilon \) is a primitive root of unity of order \( l \) (\( l \in \mathbb{N} \)), then \((kl)_\varepsilon = 0\) and \((l)_\varepsilon = 0\), for \( k \in \mathbb{Z}, k \neq 0 \) and \( 0 < i < l \).

**2 Quantum Version of the Witt Algebras**

**\( q \)-Witt algebra \( \mathcal{W}^q \).** Recall that the Witt algebra \( \mathcal{W} = \text{Der} (\mathbb{C}[x, x^{-1}]) \) consists of derivations of the algebra \( \mathbb{C}[x, x^{-1}] \) with Lie product

\[
[x^{i+1} \partial, x^{j+1} \partial] = x^{i+1} \partial (x^{j+1} \partial) - x^{j+1} \partial (x^{i+1} \partial) = (j - i)x^{i+j+1} \partial,
\]

where \( \partial = d/dx \), \( i, j \in \mathbb{Z} \). It is well-known that \( \mathcal{W}(1) = \{ x^{i+1} \partial \mid i \in \mathbb{Z}, i \geq -1 \} \) is the Witt subalgebra of Cartan type in \( \mathcal{W} \).

Let \( K \) be an arbitrary field, \( \text{char}(K) \neq 2, 3 \), and \( q \in K, q \neq 0, 1 \) be generic. Following [9], we define Jackson’s \( q \)-differential operator \( \partial_q \) over \( K[x, x^{-1}] \) by

\[
\partial_q (\mathcal{P}) = \frac{\mathcal{P}(qx) - \mathcal{P}(x)}{qx - x}, \quad \forall \mathcal{P} \in K[x, x^{-1}].
\]
Let $\tau_q$ denote an algebra automorphism of $K[x, x^{-1}]$ defined by $\tau_q(x) = qx$. The $q$-differential operator $\partial_q$, which is a linear mapping over $K$, is a $\tau_q$-derivation or skew derivation (cf. [5], [9]), namely for all $P, Q \in K[x, x^{-1}]$, we have

$$\partial_q(PQ) = \partial_q(P)Q + \tau_q(P)\partial_q(Q).$$

(3)

Let $\text{Der}_q(K[x, x^{-1}])$ denote the set of all $\tau_q$-derivations over $K[x, x^{-1}]$, and let $e_n := x^{n+1}\partial_q$, then we have

**Lemma 2.1.** (i) $\text{Der}_q(K[x, x^{-1}])$ is a free $K[x, x^{-1}]$-module of rank 1 with $\partial_q$ as its base over $K[x, x^{-1}]$, where $\partial_q(x^n) = (n)_qx^{n-1}$.

(ii) $\text{Der}_q(K[x, x^{-1}])$ is a vector space over $K$ with a basis $\{e_n \mid n \in \mathbb{Z}\}$.

(iii) If we define a $q$-bracket product $\{,\}_q$ on $\text{Der}_q(K[x, x^{-1}])$ by

$$\{e_i, e_j\}_q(x^n) := q^{i+1}e_i(e_j(x^n)) - q^{j+1}e_j(e_i(x^n)), \quad (4)$$

then we have

$$\{e_i, e_j\}_q = -\{e_j, e_i\}_q, \quad \{e_i, e_j\}_q = q^{i+1}\partial_q(x^{j+1})\partial_q - x^{j+1}\partial_q(x^{i+1})\partial_q$$

$$= [(j + 1)_q - (i + 1)_q]e_{i+j}, \quad (i \in \mathbb{Z})$$

(5)

and then the $q$-bracket product $\{,\}_q$ is bilinear over $K$ and satisfies the antisymmetry and the weighted $q$-Jacobi identities:

$$\{e_i, e_j\}_q = -\{e_j, e_i\}_q, \quad \{e_i, e_j\}_q = q^{i+1}\partial_q(x^{j+1})\partial_q - x^{j+1}\partial_q(x^{i+1})\partial_q$$

(6)

$$= 0. \quad (7)$$

Proof. (i) The first claim of (i) is clear since the ring $K[x, x^{-1}]$ is commutative and for any $P \in K[x, x^{-1}]$, $P\partial_q$ also enjoys the relation (3). According to (2), we have

$$\partial_q(x^n) = \frac{(qx)^n - x^n}{qx - x} = (n)_qx^{n-1}. \quad (8)$$

(ii) is obvious due to (i).

(iii) By (i), we have

$$\{e_i, e_j\}_q(x^n)$$

$$= (n)_q[q^{i+1}(n + j)_q - q^{j+1}(n + i)_q]x^{n+i+j}$$

$$= [(j + 1)_q - (i + 1)_q]e_{i+j}(x^n)$$

$$= [x^{i+1}\partial_q(x^{j+1})\partial_q - x^{j+1}\partial_q(x^{i+1})\partial_q](x^n).$$

The bilinearity of the $q$-bracket $\{,\}_q$ follows from the linearity of $\partial_q$, as does the usual Witt algebra. The antisymmetry (6) is clear according to its definition (5). The weighted $q$-Jacobi identity (7) is obtained from the following identity and a cyclic permutation of $(i, j, k)$.

$$\{e_i, e_j, e_k\}_q = (1 + q)[(i + k)_q - (j + 1)_q][(j + k)_q - (i + 1)_q]e_{i+j+k}$$

$$= \frac{q^2}{1 - q}[(i+k)_q + (2i+k)_q + (2j+k)_q + (2j+k)_q]$$

$$- (i+j)_q - (2i+j)_q - (2j+k)_q - (i+j+2k)_q e_{i+j+k}. \quad (9)$$


Thus we complete the proof. □

Remark. The definition (5) of \( q \)-bracket product \( \{\cdot,\cdot\}_q \) is quite similar to (1), whose bilinearity is naturally ensured. Let \( \mathcal{W}^q = (\text{Der}_q(\mathcal{K}[x,x^{-1}]),\{\cdot,\cdot\}_q) \). Obviously, when \( \mathcal{K} = \mathbb{C} \), the ordinary Witt algebra \( \mathcal{W} = \text{Der}(\mathbb{C}[x,x^{-1}]) \) is the limit of \( \mathcal{W}^q \) when \( q \) tends to 1. \( \mathcal{W}^q \) is referred to as the \( q \)-Witt algebra, which is a \( q \)-Lie algebra with \( q \)-Lie product \( \{\cdot,\cdot\}_q \) in the sense of Definitions 3.1.

\( q \)-Witt algebra \( \mathcal{W}^q(1) \). Let \( q \in \mathcal{K} \), \( q \neq 0,1 \) and be generic. Associated to the Witt subalgebra \( \mathcal{W}(1) \) of Cartan type in \( \mathcal{W} \), an intrinsic description of \( q \)-Witt algebra \( \mathcal{W}^q(1) \) can be given by introducing a so-called \( q \)-divided power structure in terms of \( q \)-binomial numbers. An advantage is that the “skew” nature of skew derivations is well-revealed over it (compare (10) with (12)).

**Definition 2.2.** An algebra \( \mathcal{A}^q(1) \) with the generators \( \{x^{(a)} \mid a \in \mathbb{Z}^+\} \) over \( \mathcal{K} \) is called a \( q \)-divided power algebra, if it has the multiplication

\[
x^{(a)} x^{(b)} = \binom{a+b}{a}_q x^{(a+b)}, \quad \forall a, b \in \mathbb{Z}^+.
\]

Similar to the modular case (cf. [14]), we define a special \( \tau_q \)-derivative of the \( q \)-divided power algebra \( \mathcal{A}^q(1) \) by:

\[
\partial_q(x^{(a)}) = x^{(a-1)},
\]

where \( \tau_q \) is an automorphism of \( \mathcal{A}^q(1) \) defined by \( \tau_q(x^{(a)}) := q^a x^{(a)} \). Denote \( \text{Der}_q(\mathcal{A}^q(1)) \) by the set of all the special \( \tau_q \)-derivatives over \( \mathcal{A}^q(1) \), set \( e_{(n)} := x^{(n+1)} \partial_q \). Similar to Lemma 2.1, we have

**Corollary 2.3.**

(i) \( \partial_q \) is a special \( \tau_q \)-derivative of the \( q \)-divided power algebra \( \mathcal{A}^q(1) \), i.e.,

\[
\partial_q(x^{(a)} x^{(b)}) = \partial_q(x^{(a)}) x^{(b)} + \tau_q(x^{(a)}) \partial_q(x^{(b)}).
\]

(ii) \( \text{Der}_q(\mathcal{A}^q(1)) \) is a vector space over \( \mathcal{K} \) with a basis \( \{e_{(i)} \mid i \geq -1\} \).

(iii) The \( q \)-Lie product \( \{\cdot,\cdot\}_q \) on \( \text{Der}_q(\mathcal{A}^q(1)) \) defined as

\[
\{e_{(i)}, e_{(j)}\}_q := x^{(i+1)} \partial_q(x^{(j+1)}) \partial_q - x^{(j+1)} \partial_q(x^{(i+1)}) \partial_q
\]

\[
= \binom{i+j+1}{i+1}_q - \binom{i+j+1}{j+1}_q e_{(i+j)}, \quad (\forall i, j \geq -1)
\]

is bilinear and satisfies the antisymmetry and the weighted \( q \)-Jacobi identities.

**Proof.** (10) follows from the operation property of \( q \)-integers and \( q \)-binomial numbers:

\[
(a)_q + q^a(b)_q = (a+b)_q,
\]

\[
\binom{a+b-1}{a-1}_q + q^a \binom{a+b-1}{a}_q = \binom{a+b}{a}_q.
\]

(ii) \& (iii) are clear from Lemma 2.1. □

Denote \( \mathcal{W}^q(1) := (\text{Der}_q(\mathcal{A}^q(1)),\{\cdot,\cdot\}_q) \). It is a \( q \)-Lie algebra (in the sense of Definitions 3.1).

**Remark.** Let \( q \) be a non-root of unity. Let \( e_n := (n+1)_q e_{(n)} \), then it follows from Corollary 2.3 (iii) that \( \mathcal{W}^q(1) \) is a \( q \)-Witt subalgebra of \( \mathcal{W}^q \). In particular, for the \( q \)-divided power algebra
In this case, there is a natural interpretation, i.e. \( x^{(a)} := \frac{1}{(a)!} x^a \), \( \forall a \in \mathbb{Z}^+ \). Since \( \partial_q(x^a) = (a)_q x^{a-1} \), the defining relation (9) is naturally satisfied. In the following, the divided power structure is crucial to the case of primitive \( l \)-th root of unity.

\( \varepsilon \)-Witt algebra \( \mathbf{W}^\varepsilon(1, 1) \). Let \( q = \varepsilon \in \mathcal{K} \) be a primitive \( l \)-th root of 1, for \( l \in \mathbb{N} \), we consider an \( \varepsilon \)-divided power subalgebra \( \mathfrak{A}^\varepsilon(1, 1) \) of \( \mathfrak{A}^\varepsilon(1) \) over \( \mathcal{K} \) as follows.

\[
\mathfrak{A}^\varepsilon(1, 1) := \langle x^{(a)} \mid 0 \leq a < l, a \in \mathbb{Z} \rangle,
\]

where \( x^{(a)}x^{(b)} = (a+b) \varepsilon x^{(a+b)} \) and \( (x^{(a)})^l = 0 \).

Similarly, we may define the special \( \tau_\varepsilon \)-derivatives \( e^{(n)} = x^{(n+1)} \partial_\varepsilon \) \( (-1 \leq n \leq l - 2) \) over the \( \varepsilon \)-divided power algebra \( \mathfrak{A}^\varepsilon(1, 1) \) and the \( \varepsilon \)-Lie product \( \{ \} \varepsilon \) of them as in Corollary 2.3. It is clear that

\[
\mathbf{W}^\varepsilon(1, 1) := \text{Der}_\varepsilon(\mathfrak{A}^\varepsilon(1, 1)) = \langle e^{(i)} \mid -1 \leq i \leq l - 2 \rangle
\]

is closed under the \( \varepsilon \)-Lie product \( \{ \} \varepsilon \), namely,

\[
\{ e^{(i)}, e^{(j)} \} \varepsilon = \begin{cases} 
\left[ (i+j+1) \varepsilon - (i+j+1) \varepsilon \right] e^{(i+j)} , & -1 \leq i + j \leq l - 2, \\
0, & \text{otherwise.}
\end{cases}
\]  

(13)

Remark. In the case of primitive \( l \)-th root of unity, the commutator relation (13) is quite analogous to that of the modular Witt algebra \( \mathbf{W}(1, 1) \) (cf. [14]). \( \mathbf{W}^\varepsilon(1, 1) \) under \( \{ \} \varepsilon \) satisfies the antisymmetry and weighted \( \varepsilon \)-Jacobi identities as in Lemma 2.1. \( \mathbf{W}^\varepsilon(1, 1) \) is called an \( \varepsilon \)-Witt algebra over \( \mathcal{K} \). In particular, \( \mathbf{W}^\varepsilon(1, 1) \) is an \( l \)-dimensional \( \varepsilon \)-Witt subalgebra of \( \mathbf{W}^\varepsilon(1) \) (in the sense of Definitions 3.1), which is similar to the modular case (cf. [14]).

Recall the notion of quantum space introduced by Manin ([10]). Let \( q = (q_{ij}) \) be an \( n \times n \) matrix with the entries in \( \mathcal{K}^* \) and \( q_{ii} = 1 \), \( q_{ij}^{-1} = q_{ji} \) \( (i \neq j) \). Let \( \mathcal{K}\{L_1, \cdots, L_n\} \) denote the free algebra generated by \( L_1, \cdots, L_n \) and let \( \mathcal{I}_q \) be the ideal of \( \mathcal{K}\{L_1, \cdots, L_n\} \) generated by the elements \( L_iL_j - q_{ij}L_jL_i \). A quantum \( n \)-space is defined as its quotient-algebra (with the same symbols \( L_i \) writting for the images under the natural homomorphism)

\[
\mathcal{K}^n_q := \mathcal{K}_q[L_1, \cdots, L_n] = \mathcal{K}\{L_1, \cdots, L_n\}/\mathcal{I}_q.
\]

The quantum \( n \)-space \( \mathcal{K}^n_q \) is too big for our purpose, we need only its quantum base space \( 1\mathcal{K}^n_q := \langle L_i \mid 1 \leq i \leq n \rangle \) satisfying the relations \( L_iL_j = q_{ij}L_jL_i \).

The following two special quantum spaces will be needed later.

Example 2.4 (Quantum \( \infty \)-space \( \mathcal{K}_q^\infty \)). Take a \( q \)-matrix \( (q^{j-i})_{i,j \in \mathbb{Z}^+} \), where \( q \in \mathcal{K}^* \) is a non-root of unity. Denote by \( \mathcal{K}^\infty_q = \mathcal{K}_q[L_0, L_1, \cdots] \) the \( \infty \)-quantum space with the relations \( L_iL_j = q^{j-i}L_jL_i \) \( (i, j \in \mathbb{Z}^+) \).

Example 2.5 (Quantum \( l \)-space \( \mathcal{K}_q^l \)). Let \( q = \varepsilon \) be a primitive \( l \)-th root of unity. Take an \( \varepsilon \)-matrix \( (\varepsilon^{j-i})_{i,j \in \mathbb{Z}(l)} \), where \( \mathbb{Z}(l) = \mathbb{Z}/(l) \). The quantum \( l \)-space \( \mathcal{K}_e^l \) is denoted by \( \mathcal{K}_e^l = \mathcal{K}_e[L_0, L_1, \cdots, L_{l-1}] \) with the relations \( L_iL_j = \varepsilon^{l-1-i}L_jL_i \) \( (0 \leq i, j \leq l - 1) \).

\( q \)-Holomorphs. For the \( q \)-Witt algebra \( \mathbf{W}^q(1) \) and the \( \varepsilon \)-Witt algebra \( \mathbf{W}^\varepsilon(1, 1) \), we construct their \( q \)-holomorph structure, which is a kind of \( q \)-extensions of \( \mathbf{W}^q(1) \) and \( \mathbf{W}^\varepsilon(1, 1) \) through the associated quantum base spaces respectively (see Definitions 3.1 (iv)). Such objects look like the "holomorph" structure in Lie algebra theory, which play a special role in representations of \( \mathbf{W}^q(1) \) and \( \mathbf{W}^\varepsilon(1, 1) \) (see section 6).
Note that the quantum base space \( 1\mathcal{K}_q^\infty \) or \( 1\mathcal{K}_\varepsilon^l \) is \( q \)- or \( \varepsilon \)-abelian respectively, namely,

\[
\{ 1\mathcal{K}_q^\infty, 1\mathcal{K}_q^\infty \}_q = 0, \quad \{ 1\mathcal{K}_\varepsilon^l, 1\mathcal{K}_\varepsilon^l \}_\varepsilon = 0.
\]

Denote

\[
\mathcal{H}^q := 1\mathcal{K}_q^\infty \bigoplus W^q(1),
\]

\[
\mathcal{H}^\varepsilon(1) := 1\mathcal{K}_\varepsilon^l \bigoplus W^\varepsilon(1, 1).
\]  

(14)

Define

\[
\{ e_{(i)}, L_j \}_q = q^{i+1} \binom{i+j}{i+1}_q L_{i+j}, \quad (i \geq -1, j \geq 0, i, j \in \mathbb{Z}),
\]

\[
\{ e_{(i)}, L_j \}_\varepsilon = \varepsilon^{i+1} \binom{i+j}{i+1}_\varepsilon L_{i+j}, \quad (-1 \leq i \leq l - 2, 0 \leq j \leq l - 1).
\]  

(15)

**Theorem 2.6.** The direct sum \( \mathcal{H}^\varepsilon(1) \) (resp. \( \mathcal{H}^q \)) forms a new \( \varepsilon \)-Lie algebra (resp. \( q \)-Lie algebra) under the \( \varepsilon \)-Lie product \( \{ \cdot, \cdot \}_\varepsilon \) (resp. the \( q \)-Lie product \( \{ \cdot, \cdot \}_q \)), called the \( q \)-holomorphic structure of \( \mathcal{H}^\varepsilon(1) \) (resp. \( \mathcal{H}^q \)). In particular, \( L_0 \) is a \( \varepsilon \)-central (resp. \( q \)-central) element in \( \mathcal{H}^\varepsilon(1) \) (resp. \( \mathcal{H}^q \)), i.e. \( \{ L_0, \mathcal{H}^\varepsilon(1) \}_\varepsilon = 0 \) (resp. \( \{ L_0, \mathcal{H}^q \}_q = 0 \)).

**Proof.** For the \( \varepsilon \)-Witt algebra \( W^\varepsilon(1, 1) \) and its associated quantum base \( l \)-space \( 1\mathcal{K}_\varepsilon^l \), we need only to show that \( \mathcal{H}^\varepsilon(1) \) is still an \( \varepsilon \)-Lie algebra in the sense of Definitions 3.1.

The antisymmetry of \( \{ \cdot, \cdot \}_\varepsilon \) for \( \mathcal{H}^\varepsilon(1) \) is obvious.

Note that \( \{ 1\mathcal{K}_\varepsilon^l, 1\mathcal{K}_\varepsilon^l \}_\varepsilon = 0 \). It suffices to verify that the weighted \( \varepsilon \)-Jacobi identity holds for the elements \( e_{(i)}, e_{(j)}, L_k \).

From the following identities

\begin{align*}
(2)_\varepsilon & \{ e_{(i)}, \{ e_{(j)}, L_k \}_\varepsilon \}_\varepsilon \\
&= \frac{\varepsilon^{i+j+2}(i+j+k)}{(i+1)\varepsilon!(j+1)\varepsilon!(k-1)\varepsilon!} (1 + \varepsilon^i)(1 + \varepsilon^j) L_{i+j+k}, \\
(2)_\varepsilon & \{ e_{(j)}, \{ L_k, e_{(i)} \}_\varepsilon \}_\varepsilon \\
&= -\frac{\varepsilon^{i+j+2}(i+j+k)}{(i+1)\varepsilon!(j+1)\varepsilon!(k-1)\varepsilon!} (1 + \varepsilon^j)(1 + \varepsilon^i) L_{i+j+k}, \\
(2)_\varepsilon & \{ L_k, \{ e_{(i)}, e_{(j)} \}_\varepsilon \}_\varepsilon \\
&= (1 + \varepsilon^k) \left[ \left( \binom{i+j+1}{j+1} \varepsilon \right) - \left( \binom{i+j+1}{i+1} \varepsilon \right) \right] \varepsilon^{i+j+1} \binom{i+j+k}{i+j+1} L_{i+j+k} \\
&= \frac{\varepsilon^{i+j+1}(i+j+k)}{(i+1)\varepsilon!(j+1)\varepsilon!(k-1)\varepsilon!} \left[ (i+1)\varepsilon - (j+1)\varepsilon \right] (1 + \varepsilon^k) L_{i+j+k},
\end{align*}

it is clear that the weighted \( \varepsilon \)-Jacobi identity holds for them.

The proof for \( \mathcal{H}^q \) is similar. \( \square \)

**Corollary 2.7.** The direct sum \( \langle L_0 \rangle \bigoplus W^q(1) \) (resp. \( \langle L_0 \rangle \bigoplus W^\varepsilon(1, 1) \)) is a split 1-dimensional \( q \)-central (resp. \( \varepsilon \)-central) extension of \( W^q \) (resp. \( W^\varepsilon(1, 1) \)). \( \square \)

**Remark.** This case suggests it might be interesting to consider the universal \( q \)-central extension of the \( q \)-Witt algebra \( W^q \). But indeed, in that case, the extension \( \langle L_0 \rangle \bigoplus W^q \) is not split any
more, just like that relationship between the usual Witt algebra \( W \) and the Virasoro algebra \( \text{Vir} \)
(see section 3).

3 q-Lie Algebras, q-PBW Theorem and q-Virasoro Algebra

q-Lie algebras. The discussion and examples in section 2 allows us to generalize and formulate a general object as follows.

Definitions 3.1. For a \( \mathbb{Z} \)-graded vector space \( \mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i \) over a field \( K \) equipped with a bilinear \( q \)-bracket product \( \{, \}_q \) (where \( q \in K, q \neq 0, 1, \dim \mathcal{L}_i < \infty \) ) satisfying \( \{ \mathcal{L}_i, \mathcal{L}_j \}_q \subseteq \mathcal{L}_{i+j} \), let \( S = \bigoplus_{i \in \mathbb{Z}} S_i \subseteq \mathcal{L} \) be a \( \mathbb{Z} \)-graded subspace of \( \mathcal{L} \).

(i) If the antisymmetry identity (6) and the weighted \( q \)-Jacobi identity (7) are fulfilled under \( \{, \}_q \) for \( x_i \in \mathcal{L}_i, \forall i \in \mathbb{Z} \), then \( \mathcal{L}^q := (\mathcal{L}, \{, \}_q) \) is called a \( q \)-Lie algebra and \( \{, \}_q \) is called the \( q \)-Lie product. In particular, \( (\mathcal{L}_0, \{, \}_q|_{\mathcal{L}_0}) \) is a usual Lie algebra and \( \{, \}_q|_{\mathcal{L}_0} \) is a usual Lie product.

(ii) If \( S \) is closed under the \( q \)-Lie product \( \{, \}_q \), then \( S^q \) is called a \( q \)-Lie subalgebra of \( \mathcal{L}^q \). Let \( S \subseteq \mathcal{L} \) be a subset of \( \mathcal{L} \), define a \( q \)-centralizer \( C_q(S) := \{ x \in \mathcal{L} \mid \{x, S\}_q = 0 \} \) of \( S \) in \( \mathcal{L} \), and a \( q \)-normalizer \( N_q(S) := \{ x \in \mathcal{L} \mid \{x, S\}_q \subseteq S \} \) of \( S \) in \( \mathcal{L} \), then both are \( q \)-Lie subalgebras, thanks to weighted \( q \)-Jacobi identities. If moreover, a \( q \)-Lie subalgebra \( S \) satisfies \( \{ S, \mathcal{L} \}_q \subseteq S \), then \( S^q \) is called a \( q \)-Lie ideal of \( \mathcal{L}^q \). In particular, if \( S, \mathcal{L} \}_q = 0 \), \( S^q \) is called a \( q \)-central subalgebra of \( \mathcal{L}^q \). If \( \mathcal{L}_0, \mathcal{L}_q = 0 \), then \( \mathcal{L}^3 \) is called \( q \)-abelian. If \( \mathcal{L}^q \) only has trivial \( q \)-Lie ideals 0 and \( \mathcal{L}^q \), then \( \mathcal{L}^q \) is said \( q \)-simple. For instance, \( W, W^q(1) \) and \( W^q(1, 1) \). If \( \mathcal{L}^q \) is a finite direct sum of \( q \)-simple ideals, then \( \mathcal{L}^q \) is \( q \)-semisimple.

(iii) For any two \( q \)-Lie algebras \( \mathcal{L}^q \) and \( \mathcal{L'}^q \), if there exists a graded vector space linear mapping \( \phi : \mathcal{L}^q \rightarrow \mathcal{L'}^q \) of degree 0 such that \( \phi(\mathcal{L}_i) \subseteq \mathcal{L}'_i \), \( \phi(\{x_i, x_j\}_q) = \{\phi(x_i), \phi(x_j)\}_q \), \( \forall x_i \in \mathcal{L}_i \) and \( x_j \in \mathcal{L}_j \), then \( \phi \) is called a \( q \)-Lie homomorphism. Then the kernel of \( \phi \) is a \( q \)-Lie ideal of \( \mathcal{L}^q \). The notions of \( q \)-Lie quotient algebra and the image of a \( q \)-Lie homomorphism also can be defined. The homomorphic fundamental theorem holds for \( q \)-Lie algebras.

(iv) Given three \( q \)-Lie algebras \( \mathcal{L}^q \), \( \mathcal{L'}^q \) and \( \overline{\mathcal{L}}^q \), if \( \mathcal{L}^q \) is a \( q \)-Lie subalgebra of \( \overline{\mathcal{L}}^q \) and there exists an exact (non-split) sequence

\[
0 \rightarrow \mathcal{L}^q \\ \rightarrow \overline{\mathcal{L}}^q \\ \rightarrow \mathcal{L}^q \\ \rightarrow 0,
\]

then the \( \overline{\mathcal{L}}^q \) is referred to as a (nontrivial) \( q \)-extension of \( \mathcal{L}^q \) through \( \mathcal{L}^q \). In particular, if \( \mathcal{L}^q \) is a \( q \)-central subalgebra of \( \overline{\mathcal{L}}^q \), then \( \overline{\mathcal{L}}^q \) is said a \( q \)-central extension of \( \mathcal{L}^q \).

(v) Let \( \mathcal{L}^q \) be a \( q \)-Lie algebra, a vector space \( \mathcal{M} \) over \( K \) is called a \( \mathbf{left} \) \( \mathcal{L}^q \)-module, if there exists a bilinear map \( \mathcal{L}^q \times \mathcal{M} \rightarrow \mathcal{M} \) such that

\[
\{x_i, x_j\}_q \cdot m = q^{i+1} x_i (x_j, m) - q^{j+1} x_j (x_i, m),
\]

for all \( x_i \in \mathcal{L}_i, x_j \in \mathcal{L}_j, m \in \mathcal{M} \). For instance, \( K[x, x^{-1}], \mathbb{A}^q(1) \) and \( \mathbb{A}^q(1, 1) \) are modules of \( W, W^q(1) \) and \( W^q(1, 1) \), respectively.

(vi) The associative algebra \( U(\mathcal{L}^q) := \mathcal{S}(\mathcal{L}^q)/\mathcal{J}_q \) is termed the (universal) enveloping algebra of \( \mathcal{L}^q \), where \( \mathcal{S}(\mathcal{L}^q) \) be the tensor algebra of \( \mathcal{L}^q \), \( \mathcal{J}_q \) be the 2-sided ideal of \( \mathcal{S}(\mathcal{L}^q) \) generated by the elements

\[
J(x_i, x_j) := q^{i+1} x_i \otimes x_j - q^{j+1} x_j \otimes x_i - \{x_i, x_j\}_q, \quad \forall x_i \in \mathcal{L}_i.
\]

The composite map \( \sigma \) of the canonical maps \( \mathcal{L}^q \rightarrow \mathcal{S}(\mathcal{L}^q) \rightarrow U(\mathcal{L}^q) \) is called the canonical map of \( \mathcal{L}^q \) into \( U(\mathcal{L}^q) \), then in \( U(\mathcal{L}^q) \), we have \( q^{i+1} \sigma(x_i) \sigma(x_j) - q^{j+1} \sigma(x_j) \sigma(x_i) = \sigma(x_i, x_j), \forall x_i, x_j \in \mathcal{L}^q \).
If $\mathcal{L}^q$ is $q$-abelian, then $\mathcal{J}_q = \mathcal{L}_q$, $U(\mathcal{L}^q)$ is a quasipolynomial algebra (cf. [5]), if moreover, dim $\mathcal{L}^q = n$, then $U(\mathcal{L}^q) = K_q^n$ is a quantum $n$-space (see section 2).

$U(\mathcal{L}^q)$ is universal in the following sense.

**Lemma 3.2.** Let $\sigma$ be the canonical map of $\mathcal{L}^q$ into $U(\mathcal{L}^q)$, let $A$ be an algebra with 1, and let $\theta$ be a linear map of $\mathcal{L}^q$ into $A$ such that

$$q^{i+1}\theta(x_i)\theta(x_j) - q^{i+1}\theta(x_j)\theta(x_i) = \theta\{x_i, x_j\}_q, \quad \forall x_i \in \mathcal{L}_i.$$ 

Then there exists a unique homomorphism $\theta'$ of $U(\mathcal{L}^q)$ into $A$ such that $\theta'(1) = 1$ and $\theta' \cdot \sigma = \theta$.

**Proof.** Since $U(\mathcal{L}^q)$ is generated by 1 and $\sigma(\mathcal{L}^q)$, $\theta'$ is unique. On the other hand, let $\phi$ be the unique homomorphism of $\mathfrak{T}(\mathcal{L}^q)$ into $A$ which extends $\theta$ with $\phi(1) = 1$. \forall x_i \in \mathcal{L}_i, we have

$$\phi(q^{i+1}x_i \otimes x_j - q^{i+1}x_j \otimes x_i - \{x_i, x_j\}_q)$$

$$= q^{i+1}\theta(x_i)\theta(x_j) - q^{i+1}\theta(x_j)\theta(x_i) - \theta\{x_i, x_j\}_q$$

$$= 0,$$

hence $\phi(\mathcal{J}_q) = 0$ and, by passage to the quotient, $\phi$ defines a homomorphism $\theta'$ of $U(\mathcal{L}^q)$ into $A$ with $\theta'(1) = 1$ and $\theta' \cdot \sigma = \theta$. □

**Remark.** The bilinearity of $q$-Lie product $\{,\}_q|\mathcal{L}^q$ in $U(\mathcal{L}^q)$ can be understood in the following sense: for any $x = \sum x_i$, $x' = \sum x'_j \in \mathcal{L}^q$, denote $\tilde{x} := \sum q^{i+1}x_i$, $\tilde{x}' := \sum q^{j+1}x'_j \in \mathcal{L}^q$, we define

$$\{x, x'\}_q := \tilde{x}x' - \tilde{x}'x,$$  

(19)

then $\{x, x'\}_q = \{\sum x_i, \sum x'_j\}_q = \sum_{i,j} \{x_i, x'_j\}_q$. So the ideal $\mathcal{J}_q$ is generated by all $J(x, y) = \tilde{y}x - \tilde{x}y - \{x, y\}_q$, $\forall x, y \in \mathcal{L}^q$.

$q$-PBW theorem. Berger ([2]) has studied a wide class of associative algebras $U$ (affine or quadratic) defined by generators and relations in which the Poincaré-Birkhoff-Witt theorem is valid. This class includes numerous recently appeared quantum algebras. Roughly speaking, our $U(\mathcal{L}^q)$ belongs to a class of affine $q$-algebras in the sense of Berger (see section 2.1 in [2]). But his proof applies only to the quadratic case (see his Definition 2.5.2 and the proof of Lemma 2.8.2). According to his remark 2.8.3, the proof of the affine case essentially follows Jacobson’s (cf. [8]). We will give a proof for $U(\mathcal{L}^q)$ by following Bergman’s idea of Diamond Lemma — reduction system ([3]).

We need some notions and notations from [3].

Fix an ordered homogeneous basis $\{x_{i_1}, x_{i_2}, \cdots\} (i_1 < i_2 < \cdots)$ of $\mathcal{L}^q$ such that $x_{i_k} \in \mathcal{L}_n$ for some $n$. Let $\{\mathcal{L}^q\}$ denote the set of monomials in $\mathfrak{T}(\mathcal{L}^q)$. Define the disordering index of a monomial $x_{j_1} \otimes \cdots \otimes x_{j_n}$ as the number of pairs $(j_i, j_k)$ such that $j_i < j_k$ but $x_{j_i} > x_{j_k}$ (cf. [8]). Thus we can partially order monomials in $\{\mathcal{L}^q\}$ by setting $\mathcal{C} \prec \mathcal{D}$ if $\mathcal{C}$ is of smaller length than $\mathcal{D}$, or if $\mathcal{C}$ is a permutation of the terms of $\mathcal{D}$ but of smaller disordering index. Clearly, this defines a semigroup partial ordering on $\{\mathcal{L}^q\}$ in the sense: $\mathcal{B} \prec \mathcal{B}'$ implies $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \prec \mathcal{A} \otimes \mathcal{B}' \otimes \mathcal{C}$ ($\mathcal{A}, \mathcal{B}, \mathcal{B}', \mathcal{C} \in \{\mathcal{L}^q\}$).

Moreover, let $S = \{\tau = (W_{\tau}, f_{\tau}) \mid W_{\tau} \in \{\mathcal{L}^q\}, f_{\tau} \in \mathfrak{T}(\mathcal{L}^q)\}$ be a reduction system of $\mathfrak{T}(\mathcal{L}^q)$. If $\forall \tau \in S$, $f_{\tau}$ is a linear combination of monomials $\prec W_{\tau}$, then the semigroup partial ordering $\prec$ is called compatible with $S$. For each $\tau \in S$, let $r_{\tau}$ denote the map on $\mathfrak{T}(\mathcal{L}^q)$ which maps each monomial $\mathcal{A} \otimes W_{\tau} \otimes \mathcal{B}$ into $\mathcal{A} \otimes f_{\tau} \otimes \mathcal{B}$ and fixes those monomials without containing subword $W_{\tau}$. Denote $r_S := \{r_{\tau} \mid \tau \in S\}$ and each $r_{\tau}$ is called a reduction of $\mathfrak{T}(\mathcal{L}_q)$. For any $\mathcal{A} \in \{\mathcal{L}^q\}$,
To further reduce the term (18), we have
\[ J(x_k, x_j, x_i) := q^{i+k} x_k, x_j q \otimes x_i - q^{2i} x_i \otimes \{ x_k, x_j \} q \]
\[ + q^{i+j} x_j, x_i q \otimes x_k - q^{2k} x_k \otimes \{ x_j, x_i \} q \]
\[ + q^{i+k} x_i, x_k q \otimes x_j - q^{2j} x_j \otimes \{ x_i, x_k \} q. \]

(20)

The following lemma is essentially due to Berger (cf. Proposition 2.4.1 [2]).

**Lemma 3.3.** The q-Jacobi sums \( J(x_k, x_j, x_i) \) is \( J_q \).

**Proof.** Note that each monomial \((J(x_k, x_j, x_i)) < x_k \otimes x_j \otimes x_i\). We have to show \( J(x_k, x_j, x_i) \in J_q \).

From (18), we have \( \{ x_i, x_j \} q = q^{i+1} x_i \otimes x_j - q^{j+1} x_j \otimes x_i - J(x_i, x_j) \), then

\[ J(x_k, x_j, x_i) = -q^{i+k} J(x_k, x_j) \otimes x_i + q^{2i} x_i \otimes J(x_k, x_j) \]
\[ - q^{i+j} J(x_j, x_i) \otimes x_k + q^{2k} x_k \otimes J(x_j, x_i) \]
\[ - q^{i+k} J(x_i, x_k) \otimes x_j + q^{2j} x_j \otimes J(x_i, x_k) \in J_q. \]

The proof is completed. \( \square \)

Denote by \( \bar{a} \) the class in \( U(L^q) \) of any element \( a \) of \( \mathcal{L}(L^q) \).

**Theorem 3.4.** Let \( \{ x_{i_1}, x_{i_2}, \ldots \} \) \((i_1 < i_2 < \cdots )\) be an ordered homogeneous basis of \( L^q \) over \( K \). Then the unit 1 and the ordered monomials \( x_{j_1} \cdots x_{j_n} \) \((j_1 \leq \cdots \leq j_n, n \geq 1)\) form a basis of \( U(L^q) \) over \( K \).

**Proof.** Take \( S = \{ \tau = \tau_{xy} : W_{\tau_{xy}} = f_{\tau_{xy}} \mid x < y, x \in L_i, y \in L_j \} \) as a reduction system of \( \mathcal{L}(L^q) \), where \( W_{\tau_{xy}} = q^{i+j} x \otimes y \in \{ L^q \}, f_{\tau_{xy}} = q^{j+i} y \otimes x - \{ y, x \} q \in \mathcal{L}(L^q) \) and the semigroup partial ordering \( < \) is compatible with \( S \). Note that the ideal generated by the differences \( W_{\tau} - f_{\tau} \) \((\tau \in S)\) is precisely \( J_q \) (since \{ \} \( q \) is antisymmetry and bilinear!). Since each reduction \( r_{\tau} \) maps each monomial \( A \otimes W_{r_{\tau}} \otimes B \) into \( A \otimes f_{r_{\tau}} \otimes B \) but fixes those monomials without containing subword \( W_{r_{\tau}} \), the images in \( U(L^q) \) of the fixed monomials under \( r_{S} \) are precisely the alleged basis.

Since each monomial of length \( n \) via certain finite number of reductions will be stable under \( r_{S} \), the above semigroup partial ordering \( < \) on \( \{ L^q \} \) satisfies the descending chain condition.

Clearly, the ambiguities of \( S \) are precisely the 5-tuples \((\tau_{x_i} x_k, \tau_{x_j} x_k, x_j, x_i)\) with \( x_i < x_j < x_k \). To see that the ambiguity is resolvable relative to \( < \), we study

\[ r_{\tau_{x_i} x_k} (q^{2k+j+1} x_k \otimes x_j \otimes x_i) - r_{\tau_{x_j} x_i} (q^{2k+j+1} x_k \otimes x_j \otimes x_i) \]
\[ = (q^{k+2j+1} x_j \otimes x_k \otimes x_i + q^{j+k} x_k \otimes \{ x_j, x_i \} q \otimes x_i) \]
\[ - (q^{2k+i+1} x_k \otimes x_i \otimes x_j + q^{2k} x_k \otimes \{ x_j, x_i \} q). \]

To further reduce the term \( q^{k+2j+1} x_j \otimes x_k \otimes x_i \), we apply first \( r_{\tau_{x_i} x_k} \), and then \( r_{\tau_{x_j} x_i} \). Similarly,
to deal with \( q^{2k+i+1} x_k \otimes x_i \otimes x_j \) we apply \( r_{\tau s_i k} \) and then \( r_{\tau s_j k} \). Thus we get

\[
\begin{align*}
(q^{2i+j+1} x_i \otimes x_j \otimes x_k + q^{i+j} \{x_j, x_i\}_q \otimes x_k \\
+ q^{2j} x_j \otimes \{x_k, x_i\}_q + q^{i+k} \{x_k, x_j\}_q \otimes x_i \\
- (q^{2i+j+1} x_i \otimes x_j \otimes x_k + q^{2i} x_i \otimes \{x_k, x_j\}_q \\
+ q^{i+k} \{x_k, x_j\}_q \otimes x_j + q^{2k} x_k \otimes \{x_j, x_i\}_q)
\end{align*}
\]

\( = J(x_k, x_j, x_i) \in J_{x_k \otimes x_j \otimes x_i}. \)

So our ambiguities are resolvable relative to \( \prec \). Hence by Theorem 1.2 of Bergman in [3], \( \mathbb{U}(\mathcal{L}^q) \) has the basis indicated.

**Corollary 3.5.** (i) The canonical map \( \sigma: \mathcal{L}^q \rightarrow \mathbb{U}(\mathcal{L}^q) \) is injective.

(ii) \( \mathbb{U}(\mathcal{L}^q) \) has no zero divisors. \( \square \)

**q-Virasoro algebra** \( \text{Vir}^q \). Here is another example of \( q \)-Lie algebras — \( q \)-Virasoro algebra. As an application of weighted \( q \)-Jacobi identity (7), we can use it to determine the structure of the universal \( q \)-central extension of the \( q \)-Witt algebra \( \mathbb{W}^q \) defined in Lemma 2.1. A standard method of computing the central extension in Lie theory (cf. [1], [7] etc.) also applies to \( q \)-Lie algebras here.

Recall that the Virasoro algebra \( \text{Vir} = \langle e_i, c \mid i \in \mathbb{Z} \rangle \) is an universal 1-dimensional central extension of the Witt algebra \( \mathbb{W} = \langle e_i \mid i \in \mathbb{Z} \rangle \) with the Lie product

\[
[e_i, e_j] = (j-i)e_{i+j} + \frac{(i-1)i(i+1)}{12} c.
\]

Define \( \text{Vir}^q = \langle e_i, L_0 \mid i \in \mathbb{Z} \rangle \) with the \( q \)-Lie product

\[
\begin{align*}
\{e_i, e_j\}_q &= 0, \\
\{e_i, e_j\}_q &= [(j+1)_q - (i+1)_q]e_{i+j} + \delta_{i+j,0} \frac{(i-1)_q(i)_q(i+1)_q}{q^1(2)_q^1(2)_q^3(3)_q} L_0.
\end{align*}
\]

**Proposition 3.6.** \( \text{Vir}^q \) is a \( q \)-Lie algebra, which is an universal 1-dimensional \( q \)-central extension of the \( q \)-Witt algebra \( \mathbb{W}^q \), i.e. there is the exact sequence

\[
0 \rightarrow \langle L_0 \rangle \rightarrow \text{Vir}^q \rightarrow \mathbb{W}^q \rightarrow 0.
\]

**Proof.** In this proof, we will derive the defining relation (22) of \( \text{Vir}^q \) using the weighted \( q \)-Jacobi identity.

Suppose that \( 0 \rightarrow \mathcal{C}^q \rightarrow \mathcal{L}^q \rightarrow \mathbb{W}^q \rightarrow 0 \) is a nonsplit \( q \)-central extension of \( \mathbb{W}^q \). Let \( \tilde{e}_i \) denote the preimages of \( e_i \). Then we have

\[
\{\tilde{e}_i, \tilde{e}_j\}_q = o(i, j)\tilde{e}_{i+j} + c(i, j),
\]

where \( o(i, j) = [(j+1)_q - (i+1)_q], c(i, j) = -c(j, i) \) and \( c(i, j) \in \mathcal{C}^q \). Since \( \{\tilde{e}_0, \tilde{e}_i\}_q = o(0, i)\tilde{e}_i + c(0, i) \). Replace \( \tilde{e}_i \) by \( \tilde{e}_i + \frac{c(0, i)}{o(0, i)} \), then we have

\[
\{\tilde{e}_0, \tilde{e}_i\}_q = o(0, i)\tilde{e}_i.
\]
For $\{\tilde{e}_1, \tilde{e}_{-1}\}_q = o(1, -1)\tilde{e}_0 + c(1, -1)$, we also can take $\tilde{e}_0$ such that $c(1, -1) = 0$, i.e.

$$\{\tilde{e}_1, \tilde{e}_{-1}\}_q = o(1, -1)\tilde{e}_0.$$ 

First, we will show that $c(i, j) = 0$, for $i + j \neq 0$ ($\forall i, j \in \mathbb{Z}$). From the following weighted $q$-Jacobi identity:

$$(2)_q\{\tilde{e}_1, \tilde{e}_{-1}\}_q = (2)_q\{\tilde{e}_0, \tilde{e}_0, \tilde{e}_0\}_q$$

we get $[(2)_{q^i} o(0, j) + (2)_{q^j} o(0, i)] c(i, j) = 2o(0, i + j) c(i, j) = 0$, i.e. $c(i, j) = 0$, if $i + j \neq 0$.

Next, we will determine $c(i, -i)$. For $r = i + j \geq 3$, using the weighted $q$-Jacobi identity:

$$(2)_q\{\tilde{e}_1, \tilde{e}_{-1}\}_q = 0\{\tilde{e}_1, \tilde{e}_{-1}\}_q$$

we obtain

$$(2)_q\{\tilde{e}_0, \tilde{e}_{0, \tilde{e}_{-1}}\}_q = 0\{\tilde{e}_0, \tilde{e}_{0, \tilde{e}_{-1}}\}_q$$

Note that $(2)_{q^{-i}} = q^{-j}(2)_{q^i}$, and set $\Delta(r) = q^r(2)_{q^r} c(r, -r)$, then (24) becomes

$$\Delta(r)(q^{-i} - q^{-j}) = \Delta(j)(q^{2i} - q^{-j}) - \Delta(i)(q^{2j} - q^{-i}).$$

Since $c(1, -1) = 0$, i.e. $\Delta(1) = 0$, take $i = 1, j = r - 1$, then we get

$$\Delta(r) = \frac{q^2 - q^{1-r}}{q^{1-r} - q^{1-r}} \Delta(r - 1) = \frac{(r + 1)_q}{(r - 2)_q} \Delta(r - 1).$$

(25)

It is easy from (25) to get

$$\Delta(i) = \frac{(i + 1)_q (i - 1)_q}{(2)_q (3)_q} \Delta(2).$$

Set $L_0 := \Delta(2)$, we obtain the required conclusion:

$$c(i, -i) = \frac{(i - 1)_q (i + 1)_q}{q^i (2)_q (2)_q (3)_q} L_0.$$ (26)

Hence, $\mathcal{C}^q = \langle L_0 \rangle$ and $\mathcal{L}^q = \text{Vir}^q$.

Finally, we shall show that the product $\{,\}_q$ defined on $\text{Vir}^q$ (cf. (21), (22)) is a $q$-Lie product.
At first, it is not hard to see that \( c(-i,i) = -c(i,-i) \) using the fact \((-i)_q = -q^{-i}(i)_q \), namely, the antisymmetry holds for the \( \{.,.\}_q \).

It remains to verify that the \( \{.,.\}_q \) satisfies the weighted Jacobi identity. In view of \( \text{Vir}^q \) being a 1-dimensional \( q \)-central extension of the \( q \)-Lie algebra \( W^q \), we only need to check it for the case when \( r = -(i+j) \neq 0 \), namely, to show the following identity:

\[
(2)_q q \{ \tilde{e}_i, \{ \tilde{e}_j, \tilde{e}_r \}_q \}_q + (2)_q q \{ \tilde{e}_j, \{ \tilde{e}_r, \tilde{e}_i \}_q \}_q + (2)_q q \{ \tilde{e}_r, \{ \tilde{e}_i, \tilde{e}_j \}_q \}_q = 0. \tag{27}
\]

Applying (22) to the left side of (27), it suffices to verify that the sum of the coefficients of \( L_0 \) in (27), \( S = S(i,j,r) + S(j,r,i) + S(r,i,j) \) is zero (since the fact that the sum of the coefficients of \( \tilde{e}_0 \) in (27) is due to the weighted Jacobi identity of \( W^q \)).

For \( k \in \mathbb{Z} \), denote \([k] := q^k + q^{-k}\), then \([-k] = [k] \). It is easy to see that the coefficient of \( L_0 \) in (27) is

\[
S(i,j,r) = \left( \frac{1}{(2)_q(3)_q} \right)^{q^{-i}(i-1)q(i+1)q((r+1)q-(j+1)q)}.
\]

Thus \( S = 0 \). This shows the \( \{.,.\}_q \) is indeed a \( q \)-Lie product and \( \text{Vir}^q \) is a \( q \)-Lie algebra.

The proof is completed. \( \Box \)

4 Algebra \( U^\varepsilon \)

From now on, our interest mainly concentrates on the situation of primitive \( l \)-th root of unity. Some of particular observations only can be carried out in this case but the results obtained in this way can be transferred to the generic case.

**Algebra \( U^\varepsilon \).** For the \( \varepsilon \)-Witt algebra \( W^\varepsilon (1,1) \) over \( \mathcal{K} \), consider its universal enveloping algebra \( U^\varepsilon := U(W^\varepsilon (1,1)) \), which contains a big central subalgebra analogous to the modular case.

Set \( H_{ij}^{(0)} = 1 \), and for \( k > 0 \),

\[
H_{ij}^{(k)} = \begin{cases} 
\varepsilon^{-k-j}{i+1 \choose j+1} \prod_{s=1}^{k} \left( (i+s+1)_{\varepsilon} - (i+s)_{\varepsilon} \right), & -1 \leq i+j \leq l-2, \\
0, & \text{otherwise}.
\end{cases}
\]

**Lemma 4.1.** For \(-1 \leq i,j \leq l-2\), we have

\begin{enumerate}
(i) \( e_{(i)} e_{(0)} = \varepsilon^{-n} (e_{(0)} - (i)_{\varepsilon}) e_{(i)} \);
(ii) \( e_{(i)} e_{(j)} = \sum_{k=0}^{n} \binom{n}{k} \varepsilon^{-n-k} e_{(j-k)_{\varepsilon}} e_{(j-k)_{\varepsilon}} \cdots \cdots e_{(0)_{\varepsilon}} e_{(0)_{\varepsilon}} e_{(0)_{\varepsilon}} \cdots \cdots e_{(0)_{\varepsilon}} e_{(0)_{\varepsilon}} \)
\]

\[
= \varepsilon^{n(j-i)} \sum_{k=0}^{n} \binom{n}{k} \varepsilon^{-n-k} H_{ij}^{(k)} e_{(j-k)_{\varepsilon}} e_{(j-k)_{\varepsilon}} e_{(j-k)_{\varepsilon}} , \quad (j \neq 0).
\]

**Proof.** (i) follows from \( e_{(i)} e_{(0)} = \varepsilon^{-l} (e_{(0)} - (i)_{\varepsilon}) e_{(i)} \), using (13).

(ii) Denote \( [e_{(i)}, e_{(j)}]_{\varepsilon} := e_{(i)_{\varepsilon}} e_{(j)} - \varepsilon^{l-i} e_{(j)_{\varepsilon}} e_{(i)}. \) Assume that the first equality of (ii) holds for \( n \). Since by (13), \( [e_{(i)}, e_{(j)}]_{\varepsilon} = * e_{(i+j)} \), then

\[
[e_{(i)}, e_{(j)}]_{\varepsilon} e_{(j)} = \varepsilon^{j-(i+k)_{\varepsilon}} e_{(j)} \cdots \cdots e_{(0)_{\varepsilon}} e_{(0)_{\varepsilon}} e_{(0)_{\varepsilon}} \cdots \cdots e_{(0)_{\varepsilon}} e_{(0)_{\varepsilon}} + \left( \cdots \cdots e_{(i)}_{\varepsilon} e_{(j)}_{\varepsilon} \right)_{\varepsilon}.
\]
and
\[
e_{(i)}e_{(j)}^{n+1} = \sum_{k=0}^{n} \binom{n}{k} \varepsilon^{(n-k)(j-i) + (i+kj)}e_{(j)}^{n+1-k}[\cdots [e_{(i)}, e_{(j)}]e_{(j)}, \cdots , e_{(j)}]e_{(j)}
\]
\[+ \sum_{k=0}^{n} \binom{n}{k} \varepsilon^{(n-k)(j-i)}e_{(j)}^{n-k}[\cdots [e_{(i)}, e_{(j)}]e_{(j)}, \cdots , e_{(j)}]e_{(j)}
\]
\[= \sum_{k=0}^{n} \varepsilon^{-kj} \binom{n}{k} \varepsilon^{(n+1-k)(j-i)}e_{(j)}^{n+1-k}[\cdots [e_{(i)}, e_{(j)}]e_{(j)}, \cdots , e_{(j)}]e_{(j)}
\]
\[+ \sum_{k=1}^{n+1} \binom{n}{k-1} \varepsilon^{(n+1-k)(j-i)}e_{(j)}^{n+1-k}[\cdots [e_{(i)}, e_{(j)}]e_{(j)}, \cdots , e_{(j)}]e_{(j)}.
\]

But
\[
\binom{n}{k-1} \varepsilon^{-kj} \binom{n}{k} \varepsilon^{-j} = \binom{n+1}{k} \varepsilon^{-j}.
\]

Hence, the first equality of (ii) holds for \(n + 1\).

The second equality of (ii) is obvious due to (13). \(\square\)

**Corollary 4.2.** For \(-1 \leq i, j \leq l - 2\), we have

(i) \([e_{(i)}, (e_{(0)} - \frac{1}{1 - \varepsilon})] = 0;\)

(ii) \([e_{(i)}, e_{(j)}] = 0 \quad \text{for} \quad j \neq 0.\)

**Proof.** (i) Set \(\tilde{e}_{(0)} := e_{(0)} - \frac{1}{1 - \varepsilon}\), since \(e_{(0)}e_{(i)} = \varepsilon^{i}e_{(i)}e_{(0)} + (i)e_{(i)}\), by (13), then \(\tilde{e}_{(0)}, e_{(i)} = \varepsilon^{i}e_{(i)}\).

(ii) is clear by virtue of Lemma 4.1 (ii) and the facts that \(e_{(i+j)} = 0 \quad \text{for} \quad -1 \leq i \leq l - 2 \quad \text{and} \quad (\frac{1}{l})e_{-i} = 0 \quad \text{for} \quad 0 < k < l.\) \(\square\)

Set \(z_{i} = e_{i}^{l} \quad (i \neq 0)\) and \(z_{0} = (e_{(0)} - \frac{1}{1 - \varepsilon})^{l}\). Then \(Z_{0} := \mathcal{K}[z_{-1}, z_{0}, \cdots , z_{l-2}]\) is a central subalgebra of \(U^{\varepsilon}\).

Following [5], we introduce a \(\mathbb{Z}_{+}^{l}\)-gradation in \(U^{\varepsilon}\) by letting \(deg(e_{(i)}) = (\delta_{i,-1}, \cdots , \delta_{i,l-2})\) \((-1 \leq i \leq l - 2\). For an integer vector \(n := (n_{-1}, \cdots , n_{l-2}), n_{i} \in \mathbb{Z}_{+}\), we set \(deg n := \sum n_{i}\) and \(e^{n} = e_{-1}^{n_{-1}} \cdots e_{l-2}^{n_{l-2}}\) and call such an element a **monomial**. Furthermore we define on the set of integer vectors the degree-lexicographic ordering, i.e. set \(n < m\) if either \(deg n < deg m\) or \(deg n = deg m\) but \(n\) is less than \(m\) in the usual lexicographic order, in this way \(\mathbb{Z}_{+}^{l}\) becomes an ordered monoid.

Denote
\[
U_{n}^{\varepsilon} := \sum_{m \leq n} \mathcal{K}e^{m},
\]
then the subspaces \(U_{n}^{\varepsilon}\) give a structure of filtered algebra relative to the ordered monoid \(\mathbb{Z}_{+}^{l}\).

Obviously, with respect to the ordered monoid \(\mathbb{Z}_{+}^{l}\), the associated graded algebra \(gr U^{\varepsilon}\) to the filtered algebra \(U^{\varepsilon}\) has the relations: \(\tilde{e}_{(i)} \tilde{e}_{(j)} = \lambda_{ij} \tilde{e}_{(i)} \tilde{e}_{(j)}\) \((\lambda_{ij} = \varepsilon^{j-i})\), which is isomorphic to a quasipolynomial algebra \(\mathcal{P}\) (cf. [5]), i.e. \(gr U^{\varepsilon} \cong \mathcal{P}\). It shows again that \(U^{\varepsilon}\) has no zero divisors since \(\mathcal{P}\) has no zero divisors (cf. [5]).
Proposition 4.3. \( U^\varepsilon \) is a maximal order.

Proof. Since \( U^\varepsilon \) has no zero divisors, one may consider \( U^\varepsilon \) as a free left \( Z_0 \)-module, and let

\[
U^\varepsilon_n := \sum_{m \leq n} Z_0 \varepsilon^m,
\]

then, similarly, \( U^\varepsilon \) has a structure of filtered algebra with coefficients in \( Z_0 \) relative to the ordered monoid \( Z_0^{(1)} \) (\( Z_0^{(1)} = Z/(l) \)) and the associated graded algebra \( gr U^\varepsilon \) is a twisted polynomial ring over \( Z_0 \). Since \( Z_0 \) is integrally closed and \( U^\varepsilon \) satisfies evidently hypotheses 1–6 in the section 6.5 [5], by Theorem 6.5 [5], we get the conclusion. \( \Box \)

Let \( Z \) be the center of \( U^\varepsilon \), since \( U^\varepsilon \) has no zero divisors, the algebra \( Q(U^\varepsilon) := Q(Z) \otimes Z U^\varepsilon \) is a division algebra of dimension \( d^2 \) over the field of fractions \( Q(Z) \) of \( Z \) and \( d \) is called the degree of \( U^\varepsilon \). Moreover, since \( U^\varepsilon \) is a maximal order in \( Q(U^\varepsilon) \), then \( Z \) is integrally closed.

Let \( A \) be an algebra over \( \mathbb{C}[q, q^{-1}] \) on generators \( x_1, \ldots, x_n \) satisfying the relations: \( x_i x_j = q^{h_{ij}} x_j x_i + P_{ij} \), for \( i > j \), \( P_{ij} \in \mathbb{C}[q, q^{-1}][x_1, \ldots, x_{i-1}] \) and \( (h_{ij}) \) is a skew-symmetric matrix over \( \mathbb{Z} \). Let \( l > 1 \) be an integer relatively prime to all elementary divisors of the matrix \( (h_{ij}) \) and let \( \varepsilon \) be a primitive \( l \)-th root of 1. Let \( A_\varepsilon = A/(q - \varepsilon) \) and assume that \( x_i^l \) are central. Let \( Z_0 = \mathbb{C}[x_1^l, \ldots, x_n^l] \), which has a canonical Poisson structure. C. de Concini and C. Procesi ([5]) suggested the following conjecture.

Conjecture 4.4. Let \( \pi \) be an irreducible representation of the algebra \( A_\varepsilon \) and let \( O_\pi \subset \text{Spec} Z_0 \) be the symplectic leaf containing the restriction of the central character of \( \pi \) to \( Z_0 \). Then the dimension of this representation is equal to \( \frac{l^n}{l^n} \text{dim} O_\pi \).

Set \( W^\varepsilon(1,1)_{(i)} = \text{span}_K \{ e_{(i)} \}, W^\varepsilon(1,1)_i = \bigoplus_{j \geq i} W^\varepsilon(1,1)_{(j)} \). So the \( \varepsilon \)-Witt algebra \( W^\varepsilon(1,1) \) is graded with \( \text{deg}(e_{(i)}) = i \), i.e. \( W^\varepsilon(1,1) = W^\varepsilon(1,1)_{-1} = \bigoplus_{i=1}^{n-2} W^\varepsilon(1,1)_{(j)} \). Let \( U^\varepsilon_i \) (\( i \geq 0 \)) denote the subalgebras generated by \( W^\varepsilon(1,1)_i \), and let \( Z_{0}^{(i)} = Z_0 \cap U^\varepsilon_i = K[z_1, \ldots, z_{l-2}] \), then the algebras \( U^\varepsilon_i \) are of the same type as \( A_\varepsilon \). In particular, for \( U^\varepsilon_1 \), we have a similar conjecture concerning its irreducible representations. However, the treatment of this problem is much more complicated than that of the modular Witt algebra.

5 Realization

Realization. In this section, an operator realization of a class of representations of \( W^\varepsilon(1,1) \) over a tensor vector space \( \mathfrak{A}^\varepsilon(1,1) \otimes K V \) is constructed, which to some extent can be compared with Shen’s mixed product realization (cf. [11]) in the modular case.

A \( W^\varepsilon(1,1) \)-module \( \mathcal{M} = \bigoplus_{i=0}^{n} \mathcal{M}_{(i)} \) is graded, if \( \varepsilon_{(i)} \mathcal{M}_{(j)} \subseteq \mathcal{M}_{(i+j)} \), where \( W^\varepsilon(1,1)_{(0)} \)-modules \( \mathcal{M}_{(0)} \) and \( \mathcal{M}_{(i)} \) are called the base space and the top space of \( \mathcal{M} \), respectively. Then we have

**Theorem 5.1.** Let \( \mathfrak{A}^\varepsilon(1,1) \) be an \( \varepsilon \)-divided power algebra and \( \tau_\varepsilon \) the automorphism of \( \mathfrak{A}^\varepsilon(1,1) \) as in section 2. Assume that \( V \) is a \( \mathfrak{A}^\varepsilon(1,1)_{(0)} \)-module and \( \rho_0 \) is the associated representation. Then the following linear map

\[
e_{(i)} \longrightarrow x^{(i+1)} \partial_\varepsilon \otimes id_V + \partial_\varepsilon (x^{(i+1)}) \tau_\varepsilon \otimes \rho_0(e_{(0)})
\]

(28)

gives a realization of graded representations of \( W^\varepsilon(1,1) \) on the vector space \( \mathfrak{A}^\varepsilon(1,1) \otimes K V \) and \( x^{(0)} \otimes V \cong V \) as \( W^\varepsilon(1,1)_{(0)} \)-modules.
Proof. For $x^{(a)} \otimes v \in \mathfrak{A}^q(1, 1) \otimes_K V$ with $\text{deg}(x^{(a)} \otimes v) = a$, (28) gives

$$\begin{align*}
e_{(i)} x^{(a)} \otimes v &= x^{(i+1)} \partial_\epsilon(x^{(a)}) \otimes v + \partial_\epsilon(x^{(i+1)}) \tau_\epsilon(x^{(a)}) \otimes e_{(0)}.v \\
&= \left(\frac{a+i}{i+1}\right) x^{(a+i)} \otimes v + \epsilon^a \left(\frac{a+i}{i}\right) x^{(a+i)} \otimes e_{(0)}.v.
\end{align*}$$

Then

$$\begin{align*}
e_{(i)} e_{(j)} x^{(a)} \otimes v &= \left(\frac{a+i+j}{i, j}\right) x^{(a+i+j)} \otimes v + \left(\frac{(a+j)e_{(a)}}{(i+1) \epsilon (j+1) \epsilon} x^{(a+i+j)} \otimes v + \frac{(a) (a+j) e_{(a)}}{(j+1) \epsilon} x^{(a+i+j)} \otimes e_{(0)}.v \\
&+ \epsilon^{2a+j} x^{(a+i+j)} \otimes e_{(0)}.v,
\end{align*}$$

where $\left(\frac{a+i+j}{i, j}\right) \epsilon = \frac{(a+i+j) e_{(a)}}{(a+i)(i) \epsilon (j) \epsilon}.$

Thus

$$\begin{align*}
(e^{i+1} e_{(i)} e_{(j)} - \epsilon^{j+1} e_{(j)} e_{(i)}) x^{(a)} \otimes v &= \left(\frac{a+i+j}{i, j}\right) \left(\frac{(j+1) \epsilon - (i+1) \epsilon}{(i+1) \epsilon (j+1) \epsilon} \right) \left(\frac{(a) \epsilon x^{(a+i+j)} \otimes v + (i+j+1) \epsilon (j+1) \epsilon x^{(a+i+j)} \otimes e_{(0)}.v}{(j+1) \epsilon}ight) \\
&= \left(\frac{(i+j+1)}{i+1} \epsilon - (i+j+1) \epsilon \right) \left(\frac{(a) \epsilon x^{(a+i+j)} \otimes v + (i+j+1) \epsilon x^{(a+i+j)} \otimes e_{(0)}.v}{(j+1) \epsilon}ight) \\
&= \left(\frac{(i+j+1)}{i+1} \epsilon - (i+j+1) \epsilon \right) e_{(i+j)} x^{(a)} \otimes v.
\end{align*}$$

This shows that (28) does yield a representation of $W^q(1, 1)$, owing to (29) and (13). □

Obviously, the proof of Theorem 5.1 also applies to the generic case.

Corollary 5.2. Let $\mathfrak{A}^q(1)$ be a $q$-divided power algebra and $\tau_q$ the automorphism of $\mathfrak{A}^q(1)$ as in section 2. Assume that $V$ is a $W^q(1)_{(0)}$-module and $\rho_0$ is the associated representation. Then the following linear map

$$e_{(i)} \mapsto x^{(i+1)} \partial_\epsilon \otimes id_V + \partial_\epsilon(x^{(i+1)}) \tau_\epsilon \otimes \rho_0(\epsilon_{(0)})$$

(30)

gives a realization of graded representations of $W^q(1)$ on the vector space $\mathfrak{A}^q(1) \otimes_K V$ and $x^{(0)} \otimes V \cong V$ as $W^q(1)_{(0)}$-modules. □

Irreducible modules. Based on the realization above, the irreducible graded $W^q(1, 1)$-modules can be explicitly described. First of all, the following fact is clear.

Lemma 5.3. For every irreducible $W^q(1, 1)_{(0)}$-module $V$, there exists, up to isomorphism, one and only one irreducible graded $W^q(1, 1)$-module $\mathcal{M}$ with $\mathcal{M}_{(0)} = V$ as its base space. □

Clearly, every irreducible $W^q(1, 1)_{(0)}$-module $V$ is 1-dimensional, i.e. $\epsilon_{(0)}.v = tv$, $t \in K$. Denote by $\mathcal{M}(t)$ the associated irreducible graded $W^q(1, 1)$-module. Note that the base (resp. top) space of $\mathfrak{A}^q(1, 1) \otimes_K V(t)$ is $V(t)$ (resp. $V((-1)_{\epsilon}(1-t))$, according to the formula (29).
Theorem 5.4.
(i) If \( t \neq 0, 1 \), then the graded \( W^\varepsilon(1, 1) \)-modules \( A^\varepsilon(1, 1) \otimes_K V(t) \) with the base spaces \( V(t) \) are irreducible.
(ii) If \( t = 0 \), then the graded \( W^\varepsilon(1, 1) \)-module \( A^\varepsilon(1, 1) \otimes_K V(0) \) with the base space \( V(0) \) is reducible and there is an exact sequence
\[
0 \rightarrow \mathcal{M}(0) \rightarrow A^\varepsilon(1, 1) \otimes_K V(0) \rightarrow \mathcal{M}(1) \rightarrow 0,
\]
where \( \mathcal{M}(0) = \langle x^{(0)} \otimes v \rangle \), \( \mathcal{M}(1) = \langle x^{(1)} \otimes v, x^{(2)} \otimes v, \ldots, x^{(t-1)} \otimes v \rangle \).
(iii) If \( t = 1 \), then the graded \( W^\varepsilon(1, 1) \)-module \( A^\varepsilon(1, 1) \otimes_K V(1) \) with the base space \( V(1) \) is reducible and there is an exact sequence
\[
0 \rightarrow \mathcal{M}(1) \rightarrow A^\varepsilon(1, 1) \otimes_K V(1) \rightarrow \mathcal{M}(0) \rightarrow 0,
\]
where \( \mathcal{M}(0) = \langle x^{(t-1)} \otimes v \rangle \), \( \mathcal{M}(1) = \langle x^{(0)} \otimes v, x^{(1)} \otimes v, \ldots, x^{(t-2)} \otimes v \rangle \).

Note that the cases (ii) and (iii) are dual to each other and the \( W^\varepsilon(1, 1) \)-module \( \mathcal{M}(1) \) is self-dual.

Proof. (i) Since \( e_{(-1)}, x^{(0)} \otimes v = 0 \), \( e_{(-1)}, x^{(m)} \otimes v = x^{(m-1)} \otimes v \), for \( m > 0 \), an arbitrary nontrivial \( W^\varepsilon(1, 1) \)-submodule of \( A^\varepsilon(1, 1) \otimes_K V \) must contain some nonzero element of the form \( x^{(0)} \otimes v \), for \( v \neq 0 \) in \( V \). By (29), we obtain \( e_{(l-2)}, x^{(0)} \otimes v = tx^{(l-2)} \otimes v \) and \( e_{(1)}, x^{(l-2)} \otimes v = (1-t)^\varepsilon x^{(l-1)} \otimes v \). So statement (i) holds.

(ii) Since \( t = 0 \), \( e_{(i)}, x^{(0)} \otimes v = 0 \), \( \forall i \geq 0 \), by (29). Then \( \langle x^{(0)} \otimes v \rangle \) is a trivial one-dimensional \( \overline{W}^\varepsilon(1, 1) \)-submodule of \( A^\varepsilon(1, 1) \otimes_K V \). And since \( e_{(l-2)}, x^{(1)} \otimes v = x^{(l-1)} \otimes v \) by (29), then \( \{x^{(i)} \otimes v, \ldots, x^{(l-1)} \otimes v\} \) is an irreducible \( W^\varepsilon(1, 1) \) quotient module. Since by (29), \( e_{(0)}, x^{(1)} \otimes v = x^{(1)} \otimes v \) and \( e_{(0)}, x^{(l-1)} \otimes v = (-1)^\varepsilon x^{(l-1)} \otimes v \). So statement (ii) is true.

(iii) Since \( t = 1 \), by (29), we have \( e_{(l-2)}, x^{(0)} \otimes v = x^{(l-2)} \otimes v \), \( e_{(i)}, x^{(l-2)} \otimes v = 0 \) for \( i \geq 2 \) and \( e_{(1)}, x^{(l-2)} \otimes v = \left(1 - t\right)^\varepsilon x^{(l-1)} \otimes v + \varepsilon^{-2} \left(1 - 2\right)^\varepsilon x^{(l-1)} \otimes v = 0 \). On the other hand, from (29), we get \( e_{(0)}, x^{(0)} \otimes v = x^{(0)} \otimes v \), \( e_{(0)}, x^{(l-2)} \otimes v = \left(1 - t\right)^\varepsilon x^{(l-2)} \otimes v + \varepsilon^{-2} x^{(l-2)} \otimes v = (-1)^\varepsilon x^{(l-2)} \otimes v \) and \( e_{(0)}, x^{(l-1)} \otimes v = \left(1 - t\right)^\varepsilon x^{(l-1)} \otimes v + \varepsilon^{-1} x^{(l-1)} \otimes v = 0 \). Then statement (iii) holds. \( \square \)

Remark. The conclusions (ii) and (iii) in the above Theorem show that there exists a certain duality between the modules \( A^\varepsilon(1, 1) \otimes_K V(0) \) and \( A^\varepsilon(1, 1) \otimes_K V(1) \).

6 q-Holomorph Structure and Representations

An approach towards more representations. In (14) and (15), we constructed the \( \varepsilon \)-holomorph structure \( H^\varepsilon(1) \) for the \( \varepsilon \)-Witt algebra \( W^\varepsilon(1, 1) \). Here we will reveal the importance of such a structure in the representation theory.

For the \( \varepsilon \)-Witt algebra \( W^\varepsilon(1, 1) \), \( \varepsilon \)-abelian algebra \( \mathcal{K}_J^\varepsilon \) and their \( q \)-holomorph \( H^\varepsilon(1) \), given two representation vector spaces \( \mathfrak{M} \) and \( \mathcal{V} \) with three associated representations:
\[
\phi : W^\varepsilon(1, 1) \rightarrow \text{End}_K(\mathfrak{M}),
\psi : \mathcal{K}_J^\varepsilon \rightarrow \text{End}_K(\mathfrak{M}),
\rho : H^\varepsilon(1) \rightarrow \text{End}_K(\mathcal{V}).
\]
and the pair \( (\phi, \psi) \) satisfies the following compatibility condition:
\[
\phi(e_{(i)})\psi(L_j) - \varepsilon^{i-1}\psi(L_j)\phi(e_{(i)}) = \binom{i + j}{i + 1} \varepsilon^j \psi(L_{i+j}). \tag{31}
\]
Then we have
**Theorem 6.1.** Suppose that $\phi$, $\psi$ and $\rho$ are as above, and set $L_{-1} = 0$.  
(i) If $\phi$ and $\psi$ satisfy (31), then the pairs $(\phi_{a\psi}, \psi)(\forall a \in K)$ afford a family of representations of $\mathcal{H}_\varepsilon(1)$, where $\phi_{a\psi}(e_{(i)}) = \phi(e_{(i)}) + a\psi(L_i)$ ($i \geq 0$).

(ii) For any fixed element $\omega \in \mathcal{H}_\varepsilon(1)$ with $\rho(\omega) \in \text{End}_K(\mathcal{V})$, if (31) holds for the pair $(\phi, \psi)$, then the triple $(\phi, \psi, \rho)$ provides a new representation $\tilde{\phi}(e_{(i)}) := \phi(e_{(i)}) \otimes \text{id}_\mathcal{V} + \psi(L_i) \otimes \rho(\omega)$ of $\mathcal{W}_\varepsilon(1, 1)$ over vector space $\mathfrak{V} \otimes_K \mathcal{V}$.

(iii) Assume that the representations $\phi$ and $\rho$ with a fixed point $\omega \in \mathcal{H}_\varepsilon(1)$ are given, $\{\psi_a\}$ is a family of representations of $\mathcal{W}_\varepsilon(1, 1)$ with the same representation vector space $\mathfrak{V}$ as $\phi$ and satisfy (31) relevant to $\phi$, then the triples $(\phi, \sum_a k_a \psi_a; \rho)$ provide a family of representations of $\mathcal{W}_\varepsilon(1, 1)$ over $\mathfrak{V} \otimes_K \mathcal{V}$, which forms a vector space structure relative to a fixed pair $(\phi, \rho)$.

**Proof.** (i) Note that $\varepsilon^{i+1}\phi(L_i)\psi(L_j) - \varepsilon^{j+1}\psi(L_j)\psi(L_i) = 0$, it is clear that the pairs $(\phi_{a\psi}, \psi)$ still satisfy (31). That $\phi_{a\psi}$ is still a representation of $\mathcal{W}_\varepsilon(1, 1)$ over $\mathfrak{V}$ follows similarly from the proof of next step (ii).

(ii) We need to show
\[
\varepsilon^{i+1}\tilde{\phi}(e_{(i)})\tilde{\phi}(e_{(j)}) - \varepsilon^{j+1}\tilde{\phi}(e_{(j)})\tilde{\phi}(e_{(i)}) = \tilde{\phi}(\{e_{(i)}, e_{(j)}\}_\varepsilon).
\]

Note that $\varepsilon^{i+1}(i_{i+1})_\varepsilon - \varepsilon^{j+1}(i_{j+1})_\varepsilon = (i_{i+1})_\varepsilon - (i_{j+1})_\varepsilon$, from (31), (13) and
\[
\tilde{\phi}(e_{(i)})\tilde{\phi}(e_{(j)}) = (\phi(e_{(i)}) \otimes \text{id}_\mathcal{V} + \psi(L_i) \otimes \rho(\omega)),
\]
\[
(\phi(e_{(j)}) \otimes \text{id}_\mathcal{V} + \psi(L_j) \otimes \rho(\omega))
\]
\[
= \phi(e_{(i)})\phi(e_{(j)}) \otimes \text{id}_\mathcal{V} + \phi(e_{(i)})\psi(L_j) \otimes \rho(\omega)
\]
\[
+ \psi(L_i)\phi(e_{(j)}) \otimes \rho(\omega) + \psi(L_i)\psi(L_j) \otimes \rho(\omega)^2,
\]
it is easy to obtain the required result:
\[
\varepsilon^{i+1}\tilde{\phi}(e_{(i)})\tilde{\phi}(e_{(j)}) - \varepsilon^{j+1}\tilde{\phi}(e_{(j)})\tilde{\phi}(e_{(i)}) = \tilde{\phi}(\{e_{(i)}, e_{(j)}\}_\varepsilon).
\]

(iii) If $\psi_a$ and $\psi_b$ satisfy (31) relevant to $\phi$, then the pairs $(k \psi_a, \phi)$ ($k \in K$) and $(\psi_a + \psi_b, \phi)$ also satisfy the same relation (31). So (iii) holds, thanks to (ii).

Thus we complete the proof.  

**Corollary 6.2.** A similar conclusion holds for the $q$-Witt algebra $\mathcal{W}^q(1)$ in the generic case, if $\mathcal{W}_\varepsilon(1, 1)$, $1\mathcal{K}_\varepsilon$, $\mathcal{H}_\varepsilon(1)$ and $\varepsilon$ in (31) are replaced by $\mathcal{W}^q(1)$, $1\mathcal{K}_q^\infty$, $\mathcal{H}^q$ and $q$, respectively.  

Here are two examples to show some relationships between the universal truncated coincided modules and those given via the $\varepsilon$-holomorph structure in Theorem 6.1.

**Example 6.3** For $\mathcal{W}_\varepsilon(1, 1)$, if we take $\mathfrak{V} = \mathbb{R}_\varepsilon(1, 1)$, $\omega = e_{(0)}$, $\mathcal{V} = K$ with $\rho(e_{(0)}) = t \in K$, and $\psi(L_i) = x^{(i)} \tau_\varepsilon$, then we get the realization given in section 5. It is a graded representation of...
$W^\varepsilon(1, 1)$ with a base space $\mathcal{K}(t) = \mathcal{K}$ over $\mathfrak{A}^\varepsilon(1, 1) \otimes_{\mathcal{K}} \mathcal{K}(t)$, where $(\mathfrak{A}^\varepsilon(1, 1) \otimes_{\mathcal{K}} \mathcal{K}(t))(0) \cong \mathcal{K}(t)$. Furthermore, if we take $\psi(L_i) = k x^{(i)} \tau_\varepsilon$, $\forall k \in \mathcal{K}$, then we get all graded $W^\varepsilon(1, 1)$-representations. In fact, the universal truncated coinduced $W^\varepsilon(1, 1)$-modules only yield the special case where the enlarging coefficient $k = 1$. In the modular case, the same fact exists (see [11, 13]).

**Example 6.4** For the generic case in Corollary 6.2, if we take $\mathfrak{V} = \mathfrak{A}^\varepsilon(1)$, $\mathfrak{V} = \mathcal{K}$, $\psi(L_i) = k x^{(i)} \tau_q$, $\omega = e_{(0)}$ and let $q$ tend to 1, then we obtain every graded representation of the usual (infinite dimensional) Witt algebra $W(1)$ of Cartan type (cf. [11, 13]).

**Remark.** Actually if we choose an arbitrary $\omega(\neq e_{(0)}) \in W^\varepsilon(1, 1)$ (resp. $W^q(1)$), we arrive at some non-graded representations. From Theorem 6.1 and Corollary 6.2, the $q$-holomorph structure does lead to some new knowledge on representations of $W^\varepsilon(1, 1)$ or $W^q(1)$.

**Acknowledgements.** I would like to express my sincere gratitude to Professor H. Strade for the hospitality and support during my stay at Hamburg. I would thank Professors Guangyu Shen and H. Strade for their helpful comments and valuable suggestions. I also acknowledge the Alexander von Humboldt Stiftung for the support of a Humboldt Research Fellowship.

**References**

1. G.M. Benkart & R.V. Moody, *Derivations, central extensions and affine Lie algebras*, Algebras Groups Geom. 3 (1986), 456–492.
2. R. Berger, *The quantum Poincaré-Birkhoff-Witt theorem*, Commun. Math. Phys. 143 (1992), 215–234.
3. G.M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. 29 (1978), 178–218.
4. H.J. Chang, *Über Wittsche Lieringe*, Abh. Math. Sem. Hamb. Univ. 14 (1941), 151–184.
5. C. De Concini & C. Procesi, *Quantum Groups*, Lecture Notes in Math. 1565 (1994), 30–141.
6. Francisco J. Narganes-Quijano, *Cyclic representations of a $q$-deformation of the Virasoro algebra*, Phys. A: Math. Gen. 24 (1991), 593–601.
7. Y. Gao, *Central extensions of nonsymmetrizable Kac-Moody algebras over commutative algebras*, Proc. Amer. Math. Soc. 121 (1) (1994), 67–76.
8. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.
9. C. Kassel, *Cyclic Homology of Differential Operators, the Virasoro Algebra and a $q$-Analogue*, Comm. Math. Phys. 146 (1992), 343–356.
10. Y.I. Manin, *Topics in Noncommutative Geometry*, Princeton Univ. Press, Princeton, NJ., 1991.
11. G.Yu. Shen, *Graded modules of graded Lie algebras of Cartan type (I) — mixed product of modules*, Scientia Sinica (Ser. A) 29:6 (1986), 570–581.
12. _______, *Graded modules of graded Lie algebras of Cartan type (II) — positive and negative graded modules*, Scientia Sinica (Ser. A) 29:10 (1986), 1009–1019.
13. _______, *Graded modules of graded Lie algebras of Cartan type (III) — Irreducible modules*, Chin. Ann. of Math. 9B(4) (1988), 404–417.
14. H. Strade & R. Farnsteiner, *Modular Lie algebras and their representations*, Pure and Applied Mathematics, vol. 116, New York, 1988.