UNIQUENESS AND EXISTENCE OF SOLUTIONS FOR A SINGULAR SYSTEM WITH NONLOCAL OPERATOR VIA PERTURBATION METHOD

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Abstract In this work, we investigate the existence and the uniqueness of solutions for the nonlocal elliptic system involving a singular nonlinearity as follows:

\[
\begin{align*}
(-\Delta)^s u &= a(x)|u|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta}, \quad \text{in } \Omega, \\
(-\Delta)^s v &= b(x)|v|^{q-2}v + \frac{1-\beta}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta}, \quad \text{in } \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundary, \(0 < \alpha < 1, 0 < \beta < 1, 2 - \alpha - \beta < p < q < p_s^* = \frac{Np}{N-2p}\), \(a, b, c \in C(\Omega)\) are non-negative weight functions with compact support in \(\Omega\), and \((-\Delta)^s\) is the fractional \(p\)-laplacian operator. We use a perturbation method combine with some variational methods in order to show the existence of a solution to the above system. We also prove the uniqueness of the solution to the system for some additional condition.

Keywords Singular nonlocal elliptic system, approximated methods, variational methods, existence of solutions, uniqueness of solutions.

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1. Introduction

The purpose of this paper is to study the following elliptic system involving nonlocal operator and singular nonlinearity:

\[
\begin{align*}
(-\Delta)^s u &= a(x)|u|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta}, \quad \text{in } \Omega, \\
(-\Delta)^s v &= b(x)|v|^{q-2}v + \frac{1-\beta}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta}, \quad \text{in } \Omega,
\end{align*}
\]

where \(\Omega\) is an open bounded domain of \(\mathbb{R}^N\) with smooth boundary, \(N > ps, s \in (0, 1), 1 < p < \infty\) and \((-\Delta)^s\) is the fractional \(p\)-laplacian operator which is defined
as
\[ (-\Delta_p)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}} dy, \quad \forall \, p \in [1, \infty) \]
with $C_{N,s}$ being the normalizing constant.

During the past few decades, the study of nonlocal elliptic PDEs involving singularity with Dirichlet boundary condition has drawn interest by many researchers, both from a pure mathematical point of view and for concrete applications, since the problems of this type are important in many fields of sciences, notably the fields of physics, probability, finance, electromagnetism, astronomy, and fluid dynamics, it also they can be used to accurately describe the jump Lévy processes in probability theory and fluid potentials for more details see [1,6] and references therein.

Set $\alpha = \beta$, $\alpha + \beta = \gamma$ and $u = v$. Then the problem (1.1) becomes the following nonlocal singular problem
\begin{align}
(-\Delta_p)^s u &= \frac{\lambda c(x)}{u^\gamma} + M |u|^{q-2} u \text{ in } \Omega, \\
u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \\
u &> 0 \text{ in } \Omega, 
\end{align}
where $N > ps$, $M \geq 0$, $c : \Omega \to \mathbb{R}$ is a nonnegative bounded function. When $M = 0$ and $p = 2$ (the purely singular problem), FANG [8] had shown that the problem (1.2) has a unique solution $u \in C^{2,\alpha}(\Omega)$ for $0 < \alpha < 1$. In [12], the multiplicity result for the problem (1.2) is proved by converting the nonlocal problem to a local problem. For $1 < p < \infty$, $M = 0$ and $\lambda = 1$ the problem (1.2) was studied by Canino et al. [5]. In [9,10], the authors established the existence and the multiplicity of weak solutions to the problem (1.2) by using the Nehari manifold method. Recently, Saoudi et al. in [14] has guaranteed the existence of at least two solutions by using min-max method with the help of modified Mountain Pass theorem. In particular, in [13], the author considered the following nonlocal problem
\begin{align}
(-\Delta_p)^s u &= \lambda a(x)|u|^{q-2}u + \frac{1-\alpha}{\alpha} c(x)|u|^{-\alpha}|v|^{1-\beta}, \quad \text{in } \Omega, \\
(-\Delta_p)^s v &= \mu b(x)|v|^{q-2}v + \frac{1-\beta}{\beta} c(x)|u|^{1-\alpha}|v|^{-\beta}, \quad \text{in } \Omega, \\
u &= v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align}
where $0 < \alpha < 1$, $0 < \beta < 1$, $2 - \alpha - \beta < p < q < p^*_s$, $p^*_s = \frac{N}{N - ps}$ is the fractional Sobolev exponent, $\lambda, \mu$ are two parameters, $a, b, c \in C(\overline{\Omega})$ are non-negative weight functions with compact support in $\Omega$. With the help of the Nehari manifold and the fiber maps (appropriately modified), the authors proved the existence of at least two non-negatives solutions of problem (1.3).

Motivated by above results, in the present work, we are interested in the existence and the uniqueness of solutions for nonlocal system (1.1) via perturbed method’s.

In order to state our result, let us introduce some notations. We define
\[ W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : u \text{ measurable} , |u|_{s,p} < \infty \right\}, \]
the usual fractional Sobolev space with the Gagliardo norm
\[ ||u||_{s,p} := \left( ||u||_p^p + ||u||_{s,p}^p \right)^{\frac{1}{p}}. \]
For a detailed account on the properties of $W^{s,p}(\mathbb{R}^N)$ we refer the reader to [11].

Denote

$$Q = \mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$$

and we define the space

$$X = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ Lebesgue measurable : } u\setminus\Omega \in L^p(\Omega) \text{ and } \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \in L^p(Q) \right\}$$

with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left( \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\,dxdy \right)^{1/p}.$$ 

Through this paper we consider the space

$$X_0 = \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \},$$

with the norm

$$\|u\|_{X_0} = \left( \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\,dxdy \right)^{1/p}.$$ 

It is readily seen that $(X_0, \|\cdot\|)$ is a uniformly convex Banach space and that the embedding $X_0 \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leq q \leq p^*_s$, and compact for all $1 \leq q < p^*_s$. The dual space of $(X_0, \|\cdot\|)$ is denoted by $(X^*, \|\cdot\|_*)$, and $\langle \cdot, \cdot \rangle$ denotes the usual duality between $X_0$ and $X^*$.

Let $X_0 = X_0 \times X_0$ be the Cartesian product of two Hilbert spaces, which is (In [6] it is claimed that $X_0$ is a Hilbert space) a reflexive Banach space endowed with the norm

$$\|(u,v)\| = \left( \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\,dxdy + \int_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}}\,dxdy \right)^{1/p}. \quad (1.4)$$

Before stating our main results, we make the following assumptions throughout this paper:

(H1) $0 < \alpha < 1, 0 < \beta < 1, 2 - \alpha - \beta < p \leq p^*_s = \frac{Np}{N-sp}$.

(H2) $a, b, c \in C(\Omega)$ are non-negative weight functions with compact support in $\Omega$.

(H3) $a = b = 0, c \in C(\Omega)$ is a non-negative weight functions with compact support in $\Omega$, $\alpha = \beta$ and $\frac{1}{2} < \alpha < 1$.

We now list out the results that we will prove in this work.

**Theorem 1.1.** Suppose that (H1)-(H2) are fulfilled. Then the system (1.1) has at least one positive solution.

**Theorem 1.2.** Suppose that (H1)-(H3) are fulfilled. Then the system (1.1) has a unique positive solution.

This paper is organized as follows: The Section 2 is devoted to study approximated system. While, existence of solution (Theorem 1.1) and uniqueness of solution (Theorem 1.2) will be presented in Section 3 and in Section 4 respectively.
2. The approximated fractional system

In this section, we introduce the following approximated system, for a fixed \( n > 1 \)

\[
\begin{align*}
(-\Delta)^s u &= a(x,|u(x)|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta} c(x) (|u| + \frac{1}{n})^{-\alpha} (|v| + \frac{1}{n})^{1-\beta} \\
(-\Delta)^s v &= b(x,|v(x)|^{q-2}v + \frac{1-\beta}{2-\alpha-\beta} c(x) (|u| + \frac{1}{n})^{1-\alpha} (|v| + \frac{1}{n})^{1-\beta}.
\end{align*}
\tag{2.1}
\]

Associated to the approximated problem (2.1), we define the functional \( E_n : \mathcal{X}_0 \to \mathbb{R} \) by

\[
E_n(u,v) = \frac{1}{p} \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}} \, dx \, dy
+ \frac{1}{q} \int_\Omega \frac{|v(x) - v(y)|^{q-2}(v(x) - v(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}} \, dx \, dy
+ \frac{1}{p} \int_\Omega (a(x)|u|^{q-2}u\phi + b(x)|v|^{q-2}v\psi) \, dx
+ \frac{1}{2-\alpha-\beta} \int_\Omega c(x)(|u| + \frac{1}{n})^{-\alpha} (|v| + \frac{1}{n})^{1-\beta} \phi \, dx
+ \frac{1}{2-\alpha-\beta} \int_\Omega c(x)(|u| + \frac{1}{n})^{1-\alpha} (|v| + \frac{1}{n})^{-\beta} \psi \, dx
\]

for all \( (\phi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \).

Before, given the first result in this section let us recall the following inequality

\[
(d + e)^m \leq d^m + e^m \tag{2.2}
\]

for all \( d, e \in \mathbb{R}^+ \) and \( m \in (0,1) \). On the other hand, using Hölder’s inequality and Sobolev inequalities, one has

\[
\int_\Omega (a(x)|u|^{q} + b(x)|v|^{q}) \, dx \leq \|a\|_{\infty} \|u\|_p^q + \|b\|_{\infty} \|v\|_p^q
\]

\[
\leq \|a\|_{\infty} \|u\|_p^q + \|b\|_{\infty} \|v\|_p^q
\]

\[
\leq \|a\|_{\infty} \|u\|_p^q + \|b\|_{\infty} \|v\|_p^q
\]
The functional
\[ E_n(u,v) \geq \frac{1}{p} ||u||^p + \frac{1}{p} ||v||^p - \frac{1}{q} \int \Omega (a(x)||u||^q + b(x)||v||^q) \, dx \]
\[ = \frac{1}{2-\alpha-\beta} \int \Omega c(x)(|u|+1)^{1-\alpha}(|v|+1)^{1-\beta} \, dx \]
\[ \geq \frac{1}{p} ||(u,v)||^p - c \Omega \frac{p^2-p}{p^*} S^{-q/p} \left( ||a||^{p/p-q} + ||b||^{p/p-q} \right) \frac{r_1}{r_1} \parallel(u,v)\parallel^q \]
\[ - \frac{1}{2-\alpha-\beta} \left[ ||c||_{r_1} ||u||^{1-\alpha} ||v||^{1-\beta} - ||c||_{r_2} ||u||^{1-\alpha} ||v||^{1-\beta} - ||c||_{r_3} ||u||^{1-\alpha} \right] \]
\[ - S^{1-\beta} ||c||_{r_3} ||v||^{1-\beta} - ||c||_{r_1} \right) \]
with
\[ r_1 = \frac{N}{(\alpha+\beta)[N-\alpha p+1]p-2N}, \quad r_2 = \frac{N}{(\alpha-1)(N-\alpha p)+Np}, \quad r_3 = \frac{N}{(\beta-1)(N-\beta p)+Np}. \]

Hence, from Lemma 2.1 combine with the Ekeland’s variational principle, we obtain the existence of the sequences \((u_k, v_k) \in \mathcal{X}_0 \) such that \( E_n(u_k, v_k) \to c \) and \( E'_n(u_k, v_k) \to 0 \) in \( \mathcal{X}_0' \) as \( k \to \infty \).

Now, we prove the following crucial result.

Lemma 2.2. the approximated system (2.1) has a nonnegative solution.

Proof. At first, it is simple to see from the coercivity of the functional energy \( E_n \), the boundedness of the sequence \( \{u_k, v_k\} \) in \( \mathcal{X}_0 \). So, there exists \( \{U_n, V_n\} \) in \( \mathcal{X}_0 \) such that
\[ (u_k, v_k) \to (U_n, V_n) \text{ weakly in } \mathcal{X}_0 \]
\[ (u_k, v_k) \to (U_n, V_n) \text{ strongly in } L^k(\Omega) \text{ for } 1 \leq k < p^*_q \]
\[ (u_k, v_k) \to (U_n, V_n) \text{ pointwise a.e. in } \Omega. \]

By dominated convergence theorem, we claim that
\[ \int_{\Omega} |u_k|^{q-2} u_k \Phi dx \to \int_{\Omega} |U_n|^{q-2} U_n \Phi dx \quad \forall \Phi \in C^\infty_c(\Omega). \tag{2.5} \]
Indeed, from [3], there exists \( l \in L^r(\mathbb{R}^N) \) such that
\[
|u_k(x)| \leq l(x), \quad |v_k(x)| \leq l(x), \quad \text{as } k \to \infty
\]
for any \( 1 \leq r < p_*^* \). The fact that, \((u_k, v_k) \to (U_n, V_n)\) pointwise a.e. in \( \Omega \). Therefore, by Dominated convergence theorem, our claim is true. Similarly,
\[
\int_{\Omega} |v_k|^{q-2}v_k\Psi dx \to \int_{\Omega} |V_n|^{q-2}V_n\Psi dx \quad \forall \Psi \in C^\infty_c(\Omega).
\]
(2.6)
On the other hand, note that \( \{u_k\} \) is bounded, by the Sobolev embedding theorem, so there exists a constant \( C > 0 \) such that \( |u_k|_{p_*^*} \leq C < \infty \). Moreover, by Hölder inequalities we have
\[
\int_{\Omega} c(x)u_k^{1-\alpha} dx \leq \|c\|_\infty \int_{\Omega} |u_k|^{1-\alpha} dx \leq \|c\|_\infty |\Omega|^{\frac{p_*^*}{p_*^*-1}} |u_k|^{1-\alpha}.
\]
(2.7)
Thus, using Eq. (2.7), there exists \( l_1 \in L^{(1-\alpha)}(\Omega) \), and up to a subsequence,
\[
c(x) \left( |u_k| + \frac{1}{n} \right)^{1-\alpha} \left( |v_k| + \frac{1}{n} \right)^{-\beta} \Psi \to c(x) \left( |U_n| + \frac{1}{n} \right)^{1-\alpha} \left( |V_n| + \frac{1}{n} \right)^{-\beta} \Psi \text{a.e.}
\]
and
\[
|c(x) \left( |u_k| + \frac{1}{n} \right)^{1-\alpha} \left( |v_k| + \frac{1}{n} \right)^{-\beta} \Psi| \leq n^\beta |c| (|l_1|^{1-\alpha} + 1) \Psi \in L^1(\Omega).
\]
Now, apply the Lebesgue’s dominated convergence theorem, we conclude that, \( \forall \, \Psi \in C^\infty_c(\Omega) \),
\[
\int_{\Omega} c(x) \left( |u_k| + \frac{1}{n} \right)^{1-\alpha} \left( |v_k| + \frac{1}{n} \right)^{-\beta} \Psi dx \to
\]
\[
\int_{\Omega} c(x) \left( |U_n| + \frac{1}{n} \right)^{1-\alpha} \left( |V_n| + \frac{1}{n} \right)^{-\beta} \Psi dx.
\]
(2.8)
Similarly, we deduce that, \( \forall \, \Phi \in C^\infty_c(\Omega) \),
\[
\int_{\Omega} c(x) \left( |u_k| + \frac{1}{n} \right)^{-\alpha} \left( |v_k| + \frac{1}{n} \right)^{1-\beta} \Phi dx \to
\]
\[
\int_{\Omega} c(x) \left( |U_n| + \frac{1}{n} \right)^{-\alpha} \left( |V_n| + \frac{1}{n} \right)^{1-\beta} \Phi dx.
\]
(2.9)
Moreover, let \( p' \) the Hölder conjugate of \( p \) given by \( p' = \frac{p}{p-1} \), then \( \{u_k(x) - u_k(y)\}^{p-2}(u_k(x) - u_k(y))/|x - y|^{(N+sp)/p'} \) is bounded in \( L^{p'}(\mathbb{R}^{2N}) \) and by pointwise convergence \( u_k \to U_n \) a.e. in \( \Omega \)
\[
\frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x - y|^{(N+sp)/p'}} \to \frac{|U_n(x) - U_n(y)|^{p-2}(U_n(x) - U_n(y))}{|x - y|^{(N+sp)/p'}} \quad \text{a.e. in } \mathbb{R}^{2N}.
\]
Furthermore, since \( (\Phi(x) - \Phi(y))/|x - y|^{(N+sp)/p} \in L^p(\mathbb{R}^{2N}) \), we deduce that
\[
\int_{Q} \frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))(\Phi(x) - \Phi(y))}{|x - y|^{N+sp}} dx \, dy
\]
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\[
\rightarrow \int_{Q} \frac{\left| U_n(x) - U_n(y) \right|^{p-2}(U_n(x) - U_n(y))(\Phi(x) - \Phi(y))}{|x - y|^{N+sp}} \, dx \, dy \quad \text{for any } \Phi \in C^\infty_c(\Omega). \tag{2.10}
\]

Using the same argument, as above, we obtain

\[
\rightarrow \int_{Q} \frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))(\Psi(x) - \Psi(y))}{|x - y|^{N+sp}} \, dx \, dy
\]

\[
\rightarrow \int_{Q} \frac{|V_n(x) - V_n(y)|^{p-2}(V_n(x) - V_n(y))(\Psi(x) - \Psi(y))}{|x - y|^{N+sp}} \, dx \, dy \quad \text{for any } \Psi \in C^\infty_c(\Omega). \tag{2.11}
\]

Therefore, from Eqs. (2.5), (2.6), (2.8), (2.9), (2.10) and (2.11), we deduce that

\[
\langle I'_n (u_k, v_k) (\Phi, \Psi) \rangle \rightarrow \langle I'_n (U_n, V_n) (\Phi, \Psi) \rangle = 0 \quad \forall (\Phi, \Psi) \in C^\infty_c(\Omega) \times C^\infty_c(\Omega).
\]

Now, we will prove that the sequence \((u_k, v_k)\) converge strongly to \((U_n, V_n)\) in \(X_0\). Indeed, since

\[
\langle I'_n (u_k, v_k) - I'_n (U_n, V_n), (u_k - U_n, v_k - V_n) \rangle \rightarrow 0,
\]

as \(k \rightarrow \infty\), we conclude that,

\[
o_k(1) = \langle I'_n (u_k, v_k) - I'_n (U_n, V_n), (u_k - U_n, v_k - V_n) \rangle
\]

\[
= \int_{Q} \frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))((u_k - U_n)(x) - (u_k - U_n)(y))}{|x - y|^{N+sp}} \, dx \, dy
\]

\[
- \int_{Q} \frac{|U_n(x) - U_n(y)|^{p-2}(U_n(x) - U_n(y))((u_k - U_n)(x) - (u_k - U_n)(y))}{|x - y|^{N+sp}} \, dx \, dy
\]

\[
+ \int_{Q} \frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))((v_k - V_n)(x) - (v_k - V_n)(y))}{|x - y|^{N+sp}} \, dx \, dy
\]

\[
- \int_{Q} \frac{|V_n(x) - V_n(y)|^{p-2}(V_n(x) - V_n(y))((v_k - V_n)(x) - (v_k - V_n)(y))}{|x - y|^{N+sp}} \, dx \, dy
\]

\[
- \int_{\Omega} a(x) \left( |u_k|^{q-2} u_k - |V_n|^{q-2} V_n \right) (u_k - U_n) \, dx
\]

\[
- \int_{\Omega} b(x) \left( |v_k|^{q-2} v_k - |V_n|^{q-2} V_n \right) (v_k - V_n) \, dx
\]

\[
- \frac{1 - \alpha}{2 - \alpha - \beta} \int c(x) \left[ \left( |u_k| + \frac{1}{n} \right)^{-\alpha} (u_k - U_n) \left( |v_k| + \frac{1}{n} \right)^{1-\beta}
\]

\[
- \left( |U_n| + \frac{1}{n} \right)^{-\alpha} (u_k - U_n) \left( |V_n| + \frac{1}{n} \right)^{1-\beta} \right] \, dx
\]

\[
- \frac{1 - \beta}{2 - \alpha - \beta} \int c(x) \left[ \left( |u_k| + \frac{1}{n} \right)^{-\alpha} (v_k - V_n) \left( |v_k| + \frac{1}{n} \right)^{-\beta}
\]

\[
- \left( |U_n| + \frac{1}{n} \right)^{-\alpha} (v_k - V_n) \left( |V_n| + \frac{1}{n} \right)^{-\beta} \right] \, dx.
\]
By Hölder inequality, it follows that

\[ \int_{\Omega} a(x) |u_k|^{q-2} u_k (u_k - U_n) \, dx \leq \|a\|_{L^{\infty}(\Omega)} \|u_k\|^q \|u_k - U_n\|_{L^r(\Omega)} \to 0 \]

as \( k \to \infty \). \hspace{1cm} (2.12)

Similarly,

\[ \int_{\Omega} b(x) |v_k|^{q-2} v_k (v_k - V_n) \, dx \leq \|b\|_{L^{\infty}(\Omega)} \|v_k\|^q \|v_k - V_n\|_{L^r(\Omega)} \to 0 \]

as \( k \to \infty \). \hspace{1cm} (2.13)

On the other hand, let \( \epsilon > 0 \), there exists \( R_\epsilon > 0 \) such that

\[ 0 \leq \int_{\Omega} c(x) |u_k - U_n|^{2(1-\alpha)} \, dx \]

\[ \leq ||c||_{L^{\frac{m}{m+\alpha-2}}} \left( \int_{B_{R_\epsilon}} |u_k - U_n|^m \, dx \right)^{\frac{2(1-\alpha)}{m}} + C_\epsilon \]

which gives that

\[ \int_{\Omega} c(x) |u_k - U_n|^{2(1-\alpha)} \, dx \to 0 \]

and

\[ c(x) \frac{1}{2(1-\alpha)} U_k \to c(x) \frac{1}{2(1-\alpha)} U_n \text{ in } L^{2(1-\alpha)}(\Omega). \] \hspace{1cm} (2.14)

Similarly, we obtain

\[ c(x) \frac{1}{2(1-\beta)} V_k \to c(x) \frac{1}{2(1-\beta)} V_n \text{ in } L^{2(1-\beta)}(\Omega). \] \hspace{1cm} (2.15)

Therefore, from Eqs. (2.14) and (2.15), there exists \( L_1 \in L^{2(1-\alpha)}(\Omega), L_2 \in L^{2(1-\beta)}(\Omega) \) and up to a subsequence

\[ c(x) \left( |u_k| + \frac{1}{n} \right)^{-\alpha} (u_k - U_n) \left( |v_k| + \frac{1}{n} \right)^{1-\beta} \to \]

\[ c(x) \left( |U_n| + \frac{1}{n} \right)^{-\alpha} (u_k - U_n) \left( |V_n| + \frac{1}{n} \right)^{1-\beta} \text{ a.e. in } \Omega, \]

\[ \leq |c| \left( |u_k|^{-\alpha} (u_k - U_n) \right) \left( |v_k|^{1-\beta} + 1 \right) \]

\[ \leq |c| \left( |u_k|^{-\alpha} (u_k - U_n) \right) \left( |v_k|^{1-\beta} + 1 \right) \]

\[ \leq |c| \left( |u_k|^{1-\alpha} (|v_k|^{1-\beta} + 1) \right) \leq |c| \left( |L_1|^{1-\alpha} |L_2|^{1-\beta} + |L_1|^{1-\alpha} \right) \in L^1(\Omega). \]

Therefore, by Dominated convergence theorem, we conclude that

\[ \int_{\Omega} c(x) \left( |u_k| + \frac{1}{n} \right)^{-\alpha} (u_k - U_n) \left( |v_k| + \frac{1}{n} \right)^{1-\beta} \, dx \to 0 \text{ as } k \to \infty. \] \hspace{1cm} (2.16)
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Using the same argument, we conclude that

\[ \int_{\Omega} C(x) \left( |v_k| + \frac{1}{n} \right)^{1-\alpha} \left( |v_k| + \frac{1}{n} \right)^{-\beta} (v_k - V_n) \, dx \to 0 \quad \text{as} \quad k \to \infty. \quad (2.17) \]

Now, from the inequality \(|a - b|^l \leq 2^{l-2} (|a|^{l-2} - |b|^{l-2}) (a - b)\) for all \(a, b \in \mathbb{R}^N\) and \(l \geq 2\), we obtain:

\[ o_k(1) = (I'_n (u_k, v_k) - I_n' (U_n, V_n), (u_k - U_n, v_k - V_n)) \geq \| (u_k - U_n) \|^p + \| v_k - V_n \|^p. \]

Now, combine Eqs. (2.12), (2.13), (2.16) and (2.17), we deduce that

\[ \| u_k - U_n \| \to 0 \quad \text{and} \quad \| v_k - V_n \| \to 0 \quad \text{as} \quad k \to \infty. \]

Hence, there exists \((U_n, V_n) \in X_0\) such that

\[ E_n (U_n, V_n) = \inf_{(u, v) \in X_0} E_n (u, v). \]

Moreover, since \(E_n(U_n, V_n) = E_n(|U_n|, |V_n|)\) and \((|U_n|, |V_n|) \in X_0\), so we may assume \((U_n, V_n) \geq 0\). Hence, \((U_n, V_n)\) is a non-trivial non-negative solution of the problem (2.1).

\[ \square \]

3. Proof of Theorem 1.1

This section is devoted to prove the existence of a solution to the problem (1.1). The proof is done in several Steps.

**Step 1:** The solutions \((U_n)\) and \((V_n)\) of the problem (2.1) are uniformly bounded.

Firstly, we have proved in section 2 that \((U_n, V_n)\) is a solution to the problem 2.1. Then, taking \((U_n, 0)\) as test function in the weak formulation of the problem (2.1), we obtain

\[ \int_Q \frac{|U_n(x) - U_n(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = \int_{\Omega} a(x)|U_n|^q \]

\[ + \frac{1 - \alpha}{2 - \alpha - \beta} \int_{\Omega} c(x)(|U_n| + \frac{1}{n})^{-\alpha} (|V_n| + \frac{1}{n})^{1-\beta} U_n \, dx \]

\[ \leq \| a \|_{\infty, \Omega} \frac{N+sp}{p+sp} \| U_n \|^q + \frac{1 - \alpha}{2 - \alpha - \beta} \int_{\Omega} c(x)(|U_n|^{1-\alpha} + 1)|V_n|^{1-\beta} \, dx. \quad (3.1) \]

In the same way, taking \((0, V_n)\) as test function in the weak formulation of the problem (2.1), we get

\[ \int_Q \frac{|V_n(x) - V_n(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = \int_{\Omega} a(x)|V_n|^q \]

\[ + \frac{1 - \alpha}{2 - \alpha - \beta} \int_{\Omega} c(x)(|U_n| + \frac{1}{n})^{-\alpha} (|V_n| + \frac{1}{n})^{-\beta} V_n \, dx \]

\[ \leq \| a \|_{\infty, \Omega} \frac{N+sp}{p+sp} \| V_n \|^q + \frac{1 - \alpha}{2 - \alpha - \beta} \int_{\Omega} c(x)(|U_n|^{1-\alpha} + 1)|V_n|^{1-\beta} \, dx. \quad (3.2) \]
Now, combine Eq. (3.1) with Eq. (3.2) and using Hölder and Sobolev inequalities, we deduce that

$$
||(U_n, V_n)||^p \leq c_0 ||(U_n, V_n)||^q + c_1 ||(U_n, V_n)||^{2-\alpha-\beta} + c_2 ||(U_n, V_n)||^{1-\alpha} + c_3 ||(U_n, V_n)||^{1-\beta} + c_4
$$

which yields that \((U_n)\) and \((V_n)\) are uniformly bounded in \(X_0\).

Now, let us prove a priori estimate in \(L^\infty(\Omega)\) of the solution \(U_n\) and \(V_n\). Define \(v_M = (M - U_n)^+\). For \(0 < t < 1\) define \(\xi(t) = I_n(U_n + tv_M, V_n + tv_M)\). Thus, using \((v_M, 0)\) as test function, we obtain

$$
\xi'(t) = \langle I_n(U_n + tv_M, V_n + tv_M), (v_M, 0) \rangle
= \langle (-\Delta_p)^s(U_n + tv_M, V_n + tv_M) - a(x)U_n + tv_M|^{q-2}(U_n + tv_M)
- \frac{1 - \alpha}{2 - \alpha - \beta} (|U_n + tv_M| + \frac{1}{\alpha})^{-\alpha} (|V_n + tv_M| + \frac{1}{\beta})^{-\beta}, v_M \rangle.
$$

Similarly,

$$
\xi'(1) = \langle I_n(U_n + v_M, V_n + v_M), (v_M, 0) \rangle
= \langle I_n(M, V_n + v_M), (v_M, 0) \rangle
= \langle (-\Delta_p)^s(M - a(x)|M|^{q-2}M
- \frac{1 - \alpha}{2 - \alpha - \beta} (|M| + \frac{1}{\alpha})^{-\alpha} (|V_n + v_M| + \frac{1}{\beta})^{-\beta}, v_M \rangle < 0
$$

for sufficiently small \(M > 0\). Moreover,

$$
-\xi'(1) + \xi'(t) = \langle (-\Delta_p)^s(U_n + tv_M) - (-\Delta_p)^s(U_n + v_M) + a(x)U_n + v_M|^{q-2}(U_n + v_M) - a(x)U_n + tv_M|^{q-2}(U_n + tv_M)
+ c(x) \frac{1 - \alpha}{2 - \alpha - \beta} (|U_n + v_M| + \frac{1}{\alpha})^{-\alpha} (|V_n + v_M| + \frac{1}{\beta})^{-\beta}
- c(x) \frac{1 - \alpha}{2 - \alpha - \beta} (|U_n + tv_M| + \frac{1}{\alpha})^{-\alpha} (|V_n + tv_M| + \frac{1}{\beta})^{-\beta}, v_M \rangle.
$$

Since \(|s|^{q-2}s + \left(|s| + \frac{1}{\alpha}\right)^{-\alpha} \left(|s| + \frac{1}{\beta}\right)^{-\beta}\) is a uniformly nonincreasing function with respect to \(x \in \Omega\) for sufficiently small \(s > 0\). Also from the monotonicity of \((-\Delta_p)^s\) we have, for sufficiently small \(M > 0\), \(0 \leq \xi'(1) - \xi'(t)\). From the Taylor series expansion, we have \(\exists 0 < \theta < 1\) such that

$$
0 \leq I_n(U_n + v_M, V_n + v_M) - I_n(U_n, V_n)
= \langle I'_n(U_n + \theta v_M, V_n + \theta v_M), v_M \rangle
= \xi'(\theta).
$$

Thus for \(t = \theta\) we have \(\xi'(\theta) \geq 0\) which is a contradiction to \(\xi'(\theta) \leq \xi'(1) < 0\). We have verified \(v_M \equiv 0\) in \(\Omega\), that is \(U_n \leq M\). Similarly, by using \((0, (M - V_n)^+)\) as test function, we obtain \(V_n \leq M\).

**Step 2:** The solutions \((U_n)\) and \((V_n)\) of the problem (2.1) are positive almost every where in \(\Omega\).
The approximated approach for a singular problem

At first, taking into account that \((U_n, V_n)\) is a solution of the problem (2.1) and taking \((U_n^{-}, 0)\) as test function, we obtain

\[
\int_{Q} \frac{|U_{n}(x) - U_{n}(y)|^{p-2}(U_{n}(x) - U_{n}(y))(U_{n}^{-}(x) - U_{n}^{-}(y))}{|x-y|^{N+sp}} \, dx \, dy \\
= \int_{\Omega} a(x)|U_{n}|^{q-2}U_{n}U_{n}^{-} - \frac{1-\alpha}{2-\alpha - \beta} \int_{\Omega} c(x)(|U_{n}| + \frac{1}{n})^{-\alpha}(|V_{n}| + \frac{1}{n})^{1-\beta}U_{n}^{-} \, dx.
\]

(3.5)

Then, by the elementary inequality \((a-b)(a_- - b_-) \leq -(a_- - b_-)^2\) we have

\[
0 \leq \int_{Q} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u_{-}(x) - u_{-}(y))}{|x-y|^{N+ps}} \, dx \, dy \\
\leq - \int_{Q} \frac{|u(x) - u(y)|^{p-2}(u_{-}(x) - u_{-}(y))^{2}}{|x-y|^{N+ps}} \, dx \, dy.
\]

(3.6)

Therefore, since the right hand side of the Eq. (3.5) is nonnegative and not equivalently to 0, we can deduce from (3.6) that \(||U_n^-|| = 0\). Which implies that \(U_n\) is nonnegative and by the strong maximum principle we conclude that \(U_n\) is positive almost everywhere in \(\Omega\). In the same manner, we can prove that \(V_n\) is positive almost everywhere in \(\Omega\).

**Step 3: The solutions \((U_n)\) and \((V_n)\) of the problem (2.1) are bounded from below.**

From step 1, we know that \((U_n)\) and \((V_n)\) are bounded in \(X_0\). Therefore, by using a standard comparison argument (see Lemma 2.1 in [2] combine with the strong maximum principle version of the fractional p-laplace operator in [4] we conclude that for all \(K \subset \subset \Omega\) there exists \(C_K\) such that \(U_n(x) \geq C_K > 0\) for a.e. \(x \in K\) and for any \(n \in \mathbb{N}\). The boundedness from below of \(V_n\) follows by the same manner.

**Step 4: The problem (1.1) has a positive weak solution.**

Firstly, by Step 1, \((U_n)\) and \((V_n)\) are bounded in \(X_0\). Thus, since the space \(X_0\) is reflexive, there exists a subsequence, still denoted by \(\{U_n\}\) and \(\{V_n\}\), which weakly converges to, say, \(U, V \in X_0\) such that

\[
U_n \rightarrow U \text{ and } V_n \rightarrow V \quad \text{weakly in } X_0 \\
U_n \rightarrow U \text{ and } V_n \rightarrow V \quad \text{strongly in } L^{r}(\Omega) \text{ for } 1 \leq r < p_\ast^* \\
U_n \rightarrow U \text{ and } V_n \rightarrow V \quad \text{pointwise a.e. in } \Omega.
\]

(3.7)

Now, since \((U_n, V_n)\) is a positive solution of the problem (2.1), one has

\[
\int_{Q} \frac{|U_{n}(x) - U_{n}(y)|^{p-2}(U_{n}(x) - U_{n}(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}} \, dx \, dy \\
+ \int_{Q} \frac{|V_{n}(x) - V_{n}(y)|^{p-2}(V_{n}(x) - V_{n}(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}} \, dx \, dy \\
= \int_{\Omega} \left( a(x)|U_{n}|^{q-2}U_{n}\phi + b(x)|V_{n}|^{q-2}V_{n}\psi \right) \, dx
\]
Similarly, we obtain
\[
\frac{1 - \alpha}{2 - \alpha - \beta} \int_{\Omega} c(x) \left( |U_n| + \frac{1}{n} \right)^{-\alpha} \left( |V_n| + \frac{1}{n} \right)^{1 - \beta} \phi \, dx
\]
\[
+ \frac{1 - \beta}{2 - \alpha - \beta} \int_{\Omega} c(x) \left( |U_n| + \frac{1}{n} \right)^{-\alpha} \left( |V_n| + \frac{1}{n} \right)^{-\beta} \psi \, dx
\]
for all \((\phi, \psi) \in X_0\). Then, by the weak convergence of \(U_n\) and \(V_n\) to \(U\) and \(V\) respectively, we have \(|U_n(x) - U_n(y)|^{p-2}(U_n(x) - U_n(y))/|x-y|^{(N+sp)/p'}\) is bounded in \(L^{p'}(\mathbb{R}^{2N})\) where \(p'\) is the Hölder conjugate of \(p\) given by \(p' = \frac{p}{p-1}\), and converges to \(|U(x) - U(y)|^{p-2}(U(x) - U(y))/|x-y|^{(N+sp)/p'}\) in \(\mathbb{R}^{2N}\) and \((U(x) - U(y))/|x-y|^{(N+sp)/p} \in L^p(\mathbb{R}^{2N})\), so
\[
\int_{Q} \frac{|U_n(x) - U_n(y)|^{p-2}(U_n(x) - U_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}} \, dx \, dy
\]
\[
\to \int_{Q} \frac{|U(x) - U(y)|^{p-2}(U(x) - U(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}} \, dx \, dy,
\]
for any \(\phi \in C_c^\infty(\Omega)\). Similarly,
\[
\int_{Q} \frac{|V_n(x) - V_n(y)|^{p-2}(V_n(x) - V_n(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}} \, dx \, dy
\]
\[
\to \int_{Q} \frac{|V(x) - V(y)|^{p-2}(V(x) - V(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp}} \, dx \, dy,
\]
for any \(\psi \in X_0\).

On the other hand, since \(U_n \to U\) in \(L^r(\text{supp}(\phi))\) and \(V_n \to V\) in \(L^r(\text{supp}(\psi))\). We deduce, using the dominated convergence theorem, that
\[
\int_{\Omega} a(x)|U_n|^{q-2}U_n \phi \, dx \to \int_{\Omega} a(x)|U|^{q-1} \phi
\]
and
\[
\int_{\Omega} b(x)|V_n|^{q-2}V_n \psi \, dx \to \int_{\Omega} b(x)|V|^{q-1} \psi.
\]
Moreover, let \(\phi \in C_c^\infty(\Omega)\) with \(\text{supp}(\phi) = K \subset \subset \Omega\). By Step 3, there exists \(C_K\) such that
\[
c(x) \left( |U_n| + \frac{1}{n} \right)^{-\alpha} \left( |V_n| + \frac{1}{n} \right)^{1 - \beta} \phi \to c(x)U^{-\alpha}V^{1 - \beta} \phi \text{ a.e.}
\]
and
\[
\left| c(x) \left( |U_n| + \frac{1}{n} \right)^{-\alpha} \left( |V_n| + \frac{1}{n} \right)^{1 - \beta} \phi \right| \leq C_K^{-\alpha}(M + 1)^{1 - \beta}|c||\phi| \in L^1(\Omega).
\]
Applying, Lebesgue’s dominated convergence theorem
\[
\int_{\Omega} c(x)|(U_n| + \frac{1}{n})^{-\alpha}(|V_n| + \frac{1}{n})^{1 - \beta} \phi \, dx \to \int_{\Omega} c(x)U^{-\alpha}V^{1 - \beta} \phi \, dx.
\]
Similarly, we obtain
\[
\int_{\Omega} c(x)|U_n|^{1-\alpha}(|V_n| + \frac{1}{n})^{-\beta} \psi \, dx \to \int_{\Omega} c(x)U^{-\alpha}V^{1 - \beta} \psi \, dx.
\]
Therefore, using Eq. (3.8), Eq. (3.8), Eq. (3.8), Eq. (3.9), Eq. (3.10) and Eq. (3.11), we conclude that

\[
\int_{Q} \frac{|U(x) - U(y)|^{p-2}(U(x) - U(y))(\phi(x) - \phi(y))}{|x - y|^{N+sp}} \, dx \, dy \\
+ \int_{Q} \frac{|V(x) - V(y)|^{p-2}(V(x) - V(y))(\psi(x) - \psi(y))}{|x - y|^{N+sp}} \, dx \, dy
\]

\[
= \int_{\Omega} \left( a(x)|U|^{q-2}U\phi + b(x)|V|^{q-2}V\psi \right) dx + \frac{1 - \alpha}{2 - \alpha - \beta} \int_{\Omega} c(x)U^{-\alpha}V^{-\beta} \phi dx \\
+ \frac{1 - \beta}{2 - \alpha - \beta} \int_{\Omega} c(x)U^{1-\alpha}V^{-\beta} \psi dx
\]

for all \((\phi, \psi) \in C_{c}^{\infty}(\Omega) \times C_{c}^{\infty}(\Omega)\). Yielding, that \((U, V)\) is a positive solution to the problem (1.1) and finished the prove of the Theorem 1.1.

4. Uniqueness of solution

Firstly, let us recall the following elementary inequality needed for the proof of uniqueness of solutions.

**Lemma 4.1** (cf. [7, Lemma 2.3]). Let \(v_1, v_2 \in W^{s,p}_0(\Omega)\setminus\{0\}\). There exists a positive constant \(c_p\), depending only on \(p\), such that

\[
\langle (-\Delta_p)^s v_1 - (-\Delta_p)^s v_2, v_1 - v_2 \rangle \geq \begin{cases}
\frac{||v_1 - v_2||^2}{||v_1||^2 + ||v_2||^2} & \text{if } 1 < p < 2 \\
\frac{||v_1 - v_2||^p}{||v_1||^p + ||v_2||^p} & \text{if } p \geq 2.
\end{cases}
\]

We begin this section by introducing the following nonlocal pure singular problem:

\[
\begin{cases}
(-\Delta_p)^s u = \frac{1}{2} c(x)|u|^{1-2\alpha}, & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]  

(4.1)

Now, we state our first result.

**Lemma 4.2.** The problem (4.1) has at most a positive solution.

**Proof.** We proceed by contradiction. Suppose that the problem (4.1) has two positives solutions \(u_1\) and \(u_2\) that is

\[
\langle (-\Delta_p)^s u_1, \Phi \rangle = \frac{1}{2} \int_{\Omega} c(x)u_1^{1-2\alpha} \Phi dx
\]

and

\[
\langle (-\Delta_p)^s u_2, \Phi \rangle = \frac{1}{2} \int_{\Omega} c(x)u_2^{1-2\alpha} \Phi dx
\]

(4.3)

for all \(\Phi\) in \(C_{c}^{\infty}(\Omega)\). Subtracting (4.2) from (4.3), we have

\[
\langle (-\Delta_p)^s u_1 - (-\Delta_p)^s u_2, \Phi \rangle = \frac{1}{2} \int_{\Omega} c(x) \left(u_1^{1-2\alpha} - u_2^{1-2\alpha}\right) \Phi dx.
\]

(4.4)
Choose $\Phi = (u_1 - u_2)$ as a test function. So, from the equation (4.4) we obtain

$$\langle (-\Delta_p)^s u_1 - (-\Delta_p)^s u_2, (u_1 - u_2) \rangle = \frac{1}{2} \int_{\Omega} c(x) \left( u_1^{1-2\alpha} - u_2^{1-2\alpha} \right) (u_1 - u_2) \, dx$$

$$\leq 0.$$  \hfill (4.5)

Hence, according to Lemma 4.1, we must have $\|u_1 - u_2\| = 0$, it follows that $u_1 = u_2$ a.e. in $\Omega$. The proof of the Lemma 4.2.

**Proof of Theorem 1.2.** Again as in Lemma 4.2, we proceed by contradiction. We suppose that $(u_1, u_2)$ be the positive solution of the system

$$\begin{cases}
(-\Delta_p)^s u = \frac{1}{2} c(x)|u|^{-\alpha}|v|^{1-\alpha}, & \text{in } \Omega, \\
(-\Delta_p)^s v = \frac{1}{2} c(x)|u|^{1-\alpha}|v|^{-\alpha}, & \text{in } \Omega, \\
u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases} \hfill (4.6)$$

That is,

$$\langle (-\Delta_p)^s u_1, \Phi \rangle = \frac{1}{2} \int_{\Omega} c(x) u_1^{1-\alpha} u_2^{1-\alpha} \Phi \, dx$$ \hfill (4.7)

and

$$\langle (-\Delta_p)^s u_2, \Phi \rangle = \frac{1}{2} \int_{\Omega} c(x) u_1^{1-\alpha} u_2^{-\alpha} \Phi \, dx$$ \hfill (4.8)

for all $\Phi$ in $C^\infty_c(\Omega)$. Now, taking $\Phi = (u_1 - u_2)$ as a test function, we have

$$0 \leq \langle (-\Delta_p)^s u_1 - (-\Delta_p)^s u_2, (u_1 - u_2) \rangle$$

$$= \frac{1}{2} \int_{\Omega} c(x) \left( u_1^{1-\alpha} - u_2^{1-\alpha} \right) (u_1 - u_2)(u_1 - u_2) \, dx$$

$$\leq 0.$$  \hfill (4.9)

So, from equation (4.9) and according to Lemma 4.1, we must have $\|u_1 - u_2\| = 0$. Therefore, we get $u_1 = u_2$ almost everywhere in $\Omega$. Hence, $(u_1, u_1)$ is the unique solution to the system (4.6). This completes the proof of the Theorem 1.2.

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