On the Fatou theorem for $\overline{\partial}_J$-subsolutions in wedges

Alexandre Sukhov*

* Univ. Lille, Laboratoire Paul Painlevé, Departement de Mathématique, 59655 Villeneuve d’Ascq Cedex, France, sukhov@math.univ-lille1.fr The author is partially supported by Labex CEMPI.

Institut of Mathematics with Computing Centre - Subdivision of the Ufa Research Centre of Russian Academy of Sciences, 45008, Chernyshevsky Str. 112, Ufa, Russia.

Abstract. We prove a version of the Fatou theorem for bounded functions with a bounded $\overline{\partial}_J$ part of the differential on wedge-type domains in an almost complex manifold.

MSC: 32H02, 53C15.

Key words: almost complex manifold, $\overline{\partial}_J$-operator, pseudoholomorphic disc, wedge-type domain, totally real manifold, the Fatou theorem.

1 Introduction

The present paper is a continuation of the work [14]. Our goal is to study the boundary behavior of certain classes of functions on almost complex manifolds with boundary. It is well-known that non-constant holomorphic functions do not exit (even locally) on an almost complex manifold $(M, J)$ (of complex dimension $> 1$) with an almost complex structure $J$ in general position. This makes natural to study the functions satisfying suitable assumptions on the $\overline{\partial}_J$-part of their differential: indeed, in this case the problem of existence does not arise. Various aspects of the boundary behavior of functions in $\mathbb{C}^n$ whose $\overline{\partial}$ (with respect to the standard complex structure) part of differential is of some prescribed growth, have been explored by several authors [8, 5, 9, 10, 2]. Their results admit important applications in Several Complex Variables.

We extend some of the well-known results on boundary values of bounded holomorphic functions (see [1, 4, 11, 2]) of several complex variables to the almost complex case. Note that our main results are new also in the case of the space $\mathbb{C}^n$ equipped with the standard complex structure. The main result is Theorem 3.2 establishing a Fatou type theorem for domains with generic corners (wedges). I also mention that, despite the fact the the obtained results concern the classes of functions much larger than the holomorphic ones, the presence of an (almost) complex structure is crucial. In particular, this is due to the fact that we are working with low-dimensional submanifolds of the boundary which are transverse to an almost complex structure (totally real manifolds). Also, pseudoholomorphic curves (introduced in [3]) are our main technical tool. We note that the main difficulty of the proof is that in the case of wedge type domains the Chirka-Lindelöf principle does not assure a non-tangential convergence. This obstacle is a principal difference with respect to the case of smooth boundaries and it considerably complicates the proof. This is the main motivation for the present paper.
The paper is organized as follows. Section 2 is preliminary and contains a brief presentation of the theory of almost complex manifolds and their properties. In Section 3 we present our main result. Section 4 contains its proof.

2 Preliminaries: almost complex manifolds and their maps

Here we briefly recall basic notions concerning almost complex manifolds; a detailed presentation is contained for example in [14]. Everywhere through this paper we assume that manifolds and almost complex structures are of class $C^\infty$ (the word "smooth" means the regularity of this class); we notice however that the main results remain true under considerably weaker regularity assumptions.

Let $M$ be a smooth manifold of real dimension $2n$. An almost complex structure $J$ on $M$ is a smooth map which associates to every point $p \in M$ a linear isomorphism $J(p) : T_pM \to T_pM$ of the tangent space $T_pM$ such that $J(p)^2 = -Id$; here $Id$ denotes the identity map of $T_pM$. Thus, every linear map $J(p)$ is a complex structure (in the usual sense of Linear Algebra) on a real vector space $T_pM$. A couple $(M, J)$ is called an almost complex manifold of complex dimension $n$.

A basic example is given by the standard complex structure $J_{st} = J_{st}(2)$ on $M = \mathbb{R}^2$; it is represented in the canonical coordinates of $\mathbb{R}^2$ by the matrix

$$ J_{st}(2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} $$

(1)

More generally, the standard complex structure $J_{st}$ on $\mathbb{R}^{2n}$ is represented by the block diagonal matrix $J_{st} = \text{diag}(J_{st}(2), \ldots, J_{st}(2))$ (here and below we drop the notation of dimension). Putting $iv := Jv$ for $v \in \mathbb{R}^{2n}$, we identify $(\mathbb{R}^{2n}, J_{st})$ with $\mathbb{C}^n$; we use the notation $z = x + iy = x + Jy$ for the standard complex coordinates $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Let $(M, J)$ and $(M', J')$ be smooth almost complex manifolds. A $C^1$-map $f : M' \to M$ is called $(J', J)$-complex or $(J', J)$-holomorphic if it satisfies the Cauchy-Riemann equations

$$ df \circ J' = J \circ df. $$

(2)

For example, a map $f : \mathbb{C}^n \to \mathbb{C}^m$ is $(J_{st}, J_{st})$-holomorphic if and only if each component of $f$ is a usual holomorphic function. In this special case the equations (2) coincide with the usual Cauchy-Riemann equations in their real form. Note that in general the first order PDE elliptic system (2) does not split on independent equations for components of $f$.

Every almost complex manifold $(M, J)$ can be viewed locally as the Euclidean unit ball $\mathbb{B}^n$ (or any other domain) in $\mathbb{C}^n$ equipped with a small (in any $C^m$-norm) almost complex deformation of $J_{st}$. The following well-known statement is often very useful.

**Lemma 2.1** Let $(M, J)$ be an almost complex manifold of complex dimension $n$. Then for every point $p \in M$, every $m \geq 0$ and $\lambda_0 > 0$ there exist a neighborhood $U$ of $p$ and a coordinate diffeomorphism $z : U \to \mathbb{B}^n$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$, and the direct image $z_\ast(J) := dz \circ J \circ dz^{-1}$ satisfies $\|z_\ast(J) - J_{st}\|_{C^m(\mathbb{B}^n)} \leq \lambda_0$.

A simple proof is contained for example in [14].
In what follows we often denote the direct image $z_\ast(J)$ of $J$ again by $J$, viewing it as a matrix representation of $J$ in the local coordinate system $(z)$. Of course, the coordinate map $z$ is $(J, z_\ast(J))$-biholomorphic. However, in general $z_\ast(J)$ does not coincide with $J_{st}$ in a neighborhood of the origin in $\mathbb{C}^n$. Recall that an almost complex structure $J$ is called integrable if $(M, J)$ is locally biholomorphic in a neighborhood of each point to an open subset of $(\mathbb{C}^n, J_{st})$. In the case of complex dimension 1 every almost complex structure is integrable. In the case of complex dimension $> 1$ integrable almost complex structures form a highly special subclass in the space of all almost complex structures on $M$. An efficient criterion of integrability is provided by the classical theorem of Newlander - Nirenberg [6]: the entries of $J$ must satisfy some PDE system.

In the special case where $M'$ has the complex dimension 1, the solutions $f$ of (2) are called $J$-complex (or $J$-holomorphic or pseudoholomorphic ) curves. Note that we view here the curves as maps i.e. we consider parametrized curves. We use the notation $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ for the unit disc in $\mathbb{C}$ (i.e. $\mathbb{B}^1$) always assuming that it is equipped with the standard complex structure $J_{st}$. Considering the equations (2) with $M' = \mathbb{D}$, we call such a map $f$ a $J$-complex disc or a pseudoholomorphic disc or just a holomorphic disc when a structure $J$ on the target space is fixed. If a disc $f$ is continuous up to the boundary $b\mathbb{D}$ of $\mathbb{D}$, then the restriction of $f$ on $b\mathbb{D}$ is called the boundary of $f$. Let $\gamma$ be a non-empty subset of $b\mathbb{D}$ and let $K$ be a subset of $M$. If $f(\gamma) \subset K$, we say that $f$ is attached or glued to $K$ along $\gamma$. In this paper $\gamma$ usually will be the upper half-circle.

A fundamental fact is that pseudoholomorphic discs always exist in a suitable neighborhood of any point of $p \in M$; furthermore, one can choose such a disc tangent to any prescribed direction $v \in T_p M$. These discs depend smoothly on deformation of $J$, $p$ and $v$. Furthermore, one can view them as a small deformation of discs in usual complex lines. This is the classical Nijenhuis-Woolf theorem (see [7]). For the proof and other applications it is convenient to rewrite the equations (2) in local coordinates similarly to the complex version of the usual Cauchy-Riemann equations.

Our considerations are local, so assume that we are in a neighborhood $\Omega$ of 0 in $\mathbb{C}^n$ with the standard complex coordinates $z = (z_1, ..., z_n)$. We assume that $J$ is an almost complex structure defined on $\Omega$ and $J(0) = J_{st}$. Let a $C^1$-map

$$z : \mathbb{D} \to \Omega,$$

$$z : \zeta \mapsto z(\zeta)$$

be a $J$-complex disc. The equations (2) can be rewritten in the equivalent form

$$z\zeta - A(z)\overline{z}\zeta = 0, \quad \zeta \in \mathbb{D}. \quad (3)$$

where we use the notation $z\zeta = \partial z/\partial \zeta$. Here a smooth map $A : \Omega \to \text{Mat}(n, \mathbb{C})$ is defined by the equality $L(z)v = A_\overline{\zeta}$ for any vector $v \in \mathbb{C}^n$ and $L$ is an $\mathbb{R}$-linear map defined by $L = (J_{st} + J)^{-1}(J_{st} - J)$. It is easy to check that the condition $J^2 = -Id$ is equivalent to the fact that $L$ is $\mathbb{C}$-linear. The matrix $A(z)$ is called the complex matrix of $J$ in the local coordinates $z$ (see [12]). Locally the correspondence between $A$ and $J$ is one-to-one. Note that the condition $J(0) = J_{st}$ means that $A(0) = 0$.

If $t$ are other local coordinates and $A'$ is the corresponding complex matrix of $J$ in the coordinates $t$, then, as it is easy to check, we have the following transformation rule:
\[ A' = (t_z A + t_{\overline{z}})(\overline{T_T + T_z A})^{-1} \]  
(4)

(see the proof in [12]).

For the convenience of readers, I sketch here the proof of the above mentioned Nijenhuis-Woolf theorem because this standard construction will be used below in the proof of the main results. Recall that for a complex function \( f \) the Cauchy-Green transform \( T f \) is defined by

\[ T f(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\omega) d\omega \wedge d\overline{\omega}}{\omega - \zeta} \]  
(5)

This classical integral operator has the following properties:

(i) \( T : C^r(D) \to C^{r+1}(D) \) is a bounded linear operator for every non-integer \( r > 0 \) (a similar property holds in the Sobolev scale). Here we use the usual Hölder norm on the space \( C^r(D) \).

(ii) \( (T f)_\overline{\zeta} = f \) i.e. \( T \) solves the \( \overline{\partial} \)-equation in the unit disc.

(iii) the function \( T f \) is holomorphic on \( \mathbb{C} \setminus \overline{D} \).

Now fix a real non-integer \( r > 1 \). Let \( z : D \to \mathbb{C}^n, z : D \ni \zeta \mapsto z(\zeta) \) be a \( J \)-complex disc. Since the operator

\[ \Psi_J : z \mapsto w = z - T_A(z)z_\overline{\zeta} \]

takes the space \( C^r(D) \) into itself, we can write the equation (3) in the form

\[ \Psi_J(z) = h \]

where \( h \) is an arbitrary holomorphic (with respect to \( J_{st} \)) vector-function. Thus, the disc \( z \) is \( J \)-holomorphic if and only if the disc \( h = \Psi_J(z) : D \to \mathbb{C}^n \) is \( J_{st} \)-holomorphic. When the norm of \( A \) is small enough (which is assured by Lemma 2.1), then the operator \( \Psi_J \) is a small deformation of the identity and by the implicit function theorem this operator is invertible. Hence we obtain a one-to-one correspondence between \( J \)-holomorphic discs and usual \( J_{st} \)-holomorphic discs. This easily implies the existence of a \( J \)-holomorphic disc in a given tangent direction through a given point of \( M \) (choosing a suitable complex linear disc as \( h \)), as well as a smooth dependence of such a disc on a deformation of a point or a tangent vector, or on an almost complex structure; this also establishes the interior elliptic regularity of discs.

Now we can define the \( \overline{\partial}_J \)-operator on an almost complex manifold \( (M, J) \). Consider first the situation when \( J \) be an almost complex structure defined in a domain \( \Omega \subset \mathbb{C}^n \); one can view this as a local coordinate representation of \( J \) in a chart on \( M \).

A \( C^1 \) function \( F : \Omega \to \mathbb{C} \) is \( (J, J_{st}) \)-holomorphic if and only if it satisfies the Cauchy-Riemann equations

\[ F_T + F_z A(z) = 0, \]  
(6)
where $F_z = (\partial F / \partial z_1, ..., \partial F / \partial z_n)$ and $F = (\partial F / \partial z_1, ..., \partial F / \partial z_n)$ are viewed as row-vectors. Generally the only solutions to (6) are constant functions unless $J$ is integrable (then $A$ vanishes identically in suitable coordinates). Note also that (6) is an overdetermined linear PDE system while (3) is a quasilinear PDE for a vector function on $\mathbb{D}$.

Every 1-differential form $\phi$ on $(M, J)$ admits a unique decomposition $\phi = \phi^{1,0} + \phi^{0,1}$. In particular, if $F : (M, J) \rightarrow \mathbb{C}$ is a $C^1$-complex function, we have $dF = dF^{1,0} + dF^{0,1}$. We use the notation $\partial J F = dF^{1,0}$ and $\overline{\partial} J F = dF^{0,1}$.

In order to write these operators explicitly in local coordinates, we find a local basis in the space of (1,0) and (0,1) forms. We view $dz = (dz_1, ..., dz_n)^t$ and $d\bar{z} = (d\bar{z}_1, ..., d\bar{z}_n)^t$ as vector-columns. Then the forms

$$\alpha = (\alpha_1, ..., \alpha_n)^t = dz - A d\bar{z}$$

and $\overline{\alpha} = d\bar{z} - \overline{A} dz$ form a basis in the space of (1,0) and (0,1) forms respectively. Indeed, it suffices to note that for 1-form $\beta$ is (1,0) (resp. (0,1)) for if and only if for every $J$-holomorphic disc $z$ the pull-back $z^* \beta$ is a usual (1,0) (resp. (0,1)) form on $\mathbb{D}$. Using the equations (8) we obtain the claim.

Now we decompose the differential $dF = F_z dz + F_{\bar{z}} d\bar{z} = \partial J F + \overline{\partial} J F$ in the basis $\alpha, \overline{\alpha}$ using (8) and obtain the explicit expression

$$\overline{\partial} J F = (F_{\bar{z}} (I - \overline{A} A)^{-1} + F_z (I - A \overline{A})^{-1} A) \overline{\alpha}$$

It is easy to check that the holomorphy condition $\overline{\partial} J F = 0$ is equivalent to (6) because $(I - A \overline{A})^{-1} A (I - \overline{A} A) = A$. Thus

$$\overline{\partial} J F = (F_{\bar{z}} + F_z A) (I - \overline{A} A)^{-1} \overline{\alpha}$$

We note that the term $(I - \overline{A} A)^{-1}$ as well as the forms $\alpha$ affect only the non-essential constants in local estimates of the $\overline{\partial} J$-operator near a boundary point which we will perform in the next sections. So we may assume that in local coordinates this operator is simply given by the left hand expression of (6).

### 3 Main result

First we introduce the main class of domains for this paper.

Let $M$ be an almost complex manifold. A generic manifold $E$ of real codimension $k$ in $M$ can be defined as

$$E = \{ p \in M : \rho_j(p) = 0, j = 1, ..., k \}$$

where $\rho_j : M \rightarrow \mathbb{R}$ are smooth real functions satisfying

$$\overline{\partial} J \rho_1 \wedge ... \wedge \overline{\partial} J \rho_k \neq 0$$

near $E$. Precisely as in the case of an integrable structure, this means that the tangent space $T_p E$ spans $T_p M$ i.e. the complex hull of $T_p E$ coincides with $T_p M$. If additionally $k = n$ (the maximal
possible value of $k$ compatible with the assumption (11), then $E$ is *totally real*. It is equivalent to the fact that for every $p$ the holomorphic tangent space $T_p E \cap J(T_p E)$ is trivial.

A domain $W = W(E)$ of the form

$$W(E) = \{ p \in M : \rho_j(p) < 0, j = 1, \ldots, k \}$$

is called a wedge with the edge $E$. Of course, if $k = 1$ we have the usual smoothly bounded domains.

Let $\Omega$ be a bounded domain in an almost complex manifold $(M, J)$. We always assume that $\Omega$ is a wedge $W(E)$ of type (12) with the edge $E$.

Fix a hermitian metric on $M$ compatible with $J$; a choice of such metric will not affect our results because it changes only constant factors in estimates. We measure all distances and norms with respect to the chosen metric.

Let $p \in E$ be a point of the edge. Fix local coordinates $z$ on $M$ near $p$ such that $p = 0$ in these coordinates. A *cone* $K \subset \Omega$ with the vertex $p$ is defined as the set of $z \in \Omega$ such in the above local coordinates $K$ is a usual circular cone with vertex at the origin and directed by some ray $l \subset \Omega$.

A non-tangential approach to $E$ at $p$ can be defined as the limit along the sets $K$. Clearly, this notion is independent of choice of local coordinates.

**Definition 3.1** A function $F : \Omega \to \mathbb{C}$ admits a non-tangential limit $L$ at $p \in E$ if

$$\lim_{K \ni z \to p} F(z) = L$$

for each cone $K \subset \Omega$ with vertex at $p$.

As above, this definition is independent of a choice of local coordinates and metrics.

The main result of the present paper is the following version of the Fatou theorem.

**Theorem 3.2** Let $(M, J)$ be an almost complex manifold of complex dimension $n \geq 1$ and $W(E)$ be a wedge with a totally real edge $E$ in $M$. Suppose that $F \in L^\infty(W(E))$ is a complex function of class $C^1$ on $W(E)$ and $\partial J F$ is bounded on $W(E)$. Then $F$ admits a non-tangential limit almost everywhere on $E$.

Of course, the interesting case arises only for $n > 1$. Note that this result is new also in the case where $M = \mathbb{C}^n$ and $J$ coincides with the standard complex structure $J_{st}$. Note also that in the case where the edge $E$ is not totally real but only a generic manifold with non-zero tangent space, the convergent regions are tangent to $E$ along the holomorphic tangent space of $E$, as usual in this type of problems (see [1, 2, 11, 14]). The assumption of the boundedness of $\partial J F$ also may be weakened. We drop the technical details focusing our presentation on the key case.

### 4 Proof of Theorem 3.2

Our approach is based on the works [1, 4, 11, 14]. The proof of Theorem contains several steps.
4.1 One-dimensional case

Recall some boundary properties of subsolutions of the $\overline{\partial}$-operator in the unit disc.

Denote by $W^{k,p}(\mathbb{D})$ the usual Sobolev classes of functions admitting generalized partial derivatives up to the order $k$ in $L^p(\mathbb{D})$ (in fact we need only the case $k = 0$ and $k = 1$). In particular $W^{0,r}(\mathbb{D}) = L^r(\mathbb{D})$. We will always assume that $p > 2$.

Denote also by $\|f\|_\infty = \sup_D |f|$ the usual sup-norm on the space $L^\infty(\mathbb{D})$ of complex functions bounded on $\mathbb{D}$.

**Lemma 4.1** Let $f \in L^\infty(\mathbb{D})$ and $f_\zeta \in L^p(\mathbb{D})$ for some $p > 2$. Then

(a) $f$ admits a non-tangential limit at almost every point $\zeta \in \partial\mathbb{D}$.

(b) if $f$ admits a limit along a curve in $\mathbb{D}$ approaching $b\mathbb{D}$ non-tangentially at a boundary point $e^{i\theta} \in b\mathbb{D}$, then $f$ admits a non-tangential limit at $e^{i\theta}$.

(c) for each positive $r < 1$ there exists a constant $C = C(r) > 0$ (independent of $f$) such that for every $\zeta_j \in \mathbb{D}$, $j = 1, 2$ one has

$$|f(\zeta_1) - f(\zeta_2)| \leq C(\|f\|_\infty + \|f_\zeta\|_{L^p(\mathbb{D})})|\zeta_1 - \zeta_2|^{1-2/p}$$

(13)

The proof is contained in [14].

Sometimes it is convenient to apply the part (c) of Lemma on the disc $\rho D$ with $\rho > 0$. Let $g \in L^\infty(\rho D)$ and $g_\zeta \in L^p(\rho D)$. The function $f(\zeta) := g(\rho \zeta)$ satisfies the assumptions of Lemma 4.1 on $\mathbb{D}$. Let $0 < \alpha < \rho$ and let $|\tau_j| < \alpha$, $j = 1, 2$. Set $\zeta_j = \tau_j/\rho$. Then $|\zeta_j| < r = \alpha/\rho < 1$, $j = 1, 2$. Applying (c) Lemma 4.1 to $f$ we obtain:

$$|g(\tau_1) - g(\tau_2)| \leq (C(r)/\rho^{1-2/p})(\|g\|_\infty + \rho \|g_\zeta\|_{L^p(\rho D)})|\tau_1 - \tau_2|^{1-2/p}$$

(14)

Note that $C = C(r) = C(\alpha/\rho)$ depends only on the quotient $r = \alpha/\rho < 1$. If $r$ is separated from 1, the value of $C$ is fixed.

4.2 Attaching discs to a totally real manifold

Our proof of Theorem 3.2 uses the properties of a family of pseudoholomorphic discs constructed in [13]. For the sake of completeness I briefly recall this construction; the proofs are contained in [13]. Note also that this construction is well known in the case of the standard complex structure.

(a) First consider the model case where $M = \mathbb{C}^n$ with $J = J_{st}$ and $E = i\mathbb{R}^n = \{x_j = 0, j = 1,..., n\}$. Denote by $W$ the standard wedge $W_0 = \{z = x + iy : x_j < 0, j = 1,..., n\}$.

Consider the family of complex lines in $\mathbb{C}^n$:

$$l : (c, t, \zeta) \mapsto (\zeta, \zeta t + ic) \in \mathbb{C}^n$$

(15)

Here $\zeta \in \mathbb{C}$; the variables $c = (c_2, ..., c_n) \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}^{n-1}_+ = \{t = (t_2, ..., t_n) \in \mathbb{R}^{n-1} : t_j > 0\}$ are viewed as parameters. Hence we write $l(c, t, \zeta) = l(c, t)(\zeta)$. Denote by $V$ the wedge $V = \mathbb{R}^{n-1}_+ \times \mathbb{R}^{n-1}_+$. Also let $\Pi = \{\text{Re} \zeta < 0\}$ be the left half-plane; its boundary $b\Pi$ coincides with the imaginary axis $i\mathbb{R}$. The following properties of the above family are easy to check:
(a1) the images $l(c,t)(b\Pi)$ form a family of real lines in $i\mathbb{R}^n = E$. For every fixed $t \in \mathbb{R}^n_+$ these lines are disjoint and
\[ \bigcup_{c \in \mathbb{R}^{n-1}} l(c,t)(b\Pi) = E. \]
In other words, for every $t$ this family (depending on the parameter $c$) forms a foliation of $E$ by parallel lines.

(a2) one has
\[ \bigcup_{(c,t) \in V} l(c,t)(\Pi) = W_0. \]

(a3) For every fixed $t \in \mathbb{R}^{n-1}_+$, one has
\[ \bigcup_{c \in \mathbb{R}^{n-1}} l(c,t)(\Pi) = E_t = \{ z \in \mathbb{C}^n : \Re (z_j - t_jz_1) = 0, j = 2, ..., n \} \cap W_0 \]
and the union is disjoint. Every $E_t$ is a real linear $(n+1)$-dimensional half-space contained in $W_0$ and $bE_t = E$.

(a4) the family $(E_t), t \in \mathbb{R}^{n-1}_+$ is disjoint in $W_0$ and its union coincides with $W_0$.

In what follows we will use these properties locally in a neighborhood of the origin. It is convenient to reparametrize the family of complex half-lines $l(c,t)$ by complex discs. Consider the Schwarz integral:
\[ S\phi(\zeta) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{\omega + \zeta}{\omega - \zeta} \phi(\omega) \frac{d\omega}{\omega} \]  
(16)
For a non-integer $r > 1$ consider the Banach spaces $C^r(b\mathbb{D})$ and $C^r(\mathbb{D})$ (with the usual Hölder norm). It is classical that $S$ is a bounded linear map in these classes of functions. For a real function $\phi \in C^r(b\mathbb{D})$ the Schwarz integral $S\phi$ is a function of class $C^r(\mathbb{D})$ holomorphic in $\mathbb{D}$; the trace of its real part on the boundary coincides with $\phi$ and its imaginary part vanishes at the origin.

In order to fill $W_0$ by complex discs glued to $i\mathbb{R}^n$ along the (closed) upper semi-circle $b\mathbb{D}^+ = \{ e^{i\theta} : \theta \in [0,\pi] \}$ we have to reparametrize the above family of complex lines. Set also $b\mathbb{D}^- := b\mathbb{D} \setminus b\mathbb{D}^+$. Fix a smooth real function $\phi : b\mathbb{D} \to [-1,0]$ such that $\phi|b\mathbb{D}^+ = 0$ and $\phi|b\mathbb{D}^- < 0$.

Consider now a real $2n$-parametric family of holomorphic discs $z^0 = (z_1^0, ..., z_n^0) : \mathbb{D} \to \mathbb{C}^n$ with components
\[ z_j^0(c,t)(\zeta) = x_j(\zeta) + iy_j(\zeta) = t_jS\phi(\zeta) + ic_j, j = 1, ..., n \]  
(17)
Here $t_j > 0$ and $c_j \in \mathbb{R}$ are parameters.

Obviously, every $z^0(c,t)(\mathbb{D})$ is a subset of $l(c,t)(\Pi)$ and $z^0(b\mathbb{D}^+) = l(c,t)(b\Pi)$. Thus, the family $z^0(c,t)$ is a (local) biholomorphic reparametrization of the family $l(c,t)$. As a consequence, the properties (a1)-(a5) also hold for the family $z^0(c,t)$. Notice also the following obvious properties of this family:

(a6) for every $j$ one has $x_j|b\mathbb{D}^+ = 0$ and $x_j(\zeta) < 0$ when $\zeta \in \mathbb{D}$ (by the maximum principle for harmonic functions).

(a7) the evaluation map $Ev_0 : (c,t,\zeta) \mapsto z^0(c,t)(\zeta)$ is one-to-one from $V \times \mathbb{D}$ to $W_0$. 

8
In the general case consider a totally real manifold $E$ and the wedge $W = W(E)$ given by (12). Applying the implicit function theorem in suitable local coordinates, one can assume that $E$ is defined by the vector equation $x = h(y)$ where $h(0) = 0$, and $dh(0) = 0$. Using the Cauchy-Green operator and the Schwarz integral, one can write a non-linear integral equation such that its solutions of the form

$$(c, t, \zeta) \mapsto z(c, t)(\zeta) \quad (18)$$

are $J$-complex discs glued to $E$ along $b\mathbb{D}^+$. Since $h(y) = o(|y|)$, the family $z(c, t)$ is a small deformation of the family $z^0(c, t)$ in any $C^m$ norm, $m > 1$. This follows by the implicit function theorem solving the above mentioned integral equation (see details in [14]). Hence, the geometric properties of obtained discs remain similar to the above model case: indeed, the properties of linear discs (a1)-(a5) are stable under small perturbations.

(b) For reader’s convenience we state explicitly the properties of the family (18).

Fix $\delta > 0$. The family $z(c, t) : \overline{\mathbb{D}} \to W$ of pseudoholomorphic discs smooth on $\overline{\mathbb{D}}$ and smoothly depending on real parameters $t = (t_2, ..., t_n)$, $t_j > 0$ and $c \in \mathbb{R}^{n-1}$, satisfies the following properties:

(b1) the images $z(c, t)(b\mathbb{D}^+)$ form a family of real curves in $E$. For every fixed $t \in \mathbb{R}_+^{n-1}$ these curves are disjoint and

$$\bigcup_{c \in \mathbb{R}^{n-1}} z(c, t)(b\mathbb{D}^+) = E.$$ 

In other words, for every $t$ this family (depending on the parameter $t$) forms a foliation of $E$. Furthermore, every disc is contained in $W = W(E)$.

(b2) one has the inclusion

$$W_\delta = \{ z : \rho_j - \delta \sum_{k \neq j} \rho_k < 0 \} \subset \bigcup_{(c, t) \in \mathcal{V}} z(c, t)(\mathbb{D}).$$

(b3) For every fixed $t \in \mathbb{R}_+^{n-1}$, the union

$$E_t := \bigcup_{c \in \mathbb{R}^{n-1}} z(c, t)(\mathbb{D}) \subset W$$

is a real generic $(n + 1)$-dimensional manifold with boundary $bE_t = E$.

(b4) the family $(E_t)$, $t \in \mathbb{R}_+^{n-1}$ is disjoint and its union contains $W_\delta$.

4.3 Around the Chirka-Lindelöf principle

Here we introduce an analog of the Chirka-Lindelöf principle [1] for wedges in almost complex manifolds. This is one the main technical tools of our proof. Note that in [1] the situation is considered in full generality (for integrable complex structures) with minimal assumptions on regularity of domains. It is observed their that in the case of domains with piecewise smooth boundaries, for a bounded holomorphic function an existence a boundary limit along a smooth curve (transverse to the boundary) does not imply an existence of a non-tangential limit at a boundary point. This is a serious difference with respect to smoothly bounded domains and one of the main technical
obstacles in the proof of our main result. Nevertheless, in the non-smooth case the convergence along a curve implies an existence of the limit along any curve with the same tangent line at a boundary point.

Let \( W = W(E) \) be a wedge with the edge \( E \) in an almost complex manifold \((M, J)\) of dimension \( > 1 \) and let \( p \) be a point of \( E \).

A curve \( \gamma : [0, 1] \rightarrow W(E) \) is called \( p\)-admissible if the following assumptions are satisfied:

(i) \( \gamma \) of class \( C^\infty[0, 1] \) and \( p = \gamma(1) \in E \);

(ii) \( \gamma \) is not tangent to any face \( \{ \rho_j = 0 \} \), \( j = 1, ..., n \) at \( p \). In particular, \( \gamma \) is not tangent to \( E \).

**Proposition 4.2** Let \( W(E) \) be a wedge with the edge \( E \) in an almost complex manifold \((M, J)\) of dimension \( > 1 \), and let \( F \) satisfies assumptions of Theorem 3.2. If \( F \) has a limit along a \( p\)-admissible curve \( \gamma_1 \) at \( p \in E \), then \( F \) has the same limit along each curve in \( W(E) \) tangent to \( \gamma_1 \) at \( p \).

**Proof.** Let \( \gamma_2 \) be another curve satisfying the assumptions (i), (ii) and such that \( \gamma_1 \) and \( \gamma_2 \) have the same tangent line at \( p \). Without loss of generality assume that \( p = 0 \) (in local coordinates).

It follows by the Nijenhuis-Woolf theorem that there exists a family \( z_t(\zeta) : D \rightarrow \mathbb{C}^n \), of embedded \( J \)-holomorphic discs near the origin in \( \mathbb{C}^n \) satisfying the following properties:

(i) the family \( z_t \) is smooth on \( \overline{D} \times [0, 1] \)

(ii) for every \( t \in [0, 1] \) the disc \( z_t \) transversally intersects each curve \( \gamma_j \) at a unique point corresponding to some parameter value \( \zeta_j(t) \in D \), \( t \in [0, 1], \ j = 1, 2 \). In other words \( \gamma_j(t) = z_t(\zeta_j(t)) \).

Furthermore, \( \zeta_1(t) = 0 \), i.e. this point is the center of the disc \( z_t \).

In the case of the standard complex structure each such a disc is simply an open piece (suitably parametrized) of a complex line intersecting transversally the both of curves \( \gamma_j \). Recall that the curves are embedded near the origin and tangent at the origin so such a family of complex lines obviously exist. The \( J \)-holomorphic discs are obtained from this family of lines by a small deformation described in the proof of the Nijenhuis-Woolf theorem in Section 2.

Furthermore, because of the condition (i), the restrictions \( F \circ z_t \) have \( \zeta \)-derivatives bounded on \( D \) uniformly with respect to \( t \). Indeed, it follows by the Chain Rule and (3) that

\[
(F \circ z)_\zeta = (F_z + F_z A)_{\zeta}\zeta
\]

and now we use the assumption that \( \partial J F \) is bounded.

Since the curves \( \gamma_j \) are tangent at the origin, we have

\[
|\zeta_2(t)| = o(1 - t)
\]

as \( t \rightarrow 1 \).

The curve \( \gamma_1 \) is admissible, so we have

\[
\text{dist}(\gamma_1(t), bW) = O(1 - t)
\]
as \( t \to 1 \). Hence, there exists \( \rho(t) = O(1 - t) \) as \( t \to 1 \) such that \( z_t(\rho(t)V) \) is contained in \( W \).

Applying (14) to the composition \( f := F \circ z_t(\zeta) \) on the disc \( \rho(t)\mathbb{D} \), we obtain (fixing \( r > 0 \))

\[
|f(0) - f(\zeta_2(t))| \leq (C/O(1 - t)^{1-2/p})(\| f \|_\infty + O(1 - t) \| f_\zeta \|_\infty) o((1 - t)^{1-2/p}) \to 0
\]

(20)
as \( t \to 1 \). Note that by (19) for every \( t \) the point \( \zeta_2(t) \) is contained in \((1/2)\rho(t)\mathbb{D}\); hence, the constant \( C \) is independent of \( t \) (see remark after (14)). This concludes the proof.

4.4 Convergence along families of rays

Under some additional assumption one can assure a non-tangential convergence. Fix local coordinates such that \( E = i\mathbb{R}^n, W = W(E) = \{x_j < 0, j = 1, ..., n\}, J(0) = J_{st}. \) Such a change of coordinates is always possible by Lemma 4.1. Through the remaining part of this paper we assume that such local coordinates are fixed.

Let \( z(c, t)(\zeta) \) be a \( J \)-complex disc constructed in the subsection 4.2. It follows by Lemma 4.1 that the composition \( F \circ z(c, t) \) admits non-tangential limits almost everywhere on \( b\mathbb{D}^+ \).

Let \( p \in E \). Assume \( F \circ z(c, t) \) admits a radial limit at point \( \zeta^0 \in b\mathbb{D}^+ \) and \( z(c, t)(\zeta^0) = p \). The image \( \gamma_p \) of this radial segment by disc \( z(c, t) \) in general is not a segment of some real line in \( \mathbb{C}^n \), but only an admissible curve. Hence it follows by Proposition 4.2 that \( F \) admits the same limit along the real ray \( l_p \subset W \) with vertex at \( p \) and tangent to \( \gamma_p \) at \( p \). It is enough to consider the case \( p = 0 \).

Lemma 4.3 Let \( \Lambda \subset W \) be the set of rays with vertex at 0. Assume that \( \Lambda \) is the uniqueness set for holomorphic (with respect to the standard structure \( J_{st} \)) functions. Suppose that \( F(z) \) admits a limit along any ray from \( \Lambda \) as \( z \to 0 \). Then \( F \) admits a limit along any ray contained in a cone \( K \subset W \). If the limits are the same for all rays from \( \Lambda \), the \( F \) admits a non-tangential limit at 0.

Proof. We present the proof in three steps.

Step 1. Consider a sequence of functions \( F_k(z) = F(z/k), k = 1, 2, ..., \) We need the following analog of the Montel compactness principle:

Lemma 4.4 The family \( (F_k) \) contains a subsequence converging uniformly on compacts in any cone \( K \) to a function \( F^0 \) holomorphic with respect to \( J_{st} \).

This is a consequence of Lemma 4.1. Indeed, it suffices to prove the equicontinuity. By the Nijenhuis-Woolf theorem any ball small enough is foliated by the psedoholomorphic discs through the center (of course, this foliation is singular at the center of the ball). Such a foliation is a small deformation of the foliation of the ball by complex lines through its center. Thus, we apply Lemma 4.1 to the restriction of \( F_k \) on each disc and conclude that the sequence \( (F_k) \) is equicontinuous on this ball. This implies the above mentioned compactness of this family and proves Lemma 4.3.

We continue the proof of Lemma 4.3. Passing to the limit we obtain that \( \mathcal{T}_{J_{st}} F^0 = 0 \) in the sense of distributions which implies that \( F^0 \) is a usual holomorphic function (with respect to \( J_{st} \)). Indeed we have

\[
|F_2(z) + A(z) F_z(z)| \leq C
\]
Hence for \( \varepsilon > 0 \) we obtain

\[
|F_{\varepsilon z}(\varepsilon z) + A(\varepsilon z)F_z(\varepsilon z)| \leq C
\]

Therefore

\[
|\varepsilon F_{\varepsilon z}(\varepsilon z) + A(\varepsilon z)\varepsilon F_z(\varepsilon z)| \leq \varepsilon C
\]

and as a consequence

\[
|(F(\varepsilon z))_{\varepsilon z} + A(\varepsilon z)(F(\varepsilon z))_z| \leq \varepsilon C
\]

Since \( F(\varepsilon z) \rightarrow F^0(z) \) converges in the sense of distributions as \( \varepsilon \to 0 \) and \( A(0) = 0 \), we obtain that \( F^0 = 0 \) in the sense of distributions and so \( F^0 \) is a usual holomorphic function.

**Step 2.** Since \( F \) admits a limit along any ray from \( \Lambda \), we obtain that \( F^0 \) is constant along such a ray. Thus given \( \alpha > 0 \) one has \( F^0(\alpha z) = F^0(z) \) for all \( z \in \Lambda \). But \( \Lambda \) is a uniqueness set. Therefore, the last identity (with fixed \( \alpha \)) holds for all \( z \in W \) and \( F^0 \) is constant already along any ray in \( W \). This implies that \( F \) admits a limit along each ray.

**Step 3.** If the limits are the same , say, \( L \), along all rays from \( \Lambda \) and \( \Lambda \) is the uniqueness set, we obtain that \( F^0 = L \) on \( \Lambda \) and hence \( F^0 \equiv L \) on \( W \). This implies Lemma.

### 4.5 Limits along rays almost everywhere

Now we study the existence of limits along rays. Fix a family \( l_z \) of rays smoothly depending on \( z \in E = i\mathbb{R}^n \) such that \( l_z \subset W \).

**Lemma 4.5** Let \( K_z \subset W \) be a family of open cones smoothly depending on \( z \), with vertex at \( z \) and directed by \( l_z \). For almost every \( z \in E \) the function \( F \) admits a limit along any ray in \( K_z \).

Note that at this moment we do not yet claim that the limits are the same independency of a ray with the same vertex.

**Proof.** We begin with the model flat case where \( J = J_{st} \). Consider the family of flat complex discs \( z(c, t) \) given by (17). Let \( S^{n-1} \) denotes the unit sphere in \( \mathbb{R}^n \). Denote by \( \Sigma \) a countable dense set in \( S^{n-1} \cap \mathbb{R}^n_+ \). Set \( \Sigma = \bigcup_{j \in \mathbb{N}} i^j \) and fix some \( \iota^j \). By Lemma 4.1 the restriction of \( F \) on every disc \( z(c, \iota^j) \) admits a non-tangential limit almost everywhere on \( bD^+ \) i.e. on the full measure subset \( Y_j \) in \( bD^+ \). Note that the image of every ray with vertex in \( bD^+ \) by \( z(c, \iota^j) \) belongs to \( \mathbb{R}^n + ic \) (and its parallel translation to 0 belongs to \( \mathbb{R}^n \)). When \( c \) runs over a neighborhood of 0, the images \( z(c, \iota^j)(Y_j) \) sweep a full measure set \( X_j \) in \( E = i\mathbb{R}^n \). The intersection \( X = \bigcap_j X_j \) of such sets is again a full measure set in \( E = i\mathbb{R}^n \). For every point \( p \in X \) and every \( j \) the function \( F \) admits a limit along the ray with vertex at \( p \) and parallel to \( \iota^j \). But \( \Sigma \) is dense in \( S^{n-1} \cap \mathbb{R}^n_+ \) so for every fixed \( p \in X \) these rays form a uniqueness set for usual holomorphic functions. Therefore, we can apply Lemma 4.3 at \( p \).

The proof in the general case of any almost complex structure \( J \) follows by a perturbation argument. Indeed, indeed, we fill \( W \) by discs \( z(c, v) \) of the form (18) constructed in subsection 4.2. In general, the images of rays by \( z(c, v) \) are not rays in \( W \), but they are the admissible curves. By Proposition 4.2 the function \( F \) admits a limit along any ray tangent to such curve at its boundary point on \( E \). Then the above argument goes through literally. This completes the proof.
4.6 Non-tangential limits

Here we conclude the proof of theorem establishing the following

**Lemma 4.6** $F$ admits non-tangential limits almost everywhere on $E$.

**Proof.** Recall again that $E = i\mathbb{R}^n$. Fix a unit vector $v$ such that $z + l$ belongs to $W$, where $z \in E$ and the ray $l$ is directed by $v$. Consider a family of cones $K_z$ with the vertex $z$ and directed by the vector $l$; we assume that $K_z$ smoothly depends on $z$. By Lemma 4.5 there exists a full measure subset $\tilde{E}$ of $E$ such that the function $F$ admits a limit $F^*$ along the ray $z + l$ with vertex at $z \in \tilde{E}$. Consider the sequence of functions $f_m(z) = \sup_{0 < t \leq 1/m} |F(z + tv) - F^*(z)|$, where $z \in \tilde{E}$ and $m > 0$, $m \in \mathbb{N}$. This sequence converges to a function 0 for almost every $z \in \tilde{E}$. Applying the Egorov theorem, we conclude that for each $\delta > 0$ there exists a subset $E_\delta \subset \tilde{E}$ such that

(i) $m(\tilde{E} \setminus E_\delta) < \delta$ (here $m(X)$ denotes the Lebesgue $n$-measure of the subset $X \subset E = i\mathbb{R}^n$).

(ii) the sequence $(f_m)$ converges uniformly to 0 on $E \setminus E_\delta$.

In particular, the functions $F(z + (1/m)v)$ are continuous on $E$ and converge to $F^*$ as $m \to \infty$. Hence, the function $F^*$ is continuous on $\tilde{E} \setminus E_\delta$ as the uniform limit of a sequence of continuous functions. Note also that (by Lemma 4.5) one can assume that the function $F$ admits a limit along any ray (not only $l$) with vertex at each point $z$ of $\tilde{E} \setminus E_\delta$. Recall that by the Lebesgue theorem almost every point of $\tilde{E} \setminus E_\delta$ is a density point with the density equal to 1. We claim that $F$ admits a non-tangential limit at such a point.

Assume that $0 \in \tilde{E} \setminus E_\delta$ is a density point. Also we may assume that the limit of $F$ along $l$ at 0 is equal to 0 i.e. $F^*(0) = 0$. Let $(z^k)$ be a sequence in an arbitrary ray $l_1$ in $K_0$, $(z^k)$ is converging to 0. We prove that there exists a subsequence $(z^{k_q})$ such that $F(z^{k_q}) \to 0$ as $q \to \infty$. This obviously implies the claim.

Set $r^k = |z^k|$. Then the point $\tilde{z} = z^k/r^k$ belongs to $l_1$, $|\tilde{z}| = 1$. Passing again to a subsequence, assume that $\tilde{z}$ is independent of $k$. Consider the sequence of functions $\tilde{F}_k(z) = F(r^kz)$. It follows from Lemma 4.4 that one can extract a subsequence $\tilde{F}_{k_q}$ (in what follows we skip the subindex $q$) converging uniformly on compacts of $K_0$ to a function $\tilde{F}$ which is a usual holomorphic function on $K_0$. Since $F$ admits a limit along every ray at the origin, the function $\tilde{F}$ is constant along every ray with vertex at the origin (as in the proof of Lemma 4.3). It suffices to show that $\tilde{F} = 0$. Indeed, in this case we obtain

$$\lim_{k \to \infty} F(z^k) = \lim_{k \to \infty} \tilde{F}_k(z^k/r^k) = \tilde{F}(\tilde{z}) = 0.$$  

Consider the set $\Lambda$ formed by the real half-lines $L = p + l$, $p \in i\mathbb{R} \setminus E_\delta$. Clearly, $\Lambda$ is contained in the subspace spanned by $E$ and $l$. $\Lambda$ is a generic (with respect to $J_{am}$) half-space contained in $W$, with the boundary $E$. Consider a sequence of points points $(q^k)$ contained in a ray $l_2 \subset \Lambda$ (with the vertex at the origin) and converging to 0. We choose the sequence $(q^k)$ such that $r^kq^k = p^k + w^k$, where $p^k \in i\mathbb{R} \setminus E_\delta$ and a vector $w^k$ belongs to the ray $l$. Since 0 is a density point, the set of rays in $\Lambda$ admitting such a sequence, is a full measure subset of $\Lambda$. The sequence $r^kq^k$ tends to 0, hence the sequences $p^k$ and $w^k$ tend to 0 as well. Therefore

$$\lim_{k \to \infty} \tilde{F}_k(q^k) = \lim_{k \to \infty} F(r^kq^k) = \lim_{k \to \infty} F(p^k + w^k).$$
In view of the uniform convergence of the sequence of functions \((f_k)\) (introduced at the beginning of the proof of Lemma) on \(\tilde{E} \setminus E_\delta\), and the continuity of the function \(F^*\) at 0, the last limit is equal to \(F^*(0) = 0\). Hence \(\tilde{F} = 0\) on the ray \(l_2\). Thus the function \(\tilde{F}\) vanishes on a full measure subset of \(\Lambda\) which is the uniqueness set for holomorphic functions. Hence \(\tilde{F} = 0\) on \(K_0\) and \(F\) tends to 0 along any ray. By Lemma 4.3 \(F\) admits a non-tangential limit 0 at the origin. Hence, \(F\) admits a non-tangential limit almost everywhere on \(\tilde{E} \setminus E_\delta\) because almost all points are density points (with the density 1).

Finally, consider the sets \(E_k = E_\delta\) for \(\delta = 1/k\). The intersection \(\Sigma = \cap_k E_k\) has the measure 0 and \(F\) admits limits almost everywhere outside \(\Sigma\), We conclude that \(F\) admits non-tangential limits almost everywhere on \(E\). This completes the proof.

References

[1] Chirka, E. *The Lindelöf and Fatou theorems in \(\mathbb{C}^n\)*, Math. U.S.S.R Sb. 21(1973), 619-641.
[2] Forstnerič, F. *Admissible boundary values of bounded holomorphic functions in wedges*, Trans. Amer. Math. Soc. 332 (1992), 583-593.
[3] Gromov, M. *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307-347.
[4] Y.Khurumov, *On the Lindelöf theorem in \(\mathbb{C}^n\)*, Dokl. Akad Nauk SSSR, 273 (1983), 1325-1328.
[5] Nagel, A., Rudin, W. *Local boundary behavior of bounded holomorphic functions*, Can. J. Math. XXX(1978), 583-592.
[6] Newlander, A., Nirenberg, L. *Complex analytic coordinates in almost complex manifolds*, Ann. Math. 65(1957), 391-404.
[7] Nijenhuis, A., Woolf, W. *Some integration problems in almost - complex and complex manifolds*, Ann. Math. 77(1963), 424-489.
[8] Nirenberg, L., Webster, S., Yang, P. *Local boundary regularity of holomorphic mappings*, Commun. Pure Appl. Math. 33(1980), 305-338.
[9] Pinchuk, S., Khansanov, S. *Asymptotically holomorphic functions and their applications*, Mat. USSR Sb. 62 (1989), 541-550
[10] Rosay, J.-P. *A propos des "wedges" et d’ "edges" et de prolongements holomorphes*, Trans. Amer. Math. Soc. 297(1986), 63-72.
[11] Sadullaev, A. *A boundary uniqueness theorem in \(\mathbb{C}^n\)*, Matem. Sb. bf 101(1976), 568-583.
[12] Sukhov, A., Tumanov, A. *Filling hypersurfaces by discs in almost complex manifolds of dimension 2*, Indiana Univ. Math. J. 57(2008), 509-544.
[13] Sukhov, A. *Pluripolar sets, real submanifolds and pseudoholomorphic discs*, J.Australian Math. Soc. (2019), 1-19, doi: 10.1017/S1446788719000119.
[14] Sukhov, A. *The Chirka-Lindelöf and Fatou type theorems for \(\partial J\)-subsolutions*, Revista Math. Iberoamericana, 36 (2020), 1469-1487.