Abstract. We describe and examine a test for shape constraints, such as monotonicity, convexity (or both simultaneously), U-shape, S-shape and others, in a nonparametric framework using partial sums empirical processes. We show that, after a suitable transformation, its asymptotic distribution is a functional of the standard Brownian motion, so that critical values are available. However, due to the possible poor approximation of the asymptotic critical values to the finite sample ones, we also describe a valid bootstrap algorithm.

1. INTRODUCTION

Hypothesis testing is one of the most relevant tasks in empirical work. Tests include the situation when the null and alternative hypothesis are assumed to belong to a parametric family of models. In a second type of tests, known as diagnostic or lack-of-fit tests, only the null hypothesis is assumed to belong to a parametric family leaving the alternative nonparametric. The later type of testing has a distinguished and long literature starting with the work of Kolmogorov, see Stephens (1992), for testing the probability distribution function and in a time series context by Grenander and Rosenblatt (1957) for testing the white noise hypothesis. In a regression model context a new avenue of work started in Stute (1997), and Andrews (1997) with a more econometric emphasis, using partial sums empirical methodology, see also Stute et al. (1998) or Koul and Stute (1999) among others. The methodology has attracted a lot of attention and it rivals tests based on a direct comparison between a parametric and a nonparametric fit to the regression model as first examined in Härdle and Mammen’s (1992) or Hong and White (1996). One advantage of tests based on partial sums empirical methodology, when compared to Härdle and Mammen’s (1992) approach, is that the former does not require the choice of a bandwidth parameter for its implementation, see also Nikabadze and Stute (1996) for some additional advantages. However, a possible drawback is that the asymptotic distribution depends, among other possible features, on the estimator of the model under the null hypothesis in a nontrivial way, as it was shown in Durbin (1973), and hence its implementation requires either bootstrap algorithms, see Stute et al. (1998), or the so-called Khmaladze’s (1981) martingale transformation, see also for earlier work and ideas Brown, Durbin and Evans (1975).

In this paper, though, we are interested on a third type of testing where neither the null hypothesis nor the alternative has a specific parametric form. These type of hypothesis testing can be denoted as testing for qualitative or shape restrictions.

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Monotonicity and monotonicity-related properties are widespread in economics and other fields. For instance, a demand function is expected to be a decreasing function of the price of a good, whereas a supply function is expected to be increasing. In single-object auctions, the equilibria analyzed commonly are those in which buyers play monotone strategy functions, see for instance Krishna (2010). In other economic relationships, it is often of importance whether the marginal returns are increasing or decreasing, which naturally amounts to convexity or concavity, respectively. It may also be of interest to analyse statistical and economic relationships that do not have a persistent shape pattern on the whole domain but rather switch the patterns once in the domain (for example, U-shaped or S-shaped relations).

To fix ideas, consider the nonparametric regression model

\[ y_i = m(x_i) + u_i, \]

with \( E[u_i|x_i] = 0 \). More specific conditions on the sequences \( \{u_i\}_{i \in \mathbb{Z}} \) and \( \{x_i\}_{i \in \mathbb{Z}} \) will be given in Condition C1 below. Our main aim is testing whether the regression function \( m(x) \) possesses some type of shape constraints such as monotonicity, convexity, or U-shape with the “switch” at some, possibly unknown, point \( s_0 \). For instance, if we were interested in the null hypothesis of (increasing) monotonicity, we might write it as

\[ H_0 : m(x_1) \leq m(x_2) \quad \text{when} \quad x_1 \leq x_2 \]

being the alternative hypothesis \( H_1 \) the negation of the null. Similarly, the null hypothesis of convexity can be written as

\[ H_0 : m(x_1) - 2m(x_2) + m(x_3) \geq 0 \quad \text{when} \quad x_1 < x_2 < x_3. \]

U-shape is the property of a function first decreasing and then increasing. So, we write the null hypothesis of U-shape with the switch at \( s_0 \) as

\[ H_0 : \begin{cases} m(x_1) > m(x_2) & \text{when} \quad x_1 < x_2 \leq s_0 \\ m(x_1) > m(x_2) & \text{when} \quad x_1 > x_2 \geq s_0. \end{cases} \]

It is worth remarking that the methodology that we introduce in Section 2 can easily accommodate testing simultaneously for several shape constraints, e.g. we can adapt our procedure to test whether the regression function is both monotone and convex. Section 5 discusses in more detail these scenarios and some extensions.

1.1. LITERATURE REVIEW ON TESTING FOR SHAPES.

There is a range of tests for the monotonicity of the conditional mean (or isotonic regressions) suggested in the statistical literature. Bowman, Jones and Gijbels (1998) propose a test analogous to Silverman’s (1981) test of multimodality in density estimation. Hall and Heckman (2000) suggest a test for monotonicity which does not require the estimation of the regression function and is based on a “running gradient” approach over short blocks. Their focus is on improving the power of a monotonicity test in marginal cases such as when the curve has a flat section or a small downwards dip. Ghosal, Sen and Van der Vaart (2000) propose a monotonicity test that involves a locally weighted version of Kendall’s tau statistic, and Chetverikov (2012) adapts their approach to the unknown smoothness of the regression function. Juditsky and Nemirovski (2002) propose to test that in white-noise models, the signal belongs to the cone of positive/increasing/convex functions. Wang and Meyer (2011) suggest a test for monotonicity or convexity
by using the constrained and unconstrained regression spline estimators, although they do not provide any asymptotic theory.

Other literature related to testing for monotonicity or convexity is Schlee (1982), Diack and Thomas-Agnan (1998), Dumbgen and Spokoiny (2001), Abrevaya and Jiang (2005), Baraud, Huet and Laurent (2005), Dette, Hoderlein and Neumayer (2016), Hoderlein et al. (2016), Birke, Neumeyer, Vogushev (2016), Ahkim, Gijbels, Verhasselt (2017). In particular, Diack and Thomas-Agnan (1998) propose tests of convexity of a smooth regression function based on least squares of hybrid splines. Baraud, Huet and Laurent (2005) develop techniques that allow to test for convexity of the regression function (together with monotonicity and positivity) in the case of Gaussian white noise errors, and Abrevaya and Jiang (2005) suggest a global test of curvature (linearity, convexity, concavity) based on a simplex statistic.

Some of the work referenced above (such as Baraud, Huet and Laurent (2005), Dumbgen and Spokoiny (2001), Juditsky and Nemirovski (2002)) focuses on the regression function in the ideal Gaussian white noise model, while our framework does not require such parametric assumptions. In the aforementioned literature some papers (such as Baraud, Huet and Laurent (2005), Diack and Thomas-Agnan (1998), Hall and Heckman (2000)) allow the explanatory variable to take only deterministic values. Ghosal, Sen and Van der Vaart (2000) and Abrevaya and Jiang (2005) treat the explanatory variable as random but either require its full stochastic independence with the unobserved regression error (Ghosal, Sen and Van der Vaart (2000)) or require the distribution of the error to be symmetric conditional on the explanatory variable (Abrevaya and Jiang (2005)), whereas we require a weaker mean independence of error condition. Some of the above-mentioned papers (such as Bowman, Jones and Gijbels (1998), Hall and Heckman (2000), Wang and Meyer (2011)) do not give any asymptotic theory. Many of the approaches are tailored to a specific type of shape and their extensions to more general shape properties do not appear straightforward (if at all possible). Moreover, these tests are often targeted to detecting specific deviations from the null hypothesis. The violations of the null can come from different sources.

The aim of this paper is to overcome some of the problems discussed above by giving, among other features, a unified framework to test shape/qualitative constraints. For that purpose, we propose a test based on a transformation, introduced in Khmaladze (1981), of the partial sums empirical process similar to that in Stute (1997). Some of the properties of the test is that it converges to a standard Brownian motion, so that critical values of standard functionals such as Kolmogorov-Smirnov, Cramér-von-Mises or Anderson-Darling are readily obtained. As a consequence, our testing procedure has the same asymptotic distribution regardless of the shape constrained under consideration. Another feature of our testing procedure is its flexibility as it is able to test simultaneously for more than one shape constrained, for instance testing for convexity and increasing. Finally, the test is very easy to implement as it requires no more than “recursive” least squares.

The remainder of the paper is organized as follows. Next section introduces and motivates our nonparametric estimator of $m(x)$ and it compares the methodology against rival nonparametric estimators. In Section 3, we describe the test and examine its statistical properties. Because the Monte-Carlo experiment suggests that the asymptotic critical values are not a good approximation of the finite sample
ones, Section 4 introduces a bootstrap algorithm. Section 5 presents a Monte-Carlo experiment and some empirical examples, whereas Section 6 concludes with a summary and possible extensions of the methodology. The proofs are confined to the Appendix A which employs a series of lemmas given in Appendix B.

2. METHODOLOGY AND REGULARITY CONDITIONS

Before we present the testing procedure for the hypothesis testing given in (1.2), or in (1.3) say, it is convenient to introduce the type of nonparametric estimator that we shall use to estimate the regression function \( m(x) \) in (1.1) under \( H_0 \) and/or \( H_1 \). Several nonparametric estimators have been proposed to estimate the model (1.1) under the null hypothesis. Indeed, the literature on isotone/monotone regressions goes back to Brunk (1955) and Wright (1981). Friedman and Tibshirani (1984) combine local averaging and isotonic regression, Mukerjee (1988) and Mammen (1991a) propose a two-step approach, which involves smoothing the data by using kernel regression estimators in the first step, and then deals with the isotonization of the estimator by projecting it into the space of all isotonic functions in the second step. Hall and Huang (2001) propose an alternative method based on tilting estimation which preserves the optimal properties of the kernel regression estimator. Finally, a different approach to isotonization is based on rearrangement methods using a probability integral transformation, see Dette, Neumeyer and Pilz (2006) or Chernozhukov, Fernandez-Val and Galichon (2009). Even though some of these techniques can potentially be considered as a first step in our testing procedure for monotonicity, the Khmaladze’s transformation in this case would be either extremely difficult or maybe even impossible to implement. However, these techniques are more narrow in their applicability than our method as they only deal with monotonicity, besides that their implementation is not trivial and/or they often lack any asymptotic theory useful for the purpose of inference.

With regard to the null hypothesis in (1.3), i.e. convexity, Hildreth (1954) proposes to estimate \( m(x) \) by least squares approach, with the consistency established by Hanson and Pledger (1976). Mammen (1991b) derives only the rate of convergence, whereas Groeneboom, Jongbloed and Wellner (2001) derive the asymptotic distribution of this estimator at a point where the second derivative of the regression function is strictly positive (the second derivative also has to be continuous in a neighbourhood of this point). The global behaviour of such an estimator is discussed in Guntuboyina and Sen (2013). Birke and Dette (2007) examine an estimator based on first obtaining unconstrained estimate of the derivative of the regression function which is isotonized and then integrated. Unlike the least squares convex estimator, their estimator is smooth. Again, these approaches could potentially be used as a first step in our testing procedure for convexity/concavity but the scope of their applicability is somewhat narrow and the Khmaladze’s transformation would be difficult to implement.

A different approach to kernels might be based on series estimation using polynomial and in particular Bernstein polynomials basis. One motivation for the latter polynomials is due to their shape-preserving property, making them a natural and appealing method of estimation in our context. However, they have an undesirable property of being highly correlated, making them difficult to utilize for our purposes – in particular, to obtain a valid Khmaladze’s transformation, which plays a key role in our results. This is discussed in more detail in Section 2.4. For that reason,
in this paper we shall employ \textit{B-splines} and/or penalized \textit{B-splines} basis known as \textit{P-splines}. As we will see later, \textit{B-splines} or \textit{P-splines} share some key features with Bernstein polynomials in that our null hypothesis can be written in terms of the coefficients of the approximation induced by the regression splines.

2.1. \textbf{B-SPLINES (or P-SPLINES)}.

\textit{B-splines} or \textit{P-splines} are constructed from polynomial pieces joined at some specific points denoted knots. Their computation is obtained recursively, see de Boor (1978), for any degree of the polynomial. It is well understood that the choice of the number of knots determines the trade-off between overfitting when there are too many knots, and underfitting when there are too few knots. The main difference between \textit{B-splines} and \textit{P-splines} is that the latter tend to employ a large number of knots but to avoid oversmoothing they incorporate a penalty function based on the second difference of the coefficients of adjacent \textit{B-splines}, in contrast to the second derivative employed in O’Sullivan (1986, 1988), see Eilers and Marx (1996).

The methodology and applications of constrained \textit{B-splines} and \textit{P-splines} are discussed by many authors, too many to review here. For a detailed discussion of \textit{B-splines}, see de Boor (1978) and Dierckx (1993). For \textit{P-splines}, a detailed coverage can be found in Eilers and Marx (1996), Bollaerts, Eilers and van Mechelen (2006). Other literature on shape-preserving splines includes, among others, Ramsay (1998), Li, Naik and Swetits (1996), Beliakov (2000), Mammen and Thomas-Agnan (1999), Turlach (2005) and Meyer (2008).

In general, the \textit{B-spline} basis of degree \(q\)

- takes positive values on the domain spanned by \(q + 2\) adjacent knots, and is zero otherwise;
- consists of \(q + 1\) polynomial pieces each of degree \(q\), and the polynomial pieces join at \(q\) inner knots;
- at the joining points, the \(q - 1\)th derivatives are continuous;
- except at the boundaries, it overlaps with \(2q\) polynomials pieces of its neighbours;
- at a given \(x\), only \(q+1\) \textit{B-splines} are nonzero.

Assume that one is interested in approximating the regression function \(m(x)\) in an interval \([a, b]\). Then the interval \([a, b]\) is split into \(L'\) equal length subintervals with \(L' + 1\) knots\(^1\) where each subinterval will be covered with \(q + 1\) \textit{B-splines} of degree \(q\). The total number of knots needed will be \(L' + 2q + 1\) (each boundary point \(a, b\) is a knot of multiplicity \(q + 1\)) and the number of \textit{B-splines} is \(L = L' + q\). Thus, \(m(x)\) is approximated by a linear combination of \textit{B-splines} of degree \(q\) with coefficients \((\beta_1, \ldots, \beta_L)\) as

\[
m_B(x; L) = \sum_{\ell=1}^{L} \beta_\ell p_{\ell,L}(x; q),
\]

and where henceforth we shall denote the knots as \(\{z^k\}, \ k = 1, \ldots, L' + 2q + 1\), where \(a = z^1 = \ldots = z^{q+1}\) and \(b = z^{L' + q + 1} = \ldots = z^{L' + 2q + 1}\).

\(^1\)Although it is possible to have nonequidistant subintervals, for simplicity we consider equally spaced knots. An alternative way to locate the knots is based on the quantiles of the \(x\) distribution.
B-splines share some properties which turns out to be very useful for our purpose. Indeed,

\[ (a) \quad \sum_{\ell=1}^{L} p_{\ell,L}(x;q) = 1 \quad \text{for all } x \text{ and } q. \]

\[ (b) \quad \frac{\partial}{\partial x} m_B(x;L) = : m'_B(x;L) = q \sum_{\ell=1}^{L-1} \frac{\Delta \beta_{\ell+1}}{z_{\ell+1+q} - z_{\ell+1}} p_{\ell+1,L}(x;q-1), \]

where \( \Delta \beta_{\ell} = \beta_{\ell} - \beta_{\ell-1} \). In particular, (a) indicates that B-splines, as is the case with Bernstein polynomials, are a partition of 1. The property (b) states that the derivative of a B-spline of degree \( q \) becomes a B-spline of degree \( q - 1 \).

Using this expression for the derivative and taking into account that the knot system effectively changes with the first and the last knots now removed (thus, the multiplicity of \( a \) and \( b \) becomes \( q \) rather than \( q + 1 \)), one can derive an expression for the second derivative. It is exactly the property (b) which makes B-splines, or P-splines, very attractive for the purpose of testing shape restrictions. Indeed, because the polynomials \( p_{\ell,L}(x;q) \) are nonnegative by construction for any \( q \), property (b) implies that monotonicity, say increasing, is guaranteed by \( \Delta \beta_{\ell} \geq 0, \ell = 2, \ldots, L \).

The conditions that guarantee convexity are slightly more involved due to the multiplicity of the boundary knots but can still be formulated as linear inequalities involving coefficients \( \beta_{\ell} \). In particular, if the \( L' \) intervals are equidistant, then the conditions for the convexity can be formulated as

\[ \Delta \beta_{\ell} \geq \Delta \beta_{\ell-1}, \quad \ell = q + 2, \ldots, L' + 1, \]

\[ (q-1)\Delta \beta_{q+1} \geq q\Delta \beta_q, \quad (q-2)\Delta \beta_q \geq (q-1)\Delta \beta_{q-1}, \ldots, \Delta \beta_3 \geq 2\Delta \beta_2, \quad (q-1)\Delta \beta_{L'+1} \leq q\Delta \beta_{L'+2}, \quad (q-2)\Delta \beta_{L'+2} \leq (q-1)\Delta \beta_{L'+3}, \ldots, \Delta \beta_{L-1} \leq 2\Delta \beta_L. \]

Because \( L \to \infty \), the constraints \( \Delta \beta_{\ell} \geq \Delta \beta_{\ell-1} \) will be increasingly more important and, thus, for simplicity of the constraint formulation, one could potentially ignore an increasingly smaller number of modified linear constraints around the boundary of the support. However, in a finite sample these constraints on the coefficients around the boundary can be important for the power of the test.

Testing for monotonicity or convexity as formulated in (1.2) or (1.3), respectively, comes down to testing for a set of constraints induced by the set of inequalities

\[ H_0 : \quad \beta_1 \leq \beta_2 \leq \ldots \leq \beta_L \]

\[ H_0 : \quad \beta_j - 2\beta_{j+1} + \beta_{j+2} \geq 0, \quad j = q, \ldots, L' - 1, \]

\[ (2.2) \]

\[ (q-1)\Delta \beta_{q+1} \geq q\Delta \beta_q, \quad (q-2)\Delta \beta_q \geq (q-1)\Delta \beta_{q-1}, \]

\[ \ldots, \Delta \beta_3 \geq 2\Delta \beta_2, \quad (q-1)\Delta \beta_{L'+1} \leq q\Delta \beta_{L'+2}, \quad (q-2)\Delta \beta_{L'+2} \leq (q-1)\Delta \beta_{L'+3}, \]

\[ \ldots, \Delta \beta_{L-1} \leq 2\Delta \beta_L \]

respectively for monotonicity and for convexity, as the polynomials \( p_{\ell,L}(x;q) \) are nonnegative for any \( q \). So, the interpretation of our null hypothesis given in (2.2) makes the testing procedure much easier to implement and also it translates the atypical null hypothesis into a more familiar formulation, see among others Ramsay (1998) or Meyer (2008). However, one major and key difference in this manuscript is that contrary to the aforementioned works, we allow and have an increasing number of restrictions, as we allow \( L \) to increase to infinity.
It is also worth mentioning that the above properties are shared with Bernstein polynomials. Indeed the approximation of \( m(x) \) by Bernstein polynomials is given by
\[
m_B(x; L) = \sum_{\ell=0}^{L} \beta_\ell \mathcal{B}_{\ell,L}(x), \quad x \in [0, 1],
\]
where \( \mathcal{B}_{\ell,L}(x) = \binom{L}{\ell} (1-x)^{L-\ell} x^\ell \) denotes the \( \ell \)th Bernstein polynomial and \( \beta_\ell = m(\ell/L) \), and hence constrains on \( m(x) \) translate into constraints on the coefficients \( \beta_\ell \). Also Bernstein polynomials satisfies a property analogous to the one given in (b). However, our main motivation not to use Bernstein polynomials comes from the observation that, contrary to B-splines or P-splines, Bernstein polynomials are highly correlated. Indeed, using results in Lee et al. (2002), the eigenvalues of the matrix \( \{ E(\mathcal{B}_{\ell_1,L}(x) \mathcal{B}_{\ell_2,L}(x)) = a_{\ell_1,\ell_2} L_{\ell_1,\ell_2=0} \} \) are
\[
\lambda_k = \frac{1}{2L+1} \left( \frac{2L+1}{L-k} \right) / \left( \frac{2L}{L} \right),
\]
which implies that \( \lambda_L \leq (2L+1)^{-1} L^{1/2} 2^{1-2L} \) since \( \left( \frac{2L}{L} \right) \geq L^{-1/2} 2^{2L-1} \). In addition, it is not difficult to see that \( \sum_{\ell_1,\ell_2} a_{\ell_1,\ell_2} a_{\ell_2,\ell_3} \rightarrow e^{-1} \) if \( |\ell_1 - \ell_2| = o(L^{1/2}) \), which yields some adverse and important technical consequences for the test proposed in Section 3.

Finally, other sieve basis could potentially be used but, the implementation of the test would be laborious. One popular sieve basis are power series \( 1, x, \ldots, x^L \) for any \( L \). However, to formulate the constrained becomes increasingly complicated when \( L \) increases and one is interested to test for either monotonicity or convexity. This can be view from the one-to-one relationship between the Bernstein polynomials basis and the power basis \( 1, x, \ldots, x^L \) for any \( L \). A similar comment applies if we want to employ the Legendre polynomial base – as it can be seen from the relationship between the Legendre and Bernstein polynomials given e.g. in Farouki’s (2000) Propositions 1 and 2, and this, again, seems an unnecessary step.

We shall mention nonetheless that there is no reason to believe that the results or the methodology introduced in this paper cannot be implemented using a power series approximation or potentially some other approximation basis, but again this route seems more arduous than using B-splines.

In the remainder of the paper, we shall assume that the domain of definition of the regressor \( X \) is in the interval \([0, 1]\). This is without loss of generality for a bounded \( X \in [a, b] \) as we can conduct a simple affine transformation of the regressor to define \( X' = (X-a)/(b-a) \) to attain the property \( X' \in [0, 1] \) and without affecting the monotonicity or any other shape property.

We now describe the first step in our methodology of testing \( H_0 \) by giving estimators of \( m(\cdot) \) under the alternative and the null hypothesis. To that end, write the B-splines as a vector of functions
\[
(2.3) \quad P_L (x) =: (p_{1,L}(x;q), \ldots, p_{L,L}(x;q))'; \quad P_k =: P_L (x_k).
\]

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2 The relations between the Bernstein basis and some other polynomial bases have been addressed in the approximation literature. E.g., Li and Zhang (1998) discuss not only the relation between the Bernstein basis and the Legendre basis but also the relation between the Bernstein basis and Chebyshev orthogonal bases.
Then, the standard series estimator of \( m(x) \) is defined as the projection of \( y \) onto the space spanned by \( P_L(x) \), that is

\[
\tilde{m}_B(x_i; L) = \tilde{b}' P_i,
\]

where \( B^+ \) denotes the Moore-Penrose inverse of the matrix \( B \).

To obtain an estimator under the null hypothesis, we conduct a linear projection subject to suitable constraints. If the null hypothesis is (1.2), then the constrained optimization problem becomes

\[
\hat{b} = (\hat{b}_1, \ldots, \hat{b}_L) = \text{arg min}_{b_1 \leq b_2 \leq \ldots \leq b_L} \sum_{i=1}^{n} \left( y_i - \sum_{\ell=1}^{L} b_{\ell} \tilde{p}_{\ell,L}(x_i; q) \right)^2,
\]

given the estimator of \( m(\cdot) \) under (1.2):

\[
\hat{m}_B(x_i; L) = \tilde{b}' P_i.
\]

If we were interested in the null hypothesis (1.3), then instead of the constrained optimization problem (2.5), we would have considered

\[
\hat{b} = (\hat{b}_1, \ldots, \hat{b}_L) = \text{arg min}_{b_1 \leq b_2 \leq \ldots \leq b_L} \sum_{i=1}^{n} \left( y_i - \sum_{\ell=1}^{L} b_{\ell} \tilde{p}_{\ell,L}(x_i; q) \right)^2
\]

It is worth pointing out that for practical purposes, it becomes more convenient to employ a different “parameterization”. More specifically, for the null hypothesis of an increasing function, denote

\[
\tilde{p}_{\ell,L}(x; q) = \sum_{j=\ell}^{L} \tilde{p}_{j,L}(x, q), \quad \ell = 1, \ldots, L.
\]

Then (2.5) is equivalent to

\[
\hat{\delta} = \text{arg min}_{\delta_1, \ldots, \delta_L, \delta_1 \leq \delta_2 \leq \ldots \leq \delta_L} \sum_{i=1}^{n} \left( y_i - \sum_{\ell=1}^{L} \delta_{\ell} \tilde{p}_{\ell,L}(x_i; q) \right)^2.
\]

Under the convexity assumption in (1.3), the optimization problem in (2.7) can be equivalently written similarly as before with \( \tilde{p}_{\ell,L}(x; q) \) having the same role as \( p_{\ell,L}(x, q) \) above and observing that \( \beta_{j} - 2\beta_{j+1} + \beta_{j+2} = \Delta \beta_{j+2} - \Delta \beta_{j+1} \) so that \( \Delta \beta_{j+2} \) plays the role of \( \beta_{j} \) before.

We now introduce our regularity conditions.

**Condition C1**: \( \{(x_i, u_i)'\}_{i \in \mathbb{Z}} \) is a sequence of independent and identically distributed random vectors, where \( x_i \) has support on \( \mathcal{X} = [0, 1] \) and its probability density function, \( f_X(x) \), is bounded away from zero. In addition, \( E[u_i|x_i] = 0, E[u_i^2|x_i] = \sigma_u^2 \), and \( u_i \) has finite \( 4t \)th moments.

**Condition C2**: \( m(x) \) is three times continuously differentiable on \( [0, 1] \).

**Condition C3**: As \( n \to \infty \), \( L \) satisfies \( L^2/n + n/L^4 = o(1) \).
Condition $C_1$ can be weakened to allow for heteroscedasticity, e.g. $E[|u_i^2| x] = \sigma_n^2(x)$ as it was done in Stute (1997). However, the latter condition complicates the technical arguments and for expositional simplicity we omit a detailed analysis of this case. However, in our empirical applications we present examples with heteroscedastic errors and illustrate how to deal with it in practice. Condition $C_3$ bounds the rate at which $L$ increases to infinity with $n$.

Condition $C_2$ is a smoothness condition on the regression function $m(x)$. It guarantees that the approximation error or bias

$$m_{\text{bias}}(x) = m_B(x; L) - m(x)$$

is $O(L^{-3})$, see Agarwal and Studden’s (1980) Theorems 3.1 and 4.1 or Zhou et al. (1998). It can be weakened to say that the second derivatives are Hölder continuous of degree $\eta > 0$. In that case, $C_3$ had to be modified to $L^2/n + n/L^{2+2\eta} = o(1)$.

In case of using $P$-splines we refer to Claeskens et al.’s (2009) Theorem 2.

2.2. THE TESTING METHODOLOGY.

Once we have presented our estimators of $m(x)$ using or not the constraints induced by the null hypothesis, we discuss the methodology to test the shape restrictions outlined in the introduction. To that end, and for clarity of exposition on the main ideas, we shall focus on the null hypothesis (2.12) or, in terms of the coefficients $\{\beta_\ell\}_{\ell=1}^L$ in (2.1), on testing the null hypothesis

$$H_0 : \beta_1 \leq \beta_2 \leq \ldots \leq \beta_L,$$

being the alternative hypothesis the negation of the null. Recall that if we use the parameterization in (2.8), (2.10) becomes

$$H_0 : \delta_0 \leq 0; \quad 2 \leq \ell \leq L.$$

(2.10), or the last displayed expression, suggests that our testing problem is the familiar testing scenario when the null hypothesis is given as a set of constraints on the parameters of the model. However the main and key difference is that, in our scenario, the number of such constraints increases with the sample size.

When testing for constraints among the parameters in a regression model, one possibility is via the (Quasi) Likelihood Ratio principle which compares the fits obtained by constrained and unconstrained estimates $\hat{m}_B(x_i; L)$ and $\tilde{m}_B(x_i; L)$ respectively. That is,

$$LR_n(L) = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{m}_B(x_i; L))^2 - \frac{1}{n} \sum_{i=1}^n (y_i - \tilde{m}_B(x_i; L))^2.$$

A second possibility is to employ the Wald principle, which involves checking if the constraints in (2.10) hold true for the estimator $\hat{b}$ of the parameters $\beta =: \{\beta_\ell\}_{\ell=1}^L$ given in (2.4). That is, if the data supports the set of inequalities

$$\hat{b}_1 \leq \hat{b}_2 \leq \ldots \leq \hat{b}_L.$$

This approach involves inequality constraints, which even when $L$ is fixed, they are not trivial to implement or even to compute the critical values based on its asymptotic distribution. However, in our scenario we have two potential technical complications. First, the number of constrains increases with the sample size $n$, which makes, from both a theoretical and practical point of view, this route to very arduous, if at all feasible. And secondly, when $b_j = b_{j+1}$, say, we are then dealing
with estimation at the boundary which implies that the asymptotic distribution cannot be Gaussian.

A third way to test for the null hypothesis is to implement the Lagrange Multiplier, $LM$, that is to check if the residuals
\begin{equation}
\hat{u}_i = y_i - \hat{m}_L(x_i; L), \quad i = 1, \ldots, n.
\end{equation}

and $x_i$ satisfies the orthogonality condition imposed by Condition $C$. That is, we might base our test on whether or not the set of moment conditions
\begin{equation}
K_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_i(x) \hat{u}_i, \quad x \in [0, 1]
\end{equation}

are significantly different from zero, where $\mathcal{I}$ is an indicator and we abbreviate $\mathcal{I}(x_i < x)$ as $\mathcal{I}_i(x)$. This approach was described and examined in Stute (1997) or Andrews (1997) with a more econometric emphasis. Tests based on $K_n(x)$ are known as testing using partial sum empirical processes. Recall that in a standard regression model the $LM$ test is based on the first order conditions
\begin{equation}
LM_n(L) = \frac{1}{n} \sum_{i=1}^{n} p_{\ell,L}(x_i; q) \hat{u}_i,
\end{equation}

which has the interpretation of whether the residuals and regressors, $p_{\ell,L}(x_i; q)$, satisfies the orthogonality condition induced by Condition $C$.

However to motivate the reasons to employ a transformation of $K_n(x)$, given in (3.9) or (3.10) below, as the basis for our test statistic, it is worth examining the structure of $K_n(x)$ given in (2.12). For that purpose, we observe that
\begin{equation}
K_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_i(x) u_i - \sum_{\ell=1}^{L} (\hat{b}_\ell - \beta_\ell) \mathcal{P}_{n,\ell}(x; q) + \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_i(x) m^{bias}(x_i),
\end{equation}

where $m^{bias}(x_i)$ was given in (2.9) and
\begin{equation}
\mathcal{P}_{n,\ell}(x; q) =: \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_i(x) p_{\ell,L}(x_i; q).
\end{equation}

Now, the third term on the right of (2.13) is $O(L^{-3})$ by Agarwal and Studden’s (1980) Theorems 3.1 and 4.1, and then Condition $C$. On the other hand, standard arguments and Condition $C$ imply that
\begin{equation}
n^{1/2} U_n(x) = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \mathcal{I}_i(x) u_i \xrightarrow{w} U(x) =: \sigma_n \mathcal{B}(F_X(x))
\end{equation}

where $\mathcal{B}(z)$ denotes the standard Brownian motion and $F_X(x)$ the distribution function of $x_i$.

Next, we discuss the contribution due to $\sum_{\ell=1}^{L} (\hat{b}_\ell - \beta_\ell) \mathcal{P}_{n,\ell}(x; q)$. In standard lack-of-fit testing problems with $L$ finite, when $K_n(x) = O_p(n^{-1/2})$, its contribution is nonnegligible as first showed by Durbin (1973) and later by Stute (1997) in a regression model context. However the proof of Theorem 1 suggests that $\sum_{\ell=1}^{L} (\hat{b}_\ell - \beta_\ell) \mathcal{P}_{n,\ell}(x; q) = O_p((L/n)^{1/2})$, so that as $L$ increases with the sample size, it yields that the normalization factor for $K_n(x)$ is of order $n^\alpha$ for some $\alpha < 1/2$.
Thus the previous arguments suggest that under our conditions, we would have that
\[
\left( \frac{n}{L} \right)^{1/2} K_n (x) =: - \left( \frac{n}{T} \right)^{1/2} \sum_{\ell=1}^{L} (\hat{b}_\ell - \beta_\ell) \mathcal{P}_{n,\ell} (x; q) (1 + o_p (1)),
\]
which results by Newey (1997) or Lee and Robinson (2016) might suggest that the left side of the last displayed expression it might converge to a Gaussian process when \( \beta_\ell > \beta_{\ell-1} \) for all \( \ell \geq 1 \). However, when \( \beta_\ell = \beta_{\ell-1} \), we would be at the boundary, which implies that the asymptotic distribution is not Gaussian, and so to obtain the asymptotic distribution of \( \left( \frac{n}{L} \right)^{1/2} \sum_{\ell=1}^{L} (\hat{b}_\ell - \beta_\ell) \mathcal{P}_{n,\ell} (x; q) \) for inference purposes appears quite difficult, if at all possible.

So, the purpose of the next section is to examine a transformation of \( K_n (x) \) such that its statistical behaviour will be free from \( \sum_{\ell=1}^{L} (\hat{b}_\ell - \beta_\ell) \mathcal{P}_{n,\ell} (x; q) \). The consequence of the transformation would then be twofold. First, we would obtain that the transformation of \( n^{1/2} K_n (x) \) is \( O_p (1) \), which leads to better statistical properties of the test, and secondly and more importantly, the test will be pivotal in the sense that \( \sigma_u^2 \) becomes the only unknown (although easy to estimate) of its asymptotic distribution. One consequence of our results is that the asymptotic distribution becomes independent of the null hypothesis under consideration.

3. KHMALADZE’S TRANSFORMATION

This section examines a transformation of \( K_n (x) \) whose asymptotic distribution is free from the statistical behaviour of \( \{\hat{b}_\ell\}_{\ell=1}^{L} \). To that end, we propose a “martingale” transformation based on ideas by Khmaladze (1981), see also Brown, Durbin and Evans (1975) for earlier work. The (linear) transformation, denoted \( T \), should satisfy that
\[
\begin{align*}
(i) \quad & n^{1/2} (Tu_n) (x) \Rightarrow \mathcal{U} (x) =: \sigma_u \mathcal{B} (F_X (x)) \\
(ii) \quad & n^{1/2} (TP_L) (x) = 0 \\
(iii) \quad & n^{1/2} (Tm^{bias}) (x) = o(1),
\end{align*}
\]
where
\[
P_L (x) := E (p_n (x; L)) = \int_0^x P_L (z) f_X (z) \, dz \quad (p_n (x; L) := \{p_{n,\ell} (x; q)\}_{\ell=1}^{L}),
\]
with \( P_L (x) \) and \( p_{n,\ell} (x; q) \) given respectively in (2.3) and (2.14).

For any \( x < 1 \), denote
\[
A_L (x) = \int_x^1 (P_L (z) P'_{L} (z)) f_X (z) \, dz.
\]
We then define the transformation \( T \) as
\[
(TW) (x) = W (x) - \int_0^x P'_{L} (z) A_L (z) \left( \int_z^1 P_L (w) W (dw) \right) f_X (z) \, dz, \quad x < 1.
\]
It is easy to see that the transformation \( T \) satisfies condition (ii) in (3.1), so that the main concerned will be to show that (i) and (iii) hold true.
However, the transformation $\mathcal{T}$ has only a theoretical value and as such, from an inferential point of view, we need to replace it by its sample analogue, which we shall denote by $\mathcal{T}_n$. To that end, for any $x \in \mathcal{X}$, define

\begin{equation}
(3.3) \quad F_n (x) = \frac{1}{n} \sum_{k=1}^{n} I_k (x)
\end{equation}

\begin{equation}
(3.4) \quad C_n (x) = \frac{1}{n} \sum_{k=1}^{n} P_k u_k J_k (x); \quad A_n (x) = \frac{1}{n} \sum_{k=1}^{n} P_k P'_k J_k (x)
\end{equation}

where $I (x \leq x_k) =: J_k (x) = 1 - I_k (x)$. In addition, we shall abbreviate

\begin{equation}
(3.5) \quad C_{n,i} =: C_n (\tilde{x}_i); \quad A_{n,i} =: A_n (\tilde{x}_i),
\end{equation}

where $\tilde{x}_i = x_i$ if $x_i + n^{-\varsigma} < z^k (x_i)$ and $= z^k (x_i)$ otherwise, with $z^k (x)$ denoting the closest knot $z^k$, $k = 1, ..., L$, bigger than $x$ and $1/2 < \varsigma < 1$. The motivation to make this “trimming” is because when $x_i$ is too close to $z^k (x)$, the $B$-spline is close but not equal to zero, which it induces some technical complications in the proof of our main results. However, in small samples this “trimming” does not appear to be needed, becoming a purely technical argument.

We define the sample analogue of $(\mathcal{T} \mathcal{W}) (x)$ as

\begin{equation}
(3.6) \quad (\mathcal{T}_n \mathcal{W}) (x) = \mathcal{W} (x) - \frac{1}{n} \sum_{i=1}^{n} P'_i A_{n,i}^{+} \int_{\tilde{x}_i}^{1} P_L (w) \mathcal{W} (dw) J_i (x).
\end{equation}

The transformation in (3.5) has a rather simple motivation. Suppose that we have ordered the observations according to $x_i$, that is $x_{i-1} \leq x_i$, $i = 2, ..., n$, which would not affect the statistical behaviour of $K_n (x)$. The latter follows by the well known argument that

\begin{equation}
(3.7) \quad \sum_{i=1}^{n} g (x_i) = \sum_{i=1}^{n} g (x_{(i)}),
\end{equation}

where $x_{(i)}$ is the $i$-th order statistic of $\{x_i\}_{i=1}^{n}$. So, we have that

\begin{equation}
K_n (x_j) = \frac{1}{n} \sum_{i=1}^{n} \tilde{u}_i J_i (x_j) =: \frac{1}{n} \sum_{i=1}^{j} \tilde{u}_i.
\end{equation}

Now, $\tilde{u}_i = u_i - P'_i A_{n,i}^{+} (0) C_n (0)$, where

\begin{equation}
(3.8) \quad v_i = u_i - P'_i A_{n,i}^{+} C_{n,i}
\end{equation}

in (3.7), so that it has a martingale difference structure as $E \left[ v_i \mid \text{past} \right] = 0$, in comparison with $\tilde{u}_i$ where $E \left[ \tilde{u}_i \mid \text{past} \right] \neq 0$. This is the idea behind the so-called (recursive) Cusum statistic first examined in Brown, Durbin and Evans (1975).
and developed and examined in length by Khmaladze (1981). Observe that (3.8) becomes the “prediction error” of \( u_i \) when we use the “last” \( j = i, \ldots, n \) observations.

Thus, the preceding argument yields the Khmaladze’s transformation

\[
(T_nK_n)(x) = M_n(x) =: \frac{1}{n^{1/2}} \sum_{i=1}^{n} v_i I_i(x).
\]

Observe that, using (3.6), we could write

\[
M_n(x) =: \frac{1}{n^{1/2}} \sum_{i=1}^{n} v_i I_i(x).
\]

Now, Lemma [4] implies that condition (ii) in (3.1) holds true when

\[
W_n(x) = \frac{1}{n} \sum_{k=1}^{n} P_k I_k(x),
\]

so the technical problem is to show that (asymptotically) conditions (i) and (iii) in (3.1) also hold true. That is, to show that

1. \( M_n(x) \xrightarrow{w} U(x) \quad x \in [0,1] \)
2. \( (T_n m_{bias})(x) = o_p\left(n^{-1/2}\right) \).

Finally, it is worth mentioning that in (3.5) we might have employed \( J_k(x) = I(x < x_k) \) instead of our definition \( J_k(x) = I(x \leq x_k) \). However because by definition of B-splines, the matrix \( A_{n,i} \), and hence \( A_L(x_i) \), might be singular, if we employed \( J_k(x) = I(x < x_k) \), then there would be not guaranteed that

\[
P_i' - P_i'A_{n,i}A_{n,i} = 0.
\]

On the other hand, Harville’s (2008) Theorem 12.3.4 yields that the last displayed equation holds true when \( J_k(x) = I(x \leq x_k) \). Now, (1) will be shown in the next theorem, whereas (2) is shown in Theorem 2.

**Theorem 1.** Under Conditions \( C_1 - C_3 \), we have that

\[
M_n(x) \xrightarrow{w} U(x) \quad x \in [0,1].
\]

Unfortunately, we do not observe \( u_i \), so that to implement the transformation we replace \( v_i \) by \( \hat{v}_i \), where \( \hat{v}_i \) is defined as \( v_i \) in (3.8) but where we replace \( u_i \) by \( \hat{u}_i \) as defined in (2.11), yielding the statistic

\[
\tilde{M}_n(x) =: \frac{1}{n^{1/2}} \sum_{i=1}^{n} \hat{v}_i I_i(x).
\]

**Theorem 2.** Assuming that \( H_0 \) holds true, under Conditions \( C_1 - C_3 \), we have that

\[
\tilde{M}_n(x) \xrightarrow{w} U(x).
\]

Denote the estimator of the variance of \( u_i \), \( \sigma_{u_i}^2 \), by

\[
\hat{\sigma}_{u_i}^2 = \frac{1}{n} \sum_{i=1}^{n} \hat{u}_i^2.
\]

**Proposition 1.** Under Conditions \( C_1 - C_3 \), we have that \( \hat{\sigma}_{u_i}^2 \xrightarrow{P} \sigma_{u_i}^2 \).
We then have the following corollary.

**Corollary 1.** Under $H_0$ and assuming Conditions $C1 - C3$, for any continuous functional $g : \mathbb{R} \to \mathbb{R}^+$,

$$g \left( \tilde{M}_n (x) / \tilde{\sigma}_u \right) \overset{d}{\to} g \left( U (x) / \sigma_u^2 \right).$$

**Proof.** The proof is standard using Theorem 2, Proposition 1 and the continuous mapping theorem, so it is omitted. □

Denoting $\tilde{n} = n - L - 2$ and $\tilde{M}_n (x_q) = \tilde{M}_{n,q}$, where $x_q = q/n$, standard functionals are the Kolmogorov-Smirnov, Cramér-von-Mises and Anderson-Darling tests defined respectively as

$$KS_n = \sup_{q=1, \ldots, \tilde{n}} \left| \tilde{M}_{n,q} / \tilde{\sigma}_u \right| \overset{d}{\to} \sup_{x \in (0,1)} |B (F_X (x))|,$$

$$CV_n = \frac{1}{\tilde{n}} \sum_{q=1}^{\tilde{n}} \tilde{M}_{n,q}^2 / \tilde{\sigma}_u^2 \overset{d}{\to} \int_0^1 B^2 (F_X (x)) \, dx,$$

$$AD_n = \frac{1}{\tilde{n}} \sum_{q=1}^{\tilde{n}} \tilde{M}_{n,q}^2 / \tilde{\sigma}_u^2 \overset{d}{\to} \int_0^1 B^2 (F_X (x)) \left( 1 - F_X (x) \right) \, dx.$$

**POWER AND LOCAL ALTERNATIVES.**

Here we describe the local alternatives for which the tests based on $\tilde{M}_n (x)$ have no trivial power. For that purpose, assume that the true model is such that

$$H_a = m(x) + \frac{n-1}{2} m_1(x),$$

where $m(x)$ is a monotonic increasing function and $m_1(x)$ is decreasing in a set $\mathcal{X}_1 \subset \mathcal{X} = (0,1)$ with Lebesgue measure greater than zero. Then, we have the following result,

**Proposition 2.** Assuming Conditions $C1 - C3$, under $H_a$ we have that

$$\tilde{M}_n (x) + L (x) \overset{w}{\Rightarrow} U (x), \quad x \in [0,1]$$

where

$$L (x) = \int_0^x \left\{ m_1 (v) - P_L^+ (v) A_L^+ (v) \int_v^1 P_L (w) m_1 (w) f_X (w) \, dw \right\} f_X (v) \, dv.$$

One consequence of Proposition 2 is that not only tests based on $\tilde{M}_n (x)$ are consistent since $L (x)$ is a nonzero function, but that it has a nontrivial power against local alternatives converging to the null at the “parametric” rate $n^{-1/2}$.

**3.1. COMPUTATIONAL ISSUES.**

This section is devoted at how we can compute our statistic. In view of the CUSUM interpretation, we shall rely on the standard recursive residuals. Observe that since $f(x)$ is continuous the probability of a tie is zero, so that we can always consider the case $x_i < x_{i+1}$.

Now with this view we have that

$$M_n (x) = \frac{1}{n^{1/2}} \sum_{i=1}^n v_i I_i (x);$$
can be written with \( v_i \) replaced by \( v_i = u_i - P_i^T A_n^+ (x) C_n (x) \) and now

\[
\left( \frac{1}{n} \sum_{k=1}^{n} P_k P_k' J_k (x_i) \right) + \frac{1}{n} \sum_{k=1}^{n} P_k u_k J_k (x_i) =: A_{n,i}^+ C_n (x_i)
\]

Then from a computational point of view is worth observing that

\[
A_{n,k}^+ = A_{n,k+1}^+ - \frac{A_{n,k+1}^+ P_k P_k' A_{n,k+1}^+}{n + P_k' A_{n,k+1}^+ P_k}
\]

and

\[
A_{n,k}^+ C_{n,k} = A_{n,k+1}^+ C_{n,k+1} + A_{n,k}^+ P_k \left( u_k - P_k' A_{n,k+1}^+ C_{n,k+1} \right)
\]

see Brown, Durbin and Evans (1975) for similar arguments. Alternatively, we could have considered the Cusum of backward recursive residuals, in which case we would have use the computational formulae,

\[
\bar{A}_{n,k+1}^+ = \bar{A}_{n,k}^+ - \frac{\bar{A}_{n,k}^+ P_k P_k' A_{n,k+1}^+}{n + P_k' A_{n,k}^+ P_k+1}
\]

and

\[
\bar{A}_{n,k+1}^+ C_{n,k+1} = \bar{A}_{n,k}^+ C_{n,k} + A_{n,k}^+ P_k+1 \left( u_k+1 - P_k' A_{n,k}^+ C_{n,k+1} \right)
\]

Of course in the previous formulas one would replace \( u_i \) by \( \hat{u}_i \) or \( y_i \).

4. BOOTSTRAP ALGORITHM

One of our motivations to introduce a bootstrap algorithm for our test(s) is that although it is pivotal, our Monte Carlo experiment suggests that they suffer from small sample biases. When the asymptotic distribution does not provide a good approximation to the finite sample one, a standard approach to improve its performance is to employ bootstrap algorithms, as they provide small sample refinements. In fact, our Monte Carlo simulation does suggest that the bootstrap, to be described below, does indeed gives a better finite sample approximation. The notation for the bootstrap is as usual and we shall implement the fast algorithm of WARP by Giacomini, Politis and White (2013) in the Monte Carlo experiment.

The bootstrap is based on the following 3 STEPS.

**STEP 1**: Compute the unconstrained residuals

\[
\tilde{u}_i = y_i - \tilde{m}_B (x_i; L), \quad i = 1, ..., n
\]

with \( \tilde{m}_B (x_i; L) \) as defined in (2.4).

**STEP 2**: Obtain a random sample of size \( n \) from the empirical distribution of \( \left\{ \tilde{u}_i - \frac{1}{n} \sum_{i=1}^{n} \tilde{u}_i \right\}_{i=1}^{n} \). Denote such a sample as \( \{ u_i^* \}_{i=1}^{n} \) and compute the bootstrap analogue of the regression model using \( \tilde{m}_B (x_i; L) \), that is

\[
y_i^* = \tilde{m}_B (x_i; L) + u_i^*, \quad i = 1, ..., n.
\]

**STEP 3**: Compute the bootstrap analogue of \( \tilde{M}_n (x) \) as

\[
\tilde{M}_n^* (x) =: \frac{1}{n^{1/2}} \sum_{i=1}^{n} \tilde{v}_i^* I_i (x)
\]
where

\[ \hat{u}_i^* = \hat{u}_i - P_i^t A_{n,i}^* C_{n,i}; \quad C_{n,i}^* =: \frac{1}{n} \sum_{k=1}^{n} P_k \hat{u}_k J_k (\bar{x}_i) \]

with \( \hat{u}_i^* = y_i^* - P_i^t A_{n,i}^* (0) C_n^* (0), \ i = 1, ..., n \).

**Theorem 3.** Under Conditions \( C1 - C3 \), we have that for any continuous function \( g : \mathbb{R} \to \mathbb{R}^+ \), (in probability),

\[ g \left( \tilde{M}_n^* (x) \right) \overset{d}{\Rightarrow} g \left( \mathcal{U} (x) \right). \]

Finally, we can replace \( \hat{u}_i^* \) by \( y_i^* \) in the computation of \( \tilde{M}_n^* (x) \). That is,

**Corollary 2.** Under Conditions \( C1 - C3 \), we have that

\[ \tilde{M}_n^* (x) - \hat{M}_n^* (x) = 0, \]

where

\[ \tilde{M}_n^* (x) =: \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left( y_i^* - P_i^t A_{n,i}^* \frac{1}{n} \sum_{k=1}^{n} P_k y_k J_k (x_i) \right) I_i (x). \]

**Proof.** The proof is immediate by Lemma \( \text{[1]} \) and it is omitted. \( \square \)

5. SOME EXTENSIONS

This section describes how our methodology could be extended to other models and/or shape constraints.

5.1. CONSTRAINTS ON SIGNS OF HIGHER-ORDER DERIVATIVES.

An expression for \( m_{B}^t (x; L) \) given in the paper earlier can be used to derive expressions of further derivatives. At every step of taking a further derivative, one has to remember to reduce the multiplicity of boundary knots by 1. It goes without saying that to impose non-trivial constraints on \( q \)-th derivative a researcher would need to consider \( B \)-splines or \( P \)-splines of degree at least \( q \).

The null hypothesis that the regression function’s \( q \)-th derivative is non-negative would lead to linear inequality constraints on the coefficients \( \beta_j \) in a \( B \)-splines approximation. If the domain of the regressor is split into equidistant intervals, then the increasing majority of these constraints will have the form

\[ \sum_{k=1}^{q} (-1)^{q-k} \binom{q}{k} \beta_{j+k} \geq 0. \]

with an increasingly smaller number of constraints having a slightly modified form due to the multiplicity of boundary knots. A special case of this, when \( q = 2 \), was illustrated in \( \text{[2]} \). In a nutshell, as the number of interior knots goes to \( \infty \) the constraints described in \( \text{[5]} \) will be increasingly dominant in capturing the restriction on non-negativity of the \( q \)-th derivative.

A researcher can impose such restrictions for several values of \( q \) simultaneously.
5.2. U-SHAPE, S-SHAPE.

*U-shaped* relationship between variables received a lot of attention in the economic and statistical literatures. For instance, inverse *U-shaped* relationships include the case of the so-called single-peaked preferences, which is an important class of preferences in psychology and economics. We stated our definition of *U-shaped* in (1.4), where the function first decreases till some switch point $s_0$ and then increases (for inverse *U-shaped* relationships the function will be first increasing and then decreasing). Our testing procedure can incorporate some variations of the definitions of *U-shaped* such as the convexity requirements or symmetry around the switch point (as e.g. implied in Simonsohn, 2017). A parametric test for *U-shape* is suggested in Lind and Melhum (2012). Simonsohn (2017) suggests a heuristic two-lines test. Kostyhak (2017) proposes a non-parametric test of *U-shaped* regression functions based on critical bandwidth, and give sufficient conditions for consistency of the test statistic. To the best of our knowledge, there is no formal approach in the literature to testing S-shapes.

We now describe our approach to testing *U-shapes*. Assume that the switch point $s_0$ is known first. Then we perform a *B-spline* estimation on each subinterval $[\bar{x}, s_0]$ and $[s_0, \bar{x}]$ imposing the constraints the regression function is decreasing on the first subinterval and increasing on the second subinterval. The number of knots, including end points, on the two subintervals is $L_1' + 1$ and $L_2' + 1$, respectively, resulting in $L_1 = L_1' + q_1$ *B-splines* base polynomials of degree $q_1$ used on the first subinterval and $L_2 = L_2' + q_2$ *B-splines* base polynomials of degree $q_2$ used on the second subinterval. Thus, with coefficients $\beta_1^{(1)}, \ldots, \beta_{L_1}^{(1)}$ and $\beta_1^{(2)}, \ldots, \beta_{L_2}^{(1)}$ for the two *B-spline* bases, the shape constraints are $\sigma_0$

$$\beta_1^{(1)} \geq \beta_1^{(2)} \geq \ldots \geq \beta_{L_1}^{(1)}, \quad \beta_1^{(2)} \leq \beta_2^{(2)} \ldots \leq \beta_{L_2}^{(2)}.$$

If, in addition, we want to guarantee that continuity of the whole curve at $s_0$, we would then impose the constraint

$$\beta_{L_1}^{(1)} = \beta_1^{(2)}.$$

To guarantee the smoothness of the curve at the switch point, in addition to (5.2), we would impose

$$\beta_{L_1}^{(1)} - \beta_{L_1-1}^{(1)} = \beta_2^{(2)} - \beta_1^{(2)}.$$

However in practice the switching point $s_0$ is not known, so that it needs to be estimated, for which there is a relatively large literature on estimation of the mode or the maximum of a regression models by nonparametric methods. See Parzen (1962) or the work by Eddy (1980, 1982) regarding the estimation of the mode, or Müller (1989) for the maximum of the regression model and Müller and Prewitt (1992) for the spectral density function. Fortunately due to the super fast rate of convergence of the estimator, say $\hat{s}_0$, to the true value $s_0$, which it can be made closer to $n^{-1}$, we envisage that whether or not $s_0$ is known will not affect the asymptotic distribution of our statistics, as it has been shown in other similar contexts, see for instance Delgado and Hidalgo (2000) or Hidalgo (2010).

Testing for *S-shape* is analogous with the difference that on the first subinterval $[\bar{x}, s_0]$ we would impose convexity constraints and on the second subinterval $[s_0, \bar{x}]$ we would impose concavity constraints. To the best of our knowledge, there are no formal statistical test in the literature for *S-shape*.
5.3. INCORPORATING OTHER COVARIATES.

Our testing procedures can be extended to situations when there are other covariates in the regression function and they are additively separable from $x_1$:

$$y_i = m(x_{1i}) + \phi(x_{2i}, \ldots, x_{di}) + u_i,$$

where no monotonicity restrictions are imposed on $\phi$. A full statistical analysis would involve imposing restrictions on the lack of the statistical relationship between $u_i$ and $x = (x_1, \ldots, x_d)$, smoothness properties of function $\phi$, the degree of $(d - 1)$-variate polynomial used to approximate $\phi$ and the rate conditions on the growth of this degree as the sample size increases. Since no shape restrictions are imposed on function $\phi$, one can employ a $(d - 1)$-variate tensor-product $B$-spline approximation of $\phi$ without any restrictions on coefficients.

6. MONTE CARLO EXPERIMENTS AND EMPIRICAL EXAMPLES

6.1. MONTE CARLO EXPERIMENTS.

In this section we present the results of several computational experiments. All the results in this section are given for cubic splines with different number of knots. We present the results for $B$-splines as well as for $P$-splines with penalties on the second differences of coefficients. The penalty parameter is chosen by cross-validation in the unconstrained estimation. In the tables “KS” refers to the Kolmogorov-Smirnov test statistic, “CvM” refers to the Cramér-von Mises test statistic and “AD” to the Anderson-Darling integral test statistic. All three test statistics are based on a Brownian bridge. $L' + 1$ denotes the number of equidistant knots on $[0, 1]$ (including 0 and 1). For example, when $L' = 6$, we consider knots $0, 1/6, 1/3, 1/2, 2/3, 5/6, 1$. In the implementation of $P$-splines in simulations, every simulation draw will give a different cross-validation parameter (we use ordinary cross validation described in Eilers and Marx, 1996). In our simulation results for each $L'$ we use a modal value of these cross-validation parameters.

**Scenario 1.** We take the following strictly monotone regression function:

$$m(x) = x^{13/4}, \quad x \in [0, 1].$$

We take $X \sim U[0, 1], \quad U \sim N(0, \sigma^2), \quad U \perp X.$

The results are summarized in Table 1. In the WARP bootstrap implementation, the demeaned residuals and $x$ are drawn independently.

**Scenario 2 (U-shape).** Now we will take the regression function defined as

$$m(x) = 10(\log(1 + x) - 0.33)^2.$$

The graph of this function is U-shaped with the switch point at $s_0 = e^{0.33} - 1$. In our simulations we take $s_0$ as known.

We take $X \sim U[0, 1], \quad U \sim N(0, \sigma^2), \quad U \perp X.$

The results are summarized in Table 2. We use two different $B$-splines – one on $[0, s_0]$ and the other on $[s_0, 1]$. We analyze the properties of the testing procedure in two approaches. In the first approach we impose additional equality restrictions for these two $B$-splines to be joined continuously at $s_0$, and in the second approach we
enforce additional equality restrictions for these two B-splines to be joined smoothly at \( s_0 \).

In the WARP bootstrap implementation, the demeaned residuals and \( x \) are drawn independently.

**Scenario 3 (analysis of power of the test).** Now we will take the regression function defined as

\[
m(x) = (10x - 0.5)^3 - \exp(-100(x - 0.25)^2)) \cdot \mathcal{I}(x < 0.5) \\
+ (0.1(x - 0.5) - \exp(-100(x - 0.25)^2)) \cdot \mathcal{I}(x \geq 0.5)
\]

and depicted in Figure 6.1.

![Figure 6.1](image)

**Figure 6.1.** Plot of the regression function in Scenario 2.

As expected, the power of the test depends on the variance of the error. We take

\[ X \sim U[0, 1], \quad U \sim \mathcal{N}(0, \sigma^2), \quad U \perp X. \]

The results are summarized in Table 3. In the WARP bootstrap implementation, the demeaned residuals and \( x \) are drawn independently.

The power of monotonicity tests based on this regression function was considered Ghosal, Sen and Van der Vaart (2000) and a similar regression function was considered in Hall and Heckman (2000). Note that Ghosal, Sen and Van der Vaart (2000) considered smaller sample sizes and also smaller standard deviation of noise with \( \sigma = 0.1 \).

**Scenario 4 (analysis of power of the test).** Now we will take the regression function defined as

\[ m(x) = x + 0.415 \exp(-ax^2), \quad a > 0. \]

and depicted in Figure 2. The left-hand side graph in Figure 2 is for the case \( a = 50 \) and the right-hand side graph in Figure 2 is for the latter case the non-monotonicity dip is smaller. These situations are considered to be challenging for monotonicity tests as these functions are somewhat close to the set of monotone functions (in any traditional metric). As expected, the power of the test depends on the value of parameter \( a \) and also depends on the variance of the error. We take

\[ X \sim U[0, 1], \quad U \sim \mathcal{N}(0, \sigma^2), \quad U \perp X. \]
The results are summarized in Table 4. In the WARP bootstrap implementation, the demeaned residuals and $x$ are drawn independently.

![Figure 2](image)

**Figure 2.** Plot of the regression function in Scenario 4. The left-hand side graph is for $a = 50$ and the right-hand side graph is for $a = 20$.

The power of monotonicity tests based on this regression function was examined in Ghosal, Sen and Van der Vaart (2000) and a similar regression function was considered in Bowman, Jones and Gijbels (1998). Note that Ghosal, Sen and Van der Vaart (2000) use smaller sample sizes and also only $a = 50$ and $\sigma = 0.1$ to analyze power implications.

6.2. **APPLICATIONS.**

1. **Hospital data** Here we use data on hospital finance for 332 hospitals in California in 2003. The data include many variables related to hospital finance and hospital utilization. We are interested in analyzing the effect of revenue derived from patients on administrative expenses.

   Figure 3 is a scatter plot of the logarithm of patient revenue and the logarithm of administrative expenses with the fitted curve obtained using cubic $B$-splines with $M = 5$ uniform knots in the range of values of the log of patient revenue under the monotonicity restriction (including the minimum and maximum points).

   We conduct tests for the following hypotheses: (a) monotonicity; (b) convexity; (c) monotonicity and convexity.

   In order to correct for heteroscedasticity of the errors, we estimate the scedastic function $\hat{\sigma}^2(x)$ using residuals obtained in the unconstrained estimation using cubic $B$-splines with the same set of knots. The scedastic function $\hat{\sigma}^2(x)$ is estimated by regressing the logarithm of the squared unconstrained residuals on a linear combination of first-order $B$-splines with 6 knots (including the end points) in the domain of the log of patient revenue.

   We then consider the constrained residuals divided by $\hat{\sigma}(x)$ when calculating $KS$, $CvM$ and $AD$ test statistics and unconstrained residuals divided by $\hat{\sigma}(x)$ when drawing bootstrap samples. After a bootstrap sample of residuals is drawn, we multiply each residual by the corresponding $\hat{\sigma}(x)$ when generating a bootstrap sample of observations of the dependent variable.

---

3The dataset has been downloaded from https://www.kellogg.northwestern.edu/faculty/dranove/htm/dranove/coursepages/mgmt469.htm. The original dataset is for 333 hospitals but we had to remove the observation that had missing information about administrative expenses.
We implement the testing procedure by conducting the Khmaladze transformation both from the right end of the support (as is described theoretically in this paper) and from the left end of the support and obtained extremely similar results. More specifically, we only report results when the transformation is conducted from the right end of the support.

In the case of $P$-splines, we use the same $B$-spline basis, take the second-order penalty and choose the penalization constant using the ordinary cross-validation criterion as in Eilers and Marx (1996). The penalty enters unconstrained optimization problems as well as constrained ones.

Tables 5–7 present results of our testing. Namely, Table 5 shows test statistics for the null hypothesis of the monotonically increasing regression function and also bootstrap critical values using both $B$-splines and $P$-splines. Table 6 presents analogous results for the null hypothesis of convexity of the regression function. Table 7 gives results for the joint null hypothesis of monotonicity and convexity.

As we can see from tables 5–7, we do not reject any of the three hypotheses even at 10% level.

2. Energy consumption in the Southern region of Russia.

The data are on daily energy consumption (in MWh) and average daily temperature (in Celsius) in the Southern region of Russia in the period from February 1, 2016 till January 31, 2018. The data have been downloaded from the official website of System Operator of the Unified Energy System of Russia.\footnote{http://so-ups.ru/}
We provide tests for U-shape with a switch at 17.6° using the outlines approach in section 5.2 and also tests for convexity. In order to correct for heteroscedasticity of the errors, we estimate the scedastic function \( \hat{\sigma}^2(x) \) using residuals obtained in the unconstrained estimation using B-splines (or P-splines, respectively). The scedastic function is estimated using cubic B-splines with 6 uniform knots (including the end points). It is estimated in the form

\[
\sigma^2(x) = \left( \sum_{k=1}^{8} c_k B_{k,s} \right)^2.
\]

Figure 4 gives scatter plots of the data together with fitted curves obtained under the U-shape constraint with the switch at \( s_0 = 17.6° \). This constraint fit is obtained in accordance with the technique in section 5.2. Namely, we consider individual B-spline fits on intervals \([x, s_0] \) and \([s_0, \pi] \), where \( x \) and \( \pi \) are respectively lowest and highest values of the temperature in the sample. On each interval we use \( L'+1 = 5 \) uniform knots (including the end points). The left-hand side figure only imposes the continuity of the fitted curve at the switch point, whereas the right-hand side figure imposes continuous differentiability.

![Figure 4. Energy consumption data. Plot of temperature and energy consumption and the constrained fit (under U-shape with the switch at 17.6°) using cubic B-spline with 5 uniform knots on each subinterval of temperature values. On the left-hand side the fitted curve is continuous at the switch point. On the right-hand side the fitted curve is continuously differentiable at the switch point.](image)

Tables 8-9 present results of our testing. Namely, table 8 shows test statistics for the null hypothesis of U-shaped regression function and also bootstrap critical values using both B-splines and P-splines in case when two B-spline curves are joined at the switch point in a continuous way. Table 9 presents analogous results for the null hypothesis of U-shaped regression function when two B-spline curves are joined at the switch point in a continuously differentiable way. Table 9 gives results for the null hypothesis of convexity. In all the cases Khmaladze’s transformation is conducted from the right end of support. He bootstrap critical values obtained on the basis of 400 bootstrap replications. As we can see, the null hypothesis of a U-shaped relationship with the switch point at 17.6° is not rejected at the 5% level by any type of the test, whereas convexity is rejected. When testing convexity we use cubic splines with \( L'+1 = 7 \) uniform knots on \([x, \pi] \) (including the end points).
7. CONCLUSION

This paper proposes a methodology for testing a wide range of shape properties of a regression function. The methodology relies on implementing Khmaladze’s transformation in a nonparametric setting when B-splines have been used to approximate the functional space under the null hypothesis. We establish that the proposed Khmaladze’s transformation eliminates the effect of nonparametric estimation and, to the best of our knowledge, this is the first implementation of Khmaladze’s transformation in a nonparametric setting.

As our main examples illustrate, we considered in this paper shape constraints that can be written as linear inequality constraints on the coefficients of the approximating regression splines. In this case the test is easy to implement and several shape properties might be tested simultaneously.

One could potentially test qualitative properties of the regression that cannot be expressed by linear inequalities on regression splines coefficients even though we expect the practical implementation of these to be more challenging. We leave it for future work.

8. APPENDIX A

For a $M \times M$ matrix $C = \{c_{kj}\}_{k,j=1}^M$, $\|C\|_F = \sum_{k,j=1}^M c_{kj}^2$ denotes the Euclidean norm and for a $M \times N$ matrix we define the spectral norm $\|G\|$ as $\lambda^{1/2} (G^t G)$, where $\lambda (H)$ denotes the maximum eigenvalue of the matrix $H$. It is worth recalling the following inequality $\|CG\|_F \leq \|C\|_F \|G\|$. Finally $K$ denotes a generic finite and positive constant.

We now introduce the following notation. We shall denote the fourth cumulant of a random variable $z$ by $\kappa_4(z)$ and for any $x_i, i = 1, ..., n$, $I\left( x^1 < x_i < x^2 \right) =: I_i(x^1,x^2)$. Also we define

\begin{equation}
A_{L,i} = A_L(\bar{x}_i), \quad \bar{P}_i = A_L^{1/2} P_i; \quad i = 1, ..., n
\end{equation}

\begin{equation}
\bar{A}_L(x) = D_L(x) A_L(x) D_L(x); \quad \bar{A}_{L,i} =: \bar{A}_L(\bar{x}_i),
\end{equation}

where $A_L(x)$ was given in (8.2) and

\begin{equation}
D_L(x) = diag (d_1(x), ..., d_L(x)), \quad d_\ell(x) = \begin{cases}
0 & \text{if} \quad x < z^{\ell-1} \\
L^{-q} (z^\ell - x)^{-q-1/2} & \text{if} \quad z^{\ell-1} \leq x < z^\ell \\
L^{1/2} & \text{if} \quad z^\ell \leq x.
\end{cases}
\end{equation}

Observe that when $x =: z^k$, that is a knot, Condition C1 yields that

\begin{equation}
\bar{A}_L(z^k) = L A_L(z^k) = diag \left( \bar{0}, B_L(z^k) \right),
\end{equation}

where $\bar{0}$ is a square $k - 1$ matrix of zeroes and the $L - k + 1$ matrix $B_L(z^k)$ is positive definite, where the elements $B_{L,\ell_1,\ell_2}(z^k)$ of the matrix $B_L(z^k)$ are zero if $|\ell_1 - \ell_2| > q$. The latter follows because there are only $q$ splines different than zero at a given value $x$. Finally, it is worth mentioning that for $x \in (z^{\ell-1},z^{\ell})$,

\begin{equation}
d_\ell(x) = K p_{\ell,L}^{-1}(x;q) (z^\ell - x)^{-1/2} \quad \text{and recalling that Harville’s (2008) Section 20.2 yields that} \quad \bar{A}_L(z^k) = diag \left( \bar{0}, B_L^{-1}(z^k) \right), \quad \text{where we abbreviate} \quad p_{\ell,L}(x;q) \quad \text{by} \quad p_{\ell,L}(x) \quad \text{in what follows.}.
\end{equation}
8.1. PROOF OF THEOREM 1

We need to show (a) the finite dimensional distributions converge to a Gaussian random variable with covariance structure given by that of \( U(x) \) and (b) the tightness of the sequence. We begin the proof of part (a) showing the structure of the covariance structure of \( \mathcal{M}_n(x) \). That is, for any \( 0 \leq x^1 \leq x^2 \leq 1, \)

\[
E(\mathcal{M}_n(x^1) \mathcal{M}_n(x^2) | X) = \frac{1}{n} \sum_{i,j=1}^n E(v_i v_j | X) I_{i}(x^1) I_{j}(x^2)
\]

\[
\overset{P}{\rightarrow} \sigma_u^2 F_X(x^1).
\]

Consider \( i < j \) first and assume, without loss of generality, that \( x_i < x_j \). When \( x_i + n^{-\varsigma} \leq z^{k(x_i)} \), that is \( \bar{x}_i = x_i \), Condition C1 implies that

\[
E(C_{n,j} u_i | X) = 0; \quad E(C_{n,i} u_j | X) = \frac{\sigma_u^2}{n} P_j
\]

and hence we obtain that

\[
E(v_i v_j | X) = \frac{\sigma_u^2}{n} \{ \bar{P}^i_n A_{n,i}^+ \bar{A}_{n,j}^+ P_j - \bar{P}_n^i A_{n,j}^+ P_j \} = 0
\]

because Harville’s (2008) Theorem 12.3.4 yields that \( A_{n,j}^+ P_j = P_j \). On the other hand when \( x_i + n^{-\varsigma} > z^{k(x_i)} \), that is \( \bar{x}_i \neq x_i \) and \( x_j > z^{k(x_i)} \), (8.4) holds true which implies (8.5). Finally when \( z^{k(x_i)} - n^{-\varsigma} < x_i, x_j \leq z^{k(x_i)} \), we obtain that

\[
E(v_i v_j | X) = \frac{\sigma_u^2}{n} P_i^1 A_{n,i}^+ \left( z^{k(x_i)} \right) P_j \mathcal{I} \left( \frac{z^{k(x_i)}}{n^\varsigma} < x_i, x_j \leq z^{k(x_i)} \right)
\]

\[
= : \vartheta_n (i, j; X).
\]

Observe that in this case we have

\[
| E(v_i v_j | X) | = O \left( Ln^{-1-2\varsigma} \right)
\]

because Lemmas 3 and 4 imply that

\[
\lambda \left( \inf_{z \in [0,1]} A_{n}^{+1/2} (x) A_n (x) A_{n}^{+1/2} (x) \right) > \delta > 0
\]

with probability approaching one, where \( \lambda(G) \) denotes the minimum eigenvalue of the matrix \( G \), Lemma 2 implies that \( E \left\| \bar{P}_i \right\|^2 = O(L) \) and

\[
E \left( \mathcal{I} \left( z^{k(x_i)} - n^{-\varsigma} < x_i \leq z^{k(x_i)} \right) \right) = O \left( n^{-\varsigma} \right).
\]

Next, when \( i = j \), proceeding as above we obtain that

\[
E(v_i v_j | X) = \sigma_u^2 \left( 1 - \frac{P_i^1 A_{n,i}^+ P_i}{n} \right).
\]

Observe that, denoting \( A_{n,i}^+ = A_{n,i} - P_i P_i^T/n \), we have that

\[
1 - \frac{P_i^1 A_{n,i}^+ P_i}{n} = 1 - 2 \frac{P_i^1 A_{n,i}^+ P_i}{n} + \left( \frac{P_i^1 A_{n,i}^+ P_i}{n} \right)^2 + \frac{P_i^1 A_{n,i}^+ A_{n,i}^+ A_{n,i}^+ P_i}{n} > 0.
\]
So, we conclude that the left side of (8.3) is, for any \(0 \leq x^1 \leq x^2 \leq 1\),
\[
\frac{\sigma_i^2}{n} \sum_{i=1}^{n-1} \left(1 - \frac{P_i'^+ A_{n,i}^+ P_i}{n}\right) \mathcal{I}_i \left(x^1\right) + \frac{\sigma_i^2}{n} \sum_{i \neq j} \vartheta_n (i,j; X) \mathcal{I}_i \left(x^1\right).
\]

The second term of (8.9) is \(o_p(1)\) because (8.6) implies that its first absolute moment is \(O\left(L/n^2\right) = o(1)\) since \(\zeta > 1/2\) and Condition C3. The first term of (8.9) is also \(o_p(1)\) as we now show. Using the inequality \(\mathcal{I}_i \left(x^1\right) \leq \sum_{k=1}^{k \left(x^1\right) - 1} \mathcal{I}_i \left(z^k; z^{k+1}\right)\), we obtain that
\[
E \left(\frac{1}{n^2} \sum_{i=1}^{n-1} \mathcal{P}_i \left(A_{n,i}^{1/2} A_{n,i}^{1/2}\right) \mathcal{I}_i \left(x^1\right)\right) \leq \sum_{k=1}^{k \left(x^1\right) - 1} \mathcal{P}_i \left(A_{n,i}^{1/2} A_{n,i}^{1/2}\right) \mathcal{I}_i \left(z^k; z^{k+1}\right) = K \sum_{k=1}^{k \left(x^1\right) - 1} \sum_{i=1}^{n} E \left\| \mathcal{P}_i \mathcal{I}_i \left(z^k; z^{k+1}\right) \right\|^2 = O \left(\frac{L}{n}\right),
\]
by (8.7) and because \(E \left\| \mathcal{P}_i \mathcal{I}_i \left(z^k; z^{k+1}\right) \right\|^2 = O(1)\) by Lemma 2, cf. (9.3). So, we conclude that (8.3) holds true as it is well known that, under Conditions C1 and then C3,
\[
\frac{\sigma_i^2}{n} \sum_{i=1}^{n} \mathcal{I}_i \left(x^1\right) \left(1 + \frac{L}{n}\right) \mathcal{P} \sigma_u^2 \mathcal{F}_X \left(x^1\right).
\]

To complete part (a), it suffices to show the asymptotic Gaussianity of \(M_n (x)\) for a fixed \(x\) due to Cramér-Wold device. First, we have already shown that
\[
\sum_{i=1}^{n} E \left(v_{in} \left(x^1\right) \mid X\right) \mathcal{P} \rightarrow 1,
\]
where by construction \(v_{in} \left(x^1\right) =: \sigma_u^{-1} \mathcal{F}_X^{-1/2} \left(x^1\right) \mathcal{I}_i \left(x^1\right) v_i / n^{1/2}\) is a martingale difference triangular array of r.v.’s. So, it remains to show the Lindeberg’s condition for which a sufficient condition is
\[
\sum_{i=1}^{n} E \left|v_{in} \left(x^1\right)\right|^4 = o(1).
\]
But the latter holds true because (8.7) yields that
\[
E \left\| \mathcal{P} A_{n,i}^+ C_{n,i} \mathcal{I}_i \left(x^1\right) \right\|^4 \leq E \mathcal{P} \left(\frac{1}{n} \sum_{j=1}^{n} \mathcal{P} u_j \mathcal{J}_j \left(\mathcal{I}_i\right) \right) \left(\frac{L^4}{n^2}\right) = O \left(\frac{L^4}{n^2}\right),
\]
since Lemma 2 implies that \(E \left\| \mathcal{P} \right\|^4 = O \left(L^2\right)\). So, (8.11) holds true by Condition C3, which concludes the proof of part (a).

(b) Tightness follows by Lemmas 4 and 5 and using the same arguments as those for the proof of Billingsley’s (1968) Theorem 22.1, see also Wu’s (2003) Lemma 14, which completes the proof of the theorem. Observe that, for any \(\delta > 0\),
\[
\delta \sum_{k \in \mathbb{Z}} \sup_{x \in \left(\delta, (k+1)\delta\right)} f_X \left(x\right) < K
\]
following Lemma 4 of Wu (2003).
8.2. PROOF OF THEOREM 2

We shall show that, uniformly in \( x \in [0, 1] \),
\[
(8.12) \quad \tilde{M}_n(x) - M_n(x) = o_p(1).
\]
Because (2.11) yields that \( \tilde{u}_i - u_i = m^{bias}(x_i) - P'_i\left(\hat{b} - \beta\right) \), we have that the left side of (8.12) is
\[
(8.13) \quad \frac{1}{n^{1/2}} \sum_{i=1}^{n} m^{bias}(x_i) I_i(x) - \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left\{ P'_i A_{n,i} \sum_{k=1}^{n} P_k m^{bias}(x_k) J_k(\tilde{x}_i) \right\} I_i(x)
\]
since the contribution due to \( P'_i(\hat{b} - \beta) \) is zero by Lemma 1.

Now the first term of (8.13) is \( o_p(1) \) by Condition C3 and that Agarwal and Studden’s (1980) Theorems 3.1 and 4.1, see also Zhou et al. (1998), yields that \( En^{bias}(x_i) = O(L^{-3}) \), whereas together with Lemma 4 and (8.7), we obtain that the second term is, with probability approaching 1, bounded by
\[
\frac{K}{L^3 n^{1/2}} \sum_{i=1}^{n} \left\{ \left\| \tilde{P}_i \right\| \frac{1}{n} \sum_{k=1}^{n} \left\| \tilde{P}_k \right\| \right\} = O_p \left( n^{1/2}/L^2 \right) = o(1)
\]
by Markov’s inequality and Lemma 2 and then Condition C3. This completes the proof of the theorem. \( \blacksquare \)

8.3. PROOF OF PROPOSITION 1

First, using (2.11), we have that
\[
\tilde{u}_i^2 = u_i^2 + m^{bias}(x_i)^2 + \left(\hat{b} - \beta\right)' P'_i P_i' \left(\hat{b} - \beta\right) + 2u_i m^{bias}(x_i) P'_i \left(\hat{b} - \beta\right) + 2m^{bias}(x_i) P'_i \left(\hat{b} - \beta\right).
\]
So, by Cauchy-Schwarz’s inequality, it suffices to show that
\[
(8.14) \quad \left(\hat{b} - \beta\right)' \frac{1}{n} \sum_{i=1}^{n} P_i P'_i \left(\hat{b} - \beta\right) \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} m^{bias}(x_i)^2
\]
are \( o_p(1) \) because \( n^{-1} \sum_{i=1}^{n} u_i^2 - \sigma_u^2 = o_p(1) \). Now, the norm of the first expression in (8.14) is \( O_p(L^{1/2}/n) = o_p(1) \) because
\[
E \left( A_L(0)^{-1/2} \frac{1}{n} \sum_{i=1}^{n} P_i u_i \right) \left( A_L(0)^{-1/2} \frac{1}{n} \sum_{i=1}^{n} P_i u_i \right)' \mid X = \frac{\sigma_u^2}{n} I_L
\]
and \( \|I_L\|_E^2 = L \). Note that by Lemma 3, \( A_n(0) \) is invertible with probability approaching 1 since \( A_L(0) \) is. On the other hand, the second expression in (8.14) is \( o_p(1) \) by Agarwal and Studden’s (1980) Theorems 3.1 and 4.1 and then Condition C3. This completes the proof of the proposition. \( \blacksquare \)

8.4. PROOF OF THEOREM 3

Define
\[
\tilde{M}_n^*(x) =: \frac{1}{n^{1/2}} \sum_{i=1}^{n} u_i^* I_i(x),
\]
where \( v_i^* = u_i^* - P_i^t A_{n,i}^+ n C_{n,i}^* \) and \( C_{n,i}^* = n^{-1} \sum_{k=1}^n P_k u_k^* J_k (\bar{x}_i) \). The proof is completed if we show that, \((in probability)\),

\[
\begin{align*}
(a) & \quad M_n^* (x) \overset{weakly}{\rightarrow} \mathcal{B} (F_X (x)) \\
(b) & \quad \sup_{x \in (0,1)} \left| \widetilde{M}_n^* (x) - M_n^* (x) \right| = o_p (1).
\end{align*}
\]

Part \((b)\) holds true trivially using Lemma 4 because \( \hat{u}_i^* - \tilde{u}_i = P_i^t (\hat{b}^* - \hat{b}) \). To show part \((a)\) it suffices to show that \((a1)\) the finite dimensional distributions converge to a Gaussian random variable with the appropriate covariance structure and \((a2)\) tightness of the sequence. The proofs proceed similarly as those in Theorem 1. Notice that in the proof of Theorem 1, we first conditioned on \( X \) and then we examined its asymptotic unconditional limit.

We begin with \((a1)\). To that end, we first examine the structure of the second moments. Proceeding as in the proof of Theorem 1, because \( E^* u_i^{*2} =: \hat{\sigma}_u^2 \), we have that

\[
E^* (M_n^* (x^1) M_n^* (x^2)) = \frac{\hat{\sigma}_u^2}{n} \sum_{i=1}^n \left( 1 - \frac{P_i^t A_{n,i}^+ P_i}{n} \right) I_i (x^1) + o_p (1)
\]

proceeding as in the proof of (8.10) and by Proposition 1.

Next, we examine the weak convergence of \( M_n^* (x) \), which due to Cramér-Wold device, it suffices to do so for a fixed \( x \). First observe that we have shown that

\[
\sum_{i=1}^n E^* (v_i^{*2} | x) \overset{P}{\rightarrow} 1,
\]

where \( v_i^{*2} =: \hat{\sigma}_u^{-1} F_X^{-1/2} (x) I_i (x) v_i^*/n^{1/2} \) is a martingale difference triangular array of r.v.’s.

So, to complete the proof, it suffices to show the Lindeberg’s condition for which a sufficient condition is \( \sum_{i=1}^n E^* |v_i^*|^4 = o_p (n^2) \), which follows proceeding as with the proof of Theorem 1 and Proposition 1.

\( (a2) \) We now examine the tightness of

\[
\frac{1}{n^{1/2}} \sum_{i=1}^n v_i^* I_i (x).
\]

The proof follows as that of Lemma 5 because again the proof there was done conditionally on \( X \) and then we examined its asymptotic unconditional limit. Indeed, as we argued in the proof of Theorem 1, part \((b)\), it suffices to show that

\[
E^* \left( \frac{1}{n^{1/2}} \sum_{i=1}^n v_i^* I_i (x^1, x^2) \right)^4 \overset{P}{\rightarrow} K \left\{ \frac{1}{n} \left( F_X (x^2) - F_X (x^1) \right) + (x^2 - x^1)^2 \sup_{x \in (x^1, x^2)} f_X^2 (x) \right\},
\]

where \( I_i (x^1, x^2) = I_i (x^2) - I_i (x^1) \). By Burkholder’s inequality implies that the left side of (8.15) is bounded by

\[
K \frac{1}{n^2} E \left( \sum_{i=1}^n (v_i^{*2} - E (v_i^{*2} | G_i^*)) I_i (x^1, x^2) \right)^2 + K \frac{1}{n^2} \left( \sum_{i=1}^n E (v_i^{*2} | G_i^*) I_i (x^1, x^2) \right)^2,
\]
where $G_i^*$ denotes the sigma algebra generated by $\{u_{i+1}^*, ..., u_i^*\}$ and $E (u_i^* | G_i^*) = \widetilde{\sigma}_u^2 + P_i^* A_{n,i}^* C_{n,i}^* C_{n,i}^* A_{n,i}^* P_i$ as it is easily seen. From here the proof proceeds as that of Lemma 8 after we observe that the only difference is that we have, say $\widetilde{\sigma}_u^2$ instead of $\sigma_u^2$, and observing that $\widetilde{\sigma}_u^2 - \sigma_u^2 = o_p (1)$ by Proposition 9 and that

$$
\kappa_4 (u^*) = \frac{1}{n} \sum_{i=1}^{n} \widetilde{\sigma}_u^4 - 3 \left( \frac{1}{n} \sum_{i=1}^{n} \widetilde{\sigma}_u^2 \right)^2 P \rightarrow E \widetilde{\sigma}_u^4 - 3 (E \widetilde{\sigma}_u^2)^2,
$$

so that the left side of (8.16) is bounded by

$$
K \left\{ \frac{1}{n} (F_X (x^2) - F_X (x^1)) + (x^2 - x^1)^2 \sup_{z \in (x^1, x^2)} f_X^2 (z) \right\} (1 + o_p (1))
$$

and the conclusion follows by using the same arguments as those for the proof of Billingsley’s (1968) Theorems 22.1, see also arguments in Wu’s (2003) Lemma 14.

\section{APPENDIX B}

In what follows we shall abbreviate $p_{\ell, L} (x; q)$ by $p_{\ell, L} (x)$ for all $\ell = 1, ..., L$.

\textbf{Lemma 1.} Any linear combination of the B-splines, $p_L (x) =: \sum_{\ell=1}^{L} a_{\ell} p_{\ell, L} (x)$, satisfies that $(T_n B_{n,L}) (x) = 0$, where $B_{n,L} (x) = n^{-1} \sum_{k=1}^{L} p_{\ell} (x_k) I_k (x)$.

\textbf{Proof.} The proof is immediate after we notice that $\int_{x_1}^{x_2} P_L (w) W_n (dw) = A_{n,i}$.

Indeed, (3.5) implies that $(T_n B_{n,L}) (x)$ is

$$
\frac{1}{n} \sum_{k=1}^{n} \left\{ p_{l} (x_k) - P_k A_{l,k}^+ \frac{1}{n} \sum_{j=1}^{n} P_j p_L (x_j) J_j (\tilde{x}_k) \right\} I_k (x)
$$

$$
= \frac{1}{n} \sum_{k=1}^{n} \left( P_k - P_k A_{l,k}^+ \frac{1}{n} \sum_{j=1}^{n} P_j P_j J_j (\tilde{x}_k) \right) a I_k (x)
$$

$$
= \frac{1}{n} \sum_{k=1}^{n} \left( P_k - P_k A_{l,k}^+ A_{l,k} \right) a I_k (x)
$$

$$
= 0,
$$

by Harville’s (2008) Theorem 12.3.4, where $a = (a_1, ..., a_L)$.

We now introduce some notation useful for the next lemmas. We shall denote $\Lambda (r) = \{ x : z^{r-1} \leq x < z^{r-n^{-c}} \}$ and $\overline{\Lambda} (r) = \{ x : z^{r-n^{-c}} \leq x < z^r \}$.

\textbf{Lemma 2.} Under Condition C1, we have that $E \left\| \hat{P}_i \right\|^s = O \left( L^{s/2} + Ln(s/2-1)c \right)$ for any $s \geq 2$, with $\hat{P}_i$ given in (8.1).

\textbf{Proof.} First,

$$
E \left\| \hat{P}_i \right\|^s = \sum_{k=1}^{L} E \left\{ \left\| \hat{P}_i \right\|^s \mathcal{T} (z^{k-1}; z^k) \right\},
$$

(9.1)
where, for each $k = 1, \ldots, L$, we have that
\begin{equation}
E \left\{ \left\| \mathbf{P}_i \right\|^2 \mathcal{I}_i (z^{k-1}; z^k) \right\} = E \left\{ \left\| \mathcal{A}_{L,i}^{1/2} D_L (\bar{x}_i) \mathbf{P}_i \right\|^2 \mathcal{I}_i (z^{k-1}; z^k) \right\}
\tag{9.2}
\end{equation}

Because $\mathcal{A}_{L,i} = : \text{diag} (0, B_{L,i})$ and $\Lambda (B_{L,i}) > 0$. The proof is now standard after observing that the first term on the right of (9.2) is bounded by
\begin{equation}
E \left\{ \left\| p_{k,L}^r (x_i) \mathcal{I} (x_i \in \Lambda (k)) \right\| \right\} + K \sum_{j=k+1}^{k+q} E \left\{ \left\| L^{s/2} p_{j,L}^r (x_i) \mathcal{I} (x_i \in \Lambda (k)) \right\| \right\}
\tag{9.3}
\end{equation}

proceeding as with the second term on the left of (9.3). \hfill \square

**Lemma 3.** Under Conditions C1 and C3, we have that
\begin{equation}
\left\| \mathcal{A}_{L,i}^{1/2} A_{n,i} \mathcal{A}_{L,i}^{1/2} - I \right\|_E \leq 0. \tag{9.4}
\end{equation}

**Proof.** We shall consider the scenario where $x_i \in \Lambda =: \{ \Lambda (r) : r = 1, \ldots, L \}$, the case when $x_i \in \overline{\Lambda} =: \{ \overline{\Lambda} (r) : r = 1, \ldots, L \}$ is handled similarly, if not easier. Because when $x_i \in \Lambda (r)$, $\bar{x}_i = x_i$, the matrix inside the norm in (9.4) is $\mathcal{A}_{L,i}^{1/2} H_{n,i} \mathcal{A}_{L,i}^{1/2}$, where
\begin{equation}
H_{n,i} =: D_L (x_i) \left( \frac{1}{n} \sum_{k=1; k \neq i}^{n} \mathbf{P}_k \mathbf{P}_k^T J_k (x_i) - A_{L,i} \right) D_L (x_i) + \frac{1}{n} D_L (x_i) \mathbf{P}_i \mathbf{P}_i^T D_L (x_i). \tag{9.5}
\end{equation}

The second term on the right of (9.5) is $o_p (1)$ as we now show. Because
\begin{equation}
p_{\ell,L} (x_k) p_{m,L} (x_k) = 0 \quad \text{if } m \geq \ell + q,
\end{equation}
Cauchy-Schwarz and then Markov’s inequalities implies that it suffices to show that
\begin{equation}
E \left\{ \frac{1}{n} \sum_{\ell=1}^{L} d_\ell^2 (x_i) p_{\ell,L}^2 (x_i) \right\} = o (1). \tag{9.7}
\end{equation}

But this is the case because, recall that $x_i \in \Lambda$, the left side is bounded by
\begin{equation}
\frac{K}{n} \sum_{r=1}^{L} E \left( \frac{1}{(z^r - x_i)^{r+q}} + \sum_{\ell=r+1}^{r+q} L p_{\ell,L}^2 (x_i) \right) \mathcal{I} (x_i \in \Lambda (r)) = O \left( \frac{L^q (1 + \zeta \log n)}{n} \right) = o (1),
\end{equation}

since $E \left( L p_{\ell,L}^2 (x_i) \mathcal{I} (x_i \in \Lambda (r)) \right) = O (1)$. It is worth observing that (9.7) also holds uniformly in $x$ since $p_{\ell,L}^2 (x) < K$ and Condition C3.
Next, the first term on the right of (9.5) is also $o_p(1)$. Indeed, because $\mathcal{A}_{L,i} = \text{diag}(0, B_{L,i})$ and $\Lambda(B_{L,i}) > 0$, it suffices to show that $\|H_{n,i}\|_E \overset{p}{\to} 0$ and more specifically, in view of (9.6), to show that
\[
(9.8) \quad \sum_{r=1}^{L} \left\{ \sum_{\ell=r}^{L} \frac{1}{n} \sum_{k=1; k \neq \ell}^{n} d_{\ell}^{2} (x_i) \left( p_{\ell,L}^{2} (x_k) J_k (x_i) - a_{i,\ell} \right) \right\} I (x_i \in \Lambda (r)) \overset{p}{\to} 0.
\]
But (9.8) holds true because, conditionally on $x_i$, when $\ell = r$, we have that Condition C1 and $I (x_i \in \Lambda (r_1)) I (x_i \in \Lambda (r_2)) = 0$ if $r_1 \neq r_2$ implies that
\[
E \left( \sum_{r=1}^{L} \frac{1}{n} \sum_{k=1; k \neq \ell}^{n} (p_{\ell,L}^{2} (x_k) J_k (x_i) - E (p_{\ell,L}^{2} (x_k) J_k (x_i))) / (z^r - x_i)^{2q+1} \right)^2
= \sum_{r=1}^{L} K_{n}^{2} \sum_{k=1; k \neq \ell}^{n} E p_{\ell,L}^{4} (x_k) J_k (x_i) - E^{2} (p_{\ell,L}^{2} (x_k) J_k (x_i))^2 I (x_i \in \Lambda (r))
= \frac{K_{n}}{n} \sum_{r=1}^{L} (z^r - x_i)^{-1} I (x_i \in \Lambda (r)) = O_{p} \left( \frac{L \log n}{n} \right),
\]
because $p_{\ell,L}^{2} (x_k) J_k (x_i) I (x_i \in \Lambda (r)) \leq K (z^r - x_k)^{q} J_k (x_i) I (x_i \in \Lambda (r))$ and Markov’s inequality, where
\[
(9.9) \quad p_{\ell,L} (x_k) = L^{-q} p_{\ell,L}^{4} (x_k)
\]
and $E \left\{ (z^r - x_i)^{-1} I (x_i \in \Lambda (r)) \right\} = O (\log n)$.

Next when $\ell > r$. Denoting $d_{\ell}^{2} (x_i) \left( p_{\ell,L}^{2} (x_k) J_k (x_i) - a_{i,\ell} \right) =: \psi_{i,\ell} (x_k)$, the second conditional moments of the left side of (9.8) is
\[
\frac{1}{n^{2}} \sum_{r=1}^{L} \left\{ \sum_{\ell_{1}, \ell_{2} = r}^{L} \sum_{k=1; k \neq \ell_{1}, \ell_{2}}^{n} E \left( \psi_{i,\ell_{1}} (x_k) \psi_{i,\ell_{2}} (x_k) \mid x_i \right) \right\} I (x_i \in \Lambda (r)) = O \left( \frac{L^{2}}{n} \right),
\]
because $a_{i,\ell_{1}} = O \left( L^{-1} \right)$, $d_{\ell}^{2} (x_i) = L$, $E p_{\ell,L}^{4} (x_k) = O \left( L^{-1} \right)$ for any $s \geq 1$ and (9.6) implies that
\[
|E \left( \psi_{i,\ell_{1}} (x_k) \psi_{i,\ell_{2}} (x_k) \mid x_i \right)| \leq K I (|\ell_{2} - \ell_{1}| \geq q) + KL I (|\ell_{2} - \ell_{1}| < q).
\]
So, (9.8) holds true for these terms by Markov’s inequality, Condition C3 and because $\sum_{r=1}^{L} I (x_i \in \Lambda (r)) = 1$, which concludes the proof of the lemma.

**Lemma 4.** Under Conditions C1 and C3, we have that
\[
\sup_{x} \left\| A_{L}^{1/2} (x) A_{n} (x) A_{L}^{1/2} (x) - I \right\|_E \overset{p}{\to} 0.
\]

**Proof.** Arguing as we did in the proof of Lemma 3, it suffices to show that
\[
(9.10) \quad \sup_{x} \| H_{n} (x) \|_E =: \sup_{1 \leq r \leq L} \left\{ \sup_{x \in \Lambda (r)} + \sup_{x \in \mathcal{X} (r)} \right\} \| H_{n} (x) \|_E \overset{p}{\to} 0.
\]
We begin examining the contribution due to the sets $\Lambda (r)$. First, as we argued in Lemma 3, we observe that (9.10) holds true if the diagonal elements of $H_{n} (x)$
converges uniformly to zero in probability. That is, it suffices to show that

\[
(9.11) \sup_{1 \leq r \leq L} \sup_{x \in \Lambda (r)} \left( \sum_{\ell = r}^{L} \frac{1}{n} \sum_{k=1}^{n} d_{\ell}^{2} (x) \left( p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (x) - E p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (x) \right) \right)^{2}
\]

\[
+ \left( \sup_{1 \leq r \leq L} \sup_{x \in \Lambda (r)} \left( \sum_{\ell = r+1}^{r+q-1} \frac{1}{n} \sum_{k=1}^{n} d_{\ell}^{2} (x) \left( p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (x) - E p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (x) \right) \right)^{2}
\]

\[
= o_{p} (1).
\]

Recall that due to the definition of B-splines, for any \( r, p_{r, L} (x_{k}) \mathcal{J}_{k} (x) \mathcal{I} (x \in \Lambda (r)) = 0 \) if \( \ell < r \).

Because \( x \in \Lambda (r) \) implies that \( p_{r, L} (x_{k}) \mathcal{J}_{k} (x) = p_{r, L} (x_{k}) \mathcal{J}_{k} (z^{r}) \) and \( d_{\ell}^{2} (x) = L \) for \( \ell \geq r + q \), the first term on the left of (9.11) is

\[
\sup_{1 \leq r \leq L} \sum_{\ell = r+q}^{L} \left| \frac{1}{n} \sum_{k=1}^{n} \left( p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (z^{r}) - E p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (z^{r}) \right) \right|^{2}
\]

\[
\leq \sum_{\ell=1}^{L} \left| \frac{1}{n} \sum_{k=1}^{n} \left( p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (z^{\ell}) - E p_{r, L}^{2} (x_{k}) \mathcal{J}_{k} (z^{\ell}) \right) \right|^{2}
\]

\[
= O_{p} \left( \frac{L}{n} \right)
\]

by Condition C1 and that \( E p_{r, L}^{2} (x_{k}) = O \left( \frac{L}{n} \right) \), and then by Markov’s inequality.

So, to complete the proof we need to show that the second term on the left of (9.11) is also \( o_{p} (1) \). We shall look at the case when \( \ell = r \), being the cases when \( \ell > r \) similarly, if not easier, handled. To that end, we first notice that this term is bounded by

\[
(9.12) \left( \sup_{1 \leq r \leq L} \sup_{x \in \Lambda (r)} \frac{1}{(z^{r} - x)^{1/2}} \left| \frac{1}{n} \sum_{k=1}^{n} \eta_{r} (x_{k}; x) - E \eta_{r} (x_{k}; x) \right| \right)^{2},
\]

where \( \eta_{r} (x_{k}; x) = \frac{\eta_{r} (x_{k}) \mathcal{J}_{k} (x)}{(z^{r} - x)^{2q+1/2}} \) with \( \eta_{r} (x_{k}) \mathcal{J}_{k} (x) \) defined in (9.9).

Consider points \( z^{r} [j] = z^{r} - (j - 1)/n, j = n^{1-\varsigma} + 1, \ldots, n/L \), so that \( z^{r} [n^{1-\varsigma} + 1] = z^{r} - n^{-\varsigma} \) and denote \( \eta_{r} (\mathcal{J}_{k} (x^{1}) - \eta_{r} (\mathcal{J}_{k} (x^{1})) \). Now if \( x \in (z^{r} [j + 1], z^{r} [j]) \), we have that

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \eta_{r} (x_{i}; x, z^{r} [j]) \right| \leq \frac{1}{n} \sum_{i=1}^{n} |\eta_{r} (x_{i}; z^{r} [j + 1], z^{r} [j])|,
\]

so that the triangle inequality yields that (9.12) is, except constants, bounded by

\[
\left( \sup_{1 \leq r \leq L} \sup_{n^{1-\varsigma} + 1 \leq j \leq n/L} \frac{1}{(z^{r} - z^{r} [j])^{1/2}} \left| \frac{1}{n} \sum_{i=1}^{n} \eta_{r} (x_{i}; z^{r} [j], z^{r} [n^{1-\varsigma} + 1]) - E (\cdot) \right| \right)^{2}
\]

\[
+ \left( \sup_{1 \leq r \leq L} \sup_{n^{1-\varsigma} + 1 \leq j \leq n/L} \frac{1}{(z^{r} - z^{r} [j])^{1/2}} \left| \frac{1}{n} \sum_{i=1}^{n} \eta_{r} (x_{i}; z^{r} [j + 1], z^{r} [j]) - E (\cdot) \right| \right)^{2}
\]

\[
(9.13) + \left( \sup_{1 \leq r \leq L} \sup_{n^{1-\varsigma} + 1 \leq j \leq n/L} \frac{1}{(z^{r} - z^{r} [j])^{1/2}} E |\eta_{r} (x_{i}; z^{r} [j + 1], z^{r} [j])| \right)^{2}.
\]
Because for any \( q > 0 \), \(^{(9.14)}\)
\[
E |\eta_r (x_i; z^r [j + 1], z^r [j])|^q \leq K \int_{z^r [j + 1]}^{z^r [j]} (z^r - x)^{-q/2} \, dx \leq K (z^r - z^r [j])^{-q/2} n^{-1},
\]
the last term of \(^{(9.13)}\) is bounded by \( K \left( \sup_{1 \leq r \leq L} |\eta_r (x_i; z^r [j + 1], z^r [j])| \right)^2 = o(1) \) because \( \varsigma < 1 \).

Next using \(^{(9.14)}\) and the inequalities \( \sup_x |g (x)|^q = \sup_x |g (x)|^q \) and \( \sup_x |c| \leq \sum \epsilon |\epsilon| \), the expectation of the second term of \(^{(9.13)}\) to the power \( p/2 \) is bounded by
\[
\sum_{r=1}^{L} \sum_{j=n^{1-c}+1}^{n/L} \frac{1}{(z^r - z^r [j])^{p/2}} E \left| \frac{1}{n} \sum_{i=1}^{n} |\eta_r (x_i; z^r [j + 1], z^r [j])| - E (\cdot) \right|^p
\]
\[= K \frac{L}{n^p} \sum_{j=n^{1-c}+1}^{n/L} j^{-p} \leq O \left( \log^{p-1} \epsilon \right) = o(1)
\]
for any \( \varsigma < 1 \) choosing \( p \) large enough. Thus, the second term of \(^{(9.13)}\) is \( o_p(1) \).

Finally, the first term of \(^{(9.13)}\). First, as we argued with the second term, Condition C1 yields that its \( p - th \) absolute moment is bounded by
\[
\sum_{r=1}^{L} \sum_{j=n^{1-c}+1}^{n/L} \frac{1}{(z^r - z^r [j])^{p/2}} E \left| \frac{1}{n} \sum_{i=1}^{n} \eta_r (x_i; z^r [j], z^r [n^{1-c} + 1]) - E (\cdot) \right|^p
\]
\[= K \frac{\log^{p/2} n}{n^{(p/2-1)(1-c)}}
\]
which is \( o(1) \) since we can always choose \( p \) large enough such that \( L = o \left( n^{p/2(1-c)} \right) \) for any \( \varsigma < 1 \) and Condition C3. Notice that we can also bound the left side of \(^{(9.14)}\) by \( K \log n \) when \( q = 2 \) there.

To complete the proof of the lemma, we need to examine the contribution due to the sets \( \mathcal{A} (r) \) into the left of \(^{(9.10)}\). However, observing that when \( x_i \in \mathcal{A} (r) \), \( J_k (x_i) = J_k (z^r) \), the proof proceeds as that of the first term on the left of \(^{(9.11)}\), and so it is omitted.

**Lemma 5.** Under Conditions C1 and C3, we have that for any \( 0 \leq x^1 < x^2 \leq 1 \),
\[
(9.15) \quad E \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n} v_i I_i (x^1, x^2) \right)^4 = K \left\{ \frac{1}{n} \left( F_X (x^2) - F_X (x^1) \right) + (x^2 - x^1)^2 \sup_{x \in (x^1, x^2)} f_X (x) \right\} (1 + o(1)).
\]
Proof: First, as we mentioned in Section 3, we notice that we can arrange the observations according to \( x \) without modifying the properties of

\[
\frac{1}{n^{1/2}} \sum_{i=1}^{n} v_i I_i (x^1, x^2),
\]

so that \( v_i \) becomes a martingale difference sequence of r.v.'s. So, Burkholder’s inequality implies that the left side of (9.15) is bounded by (9.16)

\[
\frac{K}{n^2} E \left( \sum_{i=1}^{n} (v_i^2 - E(v_i^2 | G_i)) I_i (x^1, x^2) \right)^2 + \frac{K}{n^2} \left( \sum_{i=1}^{n} E(v_i^2 | G_i) I_i (x^1, x^2) \right)^2,
\]

where \( G_i \) denotes the sigma algebra generated by \( \{u_{i+1}, ..., u_n\} \) and \( E(v_i^2 | G_i) = \sigma_u^2 + \sum_{j=1}^{n} P_i A_{n,i}^+ C_{n,i}^t A_{n,i} P_i \) as it is easily seen. We shall first examine the second term of (9.16).

By standard inequalities, that term is bounded by

\[
(9.17) \quad K \sigma_u^4 \left( \frac{1}{n} \sum_{i=1}^{n} I_i (x^1, x^2) \right)^2 + \frac{K}{n^2} \left( \sum_{i=1}^{n} P_i A_{n,i}^+ C_{n,i}^t A_{n,i} P_i I_i (x^1, x^2) \right)^2.
\]

Because Condition C1 yields

\[
\frac{1}{n} \sum_{i=1}^{n} I_i (x^1, x^2) \rightarrow F_X (x^2) - F_X (x^1),
\]

we have that the contribution of the first term of (9.17) into the left of (9.15) is

\[
(F_X (x^2) - F_X (x^1))^2 \leq K (x^2 - x^1)^2 \sup_{x \in (x^1, x^2)} f_X^2 (x).
\]

Next, the second term of (9.17) is also \( K (x^2 - x^1)^2 \sup_{x \in (x^1, x^2)} f_X (x) \), as we now show. Indeed, Cauchy-Schwarz inequality and Lemma 4, see also (8.7), yield that this term is bounded by

\[
(9.18) \quad \frac{1}{n} \sum_{i=1}^{n} \tilde{P}_i (\tilde{C}_{n,i} \tilde{A}_{n,i}^t) P_i I_i (x^1, x^2) \quad \frac{1}{n} \sum_{i=1}^{n} \tilde{P}_i \tilde{P}_i I_i (x^1, x^2),
\]

where \( \tilde{C}_{n,i} = A_{n,i}^t/n \) \( \sum_{j=i+1}^{n} P_j u_j \).

Now the second factor of (9.18) converges to

\[
(9.19) \quad E \left( \left\| \tilde{P}_i \right\|^2 I_i (x^1, x^2) \right) = O (n^\varepsilon + L) (F_X (x^2) - F_X (x^1))
\]

because

\[
(9.20) \quad \left\| \tilde{P}_i \right\|^2 = O (n^\varepsilon + L).
\]

Next, because

\[
E \left( \tilde{C}_{n,i} \tilde{C}_{n,i}^t | X \right) = \frac{\sigma_u^2}{n} \text{diag} (0, 1, ..., 1),
\]

we have that the conditional expectation of the first factor of (9.18) is (9.21)

\[
\frac{3}{n} \sum_{i=1}^{n} \left\| \tilde{P}_i \right\|^2 \mathcal{I}_i (x^1, x^2) + \frac{\kappa_n (u)}{n} \sum_{i=1}^{n} \left\| \tilde{P}_i \right\|^2 \sum_{j=i+1}^{n} \sum_{k=1}^{k_{\max}} \left\| D_{L} (x_i) P_j J_k (\tilde{x}_{i}) \right\|^4 \mathcal{I}_i (x^1, x^2).
\]
\[ \leq O(n^\lambda + L) \left( \frac{3}{n^3} \sum_{i=1}^{n} \mathcal{I}_i \left(x^1, x^2\right) + \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n^3} \sum_{j=i+1}^{n} \left\| D_L (x_i) P_j \mathcal{J}_k (\bar{x}_i) \right\|^4 \mathcal{I}_i \left(x^1, x^2\right) \right) \]

because \( \bar{\lambda} \left(D_L^{-1} (x_i) A_{n+i}^{1/2}\right) < K \) by Lemma 4 and (9.20). By standard arguments, the first term on the right of (9.21) is

\[ 3 \frac{n^\lambda + L}{n^2} (F_X (x^2) - F_X (x^1)) (1 + o_p (1)) \]

Next we examine the second term of (9.21). First we have that

\[ \| D_L (x_i) P_j \mathcal{J}_k (\bar{x}_i) \|^4 \mathcal{I}_i \left(x^1, x^2\right) \]

\[ = \sum_{k=1}^{L} \left\| D_L (x_i) P_j \mathcal{J}_k (\bar{x}_i) \right\|^4 \left\{ \mathcal{I} (x_i \in \Lambda (k)) + \mathcal{I} (x_i \in \bar{\Lambda} (k)) \right\} \mathcal{I}_i \left(x^1, x^2\right). \]

We shall examine the contribution due to the first term on the right, the second is similarly handled. Now, the expectation of a typical term of the last displayed expression is, conditionally on \( x_i \),

\[ \sum_{k=1}^{L} E \left\| D_L (x_i) P_j \mathcal{J}_k (\bar{x}_i) \right\|^4 \mathcal{I} (x_i \in \Lambda (k)) \]

\[ \leq K \sum_{k=1}^{L} \left( E \left\| p_{k,L} (x_j) \mathcal{J}_k (x_i) \right\|^4 \left\{ \mathcal{I} (x_i \in \Lambda (k)) \right\} \right) \]

\[ + L^2 \sum_{k=1}^{L} E \left( p_{k,L}^4 (x_j) \mathcal{J}_k (x_i) \right) \mathcal{I} (x_i \in \Lambda (k)) \]

\[ = \left( K \left( z^{k(x_i)} - x_i \right)^{-1} + L \right) \sum_{k=1}^{L} \mathcal{I} (x_i \in \Lambda (k)) \]

\[ = K (n^\lambda + L), \]

because \( E p_{k,L}^4 (x_j) = O \left( L^{-1}\right) \). So the second term of (9.21) is

\[ O \left( (n^\lambda + L) \frac{1}{n^2} \right) \frac{1}{n} \sum_{j=1}^{n} \mathcal{I}_i \left(x^1, x^2\right) \]

which implies that by standard arguments that it is

\[ O \left( (n^\lambda + L) \frac{1}{n^2} \right) (F_X (x^2) - F_X (x^1)) (1 + o_p (1)) . \]

Thus, gathering terms, we have that the first factor of (9.18), i.e. (9.21), is

\[ O \left( \left( \frac{n^\lambda + L}{n} \right)^2 \frac{1}{n} \right) (F_X (x^2) - F_X (x^1)) (1 + o_p (1)) . \]

So, the last displayed expression together with (9.19) implies that the second term of (9.17) is

\[ O \left( \left( \frac{n^\lambda + L}{n} \right)^3 \right) (F_X (x^2) - F_X (x^1))^2 \leq K (x^2 - x^1)^2 \sup_{x \in (x^1, x^2)} f_X^2 (x) . \]
To complete the proof we need to examine the first term of (9.16), whose first moment is

\[
\frac{K}{n^2} \sum_{i=1}^{n} E \left( \left( v_i^2 - E \left( v_i^2 \mid G_i \right) \right) I_i \left( x^1, x^2 \right) \right)^2 \leq \frac{K}{n^2} \sum_{i=1}^{n} E \left( v_i^4 I_i \left( x^1, x^2 \right) \right) \\
\leq \frac{K}{n} \left( F_X \left( x^2 \right) - F_X \left( x^1 \right) \right)
\]

by standard arguments as \( Ev_i^4 < K \) proceeding as we did with the second term of (9.16).
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| Setting | Method       | B-splines | P-splines |
|---------|--------------|-----------|-----------|
|         |              | 10%       | 5%        | 10%       | 5%        |
| $L' = 6$ | KS bootstrap | 0.0865    | 0.049     | 0.0935    | 0.0435    |
| $N = 1000$ | CvM bootstrap | 0.0895    | 0.049     | 0.097     | 0.0495    |
| $\sigma = 0.25$ | AD bootstrap | 0.101     | 0.0505    | 0.1165    | 0.0435    |
| $L' = 9$ | KS bootstrap | 0.1       | 0.053     | 0.0945    | 0.04     |
| $N = 1000$ | CvM bootstrap | 0.1165    | 0.0595    | 0.102     | 0.0455    |
| $\sigma = 0.25$ | AD bootstrap | 0.11      | 0.0535    | 0.1065    | 0.054     |
| $L' = 14$ | KS bootstrap | 0.1195    | 0.058     | 0.093     | 0.0495    |
| $N = 1000$ | CvM bootstrap | 0.1195    | 0.058     | 0.1       | 0.048     |
| $\sigma = 0.25$ | AD bootstrap | 0.114     | 0.057     | 0.1125    | 0.0635    |
| $L' = 19$ | KS bootstrap | 0.1165    | 0.0655    | 0.088     | 0.0435    |
| $N = 1000$ | CvM bootstrap | 0.111     | 0.0565    | 0.095     | 0.047     |
| $\sigma = 0.25$ | AD bootstrap | 0.1135    | 0.062     | 0.109     | 0.0545    |

Table 1. Tests for monotonically increasing regression function in Scenario 1. Rejection rates in 2000 simulations. $L' + 1$ denotes the number of equidistant knots on $[0, 1]$ (including 0 and 1). $N$ denotes the number of observations in each simulation. $\sigma$ is the standard deviation in the error distribution.
| Setting | Method | B-splines 10% | B-splines 5% | P-splines 10% | P-splines 5% | B-splines 10% | B-splines 5% | P-splines 10% | P-splines 5% |
|---------|--------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $L' = 4$ | KS     | 0.093         | 0.0495        | 0.098         | 0.05          | 0.094         | 0.051         | 0.1025        | 0.05          |
| $N = 1000$ | CvM    | 0.098         | 0.0515        | 0.1095        | 0.0485        | 0.099         | 0.054         | 0.1           | 0.0425        |
| $\sigma = 0.25$ | AD     | 0.1           | 0.05          | 0.107         | 0.0505        | 0.1025        | 0.0505        | 0.1065        | 0.0505        |
| $L' = 6$ | KS     | 0.1015        | 0.525         | 0.0905        | 0.0485        | 0.102         | 0.0515        | 0.0945        | 0.0475        |
| $N = 1000$ | CvM    | 0.0935        | 0.047         | 0.103         | 0.05          | 0.097         | 0.0455        | 0.1035        | 0.0515        |
| $\sigma = 0.25$ | AD     | 0.0905        | 0.0435        | 0.1045        | 0.0535        | 0.0885        | 0.046         | 0.1045        | 0.0575        |
| $L' = 9$ | KS     | 0.0975        | 0.046         | 0.091         | 0.049         | 0.0965        | 0.046         | 0.099         | 0.0455        |
| $N = 1000$ | CvM    | 0.088         | 0.0465        | 0.0955        | 0.0445        | 0.0885        | 0.048         | 0.1045        | 0.045         |
| $\sigma = 0.25$ | AD     | 0.089         | 0.0465        | 0.0985        | 0.053         | 0.0915        | 0.0465        | 0.1005        | 0.0515        |

**Table 2.** Tests for U-shape with the switch at $s_0 = e^{0.33} - 1$ in Scenario 2. Rejection rates in 2000 simulations. $L' + 1$ denotes the number of equidistant knots on $[0, 1]$ (including 0 and 1). $N$ denotes the number of observations in each simulation. $\sigma$ is the standard deviation in the error distribution.
| Setting | Method      | B-splines 10% | B-splines 5% | P-splines 10% | P-splines 5% |
|---------|-------------|---------------|--------------|---------------|--------------|
| $L' = 6$ | KS bootstrap | 0.998         | 0.9965       | 0.9995        | 0.999        |
| $N = 1000$ | CvM bootstrap | 0.9995       | 0.997       | 1             | 0.9965      |
| $\sigma = 0.5$ | AD bootstrap | 1             | 0.999       | 1             | 0.9975      |
| $L' = 9$ | KS bootstrap | 0.98          | 0.962       | 0.998         | 0.9935      |
| $N = 1000$ | CvM bootstrap | 0.9795       | 0.962       | 0.9955        | 0.99         |
| $\sigma = 0.5$ | AD bootstrap | 0.982        | 0.9625      | 0.995         | 0.991       |
| $L' = 12$ | KS bootstrap | 0.9185       | 0.861       | 0.995         | 0.988       |
| $N = 1000$ | CvM bootstrap | 0.904        | 0.8435      | 0.992         | 0.982       |
| $\sigma = 0.5$ | AD bootstrap | 0.914        | 0.843       | 0.991         | 0.98        |
| $L' = 19$ | KS bootstrap | 0.7195       | 0.617       | 0.989         | 0.98        |
| $N = 1000$ | CvM bootstrap | 0.6525       | 0.5295      | 0.983         | 0.967       |
| $\sigma = 0.5$ | AD bootstrap | 0.6845       | 0.533       | 0.984         | 0.964       |
| $L' = 6$ | KS bootstrap | 1             | 1           | 1             | 1           |
| $N = 1000$ | CvM bootstrap | 1             | 1           | 1             | 1           |
| $\sigma = 0.25$ | AD bootstrap | 1             | 1           | 1             | 1           |
| $L' = 9$ | KS bootstrap | 1             | 1           | 1             | 1           |
| $N = 1000$ | CvM bootstrap | 1             | 1           | 1             | 1           |
| $\sigma = 0.25$ | AD bootstrap | 1             | 1           | 1             | 1           |
| $L' = 12$ | KS bootstrap | 1             | 1           | 1             | 1           |
| $N = 1000$ | CvM bootstrap | 1             | 1           | 1             | 1           |
| $\sigma = 0.25$ | AD bootstrap | 1             | 1           | 1             | 1           |
| $L' = 19$ | KS bootstrap | 1             | 1           | 1             | 1           |
| $N = 1000$ | CvM bootstrap | 1             | 1           | 1             | 1           |
| $\sigma = 0.25$ | AD bootstrap | 1             | 1           | 1             | 1           |

**Table 3.** Tests for monotonicity in Scenario 3. Rejection rates in 2000 simulations. $L' + 1$ denotes the number of equidistant knots on [0, 1] (including 0 and 1). $N$ denotes the number of observations in each simulation. $\sigma$ is the standard deviation in the error distribution.
### Table 4. Tests for monotonicity in Scenario 3. Rejection rates in 2000 simulations. \( L' + 1 \) denotes the number of equidistant knots on \([0, 1]\) (including 0 and 1). \( N \) denotes the number of observations in each simulation. \( \sigma \) is the standard deviation in the error distribution.

| Setting  | Method | B-splines 10% | B-splines 5% | P-splines 10% | P-splines 5% |
|----------|--------|----------------|---------------|----------------|---------------|
|          |        |                |               |                |               |
| \( L' = 6 \) KS | 0.4315 | 0.343 | 0.6195 | 0.4895 | 0.1695 | 0.1015 | 0.312 | 0.2055 |
| \( N = 1000 \) CvM | 0.439 | 0.339 | 0.6435 | 0.5195 | 0.1745 | 0.098 | 0.32 | 0.207 |
| AD \( \sigma = 0.5 \) AD | 0.4855 | 0.376 | 0.6345 | 0.5115 | 0.1815 | 0.1115 | 0.291 | 0.1865 |
| \( L' = 9 \) KS | 0.387 | 0.279 | 0.612 | 0.49 | 0.179 | 0.107 | 0.3095 | 0.224 |
| \( N = 1000 \) CvM | 0.3805 | 0.277 | 0.632 | 0.5095 | 0.1795 | 0.1005 | 0.305 | 0.215 |
| \( \sigma = 0.5 \) AD | 0.4235 | 0.302 | 0.6415 | 0.518 | 0.178 | 0.099 | 0.2995 | 0.197 |
| \( L' = 12 \) KS | 0.3365 | 0.242 | 0.615 | 0.4975 | 0.16 | 0.094 | 0.3005 | 0.2125 |
| \( N = 1000 \) CvM | 0.3415 | 0.227 | 0.6315 | 0.503 | 0.163 | 0.09 | 0.3055 | 0.2205 |
| \( \sigma = 0.5 \) AD | 0.375 | 0.274 | 0.644 | 0.529 | 0.161 | 0.097 | 0.305 | 0.2125 |
| \( L' = 19 \) KS | 0.264 | 0.1775 | 0.594 | 0.4575 | 0.1645 | 0.1 | 0.312 | 0.1885 |
| \( N = 1000 \) CvM | 0.253 | 0.165 | 0.589 | 0.4635 | 0.158 | 0.0985 | 0.29 | 0.196 |
| \( \sigma = 0.5 \) AD | 0.2905 | 0.187 | 0.6375 | 0.529 | 0.16 | 0.094 | 0.309 | 0.1955 |
| \( L' = 6 \) KS | 0.965 | 0.936 | 0.992 | 0.9835 | 0.459 | 0.3735 | 0.7285 | 0.627 |
| \( N = 1000 \) CvM | 0.9645 | 0.938 | 0.9945 | 0.989 | 0.455 | 0.3625 | 0.746 | 0.6405 |
| \( \sigma = 0.25 \) AD | 0.974 | 0.9565 | 0.993 | 0.9855 | 0.456 | 0.3605 | 0.68 | 0.5715 |
| \( L' = 9 \) KS | 0.9325 | 0.89 | 0.9915 | 0.9825 | 0.394 | 0.296 | 0.6875 | 0.5635 |
| \( N = 1000 \) CvM | 0.929 | 0.885 | 0.9915 | 0.9835 | 0.387 | 0.277 | 0.683 | 0.578 |
| \( \sigma = 0.25 \) AD | 0.948 | 0.91 | 0.992 | 0.9855 | 0.391 | 0.281 | 0.665 | 0.534 |
| \( L' = 12 \) KS | 0.8915 | 0.834 | 0.9905 | 0.9825 | 0.3615 | 0.247 | 0.67 | 0.5665 |
| \( N = 1000 \) CvM | 0.887 | 0.7925 | 0.9895 | 0.9805 | 0.3545 | 0.2355 | 0.6825 | 0.5715 |
| \( \sigma = 0.25 \) AD | 0.915 | 0.866 | 0.9905 | 0.9835 | 0.364 | 0.2465 | 0.6615 | 0.5545 |
| \( L' = 19 \) KS | 0.778 | 0.674 | 0.988 | 0.9695 | 0.284 | 0.1895 | 0.656 | 0.518 |
| \( N = 1000 \) CvM | 0.759 | 0.658 | 0.986 | 0.9685 | 0.2705 | 0.177 | 0.648 | 0.522 |
| \( \sigma = 0.25 \) AD | 0.82 | 0.723 | 0.9885 | 0.98 | 0.286 | 0.1855 | 0.643 | 0.5095 |
| \( L' = 6 \) KS | 1 | 1 | 1 | 1 | 0.999 | 0.9975 | 1 | 1 |
| \( N = 1000 \) CvM | 1 | 1 | 1 | 1 | 0.9985 | 0.9975 | 1 | 1 |
| \( \sigma = 0.1 \) AD | 1 | 1 | 1 | 1 | 0.9985 | 0.9965 | 1 | 1 |
| \( L' = 9 \) KS | 1 | 1 | 1 | 1 | 0.993 | 0.985 | 0.9995 | 0.9995 |
| \( N = 1000 \) CvM | 1 | 1 | 1 | 1 | 0.9915 | 0.9825 | 0.9995 | 0.9995 |
| \( \sigma = 0.1 \) AD | 1 | 1 | 1 | 1 | 0.9915 | 0.9865 | 0.9995 | 0.9999 |
| \( L' = 12 \) KS | 1 | 1 | 1 | 1 | 0.9845 | 0.9715 | 0.9995 | 0.9995 |
| \( N = 1000 \) CvM | 1 | 1 | 1 | 1 | 0.982 | 0.9665 | 0.9995 | 0.9999 |
| \( \sigma = 0.1 \) AD | 1 | 1 | 1 | 1 | 0.9865 | 0.9725 | 0.9999 | 0.9999 |
| \( L' = 19 \) KS | 1 | 1 | 1 | 1 | 0.948 | 0.9105 | 0.999 | 0.9985 |
| \( N = 1000 \) CvM | 1 | 1 | 1 | 1 | 0.942 | 0.909 | 0.9995 | 0.9985 |
| \( \sigma = 0.1 \) AD | 1 | 1 | 1 | 1 | 0.9515 | 0.913 | 0.999 | 0.9985 |
### Table 5.
Hospital data. Test statistics and bootstrap critical values under the null hypothesis of monotonicity of the regression function. Bootstrap critical values are from 400 bootstrap replications.

| Method | B-splines | P-splines |
|--------|-----------|-----------|
|        | Test statistic | Bootstrap critical value | Test statistic | Bootstrap critical value |
|        | 10% | 5% | 10% | 5% |
| KS     | 0.8637 | 1.3393 | 1.5269 | 0.6375 | 0.7739 | 0.8683 |
| CvM    | 0.1516 | 0.4306 | 0.6406 | 0.0784 | 0.0961 | 0.1308 |
| AD     | 0.9511 | 3.5858 | 5.2313 | 0.4852 | 0.6516 | 0.8905 |
| Method | Test statistic | Bootstrap crit. value 10% | Bootstrap crit. value 5% | Test statistic | Bootstrap crit. value 10% | Bootstrap crit. value 5% |
|--------|----------------|---------------------------|--------------------------|----------------|---------------------------|--------------------------|
| KS     | 0.8017         | 1.3703                    | 1.4848                   | 0.7436         | 1.1721                    | 1.3106                   |
| CvM    | 0.1225         | 0.4810                    | 0.6070                   | 0.0768         | 0.3145                    | 0.4149                   |
| AD     | 0.6112         | 3.5323                    | 4.6403                   | 0.4509         | 1.8701                    | 2.4007                   |

Table 6. Hospital data. Test statistics and bootstrap critical values under the null hypothesis of convexity of the regression function. Bootstrap critical values are from 400 bootstrap replications.
| Method | B-splines | P-splines |
|--------|-----------|-----------|
|        | Test statistic | Bootstrap crit. value | Test statistic | Bootstrap crit. value |
|        | 10%        | 5%        | 10%        | 5%        |
| KS     | 0.7377     | 1.5335    | 1.8398     | 0.7390    | 1.5273    | 1.8322    |
| CvM    | 0.1575     | 0.7848    | 1.2445     | 0.1574    | 0.7846    | 1.2328    |
| AD     | 0.9756     | 4.3146    | 6.3153     | 0.9748    | 4.2662    | 6.2556    |

Table 7. Hospital data. Test statistics and bootstrap critical values under the null hypothesis of both monotonicity and convexity of the regression function. Bootstrap critical values are from 400 bootstrap replications.
| Method | B-splines | P-splines |
|--------|-----------|-----------|
|        | Test statistic | Bootstrap crit. value | Test statistic | Bootstrap crit. value |
|        |              | 10% | 5% |              | 10% | 5% |
| KS     | 1.6485      | 2.2762 | 2.3537 | 0.6953 | 0.9205 | 0.9947 |
| CvM    | 0.8932      | 1.768 | 1.9465 | 0.09 | 0.1969 | 0.2314 |
| AD     | 4.6227      | 9.2792 | 10.281 | 0.5627 | 1.2883 | 1.4924 |

Table 8. Energy consumption data. Test statistics and bootstrap critical values under the null hypothesis of U-shaped regression function with the switch as 17.6°. Two B-spline curves are joined continuously at the switch point. Bootstrap critical values are from 400 bootstrap replications.
| Method | Test statistic | Bootstrap critical value | Test statistic | Bootstrap critical value |
|--------|----------------|--------------------------|----------------|--------------------------|
|        | B-splines      | 10%                      | 5%            | P-splines                | 10%                      | 5%                        |
| KS     | 1.7743         | 2.3244                   | 2.4165        | 1.0594                   | 1.0777                   | 1.1417                    |
| CvM    | 1.0069         | 1.7909                   | 1.9597        | 0.2479                   | 0.2707                   | 0.3134                    |
| AD     | 5.1412         | 9.3911                   | 10.344        | 1.233                    | 1.4702                   | 1.7148                    |

Table 9. Energy consumption data. Test statistics and bootstrap critical values under the null hypothesis of U-shaped regression function with the switch as 17.6°. Two B-spline curves are joined smoothly at the switch point. Bootstrap critical values are from 400 bootstrap replications.
| Method | Test statistic | Bootstrap critical value | Test statistic | Bootstrap critical value |
|--------|----------------|--------------------------|----------------|--------------------------|
|        | B-splines      |                          | P-splines      |                          |
|        | 10%            | 5%                       | 10%            | 5%                       |
| KS     | 1.1233         | 0.8010                   | 0.9010         | 1.9018                   | 0.8398                   | 0.9144                   |
| CvM    | 0.4376         | 0.1203                   | 0.1757         | 1.2998                   | 0.1424                   | 0.1633                   |
| AD     | 3.6991         | 0.8010                   | 1.0704         | 8.5253                   | 1.1595                   | 1.8681                   |

TABLE 10. Energy consumption data. Test statistics and bootstrap critical values under the null hypothesis of convexity of the regression function. Bootstrap critical values are from 400 bootstrap replications.