2-limited broadcast domination on grid graphs

Aaron Slobodin∗, Gary MacGillivray† and Wendy Myrvold‡

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Abstract

We establish upper and lower bounds for the 2-limited broadcast domination number of various grid graphs, in particular the Cartesian product of two paths, a path and a cycle, and two cycles. The upper bounds are derived by explicit constructions. The lower bounds are obtained via linear programming duality by finding lower bounds for the fractional 2-limited multipacking numbers of these graphs.

1 Introduction

Consider a city partitioned into neighbourhoods, each of which has a communication tower capable of transmitting a radio broadcast of some non-negative strength. Given this premise, a natural question is: how can we design a broadcast scheme such that broadcasts are heard by each neighbourhood while reducing the number of communication towers transmitting?

The discrete version of this question leads to broadcast domination, which was introduced by Erwin in 2001 [7].

In a $k$-limited broadcast, each vertex $v$ of a graph $G$ is assigned a broadcast strength $f(v) \in \{0, 1, \ldots, k\}$, where $k \leq \text{rad}(G)$ and $\text{rad}(G)$ is the radius of $G$. We say that a vertex $u$ hears the broadcast from $v$ if $d(u, v) \leq f(v)$, where $d(x, v)$ is the distance between $x$ and $v$ in $G$. A broadcast $f$ is dominating if each vertex of $G$ hears the broadcast from some vertex. The cost of a broadcast $f$ is $\sum_{v \in V(G)} f(v)$. The $k$-limited broadcast domination number $\gamma_{b,k}(G)$ of $G$ is the minimum cost of a $k$-limited dominating broadcast. The $k$-limited broadcast domination number $\gamma_{b,k}(G)$ can also be formulated as an integer linear program. Let $G$ be a graph and fix $1 \leq k \leq \text{rad}(G)$. For each vertex $i \in V(G)$ and $\ell \in \{1, 2, \ldots, k\}$ let

$$x_{i,\ell} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } \ell \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The $k$-limited broadcast domination number $\gamma_{b,k}(G)$ of a graph $G$ is the cost of an optimal solution to (ILP 1.1).

Minimize: $\sum_{\ell=1}^{k} \sum_{i \in V(G)} \ell \cdot x_{i,\ell}$

Subject to: (1) $\sum_{\ell=1}^{k} \sum_{i \in V(G) \text{ s.t. } d(i,j) \leq k} x_{i,\ell} \geq 1$, for each vertex $j \in V(G)$,

(2) $x_{i,\ell} \in \{0, 1\}$ for each vertex $i \in V(G)$ and $\ell \in \{1, 2, \ldots, k\}$.

The $k$-limited broadcast domination number of a graph was first mentioned, although not explored, in [7].

Given a tree $T$ on $n$ vertices, the best possible upper bound $\gamma_{b,k}(T) \leq \lceil \frac{k+1}{k+1} \cdot \frac{n}{2} \rceil$ is shown in [2]. It gives a

∗aslobodin@uvic.ca, Department of Mathematics and Statistics, University of Victoria.
†gmacgill@uvic.ca, Department of Mathematics and Statistics, University of Victoria, research supported by NSERC funding.
‡wendym@cs.uvic.ca, Department of Computer Science, University of Victoria, research supported by NSERC funding.
bound for all graphs via spanning trees. However, these bounds are likely very far from the truth for arbitrary graphs. Specific to 2-limited broadcast domination, Yang proved that if \( G \) is a cubic graph on \( n \) vertices with no 4-cycles or 6-cycles, then \( \gamma_{b,2}(G) \leq n/3 \) [20].

For each fixed positive integer \( k \), the problem of deciding whether there exists a \( k \)-limited dominating broadcast of cost at most a given integer \( B \) is NP-complete [2]. There are polynomial time algorithms to compute \( \gamma_{b,k}(G) \) for strongly chordal graphs, interval graphs, circular arc graphs, and proper interval bigraphs [20].

The \( k \)-limited broadcast domination problem is a restriction of the broadcast domination problem (which was also introduced in [7]). The broadcast domination number of a graph can be obtained from \( \text{ILP } 1.1 \) by setting \( k = \text{rad}(G) \). The broadcast domination number of a graph with \( n \) vertices can be found in \( O(n^6) \) time [13]. There exists improved algorithms for trees [4, 5], interval graphs [3], and strongly chordal graphs [19, 21]. A survey of results on broadcast domination can be found in [14].

1.1 \( k \)-Limited Multipacking

The dual of the LP relaxation of \( \text{ILP } 1.1 \) leads to fractional \( k \)-limited multipacking. For each vertex \( i \in V(G) \), define a variable \( y_i \). The fractional \( k \)-limited multipacking packing number \( mp_{f,k}(G) \) of a graph \( G \) is the cost of an optimal solution to \( \text{LP } 1.1.1 \).

Maximize: \[ \sum_{i \in V(G)} y_i \]

Subject to:

1. \[ \sum_{i \in V(G) \text{ s.t. } d(i,j) \leq \ell} y_i \leq \ell, \text{ for each vertex } j \in V(G) \text{ and } \ell \in \{1, 2, \ldots, k\}, \text{ and } (\text{LP } 1.1.1) \]
2. \[ y_i \geq 0, \text{ for each vertex } i \in V(G). \]

The cost of an optimal solution of \( \text{LP } 1.1.1 \) when interpreted as a 0-1 ILP, is the \( k \)-limited multipacking number of a graph.

Relatively little is known about \( k \)-limited multipackings on graphs. For each fixed integer \( k \), the problem of deciding whether a given graph \( G \) has a \( k \)-limited multipacking number at least a given integer \( B \) is NP-complete [20]. But there is a known polynomial time algorithm for the \( k \)-limited multipacking number is for strongly chordal graphs [21].

The multipacking number \( mp(G) \) of a graph \( G \) is the cost of an optimal solution to \( \text{LP } 1.1.1 \) when interpreted as a 0-1 ILP, and setting \( k = \text{rad}(G) \). This problem was introduced in Teshima’s Master’s thesis [16] in 2012 (also see [18]). The \( k \)-limited multipacking problem is a restriction of the multipacking problem. The complexity of deciding whether a given graph \( G \) has a multipacking number \( mp(G) \) at least a given integer \( B \) remains an open problem. There is a \((2 + o(1))\) approximation algorithm for this decision problem on undirected graphs [1], and there are polynomial time algorithms for trees [16, 17, 21] and strongly chordal graphs [19, 21]. For further information, see [14].

1.2 Outline

After a sequence of many papers, the 1-limited broadcast domination number (i.e. the domination number) of the Cartesian product of two paths was finally determined (in [12]). This paper focuses on extending this work to determine bounds for the 2-limited broadcast domination number of the Cartesian product of two paths, a path and a cycle, and two cycles. Section 2 introduces the method of constructing 2-limited dominating broadcasts by “tiling” the graph with “broadcast tiles.” Sections 3, 4, and 5 summarize the tiling results from Chapters 2, 3 and 4 of [15] which give 2-limited dominating broadcasts on the Cartesian products of two paths, a path and a cycle, and two cycles. Section 6 utilizes fractional 2-limited multipacking to establish lower bounds for the 2-limited broadcast domination numbers of the Cartesian product of two paths, a path and a cycle, and two cycles. Section 7 includes suggested questions for future research.
Unless specified otherwise, all computations in this paper were run on a PowerEdge R7425 server equipped with two AMD EPYC processors and totalling 64 threaded CPU cores. This machine was purchased by Myrvold using NSERC funding.

2 The Tiling Method for establishing Upper Bounds

While we are able to exactly compute the 2-limited broadcast domination number of \( P_m \square P_n \), \( P_m \square C_n \), or \( C_m \square C_n \) for many values of \( m \) and \( n \), we have yet to find a general formula. Thus, we establish upper bounds. This section describes the method of “tiling” which we use in [15] to construct 2-limited dominating broadcasts on these graphs. Our description uses examples specific to \( P_4 \square P_n \). Results for all \( m \geq 2 \) and \( n \geq m \) are stated subsequently in Sections 3, 4, and 5.

Figure 1 depicts broadcast tiles \( B, B_1, B_2, B_3, B_4, \) and \( B_5 \) which can be used to construct 2-limited dominating broadcasts on \( P_4 \square P_n \) for all \( n \geq 4 \). The thick blue border indicates the border of each tile. The black circles with a black inner fill indicate vertices on each tile broadcasting at a non-zero strength. In this example, \( B \) has two vertices broadcasting at strength one (as found in the top most corners) and one vertex broadcasting at strength two (as found in the lower middle row of the center column). The thick red dotted lines indicate the broadcast ranges of the broadcasting vertices at their centers. Thick green circles with a white inner fill indicate vertices on each tile which do not hear a broadcast within the tile. Thick orange circles with an opaque orange inner fill indicate vertices exterior to the tile which can hear broadcasts from vertices on the tile.

Given the tile \( B_4 \), \( \overline{B_4} \) denotes flipping \( B_4 \) about the horizontal axis, \( B_4^\uparrow \) indicates flipping \( B_4 \) about the vertical axis, and \( \overline{B_4}^\uparrow \) indicates flipping \( B_4 \) about the horizontal and the vertical axes. The four possible states of \( B_4 \) are shown in Figure 2.

![Figure 2: Possible states of tile \( B_4 \).](image)

Given two tiles \( T_1 \) and \( T_2 \), the sequence \( T_1T_2 \) denotes placing the right side of \( T_1 \) beside the left side of \( T_2 \) such that the vertices along the right border of \( T_1 \) and the left border of \( T_2 \) are at distance one from one another. Figure 3 depicts \( B_4^\uparrow \overline{B_4} \). Observe that, in Figure 3, the undominated vertex in the upper right corner of \( B_4 \) hears the broadcast from the vertex in the upper left corner of \( \overline{B_4} \). Similarly, in Figure 3, the undominated vertex in the bottom left corner of \( B_4 \) hears the broadcast from the vertex in the bottom right corner of \( \overline{B_4} \). As all vertices of \( P_4 \square P_6 \) in Figure 3 hear a broadcast under the tiling \( B_4 \overline{B_4} \), the sequence \( B_4 \overline{B_4} \) gives a dominating broadcast scheme on \( P_4 \square P_6 \) of cost six. Therefore, \( \gamma_{b,2}(P_4 \square P_6) \leq 6 \). In fact, one can easily verify by computation that \( \gamma_{b,2}(P_4 \square P_6) = 6 \).

For each least residue of \( n \) modulo 10, Table 1 contains the associated sequences of broadcast tiles from Figure 4 which give a 2-limited dominating broadcast on \( P_4 \square P_n \). The costs of the constructions in Table 1

![Figure 1: Tiles used in constructions of 2-limited dominating broadcasts on \( P_4 \square P_n \).](image)
establish Proposition 2.1. The costs of these constructions are easily calculated from the costs of the individual broadcast tiles in Figure 1. The values of Proposition 2.1 are optimal for all $n \leq 99$ (as found by computation).

| Construction | Cost |
|--------------|------|
| $n \equiv 0 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_4$ | $8 \left( \frac{n_{10}}{10} \right) + 9 = 8 \left( \frac{n}{10} \right) + 1$ |
| $n \equiv 1 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_3$ | $8 \left( \frac{n_{10} + 1}{10} \right) + 10 = 8 \left( \frac{n - 1}{10} \right) + 2$ |
| $n \equiv 2 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_5$ | $8 \left( \frac{n_{10} - 2}{10} \right) + 11 = 8 \left( \frac{n - 2}{10} \right) + 3$ |
| $n \equiv 3 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_2$ | $8 \left( \frac{n_{10} - 3}{10} \right) + 4$ |
| $n \equiv 4 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_1$ | $8 \left( \frac{n_{10} - 4}{10} \right) + 4$ |
| $n \equiv 5 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_3$ | $8 \left( \frac{n_{10} - 5}{10} \right) + 5$ |
| $n \equiv 6 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_4$ | $8 \left( \frac{n_{10} - 6}{10} \right) + 6$ |
| $n \equiv 7 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_5$ | $8 \left( \frac{n_{10} - 7}{10} \right) + 7$ |
| $n \equiv 8 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_2$ | $8 \left( \frac{n_{10} - 8}{10} \right) + 8$ |
| $n \equiv 9 \ (\text{mod} \ 10)$: $B_1 [BB]^{n_{10}} BB_1$ | $8 \left( \frac{n_{10} - 9}{10} \right) + 8$ |

Table 1: Constructions of 2-limited dominating broadcasts on $P_4 \square P_n$ using the tiles from Figure 1.

**Proposition 2.1.** For $n \geq 4$, $\gamma_{b,2} (P_4 \square P_n) \leq 8 \left\lfloor \frac{n}{10} \right\rfloor + d(n_{10})$ where $n_{10}$ is the least residue of $n$ modulo 10 and $d(n_{10})$ is given in Table 2.

| $n_{10}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----------|---|---|---|---|---|---|---|---|---|---|
| $d(n_{10})$ | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 | 8 |

Table 2: Values of $d(n_{10})$ in the upper bound of $\gamma_{b,2} (P_4 \square P_n)$ for $n \geq 4$.

Tilings are used in Sections 3, 4, and 5 to establish upper bounds for the 2-limited broadcast domination number on all $P_n \square P_m$, $P_n \square C_m$, and $C_n \square C_m$ grid graphs, respectively. This paper does not include the broadcast tiles to prove these bounds as, when combined, they span over 100 pages. For each stated theorem, we include the section(s) of [15] which contain the appropriate broadcast tiles. Most broadcast tiles of [15] were identified by hand by examining known optimal solutions, as found by computation, and looking for recurring patterns. This manual process was greatly facilitated by automatically creating PDF visuals of known optimal solutions using a Python script (written by Slobodin).
3 Upper Bounds for $\gamma_{b,2}(P_m \square P_n)$

In this section, we establish upper bounds for the 2-limited broadcast domination number of the Cartesian product of two paths. Section 3.1 states upper bounds for $\gamma_{b,2}(P_m \square P_n)$ for $2 \leq m \leq 12$ and all $n \geq m$. These results are established using the tiling method described in Section 2. For $m, n \geq 13$, Section 3.2 describes a general construction on $P_m \square P_n$ derived from a 2-limited dominating broadcast on the Cartesian plane.

3.1 $P_{2 \leq m \leq 12} \square P_{n \geq m}$

The tilings used to establish Theorem 3.1 are in Sections 2.2 through 2.12.

**Theorem 3.1.** Fix $2 \leq m \leq 12$ and $n \geq m$. Let $x$ be the value in Table 3 dependant upon $m$ and define $n_x$ as the least residue of $n$ modulo $x$. By construction,

$$\gamma_{b,2}(P_m \square P_n) \leq b(m) + c(m, n, n_x)$$

where $b(m)$ corresponds with the linear terms in Table 4 and $c(m, n, n_x)$ corresponds with values in Table 5.

| $m$ | 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 |
|-----|----------------------------------|
| $x$ | 1, 10, 1, 16, 14, 22, 10, 18, 26, 24 |

Table 3: Value of $x$ in the upper bound of $\gamma_{b,2}(P_m \square P_n)$ for $2 \leq m \leq 12$ and $n \geq m$.

| $m$ | 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 |
|-----|----------------------------------|
| $b(m)$ | $\left\lfloor \frac{n}{2} \right\rfloor$, $\left\lfloor \frac{2n}{3} \right\rfloor$, 8, $\left\lceil \frac{n}{10} \right\rceil$, $\left\lceil \frac{n}{16} \right\rceil$, $\left\lceil \frac{n}{14} \right\rceil$, 8, $\left\lceil \frac{n}{16} \right\rceil$, 32, $\left\lceil \frac{n}{22} \right\rceil$, 16, $\left\lceil \frac{n}{10} \right\rceil$, 32, $\left\lceil \frac{n}{18} \right\rceil$, 50, $\left\lceil \frac{n}{26} \right\rceil$, 50, $\left\lceil \frac{n}{24} \right\rceil$ |

Table 4: Value of $b(m)$ in the upper bound of $\gamma_{b,2}(P_m \square P_n)$ for $2 \leq m \leq 12$ and $n \geq m$.

3.2 $P_{m \geq 13} \square P_{n \geq 13}$

This section provides upper bounds for $\gamma_{b,2}(P_m \square P_n)$ for $m, n \geq 13$ by modifying 2-limited dominating broadcasts on the Cartesian plane. Our expectation is that this method will achieve good upper bounds as, for very large $m$ and $n$, there likely exists a 2-limited dominating broadcast on $P_m \square P_n$ which, for much of the graph, resembles this 2-limited dominating broadcast on the Cartesian plane. This method was inspired by the work on the distance domination number of grids in [8].

3.2.1 2-limited dominating broadcasts on $\mathbb{Z}^2$

Let $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ denote the integer lattice. Let $G_{m,n}$ and $Y_{m+4,n+4}$ denote $P_m \square P_n$ and $P_{m+4} \square P_{n+4}$, respectively, embedded in $\mathbb{Z}^2$ with their bottom left corners at $(0,0)$ and $(-2,-2)$, respectively. That is, let

$$V(G_{m,n}) = \{(i,j) \in \mathbb{Z}^2 : 0 \leq i \leq n-1 \text{ and } 0 \leq j \leq m-1\}$$

and

$$V(Y_{m+4,n+4}) = \{(i,j) \in \mathbb{Z}^2 : -2 \leq i \leq n+1 \text{ and } -2 \leq j \leq m+1\}.$$ 

Let $\mathbb{Z}_{13} = \{0, 1, \ldots, 12\}$. Define the function $\phi : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}_{13}$ by

$$\phi : (i,j) \mapsto 3i + 2j \pmod{13}.$$ 

Fix $\ell \in \mathbb{Z}_{13}$. The set given by

$$\phi^{-1}(\ell) = \{(i,j) \in \mathbb{Z}^2 : 3i + 2j \equiv \ell \pmod{13}\}$$

5
| $n_x$ | 2   | 3   | 4   | 5   | 6   | $m$ | 7   | 8   | 9   | 10  | 11  | 12  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     | 1   | 0   | 1   | 2   | 2   | 2   | 2   | 2   | 2   | 3   | 3   |     |
| 1     | 2   | 2   | 3   | 4   | 4   | 3   | 4   | 5   | 4   |     |     |     |
| 2     | 3   | 4   | 4   | 5   | 5   | 6   | 6   | 6   | 7   |     |     |     |
| 3     | 4   | 5   | 5   | 7   | 6   | 8   | 8   | 8   | 9   |     |     |     |
| 4     | 5   | 7   | 8   | 9   | 10  | 11  | 12  |     |     |     |     |     |
| 5     | 6   | 8   | 9   | 11  | 11  | 13  | 13  | 14  | 15  |     |     |     |
| $n = 6$ | 6   | 9   | 9   | 11  | 11  | 13  | 13  | 14  | 15  |     |     |     |
| $n > 6$ | 7   | 9   | 11  | 12  | 13  | 15  | 15  | 16  | 17  |     |     |     |
| 8     | 8   | 11  | 12  | 14  | 15  | 16  | 18  | 20  |     |     |     |     |
| 9     | 8   | 11  | 13  | 15  | 16  | 19  | 19  | 21  |     |     |     |     |
| 10    | 10  | 13  | 14  | 16  | 20  | 22  | 24  |     |     |     |     |     |
| 11    | 11  | 14  | 16  | 18  | 22  | 24  | 25  |     |     |     |     |     |
| 12    | 12  | 15  | 17  | 20  | 24  | 25  | 28  |     |     |     |     |     |
| 13    | 13  | 16  | 18  | 21  | 25  | 28  | 29  |     |     |     |     |     |
| 14    | 14  | 18  | 23  | 27  | 30  | 32  |     |     |     |     |     |     |
| 15    | 15  | 24  | 29  | 31  | 34  |     |     |     |     |     |     |     |
| 16    | 16  | 25  | 31  | 33  | 36  |     |     |     |     |     |     |     |
| 17    | 17  | 27  | 32  | 35  | 38  |     |     |     |     |     |     |     |
| 18    | 18  | 28  | 37  | 40  |     |     |     |     |     |     |     |     |
| 19    | 19  | 30  | 39  | 42  |     |     |     |     |     |     |     |     |
| 20    | 20  | 31  | 41  | 45  |     |     |     |     |     |     |     |     |
| 21    | 21  | 32  | 43  | 46  |     |     |     |     |     |     |     |     |
| 22    | 22  | 33  | 44  | 49  |     |     |     |     |     |     |     |     |
| 23    | 23  | 47  | 50  |     |     |     |     |     |     |     |     |     |
| 24    | 24  | 49  |     |     |     |     |     |     |     |     |     |     |
| 25    | 25  | 50  |     |     |     |     |     |     |     |     |     |     |

Table 5: Value of $c(m, n, n_x)$ in the upper bound of $\gamma_{b,2}(P_m \Box P_n)$ for $2 \leq m \leq 12$ and $n \geq m$.

Figure 4: $G_{3,4}$, $Y_{7,8}$, and the 2-limited dominating broadcast on $\mathbb{Z}^2$ defined by $\phi^{-1}(4)$.
defines the vertices of $Z^2$ which, if broadcasting at strength two, give a 2-limited dominating broadcast on $Z^2$ [6 Lemma V.7]. Figure 4 depicts $G_{3,4}$, $Y_{7,8}$, and the broadcast vertices and their neighbourhoods given by $\phi^{-1}(4)$. The thick blue and green lines indicate the borders of $G_{3,4}$ and $Y_{7,8}$, respectively. The light blue fill and green inner fill indicate the interiors of $G_{3,4}$ and $Y_{7,8}$, respectively. The thick black lines indicate the axes of $Z^2$. To see that the broadcast set in Figure 4 is given by $\phi^{-1}(4)$ observe, for instance, that $(5, 2)$ is one such broadcaster and

$$\phi(5, 1) = 3(5) + 2(1) = 17 \equiv 4 \pmod{13} \Rightarrow (5, 1) \in \phi^{-1}(4).$$

Prior to constructing 2-limited dominating broadcasts on $P_m \square P_n$ for all $m, n \geq 13$ from 2-limited dominating broadcasts on $Z^2$, we include Example 3.2 to provide an introduction to our methodology.

**Example 3.2.** Suppose we wish to find a 2-limited dominating broadcast on $G_{13,13}$. Overlay one of the 13 possible 2-limited dominating broadcasts of $Z^2$ given by $\phi^{-1}(\ell)$ for some $\ell \in \mathbb{Z}_{13}$ on $G_{13,13}$. The broadcast vertices on $Y_{17,17}$ dominate the vertices of $G_{13,13}$. Figure 5 depicts $G_{13,13}$, $Y_{17,17}$, and the overlay of $\phi^{-1}(8)$ where the thick purple circles indicate the broadcast vertices on $Y_{17,17}$ but not $G_{13,13}$.

Moving each of the broadcast vertices exterior to $G_{13,13}$ and on $Y_{17,17}$ to the nearest point on the border of $G_{13,13}$ and reducing their broadcast strength to one produces a 2-limited dominating broadcast on $G_{13,13}$ given $\phi^{-1}(8)$.

The resulting 2-limited dominating broadcast on $G_{13,13}$ in Figure 6 establishes $\gamma_{b,2}(P_{13} \square P_{13}) \leq 35$. Note that, although this bound is not tight as $\gamma_{b,2}(P_{13} \square P_{13}) = 32$ (as determined by computation), it is also not too far from the truth. As $m$ and $n$ get large, we show that the ratio of the this upper bound and the lower bound of $\gamma_{b,2}(P_m \square P_n)$ given by Theorem 6.4 approaches one.

Example 3.2 illustrates that, given some fixed $\ell \in \mathbb{Z}_{13}$ and $m, n \geq 13$, if there are $x$ and $y$ vertices broadcasting at non-zero strength under $\phi^{-1}(\ell)$ on $G_{m,n}$ and $Y_{m+4,n+4}$, respectively, then there is a 2-limited dominating broadcast on $P_m \square P_n$ of cost $2x + (y - x)$. Our general construction, as described in the subsequent section, uses the same approach as Example 3.2 but uses a choice of $\ell \in \mathbb{Z}_{13}$ which gives a best possible construction (in terms of cost).

### 3.2.2 General Constructions

This section proves Theorem 3.4 which, with Theorem 3.1, establishes upper bounds for $\gamma_{b,2}(P_m \square P_n)$ for all $m, n \geq 2$. The proof of Theorem 3.4 uses Lemma 3.3 from [8], stated as follows.
Lemma 3.3. Let \( \ell \in \mathbb{Z}_{13} \). If either \( m \) or \( n \) is a multiple of 13, then for any \( \ell \in \mathbb{Z}_{13} \) then
\[
|\phi^{-1}(\ell) \cap V(G_{m,n})| = \frac{mn}{13}.
\]

Theorem 3.4. If \( m, n \geq 13 \) then
\[
\gamma_{b,2}(P_m \square P_n) \leq 2 \left( \frac{mn}{13} \right) + 4 \left( \frac{m+n}{13} \right) + \frac{c(m_{13}, n_{13})}{13}
\]
where \( c(m_{13}, n_{13}) \) corresponds with the values given in Table 6 where \( m_{13} \) and \( n_{13} \) are the least residues of \( m \) and \( n \) modulo 13, respectively.

| Least residue \( m_{13} \) of \( m \) modulo 13 | Least residue \( n_{13} \) of \( n \) modulo 13 |
|---------------------------------------------|---------------------------------------------|
| 0  0  1  2  3  4  5  6  7  8  9  10  11  12 | 0  1  2  3  4  5  6  7  8  9  10  11  12 |

Table 6: Value of \( c(m_{13}, n_{13}) \) for the upper bound for \( \gamma_{b,2}(P_m \square P_n) \) stated in Theorem 3.4.

Proof. Let \( G_{m,n}, Y_{m+4,n+4}, \mathbb{Z}^2, \mathbb{Z}_{13} \), and \( \phi \) be defined as in Section 3.2.1. Fix some \( \ell \in \mathbb{Z}^2 \). As \( Y_{m+4,n+4} \) is the distance two neighbourhood of \( G_{m,n} \), the set of vertices
\[
V(Y_{m+4,n+4}) \cap \phi^{-1}(\ell),
\]
By Lemma 3.3, $G$ is a 2-limited dominating broadcast of $G_{m,n}$. Let

$$B_2 = V(G_{m,n}) \cap \phi^{-1}(\ell) \quad \text{and} \quad B_1 = \phi^{-1}(\ell) \cap (V(Y_{m+4,n+4}) \setminus V(G_{m,n})).$$

For each vertex $v \in B_1$ whose broadcast is heard by a vertex in $G_{m,n}$, let $v'$ be the vertex of $G_{m,n}$ with minimum distance to $v$. Note that this choice of $v'$ is unique and that $v'$ necessarily hears a broadcast from $v$. The vertices dominated in $G_{m,n}$ by a broadcast of strength two at $v$ are a subset of the vertices dominated in $G_{m,n}$ by a broadcast of strength one at $v'$. Define the set

$$B_1' = \left\{ v' \in V(G_{m,n}) \mid v' \text{ is the vertex of } G_{m,n} \text{ which hears a broadcast from } v \text{ and is at minimum distance to } v \text{ for some } v \in B_1 \right\}.$$

Informally, $B_1'$ is the resulting collection of vertices formed by moving each vertex in $B_1$ (whose broadcast is heard by a vertex in $G_{m,n}$) to the nearest vertex in $G_{m,n}$. Each vertex in the plane hears only one broadcast under $\phi^{-1}(\ell)$ [Lemma V.7], hence each $v' \in B_1'$ hears only the broadcast from some $v \in B_1$. Therefore $v' \notin B_2$ and $B_2 \cap B_1' = \emptyset$. The broadcast $f : V(G_{m,n}) \rightarrow \{0,1,2\}$ defined by

$$f(v) = \begin{cases} 2 & \text{if } v \in B_2, \\ 1 & \text{if } v \in B_1', \\ 0 & \text{otherwise} \end{cases}$$

is a 2-limited dominating broadcast of $G_{m,n}$. The cost of $f$ is

$$2 |B_2| + |B_1'| \leq |V(G_{m,n}) \cap \phi^{-1}(\ell)| + |V(Y_{m+4,n+4}) \cap \phi^{-1}(\ell)|.$$

Note that equality holds in the above expression if and only if the broadcast from each vertex $v \in B_1$ is heard by some vertex $v' \in V(G_{m,n})$. Given this construction, it follows that

$$\gamma_{b,2}(P_m \square P_n) \leq \min_{\ell \in \mathbb{Z}_{13}} \left\{ |\phi^{-1}(\ell) \cap V(G_{m,n})| + |\phi^{-1}(\ell) \cap V(Y_{m+4,n+4})| \right\}.$$

For a fixed $\ell \in \mathbb{Z}^2$, $|\phi^{-1}(\ell) \cap V(G_{m,n})|$ is computed as follows. Let $m_{13}$ and $n_{13}$ be the least residue of $m$ and $n$ modulo 13, respectively. Partition $V(G_{m,n})$ into the following subsets:

$$G_1 = \{(i,j) \in \mathbb{Z}^2 : 0 \leq i \leq n - 1 - n_{13} \text{ and } 0 \leq j \leq m - 1\},$$

$$G_2 = \{(i,j) \in \mathbb{Z}^2 : n - n_{13} \leq i \leq n - 1 \text{ and } 0 \leq j \leq m - 1 - m_{13}\},$$

$$G_3 = \{(i,j) \in \mathbb{Z}^2 : n - n_{13} \leq i \leq n - 1 \text{ and } m - m_{13} \leq j \leq m - 1\}.$$

Figure 7 depicts $G_{16,18}$ partitioned into $G_1$, $G_2$, and $G_3$. For each $\ell \in \mathbb{Z}_{13}$,

$$|V(G_{m,n}) \cap \phi^{-1}(\ell)| = |G_1 \cap \phi^{-1}(\ell)| + |G_2 \cap \phi^{-1}(\ell)| + |G_3 \cap \phi^{-1}(\ell)|.$$

By Lemma 3.3,

$$|G_1 \cap \phi^{-1}(\ell)| = \frac{(m)(n - n_{13})}{13} \quad \text{and} \quad |G_2 \cap \phi^{-1}(\ell)| = \frac{(n_{13})(m - m_{13})}{13} \quad (3)$$

regardless of $\ell \in \mathbb{Z}_{13}$. Hence, to determine $|V(G_{m,n}) \cap \phi^{-1}(\ell)|$, what remains is to find $|G_3 \cap \phi^{-1}(\ell)|$. The same methodology can be applied to $Y_{m+4,n+4}$ by defining $m_{13}$ and $n_{13}$ as the least residues of $(m_{13} + 4)$ and $(n_{13} + 4)$ modulo 13, respectively, and partitioning $V(Y_{m+4,n+4})$ into the following subsets:

$$Y_1 = \{(i,j) \in \mathbb{Z}^2 : -2 \leq i \leq n + 1 - n_{13} \text{ and } -2 \leq j \leq m + 1\},$$

$$Y_2 = \{(i,j) \in \mathbb{Z}^2 : n + 2 - n_{13} \leq i \leq n + 1 \text{ and } -2 \leq j \leq m + 1 - m_{13}\},$$

$$Y_3 = \{(i,j) \in \mathbb{Z}^2 : n + 2 - n_{13} \leq i \leq n + 1 \text{ and } m + 2 - m_{13} \leq j \leq m + 1\}.$$
Figure 7: $G_{16,18}$ partitioned into $G_1, G_2$, and $G_3$.

Figure 8: $G_{16,18}$ partitioned into $G_1, G_2$, and $G_3$ overlayed by $Y_{20,22}$ partitioned into $Y_1, Y_2$, and $Y_3$. 
Figure 8 depicts $G_{16,18}$ partitioned into $G_1, G_2,$ and $G_3,$ (all in grey) overlayed by $Y_{20,22}$ partitioned into $Y_1, Y_2,$ and $Y_3$ (all opaque white).

By Lemma 3.3,

$$|Y_1 \cap \phi^{-1}(\ell)| = \frac{(m + 4)(n + 4 - n_{13})}{13}$$ and $$|Y_2 \cap \phi^{-1}(\ell)| = \frac{(n_{13})(m + 4 - m_{13})}{13}$$

regardless of $\ell \in \mathbb{Z}_{13}$. Finding a best choice of $\ell$ for the construction therefore reduces to determining, for each least residue of $m$ and $n$ modulo 13, the minimum values of $|G_3 \cap \phi^{-1}(\ell)| + |Y_3 \cap \phi^{-1}(\ell)|$ for $\ell \in \mathbb{Z}_{13}$. Said least values are given in Table 7 and the corresponding $\ell \in \mathbb{Z}_{13}$ is given in Table 8. In the cases where there are several such values of $\ell \in \mathbb{Z}_{13}$ for a given value in Table 7, we state the lexicographically largest in Table 8.

| Least residue $n_{13}$ of $n$ modulo 13 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|----------------------------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 0                                      | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 0 | 0   | 0   | 0   |
| 1                                      | 1 | 2 | 2 | 2 | 3 | 3 | 4 | 4 | 4 | 0 | 0   | 1   | 1   |
| 2                                      | 1 | 2 | 2 | 2 | 3 | 3 | 4 | 4 | 6 | 1 | 1   | 2   | 2   |
| 3                                      | 1 | 2 | 2 | 4 | 4 | 5 | 6 | 6 | 8 | 1 | 2   | 3   | 3   |
| 4                                      | 2 | 3 | 3 | 4 | 4 | 6 | 7 | 7 | 9 | 2 | 2   | 4   | 4   |
| 5                                      | 2 | 3 | 3 | 5 | 6 | 6 | 8 | 9 | 11| 3 | 3   | 5   | 5   |
| 6                                      | 2 | 4 | 4 | 6 | 6 | 7 | 8 | 10| 10| 12| 3   | 4   | 6   | 7   |
| 7                                      | 3 | 4 | 4 | 6 | 7 | 7 | 9 | 10| 12| 4 | 5   | 6   | 7   | 8   |
| 8                                      | 4 | 6 | 6 | 9 | 11| 12| 14| 16| 5 | 6   | 8   | 9   |     |
| 9                                      | 0 | 0 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5   | 6   | 7   | 8   |
| 10                                     | 0 | 0 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6   | 7   | 8   | 9   |
| 11                                     | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 7   | 8   | 9   | 10  |
| 12                                     | 0 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 8   | 9   | 10  | 11  |

Table 7: Minimum values of $|G_3 \cap \phi^{-1}(\ell)| + |Y_3 \cap \phi^{-1}(\ell)|$ for $\ell \in \mathbb{Z}_{13}$ for $m, n \geq 13$.

| Least residue $m_{13}$ of $m$ modulo 13 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|----------------------------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 0                                      | 4 | 10| 10 |12 |10 |12 |12 |12 |11 |10 | 4  |    |    |
| 1                                      | 11| 12 |11 |10 |12 |11 |12 | 7 |11 | 7 | 11 | 11 | 11 |
| 2                                      | 10 |12 |10 |12 |10 |12 | 1 |10 |12 |11 | 12 | 12 | 11 |
| 3                                      | 4  | 4 | 4 |12 |11 |10 |12 | 4 |11 |11 | 11 | 12 | 11 |
| 4                                      | 10 |12 |10 |12 | 1 |10 |12 | 7 | 9 |11 | 7   | 12 | 4   |
| 5                                      | 10 |10 |10 |12 |10 |12 |12 | 7 |11 | 9 | 11 | 12 | 11 |
| 6                                      | 0  | 11|10 |12 |11 |10 |12 | 7 | 9 |11 | 10 | 12 | 11 |
| 7                                      | 12 |10 |10 |12 |10 |12 | 1 |10 |12 |11 | 10 | 12 | 10 |
| 8                                      | 6  | 4 |10 |12 |10 |12 | 4 |10 |11 | 0 | 11 | 4   |     |
| 9                                      | 12 |10 |12 | 1 |10 |12 | 7 | 9 |11 | 7 | 9   | 11 | 11 |
| 10                                     | 12 |10 |12 |12 |10 |12 |12 | 1 |10 |12 |12 | 12 | 12 |
| 11                                     | 11 |10 |12 |12 |12 |12 |12 |12 |11 |12 |12 | 12 | 12 |
| 12                                     | 4  |10 |12 |12 |12 |12 | 1 |12 |12 |12 |12 | 12 | 12 |

Table 8: Lexicographically largest $\ell \in \mathbb{Z}_{13}$ corresponding with the minimum values of $|G_3 \cap \phi^{-1}(\ell)| + |Y_3 \cap \phi^{-1}(\ell)|$ for $m, n \geq 13$.

The values in Table 7 and Table 8 can be verified by an exhaustive search. Let $c (m_{13}, n_{13})$ correspond with the values given in Table 7. For a general $\ell \in \mathbb{Z}_{13}$, by Equations 3 and 4, |$\phi^{-1}(\ell) \cap V(G_{m,n})$| +
Theorem 4.1. Fix the tilings used to establish Theorem 4.1 are in [15, Section 3.2 through 3.13].

$m$ can be solved for each least residue of $p$

Section are established using the tiling method described in Section 2.

As $n$ where the values $c$ where $b$

This section establishes upper bounds for the 2-limited broadcast domination number of the Cartesian product $P_{23}$ and $3$ $(m, n) 30$

$\gamma$ of a path and a cycle. Section 4.1 states bounds for $\gamma$

$m$ $\geq \gamma$

$4.2$ states bounds for $\gamma$

All results in this section are established using the tiling method described in Section 2.

4 Upper Bounds for $\gamma_{b,2}(P_{m}C_{n})$

This section establishes upper bounds for the 2-limited broadcast domination number of the Cartesian product of a path and a cycle. Section 4.1 states bounds for $\gamma_{b,2}(P_{m}C_{n})$ for $2 \leq m \leq 22$ and $n \geq 3$. Section 4.2 states bounds for $\gamma_{b,2}(P_{m}C_{n})$ for $m \geq 23$ and $n \geq 13$. Section 4.3 states bounds for $\gamma_{b,2}(P_{m}C_{n})$ for $m \geq 23$ and $3 \leq n \leq 12$. These results require additional case work, when compared to Section 3, since $\gamma_{b,2}(P_{n}P_{n}) = \gamma_{b,2}(P_{n}P_{n})$. But $\gamma_{b,2}(P_{m}C_{n})$ is not necessarily equal to $\gamma_{b,2}(P_{n}C_{n})$. All results in this section are established using the tiling method described in Section 2.

4.1 $P_{2 \leq m \leq 22}C_{n \geq 3}$

The tilings used to establish Theorem 4.1 are in [15, Sections 3.2 through 3.13].

Theorem 4.1. Fix $2 \leq m \leq 22$ and $n \geq 3$. Let $x$ be the value in Table 9 dependant upon $m$ and define $n_{x}$ as the least residue of $n$ modulo $x$. By construction,

$$\gamma_{b,2}(P_{m}C_{n}) \leq b(m) + c(m, n, n_{x}),$$

where $b(m)$ corresponds with the terms in Table 10 and $c(m, n, n_{x})$ corresponds with values in Table 11.

$\phi^{-1}(\ell) \cap V(Y_{m+4,n+4})$ is simply

$$\begin{align*}
\frac{m(n-n_{13})}{13} + \frac{(m-m_{13})n_{13}}{13} + \frac{(m+4)(n+4-n_{13})}{13} + \frac{(m-m_{13}Y_{13})n_{13}Y_{13}}{13} + c'(m', n_{13}).
\end{align*}$$

As $n_{13}Y_{13}$ and $m_{13}Y_{13}$ are the least residues of $(m_{13}+4)$ and $(n_{13}+4)$ modulo 13, respectively, the previous equation can be solved for each least residue of $m$ and $n$ modulo 13 by computation. The result gives

$$2\left(\frac{mn}{13}\right) + 4\left(\frac{m+n}{13}\right) + c(m_{13}, n_{13})$$

where the values $c(m_{13}, n_{13})$ are given in Table 6. This proves the theorem.

Table 9: Value of $x$ in the upper bound of $\gamma_{b,2}(P_{m}C_{n})$ for $2 \leq m \leq 22$ and $n \geq 3$.

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13, ..., 22 |
|-----|---|---|---|---|---|---|---|---|----|----|----|-------------|
| $x$ | 1 | 10 | 2 | 16 | 14 | 22 | 10 | 18 | 26 | 24 | 13 |             |

Table 10: Value of $b(m)$ in the upper bound of $\gamma_{b,2}(P_{m}C_{n})$ for $2 \leq m \leq 22$ and $n \geq 3$.

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|----|----|----|
| $b(m)$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ |

| $m$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|-----|----|----|----|----|----|----|----|----|----|----|
| $b(m)$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ |
\( c(m, n, n_x) \) in the upper bound of \( \gamma_{b,2}(P_m \square C_n) \) for \( 2 \leq m \leq 22 \) and \( n \geq 3 \).

| \( n_x \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2 | 1 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 6 | 6 | 7 | 7 | 7 | 8 | 8 | 8 | 9 | 9 |
| 2 | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 8 | 8 | 9 | 9 | 9 | 10 | 11 | 11 | 11 | 12 | 13 | 14 | 15 |
| 3 | 3 | 4 | 5 | 6 | 6 | 6 | 7 | 8 | 8 | 8 | 9 | 11 | 11 | 11 | 12 | 12 | 13 | 14 | 15 | 15 | 17 | 16 |
| 4 | 4 | 5 | 6 | 6 | 6 | 7 | 8 | 8 | 9 | 10 | 10 | 10 | 10 | 12 | 12 | 13 | 14 | 14 | 15 | 16 | 16 |
| 5 | 5 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 15 | 16 | 17 | 18 | 19 | 19 | 20 | 20 | 21 | 22 | 23 |
| 6 | 5 | 7 | 8 | 9 | 10 | 10 | 12 | 12 | 14 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 20 | 21 | 21 | 22 | 23 |
| 7 | 6 | 9 | 10 | 11 | 12 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 24 | 25 | 26 | 26 | 27 |
| 8 | 7 | 10 | 11 | 12 | 14 | 15 | 16 | 18 | 19 | 20 | 22 | 23 | 24 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 |
| 9 | 8 | 11 | 13 | 14 | 16 | 18 | 19 | 21 | 22 | 24 | 26 | 27 | 29 | 30 | 32 | 34 | 34 | 36 | 36 | 37 | 37 |
| 10 | 12 | 13 | 15 | 18 | 20 | 21 | 23 | 24 | 26 | 28 | 29 | 31 | 32 | 35 | 36 | 36 | 38 | 38 | 39 | 39 | 40 |
| 11 | 14 | 15 | 18 | 21 | 23 | 25 | 27 | 28 | 30 | 32 | 33 | 36 | 37 | 39 | 39 | 41 | 41 | 42 | 42 | 44 | 44 |
| 12 | 14 | 16 | 18 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 46 | 48 | 48 | 50 | 50 |
| 13 | 16 | 18 | 20 | 24 | 26 | 28 | 29 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 46 | 48 | 48 | 50 | 50 |

Table 11: \( m \), \( n \), \( n_x \) in the upper bound of \( \gamma_{b,2}(P_m \square C_n) \) for \( 2 \leq m \leq 22 \) and \( n \geq 3 \).
4.2 \( P_{m \geq 23} \square C_{n \geq 13} \)

The tilings used to establish Theorem 4.2 are in [15, Section 3.14].

**Theorem 4.2.** If \( m \geq 23 \) and \( n \geq 13 \) then

\[
\gamma_{b,2}(P_m \square C_n) \leq 2 \left( \frac{mn}{13} \right) + \frac{4m}{13} + \frac{b(n)}{13} + \frac{c(m', n_{13})}{13}
\]

where

\[
b(n) = \begin{cases} 
0 & \text{for } n \equiv 0 \pmod{13}, \\
2n & \text{for } n \equiv 4, 7, 11, \text{ or } 12 \pmod{13}, \\
3n & \text{for } n \equiv 1, 2, 5, 6, 8, 9, \text{ or } 10 \pmod{13}, \text{ and} \\
4n & \text{for } n \equiv 3 \pmod{13}
\end{cases}
\]

and \( c(m', n_{13}) \) corresponds with the values given in Table 12 where \( m' \) and \( n_{13} \) are the least residues of \((m - 10) \) and \( n \) modulo 13, respectively.

| \( m_{13} \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----------------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| Least residue of \( n \) modulo 13 | 0 | 0 | -2 | 13 | -21 | 1 | -7 | -18 | -6 | -14 | -19 | -12 | -10 |
|                | 1 | 0 | 6 | 6 | -18 | -9 | -7 | -20 | -9 | -7 | -7 | -7 | 4 |
|                | 2 | 0 | 1 | 12 | -15 | -6 | -7 | -22 | 1 | 0 | -15 | -17 | 6 |
|                | 3 | 0 | 9 | 5 | -12 | -3 | -7 | -24 | -2 | -6 | -10 | -14 | -5 |
|                | 4 | 0 | 4 | -2 | -22 | -13 | -20 | -26 | -5 | -12 | -18 | -11 | -3 |
|                | 5 | 0 | -1 | 4 | -19 | 3 | -7 | -15 | -8 | -5 | -13 | -21 | -1 |
|                | 6 | 0 | 7 | 10 | -16 | -7 | -7 | -17 | -11 | -8 | -18 | 1 | -9 |
|                | 7 | 0 | 2 | 16 | -13 | -4 | -7 | -19 | -1 | -4 | -16 | -15 | 3 |
|                | 8 | 0 | 10 | 9 | -23 | -1 | -7 | -21 | -4 | -10 | -11 | -12 | -8 |
|                | 9 | 0 | 5 | 2 | -20 | -11 | -7 | -23 | -7 | -16 | -6 | -9 | 7 |
|                | 10 | 0 | 0 | 8 | -17 | -8 | -7 | -25 | -10 | -9 | -14 | -19 | -4 |
|                | 11 | 0 | 8 | 1 | -14 | -5 | -7 | -27 | -13 | -15 | -9 | -16 | -2 |
|                | 12 | 0 | 3 | 7 | -24 | -2 | -7 | -29 | -3 | -8 | -17 | -13 | 0 |

Table 12: Constant term \( c(m', n_{13}) \) for the upper bound for \( \gamma_{b,2}(P_m \square C_n) \) for \( m \geq 23 \) and \( n \geq 13 \).

4.3 \( P_{m \geq 23} \square C_{3 \leq n \leq 12} \)

The tilings used to establish Theorem 4.3 are in [15, Sections 3.16 through 3.25].

**Theorem 4.3.** Fix \( m \geq 23 \) and \( 3 \leq n \leq 12 \). Let \( x \) be the value in Table 13 dependant upon \( m \) and define \( n_x \) as the least residue of \( n \) modulo \( x \). By construction,

\[
\gamma_{b,2}(P_m \square C_n) \leq b(m) + c(m, n_x),
\]

where \( b(m) \) corresponds with the terms in Table 14 and \( c(m, n_x) \) corresponds with values in Table 15.

5 Upper Bounds for \( \gamma_{b,2}(C_m \square C_n) \)

This section establishes upper bounds for the 2-limited broadcast domination number of the Cartesian product of two cycles. Section 5.1 states bounds for \( \gamma_{b,2}(C_m \square C_n) \) for \( 3 \leq m \leq 25 \) and \( n \geq m \). Section 5.2 states bounds for \( \gamma_{b,2}(C_m \square C_n) \) for \( m \geq 26 \) and \( n \geq m \). All results in this section are established using the tiling method described in Section 2.
Table 13: Value of $x$ in the upper bound of $\gamma_{b,2}(P_m \Box C_n)$ for $m \geq 23$ and $3 \leq n \leq 12$.

| $m$ | 3 | 4 | 5, 6 | 7 | 8 | 9, 10 | 11, 12 |
|-----|---|---|------|---|---|-------|--------|
| $x$ | 1 | 6 | 1 | 35 | 6 | 10 | 13 |

Table 14: Value of $b(m)$ in the upper bound of $\gamma_{b,2}(P_m \Box C_n)$ for $m \geq 23$ and $3 \leq n \leq 12$.

| $m$ | 3 | 4 | 5, 6 | 7 | 8 | 9, 10 | 11 | 12 |
|-----|---|---|------|---|---|-------|----|----|
| $b(m)$ | $\frac{2m}{3}$ | $\frac{m}{6}$ | $m$ | $\frac{m}{35}$ | $\frac{m}{6}$ | $\frac{m}{10}$ | $\frac{m}{13}$ | $\frac{m}{13}$ |

Table 15: Value of $c(m, n_x)$ in the upper bound of $\gamma_{b,2}(P_m \Box C_n)$ for $m \geq 23$ and $3 \leq n \leq 12$.

| $n_x$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|----|----|----|
| $m$   | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 2  | 3  | 2  |
|       | 1 | 2 | 4 | 3 | 3 | 4 | 5 | 2  | 2  | 2  |
|       | 2 | 2 | 5 | 4 | 5 | 5 | 6 | 7  | 7  | 7  |
|       | 3 | 3 | 6 | 6 | 6 | 7 | 9 | 9  | 9  | 9  |
|       | 4 | 4 | 7 | 7 | 8 | 8 | 10| 11 | 11 | 11 |
|       | 5 | 4 | 8 | 8 | 10| 10| 12| 13 | 13 | 13 |
|       | 6 | 10| 11 |12 | 14| 15| 15| 17 | 17 | 17 |
|       | 7 | 11| 13 |13 | 15| 15| 18| 19 | 19 | 19 |
|       | 8 | 12| 15 |15 | 18| 18| 19| 21 | 21 | 21 |
|       | 9 | 13| 16 |16 | 19| 19| 21| 23 | 25 | 25 |
|       | 10| 14| 21 |21 | 23| 23| 25| 25 | 27 | 27 |
|       | 11| 16| 23 |23 | 25| 25| 27| 27 | 27 | 27 |
|       | 12| 17| 24 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 13| 18| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 14| 19| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 15| 20| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 16| 22| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 17| 23| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 18| 24| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 19| 25| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 20| 26| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 21| 28| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 22| 29| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 23| 30| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 24| 31| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 25| 32| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 26| 34| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 27| 35| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 28| 36| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 29| 37| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 30| 38| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 31| 40| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 32| 41| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 33| 42| 27 |27 | 27| 27| 27| 27 | 27 | 27 |
|       | 34| 43| 27 |27 | 27| 27| 27| 27 | 27 | 27 |

Table 15: Value of $c(m, n_x)$ in the upper bound of $\gamma_{b,2}(P_m \Box C_n)$ for $m \geq 23$ and $3 \leq n \leq 12$. 
5.1 $C_{3 \leq m \leq 25} \square C_{n \geq m}$

The tilings used to establish Theorem 4.3 are in [15, Sections 4.2 through 4.10].

**Theorem 5.1.** Fix $3 \leq m \leq 25$ and $n \geq m$. Let $x$ be the value in Table 16 dependant upon $m$ and define $n_x$ as the least residue of $n$ modulo $x$. By construction,

$$\gamma_{b,2}(C_m \square C_n) \leq b(m) + c(m,n_x),$$

where $b(m)$ corresponds with the terms in Table 17 and $c(m,n_x)$ corresponds with values in Table 18.

| $m$ | 3  | 4  | 5,6 | 7  | 8  | 9,10 | 11 | 12,13 | 14 | 15 |
|-----|----|----|-----|----|----|------|----|--------|----|----|
| $b(m)$ | $\left\lceil \frac{2m}{3} \right\rceil$ | 4 | $\frac{n}{6}$ | 8 | $\frac{n}{6}$ | $\frac{n}{13}$ | 16 | $\frac{n}{13}$ | 24 | $\frac{n}{13}$ | 31 | $\frac{n}{13}$ | 33 | $\frac{n}{13}$ |

Table 16: Value of $x$ in the upper bound of $\gamma_{b,2}(C_m \square C_n)$ for $3 \leq m \leq 25$ and $n \geq m$.

| $m$ | 16,17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|-----|-------|----|----|----|----|----|----|----|----|
| $b(m)$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ | $\frac{n}{13}$ |

Table 17: Value of $b(m)$ in the upper bound of $\gamma_{b,2}(C_m \square C_n)$ for $3 \leq m \leq 25$ and $n \geq m$.

5.2 $C_{m \geq 26} \square C_{n \geq m}$

The tilings used to establish Theorem 4.3 are in [15, Section 4.11].

**Theorem 5.2.** If $m,n \geq 26$ then

$$\gamma_{b,2}(C_m \square C_n) \leq 2 \left( \frac{mn}{13} \right) + \frac{b(m) + b(n)}{13} - c(m_{13},n_{13})$$

where

$$b(x) = \begin{cases} 
0 & \text{if } 0 \equiv x \pmod{13}, \\
4x & \text{if } 3 \equiv x \pmod{13}, \\
2x & \text{if } 4,7,11, \text{or } 12 \equiv x \pmod{13}, \\
3x & \text{if } 1,2,5,6,8,9, \text{or } 10 \equiv x \pmod{13}
\end{cases}$$

and $c(m_{13},n_{13})$ corresponds with the values given in Table 19 where $m_{13}$ and $n_{13}$ are the least residues of $m$ and $n$ modulo 13, respectively.

6 Multipacking

Sections 3, 4, and 5 established upper bounds for the 2-limited broadcast domination numbers of the Cartesian products of two paths, a path and a cycle, and two cycles. This section is devoted to determining corresponding lower bounds for the 2-limited broadcast domination numbers of these graphs. These bounds are obtained via linear programming duality by finding the lower bounds for the fractional 2-limited multipacking numbers of these graphs. All results, with the exception of those stated in Section 6.1, were found by using an exact LP
| $n_x$ | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   | 16   | 17   | 18   | 19   | 20   | 21   | 22   | 23   | 24   | 25   |
|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 1    | 2    | 1    | 3    | 2    | 2    | 4    | 5    | 4    | 5    | 5    | 6    | 6    | 7    | 7    | 7    | 8    | 8    | 8    | 8    | 9    | 9    | 9    | 9    |
| 2    | 2    | 1    | 4    | 4    | 4    | 5    | 6    | 6    | 7    | 8    | 8    | 8    | 9    | 9    | 10   | 10   | 11   | 11   | 12   | 12   | 12   | 12   |
| 3    | 3    | 1    | 5    | 6    | 5    | 7    | 8    | 8    | 10   | 10   | 11   | 12   | 13   | 13   | 14   | 14   | 15   | 15   | 16   | 16   | 17   | 17   | 17   | 18   | 18   |
| 4    | 3    | 6    | 6    | 7    | 7    | 9    | 10   | 10   | 11   | 11   | 12   | 12   | 12   | 12   | 13   | 13   | 13   | 14   | 14   | 16   | 15   | 18   | 17   | 19   | 19   | 19   |
| 5    | 4    | 7    | 8    | 9    | 10   | 11   | 11   | 11   | 13   | 14   | 14   | 15   | 16   | 17   | 18   | 18   | 19   | 20   | 21   | 22   | 23   | 24   | 24   | 25   | 25   | 25   |
| 6    | 8    | 10   | 11   | 13   | 14   | 15   | 16   | 17   | 17   | 18   | 18   | 19   | 21   | 22   | 23   | 23   | 24   | 25   | 25   | 26   | 26   | 27   | 27   | 29   | 30   | 32   |
| 7    | 9    | 12   | 13   | 14   | 14   | 16   | 16   | 18   | 19   | 20   | 21   | 22   | 23   | 24   | 25   | 25   | 27   | 27   | 29   | 30   | 32   | 34   | 34   | 36   | 37   | 39   | 40   |
| 8    | 10   | 13   | 14   | 16   | 16   | 19   | 21   | 22   | 22   | 25   | 26   | 26   | 28   | 29   | 30   | 32   | 33   | 34   | 36   | 37   | 39   | 41   | 43   | 45   | 45   | 47   | 47   | 49   |
| 9    | 12   | 15   | 16   | 18   | 20   | 21   | 23   | 25   | 26   | 28   | 29   | 31   | 33   | 34   | 36   | 37   | 39   | 41   | 43   | 45   | 47   | 49   | 50   | 50   | 51   | 51   | 51   | 52   |

Table 18: Value of $c(m, n_x)$ in the upper bound of $\gamma_{b,2}(C_m \square C_n)$ for $3 \leq m \leq 25$ and $n \geq m$. 
Table 19: Constant term $c(m_{13}, n_{13})$ in the upper bound for $\gamma_{b,2}(C_m \square C_n)$ for $m, n \geq 26$.

| Least residue $m_{13}$ of $m$ modulo 13 | Least residue $n_{13}$ of $n$ modulo 13 |
|-----------------|-----------------|
| 0 0 0 0 0 0 0 0 0 0 0 0 | 0 1 2 3 4 5 6 7 8 9 10 11 12 |
| 1 0 21 26 58 22 15 33 24 30 35 40 18 23 |
| 2 0 26 20 55 19 15 35 14 23 43 37 16 10 |
| 3 0 71 55 81 33 46 82 37 63 47 70 51 35 |
| 4 0 22 19 46 35 36 59 13 27 24 47 14 11 |
| 5 0 28 28 59 23 54 28 10 28 54 41 14 11 |
| 6 0 33 48 82 46 28 30 26 47 49 64 34 36 |
| 7 0 11 14 37 13 10 26 9 32 22 25 21 -2 |
| 8 0 17 23 50 27 28 34 32 33 26 45 30 23 |
| 9 0 35 43 60 37 41 49 22 52 34 68 28 36 |
| 10 0 27 37 70 47 41 51 25 58 42 65 26 10 |
| 11 0 18 16 51 14 10 34 21 43 28 26 26 11 |
| 12 0 23 23 35 24 36 23 -2 75 23 23 11 -2 |

Section 6.1 The Cartesian Product of Two Cycles

This section describes a general fractional 2-limited multipacking on $C_m \square C_n$ for $m, n \geq 3$. These results establish Theorem 6.1. The proof of Theorem 6.1 utilizes the LP relaxation of 2-limited broadcast domination.

Theorem 6.1. For $m, n \geq 3$,

$$\frac{2mn}{13} = mp_{f,2}(C_m \square C_n).$$

Proof. Fix $m, n \geq 3$. For each vertex $i \in V(G)$, define the variable $y_i$. Define the fractional 2-limited multipacking on $C_m \square C_n$ by $y_i = \frac{2}{13}$ for all $i \in V(C_m \square C_n)$. As $C_m \square C_n$ is vertex-transitive, for each vertex $i \in V(C_m \square C_n)$, there are at most five and thirteen vertices within distance one and two, respectively, of $i$. Given this, it is easy to verify that this assignment of values is a valid fractional 2-limited multipacking. As the cost of this multipacking is $\frac{2mn}{13}$, it follows that

$$\frac{2mn}{13} \leq mp_{f,2}(C_m \square C_n).$$

To prove the opposite inequality, consider the following fractional 2-limited dominating broadcast. For $i \in V(C_m \square C_n)$ and $k \in \{1, 2\}$ define $x_{i,1}$ and $x_{i,2}$ as the fractional broadcasts of vertex $i$ at strengths one and two, respectively. Define the fractional 2-limited broadcast by $x_{i,1} = 0$ and $x_{i,2} = \frac{1}{13}$ for all $i \in V(C_m \square C_n)$. As $m, n \geq 3$, for each $i \in V(C_m \square C_n)$, there are at least nine vertices within distance two of $i$. Given this, it is easy to verify that this assignment of values is a feasible solution to the LP relaxation of $[LP 1.1]$ for $k = 2$. The cost of this broadcast is $\frac{2mn}{13}$ which establishes

$$\gamma_{f,b,2}(C_m \square C_n) \leq \frac{2mn}{13}.$$
By the duality theorem of linear programming \( mp_{f,2}(C_m \Box C_n) = \gamma_{f,b,2}(C_m \Box C_n) \).

Combined, the results found in Section 5 and Theorem 6.1 establish the upper and lower bounds, respectively, for \( \gamma_{f,b,2}(C_m \Box C_n) \). The values of the constant terms in the upper bounds in Table 20 can be found in their respective theorems in Section 5 and are omitted here. To easily compare the upper and lower bounds in Table 20 we also, for some bounds, include a simpler lower bound. Observe that the bounds in Table 20 give optimal values of \( \gamma_{b,2}(C_m \Box C_n) = 2 \left( \frac{mn}{13} \right) \) for all \( m, n \geq 3 \) such that \( m, n \equiv 0 \pmod{13} \).

| Simpler Lower Bound | Lower Bound | \( \gamma_{b,2}(C_m \Box C_n) \) | Upper Bound |
|---------------------|-------------|-------------------------------|-------------|
| 1.3 \( \left( \frac{n}{3} \right) \) ≤ | 6 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_3 \Box C_n) \) | \( \left\lfloor \frac{2n}{3} \right\rfloor \) |
| 3.6 \( \left( \frac{n}{6} \right) \) ≤ | 8 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_4 \Box C_n) \) | \( \leq 4 \left( \frac{n}{6} \right) + O(1) \) |
| 10 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_5 \Box C_n) \) | \( \leq n \) |
| 12 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_6 \Box C_n) \) | \( \leq n + O(1) \) |
| 37.6 \( \left( \frac{n}{10} \right) \) ≤ | 14 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_7 \Box C_n) \) | \( \leq 42 \left( \frac{n}{19} \right) + O(1) \) |
| 7.3 \( \left( \frac{n}{6} \right) \) ≤ | 16 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_8 \Box C_n) \) | \( \leq 8 \left( \frac{n}{6} \right) + O(1) \) |
| 13.8 \( \left( \frac{n}{10} \right) \) ≤ | 18 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_9 \Box C_n) \) | \( \leq 16 \left( \frac{n}{10} \right) + O(1) \) |
| 15.3 \( \left( \frac{n}{10} \right) \) ≤ | 20 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{10} \Box C_n) \) | \( \leq 16 \left( \frac{n}{10} \right) + O(1) \) |
| 22 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{11} \Box C_n) \) | \( \leq 24 \left( \frac{n}{13} \right) + O(1) \) |
| 24 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{12} \Box C_n) \) | \( \leq 26 \left( \frac{n}{13} \right) + O(1) \) |
| 26 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{13} \Box C_n) \) | \( \leq 26 \left( \frac{n}{13} \right) + O(1) \) |
| 28 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{14} \Box C_n) \) | \( \leq 31 \left( \frac{n}{13} \right) + O(1) \) |
| 30 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{15} \Box C_n) \) | \( \leq 33 \left( \frac{n}{13} \right) + O(1) \) |
| 32 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{16} \Box C_n) \) | \( \leq 36 \left( \frac{n}{13} \right) + O(1) \) |
| 34 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{17} \Box C_n) \) | \( \leq 36 \left( \frac{n}{13} \right) + O(1) \) |
| 36 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{18} \Box C_n) \) | \( \leq 39 \left( \frac{n}{13} \right) + O(1) \) |
| 38 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{19} \Box C_n) \) | \( \leq 41 \left( \frac{n}{13} \right) + O(1) \) |
| 40 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{20} \Box C_n) \) | \( \leq 42 \left( \frac{n}{13} \right) + O(1) \) |
| 42 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{21} \Box C_n) \) | \( \leq 45 \left( \frac{n}{13} \right) + O(1) \) |
| 44 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{22} \Box C_n) \) | \( \leq 48 \left( \frac{n}{13} \right) + O(1) \) |
| 46 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{23} \Box C_n) \) | \( \leq 49 \left( \frac{n}{13} \right) + O(1) \) |
| 48 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{24} \Box C_n) \) | \( \leq 50 \left( \frac{n}{13} \right) + O(1) \) |
| 50 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{25} \Box C_n) \) | \( \leq 52 \left( \frac{n}{13} \right) + O(1) \) |
| 2 \( \left( \frac{n}{13} \right) \) ≤ | \( \gamma_{b,2}(C_{m \geq 26} \Box C_{n \geq 26}) \) | \( \leq 2 \left( \frac{mn}{13} \right) + O(m + n) \) |

Table 20: Upper and lower bounds for \( \gamma_{b,2}(C_m \Box C_n) \) for \( m, n \geq 3 \).

### 6.2 The Cartesian Product of a Path and a Cycle

This section describes the construction and associated costs of fractional 2-limited multipackings on \( P_m \Box C_n \) for \( m \geq 2 \) and \( n \geq 3 \). For each \( 2 \leq m \leq 22 \), an explicit fractional 2-limited multipacking is given on \( P_m \Box C_n \) for all \( n \geq 3 \). These multipackings were found by examining fractional 2-limited multipackings on \( P_m \Box C_5 \) for \( 2 \leq m \leq 22 \). These constructions are used to prove Theorem 6.2. For \( m \geq 23 \) and all \( n \geq 3 \) a fractional 2-limited multipacking is given on \( P_n \Box P_m \). This multipacking was determined by examining fractional 2-limited multipackings on \( P_{23} \Box C_5 \). This construction is used to prove Theorem 6.3.
**Theorem 6.2.** For $2 \leq m \leq 22$ and all $n \geq 3$

$$f(n) \leq mp_{f,2}(P_m \Box C_n)$$

where $f(n)$ is given in Table 21.

| $m$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $f(n)$ | $\frac{n}{2}$ | $2n$ | $4n$ | $26n$ | $29n$ | $19n$ | $212n$ |

| $m$ | 9   | 10  | 11  | 12  | 13  | 14  | 15  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $f(n)$ | $\frac{52n}{33}$ | $780n$ | $81n$ | $273n$ | $5042n$ | $2324n$ | $5690n$ |

| $m$ | 16  | 17  | 18  | 19  | 20  | 21  | 22  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $f(n)$ | $\frac{15593n}{5878}$ | $28417n$ | $103240n$ | $3896n$ | $337976n$ | $304705n$ | $548313n$ |

Table 21: Values of $f(n)$ for lower bounds for $mp_{2}(P_m \Box C_n)$ where $2 \leq m \leq 22$ and $n \geq 3$.

**Proof.** Fix $2 \leq m \leq 22$. Let $v$ be the $m \times 1$ vector as given in either Table 22 (for $2 \leq m \leq 13$) or Table 23 (for $m \geq 14$). Let $J_{m,n}$ be the all ones matrix of size $m \times n$. The product $v^T J_{m,n}$ defines a $m \times n$ matrix $M$.

| $m$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ | $\frac{4}{7}$ |
| $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ |

Table 22: Vectors used in the proof of the lower bound for $mp_{f,2}(P_m \Box C_n)$ for $2 \leq m \leq 13$.
Table 23: Vectors used in the proof of the lower bound for $mp_{f,2}(P_m \square C_n)$ for $14 \leq m \leq 22$

| $m$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|-----|----|----|----|----|----|----|----|----|----|
| 14  | 599 | 688 | 888 | 6119 | 10537 | 42 | 31248 | 26905 | 2155 |
| 185 | 2277 | 2939 | 20290 | 34873 | 185 | 31248 | 26905 | 2155 |
| 1982 | 759 | 989 | 6760 | 3262 | 13 | 9671 | 2776 | 667 |
| 2077 | 340 | 1575 | 1514 | 5211 | 187 | 15458 | 26617 | 11460 |
| 1982 | 2277 | 11756 | 10129 | 34873 | 1261 | 103415 | 178086 | 76069 |
| 351 | 45 | 2085 | 599 | 144 | 72 | 18357 | 10537 | 13098 |
| 1982 | 293 | 11756 | 3375 | 811 | 417 | 103415 | 59062 | 76069 |
| 136 | 104 | 465 | 278 | 4792 | 172 | 384 | 12238 | 10537 |
| 991 | 759 | 2939 | 20290 | 34873 | 1261 | 2795 | 89043 | 76069 |
| 317 | 40 | 464 | 119 | 5515 | 22 | 16376 | 4697 | 1129 |
| 1982 | 293 | 2939 | 756 | 34873 | 139 | 103415 | 29681 | 7132 |
| 393 | 359 | 458 | 3015 | 5454 | 65 | 1241 | 13898 | 47835 |
| 1982 | 2277 | 2939 | 20290 | 34873 | 417 | 7955 | 89043 | 306676 |
| 393 | 112 | 1791 | 504 | 5241 | 21 | 3114 | 8945 | 11559 |
| 317 | 359 | 1791 | 1591 | 5416 | 194 | 10119 | 27655 | 11920 |
| 1982 | 2277 | 11756 | 10129 | 34873 | 1261 | 2795 | 89043 | 76069 |
| 136 | 40 | 458 | 508 | 5416 | 64 | 15848 | 4572 | 11770 |
| 991 | 293 | 2939 | 3375 | 34873 | 417 | 103415 | 29681 | 76069 |
| 351 | 104 | 464 | 3151 | 5241 | 194 | 15848 | 13825 | 47109 |
| 1982 | 759 | 2939 | 20290 | 34873 | 1261 | 103415 | 89043 | 306676 |
| 297 | 45 | 405 | 119 | 5454 | 21 | 16119 | 4572 | 7169 |
| 1982 | 233 | 2939 | 756 | 34873 | 139 | 103415 | 29681 | 306676 |
| 185 | 340 | 2085 | 278 | 5515 | 65 | 3114 | 27655 | 11770 |
| 1982 | 2277 | 11756 | 10129 | 34873 | 417 | 20683 | 178086 | 76069 |
| 599 | 71 | 1757 | 509 | 4792 | 22 | 1241 | 8945 | 11920 |
| 1982 | 759 | 11756 | 3375 | 34873 | 139 | 7955 | 59062 | 76069 |
| 688 | 275 | 1514 | 144 | 34873 | 139 | 103415 | 13898 | 11559 |
| 2277 | 2939 | 888 | 631 | 5211 | 74 | 384 | 4697 | 47835 |
| 991 | 759 | 2939 | 10129 | 34873 | 1261 | 103415 | 89043 | 76069 |
| 1982 | 293 | 2939 | 756 | 34873 | 139 | 103415 | 89043 | 306676 |
| 4728 | 3262 | 187 | 18357 | 12238 | 22358 | 22358 | 22358 | 22358 |
| 10537 | 34873 | 139 | 103415 | 53082 | 42 | 9671 | 667 | 2155 |
| 1241 | 15458 | 10537 | 34873 | 139 | 103415 | 76069 | 13805 | 7132 |

$m$ refers to the number of vertices in the hypercube $P_m$, and $C_n$ refers to the cube $C_n$. The table lists the vectors used in the proof to establish the lower bound for $mp_{f,2}(P_m \square C_n)$ for $14 \leq m \leq 22$. Each entry represents a vector in $\mathbb{R}^m$ used in the proof.
where the elements within each row, respectively, have identical entries. For example if \( m = 5 \) and \( n = 6 \) then

\[
M = \begin{bmatrix}
\frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} \\
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
\frac{4}{27} & \frac{4}{27} & \frac{4}{27} & \frac{4}{27} & \frac{4}{27} & \frac{4}{27} \\
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
\frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} \\
\frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27} & \frac{8}{27}
\end{bmatrix}.
\]

For \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), define the fractional 2-limited multipacking \( f \) on \( P_m \Box C_n \) by letting the variable \( x_{i,j} \) (corresponding to the vertex in row \( i \) and column \( j \) of \( P_m \Box C_n \)) be the value in the \( i \)th row and \( j \)th column of \( M \). As the elements within each row are identical, due to symmetry, to check that \( f \) is a valid fractional 2-limited multipacking it suffices to check the entries in the first column of \( P_m \Box C_n \). The check can be done by computation. Let \( v_1, \ldots, v_m \) denote the elements of \( v \). The cost of \( f \) is \( n \sum_{i=1}^{m} v_i \). For each \( 2 \leq m \leq 22 \) the cost of \( f \) corresponds with the values in Table 21. This completes the proof.

\[ \square \]

**Theorem 6.3.** For \( m \geq 23 \) and all \( n \geq 3 \)

\[
\frac{2mn}{13} + \frac{2620n}{13767} \leq mp_{f,2}(P_m \Box C_n).
\]

**Proof.** The general case follows from the case \( m = 23 \). Letting \( v \) be the vector in Figure 9 for \( m = 23 \) and proceeding with as in the proof of Theorem 6.2 yields a fractional 2-limited multipacking on \( P_{23} \Box C_n \) for all \( n \geq 3 \). The cost of such a multipacking is

\[
(m - 16) \left( \frac{2n}{13} \right) + \frac{36508n}{13767} = 7 \left( \frac{2n}{13} \right) + \frac{36508n}{13767}.
\]

Observe that the centre most row of this multipacking on \( P_{23} \Box C_n \) is composed solely of vertices with weight 2/13. Moreover, all vertices within distance 2 of the centre row also have weight 2/13. As this row does not violate the fractional 2-limited multipacking on \( P_{23} \Box C_n \), adding additional centre rows, all of which contain vertices with weight 2/13, will yield valid fractional 2-limited multipacking on \( P_{m \geq 23} \Box C_n \). The cost of such a multipacking will be

\[
(m - 16) \left( \frac{2n}{13} \right) + \frac{36508n}{13767},
\]

which when simplified proves the theorem.

\[ \square \]

Combined, the results found in Section 4 and Theorems 6.2 and 6.3 establish the upper and lower bounds in Table 21 for the 2-limited broadcast domination number of the Cartesian products of a path and a cycle. The value of the constant terms in the upper bounds in Table 21 can be found in the respective theorems in Section 4 and are omitted here. To easily compare the upper and lower bounds in Table 21 we also, for some bounds, include a simpler lower bound.

| Simpler Lower Bound | Lower Bound | \( \gamma_{b,2}(P_m \Box C_n) \) | Upper Bound |
|---------------------|-------------|-------------------------------|-------------|
| \( \left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} \) | \( \gamma_{b,2}(P_2 \Box C_n) \) | \( \leq \left\lceil \frac{n}{2} \right\rceil \) |
| \( \left\lceil \frac{4n}{9} \right\rceil \leq \frac{4n}{9} \) | \( \gamma_{b,2}(P_3 \Box C_n) \) | \( \leq \left\lceil \frac{4n}{9} \right\rceil \) |
| \( \left\lceil \frac{2n}{9} \right\rceil \leq \frac{2n}{9} \) | \( \gamma_{b,2}(P_4 \Box C_n) \) | \( \leq 8 \left\lceil \frac{2n}{9} \right\rceil + O(1) \) |
| \( \left\lceil \frac{26n}{27} \right\rceil \leq \frac{26n}{27} \) | \( \gamma_{b,2}(P_5 \Box C_n) \) | \( \leq n + O(1) \) |
| \( 17.8 \left( \frac{n}{10} \right) \leq \frac{26n}{27} \) | \( \gamma_{b,2}(P_6 \Box C_n) \) | \( \leq 18 \left\lceil \frac{n}{10} \right\rceil + O(1) \) |

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Theorem 6.4. For $m, n \geq 23$

\[
\frac{2mn}{13} + \frac{14287568}{75254411} (m + n) - \frac{177612468}{978307343} \leq mp_{f,2}(P_m \square P_n)\n\]

Combined, the results found in Section 3 and Theorems 6.2 and 6.4 establish the upper and lower bounds, respectively, in Table 24 for the 2-limited broadcast domination numbers of the Cartesian products of two paths.
Figure 9: Vector used in the proof of the lower bound for $mp_f(P_m \square C_n)$ for $m \geq 23$
The value of the constant terms in the upper bounds in Table 25 can be found in the respective theorems in Section 3 and are omitted here. To easily compare the upper and lower bounds in Table 25 we also, for some bounds, include a simpler lower bound.

| Simpler Lower Bound | Lower Bound | $\gamma_{h,2}(P_m \square P_n)$ | Upper Bound |
|---------------------|-------------|---------------------------------|-------------|
| $\frac{n}{2}$       | $\gamma_{h,2}(P_2 \square P_n)$ | $\leq \frac{n}{2}$ | $\leq \frac{n}{2} + 1$ |
| $\frac{3}{4}$       | $\gamma_{h,2}(P_3 \square P_n)$ | $\leq \frac{3}{4}$ | $\leq \frac{n}{2}$ |
| $\frac{4m}{5}$      | $\gamma_{h,2}(P_4 \square P_n)$ | $\leq \frac{4m}{5}$ | $\leq \frac{n}{2} + O(1)$ |
| $\frac{2m}{3}$      | $\gamma_{h,2}(P_5 \square P_n)$ | $\leq \frac{2m}{3}$ | $\leq n + 1$ |
| $17.8 \left( \frac{n}{10} \right)$ | $\gamma_{h,2}(P_6 \square P_n)$ | $\leq 18 \frac{n}{10}$ | $+ O(1)$ |
| $17.7 \left( \frac{n}{11} \right)$ | $\gamma_{h,2}(P_7 \square P_n)$ | $\leq 18 \frac{n}{11}$ | $+ O(1)$ |
| $31.3 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_8 \square P_n)$ | $\leq 32 \frac{n}{13}$ | $+ O(1)$ |
| $15.7 \left( \frac{n}{10} \right)$ | $\gamma_{h,2}(P_9 \square P_n)$ | $\leq 16 \frac{n}{10}$ | $+ O(1)$ |
| $31.1 \left( \frac{n}{17} \right)$ | $\gamma_{h,2}(P_{10} \square P_n)$ | $\leq 32 \frac{n}{17}$ | $+ O(1)$ |
| $48.9 \left( \frac{n}{21} \right)$ | $\gamma_{h,2}(P_{11} \square P_n)$ | $\leq 50 \frac{n}{21}$ | $+ O(1)$ |
| $48.8 \left( \frac{n}{21} \right)$ | $\gamma_{h,2}(P_{12} \square P_n)$ | $\leq 50 \frac{n}{21}$ | $+ O(1)$ |
| $28.4 \left( \frac{n}{17} \right)$ | $\gamma_{h,2}(P_{13} \square P_n)$ | $\leq 30 \frac{n}{17}$ | $+ O(1)$ |
| $30.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{14} \square P_n)$ | $\leq 32 \frac{n}{13}$ | $+ O(1)$ |
| $32.4 \left( \frac{n}{17} \right)$ | $\gamma_{h,2}(P_{15} \square P_n)$ | $\leq 34 \frac{n}{17}$ | $+ O(1)$ |
| $34.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{16} \square P_n)$ | $\leq 36 \frac{n}{13}$ | $+ O(1)$ |
| $36.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{17} \square P_n)$ | $\leq 38 \frac{n}{13}$ | $+ O(1)$ |
| $38.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{18} \square P_n)$ | $\leq 40 \frac{n}{13}$ | $+ O(1)$ |
| $40.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{19} \square P_n)$ | $\leq 42 \frac{n}{13}$ | $+ O(1)$ |
| $42.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{20} \square P_n)$ | $\leq 44 \frac{n}{13}$ | $+ O(1)$ |
| $40.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{21} \square P_n)$ | $\leq 46 \frac{n}{13}$ | $+ O(1)$ |
| $46.4 \left( \frac{n}{13} \right)$ | $\gamma_{h,2}(P_{22} \square P_n)$ | $\leq 48 \frac{n}{13}$ | $+ O(1)$ |
| $\left[ \frac{2mn}{13} + \frac{143876}{59234411} (m + n) - O(1) \right]$ | $\gamma_{h,2}(P_m \square P_n)$ | $\leq \gamma_{h,2}(P_{m \geq 23} \square P_n)$ | $\leq \frac{2mn}{13} + \frac{143876}{59234411} (m + n) + O(1)$ |

Table 25: Upper and lower bounds for $\gamma_{h,2}(P_m \square P_n)$ for $m, n \geq 2$.

Observe that the bounds in Table 25 give optimal values for $\gamma_{h,2}(P_2 \square P_n)$ for odd $n$, optimal values for $\gamma_{h,2}(P_3 \square P_n)$, and optimal values for $\gamma_{h,2}(P_5 \square P_n)$ when $n \equiv 4$ or 9 (mod 10). The periodically optimal values for $\gamma_{h,2}(P_5 \square P_n)$ allow for an easy proof of Proposition 6.5. This proof can be found in Section 5.4 of [15].

**Proposition 6.5.** For $n \geq 2$, $\gamma_{h,2}(P_2 \square P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

**Proof.** If $n \in \{2, 3, 4, 5\}$ the claim is easily verified. Fix $n \geq 6$. Suppose $n$ is odd. As

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{n + 1}{2} = \frac{n - 1}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

by the bounds for $\gamma_{h,2}(P_3 \square P_n)$ in Theorem 6.2 and Theorem 6.1 the claim holds.

Suppose now that $n$ is even and that there exists a 2-limited dominating broadcast $f$ of $P_2 \square P_n$ of cost $\leq \left\lfloor \frac{n}{2} \right\rfloor$. Observe that a vertex broadcasting at strength one or two on $P_2 \square P_n$ can be heard by at most four or eight vertices, respectively. Therefore, given $v \in V(P_2 \square P_n)$, the broadcast from $v$ can be heard by at most 4$f(v)$ vertices. Consider the leftmost column of $P_2 \square P_n$. Any 2-limited broadcast of $P_2 \square P_n$ which dominates this column necessarily “wastes” a portion of a broadcast from a vertex. By “wastes” we mean that, in dominating
the leftmost column, there exists a vertex \( v \) whose broadcast either overlaps with the broadcast from some other vertex \( v' \) or is heard by at least one less than \( 4f(v) \) vertices. In either case, a broadcast from such a vertex is heard by at least one less than \( 4f(v) \) vertices. The same logic applies to the rightmost column of \( P_2 \square P_n \). This follows since \( n \geq 6 \). Therefore there cannot exist a vertex which dominates a vertex in the left and rightmost columns. By assumption therefore, \( f \) can dominate at most
\[
4 \left\lfloor \frac{n}{2} \right\rfloor - 2 = 4 \left( \frac{n}{2} \right) - 2 = 2n - 2 < 2n = |V(P_2 \square P_n) |
\]
vertices. This is a contradiction. \( \square \)

7 Conclusion

This paper summarizes the results from Chapters 2 through 5 of [15]. These results establish upper and lower bounds for the 2-limited broadcast domination number of the Cartesian product of two paths, a path and a cycle, and two cycles. Although we have bounded the 2-limited broadcast domination number for the Cartesian products of two paths, a path and a cycle, and two cycles, few of our bounds are tight. Additionally, given a graph \( G \), since \( \gamma_{b,k}(G) \leq \gamma_{b,k-1}(G) \), our bounds provide upper bounds for the \( k \)-limited broadcast domination numbers for these graphs where \( k \geq 2 \). However, these bounds are very likely far from the truth with respect to large grids. Given this, we present the following problems.

**Problem 7.1.** Determine \( \gamma_{b,k}(P_m \square P_n) \) for all \( m, n \geq 2 \) for \( k \geq 2 \).

**Problem 7.2.** Determine \( \gamma_{b,k}(P_m \square C_n) \) for all \( m \geq 2 \) and \( n \geq 3 \) for \( k \geq 2 \).

**Problem 7.3.** Determine \( \gamma_{b,k}(C_m \square C_n) \) for all \( m, n \geq 3 \) for \( k \geq 2 \).

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