On the quantum theory of massless spin-3/2 field in Minkowski spacetime

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Abstract

From the modern viewpoint and by the geometric method, this paper provides a concise foundation for the quantum theory of massless spin-3/2 field in Minkowski spacetime, which includes both the one-particle’s quantum mechanics and the many-particle’s quantum field theory. The explicit result presented here is useful for the investigation of spin-3/2 field in various circumstances such as supergravity, twistor programme, Casimir effect, and quantum inequality.

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1 Introduction

As is well known, relativistic quantum theory originates from the natural marriage of special relativity and quantum theory. According to the modern viewpoint, the Hilbert space for one particle quantum wave functions forms the unitary representation of the Poincare group, which is the isometric transformation group of the Minkowski spacetime. Especially, as realized on the Minkowski spacetime, the quantum wave functions need to satisfy the linear field equation of motion[1, 2, 3, 4, 5]. Furthermore, the quantum field operator, which satisfies the same linear field equation, is defined on the Fock space associated with the Hilbert space of one particle states.

Obviously, the spin-3/2 field occupies a special position in our attempts to understand nature both relativistically and quantum mechanically. It is the spin-3/2 field that turns out to be the simplest nontrivial higher spin field, which but plays a significant role in supergravity and twistor programme. By the geometric method, this paper is mainly intended to revisit the quantum theory of massless spin-3/2 field from the modern viewpoint mentioned in the beginning. In the next section, we develop the one-particle’s quantum mechanics for the massless spin-3/2 field. the many-particle’s quantum field theory for massless spin-3/2 field is presented in Section 3. We conclude with some applications and extensions in Section 4.

2 One-particle’s Quantum Mechanics for Massless Spin-3/2 Field

2.1 The Hilbert Space of One Particle States

Start with the free massless spin-3/2 field equation on the Minkowski spacetime[6]

\[ \nabla^{A'}A \phi_{ABC} = 0, \]

(1)

where \( \phi_{ABC} \) is a totally symmetric spinor field, called field strength. It is obvious that the solutions to the above equation form a complex vector space. To define an inner product on our complex vector space, we first introduce the Rarita-Schwinger potential field \( \psi^{A'}_{BC} \)[7], i.e.,

\[ \phi_{ABC} = \nabla_{AA'} \psi^{A'}_{BC}, \]

(2)

\[ \psi^{A'}_{BC} = \psi^{A'}_{(BC)}, \]

(3)

\(^{1}\)It seems that this potential field is misnamed Dirac-Fierz potential field by R. Penrose in [7].
then the field equation (11) can be rewritten in terms of $\psi^\prime A_{BC}$ as
\[ \nabla^{BB} \psi^\prime A_{BC} = 0. \] (4)

Whence a conserved current reads\[8, 9\]
\[ j_c[\psi, \psi'] = \sqrt{2} \sigma^C C \bar{\psi}^B A_{BC} \psi^\prime A_{BC}, \] (5)
then the inner product can be defined as
\[ (\phi, \phi') = (\psi, \psi') = \int_\Sigma j_a[\psi, \psi'] \epsilon_{abcd}. \] (6)

Note that the conservation of $j_a[\psi, \psi']$ implies that this inner product is independent of choice of the Cauchy surface $\Sigma^2$. Thus, for the later convenience, we choose the surface of the constant Lorentz coordinate time $x^0$ as $\Sigma$ once and for all. Moreover, Eqn.(6) can be written as
\[ (\phi, \phi') = (\psi, \psi') = \int_\Sigma (\frac{\partial}{\partial x^0})^a j_a[\psi, \psi'] \tilde{\epsilon}_{abcd}, \] (7)
where $\tilde{\epsilon}_{bcd} = (\frac{\partial}{\partial x^0})^a \epsilon_{abcd}$ is the induced spatial volume element on $\Sigma$.

In addition, by Eqn.(11), Eqn.(5), and Eqn.(7), the Stokes theorem shows that the inner product is invariant under gauge transformations\[10\]
\[ \psi^\prime A_{BC} \rightarrow \psi^\prime A_{BC} + \nabla^A B \varphi C, \]
\[ \psi^\prime A_{BC} \rightarrow \psi^\prime A_{BC} + \nabla^A B \varphi C', \] (8)
with
\[ \nabla^A B \varphi A = 0, \]
\[ \nabla^A B \varphi A' = 0. \] (9)

Thus the Hilbert space of one particle states for massless spin-3/2 field can be constructed by\[3\]
\[ H = H^+ \bigoplus \bar{H}^-, \] (10)
where $H^+(H^-)$ is the complex vector space of positive(negative) frequency solutions to the field equation (11) with respect to the Lorentz coordinate time $x^0$, and $\bar{H}^-$ is the complex conjugation space of $H^-$. That is, $\bar{H}^-$ is the complex vector space of positive frequency solutions to the field equation
\[ \nabla^{AA'} \phi_{A'B'C'} = 0. \] (11)

\footnote{This point also means the unitarity of the evolution of the free field.}
\footnote{More properly, one should complete $H$ such that it can be called a genuine Hilbert space. Actually, for the physical taste, we are a little sloppy here, ignoring the rigorous mathematics. Fortunately, there is a general completion procedure for obtaining a mathematically precise Hilbert space along with a collection of operators on it\[3\].}
2.2 Conserved Observables from the Poincare Lie Algebra

It is well known that the Poincare Lie algebra can be realized by the Killing vector fields on the Minkowski spacetime as follows

\[ P_\mu^a = i (\frac{\partial}{\partial \xi^\mu})^a, \]
\[ M_{\mu\nu}^a = i [x_\mu (\frac{\partial}{\partial x^\nu})^a - x_\nu (\frac{\partial}{\partial x^\mu})^a]. \] (12)

According to the fact that the covariant derivative commutes with the Lie derivatives via Killing vector fields, the operators from the Poincare Lie algebra, i.e.

\[ \hat{P}_\mu \phi_{ABC} = \mathcal{L}_{P_\mu^a} \phi_{ABC}, \]
\[ \hat{P}_\mu \phi_{A'B'C'} = \mathcal{L}_{P_\mu^a} \phi_{A'B'C'}, \]
\[ \hat{M}_{\mu\nu} \phi_{ABC} = \mathcal{L}_{M_{\mu\nu}^a} \phi_{ABC}, \]
\[ \hat{M}_{\mu\nu} \phi_{A'B'C'} = \mathcal{L}_{M_{\mu\nu}^a} \phi_{A'B'C'}, \] (13)

are well defined on(in) the Hilbert space of one particle states. Later, employing the Leibnitz rule, the conservation of \( j_a [\psi, \psi'] \), and the Stokes theorem, we find that the above operators are hermitian with respect to the inner product (7). In addition, since the inner product (7) is independent of the choice of \( \Sigma \), the above operators are also conserved observables. Moreover, taking into account \([\mathcal{L}_u, \mathcal{L}_v] = [u, v]\) with \( u \) and \( v \) arbitrary vector fields, we can obtain

\[ [\hat{P}_\mu, \hat{P}_\nu] = 0, \] (14)
\[ [\hat{P}_\mu, \hat{M}_{\rho\sigma}] = 2i \eta_{[\rho} \hat{P}_{\sigma]}, \] (15)
\[ [\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = 2i (\eta_{[\mu} \hat{M}_{\nu]\sigma] - \eta_{[\nu} \hat{M}_{\mu]\sigma]). \] (16)

Here, \( \hat{P}_\mu \) is the four-momentum operator. By the field equation (11) and (12), we have

\[ \hat{P}_\mu \hat{P}^\mu = -\Box = 0, \] (17)

which shows that the eigenvalue of the four-momentum operator is null. Furthermore, \( \{ \hat{L}_1 \equiv \hat{M}_{23}, \hat{L}_2 \equiv \hat{M}_{31}, \hat{L}_3 \equiv \hat{M}_{12} \} \) are the total angular momentum operators, and \( \{ \hat{K}_1 \equiv \hat{M}_{01}, \hat{K}_2 \equiv \hat{M}_{02}, \hat{K}_3 \equiv \hat{M}_{03} \} \) describe the uniform motion of center of mass.

We next introduce the Pauli-Lubanski spin vector operator

\[ \hat{S}_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} \hat{P}^\nu \hat{M}^\rho_{\lambda}, \] (18)
then we have

\[
\hat{S}^\mu \phi_{ABC} = \frac{1}{2} \epsilon^{\mu \rho \lambda} \hat{P}_\nu M_{\rho \lambda} \phi_{ABC}
\]

\[
= -\frac{1}{2} \epsilon^{\mu \rho \lambda} \left( \frac{\partial}{\partial x^\nu} \right)^a \nabla_a \left\{ [x_\rho \left( \frac{\partial}{\partial x^\lambda} \right)^b - x_\lambda \left( \frac{\partial}{\partial x^\rho} \right)^b] \nabla_b \phi_{ABC} + 3 \phi_{D(BC)U_{[\rho \lambda]}^\rho} \right\}
\]

\[
= -\frac{1}{2} \epsilon^{\mu \rho \lambda} \left( \frac{\partial}{\partial x^\nu} \right)^b \nabla_b \phi_{ABC}
\]

\[
+ [x_\rho \left( \frac{\partial}{\partial x^\nu} \right)^a (\frac{\partial}{\partial x^\lambda})^b - x_\lambda \left( \frac{\partial}{\partial x^\rho} \right)^a (\frac{\partial}{\partial x^\nu})^b] \nabla_a \nabla_b \phi_{ABC} + 3 \phi_{D(BC)U_{[\rho \lambda]}^\rho} \right\}
\]

\[
= -\frac{3}{2} \epsilon^{\mu \rho \lambda} \left( \frac{\partial}{\partial x^\nu} \right)^a \nabla_a \left[ \phi_{D(BC)U_{[\rho \lambda]}^\rho} \right].
\]

Here, we have used \( \phi_{ABC} = \phi_{(ABC)} \) in the second step, and \( \nabla_a \nabla_b = \nabla_b \nabla_a \) in the final step. In addition, \( U_{\rho \lambda A}^{\rho} \) reads

\[
\begin{align*}
U_{\rho \lambda A}^{\rho} &= \frac{1}{2} \nabla_{AA'} M_{\rho \lambda}^{DA'} = \frac{1}{2} \nabla_{AA'} \left[ x_\rho \left( \frac{\partial}{\partial x^\lambda} \right)^{DA'} - x_\lambda \left( \frac{\partial}{\partial x^\rho} \right)^{DA'} \right] = \frac{1}{2} \left( \sigma_{\rho A A'} \sigma_{\lambda A'}^D - \sigma_{\lambda A A'} \sigma_{\rho A'}^D \right).
\end{align*}
\]

Then employing Eqn. (29) in Appendix B, we have

\[
U_{\rho \lambda A}^{\rho} = \frac{i}{2} \epsilon_{\rho A C'} \epsilon_{D C'}^D,
\]

thus

\[
\hat{S}^\mu \phi_{ABC} = -\frac{3i}{4} \epsilon^{\rho \lambda \mu} \epsilon_{\rho \lambda (A[C')} \delta_{D C'}^D \left( \frac{\partial}{\partial x^\nu} \right)^a \nabla_a \phi_{[BC]}
\]

\[
= \frac{3i}{2} \left[ \sigma_{(A[C')}^\rho \sigma_{\nu D C'}^\nu \sigma_{(A[C')}^\sigma \sigma_{\mu D C'}^\mu \left( \frac{\partial}{\partial x^\nu} \right)^a \nabla_a \phi_{[BC]} \right]
\]

\[
= \frac{3i}{2} \left[ \sigma_{(A[C')}^\rho \nabla_{D C'} \phi_{[BC]} - \sigma_{(A[C')}^\mu \nabla_{D C'} \phi_{[BC]} \right]
\]

\[
= -\frac{3i}{2} \sigma_{(A[C')}^\mu \nabla_{D C'} \phi_{(ABC)} = -\frac{3}{2} \hat{P}^\mu \phi_{ABC},
\]

where we have used \( \epsilon^{abcd} \epsilon_{abcdef} = -4 \delta^{[c}[\delta_{d]} \) in the second step, and the field equation \( \Box \) has been used in the forth step. The above equation implies that those states in \( H^\pm \) are the states with the helicity \(-\frac{3}{2}\). Similarly, we can obtain that those states in \( \bar{H}^- \) are the states with the helicity \( \frac{3}{2} \), i.e.,

\[
\hat{S}^\mu \phi_{A'B'C'} = \frac{3}{2} \hat{P}^\mu \phi_{A'B'C'}. \quad (23)
\]
2.3 The Plane Wave Expansion Basis in the Coulomb Gauge

In this section, we shall employ the Rarita-Schwinger potential field in the Coulomb gauge to provide the complete orthonormal expansion basis for the Hilbert space $H$ in the momentum representation. To proceed, first let

$$\psi_a^B = \sigma_a A^C \psi_{A'C}^B,$$  \hspace{1cm} (24)

then Eqn.(4) can be written as

$$\sigma^a_{B'B} \nabla_a \psi_b^B = 0,$$  \hspace{1cm} (25)

where we have used Eqn.(3), i.e.,

$$\sigma^a_{B'B} \psi_a^B = 0.$$  \hspace{1cm} (26)

Note that Eqn.(25) and Eqn.(26) are just the Rarita-Schwinger equations for massless spin-3/2 field, which is the reason why the potential field here is called the Rarita-Schwinger potential field.

Next in the coulomb gauge, i.e.,

$$\left(\frac{\partial}{\partial x^0}\right)^a \psi_a^B = 0,$$  \hspace{1cm} (27)

and after a straightforward calculation, we obtain the complete plane wave solutions to Eqn.(25) and Eqn.(26) in the momentum representation as

$$\psi_{pa}^B(x) = \frac{1}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2|p_0|}} \tilde{\psi}_\mu^\Sigma(p)(dx^\mu)_a(\varepsilon_\Sigma)^B e^{-ip_ax^b}.$$  \hspace{1cm} (28)

Here

$$\tilde{\psi}(1, 0, 0, 1) = (0, 1, i, 0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$  \hspace{1cm} (29)

and

$$\tilde{\psi}_\mu^\Sigma(p = e^{-\lambda}, e^{-\lambda} \sin \theta \cos \varphi, e^{-\lambda} \sin \theta \sin \varphi, e^{-\lambda} \cos \theta) = \tilde{\psi}_\mu^\Sigma(-p) = (\Lambda^{-1})^\nu_\mu L^\Sigma_\Gamma \tilde{\psi}_\nu^\Gamma (1, 0, 0, 1),$$  \hspace{1cm} (30)

where

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$L^\Sigma_\Gamma = \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 & e^{i\frac{\varphi}{2}} \\ e^{-i\frac{\varphi}{2}} & \cos \frac{\varphi}{2} & -\sin \frac{\varphi}{2} \\ e^{i\frac{\varphi}{2}} & \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} \cosh \lambda & 0 & 0 & -\sinh \lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \lambda & 0 & 0 & \cosh \lambda \end{pmatrix},$$  \hspace{1cm} (31)
Thus employing the inner product (7) with the conserved current
\[ j_c[\psi, \psi'] = -\sqrt{2} \sigma_{C'C} \bar{\psi}_a C' \psi'^{\mu a C}, \]  
(32)
it can be shown that
\[ \psi_{pa}^B(x) \in H^+, \quad p_0 > 0, \]
\[ \psi_{pa}^{B'}(x) = \bar{\psi}_{-pa}^{B'}(x) \in \bar{H}^-, \quad p_0 > 0 \]
(33)
forms the complete orthonormal expansion basis for \( H \).

## 3 Many-particle’s Quantum Field Theory for Massless Spin-3/2 Field

### 3.1 The Quantum Field Operator of Many Particles System

Let \( F_a(H) \) be the anti-symmetric Fock space associated with \( H \). the annihilation and creation operators are defined on \( F_a(H) \) as usual\[3, 6\]. Then the quantum field operator is constructed as
\[ \hat{\psi}_a^B(x) = \varrho_{Ia} B(x) a(\varrho^I) + c^\dagger(\tau_I) \bar{\tau}_{Ia}^B(x), \]  
(34)
where \( \{\varrho_I\} \) and \( \{\tau_I\} \) are the complete orthonormal bases of \( H^+ \) and \( \bar{H}^- \) respectively. It can be shown that the quantum field operator constructed above is independent of the choice of the complete orthonormal basis. In particular, by the plane wave expansion basis, we have
\[ \hat{\psi}_a^B(x) = \int d^3p [a(p)\psi_{pa}^B(x) + c^\dagger(p)\psi_{-pa}^{B'}(x)], p_0 > 0, \]  
(35)
where the annihilation and creation operators satisfy the anti-commutation relations as follows
\[ \{a(p), a(p')\} = 0, \]
\[ \{a(p), a^\dagger(p')\} = \delta^3(p - p'), \]
\[ \{a^\dagger(p), a^\dagger(p')\} = 0, \]
\[ \{c(p), c(p')\} = 0, \]
\[ \{c(p), c^\dagger(p')\} = \delta^3(p - p'), \]
\[ \{c^\dagger(p), c^\dagger(p')\} = 0. \]  
(36)
3.2 Energy Momentum Tensor via the Belinfante’s Construction

In order to construct the energy momentum tensor for massless spin-3/2 field, we here resort to the Rarita-Schwinger Lagrangian\[9\]
\[
\mathcal{L} = -i\sqrt{2}[\bar{\psi}^{aB'}\sigma_{B'B}a^B\nabla_b\psi_a^B - \frac{1}{3}(\bar{\psi}^{aB'}\sigma_{aB'B}a^B\nabla_b\psi_{aB}^B + \bar{\psi}^{aB'}\sigma_{bB'B}a^B\nabla_a\psi_b^B) + \frac{2}{3}\bar{\psi}^{aB'}\sigma_{aB'B}\sigma^{BC'}\sigma_{cC'C}\nabla_b\psi^C].
\]
(37)

Since the Belinfante’s energy momentum tensor is equivalent with the metric energy momentum tensor, we here employ the Belinfante’s energy momentum tensor constructed by\[12\]
\[
T_{ab}^B = T^{(ab)}(\mathcal{C}) + \nabla_cN^{(ab)c},
\]
(38)
where the canonical energy momentum tensor
\[
T_{ab}^C = \frac{\partial \mathcal{L}}{\partial \nabla_a\psi_d^D}\nabla^b\psi_d^D - \mathcal{L}\eta_{ab},
\]
(39)
and
\[
N^{abc} = \frac{\partial \mathcal{L}}{\partial \nabla_a\psi_d^D}[(\delta_d^b\psi^c_D - \delta_d^c\psi^b_D) - \frac{1}{2}(\sigma^{bE'}\sigma^{cDE'} - \sigma^{cE'}\sigma^{bDE'})\psi_d^E].
\]
(40)
Then by the Rarita-Schwinger equation and Eqn. (50) in Appendix B, the Belinfante’s energy momentum tensor reads
\[
T_{ab}^B = -i\sqrt{2}[\bar{\psi}^{aB'}\sigma_{B'B}\nabla_b\psi_a^B - \nabla^b\psi_d^D - \nabla(a\bar{\psi}^{|dD|}\sigma^{b_{D'E}}\sigma^{a_{D'}D'}\psi_{D'}^E)](\nabla_c\psi^{|b|D'}\sigma^{a_{D'D}}\psi^{cE} - \bar{\psi}^{cD'}\sigma^{a_{D'D}}\nabla_c\psi^{bD}).
\]
(41)
Furthermore, according to\[13\], given a Killing vector field $\xi$ in the Minkowski spacetime, we have
\[
\int T_{ab}^B\xi_b\epsilon_{aefg} = \int (\frac{\partial \mathcal{L}}{\partial \nabla_a\psi_d^D}\nabla^b\psi_d^D - \xi^a\mathcal{L})\epsilon_{aefg} = \int j^a[\psi, i\mathcal{L}\psi]\epsilon_{aefg},
\]
(42)
which is obviously gauge invariant. Moreover, it can be obtained that
\[
\int :T_{ab}^B\xi_b : = \epsilon_{aefg} = a^\dagger(\rho I)a(\bar{\rho}^j)[i\mathcal{L}_\xi]^j_I + c^\dagger(\tau_I)c(\bar{\tau}^j)[i\mathcal{L}_\xi]^j_I,
\]
(43)
where
\[
[i\mathcal{L}_\xi]^j_I = (\rho_I, i\mathcal{L}_\xi\rho_J),
\]
(44)
\[
[i\mathcal{L}_\xi]^j_I = (\tau_I, i\mathcal{L}_\xi\tau_J).
\]
Especially, we have
\[
\int d^3x :\hat{T}^{0\mu} := \int d^3p \rho^{\mu}(a^\dagger(p)a(p) + c^\dagger(p)c(p)),
\]
(45)
which is our familiar result.
4 Discussions

The result obtained here provides a basis for us to investigate the Casimir effect and quantum inequality for massless spin-3/2 field, which has be reported elsewhere\cite{13, 15}. In addition, we would like to stress that the framework and method presented here are also applicable to other particles with arbitrary mass and spin such as photon\cite{16}.

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Appendix A: Notations and Conventions

Our notations and conventions follow those of \cite{3}. In particular, the ordinary vector fields are related with the spinor fields by the soldering form $\sigma_{aA'}^{A''}$. For example, the Minkowski metric $\eta_{ab} = \sigma_{aA'}^{A''} \sigma_{bB'}^{B''} \epsilon_{AB} \epsilon_{A'B'}$, the covariant derivative $\nabla_{a} = \sigma_{aA'}^{A''} \nabla_{A''}$, and the volume element compatible with the metric $\epsilon_{abcd} = \sigma_{aA'}^{A''} \sigma_{bB'}^{B''} \sigma_{cC'}^{C''} \sigma_{dD'}^{D''} \epsilon_{A''A'B'B'} \epsilon_{A''A'B'B'}$ with $\epsilon_{A''A'B'B'} \epsilon_{A''A'B'B'} = i(\epsilon_{AB} \epsilon_{A'B'} \epsilon_{B'D'} - \epsilon_{AC} \epsilon_{A'B'} \epsilon_{B'D'})$. In addition, the index is raised or lowered by $\{\epsilon_{AB}, \epsilon_{A'B'}, \eta_{ab}\}$. The d’Alembertian is defined as $\Box = \nabla_{a} \nabla^{a}$. Furthermore, the Lorentz coordinate system is specially denoted by $\{x^{\mu}| \mu = 0, 1, 2, 3\}$, and the spatial vectors are indicated by letters in boldface.
Finally, the dyad spinor basis is denoted by \( \{(\epsilon^\Sigma)^A|\Sigma = 1, 2\} \), where

\[
\epsilon^\Sigma = \epsilon^{\Sigma\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
\sigma^{\mu\Sigma} = \frac{1}{\sqrt{2}}(I, \sigma),
\]

\[
\sigma^\mu\Sigma' = \frac{1}{\sqrt{2}}(I, -\sigma),
\]

\[
\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

\[
\nabla_\mu = \partial_\mu,
\]

\[
\epsilon_{0123} = 1.
\]

(46)

**Appendix B: Some Useful Identities**

Start with the spinor formulation of the volume element

\[
\epsilon_{AA'BB'CC'DD'} = i(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'C'D'}),
\]

we have

\[
\epsilon_{abCC'D'} = \sigma_a^{A'}\sigma_b^{B'}\epsilon_{AA'BB'CC'D'}^{C'} = i(\sigma_a^{ABC'}\sigma_b^{BC'}\epsilon_{CD} - 2\sigma_{aCB'}\sigma_{bD'B'}) = i(\eta_{ab}\epsilon_{CD} - 2\sigma_{aCB'}\sigma_{bD'B'}). \tag{48}
\]

Whence a pair of identities can be obtained as

\[
\sigma_{aCB'}\sigma_{bD'B'} + \sigma_{bCB'}\sigma_{aD'B'} = \eta_{ab}\epsilon_{CD},
\]

\[
\sigma_{aCB'}\sigma_{bD'B'} - \sigma_{bCB'}\sigma_{aD'B'} = i\epsilon_{abCC'D'}. \tag{49}
\]

Furthermore, from Eqn. (49), we have

\[
\sigma_{aA'}\sigma_{cA'B'B'} + \sigma_{bA'}\sigma_{cA'B'B'} = \delta_a^c\sigma_{bA'B'} + \delta_b^c\sigma_{aA'B'} - \eta_{ab}\sigma^c_{AB'}. \tag{50}
\]
References

[1] E. P. Wigner, Ann. Math. 40: 149-204(1939).

[2] V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. 34: 211-223(1948).

[3] R. P. Geroch, Special Topics in Particle Physics(Unpublished Lecture Notes, University of Texas at Austin, 1971).

[4] S. Weinberg, The Quantum Theory of Fields: Volume I(Cambridge University Press, Cambridge, 1995).

[5] S. Weinberg, hep-th/9702027.

[6] R. M. Wald, General Relativity(University of Chicago Press, Chicago, 1984).

[7] R. Penrose, Chaos, Solitons and Fractals 10: 581-611(1999).

[8] A. Sen, Int. J. Theor. Phys. 21: 1-35(1982).

[9] W. Rarita and J. Schwinger, Phys. Rev. 60: 61(1941).

[10] R. Penrose, Proc. Roy. Soc. Lond. A284:159-203(1965).

[11] R. P. Geroch, Unpublished Notes on Lie Derivatives, private communication.

[12] H. Zhang, Commun. Theor. Phys. 44: 1007-1010(2005).

[13] C. Liang, Introduction to Differential Geometry and General Relativity, Volume II(Beijing Normal University Press, Beijing, 2001).

[14] W. Liu et al., hep-th/0604005

[15] B. Hu et al., Phys. Rev. D 73: 045015(2006).

[16] Y. Hu et al., quant-ph/0509216

[17] M. Han et al., gr-qc/0409019