The universal properties of the electroweak phase transition*

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The first order electroweak phase transition in the Standard Model turns into a regular cross-over at a critical Higgs mass \( m_{H,c} \sim 75 \text{ GeV} \). At the critical point the transition is of second order. We make a detailed investigation of the critical properties of the electroweak theory at the critical point, and we find that the transition falls into the 3d Ising universality class. The continuum limit extrapolation of the critical Higgs mass is \( m_{H,c} = 72(2) \text{ GeV} \), which implies that there is no electroweak phase transition in the Standard Model.

I. INTRODUCTION

During the last several years, an impressive amount of quantitative knowledge about the Standard Model (SM) finite temperature electroweak phase transition has been obtained through lattice Monte Carlo simulations. At small Higgs masses (\( m_H \)) the transition is of the first order. The transition becomes weaker (decreasing latent heat, surface tension) when the Higgs mass increases, and it has been found to turn into a regular cross-over when \( m_H > 75 \text{ GeV} \). At the endpoint of the first order transition line a second order phase transition appears. At this point the macroscopic behaviour of the system is determined by the universal properties of the endpoint. While the location of the endpoint and the mass spectrum near it have been studied before, the critical properties of the endpoint itself have not been resolved so far. We shall show that the universality class of the SM endpoint is of the 3d Ising type (for a full description of this work, see ref. [3]).

Near the critical point the thermodynamics of the system (susceptibilities, correlation lengths) is determined by the corresponding critical exponents, which, in turn, are determined by the universality class of the theory. Thus, the universal behaviour represents a tremendous simplification in the effective degrees of freedom of the system.

While the standard perturbative analysis can be used to resolve the first order nature of the transition at small \( m_H \), it fails completely at large values of \( m_H \) — indeed, according to perturbation theory, the transition remains of first order for all Higgs masses. Thus, the physics of the endpoint and the universal behaviour are inherently non-perturbative.

What kind of universal behaviour can one expect? Formally, the Higgs field has \( SU(2)_{\text{gauge}} \times SU(2)_{\text{custodial}} \) symmetry (in the \( SU(2) \) + Higgs theory, see below), but this remains unbroken at all temperatures. Indeed, the mass spectrum of the system has been investigated in detail both above and below the critical point, and only one scalar excitation (which couples to \( \phi \) becomes light in the neighbourhood of it. Thus, we expect the Ising-type universality to be realized, but also mean field-type or multicritical behaviour is, in principle, possible.

II. EFFECTIVE ACTION

An effective 3d \( SU(2) \) gauge + Higgs theory, obtained through dimensional reduction, accurately describes the static properties of the SM and many of its extensions at high temperatures [3]. The action of the theory is

\[
S = \int d^3x \left[ \frac{1}{4} F_\alpha^a F_\alpha^a + |D_\mu \phi|^2 + m_3^2 |\phi|^2 + \lambda_3 |\phi|^4 \right].
\]  

(1)

The physics of the theory is fixed by the dimensionful gauge coupling \( g_3^2 \) and by the dimensionless ratios

\[
x = \lambda_3 / g_3^2, \quad y = m_3^2(\mu) / g_3^2,
\]

(2)

where \( m_3^2(\mu) \) is the renormalized mass parameter in the \( \overline{\text{MS}} \) scheme. The relations of the couplings \( g_3^2, x, y \) to the full theory are computable in perturbation theory [3]. We have omitted the U(1) sector of the SM; this is justified, since the U(1) gauge boson remains massless at any temperature and does not affect the transition qualitatively.

The lattice action in standard formalism is

\[
S = \beta_0 \sum_{x;i,j} \left( 1 - \frac{1}{2} \text{Tr} P_{ij} \right) - \beta_H \sum_{x;i} \frac{1}{2} \text{Tr} \Phi^\dagger(x) U_i(x) \Phi(x + i)
\]

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\begin{align}
\sum_x \left[ \frac{1}{4} \text{Tr} \Phi^4 + \beta_R \left( \frac{1}{4} \text{Tr} \Phi^4 - 1 \right)^2 \right] \\
\equiv S_G + S_{\text{hopping}} + S_{\phi^2} + S_{(\phi^2-1)^2}.
\end{align}

Here $\Phi$ is the $2 \times 2$ matrix $\Phi = (i\sigma_2 \phi^*, \phi)$. The two actions in Eqs. (3, 4) give the same physics in the continuum limit $a \to 0$ if the three dimensionless parameters $\beta_G, \beta_H, \beta_R$ in Eq. (3) are related to the three dimensionless parameters $g_3^2a, x, y$ in Eq. (4) by the following equations:

\begin{align}
\beta_G &= \frac{4}{g_3^2a}, \\
\beta_R &= \frac{\beta_H^2}{\beta_G}, \\
y &= \frac{\beta_G^2}{8} \left( \frac{1}{\beta_H} - 3 - \frac{2x\beta_H}{\beta_G} \right) + \frac{3\Sigma\beta_G}{32\pi}(1 + 4x) \\
&+ \frac{1}{16\pi^2} \left[ \left( \frac{51}{16} + 9x - 12x^2 \right) \left( \ln \frac{3\beta_G}{2} + \zeta \right) \\
&+ 4.9941 + 5.2153x \right],
\end{align}

where $\Sigma = 3.1759115$ and $\zeta = 0.08849(1)$. The universal behaviour does not depend on the lattice spacing, which we keep fixed through fixing $\beta_G = 5$.

III. RESOLVING THE UNIVERSALITY CLASS

For concreteness, in what follows we shall discuss the universal properties of the SM by comparing it to the Ising model. However, one should bear in mind that the analysis is, by no means, limited only to verifying the Ising-type universal behaviour.

The phase diagram of the 3d SU(2)+Higgs theory is shown in Fig. 1, together with the Ising model phase diagram. Adopting now Ising-model terminology, let us call the two critical directions in Fig. 1 the $h$-like (perpendicular to the transition line) and the $t$-like (along the transition line) directions. Due to the lack of an exact order parameter, the mapping of the $h$-like and $t$-like directions of the SU(2)+Higgs model to the Ising model is non-trivial. This is illustrated in Fig. 2, where the probability density at the critical point is plotted on the $S_{(\phi^2-1)^2}$ vs. $S_{\text{hopping}}$ plane. Each point in the plot corresponds to one measurement of the observables. Middle: The same as above, after a shift and a rotation. Bottom: A density plot of the 3d Ising model on the $(-1 \times \text{energy})$ vs. magnetization plane.

This rotation closely corresponds to the rotation of the directions shown in the phase diagrams in Fig. 1. The resulting rotated operators are conjugate to the $h$-like and $t$-like couplings, and we call them the $M$-like (“magnetization”) and $E$-like (“energy”) operators. Note that all of these operators ($S_{\text{hopping}}, S_{(\phi^2-1)^2}$ and the rotated $M$-like operator) lack the explicit $M \leftrightarrow -M$ symmetry present in the Ising model. This symmetry is dynamically generated at the critical point.

However, there is no reason to restrict ourselves only to the two observables $S_{\text{hopping}}$ and $S_{(\phi^2-1)^2}$. Any number of (scalar) operators can contribute to the true $M$-like and $E$-like directions. In order to improve on the projection, it is important to consider a large number of independent operators. The central problem of the analysis is how to correctly identify the best $E$-like and $M$-like projections.
FIG. 3. The joint probability distributions $P(V_M, V_E)$ at the infinite volume critical point, for the volumes (a) $16^3$, (b) $32^3$, (c) $64^3$. It is clearly seen how the distribution becomes more symmetric for the increasing volume, and approaches the Ising model distribution of Fig. 4.

Motivated by these considerations, we use the following method:

(a) We first locate the infinite volume critical point (for details, see ref. [5]), where all of the subsequent analysis is performed.

(b) Using several volumes, we measure the fluctuation matrix $M_{ij} = \langle s_i s_j \rangle$, $s_i = S_i - \langle S_i \rangle$. We used up to 6 operators: those in Eq. (3), together with the operators

\begin{align}
S_R &= \sum_x |\Phi(x)|^2, \\
S_L &= \sum_{x,i} \frac{1}{2} \text{Tr} V(x) U(x) V(x + i),
\end{align}

where $V(x) = \Phi(x)/|\Phi(x)|$.

(c) We calculate the eigenvalues $\lambda_\alpha$ and eigenvectors $V_\alpha$ of $M_{ij}$. Some of the eigenvectors correspond to “critical” observables like $M$ or $E$, and the rest are “trivial.” They can be classified either by inspecting the probability distributions $p(V_\alpha)$ and $p(V_\alpha, V_\beta)$, or by looking at the finite volume behaviour of the eigenvalues. It is convenient to express the eigenvalues in terms of the corresponding susceptibilities: $\chi_i = \lambda_i / L^3$. For example, the $M$-like and $E$-like susceptibilities (magnetic susceptibility and heat capacity) diverge with the critical exponents as ($L$ is the linear extent of the lattice)

$$\lambda_M \propto L^{\gamma/\nu}, \quad \lambda_E \propto L^{\alpha/\nu}. \quad (9)$$

The susceptibilities corresponding to the “trivial” eigenvalues remain constant.

The method described above has much in common with the one used by Alonso et al [9] to find the $E$- and $M$-like directions at the critical point of the 4d U(1)+Higgs model, and with the method developed by Bruce and Wilding [10] for the study of the liquid-gas critical point. Both of these rely on considering only two-dimensional distributions.

IV. RESULTS

The main part of our analysis was done on 6 lattice volumes, from $16^3$ up to $64^3$, using the lattice spacing $a = 4/(g^2 \beta_G) = 4/5g^2$. All of the simulations were performed close to the critical point, and the measurements were

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shifted to the measured infinite volume critical point by histogram reweighting [5].

The fluctuation matrix analysis described in the previous section was performed either using all of the 6 terms described in the item (b) above or only the 4 terms in the action (3). In both cases the projections of the eigenvectors to the original operators remained remarkably stable as functions of the volume. This gives us confidence that the method can identify the correct critical observables and their volume dependence.

It is very illustrative to consider the joint probability distributions \( p(V_M, V_E) \). In Fig. 3 these are shown for volumes \( 16^3 \), \( 32^3 \) and \( 64^3 \) (using 6 operators). When the volume increases \( p(V_M, V_E) \) clearly approaches the Ising model distribution \( p(M, E) \), shown in Fig. 3. It is also evident that even when the SU(2)+Higgs model lacks the exact magnetization symmetry of the Ising model, this symmetry is recovered in the infinite volume limit. In fact, the shapes of the distributions \( p(V_i, V_j) \) can be used to identify the correct \( E \)-like and \( M \)-like eigenmodes.

In Figs. 3 and 4 we show the behaviour of the susceptibilities \( \chi_i \) as functions of the volume. Only \( \chi_M \) and \( \chi_E \) diverge as the volume is increased, the other 4 susceptibilities do not show any critical behaviour (note the very restricted vertical scale in Fig. 4 in comparison with Fig. 3). The behaviour of the critical susceptibilities \( \chi_M \) and \( \chi_E \) is compatible with the Ising model (with the reservation that \( \chi_E \) does not yet attain its asymptotic large volume behaviour; however, it is very close to the Ising model behaviour at similar lattice sizes).

The divergence of \( \chi_E \) implies a positive value for the critical exponent \( \alpha \). This clearly excludes \( O(N) \) models with \( N \geq 2 \) \((\alpha < 0)\) and the mean field behaviour \((\alpha = 0)\).

The results presented above were calculated using 6 operators in the fluctuation matrix analysis. The numerical results remain stable when only 4 operators are used, although in some cases small deviations from the Ising model behaviour begin to appear: for example, the joint distributions \( p(V_M, V_E) \) in Fig. 3 become slightly thicker to the \( E \)-direction. This is in itself not surprising, since in the case of 4 observables the \( E \)-like eigenmode has the smallest eigenvalue, and presumably it is very sensitive to the quality of the projection to the operator basis [5]. When the number of the operators is increased to 6, the two additional eigenvalues are smaller than the \( E \)-like one. This underlines the importance of including a large enough number of operators in the analysis.

In the simulations above the lattice spacing was fixed through \( \beta_G = 4/g_0^2a = 5 \). We have also located the critical point at \( \beta_G = 8 \), and Gürler et al. 12 have published results at \( \beta_G = 12 \) and 16. Taking into account the \( O(a) \)-correction calculated by Moore [12] we can extrapolate the critical coupling \( x_c(\beta_G) \) to the con-
continuum limit $\beta_G \to \infty$ (see Fig. 7). The final result $x_c = 0.0983(15)$ corresponds to the Standard Model Higgs mass $72(2)\text{ GeV}$. The effect of the U(1) gauge sector of the SM, which was omitted here, is much smaller than the statistical errors quoted here \[13\]. Since the experimental lower limit is $\sim 88\text{ GeV}$ \[14\], this excludes the existence of the SM phase transition. Nevertheless, a first order phase transition is still allowed in several extensions of the Standard Model, including the Minimal Supersymmetric Standard Model.

FIG. 7. The infinite volume extrapolations of $x_c$ as a function of $\beta_G$. Gürtler et al. refers to \[3\]. The $O(a)$ correction has been calculated in ref. \[12\].

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