COMPUTATIONS OF THE ORBIFOLD YAMABE INVARIANT

KAZUO AKUTAGAWA∗

ABSTRACT. We consider the Yamabe invariant of a compact orbifold with finitely many singular points. We prove a fundamental inequality for the estimate of the invariant from above, which also includes a criterion for the non-positivity of it. Moreover, we give a sufficient condition for the equality in the inequality. In order to prove it, we also solve the orbifold Yamabe problem under a certain condition. We use these results to give some exact computations of the Yamabe invariant of compact orbifolds.

1. Introduction and Main Results

There is a natural differential-topological invariant, called the Yamabe invariant, which arises from a variational problem for the functional $E$ below on a given compact smooth $n$-manifold $M$ (without boundary) of dimension $n ≥ 3$. It is well known that a Riemannian metric on $M$ is Einstein if and only if it is a critical point of the normalized Einstein-Hilbert functional $E$ on the space $\mathcal{M}(M)$ of all Riemannian metrics on $M$

$$E : \mathcal{M}(M) \to \mathbb{R}, \quad g \mapsto E(g) := \frac{\int_M R_g d\mu_g}{\text{Vol}_g(M) (n-2)/n}.$$ 

Here, $R_g, d\mu_g$ and $\text{Vol}_g(M)$ denote respectively the scalar curvature, the volume element of $g$ and the volume of $(M, g)$. Because the restriction of $E$ to any conformal class

$$C = [g] := \{e^{2f} \cdot g \mid f \in C^\infty(M)\}$$

is bounded from below, we can consider the following conformal invariant (called the Yamabe constant of $(M, C)$)

$$Y(M, C) := \inf_{\tilde{g} \in C} E(\tilde{g}).$$

A remarkable theorem [45, 43, 8, 34, 39] (cf. [9, 12, 28, 36, 40]) of Yamabe, Trudinger, Aubin, and Schoen asserts that each conformal class $C$ contains a minimizer $\tilde{g}$ of $E|C$, called a Yamabe metric (or a solution of the Yamabe problem), which is of constant scalar curvature

$$R_{\tilde{g}} = Y(M, C) \cdot \text{Vol}_g(M)^{-2/n}.$$ 

The study of the second variation of $E$ done in [21, 36] (cf. [10]) leads naturally to the definition of the following differential-topological invariant

$$Y(M) := \sup_{C \in \mathcal{C}(M)} \inf_{g \in C} E(g) = \sup_{C \in \mathcal{C}(M)} Y(M, C),$$

Date: September, 2010.

∗ supported in part by the Grants-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 21540059.
where $C(M)$ denotes the space of all conformal classes on $M$. This invariant is called the \textit{Yamabe invariant} (or $\sigma$-\textit{invariant}) of $M$ and it was introduced independently by O. Kobayashi \cite{Ko} and Schoen \cite{Sc} (see also \cite{Ko1980} \cite{Sc1980}).

In the study of Yamabe invariant, with certain geometric non-collapsing assumptions, we will often encounter \textit{Riemannian orbifolds} (or \textit{Riemannian multi-folds} more generally) as the limit spaces for sequences of Yamabe metrics (cf. \cite{Au} \cite{Sc} \cite{Sc2}). For a compact $n$-orbifold $M$ with an orbifold metric $g$, one can also define the corresponding Yamabe constant $Y(M,[g]_{orb})$ and Yamabe invariant $Y^{orb}(M)$ (see Section 2 or \cite{Sc} for details). Let $M_1$ and $M_2$ be compact $n$-orbifolds with same number of finite singularities $\{\hat{p}_1, \cdots , \hat{p}_\ell \}$ and $\{\hat{q}_1, \cdots , \hat{q}_\ell \}$ respectively. Assume that each corresponding singularities $\hat{p}_j$ and $\hat{q}_j$ have a same structure group $\Gamma_j (< O(n))$. For each $j$, let $B(\hat{p}_j)(\subset M_1)$ and $B(\hat{q}_j)(\subset M_2)$ denote respectively open geodesic balls of sufficiently small radiuses centered at $\hat{p}_j$ and $\hat{q}_j$ with fixed reference orbifold metrics. Then, the boundaries of these two balls can be naturally identified by a canonical diffeomorphism. Let

$$N := (M_1 - \cup_{j=1}^\ell B(\hat{p}_j)) \cup_Z (M_2 - \cup_{j=1}^\ell B(\hat{q}_j))$$

be the sum of $M_1 - \cup_{j=1}^\ell B(\hat{p}_j)$ and $M_2 - \cup_{j=1}^\ell B(\hat{q}_j)$ along their common boundary $Z := \partial( \cup_{j=1}^\ell B(\hat{p}_j)) = \partial( \cup_{j=1}^\ell B(\hat{q}_j))$. Note that $N$ has a canonical smooth structure as manifold. For simplicity, in Section 4, we will abbreviate the above decomposition as the generalized connected sum

$$N = M_1 \# \cup_{j=1}^\ell (S^{n-1}/\Gamma_j) M_2.$$

One of main purposes of this paper is to prove the following fundamental inequality for the estimate of the orbifold Yamabe invariant from above and a sufficient condition for the equality in this inequality. The inequality also includes a criterion for the non-positivity of the invariant:

\textbf{Theorem A.} Under the above understandings, assume that

$$Y(N) \leq 0 \text{ (resp. } < 0) \text{ and } Y^{orb}(M_2) > 0 \text{ (resp. } \geq 0).$$

Then,

$$Y^{orb}(M_1) \leq Y(N) \leq 0.$$

Moreover, if $M_1$ admits an orbifold metric $\tilde{g}$ of constant scalar curvature satisfying $E(\tilde{g}) = Y(N)$, then

$$Y^{orb}(M_1) = Y(M_1, [\tilde{g}]_{orb}) = Y(N) \leq 0.$$

On the computation of Yamabe invariants for \textit{smooth} manifolds, a first remarkable result is the following proved by Aubin \cite{Au} (cf. \cite{Sc}):

$$Y(M,C) \leq Y(S^n, [g_0]) = E(g_0) \left( = n(n-1)\text{Vol}_{g_0}(S^n)^{2/n} \right)$$

for any $C \in C(M)$, where $g_0$ is the standard metric of constant curvature one on the standard $n$-sphere $S^n$. This implies both the universal estimate for $Y(M)$ from above and the computation of $Y(S^n)$

$$Y(M) \leq Y(S^n) = n(n-1)\text{Vol}_{g_0}(S^n)^{2/n}.$$ 

Kobayashi \cite{Ko} \cite{Ko1980} and Schoen \cite{Sc} proved that

$$Y(S^{n-1} \times S^1) = Y(S^n).$$
Kazuo Akutagawa

Kobayashi also gave two kind of proof for it (see [4] for the third one), one of them especially is based on the following important inequality, called Kobayashi’s inequality:

\[ Y(M^n_1 \# M^n_2) \geq \begin{cases} \left(-\left|Y(M^n_1)\right|^{n/2} + \left|Y(M^n_2)\right|^{n/2}\right)^{2/n} \cdots Y(M^n_1), Y(M^n_2) \leq 0, \\
\min\{Y(M^n_1), Y(M^n_2)\} \cdots \text{otherwise} \end{cases} \]

for any two compact \( n \)-manifolds \( M_1, M_2 \). This has been extended to some useful surgery theorems [5, 31, 33]. On the other hand, some classification theorems for manifolds with positive scalar curvature metric [14, 15, 37, 38, 41] lead to many examples of manifolds with zero (or non-positive) Yamabe invariant, for instance, \( Y(T^n) = 0 \) for the \( n \)-torus \( T^n \) (see [32] for further development).

In 1995, LeBrun [24] computed the Yamabe invariants of smooth compact quotients of complex-hyperbolic 2-space, which was the first example of manifolds with negative Yamabe invariant. He and collaborators [16, 17, 25, 26, 27] also computed the Yamabe invariants for a large class of 4-manifolds, including Kähler surfaces \( X \) with either \( Y(X) < 0 \) or \( 0 < Y(X) < Y(S^4) \) (see [5, 7, 11] for 3-manifolds \( M^3 \) with either \( Y(M^3) < 0 \) or \( 0 < Y(M^3) < Y(S^3) \)). In particular, for any minimal complex surface of general type \( X \), he [25] computed its Yamabe invariant \( Y(X) \) to be

\[ Y(X) = -4\sqrt{2\pi} \sqrt{2\chi(X) + 3\tau(X)} < 0, \]

where \( \chi(X) \) and \( \tau(X) \) are respectively the Euler characteristic and signature of \( X \). Moreover, if \( X \) contains \((-2)\)-curves, there exist a sequence of metrics \( \{g_i\}_i \) on \( X \) and a Kähler-Einstein orbifold metric \( \tilde{g} \) on the canonical model \( X_{\text{can}} \) of \( X \) such that

\[ \lim_{i \to \infty} Y(X, [g_i]) = Y(X), \quad \lim_{i \to \infty} d_{GH}(\{X, g_i\}, \{X_{\text{can}}, \tilde{g}\}) = 0. \]

Here, \( d_{GH} \) denotes the Gromov-Hausdorff distance. This result suggests naturally the following question: “Can one describe rigorously the above fact in terms of \( Y(X_{\text{can}}, [\tilde{g}]_{\text{orb}}) \) and \( Y_{\text{orb}}(X_{\text{can}}) \)?”

The other of main purposes of this paper is to answer it.

**Theorem B.** Under the above settings, the following holds

\[ Y_{\text{orb}}(X_{\text{can}}) = Y(X_{\text{can}}, [\tilde{g}]_{\text{orb}}) = Y(X). \]

In Section 2, we recall the definition on the orbifold Yamabe invariant from [3] and explain briefly some terminologies. For the proof of Theorem A, we also recall some necessary terminologies and basic results on the Yamabe invariant of cylindrical manifolds [2]. Applying these results to the orbifold Yamabe invariant, we prove the first assertion of Theorem A. In Section 3, for the proof of the second assertion in Theorem A, we consider the existence problem of minimizers for the functional \( E \) on compact conformal orbifolds, that is, the orbifold Yamabe problem. Under a certain condition, we solve this problem. Using the solution, we can prove the second assertion. In Section 4, we give two more typical exact computations of the orbifold Yamabe invariant besides the proof of Theorem B.

**Acknowledgements.** The author would like to express his sincere gratitude to Nobuhiro Honda and Jeff Viaclovsky for helpful discussions on singularities of complex surfaces and on the orbifold Yamabe invariant respectively. He also would like to thank Claude LeBrun for useful comments.
2. The orbifold Yamabe invariant

For the sake of self-containedness, we first recall the definition of orbifolds with finitely many singular points which we discuss here [3].

**Definition 2.1.** Let $M$ be a locally compact Hausdorff space. We say that $M$ is an $n$-orbifold with singularities

$$\Sigma = \{(\tilde{p}_1, \Gamma_1), \cdots, (\tilde{p}_t, \Gamma_t)\}$$

if the following conditions are satisfied:
1. $\Sigma := \{\tilde{p}_1, \cdots, \tilde{p}_t\} \subset M$, and $M - \Sigma$ is a smooth $n$-manifold.
2. $\Gamma := \{\Gamma_1, \cdots, \Gamma_t\}$ is a collection of non-trivial finite subgroups $\Gamma_j$ of $O(n)$, each of which acts freely on $\mathbb{R}^n - \{0\}$.
3. For each $\tilde{p}_j$, there exist its open neighborhood $U_j$ and a homeomorphism $\varphi_j : U_j \to \mathbb{B}_{\tau_j}(0)/\Gamma_j$ for some $\tau_j > 0$ such that

$$\varphi_j : U_j - \{\tilde{p}_j\} \to (\mathbb{B}_{\tau_j}(0) - \{0\})/\Gamma_j$$

is a diffeomorphism. Here, $\mathbb{B}_{\tau_j}(0) := \{x = (x^1, \cdots, x^n) \in \mathbb{R}^n \mid |x| < \tau_j\}$.

We refer to the pair $(\tilde{p}_j, \Gamma_j)$ as a singular point with the structure group $\Gamma_j$ and the pair $(U_j, \varphi_j)$ as a local uniformization. To simplify the presentation, we assume, without particular mention, that an orbifold $M$ has only one singularity, i.e., $\Sigma = \{(\tilde{p}, \Gamma)\}$. Let $\varphi : U \to \mathbb{B}_r(0)/\Gamma$ be a local uniformization and $\pi : \mathbb{B}_r(0) \to \mathbb{B}_r(0)/\Gamma$ the canonical projection. We also always assume that $M$ is compact.

**Definition 2.2.** (1) A Riemannian metric $g \in \mathcal{M}(M - \{p\})$ is an orbifold metric if there exists a $\Gamma$-invariant smooth metric $\tilde{g}$ on the ball $\mathbb{B}_r(0)$ such that $(\varphi^{-1} \circ \pi)^* g = \tilde{g}$ on $\mathbb{B}_r(0) - \{0\}$. We denote by $\mathcal{M}^{orb}(M)$ the space of all orbifold metrics on $M$. In the case when $\Sigma = \{(\tilde{p}_1, \Gamma_1), \cdots, (\tilde{p}_t, \Gamma_t)\}$, the space of all orbifold metrics is defined similarly.

(2) For an orbifold metric $g \in \mathcal{M}^{orb}(M)$, its orbifold conformal class $[g]_{orb}$ is defined by

$$[g]_{orb} := [g] \cap \mathcal{M}^{orb}(M) = \{e^{2f} \cdot g \mid f \in \mathcal{C}^0(M) \cap \mathcal{C}^\infty(M - \{\tilde{p}\}), (\varphi^{-1} \circ \pi)^* f \in \mathcal{C}^\infty(\mathbb{B}_r(0))\}.$$

We denote by $\mathcal{C}^{orb}(M)$ the space of all orbifold conformal classes.

As in the smooth case, consider the normalized Einstein-Hilbert functional

$$E : \mathcal{M}^{orb}(M) \to \mathbb{R}, \quad g \mapsto \frac{\int_M R_g d\mu_g}{\text{Vol}_g(M)^{(n-2)/n}}.$$

Since the singularity has codimension at least three, Stokes’ theorem and Gauss’ divergence theorem still hold over Riemannian orbifolds. Hence, $\tilde{g}$ is a critical point of $E$ on $\mathcal{M}^{orb}(M)$ if and only if $\tilde{g}$ is an Einstein orbifold metric. Then, one can define naturally the definition of the orbifold Yamabe invariant.

**Definition 2.3.** For a conformal orbifold $(M, [g]_{orb})$, its Yamabe constant $Y(M, [g]_{orb})$ is defined by

$$Y(M, [g]_{orb}) := \inf_{\tilde{g} \in [g]_{orb}} E(\tilde{g}).$$

Moreover, the orbifold Yamabe invariant $Y^{orb}(M)$ of $M$ is also defined by

$$Y^{orb}(M) := \sup_{[g]_{orb} \in \mathcal{C}^{orb}(M)} Y(M, [g]_{orb}).$$
Before we explain some necessary terminologies on the Yamabe invariant of cylindrical manifolds, we give two comments on orbifolds with positive orbifold Yamabe invariant.

**Remark 2.4.** Let \((X, g)\) be a hyperKähler asymptotically locally Euclidean (abbreviated to ALE) 4-manifold constructed in \([22]\) (cf. \([29]\)), where \(X\) is the minimal resolution of the quotient space \(\mathbb{C}^2/\Gamma\) for a non-trivial finite subgroup \(\Gamma\) of \(SU(2)\). Then, \((X, g)\) has a smooth conformal compactification \((\hat{X}, [\hat{g}]_{\text{orb}})\) with singularity \(\{(p_\infty, \Gamma)\}\) \([13, 44]\), which has a positive Yamabe constant \(Y(\hat{X}, [\hat{g}]_{\text{orb}}) > 0\).

In \([44, \text{Theorem 1.3}]\), Viaclovsky has proved the following:

1. The orbifold Yamabe problem on \((\hat{X}, [\hat{g}]_{\text{orb}})\) has no solution. This implies that the orbifold Yamabe problem is not always solvable (see Section 3 for the solvability), in contrast with the case for smooth compact conformal manifolds.
2. He computed the orbifold Yamabe invariant of \(\hat{X}\) as \(Y_{\text{orb}}(\hat{X}) = Y(\hat{X}, [\hat{g}]_{\text{orb}}) = Y(S^4)/|\Gamma|^{1/2}\).

However, similarly to the case for smooth compact manifolds, there is not much exact computations of positive orbifold Yamabe invariants at present.

In the proof of both (1) and (2), one of key points is the following estimate, called refined Aubin’s inequality \([3, \text{Theorem B}]\)

\[
Y(M, [g]_{\text{orb}}) \leq Y_{\text{orb}}(M) \leq \min_{1 \leq j \leq \ell} \frac{Y(S^n)}{|\Gamma_j|^{2/n}}
\]

for any compact Riemannian \(n\)-orbifold \((M, g)\) with singularities \(\{(\hat{p}_1, \Gamma_1), \cdots, (\hat{p}_\ell, \Gamma_\ell)\}\).

This inequality is also crucial to give a sufficient condition for the solvability of the orbifold Yamabe problem in Section 3.

**Definition 2.5.** Let \(X\) be an open \(n\)-manifold with tame ends, i.e., it is diffeomorphic to \(W \cup_Z (Z \times [0, \infty))\), where \(W(\subset X)\) is a relatively compact open submanifold with boundary \(\partial W =: Z \cong Z \times \{0\}\) (possibly finitely many connected component). For a fixed \(h \in \mathcal{M}(Z)\), a complete Riemannian metric \(\tilde{g}\) on \(X\) is called a cylindrical metric modeled by \((Z, h)\) if there exists a global coordinate function \(t\) on \(Z \times [0, \infty)\) such that \(\tilde{g}|_{Z \times [1, \infty)}\) is the product metric \(\tilde{g}(z, t) = h(z) + dt^2\) for \((z, t) \in Z \times [1, \infty)\) (see Figure 1). Each pair \((X, \tilde{g})\) is called a cylindrical manifold and \(h\) a slice metric. We denote by \(\mathcal{M}^{h-cyl}(X)\) the space of all cylindrical metrics on \(X\) modeled by \((Z, h)\).

![Figure 1: A cylindrical manifold \((X, \tilde{g})\)](image)

For the definition of the Yamabe invariant on cylindrical manifolds, we first recall the following fact. On a compact Riemannian manifold \((M, g)\), the value of
functional $E(\tilde{g})$ for conformal metric $\tilde{g} := u^{4/(n-2)} \cdot g \in [g]$ can be rewritten by

$$E(\tilde{g}) = \int_M \left( \alpha_n |\nabla u|^2 + R_g u^2 \right) d\mu_g \left( n/(n-2) \right)^{(n-2)/n} = Q(M, \tilde{g})(u), \quad \alpha_n := \frac{4(n-1)}{n-2} > 0.$$

**Definition 2.6.** The Yamabe constant $Y(X, [\tilde{g}])$ of a cylindrical manifold $(X, \tilde{g})$ is defined by

$$Y(X, [\tilde{g}]) := \inf_{u \in C^\infty_c(X), u \neq 0} Q(X, \tilde{g})(u),$$

where $C^\infty_c(X)$ denotes the space of all smooth functions on $X$ with compact supports. Moreover, for a fixed $h \in \mathcal{M}(Z)$, the $h$-cylindrical Yamabe invariant $Y^{h-cyl}(X)$ of the open manifold $X$ with tame ends is also defined by

$$Y^{h-cyl}(X) := \sup_{\tilde{g} \in \mathcal{M}^{h-cyl}(X)} Y(X, [\tilde{g}]).$$

To simplify the presentation, we also assume, without particular mention, that each underlying manifold $X$ has only one connected tame end. In contrast with the case for compact manifolds, the constant $Y(X, [\tilde{g}])$ is not always finite. For instance, if the scalar curvature $R_h$ of slice metric $h$ is negative on $Z$, then $Y(X, [\tilde{g}]) = -\infty$.

As a complete criterion for the finiteness of $Y(X, [\tilde{g}])$, we have obtained the following [2, Lemmas 2.7, 2.9].

**Proposition 2.7.** For $h \in \mathcal{M}(Z^{n-1})$, let $L_h$ be the operator on $Z^{n-1}$ defined by

$$L_h := -\frac{4(n-1)}{n-2} \Delta_h + R_h,$$

and $\lambda(L_h)$ the first eigenvalue of $L_h$. Then, we have the following on the Yamabe constant of a cylindrical manifold $(X, \tilde{g})$ with slice metric $h$.

- If $\lambda(L_h) < 0$, then $Y(X, [\tilde{g}]) = -\infty$.
- If $\lambda(L_h) \geq 0$, then $Y(X, [\tilde{g}]) > -\infty$.
- If $\lambda(L_h) = 0$, then $0 \geq Y(X, [\tilde{g}]) > -\infty$.

We also note that the notion of the $h$-cylindrical Yamabe invariant is a natural extension of the one of the orbifold Yamabe invariant [3, Theorem 2.9].

**Proposition 2.8.** Let $M$ be a compact $n$-orbifold with singularity $\{(\bar{p}, \Gamma)\}$ (see Figure 2), and $h_0 \in \mathcal{M}(S^{n-1}/\Gamma)$ the standard metric of constant curvature one. Note that the open manifold $M - \{\bar{p}\}$ is of one tame end and $\mathcal{M}^{h_0-cyl}(M - \{\bar{p}\}) \neq \emptyset$. Then,

$$Y^{orb}(M) = Y^{h_0-cyl}(M - \{\bar{p}\}).$$

![Figure 2](image-url)
Now, we can state the key inequality for $h$-cylindrical Yamabe invariants, called refined Kobayashi’s inequality [2, Theorem 3.7].

**Theorem 2.9.** Let $N$ be a compact $n$-manifold and $Z$ a compact $(n-1)$-submanifold with trivial normal bundle. Assume that $M - Z$ has two connected components $W_1, W_2$. Let $X_1 := W_1 \cup_Z (Z \times [0, \infty))$, $X_2 := W_2 \cup_Z (Z \times [0, \infty))$ be the corresponding open $n$-manifolds with tame end $Z \times [0, \infty)$ (see Figure 3). For any $h \in \mathcal{M}(Z)$, we have

$$Y(N) \geq \begin{cases} -(Y^{h\text{-cyl}}(X_1))^{n/2} + |Y^{h\text{-cyl}}(X_2)|^{n/2})^{2/n} \cdots & \text{if } Y^{h\text{-cyl}}(X_1), Y^{h\text{-cyl}}(X_2) \leq 0, \\ \min\{Y^{h\text{-cyl}}(X_1), Y^{h\text{-cyl}}(X_2)\} \cdots & \text{otherwise.} \end{cases}$$

\[ N = W_1 \cup_Z W_2 \]

![Figure 3](image.png)

Figure 3.

Theorem 2.9 implies immediately the following.

**Corollary 2.10.** Under the same setting as in Theorem 2.9, assume that $Y(N) \leq 0$ (resp. $< 0$) and $Y^{h\text{-cyl}}(X_2) > 0$ (resp. $\geq 0$). (From Proposition 2.7, the positivity $Y^{h\text{-cyl}}(X_2) > 0$ implies automatically $\lambda(L_h) > 0$.) Then, we have

$$Y^{h\text{-cyl}}(X_1) \leq Y(N) \leq 0.$$ 

We can now prove the first assertion in Theorem A.

**Proof of the first assertion in Theorem A.** In Corollary 2.10, set $W_1 = M_1 - \sqcup_{j=1}^{\ell} B(\tilde{p}_j)$, $W_2 = M_2 - \sqcup_{j=1}^{\ell} B(\tilde{q}_j)$ and $h = h_0$ on $Z = \partial W_1 = \partial W_2 \cong \sqcup_{j=1}^{\ell} (S^{n-1}/\Gamma_j)$. Note that $X_1 = M_1 - \{\tilde{p}_1, \cdots, \tilde{p}_\ell\}$ and $X_2 = M_2 - \{\tilde{q}_1, \cdots, \tilde{q}_\ell\}$. Then, the first assertion follows directly from Proposition 2.8 and Corollary 2.10, that is,

$$Y^{orb}(M_1) = Y^{h_0\text{-cyl}}(X_1) \leq Y(N) \leq 0.$$ 

\[ \square \]
Remark 2.11. For given compact manifolds $N_1$ and $N_2$, we generally use Kobayashi’s inequality in the case for computing (or estimating) $Y(N_1 \# N_2)$ by using the values of both $Y(N_1)$ and $Y(N_2)$. In contrast with this, the generalized connected sum of compact orbifolds is often “prime” as smooth manifold. Hence, the opposite usage of (refined) Kobayashi’s inequality is also useful as Theorem A.

3. The orbifold Yamabe problem

In this section, we first prove the orbifold Yamabe problem under a certain condition.

Theorem 3.1. Let $(M, g)$ be a compact Riemannian $n$-orbifold with singularities \{$(\bar{p}_1, \Gamma_1), \cdots, (\bar{p}_\ell, \Gamma_\ell)$\}. Assume the following strict inequality:

$Y(M, [g]_{orb}) < \min_{1 \leq j \leq \ell} Y(S^j).$

Then, there exists a minimizer $\tilde{g} \in [g]_{orb}$ of the functional $E|_{[g]_{orb}}$ (called an orbifold Yamabe metric) such that the orbifold metric $\tilde{g}$ is of constant scalar curvature $R_{\tilde{g}} = Y(M, [g]_{orb}) \cdot \Vol_M(M)^{-2/n}$.

Proof. We use here the same notations as those in Definition 2.1. Without loss of generality, we may assume that $M$ has only one singularity \{$(\bar{p}, \Gamma)$\}. The method adopting here for constructing approximate solutions is similar to the one in [2, Theorem 5.2]. But, as background metric for getting both the uniform estimate of approximate solutions and the regularity of a weak solution, we will use rather the given orbifold metric $g$ itself than an asymptotically cylindrical metric $\bar{g} \in [g]|_X$ on $X := M - \{\bar{p}\}$ with $\bar{g} = r^{-2} \cdot g$ near the singularity $\bar{p}$, where $r(\cdot) := \dist_g(\cdot, \bar{p})$.

First, note that

$Y(M, [g]_{orb}) = \inf_{u \in C^\infty_c(X), u \neq 0} Q_{(X, \tilde{g})}(u).$

Let $B_\rho$ be the open geodesic ball centered at $\bar{p}$ of radius $\rho > 0$ with respect to $g$. Set

$Y_i := \inf_{u \in C^\infty_c(X - B_{1/i}), u \neq 0} Q_{(X, g)}(u)$

for $i \in \mathbb{N}$. We have that

$Y_i > Y_{i+1} > Y_{i+2} > \cdots,$

$\lim_{i \to \infty} Y_i = \inf_{u \in C^\infty_c(X), u \neq 0} Q_{(X, g)}(u) = Y(M, [g]_{orb}).$

It then follows from the strict inequality (1) and the above that there exists a large integer $i_0$ such that

$Y_i < Y(S^n)/|\Gamma|^{2/n} < Y(S^n)$ for any $i \geq i_0$.

Similarly to the case for compact manifolds without boundary, this implies that there exists a non-negative $Q_{(X - B_{1/i}, g)}$-minimizer $u_i \in C^\infty(X - B_{1/i})$ such that, for each $i \geq i_0$,

$Q_{(X - B_{1/i}, g)}(u_i) = Y_i, \quad \int_{X - B_{1/i}} u_i^{2n} d\mu_g = 1,$

$u_i = 0$ on $\partial B_{1/i}, \quad u_i > 0$ in $X - B_{1/i}$.

We denote the zero extension of each $u_i$ to $M$ by also the same symbol $u_i$. 

Suppose that the sequence \( \{u_i\} \) has a uniform \( C^0 \)-bound, that is, there exists a constant \( L > 0 \) such that
\[
\|u_i\|_{C^0(M)} \leq L \quad \text{for} \quad i \geq i_0.
\]
Under this uniform \( C^0 \)-estimate, then there exists a non-negative \( Q_{(M,g)} \)-minimizer \( u \in W^{1,2}(M; g) \) with \( \|u\|_{C^0(M)} \leq L \) such that (taking a subsequence if necessary) \( u_i \to u \) weakly in \( W^{1,2}(M; g) \), \( u_i \to u \) strongly in \( L^2(M; g) \).

Lebesgue’s bounded convergence theorem combined with the above uniform \( C^0 \)-estimate for \( \{u_i\} \) implies that
\[
\int_M u \frac{\partial u}{\partial g} \, d\mu_g = 1.
\]
By this equation and the fact that \( \{u_i\} \) is a \( Q_{(M,g)} \)-minimizing sequence, we have \( u_i \to u \) strongly in \( W^{1,2}(M; g) \).

Under the \( C^0 \)-estimate \( \|u\|_{C^0(M)} \leq L \), applying the standard elliptic \( L^p \)-estimates to the Euler-Lagrange equations for \( u \) on \( X \) and the lifting \( (\varphi^{-1} \circ \pi)^* u \) on \( \mathbb{R}^n(0) \), we obtain that \( u \in C^\infty(M) \). Here, \( u \in C^\infty(M) \) means that \( u \in C^\infty(X) \) and the lifting \( (\varphi^{-1} \circ \pi)^* u \) is smooth on \( \mathbb{R}^n(0) \). The maximum principle \([3, \text{Proposition 3.75}]\) implies that \( u > 0 \) everywhere on \( M \), and then we get an orbifold Yamabe metric
\[
\tilde{g} := u^{4/(n-2)} \cdot g \in [g]_{\text{orb}}.
\]

To complete the proof, we need only to show a uniform \( C^0 \)-estimate for the sequence \( \{u_i\} \). For each \( u_i \), take a maximum point \( q_i \in X \) of \( u_i \), and set \( m_i := u_i(q_i) \). Taking a subsequence if necessary, we then have that there exists a point \( q_\infty \in M \) such that
\[
\lim_{i \to \infty} q_i = q_\infty.
\]
Suppose that
\[
\lim_{i \to \infty} m_i = \infty.
\]
Then, we will lead to a contradiction as below.

Case 1. \( q_\infty \neq \tilde{p} \): Let \( \{V, x = (x^1, \cdots, x^n)\} \) be a geodesic normal coordinate system centered at \( q_\infty \) satisfying \( V \subset X \). We may assume that \( \{|x| < 1\} \subset V \). Set
\[
v_i(x) := m_i^{-1} \cdot u_i\left(m_i^{\frac{-2}{n-2}} \cdot x + x(q_i)\right) \quad \text{for} \quad x \in \{|x| < m_i^{\frac{-2}{n-2}}(1 - |x(q_i)|)\}.
\]
Similarly to the proof of Theorem 2.1 in \([40, \text{Chapter 5}]\), there exists a positive function \( v \in C^\infty(\mathbb{R}^n) \) such that
\[
v_i \to v \quad \text{in the} \; C^2 \text{-topology on each relatively compact domain in} \; \mathbb{R}^n.
\]

Hence, \( v \) satisfies the following:
\[
-\alpha_n \Delta_0 v = Y(M, [g]_{\text{orb}}) \cdot v^{\frac{n+2}{n-2}} \quad \text{on} \; \mathbb{R}^n,
\]
\[
\int_{\mathbb{R}^n} v^{\frac{n+2}{n-2}} \, dx \leq \liminf_{i \to \infty} \int_{V} u_i^{\frac{2n}{n-2}} \, d\mu_g \leq 1,
\]
where \( \Delta_0 \) denotes the Laplacian with respect to the Euclidean metric. This implies that \( Y(M, [g]_{\text{orb}}) \geq Y(S^n) \), and then it contradicts to the assumption (1).
Case 2: $q_{\infty} = \tilde{p}$: In this case, we consider rather the liftings $\tilde{u}_i := (\varphi^{-1} \circ \pi)^* u_i$ on $B_\tau(0)$ than $u_i$ themselves. Similarly to the above, set

$$\tilde{v}_i(x) := m_i^{-1} \cdot \tilde{u}_i(m_i^{-\frac{2}{n}} \cdot x + x(q_i)) \quad \text{for} \quad x \in \{ x \in \mathbb{R}^n \mid |x| < m_i^{-\frac{2}{n}} \tau - |x(q_i)| \}.$$  

Then, there exists a positive function $\tilde{v} \in C^\infty(\mathbb{R}^n)$ such that $\tilde{v}_i \to \tilde{v}$ in the $C^2$-topology on each relatively compact domain in $\mathbb{R}^n$.

Moreover, $\tilde{v}$ satisfies the following:

$$-\alpha_n \Delta_{\tilde{g}} \tilde{v} = Y(M, [\tilde{g}]_{\text{orb}}) \cdot \tilde{v} \quad \text{on} \quad \mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} \tilde{v} \frac{2m}{\tilde{v}} \frac{\alpha_n}{\tilde{g}} dx \leq \liminf_{i \to \infty} \int_{B_\tau(0)} \tilde{u}_i \frac{2m}{\tilde{u}_i} d\mu_{\tilde{g}} \leq |\Gamma|,$$

where $\tilde{g} := (\varphi^{-1} \circ \pi)^* g$. This implies that

$$Y(M, [\tilde{g}]_{\text{orb}}) \geq \frac{Y(S^n)}{|\Gamma|^{2/n}},$$

and then it also contradicts to the assumption (1). \hfill \Box

We can now prove the second assertion in Theorem A.

**Proof of the second assertion in Theorem A.** First, we note that

$$Y(M_1, [\tilde{g}]_{\text{orb}}) \leq Y_{\text{orb}}(M_1) \leq Y(N) \leq 0.$$  

It then follows from Theorem 3.1 and the above inequality that there exists a constant scalar curvature orbifold metric $\tilde{g} \in [\tilde{g}]_{\text{orb}}$ satisfying

$$E(\tilde{g}) = Y(M_1, [\tilde{g}]_{\text{orb}}) \leq 0.$$  

Similarly to the case for smooth conformal manifolds, the uniqueness of constant scalar curvature orbifold metrics in a non-positive orbifold conformal class [5, Lemma 2.3] implies that, up to a scaling,

$$\tilde{g} = \tilde{g}.$$  

Combining the above with the assumption $E(\tilde{g}) = Y(N)$, we then have

$$Y(N) = E(\tilde{g}) = Y(M_1, [\tilde{g}]_{\text{orb}}) \leq Y_{\text{orb}}(M_1) \leq Y(N).$$  

This implies that $Y_{\text{orb}}(M_1) = Y(M_1, [\tilde{g}]_{\text{orb}}) = Y(N)$. \hfill \Box

### 4. Exact computations

We first prove Theorem B.

**Proof of Theorem B.** First, note that the canonical model $X_{\text{can}}$ is obtained by blowing down each connected component of the union of the $(-2)$-curves in $X$ into a point. The structure of an open neighborhood of each singular point in $X_{\text{can}}$ is modeled by one of A-D-E singularities, that is, the quotient singularity $\mathbb{C}^2/\Gamma$ with a non-trivial finite subgroup $\Gamma < SU(2)$. Then, $X_{\text{can}}$ admits a Kähler-Einstein orbifold metric $\tilde{g}$ [20] satisfying

$$E(\tilde{g}) = 4\sqrt{2}\pi \sqrt{2\chi(X) + 3\tau(X)}.$$  

We denote the singularities of $X_{\text{can}}$ by $\{ (\tilde{p}_1, \Gamma_1), \cdots, (\tilde{p}_\ell, \Gamma_\ell) \}$. 

For each $\Gamma_j$, let $X_j$ denote the minimal resolution of $\mathbb{C}^2/\Gamma_j$. Then, each $X_j$ admits a hyperKähler ALE metric $h_j$ [22], and $(X_j, h_j)$ has a smooth conformal compactification $(\hat{X}_j := X_j \sqcup \{\infty_j\}, \hat{h}_j)$ with singularity $\{(\infty_j, \Gamma_j)\}$ [13, 14], which has a positive Yamabe constant $Y(\hat{X}_j, [\hat{h}_j]_{\text{orb}}) > 0$.

With these understandings, $X$ can be decomposed by

$$X = X_{\text{can}} \# \bigcup_{j=1}^l \{S^3/\Gamma_j\}(\bigcup_{j=1}^l \hat{X}_j).$$

By Theorem A, this combined with $Y(X) < 0$ and $Y^{\text{orb}}(\hat{X}_j) > 0$ implies

$$Y^{\text{orb}}(X_{\text{can}}) \leq Y(X) < 0.$$ 

Recall that the Kähler-Einstein orbifold metric $\hat{g}$ satisfies

$$E(\hat{g}) = -4\sqrt{2}\pi \sqrt{2}\chi(X) + 3\tau(X) = Y(X).$$

This gives the desired conclusion:

$$Y^{\text{orb}}(X_{\text{can}}) = Y(X_{\text{can}}, [\hat{g}]_{\text{orb}}) = Y(X).$$

Finally, we give two more typical exact computations of the orbifold Yamabe invariant.

1. Let $T$ be a complex 2-dimensional torus and $\hat{T} := T/(\text{id}, \iota)$ the quotient 4-orbifold with 16-singularities $\{(\bar{p}_1, (\text{id}, \iota)), \ldots, (\bar{p}_{16}, (\text{id}, \iota))\}$. Here, $\langle (\text{id}, \iota), \langle \hat{g}(\bar{p}_i) + \langle \hat{g}(\bar{p}_j) \rangle = 0 \rangle$. By Theorem A, we

\[ Y(\hat{T}), [\hat{g}]_{\text{orb}}) = Y(X_{\text{can}}, [\hat{g}]_{\text{orb}}) = Y(X). \]

**Proof.** Let $\mathcal{O}(-2)$ denote the complex line bundle over the complex projective line $\mathbb{C}P^1$ of degree $-2$. Then, there exists a cylindrical metric $\hat{g}$ on $\mathcal{O}(-2)$ modeled by $(S^3/(\text{id}, \iota), h_0)$ with positive scalar curvature $R_{\hat{g}} > 0$ (cf. [30, Example 4.1.27]). Hence, $(\mathcal{O}(-2), \hat{g})$ has a smooth conformal compactification $(\mathcal{O}(-2) := \mathcal{O}(-2) \sqcup \{\infty\}, \hat{g})$ with singularity $\{(\infty, (\text{id}, \iota))\}$. Note that, from the uniform positivity of $R_{\hat{g}}$ and the Sobolev embedding $W^{1,2}(\mathcal{O}(-2); \hat{g}) \hookrightarrow L^4(\mathcal{O}(-2); \hat{g})$,

$$Y(\mathcal{O}(-2), [\hat{g}]_{\text{orb}}) = Y(\mathcal{O}(-2), [\hat{g}]_{\text{orb}}) > 0.$$ 

Let $(N_1, H_1), \ldots, (N_{16}, H_{16})$ be the 16-copies of $(\mathcal{O}(-2), (\text{id}, \iota))$. With these understandings, the generalized connected sum

$$X := \hat{T} \# \bigcup_{j=1}^l (S^3/H_j)(\bigcup_{j=1}^l N_j)$$

is diffeomorphic to the Kummer surface, and hence $Y(X) = 0$. By Theorem A, we then have

$$Y^{\text{orb}}(\hat{T}) \leq Y(X) = 0.$$ 

Note that

$$E(\hat{g}_{\text{flat}}) = 0 = Y(X),$$

and hence

$$Y^{\text{orb}}(\hat{T}) = Y(\hat{T}, [\hat{g}_{\text{flat}}]_{\text{orb}}) = Y(X) = 0. \qed$$
Proposition 4.2. Let \( \Sigma \) be an exotic sphere of dimension \( n := 8k + 2 \geq 10 \) with \( \alpha(\Sigma) \neq 0 \), where \( \alpha \) is the \( \alpha \)-homomorphism from the spin cobordism group \( \Omega_n^{spin} \) to the \( KO \)-group \( KO^{-n}(pt) \cong \mathbb{Z}_2 \) (cf. [23, Chapter 2]). For any integer \( \ell \geq 2 \), set

\[
G_\ell \coloneq \{ \zeta^j I \in GL(4k + 1; \mathbb{C}) \mid j = 0, \cdots, \ell - 1 \}, \quad \zeta \coloneq \exp(2\pi \sqrt{-1}/\ell) \in \mathbb{C},
\]

where \( I \) denotes the identity matrix. The finite group \( G_\ell \) acts the \( n \)-sphere \( S^n \subset \mathbb{R}^{n+1} = \mathbb{C}^{4k+1} \times \mathbb{R} \) by

\[
A : \mathbb{C}^{4k+1} \times \mathbb{R} \to \mathbb{C}^{4k+1} \times \mathbb{R}, \quad (z, t) \mapsto A \cdot (z, t) \coloneq (A \cdot z, t) \quad \text{for} \quad A \in G_\ell.
\]

Then, the quotient space \( S^n/G_\ell \) is a compact \( n \)-orbifold with two singularities \( \{ (\tilde{p}_+ \coloneq [(0, \cdots, 0, 1)], G_\ell^+ \coloneq G_\ell), (\tilde{p}_- \coloneq [(0, \cdots, 0, -1)], G_\ell^- \coloneq G_\ell) \} \). Pushing down the standard metric \( g_0 \) on \( S^n \) to \( S^n/G_\ell \), we have an orbifold metric \( \tilde{g}_0 \) of constant curvature one on \( S^n/G_\ell \). Note that the space \((S^n/G_\ell) - \{ \tilde{p}_+, \tilde{p}_- \}, \tilde{g}_0 \) is conformal to the product space \((S^{n-1}/G_\ell) \times \mathbb{R}, \tilde{g} \coloneq h_0 + dt^2 \). Then, this combined with \( R_{\tilde{g}} = R_{h_0} = (n - 1)(n - 2) > 0 \) and the Sobolev embedding \( W^{1,2}((S^{n-1}/G_\ell) \times \mathbb{R}; \tilde{g}) \to L^{2n/(n-2)}((S^{n-1}/G_\ell) \times \mathbb{R}; \tilde{g}) \) implies that

\[
Y^{orb}(S^n/G_\ell) \geq Y(S^n/G_\ell, [\tilde{g}_0]_{orb}) = Y((S^{n-1}/G_\ell) \times \mathbb{R}, [\tilde{g}]) > 0.
\]

Proposition 4.2. \( Y^{orb}(\Sigma\#(S^n/G_\ell)) = 0 \).

Here, \( \Sigma\#(S^n/G_\ell) \) stands for the connected sum of \( \Sigma \) and \( S^n/G_\ell \) in the usual sense.

Proof. We first note the following. By results of Lichnerowicz and Hitchin (cf. [23, Chapters 2, 4]) for \( \alpha : \Omega_n^{spin} \to \mathbb{Z}_2 \), \( Y(\Sigma) \leq 0 \). On the other hand, Petean [32] proved that any simply connected compact manifold of dimension greater than \( 4 \) has a non-negative Yamabe invariant. Hence, we have

\[
Y(\Sigma) = 0.
\]

Let

\[
N_\ell \coloneq (S^n/G_\ell)\#(S^{n-1}/G_\ell^+ \cup (S^{n-1}/G_\ell^-))(S^n/G_\ell)
\]

denotes the generalized connected sum. Here, \( S^n/G_\ell \) is the same \( n \)-orbifold, but equipped with the opposite orientation. It turns out that

\[
N_\ell = (S^{n-1}/G_\ell) \times S^3,
\]

and then it is a compact spin \( n \)-manifold with positive Yamabe invariant. Then, the positivity \( Y(N_\ell) > 0 \) implies that \( \alpha([N_\ell]) = 0 \), and hence \( \alpha(\Sigma\#N_\ell) = \alpha(\Sigma) + \alpha([N_\ell]) \neq 0 \). Therefore,

\[
Y(\Sigma\#N_\ell) = 0.
\]

We now decompose \( \Sigma\#N_\ell \) as the generalized connected sum

\[
\Sigma\#N_\ell = (\Sigma\#(S^n/G_\ell))\#(S^{n-1}/G_\ell^+ \cup (S^{n-1}/G_\ell^-))(S^n/G_\ell).
\]

It then follows from Theorem A combined with (2), (3) that

\[
Y^{orb}(\Sigma\#(S^n/G_\ell)) \leq Y(\Sigma\#N_\ell) = 0.
\]

On the other hand, Kobayashi’s inequality for \( Y^{orb}(\Sigma\#(S^n/G_\ell)) \) still holds. Hence,

\[
0 = Y(\Sigma) = \min \{ Y(\Sigma), Y^{orb}(S^n/G_\ell) \} \leq Y^{orb}(\Sigma\#(S^n/G_\ell)).
\]
The inequalities (4), (5) give the desired conclusion:

\[ Y^\text{orb}(\Sigma\#(S^n/G_\ell)) = 0. \]

\[ \Box \]

References

[1] K. Akutagawa, *Yamabe metrics of positive scalar curvature and conformally flat manifolds*, Diff. Geom. Appl. 4 (1994), 239–258.

[2] K. Akutagawa and B. Botvinnik, *Yamabe metrics on cylindrical manifolds*, Geom. Funct. Anal. 13 (2003), 259–333.

[3] K. Akutagawa and B. Botvinnik, *The Yamabe invariants of orbifolds and cylindrical manifolds, and \(L^2\)-harmonic spinors*, J. Reine Angew. Math. 574 (2004), 121–146.

[4] K. Akutagawa, L. A. Florit and J. Petean, *On Yamabe constants of Riemannian products*, Comm. Anal. Geom. 15 (2007), 947–969.

[5] K. Akutagawa and A. Neves, *3-manifolds with Yamabe invariant greater than that of \(\mathbb{RP}^3\)*, J. Differential Geom. 75 (2007), 359–386.

[6] B. Ammann, M. Dahl and E. Humbert, *Smooth Yamabe invariant and surgery*, arXiv:math.DG/08041418, v3 (2008).

[7] M. T. Anderson, *Canonical metrics on 3-manifolds and 4-manifolds*, Asian J. Math. 10 (2006), 127–163.

[8] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. 55 (1976), 269–296.

[9] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, 1998.

[10] A. Besse, *Einstein Manifolds*, Springer, 1987.

[11] H. Bray and A. Neves, *Classification of prime 3-Manifolds with Yamabe invariant greater than \(\mathbb{RP}^3\)*, Ann. of Math. 159 (2004), 407–424.

[12] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), 486–490.

[13] X. Chen, C. LeBrun and B. Weber, *On conformally Kähler, Einstein manifolds*, J. Amer. Math. Soc. 21 (2008), 1137–1168.

[14] M. Gromov and H. B. Lawson Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. 111 (1980), 423–434.

[15] M. Gromov and H. B. Lawson Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. 58 (1983), 83–196.

[16] M. Gursky and C. LeBrun, *Yamabe invariants and spin\(^c\) structures*, Geom. Funct. Anal. 8 (1998), 965–977.

[17] M. Ishida and C. LeBrun, *Curvature, connected sums, and Seiberg-Witten theory*, Comm. Anal. Geom. 11 (2003), 809–836.

[18] O. Kobayashi, *On large scalar curvature*, Research Report 11, Dept. Math. Keio Univ., 1985.

[19] O. Kobayashi, *The scalar curvature of a metric with unit volume*, Math. Ann. 279 (1987), 253–265.

[20] R. Kobayashi, *A remark on the Ricci curvature of algebraic surfaces of general type*, Tohoku Math. J. 36 (1984), 385–399.

[21] N. Koiso, *On the second derivative of the total scalar curvature*, Osaka J. Math. 16 (1979), 413–421.

[22] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. 29 (1989), 605–683.

[23] H. B. Lawson Jr. and M. L. Michelson, *Spin Geometry*, Princeton Univ. Press, 1987.

[24] C. LeBrun, *Einstein metrics and Mostow rigidity*, Math. Res. Lett. 2 (1995), 1–8.

[25] C. LeBrun, *Four-manifolds without Einstein metrics*, Math. Res. Lett. 3 (1996), 133–147.
[26] C. LeBrun, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. Anal. Geom. 5 (1997), 535–553.

[27] C. LeBrun, *Kodaira dimension and the Yamabe problem*, Comm. Anal. Geom. 7 (1999), 133–156.

[28] J. Lee and T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. 17 (1987), 37–81.

[29] H. Nakajima, *Self-duality of ALE Ricci-flat manifolds and positive mass theorem*, Recent Topics in Differential and Analytic Geometry, Advanced Studies in Pure Math. 18-I (1990), 313–349.

[30] L. Nicolaescu, *Notes on Seiberg-Witten theory*, Grad. Stud. Math. 28, Amer. Math. Soc., Providence, RI, 2000.

[31] J. Petean, *Computations of the Yamabe invariant*, Math. Res. Lett. 5 (1998), 703–709.

[32] J. Petean, *The Yamabe invariant of simply connected manifolds*, J. Reine Angew. Math. 523 (2000), 225–231.

[33] J. Petean and G. Yun, *Surgery and the Yamabe invariant*, Geom. Funct. Anal. 9 (1999), 1189–1199.

[34] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Diff. Geom. 20 (1984), 479–495.

[35] R. Schoen, *Recent progress in geometric partial differential equations*, Proceedings of the International Congress of Mathematicians, (Berkeley, Calif., 1986), 121–130, Amer. Math. Soc., Providence, RI, 1987.

[36] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in Calculus of Variations, Lect. Notes in Math. 1365, 121–154, Springer, 1989.

[37] R. Schoen and S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Ann. of Math. 110 (1979), 127–142.

[38] R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979), 159–183.

[39] R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. 92 (1988), 47–71.

[40] R. Schoen and S.-T. Yau, *Lectures on Differential Geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology I, International Press, 1994.

[41] S. Stolz, *Positive scalar curvature metrics - Existence and classification questions*, Proceedings of the International Congress of Mathematicians, (Zürich, 1994), 625–636, Birkhäuser, 1995.

[42] G. Tian and J. Viaclovsky, *Moduli spaces of critical Riemannian metrics in dimension four*, Advances in Math. 190 (2005), 346–372.

[43] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 265–274.

[44] J. Viaclovsky, *Monopole metrics and the orbifold Yamabe problem*, preprint.

[45] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka J. Math. 12 (1960), 21–37.