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The Mackey-Arens and Hahn-Banach theorems for spaces over valued fields

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Abstract. Characterizations of the spherical completeness of a non-archimedean complete non-trivially valued field in terms of classical theorems of Functional Analysis are obtained.

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Spherical completeness

Throughout this paper $K = (K, \vert \cdot \vert)$ will denote a non-archimedean complete valued field with a non-trivial valuation $\vert \cdot \vert$. It is well-known that the absolute value function $\vert \cdot \vert$ of the field of the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ satisfies the following properties:

(i) $0 \leq |x|$, $|x| = 0$ iff $x = 0$,
(ii) $|x + y| \leq |x| + |y|$,
(iii) $|xy| = |x||y|$, $x, y \in \mathbb{R}$ or $x, y \in \mathbb{C}$.

If $K$ is a field, then by a valuation on $K$ we will mean a map $\vert \cdot \vert$ of $K$ into $\mathbb{R}$ satisfying the above properties; in this case $(K, \vert \cdot \vert)$ will be called a valued field. We will assume that $K$ is complete with respect to the natural metric of $K$.

It turns out that if $K$ is not isomorphic to $\mathbb{R}$ or $\mathbb{C}$, then its valuation satisfies the following strong triangle inequality, cf. e.g. [12],

(ii') $|x + y| \leq \max \{|x|, |y|\}$, $x, y \in K$.

A valued field $K$ whose valuation satisfies (ii') will be called non-archimedean and its valuation non-archimedean.

Let us first recall the following well-known result of Cantor

Theorem 0 Let $(X, \rho)$ be a metric space. Then it is complete iff every shrinking sequence of closed balls whose radii tend to zero has non-empty intersection.
Consider the set $\mathbb{N}$ of the natural numbers endowed with the following metric $\rho$ defined by $\rho(m, n) = 0$ if $m = n$ and $1 + \max\left(\frac{1}{m}, \frac{1}{n}\right)$ if $m \neq n$.

Then the metric $\rho$ is non-archimedean, i.e. $\rho(m, n) = 0$ iff either $m = n$, or $\rho(m, n) \leq \max\{\rho(m, k), \rho(k, n)\}$, for all $m, n, k \in \mathbb{N}$.

It is easy to see that every shrinking sequence of balls in $\mathbb{N}$ whose radii tend to zero has non-empty intersection; note that every ball whose radius is smaller than 1 contains exactly one point. On the other hand, the balls $B_{1+\frac{1}{1}}(1), B_{1+\frac{1}{2}}(2), \ldots$, form a decreasing sequence and their intersection is empty. This suggests the following, see Ingleton [3]:

A non-archimedean metric space $(X, \rho)$ will be said to be spherically complete if the intersection of every shrinking sequence of its balls is non-empty.

Clearly spherical completeness implies completeness; the converse fails: The space $(\mathbb{N}, \rho)$ is complete but not spherically complete. We refer to [11] and [12] for more information concerning this property.

**Theorem 1** Let $(X, \rho)$ be a non-archimedean metric space. Then $(X, \rho)$ is spherically complete iff given an arbitrary family $B$ of balls in $X$, no two of which are disjoint, then the intersection of the elements of $B$ is non-empty.

The aim of this note is to collect a few characterizations of the spherical completeness of $K$ in terms of the Mackey-Arens, Hahn-Banach and weak Schauder basis theorems, respectively, see [5], [6], [7], [12].

The Mackey-Arens and Hahn-Banach theorems

The terms "$K$-space", "topology","seminorm or norm" will mean a Hausdorff locally convex space (lcs) over $K$, a locally convex topology (in the sense of Monna) and a non-archimedean seminorm (norm), respectively. A seminorm on a vector space $E$ over $K$ is non-archimedean if it satisfies condition (ii'). Clearly the topology $\tau$ generated by a norm is locally convex. Recall that a topological vector space (tvs) $E$ over $K$ is locally convex [10] if $\tau$ has a basis of absolutely convex neighbourhoods of zero. A subset $U$ of $E$ is absolutely convex (in the sense of Monna [10]) if $\alpha x + \beta y \in U$, whenever $x, y \in U$, $\alpha, \beta \in K$, $|\alpha| \leq 1, |\beta| \leq 1$. For the basic notions and properties concerning tvs and lcs over $K$ we refer to [10], [11], [13].

A locally convex (lc) topology $\gamma$ on $(E, \tau)$ is called compatible with $\tau$, if $\tau$ and $\gamma$ have the same continuous linear functionals; $(E, \tau)^* = (E, \gamma)^*$. $(E, \tau)$ is dual-separating if $(E, \tau)^*$ separates points of $E$. If $G$ is a vector subspace of $E$, $\tau|G$ and $\tau/G$ denote the topology $\tau$ restricted to $G$ and the quotient topology of the quotient space $E/G$, respectively. If $\alpha$ is a finer l.c. topology on $E/G$, we denote by $\gamma := \tau \vee \alpha$ the weakest l.c. topology on $E$ such that $\tau \leq \gamma$, $\gamma/G = \alpha$, $\gamma|G = \tau|G$, cf. e.g. [1]. The sets $U \cap q^{-1}(V)$ compose a basis of neighbourhoods of zero for $\gamma$, where $U, V$ run over bases of neighbourhoods of zero for $\tau$ and $\alpha$, respectively, $q : E/E/G$ is the quotient map. By sup{$\tau, \alpha$} we denote the weakest l.c. topology on $E$ which is finer than $\tau$ and $\alpha$. 


By the Mackey topology $\mu(E, E^*)$ associated with a lcs $E = (E, \tau)$ we mean the finest locally convex topology on $E$ compatible with $\tau$. In [14] Van Tiel showed that every lcs over spherically complete $K$ admits the Mackey topology.

In [3] Ingleton obtained a non-archimedean variant of the Hahn-Banach theorem for normed spaces, where $K$ is spherically complete.

**Theorem 2** If $E = (E, \| \cdot \|)$ is a normed space over $K$ and $K$ is spherically complete and $D$ is a subspace of $E$, then for every continuous linear functional $g \in D^*$ there exists a continuous linear extension $f \in E^*$ of $g$ such that $\|g\| = \|f\|$. This suggests the following: A lcs $E$ will be said to have the Hahn-Banach Extension Property (HBEP) [9] if for every subspace $D$ every $g \in D^*$ can be extended to $f \in E^*$. It is known that every lcs over spherically complete $K$ has the HBEP, cf. e.g. [11].

The following theorem characterizes the spheric completeness of $K$ in terms of classical theorems of Functional Analysis; cf. also [5], [6] and [12], Theorem 4.15. The proof of our Theorem 3 uses some ideas of [4] extended to the non-archimedean case.

$I^\infty$ (resp. $c_0$) denotes the space of the bounded sequences (resp. the sequences of limit 0) with coefficients in $K$.

**Theorem 3** The following conditions on $K$ are equivalent:

(i) $K$ is spherically complete.

(ii) There exists $g \in (I^\infty)^*$ such that $g(x) = \sum_n x_n$ for every $x \in c_0$.

(iii) $(I^\infty/c_0)^* \neq 0$.

(iv) Every lcs over $K$ admits the Mackey topology.

(v) Every lcs over $K$ (resp. $K$-normed space) has the HBEP.

(vi) The completion of a dual-separating lcs over $K$ (resp. $K$-normed space) is dual-separating.

(vii) Every closed subspace of a dual-separating lcs over $K$ (resp. $K$-normed space) is weakly closed.

(viii) For every lcs over $K$ (resp. $K$-normed space) every weakly convergent sequence is convergent.

(ix) Every weak Schauder basis in a lcs over $K$ (resp. $K$-normed space) is a Schauder basis.

**Proof** By Theorem 4.15 of [12] conditions (i), (ii), (iii) are equivalent. (i) implies (iv): [14], Theorem 4.17. (i) implies (v): [3], [11]. The implications (v) implies (vi), (v) implies (vii) are obvious. (i) implies (viii): see [7], Theorem 3, [2], Proposition 4.3. (viii) implies (ix) is obvious.

(iv) implies (i): Assume that $K$ is not spherically complete and consider the space $I^\infty$ of $K$-valued bounded sequences endowed with the topology $\tau$ generated by the norm $\|x\| = \sup_n |x_n|$, $x = (x_n) \in I^\infty$. Let $f$ be a non-zero linear function on $I^\infty$ with $f|_{c_0} = 0$. Set $E := I^\infty$ and $F := c_0$. Define a linear functional $h$ on the quotient space $E/F$ by $h(q(x)) = f(x)$, where $q : E \rightarrow E/F$ is the quotient map. Let $\alpha$ be the quotient topology.
of \( E/F \). Since \((E/F, \alpha)^* = 0\), see (iii) implies (i), \( F \) is dense in the weak topology \( \sigma(E, E^*) \) (recall that \( E^* = F \), \[12\], Theorem 4.17). Observe that on \( E/F \) there exists a \( K \)-normed topology \( \beta \) such that \((E/F, \alpha) \) and \((E/F, \beta) \) are isomorphic and \( h \) is continuous in the topology \( \sup \{ \alpha, \beta \} \). Indeed, choose \( x_0 \in E/F \) such that \( h(x_0) = 2 \) and define a linear map \( T : E/F \to E/F \) by \( T(x) := x - h(x)x_0, x \in E/F \). Then \( T^2 = id \). Define \( \beta := T(\alpha) \) (the image topology). Then \( h \) is continuous in the topology \( \sup \{ \alpha, \beta \} \).

Set \( \gamma_\alpha := \sigma(E, E^*) \vee \alpha, \quad \gamma_\beta := \sigma(E, E^*) \vee \beta \). Then \( \gamma_\alpha \) and \( \gamma_\beta \) are compatible with \( \sigma(E, E^*) \), hence with \( \tau \). Assume that \( E \) admits the finest locally convex topology \( \mu \) compatible with \( \tau \). Then \( \sigma(E, E^*) \leq \sup \{ \gamma_\alpha, \gamma_\beta \} \leq \mu \).

On the other hand \( \sup \{ \gamma_\alpha, \gamma_\beta \}/F = \sup \{ \alpha, \beta \} \). Therefore \( f \) is continuous in \( \sup \{ \gamma_\alpha, \gamma_\beta \} \). Since \( f \) is not continuous in \( \sigma(E, E^*) \) we get a contradiction. The proof is complete.

(vi) implies (i) : Assume that \( K \) is not spherically complete. By the Baire category theorem we find a dense subspace \( G \) of \( E \) with \( \dim(E/G) = \dim(E/F) \), where \( E \) and \( F \) are defined as above. Indeed, let \( \{ x_s \}_{s \in S} \) be a Hamel basis of \( E \) and \( (S_n) \) a partition of \( S \) such that \( S = \bigcup_{n \in \mathbb{N}} S_n \) and \( \text{card } S_n = \text{card } S, n \in \mathbb{N} \).

For every \( n \in \mathbb{N} \), we denote by \( G_n \) the vector space generated by the elements \( x_s \) when \( s \) runs in \( \bigcup_{k=1}^n S_k \). Then we have \( E = \bigcup G_n \) and \( \dim G_n = \dim (E/G_n) = \dim E, n \in \mathbb{N} \).

Then there exists \( m \in \mathbb{N} \) such that \( G_m \) is dense in \( E \). Hence we obtain a subspace \( G \) as required. Let \( \alpha \) be a \( K \)-normed topology on \( E/G \) such that the spaces \( (E/G, \alpha) \) and \( (E/F, \tau/F) \) are isomorphic. Then the topology \( \gamma := \tau \vee \alpha \) is compatible with \( \tau \) and strictly finer than \( \tau \). Let \( E_0 \) be the completion of the dual-separating \( K \)-normed space \((E, \gamma) \). Choose \( x \in E_0 \setminus E \). There exists a sequence \( (x_n) \) in \( E \) and \( y \in E \) such that \( x_n \to x \) in \( E_0 \) and \( x_n \to y \) in \( (E, \tau) \). Then \( f(x - y) = 0 \) for all \( f \in E^*_0 \) but \( x - y \neq 0 \). This completes the proof.

(vii) implies (i) : Assume that \( K \) is not spherically complete. The space \( G \) constructed in the previous case is closed in \((E, \gamma) \) and dense in \((E, \sigma(E, E^*)) \), where \( E^* := (E, \gamma)^* \).

(v) implies (i) : Assume that \( K \) is not spherically complete. Let \( (e_n) \) be the sequence of the unit vectors in \( E \), where \( E \) is as above. Then \( e_n \to 0 \) in \( \sigma(E, E^*) \), \[13\]. Clearly \( (e_n) \) is a normalized Schauder basis in \( F \). If \( x = (x_n) \in F \), then \( x = \sum_n x_n e_n \). Set \( g(x) := \sum_n x_n \). Then \( g \) is a well-defined continuous linear functional on \( F \). Suppose that \( g \) has a continuous linear extension \( f \) to the whole space \( E \). Then \( f(e_n) \to 0 \) but \( g(e_n) = 1 \) for all \( n \in \mathbb{N} \), a contradiction.

(viii) implies (i) : See the proof of the previous implication.

(ix) implies (i) : Assume that \( K \) is not spherically complete. The sequence \( (e_n) \) is a Schauder basis in \((E, \sigma(E, E^*)) \) but it is not a Schauder basis in the original topology of \( E \). The second part of this sentence follows from the fact that \( E \) is not of countable type, cf. e.g. \[12\]. On the other hand, by Theorem 4.17 of \[12\] (and its proof) the space \( E \) is reflexive and for every \( g \in E^* \) there exists \( (a_n) \in F \) such that \( g(x) = \sum_n x_n a_n \) for every...
\( x = (x_n) \in E \). Since \((E, \sigma(E, E^*))\) is a sequentially complete lcs \([12]\), Theorem 9.6, then 
\[ \sum_{k=1}^{n} x_k e_k \] weakly converges to \( x = (x_n) \).

**Remark** In \([9]\) Martinez-Maurica and Perez-Garcia proved that whenever \( K \) is spherically complete, then the local convexity is a *three space property*, i.e. if \( E \) is an A-Banach tvs over \( K \) and \( F \) its subspace such that \( F \) and \( E/F \) are locally convex, then \( E \) is locally convex. Is the converse also true?

By \( L(E, F) \) we denote the space of all continuous linear maps between lcs \( E \) and \( F \). A topology \( \alpha \) on \( E \) will be called *compatible* with the pair \((E, L(E, F))\) if \( L((E, \alpha), F) = L(E, F) \); if \( F = \), as usual we shall say that \( \alpha \) is compatible with the dual pair \((E, E^*)\), where \( E^* := L(E, K) \).

A lcs space \( F \) will be said to have the *Mackey-Arens property* (MA-property) if for every lcs space \( E \) the finest topology \( \mu(E, L(E, F)) \) compatible with \((E, L(E, F))\) exists, \([7]\).

As we have already mentioned Van Tiel \([14]\) proved that if \( K \) is spherically complete, then \( K \) has the MA-property, i.e. every \( K \)-space \( E \) over spherically complete \( K \) admits the finest topology \( \mu(E, E^*) \) compatible with the dual pair \((E, E^*)\). We have already proved the converse: If \( K \) is not spherically complete, then \( \ell^\infty \) does not admit the Mackey topology \( \mu(\ell^\infty, (\ell^\infty)^*) \). Hence

**Corollary** \( K \) is spherically complete if and only if it has the MA-property.

On the other hand one has the following

**Theorem 4** Every spherically complete normed \( K \)-space \( F = (F, \|\cdot\|) \) has the MA-property.

We shall need the following

**Lemma 1** Let \( E, F \) be two vector spaces over \( K \), where \( F \) is endowed with a norm \( \|\cdot\| \) and \( p, q \) are seminorms on \( E \). Let \( T : E \to F \) be a linear map such that \( \|(T(x))\| \leq \max(p(x), q(x)) \). If \( F \) is spherically complete, then there exists two linear maps \( T_i : E \to F, \ i = 1, 2, \) such that \( T = T_1 + T_2 \) and \( \|(T_1(x))\| \leq p(x), \|(T_2(x))\| \leq q(x), \ x \in E \).

**Proof** Set \( P(x, x) = T(x), \ U(x, y) = \max\{p(x), q(y)\}, \ x, y \in E \). Then \( U(x, y) \) is a seminorm on \( E \times E \) and \( \|(P(x, x))\| = \|(T(x))\| \leq \max\{p(x), q(x)\} = U(x, x) \). Since \( F \) is spherically complete, then by Ingleton theorem, cf. e.g. \([6]\), Theorem 4.18, there exists a linear map \( P_0 : E \times E \to F \) extending \( P \) such that \( \|(P_0(x, y))\| \leq U(x, y), \ x, y \in E \). To complete the proof it is enough to put \( T_1(x) = P_0(x, 0), \ T_2(x) = P_0(0, x) \).

We shall also need the following lemma. Its proof uses some ideas of \([1]\) and \([4]\).

**Lemma 2** Let \( E, F \) be two dual-separating \( K \)-spaces over non-spherically complete \( K \) and such that \( F \) is complete and \( E \) is an infinite dimensional metrizable and complete. Then \( E \) admits two topologies \( \tau_1 \) and \( \tau_2 \) strictly finer than the original one of \( E \) and compatible with the pair \((E, L(E, F))\) and such that the topology \( \sup\{\tau_1, \tau_2\} \) is not compatible with \((E, L(E, F))\).
Proof: Observe that $E$ contains a dense subspace $G$ with $\dim(E/G) = \dim(l^\infty/c_0)$. Let $h$ be a non-zero linear functional on $E$ vanishing on $G$. As above we construct on $E$ two topologies $\tau_1$ and $\tau_2$ strictly finer than the original one $\tau$ of $E$ such that $\tau_j|G = \tau|G$ and $(E/G, \tau_j/G)$ is isomorphic to the quotient space $l^\infty/c_0$, $j = 1, 2$, and $h$ is continuous in $\sup\{\tau_1, \tau_2\}$. We show that the topologies $\tau_j$, $j = 1, 2$, are compatible with the pair $(E, L(E, F))$. Fix $j \in \{1, 2\}$ and non-zero $T \in L((E, \tau_j), F)$. There exists $x_0 \in E$ and $f \in F^*$ such that $f(T(x_0)) \neq 0$. Suppose that $T|G = \{0\}$. Then the map $q(x) \to f(Tx)$ defines a non-zero continuous linear functional on $(E/G, \tau_j/G)$, $q : E \to E/G$ is the quotient map. Since $l^\infty/c_0 \ast = \{0\}$, [12], Corollary 4.3, we get a contradiction. Hence $T|G$ is non-zero. Since $G$ is dense in $E$ and $\tau$ and $\tau_j$ coincide on $G$, there exists a continuous linear extension $W$ of $T$ to $E$. It is easy to see that $T = W$. Hence $T \in L(E, F)$. Finally the map $x \to h(x)y$, for fixed $y \in F$, defines a $\tau$-discontinuous linear map $H$ of $E$ into $F$ such that $H \in L((E, \sup \tau_1, \tau_2), F)$.

Proof of Theorem 4 Let $E = (E, \tau)$ be a lcs and $\mathcal{F}$ the family of all topologies on $E$ compatible with $(E, L(E, F))$. It is enough to show that the topology $\mu := \sup \mathcal{F}$ belongs to $\mathcal{F}$. Let $T : (E, \mu) \to F$ be a continuous linear map. There exist seminorms $p_j$ on $E$, $j = 1, \ldots, n$, continuous in topologies $\gamma_j$ ($\gamma_j \in \mathcal{F}$), respectively, and $M > 0$ such that $\|(T x)\| \leq M \max_{1 \leq j \leq n} p_j(x)$ for every $x \in E$. Using Lemma 1 one shows that $T$ is $\tau$-continuous.

Remarks (1) There exist complete normed $K$-spaces having the MA-property which are not spherically complete. In fact, assume that $K$ is spherically complete; then $\ell^\infty$ is spherically complete [12], p. 97; hence $\ell^\infty$ has the MA-property (by our Theorem 4). On the other hand there exists on the space $\ell^\infty$ another norm $\nu$ which is equivalent with the usual norm, such that $(\ell^\infty, \nu)$ is not spherically complete [12], p. 50 and p. 98. On the other hand the space $(\ell^\infty, \nu)$ has the MA-property.

(2) Let $E$ be an infinite dimensional normed and complete $K$-space. Since $F := \prod_n E_n / \bigoplus_n E_n$, where $E_n = E$ for every $n \in \mathbb{N}$, is spherically complete for any $K$ [12], Theorem 4.1, then by our Theorem 4 the space $F$ has the MA-property. For concrete spaces put $E = \ell^\infty$; then $F = \ell^\infty/c_0$. If $K$ is not spherically complete, then by Lemma 2 the space $\ell^\infty$ does not admit the Mackey topology $\mu(\ell^\infty, (\ell^\infty)^*)$ but $\ell^\infty/c_0$ has the MA-property. In particular there exists on $\ell^\infty$ the finest topology $\mu$ compatible with $(\ell^\infty, L(\ell^\infty, \ell^\infty/c_0))$.

(3) Let $E$ and $F$ be $K$-spaces and assume that $E$ admits the Mackey topology $\mu = \mu(E, E^*)$. Then the finest topology on $E$ compatible with $((E, \mu), L((E, \mu), F))$ exists and equals $\mu$.

(4) In [13], Corollary 7.9, Schikhof proved that for polarly barrelled or polarly bornological $K$-spaces $(E, \tau)$ where $K$ is not spherically complete, the finest polar topology $\mu(E, E^*)$ compatible with $(E, E^*)$ exists and equals $\tau$. 
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