FLUCTUATIONS OF $\beta$-JACOBI PRODUCT PROCESSES

ANDREW AHN

Abstract. We study Markov chains formed by squared singular values of products of truncated orthogonal, unitary, symplectic matrices (corresponding to the Dyson index $\beta = 1, 2, 4$ respectively) where time corresponds to the number of terms in the product. More generally, we consider the $\beta$-Jacobi product process obtained by extrapolating to arbitrary $\beta > 0$. When the time scaling is preserved, we show that the global fluctuations are jointly Gaussian with explicit covariances. For time growing linearly with matrix size, we show convergence of moments after suitable rescaling. When $\beta = 2$, our results imply that the right edge converges to a process which interpolates between the Airy point process and a deterministic configuration. This process connects a time-parametrized family of point processes appearing in the works of Akemann-Burda-Kieburg and Liu-Wang-Wang across time. In the arbitrary $\beta > 0$ case, our results show tightness of the particles near the right edge. The limiting moment formulas correspond to expressions for the Laplace transform of a conjectural $\beta$-generalization of the interpolating process.

1. Introduction

Let $F_\beta$ be the real, complex, quaternion skew field and $U_\beta(L)$ be the $L$-dimensional orthogonal, unitary, symplectic group for $\beta = 1, 2, 4$ respectively. In this article, we study Markov chains $(y^{(T)})_{T \in \mathbb{Z}_{>0}}$ formed by the squared singular values $y^{(T)} := (y_1^{(T)} \geq \cdots \geq y_N^{(T)})$ of matrices

$$Y_T := X_1 \cdots X_T$$

where $X_1, X_2, \ldots$ are independent random $N \times N$ matrices and time $T$ corresponds to the number of factors. The distributions of $X_1, X_2, \ldots$ are taken to be invariant under the right action of $U_\beta(N)$; thus $\beta$ indicates the symmetry class of our model. We focus on the case where the squared singular values of $X_T$ are distributed as the $\beta$-Jacobi ensemble and refer to the resulting Markov chain $(y^{(T)})_{T \in \mathbb{Z}_{>0}}$ as a $\beta$-Jacobi product process. The $\beta$-Jacobi ensemble depends on two parameters $\alpha > 0, M \in \mathbb{Z}_{>0}, M \geq N$. If we write $M = L - N' - N$, $\alpha = N' - N + 1$, then this is the squared singular value distribution of an $N' \times N$ submatrix $A$ of a Haar distributed $U_\beta(L)$ matrix. The matrix $A$ is sometimes referred to as a truncated orthogonal/unitary/symplectic matrix in the literature for $\beta = 1, 2, 4$.

Through an extrapolation procedure which combines ideas from [22] and [9], we extend the notion of a $\beta$-Jacobi product process to arbitrary $\beta > 0$. The general $\beta$-Jacobi ensemble is then a distribution parametrized by $\alpha > 0, M \in \mathbb{Z}_{>0}$ such that $M \geq N$ on $N$-particles $(x_1, \ldots, x_N)$ supported in $[0, 1]^N$ with density is proportional to

$$\prod_{1 \leq i < j \leq N} |x_i - x_j|^{\beta} \prod_{i=1}^{N} x_i^{(\beta/2)\alpha - 1} (1 - x_i)^{(\beta/2)(M-N+1)-1} dx_i$$

If $x^{(1)}, x^{(2)}, \ldots$ are independent random $N$-vectors where $x^{(T)}$ is distributed as the $\beta$-Jacobi ensemble with parameters $\alpha_T, M_T$, then the $\beta$-Jacobi product process with parameters $(\alpha_T, M_T)_{T \in \mathbb{Z}_{>0}}$ is a Markov chain $(y^{(T)})_{T \geq 0}$ where the distribution of $y^{(T)}$ conditioned on $y^{(T-1)}$ depends only on $x^{(T)}$. This dependence is a generalization to arbitrary $\beta > 0$ of the effect that matrix products within a symmetry class have on singular values.

Date: October 4, 2019.
In general, a $\beta$-ensemble (see e.g. [37, C20]) is a particle system $(x_1, \ldots, x_N)$ with density proportional to
\[
\prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^N w(x_i)
\]
for some weight (or potential) $w(x)$. We note that the Hermite ($w(x) = e^{-x^2/2}$), Laguerre ($w(x) = x^p e^{-x}$) and Jacobi ($w(x) = x^p (1-x)^q$) ensembles correspond to eigenvalues of classical $\mathbb{U}_\beta$-invariant matrix models (see e.g. [3, C4]). Since the former two ensembles can be realized as degenerations of the Jacobi ensemble, the Jacobi ensemble may be viewed as the most general among the classical $\beta$-ensembles.

The main objective of this article is to study the fluctuations of $(y^{(T)})_{T \to 0}$ in the limit as $N \to \infty$. In particular, we consider the following two settings: (i) global fluctuations where we do not rescale time as $N$ grows and (ii) local fluctuations at the right edge (that is, of the rightmost particles) where time grows linearly with $N$. Along the way, we prove a limit shape result for arbitrary $\beta > 0$ which extends known results [47, 16] for $\beta = 1, 2, 4$.

We show that the global fluctuations are described by a Gaussian process whose covariance function exhibits logarithmic correlation on short-scales. This is reminiscent of the Gaussian free field — a distinguished 2-dimensional, conformally invariant, log-correlated Gaussian field which appears in the fluctuations of many 2-d models from statistical mechanics.

For local fluctuations, our results are twofold. We consider the regime where time $T$ grows to infinity such that $\lim_{N \to \infty} T/N = \hat{T}$ for some $\hat{T} > 0$. First, we demonstrate that for $\beta = 2$, the fluctuations of the right edge are described by a process in time $\hat{T}$ whose fixed-time marginals interpolate between the Airy point process as $\hat{T} \to 0$ and a deterministic, “picket fence” configuration as $\hat{T} \to \infty$. Second, for arbitrary $\beta > 0$, we show tightness of the point process at the right edge where we expect to see a $\beta$-generalization of the interpolating process. Moreover, we provide exponential moment formulas for this conjectural limit process.

From the method’s perspective, our goal is to combine ideas about Macdonald processes (special processes derived from the two parameter family of Macdonald symmetric functions), $\beta$-Jacobi ensembles and products of matrices into a unified picture [4, 5, 6, 22, 9]. Our approach involves Macdonald symmetric, Jack symmetric, Heckman-Opdam hypergeometric functions, and their correspondence [22] to product processes in order to obtain moment formulas. By the asymptotic analysis of these moment formulas, we access the fluctuations of $\beta$-Jacobi product processes.

As far as the author is aware, these results are the first to describe fluctuations for products of $\beta$-ensembles beyond $\beta = 2$. For local fluctuations at $\beta = 2$, previous works [2, 31] established fixed-time convergence results to a family of point processes indexed by $\hat{T}$ for the case of Ginibre matrices. These point processes interpolate between the Airy point process and picket fence statistics as $\hat{T}$ ranges from 0 to $\infty$. Our main theorems on local fluctuations extend these results to joint convergence across time $\hat{T}$ for Ginibre and Jacobi matrices. As a consequence, the interpolating process that appears in our work links together this $\hat{T}$-parametrized family of point processes across time; we provide more details below. The appearance of this process is independent of the choice of Jacobi parameters. This suggests that there may be a wider universality class of products of matrices with generic distributions where this interpolating process appears. For $\beta > 0$ arbitrary, we find universality of limiting moment formulas which do not depend on the parameters of the $\beta$-Jacobi product process. We conjecture that there exists a $\beta$-generalization of the interpolating process. Under this conjecture, these moment formulas are expressions for the Laplace transforms of the generalized interpolating processes evaluated at positive integer values.

We now proceed to a more detailed discussion of our main results and methods. Our main results on global fluctuations are provided in Section 1.1 along with some additional background. Similarly, Section 1.2 contains some background and our main results on local fluctuations at the right edge. We conclude the introduction with a description of our methods in Section 1.3.

1.1. Global Fluctuations. Our first asymptotic regime preserves the time scaling as the number of particles $N$ grows to $\infty$. We begin with some background on known limit shape and fluctuation results under this regime.
In a series of breakthrough articles (see e.g. [46, 47]), Voiculescu established that the squared singular values of products of certain large unitarily invariant random matrices concentrate around a limit shape. These results have since been generalized to include orthogonally and symplectically invariant matrix ensembles including the \( \beta \)-Hermite, \( \beta \)-Laguerre and \( \beta \)-Jacobi ensembles for \( \beta = 1, 4 \), see e.g. [16, Theorem 5.1].

Under general assumptions on unitarily invariant random matrices \( X_1, X_2, \ldots \), the global fluctuations of the squared singular values of \( Y_T \) are known to be Gaussian due to Collins-Mingo-Sniady-Speicher [15, Theorems 7.9 and 8.3] via second-order freeness and due to Guionnet-Novak [26] via Schwinger-Dyson equations. An explicit form for the covariance was recently discovered by Gorin-Sun [24] who used a difference operators approach on multivariate Bessel functions. In particular, the authors found that the covariance can be identified by a Gaussian field related to the Gaussian free field in \( \beta \)-ensembles, discussed further below.

On a related note, a variety of authors established the Gaussianity of global fluctuations for eigenvalues of other types of product matrix ensembles. By asymptotic analysis of the Stieltjes transform, Vasilchuk [45] proved a central limit theorem for linear statistics of eigenvalues of products of certain large unitarily invariant random matrices concentrate around a limit shape. These results have since been generalized to include orthogonally and symplectically invariant matrix ensembles including the \( \beta \)-Hermite, \( \beta \)-Laguerre and \( \beta \)-Jacobi ensembles for \( \beta = 1, 4 \), see e.g. [46, Theorem 5.1].

Recently, general result of Coston-O’Rourke [17] showed Gaussian fluctuations for the eigenvalues of products \( \beta > 0 \).

As commonly done in the literature, this measure can be extended for arbitrary \( \beta \) by considering a 2-dimensional extension called the \( \beta \)-Jacobi corners process, named so as it extended the matrix corners process for \( \beta = 1, 2, 4 \). Using an approach related to ours involving Macdonald processes (discussed further in Section 1.3), they showed that the global fluctuations can be described by the Gaussian free field.

We now introduce some notation for our main results on global asymptotics.

**Definition 1.1.** For \( \beta \in \{1, 2, 4\} \), let \( Y_T \) be a right \( U_\beta(N) \)-invariant \( N \times N \) matrix with squared singular values distributed as the \( \beta \)-Jacobi ensemble with parameters \( \alpha_T > 0, M_T \in \mathbb{Z}_{\geq 0} \) (recall \( M_T \geq N \)). We say that a Markov chain \((y^{(T)})_{T \in \mathbb{Z}_{\geq 0}}\) is distributed as the \( N \)-particle \( \beta \)-Jacobi product process with parameters \((\alpha_T, M_T)_{T \geq 0}\) if \( y^{(T)} \) is distributed as the squared singular value of \( Y_T \) defined by (1.1). As commonly done in the literature, this measure can be extended for arbitrary \( \beta > 0 \); the specifics of this extension are described in Section 3 Let

\[
m_{y^{(T)}} := \frac{1}{N} \sum_{i=1}^{N} \delta_{y_{i}^{(T)}}.
\]

We first provide the limit shape theorem for arbitrary \( \beta > 0 \), extending known results for \( \beta = 1, 2, 4 \).

**Theorem 1.2** (Limit Shape). Suppose \((y^{(T)})_{T \in \mathbb{Z}_{\geq 0}}\) is distributed as the \( N \)-particle \( \beta \)-Jacobi product process with parameters \((\alpha_T := \alpha_T(N), M_T := M_T(N))_{T \in \mathbb{Z}_{\geq 0}}\) and \( \tilde{\alpha}_T \geq 0, \tilde{M}_T \geq 1 \) such that

\[
\lim_{N \to \infty} (\alpha_T/N, M_T/N) = (\tilde{\alpha}_T, \tilde{M}_T)
\]

for each \( T \in \mathbb{Z}_{\geq 0} \). Then for any positive integers \( k, T \) there exists a probability measure \( m_{(\tilde{\alpha}_T, \tilde{M}_T)}^{(T)} \) (independent of \( \beta \)) such that

\[
\lim_{N \to \infty} m_{y^{(T)}} = m_{(\tilde{\alpha}_T, \tilde{M}_T)}^{(T)}
\]
weakly in probability. Moreover, we have
\[
\int x^k \, dm_{(\alpha_\tau, M_\tau)} \tau_{\tau-1} = -\frac{1}{k} \cdot \frac{1}{2\pi i} \oint \left( \frac{v}{v + 1} \Pi_{\tau=1}^{T} \frac{v - \hat{\alpha}_\tau}{v - \hat{\alpha}_\tau - \hat{M}_\tau} \right)^k \, dv
\]
where the contour is positively oriented around the pole at $-1$ but does not enclose $\hat{\alpha}_\tau + \hat{M}_\tau$ for $1 \leq \tau \leq T$.

**Remark 1.** Note that the limit of the $\beta$-Jacobi ensemble with parameters $\alpha_\tau, M_\tau$ converges to a limit measure $m_{\hat{\alpha}_\tau, \hat{M}_\tau}$. We can express the limit of products of $\beta$-ensembles as
\[
m_{(\alpha_\tau, M_\tau)} \tau_{\tau-1} = m_{\hat{\alpha}_1, \hat{M}_1} \boxtimes \cdots \boxtimes m_{\hat{\alpha}_T, \hat{M}_T}
\]
where $\boxtimes$ denotes the *free multiplicative convolution* (see e.g. [46]).

Our main result for global asymptotics states that the fluctuations are Gaussian with an explicit covariance structure.

**Theorem 1.3 (Global Fluctuations).** Suppose $(y(T))_{T \in \mathbb{Z}_{>0}}$ is distributed as the $N$-particle $\beta$-Jacobi product process with parameters $(\alpha_T := \alpha_T(N), M_T := M_T(N))_{T \in \mathbb{Z}_{>0}}$ and $\hat{\alpha}_T \geq 0, \hat{M}_T \geq 1$ such that
\[
\lim_{N \to \infty} (\alpha_T / N, M_T / N) = (\hat{\alpha}_T, \hat{M}_T)
\]
for each $T \in \mathbb{Z}_{>0}$. Then for any positive integers $m, k_1, \ldots, k_m$ and $T_1 \geq \ldots \geq T_m$, we have that the random vector
\[
(1.4) \quad \left( \int x^{k_1} \, dm_{y(T_1)}(x) - E \int x^{k_1} \, dm_{y(T_1)}(x), \ldots, \int x^{k_m} \, dm_{y(T_m)}(x) - E \int x^{k_m} \, dm_{y(T_m)}(x) \right)
\]
converges in distribution to a Gaussian vector as $N \to \infty$ where the covariance between the $i$th and $j$th component is given by
\[
(1.5) \quad \frac{\beta}{2 \pi i} \oint \oint \frac{1}{(v_2 - v_1)^2} \left( \frac{v_1}{v_1 + 1} \Pi_{\tau=1}^{T_1} \frac{v_1 - \hat{\alpha}_\tau}{v_1 - \hat{\alpha}_\tau - \hat{M}_\tau} \right)^{k_1} \left( \frac{v_2}{v_2 + 1} \Pi_{\tau=1}^{T_2} \frac{v_2 - \hat{\alpha}_\tau}{v_2 - \hat{\alpha}_\tau - \hat{M}_\tau} \right)^{k_2} \, dv_1 \, dv_2
\]
where the $v_2$ contour encloses the $v_1$ contour, both the $v_1, v_2$ contours are positively oriented around $-1$, but the $v_1$ contour does not contain $\hat{\alpha}_\tau + \hat{M}_\tau$ for $1 \leq \tau \leq T_1$ and the $v_2$ contour does not contain $\hat{\alpha}_\tau + \hat{M}_\tau$ for $1 \leq \tau \leq T_2$.

We observe that the covariance depends on the symmetry class $\beta$ only through a factor of $\beta^{-1}$. This is a common feature among $\beta$-ensembles in the literature; compare with e.g. [6]. In the case $\beta = 2$, our result intersects that of [24]. In particular, this means that the covariance can be described in terms of a Gaussian process whose covariance function is logarithmic on short-scales, as in [6, Proof of Theorem 4.13] [24, Proof of Corollary 4.10]. This is related to the appearance of the Gaussian free field in $\beta$-ensembles which has a distinguished logarithmic covariance structure given by the Green’s kernel on the upper half plane.

**1.2. Local Fluctuations for Growing Products.** Our second asymptotic regime takes the number of products $T$ and particles $N$ to grow linearly with respect to one another. Let $T = \lim_{N \to \infty} T / N$. Under this regime, we study the fluctuations of the right edge (the rightmost particles).

A recent result due to Akemann-Burda-Kieburg [2] and Liu-Wang-Wang [31, Theorem 1.2] considers this asymptotic regime for squared singular values of $Y_T$ as defined by (1.1) with $X_1, X_2, \ldots$ taken to be $N \times N$ complex Ginibre matrices. We recall that an $N \times N$ complex Ginibre matrix is a matrix of i.i.d. complex standard Gaussians. Define $(\xi_1 := \xi_1(N) \geq \cdots \geq \xi_N := \xi_N(N))$ from the squared singular values $y^{(T)} = (y_1^{(T)}, \ldots, y_N^{(T)})$ of $Y_T$ via
\[
y_i^{(T)} = N^{T+1} \xi_i.
\]
These authors showed that in the limit $N \to \infty$, the point process $\xi_1, \xi_2, \ldots$ converges to a $\hat{T}$-parametrized limiting determinantal point process \[3\] C.4.2 for the correlation kernel of $\hat{T}_i, \hat{T}_j, \ldots$ whose correlation kernel is given by

$$K_{\hat{T}}(x, x) = \int_{1-i\infty}^{1+i\infty} \frac{1}{2\pi i} ds \int dt \frac{\Gamma(t) e^{\frac{\pi i}{2} (x^2 x^2 - x^4 s)}}{\Gamma(t) e^{\frac{\pi i}{2} (x^2 x^2 - x^4 s)}}$$

where the $t$-contour is to the left of 1, starting from $-\infty - i\epsilon$, positively looping around $0, -1, -2, \ldots$, and then going to $-\infty + i\epsilon$. We note that our description of this point process differs from that of Liu-Wang-Wang \[31\] Equation 1.8 by a translation factor of $\hat{T}/2$. Akemann-Burda-Kieburg have a different form \[2\] Equation 19 for the correlation kernel of $\hat{T}_1, \hat{T}_2, \ldots$ where their point process differs from that of ours by a multiplicative factor of $T^{2/3}$ and a translation factor of $1 + \log \hat{T}$.

The following convergence statements are given in \[31\] Theorem 3.2 with a proof sketch. As $\hat{T} \to 0$, the process $\xi_{1, \hat{T}}, \xi_{2, \hat{T}}, \ldots$ defined by

$$\xi_{1, \hat{T}} = 2^{-1/3} \hat{T}^{2/3} \xi_{1, \hat{T}} + 1 + \log \hat{T}$$

converges to the Airy point process \[3\] p232-234 defined by the correlation kernel

$$K_{\text{Airy}}(\zeta, \eta) = \frac{\text{Ai}(\zeta) \text{Ai}'(\eta) - \text{Ai}(\eta) \text{Ai}'(\zeta)}{\zeta - \eta}.$$  

As $\hat{T} \to +\infty$, we have the convergence

$$\{\hat{T}^{-1} \xi_{1, \hat{T}}\}_{i=1}^{\infty} \to \{-i \pm \frac{1}{2}\}_{i=1}^{\infty}$$

to deterministic particles, referred to as picket fence statistics due to the associated measure being a semi-infinite set of equally spaced unit delta masses. Furthermore, as $\hat{T} \to +\infty$, the fluctuations of the largest particle can be described by a Gaussian random variable. More precisely,

$$\hat{T}^{-1/2} \left( \xi_{1, \hat{T}} + \frac{\hat{T}}{2} \right) \to (\text{real}) \text{ standard Gaussian random variable.}$$

While the convergence of the second largest, third largest, etc eigenvalues are not explicitly shown in \[2, 31\], there is good evidence (e.g. \[31\] Theorem 1.1) that each should be Gaussian and a rigorous proof should be accessible through the correlation kernels. We note that Liu-Wang-Wang also consider Ginibre matrices of different rectangular sizes and show that the point process $(\xi_i^{\hat{T}})_{i=1}^{\infty}$ appears universally in this setting \[31\] Section 3.4].

Although the correlation kernel (1.7) determines $\xi^{\hat{T}} := (\xi_i^{\hat{T}})_{i=1}^{\infty}$, we use an alternative characterization via the Laplace transform. By combining our result (in particular Theorem 6.1) for the Ginibre case with that of \[31\], we obtain a formula for the Laplace transform of $\xi^{\hat{T}}$.

**Theorem 1.4** (Formula for Laplace Transform). For $c_1, \ldots, c_m > 0$, we have

$$E \left[ \prod_{i=1}^{m} \sum_{j=1}^{\infty} e^{c_i \xi_j^{\hat{T}}} \right] = \frac{(-1)^m}{(2\pi i)^m} \oint \oint \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \times \prod_{i=1}^{m} \frac{\Gamma(\xi_i - c_i)}{\Gamma(\xi_i)} \exp \left( -\hat{T} c_i + 1 \right) \frac{2 - \hat{T} c_i u_i}{c_i} \frac{du_i}{c_i}$$

where the $u_i$ contour $\partial_i$ is a positively oriented contour around $-c_i, -c_i + 1, -c_i + 2, \ldots$ which starts and ends at $+\infty$, and $\partial_i$ is enclosed by $\partial_j - c_j$ for $i < j$.

Taking $\hat{T} \to 0$ and $c_i$ to grow linearly with $\hat{T}^{-2/3}$, we can recover a formula for the Laplace transform of the Airy point process (see \[7\] Equation 14]). Taking $\hat{T} \to +\infty$ and $c_i$ to decay linearly with $\hat{T}^{-1/2}$, we obtain the Laplace transform of a real Gaussian; if instead we let $c_i$ decay as $\hat{T}^{-1}$, we obtain the Laplace
transform for picket fence statistics. In the former two cases, these limits are directly accessible from the contour integral formula. In the latter case, one must take the residue expansion.

Our main results for $\beta = 2$ consider a natural extension of $\mathbf{r}(\hat{T}) = (\mathbf{r}_1(\hat{T}), \mathbf{r}_2(\hat{T}), \ldots)$ across time $\hat{T}$.

**Definition 1.5.** By an interpolating process (that is, interpolating between the Airy point process and picket fence statistics), we mean a process $(\mathbf{r}(\hat{T}))_{\hat{T}>0} := (\mathbf{r}_1(\hat{T}), \mathbf{r}_2(\hat{T}), \ldots)$ across time $\hat{T}$.

Theorem 1.6 ($\beta = 2$ Ginibre, Edge Fluctuations). Suppose $(\mathbf{y}(T))_{T \in \mathbb{Z}_{\geq 0}}$ are the squared singular values of $X_T$ defined as in (1.1) where $X_1, X_2, \ldots$ are independent $N \times N$ complex Ginibre matrices. Then

$$\lim_{N \to \infty} \frac{1}{N^{[\hat{T}]+1}} \exp \left( \mathbf{y}(\mathcal{N} \hat{T}) \right) = \mathbf{r}(\hat{T})$$

in finite dimensional distributions across time $\hat{T} > 0$.

Theorem 1.7 ($\beta = 2$ Jacobi, Edge Fluctuations). Suppose $(\mathbf{y}(T))_{T \in \mathbb{Z}_{\geq 0}}$ are the squared singular values of $X_T$ defined as in (1.1) where $X_1, X_2, \ldots$ are independent $N \times N$ matrices with squared singular values distributed as the ($\beta = 2$) Jacobi ensemble with parameters $\alpha := \alpha(N), M := M(N)$. If $\alpha > 0, M \geq 1$ such that

$$\lim_{N \to \infty} (\alpha/N, M/N) = (\hat{\alpha}, \hat{M}),$$

then

$$\lim_{N \to \infty} \frac{1}{N} \left( \frac{M + N + \alpha - 1}{N + \alpha - 1} \right)^{[CN\hat{T}]} \exp \left( \mathbf{y}(CN\hat{T}) \right) = \mathbf{r}(\hat{T})$$

in finite dimensional distributions across time $\hat{T} > 0$, where

$$C = \left( \frac{1}{1 + \hat{\alpha}} - \frac{1}{\hat{M} + 1 + \hat{\alpha}} \right)^{-1}.$$
Theorem 1.8 ($\beta > 0$, Edge Fluctuations). Let $k_1, \ldots, k_m$ be positive integers. Suppose $(y^{(T)})_{T \in \mathbb{Z}_{>0}}$ is distributed as the $N$-particle $\beta$-Jacobi product process with parameters $(\alpha := \alpha(N), M := M(N))^T > 0$. If $\tilde{\alpha} > 0, \tilde{M} \geq 1$ such that

$$\lim_{N \to \infty} (\alpha/N, M/N) = (\tilde{\alpha}, \tilde{M}),$$

then

$$\lim_{N \to \infty} \mathbb{E} \left[ \prod_{i=1}^{m} \frac{1}{N^{k_i}} (M + N + \alpha - 1) \right]_{k_i}^{(CNT_i)} \int e^{x^k} \text{d}m_{y^{(CNT_i)}}(x)$$

$$= \frac{(-\beta/2)^{-m}}{(2\pi)^{m}} \prod_{i=1}^{m} k_i \prod_{1 \leq i \leq m, 1 \leq j \leq k_i, (i,j) < (i',j')} \left( \frac{1}{u_{i,j} - u_{i,j} + 1 - \beta/2} - \frac{1}{\beta^2 u_{i,j}} \right)$$

$$\times \prod_{i=1}^{m} \prod_{a=1}^{k_i} \frac{1}{u_{i,a+1} - u_{i,a} + 1 - \beta/2} \left( \frac{1}{e^{-\tilde{T}_i(\beta/2)} u_{i,j}} - \frac{1}{\beta^2 u_{i,j}} \right)$$

$$= \left( \frac{1}{1 + \tilde{\alpha}} - \frac{1}{\tilde{M} + 1 + \tilde{\alpha}} \right)^{-1}.$$

As in the $\beta = 2$ case, we expect the right hand side of Theorem 1.8 to describe the Laplace transform of a $\beta$-generalization of the interpolation point processes. However, it is not clear that the right hand side determines a limiting point process since the moment problem is indeterminate. Despite the indeterminacy, our result implies tightness.

Corollary 1.9. For $(y^{(T)})_{T \in \mathbb{Z}_{>0}}$ distributed as in Theorem 1.8, the finite dimensional distributions of

$$\frac{1}{N} (M + N + \alpha - 1) \right]_{k_i}^{(CNT_i)} e^{y^{(CNT)}}$$

are tight in $N$.

We conjecture that (1.8) converges to a universal limit process $(y^{(T)}(\beta))_{T > 0}$ with the property that $T \to 0$ yields the $\beta$-Airy point process and $T \to +\infty$ yields a Gaussian limit under proper rescaling. Assuming this conjecture, the right hand side of Theorem 1.8 gives formulas for Laplace transform of the conjectural point process

$$\mathbb{E} \left[ \prod_{i=1}^{m} \sum_{k=1}^{\infty} e^{k_{i}y^{(T)}(\beta)} \right]$$

evaluated at positive integer values $(k_1, \ldots, k_m) \in \mathbb{Z}_{>0}^m$. In addition, this conjecture implies that if $k = k_1 = \cdots = k_m$ are taken to grow on the order $T^{-2/3}$ as $T \to 0$, then we should obtain the Laplace transform of the $\beta$-Airy point process (see [23]) in the limit.

1.3. Method. Our method relies on deep connections between the Macdonald symmetric functions and random matrices. More specifically, we draw upon connections between (i) Macdonald processes and $\beta$-ensembles, and (ii) product processes and Macdonald symmetric functions. The key fact that we leverage is that certain observables of Macdonald processes are accessible through special operators which diagonalize the Macdonald symmetric functions. This idea was pioneered by [4] and further extended in [6, 25, 1]. The connection between Macdonald processes and $\beta$-ensembles was first observed by Borodin-Gorin [6]. They showed that the Heckman-Opdam limit (defined below) of certain Macdonald processes produce the $\beta$-Jacobi corners process. Using formulas for observables of Macdonald processes, they accessed the global
fluctuations of the \( \beta \)-Jacobi corners process. On the other hand, the connection between product ensembles and Schur symmetric functions (a special case of the Macdonald symmetric functions) was established recently by Borodin-Gorin-Strahov [9] for \( \beta = 2 \). They showed that limits of certain Schur processes yield singular value processes of products of truncated unitary matrices. Our method combines and extends these two approaches to obtain observables for products of \( \beta \)-Jacobi ensembles. We note that the observables we use are derived from a different set of operators than that of [6]. The operators we use were introduced by Negut [34] in an algebraic setting, reexpressed as contour integral formulas by Gorin-Zhang [25] for the asymptotic analysis of the \( \beta \)-Jacobi corners process and further applied to a broad class of Macdonald processes by the author [1]. These operators have the advantage of giving exact moment formulas for the \( \beta \)-Jacobi product process, thus a streamlined asymptotic analysis.

Through these connections, we establish that the \( \beta \)-Jacobi product processes are limits of certain Macdonald processes. We prove this limit statement by describing the Markov transition kernels of the \( \beta \)-Jacobi product process in terms of expectations of Jack symmetric functions and demonstrating convergence of analogous expectations on the side of Macdonald processes. Our computations are generalizable to arbitrary \( \beta > 0 \) by the extension introduced by Gorin-Marcus [22].

With this limit relation established, we obtain moment formulas for the \( \beta \)-Jacobi product process through formulas for moments of Macdonald processes. Global and local fluctuations are obtained through the moment’s method. The global asymptotic analysis uses techniques related to the approaches of [6] [25] [1]. Our method of accessing fluctuations of the right edge (as \( T, N \to \infty \)) is inspired by the method of analyzing large moments to access the edge of particle systems, pioneered by Sinai-Soshnikov [42] and further refined by Soshnikov [44] who established the universality of the Airy point process at the spectral edge of Wigner matrices. Soshnikov’s approach used moments which grow as the size of the matrices grow to obtain the Laplace transform of the Airy point process in the limit. Our approach is based on the observation that the growing order of moments can be replaced by the growing order of products in our setting. For further background on the moment’s method, see the survey [43] and references therein.

The remainder of this article is organized as follows. In Section 2, we provide the necessary background on Macdonald symmetric, Jack symmetric and Heckman-Opdam hypergeometric functions along with references for additional study. We detail the connections between these functions and the \( \beta \)-Jacobi product process in Section 3 and obtain moment formulas in Section 4 by passing through the Macdonald process formalism. In the remaining Sections 5 and 6 we prove our main theorems on global and respectively local fluctuations via asymptotic analysis of the moment formulas.

Acknowledgments

I would like to thank my advisor Vadim Gorin for suggesting this project, for useful discussions and giving feedback on several drafts. The author was partially supported by NSF Grant DMS-1664619.

2. Preliminaries on Special Functions

2.1. Symmetric Functions. Let \( \mathbb{Y} \) denote the set of partitions. We represent \( \lambda \in \mathbb{Y} \) as the nondecreasing sequence \((\lambda_1, \lambda_2, \ldots)\) of its parts. Denote by \( \ell(\lambda) \) the number of indices \( i \) such that \( \lambda_i \neq 0 \) and by \( |\lambda| \) the size the partition \( \sum_{i \geq 1} \lambda_i \). Let \( \Lambda \) denote the algebra (over \( \mathbb{C} \)) of symmetric functions in countably many variables \( x_1, x_2, \ldots \). Define \( p_0 = 1 \) and

\[
p_k = \sum_{i \geq 1} x_i^k, \; k \in \mathbb{Z}_{>0}
\]

\[
p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}, \; \lambda \in \mathbb{Y}.
\]

Then \( \{p_\lambda\}_{\lambda \in \mathbb{Y}} \) forms a linear basis of \( \Lambda \). Fixing \( 0 < q, t < 1 \), we have the scalar product

\[
\langle p_\lambda, p_\mu \rangle_{(q,t)} = \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!
\]
where \(m_i(\lambda)\) is the multiplicity of \(i\) in \(\lambda\). We remove the subscript \((q,t)\) when the dependence on \((q,t)\) is clear.

The Macdonald symmetric functions \(\{P_\lambda(X;q,t)\}_{\lambda \in \mathcal{Y}}\) are the unique (homogeneous) symmetric functions satisfying
\[
\langle P_\lambda(X;q,t), P_\mu(X;q,t) \rangle = 0
\]
for \(\lambda \neq \mu\) and with leading monomial \(x_1^{\lambda_1}x_2^{\lambda_2}\ldots\) with respect to lexicographical ordering of the powers \((\lambda_1, \lambda_2, \ldots)\). This implies that \(\{P_\lambda(X;q,t)\}_{\lambda \in \mathcal{Y}}\) forms a linear basis for \(\Lambda\). Let \(Q_\lambda(X;q,t)\) represent the multiple of \(P_\lambda(X;q,t)\) satisfying
\[
\langle P_\lambda(X;q,t), Q_\lambda(X;q,t) \rangle = 1.
\]
Given \(\mu, \nu \in \mathcal{Y}\), we may expand \(P_\mu(X;q,t)P_\nu(X;q,t)\) in the basis of Macdonald symmetric functions. By orthogonality \((2.3)\) and duality \((2.4)\), the coefficient of \(P_\lambda(X;q,t)\) is given by
\[
\langle P_\mu(X;q,t)P_\nu(X;q,t), Q_\lambda(X;q,t) \rangle
\]
and is nonzero only if
\[
|\mu| + |\nu| = |\lambda|, \ \lambda \supset \mu, \ \lambda \supset \nu
\]
where the first condition is a consequence of the homogeneity and the justification for the latter two conditions can be found in \([32]\) Chapter VI (7.4).

The skew Macdonald symmetric functions \(P_{\lambda/\mu}(X;q,t), Q_{\lambda/\mu}(X;q,t)\) are defined by
\[
\langle P_{\lambda/\mu}(X;q,t), Q_{\nu}(X;q,t) \rangle = \langle P_{\lambda}(X;q,t), Q_{\mu}(X;q,t)Q_{\nu}(X;q,t) \rangle
\]
(2.7)
\[
\langle Q_{\lambda/\mu}(X;q,t), P_{\nu}(X;q,t) \rangle = \langle Q_{\lambda}(X;q,t), P_{\mu}(X;q,t)P_{\nu}(X;q,t) \rangle.
\]
As a consequence of \((2.5)\), we have that \(P_{\lambda/\mu}(X;q,t) \neq 0\) only if \(\lambda \supset \mu\) and likewise for \(Q_{\lambda/\mu}(X;q,t)\).

Let \(\mathcal{Y}_N\) denote the set of partitions of length \(\leq N\), \(\Lambda_N\) the complex algebra of symmetric polynomials in \(N\)-variables \(x_1, \ldots, x_N\) and \(\pi_N : \Lambda \rightarrow \Lambda_N\) the restriction homomorphism which effectively takes \(x_{N+1} = x_{N+2} = \ldots = 0\). Then
\[
P_\lambda(x_1, \ldots, x_N; q,t) := \pi_N P_\lambda(X;q,t)
\]
for \(\lambda \in \mathcal{Y}_N\) form a basis for \(\Lambda_N\). We have
\[
a_1, \ldots, a_N \geq 0 \implies P_{\lambda/\mu}(a_1, \ldots, a_N; q,t) \geq 0;
\]
(2.9)
we refer to \([32]\) Chapter VI,(7.9) & (7.14)]\) for the ingredients to prove this nonnegativity. For \(A = (a_1, a_2, \ldots)\) and \(B = (b_1, b_2, \ldots)\) such that \(\sup_{1 \leq i, j \leq \infty} |a_i - b_j| < 1\), we also have the Cauchy identity
\[
\sum_\lambda P_\lambda(A; q,t)Q_\lambda(B; q,t) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{(ta_i; q)_{\infty}}{(a_i; q)_{\infty}} =: \Pi(A, B; q,t)
\]
(2.10)
(see \([32]\) Chapter VI,(4.13)]. The following key properties for the skew Macdonald symmetric functions can be found in \([32]\) Chapter VI, Section 7.4):
\[
P_{\lambda/\mu}(X;q,t) = \sum_{\mu \in \mathcal{Y}} P_{\lambda/\mu}(X;q,t)P_{\mu/\nu}(X;q,t),
\]
(2.11)
\[
\sum_{\lambda \in \mathcal{Y}} P_\lambda(X)Q_{\lambda/\mu}(Y) = \Pi(X,Y; q,t)P_{\mu}(X).
\]
(2.12)
the first equality is known as the branching rule.

We define a normalized version of the Macdonald symmetric polynomial
\[
\hat{P}_\lambda(x_1, \ldots, x_N; q,t) = \frac{P_\lambda(x_1, \ldots, x_N; q,t)}{P_\lambda(1, t, \ldots, t^{N-1}; q,t)}, \ \lambda \in \mathcal{Y}_N
\]
(2.13)
which satisfies the label-variable symmetry for \(\lambda, \mu \in \mathcal{Y}_N\)
\[
\hat{P}_\lambda(q^{\mu_1}t^{N-1}, q^{\mu_2}t^{N-2}, \ldots, q^{\mu_N}; q,t) = \hat{P}_\mu(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \ldots, q^{\lambda_N}; q,t).
\]
(2.14)
Since the normalized Macdonald symmetric functions form a linear basis for the algebra of symmetric functions, we have the existence of coefficients \( c^\lambda_{\mu \nu}(\hat{P}; q, t) \) which satisfy
\[
(2.15) \quad \hat{P}_\mu(x_1, \ldots, x_N; q, t) \hat{P}_\nu(x_1, \ldots, x_N; q, t) = \sum_\lambda c^\lambda_{\mu \nu}(\hat{P}; q, t) \hat{P}_\lambda(x_1, \ldots, x_N; q, t).
\]

### 2.2. Jack Symmetric and Heckman-Opdam Hypergeometric Functions

We now describe two degenerations of the Macdonald symmetric polynomials. For \( \theta > 0 \), the Jack symmetric polynomials are
\[
(2.16) \quad J_\lambda(x_1, \ldots, x_N; \theta) := \lim_{q \to 1} P_\lambda(x_1, \ldots, x_N; q, q^\theta), \quad \lambda \in \mathbb{Y}_N,
\]
(see [32] Chapter VI, Section 10). Let \( \hat{J} \) denote the limit obtained by replacing \( P_\lambda \) with \( \hat{P}_\lambda \).

For \( \theta > 0 \) and \( r = (r_1 > \cdots > r_N > 0 = r_{N+1} = \cdots = r_M) \), define the **Heckman-Opdam hypergeometric function** (for Type A root systems [27, 28, 36]) by
\[
t = q^\theta, \quad q = \exp(-\varepsilon), \quad \lambda = \varepsilon^{-1}(r_1, \ldots, r_N), \quad x_i = \exp(\varepsilon z_i),
\]
\[
\mathcal{F}_r(z_1, \ldots, z_M; \theta) := \lim_{\varepsilon \to 0} \varepsilon^{N(\theta-1)/2+N(M-N)} P_\lambda(x_1, \ldots, x_M; q, t)
\]
\[
\hat{\mathcal{F}}_r(z_1, \ldots, z_M; \theta) := \lim_{\varepsilon \to 0} \varepsilon^{N(\theta-1)+\theta N(1-N)/N(M-N)} Q_\lambda(x_1, \ldots, x_M; q, t)
\]
and extend the limit to \( r = (r_1 \geq \cdots \geq r_N) \).

Let
\[
\mathcal{R}^N := \{ r \in \mathbb{R}^N_{\geq 0} : r_1 \geq \cdots \geq r_N \}, \quad \mathcal{U}^N := \mathcal{R}^N \cap [0, 1]^N.
\]
We may view \( r \in \mathcal{R}^N \) as a member of \( \mathcal{R}^M \) for \( M \geq N \) via the identification with \( (r_1, \ldots, r_N, 0, \ldots, 0) \in \mathcal{R}^M \).

The Cauchy identity \((2.10)\) degenerates to
\[
(2.18) \quad \int_{\mathcal{R}^N \times \mathcal{R}^M} \hat{\mathcal{F}}_r(a_1, \ldots, a_N; \theta) \mathcal{F}_r(b_1, \ldots, b_M) \prod_{i=1}^{N(M)} \prod_{j=1}^{M} \frac{\Gamma(\theta-a_i-b_j)}{\Gamma(\theta-a_i-b_j)}
\]
for \( a_1, \ldots, a_N, b_1, \ldots, b_M \) such that \( \Re(a_i + b_j) < 0 \) for all \( i \in [1, N], j \in [1, M] \) (see [6] Proposition 6.4). The variable-index symmetry \((2.11)\) degenerates to
\[
(2.19) \quad \hat{\mathcal{F}}_r(-\lambda_1 - (N-1)\theta, -\lambda_2 - (N-2)\theta, \ldots, -\lambda_N; \theta) = \hat{J}_\lambda(\exp(-r_1), \exp(-r_2), \ldots, \exp(-r_N); \theta)
\]
for \( r \in \mathcal{R}^N \) and \( \lambda \in \mathbb{Y}_N \) (see [22], Section 2).

We also have the existence of coefficients \( c^\lambda_{\mu \nu}(\hat{P}; q, t) \), \( c^a_{\ell r}(\hat{F}; \theta) \) which satisfy
\[
\hat{J}_\mu(x_1, \ldots, x_N; \theta) \hat{J}_\nu(x_1, \ldots, x_N; \theta) = \sum_\lambda c^\lambda_{\mu \nu}(\hat{P}; q, t) \hat{J}_\lambda(x_1, \ldots, x_N; \theta)
\]
\[
\hat{F}_r(z_1, \ldots, z_N; \theta) \hat{F}_r(z_1, \ldots, z_N; \theta) = \int_s c^a_{\ell r}(\hat{F}; \theta) \hat{F}(z_1, \ldots, z_N; \theta)
\]
where
\[
(2.20) \quad c^\lambda_{\mu \nu}(\hat{P}; q, t) \to c^\lambda_{\mu \nu}(\hat{J}; \theta), \quad t = q^\theta \to 1.
\]
and \( c^a_{\ell r}(\hat{F}; \theta) \) is a (possibly signed) measure in \( s \) with total mass 1 supported in the set of \( s \in \mathcal{R}^N \) satisfying
\[
s_1 + \cdots + s_N = (r_1 + \ell_1) + \cdots (r_N + \ell_N), \quad r_N + \ell_N \leq s_i \leq r_1 + \ell_1, 1 \leq i \leq N.
\]
see [22], Section 2. For \( \theta = 1/2, 1, 2 \), the measures \( c^a_{\ell r} \) are nonnegative and therefore probability measures.
3. ∂-Jacobi Product Process

In this section, we describe the connection between the special functions discussed in the previous section with the ∂-Jacobi product process. The main result of this section (Theorem 3.12) realizes the ∂-Jacobi product processes as limits of discrete Macdonald processes, a Markov chain whose distribution has a special structure in terms of Macdonald symmetric functions. We begin by finding a description for the transition kernels of the ∂-Jacobi product process for ∂ = 1, 2, 4. We then provide the interpolation to ∂ > 0 via a limit transition from the Macdonald processes detailed below, after which point Theorem 3.12 is a straightforward consequence.

**Definition 3.1.** Suppose X, Y denote two random, independent, N × N self adjoint random matrices with right Uβ-invariant distributions. Let x and y denote the random squared singular values of X, Y respectively. Define

\[
x \boxtimes_\beta y
\]

to be the squared singular values of XY.

**Definition 3.2.** Let ∂, α > 0, and M, N ∈ \( \mathbb{Z}_{>0} \) such that M ≥ N. Denote by \( \mathbb{P}_\theta^{\alpha,M,N} \) the measure on N-particles \( x \in \mathcal{U}^N \) with density proportional to

\[
\prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\theta} \prod_{i=1}^N x_i^{\theta \alpha - 1} (1 - x_i)^{\theta(M - N) + \theta - 1} \, dx_i.
\]

**Remark 2.** Although we work with M ∈ \( \mathbb{Z}_{>0} \), one can extend the results in this section for M > 0 by analytic continuation.

This is the ∂-Jacobi ensemble with ∂ = 2θ. For θ = 1/2, 1, 2, this is the eigenvalue distribution of a certain matrix ensemble.

**Proposition 3.3** ([21 Proposition 3.8.2]). Let ∂ ∈ \{1, 2, 4\}, L, N', N ∈ \( \mathbb{Z}_{>0} \) such that N' ≥ N, L ≥ N' + N and U ∈ Uβ(L). If A is the N' × N top left rectangular submatrix of U, then the eigenvalues of \( A^*A \) are distributed as \( \mathbb{P}_\theta^{\alpha,M,N} \) with

\[
\alpha = N' - N + 1, \quad M = L - N'.
\]

The density of the ∂-Jacobi ensemble can also be given in terms of Heckman-Opdam hypergeometric functions. The following Proposition is a consequence of a general convergence statement [6, Theorem 2.8] for the ∂-Jacobi corners process; a multilevel extension of the ∂-Jacobi ensemble.

We adopt the following notation for brevity, for any \( \mu \in \mathbb{V}_N \) set

\[
\rho_\mu^N(q,t) = (q^{\mu_1}t^{N-1}, q^{\mu_2}t^{N-2}, \ldots, q^{\mu_N}),
\]

\[
\varrho_\mu^N(\theta) = -((\mu_1 + \theta(N - 1), \mu_2 + \theta(N - 2), \ldots, \mu_N),
\]

\[
(a_1, \ldots, a_N) + c = (a_1 + c, \ldots, a_N + c) \quad \text{for} \quad a_1, \ldots, a_N, c \in \mathbb{R}.
\]

The dependence on (q,t) and θ will be clear from the context so we write \( \rho_\mu^N = \rho_\mu^N(q,t) \) and \( \varrho_\mu^N = \varrho_\mu^N(\theta) \).

**Proposition 3.4.** Let θ, α > 0, and M, N ∈ \( \mathbb{Z}_{>0} \) such that M ≥ N. Let ε > 0 be a small parameter with t = q^ε, q = exp(-ε), and \( \lambda \in \mathbb{V}_N \) distributed as

\[
\text{Prob}(\lambda = \mu) = \frac{1}{\Pi(\rho_0^M, \varrho_0^M, q,t)} P_\mu(\varrho_0^N, q,t) Q_\mu(t^\alpha \rho_0^M, q,t).
\]

If \( y \sim \mathbb{P}_\theta^{\alpha,M,N} \), then \( \exp(-\varepsilon \lambda) \to y \) in distribution as \( \varepsilon \to 0 \). In particular, the density \( \mathbb{P}_\theta^{\alpha,M,N} \) has the form

\[
\frac{1}{H(\varrho_0^N, \varrho_0^M - \alpha \theta; \theta)} F_r(\varrho_0^N, \vartheta) \tilde{F}_r(\varrho_0^M - \alpha \theta; \theta) \, dr
\]

where \( y = \exp(-r) \).
Definition 3.5. Let $\beta \in \{1, 2, 4\}$ and $x^{(1)}, x^{(2)}, \ldots$ be independent, random elements of $\mathbb{R}^N$. Define the $\beta$-product process on $(x^{(T)})_{T \in \mathbb{Z}_{>0}}$ to be the random sequence $(y^{(T)})_{T \in \mathbb{Z}_{>0}}$ where $y^{(1)} := x^{(1)}$ and

\[ y^{(T+1)} = y^{(T)} \otimes_{\beta} x^{(T+1)}, \ T \in \mathbb{Z}_{>0}. \]

By the independence of $x^{(1)}, x^{(2)}, \ldots$, the $\beta$-product process is a Markov process in discrete time $T$. We compute the Markov transition probabilities of this process with $x^{(T)}$ distributed as the $\beta$-Jacobi ensemble for some set of parameters for each $T \in \mathbb{Z}_{>0}$.

Proposition 3.6. Let $\theta \in \{1/2, 1, 2\}$, $\alpha > 0$, and $M, N \in \mathbb{Z}_{>0}$ such that $M \geq N$. If $x \sim P^{\alpha,M,N}_\theta$, $y \in \mathcal{U}^N$ and $z = y \otimes_{2\theta} x$, then

\[ \mathbb{E} \hat{J}_\kappa(z; \theta) = \hat{J}_\kappa(y; \theta) \frac{H(\varrho^N_\alpha, \varrho^M_0 - \alpha \theta)}{H(\varrho^N_0, \varrho^M_0 - \alpha \theta)}, \ \kappa \in \mathbb{Y}_N. \]  

This distribution is determined by (3.8).

Proof. If $\ell, r \in \mathbb{R}^N$, then by [22, Proposition 2.2] the probability measure of the random vector $s \in \mathbb{R}^N$ defined by

\[ \exp(-s) = \exp(-\ell) \otimes_{2\theta} \exp(-r) \]

is given by $c^s_{\ell,r}(\tilde{F}; \theta)$. If we let $r \in \mathbb{R}^N$ be random such that $\exp(-r) \sim P^{\alpha,M,N}_\theta$, then for any $\kappa \in \mathbb{Y}_N$

\[ \mathbb{E} \hat{J}_\kappa(\exp(-s); \theta) = \int_r \frac{1}{H(\varrho^N_0, \varrho^M_0 - \alpha \theta)} F_r(\varrho^N_\alpha, \varrho^M_0 - \alpha \theta; \theta) \int_s \frac{1}{H(\varrho^N_0, \varrho^M_0 - \alpha \theta)} F_s(\varrho^N_\alpha, \varrho^M_0 - \alpha \theta; \theta) \hat{J}_\kappa(\exp(-s); \theta) \]

\[ = \frac{1}{H(\varrho^N_0, \varrho^M_0 - \alpha \theta)} \int_r F_r(\varrho^N_\alpha, \varrho^M_0 - \alpha \theta; \theta) \hat{J}_\kappa(\exp(-s); \theta) \]

\[ = \hat{J}_\kappa(\exp(-\ell); \theta) \frac{H(\varrho^N_\alpha, \varrho^M_0 - \alpha \theta)}{H(\varrho^N_0, \varrho^M_0 - \alpha \theta)} \]

where the first equality uses Proposition 3.4, the second and third equalities use (2.18), and the fourth equality uses (3.10). Taking $y = \exp(-\ell)$ and $z = \exp(-s)$ proves (3.8). Since the Jack symmetric functions in $N$-variables form a basis for the symmetric polynomials in $N$-variables and since $z$ is supported in $\mathcal{U}^N$, (3.8) determines the distribution of $z$. \hfill \Box

The proof of Proposition 3.6 suggests that the probability density of $s$ conditioned on $r$, where $\exp(-s) = \exp(-\ell) \otimes_{\beta} \exp(-r)$, can be expressed as

\[ \frac{1}{H(\varrho^N_0, \varrho^M_0 - \alpha \theta; \theta)} \int_r c^s_{\ell,r}(\tilde{F}; \theta) F_r(\varrho^N_\alpha, \varrho^M_0 - \alpha \theta; \theta). \]

However, $c^s_{\ell,r}$ is a measure on $s \in \mathbb{R}^N$ so that (3.11) is ill-defined. Thus without reference to a density, the integral (3.11) is ill-defined. We therefore take the approach of identifying the Markov kernel by moments.

The transition probabilities of Proposition 3.6 can be extended to arbitrary $\beta > 0$. We detail this extrapolation below. The idea is seeing the Markov kernels of Proposition 3.6 as limits of a family of kernels derived from Macdonald symmetric functions.

Definition 3.7. Let $0 < q, t < 1$, $\alpha > 0$ and $M, N \in \mathbb{Z}_{>0}$ such that $M \geq N$. Define the kernel

\[ K^{\alpha,M,N}_{q,t}(\mu, \lambda) = \frac{1}{\Pi(\rho^N_\alpha, \rho^M_0; q, t)} \sum_{\nu \in \mathbb{Y}} c^\lambda_{\mu, \nu}(\tilde{F}; q, t) P^\nu(\rho^N_\alpha; q, t) Q^\nu(\rho^M_0; q, t), \ \mu, \lambda \in \mathbb{Y}_N. \]
Expressing (2.15) as
\[ P_\mu(x_1, \ldots, x_N; q, t) = P_\mu(p_0^N; q, t) \sum_\lambda e_{\mu \lambda}^\mu(\tilde{P}; q, t) = \sum_\lambda e_{\mu \lambda}^\mu(\tilde{P}; q, t) \frac{P_\lambda(x_1, \ldots, x_N; q, t)}{P_\lambda(p_0^N; q, t)}, \]
we see that (2.4) and (2.7) imply
\[ e_{\mu \lambda}^\mu(\tilde{P}; q, t) = \frac{P_\lambda(p_0^N; q, t)}{P_\mu(p_0^N; q, t) P_\nu(p_0^N; q, t)} (Q_{\lambda / \mu}(\cdot; q, t), P_\nu(\cdot; q, t)) \]
so that
\[ K_{a, t}^{\alpha, M, N}(\mu, \lambda) = \frac{P_\lambda(p_0^N; q, t)}{P_\mu(p_0^N; q, t) P_\nu(p_0^N; q, t)} \sum_{\nu \in \Upsilon} (Q_{\lambda / \mu}(\cdot; q, t), P_\nu(\cdot; q, t)) Q_\nu(t^\alpha p_0^M; q, t) \]
\[ = \frac{1}{\Pi(p_0^N, t^\alpha p_0^M; q, t)} P_\lambda(p_0^N; q, t) Q_{\lambda / \mu}(t^\alpha p_0^M; q, t). \] (3.13)

**Lemma 3.8.** For each \( \kappa, \mu \in \Upsilon_N \), we have
\[ \sum_{\lambda \in \Upsilon} \tilde{P}_\kappa(p_0^N; q, t) K_{a, t}^{\alpha, M, N}(\mu, \lambda) = \tilde{P}_\kappa(p_0^N; q, t) \frac{\Pi(p_0^N, t^\alpha p_0^M; q, t)}{\Pi(p_0^N, t^\alpha p_0^M; q, t)}. \] (3.14)
In particular, \( K_{a, t}^{\alpha, M, N}(\mu, \cdot) \) defines a probability distribution.

**Proof.** We contract the notation by dropping the \((q, t)\) from \( P, \tilde{P}, Q \). We have
\[ \sum_{\lambda \in \Upsilon} \tilde{P}_\kappa(p_0^N; q, t) K_{a, t}^{\alpha, M, N}(\mu, \lambda) = \frac{1}{\Pi(p_0^N, t^\alpha p_0^M; q, t)} \sum_{\lambda \in \Upsilon} \tilde{P}_\lambda(p_0^N) P_\lambda(p_0^N) Q_{\lambda / \mu}(t^\alpha p_0^M) \]
\[ = \frac{1}{\Pi(p_0^N, t^\alpha p_0^M; q, t)} \sum_{\lambda \in \Upsilon} P_\lambda(p_0^N) Q_{\lambda / \mu}(t^\alpha p_0^M) \]
\[ = \tilde{P}_\mu(p_0^N) \frac{\Pi(p_0^N, t^\alpha p_0^M; q, t)}{\Pi(p_0^N, t^\alpha p_0^M; q, t)} \]
where we use (2.14) in the first equality and (2.21) in the final equality. By taking \( \kappa = (0) \), we see that \( K_{a, t}^{\alpha, M, N}(\mu, \cdot) \) defines a probability distribution. By (2.14), the lemma follows. \( \square \)

**Proposition 3.9.** Suppose \( v \in \mathcal{U}^N, q = e^{-\varepsilon}, t = q^\rho, \) and \( \mu = \mu(\varepsilon) \in \Upsilon_N \) such that \( q^\rho \to v \) as \( \varepsilon \to 0 \). If \( \lambda \sim K_{a, t}^{\alpha, M, N}(\mu, \cdot) \), then as \( \varepsilon \to 0, q^\lambda \) converges in distribution to a random variable \( u \in \mathcal{U}^N \) with probability measure \( K_{a, t}^{\alpha, M, N}(v, u) \) determined by
\[ \int_u \tilde{J}_\kappa(u; \theta) K_{a, t}^{\alpha, M, N}(v, u) = \tilde{J}_\kappa(v; \theta) \frac{H(q_0^N, q_0^M - \alpha \theta; \theta)}{H(q_0^N, q_0^M - \alpha \theta; \theta)}, \quad \kappa \in \Upsilon_N. \] (3.16)

**Proof.** By Lemma 3.8, for any \( \kappa \in \Upsilon_N \) we have the convergence
\[ \sum_{v \in \mathcal{U}^N \setminus \mathcal{Y}_N} \tilde{P}_\kappa(v_1 t^{N-1}, v_2 t^{N-2}, \ldots, v_N; q, t) K_{a, t}^{\alpha, M, N}(\mu, -\varepsilon^{-1} \log v) \to \tilde{J}_\kappa(v; \theta) \frac{H(q_0^N, q_0^M - \alpha \theta; \theta)}{H(q_0^N, q_0^M - \alpha \theta; \theta)} \] (3.17)
as \( \varepsilon \to 0 \). For any given \( \kappa \in \Upsilon_N \), we also have the uniform convergence
\[ \tilde{P}_\kappa(u_1 t^{N-1}, u_2 t^{N-2}, \ldots, u_N; q, t) \to \tilde{J}_\kappa(u_1, u_2, \ldots, u_N; \theta) \] (3.18)as \( \varepsilon \to 0 \) over \((u_1, \ldots, u_N) \in \mathcal{U}^N \). Therefore
\[ \sum_{v \in \mathcal{U}^N \setminus \mathcal{Y}_N} \tilde{J}_\kappa(v_1, \ldots, v_N; \theta) K_{a, t}^{\alpha, M, N}(\mu, -\varepsilon^{-1} \log v) \to \tilde{J}_\kappa(v; \theta) \frac{H(q_0^N, q_0^M - \alpha \theta; \theta)}{H(q_0^N, q_0^M - \alpha \theta; \theta)}. \] (3.19)
Since $\mathcal{U}^N$ is compact and the Jack symmetric functions are dense in $C(\mathcal{U}^N, \mathbb{R})$, (3.19) implies the convergence in distribution of $K_{\mu, M, N}^{\alpha, \beta}(\mu, -\epsilon^{-1} \log v)$ as probability measure on $v \in \exp(-\epsilon \mathbb{Y}_N)$ to a probability measure $K_{\mu, M, N}^{\alpha, \beta}(v, u)$ on $u \in \mathcal{U}^N$. The limiting probability measure satisfies (3.20). ∎

**Definition 3.10.** Let $\theta > 0$, $N \in \mathbb{Z}_{>0}$, $\alpha T \geq 0$, and $M_T \in \mathbb{Z}_{>0}$ such that $M_T \geq N$. The $\beta$-Jacobi product process with parameters $(\alpha T, M_T)_{T \in \mathbb{Z}_{>0}}$, $N$ is a Markov chain $(y(T))_{T \in \mathbb{Z}_{>0}}$ with state space $\mathcal{U}^N$ such that (i) $y(1) \sim P_{\alpha, M, N}^{\alpha, \beta}$ and (ii) $K_{\theta}^{\alpha, \beta}(M_T, N)(v, u)$ is the conditional probability of $y(T) = u$ given $y(T-1) = v$. We denote this probability measure by $\mathbb{P}_{\theta}^{(\alpha T, M_T)_{T \geq 0}, N}$. We refer to Markov chains of the form above as $\beta$-Jacobi product processes where $\beta = 2\theta$. Informally, we may view $y(T)$ as satisfying

$$y(T+1) = y(T) \boxtimes \beta x(T)$$

where $x(T) \sim \mathbb{P}_{\alpha, M, N}^{\alpha, \beta}$, $y(1) = x(1)$ and $\boxtimes \beta$ is some analytic extension of the operation for $\beta = 1, 2, 4$.

**Definition 3.11.** Let $\mathbb{P}_{\theta}^{(\alpha T, M_T)_{T \geq 0}, N}$ be the measure on random Young diagrams $(\lambda(T))_{T \in \mathbb{Z}_{>0}}$ such that $\ell(\lambda(T)) \leq N$,

$$\text{Prob}(\lambda^T = \mu) = \frac{1}{\Pi(\rho^{N}_{0}, \rho^{M}_{0})} P_{\mu}(\rho_{0}^{N}; q, t)Q_{\mu}(\rho_{0}^{M}; q, t),$$

$$\text{Prob}(\lambda^T = \lambda | \lambda^{T-1} = \mu) = K_{\lambda, N}^{\alpha, \beta}(\mu, \lambda), \quad T > 1.$$ 

The measure $\mathbb{P}_{\theta}^{(\alpha T, M_T)_{T \geq 0}, N}$ is a special example of a so-called (infinite) ascending Macdonald process [6]: Section 2.1].

**Theorem 3.12.** Let $\theta > 0$, $N \in \mathbb{Z}_{>0}$, $\alpha T \geq 0$, and $M_T \in \mathbb{Z}_{>0}$ such that $M_T \geq N$ for $T \in \mathbb{Z}_{>0}$. Let $\epsilon > 0$ be a small parameter with $t = \epsilon^{\theta}, q = e^{-\epsilon}$, $\lambda^T := \lambda^T(\epsilon)$ and $K_{\lambda}^{\alpha, \beta}(\mu, \lambda, N)$ be a Markov chain $(y(T))_{T \in \mathbb{Z}_{>0}}$. Then the finite dimensional distributions of $(\exp(-\epsilon \lambda^T))_{T \in \mathbb{Z}_{>0}}$ converge to those of $\mathbb{P}_{\theta}^{(\alpha T, M_T)_{T \geq 0}, N}$ as $\epsilon \to 0$.

**Proof.** We suppress the dependence on $(q, t)$ and $\theta$, though all limits we take are $\epsilon \to 0$ with $q = \exp(-\epsilon)$, $t = \epsilon^{\theta}$. Let $(y(T))_{T \geq 0} \sim \mathbb{P}_{\theta}^{(\alpha T, M_T)_{T \geq 0}, N}$. It suffices to show that

$$\mathbb{E}\tilde{P}_{\alpha, \beta}^{\mu}(\rho_{\lambda}^{N}) \cdots \tilde{P}_{\alpha, \beta}^{\mu}(\rho_{\lambda}^{M}) = \mathbb{E}\tilde{J}_{\kappa, (y(T))}$$

for any $T \geq 1$. We induct on $T$, observing that the statement for $T = 1$ follows from Proposition 3.9 and if the statement is known for $T - 1$ for some $T > 1$ then

$$\mathbb{E}\tilde{P}_{\alpha, \beta}^{\mu}(\rho_{\lambda}^{N}) \cdots \tilde{P}_{\alpha, \beta}^{\mu}(\rho_{\lambda}^{M}) = \frac{\Pi(\rho_{\lambda}^{N}, \rho_{0}^{M})}{\Pi(\rho_{\lambda}^{N}, \rho_{0}^{M}) \mathbb{E}\tilde{P}_{\alpha, \beta}^{\mu}(\rho_{\lambda}^{N}) \cdots \tilde{P}_{\alpha, \beta}^{\mu}(\rho_{\lambda}^{M})}$$

where the first equality follows from Lemma 3.8. The latter converges, by our induction hypothesis and (2.20), to

$$H(\theta, \theta^{M} - \alpha \theta) \sum_{\mu} c_{\kappa, \hat{\alpha}, \lambda}^{\mu} \mathbb{E}\tilde{J}_{\kappa, (y(T-2))} \tilde{J}_{\mu}(y(T-1))$$

which can be written as

$$\frac{H(\theta, \theta^{M} - \alpha \theta)}{H(\theta, \theta^{M} - \alpha \theta)} \mathbb{E}\tilde{J}_{\kappa, (y(T))} \cdots \tilde{J}_{\kappa, T-2}(y(T-2)) \tilde{J}_{\mu}(y(T-1)) \tilde{J}_{\mu}(y(T-1)) = \mathbb{E}\tilde{J}_{\kappa, (y(T))} \cdots \tilde{J}_{\kappa, T}(y(T))$$

by Proposition 3.10. ∎
4. Observables

The main results of this section are formulas for the joint moments of $\beta$-Jacobi product process. We provide formulas for general $\beta > 0$ and more convenient formulas for $\beta = 2$.

Given $U = (u_1, \ldots, u_k), V = (v_1, \ldots, v_l)$, define

$$
\begin{align*}
A_T(U) &:= A_T(U; \theta) := \prod_{i=1}^k \frac{u_i}{u_i + \theta N} \prod_{\tau=1}^T \frac{u_i - \theta(\alpha_\tau - 1)}{u_i - \theta(\alpha_\tau + M_\tau - 1)}, \\
B(U) &:= B(U; \theta) := \prod_{1 \leq i < j \leq k} \frac{(u_j - u_i)(u_j - u_i + 1 - \theta)}{(u_j - u_i + 1)(u_j - u_i - \theta)}, \\
C(U, V) &:= C(U, V; \theta) := \prod_{i=1}^k \prod_{j=1}^l \frac{(v_j - u_i)(v_j - u_i + 1 - \theta)}{(v_j - u_i + 1)(v_j - u_i - \theta)}.
\end{align*}
$$

**Definition 4.1.** Given $(x_1, \ldots, x_N) \in \mathbb{R}^N$, define for any $k > 0$

$$\psi_k(x_1, \ldots, x_N) = \sum_{i=1}^N x_i^k.$$ 

**Theorem 4.2.** Let $\theta, \alpha_T > 0$, $M_T, N \in \mathbb{Z}_{>0}$ such that $M_T \geq N$ for $T \in \mathbb{Z}_{>0}$, and $(y(T))_{T \in \mathbb{Z}_{>0}} \sim P_{\theta}^{(\alpha_T, M_T)}\mathbb{R}_{>0}^N$. If $T_1 \geq \cdots \geq T_m > 0$ and $k_1, \ldots, k_m > 0$ are integers, then

$$
\begin{align*}
\mathbb{E}\left[ \prod_{i=1}^m \psi_{k_i}(y(T_i)) \right] &= \frac{(\theta)^{-m}}{(2\pi i)^{k_1+\cdots+k_m}} \oint \cdots \oint_{1 \leq i < \ell \leq m} C(U_i, U_{i'}) \prod_{i=1}^m B(U_i) A_{T_i}(U_i) dU_i \\
\text{where } U_i &= (u_{i,1}, \ldots, u_{i,k_i}),
\end{align*}
$$

(1) the $u_{i,j}$ contour $\mathcal{U}_{i,j}$ is positively oriented around the point $-\theta N$ and does not enclose $\theta(\alpha_\tau + M_\tau - 1)$ for $1 \leq \tau \leq T_1$;

(2) whenever $(i, j) < (i', j')$ in lexicographical order, $\mathcal{U}_{i,j}$ is enclosed by the contour $\mathcal{U}_{i',j'} + \epsilon$ for $\epsilon \in [-\theta, 0]$;

given that such contours exist.

**Remark 3.** The existence of the contours is guaranteed for $N$ large. Since our applications are for $N$ large, the question of existence is not a hindrance.

**Theorem 4.3.** Let $\alpha_T > 0$, $M_T, N \in \mathbb{Z}_{>0}$ such that $M_T \geq N$ for $T \in \mathbb{Z}_{>0}$, and $(y(T))_{T \in \mathbb{Z}_{>0}} \sim P_{1}^{(\alpha_T, M_T)}\mathbb{R}_{>0}^N$. If $T_1 \geq \cdots \geq T_m > 0$ are integers and $c_1, \ldots, c_m \in \mathbb{R}_{>0}$, then

$$
\begin{align*}
\mathbb{E}\left[ \prod_{i=1}^m \psi_{c_i}(y(T_i)) \right] &= \prod_{i=1}^m \frac{(-c_i)^{-1}}{(2\pi i)^{c_i}} \oint \cdots \oint_{1 \leq i < \ell \leq m} \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \\
&\quad \times \prod_{i=1}^m \left( \frac{u_i + \ell - 1}{u_i + c_i + \ell - 1} \prod_{\tau = 1}^{M_\tau} \frac{u_i + c_i + \alpha_\tau - \ell + 1}{u_i - \alpha_\tau + \ell + 1} \right) du_i
\end{align*}
$$

where the $u_i$ contour $\mathcal{U}_i$ is positively oriented around $\{-c_i - \ell + 1\}_{\ell=1}^N$ but does not enclose $\alpha_\tau + \ell - 1$ for $1 \leq \ell \leq M_\tau$, $1 \leq \tau \leq T_i$, and $\mathcal{U}_i$ is contained in $\mathcal{U}_j - c_i$ for $i < j$; given that such contours exist.

**Remark 4.** We note that Theorem 4.2 holds for $\theta > 0$ arbitrary, but is restricted to taking $\psi_{k_i}$ where $k_i$ are positive integers. Moreover the contours have dimension $k_1 + \cdots + k_m$. In contrast, the contours in Theorem 4.3 for $\theta = 1$ have dimension $m$ and $k_i = c_i$ can be an arbitrary positive real. In this article, we come to these two formulas from seemingly different approaches. However, it was pointed out by E. Dimitrov through private communication that the $\theta > 0$ case is a true generalization of the $\theta = 1$ case. Via residue expansion and combinatorics, the higher dimensional contour integral formulas reduce to $m$-dimensional
The remainder of this section is devoted to the proofs of Theorems 4.2 and 4.3. The starting point is a set of contour integral formulas for exponential moments of Macdonald processes obtained in [25] and further generalized in [1] in a form well-suited for our purposes. These formulas give us expressions for the $\beta$-Jacobi product process via the connection established by Theorem 4.2.

4.1. Formulas for Macdonald Processes. We state and prove discrete analogues of Theorems 4.2 and 4.3 in the setting of Macdonald processes.

**Definition 4.4.** For $\lambda \in \mathbb{Y}_n$ define

$$p_k(\lambda; q, t) = (1 - t^{-k}) \sum_{i=1}^{n} q^{k\lambda_i} t^{k(-i+1)} + t^{-kn}$$

for integers $k > 0$. In the special case $q = t$ with $0 < t < 1$, define

$$p_t(\lambda) := p_1(\lambda; t, t) = (1 - t^{-1}) \sum_{i=1}^{n} t^{\lambda_i - i + 1} + t^{-n}.$$ 

Observe that the definitions of $p_t$ and $p_k$ are independent of $n$ as long as $\ell(\lambda) \leq n$.

If we represent an ordered $k$-tuple of variables $(z_1, \ldots, z_k)$ by $Z$, then denote $Z^{-1} := (z_1^{-1}, \ldots, z_k^{-1})$, $qZ := (qz_1, \ldots, qz_k)$, $dZ := dz_1 \cdots dz_k$. Given $Z = (z_1, \ldots, z_k)$ and $W = (w_1, \ldots, w_{\ell})$, let

$$\mathfrak{A}_T(Z) := \mathfrak{A}_T(Z; t) := \prod_{i=1}^{k} \frac{z_i - 1}{z_i - t^{-kN}} \prod_{\tau=1}^{T} \frac{1 - \rho^{\alpha - 1} z_i}{1 - t^{\alpha + M_T - 1} z_i},$$

$$\mathfrak{B}(Z) := \mathfrak{B}(Z; q, t) := \prod_{i=1}^{k} \frac{1 - q^{i-1} z_i}{z_i z_{i} - 1} \prod_{1 \leq i < j \leq k} \frac{(z_j - z_i)(z_j - z_{i})}{(z_j - qz_i)(z_j - \frac{q}{z_i})},$$

$$\mathfrak{C}(Z, W) := \mathfrak{C}(Z, W; q, t) := \prod_{i=1}^{k} \prod_{j=1}^{\ell} \frac{(w_j - z_i)(w_j - \frac{q}{z_i})}{(w_j - qz_i)(w_j - z_i)}.$$ 

**Proposition 4.5.** Let $(\lambda(T))_{T>0} \sim \mathcal{P}^{(\alpha, \tau, M_T)}_{q,t} \tau > 0$. If $T_1 \geq \cdots \geq T_m > 0$ and $k_1, \ldots, k_m > 0$ are integers, then

$$\mathbb{E} \left[ p_{k_1}(\lambda^{(T_1)}) \cdots p_{k_m}(\lambda^{(T_m)}) \right] = \frac{1}{(2\pi i)^{k_1+\cdots+k_m}} \oint \cdots \oint_{1 \leq i < j \leq m} \mathfrak{C}(Z_i, Z_j) \mathfrak{B}(Z_i) \mathfrak{A}_T(Z_i) dZ_i$$

where $Z_i = (z_{i,1}, \ldots, z_{i,k_i})$,

1. the $z_{i,j}$ contour is positively oriented around the poles at $0, t^N$ but does not enclose $t^{-\alpha_T - M_T + 1}$ for $1 \leq \tau \leq T_i$,
2. the contours satisfy $|z_{i,j}| < t|z_{i',j'}|$ for any $(i, j) < (i', j')$ in lexicographical order,

given that such contours exist.

**Remark 5.** The contours above exist for $N$ large enough as with Theorem 4.2.

**Proof.** Choose $T \geq \max(T_1, \ldots, T_m)$. Consider the measure $\mathfrak{M}$ on $\mathbb{Y}^T$ defined by

$$\mathfrak{M}(\mu^1, \ldots, \mu^T) = \frac{1}{\Pi(\rho^+; \rho^-_1, \ldots, \rho^-_T)} P_{\mu^1} \cdots \Pi(\rho^+_T) \Pi(\rho^-_1/\mu^1 \cdots (\rho^-_T)) \Pi(\mu^1/\rho^-_1) \cdots \Pi(\mu^T/\rho^-_1) \Pi(\rho^-_2) \Pi(\rho^-_3)$$

where $\rho^+ = (a_1, \ldots, a_N), \rho^-_T = (b_{\tau,1}, \ldots, b_{\tau,m})$ and $a_i, b_{\tau,j} > 0$ are chosen so that the quantity above is summable over $(\mu^1, \ldots, \mu^T) \in \mathbb{Y}^T$. Define

$$G_{\tau}(z) = \prod_{i=1}^{N} \frac{1 - t^{-1} a_i z^{-1}}{1 - a_i z^{-1}} \prod_{\tau' = 1}^{M_{\tau'}} \frac{1 - b_{\tau',j} z}{1 - t b_{\tau',j} z},$$
For such a distribution, [1] Theorem 3.8 shows that

$$
E_{\mathcal{M}}[p_{k_1}(\mu^{T_1}) \cdots p_{k_m}(\mu^{T_m})] = \frac{1}{(2\pi i)^{k_1 + \cdots + k_m}} \oint \oint \prod_{1 \leq i < i' \leq m} C(Z_i, Z_{i'}) \prod_{i=1}^m \mathcal{B}(Z_i)G_{T_i}(Z_i) dZ_i
$$

where $|Z_i| = k_i$, the $z_{ij}$ contour is positively oriented around all the poles of $G_{T_i}$ among 0, $a_1, \ldots, a_N$, but does not contain any poles of $G_{T_i}$ among $b_{\tau,1}, \ldots, b_{\tau,M_{\tau}}$ for $1 \leq \tau \leq T_i$, and $|z_{i,j}| < t|z_{i',j'}|$ for $(i,j) < (i',j')$ in lexicographical order. Setting $\rho^+ = \rho_0^N$ and $\rho^- = t^{-a} \rho_0^{M_{\tau}}$ gives the proposition. □

**Proposition 4.6.** Let $(\lambda^{(t)})_{T>0} \sim \mathcal{P}_{q,q}^{(\alpha_T,M_T)_{T>0}}$. If $T_1 \geq \cdots \geq T_m > 0$ and $0 < t_1, \ldots, t_m$, then

$$
E \left[ \prod_{i=1}^m p_t(\lambda^{(T_i)}) \right] = \frac{1}{(2\pi i)^m} \oint \oint \prod_{1 \leq i < j \leq m} (z_j - z_i)(t_iz_j - t_jz_i) \prod_{i=1}^m \prod_{\tau=1}^{T_i} \frac{1 - t_\tau^{-1}q^{\alpha_{\tau} + \ell - 1}z_i}{1 - q^{\alpha_{\tau} + \ell - 1}z_i} dz_i
$$

where the $z_i$ contour is positively oriented around 0, $\{t_\tau q^{\ell-1}N\}_{\ell=1}^{T_i}$ but does not enclose $q^{-\alpha_{\tau} - \ell + 1}$ for $1 \leq \ell \leq M_{\tau}$, $1 \leq \tau \leq T_i$, and the contours satisfy $|z_i| < |t_\tau z_j|$ for $i < j$; given that such contours exist.

**Proof.** Let $(\lambda^{(t)})_{T \in \mathbb{Z}_{>0}} \sim \mathcal{P}_{q,q}^{(\alpha_T,M_T)_{T>0}}$. By (3.13), for any integers $T_1 > \cdots > T_m > 0$, we have

$$
\mathbb{P}(\lambda^{(T)} = \mu, \lambda^{(T-1)} = \mu^2, \ldots, \lambda^{(1)} = \mu^T) = s_{\mu^T}(\rho_0^N)s_{\mu/\mu^2}(q^{\alpha_T\rho_0^T})s_{\mu^2/\mu^3}(q^{\alpha_T\rho_0^T}) \cdots s_{\mu^T}(q^{\alpha_T\rho_0^T})
$$

where $\rho_0^N := \rho_0^N(q,q) = (1, q, \ldots, q^{N-1})$. The distribution of $(\lambda^{(t)}, \ldots, \lambda^{(1)})$ can be described in terms of Schur processes (see Definition [3.1]). If $\rho = (q^{\alpha_T\rho_0^T}, \ldots, q^{\alpha_T\rho_0^T})$ and $(\nu^{1}, \ldots, \nu^{1+\sum_{i=1}^m M_i}) \sim \mathbb{P}_{\rho_0^N, \rho'}$, then

$$(\lambda^{(T)}, \lambda^{(T-1)}, \ldots, \lambda^{(1)}) \stackrel{\text{dist}}{=} (\nu^{1}, \nu^{1+M_T}, \ldots, \nu^{1+\sum_{i=1}^m M_i})$$

by the branching rule on (3.3). Now Proposition 4.6 can be seen as a special case of Theorem 3.12 □

4.2. **Excision of Poles at Zero.** The proofs of Theorems 4.2 and 4.3 are essentially taking the appropriate limit transitions of Propositions 4.4 and 4.6. However, from a technical standpoint there is one preprocessing step. Namely, we want to remove the $t^{-kN}$ term from $p_k$ (likewise $t^{-N}$ from $p_k$); under the Jacobi limit the summand $t^{-kN}$ grows faster than the term $\sum_{i=1}^N q^{\lambda_i} t^{\ell+1}$ which contains the important information. It turns out that the removal of $t^{-kN}$ corresponds to the excision of the pole at 0 in each of the contours.

**Proposition 4.7.** Let $(\lambda^{(t)})_{T>0} \sim \mathcal{P}_{q,t}^{(\alpha_T,M_T)_{T>0}}$. If $T_1 \geq \cdots \geq T_m > 0$ and $k_1, \ldots, k_m > 0$ are integers, then

$$
E \left[ \prod_{i=1}^m (p_{k_i}(\lambda^{(T_i)}) - t^{-k_iN}) \right] = \frac{1}{(2\pi i)^{k_1 + \cdots + k_m}} \oint \oint \prod_{1 \leq i < i' \leq m} C(Z_i, Z_{i'}) \prod_{i=1}^m \mathcal{B}(Z_i)G_{T_i}(Z_i) dZ_i
$$

where $Z_i = (z_{i1}, \ldots, z_{ik_i})$,

(1) the $z_{i,j}$ contour $\mathcal{Q}_{i,j}$ is positively oriented around the pole at $tN$ but does not enclose $t^{-a_{i} - M_{i} + 1}$ for $1 \leq \tau \leq T_i$;

(2) $\mathcal{Q}_{i,j}$ is contained in the contour $c\mathcal{Q}_{i',j'}$ for any $(i,j) < (i',j')$ in lexicographical order and any $c \in [t, \frac{1}{t}]$;

given that such contours exist.

**Remark 6.** There is an asymptotic version of Proposition 4.7 given by [25] Lemmas 4.14-4.17; instead of asserting equality it asserts equality after the Jacobi limit. There are overlapping ideas in the proofs of Proposition 4.7 and the asymptotic version in [25].
Proposition 4.8. Let \((\lambda(T))_{T>0} \sim \mathbb{P}_{y,q}(\alpha_T,M_T)_{T>0},\) if \(T_1 \geq \cdots \geq T_m > 0\) and \(0 < t_1, \ldots, t_m < 1\), then
\[
\mathbb{E} \left[ \prod_{j=1}^{m} (p_{t_j}(\lambda(T_j)) - t_i^{-N}) \right] = \frac{1}{(2\pi i)^m} \oint \cdots \oint \prod_{1 \leq i < j \leq m} \prod_{\tau=1}^{M_T} \frac{(z_j - z_i)(t_1z_j - t_jz_i)}{(t_1z_j - z_i)(z_j - t_jz_i)} \prod_{\tau=1}^{M_T} \frac{1 - t_1^{-1}q^{\ell-1}z_i}{1 - q^{\alpha_i+\ell-1}z_i} \frac{dz_i}{z_i}
\]
where the \(z_i\)-contour \(\gamma_i\) is positively oriented around \((t_1q^{\ell-1})_{i=1}^{N}\) but does not enclose \(q^{-\alpha_i-\ell+1}\) for \(1 \leq \ell \leq M_T\), \(1 \leq i \leq T_i\), and \(\gamma_i\) is contained in \(t_i\gamma_i\) for \(i < j\); given that such contours exist.

Example 4.1. Consider the simple case where \(m = 1\) with \(k_1 = k\) and \(T_1 = T\). We may change the contours in Proposition 4.5 to obtain the formula
\[
\mathbb{E}p_k(\lambda(T)) = \frac{1}{(2\pi i)^k} \oint_{\gamma_1} \cdots \oint_{\gamma_k} \mathbb{B}(Z)\mathbb{A}_T(Z) dZ
\]
where \(Z = (z_1, \ldots, z_k)\), \(\gamma_1 \cup \zeta_i\) is the \(z_i\)-contour such that: (i) \(\gamma_1\) is contained in \(c\gamma_j\) and \(\zeta_i\) is contained in \(c\zeta_j\) for \(1 \leq i < j \leq k\) and \(c \in [t, \frac{1}{t}]\), (ii) \(\gamma_i\) is positively oriented around the pole \(t^{N}\) but does not enclose \(\{t^{-\alpha_i-M_T+1}\}_{\tau=1}^{T}\); (iii) \(\gamma_i\) and \(\zeta_i\) are disjoint from one another.

On the other hand, Proposition 4.7 asserts that
\[
\mathbb{E}p_k(\lambda(T)) - t^{-kN} = \frac{1}{(2\pi i)^k} \oint_{\gamma_1} \cdots \oint_{\gamma_k} \mathbb{B}(Z)\mathbb{A}_T(Z) dZ.
\]
We show how to go from (4.8) to (4.9).

Proof of (4.9). Let \(\mathbb{P}(Z)\) be the power set of \(\{z_1, \ldots, z_k\}\). For each element \(\Upsilon \in \mathbb{P}(Z)\), let \(\Upsilon_i = \zeta_i\) if \(z_i \in \Upsilon\) and \(\Upsilon_i = \gamma_i\) if \(z_i \notin \Upsilon\); \(\Upsilon\) is exactly the set of \(z_i\) such that \(\Upsilon_i\) encircles 0. Let
\[
\mathcal{J}_\Upsilon = \frac{1}{(2\pi i)^k} \oint_{\Upsilon_1} \cdots \oint_{\Upsilon_k} \mathbb{B}(Z)\mathbb{A}_T(Z) dZ.
\]
Expand (4.8) so the \(z_i\)-contour is either \(\gamma_i\) or \(\zeta_i\). Thus
\[
\mathbb{E}p_k(\lambda(T)) = \sum_{\Upsilon \in \mathbb{P}(Z)} \mathcal{J}_\Upsilon.
\]
Observe that \(\mathcal{J}_{\{z_1, \ldots, z_k\}} = \mathbb{A}_T(0)^k = t^{-kN}\) by evaluating residues. Also observe that due to the \(\mathbb{B}(Z)\) term, \(\mathcal{J}_\Upsilon = 0\) unless \(\Upsilon\) is of the form \(\{z_{r+1}, \ldots, z_{r+d}\}\). Identify \(\Upsilon = \{z_{r+1}, \ldots, z_{r+d}\}\) with \((r,d)\) so that
\[
\mathbb{E}p_k(\lambda(T)) = t^{-kN} + \mathcal{J}_\emptyset + \sum_{d=1}^{k-1} \sum_{r=0}^{k-d-1} \mathcal{J}_{(r,d)}.
\]
Consider the following three cases for \((r,d)\):

(I) \(r = 0\). Evaluating the residues at 0 for \(z_1, \ldots, z_d\), we obtain that \(\mathcal{J}_{(r,d)}\) is
\[
\frac{1}{(2\pi i)^{k-d}} \oint \cdots \oint \mathbb{A}_T(0)^d \prod_{1 \leq i < j \leq r} \frac{(z_j - z_i)(z_j - z_i - \frac{d}{t}z_i)}{(z_i - z_j)(z_j - qz_i)} \prod_{d < i \leq k} \mathbb{A}_T(z_i) dz_i.
\]
We may change the contours so that
\[
\mathcal{J}_{(r,d)} = \frac{1}{(2\pi i)^{k-d}} \oint \cdots \oint \mathbb{A}_T(0)^d w_1 \cdots w_{k-d} \prod_{i=1}^{k-d} \mathbb{A}_T(w_i) dw_i
\]
where the \(w_i\)-contour is \(\gamma_i\).
(II) $0 < r < k - d$. Evaluating the residues at 0 for $z_{r+1}, \ldots, z_{r+d}$, we obtain that $\mathcal{I}_{(r,d)}$ is

$$
\frac{-1}{(2\pi i)^{k-d}} \int \cdots \int \mathcal{A}_T(0)^{d\alpha -1} \prod_{1 \leq i < k \atop i \notin \{r, r+d\}} (z_i + q) \prod_{1 \leq i < j \leq k \atop i \notin \{r, r+d\}} (z_j - z_i) \prod_{1 \leq i \leq k \atop i \notin \{r, r+d\}} \mathcal{A}_T(z_i) dz_i.
$$

Reindexing the variables via

$$(z_1, \ldots, z_r) \mapsto (w_1, \ldots, w_r),$$

$$(z_{r+1}, \ldots, z_k) \mapsto (w_{r+1}, \ldots, w_{k-d}),$$

we obtain

$$
\mathcal{I}_{(r,d)} = -\frac{1}{(2\pi i)^{k-d}} \int \cdots \int \mathcal{A}_T(0)^{d\alpha -1} \frac{1}{w_r w_{r+1}} (w_{r+1} - q t w_r) \mathcal{B}(w_1, \ldots, w_{k-d}) \prod_{i=1}^{k-d} \mathcal{A}_T(w_i) dw_i
$$

where we may take the $w_i$-contour to be $\mathcal{Y}_i$.

(III) $r = k - d$. Evaluating the residues at 0 for $z_{r+1}, \ldots, z_k$, we obtain that $\mathcal{I}_{(r,d)}$ is

$$
\frac{-1}{(2\pi i)^{k-d}} \int \cdots \int \mathcal{A}_T(0)^{d\alpha -1} \frac{1}{z_r} (z_2 - \frac{q}{z_1}) \cdots (z_r - \frac{q}{z_{r-1}}) \prod_{1 \leq i \leq r} \frac{(z_j - z_i)(z_j - \frac{q}{z_i})}{(z_j - q z_i)(z_j - \frac{q}{z_i})} \prod_{1 \leq i \leq r} \mathcal{A}_T(z_i) dz_i.
$$

This is exactly

$$
\mathcal{I}_{(r,d)} = -\frac{1}{(2\pi i)^{k-d}} \int \cdots \int \mathcal{A}_T(0)^{d\alpha -1} \mathcal{B}(w_1, \ldots, w_{k-d}) \prod_{i=1}^{k-d} \mathcal{A}_T(w_i) dw_i
$$

where the $w_i$-contour is $\mathcal{Y}_i$.

Fix $0 < d < k$. Combining the three cases and letting $W = (w_1, \ldots, w_{k-d})$, we may write

$$
\sum_{r=0}^{k-d} \mathcal{I}_{(r,d)} = \frac{\mathcal{A}_T(0)^{d}}{(2\pi i)^{k-d}} \int \cdots \int dW \cdot \mathcal{B}(W) \mathcal{A}_T(W) \left( \frac{1}{w_1} - \sum_{r=1}^{k-d-1} \frac{q^{r-1}}{w_r w_{r+1}} (w_{r+1} - q t w_r) - \frac{q^{k-d-1}}{w_{k-d}} \right)
$$

because the $w_i$-contours from each of the different cases were set as $\mathcal{Y}_i$ for $1 \leq i \leq k - d$. The integral vanishes because the parenthesized term may be rewritten as

$$
\frac{1}{w_1} + \sum_{r=1}^{k-d-1} \left( -\frac{1}{w_r t^{r-1}} + \frac{1}{w_{r+1} t^r} \right) - \frac{1}{w_{k-d} t^{k-d-1}} = 0.
$$

Therefore (4.11) simplifies to

$$
\mathbb{E} \rho_0(\lambda(T)) = t^{-kN} \mathcal{I}_0
$$

which is exactly (4.10).

The general proof of Proposition 4.7 is just a factorization into cases of the example above and an inclusion-exclusion argument.

Proof of Proposition 4.7. Define contours $\mathcal{Y}_{i,j}, \mathcal{Z}_{i,j}$ such that (i) $\mathcal{Y}_{i,j}$ is contained in $c_3(i', j')$ in lexicographical order and $c \in [t, \frac{1}{q}]$, (ii) $\mathcal{Y}_{i,j}$ is positively oriented around $t^n$ but does not enclose $t^{-\alpha - M + 1}$ for $1 \leq \tau \leq T_i$ and $\mathcal{Z}_{i,j}$ is positively oriented around 0 but does not enclose $t^{-\alpha - M + 1}$ for $1 \leq \tau \leq T_i$, (iii) $\mathcal{Y}_{m,k_m}$ and $\mathcal{Z}_{m,k_m}$ are disjoint from one another. Note that if contours $\mathcal{Y}_{i,j} (1 \leq j \leq k_i, 1 \leq i \leq m)$ satisfying (i) and (ii) exist, then the existence of contours $\mathcal{Z}_{i,j} (1 \leq j \leq k_i, 1 \leq i \leq m)$ satisfying (i), (ii), (iii) is guaranteed by choosing these contours close enough to 0.

Let $\mathcal{P}(Z_1, \ldots, Z_m)$ be the power set of $\{z_{1,j} \mid 1 \leq j \leq k_i, 1 \leq i \leq m\}$. For each element $\Upsilon \in \mathcal{P}(Z_1, \ldots, Z_m)$, let $\Upsilon_{i,j} = \mathcal{Y}_{i,j}$ if $(i, j) \in \Upsilon$ and $\Upsilon_{i,j} = \mathcal{Z}_{i,j}$ if $(i, j) \notin \Upsilon$. Let

$$
\mathcal{I}_\Upsilon = \frac{1}{(2\pi i)^{k_1 + \cdots + k_m}} \int_{\Upsilon_{1,1}} \cdots \int_{\Upsilon_{m,k_m}} \prod_{1 \leq i \leq m} \mathbb{E}(Z_i, Z_i') \prod_{i=1}^{m} \mathcal{A}(Z_i) \mathcal{A}_{\Upsilon_i}(Z_i) dZ_i.
$$
As in Example 4.1 we can use Proposition 4.3 and expand the contours as either \( Y_{i,j} \) or \( Z_{i,j} \) so that

\[
E \left[ \prod_{1 \leq i \leq m} \lambda(T_i) \right] = \sum_{T \in \mathcal{P}(Z_1, \ldots, Z_m)} \mathcal{J}_T.
\]

For an analogous reason as in Example 4.1 \( \mathcal{J}_T = 0 \) unless \( Y \cap Z_i \) has the form \( \{ \lambda_{i,r_i+1}, \ldots, \lambda_{i,r_i+d_i} \} \) for each \( 1 \leq i \leq m \) in which case we identify \( Y \) with the ordered pair \((r, d)\) where \( r = (r_1, \ldots, r_m) \) and \( d = (d_1, \ldots, d_m) \); if \( Y \cap Z_i = \emptyset \) then \( d_i = 0 \) and take \( r_i = 0 \) as convention. Let \( R_d \) denote the set of \( r \) where \( 0 \leq r_i \leq k_i - d_i \) if \( d_i > 0 \) and \( r_i = 0 \) if \( d_i = 0 \). Then

\[
E \left[ \prod_{1 \leq i \leq m} \lambda(T_i) \right] = \sum_{0 \leq d_i \leq k_i} \sum_{r \in R_d} \mathcal{J}_{(r,d)}.
\]

Fix \( 1 \leq i \leq k \). Let \( W = (w_1, \ldots, w_k) \), define \( A^{(i)}_{0,0}(W) = 1 \) and \( A^{(i)}_{0,k}(W) = \mathfrak{A}_T(0)^k = t^{-kN} \). For \( 0 < d < k \) and \( 0 \leq r \leq k - d \) define

\[
A^{(i)}_{r,d}(W) = \begin{cases} 
\mathfrak{A}_T(0)^d \frac{1}{w_1^{r-1}} & r = 0, \\
-\mathfrak{A}_T(0)^d \frac{w_{r+1}}{w_r} (w_{r+1} - \frac{q}{t} w_r) & 0 < r < k - d, \\
-\mathfrak{A}_T(0)^d \frac{k - r - d}{w_k - d} & r = k - d.
\end{cases}
\]

These were the differing parts of the integrand in Example 4.1 from the three cases where we showed

\[
\sum_{r=0}^{k-d} A^{(i)}_{r,d}(W) = 0.
\]

Consider \( \mathcal{J}_{(r,d)} \) and evaluate the residues of the \( Z_{i,j} \) contours (which are necessarily at 0) in lexicographical order of \((i, j)\). Then by similar reasoning as in Example 4.1 we have

\[
\mathcal{J}_{(r,d)} = \frac{1}{(2\pi i)^{\sum_{i=1}^m (k_i-d_i)}} \oint \cdots \oint \prod_{1 \leq i < j \leq k} \mathcal{C}(W_i, W_j) \prod_{i=1}^m \mathfrak{B}(W_i) \mathfrak{A}_T(W_i) A^{(i)}_{r_i,d_i}(W_i) dW_i
\]

where \( W_i = (w_{i,1}, \ldots, w_{i,k_i-d_i}) \) and the \( w_{i,j} \)-contour can be taken to be \( Y_{i,j} \). Notice that for any fixed \( d \), we have

\[
\sum_{r \in R_d} \prod_{i=1}^m A^{(i)}_{r_i,d_i}(W_i) = \prod_{i:d_i=0} A^{(i)}_{0,d_i}(W_i) \prod_{i:d_i>0} \sum_{r_i=0}^{k_i-d_i} A^{(i)}_{r_i,d_i}(W_i) = \prod_{i:d_i>0} \sum_{r_i=0}^{k_i-d_i} A^{(i)}_{r_i,d_i}(W_i)
\]

where the latter is zero unless we have that \( d_i = 0 \) or \( d_i = k_i \) for each \( 1 \leq i \leq m \). Thus (4.11) becomes

\[
E \left[ \prod_{1 \leq i \leq m} \lambda(T_i) \right] = \sum_{1 \leq i \leq m} \mathcal{J}_T.
\]

Given a subset \( S \subset ([1, m]) \), let

\[
\mathcal{I}_S = \frac{1}{(2\pi i)^{\sum_{i \in S} k_i}} \oint \cdots \oint \prod_{1 \leq i < j \leq k} \mathcal{C}(W_i, W_j) \prod_{i \in S} \mathfrak{B}(W_i) \mathfrak{A}_T(W_i) dW_i
\]

where the \( w_{i,j} \)-contour is \( Y_{i,j} \). Once again, by evaluating the residues of the \( Z_{i,j} \) contours in lexicographical order of \((i, j)\), we have

\[
\mathcal{J}_{\cup_{i \in S} Z_i} = \mathcal{I}_S \prod_{i \in S} \mathfrak{A}_T(0)^{k_i} = t^{-N} \sum_{i \in [1, m]} k_i \cdot \mathcal{I}_S.
\]
Since \( [4,12] \) is over those \( \mathcal{T} \) which are unions of \( Z_i \), we may write
\[
E \left[ \prod_{i=1}^{m} p_k_i (\lambda^{(T_i)}; q, t) \right] = \sum_{S \subseteq \{1, m\}} t^{-N} \sum_{i \in \{1, m\} \setminus S} k_i \cdot I_S.
\]
In particular, observe that for any \( S \subseteq \{1, m\} \),
\[
E \left[ \prod_{i \in S} p_k_i (\lambda^{(T_i)}) \right] = \sum_{S' \subseteq S} t^{-N} \sum_{i \in S \setminus S'} k_i \cdot I_{S'}.
\]
Then by inclusion-exclusion
\[
I_{\{1, m\}} = \sum_{S \subseteq \{1, m\}} (-1)^{m-|S|} t^{-N} \sum_{i \in \{1, m\} \setminus S} k_i \cdot E \left[ \prod_{i \in S} p_k_i (\lambda^{(T_i)}) \right] = E \left[ \prod_{i=1}^{m} \left( p_k_i (\lambda^{(T_i)}) - t^{-k_i N} \right) \right]
\]
which completes the proof. \( \square \)

The proof of Proposition \([4,8]\) follows in the same manner as that of Proposition \([4,7]\) we omit the proof to avoid repetition.

4.3. Jacobi Limit. Using the observables from Proposition \([4,7]\) and the Jacobi limit in Theorem \([3,12]\) we obtain observables for the \( \beta \)-Jacobi product process.

Proof of Theorem \([4,3]\). For \( \varepsilon > 0 \), let \( q = e^{-\varepsilon}, t = q^\theta \) and \( (\lambda^{(T)}; q, t)_{T \in \mathbb{Z}_+} \sim E_{q,t}^{(\sigma_T, M_T); T \geq 0, N} \). Observe that for \( k \geq 0 \) we have
\[
t^{-kN} - p_k (\lambda^{(T)}; q, t) = (e^{\theta k \varepsilon} - 1) \sum_{i=1}^{N} e^{-\varepsilon k \lambda^{(T)}(i)} e^{-\varepsilon k (i+1)}.
\]
By the convergences
\[
e^{\theta k \varepsilon}/k \theta \varepsilon \to 1, \quad e^{\theta k \varepsilon}/k \theta \varepsilon \to 1 (1 \leq i \leq N)
\]
as \( \varepsilon \to 0 \) and Theorem \([3,12]\) we have the convergence
\[
\lim_{\varepsilon \to 0} E \left[ \frac{t^{-k_1N} - p_{k_1}(\lambda^{(T_1)}; q, t)}{k_1 \theta \varepsilon} \cdots \frac{t^{-k_mN} - p_{k_m}(\lambda^{(T_m)}; q, t)}{k_m \theta \varepsilon} \right] \to E \left[ \prod_{i=1}^{m} \mathcal{P}_{k_i}(\lambda^{(T_i)}) \right]
\]
as \( \varepsilon \to 0 \), where \( (\mathcal{Y}^{(T)}; T \in \mathbb{Z}_+ \sim E_{q,t}^{(\sigma_T, M_T); T \geq 0} \).

We compute the limit on the left hand side of \([4,13]\) to obtain an expression for the right hand side. Proposition \([4,7]\) yields the contour integral formula
\[
E \left[ \prod_{i=1}^{m} \frac{t^{-k_iN} - p_{k_i}(\lambda^{(T_i)}; q, t)}{k_i \theta \varepsilon} \right] = \prod_{i=1}^{m} \left( \frac{(-k_i \theta \varepsilon)}{2 \pi i} \right)^{-1} \oint \cdots \oint \prod_{1 \leq i < i' \leq m} \mathcal{C}(Z_i, Z_i') \prod_{i=1}^{m} \mathcal{B}(Z_i) A_{T_i}(Z_i) dZ_i
\]
where \( Z_i = (z_{i,1}, \ldots, z_{i,k_i}) \),

- the \( z_{i,j} \) contour \( \mathcal{Y}_{i,j} \) is positively oriented around \( t^N \) but does not enclose \( t^{-\alpha \tau, -M_{\tau}+1} \) for \( 1 \leq \tau \leq T_i \);
- the \( \mathcal{Y}_{i,j} \) contour is contained in the contour \( \mathcal{C}^{(1, j')} \) for any \( (i, j) < (i', j') \) in lexicographical order and any \( c \in \left( t, \frac{1}{2} \right] \).

Note that we may take the \( z_{i,j} \) contours close to \( t^N \). Changing variables \( z_{i,j} = e^{U_{i,j}} \), we have
\[
E \left[ \prod_{i=1}^{m} \frac{t^{-k_iN} - p_{k_i}(\lambda^{(T_i)}; q, t)}{k_i \theta \varepsilon} \right] = \prod_{i=1}^{m} \left( \frac{(-k_i \theta)}{2 \pi i} \right)^{-1} \oint \cdots \oint \prod_{1 \leq i < i' \leq m} \mathcal{C}(e^{U_{i,j}}, e^{U_{i,j'}}) \prod_{i=1}^{m} \mathcal{B}(e^{U_{i,j}}) A_{T_i}(e^{U_{i,j}}) e^{\epsilon \sum_{j=1}^{k_i} u_{i,j}} dU_i
\]
where \( U_i = (u_{i,1}, \ldots, u_{i,k_i}) \), we denote \( e^{U_{i,j}} := (e^{U_{i,j,1}}, \ldots, e^{U_{i,j,k_i}}) \).
• the $u_{ij}$ contour $\mathcal{C}_{ij}$ is positively oriented around the pole at $-\theta N$ but does not enclose $\theta(\alpha_T + M_T - 1)$ for $1 \leq \tau \leq T_1$;

• whenever $(i, j) < (i', j')$ in lexicographical order, $\mathcal{C}_{ij}$ is enclosed by the contour $\mathcal{C}_{i', j'} + c$ for $c \in [-\theta, 1]$.

These contours are independent of $\epsilon > 0$. We have the convergences

$$\mathcal{A}_T(e^{\epsilon U_1}) \to \mathcal{A}_T(U_1), \quad \mathcal{E}^{k_1, \epsilon}\mathcal{B}(e^{\epsilon U_1}) \to k_1 \mathcal{B}(U_1), \quad \mathcal{C}(e^{\epsilon U_1, e^{U'}}) \to \mathcal{C}(U_1, U')$$

as $\epsilon \to 0$ uniformly over $u_{ij} \in \mathcal{C}_{ij}$ where $1 \leq j \leq k_i, 1 \leq i \leq m$. Therefore

$$\lim_{\epsilon \to 0} \mathbb{E} \left[ \prod_{i=1}^{m} \frac{m - k_i \cdot N - p_k(\lambda(T_k); q, t)}{k_i \theta \epsilon} \right] = \frac{(-\theta)^{-m}}{2\pi \Sigma_{i=1}^k} \oint \cdots \oint \prod_{1 \leq i < \nu \leq m} \mathcal{C}(U_1, U_{\nu}) \prod_{i=1}^{m} \mathcal{B}(U_i) \mathcal{A}_T(U_i) dU_i$$

where the $u_{ij}$ is $\mathcal{C}_{ij}$ as defined above. The theorem now follows from (4.13). \hfill \square

The proof of Theorem 4.3. We highlight some of the modifications, but omit more details to avoid repetition.

Proof of Theorem 4.3. For $\epsilon > 0$, let $q = e^{-\epsilon}$, $t_i = q^{c_i}$ for $1 \leq i \leq m$ and $(\lambda(T) := \lambda(T)(\epsilon))_{T \in \mathbb{Z}_{>0}} \sim \mathbb{P}^{(\alpha_T, M_T) \in \mathbb{Z}_{>0}, N}$. Theorem 4.3 now follows by

$$\lim_{\epsilon \to 0} \mathbb{E} \left[ \prod_{i=1}^{m} \frac{m - k_i \cdot N - p_k(\lambda(T_i))}{c_i \epsilon} \right] \to \mathbb{E} \left[ \prod_{i=1}^{m} \mathcal{P}_{c_i}(y(T_i)) \right]$$

and the change of variables $z_i = e^{\epsilon u_i}$ on the formula given by Proposition 4.8. \hfill \square

5. Global Asymptotics

In this section we prove Theorems 1.2 and 1.3. We begin by reformulating the theorems in terms of convergence of moments.

**Assumption 1.** Fix $\theta > 0$. Let $\alpha_T := \alpha_T(N) > 0, M_T := M_T(N) \in \mathbb{Z}_{>0}$ such that $M_T \geq N$ satisfy

$$(5.1) \quad \frac{\alpha_T}{N} \to \hat{\alpha}_T \quad \text{and} \quad \frac{M_T}{N} \to \hat{M}_T$$

as $N \to \infty$ for some $\hat{\alpha}_T \geq 0, \hat{M}_T \geq 1$. Let $(y(T) := y(T)(N))_{T \in \mathbb{Z}_{>0}} \sim \mathbb{P}^{(\alpha_T, M_T) \in \mathbb{Z}_{>0}, N}$.

**Theorem 5.1.** Under Assumption 1 for any positive integers $k, T$ we have

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\mathcal{P}_k(y(T))] = -\frac{1}{k} \cdot \frac{1}{2\pi i} \oint \left( \frac{v}{v + 1} \prod_{\tau=1}^{T} \frac{v - \hat{\alpha}_\tau}{v - \hat{\alpha}_\tau - \hat{M}_\tau} \right)^k dv$$

where the contour is positively oriented around the pole at $-1$ but does not enclose $\hat{\alpha}_\tau + \hat{M}_\tau$ for $1 \leq \tau \leq T$.

**Theorem 5.2.** Under Assumption 1 for any positive integers $k_1, k_2, T_1 \geq T_2$ we have

$$\lim_{N \to \infty} \text{Cov} \left( \mathcal{P}_{k_1}(y(T_1)), \mathcal{P}_{k_2}(y(T_2)) \right) = \frac{\theta^{-1}}{(2\pi i)^2} \oint \left( \frac{1}{(v_2 - v_1)^2} \prod_{\tau=1}^{T_1} \left( \frac{v_1}{v_1 + 1} \prod_{\tau=1}^{T_2} \frac{v_1 - \hat{\alpha}_\tau}{v_1 - \hat{\alpha}_\tau - \hat{M}_\tau} \right)^{k_1} dv_1 \right.$$

where the $v_2$ contour encloses the $v_1$ contour, both $v_1, v_2$ contours are positively oriented around $-1$, but the $v_1$ contour does not contain $\hat{\alpha}_\tau + \hat{M}_\tau$ for $1 \leq \tau \leq T_1$ and the $v_2$ contour does not contain $\hat{\alpha}_\tau + \hat{M}_\tau$ for $1 \leq \tau \leq T_2$.

**Theorem 5.3.** Under Assumption 1 for any positive integers $m \geq 3, k_1, \ldots, k_m, T_1, \ldots, T_m$ we have

$$\kappa \left( \mathcal{P}_{k_1}(y(T_1)), \ldots, \mathcal{P}_{k_m}(y(T_m)) \right) \to 0$$

as $N \to \infty$.

We show how these theorems imply Theorems 1.2 and 1.3.
Proof of Theorem 1.2. This follows from the convergence Theorem 5.1 and the fact that
\[ \lim_{N \to \infty} \text{Var} \left( \frac{1}{N} \mathcal{P}_k(Y^{(T)}) \right) = 0 \]
by Theorem 5.2. \( \square \)

Proof of Theorem 1.3. Theorem 5.2 implies the covariance formula (1.4). To prove asymptotic Gaussianity, first observe that for \( \nu \geq 3 \) and \( 1 \leq i_1, \ldots, i_{\nu} \leq k \), we have
\[ \kappa \left( \mathbb{P}_{k_{i_1}}(Y^{(T_{i_1})}) - \mathbb{E}[\mathbb{P}_{k_{i_1}}(Y^{(T_{i_1})})], \ldots, \mathbb{P}_{k_{i_{\nu}}}(Y^{(T_{i_{\nu}})}) - \mathbb{E}[\mathbb{P}_{k_{i_{\nu}}}(Y^{(T_{i_{\nu}})})] \right) = \kappa \left( \mathbb{P}_{k_{i_1}}(Y^{(T_{i_1})}), \ldots, \mathbb{P}_{k_{i_{\nu}}}(Y^{(T_{i_{\nu}})}) \right) \]
by (A2). By Theorem 5.3 the latter converges to 0. Lemma A.3 then implies asymptotic Gaussianity of (1.3). \( \square \)

The remainder of this section is devoted to the proofs of Theorems 5.1 to 5.3. We first state a useful lemma.

Lemma 5.4 (Corollary A.2). Let \( s \) be a positive integer. Let \( f, g_1, \ldots, g_s \) be meromorphic functions with possible poles at \( \{ P_1, \ldots, P_m \} \). Then for \( n \geq 2 \),
\[ \frac{1}{(2\pi i)^n} \oint_{\Gamma_1} \cdots \oint_{\Gamma_n} \frac{1}{(v_2 - v_1)(v_n - v_{n-1})} \prod_{i=1}^{n} f(v_i) dv_i \prod_{i=1}^{s} \left( \sum_{j=1}^{n} g_i(v_j) \right) = \frac{n^{s-1}}{2\pi i} \oint f(v) \prod_{i=1}^{s} g_i(v) dv, \]
where the contours in both sides are positively oriented around all of \( \{ P_1, \ldots, P_m \} \), and for the left hand side we require the \( v_i \) contour to contain the \( v_i \) contour for \( 1 \leq i < j \leq n \).

5.1. Proof of Theorem 5.1. By Theorem 4.2 we have
\[ \mathbb{E} \left[ \mathcal{P}_k(Y^{(T)}) \right] = \frac{-1}{(2\pi i)^k} \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \frac{1}{(v_2 - v_1 + 1 - \theta) \cdots (v_k - v_{k-1} + 1 - \theta)} \times \prod_{1 \leq i < j \leq k} \frac{(u_j - u_i)(u_j - u_i + 1 - \theta)(u_j - u_i - 1)}{(u_j - u_i + 1)(u_j - u_i - 1)} \prod_{i=1}^{k} \frac{u_i - \theta}{u_i + \theta N} \prod_{\tau=1}^{T} \frac{u_i - \theta}{u_i - \theta(\alpha_{\tau} - 1) + M_{\tau} - 1} du_i. \]

where
(1) the \( u_i \) contour \( \Gamma_i \) is positively oriented around \( -\theta N \) but does not enclose \( \theta(\alpha_{\tau} + M_{\tau} - 1) \) for \( 1 \leq \tau \leq T \);
(2) whenever \( i < j \), \( \Gamma_i \) is enclosed by the contour \( \Gamma_j + c \) for \( c \in [-\theta, 1] \).

Changing variables \( u_i = v_i \theta N \), we obtain
\[ \frac{1}{N} \mathbb{E} \left[ \mathcal{P}_k(Y^{(T)}) \right] = \frac{-1}{(2\pi i)^k} \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \frac{1}{(v_2 - v_1 + 1 - \theta) \cdots (v_k - v_{k-1} + 1 - \theta)} \times \prod_{1 \leq i < j \leq k} \frac{(v_j - v_i)(v_j - v_i + 1) - \theta}{(v_j - v_i + 1)(v_j - v_i - 1)} \prod_{i=1}^{k} \frac{v_i - \theta}{v_i + \theta N} \prod_{\tau=1}^{T} \frac{v_i - \theta}{v_i - \theta(\alpha_{\tau} + M_{\tau} - 1) + N} dv_i. \]

Then
\[ \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \mathcal{P}_k(Y^{(T)}) \right] = \frac{-1}{(2\pi i)^k} \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \frac{1}{(v_2 - v_1) \cdots (v_k - v_{k-1})} \prod_{i=1}^{k} \frac{v_i}{v_i + 1} \prod_{\tau=1}^{T} \frac{v_i - \alpha_{\tau}}{v_i - \alpha_{\tau} + M_{\tau} - 1} dv_i. \]

where
(1) the \( v_i \) contour \( \Psi_i \) is positively oriented around \( -1 \) but does not enclose \( \alpha_{\tau} + M_{\tau} \) for \( 1 \leq \tau \leq T \);
(2) whenever \( i < j \), \( \Psi_i \) is enclosed by the contour \( \Psi_j \).

The theorem now follows by applying Lemma 5.4 to the multidimensional integral above. \( \square \)
5.2. **Proof of Theorem 5.2**  By Theorem 4.2 we have

\[
\text{Cov} \left( \Psi_{k_1}(y(T_1)), \Psi_{k_2}(y(T_2)) \right) = \frac{\theta^{-2}}{(2\pi i)^{k_1+k_2}} \oint \cdots \oint (C(U_1, U_2) - 1) \prod_{i=1}^{2} B(U_i) A_{T_i}(U_i) dU_i
\]

where \( U_i = (u_{i,1}, \ldots, u_{i,k_i}) \) for \( i = 1, 2, \)

1. the \( u_{i,j} \) contour \( \mathfrak{U}_{i,j} \) is positively oriented around \(-\theta N\) but does not enclose \( \theta (\alpha_r + M_r - 1) \) for \( 1 \leq r \leq T_i; \)
2. whenever \((i, j) < (i', j')\) in lexicographical order, \( \mathfrak{U}_{i,j} \) is enclosed by the contour \( \mathfrak{W}_{i',j'} + c \) for \( c \in [-\theta, 1]. \)

Changing variables \( u_{i,j} = v_{i,j} \theta N \), we obtain

\[
\text{Cov} \left( \Psi_{k_1}(y(T_1)), \Psi_{k_2}(y(T_2)) \right) = \frac{1}{(2\pi i)^{k_1+k_2}} \oint \cdots \oint N^2 \left( \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{(v_{2,j} - v_{1,i})(v_{2,j} - v_{1,i} + \frac{1-\theta}{\theta N})}{(v_{2,j} - v_{1,i} + \frac{1-\theta}{\theta N})(v_{2,j} - v_{1,i} - \frac{1}{\theta N})} - 1 \right)
\]

\[
\times \prod_{i=1}^{k_1} \frac{1}{(v_{i,2} - v_{i,1} + \frac{1-\theta}{\theta N})} \cdots \prod_{i=1}^{k_1} \frac{1}{(v_{i,k_i - 1} - v_{i,k_i} + \frac{1-\theta}{\theta N})} \prod_{j=1}^{k_2} \frac{1}{(v_{j,1} - v_{j,2} + \frac{1-\theta}{\theta N})} \cdots \frac{1}{(v_{j,k_j - 1} - v_{j,k_j} + \frac{1-\theta}{\theta N})}
\]

\[
\times \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{1}{v_{i,j} - v_{i,j} + \frac{1-\theta}{\theta N}} \prod_{\tau=1}^{T_i} \int_{v_{i,j} - \frac{\alpha_r - 1}{\theta N}}^{v_{i,j} + \frac{\alpha_r + M_r - 1}{\theta N}} dv_{i,j}
\]

We may take the \( v_{i,j} \) contours to be independent of \( N \) for \( N \) sufficiently large. Observe that

\[
\prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{(v_{2,j} - v_{1,i})(v_{2,j} - v_{1,i} + \frac{1-\theta}{\theta N})}{(v_{2,j} - v_{1,i} + \frac{1-\theta}{\theta N})(v_{2,j} - v_{1,i} - \frac{1}{\theta N})} = \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \left( 1 + \frac{\theta^{-1}}{N^2} \cdot \frac{1}{(v_{2,j} - v_{1,i} + \frac{1-\theta}{\theta N})(v_{2,j} - v_{1,i} - \frac{1}{\theta N})} \right)
\]

\[
= 1 + \frac{\theta^{-1}}{N^2} \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \frac{1}{(v_{2,j} - v_{1,i})^2} + O \left( \frac{1}{N^3} \right)
\]

where the \( O(1/N^3) \) term is uniform over the \( v_{i,j} \) contours. Then

\[
\lim_{N \to \infty} \text{Cov} \left( \Psi_{k_1}(y(T_1)), \Psi_{k_2}(y(T_2)) \right) = \frac{\theta^{-1}}{(2\pi i)^{k_1+k_2}} \oint \cdots \oint \frac{1}{(v_{2,j} - v_{1,i})^2} \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{1}{(v_{i,j} - v_{i,j} + \frac{1-\theta}{\theta N})} \prod_{\tau=1}^{T_i} \int_{v_{i,j} - \frac{\alpha_r - 1}{\theta N}}^{v_{i,j} + \frac{\alpha_r + M_r - 1}{\theta N}} dv_{i,j}
\]

where

1. the \( v_{i,j} \) contour \( \mathfrak{W}_{i,j} \) is positively oriented around \(-1\) and does not enclose \( \hat{\alpha}_r + \hat{M}_r \) for \( 1 \leq r \leq T_i; \)
2. whenever \((i, j) < (i', j')\), \( \mathfrak{W}_{i,j} \) is enclosed by the contour \( \mathfrak{W}_{i',j'} \).

The theorem now follows by applying Lemma 5.4 twice to the multidimensional integral above. \( \square \)

5.3. **Proof of Theorem 5.3**  By Definition A.1 we have

\[
\kappa \left( \Psi_{k_1}(y(T_1)), \ldots, \Psi_{k_m}(y(T_m)) \right) = \sum_{\{s_1, \ldots, s_d\} \in \Theta_m} (-1)^{d-1} (d-1)! \prod_{\ell=1}^{d} \mathbb{E} \left[ \prod_{i \in S_\ell} \Psi_{k_i}(y(T_i)) \right]
\]

where we use the notation from the Appendix with \( \Theta_m \) denoting the collection of all set partitions of \([1, m]\). Assume \( T_1 \geq \cdots \geq T_m \) so that the conditions of Theorem 4.2 are met. Then

\[
\kappa \left( \Psi_{k_1}(y(T_1)), \ldots, \Psi_{k_m}(y(T_m)) \right) = \frac{(-\theta)^{-m}}{(2\pi i)^{\sum r k_r}} \oint \cdots \oint \tilde{C}(U_1, \ldots, U_m) \prod_{i=1}^{m} B(U_i) A_{T_i}(U_i) dU_i
\]
where $U_i = (u_{i,1}, \ldots, u_{i,k_i})$ and

$$
\tilde{C}(U_1, \ldots, U_m) = \sum_{d>0} \sum_{\{S_1, \ldots, S_d\} \in \Theta_m} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \prod_{i<j} \mathcal{C}(U_i, U_j).
$$

We note that the contours are such that

1. the $u_{i,j}$ contour $\mathfrak{U}_{i,j}$ is positively oriented around $-\theta N$ but does not enclose $\theta(\alpha_T + M_T - 1)$ for $1 \leq \tau \leq T_i$;
2. whenever $(i,j) < (i', j')$, $\mathfrak{U}_{i,j}$ is enclosed by the contour $\mathfrak{U}_{i', j'} + c$ for $c \in [-\theta, 0]$.

Let $S \subset [[1, m]]$, $\mathcal{T}(S)$ denote the set of undirected simple graphs with vertices labeled by $S$ and $\mathcal{L}(S) \subset \mathcal{T}(S)$ denote the subset of connected graphs. Given a graph $\Omega$, we denote by $E(\Omega)$ the edge set of $\Omega$. We claim that

$$
\tilde{C}(U_1, \ldots, U_m) = \sum_{\Omega \in \mathcal{L}([[1, m]])} \prod_{(i,j) \in E(\Omega)} (\mathcal{C}(U_i, U_j) - 1).
$$

Define

$$
\mathcal{K}(S) := \sum_{\Omega \in \mathcal{L}(S)} \prod_{(i,j) \in E(\Omega)} (\mathcal{C}(U_i, U_j) - 1), \quad \mathcal{E}(S) := \sum_{\Omega \in \mathcal{T}(S)} \prod_{(i,j) \in E(\Omega)} (\mathcal{C}(U_i, U_j) - 1).
$$

Then

$$
\mathcal{E}(S) = \sum_{d>0} \prod_{\ell=1}^{d} \mathcal{K}(S_{\ell})
$$

where $\Theta_S$ is the collection of partitions of $S$. By Lemma [A.4], we have

$$
\mathcal{K}(S) = \sum_{\{S_1, \ldots, S_d\} \in \Theta_S} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \mathcal{E}(S_{\ell})
$$

which agrees with the right hand side of (5.4) when $S = [[1, m]]$. Thus (5.5) follows.

Applying (5.5), (5.3) becomes

$$
k(\Psi_{k_1} (y^{(T_1)}) \cdots \Psi_{k_m} (y^{(T_m)})) = \sum_{\Omega \in \mathcal{L}([[1, m]])} (-\theta)^{-m} (2\pi i)^{\sum_{i} k_i} \prod_{(i,j) \in E(\Omega)} (\mathcal{C}(U_i, U_j) - 1) \prod_{i=1}^{m} E(U_i) A_{\mathfrak{T}_1(U_i)} dU_i
$$

$$
= \sum_{\Omega \in \mathcal{L}([[1, m]])} \mathcal{I}_\Omega.
$$

We want to show that $\mathcal{I}_\Omega \sim o(1)$ for each $\Omega \in \mathcal{L}([[1, m]])$ as $N \to \infty$. Fix $\Omega \in \mathcal{L}([[1, m]])$. Changing variables $u_{i,j} = \theta N v_{i,j}$, we obtain

$$
\mathcal{I}_\Omega = \prod_{i=1}^{m} \prod_{1 \leq a < b \leq k_i} \frac{1}{(v_{i,2} - v_{i,1} + \frac{1-\theta}{\theta N}) \cdots (v_{i,k_i} - v_{i,k_i-1} + \frac{1-\theta}{\theta N})} \left( \prod_{1 \leq a < b \leq k_i} \frac{(v_{i,b} - v_{i,a} + \frac{1-\theta}{\theta N})(v_{i,b} - v_{i,a} + \frac{1-\theta}{\theta N})}{(v_{i,b} - v_{i,a} + \frac{1-\theta}{\theta N})(v_{i,b} - v_{i,a} + \frac{1-\theta}{\theta N})} \right)
$$

$$
\times \prod_{j=1}^{k_i} \frac{v_{i,j} - \alpha - 1}{v_{i,j} + \frac{1}{\theta N}} dv_{i,j}.
$$
The contours may be chosen to be fixed for sufficiently large \( N \). In the same manner that we have \([31, \text{Theorem 3.2}]\) in the proof of Theorem 5.2, we have
\[
\prod_{\alpha=1}^{k_1} \prod_{\beta=1}^{k_j} \left( \frac{(v_{j, \beta} - v_{i, \alpha}) (v_{j, \beta} - v_{i, \alpha} + 1 - \theta)}{\theta N} \right) = 1 + \frac{1}{N^2} \sum_{\alpha=1}^{k_1} \sum_{\beta=1}^{k_j} \frac{1}{(v_{j, \beta} - v_{i, \alpha})^2} + O \left( \frac{1}{N^3} \right).
\]
This implies that \( \mathcal{I}_\Omega \sim O(N^{m-2|E(\Omega)|}) \). For any \( \Omega \in \mathcal{L}([1, m]]) \), we have that \( |E(\Omega)| \geq m - 1 \). Therefore \( \mathcal{I}_\Omega \sim o(1) \) whenever \( m \geq 3 \). This completes the proof.

6. Local Fluctuations at the Edge

In this section, we prove the main results Theorems 1.6 to 1.8 for local fluctuations. We begin by reformulating the main theorems.

**Assumption 2.** Fix \( \theta > 0 \). Let \( T_1 := T_1(N), \ldots, T_m := T_m(N) \) be integers, \( \alpha := \alpha(N) \geq 1 \) and \( M := M(N) \in \mathbb{Z}_{>0} \) such that \( M \geq N \) satisfy
\[
\frac{T_i}{N} \rightarrow \hat{T}_i, \quad \frac{\alpha}{N} \rightarrow \hat{\alpha}, \quad \frac{M}{N} \rightarrow \hat{M}
\]
as \( N \rightarrow \infty \) for some \( \hat{\alpha} \geq 0, \hat{M} \geq 1 \) and \( \hat{T}_i > 0, 1 \leq i \leq m \). Let \( (y^{(T)}(N))_{T \in \mathbb{Z}^m} \sim P_{\theta}^{(\alpha, M)^{m>0, N}} \).

**Theorem 6.1.** Let \( c_1, \ldots, c_m > 0 \) such that \( c_2 + \cdots + c_m < 1 \). Let \( T_1 := T_1(N), \ldots, T_m := T_m(N) \) be integers satisfying
\[
\frac{T_i}{N} \rightarrow \hat{T}_i
\]
as \( N \rightarrow \infty \) for some \( \hat{T}_i > 0, 1 \leq i \leq m \). If \( (y^{(T)}(N))_{T \in \mathbb{Z}^m} \) is distributed as the squared singular values of \( Y_T \) as in (1.1) where \( X_1, X_2, \ldots \) are independent \( N \times N \) complex Ginibre matrices, then
\[
\lim_{N \rightarrow \infty} \mathbb{E} \left[ \prod_{i=1}^{m} \frac{1}{N^{c^2(T_i) + 1}} \mathcal{Q}_{c_i}(y^{(T_i)}) \right] = \left( \frac{-1}{(2\pi i)^m} \right) \left( \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_i - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_i - u_i)} \right) \prod_{i=1}^{m} \Gamma \left( \frac{-u_i - c_i}{c_i} \right) \Gamma \left( \frac{-u_i}{c_i} \right) \exp \left[ -\hat{T}_i \frac{c_i(c_i + 1)}{2} - \hat{T}_i c_i u_i \right] \frac{d u_i}{c_i}
\]
where the \( u_i \) contour \( \mathbb{C}_i \) is a positively oriented contour around \( -c_i, -c_i + 1, -c_i + 2, \ldots \) which starts and ends at \( +\infty \), and \( \mathbb{C}_i \) is enclosed by \( \mathbb{C}_j - c_i \) for \( i < j \).

This gives us Theorem 1.6. By combining [31, Theorem 3.2] with Theorem 6.1, we obtain Theorem 1.8. The next theorem implies Theorem 1.7.

**Theorem 6.2.** Let \( c_1, \ldots, c_m > 0 \) such that \( c_2 + \cdots + c_m < 1 \). Under Assumption 2 with \( \theta = 1 \), we have
\[
\lim_{N \rightarrow \infty} \mathbb{E} \left[ \prod_{i=1}^{m} \frac{M + N + \alpha - 1}{N^{c^2(N + \alpha + 1)} \mathcal{Q}_{c_i}(y^{(T_i)})} \right] = \left( \frac{-1}{(2\pi i)^m} \right) \left( \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_i - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_i - u_i)} \right) \prod_{i=1}^{m} \Gamma \left( \frac{-u_i - c_i}{c_i} \right) \Gamma \left( \frac{-u_i}{c_i} \right) \exp \left[ -\hat{T}_i \left( \frac{1}{M + 1 + \alpha} \right) \left( \frac{c_i(c_i + 1)}{2} + c_i u_i \right) \right] \frac{d u_i}{c_i}
\]
where the \( u_i \) contour \( \mathbb{C}_i \) is a positively oriented contour around \( -c_i, -c_i + 1, -c_i + 2, \ldots \) which starts and ends at \( +\infty \), and \( \mathbb{C}_i \) is enclosed by \( \mathbb{C}_j - c_i \) for \( i < j \).

Our final theorem for arbitrary \( \beta > 0 \) implies Theorem 1.8.
Theorem 6.3. Let \( k_1, \ldots, k_m \) be positive integers. Under Assumption 2 we have

\[
\lim_{N \to \infty} \mathbb{E} \left[ \prod_{i=1}^{m} \frac{1}{N^{k_i}} \left( \frac{M + N + \alpha - 1}{N + \alpha - 1} \right)^{k_i} \mathcal{P}_{k_i}(\gamma^{(T_i)}) \right] = \frac{(-\beta/2)^{-m}}{(2\pi)^{\sum_{i=1}^{m} k_i}} \int \cdots \int_{0 < i, j < k} \prod_{1 \leq i < j \leq k} \left\{ \begin{array}{c}
\frac{(u_{i,j} - u_{i,j})(u_{i,j} + 1 - \beta/2)}{(u_{i,j} - u_{i,j} + 1)(u_{i,j} - u_{i,j} - \beta/2)} \\
\frac{1}{u_{i,a+1} - u_{i,a} + 1 - \beta/2} \end{array} \right\} \prod_{j=1}^{k} e^{-((1+\tilde{\alpha})^{-1} - (M+1+\tilde{\alpha})^{-1})T_j} \left( (\beta/2)^{-1} - u_{i,j} \right)\] 

where the \( u_{i,j} \) contour is positively oriented around 0, the \( u_{i,j} \) contour encloses a \( \max(\theta, 1) \) neighborhood of the \( u_{i,j} \) contour for \( (i, j) < (i', j') \) in lexicographical order.

Although Theorem 6.3 is not a perfect analogue of Theorem 6.2, for example the dimension of the contour integral formulas and the range of values for the Laplace transform, our approach for both the \( \theta > 0 \) and \( \theta = 1 \) case is the same general moment formulas (see Remark 4). The differentiation between the two cases then comes from additional structure available in the \( \theta = 1 \) case.

The remainder of this section is organized as follows. In Section 6.1 we obtain formulas for the Ginibre case via Theorem 4.2 under the appropriate limit. We then gather some lemmas in Section 6.2 to ease the presentation of the proofs of Theorems 6.1 to 6.3 in Section 6.3.

6.1. Formulas for the Ginibre Case. We obtain formulas for the joint moments of the squared singular values of complex Ginibre products by taking the appropriate limits of Theorems 4.2 and 4.3. These limit transitions rely on the following lemma:

Lemma 6.4. Suppose \((u^{(T)} := u^{(T)}(M))_{T \in \mathbb{Z}_{>0}} \sim \mathcal{P}^{(n, M)^{T>0}} \). Then \((M^T u^{(T)})_{T \in \mathbb{Z}_{>0}} \) converges as \( M \to \infty \) in finite dimensional distributions to the squared singular values \((\gamma^{(T)})_{T \in \mathbb{Z}_{>0}} \) of \( Y_T \) as in (1.1) where \( X_1, X_2, \ldots \) are independent, \( N \times N \) complex Ginibre matrices.

The lemma follows from the observation that \( M \to \infty \) corresponds to growing the size of the ambient unitary matrix. Recall that the matrices \( X_T \) in the square case can be obtained as submatrices of a Haar unitary matrix of size \( M \times N \). After renormalizing by \( 1/\sqrt{M} \) (renormalizing the singular value by \( 1/M \)) the entries of the unitary matrix behave like independent complex standard Gaussians in the limit.

Given \( U = (u_1, \ldots, u_k) \), \( V = (v_1, \ldots, v_l) \), let \( \mathcal{B}(U), \mathcal{C}(U, V) \) be defined as in (1.1) and define

\[
\tilde{\mathcal{A}}_T(U) = \theta^{-kT} \prod_{i=1}^{k} \frac{u_i^{T+1}}{-u_i - \theta N}.
\]

Proposition 6.5. Suppose \((\gamma^{(T)})_{T \in \mathbb{Z}_{>0}} \) are the squared singular values of \( Y_T \) as in (1.1) where \( X_1, X_2, \ldots \) are independent, \( N \times N \) complex Ginibre matrices. If \( T_1 \geq \cdots \geq T_m > 0 \) are integers and \( c_1, \ldots, c_m \in \mathbb{R}_{>0} \), then

\[
\mathbb{E} \left[ \prod_{i=1}^{m} \mathcal{P}_{c_i}(\gamma^{(T_i)}) \right] = \frac{\prod_{i=1}^{m} (-c_i)^{-1}}{(2\pi)^{m}} \int \cdots \int_{1 \leq i < j \leq m} \prod_{1 \leq i < j \leq m} \frac{(u_i - u_j)(u_j + c_j - u_i - c_i)}{(u_i - u_j - c_j)(u_j + c_j - u_i)} \times \prod_{i=1}^{m} \left( \prod_{k=1}^{N} \frac{u_i + \ell - 1}{u_i + c_i + \ell - 1} \left( \frac{\Gamma(1-u_i)}{\Gamma(1-u_i - c_i)} \right)^{T_i} \right) du_i
\]

where the \( u_i \) contour \( \mathcal{U}_i \) is positively oriented around \(-c_i - \ell + 1 \) \( i = 1 \) but does not enclose 1, 2, 3, \ldots, and \( \mathcal{U}_i \) is contained in \( \mathcal{U}_j - c_i \) for \( i < j \); given that such contours exist.
Proof. Suppose \( (u^{(T)})_{T \in \mathbb{Z}_{>0}} \sim \mathbb{P}_1^{(a_T, M_T)_{T > 0}, N} \) with \( a_T = 1 \) and \( M_T = 1 \) for all \( T > 0 \). By Theorem 4.3, we have

\[
\mathbb{E} \left[ \prod_{i=1}^m \mathcal{P}_{c_i}(u^{(T_i)}) \right] = \prod_{i=1}^m \left( -c_i \right)^{-1} \int \cdots \int \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \\
\times \prod_{i=1}^m \left( \prod_{i=1}^N \frac{u_i + \ell - 1}{u_i + c_i + \ell - 1} \cdot \left( \prod_{\ell=1}^{M} \frac{\ell - u_i - c_i}{\ell - u_i} \right) \right) \, du_i
\]

where the \( u_i \) contour is positively oriented around \( \{-c_i - \ell + 1\}_{i=1}^N \) but does not enclose \( 1, \ldots, M \), and \( \mathcal{U}_i \) is contained in \( \mathcal{U}_j - c_i \). Taking the Ginibre limit gives

\[
\lim_{M \to \infty} \mathbb{E} \left[ \prod_{i=1}^m \mathcal{P}_{c_i}(u^{(T_i)}) \right] = \mathbb{E} \left[ \prod_{i=1}^m \mathcal{P}_{c_i}(y^{(T_i)}) \right].
\]

This convergence, along with the following asymptotics, then implies this proposition:

\[
\lim_{M \to \infty} M^c \prod_{\ell=1}^M \frac{\ell - u - c}{\ell - u} = \frac{\Gamma(1 - u)}{\Gamma(1 - u - c)} \lim_{M \to \infty} M^c \frac{\Gamma(M + 1 - u - c)}{\Gamma(M + 1 - u)} = \frac{\Gamma(1 - u)}{\Gamma(1 - u - c)}
\]

uniformly in \( u \) on compact subsets of \( \mathbb{C} \) by Stirling’s approximation for the Gamma function. \( \square \)

6.2. Preliminary Asymptotics.

Lemma 6.6. For any \( c > 0 \), we have

\[
\log \frac{\Gamma(z + c)}{\Gamma(z)} = c \log(z + c) - \frac{1}{2} \frac{c(c + 1)}{z} + R(z)
\]

where the remainder satisfies

\[
|R(z)| \leq \frac{C_\delta}{|z|^2}
\]

over \( \{z \in \mathbb{C} : \Re z \geq 0, |z| \geq c(1 + \delta)\} \) for any fixed \( \delta > 0 \) with \( C_\delta \) some constant depending on \( \delta \).

Proof. By Stirling’s approximation, we have

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) + \frac{1}{12z} + \tilde{R}(z), \quad |\tilde{R}(z)| \leq \frac{C}{|z|^2}
\]

for some uniform constant \( C \) over \( \{z \in \mathbb{C} : \Re z \geq 0\} \), see [33, p.141]. We have

\[
\log \frac{\Gamma(z + c)}{\Gamma(z)} = \left( z + c - \frac{1}{2} \right) \log(z + c) - (z + c) + \frac{1}{12(z + c)} - \left( z - \frac{1}{2} \right) \log z - z - \frac{1}{12z} + R_1(z)
\]

\[
= \left( z - \frac{1}{2} \right) \log \left( 1 + \frac{c}{z} \right) - c + c \log(z + c) + R_2(z)
\]

where

\[
\max(|R_1(z)|, |R_2(z)|) \leq \frac{C}{|z|^2}
\]

over \( \{\Re z \geq 0\} \) for some constant \( C \). By expanding the first logarithm, we get

\[
\log \frac{\Gamma(z + c)}{\Gamma(z)} = \left( z - \frac{1}{2} \right) \left( \frac{c}{z} - \frac{1}{2} \right) - c + c \log(z + c) + R_3(z)
\]

\[
= -\frac{1}{2} \frac{c(c + 1)}{z} + c \log(z + c) + R_4(z)
\]
where for any given \( \delta > 0 \), we have
\[
\max(|R_0(z)|, |R_4(z)|) \leq \frac{C_\delta}{|z|^2}
\]
over \( \{ \Re z \geq 0, |z| \geq c(1 + \delta) \} \) for some constant \( C_\delta \) depending on \( \delta \).
\( \square \)

Define
\[
G_L(u) := \frac{L^{-c} T(L - u)}{\Gamma(L - u - c)}.
\]

Lemma 6.7. Fix \( \delta, c, \eta > 0 \) and for each \( N \in \mathbb{Z}_{>0} \), define the region
\[
\mathcal{R}_\eta^{(N)} := \{ u \in \mathbb{C} : |\Im u| \leq \eta, \quad -\eta \leq \Re u \leq N - \delta \}.
\]

Suppose \( T := T(N) \) and \( L := L(N) \geq N \) satisfy
\[
\lim_{N \to \infty} T/N \to \hat{T}, \quad \lim_{N \to \infty} L/N \to \hat{L}
\]
for some \( \hat{T} > 0 \) and \( \hat{L} \geq 1 \). Then the following statements are true.

1. For each fixed \( u \in \bigcup_{N=1}^{\infty} \mathcal{R}_\eta^{(N)} \), we have
\[
\lim_{L \to \infty} G_L(u)^T = \exp \left[ -\frac{\hat{T}}{L} \left( \frac{c(c+1)}{2} + cu \right) \right].
\]

2. If \( \hat{L} > 1 \), then
\[
|G_L(u)| \geq \exp \left( -\frac{C_{\delta,c}}{L - N} \right), \quad u \in \mathcal{R}_\eta^{(N)}
\]
for sufficiently large \( N \) and \( L \).

3. In general, we have
\[
|G_L(u)| \leq \exp \left( -\frac{\Re u}{L} + \frac{\eta}{L} \right), \quad u \in \mathcal{R}_\eta^{(N)} \cap \{ u \in \mathbb{C} : |L - u - c| \geq \delta \}
\]
for sufficiently large \( N \).

Proof. By Lemma 6.6 we have
\[
G_L(u) = \exp \left[ c \log \left( 1 - \frac{u}{L} \right) - \frac{1}{2} \frac{c(c+1)}{L - u - c} + R(L - u - c) \right]
\]
where \( R(z) \) is independent of \( L \) and satisfies
\[
|R(z)| \leq \frac{C_{\delta'}}{|z|^2}
\]
over \( \{ \Re z \geq 0, |z| \geq c(1 + \delta') \} \) for any fixed \( \delta' > 0 \). By choosing \( \delta' \) sufficiently small given our choice of \( c, \delta \), we have
\[
|R(L - u - c)| \leq \frac{C_{\delta,c}}{|L - u - c|^2}, \quad u \in \mathcal{R}_\eta^{(N)}, \quad L \geq N.
\]

If we fix \( u \in \mathcal{R}_\eta^{(N)} \), we see that (6.6) implies (6.3).

Observe that
\[
|L - u - c| \geq L - \Re u - c \geq L - N - c, \quad u \in \mathcal{R}_\eta^{(N)}
\]
for \( L \gg N \). Then (6.3) implies (6.4).

Thus we are left to prove (6.5). For this, we bound \( G_L(u) \) separately on the regions \( \mathcal{R}_\eta^{(N)} \cap \{ \Re u \leq L - N_0 \} \) and \( \mathcal{R}_\eta^{(N)} \cap \{ L - N_0 \leq \Re u \leq L \} \) where \( N_0 \) is some large number independent of \( N \) and \( L \) determined below. We have
\[
\left| \frac{L - \Re u - c}{L - u - c} \right| \leq C_{\delta,c,\eta}
\]
for \( u \in \mathfrak{R}_L^{(N)} \cap \{ |L-u| \geq \delta \} \) and in particular on \( u \in \mathfrak{R}_L^{(N)} \cap \{ \Re u \leq L - N_0 \} \) where \( N_0 \) is some fixed number. Thus
\[
\Re \left( -\frac{1}{2} \frac{c(c+1)}{L-u-c} + R(L-u-c) \right) \leq \Re \left( -\frac{1}{2} \frac{c(c+1)}{L-u-c} + \frac{c(c+1)}{L-\Re u-c} \right) + \frac{C_{\delta,c}}{|L-u-c|^2} \leq \frac{1}{2} \frac{c(c+1)}{L-\Re u-c} + \Re \left( \frac{-c(c+1)i\Re u}{(L-\Re u-c)(L-u-c)} + \frac{C_{\delta,c}}{|L-u-c|^2} \right)
\]
for \( u \in \mathfrak{R}_L^{(N)} \cap \{ \Re u \leq L - N_0 \} \) where we use (6.7) in the first inequality. By choosing \( N_0 \) sufficiently large, the first term on the right hand side dominates. Then (6.6) implies
\[
\Re \log \left( 1 - \frac{u}{L} \right) \leq \log \left( 1 - \frac{\Re u}{L} + \frac{\eta}{L} \right) \leq -\frac{\Re u}{L} + \frac{\eta}{L},
\]
we obtain
\[
|G_L(u)| \leq \exp \left( -\frac{\Re u}{L} + \frac{\eta}{L} \right), \text{ } u \in \mathfrak{R}_L^{(N)} \cap \{ \Re u \leq L - N_0 \}
\]
for sufficiently large \( N \).

We also have that (6.6) implies the bound
\[
|G_L(u)| \leq \left( \frac{N_0}{N} e^{C_{\delta,\eta,N_0}} \right)^c, \text{ } u \in \mathfrak{R}_L^{(N)} \cap \{ |L-u| \geq \delta \} \cap \{ L - N_0 \leq \Re u \leq L \}.
\]

Since
\[
\log \left( \frac{N_0}{N} + C_{\delta,\eta,N_0} \right) \leq -\frac{\Re u}{L}
\]
for large \( N \) (recall \( L \geq N \)), we obtain
\[
|G_L(u)| \leq \exp \left( -\frac{\Re u}{L} \right), \text{ } u \in \mathfrak{R}_L^{(N)} \cap \{ |L-u| \geq \delta \} \cap \{ L - N_0 \leq \Re u \leq L \}
\]
for sufficiently large \( N \). Combining (6.8) and (6.9), we obtain (6.5).

6.3. Proofs of Local Theorems. We prove Theorem 6.3 and then prove Theorems 6.1 and 6.2 simultaneously to avoid redundancy.

Proof of Theorem 6.3. By Theorem 4.2 and replacing \( u_i \) with \( u_i - \theta N \), we get (given the existence of the appropriate contours described below)
\[
\mathbb{E} \left[ \prod_{i=1}^{m} \frac{(M+N+\alpha-1)^{k_i}T_i}{N^{k_i}(N+\alpha-1)^{k_i}T_i}\Psi_{k_i}(y^{(T_i)}) \right] = \left( \frac{-\theta}{2\pi i} \sum_{k_i=1}^{\infty} \right) \oint \int \prod_{1 \leq i < \nu \leq m} C(U_i,U_{\nu}) \prod_{i=1}^{m} B(U_i) I_{\nu}(U_i) dU_i.
\]

where \( I_T := I_T^{(N)} \) is defined
\[
I_T(U) := \prod_{i=1}^{k} \frac{1 - u_i/\theta}{u_i/\theta} \left( \frac{1 - \frac{u_i}{\theta(N+\alpha-1)}}{1 - \frac{u_i}{\theta(N+\alpha+M-1)}} \right)^T
\]
for \( U = (u_1, \ldots, u_k) \). The contours in (6.10) are given as follows. The \( u_{i,j} \) contour \( \Gamma_{i,j}^{(N)} \) is positively oriented around \( 0 \), avoid the pole at \( \theta(\alpha + M - 1) \) and \( \Gamma_{i,j}^{(N)} \) is enclosed by the contour \( \epsilon_{i,j}^{(N)} + c \) for \( c \in [-\theta, 1] \) whenever \((i,j) < (i',j') \) in lexicographical order. Since \( \theta(\alpha + M - 1) \rightarrow \infty \), we may take the contour \( \epsilon_{i,j}^{(N)} \) to be fixed contours \( \Gamma_{i,j} \) for sufficiently large \( N \).
If \( U = (u_1, \ldots, u_k) \in \mathbb{C}^k \setminus \{0\} \), \( T := T(N) \) and \( \hat{T} > 0 \) such that \( T/N \to \hat{T} > 0 \), then

\[
\lim_{N \to \infty} I_T(U) = \prod_{i=1}^k \exp \left[ -\hat{T} \left( \frac{1}{1 + \alpha} - \frac{1}{1 + \alpha + M} \right) u_i/\theta \right] \cdot \frac{1}{-u_i/\theta}
\]

uniformly for \( U \) in compact subsets of \( \mathbb{C}^k \setminus \{0\} \). This completes the proof of the theorems. \( \square \)

**Proofs of Theorems 6.1 and 6.2** Throughout this proof, \( C \) is a positive constant which may vary from line to line. If \( C \) is dependent on additional parameters, we explicitly denote this dependence including the related parameter as a subscript.

Both Theorems 6.1 and 6.2 are in the setting \( \theta = 1 \), the difference being that the former is under the assumption that \( U^{(T)} \) are obtained from Ginibre matrices (we refer to this as the Ginibre case) and the latter uses Assumption 2 (which we refer to as the Jacobi case). We begin by writing our expressions in a common way. Let

\[
E^{(N)} := \left\{ \begin{array}{ll}
\mathbb{E} \left[ \prod_{i=1}^m \frac{1}{N^{c_i(T_i+1)}} \mathcal{P}_{c_i}(U^{(T_i)}) \right] & \text{for the Ginibre case,} \\
\mathbb{E} \left[ \prod_{i=1}^m (M + N - \alpha - 1)^{c_i T_i} \frac{1}{N^{M+N(N+\alpha-1)}} \mathcal{P}_{c_i}(U^{(T_i)}) \right] & \text{for the Jacobi case.}
\end{array} \right.
\]

We are interested in the limit \( E^{(N)} \) as \( N \to \infty \). By Theorem 4.3, Proposition 6.5 and replacing \( u_i \) with \( u_i - N + 1 \), we get (given the existence of the appropriate contours described below)

\[
(6.11) \quad E^{(N)} = \frac{(-1)^m}{(2\pi i)^m} \oint \cdots \oint 1 \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \prod_{i=1}^m I_{c_i T_i}(u_i) \frac{du_i}{c_i}
\]

where

\[
I_{c,T}(u) := I^{(N)}_{c,T}(u) := \left\{ \begin{array}{ll}
\frac{\Gamma(-u - c)}{\Gamma(-u)} G_N(u)^{T+1} & \text{for the Ginibre case,} \\
\frac{\Gamma(-u - c)}{\Gamma(-u)} G_N(u) \left( \frac{G_{N+\alpha-1}(u)}{G_{M+N+\alpha-1}(u)} \right)^T & \text{for the Jacobi case}
\end{array} \right.
\]

and we recall \( G_L(u) \) is defined by (6.2). The contours in (6.11) are given as follows. In the Ginibre case, the \( u_i \) contour \( \Omega_i^{(N)} \) is positively oriented around \( \{ -c_i, \ldots, -c_i + N - 1 \} \) but does not enclose \( N, N+1, N+2, \ldots, \) and \( \Omega_i^{(N)} \) is contained in \( \Omega_i^{(N)} - c_i \) for \( i < j \); given that such contours exist. In the Jacobi case, the \( u_i \) contour \( \Omega_i^{(N)} \) is positively oriented around \( \{ -c_i, \ldots, -c_i + N - 1 \} \) but does not enclose \( N + \alpha - 1, N + \alpha, \ldots, M + N + \alpha - 2 \).

The idea of the proof is to use dominated convergence where most of the work in finding a suitable dominating function for \( I_{c,T}(u) \) is done by Lemma 6.7. We divide the proof into three steps. In the first step, we choose contours \( \Omega_1^{(N)}, \ldots, \Omega_m^{(N)} \) satisfying the conditions above. In the second step, we examine the \( N \to \infty \) asymptotics of the function \( I_{c,T}(u) \) where \( c > 0 \) and \( T := T(N) \) satisfies \( \lim_{N \to \infty} T/N \to \hat{T} \) for some \( \hat{T} > 0 \). In the third step, we apply the asymptotics to (6.11) to prove the theorem.

**Step 1.** We choose \( \Omega_1^{(N)}, \ldots, \Omega_m^{(N)} \) which satisfy the conditions for (6.11). Fix \( \delta > 0 \) such that

\[
\delta(m + 1) < 1 - \sum_{j=2}^m c_j.
\]

Let \( \Omega_i^{(N)} \) be a counterclockwise contour around the boundary of the rectangle

\[
\left\{ u \in \mathbb{C} : |\Re u| \leq i, -\sum_{j=1}^{i-1} (c_j + \delta) \leq \Re u \leq N - \sum_{j=1}^m (c_j + \delta) \right\}.
\]
Then $\mathcal{U}_1^{(N)}, \ldots, \mathcal{U}_m^{(N)}$ satisfy the containment conditions, thus (6.11) is valid with this choice. The choice of bound $|3u| \leq i$ is somewhat arbitrary, we just need a bound which is increasing in $i$.

We claim that for any set of numbers $a_1, \ldots, a_m \in \mathbb{R}$, there exists a sufficiently small $\delta > 0$ such that

$$\text{dist}({a_1 + \delta, \ldots, a_m + m\delta}, \mathbb{Z}) \geq \delta.$$ 

Indeed, if $a_i$ is an integer, then this clearly holds for $\delta < 1/(2m)$. If $a_i \notin \mathbb{Z}$, then this holds for $\delta < \text{dist}(a_i, \mathbb{Z})/(2m)$. By taking $\delta$ small enough to satisfy these inequalities, the claim follows. Thus we may choose $\delta > 0$ small enough so that

$$\text{dist}(\mathcal{U}_i^{(N)}, \mathbb{Z} \cup (\mathbb{Z} - c_i)) \geq \delta, \quad 1 \leq i \leq m.$$ 

For $1 \leq i \leq m$, let $\mathcal{U}_i$ be the counterclockwise contour around the boundary of the semi-infinite region

$$\left\{ u \in \mathbb{C} : |\Im u| \leq i, -\sum_{j=1}^{i-1} (c_j + \delta) \leq \Re u \right\}.$$ 

**Step 2.** Let

$$\mathcal{R}_\eta^{(N)} := \{ u \in \mathbb{C} : |\Im u| \leq \eta, -\eta \leq \Re u \leq N - \delta \}$$

for some fixed parameter $\eta > 0$ and suppose $T := T(N)$ satisfies $\lim_{N \to \infty} T/N \to \tilde{T}$ for some $\tilde{T} > 0$. The goal is to control the term $I_{c,T}(u)$ as $N \to \infty$ on the region

$$\mathcal{R}_\eta^{(N)} \cap \{ u : \text{dist}(u, \mathbb{Z} \cup (\mathbb{Z} - c)) \geq \delta \}$$

where $\eta > 0$ is some fixed parameter. In particular, we prove that there exist constants $C_{\delta,c,\eta}, C'_{\delta,c,\eta} > 0$ such that

$$|I_{c,T}(u)| \leq C_{\delta,c,\eta} \exp \left(-C'_{\delta,c,\eta} \Re u\right) =: f_{\eta,c,T}(u), \quad u \in \mathcal{R}_\eta^{(N)} \cap \{ \text{dist}(u, \mathbb{Z} \cup (\mathbb{Z} - c)) \geq \delta \}$$

for sufficiently large $N$, and for fixed $u \in \bigcup_{N=1}^{\infty} \mathcal{R}_\eta^{(N)} \setminus (\mathbb{Z} \cup (\mathbb{Z} - c))$, we have

$$\lim_{N \to \infty} I_{c,T}(u) = g_{c,\tilde{T}}(u)$$

where

$$g_{c,\tilde{T}}(u) := \begin{cases} \frac{\Gamma(-u - c)}{\Gamma(-u)} \exp \left[-\tilde{T} \left( \frac{c(c + 1)}{2} \right) + \tilde{T} cu \right] & \text{for the Ginibre case}, \\ \frac{\Gamma(-u - c)}{\Gamma(-u)} \exp \left[-\tilde{T} \left( \frac{1}{1 + \alpha} - \frac{1}{M + 1 + \alpha} \right) \left( \frac{c(c + 1)}{2} + cu \right) \right] & \text{for the Jacobi case}. \end{cases}$$

By Euler’s reflection formula, we have

$$\frac{\Gamma(-u - c)}{\Gamma(-u)} = \frac{\sin \pi u}{\sin \pi (u + c)} \frac{\Gamma(1 + u)}{\Gamma(1 + u + c)}.$$ 

Then

$$I_{c,T}(u) = J_1(u) \cdot J_2(u) \cdot J_3(u)$$

where

$$J_1(u) := \frac{\sin \pi u}{\sin \pi (u + c)};$$

$$J_2(u) := \frac{\Gamma(1 + u)}{\Gamma(1 + u + c)};$$

$$J_3(u) := \begin{cases} G_N(u)^{T+1} & \text{for the Ginibre case}, \\ G_N(u) \left( \frac{G_{N+\alpha-1}(u)}{G_{M+N+\alpha-1}(u)} \right)^T & \text{for the Jacobi case}. \end{cases}$$

We bound each of the terms $J_1, J_2, J_3$. 

Thus, (6.17) and (6.3) imply the convergence (6.14).

(6.22)

\[ u \text{ and the contour is } U \]

for sufficiently large \( N \).

Combining the bounds (6.18), (6.19) and (6.20) implies (6.13). Moreover, from (6.16) we have

\[ |J_3(u)| \leq C_{\delta, c, \eta} \exp(-C'_{\delta, c, \eta} R u), \quad u \in \mathcal{R}_\eta^{(N)} \]

for sufficiently large \( N \).

Thus, (6.17) and (6.3) imply the convergence (6.14).

**Step 3.** Let \( \mathcal{U}_{i,1}^{(N)} \) be the vertical contour along

\[ \left\{ u \in C : |\Im u| \leq i, \Re u = N - \sum_{j=1}^{m}(c_j + \delta) \right\}, \]

alternatively the rightmost part of \( \mathcal{U}_i^{(N)} \). Let

\[ \mathcal{U}_{i,0}^{(N)} := \mathcal{U}_i^{(N)} \setminus \mathcal{U}_{i,1}^{(N)} \]

denote the complementary contour. We may rewrite (6.11) as

(6.21)

\[ E^{(N)} = \sum_{s \in \{0,1\}^m} \mathcal{I}_s \]

where

\[ \mathcal{I}_s := \frac{(-1)^m}{(2\pi i)^m} \int \cdots \int \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \prod_{i=1}^{m} I_{c_i,T}(u_i) \frac{du_i}{c_i} \]

and the contour is \( \mathcal{U}_{i,s_i}^{(N)} \) for \( 1 \leq i \leq m \) if \( s = (s_1, \ldots, s_m) \). We prove that

(6.22)

\[ \mathcal{I}_0 \rightarrow \frac{(-1)^m}{(2\pi i)^m} \int \cdots \int \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \prod_{i=1}^{m} g_{c_i,T}(u_i) \frac{du_i}{c_i}, \]

(6.23)

\[ \mathcal{I}_s \rightarrow 0, \quad s \neq 0 \]

as \( N \to \infty \), where the \( u_i \) contour in the right hand side of (6.22) is given by \( \mathcal{U}_i \) for \( 1 \leq i \leq m \). By (6.21), this completes the proof of Theorems 6.1 and 6.2.

Observe that

\[ \prod_{1 \leq i < j \leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \]

is bounded on \( (u_1, \ldots, u_m) \in \bigcup_{N=1}^{\infty}(\mathcal{U}_1^{(N)} \times \cdots \times \mathcal{U}_m^{(N)}) \). Moreover, for a suitably chosen \( \eta \), we have \( \mathcal{U}_i^{(N)} \subset \mathcal{R}_\eta^{(N)} \cap \{\text{dist}(u, \mathbb{Z} \cup (\mathbb{Z} - c_i) \geq \delta\} \). We may use the bound (6.13) so that by dominated convergence and the convergence (6.14), we obtain (6.22).

To prove (6.23), we use the boundedness of the cross-term to write

\[ |\mathcal{I}_s| \leq C \int_{\mathcal{U}_{i,s_i}^{(N)}} d|u_1| \cdots \int_{\mathcal{U}_{m,s_m}^{(N)}} d|u_m| \cdot \prod_{i=1}^{m} |I_{c_i,T}(u_i)| = C \prod_{i=1}^{m} \int_{\mathcal{U}_{i,s_i}^{(N)}} |c_i,T(u_i)| d|u|. \]
Using the bound (6.13), we have that
\[ \int_{U_{i,s_i}} |I_{c_i,T_i}(u)| \, d|u| \]
is of constant order when \( s_i = 0 \) and is \( o(1) \) when \( s_i = 1 \). Thus (6.23) follows. \( \square \)
We recall the notion of cumulants and some basic properties.

**Definition A.1.** For any positive integer \( \nu \), let \( \Theta_\nu \) be the collection of all set partitions of \([1, \nu]\), that is
\[
\Theta_\nu = \left\{ \{S_1, \ldots, S_d\} : d > 0, \bigcup_{i=1}^{d} S_i = [1, \nu], S_i \cap S_j = \emptyset \ \forall i \neq j, \ S_i \neq \emptyset \ \forall i \in [1, d] \right\}.
\]

For a random vector \( u = (u_1, \ldots, u_m) \) and any \( v_1, \ldots, v_\nu \in \{u_1, \ldots, u_m\} \), define the (order \( \nu \)) cumulant \( \kappa(v_1, \ldots, v_\nu) \) by
\[
\kappa(v_1, \ldots, v_\nu) = \sum_{\{S_1, \ldots, S_d\} \in \Theta_\nu} (-1)^{d-1}(d-1)! \prod_{t=1}^{d} \mathbb{E} \left[ \prod_{i \in S_t} v_i \right]. \tag{A.1}
\]

From the definition we see that for any random vector \( u \), the existence of all cumulants of order up to \( \nu \) is equivalent to the existence of all moments of order up to \( \nu \). Note that the cumulants of order 2 are exactly the covariances:
\[
\kappa(v_1, v_2) = \text{Cov}(v_1, v_2).
\]

We have the following alternative definition for cumulants.

**Definition A.2.** Let \( u = (u_1, \ldots, u_m) \) be a random vector. For any \( v_1, \ldots, v_\nu \in \{u_1, \ldots, u_m\} \), define the (order \( \nu \)) cumulant \( \kappa(v_1, \ldots, v_\nu) \) as
\[
\kappa(v_1, \ldots, v_\nu) = (-i)^\nu \frac{\partial^\nu}{\partial t_1 \cdots \partial t_m} \log \mathbb{E} \left[ \exp \left( \sum_{j=1}^{\nu} t_j v_j \right) \right] \bigg|_{t_1=\cdots=t_\nu=0}. \tag{A.2}
\]

For further details see [38, Section 3.1, Section 3.2] wherein the agreement between Definitions A.1 and A.2 is shown by taking the second definition and proving (A.1).

As a consequence of the second definition we have the following lemma.

**Lemma A.3.** A random vector is Gaussian if and only if all cumulants of order \( \geq 3 \) vanish.

We have the following formal versions of (A.1) and (A.2). Let \( E_{n_1, \ldots, n_\nu} \in \mathbb{C} \) with \( E_{0, \ldots, 0} = 1 \). Define the following formal power series
\[
E(t_1, \ldots, t_\nu) = \sum_{n_1, \ldots, n_\nu \geq 0} \frac{E_{n_1, \ldots, n_\nu}}{n_1! \cdots n_\nu!} t_1^{n_1} \cdots t_\nu^{n_\nu},
\]
\[
K(t_1, \ldots, t_\nu) = \log E(t_1, \ldots, t_\nu) =: \sum_{n_1, \ldots, n_\nu \geq 0} \frac{K_{n_1, \ldots, n_\nu}}{n_1! \cdots n_\nu!} t_1^{n_1} \cdots t_\nu^{n_\nu}.
\]

Let \( S \subset [1, \nu] \) and \( E(S) := E_{n_1, \ldots, n_j} \) where \( n_j = 1 \) if \( j \in S \) and 0 otherwise, and likewise define \( K(S) \). Letting \( \Theta_S \) be the collection of all set partitions of \( S \), we have
\[
K(S) = \sum_{\{S_1, \ldots, S_d\} \in \Theta_S} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} E(S_\ell) \tag{A.3}
\]
and
\[
E(S) = \sum_{\{S_1, \ldots, S_d\} \in \Theta_S} \prod_{\ell=1}^{d} K(S_\ell) \tag{A.4}
\]
are equivalent. This follows from the formal identity \( e^{K(t_1, \ldots, t_\nu)} = E(t_1, \ldots, t_\nu) \). This gives us the following lemma.
Lemma A.4. Suppose that $\tilde{K}$ and $\tilde{E}$ are functions which take values on nonempty subsets of $[[1, \nu]]$. Further suppose that

\begin{equation}
\tilde{E}(S) = \sum_{\{S_1, \ldots, S_d\} \in \Theta_S} \prod_{i=1}^{d} \tilde{K}(S_i)
\end{equation}

Then

\begin{equation}
\tilde{K}(S) = \sum_{\{S_1, \ldots, S_d\} \in \Theta_S} (-1)^{d-1} (d-1)! \prod_{i=1}^{d} \tilde{E}(S_i)
\end{equation}

**Appendix B. Observables of Schur Processes**

The goal of this appendix is to prove a contour integral formula for joint moments of Schur processes; Macdonald processes in the case $q = t$. We review several facts about Schur functions. Additional details can be found in [32, Chapters I].

For $q = t$, (2.2) is defined by

$$\langle p_\lambda, p_\mu \rangle := \langle p_\lambda, p_\mu \rangle(t,t) = \delta_{\lambda\mu} \prod_{i=1}^{\infty} t^{m_i(\lambda)} m_i(\lambda)!$$

which is independent of $0 < t < 1$. In this case, the Macdonald symmetric functions become the Schur functions

$$s_\lambda(X) := P_\lambda(X; t, t) = Q_\lambda(X; t, t), \quad s_{\lambda/\mu}(X) := P_{\lambda/\mu}(X; t, t) = Q_{\lambda/\mu}(X; t, t).$$

Thus the properties for Macdonald symmetric functions are inherited by the Schur functions. To list a few, we have that (2.3), (2.4), (2.11) imply

\begin{equation}
\langle s_\lambda(X), s_\mu(X) \rangle = \delta_{\lambda\mu},
\end{equation}

\begin{equation}
s_{\lambda/\mu}(X, Y) = \sum_{\mu \in \mathcal{Y}} s_{\lambda/\mu}(X) s_{\mu/\nu}(Y).
\end{equation}

Given a countably infinite set $X = (x_1, x_2, \ldots)$ of variables, let $\Lambda_X$ denote the algebra of symmetric functions on $X$ over $\mathbb{C}$. For sets $X^{(1)}, \ldots, X^{(n)}$ of variables, let $(X^{(1)}, \ldots, X^{(n)})$ denote the disjoint union of these sets. We have that $\{s_\lambda(X)\}_{\lambda \in \mathcal{Y}}$ forms a linear basis for $\Lambda_X$.

**Definition B.1.** Suppose $a := (a_1, \ldots, a_N) \in \mathbb{R}^n_{\geq 0}$, $b \in (b_1, \ldots, b_M) \in \mathbb{R}^M_+$ such that $a_i b_1 < 1$ for $1 \leq i \leq N$, $1 \leq j \leq M$. Let $\mathbb{S}_{a,b}$ denote the measure on $\lambda = (\lambda^1, \ldots, \lambda^M) \in \mathcal{Y}^n = \mathcal{Y} \times \cdots \times \mathcal{Y}$ (this is different from the subscripted $\mathcal{Y}_N$ notation which is the subset of $\mathcal{Y}$ containing partitions $\lambda$ with $\ell(\lambda) \leq N$) where

\begin{equation}
\mathbb{S}_{a,b}(\lambda) = \prod_{1 \leq i \leq N} \prod_{1 \leq j \leq M} (1 - a_i b_j) \cdot s_{\lambda^i}(a_1, \ldots, a_N) s_{\lambda^1/\lambda^2}(b_1) \cdots s_{\lambda^{M-1}/\lambda^M}(b_{M-1}) s_{\lambda^M}(b_M).
\end{equation}

**Remark 7.** The measure $\mathbb{S}_{a,b}$ is the Schur ($q = t$) case of the so-called ascending Macdonald process. We note that

$$\sum_{\lambda^1, \ldots, \lambda^M \in \mathcal{Y}} s_{\lambda^i}(a_1, \ldots, a_N) s_{\lambda^1/\lambda^2}(b_1) \cdots s_{\lambda^{M-1}/\lambda^M}(b_{M-1}) s_{\lambda^M}(b_M) = \prod_{1 \leq i \leq N} \prod_{1 \leq j \leq M} \frac{1}{1 - a_i b_j}$$

is a consequence of the branching rule (A.2) and the $q = t$ case of the Cauchy identity (2.10).

We prove the following contour integral formula for joint expectations of $p_t$ (recall (A.4)).
Theorem B.2. Suppose $a := (a_1, \ldots, a_N) \in \mathbb{R}_>^N$, $b \in (b_1, \ldots, b_M) \in \mathbb{R}_>^M$ such that $a_i b_j < 1$ for $1 \leq i \leq M$, $1 \leq j \leq N$. For $0 < t_1, \ldots, t_m < 1$ and $1 \leq n_1 \leq \cdots \leq n_m \leq M$, we have

$$
\mathbb{E} \left[ \prod_{i=1}^m p_{t_i}(\lambda_i^{n_i}) \right] = \frac{1}{(2\pi)^m} \oint \cdots \oint \prod_{1 \leq i < j \leq m} (z_j - z_i)(t_j z_j - t_i z_i) \prod_{i=1}^m \left( \prod_{\ell=1}^N \frac{z_{i,\ell} - a_{\ell}}{z_i - t_i a_{\ell}} \prod_{\ell=n_{i,\ell}}^M \frac{1 - t^{-1}_i b_{\ell} z_i}{1 - b_{\ell} z_i} \right) \frac{dz_i}{z_i}
$$

where the $z_i$ contour is positively oriented around $0, \{t_i a_{\ell}\}_{\ell=1}^N$ but does not encircle $\{b_{\ell}^{-1}\}_{\ell=n_{i,\ell}}^M$, and the contours satisfy $|z_i| < |t_i z_i|$ for $i < j$; given that such contours exist.

The ideas involved in the proof of Theorem B.2 are based off of the approaches of [4], [5] and [1]. In particular, the organization below follows that of [1, Section 3]. The general approach is to first prove a formal version of Theorem B.2, then specialize this formal version to obtain Theorem B.2.

We organize the subappendices below as follows. In Subappendix B.1 we provide some notions and basic facts about formal symmetric functions. This provides the setting to state and prove a formal version of Theorem B.2 in Subappendix B.2. In Subappendix B.3 we prove Theorem B.2 by specializing the formal analogue.

B.1. Preliminaries on Formal Symmetric Functions.

B.1.1. Graded Topology. Let $F$ be a field and $\mathcal{A}$ be a $(\mathbb{Z}_>)$-graded algebra over $F$. Let $\mathcal{A}_n$ denote the $n$th homogeneous component of $\mathcal{A}$. Throughout this section, let us assume that all of our graded algebras have $\dim \mathcal{A}_n < \infty$ for every $n \geq 0$.

Definition B.3. Given $a \in \mathcal{A}$, define $\operatorname{ldeg}(a)$ to be the minimum degree among the homogeneous components of $a$. The graded topology is the topology on $\mathcal{A}$ where a sequence $a_n \in \mathcal{A}$ converges to $a \in \mathcal{A}$ if and only if

$$\operatorname{ldeg}(a_n - a) \to \infty$$

as $n \to \infty$. Denote the completion of $\mathcal{A}$ under this topology by $\hat{\mathcal{A}}$.

The completion $\hat{\mathcal{A}}$ consists of formal sums $\sum_{n=1}^\infty a_n$ where $a_n \in \mathcal{A}_n$. Given two graded algebras $\mathcal{A}$ and $\mathcal{A}'$ over $F$, we give the following grading to $\mathcal{A} \otimes_F \mathcal{A}'$. If $a \in \mathcal{A}_m$ and $a' \in \mathcal{A}'_{n'}$, then $a \otimes a' \in (\mathcal{A} \otimes_F \mathcal{A}')_{m+n'}$.

For a field $F \supset \mathbb{C}$ and a graded algebra $\mathcal{A}$ over $\mathbb{C}$, denote by $\mathcal{A}[F]$ the graded algebra $\mathcal{A} \otimes_\mathbb{C} F$ over $F$. Given graded algebras $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(k)}$ over $\mathbb{C}$, we denote the completion of $(\mathcal{A}^{(1)} \otimes \cdots \otimes \mathcal{A}^{(k)})[F]$ under the graded topology by

$$\mathcal{A}^{(1)} \otimes \cdots \otimes \mathcal{A}^{(k)}[F] \quad \text{or} \quad \hat{\bigotimes}_{i=1}^k \mathcal{A}^{(i)}[F].$$

Let $\Lambda_X[F]$ denote the $F$-algebra of symmetric functions in $X = \{x_1, x_2, \ldots\}$, a set of variables, with coefficients in $F$. Take the natural grading on $\Lambda_X[F]$ in which $(\Lambda_X[F])_n$ is spanned by monomials of total degree $n$. Given disjoint ordered sets of variables $Z_1, \ldots, Z_n$ with $Z_i = (z_{i,1}, \ldots, z_{i,k_i})$, let $\mathcal{L}(Z_1, \ldots, Z_n)$ denote the field of formal Laurent series in the variables

$$\bigcup_{i=1}^n \left\{ \frac{z_{i,1}}{z_{i,2}}, \ldots, \frac{z_{i,k_i-1}}{z_{i,k_i}}, z_{i,k_i} \right\}.$$

The space $\hat{\bigotimes}_{i=1}^k \Lambda_X^{(i)}[F]$ consists of formal sums

$$\sum_{\lambda_1, \ldots, \lambda_N \in \mathcal{Y}} c_{\lambda_1, \ldots, \lambda_N} s_{\lambda_1}(X^1) \cdots s_{\lambda_N}(X^N)$$

where $c_{\lambda_1, \ldots, \lambda_N} \in F$.

For fields $\mathbb{C} \subset F_1 \subset F_2$, $1 \leq k \leq N$, there is the natural inclusion map

$$\bigotimes_{i=1}^k \Lambda_X^{(i)}[F_1] \hookrightarrow \bigotimes_{i=1}^k \Lambda_X^{(i)}[F_2].$$

We also have consistency

$$\Lambda_X[F] \otimes_F \cdots \otimes_F \Lambda_X[F] \cong \Lambda_X \otimes \cdots \otimes \Lambda_X.$$


Definition B.4. The projection map \( \pi^n_X : \hat{A}_X \to \hat{A}_{(x_1, \ldots, x_n)} \) is defined as the continuous map sending \( x_{n+1}, x_{n+2}, \ldots \) to 0 and \( x_i \) to \( x_i \) for \( i = 1, \ldots, n \).

For a field \( F \supset \mathbb{C} \) and a graded algebra \( A \) over \( \mathbb{C} \), we can extend the domain of the projection

\[
\pi^n_X : A \otimes \Lambda_X[F] \to A \otimes \Lambda_{(x_1, \ldots, x_n)}[F]
\]

by identifying with \( 1_A \otimes \pi^n_X \) then extending by continuity under the graded topology.

Definition B.5. Let \( A \) and \( A' \) be graded algebras over \( \mathbb{C} \) and \( \{a_{n,j}\}_j \) be a basis for \( A_n \) for each \( n \geq 0 \). We say that an element \( f \in A \otimes A'[F] \) is \( A \)-projective if

\[
f = \sum_{n,j} a_{n,j} \otimes \alpha'_{n,j}, \quad \alpha'_{n,j} \in A'_n
\]

such that \( \lim_{n \to \infty} \min_j \deg(\alpha'_{n,j}) = \infty \). This property is independent of the choice of basis.

Elements which are \( A \)-projective are closed under addition and multiplication and form a subalgebra of \( A \otimes A'[F] \). If \( A = \Lambda_X \), denote the algebra of \( \Lambda_X \)-projective elements by \( \mathcal{P}_X(\Lambda_X \otimes A'[F]) \).

B.1.2. Schur Pairing and Residue.

Definition B.6. Let \( A, A' \) be graded algebras over \( \mathbb{C} \). Fix a field \( F \supset \mathbb{C} \), and let the Schur pairing be the bilinear map \( \langle \cdot, \cdot \rangle_Y : (A \otimes \Lambda_X)[F] \times (\Lambda_X \otimes A')[F] \to A \otimes A'[F] \) defined by

\[
\langle a \otimes s, s' \rangle_X := \langle s, s' \rangle a \otimes b = \delta_{\lambda,a} a \otimes b.
\]

This pairing does not extend by continuity to product of the completions of \( (A \otimes \Lambda_X)[F] \) and \( (\Lambda_X \otimes A')[F] \). However, the pairing does extend continuously to

\[
\mathcal{P}_X(\Lambda_X \otimes A'[F]) \times (\Lambda_X \otimes A'[F])
\]

Definition B.7. Given an ordered set \( Z = (z_1, \ldots, z_k) \) of variables, denote by \( \oint \) \( \mathrm{d}Z : \mathcal{L}(Z) \to \mathbb{C} \) the residue operator which takes an element of \( \mathcal{L}(Z) \) and returns the coefficient of \( (z_1 \cdots z_k)^{-1} \). For \( \oint \) \( \mathrm{d}Z \) applied to \( f \in \mathcal{L}(Z) \) we write \( \oint f \mathrm{d}Z \) or \( \oint f \mathrm{d}Z f \).

As with the projection map, the residue operator can act on larger domains. For example, we can extend

\[
\oint \mathrm{d}Z : \hat{A}[\mathcal{L}(Z, W^1, \ldots, W^k)] \to \hat{A}[\mathcal{L}(W^1, \ldots, W^k)]
\]

by the action \( 1_A \otimes \oint \mathrm{d}Z \) then extension by continuity. In this case, \( \oint \mathrm{d}Z \) preserves the degree of homogeneous elements. In particular, if we replace \( \hat{A} \) with \( \Lambda_X \otimes A' \), we have that \( \oint \mathrm{d}Z \) preserves \( A \)-projectivity.

The residue operator commutes with continuous maps under the graded topology.

Lemma B.8. Let \( A, A' \) be graded algebras over \( \mathbb{C} \), and let \( \varphi : \hat{A} \to \hat{A}' \) be a continuous map which extends to a continuous map \( \hat{A}[\mathcal{L}(Z, W)] \to \hat{A}'[\mathcal{L}(Z, W)] \). Then

\[
\varphi \circ \left( \oint \mathrm{d}Z \right) = \left( \oint \mathrm{d}Z \right) \circ \varphi.
\]

Lemma B.9. Let \( A, A' \) be graded algebras over \( \mathbb{C} \), let \( f \in A \otimes \Lambda_X[\mathcal{L}(Z)] \) and \( g \in \Lambda_X \otimes A'[\mathcal{L}(W)] \). If \( f \) is \( \Lambda_X \)-projective, then

\[
\left( \oint f \mathrm{d}Z, g \right)_X = \oint (f, g)_X \mathrm{d}Z,
\]

\[
\left( f, \oint g \mathrm{d}W \right)_X = \oint (f, g)_X \mathrm{d}W.
\]

Since the residue operator preserves projectivity, the left hand sides of the equalities above are valid expressions.

Proof. For arbitrary \( g \in \Lambda_X \otimes A'[\mathcal{L}(W)] \), the map \( \langle \cdot, g \rangle_X \) is continuous on \( \mathcal{P}_X(\Lambda \otimes \Lambda_X[\mathcal{L}(Z)]) \). By Lemma B.8 (B.7) follows. For \( f \in \mathcal{P}_X(\Lambda \otimes \Lambda_X[\mathcal{L}(Z)]) \), the map \( \langle f, \cdot \rangle \) is continuous on \( \Lambda_X \otimes A'[\mathcal{L}(W)] \). By Lemma B.8 (B.8) follows. \( \square \)
**B.2. Formal Schur Processes.** Let \( X, Y \) be countable sets of variables. Define the following elements of \( \Lambda_X \otimes \Lambda_Y \)

\[
\Pi(X, Y) := \Pi(X, Y; t, t) = \prod_{x \in X, y \in Y} \frac{1}{1 - xy},
\]

\[
H(X, Y; t) := \Pi(X, Y; 0, t) = \prod_{x \in X, y \in Y} \frac{1 - txy}{1 - xy}.
\]

**Definition B.10.** Fix a positive integer \( N \) and let \( U = (U^1, \ldots, U^N) \) and \( V = (V^1, \ldots, V^N) \) be ordered \( N \)-tuples of countable sets of variables. A **formal Schur process** is a formal probability measure on \( \mathcal{Y}^N \), valued in \( \bigotimes_{i=1}^N (\Lambda_{U^i} \otimes \Lambda_{V^i}) \) with the assignment

\[
\mathbb{P}^{U, V}_{t} (\lambda) = 2^{|\lambda|-1} s_{\lambda^1}(U^1) \left( \sum_{\mu \in \mathcal{Y}} s_{\lambda^1/\mu}(V^1)s_{\lambda^2/\mu}(U^2) \right) \cdots \left( \sum_{\mu \in \mathcal{Y}} s_{\lambda^{N-1}/\mu}(V^{N-1})s_{\lambda^N/\mu}(U^N) \right) s_{\lambda^N}(V^N)
\]

where \( \lambda = (\lambda^1, \ldots, \lambda^N) \) and \( 2^{|\lambda|-1} \) is the normalization constant for which the sum over \( \lambda \in \mathcal{Y}^N \) gives unity.

From [5, Section 3],

\[
(B.9) \quad \mathcal{Z} = \prod_{1 \leq i < j \leq N} \Pi(U^i, V^j).
\]

**Theorem B.11.** The following formal identity holds for any nonzero \( t_1, \ldots, t_N \)

\[
\mathbb{E}_{\mathcal{Z}} [p_{t_1}(\lambda^1) \cdots p_{t_N}(\lambda^N)] = \frac{1}{(2\pi i)^N} \oint \frac{dz_1}{z_1} \cdots \frac{dz_N}{z_N} \prod_{1 \leq i < j \leq N} H(U^i, z_j^{-1}; t_j)^{-1} H(z_i, V^j; t_i^{-1}) \prod_{1 \leq i < j \leq N} \frac{(1 - z_i)(1 - t_i z_i)}{(1 - t_j)(1 - t_i t_j z_j)}.
\]

This theorem implies a more general corollary.

**Corollary B.12.** Let \( 1 \leq n_1 \leq \cdots \leq n_m \leq N \) and \( t_1, \ldots, t_m \neq 0 \). Then

\[
(B.10) \quad \mathbb{E}_{\mathcal{Z}} [p_{t_1}(\lambda^{n_1}) \cdots p_{t_m}(\lambda^{n_m})] = \frac{1}{(2\pi i)^m} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_m}{z_m} \prod_{a=1}^m \prod_{1 \leq i \leq n_a} \prod_{n_a < j \leq N} H(U^i, z_a^{-1}, t_a)^{-1} H(z_a, V^j; t_a^{-1}) \prod_{1 \leq a < b \leq m} \frac{(1 - z_a) (1 - t_a z_a)}{(1 - z_b) (1 - t_a t_b z_b)}.
\]

We prove Theorem [B.11] and Corollary [B.12] below.

**B.2.1. Basic Properties.** From [32, Chapter VI, Sections 2 & 4], we have the equalities

\[
\Pi(X, Y) = \sum_{\lambda \in \mathcal{Y}} s_{\lambda}(X)s_{\lambda}(Y) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} p_n(X)p_n(Y) \right),
\]

\[
H(X, Y; t) = \exp \left( \sum_{n=1}^{\infty} \frac{1 - t^n}{n} p_n(X)p_n(Y) \right).
\]

Given countable sets of variables \( X^1, X^2 \), the following splitting equality holds

\[
(B.12) \quad \Pi((X^1, X^2), Y) = \Pi(X^1, Y) \Pi(X^2, Y)
\]

and likewise for \( H(\cdot, \cdot; t) \). There is also an inversion equality

\[
(B.13) \quad H(X, Y; t)^{-1} = H(tX, Y; t^{-1})
\]
where by \( tX \) we mean the variable set \( \{tx \}_{x \in X} \).

In terms of the pairing, the formal Schur process can be expressed as

\[
\mathbb{S}_{U,V}^f(\lambda) = 2^{r-1} s_{\lambda^i}(U^1) \left( \prod_{i=1}^{N-1} (s_{\lambda^i}(V^i, Y^i), s_{\lambda^{i+1}}(Y^i, U^{i+1}))_Y \right) s_{\lambda^N}(V^N).
\]

This is an immediate consequence of the branching rule (B.2).

The \( \Pi \)'s introduced earlier also relate well with the pairing

\[
\langle \Pi(X^1, Y), \Pi(Y, X^2) \rangle_Y = \Pi(X^1, X^2).
\]

Since the power symmetric functions form an algebraic basis for \( \Lambda_X \mathcal{L}(Z) \), this relation can be further extended as follows. Take graded algebras \( \mathcal{A} \) and \( \mathcal{A}' \) over \( \mathcal{L}(Z) \) with \( Z = (z_1, \ldots, z_k) \), and sequences \( \{a_n\}, \{a'_n\} \) in \( \mathcal{A} \) and \( \mathcal{A}' \) respectively such that \( \text{ldeg}(a_n), \text{ldeg}(a'_n) \to \infty \) as \( n \to \infty \). Then

\[
\langle \exp \left( \sum_{n=1}^{\infty} \frac{a_n}{n} p_n(Y) \right), \exp \left( \sum_{n=1}^{\infty} \frac{a'_n}{n} p_n(Y) \right) \rangle_Y = \exp \left( \sum_{n=1}^{\infty} \frac{a_n a'_n}{n} \right).
\]

See [5] Proposition 2.3 for further details.

**B.2.2. Macdonald/Negut Operator.** Define the continuous linear operator \( D^X_t : \hat{\Lambda}_X \to \hat{\Lambda}_X \) by

\[
D^X_t s_\lambda(X) = p_t(\lambda)s_\lambda(X).
\]

**Proposition B.13.** Let \( X \) and \( Y \) be countable sets of variables. Then

\[
D^X_t \Pi(X, Y) = \Pi(X, Y) \frac{1}{2\pi i} \oint H(z^{-1}, X; t)^{-1} H(Y, z; t^{-1}) \frac{dz}{z}.
\]

**Proof.** From [25] Proposition 4.10, we have (B.10) where instead of a set of variables \( Y \) we have some fixed set \( \{u_1, \ldots, u_n\} \) of complex numbers, and \( X \) is still a countable set of variables. Here we have \( \oint \mathcal{L}(z) \to \hat{\Lambda}_X \). The goal is to extend this to a formal equality on \( \Lambda_X \otimes \Lambda_Y \) for \( Y \) an arbitrary countable set of variables.

We can replace (B.10) with a finite set of variables \( Y^{(n)} = \{y_1, \ldots, y_n\} \) instead of fixed complex numbers. In such a setting, we must consider the residue operator as a map \( \oint \mathcal{L}(z) \to \hat{\Lambda}_X \otimes \hat{\Lambda}_{\{y_1, \ldots, y_n\}} \).

Note that if \( f, g \in \Lambda_X \otimes \Lambda_Y \) such that \( \pi^Y_n f = \pi^Y_n g \) for all \( n \), then \( f = g \). One then sees that (B.10) holds formally for arbitrary countable sets of variables \( X, Y \). In this setting, the residue operator takes \( \Lambda_X \otimes \Lambda_Y |\mathcal{L}(z)\rangle \) to \( \Lambda_X \hat{\otimes} \Lambda_Y \).

**B.2.3. Proofs of Formal Theorems.**

**Proof of Theorem B.11.** Choose nonzero \( t_1, \ldots, t_N \) and let \( z_1, \ldots, z_N \) be variables.

1. Consider the element

\[
\sum_{\lambda} p_{t_1}(\lambda^1) \cdots p_{t_N}(\lambda^N) \mathbb{S}_{U,V}^f(\lambda) \in \bigotimes_{i=1}^{N} (\Lambda_{U^i} \otimes \Lambda_{V^i}).
\]

2. Multiply through by the normalizing constant. Reexpress the sums within \( \mathbb{S} \) in terms of Schur pairings as in (B.14)

\[
\sum_{\lambda} p_{t_1}(\lambda^1) \cdots p_{t_N}(\lambda^N) s_{\lambda^i}(U^1) \left( \prod_{i=1}^{N-1} (s_{\lambda^i}(V^i, Y^i), s_{\lambda^{i+1}}(Y^i, U^{i+1}))_Y \right) s_{\lambda^N}(V^N).
\]

Here we note the spaces which the pairings map:

\[
\langle \cdot, \cdot \rangle_Y : (\Lambda_{U^i} \otimes \Lambda_{V^i}) \times (\Lambda_{V^i} \otimes \Lambda_{U^{i+1}}) \to \Lambda_{U^i} \otimes \Lambda_{U^{i+1}}.
\]

By natural inclusions (B.4) and consistency (B.5), the domain of this pairing may be extended.
3. Bring the summation inside the pairings and the pairings inside the pairings

\[ \langle E_1, \langle E_2, \cdots, \langle E_{N-1}, E_N \cdots \rangle_{Y_{N-1}} \cdots \rangle_{Y_2} \rangle_{Y_1} \]

where

\[ E_i = \sum_{\lambda} p_{\lambda}, s_{\lambda} \cdot (Y^{i-1}, U^i) s_{\lambda} (V^i, Y^i) \]

and \( Y^0, Y^N \) are empty sets of variables. It was important to use the fact that the first argument of the \( Y^i \) Macdonald pairing is \( \Lambda_{Y^i} \)-projective which provides the continuity necessary for bringing the summations inside.

4. We can reexpress the summations in terms of the operators \( D_t \) in the residue form \( \text{(B.16)} \)

\[ E_i = D^{Y_{i-1}, U^i} (Y^{i-1}, U^i), (V^i, Y^i)) \]

\[ = \Pi((Y^{i-1}, U^i), (V^i, Y^i)) \frac{1}{2\pi i} \oint H((Y^{i-1}, U^i), z_{i-1}^{-1}; t_i) H(z_i, (V^i, Y^i); t_i^{-1}) \frac{dz_i}{z_i}. \]

5. The domain of the residue operator can be appropriately extended and consistency follows from \( \text{(B.4)} \) and \( \text{(B.5)} \). Note that the integrand in \( E_i \) remains \( \Lambda_{Y^i} \)-projective. Therefore, by \( \text{(B.7)} \) and \( \text{(B.8)} \), we may commute the residue operators with the pairings. After pulling out \( Y^i \) independent factors outside the residue operators, we obtain

\[ \frac{1}{(2\pi i)^N} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_N}{z_N} \cdot A \cdot \langle F_1, (F_2, \cdots, (F_{N-1}, F_N)_{Y_{N-1}} \cdots )_{Y_2} \rangle_{Y_1} \]

where

\[ F_i = H(Y^{i-1}, z_i^{-1}; t_i) \Pi(Y^{i-1}, V^i) H(z_i, Y^i; t_i^{-1}) \Pi((Y^{i-1}, U^i), Y^i) \]

\[ A = \prod_{i=1}^N H(U_i, z_i^{-1}; t_i) H(z_i, V_i; t_i^{-1}) \Pi(U^i, V^i). \]

Here, \( \text{(B.12)} \) and \( \text{(B.13)} \) were used to split \( H \) and \( \Pi \).

6. Apply the pairings for \( Y^i \) in decreasing order of \( i \). We claim by induction that at the \((N-i)\)th step, we have

\[ \frac{1}{(2\pi i)^N} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_N}{z_N} \cdot A \prod_{i \leq \ell \leq N} A_{\ell} \cdot \langle F_1, (F_2, \cdots, (F_i, \mathcal{F}_i)_{Y_i}, \cdots )_{Y_2} \rangle_{Y_1} \]

where

\[ A_i = \prod_{j: i < j \leq N} \frac{(1 - \frac{z_i}{z_j})(1 - \frac{t_i}{t_j})}{(1 - \frac{z_i}{z_j})(1 - \frac{t_i}{t_j})} H(z_i, V^i; t_i^{-1}) H(U^i, z_j^{-1}; t_j) \Pi(U^i, V^i), \]

\[ \mathcal{F}_i = \prod_{j: i < j \leq N} H(Y^i, z_j^{-1}; t_j) \Pi(Y^i, V^j) \]

\[ V_{i+1, N} := (V_{i+1}, \ldots, V^N). \] The base case is true since \( A_N = 1 \) and \( \mathcal{F}_{N-1} = F_N \). If we suppose \( \text{(B.20)} \) is true for \( 1 \leq i \leq N \), then

\[ \langle F_i, \mathcal{F}_i \rangle_{Y_i} = H(Y^{i-1}, z_i^{-1}; t_i) \Pi(Y^{i-1}, V^i) \]

\[ \times \left( H(z_i, V^i; t_i^{-1}) \Pi((Y^{i-1}, U^i), Y^i) \prod_{j: i < j \leq N} H(Y^i, z_j^{-1}; t_j) \Pi(Y^i, V^j) \right)_{Y_i} \]

By \( \text{(B.11)} \) and \( \text{(B.15)} \), the \( Y_i \)-bracket on the right hand side yields

\[ \prod_{j: i < j \leq N} \frac{(1 - \frac{z_i}{z_j})(1 - \frac{t_i}{t_j})}{(1 - \frac{z_i}{z_j})(1 - \frac{t_i}{t_j})} H(z_i, V^j; t_i^{-1}) H((U^i, Y^{i-1}), z_j^{-1}; t_j) \Pi((U^i, Y^{i-1}), V^j). \]
This implies
\[ \langle F_i, \mathcal{F} \rangle_{Y^i} = A_i \cdot \mathcal{F}_{i-1} \]
where we note that \( A_i \) is independent of \( \{Y^i\} \). This completes the induction. Since \( Y^0 \) is empty, in the final step we get \( \mathcal{F}_0 = 1 \). Thus, we obtain the formula
\[ \frac{1}{(2\pi i)^N} \int \frac{dz_1}{z_1} \cdots \int \frac{dz_N}{z_N} \cdot A \prod_{i=1}^{N} A_i, \]
Substituting (B.18) and (B.20), we get
\[ \frac{1}{(2\pi i)^N} \int \frac{dz_1}{z_1} \cdots \int \frac{dz_N}{z_N} \prod_{1 \leq i < j \leq N} \Pi(U_i, V_j) H(U_i, z_i^{-1}; t_i) H(z_i, V_i; t_i^{-1}) \prod_{1 \leq i < j \leq N} (1 - \frac{z_i}{z_j})(1 - \frac{t_i}{t_j}) \]
Recall that we multiplied by the normalizing constant \( \mathcal{Z} \). Dividing back by \( \mathcal{Z} \) and using (B.9), we complete the proof.

We now illustrate the main idea of the proof of Corollary B.12 via a particular example. For further details, we note that the proof is essentially identical to a corresponding extension in [5] (Theorem 3.10 to Corollary 3.11).

**Proof Idea of Corollary B.12** We consider the example of \( N = 1 \) and \( m = 2 \). Let \( t_1, t_2 \neq 0 \). Consider auxiliary variables \( S = (S^1, S^2), T = (T^1, T^2) \), and the formal expectation

\[ \mathbb{E}_{S^1,T} [p_{t_1}(1^1), p_{t_2}(1^2)] = \mathcal{Z}^{-1} \sum_{\lambda, Y \in Y} p_{t_1}(1^1)p_{t_2}(1^2)s_{\lambda}(S^1) \left( \sum_{\mu \in \mathcal{Y}} s_{\lambda'/\mu}(T^1)s_{\lambda'/\mu}(S^2) \right) s_{\lambda}(T^2) \]

where \( \mathcal{Z} \) is the normalizing factor for \( \mathbb{S}^{S,T} \). Consider the map \( \phi : \Lambda_{T^1} \otimes \Lambda_{S^2} \) defined by \( f(T^1)g(S^2) \mapsto f(0)g(0) \) for any \( f, g \in \Lambda_X \). By applying (the continuous extension of) \( \phi \) to (B.21) and rewriting \( S^1 = U \) and \( T^2 = V \), we get

\[ \mathcal{Z}^{-1} \sum_{\lambda \in \mathcal{Y}} p_{t_1}(\lambda)p_{t_2}(\lambda)s_{\lambda}(U)s_{\lambda}(V) = \mathbb{E}_{S^1,T} [p_{t_1}(\lambda)p_{t_2}(\lambda)], \]

where \( \mathcal{Z} \) is the normalizing factor for \( \mathbb{S}^{S,T} \). On the other hand, by Theorem B.11, we have a formal residue expression for (B.21). By applying \( \phi \) to this expression, we obtain (B.10) for this choice of \( N \) and \( m \).

In the general case, we consider some formal Macdonald process in a greater number of variables, apply Theorem B.11 then apply variable contractions \( \phi \) to obtain the Corollary.

**B.3. Proof of Theorem B.3.** Let \( U = (U^1, \ldots, U^{M+N}) \), \( V = (V^1, \ldots, V^{M+N}) \) be tuples of countable sets of variables. Given \( a \geq 0 \) and a countable set of variables \( U \), let \( \rho_a^U : \Lambda_U \rightarrow \mathbb{R} \) be the unital algebra homomorphism defined by

\[ \rho_a^U : p_n(U) \mapsto a^n. \]

This uniquely determines the specialization \( \rho \) because the power symmetric functions generate the algebra of symmetric functions. We may think of \( \rho \) as evaluating \( f \in \Lambda_U \) at \( (a,0,0,\ldots) \). We can certainly extend \( \rho_a^U \) to \( \Lambda_U \). Let

\[ \rho = \bigotimes_{i=1}^{N} (\rho_a^{U^i} \otimes \rho_0^{V^i}) \otimes \bigotimes_{i=1}^{M} (\rho_0^{U^{N+i}} \otimes \rho_b^{N+i}). \]

Then

\[ \rho(\mathbb{S}^{S,U,V}(\lambda)) = \mathbb{S}^{S,a,b}(\lambda). \]

This implies that for any \( 1 \leq n_1 \leq \cdots \leq n_m \leq M \) and \( t_1, \ldots, t_m \neq 0 \), we have

\[ \rho \left( \mathbb{E}_{S^1,T} \left[ \prod_{i=1}^{m} p_{t_i}(\lambda^{n_i+N}) \right] \right) = \mathbb{E}_{S^1,T} \left[ \prod_{i=1}^{m} p_{t_i}(\lambda^{n_i}) \right]. \]
By Corollary B.12, this is exactly

\[
\rho \left( \frac{1}{(2\pi i)^m} \oint \cdots \oint \prod_{1 \leq a < b \leq m} \frac{(1 - \frac{z_a}{z_b})(1 - \frac{t_a z_a}{t_b z_b})}{(1 - \frac{t_a z_b}{t_b z_a})(1 - \frac{t_b z_a}{t_a z_b})} \prod_{a=1}^{m} \prod_{1 \leq i \leq n_a, \ n_a \leq j \leq M+N} H(U^i, z_a^{-1}; t_a) \frac{d z_a}{z_a} \right)
\]

which gives precisely the contour integral formula stated by Theorem B.2. Note we used the fact that the residue operator commutes with continuous maps.
[35] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.
[36] E. M. Opdam. *Root systems and hypergeometric functions*, III, IV, Compositio Mathematica, 67 (1988), no. 1: 21-49; no. 2: 191-209.
[37] *The Oxford Handbook of Random Matrix Theory*, edited by G. Akemann, J. Baik, P. Di Francesco, Oxford University Press, 2011.
[38] G. Peccati and M. S. Taqqu. *Wiener Chaos: Moments, Cumulants and Diagrams: A Survey with Computer Implementation*. Bocconi & Springer Series. Springer-Verlag Italia Srl, 1 edition, 2011.
[39] C. E. I. Redelmeier. *Quaternionic second-order freeness and the fluctuations of large symplectically invariant random matrices*, Preprint, 2015.
[40] M. Shcherbina, *Fluctuations of linear eigenvalue statistics of beta matrix models in the multi-cut regime*, J. Stat. Phys., 151 (2013), no. 6, 1004-1034.
[41] S. Sheffield. *Gaussian free fields for mathematicians*, Probab. Theory Rel. Fields, 139 (2007), (3-4), 521-541, arXiv:0312099.
[42] Ya. Sinai and A. Soshnikov. *A refinement of Wigner’s semicircle law in a neighborhood of the spectrum edge for random symmetric matrices*, Funct. Anal. Appl., 32 (1998), no. 2, 114-131.
[43] S. Sodin. *Several applications of the moment method in random matrix theory*, Preprint, 2014, arXiv:1406.3410.
[44] A. Soshnikov. *Universality at the edge of the spectrum in Wigner random matrices*, Comm. Math. Phys., 207 (1999), no. 3, 697-733.
[45] V. Vasilchuk. *On the fluctuations of eigenvalues of multiplicative deformed unitary invariant ensembles*, Random Matrices Theory Appl., 5 (2016), no. 2, 1650007.
[46] D. Voiculescu. *Multiplication of certain non-commuting random variables*, J. Operator Theory, 18 (1987), no. 2, 223-235.
[47] D. Voiculescu. *Limit laws for random matrices and free products*, Invent. Math. 104 (1991), no. 1, 201-220.
[48] D. Voiculescu. *A strengthened asymptotic freeness result for random matrices with applications to free entropy*, Int. Math. Res. Not. 1998 (1998), no. 1, 41-63.