CONGRUENCES FOR THE FISHBURN NUMBERS

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Abstract. The Fishburn numbers, \( \xi(n) \), are defined by a formal power series expansion
\[
\sum_{n=0}^{\infty} \xi(n)q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - (1 - q)^j).
\]
For half of the primes \( p \), there is a non-empty set of numbers \( T(p) \) lying in \([0, p-1]\) such that if \( j \in T(p) \), then for all \( n \geq 0 \),
\[
\xi(pn + j) \equiv 0 \pmod{p}.
\]

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1. Introduction

The Fishburn numbers \( \xi(n) \) are defined by the formal power series
\[
\sum_{n=0}^{\infty} \xi(n)q^n = \sum_{n=0}^{\infty} (1 - q; 1 - q)_n
\]
where
\[
(A; q)_n = (1 - A)(1 - Aq)\ldots(1 - Aq^{n-1}).
\]

The Fishburn numbers have arisen in a wide variety of combinatorial settings. One can gain some sense of the extent of their applications in [9, Sequence A022493]. Namely, these numbers arise in such combinatorial settings as linearized chord diagrams, Stoimenow diagrams, nonisomorphic interval orders, unlabeled \((2 + 2)\)-free posets, and ascent sequences. They were first defined in the work of Fishburn (cf. [6, 7, 8]), and have recently found a connection with mock modular forms [4].

It turns out that the Fishburn numbers satisfy congruences reminiscent of those for the partition function \( p(n) \) [2, Chapter 1]. Surprisingly, in contrast to \( p(n) \), we shall see in Section 4 that there are congruences of the form \( \xi(pm + b) \equiv 0 \pmod{p} \) for half of all the primes \( p \). For example, for all \( n \geq 0 \),
\[
(3) \quad \xi(5n + 3) \equiv \xi(5n + 4) \equiv 0 \pmod{5},
\]
\[
(4) \quad \xi(7n + 6) \equiv 0 \pmod{7},
\]
\[
(5) \quad \xi(11n + 8) \equiv \xi(11n + 9) \equiv \xi(11n + 10) \equiv 0 \pmod{11},
\]
\[
(6) \quad \xi(17n + 16) \equiv 0 \pmod{17}, \text{ and}
\]
\[
(7) \quad \xi(19n + 17) \equiv \xi(19n + 18) \equiv 0 \pmod{19}.
\]

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These results all follow from a general result stated as Theorem 3.1 in Section 3. The next section is devoted to background lemmas. Theorem 3.1 is then proved in Section 3. In Section 4 we discuss an infinite family of primes $p$ for which these congruences hold. We conclude with some open problems.

2. Background Lemmas

The sequence of pentagonal numbers is given by

$$\{n(3n-1)/2\}_{n=-\infty}^{\infty} = \{0, 1, 2, 5, 7, 12, 15, 22, \ldots\}.$$ (8)

Throughout this work the symbol $\lambda$ will be used to designate a pentagonal number.

In our first lemma, $f(q)$ will denote an arbitrary polynomial in $\mathbb{Z}[q]$, and $p$ will be a fixed prime. Then we separate the terms in $f(q)$ according to the residue of the exponent modulo $p$.

Thus,

$$f(q) = \sum_{i=0}^{p-1} q^i \phi_i(q^p).$$ (9)

We also suppose that for every $p^{th}$ root of unity $\zeta$ (including $\zeta = 1$),

$$f(\zeta) = \sum_{\lambda} c_\lambda \zeta^\lambda$$

where the $\lambda$'s sum over some set of pentagonal numbers that includes 0. The $c$'s are thus defined to be 0 outside this prescribed set of pentagonal numbers, and the $c$'s are independent of the choice of $\zeta$.

Lemma 2.1. Under the above conditions, $\phi_j(1) = 0$ if $j$ is not a pentagonal number.

Proof. The assertion is not immediate because the $p^{th}$ roots of unity are not linearly independent. In particular, if $\zeta$ is a primitive $p^{th}$ root of unity, then

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1} = 0.$$ (p. 187)

However, we know that the ring of integers in $\mathbb{Q}(\zeta)$ has $1, \zeta, \zeta^2, \ldots, \zeta^{p-2}$ as a basis. Hence,

$$\phi_0(1)(-\zeta - \zeta^2 - \cdots - \zeta^{p-1}) + \sum_{j=1}^{p-1} \zeta^j \phi_j(1) = c_0(-\zeta - \zeta^2 - \cdots - \zeta^{p-1}) + \sum_{\lambda \neq 0} c_\lambda \zeta^\lambda.$$ (10)

Therefore, if $1 \leq j \leq p - 1$,

$$\phi_j(1) - \phi_0(1) = \begin{cases} c_\lambda - c_0 & \text{if } j \text{ is one of the designated pentagonal numbers}, \\ -c_0 & \text{otherwise} \end{cases}$$

is a linear system of $p - 1$ equations in $p$ variables $\phi_j(1), 0 \leq j \leq p - 1$. However, the $\zeta = 1$ case adds one further equation

$$\phi_0(1) + \phi_1(1) + \cdots + \phi_{p-1}(1) = \sum_{\lambda} c_\lambda.$$ (11)

We now have a linear system of $p$ equations in $p$ variables, and the determinant of the system is $p$. Hence, there is a unique solution which is the obvious solution

$$\phi_j(1) = \begin{cases} c_\lambda & \text{if } j \text{ is one of the designated pentagonal numbers}, \\ 0 & \text{otherwise}. \end{cases}$$
In the next three lemmas, we require some variations on Leibniz’s rule for taking the \( n \)th derivative of a product. Each is probably in the literature, but is included here for completeness.

**Lemma 2.2.**

\[
(q \frac{d}{dq})^n (A(q)B(q)) = \sum_{j=1}^{n} q^j c_{n,j} \left( \frac{d}{dq} \right)^j (A(q)B(q)),
\]

where the \( c_{n,j} \) are the Stirling numbers of the second kind given by \( c_{n,0} = c_{n,n+1} = 0 \), \( c_{1,1} = 1 \), and \( c_{n+1,j} = jc_{n,j} + c_{n,j-1} \) for \( 1 \leq j \leq n + 1 \).

**Proof.** The result is a tautology when \( n = 1 \). To pass from \( n \) to \( n + 1 \), we note

\[
(q \frac{d}{dq})^{n+1} (A(q)B(q)) = \frac{d}{dq} \left( (q \frac{d}{dq})^n (A(q)B(q)) \right)
\]

\[
= q \frac{d}{dq} \sum_{j=1}^{n} q^j c_{n,j} \left( \frac{d}{dq} \right)^j (A(q)B(q))
\]

\[
= q \frac{d}{dq} \sum_{j=1}^{n} q^j c_{n,j} \left( \frac{d}{dq} \right)^j (A(q)B(q))
\]

\[
\begin{aligned}
&= \sum_{j=1}^{n} jq^j c_{n,j} \left( \frac{d}{dq} \right)^j (A(q)B(q)) \\
&\quad + \sum_{j=1}^{n} q^{j+1} c_{n,j} \left( \frac{d}{dq} \right)^{j+1} (A(q)B(q))
\end{aligned}
\]

\[
= \sum_{j=1}^{n+1} q^j (jc_{n,j} + c_{n,j-1}) \left( \frac{d}{dq} \right)^j (A(q)B(q))
\]

\[
= \sum_{j=1}^{n+1} q^j c_{n+1,j} \left( \frac{d}{dq} \right)^j (A(q)B(q)).
\]

**Lemma 2.3.**

\[
\left( \frac{d}{dt} \right)^n f(qe^t) \bigg|_{t=0} = \left( q \frac{d}{dq} \right)^n f(q).
\]

**Proof.** By Lemma 2.2 with \( A(q) = f(q) \) and \( B(q) = 1 \), we see that

(10) \[
\left( q \frac{d}{dq} \right)^n f(q) = \sum_{j=1}^{n} q^j c_{n,i} f^{(j)}(q).
\]

On the other hand, we claim

(11) \[
\left( \frac{d}{dt} \right)^n f(qe^t) = \sum_{j=1}^{n} q^j e^{jt} c_{n,j} f^{(j)}(qe^t).
\]
When \( n = 1 \), this is just the chain rule applied to \( f(qe^t) \). To pass from \( n \) to \( n + 1 \), we note
\[
\left( \frac{d}{dt} \right)^{n+1} f(qe^t) = \frac{d}{dt} \left( \frac{d}{dt} \right)^n f(qe^t) = \frac{d}{dt} \sum_{j=1}^{n} q^j e^{jt} c_{n,j} f^{(j)}(qe^t) = \sum_{j=1}^{n} j q^j e^{jt} c_{n,j} f^{(j)}(qe^t) + \sum_{j=1}^{n} q^{j+1} e^{(j+1)t} c_{n,j} f^{(j+1)}(qe^t) = \sum_{j=1}^{n+1} (j c_{n,j} + c_{n,j-1}) q^j e^{jt} f^{(j)}(qe^t) = \sum_{j=1}^{n+1} c_{n+1,j} q^j e^{jt} f^{(j)}(qe^t).
\]
Comparing (11) with \( t = 0 \) to (10), we see that our lemma is established.

We now turn to the generating function for the Fishburn numbers as given by Zagier [10, page 946]. Namely,
\[
(12) \quad F(1 - q) = \sum_{n=0}^{\infty} \xi(n) q^n = \sum_{n=0}^{\infty} (1 - q; 1 - q)_n.
\]
To facilitate the study, we concentrate on
\[
(13) \quad F(q) = \sum_{n=0}^{\infty} (q; q)_n
\]
and
\[
(14) \quad F(q, N) = \sum_{n=0}^{N} (q; q)_n = \sum_{i=0}^{p-1} q^i A_p(N, i, q^p),
\]
where \( A_p(N, i, q^p) \) is a polynomial in \( q^p \). We note that if \( \zeta \) is a \( p^{th} \) root of unity
\[
(15) \quad F(\zeta) = F(\zeta, m) = F(\zeta, p - 1)
\]
for all \( m \geq p \). Furthermore,
\[
(16) \quad \left( \frac{d}{dq} \right)^r F(q) \bigg|_{q=\zeta} = \left( \frac{d}{dq} \right)^r F(q, m) \bigg|_{q=\zeta} = \left( \frac{d}{dq} \right)^r F(q, (r + 1)p - 1) \bigg|_{q=\zeta}
\]
for all \( m \geq (r + 1)p \) because \((1 - q^p)^{r+1}\) divides \((q; q)_j\) for all \( j \geq (r + 1)p \).

Similarly, for all \( m \geq (r + 1)p \),
\[
(17) \quad F^{(r)}(q) \bigg|_{q=\zeta} = F^{(r)}(q, m) \bigg|_{q=\zeta} = F^{(r)}(q, (r + 1)p - 1) \bigg|_{q=\zeta}.
\]
In the next lemma, we require a Stirling–like array of numbers $C_{N,i,j}(p)$ given by $C_{N,0,0}(p) = i^{N} (C_{0,0,0}(p) = 1), C_{N,i,N+1}(p) = 0$, and for $1 \leq j \leq N$,

$$C_{N+1,i,j}(p) = (i + j p)C_{N,i,j}(p) + pC_{N,i,j-1}(p).$$

**Lemma 2.4.**

$$\left( \frac{q}{dq} \right)^{N} F(q, n) = \sum_{j=0}^{N} \sum_{i=0}^{p-1} C_{N,i,j}(p)q^{i+jp} A_{p}^{(j)}(n, i, q^{p}).$$

**Proof.** In light of the fact that $C_{0,i,0}(p) = 1$ for all $i$, the $N = 0$ assertion is

$$F(q, n) = \sum_{i=0}^{p-1} q^{i}A_{p}(n, i, q^{p}),$$

which is just the definition of the $A$’s given in (14). To pass from $N$ to $N + 1$, we note

$$\left( \frac{q}{dq} \right)^{N+1} F(q, n) = \frac{d}{dq} \sum_{j=0}^{N} \sum_{i=0}^{p-1} C_{N,i,j}(p)q^{i+jp} A_{p}^{(j)}(n, i, q^{p})$$

$$= \sum_{j=0}^{N} \sum_{i=0}^{p-1} C_{N,i,j}(p)(i + j p)q^{i+jp} A_{p}^{(j)}(n, i, q^{p})$$

$$+ \sum_{j=0}^{N} \sum_{i=0}^{p-1} C_{N,i,j}(p)q^{i+jp} pq^{p} A_{p}^{(j+1)}(n, i, q^{p})$$

$$= \sum_{j=0}^{N} \sum_{i=0}^{p-1} ((i + j p)C_{N,i,j}(p) + pC_{N,i,j-1}(p))q^{i+jp} A_{p}^{(j)}(n, i, q^{p})$$

$$= \sum_{j=0}^{N+1} \sum_{i=0}^{p-1} C_{N+1,i,j}(p)q^{i+jp} A_{p}^{(j)}(n, i, q^{p}).$$

We now define, for any positive integer $p$, two special sets of integers:

(19) $S(p) = \{ j | 0 \leq j \leq p-1 \text{ such that } n(3n-1)/2 \equiv j \pmod{p} \text{ for some } n \}$

and

(20) $T(p) = \{ k | 0 \leq k \leq p-1 \text{ such that } k \text{ is larger than every element of } S(p) \}.$

For example, for $p = 11$, we have

$S(11) = \{ 0, 1, 2, 4, 5, 7 \}$ and $T(11) = \{ 8, 9, 10 \}$.

**Lemma 2.5.** If $i \not\in S(p)$, then

$$A_{p}(pn-1, i, q) = (1 - q)^{n} \alpha_{p}(n, i, q)$$

where the $\alpha_{p}(n, i, q)$ are polynomials in $\mathbb{Z}[q]$. 
Proof. This result is equivalent to the assertion that for $0 \leq j < n$,
\[ A_p^{(j)}(pn - 1, i, 1) = 0, \]
and by (17) we need only prove for $j \geq 0$,
\[ A_p^{(j)}((j + 1)p - 1, i, 1) = 0 \]
because $n \geq (j + 1)$.

We proceed to prove (21) by induction on $j$. When $j = 0$, we only need show that if $i \notin S(p)$,
\[ A_p(p - 1, i, 1) = 0. \]
Following [10, Section 5], we define (where $\zeta$ is now an $N$th root of unity)
\begin{align*}
F(\zeta e^t) = \sum_{n=0}^{\infty} \frac{b_n(\zeta)t^n}{n!} \\
= e^{t/24} \sum_{n=0}^{\infty} \frac{c_n(\zeta)t^n}{24^n n!} \\
= \sum_{M=0}^{\infty} \frac{t^M}{24^M M!} \sum_{n=0}^{M} \binom{M}{n} c_n(\zeta),
\end{align*}
where we have replaced Zagier’s $\xi$ with $\zeta$ to avoid confusion with $\xi(n)$. In [10, Section 5], we see that
\[ c_n(\zeta) = \frac{(-1)^n N^{2n+1}}{2n + 2} \sum_{m=1}^{N/2} \chi(m) \zeta^{(m^2-1)/24} B_{2n+2} \left( \frac{m}{N} \right), \]
where the $B$’s are Bernoulli polynomials and $\chi(m) = \left( \frac{12}{m} \right)$. Note that the only non-zero terms in the sum in (22) have
\[ \zeta^{((6m \pm 1)^2 - 1)/24} \chi(6m \pm 1) = (-1)^m \zeta^{m(3m \pm 1)/2}, \]
i.e., $c_n(\zeta)$ is a linear combination of powers of $\zeta$ where each exponent is a pentagonal number. Hence, by (22) we see that $b_n(\zeta)$ is a linear combination of powers of $\zeta$ where each exponent is a pentagonal number.

Hence, if $\zeta$ is now a $p$th root of unity,
\[ F(\zeta) = F(\zeta, p - 1) = b_0(\zeta) = \sum_{\lambda} c_{\lambda} \zeta^{\lambda}, \]
where the sum over $\lambda$ is restricted to a subset of the pentagonal numbers. On the other hand,
\[ F(\zeta) = F(\zeta, p - 1) = \sum_{i=0}^{p-1} \zeta^i A_p(p - 1, i, 1). \]
Hence, by Lemma 2.1 for $i \notin S(p)$,
\[ A_p(p - 1, i, 1) = 0 \]
which is (21) when \( j = 0 \). Now let us assume that

\[(25) \quad A_p^{(j)}(p(j + 1) - 1, i, 1) = 0\]

for \( 0 \leq j < \nu < n \). By Lemma 2.4

\[(26) \quad \left( \frac{q}{d} \frac{dq}{d} \right)^{\nu} F(q, p(\nu + 1) - 1) = \sum_{j=0}^{\nu} \sum_{i=0}^{p-1} C_{\nu,i,j}(p) \zeta^i A_p^{(j)}(p(\nu + 1) - 1, i, 1).\]

But for \( j < \nu \),

\[A_p^{(j)}(p(\nu + 1), i, 1) = A_p^{(j)}(p(j + 1) - 1, i, 1) = 0.\]

Hence the only terms in the sum in (26) where \( \zeta \) is raised to a non–pentagonal power, \( i \), arise from the terms with \( j = \nu \), namely

\[(27) \quad C_{\nu,i,\nu}(p) \zeta^i A_p^{(\nu)}(p(\nu + 1) - 1, i, 1),\]

and we note that \( C_{\nu,i,\nu}(p) \neq 0 \).

Applying Lemma 2.3 to the left side of (26), we see that by (22)

\[(28) \quad \frac{d}{dq} \frac{dq}{d} F(q) \bigg|_{q = \zeta}
= \left( \frac{d}{dq} \right)^{\nu} F(q, (\nu + 1)p - 1)
= \sum_{j=0}^{\nu} \sum_{i=0}^{p-1} C_{\nu,i,j}(p) \zeta^i A_p^{(j)}(p(\nu + 1) - 1, i, 1).\]

Recall that \( b_\nu(\zeta) \) is a linear combination of powers of \( \zeta \) where the exponents are pentagonal numbers. Hence the expression given in (27) must be zero by Lemma 2.4. Therefore,

\[A_p^{(\nu)}(p(\nu + 1) - 1, i, 1) = 0,\]

and this proves (21) and thus proves Lemma 2.5.

3. The Main Theorem

We recall from (12) that

\[\sum_{n=0}^{\infty} \xi(n)q^n = \sum_{j=0}^{\infty} (1 - q; 1 - q)_j\]

\[= 1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \sum_{h=1}^{i} (-1)^{h-1} q^h \binom{i}{h}\]

\[= 1 + \sum_{j=1}^{\infty} (q^j + O(q^{j+1})).\]

Hence,

\[(29) \quad \sum_{n=0}^{\infty} \xi(n)q^n = F(1 - q, N) + O(q^{N+1}).\]

We are now in a position to state and prove the main theorem of this paper.
Theorem 3.1. If $p$ is a prime and $i \in T(p)$ (as defined in (20)), then for all $n \geq 0$,
\[ \xi(pn + i) \equiv 0 \pmod{p}. \]

Remark 3.2. Congruences (3)–(7) are the cases $p = 5, 7, 11, 17$ and 19 of Theorem 3.1.

Proof. We begin with a simple observation derived from Lucas’s theorem for the congruence class of binomial coefficients modulo $p$ [5, page 271]. Namely if $\pi$ is any integer congruent to a pentagonal number modulo $p$, and $i \in T(p)$, then
\[ \binom{\pi}{i} \equiv 0 \pmod{p}, \]
because the final digit in the $p$–ary expansion of $\pi$ is smaller than $i$ because $i$ is in $T(p)$.

Now by Lemma 2.5, we may write
\[
F(q, pn - 1) = \sum_{i=0}^{p-1} q^i A_p(pn - 1, i, q^p)
= \sum_{i=0}^{p-1} q^i A_p(pn - 1, i, q^p) + \sum_{i=0}^{p-1} q^i (1-q^p)^n \alpha_p(n, i, q^p).
\]
So
\[
F(1-q, pn - 1) = \sum_{i=0}^{p-1} (1-q)^i A_p(pn - 1, i, (1-q)^p) + \sum_{i=0}^{p-1} (1-q)^i (1-(1-q)^p)^n \alpha_p(n, i, (1-q)^p)
:= \Sigma_1 + \Sigma_2.
\]
Now modulo $p$,
\[
\Sigma_2 = \sum_{i=0}^{p-1} (1-q)^i q^{pn} \alpha_p(n, i, 1) = O(q^{pn}).
\]
Therefore, modulo $p$,
\[
F(1-q, pn - 1) \equiv \sum_{i=0}^{p-1} (1-q)^i A_p(pn - 1, i, 1-q^p) \pmod{p}.
\]
Let us look at the terms in this sum where $q$ is raised to a power that is congruent to an element of $T(p)$. Such a term must arise from the expansion of some $(1-q)^i$ where $i \in S(p)$ because $A_p(pn - 1, 1, 1-q^p)$ is a polynomial in $q^p$.

By (30) all such terms have a coefficient congruent to 0 modulo $p$. Therefore, every term $q^j$ in $F(1-q, pn - 1)$ where $j$ is congruent to an element of $T(p)$ must have a coefficient congruent to 0 modulo $p$.

To conclude the proof, we let $n \to \infty$. \qed
4. An Infinite Set of Primes With Congruences

At this stage, one might ask whether one can identify an infinite set of primes $p$ for which congruences such as those described in Theorem 3.1 are found. The answer to this question can be answered affirmatively.

**Theorem 4.1.** Let $R = \{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22\}$. (The elements of $R$ are those numbers $r$, $0 < r < 23$, such that $(\frac{r}{23}) = -1$.) Let $p$ be a prime of the form $p = 23k + r$ for some nonnegative integer $k$ and some $r \in R$. Then $T(p)$ is not empty, i.e., at least one congruence such as those described in Theorem 3.1 must hold modulo $p$.

**Remark 4.2.** From the Prime Number Theorem for primes in arithmetic progression, we see that, asymptotically, $T(p)$ is not empty for half of the primes and $T(p)$ equals the empty set for half of the primes.

**Proof.** Assume $p$ is a prime for which $T(p)$ is empty. That means there is a pentagonal number which is congruent to $-1$ modulo $p$. Then $n(3n - 1)/2 \equiv -1 \pmod{p}$ for some integer $n$. By completing the square we then obtain $(6n - 1)^2 \equiv -23 \pmod{p}$. Thus, by contrapositive, if we know that $-23$ is a quadratic nonresidue modulo $p$, then we know that such a pentagonal number does not exist (which means $T(p)$ is not empty).

Thus, if $\left(\frac{-23}{p}\right) = -1$, then $T(p)$ is not empty. But thanks to properties of the Legendre symbol, we know

$$
\left(\frac{-23}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{23}{p}\right)
$$

$$
= (-1)^{\frac{p-1}{2}} (-1)^{\frac{23-1}{2}} \left(\frac{p}{23}\right) \quad \text{by quadratic reciprocity}
$$

$$
= (-1)^{\frac{12(p-1)}{2}} \left(\frac{r}{23}\right) \quad \text{since } p = 23k + r
$$

$$
= \left(\frac{r}{23}\right)
$$

and we want this value to be $-1$. The theorem then follows by the nature of the construction of $R$.

Thus, we clearly have infinitely many primes $p$ for which the Fishburn numbers will exhibit at least one congruence modulo $p$.

5. Conclusion

There are many natural open questions that could be answered at this point.

- First, we believe that Theorem 3.1 lists all the congruences of the form $\xi(pn + b) \equiv 0 \pmod{p}$, but we have not proved this at this time.

- Numerical evidence seems to indicate that Theorem 3.1 can be strengthened. Namely, for certain values of $j > 1$ and certain primes $p$, it appears that

$$
\xi(p^j n + b) \equiv 0 \pmod{p^j}
$$

for certain values $b$ and all $n$. 

• Numerical evidence suggests that Lemma 2.5 could be strengthened as follows: If $i \not\in S(p)$, then

$$A_p(pn - 1, i, q) = (q; q)_n \beta_p(n, i, q)$$

for some polynomial $\beta_p(n, i, q)$. That is to say, in Lemma 5, it was proved that $(1 - q)^n$ divides $A_p(pn - 1, i, q)$; it appears that the factor $(1 - q)^n$ can be strengthened to $(q; q)_n$.

• With an eye towards the recent work of Andrews and Jelínek [3], consider the power series given by

$$\sum_{n=0}^{\infty} a(n)q^n := \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)_n \left( \frac{1}{1 - q} \right)_n$$

which begins

$$1 - q + q^2 - 2q^3 + 5q^4 - 16q^5 + 61q^6 - 271q^7 + 1372q^8 - 7795q^9 + \ldots$$

We conjecture that, for all $n \geq 0$, $a(5n + 4) \equiv 0 \pmod{5}$.

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