POISSON STRUCTURE ON CHARACTER VARIETIES

INDRANIL BISWAS AND LISA C. JEFFREY

ABSTRACT. We show that the character variety for a \( n \)-punctured oriented surface has a natural Poisson structure.

Résumé. Nous démontrons que la variété des caractères d’une surface compacte orientée percée en \( n \) points est dotée d’une structure de Poisson naturelle.

1. Introduction

Let \( X \) be a compact connected oriented surface. Given any real or complex reductive Lie group \( G \), the character variety \( \mathcal{R}(X, G) \) parametrizes the equivalence classes of completely reducible \( G \)-homomorphisms of the fundamental group of \( X \). Alternatively, \( \mathcal{R}(X, G) \) parametrizes the isomorphism classes of completely reducible flat \( G \)-connections on \( X \). It is known that \( \mathcal{R}(X, G) \) has a natural symplectic structure; this symplectic structure was constructed by Atiyah–Bott and Goldman in [3], [8] respectively.

Fix finitely many points \( \{x_1, \ldots, x_m\} \) of \( X \), and fix a conjugacy class \( C_i \) in \( G \) for each \( x_i \). Let \( \mathcal{R}(X_0, G) \) be the character variety for \( X_0 = X \setminus \{x_1, \ldots, x_m\} \). Let \( \mathcal{R}(X_0, G)_C \subset \mathcal{R}(X_0, G) \) be the locus of all flat \( G \)-connections on \( X_0 \) for which the local monodromy around each \( x_i \) lies in the conjugacy class \( C_i \). It is known that this subset is equipped with a natural symplectic structure [5], [6]. When \( G \) is a compact group, and \( X \) is equipped with a complex structure, then \( \mathcal{R}(X_0, G)_C \) is the moduli space of semistable parabolic \( G_C \)-bundles [14], where \( G_C \) is the complexification of \( G \).

We prove that \( \mathcal{R}(X_0, G) \) has a natural Poisson structure (see Section 3.2). The above submanifolds \( \mathcal{R}(X_0, G)_C \) of it equipped with symplectic structure are the symplectic leaves for this Poisson structure.

This result has been known for many years – for example, see the proof given in M. Audin’s article ([4], Theorem 2.2.1). However Audin’s proof proceeds by using loop groups and central extensions of the Lie algebra of a loop group. Our proof is much simpler; one of the reasons for it is that we are able to use the known result that \( \mathcal{R}(X_0, G)_C \) are symplectic manifolds.

2. Tangent and cotangent bundles of character varieties

Let \( X \) be a compact connected oriented \( C^\infty \) surface. Fix a nonempty finite subset

\[ D := \{x_1, \ldots, x_m\} \subset X. \]
Let $X_0 := X \setminus D$ be the complement. Fix a base point $x_0 \in X_0$. For notational convenience, the fundamental group $\pi_1(X_0, x_0)$ will be denoted by $\Gamma$.

Let $G$ be a connected reductive algebraic Lie group, which is defined over $\mathbb{R}$ or $\mathbb{C}$. This implies that the Lie algebra $\mathfrak{g} := \text{Lie}(G)$ admits a $G$–invariant nondegenerate symmetric bilinear form. Fix a $G$–invariant nondegenerate symmetric bilinear form

$$B \in \text{Sym}^2(\mathfrak{g}^*) .$$  \hfill (2.1)

Consider the character variety

$$\mathcal{R} := \mathcal{R}(X_0, G) := \text{Hom}(\Gamma, G)^0 / G ,$$  \hfill (2.2)

where $\text{Hom}(\Gamma, G)^0 \subset \text{Hom}(\Gamma, G)$ is the locus of homomorphisms with completely reducible image. We note that the points of $\mathcal{R}$ correspond to the equivalence classes of homomorphisms $\rho : \Gamma \longrightarrow G$ such that the Zariski closure of $\rho(\Gamma)$ is a reductive subgroup of $G$.

Take any homomorphism $\rho : \Gamma \longrightarrow G$. Let $E_G^\rho \longrightarrow X_0$ be the corresponding principal $G$–bundle on $X_0$ equipped with a flat connection. We briefly recall the construction of the flat bundle $E_G^\rho$. Let $q_0 : (\tilde{X}_0, \tilde{x}_0) \longrightarrow (X_0, x_0)$ be the universal cover of $X_0$ for the base point $x_0$. The total space of $E_G^\rho$ is the quotient of $\tilde{X}_0 \times G$, where two points $(x_1, g_1)$ and $(x_2, g_2)$ of $\tilde{X}_0 \times G$ are identified if there is an element $\gamma \in \Gamma$ such that $x_2 = x_1 \gamma$ and $g_2 = \rho(\gamma)^{-1} g_1$ (the fundamental group $\Gamma$ acts on $\tilde{X}_0$ as deck transformations). The projection of $E_G^\rho$ to $X_0$ is given by the map $(x, g) \longmapsto q_0(x)$. The action of $G$ on $\tilde{X}_0 \times G$ given by the right–translation action of $G$ on itself produces an action of $G$ on the quotient space $E_G^\rho$, making $E_G^\rho$ a principal $G$–bundle over $X_0$. The trivial connection on the trivial principal $G$–bundle $\tilde{X}_0 \times G \longrightarrow \tilde{X}_0$ descends to a flat connection on the principal $G$–bundle $E_G^\rho$. This flat connection on $E_G^\rho$ will be denoted by $\tilde{\nabla}^\rho$.

The flat connection $\tilde{\nabla}^\rho$ induces a flat connection on every fiber bundle associated to the principal $G$–bundle $E_G^\rho$. In particular, it induces a flat connection on the adjoint vector bundle $\text{ad}(E_G^\rho)$ associated to $E_G^\rho$ for the adjoint action of $G$ on the Lie algebra $\mathfrak{g}$. This induced flat connection on $\text{ad}(E_G^\rho)$ will be denoted by $\nabla^\rho$.

Let

$$\text{ad}(E_G^\rho) \longrightarrow X_0$$  \hfill (2.3)

be the locally constant sheaf on $X_0$ given by the sheaf of covariant constant sections of the vector bundle $\text{ad}(E_G^\rho)$ for the flat connection $\nabla^\rho$. It is known that the tangent spaces of $\mathcal{R}$ defined in (2.2) have the following description: For any $\rho \in \mathcal{R}$,

$$T_\rho \mathcal{R} = H^1(X_0, \text{ad}(E_G^\rho)) ,$$  \hfill (2.4)

where $\text{ad}(E_G^\rho)$ is constructed in (2.3) [3], [3]. Since $X_0$ is oriented, this gives the following description of the cotangent space:

$$T_\rho^* \mathcal{R} = H^1(X_0, \text{ad}(E_G^\rho))^* = H^1_c(X_0, \text{ad}(E_G^\rho))^* ,$$  \hfill (2.5)

where $H^i_c$ is the compactly supported cohomology [6], [8], [5], and $\text{ad}(E_G^\rho)^*$ is the dual local system. The pairing between $H^1(X_0, \text{ad}(E_G^\rho))$ and $H^1_c(X_0, \text{ad}(E_G^\rho))^*$ is constructed in the following way:

$$H^1_c(X_0, \text{ad}(E_G^\rho))^* \otimes H^1(X_0, \text{ad}(E_G^\rho)) \longrightarrow H^2_c(X_0, \text{ad}(E_G^\rho)^* \otimes \text{ad}(E_G^\rho))$$
\[ H^2_c(X, k) = k, \quad (2.6) \]

where \( k \) is either \( \mathbb{R} \) or \( \mathbb{C} \) depending on whether the Lie group \( G \) is real or complex.

The bilinear form \( B \) in (2.1), being \( G \)-invariant, produces a fiberwise symmetric nondegenerate bilinear form \( \tilde{B} \in C^\infty(X_0, \text{Sym}^2(\text{ad}(E^\rho_G)^*)) \). This section \( \tilde{B} \) is clearly covariant constant with respect to the flat connection on \( \text{Sym}^2(\text{ad}(E^\rho_G)^*) \) induced by the above flat connection \( \nabla^\rho \) on \( \text{ad}(E^\rho_G) \) associated to \( \rho \). In other words, we have

\[ \tilde{B} \in H^0(X_0, \text{Sym}^2(\text{ad}(E^\rho_G)^*)); \]

note that \( \text{Sym}^2(\text{ad}(E^\rho_G)^*) \) coincides with the local system on \( X_0 \) defined by the sheaf of covariant constant sections of \( \text{Sym}^2(\text{ad}(E^\rho_G)^*) \). Consequently, \( \tilde{B} \) produces an isomorphism

\[ \Phi^\rho : \text{ad}(E^\rho_G) \sim \text{ad}(E^\rho_G)^*. \quad (2.7) \]

Combining (2.5) and (2.7), we have

\[ T^* \rho \mathcal{R} = H^1_c(X_0, \text{ad}(E^\rho_G)). \quad (2.8) \]

For any sheaf \( F \) on \( X_0 \), there is a natural homomorphism \( H^1_c(X_0, F) \to H^1(X_0, F) \) given by the inclusion homomorphism of the corresponding sheaves. In particular, we have a homomorphism

\[ \Phi^\rho : H^1_c(X_0, \text{ad}(E^\rho_G)) \to H^1(X_0, \text{ad}(E^\rho_G)), \quad (2.9) \]

which, using (2.4) and (2.8), gives a homomorphism \( T^*_\rho \mathcal{R} \to T^*_\rho \mathcal{R} \). This homomorphism \( T^*_\rho \mathcal{R} \to T^*_\rho \mathcal{R} \) defines an element \( \Phi^\rho \in T^*_\rho \mathcal{R} \otimes T^*_\rho \mathcal{R} \).

**Lemma 2.1.** The above element \( \Phi^\rho \) lies in the subspace \( \wedge^2 T^*_\rho \mathcal{R} \subset T^*_\rho \mathcal{R} \otimes T^*_\rho \mathcal{R} \).

**Proof.** Take \( \alpha, \beta \in H^1_c(X_0, \text{ad}(E^\rho_G)) = T^*_\rho \mathcal{R} \). For the pairing \( \langle -, - \rangle \) in (2.6), we have

\[ \langle \alpha, \tilde{\Phi}^\rho(\beta) \rangle = -\langle \beta, \tilde{\Phi}^\rho(\alpha) \rangle, \quad (2.10) \]

where \( \tilde{\Phi}^\rho \) is the homomorphism in (2.9); the isomorphism in (2.7) has been used in (2.10). The lemma follows from (2.10). \( \square \)

The above pointwise construction of \( \Phi^\rho \), being canonical, produces a section

\[ \Phi \in C^\infty(\mathcal{R}, \wedge^2 T\mathcal{R}). \quad (2.11) \]

We will show, in the next section, that this \( \Phi \) defines a Poisson structure on \( \mathcal{R} \).

### 3. Poisson structure on \( \mathcal{R} \)

#### 3.1. A criterion for Poisson structure

Let \( M \) be a smooth manifold. Let

\[ \Theta \in C^\infty(M, \wedge^2 TM) \]

be a smooth section. For any point \( x \in M \), let

\[ \Theta_x : T^*_x M \to T_x M \quad (3.1) \]

be the homomorphism defined by the equation \( w(\Theta_x(v)) = \Theta(x)(v \wedge w) \) for all \( v, w \in T^*_x M \). The image

\[ V_x := \Theta_x(T^*_x M) \subset T_x M \]
is equipped with a symplectic form. To prove this, let

$$\varphi : T^*_x M \longrightarrow V^*_x$$

be the dual of the inclusion map $V_x \hookrightarrow T_x M$, so $\varphi$ is surjective. We will prove that $\varphi$ vanishes on the subspace $\ker(\Theta_x) \subset T^*_x M$. For any $v, w \in T^*_x M$, we have

$$w(\Theta_x(v)) + v(\Theta_x(w)) = 0,$$

(3.2)

because $\Theta$ is skew-symmetric. So if $v \in \ker(\Theta_x)$, then $\varphi$ vanishes on $\ker(\Theta_x)$, and hence it descends to a homomorphism

$$\hat{\varphi}_x : \frac{T^*_x M}{\ker(\Theta_x)} = \text{image}(\Theta_x) = V_x \longrightarrow V^*_x.$$

From (3.2) it follows that $\hat{\varphi}'_x \in \wedge^2 V^*_x$, where $\hat{\varphi}'_x$ is the bilinear form on $V_x$ defined by $\hat{\varphi}_x$. Since $\varphi$ is surjective, it follows that this $\hat{\varphi}_x$ is also surjective. This implies that the bilinear form $\hat{\varphi}'_x$ is nondegenerate. So, $\hat{\varphi}'_x$ is a symplectic form on $V_x$.

The section $\Theta$ is called a Poisson structure if the Schouten–Nijenhuis bracket $[\Theta, \Theta]$ vanishes identically [1]. An equivalent formulation of this definition is the following: Given a pair of $C^\infty$ functions $f_1$ and $f_2$ on $M$, define the function

$$\{f_1, f_2\} = \Theta((df_1) \wedge (df_2)).$$

Then $\Theta$ is Poisson if and only if

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$$

(3.3)

for all $C^\infty$ functions $f_1, f_2, f_3$.

Let $\hat{\Theta} : T^* M \longrightarrow TM$ be the homomorphism given by $\Theta$, so $\hat{\Theta}(x) = \Theta_x$ for every $x \in X$.

The following proposition is in the literature already. For example, see Proposition 1.8 and Remark 1.10 of the notes [15] from E. Meinrenken’s 2017 graduate course on Poisson geometry. However we have included a proof for completeness.

**Proposition 3.1.** The section $\Theta$ is Poisson if and only if the following two hold:

1. The subsheaf $\hat{\Theta}(T^* M) \subset TM$ is closed under the Lie bracket operation.
2. For any leaf $L$ of the nonsingular locus of the integrable distribution $\hat{\Theta}(T^* M)$, let $\hat{\varphi}'_L$ be the two–form on $L$ defined by the equation

$$\hat{\varphi}'_L(x)(v_1, v_2) = \hat{\varphi}_x(v_1, v_2)$$

for all $x \in L$, and $v_1, v_2 \in T_x L$, where $\hat{\varphi}_x$ is constructed above. Then $\hat{\varphi}'_L$ is a symplectic form on $L$.

**Proof.** If $\Theta$ is Poisson, it is standard that the above two conditions hold. We shall prove the converse.

Take any point $x \in M$ where the distribution $\hat{\Theta}(T^* M)$ is nonsingular, meaning the dimension of the subspace $\hat{\Theta}(y)(T^*_y M) \subset T_y M$ is unchanged for all points $y$ in an open neighborhood of $x$. Let $L$ denote the leaf, passing through $x$, of the foliation restricted to
a sufficiently small open neighborhood $U$ of $x$ in $M$. For any two smooth functions $f, g$ defined on $U$, consider the function

$$\{f, g\} : \mathbb{L} \mapsto \mathbb{R}, \quad y \mapsto \Theta(y)(df(y), dg(y)) \in \mathbb{R}.$$ 

Let $h$ be a smooth function defined on $U$ such that $h|_L = f|_L$. We will prove that

$$\{f, g\} = \{h, g\}.$$  

(3.4)

Note that $f - h$ vanishes on $\mathbb{L}$. For notational convenience, denote the function $f - g$ by $\delta$. To prove (3.4), consider the function $\{f - h, g\}$. For any $y \in \mathbb{L}$, we have

$$\{\delta, g\}(y) = dg(y)(\Theta_y(d\delta(y))) = -d\delta(y)(\Theta_y(dg(y))),$$ 

(3.5)

where $\Theta_y$ is constructed as in (3.1). The pullback of the 1–form $d\delta$ to the submanifold $\mathbb{L} \subset U$ vanishes identically because the restriction of $\delta$ to $\mathbb{L}$ is identically zero. On the other hand, the tangent vector $\Theta_y(dg(y)) \in T_yU$ lies in the subspace $L_y \subset T_yU$ (recall that $L_y = \Theta_y(T^*_yM)$). Therefore, we have

$$d\delta(y)(\Theta_y(dg(y))) = 0.$$ 

Hence $\{\delta, g\}(y) = 0$ by (3.5). This proves (3.4).

In view of (3.4) to prove that the Poisson bracket $\{-, -\}$ satisfies the Jacobi identity in (3.3), it suffices to show that the Jacobi identity is satisfied by the Poisson bracket operation on functions on a leaf $\mathbb{L}$, where the Poisson bracket is defined using the nondegenerate two–form on the leaf given by $\hat{\varphi}_L$. But condition (2) in the proposition says that $\hat{\varphi}_L$ is symplectic on a leaf, and hence the Poisson bracket on a leaf satisfies the Jacobi identity. This completes the proof of the proposition. \hfill \Box

3.2. Application of the criterion. Using Proposition 3.1 it will be shown that $\Phi$ constructed in (2.11) is a Poisson structure on $\mathcal{R}$.

Consider the homomorphism $\Phi_1 : T^*\mathcal{R} \rightarrow T\mathcal{R}$ constructed from $\Phi$ in (2.11) as follows:

$$v(\Phi_1(\rho)(w)) = \Phi(\rho)(w \wedge v)$$

for all $v, w \in T^*_\mathcal{R} \mathcal{R}$ and $\rho \in \mathcal{R}$. So, $\Phi_1(\rho) : T^*_\mathcal{R} \mathcal{R} \rightarrow T\mathcal{R}$, $\rho \in \mathcal{R}$, coincides with the homomorphism $\hat{\Phi}_\rho$ in (2.9). Therefore, the image of $\Phi_1$ corresponds to the foliation on $\mathcal{R}$ given by loci with fixed conjugacy classes for the punctures $\{x_1, \cdots, x_m\}$. In particular, the distribution $\Phi_1(T^*\mathcal{R})$ is integrable; so the first condition in Proposition 3.1 is satisfied. On each leaf the two–form is symplectic [6], [5], [8]; so the second condition in Proposition 3.1 is also satisfied.

4. Extended moduli space

It is shown in [7] (Theorem 4.3) that the quotient of a symplectic manifold by a group action preserving the symplectic structure is a Poisson manifold. We may apply this to the symplectic manifold given in Section 2.3 of [13] (the extended moduli space, which is a symplectic quotient of the space of all connections on a vector bundle over an oriented 2-manifold by the based gauge group). The symplectic structure on the extended moduli space is given in Section 3.1 of [13].
In this section only, let \( G \) be a compact connected Lie group. The extended moduli space \( M_{\text{ext}} \) may be written as the push-out of the Lie algebra of \( G \) and the space of representations \( M = \text{Hom}(\Gamma, G) \) of the fundamental group \( \Gamma \) of a surface with one boundary component, where the map from the Lie algebra to \( G \) is the exponential map, and the map from the space of flat connections to \( G \) is the holonomy around the boundary component. In the case of one boundary component, this is summarized by the following commutative diagram. Let \( M \) be the space \( \text{Hom}(\Gamma, G) \). The symplectic structure is defined in [13] in terms of gauge equivalence classes of flat connections. The map Hol denotes the holonomy of the connection around the boundary component. The symplectic structure on \( \text{Hom}(\Gamma, G)/G \) is described from the point of view of representations of the fundamental group \( \Gamma \) in the work of Goldman [8], [9].

\[
\begin{array}{ccc}
M_{\text{ext}} & \rightarrow & \mathfrak{g} \\
\downarrow & & \exp \\
M & \xrightarrow{\text{Hol}} & G
\end{array}
\]

At a regular point, the extended moduli space is a cover of the representation space \( \text{Hom}(\Gamma, G) \) with fiber the integer lattice of \( G \) (the kernel of the exponential map).

For the case of multiple boundary components, we refer to Hurtubise-Jeffrey [11], Hurtubise-Jeffrey-Sjamaar [12] and Huebschmann [10].

The description in Section 3.1 of [13] establishes that \( M_{\text{ext}} \) is symplectic (where it is smooth). At points in \( \mathfrak{g} \) where the exponential map is a diffeomorphism, there is a local diffeomorphism between an open neighbourhood in \( M_{\text{ext}} \) and an open neighbourhood of \( \text{Hom}(\Gamma, G) \). Taking the quotient of \( M_{\text{ext}} \) by \( G \) we thus conclude that \( \text{Hom}(\Gamma, G)/G \) is a Poisson manifold.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005

E-mail address: indranil@math.tifr.res.in

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada

E-mail address: jeffrey@math.toronto.edu