First Order Chiral Phase Transition from a Six-fermion "Instanton"-Interaction

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We compute the first order chiral phase transition for an instanton motivated quark model with a local six-quark interaction. In order to compare different solutions of the gap equation we compute the bosonic effective action – a two particle irreducible free energy functional. We find that the first order transition ends for a critical current quark mass, with continuous crossover for larger quark masses. Furthermore, we investigate different possible order parameters, including a color octet condensate. We also compare our formalism with mean field theory.

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I. INTRODUCTION

The understanding of strongly interacting fermion systems is one of the great challenges for theoretical physics, ranging from condensed matter systems and ultracold atoms to strong interactions in particle physics. Non-perturbative methods need to be explored in order to deal with strong effective interactions, collective phenomena, condensates in competing channels and different characteristic physics at “microscopic” and “macroscopic” distances. In this note we deal with the bosonic effective action based on a two-particle irreducible formalism and the Schwinger-Dyson equations in a situation where the dominant interactions involve more than four fermions. We also compare with mean field theory.

Indeed, for some physical systems the higher order fermion interactions play a crucial role. As a concrete example we discuss the instanton interaction in quantum chromodynamics (QCD) with three flavors of quarks. The violation of the axial $U(1)_A$-symmetry by instanton effects induces an effective six-quark-interaction which involves all three flavors \[ U(1)_A \]. It has been speculated that this instanton mediated interaction may become dominant at a characteristic momentum scale below 1GeV. We explore here a model where the instanton mediated interaction dominates the effective theory at some given scale $\Lambda$. This scale will act as an ultraviolet cutoff for $q^2 < \Lambda^2$ which induce condensates and spontaneous chiral symmetry breaking. We concentrate on the pointlike limit of this interaction and neglect all other effective interactions between the quarks. We do not believe that this model is a sufficient description for real QCD but it points to some characteristic features and serves as an interesting demonstration for the more formal part of this work.

First investigations of strongly interacting fermionic systems are often based on Mean Field Theory (MFT) or lowest order Schwinger-Dyson equations (SDE) \[ 1 \]. For example, the recent studies of color superconductivity \[ 2 \] are mainly based on one of these methods. Both methods allow for a computation of the order parameter in systems which exhibit spontaneous symmetry breaking (SSB). However, while the SDE approach leads directly to the gap equation, the MFT approach provides naturally a free energy functional for the bosonic composite degrees of freedom introduced by partial bosonization via a Hubbard-Stratonovich transformation \[ 14, 15 \]. The minima of the free energy are determined by the field equation which corresponds to a type of gap equation. Knowledge of the free energy functional becomes necessary if the gap (or field) equation allows solutions with different order parameters and the free energy for the different solutions has to be compared.

Within the SDE approach the reconstruction of the free energy functional from the gap equation is not trivial since the information about the minima (as expressed by the gap equation) may be insufficient in order to find the whole function. For example, it is not sure that the method used in \[ 16 \] for the case of color superconductivity always works. From this point of view MFT seems superior to SDE since it directly provides a free energy. Unfortunately, MFT has also a severe disadvantage: partial bosonization is not unique and the results of the MFT calculation depend strongly on the choice of the mean field (see \[ 17 \] for the example of the Hubbard model where this has drastic consequences). Since partial bosonization is an exact procedure physical results should be independent of this choice. Indeed, the ambiguity is due to the approximation used in MFT (only fermionic fluctuations are taken into account) and is cured by more sophisticated approximations \[ 18 \]. Being a purely fermionic formulation SDE’s do not suffer from such an ambiguity\[ 2 \].

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Hence, we want to find a functional which has the SDE as its equation of motion, and which can be interpreted as a free energy. For four fermion interactions such a functional can be related to the two particle irreducible (2PI) effective action \[ \Gamma_{\text{2PI}}[\psi, G]. \] The 2PI effective action is a functional of fields and propagators \(\Gamma_{\text{2PI}}[\psi, G] \). However, for a purely fermionic system, all the information is already contained in \(\Gamma_{\text{2PI}}[0, G] \) where the “fermionic background field” is set to zero. Hence \(\Gamma_{\text{2PI}}[0, G] \) depends only on the bosonic variable \(G \), and therefore we will call it the Bosonic Effective Action (BEA) \[ \Gamma_{\text{BEA}}. \] In a suitable version the BEA is a free energy functional and the condition for its local minima precisely corresponds to the gap equation.

The work of \[ \Gamma_{\text{BEA}} \] has concentrated on effective four fermion interactions. In the present paper we want to deal with more general multi-fermion interactions. An example of such a higher order interaction is the six-fermion interaction generated by instantons in the case of three flavors and three colors \[ \Gamma_{\text{6-fermion}}. \] This interaction is \(U(1)_A\)-anomalous and solves the famous \(U(1)_A\)-problem \[ \Gamma_{\text{U(1)_A}} \] in QCD. In the simpler case of two flavors the instantons mediate a four fermion interaction which has already been studied extensively for chiral symmetry breaking \[ \Gamma_{\text{4-fermion}} \] and was considered in the early works on color superconductivity e.g. \[ \Gamma_{\text{CSM}} \]. This investigation is generalized here to three light flavors of quarks which are perhaps closer to realistic QCD. We also perform a more systematic discussion of the free energy functional. We observe that the effective interaction generated by the instantons does not only lead to interactions between color singlet effective quark-antiquark degrees of freedom which are finally associated to the usual spontaneous chiral symmetry breaking. It also produces effective interactions between color octets. In a model that goes beyond the six quark interactions the instanton effects may lead to the possibility of octet condensation and spontaneous “color symmetry breaking” \[ \Gamma_{\text{8-fermion}} \].

The paper is organized as follows. Sects. \[ \text{II and III} \] we introduce the 2PI formalism in the language of the Bosonic Effective Action (BEA). In sect. \[ \text{IV} \] we make a short comparison to MFT and hint to some possible improvements. We explicitly calculate the BEA for the six-fermion instanton interaction in sect. \[ \text{V} \] and use it to study chiral symmetry breaking. Finally, in sect. \[ \text{VI} \] we investigate color octet condensates. Sect. \[ \text{VII} \] summarizes our results and conclusions.

II. BOSONIC EFFECTIVE ACTION

In this section we briefly summarize the Schwinger Dyson equations \[ \text{II} \] and the fermionic and bosonic effective action \[ \text{II} \] and generalize it to multi-fermion interactions. In order to simplify the presentation we summarize all indices of the fermionic field in \(\tilde{\psi}_\alpha \). Here the index \(\alpha\) contains all internal indices (spin, color, flavor etc.) as well as position or momentum. Furthermore it also differentiates between \(\psi\) and \(\tilde{\psi}\). In these conventions the partition function reads

\[ Z[\eta, j] = \int D\tilde{\psi} \exp(\eta_\alpha \tilde{\psi}_\alpha + \frac{1}{2} j_{\alpha\beta} \tilde{\psi}_\alpha \tilde{\psi}_\beta - S_{\text{int}}[\tilde{\psi}]). \]  

All terms quadratic in \(\psi\) are associated to bosonic sources \(j_{\alpha\beta}\) and we investigate multifermion interactions \((n \text{ even}, n \geq 4)\)

\[ S_{\text{int}}[\tilde{\psi}] = \sum_n \frac{1}{n!} \lambda^{(n)}_{\alpha_1 \ldots \alpha_n} \tilde{\psi}_{\alpha_1} \cdots \tilde{\psi}_{\alpha_n}. \]

The generating functional of the 1PI Greens functions in presence of the bosonic sources \(j\) is defined by a Legendre transform with respect to the fermionic source term \(\eta\):

\[ \Gamma_F[\psi, j] = -W[\eta, j] + \eta_\alpha \psi_\alpha \]

where

\[ W = \ln Z[\eta, j], \quad \psi_\alpha = \langle \tilde{\psi}_\alpha \rangle = \frac{\partial W}{\partial \eta_\alpha}. \]

The fermionic effective action \(\Gamma_F\) can also be obtained by the implicit functional integral:

\[ \Gamma_F[\psi, j] = -\ln \int D\tilde{\psi} \exp(\eta_\alpha \tilde{\psi}_\alpha - S_j[\tilde{\psi} + \psi]), \]

\[ S_j[\tilde{\psi}] = \frac{1}{2} j_{\alpha\beta} \tilde{\psi}_\alpha \tilde{\psi}_\beta + S_{\text{int}}[\tilde{\psi}]. \]

This form is especially useful to derive the SDE. Taking a derivative with respect to \(\psi\) one finds

\[ \frac{\partial \Gamma_F}{\partial \psi_\beta} = -j_{\beta\alpha_2} \psi_{\alpha_2} + \sum_n \frac{\lambda^{(n)}_{\alpha_2 \ldots \alpha_n}}{F_n} \psi_{\alpha_2} \]

\times \left\{ (\Gamma_F^{(2)})^{-1}_{\alpha_3\alpha_4} \cdots (\Gamma_F^{(2)})^{-1}_{\alpha_{n-1}\alpha_n} + Z_{\alpha_3 \ldots \alpha_n} + \mathcal{O}(\psi^2) \right\}

where

\[ F_n = (n - 2)(n - 4) \cdots 2 \]

and \(\Gamma_F^{(2)}\) denotes the second functional derivative. Here \(Z_{\alpha_3 \ldots \alpha_n}\) summarizes all terms containing third and higher derivatives of \(\Gamma\). (The corresponding diagrams involve

\[ \frac{\partial \Gamma_F}{\partial \psi_\beta} = \frac{\partial \Gamma_F}{\partial \psi_\beta} \]

\[ \text{Note that in this formula } \tilde{\psi} \text{ is shifted such that } \langle \tilde{\psi} \rangle = 0 \text{ and } \eta_\alpha = -\frac{\partial \Gamma_F}{\partial \psi_\beta} \text{ depends on } \psi. \]
at least two vertices.) Taking another derivative with respect to \( \psi_\alpha \) and evaluating at \( \psi = 0 \) we find the SDE:

\[
(\Gamma_F^{(2)})_{\alpha\beta} = -j_{\alpha\beta} + \sum_n \frac{\lambda^{(n)}_{\alpha_2\alpha_3\cdots\alpha_n}}{F_n} \times \left\{ (\Gamma_F^{(2)})^{-1}{\alpha_4}_{\alpha_1} \cdots (\Gamma_F^{(2)})^{-1}{\alpha_n}_{\alpha_{n-1} \alpha_n} + Z_{\alpha_3 \cdots \alpha_n} \right\}.
\]

In this paper we are only interested in the lowest order. Using this relation we can conveniently write the SDE (13) as

\[
G^{-1}_{\alpha\beta} = -j_{\alpha\beta} + \sum_n \frac{\lambda^{(n)}_{\alpha_2\alpha_3\cdots\alpha_n}}{F_n} G_{\alpha_3 \alpha_4 \cdots \alpha_{n-1} \alpha_n}.
\]

Eq. (10) then yields a differential equation for \( \Gamma_B \)

\[
\frac{\partial \Gamma_B}{\partial G_{\alpha\beta}} = -G^{-1}_{\alpha\beta} + \sum_n \frac{\lambda^{(n)}_{\alpha_2\alpha_3\cdots\alpha_n}}{F_n} G_{\alpha_3 \alpha_4 \cdots \alpha_{n-1} \alpha_n}.
\]

By integration\(^4\) one finally finds

\[
\Gamma_B = \frac{1}{2} \text{Tr} \ln G + \sum_n \frac{\lambda^{(n)}_{\alpha_2\alpha_3\cdots\alpha_n}}{n F_n} G_{\alpha_3 \alpha_4 \cdots \alpha_{n-1} \alpha_n},
\]

the BEA at “one-vertex order”. Actually it is sometimes convenient to introduce an auxiliary effective action

\[
\Gamma_j[G, j] = \Gamma_B - \frac{1}{2} j_{\alpha\beta} G_{\alpha\beta}
\]

such that the physical propagator corresponds to the minimum of \( \Gamma_j \) (cf. Eq. (13)). The functional \( \Gamma_j \) will play the role of a suitable free energy (see [22] for details).

### III. BEA FOR LOCAL INTERACTIONS

In the following we want to consider local interactions. For clarity we now write \( x \) (or momentum \( p \)) explicitly and use latin letters for the remaining indices. The standard procedure would be the insertion of the ansatz

\[
G^{+1}_{ab}(x, y) = -j_{ab}(x, y) + \Delta_{ab}(x) \delta(x - y)
\]

into Eq. (11). This would yield the SDE for the local gap \( \Delta \). Since the BEA (13) is related to the SDE (11) by differentiation with respect to \( G \) it is not clear, however, that an effective action functional depending on \( \Delta \) can be obtained by integration with respect to \( \Delta \). In presence of several possible gaps this would require suitable “integrability conditions” for the system of gap equations. This difficulty can be avoided if we follow the construction presented in [22] and start directly from the approximate BEA (13).

With

\[
g_{ab}(x) = G_{ab}(x, x)
\]

we have

\[
\Gamma_j = \frac{1}{2} \text{Tr} \ln G + \frac{1}{2} \text{Tr}(Gj) + \int_x \sum_n \frac{\lambda^{(n)}_{\alpha_1 \cdots \alpha_n}}{n F_n} g_{\alpha_1 \alpha_2}(x) \cdots g_{\alpha_{n-1} \alpha_n}(x).
\]

For this relation it is essential that the interaction is strictly local. Furthermore, we can use the locality of the interaction in order to write Eq. (11) in the form of a local gap equation

\[
G^{-1}_{ab}(x, y) = -j_{ab}(x, y) + \Delta_{ab}(x) \delta(x - y).
\]

We will evaluate the functional \( \Gamma_j[G] \) for \( G_{\alpha\beta} \) taking values corresponding to Eq. (17). This is actually a restriction to a subspace of all possible \( G \). However, locality tells us that the extremum (solution of the SDE) is contained in this subspace.

Using \( j = -G^{-1} + \Delta \) we find (up to a shift in the irrelevant constant and using \( \Delta_{ab}(x, y) = \Delta_{ab}(x) \delta(x - y) \))

\[
\Gamma_j[g, \Delta] = \frac{1}{2} \text{Tr} \ln(-j + \Delta) - \frac{1}{2} \int_x \Delta_{ab}(x) g_{ab}(x) + \int_x \sum_n \frac{\lambda^{(n)}_{\alpha_1 \cdots \alpha_n}}{n F_n} g_{\alpha_1 \alpha_2}(x) \cdots g_{\alpha_{n-1} \alpha_n}(x).
\]

For the search of extrema of \( \Gamma_j \) it is actually convenient to treat \( \Delta \) and \( g \) as independent variables. The extremum of \( \Gamma_j[g, \Delta] \) then obeys

\[
\frac{\partial \Gamma_j[g, \Delta]}{\partial \Delta} = 0, \quad \frac{\partial \Gamma_j[g, \Delta]}{\partial g} = 0.
\]

Evaluating the derivative with respect to \( \Delta \) we recover the inverse of Eq. (17) for \( x = y \),

\[
g_{ab}(x) = (-j + \Delta)^{-1}_{ab}(x, x) = g[\Delta(x)].
\]

Inserting this functional relation into Eq. (11) leads to a gap equation for \( \Delta \). In case of a six-fermion interaction this takes, however, the form of a two-loop equation.

For \( n \)-fermion interactions with \( n > 4 \) it is more appropriate to go the other way around and first take a

\(\)

\(^4\) Note that in our notation \( \frac{\partial G_{\alpha\beta}}{\partial \epsilon_{\alpha\beta}} = \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} \).
Searching for an extremum yields Eq. (24) will be our central gap equation. We should find the effective action and its derivatives with respect to \( g \). We obtain

\[
\Delta_{ab}(x) = \sum_n \frac{\lambda^{(n)}_{ab\ldots an}}{F_n} g_{a_3a_4}(x) \cdots g_{a_{n-1}a_n}(x) = \Delta_{ab}[g(x)], \tag{21}
\]

which is precisely the value of the gap in Eq. (11). Inserting \( \Delta[g] \) into (23) we find the equation of motion corresponding

\[
\Gamma_j[g] = -\frac{1}{2} \text{Tr} \ln(-j + \Delta[g]) - \frac{1}{2} \int_x \Delta_{ab}[g(x)] g_{ab}(x) + \int_x \sum_n \frac{\lambda^{(n)}_{ab\ldots an}}{F_n} g_{a_2a_3}(x) \cdots g_{a_{n-1}a_n}(x). \tag{22}
\]

Searching for an extremum yields

\[
\frac{\partial \Gamma_j[g]}{\partial g} = \left\{ (-j + \Delta[g])^{-1} - g \right\} \frac{d\Delta[g]}{dg} = 0. \tag{23}
\]

For \( \frac{d\Delta[g]}{dg} \neq 0 \) Eq. (23) indeed corresponds to the SDE

\[
g_{ab}(x) = (-j + \Delta[g])^{-1}_{ab}(x, x). \tag{24}
\]

Eq. (24) will be our central gap equation. We should point out that possible extrema of \( \Gamma_j[g] \) corresponding to \( \frac{d\Delta[g]}{dg} = 0 \) are not solutions of the gap equation (11) and should be discarded. Finally, we also have

\[
\frac{d\Gamma_j[g]}{dg} = \frac{d\Gamma_j[g, \Delta[g]]}{dg} \tag{25}
\]

\[
= \frac{\partial \Gamma_j[g, \Delta[g]]}{\partial g} + \frac{\partial \Gamma_j[g, \Delta[g]]}{\partial \Delta} \frac{d\Delta[g]}{dg},
\]

Only as long as \( \frac{d\Delta[g]}{dg} \neq 0 \) is fulfilled we can conclude that a solution of Eq. (23) fulfills both extremum conditions (10).

The procedure proposed here is quite powerful if \( \text{Tr} \ln(-j + \Delta) \) can be explicitly evaluated as a functional of \( \Delta \). Then \( \Gamma_j[g] \) allows not only a search for the extremum (discarding those with \( \frac{d\Delta[g]}{dg} = 0 \)) but also a simple direct comparison of the relative free energy of different local extrema. This is crucial for the determination of the ground state in the case of several “competing gaps”.

Moreover, the formula (22) for \( \Gamma_j \) (and the corresponding field or gap equation) is now a one loop expression. This “one-loop” form of the equation of motion and the effective action is very close to what we would expect from MFT (cf. also the next section). In contrast to the standard SDE, which is only an equation of motion, we can use Eq. (22) to compare the values for the effective action at different solutions of the equation of motion (23), providing us with information about the stability of a given state.

The one vertex approximation (21), (22) to the bosonic effective action is the central tool of this paper

\[
\Gamma_j[g] = \int_x V(g(x)) - \frac{1}{2} \int (d^4q)/(2\pi)^4 \text{Tr} \ln(-j + \Delta[g]), \tag{26}
\]

\[
V(g(x)) = -\sum_n \frac{(n - 2)\lambda^{(n)}_{ab\ldots an}}{2nF_n} g_{a_2a_3}(x) \cdots g_{a_{n-1}a_n}(x). \tag{27}
\]

The first term may be called “classical part” and we have written the second “one loop term” as a momentum integral with \( \text{tr} \) over internal indices and \( -j + \Delta \) involving the Fourier transform of the gap (21) (see (22) for details). As mentioned already we can use \( \Gamma_j \) in order to find and compare the “local minima” of the free energy. Some care is needed, however, for the computation of susceptibilities or effective masses. Indeed, we have to be careful when considering Eq. (22) at points which are not solutions of Eq. (23). Going step by step through the procedure above, we find that if we are not at a solution of (23) we do not necessarily fulfill the ansatz (17). Therefore, at these points we are mathematically not allowed to insert the ansatz into Eq. (13). So, strictly speaking Eq. (22) only gives the value of the effective action at the solution of the equation of motion\(^5\). This is already much more than what we get from the standard SDE. Going beyond this we would also like to interpret Eq. (22) as a reasonable approximation in a small neighborhood of the solution to the equation of motion. Remembering \( g(x) = \langle \psi(x) \psi(x) \rangle \) it is suggestive to interpret \( g \) as a bosonic field. Eq. (21) gives the (non-linear) “Yukawa coupling” of \( g \) to the fermions, i.e. the relation between the gap and the bosonic field. Thus the term \( \text{Tr} \ln \) is the contribution from the fermionic loop in a background field \( g \). The second term in Eq. (26) can then be interpreted as the cost in free energy for different background fields \( g \). This interpretation allows us to use (22) to calculate the mass and the couplings of the bosonic field \( g \) approximately.

\[\text{IV. COMPARISON WITH MFT}\]

We refer here to MFT as used in most computations and obtained by partial bosonization (for a more detailed description of the procedure and its ambiguities see, e.g.

\[\text{V. ALTERNATIVE}
\]

An alternative would be to choose the gap \( \Delta \) as the “bosonic field”. Inserting Eq. (21) into Eq. (13) we could calculate a functional \( \Gamma[\Delta] \). However, as one can check there are two drawbacks. First, even for four-fermion interactions, \( \Gamma[\Delta] \) is usually unbounded from below when considering \( \Delta \rightarrow \infty \). Second, in the case of a large four-fermion coupling the “stable” solution of the field equation is usually a local maximum.

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Using the identity

\[ 1 = N \int \mathcal{D} \phi \exp(m(-\phi)\mathcal{F}) = N \int \mathcal{D} \phi \exp(m(\tilde{\psi}\psi - \phi)\mathcal{F}) \]  

(28)

we can introduce bosonic fields corresponding to fermionic bilinears \( \phi_{ab}(x) = (\tilde{\psi}_a(x)\psi_b(x)) \). The coefficient \( m \) will be chosen such that the insertion of Eq. (28) into the functional integral (11) cancels the purely fermionic interaction \( S_{\text{int}}(\tilde{\psi}) \). We have not displayed the indices in Eq. (28). Usually one restricts the choice to one or several particular index pairs \((a, b)\) (or linear combinations thereof). This corresponds to the freedom in the choice of the mean fields \( \phi_{ab}(x) \). The partially bosonized form of Eq. (2) becomes

\[ S_{\text{int}}[\phi, \tilde{\psi}] = \int_x \sum_n (-1)^{\frac{n}{2}+1} m^{(n)} \phi_{a_1 b_1} \cdots \phi_{a_n b_n} \times \left\{ \phi_{a_1 b_1}(x) \cdots \phi_{a_n b_n}(x) \right\} \]

\[ + \left[ \psi_{a_1}(x)\psi_{b_1}(x)\phi_{a_2 b_2}(x) \cdots \phi_{a_n b_n}(x) \right] \]

\[ + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) + \cdots + (1 \leftrightarrow \frac{n}{2}) \]

\[ + \cdots + O(\tilde{\psi}^{n-2}) \]

(29)

and the functional integral for the partition function (11) has now to be performed over bosonic as well as fermionic variables. We note that the coefficients \( m \) are only partly determined by \( \lambda \), i.e.

\[ m^{(n)}_{a_1 \cdots a_n} = \frac{\lambda^{(n)}_{a_1 \cdots a_n}}{n!} + S_{a_1 \cdots a_n}. \]

(30)

Here \( S \) is a sum of terms which are symmetric in at least one pair of indices. The condition (30) ensures that the partially bosonized Lagrangian is equivalent to the original fermionic one. Nevertheless, due to the anticommuting nature of the fermionic variables \( S \) gives a vanishing contribution to the purely fermionic part of the action and is therefore not fixed.

Neglecting the terms \( O(\tilde{\psi}^4) \) and performing the functional integral over the fermions provides us with the MF-effective action:

\[ \Gamma_{\text{MF}}[\phi] = \int_x V(\phi(x)) - \frac{1}{2} \text{Tr} \ln(-j + \Delta[\phi]), \]

(31)

\[ V(\phi) = \sum_n (-1)^{\frac{n}{2}+1} m^{(n)}_{a_1 \cdots b_n} \phi_{a_1 b_1} \cdots \phi_{a_n b_n}, \]

\[ \Delta[\phi|_{a_1 b_1} = (-1)^{\frac{n}{2}} \sum_n \left[ m^{(n)}_{a_1 \cdots b_n} \phi_{a_2 b_2} \cdots \phi_{a_n b_n} + \cdots + (1 \leftrightarrow \frac{n}{2}) \right]. \]

This form is strikingly similar to Eq. (29). However, the coefficients in the "classical potential" as well as in

the "gap" \( \Delta \) differ. Furthermore, we note that the coefficients \( m \) are not completely fixed due to the presence of the arbitrary symmetric part \( S \) in Eq. (31). Results can depend on the choice of \( S \) (see e.g. [14, 15]). This is the so called "Fierz ambiguity" because the addition of \( S \) corresponds to a (generalized) Fierz transformation in the fermionic language. Of course, further considerations as e.g. the stability of the initial bosonic potential might reduce the freedom in the choice of \( S \) somewhat. But, as the example of [14] shows, such restrictions are sometimes not even strong enough to get qualitatively the same phase diagram for all reasonable choices of \( S \).

Eqs. (18), (22), (23) do not suffer from such an ambiguity since in the derivation of the SDE [8] the coefficient becomes antisymmetrized and symmetric terms drop out. In [14] it was shown that the inclusion of appropriate diagrams for the bosonic fluctuations cures the Fierz ambiguity for four fermion interactions and leads to the SD-result. We believe that this holds also for higher fermion interactions. Nevertheless, the inclusion of the bosonic fluctuations needs a substantial effort. We therefore propose (22) as a natural replacement of Eq. (31) which amounts to an "optimal" choice of \( m \) for many purposes. Moreover, allowing not only for constant but also for spatially varying \( g \) we can calculate the wave function renormalizations and masses of the bosons. Again, the BEA gives unambiguous results.

Finally, let us stress again the intuitive argument for the closeness of the two approaches and explicitly for the similarity of Eqs. (22) and (31): we have \( g(x) = \langle \tilde{\psi}(x)\psi(x) \rangle \) which is exactly what one has in mind as a "mean field".

V. CHIRAL SYMMETRY BREAKING FROM A THREE-FLAVOR INSTANTON INTERACTION

In this section we want to use the method described above to study chiral symmetry breaking in an NJL-type model with a six-fermion interaction. We consider the QCD-instanton interaction with three colors and three flavors [12, 23, 24, 25, 26] in the pointlike limit. The three flavor instanton vertex can be written in the following convenient form [27]:

\[ S_{\text{inst}}[\psi] = -\frac{\zeta}{6} \int_x \epsilon_{a_1 a_2 a_3} \psi_{b_1 b_2 b_3} \left\{ \left( \bar{\psi}_{L}^{a_1} \psi_{R}^{b_1} \left( \bar{\psi}_{L}^{a_2} \psi_{R}^{b_2} \left( \bar{\psi}_{L}^{a_3} \psi_{R}^{b_3} \right) \right) \right) \right. \]

\[ - \frac{1}{8} \left( \bar{\psi}_{L}^{a_1} \lambda^2 \psi_{R}^{b_1} \left( \bar{\psi}_{L}^{a_2} \lambda^2 \psi_{R}^{b_2} \left( \bar{\psi}_{L}^{a_3} \lambda^2 \psi_{R}^{b_3} \right) \right) \right) \]

\[ - \frac{1}{8} \left( \bar{\psi}_{L}^{a_1} \psi_{R}^{b_1} \left( \bar{\psi}_{L}^{a_2} \lambda^2 \psi_{R}^{b_2} \left( \bar{\psi}_{L}^{a_3} \psi_{R}^{b_3} \right) \right) \right) \]

\[ - \left( R \leftrightarrow L \right) \}. \]

(32)
Here $\lambda^2$ are the Gell-Mann matrices with color indices acting as generators of the $SU(3)_c$ color group. The brackets $\{\}$ indicate contractions over color and spinor indices. Within QCD the coupling constant $\zeta$ should be calculated in terms of the running gauge coupling. However, already the one loop approximation involves an IR divergent integral over the instanton size. Therefore, one needs to provide a physical cutoff mechanism. To avoid this difficulty we treat $\zeta$ as a free parameter in an effective theory with ultraviolet cutoff $\Lambda$. Inspection of (38) tells us that this interaction is $U(1)_A$-anomalous with a residual $Z_3$-symmetry. This is important because we cannot restrict ourselves to real condensates from the start.

In order to extract the coupling $\lambda(6)$ we have to antisymmetrize over flavor indices $(a = 1 \ldots 3)$, color indices $(i = 1 \ldots 3)$ and Weyl spinor indices $(\chi = 1,2,L,R)$ and the indices distinguishing between $\psi$ and $\bar{\psi}$ ($s = 1,2$):

$$\lambda_{m_1 \ldots m_6} = \frac{\zeta}{12} \delta_{a_1 a_2 a_3} \delta_{a_4 a_5 a_6} \delta_{a_1 a_4} \delta_{a_2 a_5} \delta_{a_3 a_6}$$

where $m_j = (a_j, i_j, \alpha_j, \chi_j, s_j)$, $j = 1 \ldots 6$, with minus signs appropriate for total antisymmetrization.

As a first example we consider a flavor singlet, color singlet scalar chiral bilinear ($\sigma = \frac{1}{\sqrt{6}} \bar{\psi}_L \psi_R$), flavor, spin and color structure for (35) expresses the gap in terms of $\sigma$

$$\Delta[g]_{mn} = \frac{10}{9} \zeta \delta_{ab} \delta_{ij} \delta_{\alpha \beta} \left[ \sigma^2 (\delta_{\chi_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6} - \delta_{\chi_2 \delta_1 \delta_4 \delta_5 \delta_6 \delta_3} \right]$$

For the classical potential one obtains

$$V(\sigma) = \frac{20}{9} \zeta (\sigma^3 + \sigma^3).$$

For a background $\sigma$ which is constant in space we obtain the effective potential $U(x = q^2)$

$$U(\sigma) = -\frac{9}{8\pi^2} \int dx \left[ \ln(x + |M_q|^2) \right] + V(\sigma),$$

$$M_q = \frac{10}{9} \zeta \sigma^2 + m_q.$$
0 = \sigma_0 < \sigma_1 \leq \sigma_2. We know that \( U(\sigma_1) > U(\sigma_0) = 0 \) therefore \( \sigma_1 \) is not the stable solution. As can be seen from Figs. 1, 2, there is a range \( \zeta_S \leq \zeta \leq \zeta_{SSB} \) where there exist non-trivial solutions to the SDE but no spontaneous chiral symmetry breaking occurs because \( U(\sigma_2) > U(0) \). The state with \( \sigma \neq 0 \) is metastable. This holds as long as \( \zeta_S < \zeta < \zeta_{SSB} \), with \( \zeta_S \) the point of “spinodal decomposition”. A first order phase transition towards a state with nonvanishing chiral condensate \( \sigma \) occurs as \( \zeta \) is increased beyond \( \zeta_{SSB} \). Here the critical value \( \zeta_{SSB} \) for the onset of spontaneous chiral symmetry breaking denotes the value of \( \zeta \) for which the two local minima become degenerate, \( U(\sigma_2) = U(0) \). We point out that in order to calculate \( \zeta_{SSB} \) we need to know the value of \( U \), i.e. information beyond the SDE.

In Fig. 2 we have plotted the mass gap versus the six-fermion coupling strength. We observe the first order phase transition. This may be expected generically due to the “cubic term” in the classical potential. Finally, we would like to remark that in general the free energy does not follow from a naive integration of the SDE with respect to the gap \( \Delta \). We may call the result of a naive integration of the gap equation the “pseudo potential” \( \tilde{U}(\Delta) \), or, more generally, the “pseudo-free energy” \( \tilde{\Gamma}(\Delta) \). In general \( \tilde{\Gamma}[\Delta] \) is not equal to the the BEA \( \Gamma_j[G] \), not even at solutions of the SDE. Indeed, as can be seen from Fig. 2, the results for physical quantities like the effective fermion mass can differ. The underlying reason for this is that the gap \( \Delta \) is not the correct integration variable. The SDE is obtained by a \( G \)-derivative of the BEA functional \( \Gamma_j[G] \). Therefore, in order to reconstruct \( \Gamma_j[G] \), we have to integrate with respect to \( G \). As can be seen from Eq. 21, \( \Delta \) is, in general, not even a linear function of \( G \). Simple integration with respect to \( \Delta \) therefore neglects the Jacobi matrix, which is a non-trivial function of \( \Delta \) for interactions more complicated than a four-fermion interaction.

### B. Non-vanishing current quark masses \( m_q \neq 0 \)

The non-vanishing current quark mass explicitly breaks the residual \( Z_3 \)-symmetry. The effective potential \( U(\sigma, \alpha) \) does no longer depend on \( \cos(3\alpha) \) only, and we have to look at the complete complex \( \sigma \)-plane for possible extrema. Moreover, for \( m_q = 0 \), \( U(\sigma) \) is completely symmetric under \( \zeta \rightarrow -\zeta, \sigma \rightarrow -\sigma \). Therefore, we could restrict ourselves to \( \zeta \geq 0 \). For \( m_q \neq 0 \) we need to add the transformation \( m_q \rightarrow -m_q \). We can still restrict ourselves to positive \( \zeta \) but we need to consider both positive and negative \( m_q \).

In the case of \( m_q \neq 0 \) we still encounter an extremum of \( U(\sigma) \) at \( \sigma = 0 \). However, in this case it is not a solution of the SDE. It is a spurious solution due to \( \frac{d\Delta[\sigma]}{d\sigma} = 0 \). The difference to the chiral limit is that the derivative of the effective potential \( U(\sigma) \) now has only a simple zero while in the chiral limit it has a twofold zero. After dividing the field equation by \( \frac{d\Delta[\sigma]}{d\sigma} \) a simple zero remains only in the chiral limit, giving a solution of the SDE.

Although chiral symmetry is now broken explicitly we can still observe a first-order phase transition signaled by a jump in the fermion mass. The critical coupling for...
the phase transition depends on $m_q$ as depicted in Fig. 3. The critical line ends at $m_q = m_{q, \text{crit}} \approx 0.076$. For $m_q > m_{q, \text{crit}}$ we have no first order phase transition in our approximation but rather a continuous crossover. We have also plotted in Fig. 3 the jump in the constituent quark mass $|\delta M_q|$ between the phases with low and large $|\sigma|$ at the critical coupling $\zeta_{\text{SSB}}$. It vanishes at the end of the critical line at $m_{q, \text{crit}}$.

VI. COLOR-OCTET CONDENSATION

In the preceeding section we have considered only one direction in the space of all possible $g$ resulting in a phase diagram for chiral symmetry breaking. Let us now consider the more general case where we also allow for a non-vanishing expectation value in the color-octet channel, more explicitly in the color-flavor locked direction

$$g_{mn} = g_{aix\chi,sbj\tau} \left[\begin{array}{l}0 \\
\frac{1}{6} \delta_{ab} \delta_{ij} \delta_{\alpha\beta} \\
-\sigma^* (\delta_{\chi 1} \delta_{\tau 2} \delta_{s 2} \delta_{t 1} - \delta_{\chi 2} \delta_{\tau 1} \delta_{s 1} \delta_{t 2}) \\
+ \frac{1}{4} \lambda_{\alpha\beta} \chi \delta_{\alpha\beta} \\
- \chi^* (\delta_{\chi 2} \delta_{\tau 1} \delta_{s 2} \delta_{t 1} - \delta_{\chi 1} \delta_{\tau 2} \delta_{s 1} \delta_{t 2}) \end{array}\right]$$

A condensate in this direction would yield a very interesting phenomenology for QCD where the gluons acquire a mass via the Higgs mechanism and are associated with the $\rho$-mesons. A pointlike six-quark interaction seems too simple in this case for a realistic model of QCD – it does not account for the important infrared cutoff of the instanton size due to the effective gluon mass. Nevertheless, we find a study of a pointlike interaction an interesting preparation for the treatment of a more realistic model.

Following the outline of the previous section we obtain the gap

$$\Delta[g]_{mn} = -\zeta \left[\begin{array}{l}5 \\
(192\sigma^2 + 2\sqrt{6}\sigma\chi - 7\chi^2) \delta_{ab} \delta_{ij} \\
- \frac{5}{288}(2\sqrt{6}\sigma + \chi) \delta_{a1} \delta_{b1} \\
\times \delta_{\alpha\beta} \left[\delta_{\chi 1} \delta_{\tau 2} \delta_{s 2} \delta_{t 1} - \delta_{\chi 2} \delta_{\tau 1} \delta_{s 1} \delta_{t 2}\right] + (s \leftrightarrow t, \sigma \rightarrow \sigma^*, \chi \rightarrow \chi^*) \end{array}\right]$$

and the classical potential

$$V(\sigma, \chi) = \zeta \left[\begin{array}{l}20 \sigma^3 - \frac{5}{18} \sigma \chi^2 - \frac{5}{54} \chi^3 + c.c. \end{array}\right].$$

This results in the effective potential

$$U(\sigma, \chi) = V(\sigma, \chi) - \frac{1}{8\pi^2} \int dx \left[8 \ln(x + |M_8|^2) + \ln(x + |M_1|^2) \right]$$

where the singlet and octet masses for the “constituent quarks” associated with the low mass baryons are given by

$$M_1 = \frac{5}{54} \zeta (12\sigma^2 - \sqrt{6}\sigma\chi - \chi^2) + m_q$$

$$M_8 = \frac{5}{864} \zeta (192\sigma^2 + 2\sqrt{6}\sigma\chi - 7\chi^2) + m_q.$$

We start by a search of extrema of $U$ for real $\sigma$ and $\chi$. In the chiral limit every point on the line $\chi = -2\sqrt{6}\sigma$ ($M_1 = 0$ and $M_8 = 0$) has the same value of $U = 0$, and both derivatives with respect to $\sigma$ and $\chi$ vanish. However, this is one of the spurious solutions mentioned in Sect. III where $\frac{\partial U}{\partial g}$ vanishes. (In our case $\Delta = (M_1, M_8)$ and $g = (\sigma, \chi)$ are two component vectors and $\frac{\partial U}{\partial g}$ stands for the Jacobian.) Direct insertion into Eq. (24) ($\Delta(g) = 0$) shows that on this line only the point $(\sigma, \chi) = (0, 0)$ is a true solution to the SDE. Restricting both $\sigma$ and $\chi$ to be real we have not found a solution of the gap equation with $\chi \neq 0$. Thus, we have not identified a solution for which the condensate would break color symmetry but not parity.

For the most general case of complex $\sigma$ and $\chi$ things are considerably more difficult since we now have to search for an extremum of a potential which depends on four real parameters. We checked several values of the coupling constant. So far we have not found a solution which has a lower free energy than the minimum of the free energy for vanishing octet condensate $\chi = 0$. Still, we would like to point out that the potential is unbounded from below in various directions, including...
those with $\chi \neq 0$. Therefore, a physical cutoff mechanism like the one discussed in \cite{27} or a different approximation to the “classical action” which makes the potential bounded from below may provide additional solutions. In this context we stress that the instanton interaction \cite{35} should not be confounded with realistic QCD. The gluon fluctuations have been omitted here. Perhaps even more important, the effective gluon mass for nonvanishing $\chi$ should lead to an effective $\chi$-dependence of the coupling $\zeta$ \cite{27}. Also the flat direction\footnote{This flat direction is also present for $m_q \neq 0$.} of $U$ for $\chi = -2\sqrt{6}\sigma$ remains intriguing. So far we have not yet understood the reason why $U$, $M_1$ and $M_8$ all vanish simultaneously on this “line”. A small additional contribution to $U$ (for example from the gauge boson fluctuations) could lift this degeneracy and lead to a minimum of $U$ somewhat away from this line, such that $\frac{\partial U}{\partial g}$ does not vanish anymore.

Finally, let us compare the result of our SD calculation with the MFT result analogous to the computation in \cite{28}. However, we use here the corrected instanton vertex of \cite{27} (without the cutoff mechanism for the instanton interaction considered there) and no other interaction. We apply the formulae of sect. \ IV even though in our case the integral \cite{28} is not well defined. Adopting the same normalization for the MF as for the condensates $g$ in the SD calculation and taking the mean field as suggested by the brackets in \cite{35} without further Fierz transformation the “classical MF potential” is

$$V(\sigma_{MF}, \chi_{MF}) = -\zeta(\sigma^3_{MF} + \frac{1}{6}\sigma_{MF}^2\chi_{MF}^2) + \text{c.c.} \quad (47)$$

For the “one-loop” Potential $U_{MF}$ we recover the form \cite{14} but with masses

$$M^1_{1MF} = -\zeta(\sigma^2_{MF} + \frac{1}{3}\sqrt{2}\sigma_{MF}\chi_{MF} + \frac{1}{18}\chi_{MF}^2) \quad (48)$$

$$M^8_{8MF} = -\zeta(\sigma^2_{MF} - \frac{1}{12}\sqrt{6}\sigma_{MF}\chi_{MF} + \frac{1}{18}\chi_{MF}^2).$$

This quite different from our SD result Eqs. \cite{13}, \cite{15}. In particular the $\chi^3$-term in the classical potential is absent in the MFT calculation. Moreover, the sign between the $\sigma^3$-term and the $\sigma\chi^2$ is different, too. This demonstrates that the difference is more than an overall normalization of the potential or the fields. On the one side this highlights once more the importance of the Fierz ambiguity. On the other side it is not obvious if a suitable mean field formulation exists at all which would reproduce the results of the SDE. In view of all the problems of MFT we would like to argue that the SDE is clearly superior for our problem.

\section{VII. Summary and Conclusions}

Integrating the lowest order Schwinger-Dyson equation (SDE) for a multifermion-interaction we obtain the bosonic effective action at “one-vertex”-level. Within this approximation we find an “one-loop” expression for the SDE even in case of interactions involving more than four fermions. Although this gap equation is formally very similar to mean field theory, it does not suffer from the ambiguities of the latter arising from the bosonization procedure. We also propose a simple one loop formula for the free energy functional at the extrema which can be used in order to compare different local extrema.

We apply our method to a six-fermion interaction resembling the instanton induced quark vertex for three colors and flavors. We compute the solutions of the gap equation and the minimum of the free energy in dependence of the coupling strength $\zeta$. For small current quark masses $m_q$ the gap equation has several solutions, corresponding to different local extrema of the free energy. The free energy of the different extrema is compared by use of the one loop formula for the bosonic effective action. We find a first order transition to a phase with chiral symmetry breaking as the coupling of the six-quark vertex $\zeta$ increases beyond a critical value $\zeta_{SSB}$. Thereby the value of $\zeta_{SSB}$ is larger than the value $\zeta_S$, the minimal coupling for which solutions with a non-vanishing chiral order parameter exist. (Note that $\zeta_{SSB}$ equals $\zeta_S$ only in the case of the perhaps more familiar second order transition.) The critical line in the space of the current quark mass $m_q$ and $\zeta$ ends at a mass $m_q, \text{crit} \approx 0.076$ (in units of the UV cutoff near 1GeV). For $m_q > m_q, \text{crit}$ the phase diagram is characterized by a continuous crossover. We also investigate possible color octet condensates in the color-flavor-locked direction. In the approximation of a pointlike instanton induced six-quark interaction no phase with non-vanishing color octet condensate is visible. In this respect our work should be considered as a starting point for a more realistic instanton induced interaction, where the dependence of the instanton solution on the value of the octet condensate is taken into account.

We conclude that the bosonic effective action provides a practical tool for the understanding of fermionic systems where interactions involving more than four fermions play an important role. Beyond the gap equation it provides for a free energy. In contrast to mean field theory the lowest order is well defined and gives an unambiguous answer.

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