ON JET BUNDLES AND GENERALIZED VERMA MODULES

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ABSTRACT. Let $K$ be a field of characteristic zero and let $W \subseteq V$ be $K$-vector spaces of dimension $m$ and $m + n$. Let $P \subseteq \text{SL}(V) = G$ be the subgroup fixing $W$. It follows $X = G/P$ equals the grassmannian of $m$-planes in $V$. Let $\mathcal{O}_X(d)$ be the $d$’th tensor power of the line bundle $\mathcal{O}(1)$ coming from the Plücker embedding of $X$. There is an equivalence of categories between the category of finite dimensional $P$-modules and the category of $G$-linearized locally free finite rank $\mathcal{O}_X$-modules. The $l$’th jet bundle $\mathcal{P}^l_X(\mathcal{O}_X(d))$ is a $G$-linearized locally free sheaf and the aim of this paper is to describe its corresponding $P$-module using Taylor morphisms, higher direct images of $G$-linearized sheaves, filtrations of generalized Verma modules, canonical filtrations of irreducible $\text{SL}(V)$-modules and annihilator ideals of highest weight vectors and to apply this to the study of discriminants of linear systems on grassmannians. The main theorem of the paper is that the discriminant $D^l(\mathcal{O}_X(d))$ is irreducible for all $1 \leq l < d$.

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1. Introduction

Let $K$ be a field of characteristic zero and let $W \subseteq V$ be $K$-vector spaces of dimension $m$ and $m + n$. Let $P \subseteq \text{SL}(V) = G$ be the subgroup fixing $W$. It follows $X = G/P$ equals the grassmannian of $m$-planes in $V$. There is an equivalence of categories between the category of finite dimensional $P$-modules and the category of $G$-linearized locally free finite rank $\mathcal{O}_X$-modules. The $l$’th jet bundle $\mathcal{P}^l_X(\mathcal{O}_X(d))$ where $\mathcal{L} \in \text{Pic}^G(X)$ is a $G$-linearized locally free $\mathcal{O}_X$-module, and the aim of this paper is to describe its corresponding $P$-module using Taylor morphisms, higher direct images of $G$-linearized sheaves, filtrations of generalized Verma modules, canonical

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filtrations of irreducible $\text{SL}(V)$-modules and annihilator ideals of highest weight vectors.

Let $i: X \to \mathbb{P}(\wedge^m V^*)$ be the Plücker embedding and let $\mathcal{O}_X(d) = i^* \mathcal{O}(1)^\otimes d$ where $\mathcal{O}(1)$ is the tautological line bundle on $\mathbb{P}(\wedge^m V^*)$. In Theorem 5.1 we prove there is an isomorphism

$$\mathcal{P}_X^l(\mathcal{O}_X(d))(x)^* \cong U_l(g)v$$

of $P$-modules. Here $\mathcal{P}_X^l(\mathcal{O}_X(d))(x)^*$ is the dual of the fiber of the jet bundle at the distinguished point $x \in G/P$ and $U_l(g)v \subseteq H^0(X, \mathcal{O}_X(d))^*$ is the $l$'th piece of the canonical filtration of $H^0(X, \mathcal{O}_X(d))^*$. We then apply the results on jet bundles and Taylor maps to the study of discriminants of line bundles on Grassmannians. In Theorem 6.1 we prove the $l$'th discriminant $D^l(\mathcal{O}_X(d))$ is irreducible when $1 \leq l \leq d$.

The paper is organized as follows: In section two of the paper we study general properties of jet bundles of $G$-linearized locally free sheaves on homogeneous spaces $G/H$. Here $G$ is a linear algebraic group of finite type over a field $K$ and $H \subseteq G$ is a closed subgroup.

In section three of the paper we study the Taylor morphism $T^l$ for invertible sheaves on the Grassmannian. Let $\mathcal{O}_X(d)$ be the $d$'th tensor power of the tautological bundle coming from the Plücker embedding. We prove in Corollary 3.11 the Taylor morphism

$$T^l: H^0(X, \mathcal{O}_X(d)) \otimes \mathcal{O}_X \to \mathcal{P}_X^l(\mathcal{O}_X(d))$$

is a surjective map of locally free sheaves when $1 \leq l \leq d$.

In section four we study the canonical filtration $U_l(g)v$ of the irreducible $G$-module $H^0(X, \mathcal{O}_X(d))^*$. Using the universal enveloping algebra $U(g)$ and the annihilator ideal $\text{ann}(v) \subseteq U(g)$ where $v \in H^0(X, \mathcal{O}_X(d))^*$ is the highest weight vector we give in Corollary 4.14 a basis for $U_l(g)v$ as $K$-vector space. We also compute the dimension of $U_l(g)v$.

In section five we study the dual of the fiber of the jet bundle $\mathcal{P}_X^l(\mathcal{O}_X(d))(x)^*$ at the distinguished point $x \in X = G/P$ as $P$-module. Using the results obtained in the previous sections, we classify in Theorem 5.1 the fiber as $P$-module, and describe it in terms of $U_l(g)v$ - a subquotient of the generalized Verma module $U(g) \otimes L_v$ associated to the line $L_v$ spanned by the highest weight vector $v \in H^0(X, \mathcal{O}_X(d))^*$.

In section six we apply the results obtained in the previous sections and study discriminants of line bundle on the Grassmannian $X$. We prove the $l$'th discriminant $D^l(\mathcal{O}_X(d))$ is irreducible when $1 \leq l \leq d$.

The motivation for the study of the $P$-module $\mathcal{P}_X^l(\mathcal{O}_X(d))(x)^*$ is a relationship between the jet bundle and resolutions of ideal sheaves of discriminants of linear systems (see [10] Example 5.12). Using the jet bundle $\mathcal{P}_X^l(\mathcal{O}_X(d))$ one constructs a double complex

$$\mathcal{O}_X(-j) \otimes H^l(X, \wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*)$$

of sheaves on $\mathbb{P}(W^*)$ where $W = H^0(X, \mathcal{O}_X(d))$. The $l$'th discriminant $D^l(\mathcal{O}_X(d))$ of the line bundle $\mathcal{O}_X(d)$ is a closed subscheme

$$D^l(\mathcal{O}_X(d)) \subseteq \mathbb{P}(W^*)$$

and the double complex $\mathcal{O}_X(-j) \otimes H^l(X, \wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*)$ may in some cases be used to construct a resolution of the ideal sheaf of $D^l(\mathcal{O}_X(d))$. Knowledge on
the $P$-module structure of $\mathcal{P}^l_X(\mathcal{O}_X(d))(x)^*$ will give information on the problem of constructing such a resolution.

2. Jet bundles on quotients

In this section we study general properties of jet bundles on quotients $G/H$. Here $G$ is a linear algebraic group of finite type over a field and $H \subseteq G$ a closed subgroup.

**Notation:** Let $\pi : X \to S$ be a smooth and separated morphism. Let $Y = X \times_S X$ and let

$$\Delta : X \to Y$$

be the diagonal closed embedding. Let $p, q : Y \to X$ be the canonical projection maps and let $\mathcal{O}_{\Delta} = \mathcal{O}_Y/T^i_{\Delta}$ the structure sheaf of the $l$th infinitesimal neighborhood of the diagonal.

There is on $Y$ an exact sequence of $\mathcal{O}_Y$-modules

$$(2.0.1) \quad 0 \to T^i_{\Delta} \to \mathcal{O}_Y \to \mathcal{O}_{\Delta} \to 0.$$

Applying the functors $R^i p_* (- \otimes q^* \mathcal{E})$ and the formalism of derived functors to $(2.0.1)$ we get a long exact sequence of quasi coherent $\mathcal{O}_X$-modules

$$(2.0.2) \quad 0 \to p_*(T^i_{\Delta} \otimes q^* \mathcal{E}) \to p_*q^* \mathcal{E} \to p_*(\mathcal{O}_{\Delta} \otimes q^* \mathcal{E}) \to R^i p_*(T^i_{\Delta} \otimes q^* \mathcal{E}) \to \cdots .$$

By flat basechange it follows

$$R^i p_* q^* \mathcal{E} \cong \pi^* R^i p_* \mathcal{E}.$$

for all $i \geq 0$. It follows

$$p_*q^* \mathcal{E} \cong \pi^* \pi_* \mathcal{E} \otimes \mathcal{O}_X.$$

**Definition 2.1.** The quasi coherent $\mathcal{O}_X$-module

$$\mathcal{P}^l_{X/S}(\mathcal{E}) = p_*(\mathcal{O}_{\Delta} \otimes q^* \mathcal{E})$$

is the $l$th order jet bundle of $\mathcal{E}$. The morphism

$$T^l : \pi^* \pi_* \mathcal{E} \otimes \mathcal{O}_X \to \mathcal{P}^l_{X/S}(\mathcal{E})$$

is the $l$th Taylor morphism for $\mathcal{E}$.

Note: the $\mathcal{O}_X$-module $\mathcal{P}^l_{X/S}(\mathcal{E})$ has a left and right $\mathcal{O}_X$-structure. In this paper we will only consider the left $\mathcal{O}_X$-structure.

**Example 2.2.** Taylor maps on smooth projective schemes.

Let $S = \text{Spec}(K)$ and $X \subseteq \mathbb{P}^n_S$ be a smooth projective scheme of finite type over the field $K$. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. The Taylor morphism for this situation looks as follows:

$$T^l : \mathcal{H}^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{P}^l_{X/S}(\mathcal{E}).$$

Given a $K$-rational point $x \in X$ and a global section $s \in \mathcal{H}^0(X, \mathcal{E})$ it follows the Taylor map $T^l$ formally taylor expands $s$ in local coordinates at $x$:

$$T^l(x)(s) = (s(x), s'(x), \ldots, s^{(l)}(x)) \in \mathcal{P}^l_{X/S}(\mathcal{E})(x).$$

Assume $f : Y \to X$ is a smooth morphism of schemes over $S$. 
Proposition 2.3. Let $E$ be a finite rank locally free $O_X$-module. There is for all $l \geq 1$ a commutative diagram of exact sequences of locally free $O_Y$-modules

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Sym}^l(f^*\Omega^1_{X/S} \otimes f^*E) & \rightarrow & f^*P^l_{X/S}(E) & \rightarrow & f^*P^{l-1}_{X/S}(E) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Sym}^l(\Omega^1_{Y/S} \otimes f^*E) & \rightarrow & P^l_{X/S}(f^*E) & \rightarrow & P^{l-1}_{X/S}(f^*E) & \rightarrow & 0
\end{array}
$$

Proof. See [15], Proposition 2.3. 

Corollary 2.4. Assume $U \subseteq X$ is an open subscheme. It follows there is an isomorphism

$$
P^l_{X/S}(E)|_U \cong P^l_{U/S}(E|_U)
$$

of $O_U$-modules.

Proof. The inclusion $i : U \rightarrow X$ is a smooth morphism over $S$ hence the result follows from Proposition 2.3 and an induction. 

Example 2.5. Jet bundles on affine schemes.

Assume $S = \text{Spec}(A)$ and $U = \text{Spec}(B) \subseteq X$ is an open affine subscheme of $X$ and $E|_U$ the sheafification of a locally free $B$-module $E$. It follows $P^l_{X/S}(E)|_U$ is the sheafification of the $B$-module

$$
P^l_{B/A}(E) = B \otimes_A B/I^{l+1} \otimes_B E
$$

where $I \subseteq B \otimes_A B$ is the ideal of the diagonal.

Example 2.6. Jet bundles on quotients.

Let $V$ be a finite dimensional vector space over $K$. Let $H \subseteq G \subseteq \text{GL}(K, V)$ be closed subgroups. There is an action of $H$ on $G$ defined at $K$-rational points by

$$
\sigma : H(K) \times G(K) \rightarrow G(K)
$$

$$
\sigma(h, g) = gh^{-1}.
$$

The following holds: There is a quotient morphism

$$
\pi : G \rightarrow G/H
$$

(2.6.1)

and $G/H$ is a smooth quasiprojective scheme of finite type over $K$.

Let $\text{mod}^G(O_{G/H})$ denote the category of locally free $O_{G/H}$-modules of finite rank with a $G$-linearization and let $\text{mod}(H)$ denote the category of finite dimensional $H$-modules. There is an exact equivalence of categories

$$
F : \text{mod}^G(O_{G/H}) \cong \text{mod}(H)
$$

(2.6.2)

defined as follows: Assume $([\mathcal{E}, \theta]) \in \text{mod}^G(O_{G/H})$. It follows the $G$-linearization $\theta$ induces an $H$-module structure

$$
\rho(\theta) : H \rightarrow \text{GL}([\mathcal{E}, x])
$$

where $x \in G/H$ is the $K$-rational point defined by the identity $e \in G$. The functor $F$ is defined by $F([\mathcal{E}, \theta]) = (\rho(\theta), \mathcal{E}(x))$. For a proof of the claims 2.6.1 and 2.6.2 see [6].
In the following when we speak of a locally free sheaf \((\mathcal{E}, \theta) \in \text{mod}^G(\mathcal{O}_{G/H})\), and we take the fiber \(\mathcal{E}(x)\) we will use equivalence 2.6.2 when we refer to its \(H\)-module structure \(\rho(\theta)\).

Let \(X = G/H\) and let \(Y = X \times_K X\). Let \(p, q : Y \to X\) be the canonical projection morphisms. Let \((\mathcal{E}, \theta) \in \text{mod}^G(\mathcal{O}_{G/H})\). We get from Sequence 2.0.2 a long exact sequence of quasi coherent sheaves:

\[
\begin{align*}
0 & \to p_*(\mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{E}) \to H^0(G/H, \mathcal{E}) \otimes \mathcal{O}_{G/H} \to \mathcal{P}_G^{l}(\mathcal{E}) \to \\
R^1p_*(\mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{E}) & \to R^1p_*q^*\mathcal{E} \to 0
\end{align*}
\]

Here

\[
R^1p_*(\mathcal{O}_{\Delta}^{l+1} \otimes q^*\mathcal{E}) = 0
\]
since \(\mathcal{O}_{\Delta}^{l+1} \otimes q^*\mathcal{E}\) is supported on the diagonal.

There is a well defined left \(G\)-action on \(Y\) - the diagonal action - and since higher direct images preserve \(G\)-linearizations it follows the sequence \(2.6.3\) is an exact sequence of \(G\)-linearized sheaves with morphisms preserving the \(G\)-linearization. Since \(G/H\) is a homogeneous space for the \(G\)-action it follows the terms in the sequence \(2.6.3\) are locally free.

**Proposition 2.7.** Assume \((\mathcal{E}, \theta)\) is a \(G\)-linearized locally free \(\mathcal{O}_{G/H}\)-module of rank \(e\). The following holds for all \(l \geq 1\): There is an exact functor

\[
\begin{align*}
\mathcal{P}_G^{l} : \text{mod}^G(\mathcal{O}_{G/H}) & \to \text{mod}^G(\mathcal{O}_{G/H}) \\
\end{align*}
\]

There is an exact sequence of \(G\)-linearized locally free \(\mathcal{O}_{G/H}\)-modules

\[
\begin{align*}
0 & \to \text{Sym}^l(\Omega^1_{G/H}) \otimes \mathcal{E} \to \mathcal{P}_G^{l}(\mathcal{E}) \to \mathcal{P}_G^{l-1}(\mathcal{E}) \to 0.
\end{align*}
\]

The Taylor morphism

\[
\begin{align*}
T^{l} : H^0(G/H, \mathcal{E}) \otimes \mathcal{O}_{G/H} & \to \mathcal{P}_G^{l}(\mathcal{E})
\end{align*}
\]

preserves the \(G\)-linearization. Assume \(\dim(G/H) = n\). The following holds:

\[
\begin{align*}
\text{rk}(\mathcal{P}_G^{l}(\mathcal{E})) = e \left( \begin{array}{c} n + l \\ n \end{array} \right).
\end{align*}
\]

**Proof.** Assume \(\phi : (\mathcal{E}, \theta) \to (\mathcal{F}, \eta)\) is a morphism in \(\text{mod}^G(\mathcal{O}_{G/H})\). Let \(p, q : Y \to G/H\) be the canonical projection maps and let \(Y\) have the diagonal \(G\)-action. It follows \(q^*\) and \(p_*\) preserve the \(G\)-linearization. We get on \(Y\) a commutative diagram of short exact sequences of morphisms of \(G\)-linearized sheaves

\[
\begin{align*}
0 & \to \mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{E} \to \mathcal{O}_Y \otimes q^*\mathcal{E} \to \mathcal{O}_{\Delta}^{l} \otimes q^*\mathcal{E} \to 0 \\
0 & \to \mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{F} \to \mathcal{O}_Y \otimes q^*\mathcal{F} \to \mathcal{O}_{\Delta}^{l} \otimes q^*\mathcal{F} \to 0
\end{align*}
\]

Since \(p_*\) preserves the \(G\)-linearization it follows we get a morphism

\[
\mathcal{P}^l(\phi) : \mathcal{P}_G^{l}(\mathcal{E}) \to \mathcal{P}_G^{l}(\mathcal{F})
\]

of sheaves preserving the \(G\)-linearization. One checks for two composable morphisms

\[
\phi, \psi \in \text{mod}^G(\mathcal{O}_{G/H})
\]

it follows

\[
\mathcal{P}^l(\phi \circ \psi) = \mathcal{P}^l(\phi) \circ \mathcal{P}^l(\psi)
\]
hence the existence of the functor in claim 2.7.1 is clear. Since \( P_{G/H}^l(\mathcal{E}) \cong P_{G/H}^l \otimes \mathcal{E} \) and \( P_{G/H}^l \) is locally free it follows the functor is exact. It follows claim 2.7.1 is proved.

For a proof of the existence of the sequence 2.7.2 see [15], Proposition 2.2. By the argument above it follows \( P_{G/H}^l(\mathcal{E}) \) has a canonical \( G \)-linearization since \( p_* \) preserves the \( G \)-linearization. There is on \( Y \) a commutative diagram of exact sequences of \( O_Y \)-modules with a \( G \)-linearization

\[
\begin{array}{c}
0 \rightarrow I_l + 1 \rightarrow O_Y \rightarrow O_{\Delta^l} \rightarrow 0 \\
0 \rightarrow I_{\Delta} \rightarrow O_Y \rightarrow O_{\Delta^{l-1}} \rightarrow 0
\end{array}
\]

When we apply the functor \( p_* (\ - \otimes q^* \mathcal{E}) \) we get a commutative diagram of maps of \( G \)-linearized sheaves

\[
\begin{array}{c}
H^0(G/H, \mathcal{E}) \otimes O_{G/H} \rightarrow \mathcal{P}_{G/H}^l(\mathcal{E}) \\
H^0(G/H, \mathcal{E}) \otimes O_{G/H} \rightarrow \mathcal{P}_{G/H}^{l-1}(\mathcal{E})
\end{array}
\]

hence the natural morphism \( \mathcal{P}_{G/H}^l(\mathcal{E}) \rightarrow \mathcal{P}_{G/H}^{l-1}(\mathcal{E}) \) is a morphism preserving the \( G \)-linearization. It also follows the Taylor morphism preserves the \( G \)-linearization. It follows the sequence 2.7.2 is an exact sequence of \( G \)-linearized sheaves. Claim 2.7.3 also follows from this argument. We have proved claim 2.7.2 and 2.7.3.

We prove 2.7.4: Assume \( l = 1 \). We get an exact sequence

\[
0 \rightarrow \Omega^1_{G/H} \otimes \mathcal{E} \rightarrow \mathcal{P}_{G/H}^1(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow 0
\]

of locally free \( O_{G/H} \)-modules. We know \( rk(\Omega^1_{G/H}) = n \) and \( rk(\mathcal{E}) = e \) hence

\[
rk(\mathcal{P}_{G/H}^1(\mathcal{E})) = ne + e = e\left(n + 1\right).
\]

Assume \( rk(\mathcal{P}_{G/H}^{l-1}(\mathcal{E})) = e\left(\frac{n + l - 1}{n}\right) \). There is an exact sequence

\[
0 \rightarrow \text{Sym}^l(\Omega^1_{G/H}) \otimes \mathcal{E} \rightarrow \mathcal{P}_{G/H}^l(\mathcal{E}) \rightarrow \mathcal{P}_{G/H}^{l-1}(\mathcal{E}) \rightarrow 0
\]

hence

\[
rk(\mathcal{P}_{G/H}^l(\mathcal{E})) = e\left(\frac{n + l - 1}{n} \right) + e\left(\frac{n + l - 1}{n} \right) = e\left(\frac{n + l}{n} \right)
\]

and the Proposition is proved. \( \square \)

Assume \( (\mathcal{E}, \theta) \in \text{mod}^G(O_{G/H}) \) and consider \( \mathcal{P}_{G/H}^l(\mathcal{E}) \). It follows from Proposition 2.7 claim 2.7.2 there is a canonical \( G \)-linearization on \( \mathcal{P}_{G/H}^l(\mathcal{E}) \). Hence we may via equivalence 2.6.2 consider its corresponding \( H \)-module \( \mathcal{P}_{G/H}^l(\mathcal{E})(x) \). When we speak of the \( H \)-module \( \mathcal{P}_{G/H}^l(\mathcal{E})(x) \) we will always refer to this structure.
Proposition 2.8. Let $\mathfrak{m} \subseteq O_{G/H}$ be the ideal sheaf of the point $x \in G/H$. There are for all $i \geq 0$ isomorphisms of $H$-modules

\begin{align}
R^i p_* (\mathcal{I}_{\Delta} \otimes q^* \mathcal{E}) (x) & \cong H^i (G/H, \mathfrak{m}^{l+1} \mathcal{E}) \\
R^i p_* q^* \mathcal{E} (x) & \cong H^i (G/H, \mathcal{E}).
\end{align}

Proof. In the following proof when we consider a locally free sheaf $(\mathcal{E}, \theta) \in \text{mod}^G (O_{G/H})$ we will use equivalence 2.6.2 to induce an $H$-module structure on $\mathcal{E}(x)$.

Let $p : Y = G/H \times G/H \to G/H$ be defined by $p(x, y) = x$. It follows $p^{-1}(x) \cong G/H$ and we get a fiber diagram

\[
\begin{array}{ccc}
G/H & \xrightarrow{i} & G/H \\
\downarrow{\pi} & & \downarrow{p} \\
\text{Spec}(\kappa(x)) & \xrightarrow{i} & G/H
\end{array}
\]

where $i(y) = (x, y)$. There is on $Y$ an exact sequence

\[
0 \to T_{\Delta}^{l+1} \to O_Y \to O_{\Delta'} \to 0
\]

of $G$-linearized sheaves. Here $Y$ is equipped with the diagonal action. Let $q : Y \to G/H$ be defined by $q(x, y) = y$. By [5], chapter III, section 12 and equivalence 2.6.2 we get natural maps

\[
\phi^i : R^i p_* (T_{\Delta}^{l+1} \otimes q^* \mathcal{E})(x) \to R^i \pi_* (j^* (T_{\Delta}^{l+1} \otimes q^* \mathcal{E})) = H^i (G/H, \mathfrak{m}^{l+1} \mathcal{E})
\]

of $H$-modules. Let for any $O_Y$-module $\mathcal{F}$

\[
h^i(y, \mathcal{F}) = \dim_{\kappa(y)} H^i (p^{-1}(y), \mathcal{F}_y),
\]

where $\mathcal{F}_y$ is the restriction of $\mathcal{F}$ to $p^{-1}(y)$. It follows

\[
h^i(y, T_{\Delta}^{l+1} \otimes q^* \mathcal{E}) = \dim_{\kappa(y)} H^i (p^{-1}(y), T_{\Delta}^{l+1} \otimes q^* \mathcal{E})
\]

is a constant function in $y$ for $i \geq 0$. Hence from [5], chapter III, Corollary 12.9 it follows $\phi^i$ is an isomorphism of $H$-modules for all $i$. We have proved 2.8.1

Isomorphism 2.8.2 follows by a similar argument and the Proposition is proved.

We get from Proposition 2.8 the exact sequence 2.6.3 and the equivalence 2.6.2 a long exact sequence of $H$-modules

\[
0 \to H^0 (G/H, \mathfrak{m}^{l+1} \mathcal{E}) \to H^0 (G/H, \mathcal{E}) \to \mathcal{P}^l_{G/H}(\mathcal{E})(x) \to
\]

\[
H^1 (G/H, \mathfrak{m}^{l+1} \mathcal{E}) \to H^1 (G/H, \mathcal{E}) \to 0
\]

Example 2.9. Jet bundles on projective space.

Assume in the following $V$ is a $K$-vector space of dimension $n + 1$ and the characteristic of $K$ is zero. Let $L \subseteq V$ be a line and let $P \subseteq G = \text{SL}(V)$ be the parabolic subgroup fixing $L$. It follows $G/P \cong \mathbb{P}(V^*) = \mathbb{P}$ - the projective space of lines in $V$.

Let $O_P (1)$ be the tautological line bundle on $\mathbb{P}$ and let $O_P (d) = O_P (1)^{\otimes d}$. It follows $O_P (d)$ has a canonical $G$-linearization and $\text{Pic}^G (\mathbb{P}) \cong \mathbb{Z}$. It follows the jet bundle $\mathcal{P}^l_{\mathbb{P}}(O_P (d))(x)$ has a canonical $G$-linearization for all integers $d$. We may via equivalence 2.6.2 consider its $P$-module $\mathcal{P}^l_{\mathbb{P}}(O_P (d))(x)$. 

Theorem 2.10. There is for all $1 \leq l < d$ an isomorphism

\[ \mathcal{P}_l(O_p(d))(x) \cong \text{Sym}^l(V^*) \otimes \text{Sym}^{d-l}(L^*) \]

of $P$-modules.

Proof. The result is proved in [9], Theorem 2.4. \qed

Let $\pi : \mathbb{P} \rightarrow Y = \text{Spec}(K)$ be the structure morphism and let $\pi^* \text{Sym}^l(V^*)$ be the pull back of the $G$-linearized $O_Y$-module $\text{Sym}^l(V^*)$.

Corollary 2.11. There is for all $1 \leq l < d$ an isomorphism

\[ \mathcal{P}_l(O_p(d)) \cong O_p(d-l) \otimes \pi^* \text{Sym}^l(V^*) \]

of $O_p$-modules with a $G$-linearization.

Proof. Using equivalence 2.6.2 it follows the $P$-module corresponding to $O_p(d-l)$ is $\text{Sym}^{d-l}(L^*)$. It follows from Theorem 2.10 $\mathcal{P}_l(O_p(d))$ and $O_p(d-l) \otimes \pi^* \text{Sym}^l(V^*)$ have isomorphic $P$-modules. The Corollary follows since 2.6.2 is an equivalence of categories. \qed

We get for all $1 \leq l < d$ get the following formula

\[ \mathcal{P}_l(O_p(d)) \cong \bigoplus^{\binom{n+l}{n}} O_p(d-l) \]

expressing the splitting type of the jet bundle as abstract locally free $O_p$-module. This follows from Corollary 2.11 since the $O_p$-module $\pi^* \text{Sym}^l(V^*)$ corresponds to the trivial rank $\binom{n+l}{n}$ $O_p$-module. The formula 2.11.1 is well known (see [9], [12], [15], [17], [18], [19] and [20] for other proofs.)

3. On surjectivity of the Taylor morphism

In this section we study the Taylor morphism for a class of invertible sheaves on the grassmannian. We prove the Taylor morphism is surjective in most cases.

The strategy of the proof is as follows: First prove the Taylor morphism is surjective for most invertible sheaves on projective space. Embed the grassmannian into projective space using the Plücker embedding. Any invertible sheaf on the grassmannian is the pull back of an invertible sheaf on projective space via the Plücker embedding. Using the fact the grassmannian is projectively normal in the Plücker embedding and general properties of jet bundles we prove the Taylor morphism is surjective for most invertible sheaves on the grassmannian.

Notation: Let $K$ be any field and let $W \subseteq V$ be $K$ vector spaces of dimensions $m$ and $m+n$. Let $G = \text{SL}(V)$ and let $P \subseteq G$ be the subgroup fixing $W$. Let $X = G/P = G(m, m+n)$. Let

\[ i : X \rightarrow \mathbb{P}(\wedge^m V^*) \]

be the Plücker embedding and let $O_X(d) = i^*O(1)^{\otimes d}$. The grassmannian $G(m, m+n)$ has dimension $mn$. There is a bijection

\[ \{K\text{-rational points } x \in G/P \} \cong \{m\text{-planes } W_x \subseteq V \} \]

of sets. Let $\text{mod}(P)$ be the category of finite dimensional $P$-modules and morphisms and let $\text{mod}^G(O_{G/P})$ be the category of $G$-linearized locally free $O_{G/P}$-modules of
finite rank and morphisms. Recall from the previous section there is an exact equivalence of categories

\[(3.0.3) \quad \text{mod}(P) \cong \text{mod}^G(\mathcal{O}_{G/P}). \]

Let \( Y = X \times_K X \) and let \( p, q : Y \to X \) be the canonical projection morphisms. Let \( \Delta(X) \subseteq Y \) be the diagonal embedding of \( X \) and let \( \mathcal{I}_{\Delta} \subseteq \mathcal{O}_Y \) be the ideal of \( \Delta(X) \). By Definition 2.1 it follows

\[\mathcal{P}^l_X(\mathcal{O}_X(d)) = p_*(\mathcal{O}_Y/\mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{O}_X(d))\]

is the \( l \)'th sheaf of jets of \( \mathcal{O}_X(d) \). It follows from Proposition 2.7 claim 2.7.3

\[rk(\mathcal{P}^l_X(\mathcal{O}_X(d))) = \binom{mn + l}{mn}.\]

When it is clear from the context we will write \( \mathcal{P}^l(\mathcal{O}(d)) \) instead of \( \mathcal{P}^l_X(\mathcal{O}_X(d)) \).

The invertible sheaf \( \mathcal{O}_X(d) \) has a unique \( G \)-linearization for all \( d \in \mathbb{Z} \). Let the product \( Y = G/P \times_K G/P \) have the diagonal \( G \)-action.

Recall there is an exact sequence of \( G \)-linearized sheaves of \( \mathcal{O}_Y \)-modules

\[0 \to \mathcal{I}_{\Delta}^{l+1} \to \mathcal{O}_Y \to \mathcal{O}_{\Delta'} \to 0\]

on \( Y \). The functor \( p_*(\mathcal{O}_X(d)) \) is left exact and preserves the \( G \)-linearization hence we get when we use the formalism of derived functors a long exact sequence of locally free \( G \)-linearized sheaves

\[(3.0.4) \quad 0 \to p_*(\mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{O}_X(d)) \to p_*q^*\mathcal{O}_X(d) \to \mathcal{P}^l_X(\mathcal{O}_X(d)) \to \]

\[R^1 p_*(\mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{O}_X(d)) \to R^1 p_*q^*\mathcal{O}_X(d) \to 0.\]

Here \( R^1 p_*(\mathcal{O}_{\Delta'} \otimes q^*\mathcal{O}_X(d)) = 0 \) since \( \mathcal{O}_{\Delta'} \otimes q^*\mathcal{O}_X(d) \) is a sheaf supported on the diagonal.

Let \( x \in G/P \) be the \( K \)-rational point defined by the class of the identity element \( e \in G \). When we take the fiber at \( x \) of the sequence 3.0.3 and apply Proposition 2.8 we get the following exact sequence of finite dimensional \( P \)-modules

\[(3.0.5) \quad 0 \to H^0(X, m^{l+1}\mathcal{O}_X(d)) \to H^0(X, \mathcal{O}_X(d)) \to \mathcal{P}^l_X(\mathcal{O}_X(d))(x) \to \]

\[H^1(X, m^{l+1}\mathcal{O}_X(d)) \to H^1(X, \mathcal{O}_X(d)) \to \cdots\]

By Kodaira’s Vanishing Theorem it follows \( H^1(X, \mathcal{O}_X(d)) = 0 \) when \( d \geq 1 \). It follows we get an exact sequence

\[(3.0.6) \quad 0 \to H^0(X, m^{l+1}\mathcal{O}_X(d)) \to H^0(X, \mathcal{O}_X(d)) \to \mathcal{P}^l_X(\mathcal{O}_X(d))(x) \to \]

\[H^1(X, m^{l+1}\mathcal{O}_X(d)) \to 0\]

of finite dimensional \( P \)-modules. Since \( 2.8 \) is an equivalence of categories, we get an exact sequence of locally free \( G \)-linearized sheaves

\[(3.0.7) \quad 0 \to p_*(\mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{O}_X(d)) \to H^0(X, \mathcal{O}_X(d)) \otimes \mathcal{O}_X \to \mathcal{P}^l_X(\mathcal{O}_X(d)) \to \]

\[R^1 p_*(\mathcal{I}_{\Delta}^{l+1} \otimes q^*\mathcal{O}_X(d)) \to 0.\]

**Example 3.1.** Taylor maps and invertible sheaves on projective space.

Let \( E \) be an \( n+1 \)-dimensional \( K \)-vector space and let \( \mathbb{P} = \mathbb{P}(E^*) \) be the projective space of lines in \( E \). Let \( \mathcal{O}_{\mathbb{P}}(1) \) be the tautological line bundle on \( \mathbb{P} \) and let \( \mathcal{O}_{\mathbb{P}}(d) = \mathcal{O}_{\mathbb{P}}(1)^{\otimes d} \).
Lemma 3.2. The Taylor morphism
\[ T^l : H^0(P, \mathcal{O}_P(d)) \otimes \mathcal{O}_P \to \mathcal{P}^l_\mathcal{P}(\mathcal{O}_P(d)) \]
is surjective for all \( 1 \leq l \leq d \).

Proof. Let \( E = K\{e_0, \ldots, e_n\} \) and \( E^* = K\{x_0, \ldots, x_n\} \). It follows
\[ \mathbb{P}(E^*) = \text{Proj}(\text{Sym}_K(E^*)) = \text{Proj}(K[x_0, \ldots, x_n]) \]Let \( U_0 = D(x_0) = \text{Spec}(K[t_1, \ldots, t_n]) \) where \( t_i = x_i / x_0 \). There is an isomorphism
\[ \mathbb{P}(E^*) \cong \text{SL}(E) / P \]
where \( P \) is the subgroup of elements fixing a line in \( E \). Because the Taylor morphism is a map of \( \text{SL}(E) \)-linearized sheaves we may check surjectivity by restricting to the fiber of \( T^l \) at \( x \). We restrict \( T^l \) to the open set \( U_0 \):
\[ T^l|_{U_0} : K[t_1, \ldots, t_n] \otimes H^0(P, \mathcal{O}_P(d)) \to \mathcal{P}^l_{U_0}(\mathcal{O}_P(d)|_{U_0}). \]
We get a map
\[ T^l|_{U_0} : K[t_i] \otimes H^0(P, \mathcal{O}_P(d)) \to K[t_i] \otimes \{ dt_1^{d_1} \cdots dt_n^{d_n} \otimes x_0^d : 0 \leq d_i \leq n \} \]
of left \( K[t_i] \)-modules. Assume
\[ s = x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n} \]
with \( \sum d_i = d \) is a global section of \( \mathcal{O}_P(d) \). It follows \( d_0 = d - d_1 - \cdots - d_n \). On \( U_0 \) we may write
\[ s = x_0^{d_0-d_1-\cdots-d_n} x_1^{d_1} \cdots x_n^{d_n} = t_1^{d_1} \cdots t_n^{d_n} x_0^d. \]
By definition
\[ T^l(s) = 1 \otimes t_1^{d_1} \cdots t_n^{d_n} x_0^d = (t_1 + dt_1)^{d_1} \cdots (t_n + dt_n)^{d_n} \otimes x_0^d. \]
The point \( x \) is defined by \( t_1 = \cdots = t_n = 0 \) hence when we restrict \( T^l \) to the fiber at \( x \) we get the map
\[ T^l(x) : H^0(P, \mathcal{O}_P(d)) \to \mathcal{P}^l_\mathcal{P}(\mathcal{O}_P(d))(x) \]
defined by
\[ T^l(x)(s) = dt_1^{d_1} \cdots dt_n^{d_n} \otimes x_0^d. \]
Assume \( \omega = dt_1^{d_1} \cdots dt_n^{d_n} \otimes x_0^d \in \mathcal{P}^l_\mathcal{P}(\mathcal{O}_P(d))(x) \) with \( 0 \leq \sum d_i \leq n \). It follows \( d - \sum d_i \geq d - n \geq 0 \). Let \( d_0 = d - \sum d_i \). It follows \( d_0 \geq 0 \) and \( d_0 + \sum d_i = d \). It follows \( s = x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n} \in H^0(P, \mathcal{O}_P(d)) \) and
\[ T^l(x)(s) = \omega \]
and the Proposition is proved. \( \square \)

Recall we get on projective space \( P \) an exact sequence of \( \text{SL}(E) \)-linearized locally free sheaves
\[ (3.2.1) \quad 0 \to p_* (\mathcal{I}^{l+1}_\Delta \otimes q^* \mathcal{O}_P(d)) \to H^0(P, \mathcal{O}_P(d)) \otimes \mathcal{O}_P \to T^l \mathcal{P}^l_\mathcal{P}(\mathcal{O}_P(d)) \to R^1 p_* (\mathcal{I}^{l+1}_\Delta \otimes q^* \mathcal{O}_P(d)) \to R^1 p_* (q^* \mathcal{O}_P(d)) \to 0 \]
when \( 1 \leq l \leq d \).
Corollary 3.3. There is an equality
\[ R^1 p_*(T^l_\Delta \otimes q^*\mathcal{O}_p(d)) = 0 \]
when \( 1 \leq l \leq d \).

Proof. Sequence 3.2. remain exact when we take the fiber at \( x \in G/P \). Via Kodaira’s Vanishing Theorem and Proposition 2.8, claim 2.8.2 the final term becomes
\[ R^1 p_*(q^*\mathcal{O}_p(d))(x) = H^1(P, \mathcal{O}_P(d)) = 0 \]
when \( d \geq 1 \). It follows \( R^1 p_* q^*\mathcal{O}_p(d) = 0 \). Since \( T^l \) is surjective when \( 1 \leq l \leq d \) the Corollary follows. \( \square \)

We get on \( \mathbb{P} = \mathbb{P}(E^*) \) an exact sequence of \( \text{SL}(E) \)-linearized locally free sheaves
\[ 0 \to p_*(T^l_\Delta \otimes q^*\mathcal{O}_p(d)) \to H^0(P, \mathcal{O}_P(d)) \otimes \mathcal{O}_p \to T^l (\mathcal{O}_p(d)) \to 0 \]
(3.3.1) when \( 1 \leq l \leq d \).

Example 3.4. Surjectivity of Taylor maps for projectively normal schemes.

Lemma 3.5. Let \( i: Z \to W \) be a closed immersion of schemes and let \( \mathcal{E} \) be a locally free \( \mathcal{O}_W \)-module. There is a canonical surjection
\[ \phi: i^*\mathcal{P}_W^l(\mathcal{E}) \to \mathcal{P}_Z^l(i^*\mathcal{E}) \]
of \( \mathcal{O}_Z \)-modules.

Proof. Assume \( Z = \text{Spec}(A/\mathfrak{a}), W = \text{Spec}(A) \) and \( \mathcal{E} = \mathcal{E} \) where \( E \) is a locally free \( A \)-module. Let \( \mathcal{P}_W^l(\mathcal{E}) \) be the sheafification of \( A \otimes A/J^{l+1} \otimes E \) where \( J \subseteq A \otimes A \) is the ideal of the diagonal. Moreover \( \mathcal{P}_W^l(i^*\mathcal{E}) \) the sheafification of \( (A/\mathfrak{a}) \otimes (A/\mathfrak{a})/J^{l+1} \otimes (E/IE) \) where \( J \subseteq A/\mathfrak{a} \times A/\mathfrak{a} \) is the ideal of the diagonal. There is an isomorphism between \( i^*\mathcal{P}_W^l(\mathcal{E}) \) and the sheafification of
\[ A \otimes A/J^{l+1} \otimes (E/\mathfrak{a}E). \]

In this case the map \( \phi \) is the sheafification of the canonical map
\[ f: A \otimes A/J^{l+1} \otimes (E/\mathfrak{a}E) \to (A/\mathfrak{a}) \otimes (A/\mathfrak{a})/J^{l+1} \otimes (E/\mathfrak{a}E) \]
defined by
\[ f(x \otimes y \otimes \overline{\tau}) = \overline{x} \otimes \overline{y} \otimes \overline{\tau}. \]
It follows \( \phi \) is a surjective map of sheaves. This construction glue to give a morphism for any closed immersion and the Lemma is proved. \( \square \)

Assume \( i: Z \to \mathbb{P}^N_K \) is an embedding of a projective scheme \( Z \) and assume \( Z \) is projectively normal in \( i \). Assume \( \mathcal{F} \) is a coherent \( \mathcal{O}_p \)-module with
\[ T^l: H^0(P, \mathcal{F}) \otimes \mathcal{O}_p \to \mathcal{P}_p^l(\mathcal{F}) \]
surjective and let \( \mathcal{E} = i^*\mathcal{F} \).

Theorem 3.6. The Taylor morphism
\[ T^l: H^0(Z, \mathcal{E}) \otimes \mathcal{O}_Z \to \mathcal{P}_Z^l(\mathcal{E}) \]
is surjective.
Proof. Pull \( T^l \) back to \( Z \) via \( i \) to get a surjective morphism of sheaves

\[
i^*(T^l) : H^0(\mathbb{P}, \mathcal{F}) \otimes O_Z \to i^*\mathcal{P}_Z^l(\mathcal{F}).
\]

We get a commutative diagram of maps of sheaves

\[
\begin{array}{ccc}
H^0(\mathbb{P}, \mathcal{F}) \otimes O_Z & \xrightarrow{i^*(T^l)} & i^*\mathcal{P}_Z^l(\mathcal{F}) \\
\downarrow u & & \downarrow \phi \\
H^0(Z, \mathcal{E}) \otimes O_Z & \xrightarrow{T^l} & \mathcal{P}_Z^l(\mathcal{E})
\end{array}
\]

The map \( u \) is surjective since the \( Z \) is projectively normal in the embedding \( i \) and \( \phi \) is surjective by Lemma 3.11 and the Theorem is proved. \( \square \)

Example 3.7. Higher cohomology groups of coherent sheaves on \( G/P \).

Assume \( Z = G/P \) where \( P \subseteq G \) is a parabolic subgroup of a semi simple algebraic group. Assume \( i : G/P \to \mathbb{P}_K^N \) is an embedding where \( G/P \) is projectively normal in \( i \). Let \( \mathcal{O}_{G/P}(d) = i^*\mathcal{O}(d) \).

Proposition 3.8. Assume \( 1 \leq l < d \). If \( R^1 p_* q^* \mathcal{O}_Z(d) = 0 \) it follows

\[
R^1 p_* (T^{l+1} \otimes q^* \mathcal{O}_Z(d)) = 0.
\]

Proof. Let \( \mathcal{I} \subseteq \mathcal{O}_{G/P} \times G/P \) be the ideal sheaf of the diagonal. We get an exact sequence of sheaves on \( G/P \):

\[
0 \to p_* (T^{l+1} \otimes q^* \mathcal{O}_{G/P}(d)) \to p_* q^* \mathcal{O}_{G/P}(d) \to T^l \mathcal{P}^l(\mathcal{O}_{G/P}(d)) \to \phi
\]

\[
R^1 p_* (T^{l+1} \otimes q^* \mathcal{O}_{G/P}(d)) \to \psi \to R^1 p_* q^* \mathcal{O}_{G/P}(d) \to \cdots
\]

Assume \( R^1 p_* q^* \mathcal{O}_Z(d) = 0 \). Since \( G/P \) is projectively normal in \( i \) it follows from Theorem 3.6 the Taylor morphism \( T^l \) is surjective. It follows

\[
Im(T^l) = \mathcal{P}^l(\mathcal{O}_Z(d)) = Ker(\phi).
\]

It follows \( \phi(\mathcal{P}^l(\mathcal{O}_Z(d))) = 0 \) hence \( Im(\phi) = 0 = Ker(\psi) \). We get an injection

\[
R^1 p_* (T^{l+1} \otimes q^* \mathcal{O}_Z(d)) \to R^1 p_* q^* \mathcal{O}_Z(d) = 0
\]

and the Proposition is proved. \( \square \)

Assume the hypothesis from Proposition 3.8 is satisfied.

Corollary 3.9. There is for \( 1 \leq l < d \) an equality

\[
H^1(G/P, m^{l+1} \mathcal{O}_Z(d)) = 0.
\]

Proof. By Proposition 3.8 it follows

\[
R^1 p_* (T^{l+1} \otimes q^* \mathcal{O}_Z(d)) = 0.
\]

Take the fiber at the distinguished point \( x \in G/P \) to get the equality

\[
R^1 p_* (T^{l+1} \otimes q^* \mathcal{O}_Z(d))(x) = H^1(G/P, m^{l+1} \mathcal{O}_Z(d)) = 0
\]

and the Corollary is proved. \( \square \)
Note: Corollary 3.9 is a special case of Nadel’s Vanishing Theorem on the vanishing of higher cohomology groups of coherent sheaves on flag varieties. Proposition 3.8 shows there is an equation
\[ R^1 p_*(T^{l+1} \otimes q^*O_Z(d)) = 0. \]
The coherent sheaf \( R^1 p_*(T^{l+1} \otimes q^*O_Z(d)) \) is a locally free \( O_{G/P} \)-module and its fiber at \( x \) is the cohomology group
\[ H^1(G/P, m^{l+1}O_Z(d)). \]
Hence Corollary 3.9 gives a geometric proof of the vanishing of \( H^1 \) for a class of coherent sheaves on flag varieties over an arbitrary field \( K \) of characteristic zero.

Example 3.10. Taylor maps for line bundles on the grassmannian.

Let \( X = SL(V)/P = G(m,m+n) \) be the grassmannian of \( m \)-planes in \( V \) and let \( O_X(d) \) be the line bundle coming from the Plücker embedding.

Corollary 3.11. The Taylor morphism
\[ T^l : H^0(X, O_X(d)) \otimes O_X \rightarrow P^l_X(O_X(d)) \]
is surjective when \( 1 \leq l \leq d \).

Proof. The Corollary follows from Theorem 3.6 since the grassmannian is projectively normal in the Plücker embedding. \( \square \)

On \( X = G(m,m+n) \) we get an exact sequence of \( SL(V) \)-linearized sheaves
\[ 0 \rightarrow p_*(T^{l+1}_X \otimes q^*O_X(d)) \rightarrow H^0(X, O_X(d)) \otimes O_X \rightarrow T^l \rightarrow P^l_X(O_X(d)) \rightarrow 0 \]
when \( 1 \leq l \leq d \).

Corollary 3.12. On \( X = G(m,m+n) \) there is an equality
\[ R^1 p_*(T^{l+1}_X \otimes q^*O_X(d)) = 0 \]
when \( 1 \leq l \leq d \).

Proof. The sequence remain by the equivalence exact when we take the fiber at \( x \). We get by Proposition 2.8 claim
\[ R^1 p_*(q^*O_X(d))(x) = H^1(X, O_X(d)) = 0 \]
when \( d \geq 1 \) by Kodaira’s Vanishing Theorem. It follows \( R^1 p_*(q^*O_X(d)) = 0 \). By Theorem 3.11 the Taylor map \( T^l \) is surjective and the Corollary follows since 3.11.1 is an exact sequence. \( \square \)

We get an exact sequence of \( SL(V) \)-linearized sheaves
\[ 0 \rightarrow p_*(T^{l+1}_X \otimes q^*O_X(d)) \rightarrow H^0(X, O_X(d)) \otimes O_X \rightarrow T^l \rightarrow P^l_X(O_X(d)) \rightarrow 0 \]
when \( 1 \leq l \leq d \).

Corollary 3.13. On \( X = G(m,m+n) \) there is an exact sequence of \( P \)-modules
\[ 0 \rightarrow H^0(X, m^{l+1}O_X(d)) \rightarrow H^0(X, O_X(d)) \rightarrow P^l_X(O_X(d))(x) \rightarrow 0 \]
for all \( 1 \leq l \leq d \).
Proof: If we take the fiber of sequence $\delta$ we get via equivalence 2.6.2 and Proposition 2.8 an exact sequence
\[
0 \to H^0(X, m^{l+1} \mathcal{O}_X(d)) \to H^0(X, \mathcal{O}_X(d)) \to \mathcal{P}_X^l(\mathcal{O}_X(d))(x) \to 0
\]
of $P$-modules and the Corollary follows. □

4. On generalized Verma modules and canonical filtrations

In this section we study the canonical filtration $U_l(g)$ associated to the irreducible $\text{SL}(V)$-module of global sections of a line bundle $L \in \text{Pic}^G(G/P)$ on the grassmannian $G/P = G(m, m + n)$ of $m$-planes in an $m + n$-dimensional vector space. We construct a basis and compute the dimension of each term in the canonical filtration.

The strategy of the proof is as follows: There is for all $l \geq 1$ an exact sequence
\[
0 \to ann_l(v) \otimes L_v \to U_l(g) \otimes L_v \to U_l(g)v \to 0
\]
where $ann_l(v)$ is the $l$'th piece of the canonical filtration of the annihilator ideal of $v$ - the highest weight vector of $H^0(G/P, \mathcal{O}_{G/P}(d))^*$. Here $U_l(g)$ is the $l$'th piece of the canonical filtration of the universal enveloping algebra of $g = \text{Lie}(G)$. Using the theory of highest weights and the Poincare-Birkhoff-Witt Theorem we prove there is a vector space decomposition

\[
U_l(g) = U_l(n) \oplus ann_l(v)
\]
in the case when $1 \leq l < d$. Here $n \subseteq g$ is a sub Lie algebra. We then use the decomposition (4.0.1) to give a basis for $U_l(g)v$ and to calculate its dimension as vector space.

**Notation:** Let $W \subseteq V$ be vector spaces of dimension $m$ and $m + n$. Let $e_1, \ldots, e_{m+n}$ be a basis for $V$ and $e_1, \ldots, e_m$ a basis for $W$. Let $G = \text{SL}(V)$ and $P \subseteq G$ the parabolic subgroup fixing $W$. It follows there is a quotient morphism
\[
\pi : G \to G/P
\]
and there is a canonical isomorphism
\[
G/P \cong G(m, m + n)
\]
where $G(m, m + n)$ is the grassmannian of $m$-planes in $V$. Let
\[
i : G/P \to \mathbb{P}(^m V^*)
\]
be the Plücker embedding and let $\mathcal{O}_X(d) = i^* \mathcal{O}(1)^{\otimes d}$. Let $g = \text{Lie}(G)$ and $p = \text{Lie}(P)$. Let $U(g)$ be the universal enveloping algebra of $g$ and let $U_l(g)$ be its canonical filtration. Let
\[
V_\lambda = H^0(G/P, \mathcal{O}_{G/P}(d))^*
\]
be the irreducible $\text{SL}(V)$-module of dual global sections of $\mathcal{O}_{G/P}(d)$ and let $v \in V_\lambda$ be the (unique up to scalars) highest weight vector of $V_\lambda$.

In the following we use the notation from [4]. Let $g = g_- \oplus h \oplus g_+$ be the triangular decomposition of $g$ defined as follows: Elements in $g_-$ are matrices $A$ of dimension $m + n$ with trace zero. Let $g_-$ be the set of lower triangular matrices in $g$, $g_+$ the set of upper triangular matrices in $g$ and $h$ the set of diagonal matrices.
Hence \( \mathfrak{h} \) is the Lie algebra of diagonal matrices \( A \) of dimension \( m + n \) with trace zero. It follows \( \mathfrak{h} \) consists of matrices of the type

\[
A = \begin{pmatrix}
an_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & a_{m+n}
\end{pmatrix}
\]

with \( \text{tr}(A) = \sum a_i = 0 \). Let \( \mathfrak{h}^* \) be the dual of \( \mathfrak{h} \). It follows

\[
\mathfrak{h}^* = K\{L_1, \ldots, L_{m+n}\}/L_1 + \cdots + L_{m+n}
\]

where

\[
L_i(A) = a_i.
\]

Note. Given a semi simple Lie algebra \( \mathfrak{g} \) there exist many different Cartan decompositions. They are all conjugate under automorphisms of \( \mathfrak{g} \).

There is since \( K \) has characteristic zero an embedding of \( G \)-modules

\[
V_\lambda \subseteq \text{Sym}^d(\wedge^m V)^* \cong \text{Sym}^d(\wedge^m V).
\]

Let

\[
l^d = \text{Sym}^d(\wedge^m W) \subseteq \text{Sym}^d(\wedge^m V).
\]

It follows \( l^d \) is a \( P \)-stable vector. Let \( L_{ld} \subseteq \text{Sym}^d(\wedge^m V) \) be the line spanned by \( l^d \) and let \( L_v \subseteq V_\lambda \) be the line spanned by \( v \).

**Lemma 4.1.** There is an equality \( L_\nu = L_{ld} \). Moreover \( v \) is the unique highest weight vector for \( V_\lambda = H^0(X, \mathcal{O}_X(d))^* \) with highest weight

\[
\lambda = d(L_1 + \cdots + L_m).
\]

**Proof.** By the Borel-Weil-Bott Theorem it follows \( V_\lambda \) is an irreducible \( G \)-module. One checks there is an equality \( L_v = L_{ld} \) of lines, and that \( v \) is a highest weight vector for \( V_\lambda \). One also checks \( v \) has the given weight and the Lemma follows. \( \square \)

The line \( L_v \subseteq V_\lambda \) is in fact the unique \( P \)-stable line of \( V_\lambda \).

The \( l^d \)th piece \( U_l(\mathfrak{g}) \) of the canonical filtration of the enveloping algebra is a \( G \)-module via the adjoint representation. It follows \( U_l(\mathfrak{g}) \) is a \( P \)-module.

**Definition 4.2.** Let \( \text{ann}(v) \subseteq U(\mathfrak{g}) \) be the left annihilator ideal of the vector \( v \in V_\lambda \):

\[
\text{ann}(v) = \{ x \in U(\mathfrak{g}) : x(v) = 0 \}.
\]

Let \( \text{ann}_l(v) = \text{ann}(v) \cap U_l(\mathfrak{g}) \) be its canonical filtration.

It follows \( \text{ann}(v) \) is a left ideal in the associative ring \( U(\mathfrak{g}) \).

There is an exact sequence

\[
0 \to \text{ann}(v) \otimes_K L_v \to U(\mathfrak{g}) \otimes_K L_v \to V_\lambda \to 0
\]

of \( G \)-modules and an exact sequence

\[
0 \to \text{ann}_l(v) \otimes_K L_v \to U_l(\mathfrak{g}) \otimes_K L_v \to U_l(\mathfrak{g})v \to 0
\]

of \( P \)-modules. Here

\[
U_l(\mathfrak{g})v = \{ x(v) : x \in U_l(\mathfrak{g}) \} \subseteq V_\lambda
\]
is the $P$-module spanned by elements of $U_l(g)$ and the vector $v$. The $G$-module $U(g) \otimes L_v$ is the \textit{generalized Verma module} associated to the $P$-module $L_v$. The $G$-module $U(g) \otimes_K L_v$ has a canonical filtration of $P$-modules given by

$$U_l(g) \otimes L_v \subseteq U(g) \otimes L_v.$$ 

\textbf{Definition 4.3.} Let $\{U_l(g)v\}_{l \geq 0}$ be the \textit{canonical filtration} of $V_\lambda$.

The $P$-module $U_l(g)v$ depends on the $P$-stable line $L_v$ defined by $v \in V_\lambda$ which is canonically determined by the highest weight vector $v \in V_\lambda$. It follows we get a canonical filtration

$$U_1(g)v \subseteq \cdots \subseteq U_l(g)v \subseteq V_\lambda$$

of $V_\lambda$ by $P$-modules.

The Lie algebra $p$ is the sub Lie algebra of $g$ consisting of traceless matrices $M$ with coefficients in $K$ on the following form:

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where $A$ is an $m \times m$-matrix, $X$ is an $n \times m$-matrix and $B$ is an $n \times n$-matrix such that $\text{tr}(A) + \text{tr}(B) = 0$. Let $n \subseteq g$ be the sub Lie algebra consisting of matrices $M$ on the following form:

$$M = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$$

where $Y$ is an $m \times n$-matrix with coefficients in $K$. It follows there is a direct sum decomposition $g = n \oplus p$ of vector spaces.

\textbf{Proposition 4.4.} The natural map

$$f : U(n) \otimes_K U(p) \to U(g)$$

defined by

$$f(v \otimes w) = vw$$

is an isomorphism of vector spaces.

\textit{Proof.} For a proof, see [3], Proposition 2.2.9. \qed

\textbf{Definition 4.5.} Let $l \geq 1$ be an integer and define

$$U_l(n,p) = \sum_{i+j=l} U_i(n) \otimes_K U_j(p) \subseteq U(n) \otimes_K U(p).$$

Assume in the following $n$ has $\{x_1, \ldots, x_E\}$ as a basis. The line $v = l^d$ define a $p$-module

$$\rho : p \to \text{End}(v)$$

hence we get a map

$$\rho : p \to K.$$ 

If $z \in p$ is an element defined by the matrix $M$ with

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

it follows $\rho(z) = d\text{tr}(A)$. It follows we may choose a direct sum decomposition $p = p_v \oplus (x)$ where $p_v = \text{Ker}(\rho)$ and $x \in p$ has the property $\rho(x) = d$. It follows we may choose a basis for $p$ on the form $\{y_1, \ldots, y_D, x\}$ where $\rho(y_1) = 0$. 

Lemma 4.6. There is for all $l \geq 1$ an equality
\[ f(U_l(n, p)) = U_l(g). \]

**Proof.** It is clear $f(U_l(n, p)) \subseteq U_l(g)$. Assume
\[ \omega = x_1^{v_1} \cdots x_E^{v_E} x^q y_1^{u_1} \cdots y_D^{u_D} \in U_l(g) \]
with $\sum_i v_i + q + \sum_j u_j \leq l$. It follows
\[ z = x_1^{v_1} \cdots x_E^{v_E} \otimes x^q y_1^{u_1} \cdots y_D^{u_D} \in U_l(n, p) \]
and $f(z) = \omega$. The Lemma is proved. \[ \square \]

Let $1_p \in U(p)$ be the multiplicative identity element.

**Definition 4.7.** Let $l \geq 1$ be an integer. Define the following:
\[ U_l(n) \otimes 1_p = \{ x \otimes 1_p : x \in U_l(n) \} \subseteq U_l(n, p) \]
\[ W_l = \{ x \otimes w(y - \rho(y)1_p) : x \in U_l(n), y \in p, w(y - \rho(y)1_p) \in U_j(p), i+j = l \} \subseteq U_l(n, p) \]

**Proposition 4.8.** The natural map
\[ \phi : U_l(n) \otimes 1_p \otimes W_l \to U_l(n, p) \]
defined by
\[ \phi(x, y) = x + y \]
is an isomorphism of vector spaces for all $l \geq 1$.

**Proof.** We first prove the following:
\[ U_l(n) \otimes 1_p \cap W_l = \{0\} \]
for all $l \geq 1$. Assume $z \in W_l$ is a monomial with $l \geq 1$. It follows
\[ z = x \otimes w(y - \rho(y)1_p) \]
with $y \in p$. One of the following assertions is true:
(4.8.1) $z = x \otimes wy_i$
(4.8.2) $z = x \otimes w(x - d1_p)$
(4.8.3) $z = 0$
Since $x \otimes wy_i$ and $x \otimes w(x - d1_p)$ are not in $U_l(n) \otimes 1_p$ it follows $z = 0$ and the claim is proved. It follows the map $\phi$ is an injection.

We next prove the map $\phi$ is a surjection for all $l \geq 1$: By definition
\[ U_l(n, p) = \sum_{i+j=l} U_{l-i}(n) \otimes K U_i(p). \]
Pick a monomial
\[ z = x_1^{v_1} \cdots x_E^{v_E} \otimes x^q y_1^{u_1} \cdots y_D^{u_D} \in U_l(n, p). \]
If $q = u_1 = \cdots = u_D = 0$ it follows $z \in U_l(n) \otimes 1_p$. Assume $q + \sum u_i \geq 1$ and $1 \leq j \leq D$ the largest integer with $u_j \geq 1$. It follows
\[ z = x_1^{v_1} \cdots x_E^{v_E} \otimes x^q y_1^{u_1} \cdots u_j^{u_j} = \]
\[ x_1^{v_1} \cdots x_E^{v_E} \otimes x^q y_1^{u_1} \cdots y_j^{u_j-1}(y_j - \rho(y_j)1_p) \in W_l. \]
Assume $u_1 = \cdots = u_D = 0$ and $q \geq 1$. We may write
\[ x^q = d^q 1_p + y(x - d1_p) \]
for some \(y\). We get
\[
z = x_1^{v_1} \cdots x_E^{v_E} \otimes x^y = x_1^{v_1} \cdots x_E^{v_E} \otimes (d^p 1_p + y(x - d 1_p)) = x_1^{v_1} \cdots x_E^{v_E} d^l 1_p + x_1^{v_1} \cdots x_E^{v_E} \otimes y(x - d 1_p) \in U_l(n) \otimes 1_p + W_l
\]
and the Proposition is proved.

Let \(1_p \in U(g)\) be the multiplicative identity element.

**Definition 4.9.** Let
\[
\text{char}(\rho) = \{x(y - \rho(y))1_g : y \in p, x \in U(g)\}
\]
be the left character ideal of \(\rho\). Let
\[
\text{char}_l(\rho) = \text{char}(\rho) \cap U_l(g)
\]
be the canonical filtration of \(\text{char}(\rho)\).

It follows \(\text{char}(\rho) \subseteq U(g)\) is a left ideal in the associative ring \(U(g)\). By definition it follows there is an inclusion of ideals \(\text{char}(\rho) \subseteq \text{ann}(v)\). This inclusion is strict. The inclusion \(n \subseteq g\) induce a canonical injection \(U(n) \subseteq U(g)\) of associative rings. We get a canonical inclusion of filtrations \(U_l(n) \subseteq U_l(g)\) for all \(l \geq 1\).

**Lemma 4.10.** The following holds for all \(l \geq 1\):
\[
(4.10.1) \quad f(U_l(n) \otimes 1_p) = U_l(n)
\]
\[
(4.10.2) \quad f(W_l) = \text{char}_l(\rho).
\]

**Proof.** We prove claim 4.10.1 Assume \(\omega \otimes 1_p \in U_l(n) \otimes 1_p\). It follows \(\omega \in U_l(n)\). It follows
\[
f(\omega \otimes 1_p) = \omega 1_g = \omega \in U_l(n)
\]
and claim 4.10.1 follows. We prove claim 4.10.2 Assume \(z = x \otimes w(y - \rho(y))1_g \in W_l\). It follows \(x \in U_l(n)\), \(y \in p\), and \(w(y - \rho(y))1_g \in U_j(p)\) with \(i + j = l\). It follows
\[
f(z) = x w(y - \rho(y))1_g \in \text{char}_l(\rho)
\]
and claim 4.10.2 follows. The Lemma is proved.

**Theorem 4.11.** There is for all \(l \geq 1\) an isomorphism
\[
U_l(g) = U_l(n) \oplus \text{char}_l(\rho)
\]
of vector spaces.

**Proof.** The Theorem follows from Lemma 4.10 Proposition 4.8 and the fact the natural map \(f : U_l(n, p) \to U_l(g)\) is an isomorphism of vector spaces.

In the following we use the notation in \[3\] Chapter 7.2. Let \(P_{++}\) be the set of dominant weights for \(g\) and let \(\lambda \in \mathfrak{h}^*\) be the weight with
\[
L(\lambda + \delta) \cong H^0(X, \mathcal{O}_X(d))^*.
\]
Such an element \(\lambda\) is uniquely determined since the module \(L(\lambda + \delta)\) is an irreducible finite dimensional \(g\)-module and there is a one to one correspondence between \(P_{++}\) and the set of irreducible finite dimensional \(g\)-modules. Let \(B\) be a basis for the roots \(R\) of \(g\). It follows \(B = L_i - L_{i+1}\) with \(i = 1, .., m + n - 1\). Let \(\nu' \in L(\lambda + \delta)\) be the unique highest weight vector and define two left ideals \(I', I'' \subseteq U(g)\) as follows:
\[
I'' = U(g)n_+ + \sum_{h \in \mathfrak{h}} U(g)(h - \lambda(h)),
\]
\[
I' = U(g)n_+ + \sum_{h \in \mathfrak{h}} U(g)(h - \lambda(h)) + \mathcal{O}_X(d).
\]
Here we let $m_\beta = \lambda(H_\beta) + 1$ and $X_{-\beta}$ be a non-zero element of $\mathfrak{g}^{-\beta}$. It follows by Proposition 7.2.7 the ideal $\mathcal{I}'$ equals the left annihilator ideal $\text{ann}(v)$ in $U(\mathfrak{g})$ of the highest weight vector $v = \mathfrak{v}^d$. Let $\mathcal{I}' = \mathcal{I} \cap U_1(\mathfrak{g})$.

**Lemma 4.12.** For all $1 \leq l < d$ there is an equality

$$\text{char}_l(\rho) = \text{ann}_l(v)$$

of filtrations.

**Proof.** Consider the ideal $\mathcal{I}'_l$ for $1 \leq l < d$. By definition there is an inclusion $\text{char}_l(\rho) \subseteq \mathcal{I}'_l$. We prove the reverse inclusion. There is an isomorphism

$$H^0(\mathcal{X}, \mathcal{O}(d))^* \cong L(\lambda + \delta)$$

where $L(\lambda + \delta)$ is the irreducible $\mathfrak{g}$-module with highest weight $\lambda$. By Lemma 4.11 it follows $\mathfrak{v}'$ has weight $\lambda = d(L_1 + \cdots + L_m)$ in the notation of [4]. Consider the ideal $\mathcal{I}''$:

$$\mathcal{I}'' = U(\mathfrak{g})n_+ + \sum_{x \in \mathfrak{h}} U(\mathfrak{g})(x - \lambda(x)).$$

It follows $\mathcal{I}'' \subseteq \text{char}(\rho)$. Let $\beta_i \in B$ with $\beta_i = L_i - L_{i+1}, 1 \leq i \leq m + n - 1$. Let $0 \neq E_{ij} \in \mathfrak{g}^{\beta}$ and let $0 \neq H_\beta \in \mathfrak{g}^{\beta}$. One checks $\lambda(H_\beta) + 1 = 1$ if $1 \leq i \leq m - 1$, $\lambda(H_\beta) + 1 = d + 1$ if $i = m$ and $\lambda(H_\beta) + 1 = 1$ if $m + 1 \leq i \leq m + n - 1$. Let $K = \sum_{\beta \in B} U(\mathfrak{g})X_{-\beta}$ and let $K_l = K \cap U_l(\mathfrak{g})$. Let $D$ be the set of integers $i$ with $i \in \{1, \ldots, m - 1, m + 1, \ldots, m + n - 1\}$. Let $\beta_i = L_i - L_{i+1}$. It follows

$$K_l = \sum_{\beta_i, i \in D} U_{l-1}(\mathfrak{g})X_{-\beta_i}$$

and one checks $K_l \subseteq \text{char}_l(\rho)$ and the claim of the Lemma follows. \(\square\)

**Theorem 4.13.** There is for all $1 \leq l < d$ an isomorphism

$$U_l(\mathfrak{g}) \cong U_l(\mathfrak{n}) \oplus \text{ann}_l(v)$$

of vector spaces.

**Proof.** This follows from Theorem 4.11 and Lemma 4.12. \(\square\)

**Corollary 4.14.** Let $z_1, \ldots, z_D$ be a basis for $\mathfrak{n}$ where $D = mn$. It follows the set

$$(4.14.1) \quad \{z_1^{v_1} \cdots z_D^{v_D}(v) : 0 \leq \sum v_i \leq l\}$$

is a basis for $U_l(\mathfrak{g})v$ as vector space. Moreover

$$(4.14.2) \quad \text{dim}_K(U_l(\mathfrak{g})v) = \binom{D + l}{D}.\)$$

**Proof.** The natural surjection $U_l(\mathfrak{g}) \otimes L_v \to U_l(\mathfrak{g})v$ of $K$-vector spaces induce by Theorem 4.13 an isomorphism $U_l(\mathfrak{n}) \otimes L_v \cong U_l(\mathfrak{g})v$ of vector spaces. From this isomorphism and the Poincare-Birkhoff-Witt Theorem claim 4.14.1 follows. Since $\text{dim}_K(U_l(\mathfrak{n})) = \binom{D + l}{D}$ claim 4.14.2 follows and the Corollary is proved. \(\square\)

It follows we have constructed a basis for $U_l(\mathfrak{g})v$ for all $l \geq 1$ and calculated $\text{dim}_K(U_l(\mathfrak{g})v)$ as a function of $l$. \(\square\)
5. ON JET BUNDLES AND CANONICAL FILTRATIONS

In this section we study the fiber of the jet bundle as $P$-module. We prove the fiber of the $l$th jet bundle equals the $l$th term in the canonical filtration.

**Notation:** Let in the following section $W \subseteq V$ be $K$-vector spaces of dimension $m$ and $m + n$ and let $P \subseteq \text{SL}(V)$ be the parabolic subgroup fixing $W$. It follows $X^i = \text{SL}(V)/P = \mathbb{G}(m, m + n)$ is the Grassmannian of $m$-planes in $V$. Let $\mathcal{O}_X(d)$ be the line bundle coming from the Plücker embedding of $X$.

Recall the exact sequence of $P$-modules from Corollary 3.13

\[(5.0.3) \quad 0 \rightarrow H^0(X, m^{l+1}\mathcal{O}_X(d)) \rightarrow H^0(X, \mathcal{O}_X(d)) \rightarrow P^l_X(\mathcal{O}_X(d))(x) \rightarrow 0\]

when $1 \leq l \leq d$. Dualize sequence (5.0.3) to get the exact sequence

\[(5.0.4) \quad 0 \rightarrow P^l_X(\mathcal{O}_X(d))(x)^* \rightarrow H^0(X, \mathcal{O}_X(d))^* \rightarrow \psi H^0(X, m^{l+1}\mathcal{O}_X(d))^* \rightarrow 0.\]

There is by definition an isomorphism

\[H^0(X, \mathcal{O}_X(d))^* \cong V_λ\]

where $λ = d(L_1 + \cdots + L_m)$. The highest weight vector $v \in V_λ$ is given by $v = l^d$ where $l^d = \text{Sym}^d(\wedge^m W)$. We get an inclusion

\[U_l(\mathfrak{g})v \subseteq V_λ = H^0(X, \mathcal{O}_X(d))^*\]

of $P$-modules.

**Theorem 5.1.** There is for $1 \leq l < d$ an isomorphism

\[P^l_X(\mathcal{O}_X(d))(x)^* \cong U_l(\mathfrak{g})v\]

of $P$-modules.

**Proof.** Consider the exact sequence (5.0.4) of $P$-modules

\[0 \rightarrow P^l_X(\mathcal{O}_X(d))(x)^* \rightarrow H^0(X, \mathcal{O}_X(d))^* \rightarrow \psi H^0(X, m^{l+1}\mathcal{O}_X(d))^* \rightarrow 0.\]

There is an inclusion of $P$-modules

\[U_l(\mathfrak{g})v \subseteq H^0(X, \mathcal{O}_X(d))^*\]

where $v$ is the highest weight vector. Consider an element $x_1 \cdots x_i v \in U_l(\mathfrak{g})v$ with $i \leq l$ and $x_i \in \mathfrak{g}$. It follows

\[\psi(x_1 \cdots x_i v)(s) = x_1 \cdots x_i s(e)\]

for $s \in H^0(X, m^{l+1}\mathcal{O}_X(d))$. The section $s$ has a zero of order $\geq l + 1$ at $e$. Since $x_1 \cdots x_i$ acts as a differential operator of order $i$ it follows

\[x_1 \cdots x_i s \in H^0(X, m^{l+1-i}\mathcal{O}_X(d))\]

hence $x_1 \cdots x_i s$ has a zero of order $l + 1 - i$ at $e$. It follows $\psi(x_1 \cdots x_i v) = 0$ since $i \leq l$. We get $\psi(U_l(\mathfrak{g})v) = 0$ and $U_l(\mathfrak{g})v \subseteq \text{ker}(\psi) = P^l_X(\mathcal{O}_X(d))(x)^*$ since the sequence above is an exact sequence of $P$-modules. We get an inclusion

\[U_l(\mathfrak{g})v \subseteq P^l_X(\mathcal{O}(d))(x)^*\]

of $P$-modules when $1 \leq l < d$. By Corollary 4.14 it follows

\[\dim_K(U_l(\mathfrak{g})v) = \binom{mn + l}{mn} = \dim_K(P^l_X(\mathcal{O}_X(d))(x)^*\]

hence the Theorem is proved.
Note: In [2], Section 5 the authors claim they prove a similar result using different techniques. The aim of the introduction of the techniques in this paper is to use them to get more precise information on $U_l(g)v$ as $P$-module. There is work in progress (see [14]) on giving a more detailed description of the $P$-module $U_l(g)v$ and to apply this to the study of resolutions of discriminants of linear systems on Grassmannians.

Let $d = (d_1, \ldots, d_k)$ with $d_i \geq 1$. Let $E(d) = \mathcal{O}_X(d_1) \oplus \cdots \oplus \mathcal{O}_X(d_k)$ with $X = G(m, m+n)$. Let $H^0(X, \mathcal{O}_X(d_i))^* = V_{d_i} = V_i$ with highest weight vector $v_i$. Let $W = \{v_1, \ldots, v_k\} \subseteq H^0(X, E(d))^*$ be the subspace spanned by the vectors $v_i$. Let $P = \{\lambda \mid \mathcal{O}_X^{\lambda}(1) \text{ is irreducible}\}$ denote the set of irreducible $\mathcal{O}_X$-modules.

**Corollary 5.2.** There is for $1 \leq l \leq \min\{d_i\}$ an isomorphism
\[ \mathcal{P}_X^l(E(d))(x)^* \cong U_l(g)W = \bigoplus_{i=1}^k U_l(g)v_i \]
of $P$-modules.

**Proof.** We get from Theorem 5.1 the following:
\[ \mathcal{P}_X^l(E(d))(x)^* \cong \bigoplus_{i=1}^k \mathcal{P}_X^l(\mathcal{O}_X(d_i))(x)^* \cong \bigoplus_{i=1}^k U_l(g)v_i \cong U_l(g)W \]
and the Corollary follows. \(\square\)

### 6. Discriminants of Linear Systems on the Grassmannian

In this section we use the results obtained in the previous sections to prove the $l$th discriminant of any linear system on any Grassmannian is irreducible.

**Notation:** Let in the following section $X = G(m, m+n)$ be the Grassmannian of $m$-planes in an $m+n$-dimensional vector space and let $\mathcal{O}_X(1)$ be the line bundle coming from the Plucker embedding. Let $\mathcal{O}_X(d) = \mathcal{O}_X(1)^{\otimes d}$.

Let $1 \leq l \leq d$ and consider the exact sequence ([3], [12], [11]) of locally free sheaves
\[ 0 \to \mathcal{Q}_{l,d} \to H^0(X, \mathcal{O}_X(d)) \otimes \mathcal{O}_X \to \mathcal{P}_X^l(\mathcal{O}_X(d)) \to 0 \]
where
\[ \mathcal{Q}_{l,d} = p_*(I_{\Delta}^{l+1} \otimes q^*\mathcal{O}_X(d)). \]
It follows from [10], Theorem 2.5 we get a commutative diagram
\[ \begin{array}{ccc}
\mathbb{P}(\mathcal{Q}_{l,d}^*) & \xrightarrow{i} & \mathbb{P}(W^*) \times X \\
\downarrow{\pi} & & \downarrow{p} \\
D^l(\mathcal{O}_X(d)) & \xrightarrow{j} & \mathbb{P}(W^*) \\
\end{array} \]
Spec($K$)
where $W = H^0(X, \mathcal{O}_X(d))$. Here $i, j$ are closed immersions of schemes and $\pi$ is the restriction of $q$. It follows $D^l(\mathcal{O}_X(d))$ is the scheme theoretic direct image of $\mathbb{P}(\mathcal{Q}_{l,d}^*)$.

**Theorem 6.1.** The discriminant $D^l(\mathcal{O}_X(d))$ is irreducible when $1 \leq l \leq d$.

**Proof.** Since $\mathcal{Q}_{l,d}^*$ is locally free it follows from [10], Corollary 2.6 $D^l(\mathcal{O}_X(d))$ is irreducible. \(\square\)

In a series of papers (see [9], [10], [11], [12], [13] and [15]) the structure of the jet bundle $\mathcal{P}_X^l(\mathcal{O}_X(d))$ as left and right $\mathcal{O}_X$-module and left and right $P$-module has been studied using various techniques: Explicit techniques, group theoretic techniques and Lie theoretic techniques. This study is part of a project where the aim is to study discriminants of linear systems on flag varieties (see [10] and [11]).
The jet bundle $P^i_{G/P}(L)$ where $L \in \text{Pic}^G(G/P)$ and $G = \text{SL}(V), P \subseteq G$ a parabolic subgroup, may by [10] be used to define the $i$'th discriminant $D^i(L)$ of the line bundle $L$. The discriminant $D^i(L)$ is a closed subscheme

$$D^i(L) \subseteq \mathbb{P}(W^*)$$

where $W = H^0(G/P, L)$. On $G/P$ there is an exact sequence

$$0 \to Q_i \to H^0(G/P, L) \otimes \mathcal{O}_{G/P} \to P^i_{G/P}(L) \to 0$$

of locally free finite rank sheaves. We get a commutative diagram of maps of schemes

$$
\begin{array}{ccc}
\mathbb{P}(Q^*_i) & \longrightarrow & \mathbb{P}(W^*) \times G/P \\
\downarrow{\pi} & & \downarrow{p} \\
D^i(L) & \longrightarrow & \mathbb{P}(W^*) \\
i & & \\
\end{array}
$$

where $i$ is a closed immersion and $\pi$ is the restriction of the projection morphism $p$. The sheaf $Q_i$ is a locally free $\mathcal{O}_{G/P}$-module of finite rank and the map $\pi$ is a surjective generically finite morphism between irreducible schemes. The aim of the study of $P^i_{G/P}(L)$ is to use its properties to study the map $\pi$ and $D^i(L)$. We want information on the syzygies of $D^i(L)$, the singularity type of $D^i(L)$, its degree and its dimension.

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