A QUASI-LOCAL CHARACTERISATION OF $L^p$-ROE ALGEBRAS

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ABSTRACT. Very recently, Špakula and Tikuisis provide a new characterisation of (uniform) Roe algebras via quasi-locality when the underlying metric spaces have straight finite decomposition complexity. In this paper, we improve their method to deal with the $L^p$-version of (uniform) Roe algebras for any $p \in [1, \infty)$. Due to the lack of reflexivity on $L^1$-spaces, some extra work is required for the case of $p = 1$.

Mathematics Subject Classification (2010): 20F65, 46H35, 47L10.

1. Introduction

(Uniform) Roe algebras are $C^*$-algebras associated to metric spaces, which reflect coarse properties of the underlying metric spaces. These algebras have been well-studied and have fruitful applications, among which the most important ones would be the (uniform) coarse Baum-Connes conjecture and the Novikov conjecture (e.g., [31, 32, 38, 39, 40, 41]). Meanwhile, they also provide a link between coarse geometry of metric spaces and the theory of $C^*$-algebras (e.g., [1, 11, 15, 16, 19, 20, 22, 28, 29, 31, 35, 37]), and turn out to be useful in the study of topological phases of matter (e.g., [17, 10]) as well as the theory of limit operators in the study of Fredholmness of band-dominated operators (e.g., [14, 21, 30, 34]).

By definition, the (uniform) Roe algebra of a proper metric space $X$ is the norm closure of all bounded locally compact operators $T$ with finite propagation in the sense that there exists $R > 0$ such that for any $f, g \in C_b(X)$ acting on $L^2(X)$ by pointwise multiplication, we have $fTg = 0$ provided their supports are $R$-separated. Since general elements in (uniform) Roe algebras may not have finite propagation, it is usually difficult to tell what operators exactly belong to them. On the other hand, Roe [26] defined an asymptotic version of finite propagation as follows: An operator $T$ on $L^2(X)$ has finite $\varepsilon$-propagation for $\varepsilon > 0$, if there is $R > 0$ such that for any $f, g \in C_b(X)$, we have $\|fTg\| \leq \varepsilon\|f\| \cdot \|g\|$ provided their supports are $R$-separated. Operators with finite $\varepsilon$-propagation for all $\varepsilon > 0$ are called quasi-local in [25]. It is clear that limits of finite $\varepsilon$-propagation operators still...
have finite $\varepsilon$-propagation. Consequently, all operators in (uniform) Roe algebras are quasi-local.

A natural question is that whether the converse holds as well, i.e., does every locally compact quasi-local operator belong to the (uniform) Roe algebra? An affirmative answer to this question would provide a new approach to detect what operators belong to these algebras in a more practical way by estimating the norms of operator-blocks far from strips around the diagonal, and it has several immediate consequences including the followings.

The first one has its root in Engle’s work [8, Section 2], where he studied the index theory of pseudo-differential operators. He showed that the indices of uniform pseudo-differential operators on Riemannian manifolds are quasi-local, while it is unclear to him whether they live in Roe algebras, which are well-understood. Another application is in the work of White and Willett [36] on Cartan subalgebras of uniform Roe algebras. They showed that if two uniform Roe algebras of bounded geometry metric spaces with Property A are $*$-isomorphic, then the underlying metric spaces are bijectively coarsely equivalent provided that every quasi-local operator belongs to the uniform Roe algebras.

Historically, this question has been studied and partially addressed by many people including Lange and Rabinovich for $X = \mathbb{Z}^n$ [18] (in fact they worked in a more general context, see the next paragraph), Engel for $X$ is a manifold of bounded geometry with polynomial volume growth [9], Špakula and Tikuisis [33] for $X$ has straight finite decomposition complexity in the sense of [7]. To our best knowledge, this question is still open for general metric spaces.

Based on the original definitions, various versions of Roe algebras are proposed and studied by different purposes. In fact, in recent years there has been an uptick in interest in the $L^p$-version of (uniform) Roe algebras for $p \in [1, \infty)$, from the communities of both limit operator theory and coarse geometry (e.g. [30, 21, 34, 14, 3, 42]). And it is natural and important to study the same question in this context, i.e., does every locally compact and quasi-local operator belong to the $L^p$-version of (uniform) Roe algebras for $p \in [1, \infty)$?

In this paper, we improve the method of Špakula and Tikuisis [33] in order to generalise their result from the case of $p = 2$ to any $p \in [1, \infty)$. Our main result is Theorem 3.3 which answers the $L^p$-version of the question above when the metric spaces have straight finite decomposition complexity. To be a little bit more explicit on the context we are working in, we recall that Špakula and Tikuisis indeed studied a more general notion of Roe-like algebras associated to proper metric spaces (see [33, Definition 2.3]). Similarly, we extend their definition to the so-called $L^p$-Roe-like algebras in this paper. We would like to point out that our definition of $L^p$-Roe-like algebras are more general than Špakula and Tikuisis’ definition even for $p = 2$, as we drop a commutant condition in [33, Definition 2.3],

1 Notice that in the uniform case, all operators are locally compact since the Hilbert space taken in this case is just the complex number $\mathbb{C}$.
which is used in the proof of their main theorem. However, we observe that this condition is redundant for the proof of the main theorem if we replace it with Lemma 3.5 below. The reason we drop this condition is inspired by the fact that it is not fulfilled for general $L^1$-Roe-like algebras, and an obvious advantage of doing this is to allow more examples especially in the case of $p = 1$ (see Remark 2.9 and Example 2.12 for more details).

The proof of our main theorem is closely modelled on their original one in [33] at least for $p \in (1, \infty)$, except that the $L^p$-Roe-like algebras need not possess a bounded involution and von Neumann algebra techniques are invalid. Instead, we have to deal with asymmetric situation as in the proof of the implication “(iii) $\Rightarrow$ (i)” in Theorem 3.3 and provide a direct and concrete proof of Lemma 4.5.

The case of $p = 1$ is more complicated and in this case Proposition 4.1 is established, which is the most technical part of the paper and is also a generalisation of [33 Corollary 4.3]. The difficulty comes from the lack of reflexivity on $L^1$-spaces, and the trick of the proof is to consider an artificial space $L^0(X)$, which lies between $C_0(X)$ and $L^\infty(X)$. It is worth pointing out that Proposition 4.1 is based on a crucial intermediate result established in a more general setup of Banach spaces, and we hope that there might be some other applications in the future.

The paper is organised as follows: we establish the settings of the paper by recalling some background in Banach algebra theory and coarse geometry in Section 2, where various examples of $L^p$-Roe-like algebras are also provided. In Section 3, we state the main theorem and prove the relatively easier part, where the assumption of straight finite decomposition complexity is not required. In Section 4, we prove the technical tool, Proposition 4.1 and finish the remaining proof of the main theorem.

Conventions: Let $\mathfrak{X}$ be a Banach space. We denote the closed unit ball of $\mathfrak{X}$ by $\mathfrak{X}_1$. For any $a, b \in \mathfrak{X}$ and $\varepsilon > 0$, we denote $\|a - b\| \leq \varepsilon$ by $a \approx_\varepsilon b$. We also denote the bounded linear operators on $\mathfrak{X}$ by $\mathfrak{B}(\mathfrak{X})$, and the compact operators on $\mathfrak{X}$ by $\mathfrak{K}(\mathfrak{X})$.

Moreover, for a Banach algebra $A$ we define

$$A_\infty := \ell^\infty(\mathbb{N}, A) / \{(a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) : \lim_{n \to \infty} \|a_n\| = 0\},$$

which is a Banach algebra with respect to the quotient norm.

Throughout the paper, we fix a proper metric space $(X, d)$ (i.e., every bounded subset is pre-compact). Note that such a space is always locally compact and $\sigma$-compact. We also fix a Radon measure $\mu$ on $(X, d)$ with full support.

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2After we finish this paper, Špakula and the third-named author informed us that the main theorem of this paper remains true if we only require Property A rather than straight finite decomposition complexity. Their arguments include an essential application of Proposition 4.1.
2. Preliminaries

In this section, we provide the background settings of this paper by collecting several basic notions from Banach algebra theory and coarse geometry. Throughout the section, let $E$ be a (complex) Banach space and $(X, d, \mu)$ be a proper metric space with a Radon measure $\mu$ on $X$ of full support.

2.1. Banach space valued $L^p$-spaces. In this subsection, we recall some basic notions and facts on Banach space valued $L^p$-spaces.

**Definition 2.1.** Let $p \in [1, \infty]$. For a Bochner measurable function (i.e., it equals $\mu$-almost everywhere to a pointwise limit of a sequence of simple functions) $\xi : (X, \mu) \to E$, its $p$-norm is defined by
\[
\|\xi\|_p := \left( \int_X \|\xi(x)\|_E^p \, d\mu(x) \right)^{\frac{1}{p}},
\]
and its infinity-norm is defined by
\[
\|\xi\|_\infty := \text{ess sup} \{\|\xi(x)\|_E : x \in X\}.
\]

For $p \in [1, \infty)$, the space of $E$-valued $L^p$-functions on $(X, \mu)$ is defined as follows:
\[
L^p(X, \mu; E) := \left\{\xi : X \to E \mid \xi \text{ is Bochner measurable and } \|\xi\|_p < \infty \right\},
\]
where $\xi \sim \eta$ if and only if they are equal $\mu$-almost everywhere. Equipped with the $p$-norm, $L^p(X, \mu; E)$ becomes a Banach space, which is called the $L^p$-Bochner space.

We also need the following closed linear subspace of $L^\infty(X, \mu; E)$:
\[
L^0(X, \mu; E) := \left\{[\xi] \in L^\infty(X, \mu; E) \mid \forall \varepsilon > 0, \exists \text{ compact } K \subseteq X, \text{ s.t. } \|\xi\|_{X\backslash K}_\infty < \varepsilon \right\},
\]
equipped with the norm $\|\xi\|_0 := \|\xi\|_\infty$. Clearly, $L^0(X, \mu; E)$ contains $C_0(X)$ but is more flexible, as it also contains all characteristic functions of bounded subsets of the proper metric space $(X, d)$. On the other hand, $L^0(X, \mu; E)$ inherits some nice behaviours of $C_0(X)$, for example the norm of any element in $L^0(X, \mu; E)$ goes to zero when the variable goes to infinity.

In order to simplify notations, we regard $\xi$ as an element in $L^p(X, \mu; E)$ and write $L^p(X; E)$ instead if there is no ambiguity. If $X$ is discrete and equipped with the counting measure, we simply write $\ell^p(X; E)$.

If $p \in (1, \infty)$, let $q$ be the conjugate exponent to $p$ (i.e., $\frac{1}{p} + \frac{1}{q} = 1$) and if $p = 1$, we set $q = 0$ instead of $q = \infty$. It is worth noticing that the duality $L^p(X; E)^* \cong L^q(X; E')$ does not hold in general (see e.g. [2, 5, 6]), but we still have the following lemma.

**Lemma 2.2.** When $p \in (1, \infty)$, set $q$ to be its conjugate exponent and when $p = 1$, set $q = 0$. Then there is an isometric embedding $L^q(X; E^*) \to L^p(X; E)^*$ defined by
\[
\eta(\xi) := \int_X \eta(x)(\xi(x)) \, d\mu(x).
\]

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3It follows from Pettis measurability theorem that Bochner measurability agrees with weak measurability when the Banach space $E$ is separable.
where $\eta \in L^1(X; E')$ and $\xi \in L^p(X; E)$. On the other hand, there is another isometric embedding $L^p(X; E) \to L^q(X; E')$ defined by

$$\xi(\zeta) := \int_X \zeta(x)(\xi(x))d\mu(x)$$

where $\xi \in L^p(X; E)$ and $\zeta \in L^q(X; E')$.

**Proof.** We only prove the second statement, which is similar to the first one. It suffices to show that for any $\xi \in L^p(X; E)$, we have

$$\|\xi\|_p = \sup\{\|\xi(\zeta)\| : \zeta \in L^q(X; E') \text{ and } \|\zeta\|_q \leq 1\}.$$ 

It is clear that the right hand side does not exceed the left. Conversely we may assume, by the inner regularity of $\mu$, that $\xi$ is non-zero and $\xi = \sum_{i=1}^n y_i \chi_{\Omega_i}$ for some $y_i \in E$ and mutually disjoint compact subsets $\Omega_i$ in $X$. Note that $\|\xi\|_p = \sum_{i=1}^n \|y_i\|_p \mu(\Omega_i)$. For each $y_i$, choose a $y_i' \in (E')_1$ such that $y_i'(y_i) = \|y_i\|_E$. Define

$$\zeta := \sum_{i=1}^n \frac{\|y_i\|_p^{p-1}}{\|\xi\|_p} y_i' \chi_{\Omega_i}.$$ 

It is straightforward to check that $\zeta \in L^q(X; E')$ with $\|\zeta\|_q = 1$ and $\xi(\zeta) = \|\xi\|_p$ (note that when $p = 1$, we set $q = 0$). Hence, we finish the proof. \qed

Finally we recall $L^p$-tensor products (more details can be found in [4, Chapter 7] and [23, Theorem 2.16]), which will be used in Section 2.3 without further reference.

For $p \in [1, \infty)$, there is a tensor product of $L^p$-spaces with $\sigma$-finite measures such that there is a canonical isometric isomorphism $L^p(X, \mu) \otimes L^p(Y, \nu) \cong L^p(X \times Y, \mu \times \nu)$, which identifies the element $\xi \otimes \eta$ with the function $(x, y) \mapsto \xi(x)\eta(y)$ on $X \times Y$ for every $\xi \in L^p(X, \mu)$ and $\eta \in L^p(Y, \nu)$. Moreover, the following properties hold:

- Under the identification above, the linear spans of all $\xi \otimes \eta$ are dense in $L^p(X \times Y, \mu \times \nu)$.
- $\|\xi \otimes \eta\|_p = \|\xi\|_p \|\eta\|_p$ for all $\xi \in L^p(X, \mu)$ and $\eta \in L^p(Y, \nu)$.
- The tensor product is commutative and associative.
- If $a \in \mathcal{B}(L^p(X_1, \mu_1), L^p(X_2, \mu_2))$ and $b \in \mathcal{B}(L^p(Y_1, \nu_1), L^p(Y_2, \nu_2))$, then there exists a unique element

$$c \in \mathcal{B}(L^p(X_1 \times Y_1, \mu_1 \times \nu_1), L^p(X_2 \times Y_2, \mu_2 \times \nu_2))$$

such that under the identification above, $c(\xi \otimes \eta) = a(\xi) \otimes b(\eta)$ for all $\xi \in L^p(X_1, \mu_1)$ and $\eta \in L^p(Y_1, \nu_1)$. We will denote this operator by $a \otimes b$. Moreover, $\|a \otimes b\| = \|a\| \cdot \|b\|$.

- The tensor product is associative, bilinear, and satisfies $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$.

If $A \subseteq \mathcal{B}(L^p(X, \mu))$ and $B \subseteq \mathcal{B}(L^p(Y, \nu))$ are closed subalgebras, we define $A \otimes B \subseteq \mathcal{B}(L^p(X \times Y, \mu \times \nu))$ to be the closed linear span of all $a \otimes b$ with $a \in A$ and $b \in B$. 
2.2. Block cutdown maps. Now we introduce block cutdown maps, providing an approach to cut an operator into the form of block diagonals.

First, let us recall some more notions. For \( p \in [0] \cup [1, \infty] \), the multiplication representation \( \rho : C_b(X) \to \mathcal{B}(L^p(X; E)) \) is defined by pointwise multiplications: 
\[
(\rho(f)\xi)(x) = f(x)\xi(x),
\]
where \( f \in C_b(X) \), \( \xi \in L^p(X; E) \) and \( x \in X \). Without ambiguity, we write \( fT \) and \( Tf \) instead of \( \rho(f)T \) and \( T\rho(f) \), for \( f \in C_b(X) \) and \( T \in \mathcal{B}(L^p(X; E)) \), respectively. It is worth noticing that \( \mu \) has full support if and only if \( \rho \) is injective. We also recall that a net \( \{T_\alpha\} \) converges in strong operator topology (SOT) to \( T \) in \( \mathcal{B}(L^p(X; E)) \) if and only if \( \|T_\alpha(\xi) - T(\xi)\|_p \to 0 \) for any \( \xi \in L^p(X; E) \).

**Definition 2.3.** Given an equicontinuous family \( (e_j)_{j \in J} \) of positive contractions in \( C_b(X) \) with pairwise disjoint supports, define the block cutdown map \( \theta_{(e_j)_{j \in J}} : \mathcal{B}(L^p(X; E)) \to \mathcal{B}(L^p(X; E)) \) by
\[
\theta_{(e_j)_{j \in J}}(a) := \sum_{j \in J} e_j a e_j,
\]
where the sum converges in (SOT) by Lemma 2.4 below. We say that a closed subalgebra \( B \subseteq \mathcal{B}(L^p(X; E)) \) is closed under block cutdowns, if \( \theta_{(e_j)_{j \in J}}(B) \subseteq B \) for every equicontinuous family \( (e_j)_{j \in J} \) of positive contractions in \( C_b(X) \) with pairwise disjoint supports.

**Lemma 2.4.** Let \( (e_j)_{j \in J} \) and \( (f_j)_{j \in J} \) be two equicontinuous families of positive contractions in \( C_b(X) \) with pairwise disjoint supports. Then the sum \( \sum_{j \in J} f_j a e_j \) converges in (SOT) to an operator in \( \mathcal{B}(L^p(X; E)) \). Furthermore, we have:
\[
\| \sum_{j \in J} f_j a e_j \| = \sup_{j \in J} \| f_j a e_j \|.
\]

**Proof.** First of all, we prove in the case of \( p \in (1, \infty) \) and let \( q \) be the conjugate exponent to \( p \). Let \( Y_j := \text{supp}(e_j) \) and \( Z_j := \text{supp}(f_j) \). For any \( \xi \in L^p(X; E) \), any finite subset \( F \subseteq J \) and any \( \eta \in L^q(X; E^*) \) with \( \|\eta\|_q \leq 1 \), we have that
\[
\| \eta(\sum_{j \in F} f_j a e_j \xi) \| = \| \sum_{j \in F} (\eta|_{Z_j})(f_j a e_j|_{Y_j}\xi) \|
\leq \sum_{j \in F} \|\eta|_{Z_j}\|_q \cdot \| f_j a e_j(\xi|_{Y_j}) \|_p
\leq (\sum_{j \in F} \|\eta|_{Z_j}\|_q^q)^{\frac{1}{q}} \cdot (\sum_{j \in F} \| f_j a e_j(\xi|_{Y_j}) \|_p^p)^{\frac{1}{p}}
\leq \sup_{j \in F} \| f_j a e_j \| \cdot \|\xi|_{\cup_{j \in F} Y_j}\|_p.
\]
Hence, it follows from Lemma 2.2 that
\[
\left\| \left( \sum_{j \in F} f_j a e_j \right) \xi \right\|_p \leq \sup_{j \in F} \| f_j a e_j \| \cdot \|\xi|_{\cup_{j \in F} Y_j}\|_p.
\]
Since \( \|\xi|_{\cup_{j \in F} Y_j}\|_p \leq \|\xi\|_p < \infty \), we know \( \left\{ \xi|_{\cup_{j \in F} Y_j} \right\} \) is a Cauchy net. Hence, \( \sum_{j \in F} f_j a e_j \) converges in (SOT) and \( \| \sum_{j \in F} f_j a e_j \| \leq \sup_{j \in F} \| f_j a e_j \| \). On the other hand, it is clear
that \(|\sum_{j=1}^{p} f_j a_j| \geq \sup_{f_j \in \mathfrak{F}} |f_j a_j|\). Hence we finish the proof for \(p > 1\). Since the proof for the case of \(p = 1\) is more direct, we leave the details to the reader. \(\square\)

**Remark 2.5.** Note that the multiplication by \(\mathcal{C}_b(X)\) commutes with the block cut-downs, i.e., for any \(a \in \mathfrak{B}(L^p(X; E))\) and \(f \in \mathcal{C}_b(X)\), we have

\[ f\theta_{(e_j)_{j \in \mathbb{J}}}(a) = \theta_{(e_j)_{j \in \mathbb{J}}}(fa) \quad \text{and} \quad \theta_{(e_j)_{j \in \mathbb{J}}}(a)f = \theta_{(e_j)_{j \in \mathbb{J}}}(af). \]

**Definition 2.6.** Suppose \(\mathcal{X}\) is a metric family of subsets in \(\mathcal{X}\) (recall that a metric family is a set of metric spaces), and \(a \in \mathfrak{B}(L^p(X; E))\). We say that \(a\) is block diagonal with respect to \(\mathcal{X}\), if there exist an equicontinuous family \((e_j)_{j \in \mathbb{J}}\) of positive contractions in \(\mathcal{C}_b(X)\) with pairwise disjoint supports and \(\{Y_j\}_{j \in \mathbb{J}} \subseteq \mathcal{X}\), such that

\[ a = \theta_{(e_j)_{j \in \mathbb{J}}}(a), \]

and \(\text{supp}(e_j) \subseteq Y_j\). In this case, we shall denote \(a_{Y_j} := e_j a e_j\), which is called the \(Y_j\)-block of \(a\).

### 2.3. \(L^p\)-Roe-like algebras

Now we introduce \(L^p\)-Roe-like algebras, which are our main objects in this paper.

**Definition 2.7.** Let \(R \geq 0\) and \(a \in \mathfrak{B}(L^p(X; E))\). We say that

\begin{itemize}
    \item \(a\) has propagation at most \(R\), if for any \(f, f' \in \mathcal{C}_b(X)\) with \(d(\text{supp}(f), \text{supp}(f')) > R\), then \(f a f' = 0\).
    \item \(a\) has \(\varepsilon\)-propagation at most \(R\) for some \(\varepsilon > 0\), if for any \(f, f' \in \mathcal{C}_b(X)\) with \(d(\text{supp}(f), \text{supp}(f')) > R\), then \(|f a f'| < \varepsilon|\).
    \item \(a\) is quasi-local, if it has finite \(\varepsilon\)-propagation for every \(\varepsilon > 0\).
\end{itemize}

**Definition 2.8.** Let \((X, d)\) be a proper metric space equipped with a Radon measure \(\mu\) whose support is \(X\), and \(p \in [1, +\infty)\). Suppose \(E\) is a Banach space and \(B \subseteq \mathfrak{B}(L^p(X; E))\) is a Banach subalgebra such that \(\mathcal{C}_b(X) B \mathcal{C}_b(X) = B\) and is closed under block cut-downs. Define:

\begin{itemize}
    \item (i) \(\text{Roe}(X, B)\) to be the norm-closure of all the operators in \(B\) with finite propagations. \(\text{Roe}(X, B)\) is called the \(L^p\)-Roe-like algebra of \((X, d, \mu)\);
    \item (ii) \(\mathcal{K}(X, B)\) to be the norm-closure of \(\mathcal{C}_0(X) B \mathcal{C}_0(X)\) in \(\mathfrak{B}(L^p(X; E))\).
\end{itemize}

**Remark 2.9.** The definition of \(L^2\)-Roe-like algebras come from [33, Definition 2.3], in which the following extra condition is also imposed:

\(\text{(2.1)}\) \quad \([\mathcal{C}_0(X), B] \subseteq \mathcal{K}(X, B)\).

This condition is used in the proof of their main theorem, [33, Theorem 2.8, “(ii) ⇒ (iii)”]. However, it turns out to be redundant if we apply our Lemma 3.5 below. On the other hand, this condition is fulfilled by most of the well-known \(L^p\)-Roe-like algebras for \(p \in (1, \infty)\) (as we will see in the following examples), but not for \(p = 1\) (see the explanation in Example 2.12). This is exactly our starting point to explore whether condition (2.1) is necessary, and it turns out that we may omit it in Definition 2.8 without affecting the main theorem. In this way, our main result (Theorem 3.3) is a slight generalisation of [33, Theorem 2.8].
We notice that in the case of $p = 2$, it has been pointed out in [33, Remark 2.4] that $\mathcal{K}(X, B)$ is an ideal in Roe$(X, B)$ under the additional condition (2.1). Now we show that it still holds in our settings.

**Lemma 2.10.** For any $p \in [1, +\infty)$, $\mathcal{K}(X, B)$ is a closed two-sided ideal in Roe$(X, B)$.

**Proof.** It suffices to show that for any $b = f_1b_1g_1 \in C_c(X)BC_c(X)$ and $a \in B$ with finite propagation at most $R$, $ba \in \mathcal{K}(X, B)$. Take a function $g_2 \in C_c(X)$ such that $g_2$ is 1 on the compact subset $N_2(\text{supp}(g_1))$. It follows that $g_1a(1 - g_2) = 0$, which implies that $g_1a = g_1ag_2$. Hence, we have

$$ba = f_1b_1g_1a = f_1(b_1g_1a)g_2.$$

Recall that $C_b(X)BC_b(X) = B$, so we have $b_1g_1 \in B$ and $a \in B$, which implies that $ba \in C_c(X)BC_c(X)$.

Similarly, $ab \in C_c(X)BC_c(X)$ as well. So we finish the proof. □

Before we illustrate several examples of $L^p$-Roe-like algebras, let us recall the following notion related to matrix algebras.

**Definition 2.11.** Let $(X, d)$ be a discrete proper metric space and $p \in [1, +\infty)$. Denote

$$\overline{M}_X^p := \overline{C_c(X)\mathcal{B}(l^p(X))C_c(X)},$$

i.e., for any fixed point $x_0 \in X$

$$\overline{M}_X^p = \bigcup_{n \in \mathbb{N}} M_{B_n(x_0)}^p,$$

where $M_{B_n(x_0)}^p = \mathcal{B}(l^p(B_n(x_0))) \subseteq \mathcal{B}(l^p(X))$, which is the matrix algebra over the closed ball of radius $n$ and centered in $x_0$. In other words, operators in $\overline{M}_X^p$ are exactly those can be approximated by finite matrices.

Phillips studied the relation between $\overline{M}_X^p$ and compact operators $\mathcal{R}(l^p(X))$ in [24]. He showed that when $p > 1$, $\overline{M}_X^p = \mathcal{R}(l^p(X))$ ([24, Proposition 1.8]); and when $p = 1$, $\overline{M}_X^1 \subsetneq \mathcal{R}(l^1(X))$ in general as illustrated in [24, Example 1.10] (see also Example 2.12).

Now we are ready to provide various of examples of $L^p$-Roe-like algebras, which include $l^p$-uniform Roe algebras, band-dominated operator algebras, $L^p$-Roe algebras, $l^p$-uniform algebras and stable $l^p$-uniform Roe algebras.

**Example 2.12 ($l^p$-Uniform Roe Algebra).** Let $(X, d)$ be a discrete proper metric space and $p \in [1, +\infty)$. Take $E = \mathbb{C}$ to be the complex number and $B = \mathcal{B}(l^p(X))$, which is clearly closed under block cutdowns, and satisfies $C_b(X)BC_b(X) = B$. In this case, Roe$(X, B)$ is called the $l^p$-uniform Roe algebra of $X$, which is defined in [3] and denoted by $B^p(X)$, and $\mathcal{K}(X, B)$ is $\overline{M}_X^p$ introduced above. It may be worth noting that $\overline{M}_X^p$ is structurally different from $\overline{M}_X^p$ for $p > 1$.

- $p > 1$: As pointed out above, $\overline{M}_X^p = \mathcal{R}(l^p(X))$ and the condition (2.1) holds.
The algebra $\mathcal{K}(X, B)$ is in general properly contained in $\mathfrak{R}(\ell^1(X))$ (see Example 1.10 in [24]). For example, taking $X$ to the natural number $\mathbb{N}$, consider the operator $T: \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ defined by
\[
T(\xi) := \left( \sum_{n \in \mathbb{N}} \xi(n) \right) \delta_0,
\]
where $\xi \in \ell^1(\mathbb{N})$ and $\delta_0 \in \ell^1(\mathbb{N})$ is the function taking value 1 at the original point 0, and 0 elsewhere. Since $T$ has rank 1, it belongs to $\mathfrak{R}(\ell^1(\mathbb{N}))$. However, it is not hard to see that $T \notin \mathcal{K}(\mathbb{N}, \mathfrak{B}(\ell^1(\mathbb{N}))) = \overline{M}_\mathbb{N}$. Furthermore, the operator $T$ also illuminates that condition (2.1) does not hold in general, since $[\delta_0, T] \notin \mathcal{K}(\mathbb{N}, \mathfrak{B}(\ell^1(\mathbb{N}))).$

**Example 2.13 (Band-Dominated Operator Algebra).** Let $(X, d)$ be a uniformly discrete metric space of bounded geometry (in the sense that for a given $R > 0$, all closed balls $B(x, R)$ have a uniform bound on cardinalities for all $x \in X$), $p \in (1, +\infty)$ and $E$ be a Banach space. Take $B = \mathfrak{B}(\ell^p(X; E))$, which is clearly closed under block cutdowns and satisfies $C_b(X)BC_b(X) = B$. Elements in $B$ can be represented in the matrix form
\[
b = (b_{x,y})_{x,y \in X} \in \mathfrak{B}(\ell^p(X; E)), \quad \text{where } b_{x,y} \in \mathfrak{B}(E).
\]
In this case, $\text{Roe}(X, B) = \mathcal{R}_E^p(X)$, which is the algebra of band-dominated operators (see [34, Definition 2.6]) and it is clear that $\mathcal{K}(X, B) = \mathcal{K}_E^p(X)$, which is the set of all $\mathcal{P}$-compact operators on $\ell^p(X; E)$, defined in [34, Definition 2.8].

**Example 2.14 ($L^p$-Roe Algebra).** Let $(X, d)$ be a proper metric space equipped with a Radon measure $\mu$ with support $X$, and $p \in [1, +\infty)$. We say that an operator $b$ in $\mathfrak{B}(L^p(X; \ell^p(\mathbb{N}))) \cong \mathfrak{B}(L^p(X \times \mathbb{N}))$ is locally compact if for any $f \in C_0(X)$, $fb$ and $bf$ belong to $\mathfrak{R}(L^p(X \times \mathbb{N})).$

Now take $E = \ell^p(\mathbb{N})$ and $B$ to be the set of all locally compact operators in $\mathfrak{B}(L^p(X; \ell^p(\mathbb{N})))$, which is clearly closed under block cutdowns and satisfies $C_b(X)BC_b(X) = B$. The corresponding $L^p$-Roe-like algebra $\text{Roe}(X, B)$ is called the $L^p$-Roe algebra of $X$, denoted by $B^p(X)$. It is, by definition, the norm closure of all locally compact and finite propagation operators in $\mathfrak{B}(L^p(X; \ell^p(\mathbb{N}))).$ When $p > 1$, $\mathcal{K}(X, B) = \mathfrak{R}(L^p(X \times \mathbb{N})).$ However, it does not hold in general when $p = 1$.

When $X$ is discrete, the $L^p$-Roe algebra $B^p(X)$ coincides with the $\ell^p$-Roe algebra defined in [3] and in the case of $p = 2$, the $L^2$-Roe algebra is the classical Roe algebra in the literature.

**Remark 2.15.** As explained in [3], there is another version of locally compactness: we say that an operator $b$ in $\mathfrak{B}(L^p(X; \ell^p(\mathbb{N})))$ is locally compact if for any $f \in C_0(X)$, $fb$ and $bf$ belong to $\mathfrak{R}(L^p(X)) \otimes \overline{M}_\mathbb{N}^p \subseteq \mathfrak{B}(L^p(X \times \mathbb{N})).$ Note that the subalgebra $\mathfrak{R}(L^p(X)) \otimes \overline{M}_\mathbb{N}^p$ is isomorphic to the norm closure of $\bigcup_{n \in \mathbb{N}} \overline{M}_n^p(\mathfrak{R}(L^p(X))).$ Therefore, we can alternatively define another version of the $L^p$-Roe algebra of $X$ to be the norm closure of all locally compact (in this new sense) and finite propagation operators in $\mathfrak{B}(L^p(X; \ell^p(\mathbb{N}))).$ When $p > 1$, it coincides with $B^p(X)$ defined in
Example 2.14 as $\mathcal{R}(L^p(X)) \otimes \overline{M}_N \cong \mathcal{R}(L^p(X \times \mathbb{N}))$. However, it is strictly contained in $B^1(X)$ when $p = 1$.

Example 2.16 (\(\ell^p\)-Uniform Algebra). Let \((X, d)\) be a discrete metric space with bounded geometry and $p \in [1, +\infty)$. Set $E = \ell^p(\mathbb{N})$, and $B$ to be the closure of the set of all $b = (b_{x,y})_{x,y \in X} \in \mathcal{B}(\ell^p(X; \ell^p(\mathbb{N})))$ for which the rank of $b_{x,y} \in \mathcal{B}(\ell^p(\mathbb{N}))$ is uniformly bounded. Clearly, $B$ is closed under block cutdowns, and satisfies $C_b(X)BC_b(X) = B$. In this case, $\text{Roe}(X, B) = \text{UB}^p(X)$, the \(\ell^p\)-uniform algebra of $X$, introduced in [3]. When $p > 1$, we have that $\mathcal{K}(X, B) = \mathcal{R}(\ell^p(X \times \mathbb{N}))$. But it does not hold in general when $p = 1$.

Example 2.17 (Stable \(\ell^p\)-Uniform Roe Algebra). Let \((X, d)\) be a discrete metric space with bounded geometry and $p \in [1, +\infty)$. Set $E = \ell^p(\mathbb{N})$, and $B$ to be the closure of the set of all $b = (b_{x,y})_{x,y \in X} \in \mathcal{B}(\ell^p(X; \ell^p(\mathbb{N})))$ for which there exists a finite-dimensional subspace $E_b \subseteq \ell^p(\mathbb{N})$ such that $b_{x,y} \in \mathcal{B}(E_b) \subseteq \mathcal{B}(\ell^p(\mathbb{N}))$. Clearly, $B$ is closed under block cutdowns and satisfies $C_b(X)BC_b(X) = B$. In this case, $\text{Roe}(X, B) = B^p_b(X)$, the stable \(\ell^p\)-uniform Roe algebra of $X$, introduced in [3]. Moreover, $B^p_b(X) \cong B^p_b(X) \otimes \mathcal{R}(\ell^p(\mathbb{N}))$, which explains the terminology. Again when $p > 1$, $\mathcal{K}(X, B) = \mathcal{R}(\ell^p(X \times \mathbb{N}))$ and it does not hold in general when $p = 1$.

Remark 2.18. As explained in [3], there is another version of the stable \(\ell^p\)-uniform Roe algebra of $X$, defined to be the norm closure of finite propagation operators $b = (b_{x,y})_{x,y \in X} \in \mathcal{B}(\ell^p(X; \ell^p(\mathbb{N})))$ for which there exists some $k \in \mathbb{N}$ such that $b_{x,y} \in M_k(\mathbb{C}) \subseteq \mathcal{B}(\ell^p(\mathbb{N}))$. It is clear that this algebra is isomorphic to $B^p_b(X) \otimes \overline{M}_N$ for all $p \in [1, \infty)$. As before for $p > 1$, it coincides with $B^p_b(X)$ defined in Example 2.17.

Remark 2.19. In general, we have that $B^p_b(X) \subseteq B^p_b(X) \subseteq \text{UB}^p(X) \subseteq B^p(X)$. It is worth noticing that $\text{UB}^1(X)$ is not contained in the weak version of the $L^1$-Roe algebra defined in Remark 2.15. Indeed, Example 2.12 provides a rank one operator $T \in \mathcal{B}(\ell^1(\mathbb{N}))$ which does not sit in $\overline{M}_N$. Define the diagonal operator $b \in \mathcal{B}(\ell^1(X; \ell^1(\mathbb{N})))$ by $b_{x,x} := T$ for $x \in X$, and $b_{x,y} = 0$ for $x \neq y$. Clearly, $b$ is such an example as desired.

2.4. Straight finite decomposition complexity. In this subsection, we explain the notion of straight finite decomposition complexity, which will be used in the sequel.

Straight finite decomposition complexity (sFDC) was introduced in [7] as a weak version of the original notion of finite decomposition complexity (FDC), which was introduced and studied by Guentner, Tessera and Yu in their study of topological rigidity in [12]. In general, finite asymptotic dimension implies finite decomposition complexity [13, Theorem 4.1], which consequently implies straight finite decomposition complexity [7, Proposition 2.3]. Moreover, It was also shown in [7, Theorem 3.4] that straight finite decomposition complexity does
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imply Yu’s Property A. However, it is still unknown whether (FDC), (sFDC) and Yu’s Property A are all equivalent or not.

**Definition 2.20.** Let $(X, d)$ be a proper metric space and $Z, Z' \subseteq X$. Let $\mathcal{X}, \mathcal{Y}$ be metric families of subsets in $X$, and $R \geq 0$.

- $\mathcal{X}$ is uniformly bounded, if $\sup_{X \in \mathcal{X}} \text{diam}(X) < \infty$.
- Denote the $R$-neighbourhood of $Z$ by $\mathcal{N}_R(Z) := \{z \in X : d(z, Z) \leq R \}$. Set $\mathcal{N}_R(\mathcal{X}) := \{ \mathcal{N}_R(X) : X \in \mathcal{X} \}$.
- A metric family $(Y_j)_{j \in J}$ is $R$-disjoint, if $d(Y_j, Y'_j) > R$ for all $j \neq j'$. Write $\bigsqcup_{\text{R-disjoint}} Y_j$ for their union to indicate that the family is $R$-disjoint.
- $Z$ can $R$-decompose over $\mathcal{Y}$, if $Z$ can be decomposed into $Z = X_0 \cup X_1$ and $X_i = \bigsqcup_{\text{R-disjoint}} X_{ij}, \ i = 0, 1$, such that $X_{ij} \in \mathcal{Y}$ for all $i, j$.
- $\mathcal{X}$ can $R$-decompose over $\mathcal{Y}$, denoted by $\mathcal{X} \xrightarrow{R} \mathcal{Y}$, if every $Y \in \mathcal{X}$ can $R$-decompose over $\mathcal{Y}$.
- $\mathcal{X}$ has straight finite decomposition complexity, if for any sequence $0 \leq R_1 < R_2 < \cdots$, there exists $m \in \mathbb{N}$ and metric families $\{X_i\} = X_0, X_1, \ldots, X_m$, such that $\mathcal{X}_{i-1} \xrightarrow{R_i} \mathcal{X}_i$ for $i = 1, \ldots, m$, and the family $\mathcal{X}_m$ is uniformly bounded.

### 3. The main theorem

In this section, we present our main result (Theorem 3.3), which gives several different pictures of how elements in $L^p$-Roe-like algebras may look like. We also prove the relatively easier part where straight finite decomposition complexity is not required, while leaving the rest of the proof to the next section after more technical tools are developed.

To state our main theorem, we need to introduce some notions as follows.

**Definition 3.1 ([27]).** Let $(X, d)$ be a proper metric space. A function $g \in C_b(X)$ is called a Higson function (also called a slowly oscillating function), if for every $R > 0$ and $\varepsilon > 0$, there exists a compact set $A \subseteq X$ such that for any $x, y \in X \setminus A$ with $d(x, y) < R$, then $|g(x) - g(y)| < \varepsilon$. The set of all Higson functions on $X$ is denoted by $C_h(X)$.

**Definition 3.2.** [33, Definition 2.6], Let $(X, d)$ be a metric space. A bounded sequence $(f_n)_{n \in \mathbb{N}}$ in $C_b(X)$ is called very Lipschitz, if for every $L > 0$, there exists $n_0 \in \mathbb{N}$ such that $f_n$ is $L$-Lipschitz for any $n \geq n_0$. Let $\text{VL}(X)$ denote the set of all very Lipschitz bounded sequences in $C_b(X)$. Define $\text{VL}_\infty(X) := \text{VL}(X) / \{(f_n)_{n \in \mathbb{N}} \in \text{VL}(X) : \lim_{n \to \infty} \|f_n\| = 0\}$. 
It is known from [33] that $V^\infty(X)$ is a $C^*$-subalgebra of $\ell^\infty(N, C_b(X))$ and $V^\infty(X)$ is a $C^*$-subalgebra of $(C_b(X))_\infty$. In the following, we will view both $V^\infty(X)$ and $B \subseteq \mathcal{B}(L^p(X; E))$ as Banach subalgebras of $\mathcal{B}(L^p(X; E))_\infty$, and consider the relative commutant:

$$B \cap V^\infty(X)' .$$

It is clear that any operator in $\mathcal{B}(L^p(X; E))$ with finite propagation commutes with $V^\infty(X)$. Hence, by taking limits it follows that

$$(i) \implies (ii) .$$

The converse inclusion is also true provided the space $X$ has straight finite decomposition complexity and this is included in our main theorem as follows:

**Theorem 3.3.** Let $(X, d)$ be a proper metric space equipped with a Radon measure $\mu$ whose support is $X$, and $p \in [1, +\infty)$. Suppose $E$ is a Banach space and $B \subseteq \mathcal{B}(L^p(X; E))$ is a Banach subalgebra such that $C_b(X)BC_b(X) = B$ and $B$ is closed under block cutdowns. Then for $b \in B$, the following are equivalent:

(i). $[b, f] = 0$ for all $f \in V^\infty(X)$;
(ii). $b$ is quasi-local;
(iii). $[b, g] \in K(X, B)$ for any $g \in C_b(X)$.

If $X$ has straight finite decomposition complexity, then these are also equivalent to:

(iv). $b \in \text{Roe}(X, B)$.

Recall that we have already explained in Remark 2.9 that Theorem 3.3 is a slight generalisation of [33, Theorem 2.8] as condition (2.1) is not required here. Also notice that (3.1) implies that “(iv) $\implies$ (i)” holds generally and the converse implication is also true under the extra condition of straight finite decomposition complexity.

In the remaining of this section, we prove that (i), (ii) and (iii) in Theorem 3.3 are all equivalent, and leave the implication “(i) $\implies$ (iv)” to the next section, after we develop some technical tools such as Proposition 4.1.

3.1. “(i) $\iff$ (ii)”. We start with the proof of Theorem 3.3, “(i) $\iff$ (ii)”. The implication “(i) $\implies$ (ii)” follows exactly from the same arguments in [33], while the proof of “(ii) $\implies$ (i)” is slightly different from that one given in [33] due to the absence of inner products. Fortunately, since both proofs are relativity short, we include the details for the convenience of the reader.

Let us begin with the following characterisation of the condition (i) in Theorem 3.3, which is proved in [33] when $p = 2$ and actually holds for general $p$.

**Lemma 3.4.** [33] Lemma 3.1 Let $p \in [1, \infty)$, $b \in \mathcal{B}(L^p(X; E))$ and $\varepsilon > 0$. Then $||[b, f]|| < \varepsilon$ for every $f \in V^\infty(X)$ if and only if there exists some $L > 0$ such that
\(b \in \text{Commut}(L, \epsilon)\), where

\[
\text{Commut}(L, \epsilon) := \{a \in \mathcal{B}(L^p(X; E)) : \|\{a, f\}\| < \epsilon, \text{ for any } L\text{-Lipschitz } f \in C_b(X)_1\}.
\]

**Proof of Theorem 3.3.** “(i) \(\iff\) (ii).” Assume \(b \in \mathcal{B}(L^p(X; E))\) such that \([b, \text{VL}_{\infty}(X)_1] = 0\) and let \(\epsilon > 0\). By Lemma 3.4, there exists some \(L > 0\) such that \(b \in \text{Commut}(L, \epsilon)\). For any \(f, g \in C_b(X)_1\) with \(L^{-1}\)-disjoint supports, we may choose an \(L\)-Lipschitz \(h \in C_b(X)_1\) such that \(h|_{\text{supp} f} \equiv 1\) and \(h|_{\text{supp} g} \equiv 0\). In particular, \(\|\{b, h\}\| < \epsilon\). Therefore,

\[
\|fbg\| = \|fhbg\| \leq \|\{h, b\}\| + \|fhbg\| < \epsilon + 0 = \epsilon.
\]

Hence, \(b\) is quasi-local as desired.

On the other hand, we assume that for any \(\epsilon > 0\), \(b\) has finite \(\epsilon\)-propagation. Without loss of generality, we may assume that \(b\) is a contraction. Given \(\epsilon > 0\), pick \(N\) such that \(6/N < \epsilon/2\). By the hypothesis, \(b\) has \(\epsilon/(2N^2)\)-propagation at most \(R > 0\). For any \((2RN)^{-1}\)-Lipschitz \(f \in C_b(X)_1\), we claim that \(\|\{b, f\}\| < \epsilon\). In fact, take

\[
A_1 := f^{-1}([0, \frac{1}{N}]), \quad \text{and} \quad A_i := f^{-1}((\frac{i-1}{N}, \frac{i}{N}]), \quad i = 2, \ldots, N.
\]

These sets partition \(X\), and \(A_i\) is \(2R\)-disjoint from \(A_j\) for \(|i - j| > 1\). Now choose a partition of unity \(e_1, \ldots, e_N \in C_b(X)_1\) such that \(e_i\) is supported in \(N_{R/2}(A_i)\). Thus, \(\|e_i be_j\| < \epsilon/(2N^2)\). Meanwhile, we have

\[
f \approx_{1/N} \sum_{i=1}^N \frac{i}{N} e_i.
\]

Hence, it follows that

\[
\|\{f, b\}\| \leq \frac{2}{N} + \left\| \sum_{i=1}^N \frac{i}{N} e_i, b \right\|
\]

\[
= \frac{2}{N} + \left\| \left(\sum_{i=1}^N \frac{i}{N} e_i b \right) \left(\sum_{j=1}^N e_j\right) - \left(\sum_{i=1}^N e_i\right) \left(\sum_{j=1}^N \frac{j}{N} e_j b\right) \right\|
\]

\[
\leq \frac{2}{N} + \sum_{|i-j| > 1} \|e_i be_j\| + \left\| \sum_{|i-j| \leq 1} \left(\frac{i}{N} - \frac{j}{N}\right) e_i be_j \right\|.
\]

Each term in the first sum is dominated by \(\frac{\epsilon}{2N}\), hence \(\sum_{|i-j| > 1} \|e_i be_j\| < \epsilon/2\). The second sum can be broken into four sums: note that the terms vanish when \(i = j\); what remain are \(j = i + 1\) and \(j = i - 1\), and we break each of these further into even and odd parts. By Lemma 2.4, each of these terms has norm at most \(\frac{1}{N}\). Hence, we have that

\[
\|\{f, b\}\| < \frac{2}{N} + \frac{\epsilon}{2} + \frac{4}{N} < \epsilon.
\]

So we complete the proof by Lemma 3.4. \(\square\)
3.2. \textquotedbl{(i) \Leftrightarrow (iii)}\textquotedbl{.} Now we move on to Theorem 3.3 \textquotedbl{(i) \Leftrightarrow (iii)}\textquotedbl{.} Here our major work is focused on omitting condition \textcircled{2.1}, as well as providing a \textquote{non-symmetric} version of the argument given in \textcircled{33} for \(p = 2\). However, the main body of the proof is still very similar to that of the original \(p = 2\) case \textcircled{33}, so we just outline the proof and highlight the differences we make here.

First of all, we recall that the proof of \textquote{\textcircled{(i) \Rightarrow (iii)}\textquotedbl{ given in \textcircled{33} requires condition \textcircled{2.1}}:}

\[
[C_0(X), B] \subseteq \mathcal{K}(X, B).
\]

After a careful reading of the proof, we realise that it is unnecessary to assume the entire \(B\) essentially commuting with \(C_0(X)\) but only a closed subalgebra of \(B\) as shown in the following lemma:

\textbf{Lemma 3.5.} Let \(p \in [1, \infty)\) and \(B\) be a Banach subalgebra of \(\mathcal{B}(\mathcal{L}(X; E))\) such that \(C_b(X)BC_b(X) = B\). If \(b \in B\) satisfies \([b, \mathcal{V}L_{\infty}(X)] = 0\), then \([b, C_0(X)] \subseteq \mathcal{K}(X, B)\).

\textbf{Proof.} Let \(b \in B\) such that \([b, \mathcal{V}L_{\infty}(X)] = 0\). Since \(\mathcal{K}(X, B)\) is closed, we only need to prove \([b, g] \in \mathcal{K}(X, B)\) for any \(g \in C_c(X)\).

Fix a base point \(x_0 \in X\). For each \(k \in \mathbb{N}\), we may choose a \((k^{-1})\)-Lipschitz function \(f_k \in C_0(X)\) such that \(f_k\) vanishes on \(\text{supp}(g)\) and \(f_k|_{B_R(x_0)} = 1\) for some sufficiently large \(R_k > 0\). Hence, the sequence \((f_k)_{k \in \mathbb{N}} \in \mathcal{V}L_{\infty}(X)\) and \(\|g[f_k]\| \to 0\) for \(k \to \infty\) by assumption. Since \(g f_k = 0\) for any \(k \in \mathbb{N}\), it follows that

\[
\|bf_k\| = \|gbf_k\| = \|gbf_k\| = \|g[f_k]\| \leq \|g\| \cdot \|f_k\| \to 0,
\]

as \(k \to \infty\). Similarly, we have that \(\|f_k|_{V}(b, g)\| \to 0\) and \(\|f_k[b, g]f_k\| \to 0\) as \(k \to \infty\).

Moreover, we have that

\[
\|bf_k\| = (1 - f_k)[b, g](1 - f_k) \leq \|bf_k\| f_k + \|f_k[b, g]\| + \|f_k[b, g]f_k\| \to 0,
\]

as \(k \to \infty\). Since \(\text{supp}(1 - f_k) \subseteq B_{R_k}(x_0)\) and \(C_b(X)BC_b(X) = B\), it follows that \((1 - f_k)[b, g](1 - f_k) \in C_c(X)BC_c(X)\). Hence, \([b, g] \in \mathcal{K}(X, B)\). \(\square\)

Replacing condition \textcircled{2.1} by Lemma 3.5 in the original proof for \(p = 2\) \textcircled{33}, we obtain a proof of Theorem 3.3 \textquote{\textcircled{(i) \Rightarrow (iii)}\textquotedbl{ without any further changes. Hence we omit the details.

Now we outline the proof for the other direction, \textquote{\textcircled{(iii) \Rightarrow (i)}\textquotedbl{. Since \(\mathcal{L}_{\infty}\)-Roe-like algebras may not possess a bounded involuton in general, the proof becomes slightly different as explained below.

\textit{Sketch of the proof of Theorem 3.3} \textquote{\textcircled{(iii) \Rightarrow (i)}\textquotedbl{.} Fix a point \(x_0 \in X\) and we set \(B_R := B_R(x_0)\) for \(R > 0\). Let \(b \in B_1\) such that \([b, g] \in \mathcal{K}(X, B)\) for any \(g \in C_b(X)\). We assume that there exists some \(f = (f_k)_{k=1}^\infty \in \mathcal{V}L_{\infty}(X)\) such that \([b, f] \neq 0\). Take any \(\varepsilon\) such that \(0 < \varepsilon < \|bf\|\). There are only two cases:

\textit{Case I.} There exists \(R_0 > 0\) such that for all \(S > 0\), there exist infinitely many \(k \in \mathbb{N}\) for which

either \(\|\chi_{B_{R_0}}[b, f_k](1 - \chi_{B_S})\| > \frac{\varepsilon}{5}\) or \(\|(1 - \chi_{B_S})[b, f_k]\chi_{B_{R_0}}\| > \frac{\varepsilon}{5}\).
In other words, there exists $R_0 > 0$ with the following property:

1) either there exists a sequence $S_1 < S_2 < \ldots$ tending to $\infty$ such that for any $n \in \mathbb{N}$, there exist infinitely many $k \in \mathbb{N}$ such that $\|\chi_{B_{R_0}}[b, f_k](1 - \chi_{B_n})\| > \frac{\varepsilon}{5};$

2) or there exists a sequence $S_1 < S_2 < \ldots$ tending to $\infty$ such that for any $n \in \mathbb{N}$, there exist infinitely many $k \in \mathbb{N}$ such that $\|(1 - \chi_{B_n})[b, f_k]\chi_{B_{R_0}}\| > \frac{\varepsilon}{5}.$

Case II. For every $R > 0$, there exists $S > 0$ such that, for all but finitely many $k \in \mathbb{N}$, we have

$$\|\chi_{B_R}[b, f_k](1 - \chi_{B_S})\| \leq \frac{\varepsilon}{5} \quad \text{and} \quad \|(1 - \chi_{B_S})[b, f_k]\chi_{B_R}\| \leq \frac{\varepsilon}{5}.$$ 

The rest of the proof is almost identical to the original one for $p = 2$ given in [33], hence omitted to avoid too much word repetition.

\[\blacksquare\]

4. Proof of “(i) ⇔ (iv)”

In this section, we will prove the remaining case of “(i) ⇔ (iv)” in Theorem 3.3. Recall that as explained in Section 3, “(iv) ⇒ (i)” holds in general. So we will only focus on the opposite implication “(i) ⇒ (iv)”. 

A key ingredient to prove “(i) ⇒ (iv)” is to approximate a bounded operator via its block cutdowns as indicated in [33] Corollary 4.3 for the case of $p = 2$. In fact, the identical proof of [33] Corollary 4.3 works for any $p \in (1, \infty)$ but not for $p = 1$ due to the lack of reflexivity on $L^1$-spaces. Hence we need to search for a substitution of [33] Corollary 4.3, and we figure out the following crucial result, which might be of independent interest to experts in Banach space theory.

**Proposition 4.1.** Let $(X, d)$ be a proper metric space equipped with a Radon measure $\mu$ whose support is $X$ and $p \in [1, +\infty).$ Suppose $E$ is a Banach space, $a \in \mathcal{B}(L^p(X; E))$ and $a \in \text{Commut}(L, \varepsilon)$ for some $L, \varepsilon > 0.$ Let $(e_j)_{j \in J}$ be an equicontinuous family of positive contractions in $C_b(X)$ with $2/L$-disjoint supports, and define $e := \sum_{j \in J} e_j.$ Then, we have

$$\|eae - \sum_{j \in J} e_je_j\| \leq \varepsilon.$$ 

The proof of the above proposition is technical and relatively long, so we decide to postpone it to Section 4.1 for the convenience of the reader, and first show how to use the proposition to prove “(i) ⇒ (iv)”. Let us start with the following lemma, which is a consequence of Proposition 4.1 by the same proof of [33] Lemma 4.5.

**Lemma 4.2.** Let $\mathcal{Y}$ be a metric family of $X$ such that $\{X\} \xrightarrow{4L^{-1}+4} \mathcal{Y}$ for some $L > 0$, and $a \in \mathcal{B}(L^p(X; E)).$ Let $\varepsilon > 0$ be such that $a \in \text{Commut}(L, \varepsilon).$ Then we can write

$$a \approx_{8\varepsilon} a_{00} + a_{01} + a_{10} + a_{11},$$

where each $a_{ij}$ is of the form $\theta_{(g, g^\prime)(\text{ag})}$ for some $g, g^\prime \in \mathcal{C}_b(X)_1$ and some 1-Lipschitz positive $f_k \in \mathcal{C}_b(X)_1$ with disjoint supports such that each $\text{supp} f_k$ is contained in a set in $\mathcal{N}_{L^{-1}+1}(\mathcal{Y}).$
It may be worth reminding the reader that for any \( L, \varepsilon > 0 \), we denote
\
Commut(L, \varepsilon) = \{ a \in \mathcal{B}(L^p(X; E)) : ||[a, f]|| < \varepsilon, \text{ for any } L\text{-Lipschitz } f \in C_b(X)_1 \}. 
\
The next lemma follows from the previous one by an induction argument (see the proof of [33, Lemma 4.6] for more details).

**Lemma 4.3.** Let \( X \) and \( Y \) be metric families of \( X \) such that \( X \overset{4L^{-1}+4}{\twoheadrightarrow} Y \) for some \( L > 0 \), and \( a \in \mathcal{B}(L^p(X; E)) \) be block diagonal with respect to \( X \) for \( p \in [1, \infty) \). Let \( \varepsilon > 0 \) be such that \( a \in \text{Commut}(L, \varepsilon) \). Then we can write:
\[
(4.1) \quad a \approx_{8\varepsilon} a_{00} + a_{01} + a_{10} + a_{11},
\]
where each \( a_{ij} \) is of the form \( \theta(f_i) \kappa_{(g, g')} \) for some \( g, g' \in C_b(X)_1 \) and some equicontinuous positive family \( (f_i)_{i \in K} \) in \( C_b(X)_1 \) with disjoint supports, such that each \( \text{supp}(f_i) \) is contained in some set in \( \mathcal{N}_{L^{-1}+1}(Y) \). In particular:

(i) each \( a_{ii} \) is block diagonal with respect to \( N_{L^{-1}+1}(Y) \),
(ii) if \( a \in \text{Commut}(L', \varepsilon') \) for some \( L', \varepsilon' > 0 \), then each \( a_{ii} \) is in \( \text{Commut}(L', \varepsilon') \) as well, and
(iii) if \( B \subseteq \mathcal{B}(L^p(X; E)) \) is a Banach subalgebra such that \( C_b(X)_1 BC_b(X) = B \) and \( B \) is closed under block cutdowns, and if \( a \in B \), then each \( a_{ii} \) is in \( B \) as well.

**Proof of Theorem 3.3 “(i) \Rightarrow (iv)”**. Although the proof is exactly the same as the one given in [33], we decide to include it here for the completeness and show the reader how straight finite decomposition complexity is used in the proof.

Take \( b \in B \) such that it commutes with all \( f \in VL_\infty(X) \). Given \( \varepsilon > 0 \), we aim to construct a finite propagation operator in \( B \), which is \( \varepsilon \)-close to \( b \). It follows from Lemma 3.4 that for every
\[
\varepsilon_n := \varepsilon / (2 \cdot 8^n),
\]
there exists some \( L_n > 0 \) such that \( b \in \text{Commut}(L_n, \varepsilon_n) \). Set
\[
R_n := 4(L_n^{-1} + 1) + 2(L_{n-1}^{-1} + 1) + \cdots + 2(L_1^{-1} + 1).
\]
Since \( X \) has straight finite decomposition complexity, there exist metric families
\( X_0 = \{ X \}, X_1, \ldots, X_m \) such that \( X_{n-1} \overset{R_n}{\to} X_n \) for \( n \in \{1, \ldots, m\} \) and \( X_m \) is uniformly bounded. An elementary observation shows that
\[
(4.2) \quad \mathcal{N}_{(L_1^{-1}+1)\cdots(L_m^{-1}+1)}(X_m) \overset{4(L_1^{-1}+1)+\cdots+4(L_m^{-1}+1)}{\to} \mathcal{N}_{(L_1^{-1}+1)\cdots(L_m^{-1}+1)}(X_n).
\]
Thus, we can apply Lemma 4.3 inductively with \( L_n, \varepsilon_n \), the operators obtained in the previous iteration, and metric families in (4.2). After \( m \) steps, we approximate \( b \) by an operator \( b' \) which is a sum of \( 4^m \) operators in \( B \), each of which is block diagonal with respect to the bounded family \( \mathcal{N}_{(L_1^{-1}+1)\cdots(L_m^{-1}+1)}(X_m) \). Hence, operators which are block diagonal with respect to it clearly have finite propagation. Consequently, \( b' \in \text{Roe}(X, B) \). Finally, the distance between \( b \) and \( b' \) is at most
\[
8\varepsilon + 4(8\varepsilon_2 + 4(8\varepsilon_3 + 4(\ldots))) = \varepsilon
d\]
by Lemma 4.3. So we finish the proof. \( \Box \)
4.1. Approximation via block cutdowns. Finally, we complete the proof of Proposition 4.1 as promised before.

The main difficulty is the lack of reflexivity of the $L^p$-Bochner space $L^p(X; E)$ for general $p$ and general Banach space $E$ (see e.g. [2, 5, 6]), which impedes us from applying the original proof in [33] directly. Instead, we establish some substituting results in functional analysis and state them in the context of general Banach spaces, which conceivably would be of independent interests.

In the rest of this subsection, suppose $X$ is a Banach space and $\hat{X}$ is a closed subspace of the dual space $X^*$, which separates points in $X$ (i.e., for any nonzero $\xi \in X$, there exists some $\eta \in \hat{X}$ such that $\xi(\eta) \neq 0$). The inclusion $i : \hat{X} \hookrightarrow X^*$ induces a surjective adjoint map $i^* : X^{\ast\ast} \rightarrow \hat{X}^*$. Composing it with the canonical map from $X$ into its double dual $X^{\ast\ast}$, we obtain the following map

$$\tau : \hat{X} \rightarrow \hat{X}^*.$$  

It is clear that $\tau$ is injective, as $\hat{X}$ separates points in $X$.

For any $\theta \in \hat{X}^*$ and $\eta \in \hat{X}$, we use the notation $\langle \theta, \eta \rangle$ for $\theta(\eta)$. Consider the Banach space $\mathfrak{B}(X, \hat{X}^*)$ of all bounded operators from $X$ to $\hat{X}^*$, equipped with the weak* operator topology (W*OT) defined as follows: a net $\{T_\alpha\}$ converges to $T$ in $\mathfrak{B}(X, \hat{X}^*)$ if and only if for any $\xi \in X$ and any $\eta \in \hat{X}$, we have

$$\langle T_\alpha(\xi), \eta \rangle \rightarrow \langle T(\xi), \eta \rangle.$$  

The strong* topology with respect to $\hat{X}$ on $\mathfrak{B}(X)$ is defined as follows: a net $\{T_\alpha\}$ converges to $T$ in $\mathfrak{B}(X)$ if and only if for any $\xi \in X$ and any $\eta \in \hat{X}$, we have

$$\|T_\alpha(\xi) - T(\xi)\| \rightarrow 0 \quad \text{and} \quad \|T_\alpha(\eta) - T(\eta)\| \rightarrow 0.$$  

We say that $\hat{X}$ is $a^*$-invariant for $a \in \mathfrak{B}(X)$ if $a^*(\hat{X}) \subseteq \hat{X}$. In this case, the restriction $a^*|_{\hat{X}}$ belongs to $\mathfrak{B}(\hat{X})$. Hence, its adjoint $(a^*|_{\hat{X}})^* \in \mathfrak{B}(\hat{X}^*)$ as well. In order to simplify notations, we write $a^{(a^*)}$ instead of $(a^*|_{\hat{X}})^*$. Clearly, for any $\zeta \in \hat{X}^*$ and $\eta \in \hat{X}$ we have:

$$\langle a^{(a^*)}, \eta \rangle = \langle \zeta, a^* \eta \rangle.$$  

Moreover, it is easy to check that if $\hat{X}$ is $a^*$-invariant for some $a \in \mathfrak{B}(X)$, then

$$a^{(a^*)} \tau = \tau a.$$  

In other words, the following diagram commutes:

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{\tau} & \hat{X}^* \\
\downarrow^a & & \downarrow^{a^{(a^*)}} \\
\hat{X} & \xrightarrow{\tau} & \hat{X}^*. 
\end{array}$$

We say that $\hat{X}$ is $\mathcal{A}$-invariant for a subset $\mathcal{A} \subseteq \mathfrak{B}(X)$ if $\hat{X}$ is $a^*$-invariant for all $a \in \mathcal{A}$. If $G$ is a subgroup of invertible isometries in $\mathfrak{B}(X)$ and $\hat{X}$ is $G^*$-invariant,

\[\text{Note: In [33] Spakula and Tikuisis considered the weak operator topology (WOT) instead. However, (WOT) and (W*OT) agree when } \hat{X} \cong X \text{ and taking } \hat{X} := \hat{X}^*.\]
then \( u'(\hat{x}) = \hat{x} \) for all \( u \in G \). It is clear that if \( u \) is an invertible isometry, then so are \( u' \) and \( u^{(\ast)} \). Moreover, \( (u')^{-1} = (u^{-1})' \) and \( (u^{(\ast)})^{-1} = (u^{-1})^{(\ast)} \), which are denoted by \( u^{-*} \) and \( u^{(-\ast)} \), respectively. Now suppose \( (u_a) \) and \( u \) are invertible isometries in \( G \) and \( u_a \to u \) in the strong* topology with respect to \( \hat{x} \), then \( \| u_a^{-*} \eta - u^{-*} \eta \| \to 0 \) for all \( \eta \in \hat{x} \). Indeed, we have

\[
\| u_a^{-*} \eta - u^{-*} \eta \| = \| \eta - u_a^{*} u^{-*} \eta \| = \| u'(u^{-*} \eta) - u_{a}^{(\ast)}(u^{-*} \eta) \| \to 0.
\]

We have the following technical lemma, which generalises [33, Lemma 4.1].

**Lemma 4.4.** Suppose \( G \) is an abelian subgroup of the group of invertible isometries in \( \mathfrak{B}(\hat{x}) \), which is compact in the strong* topology with respect to \( \hat{x} \). Suppose \( \hat{x} \) is \( G^{-}\)-invariant, and define \( \mathcal{G} := \{ a \in \mathfrak{B}(\hat{x}, \hat{x}') : au = u^{(\ast)}a, \forall u \in G \} \).

Then there exists a unique idempotent linear contraction \( \mathcal{E}_G : \mathfrak{B}(\hat{x}, \hat{x}') \to \mathcal{G} \) with the following properties:

1) For any \( b_1, b_2 \in G \) and \( a \in \mathfrak{B}(\hat{x}, \hat{x}') \), \( \mathcal{E}_G(b_1^{(\ast)} b_2) = b_1^{(\ast)} \mathcal{E}_G(a) b_2 \).
2) The restriction of \( \mathcal{E}_G \) to the unit ball of \( \mathfrak{B}(\hat{x}, \hat{x}') \) is \( (W^*\text{OT}) \)-continuous.

In this case, for any \( a \in \mathfrak{B}(\hat{x}, \hat{x}') \), we have that

\[
(4.5) \quad \| \mathcal{E}_G(a) - a \| \leq \sup_{u \in G} \| au - u^{(\ast)}a \|.
\]

**Proof.** Since \( G \) is compact with respect to the strong* topology, we consider the normalised Haar measure \( \mu_G \) on \( G \). Fix \( a \in \mathfrak{B}(\hat{x}, \hat{x}') \), the map

\[
(G, \text{strong* topology}) \to (\mathfrak{B}(\hat{x}, \hat{x}'), W^*\text{OT})
\]

defined by \( u \mapsto u^{(-\ast)}au \) is clearly continuous. For each \( \xi \in \hat{x} \) and each \( a \in \mathfrak{B}(\hat{x}, \hat{x}') \), we may consider the following functional on \( \hat{x} \):

\[
\phi_{\xi, a} : \eta \mapsto \int_G \langle u^{(-\ast)} au \xi, \eta \rangle d\mu_G(u),
\]

whose norm is bounded by \( \| a \| \cdot \| \xi \| \). Therefore, we obtain a linear contraction \( \mathcal{E}_G : \mathfrak{B}(\hat{x}, \hat{x}') \to \mathfrak{B}(\hat{x}, \hat{x}') \) given by \( \mathcal{E}_G(a)(\xi) = \phi_{\xi, a} \) where \( \xi \in \hat{x} \) and \( a \in \mathfrak{B}(\hat{x}, \hat{x}') \).

It remains to check that \( \mathcal{E}_G \) satisfies the required properties. First of all, we show that \( \mathcal{E}_G \) has image in \( \mathcal{G} \). More precisely, \( \mathcal{E}_G(a)v = v^{(\ast)} \mathcal{E}_G(a) \) for any \( a \in \mathfrak{B}(\hat{x}, \hat{x}') \) and any \( v \in G \). Given \( \xi \in \hat{x} \) and \( \eta \in \hat{x} \), it follows from the right-invariance of the Haar measure \( \mu_G \) that

\[
\langle \mathcal{E}_G(a)v \xi, \eta \rangle = \int_G \langle u^{(-\ast)} au v \xi, \eta \rangle d\mu_G(u)
\]
\[
= \int_G \langle v^{(\ast)} u^{(-\ast)} au \xi, \eta \rangle d\mu_G(u)
\]
\[
= \int_G \langle u^{(-\ast)} au \xi, v^* \eta \rangle d\mu_G(u)
\]
\[
= \langle \mathcal{E}_G(a) \xi, v^* \eta \rangle
\]
\[
= \langle v^{(\ast)} \mathcal{E}_G(a) \xi, \eta \rangle.
\]
Hence, it follows that $E_G(a)v = v^{(w)}E_G(a)$.

Given $a \in \mathcal{B}(\hat{\mathcal{X}}, \hat{\mathcal{X}})$, $\xi \in \hat{\mathcal{X}}$ and $\eta \in \hat{\mathcal{X}}$, we have

$$\left| \langle (E_G(a) - a)\xi, \eta \rangle \right| \leq \int_G \|u^{(w)}au - a\| \cdot \|\xi\| \cdot \|\eta\| d\mu_G(u)$$

$$= \int_G \|au - u^{(w)}a\| \cdot \|\xi\| \cdot \|\eta\| d\mu_G(u)$$

$$\leq \left( \sup_{u \in G} \|au - u^{(w)}a\| \right) \cdot \|\xi\| \cdot \|\eta\|.$$ 

Hence, $(4.5)$ holds. In particular, $E_G(a) = a$ for any $a \in \mathcal{G}'$, which implies that $E_G : \mathcal{B}(\hat{\mathcal{X}}, \hat{\mathcal{X}}) \to \mathcal{G}'$ is an idempotent.

Now let us check that $E_G(b_1^{(w)}ab_2) = b_1^{(w)}E_G(a)b_2$ for any $b_1, b_2 \in G$ and $a \in \mathcal{B}(\hat{\mathcal{X}}, \hat{\mathcal{X}})$. Since $G$ is abelian, we have

$$\langle E_G(b_1^{(w)}ab_2)\xi, \eta \rangle = \int_G \langle u^{(w)}b_1^{(w)}ab_2u\xi, \eta \rangle d\mu_G(u)$$

$$= \int_G \langle b_1^{(w)}u^{(w)}ab_2u\xi, \eta \rangle d\mu_G(u)$$

$$= \int_G \langle u^{(w)}ab_2u\xi, b_1^{(w)}u\eta \rangle d\mu_G(u)$$

$$= \langle E_G(a)b_2\xi, b_1^{(w)}u\eta \rangle$$

$$= \langle b_1^{(w)}E_G(a)b_2\xi, \eta \rangle,$$

for any $\xi \in \hat{\mathcal{X}}$ and any $\eta \in \hat{\mathcal{X}}$. Hence, $E_G(b_1^{(w)}ab_2) = b_1^{(w)}E_G(a)b_2$.

In order to prove the $(W^{*}\text{OT})$-continuity of the restriction of $E_G$ to the unit ball of $\mathcal{B}(\hat{\mathcal{X}}, \hat{\mathcal{X}})$, we have to approximate the integration by finite Riemann sums uniformly in the weak$^*$ operator topology:

Indeed, fix $\xi \in \hat{\mathcal{X}}$, $\eta \in \hat{\mathcal{X}}$ and $u \in G$ and for any $\epsilon > 0$, there exists an open neighbourhood $V_u$ of $u$ in the strong$^*$ topology such that for all $v \in V_u$ and all $a \in \mathcal{B}(\hat{\mathcal{X}}, \hat{\mathcal{X}})$, we have

$$|\langle v^{(w)}au\xi, \eta \rangle - \langle u^{(w)}au\xi, \eta \rangle| < \epsilon.$$ 

Since $\{V_u : u \in G\}$ forms an open cover of $G$ and $G$ is compact in the strong$^*$ topology, there exists a finite subcover $\{V_{u_1}, \ldots, V_{u_n}\}$ of $G$. Let $W_1 = V_{u_1}$ and we put $W_k = V_{u_k} \setminus \bigcup_{i=1}^{k-1} W_i$ for $1 < k \leq n$. Without loss of generality, we may assume that $\{W_k\}_{k=1}^n$ forms a non-empty Borel partition of $G$. Take an arbitrary point $w_k$ in each $W_k$ for $k = 1, \ldots, n$. Then for any $a \in \mathcal{B}(\hat{\mathcal{X}}, \hat{\mathcal{X}})$ and $u \in W_k$, we have that

$$|\langle u^{(w)}au\xi, \eta \rangle - \langle w_k^{(w)}aw_k\xi, \eta \rangle| < 2\epsilon.$$
In particular, we have that

\[
\begin{align*}
|\langle E_G(a)\xi, \eta \rangle - \sum_{k=1}^{n} \langle w_k^{-p}a\omega_k\xi, \eta \rangle \mu_G(W_k) | \\
= \sum_{k=1}^{n} \int_{W_k} |\langle u^{-p}au\xi, \eta \rangle \mu_G(u) - \sum_{k=1}^{n} \int_{W_k} |\langle w_k^{-p}a\omega_k\xi, \eta \rangle \mu_G(u) | \\
\leq \sum_{k=1}^{n} \int_{W_k} |\langle u^{-p}au\xi, \eta \rangle - \langle w_k^{-p}a\omega_k\xi, \eta \rangle | \mu_G(u) \\
\leq \sum_{k=1}^{n} \int_{W_k} 2\varepsilon d\mu_G(u) = 2\varepsilon,
\end{align*}
\]

for all \(a \in \mathcal{B}(X, \hat{X}^*)\). Since the map \(a \mapsto \sum_{k=1}^{n} \mu_G(W_k)w_k^{-p}a\omega_k\) is continuous in the weak* operator topology, it is not hard to see that the restriction of \(E_G\) to the unit ball of \(\mathcal{B}(X, \hat{X}^*)\) is \((W^{*}\text{OT})\)-continuous as well.

Finally, we check the uniqueness of \(E_G\). If we have another \(E : \mathcal{B}(X, \hat{X}^*) \to \mathcal{L}'\) satisfying all the conditions in the lemma, then:

\[
\begin{align*}
E_G(a) &= E(E_G(a)) \quad (E \text{ fixes } \mathcal{L}') \\
&= E\left(\int_{\mathcal{L}} u^{-p}au d\mu_G(u)\right) \quad (E \text{ fixes } \mathcal{L}') \\
&= \int_{\mathcal{L}} E(u^{-p}au) d\mu_G(u) \quad (W^{*}\text{OT}-continuity on the unit ball) \\
&= \int_{\mathcal{L}} u^{-p}E(a) d\mu_G(u) \quad (\text{Property 1}) \\
&= E_G(E(a)) \\
&= E(a) \quad (E_G \text{ fixes } \mathcal{L}')
\end{align*}
\]

for all \(a \in \mathcal{B}(X, \hat{X}^*)\). Thus, \(E_G = E\) and we complete the proof. \(\Box\)

Now let us return to the setting of Proposition 4.1. Let \((X, d)\) be a proper metric space equipped with a Radon measure \(\mu\) whose support is \(X\). Let \(q\) be the conjugate exponent to \(p\) when \(p \in (1, +\infty)\), and \(q = 0\) when \(p = 1\). Suppose \(E\) is a Banach space and \((e_j)_{j \in J}\) is an equicontinuous family of positive contractions in \(C_b(X)\) with uniformly disjoint supports.

In order to apply Lemma 4.4, we put \(X = L^p(X; E)\) and \(\hat{X} = L^q(X; E^*)\). Clearly, \(\hat{X}\) is a closed subspace of the dual space \(X^*\), and separates points in \(\hat{X}\) by Lemma 2.2. For each \(j \in J\), set \(A_j = \text{supp}(e_j)\) and \(B = X \setminus \left( \bigcup_{j \in J} A_j \right)\). We consider \(p_j\) and \(q_c\) in \(\mathcal{B}(L^p(X; E))\) given by \(p_j(\xi) = \chi_{A_j} \xi\) and \(q_c(\xi) = \chi_B \xi\) for \(\xi \in L^p(X; E)\). We define that

\[
G = \left\{ \sum_{j \in J} (-1)^{\beta}p_j + (-1)^{\beta}q_c : (\alpha_j)_{j \in J} \subseteq (\mathbb{Z}/2)^J, \beta \in \mathbb{Z}/2 \right\},
\]

(4.6)
where the sum converges in (SOT) and each element in G can be presented by a function of the form $\sum_{j \in I} (-1)^{\alpha_j} \chi_A + (-1)^{\beta} \chi_B$ (in the pointwise convergence) via the faithful multiplication representation $\rho : L^\infty(X) \to \mathcal{B}(L^p(X; E))$.

Since $g^2 = id$ for all $g \in G$, G becomes a subgroup of the invertible isometry group in $\mathcal{B}(L^p(X; E))$, and clearly G is abelian. Also notice that $\hat{X}$ is $L^\infty(X)$-invariant as for any $f \in L^\infty(X) \subseteq \mathcal{B}(L^p(X; E))$ and $\eta \in \hat{X}$, we have that $f^*(\eta) = f \cdot \eta$ by pointwise multiplications as functions on $X$.

Consequently, $\hat{X}$ is $G$-invariant since $G \subseteq \rho(L^\infty(X))$. Moreover, the strong* topology on G with respect to $\hat{X}$ is compact, as it is homeomorphic to the product topology on $(\mathbb{Z}/2)^{|\beta|}$.

The next lemma is a replacement of [33, Corollary 4.2], where Špakula and Tikuisis work within the setting of von Neumann algebras. Instead, we provide a direct and concrete proof here as follows:

**Lemma 4.5.** As above, the group $G$ is defined as in (4.6) and $q$ is the conjugate exponent to $p$ when $p \in (1, \infty)$, and $q = 0$ when $p = 1$. Let $\hat{X} = L^p(X; E)$ and $\hat{X} = L^q(X; E')$. If $\mathcal{A} = \{a \in \mathcal{B}(\hat{X}, \hat{X}') : au = u^{(\alpha)}a, \forall u \in G\}$, then there exists a $(W*OT)$-continuous idempotent linear contraction $E : \mathcal{B}(\hat{X}, \hat{X}') \to \mathcal{A}$ given by the formula

$$E(x) = \sum_{j \in I} p_j^{(\alpha)} x p_j + q_j^{(\alpha)} x q_j,$$

where the sum converges in (SOT). Moreover, $E(b_1^{(\alpha)}ab_2) = b_1^{(\alpha)}E(a)b_2$ for any $b_1, b_2 \in G$ and $a \in \mathcal{B}(\hat{X}, \hat{X}')$. Consequently, we have that

$$\|E(a) - a\| \leq \sup_{u \in G} \|au - u^{(\alpha)}a\|, \quad \text{for any } a \in \mathcal{B}(\hat{X}, \hat{X}').$$

**Proof.** It is clear that $E$ is a $(W*OT)$-continuous linear map on $\mathcal{B}(\hat{X}, \hat{X}')$ and the sum defining $E$ converges in (SOT), so we leave the details to the readers.

Let us first verify that $E$ is a contraction. When $p = 1$, we have

$$\|E(x)\| \leq \sum_{j \in I} \|xp_j\|_1 + \|xq_j\|_1 \leq \|x\| \cdot \left( \sum_{j \in I} \|\chi_A\|_1 + \|\chi_B\|_1 \right) = \|x\| \cdot \|\xi\|_1,$$

for any $\xi \in L^1(X; E)$ by Lemma 2.2. It implies that $E$ is a contraction in this case.

When $p > 1$, it follows from Hölder’s inequality that

$$\left| \sum_{j \in I} p_j^{(\alpha)} xp_j \xi + q_j^{(\alpha)} xq_j \xi, \eta \right| \leq \sum_{j \in I} \|xp_j \xi\|_p \cdot \|\eta\|_q + \|xq_j \xi\|_p \cdot \|\eta\|_q \leq \|x\| \cdot \left( \sum_{j \in I} \|p_j \xi\|_p \cdot \|\eta\|_q + \|q_j \xi\|_p \cdot \|\eta\|_q \right) \leq \|x\| \cdot \left( \sum_{j \in I} \|p_j \xi\|_p \cdot \|\eta\|_q + \|q_j \xi\|_p \cdot \|\eta\|_q \right)^{\frac{1}{2}} \cdot \left( \sum_{j \in I} \|p_j \xi\|_p \cdot \|\eta\|_q \right)^{\frac{1}{2}} = \|x\| \cdot \|\xi\|_p \cdot \|\eta\|_q,$$

\(^{5}\)It is worth noting that $C_0(X, E')$ is not $L^p(X)$-invariant and this is the reason why we use $L^p(X; E')$ instead of $C_0(X, E')$ when $p = 1$.

\(^{6}\)However, it is false for $L^\infty(X; E')$ and this is the reason why we use $L^0(X; E')$ instead of $L^\infty(X; E')$ when $p = 1$. 


for any $\xi \in L'(X; E)$ and $\eta \in L'(X; E')$. This implies that
\[
\|\mathcal{E}(x)\xi\| = \left\| \sum_{j=1}^{n} p_j^{(x)} x p_j x \xi + q_c^{(x)} x q_c x \xi \right\| \leq \|x\| \cdot \|\xi\|_p
\]
by Lemma 2.2. Hence, $\mathcal{E}$ is a contraction in this case as well.

Now we show that the image of $\mathcal{E}$ sits inside $\mathfrak{g}'$. Indeed, given any $x \in \mathfrak{B}(\mathfrak{x}, \hat{\mathfrak{x}})$, any $u = \sum_{j=1}^{n} (-1)^{a_j} p_j + (-1)^{b_j} q_c \in G$, $\xi \in \mathfrak{x}$ and $\eta \in \hat{\mathfrak{x}}$, we have that
\[
\langle \mathcal{E}(x) u \xi, \eta \rangle = \left\langle \left( \sum_{j=1}^{n} p_j^{(x)} x p_j x \xi + q_c^{(x)} x q_c x \xi \right), u^{\dagger} \eta \right\rangle
\]
\[
= \sum_{j=1}^{n} \langle x p_j x \xi, p_j^* u \eta \rangle + \langle x q_c x \xi, q_c^* u \eta \rangle
\]
\[
= \sum_{j=1}^{n} (-1)^{a_j} \langle x p_j x \xi, p_j^* \eta \rangle + (-1)^{b_j} \langle x q_c x \xi, q_c^* \eta \rangle.
\]

On the other hand,
\[
\langle u^{(x)} \mathcal{E}(x) \xi, \eta \rangle = \left\langle \left( \sum_{j=1}^{n} p_j^{(x)} x p_j x \xi + q_c^{(x)} x q_c x \xi \right), u^{\dagger} \eta \right\rangle
\]
\[
= \sum_{j=1}^{n} \langle x p_j x \xi, p_j^* u \eta \rangle + \langle x q_c x \xi, q_c^* u \eta \rangle
\]
\[
= \sum_{j=1}^{n} (-1)^{a_j} \langle x p_j x \xi, p_j^* \eta \rangle + (-1)^{b_j} \langle x q_c x \xi, q_c^* \eta \rangle.
\]

Hence, $\mathcal{E}(x) u = u^{(x)} \mathcal{E}(x)$ for all $u \in G$.

Next, we show that $\mathcal{E}(x) = x$ for all $x \in \mathfrak{g}'$. In other words, $\mathcal{E}$ is an idempotent onto $\mathfrak{g}'$. Fix an $x \in \mathfrak{g}'$ and for any $u = \sum_{j=1}^{n} (-1)^{a_j} p_j + (-1)^{b_j} q_c \in G$, we have that
\[
p_j^{(x)} x p_i = (-1)^{a_j} p_j^{(x)} x u p_i = (-1)^{a_j} (p_j^{(x)} u^{(x)} x p_i = (-1)^{a_j + a_i} p_j^{(x)} x p_i.
\]

It follows that $p_j^{(x)} x p_i = 0$ for any $i \neq j$. Similarly, $p_j^{(x)} x q_c = q_c^{(x)} x p_j = 0$ for any $j \in J$. Therefore, we have that
\[
x = \left( \sum_{i=1}^{n} p_i + q_c \right)^{(x)} \left( \sum_{j=1}^{n} p_j + q_c \right) = \sum_{j=1}^{n} p_j^{(x)} x p_j x + q_c^{(x)} x q_c x = \mathcal{E}(x) \text{ for all } x \in \mathfrak{g}'.
\]

Moreover, for any $b_1, b_2 \in G$ and any $x \in \mathfrak{B}(\mathfrak{x}, \hat{\mathfrak{x}})$ we have that
\[
\mathcal{E}(b_1^{(x)} x b_2) = \sum_{j=1}^{n} p_j^{(x)} b_1^{(x)} x b_2 p_j + q_c^{(x)} b_1^{(x)} x b_2 q_c
\]
\[
= \sum_{j=1}^{n} b_1^{(x)} p_j^{(x)} x p_j x b_2 + b_1^{(x)} q_c^{(x)} x q_c x b_2
\]
\[
= b_1^{(x)} \mathcal{E}(x) b_2,
\]

where we use the fact that $b_1 p_j = p_j b_1$ and $b_1 q_c = q_c b_1$ for any $j \in J$ and $k \in \{1, 2\}$.

The final conclusion follows from the uniqueness of $\mathcal{E}$ in Lemma 4.4 and (4.5) therein. So we finish the proof. \qed
Proof of Proposition 4.7. Let the group $G$ be defined as in (4.6), and the map $\mathcal{E} : \mathcal{B}(L^p(X; E), L^q(X; E')) \to \mathcal{E}'$ be the idempotent defined in Lemma 4.5. Recall that by Lemma 2.2, the map $\tau : L^p(X; E) \to L^q(X; E')'$ defined in (1.3) is an isometric embedding, hence it induces the following isometric embedding

$$
i : \mathcal{B}(L^p(X; E)) = \mathcal{B}(L^p(X; E), L^q(X; E)) \hookrightarrow \mathcal{B}(L^p(X; E), L^q(X; E')').$$

In other words, $\iota(a) = \tau \circ a$ for any $a \in \mathcal{B}(L^p(X; E))$.

Now we define another map $\mathcal{E}' : \mathcal{B}(L^p(X; E)) \to \mathcal{B}(L^p(X; E))$ by the formula

$$\mathcal{E}'(z) = \sum_{j \in I} p_j \tau q_j + q_c z q_c$$

for $z \in \mathcal{B}(L^p(X; E))$ and the sum converges in (SOT) by Lemma 2.4. It follows easily from Equation (4.4) that the sum converges in (SOT) by Lemma 2.4.

Furthermore, we have that

$$\|\mathcal{E}'(z) - z\| = \|\iota(\mathcal{E}'(z)) - \iota(z)\| = \|\mathcal{E}(\iota(z)) - \iota(z)\| \leq \sup_{u \in G} \{\|\iota(z)u - u^{(\iota)}\|\},$$

for any $z \in \mathcal{B}(L^p(X; E))$. While for $u \in G$, it follows from Equation (4.4) that

$$\iota(z)u - u^{(\iota)} = \tau zu - u^{(\iota)}z = \tau zu - \tau uz.$$  

Combining the above facts together, we obtain that

$$\|\mathcal{E}'(z) - z\| \leq \sup_{u \in G} \{\|zu - uz\|\}.$$

Let $e := \sum_{j \in I} e_j$. Since $p_j e = e_j = ep_j$ and $q_c e = eq_c = 0$, we have that

$$\mathcal{E}'(eae) = \sum_{j \in I} p_j eae p_j + q_c eae q_c = \sum_{j \in I} e_j a e j.$$  

Also notice that for any $u = \sum_{j \in I}(-1)^{\xi_j} p_j + (-1)^{\xi_j} q_c$ in $G$, we have that $eu = ue = \sum_{j \in I}(-1)^{\xi_j} e_j$. Since $\{A_j\}_{j \in I}$ are pairwise 2/L-disjoint, there exists an L-Lipschitz map $f \in C_b(X)$ such that $f|\cdot |_{A_j} \equiv (-1)^{\xi_j} \chi_{A_j}$ for all $j$. Hence, $\|a, f\| \leq \mathcal{E}$ since $a \in \text{Commut}(L, \mathcal{E})$, and we clearly have $e_j f = fe_j = (-1)^{\xi_j} e_j$. Therefore, we obtain that

$$ueae = \left(\sum_{j \in I}(-1)^{\xi_j} e_j\right) ae = e f ae \approx e a f e = e a \left(\sum_{j \in I}(-1)^{\xi_j} e_j\right) = eae.$$  

Finally, we complete the proof by the following computation:

$$\|eae - \sum_{j \in I} e_j a e j\| = \|\mathcal{E}'(eae) - eae\| \leq \sup_{u \in G} \|eae - u^{(\iota)} eae\| \leq \mathcal{E},$$

for any $a \in \text{Commut}(L, \mathcal{E})$.  

□
Acknowledgments. The first-named author would like to thank Tomasz Kania for helpful discussions on Banach space valued $L^p$-spaces.

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