CONCORDANCE OF SPHERES IN 4-MANIFOLDS WITH AN IMMERSED DUAL SPHERE

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Abstract. Let $S_0$ and $S_1$ be two homotopic, oriented 2-spheres embedded in an orientable 4-manifold $X^4$. After discussing several operations for modifying an immersion of a 3-manifold into a 5-manifold, we discuss two concordance obstructions $f_q(S_0, S_1)$ and $s_{\text{ong}}(S_0, S_1)$ which, when defined, are defined in terms of the self-intersection set of a regular homotopy from $S_0$ to $S_1$. When $S_0$ has an immersed dual sphere, we see that under some mild topological conditions on $X$, the invariants $f_q$ and $s_{\text{ong}}$ are a complete set of concordance obstructions. This work is an adaption of the methods of Richard Stong to the context of concordances of 2-spheres.

1. Introduction

We study the question of when two homotopic, oriented 2-spheres $S_0, S_1$ embedded in a 4-manifold $X^4$ are concordant. In the case that one of the 2-spheres admits an immersed dual (that is, in the case that there is an immersed 2-sphere intersecting $S_0$ in a single point) and the 4-manifold $X$ satisfies some mild conditions then there is a complete pair of obstructions to the existence of a concordance.

These invariants are the Freedman–Quinn invariant $f_q$ (see \cite{9, 11, 7} and Section 3.1) and the Stong invariant $s_{\text{ong}}$ of Stong \cite{11} following Freedman and Quinn (see \cite{11}, \cite{1} Chapter 10), and Section 3.2. If we were to closely follow the language of Stong, we would probably call the Stong invariant the Kervaire–Milnor invariant $km$ – but this is somewhat misleading as it has very little to do with the work of Kervaire or Milnor, so we instead refer to it as the Stong invariant. The invariant stong is secondary to $f_q$, in the sense that $s_{\text{ong}}(S_0, S_1)$ cannot be defined if $f_q(S_0, S_1)$ is non-vanishing. If the spheres $S_0$ and $S_1$ are concordant, then both $f_q(S_0, S_1)$ and (when it is defined) $s_{\text{ong}}(S_0, S_1)$ must vanish. When one of $S_i$ admits an immersed dual and $X$ satisfies some mild conditions, the 2-spheres $S_0, S_1$ are concordant if and only if $f_q$ and $s_{\text{ong}}$ vanish. This follows the usual trend that stabilization (here in the form of a dual sphere, which can be introduced by various stabilization operations) greatly simplifies 4-dimensional topology.

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\footnote{This also neatly avoids any reader thinking we intended km to stand for Klug–Miller.}
The Freedman–Quinn and Stong invariants. Briefly, the Freedman–Quinn invariant \( f_q(S_0, S_1) \) is defined by considering the 1-dimensional link that is the preimage of the self-intersection set of a singular concordance between \( S_0 \) and \( S_1 \). The following definitions are of critical importance to this paper.

**Definition 1.1.** Let \( S_0 \) and \( S_1 \) be embedded, oriented spheres in a 4-manifold \( X \). A regular immersion \( H : S^2 \times I \to X^4 \times I \) with \( H(S^2 \times \{0\}) = S_0 \) and \( H(S^2 \times \{1\}) = S_1 \), with \( S^2 \times I \) oriented so that \( \partial H(S^2 \times I) \) induces the correct orientation on \( S_1 \) and the opposite orientation on \( S_0 \), is called a *singular concordance* from \( S_0 \) to \( S_1 \). If \( S_0 \) and \( S_1 \) share a basepoint \( z \in X \), and \( H \) is a singular concordance from \( S_0 \) to \( S_1 \) where there exists some \( p \in S^2 \) with \( H(p, t) = (z, t) \) for all \( t \in I \), then we call \( H \) a *based singular concordance* from \( S_0 \) and \( S_1 \).

**Remark 1.2.** If \( S_0 \) and \( S_1 \) are (based) homotopic, then the trace of a (based) regular homotopy from \( S_0 \) to \( S_1 \) yields a (based) singular concordance from \( S_0 \) to \( S_1 \). (Recall that any two homotopic, embedded 2-spheres in a 4-manifold are also regularly homotopic by work of Hirsch [3] and Smale [10] in the smooth category; see [11 Chapter 1] in the topological category.) However, we do not generally expect a singular concordance \( H \) to be the trace of a regular homotopy.

The following weakening of the usual condition that a surface in a 4-manifold is characteristic will be essential in what follows.

**Definition 1.3.** A surface \( F \) in a 4-manifold \( X \) is *s-characteristic* (or *spherically-characteristic*) if for every 2-sphere \( S \) immersed in \( X \), we have

\[
[F] \cdot [S] \equiv [S] \cdot [S] \quad (\text{mod } 2).
\]

Similarly, a 3-manifold \( M \) properly embedded in a 5-manifold \( W \) is s-characteristic if for every 2-sphere \( S \) embedded in \( W \), we have

\[
[M] \cdot [S] \equiv [S] \cdot [S] \quad (\text{mod } 2).
\]

**Observation 1.4.** If \( H \) is a singular concordance from \( S_0 \) to \( S_1 \), then \( S_0 \) (and hence \( S_1 \)) is s-characteristic in \( X \) if and only if \( H(S^2 \times I) \) is s-characteristic in \( X \times I \).

This is clear because any 2-sphere \( G \) in \( X \times I \) can be projected to an immersed 2-sphere in \( X \), and similarly any 2-sphere immersed in \( X \times 0 \) can be pushed into \( X \times I \) and isotoped to be embedded.

In particular, if there is a 2-sphere \( G \) in \( X \times I \) with trivial normal bundle that intersects \( H(S^2 \times I) \) transversely once, then \( S_0 \) (and hence also \( S_1 \)) is not s-characteristic.

Given a singular concordance \( H : S^2 \times I \to X \times I \) from \( S_0 \) to \( S_1 \), let \( L \) be the link in \( S^2 \times I \) that is the preimage of the self-intersection set of \( H \). We call \( L \) the *singular link* of \( H \). We can associate an element of \( \pi_1 X \) to every component of \( L \) via a sheet-changing based loop. The invariant \( f_q(S_0, S_1) \) counts the fundamental group elements corresponding to circles that double-cover their images under \( H \) as an element in a certain quotient of \( \mathbb{Z}[\pi_1 X]/\langle g + g^{-1}, 1 \rangle \); see Section 3.1 for a
detailed description. As we discuss in Section 3.1, in general $H$ must be a based singular concordance, although in the presence of dual spheres, this assumption is not necessary (see also [9] and [7])

The Strong stong($S_0, S_1$) invariant is more complicated but essentially counts the group elements associated to $L$ (viewed as elements in $H_1(X; \mathbb{Z}/2\mathbb{Z})$) weighted by linking numbers of the various components of $L$. A relatively complete description of an analogous invariant in the case of a 3-sphere in a 5-manifold can be found in [11]; we have further exposition of this in progress and will release it in a forthcoming paper [6]. Here is a sufficient description for a first-pass at the our main results is as follows (for a complete discussion, see Section 3.2).

(1) The invariant stong($S_0, S_1$) takes values in a quotient of $H_1(X; \mathbb{Z}/2\mathbb{Z})$. In certain cases (for example, when $\pi_3(X) = 0$), we quotient by nothing and in fact $\text{stong}(S_0, S_1) \in H_1(X; \mathbb{Z}/2\mathbb{Z})$. We discuss this further in Section 3.2.

(2) If $\text{stong}(S_0, S_1) \neq 0$, then $S_0$ and $S_1$ are not concordant.

(3) If $H$ is as above, with $S_0$ and $S_1$ s-characteristic, $\text{fq}(S_0, S_1) = 0$ and $L$ is a Hopf link in a ball in $S^2 \times I$ with the image of both knots equal, then $\text{stong}(S_0, S_1)$ is the sheet-changing group element associated to $L$ (considered in a certain quotient of $H_1(X; \mathbb{Z}/2\mathbb{Z})$).

Main Theorems. We now state the main results of the paper, delaying a more detailed discussion of the invariants until Section 3.

**Theorem 1.5** (Case where spheres are not s-characteristic). Suppose that $S_0$ and $S_1$ are embedded, oriented, homotopic 2-spheres in an orientable 4-manifold $X$ such that $S_0$ has an immersed dual sphere $G$ in $X$ (i.e., $G$ and $S_0$ intersect in a single point) and $S_0$ is not s-characteristic. Then $S_0$ and $S_1$ are concordant if and only if $\text{fq}(S_0, S_1) = 0$.

This in particular generalizes [7, Theorem 1.4]. In the case where $S_0$ is s-characteristic, we will observe a secondary obstruction stong($S_0, S_1$). Furthermore, in this case, some mild restrictions must be placed on $X$ in order for the vanishing of $\text{fq}(S_0, S_1)$ and stong($S_0, S_1$) to imply that $S_0$ and $S_1$ are concordant. The need for such conditions arises since both $\text{fq}(S_1, S_0)$ and stong($S_0, S_1$) live in quotients and the first step in attempting to find a concordance between $S_0$ and $S_1$ is to start with a singular concordance and modify it so as to make the pre-quotiented versions of $\text{fq}$ and stong simultaneously vanish for this singular concordance – which may not be possible even when $\text{fq}(S_1, S_0)$ and stong($S_0, S_1$) both vanish, without any additional hypotheses.

In what follows, the map $\mu$ is the self-intersection number of a 3-sphere in a 6-manifold (where here the 6-manifold is $X \times I \times I$) – see [9], [7], or Section 3 for additional discussion.

**Theorem 1.6** (Case where spheres are s-characteristic). Suppose that $S_0$ and $S_1$ are embedded, oriented, homotopic 2-spheres in an orientable 4-manifold $X$ such...
that \(S_0\) has an immersed dual sphere \(G\) in \(X\) and \(S_0\) is \(s\)-characteristic. Assume that \(\mu(\pi_3(X)) = 0\), e.g., by assuming either \(\pi_1X\) has no 2-torsion or \(\pi_3(X) = 0\). Then \(S_0\) and \(S_1\) are concordant if and only if \(fq(S_0, S_1) = 0\) and \(\text{stong}(S_0, S_1) = 0\).

**Remark 1.7.** In fact, since the map \(\mu\) factors through the Hurewicz homomorphism \(\pi_3(X) \to H_3(\tilde{X}; \mathbb{Z}) \cong H_3(X; \mathbb{Z}\pi_1X)\) (see [9, Lemma 4.2]), it suffices to assume that \(H_3(X; \mathbb{Z}\pi_1X)\) is trivial for the conclusion of Theorem 1.6 to hold.

We actually prove Theorem 1.6 by proving the following theorem, which is currently harder to parse.

**Theorem 1.8.** Suppose that \(S_0\) and \(S_1\) are embedded, oriented, homotopic 2-spheres in an orientable 4-manifold \(X\) such that \(S_0\) has an immersed dual sphere \(G\) in \(X\) and \(S_0\) is \(s\)-characteristic. Then \(S_0\) and \(S_1\) are concordant if and only if there exists a singular concordance \(H\) from \(S_0\) to \(S_1\) so that \(\mu(H) = 0\) and \(\Delta(H) = 0\).

We give the following applications in Section 5. The first suggests the importance of constructing spheres realizing stong.

**Theorem 1.9.** If there are 2-spheres \(S_0, S_1\) in \(B^3 \times S^1\) with \(\text{stong}(S_0, S_1) = 1\) in \(H_1(B^3 \times S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\) then there is a 2-component link of spheres in \(S^4\) that is not concordant to the unlink.

We give examples of pairs of homotopic 2-spheres with specified Stong invariant in some other 4-manifolds in Example 5.1 (actually summarizing our previous work [7] and adding some more discussion), but not in \(B^3 \times S^1\).

**Corollary 1.10** (Corollary of Theorems 1.5 and 1.6). Let \(S_0\) be an oriented embedded 2-sphere in an orientable 4-manifold \(X\) with an immersed dual sphere \(G\). Let \(\text{Concordance}(S_0)\) be the set of concordance classes of embedded spheres in \(X\) that are homotopic to \(S_0\). Suppose that \(\pi_1X\) has a finite number of 2-torsion elements.

1. Suppose \(S_0\) is not \(s\)-characteristic. Then \(\text{Concordance}(S_0)\) is finite of size at most \(2^{|T_X|}\).
2. Suppose \(S_0\) is \(s\)-characteristic and that \(\mu(\pi_3X) = 0\). Then \(\text{Concordance}(S_0)\) is finite of size at most \(2^{|T_X|} \cdot |H_1(X; \mathbb{Z}/2\mathbb{Z})|\).

Theorems 1.5, 1.6, 1.8 are greatly indebted to Richard Stong and closely follow the arguments given by Stong for the case of 3-spheres in a 5-manifold [11], and the non-\(s\)-characteristic cases follow from Freedman and Quinn [1]. Stong’s work draws many ideas from and corrects an omission in the work of Freedman and Quinn. Stong and Freedman–Quinn develop several manipulations of 3-manifolds immersed in 5-manifolds (in our setting, this will be the image of \(H\) inside of \(X \times I\)) that allow for the modification of a singular link. Our construction of the stong invariant closely follows Stong’s, including making use of his many explicit operations for modifying the immersion so as to modify the singular link in a prescribed manner. In particular, all of our moves described in Section 2 are contained in Stong’s work [11].
We provide many details, add further exposition of his work, and correct some minor errors. In addition, Propositions 3.5, 4.2, 4.8, and Lemmas 4.1, 4.7, 4.6, as well as the definition of stong are all due to Stong [11] (often drawing from [1]) in the closely related context of 3-spheres in 5-manifolds.

**Historical overview.** We now give a short overview of the previous results leading up to this work, together with an overview of some of the relevant definitions and an outline of the present work. All results hold in both the smooth and topological category. Any obstructions to concordance obstruct even locally flat concordances, and if \( S_0 \) and \( S_1 \) are smoothly embedded 2-spheres then any constructed concordance may be taken to be smooth (while if \( S_0 \) and \( S_1 \) are locally flat, then the constructed concordance can be taken only to be locally flat). This holds true for the remainder of the paper; the reader may choose to live in either the topological or smooth category.

Kervaire [5] proved that, in sharp contrast to the analogous situation in dimension 3, all embedded 2-spheres in \( S^4 \) are concordant. This was later generalized to pairs of homologous genus-\( g \) surfaces in simply-connected 4-manifolds by Sunukjian [12], to the case of homotopic 2-spheres in non-simply-connected 4-manifolds with the additional hypotheses that \( \text{fq}(S_0, S_1) = 0 \) and \( S_0 \) has a framed dual by Freedman–Quinn [1] (see also the present authors’ previous work [7] for alternate proof), and to the case that \( S_0, S_1 \) are \( \pi_1 \)-trivial homotopic positive-genus surfaces related by a homotopy \( H \) with \( \mu(H) \in \mu(\pi_3(X)) \) (which holds automatically when e.g., either \( \pi_1(X) \) has no 2-torsion or \( H_3(X; \mathbb{Z} \pi_1 X) = 0 \)) and \( S_0 \) has a framed dual sphere by the present authors [7]. In this paper, we will recover a more general version of the main theorem of [7] in the case of 2-spheres.

Schneiderman and Teichner [9] give an alternate proof of Gabai’s lightbulb theorem [2] by way of starting with a concordance between two homotopic 2-spheres \( S_0, S_1 \) in a 4-manifold \( X \) and modifying the concordance to produce an isotopy. They assume that there is another 3-sphere \( G \) embedded in \( X \) with trivial normal bundle such that each \( S_i \) intersects \( G \) transversely once, and that \( \text{fq}(S_0, S_1) = 0 \). They invoke work of Stong [11] (here making use of the Freedman–Quinn invariant vanishing) to obtain a concordance from \( S_0 \) to \( S_1 \) in \( X \times I \) such that in each \( X \times \text{pt} \) the concordance intersects \( G \times \text{pt} \) transversely once. Using these embedded dual spheres at every level, Schneiderman and Teichner are able to convert the concordance into an isotopy and thus re-prove the lightbulb theorem (which says that \( S_0 \) and \( S_1 \) are isotopic).

Our work here is an exposition of the techniques in Stong [11] and Freedman–Quinn [1] (in particular Section 10.7) together with clarification of some of the arguments and discussion of how these arguments apply to questions regarding concordance of 2-spheres in 4-manifolds.

**Outline.** In Section 2 we describe many moves for modifying a \( \pi_1 \)-trivial immersion \( H : Y^3 \to W^5 \). In Section 3 we discuss the Freedman–Quinn and Stong invariants.
In Section 4, we move to the setting of singular concordances with dual spheres and prove Theorems 1.5, 1.6, and 1.8. In Section 5 we give some examples, discuss the relationship to concordance of links of 2-spheres, and prove the finiteness (with explicit upper bounds) of the set of concordance classes of embedded 2-spheres in a given homotopy class under some conditions.

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We would like to further emphasize that many of the constructions in this paper follow the work of Stong in [11]. While we often give more details (see for example Sections 2.2, 2.3, and 2.4 on Whitney disks) or expand upon Stong’s constructive techniques while adapting them to the setting of concordance, the importance of Stong’s work in this paper cannot be overstated.

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2. Moves for modifying the immersion $H$

Given connected manifolds $Y$ and $W$, a map $H : Y \to W$ is $\pi_1$-trivial if the resulting map on $\pi_1$ is trivial. A connected submanifold $Y$ of $W$ is $\pi_1$-trivial if the inclusion map $i : Y \to W$ is $\pi_1$-trivial.

Let $Y^3$ be an oriented, compact 3-manifold, $W^5$ an orientable 5-manifold, and $H : Y \to W$ a $\pi_1$-trivial, regular immersion that restricts to an embedding $\partial Y \to \partial W$. The preimage of the self-intersection set of $H$ is a link $L$ in $Y$. We refer to $L$ as the singular link of $Y$ and the components of $L$ as singular circles. In this section, we give several techniques for modifying $H$ so as to modify its singular link. In order to assign group elements to the singular circles we will need to choose a basepoint $z$ for $Y$ and specify which we keep constant in all moves on $H$ discussed in this section. We may write $z$ to denote $H(z)$; the context should make it clear whether we mean the basepoint of $Y$ or $W$.

**Remark 2.1.** The moves we discuss in this section change the immersion $H$ while fixing $H_{\partial S^2 \times I}$ and preserving regularity and $\pi_1$-triviality of $H$.

By fixing a lift of the basepoint $z$ to the universal cover $\tilde{W}$, we may consider $[H] \in H_3(W, H(\partial Y); \mathbb{Z}\pi_1 W) = H_3(\tilde{W}, \tilde{H}(\partial \tilde{Y}); \mathbb{Z})$, where $\tilde{W}$ is the universal cover of $W$ and $\tilde{H}(\partial \tilde{Y})$ consists of the lifts of $H(\partial Y)$ to $\tilde{W}$. The moves performed to $H$ discussed in this section also preserve this homology class.

In fact, all of the moves simply change $H$ by regular homotopy except for the ambient Dehn surgery operation in Section 2.7, in which case we surger the image of $H$ along a 4-dimensional 2-handle and thus still preserve $\pi_1$-triviality and the $H_3(W, H(\partial Y); \mathbb{Z}\pi_1 W)$ homology class of the underlying immersion.

This will be important in later sections of the paper, when we will modify a singular concordance $H : S^2 \times I \to X \times I$ while needing to preserve $[H] \in H_3(X \times I, X \times \{0, 1\}; \mathbb{Z}\pi_1 X)$.

To give a 4-dimensional motivation for this discussion, consider the following situation. Let $S_0$ and $S_1$ be homotopic, oriented, embedded 2-spheres in a 4-manifold $X$. Let $H$ be a singular concordance from $S_0$ to $S_1$. For example, $H$ can be the track of a regular homotopy from $S_0$ to $S_1$. We would like to know when we can modify $H$ in the interior of $S^2 \times I$ to obtain an embedding and thus obtain a concordance between $S_0$ and $S_1$. In this case, we have $Y = S^2 \times I$ and $W = X \times I$.

2.1. **Preliminaries: the singular link.** A singular circle $A$ in the singular link $L$ of $H$ can be one of two types.

Type I: We say $A$ is *type I* if $H|_A$ is a homeomorphism. In this case, there is another singular circle $A'$ with $H(A) = H(A')$. We call $A'$ the *dual* of $A$, and refer to $A$ and $A'$ together as a *dual pair*.

Type II: We say $A$ is *type II* if $A$ double covers its image under $H$. 


There is a natural (up to taking inverses) way of assigning an element of $\pi_1 W$ to every singular circle in $L$.

**Definition 2.2.** Let $A$ be in $L$. If $A$ is type I, then let $A'$ be its dual circle; if $A$ is type II then let $A' = A$. Fix $x \in A, x' \in A'$ with $H(x) = H(x')$.

Let $\eta$ and $\eta'$ be directed arcs from the basepoint $z \in Y$ to $x$ and $x'$ (respectively), with the interiors of $\eta$ and $\eta'$ disjoint from $L$. Then $H(\eta)H(\eta')$ is a closed loop in $X$ representing an element $a \in \pi_1 W$. We call $a$ the **group element associated to** $A$ and $A'$.

Note that the choice of $\eta$ and $\eta'$ do not affect the class of $a$ in $\pi_1 W$, since $H$ is $\pi_1$-trivial. However, exchanging the roles of $x$ and $x'$ will replace $a$ by $a^{-1}$. In order to make sure $a$ is well defined, when $A$ is type I we fix an ordering of $A$ and $A'$, calling $A$ active and $A'$ inactive. Thus, when $A, A'$ are a dual type I pair, we construct a loop representing $a$ by traveling from the $z$ through $H(Y)$ to $H(A) = H(A')$ in the sheet containing $A$, and then leaving through the sheet containing $A'$ and returning to the $z$.

From now on, we will label a dual pair of circles by a capital letter and the same capital letter prime, with the unprimed letter denoting the active circle and the primed denoting the inactive circle.

**Remark 2.3.** If $A$ is type II, then we may choose $\eta, \eta'$ so that $H(\eta)H(\eta')$ is parallel to a copy of $H(A)$ (with a whisker chosen to the basepoint). Since $H$ is $\pi_1$-trivial and $A$ double covers $H(A)$ under $H$, we conclude that $a^2 = 1$. This fact, will be important in Section 4.

### 2.2. Whitney moves

We begin with the most involved move changing $H$, in which we homotope $H$ via a Whitney disk. To improve readability, we divide this subsection into several pieces: finding a disk (2.2.1), orientation restrictions on the framed boundary of a Whitney disk (2.2.2), the $\omega_2$ obstruction to a framing of a circle extending over a bounded disk (2.2.3), and some final remarks summarizing this section (2.2.4). For now, we do not assume that the image of $H$ has an immersed dual—we will add this assumption in Section 2.3 and see how it simplifies the situation.

#### 2.2.1. Finding a disk

Let $A$ and $B$ be singular circles in $L$ with corresponding fundamental group elements $a = b$. Let $A', B'$ be the duals of $A$ and $B$ (if $A$ or $B$ is type II, then $A'$ or $B'$ (respectively) is simply equal to $A$ or $B$).

Let $x \in A, x' \in A', y \in B, y' \in B'$ with $H(x) = H(x')$ and $H(y) = H(y')$. Let $\gamma_1$ be an arc in $Y$ from $x$ to $y$ that is disjoint in its interior from $L$ and let $\gamma_2$ be an arc in $Y$ from $y'$ to $x'$ that is disjoint in its interior from $L$. Let $\eta$ be an arc from the basepoint of $Y$ to $x$. Then $H(\eta)H(\gamma_1)H(\gamma_2)H(\eta)$ is a based loop in $X \times I$ representing $ab^{-1} = 1 \in \pi_1 W$ (see Figure 1). We conclude that the unbased loop $H(\gamma_1) \cup H(\gamma_2)$ is nullhomotopic.
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Figure 1. On the left, we draw two balls in $Y$ that contain points $x, y$ and $x', y'$ with $H(x) = H(x')$ and $H(y) = H(y')$. We draw arc $\gamma_1$ from $x$ to $y$ and $\gamma_2$ from $y'$ to $x'$. On the right, we see curves representing $a$ and $b$ that agree outside of the pictured region. We see that the closed curve $H(\gamma_1) \cup H(\gamma_2)$ is freely homotopic to a based curve representing $ab^{-1}$. Thus, if $a = b$, then $H(\gamma_1) \cup H(\gamma_2)$ is nullhomotopic.

Figure 2. Left: we say a disk in $X$ with boundary in $H(Y)$ is a Whitney disk if it admits a neighborhood as pictured. Right: We may perform a Whitney move along a Whitney disk to change $H$ into a different regular immersion of $Y$ with a different self-intersection set.

Similarly, if $a = b^{-1}$ and $\gamma_1$ runs from $x$ to $y'$ while $\gamma_2$ runs from $y$ to $x'$, then $H(\gamma_1) \cup H(\gamma_2)$ is a nullhomotopic loop in $X \times I$.

In either case, when $H(\gamma_1) \cup H(\gamma_2)$ is nullhomotopic it follows that there is an embedded disk $D \subset W$ with $\partial D = H(\gamma_1) \cup H(\gamma_2)$. Note that the interior of $D$ may not be disjoint from $H(Y)$.

We would like to have a local model as in Figure 2 so that we can perform the Whitney move across $D$. When this model exists, we call $D$ a Whitney disk.

After performing the Whitney move along $D$, the effect to $L$ is the following (illustrated in Figure 3).
Figure 3. We illustrate the effect that performing a Whitney move has on the singular link $L$. The pictured bands $b_1$ and $b_2$ have image under $H$ as in Figure 4 relative to the Whitney disk. The circles $A_1, \ldots, A_n$ are in correspondence with intersections of $H(Y)$ with the interior of the Whitney disk.

Figure 4. We need to know when bands $b_1$ and $b_2$ can be chosen so that $H(b_1)$ and $H(b_2)$ are as pictured.

1. We perform band surgery along $b_1$ and $b_2$ (these bands come from thickening the arcs comprising $H^{-1}(\partial D)$ via a framing of $D$; see Figure 4).
2. For each intersection of $H(Y)$ with the interior of $D$, we add a new pair of type I circles with trivial group element. One of these circles is a belt around $b_1$ while the other is unlinked from all components of $L$. The pairs of circles arising from multiple intersections of $H(Y)$ with $D$ are parallel, as in Figure 3.

Because our goal is to modify $L$ in a prescribed way, we need to know when we can preordain the bands $b_1$ and $b_2$ in $Y$ with respective cores $\gamma_1$ and $\gamma_2$ so that $H(b_1)$ and $H(b_2)$ that appear in Figure 4. That is, we need to understand what framings of $\gamma_1, \gamma_2$ give such bands.

Note that when $D$ has the local model of Figure 2 or 4, the union $H(b_1 \cup b_2)$ is an annulus (as opposed to a Möbius band) – this is the first restriction. In addition, let $l_1$ and $l_2$ be 1-dimensional subbundles of the normal bundle of $\partial D$ in $W$ so that along the core of $b_i$, the fiber of $l_i$ is tangent to $H(Y)$ and along the core of $b_{2-i}$, $l_i$ is perpendicular to $H(Y)$. When $D$ is as in Figure 2 both of these line bundles $l_1, l_2$ are trivial – this is the second restriction.

Further, $H(b_1 \cup b_2), l_1, \text{ and } l_2$ form the normal bundle of $D$ restricted to its boundary $H(\gamma_1) \cup H(\gamma_2)$ which we denote by $N_D|_{\partial D}$. Any choice of trivialization of $H(b_1 \cup b_2), l_1, \text{ and } l_2$ (in that order) yields a trivialization of $N_D|_{\partial D}$. This
trivialization must extend to a trivialization of the normal bundle of $D$ – this is the third restriction.

**Definition 2.4.** Given bands $b_1$ and $b_2$ thickening $H^{-1}(\partial D)$ for $D$ a disk in $W$, we say that $b_1$ and $b_2$ are *framed bands for $D$* when all three of the following are true.

1. $H(b_1 \cup b_2)$ is an annulus.
2. The line bundles $l_1, l_2$ are trivial.
3. A choice of trivialization of each of $H(b_1 \cup b_2), l_1, l_2$ as line bundles over $\partial D$ extends to a trivialization of the normal bundle of $D$ in $W$.

By the above discussion, if a Whitney move along $D$ using bands $b_1, b_2$ is possible, then $b_1, b_2$ must be framed bands for $D$. Moreover, so long as $b_1, b_2$ are framed bands for $D$, we may use the obtained trivialization of the normal neighborhood of $D$ to obtain the local model of Figure 2 and perform a Whitney move along $D$.

In Figure 6, we show the effect of changing either $b_1$ or $b_2$ by adding a half-twist on each of the conditions of Definition 2.4.

- Adding a half-twist to either $b_1$ or $b_2$ changes the orientability of $H(b_1 \cup b_2)$.
- Adding a half-twist to $b_i$ changes orientability of $l_i$ while preserving orientability of $l_j$ for $\{i, j\} = \{1, 2\}$.
- When $H(b_1 \cup b_2), l_1, l_2$ are all orientable, adding a full twist to one of $b_1, b_2$ changes whether an induced trivialization of $N_D|\partial D$ extends over all of $D$.

We discuss this further in Section 2.2.3.

The remainder of Section 2.2 is a study in how to know when the three conditions of Definition 2.4 hold given bands $b_1, b_2$ and a Whitney disk $D$; i.e., how to determine when $b_1, b_2$ are framed with respect to $D$. A casual reader should feel more than welcome to skip ahead to Section 2.3.

### 2.2.2. Choosing the bands to satisfy orientability conditions

For now, we provide some remarks on how to understand the restrictions that $H(b_1 \cup b_2)$ be an annulus and $L_1, L_2$ be orientable from the perspective of working within $Y$.

**Remark 2.5.** We can check whether $H(b_1 \cup b_2)$ is an annulus by orienting the curves in $L$. Orient each active and type II circle; we obtain induced orientations on inactive circles by requiring that dual pairs induce the same orientations on their image. Then $H(b_1 \cup b_2)$ is an annulus if $b_1$ and $b_2$ are either both orientation-preserving or both orientation-breaking as bands (i.e., the orientations on the singular circles can be extended across both $b_1$ and $b_2$, or neither $b_1$ nor $b_2$.) In contrast, $H(b_1 \cup b_2)$ is a Möbius band when one of $b_1, b_2$ is orientation-preserving while the other is orientation-breaking.

When checking whether $H(b_1 \cup b_2)$ is an annulus as in Remark 2.5, we can choose the orientations of the singular circles containing $x$ and $y$ independently (assuming these are not the same curve or dual curves, in which case one choice determines the other).
Remark 2.6. In Figure 5, we illustrate how, given that \( H(b_1 \cup b_2) \) is an annulus, the question of whether \( l_1 \) and \( l_2 \) are orientable is equivalent to whether two intersections have the same sign. More precisely, fix positive bases \((v_1, v_2, v_3)\) and \((v_3, v_4, v_5)\) of the tangent space of \( Y \) at a point near an end of \( b_1 \) and the corresponding end of \( b_2 \). Choose these bases so that the image of \( v_3 \) near \( b_1 \) and the image of \( v_3 \) near \( b_2 \) agree under \( H \) as in Figure 5 (hence the naming convention). Now use \( b_1 \) and \( b_2 \) to transport these bases to the opposite ends of the bands, obtaining positive bases \((w_1, w_2, w_3)\) and \((w_3, w_4, w_5)\) respectively. (Note that implicitly we use the fact that \( H(b_1 \cup b_2) \) is an annulus for this to make sense – this property tells us that the two transports of \( v_3 \) will have the same image \( w_3 \) under \( H \).) In Figure 5, we see that \( l_1 \) and \( l_2 \) are orientable when the signs of the bases \((v_1, v_2, v_3, v_4, v_5)\) and \((w_1, w_2, w_3, w_4, w_5)\) of for the tangent space of \( W \) disagree (left of Figure 5) and that \( l_1 \) and \( l_2 \) are non-orientable when the signs of these bases agree (right of Figure 5).

Or more simply, following Remark 2.5, orient the singular circle containing \( x \) near \( x \); this induces an orientation of the circle containing \( x' \) near \( x' \). These orientations induce a sign \( s \in \{+,-\} \) on the self-intersection of \( H(Y) \) near \( H(x) = H(x') \). Now we compatibly orient the singular circles near \( y \) and \( y' \) specifically so that the induced sign on the self-intersection of \( H(Y) \) near \( H(y) = H(-y) = -s \). Then \( l_1 \) and \( l_2 \) are both orientable exactly when \( b_1 \) and \( b_2 \) are both orientation preserving with respect to these local orientations.

Remark 2.6 may seem unwieldy, but essentially is an analogue of the fact that in 4-dimensional topology, a framed Whitney disk must run between self-intersection points of a surface that are of opposite sign. This condition is slightly more difficult to state in this dimension, since our immersed 3-manifold \( H(Y) \) intersects itself in circles rather than isolated points and hence the self-intersections do not come with inherent signs. Nevertheless, given a choice of sign on one circle in \( H(Y) \) and bands connecting that circle to another, we can induce a sign on the other using the bands (which is well defined exactly when the bands union is an annulus rather than a Möbius band). We then note that the bundles \( l_1, l_2 \) are orientable exactly when these signs are opposite (which is exactly the case in the 4-dimensional setting, as well).

2.2.3. The \( \omega_2 \) obstruction. Now we study the last condition of Definition 2.4 which is slightly different from the first two in that it is not a condition on orientability of a bundle. This is the condition that a certain trivialization of \( N_D|_{\partial D} \) extends over \( D \), and is analogous to the usual condition in dimension 4 of a framing on the boundary of a Whitney disk extending over the interior of the disk.

Assume that \( H(b_1 \cup b_2) \) is an annulus and that \( l_1 \) and \( l_2 \) are orientable, as in Section 2.2.2. Trivialize all three bundles to obtain a trivialization of \( N_D|_{\partial D} = H(b_1 \cup b_2) \oplus l_1 \oplus l_2 \). This trivialization gives an element of \( \pi_1(\text{SO}(3)) = \mathbb{Z}/2\mathbb{Z} \) which is trivial if the trivialization extends over all of \( D \) and nontrivial otherwise. If we alter \( b_1 \) (or similarly \( b_2 \)) by adding a full twist, the effect on the induced element
of $\pi_1(SO(3))$ is to add a full twist, changing whether the trivialization of $N_D|_{\partial D}$ extends over $D$.

**Remark 2.7.** Assume we are in the case where the bands $b_1$ and $b_2$ are such that $H(b_1 \cup b_2)$, $l_1$, and $l_2$ are all orientable, but an induced trivialization of $N_D|_{\partial D}$ does not extend over the disk $D$. Then the bands $b_1, b_2$ are not framed bands for $D$. If there is a 2-sphere $S$ embedded in $W$ with $w_2(S) = 1$, then we may obtain a new disk $D'$ by tubing $D$ to $S$. The same trivialization of $N|_{\partial D'}$ does extend over $D'$, so we conclude that $b_1$ and $b_2$ are framed bands for $D'$. This is why we write the disk with respect to which a pair of bands are framed, rather than just saying “$b_1$ and $b_2$ are framed,” without a specification of disk.

2.2.4. Final remarks on choosing bands. Recall from Section 2.2.1 that we may only perform Whitney moves along disks meeting intersection circles corresponding to the same or inverse group element.

**Remark 2.8.** Performing a Whitney move preserves the group element associated to each singular circle. That is, if a singular circle $A$ with corresponding group
element $a$ is involved in a Whitney move, then the resulting singular link has one or two components that include arcs originally in $A$. These components have group element $a$ or $a^{-1}$.

To see this, choose arcs $\eta$ from each singular circle in $L$ to the basepoint of $Y$. Then the (appropriately oriented) images of $\eta$ arcs under $H$ can be concatenated to form loops determining the group elements for each self-intersection of $H(Y)$. The images of endpoints of the arcs in $\eta$ can be taken to be disjoint from the Whitney disk. Moreover, the Whitney disk meets $H(Y)$ in arcs, so generically the boundary of the Whitney disk is disjoint from all of $\eta$. Thus, the same loops determine the group elements of singular circles for $H$ after performing the Whitney move.

**Conclusion 2.9.** When there are (not necessarily distinct) singular circles in $L$ with the same or inverse group elements $a$ and $b$, we may choose arcs $\gamma_1, \gamma_2$ between
them and use these arcs to perform a Whitney move (along a disk bounded by the images of the arcs).

- If the singular circles have inverse group element and are type I, then the arcs $\gamma_1, \gamma_2$ should connect the active circle in one pair to the inactive circle in the other pair. (This includes the case when we consider one type I pair $A, A'$ and let $\gamma_1$ and $\gamma_2$ both have ends at $A$ and $A'$.)
- If the singular circles have equal group elements and are type I, then the arcs should connect the active circle in one pair to the active circle in the other, and the inactive circle in one pair to the inactive circle in the other. (This includes the case when we consider one type I pair $A, A'$ and let $\gamma_1$ have both ends at $A$ and $\gamma_2$ have both ends at $A'$.)
- Once the arcs and Whitney disk are specified, the arcs cannot be arbitrarily thickened into bands. Instead, the twisting of the two bands is determined up to a criterion in $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. If $H(Y)$ is $s$-characteristic, then given two Whitney disks $D_1, D_2$ with the same boundary and with

$$|D_1 \cap H(Y)| \equiv |D_2 \cap H(Y)| \pmod{2},$$

these obstructions agree: bands $b_1, b_2$ are framed with respect to $D_1$ if and only if they are framed with respect to $D_2$.
- A Whitney move along a disk $D$ bounded by $H(\gamma_1) \cup H(\gamma_2)$ changes $L$ by band surgery along bands whose cores are the framed arcs $\gamma_1, \gamma_2$, and also adds a pair of dual type I circles for each intersection of $D$ with $H(Y)$. The group elements of the singular circles resulting from the band surgery are still $a$ or $a^{-1}$.

### 2.3. Clean Whitney moves.

In Section 2.2, we saw that we cannot expect the interior of a Whitney disk to be disjoint from $H(Y)$ and understood how these intersections informed the effect on the singular link $L$ of performing the Whitney move.

Let $\gamma_1, \gamma_2$ be arcs between singular circles in $Y$ so that $H(\gamma_1) \cup H(\gamma_2)$ bounds a disk $D$.

**Proposition 2.10.** Assume that there is a 2-sphere $G$ embedded in $W$ that intersects $H(Y)$ transversely in a single point. Then there is a Whitney disk $D'$ bounded by $H(\gamma_1) \cup H(\gamma_2)$ so that $D'$ does not intersect $H(Y)$.

**Proof.** The disk $D$ intersects $H(Y)$ in $n$ points for some $n$. Obtain $D'$ from $D$ by tubing $D$ to $n$ copies of $G$ near these intersections. □

**Definition 2.11.** We call a Whitney disk bounding the arcs $\gamma_1$ and $\gamma_2$ that is disjoint from $H(Y)$ clean.

**Remark 2.12.** In Proposition 2.10 if $w_2(G) = 0$, then a trivialization of $N_D|_{\partial D}$ that extends over $D$ also extends over $D'$. If $w_2(G) = 1$, then this depends on parity of $n$: If $n$ is even then the trivialization extends over $D'$ but if $n$ is odd then the trivialization does not extend over $D'$. 
When $Y$ has a dual sphere, we can say exactly when a framing of $\gamma_1, \gamma_2$ extends over a clean Whitney disk, as we explain in the following proposition.

**Proposition 2.13.**

1. If $H(Y)$ has a dual sphere $G$ with $w_2(G) = 0$, then there is a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ criterion to a framing of $\gamma_1$ and $\gamma_2$ extending to some clean Whitney disk.
2. If $H(Y)$ has a dual sphere $G$ with $w_2(G) = 1$ and if $H(Y)$ is not $s$-characteristic, then there is a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ criterion to a framing of $\gamma_1$ and $\gamma_2$ extending to some clean Whitney disk.
3. If $H(Y)$ has a dual sphere $G$ with $w_2(G) = 1$ and is $H(Y)$ is $s$-characteristic, then there is a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ criterion to a framing of $\gamma_1$ and $\gamma_2$ extending to some clean Whitney disk (and this same criterion holds for every clean Whitney disk).

**Proof.** In all of the stated cases, since $H(Y)$ has a dual sphere we know by Proposition 2.10 that there is a clean Whitney disk $D$ with boundary $H(\gamma_1) \cup H(\gamma_2)$. Let $b_1, b_2$ be band thickenings of $\gamma_1, \gamma_2$ that are framed with respect to $D$. For any other choice of Whitney disk, whether $b_1, b_2$ are framed with respect to $D$ is determined by the $\omega_2$ obstruction of Section 2.2.3.

If $\omega_2(G) = 0$, then we may twist $D$ once near its boundary to add a full twist to either $b_1$ or $b_2$. This introduces an intersection of $D$ with $H(Y)$ which we can remove by tubing $D$ to $G$. This preserves whether the induced trivialization of $N_D|_{\partial D}$ extends over $D$, settling Case (1).

If $\omega_2(G) = 1$ and $H(Y)$ is not $s$-characteristic, then let $F$ be an embedded 2-sphere in $X \times I$ with $F \cdot H(Y) \not\equiv F \cdot F \pmod{2}$. Tube $D$ to $F$ to obtain a different Whitney disk $D'$, and then tube $D'$ to a copy of $G$ for each intersection of $D'$ with $H(Y)$ to obtain a clean Whitney disk $D''$. In all, we have obtained $D''$ from $D$ by tubing on an odd number of 2-spheres with $\omega_2 = 1$ (and possibly one $\omega_2 = 0$ 2-sphere). We conclude that the bands $b_1$ and $b_2$ are not framed for $D''$, thus settling Case (2).

Finally, if $G$ is unframed and $H(Y)$ is $s$-characteristic, then let $D'$ be any other clean Whitney disk bounded by $H(\gamma_1) \cup H(\gamma_2)$. The 2-sphere $D \cup D'$ can be isotoped off $H(Y)$. Because $H(Y)$ is $s$-characteristic, then the 2-sphere $D \cup D'$ must have trivial normal bundle. Therefore, a framing of $N_D|_{\partial D}$ extends over $D$ if and only if it extends over $D'$, settling Case (3). \qed

We can use the following lemma to understand when bands $b_1, b_2$ are framed with respect to some clean Whitney disk.

**Lemma 2.14.** Assume that $H$ is $s$-characteristic in $X$ with a dual sphere $G$ and let $A$ and $A'$ be a dual pair of type I circles in $L$. Assume both $A$ and $A'$ are null-homotopic in $Y$. Then

$$\text{lk}(A, L - A) \equiv \text{lk}(A', L - A') \pmod{2}.$$ 

We delay the proof of Lemma 2.14 to the end of Section 2.7. We make note of it here to emphasize that this condition makes it quite easy to check whether bands...
Figure 7. Left: Whitney disks $D_1$ and $D_2$ meeting at $H(B)$. Right: We push $b_1 \cup b_2$ and $b'_1 \cup b'_2$ off of $B$ and $B'$ (respectively) and add a half twist to form $b_3$ and $b'_3$. The sign of the half-twist in $b'_3$ is determined by the sign of the half-twist in $b_3$ (and the directions we pushed off $B$ and $B'$), given that we want to find a disk $D_3$ agreeing with $D_1$ and $D_2$ away from $B$ that yields the bands $b_3$ and $b'_3$.

$b_1, b_2$ are framed for some clean Whitney disk when $H$ is $s$-characteristic with a dual. First of all, check that $b_1, b_2$ respect the orientation conditions of Section 2.2.2 (this is particularly easy in the case that each of $b_1, b_2$ meet only one circle, in which case we must only check that both bands are orientation-preserving). Then, check whether performing band surgery along $b_1, b_2$ would yield a singular link that satisfies Lemma 2.14. If not, add a full twist to one of $b_1$ or $b_2$ – this preserves the orientation properties, but now surgering along $b_1$ and $b_2$ yields a singular link that does satisfy Lemma 2.14. The bands $b_1, b_2$ are now framed for some clean Whitney disk by Proposition 2.13.

2.4. Gluing Whitney disks. In this subsection, we describe a procedure for gluing two Whitney disks which share a common corner to obtain a new Whitney disk.

Let $A, B, C \subset L$ be (not necessarily distinct) singular circles in $Y$. If $A$ is type I then set $A'$ to be is its dual; if $A$ is type II then set $A'$ to be $A$. Similarly choose singular circles $B'$ and $C'$.

Let $D_1$ be a Whitney disk meeting $H(A)$ and $H(B)$ and let $D_2$ be a Whitney disk meeting $H(B)$ and $H(C)$. Assume that $D_1$ and $D_2$ meet $H(B)$ in the same point, but that otherwise their boundaries are disjoint (see left of Figure 7). Then we may glue $D_1$ and $D_2$ together to obtain a new Whitney disk $D_3$ meeting $H(A)$ and $H(C)$ as follows.
Let $\eta_1$ and $\eta_1'$ be the cores of the bands $b_1$ and $b_1'$ in $Y$ whose image bounds the framed disk $D_1$. Let $\eta_2$ and $\eta_2'$ be the cores of the bands $b_2$ and $b_2'$ in $Y$ whose image bounds the framed disk $D_2$. Isotope $D_2$ near $H(B)$ so $H^{-1}(\partial D_1 \cup \partial D_2)$ form two arcs $\eta_3 = \eta_1 \cup \eta_2$ and $\eta_3' = \eta_1' \cup \eta_2'$. In Figure 7 (left) we give a prototypical schematic.

Note that if $A, B, C$ are oriented so that $b_1$ and $b_2$ are both orientation-preserving, then $b_3 = b_1 \cup b_2$ viewed as a band connecting $A$ and $C$ is not orientation-preserving.

We push $b_3$ off $B$ and add a half-twist so that $b_3$ is orientation-preserving. See the right of Figure 7.

Now push $b_3' = b_1' \cup b_2'$ slightly off $B$ as well. If we add a half-twist to $b_3'$, then $H(b_3 \cup b_3')$ will be an annulus and both $b_3$ and $b_3'$ will be orientation-preserving. Let $D_3$ be a (possibly unframed) Whitney disk for $b_3, b_3'$ obtained by plumbing together $D_1, D_2$ as in the right of Figure 7.

By the analysis of Section 2.2, $b_3$ and $b_3'$ are framed for $D_3$ up to adding a whole twist to $b_3'$. That is, for one choice of the sign of half-twist added to $b_3'$, the bands $b_3$ and $b_3'$ will be framed. (We can state this sign explicitly in terms of certain bases of points in the tangent space of $H(Y)$ and $N_H(Y)$, but we do not think this would be very instructive.)

2.5. Introducing a type II singular circle. In the definition of the Freedman–Quinn invariant, one does not count the singular circles of an immersed $Y \hookrightarrow X \times I$ that correspond to the trivial group element. In Figure 8, we produce $B \cong D^2 \times I$ immersed in $B^4 \times I$ whose singular link consists of one type II singular circle. Moreover, the boundary of $B$ is $(D \times 0) \cup (\partial D \times [0, 1]) \cup (D \times 1)$, for $D$ the unknotted disk in $B^4$ (see Figure 9). Using $B$, we may locally modify $H$: consider an immersion $H': Y \to X \times I$ whose image is obtained from that of $H$ by deleting a small standard $D^2 \times I \subset B^4 \times I$ and replacing it with a copy of $B$. Then $H'$ has a singular link obtained from that of $H$ with an additional type II circle corresponding to the trivial group element.

For the convenience of the reader, the isotopy that simplifies the Whitney disk $W'$ with boundary on immersed disk $D'$ in Figure 9 is broken into several steps:

- (a) to (b): We slide one index-1 point of $D'$ above a self-intersection (intersection/band pass of $D'$).
- (b) to (c): We cancel an index-0, index-1 pair of $D'$.
- (c) to (d): We slide another index-1 point of $D'$ above a self-intersection.
- (d) to (e): We further move the index-1 point.
- (e) to (f): We cancel an index-1, index-2 pair of $D'$.
- (f) to (g): We isotope the diagram to turn $W$ into the standard diagram of a Whitney disk.

2.6. Clasp self-intersections and finger moves. In this subsection, we reprove the following classical 3-dimensional lemma which will be applied to our singular link $L$. 

Figure 8. An illustration of $B \cong D^2 \times I$ immersed in $B^4 \times I$; here we picture only $B^4 \times [0, 1/2]$. The intersection of $B$ with $B^4 \times 0$ is an unknotted disk $D$. In $B^4 \times t$ from $t = 0$ to $t = 1/2$, $B$ consists of a trace of a trivial finger move from $D$, so that $B \cap (B^4 \times 1/2)$ is a disk $D'$ with two points of self-intersection in its interior. In $B^4 \times 1/2$, we draw a Whitney disk $W$ for $D'$. In $B^4 \times t$ from $t = 1/2$ to $t = 1$ (pictured in Figure 9), $B$ consists of a trace of a Whitney move from $D'$ along $W$. In Figure 9 we see that $B \cap (B^4 \times 1)$ is again an unknotted disk (and along the way demonstrate that $W$ is framed).

**Lemma 2.15.** Let $J$ be a link of null-homotopic components in a 3-manifold $Y^3$. Then $J$ bounds a set of immersed disks in $Y$ with only clasp intersections amongst the disks as in Figure 10.

**Proof.** Since every component of $J$ is null-homotopic, $J$ bounds an immersed collection of disks $D$. The proof proceeds by modifying these disks in the following steps.

1. Remove the branch point self-intersections of the disks in $D$.
2. Remove the circles of double points.
Figure 9. In (a), we draw the Whitney disk \( W \) with boundary on disk \( D' \subset (B^4 \times \frac{1}{2}) \) as in Figure 8. We draw \( D' \) as an immersed tangle with attached bands indicating index-1 critical point. The two self-intersections of \( D' \) are pushed into the tangle, so that it is singular. We indicate these with grey/shaded disks. See [4] for more on these diagrams of self-transverse immersed surfaces in 4-manifolds and the moves on these diagrams corresponding to isotopy or homotopy. From (a) to (g), we isotope \( D' \) and \( W \) to make \( W \) look standard. Then from (g) to (h), we perform the Whitney move along \( W \) and obtain an unknotted disk, as claimed in Figure 8.

Figure 10. A clasp intersection consists on an arc, neither of whose preimages are properly embedded. (That is, each local sheet has boundary at one end of the arc.) Note that a clasp intersection may occur between two distinct disks or as a self-intersection of one disk.

(3) Remove the triple points.
(4) Remove the ribbon singularities.

To achieve step (1), we preemptively add branch points to each disk in \( D \) so that each has an even number of branch points (Figure 11, top). In Figure 11 (middle), we illustrate how to move a branch point within a disk without introducing new
Figure 11. Step (1) of the proof of Lemma 2.15 requires us to manipulate branch points in the immersed disks until eventually cancelling them all in pairs.

Figure 12. We can break an outermost closed self-intersection circle by performing a finger move.

branch points. We can thus move a pair of branch points (of opposite signs) in $D$ to look like Figure 11 (bottom left), and then remove the pair entirely. Repeat until there are no branch point self-intersections in $D$.

For step (2), we use the finger move in Figure 12. We let $C$ be a circle of intersection in $D$ which is outermost, i.e., there is an arc $\eta$ from $C$ to $J$ in $D$ that does not meet any self-intersections of $D$ in its interior. Then we may use $\eta$ as the guiding arc for a finger move on $D$ as in Figure 12; the effect is to break the circle $C$ and replace it with an arc. We introduce no new branch points or closed circles of self-intersection to $D$. By performing this move to all of the double point circles starting with the outermost and working inward, we eventually eliminate all closed self-intersection circles of $D$. 
Figure 13. Given a triple point in $D$, we may isotope $J = \partial D$ to eliminate that triple point. This will not introduce any branch points, closed circles of self-intersection, or triple points to $D$.

Figure 14. Given a ribbon intersection in $D$, we may perform a finger move to eliminate the ribbon intersection and add two clasp intersections.

For step (3), we use isotopy of $J = \partial D$ as in in Figure 13. (If desired, one could change perspective and fix $\partial D$ and instead move the interior of $D$ via finger moves.) This move eliminates the triple point of $D$ without introducing any branch points nor closed circles of intersection. Repeat for each triple point.

For step (4), the finger move in Figure 14 is applied to each ribbon intersection. By choosing guiding arcs in $D$ whose interiors are disjoint from the self-intersections of $D$, no new triple points are created. No branch points or double point circles are created either. Now the disks $D$ have only clasp intersections.

Similar arguments to Lemma 2.15 appear in [1] and [8]. Lemma 2.15 motivates the following move, in which we achieve crossing changes of singular circles (at the cost of introducing new singular circles).

Let $\gamma \subset Y$ be a framed arc between two singular circles $A$ and $B$ (possibly with $A = B$) such that the end points of $\gamma$ are not paired by $H$. We will homotope $H$ in a small 3-ball containing $\gamma$. The effect on $L$ is to achieve a crossing change of $A$ and $B$ (along $\gamma$) and to add a dual pair of type I circles linking $A$ and $B$ as meridians.

The move is shown in Figure 15; here $\gamma$ is implicitly a horizontal arc between $A$ and $B$. We say that the finger move is a finger move along $\gamma$ for $A$ to $B$. If $A$ and $B$ correspond to group elements $a, b$ respectively, then the introduced meridians $E$ and $E'$ have associated group element $e = ab^{-1}$. 
Figure 15. Top left: the singular link bounds immersed disks including a clasp between curves $A$ and $B$. (Note that $B$ might be equal to $A$ or $A'$. Top right: we perform a finger move to the image of $H$ to remove the clasp at the cost of introducing a pair of singular circles $E, E'$ that are meridians of $A'$ and $B'$. Bottom: the finger move in the image of $H$ the achieves a crossing change of the $A$ and $B$ curves.

Remark 2.16. It is worth noting the following simpler version of the finger move. Let $\gamma$ be an arc with boundary on $H(Y)$ (away from $H(L)$ and with interior disjoint from $H$. Choose whiskers in $H(Y)$ (also away from $H(L)$) from each endpoint of $\gamma$ to the basepoint; orient these whiskers to form a based loop with $\gamma$ representing group element $g$. Then we may homotope $H$ by performing a finger move in which we push $\gamma(0)$ along $\gamma$ until introducing a new circle of self-intersection to the image of $H$ near $\gamma(1)$. The effect to $L$ is to add a dual pair of type I circles that are unknotted unlinked with each other and all other components of $L$ and associated to the group element $g$.

2.7. Ambient Dehn surgery to remove dual singular circles. In this subsection, we introduce a move that involves performing ambient Dehn surgery on $H(Y)$ in order to remove a dual pair of type I singular circles from $L$. Here, we will assume that $H(Y)$ has a dual sphere $G$. 
Suppose that $A$ and $A'$ are a dual pair of type I circles in $L$ with the property that both are unknots and $A'$ is unlinked from $L - A'$, i.e., $A'$ bounds an embedded disk $D_1$ in $Y$ that is disjoint from the other singular circles.

Let $D_1 := H(D_1)$ and thicken this disk in the two normal directions to $Y$ (normal to the sheet containing $A'$). The result is a 4-dimensional 2-handle that may be viewed as attached to $H(Y)$ along $A$. We surger $H(Y)$ along this 2-handle to obtain an immersion $H' : M \to W$ of a 3-manifold $M$. The manifold $M$ is obtained from $Y$ by integral Dehn surgery along $A$. Note that we cannot control the framing of this Dehn surgery, which we refer to as the relative framing on $A$. The self-intersection link of $H'$ agrees with that of $H$, except we have removed $A$ and $A'$.

Let $\mu$ be a meridian of $A$ in $Y$. This meridian bounds a disk $D_2$ in $Y$ meeting $A$ in one point. Let $D_2 := H(D_2)$, so that $D_2$ is now a disk in $X \times I$. The boundary of $D_2$ is naturally viewed as living in $H(Y)$, but we may also take its boundary to be in $H'(M)$.

Perturb the interior of $D_2$ to intersect $H'(M)$ transversely; this intersection consists of one point in $H'(A')$. Tube $D_2$ to the dual sphere $G$ to obtain a disk $D'_2$ whose interior is disjoint from $H'(M)$. We then again thicken $D'_2$ to obtain a 4-dimensional 2-handle, which we use to surger $H'(M')$ along $H'(\mu)$, yielding another immersion $H'' : N \to W$. The 3-manifold $N$ is obtained from $Y$ by integral Dehn surgery on a Hopf link (with components corresponding to $A$ and $\mu$). If $G$ has trivial normal bundle, then may arrange for the framing of the surgery along $\mu$ to be any integer, whereas if $G$ has nontrivial normal bundle then we may choose this framing only up to parity. Details on this framing may be found for example in [7, Section 5]. The key points to notice are:

- The choice of thickening of the disk $D_2$ to form a 4-dimensional 2-handle is not unique. By considering different thickenings, we may achieve all potential framings of some parity.
- We may twist $D_2$ once about its boundary to introduce a new intersection points between $D_2$ and $H'(M)$, and then remove this intersection by tubing $D_2$ to a copy of the dual sphere $G$. When $G$ has trivial normal bundle, this does not effect which framings of $\partial D_2$ (as a curve in $H'(M)$) extend over $D_2$: hence, we may change the induced framing of the Dehn surgery by ±1. This fails when $G$ has nontrivial normal bundle.

Thus, if $G$ has trivial normal bundle then we may ensure that $N$ is homeomorphic to $Y$. In this case, we redefine $H = H''$. The total effect on $L$ is to remove $A$ and $A'$ while leaving the other singular circles unchanged (i.e., Figure 16 with $k = 0$).

In the case where $G$ has nontrivial normal bundle but the relative framing on $A$ happens to be 0, then again $N \cong Y$. If $H(Y)$ is $s$-characteristic, the framing of the surgery along $\mu$ is forced to be odd. We can choose the framing along $\mu$ to be any odd integer $k$, and similarly redefine $H := H''$. The effect on $L$ is to remove $A$ and $A'$ while adding $k$ full twists to the singular circles that link $A$, as in Figure 16.
Figure 16. The dual singular circles $A$ and $A'$ are unknots and $A'$ (not pictured) is unlinked from all other circles. The circle $\mu$ is a meridian of $A$. We perform two ambient Dehn surgeries along disks bounded by $H(A)$ and $H(\mu)$, as pictured.

Figure 17. Left: $H(A) = H(A')$ in $W$. The horizontal and cylindrical surfaces are both cross-sections of sheets of $H(Y)$. We assume $G$ has nontrivial normal bundle. Right: We tube $H(Y)$ to a 3-sphere that bounds an Euler number $\pm 1$ disk-bundle over $G$ to obtain a new immersion, which we now call $H$. This operation does not change the singular link $L$ up to equivalence, but the relative framing induced by $A'$ on $A$ changes by $\pm 1$.

Remark 2.17. In fact, when $G$ has nontrivial normal bundle we may choose the relative framing of $A$ at the cost of changing $H(Y)$ (and hence $M''$) by ambient connect-sum with an embedded, nullhomologous 3-sphere. This is explained in [1, Page 193]. Since $G$ has nontrivial normal bundle, its normal bundle in $W$ admits a 2-dimensional subbundle whose boundary is a copy $P$ of $S^3$ with the Hopf fibration. Choose an arc $\eta$ from $A'$ to $H^{-1}(H(Y) \cap G)$ disjoint from $L$ in its interior and then connect sum $H(Y)$ to $P$ via the arc $H(\eta)$ from $H(A)$ to $P$. We redefine $H$ to be this newly obtained immersion of $Y$; the singular link is unchanged up to equivalence. (See Figure 17 for a schematic of this operation.) This changes the relative framing on $A$ by $\pm 1$, with sign depending on the choice of sign of the 2-dimensional subbundle of the normal bundle of $G$ that yielded $P$. Thus, we may assume that the relative framing on $A$ is zero.

Remark 2.18. Note that if $H'$ is obtained from $H$ by a sequence of ambient Dehn surgeries as in the section, then since $H$ is $\pi_1$-trivial, so is $H'$. This holds because
every element of $\pi_1 Y$ can be presented by a curve disjoint from the surgery curves used in the creation of $H'$. Moreover, we have $[H] = [H']$ in $H_3(W, H(\partial Y); \mathbb{Z}\pi_1 W)$ since $H, H'$ are cobordant via the trace of the 2-handle surgeries.

Making use of Remark 2.17, we are now able to prove Lemma 2.14. In order to prove this lemma, it was essential to understand relative framings and how we may manipulate them given an unframed dual. We restate the lemma for convenience.

**Lemma 2.14.** Assume that $H$ is $s$-characteristic in $X$ with a dual sphere $G$ and let $A$ and $A'$ be a dual pair of type I circles in $L$. Assume $A$ and $A'$ are both null-homotopic in $Y$. Then

$$\text{lk}(A, L - A) = \text{lk}(A', L - A') \pmod{2}.$$  

**Proof.** First we consider the simple case that $A$ and $A'$ are unknots that are unlinked from each other, so $A$ and $A'$ bound disjointly embedded disks $D_A, D_{A'}$ in $Y$ (although we expect both disks to intersect the other components of $L$). By Remark 2.17, we may change $H$ by tubing it to embedded 3-spheres, preserving $L$, so that the relative framing induced by $D_{A'}$ on $A$ is zero. Then $H(D_A) \cup H(D_{A'})$ (perturbed to be embedded) is a 2-sphere in $W$ with trivial normal bundle. Moreover, we can isotope $H(D_A) \cup H(D_{A'})$ so that it intersects $H(Y)$ transversely exactly once for every intersection of $L$ with the interiors of $D_A$ and $D_{A'}$. We conclude that $L$ intersects $D_A$ and $D_{A'}$ in the same parity – that is, we conclude that

$$\text{lk}(A, L - A) \equiv \text{lk}(A', L - A') \pmod{2},$$

as desired. \qed

**Remark 2.19.** To summarize, the move in this subsection allows us to remove a pair of dual singular circles $A, A'$ when we have that:

1. $A$ and $A'$ are unknotted and $A'$ is unlinked from all other singular circles,
2. There exists a 2-sphere $G$ in $W$ that intersects $H(Y)$ transversely in a single point.

   a. If $G$ has trivial normal bundle, then the effect to $L$ is to simply remove $A$ and $A'$, leaving the other singular circles unchanged.
   b. If $G$ has nontrivial normal bundle, then we must add some number $n$ of full twists to the circles that link $A$. If $H(Y)$ is not $s$-characteristic then we can find some other dual sphere with trivial normal bundle and arrange for $n$ to be zero, as in (2a). But if $H(Y)$ is $s$-characteristic, then we may only take $n$ to be any odd integer. We will always take $n \in \{\pm 1\}$.

3. **The Freedman–Quinn and Stong invariants**

In this section, we will review the Freedman–Quinn invariant of [1, 9] and Stong invariant of a pair of 2-spheres in a 4-manifold. We show how the vanishing of the Freedman–Quinn invariant allows us to assume that we have a singular concordance $H : S^2 \times I \to X \times I$ between the two spheres with no type II singular circles.
3.1. The Freedman–Quinn invariant and removing type II circles. We now restrict to the setting where the ambient 5-manifold is $X \times I$ for some compact, orientable 4-manifold $X$ and the 3-manifold $Y$ being immersed in $X \times I$ is $S^2 \times I$.

We begin with a definition of the self-intersection invariant $\mu$ on $\pi_3 X$, and then follow with a definition of the Freedman–Quinn invariant. See [9] and [7] for more details.

**Definition 3.1** ([1, 9]). Let $P \in \pi_3 X$ be represented by a based immersion of a 3-sphere $P : S^3 \to X \times I$ (here we are identifying $\pi_3 X = \pi_3(X \times I)$). Let $A_1, \ldots, A_n$ be the type II singular circles of $P$. For each $k$, let $g_k \in \pi_1 X$ be the group element associated to $A_k$. Since $A_k$ is type II, either $g_k$ is trivial or $g_k$ is contained in $T_X := \{g \in \pi_1 X : g^2 = 1, g \neq 1\}$.

Let $\mathbb{F}_2 T_X$ denote the vector space over the field with two elements with basis $T_X$. We then define the $\mu$ self-intersection invariant to be the map

$$\mu : \pi_3 X \to \mathbb{F}_2 T_X$$

$$P \mapsto \sum_{k|g_k \neq 1} g_k.$$

**Definition 3.2** (see also [1, 9, 7]). Let $S_0$ and $S_1$ be based-homotopic 2-spheres in a 4-manifold $X$ with basepoint $z$. Choose a based singular concordance $H : S^2 \times I \to X \times I$ from $S_0$ to $S_1$ (e.g., the trace of a based homotopy from $S_0$ to $S_1$). Let $A_1, \ldots, A_n$ be the type II singular circles of $H$. For each $k$, let $g_k \in \pi_1 X$ be the group element associated to $A_k$ (these are involutions so there is no need to distinguish between elements and their inverses). Define $\mu(H)$ to be the element of $\mathbb{F}_2 T_X$ given by

$$\mu(H) = \sum_{k|g_k \neq 1} g_k.$$

The Freedman–Quinn invariant $fq(S_0, S_1)$ of the pair $S_0, S_1$ is the element of $\mathbb{F}_2 T_X / \mu(\pi_3 X)$ represented by $\mu(H)$ for some choice of based singular concordance $H$ between $S_0$ and $S_1$.

In Definition 3.2 it is essential that $H$ is based. If $H'$ is another based immersion of $S^2 \times I$ from $S_0$ to $S_1$, then the images of $H$ and $H'$ agree at their boundaries and along the vertical arc $z \times I$. The remaining pieces of the images of $H$ and $H'$ are immersed 3-balls that together form an immersed $S^3$ in $X \times I$. We conclude that $\mu(H)$ and $\mu(H')$ differ by an element of $\mu(\pi_3 X)$ – see [9] for details.

We now deal with the issue (or rather non-issue) of basepoints in the presence of a dual sphere $G$.

**Proposition 3.3** ([9 Lemma 2.1] [7 Proposition 4.9]). Let $S_0$ and $S_1$ be homotopic 2-spheres in a 4-manifold $X^4$. Assume there is an immersed 2-sphere $G$ in $X$ with $G \cdot S_i = 1 \pmod{2}$. Let $H$ and $H'$ be (not necessarily based) singular concordances between $S_0$ and $S_1$. Then $\mu(H) = \mu(H')$ as elements of $\mathbb{F}_2 T_X / \mu(\pi_3 X)$.
Proof. Since $G \times I \cap H(S^2 \times I)$ is a 1-manifold with an odd number of endpoints in each of $S_0 \times \{0\}, S_1 \times \{1\}$, there is an arc $\eta$ in $G \times I \cap H(S^2 \times I)$ that runs from $X \times 0$ to $X \times 1$. By an isotopy in a neighborhood of $S_1 \times \{1\}$, we can arrange for $\eta$ to project to a closed loop $C$ in $X$. Since $Y$ is simply connected, $C$ represents a well-defined element of $\pi_1 X$. (Note also that since $S_1$ is simply connected, the choice of isotopy near $S_1$ that made $C$ a closed loop does not affect the group element represented by $\eta$.)

This closed loop $C$ is contained in the 2-sphere $G$ away from its self-intersections, so $C$ is nullhomotopic. We can thus isotope $H$ so that there is a vertical arc $\eta$ in $H(S^2 \times I) \cap G \times I$. We repeat from $H'$ to find a vertical arc $\eta'$ in $H'(S^2 \times I) \cap G \times I$, and then isotope these arcs to agree. The proof now follows from the discussion after Definition 3.2.

Combining Definition 3.2 and Proposition 3.3, we conclude that the Freedman–Quinn invariant $f_q(S_0, S_1)$ is well-defined if $S_0, S_1$ are homotopic (not necessarily based-homotopic) and there is an immersed 2-sphere $G$ in $X$ intersecting each of $S_i$ in an odd number of points. Note that if $H$ is a singular concordance between $S_0$ and $S_1$, and $H(S^2 \times I)$ intersects some 2-sphere $G$ transversely once, then the projection of $G$ to $X$ is an immersed sphere intersecting each $S_i$ in an odd number of points.

**Definition 3.4** ([11][9]). Let $S_0$ and $S_1$ be homotopic embedded 2-spheres in an orientable 4-manifold $X^4$ such that there exists an immersed 2-sphere $G$ in $X$ with $G \cdot S_i = 1 \pmod{2}$. Let $H : S^2 \times I \to X \times I$ be a singular concordance between $S_0$ and $S_1$. The *Freedman–Quinn invariant* $f_q(S_0, S_1)$ of the pair $(S_0, S_1)$ is defined to be $\mu(H)$ viewed as an element of $F_2 T_X / (\mu_3(\pi_3 X))$.

Note that if $S_0, S_1$ are based-homotopic, then this definition coincides with Definition 3.2.

**Proposition 3.5.** Suppose that $S_0$ and $S_1$ are homotopic embedded 2-spheres in a 4-manifold $X$ such that there exists an immersed 2-sphere $G$ in $X$ with $G \cdot S_i = 1 \pmod{2}$ and such that $f_q(S_0, S_1) = 0$. Then there exists a singular concordance $H$ between $S_0$ and $S_1$ with no type II singular circles.

The main tool in proving Proposition 3.5 is the Whitney move of Section 2.2. In brief, our strategy is to use the fact that $f_q(S_0, S_1) = 0$ to initially choose $H$ to have $\mu(H) = 0 \in F_2 T_X$, rather than $\mu(H) = 0$ only after quotienting by $\mu(\pi_3 X)$. We will then use the Whitney move to remove type II circles in pairs. The existence of such cancelling pairs is guaranteed by the assumption that $f_q(S_0, S_1) = 0$. A different approach to this argument for eliminating the type II singular circles using the hypothesis that $f_q(S_0, S_1) = 0$ is given in [7] (see in particular Figure 12).

**Proof of Proposition 3.5.** Let $H_1 : S^2 \times I \to X \times I$ be a singular concordance between $S_0$ and $S_1$. Since $f_q(S_0, S_1) = 0$, we have $\mu(H_1) = 0 \in F_2 T_X / \mu(\pi_3 X)$. Then there exists a based immersion $P : S^3 \to X \times I$ with $\mu(H_1) + \mu(P) = 0 \in F_2 T_X$. Push
the basepoint of $P$ off the basepoint of $X \times I$, interior to $H_1(S^2 \times I)$, and obtain a new immersion $H_2 : S^2 \times I \to X \times I$ by connect-summing $H_1$ and $P$. Note that the singular link $L_2$ of $H_2$ is a split union of those of $H_1$ and $P$, along with more type I circles corresponding to intersections of $H_1(S^2 \times I)$ with $P(S^3)$. Thus, $\mu(H_2) = \mu(H_1) + \mu(P) = 0 \in \mathbb{F}_2 T_X$.

By definition of $\mu$, every nontrivial 2-torsion element in $\pi_1 X$ is associated to an even number of type II singular circles in $L_2$. We must also arrange for that to be true for the trivial element. To that end, if $L_2$ has an odd number of type II circles associated to the trivial element of $\pi_1 X$, perform the move of Section 2.5 to introduce a new type II circle associated to the trivial element and denote the resulting immersion by $H_3$. Otherwise, just let $H_3 := H_2$.

Now $H_3$ is an immersion of $S^2 \times I$ whose singular link $L_3$ has the property that the type II circles of $L_3$ can be put into disjoint pairs so that in each pair, the two type II circles correspond to the same element in $T_X \cup \{1\}$.

Let $C$ and $C'$ be one such pair of type II circles in $L_3$. Let $x, x'$ be distinct points in $C$ with $H_3(x) = H_3(x')$. Let $y, y'$ be distinct points in $C'$ with $H_3(y) = H_3(y')$. Choose arcs $\gamma_1, \gamma_2$ in $Y$ from $x$ to $y$ and from $y'$ to $x'$, respectively. Take $\gamma_1, \gamma_2$ disjoint from $L_3$ in their interiors. Since $C$ and $C'$ correspond to the same element of $T_X \cup \{1\}$, $H_3(\gamma_1 \cup \gamma_2)$ is nullhomotopic (see Section 2.2.1 and note that since these are type II circles the orderings of $x, x'$ and $y, y'$ are irrelevant). Therefore, there is an embedded disk $D$ in $X \times I$ with $\partial D = H_3(\gamma_1 \cup \gamma_2)$. Frame $\gamma_1$ and $\gamma_2$ so as to yield framed bands $b_1, b_2$ for some (not necessarily clean) Whitney disk. Perform a Whitney move to obtain a new immersion which we denote by $H_4 : S^2 \times I \to X \times I$.

Recall from Section 2.2.1 that the singular link $L_4$ of $H_4$ differs from $L_3$ by surgery along the bands $b_1, b_2$, along with the potential addition of dual pairs of type I circles. By Remark 2.5, surgering $C \cup C'$ along $b_1$ and $b_2$ yields exactly two circles. These two circles in $L_4$ each include one arc in $C$ and one arc in $C'$ and have the same image under $H_4$, i.e., they are a type I dual pair. The net result of the above sequence of moves is that $H_4$ has two fewer type II circles than $H_3$, and the type II circles of $H_4$ still occur in pairs associated to the same fundamental group element. We may therefore repeat this process until obtaining an immersion $H : S^2 \times I \to X \times I$ that has no type II circles.

\[\square\]

**Remark 3.6.** Note that if $\mu(\pi_3 X) = 0$, then such a map $H$ as in Proposition 3.5 exists for any desired homology class in $H_3(X \times I, X \times \{0, 1\}; \mathbb{Z}_2 X)$ that can be represented by a singular concordance class between $S_0$ and $S_1$. This is because $\text{fq}(S_0, S_1) = 0$ implies that for any singular concordance $H'$ from $S_0$ to $S_1$, we have $\mu(H') = 0$ (rather than just vanishing in a quotient). Then we need not add any $\pi_3 X$ elements to obtain $H$ in the proof of Proposition 3.5. The other operations used in the proof preserve the homology class of the image (see Remark 2.1).

### 3.2. The Stong invariant.

In this section we define the Stong invariant for certain pairs of homotopic, oriented 2-spheres $S_0, S_1$ embedded in a 4-manifold $X$. Like the
Freedman–Quinn invariant, the Stong invariant is defined by studying the intersection set of a singular concordance between $S_0$ and $S_1$. In [11], Stong gives an argument for why this invariant is well-defined (with some additional hypotheses necessary to define the invariant at all) when the element of $H_3(X \times I, X \times \{0, 1\}; \mathbb{Z}\pi_1)$ represented by a concordance is specified (the theorem of [11] is phrased for 3-spheres in a 5-manifold but we discuss later how this extends to pairs of 2-spheres in a 4-manifold – this is also mentioned in [9]). Later in this section, we discuss a quotient of the target in which stong is well-defined without specifying the homology class represented by $H(S^2 \times I)$.

Let $S_0$ and $S_1$ be embedded, s-characteristic spheres in an orientable 4-manifold $X$ with $\text{fq}(S_0, S_1) = 0$ (here, either take $S_0, S_1$ to be based-homotopic or to be homotopic with an immersed 2-sphere intersecting each $S_i$ in an odd number of points as in Proposition 3.3). We will define an invariant $\text{stong}(S_0, S_1)$ that is valued in a quotient of $H_1(X; \mathbb{Z}/2\mathbb{Z})$. The invariant stong is secondary to $\text{fq}$, in the sense that stong cannot be defined for a pair of spheres with nonzero Freedman–Quinn invariant. Several choices will be made in defining stong. Stong [11] shows (see also [6]) that stong is independent of these choices.

Before proceeding, we need to define the relative twist number of a pair of type II components of a singular link $L$.

**Definition 3.7.** Let $A$ and $B$ be type II circles in $L$ that are associated to the same element of $\pi_1(X)$. Let $b_1, b_2$ be framed bands connecting $A$ and $B$. Let $\gamma, \gamma'$ be the two circles that result from surgering $A$ and $B$ along $b_1, b_2$. (Note that if $H(S^2 \times I)$ does not have a dual, then we may not be able to find a clean Whitney disk bounded by $H(b_1 \cup b_2)$, so we are not actually performing a Whitney move to obtain $\gamma$ and $\gamma'$. ) The *relative twist number* of $A$ and $B$ is

$$\text{tw}(A, B) := \text{lk}(\gamma, \gamma') + \text{lk}(\gamma, L - \{A, B\}) \pmod{2}.$$
A proof that $\text{tw}$ is well defined is given by Stong [11] (see also [6]). We now define the Stong invariant.

**Definition 3.8 ([11]).** Assume $fq(S_0, S_1) = 0$ for s-characteristic, oriented 2-spheres $S_0, S_1$ in a 4-manifold $X$. Let $H : S^2 \times I \to X \times I$ be a singular concordance between $S_0$ and $S_1$. Assume that either $S_0, S_1$ are based homotopic and $H$ is based or assume there is a 2-sphere $G$ in $X \times I$ intersecting $H(S^2 \times I)$ in an odd number of points.

The singular link $L$ of $H$ consists of pairs of type I components, and type II components each corresponding to an element of $T_X \cup \{1\}$.

1. For each pair of type I circles, choose one to be active and the other to be inactive. Denote the active circle by $A$, its associated group element in $\pi_1 X$ is named $a$ (i.e., the element we obtain by going into the sheet $A$ and leaving on the other sheet).
2. For each $g \in T_X$, let $L_g$ denote the collection of type II components of $L$ associated to $g$.
3. For each $h \in \pi_1 X$, let $\epsilon_h$ denote the image of $h$ in $H_1(X; \mathbb{Z}/2\mathbb{Z})$ under $\mathbb{Z}/2\mathbb{Z}$-abelianization.

Let

$$
\Delta(H) := \sum_{A \text{ type I, active}} \text{lk}(A, L - A) \epsilon_a + \sum_{g \in T_X} \sum_{B, C \in H_g} \text{tw}(B, C) \epsilon_g \in H_1(X; \mathbb{Z}/2\mathbb{Z})
$$

Let $\text{Self}(S_0)$ denote the set of singular concordances between $S_0$ and $S_0$. Note that $\text{Self}(S_0)$ has a monoid structure given by stacking, $\Delta : \text{Self}(S_0) \to H_1(X; \mathbb{Z}/2\mathbb{Z})$ is a monoid homomorphism, and since every element of $H_1(X; \mathbb{Z}/2\mathbb{Z})$ is order 2, the image $\Delta(\text{Self}(S_0))$ is a subgroup of $H_1(X; \mathbb{Z}/2\mathbb{Z})$.

Given a fixed homology class $\alpha \in H_3(X \times I, X \times \{0, 1\}; \mathbb{Z} \pi_1 X)$, the *Stong invariant* of $S_0, S_1$ relative to $\alpha$ is

$$\text{stong}(S_0, S_1; \alpha) := \Delta(H) \in H_1(X; \mathbb{Z}/2\mathbb{Z})$$

for some singular concordance $H$ between $S_0$ and $S_1$ with $[H] = \alpha$.

The *Stong invariant* of $S_0, S_1$ is

$$\Delta(H) \in H_1(X; \mathbb{Z}/2\mathbb{Z})/(\Delta(\text{Self}(S_0))).$$

We leave it as an easy exercise that for homotopic 2-spheres $S_0, S_1$, we have $\Delta(\text{Self}(S_0)) = \Delta(\text{Self}(S_1))$ and thus Definition 3.8 is symmetric in $S_0, S_1$.

The notation $\Delta$ comes from Stong [11]. The invariant $\Delta$ serves a similar role in the definition of stong that $\mu$ does for $fq$. It should not be obvious why $\Delta(H)$ and $\text{stong}(S_0, S_1; \alpha)$ are well defined, as this is a theorem of Stong (stated in a slightly different context). We plan to explain this in significant detail for concordances in [6].

Note that, if $S_0$ and $S_1$ are concordant, then $\text{stong}(S_0, S_1) = 0$. Similarly, if $S_0$ and $S_1$ are concordant via a concordance representing $\alpha$ we have $\text{stong}(S_0, S_1; \alpha) = 0$. 
While the set $\text{Self}(S_0)$ is unwieldy, $\Delta(\text{Self}(S_0))$ can often be computed in practice. For example, when $X$ is a 2-handlebody, then we have $H_3(X \times I, X \times \{0, 1\}; \mathbb{Z} \pi_1 X) = 0$ — that is, every element of $\text{Self}(S_0)$ is homologous to $S_0 \times I$, so $\Delta(\text{Self}(S_0)) = 0$. Then $\text{stong}(S_0, S_1)$ may be computed using any singular concordance.

More generally, if we can find a representative in $\text{Self}(S_0)$ from each relevant $H_3(X \times I, X \times \{0, 1\}; \mathbb{Z})$ class (which is not too daunting when this group is 0 or $\mathbb{Z}$, for example) then and modify it to represent different $H_3(X \times I, X \times \{0, 1\}; \mathbb{Z} \pi_1)$ classes and thus compute $\Delta(\text{Self}(S_0))$. We discuss this again in Example 5.1.

In Section 4.2, we will see that when $H(S^2 \times I)$ has a dual sphere, then in fact we can modify $H$ (via the moves in Section 2 and thus preserving $H_3$) and arrange for $L$ to be a Hopf link consisting of a type I pair $A, A'$. In this case, $\text{stong}(S_0, S_1; \alpha) = \epsilon_a$, and when $\epsilon_a = 1$ then it is possible to replace $H$ with an embedding.

### 3.3. Problems with positive genus.

In this section we discuss the issue of extending the invariants $f_q$ and stong to positive genus surfaces and raise a question regarding the definition of stong.

In [7], we discussed the case what we called “$\pi_1$-negligible,” homotopic, orientable genus-$g$ surfaces $F_0$ and $F_1$ embedded in a 4-manifold $X$. This was an abuse of terminology; in the setting of this paper we would say $\pi_1$-trivial rather than $\pi_1$-negligible. Unfortunately, there is an error in the proof of Lemma 6.1 and Proposition 6.2 in [7], drawn to our attention by Mark Powell. The strategy of the proof of [7, Lemma 6.1] (which [7, Proposition 6.2] relies on) was to compress an immersion $H: \Sigma_g \times I \rightarrow X \times I \times \mathbb{R}$ with $H(\Sigma_g \times 0) = H(\Sigma_g \times 1)$ along a 3-dimensional 2-handle to obtain an immersion $H': \Sigma_{g-1} \times I \rightarrow X \times I \times \mathbb{R}$ with $H'(\Sigma_{g-1} \times 0) = H'(\Sigma_{g-1} \times 1)$, eventually reducing to the understood setting of an immersion of $S^2 \times I$. Unfortunately, we can only ensure that $H(\Sigma_g \times 0) = H(\Sigma_g \times 1)$ as submanifolds, not that $H|_{\Sigma_g \times 0} = H|_{\Sigma_g \times 1}$ as a map, so there is no guarantee that our compression will take place along the same circles in $\Sigma_g \times 0$ and $\Sigma_g \times 1$. Even if these circles are the same, there is again no guarantee that the compressing disks themselves will agree, so again we cannot ensure that $H'(\Sigma_{g-1} \times 0) = H'(\Sigma_{g-1} \times 1)$.

**Question 3.9.** Can the Freedman–Quinn invariant be defined for (some) arbitrary genus based-homotopic surfaces? Similarly, can the Stong obstruction be extended to include suitable higher-genus surfaces?

The Stong invariant seems even more difficult to extend to positive-genus surfaces due to its constrained definition. The Freedman–Quinn invariant has an alternative formulation in terms of immersions into the 6-manifold $X \times I \times \mathbb{R}$ (see [9]) and lifts of immersions to the universal cover of $X \times I$ (see [7]). It seems possible that this flexibility could be useful in an attempt to study positive-genus surfaces. We are motivated to ask the following question.

**Question 3.10.** Is there a 6-dimensional definition of stong? If so, does this make it easier to prove that stong is well defined (as compared to [11])?
4. Reducing the singular link when \( H(S^2 \times I) \) has a dual sphere

In this section, we will use the moves presented in Section 2 to simplify \( H \) when \( H(S^2 \times I) \) admits a dual sphere, eventually proving Theorem 1.5. As in Section 3, we only consider immersions from \( S^2 \times I \) to \( X \times I \), rather than a general 3-manifold into a general 5-manifold (as was the case in Section 2). The arguments in this section follow those of Stong (see in particular Figures 8–16 of [11] and compare with our Figures 21–25).

4.1. Eliminating all intersections when \( H(S^2 \times I) \) is not \( s \)-characteristic.

The key difference between the situations in Theorem 1.5 and Theorem 1.6 is that when \( H(S^2 \times I) \) has a dual and is not \( s \)-characteristic, we can find a dual sphere with trivial normal bundle.

Lemma 4.1. Let \( H : Y^3 \rightarrow W^5 \) be an immersion so that there is a 2-sphere \( G \) embedded in \( W \) intersecting \( H(Y) \) transversely once. Suppose \( H(Y) \) is not \( s \)-characteristic. Then there is a 2-sphere \( G' \) embedded in \( W \) with trivial normal bundle intersecting \( H(Y) \) transversely once.

Proof. If \( G \) has trivial normal bundle, then the lemma is obviously trivial. So assume \( G \) has nontrivial normal bundle.

Since \( H(Y) \) is not \( s \)-characteristic, there is some 2-sphere \( R \) embedded in \( X \times I \) with \( R \cdot H(Y) \neq [R] \cdot [R] \) (mod 2). Let \( n \) be the geometric intersection number of \( R \) and \( H(Y) \). Let \( G_1, \ldots, G_{n-1} \) be parallel copies of \( G \). Choose disjoint arcs in \( H(Y) \) connecting the intersection points of each \( G_i \) with distinct points in \( R \cap H(Y) \). Use these arcs to surger \( R \sqcup G_1 \sqcup \cdots \sqcup G_{n-1} \) to obtain one 2-sphere \( G' \) that intersects \( H(Y) \) transversely once.

We have

\[
[G'] \cdot [G'] = ((n - 1)[G] + [R]) \cdot ((n - 1)[G] + [R])
\]

\[
= (n - 1)^2[G] \cdot [G] + [R] \cdot [R] \quad \text{(mod 2)}
\]

\[
= n - 1 + [R] \cdot [R] \quad \text{(mod 2)}
\]

\[
= 0 \quad \text{(mod 2),}
\]

since \([R] \cdot [R] \neq n \) (mod 2) by assumption. \( \square \)

Now we can prove Theorem 1.5, which we restate here for convenience.

Theorem 1.5. Suppose that \( S_0 \) and \( S_1 \) are embedded, oriented, homotopic 2-spheres in an orientable 4-manifold \( X \) such that \( S_0 \) has an immersed dual sphere \( G \) in \( X \) (i.e., \( G \) and \( S_0 \) intersect in a single point). Assume that \( S_0 \) is not \( s \)-characteristic. Then \( S_0 \) and \( S_1 \) are concordant if and only if \( f_q(S_0, S_1) = 0 \).

Proof. If \( S_0 \) and \( S_1 \) are concordant, then \( H \) can be chosen to be an embedding. Then \( f_q(S_0, S_1) = \mu(H) = 0 \). Thus, the “only if” portion of the statement is trivial.

Now suppose \( f_q(S_0, S_1) = 0 \). By Proposition 3.5 we can choose a singular concordance between \( S_0 \) and \( S_1 \) whose singular link \( L \) consists of only type I circles. Push
the 2-sphere $G$ into $X \times I$, so that it is an embedded 2-sphere intersecting the image of $H$ transversely once. By Lemma 4.1, take $G$ to have trivial normal bundle.

We use the ambient surgery technique of Section 2.7 to remove dual pairs of type I singular circles. To do this, choose immersed disks bounding the singular circles in $S^2 \times I$ that only have clasp intersections as in Lemma 2.15. Suppose that $A$ and $B$ are two singular circles (that are potentially dual or not distinct) such that the disks that they bound have a clasp intersection. In Figure 19 we illustrate the proceeding set of moves.

We perform a finger move as in Section 2.6 to remove the clasp at the cost of creating one pair of dual circles (see Section 2.6) $E$ and $E'$, with $E$ a meridian of $A'$ (the dual of $A$) and $E'$ a meridian of $B'$ (the dual of $B$). The circles $E$ and $E'$ bound meridian disks that clasp those bounded by $A'$ and $B'$. We perform another finger move to remove the clasp between $A'$ and $E$, pulling $E$ off $A'$ entirely. This introduces yet another pair of dual circles $F$ and $F'$ (again, see Section 2.6) with $F$ a meridian of $E'$ and $F'$ a meridian of $A$.

Now because $G$ is a framed dual to $H(S^2 \times I)$, we may use the ambient Dehn surgery technique of Section 2.7 (refer to Remark 2.19) to alter $H$ and remove first $E$ and $E'$, then $F$ and $F'$, completely from $L$ without otherwise altering $L$.

The total result is that we have removed a clasp from the disks bounded by $A$ and $B$ without otherwise altering the singular link $L$. By repeating this at every clasp, we can thus arrange for $L$ to be an unlink. We may again use the ambient Dehn surgery technique of Section 2.7 to remove each pair of singular circles. The resulting map from $S^2 \times I$ to $X \times I$ is thus an embedding, so $S_0$ and $S_1$ are concordant. □

4.2. Reduction to a Hopf link of singular circles. In this section, we begin the proof of of Theorem 1.6. We now take $S_0$ and $S_1$ to be $s$-characteristic, so any dual sphere to $H(S^2 \times I)$ must have nontrivial normal bundle. We will see how

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure19.png}
\caption{Eliminating a clasp from $L$ when $H(S^2 \times I)$ has a dual sphere with trivial normal bundle.}
\end{figure}
in this situation, we can still modify $H$ to reduce the number of components in its singular link $L$, eventually obtaining a type I dual pair forming a Hopf link. In the next section, we will show that when the element of $\pi_1 X$ corresponding to this last pair represents the trivial element of $H_1(X; \mathbb{Z}/2\mathbb{Z})$, then we can further remove this Hopf link to obtain a concordance from $S_0$ to $S_1$.

**Proposition 4.2.** Let $S_0, S_1$ be homotopic 2-spheres in an orientable 4-manifold $X$ with $fq(S_0, S_1) = 0$. Let $H'$ be a singular concordance from $S_0$ to $S_1$ with $\mu(H) = 0$. Suppose $S_0$ is $s$-characteristic and there exists an immersed 2-sphere in $G$ intersecting $S_0$ transversely in a single point. Then there is a singular concordance $H$ between $S_0$ and $S_1$ such that the singular link $L$ of $H$ is a type I dual pair forming a Hopf link.

**Proof.** As in the proof of Theorem 1.5 (contained in Section 4.1), we use Proposition 3.5 to choose an initial $H$ whose singular link $L$ consists of only type I circles, and we push $G$ into the interior of $X \times I$ so that it is embedded and intersects $H(S^2 \times I)$ transversely once. Since $H(S^2 \times I)$ is $s$-characteristic, $G$ must have nontrivial normal bundle.

Via Lemma 2.15, we choose a collection $D$ of immersed disks in $S^2 \times I$ bounded by each component of $L$ with the property that the intersections of $D$ are all clasp intersections. As in subSection 4.1, these clasps suggest finger moves. We modify $H$ and then refer to the result again as $H$ throughout. This modification will proceed in a number of steps; we summarize them here and will then give details about how to achieve each.

1. We perform clasping finger moves to $H$ until the singular link $L$ consists of an unlink of type I dual pairs, together with a collection of small linking meridians to the unlink that are also all type I.

2. Through a sequence of finger moves, ambient surgeries, and Whitney moves, we modify $H$ so that $L$ is a split union of Hopf links in disjoint 3-balls in $S^2 \times I$, with all components type I. We call these Hopf links $h_1, \ldots, h_n$ and note that they are partitioned into disjoint cycles by considering the equivalence classes generated by the relation, “$h_i \sim h_j$ if one of the components of $h_i$ is dual to one of the components of $h_j$.”

3. We shorten the cycles of Hopf links, obtaining a new $H$ whose singular link is a split union of Hopf links such that each Hopf link consists of one type I dual pair.

4. Finally, we modify $H$ to remove all but one of these Hopf links from $L$ – or perhaps more accurately, we merge all of the Hopf links in $L$ to form a single Hopf link.

Now we give more details on how to perform each of the above steps.

4.2.1. **Step 1.** We perform a finger move (as in Section 2.6) on each clasp of $D$. The result is to transform the components of $L$ into an unlink, at the cost of adding many pairs of dual type I circles that are meridians of the original components. We
call the original components of $L$ “long circles” and the new meridian circles “short circles,” so now $L$ consists of an unlink of long circles and an unlink of short circles, so that every short circle is a meridian of a long circle. Every component of $L$ is type I, and the dual of any long circle is long while the dual of any short circle is short.

4.2.2. Step [2]. Let $A, A'$ be a dual pair of long circles. If there are no short circles linking $A$ and $A'$, then perform ambient surgery to remove both circles. Thus, without loss of generality, assume that there is at least one short circle linking $A$, and that there are at least as many short circles linking $A$ as there are linking $A'$ (by potentially switching the roles of $A$ and $A'$).

In Figure 20, we show how we may perform a finger move and then ambient surgery to decrease the number of short circles linking $A$ by one while increasing the number of short circles linking $A'$ by one. Recall from Lemma 2.14 that the number of short circles linking $A$ has the same parity as the number of short circles linking $A'$. Therefore, by repeatedly performing this move we may arrange that $A$ and $A'$ link equal numbers of short circles.

Pick points $x, y \in A$ and $x', y' \in A'$ so that $H(x) = H(x')$ and $H(y) = H(y')$. Furthermore, pick arcs $\alpha$ from $x$ to $y$ and $\alpha'$ from $x'$ to $y'$ that are contained in the disks that demonstrate that the link of long circles is an unlink, so that the arcs cut
Figure 21. A dual pair $A$ and $A'$ of long circles in $L$. The bold arcs in $A$ and $A'$ have the same image under $H$ and each link one short circle. Bands contained in disks for $A$ and $A'$ disjoint from the other long circles give a framed Whitney disk exactly when the resulting pairs of long circles would satisfy Lemma 2.14.

By Section 2.3, we know that $\alpha, \alpha'$ can be framed to yield bands that are framed for some clean Whitney disk. Consider the band-thickenings $b$ and $b'$ of $\alpha$ and $\alpha'$ (respectively) as in Figure 21. These thickening are chosen to lie inside disjoint disks $D_1$ and $D_2$ bounded by $A$ and $A'$. It is easy to verify that the conditions of Remarks 2.5 and 2.6 are satisfied by $b, b'$, i.e., that $H(b \cup b')$ is an annulus and the bundles $l_1, l_2$ are orientable. Then by Section 2.2.3, either $b, b'$ are framed, or they would be framed if a whole twist were added to $b$. But if we were to add a twist to $b$ and perform band surgery, we would obtain a singular link violating the conclusion of Lemma 2.14, so we conclude such a Whitney move is not possible. Thus, $b, b'$ are framed for some clean Whitney disk. We perform the Whitney move, splitting $A$ into two components $A, B$ and $A', B'$, all of which we consider long circles. The circles $B$ and $B'$ each link one short circle, forming two Hopf links, and we have decreased the number of short circles linking $A$ and $A'$ by one. We repeat until each of $A$ and $A'$ link only one short circle, forming Hopf links.

4.2.3. Step (3). At the end of Step (2), the singular link $L$ consists of many Hopf links of type I circles. A cycle in $L$ consists of $m$ Hopf links whose components are $(A_1, A'_2), (A_2, A'_3), \ldots, (A_m, A'_1)$ where $A_i, A'_i$ are a dual pair. We say that such a cycle has length $m$.

We shorten a length $m > 1$ cycle as follows – see Figure 22. Perform a finger move to unclasp $A_1$ from $A'_2$. (If $n = 2$, then $A_2 = A'_m$.) This has the cost of adding two singular circles $B, B'$ that are respectively meridians of $A_2$ and $A'_1$. We perform ambient Dehn surgery to remove the pair $A_1, A'_1$, and another ambient Dehn surgery to remove the pair $A_2, A'_2$. Because $G$ has nontrivial normal bundle, in addition to removing $A_1, A'_1, A_2, A'_2$, these ambient surgeries have the effect of adding a whole twist to the remaining components of $L$ that link $A'_1$ and $A_2$. Then $A_m \cup B'$ and $B \cup A'_2$ are Hopf links. Thus, $L$ now includes Hopf links of components $(B, A'_3), (A_3, A'_4), \ldots, (A_{m-1}, A'_m), (A_m, B')$. This is a length $(m - 1)$ cycle. By repeating this argument, we can arrange for $L$ to only consist of length 1 cycles, i.e., Hopf links each consisting of one dual pair. Moreover, as we indicate in Figure 22,
Figure 22. Given a cycle of length $m > 1$, we replace it with a cycle of length $m - 1$. Top row: the length $m$ cycle. From top to bottom, we perform a finger move to unclasp $A_1$ and $A'_2$, then we perform ambient Dehn surgery to remove $A_1, A'_1$ and $A_2, A'_2$. In parentheses next to each active circle, we indicate the associated fundamental group element.

the group element associated to $B$ is $g_2g_1$, where $g_i$ is the group element associated to $A_i$. Then inductively, the group element associated to the length one cycle obtained from $(A_1, A'_2), (A_2, A'_3), \ldots, (A_m, A'_1)$ is $g_m g_{m-1} \cdots g_3 g_2 g_1$.

4.2.4. Step (4). In Figure 23, we illustrate how to perform a sequence of finger moves, a Whitney move, ambient Dehn surgery, and the cycle shortening of step (3) to replace two split Hopf links in $L$ with one Hopf link.

Remark 4.3. The Hopf link obtained in Figure 23 may be taken to have associated group element $ba$, where $a$ and $b$ are the group elements associated to $A$ and $B$, respectively. Suppose $A$ and $B$ are associated to elements $a, b \in \pi_1(X)$. Then $C$ and $D$ are each naturally associated to $c = ba$, but by exchanging the roles of $D$ and $D'$ the group element $d$ associated to $D$ is $a^{-1}b^{-1}$. (Recall that the group element associated to a type I self-intersection is defined only up to inverse due to the choice of active preimage.) The circles $E$ and $F$ are both associated to $e = f = 1$. Then the final active circle $G$ is associated to the product $f \cdot e \cdot d \cdot c \cdot b \cdot a = 1 \cdot 1 \cdot a^{-1}b^{-1} \cdot ba \cdot b \cdot a = ba$.

We repeat until $L$ consists of a single Hopf link, completing the proof of Proposition 4.2.

Remark 4.4. In the proof of Proposition 4.2, the element of $\pi_1 X$ associated to the final Hopf link is a product of all the group elements associated to the active singular circles at the end of Step (3) (at which time $L$ is a split union of Hopf links). The terms of this product may be taken to be in any order, according to a choice of in which order we merge the various Hopf links together in Steps (3) and (4).
Figure 23. In the top left, we show two dual pairs of type I circles in $L$, each of which forms a Hopf link. Following the arrows, we perform moves to $H$ to replace these two Hopf links with a single Hopf link.

**Remark 4.5.** Note that as in Remark 3.6, if $\mu(\pi_3 X) = 0$, then such a map $H$ as in Proposition 4.2 exists in any homology class in $H_3(X \times I, X \times \{0, 1\}; \mathbb{Z})$ that can be represented by a singular concordance from $S_0$ to $S_1$ (since we automatically have $\mu(H) = 0$).

**Lemma 4.6.** Let $H$ be a singular concordance between the spheres $S_0$ and $S_1$ such that $H(S^2 \times I)$ has a dual sphere $G$. Let $g$ be any element of $\pi_1 X$ representing the trivial element of $H_1(X; \mathbb{Z}/2\mathbb{Z})$. We can modify $H$ to add a Hopf link to the link of singular circles $L$ of $H$, split from the rest of $L$, consisting of a dual type I pair with active circle associated to $g$.

In order to prove Lemma 4.6, we will make use of the following group-theoretic lemma.

**Lemma 4.7.** Let $V$ be a path-connected topological space and $\mathcal{G}$ a finite set of generators for $\pi_1 V$. The kernel of the map $\pi_1 V \to H_1(V; \mathbb{Z}/2\mathbb{Z})$ contains exactly the elements of $\pi_1 V$ that can be represented by a word $w$ in the alphabet $\mathcal{G} \cup \mathcal{G}^{-1}$ (where $\mathcal{G}^{-1}$ denotes the set of inverses of elements of $\mathcal{G}$) where the total number of times $a$ appears in $w$ is equal to the number of times $a^{-1}$ appears in $w$ modulo 2 for each $a \in \mathcal{G}$. 

Figure 24. We perform a finger move along an arc representing group element \( g_i \) to introduce an unlinked pair of type I circles in \( L \) representing \( g_i \). Then we perform a finger move along bands as shown to replace this unlink with two Hopf links, each of which is a dual pair corresponding to \( g_i \).

Proof. Since \( H_1(V; \mathbb{Z}/2\mathbb{Z}) \) is abelian with every nontrivial element 2-torsion, any word in \( \mathcal{G} \cup \mathcal{G}^{-1} \) with the property that every letter and its inverse appear with the same parity must be contained in \( \ker(\pi_1(V) \to H_1(V; \mathbb{Z}/2\mathbb{Z})) \).

Now suppose \( h \in \ker(\pi_1(V) \to H_1(V; \mathbb{Z}/2\mathbb{Z})) \). The coefficient exact sequence

\[
0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0
\]

gives rise to the Bockstein exact sequence which in particular shows that the kernel of the map \( H_1(V; \mathbb{Z}) \to H_1(V; \mathbb{Z}/2\mathbb{Z}) \) is equal to the image of the map \( \mathcal{G} \cup \mathcal{G}^{-1} \) \( \mathcal{G} \cup \mathcal{G}^{-1} \) representing \( h_0 \). Then \( w := w_0w_0^2 \) is the desired word representing \( h \).

We now prove Lemma 4.6

Proof of Lemma 4.6. Let \( g_1, \ldots, g_n \) be generators for \( \pi_1X \). By Lemma 4.7, there is a word \( w \) in \( \{g_i, g_i^{-1}|1 \leq i \leq n\} \) representing \( g \) such that for each \( i \), \( g_i \) and \( g_i^{-1} \) appear in \( w \) with the same parity. Let \( n_i^+ \) and \( n_i^- \) be the number of times \( g_i \) and \( g_i^{-1} \) (respectively) appear in \( w \), so \( n_i^+ + n_i^- \) is even for each \( i \).

In Figure 24, we show how to add two split Hopf links of dual type I pairs to \( L \) with both active circles associated to \( g_i \). In words, we introduce a split 2-component unlink \( B \sqcup B' \) to \( L \) consisting of a type I pair associated to \( g_i \) via a finger move. We draw a pair of bands \( b_1, b_2 \) attached to \( B \) and \( B' \) in Figure 24 with the property that band surgery to \( B \sqcup B' \) along \( b_1 \) and \( b_2 \) yields two Hopf links split from the rest of \( L \). These bands are framed for some clean Whitney disk by the same argument we used in part 2 of the proof of Proposition 4.2. We easily check that \( b_1, b_2 \) satisfy the
orientability conditions of Remarks 2.5 and 2.6, implying that either these bands are framed or they would be framed if a whole twist were added to $b_1$, but adding a whole twist to $b_1$ and performing the band surgery would yield a link violating the conclusion of Lemma 2.14. We then modify $H$ by performing a clean Whitney move along $b_1$ and $b_2$, yielding two Hopf links both associated to $g_i$.

Perform the above move $|w|/2$ times, where $|w|$ represents the length of $w$ and we vary $i$ so that for each $i$ we introduce $n_i^+ + n_i^-$ Hopf links with active circle corresponding to $g_i$. In $n_i^-$ of these dual pairs, reverse the roles of the active and inactive circles so now these Hopf links have active circle associated to $g_i^{-1}$.

Finally, merge these $|w|$ Hopf links together to form one Hopf link split from the rest of $L$ as in the proof of Proposition 4.2 (part 5). Choose the order in which the Hopf links are merged to ensure that the active circle of the resulting Hopf link is associated to the element of $\pi_1X$ represented by $w$, i.e., $g$ (see Remarks 4.3 and 4.4).

Lemma 4.6 lets us further simplify $H$ in Proposition 4.2 in the case that the single active circle in $L$ is associated to an element of $\pi_1X$ representing the trivial element of $H_1(X; \mathbb{Z}/2\mathbb{Z})$.

**Proposition 4.8.** Let $S_0, S_1, G, H$ be as in Proposition 4.2 where the singular link $L$ is a Hopf link of type I circles with corresponding group element $g \in \pi_1X$. Suppose $g$ represents the trivial element of $H_1(X; \mathbb{Z}/2\mathbb{Z})$. Then $S_0$ and $S_1$ are concordant.

**Proof.** Using Lemma 4.6, we modify $H$ to add a second Hopf link to $L$ associated to $g^{-1}$. Merge these two Hopf links as in Figure 23 so that $L$ is a single Hopf link consisting of a dual type I pair with active circle associated to $g^{-1}g = 1$.

Again apply Lemma 4.6 to further modify $H$ to add a second Hopf link to $L$ with active circle associated to the trivial element of $\pi_1X$. Now $L$ is a split union of two Hopf links, each consisting of a dual type I pair with active circle associated to $1 \in \pi_1X$.

Consider the framed bands $b_1$ and $b_2$ attached to $L$ pictured in the top left of Figure 25. We indicate numbers $m, n$ of half-twists in $b_1, b_2$ (we can take both $m, n \in \{0, 1\}$ but this is not important for this discussion). In the top and middle row of Figure 25 we show how to perform finger moves, clean Whitney moves, and ambient surgeries to $H$ to reduce $n$ by one. Using symmetry, we can thus performing this operation until $m = n = 0$. Now perform a clean Whitney move along $b_1, b_2$, so that the singular link $L$ becomes a 2-component unlink which we can remove by ambient surgery as illustrated in the bottom row of Figure 25. The map $H$ is now an embedding, so $S_0$ and $S_1$ are concordant.

We now are in position to prove Theorem 1.8 (and then will conclude Theorem 1.6 as a corollary).

**Theorem 1.8.** Suppose that $S_0$ and $S_1$ are embedded, oriented, homotopic 2-spheres in an orientable 4-manifold $X$ such that $S_0$ has an immersed dual sphere $G$
Figure 25. In the top left we show two valid bands for a Whitney move. Here $n$ and $m$ denote numbers of half-twists. Following the arrows, we add a negative half-twist to one of the bands. By repeating this argument (perhaps mirroring or reversing signs), we obtain the bottom left diagram. Performing this Whitney move transforms the original two Hopf links into a 2-component unlink of dual type I circles, which we can remove by performing one ambient surgery.
in \( X \) and \( S_0 \) is \( s \)-characteristic. Then \( S_0 \) and \( S_1 \) are concordant if and only if there exists a singular concordance \( H \) from \( S_0 \) to \( S_1 \) so that \( \mu(H) = 0 \) and \( \Delta(H) = 0 \).

**Proof.** If \( S_0 \) and \( S_1 \) are concordant, then a concordance \( H \) between them satisfies \( \mu(H) = 0 \) and \( \Delta(H) = 0 \). By Proposition 4.2 there is a singular concordance \( H' \) from \( S_0 \) to \( S_1 \) whose singular link \( L \) is a Hopf link consisting of a dual type I pair and \( [H'] = [H] \) in \( H_3(X \times I, X \times \{0,1\}; \mathbb{Z}\pi_1X) \). Then \( \Delta(H') = \Delta(H) = 0 \). Therefore, the element of \( \pi_1(X) \) associated to the active singular circle of the singular link of \( H' \) represents 0 in \( H_1(X; \mathbb{Z}/2\mathbb{Z}) \). Applying Proposition 4.8 we find that \( S_0 \) and \( S_1 \) are concordant. \( \square \)

Theorem 1.6 follows directly from Theorem 1.8 because if \( \text{stong}(S_0, S_1) \neq 0 \) then \( \text{stong}(S_0, S_1; [H]) = 0 \) for some \( H \), and hence \( \Delta(H) = 0 \). If \( \mu(\pi_3(X)) = 0 \), then \( \text{fqi}(S_0, S_1) = 0 \) implies \( \mu(H) = 0 \) automatically.

5. **Additional discussion**

In this section, we give an example from our earlier work in [7] of a pair of 2-spheres \( S_0, S_1 \) with \( \text{stong}(S_0, S_1) \neq 0 \) and discuss the relevance of realizing the Stong invariant with obstructing sliceness of spherical links. In addition, we remark on the finiteness of certain concordance classes of knotted spheres.

**Example 5.1.** In [7], we show how to produce 2-spheres \( S_0, S_1 \) with nontrivial Stong invariant. In [7] we omit discussion of the quotient of the target by \( \Delta(\text{Self}(S_0)) \), so we discuss it in greater detail here.

Let \( X \) be a 4-manifold, \( S \) be an \( s \)-characteristic 2-sphere in \( X \), and \( \alpha \in H_1(X; \mathbb{Z}/2) \) nontrivial and not in the image of \( \text{Self}(S) \) under \( \Delta \). In practice, it is often not difficult to check that \( \alpha \) is not in \( \Delta(\text{Self}(S)) \); note that the relative long exact sequence in homology includes the following.

\[
H_3(X; \mathbb{Z}\pi_1X) \xrightarrow{\partial} H_3(X \times I, X \times \{0,1\}; \mathbb{Z}\pi_1X) \xrightarrow{\partial} H_2(X \times \{0,1\}; \mathbb{Z}\pi_1X).
\]

Therefore, given any two \( H, H' \in \text{Self}(S) \), since we have \( \partial(H) = \partial(H') \) we must have \( [H'] = [H] + \pi(x) \) for some \( x \in H_3(X; \mathbb{Z}\pi_1X) \). We could now for example consider one of the following situations.

- \( X = B^3 \times S^1 \), \( S \) any 2-sphere, \( \alpha \) the generator of \( H_1(B^3 \times S^1; \mathbb{Z}/2\mathbb{Z}) \). Since \( H_3(X; \mathbb{Z}\pi_1X) = 0 \), we have \( \Delta(\text{Self}(S)) = 0 \).

- \( X = S^3 \times S^1 \), \( S \) any 2-sphere, \( \alpha \) the generator of \( H_1(S^3 \times S^1; \mathbb{Z}/2\mathbb{Z}) \). We have \( H_3(X; \mathbb{Z}\pi_1X) \cong \mathbb{Z} \), with generator represented by an embedded 3-sphere. We conclude that every element of \( \text{Self}(S) \) is homologous (with \( \mathbb{Z}\pi_1X \) coefficients) to an embedded self-concordance of \( S \), and hence \( \Delta(\text{Self}(S)) = 0 \).

In [7] Example 7.2, we show how to produce a 2-sphere \( S_1 \) homotopic to \( S_0 := S \# \mathbb{C}P^1 \# \mathbb{C}P^1 \) embedded in \( X \# \mathbb{C}P^2 \# \mathbb{C}P^2 \) with \( \text{stong}(S_0, S_1) = \alpha \). (As usual, if \( S_0 \)
is smooth, then we may take $S_1$ to be smooth.) The 2-sphere $S_1$ is obtained from $S_0$ by performing a finger move about $\alpha$ followed by one Whitney move, chosen so that the trace of the described homotopy has singular link a Hopf link. The Whitney disk can almost be found in $X$: in \cite{7} we perform a finger move to $S$ in $X$ to obtain an immersed sphere $S'$ and exhibit a framed Whitney disk $W'$ that intersects $S'$ in two points. We remove these two intersection points by blowing up twice (with opposite signs to ensure the resulting neat Whitney disk is still framed).

Since

$$H_3(X \# \mathbb{C}P^2 \# \mathbb{C}P^2; \mathbb{Z} \pi_1(X \# \mathbb{C}P^2 \# \mathbb{C}P^2)) = H_3(X; \mathbb{Z} \pi_1 X)$$

and $\alpha \notin \Delta(\text{Self}(S))$, we also have $\alpha \notin \Delta(\text{Self}(S_0))$. Thus, $\text{stong}(S_0, S_1)$ is nontrivial in the quotient

$$H_1(X \# \mathbb{C}P^2 \# \mathbb{C}P^2; \mathbb{Z}/2\mathbb{Z})/\Delta(\text{Self}(S_0)) \cong H_1(X; \mathbb{Z}/2\mathbb{Z})/\Delta(\text{Self}(S)).$$

Thus, $S_0$ and $S_1$ are not concordant.

\textbf{Remark 5.2.} In Example \cite{5.1} if $S$ already has two embedded dual spheres with Euler numbers $2n+1$ and $-2n-1$, we can avoid blowing up. We instead have $S_0 = S$ and $S_1$ obtained from $S$ by a finger and Whitney move, with $\text{stong}(S_0, S_1) = \alpha$.

The Stong invariant is particularly interesting in light of its potential relevance to the study of concordance of links. While Kervaire \cite{5} showed that every 2-sphere in $S^3$ is concordant to the unknotted 2-sphere, it remains open whether every link of 2-spheres in $S^4$ is concordant to the unlink.

\textbf{Theorem 1.9.} If there are 2-spheres $S_0, S_1$ in $B^3 \times S^1$ with $\text{stong}(S_0, S_1) = 1$ in $H_1(B^3 \times S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ then there is a 2-component link of spheres in $S^4$ that is not concordant to the unlink.

\textbf{Proof.} Note that we have not strictly defined the Stong invariant for a pair of 2-spheres that are not based-homotopic nor have a dual sphere. However, there is a self-concordance $H$ of $S_0$ so that $H(\text{basepoint} \times I) \subset B^3 \times S^1 \times I$ projects to a loop in $B^3 \times S^1$ representing the generator of $\pi_1(B^3 \times S^1)$. (In words, take $H$ to be an isotopy that moves each $B^3$ factor once about the $S^1$ factor of $B^3 \times S^1$.)

Take $S_0$ and $S_1$ to have a common basepoint. Given any singular concordance $H'$ from $S_0$ to $S_1$, we can preconcatenate $H'$ with some number of copies of $H$ and obtain a singular concordance isotopic rel. boundary to a based singular concordance with the same singular link as $H'$. Therefore, if $\text{stong}(S_0, S_1) = 1$, implying that there is no based concordance from $S_0$ to $S_1$, then there is no concordance from $S_0$ to $S_1$ even without reference to basepoints.

Let $S$ be the unknotted 2-sphere in $B^3 \times S^1$. Since $\text{stong}(S_0, S_1) \neq 0$, we cannot have both $\text{stong}(S, S_0) = 0$ and $\text{stong}(S, S_1) = 0$. Without loss of generality, take $\text{stong}(S, S_1) \neq 0$ and redefine $S_0 := S$.

Now let $U$ be the unknotted 2-sphere in $S^4$ and identify $S^4 \setminus \nu(U)$ with $B^3 \times S^1$. Then $U \sqcup S_0$ is the 2-component unlink. Suppose there are concordances $H_U, H :
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Figure 26. A schematic of the proof of Theorem 1.9.

\[ S^2 \times I \to S^4 \times I \] with disjoint images so that \( H_U \) goes from \( U \) to \( U \) and \( H \) goes from \( S_0 \) to \( S_1 \). Let \( W = S^4 \times I \setminus \nu(\text{Im}(H_U)) \). Let \( W' \) be the intersection of \( W \) with \( S^4 \times [1 - \epsilon, 1] \) with \( \epsilon \) small so that \( W' \) can be naturally identified with \( B^3 \times S^1 \times I \). Refer to Figure 26 for a schematic.

Since \( S_1 \subset \partial W' \) is nullhomotopic in \( W' \), there is an immersed ball \( B \) in \( W' \) with boundary \( S_1 \). We could tube \( B \) to a copy of \( S_0 \) in \( \partial W' \) to obtain a concordance from \( S_0 \) to \( S_1 \), so we conclude \( \Delta(B) = 1 \). On the other hand, by including \( B \) into \( W \) and tubing \( B \) to \( S_0 \) in \( \partial W \) we also find the Stong invariant of the pair \( (S_0, S_1) \) in the boundary of the 5-manifold \( W \) is still \( \Delta(B) = 1 \in H_1(W'; \mathbb{Z}/2\mathbb{Z}) = H_1(W; \mathbb{Z}/2\mathbb{Z}) \). This contradicts the existence of \( H \).

We plan to discuss the Stong invariant in a general 5-manifold (rather than a concordance invariant defined using a product) in [6]. This is already the main focus of Stong’s work [11], so this technical point can be sidestepped by translating the above argument into his setting. In [11], one would instead define \( \text{stong}(S_1) \) to be \( \Delta(B) \) for \( B \) an immersed 3-ball bounded by \( S_1 \), rather than by considering any concordance. We conclude that because \( \text{stong}(S_1) = 1 \) when viewed as a 2-sphere in \( \partial W' \), we also have \( \text{stong}(S_1) = 1 \) when viewed as a 2-sphere in \( \partial W \) since \( H_1(W; \mathbb{Z}/2\mathbb{Z}) \) and \( H_1(W'; \mathbb{Z}/2\mathbb{Z}) \) are identified. □

It is tempting to think that Theorem 1.9 can be restated for concordance of links whose components represent nontrivial homology elements in a general simply connected 4-manifold. For instance, via Example 5.1 we can obtain 2-spheres \( S_0, S_1 \) in \( X := B^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) in the homology class \([\mathbb{C}P^1 \# \overline{\mathbb{C}P^1}]\) with \( \text{stong}(S_0, S_1) \) nontrivial. Letting \( U \) denote the unknotted 2-sphere in \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) and identifying \( X \) with \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus \nu(U) \), one can ask whether the link \( U \sqcup S_0 \) is concordant to \( U \sqcup S_1 \). We know from work of Sunukjian [12] that \( S_0 \) and \( S_1 \) are concordant inside \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \), so this is truly a question about link concordance. Unfortunately, it is not clear how to make use of the fact that \( \text{stong}(S_0, S_1) \neq 0 \) inside \( X \times I \) – this just
Corollary 1.10. Let $S_0$ be an oriented embedded 2-sphere in an orientable 4-manifold $X$ with an immersed dual sphere $G$. Let $\text{Concordance}(S_0)$ be the set of concordance classes of embedded spheres in $X$ that are homotopic to $S_0$. Suppose that $\pi_1 X$ has a finite number of 2-torsion elements.

1. Suppose $S_0$ is not $s$-characteristic. Then $\text{Concordance}(S_0)$ is finite of size at most $2^{|TX|}$.

2. Suppose $S_0$ is $s$-characteristic and that $\mu(\pi_3 X) = 0$. Then $\text{Concordance}(S_0)$ is finite of size at most $2^{|TX|} \cdot |H_1(X;\mathbb{Z}/2\mathbb{Z})|$.

Proof. Let $S_1$ and $S_2$ be embedded 2-spheres in $X$ with $f_\mathbb{Z}(S_0, S_1) = f_\mathbb{Z}(S_0, S_2)$. By concatenating singular concordances $H_1$ from $S_1$ to $S_0$ and $H_2$ from $S_2$ to $S_0$, we find $f_\mathbb{Z}(S_1, S_2) = f_\mathbb{Z}(S_1, S_0) + f_\mathbb{Z}(S_0, S_2) = 0$. Note that $G$ is a dual sphere for the singular concordance from $S_1$ to $S_2$ obtained by stacking $S_0$ to $S_1$.

Thus when $S_0$ is not $s$-characteristic, by the proof of Theorem 1.5 we find that $S_1$ and $S_2$ are concordant and obtain $|\text{Concordance}(S_0)| \leq |F_2 T_X| = 2^{|TX|}$.

Similarly, if $S_0$ is $s$-characteristic and $\mu(\pi_3 X) = 0$, we see again by applying the proof of Theorem 1.6 (using the fact that $G$ is a dual sphere to some singular concordance $H$ from $S_1$ to $S_2$ with $\mu(H) = 0$) that $S_1$ and $S_2$ are concordant exactly when $\text{stong}(S_1, S_2) = 0$. Thus $|\text{Concordance}(S_0)| \leq |F_2 T_X| \cdot |H_1(X;\mathbb{Z}/2\mathbb{Z})| = 2^{|TX|} \cdot |H_1(X;\mathbb{Z}/2\mathbb{Z})|$. \hfill $\square$

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