Floppy Membranes

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Floppy membranes are tensionless surfaces without extrinsic stiffness whose fluctuations are governed by fourth-order bending elasticity. This suppresses spiky superstructures and ensures that floppy membranes remain smooth over any distance, with Hausdorff dimension $D_H = 2$, in contrast to surfaces with stiffness which are rough on the scale of some finite persistence length.

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1. Under deformations, fluid membranes behave approximately like ideal tensionless surfaces with curvature stiffness\textsuperscript{1},\textsuperscript{2}, and have a model energy

$$H = \kappa \int d^2 \xi \sqrt{\gamma} \Tr C^2,$$

(1)

where $C_{ab}$ is the second fundamental form of the surface described by an embedding function $\mathbf{x} = \mathbf{x}(\xi^1, \xi^2)$. The symbol $g$ denotes the determinant of the induced metric $g_{ab} \equiv \partial_a \mathbf{x} \partial_b \mathbf{x}$.

Thermal fluctuations are known to soften the curvature stiffness with increasing membrane size, such that there exists a finite persistence length $\zeta = \zeta_0 \exp(4\pi \kappa / 3T)$, where $\zeta_0$ is the molecular size, beyond which the membrane loses its stiffness completely and begins exhibiting a surface tension proportional to $\zeta^{-2}$, which was initially absent. Thus the tangential correlation functions have only the range $\zeta$, and surfaces much larger than $\zeta$ appear rough. The typical behavior of bilipid vesicles can therefore be observed in the laboratory only for sizes smaller than $\zeta$. Surfaces much larger than $\zeta$ will crumple and fill the embedding space with spiky structures\textsuperscript{1,2}.

Apart from these common membranes, nature may provide us also with another type, not described by the Hamiltonian\textsuperscript{1}. If the molecules in a bilayer are strongly conical, the bending stiffness can be zero or negative\textsuperscript{2}. Such a situation can also arise for charged or dipolar molecules\textsuperscript{1}. In this case we shall speak of floppy membranes. The purpose of this note is to study the statistical properties of such floppy membranes, which will turn out to be quite different from those of ordinary membranes. In particular we shall find that, in contrast to ordinary membranes, floppy membranes without tension and stiffness are smooth over long distances. If the stiffness is negative, floppy membranes are able to form disordered superstructures similar to those recently observed in the laboratory\textsuperscript{3}. These are thought to be molten versions of the egg carton-like crystalline arrangement of local maxima and minima on the surface. This has not yet been confirmed experimentally, but was suggested by recent numerical simulations\textsuperscript{3}.

2. In order to describe floppy membranes, we must stabilize their fluctuations by adding to the energy\textsuperscript{1} a higher-gradient term. Here we shall consider only one of several possibilities, focusing our attention upon the following Hamiltonian:

$$H = r \int d^2 \xi \sqrt{\gamma} + \frac{K}{2} \int d^2 \xi \sqrt{\gamma} \Tr C^2 + \frac{1}{2m} \int d^2 \xi \sqrt{\gamma} \left[ g^{cd}D_a D_c D_b C_{bd} + Tr C^4 \right].$$

(2)

The properties of a surface with this Hamiltonian will be studied non-perturbatively in the limit of a large number $D$ of embedding dimensions.

The first term in (2) parametrizes the surface tension; the third term provides the surface with the stabilizing higher-order bending stiffness, whose parameter $m$ has the dimension (energy × surface)$^{-1}$. Although this term is irrelevant in a perturbative renormalization group analysis, it becomes relevant non-perturbatively in the limit $D \to \infty$ by a mechanism familiar from the three-dimensional Gross-Neveu model\textsuperscript{4}. The new term stabilizes the surface against growing spikes and makes it smooth over long distances.

The Hamiltonian (2) can be reformulated alternatively in terms of the tangent vectors $\partial_a \mathbf{x}$ of the surface, or in terms of the normal vectors $\mathbf{n}$:

$$H = r \int d^2 \xi \sqrt{\gamma} - \frac{1}{m} \int d^2 \xi \sqrt{\gamma} K^2 + \frac{1}{2} \int d^2 \xi \sqrt{\gamma} \mathbf{n} \left[ -\kappa D^2 + \frac{1}{m} D^4 \right],$$

(3)

where $K$ is the Gaussian curvature. This form exposes an important physical aspect of the model. As pointed out in (2), the first term in the second line is analogous to the the continuous version of the Heisenberg model of ferromagnets\textsuperscript{5}, albeit with an additional integrability condition on the $\mathbf{n}(\xi)$-field. It tries to make an ordinary membrane with a positive stiffness smooth corresponding to a ferromagnetic alignment of the normal vectors. The fact that ordinary membranes cannot be smooth over long distances has its parallel in the absence of an ordered phase in the two-dimensional Heisenberg ferromagnet. By the same analogy, we see that our new term introduces (apart from an intrinsic term $K^2$) an antiferromagnetic next-to-nearest neighbours interaction.
between normal vectors. This generates frustration, and it is due to this non-local interaction that the surface can have an ordered phase after all, although with an antiferromagnetic type of order.

To exhibit these features analytically, a formulation of the Hamiltonian (2) in terms of the tangent vectors will be most convenient:

\[ H = \frac{1}{2} \int d^2 \xi \sqrt{g} g^{ab} D_a x_\mu \left( r - \kappa D^2 + \frac{1}{m} D^4 \right) D_b x_\mu. \]  

(4)

3. We analyze the model (4) in the large-\( D \) approximation along the lines of Refs. [1][3]. To this end we introduce a Lagrange multiplier matrix \( \lambda^{ab} \) to enforce the constraint \( g^{ab} = \partial_a x \partial_b x \), extending the Hamiltonian (4) to

\[ H_{\text{ext}} = H + \frac{1}{2} \int d^2 \xi \sqrt{g} \lambda^{ab} \left( \partial_a x \partial_b x - g^{ab} \right). \]  

(5)

Then we parametrize the surface in a Gauss map by \( x(\xi) = (\xi_1, \xi_2, \phi^i(\xi)) \), \( (i = 3, \ldots, D) \), where \(-R_1/2 \leq \xi_1 \leq R_1/2, -R_2/2 \leq \xi_2 \leq R_2/2 \) and \( \phi^i(\xi) \) describe the \( (D-2) \) transverse fluctuations. In the limit of large \( D \), to be studied here, the large number of components suppresses the fluctuations of \( \lambda^{ab} \) and \( g_{ab} \). These fields take extremal values which, for large surface areas, are homogeneous and isotropic: \( g_{ab} = \rho \delta_{ab}, \lambda^{ab} = \lambda \gamma^{ab} \). Thus we may replace (5) for large \( D \) by

\[ H_{\text{ext}} = \int d^2 \xi \left[ r + \lambda (1 - \rho) \right] + \frac{1}{2} \int d^2 \xi \partial_a \phi^i \partial_a \phi^i \left( \kappa D^2 + \frac{1}{m} D^4 \right) \]  

(6)

where \( K \) represents the differential operator

\[ K (-D^2) = r + \lambda - \kappa D^2 + \frac{1}{m} D^4. \]  

(7)

Integrating out the transverse fluctuations, always for large areas, we get the free energy

\[ F = A_{\text{ext}} \left[ r + \lambda (1 - \rho) \right] + A_{\text{ext}} \frac{D - 2}{8\pi^2} \rho \int d^2 p \ln \left[ p^2 K \left( p^2 \right) \right], \]  

(8)

where, for simplicity, we have chosen natural units by setting \( \beta = 1/k_B T = 1 \) and \( A_{\text{ext}} = R_1 R_2 \) is the extrinsic, projected area in the coordinate plane. The factor \( (D-2) \) in the second term ensures that, for large \( D \), the fields \( \lambda \) and \( \rho \) are extremal and satisfy thus the saddle-point ("gap") equations

\[ 0 = f (r, \kappa, m, \lambda), \quad \rho = \frac{1}{f'(r, \kappa, m, \lambda)}. \]  

(9)

The prime denotes derivatives with respect to \( \lambda \) and the saddle-point function \( f \) is defined by

\[ f (r, \kappa, m, \lambda) = \lambda - \frac{D - 2}{8\pi} \int dp \ln \left[ p^2 K \left( p^2 \right) \right]. \]  

(10)

Inserting (4) in (8) we get \( F = (r + \lambda) A_{\text{ext}} \) showing that \( r_{ph} \equiv (r + \lambda) \) is the physical surface tension.

We now introduce the following combinations of the parameters of the model:

\[ R^2 = \frac{1}{2} \sqrt{m (r + \lambda)} + \frac{\kappa m}{4}, \quad I^2 = \frac{1}{2} \sqrt{m (r + \lambda)} - \frac{\kappa m}{4}. \]  

(11)

In terms of these, the kernel \( V \) can be written as

\[ mK \left( p^2 \right) = (R^2 + I^2)^2 + 2 \left( R^2 - I^2 \right) p^2 + p^4, \]  

(12)

from where we deduce the stability condition for the homogeneous saddle-point as being \( R^2 > 0 \), insuring that the spectrum of transverse fluctuations is positive for all \( p > 0 \). Since we shall mostly consider negative or vanishing stiffnnesses \( \kappa \) we shall also assume that \( I^2 > 0 \), so that both \( R \) and \( I \) are real. Note that the spectrum of transverse fluctuations depends drastically on the sign of the stiffness: for negative \( \kappa \) we have \( I > R \) and \( K \) develops a minimum at \( p = \sqrt{I^2 - R^2} \). Correspondingly (but for a slightly higher value of \( I/R \)), the spectrum \( E \left( p^2 \right) = p^2 K \left( p^2 \right) \) develops a roton-like minimum, which, for \( R/I \ll 1 \), lies at \( p \approx I \left( 1 - 8R^2/3I^2 \right) \), as shown schematically in Fig. 1.

![Fig. 1. The roton-like minimum in the spectrum of transverse fluctuations for \( I > R \).](image-url)

Having established the stability conditions we proceed to the evaluation of the saddle-function. This contains an ultraviolet divergent integral which must be regularized. To this end we use the standard dimensional regularization, computing the integral in \( (2 - \epsilon) \) dimensions. For small \( \epsilon \), this leads to

\[ f (r, \kappa, m, \lambda) = \lambda + \frac{1}{4\pi} \frac{(R^2 - I^2) \ln \frac{R^2 + I^2}{\Lambda^2}}{\chi^2} \]  

(13)

\[ - \frac{1}{2\pi R I} \left( \frac{\pi}{2} + \arctan \frac{I^2 - R^2}{2RI} \right), \]
where $\Lambda \equiv \exp(2/\epsilon)$ and $\mu$ is a reference scale which must be introduced for dimensional reasons. The scale $\Lambda$ plays the role of an ultraviolet cutoff, diverging for $\epsilon \to 0$.

In order to distinguish the various phases of our model we compute two correlation functions. First, we consider the orientational correlation function $g_{ab}(\xi - \xi') \equiv \langle \delta_{ab}(\xi) \delta_{a'b'}(\xi') \rangle$ for the normal components of tangent vectors to the surface. From (14) this is given by

$$g_{ab}(\xi - \xi') = \frac{\delta_{ab}}{4\pi^2} \int d^2 p \, \frac{1}{K(p^2)} \exp(\sqrt{\mu} (\xi - \xi')). \quad (14)$$

In terms of $R$ and $I$, the Fourier components can be written as

$$\frac{1}{K(p^2)} = \frac{m}{2RI} \text{Im} \frac{1}{p^2 + (R - iI)^2}, \quad (15)$$

from where we obtain the following exact result for the diagonal elements $g = g_{aa}$ of (14):

$$2\pi g(d) = \frac{m}{2RI} \text{Im} K_0((R - iI) \sqrt{\mu} d), \quad (16)$$

where $d \equiv |\xi - \xi'|$ and $K_0$ is a Bessel function of imaginary argument (13).

Secondly, we compute the scaling law of the distance $d_E$ in embedding space between two points on the surface when changing its projection $d$ on the reference plane. The exact relation between the two lengths is

$$d_E^2 = d^2 + \sum_i |\phi_i(\xi) - \phi_i(\xi')|^2. \quad (17)$$

With a computation analogous to the one leading to (16) we obtain the following behaviour:

$$d_E^2 = \begin{cases} \frac{(R^2 + I^2)}{8\pi \rho ph RI} \arctan(I/R) \alpha d^2, & d^2 \ll \frac{1}{\alpha} \\ \frac{1}{2\pi \rho ph} \left[ \ln \left( \alpha d^2/4 + c(R, I) \right) - 1 \right], & \frac{1}{\alpha} \ll d^2 \ll \frac{1}{2\pi \rho ph} \\ d^2, & d^2 \gg \frac{1}{2\pi \rho ph} \end{cases}$$

where $\alpha \equiv (R^2 + I^2)\rho$ and $c(R, I) = C + [(I^2 - R^2)/RI] \arctan(I/R)$ with $C = \text{Euler’s constant}$.

These results show that the model has three possible phases. The first is realized when there are no solutions to the saddle-point equations in the allowed range of parameters. For this choice of parameters, there exist no homogeneous, isotropic surfaces. In this phase the surfaces will form inhomogeneous structures.

If a solution to the saddle-point equations exists, two situations can be realized. For large positive stiffness $\kappa$ we have $R \gg I$, the asymptotic region is $d \gg 1/R\sqrt{\rho}$ and $I$ can be neglected. In this region we have

$$g(d) \propto \frac{1}{\sqrt{R\sqrt{\rho} d}} e^{-R\sqrt{\rho} d}, \quad (18)$$

exhibiting short-range orientational order. For short distances $d \ll 1/R\sqrt{\rho}$, the surfaces behave as two-dimensional objects. If $\rho$ becomes large we have a region $1/R\sqrt{\rho} \ll d \ll 1/\sqrt{2\pi \rho ph}$ in which $d_E$ scales logarithmically with $d$ and distances along the surface become large. The transition to this regime happens on the scale of the persistence length $d_{E\text{pl}} = 1/\sqrt{\rho ph}$. Above this scale world-sheets are crumpled, with no orientational correlations (if the tension is not large enough to dominate over the entire surface, causing $\rho \simeq 1$). This phase corresponds to the familiar behaviour of stiff membranes (3).

For large negative stiffness $\kappa$, in contrast, we have $I \gg R$, the asymptotic region is $d \gg 1/I\sqrt{\rho}$, and $R$ can be neglected in $K_0$ for $d \ll 1/R\sqrt{\rho}$. In this region we have

$$g(d) = \frac{m}{8RI} J_0(I\sqrt{\rho} d), \quad (19)$$

with $1/R\sqrt{\rho}$ playing the role of an infrared cutoff for the oscillations on the scale $1/I\sqrt{\rho}$ over which the Bessel function $J_0$ varies. We have thus a new scale (in embedding space)

$$d_{E \text{pl}}^{\text{II}} = \sqrt{\frac{I}{32RI\rho ph}}, \quad (20)$$

on which the transverse fluctuations create oscillations characterized by the “antiferromagnetic” orientational correlations (19). Crumpling takes place only if $1/R\sqrt{\rho} \ll 1/\sqrt{2\pi \rho ph}$ and the corresponding persistence length is now

$$d_{E \text{pl}}^{\text{III}} = \frac{1}{\sqrt{4\pi \rho ph}} \sqrt{\frac{\pi I}{2R} + \ln \left( \frac{I^2}{4R^2} \right)}, \quad (21)$$

which is much larger than $1/\sqrt{\rho ph}$. Otherwise, the oscillating superstructure goes over directly into the tension dominated region. In this case $d_E$ scales logarithmically with $d$ for $1/I\sqrt{\rho} \ll d \ll 1/\sqrt{2\pi \rho ph}$, and $\rho$ is large not because of crumpling but because of the oscillating superstructure. Indeed there are strong orientational correlations in this region. Note that the oscillations represent a disordered superstructure caused by fluctuations on an otherwise homogeneous and isotropic ground-state described by the solution of the saddle-point equations.

4. One might imagine that our disordered superstructure undergoes a transition to a crystalline egg-carton-type structure (3) when $R \to 0$, so that the spectrum of transverse fluctuations develops an instability at a finite value $p = I$. However this is not so, as can be seen from the explicit expression for $\rho$ obtained from (14).

$$\rho = \left[ 1 - \frac{m}{16\pi RI} \left( \frac{\pi}{2} + \arctan \frac{I^2 - R^2}{2RI} \right) \right]^{-1}. \quad (22)$$

When lowering $R$ at fixed $I$ one hits a pole where $\rho$ diverges. This means that the surface crumples before one reaches the crystal instability.
For symmetry reasons, the transition from the stiff to the disordered superstructure phase occurs on the line $R = I$, where $\kappa = 0$ and the the kernel $K$ develops its minimum.

The fourth-order bending elasticity term dominates the fluctuations of surfaces which have both vanishing bare tension and stiffness. These can be studied further analytically since, for $\kappa = 0$ ($R = I$), the saddle-point equations become polynomial, with solution

$$\lambda = \frac{m}{128} \left(1 + \sqrt{1 + 256 \frac{r}{m}}\right). \quad (23)$$

This gives

$$r_{ph} = \frac{a^2}{64} m, \quad \rho = \left(1 - \frac{1}{2a}\right)^{-1}, \quad (24)$$

where $a$ is the following function of the dimensionless parameter $r/m$:

$$a^2 = \frac{1 + 128 r/m + \sqrt{1 + 256 r/m}}{2}. \quad (25)$$

This is the previously announced result. The fourth-order bending elasticity term, although irrelevant in perturbation theory, becomes relevant non-perturbatively by contributing a term $m/64$ to the physical surface tension. This is the reason why, contrary to stiff membranes [3], floppy membranes do not crumple. The physical tension can be decreased arbitrarily with $\rho$ remaining in the range $1 \leq \rho \leq 2$. In other words one can lower the two scales $r$ and $m$ so that the range of orientational correlations $d = 1/R\sqrt{\rho} = (4/\sqrt{a m})\sqrt{a - 1/2}$ is always of the same order or larger than the inverse of the square root of the physical surface tension $1/\sqrt{r_{ph}} = 8/a\sqrt{m}$.

The corresponding $\gamma$-functions are easily obtained as

$$\gamma(r) \equiv -\Lambda \frac{d}{d\Lambda} \ln \frac{r}{\Lambda^2} = 2, \quad (26)$$

$$\gamma(m) \equiv -\Lambda \frac{d}{d\Lambda} \ln \frac{m}{\Lambda^2} = 2, \quad (27)$$

showing the absence of anomalous dimensions for $\kappa = 0$. Correspondingly, we have

$$\beta \left(\frac{r}{m}\right) = -\Lambda \frac{d}{d\Lambda} \frac{r}{m} = \frac{r}{m} \left[\gamma(r) - \gamma(m)\right] = 0, \quad (28)$$

which means that $r/m$, and thus also $a$, are renormalization group invariants.

5. In conclusion we see that the physics of floppy membranes is governed by an infrared-stable fixed-point characterized by vanishing tension and a dimensionless renormalization group invariant parameter $a^* \equiv \lim_{r \rightarrow 0, m \rightarrow 0} a(r, m)$. At this point, the surface exhibits long-range order in which the diagonal elements of the correlation functions $g_{ab}(\xi - \xi')$ do not depend on the distance, $g(d) = 4\pi^2/a^*$, and in which the length $l\xi$ scales with the distance in coordinate space like $d^2 = (\pi^2 \rho/a^*) d^2$, from which we deduce a Hausdorff dimension $D_H = 2$ for floppy membranes.

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[1] W. Helfrich, Z. Naturforsch. 28C, 693 (1973); P. G. de Gennes and C. Taupin, J. Phys. Chem. 86 (1982) 2294.
[2] For a review see: F. David, Introduction to the Statistical Mechanics of Random Surfaces and Membranes, in “Two-Dimensional Quantum Gravity and Random Surfaces”, D. Gross, T. Piran and S. Weinberg eds., World Scientific, Singapore (1992).
[3] W. Helfrich, J. Phys. (France) 46, 1263 (1985); L. Peliti and S. Leibler, Phys. Rev. Lett. 54, 1690 (1985); D. Förster, Phys. Lett. A114, 115 91986; H. Kleinert, Phys. Lett. A114, 263 (1986); M. E. S. Borelli, H. Kleinert and A. M. J. Schakel, “Derivative Expansion of One-Loop Effective Energy of Stiff Membranes with Tension”, cond-mat/9806183.
[4] F. David and E. Guitter, Nucl. Phys. B295, 382 (1988); F. David and E. Guitter, Europhys. Lett. 3, 1169 (1987).
[5] H. Kleinert, Spiky Phases of Smooth Membranes. Implications for Smooth Strings, FU-Berlin preprint 1998, cond-mat/9805305.
[6] A. G. Petrov and A. Derzhanski, J. Phys. Supp. 37, C3-155 (1976).
[7] H. Kleinert, Phys. Lett. B211, 151 (1988); B. D. Simons, J. Phys. II France 2, 1141 (1992); M. C. Diamantini, H. Kleinert and C. A. Trugenberger, Smoothening Transition of Rough Surfactant Surfaces, cond-mat/9806077.
[8] For a review see: W. Helfrich, contribution in “Handbook of Biological Physics”, vol. 1, R. Lipowski and E. Sackmann eds., Elsevier Science Publishers, Amsterdam (1995).
[9] R. Goetz and W. Helfrich, J. Phys. II (France) 6, 215 (1996).
[10] See e.g.: D. Gross, Application of the Renormalization Group to High-Energy Physics, in “Methods in Field Theory”, R. Balian and J. Zinn-Justin eds., North-Holland & World Scientific, Singapore (1981).
[11] A. M. Polyakov, Nucl. Phys. B268, 406 (1986).
[12] For a review see: J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford University Press, Oxford (1989).
[13] H. Kleinert, Phys. Rev. Lett. 58, 1915 (1987).
[14] I. Gradstheyen and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, Boston (1980).