THE HELICAL VORTEX FILAMENTS OF GINZBURG-LANDAU SYSTEM IN $\mathbb{R}^3$

LIPENG DUAN, QI GAO, AND JUN YANG

Abstract. We consider the following coupled Ginzburg-Landau system in $\mathbb{R}^3$
\[
\begin{align*}
-\varepsilon^2 \Delta w^+ + \left[ A_+ (|w^+|^2 - t^+)^2 + B (|w^-|^2 - t^-)^2 \right] w^+ &= 0, \\
-\varepsilon^2 \Delta w^- + \left[ A_- (|w^-|^2 - t^-)^2 + B (|w^+|^2 - t^+)^2 \right] w^- &= 0,
\end{align*}
\]
where $w = (w^+, w^-) \in \mathbb{C}^2$ and the constant coefficients satisfy
\[ A_+, A_- > 0, \quad B^2 < A_+ A_-, \quad t^\pm > 0, \quad t^+ t^- = 1. \]
If $B < 0$, then for every $\varepsilon$ small enough, we construct a family of entire solutions $w_\varepsilon(\tilde{z}, t) \in \mathbb{C}^2$ in the cylindrical coordinates $(\tilde{z}, t) \in \mathbb{R}^2 \times \mathbb{R}$ for this system via the approach introduced by J. Dávila, M. del Pino, M. Medina and R. Rodiac in arXiv:1901.02807. These solutions are $2\pi$-periodic in $t$ and have multiple interacting vortex helices. The main results are the extensions of the phenomena of interacting helical vortex filaments for the classical (single) Ginzburg-Landau equation in $\mathbb{R}^3$ which has been studied in arXiv:1901.02807. Our results negatively answer the Gibbons conjecture [10] for the Allen-Cahn equation in Ginzburg-Landau system version, which is an extension of the question originally proposed by H. Brezis.

Keywords: Ginzburg-Landau model, Helical Vortex Filaments, Lyapunov-Schmidt reduction

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1. INTRODUCTION

In this paper, we study the entire solutions of the following Ginzburg-Landau system in $\mathbb{R}^N$, for complex vector-valued functions $w = (w^+, w^-) : \mathbb{R}^N \to \mathbb{C}^2$:
\[
\begin{align*}
-\varepsilon^2 \Delta_{\mathbb{R}^N} w^+ + \left[ A_+ (|w^+|^2 - t^+)^2 + B (|w^-|^2 - t^-)^2 \right] w^+ &= 0, \\
-\varepsilon^2 \Delta_{\mathbb{R}^N} w^- + \left[ A_- (|w^-|^2 - t^-)^2 + B (|w^+|^2 - t^+)^2 \right] w^- &= 0,
\end{align*}
\]
Here $A_\pm > 0$, $t^\pm$, $B$ and $\varepsilon > 0$ are parameters, the notation $\Delta_{\mathbb{R}^N}$ is the Laplacian operator in $\mathbb{R}^N$. Throughout the paper we make the following assumptions concerning the constants appearing in (1.1):
\[ A_+, A_- > 0, \quad B^2 < A_+ A_-, \quad t^\pm > 0, \quad t^+ t^- = 1. \]

The corresponding energy functional to system (1.1) is
\[
E(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{1}{4 \varepsilon^2} \int_{\mathbb{R}^N} \left[ A_+ (|w^+|^2 - t^+)^2 + A_- (|w^-|^2 - t^-)^2 \right. \\
\left. + 2B (|w^+|^2 - t^+)^2 (|w^-|^2 - t^-)^2 \right].
\]
The hypothesis [H1] ensures that the potential term in the energy is positive definite and attains its minimum when $|w^\pm| = t^\pm$. From the previous work about the classical Ginzburg-Landau equation in [9], we would look for solutions which satisfy
\[
\int_{\mathbb{R}^N} \left[ A_+ (|w^+|^2 - t^+)^2 + A_- (|w^-|^2 - t^-)^2 + 2B (|w^+|^2 - t^+)^2 (|w^-|^2 - t^-)^2 \right] < +\infty.
\]
Therefore the function \( w = (w^+, w^-) \) satisfies the superconducting boundary condition
\[
|w|^2 = |w^+|^2 + |w^-|^2 \to r^+ + t^{-2} = 1, \quad \text{when } |x| \to +\infty.
\] (1.3)

When \( N = 2, 3 \) in (1.1), Ginzburg-Landau systems of this type have been introduced in physical models of \( p \)-wave superconductors [26] and two-component Bose-Einstein condensates (BEC) [3, 25]. When the coupled term \( B \) vanishes, system (1.1) is decoupled and is precisely the classical (single-component) Ginzburg-Landau equation
\[
e^2 \Delta_R u + (1 - |u|^2)u = 0, \quad u \in \mathbb{C},
\] (1.4)
which arises in the theory of superconductivity and superfluids [20] without external applied magnetic field. From the theory of superconductivity, \( |u|^2 \) is proportional to the density of superconducting electrons, i.e., \( |u|^2 \approx 1 \) corresponds to the superconducting state and \( |u|^2 \approx 0 \) corresponds to the normal state. The zero set of \( u \), which the superconductivity is not present, is said to be vortex set or vorticity. The solutions with vortex lines or vortex filaments to the system (1.4) are the main objects in this paper.

1.1. The case \( B = 0 \). For the classical Ginzburg-Landau equation in a smooth bounded domain \( \Omega \subset \mathbb{R}^N \),
\[
\begin{cases}
e^2 \Delta u + (1 - |u|^2)u = 0 & \text{in } \Omega \subset \mathbb{R}^N, \quad u \in \mathbb{C}, \\
u = f & \text{on } \partial \Omega,
\end{cases}
\] (1.5)
the solutions with vortices have been widely studied in the past years.

For \( N = 2 \), F. Bethuel, H. Brezis, and F. Hélein [7] considered the asymptotic behavior of \( u_\epsilon \) with a boundary condition \( f : \partial \Omega \to S^1 \) of degree \( k \geq 1 \). They showed that there exists a subsequence \( \{\epsilon_j\} \) and exactly \( k \) points \( b_1, \cdots, b_k \) in \( \Omega \) such that
\[
u_j(x) \to u = e^{i\varphi(x)} \prod_{l=1}^k \frac{x - b_l}{|x - b_l|} \quad \text{in } C_{loc}(\Omega \setminus \{b_1, \cdots, b_k\}), \quad \text{as } \epsilon_j \to 0,
\] (1.6)
where \( \varphi \) is a harmonic function satisfying \( \varphi|_{\partial \Omega} = f \). Furthermore, they also showed that the \( k \)-tuple \( (b_1, \cdots, b_k) \) globally minimizes the renormalized energy functional \( \mathcal{W}(b_1, \cdots, b_k) \) which is characterised by
\[
\mathcal{W}(b_1, \cdots, b_k) = \lim_{r \to 0} \int_{\Omega \cup \bigcup_{j=1}^k B_r(b_j)} \left[ |\nabla u|^2 - k\pi |\ln r| \right].
\] (1.7)

And in [17], an implicit expression of \( \mathcal{W}(b_1, \cdots, b_k) \) in terms of Green’s function was given, see also [13, 36] for similar results. One better approximation than \( u \) in (1.6) is
\[
e^{i\varphi(x)} \prod_{l=1}^k \omega \left( \frac{x - b_l}{\epsilon} \right),
\] (1.8)
where \( \omega \) is the standard degree \( +1 \) vortex solution of the following Ginzburg-Landau equation in \( \mathbb{R}^2 \)
\[
\Delta u + (1 - |u|^2)u = 0, \quad u \in \mathbb{C}.
\] (1.9)
In other words, equation (1.9) has a unique solution of the form
\[
\omega(\tilde{z}) = e^{i\theta}U(r), \quad \tilde{z} = re^{i\theta}, \quad U \in \mathbb{R},
\] (1.10)
where \( U > 0 \) is a solution of
\[
\begin{cases}
U'' + \frac{U'}{r} - \frac{U}{r^2} + (1 - U^2)U = 0 & \text{in } (0, \infty), \\
U(0^+) = 0, \quad U(+\infty) = 1,
\end{cases}
\]
see [11, 21]. On the other hand, a natural question is to find solutions of (1.5) with vortices at other critical points of the renormalized energy functional \( \mathcal{W}(b_1, \cdots, b_k) \) (see [6, 17, 28, 29, 30, 33-34] and the references therein for research on this problem). In [33], F. Pacard and T. Rivière constructed solutions of (1.5) with vortices of combined degree \( \pm 1 \) and showed that the vortex points converge to non-degenerate critical point of corresponding renormalized energy. In [17], M. del Pino, M. Kowalczyk and M. Musso considered the Ginzburg-Landau equation (1.5) with added zero Neumann boundary condition and constructed solutions by
using Lyapunov-Schmidt reduction without non-degenerate assumption on the critical point of renormalized energy.

In the high dimension $N \geq 3$, the locations of vortices to the classical Ginzburg-Landau with suitable boundary conditions and energy levels do not occur at points, but along a generalized minimal sub-manifold structure with co-dimension 2, which is naturally interpreted as “vortex sub-manifold”. In $\mathbb{R}^3$, the structure of vortices should typically be the form of curves which are called vortex filaments. In the work [10], M. del Pino and M. Kowalczyk studied (1.5) in a cylinder $\Omega = B_R(0) \times (0, 2\pi)$ in $\mathbb{R}^3$ and formally gave an approximation of solution to (1.5) with helical vortex filaments structure. Moreover, they conjectured in [16] that (1.5) has a solution $u_3$ of the form

$$u_3 \approx e^{i\varphi(\tilde{z}, t)} \prod_{j=1}^{k} \omega \left( \frac{\tilde{z} - g_j(t)}{\epsilon} \right), \quad (\tilde{z}, t) \in \mathbb{R}^3,$$

(1.11)

where $\varphi$ is a harmonic function satisfying the boundary condition and $g = (g_1, \cdots, g_k)$ represents $k$ curves $: t \to (g_j(t), t), j \leq k$. For simplicity, the authors assumed that the $k$ curves are $2\pi$-periodic in $t$. The asymptotic expansion of energy functional $E_\epsilon(u_3)$ of (1.3) is

$$I_\epsilon(g) := E_\epsilon(u_3) \approx 2\pi \times k \pi | \log \epsilon | + I_\epsilon(g),$$

where

$$E_\epsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\epsilon^2} \int_\Omega (1 - |u|^2)^2,$$

and

$$I_\epsilon(g) := \pi \int_0^{2\pi} \left( |\log \epsilon| \frac{1}{2} \sum_{j=1}^{k} |g_j'(t)|^2 - \sum_{j \neq l} \log |g_j(t) - g_l(t)| \right) dt.$$

Therefore, the asymptotic expressions for equilibrium location in (1.11) are

$$g_j(t) \approx \frac{1}{\sqrt{|\log \epsilon|}} \sqrt{k - 1} e^{itj} e^{2i(j-1)\pi/k}, \quad j = 1, \cdots, k,$$

(1.12)

and they can be obtained by solving the following Euler-Lagrange equations of $I_\epsilon(g)$

$$g(t) = \frac{1}{\sqrt{|\log \epsilon|}} \tilde{g}(t), \quad \tilde{g} = (\tilde{g}_1, \cdots, \tilde{g}_k), \quad -\tilde{g}_j''(t) = 2 \sum_{i \neq j} \frac{\tilde{g}_j(t) - \tilde{g}_i(t)}{|\tilde{g}_j(t) - \tilde{g}_i(t)|^2}.$$

Recently, J. Dávila, M. del Pino, M. Medina and R.Rodiac in [12] have rigorously proved this conjecture in $\mathbb{R}^3$ by constructing a family of entire solutions $u_\epsilon(z, t)$ as $\epsilon \to 0$ with the form

$$u_\epsilon(\tilde{z}, t) \approx \prod_{j=1}^{k} \omega \left( \frac{\tilde{z} - g_j(t)}{\epsilon} \right), \quad (\tilde{z}, t) \in \mathbb{R}^3.$$

Now we recall the main results provided in [12].

\textbf{Theorem 1.1.} ([12]). For every integer $k \geq 2$ and $\epsilon > 0$ sufficiently small, there exists a solution $u_\epsilon(\tilde{z}, t) \in C$ of (1.4) with $N = 3$, $2\pi$-periodic in the $t$-variable, with the following asymptotic profile:

$$u_\epsilon(\tilde{z}, t) = \prod_{j=1}^{k} \omega \left( \frac{\tilde{z} - g_j(t)}{\epsilon} \right) + \varphi_\epsilon(\tilde{z}, t),$$

where $g_j(t)$ is $2\pi$-periodic with the asymptotic behavior (1.12) and

$$|\varphi_\epsilon(\tilde{z}, t)| \leq \frac{C}{|\log \epsilon|}.$$
It is worth pointing out that the solutions \( u_\epsilon(\tilde z, t) \) constructed in [12] satisfy
\[
\lim_{|\tilde z| \to +\infty} |u_\epsilon(\tilde z, t)| = 1 \quad \text{uniformly in } t, \tag{1.13}
\]
and still depend on \( t \). As we know that H. Brezis has postulated the following Gibbons conjecture: whether a solution \( u_\epsilon(\tilde z, t) \in \mathbb{C} \) of classical Ginzburg-Landau with the uniform convergence condition (1.13) must necessarily be a function in \( \tilde z \). However, the results obtained in [12] actually negatively answered the conjecture proposed by H. Brezis when \( N = 3 \). The reader can refer to [10], [19] for more information about the Gibbons conjecture.

### 1.2. The case \( B \neq 0 \)

When \( N = 2 \), with space rescaling, we can reduce the system (1.14) into the following one:
\[
\begin{align*}
-\Delta w^+ &+ \left[A_+ \left(|w^+|^2 - t^+\right) + B \left(|w^-|^2 - t^-\right)\right] w^+ = 0, \\
-\Delta w^- &+ \left[A_- \left(|w^-|^2 - t^-\right) + B \left(|w^+|^2 - t^+\right)\right] w^- = 0,
\end{align*}
\tag{1.14}
\]

The two-component model (1.14) was studied by [1] and [2] in the “balanced” case, \( A_+ = A_- \) and \( t^+ = t^- = 1/\sqrt{3} \). S. Alama, L. Bronsard and P. Mironescu in [1] and [2] proved the existence, uniqueness, monotonicity and stability of radial solutions with symmetric vortices. Furthermore, under the assumption (1.14), S. Alama and Q. Gao obtained the existence, uniqueness, asymptotic behaviors and quantization results of symmetric equivariant vortex solution with degree pair \((n^+, n^-)\) in the form
\[
w^+_n(\ell, \theta) = W^+_n(\ell) e^{in \theta}, \quad w^-_n(\ell, \theta) = W^-_n(\ell) e^{-in \theta},
\]
to the system (1.14), where \((\ell, \theta)\) are the polar coordinates (see [3]).

In order to give our results in a more precise way, we use the notation in the rest of this paper to denote the radially symmetric solution with vortices of degree pair \((n_+, n_-) = (+1, +1)\) as below:
\[
w(\tilde z) = (w^+(\tilde z), w^-(\tilde z)) \quad \text{where} \quad w^\pm(\tilde z) = W^\pm(\ell)e^{i\theta}, \quad \tilde z = \ell e^{i\theta}.
\]

Then we get the corresponding ODE system
\[
\begin{align*}
-W^{\prime\prime} - \frac{1}{\ell}W^{\prime\prime} + \frac{1}{\ell^2}W^+ &+ \left[A_+ (W^+)^2 - t^+\right] W^+ = 0, \\
-W^{\prime\prime} - \frac{1}{\ell}W^{\prime\prime} + \frac{1}{\ell^2}W^- &+ \left[A_- (W^-)^2 - t^-\right] W^- = 0,
\end{align*}
\tag{1.15}
\]

where \( \ell \in [0, +\infty) \). The facts below about the properties of \( W = (W^+, W^-) \) are given in [3].

**Lemma 1.2.** ([3]). Suppose \( W = (W^+, W^-) \) is a solution of the ODE system (1.15). Then \( W^+ \) and \( W^- \) satisfy the following:
\[
\begin{align*}
W^\pm &\in C^\infty(0, \infty), \quad 0 < W^\pm < t^\pm \text{ for all } \ell > 0, \\
W^\pm &\sim \ell \text{ as } \ell \to 0, \quad W^\pm \sim t^\pm - \frac{2B}{A_+ - A_-} \text{ as } \ell \to \infty, \\
W^{\prime\prime} &> 0 \text{ when } B < 0, \quad W^{\prime\prime} \sim \frac{c_+}{\ell} \text{ as } \ell \to \infty,
\end{align*}
\tag{1.16}
\]

where
\[
c_+ = \frac{A_+ - B}{(A_+ A_- - B^2)t^+}.
\]

Moreover, \((W^+, W^-)\) is the unique solution of (1.15). \qed

The linear Ginzburg-Landau operator of (1.14) around the standard vortex \( w = (w^+, w^-) \) is given by
\[
L_0(\phi) := (L^+_0, L^-_0)(\phi),
\]
with
\[
L^\pm_0(\phi) = \Delta \phi^\pm + \left[A_\pm (t^{\pm2} - W^{\pm2}) + B(t^{\mp2} - W^{\mp2})\right] \phi^\pm
- 2A_\pm \text{Re}(w^\mp \bar{\phi}^\pm)w^\pm - 2B \text{Re}(w^\mp \bar{\phi}^\pm)w^\pm.
\tag{1.17}
\]
S. Alama and Q. Gao in [3] also studied the stability of \( w(\tilde{z}) \) in the sense of spectrum of the linear operator \( L_0(\phi) \) in a disk of \( \mathbb{R}^2 \). Furthermore, in [18], L. Duan and J. Yang studied the linearized operator \( L_0 \). Under an extra assumption \( B < 0 \), they proved the non-degeneracy result of degree pair \((+1, +1)\) vortex solution in a natural Hilbert space \( \mathcal{H} \) endowed with the norm \( \| \cdot \|_{\mathcal{H}} \) in the following form

\[
\| \phi \|_{\mathcal{H}} = \int_{\mathbb{R}^2} (|\nabla \phi^+|^2 + |\nabla \phi^-|^2) + \int_{\mathbb{R}^2} \left[ A_+ (t^{+2} - W^{+2}) - B(t^{-2} - W^{-2}) \right] |\phi^+|^2 \\
+ \int_{\mathbb{R}^2} \left[ A_+ (t^{-2} - W^{-2}) - B(t^{+2} - W^{+2}) \right] |\phi^-|^2.
\]  

(1.18)

For convenience of reader, we present the non-degeneracy result in the following lemma.

**Lemma 1.3.** ([18]). Assume that (H1) and \((H2)\) hold. Suppose that \( L_0(\phi) = 0 \) for some \( \phi \in \mathcal{H} \). Then we have

\[
\phi = c_1 \frac{\partial w}{\partial x_1} + c_2 \frac{\partial w}{\partial x_2},
\]

for certain constants \( c_1 \) and \( c_2 \). \( \square \)

Note that we also get the non-degeneracy result for \( w \) with the degree pair \((-1, -1), (-1, 1) \) or \((1, -1)\) in [18] under the same constraints. Furthermore, in the Appendix, we show that the operator \( L_0(\phi) \) does have one kernel if \( \phi \) enjoys some symmetry and decay assumptions, see Lemmas [8.1-8.2].

There are extensive researches on solutions with vortex structures to the classical Ginzburg-Landau equation with suitable boundary conditions, see [8, 23, 27, 31, 34]. However, for the Ginzburg-Landau system, much less is known about the vortices, especially for solutions with vortex filaments in the case with higher dimension \( N \geq 3 \). The non-degeneracy arguments provided in [18] for the vortex solutions with standard degree pair \((+1, +1)\) to the system (1.14) make it possible to construct vortex solutions by using the singular perturbation methods. Inspired by [12], we construct new solutions of (1.1) with helical vortex filaments in \( \mathbb{R}^3 \). We now present the first result as follows:

**Theorem 1.4.** Assume that (H1) and \( B < 0 \) hold. For any integer \( k \geq 2 \) and \( \epsilon \) small enough, there exists a solution \( w_\epsilon(\tilde{z}, t) = (w_\epsilon^+(\tilde{z}, t), w_\epsilon^-(\tilde{z}, t)) \in \mathbb{C}^2 \) of (1.1) with \( N = 3 \) in the form

\[
w_\epsilon(\tilde{z}, t) = \left( (t^+)^{1-k} \prod_{j=1}^k w^+(\tilde{z} - \frac{g_{\epsilon,j}^+(t)}{\epsilon}), (t^-)^{1-k} \prod_{j=1}^k w^-(\tilde{z} - \frac{g_{\epsilon,j}^-(t)}{\epsilon}) \right) + \phi_\epsilon(\tilde{z}, t),
\]

(1.19)

where \( \phi_\epsilon(\tilde{z}, t) = (\phi_\epsilon^+(\tilde{z}, t), \phi_\epsilon^-(\tilde{z}, t)) \in \mathbb{C}^2 \) and all \( g_{\epsilon,j}^\pm(t) \)'s are \( 2\pi \) periodic in variable \( t \) which are given by

\[
g_{\epsilon,j}^\pm(t) \approx \frac{1}{\sqrt{\ln \epsilon}} \frac{1}{\sqrt{k - 1 + e^{2i\epsilon}(\epsilon + 1)k}}, \quad \text{for} \ j = 1, \cdots, k.
\]

(1.20)

Furthermore, we also have

\[
|\phi_\epsilon^\pm| \leq \frac{D}{\ln \epsilon}, \quad \text{for some constant} \ D > 0,
\]

(1.21)

and

\[
\lim_{|\tilde{z}| \to +\infty} |w_\epsilon(\tilde{z}, t)| = \lim_{|\tilde{z}| \to +\infty} \sqrt{|w_\epsilon^+(\tilde{z}, t)|^2 + |w_\epsilon^-(\tilde{z}, t)|^2} = 1, \quad \text{uniformly in} \ t.
\]

(1.22)

\( \square \)

Note that, due to (1.22), the solutions given in Theorem 1.4 satisfies the asymptotic property (1.3). Theorem 1.4 also gives a negative answer to an analogue of Gibbons conjecture (or De Giorgi conjecture) for the Ginzburg-Landau system.

In Theorem 1.4 we use the degree pair \((+1, +1)\) vortex solution \( w \) as the building block to construct solutions. Thus the solutions constructed in Theorem 1.4 consist in \( k \) vortex helices of degree pair \((+1, +1)\). Note that in [18], the authors showed that \( \tilde{w} = (w^+, \tilde{w}^-) \), the degree pair \((-1, -1)\) vortex solution of (1.14), is non-degenerate. Taking \( \tilde{w} \) as the building block and using the same method, under the same conditions
in Theorem 1.4 we can construct solution \( \hat{w}_\epsilon(\tilde{z}, t) \) which consists in \( k \) vortex helices of degree pair \((1, -1)\) with the form

\[
\hat{w}_\epsilon(\tilde{z}, t) \approx \left( (t^+)^{1-k} \prod_{j=1}^{k} w^+ \left( \frac{\tilde{z} - \tilde{g}_{\epsilon,j}(t)}{\epsilon} \right), (t^-)^{1-k} \prod_{j=1}^{k} w^- \left( \frac{\tilde{z} - \tilde{g}_{\epsilon,j}(t)}{\epsilon} \right) \right).
\]

(1.23)

Analogous to the work in [12], we also consider solutions to (1.1) which consist in \( \geq 4 \) vortex helices of degree pair \((+1, +1)\) rotating around a straight vortex filament of degree pair \((-1, -1)\). The result is as below.

**Theorem 1.5.** Assume that (H1) and \( B < 0 \) hold. For any integer \( k \geq 4 \), there exists \( \epsilon_0 > 0 \) small such that for every \( \epsilon < \epsilon_0 \), there exists a solution \( w_\epsilon \) solving (1.1) with \( N = 3 \). Furthermore, the solution can be written in the form

\[
w_\epsilon(\tilde{z}, t) \approx \left( (t^+)^{-k} \prod_{j=1}^{k} w^+ \left( \frac{\tilde{z} - \tilde{g}_{\epsilon,j}(t)}{\epsilon} \right), (t^-)^{-k} \prod_{j=1}^{k} w^- \left( \frac{\tilde{z} - \tilde{g}_{\epsilon,j}(t)}{\epsilon} \right) \right),
\]

(1.24)

where \( \tilde{g}_{\epsilon,j}^\pm \) is \( 2\pi \) periodic in \( t \) and satisfies the asymptotic behavior

\[
\tilde{g}_{\epsilon,j}^\pm(t) \approx \frac{1}{\sqrt{\ln \epsilon}} \sqrt{k - 3e^{it}e^{\frac{2(j+1)\pi}{k}}}, \quad \text{for } j = 1, \ldots, k.
\]

\( \Box \)

Theorems 1.4 and 1.5 establish the existence of multiple helical vortex filaments for Ginzburg-Landau system (1.1) in \( \mathbb{R}^3 \). In fact, the existence of a solution with a multiple-helices vortex structure is an interesting question. In classical fluid and Bose-Einstein condensates, experiments and numerical methods show that the possible existence of a large number of interacting helical vortex filaments, see [5, 22]. The reader can also refer to [13, 14, 24, 32, 37] for the constructions of solutions with helical vortex structure, involving Schrödinger map equation, Gross-Pitaevskii equation and Euler equation.

For convenience, we just provide the proof of Theorem 1.4 in the present paper and Theorem 1.5 can be proved by an exactly similar way. Here is the outline of procedure.

- Using the screw symmetry in (1.20), we reduce the original problem (1.1) to a 2-dimensional one, see (2.8). This step will be done in Section 2.1.

- Next, we will construct an approximation to the real solutions of (2.8) which is denoted by \( v_d = (v^+_d, v^-_d) \), see (2.9), (2.10).

- Note that the coupling between the real part and imaginary part in the linearized problem of Ginzburg-Landau system (1.14) around \( v_d \) would make the setting very complicated. We would use the idea in [17] to overcome this obstacle. More precisely speaking, for a perturbation,

\[
\psi = (\psi^+, \psi^-) = (\psi_1^+, i\psi_2^+, \psi_1^-, i\psi_2^-) \in \mathbb{C}^2
\]

with symmetry (2.20), we take the solution in the form of (2.14). Then we rewrite the original problem to a perturbation of the linearized problem (see (2.22))

\[
\mathcal{L}_\epsilon \psi + \mathcal{R} + \mathcal{N}(\psi) = 0,
\]

where \( \mathcal{L}_\epsilon = \left( \mathcal{L}^+_\epsilon, \mathcal{L}^-_\epsilon \right) \) is the linearized operator, \( \mathcal{R} = \left( \mathcal{R}^+, \mathcal{R}^- \right) \) is the error term, and \( \mathcal{N}(\psi) = \left( \mathcal{N}^+(\psi), \mathcal{N}^-(\psi) \right) \) is the nonlinear term of higher order. When far away from the vortices, the linearized operator \( \mathcal{L}_\epsilon \psi \) behaves as below

\[
\mathcal{L}_\epsilon \psi \approx \begin{cases}
\Delta \psi^+ - 2iA_k |v^+_d|^2 \text{Im}(\psi^+) - 2iB |v^+_d|^2 \text{Im}(\psi^-) \\
\Delta \psi^- - 2iA_k |v^-_d|^2 \text{Im}(\psi^+) - 2iB |v^-_d|^2 \text{Im}(\psi^-).
\end{cases}
\]

Although there are also coupling between the “+” part and “−” part, the operator \( \mathcal{L}_\epsilon \psi \) is good far away from vortices due to the positive definite assumption (H1). On the other hand, when near the vortices, from Lemma 2.2, Lemmas 8.1, 8.2 we would deal with the problem by setting up a projected nonlinear problem, see (4.1).
• Section 3 is devoted to estimating the error in a suitable weighted norm.
• We study and solve the projected nonlinear problem (4.1) by using a fixed point argument. This step is more or less standard and will be done in Sections 4-6.
• In order to finish the proof of Theorem 1.4, in Section 7 the reduction process will be applied to force the vanishing of Lagrange multiplier $c$ in (4.1).

In order to finish the proof of Theorem 1.4, in Section 7 the reduction process will be applied to force the vanishing of Lagrange multiplier $c$ in (4.1).

Then a function with screw-symmetry can be expressed as a function of two variables, i.e., for any $(\tilde{z},t)$.

Notation and Definitions: We end this introduction by giving some notation and definitions. For any complex-valued vector function $F = (F^+, F^-) \in \mathbb{C}^2$, we define its conjugation

$$\mathcal{T} = (F^+, F^-).$$

Next, we give the definition of screw-symmetry in the coordinates $(r, \theta, t)$ with $(\tilde{z}, t) = (re^{i\theta}, t)$.

**Definition 1.6.** We say that a function $u = (u^+, u^-) : \mathbb{R}^3 \to \mathbb{C}^2$ has screw-symmetry if

$$u(r, \theta + h, t + h) = u(r, \theta, t),$$

i.e.,

$$u^\pm(r, \theta + h, t + h) = u^\pm(r, \theta, t),$$

for any $h \in \mathbb{R}$. \hfill $\square$

## 2. Formulation of the Problem

### 2.1. Reduction to a 2-dimensional problem by using screw-symmetry.

By using the screw-symmetry, we will transform the problem (1.14), which is of 3 dimensional in nature, to a 2 dimensional one. For simplicity, we only treat the case $k = 2$ in Theorem 1.4 and the arguments for general $k > 2$ can be easily adapted.

Notice that the condition (1.25) in Definition 1.4 is equivalent to

$$u(r, \theta, t + h) = u(r, \theta - h, t)$$

for any $h \in \mathbb{R}$.

Then a function with screw-symmetry can be expressed as a function of two variables, i.e., for any $(r, \theta, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R},$

$$u(r, \theta, t) = u(r, \theta - t, 0) =: \tilde{u}(r, \theta - t).$$

We will apply these facts to make a setting to reduce the dimension.

Here is the observation in [12]. We can write the standard degree pair $(+1, +1)$ vortex solution for (1.14) in polar coordinates $(\ell, \theta)$, i.e.,

$$w(\tilde{z}) = \left(w^+(\tilde{z}), w^-(-\tilde{z})\right) = \left(W^+(r)e^{i\theta}, W^-(r)e^{i\theta}\right),$$

and consider a function $w_d$ in the form

$$w_d(\tilde{z}, t) = \left(\tilde{w}_d^+(\tilde{z}, t), \tilde{w}_d^-(-\tilde{z}, t)\right),$$

(2.1)
where

\[
    w^{\pm}_d(\tilde{z}, t) = (t^{\pm})^{-1} \prod_{j=1}^{2} w^{\pm} \left( \frac{\text{Re}(\tilde{z}) - d \cos(t + (j - 1)\pi)}{\epsilon}, \frac{\text{Im}(\tilde{z}) - d \sin(t + (j - 1)\pi)}{\epsilon} \right).
\]  

(2.2)

It can be checked that

\[
    w_d(r, \theta, t + h) = \left( e^{2i\theta} w^+_d(r, \theta - h, t), e^{2i\theta} w^-_d(r, \theta - h, t) \right) = e^{2i\theta} w_d(r, \theta - h, t),
\]

for any \( h \) in \( \mathbb{R} \). That is, \( w_d \) does not have screw-symmetry but \( \tilde{w}_d(r, \theta, t) := e^{-2it} w_d(r, \theta, t) \) fulfills the symmetry.

The above arguments suggest that we should look for a solution \( w_e \) of (1.1) that can be written as

\[
    w_e(r, \theta, t) = \left( w^+_e(r, \theta, t), w^-_e(r, \theta, t) \right) = \left( e^{2it} \tilde{w}^+_d(r, \theta, t), e^{2it} \tilde{w}^-_d(r, \theta, t) \right)
\]

(2.3)

with \( \tilde{w}^\pm \) being screw-symmetric. Thus

\[
    \tilde{w}^\pm(r, \theta, t) = u^\pm(r, \theta - t),
\]

(2.4)

where \( u^\pm : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{C} \) is 2\( \pi \)-periodic in the second variable. Denoting

\[
    u^\pm(r, s) = u^\pm(r, \theta - t)
\]

with \( s := \theta - t \),

we can see that

\[
    \partial_r u^\pm_1 = e^{2it} \partial_r u^\pm(r, s), \quad \partial_r^2 u^\pm_1 = e^{2it} \partial_r^2 u^\pm(r, s), \quad \partial_\theta u^\pm_1 = e^{2it} \partial_\theta u^\pm(r, s), \quad \partial_\theta^2 u^\pm_1 = e^{2it} \partial_\theta^2 u^\pm(r, s), \quad \partial_{ss} u^\pm_1 = e^{2it} \partial_{ss} u^\pm(r, s), \quad \partial_{tt} u^\pm_1 = e^{2it} \partial_{tt} u^\pm(r, s),
\]

\[
    \partial_r u^\pm_1 = [2iu^\pm - \partial_r u^\pm]e^{2it}, \quad \partial^2 u^\pm_1 = [\partial^2_{ss} u^\pm - 4i\partial_s u^\pm - 4u^\pm]e^{2it}.
\]

Recalling the expression of the Laplacian operator in cylindrical coordinates as below

\[
    \partial^2_r + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial^2_\theta + \partial^2_{tt},
\]

we conclude that \( w_e \) is a solution of (1.1) if and only if \( u = (u^+, u^-) \) is a solution of the following equations

\[
    \epsilon^2 \left( \partial^2_{rr} u^\pm + \frac{1}{r} \partial_r u^\pm + \frac{1}{r^2} \partial^2_\theta u^\pm + \partial^2_{ss} u^\pm - 4i\partial_s u^\pm - 4u^\pm \right) + \left( A_\pm (t^\pm - |u^\pm|^2) + B (t^\mp - |u^\mp|^2) \right) u^\pm = 0 \quad \text{in } \mathbb{R}^+_e \times \mathbb{R}.
\]

(2.5)

We will work in the rescaled coordinates, that is, we define

\[
    v^\pm(r, s) := u^\pm(\epsilon r, s),
\]

(2.7)

and search for a solution to the system of equations

\[
    \partial^2_{rr} v^\pm + \frac{1}{r} \partial_r v^\pm + \frac{1}{r^2} \partial^2_\theta v^\pm + \epsilon^2 (\partial^2_{ss} v^\pm - 4i\partial_s v^\pm - 4v^\pm)
\]

\[
    + \left( A_\pm (t^\pm - |v^\pm|^2) + B (t^\mp - |v^\mp|^2) \right) v^\pm = 0 \quad \text{in } \mathbb{R}^+_e \times \mathbb{R}.
\]

(2.8)

From now on we will work in the plane \( \mathbb{R}^2 \) and use the notation \( z = x_1 + ix_2 = re^{i\theta} \). We denote by \( \Delta \) the Laplacian operator in 2-dimensional space, i.e.,

\[
    \Delta = \partial^2_{xx} + \partial^2_{zz} = \partial^2_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial^2_\theta.
\]

The approximation of a real solution to (2.8) can be chosen in the form

\[
    v_d(z) = (v^+_d(z), v^-_d(z))
\]

(2.9)

with

\[
    v^\pm_d = (t^\pm)^{-1} \prod_{j=1}^{2} w^\pm \left( z - de^{i(j - 1)\pi} \right),
\]

(2.10)
In the above formulae, we have denoted the linear operator in the form \( e_j = e^{i(\pi - 1)\theta} \), for some \( \delta = O(1) \) to be determined later and \( w = (w^+, w^-) \) is the standard degree \((+1, +1)\) vortex solution of equation \((1.1)\). For convenience, we use the notation

\[ e_j = e^{i(\pi - 1)\theta}, \quad j = 1, 2. \]  
(2.11)

**Remark 2.1.** Starting from \((2.9)\) and following the transformations in \((2.3)\) and \((2.7)\), we claim that by some tedious but straightforward calculations, we can get

\[ e^{2it} \prod_{j=1}^{2} w^\pm \left( \frac{z}{\epsilon} - \delta e^{2i(k-1)\theta} \right) = \prod_{j=1}^{2} w^\pm \left( \frac{z}{\epsilon} - \delta e^{i(\pi - 1)\theta} e^{2i(k-1)\theta} \right), \]

with

\[ z = re^{i(\theta - t)}, \quad \tilde{z} = re^{i\theta}. \]

Equivalently, we get an approximation which has 2 vortex helices with degree pair \((+1, +1)\) to the system \((1.1)\) with the form

\[ \left( (t^+)^{-1} \prod_{j=1}^{2} w^\pm \left( \frac{z}{\epsilon} - \delta e^{it} e^{2i(k-1)\theta} \right), (t^-)^{-1} \prod_{j=1}^{2} w^\pm \left( \frac{z}{\epsilon} - \delta e^{it} e^{2i(k-1)\theta} \right) \right). \]  
(2.12)

For the general integer \( k > 2 \), the corresponding approximation to the solution of system \((2.8)\) would be

\[ \left( (t^+)^{1-k} \prod_{j=1}^{k} w^\pm \left( z - \delta e^{it} e^{2i(k-1)\theta} \right), (t^-)^{1-k} \prod_{j=1}^{k} w^\pm \left( z - \delta e^{it} e^{2i(k-1)\theta} \right) \right). \]  
(2.13)

The asymptotic property \((1.22)\) of our solutions holds from the construction and Lemma 1.2. The functions in \((2.12)\) and \((2.13)\) will provide the profile of the solution \( w_\delta \) in Theorem 1.4. \(\square\)

**2.2. Additive-multiplicative perturbation.** Let

\[ S(v) := \left( S^+(v), S^-(v) \right), \quad \text{for} \quad v = (v^+, v^-), \]

with

\[ S^\pm(v) = \Delta v^\pm + \epsilon^2 (\partial^2_{ss} v^\pm - 4i\partial_s v^\pm - 4v^\pm) + \left[ A_\pm (t^\pm - |v^\pm|^2) + B(t^\mp - |v^\mp|^2) \right] v^\pm. \]

The operator \( S \) can be decomposed as

\[ S = S_0 + S_1, \]

where

\[ S_0(v) = \left( S_0^+(v), S_0^-(v) \right), \quad S_1(v) = \left( S_1^+(v), S_1^-(v) \right), \]

with

\[ S_0^\pm(v) = \Delta v^\pm + \left[ A_\pm (t^\pm - |v^\pm|^2) + B(t^\mp - |v^\mp|^2) \right] v^\pm, \]

\[ S_1^\pm(v) = \epsilon^2 (\partial^2_{ss} v^\pm - 4i\partial_s v^\pm - 4v^\pm). \]

Therefore, the equation \((2.8)\) can be written as

\[ S(v) = 0. \]  
(2.14)

We first try to search for a solution of \((2.14)\) with the form

\[ v = v_d + \phi, \quad \text{for some} \quad \phi = (\phi^+, \phi^-) \text{ small}. \]

We see that

\[ S_0(v_d + \phi) = S_0(v_d) + \mathcal{L}_0(\phi) + \mathcal{N}_0(\phi), \]

\[ S_1(v_d + \phi) = S_1(v_d) + S_1(\phi). \]

In the above formulae, we have denoted the linear operator in the form

\[ \mathcal{L}_0(\phi) = \left( \mathcal{L}_0^+, \mathcal{L}_0^- \right)(\phi), \]
with
\[
\mathcal{L}_0^\pm(\phi) = \Delta \phi^\pm + \left[ A_\pm(t^\pm^2 - |v_d^\pm|^2) + B(t^\mp^2 - |v_d^\mp|^2) \right] \phi^\pm
- 2A_\pm \text{Re}(v_d^\pm \phi^\mp)v_d^\pm - 2B\text{Re}(v_d^\mp \phi^\mp)v_d^\pm,
\]
and the nonlinear operator as the following
\[
\mathcal{N}_0(\phi) = \left( \mathcal{N}_0^+, \mathcal{N}_0^- \right)(\phi),
\]
where
\[
\mathcal{N}_0^\pm(\phi) = -2A_\pm \text{Re}(v_d^\pm \phi^\mp)\phi^\pm - A_\pm |\phi^\pm|^2(v_d^\pm + \phi^\pm)
- 2B\text{Re}(v_d^\mp \phi^\mp)\phi^\pm - B|\phi^\mp|^2(v_d^\pm + \phi^\pm).
\]
We note that, due to the strong coupling effect in the following terms
\[
- 2A_\pm \text{Re}(v_d^\pm \phi^\mp)v_d^\pm - 2B\text{Re}(v_d^\mp \phi^\mp)v_d^\pm,
\]
it is very complicated to handle the linear operator \( \mathcal{L}_0(\phi) \).

So, we follow the idea in the work of \cite{17} to look for a solution of (2.14) in the following form
\[
v = (v^+, v^-), \quad v^\pm = \eta(v_d^\pm + (1 + i\psi^\pm) + (1 - \eta)v_d^\mp e^{i\psi^\mp}, \quad (2.15)
\]
where \( v_d = (v_d^+, v_d^-) \) is the ansatz (2.9) and \( \psi = (\psi^+, \psi^-) = (\psi_1^+ + i\psi_2^+, \psi_1^- + i\psi_2^-) \) is the unknown perturbation. The cut-off function \( \eta \) is defined as
\[
\eta(z) = \eta_1(|z - \tilde{d}|) + \eta_1(|z + \tilde{d}|), \quad z \in \mathbb{C} \cong \mathbb{R}^2,
\]
and \( \eta_1 : \mathbb{R} \to [0, 1] \) is a smooth cut-off function such that
\[
\eta_1(t) = 1 \text{ for } t \leq 1 \text{ and } \eta_1(t) = 0 \text{ for } t \geq 2. \quad (2.16)
\]
Our goal is to rewrite (2.14) in the form
\[
\mathcal{L}_\epsilon \psi + \mathcal{R} + \mathcal{N}(\psi) = 0 \quad (2.17)
\]
by identifying the linear operator \( \mathcal{L}_\epsilon = \left( \mathcal{L}_\epsilon^+, \mathcal{L}_\epsilon^- \right) \), the error term \( \mathcal{R} = \left( \mathcal{R}^+, \mathcal{R}^- \right) \) and the nonlinear term \( \mathcal{N}(\psi) = \left( \mathcal{N}^+(\psi), \mathcal{N}^-(\psi) \right) \). This will be done in the sequel and the result is given in (2.22).

Rewrite \( v \) in (2.15) as
\[
v = v_d + \phi = (v_d^+, v_d^-) + (\phi^+, \phi^-)
\]
with
\[
\phi^\pm = iv_d^\pm \psi^\pm + \gamma(\psi^\pm), \quad \gamma(\psi^\pm) = (1 - \eta)v_d^\pm(e^{i\psi^\pm} - 1 - i\psi^\pm).
\]
Then
\[
S_0(v) = S_0(v_d) + \mathcal{L}_0\left( (iv_d^+ \psi^+, iv_d^- \psi^-) \right) + \mathcal{L}_0\left( (\gamma(\psi^+), \gamma(\psi^-)) \right)
+ \mathcal{N}_0\left( (iv_d^+ \psi^+, iv_d^- \psi^-) + (\gamma(\psi^+), \gamma(\psi^-)) \right),
\]
\[
S_1(v) = S_1(V_d) + S_1\left( (iv_d^+ \psi^+, iv_d^- \psi^-) \right) + \mathcal{L}_0\left( (\gamma(\psi^+), \gamma(\psi^-)) \right).
\]
Explicit expressions of some operators in the above will be derived one by one.
• By some computations, we start from \( S_0(v) \) and consider
\[
\mathcal{L}_0\left( (iv_d^+ \psi^+, iv_d^- \psi^-) \right) = \left( \mathcal{L}_0^+, \mathcal{L}_0^- \right) \left( (iv_d^+ \psi^+, iv_d^- \psi^-) \right)
\]
where
\[
\mathcal{L}_0^\pm\left( (iv_d^+ \psi^+, iv_d^- \psi^-) \right)
\]
\[
\Delta (iv_d^\pm \psi^\pm) = \Delta (iv_d^\pm \psi^\pm) + [A_\pm (t^\pm - |v_d|^2)] (iv_d^\pm \psi^\pm) \\
- 2A_\pm \text{Re} \left( \frac{\Delta (iv_d^\pm \psi^\pm)}{v_d^\pm} \right) v_d^\pm - 2B \text{Re} \left( \frac{\Delta (iv_d^\pm \psi^\pm)}{v_d^\pm} \right) v_d^\pm \\
= iv_d^\pm \left( \frac{\Delta S_0^\pm (v_d)}{v_d^\pm} + [A_\pm (t^\pm - |v_d|^2)] \right) \psi^\pm \\
+ iv_d^\pm \left( \Delta \psi^\pm + \frac{\nabla v_d^\pm \nabla \psi^\pm}{v_d^\pm} - 2iA_\pm |v_d^\pm|^2 \text{Im}(\psi^\pm) - 2iB |v_d^\pm|^2 \text{Im}(\psi^\mp) \right) \\
= iv_d^\pm \left( \frac{\Delta S_0^\pm (v_d)}{v_d^\pm} + [A_\pm (t^\pm - |v_d|^2)] \right) \psi^\pm \\
+ iv_d^\pm \left( \Delta \psi^\pm + \frac{\nabla v_d^\pm \nabla \psi^\pm}{v_d^\pm} - 2iA_\pm |v_d^\pm|^2 \text{Im}(\psi^\pm) - 2iB |v_d^\pm|^2 \text{Im}(\psi^\mp) \right) \\
= iv_d^\pm \left( \frac{S_0^\pm (v_d)}{v_d^\pm} \psi^\pm + \tilde{L}_0^\pm (\psi) \right), \\
\text{weak effect} \\
\tilde{L}_0^\pm (\psi) = \Delta \psi^\pm + \frac{\nabla v_d^\pm \nabla \psi^\pm}{v_d^\pm} - 2iA_\pm |v_d^\pm|^2 \text{Im}(\psi^\pm) - 2iB |v_d^\pm|^2 \text{Im}(\psi^\mp). \\
\] (2.18)

Thus we have
\[
S_0^\pm (v) = S_0^\pm (v_d) + \mathcal{L}_0^\pm \left( (iv_d^\pm \psi^\pm, iv_d^\mp \psi^-) \right) + \mathcal{L}_0^\pm \left( (\gamma(\psi^\pm), \gamma(\psi^-)) \right) \\
+ \mathcal{N}_0^\pm \left( (iv_d^\pm \psi^\pm, iv_d^\mp \psi^-) + (\gamma(\psi^\pm), \gamma(\psi^-)) \right) \\
= iv_d^\pm \left( - \frac{S_0^\pm (v_d)}{v_d^\pm} + S_0^\pm (v_d) \psi^\pm + \tilde{L}_0^\pm (\psi) - i \frac{1}{v_d^\pm} \left( \gamma(\psi^\pm), \gamma(\psi^-) \right) \right) \\
- \frac{1}{v_d^\pm} \mathcal{N}_0^\pm \left( (iv_d^\pm \psi^\pm, iv_d^\mp \psi^-) + (\gamma(\psi^\pm), \gamma(\psi^-)) \right). \\
\]

Specially, when far away from the vortices, \( v = (v_d^+ e^{i\psi^+}, v_d^- e^{i\psi^-}) \). This implies that
\[
S_0(v) = \left( S_0^+ \left( (v_d^+ e^{i\psi^+}, v_d^- e^{i\psi^-}) \right), S_0^- \left( (v_d^+ e^{i\psi^+}, v_d^- e^{i\psi^-}) \right) \right), \\
\] (2.19)
and
\[
S_0^\pm \left( (v_d^+ e^{i\psi^+}, v_d^- e^{i\psi^-}) \right) \\
= \Delta (v_d^\pm e^{i\psi^\pm}) + \left[ A_\pm (t^\pm - |(v_d^\pm e^{i\psi^\pm})|^2) \right] (v_d^\pm e^{i\psi^\pm}) \\
= iv_d^\pm e^{i\psi^\pm} \left( - i \frac{S_0^\pm (v_d)}{v_d^\pm} + \tilde{L}_0^\pm (\psi) + \tilde{N}_0^\pm (\psi) \right),
\]
where

$$\hat{N}^\pm_0(\psi) = i(\nabla \psi^\pm)^2 + iA_\pm |v_d^\pm|^2(e^{-2\psi_0^\pm} - 1 + 2\psi_2^\pm) + iB|v_d^\mp|^2(e^{-2\psi_2^\mp} - 1 + 2\psi_2^\mp).$$

- We consider
  \[
  S_1\left((iv_d^\pm \psi^+, iv_d^- \psi^-)\right) = \left(S_1^\pm\left((iv_d^\pm \psi^+, iv_d^- \psi^-)\right), S_1^\mp\left((iv_d^\pm \psi^+, iv_d^- \psi^-)\right)\right),
  \]
  and it follows that
  \[
  S_1^\pm\left((iv_d^\pm \psi^+, iv_d^- \psi^-)\right) = iv_d^\pm \left(S_1^\pm(v_d) + \hat{L}_1^\pm(\psi)\right)
  \]
  with
  \[
  \hat{L}_1^\pm(\psi) = e^2(\partial_s^2 \psi^\pm + \frac{2\partial v_d^\pm}{v_d^\pm} \partial_s \psi^\pm - 4i\partial_s \psi^\pm).
  \]

Specially, we have that, when far away from the vortices

\[
S_1(v) = \left(S_1^\pm\left((v_d^\pm e^{i\psi^+}, v_d^\mp e^{i\psi^-})\right), S_1^\mp\left((v_d^\pm e^{i\psi^+}, v_d^\mp e^{i\psi^-})\right)\right),
\]
with

\[
S_1^\pm\left((v_d^\pm e^{i\psi^+}, v_d^\mp e^{i\psi^-})\right) = iv_d^\pm e^{i\psi^\pm} \left[-i \frac{S_1^\pm(v_d)}{v_d^\pm} + \hat{L}_1^\pm(\psi) + e^2 i(\partial_s \psi^\pm)^2\right].
\]

In order to draw a conclusion, a new cut-off function will be introduced as the following

$$\tilde{\eta}(z) = \eta_1(|z - \tilde{d}| - 1) + \eta_1(|z + \tilde{d}| - 1)$$

with $\eta_1$ defined in (2.16). Then the problem (2.14) can be rewritten as

\[
0 = \tilde{\eta} iv_d^\pm \left[-i \frac{S_0^\pm(v_d)}{v_d^\pm} + \hat{L}_0^\pm(\psi) + \frac{S_1^\pm(v_d)}{v_d^\pm} \psi^\pm - i \frac{S_1^\mp(v_d)}{v_d^\pm} + \hat{L}_1^\pm(\psi) + S_1^\pm(v_d) \psi^\pm \right]
- \frac{i}{v_d^\pm} L_0^\pm\left((\gamma(\psi^+), \gamma(\psi^-))\right) - \frac{i}{v_d^\pm} N_0^\pm\left((iv_d^\pm \psi^+, iv_d^- \psi^-) + (\gamma(\psi^+), \gamma(\psi^-))\right)
- \frac{i}{v_d^\pm} S_1^\pm\left((\gamma(\psi^+), \gamma(\psi^-))\right)
+ (1 - \tilde{\eta}) iv_d^\pm e^{i\psi^\pm} \left[-i \frac{S_1^\pm(v_d)}{v_d^\pm} + \hat{L}_1^\pm(\psi) + \hat{N}_0^\pm(\psi) - i \frac{S_1^\mp(v_d)}{v_d^\pm} + \hat{L}_1^\pm(\psi) + e^2 i(\partial_s \psi^\pm)^2\right].
\]

By the definitions of $\mathcal{L}, \mathcal{R}$ and $\mathcal{N}$, we can have a simpler representation of the above equation for (2.14):

\[
\mathcal{L}_\psi \psi + \mathcal{R} + \mathcal{N}(\psi) = 0, \quad \text{i.e.} \quad \mathcal{L}_\psi^\pm \psi + \mathcal{R}_\psi^\pm + \mathcal{N}_\psi^\pm = 0,
\]

where

\[
\mathcal{L}_\psi^\pm(\psi) := (\hat{L}_0^\pm + \hat{L}_1^\pm)(\psi) + \tilde{\eta} \frac{S_1^\pm(v_d)}{v_d^\pm} \psi^\pm, \quad \mathcal{R}_\psi^\pm := -i \frac{S(v_d^\pm)}{v_d^\pm},
\]

\[
\mathcal{N}_\psi^\pm(\psi) := \tilde{\eta} \left(\frac{1}{\tilde{\eta} + (1 - \tilde{\eta})e^{i\psi^\pm}} - 1\right) \frac{S_1^\pm(v_d)}{v_d^\pm} \psi^\pm
- \frac{i}{v_d^\pm} \tilde{\eta} \frac{\tilde{\eta}}{\tilde{\eta} + (1 - \tilde{\eta})e^{i\psi^\pm}} \left[L_0^\pm\left((\gamma(\psi^+), \gamma(\psi^-))\right) + S_1^\pm\left((\gamma(\psi^+), \gamma(\psi^-))\right)\right]
+ \tilde{\eta} \left(i \frac{S_1^\pm(v_d)}{v_d^\pm} + (\gamma(\psi^+), \gamma(\psi^-))\right).\]
Recall that \( L \) and then let \( \tilde{L} \) and define

\[
N_0^\pm(\psi) + e^2 i (\partial_s \psi^\pm)^2.
\]

From (2.18) and (2.20), we know that

\[
L_c^\pm(\psi) = \Delta \psi^\pm + \frac{\nabla v_d^\pm \nabla \psi^\pm}{v_d^\pm} - 2i A \psi^\pm \frac{\partial v_d^\pm |\psi^\pm|^2}{v_d^\pm} - 2i B |v_d^\mp|^2 \text{Im}(\psi^\mp)
\]

\[
+ e^2 \left( \partial_s \psi^\pm + \frac{2i v_d^\pm}{v_d^\pm} \partial_s \psi^\pm - 4i \partial_s \psi^\pm \right) + \frac{N^\pm(v_d)}{v_d^\pm} \psi^\pm.
\]

Note that when \( |z \mp d| \geq 3 \) the nonlinear terms take the form

\[
N^\pm(\psi) = N_0^\pm(\psi) + e^2 i (\partial_s \psi^\pm)^2
\]

\[
= i(\nabla \psi^\pm)^2 + i A |v_d^\pm|^2 (e^{-2\psi^\pm} - 1 + 2 \psi^\pm) + i B |v_d^\mp|^2 (e^{-2\psi^\mp} - 1 + 2 \psi^\mp) + e^2 i (\partial_s \psi^\pm)^2.
\]

2.3. Another form of the equation near each vortex. Note that the expressions of the terms \( S_0(v) \) and \( S_1(v) \) in the outer region (far from the vortices) are given in (2.19) and (2.21). In this part, we will write the operator \( L_c \psi \) in another form when it is near the vortices so that we can know it better.

In order to do so, we first introduce some notation. We denote

\[
z_j = z - de^{i(2j-1)\pi} \quad \text{for } j = 1, 2,
\]

and then let

\[
\Phi_j(z) = (\Phi_j^+(z), \Phi_j^-(z)), \quad \Phi_j^\pm(z) = i w^\pm(z_j) \psi^\pm(z).
\]

Furthermore, when near the vortex center, we also assume

\[
\Phi(z) = (\Phi^+(z), \Phi^-(z)), \quad \Phi^\pm(z) = i v_d^\pm(z) \psi^\pm(z),
\]

\[
\Omega_j(z) = (\Omega_j^+(z), \Omega_j^-(z)), \quad \Omega_j^\pm(z) = \frac{v_d^\pm(z)}{w^\pm(z_j)},
\]

\[
\Omega := S(v_d), \quad \Omega^\pm = S^\pm(v_d).
\]

By some calculations, the following identity holds

\[
i w^\pm(z_j) L_c^\pm(\psi) = \frac{L_d^\pm(\Phi)}{\Omega_j^\pm}(z_j) + (\eta - 1) \frac{E^\pm}{v_d^\pm} \Phi_j^\pm(z_j)
\]

\[
= \frac{L_d^\pm \left( (\Phi_j^+ \Omega_j^+, \Phi_j^- \Omega_j^-) \right)}{\Omega_j^\pm}(z_j) + (\eta - 1) \frac{E^\pm}{v_d^\pm} \Phi_j^\pm(z_j),
\]

where

\[
L_d^\pm(\Phi) = \Delta \Phi^\pm + \left( A_\pm (t^\pm_\psi |v_d^\pm|^2) + B (t^\mp_\psi |v_d^\mp|^2) \right) \Phi^\pm
\]

\[
- 2A \text{Re} \left( v_d^\pm \Phi^\mp \right) v_d^\pm - 2B \text{Re} \left( v_d^\mp \Phi^\mp \right) v_d^\pm + e^2 (\partial_s \Phi^\pm - 4i \partial_s \Phi^\pm - 4 \Phi^\pm).
\]

Recall that

\[
\Phi = (iw_d^\pm \psi^+, iv_d^\mp \psi^-),
\]

and define

\[
L_{d,j}(\Phi) = \left( L_{d,j}^+, L_{d,j}^- \right)(\Phi), \quad \text{with } L_{d,j}^\pm(\Phi) := iw^\pm(z_j) L_c^\pm(\psi).
\]

In order to describe the properties of \( L_c^\pm(\psi) \) near the vortex, we give the following lemma.
Lemma 2.2. When near the vortices, we can see that the linear operator $L_{d,j}(\Phi) = (L^+_{d,j}, L^-_{d,j})(\Phi)$ defined in (2.28) is a small perturbation of $L^0(\Phi)$, where $L^0$ is the linearized Ginzburg-Landau operator around $(w^+(z_j), w^-(z_j))$, i.e.,

$$L^0(\phi) := (L^0_+(\phi), L^0_-(\phi)) \text{ with } \phi = (\phi^+, \phi^-),$$

and

$$L^0_\pm(\phi) = \Delta \phi^\pm + \left[ A_\pm(t^\pm - |x^\pm|)^2 + B(t^\mp - |x^\mp|)^2 \right] \phi^\pm - 2A_\pm \text{Re} \left( w^\mp \overline{\phi^\pm} \right) w^\pm(z_j) - 2B \text{Re} \left( w^\mp \overline{\phi^\pm} \right) w^\pm(z_j).$$

(2.24)

Proof. From the definition, we know that

$$L^\pm_{d,j}(\Phi) = \frac{L^\pm_{d,j}((\Phi^+_j \Omega^+_j, \Phi^-_j \Omega^-_j))}{\Omega^\pm_j} + (\eta_1 - 1) \frac{E^\pm_\pm_{d} \Phi^\pm_\pm_{d,j}}{v^\pm_\pm_{d}}$$

$$= \frac{1}{\Omega^\pm_j} \left\{ \Delta(\Phi^+_j \Omega^+_j) + \left[ A_\pm(t^\pm - |v^\pm_\pm|^2) + B(t^\mp - |v^\mp_\pm|^2) \right] \Phi^\pm_\pm_{d,j} \right. \right.$$}

$$- 2A_\pm \text{Re} \left( v^\pm_\pm \overline{(\Phi^+_j \Omega^+_j)} \right) v^\pm_\pm_{d,j} - 2B \text{Re} \left( v^\pm_\pm \overline{(\Phi^+_j \Omega^+_j)} \right) v^\pm_\pm_{d,j}$$

$$+ \epsilon^2 \left[ \partial_{ss} \Phi^\pm_\pm_{d,j} + 4i \partial_s \Phi^\pm_\pm_{d,j} - 4 \Phi^\pm_\pm_{d,j} \right] + (\eta_1 - 1) \frac{E^\pm_\pm_{d} \Phi^\pm_\pm_{d,j}}{v^\pm_\pm_{d}}$$

$$= \frac{1}{\Omega^\pm_j} \left( \Phi^\pm_\pm_{d,j} \Delta \Omega^\pm_j + 2 \nabla \Phi^\pm_\pm_{d,j} \nabla \Omega^\pm_j + \Omega^\pm_j \Delta \Phi^\pm_\pm_{d,j} \right) + (\eta_1 - 1) \frac{E^\pm_\pm_{d} \Phi^\pm_\pm_{d,j}}{v^\pm_\pm_{d}}$$

$$+ \left[ A_\pm(t^\pm - |v^\pm_\pm|^2) + B(t^\mp - |v^\mp_\pm|^2) \right] \Phi^\pm_\pm_{d,j}$$

$$+ \frac{1}{\Omega^\pm_j} \left[ - 2A_\pm \text{Re} \left( v^\pm_\pm \overline{(\Phi^+_j \Omega^+_j)} \right) v^\pm_\pm_{d,j} - 2B \text{Re} \left( v^\pm_\pm \overline{(\Phi^+_j \Omega^+_j)} \right) v^\pm_\pm_{d,j} \right]$$

$$+ \epsilon^2 \frac{1}{\Omega^\pm_j} \left[ \Omega^\pm_j \partial_{ss} \Phi^\pm_\pm_{d,j} + \Phi^\pm_\pm_{d,j} \partial_{ss} \Omega^\pm_j + 2 \partial_s \Omega^\pm_j \partial_s \Phi^\pm_\pm_{d,j} - 4i \partial_s \Omega^\pm_j \Phi^\pm_\pm_{d,j} + \partial_s \Phi^\pm_\pm_{d,j} \right]$$

$$:= \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + (\eta_1 - 1) \frac{E^\pm_\pm_{d} \Phi^\pm_\pm_{d,j}}{v^\pm_\pm_{d}},$$

with

$$\mathfrak{A}_1 = \Delta \Phi^\pm_\pm_{d,j} + \left[ A_\pm(t^\pm - |v^\pm_\pm|^2) + B(t^\mp - |v^\mp_\pm|^2) \right] \Phi^\pm_\pm_{d,j}$$

$$- 2A_\pm \text{Re} \left( v^\pm_\pm \overline{(\Phi^+_j \Omega^+_j)} \right) w^\pm(z_j) - 2B \text{Re} \left( v^\pm_\pm \overline{(\Phi^+_j \Omega^+_j)} \right) w^\pm(z_j),$$

$$\mathfrak{A}_2 = \epsilon^2 \frac{1}{\Omega^\pm_j} \left[ \Omega^\pm_j \partial_{ss} \Phi^\pm_\pm_{d,j} + \Phi^\pm_\pm_{d,j} \partial_{ss} \Omega^\pm_j + 2 \partial_s \Omega^\pm_j \partial_s \Phi^\pm_\pm_{d,j} - 4i \partial_s \Omega^\pm_j \Phi^\pm_\pm_{d,j} + \partial_s \Phi^\pm_\pm_{d,j} \right],$$

$$\mathfrak{A}_3 = \frac{1}{\Omega^\pm_j} \left[ \Delta \Omega^\pm_j + \epsilon^2 \partial_{ss} \Omega^\pm_j - 4ie^2 \partial_s \Omega^\pm_j \right] \Phi^\pm_\pm_{d,j}.$$

For $\mathfrak{A}_1$, we have

$$\mathfrak{A}_1 = L^0_\pm(\Phi_\pm_{d,j}) + \left[ A_\pm(|w^\mp(z_j)|^2 - |v^\pm_\pm|^2) + B(|w^\pm(z_j)|^2 - |v^\mp_\pm|^2) \right] \Phi^\pm_\pm_{d,j} + 2A_\pm \text{Re} \left( w^\pm(z_j) \overline{\Phi^\pm_\pm_{d,j}} \right) w^\pm(z_j) + 2B \text{Re} \left( w^\mp(z_j) \overline{\Phi^\pm_\pm_{d,j}} \right) w^\pm(z_j).$$
\[-2A_\pm \text{Re} \left( v_d^\pm (\Phi_j^\pm \Omega_j^\pm) \right) w^\pm(z_j) - 2B \text{Re} \left( v_d^\pm (\Phi_j^\pm \Omega_j^\pm) \right) w^\pm(z_j) \]
\[= L^0_{d,j}( \Phi_j ) + \left[ A_\pm \left( |w^\pm(z_j)|^2 - |v_d^\pm|^2 \right) + B \left( |w^\mp(z_j)|^2 - |v_d^\mp|^2 \right) \right] \Phi_j^\pm \]
\[+ 2A_\pm (1 - |\Omega_j^\pm|^2) \text{Re} \left( w^\pm(z_j) \Phi_j^\mp \right) w^\pm(z_j) + 2B (1 - |\Omega_j^\mp|^2) \text{Re} \left( w^\mp(z_j) \Phi_j^\mp \right) w^\pm(z_j). \]

Since
\[E^\pm = S^\pm(v_d)\]
\[= \Delta(w^\pm(z_j)\Omega_j^\pm) + \left[ A_\pm (1 - |w^\pm(z_j)|^2 |\Omega_j^\pm|^2) + B (1 - |w^\mp(z_j)|^2 |\Omega_j^\mp|^2) \right] w^\pm(z_j)\Omega_j^\pm \]
\[+ c^2 \left[ \partial_s^2 (w^\pm(z_j)\Omega_j^\pm) - 4i \partial_s (w^\pm(z_j)\Omega_j^\pm) - 4w^\pm(z_j)\Omega_j^\pm \right] \]
\[= w^\pm(z_j) \left[ \Delta \Omega_j^\pm + c^2 \partial_s^2 \Omega_j^\pm - 4i c^2 \partial_s \Omega_j^\pm \right] - 4c^2 w^\pm(z_j)\Omega_j^\pm \]
\[+ \left[ A_\pm |w^\pm(z_j)|^2 (1 - |\Omega_j^\pm|^2) + B |w^\mp(z_j)|^2 (1 - |\Omega_j^\mp|^2) \right] w^\pm(z_j)\Omega_j^\pm \]
\[+ 2 \nabla w^\pm(z_j) \nabla \Omega_j^\pm + 2c^2 \partial_s w^\pm(z_j) \partial_s \Omega_j^\pm + c^2 \left[ \partial_s w^\pm(z_j) - 4i \partial_s w^\mp(z_j) \right] \Omega_j^\pm, \]

we therefore conclude that
\[\Delta \Omega_j^\pm + c^2 \partial_s^2 \Omega_j^\pm - 4i c^2 \partial_s \Omega_j^\pm \]
\[= \frac{E^\pm}{w^\pm(z_j)} + 4c^2 \Omega_j^\pm - \frac{2 \nabla w^\pm(z_j) \nabla \Omega_j^\pm + 2c^2 \partial_s w^\pm(z_j) \partial_s \Omega_j^\pm}{w^\pm(z_j)} \]
\[\quad - \left[ A_\pm |w^\pm(z_j)|^2 (1 - |\Omega_j^\pm|^2) + B |w^\mp(z_j)|^2 (1 - |\Omega_j^\mp|^2) \right] \Omega_j^\pm \]
\[\quad - c^2 \frac{\Omega_j^\pm \left[ \partial_s w^\pm(z_j) - 4i \partial_s w^\pm(z_j) \right]}{w^\pm(z_j)}. \]

Then we can get
\[\mathcal{A}_3 = \frac{1}{\Omega_j^\pm} \left[ \Delta \Omega_j^\pm + c^2 \partial_s^2 \Omega_j^\pm - 4i c^2 \partial_s \Omega_j^\pm \right] \Phi_j^\pm \]
\[= \left\{ \frac{E^\pm}{v_d^\pm} + 4c^2 - \frac{2 \nabla w^\pm(z_j) \nabla \Omega_j^\pm + 2c^2 \partial_s w^\pm(z_j) \partial_s \Omega_j^\pm}{v_d^\pm} \right. \]
\[\quad - c^2 \frac{\left[ \partial_s w^\pm(z_j) - 4i \partial_s w^\pm(z_j) \right]}{w^\pm(z_j)} \left. \right\} \Phi_j^\pm \]
\[\quad - \left[ A_\pm |w^\pm(z_j)|^2 (1 - |\Omega_j^\pm|^2) + B |w^\mp(z_j)|^2 (1 - |\Omega_j^\mp|^2) \right] \Phi_j^\pm. \]

Combining all the above calculations, we obtain
\[L^\pm_{d,j}( \Phi_j ) = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \eta_1 (\eta_1 - 1) \frac{E^\pm}{v_d^\pm} \Phi_j^\pm \]
\[= L^0_{d,j}( \Phi_j ) + 2A_\pm (1 - |\Omega_j^\pm|^2) \text{Re}(w^\pm(z_j) \Phi_j^\mp) w^\pm(z_j) + 2B (1 - |\Omega_j^\mp|^2) \text{Re}(w^\mp(z_j) \Phi_j^\mp) w^\pm(z_j) \]
\[+ c^2 \left[ \partial_s^2 \Phi_j^\pm - 4i \partial_s \Phi_j^\pm - 4 \Phi_j^\pm \right] + \frac{1}{\Omega_j^\pm} \left[ 2 \nabla \Phi_j^\pm \nabla \Omega_j^\pm + 2c^2 \partial_s \Omega_j^\pm \partial_s \Phi_j^\pm \right]. \]
(2.25)

We point out that for \(|z_j| < 3\), from Lemma 1.2, there holds
\[
|\Omega_j^\pm(z_j)| = 1 + O_\epsilon(\epsilon^2 \ln \epsilon), \quad \nabla \Omega_j^\pm(z_j) = O_\epsilon(\epsilon \sqrt{\ln \epsilon}), \quad |\Delta \Omega_j^\pm(z_j)| = O_\epsilon(\epsilon^2 \ln \epsilon).
\]
Therefore, the linear operator \(L_{\alpha,j}^\pm\) is a small perturbation of \(L_{\alpha}^\pm\) when near the vortices.

2.4. **Symmetry assumptions on the perturbation.** At the last of this part, we point out that we would make some symmetry assumptions for the perturbation in our problem. In fact, from the expression of \(v_d = (v_d^+, v_d^-)\) in (2.11), we know that, under the coordinates \(z = x_1 + ix_2\), there hold
\[
v_d^\pm(x_1, x_2) = \overline{v_d^\pm}(x_1, x_2), \quad v_d^\pm(x_1, -x_2) = \overline{v_d^\pm}(x_1, x_2).
\]
Note that these symmetries are compatible with the solution operator \(S(v)\):
\[
\text{if } S^\pm(v) = 0 \text{ and } u(z) = (u^+, u^-)(z) := (\overline{v}^+, \overline{v}^-)(x_1, -x_2) \text{, then } S(u) = 0,
\]
and the same property holds for \(u(z) = (u^+, u^-)(z) := (v^+(-x_1, x_2), v^-(-x_1, x_2))\). Then we naturally assume that the perturbation \(\psi\) satisfies
\[
\psi(x_1, -x_2) = -\overline{\psi}(x_1, x_2), \quad \psi(-x_1, x_2) = -\overline{\psi}(x_1, x_2).
\]

3. **Error estimate and Fourier decomposition**

3.1. **Error estimate.** Recall the approximation \(v_d\) of \(S(v) = 0\) in (2.9)-(2.10). Suppose
\[
z - e_j = \ell_j e^{i\theta_j} \quad \text{for } j = 1, 2,
\]
where the \(e_j\)'s are given in (2.11). We can rewrite \(v_d\) in the new coordinates \((\ell_j, \theta_j)\) as
\[
v_d = (v_d^+, v_d^-), \quad v_d^\pm(z) = (t^\pm)^{-1} W^\pm(\ell_j) W^\pm(\ell_j) e^{i(\theta_j + \theta_j)}.
\]
In this section, we would give an accurate estimation of \(S(v_d)\). Let
\[
S(v_d) = (S^+(v_d), S^-(v_d)) := i(v_d^+ R^+, v_d^- R^-),
\]
and
\[
S^\pm(v_d) = S_0^\pm(v_d) + S_1^\pm(v_d) := iv_d^+ R_0^\pm + iv_d^- R_1^\pm,
\]
with
\[
S^\pm(v_d) = iv_d^\pm R_0^\pm.
\]
To estimate the size of error \(R = (R^+, R^-)\), we introduce the norm \(\| \cdot \|_{\ast, \ast}\). For given \(0 < \alpha, \sigma < 1\), \(\mathcal{H} = (\mathcal{H}^+, \mathcal{H}^-)\), we define the norm
\[
\|\mathcal{H}\|_{\ast, \ast} = \|\mathcal{H}^+\|_{\ast, 1} + \|\mathcal{H}^-\|_{\ast, 1},
\]
with
\[
\|\mathcal{H}^\pm\|_{\ast, 1} := \sum_{j=1}^2 \|v_d^\pm \mathcal{H}^\pm\|_{C^\alpha(\ell_j < 3)} + \sup_{\ell_1 > 2, \ell_2 > 2} \left[ \frac{|\text{Re}(\mathcal{H}^\pm)|}{l_1^{-2} + l_2^{-2} + \epsilon^2} + \frac{|\text{Im}(\mathcal{H}^\pm)|}{l_1^{-2+\sigma} + l_2^{-2+\sigma} + \epsilon^{-\sigma}} \right]
\]
\[
\sup_{2 < \ell_1 < 2R_1, 2 < \ell_2 < 2R_1} \left[ \frac{|\text{Re}(\mathcal{H}^\pm)|_{\alpha, B_{\ell_j/2}(z)}}{l_1^{-2-\alpha} + l_2^{-2-\alpha}} + \sup_{2 < \ell_1 < 2R_2, 2 < \ell_2 < 2R_2} \frac{|\text{Im}(\mathcal{H}^\pm)|_{\alpha, B_{\ell_j}(z)}}{l_1^{-2+\sigma} + l_2^{-2+\sigma}},
\]
where \(\mathbb{R}_\epsilon = \epsilon^{-\sigma}\) for some \(\alpha_0\) not large such that \(\mathbb{R}_\epsilon \leq \frac{1}{2} \ell_1\), and \(\| \cdot \|_{C^\alpha(D)}, [\cdot]_{\alpha, D}, \| \cdot \|_{C^\alpha(D)}\) are defined as
\[
\|f\|_{C^\alpha(D)} = \|f\|_{C^{\alpha,D}}, \quad [f]_{\alpha, D} = \sup_{x, y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},
\]
Proposition 3.1. Recall that \( \mathcal{R} = (\mathcal{R}^+, \mathcal{R}^-) \) with \( \mathcal{R}^\pm = -i \frac{S^\pm(v_d)}{v_d} \).

The following estimate for \( \mathcal{R} \) holds

\[
\|\mathcal{R}\|_{s,s} \leq \frac{C}{\ln \epsilon},
\]

for \( f : \mathbb{R}^2 \to \mathbb{R} \).

Proof. Without loss of generality, we assume \( x_1 > 0 \). For \( S^\pm(v_d) \), we recall that \( S^\pm(v_d) = S^\pm_0(v_d) + S^\pm_1(v_d) \),

with

\[
S^\pm_0(v_d) = \Delta v^\pm_d + \left[ A_\pm(t^\pm - |v_d^\pm|^2) + B(t^{-2} - |v_d^\pm|^2) \right] v^\pm_d,
\]

and

\[
S^\pm_1(v_d) = \epsilon^2(\theta^{ss}v^\pm_d - 4i\partial_s v^\pm_d - 4v^\pm_d).
\]

Denote

\[
w^\pm_a(z) := w^\pm(z - \hat{d}), \quad w^\pm_b(z) := w^\pm(z + \hat{d}).
\]

Since \( x_1 > 0 \), it follows from Lemma 12 that

\[
(t^\pm)^{-1}|w^\pm_b(z)| = 1 + O_\epsilon(\epsilon^2|\ln \epsilon|).
\]

Substituting (3.1) into the expressions of \( S^\pm_0(v_d) \), we get

\[
S^\pm_0(v_d) = (t^\pm)^{-1} \left\{ \Delta(w^\pm_a w^\pm_b) + \left[ A_\pm(t^\pm - (t^\pm)^{-2}|w^\pm_a|^2|w^\pm_b|^2) + B(t^\pm - (t^\pm)^{-2}|w^\pm_a|^2|w^\pm_b|^2) \right] w^\pm_a w^\pm_b \right\}
\]

\[
eq (t^\pm)^{-1} \left\{ w^\pm_b \Delta w^\pm_a + w^\pm_a \Delta w^\pm_b + 2\nabla w^\pm_a \nabla w^\pm_b \right\}
\]

\[
+ \left[ A_\pm(t^\pm - (t^\pm)^{-2}|w^\pm_a|^2|w^\pm_b|^2) + B(t^\pm - (t^\pm)^{-2}|w^\pm_a|^2|w^\pm_b|^2) \right] w^\pm_a w^\pm_b \right\}
\]

\[
= -(t^\pm)^{-1} \left\{ A_\pm(t^\pm - |w^\pm_a|^2) + B(t^\pm - |w^\pm_b|^2) \right\} w^\pm_a w^\pm_b
\]

\[- (t^\pm)^{-1} \left[ A_\pm(t^\pm - |w^\pm_a|^2) + B(t^\pm - |w^\pm_b|^2) \right] w^\pm_a w^\pm_b
\]

\[+ (t^\pm)^{-1} \left[ A_\pm(t^\pm - (t^\pm)^{-2}|w^\pm_a|^2|w^\pm_b|^2) + B(t^\pm - (t^\pm)^{-2}|w^\pm_a|^2|w^\pm_b|^2) \right] w^\pm_a w^\pm_b
\]

\[+ (t^\pm)^{-1} \left[ (W^\pm(\ell_1)e^{i\theta_1}\nabla \ell_1 + iW^\pm(\ell_1)e^{i\theta_1}\nabla \theta_1) \cdot \left( W^\pm(\ell_1)e^{i\theta_2}\nabla \ell_2 + iW^\pm(\ell_1)e^{i\theta_2}\nabla \theta_2 \right) \right]
\]

\[= (t^\pm)^{-1} \left[ A_\pm(|w^\pm_a|^2 - t^\pm) (1 - (t^\pm)^{-2}|w^\pm_b|^2) + B(|w^\pm_a|^2 - t^\pm) (1 - (t^\pm)^{-2}|w^\pm_b|^2) \right] w^\pm_a w^\pm_b
\]

\[+ (t^\pm)^{-1} \left[ W^\pm(\ell_1)W^\pm(\ell_2) \left( \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \right) - W^\pm(\ell_1)W^\pm(\ell_2) \cos \theta_1 \cos \theta_1 - \sin \theta_1 \sin \theta_2 \right]
\]

\[+ iW^\pm(\ell_1)W^\pm(\ell_2) \cos \theta_2 \sin \theta_1 - \cos \theta_1 \sin \theta_2 \].
where we have the decompositions
\[ S^\pm_1(v_d) = \frac{\tilde{d}^2}{\ln \epsilon} w^\pm_{a,x_2} w^\pm_2 + \frac{d e}{\sqrt{\ln \epsilon}} w^\pm_{a,x_1} w^\pm_2 + \Gamma^\pm, \] \tag{3.7}

where \( \Gamma^\pm \) are the terms given in \(3.9\). Moreover, there hold
\[ \text{Re} \int_{B_d(e_1)} w^\pm_{a,x_2} \frac{\Gamma^\pm}{w^\pm_2} = 0 \quad \text{and} \quad \text{Re} \int_{B_d(e_1)} \frac{\Gamma^\pm}{w^\pm_2} \frac{w^\pm_{a,x_1}}{w^\pm_2} = O_{\epsilon} \left( \frac{e}{\sqrt{\ln \epsilon}} \right). \] \tag{3.8}

Proof. We can show easily that the first equality of \(3.8\) holds by orthogonality. From \(3.9\), we have
\[ S^\pm_1(v_d) = \frac{\tilde{d}^2 W^\pm(\ell_2) e^{i(\theta_1 + \theta_2)}}{\ln \epsilon \ell_2} \left[ W^\pm(\ell_2) \sin^2 \theta_2 + \left( \cos^2 \theta_2 + 2 i \cos \theta_1 \sin \theta_1 \right) \left( \frac{W^\pm(\ell_1)}{\ell_1} - \frac{W^\pm(\ell_1)}{\ell_2^2} \right) \right] \]
\[ + \frac{d e}{\sqrt{\ln \epsilon \ell_2}} \left[ \cos \theta_1 W^\pm(\ell_1) W^\pm(\ell_2) - i W^\pm(\ell_1) W^\pm(\ell_2) \sin \theta_1 \sin \theta_2 \right] + \Gamma^\pm, \]

where
\[ \Gamma^\pm = \left\{ \frac{\tilde{d}^2}{\ln \epsilon} \left[ W^\pm(\ell_2) W^\pm(\ell_1) \sin^2 \theta_2 - W^\pm(\ell_1) W^\pm(\ell_2) \sin \theta_1 \sin \theta_2 \right] \right\}. \]
process in Section 7.

□

From the decompositions above, we know that (3.7) and (3.8) hold.

Let

\begin{align}
&+ \frac{\cos^2 \theta_2}{\ell_2} W^{\pm} (\ell_2) W^{\pm} (\ell_1) - \left( \frac{-2 \cos \theta_1 \cos \theta_2}{\ell_1 \ell_2} + \frac{\cos^2 \theta_2}{\ell_2^2} \right) W^{\pm} (\ell_1) W^{\pm} (\ell_2) \\
&+ \frac{\epsilon d}{\sqrt{\ln \epsilon}} \left( - \cos \theta_2 W^{\pm} (\ell_2) W^{\pm} (\ell_1) \right) (t^\pm)^{-1} e^{i(\theta_1 + \theta_2)} \\
&+ i \left[ \frac{\epsilon d}{\sqrt{\ln \epsilon}} \sin \theta_2 W^{\pm} (\ell_2) W^{\pm} (\ell_1) - \frac{2 \epsilon d^2}{\ln \epsilon} \left( \frac{\sin \theta_2 \cos \theta_2}{\ell_2^2} \right) W^{\pm} (\ell_1) W^{\pm} (\ell_2) \\
&- \frac{2 \epsilon d^2}{\ln \epsilon} \cos \theta_2 \left( \sin \theta_1 W^{\pm} (\ell_1) W^{\pm} (\ell_2) - \sin \theta_2 W^{\pm} (\ell_2) W^{\pm} (\ell_1) \right) \\
&- \frac{2 \epsilon d^2}{\ln \epsilon} \sin \theta_2 W^{\pm} (\ell_2) W^{\pm} (\ell_1) \right] (t^\pm)^{-1} e^{i(\theta_1 + \theta_2)}. \tag{3.9}
\end{align}

From the decompositions above, we know that (3.7) and (3.8) hold.

In Lemma 3.2, we decompose $S^+_1 (v_d)$ into two parts: one is $\frac{\epsilon^2}{\ln \epsilon} w^+_a x_2 w^+_b$ which is in big size but has a symmetry that makes it orthogonal to the kernel $w^+_a x_1$, the other one is $- \frac{\epsilon^2}{\ln \epsilon} w^+_a x_1 w^+_b + \Gamma^\pm$ which has smaller size. Similar decomposition also holds for $S^-_1 (v_d)$. Lemma 3.2 will be used in the reduction process in Section 7.

3.2. Fourier series for the error. Next, we shall make a more precise decomposition for the error $\mathcal{R}$. Recall the relations

\[ z - e_j = \ell_j e^{i\theta_j} \quad \text{for } j = 1, 2, \]

where $e_j$’s are given in (2.11). For given $\mathcal{H} = (\mathcal{H}^+, \mathcal{H}^-) \in \mathbb{C}^2$ satisfying

\[ \mathcal{H}(\infty) = -\overline{\mathcal{H}(z)}, \quad \text{i.e., } \mathcal{H}^\pm(\infty) = -\overline{\mathcal{H}^\mp(z)}, \]

we have the following local decomposition for $\mathcal{H}^\pm$ when near $e_j$, $j = 1, 2$,

\[ \mathcal{H}^\pm(\ell_j) = \sum_{k=0}^{\infty} n^\pm_{k,j}(\ell_j, \theta_j), \tag{3.10} \]

and

\[ n^\pm_{k,j}(\ell_j, \theta_j) = n^\pm_{k,j}(\ell_j) \sin(k\theta_j) + i n^\pm_{k,j}(\ell_j) \cos(k\theta_j), \quad n^\pm_{k,j}(\ell_j), n^\pm_{k,j}(\ell_j) \in \mathbb{R}. \tag{3.11} \]

Then define

\[ n^\pm_{k,j}(\ell_j, \theta_j) = \sum_{k \text{ is even}} n^\pm_{k,j}(\ell_j), n^\pm_{k,j}(\ell_j) = \sum_{k \text{ is odd}} n^\pm_{k,j}(\ell_j), \tag{3.12} \]

\[ \mathcal{H}_{c,j} = (\mathcal{H}^+_{c,j}, \mathcal{H}^-_{c,j}), \quad \mathcal{H}_{o,j} = (\mathcal{H}^+_{o,j}, \mathcal{H}^-_{o,j}). \tag{3.13} \]

Let $\mathcal{R}_j z$ denote the reflection across the real line $\text{Re}(z) = (-1)^{j+1} \hat{d}$, then

\[ \mathcal{R}_j z = 2e_j - \text{Re}(z) + i \text{Im}(z), \quad \text{or } \mathcal{R}_j z = \ell_j e^{i(\pi - \theta_j)} + e_j. \tag{3.14} \]

Combining (3.14) and (3.11)-(3.13), we get

\[ n^\pm_{c,j}(\mathcal{R}_j z) = -n^\pm_{c,j}(z), \quad n^\pm_{o,j}(\mathcal{R}_j z) = n^\pm_{o,j}(z), \]

\[ n^\pm_{c,j}(\mathcal{R}_j z) = -n^\pm_{c,j}(z), \quad n^\pm_{o,j}(\mathcal{R}_j z) = n^\pm_{o,j}(z), \]

and thus we can define equivalently

\[ n^\pm_{o,j}(z) = \frac{1}{2} \left[ n^\pm(z) + \mathcal{H}^\pm(\mathcal{R}_j z) \right], \quad n^\pm_{c,j}(z) = \frac{1}{2} \left[ n^\pm(z) - \mathcal{H}^\pm(\mathcal{R}_j z) \right], \]
Note that, in the coordinates $(\ell_j, \theta_j)$, the functions $H_{o,j}$ introduced in (3.11)–(3.13) are the odd parts of $H$ in local domains $B_{\tilde{\rho}}(e_j)$, $j = 1, 2$. Furthermore, we would define functions $H_{o}^{+}, H_{o}^{-}$ which stand for the odd part of $H^{+}, H$ globally. Define the cut-off functions $\eta_{j,R}$ for $j = 1, 2$ as the following

$$\eta_{j,R}(z) = \eta_j \left( \frac{|z - e_j|}{2} \right),$$  \hspace{1cm} (3.15)

where $\eta_j$ is the cut-off functions defined in (2.16) and $R$ is a constant to be chosen later. For any given $H = (H^+, H^-) \in \mathbb{R}^2$, we define

$$H_o = (H_{o}^+, H_{o}^-), \quad \mathrm{with} \quad H_{o}^{\pm} = \eta_{1, R} H_{o,1}^{\pm} + \eta_{2, R} H_{o,2}^{\pm},$$  \hspace{1cm} (3.16)

where $R_c = \frac{c_0}{e |\ln\epsilon|}$ for some $c_0$ not large such that $R_c \leq \frac{1}{2} \tilde{d}$. On the other hand, define the global functions

$$H_e = H - H_o, \quad \mathrm{with} \quad H_{e}^{\pm} = H_{e}^{\pm} - H_{o}^{\pm}.$$  \hspace{1cm}

We note that, using the decompositions introduced above, we could decompose the error function

$$\mathcal{R} = (\mathcal{R}^+, \mathcal{R}^-) = \left( \frac{S^+(v_d)}{iv_d}, \frac{S^-(v_d)}{-iv_d} \right)$$

in odd Fourier modes and even Fourier modes, see Proposition 3.3. The odd Fourier modes of the error have much smaller sizes than even ones. However, there exists one difficulty: some terms of odd Fourier modes have a good size but decay slowly. Using the idea in [12], we introduce the semi-norm $| \cdot |_{\sharp \sharp}$ for these slow decay terms. For any $H = (H^+, H^-) = (H_{11}^+ + iH_{12}^+, H_{11}^- + iH_{12}^-) \in \mathbb{C}^2$, the semi-norm $|H|_{\sharp \sharp}$ is defined as

$$|H|_{\sharp \sharp} = |H^+|_{\sharp \sharp} + |H^-|_{\sharp \sharp},$$  \hspace{1cm} (3.17)

where

$$|H^+|_{\sharp \sharp} = \sum_{j=1}^{2} \|v_d^+ H^+\|_{C^{0,\alpha}(\ell_j < 2)} + \sup_{2 < \ell_1 < R_c, 2 < \ell_2 < R_c} \left[ \frac{|H_{11}^+|}{\ell_1 - \ell_2} + \frac{|H_{12}^+|}{\ell_1 - \ell_2} + \frac{|H_{12}^+|}{\ell_1 - \ell_2} + \frac{|H_{11}^+|}{\ell_1 - \ell_2} \right].$$  \hspace{1cm} (3.18)

Now we give a precise decomposition for the error $\mathcal{R}$.

**Proposition 3.3.** Recall that $v_d$ is given by (2.9) and $S = S(v_d) = (iv_d^+ \mathcal{R}^+, iv_d^- \mathcal{R}^-), \quad \mathcal{R} = (\mathcal{R}^+, \mathcal{R}^-).$

Then we can write

$$\mathcal{R} = \mathcal{R}_o + \mathcal{R}_e = (\mathcal{R}_o^+, \mathcal{R}_o^-) + (\mathcal{R}_e^+, \mathcal{R}_e^-), \quad \mathcal{R}_o = \mathcal{R}_o^+ + \mathcal{R}_o^- = (\mathcal{R}_o^{+\alpha}, \mathcal{R}_o^{-\alpha}) + (\mathcal{R}_o^{+\beta}, \mathcal{R}_o^{-\beta}),$$

i.e,

$$\mathcal{R}^\pm = \mathcal{R}_o^\pm + \mathcal{R}_e^\pm, \quad \mathcal{R}_o^{\pm} = \mathcal{R}_o^{+\alpha} + \mathcal{R}_o^{-\alpha},$$

with $\mathcal{R}_o, \mathcal{R}_o^\pm$ defined analogously to (3.10). Moreover, the symmetry $\mathcal{R}_o(\mathcal{R}_j z) = \mathcal{R}_o(z)$ in $B_{R_c}(e_1) \cup B_{R_c}(e_2)$ is true and the following estimates are valid

$$\|\mathcal{R}_e\|_{\ast \ast} + \|\mathcal{R}_o\|_{\ast \ast} \leq C\frac{\epsilon}{|\ln \epsilon|}, \quad \|\mathcal{R}_o^\alpha\|_{\sharp \sharp} \leq C\frac{\epsilon}{\sqrt{|\ln \epsilon|}}, \quad \|\mathcal{R}_o^\beta\|_{\ast \ast} \leq C\epsilon \sqrt{|\ln \epsilon|}.$$  \hspace{1cm}

**Proof.** Following Proposition 3.1 we immediately get that

$$\|\mathcal{R}_e\|_{\ast \ast} + \|\mathcal{R}_o\|_{\ast \ast} \leq \frac{C}{|\ln \epsilon|}.$$  \hspace{1cm}

For every $j \in \{1, 2\}$, we define

$$r_{o,1}^\pm := -i \frac{\tilde{d}^2}{|\ln \epsilon|(|t|^{\pm})} \left( \frac{2 \cos \theta_1 \cos \theta_2}{\ell_1 \ell_2} + \frac{\cos \theta_2}{\ell_2^2} \right) + \left( \frac{\tilde{d} \epsilon}{\sqrt{|\ln \epsilon|(|t|^{\pm})}} \frac{-\sin \theta_1}{\ell_1} - \frac{2 \tilde{d}^2}{|\ln \epsilon|(|t|^{\pm})} \frac{\sin \theta_2 \cos \theta_2}{\ell_2^2} \right),$$

$$r_{o,2}^\pm := -i \frac{\tilde{d}^2}{|\ln \epsilon|(|t|^{\pm})} \left( \frac{2 \cos \theta_1 \cos \theta_2}{\ell_1 \ell_2} + \frac{\cos \theta_1}{\ell_1^2} \right) + \left( \frac{\tilde{d} \epsilon}{\sqrt{|\ln \epsilon|(|t|^{\pm})}} \frac{-\sin \theta_2}{\ell_2} - \frac{2 \tilde{d}^2}{|\ln \epsilon|(|t|^{\pm})} \frac{\sin \theta_1 \cos \theta_1}{\ell_1^2} \right).$$

\[ R_{o,j}^{\pm,\alpha} := \frac{1}{2} \left( r_{o,j}^{\pm} + r_{o,j}^{\pm} (\mathcal{A}_j z) \right), \] (3.19)

and

\[ R_{o}^{\pm} := (R_{o}^{+,\alpha}, R_{o}^{-,\alpha}), \quad \text{with} \quad R_{o}^{\pm,\alpha} := \eta_{1,\pm} R_{o,1}^{\pm,\alpha} + \eta_{2,\pm} R_{o,2}^{\pm,\alpha}. \]

Note that the terms in \( r_{o,j}^{\pm} \) are all slow decay terms.

We claim that the components of functions \( R_{o,j}^{+,\alpha} \) are all odd mode terms when they lie locally in domain \( \mathcal{B}_{\Re \epsilon}(e_j) \). In fact, for example, taking \( j = 1 \), when \( |z - e_1| \leq \Re \epsilon \), by rewriting \( r_{o,1}^{\pm} \) in Fourier series in \( \theta_1 \), there hold

\[
\frac{2 \cos \theta_1 \cos \theta_2}{\ell_1 \ell_2} = \frac{2 \cos \theta_1 (x_1 + \hat{d})}{\ell_1 \ell_2^2} = \frac{2 \cos \theta_1 \cos \theta_1 + 4 \cos \theta_1 \hat{d}}{\ell_1 \ell_2^2} = \frac{1}{\ell_1^2 + d^2 + 4d \ell_1 \cos \theta_1} \left( 2 \cos^2 \theta_1 + \frac{4 \cos \theta_1 \hat{d}}{\ell_1} \right) = \sum_{m=0}^{\infty} a_m(\ell_1) \cos^m \theta_1,
\]

and

\[
\frac{\cos^2 \theta_2}{\ell_2^2} = \frac{(x_1 - \hat{d} + 2\hat{d})^2}{\ell_2^4} = \frac{1}{\ell_1^2 + d^2 + 4d \ell_1 \cos \theta_1} \left( \ell_1^2 \cos^2 \theta_1 + 2d \ell_1 \cos \theta_1 + 4d^2 \right) = \sum_{m=0}^{\infty} b_m(\ell_1) \cos^m \theta_1,
\]

\[
\frac{\sin \theta_2 \cos \theta_2}{\ell_2^2} = \frac{(x_1 - \hat{d} + 2\hat{d})x_2}{\ell_2^4} = \frac{(x_1 - \hat{d})x_2 + 2\hat{d}x_2}{\ell_2^4} = \sin \theta_1 \sum_{m=0}^{\infty} c_m(\rho_1) \cos^m \theta_1,
\]

where \( a_m, b_m, c_m \) are bounded smooth functions. Therefore, we can rewrite \( r_{o,1}^{\pm} \) as

\[
r_{o,1}^{\pm}(z) = -i \frac{\hat{d}^2}{|\ln \epsilon(t)_deriv}} \left( \sum_{m=0}^{\infty} \left[ a_m(\ell_1) \cos^m \theta_1 + b_m(\ell_1) \cos^m \theta_1 \right] \right) (3.20)
\]

\[
\quad \quad + \left( \frac{\hat{d} \epsilon}{|\ln \epsilon(t)_deriv}} - \frac{\sin \theta_1}{\ell_1} - \frac{2 \hat{d}^2}{|\ln \epsilon(t)_deriv}} \sin \theta_1 \sum_{m=0}^{\infty} c_m(\rho_1) \cos^m \theta_1 \right). (3.21)
\]

By the analysis above and recalling

\[ \mathcal{A}_1 z = \ell_1 e^{i(\pi - \theta_1)} + e_1, \]

we can easily show that

\[
r_{o,1}^{\pm}(\mathcal{A}_1 z) = -i \frac{\hat{d}^2}{|\ln \epsilon(t)_deriv}} \left( \sum_{m=0}^{\infty} (-1)^{m+1} \left[ a_m(\ell_1) \cos^m \theta_1 + b_m(\ell_1) \cos^m \theta_1 \right] \right)
\]

\[
\quad \quad + \left( \frac{\hat{d} \epsilon}{|\ln \epsilon(t)_deriv}} - \frac{\sin \theta_1}{\ell_1} - \frac{2 \hat{d}^2}{|\ln \epsilon(t)_deriv}} \sin \theta_1 \sum_{m=0}^{\infty} (-1)^m c_m(\rho_1) \cos^m \theta_1 \right). (3.22)
\]
Therefore, from (3.19), (3.20), and (3.22), we know that the claim above is true. Furthermore, we can also check that $R_0^\alpha$ and $R_0^\beta := R_0 - R_0^\alpha$ satisfy the desired properties.

\section{The resolution of projected linear problem}

For convenience, we first introduce some symmetries for the functions $H, \psi \in \mathbb{C}^2$:

\begin{equation}
H(z) = -\overline{H(z)},
\end{equation}

\begin{equation}
H(\mathcal{R}_j z) = -\overline{H(z)}, \quad |z - e_j| < 2\mathbb{R}_e, \quad j = 1, 2,
\end{equation}

\begin{equation}
H(\mathcal{R}_j z) = \overline{H(z)}, \quad |z - e_j| < 2\mathbb{R}_e, \quad j = 1, 2,
\end{equation}

\begin{equation}
\psi(x_1, -x_2) = -\overline{\psi(x_1, x_2)}, \quad \psi(-x_1, x_2) = -\overline{\psi(x_1, x_2)}.
\end{equation}

From the analysis in Lemma 2.2, we know that the linear operator $L_{d,\ell}^\pm(\Phi) := iw^\pm(z - e_j)\mathcal{L}_e^\pm(\psi)$ has one nontrivial kernel for $\psi$ which satisfies the symmetry (S4) in the vortex region. Therefore, in order to solve the problem (2.22), we should consider the nonlinear projected problem for $\psi$ satisfies the symmetry (S4)

\begin{equation}
\begin{cases}
L_e^\pm(\psi) = \mathcal{R}^\pm - N^\pm(\psi) + \varepsilon \sum_{j=1}^2 \frac{\chi_j}{iw^\pm(z - e_j)}(-1)^j w_{x_1}^\pm(z - e_j) & \text{in } \mathbb{R}^2, \\
\Re \int_{B(0,4)} \chi \left[ \overline{\phi_j^+ w_{x_1}^+} + \overline{\phi_j^- w_{x_1}^-} \right] = 0, \text{ with } \phi_j^\pm(z) = iw^\pm(z)\psi^\pm(z + e_j), \quad j = 1, 2,
\end{cases}
\end{equation}

where

\begin{equation}
\chi(z) := \eta_1 \left( \frac{|z|}{2} \right), \quad \chi_j(z) := \eta_1 \left( \frac{\ell_j}{2} \right) = \eta_1 \left( \frac{|z - e_j|}{2} \right),
\end{equation}

and $\eta_1$ is a smooth cut-off function defined in (2.11). The resolution theory of a linear projected problem will be first provided for solving (4.1). Then an application of the Contraction Mapping Principle will give the existence of solutions.

\subsection{The linear resolution theory}

The main objective of this part is to set up the resolution theory of a linear projected problem. For any $H$ satisfies (S4), we first consider the projected problem

\begin{equation}
\begin{cases}
L_e^\pm(\psi) = H^\pm + \varepsilon \sum_{j=1}^2 \frac{\chi_j}{iw^\pm(z - e_j)}(-1)^j w_{x_1}^\pm(z - e_j) & \text{in } \mathbb{R}^2, \\
\Re \int_{B(0,4)} \chi \left[ \overline{\phi_j^+ w_{x_1}^+} + \overline{\phi_j^- w_{x_1}^-} \right] = 0, \text{ with } \phi_j^\pm(z) = iw^\pm(z)\psi^\pm(z + e_j), \quad j = 1, 2,
\end{cases}
\end{equation}

For solving (4.2), at first we shall get a priori estimates expressed in suitable norms. Define for $\psi = (\psi^+, \psi^-) : \mathbb{R}^2 \to \mathbb{C}^2$ the norms for fixed small $\alpha > 0, \sigma > 0$,

\begin{equation}
\|\psi\|_* = \|\psi^+\|_{*, 1} + \|\psi^-\|_{*, 1},
\end{equation}

\begin{equation}
\|\psi^\pm\|_{*, 1} = \sum_{j=1}^2 \|v_j^\pm \psi^\pm\|_{C^2(\ell_j < 3)} + \|\Re(\psi^\pm)\|_{*, 1, \Re} + \|\Im(\psi^\pm)\|_{*, 1, \Im},
\end{equation}

where, by the relations

\begin{equation}
\Re(\psi^\pm) = \psi^+_1, \quad \Im(\psi^\pm) = \psi^+_2,
\end{equation}
we have denoted

\[ \|\psi^\pm_1\|_{1,Re} = \sup_{t_1 > 2, t_2 > 2} |\psi^\pm_1| + \sup_{2 < t_1 < \frac{2}{t_2}, 2 < t_2 < \frac{2}{t_1}} \frac{|\nabla \psi^\pm_1|}{\ell_1 + \ell_2} + \sup_{r > \frac{1}{2}} \frac{1}{\epsilon} (|\partial_r \psi^\pm_1| + |\partial_s \psi^\pm_1|) \]

\[ + \sup_{2 < t_1 < \frac{2}{t_2}, 2 < t_2 < \frac{2}{t_1}} \frac{|D^2 \psi^\pm_1|}{\ell_1^{2 + \sigma} + \ell_2^{2 + \sigma}} + \sup_{2 < t_1 < \frac{2}{t_2}, 2 < t_2 < \frac{2}{t_1}} \frac{|D^2 \psi^\pm_2|}{\ell_1^{1 + 2 - \sigma} \ell_2^{2 + \sigma}}, \tag{4.5} \]

\[ \|\psi^\pm_2\|_{1,Im} = \sup_{t_1 > 2, t_2 > 2} |\psi^\pm_2| + \sup_{2 < t_1 < \frac{2}{t_2}, 2 < t_2 < \frac{2}{t_1}} \frac{|\nabla \psi^\pm_2|}{\ell_1^{1 + 2 - \sigma} + \ell_2^{2 + \sigma}} + \sup_{r > \frac{1}{2}} \frac{1}{\epsilon} (|\partial_r \psi^\pm_2| + |\partial_s \psi^\pm_2|) \]

\[ + \sup_{2 < t_1 < \frac{2}{t_2}, 2 < t_2 < \frac{2}{t_1}} \frac{|D^2 \psi^\pm_2|}{\ell_1^{1 + 2 - \sigma} \ell_2^{2 + \sigma}} + \sup_{2 < t_1 < \frac{2}{t_2}, 2 < t_2 < \frac{2}{t_1}} \frac{|D^2 \psi^\pm_2|}{\ell_1^{1 + 2 - \sigma} \ell_2^{2 + \sigma}}. \tag{4.6} \]

We next give the solvability of linear projected problem (4.2).

**Proposition 4.1.** There exists a constant \( C > 0 \) depending only on \( \alpha, \sigma \in (0,1) \), such that the following hold: if \( \mathcal{H} \) satisfies \( \mathcal{H} \) and

\[ \|\mathcal{H}\|_{ss} < +\infty, \]

then for \( \epsilon > 0 \) sufficiently small there exists a unique solution \( T_\epsilon(\mathcal{H}) = (T^1_\epsilon(\mathcal{H}), T^2_\epsilon(\mathcal{H})) = (\psi_\epsilon, c_\epsilon) \) to (4.2). Furthermore, there holds

\[ \|\psi_\epsilon\|_s \leq C \|\mathcal{H}\|_{ss}. \]

**Proof.** We delay the proof of this proposition to § 5.1. \( \square \)

### 4.2. More precise estimates and decompositions for \( \psi \)

The resolution theory of (4.2) provided in Proposition 4.1 implies that we can find a solution of (4.1) in the region

\[ \mathcal{A} := \left\{ \psi : \psi \text{ satisfies } (S4), \Re \int_{B(0,4)} \chi \left[ \phi_j^+ w_{x_1}^+ + \phi_j^- w_{x_1}^- \right] = 0, \ j = 1, 2, \text{ and } \|\psi\|_s \leq \frac{C}{|\ln \epsilon|} \right\}. \]

In fact, from Proposition 4.1 the problem (4.1) is equivalent to the following fixed point problem

\[ \psi = T_\epsilon \left( \mathcal{R} + \mathcal{N}(\psi) \right) =: G_\epsilon(\psi). \]

The existence of \( \psi \) to (4.1) follows by the Contraction Mapping Principle, see Proposition 6.1 in Section 6 for more details.

Next, to find a real solution of (2.17), we need to solve the reduced problem by finding a suitable \( \tilde{\alpha} \) such that the multiplier \( c \) in (4.1) is identical zero when \( \epsilon \) is sufficiently small. However, since the estimates for \( \psi \) are not delicate enough, we can not carry out reduction procedure such that \( c \) is zero. In order to get around the technical difficulty, we are led to using the idea doing more precise decompositions and estimates for the perturbation \( \psi \). This was first introduced by J. Dávila, M. del Pino, M. Medina and R. Rodiac in [12] and [13] to construct vortex helical filaments of classical (single component) Ginzburg-Landau equation and Gross-Pitaevskii equation.

For \( \psi = (\psi^+, \psi^-) \in \mathbb{C}^2 \) satisfying (S4), we decompose \( \psi \) in Fourier series in \( \theta_j \) as in (3.10) and define

\[ \psi_{\epsilon,j}^\pm = \sum_{k \text{ even}} \psi_{k,j}^\pm, \quad \psi_{o,j}^\pm = \sum_{k \text{ odd}} \psi_{k,j}^\pm, \ j = 1, 2. \]

Similar to the definitions of \( \mathcal{H}_o \) and \( \mathcal{H}_e \) in Section 3, we define

\[ \psi_o = (\psi_o^+, \psi_o^-), \quad \psi_e = (\psi_e^+, \psi_e^-), \]

\[ \psi_{\epsilon} = (\psi_{\epsilon}^+, \psi_{\epsilon}^-), \quad \psi_{\epsilon}^\pm := \psi_{\epsilon}^\pm - \psi_o^\pm, \]

where \( \eta_1, \eta_2 \in \mathbb{R}^+ \) are two cut-off functions given by the form in (3.15).
We recall some facts about the error in the form
\[ \mathcal{R} = (\mathcal{R}^+, \mathcal{R}^-) = \left( \frac{S^+(v_d)}{i\nu_d^+}, \frac{S^-(v_d)}{i\nu_d^-} \right). \]

There are some odd Fourier modes terms which have good size but decay slowly. Therefore, we have introduced the semi-norm \( | \cdot |_{\text{Lip}} \) in (3.17) for these slow decay terms. As we have stated, by taking \( \mathcal{H}^\pm = -\mathcal{R}^\pm - N^\pm(\psi) \) in Proposition 4.1 we can solve the nonlinear projected problem (4.1) in Proposition 4.1. And then, in order to carry out the reduction process, we will establish Proposition 4.2. To prove Proposition 4.2, we would have to solve some problems like \( \Delta \psi^\pm \approx O \left( \frac{1}{\ell^2} \right) \) and thus the functions \( \psi^\pm \) would grow logarithmically up to a certain distance, see Lemma 5.2 and Lemma 8.3 for more explanations. As a result of these, we need the following semi-norm to capture the behaviors of \( \psi = (\psi^+, \psi^-) = (\psi^+_1 + i\psi^+_2, \psi^+_1 - i\psi^+_2) \):
\[ |\psi|_2 = |\psi^+_1|_{2,1} + |\psi^-|_{2,1} \tag{4.9} \]

with
\[ |\psi^\pm|_{2,1} = \sum_{j=1}^2 \| \log \epsilon^{-1} \|V\psi^\pm\|_{C^{2,\alpha}(\ell_j < 3)} + \|\psi^\pm|_{2,1,\text{Re}} + \|\psi^\pm|_{2,1,\text{Im}}, \tag{4.10} \]

where, by the relations
\[ \text{Re}(\psi^\pm) = \psi^\pm_1, \quad \text{Im}(\psi^\pm) = \psi^\pm_2, \]
we have denoted
\[ |\psi^\pm|_{2,1,\text{Re}} = \sup_{2<\ell_j<\ell_\eta, j=1,2} \left[ \frac{|\psi^\pm_1|}{\ell_1 \log(2\ell_\eta/\ell_1) + \ell_2 \log(2\ell_\eta/\ell_2)} + \frac{|\nabla \psi^\pm_1|}{\log(2\ell_\eta/\ell_1) + \log(2\ell_\eta/\ell_2)} \right], \tag{4.11} \]
and
\[ |\psi^\pm|_{2,1,\text{Im}} = \sup_{2<\ell_j<\ell_\eta, j=1,2} \left[ \frac{|\psi^\pm_2|}{\ell_1^{1+\sigma} + \ell_2^{1+\sigma} + \ell_1^{-1} \log(2\ell_\eta/\ell_1) + \ell_2^{-1} \log(2\ell_\eta/\ell_2)} \right]. \tag{4.12} \]

We recall the semi-norm \( | \cdot |_{\text{Lip}} \) in (3.17) and the reflection mapping \( \mathcal{R} z \) in (3.14). We finally give a crucial proposition in this paper. In some sense, it is the “heart” of the present paper.

**Proposition 4.2.** Suppose that \( \mathcal{H} \) satisfies the symmetries (S1) and \( \|\mathcal{H}\|_\infty < \infty \). \( \mathcal{H}_o \) is the function defined in (3.16) and it can be decomposed as
\[ \mathcal{H}_o = \mathcal{H}_o^\alpha + \mathcal{H}_o^\beta, \]
where \( |\mathcal{H}_o^\alpha|_{\text{Lip}} < \infty \), and the terms \( \mathcal{H}_o^\alpha, \mathcal{H}_o^\beta \) satisfy (S3), i.e.,
\[ \mathcal{H}_o^\alpha(\mathcal{R} z) = \mathcal{H}_o^\alpha(z), \quad |z - e_j| < \ell_\eta, \quad j = 1, 2 \quad \text{and} \quad \kappa = \alpha, \beta, \]
and have supports in \( B_{2\ell_\eta}(e_1) \cup B_{2\ell_\eta}(e_2) \). Then for any solution \( \psi = \psi_c + \psi_o \) of (1.2) with \( \psi_o \) defined by (4.12), \( \psi_o \) can be decomposed as
\[ \psi_o = \psi_o^\alpha + \psi_o^\beta, \]
where \( \psi_o^\alpha \) are supported in \( B_{\ell_\eta}(e_1) \cup B_{\ell_\eta}(e_2) \) with \( \kappa = \alpha, \beta \). Furthermore, the following estimates hold
\[ |\psi_o^\alpha|_{2} \lesssim \|\mathcal{H}_o^\alpha\|_{\text{Lip}} + \epsilon \log \epsilon^{-1} \|\mathcal{H}_o^\alpha\|_{\infty} \|\mathcal{H} - \mathcal{H}_o\|_\infty, \]
\[ \|\psi_o^\alpha\|_\infty \lesssim \|\mathcal{H}_o^\alpha\|_\infty, \]
\[ \|\psi_o^\alpha\|_\infty + \|\psi_o^\beta\|_\infty \lesssim \|\mathcal{H}\|_\infty + \|\mathcal{H}_o\|_\infty + \|\mathcal{H}_o^\beta\|_\infty, \]
and
\[ \psi_o^\alpha(\mathcal{R} z) = \overline{\psi_o^\alpha}(z), \quad |z - e_j| < \ell_\eta, \quad j = 1, 2 \quad \text{and} \quad \kappa = \alpha, \beta. \]

**Proof.** The proof of Proposition 4.2 will be provided in § 5.2. \( \square \)
5. PROOF OF LINEAR THEORY

5.1. Proof of Proposition 4.1. We now consider the following: for given $\mathcal{H}$ satisfying (5.1), find $\psi$ such that

$$\begin{cases}
\mathcal{L}^\pm_\epsilon(\psi) = \mathcal{H}^\pm \quad \text{in } \mathbb{R}^2, \\
\Re \int_{B(0,4)} \chi [\overline{\phi_j^+} w_{x_1}^+ + \phi_j^- w_{x_1}^-] = 0, \quad \text{with } \phi_j^+(z) = iu^+(z)\psi^+(z + e_j),
\end{cases} \quad (5.1)$$

Recall the definitions of norms $\| \cdot \|$ in (3.2)-(3.3) and $\| \cdot \|_\star$ in (3.2)-(3.3). We shall first give a priori estimate for the problem (5.1).

**Lemma 5.1.** There exists a constant $C > 0$, depending only on $\alpha \in (0,1), \sigma \in (0,1)$, such that for all $\epsilon$ sufficiently small and any solution $\psi$ of (5.1) with $\|\psi\|_\star < \infty$ one has

$$\|\psi\|_\star \leq C\|\mathcal{H}\|_\star. \quad (5.2)$$

**Proof.** First, we introduce the norms

$$\|\psi\|_\star = \|\psi^+\|_\star,0 + \|\psi^-\|_\star,0,$$

and

$$\|\mathcal{H}\|_\star,0 = \|\mathcal{H}^+\|_\star,0 + \|\mathcal{H}^-\|_\star,0,$$

where we have set the notation by the following

$$\|\psi^\pm\|_\star,0 = \sum_{j=1}^2 \|v^\pm_d \mathcal{H}^\pm\|_{C_0(\ell_j < 3)} + \|\psi^\pm\|_\star,0,Re + \|\psi^\pm\|_\star,0,Im,$$

$$\|\psi^\pm\|_\star,0,Re = \sup_{\ell_1 > \ell_2 > \ell_2} |\psi^\pm_1| + \sup_{2<\ell_1<\ell_2 < \ell_2} \frac{|\nabla \psi^\pm_1|}{\ell_1^{-1} + \ell_2^{-1}} + \sup_{r > \ell_1} \left\{ \frac{1}{r} |\partial_\ell \psi_1^\pm| + |\partial_\epsilon \psi_1| \right\},$$

$$\|\psi^\pm\|_\star,0,Im = \sup_{\ell_1 > \ell_2 > \ell_2} \frac{|\psi^\pm_2|}{\ell_1^{1+\sigma} + \ell_2^{1+\sigma}} + \sup_{2<\ell_1<\ell_2 < \ell_2} \frac{|\nabla \psi^\pm_2|}{\ell_1^{2+1+\sigma} + \ell_2^{2+1+\sigma}}$$

$$+ \sup_{r > \ell_1} \left\{ \epsilon^{\sigma-2} |\partial_\ell \psi_2| + \epsilon^{\sigma-1} |\partial_\epsilon \psi_2| \right\},$$

and

$$\|\mathcal{H}\|_\star,0 := \sum_{j=1}^2 \|v^\pm_d \mathcal{H}^\pm\|_{C_0(\ell_j < 3)} + \sup_{\ell_1 > \ell_2 > \ell_2} \left[ \frac{|\Re(\mathcal{H}^\pm)|}{\ell_1^{-2} + \ell_2^{-2}} + \frac{|\Im(\mathcal{H}^\pm)|}{\ell_1^{-2+\sigma} + \ell_2^{-2+\sigma}} \right].$$

We claim that

$$\|\psi\|_\star \leq C\|\mathcal{H}\|_\star. \quad (5.3)$$

By using Schauder theory, we then can get the full estimate (5.2).

In [12], the authors gave a proof of (5.3) for the complex-valued scalar case ($B = 0$). We adopt the methods and technology from [12]. In the present case, we need to deal with some new problems caused by the coupled terms due to that $B \neq 0$. For convenience, we only highlight the difference.

We argue by contradiction. Assume that there exist sequences of the parameter epsilon $\{\epsilon_n\}$ approaching $0^+$ and solutions $\{\psi^n\}$ in such a way that (5.1) is valid with $\mathcal{H} = \mathcal{H}^{(n)}$ and the following estimates are true

$$\|\psi^n\|_\star = 1, \quad \|\mathcal{H}^{(n)}\|_\star = o_n(1). \quad (5.4)$$

Since the linearized operator $\mathcal{L}_\epsilon = (\mathcal{L}^+_\epsilon, \mathcal{L}^-_\epsilon)$ has different asymptotic behaviors in different domains, therefore, in order to obtain a contradiction, we will divide the analysis into two parts.
In the region near the vortices, by a similar method to the proof of Lemma 5.1 in [12], we can show that
\[
\phi_j^o = (i w^+(\psi^o)^+, i w^-(\psi^o)^-)(z + e_j) \rightarrow 0, \quad \text{in } C^2_{loc}(\mathbb{R}^2, \mathbb{C}^2).
\]
(5.5)

Here we point out that we have used the non-degeneracy results in Lemma 8.2 in the proof of (5.5). The linearized operator \(L_0(\phi)\) does have one kernel element \(\psi_{x_1}\) when the perturbations satisfy some decays and symmetry constraints in that lemma. Then the orthogonality in (5.1) will imply the validity of (5.5).

Next we consider the case far away from the vortices. Here we assume \(x_1 > 0\) and \(\ell_1 > R_0\). For \(\mathcal{H} = (\mathcal{H}^+, \mathcal{H}^-)\), we express \(L_\varepsilon^\pm(\psi) = \mathcal{H}^\pm\) as the following
\[
\Delta \psi^\pm + \frac{\nabla v_d^\pm \nabla \psi^\pm}{v_d^\pm} - 2i A_s |v_d| \psi^\pm - 2i B|v_d|^2 \psi_{x_2}^\pm
\]
\[+ \epsilon^2 \left( \partial_s^2 \psi^\pm + \frac{2\partial_s v_d^\pm}{v_d^\pm} \partial_s \psi^\pm - 4i \partial_s \psi^\pm \right) = \mathcal{H}^\pm.
\]
(5.6)

Noting that
\[
\psi^\pm = \psi_1^\pm + i \psi_2^\pm, \quad \mathcal{H}^\pm = \mathcal{H}_1^\pm + i \mathcal{H}_2^\pm,
\]
we now try to simplify the above system (5.6) for \((\psi^+, \psi^-)\). We rewrite the system (5.6) to the following one in \(\mathbb{R}^2\)
\[
\begin{align*}
\Delta \psi_1^+ + \epsilon^2 \partial_s^2 \psi_1^+ &= \mathcal{H}_1^+, \\
\Delta \psi_2^- + \epsilon^2 \partial_s^2 \psi_2^- &= \mathcal{H}_2^-,
\end{align*}
\]
(5.7)
with
\[
\mathcal{H}_1^+ = \mathcal{H}_1^+ - \frac{\nabla W^+(\ell_1)}{W^+(\ell_1)} \nabla \psi_1^+ - \nabla (\theta_1 + \theta_2) \psi_2^+, \\
- 2\epsilon^2 \left[ \frac{\partial_s W^+(\ell_1)}{W^+(\ell_1)} \partial_s \psi_1^+ - \partial_s (\theta_1 + \theta_2) \partial_s \psi_2^+ \right] - 4\epsilon^2 \partial_s \psi_2^+, \\
\mathcal{H}_2^- = \mathcal{H}_2^- - \frac{\nabla W^-(\ell_1)}{W^-(\ell_1)} \nabla \psi_2^- - \nabla (\theta_1 + \theta_2) \psi_1^-, \\
- 2\epsilon^2 \left[ \frac{\partial_s W^-(\ell_1)}{W^-(\ell_1)} \partial_s \psi_2^- - \partial_s (\theta_1 + \theta_2) \partial_s \psi_1^- \right] + 4\epsilon^2 \partial_s \psi_1^- + 2A_\pm (|v_d|^2 - f^2) \psi_{x_2}^\pm + 2B(|v_d|^2 - f^2) \psi_{x_2}^\pm.
\]

Since the matrix \(M = \begin{pmatrix} A_+ |t|^2 & B|t|^2 \\ B|t|^2 & A_- |t|^2 \end{pmatrix}\) is positive definite, there then exists an invertible matrix \(C\) such that
\[
C^T C = I, \quad 2C M C^T = \text{diag}(\lambda^+, \lambda^-) \quad \text{with } \lambda^+, \lambda^- > 0.
\]
(5.8)

Now we define the new vector functions
\[
\tilde{\psi}_1 = (\tilde{\psi}_1^+, \tilde{\psi}_1^-)^T, \quad \tilde{\psi}_2 = (\tilde{\psi}_2^+, \tilde{\psi}_2^-)^T := C^T (\psi_2^+, \psi_2^-)^T,
\]
(5.9)
and
\[
H_2 = (\mathcal{H}_2^+, \mathcal{H}_2^-) := C^T (\mathcal{H}_2^+, \mathcal{H}_2^-)^T, \quad H_1 = (\mathcal{H}_1^+, \mathcal{H}_1^-) := (\mathcal{H}_1^+, \mathcal{H}_1^-)^T.
\]
(5.10)
Then if $|z - e_j| > R_0$, the system (5.7) can be rewritten as

\[ (\Delta + \epsilon^2 \partial_z^2) \tilde{\psi} = \mathbb{H}_1, \quad (5.11) \]

\[ (\Delta + \epsilon^2 \partial_z^2) \tilde{\psi}_2 - \text{diag}(\lambda^+, \lambda^-) \tilde{\psi}_2 = \mathbb{H}_2. \quad (5.12) \]

For the system (5.11)--(5.12), we will use the barrier arguments based on the maximum principle provided in Lemmas 5.3 and 5.4. We obtain that, by some calculations,

\[ \|\mathbb{H}_2\| \leq C \left( \|\mathcal{H}\|_{**0} + R_0^{-\sigma} + \epsilon^a \right) \left( \epsilon_1^{a-2} + \epsilon^{2-\sigma} \right). \]

By the comparison principle, elliptic estimates, and choosing the barrier as

\[ B_2 = \hat{C} \left( \|\mathcal{H}\|_{**} + R_0^{-\sigma} + \epsilon^a + \|\tilde{\psi}_2\|_{L^\infty(B_{R_0}(d))} \right) \left( \epsilon_1^{a-2} + \epsilon^{2-\sigma} \right), \]

for some fixed large constant $\hat{C} > 0$, we obtain

\[ \|\tilde{\psi}_2\|_{**,0,R} \leq C \left( \|\mathcal{H}\|_{**,0} + R_0^{-\sigma} + \epsilon^a + \|\tilde{\psi}_2\|_{L^\infty(B_{R_0}(d))} \right). \]

On the other hand, by a similar argument as in [12], we have

\[ \|\tilde{\psi}_1\|_{**,0,Re} \leq C \left( \|\mathcal{H}\|_{**,0} + R_0^{-\sigma-1} + \epsilon^{1-\sigma} + \|\tilde{\psi}_1\|_{L^\infty(B_{R_0}(d))} \right). \]

Meanwhile, using (5.5) and $\|\mathcal{H}^{(t)}\|_{**,0} = o_n(1)$, we can get

\[ \|\tilde{\psi}\|_{**} \leq C \|\mathcal{H}\|_{**,}, \quad \text{with} \quad \tilde{\psi} = (\tilde{\psi}_1 + i\tilde{\psi}_2, \tilde{\psi}_1 - i\tilde{\psi}_2). \]

Since $C^T$ is invertible, we directly get

\[ \|\psi\|_{**} \leq C \|\mathcal{H}\|_{**}. \]

Therefore, by combining the above results we derive an estimate contradicting to (5.4).

We next give the proof of Proposition 4.1.

**Proof of Proposition 4.1.** For given $M > 100d$, we first consider the problem (4.2) locally as the following

\[
\begin{cases}
\mathcal{L}_x^+(\psi) = \mathcal{H}_x^+ + c \sum_{j=1}^2 \frac{\chi_j}{i\omega_j(z - e_j)} ( -1)^j w_{x_z}^\pm(z - e_j) & \text{in } B_M(0), \\
\text{Re} \int_{B(0,4)} \chi \left[ \phi_j^+ w_{x_z}^+ + \phi_j^- w_{x_z}^- \right] = 0, & \text{with } \phi_j^\pm(z) = i w^\pm(z) \psi^\pm(z + e_j), \\
\psi \text{ satisfies the symmetry (5.4)} \quad \text{and } \psi = 0 \text{ on } \partial B_M(0). 
\end{cases}
\]

Let

\[ \mathcal{B} := \left\{ \phi = (i v_d^+ \psi^+, i v_d^- \psi^-) \in H_0^1(B_M(0), \mathbb{C}^2) : \psi \text{ satisfies (5.4)} \right\}. \]

We endow the space $\mathcal{B}$ with the inner product

\[ [\phi, \varphi]_{\mathcal{B}} := \text{Re} \int_{B_M(0)} \left[ (\nabla \phi^+ \cdot \nabla \varphi^- + 2 \partial_x \phi^+ \cdot \partial_x \varphi^-) + (\nabla \phi^- \cdot \nabla \varphi^+ + 2 \partial_x \phi^- \cdot \partial_x \varphi^+) \right], \quad (5.14) \]

for any $\phi, \varphi \in \mathcal{B}$. Due to the Poincaré inequality, we know that the space equipped with the topology is a subspace of $H_0^1(B_M(0), \mathbb{C}^2)$. By some calculations, the following identities hold

\[ \Delta \Phi^\pm + \left[ A_\pm \left( \epsilon_{ss}^2 + |v_{d}^\pm|^2 \right) + B(\epsilon_{ss}^2 - |v_{d}^\pm|^2) \right] \Phi^\pm + (\eta_1 - 1) \frac{\Phi^\pm}{v_{d}^\mp} \Phi_j^\pm(z) \]

\[ - 2A_\pm \text{Re} \left( v_{d}^\mp \bar{\Phi}^\pm \right) v_{d}^\pm - 2B \text{Re} \left( v_{d}^\mp \bar{\Phi}^\pm \right) v_{d}^\pm + \epsilon^2 (\partial_{ss}^2 \Phi^\pm - 4i \partial_x \Phi^\pm - 4 \Phi^\pm) \]
For any test function \( \varphi = (\varphi^+, \varphi^-) \), we rewrite these equations in the sense of distribution

\[
- \text{Re} \int_{B_M(0)} \left[ (\nabla \Phi^+ \nabla \varphi^+ + \epsilon^2 \partial_\delta \Phi^+ \partial_\delta \varphi^+) + (\nabla \Phi^- \nabla \varphi^- + \epsilon^2 \partial_\delta \Phi^- \partial_\delta \varphi^-) \right]

- \epsilon^2 \text{Re} \int_{B_M(0)} \left[ (4i \partial_x \Phi^+ - 4\Phi^+) \varphi^+ + (4i \partial_x \Phi^- - 4\Phi^-) \varphi^- \right]

- \text{Re} \int_{B_M(0)} 2A_{\pm} \text{Re}(v_d^+ \Phi^+) v_d^+ \varphi^+ + 2B \text{Re}(v_d^- \Phi^-) v_d^- \varphi^-

- \text{Re} \int_{B_M(0)} 2A_{\pm} \text{Re}(v_d^- \Phi^-) v_d^- \varphi^- + 2B \text{Re}(v_d^+ \Phi^+) v_d^+ \varphi^+

+ \text{Re} \int_{B_M(0)} \left( A_+ (t^2 - |v_d^+|^2) + B (t^2 - |v_d^-|^2) \right) + (\eta_1 - 1) \frac{E^+}{v_d^+} \Phi^+ \varphi^+

+ \text{Re} \int_{B_M(0)} \left( A_- (t^2 - |v_d^-|^2) + B (t^2 - |v_d^+|^2) \right) + (\eta_1 - 1) \frac{E^-}{v_d^-} \Phi^- \varphi^-

= \text{Re} \int_{B_M(0)} \left( iv_d^+ \mathcal{H}^+ + iv_d^+ c \sum_{j=1}^2 \frac{\chi_j}{i w^+(z - e_j)} (-1)^j w^+_{x_j}(z - e_j) \right) \varphi^+

+ \text{Re} \int_{B_M(0)} \left( iv_d^- \mathcal{H}^- + iv_d^- c \sum_{j=1}^2 \frac{\chi_j}{i w^-(z - e_j)} (-1)^j w^-_{x_j}(z - e_j) \right) \varphi^-.

(5.15)

Denote by linear operator \( \langle \mathcal{K}(\Phi), \cdot \rangle_\mathcal{B} \) on \( \mathcal{B} \) as

\[
- \epsilon^2 \text{Re} \int_{B_M(0)} \left[ (4i \partial_x \Phi^+ - 4\Phi^+) \varphi^+ + (4i \partial_x \Phi^- - 4\Phi^-) \varphi^- \right]

- \text{Re} \int_{B_M(0)} 2A_{\pm} \text{Re}(v_d^+ \Phi^+) v_d^+ \varphi^+ + 2B \text{Re}(v_d^- \Phi^-) v_d^- \varphi^-

- \text{Re} \int_{B_M(0)} 2A_{\pm} \text{Re}(v_d^- \Phi^-) v_d^- \varphi^- + 2B \text{Re}(v_d^+ \Phi^+) v_d^+ \varphi^+

+ \text{Re} \int_{B_M(0)} \left( A_+ (t^2 - |v_d^+|^2) + B (t^2 - |v_d^-|^2) \right) + (\eta_1 - 1) \frac{E^+}{v_d^+} \Phi^+ \varphi^+

+ \text{Re} \int_{B_M(0)} \left( A_- (t^2 - |v_d^-|^2) + B (t^2 - |v_d^+|^2) \right) + (\eta_1 - 1) \frac{E^+}{v_d^-} \Phi^- \varphi^-

= : \langle \mathcal{K} \Phi, \varphi \rangle_\mathcal{B}.

Also, we denote by the linear operator \( \langle \mathcal{S}(\Phi), \cdot \rangle_\mathcal{B} \) on \( \mathcal{B} \) as

\[
\text{Re} \int_{B_M(0)} \left( iv_d^+ \mathcal{H}^+ + iv_d^+ c \sum_{j=1}^2 \frac{\chi_j}{i w^+(z - e_j)} (-1)^j w^+_{x_j}(z - e_j) \right) \varphi^+.
\]
we can get the expression for $c$ when $H$ is rewritten into the following form

$$\text{Step 1: Lemma 5.2 in [12] for more details.}$$

**Proof.** We give some technical essentials and the outline of the proof. The reader can refer to the proof of Lemma 5.1. Suppose $\epsilon$ is very small such that the following estimate holds

$$\|\psi\|_\ast \leq C \|\mathcal{H}\|_{\ast\ast},$$

and thus there exists only trivial solution when $\mathcal{H}_j^\pm = 0$. It is worth noting that we now apply the norms $\|\cdot\|_\ast$, $\|\cdot\|_{\ast\ast}$ restricted on the bounded domains. Then the existence of (5.13) holds and from Lemma 5.1 there exists a constant $C$ independent of $\mathcal{M}$ such that the following estimate holds

$$\|\psi_M\|_\ast \leq C \|\mathcal{H}\|_{\ast\ast}.$$
such that the main terms in the linear operator of $\hat{\psi}_2 = (\hat{\psi}_2^+, \hat{\psi}_2^-)$ can be decoupled in the region far away from the vortices, see (5.12).

**Step 2:** When near the vortices, we just proceed as in the proof of Lemma 5.2 in [12]. We point out that in this step we use Lemma 2.2 and the non-degeneracy results in Lemma 8.3. On the other hand, when far away from the vortices, we handle the system (5.11)-(5.12) by the maximum principle, see Lemmas 8.3-8.4 in the Appendix. In this step, we establish the estimate

$$|\hat{\psi}|_2 \leq C (|\mathcal{H}|_{\infty} + \epsilon \log \epsilon |\mathcal{H}|_{\ast \ast}).$$

**Step 3:** Finally, since $C^T$ is invertible, we get that (5.18) holds.

In order to prove Proposition 4.2, we also need the following lemma. Here we recall the reflection $\mathcal{R}_j(z)$ defined in (3.14) and suppose that the function $H$ enjoys the local symmetry (S2) and the following

$$\mathcal{H}(\mathcal{R}_j z) = -\mathcal{H}(z), \quad |z - e_j| < 2 \Re e, \quad \text{for } j = 1, 2.$$  \hfill (S2)

We note that the function $H$ satisfies the local symmetry (S2) consisting in even Fourier modes.

**Lemma 5.3.** Suppose that $H$ enjoys the symmetry (S2) and (S4) and

$$||H||_{\ast \ast} \leq +\infty.$$

For any solution $\psi$ of (5.1) with $||\psi||_{\ast} \leq +\infty$, there exist $\psi_s, \psi_*$ such that

$$\psi = \psi_s + \psi_*,$$

where $\psi_s$ satisfies the following symmetry

$$\psi_s(\mathcal{R}_j z) = -\overline{\psi_s(z)}, \quad |z - e_j| < \Re e.$$

Moreover, there exists a constant $C$ such that for all $\epsilon$ small, the following estimates hold

$$||\psi_s||_{\ast} + ||\psi_*||_{\ast} \leq C ||\mathcal{H}||_{\ast \ast},$$

$$||\psi_*||_2 \leq C \epsilon \log \epsilon |\mathcal{H}|_{\ast \ast}.$$

**Proof.** We decompose the operator $\mathcal{L}_e = (\mathcal{L}_e^+, \mathcal{L}_e^-)$ as

$$\mathcal{L}_e^\pm = \mathcal{L}_{e,s,j}^\pm + \mathcal{L}_{e,r,j}^\pm,$$

with $\mathcal{L}_{e,s,j}$ preserving the symmetry (S2). More specifically, we have the expressions of $\mathcal{L}_{e,s,j}, \mathcal{L}_{e,r,j}$ in the coordinates $(\ell_j, \theta_j)$

$$\mathcal{L}_{e,s,j}^\pm(\psi) = \Delta \psi^\pm + 2 \frac{\nabla w_a \nabla \psi^\pm}{w_a} - 2i A_\pm |w_a^\pm|^2 \text{Im}(\psi^\mp) - 2i B |w_a^\mp|^2 \text{Im}(\psi^\mp)$$

$$+ \epsilon^2 \left[ d^2 \partial_{\ell_1}^2 \psi \sin^2 \theta_1 + d^2 \partial_{\ell_1}^2 \psi^\pm \sin \theta_1 \cos \theta_1 + \partial_{\partial_\ell_1 \ell_1} \psi^\pm \left( 1 + \frac{d^2}{\ell_1^2} \cos^2 \theta_1 \right) 

+ \partial_{\ell_1} \psi^\pm \frac{d^2}{\ell_1^2} \cos \theta_1 - 2 \partial_{\theta_1} \psi^\pm \frac{d^2}{\ell_1} \sin \theta_1 \cos \theta_1 \right]$$

$$+ \frac{\partial^2 m}{|\ln \epsilon|} W^\pm(\ell_1) \sin^2 \theta_1 \psi^\pm,$$

and

$$\mathcal{L}_{e,r,j}^\pm(\psi) = 2 \frac{\nabla w_b \nabla \psi^\pm}{w_b} - 2i A_\pm (|v_d^\pm|^2 - |w_a^\pm|^2) \text{Im}(\psi^\mp) - 2i B (|v_d^\mp|^2 - |w_a^\mp|^2) \text{Im}(\psi^\mp)$$

$$+ \epsilon^2 \left[ 2 d^2 \partial_{\ell_1} \psi \sin \theta_1 + 2 \partial_{\theta_1} \psi^\pm \frac{d}{\ell_1} \cos \theta_1 + \partial_{\ell_1} \psi^\pm \frac{d}{\ell_1} \cos \theta_1 - \partial_{\theta_1} \psi^\pm \frac{d}{\ell_1} \sin \theta_1 \right]$$
Note that, due to the local symmetry (S2), the expressions of $L$ and consider the following

\begin{equation*}
\mu \left( \frac{S^\pm(v_d)}{v_d} - \frac{d^2}{d \ln \epsilon} W^\pm(\ell_1) \sin^2 \theta_1 \right) \psi^\pm,
\end{equation*}

with

\begin{align*}
w_a^\pm(z) &= W^\pm(\ell_1)e^{i\xi}, & w_b^\pm(z) &= W^\pm(\ell_2)e^{i\eta}.
\end{align*}

The expressions of $L_{c,s,2}(\psi)$ and $L_{c,r,2}(\psi)$ are similar.

Next, we consider the following problem

\begin{equation*}
L_{c,s,j}(\psi_{s,j}) = \mathcal{H}^\pm \eta_{j,2R}, \quad \text{in } \mathbb{R}^2,
\end{equation*}

\begin{equation*}
\begin{align*}
&\text{Re} \int_{B(0,4)} \chi \left[ \overline{\phi_{s,j}^+ w_{x_1}^+} + \overline{\phi_{s,j}^- w_{x_1}^-} \right] = 0, \quad \text{with } \phi_{s,j}^\pm(z) = iw^\pm(z) \psi_{s,j}^\pm(z + e_j), \\
&\psi_{s,j}^\pm \text{satisfies } \psi_{s,j}^\pm(z) = -\psi_{s,j}^\mp(z).
\end{align*}
\end{equation*}

Note that, due to the local symmetry (S2), $H$ is orthogonal to the kernel locally, thus the multiplier $c$ vanishes. Recall that $L_{c,s,j} = (L_{c,s,j}^+, L_{c,s,j}^-)$ preserves symmetry (S2) and $H$ satisfies (S2). Then for $\psi_{s,j}$, the symmetry (S2) and the following estimate both hold

\begin{equation*}
\| \psi_{s,j} \| \leq C \| H \|_{**}.
\end{equation*}

Let

\begin{equation*}
\psi_s = \eta_{1,4R} \psi_{s,1} + \eta_{2,4R} \psi_{s,2},
\end{equation*}

then

\begin{equation*}
\| \psi_s \| \leq C \| H \|_{**}.
\end{equation*}

Define $\tilde{H} = (\mathcal{H}^+, \mathcal{H}^-)$ with

\begin{equation*}
\tilde{H} := H - L_{c,r,1}(\eta_{1,4R}\psi_{s,1}) - L_{c,s,1}(\eta_{1,4R}\psi_{s,1}) - L_{c,r,2}(\eta_{2,4R}\psi_{s,2}) - L_{c,s,2}(\eta_{2,4R}\psi_{s,2}),
\end{equation*}

and consider the following

\begin{equation*}
\begin{align*}
L_c^\pm(\hat{\psi}^\pm) &= \tilde{H}^\pm + c \sum_{j=1}^2 \frac{\lambda_j}{iw^\pm(z - e_j)}(-1)^j w_{x_1}^\pm(z - e_j), \quad \text{in } \mathbb{R}^2, \\
\text{Re} \int_{B(0,4)} \chi \left[ \overline{\phi_j^+ w_{x_1}^+} + \overline{\phi_j^- w_{x_1}^-} \right] = 0, \quad \text{with } \phi_j^\pm(z) = iw^\pm(z) \hat{\psi}_{s,j}^\pm(z + e_j),
\end{align*}
\end{equation*}

(5.19)

$\tilde{H}$ satisfies the symmetry (S2).

By the solvability in Proposition 4.1 and the a priori estimate in Lemma 5.2 we can get a solution $\hat{\psi}$ of (5.19) with the estimates

\begin{align*}
\| \hat{\psi} \| &\leq C \| \tilde{H} \|_{**}, \\
|\hat{\psi}_{s,j}^\pm| &\leq C(\| \tilde{H} \|_{**} + \epsilon \| \log \epsilon \|_{**} H \|_{**}), \\
|\tilde{H} |_{**} &\leq C \epsilon \| \log \epsilon \|_{**} H \|_{**}.
\end{align*}

In fact, we have

\begin{equation*}
\hat{\psi} = \psi_{s,j} + \hat{\psi}_{s,j}^\pm,
\end{equation*}

so we take

\begin{equation*}
\psi_{s,j} := \hat{\psi}_{s,j}.
\end{equation*}

From the analysis above, $\psi_{s,j}$ and $\psi_s$ satisfy the estimates in this Lemma.

Finally, we give the proof of Proposition 4.2.
Proof of Proposition 4.2. By combining the a priori estimates in Lemma 5.2 and Lemma 5.3 and the decompositions in Lemma 5.3, we can prove Proposition 4.2. More technical details can be found in [12]. □

6. Solving the projected nonlinear problem

In this section, we would deal with the projected problem (4.1). The resolution theory is provided in the following:

Proposition 6.1. There exists a number \( D > 0 \), depending only on \( \alpha, \sigma \in (0, 1) \), such that for any \( \epsilon \) sufficiently small, problem (4.1) has a solution \((\psi_\epsilon, c)\) which satisfies

\[
\|\psi_\epsilon\|_* \leq D \frac{1}{|\ln \epsilon|},
\]

and \( \psi_\epsilon \) is a continuous function of the parameter \( \hat{d} \). Furthermore, we have the decomposition

\[
\psi_\epsilon = \psi_{\epsilon_o} + \psi_{\epsilon_e} = \psi_{\epsilon_o}^\alpha + \psi_{\epsilon_o}^\beta + \psi_{\epsilon_e}
\]

and the estimates

\[
|\psi_{\epsilon_o}^\alpha|_* + \|\psi_{\epsilon_o}^\beta\|_* \leq D \epsilon \sqrt{|\ln \epsilon|}, \quad |\psi_{\epsilon_o}^\alpha|_* \leq D \epsilon \sqrt{|\ln \epsilon|}.
\]

Proof. We consider the following closed, bounded domain for functions with the form \( \psi_\epsilon = \psi_{\epsilon_o}^\alpha + \psi_{\epsilon_o}^\beta + \psi_{\epsilon_e} \):

\[
\hat{A} := \left\{ \psi: \psi \text{ satisfies } (S4), \quad \Re \int_{B(0,4)} \chi[\phi_j w_{x_1} + \phi_j w_{x_2}] = 0, \quad j = 1, 2, \right\}
\]

\[
\|\psi\|_* \leq C \frac{1}{|\ln \epsilon|}, \quad |\psi_{\epsilon_o}^\alpha|_* \leq D \epsilon \sqrt{|\ln \epsilon|}, \quad |\psi_{\epsilon_o}^\beta|_* \leq D \epsilon \sqrt{|\ln \epsilon|}.
\]

From Proposition 4.1 the problem (4.1) is equivalent to finding a fixed point of

\[
\psi = T_\epsilon \left( R + N(\psi) \right) =: G_\epsilon(\psi)
\]

in the domain \( \hat{A} \). Now what we shall show is that, for \( \epsilon \) small enough, the map \( G_\epsilon \) is a contraction from \( \hat{A} \) to itself. We divide this proof into several steps.

Step 1: We claim that

\[
\|G_\epsilon(\psi)\|_* \leq D \frac{1}{|\ln \epsilon|}, \quad \text{for any } \psi \in \hat{A}.
\]

The readers can refer to [12] for the proof of this step.

Step 2: We claim that, for any \( \psi \in \hat{A} \), we have the decompositions and estimates

\[
G_\epsilon(\psi) = G_{\epsilon_o}(\psi) + G_{\epsilon_e}(\psi) = G_{\epsilon_o}^\alpha(\psi) + G_{\epsilon_o}^\beta(\psi) + G_{\epsilon_e}(\psi),
\]

\[
|G_{\epsilon_o}^\alpha(\psi)|_H \leq D \epsilon \sqrt{|\ln \epsilon|}, \quad \|G_{\epsilon_o}^\beta(\psi)\|_* \leq D \epsilon \sqrt{|\ln \epsilon|}.
\]

The proofs of (6.1)-(6.2) rely on the following two facts.

\[\blacklozenge\] From Proposition 3.3 we have

\[
R = R_o + R_e = R_o^\alpha + R_o^\beta + R_e,
\]

with

\[
|R_o^\alpha|_* \leq D \epsilon \sqrt{|\ln \epsilon|}, \quad \|R_o^\beta\|_* \leq D \epsilon \sqrt{|\ln \epsilon|}.
\]

\[\blacklozenge\] Next, we claim that we can decompose \( N(\psi) \) into the following three parts

\[
N(\psi) = (N(\psi))_o^\alpha + (N(\psi))_e^\beta + (N(\psi))_e.
\]
with the estimates
\[
\|N(\psi)\| \leq \frac{D\epsilon}{\ln \epsilon}
\]
These can be verified in the following way. Recall that
\[
N_j(\psi) = N_j^+ \psi + N_j^- \psi,
\]
with estimates in (6.4) for real scalar functions. Now we express the product of two real scalar functions
\[
N_j(\psi) = N_j^+ (\psi) N_j^- (\psi)
\]
Without loss of generality, express \(N_j^+\), \(N_j^-\) in the coordinates \((\ell_1, \theta_1) = (\ell, \theta)\) when \(|z - \tilde{d}| > 3\) and \(\text{Re}(z) > 0\)
\[
N_j^+ = 2(\partial_\psi \partial_\theta \psi + 2(\partial_\psi \partial_\theta \psi)) \left( \epsilon^2 + \frac{2\epsilon}{\ell^2} \right)
\]
and
\[
N_j^- = -2(\partial_\psi \partial_\theta \psi - (\partial_\psi \partial_\theta \psi)) \left( \epsilon^2 + \frac{2\epsilon}{\ell^2} \right)
\]
We will use the approach introduced in [12]. Suppose that the real scalar functions \(f, g\) satisfy the following decompositions
\[
f = f_o + f_e = f_o^+ + f_o^- + f_e, \quad g = g_o + g_e = g_o^+ + g_o^- + g_e,
\]
with estimates
\[
\|f_o\|, \|g_o\| \leq \frac{D\epsilon}{\ln \epsilon}
\]
where \(|\cdot|_{\ell, 1}\) is the semi-norm defined in (4.11)-(4.12) for real scalar functions and \(\|\cdot\|_{+, 1}\) is the norm defined in (4.1) for real scalar functions. Now we express the product of two real scalar functions \(f, g\) as the following
\[
f g = (f_o \alpha + f_o \beta + f_e) (g_o \alpha + g_o \beta + g_e)
\]
\[
= f_o g_o + f_o \alpha g_o + f_o g_o \alpha + f_o \beta g_o + f_e g_o \beta + f_e g_o \beta + f_e g_o \beta + f_e g_o \beta
\]
\[
= (f(e) g_o) \alpha + (f(e) g_o) \beta
\]
where
\[
(f(e) g_o) = f_e g_o, \quad (f(e) g_o) = f_o g_o + f_o \alpha g_o + f_o \beta g_o, \quad (f(e) g_o) = f_o g_o + f_e g_o \beta.
\]
Using the expressions of \(N_j^+, N_j^-\) in (6.5)-(6.6), the decompositions in (6.7)-(6.8), when \(|z - \tilde{d}| > 3\), we have
\[
N_j^+ = N_j^+ (\epsilon) \psi + (N_j^+ \alpha) \psi + (N_j^+ \beta) \psi, \quad j = 1, 2,
\]
and
\[
\|N_j^+ \| \leq C \left( \ln \epsilon \|\psi\| \psi \| + \psi \| \| + \frac{D\epsilon}{\ln \epsilon} \|\psi\| \right)
\]
Integrating the equations (7.1) against \( w \) and we omit it here. Readers can refer to [12] for more details.

Combining the analysis and estimates above, we conclude that (6.1) and (6.2) hold.

The computations of all terms in (7.2) will be carried out in the sequel.

Note that the parameter \( \hat{\epsilon} \) will determine the locations of vortices. As a standard step in the reduction method to make the Lagrange multiplier \( c \) vanish in (4.1), we will set up an equation involving \( \hat{\epsilon} \) in such a way that \( c = 0 \).

Without loss of generality, we consider the following equivalent problem of (4.1)

\[
\begin{align*}
    iw^\pm(z) \left[ \mathcal{L}_\kappa^\pm(\psi) + \mathcal{R}^\pm + \mathcal{N}^\pm(\psi) \right] (z + \hat{d}) &= c^\kappa w^\pm_{z_1}, \quad \text{for } z \in B_{\mathbb{R}_+}(0).
\end{align*}
\]

Integrating the equations (7.1) against \( w^\pm_{z_1} \) respectively, we get

\[
\sum_{\kappa \in \{+, -\}} \operatorname{Re} \int_{B_{\mathbb{R}_+}(0)} \left( \frac{\hat{\epsilon}}{i w^\pm_{z_1}(z) w^\kappa(z) \left[ \mathcal{L}_\kappa^\pm(\psi) + \mathcal{R}^\kappa + \mathcal{N}^\kappa(\psi) \right] (z + \hat{d})} \right) = c \int_{B_{\mathbb{R}_+}(0)} \chi\left(|w^+_{z_1}|^2 + |w^-_{z_1}|^2\right) = cc^*.
\]

Since \( c^* \sim C \), we know easily that \( c = 0 \) is equivalent to the following relation

\[
\sum_{\kappa \in \{+, -\}} \operatorname{Re} \int_{B_{\mathbb{R}_+}(0)} \left( \frac{\hat{\epsilon}}{i w^\pm_{z_1}(z) w^\kappa(z) \left[ \mathcal{L}_\kappa^\pm(\psi) + \mathcal{R}^\kappa + \mathcal{N}^\kappa(\psi) \right] (z + \hat{d})} \right) = 0.
\]

The computations of all terms in (7.2) will be carried out in the sequel.
We recall the following results introduced in Lemma 2.2
\[ iw^\pm(z)L_\epsilon^\pm(\psi)(z + \tilde{d}) := L_0(\Phi_1) + \mathcal{F}_{1,\pm}(z) \]
where \( L_0 = (L_0,+, L_0,-) \) is the linearization of Ginzburg-Landau equation \([1.14]\) around \( w \), and
\[
\mathcal{F}_{1,\pm}(z) = 2A_\pm(1 - |\Omega_1^\pm|^2)Re\left(w^\pm(z)\tilde{\Phi}_1^\pm\right)w^\pm(z) + 2B(1 - |\Omega_1^\pm|^2)Re\left(w^\mp(z)\tilde{\Phi}_1^\mp\right)w^\pm(z)
+ \epsilon^2\left[\partial_{ss}\Phi_j^\mp - 4i\partial_s\Phi_j^\mp - 4\Phi^\mp_j\right] + \frac{1}{\Omega_1^\pm}\left[2\nabla\Phi_j^\pm\nabla\Omega_1^\pm + 2\epsilon^2\partial_s\Omega_1^\pm\partial_s\Phi_j^\pm\right]
+ \left\{4\epsilon^2 - \frac{2\nabla w^\pm(z)\nabla\Omega_1^\pm + 2\epsilon^2\partial_s w^\pm(z)\partial_s\Omega_1^\pm}{v^\pm_d}\right\} + \frac{\eta_1}{v^\pm_d} \Phi_j^\pm,
\]
and
\[
\Phi_j^\pm(z) = iw^\pm(z)\psi^\pm(z + \tilde{d}), \quad \Omega_1^\pm(z + 2\tilde{d}) = w^\pm(z + 2\tilde{d}), \quad \text{for } z \in B_{\mathfrak{R},0}. \]

We first consider
\[
\sum_{\kappa\in\{+,-\}}\text{Re}\int_{B_{\mathfrak{R}}(0)} i\overline{w^\kappa_{x_1}}(z)w^\kappa(z)L_\epsilon^\kappa(\psi)(z + \tilde{d})
= \sum_{\kappa\in\{+,-\}}\text{Re}\int_{B_{\mathfrak{R}}(0)} \overline{w^\kappa_{x_1}}(z)\left[L_0,\kappa(\Phi_1) + iw^\kappa(z)\mathcal{F}_{1,\kappa}(z)\right]
= \sum_{\kappa\in\{+,-\}}\text{Re}\int_{B_{\mathfrak{R}}(0)} \overline{\Phi_1^\kappa}(z)L_0,\kappa(w^\kappa_{x_1}) + \sum_{\kappa\in\{+,-\}}\text{Re}\int_{\partial B_{\mathfrak{R}}(0)} \left[\frac{\partial\Phi_1^\kappa}{\partial\nu}\overline{w^\kappa_{x_1}} - \overline{\Phi_1^\kappa}\frac{\partial w^\kappa_{x_1}}{\partial\nu}\right]
+ \sum_{\kappa\in\{+,-\}}\text{Re}\int_{B_{\mathfrak{R}}(0)} i\overline{w^\kappa_{x_1}}(z)w^\kappa(z)\mathcal{F}_{1,\kappa}(z)
:= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\]

We easily get \( \mathcal{J}_1 = 0 \) since \( L_0(w_{x_1}) = 0 \). For \( \mathcal{J}_2 \), using the decay properties of \( W^\pm \) in Lemma 1.2 and the definition of \( \| \cdot \|_* \), we get
\[
\mathcal{J}_2 \leq C\epsilon|\ln\epsilon|^2\|\psi\|_* = O\left(\frac{\epsilon}{|\ln\epsilon|^2}\right).
\]
Using the expression of \( \mathcal{F}_{1,\pm}(z) \) in (7.3), by some tedious but straightforward calculations, we can get
\[
\mathcal{J}_3 = O\left(\frac{\epsilon}{|\ln\epsilon|^2}\right).
\]
Therefore, we have
\[
\sum_{\kappa\in\{+,-\}}\text{Re}\int_{B_{\mathfrak{R}}(0)} i\overline{w^\kappa_{x_1}}(z)w^\kappa(z)L_\epsilon^\kappa(\psi)(z + \tilde{d}) = O\left(\frac{\epsilon}{|\ln\epsilon|^2}\right).
\]

Next, we consider the third term of (7.2). We decompose locally
\[
\mathcal{N}^\pm(\psi) = \mathcal{N}_e^\pm(\psi) + \mathcal{N}_o^\pm(\psi)
= (\mathcal{N}_1^\pm)_e + i(\mathcal{N}_2^\pm)_o + (\mathcal{N}_1^\pm)_o + i(\mathcal{N}_2^\pm)_e
\]
where \((N_j^\pm)\) and \((N_j^\pm)\) for \(j = 1, 2\) are defined in Section 3. Then
\[
\sum_{\kappa \in \{+, -\}} \Re \int_{B_{\kappa}(0)} \left( i w_{x_1}^\kappa (z) w^\kappa (z) \mathcal{N}_j^\kappa (\psi) (z + \tilde{d}) \right)
= \sum_{\kappa \in \{+, -\}} \Re \int_{B_{\kappa}(0)} \left( i w_{x_1}^\kappa (z) w^\kappa (z) \mathcal{N}_0^\kappa (\psi) (z + \tilde{d}) \right)
= \sum_{\kappa \in \{+, -\}} \Re \int_{B_{\kappa}(0)} i W^\kappa \left( W^\kappa \cos \theta + i \frac{\sin \theta}{\kappa} W^\kappa \right) \mathcal{N}_0^\kappa (\psi) (z + \tilde{d})
= \sum_{\kappa \in \{+, -\}} \int_{B_{\kappa}(0)} W^\kappa \left( - \frac{\sin \theta}{\kappa} W^\kappa (N_1^\kappa (\psi)) (z + \tilde{d}) - W^\kappa \cos \theta (N_2^\kappa (\psi)) (z + \tilde{d}) \right)
\]
where \(z = re^{i\theta}\) and the second equality hold due to orthogonality. For the estimates of the terms \((N_1^\pm)\) and \((N_2^\pm)\), from (6.3) – (6.8), we have
\[
| (N_1^\pm) (z + \tilde{d}) | \leq \frac{C}{1 + r^2} \left( \| \psi_0^\pm \| \| \psi \| + \| \ln | \psi \| \| \psi_0^\pm \|_2 + | \psi_0^\pm |_2 \right)
\leq \frac{C \epsilon}{(1 + r^2)} | \ln \epsilon |^{\frac{1}{2}},
\]
and
\[
| (N_2^\pm) (z + \tilde{d}) | \leq C \left( \| \psi_0^\pm \| \| \psi \| + \| \ln | \psi \| \| \psi_0^\pm \|_2 + | \psi_0^\pm |_2 \right) \leq \frac{C \epsilon}{| \ln \epsilon |^{\frac{1}{2}}}. \tag{7.5}
\]
Therefore, we have
\[
\sum_{\kappa \in \{+, -\}} \Re \int_{B_{\kappa}(0)} \left( i w_{x_1}^\kappa (z) w^\kappa (z) \mathcal{N}_j^\kappa (\psi) (z + \tilde{d}) \right) = O \left( \frac{\epsilon}{| \ln \epsilon |^{\frac{1}{2}}} \right). \tag{7.5}
\]
At last, we consider
\[
\sum_{\kappa \in \{+, -\}} \Re \int_{B_{\kappa}(0)} \left( i w_{x_1}^\kappa (z) w^\kappa (z) \mathcal{R}_j^\kappa (z + \tilde{d}) \right)
= \sum_{\kappa \in \{+, -\}} \sum_{j \in \{0, 1\}} \Re \int_{B_{\kappa}(0)} \left( i w_{x_1}^\kappa (z) w^\kappa (z) \mathcal{R}_j^\kappa (z + \tilde{d}) \right)
:= \mathcal{T}_0 + \mathcal{T}_1,
\]
where
\[
\mathcal{T}_j = \sum_{\kappa \in \{+, -\}} \Re \int_{B_{\kappa}(0)} \left( i w_{x_1}^\kappa (z) w^\kappa (z) \mathcal{R}_j^\kappa (z + \tilde{d}) \right), \quad S_j^\pm (v_d) = iv_d^\pm \mathcal{R}_j^\pm, \quad \text{for } j = 0, 1.
\]
For \(\mathcal{T}_1\), by Lemma 3.2 we have
\[
\mathcal{T}_1 = \frac{\epsilon \tilde{d}}{| \ln \epsilon |^{\frac{1}{2}}} \sum_{\kappa \in \{+, -\}} (t^\kappa)^{-1} \int_{B_{\kappa}(0)} | w_{x_1}^\kappa |^2 + O \left( \frac{\epsilon}{| \ln \epsilon |^{\frac{1}{2}}} \right)
= \hat{d} | \ln \epsilon |^{\frac{1}{2}} \sum_{\kappa \in \{+, -\}} (t^\kappa)^{-1} \int_0^{2\pi} \int_0^{\frac{1}{2}} \frac{(W^\pm)^2 \sin^2 \theta}{r} \, dr \, d\theta + O \left( \frac{\epsilon}{| \ln \epsilon |^{\frac{1}{2}}} \right)
= (t^+ + t^-) \hat{d} | \ln \epsilon |^{\frac{1}{2}} \pi (1 + o_\epsilon (1)).
\]
Here we have used the facts
\[ W^\pm = t^\pm + O\left(\frac{1}{r^2}\right), \quad \text{as } r \to +\infty. \]

On the other hand, for \( T_0 \), by the expression of \( S^\pm_\delta(v_d) \) in (3.5), we have
\[
T_0 = 2 \sum_{\kappa \in \{+, -\}} (t^\kappa)^{-1} \text{Re} \int_{\{t_1 \leq R_\epsilon \}} \left[ \frac{\nabla w^\kappa_b \nabla w^\kappa_b}{w^\kappa_b} (z + d) \right] w^\kappa_{x_1} + O\left(\frac{\epsilon}{\ln \epsilon}\right)
\]
\[
= 2 \sum_{\kappa \in \{+, -\}} (t^\kappa)^{-1} \text{Re} \int_{\{t_1 \leq R_\epsilon \}} \left[ \frac{\partial x_1 w^\kappa_b \partial x_1 w^\kappa_b + \partial x_2 w^\kappa_b \partial x_2 w^\kappa_b}{w^\kappa_b} \right] w^\kappa_{a,x_1} + O\left(\frac{\epsilon}{\ln \epsilon}\right).
\]

Here we have used the notation
\[ w^\pm (z - d) = w^+_a(z), \quad w^\pm (z + d) = w^+_b(z). \]

Similar to the calculations in [12], by the properties of \( w^\pm \) in Lemma [12] we can easily get that
\[ 2 \sum_{\kappa \in \{+, -\}} \text{Re} \int_{\{t_1 \leq R_\epsilon \}} \frac{\partial x_1 w^\kappa_b \partial x_1 w^\kappa_b}{w^\kappa_b} w^\kappa_{x_1} = O\left(\epsilon^2 |\ln \epsilon|\right). \]

Finally we consider the term
\[
2 \sum_{\kappa \in \{+, -\}} \text{Re} \int_{\{t_1 \leq R_\epsilon \}} \frac{\partial x_2 w^\kappa_b \partial x_2 w^\kappa_b}{w^\kappa_b} w^\kappa_{x_1}
\]
\[
= -2 \sum_{\kappa \in \{+, -\}} \int_{\{t_1 \leq R_\epsilon \}} \frac{W^\pm(t_1) W^\pm(t_1)}{t_2} \cos \theta_2 \, d\ell_1 \, d\theta_1 + O\left(\epsilon^2 |\ln \epsilon|\right).
\]

Note that
\[ \frac{\cos \theta_2}{t_2} = \frac{1}{2d} \left(1 + o_\epsilon(1)\right), \]
and
\[ 2 \int_{\{t_1 \leq R_\epsilon \}} W^\pm(t_1) W^\pm(t_1) \, d\ell_1 \, d\theta_1 = 2\pi \times \frac{1}{2} \left(|W^\pm(R_\epsilon)|^2 - |W^\pm(0)|^2\right)
\]
\[ = \pi \left(t^\pm + o_\epsilon(1)\right). \]

The above estimates will give
\[ T_0 = -(t^+ + t^-) \frac{|\ln \epsilon|^{\frac{1}{2}} \pi}{d} \left(1 + o_\epsilon(1)\right). \]

Therefore, by collecting the estimates of \( T_0 \) and \( T_1 \), we obtain
\[
\sum_{\kappa \in \{+, -\}} \text{Re} \int_{B_R(0)} \left( \frac{w^\kappa_a}{w^\kappa_b} (z) w^\kappa_b (z + d) \right) = (t^+ + t^-) \left[ \pi \hat{d} \epsilon |\ln \epsilon|^{\frac{1}{2}} - \frac{\epsilon |\ln \epsilon|^{\frac{1}{2}} \pi}{d} \right] \left(1 + o_\epsilon(1)\right). \quad (7.6)
\]

In conclusion, the substitutions of (7.4)–(7.6) into (7.2) will imply that: \( c = 0 \) if and only if
\[
\left[ \pi \hat{d} \epsilon |\ln \epsilon|^{\frac{1}{2}} - \frac{\epsilon |\ln \epsilon|^{\frac{1}{2}} \pi}{d} \right] \left(1 + o_\epsilon(1)\right) = 0, \quad (7.7)
\]
where the functions in \( o_\epsilon(1) \) are continuous with respect to the parameter \( \hat{d} \) since \( \psi_\epsilon \) is continuous on \( \hat{d} \).

Therefore we obtain that there exists \( \hat{d} = 1 + o(1) \) such that (7.7) holds. By combining the arguments in Sections [214], we complete the proof of Theorem [1.4].
8. Appendix

8.1. The linear operator of Ginzburg-Landau system in $\mathbb{R}^2$ around the standard vortex $w$. The following non-degeneracy results will be used in the proof of Lemma 5.1 and Lemma 5.2. We first recall the operator $L_0(\phi) = (L_0^+, L_0^-)(\phi)$, with

$$L_0^\pm(\phi) = \Delta \phi^\pm + \left[A_\pm (t^\pm - W^\pm) + B(t^\mp - W^\mp)\right] \phi^\pm - 2A_\pm \text{Re}(w^\pm \overline{\phi}^\pm) w^\pm - 2B \text{Re}(w^\mp \overline{\phi}^\mp) w^\mp.$$

**Lemma 8.1.** For any $\phi = (\phi^+, \phi^-) \in L^\infty(\mathbb{R}^2, \mathbb{C}^2)$ satisfying the symmetry $\phi(\overline{z}) = \overline{\phi(z)}$, suppose that

$$|\psi_1| + (1 + |z|)|\nabla \psi_1| \leq C, \quad |\psi_2| + |\nabla \psi_2| \leq \frac{C}{1 + |z|}, \quad \text{for } |z| > 1,$$

with $\phi^\pm = iw^\pm \psi^\pm$ and $\psi^\pm = \psi_1^\pm + iw_2^\pm$ with $\psi_1^\pm, \psi_2^\pm \in \mathbb{R}$. Then $L_0(\phi) = 0$ if and only if

$$\phi = c_1 u_{x_1}$$

for some real constant $c_1$.

**Proof.** Taking $\phi = (iw^+ \psi^+, iw^- \psi^-)$ into $L_0(\phi) = 0$, we can get

$$\Delta \psi^\pm + 2\nabla w^\pm \nabla \psi^\pm - 2iA_\pm |w^\pm|^2 \psi^\pm - 2iB |w^\mp|^2 \psi^\mp = 0, \quad \text{in } B(0, 1)^c.$$

This reads in the coordinates $(r, \theta)$

$$0 = \Delta \psi_1^\pm + \frac{2W^\pm'}{W^\pm} \partial_r \psi_1^\pm + \frac{2}{r^2} \partial_\theta \psi_2^\pm \quad \text{in } B(0, 1)^c, \quad (8.3)$$

$$0 = \Delta \psi_2^\pm + \frac{2W^\pm'}{W^\pm} \partial_r \psi_2^\pm + \frac{2}{r^2} \partial_\theta \psi_1^\pm - 2A_\pm t^\pm \psi_2^\pm - 2B t^\mp \psi_2^\mp \quad \text{in } B(0, 1)^c. \quad (8.4)$$

Rewrite (8.4) as

$$0 = \Delta \psi_2^\pm + \frac{2W^\pm'}{W^\pm} \partial_r \psi_2^\pm + \frac{2}{r^2} \partial_\theta \psi_1^\pm - 2A_\pm t^\pm \psi_2^\pm - 2B t^\mp \psi_2^\mp$$

$$- 2A_\pm ((W^\pm)^2 - t^\pm) \psi_2^\pm - 2B ((W^\mp)^2 - t^\mp) \psi_2^\mp \quad \text{in } B(0, 1)^c.$$

Recall (8.8)

$$C^T C = I, \quad 2CM C^T = \text{diag}(\lambda^+, \lambda^-) \quad \text{with } \lambda^+ > 0,$$

and define new vector functions

$$\tilde{\psi}_j = (\tilde{\psi}_j^+, \tilde{\psi}_j^-)^T \equiv C^T (\psi_j^+, \psi_j^-)^T, \quad \text{for } j = 1, 2,$$

$$\mathcal{W}_1 = (\mathcal{W}_1^+, \mathcal{W}_1^-) = C^T \left(\frac{2W^+/W^+}{W^+} \partial_r \psi_1^+, \frac{2W^/-W^-}{W^-} \partial_r \psi_1^-\right),$$

$$\mathcal{W}_2 = (\mathcal{W}_2^+, \mathcal{W}_2^-) = C^T \left(\frac{2W^+/W^+}{W^+} \partial_r \psi_2^+, \frac{2W^/-W^-}{W^-} \partial_r \psi_2^-\right),$$

$$\mathcal{A} = (\mathcal{A}^+, \mathcal{A}^-) := -2C^T \left(A_+ ((W^+)^2 - t^+)^2 \psi_2^+ - A_-(W^-)^2 - t^-)^2 \psi_2^+\right),$$

and

$$\mathcal{B} = (\mathcal{B}^+, \mathcal{B}^-) := -2C^T \left(B((W^-)^2 - t^-)^2 \psi_2^- - B(W^+)^2 - t^+)^2 \psi_2^+\right).$$

Then the system (8.3)-(8.4) can be transformed to

$$0 = \Delta \tilde{\psi}_1 + \mathcal{W}_1 + \frac{2}{r^2} \partial_\theta \tilde{\psi}_2 \quad \text{in } B(0, 1)^c, \quad (8.5)$$

$$0 = \Delta \tilde{\psi}_2 + \frac{2}{r^2} \partial_\theta \tilde{\psi}_1 - \text{diag}(\lambda^+, \lambda^-) \tilde{\psi}_2 + \mathcal{A} + \mathcal{B} \quad \text{in } B(0, 1)^c. \quad (8.6)$$
Using the estimate (8.1) and the asymptotic for $W^\pm$, we have
\[\left| -\frac{2}{r^2} \partial_\theta \hat{v}_1 + \mathcal{W}_2 + \mathcal{A} + \mathcal{B} \right| \leq \frac{C}{1 + r^2} \text{ in } B(0, 1)^c.\]

Then by a barrier argument and elliptic estimates on (8.6), we can get
\[|\hat{v}_1| + |\nabla \hat{v}_1| \leq \frac{C}{1 + r^2}, \quad (8.7)\]
which yields that
\[|\hat{v}_2| + |\nabla \hat{v}_2| \leq \frac{C}{1 + r^2}. \quad (8.8)\]

On the other hand, considering the equation (8.5), together with the barrier argument and elliptic estimates, we obtain
\[|\hat{v}_1| + (1 + |z|)|\nabla \hat{v}_1| \leq \frac{C}{(1 + |z|)^\alpha}, \quad |\psi_1| + |\nabla \psi_1| \leq \frac{C}{(1 + |z|)^\alpha}, \quad |\psi_2| + |\nabla \psi_2| \leq \frac{C}{1 + r^2}, \quad (8.8)\]
for any $\alpha \in (0, 1)$. Since $C$ is invertible, we can establish the estimates for $\psi_1, \psi_2$
\[|\psi_1| + (1 + |z|)|\nabla \psi_1| \leq \frac{C}{(1 + |z|)^\alpha}, \quad (8.8)\]

By (8.8) and the decay of $W^\pm, W^\pm'$, we conclude
\[
\begin{align*}
\int_{\mathbb{R}^2} (|\nabla \phi^+|^2 + |\nabla \phi^-|^2) &+ \int_{\mathbb{R}^2} \left[ A_+ (t^{+2} - W^+ + 2) - B (t^{-2} - W^- + 2) \right] |\phi^+|^2 \\
&+ \int_{\mathbb{R}^2} \left[ A_+ (t^{-2} - W^- + 2) - B (t^{+2} - W^+ + 2) \right] |\phi^-|^2 < +\infty.
\end{align*}
\]

Then Lemma 1.3 implies that
\[\phi = c_1 w_{x_1} + c_2 w_{x_2}, \quad \text{for some constant } c_1, c_2.\]
Using the symmetry $\phi(z) = \overline{\phi(z)}$, we conclude that (8.2) holds.

\[\square\]

**Lemma 8.2.** For any $\phi \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ satisfying the symmetry $\phi(z) = \overline{\phi(z)}$, suppose that
\[|\psi_1| + (1 + |z|)|\nabla \psi_1| \leq C(1 + |z|)^\alpha, \quad |\psi_2| + |\nabla \psi_2| \leq \frac{C}{1 + |z|} \quad \text{for } |z| > 1, \quad (8.2)\]
for some $\alpha < 3$, where $\phi = (iw^+ \psi^+, iw^- \psi^-)$ and $\psi^\pm = \psi_1^\pm + iw_{x_2}^\pm$ with $\psi_1^\pm, \psi_2^\pm \in \mathbb{R}$. Then $L_0(\phi) = 0$ if and only if
\[\phi = c_1 w_{x_1} \quad (8.2)\]
for some real constant $c_1$.

**Proof.** Using Lemma 8.1 some Fourier analysis and ODE theory, we can prove this lemma. The proof is similar as that in Lemma 7.3 in [12]. We omit it here for conciseness.

\[\square\]

### 8.2. Elliptic estimates used in the linear theory.
Recall the polar coordinate notation $z = re^{is}, r > 0, s \in \mathbb{R}$. For the convenience of the reader, we provide some useful lemmas from [12].

**Lemma 8.3.** ([12]) Let $u : \mathbb{R} \times \mathbb{R}^*_+ \to \mathbb{R}$ be a bounded function which is in $C^2(\mathbb{R} \times \mathbb{R}^*_+) \cap C^0(\mathbb{R} \times \mathbb{R}^*_+)$ and satisfy
\[
\begin{cases}
\Delta u + \epsilon^2 \partial^2_{s^2} u &\geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^*_+, \\
u &\leq 0 \text{ on } \mathbb{R} \times \{0\}.
\end{cases}
\]
Then $u \leq 0$ in $\mathbb{R} \times \mathbb{R}^*_+$. \[\square\]
Lemma 8.4. (12) Let \( u : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \) be a bounded function which is in \( C^2(\mathbb{R} \times \mathbb{R}^+) \cap C^1(\mathbb{R} \times \mathbb{R}^+) \) and \( c \geq 0 \). If \( u \) satisfies
\[
\begin{aligned}
\Delta u + \epsilon^2 \partial^2_{ss} u - cu & \geq 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+ , \\
\partial_
u u & \leq 0 \quad \text{on} \quad \mathbb{R} \times \{0\},
\end{aligned}
\]
then \( u \leq 0 \) in \( \mathbb{R} \times \mathbb{R}^+ \).
\( \Box \)

Let \( R_0 > 0 \) be fixed with \( R_0 < R_c < \epsilon^{-1} \), and \( \Omega, \Omega' \) be two regions, respectively:
\[
\Omega = \{ z \in \mathbb{R}^2 \mid R_0 < |z| < R_c \}, \quad \Omega' = \{ z \in \mathbb{R}^2 \mid 2R_0 < |z| < \frac{1}{2}R_c \}.
\]

We adopt the following lemma presented in (12).

Lemma 8.5. (12) Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be such that \( f(z) = -f(z) \) and \( |f(z)| \leq \frac{1}{16} \). Let \( u \) be a solution of
\[
\Delta u + \epsilon^2 \partial^2_{ss} u = f \quad \text{in} \quad \Omega,
\]
such that \( u(\bar{z}) = -u(z) \) and
\[
|u(z)| \leq R_0 \ln \epsilon, \quad \forall z \text{ with } |z| = R_0,
\]
\[
|u(z)| \leq R_c, \quad \forall z \text{ with } |z| = R_c.
\]

Then there is a constant \( C \) such that
\[
|u(z)| \leq C|z| \log \left( \frac{2R_c}{|z|} \right), \quad \forall z \in \Omega'.
\]
\( \Box \)

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