On the Alberti-Uhlmann Condition for Unital Channels

Sagnik Chakraborty, 1, * Dariusz Chruściński, 1, † Gniewomir Sarbicki, 1, ‡ and Frederik vom Ende 2, 3, §

1 Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziądzka 5/7, 87aA$100 Toruń, Poland
2 Department of Chemistry, Technische Universität München, 85747 Garching, Germany
3 Munich Centre for Quantum Science and Technology (MCQST), Schellingstr. 4, 80799 München, Germany

We address the problem of existence of completely positive trace preserving (CPTP) maps between two sets of density matrices. We refine the result of Alberti and Uhlmann and derive a necessary and sufficient condition for the existence of a unital channel between two pairs of qubit states which ultimately boils down to three simple inequalities.

I. INTRODUCTION

A fundamental aspect of resource theories is finding conditions which characterize the possibility of state-conversion via “allowed” operations. In quantum thermodynamics, for example, one usually asks whether a state \( \tau \) can be generated from an initial state \( \rho \) via a Gibbs-preserving channel, that is, a completely positive and trace-preserving (CPTP) map which leaves the Gibbs-state of the system invariant. \([5, 11, 12, 16, 23, 25]\)

Recall that a linear map \( T : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) is completely positive if its amplification \( id \otimes T : M_n(M_m(\mathbb{C})) \rightarrow M_n(M_m(\mathbb{C})) \) is positive \([20, 22]\).

The above is just a special case of the following more general question: given two pairs of quantum states \( \{\tau_1, \tau_2\}, \{\rho_1, \rho_2\} \) can one find a characterization (which is non-trivial and possibly simple to verify) for the existence of a quantum channel which transforms both \( \rho_j \) into \( \tau_j \)? While this problem is fully solved in the classical case (for an overview on the equivalent conditions see \([12, \text{Ch. V}] \) or \([24, \text{Prop. 4.2}] \), in the quantum realm this remains unanswered, with the qubit case being the notable exception: In a seminal paper \([1]\) Alberti and Uhlmann derived a necessary and sufficient condition for such a simultaneous state conversion. The Alberti-Uhlmann condition reads

\[
\|\rho_1 - t\rho_2\|_1 \geq \|\tau_1 - t\tau_2\|_1 \quad \forall t \geq 0, \tag{1}
\]

where \( \|A\|_1 = \text{tr} \sqrt{A^*A} \) denotes the trace-norm \([19]\). Note that this equation is equivalent to

\[
\|p_1\rho_1 - p_2\rho_2\|_1 \geq \|p_1\tau_1 - p_2\tau_2\|_1, \tag{2}
\]

for all \( p_1, p_2 \geq 0 \) and \( p_1 + p_2 = 1 \). Due to Helstrom theory \([14]\) condition \(2\) means that distinguishability of states \( \{\rho_1, \rho_2\} \) given with probabilities \( \{p_1, p_2\} \) is not less than distinguishability of states \( \{\tau_1, \tau_2\} \) (occurring with the same probabilities). For the sake of computation \(1\) can be reduced to finitely many inequalities via the formulæ

\[
\|A\|^2 = \text{tr}(A^*A) + 2|\text{det}(A)|
\]

\[
\text{det}(A + B) = \text{det}(A) + \text{det}(B) + \text{tr}(A^\#B) \tag{3}
\]

for all \( A, B \in M_2(\mathbb{C}) \) where \( (\cdot)^\# \) denotes the adjugate \([3]\): then the function resulting from \(1\) is piecewise quadratic in \( t \) so non-negativity reduces to certain conditions on the coefficients (with respect to \( t \)).

It is well known that every positive trace-preserving map (PTP) is a contraction in the trace-norm and hence Alberti-Uhlmann is necessary for the existence of a PTP map \( T \). Interestingly, this condition is sufficient in two dimensions—even for the existence of a quantum channel—and fails to be sufficient for dimension three and larger \([13, \text{Ch. VII.B}]\).

In \([9]\), Chefles et al. generalized the problem to input and output sets \( \{\rho_1, \ldots, \rho_n\} \) and \( \{\tau_1, \ldots, \tau_n\} \), respectively, with arbitrary dimension and arbitrary value of \( n \), under the constraint that at least one of the two sets must be a set of pure states. They derived conditions for the existence of a CPTP map between the sets in terms of the Gram matrices of the two sets. A result for arbitrary (non-pure) states was derived by Huang et al. \([17]\) where their characterization (of existence of a CPTP map) goes via the existence of some more abstract decomposition of the initial and target states.

More interestingly, they considered the case of qubit states \( \{\rho_1, \rho_2, \rho_3\}, \{\tau_1, \tau_2, \tau_3\} \) under the generic assumption of the input states being pure (cf. footnote \([26]\)). However the characterization they derived, while verifiable with standard software, seems to not generalize to a condition for arbitrary input states and, moreover, seems to not lead to much physical insight.

In \([13]\), Heinosuuri et al. considered a slightly different variant of the problem and studied the conditions for existence of only CP maps between two sets of quantum states. Moreover they gave a fidelity characterization for the existence of a CPTP transformation on pairs of qubit states, consisting of only a finite number of conditions.

In a more recent paper \([11]\), Dall’Arno et al. derived a condition for the existence of a CPTP map when the input set is a collection of qubit states which can be, through a simultaneous unitary rotation, written as real matrices. They study the problem from the perspective

* Electronic address: sagnik@fizyka.umk.pl
† Electronic address: darch@fizyka.umk.pl
‡ Electronic address: gniewko@fizyka.umk.pl
§ Electronic address: frederik.vom-ende@tum.de
of quantum statistical comparison and derive that if the testing region of the input real states include the testing region of the output states then there exists a CPTP map connecting them. For further analysis on the relation of this problem with quantum statistical comparison see [6, 7].

In this paper we refine the original Alberti-Uhlmann problem asking about the existence of a unital channel, that is, $T$ maps $\rho_k$ to $\tau_k$ and additionally $T(1) = 1$. Original condition (1) guarantees the existence of a CPTP map but does say nothing whether $T$ is unital. Clearly, condition (1) is again necessary but no longer sufficient. Note, that the map $T$ is uniquely defined only on an at most 3-dimensional subspace $\mathcal{M}$ spanned by $\{1, \rho_1, \rho_2\}$ and one asks whether this map can be extended to the whole algebra $M_2(\mathbb{C})$ such that the extended map is CPTP and unital. Extension problems such as this one were already considered by many authors before [20]. The classical result of Arveson [4] says that if $\mathcal{M}$ is an operator system in $B(\mathcal{H})$, that is, $\mathcal{M}$ is a linear subspace closed under hermitian conjugation and containing $1$, and if $\Phi: \mathcal{M} \to B(\mathcal{H})$ is completely positive unital map, then it can be extended to a unital completely positive map $\hat{\Phi}: B(\mathcal{H}) \to B(\mathcal{H})$. Note, however, that this result says nothing about trace-preservation. Hence, even if the unital map $\Phi$ is trace-preserving the unital extension $\hat{\Phi}$ need not be trace-preserving. Actually, unitality may be relaxed by assuming that the hermitian subspace $\mathcal{M}$ contains a strictly positive operator [13]. Interestingly, it was shown [18] that if $\mathcal{M}$ is spanned by positive operators and $\Phi$ is completely positive then there exists a completely positive extension $\hat{\Phi}$.

The main result of this paper reads as follows.

**Theorem 1.** Let qubit states $\{\rho_1, \rho_2\}$, $\{\tau_1, \tau_2\}$ be given. The following statements are equivalent.

(i) There exists a unital quantum channel mapping $\{\rho_1, \rho_2\}$ into $\{\tau_1, \tau_2\}$.

(ii) For all $\alpha, \beta, \gamma \in \mathbb{R}$

$$\|\alpha 1 + \beta \tau_1 + \gamma \tau_2\|_1 \leq \|\alpha 1 + \beta \rho_1 + \gamma \rho_2\|_1$$

(iii) For all $\beta, \gamma \in \mathbb{R}$

$$\|\frac{1}{\beta} + \frac{\gamma}{\beta} \tau_1 + \gamma \tau_2\|_1 \leq \|\frac{1}{\beta} \rho_1 + \gamma \rho_2\|_1$$

(iv) For all $t \in \mathbb{R}$ one has $\tau_1 - t \tau_2 \prec \rho_1 - t \rho_2$.

Here $\prec$ denotes classical matrix majorization which is usually defined via the comparison of eigenvalues and is well-known to be equivalent to the existence of a unital CPTP map which maps the right to the left-hand side (refer to, e.g., [2, Ch. 7]).

Clearly, the original Alberti-Uhlmann condition (1) provides now only the necessary condition corresponding to $\alpha = 0$ in (4). This condition readily serves as the necessary condition for existence of the unital channel as the trace-norm is contractive under the action of any PTP map [21]. While the conditions in Theorem 1 give conceptional insight one can also reduce the problem three easy-to-verify conditions.

**Theorem 2.** There exists a unital quantum channel mapping qubit states $\{\rho_1, \rho_2\}$ into $\{\tau_1, \tau_2\}$ if and only if

$$\det(\rho_j) \geq \det(\rho_i)$$

for $j = 1, 2$ as well as

$$\left(\text{tr}(\rho_j^\#) - \text{tr}(\tau_j^\#)\right)^2 \leq 4\left(\text{det}(\tau_1) - \text{det}(\rho_1)\right)\left(\text{det}(\tau_2) - \text{det}(\rho_2)\right)$$

where $(\cdot)^\#$ denotes the adjugate [15, Ch. 0.8], [27].

**II. PROOF OF ALBERTI-UHLMANN CONDITIONS FOR UNITAL MAPS**

Let $\mathcal{M} := \text{span}\{1, \rho_1, \rho_2\}$ and $\mathcal{N} := \text{span}\{1, \tau_1, \tau_2\}$. The linear map $T: \mathcal{M} \to \mathcal{N}$ mapping $\rho_k$ to $\tau_k$ by construction is unital and preserves trace and hermiticity.

**Proposition 1.** There exist unitary $U, V \in M_2(\mathbb{C})$ such that the following statements hold.

(i) $U^\dagger \rho_1 U, U^\dagger \rho_2 U, V^\dagger \tau_1 V, V^\dagger \tau_2 V \perp \sigma_z$, that is, $U^\dagger M U, V^\dagger N V \subseteq \text{span}\{1, \sigma_x, \sigma_y\}$.

(ii) The adjusted map

$$S: \text{span}\{1, \sigma_x, \sigma_y\} \to \text{span}\{1, \sigma_x, \sigma_y\}$$

$$\rho \mapsto V^\dagger T(U \rho U^\dagger)V$$

is well-defined, linear, unitary (hence trace-preserving) and satisfies $S(\sigma_x) = a \sigma_x$ and $S(\sigma_y) = b \sigma_y$ for some $a, b \in \mathbb{R}$.

**Proof.** Since $\mathcal{M}$ is an (at most) 3-dimensional subspace of $L(\mathcal{H}_2)$ there exists an operator orthogonal to $\mathcal{M}$ w.r.t. the Hilbert-Schmidt inner product. Being orthogonal to $1$ it is traceless and hence choosing an appropriate basis it equals $\sigma_z$. Analogously, choosing an appropriate basis in the output Hilbert space makes $\mathcal{N}$ orthogonal to $\sigma_z$. More precisely this yields $U, V \in M_2(\mathbb{C})$ unitary such that (i) holds and $S(\rho) := V^\dagger T(U \rho U^\dagger)V$—because it preserves hermiticity and the trace—satisfies

$$S(\sigma_x) = \begin{bmatrix} 0 & z_x & 0 \\ \frac{z_x}{z_y} & 0 & 0 \\ 0 & \frac{z_y}{z_x} & 0 \end{bmatrix},$$

$$S(\sigma_y) = \begin{bmatrix} 0 & z_y & 0 \\ \frac{z_y}{z_x} & 0 & 0 \\ 0 & \frac{z_x}{z_y} & 0 \end{bmatrix},$$

with $z_x, z_y \in \mathbb{C}$ so $S$ acts on $(1, \sigma_x, \sigma_y) \in (M_2(\mathbb{C}))^3$ as

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix}$$

for some $A = (a_{jk})^2_{j,k=1} \in \mathbb{R}^{2 \times 2}$. Applying the singular value decomposition to $A$ one finds orthogonal matrices $W_1, W_2 \in M_2(\mathbb{R})$ such that $W_1^\dagger A W_2 = \text{diag}(a, b)$.
with $a, b \geq 0$ [15, Thm. 7.3.5]. But every orthogonal $2 \times 2$ matrix is a rotation matrix (possibly up to a composition with $\sigma_z$) [15, p. 68] so because the channel $S_\phi(\rho) := U'_\phi \rho U_\phi$ with $U_\phi$ = diag$(1, e^{i\phi})$, $\phi \in \mathbb{R}$ leaves span$\{1, \sigma_x, \sigma_y\}$ invariant and acts on $(1, \sigma_x, \sigma_y) \in (M_2(\mathbb{C}))^3$ as

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{bmatrix},
\]

there exist $\phi, \theta \in \mathbb{R}$ such that $S_\phi \circ \tilde{S} \circ S_{-\theta} = \text{diag}(1, a, b)$ or $\text{diag}(1, a, -b)$. Therefore the matrices we were looking for are $U = \hat{U} U_\theta$, $V = \hat{V} V_\phi$ which concludes the proof. \(\square\)

In other words Proposition 1 guarantees the existence of unitary channels which rotate both domain and codomain of $T$ into the subspace span$\{1, \sigma_x, \sigma_y\}$ and, at the same time, diagonalize $T$. From the latter one has the obvious inference:

**Proposition 2.** The map $S : \text{span}\{1, \sigma_x, \sigma_y\} \to L(H_2)$ is positive iff $|a| \leq 1$ and $|b| \leq 1$.

In abuse of notation we henceforth write $\mathcal{M}$ for span$\{1, \sigma_x, \sigma_y\}$.

Now, to extend $S$ from the 3-dim. subspace $\mathcal{M}$ to the full space $L(H_2)$ one needs the action of $S$ on $\sigma_z$. Due to hermiticity and the trace preservation condition, one has in general

\[
S(\sigma_z) = x \sigma_x + y \sigma_y + z \sigma_z,
\]

with $x, y, z \in \mathbb{R}$.

**Theorem 3.** There exists a positive trace-preserving extension of $S : \mathcal{M} \to L(H_2)$ iff $|a| \leq 1$ and $|b| \leq 1$.

**Proof.** The map $S : \mathcal{M} \to L(H_2)$ maps a density matrix represented by a Bloch vector $\vec{r} = (r_x, r_y, r_z)$ to

\[
S(1 + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z) = 1 + ar_x \sigma_x + br_y \sigma_y + r_z (x \sigma_x + y \sigma_y + z \sigma_z).
\]

Therefore the action on the Bloch vector is realized as $\vec{r} \to M \vec{r}$, where

\[
M = \begin{bmatrix}
a & 0 & x \\
0 & b & y \\
0 & 0 & z
\end{bmatrix}.
\]

The map $S$ is positive iff it maps Bloch ball to Bloch ball, that is, if the matrix $M$ is contraction. Its supremum norm reads

\[
\|M\|_{\text{sup}} \geq \max \left\{ \left\| \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix} \right\|_{\text{sup}}, |z| \right\} = \max\{|a|, |b|, |z|\},
\]

and the inequality is saturated for $x = y = 0$. Hence, whenever $|a| \leq 1$ and $|b| \leq 1$, there exist $x, y, z$ defining the extension (7).

**Remark 1.** Note, that the simplest extension corresponds to $S(\sigma_z) = 0$, i.e. $x = y = z = 0$. Actually, it was shown [8] that if $\mathcal{M}$ is a 3-dim. operator system, then there exists unital positive trace-preserving projector $\Pi : L(H_2) \to \mathcal{M}$. Hence, $S \circ \Pi$ defines unital positive trace-preserving extension of $S : \mathcal{M} \to L(H_2)$. Interestingly, it was proved [8] that there is no completely positive trace-preserving projector $\Pi : L(H_2) \to \mathcal{M}$.

**Theorem 4.** There exists a completely positive trace-preserving extension of $S : \mathcal{M} \to L(H_2)$ iff $|a| \leq 1$ and $|b| \leq 1$.

**Proof.** One has

\[
S(0\langle 0|) = S\left(\frac{1}{2}(1 + \sigma_z)\right) = \frac{1}{2}(1 + x \sigma_x + y \sigma_y + z \sigma_z)
\]

(11)

\[
S(\langle 0| 1\rangle) = S\left(\frac{1}{2}(\sigma_x + i \sigma_y)\right) = \frac{1}{2}(a \sigma_x + ib \sigma_y)
\]

(12)

\[
S(\langle 1| 0\rangle) = S\left(\frac{1}{2}(\sigma_x - i \sigma_y)\right) = \frac{1}{2}(a \sigma_x - ib \sigma_y)
\]

(13)

\[
S(1\langle 1|) = S\left(\frac{1}{2}(1 - \sigma_z)\right) = \frac{1}{2}(1 - x \sigma_x - y \sigma_y - z \sigma_z)
\]

(14)

and hence the corresponding Choi matrix

\[
C = \sum_{i,j=1}^{2} |i\rangle\langle j| \otimes S(|i\rangle\langle j|),
\]

reads

\[
C = \frac{1}{2} \begin{bmatrix}
1 + z & x - iy & 0 & a + b \\
x + iy & 1 - z & a - b & 0 \\
0 & a - b & 1 - z & -x + iy \\
a + b & 0 & -x - iy & 1 + z
\end{bmatrix}.
\]

(15)

The extended map $S$ is completely positive iff $C \succeq 0$ [10]. A necessary condition for its positivity will be given by positivity of its two main minors:

\[
\begin{bmatrix}
1 - z & a - b \\
a - b & 1 - z
\end{bmatrix} \succeq 0,
\]

(16)

\[
\begin{bmatrix}
1 + z & a + b \\
a + b & 1 + z
\end{bmatrix} \succeq 0,
\]

which is equivalent to the following conditions for $z$:

\[
|a + b| - 1 \leq z \leq 1 - |a - b|.
\]

(17)

Clearly, there exists nontrivial solution for $z$ iff

\[
2 \geq |a - b| + |a + b| = 2 \max\{|a|, |b|\}.
\]

(18)

Observe, that for $x, y = 0$ the matrix (15) becomes a direct sum of blocks (16), hence the necessary condition becomes also sufficient. \(\square\)

**Corollary 1.** A completely positive trace extension exists if and only if a positive extension exists.
Now, we are ready to prove the main theorem.

**Proof of Thm. 1.** (i) $\Rightarrow$ (ii) was shown before. (ii) $\Rightarrow$ (i): Assume w.l.o.g. $\dim(M) = 3$ (dim$(M) = 1$ is trivial and dim$(M) = 2$ can be traced back to original Alberti-Uhlmann). Let us observe that the very condition (4) is equivalent to

$$\|\alpha \mathbb{I} + \beta \alpha \sigma_x + \gamma \beta \sigma_y\|_1 \leq \|\alpha \mathbb{I} + \beta \alpha \sigma_x + \gamma \beta \sigma_y\|_1$$

(19)

for all $\alpha, \beta, \gamma \in \mathbb{R}$. Indeed, assume (4), i.e. for all $\alpha, \beta, \gamma$

$$\|\alpha \mathbb{I} + \beta \mathbb{V}^\dagger \tau_1 \mathbb{V} + \gamma \mathbb{V}^\dagger \tau_2 \mathbb{V}\|_1 = \|\alpha \mathbb{I} + \beta \mathbb{\alpha}_1 + \gamma \mathbb{\alpha}_2\|_1$$

$$\leq \|\alpha \mathbb{I} + \beta \mathbb{\rho}_1 + \gamma \mathbb{\rho}_2\|_1$$

$$= \|\alpha \mathbb{I} + \beta \mathbb{\rho}_1 \mathbb{U} + \gamma \mathbb{\rho}_2 \mathbb{U}\|_1$$

(19)

with $U, V$ being the unitary matrices from Prop. 1, using unitary equivalence of the trace norm. Writing

$$U^\dagger \rho_j U = \frac{1}{2} + r_{xj} \sigma_x + r_{yj} \sigma_y$$

for some $r_{xj}, r_{yj} \in \mathbb{R}$ yields

$$V^\dagger \tau_j V = \frac{1}{2} + a r_{xj} \sigma_x + b r_{yj} \sigma_y .$$

Thus the trace norm condition (4) becomes:

$$\|\alpha^{\prime} \mathbb{I} + \beta^{\prime} \alpha \sigma_x + \gamma^{\prime} \beta \sigma_y\|_1 \leq \|\alpha^{\prime} \mathbb{I} + \beta^{\prime} \alpha \sigma_x + \gamma^{\prime} \beta \sigma_y\|_1 ,$$

(20)

where

$$\begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & r_{1x} & r_{2x} \\ 0 & r_{1y} & r_{2y} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} .$$

Now, observing that

$$\det \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & r_{1x} & r_{2x} \\ 0 & r_{1y} & r_{2y} \end{bmatrix} = \det \begin{bmatrix} r_{1x} & r_{2x} \\ r_{1y} & r_{2y} \end{bmatrix} \neq 0$$

because dim$(M) = 3$. Hence condition (4) for all $\alpha, \beta, \gamma$ is equivalent to (20) for all $\alpha', \beta', \gamma'$, that is, it is equivalent to (19) for all $\alpha, \beta, \gamma$.

Now, the last step is to prove (19). Choosing $(\alpha, \beta, \gamma) = (0, 1, 0)$ and $(\alpha, \beta, \gamma) = (0, 0, 1)$ in (19) implies $|a| \leq 1$ and $|b| \leq 1$ so there exists a CPTP extension of $S$ by Theorem 4. Finally $S$ and $T$ differ by unitary channels so if one of them has a P or CP extension then this is true for the other, as well.

(ii) $\Rightarrow$ (iii) is readily verified using homogeneity and continuity of the norm.

(iv) $\Rightarrow$ (iii): By [2, Thm. 7.1] and (1) condition (iv) is equivalent to $\|\frac{1}{2} - s(\tau_1 - t\tau_2)\|_1 \leq \|\frac{1}{2} - s(\rho_1 - t\rho_2)\|_1$ for all $s, t \in \mathbb{R}$ which by the same argument as before is equivalent to (iii).

Condition (iii) from Theorem 1 can be geometrically interpreted as the linear subspace span$\{\tau_1, \tau_2\}$ being to be closer to the maximally mixed state $\frac{I}{2}$ than span$\{\rho_1, \rho_2\}$.

**Proof of Thm. 2.** From the eigenvalue formula for $2 \times 2$ matrices it is easy to see that if $A, B \in M_2(\mathbb{C})$ are hermitian and of same trace then $A < B$ if det$(A) \geq$ det$(B)$. Using (3) (iv) is equivalent to

$$t^2(\text{det}(\tau_2) - \text{det}(\rho_2)) - t(\text{tr}(\rho_1^\# \rho_2) - \text{tr}(\tau_1^\# \tau_2)) + (\text{det}(\tau_1) - \text{det}(\rho_1)) \geq 0$$

for all $t \in \mathbb{R}$. But a parabola $t \mapsto at^2 + bt + c$ is non-negative iff $a, c \geq 0$ and $b^2 \leq 4ac$ which concludes the proof.

\[ \square \]

### III. EXAMPLE

We now present an example where for an input and output pair of qubit states, although a CPTP extension exists, there is no unital CPTP extension. Thus this example clearly emphasizes the necessity of Theorem 1. Consider the following map

$$\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \rightarrow \begin{bmatrix} \rho_{11} + (1 - p) \kappa \rho_{12} \\ \kappa \rho_{21} \\ \rho_{22} \end{bmatrix} ,$$

(21)

which is CPTP whenever $1 \geq p \geq \kappa^2$. Let $p = \kappa = \frac{1}{2}$, and consider two density matrices:

$$\rho_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad \rho_2 = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$$

(22)

which are mapped to

$$\tau_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad \tau_2 = \begin{bmatrix} .6 & .2 \\ .2 & .4 \end{bmatrix} .$$

(23)

The Alberti-Uhlmann condition is obviously satisfied since by constriction the $\tau_k$ are related to the $\rho_k$ via a CPTP map. Now let us check whether a unital extension exists via one of the equivalent conditions. By considering $\alpha = -0.5$ and $\beta = \gamma = 1$ in (19) one finds

$$\alpha \mathbb{I} + \beta \mathbb{\alpha}_1 + \gamma \mathbb{\alpha}_2 = \begin{bmatrix} 1.1 & .2 \\ .2 & -1 \end{bmatrix}$$

(24)

and

$$\alpha \mathbb{I} + \beta \rho_1 + \gamma \rho_2 = \begin{bmatrix} .7 & .4 \\ .4 & .3 \end{bmatrix}$$

(25)

and their trace norms are $\sqrt{1.6}$ and 1 respectively. Hence there is no unital channel mapping $\rho_k$ to $\tau_k$. Alternatively one can check that condition (6) reads

$$.16 = (.8 - .4)^2 \leq 4 \cdot .2 = 0 ,$$
a contradiction. Indeed this condition allows us to answer the following question: how does $\tau_2$ have to be modified to guarantee a simultaneous unital state transformation? Because of $\det(\rho_1) = \det(\tau_1) = 0$ the transition by (6) is possible in a unital manner if $\text{tr}(\tau_1^{#}\tau_2) = .8$ so

$$\tau_2 = \begin{bmatrix} .2 & z \\ \overline{z} & .8 \end{bmatrix}$$

with $|z| \leq A$. Thus the only allowed channels in this scenario are those which relax (21) with $1 = p \geq \kappa^2$ and rotate the off-diagonals (unitaries of the form $\text{diag}(1, e^{i\phi})$).

**IV. CONCLUSIONS & OUTLOOK**

In this paper we derived necessary and sufficient conditions for the existence of a unital quantum channel mapping a pair of qubit states $\{\rho_1, \rho_2\}$ into $\{\tau_1, \tau_2\}$. These conditions connect the problem to trace-norm inequalities (in the spirit of Alberti-Uhlmann (1) which is reproduced by setting $\alpha = 0$ in (19)) and majorization on matrices. Moreover we reduced the infinite family of conditions to just three inequalities which are simple enough to be verified with pen and paper. We also provided an example of two pairs of qubit states which satisfy the Alberti-Uhlmann condition, that is, there exists a quantum channel mapping $\rho_k$ to $\tau_k$, but condition (4) is violated which implies that there is no unital channel between $\rho_k$ and $\tau_k$.

We expect that our result will encourage more research in this direction and shed light on finding more general closed form conditions for existence of channels between sets of quantum states. Possible next steps could focus on the case of the input set consisting of any three linearly independent qubit states or—in spirit of thermodynamics and general $D$-majorization [24]—how to modify Theorem 1 & 2 if the fixed point of the channel is not the identity but an arbitrary Gibbs state (i.e. an arbitrary positive-definite state $D$).

**Acknowledgements**

The authors are grateful to Teiko Heinosaari, Antonella De Pasquale, Namit Anand and Francesco Buscemi for fruitful discussion and constructive comments. S.C., D.C. and G.S. were supported by the Polish National Science Centre project 2018/30/A/ST2/00837. F.v.E. is supported by the Bavarian excellence network ENB via the International PhD Programme of Excellence *Exploring Quantum Matter* (EXQM).

[1] P. Alberti and A. Uhlmann, Rep. Math. Phys. 18, 163 (1980).
[2] T. Ando, Lin. Alg. Appl. 118, 163–248 (1989).
[3] G.W. Stewart, Lin. Alg. Appl. 283, 151–164 (1998).
[4] W. B. Arveson, Acta Math. 123, 141 (1969).
[5] F. Brandão, M. Horodecki, Ng Nelly, J. Oppenheim, S. Wehner, Proc. Natl. Acad. Sci. U.S.A. 112 (11), 3275–3279 (2015).
[6] F. Buscemi, arXiv:1505.00535 (2015).
[7] F. Buscemi, D., Sutter, M. Tomamichel, Quantum 3, 209 (2019).
[8] S. Chakraborty and D. Chruściński, Phys. Rev. A 99, 042105 (2019).
[9] A.Chefles, R. Jozsa, and A. Winter, Int. J. Quantum. Inform. 2, 11 (2004).
[10] M.D. Choi, Lin. Alg. Appl. 10, 285–290 (1975).
[11] M. Dall’Arno, F. Buscemi, V. Scarani, Quantum 4, 233 (2020).
[12] G. Gour, M.P. MÄjdler, V. Narasimhachar, R.W. Spekkens, N.Y. Halpern, Phys. Rep 583, 1–58 (2015).
[13] T. Heinosaari, M. A. Jivulescu, D. Reeb, and M. M. Wolf, J. Math. Phys. 53, 102208 (2012).
[14] C. Helstrom, J. Stat. Phys. 1, 231 (1969).
[15] R.A. Horn, C.R. Johnson, Matrix Analysis (Cambridge University Press, Cambridge, England, 1987).
[16] J. Oppenheim and M. Horodecki, Nat. Commun. 4, 2059 (2013).
[17] Z. Huang, C.-K. Li, E. Poon, and N.-S. Sze, J. Math. Phys. 53, 102209 (2012).
[18] A. Jencova, J. Math. Phys. 53, 012201 (2012).
[19] M A. Nielsen and I. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2002).
[20] V. Paulsen, Completely Bounded Maps and Operator Algebras (Cambridge University Press, Cambridge, England, 2003).
[21] D. Pérez–García, M.M. Wolf, D. Petz, M.B. Ruskai, J. Math. Phys. 47, 083506 (2006).
[22] E. StÀyrmer, Positive Linear Maps of Operator Algebras, Springer Monographs in Mathematics (Springer, New York, 2013).
[23] F. vom Ende, arXiv:2003.04164 (2020).
[24] F. vom Ende and G. Dirr, arXiv:1911.01061 (2020).
[25] Sometimes one limits the problem from Gibbs-preserving channels to the smaller set of thermal operations which are of a specific form compliant with the underlying resource theory. However we for this manuscript will focus on general CPTP maps.
[26] For arbitrary qubit states $\rho_1, \rho_2$ there exists $c \in [0,1]$ such that $\hat{\rho}_1 := \frac{\rho_1 + c\rho_2}{1+c}$ is a pure state. Thus the problem $\{\rho_1, \ldots, \rho_n\} \rightarrow \{\tau_1, \ldots, \tau_n\}$ by linearity of the desired channel can be reduced to $\{\hat{\rho}_1, \ldots, \hat{\rho}_n\} \rightarrow \{\tilde{\tau}_1, \ldots, \tilde{\tau}_n\}$.
[27] Recall that in two dimensions one has the simple identities $\{(a,b),\{c,d\}\}^\# = \{(d,-b),\{-c,a\}\}$ as well as $\text{det}(A+B) = \text{det}(A) + \text{det}(B) + \text{tr}(A^#B)$ for all $a,b,c,d \in \mathbb{C}$, $A,B \in M_2(\mathbb{C})$. 
