Braids of the $N$-body problem II: carousel solutions by cabling central configurations

Marine Fontaine$^1$ · Carlos García-Azpeitia$^2$

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Abstract
We prove the existence of relative periodic solutions of the planar $N = \sum_{j=1}^{n} k_j$-body problem starting with $n$ bodies moving close to a non-degenerate central configuration and replacing each of them with clusters of $k_j$ bodies that move close to a small central configuration. We name these solutions carousel solutions. The proof relies on blow-up techniques for variational methods used in our previous work (Fontaine and García-Azpeitia in Non-linearity 34(2):822, 2021).

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1 Introduction

The existence of braids in the $N$-body problem has been intensively studied since the pioneering work of Poincaré. In the case of strong forces, the classical approach exploits the fact that the Euler action functional blows up at any orbit belonging to the boundary of a braid class, which implies the existence of minimizers for tied braid classes by using the direct method of the calculus of variations, see [21, 31] and references therein. Later on, Moore in [32] found braids for gravitational and weak forces by making numerical continuations of the aforementioned braids for strong forces.

For gravitational forces, the existence of braids in the 3-body problem ($N = 2 + 1$) was established in the classical setting of the Sun-Earth-Moon system. These solutions are obtained by replacing one body with two in a circular motion of the 2-body problem. There...
Fig. 1 Carousel solutions of the 12-body problem. The four bodies in a polygonal configuration are replaced by clusters of three bodies of equal masses arranged at the vertices of an equilateral triangle (Lagrange triangular configuration)

is a large literature related to this problem [8, 23, 24, 34, 39]. An extension of these solutions to the 4-body problem ($N = 3 + 1$), where one body in the Lagrange triangular configuration is replaced by two, was obtained in [11]. This result was then extended in [29] to the general case $N = n + 1$, where one body in a nondegenerate central configuration is replaced by two, making use of symplectic scalings and the Implicit Function Theorem.

In our previous work [17], we established a new approach based on blow-up techniques to construct solutions of the $N = n + 1$-body problem in $E = \mathbb{R}^{2d}$ for central forces (weak, gravitational and strong), where one body in a nondegenerate central configuration of $n$ bodies is replaced by two. For such solution, two of the bodies rotate uniformly around their center of mass. The other $n - 1$ bodies and the center of mass of the pair remain, at each time, close to a central configuration. When $d = 1$, these solutions are braids obtained by replacing a strand in a braid by a new braid, a process called cabling by Moore in [32].

In the present article, we generalize the construction in [17] to replace several bodies in a non-degenerate central configuration by clusters of bodies arranged themselves in small central configurations. We name these new solutions carousel solutions (Fig. 1). Due to the higher level of complexity that involves dealing with multiple clusters, we only treat the case when the motion takes place in the plane ($d = 1$). One may consider symmetry constraints as in [17] to possibly extend these results to higher even-dimensional spaces.

**Model: the N-body and N-vortex filament problems**

We first introduce a multi-index notation to describe the positions of the bodies in their respective cluster. The positions are given by vectors $q_{j,k}$ in $E = \mathbb{R}^2$ for $k = 1, \ldots, k_j$ and $j = 1, \ldots, n$. To each $q_{j,k}$ we attach a positive mass $m_{j,k} > 0$. The index $j$ represents the cluster that contains $k_j$ bodies. Setting $N = \sum_{j=1}^n k_j$, the equations of motion of the $N$-body problem are
Without loss of generality, we suppose that $k_j > 1$ for $j = 1, \ldots, n_0$ and $k_j = 1$ for $j = n_0 + 1, \ldots, n$. That is, the $j$-cluster contains only one body when $j = n_0 + 1, \ldots, n$.

The relevant cases from the physical point of view are the gravitational potential ($\alpha = 2$) and the logarithmic potential ($\alpha = 1$). In the latter case, Eq. (1.1) governs the approximate interaction of $N$ steady vortex filaments in fluids (Euler equation) [4, 10, 25, 26], Bose–Einstein condensates (Gross–Pitaevskii equation) and superconductors (Ginzburg–Landau equation) [9, 13]. Although it is worth mentioning that there are some subtle differences in the equations for steady vortex filaments with respect to Eq. (1.1) . Namely, the role of the masses in the filament model is played by the circulations, which can be negative, quantized or weighted by different factors representing the vortex core of filaments.

**Main result: existence of carousel solutions**

We construct carousel solutions starting from multiple central configurations. For $j = 1, \ldots, n_0$ the $j$-cluster is close to a central configuration of $k_j$-bodies $a_j = (a_j, 1, \ldots, a_j, k_j)$ such that

\[
m_{j,k} a_{j,k} = \sum_{k'=1 \atop k' \neq k}^{k_j} m_{j,k} m_{j,k'} \frac{a_{j,k} - a_{j,k'}}{\|a_{j,k} - a_{j,k'}\|^{\alpha+1}} \quad k = 1, \ldots, k_j.
\]

(1.2)

We assume that each central configuration $a_j$ has zero center of mass. For $j = n_0 + 1, \ldots, n$ we assume that the $j$-clusters are made of a single body. Our main result states that we can replace $n_0$ bodies in a central configuration of $n$-bodies by clusters of $k_j$-bodies close to the central configurations $a_j$. For this purpose, we start from a central configuration $a_0 = (a_0, 1, \ldots, a_0, n)$ with masses $M_j = \sum_{k=1}^{k_j} m_{j,k}$ given by the total masses of the $j$-clusters for $j = 1, \ldots, n_0$. This means that

\[
M_j a_{0,j} = \sum_{j'=1 \atop j' \neq j}^{n_0} M_{j'} M_{j'} \frac{a_{0,j} - a_{0,j'}}{\|a_{0,j} - a_{0,j'}\|^{\alpha+1}} \quad j = 1, \ldots, n_0.
\]

(1.3)

We also assume, without loss of generality, that the central configuration $a_0$ has zero center of mass.
Specifically, in Theorem 3.8 we construct the carousel solutions of (1.1) starting from \( n_0 \) central configurations \( a_1, \ldots, a_{n_0} \) satisfying (1.2) and \( a_0 \) satisfying (1.3). We suppose that \( a_j \) are \( 2\pi p_j \)-nondegenerate for \( p_1, \ldots, p_{n_0} \in \mathbb{Z} \setminus \{0\} \) fixed integers (Definition 3.3), and also that \( a_0 \) is nondegenerate (Definition 3.6). We then prove that, for every sufficiently small \( \varepsilon \), there are at least \( n_0 + 1 \) solutions of the \( N \)-body problem (1.1) with components of the form

\[
q_{j,k}(t) = \exp(tJ)u_{0,j}(\nu t) + r_j \exp(t\omega_j J)u_{j,k}(\nu t), \quad j = 1, \ldots, n_0, \quad k = 1, \ldots, k_j
\]

\[
q_{j,1}(t) = \exp(tJ)u_{0,j}(\nu t), \quad j = n_0 + 1, \ldots, n.
\]

(1.4)

The matrix \( J \) is a complex structure on \( E \). The paths \( (u_{0,1}(s), \ldots, u_{0,n}(s)) \) and \( (u_{j,1}(s), \ldots, u_{j,k_j}(s)) \) remain \( \varepsilon \)-close in a space of \( 2\pi \)-periodic paths to the central configurations \( a_0 \) and \( a_j \) for \( j = 1, \ldots, n_0 \) respectively. The amplitudes and the frequencies of rotation of the clusters

\[
r_j = \left(1 + p_j \nu\right)^{-2/(a+1)} \quad \text{and} \quad \omega_j = 1 + p_j \nu
\]

are controlled uniformly by the parameter \( \nu \), where

\[
\nu = \varepsilon^{-(a+1)/2} - 1
\]

represents the frequency of the perturbation from the arrangement of the central configurations in rigid motion. Furthermore, the solutions are determined by different orientating phases \( \vartheta_j \in S^1 \) such that \( u_{j,k}(s) = \exp(\vartheta_j J) a_{j,k} + \mathcal{O}(\varepsilon) \) for \( j = 1, \ldots, n_0 \). Thus the result can be rephrased as

\[
q_{j,k}(t) = \exp(tJ)a_{0,j} + r_j \exp((t\omega_j + \vartheta_j J)a_{j,k} + \mathcal{O}(\varepsilon)) \quad j = 1, \ldots, n_0, \quad k = 1, \ldots, k_j
\]

\[
q_{j,1}(t) = \exp(tJ)a_{0,j} + \mathcal{O}(\varepsilon) \quad j = n_0 + 1, \ldots, n,
\]

(1.5)

where \( \mathcal{O}(\varepsilon) \) are quasi-periodic functions of order \( \varepsilon \) and the parameter \( \varepsilon \) provides a uniform measure of the shrink of each cluster because the amplitudes are \( r_j = \mathcal{O}(\varepsilon) \).

The solutions that we obtain are quasi-periodic if \( \nu \notin \mathbb{Q} \). If the frequency \( \nu = p/q \) is rational, the frequencies \( \omega_j = (q + p_j p)/q \) are rational as well. For this case our theorem implies that for any fixed integer \( q > 0 \), there is some integer \( p_0 > 0 \) such that, for each integer \( p > p_0 \), the components \( q_{j,k}(t) \) are \( 2\pi q \)-periodic. These are braid solutions, where the centers of mass of the \( n \) clusters (close to the central configuration \( a_0 \)) wind around the origin \( q \) times, while the configuration of \( k_j \) bodies in each cluster winds around its center of mass \( q + p_j p \) times (Figure 1). The sign of \( p_j \) determines whether the \( j \)-cluster has a prograde or a retrograde rotation with respect to the whole system. When the rotation is...
prograde \((p_j > 0)\), the cluster rotates in the same direction as the main relative equilibrium. When the rotation is retrograde \((p_j < 0)\), the cluster rotates in the opposite direction.

Our proof relies on the assumption that the central configurations \(a_j\) for \(j = 1, \ldots, n_0\) are \(2\pi p_j\)-nondegenerate. In simple terms, the \(2\pi p_j\)-nondegeneracy condition means that the group orbit of \(a_j\) is a nondegenerate critical manifold for the action functional of the \(k_j\)-body problem defined in the space of \(2\pi p_j\)-periodic paths (Definition 3.3). In Sect. 4 we prove that this condition holds for an infinite number of \(k_j\)-polygonal configurations with equal masses when \(\alpha \in (1, 1 + \delta)\) for a positive small \(\delta\). In the case \(\alpha \geq 1\) (but \(\alpha \neq 2\)) we show that this condition holds for the Lagrange triangular configuration with different masses provided they satisfy the relation

\[
27 \frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} > 9 \left(\frac{3 - \alpha}{1 + \alpha}\right)^2. \tag{1.6}
\]

We can therefore always replace any body in a nondegenerate central configuration by regular \(k_j\)-polygons. We conjecture that the \(2\pi p_j\)-nondegeneracy condition is generic for central configurations in the case \(\alpha \geq 1\) \((\alpha \neq 2)\).

When \(\alpha = 2\), all the central configurations are \(2\pi p_j\)-degenerate due to the existence of the elliptic homographic solutions generated by the central configuration. This implies that our result cannot be directly extended to the case \(\alpha = 2\). In Theorem 3.9 we extend our result to the gravitational case under two additional assumptions: (i) we can divide only a central body in a central configuration which is symmetric under \(2\pi\)-rotations with \(m \geq 2\) and (ii) the central configuration \(a_1\) of the 1-cluster needs to be \(2\pi/m\)-nondegenerate (Definition 3.5). There are plenty of central configurations satisfying the first assumption such as the Maxwell configuration [37] and the nested polygonal configurations with a center appearing in [18, 22, 30]. Regarding the second assumption, we prove in Sect. 4 that the \(2\pi/m\)-nondegeneracy condition of a central configuration holds for the \(k_1\)-polygonal configurations with equal masses for \(k_1 = 4, \ldots, 1000\). We also show that this condition also holds true in the case of the Lagrange triangular configuration when the masses satisfy the inequality (1.6) with \(\alpha = 2\). Therefore, we can always replace the central body in a symmetric central configuration by a regular \(k_1\)-polygon. We conjecture that the \(2\pi/m\)-nondegeneracy condition is generic for central configurations.

**Method: perturbation of nondegenerate critical manifolds**

Our method starts by writing down the Euler–Lagrange equations with respect to the action functional \(A\) of the \(N\)-body problem. We implement several changes of coordinates in configuration space that involve Jacobi-like coordinates, rotating frames and scalings of the amplitudes of the clusters. We extend the action functional \(A\) to new coordinates

\[
u = (u_0, u_1, \ldots, u_n) \in E^n \times E^{k_1} \times \cdots \times E^{k_n},
\]

where the action functional splits into two terms \(A(u) = A_0(u) + H(u)\), and the Euler–Lagrange equations of \(A_0(u)\) are uncoupled in the components \(u_j\) and are given simply by the \(k_j\)-body problem in \(u_j\) for \(j = 1, \ldots, n_0\) and the \(n\)-body problem for \(u_0\). The action of the \(N\)-body problem corresponds to the restriction of \(A(u)\) to the product of the subspaces \(E_j\) defined in (2.4) as the subspaces of zero center of mass. In particular, we have the convention that \(u_j = 0 \in E_j\) for the clusters \(j = n_0 + 1, \ldots, n\). The coupling term \(H = O(\varepsilon)\) is small of order \(\varepsilon\) and encodes the interactions of the different clusters in the new coordinates, which depends on \(\varepsilon\) through the parameters \(r_j, \omega_j, \nu\) introduced before.
The Euler–Lagrange equations of $\mathcal{A}$ are set in a gradient form defined in a subspace of $2\pi$-periodic paths denoted by $X$. We fix a configuration

$$u_a = (a_0, a_1, \ldots, a_n)$$

where $a_0$ is a central configuration (1.3) of the $n$-body problem, $a_j$ is a central configuration (1.2) of the $k_j$-body problem for $j = 1, \ldots, n_0$, and, according to our convention, $a_j = 0$ for $j = n_0 + 1, \ldots, n$. It turns out that the configuration $u_a$ is a critical point of $\mathcal{A}_0$. Furthermore, the functional $\mathcal{A}_0(u)$ is invariant under the action of the torus group $G = U(1)^{n+1}$ on $X$ defined by

$$(g_0, g_1, \ldots, g_n) \cdot (u_0, u_1, \ldots, u_n) = (g_0 u_0, g_1 u_1, \ldots, g_n u_n).$$

where $(g_0, g_1, \ldots, g_n) \in U(1)^{n+1}$. The action of $g_0$ rotates the $n$-body problem consisting of the $n$ centers of mass of the clusters about the origin. The action of each $g_j$ rotates the configuration of the $j$-cluster about its center of mass for $j = 1, \ldots, n_0$ and acts trivially on $u_j = 0$ for $j = n_0 + 1, \ldots, n$.

It follows that the group orbit $G(u_a)$ is a critical manifold of $\mathcal{A}_0$. The perturbation $\mathcal{H} = \mathcal{O}(\varepsilon)$ breaks the $G$-symmetry, in the sense that it is invariant under the action of the diagonal subgroup $H = U(1)$ of $G$. Our theorem is obtained by proving the persistence of $H$-orbits of critical points of the perturbed action $\mathcal{A} = \mathcal{A}_0 + \mathcal{H}$ in a tubular neighbourhood of the critical manifold $G(u_a)$ in $X$. The core of the proof (Sect. 3) relies on three Lyapunov–Schmidt reductions: a reduction to finite dimension, a reduction that regularizes the functional, and a reduction that uses Palais-slice coordinates near the orbit $U(1)^n(a_1, \ldots, a_n)$. We conclude that finding critical points of $\mathcal{A}$ in a neighbourhood of $G(u_a)$ is equivalent to doing so for some regular function

$$\Psi'_\varepsilon : U(1)^{n_0} \rightarrow \mathbb{R}$$

defined on the compact manifold $U(1)^{n_0}$ which represents the orbit of $(a_1, \ldots, a_n)$ under the action of the group $U(1)^n$. A major difference regarding our previous work [17] is that to obtain $\Psi'_\varepsilon$ we need to solve first the component $u_0$. The delicate part of the procedure is finding uniform estimates in $\varepsilon$ because the functional $\mathcal{A}_0$ explodes when $\varepsilon \to 0$ at different scales. After the reduction, the theorem is easily obtained as a consequence of the fact that the Lyusternik-Schnirelmann category of the compact manifold $U(1)^{n_0}$ is $n_0 + 1$, which gives a lower bound for the number of critical points of $\Psi'_\varepsilon$. To each of these critical points corresponds a different way of orienting the central configurations $a_j$ for $j = 1, \ldots, n_0$ with respect to the central configuration $a_0$. Alternately, from Morse theory it follows that the functional has generically $2^{n_0}$ critical points.

It is worth mentioning that many authors have analyzed the persistence of solutions near a nondegenerate critical manifold under perturbation. In our case, the critical manifold of $\mathcal{A}_0$ is the group orbit $G(u_a)$ and the perturbation $\mathcal{H}$ is $H$-invariant. The perturbation of nondegenerate critical manifolds consisting of group orbits has been studied previously in [2, 7, 12, 16, 27, 40], and references therein. In the context of the $n$-body problem, the papers [33] and [1] analyze the breaking of symmetries of the critical manifold of periodic solutions of the Kepler problem when a non-radial external force is introduced. Our work uses ideas already appearing in all those works, although a remarkable difference is that, in our case, the functional $\mathcal{A}_0$ explodes as $\varepsilon \to 0$ at different scales. We solve this problem by means of a procedure that is similar to our previous work [17] and was motivated by the blow-up methods appearing in [5] and [6].
Blow-up techniques have a long history. For instance, they were used by Floer and Weinstein in [15] to find single-soliton standing waves of the nonlinear Schrödinger equation in dimension one. The critical manifold in that case is non-compact due to the action of the group $\mathbb{R}$. Later on, these ideas were used to obtain multi-bump solitons for Schrödinger equations in higher dimension - the interested reader can consult the large bibliography in [3]. We think that it is possible to use similar methods to extend our results to the gravitational case by taking into consideration the non-compact critical manifold of all the elliptic homographic solutions generated by a central configurations.

2 Problem setting for carousels

Let $E = \mathbb{R}^2$ with inner product $\langle \cdot, \cdot \rangle$. We consider $N = \sum_{j=1}^{n} k_j$ bodies moving in $E$ under the influence of a central force field. We use a multi-index notation which simplifies greatly the treatment of the problem. The positions of the bodies in $E$ are denoted by $q_{j,k}$, where $k = 1, \ldots, k_j$ and $j = 1, \ldots, n$. To each of the positions we attach positive masses $m_{j,k} > 0$. The index $j$ represents the cluster of bodies that contains $k_j$ bodies.

We define the kinetic energy and the potential function by

$$K = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{k_j} m_{j,k} \| \dot{q}_{j,k} \|^2$$

$$U = \frac{1}{2} \sum_{(j',k'),(j,k)} m_{j,k} m_{j',k'} \phi_\alpha(\| q_{j,k} - q_{j',k'} \|)$$

where $\| \dot{q}_{j,k} \|^2 = \langle \dot{q}_{j,k}, \dot{q}_{j,k} \rangle$ and $\phi_\alpha$ is a function such that $\phi'_\alpha(r) = -r^{-\alpha}$. The factor $1/2$ in the potential $U$ appears due to the double sum of the same term. The Newtonian potential corresponds to $\phi_2(r) = 1/r$ and the vortex filament potential corresponds to $\phi_1(r) = -\ln(r)$.

Newton’s laws of motion are:

$$m_{j,k} \ddot{q}_{j,k} = \nabla_{q_{j,k}} U = - \sum_{(j',k'),(j,k')} m_{j,k} m_{j',k'} \frac{q_{j,k} - q_{j',k'}}{\| q_{j,k} - q_{j',k'} \|^{\alpha+1}},$$

$$k = 1, \ldots, k_j, \quad j = 1, \ldots, n.$$

Let $\mathcal{L} = K + U$ be the Lagrangian of the system. The action functional

$$\mathcal{A}(q) = \int_0^T \mathcal{L}(q(t), \dot{q}(t)) \, dt$$

is taken over the Sobolev space $H^1([0,T], E^N)$ of paths $q : [0,T] \to E^N$ such that $q$ and its first derivative $\dot{q}$ are square integrable in the sense of distributions.

2.1 Jacobi-like coordinates

Similarly to [17] we make a change of coordinates so that the action functional splits into two terms $\mathcal{A}_0 + \mathcal{H}$. The Euler–Lagrange equations of $\mathcal{A}_0$ give rise to a set of uncoupled $k_j$-body problems and an $n$-body problem. This procedure is a generalization of [17] which allows us to produce carousel solutions of the $N$-body problem. For this purpose we define new

\[ \text{carousel solutions} \]
coordinates

\[ q_{j,k} = Q_{0,j} + Q_{j,k}, \quad k = 1, \ldots, k_j, \quad j = 1, \ldots, n \]  

(2.1)

which are overdetermined in the coordinates \( Q_{0,j} \) and \( Q_{j,k} \). To address this matter we choose components \( Q_{j,k} \) satisfying the \( n \) constraints

\[ \sum_{k=1}^{k_j} m_{j,k} Q_{j,k} = 0, \quad j = 1, \ldots, n. \]  

(2.2)

Let \( Q = (Q_0, Q_1, \ldots, Q_n) \), where \( Q_0 = (Q_{0,1}, \ldots, Q_{0,n}) \in E^n \) represent the positions of the centers of mass of the \( n \) clusters, and each \( Q_j = (Q_{j,1}, \ldots, Q_{j,k_j}) \) \( \in E^{k_j} \) represents the positions of the \( k_j \) bodies in the \( j \)-cluster. Denoting by \( M_j = \sum_{k=1}^{k_j} m_{j,k} \) the total mass of the \( j \)-cluster, we use (2.1) and (2.2) to write the center of mass of the whole system as

\[ \sum_{j=1}^{n} \sum_{k=1}^{k_j} m_{j,k} q_{j,k} = \sum_{j=1}^{n} M_j Q_{0,j}. \]  

(2.3)

From now on, we impose the centers of mass of the \( n \) clusters and the center of mass of the \( n \)-body problem to be zero by considering the coordinates such that

\[ Q = (Q_0, Q_1, \ldots, Q_n) \in E_{\text{red}}^{n+N} := E_0 \times E_1 \times \cdots \times E_n, \]

where the subspaces \( E_j \) are given by

\[ E_0 := \left\{ Q_0 \in E^n : \sum_{j=1}^{n} M_j Q_{0,j} = 0 \right\}, \quad E_j := \left\{ Q_j \in E^{k_j} : \sum_{k=1}^{k_j} m_{j,k} Q_{j,k} = 0 \right\}, \]

(2.4)

for \( j = 1, \ldots, n \). Since there is only one body in a \( j \)-cluster for \( j = n_0 + 1, \ldots, n \), the constraint (2.2) with \( k_j = 1 \) implies that \( Q_j \in E_j = \{0\} \). Hence the position of a single body is determined by \( q_{j,1} = Q_{0,j} \) with \( M_j = m_{j,1} \).

The Jacobi-like coordinates allow us to write the action functional as \( A(Q) = A_0(Q) + H(Q) \). It is important to remark that we consider this functional extended to \( H^1([0,T], E^{N+N}) \), and that the solutions to the \( N \)-body problem are the solutions of the
action $\mathcal{A}(Q)$ with the holonomic constraint that $Q \in E_{red}^{n+N}$. The first term $\mathcal{A}_0$ is called the unperturbed functional. The remaining part $\mathcal{H}$ is a coupling term invariant by linear isometries which becomes small when the norms $\|Q_{j,k}\|$ simultaneously become small for every $k, j$.

**Proposition 2.1** In Jacobi-like coordinates, the action functional becomes

$$\mathcal{A}(Q) = \mathcal{A}_0(Q) + \mathcal{H}(Q) = \int_0^T \sum_{j=0}^n \mathcal{L}_j(Q_j(t), \dot{Q}_j(t))dt + \int_0^T h(Q(t))dt,$$

with

$$\mathcal{L}_0(Q_0, \dot{Q}_0) = K_0(\dot{Q}_0) + U_0(Q_0) = \frac{1}{2} \sum_{j=1}^n M_j \|\dot{Q}_{0,j}\|^2 + \sum_{j,j'=1}^n M_j M_{j'} \phi_\alpha (\|Q_{0,j} - Q_{0,j'}\|),$$

(2.5)

$$\mathcal{L}_j(Q_j, \dot{Q}_j) = K_j(\dot{Q}_j) + U_j(Q_j) = \frac{1}{2} \sum_{k=1}^{k_j} m_{j,k} \|\dot{Q}_{j,k}\|^2 + \sum_{k,k'=1}^{k_j} m_{j,k} m_{j,k'} \phi_\alpha (\|Q_{j,k} - Q_{j,k'}\|),$$

(2.6)

for $j = 1, \ldots, n$ and

$$h(Q) = \sum_{j,j'=1}^n \sum_{k \in K_j} \sum_{j < j'} \sum_{k' \in K_{j'}} m_{j,k} m_{j',k'} \phi_\alpha (\|Q_{0,j} - Q_{0,j'}\|)$$

$$+ (Q_{j,k} - Q_{j',k'}) \|) - \phi_\alpha (\|Q_{0,j} - Q_{0,j'}\|),$$

(2.7)

where $K_j = \{1, \ldots, k_j\}$ is the set of indices for the $j$-cluster.

**Proof** It suffices to show that $\mathcal{L}(Q, \dot{Q}) = \sum_{j=0}^n \mathcal{L}_j(Q_j, \dot{Q}_j) + h(Q)$. We first compute the kinetic energy $K$ in these new coordinates. We get

$$K = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^{k_j} m_{j,k} \|\dot{Q}_{j,k}\|^2 = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^{k_j} m_{j,k} (\|\dot{Q}_{0,j}\|^2 + \|\dot{Q}_{j,k}\|^2 + 2\langle \dot{Q}_{0,j}, \dot{Q}_{j,k} \rangle).$$

Using the constraint (2.2), it follows that $\sum_{k=1}^{k_j} m_{j,k} \langle \dot{Q}_{0,j}, \dot{Q}_{j,k} \rangle = 0$. Since the total mass of the $j$-cluster is $M_j$, we get

$$K = \frac{1}{2} \sum_{j=1}^n M_j \|\dot{Q}_{0,j}\|^2 + \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^{k_j} m_{j,k} \|\dot{Q}_{j,k}\|^2.$$

The potential in the new coordinates becomes

$$U = \frac{1}{2} \sum_{j=1}^n \sum_{k,k' \in K_j} m_{j,k} m_{j,k'} \phi_\alpha (\|Q_{j,k} - Q_{j,k'}\|)$$

$$+ \sum_{j,j'=1}^n \sum_{k \in K_j} \sum_{k' \in K_{j'}} m_{j,k} m_{j',k'} \phi_\alpha (\|Q_{j,k} - Q_{j',k'}\|).$$

Since the term $\sum_{j,j'=1}^n \sum_{k \in K_j} \sum_{k' \in K_{j'}} m_{j,k} m_{j',k'} \phi_\alpha (\|Q_{j,k} - Q_{j',k'}\|)$ is a coupling term invariant by linear isometries, it becomes small when the norms $\|Q_{j,k}\|$ simultaneously become small for every $k, j$. Therefore, the potential also becomes small when the norms $\|Q_{j,k}\|$ simultaneously become small for every $k, j$.

Hence, we have shown that the action functional $\mathcal{A}(Q)$ with the holonomic constraint becomes small when the norms $\|Q_{j,k}\|$ simultaneously become small for every $k, j$. This completes the proof of Proposition 2.1.
The Lagrangian is of the form

\[ L = \frac{1}{2} \sum_{j, j' = 1}^{n} \sum_{k, k' \in K_j} m_{j,k} m_{j', k'} \phi_\alpha \left( \| \left( Q_{0,j} - Q_{0,j'} \right) + \left( Q_{j,k} - Q_{j', k'} \right) \| \right). \]

The variations for \( M_j \) in (2.5) and (2.6) by the endomorphisms \( \mathcal{M}_0 \in \text{End}(E^n) \) by

\[ \mathcal{M}_j = m_{j, 1} I + \cdots + m_{j, k_j} I \quad \text{and} \quad \mathcal{M}_0 = M_1 I + \cdots + M_n I. \]

The variations for \( \mathcal{A}_0 \) with respect to \( \delta v_j \) are

\[ \frac{\delta \mathcal{A}_0}{\delta v_j} = \mathcal{M}_j \left( I_{k_j} \partial_t + \omega_j J_{k_j} \right)^2 v_j + \nabla v_j U_j(v), \quad j = 1, \ldots, n, \]
The potentials

\[
U_0(v) = \sum_{j, j' = 1, j < j'}^n M_j M_{j'} \phi_\alpha (\|v_{0,j} - v_{0,j'}\|) \quad \text{and} \quad U_j(v) = \sum_{k, k' = 1}^{k_j} m_{j,k} m_{j,k'} \phi_\alpha (\|v_{j,k} - v_{j,k'}\|)
\]

where obtained in Proposition 2.1. The variations (2.9) correspond to \(k_j\)-body problems in rotating frame, and (2.10) corresponds to an \(n\)-body problem in rotating frame.

**Definition 2.1** The amended potential \(V_j : E^{k_j} \to \mathbb{R}\) for the \(k_j\)-body problem is defined by

\[
V_j(v_j) = \frac{1}{2} \left\| M_j^{1/2} v_j \right\|^2 + U_j(v_j).
\]

A configuration \(a_j \in E_j\) is a central configurations of the \(k_j\)-body problem (with frequency one and zero center of mass) if it is a critical point of \(V_j\).

A central configuration \(a_j \in E_j\) of the \(k_j\)-body problem satisfies the equation \(M_j a_j = \nabla v_j U_j(a_j)\). Similarly, a central configuration \(a_0 \in E_0\) of the \(n\)-body problem satisfies \(M_0 a_0 = \nabla v_0 U_0(a_0)\). These equations are equivalent to (1.2) and (1.3).

### 2.3 Time and space scaling

Given a central configuration \(a_j = (a_{j,1}, \ldots, a_{j,k_j})\) of the \(k_j\)-body problem, the scaling \(v_j = r_j a_j\) is a central configuration if the frequency \(\omega_j\) satisfies \(\omega_j^2 = r_j^{-(\alpha + 1)}\). Indeed

\[
M_j \left( I_{k_j} \partial_t + J_{k_j} \right)^2 v_j = -\omega_j^2 r_j M_j a_j = -\omega_j^2 r_j^{\alpha + 1} \nabla v_j U_j(v_j).
\]

For \(v \in \mathbb{R}\), we define new coordinates \(u_0 = (u_{0,1}, \ldots, u_{0,n})\) and \(u_j = (u_{j,1}, \ldots, u_{j,k_j})\) by setting

\[
v_j(t) = r_j u_j(vt), \quad j = 1, \ldots, n
\]

\[
v_0(t) = u_0(vt).
\]

We shall rewrite the action functional \(A = A_0 + H\) with respect to the \(u\)-coordinates and find some restrictions on the set of frequencies \(v\) and \(\omega_j\) so that the functional \(A\) is \(2\pi\)-periodic with respect to the new time-parameter \(s = vt\). We first start looking at the unperturbed functional \(A_0\). The \(u\)-coordinates are related to the \(q\)-coordinates as follows

\[
q_{j,k}(t) = \exp(t J) u_{0,j}(vt) + r_j \exp(\omega_j t J) u_{j,k}(vt),
\]

for \(k = 1, \ldots, k_j\) and \(j = 1, \ldots, n\), with the constraint \(u \in E^{n+\mathbb{N}}_{\text{red}}\).

**Proposition 2.2** The unperturbed functional \(A_0(u)\) is given by

\[
A_0(u) = \int_0^{vT} \left( L_0(u_0, \dot{u}_0) + \sum_{j=1}^n r_j^{1-\alpha} L_j(u_j, \dot{u}_j) \right) ds
\]

(2.13)
where

\[
\mathcal{L}_j(u_j, \dot{u}_j) = \frac{1}{2} \left\| \mathcal{M}_j^{1/2} \left( \frac{v}{\omega_j} \partial_s + \mathcal{J}_k \right) u_j(s) \right\|^2 + U_j(u_j)
\]

\[
\mathcal{L}_0(u_0, \dot{u}_0) = \frac{1}{2} \left\| \mathcal{M}_0^{1/2} (v \partial_s + \mathcal{J}_n) u_0(s) \right\|^2 + U_0(u_0).
\]

\textbf{Proof} When \( \alpha > 1 \) the potential \( \phi_\alpha \) is homogeneous of degree \( 1 - \alpha \). Then \( U_j(v_j) = r_j^{1-\alpha} U_j(u_j) \) and the kinetic energy is

\[
\left\| \mathcal{M}_j^{1/2} (\partial_t + \omega_j \mathcal{J}_k) v_j(t) \right\|^2 = \left\| r_j^{1-\alpha} \mathcal{M}_j^{1/2} \left( \frac{v}{\omega_j} \partial_s + \mathcal{J}_k \right) u_j(s) \right\|^2.
\]

For \( j = 0 \) we have \( U_0(v_0) = U_0(u_0) \) and the kinetic energy is

\[
\left\| \mathcal{M}_0^{1/2} (\partial_t + \mathcal{J}_n) v_0(t) \right\|^2 = \left\| \mathcal{M}_0^{1/2} (v \partial_s + \mathcal{J}_n) u_0(s) \right\|^2.
\]

The case \( \alpha = 1 \) is similar, but now for \( j = 1, \ldots, n \) we have \( U_j(v_j) = U_j(u_j) \) and

\[
\left\| \mathcal{M}_j^{1/2} (\partial_t + \omega_j \mathcal{J}_k) v_j(t) \right\|^2 = \left\| \mathcal{M}_j^{1/2} \left( \frac{v}{\omega_j} \partial_s + \mathcal{J}_k \right) u_j(s) \right\|^2.
\]

The result follows by rescaling \( \mathcal{A} \) and adding a constant to it. \( \square \)

\textbf{Proposition 2.3} The nonlinear term \( h(u(s), s) \) of \( \mathcal{H}(u) = \int_0^{2\pi} h(u(s), s)ds \) is \( 2\pi \)-periodic with respect to the time variable \( s \) if and only if

\[
\omega_j = 1 + pjv, \quad p_j \in \mathbb{Z}, \tag{2.15}
\]

for each \( j = 1, \ldots, n \). In this case,

\[
h(u(s), s) = \sum_{j \neq j'} \sum_{k \in K_j} \sum_{k' \in K_{j'}} m_{j,k} m_{j',k'} \phi_\alpha (\| u_{0,j}(s) - u_{0,j'}(s) \| + r_j \exp (p_{j,k} \mathcal{J}) u_{j,k}(s) - r_{j'} \exp (p_{j',k'} \mathcal{J}) u_{j',k'}(s)) - \sum_{j \neq j'} \sum_{k \in K_j} \sum_{k' \in K_{j'}} m_{j,k} m_{j',k'} \phi_\alpha (\| u_{0,j}(s) - u_{0,j'}(s) \|).
\]

\textbf{Proof} In (2.7) the terms \( \phi_\alpha (\| (Q_{0,j}(t) - Q_{0,j'}(t)) + (Q_{j,k}(t) - Q_{j',k'}(t)) \|) \) become

\[
\phi_\alpha \left( \| u_{0,j}(s) - u_{0,j'}(s) + r_j \exp \left( \frac{\omega_j - 1}{v} s \mathcal{J} \right) u_{j,k}(s) - r_{j'} \exp \left( \frac{\omega_j - 1}{v} s \mathcal{J} \right) u_{j',k'}(s) \| \right),
\]

and the terms \( \phi_\alpha (\| Q_{0,j}(t) - Q_{0,j'}(t) \|) \) become \( \phi_\alpha (\| u_{0,j}(s) - u_{0,j'}(s) \|) \). It follows that the integrand \( h(u(s)) \) is \( 2\pi \)-periodic with respect to \( s \) if and only if, for all \( j = 1, \ldots, n \), the frequency \( (\omega_j - 1)/v \) is an integer. We can thus fix \( n \) integers \( p_j \) such that \( \omega_j = 1 + pjv \). Replacing \( (\omega_j - 1)/v \) by \( p_j \) in the above expression yields (2.16). \( \square \)
Thus, we obtain the action functional

\[ A(u) = A_0(u) + \mathcal{H}(u) = \int_0^{2\pi} \left( L_0(u_0, \dot{u}_0) + \sum_{j=1}^n r_j^{1-\alpha} L_j(u_j, \dot{u}_j) \right) ds + \int_0^{2\pi} h(u(s), s) ds, \]

(2.17)

with \( h(u(s), s) \) as in (2.16) and \( L_0(u_0, \dot{u}_0) \) and \( L_j(u_j, \dot{u}_j) \) as in (2.14). At this point, the frequency \( \nu \in \mathbb{R} \) is still a free parameter. In analogy with [17], we choose the frequency \( \nu \) as a function of \( \varepsilon \) such that the relation \( r_j = \varepsilon \) holds in the case that \( p_j = 1 \). Thus, we fix arbitrary integers \( p_1, \ldots, p_n \in \mathbb{Z} \) and impose the following conditions

(A) \( \omega_j = 1 + p_j \nu \) and \( r_j = (1 + p_j \nu)^{-2/(\alpha+1)} \) for each \( j = 1, \ldots, n \).

(B) \( \nu = \varepsilon^{-(\alpha+1)/2} - 1 \) for some \( \varepsilon > 0 \).

These conditions allow us to express the parameters \( \omega_j, r_j, \nu \) as functions of \( \varepsilon \) with

\[ r_j = (1 + p_j \nu)^{-2/(\alpha+1)} = p_j^{-2/(\alpha+1)} \nu + O\left(\varepsilon^{(\alpha+3)/2}\right) = p_j^{-2/(\alpha+1)} \nu + O(\varepsilon^2), \]

\[ \omega_j / \nu = p_j + 1 / \nu = p_j + O(\varepsilon^{(\alpha+1)/2}) = p_j + O(\varepsilon). \]

(2.18)

We look for solutions of the \( N \)-body problem as critical points of \( A(u) \) on some collision-less open set \( \Omega \) for small \( \varepsilon \).

### 2.4 Gradient formulation and symmetries

The space \( H^1(S^1, E^{n+N}) \) is identified with its dual \( H^1(S^1, E^{n+N})^* \) by the Riesz representation Theorem. This allows us to define the gradient operator by the relation \( \nabla A = (-\partial_s^2 + 1)^{-1} \delta A \). We use the gradient formulation \( \nabla A = \nabla A_0 + \nabla \mathcal{H} \), where \( \nabla \mathcal{H} = O(\varepsilon) \) is a compact operator and \( \nabla A_0 \) is given by

\[ \nabla_{u_j} A_0(u) = (-\partial_s^2 + 1)^{-1} r_j^{-1} \left( -M_j \left( \frac{\nu}{\omega_j} I_{k_j} \partial_s + J_{k_j} \right)^2 u_j + \nabla_{u_j} U_j(u_j) \right), \]

(2.19)

\[ \nabla_{u_0} A_0(u) = (-\partial_s^2 + 1)^{-1} \left( -M_0 \left( \nu I_n \partial_s + J_n \right)^2 u_0 + \nabla_{u_0} U_0(u_0) \right). \]

(2.20)

We know that these equations admit the constant solution \( u_a = (a_0, a_1, \ldots, a_n) \) where \( a_0 \in E_0 \) is a central configuration of the \( n \)-body problem, each \( a_j \in E_j \) is a central configuration of the \( k_j \)-body problem for \( j = 1, \ldots, n_0 \) and \( a_j \in E_j = \{0\} \) for \( j = n_0 + 1, \ldots, n \).

Due to the presence of symmetries, there is in fact a group orbit of solutions generated by \( u_a \). The functional \( A_0(u) \) is invariant under the action of the torus \( G = U(1)^{n+1} \) defined by

\[ (g_0, g_1, \ldots, g_n) \cdot (u_0, u_1, \ldots, u_n) = (g_0 u_0, g_1 u_1, \ldots, g_n u_n). \]

where \( (g_0, g_1, \ldots, g_n) \in U(1)^{n+1} \). The action of \( g_0 \) rotates the \( n \)-body problem consisting of the \( n \) centers of mass of the clusters about the origin. The action of each \( g_j \) rotates the \( k_j \)-bodies in each cluster about their center of mass for \( j = 1, \ldots, n_0 \) and acts trivially otherwise. The coupling term \( \mathcal{H} \) breaks this symmetry in the sense that the perturbed functional \( A \) is invariant under the diagonal subgroup \( H = \tilde{U}(1) \). By \( G \)-equivariance of the equations (2.19) and (2.20), the group orbit \( G(u_a) \) is an orbit of solutions.
2.5 Euler–Lagrange equations with holonomic constraints

Let $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ be the standard parametrization of the circle and denote by

$$X = H^1(S^1, E_{\text{red}}^{n+N}) \subset H^1(S^1, E^{n+N})$$

the real Hilbert space of $2\pi$-periodic paths in $E_{\text{red}}^{n+N}$. The solutions of the $N$-body problem are the critical points of $\mathcal{A}$ restricted to $X$. That is, the system of equations of the $N$-body problems is the gradient of $\mathcal{A}$ projected onto $X$

$$P_X \nabla \mathcal{A}(u) = 0. \quad (2.21)$$

Solving this system is equivalent to finding the critical points of $\mathcal{A}$ with the holonomic constraints

$$g^0_0(u_0) = \sum_{j=1}^n M_j u_{0,j} \cdot e_s = 0, \quad g_j^s(u_j) = \sum_{k=1}^{k_j} m_{j,k} u_{j,k} \cdot e_s = 0, \quad s = 1, 2.$$

The explicit projection $P_X : H^1(S^1, E^{n+N}) \to X$ is given in components $u_j$ by

$$P_X(u_j) = u_j - \frac{2}{\varepsilon} \sum_{s=1}^2 \frac{\langle u_j, \nabla u_j g_j^s(u_j) \rangle}{\|\nabla u_j g_j^s(u_j)\|^2} \nabla u_j g_j^s(u_j), \quad j = 0, \ldots, n. \quad \text{(2.22)}$$

Therefore, the explicit system of equations, equivalent to (2.21), is

$$\nabla u_j \mathcal{A}(u) = \sum_{s=1}^2 \frac{\langle \nabla u_j \mathcal{A}(u), \nabla u_j g_j^s(u_j) \rangle}{\|\nabla u_j g_j^s(u_j)\|^2} \nabla u_j g_j^s(u_j), \quad j = 0, \ldots, n.$$ with the left hand side given in (2.20) and (2.19). We study these equations in a collision-less tubular neighbourhood $\Omega \subset X$ of the orbit $G(u_a)$.

Remark 2.2 One may consider also the augmented action with the holonomic constraints

$$\mathcal{A}^a(u^a) = \mathcal{A}_0(u) + \mathcal{H}(u) + \sum_{j=0}^n \sum_{s=1}^2 \lambda_j^s \cdot g_j^s,$$

where $u^a = (u, \lambda_0, \ldots, \lambda_n) \in E^{n+N} \times \mathbb{R}^{2(n+1)}$. Solving the action for the augmented system $\nabla \mathcal{A}^a(u^a) = 0$ is equivalent to solving $\nabla u_j \mathcal{A}^a(u^a) = P_X \nabla u_j \mathcal{A}(u) = 0$ with $u \in E_{\text{red}}^{n+N}$ (because $\nabla u_j \mathcal{A}^a(u^a) = g_j^s(u^a) = 0$). For each solution $u_a$ such that $\nabla u_j \mathcal{A}_0(u_a) = 0$ there is a unique $u^a_0$ such that $\nabla u^a_0 \mathcal{A}^a(u^a_0) = 0$, and a similar procedure may be implemented for the orbit of $u^a_0$ with the Lagrange multipliers $\lambda_j$ given as variables.

Remark 2.3 We may consider also the alternative of writing directly the action in coordinates for $E_{\text{red}}^{n+N}$. That is, we may fix coordinates $w_j = (w_{j,1}, \ldots, w_{j,k_j-1})$ on $E_j$ such that $u_j \in E_j$ viewed as an element of $E^{k_j}$ can be written of the form $u_j = \Lambda_j w_j$ for some $k_j \times (k_j - 1)$ matrix $\Lambda_j$. Note that

$$\left\langle \nabla u_j \mathcal{A}_0(u_j), \delta u_j \right\rangle = \left\langle \Lambda_j^T \nabla u_j \mathcal{A}_0(\Lambda_j w_j), \delta w_j \right\rangle$$

where the first inner product is taken on the ambient space $H^1([0, 2\pi], E^{k_j})$ and the second on the reduced space $H^1([0, 2\pi], E_j)$. The reduced Euler–Lagrange equations on $X$ are then

$$\Lambda_j^T \nabla u_j \mathcal{A}_0(\Lambda_j w_j) = 0, \quad j = 0, \ldots, n - 1, \quad (2.23)$$
which is the method adopted in our previous paper [17] for \( j = 1 \) and \( k_j = 2 \). For this case the Euler–Lagrange equations of the 2-body problem in rotating frame are

\[
-M_1 \left( \frac{v}{\omega_1} I_{k_1} \partial_s + J_{k_1} \right) u_1 + \nabla u_1 U_1(u_1) = 0
\]

where the positions are denoted \( u_1 = (u_{1,1}, u_{1,2}) \) and \( M_1 \) is the diagonal matrix whose entries are the masses of the two bodies \( m_{1,1} \) and \( m_{1,2} \). We can parametrize the reduced space \( E_1 \) by the relative position \( w_1 = u_{1,1} - u_{1,2} \), so that \( u_1 = \Lambda_1 w_1 \) with \( \Lambda_1 = \left[ \lambda_{1,1} I \lambda_{1,2} I \right] \), and

\[
\lambda_{1,1} = \frac{m_{1,2}}{m_{1,1} + m_{1,2}}, \quad \lambda_{2,1} = -\frac{m_{1,1}}{m_{1,1} + m_{1,2}}.
\]

Conjugating equation (2.24) by \( \Lambda_1^T \) on the left and \( \Lambda_1 \) on the right yields the Kepler problem in rotating frame

\[
-M_0 \left( \frac{v}{\omega_1} \partial_s + J \right) w_1 - m_{1,1} m_{1,2} w_1 \| w_1 \|^{-\alpha + 1}_\nu = 0,
\]

where \( M_0 = \frac{m_{1,1} m_{1,2}}{m_{1,1} + m_{1,2}} \) is the reduced mass.

### 3 Lyapunov–Schmidt reduction

In this section we reduce the problem to finite dimension by writing the paths in Fourier series and applying a Lyapunov–Schmidt reduction. We have that

\[
X = \left\{ u \in L^2(S^1, E_\text{red}^{n+N}) \mid \sum_{\ell \in \mathbb{Z}} (\ell^2 + 1) \| \hat{u}_\ell \|^2 < \infty \right\},
\]

where \( (\hat{u}_\ell) \) is the sequence of Fourier coefficients in \( (E_\text{red}^{n+N})^C = E_\text{red}^{n+N} \oplus i E_\text{red}^{n+N} \) satisfying \( \hat{u}_\ell = \overline{\hat{u}}_{-\ell} \). That is, \( u \in X \) has Fourier series \( u = \sum_{\ell \in \mathbb{Z}} \hat{u}_\ell e_\ell \) where \( e_\ell : S^1 \to \mathbb{C} \) is given by \( e_\ell(s) = e^{i\ell s} \). We can then write \( X = X_0 \oplus W \) and any element \( u \in X \) decomposes uniquely as \( u = \xi + \eta \) with

\[
\xi = \hat{u}_0, \quad \eta = \sum_{\ell \neq 0} \hat{u}_\ell e_\ell.
\]

The system of equations \( P_X \nabla A(\xi + \eta) = 0 \) splits into

\[
P_{X_0} \nabla A(\xi + \eta) = 0 \in X_0,
\]

\[
P_W \nabla A(\xi + \eta) = 0 \in W,
\]

where \( P_{X_0} : H^1(S^1, E_\text{red}^{n+N}) \to X_0 \) is the canonical projection from \( P : X \to X_0 \) given by \( Pu = \xi \), composed with \( P_X \) in (2.22). The projection \( P_W : H^1(S^1, E_\text{red}^{n+N}) \to W \) is defined as the canonical projection from \( (I - P) : X \to W \) given by \( (I - P)u = \eta \), composed with \( P_X \).

The Lyapunov–Schmidt reduction requires solving the equation \( P_W \nabla A(\xi + \eta) = 0 \). For this purpose, we define an operator \( F_\epsilon : \Omega \subset X \to W \) by

\[
F_\epsilon(\xi, \eta) := D_\epsilon P_W \nabla A(\xi + \eta)
\]
where $\mathcal{D}_\varepsilon \in \text{End}(E^{n+N})$ is the block diagonal matrix
\begin{equation}
\mathcal{D}_\varepsilon = v^{-2}\mathcal{I}_n \oplus r_1^{\alpha-1}\mathcal{I}_{k_1} \oplus \cdots \oplus r_n^{\alpha-1}\mathcal{I}_{k_n},
\end{equation}
with $r_j^{\alpha-1} = \mathcal{O}(\varepsilon^{\alpha-1})$ and $v^{-2} = \mathcal{O}(\varepsilon^{\alpha+1})$. Since $\mathcal{D}_\varepsilon$ is block diagonal, it commutes with $P_W$ and we get
\begin{equation}
F_\varepsilon(\xi, \eta) = P_W \mathcal{D}_\varepsilon \nabla A(\xi + \eta).
\end{equation}

Solving (3.1) is equivalent to solving $F_\varepsilon(\xi, \eta) = 0$ for $\varepsilon \neq 0$ because $\mathcal{D}_\varepsilon$ is an isomorphism. The operator $F_\varepsilon(\xi, \eta)$ is continuous at $\varepsilon = 0$ because $\lim_{\varepsilon \to 0} (v/\omega_j)^2 = (1/p_j)^2$. The limit
\begin{equation}
F_0(\xi, \eta) = \lim_{\varepsilon \to 0} P_W \mathcal{D}_\varepsilon \nabla A_0(\xi + \eta)
\end{equation}
is well defined since $\mathcal{D}_\varepsilon \nabla \mathcal{H} = \mathcal{O}(\varepsilon)$. Furthermore, $F_0(gu_a, 0) = 0$ for all $g \in G$ by equivariance of the unperturbed gradient. Solving $F_\varepsilon(\xi, \eta) = 0$ requires the derivative $\partial_\eta F_0[(gu_a, 0)]$ to be invertible on $W$. Although this is true when $\alpha \neq 2$, the operator is not invertible on the whole space $W$ when $\alpha = 2$. We shall then treat these cases separately.

**Remark 3.1** Alternatively, we could have used the matrix
\begin{equation}
\mathcal{D}'_\varepsilon = \varepsilon^{\alpha+1}\mathcal{I}_n \oplus \varepsilon^{\alpha-1}\mathcal{I}_{k_1} \oplus \cdots \oplus \varepsilon^{\alpha-1}\mathcal{I}_{k_n}
\end{equation}
according to [17]. Both regularizations allow us to perform the same reduction, the only difference is that the scaling matrix $\mathcal{D}_\varepsilon$ depends on $p_j$’s.

**The case $\alpha \neq 2$**

The linearization of (3.3) at $u_a$ is given by
\begin{equation}
\partial_\eta F_0[(u_a, 0)] = \lim_{\varepsilon \to 0} P_W \mathcal{D}_\varepsilon \nabla^2 A_0[u_a] |_{W}.
\end{equation}
The Hessian operator of $A_0$ at the critical point is block diagonal of the form
\begin{equation}
\nabla^2 A_0[u_a] = \nabla^2_{u_0} A_0[a_0] \oplus \nabla^2_{u_1} A_0[a_1] \oplus \cdots \oplus \nabla^2_{u_n} A_0[a_n].
\end{equation}
The blocks are derived using (2.19), (2.20) and given by
\begin{align*}
\nabla^2_{u_0} A_0[a_j] &= (-\partial_s^2 + 1)^{-1}r_j^{\alpha-1} \left(- (v/\omega_j)^2 \mathcal{M}_j \partial_s^2 - 2 (v/\omega_j) \mathcal{M}_j \mathcal{J}_k \partial_s + \nabla^2_{u_j} V_j[a_j] \right), \\
\nabla^2_{u_{k+1}} A_0[a_0] &= (-\partial_s^2 + 1)^{-1}v^2 \left(-\mathcal{M}_0 \partial_s^2 - 2v^{-1} \mathcal{M}_0 \mathcal{J}_n \partial_s + v^{-2} \nabla^2_{u_0} V_0[a_0] \right),
\end{align*}
where $V_j$ are the amended potentials in (2.11).

**Definition 3.2** We define the regularized action for the $j$-cluster by $A_j(u_j) = \int_0^{2\pi} L_j(u_j, \dot{u}_j)ds$ where
\begin{equation}
L_j(u_j, \dot{u}_j) = \lim_{\varepsilon \to 0} \mathcal{L}_j(u_j, \dot{u}_j) = \frac{1}{2} \left\| \mathcal{M}_j^{1/2} \left( \frac{1}{p_j} \partial_s + \mathcal{J}_k \right) u_j(s) \right\|^2 + U_j(u_j).
\end{equation}
Since \( \lim_{\varepsilon \to 0} \left( \omega_j / \nu \right) = 1 / p_j \), we obtain
\[
\lim_{\varepsilon \to 0} \sqrt[\varepsilon]{r_{ij}} \nabla_{u_j} A_0(u_j) = \nabla_{u_j} A_j(u_j),
\]
and
\[
\partial \eta F_0[(u_a, 0)] = P_W \left( -(-\delta^2 + 1)^{-1} \mathcal{M}_0 \delta^2 + \nabla^2_{u_1} A_1[a_1] + \cdots + \nabla^2_{u_n} A_n[a_n] \right)_W.
\]
where
\[
\nabla^2_{u_j} A_j[a_j] = (-\delta^2 + 1)^{-1} \left( -(1/p_j)^2 \mathcal{M}_j \delta^2_\eta - 2 \left( \frac{1}{p_j} \right) \mathcal{M}_j \mathcal{J}_k \delta_\eta + \nabla^2_{u_j} V_j[a_j] \right).
\]
Since \( \eta = \sum_{\ell \neq 0} \hat{u}_\ell e_\ell \in W \), we can write
\[
\partial \eta F_0[(u_a, 0)] \eta = \sum_{\ell \neq 0} \hat{T}_\ell \hat{u}_\ell e_\ell, \tag{3.5}
\]
where the matrix \( \hat{T}_\ell \in \text{End}(E^{n+N}_{\text{ms}}) \) is block diagonal of the form
\[
\hat{T}_\ell = \hat{T}_{\ell, u_0} \oplus \hat{T}_{\ell, u_1} \oplus \cdots \oplus \hat{T}_{\ell, u_n}. \tag{3.6}
\]
These blocks are given explicitly by
\[
\hat{T}_{\ell, u_0} = \frac{\ell^2}{\ell^2 + 1} P_{E_0} \mathcal{M}_0 |_{E_0},
\]
\[
\hat{T}_{\ell, u_j} = \frac{1}{1 + \ell^2} P_{E_j} \left( \left( \frac{\ell}{p_j} \right)^2 \mathcal{M}_j \mathcal{J}_k + \nabla^2_{u_j} V_j[a_j] \right) |_{E_j}, \tag{3.7}
\]
where \( \hat{T}_{\ell, u_0} \in \text{End}(E^{C}_{E_0}) \) and \( \hat{T}_{\ell, u_j} \in \text{End}(E^{C}_{E_j}) \).

**Definition 3.3** The central configuration \( a_j \in E_j \) is \( 2\pi p_j \)-nondegenerate if the group orbit \( U(1)(a_j) \) is a nondegenerate critical manifold of the functional \( A_j(u_j) \) defined by (3.4) in the space \( H^1(S^1, E_j) \).

The orbit \( U(1)(a_j) \) is called a nondegenerate critical manifold of the functional \( A_j(u_j) \) if the kernel of the Hessian at \( a_j \) in \( H^1(S^1, E_j) \) is span(\( \mathcal{J}_k, a_j \)) is orthogonal to span(\( \mathcal{J}_k, a_j \)) if and only if \( u = \sum_{\ell \in \mathbb{Z}} \hat{u}_\ell e_\ell \) with \( \hat{u}_0 \) orthogonal to span(\( \mathcal{J}_k, a_j \)) in \( E_j \), this condition is equivalent to the assumption that the blocks \( \hat{T}_{\ell, u_j} \) are invertible in \( E^{C}_{E_j} \) for \( \ell \neq 0 \) and \( \hat{T}_{0, u_j} \) is invertible in the complement to span(\( \mathcal{J}_k, a_j \)) in \( E_j \).

**Lemma 3.1** Assume that \( \alpha \neq 2 \) and \( a_j \) is \( 2\pi p_j \)-nondegenerate for \( j = 1, \ldots, n_0 \). Then the operator \( \partial \eta F_0[(gu_a, 0)] \) is invertible on \( W \) for all \( g \in G \), i.e. there is a constant \( c > 0 \) such that
\[
\| \partial \eta F_0[(gu_a, 0)]^{-1} \eta \| \leq c \| \eta \| \text{ for each } \eta \in W, \ g \in G.
\]

**Proof** For \( \ell \neq 0 \), the matrix \( \hat{T}_\ell \) in (3.6) is always invertible and so are the blocks \( \hat{T}_{\ell, u_j} \) when \( \ell \neq 0 \) by assumption that \( a_j \) is \( 2\pi p_j \)-nondegenerate for \( j = 1, \ldots, n_0 \). This implies that the operator \( \partial \eta F_0[(u_a, 0)] \) is invertible on \( W \) with
\[
\partial \eta F_0[(u_a, 0)]^{-1} \eta = \sum_{\ell \neq 0} \hat{T}_\ell^{-1} \hat{u}_\ell e_\ell, \ \eta \in W.
\]
\[ \hat{T}_{\ell,uo_j} \to \left(1/p_j\right)^2 P_{E_j}M_j|_{E_j} \] for \( j = 1, \ldots, n \) and \( \hat{T}_{\ell,uo_1} \to P_{E_0}M_0|_{E_0} \) when \( \ell \to \infty \), it follows that
\[ \|\partial_\eta F_0[(uo_1, 0)]^{-1}\eta\| \leq c\|\eta\| \] (3.8)
where \( c > 0 \) is a constant such that any eigenvalue \( \lambda \) of \( \hat{T}_{\ell} \) in (3.6) satisfies \( |\lambda| \geq c^{-1} \).

Note that the Hessian \( \nabla^2A_0[gu_a] \) is conjugated to \( \nabla^2A_0[u_a] \) as \( \nabla A_0 \) is \( G \)-equivariant. This also holds for the constrained gradient because the \( G \)-action preserves the constraints. Hence \( \partial_\eta F_0[(gu_a, 0)] \) and \( \partial_\eta F_0[(u_a, 0)] \) are conjugated and the estimate (3.8) holds when replacing \( u_a \) by \( gu_a \) because \( G \) acts by isometries.

**Remark 3.4** To explain further the meaning of the \( 2\pi p_j \)-nondegeneracy condition, we can consider the Hamiltonian system with Hamiltonian

\[ H_j(u_j, \pi_j) = K_j - U_j \]

for the \( k_j \)-body problem, where \( \pi_j = \delta_{\eta_j}L \) is obtained from the Lagrangian \( L_j = K_j + U_j \) defined in the space \( E_j \) by the Legendre transformation. The relative equilibrium \( a_j \) is linearly stable if the eigenvalues of the linearized Hamiltonian vector field \( 3^{-1}\nabla^2H_j[(a_j, 0)] \) are all purely imaginary, except by the double zero-eigenvalue corresponding to the generator of the group orbit \( (J_ju_j, 0) \). On the other hand, our \( 2\pi p_j \)-nondegenerate condition for \( a_j \) can be verified similarly according to the equivalent condition that the matrix \( 3^{-1}\nabla^2H_j[(a_j, 0)] \) has no eigenvalues of the form \( 2\pi i\ell \) with \( \ell \in \mathbb{Z} \), except a double zero-eigenvalue corresponding to the generator of the \( U(1) \)-orbit of \( (a_j, 0) \).

Unfortunately, the \( 2\pi p_j \)-nondegenerate condition has not been verified before in the literature for central configurations. In order to complement our result, we verify this condition for an infinite number of polygonal configurations in Sect. 4. We conjecture that, for \( \alpha \neq 2 \), the condition of being \( 2\pi p_j \)-nondegenerate holds for a generic set of central configurations in a set of parameters of masses.

**Gravitational case \( \alpha = 2 \)**

When \( \alpha = 2 \) all the central configurations \( a_j \) are \( 2\pi p_j \) degenerate due to the existence of elliptic homographic solutions, and the matrices \( \hat{T}_{\ell,uo_j} \) are never invertible for \( \ell = \pm p_j \).

To study the case \( \alpha = 2 \), we distinguish different types of symmetric configurations under \( 2\pi/m \)-rotations at the origin. Examples of symmetric configurations that we can braid are the Maxwell configuration and nested polygonal configurations. In these cases, we can only divide the central body. Thus, we require the additional assumptions listed below.

**(C0)** We consider the \( N \)-body problem with \( N = n + k_1 - 1 \), i.e.

\[ k_1 > 1, \quad k_2 = \cdots = k_n = 1. \]

Then \( E_j = \{0\} \) for \( j = 2, \ldots, n \) and \( E_{n+N}^{ad} = E_0 \times E_1 \). A path \( u \in X \) is then written as \( u = (u_0, u_1) \). Denote by \( S_n \) the permutation group of \( n \) letters. We need a group of symmetries \( \Gamma \) that allows dealing with the resonances. We consider \( \Gamma < \mathbb{Z}_m \times S_n \) to be the discrete subgroup generated by the element \((\theta, \sigma)\) such that

\[ \theta = 2\pi/m \in \mathbb{Z}_m, \quad \sigma^m = (1) \in S_n, \quad \sigma(1) = 1, \]

and that acts on the components of \( u = (u_0, u_1) \) as follows:

\[ (\theta, \sigma)u_1(s) = u_1(s + \theta), \]
\((\theta, \sigma)u_0(s) = (\exp(-\theta J)u_{0,\sigma(1)}(s + \theta), \ldots, \exp(-\theta J)u_{0,\sigma(n)}(s + \theta))\).

**(C1)** The next assumption is that the masses for the bodies \(q_{j,1}\) for \(j = 2, \ldots, n\) satisfy

\[ m_{j,1} = m_{\sigma(j),1}, \quad j = 2, \ldots, n. \]

For the sake of simplicity, we also assume that \(p_1 = 1\), which along with conditions (A)–(B) imply that

\[ r_1 = \epsilon, \quad \omega_1 = \epsilon^{-(\alpha+1)/2}, \quad v = \epsilon^{-(\alpha+1)/2} - 1. \]

**Lemma 3.2** For \(\alpha = 2\), under the assumptions (C0)–(C1), the functional \(A\) is \(\Gamma\)-invariant.

**Proof** Since the variables \(u_j(s)\) for \(j = 0, \ldots, n\) are uncoupled in \(A_0\), it is an immediate consequence of the assumptions that the functional \(A_0\) is \(\Gamma\)-invariant. It remains to show that the coupling term \(\mathcal{H}\) is \(\Gamma\)-invariant. By assumptions (C0)–(C1), the integrand \(h\) is obtained from (2.16) after setting \(u_{j,k} = 0\) whenever \(j \geq 2\), and \(p_1 = 1\). We get

\[ h(u(s)) = \sum_{j'=2}^{n} \sum_{k \in K_1} m_{1,k} m_{j',1} \left( \phi_\alpha \left( u_{0,1}(s) - u_{0,j'}(s) + r_1 \exp(sJ) u_{1,k}(s) \right) \right) \]

\[ - \phi_\alpha \left( u_{0,1}(s) - u_{0,j'}(s) \right) \right). \]

Set \(s' = s + \theta\). Since \(\sigma(1) = 1\) and the norms are invariant by rotations, we obtain

\[ h((\theta, \sigma)u(s)) = \sum_{j'=2}^{n} \sum_{k \in K_1} m_{1,k} m_{\sigma(j'),1} \left( \phi_\alpha \left( u_{0,1}(s') - u_{0,\sigma(j')}(s') + r_1 \exp(s'J) u_{1,k}(s') \right) \right) \]

\[ - \phi_\alpha \left( u_{0,1}(s') - u_{0,\sigma(j')}(s') \right) \right) = h(u(s')) \]

by re-indexing the sum at the end. Finally,

\[ \mathcal{H}((\theta, \sigma)u) = \int_0^{2\pi} h((\theta, \sigma)u(s))ds = \int_0^{2\pi+\theta} h(u(s'))ds' = \mathcal{H}(u). \]

Thus, by the Palais Principle of Symmetric Criticality [35], we can restrict the study of critical points to the fixed point space \(X^\Gamma\). Notice that a path \(u = (u_0, u_1)\) belongs to \(X^\Gamma\) if and only if \(u(s) = (\theta, \sigma)u(s)\). Thus

\[ X^\Gamma = H^1(S^1, E_0)^\Gamma \oplus H^1(S^1, E_1)^\Gamma, \]

where \(H^1(S^1, E_1)^\Gamma\) is the Sobolev space of \(2\pi/m\)-periodic functions in \(E_1\) and \(H^1(S^1, E_0)^\Gamma\) is the subspace of paths \(u_0\) satisfying the symmetry \(u_{0,\sigma(j)} = \exp(-\theta J)u_{0,\sigma(j)}(s + \theta)\).

**(C2)** The last assumption (to ensure that \(u_a \in X^\Gamma\)) is that the central configurations \(a_0 \in E_0\) satisfies the property

\[ a_{0,j} = \exp(-\theta J)a_{0,\sigma(j)}. \quad (3.9) \]

Since \(\sigma^m = 1\) and \(\theta = 2\pi/m\), this condition implies that the central configuration \(a_0\) is symmetric by \(2\pi/m\)-rotations in the plane, and since \(\sigma(1) = 1\) that

\[ a_{0,1} = \exp(-\theta J)a_{0,1} = 0. \]

This condition holds true in many symmetric configurations: Maxwell configuration, nested polygons with a center [30] and spiderwebs with a center [22].
Thus $a_1$ is a $2\pi/m$-nondegenerate if the kernel of the Hessian of $A_1$ at $a_1$ in $H^1(S^1, E_1)^\Gamma$ is $\text{span}(J_{k_1}a_1)$. This weaker condition is equivalent to the hypothesis that the matrices $\hat{T}_{\ell,u_1}$ are invertible in $E_1^\Gamma$ for $\ell \in m\mathbb{Z}/\{0\}$ and $\hat{T}_{0,u_1}$ is invertible in the orthogonal complement to $\text{span}(J_{k_1}a_1)$ in $E_1$. We conjecture that for $\alpha = 2$, the condition of being $2\pi/m$-nondegenerate holds for a generic set of central configurations in the set of parameters. This condition is verified for the $k$-polygonal configuration in Sect. 4 for $k = 4, \ldots, 1000$.

**Lemma 3.3** Assume $\alpha = 2$. Under the conditions (C0)–(C2), if $a_1$ is $2\pi/m$-nondegenerate for $m \geq 2$, the statement of Lemma 3.1 holds true after replacing $W$ by the fixed point space $W^\Gamma$.

**Proof** By conditions (C0)–(C1),

$$\hat{T}_{\ell} = \hat{T}_{\ell,u_0} \oplus \hat{T}_{\ell,u_1}$$

because $u = (u_0, u_1)$. Since $a_1$ is $2\pi/m$-nondegenerate, the matrix $\hat{T}_{\ell,u_1}$ is invertible for the Fourier modes $\ell = 0, \pm m, \pm 2m, \ldots$. A path $u = (u_0, u_1)$ belongs to $X^\Gamma$ if $u_1$ is $2\pi/m$-periodic. In particular, the Fourier expansion of $u_1$ is fixed by $\Gamma$ only if

$$\hat{u}_{1,\ell} = 0 \text{ for } \ell \neq 0, \pm m, \pm 2m, \ldots$$

and hence the operator $\partial_0 F_0[(u_a, 0)]$ is invertible on $W^\Gamma$. The argument in the proof of Lemma 3.1 applies now in the fixed point space. Moreover, note that the group action of $G$ commutes with that of $\Gamma$. Hence by (C2), the orbit $G(u_a)$ belongs to $X_0^\Gamma \subset X^\Gamma$ and the functional $A_0$ restricted to $X^\Gamma$ is still $G$-invariant. \hfill $\square$

**Lyapunov–Schmidt reduction**

Because of Lemmas 3.1, 3.2 and 3.3, we can perform a Lyapunov–Schmidt reduction as in Theorem 3.2, 3.3 and 3.4 in [17].

**Theorem 3.4** (Lyapunov–Schmidt reduction and uniform estimates) Under conditions (A)–(B) and $\alpha \neq 2$, there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there is a $G$-invariant neighbourhood $\mathcal{V} \subset X_0$ of $u_a$ and an analytic $H$-equivariant mapping $\varphi_\varepsilon : \mathcal{V} \subset X_0 \rightarrow W$ such that $P_\mathcal{V} \nabla A(\xi + \eta) = 0$ for $\xi, \eta \in \mathcal{V}$ if and only if $\eta = \varphi_\varepsilon(\xi)$. The system reduces to the finite-dimensional system $P_{X_0} \nabla \Psi_\varepsilon(\xi) = 0$ in $\mathcal{V}$ where $\Psi_\varepsilon(\xi) = A(\xi + \varphi_\varepsilon(\xi))$, and $P_{X_0}$ is the projection of $X$ to $X_0$ composed with $P_X$. Furthermore, for each $\xi \in \mathcal{V}$ and $g \in G$, the following estimate holds:

$$\|\varphi_\varepsilon(\xi)\| \leq N_1(\varepsilon + \|\xi - gu_a\|^2). \quad (3.10)$$

If $\alpha = 2$, the same result holds under the additional conditions (C0)–(C2) after replacing $X_0$ and $W$ by their $\Gamma$-fixed point spaces.

We set $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$ where $\xi_j \in E_j$ are coordinates for the $k_j$-body problems, and $\xi_0 \in E_0$ are coordinates for the $n$-body problem formed by the centers of mass of the
of the form \( \Psi_\varepsilon = \mathcal{V} \subset X_0 \to \mathbb{R} \) is of the form
\[
\Psi_\varepsilon(\xi) = \mathcal{A}_0(\xi) + \mathcal{N}(\xi)
\]
where
\[
\mathcal{A}_0(\xi) = 2\pi \left( V_0(\xi_0) + \sum_{j=1}^{n} r_j^{1-\alpha} V_j(\xi_j) \right).
\]
is \( G \)-invariant, and
\[
\mathcal{N}(\xi) = \mathcal{A}_0(\xi + \varphi_\varepsilon(\xi)) - \mathcal{A}_0(\xi) + \mathcal{H}(\xi + \varphi_\varepsilon(\xi))
\]
is \( H \)-invariant and satisfies the estimate
\[
\| \mathcal{C}_\varepsilon \nabla \mathcal{N}(\xi) \| \leq N(\varepsilon + \| \xi - g u_\varepsilon \|^2),
\]
where \( \mathcal{C}_\varepsilon = \mathcal{I}_n \oplus r_1^{\alpha-1} \mathcal{I}_k_1 \oplus \cdots \oplus r_n^{\alpha-1} \mathcal{I}_k_n \) and \( g \in G \). The proof of this estimate follows the same steps as in Theorem 3.5 in our previous work \[17\].

**Symmetry reduction**

Even if the problem is now finite dimensional, it is still not possible to continue the solutions of \( P_{X_0} \nabla \Psi_\varepsilon(\xi) = 0 \) from \( \varepsilon = 0 \) because \( \Psi_\varepsilon \) still blows up when \( \varepsilon \to 0 \). We obtain a regular function by passing to the quotient space under the action of \( H \) on \( \mathcal{V} \). Accordingly we write
\[
\xi' = (\xi_1, \ldots, \xi_n) \in E' = E_1 \times \cdots \times E_n
\]
and \( a' = (a_1, \ldots, a_n) \in E' \). The gradient equation \( P_{X_0} \nabla \Psi_\varepsilon(\xi_0, \xi') = 0 \) splits into two parts
\[
P_{E_0} \nabla \Psi_\varepsilon(\xi_0, \xi') = 0 \quad \text{and} \quad P_{E'} \nabla \Psi_\varepsilon(\xi_0, \xi') = 0.
\]
We perform a second reduction to express the regular part \( \xi_0 \) with respect to the singular part \( \xi' \). We solve \( \xi_0(\xi', \varepsilon) \) from the equation \( P_{X_0} \nabla \Psi_\varepsilon(\xi_0, \xi') = 0 \).

The group \( G = G_0 \times G' \) with \( G_0 = U(1) \) and \( G' = U(1)^n \) acts diagonally on \( E_0 \times E' \). We define the one-codimensional subspace of the regular part \( E_0 \),
\[
E_0 = \left\{ \xi_0 = (\rho_0, e^{i\theta_0,1}, \ldots, \rho_0, e^{i\theta_0,n}) \mid \sum_{j=1}^{n} \theta_0,j = 0 \right\}.
\]
For every \( \xi_0 \in E_0 \) we can find \( h \in H = \overline{U(1)} \) such that \( \xi_0 = h \xi_0 \) for some \( \xi_0 \in E_0 \). Setting \( \xi' = h \xi' \) one uses \( H \)-invariance to get
\[
\Psi_\varepsilon(\xi_0, \xi') = \Psi_\varepsilon(h \xi_0, h \xi') = \Psi_\varepsilon(\xi_0, \xi').
\]
The function \( \Psi_\varepsilon \) now only depends on \( \xi_0(\xi', \varepsilon) \in E_0 ' \times E' \) and (3.13) become
\[
P_{E_0} \nabla \Psi_\varepsilon(\xi_0, \xi') = 0 \quad \text{and} \quad P_{E'} \nabla \Psi_\varepsilon(\xi_0, \xi') = 0.
\]

**Definition 3.6** The central configuration \( a_0 \in E_0 \) is **nondegenerate** if the orbit \( U(1)(a_0) \) is a nondegenerate critical manifold of the amended potential \( V_0|_{E_0} : E_0 \to \mathbb{R} \) defined in (2.11).

In this case, the kernel of \( P_{E_0} \nabla^2 V_0(a_0)|_{E_0} \) is generated by \( \mathcal{J}_n a_0 \). This is equivalent to the assumption that the matrix \( \hat{T}_{0,a_0} \) in (3.6) is invertible in a complement to \( \text{Span}(\mathcal{J}_n a_0) \) in \( E_0 \).
Theorem 3.5 Suppose that $a_0 \in E_0$ is nondegenerate. Then for $\varepsilon \in (0, \varepsilon_0)$, the critical points of $\Psi_\varepsilon(\zeta_0, \zeta')$ in the (possibly smaller) neighbourhood $\mathcal{V}$ are in one to one correspondence with the critical points of the function $\Psi'_\varepsilon : \mathcal{V}' \subset E' \to \mathbb{R}$ given by

$$
\Psi'_\varepsilon(\zeta') = \sum_{j=1}^{n} (r_j/\varepsilon)^{1-\alpha} V_j(\zeta_j) + N'(\zeta'),
$$

where $\mathcal{V}' \subset E'$ is a neighbourhood of the orbit $G'(a')$, $V_j(\zeta_j)$ is the amended potential and

$$
N'(\zeta') = \varepsilon^{\alpha-1} \left( \frac{1}{2\pi} A_0(\zeta_0(\zeta', \varepsilon), \zeta') - \sum_{j=1}^{n} r_j^{1-\alpha} V_j(\zeta_j) \right) + \frac{\varepsilon^{\alpha-1}}{2\pi} N(\zeta_0(\zeta', \varepsilon), \zeta'),
$$

where $\zeta_0(\cdot, \varepsilon) : \mathcal{V}' \subset E' \to \mathbb{R}$ is unique such that $P_{E_0'} \nabla \Psi_\varepsilon(\zeta_0(\zeta', \varepsilon), \zeta') = 0$.

Proof Consider the equations obtained in (3.15)

$$
P_{E_0'} \nabla \Psi_\varepsilon(\zeta_0, \zeta') = 0 \quad \text{and} \quad P_{E'} \nabla \Psi_\varepsilon(\zeta_0, \zeta') = 0.
$$

The uniform estimate

$$
\| P_{X_0} C_\varepsilon \nabla N(\zeta_0, \zeta') \| \leq \| C_\varepsilon \nabla N(\zeta_0, \zeta') \| \leq N(\| \zeta' - g'a' \|^2 + \| \zeta_0 - a_0 \|^2 + \varepsilon)
$$

implies that

$$
\lim_{\varepsilon \to 0} P_{E_0'} C_\varepsilon \nabla N(a_0, g'a') = 0
$$

and $\lim_{\varepsilon \to 0} P_{E_0'} C_\varepsilon \nabla^2 N[a_0, g'a'] = 0$ for each $g' \in G'$. This is because the scaling matrix $C_\varepsilon$ acts as the identity on the component $\zeta_0$. In particular, $P_{E_0'} \nabla \Psi_0(a_0, g'a') = 0$ and the Hessian

$$
P_{E_0'} \nabla^2 \Psi_0[a_0, g'a']|_{E_0'} = P_{E_0'} \nabla^2 V_0|_{E_0'}
$$

is non-singular on $E_0'$ by the assumption that $a_0$ is nondegenerate. Using the implicit function theorem and the compactness of $G'$, there is, for each $\varepsilon$ sufficiently small, a smooth function $\zeta_0(\zeta', \varepsilon)$ defined on a neighbourhood $\mathcal{V}' \subset E'$ of the orbit $G'(a')$ such that

$$
P_{E_0'} \nabla \Psi_\varepsilon(\zeta_0(\zeta', \varepsilon), \zeta') = 0
$$

(3.17)
on this neighbourhood and $\zeta_0(a', 0) = a_0$. Hence, when we fix $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0$ possibly smaller, and take a smaller neighbourhood $\mathcal{V} \subset E'_0 \times E'$, the critical points of $\Psi_\varepsilon(\zeta_0, \zeta')$ in $\mathcal{V}$ are in one to one correspondence with the critical points of the function $\Psi'_\varepsilon : \mathcal{V}' \subset E' \to \mathbb{R}$ given by

$$
\Psi'_\varepsilon(\zeta') := \frac{\varepsilon^{\alpha-1}}{2\pi} \left( A_0(\zeta_0(\zeta', \varepsilon), \zeta') + N(\zeta_0(\zeta', \varepsilon), \zeta') \right) = \sum_{j=0}^{n} (r_j/\varepsilon)^{1-\alpha} V_j(\zeta_j) + N'(\zeta').
$$

These are solutions of the equation $P_{E'} \nabla \Psi'_\varepsilon(\zeta') = 0$. Note that the regularizing factor $\varepsilon^{\alpha-1}/2\pi$ leaves the equation (3.17) unchanged when we fix $\varepsilon > 0$. $\Box$
Lemma 3.6 There are constants $N', N'' > 0$ such that, for each $g' \in G'$,
\[
\|\zeta_0(\zeta', \varepsilon) - a_0\| \leq N'(\varepsilon + \|\zeta' - g'a'\|^2) \quad \text{and} \quad \|P_{E'}\nabla_N(\zeta')\| \leq N''(\varepsilon + \|\zeta' - g'a'\|^2).
\]

Proof We first write the Taylor expansion of the operator
\[
P_{E'_0} \nabla A_0(\zeta_0(\zeta', \varepsilon), \zeta') = 2\pi P_{E'_0} \nabla V_0(\zeta_0(\zeta', \varepsilon))
\]
around $\zeta_0 = a_0$. For simplicity, we omit the dependence of $\varepsilon$ in the function $\zeta_0(\zeta', \varepsilon)$. We may shrink the neighbourhood $\mathcal{V}'$ such that for each $\zeta' \in \mathcal{V}'$ we get
\[
\|P_{E'_0} \nabla V_0(\zeta_0(\zeta')) + P_{E'_0} \nabla^2 V_0[a_0](\zeta_0(\zeta') - a_0)\| \leq N'_1\|\zeta_0(\zeta') - a_0\|^2
\]
for some constant $N'_1$. By the reverse triangle inequality,
\[
\|P_{E'_0} \nabla^2 V_0[a_0](\zeta_0(\zeta') - a_0)\| \leq \|P_{E'_0} \nabla V_0(\zeta_0(\zeta'))\| + N'_1\|\zeta_0(\zeta') - a_0\|^2. \tag{3.18}
\]

Since $\zeta_0(\zeta')$ solves uniquely the equation $P_{E'_0} \nabla \Psi(\zeta_0(\zeta'), \zeta') = 0$, we have
\[
2\pi P_{E'_0} \nabla V_0(\zeta_0(\zeta')) = -P_{E'_0} \nabla \mathcal{N}(\zeta_0(\zeta'), \zeta').
\]

The estimates for $\mathcal{N}$ in (3.11) yields
\[
2\pi \|P_{E'_0} \nabla V_0(\zeta_0(\zeta'))\|^2 \leq \|\nabla \mathcal{N}(\zeta_0(\zeta'), \zeta')\|^2 \leq N\left(\varepsilon + \|\zeta_0(\zeta') - a_0\|^2 + \|\zeta' - a'\|^2\right)
\]
and (3.18) becomes
\[
\|P_{E'_0} \nabla^2 V_0[a_0](\zeta_0(\zeta') - a_0)\| \leq N'_2(\varepsilon + \|\zeta_0(\zeta') - a_0\|^2 + \|\zeta' - a'\|^2)
\]
for some $N'_2$. Since $P_{E'_0} \nabla^2 V_0[a_0]|_{E'_0}$ is invertible, there is $c > 0$ such that $\|P_{E'_0} \nabla^2 V_0[a_0]|_{E'_0}\| \geq 2c$. Therefore
\[
\|\zeta_0(\zeta') - a_0\|(2c - N'_2\|\zeta_0(\zeta') - a_0\|) \leq N'_2(\varepsilon + \|\zeta' - a'\|^2).
\]
We can take $\mathcal{V}'$ smaller such that $\|\zeta_0(\zeta') - a_0\| \leq c/N'_2$ and then
\[
c\|\zeta_0(\zeta') - a_0\| \leq N'_2(\varepsilon + \|\zeta' - a'\|^2).
\]
We now set $N' = N'_2/c$. This allows us to write $\zeta_0(\zeta') = a_0 + R_\varepsilon(\zeta')$, where $R_\varepsilon(\zeta')$ is the remainder which satisfies by the above
\[
\|R_\varepsilon(\zeta')\| \leq N'(\varepsilon + \|\zeta' - a'\|^2). \tag{3.19}
\]
To obtain the second estimate, we use the definition (3.16) and we replace $\zeta_0(\zeta')$ by $a_0 + R_\varepsilon(\zeta')$. By (3.11), the first terms of (3.16) become
\[
\frac{e^{a-1}}{2\pi} A_0(a_0 + R_\varepsilon(\zeta'), \zeta') - \sum_{j=1}^{n} (r_j/\varepsilon)^{1-\alpha} V_j(\zeta_j) = e^{a-1} V_0(a_0 + R_\varepsilon(\zeta')).
\]
Applying the Mean Value Theorem, there is some $\mu \in [0, 1]$ such that
\[
P_{E'} \nabla V_0(a_0 + R_\varepsilon(\zeta')) = P_{E'} \nabla^2 V_0(a_0 + \mu R_\varepsilon(\zeta')) (R_\varepsilon(\zeta')),
\]
and there is a constant $e$ such that
\[
\|P_{E'} \nabla V_0(a_0 + R_\varepsilon(\zeta'))\| \leq e \|R_\varepsilon(\zeta')\| \tag{3.20}.
\]
We finally get
\[ \| P_E \nabla N(\zeta') \| \leq \epsilon^{\alpha-1} e \| R(\zeta') \| + N(\epsilon + \| \zeta' - a \|^2 + \| R_\epsilon(\zeta') \|^2) \leq N''(\epsilon + \| \zeta' - a \|^2) \]
for some $N''$ after using the uniform estimate (3.11) and (3.19). The arguments can be repeated replacing $a'$ by $g'a'$ for each $g' \in G'$ and the neighbourhood $\mathcal{V}'$ can be taken as a neighbourhood of the orbit $G'(a')$ by compactness of $G'$.

\[ \square \]

**Lyusternik–Schnirelmann application**

We now show that the function
\[ \Psi'_\epsilon(\zeta') = \sum_{j=1}^{n} \left( r_j/\epsilon \right)^{1-\alpha} V_j(\zeta_j) + N'(\zeta') \quad (3.21) \]
has critical points in the neighbourhood $\mathcal{V}' \subset E'$ of the orbit $G'(a')$.

**Theorem 3.7** Suppose that $a_0$ is a nondegenerate central configuration and $a_j$ is a $2\pi p_j$-nondegenerate central configuration for each $j = 1, \ldots, n$. Then, for each $\epsilon \in (0, \epsilon_0)$, there is a neighbourhood $\mathcal{V}' \subset E'$ of the orbit $G'(a')$ such that the number of critical points of the function $\Psi'_\epsilon : \mathcal{V}' \to \mathbb{R}$ is bounded below by $\text{Cat}(G'/K)$ where $K$ is the stabiliser of $a'$.

**Proof** The critical points of the function $\Psi'_\epsilon(\zeta')$ restricted to the subspace $E' \subset E^N$ are solutions of $P_E \nabla \Psi'_\epsilon(\zeta') = 0$. Using the expansion (2.18), we obtain $\lim_{\epsilon \to 0} \left( r_j/\epsilon \right)^{1-\alpha} = p_j^{2(\alpha-1)/(\alpha+1)}$, and
\[ P_E \nabla \Psi'_0(a') = \left( p_j^{2(\alpha-1)/(\alpha+1)} P_E \nabla V_j(a_j) + \lim_{\epsilon \to 0} P_E \nabla N'(a') \right). \]

By the second estimate in Lemma 3.6, $\| \lim_{\epsilon \to 0} P_E \nabla \zeta' \cdot N'(\zeta') \| \leq N'\| \zeta' - a' \|^2$, we have
\[ \lim_{\epsilon \to 0} P_E \nabla N'(a') = 0, \quad \lim_{\epsilon \to 0} P_E \nabla^2 N'[a'(\zeta')]_{E'} = 0. \]
Since $a_j$ is a critical point of $V_j$ for each $j = 1, \ldots, n$, we get $P_E \nabla V_j(a_j)$ and $P_E \nabla \Psi'_0(a') = 0$. Furthermore, the same estimate implies that the Hessian of $\Psi'_0$ with respect to $E'$ is
\[ P_E \nabla^2 \Psi'_0[a'(\zeta')]_{E'} = p_1^{2(\alpha-1)/(\alpha+1)} \hat{T}_{0,a_1} \oplus \cdots \oplus P_n^{2(\alpha-1)/(\alpha+1)} \hat{T}_{0,a_n}, \]
where
\[ \hat{T}_{0,a_j} = P_E \nabla^2 V_j(a_j)|_{E_j} \in \text{End}(E_j). \]

By the $2\pi p_j$-nondegeneracy assumption on $a_j$, each block $\hat{T}_{0,a_j}$ is non-singular on a subspace $W_j$ complementary to $T_{a_j} U(1)(a_j)$ in $E_j$. Consequently, $W = \bigoplus_{j=1}^{n} W_j$ is a complement to $T_{a_j} G'(a')$ in $E'$. The argument above is valid if we replace $a'$ by $g'a'$ for any $g' \in G'$. A standard application of the Palais-Slice coordinates as in [17] allows us to express the normal coordinates in $W$ in terms of the coordinates along the group orbit $G'(a')$ after taking $\epsilon_0$ and $\mathcal{V}'$ possibly smaller. For $\epsilon \in (0, \epsilon_0)$, the solutions of $P_E \nabla \Psi'_\epsilon(\zeta') = 0$ in $\mathcal{V}'$ are in one to one correspondence with the critical points the function $\Psi'_\epsilon : G'(a') \to \mathbb{R}$. By the Lyusternik-Schnirelmann theorem for compact manifolds [14, 28], the number of critical points of this function is bounded below by $\text{Cat}(G'/K)$ where $K$ is the stabiliser of $a'$.

\[ \square \]
Existence of carousel solutions

We state our main result regarding the existence of carousel solutions of the $N$-body problem. In summary, we proved that the search for the periodic solutions near the orbit $G(u_0)$ with $u_{a_0} = (a_0, a_1, \ldots, a_n)$ reduces to determining whether the function (3.21) admits critical points. Recall that we assume that $k_j > 1$ for $j = 1, \ldots, n_0$ and $k_j = 1$ for $j = n_0 + 1, \ldots, n$. Thus $K = U(1)^{n-n_0}$, because $G' = U(1)^n$ acts trivially on each subspace $E_j = \{0\}$ for $j = n_0 + 1, \ldots, n$. By Theorem 3.7, there are at least $\text{Cat}(G'/K) = n_0 + 1$ solutions.

**Theorem 3.8** (Carousel for non-gravitational potentials) Set $\alpha \neq 2$ and $k_j > 1$ for $j = 1, \ldots, n_0$ and $k_j = 1$ for $j = n_0 + 1, \ldots, n$. Fix integers $p_1, \ldots, p_{n_0} \in \mathbb{Z} \setminus \{0\}$ and choose the frequencies $\omega_j, \nu$ and the amplitudes $r_j$ according to the conditions (A)–(B). Suppose that $a_0$ is a nondegenerate central configuration of the $n$-body problem and $a_j$ is a $2\pi p_j$-nondegenerate central configuration of the $k_j$-body problem for $j = 1, \ldots, n_0$. Then for every sufficiently small $\epsilon$, there are at least $\text{Cat}(G'/K) = n_0 + 1$ solutions of the $N$-body problem (1.1) with components of the form

$$q_{j,k}(t) = \exp(tJ)u_{0,j}(\nu t) + r_j \exp(t\omega_j J)u_{j,k}(\nu t), \quad j = 1, \ldots, n_0, \quad k = 1, \ldots, k_j$$

$$q_{j,1}(t) = \exp(tJ)u_{0,j}(\nu t) \quad \text{for } j = n_0 + 1, \ldots, n.$$

where $u_{0,j}(\nu t) = a_{0,j} + O_X(\epsilon)$ and $u_{j,k}(\nu t) = e^{\theta_j J}a_{1,k} + O_X(\epsilon)$ for $j = 1, \ldots, n_0$ with some phases $\theta_j \in S^1$.

These solutions are quasi-periodic if $\nu \notin \mathbb{Q}$, and periodic if $\nu \in \mathbb{Q}$. In the case that $\nu = (p - q)/q$ is rational, then $\varepsilon^{-(\alpha+1)/2} = p/q$ and $\omega_j = (q + p_j p)/q$ are rational. Thus, for any fixed integer $q > 0$, there is some $p_0 > 0$ such that, for each $p > p_0$, the components $q_j(t)$ are $2\pi q$-periodic. In these solutions the centers of mass of $n$ clusters (close to the central configuration $a_0$) wind around the origin $q$ times, while each $j$-cluster winds around its center of masses $q + p_j p$ times. The sign of the frequency $\omega_j$ is determined by $p_j$ and represents whether the $j$-cluster has prograde or retrograde rotation with respect to the whole system. In the case of rational $\nu = (p - q)/q$, the prograde ($p_j > 0$) refers to the case that the cluster rotates in the same direction as the main relative equilibrium, while retrograde ($p_j < 0$) refers to the case that the cluster rotates in the opposite direction.

The strategy still applies for the gravitational potential under extra assumptions. In this case, we have from condition (C1) that

$$r_1 = \varepsilon, \quad \omega_1 = \varepsilon^{-(\alpha+1)/2}, \quad \nu = \varepsilon^{-(\alpha+1)/2} - 1.$$

**Theorem 3.9** (Carousel for gravitational potentials) Set $\alpha = 2$, assume conditions (C0)–(C2) and choose the frequencies $\omega_1, \nu$ and the amplitude $r_1$ according to (C1). Suppose that $a_0$ is a nondegenerate central configuration of the $n$-body problem, and $a_1$ is a $2\pi/m$-nondegenerate central configuration. Then for every sufficiently small $\varepsilon$, there are at least $\text{Cat}(G'/K) = 2$ solutions of the $N$-body problem with components of the form

$$q_{1,k}(t) = \exp(tJ)u_{0,1}(\nu t) + r_1 \exp(t\omega_1 J)u_{1,k}(\nu t),$$

$$q_{j,1}(t) = \exp(tJ)u_{0,j}(\nu t) \quad \text{for } j = 2, \ldots, n.$$

where $u_{0,j}(\nu t) = a_{0,j} + O_X(\varepsilon)$ and $u_{1,k}(\nu t) = e^{\theta_1 J}a_{1,k} + O_X(\varepsilon)$ for some phase $\theta_1 \in S^1$. Furthermore, each $u_{1,k}(s)$ is $2\pi/m$-periodic with $m \geq 2$ and

$$u_{0,j}(s) = \exp(-\theta J)u_{0,1}(s + \theta) \quad \text{(3.22)}$$

where $(\theta, \sigma)$ is the generator of the discrete symmetry group $\Gamma$ introduced in (C0).
Example 3.7 The particular case $k_1 = 2$ and $k_2, \ldots, k_n = 1$ is studied in [17] in the general setting $E = \mathbb{R}^d$. This follows from the fact that the Euler–Lagrange equations (2.9) for $j = 1$ are equivalent to a 2-body problem in a rotating frame of frequency $\omega$. That is, the system for $u_1 \in E_1$ can be parametrized by $w_1 \in E$ as

$$E_1 = \left\{ u_1 = (u_{1,1}, u_{1,2}) \in E^2 : u_{1,1} = \frac{m_{1,2}}{m_{1,1} + m_{1,2}} w_{1}, u_{1,2} = - \frac{m_{1,1}}{m_{1,1} + m_{1,2}} w_{1} \right\}.$$ 

In these coordinates $(u_0, w_1) \in E_0 \times E$ the action $A_0(u_0, u_1)$ becomes

$$A_0(u_0, w_1) = \int_0^{2\pi} \left( \mathcal{L}_0(u_0, \dot{u}_0) + r_1^{1-\alpha} \mathcal{L}_1(w_1, \dot{w}_1) \right),$$

where

$$\mathcal{L}_1(w_1, \dot{w}_1) = \frac{m_{1,1} m_{1,2}}{m_{1,1} + m_{1,2}} \| (\partial_t + \omega J) w_1 \|^2 + m_{1,1} m_{1,2} \phi_\alpha(\|w_1\|),$$

which corresponds to the Lagrangian of the Kepler problem in a rotating frame of frequency $\omega$. We are then left with a Kepler problem for $\mathcal{L}_1(w_1, \dot{w}_1)$ and an $n$-body problem for $\mathcal{L}_0(u_0, \dot{u}_0)$. We can normalize the total mass of the cluster by setting $m_{1,1} + m_{1,2} = 1$. The solutions (2.12) are of the form

$$q_{1,1}(t) = \exp(tJ)u_{0,1}(vt) + \varepsilon m_{1,2} \exp(\omega_1 t J) w_1(vt),$$

$$q_{1,2}(t) = \exp(tJ)u_{0,1}(vt) - \varepsilon m_{1,1} \exp(\omega_1 t J) w_1(vt),$$

$$q_{j,1}(t) = \exp(tJ)u_{0,j}(vt), \quad j = 2, \ldots, n,$$

which are exactly the solutions obtained in [17] for the particular case $E = \mathbb{R}^2$.

4 The $2\pi p$-nondegeneracy property of the $k$-polygon

In this section we verify the $2\pi p$-nondegeneracy property of a polygonal central configuration (with frequency one) for $k$ bodies with masses equal to one. We shall denote by $E_{\text{red}}^k$ the subspace of $E^k$ of central configurations with center of mass fixed at the origin. With our previous notations, we have $E_j = E_{\text{red}}^{kj}$. By (3.7), a central configuration $a \in E_{\text{red}}^k$ is $2\pi p$-nondegenerate if the block $\hat{T}_{\ell,u} \in \text{End}(E_{\text{red}}^k)^C$ given by

$$\hat{T}_{\ell,u} = (\ell^2 + 1)^{-1} P_{E_{\text{red}}^k} M_a(\ell/p)|_{E_{\text{red}}^k} \in \text{End}(E_{\text{red}}^k)^C$$

is a non-singular matrix. Here

$$M_a(\lambda) = \lambda^2 I - 2i\lambda J + \nabla_a^2 V[a] \in \text{End}(E^k)^C$$

and $V$ is the amended potential

$$V(u) = \frac{1}{2} \sum_{j=1}^k \|u_j\|^2 + \sum_{1 \leq i < j \leq k} \phi_\alpha(\|u_j - u_i\|).$$

Specifically, the $2\pi p$-nondegeneracy property of $a$ is equivalent to the conditions:

(a) $\hat{T}_{\ell,u}$ is invertible for all $\ell \neq 0$,

(b) $\hat{T}_{0,u}$ has a one dimensional kernel generated by $J_{ka}$.
4.1 Spectrum of the polygonal configuration

First we find the ratio of the polygonal relative equilibrium with frequency one.

**Proposition 4.1** The polygonal configuration

\[ a = (s_1)^{\frac{1}{2\pi}} (\exp(J\zeta)e_1, \ldots, \exp(kJ\zeta)e_1), \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]  

(4.3)

is a central configuration of frequency one, where \( \zeta = 2\pi/k \) and

\[ s_1 = \frac{1}{2\alpha} \sum_{j=1}^{k-1} \frac{1}{\sin^{\alpha-1}(j\zeta/2)}. \]

**Proof** We shall show that \( a \) is a critical point of the amended potential \( V \). Consider the function

\[ \tilde{V}(u; \omega) = \frac{\omega}{2} \sum_{j=1}^{k} \|u_j\|^2 + \sum_{1 \leq i < j \leq k} \phi_\alpha (\|u_j - u_i\|). \]  

(4.4)

It satisfies \( \tilde{V}(ru; 1) = V(u) \) and has the scaling property

\[ \nabla \tilde{V}(ru; r^{-(\alpha+1)}\omega) = r^{-\alpha} \nabla \tilde{V}(u; \omega) \quad \text{for any} \quad r > 0. \]  

(4.5)

Thus \((\tilde{a}; \omega)\) is critical point of \( \tilde{V} \) if and only if \((r\tilde{a}; r^{-(\alpha+1)}\omega)\) is a critical points of \( \tilde{V} \) for any \( r > 0 \). By [18], the unitary polygon

\[ \tilde{a} = (\exp(J\zeta)e_1, \ldots, \exp(kJ\zeta)e_1), \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

with \( \omega = s_1 \) is a critical point of \( \tilde{V} \). It follows that \((a; 1)\) where \( a = (s_1)^{\frac{1}{2\pi}} \tilde{a} \) is a critical point of \( \tilde{V}(u; 1) = V(u) \). In other words, \( a \) is a central configuration with frequency one. \( \square \)

**Block diagonalisation**

Let \( S_k \) be the permutation group of \( k \) letters. The group \( G = S_k \times SO(2) \) acts on \( E^k \) by

\[ (\sigma, \theta) \cdot (u_1, \ldots, u_k) = (\exp(-J\theta)u_{\sigma(1)}, \ldots, \exp(-J\theta)u_{\sigma(k)}). \]

The amended potential (4.2) is \( G \)-invariant. Let \( a \in E^k \) be the polygonal configuration (4.3). Its stabiliser is the subgroup \( G_a = C_k \) generated by the element \((\sigma, \zeta) \in G \) where \( \sigma = (1 \ldots k) \) and \( \zeta = \frac{2\pi}{k} \). The \( G_a \)-equivariant property of the Hessian \( \nabla_a^2 V[a] \) is used in Proposition 7 of [18] to find the irreducible representations of \((E^C)^k\). By Schur’s lemma the Hessian of \( V \) is equivalent to a block diagonal matrix with \( k \) blocks corresponding to the isotypic components.

**Definition 4.1** For \( j = 1, \ldots, k \), we define isomorphisms \( T_j : E^C \to W_j \) by

\[ T_j(w) = \frac{1}{\sqrt{k}} (\exp((ijI + J)\zeta)w, \ldots, \exp(k(ijI + J)\zeta)w), \]  

(4.6)

where

\[ W_j = \{ (\exp((ijI + J)\zeta)w, \ldots, \exp(k(ijI + J)\zeta)w) \mid w \in E^C \} \subset (E^C)^k. \]
Specifically, in [18] is proved that the subspaces $W_j$ are the isotypic components under the action of $G_a$. The group $G_a$ acts on each subspace $W_j$ by rotating each component by $\exp(i j \zeta J)$. Since the subspaces $W_j$ are mutually orthogonal, the endomorphism $P \in \text{End}((E^C)^k)$ defined by

$$P(w_1, \ldots, w_k) = \sum_{j=1}^{k} T_j(w_j)$$

is orthogonal. Since $P$ rearranges the coordinates of the isotypic decomposition, it follows by Schur’s Lemma that

$$P^{-1} \nabla_u^2 V[a] P = B_1 \oplus \cdots \oplus B_k$$

(4.7)

where each $B_j \in \text{End}(E^C)$ satisfies

$$\nabla_u^2 V[a] T_j(w) = T_j(B_j w).$$

(4.8)

Define

$$s_j = \frac{1}{2\alpha} \sum_{i=1}^{k-1} \sin^2 \left( j l \frac{\zeta}{2} \right), \quad \zeta = \frac{2\pi}{k}.$$

(4.9)

Following [18], the numbers $s_j$ have the following properties: They are $k$-periodic, that is $s_{k+j} = s_j$. They are symmetric with respect to $[k/2]$, that is $s_{[k/2]-j} = s_{[k/2]-j}$. They increase as $j$ increases, that is

$$s_{j+1} > s_j, \quad 0 \leq j \leq n/2.$$

In particular, $s_j > s_0 = 0$ for $j = 1, \ldots, [n/2]$ with its maximum attained at $j = [n/2]$.

**Proposition 4.2** [Normal form of the amended potential] Each endomorphism $B_j$ is a matrix of the form

$$B_j = (1 + \alpha_j) I - \beta_j R - \gamma_j i J$$

(4.10)

where $I$ is the identity matrix,

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the coefficients are given by

$$\alpha_j = \frac{\alpha - 1}{4s_1} (s_{j+1} + s_{j-1}), \quad \beta_j = \frac{\alpha + 1}{2s_1} (s_j - s_1), \quad \gamma_j = \frac{\alpha - 1}{4s_1} (s_{j+1} - s_{j-1}).$$

(4.11)

**Proof** By the scaling property (4.5) of the amended potential $\tilde{V}$ given in (4.4), we have

$$\nabla_u^2 V[a] = \nabla_u^2 \tilde{V}[(a; 1)] = \frac{1}{s_1} \nabla_u^2 \tilde{V}[(\tilde{a}; s_1)].$$

(4.12)

By (4.8) we get

$$\nabla_u^2 V[a] T_j(w) = \frac{1}{s_1} \nabla_u^2 \tilde{V}[(\tilde{a}; s_1)] T_j(w) = T_j \left( \frac{1}{s_1} \tilde{B}_j w \right).$$

The matrices $\tilde{B}_j$ are computed in [18] and are given by

$$\tilde{B}_j = (s_1 + \tilde{a}_j) I - \tilde{\beta}_j R - \tilde{\gamma}_j i J$$

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where
\[ \tilde{\alpha}_j = \frac{\alpha - 1}{4}(s_{j+1} + s_{j-1}), \quad \tilde{\beta}_j = \frac{\alpha + 1}{2}(s_j - s_1), \quad \tilde{\gamma}_j = \frac{\alpha - 1}{4}(s_{j+1} - s_{j-1}). \]

The result follows from \( \nabla^2_w V[a] T_j(w) = T_j(B_j(w)) \) with \( B_j = \frac{1}{s_1} \tilde{B}_j. \)

We can now find a block diagonalisation of the matrix
\[ M_a(\lambda) = \lambda^2 I - 2i\lambda J + \nabla^2_w V[a] \]
that appears in (4.1). With respect to the isotypic decompositions, we get by (4.7)
\[ P^{-1} M_a(\lambda) P = m_1(\lambda) \oplus \cdots \oplus m_k(\lambda), \]
where each block \( m_j(\lambda) \in \text{End}(E^C) \) is given by
\[ m_j(\lambda) = \lambda^2 I - 2\lambda i J + B_j. \] (4.13)

Zero eigenvalues arising from symmetries

The analysis of the spectrum of the blocks \( m_j(\lambda) \) for \( j = 1, k - 1, k \) is special because the block \( m_k(\lambda) \) contains the generator of the \( \text{SO}(2) \)-orbit and the blocks \( m_1(\lambda) \) and \( m_{k-1}(\lambda) \) contain vectors in the orthogonal complement to \( (E^k_{\text{red}})^C \). These cases are treated separately in Lemmas 4.3 and 4.4.

**Lemma 4.3** Given the block diagonal matrix \( P^{-1} M_a(\lambda) P = m_1(\lambda) \oplus \cdots \oplus m_k(\lambda), \)

1. The block \( m_k(\ell/p) \) is invertible for all \( \ell \in \mathbb{Z}/[0] \) when \( p (3 - \alpha)^{1/2} \notin \mathbb{N}. \)
2. The block \( m_k(0) \in \text{End}(E^C) \) is invertible on the orthogonal complement of the line spanned by \( e_2 = Je_1 \) which is a generator of the space tangent to the \( \text{SO}(2) \)-orbit at \( a. \)

**Proof** 1. The coefficients in (4.11) have the property \( s_j = s_{-j} = s_{k-j} \) and \( s_k = 0. \) The coefficients (4.11) are then given by
\[ \alpha_k = \frac{\alpha - 1}{2}, \quad \beta_k = -\frac{\alpha + 1}{2}, \quad \gamma_k = 0. \]

From (4.10) we get
\[ m_k(\lambda) = \lambda^2 I - 2\lambda i J + \frac{\alpha + 1}{2}(I + R) = \begin{pmatrix} \lambda^2 + 1 & 2\lambda i \\ -2\lambda i & \lambda^2 \end{pmatrix} \]
which is the linearisation of the Kepler problem as in [17] and corresponds to the invariant manifold of homographic solutions. This matrix has eigenvalues
\[ \mu_k^\pm(\lambda) = \frac{\alpha + 1}{2} + \lambda^2 \pm \frac{1}{2} \sqrt{(\alpha + 1)^2 + 16\lambda^2}. \]

The eigenvalue \( \mu_k^+(\lambda) \neq 0 \) for all \( \lambda \) and \( \mu_k^-(\lambda) \neq 0 \) if \( \lambda \notin \{0, \sqrt{3 - \alpha}\}. \) In our analysis, \( \lambda = \ell/p \) where \( \ell, p \) are integers and \( \ell \neq 0. \) In particular, the operator is invertible if we suppose that \( p (3 - \alpha)^{1/2} \notin \mathbb{N}. \) When \( \alpha = 2, \) note that the eigenvalues \( \mu_k^+(\ell/p) \) for \( \ell = \pm p \) are equal to zero due to the existence of the homographic elliptic orbits of the gravitational \( k \)-body problem.
2. Since \( a = s_1^{-1} \exp(J\zeta)e_1, \ldots, \exp(kJ\zeta)e_1 \), the generator of the tangent space of its \( SO(2) \)-orbit is

\[
J_ka = s_1^{-1} (J \exp(J\zeta)e_1, \ldots, J \exp(kJ\zeta)e_1) = s_1^{-1} T_k(e_2)
\]

since \( J e_1 = e_2 \). Therefore, the matrix \( m_k(0) = \text{diag}(\alpha + 1, 0) \) is singular only on the space tangent to the \( U(1) \)-orbit of \( a \) which is generated by \( e_2 \) in \( E^C \).

\[
\Box
\]

We now analyze the spectrum of \( m_1(\lambda) \) and \( m_{k-1}(\lambda) \). In our analysis, the matrix \( M_a(\lambda) \) is an endomorphism of the subspace reduced by the symmetry of translations \((E^k_{\text{red}})^C\). Therefore, the spectrum of the blocks \( m_1(\lambda) \) and \( m_{k-1}(\lambda) \) must be analyzed on the subspaces \( T_j^{-1}((W_j)_{\text{red}}) \) for \( j = 1 \) and \( j = k - 1 \) respectively. We use the notation

\[
(W_j)_{\text{red}} = W_j \cap (E^k_{\text{red}})^C.
\]

**Lemma 4.4** If \( \alpha > 1 \), the matrix \( m_j(\lambda) \) restricted to the subspace \( T_j^{-1}((W_j)_{\text{red}}) \) for \( j = 1, k - 1 \) is invertible for any \( \lambda \).

**Proof** Case \( j = 1 \). By (4.13) and Proposition 4.2, the coefficients in (4.10) are

\[
\alpha_1 = \gamma_1 = \frac{\alpha - 1}{4s_1} s_2 \quad \beta_1 = 0
\]

and then

\[
m_1(\lambda) = (\lambda^2 + 1 + \alpha_1)I - (2\lambda + \alpha_1) i J = \left( \begin{array}{cc} \lambda^2 + 1 + \alpha_1 & (2\lambda + \alpha_1) i \\ - (2\lambda + \alpha_1) i & \lambda^2 + 1 + \alpha_1 \end{array} \right).
\]

The eigenvalues of \( m_1(\lambda) \) are \( \mu_1(\lambda) = (\lambda - 1)^2 \) with eigenvector \( w_1 = (1, i) \) and \( \mu_2(\lambda) = (\lambda + 1)^2 + 2\alpha_1 \) with eigenvector \( w_2 = (1, -i) \). The first eigenvalue \( \mu_1(\lambda) \) vanishes when \( \lambda = 1 \), but we will show that the corresponding eigenvector \( w_1 \) does not belong to \( T_1^{-1}((W_1)_{\text{red}}) \).

By (4.6)

\[
T_1(w_1) = \frac{1}{\sqrt{k}} (\exp((iI + J)\zeta) w_1, \ldots, \exp(k(iI + J)\zeta) w_1).
\]

Observe that

\[
\exp(j\zeta J) w_1 = \begin{pmatrix} \cos j\zeta - \sin j\zeta \\ \sin j\zeta & \cos j\zeta \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos (j\zeta) - i \sin (j\zeta) \\ i \cos (j\zeta) + \sin (j\zeta) \end{pmatrix} = e^{-ij\zeta} I w_1
\]

for \( j = 1, \ldots, k \) from which it follows that

\[
T_1(w_1) = \frac{1}{\sqrt{k}} (w_1, \ldots, w_1).
\]

In particular, \( T_1(w_1) \) does not belong to \( (W_1)_{\text{red}} \). The matrix \( m_1(\lambda) \) restricted to \( T_1^{-1}((W_1)_{\text{red}}) \) is given by \( (\lambda + 1)^2 + 2\alpha_1 \), which is invertible for \( \alpha > 1 \).
Case $j = k - 1$.

By (4.13) and Proposition 4.2, the coefficients in (4.10) are
\[ \alpha_{k-1} = -\gamma_{k-1} = \frac{\alpha - 1}{4s_1} s_2 = \alpha_1 \quad \beta_1 = 0 \]
and then
\[ m_{k-1}(\lambda) = \left( \begin{array}{cc} \lambda^2 + 1 + \alpha_1 & (2\lambda - \alpha_1) i \\ -2(\lambda - \alpha_1) i & \lambda^2 + 1 + \alpha_1 \end{array} \right) \]
are $\mu_1(\lambda) = (\lambda + 1)^2$ with eigenvector $w_1 = (1, -i)$ and $\mu_2(\lambda) = (\lambda - 1)^2 + 2\alpha_1$ with eigenvector $w_2 = (1, i)$. Since $\exp(\jmath \zeta J) w_1 = e^{\jmath \zeta J} I w_1$ for $j = 1, \ldots, k$, we get
\[ T_{k-1}(w_1) = \frac{1}{\sqrt{k}} (w_1, \ldots, w_1). \]
In particular, $T_{k-1}(w_1)$ does not belong to $(W_{k-1})_{red}$. The matrix $m_{k-1}(\lambda)$ restricted to $T_{k-1}^{-1}((W_{k-1})_{red})$ is given by $(\lambda - 1)^2 + 2\alpha_1$, which is invertible for $\alpha > 1$.  \(\Box\)

4.2 The $2\pi p$-nondegeneracy property of the $k$-polygon for weak forces

We now study the special case of weak forces.

**Proposition 4.5** Assume that
\[
(i) \quad p(3 - \alpha)^{1/2} \notin \mathbb{N} \quad \text{(ii) } \quad p^2 \frac{j(k - j)}{k - 1} \notin \mathbb{N} \quad j = 2, \ldots, k - 2.
\]
Then there exists $\delta > 0$ such that the $k$-polygon is $2\pi p$-nondegenerate for any $\alpha \in (1, 1 + \delta)$.

**Proof** By Proposition 4.3 and assumption (i), the block $m_k(\ell/p)$ is invertible for $\ell \neq 0$ and $m_k(0)$ is invertible in an orthogonal complement to the generator of the orbit. By Proposition 4.4 and the assumption that $\alpha > 1$, the blocks $m_j(\ell/p)$ restricted to $T_j^{-1}((W_j)_{red})$ are invertible for $j = 1, k - 1$. It remains to show that the blocks $m_j(\ell/p)$ are invertible for all $j = 2, \ldots, k - 2$ and $\ell \in \mathbb{Z}$. In the logarithmic case $\alpha = 1$ the numbers $s_j$ can be computed explicitly [19] and are given by $s_j = j(k - j)/2$. In this case, the coefficients in (4.11) are
\[ \alpha_j = \gamma_j = 0, \quad \beta_j = s_j/s_1 - 1. \]
We can compute the matrix (4.13) explicitly and the determinant is
\[ \det (m_j(\lambda)) = (\lambda^2 - 1)^2 - \beta_j^2. \]
Thus $\det (m_j(\lambda)) \neq 0$ if and only if
\[ \lambda^2 \neq \frac{s_j}{s_1} = \frac{j(k - j)}{k - 1}. \]
Therefore, the blocks $m_j(\ell/p)$ are invertible if and only if assumption (ii) holds. Notice that there is $\ell_0 > 0$ such that the blocks $m_j(\ell/p)$ are always invertible for $|\ell| > \ell_0$, which reflects the compactness nature of the operators. By continuity of $m_j(\ell/p)$ with respect to $\alpha$, there is a $\delta > 0$ such that for any $\alpha \in (1, 1 + \delta)$, the remaining blocks $m_j(\ell/p)$ for $|\ell| < \ell_0$ are invertible for all $j = 2, \ldots, k - 2$. The result follows.  \(\Box\)
For \( \alpha \in (1, 1+\delta) \) and \( k \in \mathbb{N} \) set
\[
C_{k,\alpha} = \left\{ p \in \mathbb{N} : p \notin (k - 1)\mathbb{N} \cup (3 - \alpha)^{-1/2}\mathbb{N} \right\}.
\]
For each prime \( k \) and \( p \in C_{k,\alpha} \) we have that conditions (i) and (ii) hold. This follows from
the fact that \( k - 1 \) does not divide \( p^2 \) if \( p \in C_{k,\alpha} \), neither does \( j(k - j) \) for \( j = 2, \ldots, k - 1 \)
because \( k - 1 \) is prime. When \( (3 - \alpha)^{-1/2} \) is irrational, the set \( C_{k,\alpha} \) consists of the integers \( p \) that
are not divided by \( k - 1 \). If \( (3 - \alpha)^{-1/2} = p/q \) is rational, then \( (3 - \alpha)^{-1/2} > 1/2 \)
for \( \alpha \in (1, 1+\delta) \), \( p > 1 \) and \( C_{k,\alpha} = \{ p \in \mathbb{N} : p \notin (k - 1)\mathbb{N} \cup p\mathbb{N} \} \) is the infinite set of
integers \( p \) that are not multiples of \( k - 1 \) or \( p \). In both cases \( C_{k,\alpha} \) is an infinite set. We have
the following theorem,

**Theorem 4.6** There is a small \( \delta > 0 \) such that for any \( \alpha \in (1, 1+\delta) \), the \( k \)-polygon is
\( 2\pi p \)-nondegenerate if \( k - 1 \) is prime and \( p \) is chosen from the infinite set \( C_{k,\alpha} \).

### 4.3 The \( n \)-polygon property of the \( np \)-polygon for gravitational forces

In this section we verify the \( 2\pi/m \)-nondegeneracy property in the gravitational case. Before
proving this property, we use computer-assisted proofs to validate that there are no integers \( \ell \in \mathbb{N} \)
such that \( \det (m_j(\ell)) = 0 \) for \( j = 2, \ldots, k - 2 \).

**Proposition 4.7** For each \( k \) from 4 to 1000, the polynomial
\[
P_j(\lambda) = \det (m_j(\lambda))
\]
has no integer roots for \( j = 2, \ldots, k - 2 \).

**Proof** When \( \alpha = 2 \), we get by (4.11)
\[
\alpha_j - \gamma_j = \frac{s_j-1}{2s_1} \geq 0 \quad \alpha_j + \gamma_j = \frac{s_j+1}{2s_1} \geq 0.
\]
Since
\[
P_j(\lambda) = ((\lambda - 1)^2 + \alpha_j - \gamma_j)((\lambda + 1)^2 + \alpha_j + \gamma_j) - \beta_j^2 \\
\geq (\lambda - 1)^2 (\lambda + 1)^2 - \beta_j^2 \geq (\lambda^2 - 1)^2 - \beta_j^2
\]
the polynomial has no roots for \( \lambda^2 - 1 \geq \beta_j^2 \). The result is obtained by validating rigorously,
using interval arithmetics in the package INTLAB in MATLAB, that \( P_j(\ell) \neq 0 \) for \( \ell = 0, \ldots, \sqrt{\beta_j^2 + 1} \) and \( j = 2, \ldots, k - 2 \). \( \square \)

**Theorem 4.8** For \( \alpha = 2 \), the \( k \)-polygon is \( 2\pi/m \)-nondegenerate with \( m > 1 \) for any \( k = 4, \ldots, 1000 \). In addition, there is a small \( \delta > 0 \) such that for all \( \alpha \in (2 - \delta, 2 + \delta) \cup \{2\} \),
the \( k \)-polygon is \( 2\pi \)-nondegenerate for any \( k = 4, \ldots, 1000 \).

**Proof** By Proposition 4.3 and the fact that \( \sqrt{3 - \alpha} \) is integer only if \( \alpha = 2 \) with \( \sqrt{3 - \alpha} = 1 \),
the block \( m_k(\ell) \) is invertible for \( \ell \in m\mathbb{Z} \) if \( \alpha = 2 \) and for \( \ell \in \mathbb{Z} \) if \( \alpha \in (2 - \delta, 2 + \delta) \setminus \{2\} \).
The blocks \( m_j(\ell) \) restricted to \( T_j^{-1}(\{(W_j)_{\text{red}}\}) \) are always invertible for \( j = 1, k - 1 \) by
Proposition 4.4, Proposition 4.7 and the continuity with respect to \( \alpha \) imply that the blocks
\( m_j(\ell) \) are invertible for all \( \alpha \in (2 - \delta, 2 + \delta) \), \( j = 2, \ldots, n - 2 \) and \( \ell \in \mathbb{Z} \). The result follows.
\( \square \)
4.4 The $2\pi p$-nondegeneracy property of the Lagrange triangle for different masses

Since the conditions of Remark 3.4 for the Hamiltonian is equivalent to the $2\pi p$-nondegeneracy property, we can use previous computations made for the analysis of the stability of the Lagrange triangular configuration. In the case $\alpha = 2$, according to the Gascheau result regarding the 3-body problem [38], the linearised Hamiltonian system at the Lagrange triangular configuration with masses $m_j$ has four pairs of zero-eigenvalues and four complex roots off the imaginary axis (leading to instability) when

$$\beta = 27 \frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} > 1.$$  

The four pairs of zero-eigenvalues correspond to the center of mass and the homographic elliptic orbits. The linearization of the homographic elliptic orbits leads to our block $m_k(\lambda)$. Therefore, the Lagrange triangular configuration is $2\pi/m$-nondegenerate for any $m > 1$ when $\beta > 1$.

In the case $\alpha \neq 2$, after fixing the center of mass equal to zero, there are four pairs of non-zero eigenvalues. Two pairs correspond to the linearization in the submanifold of homographic solutions having the block of the Kepler problem in our analysis $m_k(\lambda)$. This block is $2\pi p$-nonresonant when $p \notin (3 - \alpha)^{-1/2}\mathbb{N}$. The other two pairs of eigenvalues can be found explicitly in [36], and correspond in our setting to

$$\lambda_1^\pm = \pm \frac{1}{6} i \sqrt{18(1 - \alpha) + 6\sqrt{9(\alpha - 1)^2 - \beta(\alpha + 3)^2}}$$

$$\lambda_2^\pm = \pm \frac{1}{6} i \sqrt{18(1 - \alpha) - 6\sqrt{9(\alpha - 1)^2 - \beta(\alpha + 3)^2}}.$$  

These eigenvalues are off the imaginary axis when

$$\beta > 9 \left( \frac{3 - \alpha}{1 + \alpha} \right)^2. \quad (4.14)$$

Therefore, we conclude that for $\alpha \neq 2$, the Lagrange triangular configuration is $2\pi p$-nondegenerate when the inequalities $p \notin (3 - \alpha)^{-1/2}\mathbb{N}$ and (4.14) hold. Furthermore, since the eigenvalues are analytic in $\alpha$ and $\beta$, the Lagrange triangular configuration is generically $2\pi p$-nondegenerate for the homogeneous exponent $\alpha$ and the set of masses $m_j$. Observe that for equal masses $m_j = 1$ the inequality (4.14) does not hold precisely for $\alpha = 1$. This is the degeneracy found in the previous analysis for the blocks $m_1$ and $m_{n-1}$ in the case of $\alpha = 1$.

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