GEOMETRIC GALOIS THEORY, NONLINEAR NUMBER FIELDS AND A GALOIS GROUP INTERPRETATION OF THE IDELE CLASS GROUP

REVISED VERSION

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In memory of Egidio Barrera.

ABSTRACT. This paper concerns the description of holomorphic extensions of algebraic number fields. We define a hyperbolized adele class group \( \hat{\mathcal{S}}_K \) for every algebraic number field \( K \) and consider the Hardy space \( H_*([K]) \) of graded-holomorphic functions on \( \hat{\mathcal{S}}_K \). We show that the subspace \( N_*([K]) \) of the projectivization \( P^H_*([K]) \) defined by the functions of non-zero trace possesses two partially-defined operations \( \oplus \) and \( \otimes \), with respect to which there is canonical monomorphism of \( K \) into \( N_*([K]) \). We call \( N_*([K]) \) a nonlinear field extension of \( K \). We define Galois groups for nonlinear fields and show that \( \text{Gal}(N_*([L])/N_*([K])) \cong \text{Gal}(L/K) \) if \( L/K \) is Galois. If \( \mathbb{Q}^{ab} \) denotes the maximal abelian extension of \( \mathbb{Q} \), \( C_{\mathbb{Q}} \) the idele class group and \( \overline{N}([\mathbb{Q}^{ab}]) = P^H_*([K]) \) is the full projectivization then we show that there exist embeddings of \( C_{\mathbb{Q}} \) into \( \text{Gal}_\oplus(\overline{N}([\mathbb{Q}^{ab}]/\mathbb{Q})) \) and \( \text{Gal}_\otimes(\overline{N}([\mathbb{Q}^{ab}]/\mathbb{Q})) \), the “Galois groups” of automorphisms preserving \( \oplus \) resp. \( \otimes \) only.

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1. Introduction

Since the introduction of a global understanding of number theory (Gauss, Galois) and geometry (Riemann), the idea that the two subjects exist in parallel duality has exercised a tremendous pull on mathematical thought. An indication of this conjectural relationship can be found in the equivalence between (coverings of) Riemann surfaces and (extensions of) their fields of meromorphic functions, based on which, one can formulate the following meta-principle:

To every algebraic number field \( K / \mathbb{Q} \), there exists a “Riemann surface” \( \Sigma_K \) for which

\[
\text{Mer}(\Sigma_K) \cong K.
\]

If \( L / K \) is a Galois extension, then \( \Sigma_L \) is a Galois covering of \( \Sigma_K \) and

\[
\text{Deck}(\Sigma_L / \Sigma_K) \cong \text{Gal}(L/K).
\]

If we were interested in an extension \( E \) over the function field \( \mathbb{F}_{p^n}[X] \), then this principle is, suitably interpreted, correct: there exists a curve over \( \mathbb{F}_{p^n} \) whose function field is \( E \). This observation was used by Weil to prove the Riemann hypothesis for function fields over finite fields \([15]\).

In the case of a number field \( K \), the principle as stated above must be modified, as any reasonable notion of “field of meromorphic functions” must contain the field of constants \( \mathbb{C} \), which cannot be a subfield of \( K \). Thus we might instead ask that \( K \) generate \( \text{Mer}(\Sigma_K) \) over \( \mathbb{C} \). The additional freedom afforded by the use of \( \mathbb{C} \)-coefficients motivates a second meta-principle:

If \( K_{ab} \) is the maximal abelian extension of \( K \) and \( \mathcal{C}_K \) is the idele class group of \( K \), then there is a monomorphism

\[
\mathcal{C}_K \hookrightarrow \text{Gal}(\text{Mer}(\Sigma_{K_{ab}})/K).
\]

This second meta-principle – which is true for function fields over finite fields upon using rational function fields in place of meromorphic function fields – has an important place in Weil’s approach to the classical Riemann hypothesis, as the following often quoted passage reveals \([16]\):

“La recherche d’une interprétation pour \( \mathcal{C}_K \) si \( K \) est un corps de nombres, analogue en quelque manière à l’interprétation par un groupe de Galois quand \( K \) est un corps de fonctions, me semble constituer l’un des problèmes fondamentaux de la théorie des nombres à l’heure actuelle; il se peut qu’une telle interprétation renferme la clef de l’hypothèse Riemann...”

Weil’s speculation regarding a Galois group interpretation of \( \mathcal{C}_K \) has inspired new approaches to the Riemann hypothesis, e.g. especially that of Alain Connes \([3]\). In this paper, we shall give a certain expression to these principles through a hyperbolized version of the adele class group of \( K \).

Let us consider first a field \( K \) of finite degree \( d \) over \( \mathbb{Q} \). To \( K \) we may associate the adele class group \( \hat{\mathcal{A}}_K = \mathcal{A}_K / \mathcal{O}_K \) where \( \mathcal{O}_K \) is the ring of integers of \( K \). A \( d \)-dimensional solenoid whose leaves are dense and isomorphic to a product of the form

\[
K_\infty = \mathbb{R}^r \times \mathbb{C}^s
\]

where \( K \) has \( r \) real places and \( 2s \) complex places. From \( \hat{\mathcal{A}}_K \) we construct a hyperbolization \( \hat{\Sigma}_K \) where leaves are polydisks isomorphic to \( \mathbb{H}^d \) and whose distinguished boundary is \( \hat{\mathcal{A}}_K \). See §§2\,3 and \([6]\) for more details.

For a number field \( \mathcal{N} / \mathbb{Q} \) of infinite degree, such as the maximal abelian extension \( K_{ab} \), the notion of adele class group has not yet, to our knowledge, been defined. In
order to redeem the desired properties found in the adele class group of a finite field extension, it is necessary to consider a pair of adele class groups that work in tandem. Representing \( \mathcal{K} = \lim_{\rightarrow} K_\lambda \), where the \( K_\lambda \) are finite degree extensions of \( \mathbb{Q} \), the (ordinary) adele class group \( \hat{\mathcal{K}} \) is formed from \( \lim_{\rightarrow} \hat{K}_\lambda \) by completing its canonical leaf-wise euclidean metric, see §4. Since \( \hat{\mathcal{K}} \) is not locally-compact, we consider in §5 a compactification \( \hat{\mathcal{K}} \) called the proto adele class group of \( \mathcal{K} \), arising as the inverse limit of the trace maps. We use the hilbertian \( \hat{\mathcal{K}} \) to define the hyperbolization \( \hat{\mathcal{K}} \), and the compact \( \hat{\mathcal{K}} \) to provide Fourier analysis. See §§4, 5.

The origin of the notion of a nonlinear field comes from an enhanced understanding of the character group \( \text{Char}(\hat{\mathcal{K}}) \), described in §7. The character group possesses an additional operation making it a field, and there is a canonical isomorphism

\[
\text{Char}(\hat{\mathcal{K}}) \cong K.
\]

If we denote by \( \mathcal{T}_K = K_\infty/O_K \) the Minkowski torus, the above isomorphism identifies

\[
\text{Char}(\mathcal{T}_K) \cong \mathcal{O}_K^{-1} \rightarrow O_K
\]

where \( \mathcal{O}_K^{-1} \) is the inverse different. Thus \( \text{Char}(\mathcal{T}_K) \) has the enhanced structure of an \( O_K \)-module canonically extending the ring \( O_K \). The same is true for an infinite field extension \( \mathcal{K} \) that is Galois over \( \mathbb{Q} \) if we use the proto adele class group \( \hat{\mathcal{K}} \).

Let \( f \in L^2(\hat{\mathcal{K}}, \mathbb{C}) \) be the Hilbert space of square integrable complex-valued functions with respect to normalized Haar measure. By Fourier theory, \( f \) has the development

\[
f = \sum a_a \phi_a
\]

where \( a \in K, a_a \in \mathbb{C} \) and \( \phi_a \in \text{Char}(\hat{\mathcal{K}}) \), and so the isomorphism (1) defines an inclusion \( K \hookrightarrow L^2(\hat{\mathcal{K}}, \mathbb{C}) \). Cauchy (point-wise) multiplication of functions, when defined, is denoted \( f \circ g \) since it restricts to \( +_K \) in \( K \). The operation corresponding to \( \times_K \) is the Dirichlet product \( f \circ g \), and through it \( L^2(\hat{\mathcal{K}}, \mathbb{C}) \) acquires the structure of a partial double group algebra, where partial refers to the fact that the two operations are only defined when square integrability is conserved. The departure from ordinary field theory begins with the observation that Dirichlet multiplication does not distribute over Cauchy multiplication, or to put it differently, the extension of multiplication in \( K \) to \( L^2(\hat{\mathcal{K}}, \mathbb{C}) \) no longer defines a Cauchy bilinear operation. See §8.

An interpretation of \( L^2(\hat{\mathcal{K}}, \mathbb{C}) \) as boundary values of holomorphic functions on \( \hat{\mathcal{K}} \) comes through the introduction of graded holomorphicity, treating, in the style of conformal field theory, each notion of holomorphicity on the same footing as the classical one. Suppose first that \( K \) is totally real. Denote by \( \Theta_K = (-,+)^d \) the sign group. To each \( \theta \in \Theta_K \) we associate the Hardy space \( H_0[K] \) of \( \theta \)-holomorphic functions on \( \hat{\mathcal{K}} \), and every \( f \in L^2(\hat{\mathcal{K}}, \mathbb{C}) \) determines a \( 2^d + 1 \) tuple \((F_0;F_\theta)\) consisting of \( \theta \)-holomorphic components and the constant term \( F_0 = a_0 \). In particular, there is an isomorphism of Hilbert spaces

\[
H_\ast[K] := \bigoplus_\theta H_0[K] \oplus \mathbb{C} \cong L^2(\hat{\mathcal{K}}, \mathbb{C}).
\]

The graded Hilbert space \( H_\ast[K] \) inherits the partially-defined operations of \( \oplus \) and \( \otimes \), where the Dirichlet product has a homogeneous decomposition with respect to the grading of \( H_\ast[K] \). See §5.
When $K$ is totally complex we consider, for each complex place, a pair of order four signings: the singular complex sign group $\Theta^c = \{\sqrt{\cdot}, -\sqrt{\cdot}, \sqrt{\cdot}, -\sqrt{\cdot}\}$, which signs points on $(\mathbb{R} \cup i\mathbb{R}) - 0$ according to the axial component that they belong to, and the complex sign set $\Omega = (\sqrt{\cdot}, e^\omega, -\sqrt{\cdot}, e^\omega)$ which signs points in $\mathbb{C} - (\mathbb{R} \cup i\mathbb{R})$ according to the quadrant they belong to. If we denote by $\mathring{\Omega} = \Theta^c \cup \Omega$ then there is a subset $\mathring{\Omega}_K \subset \mathring{\Omega}$ with respect to which we may define, for each $\vartheta \in \mathring{\Omega}_K$, a Hardy space $H_{\vartheta}[K]$ of $\vartheta$-holomorphic functions on $\mathring{\Sigma}_K$. We obtain in analogy with the real case an isomorphism
\[ H_{\vartheta}[K] := \bigoplus_{\vartheta} H_{\vartheta}[K] \cong \mathbb{C} \cong L^2(\mathring{\Sigma}_K, \mathbb{C}). \]

See [5.2] In the hybrid case where $K$ has both real and complex planes we obtain a direct sum $H_{\vartheta}[K] = H_{\vartheta}^c[K] \oplus H_{\vartheta}^c[K]$ consisting of the $\vartheta$-holomorphic and $\vartheta$-holomorphic parts.

Since the vector space structure of $H_{\vartheta}[K]$ does not descend to $K$, it is natural to discard it by projectivizing. After removing the functions whose trace is defined and equal to zero, we obtain an infinite-dimensional affine subspace
\[ N_{\vartheta}[K] \subset PH_{\vartheta}[K] \]
endowed with a partial double semigroup structure induced from $\oplus$ and and $\otimes$, satisfying the following properties:

1. Let $\mathbb{C}[K]$ be the field algebra (double group algebra) associated to $K$ and let $N_{\vartheta}^0[K] \subset \mathbb{C}[K]$ denote the sub double semigroup of elements of non zero trace, graded according to the same scheme described above. Then there is a graded monomorphism $N_{\vartheta}^0[K] \hookrightarrow N_{\vartheta}[K]$ with dense image.
2. The identity $id_{\vartheta}$ is a universal annihilator for the product $\otimes$: for all $f \in N_{\vartheta}[K]$ for which $f \otimes id_{\vartheta}$ is defined, $f \otimes id_{\vartheta} = id_{\vartheta}$. We call a topological partial double semi-group satisfying these properties an (abstract) nonlinear number field over $K$. The ring of integers $O_K$ of $K$ generates in turn the nonlinear ring of integers $N_{\vartheta}[O_K] \subset N_{\vartheta}[K]$, a nonlinear extension of $O_K$. On a dense subset of $N_{\vartheta}[K]$, every element may be regarded as a Dirichlet quotient of elements of $N_{\vartheta}[O_K]$. See [9].

An automorphism of the nonlinear number field $N_{\vartheta}[K]$ is defined to be the restriction of a graded Fubini-Study isometry preserving the operations $\oplus$ and $\otimes$. Given $K$ an algebraic number field, denote by
\[ \text{Gal}(N_{\vartheta}[K]/K) \]
the group of automorphisms of $N_{\vartheta}[K]$ fixing $K$; if $L/K$ is Galois, denote by
\[ \text{Gal}(N_{\vartheta}[L]/N_{\vartheta}[K]) \]
the group of automorphisms of $N_{\vartheta}[L]$ fixing $N_{\vartheta}[K]$.

**Theorem.** For all $K$, $\text{Gal}(N_{\vartheta}[K]/K)$ is trivial. If $L/K$ is Galois, then
\[ \text{Gal}(N_{\vartheta}[L]/N_{\vartheta}[K]) \cong \text{Gal}(L/K). \]

We consider finally the case $K = \mathbb{Q}$. In order to interpret the idele class group $C_{\mathbb{Q}}$ as a Galois group within this framework, the operations $\oplus$ and $\otimes$ must be decoupled. Let us consider $N_{\vartheta}[\mathbb{Q}] = PH_{\vartheta}[\mathbb{Q}]$, an abstract nonlinear number field containing $N_{\vartheta}[\mathbb{Q}]$. We show that there exist flows
\[ [\Phi]: \mathbb{Q}_\infty \rightarrow \text{Gal}_\vartheta(N_{\vartheta}[\mathbb{Q}]/\mathbb{Q}), \quad [\Psi]: \mathbb{Q}_\infty \rightarrow \text{Gal}_\vartheta(N_{\vartheta}[\mathbb{Q}]/\mathbb{Q}), \]
where $\text{Gal}_a, \text{Gal}_b$ denote the groups of automorphisms of $\bar{\mathbb{N}}_1[\mathbb{Q}]$ preserving $a, \otimes$ only. Using the above theorem with $L = \mathbb{Q}^{ab}$ leads to the following

**Theorem.** There are monomorphisms

$C_0 \hookrightarrow \text{Gal}_a(\bar{\mathbb{N}}_1[\mathbb{Q}^{ab}]/\mathbb{Q}), \quad C_0 \hookrightarrow \text{Gal}_b(\bar{\mathbb{N}}_1[\mathbb{Q}^{ab}]/\mathbb{Q})$.

These results are proved in §10

**Remark:** We have recently written some notes [5] which expand on the ideas described in this paper.

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## 2. Solenoids

We review here a notion fundamental to this paper. References: [11], [6], [7], [10].

Let $T$ be a 2nd countable, Hausdorff space. An $n$-lamination is a 2nd countable, Hausdorff space $L$ equipped with a maximal atlas of homeomorphisms

$$\{\phi_a : L \supset U_a \rightarrow V_a \subset \mathbb{R}^n \times T\},$$

in which every overlap $\phi_a \beta = \phi_\beta \circ \phi_a^{-1}$ is of the form

$$\phi_{a\beta}(x, t) = (h_{a\beta}(x, t), f_{a\beta}(t)),$$

where $t \mapsto h_{a\beta}(\cdot, t)$ is a continuous family of homeomorphisms and $f_{a\beta}$ is a homeomorphism. We call $L$ a foliation if $T = \mathbb{R}^k$, a solenoid if $T$ is a Cantor set.

Let $\phi$ be a chart, $D \subset \mathbb{R}^n$ an open disk, $T' \subset T$ open. A flowbox is a subset of the form $\phi^{-1}(D \times T')$, a flowbox transversal a subset of the form $\phi^{-1}(x \times T')$ and a plaque a subset of the form $\phi^{-1}(D \times \{t\})$. A leaf is a maximal continuation of overlapping plaques: by definition, an $n$-manifold. A riemannian metric on a smooth lamination is a family $Y = \{\gamma_t\}$ of smooth riemannian metrics, one on each leaf $\ell$, which when restricted to a flowbox gives a continuous family $t \mapsto \gamma_t$ of smooth metrics. If $L$ has the structure of a topological group such that the multiplication and inversion maps take leaves to leaves, $L$ is called a topological group lamination. If in addition $L$ is smooth and the multiplication and inversion maps are smooth along the leaves, $L$ is called a Lie group lamination.

Let $B$ be a manifold, $F$ a 2nd countable Hausdorff space, $\rho : \pi_1 B \rightarrow \text{Homeo}(F)$ a representation. The quotient

$$L_\rho = \left[\tilde{B} \times F\right]/\pi_1 B$$

by the action $\alpha \cdot (\tilde{x}, t) = (\alpha \cdot \tilde{x}, \rho_\alpha(t))$ is a lamination called the suspension of $\rho$. The projection $L_\rho \rightarrow B$ displays $L_\rho$ as a fiber bundle with fiber $F$, and the restriction of the projection to any leaf is a covering map.

The solenoids considered in this paper are modeled on the following example. Let $B = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the $d$-torus and let $F = \mathbb{Z}^d = \mathbb{N}^d/\mathbb{Z}^d$ be the profinite completion of $\mathbb{Z}^d$. Let

$$\rho : \pi_1 \mathbb{T}^d \cong \mathbb{Z}^d \rightarrow \text{Homeo}(\mathbb{Z}^d), \quad \rho_\alpha(\mathbf{m}) = \mathbf{m} - \mathbf{n}$$

for $\mathbf{n} \in \mathbb{Z}^d$ and $\mathbf{m} \in \mathbb{Z}^d$. The associated suspension is called the $d$-dimensional torus solenoid $\mathbb{S}^d$. Its leaves are its path components, are dense, and may be identified
with $\mathbb{R}^d$. Moreover, $\mathbb{F}^d$ is a compact, abelian, Lie group solenoid, and the additive group $\mathbb{R}^d$ sits inside as the path-component/leaf containing the identity.

One can extend the definition of a lamination to include infinite dimensional leaves modeled on a locally convex topological vector space $V$. If $V$ is a Hilbert space, we say we have a Hilbert space lamination. If $V = \mathbb{R}^d$ with the Tychonoff topology, we call it an $\mathbb{R}^d$-lamination. For Hilbert space laminations, one may use the Fréchet derivative to define smoothness, for $\mathbb{R}^d$-laminations one uses the Gâteaux derivative. One may make sense of all other notions discussed above in the infinite-dimensional setting.

3. Adele Class Groups I: Finite Field Extensions

The material here is classical and can be found in standard texts on algebraic number theory: [2], [8], [11], [13]. We review it in order to fix notation.

Let $K$ be an algebraic number field of degree $d$ over $\mathbb{Q}$, $O_K$ its ring of integers. If $K$ is Galois over $\mathbb{Q}$ we denote by $\text{Gal}(K/\mathbb{Q})$ the Galois group. By a local field, we mean a locally compact field. A local field of characteristic 0 is either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$, the field of $p$-adic numbers.

Let $v : K \to K_v$ be an embedding where $K_v$ is a local field (necessarily of characteristic 0) and $v(K)$ is dense. Embeddings that are related by a continuous isomorphism of target fields different from complex conjugation are deemed equivalent, and an equivalence class of embedding is called a place. When $K/\mathbb{Q}$ is Galois, we view $\sigma \in \text{Gal}(K/\mathbb{Q})$ as acting on the left of the places via $\sigma v := v \circ \sigma$. In practice, we shall use the word place to mean a representative of an embedding equivalence class, and we will write $q_v$ for $v(q)$, $q \in K$.

When $K_v$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, $v$ is said to be a real or complex infinite place. If $K$ has a complex place $\mu$, then there is an element $\sigma_{\text{conj}} \in \text{Aut}(K)$, called $\text{C}$-conjugation, such that $\sigma_{\text{conj}} \mu = \overline{\mu}$: so the complex places come in conjugate pairs $(\mu, \overline{\mu})$. We have $d = r + 2s$ where $r =$ the number of real places, $2s =$ the number of complex places. Let $\mathcal{P}_\infty = \{v_1, \ldots, v_r, \mu_1, \ldots, \mu_s, \overline{\mu}_1, \ldots, \overline{\mu}_s\}$, where $v_j$, $j = 1, \ldots, r$ are the real places and the $\mu_k, \overline{\mu}_k$, $k = 1, \ldots, s$, are the complex places. When $K/\mathbb{Q}$ is Galois, the places are either all real or all complex, in which case we will refer to $K$ as being either real or complex.

If $v$ is not infinite, it is said to be finite. The set of finite places will be denoted $\mathcal{P}_{\text{fin}}$. If $v \in \mathcal{P}_{\text{fin}}$, then $K_v$ has a maximal open subring $O_v$: its ring of integers. We note that $K_v$ is locally Cantor (totally-disconnected, perfect and locally compact) and $O_v$ is Cantor.

Denote

$$K_\infty = \{(z_v) \in \mathbb{C}^{\mathcal{P}_\infty} = \mathbb{C}^d \mid \hat{z}_v = z_v\} \cong \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^d.$$  

Note that $K_\infty$ has a canonical inner-product induced from the hermitian inner product on $\mathbb{C}^d$, which decomposes as the usual inner product on $\mathbb{R}^r$ and the usual hermitian inner-product on $\mathbb{C}^s$. $K$ embeds diagonally into $K_\infty$ via $q \mapsto (q_v)$ and the image of $O_K$ is a lattice in $K_\infty$. We shall identify $K$ and $O_K$ with their images in $K_\infty$. The quotient

$$\mathbb{T}_K = K_\infty/O_K$$

is called the Minkowski torus, a torus of (real) dimension $d$. When $K/\mathbb{Q}$ is totally complex, $\mathbb{T}_K$ also has the structure of a complex torus.
The ring of finite adeles is the restricted product
\[ \mathbb{A}_K^{\text{fin}} = \prod_{v \in \mathcal{P}_{\text{fin}}} K_v \]
with respect to the $O_v$, $v \in \mathcal{P}_{\text{fin}}$. By definition, this is the set of all tuples $(q_v)$ in which $q_v \in O_v$ for almost every $v \in \mathcal{P}_{\text{fin}}$. $\mathbb{A}_K^{\text{fin}}$ is a locally Cantor topological ring. The adele ring is defined
\[ \mathbb{A}_K = \mathbb{K}_\infty \times \mathbb{A}_K^{\text{fin}}. \]
$\mathbb{A}_K$ is a locally compact ring, and a solenoid as well since it is locally homeomorphic to $(\text{euclidean}) \times (\text{Cantor})$. $K$ embeds diagonally in $\mathbb{A}_K$ as a discrete co-compact subgroup with respect to addition. The quotient
\[ \hat{\mathbb{S}}_K = \mathbb{A}_K/(K,+). \]
is called the adele class group associated to $K$.

Given an ideal in $O_K$, denote by $\mathbb{T}_a$ the quotient $\mathbb{K}_\infty/a$, a $d$-dimensional torus covering $\mathbb{T}_K$.

**Proposition 1.** $\hat{\mathbb{S}}_K$ is a $d$-dimensional euclidean Lie group solenoid, isomorphic to

1. The inverse limit of euclidean Lie groups
   \[ \lim_{\rightarrow -} \mathbb{T}_a, \]
   where $a$ ranges over all ideals in $O_K$.
2. The suspension
   \[ (\mathbb{K}_\infty \times \hat{\mathbb{O}}_K)/O_K, \]
   where $\hat{\mathbb{O}}_K = \lim_{\rightarrow -} O_K/a$.

**Proof.** For item (1), see [3], pg. 67. Given $(z, \hat{y}) \in \mathbb{K}_\infty \times \hat{\mathbb{O}}_K$, note that $z$ defines an element $\hat{z} = (z_a)$ of $\lim_{\rightarrow -} \mathbb{T}_a$ by projection on to each of the factors. Moreover, $\hat{y}$ is by definition a coherent sequence $\{y_a\}$ of deck transformations of the coverings $\mathbb{T}_a \rightarrow \mathbb{T}_K$.

Then the association $(z, \hat{y}) \mapsto \hat{y} \cdot \hat{z} = (y_a(z_a))$ defines a homomorphism $\mathbb{K}_\infty \times \hat{\mathbb{O}}_K \rightarrow \hat{\mathbb{S}}_K$ which identifies precisely the $O_K$-related points, and descends to an isomorphism of $(\mathbb{K}_\infty \times \hat{\mathbb{O}}_K)/O_K$ with $\lim_{\rightarrow -} \mathbb{T}_a$.

By Proposition 1, it follows that the path-component of 0 is a leaf canonically isometric to $\mathbb{K}_\infty$; the restriction of the projection $\hat{\mathbb{S}}_K \rightarrow \mathbb{T}_K$ to $\mathbb{K}_\infty$ is the quotient by $O_K$. We will endow $\hat{\mathbb{S}}_K$ with the riemannian metric $\rho$ along the leaves induced from the inner-product on $\mathbb{K}_\infty$.

For example, if $K = \mathbb{Q}$, then $\mathbb{T}_0 = \mathbb{S}^1$ and $\hat{\mathbb{S}}_0$ is the classical 1-dimensional solenoid obtained as the inverse limit of circles $\mathbb{R}/m\mathbb{Z}$ under the natural covering homomorphisms.

We now suppose that $K/\mathbb{Q}$ is Galois and describe the actions of the Galois groups. If $z = (z_v) \in K_\infty$ and $\sigma \in \text{Gal}(K/\mathbb{Q})$ then $\sigma(z) = (z_{\sigma v})$. Alternatively, we can write
\[ K_\infty \equiv \mathbb{R} \otimes \mathbb{Q} K \quad \mathbb{A}_K \equiv \mathbb{A}_Q \otimes \mathbb{Q} K \]
and the action of $\sigma$ on $K_\infty$ and on $\mathbb{A}_K$ is via $x \otimes q \rightarrow x \otimes \sigma(q)$, where $x \in \mathbb{R}$ or $\mathbb{A}_Q$ as the case may be. In any event it is clear that $\text{Gal}(K/\mathbb{Q})$ acts orthogonally. The action of $\sigma$ on $\mathbb{A}_K$ may be understood as the product of the actions on $K_\infty$ and $\mathbb{A}_Q^{\text{fin}}$.

Note that the image of $O_K$ resp. $K$ is preserved by $\sigma$ and we induce (leafwise) isometric isomorphisms
\[ \sigma : \mathbb{T}_K \rightarrow \mathbb{T}_K \quad \delta : \hat{\mathbb{S}}_K \rightarrow \hat{\mathbb{S}}_K, \]
which are intertwined by the projection \( p : \hat{\mathbb{T}}_K \to \mathbb{T}_K \) in that \( p \circ \sigma = \sigma \circ p \). This leads to representations
\[
\rho : \text{Gal}(K/Q) \to \text{Isom}(\mathbb{T}_K), \quad \hat{\rho} : \text{Gal}(K/Q) \to \text{Isom}(\hat{\mathbb{T}}_K),
\]
where \( \text{Isom}(\cdot) \) means the group of isometric isomorphisms.

Let \( L/K \) be a finite extension of number fields. Any place of \( L \), finite or not, defines one on \( K \) by restriction. We thus obtain injective inclusions of vector spaces resp. rings
\[
K_\infty \hookrightarrow L_\infty, \quad \mathbb{A}^\text{fin}_K \hookrightarrow \mathbb{A}^\text{fin}_L, \quad \hat{\mathbb{A}}_K \hookrightarrow \mathbb{A}_L
\]
which scale inner-products/metrics by a factor of \( \deg(L/K) \). These maps in turn induce injective homomorphisms
\[
(2) \quad \mathbb{T}_K \hookrightarrow \mathbb{T}_L, \quad \hat{\mathbb{T}}_K \hookrightarrow \hat{\mathbb{T}}_L
\]
which scale metrics by \( \deg(L/K) \) and whose images are, in case \( L/K \) is Galois, fixed by the action of \( \text{Gal}(L/K) \).

We may also define maps in the opposite direction through the trace map \( \text{Tr}_{L/K} : L \to K \), which, when \( L/K \) is Galois, is given by
\[
\text{Tr}_{L/K}(a) = \sum_{\sigma \in \text{Gal}(L/K)} \sigma(a),
\]
and which has the property that \( \text{Tr}_{L/K}(O_L) \subset O_K \). The trace map extends to a linear map \( \text{Tr}_{L/K} : L_\infty \to K_\infty \) as follows: if \( \text{Tr}_{L/K}(z) = w = (w_\nu) \) then
\[
w_\nu = \sum_{\nu'} z_{\nu'}
\]
where the sum is over \( \nu' \) that restrict to \( \nu \). Note that if \( \nu \) is a real and \( \nu' \) is complex, then complex conjugation enters into the above sum, so that the contribution from \( z_{\nu'} \) is \( 2\text{Re}(z_{\nu'}) \). We thus induce epimorphisms
\[
\text{Tr}_{L/K} : \mathbb{T}_L \to \mathbb{T}_K, \quad \hat{\text{Tr}}_{L/K} : \hat{\mathbb{T}}_L \to \hat{\mathbb{T}}_K
\]
which are, in case \( L/K \) is Galois, \( \text{Gal}(L/K) \)-equivariant: \( \text{Tr}_{L/K} \circ \sigma = \text{Tr}_{L/K} \) and \( \hat{\text{Tr}}_{L/K} \circ \sigma = \hat{\text{Tr}}_{L/K} \) for all \( \sigma \in \text{Gal}(L/K) \).

For the extension \( K/Q \) we will write the associated trace map \( \text{Tr} = \text{Tr}_K/Q \), referring to it as the absolute trace. The ideal
\[
\mathfrak{d}_K = \{ a \in K | \text{Tr}(a \cdot O_K) \subset \mathbb{Z} \}^{-1} \subset O_K
\]
is called the absolute different.

4. Adele Class Groups II: Infinite Field Extensions

Let \( \mathcal{K} \) be a field occurring as a direct limit \( \lim_{\to} K_\lambda \) of fields \( K_\lambda/Q \) of finite degree e.g. \( \bar{Q} = \text{the algebraic closure of } Q \) or \( Q^{ab} = \text{the maximal abelian extension of } Q \). We may associate to \( \mathcal{K} \) (non locally-compact) abelian groups by taking the induced direct limits of inclusions \( \mathcal{K}_\lambda \):
\[
(3) \quad \lim_{\to} K_\lambda, \quad \lim_{\to} \hat{K}_\lambda.
\]
The projections \( \hat{K}_\lambda \to K_\lambda \) induce a projection \( \hat{\mathbb{T}}_K \to \lim_{\to} \mathbb{T}_K \). If the \( K_\lambda \) are Galois over \( Q \), then the inverse system of Galois groups \( \{ \text{Gal}(K_\lambda/Q) \} \) acts compatibly on the direct systems of tori and solenoids, inducing an action of the profinite Galois group \( \text{Gal}(\mathcal{K}/Q) = \lim_{\to} \text{Gal}(K_\lambda/Q) \) on each of the spaces appearing in (3).
Consider also
\[ \lim_{\rightarrow} (K_\lambda), \quad \lim_{\rightarrow} A_{K_\lambda}, \quad \lim_{\rightarrow} A_{K_\lambda}, \]
the first a (non locally-compact) topological vector space, the last two (non locally-compact) topological rings. Note that we may identify
\[ \lim_{\rightarrow} A_{K_\lambda} \sim \lim_{\rightarrow} (K_\lambda) \times \lim_{\rightarrow} A_{K_\lambda}, \]
since the direct limit maps preserve the solenoid product structure (euclidean) \times (locally Cantor) of the adele spaces. There are natural inclusions
\[ O_{K_\lambda} \hookrightarrow \lim_{\rightarrow} (K_\lambda), \quad K_{\lambda} \hookrightarrow \lim_{\rightarrow} A_{K_\lambda}, \]
and
\[ \lim_{\rightarrow} T_{K_\lambda} \equiv \left( \lim_{\rightarrow} (K_\lambda) \right) / O_{K_\lambda}, \quad \lim_{\rightarrow} \hat{S}_{K_\lambda} \equiv \left( \lim_{\rightarrow} A_{K_\lambda} \right) / K_{\lambda}. \]
It follows that \( \lim_{\rightarrow} \hat{S}_{K_\lambda} \) is a lamination, each leaf of which may be identified with \( \lim_{\rightarrow} (K_\lambda) \). Although \( \lim_{\rightarrow} A_{K_\lambda} \) is totally disconnected and perfect, it is not locally compact since the direct limit maps are not open. Thus \( \lim_{\rightarrow} \hat{S}_{K_\lambda} \) is not a solenoid.

An inner-product on the topological vector space \( \lim_{\rightarrow} (K_\lambda) \) is defined by scaling the canonical inner-product on each summand \( (K_\lambda) \) by \( (\deg(K_\lambda/Q))^{-1} \) and taking the direct limit. The action of \( O_{K_\lambda} \) on \( \lim_{\rightarrow} (K_\lambda) \) preserves this inner-product and we induce a riemannian metric on \( \lim_{\rightarrow} T_{K_\lambda} \). Similarly, the action of \( K_{\lambda} \) on \( \lim_{\rightarrow} A_{K_\lambda} \) is isometric along the factor \( \lim_{\rightarrow} (K_\lambda) \), and we induce a leaf-wise riemannian metric on \( \lim_{\rightarrow} \hat{S}_{K_\lambda} \).

The completion of \( \lim_{\rightarrow} (K_\lambda) \) is a Hilbert space denoted \( \mathcal{H}_\infty \). Write
\[ \mathcal{H}_{\mathcal{K}} = \mathcal{H}_\infty \times \lim_{\rightarrow} A_{K_\lambda}. \]
We define the **hilbertian torus** resp. the **hilbertian adele class group** of \( \mathcal{K} \) by
\[ T_{\mathcal{K}} \equiv \mathcal{H}_\infty / O_{\mathcal{K}}, \quad \hat{S}_{\mathcal{K}} \equiv \mathcal{H}_{\mathcal{K}} / \mathcal{K}. \]
Thus \( \hat{S}_{\mathcal{K}} \) is a Hilbert space lamination whose leaves are isometric to \( \mathcal{H}_\infty \). When the system is Galois, the Galois actions on the summands of the direct limits preserve the scaled inner-products, and we obtain representations
\[ \text{Gal}(\mathcal{K}/Q) \to \text{Isom}(T_{\mathcal{K}}), \quad \text{Gal}(\mathcal{K}/Q) \to \text{Isom}(\hat{S}_{\mathcal{K}}). \]

Any ideal \( a \subset O_{\mathcal{K}} \) can be realized as the direct limit of ideals \( a_\lambda = a \cap O_{K_\lambda} \). The quotient \( T_{\mathcal{K}} / a \) will be referred to as the hilbertian torus of \( a \). Consider the inverse limit
\[ \hat{O}_{\mathcal{K}} = \lim_{\rightarrow} O_{\mathcal{K}} / a \]
as \( a \) ranges over ideals in \( O_{\mathcal{K}} \). Since ideals in \( O_{\mathcal{K}} \) need not have finite index, \( \hat{O}_{\mathcal{K}} \) is not compact, or even locally compact. Nevertheless,

**Proposition 2.** \( \hat{O}_{\mathcal{K}} \equiv \lim_{\rightarrow} \hat{O}_{K_\lambda} \).
Proof. Observe first that for any ideal \( a \subset O_{\mathcal{K}} \), we have \( O_{\mathcal{K}}/a \cong \lim_{\to} O_{K_{\lambda}}/a_{\lambda} \). If \( b \subset a \), we have a commutative diagram:

\[
\begin{array}{ccc}
O_{\mathcal{K}}/a & \to & O_{\mathcal{K}}/b \\
\cup & & \cup \\
O_{K_{\lambda}}/a_{\lambda} & \to & O_{K_{\lambda}}/b_{\lambda}
\end{array}
\]

This allows us to interchange direct and inverse limits and write:

\[
\hat{O}_{\mathcal{K}} = \lim_{\to} O_{\mathcal{K}}/a \cong \lim_{\to} \left( \lim_{\to} O_{K_{\lambda}}/a_{\lambda} \right) \cong \lim_{\to} \hat{O}_{K_{\lambda}}
\]

and the result follows. \( \square \)

**Proposition 3.** \( \hat{\mathcal{S}}_{\mathcal{K}} \) is isomorphic to

1. The inverse limit of hilbertian tori
   \[ \lim_{\to} \Gamma_a \]
   as a ranges over ideals in \( O_{\mathcal{K}} \).
2. The suspension
   \[ \left( \mathcal{H}_{\infty} \times \hat{O}_{\mathcal{K}} \right)/O_{\mathcal{K}} \]

Proof. The spaces appearing in items (1) and (2) are isomorphic by an argument identical to that appearing in Proposition 1. That \( \hat{\mathcal{S}}_{\mathcal{K}} \) is isomorphic to the inverse limit appearing in (1) follows from the same limit interchange argument used in the proof of Proposition 2. \( \square \)

Unfortunately, neither \( \Gamma_{\mathcal{K}} \) nor \( \hat{\mathcal{S}}_{\mathcal{K}} \) are locally compact. This creates complications, since harmonic analysis plays a fundamental role in the linking of the arithmetic of \( \mathcal{K} \) and the algebra of the Hilbert space of \( L^2 \) functions on \( \hat{\mathcal{S}}_{\mathcal{K}} \). For this reason, we consider in the next section compactifications coming from inverse limits of tori and solenoids.

### 5. Proto Adele Class Groups

Let \( \mathcal{K} = \lim K_{\lambda} \) be a direct limit of finite extensions over \( Q \). The trace maps induce an inverse limit

\[ \hat{\mathcal{K}} = \lim K_{\lambda}, \]

an abelian group with respect to addition. This system restricts to one of integers, however:

**Theorem 1.** Let \( \mathcal{K} \) be a field containing \( Q_{\text{ab}} \). Then

\[ \lim O_{K_{\lambda}} = \{0\}. \]

Proof. Let \( \omega \) be a primitive \( m \)th root of unity and consider the cyclotomic extension \( K = Q(\omega) \). Since \( K \) is abelian, by assumption it occurs in the direct system defining \( \mathcal{K} \). The ring of integers \( O_K \) is generated by 1 and \( \omega^j \), where \( 1 \leq j \leq d - 1 \) and \( d \) is the degree of \( K/Q \). If we take say \( m = 2^k \), then \( d = 2^{k-1} \) and

\[
\text{Tr}(\omega^j) = \sum_{\sigma \in \text{Gal}(K/Q)} \sigma(\omega^j) = 0
\]
for each \( j \geq 1 \). On the other hand, \( \text{Tr}(1) = d \), so it follows that \( \text{Tr}(O_K) = (d) \subset Z \). Since \( d \) can be taken arbitrarily large, this means that the only coherent sequence that we may form in the inverse limit of rings of integers is \((0,0,\ldots)\).

\( \square \)

**Note 1.** It is worth pointing out that normalizing the trace map by dividing by the degree would not produce a non-trival limit. Indeed, if we let \( \omega \) be a primitive \( p \)th root of unity, \( p \) a prime \( > 2 \), then \( \text{Tr}(O_K) = Z \). Normalizing would take us out of the integers. Thus, all that survives in the trace inverse limit is a kind of “scaled” additive number theory.

The trace inverse limits \( \hat{T}_K = \lim_{\leftarrow} T_{K_1} \), \( \hat{S}_K = \lim_{\leftarrow} S_{K_1} \) are called, respectively, the proto-torus and the proto-adele class group of \( \mathcal{K} \). Each is a compact abelian group, being inverse limits of the same. The trace maps are natural with respect to the epimorphisms \( \hat{S}_{K_1} \rightarrow \hat{T}_{K_1} \) and induce in turn an epimorphism \( \hat{\mathcal{A}}_K \rightarrow \hat{T}_K \).

Consider as well the inverse limits

\[ \hat{\mathcal{K}} = \lim_{\leftarrow} (\mathcal{K}_\lambda)_\infty, \quad \hat{\mathcal{A}}_K = \lim_{\leftarrow} \hat{\mathcal{A}}_{K_1}, \quad \hat{\mathcal{S}}_K = \lim_{\leftarrow} \hat{\mathcal{S}}_{K_1}. \]

Observe that \( \hat{\mathcal{K}} \) is a locally convex, (non locally compact) topological vector space whose topology is induced from an embedding in \( \mathbb{R}^\omega \). Moreover, \( \hat{\mathcal{A}}_K \) and \( \hat{\mathcal{S}}_K \) are abelian topological groups only. Note that

\[ \hat{\mathcal{A}}_K \cong \hat{\mathcal{K}} \times \hat{\mathcal{A}}_K^\text{fin}. \]

The space \( \hat{\mathcal{A}}_K^\text{fin} \) is totally-disconected, perfect but not locally compact. There is a natural inclusion \( \hat{\mathcal{K}} \hookrightarrow \hat{\mathcal{A}}_K\) and

\[ \hat{\mathcal{S}}_K \equiv \hat{\mathcal{A}}_K / \hat{\mathcal{K}}. \]

Since the action of \( \hat{\mathcal{K}} \) locally preserves the product structure, \( \hat{\mathcal{S}}_K \) is an \( \mathbb{R}^\omega \)-lamination. When the system is Galois, the trace system maps are compatible with the Galois actions and we obtain representations of \( \text{Gal}(\mathcal{K}/Q) \) on \( \hat{T}_K \) and on \( \hat{S}_K \), acting by (leaf-preserving) topological isomorphisms.

**Theorem 2.** Let \( \mathcal{K} = \lim K_\lambda \) be an infinite field extension containing \( Q^{ab} \). Then the covering homomorphisms \( (K_\lambda)_\infty \rightarrow T_{K_1} \) induce a continuous monomorphism \( \hat{\mathcal{K}} \rightarrow \hat{T}_K \).

**Proof.** The inverse limits giving rise to each of \( \hat{\mathcal{K}} \) and \( \hat{T}_K \) also give rise to a continuous homomorphism \( \hat{\mathcal{K}} \rightarrow \hat{T}_K \). An element in the kernel must be a coherent sequence of algebraic integers \( i.e. \) an element of \( \lim O_{K_1} \). By Theorem [1] the latter is trivial and the map is an injective. \( \square \)

The map \( \hat{\mathcal{K}} \rightarrow \hat{T}_K \) is not surjective since the fibers of the projections \( (K_\lambda)_\infty \rightarrow T_{K_1} \) are not compact. Nonetheless, the image of \( \hat{\mathcal{K}} \) is dense in the compact \( \hat{T}_K \). Thus \( \hat{T}_K \) is laminated by the cosets of the image of \( \hat{\mathcal{K}} \). In fact,

**Theorem 3.** Let \( \mathcal{K} = \lim K_\lambda \) be an infinite field extension containing \( Q^{ab} \). Then the trace map inverse systems induce an isomorphism

\[ \hat{\mathcal{S}}_K \rightarrow \hat{T}_K \]
of topological groups.

Proof. For each $K_\lambda$, the kernel of $\hat{S}_{K_\lambda} \to \hat{T}_{K_\lambda}$ is $\hat{O}_{K_\lambda}$, hence the kernel of $\hat{S}_{\hat{K}} \to \hat{T}_{\hat{K}}$ is $\lim \hat{O}_{K_\lambda} = 0$. Since the fibers of the projections $\hat{S}_{K_\lambda} \to \hat{T}_{K_\lambda}$ are compact, the induced map of limits $\hat{S}_{\hat{K}} \to \hat{T}_{\hat{K}}$ is surjective. Since each map $\hat{S}_{K_\lambda} \to \hat{T}_{K_\lambda}$ is open with compact fibers, the limit map $\hat{S}_{\hat{K}} \to \hat{T}_{\hat{K}}$ is also open.

When $Q^{ab} \subset \hat{K}$, there are no exact analogues of Propositions 1 and 3 due to the lack of a notion of integers in $\hat{K}$. On the other hand, given any subfield $K \subset \hat{K}$ of finite degree over $Q$, one can find “level-$K$” suspension structures on $\hat{T}_{\hat{K}} \cong \hat{S}_{\hat{K}}$. Specifically, it is not difficult to see that the kernel of the projection $\hat{T}_{\hat{K}} \to \hat{T}_K$ is isomorphic to

$$\hat{O}_K \times \hat{T}_{\hat{K}/K},$$

where $\hat{T}_{\hat{K}/K}$ is the inverse limit of tori occurring as the connected components of $0$ of the kernels of the projections $\hat{T}_L \to \hat{T}_K$. Then

$$\hat{T}_{\hat{K}} \cong \left( K_\infty \times \left( \hat{O}_K \times \hat{T}_{\hat{K}/K} \right) \right) / O_K.$$

These representations are related by homeomorphisms induced by $\text{Tr} : L \to K$, but they do not survive the trace inverse limit. Similar “level-$K$” suspension representations are available for $\hat{S}_{\hat{K}}$ as well.

The relationship between the proto constructions of this chapter with the hilbertian constructions of the previous chapter is described by the following

**Theorem 4.** There are canonical inclusions

$$\hat{T}_{\hat{K}} \hookrightarrow \hat{S}_{\hat{K}}, \quad \hat{S}_{\hat{K}} \hookrightarrow \hat{K}_\infty$$

with dense images, which are $\text{Gal}(\hat{K}/Q)$-equivariant in case the defining system is Galois.

Proof. We begin by defining the inclusion $\lim (K_\lambda)_\infty \hookrightarrow \hat{K}_\infty$. Let $z_{\lambda_0} \in (K_\lambda)_\infty$ and define $z_{\lambda_0} \rightarrow (z_\lambda)$ as follows. If $\lambda$ is an index below $\lambda_0$, project $z_{\lambda_0}$ by the appropriate trace map to get the $z_\lambda$ coordinate. If $\lambda$ is above $\lambda_0$, we map $z_{\lambda_0}$ into $(K_\lambda)_\infty$ by the inclusion $(K_\lambda)_\infty \hookrightarrow (K_\lambda)_\infty$ and scale by $1/d$ where $d$ is the degree of $K_\lambda/K_{\lambda_0}$. This prescription defines a coherent sequence hence an element of $\hat{K}$. This map clearly defines an injective homomorphism of vector spaces $\lim (K_\lambda)_\infty \hookrightarrow \hat{K}_\infty$. By virtue of the scaling, this map is also continuous with regard to the inner-product defining $\hat{K}_\infty$. Now let $(z_\lambda) \subset \lim (K_\lambda)_\infty$ be a Cauchy sequence defining an element of $\hat{K}_\infty$. For every index $\beta$, if we let $\text{Tr}_{\lambda_0, \beta}$ denote the trace map from $(K_{\lambda_0})_\infty$ to $(K_{\lambda_0})_\infty$, then $\{\text{Tr}_{\lambda_0, \beta}(z_\lambda)\}$ is Cauchy in $(K_\beta)_\infty$. But this means precisely that the image of $(z_\lambda)$ in $\hat{K}_\infty$ is convergent. Thus the inclusion $\lim (K_\lambda)_\infty \hookrightarrow \hat{K}_\infty$ extends to $\hat{K}_\infty$. The other inclusions are induced by this one. That the images are dense follows from the definition of the above map: given any element of $\hat{K}_\infty$ and any level $\lambda$, there is an element of $\hat{K}_\infty$ agreeing up to level $\lambda$.

6. **Hyperbolizations**

Let $K/Q$ be finite. For each real place $\nu$ one pairs $K_\nu \approx \mathbb{R}$ with a single factor of $(0, \infty)$ to form the upper half-plane factor $\mathbb{H}_\nu = \mathbb{R} \times i(0, \infty)$, which we equip with the hyperbolic metric in the usual way.
For a complex place pair \((\mu, \bar{\nu})\), we must alter slightly this prescription in order to take into account the complex algebra of the factor
\[
\mathbb{C}_{(\mu, \bar{\nu})} = \{(x, \bar{x}) \mid x \in \mathbb{C}\} \cong \mathbb{C}.
\]
In order to do this, it is convenient to regard the factor \((0, \infty) \times (0, \infty)\) to be attached to \(\mathbb{C}_{(\mu, \bar{\nu})}\) as being complex. Thus let \(\mathbb{B}\) be the complex quarter space
\[
\mathbb{B} = \{b = s + it \in \mathbb{C} \mid 0 < s, t\}
\]
and define
\[
\mathbb{H}_{(\mu, \bar{\nu})} = \{(z, \bar{z}) \times (b, -\bar{b}) \mid z \in \mathbb{C}, b \in \mathbb{B}\} \subset (\mathbb{C} \times \mathbb{B}) \times (\overline{\mathbb{C}} \times (-\mathbb{B})) \subset \mathbb{C}^2 \times \mathbb{C}^2.
\]
Let \(-i\mathbb{H}^2\) denote the right half-plane. Then we may identify \(\mathbb{H}_{(\mu, \bar{\nu})}\) with \(\mathbb{H}^2 \oplus -i\mathbb{H}^2\) via the map
\[
(z, \bar{z}) \times (b, -\bar{b}) \rightarrow \frac{1}{2}(z + \bar{z} + b - \bar{b}, z - \bar{z} + b + \bar{b}) = (x + it, s + iy) \equiv (u, v)
\]
for \(z = x + iy\). We give \(\mathbb{H}_{(\mu, \bar{\nu})}\) the product hyperbolic metric coming from this identification, and also write
\[
(4) \quad \mathbb{H}_{(\mu, \bar{\nu})} = \mathbb{H}_\mu \oplus -i\mathbb{H}_\mu
\]
so as to be able to refer to the upper and right half-planes associated to \((\mu, \bar{\nu})\). In the sequel, a function defined in \(\mathbb{H}_{(\mu, \bar{\nu})}\) will be considered holomorphic if it is holomorphic in each of the variables \((u, v)\) and that in the \((\mu, \bar{\nu})\) variables, acts via
\[
(u, v) \rightarrow (u, \bar{v}),
\]
thus defining an (orientation-reversing) isometry of \(\mathbb{H}_{(\mu, \bar{\nu})}\).

Finally, we define the hyperbolization
\[
\mathbb{H}_K = \mathbb{H}^R_K \times \mathbb{H}^C_K := \prod \mathbb{H}_v \times \prod \mathbb{H}_{(\mu, \bar{\nu})} = \prod \mathbb{H}_v \times (\prod \mathbb{H}_\mu \oplus -i\mathbb{H}_\mu) \cong K_\infty \times (0, \infty)^d.
\]
Thus \(\mathbb{H}_K\) has the structure of a \(d\)-dimensional complex polydisk equipped with the product riemannian metric. For \(v\) a real place we write \(\tau_v = x_v + it_v\) for a point of \(\mathbb{H}_v\), and for \(\mu\) complex we write
\[
(\kappa_{\mu}, \bar{\kappa}_{\mu}) = ((z_\mu, \bar{b}_\mu), (\bar{z}_\mu, -\bar{b}_\mu)) \equiv (u_\mu, v_\mu).
\]
We write points of \(\mathbb{H}_K\) in the form \(\rho = \tau \times \kappa\) where \(\tau = (\tau_1, \ldots, \tau_r) \in \mathbb{H}^R_K\) are the coordinates of the “real hyperbolization” and \(\kappa = ((\kappa_1, \bar{\kappa}_1), \ldots, (\kappa_d, \bar{\kappa}_d)) \in \mathbb{H}^C_K\) are those of the “complex hyperbolization”. When \(K/Q\) is Galois we have \(\mathbb{H}_K = \mathbb{H}^R_K\) or \(\mathbb{H}^C_K\) depending on whether \(K\) is real or complex, and the action of \(\text{Gal}(K/Q)\) extends to an isometric action on \(\mathbb{H}_K\) by acting trivially in the extended coordinates.

The subgroups \(O_K\) and \(K\) of \(K_\infty\), viewed as groups of translations, extend to translations of \(\mathbb{H}_K\) that are isometries. The quotients
\[
(5) \quad \mathfrak{T}_K = \mathbb{H}_K/O_K \cong (\Delta^*)^d, \quad \bar{\mathfrak{S}}_K = \left(\mathbb{H}_K \times \mathbb{A}^{\mathrm{fin}}_K\right)/K \cong \mathfrak{S}_K \times (0, \infty)^d
\]
are referred to as the hyperbolized torus and the hyperbolized adele class group of \(K\). (In the above, \(\Delta^*\) denotes the punctured hyperbolic disk.) For \(L/K\) a finite extension of finite degree extensions of \(Q\), the canonical inclusions \(\mathfrak{T}_K \hookrightarrow \mathfrak{T}_L\) and \(\bar{\mathfrak{S}}_K \hookrightarrow \bar{\mathfrak{S}}_L\) are isometric up to the scaling factor \(\text{deg}(L/K)\). If \(L/K\) is Galois, the action of \(\text{Gal}(L/K)\) on \(\mathfrak{T}_L\) and \(\bar{\mathfrak{S}}_L\) extends by isometries to both of \(\mathfrak{T}_L\) and \(\bar{\mathfrak{S}}_L\), restricting to the identity on the images of \(\mathfrak{T}_K\) and \(\bar{\mathfrak{S}}_K\).
In the case of an infinite field extension \( \mathcal{K} \), one follows the prescription of the preceding paragraphs using the hilbertian torus and solenoid \( \mathcal{T}_\mathcal{K} \) and \( \tilde{S}_\mathcal{K} \). Thus \( \mathcal{H}_\mathcal{K} \) is an infinite product of hyperbolic planes. We obtain an infinite-dimensional hyperbolized torus \( \mathcal{T}_\mathcal{K} \) and hyperbolized solenoid \( \tilde{S}_\mathcal{K} \) upon quotient by \( O_\mathcal{K} \) and \( \mathcal{K} \).

### 7. The Character Field

For \( G \) a locally compact abelian group, by a character we mean a continuous homomorphism \( \chi : G \to U(1) \), where \( U(1) \subset \mathbb{C} \) is the unit circle. An operation \( \oplus \) on characters is defined by multiplying their values, and the set of characters

\[ \text{Char}(G) = \text{Hom}_{cont}(G,U(1)) \]

is itself a locally compact abelian group. In the case of interest, the projection \( \hat{\mathcal{S}}_K \to T_K \) induces an inclusion

\[ \text{Char}(T_K) \hookrightarrow \text{Char}(\hat{\mathcal{S}}_K) \]

and so we will view \( \text{Char}(T_K) \) as a subgroup of \( \text{Char}(\hat{\mathcal{S}}_K) \). If \( \mathcal{K} = \varprojlim K_i \) is an infinite extension over \( \mathbb{Q} \), we have a similar inclusion

\[ \text{Char}(\hat{T}_\mathcal{K}) \hookrightarrow \text{Char}(\hat{\mathcal{S}}_K) \]

and corresponding convention.

The purpose of this section is to give a proof of the well-known identification \( \text{Char}(\hat{\mathcal{S}}_K) \cong (K,+), \) and show that this identification may be used to view \( \text{Char}(\hat{\mathcal{S}}_K) \) as a field, in a way which is natural with respect to the trace maps.

#### Lemma 1

Let \( K/\mathbb{Q} \) be a finite extension. If \( w \in K_\infty \) has the property that

\[ \text{Tr}(wK) \subset \mathbb{Q}, \]

then \( w \in K \).

**Proof:** First suppose that \( K \) is Galois over \( \mathbb{Q} \). Let \( \alpha_1, \ldots, \alpha_d \) be an integral basis of \( K \), and let \( A \) be the \( d \times d \) invertible matrix whose \( ij \)-element is \( v_j(\alpha_i) \) where the \( v_1, \ldots, v_d \) are the places of \( K \). Then the hypothesis \( \text{Tr}(wK) \subset \mathbb{Q} \) implies that \( Aw = q \in \mathbb{Q}^d \) or \( w \in A^{-1}\mathbb{Q}^d \). Since \( K \) is Galois, it is normal, hence all of the entries of \( A \) belong to \( K \) (see [9]), thus the coordinates of \( w \) belong to \( K \). Suppose that nevertheless \( w \notin K \). Then there exists some element \( \sigma_0 \in \text{Gal}(K/\mathbb{Q}) \) and a coordinate \( w_{v_0} \) such that \( \sigma_0(w_{v_0}) \neq w_{\sigma_0 v_0} \). Let \( A^v \) denote the column indexed by a place \( v \) (i.e. the vector \( (v(\alpha_1), \ldots, v(\alpha_d))\) so that

\[ \sum w_i A_i^v = q \in \mathbb{Q}^d. \]  

(6)

Acting by \( \sigma_0 \) on this equation fixes \( q \), permutes the column vectors \( A^v \), \( \sigma_0(A^v) = A^{\sigma_0 v} \), but does not similarly permute the entries of \( w \). Thus there exists a vector \( w' \neq w \), defined \( w'_{\sigma_0 v} = \sigma_0(w_v) \), for which \( Aw' = q \), implying that the kernel of \( A \) is nontrivial (since \( A(w' - w) = 0 \) and \( w' - w \neq 0 \), contradiction.

Now suppose that \( K/\mathbb{Q} \) is not Galois. As in the previous paragraph, fix the basis \( \alpha_1, \ldots, \alpha_d \) and note that the columns of the embedding matrix \( A \) continue to satisfy the vector equation (5). Let \( L/\mathbb{Q} \) be a finite Galois extension containing the images of \( K \) by its places: for example, if one writes \( K = \mathbb{Q}(a) \) one can take \( L \) to be the splitting field of the minimal polynomial of \( a \). Note then that \( L/K \) is Galois. Then \( A \) has entries belonging to \( L \), as does its inverse, so that each coordinate of \( w \) belongs to \( L \). Consider the usual diagonal embedding of vector spaces \( i : K_\infty \to L_\infty \) defined by

\[ i(w)_\mu = w_v \]
for each $\mu$ an $L$-place with restriction $\mu|_K = \nu$. In addition, we have the scaled embedding $i: K_\infty \rightarrow L_\infty$.

$$i(w)_\mu = \frac{w_\nu}{\text{mult}(\nu)},$$

where $\mu|_K = \nu$ and $\text{mult}(\nu)$ is the number of places having restriction to $K$ equal to $\nu$. The basis $a_1, \ldots, a_d$ defines a $d \times (d[L:K])$ matrix, denoted $B$, whose $\mu$th-column is $B^{\mu} = (i(a_1)_\mu, \ldots, i(a_d)_\mu)^T$ (here we are identifying the basis elements with vectors in $K_\infty$). Since $Aw = q \in \mathbb{Q}^d$, we have $Bi(w) = q \in \mathbb{Q}^d$ and we have the analogue of the vector equation (6):

$$\sum i(w)_\mu B^{\mu} = q \in \mathbb{Q}^d.$$  

Note that if $\mu$ and $\mu'$ have the same restriction $\nu$ to $K$ then $B^{\mu} = B^{\mu'}$.

Suppose that $w$ does not belong to $K$. Then $i(w)$ does not belong to $L$: for if $i(w) = \beta \in L$, then by definition of $i$, for all $\mu$ extending the identity place on $K$, we must have $\mu(\beta) = \beta$. But the places extending the identity place on $K$ comprise the Galois group $\text{Gal}(L/K)$, hence $\beta$ belongs to the fixed field of $\text{Gal}(L/K)$ i.e. $i(w) \in K$ viewed as a subfield of $L_\infty$. But then this would imply that $w \in K$ contrary to our hypothesis.

Since $i(w)$ does not belong to $L$, there exists $\sigma_0 \in \text{Gal}(L/\mathbb{Q})$ and a coordinate $\mu_0$ such that

$$\sigma_0(i(w))_{\mu_0} \neq i(w)_{\sigma_0 \mu_0}.$$  

Then as before we deduce a vector $y$, defined $y_{\sigma_0 \mu} := \sigma_0(i(w))_{\mu}$, and for which $By = q$. This vector $y$ satisfies $y_{\mu_1} = y_{\mu_2}$ whenever $\mu_1$ and $\mu_2$ have the same restriction $\nu$ to $K$, so that it must belong to the image of the diagonal embedding $i$. Thus there exists a vector $w' \in K_\infty$ distinct from $w$ and which satisfies $Aw' = q$, again contradicting the invertibility of $A$.

Recall that the inverse different of a finite degree algebraic number field $K/\mathbb{Q}$ is the $O_K$-module

$$\frak{d}_K^{-1} = \{a \in K| \text{Tr}_{K/\mathbb{Q}}(aO_K) \subset \mathbb{Z}\} \supset O_K.$$  

Note that if $L/K$ is finite and $a \in \frak{d}_K^{-1}$ then for all $\beta \in O_L$,

$$\text{Tr}_{L/\mathbb{Q}}(a\beta) = \text{Tr}_{K/\mathbb{Q}}(a \cdot \text{Tr}_{L/K}(\beta)) \subset \mathbb{Z}$$

(since $\text{Tr}_{L/K}(\beta) \subset O_K$) so that $\frak{d}_K^{-1} \subset \frak{d}_L^{-1}$. Then if $\mathcal{K}/\mathbb{Q}$ is an infinite degree algebraic extension, we define

$$\frak{d}_{\mathcal{K}}^{-1} = \lim \frak{d}_K^{-1}$$

where the $K \subset \mathcal{K}$ range over finite subextensions of $\mathbb{Q}$.

**Theorem 5.** Suppose that

- $K$ is a finite field extension over $\mathbb{Q}$. Then $\text{Char}(\hat{S}_K)$ possesses a second operation $\otimes$ making it a field and for which their is a canonical isomorphism

$$\text{Char}(\hat{S}_K) \cong K$$

which is natural with respect to the trace maps. This isomorphism identifies $\text{Char}(\hat{T}_K)$ with the inverse different $\frak{d}_{\mathcal{K}}^{-1}$ and in particular, there is a canonical embedding of the ring $O_K$ in $\text{Char}(\hat{T}_K)$.
\[ \mathcal{K} = \lim_{\rightarrow} K_i \text{ is an infinite extension over } \mathbb{Q}. \text{ Then } \text{Char}(\hat{\mathcal{S}}_{K}) \text{ possesses a second operation } \otimes \text{ making it a field and for which their is a canonical isomorphism} \]

\[ \text{Char}(\hat{\mathcal{S}}_{K}) \cong \mathcal{K} \]

which is natural with respect to the trace maps. This isomorphism identifies \(\text{Char}(\hat{\mathcal{T}}_{K})\) with the inverse different \(d_{\mathcal{K}}^{-1}\). In particular, there is a canonical embedding of the ring \(O_{\mathcal{K}}\) in \(\text{Char}(\hat{\mathcal{T}}_{K})\). In particular, there is a canonical embedding of the ring \(O_{\mathcal{K}}\) in \(\text{Char}(\hat{\mathcal{T}}_{K})\).

**Note 2.** The statement that \(\text{Char}(\hat{\mathcal{S}}_{K}) \cong (K, +)\) is classical and has been known since the time of Tate’s thesis [14].

**Proof.** We begin with \(a\). Consider the character on \(K_{\infty}\) defined

\[ \phi^K_{\infty}(z) = \exp(2\pi i \text{Tr}(z)). \]

Note that \(\phi^K_{\infty}\) extends continuously to a character \(\phi^K\) of \(\hat{\mathcal{S}}_{K}\). Indeed, \(\phi^K_{\infty}\) is invariant w.r.t. translation by \(O_{\mathcal{K}}\) and so induces a character \(\mathcal{T}_{K} \rightarrow U(1)\), which, when composed with the projection \(\hat{\mathcal{S}}_{K} \rightarrow \mathcal{T}_{K}\), defines the extension \(\phi^K\). Note that for any finite degree extension \(L/K\) we have \(\phi^L = \phi^K \circ \hat{\text{Tr}}_{L/K}\). For each \(a \in K\) define \(\phi^K_a\) by

\[ \phi^K_a(\hat{z}) = \phi^K(a\hat{z}) \]

(here \(a\hat{z}\) is defined for all \(z \in \hat{\mathcal{S}}_{K}\) since \(\hat{\mathcal{S}}_{K}\) is a \(K\)-vector space). Then the map

(7)

\[ a \mapsto \phi^K_a \]

defines a monomorphism \((K, +) \rightarrow \text{Char}(\hat{\mathcal{S}}_{K})\). We will show that it is an isomorphism. So let \(\phi : \hat{\mathcal{S}}_{K} \rightarrow U(1)\) be any character. Then the restriction to the leaf through 0, which is just \(K_{\infty}\), is of the form

\[ \phi(\exp(2\pi i \text{Tr}(wz))) \]

for some \(w \in K_{\infty}\). On the other hand, we may also restrict \(\phi\) to the transversal through 0, which is just \(\hat{O}_{K} \subset \hat{\mathcal{S}}_{K}\), where it must factor through some finite quotient \(O_{K}/a\). It follows that \(\phi\) is the pullback of a character \(\mathcal{T}_{a} = K_{\infty}/a \rightarrow U(1)\)

\[ \mathcal{T}_{a} = K_{\infty}/a \rightarrow U(1) \]

for some ideal \(a \subset O_{K}\). In particular, \(\text{Tr}(wa) \subset \mathbb{Z}\) which implies that \(\text{Tr}(waK) \subset \mathbb{Q}\), so by Lemma \(1\) \(w = a \in K\) for some \(a\) and \(\phi = \phi^K_{a}\). Thus the map \(a \mapsto \phi^K_{a}\) defines an isomorphism between \((K, +)\) and \(\text{Char}(\hat{\mathcal{S}}_{K})\). One then pushes forward the product operation of \(K\) to define \(\otimes\). Naturality is an expression of the commutative diagram

\[ L \xrightarrow{=} \text{Char}(\hat{\mathcal{S}}_{L}) \]

\[ \cup \]

\[ K \xrightarrow{=} \text{Char}(\hat{\mathcal{S}}_{K}) \]

whose commutativity follows from the fact that for all \(a \in K\),

\[ \hat{\text{Tr}}_{L/K}(\phi^K_a) = \phi^K_a \circ \hat{\text{Tr}}_{L/K} = \phi^L_a. \]

The map \(7\) identifies \(\text{Char}(\mathcal{T}_{K})\) with \(d_{\mathcal{K}}^{-1} \supset O_{K}\) so that the restriction of \(\otimes\) to \(\text{Char}(\mathcal{T}_{K})\) makes the latter an \(O_{K}\)-module isomorphic to \(d_{\mathcal{K}}^{-1}\).
To prove $h$, recall that if $L/K$ is a finite extension of number fields finite over $\mathbb{Q}$, then the trace projection $\tilde{\operatorname{Tr}}_{L/K}: \hat{S}_L \to \hat{S}_K$ induces an inclusion $\tilde{\operatorname{Tr}}_{L/K}^*: \operatorname{Char}(\hat{S}_K) \to \operatorname{Char}(\hat{S}_L)$ of fields. Then the direct limit

$$
\lim_{\to} \operatorname{Char}(\hat{S}_{K_i})
$$

is a field isomorphic to $\mathcal{K}$. The direct limit (8) has dense image in $\operatorname{Char}(\hat{S}_{K})$, but the latter is discrete since $\hat{S}_K$ is compact, so they are equal. Thus every character $\phi$ is the pull-back of one defined on $\hat{S}_K$ where $K/\mathbb{Q}$ is of finite degree. Again, this isomorphism is trace natural. It identifies $\operatorname{Char}(\hat{S}_K)$ with $\mathcal{K}$ and so as in the finite extension case, the former has a module structure over the subring corresponding to $O_{\mathcal{K}}$. 

Note that for $K = \mathbb{Q}$, we have $\operatorname{Char}(\mathbb{Q})$ is a ring since $\mathcal{K} = \mathbb{Z}$. On the other hand, since $\mathcal{K} \neq O_K$ for $K \neq \mathbb{Q}$, we see that $\operatorname{Char}(\mathbb{Q})$ is strictly larger than $O_K$; it contains $O_K$ as a ring, and is to be viewed as an $O_K$-module extension of $O_K$. If $\mathcal{K} \supset \mathbb{Q}^{ab}$, then $\mathcal{K} = \mathcal{K}$ and $\operatorname{Char}(\mathbb{Q}) = \operatorname{Char}(\hat{S}_{\mathcal{K}})$, which was anticipated by Theorem 3.

8. Graded Holomorphic Functions

Let $\mu$ be the unit mass Haar measure on $\hat{S}_K$ and let $L^2(\hat{S}_K, \mathbb{C})$ be the associated space of square integrable complex valued functions on $\hat{S}_K$. The characters $\{\phi_a\}$ form a complete orthonormal system in $L^2(\hat{S}_K, \mathbb{C})$ and so every element $f \in L^2(\hat{S}_K, \mathbb{C})$ has the development

$$
f = \sum a_a \phi_a,
$$

where $\{a_a\} \in l^2(K)$, $\phi_a \in \operatorname{Char}(\hat{S}_K)$ and equality is taken w.r.t. the $L^2$ norm. We note that since we are in a Hilbert space, by Parseval’s Lemma, we have $\|f\|^2 = \sum |a_a|^2$ and so the sum converges with respect to any well-ordering of the indexing set $K$. The space $L^2(\mathbb{Q}_K, \mathbb{C})$ may be identified with the subspace of $L^2(\hat{S}_K, \mathbb{C})$ whose Fourier series satisfy $a_a = 0$ for $a \notin \mathcal{K}$. If we restrict $f$ to the dense leaf $K_{\infty}$, then we may identify $\phi_a(z) = \exp(2\pi i \operatorname{Tr}(az)) = \xi^a$ and write $f$ in the form of an $L^2$ Puiseux series

$$
f(\xi) = \sum a_a \xi^a.
$$

Let $f, g \in L^2(\hat{S}_K, \mathbb{C})$ be given by the developments $f = \sum a_a \phi_a$, $g = \sum b_a \phi_a$. The Cauchy product $f \ast g$ is defined as the $L^2$ extension of the point-wise product of continuous functions, that is,

$$
f \ast g = \sum_a c_a \phi_a = \sum_a \left( \sum_{a_1a_2 = a} a_1b_2 \right) \phi_a,
$$

provided that the sum on the right converges. In this regard, note that $c_a$ is equal to the inner-product $\langle f, g \rangle$, where $\sigma_{a,b} = \sum_{b \in K} b_{a-b} \phi_b \in L^2(\hat{S}_K, \mathbb{C})$, hence is an unambiguously defined complex number. Thus the Cauchy product is defined whenever the sequence $\{c_a\}$ belongs to $l^2(K)$. Given $f \in L^2(\hat{S}_K, \mathbb{C})$, we denote by $\operatorname{Dom}_a(f)$ the set of $g \in L^2(\hat{S}_K, \mathbb{C})$ for which $f \ast g$ is defined.

The Dirichlet product $f \circ g$ will be defined by the development

$$
f \circ g = \sum_a d_a \phi_a = \sum_a \left( \sum_{a_1a_2 = a} a_1b_2 \right) \phi_a,
$$

where $\{d_a\} \in l^2(K)$. The space $L^2(\hat{S}_K, \mathbb{C})$ may be identified with the subspace of $L^2(\hat{S}_K, \mathbb{C})$ whose Fourier series satisfy $d_a = 0$ for $a \notin \mathcal{K}$. If we restrict $f$ to the dense leaf $K_{\infty}$, then we may identify $\phi_a(z) = \exp(2\pi i \operatorname{Tr}(az)) = \xi^a$ and write $f$ in the form of an $L^2$ Puiseux series

$$
f(\xi) = \sum a_a \xi^a.
$$

Let $f, g \in L^2(\hat{S}_K, \mathbb{C})$ be given by the developments $f = \sum a_a \phi_a$, $g = \sum b_a \phi_a$. The Cauchy product $f \ast g$ is defined as the $L^2$ extension of the point-wise product of continuous functions, that is,
providing that it converges. Let us say a few words about what this means. Here, we note that for \( \alpha \neq 0 \), we have that \( d_\alpha = \langle f, \tau_\alpha \rangle \) where \( \tau_\alpha \bar{g} = \sum_{\beta \in \mathcal{K}} b_{\alpha \beta}^{-1} \phi_\beta \), hence is well-defined. Consider also the formal expression

\[
d_0 = a_0 \sum_{\alpha \in \mathcal{K}^*} b_\alpha + b_0 \sum_{\alpha \in \mathcal{K}^*} a_\alpha + a_0 b_0.
\]

Then we say that the Dirichlet product \( f \otimes g \) converges when \( d_0 \) converges absolutely and the sequence \( \{d_\alpha\} \) defines an element of \( \ell^2(\mathcal{K}) \). We denote by \( \text{Dom}_a(f) \) the set of functions having defined Dirichlet product with \( f \).

In terms of the \( \xi \) parameter:

\[
(f \otimes g)(\xi) = \sum_{\alpha} a_\alpha \cdot g(\xi^\alpha) = \sum_{\alpha} b_\alpha \cdot f(\xi^\alpha),
\]

showing that Dirichlet multiplication is commutative and distributive over ordinary addition + of functions. It follows that \( L^2(\hat{\mathcal{S}}_K, \mathbb{C}) \) is a partial algebra with respect to each of the operations \( \oplus \) and \( \otimes \) separately: here, partial refers to the fact that \( \oplus \) and \( \otimes \) are only partially defined, and when they are, the usual axioms of group algebra hold for each operation. When \( \mathcal{K} \) is of infinite degree over \( \mathbb{Q} \), the relevant discussion applies to the space of square integrable functions on the proto solenoid \( L^2(\hat{\mathcal{S}}_K, \mathbb{C}) \).

**Note 3.** Suppose that \( K = \mathbb{Q} \) and \( f, g \in C_0(\mathbb{Q}, \mathbb{C}) \subset C_0(\hat{\mathbb{Q}}, \mathbb{C}) \) with \( a_n = b_n = 0 \) for \( n \leq 0 \). Then \( f \) and \( g \) define via a Mellin-type transform convergent Dirichlet series

\[
D_f(y) = \sum_{n} a_n n^{-2\pi i y} \quad \text{and} \quad D_g(y) = \sum_{n} b_n n^{-2\pi i y},
\]

\( y \in \mathbb{R} \), and

\[
D_{f \otimes g} = D_f \oplus D_g.
\]

Thus the algebra of zeta functions, L-functions, Dirichlet series etc. is codified by the Dirichlet product.

Now let \( \hat{\mathcal{S}}_K \) be the hyperbolized adele class group defined in §6. \( \hat{\mathcal{S}}_K \) is a lamination whose leaves are \( d \)-dimensional polydisks, and so we say that a continuous function \( F : \hat{\mathcal{S}}_K \to \mathbb{C} \) is holomorphic if its restriction \( F|_L \) to each leaf \( L \) is holomorphic, or equivalently (since all leaves are dense), if its restriction to the canonical leaf \( \mathbb{H}_K = \mathbb{H}_K^\mathbb{R} \times \mathbb{H}_K^\mathbb{C} \) is holomorphic. We recall that holomorphicity in the factor \( \mathbb{H}_K^\mathbb{C} \) is defined in terms of the “upper half-plane \( \times \) right half-plane” multi-variable \( u \times v \).

For each \( t \in (0, \infty)^d \) let \( \hat{\mathcal{S}}_K(t) \subset \hat{\mathcal{S}}_K \) be the subspace of points having extended coordinate \( t \) with respect to the decomposition (2). Since \( \hat{\mathcal{S}}_K(t) \approx \hat{\mathcal{S}}_K \) we may put on \( \hat{\mathcal{S}}_K(t) \) the unit mass Haar measure and define for \( F, G : \hat{\mathcal{S}}_K \to \mathbb{C} \) the pairing

\[
(F, G)_t = \int_{\hat{\mathcal{S}}_K(t)} FG d\mu.
\]

**Definition 1.** The **Hardy space** associated to \( K/\mathbb{Q} \) is the Hilbert space

\[
\mathcal{H}(K) = \left\{ F : \hat{\mathcal{S}}_K \to \mathbb{C} \mid F \text{ is holomorphic and } \sup_t (F, F)_t < \infty \right\},
\]

with inner-product \( \langle \cdot, \cdot \rangle_t = \sup_t \langle \cdot, \cdot \rangle_t \).

Evidently any \( F \in \mathcal{H}(K) \) has an a.e. defined \( L^2 \) limit for \( t \to 0 \), and such a limit defines an element

\[
\partial F := f \in L^2(\hat{\mathcal{S}}_K, \mathbb{C}).
\]
Using the Fourier development available there, we may write the restriction $F|_{H_K}$ as follows: if $\rho = \tau \times \kappa \in H_K = \mathbb{H}^R_K \times \mathbb{H}^C_K$ (see §6 for the relevant notation) then

$$F|_{H_K}(\rho) = \sum a_\alpha \exp(2\pi i \text{Tr}(\alpha \cdot \rho)) = \sum a_\alpha \exp(2\pi i \text{Tr}(\alpha \cdot \tau)) \exp(2\pi i \text{Tr}(\alpha \cdot \kappa))$$

where

$$\exp(2\pi i \text{Tr}(\alpha \cdot \tau)) = \prod_{\nu} \exp(2\pi i \alpha, x_\nu) \exp(-2\pi \alpha, t_\nu)$$

and

$$\exp(2\pi i \text{Tr}(\alpha \cdot \kappa)) = \prod_{(\mu, \bar{\mu})} \exp(4\pi i (\text{Re}(\alpha, z_\mu)) \exp(-4\pi i \text{Im}(\alpha, b_\mu)).$$

We will sometimes switch to power series notation and shorten this to

$$F|_{H_K}(\rho) = \sum a_\alpha \zeta^\alpha = \sum a_\alpha \xi^\alpha \eta^a,$$

where $\rho = \exp(2\pi i \text{Tr}(\rho))$, $\xi = \exp(2\pi i \text{Tr}(\tau))$ and $\eta = \exp(2\pi i \text{Tr}(\kappa))$.

By the positive cone in $K_\infty$ we shall mean the set of $x = (x_v; (z_\mu, \bar{z}_\mu)) \in K_\infty$ for which $x_v > 0$ for all $v$ real and $(z_\mu, \bar{z}_\mu) \in \mathbb{B} \times \mathbb{B}$ for each complex place pair $(\mu, \bar{\mu})$. It is clear then that for the series of $(10)$ to define elements of $H[K]$, we must have that $a_\alpha = 0$ for $\alpha$ not contained in the positive cone of $K_\infty$. We have that $\|F\|_2^2 = \sum |a_\alpha|^2$ and hence the correspondence $F \mapsto \partial F$ yields an isometric inclusion of Hilbert spaces

$$H[K] \hookrightarrow L^2(\hat{S}_K, \mathbb{C}).$$

Note 4. In (12), observe that when $a$ is real and positive (i.e. all of its place coordinates are real and positive), then $\eta^a$ is a holomorphic function purely of the upper half plane variable $u = (\mu, \bar{\mu}) \in \prod \mathbb{H}_K$ of $H_K$ i.e. is constant in the right plane variable $v = (v_\mu) \in \prod (-\mathbb{H}_K)$. Similarly, when $a$ belongs to the positive imaginary axis, $\eta^a$ only depends on $v$. In particular, this shows that $F|_{H_K}$ is holomorphic with respect to the multi-variable $u \times v \in H_K$.

In order to build up from $H[K]$ a kind of holomorphic extension of $K$, it will be necessary for us to be able to interpret all elements of $L^2(\hat{S}_K, \mathbb{C})$ as boundaries of holomorphic functions. We shall thus expand upon the $\mathbb{Z}/2\mathbb{Z}$-graded (super) convention, which has the virtue of regarding holomorphic and anti-holomorphic functions on equal footing.

8.1. The Totally Real Case. We first consider the case $K = \mathbb{Q}$. Let $\Theta = \{-, +\} \subseteq \mathbb{Z}/2\mathbb{Z}$ be the sign group and denote by $H^+ = \mathbb{R} \times (-\infty, 0)$ the hyperbolic lower half-plane, $c_- : H \to H^+$ complex-conjugation. Then every element $f = \sum a_q \zeta^q \in L^2(\hat{S}_K, \mathbb{C})$ determines a triple $(F_-, F_0, F_+)$

$$F_+(\tau) = \sum_{q > 0} a_q \exp(2\pi i q\tau), \quad F_0 = a_0, \quad F_-(\tau) = \sum_{q < 0} a_q \exp(2\pi i q c_-(\tau)).$$

The functions $F_+(z)$ and $F_-(z)$ are viewed as elements of the Hardy spaces $H_+[\mathbb{Q}] = H[\mathbb{Q}]$ and $H_-[\mathbb{Q}] = \text{Hardy space of anti-holomorphic functions on } \hat{S}_K$.

Now let $K/\mathbb{Q}$ be a totally real extension of degree $d$. Let

$$\Theta_K = \{-, +\}^d \cong (\mathbb{Z}/2\mathbb{Z})^d$$

and write $C_K = K_\infty \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^d$, whose points are written $\tau = (x_v, i t_v, \ldots, x_v + i t_v)$. For each $\theta \in \Theta_K$, define

$$H^K_\theta = \{ \tau \in C_K \mid \{\text{sign}(t_v), \ldots, \text{sign}(t_v)\} = \theta\}.$$
We associate to $\theta$ a conjugation $c_{\theta} : \mathbb{C}_K \to \mathbb{C}_K$, where the coordinates of $c_{\theta}(\tau) = \tau'$ satisfy
\[ x'_{ij} + it'_{ij} = x_{ij} + \theta_j i t_{ij} \]
for $j = 1, \ldots, d$. The conjugation maps are not holomorphic in $\tau$ and satisfy for all $\theta, \theta_1, \theta_2 \in \Theta_K$:
\[ c_{\theta}(i^2 K) = i^2 \theta K \quad \text{and} \quad c_{\theta_1} \circ c_{\theta_2} = c_{\theta_2} \circ c_{\theta_1}. \]
A $\theta$-holomorphic function is one of the form $F \circ c_{\theta}$, where $F : \mathbb{C}_K \to \mathbb{C}$ is holomorphic.

Denote by $K^\theta$ those elements $a \in K$ whose coordinates with respect to the embedding $K \hookrightarrow K_\infty$ satisfy
\[ (\text{sign}(a_{v_1}), \ldots, \text{sign}(a_{v_d})) = \theta \]
Note that
\[ K^{\theta_1} \cdot K^{\theta_2} \subset K^{\theta_2}. \]
Then every element $f = \sum a_c c^a \in L^2(\hat{\mathbb{S}}_K, \mathbb{C})$ determines a $(2^d + 1)$-tuple $(F_\theta : F_0)$, where for each $\theta \in \Theta_K$, $F_\theta : \hat{\mathbb{S}}_K \to \mathbb{C}$ is defined as the unique extension to $\hat{\mathbb{S}}_K$ of the following $\theta$-holomorphic function on $H_K$:
\[ F_\theta(f) = \sum_{a \in \mathbb{R}^d} a_a \exp \left(2\pi i \text{Tr}(\alpha \cdot c_{\theta}(f))\right) = \sum_{a \in \mathbb{R}^d} a_a \exp \left(2\pi i \text{Tr}(\alpha \cdot c_{\theta}(f))\right) \]
where $c_{\theta}(\xi) \equiv \exp(2\pi i \text{Tr}(c_{\theta}(f)))$. The term $F_0$ is the constant function $a_0$.

The Hardy space of $\theta$-holomorphic functions is denoted $H_\theta[K]$. The space of $(2^d + 1)$-tuples is viewed as a graded Hilbert space:
\[ H_\theta[K] = \left( \bigoplus_{\theta} H_\theta[K] \right) \oplus \mathbb{C}, \]
whose inner-product is the direct sum of the inner-products on each of the summands. We will often write $H[K]$ for the summand of $H_\theta[K]$ corresponding to $\theta = (+, \ldots, +)$, i.e., the Hardy space of functions holomorphic on $\hat{\mathbb{S}}_K$ in the ordinary sense.

The Cauchy and Dirichlet products are defined on $H_\theta[K]$ via boundary extensions e.g. $F \otimes G$ is defined to be the unique element of $H_\theta[K]$ whose boundary is $\partial F \otimes \partial G$, provided the latter is defined. The Cauchy product does not generally respect the grading, but the Dirichlet product has the following graded decomposition law:
\[ (F \otimes G)_\theta = \sum_{\theta = \theta_1 \theta_2} F_{\theta_1} \otimes G_{\theta_2} \]
\[ (F \otimes G)_0 = F(1)G(1) - F_0 G_0. \]

8.2. **The Totally Complex Case.** Now let us suppose that $K/Q$ is totally complex of degree $d = 2s$. In order to extend the ideas in the previous paragraphs in a way compatible with the complex places, we develop a complex theory of signs. We consider the singular sign group
\[ \Theta^C = \{ \sqrt{-}, -\sqrt{-}, \sqrt{+} \} \equiv \mathbb{Z}/4\mathbb{Z} \]
and say that $z \in (\mathbb{R} \cup i\mathbb{R}) - 0$ has singular sign $\sqrt{-}, -\sqrt{-}, \sqrt{+}$ according to whether $z$ belongs to $i\mathbb{R}_+, -\mathbb{R}_+, -i\mathbb{R}_+$, or $\mathbb{R}_+$.

For points which do not belong to $\mathbb{R} \cup i\mathbb{R}$, we introduce the complex sign set
\[ \Omega = \{ \sqrt{-}, -\sqrt{-}, \sqrt{+} \}, \]
viewed as a $\Theta^C$-set in the obvious way. We say that $z \in \mathbb{C} - (\mathbb{R} \cup i\mathbb{R})$ has complex sign $\sqrt{-}, -\sqrt{-}, \sqrt{+}$ according to whether $z$ belongs to $i\mathbb{B}, -\mathbb{B}, -i\mathbb{B}$ or $\mathbb{B}$, where $\mathbb{B}$ is as before the quarter plane of $z = x + iy$ with $x, y > 0$. Every $\omega \in \Omega$ can be written uniquely in the form
\[ \omega = \theta \epsilon \]
for \( \theta \in \Theta^c \), and the map

\[ e : \Omega \to \Theta^c, \quad e(\theta \varepsilon) = \theta \]

is an isomorphism of \( \Theta^c \)-sets. The singular sign group \( \Theta^c \) may be viewed intermediate to the “signless” element 0 and the complex sign \( \Theta^c \)-set \( \Omega \).

If \( z, z' \) have complex signs \( \omega, \omega' \) then the product can have sign \( e(\omega)e(\omega')\varepsilon \), or else it can “overflow” into the neighboring signs: the singular sign \( \sqrt{-e(\omega)e(\omega')} \) and the complex sign \( \sqrt{-e(\omega)e(\omega')}^c \): explaining why we view \( \Omega \) as no more than a \( \Theta^c \)-set.

We view the union

\[ \mathcal{U} := \Theta^c \cup \Omega \]

as an abelian “multi-valued group” i.e. a set equipped with the abelian set-valued product

\[ \mathcal{U} \times \mathcal{U} \to 2^{\mathcal{U}} \]

with identity the sign +. We write for \( \theta, \theta_1, \theta_2 \in \mathcal{U} \)

\[ \theta \in \theta_1 \cdot \theta_2 \]

to indicate that \( \theta \) is among the possible signs that the product \( z_1z_2 \) can assume when

\[ \text{sign}(z_1) = \theta_1, \text{sign}(z_2) = \theta_2. \]

Note that \( |\theta_1 \cdot \theta_2| > 1 \) only when \( \theta_1, \theta_2 \in \Omega \), in which case \( |\theta_1 \cdot \theta_2| = 3 \). The function \( e \) of the previous paragraph extends to \( \mathcal{U} \) by the identity in \( \Theta^c \).

Consider now the set

\[ \mathcal{U}_K = \left\{ \theta = (\theta_j) \in \mathcal{U}^s \mid \exists \alpha \in K \text{ such that } \text{sign}(\alpha_{\mu_j}) = \theta_j \text{ for } j = 1, \ldots, s \right\}. \]

Notice that we are only using the first element \( \mu_j \) of each complex place pair \( (\mu_j, \bar{\mu}_j) \) to define the sign, and that we have that for all \( K \) the sign \( (+, \cdots, +) \in \mathcal{U}_K \). The set-valued product of components induces a set-valued product

\[ \mathcal{U}_K \times \mathcal{U}_K \to 2^{\mathcal{U}_K}. \]

The map

\[ e : \mathcal{U}_K \to (\Theta^c)^s \]

is defined \( e(\theta) = (e(\theta_j)) \).

If we define the type of \( \theta \in \mathcal{U}_K \) to be the vector \( t(\theta) \) whose \( j \)th component is 1 or \( \varepsilon \) depending on whether \( \theta_j \in \Theta^c \) or \( \Omega \), then

\[ \theta = e(\theta) \cdot t(\theta). \]

We will denote \( \varepsilon = (\varepsilon, \ldots, \varepsilon) \) and 1 = (1, \ldots, 1), and say that \( \theta \) is complex homogeneous (singular homogeneous) if \( t(\theta) = \varepsilon \) (\( t(\theta) = 1 \)). If \( \theta \) is singular homogeneous, we will sometimes write \( \theta = \theta \). The singular homogeneous elements form a subgroup \( \Theta^c_K \) with respect to which \( \mathcal{U}_K \) is a \( \Theta^c_K \)-set.

**Example 1.** Let \( K \) be the splitting field for \( X^3 - 2 \). Then the sign of any root is (modulo ordering of the places) \( (+, \sqrt{-\varepsilon}, -\varepsilon) \), so that \( \mathcal{U}_K \) has inhomogeneous triples. On the other hand, if \( K \) is the splitting field of \( X^3 - 1 \) then all elements of \( K \) are homogeneous. This is also true of the Gaussian numbers \( \mathbb{Q}(i) \), where \( \mathcal{U}_{\mathbb{Q}(i)} = \mathcal{U} \).

For each \( \theta \in \mathcal{U}_K \) we will associate a product of half spaces. For \( \omega \in \Omega \) and each place pair \( (\mu, \bar{\mu}) \) write

\[ \mathcal{H}^{\theta}_{(\mu, \bar{\mu})} = \left\{ (\kappa_{\mu}, \bar{\kappa}_{\mu}) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid \text{sign}(\bar{b}_{\mu}) = \omega \right\} \]
where we recall that $\kappa_\mu = (z_\mu, b_\mu)$, see \[8\]. Notice that when $\omega = +\epsilon$, $\mathbb{H}_\infty^\mu \supset \mathbb{H}_\infty^\mu(\mu, \bar{\omega})$.

We will view these spaces as products of half-spaces in the same way that we viewed $\mathbb{H}_\infty(\mu, \bar{\omega})$ as a product of half-spaces, see \((4)\) in \[8\]. For example, when $\omega = \sqrt{-\epsilon}$, $b_\mu = s_\mu + it_\mu$ satisfies $s_\mu < 0$ and $t_\mu > 0$ so that we obtain the product of the upper half-plane with the left half-plane:

$$\mathbb{H}_\infty^\mu = \mathbb{H}_\mu \oplus i\mathbb{H}_\mu.$$  

The points of the first factor are denoted $u_\mu = x_\mu + it_\mu$ and those of the second factor by $v_\mu = s_\mu + iy_\mu$. In the same fashion, we have

$$\mathbb{H}_\infty^\mu(\mu, \bar{\omega}) = -\mathbb{H}_\mu \oplus i\mathbb{H}_\mu, \quad \mathbb{H}_\infty^\mu = -\mathbb{H}_\mu \oplus -i\mathbb{H}_\mu.$$  

Also, for each singular sign $\theta \in \Theta^C$, we write

$$\mathbb{H}_\infty^\mu(\mu, \bar{\omega}) := \mathbb{H}_\infty^\mu(\mu, \bar{\omega}).$$

Then for $\theta \in \mathcal{U}_K$ we define the $\theta$-hyperbolic plane as

$$\mathcal{H}_K^\theta := \prod_{j=1}^s \mathbb{H}_\infty^\mu(\mu, \bar{\omega}).$$

We now define conjugation maps relating these planes. For $\kappa_\mu = (z_\mu, b_\mu) \in C^2$ and $\theta \in \Theta^C$, we define

$$c_\theta(z_\mu, b_\mu) = (z_\mu, \theta b_\mu).$$

This induces a map of $\mathbb{H}_\infty^\mu(\mu, \bar{\omega}) = \{(z, \bar{z}) \times (\bar{b}, b) | \bar{z} \in C, b \in \mathbb{B}\}$ such that

$$c_\theta(\mathbb{H}_\infty^\mu(\mu, \bar{\omega})) = \mathbb{H}_\infty(\mu, \bar{\omega}) \supset \mathbb{H}_\infty(\mu, \bar{\omega}).$$

Notice that the conjugation maps are not (holomorphic) functions of the half-plane variables $u_\mu = x_\mu + it_\mu, u_\mu = s_\mu + iy_\mu$. For example, when $\theta = \sqrt{-\epsilon}$,

$$c_\theta(x_\mu + it_\mu, s_\mu + iy_\mu) = (x_\mu + is_\mu, -t_\mu + iy_\mu).$$

We have

$$c_{\theta_1} \circ c_{\theta_2} = c_{\theta_1 \theta_2}$$

for all $\theta_1, \theta_2 \in \Theta^C$.

For each $\theta \in \mathcal{U}_K$ we associate a conjugation $c_\theta : C_K \to C_K$ where the coordinates of $c_\theta(\kappa) = \kappa'$ are determined by

$$k_{\mu} = (z_{\mu}, b_{\mu}')(= (z_{\mu}, e(\theta) b_{\mu})).$$

for $k = 1, \ldots, s$. Notice that $c_\theta$ restricts to a map $c_\theta : \mathbb{H}_K \to \mathbb{H}_K^\theta$. A function $F : C_K \to C$ of the shape $F = G \circ c_\theta$ where $G : C_K \to C$ is holomorphic, will be called $\theta$-holomorphic.

For each $\theta \in \mathcal{U}_K$, denote by $K^\theta$ those elements $\alpha$ whose coordinates with respect to the embedding $K \hookrightarrow K_\infty$ satisfy

$$\text{sign}(\alpha_\mu), \ldots, \text{sign}(\alpha_\mu) = \theta.$$

For $\theta \in \Theta^C$ and arbitrary $\theta \in \mathcal{U}_K$ we have

$$K^\theta \cdot K^\theta \subset K^\theta.$$  

In general, we cannot expect a law of the genre \((15)\) since the complex signs do not form a group. We have instead the multi-valued law:

$$K^\theta \cdot K^\theta \subset \bigcup_{\theta \in \Theta \cdot \theta_2} K^\theta.$$
Every element \( f \in L^2(\mathbb{S}_K, \mathbb{C}) \) now determines a \((|\mathcal{U}_K| + 1)\)-tuple \((F_\theta; F_0)\) as follows. For each \( \theta \in \mathcal{U}_K \), \( F_\theta : \mathbb{S}_K \to \mathbb{C} \) is defined as the extension to \( \hat{\mathbb{S}}_K \) of the following function on \( H_K \):

\[
F_\theta(k) = \sum_{a \in K^\theta} a_a \exp(2\pi i \text{Tr}(ac_\theta(k))) \equiv \sum_{a \in K^\theta} a_a(c_\theta(\eta))^a.
\]

where \( c_\theta(\eta) \equiv \exp(2\pi i \text{Tr}(c_\theta(k))) \). Observe that the comments found in Note 4 imply that whenever the sign coordinate \( \Theta_j \) is singular i.e. \( \theta_j \in \Theta^C \) then \( F_\theta \) is constant in one of the corresponding half-plane coordinates \( u_{\mu_1}, u_{\mu_2} \). This explains why we call the signs in \( \Theta^C \) “singular”.

We refer to \( F_\theta \) as the \( \theta \)-holomorphic component of \( f \), and the Hardy space of \( \theta \)-holomorphic functions is denoted \( H_\theta[K] \). We obtain a graded Hilbert space:

\[
H_*[K] = \left( \bigoplus_{\theta} H_\theta[K] \right) \oplus \mathbb{C},
\]

whose inner-product is the direct sum of the inner-products on each of the summands. We will write as in the real case \( H[K] \) for the summand of \( H_*[K] \) corresponding to \( \theta = \mathbf{1} \).

The Cauchy and Dirichlet products are defined on \( H_*[K] \) via boundary extensions just as in the real case. When \( \theta \in \Theta_1 \cdot \Theta_2 \) we write

\[
(F_{\theta_1} \otimes G_{\theta_2})|_\theta
\]

for the projection of \( F_{\theta_1} \otimes G_{\theta_2} \) onto the sub series indexed by \( a \in K^\theta \). Then we have the following graded decomposition law generalizing that described in (13) for the totally real case:

\[
(F \otimes G)_\theta = \sum_{\theta \in \Theta_1 \cdot \Theta_2} (F_{\theta_1} \otimes G_{\theta_2})|_{\theta}, \quad (F \otimes G)_0 = F(1)G(1) - F_0G_0.
\]

Now consider \( K/\mathbb{Q} \) a general finite extension. The we obtain a decomposition of the form

\[
H_*[K] = H_*[K] \oplus H_*[K]
\]

corresponding to the real and complex places (graded accordingly) so that in particular, every \( f \in L^2(\mathbb{S}_K, \mathbb{C}) \) determines a \((2^r + |\mathcal{U}_K| + 1)\)-tuple of functions.

Note 5. Let \( K/\mathbb{Q} \) be Galois. Then the action of the Galois group \( \text{Gal}(K/\mathbb{Q}) \) induces a well-defined action on the sign group \( \Theta_K \) (when \( K \) is real) and on the “multi-valued sign group” \( \mathcal{U}_K \) (when \( K \) is complex).

For \( K \) real or complex, we have a sub partial double group algebra \( H_*[O_K] \) of \( H_*[K] \) defined by Fourier series whose indices belong only to \( O_K \). The Hilbert space of graded holomorphic functions pulled back from the Minkowski hyperbolized torus \( \mathbb{T}_K \) are denoted

\[
H_*[\mathbb{T}_K] := H_*[\partial_K^{-1}],
\]

where \( \partial_K^{-1} \) is the inverse different. This sub Hilbert space is closed with respect to the Cauchy product (where it is defined), and is closed with respect to the action by Dirichlet multiplication with elements of \( H_*[O_K] \) (whenever such products are defined). When \( K = \mathbb{Q} \), then we have \( H_*[\mathbb{Z}] = H_*[\mathbb{T}_\mathbb{Q}] \) is a partial double algebra with respect to the Cauchy and Dirichlet products.
We now indicate how to extend this construction to infinite field extensions $\mathcal{K}/\mathbb{Q}$. Here we use the hyperbolized adele class group $\hat{C}\mathcal{X}$ (associated to $\hat{C}\mathcal{X}$) in conjunction with the proto adele class group $\hat{C}\mathcal{X}$. A continuous function $F : \hat{C}\mathcal{X} \to \mathbb{C}$ is holomorphic if its restriction to any of the dense leaves is holomorphic. In particular, we note that the restriction $F|_{H_{\mathcal{X}}}$ is holomorphic if $F$ is holomorphic separately in each factor of the polydisk decomposition $H_{\mathcal{X}} \approx \prod H_{\nu} \times \prod H_{(\mu, \nu)}$. As in the case of a finite field extension, we define $\hat{C}\mathcal{X}(t)$ as the subset of $\hat{C}\mathcal{X}$ having extended coordinate $t \in (0, \infty)^{\infty}$. The proto compactification of $\hat{C}\mathcal{X}(t)$ is a lamination $\hat{C}\mathcal{X}(t)$ homeomorphic to $\hat{C}\mathcal{X}$. If we let

$$\hat{C}\mathcal{X} = \hat{C}\mathcal{X} \times (0, \infty)^{\infty} = \bigcup \hat{C}\mathcal{X}(t),$$

the Hardy space $H[\mathcal{K}]$ is defined as the space of holomorphic functions $F : \hat{C}\mathcal{X} \to \mathbb{C}$ having a continuous extension $\hat{F} : \hat{C}\mathcal{X} \to \mathbb{C}$ and for which the norm $\|\hat{F}\|_{t}^{2} = \int_{\hat{C}\mathcal{X}(t)} |\hat{F}|^{2} d\mu$ is uniformly bounded in $t$ (where $d\mu$ is induced from the Haar measure $\mu$ on $\hat{C}\mathcal{X} \approx \hat{C}\mathcal{X}(t)$). As in the case of the finite dimensional case, $H[\mathcal{K}]$ is a Hilbert space with respect to the supremum of the integration pairings on each $\hat{C}\mathcal{X}(t)$. The rest of the development follows that of the finite extension case, where the grading is defined for the real and complex places separately.

We summarize the above remarks in the following

**Theorem 6.** Let $\mathcal{K}$ be a (possibly infinite degree) algebraic number field over $\mathbb{Q}$. Then $H_{r}[\mathcal{K}]$ is a graded Hilbert space equipped with the structure of a partial double $\mathbb{C}$-algebra with respect to the operations of $\oplus$ and $\otimes$.

### 9. Nonlinear Number Fields

Let $K/\mathbb{Q}$ be a number field of finite degree over $\mathbb{Q}$. Let $C[K]$ denote the vector space of formal, finite $\mathbb{C}$-linear combinations of elements in $K$ i.e. expressions of the form $\sum a_{a} \alpha$ where $\alpha \in K$ and $a_{a} \in \mathbb{C}$, zero for all but finitely many $\alpha$. The operations $+_{K}$ and $\times_{K}$ extend linearly to $C[K]$ yielding two operations, written $\oplus$ and $\otimes$, which define on $C[K]$ two algebra structures. We refer to $C[K]$ as the field algebra generated by $K$. To avoid confusion, we use the notation $id_{\oplus} = 0_{K}$ and $id_{\otimes} = 1_{K}$; 0 will denote the vector space identity, the element of $C[K]$ for which $a_{a} = 0$ for all $a$. There exists a canonical double algebra monomorphism generated by $a \mapsto \zeta^{a}$,

$$C[K] \hookrightarrow L^{2}(\hat{C}K, \mathbb{C}) \cong H_{r}[K]$$

having dense image. The subspace $C[O_{K}]$ of $C$-linear combinations of integers is closed with respect to both algebra structures. These algebra structures are not compatible in any familiar sense as the operations $\oplus$ and $\otimes$ do not obey the distributive law.

We define a linear map $T : C[K] \to \mathbb{C}$ by

$$T(f) = \sum a_{a} \in \mathbb{C}.$$

Notice that $f \otimes id_{\oplus} = T(f) \cdot id_{\oplus}$. The vector space $I_{K} = \text{Ker}(T)$ is an ideal in $C[K]$ with respect to the operations of $\oplus$ and $\otimes$ and the set-theoretic difference

$$C^{*}[K] := C[K] - I_{K}$$
is preserved by both of $\oplus$ and $\otimes$. Denote a typical element of $\mathbb{C}^*[K]$ by $f^*$. Then for any $f^*$ we have

$$f^* \otimes \text{id}_e^* \in \mathbb{C}^* \cdot \text{id}_e^*.$$  

Let $N^0[K]$ denote the image of $\mathbb{C}^*[K]$ in the complex projectivization $\mathbb{P}\mathbb{C}[K]$ of $\mathbb{C}[K]$. By virtue of (18), the operations $\oplus$ and $\otimes$ descend to $N^0[K]$, making it a \textit{double semigroup}: a set with two semigroup structures having no \textit{a priori} compatibility. We denote its elements $[f]$. The sub double semigroup $N^0[O_K]$ is defined similarly.

Note that the element $[\text{id}_e]$ behaves very much like the zero in a field in that it is a universal annihilator with respect to $\otimes$; for all $[f] \in N^0[K]$,

$$[f] \otimes [\text{id}_e] = [\text{id}_e].$$

Furthermore, the natural inclusions $O_K \hookrightarrow \mathbb{C}[O_K]$ and $K \hookrightarrow \mathbb{C}[K]$ induce homomorphisms $O_K \twoheadrightarrow N^0[O_K]$ and $K \twoheadrightarrow N^0[K]$. These echoes with field theory make the double semigroup $N^0[K]$ a natural paradigm for the ensuing development of nonlinear fields. We remark that the preceding comments hold without change for an infinite algebraic extension $\mathcal{K}/\mathbb{Q}$.

Motivated by the monomorphism $K \subset \mathbb{C}[K] \hookrightarrow \mathcal{H}_*[K]$ we set out to create from $\mathcal{H}_*[K]$ something akin to a field extension of $K$ by graded holomorphic functions. In this connection, we note that the operations $\oplus$ and $\otimes$ restrict to $+_{K}$ and $\times_{K}$ on $K$, and on the other hand, the vector space operations of point-wise addition and scalar multiplication do not preserve $K$. Accordingly, we discard the vector space structure by projectivizing, retracing the steps in the construction of $N^0[K]$.

Here, the trace operator $T$ is not defined on all of $\mathcal{H}_*[K]$. It is unambiguously defined on the subspace of functions $F$ having boundary $\partial F$ lying in the subspace $L^1_{\mathbb{C}}(K) \cap L^2_{\mathbb{C}}(K)$ of functions whose Fourier coefficients $\{a_n\}$ are absolutely summable. For such elements $F \in \mathcal{H}_*[K]$ we define $T(F) = \sum a_n = F(1)$ and denote $l_K = \text{Ker}(T)$. We note that $l_K$ is not closed in in $\mathcal{H}_*[K]$ and is in fact dense there. The set theoretic difference

$$\mathcal{H}_*[K] := \mathcal{H}_*[K] - l_K$$

inherits the grading by restriction. The associated quotient by $\mathbb{C}^*$, denoted

$$N_*[K],$$

is an infinite dimensional subspace of the full projectivization $\mathbb{P}\mathcal{H}_*[K]$ which inherits $\oplus, \otimes$ as partially defined operations. The grading gives rise to an arrangement of subspaces

$$(N_0[K])$$

where $N_0[K] = \mathbb{P}\mathcal{H}_0[K] \cap N_*[K]$. The canonical monomorphism

$$N^0[K] \hookrightarrow N_*[K]$$

induced by $a \mapsto \rho^a$, has dense image, and the elements of $N^0[K]$ may be Cauchy or Dirichlet multiplied with any element of $N_*[K]$. We also have a monomorphism $K \hookrightarrow N_*[K]$. Elements of $N_*[K]$ will be denoted by $[F]$. The sub partial double semigroup $N_*[O_K]$ is defined similarly. These remarks are valid without change for an infinite field extension $\mathcal{K}/\mathbb{Q}$.

The grading of $N_*[K]$ induces one on $N^0[K]$ and so we write $N_0^0[K]$.

\textbf{Definition 2.} A \textbf{nonlinear number field} is a graded topological abelian partial double semigroup $S$, with respect to two operations $\oplus$ and $\otimes$ such that
(1) There exists a (possibly infinite degree) algebraic number field \( \mathcal{K}/\mathbb{Q} \) and a graded double semigroup monomorphism \( \iota : \mathbb{N}^0[\mathcal{K}] \hookrightarrow S_x \) having dense image.

(2) The identity \( \text{id}_\varnothing \) is a universal annihilator for \( \otimes \): for all \( F \in \text{Dom}_\varnothing(\text{id}_\varnothing) \), \( F \otimes \text{id}_\varnothing = \text{id}_\varnothing \).

The closure \( O \) of the image \( \iota(\mathbb{N}^0[O_{\mathcal{K}}]) \) is called the \textbf{nonlinear ring of integers}.

The qualitative “nonlinear” refers to the fact that distributivity in an ordinary field is equivalent to the fact that the multiplication map is a bilinear operation. For \( \mathcal{K}/\mathbb{Q} \) be a possibly infinite degree extension, notice that both \( \mathbb{N}^0[\mathcal{K}] \) and \( N_\mathcal{K}[\mathcal{K}] \) are nonlinear number fields. In addition the following are also nonlinear number fields:

- \( \mathbb{N}[\mathcal{K}] = \mathbb{P} H_\mathcal{K}[\mathcal{K}] \) is the full projectivization of \( H_\mathcal{K}[\mathcal{K}] \).
- Let \( W_\mathcal{K}[\mathcal{K}] \) be the projectivization of the subspace of \( H_\mathcal{K}[\mathcal{K}] \) consisting of absolutely convergent series with non-zero trace. Then \( W_\mathcal{K}[\mathcal{K}] \) is the \textbf{Wienriev nonlinear number field} associated to \( \mathcal{K} \): a full (and not partial) sub-semigroup of \( N_\mathcal{K}[\mathcal{K}] \).

We have inclusions \( \mathcal{K} \subset \mathbb{N}^0[\mathcal{K}] \subset N_\mathcal{K}[\mathcal{K}] \subset W_\mathcal{K}[\mathcal{K}] \subset \mathbb{N}[\mathcal{K}] \), the last three of which are dense.

\textit{Note 6.} In view of \textit{Note 3} all of the arithmetic of classical (single variable) zeta functions, Dirichlet series, \( L \)-functions, etc. is contained in \( N_\mathcal{K}[\mathcal{K}] \).

\textit{Note 7.} The set of Cauchy units \( U_\mathcal{K}[\mathcal{K}] \) form a dense subset of \( N_\mathcal{K}[\mathcal{K}] \), since it contains all classes represented by real analytic nonvanishing functions of \( \hat{S}_\mathcal{K} \). It is an interesting question as to whether the set of Dirichlet units \( U_\mathcal{K}[\mathcal{K}] \) is also dense, and whether there exists an algorithmic procedure to determine the coefficients of a Dirichlet inverse analogous to the classical Möbius inversion formula.

The following theorem says that on a dense subset, \( N_\mathcal{K}[\mathcal{K}] \) is the “nonlinear field of fractions” of \( N_\mathcal{K}[O_{\mathcal{K}}] \):

\textbf{Theorem 7.} Let \( \mathcal{K} \) be a (possibly infinite degree) number field. Then there is a dense subset \( \mathcal{P} \subset N_\mathcal{K}[\mathcal{K}] \) such that for all \( [F] \in \mathcal{P} \), there exists \( [A] \in N_\mathcal{K}[O_{\mathcal{K}}] \) with

\[ [A] \otimes [F] \in N_\mathcal{K}[O_{\mathcal{K}}]. \]

\textit{Proof.} Let \( N_\mathcal{K}[\mathcal{K}]_{\mathcal{P}} \) be the subspace associated to functions whose nonzero Fourier coefficients are indexed by \( a \) in some fractional ideal \( O_{\mathcal{K}} a^{-1} \subset \mathcal{K} \), where \( a \in O_{\mathcal{K}} \). Consider the sub double semigroup \( N_\mathcal{K}[a] \). Then given \( [F] \in N_\mathcal{K}[\mathcal{K}]_{\mathcal{P}} \) whose nonzero Fourier coefficients are indexed by \( O_{\mathcal{K}} a^{-1} \), there exists \( [A] \in N_\mathcal{K}[a] \) such that \( [A] \otimes [F] \in N_\mathcal{K}[O_{\mathcal{K}}] \). \( \square \)

10. \textbf{Galois Groups and the Action of the Idele Class Group of} \( \mathbb{Q} \)

Let \( \mathcal{K}/\mathbb{Q} \) be a possibly infinite degree algebraic number field Galois over \( \mathbb{Q} \). In this case, \( \mathcal{K} \) is either totally real or totally complex: in either case we denote by \( i_\mathcal{K} \) the sign set. We will also not distinguish \( \mathcal{K} \) from its images in \( H_\mathcal{K}[\mathcal{K}] \) or \( N_\mathcal{K}[\mathcal{K}] \). We will prefer here to represent elements of \( \mathcal{K} \) using the power series notation \( \varphi^a = \exp(2\pi i \text{Tr}(\varphi \cdot a)) \) (as opposed to the character notation \( \phi_a \)), which has the advantage of allowing us to interpret Dirichlet multiplication with monomials in terms of composition:

\[ [F] \otimes [\varphi^a] = [F(\varphi^a)]. \]
We denote as in the previous section $N_\ast[\mathcal{H}] = \mathbb{P}H_\ast[\mathcal{H}]$.

Equip $N_\ast[\mathcal{H}]$ with the Fubini-Study metric associated to the inner-product on $H_\ast[\mathcal{H}]$. A nonlinear automorphism $\gamma : N_\ast[\mathcal{H}] \rightarrow N_\ast[\mathcal{H}]$ is the restriction of a graded Fubini-Study isometry of $N_\ast[\mathcal{H}]$ respecting the operations $\oplus$, $\otimes$ whenever they are defined: that is,

$$\gamma([F] \oplus [G]) = \gamma([F]) \oplus \gamma([G]), \quad \gamma([F] \otimes [G]) = \gamma([F]) \otimes \gamma([G])$$

and for some permutation $\iota$ of $\mathcal{U}_\mathcal{H}$, $\gamma(\mathbb{P}H_\theta[\mathcal{H}]) = \mathbb{P}H_{\iota(\theta)}[\mathcal{H}]$ for all $\theta \in \mathcal{U}_\mathcal{H}$.

For example, let $\mathcal{L}/\mathcal{K}$ be Galois. Then the Galois group $\text{Gal}(\mathcal{L}/\mathcal{K})$ acts on $H_\ast[\mathcal{L}]$ by:

$$\sum a_\alpha \rho^\alpha \mapsto \sum a_\alpha \rho^{\sigma(\alpha)} = \sum a_{\sigma^{-1}(\alpha)} \rho^\alpha$$

for $\sigma \in \text{Gal}(\mathcal{L}/\mathcal{K})$. This action permutes the coefficient set, hence acts by isometries. Viewing the action on $\mathcal{L}_\infty$, where it simply permutes coordinates, we see that there is an induced action on the sign set $\mathcal{U}_\mathcal{L}$, so that for any $\theta \in \mathcal{U}_\mathcal{L}$, we have $\sigma(\mathcal{L}_\theta) = \mathcal{L}_{\iota(\theta)}$. Thus $\sigma$ permutes the grading of $H_\ast[\mathcal{L}]$. Finally, $\sigma$ preserves the elements of trace zero and commutes with multiplication by elements of $\mathbb{C}^\ast$, hence induces a nonlinear automorphism

$$\sigma : N_\ast[\mathcal{L}] \rightarrow N_\ast[\mathcal{L}]$$

that is trivial on $N_\ast[\mathcal{K}]$.

If $\mathcal{L}/\mathcal{K}$ is a field extension of number fields of possibly infinite degree over $\mathbb{Q}$, denote by

$$\text{Gal}(N_\ast[\mathcal{L}]/N_\ast[\mathcal{K}])$$

the group of nonlinear automorphisms of $N_\ast[\mathcal{L}]$ fixing the sub nonlinear field $N_\ast[\mathcal{K}]$, and by

$$\text{Gal}(N_\ast[\mathcal{K}]/\mathcal{K})$$

the group of nonlinear automorphisms of $N_\ast[\mathcal{K}]$ fixing $\mathcal{K}$.

We recall the following theorem of Wigner [17].

**Wigner’s Theorem.** Let $\mathcal{H}$ be a complex Hilbert space, $\mathbb{P}\mathcal{H} = (\mathcal{H} \setminus \{0\})/\mathbb{C}^\ast$ its projectivization. Let $[h] : \mathbb{P}\mathcal{H} \rightarrow \mathbb{P}\mathcal{H}$ be a bijection preserving the Fubini-Study metric. Then $[h]$ is the projectivization of a unitary or anti-unitary linear map $h : \mathcal{H} \rightarrow \mathcal{H}$.

**Theorem 8.** Let $\mathcal{K}$ be a (possibly infinite degree) number field. Then

$$\text{Gal}(N_\ast[\mathcal{K}]/\mathcal{K}) \cong \{1\}.$$

**Proof.** Let $\sigma \in \text{Gal}(N_\ast[\mathcal{K}]/\mathcal{K})$. By Wigner’s Theorem, $\sigma$ is the projectivization of an (anti) unitary linear map

$$\tilde{\sigma} : H_\ast[\mathcal{K}] \rightarrow H_\ast[\mathcal{K}].$$

Since $\sigma$ fixes $\mathcal{K}$, there exist multipliers $\lambda_\alpha \in U(1)$ with

$$\tilde{\sigma}(\rho^\theta) = \lambda_\alpha \cdot \rho^\theta.$$

But $\sigma$ respects the Cauchy and Dirichlet products, wherein we must have that $\lambda$ is simultaneously an additive and multiplicative character:

$$\lambda_{\alpha_1 + \alpha_2} = \lambda_{\alpha_1} \lambda_{\alpha_2} = \lambda_{\alpha_1 \alpha_2},$$

clearly possible only for $\lambda$ trivial. 

\[\square\]
Lemma 2. Let \( \mathcal{K} \) be a (possibly infinite degree) algebraic number field Galois over \( \mathbb{Q} \). Let \( \theta \in \overline{\mathcal{K}} \), let \( F \in \mathbb{H}_\theta[\mathcal{K}] \) satisfy the functional equation
\[
F(q) = (F(q))^r
\]
for all \( r \in \mathbb{R}_+ \). Then \( F \in \mathcal{K} \) i.e. there exists \( \alpha \in \mathcal{K} \) with \( F(q) = \alpha^q \).

Proof. We first consider the case of \( K = \mathbb{Q} \). In this totally real case we write \( F \) as a function of the parameter \( \xi = \exp(2\pi i \tau) \) where \( \tau = x + iy \in \mathbb{H}_0 \). Assume first that \( F \) is holomorphic and non constant i.e. \( F \in \mathbb{H}[\mathbb{Q}] = \) Hardy space of holomorphic functions. We will show that \( F(\xi) = \xi^q \) for some \( q \in \mathbb{Q}_+ \). Let us return to viewing \( F \) as a holomorphic function of the half-plane variable \( \tau \). Let \( \tau_0 \in \mathbb{H}_0 \subset \mathbb{H}_Q \) be such that \( F(\tau_0) \neq 0 \). Then there exists a complex number \( \alpha \) with \( \exp(2\pi i \alpha \tau_0) = F(\tau_0) \). By (19), the functions \( F(\tau) \) and \( \exp(2\pi i \alpha \tau) \) agree on the ray \( \mathbb{R}_+ \cdot \tau_0 \) hence by holomorphicity, they coincide. Since \( F \) is a function of \( \mathbb{H}_Q \), it follows that \( \alpha = q \in \mathbb{Q}_+ \) hence \( F(\xi) = \xi^q \). For \( F \in \mathbb{H}_-[\mathbb{Q}] = \) Hardy space of anti-holomorphic functions, the argument is the same: we use the anti-holomorphic exponential \( \exp(2\pi i \alpha \tau) \) and the fact that anti-holomorphic functions agreeing on a co-dimension 1 subspace coincide.

Now let \( K/\mathbb{Q} \) be totally real of finite degree. Without loss of generality we may assume that \( F \in \mathbb{H}[K] = \) the Hardy space of holomorphic functions. Let \( \Delta \subset \mathbb{H}_K \) be the diagonal hyperbolic sub-plane, which is the dense leaf of the image of \( \hat{\mathbb{H}}_Q \) under the diagonal embedding \( \hat{\mathbb{H}}_Q \to \hat{\mathbb{H}}_K \). Then by the previous paragraph, the restriction of \( F \) to \( \Delta \) is an exponential
\[
\exp(2\pi i q \tau)
\]
for \( q \in \mathbb{Q}_+ \). Let \( \Delta_1 \subset \mathbb{H}_K \) be the diagonal
\[
\Delta_1 = \{(\tau, \ldots, \tau, \bar{\tau}) | \tau, \bar{\tau} \in \mathbb{H}\} \supset \Delta.
\]
For each \( \tau \) fixed, we can (using the same argument employed in the previous paragraph) write the function \( \bar{\tau} \mapsto F(\tau, \ldots, \tau, \bar{\tau}) \) as
\[
F(\tau, \ldots, \tau, \bar{\tau}) = \exp(2\pi i q \tau) \cdot \exp(2\pi i \beta(\tau) \bar{\tau})
\]
where \( \beta(\tau) \in \mathbb{C} \). Moreover we have
\[
F(r \cdot (\tau, \ldots, \tau, \bar{\tau})) = \exp(2\pi i q r \tau) \cdot \exp(2\pi i \beta(\tau) r \bar{\tau})
\]
by hypothesis. As we vary \( r \), \( \beta(\tau) \) varies holomorphically and we obtain that on a real codimension 1 subspace of \( \Delta_1 \),
\[
F(\tau, \ldots, \tau, \bar{\tau}) = F_1(\tau, \ldots, \tau, \bar{\tau}) := \exp(2\pi i q \tau) \cdot \exp(2\pi i \beta(\tau) \bar{\tau})
\]
hence they are equal on \( \Delta_1 \). But this means that \( F_1 \) must also satisfy the functional equation \( F_1(r \cdot (\tau, \ldots, \tau, \bar{\tau})) = F_1(\tau, \ldots, \tau, \bar{\tau}) \) which implies that \( \beta(\tau) = \beta \) is a constant.

Inductively, we obtain that \( F \) restricted to \( \mathbb{H}_K \) is an exponential function, and since \( F \) is a function of \( \hat{\mathbb{H}}_K \), this restriction must be of the form \( \xi^\alpha = \exp(2\pi i \text{Tr}(\alpha \cdot \tau)) \) for \( \alpha \in K_+ \). The case of a totally complex field extension, one of mixed type, or one of infinite degree, is handled in exactly the same manner.

\[ \square \]

Theorem 9. Let \( \mathcal{L}/\mathcal{K} \) be a Galois extension of (possibly infinite degree) algebraic number fields. Then
\[
\text{Gal}(N_1[\mathcal{L}]/N_1[\mathcal{K}]) \cong \text{Gal}(\mathcal{L}/\mathcal{K}).
\]
Proof. Let $\sigma \in \text{Gal}(\mathbb{N},[\mathcal{L}]/\mathbb{N},[\mathcal{K}])$. We begin by showing that $\sigma(\mathcal{L}) = \mathcal{L}$, where $\mathcal{L}$ is identified with the field of monomials $[\mathcal{g}^\alpha]$, $\alpha \in \mathcal{L}$. Note first that we have already $\sigma(\mathcal{K}) = \mathcal{K}$. Since $\sigma(\mathcal{L})$ is a field, all of its elements obey the distributive law. Thus, given any $[F] \in \sigma(\mathcal{L})$, since $\sigma([\mathcal{g}^m]) = [\mathcal{g}^m] \in \mathcal{K}$, we have

$$[F([\mathcal{g}^m])] = [F] \otimes [\mathcal{g}^m] = ([F] \otimes [\mathcal{g}]) \oplus \cdots \oplus ([F] \otimes [\mathcal{g}]) = [F\mathcal{g}] \oplus \cdots \oplus [F\mathcal{g}] = [F]^m,$$

where $[F]^m$ denotes the Cauchy $m$-th power of $[F]$. In fact, the same argument shows that for any $m/n \in \mathbb{Q}_+$,

$$([F^{m/n}])^n = [F]^m.$$ We may thus find $F \in [F]$ satisfying the functional equation $F([\mathcal{g}]) = (F([\mathcal{g}]))^q$ for all $q \in \mathbb{Q}_+$. By continuity, this extends to $\mathbb{R}_+$. Note that since $\sigma$ respects the grading and each element of $\mathcal{L}$ is homogeneous (is contained in a fixed projective summand) then $[F]$ is also homogeneous. Thus, by Lemma 2 it follows that $F \in \mathcal{L}$ and $\sigma(\mathcal{L}) = \mathcal{L}$.

We induce in this way a homomorphism

$$\Pi : \text{Gal}(\mathbb{N},[\mathcal{L}]/\mathbb{N},[\mathcal{K}]) \to \text{Gal}(\mathcal{L}/\mathcal{K}), \quad \Pi(\sigma) = \sigma|_{\mathcal{L}}.$$ Note that $\Pi$ is clearly onto, as we have already observed that any $\sigma \in \text{Gal}(\mathcal{L}/\mathcal{K})$ generates an automorphism of $\mathbb{N},[\mathcal{L}]/\mathbb{N},[\mathcal{K}]$ fixing $\mathbb{N},[\mathcal{K}]$ via $\mathcal{g}^\alpha \mapsto \mathcal{g}^{\sigma(\alpha)}$. Suppose now that $\Pi(\sigma) = 1$ for $\sigma$ in $\text{Gal}(\mathbb{N},[\mathcal{L}]/\mathbb{N},[\mathcal{K}])$. Then $\sigma \in \text{Gal}(\mathbb{N},[\mathcal{L}]/\mathcal{L})$, but by Theorem $\S$ the latter group is trivial. \hfill \qed

We now concentrate on the case of a finite Galois extension $K/\mathbb{Q}$ and consider each of the operations $\oplus$ and $\otimes$ separately. We will work with the nonlinear number field $\mathbb{N},[K] = \mathbb{P}\mathbb{H},[K]$. Let $\text{Gal}_\mathbb{Q}(\mathbb{N},[K]/K)$ denote those isometries fixing $K$ and homomorphic with respect to $\oplus$ only. $\text{Gal}_\mathbb{Q}(\mathbb{N},[K]/K)$ is defined similarly.

Denote by $U[\mathbb{H},[K]]$ the group of unitary operators of $\mathbb{H},[K]$. The action of $r \in K_\infty$ by translation in $\mathbb{H},[K]$, $z \mapsto z + r$, induces an action on $\mathbb{H},[K]$ by

$$\Phi_r(F) = \sum a_{\alpha} \exp(2\pi i \text{Tr}(ar)) \mathcal{g}^\alpha$$

for $F = \sum a_{\alpha} \mathcal{g}^\alpha$, yielding a faithful representation

$$\Phi : K_\infty \to U[\mathbb{H},[K]].$$

**Proposition 4.** The projectivization $[\Phi]$ of $\Phi$ defines a monomorphism

$$[\Phi] : K_\infty \to \text{Gal}_\mathbb{Q}(\mathbb{N},[K]/K).$$

**Proof.** For $[\mathcal{g}^\alpha] \in K$ and $r \in K_\infty$, $[\Phi]_r([\mathcal{g}^\alpha]) = [\exp(2\pi i \text{Tr}(ar)) \mathcal{g}^\alpha] = [\mathcal{g}^\alpha]$. For any $[F]$, $[G] \in \mathbb{N},[K]$, let $[f]$, $[g]$ be the projective classes of their boundary functions. Then $[\Phi]_r([F] \oplus [G])$ is the element of $\mathbb{N},[K]$ whose boundary function is

$$[\Phi]_r([f] \oplus [g]) = \left[ \sum_{\alpha} \left( \sum_{a_1 + a_2 = \alpha} a_{a_1} b_{a_2} \exp(2\pi i \text{Tr}(ar)) \zeta^\alpha \right) \right]$$

where $[\Phi]_r((f) \oplus (g)) = [\Phi]_r((f)) \oplus [\Phi]_r((g))$, which is the boundary function of $[\Phi]_r((F) \oplus (G))$. \hfill \qed
In like fashion, we may define a flow on \( \hat{N}_e[K] \) respecting \( \Phi \) as follows. For a vector \( z \in K_\infty \) we denote by \( \log |z| \) the vector 
\[
[\log |z_{v_1}|, \ldots, \log |z_{v_d}|]
\]
when \( K \) is real, or when \( K \) is complex 
\[
[\log |z_{\mu_1}|, \log |z_{\mu_1}|, \ldots, \log |z_{\mu_1}|, \log |z_{\mu_1}|].
\]
Then for \( F = \sum a_\alpha q^\alpha \) we define 
\[
\Psi_r(F) = \sum_{\alpha \in K} a_\alpha \exp\left(2\pi i r \log |\alpha|\right)q^\alpha.
\]
This defines a faithful representation 
\[
\Psi : K_\infty \rightarrow U(H_e[K]).
\]
The following is proved exactly as Proposition 4.

**Proposition 5.** The projectivization \([\Psi]\) of \( \Psi \) defines a monomorphism 
\[
[\Psi] : K_\infty \rightarrow \text{Gal}_e(\hat{N}_e[K]/K).
\]

Recall that the idele class group of \( \mathbb{Q}, C_Q \), may be identified with \( \mathbb{R}^*_+ \times \text{Gal}(\hat{Q}_{ab}/\mathbb{Q}) \).

We have the following Corollary to Propositions 4 and 5.

**Corollary 1.** There are monomorphisms
\[
C_Q \hookrightarrow \text{Gal}_e(\hat{N}_e(\mathbb{Q})/\mathbb{Q}), \quad C_Q \hookrightarrow \text{Gal}_e(\hat{N}_e(\mathbb{Q}_{ab})/\mathbb{Q}).
\]

We end by noting that the above structures have an interpretation within the von Neumann depiction of quantum mechanics. The space \( \hat{N}_e[K] \) may be viewed as the space of states of a quantum mechanical system with \( d \) degrees of freedom. We view each of the flows \( \Phi \) and \( \Psi \) as generating the coordinate observables of two distinct physical systems and write
\[
\Phi_r = \exp(2\pi i \langle H_\theta, r \rangle), \quad \Psi_r = \exp(2\pi i \langle H_\theta, r \rangle),
\]
where \( H_\theta = \text{diag}(q)_{q \in K} \) and \( H_\theta = \text{diag}(\log |q|)_{q \in K} \) are the associated (vector-valued) Hamiltonian operators. The set of stationary states for each of \( H_\theta \) and \( H_\theta \) is the field \( K \).

Each \( [F] \in \hat{N}_e[K] \) defines a Cauchy multiplication operator \( \hat{M}_e([F]) \) and a Dirichlet multiplication operator \( M_e([F]) \). It is easy to see that if \( [F] \) has a representative \( F \) whose Fourier coefficients are real, then \( M_e([F]) \) defines an observable i.e. the projectivization of a self-adjoint operator of \( H_e[K] \). This is not true of \( M_e([F]) \) due to the fact that the Haar measure on \( \hat{S}_K \) – which we use to define the Hardy inner-product – is invariant with respect to addition but not multiplication. This suggests that regarding the system defined by \( H_\theta \), it may be more natural to use the Hardy space of functions holomorphic on a hyperbolized idele class group \( C_K \) (with its multiplicatively invariant Haar measure). A formal computation shows that the operators \( M_e([F]) \) are self-adjoint for the “idelic” Hardy inner-product when the Fourier coefficients of some representative \( F \) are real. We suspect that the eigenvalues of \( M_e([F]) \) are equal or related in a straightforward manner to the imaginary parts of the zeros of a meromorphic extension of a Dirichlet type series corresponding to \( [F] \).
11. APPENDIX: ERRATA TO THE PUBLISHED VERSION [4p]

In what follows, the revised version (that is, this version) is denoted [R].

1. The definition given of nonlinear number field was erroneous. On page 582, line 15 of [4p] the definition of $C^*[K]$ should read:

\[(20)\]

$$C^*[K] = C[K] - I_K$$

and not \(C[K]/I_K\). This correction must be carried out in the more general definition e.g. on page 586 line 30 of [4p] it should read $H^*_I[K] = H_*[K] - I_K$, where $I_K$ is the kernel of the trace map on its domain of definition. See §9 of [R].

2. The discussion of the Dirichlet product structure of $\text{Char}(\hat{S}_K)$ (part 1 of Theorem 5, page 581 of [4p]) was incomplete. See §7 of [R].

3. In part 2 of Theorem 5 of [4p], the additive group $(\mathcal{O}_K, +)$ was mistakenly identified with the character group $\text{Char}(\mathbb{T}_K)$ of the Minkowski torus $\mathbb{T}_K$. Rather, it is the inverse different $d^{-1}_K \supset \mathcal{O}_K$ which is identified with $\text{Char}(\mathbb{T}_K)$. The character group $\text{Char}(\mathbb{T}_K)$ is thus an $\mathcal{O}_K$-module extension of $\mathcal{O}_K$. See See §7 of [R].

4. On line 15, page 583 of [4p], it was incorrectly asserted that the Cauchy and Dirichlet products are fully defined on the Hilbert space $L^2(\hat{S}_K, \mathbb{C})$. These operations only extend partially so that for each element $f \in L^2(\hat{S}_K, \mathbb{C})$ one must specify the Cauchy and Dirichlet domains $\text{Dom}_\delta(f)$, $\text{Dom}_\delta(f)$ of elements $g$ for which $f \otimes g$ resp. $f \oplus g$ make sense. The definition of nonlinear number field, which appears in Definition 2 on page 587 of [4p] must be adjusted accordingly by replacing everywhere the phrase “double semigroup” by “partial double semigroup” to take into account this correction. See §§8,9 of [R].

5. The proof of Lemma 1 on page 589 of [4p] was incorrect. A correct proof can be found in §10 of [R] (where it is known as “Lemma 2”).

6. The discussion of the idele class group found on pages 591-592 of [4p] is only valid for $K = \mathbb{Q}$, so that the idele class group $C_K$ should be replaced by $C_\mathbb{Q}$.

7. Apart from implementing the above corrections, §6 of the revised version [R] contains an enhancement of the hyperbolization $\mathbb{H}_K$ (page 579-580 of [4p]). Furthermore, the $\theta$-holomorphic grading of functions on $\mathbb{H}_K$ (described on page 585 of [4p]) as been expanded by a complex sign set along the complex places. This is described in §8.2 of [R].

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