SUSPENSION FLOWS IN PIPELINE WITH PARTIAL PHASE SEPARATION (*)

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Abstract. The formulation of a model for the evolution of the flow of a solid-liquid mixture (coal-water) in a horizontal pipeline with partial phase separation is the aim of this work. Problems of instabilities due to complex eigenvalues, observed in previous models, seem to be completely solved in the present model, in which we give the genesis of the different terms written in the equations, coming from the natural definition of mass and momentum balance, and the consequent proof of well-posedness of the obtained PDE system with boundary-Cauchy data.

The model describes a three-layer flow. Most of the material is carried by the upper layer, while the bottom layer consists of an immobile sediment. The intermediate layer grows to a maximum thickness and has the role of regulating the mass exchange between the extreme layers.

In the last section we present some simulations for a particular choice of flow regime, and boundary-Cauchy data, that were suggested by experimental results provided by Snamprogetti (Fano, Italy).

keywords: suspensions, phase-separation, turbulent flow, hyperbolic, well-posedness

1. The physical problem

In this paper we formulate a mathematical model for the flow in a horizontal pipeline of a solid-water mixture with a progressive phase separation due to gravity.

The specific case we have in mind is the one of “dilute” coal-water suspensions and was suggested to us by Snamprogetti (Fano, Italy). The fresh mixture has a coal concentration of about 50% (in weight) and a particle size distribution centered at $0.205\text{mm}$ ($\leq 1.25\text{mm}$). The rheological properties of such a system are totally different from the ones of coal-water slurries that have been studied extensively in a number of papers ([2], [6], [7], [8], [17], [20], [23]).

Indeed slurries, which can have coal concentrations up to 70% and are prepared using smaller particles, are stable at rest thanks to the action of chemical additives and exhibit partial sedimentation under stress, mainly because of the presence of impurities that are not affected by the additive and are no longer sustained by the slurry yield stress in dynamical conditions.

The situation here is completely different, because the liquid and solid components tend to separate under gravity both at rest, and in dynamical conditions.

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The approach we are going to follow is in the typical framework of stratified flows: in any transversal cross section of the pipe we have three layers, whose motion and composition are described by averaged quantities.

The upper layer carries most of the material, while the bottom layer is immobile (a stationary deposit that should be avoided in practice). The middle layer has a density intermediate between the lighter, fast flowing, upper phase, and the sediment. We suppose that at the beginning just one phase exists, having the same properties of the material entering the pipeline. Therefore the system goes through a first stage in which there is no immobile layer, while the denser flowing layer grows up to a thickness $\Delta$, depending on the pipe discharge. Once this value of the thickness is reached, the stationary sediment appears and the intermediate layer moves upwards, keeping its thickness $\Delta$ and its concentration, and transmitting solid material from the main phase to the sediment.

The idea of the intermediate layer is inspired to a model that has successfully explained sedimentation in slurries (see [5], [16]), although as we said, the general properties are quite different in the two cases.

The scheme utilizing piecewise constant velocities over a cross section, although very convenient and frequently used in pipeline modeling, is already largely approximated, so that, taking the density constant in each layer, is not only a reasonable assumption, but it looks the only one really consistent with the general setting of the problem. In fact, the way density varies, over a cross section, can be measured by means of a Gamma-Densimeter, however the accuracy is not such that one could look for a more sophisticated model.

Previous models in the literature describing flows of mixtures (see [4], [13], [15], [18], [19], [20], [21], [22], [24]) could not be adapted to our case for different reasons, so that the model we are going to illustrate is, as far as we know, an original contribution towards the study of non-steady suspensions flows with phase separation.

2. The two-layer flow model

Let us deal first with the initial stage of the flow in which we have just two layers.

We denote by $x$ the coordinate along the flow. At any section $x = \text{const.}$, the two layers occupy two regions of areas $A_1$ (the upper layer), $A_2$ (the lower), separated by a straight line of length $S_i$.

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1 Problems of ill-posedness in [19] were pointed out (see Remark 3.1 in [13], while the stationary models present in the literature are not interesting in our case, since stationary flow is reached at such large distances from the entrance of the pipeline, that it is not observable in practice. In addition, it would correspond to the situation in which only water is transported by the main flow, which is of course of no practical use.
perimeters of the outer boundaries are $S_1$, $S_2$, respectively. In the stage we are considering the thickness of the lower layer is less than $\Delta$ (see fig. 2.1).

![Two-layer flow model, vertical section](image)

Figure 2.1. Two-layer flow model, vertical section

The flow model is based on mass and momentum conservation accounting for the exchange of liquid and solid between the two layers.

The basic quantities entering the model are, besides $A_1$, $A_2$,

- $\alpha$: solid volume fraction in layer 1;
- $\beta$: solid volume fraction in layer 2;
- $U$: velocity of layer 1;
- $V$: velocity of layer 2.

The solid volume transfer rate from layer 1 to layer 2 will be assumed to be $\alpha A_1 \psi$, where $\psi$ is a positive constant. The corresponding quantity for the liquid component can be written $(1-\alpha)A_1 \varphi$ where $\varphi$ turns out to be a function of $\alpha$, $\beta$, $\psi$, as we shall see.

2.1. Mass balance. Both the components are incompressible, so we have the following volume conservation laws:

- **LAYER 1-solid**

  \begin{equation}
  \partial_t (\alpha A_1) + \partial_z (\alpha A_1 U) = -\alpha A_1 \psi
  \end{equation}

- **LAYER 1-liquid**

  \begin{equation}
  \partial_t [(1-\alpha)A_1] + \partial_z [(1-\alpha)A_1 U] = -(1-\alpha)A_1 \varphi
  \end{equation}

- **LAYER 2-solid**

  \begin{equation}
  \partial_t (\beta A_2) + \partial_z (\beta A_2 V) = \alpha A_1 \psi
  \end{equation}
- LAYER 2-liquid

\[(2.4)\]
\[\partial_t [(1 - \beta)A_2] + \partial_x [(1 - \beta)A_2V] = (1 - \alpha)A_1\varphi\]

Clearly \(A_1\) and \(A_2\) are related by:

\[(2.5)\]
\[A_1 + A_2 = A\]

Also the two velocities \(U, V\) are not independent, since summing up equations \((2.1), (2.2), (2.3), (2.4),\)
we obtain that the discharge \(Q\) along the pipe does not depend on \(x\), as a natural consequence of the
incompressibility of the mixture:

\[(2.6)\]
\[\partial_x (A_1U + A_2V) = 0,\]

so that \(Q\) may depend on \(t\) only, thus:

\[(2.7)\]
\[V = \frac{Q(t) - A_1U}{A_2}, \text{ if } A_2 > 0\]

The above equations are not enough to describe the evolution of the bottom layer. We assume that the
solid concentration in it is constant, i.e.:

\[(2.8)\]
\[\beta = \text{const.}\]

Thus we can divide eq. \((2.3)\) by \(\beta\) and \((2.4)\) by \((1 - \beta)\) and realise that

\[(2.9)\]
\[\varphi = \frac{\alpha}{1 - \alpha} \frac{1 - \beta}{\beta} \psi\]

and that \((2.3), (2.4)\) represent the same equation, as well as \((2.3)\) is a consequence of \((2.1), (2.3)\) and
\((2.9)\)

Therefore we reformulate all the previous conservation laws using two equations only in the three
unknowns \(\alpha, A_1, U\). For instance we can replace \(A_2\) and \(V\) in \((2.3)\) using the expressions \((2.5), (2.7)\),

\[(2.10)\]
\[\partial_t A_1 + \partial_x (A_1U) = -\frac{\alpha}{\beta} A_1\psi,\]

and then we can modify \((2.1)\) as follows:

\[(2.11)\]
\[\partial_t \alpha + U\partial_x \alpha = -\alpha \left(1 - \frac{\alpha}{\beta}\right) \psi\]

We need one more equation which has to be obtained from the balance of momentum.

Before deriving the missing equation, we define the average densities \(\rho_1, \rho_2\) in the two layers:

\[(2.12)\]
\[\rho_1 = \alpha \rho_s + (1 - \alpha) \rho_w\]

\[(2.13)\]
\[\rho_2 = \beta \rho_s + (1 - \beta) \rho_w\]
\( \rho_s \) and \( \rho_w \) are the densities of the solid and of the liquid components.

While \( \rho_2 \) is constant, \( \rho_1 \) varies. From (2.1) and (2.2), we obtain the global volume balance in \( A_1 \):

\[
\frac{\partial t}{A_1} + \frac{\partial x}{A_1} (A_1 U) = -\alpha \beta A_1 \psi
\]

after using (2.9). Similarly we have

\[
\frac{\partial t}{A_2} + \frac{\partial x}{A_2} (A_2 V) = \frac{\alpha}{\beta} A_1 \psi,
\]

which is nothing but (2.3) divided by \( \beta \).

Now, multiplying (2.1) by \( \rho_s \), (2.2) by \( \rho_w \), and adding the results, we get:

\[
\frac{\partial t}{\rho_1 A_1} + \frac{\partial x}{\rho_1 A_1 U} = -A_1 \rho_2 \frac{\alpha}{\beta} \psi,
\]

and since the left hand side is

\[- \rho_1 \frac{\alpha}{\beta} \psi + A_1 \left( \frac{\partial \rho_1}{\partial t} + U \frac{\partial \rho_1}{\partial x} \right),\]

we deduce how \( \rho_1 \) varies along the flow:

\[
D_1 \rho_1 = -\frac{\alpha}{\beta} \psi (\rho_2 - \rho_1)
\]

where, here and in the following, \( D_1 \) and \( D_2 \), denote the Lagrangian derivatives along the respective flows:

\[
D_1 = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x}
\]

As we are in the case \( \alpha < \beta \) and \( \rho_s > \rho_w \), and

\[
\rho_2 - \rho_1 = (\beta - \alpha) (\rho_s - \rho_w),
\]

we find out that (from (2.17)) \( \rho_1 \) is strictly decreasing along the flow, as long as \( \alpha > 0 \).

\subsection*{2.2. Momentum balance.}

Now we write the following pair of momentum balance equations per unit length:

- **LAYER 1**

\[
D_1 (A_1 \rho_1 U) = A_1 G - \tau_1 S_1 \pm \tau_i S_i + UD_1 (\rho_1 A_1)
\]

- **LAYER 2**

\[
D_2 (A_2 \rho_2 V) = A_2 G - \tau_2 S_2 \pm \tau_i S_i - \rho_2 UD_2 A_1
\]

where \( G \) denotes the absolute value of the pressure gradient, \( \tau_1, \tau_2 \) are the stresses at the respective wall portions, \( \tau_i \) is the interfacial stress, with “+”, in (2.21) if \( U > V \). We will come back to the form of such stresses later on.
Note that since $D_j A_1 < 0$ for $j = 1, 2$ and $D_1 \rho_1 < 0$ (see (2.17)), we have a momentum loss in (2.20) and a momentum gain in (2.21).

The genesis of the loss and gain terms is the following. Consider the motion of volume element $A_1 dx$ of the upper layer. In the time element $dt$ it will transfer the mass element $|D_1(\rho_1 A_1)| dx dt$ to the other layer, with the corresponding (negative) momentum transfer $UD_1(\rho_1 A_1)$, per unit length and unit time.

If we follow the motion of an element $A_2 dx$ of the lower layer, it will acquire the volume $D_2 A_2 dx dt$ in the interval time $dt$. The corresponding mass increase is obtained multiplying by $\rho_2$ (constant) the volume accretion, and by $U$, in order to get the momentum variation. Remember that the matter coming from the upper layer, had a flow velocity $U$ and that it is slowed down to $V$ after passing the interface.

We can rewrite (2.20) and (2.21) in a more expressive way:

\begin{align}
\rho_1 A_1 D_1 U &= A_1 G - \tau_1 S_1 \mp \tau_i S_i \\
\rho_2 A_2 D_2 V &= A_2 G - \tau_2 S_2 \pm \tau_i S_i + \rho_2 (U - V) D_2 A_2
\end{align}

**Remark 2.1.** The first equation says that the "donor" system obeys Newton’s law. The second equation says that the accepting system experiences an extra inertia force due to the relative momentum of the transferred element.

Note that, due to the relative motion of the layers, the mass element $|D_1(\rho_1 A_1)| dx dt$ is distributed among different elements of the lower layer. Similarly, the mass increase $\rho_2 D_2 A_2 dx dt$ is the sum of the contribution of different elements of the upper layer. Therefore (2.22), (2.23) should not be used to get the overall momentum balance.

### 2.2.1. The stress terms

In (2.20), (2.21) we take account of the wall and interface shear stress. Generally speaking, we introduce wall-specific shear stress (see [1]):

\begin{equation}
\tau = \frac{\lambda}{2} \rho v^2
\end{equation}

where

\begin{equation}
\lambda = c \left( \frac{D_v}{v} \right)^{-n}
\end{equation}

is the friction factor (see [12], [1]), $v$ is the phase velocity, $\rho$ is the average phase density, $D$ is the equivalent hydraulic diameter, namely (11):

\footnote{Decompose the mass element $\rho_1 A_1 dx$ into $(\rho_1 + D_1 \rho_1 dt)(A_1 + D_1 A_1 dt)dx$, which is still in layer 1 after $dt$ and the complementary part $-D_1 \rho_1 dt(A_1 + D_1 A_1 dt)dx - \rho_1 D_1 A_1 dt dx$ (which is $O(dt)$). The main portion moves according to $(\rho_1 + D_1 \rho_1 dt)(A_1 + D_1 A_1 dt)D_1 U = \text{external forces} + \text{interaction with the } O(dt) \text{ portion which gives (2.22) as } dt \to 0.$}
with usual notations, and the coefficients \( c, n \) can be evaluated, in the case of Newtonian fluids, for stratified flow, depending on the flow regime (laminar, turbulent) as:

| reg. | \( c \) | \( n \) |
|------|------|------|
| Lam. | 16   | 1    |
| Turb.| 0.046| 0.2  |

In \((2.25)\), \( \nu \) is the kinematic viscosity:

\[
(2.26) \quad \nu = \frac{\eta}{\rho}
\]

In the expression of the interface stress, \( v \) has to be interpreted as the relative velocity \((U - V) = W\) of the two layers so that:

\[
(2.27) \quad \tau_i = \frac{\lambda_i}{2} \rho_i W^2
\]

Here \( \lambda_i \) and \( \rho_i \) are the friction factor and the density of the faster phase.

2.2.2. A model for the viscosity. The viscosity of the mixture is related to solid fraction. For this reason we select a particular behaviour of \( \eta(\alpha) \), considering, on one hand, the typical value of the viscosity of water (\(0.01 \text{gcm}^{-1}\text{s}^{-1}\)), corresponding to \( \alpha = 0 \), and, on the other hand, the value of the viscosity measured for a suspension with solid fraction 0.5, typically \(0.35 \text{gcm}^{-1}\text{s}^{-1}\) (Snamprogetti). Since the bottom layer with solid fraction 0.7, is at rest, we impose that viscosity increases considerably beyond the value 0.5 for \( \alpha \). These considerations suggest the following choice of \( \eta(\alpha) \):

\[
(2.28) \quad \eta(\alpha) = \eta_w \left( \frac{\eta_2}{\eta_w} \right)^{\alpha \beta^b}
\]

(see Fig. 2.2) where \( \eta_w \) is the viscosity of water and \( b \) is a parameter to be determined in such a way that \( \frac{\eta}{\eta_w} = 35 \), for \( \alpha = 0.5 \), and \( 10^3 \), for \( \alpha = \beta \) (= 0.6).

**Remark 2.2.** The selection above of \( \eta(\alpha) \) is extrapolated from just two experimental data, so that it can be used to obtain just a qualitative, although quite realistic, description of the system.
3. The Final Equations

Using the definition of $D_1$, $D_2$ in (2.18), dividing (2.22) by $A_1$ and (2.23) by $A_2$ and subtracting the second equation from the first, we obtain:

\[(3.1) \quad (A_2 \rho_1 + A_1 \rho_2) \partial_t U + (A_2 \rho_1 U + A_1 \rho_2 V) \partial_x U = \rho_2 \dot{Q} - \frac{A_2}{A_1} \tau_1 S_1 + \tau_2 S_2 \mp \tau_i S_1 \left( \frac{A_2}{A_1} + 1 \right)\]

which is the third differential equation of the model (together with (2.11), (2.10)), in the three unknowns $\alpha, A_1, U$. Equation (2.7) gives

\[V = V(\alpha, A_1, U)\]

while $\beta$ is given.

We can easily rewrite (2.11), (2.10), (3.1) in matrix form, introducing $M$, $N$, $3 \times 3$ matrices and $\Omega, F$, two column vectors:

\[(3.2) \quad \Omega = \begin{pmatrix} \alpha \\ A_1 \\ U \end{pmatrix} \quad F = \begin{pmatrix} -\alpha \left( 1 - \frac{\alpha}{\beta} \right) \psi \\ -A_1 \frac{\alpha}{\beta} \psi \\ f \end{pmatrix}\]

where

\[(3.3) \quad f = \rho_2 \dot{Q} - \frac{A_2}{A_1} \tau_1 S_1 + \tau_2 S_2 \mp \tau_i S_1 \left( \frac{A_2}{A_1} + 1 \right)\]

\[(3.4) \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_2 \rho_1 + A_1 \rho_2 \end{pmatrix} \quad N = \begin{pmatrix} U & 0 & 0 \\ 0 & U & A_1 \\ 0 & 0 & A_2 \rho_1 U + A_1 \rho_2 V \end{pmatrix}\]
so that:

\[(3.5) \quad M \partial_t \Omega + N \partial_x \Omega = F\]

**Theorem 3.1 (Well-posedness).** The problem \[(3.5)\] with boundary-Cauchy data:

\[(3.6) \quad \begin{align*}
\Omega(0, t) &= \Omega^0; \\
\Omega(x, 0) &= \Omega^0;
\end{align*}\]

is well-posed.

**Proof.** We begin by showing that \[(3.5)\] is hyperbolic (see [3], [9], [11]). \(M\) is invertible, so multiply \[(3.5)\] on both sides by \(M^{-1}\) and obtain the normal form of hyperbolic systems of PDE's

\[(3.7) \quad \partial_t \Omega + L \partial_x \Omega = H\]

where now:

\[(3.8) \quad L = M^{-1}N = \begin{pmatrix}
U & 0 & 0 \\
0 & U & A_1 \\
0 & 0 & \frac{A_2 \rho_1 U + A_1 \rho_2 V}{A_2 \rho_1 + A_1 \rho_2}
\end{pmatrix}\]

and

\[(3.9) \quad H = M^{-1}F = \begin{pmatrix}
-\alpha \left(1 - \frac{\alpha}{\beta}\right) \psi \\
-A_1 \frac{\alpha}{\beta} \psi \\
h
\end{pmatrix}\]

where

\[(3.10) \quad h = \frac{1}{A_2 \rho_1 + A_1 \rho_2} \left(\rho_2 \dot{Q} - \frac{A_2}{A_1} \tau_1 S_1 + \tau_2 S_2 + \tau_i S_1 \left(\frac{A_2}{A_1} + 1\right)\right)\]

As it can be easily seen, \(L\) has real positive eigenvalues:

\[(3.11) \quad \begin{align*}
\lambda_1, \lambda_2 &= U \\
\lambda_3 &= \frac{A_2 \rho_1 U + A_1 \rho_2 V}{A_2 \rho_1 + A_1 \rho_2}
\end{align*}\]

Moreover, as the three eigenvalues are positive and finite, the boundary-Cauchy data on the two axes \((x = 0, \ t = 0)\), are given on time-like segments (see [3], [11]), and the well-posedness of the problem is guaranted. \(\square\)

**Remark 3.1.** Ill-posedness of previous models formulated for similar non-steady, multiphase flow (see [15]) had already been pointed out in different papers (see [13], [18], [21]) where different methods to bypass ill-posedness were proposed, based on the introduction of surface tension terms. Such a procedure, which would be not applicable to our case, is not necessary in our model which is consistently formulated as a correct evolution problem in a natural way.
3.1. **Non-dimensional variables.** Before deriving the complete model with the introduction of the third layer, let us define a typical length-scale \( L_c \) (e.g. the length of the pipeline, between two pumping stations) and a typical time-scale \( t^0 \) so to use non-dimensional variables; the time-scale will be chosen to be \( \frac{L_c}{U^0} \), with \( U^0 \) the velocity of the fresh mixture at the entrance of the pipe. In particular \( L_c \) will be the average distance between two pumping stations (\( \approx 100 km \)), while \( U^0 \) will be \( 2.0 ms^{-1} \) in accord with the expected total flow rate (\( \approx 1424 m^3 h^{-1}, (0.4 m^3 s^{-1}) \)), with a section diameter of \( 0.5 m \).

\[
\tau = \frac{t}{t^0}, \quad \xi = \frac{x}{L_c}
\]

Now we can use non-dimensional unknowns:

\[
a_{1,2} = \frac{A_{1,2}}{A}, \quad v_{1,2} = \frac{(U, V)}{U^0}
\]

We use also non-dimensional densities, scaled with the density of water \( \rho_w \):

\[
\delta_{1,2} = \frac{\rho_{1,2}}{\rho_w},
\]

non-dimensional cross-sections perimeters, scaled by the perimeter of the pipeline:

\[
\sigma_{1,2,i} = \frac{S_{1,2,i}}{S},
\]

non-dimensional specific shear stresses

\[
\mu_{1,2,i} = \frac{\tau_{1,2,i}}{\lambda_{1^0} \rho_w U^0},
\]

where \( \lambda_{1^0} \) is the friction factor of the wall shear-stress for the upper layer, evaluated at the boundary.

We can rewrite the system (3.7) in a completely non dimensional form:

\[
\partial_\tau \mathbf{U} + A \partial_\xi \mathbf{U} = \mathbf{S}
\]

where:

\[
\mathbf{U} = \begin{pmatrix} \alpha \\ a_1 \\ v_1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \alpha \left(1 - \frac{\alpha}{\beta} \right) t^0 \psi \\ \frac{\alpha}{\beta} a_1 t^0 \psi \\ s \end{pmatrix}
\]

with

\[
s = \frac{1}{a_2 \delta_1 + a_1 \delta_2} \left[ \lambda_{1^0} \frac{2\pi R L_c}{A} \left( -\mu_1 \frac{a_2}{a_1} \sigma_1 + \mu_2 \sigma_2 \mp \mu_i \left( \frac{a_2}{a_1} + 1 \right) \sigma_i \right) \right]
\]

and

\[
A = \begin{pmatrix} v_1 & 0 & 0 \\
0 & v_1 & a_1 \\
0 & 0 & \frac{\delta_1 a_2 v_1 + \delta_2 a_1 v_2}{\delta_1 a_2 + \delta_2 a_1} \end{pmatrix}
\]
Remark 3.2. Here and in the following we considered constant the total volume discharge along the pipe, i.e.

(3.21) \[ Q = AU^0 \]

4. The three-layer flow model

Let us now introduce the third layer, seen, as we said in sec. [1] as a stationary deposit with constant solid volume fraction \( \gamma \), and let us write the volume conservation equations, introducing the new transfer rates \( \bar{\psi}, \bar{\varphi} \) of the solid and liquid components from layer 2 to layer 3. Then, the volume conservation equations will be:

- **LAYER 1-solid:**
  \[
  \partial_t (\alpha A_1) + \partial_x (\alpha A_1 U) = -\alpha A_1 \psi 
  \]

- **LAYER 1-liquid:**
  \[
  \partial_t [(1 - \alpha) A_1] + \partial_x [(1 - \alpha) A_1 U] = -(1 - \alpha) A_1 \varphi 
  \]

- **LAYER 2-solid**
  \[
  \partial_t (\beta A_2) + \partial_x (\beta A_2 V) = \alpha A_1 (\psi - \bar{\psi}) 
  \]

- **LAYER 2-liquid**
  \[
  \partial_t [(1 - \beta) A_2] + \partial_x [(1 - \beta) A_2 V] = (1 - \alpha) A_1 (\varphi - \bar{\varphi}) 
  \]
• LAYER 3-solid

\[ \partial_t (\gamma A_3) = \alpha A_1 \overline{\varphi} \]  

• LAYER 3-liquid

\[ \partial_t [(1 - \gamma)A_3] = (1 - \alpha)A_1 \overline{\varphi} \]

**Remark 4.1.** The reason why the quantity of matter passing from the middle layer to the stationary deposit is proportional to \( \alpha A_1 \), and not to the corresponding factor of the middle layer \( \beta A_2 \), in (4.5), is due to the fact that we suppose the thickness \( \Delta \) of the middle layer constant. From this it follows that the transfer rate across the intermediate layer is driven by the transfer of matter between the upper and the middle layer.

In this way, the mass transfer to the stationary deposit stops, as well as the one from the upper to middle layer, only when the upper layer is completely empty of solid \((\alpha = 0)\). This is the condition of complete phase separation and consequent steady flow, which is reached asymptotically by our system, although in a region of not practical interest, since it is far from the operating conditions of a plant.

Now we have

\[ A = A_1 + A_2 + A_3 \]  

and

\[ \beta, \gamma = \text{const.} \]

Summing up equations from (4.1) to (4.6), we still have the total volume conservation:

\[ \partial_x (A_1 U + A_2 V) = 0 \]

which means again:

\[ V = \frac{Q - A_1 U}{A_2} \]  

if \( A_2 > 0 \)

Dividing (4.5) by \( \gamma \) and (4.6) by \((1 - \gamma)\), after using (4.8), and equating the results, we get:

\[ \overline{\varphi} = \frac{\alpha}{1 - \alpha} \frac{1 - \gamma}{\gamma} \overline{\psi} \]

which replaces (2.9).

Dividing (4.3) by \( \beta \) and (4.4) by \((1 - \beta)\), and equating the results, we have now:

\[ \varphi = \frac{1 - \beta}{\beta} \frac{\alpha}{1 - \alpha} \psi + \frac{\alpha}{1 - \alpha} \left( \frac{1 - \gamma}{\gamma} - \frac{1 - \beta}{\beta} \right) \overline{\psi} \]
Note that, being the thickness of the middle layer constant, we can express

\[(4.13)\quad A_2 = A_2(A_3, \Delta)\]

from which it follows:

\[(4.14)\quad \frac{\partial A_2}{\partial (x,t)} = \frac{\partial A_2}{\partial A_3} \frac{\partial A_3}{\partial (x,t)}\]

**Remark 4.2** (On the thickness of the middle and bottom layer). *To simplify the equation and without losing physical meaning, we will suppose that the thickness of the sedimentation layers can be considered small enough to have (see Fig. 4.1):*

\[(4.15)\quad \sin \phi_i \simeq \phi_i - \frac{\phi_i^3}{6}, \quad i = 1, 2\]

*With this hypothesis, we will have:*

\[(4.16)\quad A_2 \simeq \frac{2}{3} R^2 (\phi_1^3 - \phi_2^3)\]

and

\[(4.17)\quad A_3 \simeq \frac{2}{3} R^2 \phi_2^3\]

*while \(\phi_1\) and \(\phi_2\) will be related by:*

\[(4.18)\quad \Delta \simeq \frac{1}{2} R (\phi_1^2 - \phi_2^2)\]

*This assumption can be made without any loss of physical meaning since we know that the presence of the stationary deposit is a damage for the correct use of the plant. Therefore, in practical applications in plant, its thickness must be kept below a small fraction of the pipe radius.*

From \[(4.7)\] we get

\[(4.19)\quad \frac{\partial A_3}{\partial (x,t)} = - \frac{\partial A_1}{\partial (x,t)} - \frac{\partial A_2}{\partial (x,t)}\]

Using \[(4.14)\] in \[(4.19)\], we have then

\[(4.20)\quad \frac{\partial A_3}{\partial (x,t)} = - \frac{1}{1 + \frac{\partial A_2}{\partial A_3}} \frac{\partial A_1}{\partial (x,t)}\]

Now substitute \[(4.20)\] in \[(4.5)\]:

\[(4.21)\quad \psi = -\frac{\gamma}{\alpha} \frac{1}{1 + \frac{\partial A_2}{\partial A_3}} \frac{\partial_1 A_1}{A_1}\]

which completes the relation between \(\phi\) and \(\psi\) in \[(4.12)\].
Remark 4.3. We will actually suppose that the transfer of matter from the middle layer to stationary deposit starts only when the thickness of the middle layer has reached the constant value $\Delta$. We will suppose in particular that:

\[
\overline{\psi} = -\frac{\gamma}{\alpha} \frac{1}{1 + \frac{\partial A_2}{A_1} \frac{\partial A_1}{A_3}} \Gamma(h)
\]

where $\Gamma(h)$ is the step function of the thickness $h$ of the middle layer:

\[
\Gamma(h) = \begin{cases} 
0 & \text{if } h \leq \Delta \\
1 & \text{if } h > \Delta 
\end{cases}
\]

Using (4.3) we can rewrite (4.3) as:

\[
\partial_t A_2 + \partial_x (A_2 V) = \frac{\alpha}{\beta} A_1 (\psi - \overline{\psi})
\]

and using (4.14) together with (4.20), (4.21) and the definition of $V$, (4.10), we get:

\[
\frac{\partial A_2}{A_3} + \frac{\gamma}{\alpha} \frac{\partial A_2}{A_3} \partial_x A_1 + \partial_x (A_1 U) = -A_1 \frac{\alpha}{\beta} \psi
\]

Now we divide (4.1) by $\alpha$ and subtract (4.24) from the resulting equation:

\[
\frac{A_1}{\alpha} \partial_t \alpha + \left( 1 - \frac{\partial A_2}{A_3} \frac{\gamma}{\beta} \frac{\partial A_2}{A_3} \partial_x A_1 + \frac{A_1 U}{\alpha} \partial_x \alpha = -A_1 \left( 1 - \frac{\alpha}{\beta} \right) \psi
\]

that we can rewrite:

\[
\partial_t \alpha - \alpha \left( 1 - \frac{\partial A_2}{A_3} \frac{\gamma}{\beta} \frac{\partial A_2}{A_3} \partial_x A_1 + U \partial_x \alpha = -\alpha \left( 1 - \frac{\alpha}{\beta} \right) \psi
\]

We will take (4.26) and (4.24) as the first two equations of the final model in the unknowns $\alpha, A_1, U$.

Remark 4.4. Equation (4.26) can be rewritten, using the usual definition of the derivatives along the flow,

\[
D_1 \alpha = -\alpha \left( 1 - \frac{\alpha}{\beta} \right) \psi + \alpha \frac{\gamma}{\beta} \frac{1}{1 + \frac{\partial A_2}{A_3}} \partial_x A_1
\]

and, since $\partial_x A_1 \leq 0$, $\alpha < \beta$ and $\beta < \gamma$, $\alpha$ is strictly decreasing along the flow.

4.1. Momentum balance. In the following we will use the same notations as in sec.2 with the changes due to the introduction of the stationary deposit, as described in Fig. 4.1.

The momentum exchange will be now between the upper and the middle layer, with exactly the same terms as before and a term of momentum loss of the middle layer, in favour of the stationary deposit.
The momentum transferred to the stationary deposit is completely absorbed by the pipe wall. With this in mind, let us write down the two equations of momentum balance for the layers in motion.

- **LAYER 1:**
  \[
  D_1(A_1 \rho_1 U) = A_1 G - \tau_1 S_1 + \tau_{12} S_{12} + UD_1(\rho_1 A_1)
  \]

- **LAYER 2:**
  \[
  D_2(A_2 \rho_2 V) = A_2 G - \tau_2 S_2 - \tau_{12} S_{12} - \tau_{23} S_{23} - \rho_2 UD_2 A_1 - \rho_2 V D_2 A_3
  \]

Using (4.20), (4.10) in (4.29), dividing (4.28) by \(A_1\) and (4.29) by \(A_2\), and subtracting the first from the second, we get the third equation of the complete model:

\[
\rho_2 V^1 + \partial A_2 \partial A_3 \partial t A_1 + (A_2 \rho_1 + A_1 \rho_2) \partial t U + \frac{\rho_2 V^2}{1 + \frac{\partial A_2}{\partial A_3}} \partial_x A_1 + (A_2 \rho_1 U + A_1 \rho_2 V) \partial_x U = 0
\]

Before going further with the calculations, let us reduce equations (4.26), (4.24), (4.30) to non-dimensional form, with the same scale factors as before:

\[
\frac{\gamma - 1}{\beta} \partial_a a_1 + (a_2 \delta_1 + a_1 \delta_2) \partial_a v_1 +
\]

\[
\frac{\gamma - 1}{\beta} \partial_a a_1 + (a_2 \delta_1 + a_1 \delta_2) \partial_a v_1 +
\]

\[
= \lambda_1^0 \frac{2L_c}{R} \left( \frac{a_2}{a_1} \mu_1 \sigma_1 + \mu_2 \sigma_2 \mp \left( \frac{a_2}{a_1} + 1 \right) \mu_{12} \sigma_{12} + \mu_{23} \sigma_{23} \right)
\]

The boundary-Cauchy data, will be a constant vector:

\[
U^0 = \begin{pmatrix} \alpha^0 \\ 1 \end{pmatrix}
\]

**Theorem 4.1** (Well-posedness). The system of PDE (4.31), (4.32), (4.33) with boundary-Cauchy data:

\[
\begin{cases}
U(0, \tau) = U^0 \\
U(\xi, 0) = U^0
\end{cases}
\]
is well-posed in the range of validity of the assumption (4.15).

**Proof.** The system of equations (4.31) (4.32) (4.33) can be reduced again to normal matrix form:

\[
\partial_t \mathbf{U} + L \partial_\xi \mathbf{U} = \mathbf{S}
\]

where, as usual:

\[
\mathbf{U} = 
\begin{pmatrix}
\alpha \\
a_1 \\
v_1
\end{pmatrix}
\quad \mathbf{S} = 
\begin{pmatrix}
-\alpha \left( 1 - \frac{\partial a_2}{\partial a_3} + 1 \frac{\alpha}{\partial a_3} + \frac{\gamma \beta}{\partial a_3} \right) t^0 \psi \\
-\frac{\partial a_2}{\partial a_3} + 1 \frac{\gamma \beta}{\partial a_3} t^0 \psi \\
-\frac{\partial a_2}{\partial a_3} + 1 \frac{\gamma \beta}{\partial a_3} t^0 \psi
\end{pmatrix}
\]

with

\[
s = \frac{1}{\partial a_2 \partial a_3} \frac{\alpha}{\partial a_3} a_1 \delta_2 v_2 t^0 \psi +
\frac{\lambda_1}{a_2 \delta_1 + a_1 \delta_2} \frac{2 L_{\xi}}{R} \left( -\frac{\alpha}{a_1} \mu_1 \sigma_1 + \frac{\alpha}{a_1} \mu_2 \sigma_2 + \left( \frac{\alpha}{a_1} + 1 \right) \mu_{12} \sigma_{12} + \mu_{23} \sigma_{23} \right)
\]

and where the matrix \(L\) is now:

\[
L = 
\begin{pmatrix}
\frac{\gamma}{\beta} - 1 + \alpha v_1 & \frac{\gamma}{\beta} - 1 + \alpha \frac{a_1}{a_3} & v_1 \\
\frac{\gamma}{\beta} a_1 \frac{a_2}{a_3} + \frac{\gamma}{\beta} & \frac{\gamma}{\beta} a_1 \frac{a_2}{a_3} + \frac{\gamma}{\beta} & \frac{\gamma}{\beta} a_1 \frac{a_2}{a_3} + \frac{\gamma}{\beta} \\
0 & 1 + \frac{\gamma}{\beta} a_1 \frac{a_2}{a_3} + \frac{\gamma}{\beta} & 1 + \frac{\gamma}{\beta} a_1 \frac{a_2}{a_3} + \frac{\gamma}{\beta} \\
0 & \frac{\gamma}{\beta} a_1 \frac{a_2}{a_3} + \frac{\gamma}{\beta} & \frac{\gamma}{\beta} a_1 \frac{a_2}{a_3} + \frac{\gamma}{\beta}
\end{pmatrix}
\]

with:

\[
k = \frac{\delta_2 v_2}{a_2 \delta_1 + a_1 \delta_2} \frac{1}{\frac{\gamma}{\beta} - 1} ,
\]

\[
\left\{ \begin{array}{c}
l_{32} = -\frac{\gamma}{\beta} - 1 + \frac{\partial a_2}{\partial a_3} + 1 \left( v_1 - v_2 \frac{\partial a_2}{\partial a_3} + \frac{\gamma}{\beta} \right) \\
l_{33} = \frac{a_2 \delta_1 v_1 + a_1 \delta_2 v_2}{a_2 \delta_1 + a_1 \delta_2} - \frac{\gamma}{\beta} - 1 \frac{\partial a_2}{\partial a_3} + 1 \end{array} \right.
\]

It is easy to see that, defined a function

\[
\epsilon = \frac{\gamma}{\beta} - 1 + \frac{\partial a_2}{\partial a_3} + \frac{\gamma}{\beta} ,
\]

\[
(4.41)
\]
the matrix $L$ can be rewritten

\[(4.42) \quad L = A + \epsilon C\]

where $A$ is the matrix of the two-layer model

\[(4.43) \quad A = \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_1 & a_1 \\ 0 & 0 & -a_1 \end{pmatrix} + \frac{\delta_1 a_2 v_1 + \delta_2 a_1 v_2}{\delta_1 a_2 + \delta_2 a_1}\]

and $C$ is

\[(4.44) \quad C = \begin{pmatrix} 0 & \frac{\alpha v_1}{a_1} & \alpha \\ 0 & -\frac{v_1}{a_1} & -a_1 \\ 0 & -k & \frac{\partial a_2}{\partial a_3} + \frac{\gamma}{\beta} \end{pmatrix}\]

so (see Remark 1.6), for small $\epsilon$, the introduction of the stationary deposit can be seen as a small perturbation of the two-layer flow model.

We want to show that the equation (4.36) is hyperbolic. The first eigenvalue of $L$ is easily seen to be $v_1$, the other two are the zeros of the polynomial:

\[(4.45) \quad p_L(\lambda) = (l_{22} - \lambda)(l_{33} - \lambda) - l_{32} l_{23}\]

To show that the eigenvalues are real, we show that:

\[(4.46) \quad \Delta_L^2 = (l_{22} + l_{33})^2 - 4(l_{22} l_{33} - l_{32} l_{23}) \geq 0\]

in the hypothesis of small stationary deposit.

This can be rewritten:

\[(4.47) \quad \Delta_L^2 = (l_{22} - l_{33})^2 + 4l_{23} l_{32} \geq 0\]

which means:

\[(4.48) \quad \Delta_L^2 = \left[ (1 - \epsilon) v_1 - \lambda_3^A + \epsilon a_1 k \right]^2 - 4\epsilon a_1 k (v_1 (1 - \epsilon) - v_2)\]

where we used:

\[(4.49) \quad \lambda_3^A = \frac{a_1 \delta_2 v_2 + a_2 \delta_1 v_1}{a_1 \delta_2 + a_2 \delta_1}\]

the third eigenvalue of the matrix $A$.

Let:

\[(4.50) \quad \omega = \frac{a_1 \delta_2}{a_1 \delta_2 + a_2 \delta_1}\]
\[(4.51) \quad \Gamma = \frac{\beta}{\gamma - \beta}\]

Now we can rewrite eq. (4.48):
\[(4.52) \quad \Delta L^2 = \omega^2 \left[ (1 - \Gamma \epsilon)^2 + 4 \frac{\Gamma}{\omega} \right] v_2^2 - 2 \omega v_1 \left[ (1 - \Gamma \epsilon) \left( \omega - \epsilon \right) + 2 \epsilon \Gamma (1 - \epsilon) \right] v_2 + v_1^2 (\omega - \epsilon)^2\]

Calling
\[(4.53) \quad \tau = \Gamma \epsilon,\]
\[(4.52)\] can be rewritten in the more compact way:
\[(4.54) \quad \Delta L^2 = v_1^2 (\omega - \epsilon)^2 \left\{ \left( \frac{\omega}{\omega - \epsilon} \right)^2 \left[ (1 - \tau)^2 + 4 \frac{\tau}{\omega} \right] u^2 - 2 \left( \frac{\omega}{\omega - \epsilon} \right) \left[ (1 - \tau) + \frac{2 \tau (1 - \epsilon)}{\omega - \epsilon} \right] u + 1 \right\}\]

where
\[(4.55) \quad u = \frac{v_2}{v_1}\]

Therefore, the zeros of \(\Delta L^2\) correspond to the zeros of a polynomial in \(u\) of the form:
\[\left( \frac{\omega}{\omega - \epsilon} \right)^2 a u^2 - 2 \left( \frac{\omega}{\omega - \epsilon} \right) b u + 1\]

that has solutions for
\[u_{1,2} = \left( \frac{\omega - \epsilon}{\omega} \right) \frac{b \pm \sqrt{b^2 - a}}{a}\]

After some algebra we will see that these solutions are in fact out of the region in which \(v_2\) varies.

Neglecting the \(\epsilon^3\) terms (see Rem. 4.6), we find that:
\[(4.56) \quad b^2 - a \simeq 4 \frac{\tau^2}{\omega^2} \left[ (1 - \omega) \left( \frac{1}{\Gamma} + 1 \right) \right]\]

So, neglecting the \(\epsilon^2\) terms in \(b\) and \(a\), we end up with:
\[(4.57) \quad u_{1,2} \simeq \left( \frac{\omega - \epsilon}{\omega} \right) \frac{1 - \tau + \frac{2 \tau}{\omega} \pm \frac{2 \tau}{\omega} \sqrt{\left( 1 - \omega \right) \left( \frac{1}{\Gamma} + 1 \right)}}{1 + 2 \tau \left( \frac{2}{\omega} - 1 \right)}\]

where \(\epsilon, \tau << 1\) (see Rem. 4.6).

From the definition of \(\omega\), we see that:
\[(4.58) \quad 1 - \omega = \frac{a_2 \delta_1}{a_2 \delta_1 + a_1 \delta_2}\]

which is of order \(\varphi_1^3 - \varphi_2^3\) (see eq. 4.10), so that in the expression (4.57), the difference between the two zeros \(u_1\) and \(u_2\) is small in comparison with \(u_1, u_2\), which are both close to
\[(4.59) \quad \tau \simeq 1 - \tau \left( 1 + 2 \frac{a_2 \delta_1}{a_1 \delta_2} \right),\]

the difference being of the order of \(\varphi_2 \varphi_1^{2/3}\).
From the last we can actually appreciate that \( \pi \) has to be close to 1, to make \( \Delta_2^2 \) negative, conditions that will never be reached in the hypothesis (4.15) (see Remark 4.5).

**Remark 4.5 (On the validity of Theorem 4.1).** It is clear that the interval in which \( \Delta_2^2(u) \) is negative, is very close to \( u = 1 \), conditions in which we are not interested (see Rem. 4.7), corresponding to large phase separation (see Rem. 4.6), negligible in practical applications.

From asymptotic study (see App. A) we know that \( v_2 \), after a jump-like behaviour for \( \xi << 1 \), starts increasing, pushed by the faster, less dense, upper phase, but remaining considerably smaller than \( v_1 \), also increasing.

**Remark 4.6 (On \( \epsilon \)).** It is easy to see that, for \( a_3 << 1 \),

\[
\epsilon \simeq \frac{\varphi_2}{\varphi_2 + \Gamma \left( \frac{\varphi_2^2 + 2\frac{\Delta R}{R}}{2} \right)^{1/2}}
\]

with \( \varphi_2 \), the angle subtended by \( a_3 \) (see Fig. 4.1).

Therefore \( \epsilon << 1 \) if

\[
\frac{\varphi_2}{\Gamma \left( \frac{2\frac{\Delta R}{R}}{2} \right)^{1/2}} << 1,
\]

which defines what we mean by a thin stationary deposit.

**Remark 4.7.** The case in which \( a_3 \) is not small, like, e.g., in the case of large phase separation, should be treated differently and will not be considered in this model. For example in problems of restart, the pressure gradient, that we considered the same for the two layers in motion, is different at different levels, during a transient phase.

5. **On the boundary-Cauchy data**

In sections 3 and 4.1 we introduced the boundary-Cauchy data. In particular, in theorems 3.1 and 4.1 we imposed constant data on both axes (\( \xi = 0, \tau = 0 \)).

It has to be noted that this choice was not the only possible. We chose that at the entrance the condition of the mixture is, at every time \( t \), the same as it was in the whole pipe at time \( t = 0 \).

Note that the regularity of the boundary-Cauchy data influences the regularity of solution (see [11], Theorem 2.1, pag 71). In our case, being the functions \( L \) and \( S \) in (4.36) regular and choosing constant the boundary-Cauchy data, the solution will be regular too.
Our choice will be therefore:

\( U^0 = \begin{pmatrix} \alpha^0 \\ 1 \\ 1 \end{pmatrix} \) 

This corresponds to the choice we made of having at time \( t = 0 \) and at any \( t \), at the entering cross section, the pipe full of matter with solid fraction \( \alpha^0 \) and velocity \( U^0 \), namely:

\( \alpha^0 = 0.5 \quad A_1^0 = A \quad U^0 = 2 \text{m/s} \)

5.1. On \( V^0 \). Some remarks have to be made for the value of the velocity of the middle layer at the entrance.

From eq. (2.7) we see that in fact \( V \) is defined only for \( A_2 \neq 0 \). However this is not true in our conditions, for \( x = 0 \). We can calculate the value \( V^0 \) as:

\( V^0 = \lim_{x \to 0} \frac{Q(t) - A_1 U}{A_2} \)

To do this we have to take into account the different terms in the source function in eq. (2.22) and (2.23), evaluated in the vicinity of \( x = 0 \). All the terms of the right-hand side vanish, linearly with \( A_2 \) or with \( S_i \), as \( A_2 \to 0 \), apart from the wall shear-stress term \( \tau_2 S_2 \) vanishing only in turbulent regime. In this case, \( V^0 \) turns out to be equal to \( U^0 \), so the second layer appears with the same velocity as the fresh mixture.

On the other hand, choosing the laminar regime, we get \( V^0 = 0 \), which is however consistent with the model only if \( \psi \) is taken no longer constant, but a function of \( x \), vanishing at \( x = 0 \).

We put in the Appendix the study of the asymptotics near the origin, in which we distinguish the different cases of flow regime.

In the numerical simulations we chose the turbulent-turbulent regime because it fits more naturally the model with \( \psi \) constant.

A possible generalization could be the introduction of a transition from one regime to another in the first segment of the pipe.

6. The numerical results

To evaluate the solution of eq. (4.36) we elaborated a numerical code, based on an explicit approximation method. Eq. (4.36) is discretised as

\( u[i + 1][x_j] - u[i][x_j] = \Delta t \left\{ L[i][x_j] \frac{(u[i][x_j] - u[i][x_{j-1}])}{\Delta x} \right\} + H[i][x_j] \)

where we indicate with \( u[i][x_j] \) the solution of (4.36) evaluated at \( i \)th time step, in the \( j \)th cell of the spatial domain.
This finite differences method of approximation explicit in time (Euler explicit) with a judicious choice of the dimension of the cells of the space-time grid, gives good confidence of stability and quite acceptable approximation, and allows the simulations to be run on a normal PC. In the numerical code we inserted a control for the dimension of the space-time grid, in order to satisfy the Courant-Friedrichs-Lewy condition (see [14]) during the whole run.

The solution of equation (4.36) can be plotted at different time steps as a function of the normalized distance from the entrance $\xi (= x/L_c)$. We plot, in the graphs below different quantities: $\alpha$, $v_1$ and $1 - a_1$, solutions of (4.36), as well as some important ones, related to the solution vector ($U$), namely $v_2$, $a_3$.

In Fig. 6.1 we show the result of a simulation of the evolution of the system in 100000 sec., as we said, in turbulent-turbulent regime. We see the formation of the stationary deposit after the achievement of the maximum thickness of the middle layer. Here we chose $U^0 = 0.5m/s$ and $L_c = 100km$, with a maximum thickness $\Delta = 5cm$.

3The condition states that, given the eigenvalues of the characteristic matrix, $\lambda_p$, and being $h$ (along $x$) and $k$ (along $t$) the dimensions of the cells in the space-time grid, necessary condition for the convergence of an approximate method for PDE systems, is that:

$$\left| \frac{\lambda_p k}{h} \right| \leq 1$$

In the case of the complete system (see sec. 4), we calculate numerically the eigenvalues of the characteristic matrix and we reject the time steps at which the CFL condition is not satisfied. For a detailed discussion on the convergence of this approximation method, see [10].
In similar conditions, but with a maximum thickness $\Delta = 8\,\text{cm}$, we do not see any stationary deposit (see Fig. 6.2).

![Figure 6.2. Numerical simulation of the three layers flow model for half a characteristic time ($t^0 = 200000\,\text{sec.}$), $\Delta = 8\,\text{cm}$](image)

With the initial velocity $U^0 = 2\,\text{m/s}$ and $\Delta = 8\,\text{cm}$, again, no stationary deposit is present on a length of $100\,\text{km}$ (see Fig. 6.3), while the stationary deposit is visible in similar conditions, but with $\Delta = 5\,\text{cm}$ (See Fig. 6.4). From the above simulations we see the dependence of the deposit upon the thickness, $\Delta$, of the middle layer and upon the input velocity $U^0$. The stationary deposit is, as we expect, thinner in the case of a larger input velocity and disappears completely in figures 6.2 and 6.3 for large value of $\Delta$.

Note the jump-like behaviour of $v_2$ near the origin which simulates the real behaviour $v_2 \sim x^{0.2}$ (see App. A).

We have also the convergence, observed for bigger values of the length-scale (out of the physical length of $100\,\text{km}$) to an asymptotic solution of steady flow, corresponding to the condition of complete phase separation. Our model converges then to the steady models found in the literature, in [1], [20], [22].

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Figure 6.3. Numerical simulation of the three layers flow model for two characteristic times ($t^0 = 50000sec.$), $\Delta = 8cm$

Figure 6.4. Numerical simulation of the three layers flow model for two characteristic times ($t^0 = 50000sec.$), $\Delta = 5cm$

from Università di Milano, for his constant assistance in the development of the numerical simulations and Prof. A.Fasano from Università di Firenze for his advise.
Appendix A. Asymptotic behaviour

We try here to evaluate the behaviour of the solutions of the system of PDE’s we obtained, in the vicinity of the origin.

Here we neglect the dependence of the solution upon \( t \), situation that, as we observed in the development of the numerical simulations, can be actually reached in a time of the same order of the characteristic time \( t^0 \).

Let us suppose then that, for small \( x \), the solution vector \( \Omega \) has the expansion:

\[
\begin{align*}
A_1 &= A - k x^m - k_0 x \\
A_2 &= k x^m + k_0 x \\
U &= U^0 + h_1 x^{m_1} + h_1^0 x \\
V &= V^0 + h_2 x^{m_2} + h_2^0 x
\end{align*}
\]

with \( m, m_1, m_2 < 1 \).

From eq. (2.10), we get

\[
\partial_x (A_1 U) \simeq -\frac{\alpha^0}{\beta} A \psi
\]

From the conservation of total volume:

\[
\partial_x (A_1 U) = -\partial_x (A_2 V)
\]

Using (A.1) in (A.2), equating the terms of the same power, we get:

\[
\begin{cases}
Ah_1^0 - k_0 U^0 = -A \frac{\alpha^0}{\beta} \psi \\
h_1 = k = 0
\end{cases}
\]

For \( m + m_1 > 1 \), while the case \( m + m_1 < 1 \) is not possible.

If, in particular, \( m + m_1 = 1 \), these conditions change a little in the case \( m = m_1 \), while nothing changes in the other cases:

\[
\begin{cases}
-k h_1 + Ah_1^0 - k_0 U^0 = -A \frac{\alpha^0}{\beta} \psi \\
Ah_1 - U^0 k = 0
\end{cases}
\]

Using now (A.3) in the same way:

\[
\begin{cases}
0 = k_0 V^0 \\
kV^0 = 0
\end{cases}
\]

(A.7)

\[
\begin{cases}
k_0 V^0 = A \frac{\alpha^0}{\beta} \psi \\
kV^0 = 0
\end{cases}
\]

\[
\begin{cases}
k_0 V^0 = A \frac{\alpha^0}{\beta} \psi \\
kV^0 = 0
\end{cases}
\]

(A.8)
Note that, in the case $m + m_2 \neq 1$, it cannot be $V^0 = 0$, while, with $m + m_2 = 1$ both, $V^0 = 0$ and $V^0 \neq 0$, are acceptable.

We can do the same with the two equations of momentum balance (2.22) (2.23); from the first we get:

(A.8) \[ \rho_1^0 AU^0 (m_1 h_1 x^{m_1 - 1} + h_1^0) \simeq AG - S \tau_1^0 - \tau_i S_i \]

Then it has to be:

(A.9) \[ h_1 = 0 \]

from which it follows (see (A.4) and (A.6))

(A.10) \[ k = 0 \]

in all the cases.

Moreover, as $\tau_i S_i$ is small near the origin,

(A.11) \[ h_1^0 = \frac{AG - S \tau_1^0}{\rho_1^0 AU^0} \]

Then the behaviour of $U$ around the origin depends on the pressure gradient.

From (2.23) we get:

\[
\rho_2 k_0 x (V^0 + h_2^0 x) (h_2^0 + m_2 h_2 x^{m_2 - 1}) \simeq k_0 x G - \tau_2 S_2 + \tau_i S_i + \rho_2 (U^0 - V^0 - h_2 x^{m_2} + (h_1^0 - h_2^0) x) (V^0 + h_2 x^{m_2} + h_2^0 x) k_0
\]

(A.12)

All the terms vanish in $x = 0$, apart from

\[ \rho_2 (U^0 - V^0) V^0 k_0 \]

then, it has to be either

\[ V^0 = 0 \]

or

\[ V^0 = U^0 \]

In the case $V^0 = U^0$, equating the linear terms we get also:

(A.13) \[ h_2^0 = \frac{G}{2 \rho_2 U^0} + \frac{h_1^0}{2} \]

and, as $k$ is zero, from (A.4) and (A.6) we have

(A.14) \[ k_0 = \frac{A}{U^0} \left( h_1^0 + \frac{a_0}{\beta} \psi \right) \]

Now let us call $\varphi$ the angle subtended by $A_2$, (see fig. 2.1).
It can be seen that, for small $\varphi$,

(A.15) \[ \tau_2 S_2 \simeq \omega_2 V^{2-n} \varphi^{1-2n} \]

where $\omega_2$ is a positive constant, and $n$ is the power appearing in the friction factor, (see sec. 2.2.1), depending on the flow regime.

Note that, to prevent this term from becoming singular in the origin, we need $V \to 0$ as $x \to 0$, if $n > \frac{1}{2}$.

We can rewrite (A.16)

(A.16) \[ \tau_2 S_2 \simeq \omega_2' V^{2-n} x^{1-2n} \]

where

\( \omega_2' = \left( \frac{3 k_0}{4 R^2} \right)^\frac{1}{3} \omega_2 \)

In the case $n \leq \frac{1}{2}$, it has to be then,

\[ m_2 = \frac{1 - 2n}{3} \]

from which it follows that:

(A.17) \[ h_2 = -\frac{\omega_2' U_0^{1-n}}{\rho k_0 (m_2 + 1)} \]

So we have $A_{1,2}$ and $U$, linear in $x$, $V \simeq U^0 + h_0^0 x + h_2 x^{1-2n}$, with $h_2$ negative.

The case $V^0 = 0$ is actually not consistent with the assumption $\psi = \text{const}$.

Let us see how we have to modify it in order to make this behaviour (suggested by the choice of laminar regime, $n = 1$) acceptable.

The dominant terms are,

- on the left-hand side:
  \[ \rho k_0 h_2^0 x^{m_2} \]

- on the right-hand side:
  \[ \tau_2 S_2 \simeq \omega_i' h_2 x^{m_2 - \frac{1}{2}} \]
  \[ \tau_i S_1 \simeq \omega_i U_0^0 x^{1+\frac{4}{3}} \]
  \[ \rho_2 U_0^0 k_0 h_2 x^{m_2} \]
  \[ \rho_2 k_0 h_2 x^{2m_2} \]

Then, if $h_2 \neq 0$, the $\tau_2 S_2$ term is balanced only if $m_2 = \frac{2}{3}$. This requires also:

(A.18) \[ h_2 = \frac{\omega_i}{\omega_i' U_0^0} \]

(A.19) \[ h_0^0 = U^0 \]
We said also:

\[ \partial_x (A_2 V) \simeq A_0^\alpha \beta \psi \]

which means

\[ k_0 h_2 (m_2 + 1) x^{m_2} + 2 k_0 h_2^0 x \simeq A_0^\alpha \beta \psi \]  

(A.20)

The matching above requires a different choice for \( \psi \). For example we could say that,

\[ \psi(x) = \begin{cases} 
\psi^0 \frac{x}{x_0} & x < x_0 \\
\psi^0 & x \geq x_0 
\end{cases} \]

(A.21)

In this way it has to be \( h_2 = 0 \) and

\[ h_2^0 = \frac{A \psi^0 \alpha^0}{2 k_0 x_0 \beta} \]

(A.22)

and also \( V \) is linear around the origin; otherwise, if \( h_2 \neq 0 \), we must have:

\[ \psi \simeq \psi^0 x^{m_2} \]

(A.23)

and:

\[ h_2 = \frac{A \alpha^0 \psi^0}{k_0 (m_2 + 1)} \]

(A.24)

which is consistent with (A.18), only if we take:

\[ \psi^0 = \frac{\omega_i}{\omega_2' U_0'^2} \frac{k_0 (m_2 + 1)}{A_0^\alpha \beta} \]

(A.25)

Now in (A.4) we would have

\[ k_0 = \frac{A h_1^0}{U_0} \]

(A.26)

and \( \alpha, A_1, U \), would stay linear.
References

[1] Neima Brauner and David Moalem Marom. Two-phase liquid-liquid stratified flow. *PCH Physical Chemical Hydrodynamics*, pages 487–506, 1989.

[2] E. Comparini and E. De Angelis. Flow of Bingham fluid in a concentric cylinder viscometer. *Adv. in Math. Sc. and Appl.* 6, pages 97–116, 1996.

[3] R. Courant and D. Hilbert. *Methods of Mathematical Physics*. Pure and Appl. Math. Wiley Interscience, 1962.

[4] P. Doron and D. Barnea. Multiphase-flow. *Multiphase Flow*, 22, pages 274–282, 1996.

[5] A. Fasano. A mathematical model for the dynamics of sediments in a pipeline. In H. Neunzert eds. *Wiley & Teubner*, editor, *Progress in Industrial Mathematics, ECMI 94*, 1994, pp. 241-249.

[6] A. Fasano. Pipelining of liquid fuel with peculiar rheological properties. In J. Bergh, editor, *2nd Alan Taylor Lecture Progr. in Ind. Mathematics ECMI 98*, 1998.

[7] A. Fasano, C. Manni, and M. Primicerio. Modelling the dynamics of fluidizing agents in coal slurries. In Science Press Shuitie Xiao, Xiang-Cheng Hu eds, editor, *Proc. Int Sym. "Nonlinear Problems in engineering and Science"*, 1991. pp. 64-71.

[8] A. Fasano and M. Primicerio. Modelling the rheology of coal-water slurries. In H. Wacker, editor, *Proc. 4th ECMI Conference*, pages 269–274, 1990.

[9] P. Garabedian. *Partial Differential Equations*. New York: Wiley, 1964.

[10] E. Isaacson and H. B. Keller. *Analysis of Numerical Methods*. John Wiley and Sons Inc., New York, 1966.

[11] Alan Jeffrey. *Quasilinear Hyperbolic systems and waves*, volume 5 of *Lecture notes in mathematics*. Pitman Publishing, 1976.

[12] L. Landau and M Lifshits. *Fluid Mechanics*. Course of theoretical physics. Pergamon Press, 1959.

[13] S. J. Lee, K. S. Chang, and K. Kim. Pressure wave speeds from characteristics of two fluids, two-phase hyperbolic equation system. *Int. Journal of Multiphase Flow*, 24:855–866, 1998.

[14] Randall J. LeVeque. *Numerical Methods for Conservation Laws*. Lectures in Mathematics, ETH Zürich. Birkhäuser-Berlin, 1992.

[15] H. W. Liepmann and A. Roshko. *Elements of Gas-dynamics*. John Wiley and Sons Inc., New York, 1957.

[16] A. Mancini. Evolution of sedimentation profiles in the transport of c. w. s. through a pipeline. In Teubner M. Brons et al. eds., editor, *Progress in Industrial Math. (ECMI 96)*, pages 442–449, 1997.

[17] M. Primicerio. Dynamics of slurries. In H. Neunzert, editor, *Proc. of the 2nd European Symposium on Mathematics in Industry, Oberwolbach*, pages 121–134, 1987.

[18] John D. Ramshaw and John A. Trapp. Characteristics, stability, and short-wavelength phenomena in two-phase flow equation systems. *Nucl. Sc. and Eng.*, 66, pages 93–102, 1978.

[19] A/S Scandpower. Olga user course. user’s manual, 1992.

[20] C. A. Shook. Slurry transport course. internal report-Snamprogetti, 1998.

[21] C. A. Shook and A. S. Bartosik. Particle wall-stresses in vertical slurry flow. *Powder Technology*, 81, pages 117–124, 1994.

[22] C. A. Shook and C. A. Roco. *Slurry flow:Principles and Practice*. Butterworth/Heinemann, 1991.

[23] Snamprogetti. Persol-calcolo di carico e velocità critiche di sospensioni solido/liquido. user’s manual, 1997.

[24] H. Bruce Stewart. Stability of two-phase flow calculation using two-fluid models. *J. of Comp. Phys. 33*, pages 259–270, 1979.