THE ACTION OF A MIRABOLIC SUBGROUP ON A SYMMETRIC VARIETY

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Abstract. Let $F$ be a local field of characteristic zero. Let $E$ be a quadratic field extension of $F$. We show that any $P \cap \text{GL}_n(E)$-invariant linear functional on a $\text{GL}_n(E)$-distinguished irreducible smooth admissible representation of $\text{GL}_n(F)$ is also $\text{GL}_n(E)$-invariant, where $P$ is the standard mirabolic subgroup of $\text{GL}_n(F)$.

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1. Introduction

Let $F$ be a local field of characteristic zero. Let $E = F[\delta]$ be a quadratic field extension of $F$ with $\delta^2 \in F^\times \setminus (F^\times)^2$. Let $\text{Mat}_{n,n}(F)$ (resp. $\text{Mat}_{n,n}(E)$) denote the set of all $n \times n$ matrices over $F$ (resp. $E$). Let $\text{GL}_n(F)$ act on $\text{Mat}_{n,n}(F)$ by inner conjugation. Let $P_F$ be the mirabolic subgroup of $\text{GL}_n(F)$ consisting of matrices with last row vector $(0, \cdots, 0, 1)$. Bernstein [Ber84] proved that any $P_F$-invariant distribution on $\text{Mat}_{n,n}(F)$ must be $\text{GL}_n(F)$-invariant when $F$ is non-archimedean. Baruch [Bar03] proved that any $P_F$-invariant eigendistribution (with respect to the center of the the universal enveloping algebra of $\text{gl}_n(F)$) is $\text{GL}_n(F)$-invariant when $F$ is archimedean, which has been proved completely in [AG09b, SZ12]. It is expected that there is a more general phenomenon related to the mirabolic subgroup $P_F$. Let $H_{p,n-p} = \text{GL}_p(F) \times \text{GL}_{n-p}(F)$. Gurevich [Gur17] investigated the role of the mirabolic subgroup $P_F$ of $\text{GL}_n(F)$ on the symmetric variety $\text{GL}_n(F)/\text{H}_{p,n-p}$ when $F$ is non-archimedean. Then Gurevich proved that any $H_{1,n-1} \cap P_F$-invariant linear functional on an $H_{1,n-1}$-distinguished irreducible smooth representation of $\text{GL}_n(F)$ is also $H_{1,n-1}$-invariant (see [Gur17 Theorem 1.1]). It is expected that it holds for all $H_{p,n-p}$. The case when $n-p = p+1$ has been verified in [Lu20] if $F$ is non-archimedean (see [Lu20 Theorem 6.3]). Let $P_E$ denote the mirabolic subgroup of $\text{GL}_n(E)$. Then $P_E \cap \text{GL}_n(F) = P_F$. Offen and Kemarsky proved that any $P_E \cap \text{GL}_n(F)$-invariant linear functional on a $\text{GL}_n(F)$-distinguished irreducible smooth representation of $\text{GL}_n(E)$ is also $\text{GL}_n(F)$-invariant. (See [OH11 Theorem 3.1] for the $p$-adic case and [Kem15 Theorem 1.1] for the archimedean case.) This paper studies the role of the mirabolic subgroup $P$ of $\text{GL}_{2n}(F)$ on the symmetric variety $\text{GL}_{2n}(F)/\text{GL}_n(E)$.

There is a natural group embedding $\text{GL}_{2n}(F) \hookrightarrow \text{GL}_{2n}(E)$ such that each element in the image of $\text{GL}_{2n}(F)$ is of the form

$$
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}
$$

where $A, B \in \text{Mat}_{n,n}(E)$ and the Galois action on $\text{Mat}_{n,n}(E)$ is given by $A \mapsto \bar{A}$.

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Let \( \theta \) be an involution of \( \text{GL}_{2n}(F) \) given by

\[
\theta : g \mapsto \left( \begin{array}{cc} \delta & -\delta \\ -\delta & \delta \end{array} \right) g \left( \begin{array}{cc} \delta & -\delta \\ -\delta & \delta \end{array} \right)^{-1}
\]

for \( g \in \text{GL}_{2n}(F) \). Then the fixed points of \( \theta \) in \( \text{GL}_{2n}(F) \) coincide with \( \text{GL}_n(E) \). Denote by \( \mathfrak{gl}_{2n}(F) \) the Lie algebra of \( \text{GL}_{2n}(F) \). Then any \( g \) in \( \mathfrak{gl}_{2n}(F) \) is of the form

\[
\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}
\]

where \( a, b \in \text{Mat}_{n,n}(E) \). Let \( \mathfrak{p} \) denote the Lie algebra of \( P \) which is given by

\[
\left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a = (a_{i,j}), b = (b_{i,j}) \text{ for } i, j \in \{1, 2, \ldots, n\}, a_{n,j} = \bar{b}_{n,j} \text{ for } j \in \{1, 2, \ldots, n\} \right\}.
\]

Then \( \mathfrak{p} \cap \mathfrak{gl}_n(E) = \mathfrak{p}_E \), where \( \mathfrak{p}_E \) is the Lie algebra of \( P_E \).

The main result in this paper is the following:

**Theorem 1.1.** Any \( P \cap \text{GL}_n(E) \)-invariant linear functional on a \( \text{GL}_n(E) \)-distinguished irreducible smooth admissible representation \( \pi \) of \( \text{GL}_{2n}(F) \) is also \( \text{GL}_n(E) \)-invariant.

The geometry of closed \( \text{GL}_n(E) \)-orbits on the symmetric space \( \text{GL}_{2n}(F)/\text{GL}_n(E) \) is well known due to Guo [Guo97] and Carmeli [Car15]. Then we will use the Harish-Chandra descent techniques developed in [AG09a] to show the following identity of distributions

\[
\mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E))^{\text{GL}_n(E) \cap P} = \mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E))^{\text{GL}_n(E)}
\]

(see §4.1). Together with the injective map

\[
A_\pi : \pi^* \otimes (\pi^V)^* \rightarrow \mathcal{D}(\text{GL}_{2n}(F))
\]

(defined in Lemma 4.3, 4.4) will lead to a proof of Theorem 1.1. In fact, we will prove that any element in \( \mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E))^{\text{GL}_n(E) \cap P} \) is invariant under transposition. Here we identify the symmetric variety \( \text{GL}_{2n}(F)/\text{GL}_n(E) \) with the space of matrices

\[
X_n = \{ g \in \text{GL}_{2n}(F) : g\theta(g) = 1 \}
\]

and the transpose acts on \( X_n \). Thus the transpose acts on \( \mathcal{D}(X_n) = \mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E)) \) as well.

**Theorem 1.2.** One has \( \mathcal{D}(X_n)^{\text{GL}_n(E) \cap P} = \mathcal{D}(X_n)^{\text{GL}_n(E)} \).

The key idea in the proof of Theorem 1.2 is to reduce a question on the distribution spaces of \( X_n = \text{GL}_{2n}(F)/\text{GL}_n(E) \) to that of distributions on its tangent space.

We may identify the linear version of \( X_n = \text{GL}_{2n}(F)/\text{GL}_n(E) \) with the space of matrices

\[
\begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix}
\]

for \( x \in \text{Mat}_{n,n}(E) \) (see [Guo97]), denoted by \( L_n \). Let \( \mathcal{C}(L_n) \) denote the tempered generalized functions on \( L_n \). Let \( \text{GL}_n(E) \) act on \( L_n \) by the twisted conjugation, i.e.,

\[
\begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix} \cdot \begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix} = \begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix} \begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix}^{-1} = \begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix}
\]

for \( A \in \text{GL}_n(E) \) and let \( \mathfrak{gl}_n(E) \) act on \( L_n \) by its differential. More precisely,

\[
\begin{pmatrix} a & x \\ \bar{x} & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix} = \begin{pmatrix} A & x \\ \bar{x} & \bar{A} \end{pmatrix} \begin{pmatrix} a & x \\ \bar{x} & \bar{a} \end{pmatrix}
\]

for \( a, x \in \text{Mat}_{n,n}(E) \). Then there is an analogue for the Lie algebra version.

**Theorem 1.3.** One has

\[
\mathcal{C}(L_n)^{\text{GL}_n(E) \cap P} = \mathcal{C}(L_n)^{\text{GL}_n(E)}.
\]
Remark 1.4. Here we study the tempered generalized functions space $\mathcal{C}(L_n)$ instead of the generalized functions or distributions $\mathcal{D}(L_n)$ on $L_n$ because we will use the Fourier transform on $\mathcal{C}(L_n)$. Moreover, there is not so much difference between $\mathcal{C}(L_n)$ and $\mathcal{D}(L_n)$ due to [AG09a Theorem 4.0.2].

There is a brief introduction to the proof of Theorem 1.3. We will use the result of Aizenbud-Gourevitch (see Theorem 2.2) to reduce the problem on the tempered generalized functions supported on the nilpotent cone. If $F$ is non-archimedean, then we will pick up a $\mathfrak{sl}_2(F)$-triple $\{h, e, f\}$ (see §3.1) and use Chen-Sun’s method [CS20] to study some special nilpotent orbits $O \ni \mathfrak{e}$. If $F = \mathbb{R}$, then we will use the machine of $D$-modules to show the vanishing theorem. Note that $GL_{n-1}(E)$ is a proper subgroup of $P_E = GL_n(E) \cap P$. It turns out that each $GL_{n-1}(E)$-invariant tempered generalized function on $L_n$ supported on $O$ is invariant under transposition (see Theorem 3.1), which implies Theorem 1.3.

The paper is organized as follows. In §2, we introduce some notation from the algebraic geometry. Then we will use Chen-Sun’s method (resp. the machine of $D$-modules) to prove Theorem 1.3 when $F$ is non-archimedean (resp. $F = \mathbb{R}$) in §3. In §4.1, we will give a proof to Theorem 1.2. The proof of Theorem 1.1 will be given in §4.2.

2. Preliminaries and notation

Let $X$ be an $\ell$-space (i.e. locally compact totally disconnected topological spaces) if $F$ is non-archimedean or a Nash manifold (see [AG09a]) if $F = \mathbb{R}$. Let $\mathcal{C}(X)$ denote the tempered generalized functions on $X$. Let a reductive group $G(F)$ act on an affine variety $X$. Let $x \in X$ such that its orbit $G(F)x$ is closed in $X$. We denote the normal bundle by $N^X_{G(F)x,x}$. Let

$$G_x := \{g \in G(F) | gx = x\}$$

be the stalarizer subgroup of $x$.

Theorem 2.1. [AG09a Theorem 3.1.1] Let $G(F)$ act on a smooth affine variety $X$. Let $\chi$ be a character of $G(F)$. Suppose that for any closed orbit $Gx$ in $X$, we have

$$\mathcal{C}(N^X_{G(F)x,x})^{G_x,\chi} = 0.$$

Then

$$\mathcal{C}(X)^{G(F),\chi} = 0.$$

If $X$ is a finite dimensional representation of $G(F)$, then we denote the nilpotent cone in $X$ by

$$\Gamma(X) := \{x \in X | G(F)x \ni 0\}.$$

Let $Q_G(X) := X / X^G$ and $R_G(X) := Q(X) \setminus \Gamma(X)$.

Theorem 2.2. Let $X$ be a finite dimensional representation of a reductive group $G(F)$. Let $K \subset G(F)$ be an open subgroup and let $\chi$ be a character of $K$. Suppose that for any closed orbit $G(F)x$ such that

$$\mathcal{C}(R_G(N^X_{G(F)x,x}))^{K_x,\chi} = 0$$

we have

$$\mathcal{C}(Q_G(N^X_{G(F)x,x}))^{K_x,\chi} = 0.$$

Then $\mathcal{C}(X)^{K_x,\chi} = 0$.

Proof. See [AG09a Corollary 3.2.2].

2.1. $D$-modules and singular support. In this subsection, assume that $F = \mathbb{R}$. Let $X$ be a Nash manifold. Denote by $S(X)$ the space of Schwartz functions on $X$. Denote by $\mathcal{C}(X)$ the linear dual space to $S(X)$, i.e. the tempered generalized functions on $X$. All the materials in this subsection come from [AG09a, AG09b, Aiz13].
2.1. Coisotropic variety. Let $M$ be a smooth algebraic variety and $\omega$ be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it $M$-coisotropic if $T_z Z \supseteq (T_z Z)^\perp$ for a generic smooth point $z \in Z$, where $(T_z Z)^\perp$ denotes the orthogonal complement to $T_z Z$ in $T_z M$ with respect to $\omega$. Note that every non-empty $M$-coisotropic variety is of dimension at least $\frac{1}{2}\dim M$. For a smooth algebraic variety $X$, we always consider the standard symplectic form on the cotangent bundle $T^* X$. Also, we denote by $p_X : T^* X \to X$ the standard projection.

**Lemma 2.3.** Let $X$ be a smooth algebraic variety. Let a group $G$ act on $X$ which induces an action on $T^* X$. Let $S \subset T^* X$ be a $G$-invariant subvariety. Then the maximal $T^* X$-coisotropic subvariety of $S$ is also $G$-invariant.

Let $Y$ be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety. Let $R \subset T^* Y$ be any subvariety. We define the restriction $R|_Z := i^*(R)$ of $R$ to $Z$ in $T^* Z$, where $i : Z \to Y$ is the embedding.

**Lemma 2.4.** Let $R \subset T^* Y$ be a coisotropic subvariety. Assume that any smooth point $z \in Z \cap p_Y(R)$ is also a smooth point of $p_Y(R)$ and we have $T_z (Z \cap p_Y(R)) = T_z (Z) \cap T_z (p_Y(R))$. Then $R|_Z$ is $T^* Z$-coisotropic.

**Corollary 2.5.** Let $Y$ be a smooth algebraic variety. Let an algebraic group $H$ act on $Y$. Let $q : Y \to B$ be an $H$-equivariant morphism. Let $O \subset B$ be an orbit. Consider the natural action of $H$ on $T^* Y$ and let $R \subset T^* Y$ be an $H$-invariant subvariety. Suppose that $p_Y(R) \subset q^{-1}(O)$. Let $x \in O$. Denote $Y_x := q^{-1}(x)$. Then if $R$ is $T^* Y$-coisotropic then $R|_{Y_x}$ is $T^* (Y_x)$-coisotropic. Thus if $R|_{Y_x}$ has no (non-empty) $T^* (Y_x)$-coisotropic subvarieties then $R$ has no (non-empty) $T^* Y$-coisotropic subvariety.

2.1.2. Singular support. Let $X$ be a smooth algebraic variety. Let $D_X$ denote the algebra of polynomial differential operators on $X$. Let $GrD_X$ be the associated graded algebra of $D_X$. Then $GrD_X \cong O(T^* X)$.

Let $\xi \in \mathcal{E}(X)$. Denoted by $SS(\xi)$ the singular support of the right $D_X$-module generated by $\xi$. Then $SS(\xi) \subset T^* X$ is nothing but the zero set of $Gr(Ann_{D_X} \xi)$ where $Ann_{D_X} \xi$ is the annihilator of $\xi$. (See Appendix B] for more details.)

Let $V$ be a vector space over $F$. Let $B$ be a non-degenerate bilinear form on $V$. Then $B$ defines Fourier transform with respect to the self-dual Haar measure on $V$, denoted by $\mathcal{F}_V$. For any Nash manifold $M$, we also denote by

\[ \mathcal{F}_V : \mathcal{E}(M \times V) \to \mathcal{E}(M \times V) \]

the fiberwise Fourier transform or partial Fourier transform. Consider $B$ as a map $B : V \to V^*$. Identify $T^*(X \times V)$ with $T^* X \times V \times V^*$. We define

\[ F_V : T^*(X \times V) \to T^*(X \times V) \]

by $F_V(x, v, \phi) = (x, -B^{-1}\phi, Bv)$.

**Proposition 2.6.** (i) Let $\xi \in \mathcal{E}(X)$. Then the Zariski closure of $Supp(\xi)$ is $p_X(\mathbb{S}(\xi))$.

(ii) Let $\xi \in \mathcal{E}(X)^G$. Then $SS(\xi) \subset \{(x, \phi) \in T^* X | \phi(\alpha(x)) = 0 \text{ for all } \alpha \in g\}$.

(iii) Let $(V, B)$ be a quadratic space. Let $X$ be a smooth algebraic variety. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in $V$. Suppose that $Supp(\xi) \subset Z$. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X \times V}(Z))$.

(iv) Let $X$ be a smooth algebraic variety. Let $\xi \in \mathcal{E}(X)$. Then $SS(\xi)$ is coisotropic.

2.1.3. Distributions on non-distinguished nilpotent orbits. Let $V$ be an algebraic finite dimensional representation of a reductive group $G$. Let $\Gamma(V)$ be the nilpotent cone of $V$.

**Definition 2.7.** Suppose that there is a finite number of $G$-orbits in $\Gamma(V)$. Let $x \in \Gamma(V)$. We call it $G$-distinguished if its conormal bundle $CN_{Gx,x}^G \subset \Gamma(V^*)$. We will call a $G$-orbit $G$-distinguished if all its elements are $G$-distinguished.

In the case when $G = GL_n(\mathbb{R})$ and $V = Mat_{n,n}(\mathbb{R})$ the set of $G$-distinguished elements is exactly the set of regular nilpotent elements.
Let $H$ be the set of distinguished elements in $\text{Supp}$ and $\tilde{\text{Supp}}$.

Theorem 3.1. We have $n$ generate the whole group $\text{GL}_n$ for $x \in \text{Mat}_{n,n}(E) \subset \mathfrak{gl}_2n(F)$.

Let $H_n := \text{GL}_n(E)$. Denote $\tilde{H}_n := H_n \times \langle \sigma \rangle$ where $\sigma$ acts on $H_n$ by the involution $a \mapsto (a^{-1})^t (\bar{a}^{-1})^t$.

The group $\tilde{H}_n$ acts on $L_n$ by $a \tilde{x} = ax \bar{a}$ and $x^t = \sigma$ for $x \in L_n$. Let $\chi$ be the sign character of $\tilde{H}_n$, i.e. $\chi|_{H_n}$ is trivial and $\chi(\sigma) = -1$.

Let $H_{n-1}$ be a natural subgroup of $H_n$ through the embedding $g \mapsto \left( \begin{array}{cc} g & 0 \\ 0 & 1 \end{array} \right)$ for $g \in \text{GL}_{n-1}(E)$. Then $H_{n-1}$ is a proper subgroup of $P_E = P \cap \text{GL}_n(E)$.

Theorem 3.1. We have $\mathcal{C}(L_n)^{\tilde{H}_{n-1}, \chi} = 0$.

Then Theorem 1.3 follows from Theorem 3.1 due to the fact that the subgroups $P_E$ and its transpose $P_E^t$ generate the whole group $\text{GL}_n(E)$.

Consider the decomposition $L_n = L_{n-1} \oplus V \oplus V^* \oplus E$ of $\tilde{H}_{n-1}$-spaces, where $\tilde{H}_{n-1}$ acts on $E$ trivially and acts on $L_{n-1} \oplus V \oplus V^*$ via $g \cdot (x,v,v^*) = (gxg^{-1}, g^* v, v^* g^{-1})$ for $x \in L_{n-1} = \text{Mat}_{n-1,n-1}(E), v \in V$ and $v^* \in V^*$, $V^*$ is the linear dual space of $V$ and $\text{dim}_E V = n - 1$.

Consider the nilpotent cone in $L_{n-1} \oplus V \oplus V^*$. (See [Aiz13 §6.1].) For the proof of Theorem 3.1 we will prove the following.

Theorem 3.2. One has $\mathcal{C}_{\tilde{N}_n}(L_{n-1} \oplus V \oplus V^*)^{\tilde{H}_{n-1}, \chi} = 0$. 

Corollary 2.9. Let $\xi \in \mathcal{C}(W)$ and suppose that $\text{Supp}(\xi) \subset \Gamma(V)$ and $\text{Supp}(\tilde{\xi}(\xi)) \subset \Gamma(V^*)$. Then the set of distinguished elements in $\text{Supp}(\xi)$ is dense in $\text{Supp}(\xi)$.
Then Theorem 3.1 follows from Theorem 3.2.

Define a non-degenerate symmetric $F$-bilinear form on $\mathfrak{gl}_{2n}(F)$ by
\[
(z, w)_{\mathfrak{gl}_{2n}(F)} := \text{the trace of } zw \text{ as a } F\text{-linear operator.}
\]

Note that the restriction of this bilinear form on $L_n$ is still non-degenerate. Fix a non-trivial unitary character $\psi$ of $F$. Denote by
\[
\tilde{\mathfrak{f}} : \mathcal{C}(L_n) \rightarrow \mathcal{C}(L_n)
\]
the Fourier transform which is normalized such that for every Schwartz function $\varphi$ on $L_n$,
\[
\tilde{\mathfrak{f}}(\varphi)(z) = \int_{L_n} \varphi(w)\psi((z, w)_{\mathfrak{gl}_{2n}(F)})dw
\]
for $z \in L_n$, where $dw$ is the self-dual Haar measure on $L_n$. If $L_n$ can be decomposed into a direct sum of two quadratic subspaces $U_1 \oplus U_2$ such that each $U_i$ is non-degenerate with respect to $(\cdot , \cdot )|_{U_i}$, then we may define the partial Fourier transform
\[
\tilde{\mathfrak{f}}_{U_i}(\varphi)(x, y) = \int_{U_i} \varphi(z, y)\psi((x, z)|_{U_i})dz
\]
for $x \in U_1, y \in U_2$ and $\varphi \in \mathcal{C}(U_1 \oplus U_2)$. Similarly for $\tilde{\mathfrak{f}}_{U_2}(\varphi)$. It is clear that the Fourier transform $\tilde{\mathfrak{f}}$ intertwines the action of $\tilde{H}_{n-1}$. Thus we have the following lemma.

**Lemma 3.3.** The Fourier transform $\tilde{\mathfrak{f}}$ preserves the space $\mathcal{C}(L_{n-1} \oplus V \oplus V^*)\tilde{H}_{n-1}\cdot x$.

**3.1. Reduction within the null cone.** Recall that
\[
\mathcal{N}_n := \{ (x, v, v^*) \in L_{n-1} \oplus V \oplus V^* \mid (xx)^{n-1} = 0 \text{ and } v^*(\bar{x}x)^k\bar{v} = 0 = v^*(\bar{x}x)^k\bar{v} \text{ for all non-negative integer } k \}
\]
is the nilpotent cone in $L_{n-1} \oplus V \oplus V^*$. Let
\[
\mathcal{N} := \{ x \in \text{Mat}_{n-1,n-1}(E) : xx \text{ is nilpotent} \}.
\]

**Lemma 3.4.** \cite[Lemma 2.3]{Gru97} For any $x \in \mathcal{N}$, there exists a $g$ in $H_{n-1}$ such that the twisted conjugate $g \bar{x} g^{-1}$ of $x$ is in its Jordan normal form.

Following \cite[Proposition 3.9]{CS20}, we shall prove the following proposition when $F$ is non-archimedean in this subsection.

**Proposition 3.5.** Let $f$ be a $H_{n-1}$-invariant generalized function on $L_{n-1} \oplus V \oplus V^*$ such that $f$, its Fourier transform $\tilde{\mathfrak{f}}(f)$ and its partial Fourier transforms $\tilde{\mathfrak{f}}_{V \oplus V^*}(f), \tilde{\mathfrak{f}}_{L_{n-1}}(f)$ are all supported on $\mathcal{N}_n$. Then $f = 0$.

**Remark 3.6.** We will postpone the proof of Proposition 3.5 when $F = \mathbb{R}$ until the next subsection, which involves the machine of D-modules.

Let $\mathcal{O}$ be an $H_{n-1}$-orbit in $\mathcal{N}_n$. Pick $(e, v_0, v_0^*) \in \mathcal{O}$. Then $e \in \mathcal{N}$. Moreover, we may assume that $x \in \text{Mat}_{n-1,n-1}(F)$ due to Lemma 3.4. Recall that every $e \in L_{n-1}$ can be extended to a $\mathfrak{sl}_2$-triple $\{h, e, f\}$ (see \cite[Proposition 4]{KR71}) in the sense that
\[
[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h
\]
where $f \in \mathcal{N}$ and $h \in \mathfrak{h}_{n-1}$, where $\mathfrak{h}_{n-1} = \text{gl}_{n-1}(E)$ is the Lie algebra of $H_{n-1}$. Furthermore, we may assume that $f, h \in \text{Mat}_{n-1,n-1}(F)$ due to Lemma 3.4. Let $L^e_{n-1}$ denote the elements in $L_{n-1}$ annihilated by $e$ under the adjoint action of the triple $\{h, e, f\}$ on $L_{n-1}$. Then
\[
L_{n-1} = [\mathfrak{h}_{n-1}, e] + L^e_{n-1}.
\]

Denote by $\mathcal{C}_{\mathcal{O}}(L_{n-1} \oplus V \oplus V^*)$ the space of the tempered generalized functions on $(L_{n-1} \oplus V \oplus V^*) \setminus \partial \mathcal{O}$ with support in $\mathcal{O}$, where $\partial \mathcal{O}$ is the complement of $\mathcal{O}$ in its closure in $L_{n-1} \oplus V \oplus V^*$. (See \cite[Notation 2.5.3]{AG99}.) We will use similar notation without further explanation.

Let $F^*$ act on $\mathcal{C}(L_{n-1} \oplus V \oplus V^*)$ by
\[
(t \cdot f)(x, v, v^*) = f(t^{-1}x, t^{-1}v, t^{-1}v^*)
\]
for $t \in F^\times$, $x \in L_{n-1}$, $v \in V$, $v^* \in V^*$ and $f \in C(L_{n-1} \oplus V \oplus V^*)$. The orbit $O$ is invariant under dilatation and so $F^\times$ acts on $C_{\mathfrak{sl}_2}(L_{n-1} \oplus V \oplus V^*)$ as well.

Suppose that the $\mathfrak{sl}_2(F)$-triple $\mathfrak{sl}_2(F)$ integrates to an algebraic homomorphism

$$\text{SL}_2(F) \rightarrow \text{GL}_{2n-2}(F).$$

Denote by $D_t$ the image of $\left( \begin{array}{cc} t & \cdot \\ t^{-1} & \cdot \end{array} \right)$ in $H_{n-1} \cap \text{GL}_{2n-2}(F)$. Let

$$T := \{(D_t, t^{-2}) \in H_{n-1} \times F^\times | t \in F^\times\}$$

be a closed subgroup in $H_{n-1} \times F^\times$ which fixes the element $e$. Define a quadratic form on $V \oplus V^*$ as follows:

$$(v, v^*) \mapsto tr_{E/F}(v^*(\bar{v}))$$

for $v \in V$ and $v^* \in V^*$ which induces a $F$-bilinear form $(-, -)$ on $(V \oplus V^*) \times (V \oplus V^*)$. Define

$$V(e) := \{(v, v^*) \in V \oplus V^* | (e, v, v^*) \in O \text{ and } (h \cdot (v, v^*), (v, v^*)) = 0 \text{ for all } h \in \langle D_t \rangle\}.$$  

The following lemma is similar to [CS20 Lemma 3.13].

**Lemma 3.7.** Let $\eta$ be an eigenvalue for the action of $F^\times$ on $C_{\mathfrak{sl}_2}(L_{n-1} \oplus V \oplus V^*)$. Let $| \cdot |$ denote the absolute value of $F^\times$. Then $\eta^2 = |t|^{r(2-h)|t|_{F_{n-1}}^{-4(n-1)}}$.

**Remark 3.8.** There is a more general version of Lemma 3.7, see Theorem 3.10

**Proof.** Consider the map

$$(3.2) \quad H_{n-1} \times F^\times \times (F_{n-1}^\times \oplus V \oplus V^*) \rightarrow L_{n-1} \oplus V \oplus V^*$$

via $(h, \xi, v, v^*) \mapsto h.(e + \xi + v + v^*)$ for $\xi \in L_{n-1}^{\times}, h \in H_{n-1} \times F^\times, v \in V$ and $v^* \in V^*$, which is submersive at every point of $H_{n-1} \times F^\times \times \{0, v_0, v_0^*\}$. Moreover, $H_{p,p} \times F^\times \times \{0, v_0, v_0^*\}$ is open in the inverse image of $O = (H_{n-1} \times F^\times) \cdot (e, v_0, v_0^*)$ under the map (3.2). (See [CS20 Page 18].) Thanks to [JSZ11 Lemma 2.7], the restriction map yields an injective linear map

$$C_{\mathfrak{sl}_2}(L_{n-1} \oplus V \oplus V^*)^{H_{p,p} \times F^\times, 1 \times \eta} \rightarrow C_{\mathfrak{sl}_2}(O) \times E(e)(F_{n-1}^\times \oplus V \oplus V^*)^{T, 1 \times \eta}$$

where $1 \times \eta|_T((D_t, t^{-2})) = \eta(t)^{-2}$ and

$$E(e) := \{(v, v^*) \in V \oplus V^* | v^*(x_0^{2}\bar{v}) = v^*x_0^{2k+1}v \text{ for all non-negative integers } k\}$$

for $e = \begin{pmatrix} x_0 & x_0 \\ 0 & 0 \end{pmatrix} \in L_{n-1}$. It is easy to see that the representation $C_{\mathfrak{sl}_2}(O)(L_{n-1}^{\times})$ of $T$ is complete reducible and every eigenvalue has the form

$$(D_t, t^{-2}) \mapsto |t|^{r(2-h)|t|_{F_{n-1}}^{-1}}.$$ 

Thus

$$\eta(t)^2 = |t|^{r(2-h)|t|_{F_{n-1}}^{-1}} \gamma^{-1}(t)$$

for any $t \in F^\times$, where $\gamma$ is an eigenvalue for the action of $T$ on $C_{\mathfrak{sl}_2}(O)(V \oplus V^*)$. In order to compute $\gamma$, we will restrict $\gamma$ to a smaller subspace $C_{\mathfrak{sl}_2}(e)(V \oplus V^*)$ of $C_{\mathfrak{sl}_2}(O)(V \oplus V^*)$.

Define a symplectic form on $(V \oplus V^*) \times (V \oplus V^*)$ as follow

$$\langle (x_1, y_1), (x_2, y_2) \rangle := \langle x_1, y_2 \rangle - \langle y_1, x_2 \rangle$$

where $x_i, y_i \in V \oplus V^*$. Then $V \oplus V^*$ is a maximal isotropic subspace. Consider the Weil representation on $\text{Mp}_{4n-4}(F) = \text{Mp}(V \oplus V^*) \times (V \oplus V^*)$, $\langle -, - \rangle$. Under the Weil representation $\omega_\psi$,

$$\begin{pmatrix} A \delta(A^{-1}) \varphi(x) = |\det A|^{1/2} \varphi(A^{-1}x), & \text{for } A \in \text{GL}_{2n-2}(F), \\ \omega_\psi \begin{pmatrix} N_1 \delta(N_1) \varphi(x) = \psi((N_x(x)) \varphi(x), & \text{for } N = N_1. 
\end{pmatrix}$$
for \( \varphi \in S(V \oplus V^*) \) and \( x \in V \oplus V^* \). We may extend \( \omega_\varphi \) from the Schwartz space \( S(V \oplus V^*) \) to the tempered general function space \( \mathcal{C}(V \oplus V^*) \). Note that

\[
\begin{pmatrix}
X \\
X^{-1}
\end{pmatrix} = 
\begin{pmatrix}
I_n & -X \\
X & I_n
\end{pmatrix}
\begin{pmatrix}
I_n & 1 - X \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
I_n & 1 \\
1 & I_n
\end{pmatrix}
\]

holds for any \( X \in \text{GL}_n(F) \). Here we only need the case that \( X \) is a diagonal matrix. Denote \( D_t = \begin{pmatrix} A_t & B_t \\
B_t^{-1}
\end{pmatrix} \) and \( X_t = \begin{pmatrix} A_t & \\
A_t^{-1}
\end{pmatrix} \). Then the action of \( D_t \) on \( V \oplus V^* \) is given by

\[
(v, v^*) \mapsto (A_t v, v^* A_t^{-1}).
\]

It is obvious that

\[
\omega_\varphi \left( \begin{pmatrix} 1_{2n-2} & X_t \\
1_{2n-2}
\end{pmatrix} \right) f(v, v^*) = \psi((A_t v, v^* A_t^{-1}), (v, v^*)) f(v, v^*) = f(v, v^*)
\]

for any \( f \in \mathcal{C}(V(V) \oplus V^*) \). Then \( \begin{pmatrix} 1_{2n-2} & X_t \\
1_{2n-2}
\end{pmatrix} \) acts on \( \mathcal{C}_V^V(V \oplus V^*) \) trivially and so is \( \begin{pmatrix} 1_{2n-2} & X_t^{-1} \\
1_{2n-2}
\end{pmatrix} \).

Thus \( D_t \) does not contribute to \( \gamma \). Therefore \( \gamma \) has the form

\[
(D_t, t^{-2}) \mapsto |t|^{-2} |\frac{\dim F(V \oplus V^*)}{2} = |t|^{4-4n}
\]

for any \( t \in \bar{F}^\times \). \( \square \)

Consider \( \text{Mat}_{n-1, n-1}(F) \) as a representation of \( \mathfrak{sl}_2(F) \)- triple \((3.1)\). Decompose it into irreducible representations

\[
\text{Mat}_{n-1, n-1}(F) = \bigoplus_{i=1}^d V_i.
\]

Let \( \lambda_i \) be the highest weight of \( V_i \).

**Lemma 3.9.** Assume \( n \geq 3 \). One has

\[
2(n-1)^2 + 3 < \text{tr}(2 - h)|_{L_{n-1}^\mathfrak{t}} < 4(n-1)^2
\]

**Proof.** It is easy to see that \( (n-1)^2 = \sum_{i=1}^d (\lambda_i + 1) = \sum_i \lambda_i + d \). Note that

\[
\text{tr}(2 - h)|_{L_{n-1}^\mathfrak{t}} = 2(d + \sum_i \lambda_i).
\]

Therefore \( \text{tr}(2 - h)|_{L_{n-1}^\mathfrak{t}} - 2(n-1)^2 = 2d > 3 \) due to the fact that \( d \geq 2 \). \( \square \)

Let \( Q \) be a quadratic form on \( L_{n-1} \oplus V \oplus V^* \) defined by

\[
Q(x, v, v^*) = \text{tr}(x\bar{x}) + \text{tr}_{E/F} v^*(\bar{v})
\]

for \( x \in \text{Mat}_{n-1, n-1}(E) = L_{n-1}, v \in V \) and \( v^* \in V^* \). Denote by \( Z(Q) \) the zero locus of \( Q \) in \( L_{n-1} \oplus V \oplus V^* \). Then \( Z_n \subset Z(Q) \subset L_{n-1} \oplus V \oplus V^* \). Recall the following homogeneity result on tempered generalized functions due to Aizenbud-Gourevitch. (See [AG09a, Theorem 5.1.5], )

**Theorem 3.10.** Let \( I \) be a non-zero subspace of \( \mathcal{C}(Z(Q)(L_{n-1} \oplus V \oplus V^*)) \) such that for every \( f \in I \), one has that \( \mathfrak{g}(f) \in I \) and \( (\psi \circ Q) : f \in I \) for all unitary character \( \psi \) of \( F \). Then \( I \) is a completely reducible \( \mathfrak{g}^\times \)-subrepresentation of \( \mathcal{C}(L_{n-1} \oplus V \oplus V^*), \) and it has an eigenvalue of the form \( | - \frac{\dim E(L_{n-1} \oplus V \oplus V^*))}{2} \).

Now we are prepared to prove Proposition 3.5 when \( F \) is non-archimedean.

**Proof of Proposition 3.5 when \( F \) is non-archimedean.** Denote by \( I \) the space of all tempered generalized functions \( f \) on \( L_{n-1} \oplus V \oplus V^* \) with the properties in Proposition 3.5. Assume by contradiction that \( I \) is nonzero. If \( n-1 = 1 \), i.e. \( n = 2 \), then

\[
\mathcal{N}_2 \cong \{0\} \oplus (E \oplus \{0\} \cup \{0\} \oplus E)
\]
and so Proposition 3.5 follows from [Aiz13, Lemma 6.3.4] that if there exists an
\[ f \in \mathcal{C}_N((E \oplus E \oplus E)^E) \]
such that both \( \mathfrak{F}_{L_{n-1}}(f) \) and \( \mathfrak{F}_{V \oplus V^*}(f) \) are supported on \( N_n \), then \( f = 0 \). Here the action of \( E^E \) is given by
\[ g \cdot (x, v, v^*) = (gx\bar{g}^{-1}, gv, v^*\bar{g}^{-1}) \]
where \( g \in E^E, x, v, v^* \in E \). Assume that \( n \geq 3 \). Then by Lemma 3.7 and Theorem 3.10 one has
\[ \dim F(L_{n-1} \oplus V \oplus V^*) = \text{tr}(2 - h)|_{L_{n-1}} + 4n - 4 \]
and so
\[ \text{tr}(2 - h)|_{L_{n-1}^*} = 2(n - 1)^2 \]
which contradicts the equality (3.3). This finishes the proof. \( \square \)

3.2. Proof of Proposition 3.5 when \( F = \mathbb{R} \). This subsection focuses on the proof of Proposition 3.5 when \( F = \mathbb{R} \). We will follow [Aiz13] §6 to prove that \( SS(f) \) is not coisotropic for any non-zero tempered generalized function \( f \) satisfying the conditions in Proposition 3.5, which implies that \( f \) must be zero.

Recall that \( N = \{ x \in \text{Mat}_{n-1,n-1}(\mathbb{C})| x \bar{x} \text{ is nilpotent} \} \) and
\[ N_n = \{ (x, v, v^*) \in L_{n-1} \oplus V \oplus V^*| x \in N, v^*(\bar{x}x)^k = 0 = v^*(\bar{x}x)xv \text{ for all non-negative integer } k \}. \]

Define
\[ S = \left\{ ((A_1, v_1, v_1^*), (A_2, v_2, v_2^*)) \in N_n^2 \text{ for any } i, j \in \{1, 2\} \text{ and } (A_1, v_1, v_1^*) \perp (A_2, v_2, v_2^*) \text{ for any } \alpha \in \mathfrak{gl}_{n-1}(\mathbb{C}) \right\}. \]

Note that the orthogonality condition can be replaced by \( A_1A_2 - A_2A_1 + v_1\bar{v}_2 - \bar{v}_1v_2 = 0 \). Let \( O_1, O_2 \subset N \) be any two nilpotent orbits. Set
\[ U(O_1, O_2) := \{ (A_1, v_1, v_1^*), (A_2, v_2, v_2^*) \in S| A_1 \in O_1 \text{ and } (v_1, v_1^*) \notin (V \times \{0\}) \cup (\{0\} \times V^*) \}. \]

Proposition 3.11. Let \( O = H_{n-1} \cdot e \) with \( e \) regular. The set \( U(O, O') \) does not contain any (non-empty) coisotropic subvariety.

Proof. It suffices to show that \( R_e := U(O, O')|_{\{e\} \times V \times V^*} \) is not \( T^*(V \times V^*) \)-coisotropic. It is easy to obtain that \( e^{n-1}v_1 = 0 \). Otherwise \( v_1 = 0 \). Similarly \( e^{n-2}v_2 \) and \( v_1^*e^{n-1} \) are all zero. Thus \( R_e \) is not \( T^*(V \times V^*) \)-coisotropic due to [Aiz13] Lemma 6.4.4 and the relation \( eA - Ae + v_1\bar{v}_2 - \bar{v}_1v_2 = 0 \) for \( A \in N \). \( \square \)

Proof of Proposition 3.5 when \( F = \mathbb{R} \). Suppose that \( (x, v, v^*) \in \text{Supp}(f) \). Then we may assume that \( x = e \in \text{Mat}_{n-1,n-1}(\mathbb{R}) \cap N \) due to Lemma 3.4. Note that \( SS(f) \) is \( T^*(L_{n-1} \oplus V \oplus V^*) \)-coisotropic. Since both \( f \) and \( \mathfrak{F}_{L_{n-1}}(f) \) are \( H_{n-1} \)-invariant, \( e \) is \( H_{n-1} \)-distinguished (see §2.1.3), i.e. \( e \) is regular nilpotent. Moreover, \( SS(f) \subset S \). Thanks to Proposition 3.11
\[ SS(f) \subset \left( N \times (V \times \{0\}) \cup (\{0\} \times V^*) \right) \times (N \times (V \times \{0\}) \cup (\{0\} \times V^*)). \]
Thus \( \text{Supp}(f) \subset N \times (V \times \{0\}) \cup (\{0\} \times V^*) \). Due to [Aiz13] Lemma 6.3.4, \( f \) must be zero. This finishes the proof. \( \square \)

Remark 3.12. One may follow [Aiz13] §6 to give a uniform proof for Proposition 3.5 which involves more techniques and more notation when \( F \) is non-archimedean.

3.3. Proof of Theorem 3.1. In this subsection, we shall give the proof of Theorem 3.1.

Let \( (G, H, \theta) \) be a symmetric pair. Consider the action of \( H \times H \) on \( G \) by left and right translations and the action of \( H \) on \( L_n \) by conjugation. Let \( g \in G(F) \) such that \( H(F)gH(F) \) is closed in \( G(F) \). Let \( x = g\theta(g^{-1}) \). Then \( G_x, H_x, \theta|_{G_x} \) is a descendant of \( (G, H, \theta) \). (See [AG19a, §7.2].)

Lemma 3.13. [Car15] Theorem 6.15 Every descendant of the pair \( (\text{GL}_{2n}, \text{GL}_n \cdot \text{GL}_r) \) is a product of pairs of the form \( (\text{GL}_r \cdot \text{GL}_n \cdot \text{GL}_r, \text{GL}_r \cdot \text{GL}_n \cdot \text{GL}_r \cdot \text{GL}_r) \) and \( (\text{GL}_{2r}, \text{GL}_r \cdot \text{GL}_r) \) for some \( r < n \), where \( L_2 \) is a finite field extension over \( F \) and \( L_1 \) is a quadratic extension of \( L_2 \).
Proof of Theorem 3.2 It is enough to show that
\[ \mathcal{C}_{\mathcal{N}_n}(L_{n-1} \oplus V \oplus V^*) \tilde{H}_{n-1,x} = 0. \]
Pick any nilpotent orbit \( \mathcal{O} \) in \( \mathcal{N}_n \). Thanks to Lemma 3.3 and Proposition 3.5 we have
\[ \mathcal{C}_\mathcal{O}(L_{n-1} \oplus V \oplus V^*) \tilde{H}_{n-1} = 0. \]
This finishes the proof. \( \square \)

Finally, we give the proof of Theorem 3.1.

Proof of Theorem 3.1 It is well-known that
\[ \mathcal{C}(\text{Mat}_{r+1, r+1}(F))^{\text{GL}_r(F) \cdot \chi} = 0 \]
where \( \text{GL}_r(F) = \text{GL}_r(F) \rtimes \langle \sigma \rangle \) and \( \sigma \) acts on \( \text{GL}_r(F) \) by
\[ \sigma \cdot g = (g^t)^{-1} \]
for \( g \in \text{GL}_r(F) \). (See [AGRS10] for the non-archimedean case and [AG09b, SZ12] for the archimedean case.) We will show that
\[ \mathcal{C}(L_{n-1} \oplus V \oplus V^*) \tilde{H}_{n-1,x} = 0. \]
Applying Theorem 2.2 and Lemma 3.13, it suffices to show that
\[ \mathcal{C}(R(L_{r-1} \oplus V \oplus V^*)) \tilde{H}_{r-1,x} = 0 \Rightarrow \mathcal{C}(Q(L_{r-1} \oplus V \oplus V^*)) \tilde{H}_{r-1,x} = 0 \]
for all \( r \). Thus
\[ \mathcal{C}_{\mathcal{N}_r}(L_{r-1} \oplus V \oplus V^*) \tilde{H}_{r-1,x} = 0 \]
(see Theorem 3.2) implies \( \mathcal{C}(L_{n-1} \oplus V \oplus V^*) \tilde{H}_{n-1,x} = 0. \)
Recall that \( L_n = L_{n-1} \oplus V \oplus V^* \oplus E \) and \( \tilde{H}_{n-1} \) acts on \( E \) trivially. Therefore, \( \mathcal{C}(L_n) \tilde{H}_{n-1,x} = 0 \) by Localization Principle (see [AG09a, Appendix D]). \( \square \)

4. Proof of Theorem 1.1

Following [Oh11, Kem15], we will give the proof of Theorem 1.1 in this section.

4.1. Proof of Theorem 1.2 This subsection focuses on the proof of Theorem 1.2. Define
\[ \mathcal{H}_n := \text{GL}_n(E) \times \text{GL}_n(E) \]
and \( \tilde{\mathcal{H}}_n = \mathcal{H}_n \rtimes \langle \sigma \rangle \), where the action is given by
\[ \sigma(g_1, g_2) = ((g_2^{-1})^t, (g_1^{-1})^t) \]
for \( g_i \in \text{GL}_n(E) \). Let \( \tilde{\mathcal{H}}_n \) act on \( \text{GL}_{2n}(F) \) by
\[ (g_1, g_2) \cdot x = g_1 x g_2^{-1} \]
and \( \sigma \cdot x = x^t \) for \( g_i \in \text{GL}_n(E) \) and \( x \in \text{GL}_{2n}(F) \), which induces an action of \( \tilde{\mathcal{H}}_n \) on \( \mathcal{C}(\text{GL}_{2n}(F)) \). Let \( \mathcal{H}_{n,x} \) (resp. \( \tilde{\mathcal{H}}_{n,x} \)) denote the stabilizer of \( x \) in \( \mathcal{H}_n \) (resp. \( \tilde{\mathcal{H}}_n \)).

Lemma 4.1. [Guo96, Proposition 1.2] The double cosets \( H_n x H_n \), where
\[ x \theta(x^{-1}) = \begin{pmatrix} A & 0 & 0 & B_A & 0 & 0 \\ 0 & -1_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_q & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & -1_p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_q \end{pmatrix}, \]
exhaust all closed orbits in \( \text{GL}_{2n}(F) \), where \( A \in \text{Mat}_{\nu, \nu}(F) \) is semisimple without eigenvalues \( \pm 1 \), \( \nu + p + q = n \), \( 1_p, 1_q, 1_v \) are identity matrices and \( B_A \in \text{Mat}_{\nu, \nu}(E) \) satisfies \( A^2 - 1_\nu = B_A B_A \) and \( AB_A = B_A A \).

Lemma 4.2. One has \( \mathcal{C}((\text{GL}_{2n}(F))^{P_E \times \text{GL}_n(E)}) = \mathcal{C}(\text{GL}_{2n}(F))^{\text{GL}_n(E) \times \text{GL}_n(E)} \).
Note that \( \mathcal{C}(GL_{2n}(F))^{K \times GL_n(E)} \cong \mathcal{C}(GL_{2n}(F)/GL_n(E))^K \) for any subgroup \( K \) of \( GL_{2n}(F) \) (see [Kem15, Lemma 3.7]). Thus it suffices to show that any element in \( \mathcal{C}(GL_{2n}(F)/GL_n(E))^{P_E} \) is invariant under transposition. Indeed, we shall prove that
\[
\mathcal{C}(GL_{2n}(F)/GL_n(E))^T_{n-1} \chi = 0.
\]
Applying Theorem 2.1, it is enough to show that
\[
(4.1) \quad \mathcal{C}(H_n \times H_{n, x})^T_{n-1} \chi = 0
\]
for any closed orbit \( H_n \times H_{n, x} \) in \( GL_{2n}(F) \). Note that if \( x \in H_n \), then \( N_{H_n \times H_{n, x}} \cong L_n \) and \( H_{n, x} \cong GL_n(E), H_{n, x} \cap (H_{n-1} \times GL_n(E)) \cong H_{n-1} \). Then (4.1) follows from Theorem 3.1. According to Lemma 4.1 we separate the proof into two cases.

1. If \( x = \begin{pmatrix} 1_p & 1_q \\ 1_p & 1_q \end{pmatrix} \) with \( p + q = n \), then
\[
N_{H_n \times H_{n, x}} \cong \frac{gl_{2n}(F)}{gl_n(E) + Ad_x gl_n(E)} \cong Mat_{p, p}(E) \oplus L_q,
\]
\( H_{n, x} \cong GL_p(E) \times GL_q(E) \) and
\( H_{n, x} \cap (H_{n-1} \times GL_n(E)) \cong GL_p(E) \times H_{q-1} \).

The action of \( H_{n, x} \) on \( Mat_{p, p}(E) \oplus Mat_{q, q}(E) \) is given by
\[
(g_1, g_2) \cdot (x, y) = (g_1 xg_1^{-1}, g_2 yg_2^{-1})
\]
for \( g_1 \in GL_p(E), g_2 \in GL_q(E), x \in Mat_{p, p}(E) \) and \( y \in Mat_{q, q}(E) \). Thus (4.1) follows from Theorem 3.1.

2. If \( x \) satisfies
\[
x \theta(x^{-1}) = \begin{pmatrix} A & \nu_n \\ \nu_n^t & A \end{pmatrix},
\]
then we may assume that \( A \) is a scalar and \( A^2 \neq 1_n \). It is easy to see that \( N_{H_n \times H_{n, x}} \cong Mat_{\nu, \nu}(F) \oplus Mat_{n-\nu, n-\nu}(E), H_{n, x} \cong GL_{n}(F) \times GL_{n-\nu}(E) \) and
\( H_{n, x} \cap (H_{n-1} \times GL_n(E)) \cong GL_{\nu}(F) \times H_{n-1-\nu} \).

The action of \( H_{n, x} \) on \( Mat_{\nu, \nu}(F) \oplus Mat_{n-\nu, n-\nu}(E) \) is given by
\[
(g_1, g_2) \cdot (x, y) = (g_1 xg_1^{-1}, g_2 yg_2^{-1})
\]
for \( g_1 \in GL_{\nu}(F), g_2 \in GL_{n-\nu}(E), x \in Mat_{\nu, \nu}(F) \) and \( y \in Mat_{n-\nu, n-\nu}(E) \). In a similar way, (4.1) holds.

This finishes the proof.

Proof of Theorem 1.2. Recall that \( X_n = GL_{2n}(F)/GL_n(E) \). From the proof of Lemma 4.2, we obtain that
\[
\mathcal{D}(X_n)^T_{n-1} \chi = 0.
\]
From a general principle of ”distribution versus Schwartz distribution” (see [AG09, Theorem 4.0.2]), the equality (4.2) implies
\[
\mathcal{D}(X_n)^T_{n-1} \chi = 0.
\]
Note that \( H_{n-1} \subset P_E \) and that the mirabolic subgroup \( P_E \) and its transpose \( P_E^t \) generate \( GL_n(E) \). Thus one has \( \mathcal{D}(X_n)^{P_E} = \mathcal{D}(X_n)^{GL_n(E)} \). This finishes the proof.

\[ \square \]
4.2. Proof of Theorem 1.1. This subsection focuses on the proof of Theorem 1.1. Let us recall the following lemma appearing in [Off11, Kem15].

Lemma 4.3. [Kem15 Corollary 3.5] Let \( \pi \) be an irreducible smooth admissible representation of \( \text{GL}_{2n}(F) \). Let \( \pi^\vee \) (resp. \( \pi^* \)) be the contragredient (resp. linear dual) of \( \pi \). Then there exists an injective morphism from \( \pi^* \otimes (\pi^\vee)^* \) to the space \( \mathcal{D}(\text{GL}_{2n}(F)) \) consisting of all distributions on \( \text{GL}_{2n}(F) \) as \( \text{GL}_{2n}(F) \times \text{GL}_{2n}(F) \)-modules, denoted by \( A_\pi \).

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Note that
\[
\mathcal{D}(X_n)^K \cong \mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E))^K \cong \mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E))(\text{GL}_{2n}(F)/\text{GL}_n(E))
\]
for any subgroup \( K \) of \( \text{GL}_n(E) \). Thus Theorem 2 implies that
\[
\mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E)) = \mathcal{D}(\text{GL}_{2n}(F)/\text{GL}_n(E))(\text{GL}_{2n}(F)/\text{GL}_n(E)).
\]
Let \( \pi \) be a \( \text{GL}_n(E) \)-distinguished representation of \( \text{GL}_{2n}(F) \). Then its contragredient representation \( \pi^\vee \) is also \( \text{GL}_n(E) \)-distinguished. Denote by \( \pi^* \) the linear dual of \( \pi \). Take two non-zero linear forms \( \mu \in (\pi^*)^P \) and \( \lambda \in ((\pi^\vee)^*)^GL_n(E) \). Then Lemma 4.3 implies
\[
0 \neq A_\pi (\mu \otimes \lambda) \in \mathcal{D}(\text{GL}_{2n}(F))^P \times \text{GL}_n(E)
\]
which is \( \text{GL}_n(E) \times \text{GL}_n(E) \)-invariant as well by the identity 4.4. Since \( A_\pi \) is injective due to Lemma 4.3, \( \mu \otimes \lambda \in (\pi^* \otimes (\pi^\vee)^*)^\text{GL}_n(E) \times \text{GL}_n(E) \). Therefore \( \mu \in (\pi^*)^\text{GL}_n(E) \). This finishes the proof of Theorem 1.1.

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