On an Incompleteness in the General-Relativistic Description of Gravitation

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The recently introduced mechanism of flavor-oscillation clocks has been used to emphasize observability of constant gravitational potentials and thereby to question completeness of the theory of general relativity. An inequality has been derived to experimentally test the thesis presented.

1 Introduction

The gradients of the gravitational potentials are well known to play a major role in the understanding of motion of the cosmic bodies. Especially, in the weak-field limit of Einstein’s theory of gravitation, they are responsible for the description of, say, the planetary orbits. In contrast to that, importantly, in the same limit, there are quantum mechanical effects which depend upon the gravitational potentials themselves. For example, it was recently shown that in performing a quantum mechanical linear superposition of different mass eigenstates of neutrinos belonging to different lepton generations, one may create a so called “flavor oscillation clock” that has the remarkable property to redshift precisely as required by the Einstein’s theory of gravitation.

In the present study we demonstrate that such clocks in principle allow to measure the essentially constant gravitational potential of the local clusters of the galaxies. Taken to its logical conclusion this observation results in the question on the completeness of Einstein’s theory of gravitation.

In this communication we systematically explore this question. We come to the conclusion that, while the gravitationally induced accelerations vanish in a terrestrial free fall, the gravitationally-induced phases of the flavor-oscillation clocks do not. This communication is organized as follows. In the next section we define the context of this paper. In Sec. III the incompleteness-establishing inequality is derived. Section IV outlines a possible experiment to test the inequality. The final Section contains some concluding remarks and summarizes the presented thesis.
2 Gradientless Gravitational Potentials

As is well known, the solar system is embedded in the essentially constant gravitational potential of the local cluster of the galaxies, the so called Great attractor. This gravitational potential, denoted by $\Phi_{GA}$ in the following, may be estimated over the entire solar system to be

$$\text{Solar system: } \Phi_{GA} = \frac{1}{c^2} \phi_{GA} = -3 \times 10^{-5}. \quad (1)$$

For the present communication the precise value of $\Phi_{GA}$ is not important, but what is more relevant is that it is constant over the entire region of the solar system to an exceedingly large accuracy of 1 part in $R_{GA-S}/\Delta R_S$. Here $\Delta R_S$ represents the spatial extent of the Solar system, and $R_{GA-S}$ is the distance of the Solar system from the Great attractor. Taking $\Delta R_S$ to be of the order of Pluto's semi-major axis (i.e. approximately 40 AU), and $R_{GA-S}$ to be about 40 Mpc, we obtain $R_{GA-S}/\Delta R_S \sim 10^{11}$. For comparison, the terrestrial and solar potentials on their respective surfaces are of the order $\Phi_E = -6.95 \times 10^{-10}$, $\Phi_S = -2.12 \times 10^{-6}$, and therefore much smaller as compared to $\Phi_{GA}$. Nonetheless, they carry significantly larger gradients over the relevant experimental regions.

Yet, the constant potential of the Great attractor that pervades the entire solar system is of no physical consequence within the general-relativistic context (apart from it being responsible for the overall local motion of our galaxy). Even the parenthetically observed motion disappears if we hypothetically and uniformly spread the matter of the galactic cluster into a spherical mass to concentrically surround the Earth. Such a massive shell in its interior provides an example of the gradientless contribution to the gravitational potential that we have in mind.

A terrestrial freely-falling frame which measures accelerations to an accuracy of less than 1 part in about $10^{11}$ is completely insensitive to this constant potential. Similarly, since the planetary orbits are determined by the gradient of the gravitational potential they too remain unaffected by this potential. Nonetheless, in what follows we shall show that quantum mechanical systems exist that are sensitive to $\Phi_{GA}$. The simplest example for such a system is constructed in performing a linear superposition of, say, two different mass eigenstates (see Eqs. (9) and (10) below).

In the next section, $\Phi_{GA}$ shall be considered as a physical and gradientless gravitational potential as idealized in the example indicated above. This potential is to be distinguished from the usual “constant of integration” or the “potential at spatial infinity.”
3 An Inequality on the Incompleteness of General Relativity

In the following we will exploit the weak-field limit of gravity as being introduced on experimental grounds. The phrase “weak-field limit” refers to the experimentally established limit in the weak gravitational fields, rather than to the limit of a specific theory. Further, though not necessary, for the sake of the clarity of presentation we shall work in the non-relativistic domain and neglect any rotation that the gravitational source may have. This assumption shall be implicit throughout this communication. The arguments shall be confined to the system composed of the Earth and the Great attractor, and are readily extendable to more general situations.

For the measurements on Earth the appropriate general-relativistic space-time metric is

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left( 1 - \frac{2GM}{c^2 r} \right) dt^2 - \left( 1 + \frac{2GM}{c^2 r} \right) d\vec{r}^2, \]

(2)

where \( M \) is the mass of the Earth, \( r \) refers to the distance of the experimental region from Earth’s center, and \( d\vec{r}^2 = (dx^2 + dy^2 + dz^2) \). The conceptual basis of the theory of general relativity asserts that the flat space-time metric

\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{r}^2 \]

(3)

is measured by a freely falling observer on Earth (or, wherever the observer is). In this framework, a stationary observer on the Earth may define a gravitational potential according to

\[ \phi_E(\vec{r}) = \frac{c^2}{2} (g_{00} - \eta_{00}) = -\frac{c^2}{2} (g_{jj} - \eta_{jj}), \quad j = 1, 2, 3 \text{ (no sum)}. \]

(4)

One immediately suspects that such a description may not incorporate the full physical effects of such physical potentials as \( \phi_{GA} \) even though this conclusion is consistent with the classical wisdom. Indeed, the classical equation of motion consistent with the approximation in Eq. (4)

\[ m \frac{d^2 \vec{r}}{dt^2} = -m g \vec{\nabla} \phi_E(\vec{r}), \]

(5)

is invariant under the transformation

\[ \phi(\vec{r})_E \rightarrow \phi_E(\vec{r}) = \phi_{GA} + \phi_E(\vec{r}). \]

(6)

For this reason \( \phi_{GA} \) has no apparent effect on the planetary orbits.
In the quantum realm the appropriate equation of motion is the Schrödinger equation with a gravitational interaction energy term

$$\left[-\left(\frac{\hbar^2}{2m_i}\right)\nabla^2 + m_g\phi_{\text{grav}}(\vec{r})\right]\psi(t, \vec{r}) = i\hbar\frac{\partial\psi(t, \vec{r})}{\partial t}, \quad (7)$$

as has been confirmed experimentally in the classic neutron interferometry experiments of Collela, Overhauser, and Werner. Equation (7) is not invariant under the transformation of the type (6).

Moreover, this lack of invariance does not disappear in the relativistic regime where an appropriate relativistic wave equation, such as the Dirac equation, must be considered. Therefore, the gravitational potential that appears in Eq. (7) cannot be identified with $\phi_E(\vec{r})$ of Eq. (4). To treat the contributions from the Great attractor and the Earth on the same footing of physical reality, the following identification has to be made:

$$\phi_{\text{grav}}(\vec{r}) \equiv \varphi_E(\vec{r}) = \phi_{\text{GA}} + \phi_E(\vec{r}). \quad (8)$$

A second observation to be made is to note that while by setting $m_i = m_g$ in Eq. (5) the resulting equation becomes independent of the test-particle mass, this is not so for the quantum mechanical equation of motion (7).

These two distinctions between the classical- and quantum-evolutions lead to the conclusion that the theory of general relativity for the description of gravitation cannot be considered complete. The gravitational potentials as defined via $g_{\mu\nu}(\vec{r})$ carry an independent physical significance in the quantum realm, a situation that is reminiscent on the significance of the gauge potential in electrodynamics as revealed by the Aharonov-Bohm effect.

The statement on the incompleteness of general relativity is best illustrated on the example of a "flavor-oscillation clock."

$$|F_a\rangle = \cos(\theta)|m_1\rangle + \sin(\theta)|m_2\rangle, \quad (9)$$

$$|F_b\rangle = -\sin(\theta)|m_1\rangle + \cos(\theta)|m_2\rangle. \quad (10)$$

In the linear superposition of the mass eigenstates we assume (only for simplicity) that both $|m_1\rangle$ and $|m_2\rangle$ carry vanishingly small three momentum (i.e. are at rest).

By studying the time-oscillation between the flavor states $|F_a\rangle$ and $|F_b\rangle$ one discovers that this system can be characterized by the flavor-oscillation frequency

$$\Omega_{a\rightarrow b} = \frac{(m_2 - m_1)c^2}{2\hbar}. \quad (11)$$
The superscript on $\Omega^\infty_{a \leftrightarrow b}$ is to identify this frequency with a clock at the spatial infinity from the gravitational sources under consideration (see below).

Now consider this flavor-oscillation clock to be immersed into the gravitational potential $\varphi_E(\vec{r})$. Then each of the mass eigenstates picks up a different phase because the gravitational interaction is of the form $m \times \varphi_E(\vec{r})$. As a result, one finds that the new flavor-oscillation frequency, denoted by $\Omega'_{a \leftrightarrow b}$, is given by

$$\Omega'_{a \leftrightarrow b} = \left( 1 + \frac{\varphi_E(\vec{r})}{c^2} \right) \Omega^\infty_{a \leftrightarrow b}. \quad (12)$$

This equation is valid for an observer fixed in the global coordinate system attached to the Earth.

Equation (12) would have been the standard gravitational red shift expression if the $\varphi_E(\vec{r})$ was replaced by $\phi_G(\vec{r})$. Freely falling frames ($F$) do not carry fastest moving clocks, they carry clocks that are sensitive to potentials of the type $\phi_GA$. A freely falling frame in Earth’s gravity only annuls the gradients of the gravitational potential while preserving all its constant pieces such as $\phi_GA$. In denoting by $\Omega^F_{a \leftrightarrow b}$ the frequency as measured in a freely falling frame on Earth, one is led to

$$\Omega^F_{a \leftrightarrow b} = \left( 1 + \frac{\phi_GA}{c^2} \right) \Omega^\infty_{a \leftrightarrow b}. \quad (13)$$

From a physical point of view, $\phi_GA$ represents contributions from all cosmic-matter sources. However, all these contributions carry the same sign. In addition, in the context of the cosmos, $\Omega^\infty_{a \leftrightarrow b}$ becomes a purely theoretical entity. Nevertheless, as shown below, $\Omega^\infty_{a \leftrightarrow b}$ does have an operational meaning.

As a consequence, the following incompleteness-establishing inequality is found,

$$\Omega^F_{a \leftrightarrow b} < \Omega^\infty_{a \leftrightarrow b}. \quad (14)$$

This is the primary result of our communication.

4 Outline of an Experiment

To experimentally test the incompleteness of the general-relativistic description of gravitation and measure the essentially constant gravitationally potential in the solar system, we rewrite Eqs. (12) and (13) into (to first order in the potentials)

$$\frac{\Omega'_{a \leftrightarrow b}}{\Omega^F_{a \leftrightarrow b}} = 1 + \frac{\phi_E(\vec{r})}{c^2}, \quad (15)$$
\[ \frac{\Omega'_{a \leftrightarrow b}}{\Omega_\infty_{a \leftrightarrow b}} = \frac{\phi_{GA}}{c^2} + \left(1 + \frac{\phi_E(\vec{r})}{c^2}\right). \]  

Equation (15) shows how the \( \phi_{GA} \)-dependence disappears in \( \Omega'_{a \leftrightarrow b}/\Omega_\infty_{a \leftrightarrow b} \). Equation (16), however, indicates that by systematically measuring \( \Omega'_{a \leftrightarrow b} \) as a function of \( \vec{r} \), e.g. for an atomic system prepared as a linear superposition of different energy eigenstates, one can decipher existence of \( \phi_{GA} \). Because all terrestrial clocks are influenced by the same \( \phi_{GA} \)-dependent constant factor, it is essential that the flavor-oscillation clocks under consideration integrate the accumulated phase over different paths, thus probing different \( \phi_E(\vec{r}) \), and then return to the same spatial region in order that all the data interpretation refers to the same time standard. Such an integration is easily accommodated in Eq. (16). One would then make a two parameter fit in \( \{\Omega_\infty_{a \leftrightarrow b}, \phi_{GA}\} \) to a large set of the closed-loop integrated data on \( \{\Omega'_{a \leftrightarrow b}(\vec{r}), \phi_E(\vec{r})\} \). Explicitly

\[ \oint_\Gamma \Omega'_{a \leftrightarrow b}(\vec{r})d\ell(\vec{r}) = \Omega_\infty_{a \leftrightarrow b} \left(1 + \frac{\phi_{GA}}{c^2}\right) \int_\Gamma d\ell(\vec{r}) + \frac{\Omega_\infty_{a \leftrightarrow b}}{c^2} \int_\Gamma \phi_E(\vec{r})d\ell(\vec{r}), \]  

where \( d\ell(\vec{r}) \) is the differential length element along the closed path \( \Gamma \). By collecting the data on the “accumulated phase” \( \oint_\Gamma \Omega'_{a \leftrightarrow b}(\vec{r})d\ell(\vec{r}) \) and the “probed gravitational potential” \( \int_\Gamma \phi_E(\vec{r})d\ell(\vec{r}) \) for a set of \( \Gamma \), and fitting a straight line, one may extract \( \{\Omega_\infty_{a \leftrightarrow b}, \phi_{GA}\} \). Rigorously speaking, what one obtains is \( \Omega_\infty_{a \leftrightarrow b} \) and the constant \( \phi_{GA} \) as modified by other cosmic contributions. Further, these additional contributions may include extra general-relativistic contributions from the yet-unknown interactions that may couple to the various parameters associated with the superimposed quantum states.

A simple consideration on the magnitude of various gravitational potentials involved and the accuracy of clocks based on quantum superpositions of atomic states leads to the tentative conclusion that the suggested experiment is feasible within the existing technology. In this regard note is taken that various ionic and atomic clocks have reached an accuracy of 1 part in \( 10^{15} \) with a remarkable long term stability. In addition, workers in this field are optimistic that a several orders of magnitude improvement may be expected in the next few years (see, e.g. Barbara Levi’s recent coverage of this subject in the February 1998 issue of Physics Today).

### 5 Concluding remarks and Summary

In the present study we emphasized observability of the constant potential of the Great attractor by means of flavor oscillation clocks. While in a classical context, the force \( \vec{F} = -m_g \vec{\nabla} \phi(\vec{r}) \) experienced by an object is independent of
gradientless gravitational potentials such as $\phi_{GA}$, the frequency of the flavor oscillation clocks depends directly on $\phi_{GA}$ [in addition to $\phi_E(\vec{r})$].

The above considerations suggest that in a free fall the space-time interval (at least in the quantum context) is given by

$$ds^2 = \chi_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{2\phi_{GA}}{c^2}\right)dt^2 - \left(1 - \frac{2\phi_{GA}}{c^2}\right)d\vec{r}^2.$$  \hspace{1cm} (18)

Simultaneously, Eq. (2) should be replaced by

$$ds^2 = \psi_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{2\phi_E(\vec{r})}{c^2}\right)dt^2 - \left(1 - \frac{2\phi_E(\vec{r})}{c^2}\right)d\vec{r}^2,$$  \hspace{1cm} (19)

with Eq. (3) remaining valid at “spatial infinity.” Such a modification is perfectly justified because of the linearity of the weak-field limit (where one is able to formulate the physics in terms of the additive gravitational potentials).

Within the considered framework and approximations, the space-time curvatures derived from $g_{\mu\nu}$ and $\psi_{\mu\nu}$ are identical.

The reported incompleteness in the theory of general relativity for the description of gravitation reveals certain similarities to the Aharonov-Bohm effect. Indeed, in the Aharonov-Bohm effect an observable phase arises in a region with vanishing field strength tensor $F^{\mu\nu}(\vec{r})$, (i.e. in a region with vanishing 4-curl of the gauge potential $A^{\mu}(\vec{r})$). In the effect reported here, an observable phase arises in a region where the contributions of the $\phi_{GA}$-type constant potentials to the curvature tensor $R^{\mu\nu\sigma\lambda}(\vec{r})$ vanish. Both of the effects mentioned above, illustrate the circumstance that in quantum mechanical processes the gauge field $A^{\mu}(\vec{r})$ and the gravitational potential $g^{\mu\nu}(\vec{r})$ may be favored over the corresponding fields strength tensor $F^{\mu\nu}(\vec{r})$, and the curvature tensor $R^{\mu\nu\sigma\lambda}(\vec{r})$, respectively.

However, since the number of the independent degrees of freedom of $A^{\mu}(\vec{r})$ is quite different from that of $g^{\mu\nu}(\vec{r})$, the analogy between the Aharonov-Bohm effect and the one considered here is not complete.

In summary, the local galactic cluster, the Great attractor, embeds us in a dimensionless gravitational potential of about $-3 \times 10^{-5}$. In the solar system this potential is constant to about 1 part in $10^{11}$. Consequently, planetary orbits remain unaffected. However, this is not so for the flavor-oscillation clocks. In a terrestrial free fall the gravitationally induced accelerations vanish, but the gravitationally induced phases of the flavor-oscillation clocks do not. We argued that there exists an element of incompleteness in the general-relativistic description of gravitation. The arrived incompleteness may be subjected to an experimental test by verifying the inequality derived here.
The origin of the reported incompleteness lies in the implicit general-relativistic assumption on the equivalence of the space-time metric as measured by a freely falling observer in the vicinity of a gravitating source (which in turn is embedded in a $\Phi_{GA}$-type constant gravitational potential) and the space-time metric as measured by an observer at the “spatial infinity.”

Acknowledgments

This paper represents an evolution of thoughts originally presented at the festivities. After this work was completed George Matsas (then at the University if Chicago, and now at ITF, Sao Paulo) showed to me a 1991 page from his research notes which hinted at the conclusion now arrived here. I became aware of his considerations in the July of 1998 while visiting Sao Paulo.

This essay is adapted from a recent publication of the present author to honor Professor Sachs.

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