Zero-inflated explanatory variables are common in fields such as ecology and finance. In this paper we address the problem of having excess of zero values in some explanatory variables which are subject to multioutcome lasso-regularized variable selection. Briefly, the problem results from the failure of the lasso-type of shrinkage methods to recognize any difference between zero value occurring either in the regression coefficient or in the corresponding value of the explanatory variable. This kind of confounding will obviously increase number of false positives - all non-zero regression coefficients do not necessarily represent real outcome effects.

We present here the adaptive LAD-lasso for multiple outcomes which extends the earlier work of multivariate LAD-lasso with adaptive penalization. In addition of well known property of having less biased regression coefficients, we show here how the adaptivity improves also method’s ability to recover from influences of excess of zero values measured in continuous covariates.

**Keywords** Multivariate analysis; $p > n$ regression; Penalized regression; Robust Procedures; Variable Selection

### 1 Introduction

In high-dimensional regression problems the number of parameters is often larger than the number of individuals in the sample. Ordinary least squares and other conventional estimation methods don’t work in such situations. Simultaneous estimation and variable selection using lasso [Tibshirani, 1996, Li and Sillanpää, 2012] is a popular shrinkage-estimation approach to obtain sparse estimates for regression coefficients in high-dimensional regression problems. However, the generally known drawback of shrinkage-inducing methods, including lasso, is that they improve estimation accuracy but also introduce downward bias to the estimates. To alleviate this problem, a re-estimation procedure has been proposed where effects of selected predictors are re-estimated using no penalty [Efron et al., 2004, Meinshausen, 2007]. Adaptive lasso has been also presented [Zou, 2006], so that the selected predictors are subject of reduced penalty and unselected positions obtain heavier penalty. This is performed either by using two-stage or iterative estimation strategy. See also related work of thresholded lasso [Zhou, 2010, van de Geer et al., 2011].
We have pinpointed a potential problem in LAD-lasso or lasso method when there are excess of real zero values in some covariates. When the outcome distributions are skewed or they contain some outlying observations, it is possible to apply robust LAD regression instead of ordinary regression. In lasso context, LAD-lasso has been proposed for univariate and multioutcome cases [Möttönen and Sillanpää 2015, Li et al. 2015]. The multioutcome LAD-lasso is related to the group lasso [Yuan and Lin 2006] because the objective function of multioutcome LAD-lasso also contains group lasso penalty. As common shrinkage-inducing methods, also LAD-lasso methods are suffering from downward bias of the estimates. To alleviate this, adaptive LAD-lasso has been proposed for univariate LAD-lasso [Arslan, 2012]. In this paper we study the multioutcome LAD-lasso. Note that adaptive group lasso has been presented in Wang and Leng [2008].

We have pinpointed a potential problem in LAD-lasso or lasso method when there are excess of real zero values in some covariates. We describe this problem here and provide also some potential solutions. Shortly, the problem occurs as the lasso-type of shrinkage methods do not recognize any difference between zero value occurring either at the regression coefficient or in the corresponding value of the covariate. Now there is a danger of misinterpretation how use of the adaptive LAD-lasso will minimize this kind of confounding to happen in practice.

2 Multioutcome LAD-lasso

Consider multiple regression model with multiple outcomes

$$Y = XB + E,$$

where $Y = (y_1, \ldots, y_n)'$ is an $n \times p$ matrix of $n$ observed values of $p$ outcome variables, $X = (x_1, \ldots, x_n)'$ is an $n \times q$ matrix of $n$ observed values of $q$ explanatory variables, $B$ is a $q \times p$ matrix of regression coefficients, and $E = (e_1, \ldots, e_n)'$ is an $n \times p$ matrix of residuals. We further assume that $e_1, \ldots, e_n$ is a random sample of size $n$ from a $p$-variate distribution centered at the origin.

The multivariate lasso estimation method is based on the penalized objective function

$$\frac{1}{n} \sum_{i=1}^{n} \| y_i - B'x_i \|^2 + \lambda \sum_{j=2}^{q} \| \beta_j \| \tag{1}$$

(See e.g. Turlach et al. 2005, Yuan and Lin 2006 and Yuan et al. 2007). The minimizer of the objective function $\hat{B}$ gives now multiivariate lasso estimate for the regression coefficient matrix $B$. Note that equation $\hat{B}$ is also an objective function of group lasso where each of $q - 1$ explanatory variables (the intercept terms are omitted) have $p$ regression coefficients (one for each response variable) which form an group.

The multivariate lasso method gives sparse solutions but it is obviously not very robust. You will get a more robust version by minimizing the penalized objective function (where the squared norms are replaced with norms)

$$\frac{1}{n} \sum_{i=1}^{n} \| y_i - B'x_i \| + \lambda \sum_{j=2}^{q} \| \beta_j \| \tag{2}$$

with respect to the coefficient matrix $B$ [Möttönen and Sillanpää 2015, Li et al. 2015]. Denote the minimizing value of $\hat{B}$ by $\hat{B}_{LL}$. It is easily seen that if we define

$$\begin{cases} (y^*_i, x^*_i) , & \text{if } i = 1, \ldots, n, \\ (0, n\lambda e_{i-n+1}) , & \text{if } i = n+1, \ldots, n+q-1, \end{cases}$$

the objective function $\hat{B}$ reduces to the LAD estimation objective function [Oja, 2010]

$$\frac{1}{n+q-1} \sum_{i=1}^{n+q-1} \| y^*_i - B'x^*_i \|, \tag{3}$$

which shows that we can use any multioutcome LAD regression estimation routine to find the multioutcome LAD-lasso estimate $\hat{B}_{LL}$. You can, for example, use the function mv.l1lm of the R-package MNM [Nordhausen et al., 2016, Nordhausen and Oja 2011].
3 Multioutcome adaptive LAD-lasso

The multioutcome LAD-lasso estimate for a fixed $\lambda$ can be defined as

$$\hat{B}^*(\lambda) = \arg\min_{\beta} \left[ \frac{1}{n} \sum_{i=1}^{n} \|y_i - B'x_i\| + \lambda \sum_{j=2}^{q} \|\beta_j\| \right].$$

The question then arises how to choose the tuning parameter $\lambda$. If we are mainly concerned about recovering the right model, then you can use, for example, Akaike’s information criterion (AIC) or Bayesian information criterion (BIC) \[. On the other hand, if we are mainly concerned about prediction accuracy, then cross-validation technique is often a good choice.

Let $\lambda_0^* = \arg\min_{\lambda} \text{CV}(\hat{B}^*(\lambda))$ be the value of the tuning parameter $\lambda$ which minimizes the cross-validation criterion (or alternatively AIC or BIC criterion) function for the multioutcome LAD-lasso. The multioutcome LAD-lasso estimate can then be defined as

$$\hat{B}^* = \hat{B}^*(\lambda_0^*).$$

It has been shown that lasso-estimation tends to underestimate the regression coefficients and the same is true for the multioutcome LAD-lasso estimate $\hat{B}^*$. Zou [2006] proposed an adaptive lasso method which gives an estimate whose bias is smaller than that of the standard lasso. The adaptive method can also be used in the LAD-lasso [Arslan, 2012]. We extend this method here for multioutcome LAD-lasso case. The multivariate adaptive LAD-lasso estimate for tuning parameters $\lambda_j, j = 2, \ldots, q$ can be defined as

$$\hat{B}(\lambda) = \arg\min_{\beta} \left[ \frac{1}{n} \sum_{i=1}^{n} \|y_i - B'x_i\| + \lambda \sum_{j=2}^{q} \lambda_j \|\beta_j\| \right],$$

where

$$\lambda_j = \frac{\lambda}{\|\beta_j^*\| + 1/n}, \quad j = 2, \ldots, q$$

and $\beta_j^*$ is the $j$th row of the multioutcome LAD-lasso estimate $\hat{B}^* = \hat{B}^*(\lambda_0^*)$. If we denote $w_j^* = (\|\beta_j^*\| + 1/n)^{-1}$, then the adaptive multioutcome LAD-lasso estimate can be written in the form

$$\hat{B}(\lambda) = \arg\min_{\beta} \left[ \frac{1}{n} \sum_{i=1}^{n} \|y_i - B'x_i\| + \lambda \sum_{j=2}^{q} w_j^* \|\beta_j\| \right] = \arg\min_{\beta} \left[ \frac{1}{n} \sum_{i=1}^{n} \|y_i - B'x_i\| + \frac{1}{n} \sum_{j=2}^{q} \|0 - B' w_j^* e_j\| \right]$$

which further implies that it can be written in the multioutcome LAD regression form

$$\hat{B}(\lambda) = \arg\min_{\beta} \left[ \frac{1}{n+q-1} \sum_{i=1}^{n+q-1} \|y_i^* - B'x_i^*\| \right],$$

where

$$\left( y_i^* \atop x_i^* \right) = \begin{cases} \left( y_i \atop x_i \right), & \text{if } i = 1, \ldots, n, \\ \left( 0 \atop n \lambda w_j^* e_{i-n+1} \right), & \text{if } i = n+1, \ldots, n+q-1, \end{cases}$$

Let $\lambda_0 = \arg\min_{\lambda} \text{CV}(\hat{B}(\lambda))$ \[. For a BIC-like criterion in multioutcome LAD-lasso context, see Möttönen and Sillanpää [2015].
be the value of the tuning parameter which minimizes the cross-validation criterion function for the estimate \( \hat{\mathbf{B}}(\lambda) \). *Adaptive multioutcome LAD-lasso estimate* is then

\[
\hat{\mathbf{B}} = \hat{\mathbf{B}}(\lambda_0).
\]

Since the performance of the adaptive LAD-lasso might be sensitive to the initial weights \( \gamma_j \), the following iterative estimation procedure can give more stable results:

1. Find the initial multioutcome LAD-lasso estimate \( \hat{\mathbf{B}}^* = \hat{\mathbf{B}}(\lambda_0^*) \), where \( \lambda_0^* \) minimizes the cross-validation criterion function.

2. Calculate \( w_j^* = (\|\beta_j^*\| + 1/n)^{-1} \), \( j = 2, \ldots, q \).

3. Find \( \hat{\mathbf{B}} = \hat{\mathbf{B}}(\lambda_0) \), where \( \lambda_0 \) minimizes the cross-validation criterion function.

4. Calculate \( w_j = (\|\beta_j\| + 1/n)^{-1} \), \( j = 2, \ldots, q \).

5. Iterate through steps (S3)-(S4) until convergence measured as

\[
\frac{\|\hat{\mathbf{B}}(s) - \hat{\mathbf{B}}(s-1)\|}{\|\hat{\mathbf{B}}(s-1)\|}
\]

is obtained.

### 4 Simulation studies

In this section we present two simulated data sets to provide insights to the multioutcome adaptive LAD-lasso in certain scenarios. R-code for the simulations can be found on github [Lähderanta and Möttönen 2021].

#### 4.1 Bias reduction

For this study, we simulated new phenotypes with trivariate traits using the public genotype data set from the 12th QTL-MAS workshop in Uppsala, Sweden, in 2008. The original genotype data set contains 5,865 individuals and 6,000 markers. We took a random sample of size 300 individuals and chose every 30th marker with a resulting total number of 200 markers. We then generated trivariate traits by using the multivariate multiple regression model

\[
\mathbf{Y} = \mathbf{X} \mathbf{B} + \mathbf{E},
\]

where \( \mathbf{Y} \) is a \( 300 \times 3 \) matrix of trivariate traits, \( \mathbf{X} \) is a \( 300 \times 200 \) matrix with \( ij \)th element

\[
x_{ij} = \begin{cases} -1, & \text{if indiv. } i \text{ is homozygote (11) at marker } j, \\ 0, & \text{if indiv. } i \text{ is heterozygote at marker } j, \\ 1, & \text{if indiv. } i \text{ is homozygote (22) at marker } j, \end{cases}
\]

\( \mathbf{B} \) is a \( 200 \times 3 \) matrix with four QTLs indicated as non-zero rows

\[
\beta_5' = (100, 100, 100), \quad \beta_7' = (0, 50, 100)
\]

\[
\beta_{100}' = (5, 10, 15) \quad \text{and} \quad \beta_{150}' = (3, 3, 3),
\]

and \( \mathbf{e} \) is a \( 300 \times 3 \) matrix with i.i.d. rows distributed as

\[
N_3 \left( \begin{array}{ccc} 0 \\ 0 \\ 0 \end{array} \right), \quad \left( \begin{array}{ccc} 1.0 & 0.5 & 0.3 \\ 0.5 & 1.0 & 0.2 \\ 0.3 & 0.2 & 1.0 \end{array} \right).
\]

We estimated the tuning parameter \( \lambda \) by using 5-fold cross-validation. The Figure 1 shows the marker effects \( \|\beta_j\|, j = 1, \ldots, 200 \). We see that the LAD-lasso method correctly finds all four QTLs.

We then studied the bias of the LAD-lasso estimates of the non-zero coefficient vectors \( \beta_{50}, \beta_{75}, \beta_{100} \) and \( \beta_{150} \):

\[
\begin{pmatrix}
\hat{\beta}_{50}' - \beta_{50}' \\
\hat{\beta}_{75}' - \beta_{75}' \\
\hat{\beta}_{100}' - \beta_{100}' \\
\hat{\beta}_{150}' - \beta_{150}'
\end{pmatrix} = \begin{pmatrix}
-0.15 & -0.19 & -0.08 \\
-0.09 & -0.08 & -0.13 \\
0.00 & 0.04 & -0.12 \\
-0.29 & -0.22 & -0.34
\end{pmatrix}
\]
Figure 1: Marker effects of the multioutcome LAD-lasso estimates. The red ticks show the locations of the four QTLs (markers 50, 75, 100 and 150).

Figure 2: Marker effects of adaptive multioutcome LAD-lasso estimates. The red ticks show the locations of the four QTLs (markers 50, 75, 100 and 150).
We see that the LAD-lasso estimates are noticeably biased.

Then we used the adaptive multioutcome LAD-lasso estimation method. The Figure 2 shows the marker effects $\|\beta_j\|$, $j = 1, \ldots, 200$. We see that the adaptive multioutcome LAD-lasso method correctly finds the QTLs.

The biases of the non-zero coefficient vectors were in this case

$$
\begin{pmatrix}
\hat{\beta}'_{50} - \beta'_{50} \\
\hat{\beta}'_{75} - \beta'_{75} \\
\hat{\beta}'_{100} - \beta'_{100} \\
\hat{\beta}'_{150} - \beta'_{150}
\end{pmatrix}
= 
\begin{pmatrix}
-0.06 & -0.10 & 0.01 \\
0.00 & 0.12 & 0.17 \\
0.03 & 0.15 & 0.02 \\
-0.17 & -0.09 & -0.21
\end{pmatrix}
$$

We can see that the estimates are now less severely biased. The Figure 3 shows that the biases of the adaptive regression estimates are scattered around zero but the non-adaptive regression estimates are scattered around -0.15.

We then constructed a simple robustness study. We multiplied $y_{10}$ and $y_{292}$ by 100 and calculated the adaptive multioutcome LAD-lasso estimates. The Figure 4 shows that the adaptive method finds the QTL’s also in the contaminated data case.

![Figure 3: Box-plots of the biases of the regression coefficient estimates](image)

The biases of the regression coefficients are now

$$
\begin{pmatrix}
\hat{\beta}'_{50} - \beta'_{50} \\
\hat{\beta}'_{75} - \beta'_{75} \\
\hat{\beta}'_{100} - \beta'_{100} \\
\hat{\beta}'_{150} - \beta'_{150}
\end{pmatrix}
= 
\begin{pmatrix}
-0.08 & -0.12 & -0.00 \\
-0.02 & 0.15 & 0.16 \\
0.01 & 0.11 & 0.00 \\
-0.20 & -0.11 & -0.23
\end{pmatrix}
$$

which indicates that the outliers had only a minor effect on the biases.

4.2 Excess of zeros

In this study, we simulate a different data set to demonstrate the excess of zeros scenario with multioutcome adaptive LAD-lasso and regular LAD-lasso.
Figure 4: Marker effects of adaptive multioutcome LAD-lasso estimates for the contaminated data. The red ticks show the locations of the four QTLs (markers 50, 75, 100 and 150).

Multiple data sets are shown to illustrate the performance of algorithms in wide variety of situations. We generate two types of data sets, one with high number of observations compared to covariates \((n = 100, q = 10)\), and contrarily one with high number of covariates when compared to observations \((n = 25, q = 50)\). Moreover, we alternate the proportion of zeros in the covariates from 0.1 to 0.4 and apply two different error \(E\) distributions: uniform and asymmetric Laplace. When all the combinations of these properties are consider, in total of 16 different types of scenarios are examined. In each scenario we simulate 100 data sets to further evaluate the impact of the methods. The data sets and the alternating proportion of zeros is shown in Figure 5.

The data sets are generated with multioutcome regression model

\[
Y = XB + E, 
\]

with matrix \(X\) of a size \(n \times q\), response matrix \(Y\) of a size \(n \times 2\). The observed values \(x_{ij}\) are simulated from a normal distribution such as

\[
x_{ij} \sim \begin{cases} 
N(3, 1), & \text{if } u_{ij} > p_{zeros}, \\
0, & \text{else},
\end{cases}
\]

where \(u_{ij} \sim Unif(0, 1)\) and \(p_{zeros}\) is the proportion of zeros.

\(B\) is a \(q \times 2\) matrix where

\[
b_{jk} \sim \begin{cases} 
N(0, 1), & \text{when } j = 1, 2, 3, \\
0, & \text{else},
\end{cases}
\]

\(E\) is a \(n \times 2\) matrix where

\[
e_{jk} \sim Unif(0, 1)
\]
or
Figure 5: Simulated data replicates in the excess of zeros study with different proportions of zeros.

\[ e_{jk} \sim \text{ALaplace}(\mu = [3, 6], \Sigma = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}) \]

depending of the choice for error distribution. Above, ”ALaplace” stands for Asymmetric Laplace distribution.

From the simulated data sets, we can observe that the adaptive LAD-lasso is superior to non-adaptive in every scenario, when we compare the correctly found zero coefficients (Figure 6). In scenario \( q > n \) the difference between the methods is much smaller.

5 Concluding remarks

We have shown here that the multioutcome adaptive LAD-lasso is a versatile robust tool and is capable of reducing bias from effect estimates as well as alleviating the influence from the problem of excess of zero values in continuous covariates. The competitive performance of adaptive version compared to multioutcome LAD-lasso has been illustrated with several examples. In the future, it would be interesting to develop a robust BIC criterion for multivariate LAD-lasso context, in order to be able to account for linear dependencies between outcomes.

6 Acknowledgments

This research is supported by the Infotech Oulu spearhead project funding.
Figure 6: Percentage of correctly found zero coefficients with adaptive LAD-lasso and regular LAD-lasso in multivariate context.

References

Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B*, 58(1):267–288, 1996. URL [http://www.jstor.org/stable/2346178](http://www.jstor.org/stable/2346178).

Zitong Li and Mikko J. Sillanpää. Overview of lasso-related penalized regression methods for quantitative trait mapping and genomic selection. *Theoretical and Applied Genetics*, 125(3):419–435, 2012. URL [http://dx.doi.org/10.1007/s00122-012-1892-9](http://dx.doi.org/10.1007/s00122-012-1892-9).

Bradley Efron, Trevor Hastie, Iain Johnstone, and Robert Tibshirani. Least angle regression. *The Annals of Statistics*, 32(2):407–499, 2004. URL [http://dx.doi.org/10.1214/009053604000000067](http://dx.doi.org/10.1214/009053604000000067).

Nicolai Meinshausen. Relaxed lasso. *Computational Statistics & Data Analysis*, 52(1):374–393, 2007. URL [http://dx.doi.org/10.1016/j.csda.2006.12.019](http://dx.doi.org/10.1016/j.csda.2006.12.019).

Hui Zou. The adaptive lasso and its oracle properties. *Journal of the American Statistical Association*, 101(476):1418–1429, 2006. URL [http://dx.doi.org/10.1198/016214506000000735](http://dx.doi.org/10.1198/016214506000000735).

Shuheng Zhou. Thresholded lasso for high dimensional variable selection and statistical estimation. *arXiv.org Preprint*, 2010. URL [http://arxiv.org/pdf/1002.1583v2.pdf](http://arxiv.org/pdf/1002.1583v2.pdf).

Sara van de Geer, Peter Bühlmann, and Shuheng Zhou. The adaptive and the thresholded lasso for potentially misspecified models (and a lower bound for the lasso). *Electronic Journal of Statistics*, 5:688–749, 2011. URL [http://dx.doi.org/10.1214/10-EJS624](http://dx.doi.org/10.1214/10-EJS624).

Hansheng Wang, Guodong Li, and Guohua Jiang. Robust regression shrinkage and consistent variable selection through the LAD-lasso. *Journal of Business & Economic Statistics*, 25:347–355, 2007. URL [http://dx.doi.org/10.1198/073500106000000251](http://dx.doi.org/10.1198/073500106000000251).

Jyrki Möttönen and Mikko J. Sillanpää. Robust variable selection and coefficient estimation in multivariate multiple regression using LAD-lasso. In Klaus Nordhausen and Sara Taskinen, editors, *Modern Nonparametric, Robust and Multivariate Methods -Festschrift in Honour of Hannu Oja*, pages 235–247. Springer, 2015. URL [http://dx.doi.org/10.1007/978-3-319-22404-6_14](http://dx.doi.org/10.1007/978-3-319-22404-6_14).

Zitong Li, Jyrki Möttönen, and Mikko J. Sillanpää. A robust multiple-locus method for quantitative trait locus analysis of non-normally distributed multiple traits. *Heredity*, 115:556–564, 2015. URL [http://dx.doi.org/10.1038/hdy.2015.61](http://dx.doi.org/10.1038/hdy.2015.61).
Ming Yuan and Yi Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society. Series B*, 68(1):49–67, 2006. URL http://dx.doi.org/10.1111/j.1467-9868.2005.00532.x.

Olcay Arslan. Weighted LAD-lasso method for robust parameter estimation and variable selection in regression. *Computational Statistics & Data Analysis*, 56(6):1952–1965, 2012. URL http://dx.doi.org/10.1016/j.csda.2011.11.022.

Hansheng Wang and Chenlei Leng. A note on adaptive group lasso. *Computational Statistics & Data Analysis*, 52 (12):5277–5286, 2008. URL http://dx.doi.org/10.1016/j.csda.2008.05.006.

Berwin A. Turlach, William N. Venables, and Stephen J. Wright. Simultaneous variable selection. *Technometrics*, 47 (3):349–363, 2005. URL http://dx.doi.org/10.1198/004017005000000139.

Ming Yuan, Ali Ekici, Zhaosong Lu, and Renato Monteiro. Dimension reduction and coefficient estimation in multivariate linear regression. *Journal of the Royal Statistical Society. Series B*, 69(3):329–346, 2007. URL http://www.jstor.org/stable/4623272.

Hannu Oja. *Multivariate Nonparametric Methods with R - An Approach Based on Spatial Signs and Ranks*. Springer, New York, 2010. URL http://dx.doi.org/10.1007/978-1-4419-0468-3.

Klaus Nordhausen, Jyrki Möttönen, and Hannu Oja. MNM: Multivariate nonparametric methods - An approach based on spatial signs and ranks, 2016. URL http://CRAN.R-project.org/package=MNM, R package version 1.0-2.

Klaus Nordhausen and Hannu Oja. Multivariate $L_1$ methods: The package MNM. *Journal of Statistical Software*, 43 (5):1–28, 2011. URL http://dx.doi.org/10.18637/jss.v043.i05.

Tero Lähderanta and Jyrki Möttönen. Multoutcome LAD-lasso. https://github.com/terolahderanta/multioutcome_lad_lasso, 2021.