Quantum quench in matrix models: Dynamical phase transitions, Selective equilibration and the Generalized Gibbs Ensemble

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Abstract

Quantum quench dynamics is considered in a one dimensional unitary matrix model with a single trace potential. This model is integrable and has been studied in the context of non-critical string theory. We find dynamical phase transitions, and study the role of the quantum critical point. In course of the time evolutions, we find evidence of selective equilibration for a certain class of observables. The equilibrium is governed by the Generalized Gibbs Ensemble (GGE) and differs from the standard Gibbs ensemble. We compute the production of entropy which is $O(N)$ for large $N$ matrices. An important feature of the equilibration is the appearance of an energy cascade, reminiscent of the Richardson cascade in turbulence, where we find flow of energy from initial long wavelength modes to progressively shorter wavelength excitations. We discuss possible implication of the equilibration and of GGE in string theories and higher spin theories. In another related study, we compute time evolutions in a double trace unitary matrix model, which arises as an effective theory of D2 branes in IIA string theory in the confinement phase. We find similar equilibrations and dynamical transitions in this matrix model. The dynamical transitions are related to Gregory-Laflamme transitions in string theory and are potentially connected with the issue of appearance of naked singularities.
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1 Introduction and Summary

There has been a lot of progress recently in understanding dynamical processes in statistical mechanics models as well as in large $N$ gauge theories. An important development in statistical mechanics has been an extensive study of equilibration in integrable systems \cite{1}. In these studies it is found that equilibration happens only for a certain class of observables (whose characterization remains an open problem), and when it does, it is not described by the standard Gibbs ensemble but rather by a generalized Gibbs ensemble (GGE) which has ‘chemical potentials’ for each conserved charge. There are a plethora of examples in string theory in which integrable field theories appear, including many which have gravity duals (see below). A question naturally appears whether we can understand equilibration in these models via GGE, and if so, how to interpret these in gravity. One of the purposes of this paper is to initiate a study in this direction by studying Matrix Quantum Mechanics (MQM) models, which appear in a variety of contexts in string theory, as discussed below.

Besides the above issue, MQM offers simple toy models to study dynamical phase transitions. Compared to some of the previously studied statistical mechanics models, MQM has an advantage that it has a simple large $N$ limit ($N$ is the rank of the matrix) which is described by a semiclassical Fermi liquid. The dynamics of the eigenvalue density of the matrix (whose gaps characterize the thermodynamic phases) is inferred in this limit by the classical motion of these droplets. In this paper, we study dynamics of a quantum quench, both for finite $N$ (numerically) and in the semiclassical large $N$ limit.

The MQM (matrix quantum mechanics) we will study is the following \(^1\)

\[ Z = \int DU(t) e^{i N S}, \quad S = \int dt \left[ \frac{1}{2} \text{Tr} \left( \partial_t U \right)^2 - V(U) \right], \quad (1) \]

\(^1\)In AdS/CFT and similar studies, thermal equilibrium has often been described in terms of black holes, which normally possess only a few parameters. A GGE, with an infinite number of chemical potentials, presents a novelty \textit{vis-a-vis} this standard paradigm (see Section \ref{section4} for a detailed discussion).

\(^2\)We will restrict ourselves to excitations in the singlet sector of the model. This can be precisely achieved \(^2\) by coupling to a gauge field $A_0$: $\partial_t U \rightarrow D_t U$, $D_t = \partial_t - i A_0$ or by restricting to low energy dynamics \(^3\) which cuts off the non-singlets. For our purposes, we may consider (20) and (35) as the definition of the single-trace and double-trace models respectively.
We will focus on two cases:

(i) the single-trace model, given by
\[ V(U) = \frac{a}{2} (\text{Tr} U + \text{Tr} U^\dagger), \]
and (ii) the double-trace model, given by
\[ V(U) = -\frac{\xi}{N} (\text{Tr} U)(\text{Tr} U^\dagger). \]

Here \( U(t) \) is a unitary matrix of rank \( N \). The model (2) was first introduced in [2] in the context of 2+1 dimensional gauge theory. We will also discuss a related hermitian matrix model (25) which shares similar universal features with (2). We will discuss, in Section 4 string theory/gravity duals of these models, in particular the connection between (2) and the non-critical 2D string theory, and (3) and the near-horizon D2 brane geometry.

We list below a few highlights of our results:

(i) We find evidence of equilibration\(^5\) in the time-evolution of certain observables (\( e.g. \) moments of the eigenvalue distribution) in both the MQM examples above. In case of the single-trace models, which is integrable, we find excellent agreement of the equilibrium configuration with that predicted by GGE (see Fig. 1). We also give simple examples of observables which do not equilibrate.

(ii) We are able to give an interpretation of the equilibration in a special case where an initial long wavelength perturbation dissipates into progressively shorter and shorter wavelength perturbations, as in the Richardson-Kolmogorov cascade in turbulence. (See Fig. 2).

(iii) A surprising conclusion of our semiclassical analysis of the quantum quench is that dynamical transitions from the gapless phase to a gapped phase is possible, but not the other way around. In terms of the semiclassical Fermi liquid description (see Section 2.4) the interpretation of this is that Fermi liquid droplets cannot split. In the gravity context, such as in a Gregory-Laflamme transition between a black string and a black hole, these observations have a potential relation to a change of topology of the horizon. See Section 4.3 for more details.

The rest of the paper is organized as follows. In Section 2 we discuss the integrable single trace model and its dynamics, including equilibration and GGE, entropy production, energy cascades and dynamical phase transitions. In Section 3 we discuss the double trace model, including its relation to gauge theory, dynamical phase transitions and equilibration/oscillations for various initial conditions. In Section 4 we discuss implication of our results for the string theory/gravity duals of these models and possible generalization to a larger class of models. In Appendix A we discuss the fermion picture of the matrix models, and the semiclassical Fermi liquid description at large \( N \). In Appendix B we discuss the notion of the generalized Gibbs

\(^5\)For other, recent, discussions of quantum quench in AdS/CFT, see [4] and references therein.

\(^4\)The movies for our time evolutions are available on http://www2.yukawa.kyoto-u.ac.jp/~mtakeshi/MQM/index.html

\(^3\)For some other works on thermalization in matrix models, see Refs [5] [6] [7], which consider BFSS and pp-wave matrix models.
ensemble in integrable models and its application to the single trace matrix quantum mechanics. In Appendix C we discuss the formalism of quantum quench dynamics in these models, and present explicit formulae for computing various relevant time-dependent quantities. In Appendix D we discuss a common limiting case of both the single-trace and double-trace models in which the fermions are free and are without a potential; the density profile has a nontrivial nonlinear dynamics and shows, in this case, a power law decay to equilibrium. In Appendix E we present some details of our numerical analysis used to compute time evolution.

2 Single trace model

The single trace model (2), as explained in Appendix A and in the references cited there, can be equivalently formulated in terms of $N$ noninteracting fermions moving in a one dimensional spatial circle with an external potential $V(\theta) = a \cos \theta$. The hamiltonian, in the second quantized form, is (see (20))

$$H = \int_{-\pi}^{\pi} d\theta \psi^\dagger(\theta,t)h\psi(\theta,t), \quad h = -\frac{1}{2N^2} \partial^2_\theta + V(\theta), \quad V(\theta) = a \cos \theta.$$  \hspace{1cm} (4)

The fermion density

$$\rho(\theta,t) = \frac{1}{N} \psi^\dagger(\theta,t)\psi(\theta,t), \quad \int_{-\pi}^{\pi} d\theta \rho(\theta,t) = 1,$$

is the same as the eigenvalue density of $U(t)$. Depending on the parameter $a$, there are two different static phases (ground states) in the system at large-$N$: (i) gapless for $a < a_c = \frac{\pi^2}{64}$, and (ii) gapped for $a > a_c$, where the ‘gap’ refers to that in the equilibrium value of $\rho(\theta)$ (see Fig. 4 for various shapes of $\rho(\theta)$). The quantum critical transition at $a_c$ is a third-order transition \cite{2} \cite{6} (see below (24)).

In what follows, we will analyze various types of dynamics of the above system:

(i) We study time evolution of some arbitrarily chosen initial state (corresponding to a suitable density profile) under the hamiltonian (4).

(ii) Quantum Quench Dynamics (QQD) (see Section C for details): we start, in the far past, with the ground state of the Hamiltonian (4) with an initial parameter $a = a_i$; we then change $a$ suddenly to $a_f$ at $t = 0$, and study the $t \geq 0$ evolution of this initial state and various expectation values under the hamiltonian (4) with parameter $a = a_f$. We can regard this as a special case of (i) where the initial state is provided by the ground state of the system in the far past.

In this section, we just show the results of these dynamics, and the details of the calculations are summarised in appendixes.

This is similar to the Gross-Witten-Wadia transition \cite{8} \cite{9} of zero-dimensional single-trace unitary matrix model.
2.1 Integrability and equilibration

Since the Hamiltonian (4) describes free fermions, it is clearly integrable. Indeed we can find the conserved quantities as follows (see Appendix B for details). Let us expand the fermion field as

$$\psi(\theta) = \sum_{m=0}^{\infty} c_m \varphi_m(\theta), \quad h \varphi_m(\theta) = \epsilon_m \varphi_m, \quad (5)$$

where $$\varphi_m$$ is the $$m$$-th eigenfunction of the single particle Hamiltonian $$h$$ (4) with the eigenvalue $$\epsilon_m$$ and $$c_m$$ is the corresponding annihilation operator. The fermion occupation number operators

$$N_m = c_m^\dagger c_m, \quad (m = 0, 1, \ldots), \quad (6)$$

are clearly all conserved, and all independent in the $$N \to \infty$$ limit.

A first guess about a free system such as this would be that no observable can possibly equilibrate since there presumably cannot be any dissipation. Indeed, such a guess would seem to have a lot of merit. Firstly, there are infinite number of charges, such as the $$N_m$$, which, given any initial state however complicated, clearly stay constant as the system evolves. Secondly, it is easy to construct a whole series of observables which do not remain constant but keeps oscillating for ever; as an example consider the operator

$$\hat{O}_{m,n}(t) = c_m^\dagger c_n \exp[-i(\epsilon_m - \epsilon_n)t/\hbar], \quad m \neq n. \quad (7)$$

Clearly, given any state $$|\Psi(t)\rangle$$, the expectation value of the above operator evolves as (using the Heisenberg picture)

$$\langle \hat{O}_{m,n}(t) \rangle \equiv \langle \Psi(0)|\hat{O}_{m,n}(t)|\Psi(0)\rangle = V_{m,n} \exp[-i(\epsilon_m - \epsilon_n)t/\hbar], \quad V_{m,n} \equiv \langle \Psi(0)|c_m^\dagger c_n|\Psi(0)\rangle. \quad (8)$$

Thus, the expectation value keeps oscillating ad infinitum (unless $$V_{m,n} = 0$$ in the given initial state, in which case the value remains zero). It is obvious how to generalize such examples, e.g. by combining many creation-annihilation pairs.

**Behaviour of $$\rho(\theta,t)$$:** In the light of the above discussion, we find surprises when we consider the time evolution of the density $$\rho(\theta,t)$$ or its various ‘moments’, defined by

$$\langle \rho_n(t) \rangle = \int d\theta \langle \rho(\theta, t) \rangle \cos(n\theta). \quad (9)$$

In (57) a finite-$$N$$ expression for $$\langle \rho(\theta, t) \rangle$$ is presented for a quantum quench (defined briefly below (4) and discussed in detail in Section C). Using this expression, it is easy to compute the time-dependence of various moments $$\rho_n(t)$$. In Fig.1 (a) we have plotted $$\rho_1(t)$$ for $$N = 120$$. In the same figure we have also plotted the $$N \to \infty$$ limit of $$\rho_1(t)$$, computed using (9) and the
expression (33), the latter being derived using the large $N$ semiclassical limit of the fermion theory (4). From these plots, the evidence of equilibration is quite clear; while for finite $N$, there appears to be relaxation followed by a recurrence (with the recurrence time increasing with $N$ as shown in Fig. 1(b)), at large $N$, $\rho_1(t)$ appears to relax to a certain value.

![Figures 1(a) and 1(b)]

Figure 1: The time evolution of the first moment $|\langle \rho_1(t) \rangle|$ (defined in (9)) for a quantum quench is shown. We take $a_i = 0.8a_c$ initially and change it to $a_f = 1.2a_c$ instantaneously at $t = 0$. (a) The (red) dashed line shows the result for $N = 120$, which appears to show Poincare cycles. The Poincare cycles seem to grow linearly with $N$ as shown in (b). The (blue) solid line in (a) is for $N = \infty$ (computed using the droplet formalism, see Section A and E); it shows equilibration and absence of Poincaré cycles. The mean value of $|\langle \rho_1(t) \rangle|$ for finite $N$ and its asymptotic value for $N = \infty$ are seen to have excellent agreement with each other and, in turn, with $|\langle \rho_1 \rangle_{\text{GGE}}|$ as computed in GGE, (see Section B for details of the computation), whereas these values all differ significantly from the ensemble average in the standard Gibbs ensemble. We have verified that the $N$ dependence of the $\langle \rho_1 \rangle_{\text{GGE}}$ is small.

**Understanding of the equilibration in terms of GGE:** Since in view of the discussion around Eqn. (8), the above equilibration is surprising, it is useful to look for similar phenomena in other examples. Fortunately, in recent years several examples of selective equilibration have been found in integrable systems (see [1] for a review), in which a certain class of observables has been shown to equilibrate in such integrable systems (which include the hard core bosonic lattice and the transverse field Ising model), and the equilibrium configuration is characterized not in terms of the standard thermal (Gibbs) ensemble, but by a Generalized Gibbs Ensemble (GGE) which keeps track of the infinite number of conserved quantities by means of an infinite number of chemical potentials.\(^7\) In our present model, the GGE is defined by the density matrix

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp \left( -\sum_m \mu_m N_m \right),$$

\(^7\)The infinite number of chemical potentials can sometimes be thought of as a separate ‘temperature’ for every mode, see, e.g. [10].
where $\mu_m$ is the chemical potential corresponding to the conserved fermion number $N_m$ (see sections 3 and C.1 for details). Using this density matrix, we are able to compute the postulated equilibrium value of the density $\langle \rho(\theta) \rangle_{\text{GGE}} = \text{Tr}(\rho_{\text{GGE}} \rho(\theta))$ and hence, through (9), that of the various moments $\langle \rho_n \rangle_{\text{GGE}}$. In Fig. 1 we have shown the value of $\langle \rho_1 \rangle_{\text{GGE}}$; we find that the large $t$ asymptotic value of $\langle \rho_1(t) \rangle$ computed at large $N$ fits very well with the GGE-value. For comparison, we have also displayed in the same figure the value of $\rho_1$ computed according to the Gibbs ensemble, which differs significantly from both the asymptotic value of $\langle \rho_1(t) \rangle$ as well as $\langle \rho_1 \rangle_{\text{GGE}}$. We have verified these statements for higher moments $\rho_2, \rho_3$ etc. as well. We find, therefore, that the moments $\rho_n(t)$ belong to the class of observables in our system which equilibrate, and the equilibrium configuration can be quantitatively understood in terms a GGE.

So far we have seen two classes of observables. One, like (7), never equilibrates and another, like $\rho_n$ above, equilibrate to values described by GGE. It is, therefore, a subtle issue whether the whole system should be regarded as equilibrated or not. Indeed the criterion for equilibration in quantum systems has a long history (see, e.g. [1] for a recent history). One way to define it is considering only macroscopic quantities.

In our model, we should regard observables like (7) as microscopic, since they connect a given energy eigenstate to a few (only one in the precise example of (7)) eigenstate(s), and should regard $\rho_n$ as macroscopic, since they connect a given eigenstate to an infinite number of other eigenstates (these observables are also moments of the fermion density which is clearly a macroscopic, collective, field). It appears, therefore, that the following equation is satisfied for macroscopic observables $O$

\[
\text{Tr}(\rho_{\text{true}} O(t)) \rightarrow \text{Tr}(\rho_{\text{GGE}} O(t)), \quad \text{as } t \rightarrow \infty,
\]

where $\rho_{\text{true}} = |\Psi(0)\rangle\langle \Psi(0)|$ is the density matrix corresponding to the initial state at $t = 0$. Thus, according to the above criterion, we conclude that our model equilibrates to GGE.

2.2 Energy Cascade: an interpretation of the dissipation in the integrable model

To get more insight into the phenomenon of equilibration, let us consider the special case of $V = 0$ (i.e. $a = 0$) in (4). In stead of a quantum quench type initial configuration, let us in this

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8The present example provides evidence for relaxation for a quantum quench dynamics. However, we will find later the same phenomenon with other initial conditions; in particular see Section D where we provide an analytic proof of such relaxation in a special case.

9Another way is to define it in terms of long time averages of macroscopic quantities, as in the discussion of ergodic motion.

10Usually a criterion like eq. (10) is used for the standard Gibbs ensemble. Here we are proposing a generalization of that to the generalized Gibbs ensemble.
case start with an initial state $|\Psi_0\rangle$ which corresponds to a sinusoidal semiclassical configuration (see (27) for notation)\(^{11}\)

$$\rho(\theta, t = 0) = \frac{1}{2\pi} (1 + 2b \cos(n\theta)), \quad P(\theta, t = 0) = 0.$$  \hspace{1cm} (11)

The initial condition implies that only $\rho_n$ is excited at $t = 0$ with an amplitude $b$. By using the technique detailed in Appendix D\(^{11}\), we can compute the time evolution of the system (see Figure 13), and compute expressions for the moments $\rho_n(t)$ analytically. For instance, the late time behaviour of the first two moments (for the choice $n = 1$ in (11)) is given by

$$\rho_1(t) \to \frac{2}{t^{3/2}} \sqrt{\frac{2}{\pi b}} \cos \left( bt - \frac{3\pi}{4} \right) \cos \left( \frac{t}{4\pi} \right), \quad (t \to \infty),$$  \hspace{1cm} (12)

and

$$\rho_2(t) \to \frac{1}{t^{3/2}} \frac{1}{\sqrt{\pi b}} \cos \left( 2bt - \frac{\pi}{4} \right) \sin \left( t \right), \quad (t \to \infty).$$  \hspace{1cm} (13)

Indeed it turns out that ALL the moments die out as $t^{-3/2}$ (see Fig. 2\(^{12}\)).

It would appear surprising at first sight that ALL the moments die away!, leaving us apparently with the question ‘where does the energy go?’ Of course, since we have a conservative

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\(^{11}\)This kind of initial condition is chosen here only for simplicity; we expect the same qualitative behaviour for the quantum quench initial conditions as well.

\(^{12}\)The power-law relaxation may be a special feature of the simple $a = 0$ model. In the more general case, such as in Fig. \(\text{III}\) the relaxation appears to have a characteristic time scale.
model, even integrable to boot, energy cannot really ‘go’ anywhere. It turns out, just as studies in turbulence and various other areas have taught us, that there is a flow of energy from one set of ‘modes’ to another set of ‘modes’. To be precise, as the longer wavelength modes die, shorter wavelength modes start getting excited, and as they too die away, even shorter wavelengths get excited, etc. This phenomenon is similar to the Richardson-Kolmogoroff type cascade in turbulence, and provides a mechanism of dissipation in our integrable system.

2.3 Entropy

As we remarked above, our model equilibrates, in the sense of eq.(10). Since the initial density matrix $\rho_{\text{true}}$ in that equation describes a pure state, with zero von Neumann entropy, while the final density matrix on the right hand side describes a mixed state, with non-zero von Neumann entropy (see (44)), we can say that a non-zero entropy has been generated during the time evolution.

We should, of course, clarify here that (10) cannot be interpreted to mean $\rho_{\text{true}}(t) \equiv e^{-iHt} \rho_{\text{true}} e^{iHt} \xrightarrow{t \to \infty} \rho_{\text{GGE}}$! The correct interpretation of (10) is that the macroscopic observables $O(t)$ are suitably coarse-grained to smooth out the difference between $\rho_{\text{true}}$ and $\rho_{\text{GGE}}$ after a sufficiently long time. It is in these sense that we can regard that the entropy $S_{\text{GGE}}$ has been produced.

Amount of entropy production: The von Neumann entropy of the final state, $S_{\text{GGE}} = -\text{Tr} \rho_{\text{GGE}} \ln \rho_{\text{GGE}}$, is computed in (44). Using this equation and (59) we can compute the entropy $S_{\text{GGE}}$ characterizing the GGE for a quantum quench. In Fig. 3 we show the result of this computation as a function of $N$. From the plots we can see that $S_{\text{GGE}} \propto N$\footnote{We have also verified that in units of (4), the conserved energy of the system is $E = O(N)$ and the temperature is $T = O(1)$.}. This could be roughly expected from the fact that there are $N$ non-interacting particles in the system. We discuss below an analytic calculation of the entropy. In Section 4 we discuss this $O(N)$ entropy in terms of various string theory duals.

An analytic calculation of entropy We will now present an analytic, large $N$, computation of the entropy $S_{\text{GGE}}$ in the $V = 0$ case argued in section 2.2. As shown in Appendix D we can evaluate the entropy as

$$S = 2.36 \, b \, N$$

(14)

Note that $S$ is exactly linear in $N$ (as well as in the amplitude $b$ of the initial perturbation (11)).
2.4 Dynamical phase transitions

We mentioned above that if we tune the parameter $a$ in Fig. 4, say from zero upwards, we encounter a third order quantum phase transition at $a = a_c = \pi^2/64$ from a gapless phase to a gapped phase. What happens if we make this change of $a$ dynamically, say from $a_i < a_c$ to $a_f > a_c$? If we prepared the system in the ground state initially (corresponding to $a = a_i$), does it eventually relax to the final ground state (corresponding to $a = a_f$)? Do we observe dynamical appearance of a gap at a certain time? What role does the critical point $a_c$ play, if any? What happens if we make the change from $a_i > a_c$ to $a_f < a_c$?

The general setup for addressing these questions is to consider a dynamical variation with a tunable ‘ramp speed’, e.g. in the Kibble-Zurek transition (see, e.g., the review [1]). However, in the present work, we will consider two special cases, namely (a) the quantum quench dynamics discussed in the previous section, in which the change from $a_i$ to $a_f$ happens instantaneously, and (b) the adiabatic case, in which the change happens at a rate slower than the scale set by energy level spacings. We will first address the question of appearance/disappearance of a gap in the final state in case we cross the critical value $a_c$ from an initially gapless/gapped phase.

2.4.1 One-way phase transitions

Fig. 4 shows, for specific values of $a_i, a_f$, the large $t$ asymptotic density $\rho(\theta, t)$ for quantum quench dynamics (QQD), calculated from (57), as well as the same quantity for adiabatic

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\[14\] In this section, we use GGE to evaluate the asymptotic values of the observables in the QQD, although, at finite $N$, the actual values keep oscillating slightly around the GGE predictions as shown in Fig. 1 (b).

\[15\] Since the energy level spacings become zero at large-$N$, the adiabatic approximation works only if we tune the change of $a$ slower than $O(N)$.
Figure 4: A plot of density $\langle \rho(\theta) \rangle$ vs $\theta$ at $N = 120$. In the left panel, $a_f > a_c > a_i$; the green curve shows the initial gapless density as a function of $\theta$, the blue curve shows the gapped density corresponding to the final ground state for $a = a_f$, and the red curve shows the large $t$ asymptotic value of $\rho(\theta, t)$ in quantum quench dynamics. Since no gap appears, no dynamical phase transition occurs in the quench case. In the right panel, $a_f < a_c < a_i$, the initial density is gapped, whereas both the final configuration in the actual dynamics as well as the final ground state show a gapless phase, showing the presence of a dynamical phase transition in this case.

dynamics (where the time-dependent wavefunction is approximated by the instantaneous ground state), as calculated in (62). We find that there is a rather remarkable ‘time-asymmetry’ in the QQD, viz. that while a gapped → gapless transition is possible, a gapless → gapped transition is impossible. In other words, in the context of QQD, while a gap can close dynamically, it cannot open dynamically.\footnote{Of course, this is not a real time-asymmetry in the sense that, for $a_i < a_c < a_f$ if we start from the rather complicated, gapless, final pure state, and reverse the direction of time, we will indeed get back the initial gapped pure state. However, in the context of QQD, we always start from a ground state, in either direction, leading to the apparent time-asymmetry.}

In Fig. 5 we present more details about the gap at varying values of the $a$-parameter. As seen in Fig. 4, for $a > a_c$ the gap in the ground state density $\rho(\theta)$ opens around $\theta = 0$ (because the potential $a \cos \theta$ draws the fermions from $\theta = 0$ towards $\theta = \pi$, as explained below (24)). Thus, the value of the density at $\theta = 0$ can be regarded as an order parameter which is non-zero in the gapless phase and zero in the gapped phase.\footnote{These statements refer to the large $N$ limit. For finite $N$, or in a double scaled limit, a non-zero fermion density exists in the ‘gapped region’.} In Fig. 4 we present this order parameter for various values of $a_f$ for QQD after relaxation and compare it with the ground state. The one-way nature of transition is quite clear in this plot as well. Note also the progressive departure from adiabaticity as $a_f$ is varied, starting from $a_i$, across $a_c$ to the other phase (this is more prominent in the left panel where we study the appearance of a gap). We will have more to say shortly about the departure from adiabaticity.
Figure 5: A plot of $\rho(\theta = 0)$ for various values of $a_f$ at $N = 120$. In the left panel, we begin in the gapless phase with $a_f = a_i < a_c$, and consider a continuous sequence of QGD evolutions in which $a_f$ is progressively increased across criticality to $a_f > a_c$: the ‘adiabatic’ curve, calculated according to (62), shows transition to a gapped phase (with $\rho(\theta = 0) \rightarrow 0$), whereas the actual dynamical evolution under quantum quench dynamics (QGD) shows that no gap appears dynamically ($\rho(\theta = 0)$ remains non-zero). In the right panel, both the adiabatic and QGD curves show disappearance of the initial gap, confirming the presence of a dynamical gapped $\rightarrow$ gapless transition. Note that the gap does not disappear at $a_c$ even in the ground state (adiabatic case). This is due to a finite $N$ effect.

2.4.2 A topological interpretation of the one-way transition from Fermi liquid drops

The apparent irreversibility in dynamical phase transitions, found above, turns out to have a beautiful semiclassical interpretation from the semiclassical Fermi liquid picture. For simplicity, let us consider the large $N$ limit of the single-trace hermitian model (25), which corresponds to replacing the potential $a \cos \theta$ in (4) by $-a\theta^2/2$ (defined together with a hard cut-off $\theta = \pm \theta_m$). Since in these models $\hbar = 1/N$, the large $N$ limit, given by (26), is given by the semiclassical picture of a Fermi liquid in phase space, which is described by a phase space density $u(\theta, p, t)$ which is either 0 or 1, and the filled regions $u = 1$ are described as ‘droplets’. There is a simple description of quantum quench dynamics in this picture, as already alluded to in Section 2.1 and described in detail in Section A.

As we describe in Fig. 6 the time evolution from the gapless phase towards the gapped phase involves starting with a single connected Fermi droplet in the Fermi sea. The evolution of the droplet follows the equation (31). It is not difficult to see that the hyperbolic motion of the fermions (in which fermions belonging to different hyperbolas move at different speeds) leads to squeezing and stretching of the droplet, although it can never split. Since $\rho(\theta, t) = \int \frac{dp}{2\pi} u(\theta, p, t)$, which can be interpreted as a kind of projection on to the $\theta$-axis, the impossibility of droplet splitting in phase space translates to the impossibility of a gap-opening transition in $\theta$-space. In
other words, a continuous \( u(\theta, p, t) \) means that there will always be some fermions in the phase space along the \( p \)-axis, hence \( \rho(\theta = 0) \) will remain non-zero. We explicitly see this in part I of the figure. On the other hand, when we consider time evolution from a gapped phase towards a gapless phase, even though the initially disconnected Fermi sea remains disconnected in phase space (again working with the same evolution equation (31)), we find that the projection on the \( \theta \)-axis can become gapless, thus ensuring the presence of a \( \text{gapped} \rightarrow \text{gapless} \) transition.

**Relation to Gregory-Laflamme transition:** Note that the above argument is robust and ensures that the asymmetric nature of dynamical transitions will not depend on the details of the potential \( V(U) \) in (1) (for example, it persists for the double trace potential, as we will see in Section 3). Now the choice of \( V(U) \) dictates the rate at which the phase space fluid is squeezed due to the dynamics of individual particles; what happens when \( V(U) \) is such that this squeezing effect is very strong? In this case, the neck will be stretched thin and its thickness will shortly become \( O(1/N) = O(\hbar) \). At this stage, the semiclassical droplet description will clearly be violated, and we will need to use the full quantum mechanical description. As discussed in Section 4.4, a gap-opening transition in the double trace matrix model is related to a Gregory-Laflamme transition in the dual string theory. In a GL transition, classical general relativity is not valid when the size of the ‘neck’ (of a black string or of a solitonic string) becomes Planck size. In case of the black string \( \rightarrow \) black hole transition, the breaking of the neck involves appearance of a naked singularity. In view of the discussion above, this singularity would be related to the size \( O(1/N) \) neck region in the matrix model. Thus we expect that the understanding of matrix models would be important to understand the issue of the Gregory-Laflamme transition. We will discuss this issue further in Section 4.4.

### 2.4.3 More on departure from adiabaticity

In this subsection, we come back to the issue of the role played by the quantum critical point in quantum quench dynamics (QQD). This has been discussed in several contexts: e.g. (a) how the asymptotic density of excitations scales as a function of the parametric distance from criticality when the initial hamiltonian is at criticality (see [11]), or (b) how the asymptotic QQD value of some observable or order parameter differs from the adiabatic value, as a function of (in our notation) \( a_f - a_c \) (for given fixed \( a_i \)). In the above subsections, we found many examples of differences between QQD and adiabatic evolution. We explore this aspect more in this subsection by studying the behaviour of \( \rho_1 \). We find that as \( a_f \) is taken further away from \( a_i \) towards the other side of the quantum critical point, the departure from adiabaticity becomes appreciable from near \( a_f \sim a_c \). We have found similar behaviour for the higher moments \( \rho_2, \rho_3 \) etc. as well. This phenomenon of departure from adiabaticity around the critical point is similar
Figure 6: Quantum quench dynamics of the filled Fermi sea in the phase space in the hermitian model (25).

The dashed lines are \( p = \pm \sqrt{a_f \theta} \), which are the separatrices for the single particle motion. In (I), the left panel shows the shape of the initial filled Fermi sea, corresponding to the ground state for \( a = a_i < a_c \). This is a gapless phase, hence the Fermi level is above the top and the droplet is a single connected one. The right panel shows an intermediate time-evolved configuration. After the quench, the modified Hamiltonian stretches the droplet, but it cannot split it, hence the eigenvalue density, which is a certain projection on the horizontal axis (see (27)), also cannot split. In (II), left panel, the initial \( a = a_i > a_c \) is in the gapped phase. The Fermi level is below the top, hence the left and right ‘wells’ are both filled up to the Fermi level, while remaining disconnected. Hence the filled Fermi sea has two disconnected segments (therefore, the eigenvalue distribution is gapped). The right panel shows the time-evolved configuration (according to \( a = a_f < a_c \)), which is such that although the droplet still consists of two disconnected parts in phase space, the projection on the horizontal axis is gapless, hence the eigenvalue density is gapless.
Figure 7: A plot of (the asymptotic value of) $\rho_1$ as a function of $a_f$. The setup is the same as that of Fig. 5. In the left panel, we start with the gapless phase $a_f = a_i = 0.8a_c$, and consider QGD experiments for progressively higher values of $a_f$ till we go across $a_f = a_c$ towards $a_f = 1.2a_c$. The blue curve shows the adiabatic value of $\rho_1$ (which corresponds to the ground state of the $a_f$-hamiltonian), while the red curve shows the large $t$ asymptotic value of $\rho_1(t)$, which agrees with the GGE result. The two curves start showing significant departure from around $a_f = a_c$. In the right panel, we start with the gapped phase with $a_f = a_i = 1.2a_c$ and end up with $a_f = 0.8a_c$; the colour coding is similar. Once again we find that departure from adiabaticity starts from around $a_f = a_c$. This figure is similar to, e.g., Fig. 3 of [12], where a similar pattern of departure from adiabaticity is reported for quantum quench in a Hubbard model.

This points to some possible universal role played by the quantum critical point even in dynamical phenomena.

3 Double trace model

Like the single trace model, the double trace model (3), can also be described (see (35)) in terms of $N$ fermions moving in one spatial dimension. The hamiltonian is given by

$$H = \int d\theta \frac{1}{N^2} \partial_\theta \psi^\dagger(\theta) \partial_\theta \psi(\theta, t) + H_{\text{int}},$$
$$H_{\text{int}} = \frac{\xi}{N} \int dt d\theta d\theta' [\psi^\dagger(\theta, t) \psi(\theta, t) \cos(\theta - \theta') \psi^\dagger(\theta', t) \psi(\theta', t)].$$

The difference from the single trace case is that the fermions are now interacting with a mutual $\xi \cos(\theta - \theta')$ potential. Since this is an interacting system, it is not obviously integrable; we will not explore the question of integrability in this work.

Before going to explore dynamics in this system, we will briefly review its connection to 2D adjoint scalar QCD and string theory.

\textsuperscript{18}The departure from the adiabatic curve does not exactly coincide with criticality in [12] or in our work here, an effect which is more pronounced for higher moments.
The 2D adjoint scalar QCD: Consider the following two dimensional gauge theory on a $S^1$ \cite{13},

$$
S = \int dt \int_0^L dx \text{Tr} \left( \frac{1}{2g^2} F_{tx}^2 + \sum_{I=1}^D \frac{1}{2} (D_\mu Y^I)^2 + \sum_{I,J} \frac{g'}{4} [Y^I, Y^J] [Y^I, Y^J] \right). \quad (16)
$$

Here $L$ is the period of the $S^1$. This theory is related to a dimensional reduction of $D + 2$ dimensional pure Yang-Mills theory (which, for $D = 8$, can be related to $N$ D2 branes compactified on a Scherk-Schwarz circle\footnote{If we derive the model (16) from the higher dimensional gauge theory or D branes, additional terms are induced through quantum loop effects. For this reason, the coupling at the two dimensional gauge field $g$ and the commutator interaction $g'$ have been distinguished in the model (16). The most relevant term in these loop corrections in weak coupling is the mass term for the adjoint scalar. We omit this mass in this paper as argued in \cite{13}.}). The thermodynamical phase structure of this model is as shown in Fig. 8. At sufficiently large $\beta$, (at least above the critical line $CBDO$ so that the system is confined, $\text{Tr} V = 0$), the adjoint scalars would have a mass gap $\Delta$ and are integrated out through the $1/D$ expansion \cite{13} (see also \cite{14, 15}). The theory is then described by the following effective action

$$
S/DN^2 = \int dt \left[ \frac{1}{2N} \text{Tr} \left( |\partial_t U|^2 \right) - \frac{\Delta}{\pi \lambda L} \sum_{n=1}^\infty \frac{K_1(n\Delta L)}{n} \left| \frac{1}{N} \text{Tr} U^n \right|^2 \right], \quad (17)
$$

where $U(t) = P \exp[i \oint A_1(t) dx]$, and $K_1$ is the modified Bessel function of the second kind.

In the following subsections, we would mainly consider dynamical phase transitions near the critical line $AB$ in Fig. 8. In this case, we can ignore the higher modes ($n > 1$) in the infinite sum in the potential term, and obtain the double-trace model (3) or (15). Thus the model (15) describes the dynamics of the spatial Polyakov loop $U(t)$ of the QCD (16) in the confined phase.

Here $\xi$ in (15) is a function of the parameters in the QCD, and importantly it monotonically decreases as $L$ increases \cite{13}.

**Phases:** The parameter $\xi$ determines the phase structure of the theory (15). For high enough $\xi$ ($\xi > \xi_2$), the attractive potential $S_{int}$ dominates and the system is gapped. For low enough $\xi$ ($\xi < \xi_2$), the kinetic term dominates and the system is gapless. See Fig. 8 (b) for more details.

### 3.1 Dynamical evolution

In case of the double trace model, several solutions have been derived for a given $\xi$ ($> \xi_0$) as shown in Fig. 8 (b) but the ground state is always unique. Hence we naively expect that the unstable or meta-stable solutions may evolve toward the ground state if we add sufficient perturbations to these solutions. We will explore the question of possible dynamical phase transitions.
\( \beta \)

\[
\begin{align*}
\text{Tr} U & \neq 0 \\
\text{Tr} V & \neq 0
\end{align*}
\]

(\text{IA})

(a) Phase structure of 2D adjoint scalar QCD \[ (16) \].

Figure 8: (a) Phases of the 2D QCD model \([16]\) are shown for various values of the inverse temperature \( \beta \) and spatial size \( L \). The phases are characterised by the expectation values of the temporal Polyakov loop \( V \) and the spatial Wilson loop \( U \). We are interested in the limit of zero temperature (non-compact time direction), in which the Polyakov loop \( V = 0 \); the only phase boundary of interest here is the line \( AB \), across which the spatial Wilson line \( U \) changes. The theory in this region is described by (3) or (15).

(b) The energy \( H = E \) of (15), computed using large \( N \) Fermi liquid picture, is plotted for various values of \( \xi \) (see [13, 16] for details). To connect with diagram (a), note that \( \xi \) is a monotonically decreasing function of \( L \). Solid lines represent the stable phase (the solution with the lowest value of \( E \) for a given \( \xi \)) as well as metastable phases, while the dotted lines represent unstable phases. At \( \xi_2 = 0.237 \) there is a 1st-order transition from a stable uniform distribution to a stable gapped one. Below \( \xi_0 = 0.227 \) the metastable gapped phase does not exist, similarly above \( \xi_3 = 1/4 \) the metastable uniform phase does not exist. At \( \xi_1 = 0.231 \) there is a Gross-Witten-Wadia (GWW) transition among two unstable phases.

transition as well as the question of relaxation in this process. In this section, we use the semiclassical approximation and consider only the large-\( N \) case. (See the details in Appendix \([E]\).)

### 3.1.1 Dynamical evolution from the uniform gapless phase

We will consider the time evolution of the uniform phase. In terms of the droplet picture (Section [A]), the uniform phase is given by

\[
\rho(\theta) = 1/(2\pi), \quad \mathcal{P} = 0. \tag{18}
\]

This configuration is always a solution of the model \([15]\), since the force term vanishes for particles located uniformly on the circle. In order to have non-trivial dynamics, we add a small perturbation to this configuration.

Before we discuss the actual results of the evolution below, let us pause and ask what to expect. If \( \xi \) in \([15]\) was small enough, say well below \( \xi_2 = 0.237 \), for such values of \( \xi \) the uniform phase is the ground state. Thus, we expect that for small enough amplitude of the perturbation, the system will revert back to the uniform phase. Indeed for \( \xi = 0 \), discussed in Section [2.2] we find this to be explicitly true.
On the other hand, we are interested here in values of $\xi$ above $\xi_2$ where the ground state is the gapped one. Thus, we will expect the perturbation to be unstable towards formation of a gapped phase.

**Order parameter:** Since all moments \( \rho \) of the density vanish in the uniform phase, any of these, say $\rho_1$, can be considered as an order parameter, which would change from $\rho_1 = 0 \to \rho_1 \neq 0$ if the distribution changes from uniform to a non-uniform solution. Note that we consider $Z_2$ symmetric $(\theta \leftrightarrow -\theta)$ perturbations only in this article. Then we can take $\rho_1 = \text{Tr}U/N$ by a gauge choice.

We will now discuss the actual result of the time-evolution of a perturbation according to (76). In Fig. 9 the result is presented for $\xi = 0.260$, at which the uniform solution (18) is unstable. Then $|\rho_1|$ develops to a non-zero value (see the right panel of the figure), and we obtain a non-uniform distribution. However the distribution does not develop a gap, unlike the naive expectation above. Indeed, just as in Fig. 6, part (I), the gapless $\to$ gapped transition is once again not possible since the droplet in phase space does not split. In addition to all this, we also find evidence for equilibration in $|\rho_1|$ (see the right panel) which indicates relaxation of the phase space configuration to one with a thin stable neck.

\[
\begin{array}{c}
\begin{array}{c}
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\includegraphics[width=0.3\textwidth]{fig9a}
\end{array}
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\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig9b}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{fig9c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Figure 9: Time evolution of the uniform solution (18) with a small perturbation at $\xi = 0.260$. The left panel shows the initial configuration. The configuration approaches a gapped phase, but develops a thin neck, like in Part (I) of Fig. 6, and the gap never develops. The right-most panels show the behaviour of the order parameter $\rho_1$ with time. It shows evidence of an equilibration.

### 3.1.2 Time evolution from the gapped phase

The discussion is analogous to the previous case. We consider a meta-stable gapped state at $\xi < \xi_2$ (Phase III in Fig. 8 (b)) and perturb the system.

The result at $\xi = 0.230$ is shown in Fig. 10. A dynamical $gapped \to gapless$ transition takes place. Thus, like in the single trace case, we once again see a one-way dynamical phase transition,

\^{20}The ‘neck’ avoids a gap-opening transition, and consequently avoids a dynamical singularity. It is important to explore this further, particularly in the context of the double scaled matrix model. For similar issues related to smoothening of singularities through $1/N$ effects, please see \[17] \[18].
gapped to gapless, but not gapless to gapped. This once again, has a clear interpretation in terms of phase space.

Note that if the perturbation is not sufficient, the state reverts back to the original gapped configuration, since the configuration is meta-stable.

Figure 10: The figure on the left shows a slightly perturbed meta-stable gapped distribution at $t = 0$. The value of $\xi$ is 0.23. The figure in the middle shows a gapless distribution at $t = 400$. The figure on the right depicts $\rho_1(t)$ as it changes from 0.4 at $t = 0$ to 0.

### 3.1.3 Appearance of exotic states

We have considered the time evolutions of unstable/meta-stable solutions by adding perturbations. However if we add sufficiently large perturbations, we observe appearance of exotic asymptotic states. See Fig. 11, e.g., where in the middle panel two blobs of fermions have appeared, which are moving towards the left and the right respectively. As time progresses, the two blobs keep executing a period motion and in coordinate space, this means that two density peaks approach, merge and separate out, and repeatedly pass through each other like colliding solitons (remember that in our model the fermions are moving in a circle).

Figure 11: Time evolution of the unstable uniform solution with a large perturbation at $\xi = 0.260$. The particles split into two well-defined blobs in phase space. The upper blob moves to the right while the lower blob moves to the left, and they keep moving past each other periodically. In coordinate space, the blobs overlap; this implies that two density peaks periodically pass through each other like colliding solitons.

Another exotic time evolution can be seen if we consider the higher modes of the potential (17). We assume appropriate values of $\Delta$ and small $L$ such that $\xi$ is large, consequently the gapped solution is stable and the uniform one (18) is unstable. In this setup, the unstable
uniform solution \(^{(18)}\) evolves as shown in Fig. 12. In this case, the dynamical phase transition consists of two steps. First the unstable uniform state decays to a 2-peak state (where the ‘peaks’ refer to those in the coordinate space density \(\rho(\theta)\)). This 2-peak state remains for some time, and finally decays to the one-peak state. We have also observed appearance of various multi-peak intermediate states if we change the initial perturbations.

Indeed the existence of unstable multi-peak solutions for the small \(L\) region in Fig. 8 (a) and the possibility of their appearance during the decays of the unstable uniform solution have been predicted in \(^{(19)}\). Our result confirms this conjecture. Interestingly similar time evolutions have been observed in the decay of black string in the Gregory-Laflamme transition also \(^{(20)}\; \text{21}\). An unstable black string does not always decay directly to a single-black hole, which is thermodynamically most stable, but to multiple black holes in some cases.

![Figure 12: The appearance of the 2 peak state during the time evolution of the uniform configuration.](image)

Figure 12: The appearance of the 2 peak state during the time evolution of the uniform configuration. We take \(\xi = 5.0\) and \(\Delta L = 1.0\). The strong signal of \(\rho_2\) indicates the two peaks in the eigenvalue. First the uniform state decays to the 2-peak state around \(t = 1\), and then it decays to one peak state around \(t = 15\). (We do not plot the one peak solution here but it is similar to Fig. [9].)

Note that the two-peak state that appears in Fig. 12 is qualitatively different from the breather-like asymptotic state in Fig. 11. The former state is approximately static (with two distinct, long-lived density peaks around \(\theta = \pm \pi/2\)) until it decays to a single-peak state, whereas the latter state is approximately stationary and the two peaks merge and separate out repeatedly in an oscillatory fashion. (See the movies for these evolutions from the link in footnote 11.)

As we have seen in section 2 the system is integrable if the double trace interactions are turned off. One open question is whether this interaction breaks the integrability. The appearances of these exotic asymptotic/intermediate states may shed light on this issue. Especially the exotic asymptotic states in Fig. 11 may imply an attractor structure in this system. We will come back to this issue in a future work.


4 Discussions

The results of our paper have already been summarised in Section 1 and described in Sections 2 and 3. In this section, we will describe some related issues, speculations and outstanding questions.

4.1 Duality to 2D string theory and $W_\infty$ algebra

The model defined by Eqs. (25), for fixed $a$, is essentially the $c = 1$ matrix model. This model (in its double-scaled form: $(a - a_c)N = \mu$), has a string theory ‘dual’, namely the two-dimensional bosonic string theory, which is characterized by a massless scalar field (called ‘tachyon’) on a flat spacetime and linear dilaton background (see, e.g. the review of [22]). The tachyon field fluctuation is given by a non-linear and non-local transformation of the matrix model scalar field fluctuation (see, e.g., [23, 24, 25] for more details). The filled Fermi sea, $\rho(\theta) = \frac{1}{\pi} \sqrt{2(\mu - V(\theta))}$, $P = 0$, is a solution of (28), and corresponds to the tachyon vacuum in string theory. In quantum quench dynamics, in the matrix model description, we start with an unstable density perturbation, which evolves in time under the restoring force towards the above-mentioned stable configuration. Since the motion is conservative (in fact, it has an infinite number of conserved quantities), the configuration cannot reach the stable configuration; instead there is an oscillatory motion, in which lower frequency modes transfer energy to the higher frequency ones, leading to an energy cascade as discussed in Section 2.2.

It would be interesting to map this oscillation to string theory tachyon modes by applying the transformation alluded to above. We expect that qualitatively it would be a similar phenomenon to the one found in the matrix model, with a finite energy extended configuration giving rise to proliferation of high frequency modes, asymptotically approximated by a general Gibbs ensemble (GGE) in spacetime. Note that in the 2D string theory too, there are an infinite number of conserved charges, which are the diagonal elements of an infinite dimensional algebra, viz. the $W_\infty$ algebra (see [26]), thus making the GGE a natural choice of ensemble. It is an interesting question whether the GGE can be represented by a new field configuration such as a non-trivial solitonic configuration, or a black hole. We discuss the black hole scenario in some detail in the subsection 4.3 below.

In [28], a different, type 0, 2D string theory is proposed as dual to the (gauged) hermitian model (25), where the singlet condition is ensured by a gauge covariant kinetic term $\text{Tr}((\partial_t + i[A_0, M])^2$). In the context of the type 0B theory, there are two massless scalars, the tachyon

\textsuperscript{\footnote{We show this in the unitary matrix model, but we find similar behaviour also in the Hermitian matrix model which is directly identified as the $c = 1$ matrix model.}}

\textsuperscript{\footnote{It has been argued that inflaton oscillations around its vacuum configuration, which is analogous to the tachyon oscillations described above, can produce primordial black holes (see [27], for example).}}

21

22
and an axion. We expect many of the comments in the previous paragraph to go through in this case as well.

4.2 Gravity duals of other integrable systems

The phenomena of selective equilibration and emergence of GGE studied in this paper would be of interest in many other examples of integrable systems with gravity duals. We mention below a few of them.

**D1-D5 system:** The D1-D5 system (reviewed in, e.g. [29]) is described, at a certain point \( P \) of its moduli space, by a free 2D CFT (with target space an orbifold of the type \( M^N/S(N) \), where \( M = T^4 \) or \( K3 \), and \( N = N_1 N_5 \)). The usual D1-D5 supergravity solution is located at a different point \( Q \) of the moduli space, which can be reached by introducing a marginal interaction \( \lambda \int O \) to the free CFT at \( P \). There are certain nonrenormalization theorems which ensure the protection, under the \( \lambda \)-deformation, of a number of quantities including black hole entropy and absorption cross-section. In [30] a certain response function in the \( \lambda = 0 \) integrable theory is exactly computed and shown to agree with its gravity dual, namely the absorption cross-section of the two-charge black hole (at the \( \lambda = \lambda_0 \) supergravity point). The agreement is found only in the large \( N \) limit and using a coarse-grained probe. The field theory calculation seems to be well-suited for a GGE description.

**Giant gravitons:** The half-BPS configurations of \( \mathcal{N} = 4 \) SYM on \( S^3 \times R \) are described by a complex matrix \( QM \) in a harmonic oscillator potential which can be described by free fermions [31, 32] in a manner similar to that described in this paper. The gravity duals are the well-known LLM geometries [33], which can be described in terms of giant gravitons [34, 35, 36] in the probe approximation.

**Higher spin duals of 2D CFT’s:** Recently, [37] has proposed a duality between the large \( N \) limit of certain 2D CFT’s called the \( W_N \) minimal models (represented by the coset WZW models \( SU(N)_k \times SU(N)_1/SU(N)_{k+1} \)) and a one-parameter higher spin (Vasiliev) theory parametrized by \( \lambda = N/(k+N) \) fixed, \( k, N \to \infty \) (see, e.g. the review [38]). The \( \lambda = 0 \) limit corresponds to \( N \) 2D relativistic free (complex) fermions and is very similar to the system described in this paper.

**D2 branes:** The gauge theory system [16] was considered in [13], where a gravity dual is discussed (see Section 5 of that paper) in terms of the near horizon geometry of D2 branes compactified on a ‘Scherk-Schwarz circle’ (i.e. a small spatial circle of size \( L_2 \) with antiperiodic
fermions). The other two directions are periodic, with sizes $\beta, L$, which characterize the phase diagram of the model is given in Fig. 8 (a). As discussed in [13], the low temperature phase $\beta > \beta_{cr}$ of the model is described by a soliton solution wrapping the temporal circle uniformly, while the high temperature (‘deconfinement’) phase, $\beta < \beta_{cr}$, is given by a localized soliton. We clearly find evidence of equilibration in the low temperature phase of the field theory in Fig. 9 (a). It is also known that the deconfinement phase shows dissipation. It would be interesting to understand such equilibration in terms of the solitonic geometries which obviously do not have a horizon.

$\mathcal{N} = 4$ super Yang Mills theory: This is clearly one of the most important examples of integrable models in AdS/CFT [39]. The giant gravitons described above describe a special sector of this model. More generally, operators of $\mathcal{N} = 4$ super Yang Mills theory whose energies do not scale with $N$, can be mapped to states in integrable spin chains which possess an infinite number of conserved charges. For energies $E \sim O(N^2)$ the integrability does not persist; thus, a generic perturbation equilibrates, and is described in the gravity dual by black hole formation (see, e.g., [40]). The present discussion opens the possibility of equilibration of certain perturbations in the integrable subsector and emergence of a GGE; it would clearly be of interest to find a gravitational dual of such an ensemble, if any.

4.3 Black holes and $O(N)$ entropy

It is tempting to speculate that the equilibration, discussed above for the matrix QM model, and possibly generalizable to other integrable field theories, is described in terms of formation of black holes in a dual theory. In many known examples of AdS/CFT, equilibration does have a natural interpretation in terms of formation of black holes [38].

Such a scenario, however, must pass the following stringent tests:

(i) Black holes are typically not associated with integrable models; while the latter have an infinite number of conserved charges, black holes typically have only a few (cf. uniqueness theorems). Since a GGE [38] is characterized by an infinite number of chemical potentials, any black hole describing a GGE must also have an infinite number of parameters.

(ii) The entropy of the GGE must match the entropy of the known black holes.

(iii) Besides the static black hole solution, there must be a dynamical collapse solution which would capture the equilibration process. The black hole formation time should then match the time of relaxation to equilibrium described by the GGE.

(iv) A non-extremal black hole will Hawking radiate; there must be an interpretation of the.

\[23\] or, in the context of a probe approximation, in terms of the damping of quasinormal fluctuations in a black hole background.
two-stage process—formation and evaporation—in the field theory description.

(v) In case we seek a description of equilibration and dissipation in terms of absorption of energy by a fixed black hole background in the dual geometry.

2D: As mentioned above, the $c = 1$ matrix model (25), is dual to the 2D bosonic string theory, in the linear dilaton background. This theory, as described by its massless fields (the metric, dilaton and the massless ‘tachyon’), does have the MSWW black hole solution [41, 42] (see [43, 44] for superstring generalizations). Let us discuss the above points to see whether this black hole can be relevant for a description of the GGE found in the present paper.

Point (i): the MSWW black hole solution, is indeed the unique solution of the effective field theory of massless modes of 2D string theory, and is specified by only the mass parameter $24$. However, this solution can be extended to an infinite-parameter generalization in 2D string field theory [46]. Each such black hole possesses an infinite number of specified $W_\infty$ charges (corresponding to charges of higher mass non-propagating gauge fields), and could potentially correspond to a given GGE with fixed values of the chemical potentials.

Point (ii): the entropy of the MSWW black hole was found to be (see [45]) of the order of $1/g_{str}^2 \sim N^2$, whereas the entropy of the GGE in Section 2.3 turns out to be of order $N$.

Point (iii): [47] has argued that if we wish to form the MSWW black hole (or its type 0B counterpart) by collapsing a tachyon shell, physics becomes strongly coupled even before a horizon is formed. In the type 0B black hole, one can instead consider collapse of an axion shell; in this case, there is no strong coupling problem but the shell is scattered back to infinity from near the incipient horizon and a black hole is not formed.

Point (iv): Reference [47] performs a matrix model computation to rule out any Hawking radiation from a possible gravitational collapse of axionic matter.

Point (v): a fixed black hole background in type 0A theory was argued in [43] to be dual to a deformed matrix model [48]. One could in principle consider fermionizing such a model (as in (20)), explore dissipation in this, and attempt to explain this in terms of black hole absorption. However, doubts have been raised in [49] about the duality, as the entropy of the matrix model is computed there to be $O(1)$ whereas the entropy of the black hole is $O(N^2)$. The suggested dual matrix model, according to [49], which reproduces an $O(N^2)$ entropy, is the one proposed in [50]; however, in this model, the off-diagonal matrix elements play a crucial role, and the model is not (at least in any obvious sense) integrable any more. One could also consider the proposal in [24] for construction of a black hole background in terms of an asymmetrically filled Fermi sea. We leave further studies of these issues for the future.

$24$The charged cousins described in [43, 45] require fluxes which can come from superstring or type 0 theories.
Higher spin black holes in AdS$_3$: We have already discussed above the duality between large $N$ 2D CFT’s ($W_N$ minimal models) and higher spin (Vasiliev) theories in AdS$_3$. In the limit of $\lambda = 0$, the field theory is that of $N$ number of complex free fermions, and is likely to equilibrate as in this paper, with emergence of GGE. As in case of the 2D gravity duals, the AdS$_3$ also has black holes, the most well-known being the BTZ black hole. Indeed, it has been found in [51] that the BTZ black hole has spin three and spin four generalizations. It would be very interesting to investigate if this result generalizes to arbitrarily high spin, in which case they would be natural candidates for a dual description of GGE. These black holes have an entropy of $O(N)$, matching the central charge $c \sim N$ of the field theory (see [52], e.g.), which agree with the $O(N)$ entropy of the GGE computed in this paper.

The superstar geometry: In the context of the matrix model-LLM-giant graviton duality mentioned earlier, thinned out phase space configurations such as (73) in which the fermion occupation numbers are fractional have been studied. The resulting LLM geometries are singular and are called superstar geometries. These geometries have zero horizon area, but a Bekenstein entropy can be assigned to a stretched horizon [53], which turns out to be $O(N)$ just like the standard Boltzmann formula for entropy in these systems, and agrees with the GGE estimate of entropy.

D1-D5: The possible relevance of GGE to the two-charge D1-D5 black hole is already mentioned in the previous subsection. The entropy of the two-charge black hole $S \sim \sqrt{Q_1 Q_5} \sim 1/g_{str}$ appears to agree with the $O(N)$ entropy of GGE.

4.4 Topology change and the Gregory-Laflamme phase transition

The uniform and gapped phases of the double trace model [3] correspond to the large $L$ and small $L$ sides of the phase boundary AB in Fig. 8 (a). These phases, by definition, refer to the uniform and clumped eigenvalue distributions of the spatial Wilson line $U$, which arises from the gauge of theory of D2 branes wrapped on a small Scherk-Schwarz circle. (see page 22). What do these phases mean in gravity? This question was investigated in detail in [19] [54] [55] [56] (see [14] [57] for other related references). The Euclidean near-horizon description of such a system at large $\beta$ and $L$ (sizes of the temporal and spatial circles, respectively) is that of a D2 soliton. If we T-dualize this system along the large spatial circle, we get a D1-brane soliton which is uniformly smeared along a small dual circle $L' \propto 1/L$. This is the description of the phase to the right of the line AB in Fig. 8 (a); now as we decrease $L$, the dual circle $L'$ starts to increase and the uniformly smeared D1 soliton starts getting stretched and as $L'$ increases beyond a certain critical size, it becomes unstable towards breaking and forming a localized
soliton. In [58], Gregory and Laflamme predicted such a transition as a thermodynamic phase transition[59, 60, 61, 62]. Thus, the dynamical gapless → gapped transition in our model translates to a dynamical GL transition from a uniform to a localized soliton. In particular, the observed non-existence of such a transition in the matrix model, ultimately due to non-splitting of Fermi liquid droplets, implies impossibility of a dynamical GL transition from a 1-brane (uniform soliton) to a 0-brane (split solution).

The original context of the GL transition was in the context of a clumping transition of a black string into a black hole. The question about whether this can happen dynamically is related to the issue of cosmic censorship and the appearance of a naked singularity, and has been discussed extensively in the literature [20, 21, 63, 64, 65, 66]. In a supersymmetric variant of (16), the phase transition discussed in the previous paragraph would correspond to such a transition from a black string to a black hole. Since the technique developed in this paper appears to be fairly robust, dynamical transition in this model could share similar features, and could shed light on the issue of dynamical appearance of a naked singularity. See also the related discussion in page [13].

4.5 Non-ergodic time evolution in confinement geometries

Very recently, non-linear instabilities of global AdS geometries under perturbations are actively studied [67, 68, 69, 70]. There, depending on initial perturbations, the perturbed geometries evolve to small black holes, boson stars, or geons [71]. Since the final states depend on initial conditions, the time evolutions are non-ergodic. Since global AdS would correspond to a confinement phase in the dual gauge theory, the appearance of these non-ergodic evolutions sounds reasonable. Especially, they would be related to the non-ergodic time evolutions in the D2 brane model (16) in the confinement phase, as argued in section 3. Since the gravity dual of the ground state (which is in confinement phase), is a soliton geometry, as shown in [13] based on the Witten’s holographic QCD, we presume that the non-ergodic time evolutions would be observed in this soliton geometry too. Then we can directly compare the time evolutions in the confinement phase in the gauge theory and the time evolutions in the soliton geometry. This would be an interesting new direction of the gauge/gravity correspondence, since we can calculate both explicitly.

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A Fermionic formulation of MQM

In this appendix, we will discuss the fermionic formulation of MQM. For a general review, see, e.g. [22, 72].

A.1 Single trace model

Let us first consider the single-trace model [2]. The singlet sector of this model (see footnote 2) is described by the following $N$-fermion action and hamiltonian

$$S = \int dt \sum_{i=1}^{N} \left( \frac{\dot{\theta}_i^2}{2} - a \cos \theta_i \right),$$

$$H = \sum_{i=1}^{N} h(\theta_i, \partial \theta_i), \quad h = -\frac{\hbar^2}{2} \partial^2 \theta + V(\theta), \quad V(\theta) = a \cos \theta, \quad \hbar = 1/N. \quad (19)$$

The positions $\theta_i$ are defined on a circle, and arise from the eigenvalues of $U = V \text{diag}[e^{i\theta_i}] V^{-1}$; the fermionic nature arises from the measure (after integrating out the $V_{ij}$’s) which vanishes for coincident eigenvalues. The identification of $\hbar$ as $1/N$ comes from the partition function in (1).

A second-quantized version of the above action is written down:

$$S = \int dt \int d\theta \ \psi^\dagger(\theta, t) \left[ -i \hbar \partial_t - h(\theta, \partial \theta) \right] \psi(\theta, t), \quad \int d\theta \ \psi^\dagger(\theta, t) \psi(\theta, t) = N. \quad (20)$$

It is useful to describe states of the system in terms of expectation value of the phase space (Wigner) distribution

$$\hat{u}(\theta, p, t) = \int d\eta \ \psi^\dagger \left( \theta + \frac{\eta}{2}, t \right) \psi \left( \theta - \frac{\eta}{2}, t \right) e^{i p \eta / \hbar}. \quad (21)$$

$^{25}$For specific details of the material presented here, see, e.g. [73, 74, 75, 76, 77, 78, 79].
We define the position space density as
\[
\rho(\theta, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i(t)) = \frac{1}{N} \psi^\dagger(\theta, t) \psi(\theta, t) = \int \frac{dp}{2\pi\hbar} \bar{u}(\theta, p, t).
\] (22)

The normalization condition in (20) implies that
\[
\int d\theta \rho(\theta, t) = \int d\theta dp \frac{2}{\pi} u(\theta, p, t) = 1.
\] (23)

**Large-\(N\) limit** Since \(\hbar = 1/N\), the large \(N\) limit reduces to
\[
N \to \infty, \quad \hbar \to 0, \quad N\hbar = 1.
\] (24)

In this limit, the fermion configurations are represented by a semiclassical Fermi liquid with a smooth phase space density and position space density; the expectation value \(u(\theta, p, t)\) is either 0 or 1 depending on whether the phase space cell (of vanishing area \(2\pi\hbar\)) is occupied by a fermion or not. The regions with \(u = 1\) are called droplets of fermi liquid.

**Phases:** Classically, due to the potential \(V(\theta) = a \cos \theta\) in (19) the particles tend to clump around \(\theta = \pi\); quantum mechanically however the states of the system are quantized and Pauli exclusion tends to spread the fermions up. The competing tendencies dictate the qualitative nature of the ground state: (a) For \(a > a_c = \pi^2/64\), the potential is strong and the eigenvalues are clumped around \(\theta = \pi\). Pauli exclusion forces the fermions to occupy \(N\) levels, but the Fermi level is below the top of the potential. This leads to a gap in \(\rho(\theta)\) around \(\theta = 0\). This is called the *gapped phase*. (b) For \(a < a_c\) the potential is weak and the eigenvalues are spread out (non-uniformly). The Fermi level is above the top of the potential and there are no gaps in \(\rho(\theta)\). This phase is called a *gapless phase*. The phase transition at \(a = a_c\) is characterized by the closing of a gap, when the Fermi level comes up and touches the top of the potential. This is a third order phase transition and was found in [2].

**Double scaling:** It is clear from the above that the position of the fermi level \(\mu\), measured from the top of the potential, is \(O(a - a_c)\). It turns out that the singularity of the free energy is the form \(F_{\text{sing}}(\mu N) \propto f_0(\mu N)^2 + f_1 + f_2(\mu N)^{-2} + \ldots\). The double scaling limit is defined by \(N \to \infty, \mu \to 0\), such that \(\mu N\) is held fixed. Our result about the entropy \(S \sim O(N)\) is without taking into account such a double scaling.

**The hermitian model:** The double scaling limit captures the universal part of the potential near the top: \(\theta \approx 0\), whose essential features can be equally simply reproduced, by replacing
cos \theta by \(1 - \theta^2/2\) in \(\theta\); dropping the 1, the potential becomes \(V(\theta) = -a\theta^2/2\), which is equivalent to the (singlet sector of the) hermitian matrix model

\[
Z_H = \int DM(t) \ e^{iNS}, \quad S = \int dt \left[ \frac{1}{2} \Tr M^2 - V(M) \right], \quad V(M) = -\frac{a}{2} \Tr M^2. \quad (25)
\]

This model is called the \(c = 1\) matrix model.\(^{26}\) The double-scaled \(c = 1\) model is mapped to two-dimensional bosonic string theory (see the review \[22\]) by identifying \(N \mu = 1/g_{\text{str}}\). A similar duality to two-dimensional type 0 theories also exists (see the review \[72\]).

**Classical droplet motion:** Suppose that the Fermi liquid configuration is that of a single droplet (e.g. the Fermi sea or a continuous distortion of it), characterized by specifying two functions \(p_{\pm}(\theta, t)\), which demarcate the upper/lower boundaries of the droplet at a specific value of \(\theta\) at time \(t\), i.e

\[
u(\theta, p, t) = \theta(p_{+}(t) - p)\theta(p - p_{-}(t)). \quad (26)
\]

Using this, we get the following formulae for density, momentum density and energy density:

\[
\rho(\theta, t) = \int \frac{dp}{2\pi} u(\theta, p, t) = \frac{1}{2\pi} \left( p_{+}(\theta, t) - p_{-}(\theta, t) \right),
\]

\[
\rho(\theta, t) \mathcal{P}(\theta, t) = \int \frac{dp}{2\pi} pu(\theta, p, t) = \frac{1}{2\pi} \left( p^2_{+}(\theta, t) - p^2_{-}(\theta, t) \right)/2,
\]

\[
H = \int \frac{d\theta}{2\pi} \frac{dp}{2\pi} \ h(p, \theta) u(\theta, p, t) = \int d\theta \int_{p_{+}(\theta, t)}^{p_{-}(\theta, t)} \frac{dp}{2\pi} \left( p^2/2 + V(\theta) \right), \quad (27)
\]

Here \(V(\theta) = a \cos \theta\) for the unitary matrix model \[2, 20\], whereas \(V(\theta) = -a\theta^2/2\) for the hermitian matrix model \[25\]. It turns out that \(\rho(\theta, t), \mathcal{P}(\theta, t)\) are canonically conjugate variables, with Poisson bracket

\[
\{\rho(\theta, t), \mathcal{P}(\theta', t)\} = -\partial_{\theta'} \delta(\theta - \theta').
\]

This can be derived from the anticommutation relation of the fermion field, or from the symplectic \(W_{\infty}\) structure of the \(u(\theta, p)\)-fields (see \[73, 79\]). The equation of motion of the ‘collective fields’ \(\rho, \mathcal{P}\) or \(p_{\pm}\) follow from the above hamiltonian and Poisson brackets (see comments below \[32\] for limitations of the collective field description):

\[
\dot{\rho}(\theta, t) = -\partial_{\theta}(\rho(\theta, t) \mathcal{P}(\theta, t)), \quad \dot{\mathcal{P}}(\theta, t) = -\partial_{\theta} \left[ \frac{1}{2} \mathcal{P}^2 + \frac{\pi^2}{2} \rho^2 + V(\theta) \right]. \quad (28)
\]

\(^{26}\)The periodic boundaries \(\theta = \pm \pi\) need to be supplanted by \(\theta = \pm \theta_{m}\) to make the model well-defined; the choice of \(\theta_{m}\) is not particularly relevant and does not affect critical/universal quantities.
These equations of motion (EOM) can alternatively be derived from that of the phase space density
\[
\left( \partial_t - \frac{\partial h}{\partial \theta} \partial_p + \frac{\partial h}{\partial p} \partial_\theta \right) u(\theta, p, t) = 0, \tag{29}
\]
which, viewed from the particle-fixed frame, is simply the particle EOM, and can be regarded as the Euler equation or the dissipationless Boltzmann equation. This viewpoint provides an explicit solution of the equation of motion:
\[
u(\theta, p, t) = \nu_0(\nu_0(\theta, p, t), p_0(\theta, p, t)), \tag{30}
\]
where \((\theta_0, p_0)\) is the unique initial phase space point at \(t = 0\) which reaches \((\theta, p)\) at time \(t\) through the dynamical evolution (19), and \(\nu_0\) is the phase space density at \(t = 0\). For the hermitian model (25), \(V = -a\theta^2/2\), and (30) becomes
\[
u(\theta, p, t) = \nu_0 \left( \theta \cosh(\sqrt{a}t) - \frac{p}{\sqrt{a}} \sinh(\sqrt{a}t), -\theta \sqrt{a} \sinh(\sqrt{a}t) + p \cosh(\sqrt{a}t) \right), \tag{31}
\]
which can be easily verified to be a solution of (29) for \(h = p^2/2 - a\theta^2/2\). In the special case of \(a = 0\) (free fermions with zero potential), we get
\[
u(\theta, p, t) = \nu_0(\theta - pt, p). \tag{32}
\]
Note that the collective field equations of motion (28) depend on the assumption of a quadratic profile (26), which are clearly violated by ‘folds’ e.g. in Fig. 6 and Fig. 13. As a result, they are not valid for long times (see [78, 79]). In our actual analysis where we are specially interested in long time evolutions, we do not use (28) but use (29) or equivalently the particle equations of motion in Appendix E.

Given the solution (30), and the relation (22) between \(\rho\) and \(\nu\), we get the following result for \(\rho(\theta, t)\) at \(N \to \infty\):
\[ho(\theta, t) = \int \frac{dp}{2\pi} \nu_0(\nu_0(\theta, p, t), p_0(\theta, p, t)). \tag{33}
\]
This can be numerically implemented by sprinkling fermions uniformly in the support of the original function \(\nu_0\) and let each fermion evolve according to the EOM of (19).
A.2 Double trace model

Using the fermionization techniques discussed in the previous subsection, the action (3) can be written as (with $\hbar = 1/N$, as before)

$$S = \int dt \left[ \sum_{i=1}^{N} \frac{\dot{\theta}_i^2}{2} - \frac{\xi}{N} \sum_{i,j} \cos(\theta_i - \theta_j) \right]$$

$$= \int dt \ d\theta \ \psi^\dagger(\theta, t)[-i\hbar \dot{\theta} - h(\theta, \partial_\theta)]\psi(\theta, t) - H_{\text{int}}, \quad h(\theta, \partial_\theta) = -\hbar^2 \partial_\theta^2,$$

$$H_{\text{int}} = \frac{\xi}{N} \int dt \ d\theta \ d\theta' \left[ \psi^\dagger(\theta, t)\psi(\theta, t) \cos(\theta - \theta')\psi^\dagger(\theta', t)\psi(\theta', t) \right].$$

The definition of $u(\theta, p, t), \rho(\theta, t), P(\theta, t), p_\pm(\theta, t)$ all remain as before.

$$H = \int d\theta \left[ \frac{1}{2} \rho P^2 + \frac{\pi^2}{6} \rho^3 \right] + \frac{\xi}{N} \int d\theta \ d\theta' \rho(\theta) \cos(\theta - \theta')\rho(\theta').$$

Because of the limitation of the collective field description in the presence of folds (see comments below (32)) for computation of the time evolution we use the equation of motion of the phase space density or equivalently the particle equations of motion as in Appendix E.

B Integrability and GGE

The single-trace model (19), (20) is clearly integrable, since it consists of $N$ free particles. Classically one can consider the energy $\epsilon_i$ of each particle, $i = 1, \ldots, N$, to be an ‘action’ variable. Quantum mechanically, each linearly independent $N$-fermion state can be specified by saying which single-particle energy levels, $\epsilon_i$, are occupied. The quantities

$$I_p = \sum_{i=1}^{N} \epsilon_i^p,$$

are all constant in time, and commute with each other. For finite $N$, only $N$ of them, $I_p$ ($p = 1, \ldots, N$) is enough to determine the energy levels $\epsilon_i$ (up to some discrete ambiguities which we shall ignore), hence $I_{N+1}, I_{N+2}, \ldots$ are all dependent on the first $N I_p$’s. In the $N \to \infty$ limit, of course, the charges (37) are all independent.

We can define the Generalized Gibbs Ensemble (GGE) in terms of the $N$ conserved charges as follows:

$$\varrho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp[-\sum_p \beta_p I_p], \quad Z_{\text{GGE}} = \text{Tr} \exp[-\sum_p \beta_p I_p].$$

Here $\beta_p$ are $N$ chemical potentials. For more details, see [1].
In the present discussion, it is convenient to introduce the fermion occupation numbers $N_m$, $m = 0, 1, ..., \infty$, where $N_m = 1$ if the single particle energy level $\epsilon_m$ is occupied, and $= 0$ if it is not (note that our fermions are spinless, and that the energy levels are non-degenerate). These can be defined in terms of the creation/annihilation operators through the equations

$$
\psi(\theta) = \sum_m c_m \phi_m(\theta), \quad h(\theta, \partial) \phi_m(\theta) = \epsilon_m \phi_m(\theta), \quad \text{(39)}
$$

$$
N_m \equiv c^\dagger_m c_m. \quad \text{(40)}
$$

In terms of the $N_m$'s, $I_p = \sum_m \epsilon_m^p N_m$. For $N = \infty$, in stead of (37), one can use $N_m$'s as the (independent) conserved charges, which leads to the following definition of the GGE:

$$
\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp\left[ - \sum_m \mu_m N_m \right], \quad Z_{\text{GGE}} = \text{Tr} \exp\left[ - \sum_m \mu_m N_m \right]. \quad \text{(41)}
$$

With this definition, we get, by explicit evaluation (see [80]),

$$
\ln Z = \sum_m \ln(1 + e^{-\mu_m}), \quad \langle N_m \rangle_{\text{GGE}} = - \frac{\partial}{\partial \mu_m} \ln Z = \frac{1}{e^{\mu_m} + 1}, \quad \text{(42)}
$$

which gives a remarkably simple Fermi-Dirac distribution which involves only one chemical potential. The chemical potentials are now trivially found:

$$
\mu_m = \ln \left( \frac{1}{\langle N_m \rangle_{\text{GGE}}} - 1 \right). \quad \text{(43)}
$$

The entropy of the GGE is given by

$$
S_{\text{GGE}} = - \text{Tr} \rho_{\text{GGE}} \ln \rho_{\text{GGE}} = - \sum_{m=0}^{\infty} \left[ \langle N_m \rangle_{\text{GGE}} \ln \langle N_m \rangle_{\text{GGE}} + (1 - \langle N_m \rangle_{\text{GGE}}) \ln(1 - \langle N_m \rangle_{\text{GGE}}) \right]. \quad \text{(44)}
$$

The GGE hypothesis is that, if we start from an appropriate excited state $|\Psi(0)\rangle$, there exists a certain class of observables $O$ such that, as $t \to \infty$

$$
\langle \Psi(t)|O|\Psi(t)\rangle \to \text{Tr} (\rho_{\text{GGE}} O), \quad \text{(45)}
$$

where the GGE is defined by the charges

$$
\langle N_m \rangle_{\text{GGE}} = \langle \Psi(0)|N_m|\Psi(0)\rangle = \langle \Psi(t)|N_m|\Psi(t)\rangle, \quad \text{(46)}
$$

or equivalently by the chemical potentials $\mu_m$, determined through (43).

Note that if the initial state $|\Psi(0)\rangle$ is an eigenstate of the number operators $N_m$, then by (46), $\langle N_m \rangle = 0$ or 1. The GGE constructed from such an initial state has vanishing entropy $S_{\text{GGE}} = 0$, according to (44).
B.1 Verification of the GGE hypothesis for $\rho(\theta)$

We wish to verify (45) where the observable is the density variable

$$O(t) = \rho(\theta,t) = \sum_{m,n} c_m^\dagger(t) c_n(t) \phi_m^*(\theta) \phi_n(\theta),$$

where $c_m(t) = c_m \exp[i \epsilon_m t/\hbar]$, $c_m^\dagger(t) = c_m^\dagger \exp[-i \epsilon_m t/\hbar]$. The LHS of (45) becomes, in the Heisenberg picture,

$$\langle \rho(\theta,t) \rangle = \sum_{m,n} \langle c_m^\dagger c_n \phi_m^*(x) \phi_n(x) \exp[-i(\epsilon_m - \epsilon_n)t/\hbar] \rangle,$$

(47)

where $\langle ... \rangle = \langle \Psi(0)|...|\Psi(0) \rangle$ represents expectation value in the initial state $|\Psi(0)\rangle$. We will evaluate this expression in (57) where the initial state is prepared according to QQD. The large $N$ limit of (47) is given by the semiclassical expression (33).

To compute the RHS of (45) for $O = \rho(\theta)$, first note that

$$\langle c_m^\dagger(t)c_n(t) \rangle_{\text{GGE}} = \delta_{mn} \langle N_m \rangle_{\text{GGE}},$$

(48)

where $\langle ... \rangle_{\text{GGE}} = \text{Tr}(\rho_{\text{GGE}}[...])$. Using this and (39), we get

$$\langle \rho(\theta) \rangle_{\text{GGE}} = \sum_m \langle N_m \rangle_{\text{GGE}} |\phi_m(\theta)|^2.$$  

(49)

Note that this is independent of time, as expected. Here $\langle N_m \rangle_{\text{GGE}}$ are to be determined from their values in the initial state $|\Psi(0)\rangle$

$$\langle N_m \rangle_{\text{GGE}} = \langle \Psi(0)|N_m|\Psi(0)\rangle.$$  

(50)

Thus,

$$\langle \rho(\theta) \rangle_{\text{GGE}} = \sum_m \langle \Psi(0)|N_m|\Psi(0)\rangle |\phi_m(\theta)|^2.$$  

(51)

Since these uniquely determine the chemical potentials through (43), the GGE is specified completely.

Verifying the GGE hypothesis amounts to showing that (47) tends to (51) as $t \to \infty$. In Section C we will compute both these expressions, and in Section 2.1 present the result in support of the hypothesis.
B.2 Phase space density in GGE

In this section, we will compute \( \langle \hat{u}(\theta, p) \rangle_{\text{GGE}} \), where \( \hat{u}(\theta, p) \) is the Wigner phase space density (21). By using (39) and (48), it is easy to show that

\[
\langle \hat{u}(\theta, p) \rangle_{\text{GGE}} = \sum_m \langle N_m \rangle_{\text{GGE}} u_m(\theta, p),
\]

where \( \langle N_m \rangle_{\text{GGE}} \) is given by (50). In Section D.1 we evaluate this expression in the semiclassical limit.

C Quantum quench dynamics (QQD)

Suppose we consider a sudden change of the parameter \( a \) in the single-particle hamiltonian (19), from \( a_i \) at \( t < 0 \) to \( a_f \) at \( t \geq 0 \). Time evolution works as follows. We consider the ground state \( |F'\rangle \) of the system (Fermi sea in this case) appropriate to the parameter \( a_i \) for \( t < 0 \). Quantum quench dynamics refers to the sudden approximation in which the system evolves for \( t \geq 0 \) according to the new parameter \( a_f \), from an initial state equal to the old Fermi sea

\[
|\Psi(0)\rangle = |F'\rangle.
\]

Clearly the system is integrable after \( t = 0 \) and our discussion in the previous section applies (we need to identify the \( a \) parameter with \( a_f \)).

We wish to compute the time-dependence of

\[
\langle \rho(\theta, t) \rangle \equiv \langle F'|\rho(\theta, t)|F'\rangle = \langle \Psi(t)|\rho(\theta)|\Psi(t)\rangle,
\]

for \( t \geq 0 \). Here the second expression is in the Heisenberg picture (in which the state remains the initial state (53)), while the third expression is in the Schroedinger picture in which \( |\Psi(t)\rangle \) is the time evolution of (53) according to the new Hamiltonian.

To proceed, let us express the old Fermi sea \( |F'\rangle \) in terms of eigenstates in terms of the new number operators \( N_m \). Suppose we denote the eigensystem of the old hamiltonian (with \( a_i \)) as \( \phi'_m, \epsilon'_m \). The functions \( \phi'_m \) can clearly be written in the new eigenbasis \( \phi_n \): thus

\[
\phi'_m = \sum_m B_{mn} \phi_n, \quad \phi_n = \sum_m \phi'_m B^*_{mn}.
\]

Just like (39), the Fermi field can be equally well expanded in terms of \( \phi'_m \): thus

\[
\psi(\theta) = \sum_n c_n \phi_n(\theta) = \sum_m c'_m \phi'_m,
\]

(55)
we get
\[ c'_m = B_m c_n, c'_n = c'_m B_m, c_n = c_m B_m, c'_n = c'_m B_m. \] (56)

Using the expression (47) we get
\[ \langle \rho(\theta,t) \rangle = \sum_{m,n=1}^{N} \sum_{r=1}^{N} B_{rm} B_{rn} \phi_m(x) \phi_n(x) \exp[-i(\epsilon_m - \epsilon_n)t/\hbar], \] (57)

where we have used (56) and the defining property of the old Fermi-sea:
\[ \langle F' | c'_n c'_m | F' \rangle = N_{nk}', \quad N_{nk}' = f_n \delta_{nk}, \quad f_n = 1 (n = 1, \ldots, N), \quad f_n = 0 (n > N). \] (58)

We explore (57) numerically in the text for various values of $N$. The $N \to \infty$ limit can be computed in the semiclassically in the droplet picture as in (33).

**C.1 GGE for QQD**

We wish to compute $\langle \rho(\theta) \rangle_{\text{GGE}}$ in the context of QQD, using the general expression (51), for the initial state (53).

The expression (50) becomes
\[ \langle N_m \rangle_{\text{GGE}} = \langle F'_n c'_m c_m | F' \rangle = \langle F'_n c'_m B'_m c'_k B_B | F' \rangle = \sum_{n=1}^{N} |B_{nm}|^2, \] (59)

where we have used (58) and (56). Using this expression in (49), we now get
\[ \langle \rho(\theta,t) \rangle_{\text{GGE}} = \sum_{m,n=1}^{N} |B_{nm}|^2 |\phi_m(\theta)|^2. \] (60)

In Section 2.1 we verify the hypothesis (45) for the fermion density $\rho(\theta,t)$ (more precisely, its various moments (9)), by comparing the large time behaviour of (57) (or (33)) and (60).

**C.2 Adiabatic time evolution**

Suppose we have a time-dependent Hamiltonian $H(t)$, e.g. (20) with $a = a(t)$-parameter, or (35) with $\xi = \xi(t)$-parameter. Suppose the time-dependence starts after some time $t_0$; we wish to study the time-evolution of a wavefunction $|\Psi(t)\rangle$, which at $t_0$ is an eigenstate: $H(t_0) |\Psi(t_0)\rangle = E_0 |\Psi(t_0)\rangle$. QQD is an example in which the time-variation is in the form a step function at $t = 0$. An adiabatic time evolution, on the other hand, refers to a sufficiently slow time-variation such that one can approximate $|\Psi(t)\rangle$ as the instantaneous eigenstate of $H(t)$ with eigenvalue $E(t)$; this notion is well-defined in the absence of level crossing, which can be ensured if the
spectrum is discrete which sets a finite time scale. In the examples considered in this paper, we are often interested in the time-evolution of the ground state, and by an adiabatic expectation value $v_{\text{adiab}}(t)$ we will always mean expectation value in the instantaneous ground state $v_{\text{inst}}(t) = \langle \Psi_{\text{ground,inst}}(t) | O | \Psi_{\text{ground,inst}}(t) \rangle$.  

In the context of (20), the ground state of a specific value $a(t_1) \equiv a_1$ is clearly given by, in an obvious notation where we denote all instantaneous quantities by an additional subscript $(...)_{1}$

$$
\langle F_1 | c_{1,m}^\dagger c_{1,n}^\dagger | F_1 \rangle = \delta_{mn} f_{1,m}, \quad f_{1,m} = \sum_{n=1}^{N} \delta_{nm}.
$$

(61)

By expanding $\psi(\theta)$ in this new basis, we get

$$
\langle \rho(\theta) \rangle_{\text{adiab}} \equiv \langle F_0' | \rho(\theta) | F_0' \rangle = \sum_{m=1}^{N} \phi_m'(\theta)^2.
$$

(62)

The time dependence of the adiabatic quantity is reflected in the appearance of the instantaneous eigenfunctions.

### D An analytic calculation of relaxation

In this section, we will consider time evolution in the single-trace model (2) with $a = 0$. (This model is also equivalent to $\xi = 0$ in (3).) In this case, the ground state is given by $p_{\pm}(\theta) = \pm 1/2$ (see (27) for notation). Suppose we perturb the ground state configuration at $t = 0$ by a sinusoidal deformation

$$
p_{\pm}(\theta, t = 0) = \frac{1}{2} + b \cos(n\theta), \quad (|b| \leq 1/2), \quad p_{-}(\theta, t = 0) = -p_{+}(\theta, t = 0),
$$

(63)

where $b$ is the amplitude of the perturbation (see Fig. [13]). We wish to evaluate the time evolution of this perturbation using the action (20) with $a = 0$.

We will evolve the configuration as in (32), which essentially says that $u_t(\theta_0 + p_0t, p_0) = u_0(\theta_0, p_0)$; in other words, to get any point of the evolved Fermi surface at time $t$, we should simply evolve points on the initial Fermi surface according to the equation of motion of (19) with $a = 0$, which is given by

$$
\dot{p} = 0, \quad \dot{\theta} = p.
$$

(64)

---

At a critical point, the spectrum is gapless; there we will simply define $v_{\text{adiab}}(t) \equiv v_{\text{inst}}(t)$.  

Note that this corresponds to $\rho(\theta) = \frac{1}{2\pi}, \mathcal{P}(\theta) = 0$, which obviously satisfies (28) for $V = 0$.  

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36
In order to solve this equation for points on the Fermi surface, it is convenient to employ a parameter \( s \in [-\pi, \pi] \) and rewrite the initial configuration (63) as

\[
p_+(s, t = 0) = \frac{1}{2} + b \cos(ns), \quad \theta(s, t = 0) = s.
\]

Then from (64) we obtain the Fermi surface at time \( t \) as

\[
p_+(s, t) = \frac{1}{2} + b \cos(ns), \quad \theta(s, t) = s + \left(\frac{1}{2} + b \cos(ns)\right) t.
\]

This solution with \( n = 1 \) is plotted in the middle of Fig. 13; the ‘folds’ appear because of the stretching caused by different speeds of fermions at different heights from the \( \theta \)-axis. As time progresses, more or more ‘folds’ keep appearing.

In the perturbation (63), only \( \rho_n \) is excited and other \( \rho_m \) are zero. We evaluate how these modes behave as \( t \) increases. From the solution (66), we calculate the \( m \)-th moment as

\[
\rho_m(t) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \cos(m\theta) \left( p_+ - p_- \right) = \int_{-\pi}^{\pi} ds \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \cos(m\theta(s, t)) \left( p_+ - p_- \right)
\]

\[
= \int_{-\pi}^{\pi} ds \left( 1 - nb t \sin(ns) \right) \cos\left( ms + mt \left(\frac{1}{2} + b \cos(ns)\right)\right) \left(\frac{1}{2} + b \cos(ns)\right)
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi} ds \cos(ms) \cos\left( mt \left(\frac{1}{2} + b \cos(ns)\right)\right) \left(\frac{1}{2} + b \cos(ns)\right)
\]

\[
+ \frac{2}{\pi} \int_{0}^{\pi} ds \ nbt \sin(ns) \sin(ms) \sin\left( mt \left(\frac{1}{2} + b \cos(ns)\right)\right) \left(\frac{1}{2} + b \cos(ns)\right).
\]

These integrals will be given by the Bessel function \( J_k \). For example, if \( n = 1 \), the integral (67) with \( m = 1 \) and 2 are evaluated as

\[
\rho_1(t) = \frac{2}{t} J_1(bt) \cos\left(\frac{t}{2}\right), \quad \rho_2(t) = -\frac{1}{t} J_2(2bt) \sin(t).
\]

\footnote{From now, we omit \( p_- \), since we can derive it trivially from \( p_+ \).}

\footnote{The expression in the first line in this equation is crude, since the droplet has many folds at a late time as shown in the center of Fig. 13 and we need to subtract the white region below \( p_+ \). However after the change of the variable from \( \theta \) to \( s \), this subtraction is automatically involved.}
For large $t$, they decay with $t^{-3/2}$ as
\[ \rho_1(t) \rightarrow \frac{2}{t^{3/2}} \sqrt{\frac{2}{\pi b}} \cos \left( bt - \frac{3\pi}{4} \right) \cos \left( \frac{t}{4\pi} \right), \quad (t \rightarrow \infty), \] (69)
and
\[ \rho_2(t) \rightarrow \frac{1}{t^{3/2}} \frac{1}{\sqrt{\pi b}} \cos \left( 2bt - \frac{\pi}{4} \right) \sin (t), \quad (t \rightarrow \infty). \] (70)
Other $\rho_m(t)$ also exhibits a similar decay. In Section 2.2 we will compare the time-evolution of various modes and find a picture of an energy cascade in which lower frequency modes die out before the higher frequency modes, causing a transfer of energy from higher to lower frequencies, in a manner similar to energy cascades in turbulence.

### D.1 Asymptotic and GGE form of the Fermi surface

As we mentioned below (66), the asymptotic form of the droplet consists of an infinite number of ‘folds’ which eventually densely fill an entire band. When suitably coarse-grained (the ‘grain’ size can be finer and finer as time progresses), this describes a thinned out Fermi liquid, as schematically plotted on the right of Fig. 13.

Since we have already verified the GGE hypothesis of our system for the density variable, let us describe the above ‘thinned out’ Fermi liquid in terms of the GGE. We begin with the GGE value (52) of the phase space density. The single-particle Hilbert space can be labelled by the momentum wavefunctions
\[ \phi_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}, \quad m = 0, \pm 1, \pm 2, \ldots, \]
which trivially gives $u_m(\theta, p) = \delta(m - p/\hbar)$, and hence by (52) (keeping the notation $\langle \ldots \rangle_{\text{GGE}}$ implicit)
\[ u(\theta, p) = \sum_m N_m \delta(m - p/\hbar) = N_{p/\hbar} \equiv \tilde{\rho}(p) \frac{1}{2\pi}. \] (71)
We can identify $\tilde{\rho}(p)$ as the momentum-space density
\[ \tilde{\rho}(p) \equiv \int d\theta u(\theta, p). \] (72)
We can compute $N_m$’s from a given initial quantum state by using (50). However, the calculation is much better done semiclassically, when the initial state is given in terms of a classical droplet configuration (63). Note, first of all, that the momentum-space density (72) in the present case is a constant of motion. This is easy to see since $h = p^2/2$ and all fermions have a constant
momentum so the momentum density remains constant.\footnote{\textsuperscript{31}It can also be seen from \textsuperscript{(29)} and \textsuperscript{(72)}.} We can, therefore, compute the momentum density from the initial configuration \textsuperscript{(63)}. Note that from \textsuperscript{(63)} and \textsuperscript{(26)} we can find out the initial value of $u(\theta, p)$ and hence $\bar{\rho}(p)$ by using \textsuperscript{(72)}. The result of this calculation is \footnote{\textsuperscript{32}Geometrically the RHS of \textsuperscript{(72)} is given by $\theta_+(p) - \theta_-(p)$ where $\theta_\pm(p)$ are the counterparts of $p_\pm(\theta)$ in \textsuperscript{(20)}.}

$$u(\theta, p) = \frac{1}{2\pi} \bar{\rho}(p) = \begin{cases} 0 & |p| > \frac{1}{2} + b, \\ 1 & |p| < \frac{1}{2} - b, \\ \frac{1}{\pi} \cos^{-1} \left( \frac{1}{b} \left( p - \frac{1}{2} \right) \right) & \text{otherwise}, \end{cases} \quad (73)$$

which gives us a quantitative representation of a ‘thinned out’ Fermi liquid on the right panel of Fig. \textsuperscript{13}.

\subsection*{D.2 Calculation of Entropy}

We will now calculate the entropy $S_{GGE}$. As we show in Appendix \textsuperscript{D.1} in the large $N$ droplet description, the $\langle N_m \rangle$’s are given by \textsuperscript{(71)}. Combining with \textsuperscript{(73)}, and \textsuperscript{(44)}, we get

$$S = -\sum_{m=0}^{\infty} \left[ \langle N_m \rangle_{\text{GGE}} \ln \langle N_m \rangle_{\text{GGE}} + (1 - \langle N_m \rangle_{\text{GGE}}) \ln(1 - \langle N_m \rangle_{\text{GGE}}) \right]$$

$$= -2 \int_{0}^{\infty} \frac{dp}{\hbar} [u(p) \ln u(p) + (1 - u(p)) \ln(1 - u(p))] = 2.36 \ b \ N \quad (74)$$

In the first equality, we have inserted a factor of 2 since the negative $p$-axis contributes the same amount. Also we have used $m = p/\hbar = Np$ from \textsuperscript{(71)}. In the last step, note that only $\frac{1}{2} - b < p < \frac{1}{2} + b$ contributes to the integral, for which we use the value of $u(\theta, p) \equiv u(p)$ in the third line of \textsuperscript{(73)}. Note that $S$ is exactly linear in $N$ (as well as in the amplitude $b$ of the initial perturbation \textsuperscript{(11)}).

\section*{E Details of our numerical analysis}

We explain the details of our computations in Section \textsuperscript{2} and \textsuperscript{3}.

\textbf{Exact computation} We use an exact computation for the single trace matrix model \textsuperscript{(2)} at finite $N$. In this case, the eigen function $\phi_m(\theta)$ is given by the Mathieu function. We numerically evaluate this function by using GCC with the GNU Scientific Library.
Semiclassical computation For the single trace matrix model (2) at large-N and the double trace matrix model (3), we use a semiclassical approximation. At the strict large-N, the dynamics of the matrix models are described by the droplet $u(\theta, p, t)$ in the phase space. We discretize the support of the initial droplet by $\hat{N}$ uniformly distributed points (we take $\hat{N}$ to be $O(10^4)$.) Each of these points is then evolved according to the classical equation of motion

$$\dot{\theta}_i = p_i, \quad \dot{p}_i = a \sin \theta_i, \tag{75}$$

for the single trace model (2), and according to the classical equation of motion

$$\dot{\theta}_i = p_i, \quad \dot{p}_i = -\xi \sum_j \sin(\theta_i - \theta_j), \tag{76}$$

for the double trace model (3). Here $\theta_i$ and $p_i (i = 1, \ldots, \hat{N})$ are the points of the discretized droplet. Evolving these points in this fashion is equivalent to a discretization of the Euler equation (29). By using the Runge-Kutta method, we solve these equations with a given initial condition. Note that the above use of the classical equations means that we are in the strict large-N limit. As a result, the number $\hat{N}$, which is simply a measure of discretization of the classical phase space, is not related to $N$ in the original matrix model (while $1/N$ measures quantum effects, $1/\hat{N}$ measures some discretization error of classical equations).

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33In the case of the single trace model, the motion of the surface of the droplet is sufficient to determine the dynamics of the model, since the particles are free. In addition, analytic solutions of the equation of motion (75) are given by elliptic functions, and can be used to significantly improve the calculation of droplet motion. Indeed we use this idea to plot Fig. 6 in the hermitian matrix model (25).
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