A DANZER SET FOR AXIS PARALLEL BOXES

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ABSTRACT. We present concrete constructions of discrete sets in $\mathbb{R}^d$ ($d \geq 2$) that intersect every aligned box of volume 1 in $\mathbb{R}^d$, and which have optimal growth rate $O(T^d)$.

1. Introduction

A set $D \subseteq \mathbb{R}^d$ is called a Danzer set if there exists an $s > 0$ such that $D$ intersects every convex set of volume $s$. The question whether a discrete Danzer set in $\mathbb{R}^d$ of growth rate $O(T^d)$ exists is due to Danzer, see $[CFG, Go, GL]$, and has been open since the sixties.

There are several ways to weaken this question. One is to weaken the Danzer property in the following sense. We say that $Y \subseteq \mathbb{R}^d$ is a dense forest if there is a function $\varepsilon = \varepsilon(T) \xrightarrow{T \to \infty} 0$ so that for every $x \in \mathbb{R}^d$ and for every direction $v \in S^{d-1}$, the distance between $Y$ and the line segment of length $T$ which starts at $x$ and proceeds in direction $v$ is less than $\varepsilon(T)$. Intuitively, as it was presented in $[Bi]$, $T$ is the maximal distance that a man can see when standing in a forest with a trunk of radius $\varepsilon$ located at each element of $Y$. Clearly a Danzer set is just a dense forest for which $\varepsilon(T) = O(T^{d-1})$. Constructions of dense forests of growth rate $O(T^d)$ are given in $[Bi, SW]$.

One other interesting direction is to look for Danzer sets with faster growth rates. A Danzer set of growth rate $O(T^d \log T)^{d-1}$ is given in $[BW]$; this bound was improved recently in $[SW]$ by a probabilistic construction that gives growth rate $O(T^d \log T)$.

Another approach in trying to weaken the Danzer problem is by hitting a smaller family of sets, instead of all the convex sets. John’s theorem $[Jo]$ implies that replacing convex sets by boxe$^1$ gives an equivalent question. In this note we consider a question that arises naturally from the Danzer problem. We say that $D \subseteq \mathbb{R}^d$ is an align-Danzer set if there is an $s > 0$ such that $D$ intersects every aligned box of volume $s$. In our main results, Theorem 1.1 and Theorem 1.3 below, we present simple constructions for align-Danzer sets in

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$^1$A box in $\mathbb{R}^d$ is the image of an aligned box $[a_1, b_1] \times \cdots \times [a_d, b_d]$ under an orthogonal matrix.
$\mathbb{R}^d$ of growth rate $O(T^d)$. The second of these constructions is not new, but the viewpoint of seeing it as connected with Danzer’s problem is new.

We denote by $\{0,1\}^{\mathbb{Z}}_{\text{Fin}}$ the subset of $\{0,1\}^{\mathbb{Z}}$ consisting of those bi-infinite sequences that contain only finitely many 1s.

**Theorem 1.1.** The set

$$D \overset{\text{def}}{=} \left\{ \left( \pm \sum_{n \in \mathbb{Z}} a_n 2^n, \pm \sum_{n \in \mathbb{Z}} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0,1\}^{\mathbb{Z}}_{\text{Fin}} \right\}$$

is an align-Danzer set in $\mathbb{R}^2$ of growth rate $O(T^2)$.

Although the set $D$ in Theorem 1.1 is given very explicitly, and the proof is by elementary means, it only solves the problem in dimension 2, and no simple higher-dimensional extension comes to mind. To solve the problem in higher dimensions we use a dynamical approach.

For a fixed $d \geq 2$ let $A \subseteq SL_d(\mathbb{R})$ be the subgroup of diagonal matrices with positive entries, and let $\Omega$ be the space of all lattices in $\mathbb{R}^d$.

**Definition 1.2 ([Sk, p.6]).** A lattice $\Lambda \in \Omega$ is *admissible* if its orbit under $A$ is precompact in $\Omega$.

**Theorem 1.3** (Corollary of [Sk, Theorem 1.2]). For every $d \geq 2$ there exists an admissible lattice in $\mathbb{R}^d$, and every admissible lattice is an align-Danzer set.

Although Theorem 1.3 is a direct consequence of [Sk, Theorem 1.2], we provide the proof since it is elementary.

As a direct consequence we reprove a result in computational geometry, that follows from a result of Halton on low discrepancy sequences, see [Ha]. We remark that Corollary 1.4 is not stated in [Ha], but it is well known in the computational geometry and combinatorics communities that Halton’s construction satisfies it.

**Corollary 1.4.** For every $\varepsilon > 0$ there are $\varepsilon$-nets of optimal sizes $O(1/\varepsilon)$ for the range space $(X, \mathcal{R})$, where $X = [0,1]^d$ and $\mathcal{R} = \{\text{aligned boxes}\}$.

This Corollary follows directly from the above Theorems by restricting to a bounded cube and rescaling to $[0,1]^d$. We refer to [AS, Ma] for a more comprehensive reading about the notions in Corollary 1.4.

**Remark 1.5.** Align Danzer sets in $\mathbb{R}^d$ of growth rate $O(T^d)$ can also be constructed by modifying the proof of [SW, Theorem 1.4] to work for aligned boxes and then combining with the result of Halton [Ha]. Nonetheless, our constructions here are simple and the proofs are straightforward.
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2. **Proof of Theorem 1.1**

**Proof of Theorem 1.1.** We first show that $D$ intersects every aligned box of volume 64. It suffices to show that

$$D_+ \overset{\text{def}}{=} \left\{ \left( \sum_{n \in \mathbb{Z}} a_n 2^n, \sum_{n \in \mathbb{Z}} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0, 1\}_F^{\mathbb{Z}} \right\}$$

intersects every aligned box of volume 16 that sits in $\mathbb{R}_+^2 = [0, \infty)^2$.

Let $R \subseteq \mathbb{R}_+^2$ be an aligned box of volume 16, and denote its lower left vertex by $(x, y)$. Let $t > 0$ be such that the lower right and the upper left vertices of $R$ are $(x + t, y)$ and $(x, y + \frac{16}{t})$ respectively. We define a sequence $(a_n)_{n \in \mathbb{Z}} \in \{0, 1\}_F^{\mathbb{Z}}$ so that $\left( \sum_{n \in \mathbb{Z}} a_n 2^n, \sum_{n \in \mathbb{Z}} a_n 2^{-n} \right) \in R$.

For each integer $k$, we denote by $\{0, 1\}_{F_{\leq k}}^{\mathbb{Z}}$ and $\{0, 1\}_{F_{< k}}^{\mathbb{Z}}$ the subsets of $\{0, 1\}_{F_{\leq k}}^{\mathbb{Z}}$ and $\{0, 1\}_{F_{< k}}^{\mathbb{Z}}$, respectively, consisting of those sequences that contain only finitely many 1s. Here $\{0, 1\}_{F_{\leq k}}^{\mathbb{Z}}$ is the set of all sequences in $\{0, 1\}^{\mathbb{Z}}$ of the form $(a_k, a_{k+1}, \ldots)$, and $\{0, 1\}_{F_{< k}}^{\mathbb{Z}}$ is the set of all sequences in $\{0, 1\}^{\mathbb{Z}}$ of the form $(\ldots, a_{k-2}, a_{k-1})$.

Let $k \in \mathbb{Z}$ be such that $2^k \leq \frac{t}{2} < 2^{k+1}$. Observe that $\sum_{n \leq k} a_n 2^n < 2^k \leq \frac{t}{2}$ for any sequence $(a_n)$ in $\{0, 1\}_{F_{\leq k}}^{\mathbb{Z}}$, and that the interval $(x, x + \frac{t}{2})$ intersects the set

$$2^k \mathbb{N} = \left\{ \sum_{n \geq k} a_n 2^n : (a_n) \in \{0, 1\}_{F_{\leq k}}^{\mathbb{Z}} \right\}.$$

Then we may choose the $a_n$s for $n \geq k$ so that $\sum_{n \geq k} a_n 2^n \in (x, x + \frac{t}{2})$, and thus for any choice of the $a_n$s for $n < k$ (and in particular for the choice described above) we have $\sum_{n \in \mathbb{Z}} a_n 2^n \in (x, x + t)$.

The analysis of the $y$ coordinate is similar. Here $2^{-k-1} \frac{8}{t} \leq 2^{-k}$, and therefore $2^{-k+1} \frac{8}{t} \leq 2^{-k+2}$. We have $\sum_{n \geq k} a_n 2^{-n} < 2^{-k+1} \frac{8}{t}$ for any sequence $(a_n)$ in $\{0, 1\}_{F_{\leq k}}^{\mathbb{Z}}$, and the interval $(y, y + \frac{8}{t})$ intersects the set

$$2^{-k+1} \mathbb{N} = \left\{ \sum_{n < k} a_n 2^{-n} : (a_n) \in \{0, 1\}_{F_{< k}}^{\mathbb{Z}} \right\}.$$

Then we may choose the $a_n$s for $n < k$ so that $\sum_{n < k} a_n 2^{-n} \in (y, y + \frac{8}{t})$, and thus for any choice of the $a_n$s for $n \geq k$ (and in particular for the choice described above) we have $\sum_{n \in \mathbb{Z}} a_n 2^{-n} \in (y, y + \frac{16}{t})$. 


It is left to show that $D$ (or $D_+$) is of growth rate $O(T^2)$. To see that, consider the set

$$B \overset{\text{def}}{=} \left\{ \left( \sum_{n \geq 0} a_n 2^n, \sum_{n < 0} a_n 2^{-n} \right) \in \mathbb{R}^2 : (a_n) \in \{0, 1\}^\mathbb{Z}_{\text{fin}} \right\}.$$

Observe that the mapping $g : D_+ \to B$ which is defined in the obvious way by

$$\left( \sum_{n \in \mathbb{Z}} a_n 2^n, \sum_{n \in \mathbb{Z}} a_n 2^{-n} \right) \mapsto \left( \sum_{n \geq 0} a_n 2^n, \sum_{n < 0} a_n 2^{-n} \right)$$

is a bijection, and for any $(x, y) \in D_+$ we have $\| (x, y) - g(x, y) \|_2 \leq \sqrt{5}$ (where $\|\cdot\|_2$ denotes the Euclidean norm). But since $B = \mathbb{N} \times 2\mathbb{N}$, the assertion follows. □

**Remark 2.1.** We want to stress that $D$ is not a Danzer set in $\mathbb{R}^2$ and not even a dense forest. To see it, observe that symmetric sequences $(a_n)$ correspond to points on the line $y = x$. On the other hand, non-symmetric sequences correspond to points $(x, y)$ with $|x - y| > 1$, and in particular $D$ misses a neighborhood of the line $y = x + \frac{1}{4}$.

### 3. Proof of Theorem 1.3

Fix $d \geq 2$. Let $V = \{ t \in \mathbb{R}^d : \sum_{i=1}^d t_i = 0 \}$, and for each $t \in V$ let $g_t \in SL_d(\mathbb{R})$ be the diagonal matrix whose entries are $e^{t_i}$. Then $t \mapsto g_t$ is a homomorphism.

**Proof of Theorem 1.3.** Let $K$ be a totally real number field of degree $d$, and let $\mathcal{O}_K$ be its ring of integers. Let $\phi_1, \ldots, \phi_d : K \to \mathbb{R}$ be the Galois embeddings of $K$ into $\mathbb{R}$, and let $\Phi : K \to \mathbb{R}^d$ be their direct sum. Then $\Lambda \overset{\text{def}}{=} \Phi(\mathcal{O}_K)$ is a lattice in $\mathbb{R}^d$. To see that $\Lambda$ is admissible, fix $x = \Phi(\alpha) \in \Lambda$, and observe that if $x \neq 0$,

$$\prod_{i=1}^d |x_i| = \prod_{i=1}^d |\phi_i(\alpha)| = |N(\alpha)| \in \mathbb{Z} \setminus \{0\}.$$

Here $N$ denotes the norm in the field $K$. In particular, $\prod_{i=1}^d |x_i| \geq 1$ and thus $\prod_{i=1}^d |e^{t_i}x_i| \geq 1$ for all $t \in V$. It follows that $|e^{t_i}x_i| \geq 1$ for some $i = 1, \ldots, d$ and thus $\| g_t x \| \geq 1$. Since $t, x$ were arbitrary this shows that $\Lambda$ is admissible.

For the second part of the proof, let $\Lambda$ be an admissible lattice in $\mathbb{R}^d$. Let $R$ be an aligned box disjoint from $\Lambda$. Then there exists $t \in V$ such that $g_t R$ is a cube. By assumption $g_t \Lambda$ is in a compact subset $K \subseteq \Omega$, hence it follows from Mahler’s compactness criterion that the codiameter of $g_t \Lambda$ is bounded above by a constant independent of $t$. But since $g_t R$ is disjoint from $g_t \Lambda$, the edge length of the cube $g_t R$ is bounded above by the codiameter of $g_t (\Lambda)$. 

Thus both the radius and the volume of $g_t R$ are bounded above by a constant independent of $t$. Since $\text{Vol}(R) = \text{Vol}(g_t R)$, the proof is complete. □

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