A geometric simulation theorem on direct products of finitely generated groups

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Abstract

We show that every effectively closed action of a finitely generated group $G$ over a Cantor set can be obtained as a topological factor of the $G$-subaction of a $(G \times H_1 \times H_2)$-subshift of finite type for any choice of infinite and finitely generated groups $H_1, H_2$. As a consequence, we obtain that every group of the form $G_1 \times G_2 \times G_3$ admits a non-empty strongly aperiodic SFT subject to the condition that each $G_i$ is finitely generated and has decidable word problem. As a corollary of this last result we prove the existence of a non-empty strongly aperiodic SFT in the Grigorchuk group.

1 Introduction

Dynamical systems can be often quite difficult to study and several tools have been developed in order to better understand them. A particularly interesting approach to do so is to restrict to subclasses of dynamical systems that can be defined using a finite amount of information. With this point of view, a reasonably large class of dynamical systems is that of effectively closed systems. Informally speaking, effectively closed systems are those where both the configurations in the system and the action can be described completely by a Turing machine. Even though these systems admit finite presentations, they remain quite complicated.

A natural question is whether an effectively closed system can be obtained as a subaction of another system which admits a simpler description. This question is motivated by the following: consider the class of subshifts of finite type (SFT), that is, the sets of colorings of a group which respect a finite number of local constrains –in the form of a finite list of forbidden patterns– and are equipped with the shift action. It can be easily shown that any system obtained as a restriction of the shift action to a subgroup is not necessarily an SFT, but under weak assumptions, such as the groups being recursively presented, we obtain that the subaction is still an effectively closed dynamical system, see [Hoc09] for the $\mathbb{Z}^d$ case.
For the class of $\mathbb{Z}^d$-SFTs there is still no characterization of which dynamical systems can arise as their subactions, there are in fact some effective $\mathbb{Z}$-dynamical systems that cannot appear as a subaction of a $\mathbb{Z}^d$-SFT [Hoc09] or more generally, of any shift space. Nevertheless, in that same article, Hochman proves every effectively closed $\mathbb{Z}^d$-action over a Cantor set $T : \mathbb{Z}^d \curvearrowright X$ admits an almost trivial isometric extension which can be realized as the subaction of a $\mathbb{Z}^{d+2}$-SFT. This result has subsequently been improved for the expansive case independently in [AS13] and [DRS10] showing that every effectively closed $\mathbb{Z}$-subshift can be obtained as the projective subdynamics of a sofic $\mathbb{Z}^2$-subshift.

This type of results give powerful techniques to prove properties about the original systems. An example is the characterization of the set of entropies of $\mathbb{Z}^2$-SFTs as the set of right recursively enumerable numbers [HM10]. More recently, in an article of Sablik and the author [BS17], Hochman’s theorem was extended to groups which are of the form $G = \mathbb{Z}^2 \rtimes H$. More specifically, it was shown that for every finitely generated group $H$, homomorphism $\varphi : H \to \text{Aut}(\mathbb{Z}^2)$ and $H$-effectively closed dynamical system $(X, T)$ one can construct a $(\mathbb{Z}^2 \rtimes H)$-SFT whose $H$-subaction is an extension of $(X, T)$. As a consequence of this result, a new class of groups admitting strongly aperiodic subshifts of finite type was found.

All of these previous results have a common denominator: they involve the use of a $\mathbb{Z}$ component in the group where the simulation is carried. In particular, a natural question would be to ask whether it is possible to obtain a realization result in a periodic group.

The purpose of this article is to prove a simulation theorem which does not involve a $\mathbb{Z}$ component. Specifically,

**Theorem 3.1.** Let $G$ be a finitely generated group and $(X, T)$ an effectively closed $G$-dynamical system. For every pair of infinite and finitely generated groups $H_1, H_2$ there exists a $(G \times H_1 \times H_2)$-SFT whose $G$-subaction is an extension of $(X, T)$.

This result is obtained through the combination of two different techniques already present in the literature. On the one hand, we use Toeplitz configurations to encode the dynamical system $(X, T)$ in an effectively closed $\mathbb{Z}$-subshift and then extend this object to a $\mathbb{Z}^2$-sofic subshift through the theorems of [AS13, DRS10]. This has already been used in [BS17]. On the other hand, we employ a technique previously used by Jeandel [Jea15b] to force grid structures through local rules. Namely, a theorem by Seward [Sew14] shows that a geometric analogue of Burnside’s conjecture holds, namely, that every infinite and finitely generated group admits a translation-like action by $\mathbb{Z}$. From a graph theoretical perspective, this means that the group admits a set of generators such that its associated Cayley graph can be covered by disjoint bi-infinite paths. We use that theorem and Jeandel’s technique to geometrically embed a two-dimensional grid into $H_1 \times H_2$ and create the necessary structure to prove our main result.

In the case where the $G$-dynamical system is a subshift, we can give a stronger result.
Theorem 4.1. Let $G$ be a recursively presented and finitely generated group and $Y$ an effectively closed $G$-subshift. For every pair of infinite and finitely generated groups $H_1, H_2$ there exists a sofic $(G 	imes H_1 	imes H_2)$-subshift $X$ such that:

- the $G$-subaction of $X$ is conjugate to $Y$.
- the $G$-projective subdynamics of $X$ is $Y$.
- The shift action $\sigma$ restricted to $H_1 \times H_2$ is trivial on $X$.

It is known that every $\mathbb{Z}$-SFT contains a periodic configuration [LM95]. More generally, any SFT defined over a finitely generated free group also has a periodic configuration [Pia08]. However, it was shown by Berger [Ber66] that there are $\mathbb{Z}^d$-SFTs which are strongly aperiodic, that is, such that the shift acts freely on the set of configurations. This result has been proven several times with different techniques [Rob71, Kar96, JR15] giving a variety of constructions. However, the problem of determining which is the class of groups which admit strongly aperiodic SFTs remains open. Amongst the groups that do admit strongly aperiodic SFTs are: $\mathbb{Z}^d$ for $d > 1$, hyperbolic surface groups [CGS15], Osin and Ivanov monster groups [Jea15a], the direct product $G \times \mathbb{Z}$ for a particular class of groups $G$ which includes Thompson’s $T$ group and $\text{PSL}(\mathbb{Z}, 2)$ [Jea15a] and groups of the form $\mathbb{Z}^2 \rtimes G$ where $G$ is finitely generated and has decidable word problem [BS17]. It is also known that no group with two or more ends can contain strongly aperiodic SFTs [Coh17] and that recursively presented groups which admit strongly aperiodic SFTs must have decidable word problem [Jea15a].

As an application of Theorem 3.1 we present a new class of groups which admit strongly aperiodic SFTs, that is:

Theorem 4.4. For any triple of infinite and finitely generated groups $G_1, G_2$ and $G_3$ with decidable word problem, their direct product $G_1 \times G_2 \times G_3$ admits a non-empty strongly aperiodic subshift of finite type.

A result by Carroll and Penland [CP15] shows that having a non-empty strongly aperiodic subshift of finite type is a commensurability invariant of groups. Putting this together with Theorem 4.4 we deduce that any finitely generated group with decidable word problem which is commensurable to its square also has the property. In particular, as the Grigorchuk group is commensurable to its square, this yields the existence of non-empty strongly aperiodic subshifts of finite type in the Grigorchuk group.

Corollary 4.8. There exists a non-empty strongly aperiodic subshift of finite type defined over the Grigorchuk group.

This strengthens Jeandel’s result from [Jea15b] where the Grigorchuk group was shown to admit a weakly aperiodic SFT, that is, a subshift such that the orbit of every configuration under the shift action is infinite.
2 Preliminaries

Consider a group $G$ and a compact topological space $X$. The tuple $(X, T)$ where $T : G \times X \to X$ is a left $G$-action by homeomorphisms is called a $G$-dynamical system. Let $(X, T)$ and $(Y, S)$ be two $G$-dynamical systems. We say $\phi : X \to Y$ is a topological morphism if it is continuous and $\phi \circ T^g = S^g \circ \phi$ for all $g \in G$.

A surjective topological morphism $\phi : X \to Y$ is a topological factor and we say that $(Y, S)$ is a factor of $(X, T)$ and that $(X, T)$ is an extension of $(Y, S)$. When $\phi$ is a bijection and its inverse is continuous we say it is a topological conjugacy and that $(X, T)$ is conjugated to $(Y, S)$.

In what follows, we consider the space $X$ to be a Cantor set equipped with the product topology and $G$ to be a finitely generated group. Without loss of generality, we consider $X$ to be a closed subset of $\{0, 1\}^N$ and for a word $w = w_0, \ldots, w_n \in \{0, 1\}^*$ we denote by $[w]$ the set of all points $x \in \{0, 1\}^N$ such that $x_i = w_i$ for $i \leq n$. Let $S \subset G$ a finite set of generators of $G$. An effectively closed $G$-dynamical system is a $G$-dynamical system $(X, T)$ where:

1. $X \subset \{0, 1\}^N$ is a closed effective subset: $X = \{0, 1\}^N \setminus \bigcup_{i \in I} [w_i]$ where $\{w_i\}_{i \in I} \subset \{0, 1\}^*$ is a recursively enumerable language. That means that $X$ is the complement of a union of cylinders which can be enumerated by a Turing machine.

2. $T$ is an effectively closed action: there exists a Turing machine which on entry $s \in S$ and $w \in \{0, 1\}^*$ enumerates a sequence of words $(w_j)_{j \in J}$ such that $T^s([w]) = \{0, 1\}^N \setminus \bigcup_{j \in J} [w_j]$.

The idea behind the definition is the following: There is a Turing machine $T$ which given a word $q \in S^*$ representing an element of $G$ and $n$ coordinates of $x \in X \subset \{0, 1\}^N$ returns a sequence of sets which yield successive approximations of the image of $x$ by $T^s$.

Let $A$ be a finite alphabet and $G$ a finitely generated group. The set $A^G = \{x : G \to A\}$ equipped with the left group action $\sigma : G \times A^G \to A^G$ given by: $(\sigma^h(x))_g = x_{h^{-1}g}$ is the full $G$-shift. The elements $a \in A$ and $x \in A^G$ are called symbols and configurations respectively. We endow $A^G$ with the product topology, therefore obtaining a compact metric space. The topology is generated by the metric $d(x, y) = 2^{-\inf\{|g| \mid g \in G : x_g \neq y_g\}}$ where $|g|$ is the length of the smallest expression of $g$ as the product of some fixed set of generators of $G$. This topology is also generated by the clopen subbase given by the cylinders $[a]_g = \{x \in A^G \mid x_g = a \in A\}$. A support is a finite subset $F \subset G$. Given a support $F$, a pattern with support $F$ is an element $p \in A^F$, i.e. a finite configuration and we write $\text{supp}(p) = F$. Analogously to words, we denote the cylinder generated by $p$ centered in $g$ as $[p]_g = \bigcap_{h \in F} [p_h]_{gh}$. If $x \in [p]_g$ for some $g \in G$ we write $p \subset x$ to say that $p$ appears in $x$.

A subset $X$ of $A^G$ is a $G$-subshift if and only if it is $\sigma$-invariant – for each $g \in G$, then $\sigma^g(X) \subset X$ and closed for the product topology. Equivalently, $X$ is a $G$-subshift if and only if there exists a set of forbidden patterns $F$ that...
defines it:

\[ X = X_\mathcal{F} := \mathcal{A}_G \setminus \bigcup_{p \in \mathcal{F}, g \in G} [p]_g. \]

Said otherwise, the \( G \)-subshift \( X_\mathcal{F} \) is the set of all configurations \( x \) such that no \( p \in \mathcal{F} \) appears in \( x \).

If the context is clear enough, we will drop the group \( G \) from the notation and simply refer to a subshift. A subshift \( X \subseteq \mathcal{A}_G \) is \textit{sofic} if there exists a recursive set of forbidden patterns \( \mathcal{F} \) such that \( X = X_\mathcal{F} \).

We say a \( \mathcal{A}_G^2 \)-subshift is \textit{nearest neighbor} if there exists a set of forbidden patterns \( \mathcal{F} \) defining it such that each \( p \in \mathcal{F} \) has support \( \{(0,0), (1,0)\} \) or \( \{(0,0), (0,1)\} \). While there are \( \mathcal{A}_G^2 \)-SFTs which are not nearest neighbor, each \( \mathcal{A}_G^2 \)-SFT is topologically conjugate to a nearest neighbor \( \mathcal{A}_G^2 \)-subshift through a higher block recoding, see for instance [LM95] for the 1-dimensional case. What is more, for every sofic \( \mathcal{A}_G^2 \)-subshift \( Y \) we can jointly extract a nearest neighbor \( \mathcal{A}_G^2 \)-SFT extension \( \tilde{X} \) and a 1-block topological factor \( \tilde{\phi} : \tilde{X} \to Y \), that is, a topological factor such that there exists a local recoding of the alphabet \( \tilde{\Phi} : \mathcal{A}_\tilde{X} \to \mathcal{A}_Y \) such that for each \( x \in \tilde{X} \) we have \((\tilde{\phi}(x))_z = \tilde{\Phi}(x_z)\).

Let \( H \leq G \) be a subgroup and \((X, T)\) a \( G \)-dynamical system. The \( H \)-subaction of \((X, T)\) is \((X, T_H)\) where \( T_H : H \curvearrowright X \) is the restriction of \( T \) to \( H \), that is, for each \( h \in H \), then \((T_H)^h(x) = T^h(x)\). In the case of a subshift \( X \subset \mathcal{A}_G \) there is also the different notion of projective subdynamics. The \( H \)-projective subdynamics of \( X \) is the set \( \pi_H(X) = \{y \in \mathcal{A}_H \mid \exists x \in X, \forall h \in H, y_h = x_h\} \). It is important to remark that subactions do not preserve expansivity, so in particular a subaction of a subshift is not necessarily a subshift. Nevertheless, the projective subdynamics of a subshift \( \pi_H(X) \) is always an \( H \)-subshift.

### 3 Simulation without \( \mathbb{Z} \)

The purpose of this section is to prove the following result.

**Theorem 3.1.** Let \( G \) be a finitely generated group and \((X, T)\) an effectively closed \( G \)-dynamical system. For every pair of infinite and finitely generated groups \( H_1, H_2 \) there exists a \((G \times H_1 \times H_2)\)-SFT whose \( G \)-subaction is an extension of \((X, T)\).

The general scheme of the proof is the following: First, in Section 3.1 we use Toeplitz sequences to encode the elements of \( X \) and the effectively closed action \( T : G \curvearrowright X \) into an effectively closed \( \mathbb{Z} \)-subshift \( \text{Top}_{1D}(X, T) \). Subsequently, we extend \( \text{Top}_{1D}(X, T) \) to a \( \mathbb{Z}^2 \)-subshift by repeating its rows periodically in the vertical direction. Using a known simulation theorem [AS13, DRS10] we
conclude that this object, which we call $\text{Top}^2(X, T)$, is a sofic $\mathbb{Z}^2$-subshift from which we extract a nearest neighbor SFT extension $\hat{\text{Top}}(X, T)$.

The next step is presented in Section 3.2 where we construct an SFT $\Omega$ in $H_1 \times H_2$ with the property that each $\omega \in \Omega$ induces a bounded $\mathbb{Z}^2$-action over $H_1 \times H_2$. We use a result by Seward [Sew14] to guarantee that for a specific choice of generators of $H_1$ and $H_2$, then there is at least one $\omega \in \Omega$ inducing a free action. We use these actions as replacements of two-dimensional grids, which we then proceed to use to embed configurations of a nearest neighbor $\mathbb{Z}^2$-subshift.

Finally, in Section 3.3 we use the simulated grids in $H_1 \times H_2$ to encode $\hat{\text{Top}}(X, T)$. This yields an $(H_1 \times H_2)$-SFT which factors onto a sofic $(H_1 \times H_2)$-subshift where every grid codes an element $x \in X$ and its image under $T$ by the generators of $G$. We then proceed to extend this object to a $G \times H_1 \times H_2$ subshift of finite type by forcing every $G$-coset to have exactly the same grid structure and by linking the $\text{Top}_{1D}(X, T)$ layers through local rules.

We finish this section by defining the factor code and showing that it satisfies the required properties.

### 3.1 Encoding an effectively closed dynamical system using Toeplitz configurations

Let $T : G \curvearrowright X$ be an effectively closed action of a finitely generated group $G$ over $X \subset \{0, 1\}^\mathbb{N}$. Here we show how to encode $T$ into an effectively closed $\mathbb{Z}$-subshift. This section presents the same ideas as that of [BS17], although here we will only treat a special simplified case which is enough for our purposes.

A configuration $\tau \in A^\mathbb{Z}$ is said to be Toeplitz if for every $m \in \mathbb{Z}$ there is $p > 0$ such that $\tau_m = \tau_{m+kp}$ for each $k \in \mathbb{Z}$. These configurations were initially defined by Jacobs and Keane [JK69] for one-sided dynamical systems and are quite useful to encode information in a recurrent way. Indeed, consider the function $\Psi : \{0, 1\}^\mathbb{N} \to \{0, 1, \$\}^\mathbb{Z}$ given by:

$$\Psi(x)_j = \begin{cases} x_n & \text{if } j = 3^n \mod 3^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

For instance, if we write $x = x_0x_1x_2x_3\ldots$ we obtain,

$$\Psi(x)_{\{0,\ldots,30\}} = \$x_0\$x_1x_0\$x_2x_0\$x_3x_0x_0\$x_4x_0\$x_5x_0\$x_6x_0\$x_7x_0\$x_8x_0\$x_9x_0\$x_{10}x_0\$x_{11}x_0\$x_{12}x_0\$x_{13}x_0\$x_{14}x_0\$x_{15}x_0\$x_{16}x_0\$x_{17}x_0\$x_{18}x_0\$x_{19}x_0\$x_{20}x_0\$x_{21}x_0\$x_{22}x_0\$x_{23}x_0\$x_{24}x_0\$x_{25}x_0\$x_{26}x_0\$x_{27}x_0\$x_{28}x_0\$x_{29}x_0\$x_{30}x_0.$$

Technically speaking, $\Psi(x)$ is not Toeplitz as $m = 0$ fails to satisfy the requirement, however, every other $m \in \mathbb{Z} \setminus \{0\}$ does. For $x = (x_i)_{i \in \mathbb{N}} \in \{0, 1\}^\mathbb{N}$ let $\sigma(x) \in \{0, 1\}^\mathbb{N}$ be the one-sided shift defined by $\sigma(x)_i = x_{i+1}$ and note that for every $j \in \mathbb{Z}$ we have that:

$$\Psi(x)_{3j} = \Psi(\sigma(x))_j, \quad \Psi(x)_{3j+1} = x_0, \quad \text{and} \quad \Psi(x)_{3j+2} = \$.$$ 

The important property of $\Psi(x)$ is that $x$ can be recognized locally from any configuration $y \in \text{Orb}_{\sigma}(\Psi(x))$. Indeed, each subword of length 3 in $\Psi(x)$ is
a cyclic permutation of a word of the form $ax_0$ where $a \in \{0, 1, \}$, therefore
$x_0$ can be recognized as it is the only non-$\$ symbol which is followed by a $\$. Similarly, any word of length $9$ can be used to decode $x_0, x_1$ and generally, a word of length $3^n$ is sufficient to decode $x_0, x_1, \ldots x_{n-1}$. As any $y \in \text{Orb}_G(\Psi(x))$ must coincide in arbitrarily large blocks with a shift of $\Psi(x)$ we have that this property holds for every configuration in $\text{Orb}_G(\Psi(x))$.

Let $(X, T)$ be a $G$-dynamical system. We use the encoding $\Psi$ defined above to construct an effectively closed $\mathbb{Z}$-subshift $G_{1D}(X, T)$ which encodes the configurations of $X$ and the action of $T$ around a unit ball in $G$. Formally, let $S \subset G$ be a finite and symmetric ($S^{-1} \subset S$) set of generators of $G$ which contains the identity.

Given a word $w \in (\{0, 1, \}$)$^S)$, we denote its $s$-th coordinate by $w(s) \in \{0, 1, \}$$. We define $G_{1D}(X, T)$ as the $\mathbb{Z}$-subshift over the alphabet $\{0, 1, \}$ given by the set of forbidden words $F_T$, where $F_T = \bigcup_{m \in N} \mathcal{F}_n$ and $\mathcal{F}_n$ is the set of words $w$ of length $3^n+1$ over the alphabet $\{0, 1, \}$ which are accepted by the following algorithm.

For each $s \in S$ do the following: fix $j := 0$ and check if there exists $b \in \{0, 1\}$ such that each string of three contiguous symbols appearing in $w(s)$ is a cyclic permutation of $ab$ for some $a \in \{0, 1, \}$. If no such $b$ exists, accept the word as forbidden, otherwise, define $u(s)_j = b$, set $j := j + 1$ and repeat with the string formed by the $a$’s until either a problem is found (and thus the word is forbidden) or a valid symbol $u(s)_m$ is found for every $s \in S$ and $m \in \{0, \ldots, n\}$.

Finally, if this stage of the algorithm is reached without accepting the word, we can construct words $u(s) := u(s)_0u(s)_1 \ldots u(s)_n$. In parallel, for each $s \in S$, run simultaneously the following two procedures:

- Check whether $[u(s)] \subset \{0, 1\}^N \setminus X$ and accept if this is the case.
- Check whether $[u(s)] \subset \{0, 1\}^N \setminus T^*(\{u(1_G)\})$ and accept if this is the case.

These two last algorithms do exist in the case where $(X, T)$ is an effectively closed $G$-dynamical system. In particular, this shows that in this case $F_T$ is recursively enumerable and thus $G_{1D}(X, T)$ is effectively closed. We state this formally in Proposition 3.2.

**Proposition 3.2.** If $(X, T)$ is an effectively closed $G$-dynamical system, then $G_{1D}(X, T)$ is an effectively closed $\mathbb{Z}$-subshift.

Consider $y \in G_{1D}(X, T)$. By definition none of the words of length $3^n$ appearing in $y$ belongs to $F_T$ and thus any $u \subset y(s)$ of length $3^n$ defines a unique word $u_w \in \{0, 1\}^n$ by the decoding process. Moreover, one can easily verify with the definition of $F_T$ by using a word of length $3^n+1$ that $u_w$ does not depend on the specific choice $w \subset y(s)$ but only on $y(s)$. We can therefore define a family of functions $\gamma_n : S \times G_{1D}(X, T) \to \{0, 1\}$ by

$$\gamma_n(s, y) := (u_{y(s)}(1, \ldots, 3^n+1)_n$$.

Said otherwise, $\gamma_n$ recovers the $n$-th symbol from the $s$-th component of $y \in G_{1D}(X, T)$. Furthermore, we can use the $\gamma_n$ to construct a function
\[ \gamma : S \times \text{Top}_{1D}(X, T) \to \{0, 1\}^\mathbb{N} \text{ defined by } \gamma(s, y)_n := \gamma_n(s, y). \] By definition of \( F_T \) we have that for each \( n \in \mathbb{N} \) then:

\[ [\gamma(s, y)_0, \ldots, \gamma(s, y)_n] \cap X \neq \emptyset, \]

\[ [\gamma(s, y)_0, \ldots, \gamma(s, y)_n] \cap T^s([\gamma(1G, y)_0, \ldots, \gamma(1G, y)_n]) \neq \emptyset. \]

From the fact that \( X \) is compact and \( T \) continuous, we deduce that \( \gamma(s, y)_n \in X \) and \( T^s(\gamma(1G, y)) = \gamma(s, y) \). Also, if we define \( \tilde{\Psi}(x) \) as the configuration in \( (\{0, 1, $\}$)^S \) such that \( (\tilde{\Psi}(x))_n = \Psi(T^s(x))_n \) we can verify that \( \tilde{\Psi}(x) \in \text{Top}_{1D}(X, T) \) and \( \gamma(s, \tilde{\Psi}(x)) = T^s(x) \). Therefore \( \gamma : S \times \text{Top}_{1D}(X, T) \to X \) is onto. Finally, from the fact that each \( \gamma_n \) only depends on an arbitrary subword of length \( 3^n \) of \( y \), we obtain that \( \gamma(s, y) = \gamma(s, \sigma^m(y)) \) for each \( m \in \mathbb{Z} \) and that \( \gamma \) is continuous.

Next we will make use of a known simulation theorem to lift our effectively closed \( \mathbb{Z} \)-subshift up to a sofic \( \mathbb{Z}_2 \)-subshift.

**Theorem 3.3 ([AS13, DRS10]).** Let \( X \) be an effectively closed \( \mathbb{Z} \)-subshift. There exists a sofic \( \mathbb{Z}_2 \)-subshift \( Y \) such that for every \( y \in Y \) we have \( \sigma^{(0,1)}(y) = y \) and \( \pi_{(\mathbb{Z},0)}(Y) = X \).

Using Theorem 3.3 we obtain a sofic \( \mathbb{Z}_2 \)-subshift which we call \( \text{Top}_{2D}(X, T) \). As every row in a configuration in \( \text{Top}_{2D}(X, T) \) is the same, we can naturally extend the definition of \( \gamma \) to this subshift by restricting to \( \mathbb{Z} \times \{0\} \). We resume the important points of all that has been constructed in this subsection in the following lemma.

**Lemma 3.4.** There exists a sofic \( \mathbb{Z}_2 \)-subshift \( \text{Top}_{2D}(X, T) \) and a continuous onto function \( \gamma : S \times \text{Top}_{2D}(X, T) \to X \) with the following properties:

1. For any \( z \in \mathbb{Z}_2 \), \( \gamma(s, y) = \gamma(s, \sigma^z(y)) \).
2. \( \gamma(s, y) = T^s(\gamma(1G, y)) \).

### 3.2 Finding a grid in \( H_1 \times H_2 \)

The second ingredient of the proof uses the notion of translation-like action introduced by Whyte [Why99]. Instead of having the rigidity of a proper translation, translation-like actions only ask for the action to be free and that given an element of the acting group \( g \) then the image by \( g \) does not go arbitrarily far. Formally,

**Definition 3.1.** A left action of a group \( G \) over a metric space \( (X, d) \) is translation-like if and only if it satisfies:

1. \( G \curvearrowright X \) is free, that is, \( gx = x \) implies \( g = 1_G \).
2. For every \( g \in G \) the set \( \{d(gx, x) \mid x \in X\} \) is bounded.
This notion gives a proper setting to define geometric analogues of classical disproved conjectures in group theory concerning subgroup containments. For instance, the Burnside conjecture and the Von Neumann conjecture can be reinterpreted geometrically as the question of whether every infinite and finitely generated group admits a translation-like action by \( \mathbb{Z} \) or by a non-abelian free group respectively.

In what concerns our study, we are only going to make use of the following result from Seward [Sew14] which is the geometric version of the Burnside conjecture:

**Theorem 3.5** ([Sew14], Theorem 1.4). *Every finitely generated infinite group admits a translation-like action of \( \mathbb{Z} \).*

Theorem 3.5 has already been used by Jeandel in [Jeand15b] to show that the Domino problem—that is, the problem of deciding whether a finite set of forbidden patterns defines a non-empty subshift—is undecidable for groups of the form \( H_1 \times H_2 \) where both \( H_i \) are infinite and finitely generated. He also showed that groups containing such a product as a subgroup have undecidable Domino problem and admit weakly aperiodic SFTs, that is, an SFT such that every configuration has an infinite orbit. Here we make use of the same technique to prove our result.

Let \( S_1, S_2 \) be finite and symmetric sets of generators for \( H_1 \) and \( H_2 \) respectively which contain the identity. Consider the alphabet \( B = S_1 \times S_1 \times S_2 \times S_2 \). For \( b = (s_1, s_2, s_3, s_4) \) we write

\[
\begin{align*}
in_1(b) &= s_1, & \text{out}_1(b) &= s_2, \\
in_2(b) &= s_3, & \text{out}_2(b) &= s_4.
\end{align*}
\]

We can think of \( \text{in}_i \) and \( \text{out}_i \) as arrows pointing towards the left and right neighbor in a path.

We define a subshift \( \Omega \subset B^{H_1 \times H_2} \) by the set of forbidden patterns \( \mathcal{F}_\Omega \) such that \( p \in \mathcal{F}_\Omega \) if and only if \( p \) is one of the following patterns:

1. \( \text{supp}(p) = \{(1_{H_1}, 1_{H_2}), (s, 1_{H_2})\} \) where \( s \in S_1 \) and \( \text{in}_2(p(1_{H_1}, 1_{H_2})) \neq \text{in}_2(p(s, 1_{H_2})) \) or \( \text{out}_2(p(1_{H_1}, 1_{H_2})) \neq \text{out}_2(p(s, 1_{H_2})) \).
2. \( \text{supp}(p) = \{(1_{H_1}, 1_{H_2}), (1_{H_1}, s)\} \) where \( s \in S_2 \) and \( \text{in}_1(p(1_{H_1}, 1_{H_2})) \neq \text{in}_1(p(1_{H_1}, s)) \) or \( \text{out}_1(p(1_{H_1}, 1_{H_2})) \neq \text{out}_1(p(1_{H_1}, s)) \).
3. \( \text{supp}(p) = \{(1_{H_1}, 1_{H_2}), (s, 1_{H_2})\} \) where \( s \in S_1 \) and either
   - \( \text{out}_1(p(1_{H_1}, 1_{H_2})) = s \) but \( \text{in}_1(p(s, 1_{H_2})) \neq s^{-1} \)
   - or \( \text{in}_1(p(1_{H_1}, 1_{H_2})) = s \) but \( \text{out}_1(p(s, 1_{H_2})) \neq s^{-1} \).
4. \( \text{supp}(p) = \{(1_{H_1}, 1_{H_2}), (1_{H_1}, s)\} \) where \( s \in S_2 \) and
   - \( \text{out}_2(p(1_{H_1}, 1_{H_2})) = s \) but \( \text{in}_2(p(1_{H_1}, s)) \neq s^{-1} \)
   - or \( \text{in}_2(p(1_{H_1}, 1_{H_2})) = s \) but \( \text{out}_2(p(1_{H_1}, s)) \neq s^{-1} \).
In simpler words, the first and second rules just codify that when moving in a component $H_i$, then the arrows associated to the other component do not change. For instance, each $\omega \in \Omega$ satisfies that $\text{in}_1(\omega(h_1,h_2)) = \text{in}_1(\omega(h_1,1h_2))$ for each $h_2 \in H_2$.

The third and fourth rules code the fact that in any configuration $\omega \in \Omega$, if one looks at a particular position $(h_1, h_2)$, follows an arrow in a component and then in the new position follows the inverse arrow, one comes back to the starting position. In particular, we can interpret $\omega \in \Omega$ as a product of two directed graphs with both in-degree and out-degree 1, that is, a collection of cycles and bi-infinite paths. Here $\text{in}_i(\omega(h_1,h_2))$ codes the incoming arrow in the $i$-th component while $\text{out}_i(\omega(h_1,h_2))$ codes the corresponding outgoing arrow. With this graph product interpretation, each connected component of some $\omega \in \Omega$ can be either a product of two finite cycles, a product of a cycle and an infinite path, or a product of two infinite paths. Figure 1 shows how these grids look like in the case where both $H_1$ and $H_2$ are covered by bi-infinite paths.

![Figure 1: Finding a grid in $H_1 \times H_2$](image)

Clearly $\Omega$ is an $(H_1 \times H_2)$-SFT as $S$ is finite. We can also give a group theoretical interpretation of $\Omega$ as a family of $\mathbb{Z}^2$ actions over $H_1 \times H_2$, namely, given $\omega \in \Omega$ we can define $\eta(\omega) : \mathbb{Z}^2 \curvearrowright H_1 \times H_2$ by:

$$
\eta(\omega)^{1,0}(h_1, h_2) := (h_1 \text{out}_1(\omega(h_1,h_2)), h_2),
$$

$$
\eta(\omega)^{0,1}(h_1, h_2) := (h_1, h_2 \text{out}_2(\omega(h_1,h_2))),
$$

$$
\eta(\omega)^{-1,0}(h_1, h_2) := (h_1 \text{in}_1(\omega(h_1,h_2)), h_2),
$$

$$
\eta(\omega)^{0,-1}(h_1, h_2) := (h_1, h_2 \text{in}_2(\omega(h_1,h_2))).
$$

The first and second rules force $\eta(\omega)$ to be commuting, while the third and fourth rules make the inverses well defined.

Let $X \subset (A_X)^{\mathbb{Z}^2}$ be a $\mathbb{Z}^2$-subshift given by the nearest neighbor set of forbidden patterns $F_X$. We define the $(H_1 \times H_2)$-SFT $\Omega(X) \subset \Omega \times A_X^{H_1 \times H_2}$ by adding the forbidden patterns $F_{\Omega(X)}$. Write $p = (p_1, p_2) \in B^{\text{supp}(p)} \times (A_X)^{\text{supp}(p)}$ and define $F_{\Omega(X)}$ as the set of patterns $p$ such that supp$(p) = S_1 \times S_2$ and the following condition holds:
Let $s_1 = \text{out}_1(p_1(1_{H_1}, 1_{H_2}))$ and $s_2 = \text{out}_2(p_1(1_{H_1}, 1_{H_2}))$. Define $q \in (A_X)^{((0,0),(1,0))}$ and $r \in (A_X)^{((0,0),(0,1))}$ by setting:

$$
q(0,0) = p_2(1_{H_1}, 1_{H_2}) \quad q(1,0) = p_2(s_1, 1_{H_2})
$$

$$
r(0,0) = p_2(1_{H_1}, 1_{H_2}) \quad r(0,1) = p_2(1_{H_1}, s_2)
$$

We have $p \in \mathcal{F}_{\Omega(X)}$ if and only if $q \in \mathcal{F}_X$ or $r \in \mathcal{F}_X$. In other words, we forbid patterns where forbidden patterns from $\mathcal{F}$ and let $Y$ identified to $(0, 0)$. By definition of $\mathcal{F}_{\Omega(X)}$, we have that $\mathcal{F}(\omega, y)$ satisfies the requirements. Indeed, we can define a function $\mathcal{C} : \Omega(X) \times H_1 \times H_2 \rightarrow (A_X)^{\mathbb{Z}^2}$ by setting for each $z \in \mathbb{Z}^2$

$$
\mathcal{C}(\omega, y, h_1, h_2)_z := y_{\eta(\omega)^z(h_1, h_2)}.
$$

In simpler words, $\mathcal{C}(\omega, y, h_1, h_2)$ is the configuration obtained by reading $y$ along the two-dimensional grid defined by $\eta(\omega)$ where the pair $(h_1, h_2)$ is identified to $(0, 0)$. By definition of $\mathcal{F}_{\Omega(X)}$, we have that $\mathcal{C}(\omega, y, h_1, h_2) \in \mathcal{F}_X$ as no forbidden patterns from $\mathcal{F}_X$ can occur.

In particular, if the action $\eta(\omega)$ is not free there exist $(h_1, h_2)$ and $z \in \mathbb{Z}^2$ such that $\eta(\omega)^z(h_1, h_2) = (h_1, h_2)$ and this implies that $\mathcal{C}(\omega, y, h_1, h_2)_z = \mathcal{C}(\omega, y, h_1, h_2)_{(0,0)}$. In particular, if $R$ is a nearest neighbor strongly aperiodic $\mathbb{Z}^2$-subshift, for instance, a nearest neighbor recoding of the Robinson tiling [Rob71] we have that every pair $(\omega, y) \in \Omega(R)$ must satisfy that $\eta(\omega)$ is free.

**Proposition 3.6.** There exist finite and symmetric sets of generators $S_1$ and $S_2$ for $H_1$ and $H_2$ respectively which contain the identity and such that $\Omega$ contains a configuration $\omega$ such that $\eta(\omega)$ is free.

**Proof.** By Theorem 3.5 there exist translation-like actions $f_1 : \mathbb{Z} \rightharpoonup H_1$ and $f_2 : \mathbb{Z} \rightharpoonup H_2$. Consider generator metrics $d_1$ and $d_2$ on $H_1$ and $H_2$ respectively and define for $i \in \{1, 2\}$

$$
S'_i := \left\{ s \in H_i \mid d_i(1_{H_i}, s) \leq \sup_{h \in H_i} d_i(h, f_i(1, h)) \right\}.
$$

As $f_i$ is a translation-like, we know that the distance from $f_i(1, h)$ to $h$ is uniformly bounded, therefore $S'_i$ is finite. Let $S_i$ be a finite set of generators of $H_i$ with the desired properties and which contains $S'_i$. We claim that $S_1, S_2$ satisfy the requirements. Indeed, we can define $\omega \in \Omega$ by:

$$
\omega(h_1, h_2) := (f_1(-1, h_1), f_1(1, h_1), f_2(-1, h_2), f_2(1, h_2)).
$$

We verify directly that $\eta(\omega)^{(1,0)} = f_1$ and $\eta(\omega)^{(0,1)} = f_2$. As both $f_i$ are free, we have that $\eta(\omega)$ is free. 

**Proposition 3.7.** Suppose $\Omega$ contains a configuration $\omega$ such that $\eta(\omega)$ is free and let $Y$ be a nearest neighbor $\mathbb{Z}^2$-subshift. Then for each $c \in Y$ there exists $(\omega, y) \in \Omega(Y)$ such that for every $(h_1, h_2) \in H_1 \times H_2$ then $\mathcal{C}(\omega, y, h_1, h_2) \in \text{Orb}_\omega(c)$ and $c = \mathcal{C}(\omega, y, 1_{H_1}, 1_{H_2})$. 

11
Proof. We begin by fixing an arbitrary starting point in every grid, formally, define the equivalence relation over \( H_1 \times H_2 \) where \((h_1, h_2) \sim (h'_1, h'_2)\) if and only if there is \( z \in \mathbb{Z} \) such that \( \eta(\omega)^2(h_1, h_2) = (h'_1, h'_2) \). Let \((h'_1, h'_2)_{i \in I}\) be a representing set of \((H_1 \times H_2)/\sim\) which contains \((1_{H_1}, 1_{H_2})\). We define \( y \in \mathcal{A}_{H_1 \times H_2}' \) by

\[ y_{\eta(\omega)}(h'_1, h'_2) := c_z. \]

By definition of \( \sim \) and freeness of \( \eta(\omega) \), \( y \) is well defined over all of \((H_1 \times H_2)\). Moreover, by definition of \( \mathcal{F}_{\Omega(Y)} \), we have that \((\omega, y) \in \Omega(Y)\).

Let \( z \in \mathbb{Z} \) and \( i \in I \) such that \((h_1, h_2) = \eta(\omega)^2(h'_1, h'_2)\). We have \( \mathcal{C}(\omega, y, h_1, h_2) = \sigma^{-z}(c) \in \text{Orb}_\sigma(c) \). Finally, as \((1_{H_1}, 1_{H_2})\) belongs to the representing set we obtain \( \mathcal{C}(\omega, y, 1_{H_1}, 1_{H_2}) = \sigma^{(0,0)}(c) = c \). \( \square \)

We collect everything we need from this subsection in the next lemma.

**Lemma 3.8.** For every non-empty nearest neighbor \( \mathbb{Z}^2 \)-subshift \( Y \), there exists a \((H_1 \times H_2)\)-subshift \( \Omega(Y) \subset \Omega \times \mathcal{A}_{H_1 \times H_2}' \) such that:

1. \( \Omega(Y) \) is a non-empty SFT.
2. For \((\omega, y) \in \Omega(Y)\) and \((h_1, h_2) \in H_1 \times H_2\) then \( \mathcal{C}(\omega, y, h_1, h_2) \in Y \).
3. There exists \( \omega \in \Omega \) such that \( \eta(\omega) \) is free.
4. For each \( c \in Y \) and \( \omega \in \Omega \) such that \( \eta(\omega) \) is free, there exists \((\omega, y) \in \Omega(Y)\) such that for every \((h_1, h_2) \in H_1 \times H_2\) then \( \mathcal{C}(\omega, y, h_1, h_2) \in \text{Orb}_\sigma(c) \) and \( c = \mathcal{C}(\omega, y, 1_{H_1}, 1_{H_2}) \).

### 3.3 Proof of Theorem 3.1

Consider first the subshift \( \text{Top}_{20}(X, T) \) from Lemma 3.4. One can extract a nearest neighbor \( \mathbb{Z}^2 \)-SFT extension \( \text{Top}(X, T) \) and a 1-block map \( \hat{\phi} : \text{Top}(X, T) \to \text{Top}_{20}(X, T) \) defined by a local function \( \Phi \). We can thus construct the \((H_1 \times H_2)\)-SFT \( \Omega(\text{Top}(X, T)) \) from Lemma 3.8. Let \( \mathcal{F}_{\Pi} \) be a finite set of forbidden patterns defining \( \Omega(\text{Top}(X, T)) \).

Recall that \( S \) is the finite set of generators of \( G \) with which \( \text{Top}_{10}(X, T) \) was defined. We construct a \((G \times H_1 \times H_2)\)-subshift \( \text{Final} \) using the alphabet of \( \Omega(\text{Top}(X, T)) \) and the following forbidden patterns \( \mathcal{F} := A_1 \cup A_2 \cup A_3 \):

1. For each \( q \in \mathcal{F}_{\Pi} \) let \( p \) be the pattern with \( \text{supp}(p) = \{1_G\} \times \text{supp}(q) \) such that for every \((h_1, h_2) \in \text{supp}(q)\) we have \( p(1_G, h_1, h_2) = q(h_1, h_2) \). We define \( A_1 \) as the set of all such \( p \).

2. We define \( A_2 \) as the set of patterns \( p \) which satisfy that \( \text{supp}(p) = \{(1_G, 1_{H_1}, 1_{H_2}), (s, 1_{H_1}, 1_{H_2})\} \) for some \( s \in S \) such that if we write \( p = (p_1, p_2) \) then:

\[ p_1(1_G, 1_{H_1}, 1_{H_2}) \neq p_1(s, 1_{H_1}, 1_{H_2}) \]

12
3. We define $A_3$ as the set of patterns $p$ which satisfy that $\text{supp}(p) = \left\{(1_G,1_{H_1},1_{H_2}),(s^{-1},1_{H_1},1_{H_2})\right\}$ for some $s \in S$ such that if we write $p = (p_1,p_2)$ then:

$$\tilde{\Phi}(p_2(1_G,1_{H_1},1_{H_2}))(s) \neq \tilde{\Phi}(p_2(s^{-1},1_{H_1},1_{H_2}))(1_G).$$

In simpler words: $A_1$ forces each $(H_1 \times H_2)$-coset to contain a configuration of $\Omega(\mathsf{Top}(X,T))$, $A_2$ constrains $\text{Final}$ such that in every $(H_1 \times H_2)$-coset the first component (the $\omega$) is the same and $A_3$ links the different $(H_1 \times H_2)$-cosets indexed by $G$ forcing them to respect the action of $T$. This last rule is illustrated on Figure 2 where the grids are induced by $\omega$ and appear somewhere in $\{1_G\} \times H_1 \times H_2$ and $\{s_1^{-1}\} \times H_1 \times H_2$. In each of the two grids a configuration from $\mathsf{Top}(X,T)$ is encoded. The forbidden patterns from $A_3$ code the fact that if we project these configurations to $\mathsf{Top}_{20}(X,T)$, then, if the configurations indexed by $1_G$ and $s_1^{-1}$ code respectively $x$ and $y$, we will forcefully have that $y = T^{s_1}(x)$.

![Figure 2: The set $A_3$ links the different cosets by forcing that in all configurations as above then $\Psi(y) = \Psi(T^{s_1}(x))$.](image)

As the union of the supports of the patterns in $\mathcal{F}$ is bounded, we have that $\text{Final}$ is an SFT. We claim that its $G$-subaction is an extension of $(X,T)$. Indeed, consider the map $\varphi : \text{Final} \to X$ defined as follows: for $(\omega,y) \in \text{Final}$ let $(\omega',y') := (\omega,y)|_{\{1_G,H_1,H_2\}}$, then

$$\varphi(\omega,y) = \gamma\left(1_G,\hat{\phi}(\mathcal{C}(\omega',y',1_{H_1},1_{H_2}))\right).$$

Note that as $A_1 \subset \mathcal{F}$, then $(\omega',y') \in \Omega(\mathsf{Top}(X,T))$ and thus the configuration $\mathcal{C}(\omega',y',1_{H_1},1_{H_2}) \in \mathsf{Top}(X,T)$ as stated in Lemma 3.8. In turn, $\hat{\phi}(\mathcal{C}(\omega',y',1_{H_1},1_{H_2})) \in \mathsf{Top}_{20}(X,T)$ and thus $\gamma$ and $\varphi$ are also well defined. Moreover, in order to compute the first $n$ coordinates of $\varphi(\omega,y)$ it suffices to know the values of $\mathcal{C}(\omega',y',1_{H_1},1_{H_2})$ restricted to a ball of diameter $3^n$ of $\mathbb{Z}^2$. And in turn, it suffices to know $(\omega',y')$ restricted to the ball of diameter $3^n$ of
$H_1 \times H_2$ with respect to the generators $S_1 \times S_2$. This means that $\phi(\omega, y)$ is continuous. In order to conclude we need to show that $\varphi$ is onto and that it interchanges the subaction $\sigma_G$ with $T$.

Claim 3.1. For every $(\omega, y) \in \text{Final}$ and $g \in G$, $\varphi(\sigma^g(\omega, y)) = T^g(\varphi(\omega, y))$.

Proof. Clearly, it suffices to show the property for each $s \in S$. Let $(\omega, y) \in \text{Final}$ and denote $(\omega^0, y^0) := (\omega, y)|_{(1_G, H_1, H_2)}$ and $(\omega^1, y^1) := (\omega, y)|_{(s^{-1}, H_1, H_2)}$. As no patterns from $A_2$ appear in $(\omega, y)$, we have that $\omega^0 = \omega^1$. Denote both of them by $\omega$, using this we get that on the one hand,

$$\hat{\phi}(C(\omega^1, y^1, 1_{H_1}, 1_{H_2})) = \hat{\Phi}(C(\omega, y^1, 1_{H_1}, 1_{H_2})) = \hat{\Phi}(y_{\hat{\eta}(\omega)^*(1_{H_1}, 1_{H_2}))}.$$  

And on the other hand,

$$\hat{\phi}(C(\omega^0, y^0, 1_{H_1}, 1_{H_2})) = \hat{\Phi}(C(\omega, y^0, 1_{H_1}, 1_{H_2})) = \hat{\Phi}(y_{\hat{\eta}(\omega)^*(1_{H_1}, 1_{H_2}))}.$$ 

Putting the previous equations together with the fact that no patterns from $A_3$ appear in $(\omega, y)$, we obtain

$$\hat{\phi}(C(\omega^1, y^1, 1_{H_1}, 1_{H_2}))(1_G) = \hat{\phi}(C(\omega^0, y^0, 1_{H_1}, 1_{H_2}))(s).$$

Finally, a simple computation yields:

$$\varphi(\sigma^s(\omega, y)) = \gamma(1_G, \hat{\phi}(C(\omega^1, y^1, 1_{H_1}, 1_{H_2})))$$

$$= \gamma(s, \hat{\phi}(C(\omega^0, y^0, 1_{H_1}, 1_{H_2})))$$

$$= T^s(\gamma(1_G, \hat{\phi}(C(\omega^0, y^0, 1_{H_1}, 1_{H_2}))))$$

$$= T^s(\varphi(\omega, y)).$$

Where the penultimate equality is from Lemma 3.4. \qed

Claim 3.2. $\varphi$ is onto.

Proof. Let $x \in X$. For $g \in G$, choose $c^g \in \tilde{\text{Top}}(X, T)$ as a preimage under $\tilde{\phi}$ of the vertical extension of $\tilde{\Psi}(T^g(x)) \in \tilde{\text{Top}}_d(X, T)$. Also, choose $\tilde{\omega} \in \Omega$ such that $\eta(\tilde{\omega})$ is free. By Lemma 3.8 there exists a pair $(\tilde{\omega}, y^0) \in \Omega(\tilde{\text{Top}}(X, T))$ such that $C(\tilde{\omega}, y^0, 1_{H_1}, 1_{H_2}) = c^0$. Moreover, by fixing a set of representatives $(h_1^1, h_2^1) \in I$ of $(H_1 \times H_2)/\sim$ as in Proposition 3.7, we can ask that for each $(h_1, h_2) \in H_1 \times H_2$ then $C(\tilde{\omega}, y^0, h_1, h_2) = \sigma^{-z}(c^g)$ for the unique $z = z(h_1, h_2) \in \mathbb{Z}^2$ such that
there is an \( i \in I \) satisfying \( \eta(\bar{\omega})^2(h_1^i, h_2^i) = (h_1, h_2) \). We define the \( G \times H_1 \times H_2 \) configuration \((\omega, y)\) as follows:

\[
(\omega, y)_{(g, h_1, h_2)} := (\bar{\omega}(h_1, h_2), (y^{-1}_g)_{(h_1, h_2)}).
\]

By definition, we have

\[
\varphi(\omega, y) = \gamma(1_G, \hat{\Phi}(\sigma_G, 1, 1)) = \gamma(1_G, \hat{\Phi}(c^1_G)) = \gamma(1_G, \Psi(x)) = x.
\]

Therefore, it suffices to show that \((\omega, y) \in \text{Final}\). As every \((H_1 \times H_2)\)-coset contains a configuration from \(\Omega(\text{Top}(X, T))\), no patterns from \(A_1\) appear. Also, as the first component is always \(\bar{\omega}\), we have that no patterns from \(A_2\) appear. Finally, we have that for every \(g \in G\) and \(s \in S\) then \(y_g^{-1}(h_1, h_2) = (y^g)_{(h_1, h_2)} = (c^g)\), and \(y_{g^{-1}s^{-1}, h_1, h_2} = (y^g)_{(h_1, h_2)} = (c^g)\).

Therefore if we write \(z = (z_1, z_2)\) we have \(\hat{\Phi}((c^g)_z) = \Psi(T^g(x))z_1\) and \(\hat{\Phi}((c^g)_z) = \Psi(T^g(x))z_1\). In particular as

\[
\Psi(T^g(x))(s) = \Psi(T^g(T^g(x))) = \Psi(T^{hg}(x)) = \Psi(T^g(x))(1_G)
\]

we get that \(\hat{\Phi}((c^g)_z)(s) = \hat{\Phi}((c^g)_z)(1_G)\) and thus,

\[
\hat{\Phi}(y_{g^{-1}, h_1, h_2})(s) = \hat{\Phi}(y_{g^{-1}s^{-1}, h_1, h_2})(1_G).
\]

This implies that no patterns from \(A_3\) appear. Therefore \((\omega, y) \in \text{Final}\). \(\square\)

Adding up both claims and the previously proven properties of \(\varphi\), we obtain that \(\varphi : (\text{Final}, \sigma_G) \twoheadrightarrow (X, T)\) is a topological factor. This proves Theorem 3.1.

## 4 Consequences and remarks

In this last section we explore some consequences of Theorem 3.1. In the case of expansive actions, we can give more detailed information about the factor. Specifically, we show that if \(G\) is a recursively presented group, then every effectively closed \(G\)-subshift can be realized as the projective subdynamics of a sofic \((G \times H_1 \times H_2)\)-subshift. Moreover, we prove that the sofic subshift can be picked in such a way that it is invariant under the shift action of \(H_1 \times H_2\).

This result is particularly helpful for the next part where we show that any group that can be written as the direct product of three infinite and finitely generated groups with decidable word problem admits a non-empty strongly aperiodic SFT.

Finally, we close this section by showing how the previous result can be used to prove the existence of non-empty strongly aperiodic subshifts in the Grigorchuk group.
4.1 The case of effectively closed expansive actions

The subshift $\text{Final}$ constructed in the proof of Theorem 3.1 satisfies the required properties, however, it has an undesirable perk. Namely, it might happen that for $(\omega, y) \in \text{Final}$ then $\varphi(\omega, y) \neq \varphi^{(1_{G}, h_{1}, h_{2})}(\omega, y)$ for some $(h_{1}, h_{2}) \in H_{1} \times H_{2}$. The reason is that in $\Omega$ there might be many different grids and a priori there is no restriction forcing them to contain shifts of the same configuration.

The natural approach to get rid of this perk is to use the functions $\gamma_{n}$ to impose in every $(H_{1} \times H_{2})$-coset that the first $n$-coordinates of the coded configuration are the same everywhere. While we show that this works perfectly for an expansive system, it naturally fails in the case where $(X, T)$ is equicontinuous, see [Hoc09], Proposition 6.1. This makes expansive systems particularly interesting in this construction, specially in the proof of Theorem 4.4 where we show that every triple direct product of finitely generated groups with decidable word problem admit strongly aperiodic SFTs.

Given a group $G$ generated by $S$ and a finite alphabet $A$ a pattern coding $c$ is a finite set of tuples $c = \{(w_{i}, a_{i})\}_{i \in I}$ where $w_{i} \in S^{*}$ and $a_{i} \in A$. A set of pattern codings $C$ is said to be recursively enumerable if there is a Turing machine which takes as input a pattern coding $c$ and accepts it if and only if $c \in C$. A subshift $X \subset \mathcal{A}^{G}$ is effectively closed if there is a recursively enumerable set of pattern codings $C$ such that:

$$X = X_{C} := \bigcap_{g \in G, c \in C} \left( \mathcal{A}^{G} \setminus \bigcap_{(w, a) \in c} [a]_{gw} \right).$$

**Theorem 4.1.** Let $G$ be a recursively presented and finitely generated group and $Y$ an effectively closed $G$-subshift. For every pair of infinite and finitely generated groups $H_{1}, H_{2}$ there exists a sofic $(G \times H_{1} \times H_{2})$-subshift $X$ such that:

- The $G$-subaction of $X$ is conjugate to $Y$.
- The $G$-projective subdynamics of $X$ is $Y$.
- The shift action $\sigma$ restricted to $H_{1} \times H_{2}$ is trivial on $X$.

**Proof.** Let $S \subset G$ be a finite generating set and consider a recursive bijection $\varphi : \mathbb{N} \to S^{*}$ where $S^{*}$ is the set of all words on $S$. As $G$ is recursively presented, then its word problem $\text{WP}(G) = \{w \in S^{*} \mid w =_{G} 1_{G}\}$ is recursively enumerable and there is a Turing machine $\mathcal{M}$ which accepts a pair $(n, n') \in \mathbb{N}^{2}$ if and only if $\varphi(n) = \varphi(n')$ as elements of $G$. For simplicity, fix $\varphi(0)$ to be the empty word representing $1_{G}$.

Let $Y \subset \mathcal{A}^{G}$ be the effectively closed $G$-subshift of the statement, $\kappa := \lfloor \log_{2}(|A|) \rfloor$ and $\nu : A \to \{0, 1\}^{\kappa}$ a 1-to-1 map. We can define a function $\rho : Y \to \{0, 1\}^{\kappa}$ by

$$\rho(y)_{n} = (\nu(\varphi([n]_{\kappa}))_{n} \mod \kappa.$$

Here $\varphi([n]_{\kappa}) \in S^{*}$ is identified as an element of $G$. Consider the set $Z = \rho(Y)$ and the $G$-action $T : G \rhd Z$ defined by $T^{g}(\rho(y)) = \rho(\sigma^{g}(y))$. Clearly
\( \rho \) is a topological conjugacy between \((Y, \sigma)\) and \((Z, T)\). We claim that \((Z, T)\) is an effectively closed \(G\)-dynamical system.

Indeed, let \( w \in \{0, 1\}^* \). A Turing machine which accepts \( w \) if and only if \([w] \in \{0, 1\}^n \setminus Z\) is given by the following scheme: First, if for some \( n < |w|/\kappa \) we have that \( w_{\kappa n}, \ldots, w_{\kappa n+\kappa-1} \) does not belong to \( v(A) \) accept \( w \). Then for each pair \((\kappa n, \kappa n')\) in the support of \( w \) run \( M \) in parallel over the pair \((n, n')\).

If \( M \) accepts for a pair such that \( w_{\kappa n}, \ldots, w_{\kappa n+\kappa-1} \neq w_{\kappa n'}, \ldots, w_{\kappa n'+\kappa-1} \) then accept \( w \) (this means that \( w \) did not codify a configuration in \( A^Z \) as two different \( n, n' \) codifying the same group element have different symbols assigned to them). Also, in parallel, use the algorithm recognizing a maximal set of forbidden patterns for \( Y \) (this exists by [ABS17], Lemma 2.3) over the pattern coding

\[
c_w = \langle \varphi(n), v^{-1}(w_{\kappa n}, \ldots, w_{\kappa n+\kappa-1}) \rangle_{n<|w|/\kappa}.
\]

This eliminates all \( w \) which codify configurations containing forbidden patterns in \( Y \). For the analogue algorithm for \( T^*(\{w\}) \) just note that as \( G \) is recursively presented, the set of pairs \((n, m)\) such that \( \varphi(n) =_G s \varphi(m) \) also form a recursively enumerable set. Therefore \( T^*(\{w\}) \) also admits the required algorithm.

We use Theorem 3.1 to construct the \((G \times H_1 \times H_2)\)-SFT \( \text{Final} \). We further restrict \( \text{Final} \) with an extra set of forbidden patterns \( A_4 \). Let \( B_n \) be the ball of size \( n \) in \( H_1 \times H_2 \) with respect to the metric induced by the set of generators \( S_1 \times S_2 \) used to construct \( \Omega \) from Lemma 3.8. Let \( p \) be a pattern with support \( \{1_G\} \times B_3^{m+1} \) and let \((\omega, y) \in [p]\). By definition of \( \gamma_m \), we have that for each \((s_1, s_2) \in S_1 \times S_2\), \( \gamma_m(1_G, \phi(C(\omega, y, s_1, s_2))) \) only depends on the ball of size \( 3^m \) around \((s_1, s_2)\), therefore all of these functions depend only on \( p \). Denote them by \( \gamma_m(1_G, p, s_1, s_2) \).

For each \( m \in \{0, \ldots, \kappa - 1\} \) we put in \( A_4 \) all the patterns \( p \) with support \( \text{supp}(p) = \{1_G\} \times B_3^{m+1} \) such that there exists \((s_1, s_2) \in S_1 \times S_2\) satisfying

\[
\gamma_m(1_G, p, s_1, s_2) \neq \gamma_m(1_G, p, 1H_1, 1H_2).
\]

In other words, we force the first \( \kappa \) symbols coded in every simulated grid to coincide. As the size of the support of these patterns is bounded, \( A_4 \) is finite and \( \text{Final} \) defined by forbidding the patterns in \( A_4 \) is still an SFT. Moreover, the configuration constructed in Claim 3.2 clearly satisfies these constrains so \( \varphi \) is still onto.

Finally, define a map \( \hat{\varphi} : \text{Final} \rightarrow A^{G \times H_1 \times H_2} \) by

\[
\hat{\varphi}(\omega, y)_{(g, h_1, h_2)} := v^{-1} \left( \varphi(g, h_1, h_2)^{-1}(\omega, y) \right)_{(0, \ldots, \kappa - 1)}.
\]

Let \( X := \hat{\varphi}(\text{Final}) \). The function \( \varphi(g, h_1, h_2)^{-1}(\omega, y) \) depends only on a finite support (a ball of size \( 3^\kappa \) around the identity for instance) and clearly commutes with the shift. Therefore \( \hat{\varphi} \) is indeed a topological factor and thus \( X \) is a sofic subshift. Also, by definition of \( A_4 \) and the fact that \( H_1 \times H_2 \) is generated by \( S_1 \times S_2 \) we obtain that \( \hat{\varphi} \) does not depend on \((h_1, h_2)\) and thus \( H_1 \times H_2 \) acts trivially on \( X \).
Finally, the projective subdynamics \( \pi_G(X) \) clearly satisfy that \( \pi_G(X) \subset Y \). Let \( y \in Y \) and consider \((\omega, r)\) as in Claim 3.2 such that \( \varphi(\sigma^g(\omega, r)) = T^g(\rho(y)) \). By construction \((\omega, r) \in \text{Final} \) and thus we can furthermore say that \( \varphi(\sigma^{(g_1, h_1, h_2)}(\omega, r))|_{0, \ldots, n-1} = T^g(\rho(y))|_{0, \ldots, n-1} \). We deduce that
\[
\hat{\varphi}(\omega, r)_{(g_1, h_1, h_2)} = v^{-1} \left( \varphi(\sigma^{(g_1, h_1, h_2)}(\omega, r))|_{0, \ldots, n-1} \right) = v^{-1} \left( T^g(\rho(y))|_{0, \ldots, n-1} \right) = v^{-1} \left( (\rho(\sigma^g(\omega)))|_{0, \ldots, n-1} \right) = v^{-1}(v(y)) = y_g.
\]

Therefore \( \pi_G(X) = Y \). As \( H_1 \times H_2 \) acts trivially on \( X \) every configuration is a periodic extension of some \( y \in Y \). Hence the subduction \((X, \sigma_G)\) is conjugate to \((Y, \sigma)\). \( \square \)

### 4.2 Strongly aperiodic SFTs in triple direct products

Next we show how Theorem 4.1 can be applied to produce strongly aperiodic subshifts of finite type. Recall that a \( G \)-subshift \((X, \sigma)\) is strongly aperiodic if the shift action is free, that is, \( \forall x \in X, \sigma^g(x) = x \implies g = 1_G \).

**Lemma 4.2.** Let \( G_i \) for \( i \in \{1, 2, 3\} \) be infinite and finitely generated groups such that there exists a non-empty effectively closed subshift \( Y_i \subset A^{G_i} \) which is strongly aperiodic. Then \( G_1 \times G_2 \times G_3 \) admits a non-empty strongly aperiodic SFT.

**Proof.** Recall the following general property of factor maps. Suppose there is a factor \( \phi : (X, T) \to (Y, S) \) and let \( x \in X \) such that \( T^g(x) = x \). Then \( S^g(\phi(x)) = \phi(T^g(x)) = \phi(x) \in Y \). This means that if \( S \) is a free action then \( T \) is also a free action. In particular, it suffices to exhibit a non-empty strongly aperiodic subshift to conclude.

By Theorem 4.1 we can construct for each \( i \in \{1, 2, 3\} \) a non-empty sofic \((G_1 \times G_2 \times G_3)\)-subshift \( X_i \) whose \( G_i \)-subaction is conjugate to \( Y_i \) and is invariant under the action of \( G_j \times G_k \) with \( i \notin \{j, k\} \). Let \( X = X_1 \times X_2 \times X_3 \). We claim that \( X \) is a non-empty strongly aperiodic sofic subshift.

Clearly, \( X \) is a non-empty sofic subshift. Let \( g = (g_1, g_2, g_3) \in G_1 \times G_2 \times G_3 \) and \( x \in X \) such that \( \sigma^{(g_1, g_2, g_3)}(x) = x \). Write \( x = (x_1, x_2, x_3) \) and note that \( \sigma^{(g_1, g_2, g_3)} = \sigma^{(g_1, 1_{G_2}, 1_{G_3})} \circ \sigma^{(1_{G_1}, g_2, 1_{G_3})} \circ \sigma^{(1_{G_1}, 1_{G_2}, g_3)} \).

As \( X_i \) is invariant under \( G_j \times G_k \), we have that
\[
(x_1, x_2, x_3) = \sigma^{(g_1, g_2, g_3)}(x_1, x_2, x_3) = (\sigma^{(g_1, 1_{G_2}, 1_{G_3})}(x_1), \sigma^{(1_{G_1}, g_2, 1_{G_3})}(x_2)) = \sigma^{(1_{G_1}, 1_{G_2}, g_3)}(x_3)).
\]
On the other hand, as the $G_i$-subaction is conjugate to $Y_i$ which is strongly aperiodic, we conclude that $g_i = 1_{G_i}$. Therefore $g = 1_{G_1 \times G_2 \times G_3}$. As the choice of $x$ was arbitrary, this shows that $X$ is strongly aperiodic.

Lemma 4.2 requires the existence of a non-empty effectively closed and strongly aperiodic $G_i$-subshift. Luckily, these objects always exist whenever the word problem of the group is decidable. Furthermore, in the class of recursively presented groups, non-empty effectively closed subshifts which are strongly aperiodic exist if and only if the word problem of the group is decidable. This is proven in [Jea15a] and [ABT15] and can be formally stated as follows.

**Lemma 4.3** ([ABT15] Theorem 2.8). Let $G$ be a recursively presented group. There exists a non-empty, effectively closed and strongly aperiodic $G$-subshift if and only if the word problem of $G$ is decidable.

The only if part of this proof is a result by Jeandel [Jea15a] and is basically the fact that a strongly aperiodic SFT (or more generally, an effectively closed strongly aperiodic subshift) in a recursively presented group gives enough information to recursively enumerate the complement of the word problem of the group. Conversely, the existence part of the proof of Lemma 4.3 relies on a proof by Alon, Grytczuk, Haluszczak and Riordan [AGHR02] which uses Lovász local lemma to show that every finite regular graph of degree $\Delta$ can be vertex-colored with at most $(2e^{16} + 1)\Delta^2$ colors in a way such that the sequence of colors in any non-intersecting path does not contain a square word. Using compactness arguments this result is extended to Cayley graphs $\Gamma(G, S)$ of finitely generated groups where the bound takes the form $2^{19}|S|^2$ colors where $|S|$ is the cardinality of a set of generators of $G$. One can also show that the set of square-free vertex-colorings of $\Gamma(G, S)$ yields a strongly aperiodic subshift, which is thus non-empty if the alphabet has at least $2^{19}|S|^2$ symbols. In the case where $G$ has decidable word problem, a Turing machine can construct a representation of the sequence of balls $B(1_G, n)$ of the Cayley graph and enumerate a codification of all patterns containing a square colored path.

Adding up Lemma 4.2 and Lemma 4.3 gives us the following result.

**Theorem 4.4.** For any triple of infinite and finitely generated groups $G_1, G_2$ and $G_3$ with decidable word problem, then $G_1 \times G_2 \times G_3$ admits a non-empty strongly aperiodic subshift of finite type.

Note that the hypothesis of having decidable word problem is necessary, if not, any finitely generated and recursively presented $G_i$ with undecidable word problem gives a counterexample by Lemma 4.3. On the other hand, to the best of the knowledge of the author, there are no known examples of a group of the form $G_1 \times G_2$ where both $G_i$ are infinite, finitely generated, have decidable word problem, and $G_1 \times G_2$ does not admit a strongly aperiodic SFT. Therefore, it is possible that Theorem 4.4 can be improved in that direction.
4.3 A strongly aperiodic SFT in the Grigorchuk group

Here we exhibit a class of groups which admit strongly aperiodic SFTs. In particular, this class contains the Grigorchuk group [Gri85]. In order to present this result we need to recall the notion of commensurability.

Definition 4.1. We say two groups \( G, H \) are commensurable if they have isomorphic subgroups of finite index. Namely, \( G' \leq G, H' \leq H \) such that \( |G : G'| < \infty, |H : H'| < \infty \) and \( G' \cong H' \).

A result by Carroll and Penland [CP15] establishes that the group property of admitting a non-empty strongly aperiodic SFT is invariant under commensurability. In their article they say that a group \( G \) is weakly aperiodic if it does not admit a non-empty strongly aperiodic SFT.

Theorem 4.4 implies that \( G \) is a non-empty strongly aperiodic subshift of finite type. We suppose from now that \( G \) is infinite, finitely generated and has decidable word problem.

Theorem 4.5 ([CP15], Theorem 1). Let \( G_1 \) and \( G_2 \) be finitely generated commensurable groups. If \( G_1 \) is weakly periodic, then \( G_2 \) is weakly periodic.

With this result in hand, we can show the following:

Theorem 4.6. Let \( G \) be a finitely generated group with decidable word problem such that \( G \) is commensurable to \( G \times G \). Then \( G \) admits a non-empty strongly aperiodic subshift of finite type.

Proof. If \( G \) is finite, the result is immediate as \( X = \{ x \in \{a, b\}^G \mid |x^{-1}(a)| = 1 \} \) is a non-empty strongly aperiodic subshift of finite type. We suppose from now on that \( G \) is infinite. If \( G \) is commensurable to \( G \times G \), then there exists \( H_1 \leq G, H_2 \leq G \times G \) of finite index such that \( H_1 \cong H_2 \). In particular, if we define \( H_1 = G \times H_1 \) and \( H_2 = G \times H_2 \) we get that \( H_1 \cong H_2, H_1 \) is a finite index subgroup of \( G \times G \) and \( H_2 \) is a finite index subgroup of \( G \times G \times G \). Therefore \( G \times G \) and \( G \times G \times G \) are commensurable. As commensurability is an equivalence relation in the class of groups we get that \( G \) is commensurable to \( G \times G \times G \).

As \( G \) is infinite, finitely generated and has decidable word problem, Theorem 4.4 implies that \( G \times G \times G \) is not weakly periodic. It follows by Theorem 4.5 that \( G \) is not weakly periodic as well and thus admits a non empty strongly aperiodic SFT.

The Grigorchuk group [Gri85] is a famous example of an infinite and finitely generated group of intermediate growth which contains no isomorphic copy of \( \mathbb{Z} \) and has decidable word problem. It can be defined as the group generated by the involutions \( a, b, c, d \) over \( \{0, 1\}^\mathbb{N} \) as follows, let \( x = x_0x_1x_2 \ldots \) and

\[
\begin{align*}
a(x) &= \begin{cases} 
1x_1x_2 \ldots \text{ if } x_0 = 0 \\
0x_1x_2 \ldots \text{ if } x_0 = 1
\end{cases} \\
b(x) &= \begin{cases} 
0a(x_1x_2 \ldots) \text{ if } x_0 = 0 \\
1c(x_1x_2 \ldots) \text{ if } x_0 = 1
\end{cases} \\
c(x) &= \begin{cases} 
0a(x_1x_2 \ldots) \text{ if } x_0 = 0 \\
1d(x_1x_2 \ldots) \text{ if } x_0 = 1
\end{cases} \\
d(x) &= \begin{cases} 
0x_1x_2 \ldots \text{ if } x_0 = 0 \\
1b(x_1x_2 \ldots) \text{ if } x_0 = 1
\end{cases}
\end{align*}
\]
These four actions can be represented in the Mealy automaton of Figure 3. Here an arrow of the form $i \rightarrow j$ means: “replace $i$ by $j$ and follow the arrow”. To compute the image of $x \in \{0, 1\}^\mathbb{N}$ under one of these involutions, start at the corresponding node and follow the arrow of the form $x_0 \rightarrow i$. Replace $x_0$ by $i$ and continue with $x_1$ and so on.

Besides the remarkable aforementioned properties, the Grigorchuk group is commensurable to its square.

**Lemma 4.7** ([CSC09] Lemma 6.9.11). The Grigorchuk group $G$ is commensurable to $G \times G$.

Therefore, we can apply Theorem 4.6 to obtain:

**Corollary 4.8.** There exists a non-empty strongly aperiodic subshift of finite type defined over the Grigorchuk group.

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