Quantum criticality of the two-channel pseudogap Anderson model: universal scaling in linear and non-linear conductance

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Abstract

The quantum criticality of the two-lead two-channel pseudogap Anderson impurity model is studied. Based on the non-crossing approximation (NCA) and numerical renormalization group (NRG) approaches, we calculate both the linear and nonlinear conductance of the model at finite temperatures with a voltage bias and a power-law vanishing conduction electron density of states, \( \rho(\omega) \propto |\omega - \mu_F|^r (0 < r < 1) \) near the Fermi energy \( \mu_F \). At a fixed lead-impurity hybridization, a quantum phase transition from the two-channel Kondo (2CK) to the local moment (LM) phase is observed with increasing \( r \) from \( r = 0 \) to \( r = r_c < 1 \). Surprisingly, in the 2CK phase, different power-law scalings from the well-known \( T \) or \( V \) form is found. Moreover, novel power-law scalings in conductances at the 2CK-LM quantum critical point are identified. Clear distinctions are found on the critical exponents between linear and non-linear conductance at criticality. The implications of these two distinct quantum critical properties for the non-equilibrium quantum criticality in general are discussed.

Keywords: quantum criticality, two-channel Kondo physics, quantum phase transitions, non-equilibrium quantum transport

1. Introduction

Quantum phase transitions (QPTs) [1], the zero-temperature phase transitions due to quantum fluctuations, are of fundamental importance in condensed matter systems. Near the transitions, these systems show non-Fermi liquid behaviors manifested in universal power-law scaling in all thermodynamic observables. Recently, QPTs in quantum impurity problems [2], have attracted much attention recently due to their relevance for the nano-systems, such as: quantum dots [3], realized experimentally. The well-known Kondo effect [4], the antiferromagnetic spin correlations between impurity and conduction electrons, plays a crucial role in understanding their low temperature behaviors. New scaling laws are expected to occur when the QPTs are associated with the Kondo breakdown in these systems either in equilibrium (at finite temperatures) or under non-equilibrium conditions (at finite voltage bias). Particular interest lies in QPTs out of equilibrium [5] where distinct universal scalings are expected in contrast to the counterparts in equilibrium.

A fascinating playground to address this issue is the exotic two-channel Kondo (2CK) [6–11] systems with non-Fermi liquid ground state due to overscreening of \( s = 1/2 \) impurity spin by two independent conduction reservoirs. Much
of the theoretical effort has been made for the 2CK physics, including: via Bethe ansatz [12], conformal field theory [13], bosonization [14] and the numerical renormalization group (NRG) [15]. Experimentally, the 2CK ground state has been realized in semiconductor quantum dots [16], magnetically doped nanowires, and metallic glasses [17, 18]. Recently, Kondo physics in magnetically doped graphene has attracted much attention for the possible 2CK physics as well as the pseudogap local density of states (LDOS) $\rho_\omega$ which vanishes linearly due to the Dirac spectrum: $\rho_\omega \propto |\omega|^r$ with $r = 1$ [19–21, 23]. This leads to QPT from the Kondo screened phase to the unscreened local moment (LM) phase with decreasing Kondo correlation due to insufficient DOS of conduction electrons. In fact, such transitions exist in the more general framework of pseudogap Kondo (or Anderson) models with power-law exponent $0 < r < 1$, which have been extensively studied [24–28]. However, relatively less is known on the more exotic 2CK-LM QPT in the pseudogap two-channel Anderson (Kondo) models [29, 30], especially in transport properties in the two-lead setup. In particular, three fundamentally important issues are yet to be addressed: (1) Does the equilibrium conductance in the 2CK phase still exhibit the $\sqrt{T}$ scaling as in the case of $r = 0$? (2) What are the scaling properties near 2CK-LM QPT in conductances both in equilibrium and out of equilibrium? (3) Does the scaling in non-equilibrium conductance at criticality show distinct behaviors from its equilibrium counterpart [5, 23, 31]?

Motivated by these developments, we address the above issues on the 2CK-LM quantum phase transition in the two-lead two-channel pseudogap Anderson model both in and out of equilibrium in a Kondo quantum dot subject to a voltage bias across the impurity and finite temperature. By studying experimentally accessible steady-state transport, we search for universal scalings in both linear and non-linear conductance in the 2CK regime and near criticality. Two different theoretical approaches are applied here: (1) a large-N method based on the non-crossing approximation (NCA) [8, 32–34], a reliable approach for multi-channel Kondo systems with non-Fermi liquid ground states, and (2) the NRG approach [35], a well-established method for Anderson/Kondo impurity systems. A fundamentally important but less addressed issue—non-equilibrium quantum criticality—is emphasized here by identifying and comparing different universal scaling behaviors between equilibrium (zero bias) and nonequilibrium (finite bias) conductances near the 2CK-LM transition [36].

2. The NCA approach

2.1. The model Hamiltonian

We start from the original Hamiltonian [22, 37]

$$H = \sum_{\sigma} E_\sigma \chi_{\sigma, \sigma} + \sum_{\tau} E_\tau \chi_{\tau, \tau} + \sum_{k\alpha} \epsilon_k \chi_{k\alpha}^\dagger \chi_{k\alpha}^\sigma + \sum_{k\alpha} [V^\alpha_{\sigma\tau} \chi_{\sigma, \alpha}^\dagger \chi_{\tau, \alpha} + \text{h.c.}]$$

where the Hubbard operator $\chi_{\sigma, \sigma} = |\sigma\rangle \langle \sigma|$ induces a transition from the impurity state $|\tau\rangle$ containing $N$ electrons and with energy $E_\tau$ to an impurity state $|\sigma\rangle$ containing $N + 1$ electrons with energy $E_\sigma$. The impurity part of the Hamiltonian comprises only fluctuations between the ground states of the two valence configuration; all other local states are omitted due to the local Coulomb interaction. For derivation of such models we refer to [22, 38] and the review on multi-orbital Kondo models [17]. Introducing the energy difference $e_\sigma = E_\sigma - E_\tau$ for energy degenerate multiplets $|\tau\rangle$ and $|\sigma\rangle$ and the slave-boson representation of the transition operator $X_{\sigma, \alpha} = f_\sigma^\dagger b_\alpha$, allows us to map equation (1) to (2) (see below).

The Hamiltonian of the two-lead two-channel single-impurity pseudogap Anderson model defined above can be formulated within the NCA [8, 32–34, 38], a large-N approach based on the $SU(N) \times SU(M)$ generalization of the SU(2) model with $N \rightarrow \infty, M \rightarrow \infty$ being the number of degenerate flavors of spins $\sigma = \uparrow, \downarrow, \cdots N$ and the number of Kondo screening channels $K = 1, 2, \cdots M$. In the physical SU(2) two-channel Kondo system, $N = M = 2$. The two leads are described by a power-law vanishing DOS at Fermi energy defined as $\rho_\omega(\omega) \sim |\omega|^r (\Theta(\omega) - |\omega|)$ with $0 < r < 1$, where $D = 1$ is the bandwidth cutoff. Graphene and high $T_c$-cuprate superconductors are possible realizations of the pseudogap leads with $r = 1$, while semiconductors with soft gaps are candidates with $0 < r < 1$.

The Hamiltonian of the model via NCA in the large-U limit $U \rightarrow \infty$ has the following form in the pseudofermion slave-boson representation [8, 33]:

$$H = \sum_{k\alpha} (\epsilon_k - \mu_\alpha) \chi_{k\alpha}^\dagger \chi_{k\alpha}^\sigma + e_\sigma \sum_{\alpha} f_\sigma^\dagger f_\sigma + \sum_{k\alpha} (V_{k\alpha} f_\sigma^\dagger b_\alpha \chi_{k\alpha}^\sigma + \text{h.c.})$$

(2)

where $\mu_{\alpha=L,R} = \pm V/2$ is the chemical potential of lead $\alpha = L/R$. The operators $\chi_{k\alpha}^\dagger$ ($\chi_{k\alpha}^\sigma$) create (destroy) an electron in the leads with momentum $k$. Spin flavors are represented by $\sigma, \sigma' = \uparrow, \downarrow, \cdots N$ and $\tau, \tau' = 1, \cdots M$ corresponds to $M$ independent electron reservoirs. Here, $N = M = 2$. The $V_{k\alpha}$ term represents the hybridization strength between the conduction electrons and the impurity. Here, the local (impurity) electron operator $d_{\alpha\tau}^\dagger$ is decomposed in the pseudofermion representation as a product of pseudofermion $f_\sigma^\dagger$ and a slave-boson $b_\alpha$: $d_{\alpha\tau}^\dagger = f_\sigma^\dagger b_\alpha$, subject to the local constraint $Q = \sum_{\sigma} f_\sigma^\dagger f_\sigma + \sum_\alpha b_\alpha b_\alpha = 1$ via the Lagrange multiplier $\lambda$ to ensure single occupancy on impurity.

We employ the NCA to address the equilibrium and nonequilibrium transport at criticality based on equation (2). This approach has been known to correctly capture the non-Fermi liquid properties of the two-channel Anderson model [4, 17, 30, 31, 39]. Recently, it has been generalized to address the 2CK-LM crossover in non-equilibrium transport in a voltage-biased two-channel pseudogap Anderson model with $r = 1$, relevant for graphene [23]. We generalize this approach here further to the voltage-biased two-channel pseudogap Anderson model with $0 < r < 1$. Note that though the NCA formalism in our case is the same as that in [23] except for the value of $r$, we find surprising new scaling behaviors in conductances
both in and out of equilibrium both in the 2CK phase and near 2CK-LM QCP.

To facilitate our discussions, let us briefly summarize the NCA formalism for the 2CK pseudogap Anderson model in $U \to \infty$ limit. Within NCA, Green’s functions for the conduction electrons $G_{\sigma,i\lambda}(t)$, pseudo-fermions $G_{\sigma f}(t)$, and the slave-bosons $D(t)$ are given by [8, 33]:

$$G_{\sigma,i\lambda}(t) = -i\theta(t) < c_{\sigma i}(t), c^\dagger_{\sigma i}(0) >_\lambda,$$
$$= -i\theta(t)[D^-(t)G_{\sigma f}(t) - D^+(t)G_{\sigma f}^d(t)].$$

$$G_{\sigma f}^\dagger \equiv -i < f_{\sigma}(t) f^\dagger_{\sigma}(0) >_\lambda, \quad G_{\sigma f} \equiv i < f^\dagger_{\sigma}(0) f_{\sigma}(t) >_\lambda,$$

and

$$D^+ \equiv -i < b_{\lambda i}(t) b_{\lambda i}^\dagger(0) >_\lambda, \quad D^- \equiv i < b^\dagger_{\lambda i}(0) b_{\lambda i}(t) >_\lambda,$$

where the notation $< \text{and} >$ represents lesser and greater Green functions. The lesser (greater) Green functions can be written as [8, 33]:

$$D^-(\omega) = D^+(\omega) \Pi^-(\omega) D^+(\omega),$$

$$G_{\sigma f}^\dagger(\omega) = G_{\sigma f}(\omega) \Pi^+(\omega) G_{\sigma f}^d(\omega),$$

where $D^\pm(\omega)$ and $G_{\sigma f}^\pm(\omega)$ are advanced (retarded) Green functions of the boson and fermion, respectively, and the subscript $\lambda$ and the slave-bosons impurity via Lagrange multiplier $\lambda$ when evaluating these correlation functions. The $\Pi^- (\omega)$ and $\Sigma^+(\omega)$ are the self-energies of the slave-boson and pseudogap lesser (greater) Green functions, respectively. The NCA expressions for the lesser self-energy of the pseudofermion, $G_{\sigma f}^\dagger(\omega) = \Sigma^-(\omega) G_{\sigma f}(\omega) \Pi^+(\omega)$, and slave-boson, $D^-(\omega) = \Pi^-(\omega) D^+(\omega)$, are [8, 23, 33]:

$$\Sigma^\pm_{\sigma f}(\omega) = \frac{2}{\pi} \sum_{\alpha} \int d\epsilon \Gamma_{\alpha}(\epsilon - \omega - \mu_\alpha) f(\epsilon - \omega - \mu_\alpha) D^\pm(\epsilon),$$

$$\Pi^\pm(\omega) = \frac{2}{\pi} \sum_{\alpha} \int d\epsilon \Gamma_{\alpha}(\epsilon - \omega - \mu_\alpha) f(\epsilon - \omega - \mu_\alpha) G_{\sigma f}^\mp(\epsilon).$$

Here, we define an effective hybridization with a power-law spectral density: $\Gamma_{\alpha}(\omega) = \pi \sum_\nu |V_{\nu \sigma}|^2 \delta(\omega - \epsilon_{\nu \sigma}) \propto |\omega|^\gamma$. The spectral density of the conduction bath is given by $\rho_{\alpha f}(\omega) = -\frac{1}{\pi} \mathrm{Im} \Gamma_{\alpha}(\omega) = \frac{\Gamma_{\alpha}(\omega)}{\Gamma_{\alpha}} = \frac{r + 1}{\pi \Gamma_{\alpha}} |\omega|^\gamma \theta(D - |\omega - \mu_{\alpha}|)$ with $\Gamma_{\alpha} = \int \Gamma_{\alpha}(\omega) d\omega$ and $D = 1$ being the bandwidth of the conduction bath. Here, $f(\omega) = \frac{1}{1 + e^{\omega/T}}$ is the Fermi function.

The relation between the greater Green function and the retarded Green function are, $D^-(\omega) = 2\mathrm{Im} D^+(\omega)$, $G_{\sigma f}^\dagger = 2\mathrm{Im} G_{\sigma f}$, where the self-energies, $\Pi^-(\omega) = 2\mathrm{Im} \Pi^+(\omega)$ and $\Sigma^+_{\sigma f}(\omega) = 2\mathrm{Im} \Sigma^-_{\sigma f}(\omega)$. The NCA expressions for the self-energies of the retarded Green functions for pseudofermion $G_{\sigma f}(\omega) = [\omega - \epsilon_{\sigma} - \Sigma^+(\omega)]^{-1}$ and for slave-boson $D^-(\omega) = [\omega - \Pi^-(\omega)]^{-1}$ are given by [8, 23, 33]:

$$\Sigma^+(\omega) = \frac{2}{\pi} \sum_{\alpha} \int d\epsilon \Gamma_{\alpha}(\epsilon - \omega - \mu_\alpha) f(\epsilon - \omega - \mu_\alpha) G_{\sigma f}(\epsilon).$$

$$\Pi^+(\omega) = \frac{2}{\pi} \sum_{\alpha} \int d\epsilon \Gamma_{\alpha}(\epsilon - \omega - \mu_\alpha) f(\epsilon - \omega - \mu_\alpha) G_{\sigma f}^d(\epsilon).$$

The physical impurity spectral function, $\rho_{\alpha f}(\omega, V)$, is obtained via the convolution of the pseudo-fermion and slave-boson Green function based on equations (7)–(9) and (10) as [23, 38]:

$$\rho_{\alpha f}(\omega, V) = \frac{i}{2\pi^2 Z} \int d\epsilon \{\mathrm{Im} D^+(\epsilon) G_{\sigma f}^d(\omega + \epsilon) - D^+(\epsilon) \mathrm{Im} G_{\sigma f}^d(\omega + \epsilon)\}.$$

The normalization factor $Z = \frac{1}{2\pi} \int d\omega [M \times D^+(\omega) - N \times G^+(\omega)]$ enforces the constraint, $<Q> = 1$ with $M = N = 2$ here. The current going through the impurity therefore reads [23]:

$$I(V, T) = \frac{2e}{h} \sum_{\alpha} \int d\omega [2\Gamma_{\alpha}(\omega) \Gamma_{\alpha}(\omega) - \frac{\partial f(\omega)}{\partial \omega}] \rho_{\alpha f}(\omega, V, T) \times [f(\omega + eV/2) - f(\omega - eV/2)],$$

where $\Gamma_{\alpha}(\omega) = \Gamma_{\alpha}(\omega - \mu_{\alpha})$ with $\alpha = L, R$. The equilibrium (linear) conductance is directly obtained via

$$G(0, T) = \frac{2e^2}{h} \sum_{\alpha} \int d\omega [2\Gamma_{\alpha}(\omega) \Gamma_{\alpha}(\omega) - \frac{\partial f(\omega)}{\partial \omega}] \rho_{\alpha f}(\omega, V = 0).$$

And the nonlinear conductance $G(V)$ is given by $\frac{dI(V)}{dV}$. Equations (6)–(10) form a self-consistent set of Dyson’s equations within NCA. We solve these equations self-consistently and evaluate equations (11)–(13) based on these solutions.

Note that via equations (11) and (12) and the power-law spectrum of the effective hybridization $\Gamma_{\alpha}(\omega)$, we expect that the conductance shows power-law behaviors as a function of $T$ or $V$. It is therefore interesting to clarify the dependence of the power-law exponents in conductance on the pseudogap bath exponent $r$ both in the 2CK and in the 2CK-LM quantum critical regime. To this end, we access the critical point by tuning $r$ (instead of conventional ways by tuning hybridization $\Gamma_{\alpha}$ or $\epsilon_d$).

2.2. Results

2.2.1 Quantum critical point at 2CK-LM phase transition: impurity spectral function. The existence of a quantum critical point (QCP) separating 2CK for $r < r_c$ from the LM for $r > r_c$ phases exists in the PSG Anderson model has been studied extensively [29, 30]. The generic phase diagram is shown as in figure 1. We focus here on the transport properties for the two-lead setup near criticality both in and out of equilibrium, especially on the distinct non-equilibrium quantum critical properties (see figure 1). In equilibrium ($\mu_{\alpha} = 0$), the 2CK-LM phase transition is studied here by tuning $r$ of the pseudogap power-law DOS of the leads with fixed hybridization parameter $\Gamma_{\alpha}$ and the impurity energy $\epsilon_d$. The value of $r_c$ is extracted from the local impurity spectral function $\rho_{\alpha f}(\omega, V = 0)$ via solving the self-consistent Dyson
equations. Since \( \mu, \epsilon_D \neq 0 \), the 2CK pseudogap Anderson model considered here shows particle-hole (ph) asymmetry, giving rise to an overall asymmetric impurity spectral function with respect to the Fermi energy (see figure 2). For \( r = 0 \), the NCA approach to Anderson models is known to give rise to a non-Lorenzian power-law cusp in impurity spectral function, \( \rho_0(\omega) \sim \omega^{-\mu} \) [17, 40]. For \( 0 < r < r_c \), \( \rho_0(\omega) \) in our two-channel pseudogap Anderson model exhibits a power-law singularity near \( \omega = 0 \): \( \rho_0(\omega) \sim \omega^{-\mu} \) [30].

With increasing \( r \), the Kondo peak gets narrower with reduced spectral weight. For \( r \geq r_c \) however, the Kondo peak is strongly suppressed, and the ground state is in the LM phase. The critical value \( r_c \approx 0.115 \) for \( \Gamma_{\text{c}} \sim 0.3D, \epsilon_D \sim -0.2D \) (see figure 2). The spectral weight of the Kondo peaks gets further suppressed with increasing \( r \) until it completely disappears. At a finite bias voltage for \( r < r_c \), the impurity local DOS shows split Kondo peaks at \( \omega = \pm V/2 \) (see figure 3). Note that the non-zero local DOS of \( \rho_0(\omega = 0) \) for \( r > r_c \) is due to the limitation of the lowest temperature \( T_0 \sim 5 \times 10^{-7}D \) we can reach numerically. As \( T \to 0 \) and for \( r \geq r_c \), we expect a complete dip developed in local DOS such that \( \rho_0(\omega = 0) \sim 0 \).

2.2.2. Universal scaling in linear (equilibrium) conductance in 2CK regime and near criticality. Clearer signatures of the 2CK-LM quantum phase transition can be obtained via linear and non-linear conductances. First, we analyze the linear (equilibrium) conductance at finite temperatures but zero bias voltage \( G(V = 0, T) \). Figure 4 shows \( G(0, T) \) for different \( r \) with \( \Gamma = 0.28D, \mu_0 = 0, \epsilon_D = -0.3D, \) and \( D = 1 \). For \( r = 0 \) it is well known that \( G(0, T) \) follows the 2CK scaling function [8]: \( G(0, T) - G(0, 0) = B_j \sqrt{T/T_{2\text{CK}}} \) for \( T < T_{2\text{CK}} \) with \( T_{2\text{CK}} \sim 3 \times 10^{-3}D \) being the 2CK Kondo temperature at \( r = 0 \) and \( B_j \) being a non-universal constant. Here, we set \( G(0, 0) \approx G(0, T_0 \approx 5 \times 10^{-7}D) \). For \( 0 < r < r_c \), however, the 2CK \( \sqrt{T} \) scaling in \( G(0, 0) - G(0, T) \) ceases to exist. As shown in figure 4, the \( \sqrt{T} \) behavior in \( G(0, 0) - G(0, T) \) is
We propose that this deviation from the conventional via NCA for being the correlation length exponent: $T_0$, $T$ can not be manifested in close to criticality as a function of shows a universal power-law in $\sigma_{/\nu}$. Based on the above analysis, we divide $c$ for $2CK$. As a result, gives: $GT$ and $T$, gives: $c$, exhibits a linear relation to $c$ exhibits a linear relation to $c$.

Figure 5. The $\sqrt{T}$ scaling of equilibrium (linear) conductance $(G(0,0) - G(0, T))B_T$, in units of $2e^2/h$ versus $T/T_{2CK}^0$ via NCA for various $r$ with $\Gamma = 0.28D$, $\epsilon_d = -0.2D$ with $T_{2CK}$ being the two-channel Kondo temperature for $r = 0$. Here, $B_T$ are non-universal functions of $r$. The dashed line is a fit to $\sqrt{T}$ behavior.

Figure 6. Universal scaling in linear conductance $G(0, T/T^*)$ (in units of $2e^2/h$ and normalized to $\Sigma_0(T/T^*)^{\nu}$) as a function of temperature $T/T^*$ via NCA near 2CL-LM quantum phase transition for various values of $r$. Parameters are the same as in figure 4. Here, $T^*$, $\sigma_T$, and $\Sigma_0$ are defined in the text. The magenta dashed line is a fit to $T^{2.23}$. Top inset: the power-law exponent $\alpha_T$ in linear conductance $G_{QCP}(T)$ close to criticality as a function of $|r - r_c|$. Solid line is a fit to a linear relation: $\alpha_T = \alpha_T^* - \alpha_T^{|r - r_c|}$ with $\alpha_T^* = -0.145$, $\alpha_T^* = 1.25$. Bottom inset: crossover temperature $T^*$ versus $|r - r_c|$. The red dashed line is a fit to $|r - r_c|^4$.

clear for $r = 0$, but it deviates from $\sqrt{T}$ more with increasing $r < r_c$. We propose that this deviation from the conventional 2CK behavior for $T < T_{2CK}$ for $0 < r < r_c$ can be due to the following two scenarios: (i) the emergence of distinct universal power-law scaling behaviors when the system approaches the 2CK-LM quantum critical regime for $T > T^*$ with $T^*$ being the crossover energy scale above which quantum critical behaviors are observed, and (ii) the existence of a distinct 2CK scaling form other than $\sqrt{T}$ in conductance in low temperature regime $T < T_{2CK}$ for $0 < r < r_c$ with $T_{2CK}$ being the two-channel Kondo temperature for $0 < r < r_c$. As indicated in figure 5, the generic $2CK$ behaviors for $0 < r < r_c$ can not be manifested in $G(0, 0) - G(0, T)$ as a universal power-law in $T$ as does for the $r = 0$ case. Therefore, instead of analyzing $G(0, 0) - G(0, T)$, we address the above two scenarios below by trying the possible scaling behaviors of $G(0, T)$ itself.

First, as $r \to r_c$, the existence of a quantum critical regime for $T > T^*$ requires the divergence of the correlation length $\xi$ in a power-law fashion: $\xi \propto |r - r_c|^{-\nu} \to \infty \ [1, 2]$. As a result, all thermal dynamical observables, including conductances, are expected to exhibit universal scaling properties. As shown in figure 6, we indeed find numerically that the linear conductance $G_{QCP}(0, T)$ shows a universal power-law in $T$ near criticality within a temperature range of (approximately) $5 \times 10^{-7}D < T < 5 \times 10^{-4}D$ as:

$$G_{QCP}(0, T) \propto T^{\beta_T - \alpha_T |r - r_c|},$$  \hspace{1cm} (14)

where the exponent $\sigma_T$ exhibits a linear relation to $|r - r_c|$ with $\beta_T \approx -0.145$ and $\alpha_T \approx 1.25$ being non-universal constant pre-factors dependent on $\Gamma$ and $\epsilon_d$, and $T^{\beta_T} = T^{\nu \epsilon}$ as $r = r_c \sim 0.115$. Based on the above analysis, we divide $G(0, T)$ by the power-law function $T^{\nu \epsilon}$, gives:

$$\tilde{G}(0, T) = G(0, T)/T^{\nu \epsilon} = \frac{G(0, T)}{T^{\beta_T - \alpha_T |r - r_c|}},$$  \hspace{1cm} (15)

where $\tilde{G}_{QCP}(0, T)$ in the quantum critical region becomes a constant: $\tilde{G}_{QCP}(0, T) \sim \tilde{G}_{QCP}$. The universal scaling function $\tilde{G}(0, T)$ in linear conductance is obtained by rescaling $\tilde{G}(0, T)$ by a non-universal factor $\nu_T$ and $T$ by the crossover energy scale $T^*$ (see figure 6):

$$\tilde{G}(0, T^*) = \left(\frac{T^*}{T}\right)^{\beta_T - \alpha_T |r - r_c|} \tilde{G}(0, T)/T^{\nu \epsilon},$$  \hspace{1cm} (16)

where $T^*$ is proportional to the inverse of correlation length $1/\xi_T \propto |r - r_c|^\nu$, vanishing in a power-law of $|r - r_c|$ with the exponent $\nu_T$ being the correlation length exponent.
\[ T^* \propto \frac{1}{T_c} = D |r - r_c|^\nu_T \]  

with \( \nu_T \) being the correlation length exponent corresponding to the power-law exponent of crossover scale \( T^* \) versus \( |r - r_c| \). We find \( \nu_T \approx 4 \) here. (see inset of figure 6). As \( r \to r_c \) from below, we find a perfect data collapse in \( \hat{G}(0, \frac{T}{T_c}) \approx \text{const.} \) for \( 1 < T/T^* < 10^3 \), indicating the range of quantum critical region. Surprisingly, at low temperatures \( T < T_{2CK} \approx 0.17^* \) where the system is governed by the 2CK phase, we find the above function \( \hat{G}(0, \frac{T}{T_c}) \) exhibits as a distinct universal power-law scaling function (see figure 6):

\[ \hat{G}(0, \frac{T}{T_{2CK}}) \propto \left( \frac{T}{T_{2CK}} \right)^{\sigma_{2CK}} \]  

with \( \sigma_{2CK} \approx 0.23 \) for the parameters we set (or equivalently \( \hat{G}(\frac{T}{T_{2CK}}) \propto \left( \frac{T}{T_{2CK}} \right)^{\nu_T + \sigma_T} \)). We may therefore regard this low temperature power-law behavior in \( G(0, T) \) as the distinct 2CK scaling in equilibrium (linear) conductance for the pseudogap two-channel Anderson model. Note that the constants \( \sigma_{2CK}, \nu_T, \) and \( \sigma_T \) in the exponent of equation (18) are in general functions of various parameters in our model Hamiltonian, and shall be addressed elsewhere. Nevertheless, we would like to emphasize that the power-law in the temperature dependence of \( G(0, T) \) is found to be a generic feature of 2CK phase in the pseudogap Anderson model.

### 2.2.3. Universal scaling in non-linear (non-equilibrium) conductance in the 2CK regime and near criticality

We now add the bias voltage in the leads and study the scalings in non-equilibrium conductance near the 2CK-LM quantum critical point. It is generally expected that the scaling behaviors of non-linear conductance near criticality are distinct from that in equilibrium [5]. The non-linear conductance \( G(V, T_0) \) is obtained at a fixed low temperature \( T_0 = 5 \times 10^{-7}D \), symmetrical hybridization \( \Gamma_1 = \Gamma_2 = 0.3D \), and \( \epsilon_d = -0.3D \).
where $\tilde{G}(V, T_0)$ becomes a constant in the critical regime. We may further define the universal scaling function $\tilde{G}(V, T_0)$ for non-linear conductance via the following re-scalings: $V \rightarrow V/V^*$, $\tilde{G}(V, T_0) \rightarrow \tilde{G}(V, T_0)/\Sigma_\circ$ (see figure 9):

$$\tilde{G}(V, T_0) = \frac{G(V, T_0)}{\Sigma_\circ (V/V^*)^{\beta_\nu - \alpha_\nu |r - r_c|}} \tag{21}$$

where $V^*$ is the crossover energy scale, and $\Sigma_\circ$ a non-universal constant pre-factor. Here, we find the exponent $\alpha_\nu$ depends linearly on $|r - r_c|$ with $\beta_\nu \approx 0$, $\alpha_\nu \approx 2.5$ by the best fit of the data (see inset of figure 9), and the crossover scale $V^*$ is inversely proportional to the correlation length $\xi_r$ with a power-law dependence on the distance to criticality: $V^* \propto \frac{1}{\xi_r} \propto |r - r_c|^{\nu_\nu}$ with $\nu_\nu \approx 0.5$ (see inset of figure 9). Note that these critical exponents out of equilibrium are distinct from those in equilibrium, and can be considered as characteristics of non-equilibrium quantum criticality. The distinction between equilibrium and nonequilibrium quantum critical properties is expected due to the different role played by the temperature and voltage bias near criticality, leading to different behaviors in decoherence rate $\Gamma^*$ (the broadening of impurity DOS) in equilibrium $\Gamma^*$ versus out of equilibrium $\Gamma^*_r$ [5, 41].

For $T_0 < V < V^*$, the system approaches 2CK state at a characteristic energy scale $V \sim V_{2CK} \approx 0.1V^*$ below which a universal power-law scaling is observed (see figure 9):

$$\tilde{G}(V, T_0)_{2CK} \propto \left( \frac{V}{V_*} \right)^{\sigma_{2CK}} \tag{22}$$

where the exponent $\sigma_{2CK} \approx 0.23$. In the low bias limit $V < T_0$, conductance reaches a constant equilibrium value $G(0, T_0)$ and therefore deviates from the 2CK scaling (see figure 9).

Figure 10. The equilibrium zero temperature impurity spectral function versus energy $\rho(\omega)$ via NRG in units of the band width $D$, with hybridization strength $\Gamma = 0.3D$, and the impurity level $\epsilon_d = -0.3D$. For $r < r_c$ the Kondo peaks form. At $r_c \approx 0.12$, $\rho(\omega)$ becomes flat near $\omega = 0$, whereas a dip at the Fermi level develops for $r > r_c$, signature of the local moment (LM) phase.

Figure 11. The equilibrium conductance $G(0, T)$ via NRG zero temperature impurity spectral function versus energy $\rho(\omega)$ in units of the band width $D$, with hybridization strength $\Gamma = 0.3D$, and the impurity level $\epsilon_d = -0.3D$.

Figure 12. The $\sqrt{T}$ scaling of equilibrium (linear) conductance $G(0, 0) - G(0, T)/B$, via NRG (in units of $2e^2/h$) versus $T/T_{2CK}^0$ for various $r$ with $\Gamma = 0.3D$, $\epsilon_d = -0.3D$ with $T_{2CK}^0 \approx 10^{-3}D$ being the two-channel Kondo temperature for $r = 0$. Here, $B_r$ are non-universal functions of $r$. The dashed line is a fit to $\sqrt{T}$ behavior.

Note that we find equation (22) for non-equilibrium conductance to exhibit the same form as that in equilibrium in equation (18) with the same exponent $(\sigma_{2CK}^* = \sigma_{2CK} \approx 0.23)$. Further studies are required to clarify if this relation holds in general. Though the exact value (0.23) of the above exponents depend in general on physical parameters, such as: $\epsilon_d$ and $\Gamma$, equations (18) and (22) correctly reproduce the well-known $\sqrt{V}$ and $\sqrt{T}$ behaviors for $r = 0$ in the 2CK regime in $G(0, T_0) = G(V, T_0)$ and $G(0, T_0) - G(0, T)$, respectively.

3. The NRG approach

In order to confirm the generic and robust nature of the NCA results obtained above, we perform the calculation via a different approach—via the numerical renormalization group (NRG) [4, 35, 42], a widely used approach to the Kondo
impurity problem. This approach has been generalized to study pseudogap Anderson and Kondo models. The NRG Hamiltonian has the same form as shown in equation (2) with $M = N = 2$ [40]. The discretization parameter is set to be $\Lambda = 1.8$, and the lowest 1200 states are kept in each NRG iteration, allowing the convergence of results within 0.1% in error.

3.1. Results

The results for the equilibrium impurity spectral function at ground state $\rho(\omega)$ and the equilibrium conductance $G(0, T)$ are summarized below.

3.1.1. Impurity spectral function. As shown in figure 10, the impurity spectral function $\rho(\omega)$ via NRG exhibits the same qualitative feature as that obtained via NCA (see figure 2): the Kondo peak occurs for $r < r_c$, while the dip develops in the local moment phase with $r > r_c$. The value of $r_c \approx 0.12$ via NRG is in excellent agreement with that via NCA ($r_c \approx 0.115$). Some difference in detailed profiles of $\rho(\omega)$ between NRG and NCA are observed though: the peaks and dips in $\rho(\omega)$ near the Fermi level are sharper via NRG, while they are less pronounced in the NCA results. This difference is likely due to the frequency mesh used to solve the NCA equations.

3.1.2. Equilibrium conductance in the 2CK regime and near criticality. We further compute the linear (equilibrium) conductance $G(0, T)$ in the 2CK and 2CK-LM quantum critical regime for $0 \leq r < r_c$. As shown in figure 11, the profile of $G(0, T)$ via NRG agrees qualitatively well with that via NCA (see figure 4) although quantitative differences are noticed and expected. Next, we examine the $\sqrt{T}$ scaling in conductance.

As shown in figure 12, the NRG result for $G(0, 0) - G(0, T)$ in the 2CK regime ($T < T_{2CK}^{2CK} \approx 10^{-4}D$) shows $\sqrt{T}$ scaling for $r = 0$, and starts to deviate from $\sqrt{T}$ scaling for $0 < r < r_c$, the same qualitative feature as that shown in figure 5 via the NCA approach. Following the approach in section 2.2, we perform the scaling on $G(0, T)$ itself. As shown in figure 13, near the 2CK-LM critical point, we obtain the same scaling form as equation (14) with $\sigma_\rho = \beta_\rho - \alpha_\rho |r - r_c|$ and $\beta_\rho \approx -0.10, \alpha_\rho \approx 1.18$. The crossover temperature shows a power-law scaling near criticality as: $T^* \sim |r - r_c|^{1.18}$. In the 2CK phase, the same power-law scaling form as equation (18) is observed for $T < T_{2CK} \sim 0.1T^*$ with slightly different exponent $\sigma_\rho T_{2CK}^{2CK} \approx 0.17$ (see figure 13).

4. Discussion and conclusions

Before we conclude, we would like to make a few remarks. First, on the origin of the power-law scalings, we found in our model system near the 2CK-LM transition. As mentioned in section 2.2, in the 2CK phase the local density of states in equilibrium $\rho(\omega)$ shows a power-law singularity at $0 = \omega(\omega) \sim \omega^{-\alpha}$. Via equations (12) and (13), this power-law singularity in the local density of states is expected to give rise to the power-law increase in linear and non-linear conductances $G(0, T)$ and $G(V, T)$ (for $V > T$) with decreasing temperature and voltage bias, respectively. This generic power-law feature persists both in the 2CK and in the 2CK-LM quantum critical regime, and explains the power-law scalings that we found above for $G(0, T)$ and $G(V, T)$ with the exponents being functions of $|r - r_c|$.

Second, we access the 2CK-LM quantum critical point via varying the bath power-law exponent $r$, a non-local parameter. This is somewhat different from the more conventional approach via varying local parameters, such as: $\Gamma_0, \epsilon_d$. Nevertheless, we would like to point out that the introduction of the pseudogap leads in our model is equivalent to replacing the constant hybridization $\Gamma_\rho$, a local parameter, by an effective one with a power-law (pseudogap) spectral density while keeping the density of states of conduction bath constant (see equations (6) and (7)). Hence, tuning $r$ effectively changes the spectral density of the lead-impurity hybridization, a local parameter.

Finally, it is a challenging task to realize a generic pseudogap fermionic bath with $0 < r < 1$ in solid state systems. Nevertheless, due to the advance in nano-technology, it is possible to realize the two-lead Anderson model in the $U \to \infty$ limit in a quantum dot coupled to two Ohmic dissipative nano-wires via side-gate voltages where the dissipative bath gives rise to a power-law (pseudogap) suppression in the tunneling density of states (or the hybridization $\Gamma_\rho$). (The spinless version of the setup has been realized in [44].) For the two-channel leads, this effective system leads to the same set of equations as shown in equations (7)–(11), and therefore gives the same transport properties. The Hamiltonian of such an effective system in the simpler case of two single-channel leads reads:
$H = H_{\text{leads}} + H_d + H_T + H_{\text{Env}}$

$H_{\text{leads}} = \sum_{\sigma} \sum_{k} \varepsilon_{\sigma}(e_{\sigma}c_{\sigma k} - \mu_{\sigma})$

$H_d = \varepsilon_d \sum_{\sigma} f_{\alpha} f_{\sigma}^\dagger$

$H_T = V_i \sum_{k \sigma} \left(e^{-i2\phi} b_{\sigma k L} + \text{h.c.} \right)$

$+ V_R \sum_{k \sigma} \left(e^{i2\phi} b_{\sigma k R} + \text{h.c.} \right)$

$H_{\text{Env}} = \frac{Q^2}{2C} + \sum_{k=1}^{N} \left( \frac{q_{k}^2}{2C_k} + \frac{(\phi)}{e} \frac{1}{2L_k} (\varphi - \varphi_k)^2 \right)$

with $\mu_{LR} = \mp V/2$. The Hamiltonian of the environment, $H_{\text{Env}}$ [45–47] is represented by harmonic oscillators described by inductances and capacitances with frequencies given by $\omega_k = \sqrt{L_k/C_k}$. Here, the charge $Q$ and the capacitance $C$ are defined as: $Q = C_L Q_L + C_R Q_R/(C_L + C_R)$ with $C_L = C_R = C$ being the capacitance and $Q_{LR}$ the charge on the left/right tunneling junction, respectively. These oscillators are then bilinearly coupled to the phase operator $\varphi$ (canonically conjugate to the charge operator $Q_{LR} = Q_L - Q_R$ on the dot) through the oscillator phase. The phase operator $\varphi$ induces phase fluctuations of the tunneling amplitude between the dot and the LR lead [44, 47, 48]. The dissipative environment has the following action [48]:

$$S_{\text{Env}} = \frac{1}{2\beta R} \sum_n |\omega_n| |\varphi(\omega_n)|^2$$

where $\omega_n$ represents for the Matubara frequency of the harmonic oscillator (boson) fields, $r \equiv R/R_K$ is the dimensionless impedance of the leads with $R_K = h/e^2$ being the quantum resistance. The action $S_{\text{Env}}$ gives the following correlator of the $\varphi$ fields:

$$\langle e^{-i\omega \phi(0)} e^{i\omega \varphi(t)} \rangle \sim \frac{1}{t^\eta}.$$  

The above correlator effectively introduces a power-law vanishing spectral density of the tunneling (hybridization) between lead and dot when evaluating the self-energies of the pseudo-fermion $f_{\alpha}$ and slave-boson $b_{\sigma}$ fields: $\Gamma_{\alpha}(\omega) = \pi |V_{\alpha}^2 \omega \Theta(D - |\omega - \mu_{\alpha}|))$ (see also equations (7) and (8)). This dissipative single-channel Anderson model can be straightforwardly generalized to the two-channel one. Hence, by tuning the dissipation strength $0 < r < 1$, we effectively change the power-law exponent $r$ of the pseudogap fermionic bath in the original model. Our approach to the 2CK-LM quantum critical point via tuning $r$ can potentially be realized in such systems.

In conclusion, we have studied quantum phase transitions in and out of equilibrium between two-channel Kondo and local moment phases in the two-channel pseudogap Anderson model where the conduction leads show a power-law vanishing density of states with exponent $0 < r < 1$. Via non-crossing approximation (NCA), we solved self-consistently for the impurity Green’s function, and therefore determined the linear and non-linear conductances. The 2CK-LM quantum criticality is reached by varying the power-law exponent $r$ of the pseudogap conduction bath with fixed lead-dot hybridizations and chemical potentials. The impurity local density of states in equilibrium exhibits the pronounced Kondo peak at the Fermi level for $r < r_c$; while the Kondo peak splits and a dip develops for $r > r_c$ in the local moment regime. The linear $G(V = 0, T)$ and non-linear $G(V, T_0)$ $(T_0 \sim 5 \times 10^{-2}D)$ conductances show distinct universal power-law scalings near criticality with different critical exponents. Furthermore, in the 2CK regime $T, V < T_{\text{CK}}$, we also find different characteristic power-law scalings in $G(T)$ and $G(V, T_0)$ compared to the well-known $\sqrt{T}, \sqrt{V}$ scalings for $G(0, 0) - G(0, T)$ and $G(0, V) - G(V, T_0)$, respectively. An independent confirmation of our NCA results is obtained by further computing the local density of states in equilibrium and the linear conductances via the NRG approach. Good agreement is found for the equilibrium properties between NCA and NRG. Our results provide further insights on two-channel Kondo physics and on the non-equilibrium quantum criticality in nano-systems. Further analytic and numerical investigations are needed to address issues on the mechanism behind the different scalings between equilibrium and nonequilibrium conductances.

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