AUTOMORPHISM GROUPS OF FINITE-DIMENSIONAL
ALGEBRAS ACTING ON SUBALGEBRA VARIETIES

ALEX SISTKO

Abstract. Let $k$ be an algebraically-closed field, and $B$ a unital, associative $k$-algebra with $n := \dim_k B < \infty$. For each $1 \leq m \leq n$, the collection of all $m$-dimensional subalgebras of $B$ carries the structure of a projective variety, which we call $\text{AlgGr}_m(B)$. The group $\text{Aut}_k(B)$ of all $k$-algebra automorphisms of $B$ acts regularly on $\text{AlgGr}_m(B)$. In this paper, we study the problem of explicitly describing $\text{AlgGr}_m(B)$, and classifying its $\text{Aut}_k(B)$-orbits. Inspired by recent results on maximal subalgebras of finite-dimensional algebras, we compute the homogeneous vanishing ideal of $\text{AlgGr}_{n-1}(B)$ when $B$ is basic, and explicitly describe its irreducible components. We show that in this case, $\text{AlgGr}_{n-1}(B)$ is a finite union of $\text{Aut}_k(B)$-orbits if $B$ is monomial or its Ext quiver is Schur, but construct a class of examples to show that these conditions are not necessary.

1. Introduction

Within contemporary algebra, finite-dimensional associative algebras undoubtedly play a critical role. For instance, their (finite-dimensional) representation theory can be reasonably said to unite the (finite-dimensional) representation theories of groups, quivers, and other categories of continuing interest. But they can also be studied directly, ex. as a subcategory of the category of associative rings, or algebras over a fixed base field. We might call this direct approach “ring-theoretic,” although this title perhaps obscures the difficulty and importance of the problems that arise, since noncommutative ring theorists do not seem particularly interested in finite-dimensional objects. For instance, this approach is necessary to understand automorphism groups of finite-dimensional algebras, which can be related to important symmetries of their module categories [3], [13]. It is also necessary for the study of varieties of algebras of a fixed dimension, or for related classification problems [12], [13].

To remedy this hopelessness, we can take a relative point of view. Instead of trying to classify $n$-dimensional algebras over a field $k$, we can start with such an algebra $B$, and show how conditions on $B$ influence its quotient algebras and subalgebras. Of the two, quotients appear to be better-behaved than subalgebras. Nevertheless, subalgebras of finite-dimensional algebras still remain important to contemporary mathematics: consider the theory of split and separable extensions [32], or the work of Motzkin and Taussky [28] [29] on commutative subalgebras of matrix rings. In particular, maximal subalgebras of finite-dimensional algebras,

2000 Mathematics Subject Classification. Primary 16W20; Secondary 16S99, 16U60, 14L30, 14N99

Keywords finite-dimensional algebra, subalgebra, maximal subalgebra, automorphism group, subalgebra variety, quiver.
possibly with respect to a certain property (ex. commutativity), have been a subject of interest for decades. Maximal commutative subalgebras of $M_n(k)$ have been studied extensively, for instance in [37], [23], [27], [28], [29], [18], [25]. Maximal subalgebras of central simple algebras were classified by Racine in [33], [34]. Independently, Agore [1] used a geometric argument of Gerstenhaber [17] to bound the maximal dimension of a $C$-subalgebra in $M_n(C)$ by $n^2 - n + 1$. More recently, the author and Iovanov [22] found general structure theorems for maximal subalgebras of finite-dimensional algebras, provided explicit classifications for semisimple algebras and basic algebras, and connected their theory to split and separable extensions of algebras.

This paper was inspired by a question raised during the writing of [22]: if $B$ is a finite-dimensional algebra, and $A, A' \subset B$ are two isomorphic subalgebras, under what conditions do we know that $A' = \psi(A)$ for some $k$-algebra automorphism $\psi \in \text{Aut}_k(B)$? More generally, can we classify $\text{Aut}_k(B)$-orbits of subalgebras of $B$, and relate them to isoclasses of subalgebras? To study this question, it seems natural to try and exploit the structure of $\text{Aut}_k(B)$, which is a linear algebraic group. For each $m \leq \dim_k B$, we define a (projective) $k$-variety $\text{AlgGr}_m(B)$ which parametrizes $m$-dimensional subalgebras, and on which $\text{Aut}_k(B)$ acts regularly. We then seek to understand geometric properties of $\text{AlgGr}_m(B)$, and properties of the $\text{Aut}_k(B)$-action upon it.

The idea of studying subalgebras through the variety $\text{AlgGr}_m(B)$ was partially inspired from recent work in representation theory. Taking $B = kQ/I$ for some quiver $Q$ and admissible ideal $I$, representation theorists have long studied affine and projective varieties parametrizing $B$-modules of a fixed dimension vector $d = (d_1, \ldots, d_n)$. For exposition, see [4], [15], [21]. The affine theory appears to extend at least as far back as Gabriel’s theorem on representation-finite hereditary algebras [21], while the projective theory was developed by Bongartz and Huisgen-Zimmermann in [5], [6]. A common unifying principle is the construction of moduli spaces in the spirit of King [24]. On the algebra side, Gabriel [14] constructs an affine variety $\text{Alg}(n)$ which parametrizes $n$-dimensional algebras over a fixed field $k$. In a broad sense, one might hope that $\text{AlgGr}_m(B)$ provides a projective context to study questions related to Gabriel’s $\text{Alg}(n)$, in analogy to the work described above. However, $\text{AlgGr}_m(B)$ is in some sense fundamentally different, with its focus on subalgebras of a fixed algebra $B$, rather than all algebras of a fixed dimension. But again, this relative perspective has an analogy in geometric representation theory, where others have used projective varieties to study submodules of a fixed dimension vector. Indeed, if $M$ is a $d$-dimensional $kQ$-module and $e = (e_1, \ldots, e_n)$ is a dimension vector satisfying $e_i \leq d_i$ for all $1 \leq i \leq n$, the Quiver Grassmannian $\text{Gr}_e^Q(M)$ of all $e$-dimensional $kQ$-submodules of $M$ has provoked recent interest. These varieties first appeared in the work of Schofield [39] and Crawley-Boevey [11]. Subsequent work can be seen from Reineke [35], Caldero-Reineke [7], and Cerulli Irelli-Feigin-Reineke [8], [9], [10].

The paper is organized as follows. Section 2 supplies the main definitions used in this paper, and briefly reviews relevant results from [22]. In section 3 we introduce the variety $\text{AlgGr}_m(B)$ of $m$-dimensional subalgebras of $B$, and discusses its basic properties. Section 4 contains the main results of this paper. In it, we give an explicit realization for $\text{AlgGr}_{\dim_k B - 1}(B)$ where $B$ is a basic, including a description of its irreducible components. More specifically, we prove the following:
**Theorem 1.1.** Let $B = kQ/I$, for some quiver $Q$ and admissible ideal $I$. Let $\dim_k B = n$. Then $\text{AlgGr}_{n-1}(B)$ only depends on the Ext-quiver of $B$. Furthermore, there exists a closed embedding $\text{AlgGr}_{n-1}(B) \hookrightarrow \mathbb{P}^N$, for some natural number $N$, under which the irreducible components of $\text{AlgGr}_{n-1}(B)$ become linear subspaces. The homogeneous vanishing ideal of $\text{AlgGr}_{n-1}(B)$ with respect to this embedding is explicitly computed, and its irreducible components are explicitly identified.

See theorem 4.1 for a description of the vanishing ideal and corollary 4.2 for the description of the irreducible components. As a consequence, we have the following:

**Corollary 1.2.** If $B$ is basic, then $\text{AlgGr}_{n-1}(B)$ is irreducible if and only if its Ext-quiver is either an m-loop quiver, and m-Kronecker quiver, or two isolated vertices.

See definition 4.2 for all relevant terminology.

Finally, section 5 discusses the action of $\text{Aut}_k(B)$ on $\text{AlgGr}_{n-1}(B)$ in the basic case. In particular, we study conditions under which $\text{AlgGr}_{n-1}(B)$ is a finite union of $\text{Aut}_k(B)$-orbits. Using results from [22], we prove the following theorem:

**Theorem 1.3.** Let $B$, $Q$, and $n$ be as before. Let $H_B$ be the group of all $B$-automorphisms fixing the vertices of $Q$. Then $\text{AlgGr}_{n-1}(B)$ is a finite union of $\text{Aut}_k(B)$-orbits if and only if for every pair $(u,v) \in Q_0 \times Q_0$ with $uJ(B)/J(B)^2v \neq \{0\}$, $\mathbb{P}((uJ(B)/J(B)^2v)^*)$ is a finite union of $H_B$-orbits.

See Theorem 5.2 for details. In the above, $(uJ(B)/J(B)^2v)^*$ denotes the vector space dual. As a consequence, we show that $\text{AlgGr}_{n-1}(B)$ is a finite union of $\text{Aut}_k(B)$-orbits in the case that either $Q$ is Schur, or $B$ is monomial. We end this section by constructing a class of $B$ satisfying this finite orbit property, but for which $B$ is not generally monomial and its Ext-quiver is not generally Schur.

### 2. Background

Unless otherwise stated, $k$ will denote an algebraically-closed field. All algebras are unital, associative, finite-dimensional $k$-algebras. Our terminology on bound quiver algebras essentially comes from [2]. Let $Q$ be a finite quiver with vertex set $Q_0$, arrow set $Q_1$, and source (resp. target) function $s$ (resp. $t$) : $Q_1 \rightarrow Q_0$. Let $kQ$ denote the path algebra of $Q$, and let $J(Q)$ denote the two-sided ideal in $kQ$ generated by $Q_1$. By a slight abuse of notation, for any $u, v \in Q_0$ we let $uQ_1v$ denote the set of arrows in $Q$ with source $u$ and target $v$, and we let $ukQ_1v$ denote their $k$-span inside $kQ$. Define $d(u,v) = \dim_k ukQ_1v$. Note that if $uQ_1v = \emptyset$, then $ukQ_1v = \{0\}$ and $\text{GL}(ukQ_1v)$ is the trivial group. Similar to [20], we define $V^2(Q) = \{(u,v) \in Q_0 \times Q_0 \mid uQ_1v \neq \emptyset\}$. The no-double edge graph of a quiver $Q$ is the (undirected) graph with vertex set $Q_0$, and an edge between $u$ and $v$ if and only if $uQ_1v \cup vQ_1u \neq \emptyset$.

A basic algebra is an algebra of the form $B = kQ/I$, where $I$ is an admissible ideal of $kQ$, i.e. an ideal satisfying $J(Q)^2 \supset I \supset J(Q)^\ell$ for some $\ell \geq 2$. Note that $B = kQ_0 \oplus J(B) = kQ_0 \oplus J(Q)/I$, and that $kQ_0 \cong B/J(B) \cong k[Q_0]$. In fact, the Wedderburn-Malcev theorem tells us that this decomposition is unique in the following sense: for all subalgebras $B_0 \subset B$ isomorphic to $kQ_0$, there is an $x \in J(B)$ such that $(1+x)B_0(1+x)^{-1} = kQ_0$ (see, for instance, [30]).
For $n \geq 2$, we define $T_n Q := kQ/J(Q)^n$, the $n$th truncated quiver algebra associated to $Q$. From here on out, we put total orderings on $Q_0$ and $Q_1$; write $Q_0 = \{v_0 < \ldots < v_{n_0}\}$ for some $n_0 \in \mathbb{Z}_{\geq 0}$ and $Q_1 = \{\alpha_1 < \ldots < \alpha_{n_1}\}$ for some $n_1 \in \mathbb{N}$. Note that for the sake of stating the results with a minimum of caveats, the author assumes $Q_1 \neq \emptyset$ for our generic quiver $Q$; the theorems are still valid, albeit trivial, for the $Q_1 = \emptyset$ case. We also assume that the total ordering on $Q_1$ satisfies the following condition: if $uQ_1v \neq \emptyset$, then the arrows in $uQ_1v$ form an interval within the total ordering of $Q_1$ (i.e. there exist indices $1 \leq u\_v \leq n_1$ and $n\_uv \geq 0$ such that $uQ_1v = \{\alpha_{u\_v}, \alpha_{u\_v+1}, \ldots, \alpha_{u\_v+n\_uv}\}$).

We let $\text{Aut}_k(B)$ denote the group of all $k$-algebra automorphisms of $B$. It is a Zariski-closed subgroup of $\text{GL}(B)$, and hence a linear affine algebraic group. Our notation for subgroups of $\text{Aut}_k(B)$ is borrowed from the notation in [31], [19], [20]. Given two subalgebras $A$ and $A'$ of $B$ and a subgroup $G \leq \text{Aut}_k(B)$, we say they are $G$-conjugate in $B$ if there exists a $\phi \in G$ such that $\phi(A) = A'$. If $G = \text{Aut}_k(B)$, we just say that they are conjugate. A maximal subalgebra of $B$ is a proper subalgebra which is contained in no other proper subalgebras. For a unit $u \in B^\times$, we let $\iota_u$ denote the corresponding inner automorphism, i.e. the map $\iota_u(x) = u x u^{-1}$ for all $x \in B$. We let $\text{Inn}(B)$ denote the group of all inner automorphisms, and $\text{Inn}^*(B) = \{\iota_{1+x} \mid x \in J(B)\}$ denote the group of unipotent inner automorphisms. For a quiver $Q$, we define $S_Q$ to be the permutations $\sigma$ of $Q_0$ satisfying $|\sigma(u)Q_1\sigma(v)| = |uQ_1v|$ for all $u, v \in Q_0$. This is a quotient of $\text{Aut}(Q)$, and with our ordering of $Q_1$, it acts as a subgroup of $\text{Aut}_k(T_n Q)$ by defining it on arrows as follows: for $\sigma \in S_Q$, and $(u, v) \in V^2(Q)$ with $uQ_1v = \{\alpha_1, \ldots, \alpha_{i+1}\}$ and $\sigma(u)Q_1\sigma(v) = \{\alpha_j, \ldots, \alpha_{j+\ell}\}$, $\sigma(\alpha_{i+1}) = \alpha_{j+\ell}$.

Most of the constructions outlined in the next section work for arbitrary $B$, but basic algebras will be the primary focus of this paper. There is a simple reason for this: subalgebras of basic algebras are particularly well-behaved. For instance, maximal subalgebras of basic algebras can be classified explicitly up to $\text{Inn}^*(B)$-conjugation. Indeed, theorem 4.1 of [22] shows the following:

**Theorem 2.1.** Let $B = kQ/I$ be a basic algebra over an algebraically-closed field $k$. Let $A \subset B$ be a maximal subalgebra. Consider the following two classes of maximal subalgebras of $B$:

For a two-element subset $\{u, v\} \subset Q_0$, we define

$$A(u + v) := k(u + v) \oplus \bigoplus_{w \in Q_0 \setminus \{u, v\}} kw \oplus J(B).$$

For an element $(u, v) \in V^2(Q)$ and a codimension-1 subspace $U \leq ukQ_1v$, we define

$$A(u, v, U) := kQ_0 \oplus U \oplus \bigoplus_{(w, y) \in Q_0^2 \setminus \{(u, v)\}} wkQ_1y \oplus J(B)^2.$$ 

Then there exists a unipotent inner automorphism $\iota_{1+x} \in \text{Inn}^*(B)$ such that either $\iota_{1+x}(A) = A(u + v)$ or $\iota_{1+x}(A) = A(u, v, U)$, for some appropriate choice of $u, v,$ and possibly $U$.

Following the terminology in [22], if $A$ is $\text{Inn}^*(B)$-conjugate to a subalgebra of the form $A(u + v)$, then we say that $A$ is of separable type. If $A$ is $\text{Inn}^*(B)$-conjugate to an algebra of the form $A(u, v, U)$, then we say that $A$ is of split type. Theorem 2.1
has immediate consequences for subalgebras of basic algebras. These properties are not difficult to demonstrate, but since they are conceptually useful, we list them in a corollary below for future reference:

**Corollary 2.2.** Let $B$ be a basic $k$-algebra of dimension $n$, and let $A \subset B$ be a subalgebra. Then $A$ satisfies the following:

1. $A$ is also a basic algebra.
2. If $A$ is a maximal subalgebra, then $\dim_k A = n - 1$.
3. If $A$ is a maximal subalgebra, then $J(A)$ is a $B$-subbimodule of $J(B)$, $J(A) = A \cap J(B)$, and $J(B)^2 \subset J(A)$.
4. More generally, if $m = \dim_k A$, then $J(B)^{2(n-m)} \subset A$.

**Proof.** Properties 1-3 follow immediately from Theorem 2.1. Property 4 follows from 1-3 by induction on $n - m$. \(\square\)

For arbitrary $B$, properties 1 and 2 are false. Property 3 only holds for a suitable generalization of split-type subalgebras, although it follows from [22] that, in general, $J(B)^2 \subset J(A)$. Similar reasoning also tells us that if we have a sequence $A_m \subset A_{m-1} \subset \ldots \subset A_1 \subset A_0 = B$, such that $A_i$ is a maximal subalgebra of $A_{i-1}$ for all $i$, then $J(B)^{2m} \subset A_m$.

**Note:** In spite of this paper’s focus on basic algebras, there are good reasons to study $\text{AlgGr}_m(B)$ when $B$ is not basic. In the next section, for instance, we will see why subalgebras of $M_n(k)$ might be of particular interest.

### 3. Varieties of Subalgebras

Set $n = \dim_k B$, and fix some $1 \leq m \leq n$. For any two finite-dimensional $k$-vector spaces $V$ and $W$, we let $\text{Hom}_k^2(V,W)$ denote the collection of injective $k$-linear maps $V \to W$. We think of $\text{Hom}_k^2(V,W)$ as $(\dim_k W) \times (\dim_k V)$-matrices of full rank as needed. $GL(V)$ acts on the right of $\text{Hom}_k^2(V,W)$, and the projection $\text{Hom}_k^2(V,W) \to \text{Hom}_k^2(V,W)/GL(V)$ is a geometric quotient in the category of schemes. Of course, $\text{Hom}_k^2(V,W)/GL(V)$ is just $\text{Gr}_{\dim_k V}(W)$, the Grassmannian of $\dim_k V$-dimensional subspaces of $W$.

**Definition 3.1.** Let $\text{AlgGr}_m(B)$ denote the collection of all $m$-dimensional subalgebras of $B$, considered as a subset of $\text{Gr}_m(B)$. Note that if $B$ is basic, then $\text{AlgGr}_{n-1}(B)$ is the collection of all maximal subalgebras of $B$.

**Lemma 3.2.** $\text{AlgGr}_m(B)$ is closed in $\text{Gr}_m(B)$. In particular, $\text{AlgGr}_m(B)$ is a projective variety.

**Proof.** Choose a basis of $B$ to identify $U := \text{Hom}_k^2(k^m,B)$ with the open subset of $M_{n,m}(k)$ consisting of matrices with full rank. Let $\pi : U \to \text{Gr}_m(B)$ denote the corresponding quotient map. Since $\pi$ is a geometric quotient, it is enough to show that $X := \pi^{-1}(\text{AlgGr}_m(B))$ is closed in $U$. Now, $X$ is the collection of all matrices $A \in U$ satisfying the following:

1. $1 \in \text{col} A$.
2. $\text{Im}(\mu_B |_{\text{col} A \otimes \text{col} A}) \subset \text{col} A$.
Here, col\(A\) denotes the column space of \(A\), and \(\mu_B : B \otimes B \to B\) is the multiplication map of \(B\) (considered as a map \(k^{\mu} \to k^{\mu}\) when needed). It is standard to check that these conditions are Zariski-closed, from which the claim follows. \(\square\)

At this point, we wish to take a quick look at \(\text{AlgGr}_m(B)\) for \(B = M_d(k)\), where \(d > 1\). This example will show us two things: first, it will show us that these varieties encode interesting algebraic data; and second, it will give us a sense of how difficult it might be to provide a uniform description of \(\text{AlgGr}_m(B)\) for arbitrary \(m\) and \(B\).

**Definition 3.3.** For any \(A \in \text{AlgGr}_m(B)\), define \(\text{Iso}(A, B) = \{A' \in \text{AlgGr}_m(B) \mid A' \cong A\ as\ k\text{-algebras}\}\).

**Proposition 3.4.** Let \(d > 1\), and let \(1 \leq m \leq d^2\). Then points of \(\text{AlgGr}_m(M_d(k))\) may be identified with \(d\)-dimensional faithful representations of \(m\)-dimensional algebras, in such a way that the following hold, for each \(A \in \text{AlgGr}_m(M_d(k))\):

1. \(\text{Iso}(A, M_d(k))\) is the collection of all faithful \(d\)-dimensional \(A\)-modules;
2. \(A' \in \text{Iso}(A, M_d(k))\) is conjugate to \(A\) if and only if they correspond to isomorphic \(A\)-modules.

**Proof.** Let \(A\) be an \(m\)-dimensional \(k\)-algebra. Then a \(d\)-dimensional faithful module is the same as a \(d\)-dimensional vector space \(V\), along with a \(k\)-algebra injection \(A \hookrightarrow \text{End}_k(V)\). After fixing a basis for \(V\), this is the same as a \(k\)-algebra injection \(A \hookrightarrow M_d(k)\). A different faithful \(d\)-dimensional representation, call it \(V'\), will yield another injection \(A \hookrightarrow \text{End}_k(V') \cong M_d(k)\), whose image is isomorphic to \(A\). Every subalgebra of \(M_d(k)\) isomorphic to \(A\) arises in the way, and so elements of \(\text{AlgGr}_m(M_d(k))\) correspond bijectively to \(d\)-dimensional faithful representations of \(m\)-dimensional algebras. For claim (1), recall that \(\text{Aut}_k(M_d(k)) = \text{Inn}(M_d(k)) \cong GL_d(k)/k^\times\) by the Skolem-Noether Theorem. If \(g \in GL_d(k)\), then restriction along \(\phi := t_g \ |_A : A \to gAg^{-1}\) replaces the faithful \(A\)-module \(k^d\) by the faithful \(A\)-module \(\text{Res}_g(k^d)\). But \(\text{Res}_g(k^d) \cong k^d\) as \(A\)-modules under \(g\), considered as an endomorphism of \(k^d\). Conversely, suppose that \(A' \in \text{AlgGr}_m(M_d(k))\) is given, such that \(A \cong A'\) and the \(A\)-module structure on \(k^d\) induced by \(A'\) is isomorphic to the structure defining \(A\) itself. This means that there exists a \(k\)-algebra isomorphism \(\psi : A \to A'\) such that \(\text{Res}_\psi(k^d) \cong k^d\) as \(A\)-modules. In other words, there is an invertible \(k\)-linear map \(g \in GL_d(k)\) such that for all \(a \in A\) and \(v \in k^d\), \(g(\psi(a)v) = ag(v)\). But since \(k^d\) is faithful, this implies \(g\psi(a) = ag\) as endomorphisms on \(k^d\), and hence \(\psi = t_{g^{-1}} \ |_A\). \(\square\)

**Note:** Theorem 3.3 implies that \(\text{AlgGr}_m(B)\) is not always a finite union of \(\text{Aut}_k(B)\)-orbits. For instance, consider the three-dimensional algebra \(A = k[x, y]/(x^2, xy, y^2)\). \(A\) has infinite representation type. In fact, it is easy to see that for each \(\lambda \in k\), the map \((x, y) \mapsto \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) ; \left(\begin{array}{cc} 0 & \lambda \\ 0 & 0 \end{array}\right)\) defines a 2-dimensional indecomposable \(A\)-module \(M_\lambda\), and \(M_\lambda \cong M_\mu\) if and only if \(\lambda = \mu\). But then \(\{M_\lambda \oplus A\}_{\lambda,\mu \in k}\) is an infinite collection of pairwise non-isomorphic, 5-dimensional faithful \(A\)-modules. In particular, \(\text{AlgGr}_5(M_2(k))\) is not a finite union of \(\text{Aut}_k(M_2(k))\)-orbits. It may amuse the reader to note that \(\text{AlgGr}_3(M_2(k))\) is a finite union of \(\text{Aut}_k(M_2(k))\)-orbits.
Isomorphism classes of subalgebras of $B$ are generally not open or closed, but they are constructible in $\text{AlgGr}_m(B)$. This may not surprise the reader, since they are clearly $\text{Aut}_k(B)$-invariant, and hence a union of $\text{Aut}_k(B)$-orbits. Note, however, that the example above shows that this fact alone does not imply constructibility, since isoclasses might be (uncountably) infinite unions of $\text{Aut}_k(B)$-orbits. To prove the constructibility of isoclasses, we rely on a well-known theorem of Chevalley:

**Theorem 3.5** (Chevalley). Let $f : X \to Y$ be a morphism of schemes. Assume that $f$ is quasi-compact and locally of finite presentation, and that $Y$ is quasi-compact and quasi-separated. Then the image of every constructible subset of $X$ under $f$ is a constructible subset of $Y$.

It is easy to check that $\pi : U \to \text{Gr}_m(B)$ of lemma 3.2 satisfies these conditions. To prove the constructibility of isoclasses, we introduce a few extra definitions.

**Definition 3.6.** Let $\pi : U \to \text{Gr}_m(B)$ be the quotient map from before. For any $A \in \text{AlgGr}_m(B)$, an element $F \in \pi^{-1}(A)$ is called a frame of $A$.

**Definition 3.7.** Let $F$ be a fixed frame for $A \in \text{AlgGr}_m(B)$, with columns $f_1, \ldots, f_m$. Define $I(F)$ to be the group of all $g \in \text{GL}(B)$ such that $g \cdot B \cdot (f_i, f_j) = B \cdot (g \cdot f_i, g \cdot f_j)$ for all $i, j \in \{1, \ldots, m\}$.

Of course, $I(F)$ is a closed subgroup of $\text{GL}(B)$, and $I(F)$ contains $\text{Aut}_k(B)$. Note that $I(F)$ depends a priori on the frame chosen for $A$.

**Proposition 3.8.** Fix $A \in \text{AlgGr}_m(B)$ and $F \in \pi^{-1}(A)$ with columns $f_1, \ldots, f_m$. Then $A' \in \text{Iso}(A, B)$ if and only if for any frame $F'$ of $A'$, there exist $g \in I(F)$ and $\phi \in \text{GL}_m(k)$ such that $F' = g \cdot F \cdot \phi^{-1}$. Consequently, $\text{Iso}(A, B)$ is a constructible subset of $\text{AlgGr}_m(B)$.

**Proof.** It is clear that $A' \cong A$ if any frame of $A'$ is of the form $g \cdot F \cdot \phi^{-1}$, for some $g \in I(F)$ and $\phi \in \text{GL}_m(k)$. So assume that $\gamma : A \to A'$ is a $k$-algebra isomorphism. Then $\gamma(F) := \{\gamma(f_1) \mid \cdots \mid \gamma(f_m)\}$ is a frame for $A'$. Since any two frames for $A'$ are in the same $\text{GL}_m(k)$-orbit of $U$, it is enough to show that $\gamma(F) = g \cdot F$ for some $g \in I(F)$. Pick ordered bases $\beta = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_n\}$ and $\gamma(\beta) := \{\gamma(f_1), \ldots, \gamma(f_m), h_{m+1}, \ldots, h_n\}$ for $B$, and define $\gamma(f_i) = \gamma(f_i)$ for $1 \leq i \leq m$ and $\gamma(f_{m+j}) = h_{m+j}$ for $1 \leq j \leq n - m$. Then if $g$ denotes the matrix corresponding to $\gamma$ with respect to the fixed basis chosen for $B$, $g \in I(F)$ and $\gamma(F) = g \cdot F$. From this, it follows that $\pi^{-1}(\text{Iso}(A, B)) = [I(F) \times \text{GL}_m(k)] \cdot F$ is constructible. By theorem 3.5, $\text{Iso}(A, B)$ is constructible as well. \qed

We now begin our study of subalgebra varieties of basic algebras. We first note a straightforward consequence of corollary 2.2.

**Lemma 3.9.** Let $B = kQ/I$. Then $\text{AlgGr}_m(B) \cong \text{AlgGr}_m(B/J(B)^2(n-m))$ as a projective variety, where $\dim_k B/J(B)^2(n-m) = n + m$.

**Proof.** By corollary 2.2, $J(B)^2(n-m) \subset A$ for all $A \in \text{AlgGr}_m(B)$. Therefore, the correspondence $A \leftrightarrow A/J(B)^2(n-m)$ provides an equivalence $\text{AlgGr}_m(B) \cong \text{AlgGr}_m(B/J(B)^2(n-m))$, as we wished to show. \qed

**Note:** Applying lemma 3.9 to maximal subalgebras of $B = kQ/I$, we find that $\text{AlgGr}_{n-1}(B)$ is isomorphic to $\text{AlgGr}_{\dim_{\text{Gr}_m(B)} B/J(B)^2 - 1}(B/J(B)^2)$. But note that the algebra $B/J(B)^2$ is isomorphic to $T_2Q$, so that $\text{AlgGr}_{n-1}(B)$ only depends on the underlying quiver of $B$. Therefore, we make the following definition:
Definition 3.10. \( \text{msa}(Q) := \text{AlgGr}_{n-1}(T_2Q) \) where \( \dim_k T_2Q = n \). We can think of \( \text{msa}(Q) \) as the variety of maximal subalgebras of any algebra of the form \( B = kQ/I \), for some admissible ideal \( I \subset kQ \). The action of \( \text{Aut}_k(B) \) on \( \text{msa}(Q) \) factors through the projection map \( \text{Aut}_k(B) \to \text{Aut}_k(B/J(B)^2) \cong \text{Aut}_k(T_2Q) \).

Lemma 3.9 shows that \( \text{AlgGr}_{m}(B) \cong \text{AlgGr}_{m'}(B') \) does not imply \( m = m' \) or \( B \not\cong B' \). This problem is particularly acute for maximal subalgebras of basic algebras. However, in section 5 we discuss several interesting conditions on an admissible ideal \( I \subset kQ \), which guarantee that \( \text{msa}(Q) \) is at least a finite union of \( \text{Aut}_k(B) \)-orbits.

Problem 3.11. Suppose that \( B \) is a finite-dimensional algebra, and that \( G \) is a subgroup of \( \text{Aut}_k(B) \). Classify the \( G \)-orbits of \( \text{AlgGr}_m(B) \).

This problem is too general for us to expect any reasonable answer, even for basic algebras. However, in section 5 we discuss several interesting conditions on an admissible ideal \( I \subset kQ \), which guarantee that \( \text{msa}(Q) \) is at least a finite union of \( \text{Aut}_k(B) \)-orbits.

4. The Structure of \( \text{msa}(Q) \)

In this section, we provide an explicit description for the variety \( \text{msa}(Q) \) introduced in the previous section. Specifically, we construct a closed embedding \( \text{msa}(Q) \to \mathbb{P}^N \) for \( N = |Q_0| + |Q_1| - 2 \), and find a finite set of generators for the homogeneous vanishing ideal of \( \text{msa}(Q) \) in \( k[\mathbb{P}^N] \). We then describe the irreducible components of \( \text{msa}(Q) \).

For the next theorem, let \( B = T_2Q \) for some quiver \( Q \), with \( Q_0 = \{v_0, v_1, \ldots, v_{n_0}\} \), and \( Q_1 = \{a_1, \ldots, a_{n_1}\} \). Let \( Q_0 \) and \( Q_1 \) be ordered as in Section 2. Extend these orderings to a total ordering on \( Q_0 \cup Q_1 \) by defining \( v < \alpha \) for all \( v \in Q_0 \) and \( \alpha \in Q_1 \). We will freely treat subsets of \( Q_0 \cup Q_1 \) as totally ordered sets under the induced ordering. Let \( V = \text{span}_k(Q_0 \setminus \{v_0\} \cup Q_1) \), and let \( R = k[V] = k[v_1, \ldots, v_{n_0}, a_1, \ldots, a_{n_1}] \) denote the homogeneous coordinate ring of the isomorphic projective varieties:

\[
\text{Gr}_{n_0+n_1-1}(V) \cong \mathbb{P}(V^*) \cong \mathbb{P}(V).
\]

Recall that if we consider an element \( x \in Q_0 \setminus \{v_0\} \cup Q_1 \) as an element of \( R \), then the statement \( \text{“} x \text{ vanishes on } A \in \text{Gr}_{n_0+n_1-1}(V) \text{”} \) is equivalent to the statement \( \text{“} x(F) = 0 \text{ for any frame } F \text{ of } A \text{”} \). Here, a frame of \( A \) is simply an \( (n_0 + n_1) \times (n_0 + n_1 - 1) \)-matrix whose columns form an ordered basis for \( A \), and \( x(F) \) is the \((n_0 + n_1 - 1) \times (n_0 + n_1 - 1)\)-minor of \( F \) formed by deleting the \( x^{th} \)-row.

Note that if \( F' \) is obtained from \( F \) by column operations, then \( x(F) = 0 \) implies \( x(F') = 0 \). Also recall that \( \text{Gr}_{n_0+n_1-1}(V) \) can be decomposed into Schubert cells indexed by \((n_0 + n_1 - 1)\)-element subsets of \( Q_0 \setminus \{v_0\} \cup Q_1 \). For our purposes, it will be easier to index the Schubert cells by 1-element subsets instead, i.e. by taking the complement of a given \((n_0 + n_1 - 1)\)-element subset. With this convention, if \( x \in Q_0 \setminus \{v_0\} \cup Q_1 \) is any element, then its corresponding Schubert cell is the collection of all \( A \in \text{Gr}_{n_0+n_1-1}(V) \) with the following property: any frame \( F \) of \( A \) is column-equivalent to a reduced-column echelon matrix with pivots on every row except for the \( x \)-row.
The basis $Q_0 \cup Q_1$ for $B$ allows us to fix a $k$-vector space isomorphism $B \to B^*$. The dual vector of $x \in Q_0 \cup Q_1$ will be denoted $x^*$. More generally, for any vector $x = \sum_{\alpha \in Q_0} \lambda \alpha + \sum_{\alpha \in Q_1} \lambda_\alpha \alpha$ in $B$, we let $x^*$ denote its image $\sum_{v \in Q_0} \lambda v^* + \sum_{\alpha \in Q_1} \lambda_\alpha \alpha^*$ in $B^*$. Finally, if $S$ is any finite set, and $\sigma : S \to \mathbb{Z}_{\geq 0}$ is function taking values in the non-negative integers, we let $|\sigma| = \sum_{x \in S} \sigma(x)$ and $\text{supp}(\sigma) = \{x \in S \mid \sigma(x) \neq 0\}$.

**Theorem 4.1.** There is a closed embedding $\mathfrak{msa}(Q) \hookrightarrow \text{Gr}_{n_0+n_1-1}(V)$, whose image is the vanishing set of the ideal $X = X_0 + X_{1/2} + X_1 \subset \mathbb{R}$, where:

1. $X_0$ is the ideal generated by the set $\{v_i^2 v_j - (-1)^{j-i-1} v_i v_j^2 \mid 1 \leq i < j \leq n_0\}$.
2. $X_{1/2}$ is the ideal generated by the sets $\{v_i v_j \mid v_j \notin \{s(\alpha_i), t(\alpha_i)\}\}$, $\{(v_k - (-1)^{k-1} v_j_\alpha_\alpha_1 \mid 1 \leq j < k \leq n_0, \alpha \in v_j Q_1 v_k\}$, and $\{v_\alpha \mid s(\alpha) = v = t(\alpha)\}$.
3. $X_1$ is the ideal generated by the set $\{v_i v_j \mid \alpha \notin \alpha_j\}$.

**Proof.** Let $v_0^* : B \to k$ denote the algebra character afforded by the simple $B$-module at $v_0$, and let $L : B \to \ker v_0^* = V$ be the linear transformation $L(x) = x - v_0^*(x) \cdot 1$. Then $L$ is surjective with $\ker L = k \cdot 1$, and so it induces an isomorphism $\mathcal{T} : B/k \cdot 1 \to V$. Dualizing, we get an isomorphism $\mathcal{T}^* : V^* \to (B/k \cdot 1)^*$, which induces an isomorphism $\mathcal{P}(V^*) \cong \mathcal{P}((B/k \cdot 1)^*) \cong \text{Gr}_{n_0+n_1-1}(B/k \cdot 1)$. But since every subalgebra is unital, the map $A \to A/k \cdot 1$ induces a closed immersion $\mathfrak{msa}(Q) \hookrightarrow \text{Gr}_{n_0+n_1-1}(B/k \cdot 1)$, and hence a closed immersion $\mathfrak{msa}(Q) \to \mathcal{P}(V^*)$. Note that $V : V \subset V$ as a subset of $B$. And that the image of $\mathfrak{msa}(Q)$ in $\mathcal{P}(V^*) \cong \text{Gr}_{n_0+n_1-1}(V)$ is simply the set of all multiplicatively closed $(n_0 + n_1 - 1)$-dimensional subspaces of $V$. By a slight abuse of notation, we will sometimes identify $A \in \mathfrak{msa}(Q)$ with its image $L(A) \subset V$.

Under this identification, we first show that every element of $X$ vanishes on $\mathfrak{msa}(Q)$. To show that $X_0$ vanishes on $\mathfrak{msa}(Q)$, we first consider $v_i(A(v_j + v_k))$, where $1 \leq i \leq n_0$ and $0 \leq j < k \leq n_1$. If $j = 0$, then $L(A(v_0 + v_k))$ has $Q_0 \setminus \{v_0, v_k\} \cup Q_1$ as an ordered basis; by performing column operations on any frame for $L(A(v_0 + v_k))$, we can transform it into a frame with precisely these columns. But then the $v_k$-row in the frame is zero, so that if $i \neq k$, $v_i(A(v_0 + v_k)) = 0$. This immediately implies that $v_i v_j v_p(A(v_0 + v_k)) = 0$ for any triple $(i, j, p)$ of pairwise distinct positive integers. Similarly, for all $1 \leq i < j < n_0$, either $i \neq k$ or $j \neq k$, and so $[v_i^2 v_j - (-1)^{j-i-1} v_i v_j^2](A(v_0 + v_k)) = 0$. Otherwise $j > 0$, and $A(v_j + v_k)$ has a basis of the form $Q_0 \setminus \{v_0, v_j, v_k\} \cup \{v_j + v_k\} \cup Q_1$. We can totally order this basis by defining $v_i < v_j + v_k$ for $i < j$ and $v_j + v_k < v_i$ for $i > j$. Then $v_i(A(v_j + v_k)) = 0$ for all $i \notin \{j, k\}$, since the $v_j$-row equals the $v_k$-row in this frame. Again, we find that $[v_i v_j v_p](A(v_j + v_k)) = 0$ for any triple $(i, j, p)$ of pairwise distinct positive integers, and that $[v_i^2 v_p - (-1)^{p-i-1} v_i v_p^2](A(v_j + v_k)) = 0$ if $i, p \notin \{j, k\}$. A direct computation shows that if $j < k$, then $v_k(A(v_j + v_k)) = 1$ and $v_j(A(v_j + v_k)) = (-1)^{k-j-1}$ with the specified frame. Hence, $[v_i^2 v_k - (-1)^{k-j-1} v_i v_k^2](A(v_j + v_k)) = 0$. In other words, the generators of $X_0$ vanish on maximal subalgebras of separable type.

Now consider $v_i(A)$, for $1 \leq i \leq n_0$ and $A$ a maximal subalgebra of split type. $A$ is $\text{Inn}^n(B)$-conjugate to an algebra of the form $A(v_j, v_k, W)$, for $v_j, v_k \in Q_0$. If $i \notin \{j, k\}$, then $v_i \in L(A)$, and so there is a frame for $A$ with $v_i$ as a column. The minor of this frame formed by deleting the $v_i$-row contains a zero column, and so
\(v_i(A) = 0\). Immediately, we see that \([v_iv_jv_p](A) = 0\) for any three distinct vertices, and \([v_i^2v_p - (-1)^{p-1}v_jv_k^2](A) = 0\) if \(\{i, p\} \neq \{j, k\}\). So it only remains to check the case \(j \neq k\) and \(\{i, p\} = \{j, k\}\). Note that \(A\) must belong to a Schubert cell corresponding to an arrow \(v_j \overset{\alpha_p}{\longrightarrow} v_k\). This means that \(L(A)\) contains a basis of the form \(\gamma = Q_0 \setminus \{v_0, v_j, v_k\} \cup \{v_j + \lambda v_p, v_k - \lambda v_p\} \cup Q_1 \setminus v_j Q_1 v_k \cup \beta\), where \(\beta\) is a basis for \(W\). Note that if \(v_j = v_k\), we necessarily have \(\lambda = 0\). In fact, if \(\{\alpha_1, \ldots, \alpha_{p-1}, \alpha_p, \ldots, \alpha_q\}\) are the (ordered) arrows from \(v_j\) to \(v_k\), then \(\beta\) can be taken to be of the form \(\beta = \{\alpha_i + \lambda_1 \alpha_p, \ldots, \alpha_{p-1} + \lambda_{p-1} \alpha_p, \alpha_{p+1}, \ldots, \alpha_q\}\). Ordering this basis for \(L(A)\) in the obvious way, we obtain a frame for \(L(A)\) satisfying \(v_j(A) = (-1)^{k-j-1}v_k(A)\). Hence, \(v_i^2v_j - (-1)^{k-j-1}v_jv_k^2(A) = 0\). This shows that elements of \(X_0\) belong to the vanishing ideal of \(\text{msa}(Q)\).

Observe that \(\alpha_i(A(v_j + v_k)) = 0\) because \(\alpha_i \in L(A(v_j + v_k))\). So we only need to check the remaining relations on maximal subalgebras of split type. To show that elements of \(X_{1/2}\) vanish on \(\text{msa}(Q)\), suppose that \(A\) is conjugate to \(A(v_j, v_k, W)\) as before, and take \(\gamma\) to be the ordered basis described above. If \(v_p \notin \{v_j, v_k\}\) then we have already seen \([\alpha_i v_p](A) = 0\), so we may assume that \(v_p \in \{v_j, v_k\}\). Then if \(\alpha_i v_p \in X_{1/2}\), \(v_p\) must satisfy \(v_p \notin \{s(\alpha_i), t(\alpha_i)\}\). This implies that \(\alpha_i\) is not parallel to \(\alpha_p\). Then \(\alpha_i \in L(A)\) and \(\alpha_i(A) = 0\). We must also show that for all arrows \(\alpha \in v_j Q_1 v_k\), \([v_k - (-1)^{k-j-1}v_j \alpha](A) = 0\). But this is clear, since \(v_j(A) = (-1)^{k-j-1}v_k(A)\). Finally, suppose that \(\alpha \in Q_1\) is a loop at \(v\). If \((v_j, v_k) \neq (v, v)\) then of course \([v\alpha](A) = 0\). Otherwise \((v_j, v_k) = (v, v)\). In this case, \(v \in L(A)\), as noted above, so that \(v(A) = 0\) and hence \([v\alpha](A) = 0\). We conclude that every element of \(X_{1/2}\) vanishes on \(\text{msa}(Q)\). A similar argument shows that every element of \(X_1\) also vanishes on \(\text{msa}(Q)\).

All that remains to be shown is that any homogeneous polynomial vanishing on \(\text{msa}(Q)\) belongs to \(X\). Let \(f \in R\) be a homogeneous polynomial of degree \(d\) vanishing on \(\text{msa}(Q)\). Then we can write \(f = \sum \phi_\sigma \mu_\sigma\), where:

1. the sum runs through all functions \(\sigma : Q_0 \setminus \{v_0\} \cup Q_1 \rightarrow \mathbb{Z}_{>0}\) satisfying \(|\sigma| = d\),
2. each \(\phi_\sigma\) is an element of \(k\), and
3. \(\mu_\sigma := \prod_{x \in Q_0 \setminus \{v_0\} \cup Q_1} x^{\sigma(x)}\).

Then for all \(1 \leq i \leq n_0\), \(0 = f(A(v_0 + v_i)) = \sum \phi_\sigma \mu_\sigma (A(v_0 + v_i) = \phi_{d-v_i} v_i (A(v_0 + v_i))^d\). There is a frame for \(L(A(v_0 + v_i))\) in which \(v_i (A(v_0 + v_i)) = 1\), and so this implies \(\phi_{d-v_i} = 0\) for all \(1 \leq i \leq n_0\).

For all \(1 \leq j < k \leq n_0\), \(0 = f(A(v_j + v_k)) = \sum_{i=1}^{d-1} \phi_{v_i v_j + (d-i) v_k} (A(v_j + v_k))^i v_k (A(v_j + v_k))^d-i\).

Note that the previous argument allowed us to exclude \(i = 0\) and \(i = d\). Since we can choose a frame for \(L(A(v_j + v_k))\) in which \(v_j (A(v_j + v_k)) = (-1)^{k-j-1}\) and \(v_k (A(v_j + v_k)) = 1\), we see that this relation reduces to
\[
0 = \sum_{i=1}^{d-1} \phi_{v_i v_j + (d-i) v_k} (-1)^{(k-j-1)i},
\]
which in turn holds if and only if
Every other term $\phi_{\sigma} \mu_{\sigma}$ in $f$ satisfying $\sigma(Q_1) = 0$ must involve three distinct vertices, and all such terms already belong to $X_0$. So without loss of generality, we may assume that $f$ is contained in the ideal of $R$ generated by $Q_1$. Of course, we may also subtract off any terms in $f$ that are already in $X_{1/2}$ or $X_1$. So in fact, we may assume that if $\phi_{\sigma} \neq 0$, then there exist $v_j, v_k \in Q_0 \setminus \{v_0\}$ such that $\sigma(v_j Q_1 v_k) \neq 0$ and $\text{supp}(\sigma) \subseteq \{v_j, v_k\} \cup v_j Q_1 v_k$. Suppose that $v_j Q_1 v_k = \{\alpha_i, \ldots, \alpha_q\}$. Let $\lambda$ be a function $\{j\} \cup \{i, \ldots, q - 1\} \to \mathbb{R}^\times$, and let $A(\lambda)$ be the maximal subalgebra with basis $Q_0 \setminus \{v_j, v_k\} \cup v_j + \lambda(j)\alpha_q, v_k - \lambda(j)\alpha_q \cup Q_1 \setminus v_j Q_1 v_k \cup \{\alpha_i + \lambda(i)\alpha_q, \ldots, \alpha_{q-1} + \lambda(q-1)\alpha_q\}$ (again, if $v_j = v_k$ we assume $\lambda(j) = 0$). Then evaluating at the corresponding frame yields

$$0 = f(A(\lambda)) = \sum \phi_q v_j^\ast + r_v k + r(\lambda(j)(-1)^{n_0 + q - j}q(\lambda(j)(-1)^{n_0 + q - k + 1})^\tau \lambda^\tau,$$

where:

1. the sum ranges over all triples $(q, r, \tau)$ such that $\text{supp}(\tau) \subseteq v_j Q_1 v_k$ and $|\tau| = d - q - r$, and
2. 

$$\lambda^\tau := \prod_{i \leq \ell \leq q - 1} \alpha_{\ell}(A(\lambda))^{\tau(\alpha_{\ell})} = \prod_{i \leq \ell \leq q - 1} \lambda(\ell)^{\tau(\alpha_{\ell})}.$$

Since this must hold for all $\lambda$, it follows that for all fixed $\tau$,

$$0 = \sum_{i=0}^{d-|\tau|} \phi_{(d-|\tau|-i)v_j^\ast + i + v_k^\ast + \tau(1-j-k)i},$$

where we have gathered terms from the powers of $v_j(A(\lambda))$ and $v_k(A(\lambda))$, and have cancelled constant powers of $-1$ from both sides of the equation. When $|\tau| < d$, this implies that the sum of all monomials in $f$ containing a vertex and an arrow lie in $X_0 + X_{1/2}$. Hence, we have reduced to the following case: $f = \sum \phi_{\sigma} \mu_{\sigma}$, where $\phi_{\sigma} \neq 0$ implies that for some $v_j, v_k \in Q_0$, $\text{supp}(\sigma) \subseteq v_j Q_1 v_k$. But applying $A(\lambda)$ to such a polynomial yields

$$0 = \sum_{\text{supp}(\sigma) \subseteq v_j Q_1 v_k} \phi_{\sigma} \mu_{\sigma}(A(\lambda)).$$

Notice that

$$\mu_{\sigma}(A(\lambda)) = \prod_{\ell \in \{j\} \cup \{i, \ldots, q\}} \lambda(\ell)^{\sigma(\alpha_{\ell})}.$$ 

Since this holds for all such $\lambda$, it follows that for all $\sigma$ with $\text{supp}(\sigma) \subseteq v_j Q_1 v_k$, $\phi_{\sigma} = 0$. Applying this to all pairs of vertices $v_j, v_k \in Q_0$, we conclude that $f = 0$.

In other words, any polynomial vanishing on $\text{msa}(Q)$ must lie in $X_0 + X_{1/2} + X_1 = X$, as we wished to show. 

For the following Corollary, we recall the definitions of some standard quivers:
Definition 4.2. For any \( m \in \mathbb{N} \), the \( m \)-loop quiver is the quiver with vertex set \( \{1\} \) and arrow set \( \{\alpha_1, \ldots, \alpha_m\} \):

\[
\begin{array}{c}
\cdots \quad \alpha_1 \quad \cdots \quad \alpha_m \\
\circ \\
\end{array}
\]

We denote this quiver by \( L_m \). The \( m \)-Kronecker quiver is the quiver with vertex set \( \{s, t\} \) and arrow set \( Q_1 = sQ_1t = \{\alpha_1, \ldots, \alpha_m\} \):

\[
\begin{array}{c}
\alpha_1 \\
\cdots \\
\alpha_m \\
\circ \\
\circ \\
s \\
t \\
\end{array}
\]

We denote this quiver by \( K_m \). Finally, the \( m \)-isolated vertices quiver is just the quiver with vertex set \( \{1, \ldots, m\} \) and arrow set \( Q_1 = \emptyset \).

Example 4.3. It is easy to check that any subspace of \( k\mathbb{K}_m \) is a subalgebra if and only if it contains 1. Hence, \( \text{msa}(\mathbb{K}_m) = \text{Gr}_m(k\mathbb{K}_m/k \cdot 1) \cong \mathbb{P}^m \). Similarly, any subspace of \( J(T_2\mathbb{L}_m) \) is multiplicatively closed, and so \( \text{msa}(\mathbb{L}_m) = \text{Gr}_{m-1}(J(T_2\mathbb{L}_m)) \cong \mathbb{P}^{m-1} \). If \( Q \) is two isolated vertices, then \( k \cdot 1 \) is the only proper subalgebra and \( \text{msa}(Q) \) is a point. It turns out that for all other \( Q \), \( \text{msa}(Q) \) is reducible.

Corollary 4.4. Let \( Q \) be a finite quiver. Identify \( \text{msa}(Q) \) with the variety of maximal subalgebras of \( T_2Q \). Then the irreducible components of \( \text{msa}(Q) \) may be described explicitly as follows:

1. For each two-element subset \( \{s, t\} \subset Q_0 \) such that \( sQ_1t \cup tQ_1s = \emptyset \), the singleton \( \{A(s + t)\} \) is an irreducible component of dimension 0;
2. For each \((s, t) \in V^2(Q)\) with \( s \neq t \), the collection
   \[
   \bigcup_{U \in \mathbb{P}((skQ_1t)^*)} [\text{Inn}^*(B) \cdot A(s, t, U)] \cup \{A(s + t)\}
   \]
   is an irreducible component of dimension \( |sQ_1t| \);
3. For each \((s, s) \in V^2(Q)\), the collection
   \[
   \bigcup_{U \in \mathbb{P}((skQ_1s)^*)} [\text{Inn}^*(B) \cdot A(s, s, U)]
   \]
   is an irreducible component of dimension \( |sQ_1s| - 1 \).

These irreducible components are all projective spaces, and the dimension of \( \text{msa}(Q) \) is the maximum of the dimensions described above. Furthermore, \( \text{msa}(Q) \) is an irreducible projective variety if and only if \( Q \) is an \( m \)-Kronecker quiver, an \( m \)-loop quiver, or the two isolated vertices quiver.

Proof. It is clear that the singletons in (1) are closed irreducible subsets of \( \text{msa}(Q) \). So suppose \((v_j, v_\ell) \in V^2(Q)\) is given with \( j \neq \ell \). Assume \( j < \ell \). We claim that

\[
\bigcup_{U \in \mathbb{P}((v_jkQ_1v_\ell)^*)} \text{Inn}^*(B)A(v_j, v_\ell, U) \cup \{A(v_j + v_\ell)\}
\]
is the vanishing set of a homogeneous prime ideal \( P \). Indeed, it is easy to check using the proof of theorem 4.1 that \( P = X + (Q_0 \setminus \{v_0, v_1\}, Q_1 \setminus v_jQ_1v_j) \) if \( j = 0 \), and \( P = X + (v_j - (-1)^{t-j-1}v_t, Q_0 \setminus \{v_0, v_j, v_t\}, Q_1 \setminus v_jQ_1v_t) \) otherwise. In either case, we find that \( k[\text{msa}(Q)]/P \) is isomorphic to the polynomial ring \( k[v_j, v_1v_j] \). The case \( \ell < j \) is similar. Finally, suppose that \( v_j \in Q_0 \) satisfies \( v_jQ_1v_j \neq 0 \). Then \( \bigcup_{U \in \mathbb{P}(v_jkQ)} \text{Im}^*(B)A(v_j, v_j, U) \) is the vanishing set of \( P = X + (Q_0 \setminus \{v_0\}, Q_1 \setminus v_jQ_1v_j) \) and \( k[\text{msa}(Q)]/P \cong k[v_j] \).

Every element of \( \text{msa}(Q) \) is contained in the union of the closed irreducible subsets described above. Since there are no containment relations between these sets, it follows that these are the irreducible components of \( \text{msa}(Q) \). From our description, everything is clear but the criterion for when \( \text{msa}(Q) \) is itself irreducible. In example 4.3 we verified that \( \text{msa}(Q) \) is irreducible when \( Q \) is an \( m \)-Kronecker quiver, an \( m \)-loop quiver, or the union of two isolated vertices. So we only need to show that for every other \( Q \), \( \text{msa}(Q) \) is reducible. If \( Q \) is not connected, and is not the union of two isolated vertices, then (1)-(3) yield more than one irreducible component, so we may assume that \( Q \) is connected. If there exist non-parallel arrows in \( Q \), then \( X_1 \) is not the zero ideal, and \( k[\text{msa}(Q)] \) is not a domain. So all arrows in \( Q \) must be parallel, which implies that either \( Q \) is a Kronecker quiver or an \( n \)-loop quiver. \( \square \)

**Example 4.5.** Let

\[
Q = \begin{array}{c}
\node{t} \node{s} \node{\gamma} \\
\alpha_1 \\
\alpha_2 \\
\beta
\end{array}
\]

and take \( R = k[t, \alpha_1, \alpha_2, \beta, \gamma] \). Then \( \text{msa}(Q) \) is the vanishing set of the ideal \( X = (\gamma t, \alpha_1 \beta, \alpha_2 \beta, \alpha_1 \gamma, \alpha_2 \gamma, \gamma \beta) \). \( V^2(Q) \) consists of the three points \((s, t), (t, s), \) and \((t, t)\). Each of these corresponds to an irreducible component of dimension 2, 1, and 0, respectively.

5. Orbits and Isoclasses in \( \text{msa}(Q) \)

Let \( B = kQ/I \) for some admissible ideal \( I \). Taking the Wedderburn-Malcev decomposition \( B = kQ_0 \oplus J(Q)/I \) for \( B \) discussed in section 2 we define the following subgroups of \( \text{Aut}_k(B) \):

\[
H_B = \{ \phi \in \text{Aut}_k(B) \mid \phi(Q_0) = Q_0 \},
\]

\[
H_B = \{ \phi \in \text{Aut}_k(B) \mid \phi|_{Q_0} = \text{id}_{Q_0} \},
\]

\[
V_{l}(Q,I) = \{ \phi \in H_B \mid \phi(ukQ_1v) = ukQ_1v \text{ for all } (u, v) \in V^2(Q) \}.
\]

These definitions are chosen to be consistent with the terminology of [19]. The group \( V_{l}(Q,I) \) is called the group of linear changes of variables. In definition 3.10 we note that the action of \( \text{Aut}_k(B) \) on \( \text{msa}(Q) \) factors through the map \( \text{Aut}_k(B) \rightarrow \text{Aut}_k(B/J(B)^2) = \text{Aut}_k(T_2Q) \). Hence, \( \text{Aut}_k(B) \) acts on \( \text{msa}(Q) \) through a subgroup of \( \text{Aut}_k(T_2Q) \). This allows us to think of \( \text{Aut}_k(T_2Q) \) as a collection of “extra symmetries” of \( \text{msa}(Q) \), beyond those afforded by \( \text{Aut}_k(B) \).
Thankfully, the structure theory of Aut\(_k(T_2Q)\) is fairly straightforward, so we recall it below.

Write \( G = H_{T_2Q} \), and consider \( S_Q \) as a subgroup of Aut\(_k(T_2Q)\). Then there exist short exact sequences

\[
\begin{align*}
(1) & \quad 1 \to \text{Inn}^*(T_2Q) \to \text{Aut}_k(T_2Q) \to \text{Aut}_k(T_2Q)/\text{Inn}^*(T_2Q) \to 1, \\
(2) & \quad 1 \to G \to \text{Aut}_k(T_2Q)/\text{Inn}^*(T_2Q) \to S_Q \to 1,
\end{align*}
\]

both of which split on the right. First we show that, as an abstract group, Aut\(_k(T_2Q)\) is generated by \( \text{Inn}^*(T_2Q) \), \( G \), and \( S_Q \). Let \( \phi : T_2Q \to T_2Q \) be an automorphism. Then there exists an inner automorphism \( \iota_{\psi} \in \text{Inn}(T_2Q) \) such that \( \psi = \iota_{\psi} \) maps \( Q_0 \) to itself (see, for instance [22]). Furthermore, if we write \( u = \sum_{v \in Q_0} \lambda_v v + \sum_{\alpha \in Q_1} \lambda_\alpha \alpha \), then \( u = (\sum_{v \in Q_0} \lambda_v v)(1 + \sum_{\alpha \in Q_1} \lambda_\alpha \alpha) \), and \( u \sum_{v \in Q_0} \lambda_v v \) fixes each vertex. Hence, we may assume that \( u = 1 + x \), where \( x \in J(T_2Q) = kQ_1 \). Then for all \( s, t \in Q_0 \), \( \psi \) maps the standard basis of \( skQ_1t \) to a basis of \( \psi(s)kQ_1\psi(t) \). Therefore, there exists a \( g \in G \) such that \( \tau := gv \) maps the arrows from \( s \to t \) to the arrows \( \psi(s) \to \psi(t) \). In other words, \( \tau \in \text{Aut}(Q) \).

To verify sequence (1), suppose that \( t_{1+x} \in \text{Inn}^*(T_2Q) \cap G \cdot S_Q \). Since \( t_{1+x}(v) = v \) (mod \( kQ_1 \)) for all \( v \in Q_0 \), we must have \( t_{1+x} \in G \). But then \( t_{1+x}(v) = v \) for all \( v \in Q_0 \), and \( t_{1+x}(\alpha) = \alpha \) for all \( \alpha \in Q_1 \). In other words, \( t_{1+x} = \text{id} \), so \( \text{Aut}_k(T_2Q)/\text{Inn}^*(T_2Q) \cong G \cdot S_Q \). This implies that sequence (1) splits on the right.

For sequence (2), simply note that \( G \) is normal in \( G \cdot S_Q \), and that \( G \cap S_Q = \{ \text{id} \} \).

The subgroups \( G \) and \( \text{Inn}^*(T_2Q) \) are easy to explicitly describe. We start with \( G \): since an element of \( G \) fixes each vertex, it sends \( ukQ_1v \) to itself for all \( u, v \in Q_0 \). The restriction to \( ukQ_1v \) is just an invertible linear map. Hence, there is an inclusion

\[
G \hookrightarrow \prod_{u, v \in Q_0} \text{GL}(ukQ_1v),
\]

which is surjective since \( T_2Q \) has radical square zero. For \( \text{Inn}^*(T_2Q) \), it is straightforward to check that the map \( x \mapsto t_{1+x} \) is an isomorphism of groups \( (kQ_1)_a \cong \text{Inn}^*(T_2Q) \), where \( (kQ_1)_a \) denotes the additive group of \( kQ_1 \). Now we can reduce the problem of describing Aut\(_k(T_2Q)\)-orbits on \( \text{msa}(Q) \) to the action of the finite group \( S_Q \) on certain finite sets:

**Proposition 5.1.** Aut\(_k(T_2Q)\)-orbits on \( \text{msa}(Q) \) may be described as follows:

1. Orbits containing a maximal subalgebra of split type are in bijection with \( S_Q \)-orbits on \( V^2(Q) \).
2. Orbits containing a maximal subalgebra of separable type are in bijection with \( S_Q \)-orbits on \( \{ (s, t) \subset Q_0 : s \neq t \} \).

**Proof.** \( \text{Inn}^*(T_2Q) \)-orbits on \( \text{msa}(Q) \) are already described by theorem [21] so it remains to see which \( \text{Inn}^*(T_2Q) \)-orbits lie in the same \( G \)-orbit, and which of those lie in the same \( S_Q \)-orbit. Any element \( g \in G \) fixes the maximal subalgebras of separable type, and sends \( A(s, t, V) \) to \( A(s, t, gV) \). Since the action of \( \text{GL}(skQ_1t) \) on \( \mathbb{P}((skQ_1t)^*) \) is transitive, it follows that \( A(s, t, U) \) and \( A(s, t, W) \) lie in the
same \( \text{Aut}_k(T_2Q) \)-orbit, for any two \( U, W \in \mathbb{P}((skQ_1)^t) \). So the \( \text{Aut}_k(T_2Q) \) orbits are determined by the action of \( S_Q \) on vertices. For \( \sigma \in S_Q \), \( \sigma(A(s, t, U)) = A(\sigma(s), \sigma(t), \sigma(U)) \) and \( \sigma(A(u + v)) = A(\sigma(u) + \sigma(v)) \). Claims (1) and (2) are now clear. \( \square \)

Let \( B = kQ/I \). Then the map \( H_B \to H_{T_2Q} = \prod_{u, v \in Q_0} \text{GL}(ukQ_1v) \) has closed image. Let \( H_B(u, v) \) denote the image of the corresponding map \( \pi_{uv} : H_B \to H_{T_2Q} \to \text{GL}(ukQ_1v) \).

**Theorem 5.2.** Let \( B = kQ/I \). Then there are finitely many \( \text{Aut}_k(B) \)-orbits on \( msa(Q) \) if and only if for each \( (u, v) \in V^2(Q) \), there are finitely-many \( H_B(u, v) \)-orbits on \( Gr_{d(u,v)-1}(ukQ_1v) \).

**Proof.** There are \( \binom{|Q_0|}{2} \) orbits corresponding to maximal subalgebras of separable type, so it suffices to prove the equivalence on maximal subalgebras of split type. To prove sufficiency, suppose that \( A(u, v, W) \) is a maximal subalgebra of split type, and that for all \( (u, v) \in V^2(Q) \), \( Gr_{d(u,v)-1}(ukQ_1v) \) is a finite union of \( H_B(u, v) \)-orbits. Consider \( A(u, v, hW) \), where \( h \) is an element of \( H_B(u, v) \). Picking \( h \in H_B \) with \( h = \pi_{uv}(h) \), we see that \( A(u, v, hW) = A(u, v, \pi_{uv}(h)W) = hA(u, v, W) \). In other words, \( A(u, v, W) \) is conjugate to \( A(u, v, hW) \). This implies that there are finitely-many orbits in \( msa(Q) \) under the action of the group generated by \( \text{Inn}^*(B) \) and \( H_B \).

It remains to prove necessity. First, note that

\[ \text{Aut}_k(B) = \text{Inn}^*(B)\hat{H}_B = \hat{H}_B \text{Inn}^*(B). \]

Suppose that \( A_1 \) and \( A_2 \) are maximal subalgebras of split type lying in the same \( \text{Aut}_k(B) \) orbit. We know that there exist triples \( (u_1, v_1, W_1) \) and \( (u_2, v_2, W_2) \) such that \( A_i \) is \( \text{Inn}^*(B) \)-conjugate to \( A(u_i, v_i, W_i) \), for \( i = 1, 2 \). So we may assume \( A(u_2, v_2, W_2) = \hat{h}A(u_1, v_1, W_1) \), for some \( \hat{h} \in \text{Inn}^*(B) \) and \( h \in H_B \). Then \( \hat{h}A(u_1, v_1, W_1) \) is \( \text{Inn}^*(B) \)-conjugate to \( A(u_2, v_2, W_2) \), and so their Jacobson radicals coincide. If we write \( J(u, v, W) \) for the Jacobson radical of \( A(u, v, W) \), then this means \( \hat{h}J(u_1, v_1, W_1) = J(u_2, v_2, W_2) \). Conversely, \( \hat{h}J(u_1, v_1, W_1) = J(u_2, v_2, W_2) \) implies \( \hat{h}A(u_1, v_1, W_1) = A(u_2, v_2, W_2) \), since \( \hat{h} \) permutes \( Q_0 \subset A(u_1, v_1, W_1) \cap A(u_2, v_2, W_2) \). It follows that \( A(u_1, v_1, W_1) \) lies in the same \( \text{Aut}_k(B) \)-orbit of \( A(u_2, v_2, W_2) \), if and only if \( J(u_1, v_1, W_1) \) and \( J(u_2, v_2, W_2) \) lie in the same \( H_B \)-orbit of codimension-1 \( B \)-subbimodules of \( J(B) \). Hence, there are finitely-many \( H_B \)-orbits of codimension-1 \( B \)-subbimodules of \( J(B) \). But \( H_B \) is a finite-index normal subgroup of \( \hat{H}_B \) by proposition 13 of [19], and so each \( H_B \) orbit is a finite union of \( H_B \)-orbits. Furthermore, the \( H_B \)-orbits of codimension-1 subbimodules can be expressed as the union of the \( H_B(u, v) \)-orbits of \( Gr_{d(u,v)-1}(ukQ_1v) \) over the pairs \( (u, v) \in V^2(Q) \). So for each fixed \( (u, v) \in V^2(Q) \), \( Gr_{d(u,v)-1}(ukQ_1v) \) is a finite union of \( H_B(u, v) \)-orbits. \( \square \)

**Corollary 5.3.** Let \( B = kQ/I \). Then \( msa(Q) \) is a finite union of \( \text{Aut}_k(B) \)-orbits if one of the following conditions holds:

1. \( Q \) is Schur.
2. \( I \) is monomial.

**Proof.** If \( Q \) is Schur, then \( d(u, v) = 1 \) for all \( (u, v) \in V^2(Q) \) and each Grassmannian \( Gr_{d(u,v)-1}(ukQ_1v) \) is a point. If \( I \) is monomial, then for all \( (u, v) \in V^2(Q) \), \( H_B(u, v) \)
contains the invertible diagonal matrices, which decompose $ukQ_1v$ into finitely-
many orbits.

Of course, these conditions are not necessary for $\text{msa}(Q)$ to be a finite union of
$\text{Aut}_k(B)$-orbits. We now construct a class of (generally) non-Schur, non-monomial
$B$ which decompose $\text{msa}(Q)$ as a finite union of $\text{Aut}_k(B)$-orbits.

**Definition 5.4.** Let $Q$ be a quiver. Then we define $r_2(Q)$ to be the quiver with
vertex set $(r_2Q)_0 = V^2(Q)$, for which there is an arrow $(u, v) \overset{\alpha}{\to} (u', v')$ if and only
if $v = v'$.

**Definition 5.5.** Let $\lambda \in kQ$ be any element. We call $\lambda$ an
$r_2$-element if there exist
$(u, v), (v, w) \in V^2(Q)$ such that $\lambda = \sum_{\alpha, \beta} \lambda_{\alpha\beta}$, where the sum ranges over all
$\alpha \in uQ_1v$ and $\beta \in vQ_1w$, and $\lambda_{\alpha\beta} \in k$. If $(u, v) \overset{\alpha}{\to} (v, w)$ is the corresponding
arrow in $r_2Q$, then we say that $\lambda$ has
type $\alpha$.

**Note:** For any quiver $Q$, $r_2(Q)$ is always a Schur quiver. For any arrow $\alpha \in r_2Q$,
$0$ is an $r_2$-element of type $\alpha$.

**Definition 5.6.** An admissible ideal $I \subset kQ$ is called an
$r_2$-ideal if it is generated
by a set $\{\lambda_\alpha \mid \alpha \in (r_2Q)_1\}$ such that for all arrows $\alpha \in r_2Q$, $\lambda_\alpha$ is an $r_2$-element
of type $\alpha$.

Let $I$ be an $r_2$-ideal of $kQ$. Fix a generating set $\lambda = \{\lambda_\alpha \mid \alpha \in (r_2Q)_1\}$ of $I$, such
that each $\lambda_\alpha$ is an $r_2$-element of type $\alpha$. If $\alpha$ is an arrow from $(u, v)$ to $(v, w)$,
write $\lambda_\alpha = \sum_{\gamma, \delta} \lambda_{\alpha\gamma}\delta$, for $\gamma \in uQ_1v$ and $\delta \in vQ_1w$. Then $\lambda$ gives rise to a
representation of $r_2Q$, which we term $V_\lambda$. To each $(u, v) \in V^2(Q)$, we attach the
vector space $V_\lambda(u, v) := ukQ_1v$. The map $V_\lambda(u, v) \xrightarrow{V_\lambda(\alpha)} V_\lambda(u, v)$ is then defined
by
\[
\gamma \mapsto \sum_{\delta \in vQ_1w} \lambda_{\gamma\delta}.
\]

We let $d$ denote the dimension vector of this representation, i.e. $d : V^2(Q) \to \mathbb{N}$ is
the function $d(u, v) = d(u, v) = \dim_k ukQ_1v$.

**Note:** To be consistent with the multiplication in the path algebra, these representa-
tions should be considered right $r_2(Q)$-modules. In the example below, therefore,
all vector spaces should be understood as row vectors, with matrices acting on the
right.

**Example 5.7.** Let $Q$ be the quiver

\[
Q = \begin{array}{c}
\bullet \quad v_1 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\beta \\
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\bullet \quad v_2
\end{array}
\]

Then $r_2(Q)$ is the quiver
Consider the ideals $I_1 = (\alpha_1 \gamma_1 - \alpha_1 \gamma_2 - \alpha_2 \gamma_1 + \alpha_2 \gamma_2 + \pi \alpha_3 \gamma_2)$, $I_2 = (\alpha_1 \gamma_2 + \alpha_2 \gamma_1, \beta \gamma_1)$, $I_3 = (\alpha_1 \gamma_2 + \alpha_2 \gamma_1, \alpha_1 \gamma_1 + \alpha_2 \gamma_2)$. Then $I_1$ and $I_2$ are $r_2$-ideals of $kQ$, but $I_3$ is not. The representation corresponding to the $r_2$-generating set $\lambda = \{\alpha_1 \gamma_1 - \alpha_1 \gamma_2 - \alpha_2 \gamma_1 + \alpha_2 \gamma_2 + \pi \alpha_3 \gamma_2\}$ is

$$V_\lambda = \begin{pmatrix} 0 & 0 \\ k & k^2 & k^3 \end{pmatrix}.$$

and the representation corresponding to the $r_2$-generating set $\mu = \{\alpha_1 \gamma_2 + \alpha_2 \gamma_1, \beta \gamma_1\}$ is

$$V_\mu = \begin{pmatrix} 1 & 0 \\ k & k^2 & k^3 \end{pmatrix}.$$

Let $B = kQ/I$. Then a linear change of variables $g \in V_{1(Q)} \leq H_B$ can be identified with an element $g = (g_{uv}) \in \prod_{u,v \in Q_0} \text{GL}(ukQ_1v)$ satisfying $g(I) = I$. Note that $g\lambda_\alpha$ is an $r_2$-element of type $\alpha$, and that since $I$ is an $r_2$-ideal, the condition $g(I) = I$ means $g\lambda_\alpha \in k\lambda_\alpha \setminus \{0\}$. From this, it is easy to see that this is equivalent to $g_{uv}V_\lambda(\alpha)g_{uv}^T = \chi(\alpha)V_\lambda(\alpha)$, for some $\chi(\alpha) \in k^\times$ (where $T$ denotes the usual transpose of a matrix). We are now ready to build our examples. For the next proposition, we introduce the following notation: if $W$ is a vector space, then we let $O(W)$ denote the orthogonal group of $W$, i.e. the group of all matrices $g$ satisfying $g^T g = I$.

**Proposition 5.8.** Let $Q$, $I$, $B$, and $\lambda$ be as above. Suppose that the no-double-edge graph of $Q$ is a disjoint union of oriented trees. Suppose further that there exists a $\phi = (\phi_{uv}) \in \prod_{u,v \in Q_0} \text{GL}(ukQ_1v)$ satisfying $\phi(I) \leq H_B$ and that since $I$ is an $r_2$-ideal, the condition $\phi(I) = I$ means $\phi\lambda_\alpha \in k\lambda_\alpha \setminus \{0\}$. From this, it is easy to see that this is equivalent to $\phi_{uv}V_\lambda(\alpha)\phi_{uv}^T = \chi(\alpha)V_\lambda(\alpha)$, for some $\chi(\alpha) \in k^\times$ (where $T$ denotes the usual transpose of a matrix). We are now ready to build our examples. For the next proposition, we introduce the following notation: if $W$ is a vector space, then we let $O(W)$ denote the orthogonal group of $W$, i.e. the group of all matrices $g$ satisfying $g^T g = I$.

**Proof.** By theorem 5.2, it suffices to show that for all $(u,v) \in V^2(Q)$, $ukQ_1v$ is a finite union of $H_B(u,v)$-orbits. We show that in fact, $ukQ_1v$ is a finite union of $\pi_{uv}(V_{1(Q)})$-orbits. Since the no-double-edge graph of $Q$ is a disjoint union of oriented trees, so is $r_2Q$. As before, $V_{1(Q)}$ can be identified with the elements $g = (g_{uv}) \in \prod_{u,v \in Q_0} \text{GL}(ukQ_1v)$ satisfying $g_{uv}V_\lambda(\alpha)g_{uv}^T = \chi(\alpha)V_\lambda(\alpha)$ for all arrows $\alpha \in r_2Q$. Now, $W(\alpha) = \phi_{uv}V_\lambda(\alpha)\phi_{uv}^{-1}$ for all $\alpha \in (r_2Q)_1$. Since $\phi_{uv}^{-1} = \phi_{uv}^T$ for all $(u,v) \in V^2(Q)$, $V_{1(Q)}$ is conjugate in $\prod_{u,v \in Q_0} \text{GL}(ukQ_1v)$ to the collection of all $g$ satisfying $g_{uv}W(\alpha)g_{uv}^T = \chi(\alpha)W(\alpha)$ for all $\alpha$, and some $\chi(\alpha) \in k^\times$. Call this group $G$. Pick a function $\mu : uQ_1v \to k^\times$, and let $g_{uv}^\mu \in \text{GL}(ukQ_1v)$ be the map $g_{uv}^\mu(\gamma) = \mu(\gamma)\gamma$. We claim that there exists a $g \in G$ such that $g_{uv} = g_{uv}^\mu$. To construct this map, pick $\alpha \in (r_2Q)_1$ with $s(\alpha) = (u,v)$. Let

$$r_2(Q) = (v_2, v_3) (v_3, v_4) (v_1, v_3).$$
\( t(\alpha) = (v, w) \). Then \( g_{w,v}^u W(\alpha) \) rescales the rows of \( W(\alpha) \), and so there exists an appropriate diagonal matrix \( g_{w,v}^u \in \text{GL}(uvkQ_1v) \) such that \( g_{w,v}^u W(\alpha) g_{w,v}^{T u} = W(\alpha) \).

For each \( \beta \) with \( t(\beta) = (u, v) \), we can find another diagonal matrix \( g_{u,v}^{(\beta)} \) satisfying \( g_{u,v}^{(\beta)} W(\alpha) g_{u,v}^{-T} = W(\alpha) \). Now repeat this procedure for the targets of the \( \alpha \)'s and the sources of the \( \beta \)'s. Since the connected component of \( r_2Q \) containing \((u, v)\) is an oriented tree, we can iterate this process a finite number of times to build \( g \) on this connected component. We can finish the construction of \( g \) by defining it to be the identity on the remaining components. We have shown that \( \pi_{uv}(G) \) contains the diagonal matrices. But \( \pi_{uv}(G) = \phi_{uv} \pi_{uv}(Vl_{Q(I)}) \phi_{uv}^{-1} \), so that \( \pi_{uv}(Vl_{Q(I)}) \), and hence \( H_B(u, v) \), must contain a maximal torus of \( \text{GL}(ukQ_1v) \). Then \( ukQ_1v \) is a finite union of \( H_B(u, v) \)-orbits, as we wished to show. \( \square \)

**Note:** For any quiver \( Q \), the condition that an \( r_2 \)-ideal \( \lambda \) be monomial is equivalent to the condition that each matrix in \( V_\lambda \) has at most one non-zero entry. But the map

\[
\{ r_2 \text{-generating sets of } r_2 \text{-ideals of } kQ \} \to \text{rep}_2(r_2(Q))
\]

\[\lambda \mapsto V_\lambda \]

is a surjection. Hence, this allows us to easily construct examples of non-monomial bound quiver algebras \( kQ/I \), whose Ext-quiver is not Schur, with the property that \( \text{mao}(Q) \) is a finite union of \( \text{Aut}_k(kQ/I) \)-orbits.

**Example 5.9.** Let \( Q, I_2, \) and \( \mu \) be as in example 5.7. Then \( V_\mu \) satisfies the conditions of proposition 5.8 and so \( \text{mao}(Q) \) is a finite union of \( \text{Aut}_k(kQ/I_2) \)-orbits. Of course, \( Q \) is not Schur, and \( kQ/I_2 \) is not a monomial algebra.

**References**

[1] Agore, A. L. *The Maximal Dimension of Unital Subalgebras of the Matrix Algebra*, Forum Math. 29, no. 1, (2017), 1-5.

[2] Assem I., Simson D., Skowroński, A. *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Mathematical Society Student Texts 65, Cambridge University Press, New York (2006).

[3] Bass H. *Algebraic K-Theory*, Benjamin, New York, 1968.

[4] Bongartz K., *On Degenerations and Extensions of Finite-Dimensional Modules*, Advances in Math. 121, no. 2, (1996), 245-287.

[5] Bongartz K., Huisgen-Zimmermann B., *The Geometry of Uniserial Representations of Algebras II. Alternate Viewpoints and Uniqueness*, J. Pure Appl. Algebra 157, (2001), 23-32.

[6] Bongartz K., Huisgen-Zimmermann B., *Varieties of Uniserial Representations IV. Kinship to Geometric Quotients*, Trans. Amer. Math. Soc. 353, (2001), 2091-2113.

[7] Caldero P., Reineke M. *On the Quiver Grassmannian in the Acyclic Case*, J. Pure Appl. Algebra 212, no. 11, (2008), 2369-2380.

[8] Cerulli Irelli G., Feigin E., Reineke M. *Quiver Grassmannians and Degenerate Flag Varieties*, Algebra Number Theory 6, (2012), 165-193.

[9] Cerulli Irelli G., Feigin E., Reineke M. *Degenerate Flag Varieties: Moment Graphs and Schröder Numbers*, J. Algebraic Combin. 38, (2013), 159-189.

[10] Cerulli Irelli G., Feigin E., Reineke M. *Desingularization of Quiver Grassmannians for Dynkin Quivers*, Adv. Math. 245, no. 1, (2013), 182-207.

[11] Crawley-Boevey, W. W. *Maps Between Representations of Zero-Relation Algebras*, J. Algebra 126, no. 2, (1989), 259-263.

[12] de Graaf, W. A. *Classification of Nilpotent Associative Algebras of Small Dimension*, Int. J. Algebra Comput., 28, no. 1, (2018), 133-161.

[13] Fröhlich A., *The Picard Group of Noncommutative Rings, in Particular of Orders*, Trans. AMS, 180, (1973), 1-45.
[14] Gabriel P. Finite Representation Type is Open. In: Dlab V., Gabriel P. (eds) Representations of Algebras. Lecture Notes in Mathematics 488, Springer, (1975).
[15] Geiss C., Introduction to Moduli Spaces Associated to Quivers (With an Appendix by L. Le Bruyn and M. Reineke), in Trends in Representation Theory of Algebras and Related Topics, Contemporary Mathematics 406, Amer. Math. Soc., (2006), 31-50.
[16] Gerstenhaber M. On the Deformation of Rings and Algebras, Ann. of Math. (2) 79, no. 1, (1964), 59-103.
[17] Gerstenhaber M. On Nilalgebras and Linear Varieties of Nilpotent Matrices I, Amer. J. Math. 80, no. 3, (1958), 614-622.
[18] Gerstenhaber M. On Dominance and Varieties of Commuting Matrices, Ann. of Math. 73, no. 2, (1961), 324-348.
[19] Guil-Asensio F., Saorín M. The Group of Outer Automorphisms and the Picard Group of an Algebra, Algebr. Represent. Theory 2, (1999), 313-330.
[20] Guil-Asensio F., Saorín M. The Automorphism Group and the Picard Group of a Monomial Algebra, Comm. Algebra 27, no. 2, (1999), 857-887.
[21] Huisgen-Zimmermann, B. Fine and Coarse Moduli Spaces in the Representation Theory of Finite-Dimensional Algebras, in Expository Lectures in Representation Theory, Contemporary Mathematics 607, Amer. Math. Soc., (2014), 1-34.
[22] Iovanov M. C., Sistko A. Maximal Subalgebras of Finite-Dimensional Algebras, (2017). Accessed online at https://arxiv.org/abs/1705.00762.
[23] Jacobson N. Schur’s Theorems on Commutative Matrices, Bull. Amer. Math. Soc. 50 (1944), 431-436.
[24] King A., Moduli of Representations of Finite-Dimensional Algebras, Quart. J. Math. Oxford (2) 45, no. 180, (1994), 515-530.
[25] Laffey T. J. The Minimal Dimension of Maximal Commutative Subalgebras of Matrix Algebras, Linear Algebra Appl. 71 (1985), 199-212.
[26] Lakshmibai V., Brown J. The Grassmannian Variety: Geometric and Representation-Theoretic Aspects, Developments in Mathematics 47, Springer, (2015).
[27] Mirzakhani M. A Simple Proof of a Theorem of Schur, Amer. Math. Monthly 105 (1998), 260-262.
[28] Motzkin T., Taussky O. Pairs of Matrices with Property L, Trans. Amer. Math. Soc. 73 (1952), 108-114.
[29] Motzkin T., Taussky O. Pairs of Matrices with Property L (II), Trans. Amer. Math. Soc. 80 (1955), 387-401.
[30] Pierce R.S., Associative Algebras, Graduate Texts in Mathematics 88, Springer, (1982), 436.
[31] Pullack D. R. Algebras and their Automorphism Groups, Comm. Algebra 17, no. 8, (1989), 1843-1866.
[32] Rafael M. D., Separable functors revisited, Comm. Algebra 18 (1990), 1445-1459.
[33] Racine M. L. On Maximal Subalgebras, J. Algebra 30, no. 1-5, (1974), 155-180.
[34] Racine M. L. Maximal Subalgebras of Central Separable Algebras, Proc. Amer. Math. Soc. 68, no. 1, (1978), 11-15.
[35] Reineke M. Every Projective Variety is a Quiver Grassmannian, Algr. Represent. Theory 16, no. 5, (2013), 1313-1314.
[36] Schofield A., General Representations of Quivers, Proc. London Math. Soc. (3) 65, no. 1, (1992), 46-64.
[37] Schur I. Zur Theorie Vertauschbarer Matrizen, J. Reine Angew. Math. 130, (1905), 66-76.