Computing mixed Schatten norm of completely positive maps

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Abstract

Computing $p \rightarrow q$ norm for matrices is a classical problem in computational mathematics and power iteration is a well known method for computing $p \rightarrow q$ norm for a matrix with nonnegative entries. Here we define an equivalent iteration method for computing $S_p \rightarrow S_q$ norm for completely positive maps where $S_p$ is the Schatten $p$ norm. We generalize almost all of definitions, properties, lemmas etc. in the matrix setting to completely positive maps and prove an important theorem in this setting.

1 Introduction

Given a matrix $A \in \mathbb{R}^{m \times n}$ we define $\|A\|_{p \rightarrow q}$ as

$$\|A\|_{p \rightarrow q} = \max \frac{\|Ax\|_q}{\|x\|_p}$$

where the $\|\cdot\|_p$ and $\|\cdot\|_q$ are $\ell^p$ norms. Computing $\|A\|_{p \rightarrow q}$ is a classical problem in computational mathematics. The best known method for this is a nonlinear power method, introduced by Boyd in [1] and then further analyzed and extended for instance in [15, 16, 17, 18]. However the best known result about this problem is the following

**Theorem 1.1** (Theorems 3.2 and 3.3, [2]). Let $A \in \mathbb{R}^{m \times n}$ be a matrix with non-negative entries and suppose that $A^T A$ has at least one positive entry per row. If $1 < q \leq p < \infty$, then, every positive critical point of $f_A(x) = \|Ax\|_q / \|x\|_p$ is a global maximizer. Moreover, if either $p > q$ or $A^T A$ is irreducible, then $f_A$ has a unique positive critical point $x^+$ and the power sequence

$$x_0 = \frac{x_0}{\|x_0\|_p}, \quad x_{k+1} = J_p^+ (A^T J_q(Ax_k)), \quad k = 0, 1, 2, \ldots$$

converges to $x^+$ for every positive starting point.
We can consider completely positive linear maps between complex matrix spaces (see [3]) as a generalization for positive matrices so it’s interesting to investigate this results for completely positive maps. That means in this note we are trying to find $S_p \rightarrow S_q$ norm of a completely positive map that defines as follows

$$\|\Phi\|_{S_p \rightarrow S_q} = \max \frac{\|\Phi(A)\|_{S_q}}{\|A\|_{S_p}}$$

where the max function is on the set of all $n \times n$ Hermitian matrices and $S_p$ and $S_q$ are Schatten norms.

We generalize the power method that is defined for positive matrices, to completely positive maps and prove this method converges to the value of $S_p \rightarrow S_q$ norm.

## 2 $H^n_+$ cone and completely positive maps preliminaries

In this section we talk about positive semidefinite cone and completely positive maps properties. Let $H^n$ be the set of all Hermitian $n \times n$ matrices and $H^n_+$ (resp. $H^n_{++}$) be the set of positive semidefinite (resp. positive definite) Hermitian $n \times n$ matrices. Also we use $A^\dagger$ for representing conjugate transpose of matrix $A$.

**Definition 2.1. (completely positive map)** Let $\Phi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{m \times m}$ be a function on the set of complex matrices then $\Phi$ is a completely positive map if and only if there exist matrices $V_i \in \mathbb{C}^{m \times n}$ such that

$$\Phi(A) = \sum_{i=1}^{k} V_i A V_i^\dagger, \quad k \leq nm$$

Also represent the transpose of $\Phi$ by $\Phi^*$ as

$$\Phi^* : \mathbb{C}^{m \times m} \rightarrow \mathbb{C}^{n \times n}, \quad \Phi^*(A) = \sum_{i=1}^{k} V_i^\dagger A V_i$$

**note.** It’s not the original definition but these are equivalent (see [3])

For a completely positive map we define $S_p \rightarrow S_q$ norm as

$$\|\Phi\|_{S_p \rightarrow S_q} = \max \frac{\|\Phi(A)\|_{S_q}}{\|A\|_{S_p}}$$

where $S_p$ is the Schatten $p$-norm and defines as

$$\|A\|_{S_p} = tr(|A|^p)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{n} \sigma_i(A)^p\right)^{\frac{1}{p}}$$
where $|A| = \sqrt{A^\dagger A}$ and $\sigma_i$-s are singular values of $A$.

**Definition 2.2.** (proper cone) A cone $K$ in a vector space on field $\mathbb{R}$ is called proper cone if it satisfies these conditions:

- $K$ is convex.
- $K$ is closed.
- $K$ is solid, which means it has nonempty interior.
- $K$ is pointed, which means that it contains no line (or equally $x \in K, -x \in K \Rightarrow x = 0$)

By [8] we know that $H_n^+$ is a proper cone in space of complex matrices so we can define a partial ordering on matrices with respect to $H_n^+$ (for instance see [9], chapter 2). So we use the notation $A \succeq 0$ for saying $A$ is positive semidefinite and $A \succ 0$ means $A$ is positive definite also $A \succeq B$ means $A - B$ is positive semidefinite or equally $A - B \in H_n^+$. Also for vectors $x$ and $y$ the notation $x \geq y$ means $x - y \in \mathbb{R}_n^+$ (respectively $x > y$ means $x - y \in \mathbb{R}_n^{++}$).

**Definition 2.3.** For a positive semidefinite matrix $A$ let $A = Q\Lambda Q^\dagger$ be the eigendecomposition of $A$ then for a positive real number $p$ we define $A^p$ as

$$A^p = QA^pQ^\dagger$$

where $A^p$ is the diagonal matrix that has $p$-th power of eigenvalues of $A$ as it's diagonal entries.

**Lemma 2.1.** For $A, B \in H_n^+$ and completely positive map $\Phi$ we have the following properties:

1. If $A \succeq B$ and consider $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are $A$'s and $B$'s eigenvalues respectively then we have $\forall i : \lambda_i \geq \gamma_i$.

2. If $A \succeq B$ then $\Phi(A) \succeq \Phi(B)$.

**Proof.** For first part consider $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_n\}$ are set of eigenvectors of $A, B$ respectively such that $Av_i = \lambda_i v_i, Bu_i = \gamma_i u_i$ so these are orthonormal basis for space of complex vectors. Now for proving $\lambda_i \geq \gamma_i$ consider $P = \{c_1v_1 + c_1v_{i+1} + \cdots + c_nv_n : \forall j c_j \in \mathbb{C}\}$ and $Q = \{d_1u_1 + d_2u_2 + \cdots + d_iu_i : \forall j d_j \in \mathbb{C}\}$ and $T = P \cap Q$ the intersection of spanned spaces by last $n - i + 1$ eigenvectors of $A$ and first $i$ eigenvectors of $B$. Note that $\text{dim}(P) = n - i + 1$ and $\text{dim}(Q) = i$ so $\text{dim}(T) > 0$ and it means there exist $x \neq 0, \|x\| = 1$ in $T$, for this $x$ it’s easy to see $x^\dagger Ax \leq \lambda_i$ and $x^\dagger Bx \geq \gamma_i$ but $A \succeq B$ implies $x^\dagger Ax \geq x^\dagger Bx$ so $\lambda_i \geq \gamma_i$.

For the second part note that $\Phi$ is linear. \qed
Hilbert projective metric is a metric that is defined on rays in a real Banach space and is introduced by Hilbert in [2] and first time it defined on positive semidefinite cone by C. Liverani and M. P. Wojtkowski in [10]. Consider K is a proper cone, for \( x, y \in K \) we use notation \( x \sim y \) if there exist positive real numbers \( c \) and \( C \) such that \( cy \preceq_K x \preceq_K Cy \) and that means \( x - cy, Cy - x \in K \). It’s easy to see \( \sim \) is an equivalency relation and the equivalency classes are called parts of K. Also the function \( M : K \times K \to \mathbb{R} \) is defined as \( M(x/y) = \inf \{ \lambda > 0 : x \preceq_K \lambda y \} \) then the Hilbert projective metric \( d : K \times K \to \mathbb{R} \) defines as follows

\[
d(x, y) = \begin{cases} 
\ln M(x/y)M(y/x) & x \sim y \\
0 & x = y = 0 \\
\infty & \text{otherwise}
\end{cases}
\]

It’s easy to see that for \( K = H^n_+ \) if \( A, B \succ 0 \) then

\[
d(A, B) = \ln \left( \left\| B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \right\| \left\| A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right\| \right)
\]

where \( \| A \| \) equals greatest eigenvalue of A. The Hilbert projective metric is a metric on rays that means \( d(A, B) = d(\alpha A, \beta B) \) for every positive \( \alpha, \beta \) you can find more properties about Hilbert metric in [2].

The proof of our main theorem is based on the Banach contraction principle. Thus, for a map \( \Phi : \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m} \) we consider the Birkhoff contraction ratio \( \kappa(\Phi) \in [0, \infty] \) of \( \Phi \), defined as the smallest Lipschitz constant of \( \Phi \) with respect to \( d \) (see [11]):

\[
\kappa(\Phi) = \inf \{ C > 0 : d(\Phi(A), \Phi(B)) \leq Cd(A, B), \forall A, B \in H^n_+ \text{ such that } A \sim B \}
\]

where if there exist \( A, B \in H^n_+ \) such that \( A \sim B \) and \( \Phi(A) \sim \Phi(B) \) then \( \kappa(\Phi) = \infty \) but we know that this case never happen for completely positive maps because for linear maps we have \( \kappa(\Phi) \leq 1 \). Moreover one can easily show that if \( \Phi \) is a completely positive map then \( A \sim B \) implies \( \Phi(A) \sim \Phi(B) \).

**Theorem 2.1. (Birkhoff-Hopf,[12])** Let \( \Phi : \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m} \) be a completely positive map then we have:

\[
\kappa(\Phi) = \tanh \left( \frac{\Delta(\Phi)}{4} \right)
\]

where \( \Delta \) is the diameter of \( \Phi \) and defines as follows

\[
\Delta(\Phi) = \sup \{ d(\Phi(A), \Phi(B)) : A, B \in H^n_+, A \sim B \}
\]

and with the convention of \( \tanh(\infty) = 1 \).

So the Bikhoff-Hopf theorem tells us the contraction ratio of a completely positive map is always less than or equal to 1 and it equals 1 if and only if \( \Delta(\Phi) = \infty \).
3 Nonlinear Perron Frobenius theorem for \(\|\Phi\|_{S_p \rightarrow S_q}\)

In this section we generalize the approach, includes structure, lemmas, theorems, etc, as [13] chapter 4 for proving an important theorem (Theorem 3.1) in computing \(S_p \rightarrow S_q\) norm for completely positive maps.

For the Schatten \(p\)-norm, derivative \(J_{S_p}\) can be represented as follows (see [14]):

\[
J_{S_p}(A) = \{B : \langle A, B \rangle = \|A\|_{S_p}, \|B\|_{S_p^*} = 1\}
\]

where \(\langle A, B \rangle = \text{tr}(A^\dagger B)\) is the Frobenius inner product on matrices. Also we know that because the Schatten norm is Fréchet differentiable so \(J_{S_p}\) is single valued also it's easy to see that for positive semidefinite matrices, \(J_{S_p}\) satisfies following equation (see [5])

\[
J_{S_p}(A) = Q \Lambda^{-1} Q^\dagger \|Q \Lambda^{-1} Q^\dagger\|_{S_p^*}
\]

where \(S_p^*\) is the dual norm such that \(\frac{1}{p} + \frac{1}{p^*} = 1\).

**Lemma 3.1.** Given matrix \(A\) with \(\|A\|_{S_p} = 1\), \(f_\Phi(A) = \|\Phi(A)\|_{S_q} / \|A\|_{S_p} \neq 0\). \(A\) is a critical point of function \(f_\Phi\) if and only if it is a fixed point of \(S_\Phi(A) = J_{S_p^*}(\Phi^* J_{S_q}(\Phi(A)))\).

**Proof.** At first assume \(A\) is a critical point of \(f_\Phi\) then with differentiation we find

\[
\Phi^* J_{S_q}(\Phi(A)) = f_\Phi(A)J_{S_p}(A)
\]

then by applying the \(J_{S_p^*}\) function to above equation we have the following

\[
J_{S_p^*}(\Phi^* J_{S_q}(\Phi(A))) = \frac{A}{\|A\|_{S_p}}
\]

because \(J_{S_p}(J_{S_p^*}(A)) = A/\|A\|_{S_p}\) and \(J_{S_p}(\alpha A) = J_{S_p}(A)\) so by the assumption \(\|A\|_{S_p} = 1\) we conclude \(A\) is a fixed point of \(S_\Phi\). Now assume \(A\) is fixed point of \(S_\Phi\) then we have \(J_{S_p^*}(\Phi^* J_{S_q}(\Phi(A))) = A\) so there exist \(\lambda > 0\) such that \(\lambda \Phi^* J_{S_q}(\Phi(A)) = J_{S_p}(A)\) so by definition of \(J_{S_p}(A)\) we have

\[
\lambda^{-1} = \langle A, \Phi^* J_{S_q}(\Phi(A)) \rangle = \langle \Phi(A), J_{S_q}(\Phi(A)) \rangle = \|\Phi(A)\|_{S_q} = f_\Phi(A) \quad (1)
\]

where the second equality holds because for linear function \(\Phi\) we have \(\langle \Phi(A), B \rangle = \langle A, \Phi^*(B) \rangle\) and the last equality holds because \(\|A\|_{S_p} = 1\) so from (1) we conclude \(A\) is a critical point of \(f_\Phi\).

**Definition 3.1.** (see [6], Proposition C.6) Let \(\Phi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{m \times m}\) be a completely positive map then we call it fully indecomposable if for all singular, but nonzero \(A \succeq 0\), \(\text{rank}(\Phi(A)) > \text{rank}(A)\) and \(\Phi\) is fully indecomposable if and only if \(\Phi^*\) is fully indecomposable.
Lemma 3.2. Let $\Phi : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times m}$ be a completely positive map and $P$ be a part of positive semidefinite cone such that for every $A \in P$ we have $\Phi^*(\Phi(A)) \in P$. If $\kappa(\Phi) \leq \tau < 1$ then $J_{S_\Phi}(\Phi^*(\Phi(A)))$ has a unique fixed point $X$ in $P$ and the following power method converges to $X$ for any starting point $A \in P$:

$$A_0 = A, \quad A_{k+1} = S_\Phi(A_k), \quad k = 1, 2, \ldots$$

Proof. Note that if $A \in P$ then $J_{S_\Phi}(A) \in P$ because $P$ is a part of $H^n_+$ and $A \sim J_{S_\Phi}(A)$ so the assumption $\Phi^*(\Phi(A)) \in P$ implies $S_\Phi(A) \in P$ so by assumption $\kappa(\Phi) \leq \tau < 1$ we can use the Banach fixed point theorem for complete metric space $(M, d)$ where $M = P \cap \{ A \succeq 0 : \|A\|_{S_p} = 1 \}$ (for proof see [7]) and conclude $S_\Phi$ has a unique fixed point $X$ in $P$ and the power method converges to $X$ for any starting point. 

Note that it’s not enough to say the power method always converges to the maximizer of $f_\Phi$ because now it’s possible for $f_\Phi$ to have more than one critical points because we only proved in every part of $H^n_+$ it has at most one critical point and for our claim we need some more assumptions that you can see in following lemmas.

Lemma 3.3. For any completely positive map $\Phi$ the global maximum of $f_\Phi$ is attained in $H^n_+.$

Proof. We know that $\|A\|_{S_p} = \|A\|_{S_\Phi}$ and also $\|\Phi(A)\|_{S_\Phi} \leq \|\Phi(|A|)\|_{S_p}$ for any $A \in H^n$ because for every Hermitian matrix $A,$ eigenvalues of $|A|$ are absolute values of eigenvalues of $A$ so $A \preceq |A|$ and by Lemma 2.1 we have $\Phi(A) \preceq \Phi(|A|)$ and then because of monotonicity of Schatten norms we have $\|\Phi(A)\|_{S_p} \leq \|\Phi(|A|)\|_{S_p}.$ So we have the following

$$f_\Phi(A) = \|\Phi(A)\|_{S_\Phi} - \|A\|_{S_p} = \|\Phi(A)\|_{S_\Phi} \leq \|\Phi(|A|)\|_{S_p} = f_\Phi(|A|)$$

so if $A$ is a maximizer of $f_\Phi$ then $f_\Phi(A) \leq f_\Phi(|A|)$ which concludes the proof. 

Lemma 3.4. For every $A \in H^n_+, p \in \mathbb{R}_+^N$ we have $S_\Phi(A) \sim \Phi^*\Phi(A)$.

Proof. It’s easy to see $A \sim J_{S_\Phi}(A)$ and if $A \sim B$ then $\Phi(A) \sim \Phi(B)$ so we have

$$\Phi(A) \sim J_{S_\Phi}(\Phi(A)) \Rightarrow \Phi^*\Phi(A) \sim \Phi^*J_{S_\Phi}(\Phi(A)) \Rightarrow \Phi^*\Phi(A) \sim S_\Phi(A)$$

Lemma 3.5. Let $\Phi : \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m}$ be a completely positive map and suppose that $\Phi^*\Phi$ is fully indecomposable then $S_\Phi(A) \in H^n_{++}$ for every $A \in H^n_{++}$ and every positive semidefinite critical point $A$ of $f_\Phi$ is positive definite.
Proof. For proving \( S_\Phi(H^n_{++}) \subseteq H^n_{++} \) consider there exist a matrix \( A \in (H^n_{++}) \) such that \( S_\Phi(A) \not\subseteq (H^n_{++}) \) then \( \text{rank}(S_\Phi(A)) \neq \text{rank}(A) \) but from Lemma 3.3 it’s known \( S_\Phi(A) \sim \Phi^*\Phi(A) \) so \( \text{rank}(\Phi^*\Phi(A)) = n \) and it’s impossible because of the assumption \( \Phi^*\Phi \) is fully indecomposable. Now consider \( S_\Phi(A) = A \) and \( A \in H^n_{++} \) so \( \Phi^*\Phi(A) \sim A \) and this implies \( \text{rank}(A) = n \) so \( A \) is positive definite.

Theorem 3.1. Let \( \Phi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{m \times m} \) be a completely positive map and \( \Phi^*\Phi \) fully indecomposable. If \( \kappa(S_\Phi) \leq \tau < 1 \) then \( f_\Phi \) has a unique critical point \( X \in H^n_{++} \) and \( f_\Phi(X) = \|\Phi\|_{S_p \rightarrow S_q} \) and \( X \) is positive definite. Moreover the following power method converges to \( X \) for any starting point \( A \in H^n_{++} \)

\[
A_0 = A, \quad A_{k+1} = S_\Phi(A_k), \quad k = 1, 2, \ldots
\]

Proof. From Lemma 3.3 we know that \( f_\Phi \) has a minimizer \( X \in H^n_{++} \) and from Lemma 3.5 we know \( X \) is positive definite also from Lemma 3.2 we know that the power method converges to fixed point of \( S_\Phi \) and from Lemma 3.4 it’s the unique maximizer of \( f_\Phi \) in \( H^n_{++} \) so we are done.

Corollary 3.1. By definition of \( \kappa \) it’s clear \( \kappa(S_\Phi) \leq \kappa(J_{S_{pp}})\kappa(\Phi^*)\kappa(J_{S_q})\kappa(\Phi) \) on the other hand we have \( \kappa(J_{S_q}) = p - 1 \) so \( \kappa(S_\Phi) \leq \kappa(\Phi^*)\kappa(\Phi)\frac{p-1}{p} \) so from Theorem 3.1 the convergence to \( S_\Phi \) in \( H^n_{++} \) is proven for case \( p > q \).

So this result is close to result for classical setting of computing \( p \rightarrow q \) norm for nonnegative matrices (Theorem 1.1) and we can see the similarity between irreducibility in matrices and fully indecomposability in completely positive maps but it’s remained to generalize the result for \( p = q \) case, this case isn’t proven yet because it’s possible to have \( \kappa(\Phi^*) = \kappa(\Phi) = 1 \) then \( S_\Phi \) isn’t a contraction and Banach fixed point theorem isn’t useful.

So it’s interesting to find the biggest set of completely positive maps \( \Phi \) with \( \kappa(\Phi) < 1 \). However we finish this note by a generalization for the case \( p = q \) for positively improving maps.

Definition 3.2. Let \( \Phi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{m \times m} \) be a completely positive map then it is positively improving if for every \( A \in H^n_{++} \) we have \( \Phi(A) \in H^n_{++} \).

Theorem 3.2. If \( \Phi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{m \times m} \) be a positively improving completely positive map then \( \kappa(\Phi) < 1 \).

Proof. From Birkhoff-Hopf theorem it’s enough to prove \( \Delta(\Phi) < \infty \) and also it’s enough to prove \( \left\| \Phi(B)^{-\frac{1}{2}}\Phi(A)\Phi(B)^{-\frac{1}{2}} \right\| < \infty \) for every \( A, B \in H^n_{++} \) with \( \text{trace}(A) = \text{trace}(B) = 1 \) because Hilbert metric is defined on rays and also by definition \( \Phi(B), \Phi(A) \) are positive definite so \( \Phi(B)^{-\frac{1}{2}} \) exist. It’s known for every \( A \) and \( B \in H^n_{++} \) we have \( \|AB\| \leq \|A\| \|B\| \) so \( \left\| \Phi(B)^{-\frac{1}{2}}\Phi(A)\Phi(B)^{-\frac{1}{2}} \right\| \leq \lambda_{\max}(\Phi(A))/\lambda_{\min}(\Phi(B)) \) where \( \lambda_{\max} \) and \( \lambda_{\min} \) are the largest and the smallest eigenvalues. So it’s enough to show there exist numbers \( c, C > 0 \) such that

\[ n \geq \frac{\text{trace}(A)}{\lambda_{\min}(\Phi(B))} \leq C \]
\(\lambda_{\text{max}}(\Phi(A)) < C\) and \(\lambda_{\text{min}}(\Phi(A)) > c\) for any \(A \in H^n_+\) with \(\text{trace}(A) = 1\), for proving this note that \(\Phi\) is a continuous function on the compact set \(\{A : A \in H^n_+, \text{trace}(A) = 1\}\) so the image of \(\Phi\) is also compact so \(\lambda_{\text{max}}(\Phi(A)) < C\) and also we know the image is a subset of inside of positive semidefinite cone that contains it’s boundary so the image of \(\Phi\) has a positive distance \(c > 0\) with the boundary and it means for every matrix \(A\) the matrix \(\Phi(A) - cI\) is positive definite and it implies \(\lambda_{\text{min}}(\Phi(A)) > c\). \(\Box\)

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