Multi-Embedding of Metric Spaces

Yair Bartal† Manor Mendel‡

February 7, 2008

Abstract

Metric embedding has become a common technique in the design of algorithms. Its applicability is often dependent on how high the embedding’s distortion is. For example, embedding finite metric space into trees may require linear distortion as a function of its size. Using probabilistic metric embeddings, the bound on the distortion reduces to logarithmic in the size.

We make a step in the direction of bypassing the lower bound on the distortion in terms of the size of the metric. We define “multi-embeddings” of metric spaces in which a point is mapped onto a set of points, while keeping the target metric of polynomial size and preserving the distortion of paths. The distortion obtained with such multi-embeddings into ultrametrics is at most $O(\log \Delta \log \log \Delta)$ where $\Delta$ is the aspect ratio of the metric. In particular, for expander graphs, we are able to obtain constant distortion embeddings into trees in contrast with the $\Omega(\log n)$ lower bound for all previous notions of embeddings.

We demonstrate the algorithmic application of the new embeddings for two optimization problems: group Steiner tree and metrical task systems.

1 Introduction

Finite metric spaces and their analysis play a significant role in the design of combinatorial algorithms. Many algorithmic techniques were introduced in recent years concerning and using metric spaces and their approximate embedding in other spaces, see the surveys [20, 21] for an overview of this topic.

Definition 1. An embedding of a metric space $M = (V_M, d_M)$ into a metric space $N = (V_N, d_N)$ is a mapping $\phi : V_M \rightarrow V_N$. The embedding is called non-contractive if for all $u, v \in V_M$, $d_M(u, v) \leq d_N(\phi(u), \phi(v))$ and has distortion at most $\alpha$ if in addition for all $u, v \in V_M$, $d_N(\phi(u), \phi(v)) \leq \alpha \cdot d_M(u, v)$. A non-contractive embedding whose distortion is at most $\alpha$ is called $\alpha$-embedding.

The general framework for applying metric embeddings in optimization problems is to embed a given metric spaces into a metric space from some “nice” family and then apply an algorithm for that space. As a result, the approximation ratio increases by a factor equal to the embedding’s distortion.

* A preliminary version of this paper appeared in [10].
† School of Computer Science, Hebrew University, Jerusalem 91904, Israel. Email: yair@cs.huji.ac.il. Supported in part by a grant from the Israeli Science Foundation (195/02).
‡ School of Computer Science, Hebrew University, Jerusalem 91904, Israel. Email: mendelma@gmail.com. Supported by the Landau Center and a grant from the Israeli Science Foundation (195/02).
Among others, embeddings into low dimensional normed spaces \([12, 24]\) as well as probabilistic embeddings into trees \([2, 3, 15, 4]\) have many algorithmic applications. In both cases the distortions of the embeddings are logarithmic in the size of the metric. Unfortunately, there is a matching lower bound on the distortion of these embeddings as well, which sets a limit to their applicability. This paper presents a partial remedy for this problem.

Tree metrics, and in particular ultrametrics, seem a natural choice as a target class of “simple” metric spaces. Unfortunately, standard embedding is not useful when the target space is a tree metric. Embedding arbitrary metric spaces into trees requires distortion linear in the size of the metric space \([27]\). Probabilistic embedding \([2]\) provides a way to bypass this problem:

**Definition 2 (Probabilistic Embeddings).** A metric space \(M = (V_M, d_M)\) is \(\alpha\)-probabilistically embedded in a set of metric spaces \(S\) if there exists a distribution \(D\) over \(S\) and for every \(N \in S\), a non-contractive embedding \(\phi_N : V_M \rightarrow V_N\), such that for all \(u, v \in V_M\), \(\mathbb{E}_{N \in D}[d_N(\phi_N(u), \phi_N(v))] \leq \alpha \cdot d_M(u, v)\).

Using probabilistic embeddings, it is possible to obtain much better bounds on the distortion \([1, 2, 3, 15, 4]\). The following bound is shown in \([15, 4]\):

**Theorem 1.** Any metric space on \(n\) points can be \(O(\log n)\) probabilistically embedded in a set of \(n\)-point ultrametrics. Moreover, the distribution can be sampled efficiently.

Theorem 1 found many algorithmic applications in approximation algorithms, online algorithms, and distributed algorithms, see for example \([2, 17, 16, 23, 7]\). The bound on the distortion in Theorem 1 is tight even for probabilistic embeddings into tree metrics for which there is an \(\Omega(\log n)\) lower bound \([2]\).

Theorem 1 was originally formulated for a class of metric spaces defined by the following natural generalization of ultrametrics:

**Definition 3 ([2]).** For \(k \geq 1\), a \(k\)-hierarchically well-separated tree (\(k\)-HST) is a metric space defined on the leaves of a rooted tree \(T\). To each vertex \(u \in T\) there is associated a label \(\Delta(u) \geq 0\) such that \(\Delta(u) = 0\) if and only if \(u\) is a leaf of \(T\). The labels are such that if a vertex \(v\) is a child of a vertex \(u\) then \(\Delta(v) \leq \Delta(u)/k\). The distance between two leaves \(x, y \in T\) is defined as \(\Delta(\text{lca}(x, y))\), where \(\text{lca}(x, y)\) is the least common ancestor of \(x\) and \(y\) in \(T\).

The definition of finite ultrametric is the same as a 1-HST. Any \(k\)-HST is therefore, in particular, an ultrametric and any finite ultrametric can be \(k\)-embedded in some \(k\)-HST \([3]\). We can therefore restrict our attention to ultrametrics, while all results generalize to \(k\)-HSTs.

The main contribution of this paper is in offering a new type of metric embedding that makes it possible to bypass lower bounds for the standard and even probabilistic metric embeddings. There are two key observations that lead to this new type of embedding. The first is that in some applications it is natural to match a point onto a set of points in the target metric space. Motivated by two applications of Theorem 1 the group Steiner tree problem (henceforth, GST), and the metrical task systems problem (henceforth, MTS), we propose the following definition:

**Definition 4 (Multi Embedding).** A multi embedding of \(M\) in \(N\) is a partial surjective function \(f\) from \(N\) on \(M\), i.e. each point \(x \in M\) is embedded into a non-empty set \(f^{-1}(x)\). Points in \(f^{-1}(x)\) are called representatives of \(x\) in \(N\).
The role of \( f^{-1} \) in Definition 4 is analogous to the role of \( \phi \) in the Definitions 1 and 2 of embedding and probabilistic embedding. Another way to define multi embedding is by \( \phi : M \rightarrow 2^N \), in which \( \phi(u) \cap \phi(v) = \emptyset \) for every \( u \neq v \). In our notation we have \( \phi(x) = f^{-1}(x) \). Since the \( f \) notation will be more convenient, henceforth we will exclusively use it.

The second observation is that for many applications, including those mentioned above, there is no need to approximate the original distance for every pair of representatives. What is really needed is that every path in the original space will be approximated well by some path in the target space.

A path in a metric space is an arbitrary finite sequence of points in the space. The length of a simple path \( p = \langle u_1, u_2, \ldots, u_m \rangle \) in a metric space \( M = (V, d) \) is defined as \( \ell(p) = \sum_{i=1}^{m-1} d(u_i, u_{i+1}) \).

**Definition 5 (Path Distortion).** A multi-embedding \( f \) of \( M \) in \( N \), is called non-contractive if for any \( u, v \in N \), \( d_N(u, v) \geq d_M(f(u), f(v)) \). The path-distortion of a non-contractive multi-embedding of \( M \) in \( N \), \( f : N \rightarrow M \), is the infimum over \( \alpha \), for which any path \( p = \langle u_1, u_2, \ldots, u_m \rangle \) in \( M \), has a path \( p' = \langle u'_1, u'_2, \ldots, u'_m \rangle \) in \( N \) such that \( f(u'_i) = u_i \) and \( \ell(p') \leq \alpha \cdot \ell(p) \).

A multi embedding whose path-distortion is at most \( \alpha \) is called \( \alpha \)-path embedding.

A crucial parameter for multi-embeddings is the size of the target space \( \Gamma \). In general, it will be desirable that \( \Gamma \) will be polynomial in the size of the source space. In fact, if \( \Gamma = \infty \) then there is a simple 1 path embedding of any finite metric space by trees: Take all finite paths, convert each path to a simple path (by duplicating points, if necessary), and put them under a single root with an edge of length half the diameter. This motivates a study of the trade-off between \( \Gamma \) and the path to a simple path (by duplicating points, if necessary), and put them under a single root with a path in the target space.

In Section 4 we study path embedding of expander graphs and the hypercube into tree metrics. We show e.g. that an \( n \)-point Ramanujan graphs have 3-path-embedding into tree metrics of size \( \Gamma(n) \leq \text{poly}(n) \). This is in sharp contrast to the status of expander graphs for previous notions of embeddings for which they are considered “worst case” examples with \( \Omega(\log n) \) distortion \[24\]. These results directly imply nearly tight results on the approximation ratio for GST on expander graphs and hypercubes.

We consider multi embeddings when the class of target metric spaces are ultrametrics. First, we observe that probabilistic embedding into ultrametrics directly implies a bound for path embedding by putting all the trees in \( S \) (the set used in the probabilistic embedding) under a common new root. This results with an \( \alpha \)-path embedding into an ultrametric of size \( |S| \). Using the bound of \[13\] on the number of ultrametrics needed in Theorem \[1\] we obtain an \( O(\log n) \) path embedding into an ultrametric of size \( O(n^2 \log n) \).

An important parameter of the metric spaces appearing in practice is the aspect ratio of the metric, which is the ratio between the diameter and minimum non-zero distance in the metric space. It will be convenient for us to assume that the minimum distance is 1, and so the aspect ratio becomes the diameter. It turns out that the aspect ratio of the metric plays a significant role in the path distortion of multi-embeddings. In Section 3 we prove:

**Theorem 2.** Fix \( \beta > 1 \). For any metric space \( M = (V, d) \) on \( |V| = n \) points and aspect ratio \( \Delta \), there exists an efficiently constructible multi-embedding into an ultrametric of size \( n^{2\beta} \), whose path

\[\text{1In a preliminary version of this paper \[16\], we also introduced the notion of probabilistic multi-embedding. Using that notion we were able to show probabilistic multi-embedding into ultrametrics of polynomial size and path distortion \( O(\log n \log \log \log n) \). Since an \( O(\log n) \) bound now follows from Theorem \[1] \[16] \[14], we have decided to drop the probabilistic multi-embedding result from this version.}
distortion is at most

\[ O_\beta(\min\{\log n \cdot \log \log n, \ \log \Delta \cdot \log \log \Delta\}) \].

Our construction beats the probabilistic embedding based constructions on metrics with small aspect ratio. Expander graphs are examples where a lower bound of \( \Omega(\Delta) \) exists on probabilistic embedding using trees \([24]\).

The constructions of multi-embeddings are in a sense dual to Ramsey-type theorems for metric spaces \([6, 8]\), where the goal is to find a large subset which is well approximated by some ultrametric.

We also provide a simple example in which Theorem 2 is almost tight: Any \( \alpha \)-path embedding into ultrametrics of a simple unweighted path of length \( n \) has \( \alpha = \Omega(\log n) \). It follows, in particular, that any \( \alpha \)-path embedding into ultrametrics of the metric defined by an unweighted graph of diameter \( \Delta \) has \( \alpha = \Omega(\log \Delta) \). Path embedding is motivated by two intensively-studied algorithmic minimization problems: GST and MTS, mentioned above. For both, the best known algorithms use probabilistic embedding into trees/ultrametrics. In Section 2 we prove that in order to reduce these problems to other metric spaces it is sufficient to use path embedding. We therefore achieve improved algorithms for these problems whenever the path embedding distortion beats that of probabilistic embedding, and in particular, when the underlying metric is of small aspect ratio.

## 2 Applications

In this section we define MTS and GST show that path distortion of multi-embeddings reduces these problems to similar problems with different underlying metrics.

Metrical Task Systems (MTS) \([11]\) was introduced as a framework for many online minimization problems. A MTS on metric space \( M = (S, d) \), \( |S| = n \), is defined as follows. A “system” has a set of \( n \) possible internal states \( S \). It receives a sequence of tasks \( \sigma = \tau_1 \tau_2 \cdots \tau_m \). Each task \( \tau \) is a vector \( \tau : S \rightarrow \mathbb{R}^+ \cup \{\infty\} \) of nonnegative costs for serving \( \tau \) in each of the internal states of the system. The system may switch states (say from \( u \) to \( v \)), paying a cost equal to the distance \( d(u, v) \) in \( M \), and then pays the service cost \( \tau(v) \) associated with the new state. The major limiting factor for the system is the requirement to process the sequence in an online fashion, i.e., serving each task without knowing the future tasks.

As customary in the analysis of online algorithms, MTS is analyzed using the notion of competitive ratio. A randomized online algorithm \( A \) is called \( r \)-competitive if there exists some constant \( c \) such that for any task sequence \( \sigma \), \( E[\text{cost}_A(\sigma)] \leq r \cdot \text{cost}_{\text{Opt}}(\sigma) + c \), where \( \text{cost}_A(\sigma) \) is the random variable of the cost for serving \( \sigma \) by \( A \), and \( \text{cost}_{\text{Opt}}(\sigma) \) is the optimal (offline) cost for serving \( \sigma \). The current best online algorithm for the MTS problem in \( n \)-point metric spaces is \( O(\log^2 n \log \log n) \) competitive \([10, 13]\) (an improvement of \([5, 3]\)). Both papers \([5, 10]\) actually solve the MTS problem for ultrametrics, and then reduce arbitrary metric spaces to ultrametrics using Theorem 1. We next show that path embedding suffices:

**Proposition 1.** Assume that a metric space \( M \) is \( \alpha \)-path embedded in \( N \). Assume also that \( N \) has an \( r \)-competitive MTS algorithm. Then there is an \( \alpha r \)-competitive algorithm for \( M \).

**Proof.** We construct an online algorithm \( A \) for \( M \) as follows: Let \( A_N \) be an \( r \)-competitive online algorithm for \( N \), and \( f : N \rightarrow M \) an \( \alpha \) path embedding of \( M \) in \( N \). The task sequence \( \sigma \) is translated to a task sequence \( \sigma^N \) for \( N \) task by task as follows. A task \( \tau \) for \( M \) is translated into a task \( \tau^N \) for \( N \) such that \( \tau^N(u') = \tau(f(u')) \). \( A \) maintains the invariant that if \( A_N \) is in state \( v' \), then \( A \) is in state \( f(v') \).
It is easy to verify that $\text{cost}_A(\sigma) \leq \text{cost}_{A_N}(\sigma^N)$, since the service costs are the same, and the distances in $N$ are larger. Consider $\text{Opt}(\sigma)$, it defines a path $p$ of serving $\sigma$ in $M$. Thus there exists a path $p^N$ as in the statement of Definition [3]. The path $p^N$ is the way $\sigma^N$ would be served in $N$. In this way, since $f(p^N) = p$, the service costs in $N$ are the same as the services costs in $M$, and $\ell(p^N) \leq \alpha(\ell(p))$. Thus $\text{cost}_{\text{Opt}}(\sigma^N) \leq \alpha \cdot \text{cost}_{\text{Opt}}(\sigma)$. Summarizing:

$$E[\text{cost}_A(\sigma)] \leq E[\text{cost}_{A_N}(\sigma^N)] \leq \alpha \cdot \text{cost}_{\text{Opt}}(\sigma) + c.$$

Corollary 2. There is an $O(\log \Delta \log \log \Delta \cdot \log n \log \log n)$-competitive randomized MTS algorithm for MTS defined on metric spaces with diameter $\Delta$.

Proof. Apply Theorem [2] on the original metric and obtain an $O(\log \Delta \log \log \Delta)$ path embedding into an ultrametric of size $\Gamma(n) = \text{poly}(n)$. This ultrametric has $O(\log \Gamma(n) \log \log \Gamma(n))$ competitive algorithm [16]. Now apply Proposition [1] to obtain the claim. □

The Group Steiner Tree Problem (GST) [29] can be stated as follows: Given a graph $G = (V, E)$ on $n$ vertices with a weight function $c : E \rightarrow \mathbb{R}_+$, and subsets of the vertices $g_1, \ldots, g_k \subset V$ (called groups), the objective is to find a minimum weight subtree $T$ of $G$ that contains at least one vertex from each $g_i$, $i \in [k]$. Under certain standard complexity assumptions, this is hard to approximate by a factor better than $\max\{\log^{2-\varepsilon} k, \log^{2-\varepsilon} n\}$ [19]. The current best upper bound on the approximation factor is $O(\log^2 n \log k)$ [17, 15]. In [17], an $O(\log n \log k)$ approximation algorithm for tree metrics is given, and the general case is reduced to tree metrics using Theorem [11]. Again, we show that it is actually sufficient to use multi embedding for this problem.

As a first step we observe that the problem can be easily cast in terms of metric spaces instead of graphs: Given a graph $G = (V, E)$ with weights $w : E \rightarrow \mathbb{R}_+$, let $M = (V, d)$ be the shortest path metric induced by $G$ and $w$ on $V$. A tree $T$ in $M$ can be transformed into a tree $\hat{T}$ in $G$ such that the total weight in $\hat{T}$ is not larger than the total weight in $T$, and $\hat{T}$ contains all the vertices in $T$. This is done by replacing each edge in $T$ by the shortest path between its endpoints in $G$, and taking a spanning tree of the resulting subgraph. It therefore suffices to solve GST on metric spaces.

Proposition 3. Assume that a metric space $M$ is $\alpha$ path embedded in a metric space $N$. Assume in addition that there is a [randomized] polynomial time $r$ approximation algorithm for any GST instance with $k$ groups defined on $N$. Then there exists a [randomized] polynomial time $2\alpha r$ approximation algorithm for any GST instance with $k$ groups defined on $M$.

Proof. We construct an approximation algorithm $A$ for the instance $\sigma = (M; g_1, \ldots, g_k)$ as follows. Denote by $f : N \rightarrow M$ the $\alpha$ path embedding of $M$ in $N$. Consider the following instance of GST: $\sigma_N = (N; f^{-1}(g_1), \ldots, f^{-1}(g_k))$. Let $A_N$ be an $r$-approximation algorithm for $\sigma_N$. Let $T_N = A_N(\sigma_N)$ be the tree constructed by $A_N$. Denote by $f(T_N)$ the image graph of $T_N$. I.e., if $T_N = (V_N, E_N)$, then $f(T_N) = (f(V_N), \{f(u)f(v) | uv \in E_N\})$. The graph $f(T_N)$ is a connected and its weight is at most the weight of $T_N$. It also spans at least one representative form each group. Algorithm $A$ returns a spanning tree of $f(T_N)$. This tree is a feasible solution and it satisfies $\text{cost}_A(\sigma) \leq \text{cost}_{A_N}(\sigma_N)$.

Consider the tree $\text{Opt}(\sigma)$, double each edge in $\text{Opt}(\sigma)$ and take an Euler tour $p$ of this graph. There exists a path in $N$, $p_N$, as in the statement of Definition [3] such that $f(p_N) = p$. The path
\( p_N \) is a connected graph and spans at least one representative from each group \( f^{-1}(g_j) \). As the weight of \( p \) is twice the weight of \( \text{OPT}(\sigma) \), we have

\[
\text{cost}_{\text{OPT}_N}(\sigma_N) \leq \ell(p_N) \leq \alpha\ell(p) \leq 2\alpha\text{cost}_{\text{OPT}}(\sigma).
\]

Summarizing:

\[
\mathbb{E}[\text{cost}_A(\sigma)] \leq \mathbb{E}[\text{cost}_{A_N}(\sigma_N)] \leq \tau\text{cost}_{\text{OPT}_N}(\sigma_N) \leq 2\alpha r\text{cost}_{\text{OPT}}(\sigma).
\]

Corollary 4. There is an polynomial time \( O(\log \Delta \log \log \Delta \log n \log k) \) approximation algorithm for GST on metric spaces with diameter \( \Delta \).

### 3 Multi Embedding into Ultrametrics

The following theorem is a restatement of Theorem 2 in a more general form.

**Theorem 3.** Given any metric space \( M = (V, d) \) on \( |V| = n \) points and diameter \( \Delta \), for any \( t \in \mathbb{N}, M \) is \( O(t \min \{\log \Delta, \log n\}) \) path embedded into an efficiently constructible ultrametric of size \( \Gamma \leq n^\beta \), where \( \beta = \min\{((\log n)^{1/t}, [t \log(4\Delta)]^{2/t}\} \).

**Proof.** The construction of the multi-embedding is motivated by the construction of subspaces approximating ultrametric in [3] [4], but instead of deleting points, we duplicate them. We then prove the bounds on the path distortion.

Let \( \Delta \) be the diameter of \( M \). Let \( x \) and \( \bar{x} \) be two points realizing the diameter of \( M \), and assume without loss of generality that \( |\{y \in M : d(x, y) < \Delta/4\}| \leq n/2 \) (otherwise, switch the roles of \( x \) and \( \bar{x} \)). Define a series of sets \( A_0 = \{x\} \), and for \( i \in \{1, 2, \ldots, t\} \), \( A_i = \{y \in M : d(x, y) < i\Delta/4t\} \), and “shells” \( S_i = A_i \setminus A_{i-1} \). Let \( |V| = n \) and let \( \epsilon_i = |A_i|/n \).

The algorithm for constructing the multi-embedding works as follows: Choose a shell \( S_i, i \in [t] \). Recursively, construct a multi-embedding of the sub space \( A_i \) into an ultrametric \( T_1 \) and a multi-embedding of the subspace \( V \setminus A_{i-1} \) into an ultrametric \( T_2 \). To construct the multi-embedding for \( M \), we construct an ultrametric \( T \) with root labelled with \( \Delta \), and two children, one is \( T_1 \) and the other is \( T_2 \). This is a multi-embedding since the points in \( S_i \) are essentially being “duplicated” at this stage. Note that this is a non-contractive multi-embedding.

Next we prove an upper bound on the size of the resulting ultrametric \( T \), assuming that the shell was chosen carefully enough. The bound we prove is \( n^\beta \), where \( \beta = \min\{((\log n)^{1/t}, [t \log(4\Delta)]^{2/t}\} \).

We begin with the first bound. Let \( \beta = \beta(n) = (\log n)^{1/t} \). The proof proceeds by induction on \( n \) (whereas \( t \) is fixed). There must exist \( i \in [t] \) such that \( \epsilon_{i-1} \geq \epsilon_i^\beta \). Indeed, note that \( n^{-1} \leq \epsilon_0 < \epsilon_{t+1} \leq 1/2 \). Assume for the contrary that \( \epsilon_{i-1} < \epsilon_i^\beta \) for all \( i \in [t] \), then

\[
\epsilon_0 < \epsilon_1^\beta < \cdots \epsilon_t^\beta \leq \left(\frac{1}{2}\right)^{\log n} = \frac{1}{n},
\]

which is a contradiction. Therefore we can fix \( i \) such that \( \epsilon_{i-1} \geq \epsilon_i^\beta \). Inductively, assume that the recursive process results in at most \( \epsilon_{i-1}^\beta \leq \epsilon_i^\beta \) leaves in \( T_1 \) and at most \( (1 - \epsilon_{i-1})n^\beta((1-\epsilon_{i-1})n) \leq (1 - \epsilon_{i-1})n^\beta \) leaves in \( T_2 \). So \( |T| \leq (\epsilon_i^\beta + (1 - \epsilon_{i-1})^\beta)\). Since \( \epsilon_{i-1} \geq \epsilon_i^\beta \), we have \( \epsilon_i^\beta + (1 - \epsilon_{i-1})^\beta \leq \epsilon_{i-1} + (1 - \epsilon_{i-1}) = 1 \) and we are done.
We next prove the second bound. Let $\beta = \beta(\Delta) = [t \log(4\Delta)]^{2/t}$. The proof is by induction on (the rounded value of) $\Delta$. We claim that
\[ \exists i \in [t] \quad \varepsilon_{i-1} \geq \varepsilon_i^{\beta(\Delta)/2} n^{\beta(\Delta)/2 - \beta(\Delta)}. \quad (1) \]
Indeed, assume for the contrary that no such $i$ exists. Set $a = \log(2\Delta) \geq 1$, so that $\beta(\Delta/2) = (ta)^{2/t}$ and $\beta(\Delta) = [(t(a + 1))^{2/t} - (ta)^{2/t}]$. Denote $b = n^{(ta)^{2/t} - [(t(a + 1))^{2/t}] + c = (ta)^{2/t}}$. The opposite of (1) then becomes $\varepsilon_{i-1} < \varepsilon_i^c b$, for any $i \in [t]$. Iterating this $t$ times we get:
\[
\frac{1}{n} = \varepsilon_0 < \varepsilon_i^{c^t b^{1+c^2+\ldots+c^{t-1}}} \leq \varepsilon_i^{c^t b^{c-t-1}} \leq b^{c-t}.
\]
So that:
\[ n^{(ta)^{2/t} - [(t(a + 1))^{2/t} - (ta)^{2/t}]} < n, \]
but this is a contradiction, since an application of the mean value theorem implies the existence of $\xi \in [a, a + 1]$, for which
\[
(ta)^{2/t} [(t(a + 1))^{2/t} - (ta)^{2/t}] = (ta)^{2/t} [2t^{-1}{2/t}\xi^{-1}2/t] = 2(ta) \left( \frac{a}{\xi} \right)^{1-2/t} \geq ta \geq 1.
\]
Choose an index $i \in \{1, \ldots, t\}$ satisfying (1). Since $i \leq t$, $\Delta(A_i) \leq \Delta/2$. The choice of the index $i$, and using the inductive hypothesis, gives the required lower bound on the cardinality of $T$ since:
\[ |T| \leq (\varepsilon_i n)^{\beta(\Delta)/2} + [(1 - \varepsilon_i n)]^{\beta(\Delta)} \leq \varepsilon_i n^{\beta(\Delta)} + (1 - \varepsilon_i n) n^{\beta(\Delta)} \leq n^{\beta(\Delta)}. \]

We note that the running time of the algorithm above is $O(n^2)$ on each vertex in the tree and therefore $O(n^{2+2})$ for the whole tree. A slight variation on this algorithm (and a more careful analysis) has an $O(n^{\max\{\beta, 2\}})$ running time.

The multi-embedding described above has the following properties:

1. The multi-embedding is non-contractive.

2. The tree structure defining the ultrametric is a binary tree.\(^2\)

Let $u$ be an internal vertex in the binary tree defining the ultrametric, and $T$ the subtree rooted at $u$. We can rename the subtrees rooted with the children of $u$, as $T_1$ and $T_2$ such that:

3. Let $x$ and $y$ two points in $M$. If $\emptyset \neq f^{-1}(x) \cap T < T_1$ and $\emptyset \neq f^{-1}(y) \cap T < T_2$, then $d(x, y) \geq \Delta(T)/4t$.

4. $|f(T_1)| \leq |f(T)|/2$.

5. $\Delta(T_1) \leq \Delta(T)/2$.

We next show, using the properties above, that the path distortion of this multi embedding is at most $8t \log \min \{n, \Delta\}$. Let $p = \langle u_1, u_2, \ldots, u_m \rangle$ be a path in $M$ whose length is $\ell(p)$. We
construct a path \( \bar{p} \) on the leaves of \( T \) whose length satisfies \( \ell(\bar{p}) \leq 8t \log \min\{n, \Delta\} \cdot \ell(p) \). The proof proceeds by induction on the height of the tree defining the ultrametric.

We partition \( p \) into sub-paths as follows. Define a sequence of indices and a sequence of sub-trees of the root: Let \( j_1 = 1 \) and let \( T_1 \in \{T_1, T_2\} \) be the subtree of the root that includes the longest prefix of \( p \). Assume inductively that we have already defined \( j_{i-1} \) and \( \hat{T}_{i-1} \in \{T_1, T_2\} \). Define \( j_i \) to be the minimum index such that \( u_{j_i} \) is the first point in \( p \) after \( u_{j_{i-1}} \) with no representative in \( \hat{T}_{i-1} \). Let \( \hat{T}_i \in \{T_1, T_2\} \) be the other subtree of the root. Assume this process is finished with \( j_s, \hat{T}_s \). Next we define another sequence of indices: \( k_s = m \), for \( i < s \) we define \( k_i \) to be the largest number, smaller than \( j_{i+1} \), such that \( u_{k_i} \) does not have a representative in \( \hat{T}_{i+1} \). By the construction of \( \hat{T}_i \), we have that \( j_i \leq k_i \) and \( u_{k_i} \) has a representative in \( \hat{T}_i \). See Figure 1 for example of such partition. We have partitioned \( p \) into sub-paths \( \langle u_{j_1}, \ldots, u_{k_1} \rangle_i \) and \( \langle u_{k_1}, \ldots, u_{j_{i+1}} \rangle_i \). Informally, a sub-path \( \langle u_{j_i}, \ldots, u_{k_i} \rangle \) will be realized in \( \hat{T}_i \), while sub-path \( \langle u_{k_i}, \ldots, u_{j_{i+1}} \rangle \) will be realized in \( T_1 \).

More formally, let \( L = \ell(p) \), \( \tilde{L}_{i,j} = \ell(\langle u_{i}, u_{i+1}, \ldots, u_{j} \rangle) \), \( n = |f(T)| \), \( n_1 = |f(T_1)| \), \( n_2 = |f(T_2)| \), and \( \Delta = \Delta(T) \), \( \Delta_1 = \Delta(T_1) \), and \( \Delta_2 = \Delta(T_2) \). We construct by induction on the tree structure \( T \) a path \( \bar{p} \) in \( T \) whose length satisfies \( \tilde{L} = \ell(\bar{p}) \leq 8t \log \min\{\Delta, n\} \cdot L \).

By the induction hypothesis it is possible to construct for any \( i \), a path in \( \hat{T}_i \) of representatives of \( \langle u_{j_i}, \ldots, u_{k_i} \rangle \) whose length is \( \tilde{L}_{j_i,k_i} \leq L_{j_i,k_i} \cdot 8t \log \min\{n, \Delta\} \).

Next, for any \( i \), we construct a path of representatives of \( \langle u_{k_i+1}, \ldots, u_{j_{i+1}} \rangle \). Note that \( u_{k_i+1}, \ldots, u_{j_{i+1}-1} \) have representatives in both \( T_1 \) and \( \hat{T}_{i+1} \). Therefore, we construct inductively a path from a representative of \( u_{k_i+1} \) to a representative of \( u_{j_{i+1}-1} \) in \( T_1 \), so \( \tilde{L}_{k_i,j_{i+1}} \leq \tilde{L}_{k_i,j_{i+1}} \cdot 8t \log \min\{n_1, \Delta_1\} \). We then connect the representative of \( u_{k_i} \) with the representative of \( u_{k_i+1} \) and the representative of \( u_{j_{i+1}-1} \) with the representative of \( u_{j_{i+1}} \), each such edge is of length at most the diameter of \( T, \Delta \). We have therefore constructed a path of representatives of \( \langle u_{k_i}, \ldots, u_{j_{i+1}} \rangle \) whose length is \( \tilde{L}_{k_i,j_{i+1}} + 2\Delta \).

Since \( u_{k_i} \) does not have a representative in \( \hat{T}_{i+1} \) and \( u_{j_{i+1}} \) does not have representative in \( \hat{T}_i \), we conclude using property (3) above, that \( d_M(u_{k_i}, u_{j_{i+1}}) \geq \Delta / 4t \), and so \( \Delta \leq 4t \cdot \tilde{L}_{k_i,j_{i+1}} \). To

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2Note that more generally, any ultrametric can be defined by a binary tree.
Proposition 5. Consider the metric defined by a simple $N$-point path. Then any $\alpha$ path-embedding of this metric in an ultrametric must have $\alpha = \Omega(\log n)$.

Proof. Let $M = \{v_1, v_2, \ldots, v_n\}$ be the metric space on $n$ points such that $d_M(v_i, v_j) = |i - j|$. We prove that for any non-contractive multi-embedding into an ultrametric $T$, any path of representatives of $\langle v_1, v_2, \ldots, v_n \rangle$ is of length at least $g(n) = \frac{n}{2} \log n$.

We conclude, 

$$\bar{L} = \sum_{i=1}^{s} L_{j_i, k_i} + \sum_{i=1}^{s-1} L_{k_i, j_{i+1}}$$

$$\leq 8t \log \min\{n, \Delta\} \cdot \left( \sum_{i=1}^{s} L_{j_i, k_i} + \sum_{i=1}^{s-1} L_{k_i, j_{i+1}} \right)$$

$$= 8t \log \min\{n, \Delta\} L.$$  

We end the discussion on multi embedding into ultrametrics with the following impossibility result.

Proposition 5. Consider the metric defined by a simple $N$-point path. Then any $\alpha$ path-embedding of this metric in an ultrametric must have $\alpha = \Omega(\log n)$.

Proof. Let $M = \{v_1, v_2, \ldots, v_n\}$ be the metric space on $n$ points such that $d_M(v_i, v_j) = |i - j|$. We prove that for any non-contractive multi-embedding into an ultrametric $T$, any path of representatives of $\langle v_1, v_2, \ldots, v_n \rangle$ is of length at least $g(n) = \frac{n}{2} \log n$.

The proof proceeds by induction on $n$. For $n = 1$ the claim is trivial. For $n > 1$, let $\langle v_1', v_2', \ldots, v_n' \rangle$ be a path of representatives in $T$. Let $u = \text{lca}_T(v_1', v_n')$, $\Delta(u) = d_T(v_1', v_n') \geq d_M(v_1, v_n) = n - 1$. Let $T_1$ be the subtree of the child of $u$ that contains $v_1'$. $T_1$ does not contains $v_n'$. Let $i_1 < n$ be the maximal $i$ such that $\{v_1', \ldots, v_i'\} \subset T_1$. As $i_1 + 1$ is not contained in $T_1$, it must be that $d_T(v_i', v_{i+1}') \geq \Delta(u) \geq n - 1$. By the induction hypothesis

$$\ell_T(\langle v_1', \ldots, v_i' \rangle) \geq g(i_1),$$

$$\ell_T(\langle v_{i+1}', \ldots, v_n' \rangle) \geq g(n - i_1).$$

Since $g$ is a convex function, $(g(i_1) + g(n - i_1))/2 \geq g((i_1 + (n - i_1))/2) = g(n/2)$. We conclude

$$\ell_T(\langle v_1', \ldots, v_n' \rangle) \geq g(i_1) + (n - 1) + g(n - i_1)$$

$$\geq 2g(n/2) + n - 1 = 2\frac{n}{2} \log \frac{n}{2} + (n - 1) \geq \frac{n}{2} \log n.$$  

4 Multi-Embedding into Trees

In this section we consider multi embeddings into arbitrary tree metrics. We only have preliminary results. Specifically, we only consider two important types of metric spaces: expander graphs and the discrete cube with the Hamming metric, for which we obtain better results. For both of them the preceding sections proved an upper bound of $O(\log \log n \log \log \log n)$ and a lower bound of...
Lemma 8. To obtain a better approximation algorithm for GST.

We begin with the observation that for $\Gamma = \infty$ it is easy to obtain 1 path embedding of any finite metric space into trees. This is achieved by defining an infinite tree metric as follows: joining all possible finite paths with a common root, where the first node in the path is connected with an edge weight of $\Delta/2$ to the root. Moreover,

**Proposition 6.** Given a metric space $M$ defined by an unweighted graph of maximum degree $d$ and diameter $\Delta$, and let $s \in \mathbb{N}$. Then $M$ can be $(2 + \frac{\Delta}{s})$-path-embedded into a tree metric of size $nd^s$.

*Proof.* Along the lines of the construction described above, we take all paths of length $s$, and join these with a common root, where the first node in the path is connected with an edge weight of $\Delta$. Obviously, there are at most $nd^s$ such paths. Notice that our choice of weights to the edges adjacent to the root guarantees that distances in the resulting tree are no smaller than the original distances. We next claim that the path distortion is at most $(2 + \frac{\Delta}{s})$. To see this, consider a path $p = \langle v_1, \ldots, v_t \rangle$ of length $\ell$. We partition $p$ into sub-paths of length $s$: $p = p_1p_2 \cdots p_t$, where $t = \lceil \ell/s \rceil$, $p_j = \langle v_{js+1}, \ldots, v_{js+1} \rangle$ for $j < t$, and $p_t = \langle v_{(t-1)s+1}, \ldots, v_{\ell} \rangle$. Now the sub-path $p_j$ is mapped to the appropriate path in the tree. Note that the length of the image path is $2\ell + (t-1)\Delta$.

This simple fact is particularly interesting for its implication for expander graphs. Let $G$ be an $(n, d, \gamma d)$ graph, i.e., a $d$-regular, $n$-vertex graph whose second eigenvalue in absolute value is at most $\gamma d$. It is known [13] that such a graph has diameter at most $1 + \log_{1/\gamma} n$, and so we obtain:

**Corollary 7.** Any $(n, d, \gamma d)$-graph has $3$ path embedding in a tree of size $dn^{1+\log_{1/\gamma} d}$.

We also note that for the trees constructed in the proof of Proposition 6 it is particularly easy to obtain a better approximation algorithm for GST.

**Lemma 8.** Consider a tree metric $M = (V, d)$, where $V = P_1 \cup P_2 \cup \cdots \cup P_t$, $P_i = \langle v^i_1, v^i_2, \ldots, v^i_s \rangle$ is an unweighted simple path of length $s$, and $d(v^i_j, v^i_{j+1}) = \Delta$ for $i \neq j$. Then an instance of the GST with $k$ groups defined on $M$ has $(1 + \frac{\Delta}{\ell^2})(1 + \ln k)$ approximation algorithm.

*Proof.* Consider a GST instance $g_1, \ldots, g_k$ defined on $M$. We first check whether there is a solution that is completely contained in one $P_i$. This can be checked in polynomial time by noting that if an optimal solution is contained in one $P_i$ then it is an interval. Thus all is needed to be checked are $\ell(s+1)$ intervals.

Otherwise, the optimal solution intersects $t > 1$ of the paths $P_1, \ldots, P_t$. Define a Hitting Set instance whose ground set is $\{P_1, \ldots, P_t\}$ and the subsets are $g'_1, \ldots, g'_k$, where $g'_i = \{P_j; P_j \cap g_i \neq \emptyset\}$. It follows that the optimal cost of the hitting set problem is at least $t-1$. The Hitting Set problem has a polynomial time $1 + \ln k$ approximation algorithm [22, 25]. Let $S$ be the approximate solution for the hitting set. We define a solution for the GST instance by taking a natural path over $\bigcup\{P_i; P_i \in S\}$. Note that its length is at most $(\Delta + 2s)|S| \leq (\Delta + 2s)(t-1)(1 + \ln k)$. But the cost of the optimal GST algorithm is at least $(t-1)\Delta$.

**Corollary 9.** For fixed $d > \lambda$, There exist constants $c = c_{d,\lambda}$, $C = C_{d,\lambda}$, and polynomial $p(t) = p_{d,\lambda}(t)$ such that GST on $(n, d, \lambda)$ graphs has $p(n)$-time $(C\log k)$ approximation algorithm, and it is NP-hard to approximate within a factor of $c\log k$. 


Proof. The approximation algorithm follows from Proposition 6 and Lemma 8 by setting $s = \Delta$. The hardness result follows since an $(n, d, \lambda)$ graph contains a subset of $n^{\Omega_{d,\lambda}(1)}$ points that is $O_{d,\lambda}(1)$ approximated by an equilateral space \[^8\]. GST on equilateral space is equivalent to a standard Hitting Set problem, which is NP-hard to approximate within a factor of $c \ln k$ \[^{28,26}\]. Usage of points not in this subspace (“Steiner points”) can improve the approximation factor by at most a factor of two \[^{20,18}\].

We next examine multi-embedding of the $h$-dimensional hypercube with $n = 2^h$ vertices. Using Proposition \[^4\] with $s = h / \log h$, and using $d = h$ and $\Delta = h$, we obtain $(\log \log n + 2)$ path embedding into trees of size $\Gamma(n) \leq n^2$. Using Lemma \[^8\] it also implies a polynomial time $O(\log k \log \log n)$ approximation algorithm to GST on the cube. Similarly to the expander graphs, it is hard to approximate instances of GST on the cube to within a $c \log k$ factor, for some constant $c > 0$, since the cube contains a subset of $n^{\Omega(1)}$ points that is $O(1)$ approximated by an equilateral space.

5 Discussion

An interesting open problem is to determine worst case bounds for path distortion of multi embedding into trees of polynomial size. As indicated by the case of expander graphs, such bounds may be better than those for ultrametrics.

Results on multi-embedding into trees directly reflect on the approximability of GST. As shown for expanders and hypercubes, it is possible that for special classes of metric spaces, a combination of improved path embedding and a specialized solution would yield (nearly) tight upper bounds. Our approach to show the (near) tightness of the results in those cases stems from metric Ramsey-type considerations (i.e. the existence of large approximately equilateral subspace). Such considerations are in fact more general and may lead to more results of this flavor. \[^3\]

Multi-embedding into ultrametrics, also implies multi embedding into $\ell_p^d$, where $d = O(\log n)$ with similar path distortion \[^9\]. It is natural to ask whether better path distortion is possible for multi-embedding into $\ell_1$ or $\ell_2$. Further study of multi-embeddings in other settings and their applications seems an attractive direction for future research.

Acknowledgments

We thank Robi Krauthgamer and Assaf Naor for fruitful discussions.

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\[^3\] In \[^8\] it is shown that any metric space contains a “large” subspace which is approximately an ultrametric, or a $k$-HST. Such trees were used in \[^19\] to prove inapproximability results for GST. It is plausible that these techniques can be combined to obtain tight bounds for GST in specific metric spaces.
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