We find a semi-algebraic description of the Minkowski sum $A_{3,n}$ of $n$ copies of the bounded twisted cubic $\{(t, t^2, t^3) \mid -1 \leq t \leq 1\}$ for each integer $n \geq 3$. These descriptions provide efficient membership tests for the sets $A_{3,n}$. These membership tests in turn can be used to resolve some instances of the underdetermined matrix moment problem, which was formulated by Michael Rubinstein and Peter Sarnak in order to study problems related to $L$-functions and their zeros.

1. Introduction

The zeros of $L$-functions are known to be able to describe various geometrical and arithmetical objects and are the subjects of several conjectures (cf. [1–3]). For example, the Generalized Riemann Hypothesis conjectures that all non-trivial zeros of an $L$-function have real part $\frac{1}{2}$ and the Grand Simplicity Hypothesis asserts that the imaginary parts of zeros of Dirichlet $L$-functions are linearly independent over $\mathbb{Q}$ (cf. [5]). $L$-functions can also be encountered in proofs of the Prime Number Theorem (cf. [6]) and in primality tests (cf. [4]).

Let $\pi$ be an automorphic cusp form and let $L(s, \pi)$ be its standard $L$-function. A conjecture, which has been verified in many cases, states that under certain conditions the function $L(s, \pi)$ has an analytic continuation $\Lambda(s, \pi)$ that satisfies the functional equation

$$\Lambda(1 - s, \pi) = W(\pi)N_\pi^{s-1/2}\Lambda(s, \pi),$$

where $N_\pi$ is the conductor of $\pi$ and $W(\pi)$ is either 1 or $-1$. The sign $W(\pi) \in \{\pm 1\}$ is called the root number of $\pi$. The problems of computing root numbers and counting the zeros of $L$-functions reduce to the following problem with $\log N_\pi \approx n$.

**Problem 1.1** (The underdetermined matrix moment problem). Determine the possible sets of eigenvalues of a real orthogonal $(2n+1) \times (2n+1)$ matrix $A$ given its first $k \leq n$ moments $\text{tr}(A), \text{tr}(A^2), \ldots, \text{tr}(A^k)$.

This problem is the object of study in the paper [7] by Michael Rubinstein and Peter Sarnak. For the full background and relevance of the problem, we refer to this paper.
Let \( A \) be a real orthogonal \((2n + 1) \times (2n + 1)\) matrix. Then its eigenvalues are
\[
\det(A), e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n}
\]
for some \( \theta_1, \ldots, \theta_n \in [0, \pi] \). And conversely, any such sequence is the spectrum of a real orthogonal \((2n + 1) \times (2n + 1)\) matrix. We have
\[
\text{tr}(A^j) = \det(A)^j + 2 \sum_{i=1}^{n} \cos(j\theta_i)
\]
and \( \cos(j\theta_i) = T_j(\cos(\theta_i)) \) for all integers \( j \geq 1 \), where \( T_j \) is the \( j \)-th Chebyshev polynomial of the first kind. The polynomial \( T_j(x) \) has degree \( j \). So, given \( \det(A) \) and \( \text{tr}(A), \text{tr}(A^2), \ldots, \text{tr}(A^k) \) for some integer \( k \leq n \), we can compute \( \sum_{i=1}^{n} \cos(\theta_i)^j \) for each \( j \in \{1, \ldots, k\} \) using Gaussian elimination on the coefficient vectors of \( T_1, \ldots, T_k \). As \( \det(A) \in \{ \pm 1 \} \) only has finitely many possible values, we write \( t_i = \cos(\theta_i) \in [-1, 1] \) and see that Problem 1.1 reduces to the following problem.

**Problem 1.2** (The moment curve problem). Determine the set
\[
\left\{ (t_1, \ldots, t_n) \in [-1, 1]^n \left| \sum_{i=1}^{n} t_i^j = x_j \text{ for all } j \in \{1, \ldots, k\} \right. \right\}
\]
given the real numbers \( x_1, \ldots, x_k \in \mathbb{R} \).

This problem was also formulated by Michael Rubinstein and Peter Sarnak. Note that given the first \( k \) power sums of \( t_1, \ldots, t_n \), we can compute all symmetric polynomial expressions in \( t_1, \ldots, t_n \) of degree at most \( k \). So if \( k = n \), then we are able to compute the coefficients of the polynomial \((x - t_1) \cdots (x - t_n)\), which not only allows us to recover \( t_1, \ldots, t_n \), but also shows that \( t_1, \ldots, t_n \in \mathbb{C} \) are unique up to reordering. So we are interested in the case where \( k < n \). In this case, Michael Rubinstein and Peter Sarnak propose the following strategy: consider the set \( C_k := \{(t, t^2, \ldots, t^k) \mid -1 \leq t \leq 1\} \subseteq \mathbb{R}^k \) and define
\[
\mathcal{A}_{k,n} := C_k + C_k + \cdots + C_k
\]
to be the Minkowski sum of \( n \) copies of \( C_k \) for each integer \( n \geq 1 \). Then we can determine the set of tuples \((t_1, \ldots, t_n) \in [-1, 1]^n\) such that
\[
\sum_{i=1}^{n} t_i^j = x_j
\]
for all \( j \in \{1, \ldots, k\} \) recursively by first computing the set \( t_n \in [-1, 1] \) such that
\[
(x_1, x_2, \ldots, x_k) = (t_n, t_n^2, \ldots, t_n^k) \in \mathcal{A}_{k,n-1}.
\]
In order to do the latter, we need an efficient membership test for the set \( \mathcal{A}_{k,n} \) for all \( n > k \). For \( k \in \{1, 2\} \), this is easy. In general, one way to get an efficient membership test, would be to describe the sets \( \mathcal{A}_{k,n} \) implicitly using only equalities and inequalities involving polynomial expressions in \( x_1, \ldots, x_k \) and unions. In other words, using semi-algebraic descriptions of the sets \( \mathcal{A}_{k,n} \). In this paper, we provide exactly such descriptions in the case that \( k = 3 \).
2. Main results

Let \( n \geq 3 \) be a positive integer. Our first result describes the boundary of \( A_{3,n} \). We need this result in order to prove the Main Theorem. However, it also provides us with a piecewise parametrization, which is useful for rendering a visualization of \( A_{3,n} \). See Figure 1 for an example.

In order to state the result, we define the following sets.

- Take \( C_{k,a}^+ = \left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + t \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \mid -1 \leq s \leq t \leq 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \),

- \( C_{\ell,b}^- = \left\{ \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \mid -1 \leq s \leq t \leq 1 \right\} + b \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \).
Theorem 2.1. We also let
\[
\begin{align*}
\text{for all positive integers } k, \ell \geq 1 \text{ and } a, b \geq 0. \\
\text{• Take } B_+^n = \bigcup_{k=1}^{n-1} C_{k,n-k-1}^+ \text{ and } B_-^n = \bigcup_{\ell=1}^{n-1} C_{\ell,n-\ell-1}^-.
\end{align*}
\]
\[
\text{Let } B_n^0 \text{ be the set consisting of all points } (x, y) \in \mathbb{R}^2 \text{ such that } ny \geq x^2 \text{ and } y \leq n-1 + (x + 2i - (n-1))^2 
\]
for each \(i \in \{0, \ldots, n-1\}\). We also let \(\pi: \mathbb{R}^3 \to \mathbb{R}^2\) be the projection map sending \((x, y, z) \mapsto (x, y)\).

**Theorem 2.1.**

(a) The boundary of \(A_{3,n}\) is the union of \(B_n^+\) and \(B_n^-\).

(b) We have \(\pi(B_n^+) = \pi(B_n^-) = B_n^0\).

(c) Let \((x, y) \in B_n^0\) be a point. Then there exist unique \(z^+, z^- \in \mathbb{R}\) such that \((x, y, z^+) \in B_n^+\) and \((x, y, z^-) \in B_n^-\). We have \(z^+ \geq z^-\). Moreover, equality holds precisely when the point \((x, y)\) lies on the boundary of \(B_n^0\).

In Section 3, we find semi-algebraic descriptions of (in particular) \(C_{k,a}^+\) and \(C_{\ell,b}^-\).

To write these descriptions down, we define
\[
A_{k\ell} = k\ell(k + \ell)^2, \\
B_{k\ell}(x, y) = 2k\ell(2x^2 - 3(k + \ell)y), \\
C_{k\ell}(x, y) = x^6 - 3(k + \ell)x^4y + 3(k^2 + k\ell + \ell^2)x^2y^2 - (k - \ell)^2(k + \ell)y^3, \\
D_{k\ell}(x, y) = (k + \ell)y - x^2, \\
f_{k\ell}(x, y, z) = A_{k\ell}z^2 + B_{k\ell}(x, y)z + C_{k\ell}(x, y)
\]
for all positive integers \(k, \ell \geq 1\). Note here that
\[
B_{k\ell}^2 - 4A_{k\ell}C_{k\ell} = 4k\ell(\ell - k)^2D_{k\ell}^2
\]
for all \(k, \ell\). We then use these descriptions together with the previous theorem to prove our main result.

**Main Theorem.** Take the following sets:

\[
X_1 = \bigcup_{k=1}^{n-1} \begin{cases}
  \left(\begin{array}{c}
x \\
y \\
z
\end{array}\right) + (n - k - 1) \begin{array}{c}1 \\
1 \\
1
\end{array} \in \mathbb{R}^3 & y \leq k + (x + k)^2 \\
y \geq (k + 1)^{-1}x^2 & y \leq 1 + k^{-1}(x - 1)^2 \\
z \leq -\frac{B_{k\ell}(x, y)}{2A_{k\ell}} & \end{cases}
\]

\[
X_2 = \bigcup_{k=1}^{n-1} \begin{cases}
  \left(\begin{array}{c}
x \\
y \\
z
\end{array}\right) + (n - k - 1) \begin{array}{c}1 \\
1 \\
1
\end{array} \in \mathbb{R}^3 & y \leq k + (x + k)^2 \\
y \geq (k + 1)^{-1}x^2 & y \leq 1 + k^{-1}(x - 1)^2 \\
f_{k\ell}(x, y, z) \leq 0 & \end{cases}
\]

\[
Y_1 = \bigcup_{\ell=1}^{n-1} \begin{cases}
  \left(\begin{array}{c}
x \\
y \\
z
\end{array}\right) + (n - \ell - 1) \begin{array}{c}-1 \\
1 \\
-1
\end{array} \in \mathbb{R}^3 & y \leq \ell + (x - \ell)^2 \\
y \geq (\ell + 1)^{-1}x^2 & y \leq 1 + \ell^{-1}(x + 1)^2 \\
z \geq -\frac{B_{\ell\ell}(x, y)}{2A_{\ell\ell}} & \end{cases}
\]

\[
Y_2 = \bigcup_{\ell=1}^{n-1} \begin{cases}
  \left(\begin{array}{c}
x \\
y \\
z
\end{array}\right) + (n - \ell - 1) \begin{array}{c}-1 \\
1 \\
-1
\end{array} \in \mathbb{R}^3 & y \leq \ell + (x - \ell)^2 \\
y \geq (\ell + 1)^{-1}x^2 & y \leq 1 + \ell^{-1}(x + 1)^2 \\
f_{\ell\ell}(x, y, z) \leq 0 & \end{cases}
\]

Then we have \(A_{3,n} = (X_1 \cup X_2) \cap (Y_1 \cup Y_2)\).
Structure of the paper. In the next section, we prove a result about representations of points on the boundary of $A_{3,n}$. In the section after that, we use this result to show that the semi-algebraic components of the boundary do not intersect in their interior. We also find semi-algebraic descriptions for these components. In Section 5, we prove that certain semi-algebraic sets are not contained in the boundary of $A_{3,4}$ and derive from this one half of Theorem 2.1(a). And finally, in Section 6, we study the sets $B_n^+$ and $B_n^-$ in more detail and prove Theorem 2.1 and the Main Theorem. We conclude the paper by discussing some obstacles to our approach for higher dimensions.

Acknowledgments. The problem that this paper solves was brought to our attention by Bernd Sturmfels during the graduate student meeting on applied algebra and combinatorics held in Leipzig on 18–20 February 2019. We would like to thank him for doing so and we would like to thank the organizers of this meeting for making it possible. We would also like to thank Peter Sarnak for explaining the origin and relevance of the problem to us.

3. Representations of points on the boundary of $A_{3,n}$

The goal of this section is to prove the following theorem.

**Theorem 3.1.** Let $p \in \mathbb{R}^3$ be a point on the boundary of $A_{3,n}$ and write

$$p = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} + \cdots + \begin{pmatrix} t_n \\ t_n^2 \\ t_n^3 \end{pmatrix}$$

for some tuple $(t_1, \ldots, t_n) \in [-1,1]^n$. Then the set $\{t_1, \ldots, t_n\} \setminus \{-1,1\}$ has at most two elements. Furthermore, the tuple $(t_1, \ldots, t_n)$ is unique up to permutation of its entries.

**Proof.** Consider the map

$$\varphi: [-1,1]^n \to \mathbb{R}^3,$$

$$(t_1, \ldots, t_n) \mapsto \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} + \cdots + \begin{pmatrix} t_n \\ t_n^2 \\ t_n^3 \end{pmatrix},$$

and write $p = \varphi(t_1, \ldots, t_n)$ for some $t_1, \ldots, t_n \in [-1,1]^n$. The Jacobian of $\varphi$ at the point $(t_1, \ldots, t_n)$ is

$$\begin{pmatrix} 1 & \cdots & 1 \\ 2t_1 & \cdots & 2t_n \\ 3t_1^2 & \cdots & 3t_n^2 \end{pmatrix}$$

and so we see that

$$\begin{pmatrix} 1 \\ 2t_i \\ 3t_i^2 \end{pmatrix} \in T_p A_{3,n}$$

for all $i \in \{1, \ldots, n\}$ with $t_i \in (-1,1)$. Since the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 2r & 2s & 2t \\ 3r^2 & 3s^2 & 3t^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ r & s \\ r^2 & s^2 & t^2 \end{pmatrix}$$




has rank 3 when \( r < s < t \), we see that the set \( \{ t_1, \ldots, t_n \} \setminus \{-1, 1\} \) has at most two elements. Write \( p = \varphi(s_1, \ldots, s_n) \) for some \( s_1, \ldots, s_n \in [-1, 1] \). Then we also see that the set \( \{ s_1, \ldots, s_n, t_1, \ldots, t_n \} \setminus \{-1, 1\} \) has at most two elements. So it suffices to prove that if

\[
p = k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},
\]

for some \(-1 < s < t < 1\) and \( k, \ell, a, b \geq 0 \) with \( k + \ell + a + b = n \), then \( k, \ell, a, b \) are completely determined by \( p, s \) and \( t \). As we have

\[
\binom{n}{p} = k \begin{pmatrix} 1 \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} 1 \\ t^2 \\ t^3 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},
\]

this easily follows from the fact that the four vectors on the right are linearly independent. \( \square \)

4. The semi-algebraic components of the boundary of \( \mathcal{A}_{3,n} \)

From Theorem 3.1 it immediately follows that the boundary of \( \mathcal{A}_{3,n} \) is contained in the union of the sets

\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \middle| -1 \leq s \leq t \leq 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}
\]

over all integers \( k, \ell, a, b \geq 0 \) such that \( k + \ell + a + b = n \). However, Theorem 3.1 tells us more.

**Proposition 4.1.** Let \( k, \ell \geq 1 \) and \( a, b \geq 0 \) be integers such that \( k + \ell + a + b = n \). Then the set

\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \middle| -1 < s < t < 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}
\]

either is entirely contained in the boundary of \( \mathcal{A}_{3,n} \) or does not intersect the boundary of \( \mathcal{A}_{3,n} \) at all.

**Proof.** Take integers \( k', \ell', a', b' \geq 0 \) such that \( k' + \ell' + a' + b' = n \). Then it follows from the uniqueness part of the statement of Theorem 3.1 that the intersection of

\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \middle| -1 < s < t < 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}
\]

with

\[
\left\{ k' \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell' \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \middle| -1 \leq s \leq t \leq 1 \right\} + a' \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}
\]

and the boundary of \( \mathcal{A}_{3,n} \) is empty when \( (k', \ell', a', b') \neq (k, \ell, a, b) \). Next, take \(-1 < s_1 < t_1 < 1 \) and \(-1 < s_2 < t_2 < 1 \) such that

\[
k \begin{pmatrix} s_1 \\ s_1^2 \\ s_1^3 \end{pmatrix} + \ell \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} = k \begin{pmatrix} s_2 \\ s_2^2 \\ s_2^3 \end{pmatrix} + \ell \begin{pmatrix} t_2 \\ t_2^2 \\ t_2^3 \end{pmatrix}.
\]
Then one can check that \((s_1, t_1) = (s_2, t_2)\) using \(4 \times 4\) Vandermonde matrices. So the map

\[
\{(s, t) \mid -1 \leq s \leq t \leq 1\} \rightarrow \mathbb{R}^3, \\
(s, t) \mapsto k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ -1 \end{pmatrix},
\]

is injective. This together with the earlier statement implies the proposition. □

We are only interested in the set

\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \mid -1 \leq s \leq t \leq 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ -1 \end{pmatrix}
\]

for \(k, \ell \geq 1\) since it has dimension at most 1 otherwise. Recall that

\[
A_{k\ell} = k\ell(k + \ell)^2, \\
B_{k\ell}(x, y) = 2k\ell x(2x^2 - 3(k + \ell)y), \\
C_{k\ell}(x, y) = x^3 - 3(k + \ell)x^2y + 3(k^2 + k\ell + \ell^2)x^2y^2 - (k - \ell)^2(k + \ell)y^3, \\
D_{k\ell}(x, y) = (k + \ell)y - x^2, \\
f_{k\ell}(x, y, z) = A_{k\ell}z^2 + B_{k\ell}(x, y)z + C_{k\ell}(x, y)
\]

and \(B^2_{k\ell} - 4A_{k\ell}C_{k\ell} = 4k\ell(\ell - k)^2D^2_{k\ell}\) from Section 2. Our second goal for this section is to prove the following proposition and theorem.

**Proposition 4.2.** If \(k = \ell\), then

\[
f_{k\ell}(x, y, z) = A_{k\ell} \left( z + \frac{B_{k\ell}(x, y)}{2A_{k\ell}} \right)^2
\]

decomposes the polynomial \(f_{k\ell}\) into irreducible factors over \(\mathbb{Q}\). If \(k \neq \ell\), then \(f_{k\ell}\) is irreducible over \(\mathbb{C}\).

**Theorem 4.3.** The set

\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \mid -1 \leq s \leq t \leq 1 \right\}
\]

consists of all points

\[
(x, y, z) \in \left[ -(k + \ell), (k + \ell) \right] \times \left[ 0, (k + \ell) \right] \times \left[ -(k + \ell), (k + \ell) \right]
\]

such that \(f_{k\ell}(x, y, z) = 0\), the inequalities

\[
0 \leq k\ell D_{k\ell}(x, y) \leq k^2(k + \ell + x)^2, \ell^2(k + \ell - x)^2
\]

hold and in addition the following requirements are met:

- If \(k < \ell\), then the inequality \(z \leq -\frac{B_{k\ell}(x, y)}{2A_{k\ell}}\) must hold.
- If \(k = \ell\), then equation \(z = -\frac{B_{k\ell}(x, y)}{2A_{k\ell}}\) must hold.
- If \(k > \ell\), then the inequality \(z \geq -\frac{B_{k\ell}(x, y)}{2A_{k\ell}}\) must hold.
For the remainder of the section, we fix integers \( k, \ell \geq 1 \) and we write
\[
A = A_{k\ell}, B = B_{k\ell}, C = C_{k\ell}, D = D_{k\ell}, f = f_{k\ell}
\]
in order to simplify the used notation.

**Proof of Proposition 4.2.** The first statement is easy. Assume that \( k \neq \ell \). To prove that \( f \) is irreducible under this assumption, note that \( f \) is homogeneous with respect to the grading where \( \deg(x) = 1 \), \( \deg(y) = 2 \) and \( \deg(z) = 3 \). It follows that if \( f \) is reducible, then
\[
Az^2 + B(x, y)z + C(x, y) = f = A(z + ax^3 + bxy)(z + cx^3 + dxy)
\]
for some \( a, b, c, d \). However, this would imply that the coefficient
\[-(k - \ell)^2(k + \ell)\]
of \( C \) at \( y^3 \) equals 0. This is a contradiction. So \( f \) is irreducible. \( \square \)

**Proof of Theorem 4.3.** Note that we have \( x, z \in [-(k + \ell), (k + \ell)] \) and \( y \in [0, (k + \ell)] \) for all points
\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]
be a point and find out when it is contained in
\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \mid -1 \leq s \leq t \leq 1 \right\}.
\]
So we let
\[
(x, y, z) \in [-(k + \ell), (k + \ell)] \times [0, (k + \ell)] \times [-(k + \ell), (k + \ell)]
\]
be a point and find out when it is contained in
\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \mid -1 \leq s \leq t \leq 1 \right\}.
\]
We start by looking at the first two coordinates. So we solve the system of equations
\[
\begin{align*}
x &= ks + \ell t, \\
y &= ks^2 + \ell t^2
\end{align*}
\]
under the conditions that \(-1 \leq s \leq t \leq 1\). Solving the system, we find that
\[
(ks, \ell t) = \left( \frac{kx \pm \sqrt{k\ell D(x, y)}}{k + \ell}, \frac{tx \mp \sqrt{k\ell D(x, y)}}{k + \ell} \right).
\]
So we need to assume that \( k\ell D(x, y) \geq 0 \). Adding the condition \( s \leq t \), we get
\[
(ks, \ell t) = \left( \frac{kx - \sqrt{k\ell D(x, y)}}{k + \ell}, \frac{tx + \sqrt{k\ell D(x, y)}}{k + \ell} \right)
\]
and so the conditions \(-1 \leq s \) and \( t \leq 1 \) translate to
\[
\sqrt{k\ell D(x, y)} \leq k(k + \ell + x), \ell(k + \ell - x).
\]
As \( x \in [-(k + \ell), (k + \ell)] \), these conditions are equivalent to
\[
k\ell D(x, y) \leq k^2(k + \ell + x)^2, \ell^2(k + \ell - x)^2.
\]
Now, also consider the third coordinate \( z = ks^3 + \ell t^3 \). One can check that 
\( f(x, y, z) = 0 \). So if \( k = \ell \), then we have
\[
z = \frac{-B(x, y)}{2A}
\]
by Proposition 4.2 and we are done. So assume that \( k \neq \ell \). Then there are a priori two possibilities for \( z \) given \( x \) and \( y \). However, given \( s \) and \( t \), it becomes clear that only one possibility remains. So we just need to find an inequality that selects the correct root of \( f(x, y, \cdot) \). One can check that
\[
k^2\ell^2(k + \ell)^3 \left( z - \frac{-B(x, y)}{2A} \right) = (k^2 - \ell^2) \sqrt{k\ell D(x, y)}^3.
\]
So we find that \( z \leq \frac{-B(x, y)}{2A} \) when \( k < \ell \) and \( z \geq \frac{-B(x, y)}{2A} \) when \( k > \ell \). This concludes the proof. □

5. The boundary of \( A_{3,4} \) versus the boundary of \( A_{3,n} \)

By Theorem 3.1, we know that the boundary of \( A_{3,n} \) is contained in the union of the sets
\[
\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \middle| -1 \leq s \leq t \leq 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}
\]
over all integers \( k, \ell \geq 1 \) and \( a, b \geq 0 \) such that \( k + \ell + a + b = n \). The goal of this section is to eliminate unnecessary conjugands from this union and show that the boundary of \( A_{3,n} \) is contained in \( B_n^+ \cup B_n^- \). Our tools for doing so will be Theorem 3.1 and Proposition 4.1. We start with the case \( n = 4 \) and then use it to prove what we want for general \( n \geq 3 \). For \( n = 4 \), we need to eliminate the cases
\[
(k, \ell, a, b) = (1, 1, 1, 1), (1, 2, 1, 0), (2, 1, 0, 1), (2, 2, 0, 0).
\]
We start with the case \( k, \ell, a, b = (2, 2, 0, 0) \).

Lemma 5.1. Take \(-1 < s < t < 1\). Then the point
\[
p = 2(s, s^2, s^3) + 2(t, t^2, t^3)
\]
does not lie on the boundary of \( A_{3,4} \).

Proof. Consider the system of equations
\[
2s + 2t = t_1 + t_2 + t_3 + t_4,
2s^2 + 2t^2 = t_1^2 + t_2^2 + t_3^2 + t_4^2,
2s^3 + 2t^3 = t_1^3 + t_2^3 + t_3^3 + t_4^3
\]
with the additional conditions that \(-1 < t_1, t_2, t_3, t_4 < 1\) are pairwise distinct. If this system has a solution that satisfies the additional conditions, then the point \( p \) cannot lie on the boundary of \( A_{3,4} \) by Theorem 3.1. It turns out that such a solution
(t_1, t_2, t_3, t_4) can even be found when we assume that t_1 + t_2 = t_3 + t_4. Indeed, let 0 \neq \alpha \neq \beta \neq 0 be such that |\alpha|, |\beta| < 1 - \frac{1}{2}|s + t| and \alpha^2 + \beta^2 = \frac{1}{2}(s - t)^2. Then

\[(t_1, t_2, t_3, t_4) = \left( s + t + \alpha, s + t - \alpha, s + t + \beta, s + t - \beta \right) \]

is a solution to the system equalities so that \(-1 < t_1, t_2, t_3, t_4 < 1\) are pairwise distinct. One can check that |\alpha|, |\beta| < 1 - \frac{1}{2}|s + t| and \alpha^2 + \beta^2 = \frac{1}{2}(s - t)^2 for \alpha, \beta = \pm \frac{1}{2}(s - t). Here we use that |s - t| + |s + t| \leq 2 \cdot \max(|s|, |t|) < 2. It follows that for any point (\alpha, \beta) on the circle given by

\[\alpha^2 + \beta^2 = \frac{1}{2}(s - t)^2\]

that is sufficiently close to \((\frac{1}{2}(s - t), \frac{1}{2}(s - t))\) also satisfies these conditions. So to conclude the proof, we simply let (\alpha, \beta) be such a point with 0 \neq \alpha \neq \beta \neq 0. \quad \Box

From the lemma follows that

\[2 \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \frac{t}{t^3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]

is not contained in the boundary of \(A_{3,4}\). The next lemma takes care of the remaining cases.

**Lemma 5.2.** For each tuple \((k, \ell, a, b) \in \{(1,1,1,1), (1,2,1,0), (2,1,0,1)\}\), the set

\[\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \mid -1 \leq s \leq t \leq 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]

is not entirely contained in the boundary of \(A_{3,4}\).

**Proof.** We prove the lemma case by case and use Lemma 5.1

- For \((k, \ell, a, b) = (1, 1, 1, 1)\), take \(-1 < s < 1\) and \(t = \sqrt{(1 + s^2)/2}\). Then we see that the point

\[2 \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + 2 \begin{pmatrix} -t \\ t^2 \\ -t^3 \end{pmatrix} = \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \begin{pmatrix} -s \\ s^2 \\ -s^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]

does not lie on the boundary of \(A_{3,4}\).

- For \((k, \ell, a, b) = (1, 2, 1, 0)\), take \(-1 < s < 1\) and \(t = (s + 1)/2\). Then

\[\begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + 2 \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} u \\ u^2 \\ u^3 \end{pmatrix} + 2 \begin{pmatrix} v \\ v^2 \\ v^3 \end{pmatrix} \]

for

\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(s + 1) + \frac{1}{2\sqrt{2}}(1 - s) \\ \frac{1}{2}(s + 1) - \frac{1}{2\sqrt{2}}(1 - s) \end{pmatrix}
\]

and hence this point does not lie on the boundary of \(A_{3,4}\).

- For \((k, \ell, a, b) = (2, 1, 0, 1)\), take \(-1 < t < 1\) and \(s = (t - 1)/2\). Then we similarly find that the point

\[2 \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]

does not lie on the boundary of \(A_{3,4}\).
that the boundary of back to considering

Let Lemma 5.3.

gives us four cases to eliminate.

Then

\[
\begin{cases}
  k \left( \frac{s}{s^2} \right) + \ell \left( \frac{t^2}{t^3} \right) & -1 \leq s \leq t \leq 1 \\
\end{cases}
\]

is not contained in the boundary of \(A_{3,n}\).

Proof. Let \(-1 < s < t < 1\) be real numbers and let \(k', \ell' \geq 1\) and \(a', b' > 0\) be integers with sum \(n'\) such that \(k' \leq k, \ell' \leq \ell, a' \leq a\) and \(b' \leq b\). If the point

\[k'(s, s^2, s^3) + \ell'(t, t^2, t^3) + a'(1, 1, 1) + b'(-1, 1, -1)\]

does not lie on the boundary of \(A_{3,n'}\), then the point

\[k(s, s^2, s^3) + \ell(t, t^2, t^3) + a(1, 1, 1) + b(-1, 1, -1)\]

cannot lie on the boundary of

\[A_{3,n'} + (k - k')(s, s^2, s^3) + (\ell - \ell')(t, t^2, t^3) + (a - a')(1, 1, 1) + (b - b')(-1, 1, -1)\]

and hence it can also not lie on the boundary of \(A_{3,n}\). So it suffices to find such \(s, t, k', \ell', a'\) and \(b'\). We do this case by case.

1. By Lemma 5.1 it can be done with \((k', \ell', a', b') = (2, 2, 0, 0)\).
2. By Lemma 5.2 it can be done with \((k', \ell', a', b') = (1, 1, 1, 1)\).
3. By Lemma 5.2 it can be done with \((k', \ell', a', b') = (1, 2, 1, 0)\).
4. By Lemma 5.2 it can be done with \((k', \ell', a', b') = (2, 1, 0, 1)\).

We can now prove half of the statement of Theorem 2.1(a).

Lemma 5.4. The boundary of \(A_{3,n}\) is contained in the union of \(B^+_n\) and \(B^-_n\).

Proof. The lemma follows from Theorem 3.1 Proposition 4.1 and Lemma 5.3 \(\square\)

6. The sets \(B^+_n\) and \(B^-_n\)

In this final section, we prove Theorem 2.1 and the Main Theorem. Recall the following notation from Section 2.
We start by listing some properties of Proposition 6.1. Let $\pi$ and $\{\alpha_k\}$ be as defined in the proposition.

(a) The map

$$\alpha_k: \{(s,t) \mid -1 \leq s \leq t \leq 1\} \rightarrow \pi(C_{k,n-k-1}^+),$$

$$(s,t) \mapsto k \begin{pmatrix} s \\ s^2 \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix} + (n-k-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

is a bijection.

(b) The boundary of $\pi(C_{k,n-k-1}^+)$ is the union of the following three sets:

$$\{\alpha_k(-1,t) \mid -1 \leq t \leq 1\}, \{\alpha_k(s,s) \mid -1 \leq s \leq 1\}, \{\alpha_k(s,1) \mid -1 \leq s \leq 1\}.$$

(c) We have $\pi(B_n^+) = B_n^\alpha$.

(d) The projection map

$$B_n^+ \rightarrow B_n^\alpha,$$

$$(x,y,z) \mapsto (x,y),$$

is a bijection.

**Proof.** To see (a), note that the map clearly is surjective. For injectivity, one has to solve $\alpha_k(s,t) = (x,y)$ for $s,t$ under the condition that $s \leq t$. This yields at most one solution for all $(x,y)$. For (b), note that the Jacobian of the map $\alpha_k$ has full rank at all points $(s,t)$ with $-1 < t < s < 1$. From (b) follows that the boundary of $\pi(B_n^+)$ is the union of

$$\begin{cases} n \begin{pmatrix} s \\ s^2 \end{pmatrix} \mid -1 \leq s \leq 1 \end{cases}$$

and

$$\begin{cases} i \begin{pmatrix} -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ t^2 \end{pmatrix} + (n-i-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid -1 \leq t \leq 1 \end{cases}$$

for $i = 0, \ldots, n-1$. So the set itself is indeed given by the inequalities defining $B_n^\alpha$.

Finally, to see (d), it suffices to note that $\pi(C_{k,n-k-1}^+ \cap C_{k+1,n-k}^+)$ is equal to

$$\begin{cases} (k+1) \begin{pmatrix} s \\ s^2 \end{pmatrix} + (n-k-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid -1 \leq s \leq 1 \end{cases}$$

for $k = 1, \ldots, n-2$. \qed
Proposition 6.2. Let $1 \leq \ell \leq n - 1$ be an integer.

(a) The map 
\[ \beta_\ell: \{(s, t) \mid -1 \leq s \leq t \leq 1\} \to \pi(C^-_{\ell, n-\ell-1}), \]
\[ (s, t) \mapsto \left(\frac{s}{s^2} + \ell \left(\frac{t}{t^2}\right) + (n - \ell - 1)\left(-1\right)\right), \]

is a bijection.

(b) The boundary of $\pi(C^-_{\ell, n-\ell-1})$ is the union of the following three sets:
\[ \{\beta_\ell(-1, t) \mid -1 \leq t \leq 1\}, \{\beta_\ell(t, t) \mid -1 \leq t \leq 1\}, \{\beta_\ell(s, 1) \mid -1 \leq s \leq 1\}. \]

(c) We have $\pi(B^-_n) = B^+_n$.

(d) The projection map 
\[ B^-_n \to B^+_n, \]
\[ (x, y, z) \mapsto (x, y), \]

is a bijection.

Proof. The proofs are similar to those of Proposition 6.1. \[\square\]

The decomposition of $B^+_n$ as a union of the projections of $C^+_{1, n-2}, \ldots, C^+_{n-1, 0}$ is visualized in Figure 2. We note that the decomposition of $B^-_n$ as a union of the projections of $C^-_{1, n-2}, \ldots, C^-_{n-1, 0}$ looks similar but is mirrored along the vertical axis.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The set $B^+_n = \pi(C^+_{1, n-2}) \cup \cdots \cup \pi(C^+_{n-1, 0})$.}
\end{figure}

We can now prove Theorem 2.1.

Proof of Theorem 2.1. We already know that (b) holds by Propositions 6.1 and 6.2. We know that $B^+_n, B^-_n \subset A_{3,n}$, we know that the boundary of $A_{3,n}$ is contained in $B^+_n \cup B^-_n$ by Lemma 5.4 and we know that the projection maps 
\[ B^+_n \to B^+_n, \]
\[ (x, y, z) \mapsto (x, y), \]
\[ B^-_n \to B^-_n, \]
\[ (x, y, z) \mapsto (x, y), \]

are bijections by Propositions 6.1 and 6.2. Together these statements imply (a). Let $(x, y) \in B^+_n$ be a point. Then there exist unique numbers $z^+, z^- \in \mathbb{R}$ such that
Suppose that $z^+ = z^-$. Then
\[
\begin{pmatrix}
x
y
z
\end{pmatrix} = k \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} + \ell \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]
for some integers $k, \ell \geq 1$ and $a, b \geq 0$ with $k + a = \ell + b = n - 1$ and numbers $-1 \leq s_1, t_1 \leq 1$ and $-1 \leq s_2, t_2 \leq 1$. From (c) and Theorem 3.1 follow that
\[
\begin{pmatrix}
x
y
z
\end{pmatrix} = n \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \text{ or } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = i \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + (n - i - 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
for some $-1 \leq t \leq 1$ and $i \in \{0, \ldots, n - 1\}$. To see this, split into cases where $a, b$ are zero or non-zero. So we see that $z^+ = z^-$ if and only if $(x, y)$ lies on the boundary of $\mathcal{B}_n$. This implies in particular that either $z^+ \geq z^-$ for all $(x, y) \in \mathcal{B}_n$ or $z^+ \leq z^-$ for all $(x, y) \in \mathcal{B}_n$. To see that the former is the case, consider the point $(x, y) = (0, 1) \in \mathcal{B}_n$. The equations
\[
\begin{pmatrix}
0 \\
1 \\
z^+
\end{pmatrix} = k \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} + \ell \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},
\]
\[
\begin{pmatrix}
x \\
y \\
z^-
\end{pmatrix} = n \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
for integers $k, \ell, a, b \geq 0$ such that $k + a = \ell + b = n - 1$ and numbers
\[
-1 \leq s_1, t_1 \leq 1,
\]
\[
-1 \leq s_2, t_2 \leq 1
\]
yield $a = b = 0$, $k = \ell = n - 1$ and
\[
s_1 = -\frac{1}{\sqrt{n(n-1)}}, \quad s_2 = -(n - 1)t_2,
\]
\[
t_1 = -(n - 1)s_1, \quad t_2 = \frac{1}{\sqrt{n(n-1)}}
\]
and hence
\[
z^+ = (n - 1)s_1^3 + t_1^3 > 0 > s_2^3 + (n - 1)t_2^3 = z^-
\]
for $(x, y) = (0, 1)$. So $z^+ \geq z^-$ for all $(x, y) \in \mathcal{B}_n$. \qed

Finally, we use Theorem 2.1 to prove the Main Theorem.

**Proof of Main Theorem.** Let $(x, y, z) \in \mathbb{R}^3$ be a point. Then $(x, y, z) \in \mathcal{A}_{3,n}$ if and only if $(x, y) \in \mathcal{B}_n$ and $z^+ \geq z \geq z^-$ where $z^+, z^- \in \mathbb{R}$ are such that $(x, y, z^+) \in \mathcal{B}_n^+$ and $(x, y, z^-) \in \mathcal{B}_n^-$. The following are equivalent:

(a) We have $(x, y) \in \mathcal{B}_n^+$.

(b) We have $(x, y) \in \pi(\mathcal{C}_{k,n-k-1})$ for some $k \in \{1, \ldots, n - 1\}$.

(c) We have $(x, y) \in \pi(\mathcal{C}_{\ell,n-\ell-1})$ for some $\ell \in \{1, \ldots, n - 1\}$.

Take $k, \ell \in \{1, \ldots, n - 1\}$. Then, using Lemma 6.1 we see that
\[
\pi(\mathcal{C}_{k,n-k-1}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \left| \begin{array}{l}
y \leq n - 1 + (x + k - (n - k - 1))^2 \\
y \geq n - k - 1 + (k + 1)^{-1}(x - (n - k - 1))^2 \\
y \leq n - k + k^{-1}(x - (n - k))^2
\end{array} \right. \right\}
\]
and we similarly get
\[
\pi(C^\ell_{n-k-1}) = \begin{cases} 
(x, y) \in \mathbb{R}^2 & y \leq n - 1 + (x - k + (n - k - 1))^2 \\
y \geq n - k - 1 + (\ell + 1)^{-1}(x + (n - k - 1))^2 & y \leq n - k + (n - k)^2 
\end{cases}
\]
using Lemma 6.2. Assume that \((x, y) \in B^\ell_n\) and that \(k, \ell\) are as in (b) and (c). Then
\[
f_{k1}(x - (n - k - 1), y - (n - k - 1), z^+ - (n - k - 1)) = 0,
\]
\[
(n - k - 1) + \frac{-B_{k1}(x - (n - k - 1), y - (n - k - 1))}{2A_{k1}} \leq z^+
\]
by Theorem 4.3. So \(z \leq z^+\) when
\[
z \leq (n - k - 1) + \frac{-B_{k1}(x - (n - k - 1), y - (n - k - 1))}{2A_{k1}} =: \theta
\]
or
\[
f_{k1}(x - (n - k - 1), y - (n - k - 1), z - (n - k - 1)) \leq 0.
\]
Note here that the polynomial \(f_{k1}(x, y, -)\) has degree 2 in \(z\), that its leading coefficient is positive, that \(z^+\) is its highest root and that it attains its minimum at \(\theta\). This is visualized in Figure 3. We also have
\[
f_{1\ell}(x + (n - \ell - 1), y - (n - \ell - 1), z^- + (n - \ell - 1)) = 0,
\]
\[
-(n - \ell - 1) + \frac{-B_{1\ell}(x + (n - \ell - 1), y - (n - \ell - 1))}{2A_{1\ell}} \geq z^-
\]
by Theorem 4.3 and from this we conclude that \(z \geq z^-\) if and only if
\[
z \geq -(n - \ell - 1) + \frac{-B_{1\ell}(x + (n - \ell - 1), y - (n - \ell - 1))}{2A_{1\ell}}
\]
or
\[
f_{1\ell}(x + (n - \ell - 1), y - (n - \ell - 1), z + (n - \ell - 1)) \leq 0.
\]
This leads to the semi-algebraic description of the Main Theorem. \(\square\)

---

**Figure 3.** Visualization of the condition \(z \leq z^+\) in the proof of the Main Theorem. The parabola represents the function sending \(z\) to \(f_{k1}(x - (n - k - 1), y - (n - k - 1), z - (n - k - 1))\).
7. Higher Dimensions

The Main Theorem provides a semi-algebraic description of the set $A_{3,n}$ for each integer $n \geq 3$. So, a natural question to ask is: can we use the same proof strategy to find a semi-algebraic description of the sets $A_{k,n}$ for $k > 3$? At the moment, there still are some obstacles to doing so, which we will discuss in this section.

Following the same strategy as for $k = 3$, we would again start by trying to find a description of the boundary of $A_{k,n}$. One can check that the statement and proof of Theorem 3.1 carry over in a straightforward fashion for $k > 3$, which yields a superset of the boundary. However, the proof of Proposition 4.1 does not directly generalize since injectivity of the parametrization map is not obvious. But, if the proposition still holds, the main obstacle to overcome is, in our opinion, finding an analogue of Theorem 4.3. For $k = 4$, this means we need to solve the following problem.

**Problem 7.1.** Determine a semi-algebraic description of the set

$$
\left\{ \ell_1 \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \\ t_1^4 \end{pmatrix} + \ell_2 \begin{pmatrix} t_2 \\ t_2^2 \\ t_2^3 \\ t_2^4 \end{pmatrix} + \ell_3 \begin{pmatrix} t_3 \\ t_3^2 \\ t_3^3 \\ t_3^4 \end{pmatrix} \mid -1 \leq t_1 \leq t_2 \leq t_3 \leq 1 \right\}
$$

given the integers $\ell_1, \ell_2, \ell_3 \geq 1$.

These sets are expected to be the building blocks for the boundary of $A_{4,n}$, so a solution to this problem seems essential if we want to apply the same approach we used for $A_{3,n}$. Using elimination theory, we find that the Zariski closure of this set is a hypersurface defined by a single polynomial $f_{\ell_1, \ell_2, \ell_3}(x_1, x_2, x_3, x_4)$. This polynomial is homogeneous of degree 24 with respect to the grading where $\deg(x_i) = i$ and has 169 terms. Its coefficients are symmetric polynomials in $\ell_1, \ell_2, \ell_3$ of degree up to 18. When $\#(\ell_1, \ell_2, \ell_3) \leq 2$, the polynomial is a square. And, we have

$$f_{\ell_1, \ell_2, \ell_3}(x_1, x_2, x_3, x_4) = (-x_1^4 + 6\ell_1 x_2^2 x_2 - 3\ell_2 x_2^2 - 8\ell_3 x_2 x_3 + 6\ell_3 x_4)^6.$$

This suggests that we should first solve

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \ell_1 \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \\ t_1^4 \end{pmatrix} + \ell_2 \begin{pmatrix} t_2 \\ t_2^2 \\ t_2^3 \\ t_2^4 \end{pmatrix} + \ell_3 \begin{pmatrix} t_3 \\ t_3^2 \\ t_3^3 \\ t_3^4 \end{pmatrix}
$$

for $t_1, t_2, t_3$ and then solve $f_{\ell_1, \ell_2, \ell_3}(x_1, x_2, x_3, x_4) = 0$ for $x_4$. As this only involves solving polynomial equations of degree $\leq 4$, this is theoretically doable. The problem however is to express the inequalities $-1 \leq t_1 \leq t_2 \leq t_3 \leq 1$ as polynomial inequalities in $x_1, x_2, x_3, x_4$.

As an example, consider the case $\ell_1 = \ell_2 = \ell_3 = 1$. In this case, the set is contained in the hypersurface given by the equation

$$x_1^4 - 6x_1^2 x_2 + 3x_2^2 + 8x_1 x_3 - 6x_4 = 0,$$

which allows to eliminate the coordinate $x_4$. So here, the problem consists of finding a semi-algebraic description of the set

$$
\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} + \begin{pmatrix} t_2 \\ t_2^2 \\ t_2^3 \end{pmatrix} + \begin{pmatrix} t_3 \\ t_3^2 \\ t_3^3 \end{pmatrix} \mid -1 \leq t_1 \leq t_2 \leq t_3 \leq 1 \right\}
$$
given \( x_1, x_2, x_3 \in \mathbb{R} \).

If we can solve Problem 7.1, we still need to find analogues for the results in Sections 5 and 6. For the results of Section 5, one might hope that the projection map which forgets the last coordinate again plays an important role. As for the results of Section 6, these results relied on our complete understanding of the roots and extrema of parabolas. So to generalize these results, we probably need a similar level of understanding in the cases of cubics and quartics, which for now seems out of reach.

**References**

[1] A. Chang, D. Mehrle, S. J. Miller, T. Reiter, J. Stahl, D. Yott, *Newman’s conjecture in function fields*, J. Number Theory 157 (2015), pp. 154–169.

[2] Z.-L. Dou, Q. Zhang, *Six Short Chapters on Automorphic Forms and L-functions*, Springer-Verlag Berlin Heidelberg (2012).

[3] P.-C. Hu, A.-D. Wu, *Zero distribution of Dirichlet L-functions*, Ann. Acad. Sci. Fenn. Math. 41 (2016), pp. 775–788.

[4] G. L. Miller, *Riemann’s Hypothesis and Tests for Primality*, J. Comput. Syst. Sci. 13 (1976), no. 3, pp. 300–317.

[5] S. J. Miller, *An orthogonal test of the L-functions Ratios conjecture*, Proc. London Math. Soc. 99 (2009), no. 2, pp. 484–520.

[6] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Größe*, Monatsberichte der Berliner Akademie (November 1859).

[7] M. O. Rubinstein, P. Sarnak, *The underdetermined matrix moment Problem I*, in preparation.