DESIGN VIA (3, 3)-ISOGENY ON JACOBIANS OF GENUS 2 CURVES

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Abstract. We give an invariant of the curve $C$ of genus 2 with a maximal isotropic $(\mathbb{Z}/3)^2$ in $J[3]$, where $J$ is the Jacobian variety of $C$, and develop the theory required to perform descent via (3, 3)-isogeny. We apply this to several examples, where it can shown that non-reducible Jacobians have nontrivial 3-part of the Tate-Shafarevich group.

1. Introduction

In this article we consider curves $C$ of genus 2 over a field $k$ of characteristic not 2 or 3, that have special structure in the 3-torsion of their Jacobians $J$. In particular, we consider the situation where $J[3](k)$ contains a group $\Sigma(k)$ of order 9. As we show in Section 2, such a curve $C$ can be given by a model of the form

$$y^2 = F(x) = G(x)^2 + \lambda H(x)^3,$$

where $G(x)$ is cubic and $H(x)$ is quadratic in $x$. The divisor $D = [(x, G(x)) + (x', G(x')) - \kappa]$, where $x, x'$ are the roots of $H(x)$ and $\kappa$ is a canonical divisor, represents a point in $J[3](k)$, as can be seen from the fact that $3D$ is linearly equivalent to the divisor of the function $y - G(x)$.

The curves of interest to us can be expressed as such a model in several ways. As we show in Lemma 3, the Weil pairing of the 3-torsion points can be easily expressed in terms of the corresponding polynomials $G(x)$ and $H(x)$. This allows us to fully describe the genus 2 curves that have a subgroup $\Sigma(k) \subset J[3](k)$ of size 9 on which the Weil pairing is trivial.

In Section 3 we phrase the question of classification of such curves in terms of partial level structures on principally polarized abelian surfaces. The relevant moduli space is $A_2(\Sigma)$, with $\Sigma = \mathbb{Z}/3 \times \mathbb{Z}/3$. In Section 4 we determine a genus 2 curve $C_{rst}$ over $k(r,s,t)$ such that a sufficiently general curve of the type we are interested in, can be obtained by specializing $r,s,t$. Our construction identifies $k(r,s,t)$ with the function field of the moduli space $A_2(\Sigma)$, thereby giving a particularly explicit proof of its rationality. Furthermore, we observe that if $J$ is equipped with a $\Sigma$-level structure, then the quadratic twist $J^{(d)}$ is naturally equipped with a $\Sigma^{(d)}$ level structure. Thus we find that $A_2(\Sigma)$ and $A_2(\Sigma^{(d)})$ are naturally isomorphic.

Since $\Sigma \subset J[3]$ is maximally isotropic with respect to the Weil pairing, we have that $\bar{J} = J/\Sigma$ is also principally polarized and in fact has a $\Sigma^{\vee}$-level structure. In Section 5 we identify a curve $\bar{C}_{rst}$ such that its Jacobian $\bar{J}_{rst}$ is $J_{rst}/\Sigma$. In fact, we observe that $\Sigma^{\vee} = \Sigma^{(-3)}$ and hence that the quadratic twist $\bar{C}_{rst}^{(-3)} \simeq C_{r's't'}$, where $(r', s', t') = \psi_0(r, s, t)$ is a birational transformation that we explicitly determine. While the final formulas we find are quite manageable, the proof that they are correct requires some significant computation.
In Theorem \[13\] and Corollary \[14\] we observe some remarkable relations between the models for \(C_{rst}\) and \(\tilde{C}_{rst}\). We also identify the natural action of \(\text{PGL}_2(\mathbb{F}_3)\) on \(\mathcal{A}_2(\Sigma)\), as well as the involution \(C_{rst} \to \tilde{C}_{rst}^{(-3)}\), as automorphisms of \(k(r,s,t)\).

In Section \[8\] we get to the original motivation of this paper. If we take \(J\) to be a Jacobian of a genus 2 curve with a \(\Sigma\) level structure over a number field \(k\), then it is particularly easy to compute interesting information about \(J(k)\) and \(\text{III}(J/k)[3]\) via \((3,3)\)-isogeny descent, using ideas from \[12\] and \[5\]. This allows us to give examples of various absolutely simple abelian surfaces with interesting structures in \(\text{III}(J/k)[3]\), see Examples \[19\] \[20\] \[21\]

2. Three-torsion on genus two Jacobians

Let \(k\) be a field of characteristic different from 2,3 and let \(C\) be a smooth projective curve of genus 2 over \(k\) given by an affine model

\[C: y^2 = F(x),\]

where \(F(x) \in k[x]\) is of degree 6 (this only (mildly) restricts the admissible \(C\) if \(k\) is a finite field of 5 elements). Let \(J\) be the Jacobian of \(C\). The group \(J(k)\) is isomorphic to the group of divisor classes of \(k\)-rational degree 0 divisors on \(C\). Since \(C\) is of genus 2, every degree 0 class contains a representative of the form

\[D - \kappa,\]

where \(D\) is an effective divisor of degree 2 and \(\kappa\) is an effective canonical divisor. Furthermore, for any non-principal class, the divisor \(D\) is unique. For \(\kappa\) we have choice, since effective canonical divisors are exactly the fibers of the hyperelliptic double cover \(x: C \to \mathbb{P}^1\). We write \(\iota: C \to C\) for the hyperelliptic involution, i.e., \(\iota(x,y) = (x,-y)\).

**Lemma 1.** Let \(\Sigma \subset \text{Pic}(C/k)[3]\) be a subgroup of size 9. Then \(\Sigma\) can be generated by a pair of divisors

\[D_1 - \kappa_1, D_2 - \kappa_2\]

where \(D_1, D_2\) are effective divisors of degree 2 and \(\kappa_1, \kappa_2\) are effective canonical divisors, with the supports of \(D_1, D_2, \kappa_1, \kappa_2\) pairwise disjoint. A fortiori, we can ensure that \(x_*(D_1)\) and \(x_*(D_2)\) are disjoint.

**Proof.** First, note that we can choose \(\kappa_1\) and \(\kappa_2\) to be any fiber of \(x: C \to \mathbb{P}^1\) over points in \(\mathbb{P}^1(k)\). Since \(k\) has characteristic 0 or at least 5, we know that \#(\mathbb{P}^1(k)) \geq 6. Since \(x_*(D_1 + D_2)\) is supported on at most 4 rational points, we can choose \(\kappa_1, \kappa_2\) with disjoint support from \(D_1, D_2\).

It remains to show that we can choose \(D_1, D_2\) with disjoint support. Since these divisors are uniquely determined, we lose no generality by assuming that \(k\) is algebraically closed. Therefore, we assume that \(\Sigma\) is generated by the classes

\[T_1 = [P_1 + Q_1 - \kappa_1] \text{ and } T_2 = [P_2 + Q_2 - \kappa_2],\]

where \(P_1, Q_1, P_2, Q_2 \in C(k)\). We want to ensure that \(\{P_1, \iota P_1, Q_1, \iota Q_1\}\) and \(\{P_2, \iota P_2, Q_2, \iota Q_2\}\) are disjoint. If they are not, we can assume that \(P_1 = P_2\) and it follows that

\[T_3 = T_1 - T_2 = [Q_1 + \iota Q_2 - \kappa].\]

A straightforward computation shows that if \(T_3 = T_1 + T_2 = [P_4 + Q_4 - \kappa]\) has \(P_4 = P_1\) or \(P_4 = \iota P_1\) then either \(T_4 = [2P_1 - \kappa]\) and \(T_3 = [2Q_1 - \kappa]\), so that choosing \(T_3, T_4\) works, or \((T_1, T_2)\) is not of size 9. Therefore, one of the choices \((T_1, T_2)\) or \((T_1, T_3)\) or \((T_3, T_4)\) satisfies our criteria.

**Example 2.** At this point it may be worth noting that rather exceptional configurations of support for 3-torsion do occur. For instance, for

\[C: y^2 = x^6 + rx^3 + 1\]
we see that $T_1 = [(0,1) + \infty^+ - \kappa]$ and $T_2 = [(0,1) + \infty^- - \kappa]$ are 3-torsion points. In fact, it is straightforward to check that any genus 2 Jacobian with two independent 3-torsion points such that the group generated by those 3-torsion points is supported on only 4 points of the curve must be isomorphic to a curve of this form.

**Lemma 3.** Let $C$ be a curve of genus 2. Then $\text{Pic}(C/k)[3]$ has a subgroup $\Sigma$ of size 9 if and only if $C$ admits a model of the form
\[
C: y^2 = F(x) = G_1(x)^2 + \lambda_1 H_1(x)^3 = G_2(x)^2 + \lambda_1 H_2(x)^3,
\]
where $H_1, H_2, G_1, G_2, F \in k[x]$ and $\lambda_1, \lambda_2 \in k^\times$ and $H_1, H_2$ are of degree 2 and $\gcd(H_1, H_2) = 1$.

**Proof.** Suppose that $T \in \text{Pic}(C/k)[3]$ is non-trivial. We assume that $T = [D - \kappa_\infty]$, where $\kappa_\infty$ is the fiber above $x = \infty$ and $D$ is an effective divisor with support disjoint from $\kappa_\infty$. Then $x_\ast(D)$ can be described by $H(x) = 0$, where $H(x) \in k[x]$ is a quadratic monic polynomial. Since $3T$ is the principal class, there is a function $g \in k(C)$ such that
\[
\text{div}(g) = 3D - 3\kappa_\infty
\]
and it is straightforward to check that we must have $g = y - G(x)$ for some $G(x) \in k[x]$, with $\deg(G) \leq 3$. It follows that
\[
y^2 = F(x) = G(x)^2 + \lambda H(x)^3,
\]
and conversely, that any such decomposition of $F(x)$ gives rise to a 3-torsion point $T = [D - \kappa]$, where $D$ is the effective degree 2 divisor described by the vanishing of $\{y - G(x), H(x)\}$. The class $2T = -T$ is then described by the vanishing of $\{y + G(x), H(x)\}$.

The existence of $\Sigma$ as stated in the lemma would lead to 4 decompositions of the type described and simple combinatorics shows that not all quadratic polynomials $H(x)$ featuring in them can be equal. This proves the lemma. \qed

The torsion subgroup scheme $J[3]$ comes equipped with a non-degenerate, bilinear, alternating Weil pairing
\[
e_3: J[3] \times J[3] \to \mu_3,
\]
where $\mu_3$ is the group scheme representing the cube roots of unity.

We say that a subgroup $\Sigma \subset J[3]$ is *isotropic* if $e_3$ restricts to the trivial pairing on $\Sigma$. If $\Sigma$ is of degree 9 then $\Sigma$ is *maximal isotropic*, meaning $\Sigma$ is not properly contained in an isotropic subgroup. The nondegeneracy of $e_3$ then induces an isomorphism $J[3]/\Sigma \to \Sigma^\vee = \text{Hom}(\Sigma, \mu_3)$. In fact, we have a direct sum decomposition $J[3] = \Sigma \times \Sigma^\vee$.

In particular, if $\Sigma = \mathbb{Z}/3 \times \mathbb{Z}/3$ then we have $\Sigma^\vee = \mu_3 \times \mu_3$.

**Lemma 4.** Let
\[
C: y^2 = F(x) = G_1(x)^2 + \lambda_1 H_1(x)^3 = G_2(x)^2 + \lambda_2 H_2(x)^3
\]
be a genus 2 curve with $H_1, H_2$ quadratic monic polynomials and $H_1 \neq H_2$. For $i \in \{1, 2\}$, let $D_i = \{y - G_i(x), H_i(x)\}$ and let $T_i = [D_i - \kappa] \in \text{Pic}(C/k)[3]$. Then
\[
e_3(T_1, T_2) = \frac{\lambda_2 \text{Res}(G_2 - G_1, H_2)}{\lambda_1 \text{Res}(G_1 - G_2, H_1)}
\]

**Proof.** We choose canonical divisors $\kappa_1$ and $\kappa_2$ above $x = r_1$ and $x = r_2$ respectively, such that the divisors $D_1 - \kappa_1$ and $D_2 - \kappa_2$ have disjoint support. We have the functions
\[
g_i = \frac{y - G_i}{(x - r_i)^3} \quad \text{with} \quad \text{div}(g_i) = 3D_i - 3\kappa_i.
\]
We can compute the pairing via
\[ e_3(T_1, T_2) = \frac{g_1(D_2 - \kappa_2)}{g_2(D_1 - \kappa_1)}, \]
where evaluating a function on a divisor is defined as \( g(\sum n_P P) = \prod g(P)^{n_P}. \) Evaluating \( y - G_1(x) \) at \( D_2 \) means evaluating \( G_2(x) - G_1(x) \) at the roots of \( H_2(x) \), yielding \( \text{Res}(G_2 - G_1, H_2). \) Evaluating \( (x - r_1) \) at \( D_2 \) yields \( H_2(r_1). \) Noting that \( \kappa_2 = (r_2, \sqrt{F(r_2)}) + (r_2, -\sqrt{F(r_2)}) \), we see that
\[ g_1(\kappa_2) = \frac{G_1(r_2)^2 - F(r_2)}{(r_2 - r_1)^6} = \frac{-H_1(r_2)^3}{(r_2 - r_1)^6} \]
and hence that
\[ g_1(D_2 - \kappa_2) = \frac{\text{Res}(G_2 - G_1, H_2)(r_2 - r_1)^6}{-\lambda_1 H_1(r_2)^3 H_2(r_1)^3}. \]
Symmetry yields the result stated in the lemma.

We can characterize when \( e_3(T_1, T_2) = 1 \) in terms of the polynomials \( G_i, H_i \) in the following way. First note that
\[ G_2^2 - G_1^2 = \lambda_1 H_1^3 - \lambda_2 H_2^3. \]
Writing \( \alpha_1, \alpha_2, \alpha_3 \) for the cube roots of \( \lambda_2/\lambda_1 \) in an algebraic closure of \( k \), we find that
\[ (G_2 - G_1)(G_2 + G_1) = \lambda_1 (H_1 - \alpha_1 H_2)(H_1 - \alpha_2 H_2)(H_1 - \alpha_3 H_2). \]
It follows that the roots of the quadratic polynomials \( H_1 - \alpha_i H_2 \) are the same as the roots of \( G_2 - G_1 \) and \( G_2 + G_1 \). The way in which they distribute determines the Weil pairing.

**Lemma 5.** The pairing \( e_3(T_1, T_2) = 1 \) if and only if none of the polynomials \( (H_1 - \alpha_i H_2) \) divide \( G_2 - G_1 \).

**Proof.** First suppose that \( G_2 - G_1 = L_1(H_1 - \alpha_1 H_2) \). Then
\[ e_3(T_1, T_2) = \frac{\lambda_2 \text{Res}(G_2 - G_1, H_2)}{\lambda_1 \text{Res}(G_1 - G_2, H_1)} = \frac{\lambda_2 \text{Res}((H_1 - \alpha_1 H_2)L_1, H_2)}{\lambda_1 \text{Res}((H_1 - \alpha_1 H_2)L_1, H_1)} = \frac{\lambda_2 \text{Res}(L_1, H_2)}{\lambda_1 \alpha_1^2 \text{Res}(L_1, H_1)}. \]
Observe that \( L_1 \) must divide one of the other factors \( (H_1 - \alpha_j H_2), \) say for \( j = 2 \). Therefore, \( \text{Res}(L_1, H_1) = \text{Res}(L_1, \alpha_2 H_2) \). Since \( \text{deg}(L_1) = 1 \) and \( \lambda_2/\lambda_1 = \alpha_3^d \) we obtain
\[ e_3(T_1, T_2) = \alpha_1 \frac{\text{Res}(L_1, H_2)}{\text{Res}(L_1, H_1)} = \alpha_1 \frac{\text{Res}(L_1, H_2)}{\text{Res}(L_1, \alpha_2 H_2)} = \frac{\alpha_1}{\alpha_2}, \]
which is indeed a primitive cube root of unity.

In the remaining situation we have \( G_2 - G_1 = L_1 L_2 L_3 \), where, for \( i \in \{1, 2, 3\} \), the polynomial \( L_i \) divides \( H_1 - \alpha_i H_2 \), and hence \( \text{Res}(L_i, H_1) = \text{Res}(L_i, \alpha_i H_2) \). We obtain that
\[ \text{Res}(G_2 - G_1, H_1) = \alpha_1 \alpha_2 \alpha_3 \text{Res}(L_1 L_2 L_3, H_2) = \frac{\lambda_2}{\lambda_1} \text{Res}(G_2 - G_1, H_2), \]
which indeed implies that \( e_3(T_1, T_2) = 1 \).

**3. Level structure on principally polarized abelian surfaces**

Let \( n \) be a positive integer. We need the analogues of the modular curves \( Y(n), Y_1(n) \) and \( Y_0(n) \) for genus 2 Jacobians. The theory is most conveniently stated in terms of slightly more general objects, namely \textit{principally polarized abelian surfaces} (PPAS). These include direct products of elliptic curves, equipped with the product polarization. The advantage is that the category of PPAS is closed under polarized isogenies.

Let \( k \) be a field of characteristic not dividing \( 6n \). We write \( \mathcal{A}_2(1) \) for the moduli space of isomorphism classes of PPAS over \( k \). The coarse moduli space \( \mathcal{A}_2(1) \) is a 3-dimensional variety. Note
however that, for any extension $L$ of $k$, the set $A_2(1)(L)$ of $L$-rational points of $A_2(1)$ corresponds to $L$-rational isomorphism classes that need not contain an $L$-rational abelian surface; similarly two abelian surfaces defined over $L$ that are isomorphic over $\mathbb{F}_L$ need not be isomorphic over $L$.

Let $J$ be a principally polarized abelian surface over $k$. Then $J[n](\overline{k})$ has a non-degenerate bilinear alternating Galois covariant Weil-pairing $J[n] \times J[n] \rightarrow \mu_n$. A partial level $n$ structure on $J$ consists of a finite étale group scheme $\Sigma$ with a pairing $\Sigma \times \Sigma \rightarrow \mu_n$ and an injective homomorphism $\Sigma \rightarrow J[n]$ that is compatible with the pairings. An isomorphism between $(J, \Sigma \rightarrow J[n])$ and $(J', \Sigma \rightarrow J'[n])$ is an isomorphism $\phi : J \rightarrow J'$ of PPAS such that the composition of $\Sigma \rightarrow J[n]$ with $\phi$ yields $\Sigma \rightarrow J'$. We write $A_2(\Sigma)$ for the moduli space of PPAS equipped with a partial level $n$-structure involving $\Sigma$. If we take for example a sample abelian surface $J_0$ and set $\Sigma = J_0[n]$ then $A_2(\Sigma)$ is the moduli space of PPAS with full level $n$ structure.

We will work with the case $n = 3$ and $\Sigma = \mathbb{Z}/3 \times \mathbb{Z}/3$ with trivial pairing (i.e., $\Sigma$ is isotropic with respect to its pairing). Note that the automorphism group of $\Sigma$ is the full $GL_2(\mathbb{F}_3)$. However, since on every abelian surface $J$ the involution $-1 : J \rightarrow J$ induces the automorphism $-1$ on level structures, we deduce that $(J, \Sigma \rightarrow J)$ and $(J, -\Sigma \rightarrow J)$ are isomorphic as level $n$ structures on $J$. Therefore we obtain an action of $PGL_2(\mathbb{F}_3)$ on $A_2(\Sigma)$.

In our case $\Sigma \subset J[3]$ is maximal isotropic, so the nondegeneracy of $e_3$ yields that $J[3]/\Sigma \simeq \Sigma' = Hom(\Sigma, \mu_3)$. The fact that $\Sigma$ is maximal isotropic also means that the principal polarization on $J$ induces a principal polarization on the isogenous abelian surface $J/\Sigma$. Furthermore, the Weil pairing determines an injection $\Sigma' \rightarrow (J/\Sigma)[3]$. Thus we see that our maximal isotropic level structure leads to an isogeny $J \rightarrow J/\Sigma$, inducing an isomorphism $A_2(\Sigma) \rightarrow A_2(\Sigma')$, whose inverse is induced by the dual isogeny, using the principal polarizations to identify $J$ and $J/\Sigma$ with their duals.

The negation automorphism on $J$ also gives rise to a quadratic twisting operation. We write $J^{(d)}$ for the twist of $J$ by the quadratic character of discriminant $d$. A level structure under twisting gives rise to a twisted level structure $\Sigma^{(d)} \rightarrow J^{(d)}$, where $\Sigma^{(d)}$ is the quadratic twist of $\Sigma$. This gives rise to an isomorphism $A_2(\Sigma) \rightarrow A_2(\Sigma^{(d)})$.

We finish by making some observations about the covering degrees of the various moduli spaces of level structures we have introduced. Let us write $A_2(3)$ for the space corresponding to full level 3 data (say, for $\mathbb{Z}/3 \times \mathbb{Z}/3 \times \mu_3 \times \mu_3$ with obvious pairing). This has a full $PSp_4(\mathbb{F}_3)$ acting on it. The subgroups fixing a maximal isotropic space are all conjugate and have the group structure $(\mathbb{Z}/3)^3$. There are 40 of them. We see that $A_2(3) \rightarrow A_2(\Sigma)$ is finite of degree 27. The cover $A_2(\Sigma) \rightarrow A_2(1)$ is of degree $40 \cdot 24$, determined by the choice of isotropic space times the size of $PGL_2(\mathbb{F}_3)$.

The variety $A_2(3)$ is very well-known. Its completion is the Burkhardt quartic, defined by the homogeneous equation

$$t^4 - t(w^3 + x^3 + y^3 + z^3) + 3wxyz = 0,$$

as is described in [6, 8]. In particular, it, and its finite quotients, are absolutely irreducible.

Note that most PPAS are Jacobians of genus 2 curves in the sense that outside a proper closed subvariety of $A_2(1)$, any point can be represented by the Jacobian of a curve of genus 2. In what follows we will determine a genus 2 curve $C_{rst}$ over the function field of $A_2(\Sigma)$ together with a level structure on its Jacobian that makes it correspond to the generic point on $A_2(\Sigma)$.

4. Parametrisation of genus 2 curves with a maximal isotropic $(\mathbb{Z}/3)^2$ in $J[3]$

Let $k$ be a field of characteristic different from 2, 3. In this section we derive a genus 2 curve $C_{rst}$ over $k(r, s, t)$ with two non-trivial divisor classes $T_1, T_2 \in \text{Pic}(C_{rst})/k(r, s, t))^3$ with $T_1 \neq \pm T_2$ and $e_3(T_1, T_2) = 1$. This specifies a $\Sigma$-level structure on the Jacobian $J_{rst}$ of $C_{rst}$. In the process we will see that any sufficiently general Jacobian with a $\Sigma$-level structure over $k$ can be obtained via
specialization from \( \mathcal{J}_{rst} \). This identifies \( k(r,s,t) \) with the function field of \( \mathcal{A}_2(\Sigma) \), verifying that this moduli space is indeed rational.

We use the notation from Lemmas 8 and 9. We consider the algebra \( k[\alpha] = k[t]/(t^3 - \lambda_2/\lambda_1) \), which is only a field if \( \lambda_2/\lambda_1 \) is not a cube in \( k \), but at least will always be an étale algebra because \( \lambda_1, \lambda_2 \neq 0 \). We write \( \text{Nm} = \text{Norm}_{k[x,\alpha]/k[x]} \). We have

\[
(G_2 - G_1)(G_2 + G_1) = \lambda_1 \text{Nm}(H_1 - \alpha H_2).
\]

From Lemma 8 it follows that

\[
H_1 - \alpha H_2 = LM \text{ for some } L, M \in k[\alpha, x]
\]

and that for some \( c \in k^\times \) we have

\[
G_2 - G_1 = \frac{1}{c} \text{Nm}(M), \quad G_1 = \frac{1}{c}(c\lambda_1 \text{Nm}(L) - \frac{1}{c} \text{Nm}(M)), \quad G_2 = \frac{1}{c}(c\lambda_1 \text{Nm}(L) + \frac{1}{c} \text{Nm}(M)).
\]

We observe that

\[
(cy)^2 = (cG_1)^2 + (c^2 \lambda_1)H_1^2 = (cG_2)^2 + (c^2 \lambda_2)H_2^2,
\]

so by adjusting the values of \( \lambda_1, \lambda_2 \) we can assume \( c = 1 \). This shows that the isomorphism class of \( C \) is determined by \( \lambda_1, \lambda_2, L, M \).

Furthermore, if \( k \) has sufficiently many elements we can ensure that \( L \) does not vanish at \( x = \infty \) and that \( L \) is monic, so that

\[
L = x - (l_0 + \alpha l_1 + \alpha^2 l_2).
\]

The fractional linear transformation

\[
x \mapsto \frac{l_1 x + t l_2^2 - l_0 l_1}{t_2 x + t_1^2 - l_0 l_2},
\]

with determinant \( l_1^3 - t l_2^3 \) sends \( l_0 + l_1 \alpha + l_2 \alpha^2 \) to \( \alpha \). One can check that if \( l_1^3 - t l_2^3 = 0 \) then either \( F \) has a repeated root, and hence our data does not specify a genus 2 curve, or \( l_1 = l_2 = 0 \). In the latter case, \( L \) is already defined over \( k \), so via \( x \mapsto x - l_0 \) we can move its root to 0. This shows that via a fractional linear transformation, we can assume that

\[
L = x - u \alpha \quad \text{and} \quad M = (c_0 + c_1 \alpha + c_2 \alpha^2)x - (m_0 + m_1 \alpha + m_2 \alpha^2),
\]

where \( u = 0 \) corresponds to the case where \( l_1 = l_2 = 0 \). In order for \( LM \) to be of the form \( H_1 - \alpha H_2 \), with \( H_1, H_2 \in k[x] \), we need

\[
c_2 = 0, \quad m_1 u = 0, \quad m_2 = -c_1 u.
\]

We set \( \lambda_1 = s, \lambda_2 = st \) and observe that \( c^2 F(x) \) is homogeneous with respect to the following gradings.

| weights | s | t | c₀ | c₁ | c₂ | m₀ | m₁ | m₂ | u | x | c |
|----------|---|---|----|----|----|----|----|----|---|---|---|
| 3        | 0 | -1| -1 | -1 | 0  | 0  | 0  | 1  | 1 | -3 | 1 |
| 0        | 3 | 0 | -1 | -2 | 0  | -1 | -2 | -1 | 0 | 0  | 0 |
| 0        | 0 | 0 | 0  | 0  | 1  | 1  | 1  | 1  | -3|

We can solve (5) via either \( u = 0 \) or via \( m_1 = 0 \). For \( u = 0 \) we find that \( H_1 \) and \( H_2 \) both have a root at \( x = 0 \). By Lemma 10 we know we can avoid this case by changing the basis for the 3-torsion subgroup. Thus we see that any \( \text{PGL}_2(\mathbb{F}_3) \)-orbit has a representative that avoids this locus.

For the other case we take the affine open described by

\[
(s, t, c_0, c_1, c_2, m_0, m_1, m_2, u) = (s, t, 1, -1, 0, -r, 0, 0, 1),
\]

leading to

\[
H_1 = x^2 + r x + t, \quad H_2 = x^2 + x + r, \quad \lambda_1 = s, \lambda_2 = st.
\]
It is instructive to record which cases are excluded by the choices that we make here. We use the gradings to scale \( c_0 = 1, c_1 = -1, u = 1 \), so any curves that require any of these parameters to be 0 are ruled out. The gradings immediately show that setting any of \( c_0, c_1, u \) to 0 yields at most a 2-dimensional family of curves. For \( u = 0 \), we have seen that \( H_1, H_2 \) have a common root. Furthermore, the gradings show that this forms at most a 2-dimensional family up to isomorphy. For \( c_0 = 0 \), we see that \( H_1 \) has a root at \( \infty \) and for \( c_1 = 0 \) we see that \( H_2 \) has a root at infinity. Note that if \( u \neq 0 \), we have applied a linear transformation to normalize the form of \( L \), so \( \infty \) has geometric meaning. For instance, Example 2 describes a 1-dimensional family of curves that lie in this locus.

To summarize, we have established the following theorem, where we have scaled \( s \) by 4 to avoid some denominators in coefficients.

**Theorem 6.** Let \( k \) be a field of characteristic different from 2, 3 and suppose that \((C, T_1, T_2)\) consists of a genus 2 curve \( C \) over \( k \) and \( T_1, T_2 \in \text{Pic}(C/k)[3] \) such that \(#(T_1, T_2) = 9 \) and \( e_3(T_1, T_2) = 1 \). If the specified data is sufficiently general then \((C, T_1, T_2)\) is isomorphic to a suitable specialization of \( r, s, t \) in the family described by the following data.

\[
H_1 = x^2 + rx + t \\
\lambda_1 = 4s \\
G_1 = (s - st - 1)x^3 + 3s(r - t)x^2 + 3sr(r - t)x - st^2 + sr^3 + t \\
H_2 = x^2 + x + r \\
\lambda_2 = 4st \\
G_2 = (s - st + 1)x^3 + 3s(r - t)x^2 + 3sr(r - t)x - st^2 + sr^3 - t \\
H_3 = sx^2 + (2sr - st - 1)x + sr^2 \\
\lambda_3 = 4t/(st + 1)^2 \\
G_3 = ((s^2t^2 - s^2t + 2st + s + 1)x^3 + (3s^2t^2 - 3s^2tr + 3st + 3sr)x^2 \\
\quad + (3s^2tr - 3s^2tr^2 + 3str + 3sr^2)x + s^2t^3 - s^2tr^3 + 2st^2 + sr^3 + t)/(st + 1) \\
H_4 = (str - st - sr^2 + sr + r)x^2 + (st^2 - str - st - sr + 2sr^2 + t)x + st^2 - str^2 - str + sr^3 + t \\
\lambda_4 = 4st/(st^2 - 3str + st + sr^3 + t)^2 \\
G_4 = ((s^2t^3 - 3s^2t^2r - s^2tr^3 + 6s^2tr^2 - 3s^2tr + s^2t - s^2r^3 + 2st - 3str + 2st - sr^3 + t)x^3 \\
\quad + (3s^2t^3 - 6s^2t^2r - 3s^2tr + 9st + 3s^2tr^2 + 6s^2tr - 3s^2r^4 + 3st - 6str^2 + 3str)x^2 \\
\quad + (-3s^2tr^3 + 6s^2tr^2 + 3s^2tr^4 + 3s^2tr^3 + 3s^2tr - 3s^2r^5 - 3st^2r + 6st - 3str^2)x \\
\quad - s^2t^4 + 3s^2t^3r + s^2t^3 - 6s^2tr^2 + 3s^2tr^4 + s^2tr^3 - s^2r^6 - 2st^3 + 3st^2r + st^2 - 2str^3 - t^2)/(st^2 - 3str + st + sr^3 + t).
\]

Here \( C_{rst} \), \( y^2 = F_{rst}(x) = G_i^2 + \lambda_iH_i^3 \) for \( i = 1, 2, 3, 4 \) and \( T_i = \{|H_i(x) = 0, y - G_i(x) = 0\} - \kappa \) and \( T_3 = T_1 + T_2 \) and \( T_4 = T_1 - T_2 \).

For future reference we note that

\[
\text{Disc}(F_{rst}) = -2^{12}3^65^3\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7^3,
\]
where
\[
\begin{align*}
\delta_1 &= s \\
\delta_2 &= t \\
\delta_3 &= st + 1 \\
\delta_4 &= r^3 - 3rt + t^2 + t \\
\delta_5 &= r^3s - 3rst + st^2 + st + t \\
\delta_6 &= r^3s^2 - 3rs^2t - 3rs + s^3t^2 + s^2t + 2st + s + 1 \\
\delta_7 &= r^3s^2t - 3rs^2t^2 - 3rst + s^2t^3 + s^2t^2 + 2st^2 + t.
\end{align*}
\] (7)

\textbf{Remark 7.} Note that \(c_{rst}^{(d)} \cdot dy^2 = F_{rst}(x)\) has exactly the same property for \(\Sigma^{(d)}\)-level structure.

5. The Isogeny

We consider the curve \(C_{rst}\) as defined in Theorem 8 and its Jacobian \(J_{rst}\). In this section we determine a curve \(\tilde{C}_{rst}\) whose Jacobian \(\tilde{J}_{rst}\) is isogenous to \(J_{rst}\) via the isogeny \(J_{rst} \rightarrow J_{rst}/\Sigma\). We do so by determining the corresponding map between their Kummer surfaces.

5.1. Isogenies and the quartic model of the Kummer surface. Let \(J\) be a principally polarized abelian surface with theta divisor \(\Theta_J\) and suppose that \(\Sigma \subset J[3]\) is maximal isotropic. We consider \(\tilde{J} = J/\Sigma\) and the isogeny \(\phi: J \rightarrow \tilde{J}\). By [10] Proposition 16.8 there exists a principal polarization on \(\tilde{J}\) with theta divisor \(\Theta_{\tilde{J}}\) such that \(\phi^*(\Theta_{\tilde{J}}) = 3\Theta_J\).

The classical theory of theta divisors gives us that \(O_J(2\Theta_J)\) is 4-dimensional and that the induced map \(J \rightarrow \mathbb{P}^3\) yields a quartic model of the Kummer surface \(K = J/\langle -1 \rangle\). We write \(\tilde{K}\) for the Kummer surface of \(\tilde{J}\). Similarly, the linear system \(O_J(6\Theta_J)\) provides a quartic model of \(\tilde{K}\).

\textbf{Remark 8.} Note that this construction requires that we choose \(2\Theta_J\) to be defined over the base field \(k\). If \(k\) is not algebraically closed, then there might not exist a \(k\)-rational divisor \(\Theta_J\) in its class. For an abelian surface, however, \(2\Theta_J\) is always linearly equivalent to a \(k\)-rational divisor.

The isogeny induces a map \(K \rightarrow \tilde{K}\) in the following way. First note that \(\phi^*O_J(2\Theta_J) \subset O_J(6\Theta_J)\). The involution \(-1: J \rightarrow J\) induces a linear map on \(O_J(6\Theta_J)\). We write \(O_J(6\Theta_J)\) for the fixed subspace. Similarly, the translation action of \(\Sigma\) on \(J\) induces a linear action on the same space. We write \(O_J(6\Theta_J)\) for its fixed space. It is straightforward to check that
\[
\phi^*O_J(2\Theta_J) = O_J(6\Theta_J) \Theta_J \cap O_J(6\Theta_J)\).
\]

If \(\xi_0, \xi_1, \xi_2, \xi_3\) forms a basis for \(O_J(2\Theta_J)\) then \(O_J(6\Theta_J)\) is generated by the cubic forms in \(\xi_0, \xi_1, \xi_2, \xi_3\). Thus we see that the isogeny \(\phi: J \rightarrow \tilde{J}\) induces map \(K \rightarrow \tilde{K}\) which, between the quartic models, is given by cubic forms.

5.2. Choice of model for Kummer surfaces. Let \(C\) be a curve of genus 2 given by a model
\[
C: y^2 = f_6x^6 + f_5x^5 + \cdots + f_0,
\]
with Jacobian \(J\). We follow [7] p.17 and choose a particular basis for \(O_J(\Theta_J)\). We describe \(\xi = \xi(x) = \xi_0, \ldots, \xi_3\) as functions on \(J\) in terms of a divisor class \(D = [(x_1, y_1) + (x_2, y_2) - \kappa]\) on \(C\) as follows.

\[
\begin{align*}
\xi_0 &= 1, \quad \xi_1 = x_1 + x_2, \quad \xi_2 = x_1x_2, \quad \xi_3 = \frac{\Phi(\xi_0, \xi_1, \xi_2) - 2y_1y_2}{\xi_1^2 - 4\xi_0\xi_2}, \quad \text{where} \\
\Phi(\xi_0, \xi_1, \xi_2) &= 2f_0\xi_0^3 + f_1\xi_0^2\xi_1 + 2f_2\xi_0^2\xi_2 + f_3\xi_0\xi_1\xi_2 + 2f_4\xi_0\xi_2^2 + f_5\xi_2^2\xi_1 + 2f_6\xi_2^3.
\end{align*}
\] (8)
Note that for a Mumford representation \( D = \{ x^2 - \xi_1 x + \xi_2, y - g_0 - g_1 x \} \) we have \( y_1 y_2 = g_0^2 + g_0 g_1 \xi_1 + g_1^2 \xi_2 \), so one can compute these coordinates readily from such a representation.

The quartic equation for the model of \( K \) arising from these coordinates has the shape

\[
K: (\xi_1^2 - 4\xi_0 \xi_2)^2 + \Phi(\xi_0, \xi_1, \xi_2) \xi_3 + \Psi(\xi_0, \xi_1, \xi_2) = 0,
\]

where \( \Psi(\xi_0, \xi_1, \xi_2) \) is a quartic form we do not need explicitly here. The important observation is that one can read off the coefficients \( f_0, \ldots, f_6 \) directly from \( \Phi \) and thus recover \( C \) from it.

In order to produce \( \Sigma \)-invariant forms on \( K \), we use biquadratic forms from [7] p.23], arising from the addition structure on \( J \). For \( i, j = 0, \ldots, 3 \) we have forms \( B_{i,j} \in k[\xi_0, \ldots, \xi_3, \xi'_0, \ldots, \xi'_3] \), biquadratic in \( (\xi_0, \ldots, \xi_3) \) and \( (\xi'_0, \ldots, \xi'_3) \) such that for points \( D_1, D_2 \) on \( J \) we have, as projective matrices,

\[
R_{ij}(\xi_0, \ldots, \xi_3) = B_{ij}(\xi_0, \ldots, \xi_3, \xi(T_1)) \quad \text{and} \quad S_{ij}(\xi_0, \ldots, \xi_3) = B_{ij}(\xi_0, \ldots, \xi_3, \xi(T_2)).
\]

We see that the cubic forms

\[
R_{ijk} = \xi_i R_{jk} + \xi_j R_{ki} + \xi_k R_{ij} \quad \text{with} \quad i, j, k \in \{1, \ldots, 4\}
\]

are invariant under translation by \( T_1 \) and similarly that the forms

\[
S_{ijk} = \xi_i S_{jk} + \xi_j S_{ki} + \xi_k S_{ij} \quad \text{with} \quad i, j, k \in \{1, \ldots, 4\}
\]

are invariant under translation by \( T_2 \). For \( C = C_{rst} \) the \( R_{ijk} \) and \( S_{ijk} \) each generate spaces of dimension 8 that intersect in a space of dimension 4. This intersection provides us with an explicit description of \( \phi^*(O_{J}(2\Theta_{\bar{J}})) \).

Generally, we expect \( \bar{J} \) to be the Jacobian of a curve of genus 2, say \( \bar{C} \). We can try to find a basis \( \tilde{\xi}_0, \ldots, \tilde{\xi}_3 \) for \( \phi^*(O_{J}(2\Theta_{\bar{J}})) \) that is the pullback of a basis of the type described by [8]. We can then read off the curve \( \bar{C} \), at least up to quadratic twist, from the resulting equation for \( \bar{K} \).

The basis choice can largely be characterized by the order of vanishing of each \( \tilde{\xi}_i \) at the identity element. This leads us to conclude that, up to scalar multiples, we should take the basis choice

\[
\begin{align*}
\tilde{\xi}_0 &= (1\xi_0 + 0\xi_1 + 0\xi_2)\xi_3 + \cdots, \\
\tilde{\xi}_1 &= (0\xi_0 + 1\xi_1 + 0\xi_2)\xi_3 + \cdots, \\
\tilde{\xi}_2 &= (0\xi_0 + 0\xi_1 + 1\xi_2)\xi_3 + \cdots.
\end{align*}
\]

The determination of \( \tilde{\xi}_3 \) is a little more involved. The resulting forms for \( \bar{C} = \bar{C}_{rst} \) are too voluminous to reproduce here, but we have made them available electronically at [4]. Via interpolation we find the following result.

**Theorem 9.** Let \( C_{rst} \) be as described by Theorem [6] Then \( \bar{J}_{rst} = J_{rst}/\Sigma \) is the Jacobian of the genus 2 curve

\[
\bar{C}_{rst}: -3y^2 = \bar{G}_4^2 + \bar{\lambda}_4 \bar{H}_4,
\]

with

\[
\begin{align*}
\bar{G}_4 &= \Delta ((s - st - 1)x^3 + 3s(r - t)x^2 + 3rs(r - t)x + (r^3s - st^2 - t)) , \\
\bar{H}_4 &= (r - 1)(rs - st - 1)x^2 + (r^3s - 2r^2s + rst + r - st^2 + st - t)x - (r^2 - t)(rs - st - 1), \\
\bar{\lambda}_4 &= 4\Delta st,
\end{align*}
\]
where
\[
\Delta = r^6s^2 - 6r^4s^2t - 3r^4s + 2r^3s^2t^2 + 2r^3s^2t - 3r^3st + r^3s + r^3 + 9r^2s^2t^2 + 6r^2st
\]
\[-6rs^2t^3 - 6rs^2t^2 - 9rst^2 - 3rst - 3st - 2s^2t^2 + 2s^2t + 2st^2 + t^2 + t.
\]

5.3. **Proof of Theorem** $\mathbf{9}$. Since we have completely explicit descriptions of $C_{rst}$ and $\tilde{C}_{rst}$, we can write down explicit quartic models
\[
\mathcal{K}: Q = (\xi_1^2 - 4\xi_0\xi_2)\xi_3^2 + \Phi(\xi_0, \xi_1, \xi_2)\xi_3 + \Psi(\xi_0, \xi_1, \xi_2) = 0
\]
\[
\tilde{\mathcal{K}}: \tilde{Q} = (\xi_1^2 - 4\xi_0\tilde{\xi}_2)\tilde{\xi}_3^2 + \tilde{\Phi}(\xi_0, \tilde{\xi}_1, \tilde{\xi}_2)\tilde{\xi}_3 + \tilde{\Psi}(\xi_0, \tilde{\xi}_1, \tilde{\xi}_2) = 0.
\]

Furthermore, we have an explicit description at $\mathbf{4}$ of the map between them by expressions that give $\xi_0, \ldots, \xi_3$ as cubic forms in $\xi_0, \ldots, \xi_3$. We already know that $\tilde{\mathcal{K}}$ is irreducible, because it is the Kummer surface of a Jacobian. Therefore, to check if $\tilde{\mathcal{K}}$ is indeed the image of $\mathcal{K}$, we only need to substitute the cubic forms into the equation for $\tilde{\mathcal{K}}$ and check that the resulting degree 12 equation is divisible by the quartic equation for $\mathcal{K}$. This is doable for specific specializations of $r, s, t$ in $\mathbb{Q}$, but the computers at our disposal were not able to do this directly.

We note that $Q$ and $\tilde{Q}$ (after substitution of the cubic forms), are polynomials in $r, s, t, \xi_0, \ldots, \xi_3$, of degrees 10 and 2 in $\xi_3$ respectively. Hence, long division yields unique polynomials $\sigma \in \mathbb{Q}[r, s, t, \xi_0, \ldots, \xi_3]$ and $\rho_0, \rho_1 \in \mathbb{Q}[r, s, t, \xi_0, \xi_1, \xi_2]$ such that
\[
(\xi_1^2 - 4\xi_0\xi_2)^9\tilde{Q} = \sigma Q + \rho_1 \xi_3 + \rho_0.
\]

We want to prove that $\rho_1$ and $\rho_0$ are identically zero. To this end, we analyse the appropriate Newton polygons (or do the required computation using polynomials with coefficients truncated to the appropriate leading terms) to verify that $\rho_0, \rho_1$ are of degrees at most 102, 67, 36 in $r, s, t$. Hence, if we check that $Q$ indeed divides $\tilde{Q}$ for a grid of $103 \times 68 \times 37$ values for $(r, s, t)$ then a straightforward interpolation argument shows that $\rho_0, \rho_1$ must indeed be identically 0. This is something that can easily be verified by a computer in less than 3 hours.

This computation shows that $\tilde{\mathcal{K}}$ is indeed the Kummer surface of $\tilde{J} = J/\Sigma$ and hence that $\tilde{\mathcal{C}}_{rst}$ is correct up to quadratic twist. Recall from Section $\mathbf{3}$ that $J/\Sigma$ comes equipped with a $\Sigma'$-level structure. In our case, we have that $\Sigma = (\mathbb{Z}/3)^2$, so $\Sigma' = (\mu_3)^2 = \Sigma^{(-3)}$. Thus, it follows that $\tilde{\mathcal{J}}_{rst}^{(-3)}$ should have a $\Sigma$-level structure itself such that the isogeny corresponding to it brings us back to $\mathcal{J}_{rst}^{(-3)}$.

**Lemma 10.** Let $C_{rst}$ be as in Theorem $\mathbf{9}$ let $\tilde{C}_{rst}$ be as in Theorem $\mathbf{9}$ and let $\tilde{C}_{rst}^{(-3)}$ be the quadratic twist of $\tilde{C}_{rst}$ by $-3$, using the notation in Remark $\mathbf{7}$. Define $\psi_0$ by
\[
\psi_0(r, s, t) = \left(\frac{-s(r - 1)(r^2 - t)(\delta_5 - r)}{(rs - st - 1)^2 \delta_4}, \frac{(rs - st - 1)^3 \delta_2^2}{st(r - 1)^3 \Delta}, \frac{s^2(r - 1)^3(r^2 - t)^3}{(rs - st - 1)^3 \delta_4^2}\right).
\]

Then $C_{r's't'}$ is birationally equivalent to $C_{rst}^{(-3)}$, where $(r', s', t') = \psi_0(r, s, t)$. Furthermore, as a rational map we have $\psi_0(\psi_0(r, s, t)) = (r, s, t)$. The $\Sigma^{(-3)}$ level structure induced on $\tilde{J}_{rst}$ determines the kernel of the dual isogeny $\bar{\tilde{J}}_{rst} \to \tilde{J}_{rst}^{(-3)}$.

**Proof.** One can check directly that $C_{r's't'}$ is birationally equivalent to $\tilde{C}_{rst}^{(-3)}$ under the transformation
\[
\theta_0 : (x, y) \mapsto \left(\frac{-(r^3 - 3rt + t^2 + t)(r^2 - t - 1)^2 s}{(r^2 - t - 1)^2 s}, \frac{r - t}{r - 1}, \frac{\Delta (rs - st - 1)^3(r^3 - 3rt + t^2 + t)^2}{s^2(r - 1)^3(r^2 - t)^3}(r - 1)^2 y\right).
\]

This naturally marks some $\Sigma^{(-3)}$ level structure on $\tilde{J}_{rst}$. Note that it even does so over $\mathbb{Q}$, where we have no primitive cube root of unity. The Weil pairing implies that on $\tilde{J}_{rst}^{(-3)}$, any two $\Sigma$ level
structures must differ by a unique automorphism of $\Sigma$. It follows that the same holds for $\Sigma^{(-3)}$ level structures on $\mathcal{J}_{rst}$ itself.

\textbf{Remark 11.} A little more is true than we prove in Lemma 10: we have a \textit{natural} $\Sigma^\vee$ structure on $\mathcal{J}_{rst}$. In Lemma 15 we identify this and in Lemma 16 we identify the corresponding involution on $k(r,s,t)$, which is not \textit{quite} $\psi_0$ as listed above. We selected $\psi_0$ because the corresponding transformation $\theta_0$ is easy to write down.

5.4. \textbf{Additional relations.} At this point, we have what we require for the applications of the next sections, since we only need the $\Sigma$ level structure up to $\Sigma$-automorphism. We shall devote the remainder of the section to a more concise description of $\tilde{\mathcal{C}}_{rst}$, which will also give the full natural $\Sigma^\vee$ level structure on $\mathcal{J}_{rst}$.

\textbf{Remark 12.} In the case of isotropic $(\mathbb{Z}/2)^2$-level structure, there is a very satisfying expression for the isogenous abelian variety in the general case (i.e., when it is a Jacobian), described in \cite{2} (see also \cite{7}, Chapter 9). An isotropic $(\mathbb{Z}/2)^2$-level structure on the Jacobian of a genus 2 curve can be expressed by a model of the curve of the form

$$C: y^2 = q_1(x)q_2(x)q_3(x),$$

where each $q_i$ is a quadratic polynomial in $x$. One forms a $3 \times 3$ matrix whose columns are the coefficients of $q_1, q_2, q_3$. If the determinant $\Delta$ of this matrix is nonzero, then the isogenous surface is a Jacobian and the associated curve can be expressed as

$$\tilde{C}: y^2 = \Delta \tilde{q}_1(x)\tilde{q}_2(x)\tilde{q}_3(x),$$

where the coefficients of the $\tilde{q}_i$ are easily expressible in terms of the cofactors of this same $3 \times 3$ matrix. Of particular note is that the curve is (naturally) again of the same form. Indeed, it is straightforward to verify that the same operation applied twice gives us back a model that is isomorphic to the curve we started with and that the quadrics satisfy the peculiar relation

$$q_1(x)\tilde{q}_1(\tilde{x}) + q_2(x)\tilde{q}_2(\tilde{x}) + q_3(x)\tilde{q}_3(\tilde{x}) = \Delta(x - \tilde{x})^2.$$

In fact, this relation is the basis for the $(2, 2)$ correspondence between $C$ and $\tilde{C}$ that gives rise to the polarized isogeny between their Jacobians.

One might hope to find a similar relation in our case. Indeed, the general theory implies there is a correspondence between $\mathcal{C}_{rst}$ and $\tilde{\mathcal{C}}_{rst}$ giving rise to the polarized isogeny between $\mathcal{J}_{rst}$ and $\tilde{\mathcal{J}}_{rst}$. However, that general theory only predicts an $(18, 2)$ correspondence which lacks the desired symmetry and does not seem inviting from a computational point of view. The following theorem gives a possibly more attractive relation between the models for $\mathcal{C}_{rst}$ and $\tilde{\mathcal{C}}_{rst}$ expressing the level structures on their Jacobians.

\textbf{Theorem 13.} For $j = 1, \ldots, 4$ let $H_j(x) = h_{2j}x^2 + h_{1j}x + h_{0j}$ be as in Theorem 7 for $i = 1, \ldots, 7$, let $\delta_i$, be as in \cite{7}. Define the matrix $A$ by

\begin{equation}
A = \begin{pmatrix}
h_{21} & h_{22} & h_{23} & h_{24} \\
h_{11} & h_{12} & h_{13} & h_{14} \\
h_{01} & h_{02} & h_{03} & h_{04} \\
\frac{r}{2} & \frac{1}{2} & \frac{\delta_1}{2} & \frac{\delta_2}{2}
\end{pmatrix}.
\end{equation}
Then \( \det(A) = \Delta \) as in (11). Let \( M \) be the cofactor matrix of \( A \), i.e., \( M^T = \det(A)A^{-1} \) and let \( M_{i,j} \) be its entry in the \( i \)-th row and \( j \)-th column. Define

\[
\tilde{A} = \begin{pmatrix}
M_{31} & M_{32} & M_{33} & M_{34} \\
-2M_{21} & -2M_{22} & -2M_{23} & -2M_{24} \\
M_{11} & M_{12} & M_{13} & M_{14} \\
\frac{1}{2}M_{41} & \frac{1}{2}M_{42} & \frac{1}{2}M_{43} & \frac{1}{2}M_{44}
\end{pmatrix},
\]

and

\[
\tilde{H}_j(x) = \tilde{A}_{1j}x^2 + \tilde{A}_{2j}x + \tilde{A}_{3j}, \quad \text{for } j = 1, \ldots, 4,
\]

so that the \( \tilde{H}_j(x) \) bear the same relationship to the first three rows of \( \tilde{A} \) as the \( H_j(x) \) bear to \( A \). Also define

\[
\tilde{\lambda}_1 = \lambda_1\Delta/\delta_6^2, \quad \tilde{\lambda}_2 = \lambda_2\Delta/\delta_7^2, \quad \tilde{\lambda}_3 = \lambda_3\Delta\delta_8^2/\delta_4^2, \quad \tilde{\lambda}_4 = \lambda_4\Delta\delta_9^2.
\]

Finally define

\[
\tilde{G}_4(x) = \Delta(G_1(x) - 2t),
\]

and define \( \tilde{G}_1(x), \tilde{G}_2(x), \tilde{G}_3(x) \) up to \( \pm \) (which is all we require for these) to be such that

\[
\tilde{G}_i(x)^2 = \tilde{G}_1(x)^2 + \tilde{\lambda}_i\tilde{H}_i(x)^3 - \tilde{\lambda}_i\tilde{H}_i(x)^3, \quad \text{for } i = 1, 2, 3.
\]

Then the curve \( \tilde{C}_{rst} \) of Theorem 13 is the same as

\[
\tilde{C}_{rst}: y^2 = -3\left(\tilde{G}_i(x)^2 + \tilde{\lambda}_i\tilde{H}_i(x)^3\right), \quad \text{for } i = 1, \ldots, 4.
\]

An immediate consequence the relationship between the matrices \( A, \tilde{A} \) is the following identity, which is strikingly similar to the identity (9.2.5) of [7] for the Richelot isogeny.

**Corollary 14.** Let the \( H_i(x), \tilde{H}_i(x) \) be as in Theorem 13. Then

\[
H_1(x)\tilde{H}_1(\tilde{x}) + H_2(x)\tilde{H}_2(\tilde{x}) + H_3(x)\tilde{H}_3(\tilde{x}) + H_4(x)\tilde{H}_4(\tilde{x}) = \Delta(x - \tilde{x})^2.
\]

**Proof.** We first note that, if we take the matrix \( \tilde{A} \) of (13), divide the second row by \(-2\), then swap the first and third rows, and then take the transpose, we obtain \( A^{\text{adj}} \), the adjugate of the matrix \( A \) of (12). We recall the standard identity \( AA^{\text{adj}} = \Delta I_4 \) from linear algebra and note that, in the left hand side of (19), the coefficients of \( x^2, \tilde{x}^2 \) are equal to diagonal entries of \( AA^{\text{adj}} \) and so equal \( \Delta \). Similarly, the coefficient of \( x\tilde{x} \) is \(-2 \) times a diagonal entries of \( AA^{\text{adj}} \) and so equals \(-2\Delta \). The remaining coefficients on the left hand side of (19) equal non-diagonal entries of \( AA^{\text{adj}} \), and so are all \( 0 \), as required. \( \square \)

We can also formulate the relationship between the \( H_i \) and the \( \tilde{H}_i \) in more intrinsic terms. Let \( T_1, T_2, T_3, T_4 \in J_{rst}[3] \) corresponding to \( H_i \) such that \( T_1 + T_2 = T_3 \) and \( T_1 - T_2 = T_4 \) and let \( \tilde{T}_1 \) and \( \tilde{H}_i \) be related analogously. Suppose we have a basis \( T_1, T_2, U_1, U_2 \) for \( J_{rst}[3] \) such that the Gram matrix of the Weil pairing (with values written additively) is

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

Then \( U_1, U_2 \) naturally provides a basis choice for \( J_{rst}[3]/\Sigma = \Sigma' \). We find that \( \phi(U_1) = \pm \tilde{T}_1 \) and that \( \phi(U_2) = \pm \tilde{T}_2 \), with the same choice of sign (which is an ambiguity in \( U_1, U_2 \) already). This provides the following.
Lemma 15. The labelling for the $\tilde{T}_i$ naturally marks the $\Sigma^\vee$ level structure on $\tilde{J}_{rst}$ in the sense that for $i = 1, \ldots, 4$, we have

$$e_3(T_i, U) = 0 \text{ for any } U \in \tilde{J}_{rst}[3] \text{ such that } \phi(U) = \tilde{T}_i.$$ 

Proof. We only have to check this statement for a particular specialization of $r, s, t$, for instance over a finite field where the full 3-torsion is pointwise defined. We have a completely explicit description of the map $\phi$ on the Kummer surfaces, which is sufficient to determine the appropriate elements $U$. One can then just verify the claim in the lemma by exhaustion. It is straightforward to check that the condition given indeed uniquely determines the structure (up to sign).

□

5.5. Automorphisms. Recall from Section 4 that $A_2(\Sigma)$ has $\text{PGL}_2(\mathbb{F}_3)$ acting on it. Furthermore, because in our case we have $\Sigma^\vee \simeq \Sigma^{(-3)}$, we get an additional automorphism $J \mapsto \tilde{J}^{(-3)}$. We identified the effect of this last automorphism on $r, s, t$ in Lemma 10. Here we describe generators for the other automorphisms as well. Note that $\psi_0$ is only a rational map, because the abelian surface $\tilde{J}$ need not be a Jacobian if $J$ is: $\tilde{J}$ may be a product of elliptic curves. In addition, while we have seen in Section 3 that every Jacobian with a $\Sigma$ level structure admits a model that is a specialization of $J_{rst}$, this may involve a change of basis. Thus, we should also expect $\text{PGL}_2(\mathbb{F}_3)$ to only act birationally on $(r, s, t)$.

Lemma 16. The following transformations

$$\psi_1(r, s, t) = \left( \frac{t}{r^2}, \frac{r^3 s}{t}, \frac{t^2}{r^3} \right),$$

$$\psi_2(r, s, t) = \left( r, \frac{1}{s(r^3 - 3rt + t^2 + t)}, t \right),$$

$$\psi_3(r, s, t) = \left( r, \frac{t(st + 1)}{r^3}, \frac{r^3 s}{st + 1} \right),$$

have the property that, for each $i = 1, 2, 3$, if $(r', s', t') = \psi_i(r, s, t)$ then $C_{r's't'}$ is birationally equivalent to $C_{rst}$. The group generated by $\psi_1, \psi_2, \psi_3$ is isomorphic to $\text{PGL}_2(\mathbb{F}_3)$. Furthermore, if $\psi_0$ is as in Lemma 10 then the group generated by $\psi_0, \psi_1, \psi_2, \psi_3$ is isomorphic to $\mathbb{Z}/2 \times \text{PGL}_2(\mathbb{F}_3)$.

Proof. We first note that, for each $i = 1, 2, 3$, if $(r', s', t') = \psi_i(r, s, t)$ then $C_{r's't'}$ is birationally equivalent to $C_{rst}$ under $\theta_i$, where

$$\theta_1(x, y) = \left( \frac{t}{rx}, \frac{ty}{x^3} \right),$$

$$\theta_2(x, y) = \left( \frac{(r-t)x + (r^2-t)}{(r-1)x + (t-r)}, \frac{s(r^3 - 3rt + t^2 + t)^2y}{(rx - x + t - r)^3} \right),$$

$$\theta_3(x, y) = \left( \frac{-rx}{x + r}, \frac{r^3 y}{(x + r)^3} \right).$$

It can also be checked that $\psi_1, \psi_2, \psi_3$ permute the roles of the $H_i$ and correspond, respectively, to:

$$(H_1, H_2, H_3, H_4) \leftrightarrow (H_2, H_1, H_3, H_4),$$

$$(H_1, H_2, H_3, H_4) \leftrightarrow (H_1, H_2, H_4, H_3),$$

$$(H_1, H_2, H_3, H_4) \leftrightarrow (H_3, H_2, H_1, H_4).$$

It follows that they can be identified with the transpositions $(12), (34), (13)$ in $S_4$, which generate all of $S_4$. Hence the group generated by $\psi_1, \psi_2, \psi_3$ is isomorphic to $S_4$ which, in turn, is isomorphic to $\text{PGL}_2(\mathbb{F}_3)$. Note also that $\psi_1, \psi_2, \psi_3$ give the same permutation of the roles of the $\tilde{H}_i$. We finally
note that \( \psi_0 \) corresponds to \((H_1, H_2, H_3, H_4) \leftrightarrow (\tilde{H}_1, \tilde{H}_3, \tilde{H}_2, \tilde{H}_1) \) and can be replaced by \( \psi'_0 = \psi_3 \psi_1 \psi_2 \psi_3 \psi_0 \) which corresponds to \((H_1, H_2, H_3, H_4) \leftrightarrow (\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4) \), and is given explicitly by

\[
\psi'_0(r, s, t) = \left( \frac{-(r^2 - t)(rs - st - 1)(\delta_5 - r)}{(r - 1)^2\delta_7}, \frac{(r - 1)^3s\delta_6^2}{(rs - st - 1)^3\delta_5\Delta}, \frac{t(rs - st - 1)^3(\delta_5 - r)^3}{(r - 1)^3\delta_6^2\delta_7} \right).
\]

This is an involution and it commutes with \( \psi_1, \psi_2, \psi_3 \). Hence the group generated by the maps \( \psi_0, \psi_1, \psi_2, \psi_3 \), which is the same as the group generated by \( \psi'_0, \psi_1, \psi_2, \psi_3 \), must be isomorphic to the group \( \mathbb{Z}/2 \times \text{PGL}_2(\mathbb{F}_3) \).

6. ISOGENY DESCENT

Galois cohomology associates to an isogeny

\[
0 \to J[\phi] \to J \xrightarrow{\phi} \tilde{J} \to 0
\]

between abelian varieties over a field \( k \) an exact sequence

\[
0 \to \tilde{J}(k) \xrightarrow{\phi J(k)} H^1(k, J[\phi]) \to H^1(k, J).
\]

For \( k \) a number field and \( v \) a place of \( k \), we consider the completion \( k_v \) and its separable closure \( k_v^{sep} \) and identify \( \text{Gal}(k_v^{sep}/k_v) \) with a relevant decomposition group inside \( \text{Gal}(k^{sep}/k) \). This allows us to consider restriction maps \( \text{res}_v: H^1(k, .) \to H^1(k_v, .) \). Writing \( \gamma_v \) for the relevant connecting homomorphism over the base field \( k_v \), this allows us to define the Selmer group

\[
\text{Sel}^\phi(J/k) = \{ \delta \in H^1(k, J[\phi]) : \text{res}_v(\delta) \in \text{im} \gamma_v \text{ for all places } v \text{ of } k \}.
\]

The Selmer group contains the image of \( \gamma \). If this containment is strict then part of the Selmer group represents non-trivial elements in \( \text{III}(J/k) \). To be precise, we have

\[
0 \to \tilde{J}(k) \xrightarrow{\phi J(k)} \text{Sel}^\phi(J/k) \to \text{III}(J/k)[\phi] \to 0.
\]

Therefore, the computation of Selmer groups can be used to exhibit non-trivial elements in Tate-Shafarevich groups. This is taking a historically backward view, since originally Tate-Shafarevich groups were introduced as a means to measure the failure of Selmer groups to provide sharp bounds on the size of \( \tilde{J}(k)/\phi J(k) \).

Let \( \mathbb{G}_m \) be the multiplicative group scheme over a field \( k \) of characteristic not dividing 2. For \( d \in k^\times \), we write \( \mathbb{G}_m^{(d)} \) for the quadratic twist by \( d \) of the multiplicative group. It is the group scheme that fits in the short exact sequence

\[
1 \to \mathbb{G}_m^{(d)}(L) \to L[\sqrt{d}]^\times \xrightarrow{\text{Norm}} L^\times \to 1
\]

for any extension \( L \) of \( k \). Similarly, for a positive integer \( n \), we write \( \mu_n^{(d)} \subset \mathbb{G}_m^{(d)} \) for the kernel of the morphism \( x \to x^n \).

We begin by stating the following slight generalization of a classical result from Kummer theory.

**Lemma 17.** Let \( n > 0 \) be odd, let \( k \) be a field of characteristic not dividing \( 2n \) and let \( \mu_n \) be the \( \text{Gal}(k^{sep}/k) \)-module of \( n \)-th roots of unity in \( k^{sep} \). For \( d \in k^\times \), we have

\[
H^1(k, \mu_n^{(d)}) = \frac{\mathbb{G}_m^{(d)}(k)}{\mathbb{G}_m^{(d)}(k)^n}.
\]
Lemma 18. Let $\tilde{\phi}: \tilde{J} \to J$ be a polarized isogeny between principally polarized abelian surfaces with kernel $\Sigma(-3d) = \mu_3^{(d)} \times \mu_3^{(d)}$. Suppose that $J$ is the Jacobian of a genus 2 curve of the form
\[
C: y^2 = -3d(G_1(x)^2 + \lambda_1 H_1(x)^3) = -3d(G_2(x)^2 + \lambda_2 H_2(x)^3),
\]
where the 3-torsion subgroup with generators supported at $H_1(x) = 0$ and $H_2(x) = 0$ is the kernel of an isogeny $\phi: J \to \tilde{J}$ such that $\phi \circ \tilde{\phi} = 3$. Then the connecting homomorphism
\[
\tilde{\gamma}: \tilde{J}(k) \to H^1(k, \Sigma(-3d)) = \frac{\mathbb{G}_m^{(d)}(k)}{(\mathbb{G}_m^{(d)}(k))^3} \times \frac{\mathbb{G}_m^{(d)}(k)}{(\mathbb{G}_m^{(d)}(k))^3}
\]
is induced by the partial map
\[
C \longrightarrow \mathbb{G}_m^{(d)} \times \mathbb{G}_m^{(d)}
\]
\[(x, y) \mapsto \left( y - \sqrt{d}G_1(x), \ y - \sqrt{d}G_2(x) \right) \]

Proof. This is a direct application of the theory developed in [12] and [3].

From Theorems 9 and 9 we can obtain isogenies $\phi: J \to \tilde{J}$ and $\tilde{\phi}: \tilde{J} \to J$ with kernels $\Sigma = \mathbb{Z}/3 \times \mathbb{Z}/3$ and $\Sigma' = \mu_3 \times \mu_3$ respectively. We use Lemma 18 to compute $\text{Sel}^{\phi}(J/Q)$ and $\text{Sel}^{\tilde{\phi}}(\tilde{J}/Q)$. As suggested by the lemma, we represent the cohomology classes by elements of $\mathbb{Q}(\sqrt{-3})^\times /\mathbb{Q}(\sqrt{-3})^\times 3$ and $\mathbb{Q}(\sqrt{-3})^\times /\mathbb{Q}(\sqrt{-3})^\times 3$ respectively.

We take $S$ to be the set of primes consisting of 3 and the primes of bad reduction of $C$. By [5, Proposition 9.2], the Selmer groups lie in the subgroups that are unramified outside $S$. We can represent those using $S$-units in $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}$. This already provides us with explicit finite groups that contain the Selmer groups. The remaining conditions come from the local images at $v \in S$ (note that $\mathbb{R}^\times /\mathbb{R}^\times 3$ is trivial, so the archimedian place does not provide any information). With the explicit description of the maps $\gamma_v$ and $\tilde{\gamma}_v$ we can generate elements in their images. Using [12, Lemma 3.8, Proposition 3.9] and some basic basic diagram chasing we have
\[
\frac{\# \tilde{J}(Q_v)}{\# \tilde{\phi}J(Q_v)} \times \frac{\# J(Q_v)}{\# \phi J(Q_v)} = \frac{\# \Sigma(Q_v) \# \Sigma'(Q_v)}{|3|_p^2} = \begin{cases} 81 & \text{if } v = 3 \text{ or } v \equiv 1 \pmod{3} \\ 9 & \text{if } v \equiv 2 \pmod{3} \end{cases},
\]
so we know when we have found enough elements to generate the entire image. By explicitly computing the restriction maps $\mathbb{Q}^\times /\mathbb{Q}^\times 3 \to \mathbb{Q}_v^\times /\mathbb{Q}_v^\times 3$ and similarly for $\mathbb{Q}(\sqrt{-3})$, we can compute the Selmer groups using essentially the definition in (20).

We now apply the above theory to several examples, which combine the information of standard 2-descent and the above descent via $\Sigma$-isogeny and $\Sigma'$-isogeny. In particular, this will give the first examples of nontrivial 3-part of the Tate-Shafarevich group on a non-reducible abelian surface.

The first example illustrates a situation where a 3-isogeny descent can be used to obtain a sharper rank bound than one can get from a 2-descent and hence exhibit some non-trivial 2-torsion elements in the Tate-Shafarevich group. There are alternative methods to do this, which show in the process that there is no 4-torsion, but obtaining unconditional results through these is too computationally expensive at present. The computations involved in these examples are easily reproduced using Magma [1] and the software we have made available at [4].

Example 19. Let $J$ be the Jacobian of the curve
\[
C_{-3,-3,-3}: y^2 = (12x^3 - 105)^2 - 12(x^2 - 3x - 3)^3.
\]
Then $J(Q) \simeq (\mathbb{Z}/3)^2 \times \mathbb{Z}$ and $\text{III}(J/Q)[6] \simeq (\mathbb{Z}/2)^3$. 

Furthermore, conditional on the Generalized Riemann Hypothesis for a certain degree 12 number field, we have 
$$III(J/Q)[6^\infty] = (\mathbb{Z}/2)^3.$$ 

**Proof.** Using the isogenies $\phi: J \to \tilde{J}$ and $\tilde{\phi}: \tilde{J} \to J$ we find

$$\text{Sel}^5(\tilde{J}/Q) = (\mathbb{Z}/3)^4$$ and $$\text{Sel}^6(J/Q) = 0.$$ 

We already know that $J[3](Q)$ is non-trivial and further computation shows that $J(Q)^{\text{tors}} = (\mathbb{Z}/3)^2$. 

A height computation shows that divisors supported at 101$x^2 + 21x + 147 = 0$ and at $x^2 - 6x - 45 = 0$ 

generate independent classes in $J(Q)$. From this one can deduce the structure of $J(Q)$ and that $III(J/Q)[\phi]$ 

and $III(\tilde{J}/Q)[\tilde{\phi}]$ are trivial. From $3 = \tilde{\phi} \circ \phi$ it follows that $III(J/Q)[3]$ is trivial as well. 

A 2-descent on $J$ yields $\text{Sel}^2(J/Q) = (\mathbb{Z}/2)^5$, which shows that $III(J/Q)[2] = (\mathbb{Z}/2)^3$. Indeed $J$ 

is odd in the sense of $\text{II}$. 

In order to prove there is no 4-torsion in $III(J/Q)$ we observe that $J(\mathbb{Q})(\mathbb{Q})^\ast$ has rank at least $3$, 

which can be shown by exhibiting enough points and a height pairing computation. One can compute that $\text{Sel}^2(J/Q(\sqrt{-2})) = (\mathbb{Z}/2)^5$, provided one verifies a certain class group computation 

for which one presently requires the Generalized Riemann Hypothesis. Since the rank of $J(Q)(\sqrt{-2})$ 

is the sum of the ranks of $J(Q)$ and $J(\mathbb{Q})(\mathbb{Q})^\ast$, one concludes that $III(J/Q(\sqrt{-2}))[2]$ is trivial. 

Since the restriction map $III(J/Q) \to III(J/Q(\sqrt{-2}))$ can only kill elements of order 2, the statement follows. 

It is worth noting that $\text{Sel}^2(J/Q) = (\mathbb{Z}/2)^5$, so $III(J(\mathbb{Q})(\mathbb{Q})^\ast)[2^\infty] = (\mathbb{Z}/2)^2$. 

In fact, using a visibility argument $\text{III}$ we find that $J(\mathbb{Q})(\mathbb{Q})^\ast[2] = Z^3$ 

and that $\text{Sel}^2(J/Q(\sqrt{-2})) = (\mathbb{Z}/3)^2$. 

It follows that $III(J/Q(\sqrt{-2}))[2] = 0$. Since the restriction map $III(J/QQ) \to III(J/Q(\sqrt{-2}))$ 

can only kill 2-torsion, the statement in the example follows. 

**Example 20.** Let $\tilde{J}$ be the Jacobian of the curve 

$$C_{-2,1,2}: y^2 = -48(83x^3 + 498x^2 - 996x + 581)^2 - 3984(15x^2 - 26x + 10)^3.$$ 

Then $\tilde{J}(\mathbb{Q}) \cong \mathbb{Z}$ and $III(\tilde{J}/Q)[3] \cong (\mathbb{Z}/3)^2$. 

**Proof.** We find

$$\text{Sel}^5(\tilde{J}/Q) = (\mathbb{Z}/3)^5$$ and $$\text{Sel}^6(J/Q) = 0.$$ 

With $J(Q)[3] = (\mathbb{Z}/3)^2$ and $\tilde{J}(Q)[3] = 0$, this implies that $\tilde{J}(Q)$ is of rank at most 3. 

From a 2-descent we find $\text{Sel}^2(J/Q) = (\mathbb{Z}/2)$. Furthermore, we find a non-torsion point in $J(Q)$, 

so we find that $J(Q) = (\mathbb{Z}/3)^2 \times \mathbb{Z}$ and $\tilde{J}(Q) = \mathbb{Z}$. Combined with the result above, this yields that 

$III(\tilde{J}/Q)[3] = (\mathbb{Z}/3)^2$. 

**Example 21.** The Jacobian $\tilde{J}$ of the curve 

$$C_{-2,1,2}: y^2 = -48(706x^3 + 2118x^2 + 4236x + 353)^2 + 16944(5x^2 - 14x - 30)^3$$

has 6-torsion in $III(\tilde{J})$ and $\tilde{J}(Q) = \{0\}$. 

**Proof.** Let $J$ be the Jacobian of 

$$C_{2,-1,2}: y^2 = 12x^6 + 72x^5 + 312x^4 + 688x^3 + 768x^2 + 192x + 68.$$ 

A direct computation shows that $\text{Sel}^5(\tilde{J}/Q) = (\mathbb{Z}/3)^4$ and $\text{Sel}^6(J/Q) = 0$. The torsion $J[3](Q) = (\mathbb{Z}/3)^2$ explains two factors, so either $J(Q)$ is of rank 2 or $III(J/Q)[3]$ is non-trivial. 

Similarly, a 2-descent shows that $\text{Sel}^2(J/Q) \cong \text{Sel}^2(J/Q) = (\mathbb{Z}/2)^2$. Further computation shows that $\text{Sel}^2(J(\mathbb{Q})(\sqrt{3})) = (\mathbb{Z}/2)^3$ and that $J(\mathbb{Q})(\sqrt{3}) \cong \mathbb{Z}^3$. Further computation shows that $\text{Sel}^2(J(\mathbb{Q})(\sqrt{3})) = (\mathbb{Z}/2)^3$ as well, so $J(\mathbb{Q})(\sqrt{3}) \cong \mathbb{Z}^3$ as well. It follows that $J(Q) \cong (\mathbb{Z}/3)^2$ 

and that $III(J/Q)[2^\infty] = III(\tilde{J}/Q)[2^\infty] = (\mathbb{Z}/2)^2$. 

\textbf{\textit{16}}
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