Toward a Comprehensive Impossibility Result for String Stability

Arash Farnam and Alain Sarlette

Abstract—We provide a comprehensive impossibility result toward achieving string stability, i.e., keeping local relative errors in check with local controllers independently of the size of a chain of subsystems. We significantly extend the existing results, from the linear time-invariant (LTI) setting to any homogeneous controllers that can be nonlinear, time varying, and locally communicating. We prove this impossibility for a set of definitions with various norm choices, including the $L_2$-type standard and a bounded-input implies bounded-output (BIBO) type criterion. All the results hold for a general discrete-time controller, which should cover most applications.

Index Terms—Decentralized control, fault tolerant control, non-linear control systems, networked control systems, traffic control.

I. INTRODUCTION

String stability roughly requires that a chain of subsystems (e.g., vehicles), controlled by local feedback from relative position measurements and subject to bounded local perturbations, keeps local deformations bounded independently of the size of the chain. In the absence of further variables intervening in the system, local deformations are defined as the deviations of the distances between consecutive vehicles from a constant target value. The topic has gathered attention from the observation that, for second-order integrator subsystems, by using any linear time-invariant (LTI) controller reacting to their predecessor’s distance, the amplification of some disturbance components along the chain is unavoidable [1], [2]. The historical application is a chain of acceleration-controlled vehicles, but other distributed systems like mechanical structures [3], [4], or more fundamental models, should also benefit from string stability insight. This has initiated a rich line of results such as the following.

1) Establishing various impossibility results and scalings for LTI controllers [1], [5], [6], mostly with subsystems reacting to their immediate predecessor and immediate follower. Often PD controllers are used as a proxy for bandwidth limitation, but problems associated with integral control have also been characterized [7].

2) Designing a passive LTI controller, with symmetric coupling to the predecessor and follower, to keep in check, at least in some sense, the effect of disturbances acting on the leading subsystem [3].

3) Adding dependence on absolute velocity into the pure double integrator, to overcome the string instability issue [8]–[10]. This can take the form of strong enough drag, which, e.g., for terrestrial vehicle applications are easy to ensure but would be in tradeoff with fuel efficiency; in other applications, ensuring strong drag may not always be trivial. Alternatively, the “time-headway” spacing policy makes the target intervehicle distance dependent on absolute velocity. For terrestrial vehicles, again, this appears easy to achieve, but questionable in terms of chain performance (possibly big spacing at high velocities, instead of moving efficiently like a big rigid body at all speeds); for other applications, the necessity to measure absolute velocity of each subsystem with respect to a common reference may even pose sensing questions.

4) Adding local communication between subsystems, as in adaptive cruise control [11], [12]. Remarkably, the existing studies with this approach all incorporate a dependence on absolute velocity, as mentioned in the previous item. In a recent paper [13], a variation on time headway is proposed for a generic controlled system, and, in the nominal case, the controller uses both absolute state information and communication with the leader.

We can further refer the reader to a recent review in [14]. Research has also been carried out on properties related to, yet different from, string stability. Without being comprehensive, we can mention the scaling of linear network eigenvalues with network size [15], the poor robustness of large linear networks under distributed sensing [16], and the coherence of linear networks in presence of stochastic noise [17].

This literature leaves several questions open. How much does each added element (dependence on absolute velocity, local communication capabilities, coupling symmetrically, or asymmetrically to one or several vehicles ahead and/or behind) actually contribute toward solving the string stability issue? How much can be gained with more involved, e.g., nonlinear controller designs? What is possible with digital, quantized controllers, and realistic modern communication? How much is the LTI setting actually limiting?

The objective of the this note is to answer these questions toward the more precise understanding of string stability and related issues. As a small variation, we consider a discrete-time controller, which is closer to digital controller implementations and incorporates related “natural” constraints in a direct way; although, probably, most current controllers are digital and discrete-time anyways, the conclusions should carry over in practice to the continuous-time setting as we discuss below. We work only with relative state measurements between neighboring subsystems; e.g., we do not allow controller dependence on absolute velocity. In accordance with the existing literature, we consider an idealized model where input disturbances must be countered while assuming perfect measurements and communication. Our main result is as follows.

We establish that by enabling nonlinear controllers, any couplings to a few vehicles in front and behind, any local communication (e.g., nonlinear, quantized, and event-based), and controller dependence on the chain length, all together do not allow to design a controller that would achieve string stability with respect to bounded disturbances acting on the subsystems of the chain. We prove this for...
several variants of the string stability measure, and with only main
the constraints: 1) the controller uses relative state measurements
between neighboring vehicles; 2) the controller is homogeneous;
i.e., each vehicle (except boundaries) reacts in the same way to its
neighbors; and 3) controller discretization step $\Delta t$ remains bounded
away from zero despite increasing chain length.

The proof comes down to identifying a general counterexample,
then working out rather basic computations; we, hence, believe that
the contribution mainly rests on the unprecedented generality of the
conclusions. Essentially, string instability in a chain of second-order
integrators is an unavoidable property of distributed sensing, for a
(much) larger class of controllers than LTI.

This note is organized as follows. Section II describes the setting
and several precise definitions of string stability. Section III gives
our main result, about a (very) general impossibility of string stability;
it ends with a short discussion on discrete-time versus continuous-
time controllers, and with an illustrative simulation of the system
behavior under a “bad” disturbance. Section IV concludes this note
with an outlook on the remaining open points and possible further
implications.

II. PROBLEM DEFINITION

A. Model Setting

We consider a chain of undamped second-order integrators, called
“vehicles,” for concreteness but which may represent any other (typi-
ically mechanical) subsystems coupled with local interactions. In
the literature, the following continuous-time system

$$
\dot{x}_k(t) = u_k(t)
$$

where $x_k$ is the position and $u_k$ the input signal of subsystem $k$ at time $t$,
is the standard starting point for string (in)stability [1]–[6], [14],
with continuous-time, e.g., LTI controllers defining the $u_k$. We, here,
consider its (exact) equivalent under discrete-time control. We assume
that each subsystem is controlled using a discrete-time control logic,
thus computing at each time $t = n \Delta t$ with $n \in \mathbb{Z}$, a control signal
that will be applied during the whole interval $(t, t + \Delta t)$ as input to each
double integrator; thus, $\Delta t$ is the time increment of the discrete-time
controller. We can, then, integrate exactly the second-order dynamics
over the time interval $(t, t + \Delta t)$ to obtain

$$
\begin{align*}
\xi_k(t + \Delta t) &= \xi_k(t) + \int_t^{t + \Delta t} \Delta \xi_k(t) \, dt \\
\Delta \xi_k(t) &= \left( u_k(t) + u_{k+1}(t) \right) + d_{k+1}(t)
\end{align*}
$$

where $\Delta \xi_k(t)$, $\xi_k$ denote position and velocity respectively; the control inputs $u_k, u_{k+1}$ result from integrating the
control signal applied during the interval $(t, t + \Delta t)$, respectively.

A key constraint is that the feedback signals $u_{k,1}, u_{k,2}$ applied be-
tween $t$ and $t + \Delta t$ must be based on only relative displacement measure-
ments $e_k = x_{k-1} - x_k$ between consecutive vehicles in the chain,
including possibly relative speeds $\dot{e}_k = v_{k-1} - v_k$, at time $t$. We,
then, consider the following general controller, where $e_k, \dot{e}_k$ denotes for

$$
e > k \text{ the set of values } e_k, e_{k+1}, \ldots, e_{\ell}:
$$

$$
\begin{align*}
u_{k,1} &= f_1(e_{(k-m_1)}(k + m_2), \hat{e}_{(k-m_1)}(k + m_2)) \\
u_{k,2} &= f_2(e_{(k-m_1)}(k + m_2), \hat{e}_{(k-m_1)}(k + m_2)) \\
\end{align*}
$$

where $m_1, m_2$ are the sets of integers from $k$ to $k + m_1$ and from $k + 1$ to $k + m_2$.

The following are the assumptions implied by the model.

1) The controllers must be homogeneous along the chain; i.e., the
functions $f_1, f_2, g_1, g_2, h$ do not depend on vehicle index $k$,
and the internal variables are initialized with the same default values
for each $k$.

2) The digital controller has a finite update time $\Delta t$, fixed indepen-
dently of the chain length $N$.

3) The control commands are based on relative state information only;
i.e., there is no dependence on absolute states that would be ob-
tained with respect to some common reference, like absolute
velocity.

The first point can be relaxed to approximately homogeneous in
our proof; future work, if deemed relevant, could address the fully
homogeneous setting, e.g., with adversarial methods. The second point
is discussed after the main result. Regarding the third point, we recall
that the case with control using absolute velocity has been solved [8],
[9]. It is not clear, however, whether relying solely on absolute velocity
for achieving string stability is acceptable in all practical contexts.
Avoiding to rely on any absolute information also fits the original
academic question.
B. Control Objective

The goal is to achieve string stability. Roughly said, this means avoiding the performance getting unboundedly worse when the chain length grows. Formally, several definitions have been proposed [1]–[3], [6], [7], [12], [14]. Their common point is to focus on stabilizing the $e_k$, i.e., the distances between consecutive vehicles. This makes sense, e.g., for collision-avoidance, for maintaining a tight platoon, or for avoiding too extreme accelerations on the last vehicles of the chain, and is a weaker requirement than controlling the position of every vehicle with respect to the leader. Definitions vary in the norms used to consider the distribution over vehicles, the norm over the time of the signals, and the type of disturbance, namely, as signals or as initial conditions [12], [14]. We, here, translate the definitions in the presence of input disturbances to discrete-time control.

**Definition 1:** For positive integers $p, q$, the $\ell_{p,q}$ string stability requires that there exist $C_1, C_2 > 0$ independent of $N$ such that: for any disturbances satisfying

$$\sum_{s=1,2} \sum_{k=0}^{N} \left( \sum_{n \in \mathbb{Z}} |d_{k,s}(n \Delta t) \frac{1}{\Delta t^p} \Delta t \right)^{q/p} < C_1,$$

it is ensured that $\sum_{n \in \mathbb{Z}} |e_j(n \Delta t)|^p \Delta t < C_2$ for each $j$.

**Definition 2:** For positive integers $p, q$, the $(\ell_p, \ell_q)$ string stability requires that there exist $C_1, C_2 > 0$ independent of $N$ such that: for any disturbances satisfying

$$\sum_{s=1,2} \sum_{k=0}^{N} \left( \sum_{n \in \mathbb{Z}} |d_{k,s}(n \Delta t) \frac{1}{\Delta t^p} \Delta t \right)^{q/p} < C_1,$$

it is ensured that $\sum_{k=0}^{N} \left( \sum_{n \in \mathbb{Z}} |e_j(n \Delta t)|^p \Delta t \right)^{q/p} < C_2$.

**Definition 3:** For positive integer $p$, the $(\ell_p, \ell_\infty)$ string stability requires that there exist $C_1, C_2 > 0$ independent of $N$ such that: for any disturbances satisfying

$$\sum_{n \in \mathbb{Z}} |d_{k,s}(n \Delta t) \frac{1}{\Delta t^p} | \Delta t < C_1, \quad s = 1, 2 \quad \text{for all } k$$

it is ensured that $\sum_{n \in \mathbb{Z}} |e_j(n \Delta t)|^p \Delta t < C_2$ for all $j$.

**Definition 4:** The $(\ell_\infty, \ell_\infty)$ string stability requires that there exist $C_1, C_2 > 0$ independent of $N$ such that: for any disturbances satisfying $|d_{k,s}(t) \Delta t^p < C_1$ for $s = 1, 2$ and all $k, t$, it is ensured that $|e_j(t)| < C_2$ for all $j, t$.

The $\Delta t$ factors are introduced because $d_{k,1}$ and $d_{k,2}$ are supposed to result from integrating (respectively, once and twice) a continuous-time signal $d_k(t)$ over the time interval $(t, t + \Delta t)$. In principle, we could just choose units such that $\Delta t = 1$, but we keep $\Delta t$ for later discussion. The main point about string stability is to satisfy the constraints with constants independent of the number of vehicles $N$.

Definition 1 is the weakest since it imposes a bound on the vector norm of disturbance inputs, but in return, it only requires each individual $e_j$ to have a bounded signal norm. It is a necessary condition for achieving stronger versions of string stability, and it has been considered a lot in the literature; e.g., until recently this was the only proven working definition for a controller depending on absolute velocity, as we exclude here [8]–[10]. The other definitions each consider the same norm on input disturbances and on output errors, either summing them over vehicles (see Definition 2) or not (Definitions 3 and 4). The most popular norm has been $p = q = 2$; especially the time-integration with a 2-norm has attracted a lot of attention, thanks to its equivalent formulation in frequency domain. Definition 4 changes the treatment of time to formulate a bounded-input implies bounded-output (BIBO) type version of string stability. It seems to have attracted little attention in the literature, probably because most of the literature has focused on LTI frequency-domain approaches. However, we would argue that this version is closest to practical concerns; and, as we will show that the string instability issue is not limited to the LTI setting, we will use tools that can treat this definition explicitly too. In the particular context of (1) and (2), or when restricting the controllers, it might well be that some of the above definitions hold strictly together; in the absence of further evidence, we will treat them all.

**Remark 1:** The definitions were initially stated in the linear context, where $C_1$ and $C_2$ can be rescaled such that it makes no difference in which order they are chosen (e.g., variants like “for each $C_1$, there exists a $C_2$” become equivalent to our statement). In the nonlinear context, this might differ, and we have chosen the weaker constraint; thus, our impossibility results will also hold for stronger variants.

**Remark 2:** The disturbance model appears quite standard: Each sub-system is subject to some disturbance input, bounded in some sense, and the consequences on the system must be bounded. A possible point of discussion though is that this standard definition of string stability checks system behavior under the worst disturbance satisfying the constraints. If the worst disturbance can get rejected, then we have a strong guarantee, and this is the typical approach for studying system stability. However, if the behavior under the worst disturbance does not behave well as $N$ gets large, it might also be that this worst disturbance becomes increasingly unlikely among all the possible combinations of disturbance signals. In other words, in a probabilistic model of all the possible input disturbance signals, it may be that the probability to have a problematic input signal converges to 0 as $N$ increases to infinity. More positive results have indeed been obtained using such a probabilistic model, although not for a chain of subsystems; see e.g. [17]. We, here, stick to the standard definition, thus considering the worst bounded input signal.

III. MAIN IMPOSSIBILITY RESULT

We now prove that Definitions 1–4 are all impossible to be satisfied even with a general controller as allowed in (2). The main idea of the proof is to construct a disturbance input that is badly countered by any distributed controller. We take advantage of a simple construction that focuses on exactly solving the central part of the chain only, in order to give a lower bound on the induced error. A simulation illustrating the behavior of the system under this construction can be found in Section III-D.

A. Badly Countered Disturbance Situation

Consider disturbances of the following form:

$$d_{k,1}(t) = d_{k,2}(t) = 0 \quad \text{for all } t < 0, \quad k = 0, 1, \ldots, N$$
$$d_{k,1}(t) = \frac{a \Delta t}{\Delta t^p} \quad \text{for all } t = 0, \Delta t, \ldots, T$$
$$d_{k,2}(t) = \frac{a \Delta t}{\Delta t^p} \quad \text{for all } t = 0, 1, \ldots, N$$

with constants $a > 0$ and $T > 0$ to be specified later.

To compute the evolution of the system under these disturbances, we exploit the finite propagation speed of signals along the chain—namely at most $(m_1, m_2)$ vehicles per time step—to restrict our attention to a central subset of vehicles.

1) Consider the evolution of $e_k$ and $\dot{e}_k$ over one time step, when $N + 1$ vehicles all start with the same state $x_k(0) = 0$ for all $k$, and with controllers initialized at $\xi_k = c_{k,+} = c_{k,-} = 0$ for all $k$. 
We get
\[
e_k(\Delta t) = e_k(0) + \Delta t \dot{e}_k(0) + u_{k-1,2}(0) - u_k,2(0)
+ d_{k-1,2}(0) - d_{k,2}(0)
= u_{k-1,2}(0) - u_{k,2}(0) + \alpha \Delta t^2/N
\]
\[
\dot{e}_k(\Delta t) = \dot{e}_k(0) + u_{k-1,1}(0) - u_k,1(0)
+ d_{k-1,1}(0) - d_{k,1}(0)
= u_{k-1,1}(0) - u_k,1(0) + \alpha \Delta t/N.
\]
Since \(e_k(0)\) and \(\dot{e}_k(0)\) are all equal, it is clear that the control inputs are all equal too; i.e., \(u_{k-1,1} = u_k,1\) and \(u_{k-1,2} = u_k,2\), at least for all the vehicles with \(m_1 < k < N - m_2\). Completely irrespectively of the controller chosen, we, thus, have the following for all \(m_1 < k < N - m_2\):
\[
e_k(\Delta t) = \alpha \Delta t^2/N \quad \text{and} \quad \dot{e}_k(\Delta t) = \alpha \Delta t/N.
\]
Also, \(c_{k,2}(\Delta t), c_{k,1}(\Delta t), \) and \(\xi_k(\Delta t)\) will be equal.

2) Now consider a time \(t = n\Delta t\) for some integer \(n > 0\) and assume that all the state variables satisfy equalities \(e_k(t) = e_j(t), \dot{e}_k(t) = \dot{e}_j(t), c_{k,j}(t) = c_{j,k}(t), \) and \(\dot{\xi}_k(t) = \dot{\xi}_j(t)\), for all \(j, k\) in \([N_{\text{lead}}, N - N_{\text{null}}]\), for some integers \(N_{\text{lead}}, N_{\text{null}} > 0\). Slightly extending the above example, we get
\[
e_k(t + \Delta t) = e_k(t) + \Delta t \dot{e}_k(t) + u_{k-1,2}(t) - u_k,2(t)
+ d_{k-1,2}(t) - d_{k,2}(t)
= e_k(t) + \Delta t \dot{e}_k(t) + \alpha \Delta t^2/N = e_j(t + \Delta t)
\]
for all \(j, k \in [N_{\text{lead}} + m_1, N - (N_{\text{null}} + m_2)]\)
\[
\dot{e}_k(t + \Delta t) = \dot{e}_k(t) + \alpha \Delta t/N = \dot{e}_j(t + \Delta t)
\]
for all \(j, k \in [N_{\text{lead}} + m_1, N - (N_{\text{null}} + m_2)]\) (4) and similarly we maintain \(c_{k,j}(t) = c_{j,k}(t), c_{k,j'}(t) = c_{j,k'}(t), \) and \(\dot{\xi}_k(t) = \dot{\xi}_j(t)\) for that subset of vehicles.

By iterating this argument, we get the following property.

Lemma 1: Consider the system (1) and (2) subject to the particular disturbance (3) and zero initial conditions. Then, for any (well-defined) controller, the solution satisfies
\[
e_k(t) = t(\Delta t) \alpha / (2N)
\]
\[
\dot{e}_k(t) = t \alpha/N
\]
for all \(t \in [0, T]\) and all \(k \in \left(\frac{t}{\Delta t}, m_1, N - \frac{t}{\Delta t}, m_2\right)\).

Proof: The main argument is provided by the explanations preceding the statement. From (4), \(e_k\) is obtained as a sum of \(t/\Delta t\) times the bias \(\Delta t/N\). Then, replacing this into the expression of \(e_k\) in (4), one observes that the increment of \(e_k\) at time \(n = t/\Delta t\) is linear in \(n\), so the standard formula for a linearly progressing series gives the result. 

To be useful at time \(t\), the solution (5) of Lemma 1 should cover at least one vehicle; i.e., \(N - \frac{t}{\Delta t}(m_1 + m_2) \geq 1\). For fixed \(m_1, m_2,\) and \(\Delta t\), we can ensure to have a valid solution for at least \(N/2\) vehicles over the interval \([0, T]\), by taking \(T = \frac{N}{2\alpha \Delta t/m_2}\). As \(m_1\) and \(m_2\) are constants independent of \(N\), we essentially suggest to select the duration \(T\) of the “bad” input disturbance to be of order \(N \Delta t\).

B. Consequences for String Stability

We now investigate what the above construction implies for string stability. First, take Definition 1. For the proposed disturbance, the condition on \(d_{k,1}, d_{k,2}\) becomes
\[
2T^{q/p} \left(\frac{p}{q}\right) \sum_{k=0}^{N} k^q < C_1.
\]
For large \(N\), the dominating term in the sum is \(N^{q+1}\) such that we need in fact \(2T^{q/p} \alpha^q N < C_1\) or, in other words, \(\alpha\) of order \(1 / (N^{1/2}T^{1/p})\). This fixes the allowed disturbance amplitude as a function of its duration and of \(N\).

For the vehicles covered by Lemma 1, we then have
\[
\sum_{n \leq Z} |e_j(n \Delta t)|^p \Delta t \geq \sum_{n = 0}^{T/\Delta t} |e_j(n \Delta t)|^p \Delta t
\]
\[
\geq \left( \frac{T}{n} \right)^{p/\Delta t} \sum_{n = 0}^{T/\Delta t} (n \Delta t)^{2p} \Delta t
\]
\[
\geq \left( \frac{T}{n} \right)^{p/\Delta t} \Delta t^{2p+1} \left( \frac{1}{\Delta t} \right)^{2p+1}.
\]
Toward the last line, we have again used the dominating term in the sum. Combining this with the just obtained bound on \(\alpha\) and with taking \(T\) of order \(N \Delta t\) as suggested at the end of Section III-A, we obtain that \(\sum_{n \leq Z} |e_j(n \Delta t)|^p \) is at least of order \(N^{p/q-1} \Delta t^{2p+1}\).

A similar argument can be repeated for the other definitions of Section II, yielding the following results.

Theorem 1: For the system (1) and (2), there exist disturbances \(d_{k,1}\) and \(d_{k,2}\) satisfying the required respective bounds according to the definitions in Section II and such that, irrespectively of any (well-defined) controller choice, for large \(N\)

\[\text{Definition 1:} \sum_{n \leq Z} |e_j(n \Delta t)|^p \text{ grows as } N^{p/q-1} \Delta t^{2p};
\]

\[\text{Definition 2:} \sum_{n \leq Z} \left( \sum_{e} |e_j(n \Delta t)|^p \right)^q \text{ grows as } N^{q} \Delta t^{2q};
\]

\[\text{Definition 3:} |e_j(n \Delta t)|^p \text{ grows as } N^{q} \Delta t^{2p};
\]

\[\text{Definition 4:} |e_j(t)| \text{ grows as } N^{q} \Delta t^2.
\]

Proof: We will always assume \(T\) of order \(N \Delta t\) and consider the output errors for \(t \in [0, T]\). The computation for Definition 1 is given above. For Definition 2, it is the same, but taking the power \(q/p\) and summing the disturbance over the number of vehicles for which Lemma 1 is valid—this can be of order \(N\) as mentioned in the last paragraph of Section III-A. For Definition 3, the disturbance can be larger, i.e., \(\alpha\) of order \(1/T^{1/p}\) and with respect to Definition 1 computation, this adds a factor \(N^{p/q}\) to the output error. For Definition 4, we can have \(\alpha\) of order \(1\) and since the result of Lemma 1 is valid for \(T\) of order \(N \Delta t\), we can have \(e_k(t)\) of order \(T(T + 1)\) \(\alpha/N \sim N^2 \Delta t^2/N\).

For \(\Delta t\) fixed and \(N\) growing to infinity, this result establishes the impossibility to satisfy any of the definitions of string stability given in Section II, except possibly Definition 1 with \(q = 1\) (which does not appear to have any practical significance; see comments about the definitions in Section II). This impossibility is established in a very general setting, allowing unidirectional or bidirectional symmetric or asymmetric coupling, looking a number of vehicles ahead and behind (as long as that number is independent of \(N\)), communicating with neighbors with any encoding/decoding schemes with possibly packets and event-based logic, and processing all this in an arbitrary nonlinear control system with memory. In particular, even perfect local communication among the vehicles is not sufficient, on its own, to ensure string stability. It is, thus, no wonder that vehicle chain controllers with realistic communication channels have so far required an additional feedback from absolute velocity to achieve string stability [11]–[13].

C. How Telling is the Discrete-Time Controller Setting?

The reader will have noticed that the above impossibility breaks down if we let \(\Delta t\) converge to zero fast enough as \(N\) grows to infinity.
While this does not look like a practical solution, it may express a relevant tradeoff, and, it does create a gap with the pure continuous-time literature. We will, thus, briefly comment on the comparison of this result with the literature on continuous-time, typically LTI systems.

First, note that since our result only follows a sufficient construction, Theorem 2 proves that it is necessary—yet possibly not even sufficient—to let $\Delta t$ go to zero with increasing $N$ in order to satisfy string stability. And indeed, for example, PD coupling with nearest neighbors, we know from LTI results that our disturbance would yield string instability in continuous-time too. This may not be too surprising as a low-frequency disturbance appears to cause most of the problem. Our argument leading to Theorem 2 thus indeed appears to be quite too optimistic still. Now, let us try to discuss which controller features would typically go with a very small $\Delta t$.

1) One obvious effect of smaller $\Delta t$ is faster communication across the vehicle chain. If one could communicate arbitrarily fast, perfectly, and without measurement errors, then each vehicle $k$ could get very fast knowledge of $e_1 + e_2 + \cdots + e_k = x_k - x_0$. One can, then, obviously achieve string stability: Just control each $x_k - x_0$ independently to stabilize each vehicle with respect to the leader. The “distributed system” setting and chain size $N$ play no role anymore. Of course, this idealized situation is unrealistic. In reality, the precision of a message (and, in fact, of a measurement) is in a clear tradeoff to update speed. As long as the communication bandwidth per signal remains bounded when $N$ increases, the imperfections resulting from smaller $\Delta t$ are likely to counterbalance the apparent benefits of smaller $\Delta t$ from our perfect communication model.

2) Setting sensing and communication aside, in practice, the controller’s discretization step $\Delta t$ is chosen as the desired dwell-time before vehicle $k$ reacts to a measurement; thus, in practice, $\Delta t$ converging to zero would mean the controller bandwidth tending to infinity, pointing toward controllers with gain increasing as a function of $N$. It is known indeed that academically speaking, this can provide string stability: In continuous-time, without communication, an LTI controller whose gain increases fast enough with $N$, can ensure string stability. However, as the control gain keeps increasing toward infinity, the effects of unmodeled system limitations and imperfections cannot be neglected forever, and practical problems are likely to appear.

3) Theorem 2, thus, shows anyways that string stability is, at best, not robust to time-discretization. This is important to know toward system simulations, where situations that work only for infinitesimal $\Delta t$ are quickly considered nonrobust for all practical purposes. In a sense, testing robustness to finite $\Delta t$ can even be mathematically compared to the traditional requirement of “no poles cancellation” in the continuous-time setting. Indeed, allowing a decreasingly small $\Delta t$ without any measurement noises can be compared to allowing the precise computation of $\lim_{\Delta t \to 0} \sum_{i=1}^{(t+\Delta t) - (t-1)} \frac{2\pi i}{\Delta t}$ for a signal $s$, i.e., evaluating pure derivatives. For a double-integrator, this implies the possibility of pole cancellation at zero frequency, which is almost always excluded.

4) The dependence on $\Delta t$ is rooted in the fact that we analyze the system before the signals from the edges of the chain reach all the vehicles and make a detailed analysis harder. This does not mean, of course, that the vehicle chain would automatically be stabilized as soon as the signals from the edges have crossed the chain; see, e.g., the simulations in Section III-D. In this sense, it appears that the boundary controllers would play a key role toward string stabilizing the system with infinitesimal $\Delta t$, similarly to PDE control.

These arguments give strong indications to the conjecture that string stability would be impossible with any “reasonable” homogeneous, possibly nonlinear, and communicating controllers, in continuous-time too. At this point of detail, we might argue as well that the digital-controller model is, in fact, closer to applications, than the traditional continuous-time one.

Remark 3: To further connect this result to the existing work, we can look at how the chain reacts to the disturbances acting on the first subsystem only. This has indeed been considered in several continuous-time LTI studies, which we first review now. For unidirectional coupling, the reaction to leader-disturbances is sometimes viewed as a major indicator of general behavior [1], [2]. For bidirectional chains under symmetric coupling, although impossibility results are known from e.g., [7], under the condition of disturbance restricted to the leading vehicle, $\ell_{2,2}$ string stability has been established in [3]. Pushing further the idea of [15] and of [18] and [19] about the possible advantages of slight mistuning in the controller symmetry, we have proved in [20] that a sufficiently asymmetric bidirectional PD controller is sufficient to ensure also the stronger versions of string stability, with respect to the disturbances restricted to a fixed number of leading vehicles. The proof uses an analytic almost-inversion of the system equations based on forward and backward flows, loosely inspired from [21].

This line of work can be related in two ways to the result of this paper. First, we have checked that a discrete-time version of our string stability result in [21] can be worked out perfectly well for the model (1) and (2), see [22]. This suggests that our discrete-time model does enable positive results when the continuous-time model does, i.e., it adds evidence in favor of (1) and (2) not being essentially more constraining than the more standard continuous-time approach. Second, the disturbance proposed in Section III-A is extensively distributed along the vehicle chain, as a function of $N$. This is consistent with a picture of two regimes: When disturbances act on a few vehicles (at known places!), it may be possible to reject them in a string stable way; however, when they are distributed along the whole chain, there is no way to achieve string stability on the basis of relative measurements only.

D. Illustrative Simulation

We can, of course, only illustrate the string instability for a particular choice of controller. However, trusting in the simple analysis in Section III-A, our main argument is independent of the controller choice. Thus, we will just show how indeed the solution of Lemma 1 appears for a simple linear controller without communication. We choose this simplicity to avoid selecting too many elements in the controller design “arbitrarily”—since according to Theorem 2, any attempt is anyways doomed to fail. Complementarily, the simulation shows what happens when the “boundary effects” have propagated throughout the chain, i.e., when the solution of Lemma 1 is not valid anymore; this will depend on the controller choice, but it falls outside the scope of this paper. We, thus, suggest that the reader should not draw too strong conclusions from what happens outside the scope of Lemma 1 with this particular controller.

We take a hint from [18] and [19] and select a PD controller having bidirectional coupling, with the gain on position feedback symmetric toward the preceding and following vehicle, but with the gain on velocity asymmetric. Considering a simple sample-and-hold digital actuation, we will, thus, assume that $u_k(\tau) = f(t, k) := b_1(v_{k-1}(t) - v_k(t)) + b_2(v_{k+1}(t) - v_k(t)) + a(x_{k-1}(t) - x_k(t)) + a(x_{k+1}(t) - x_k(t))$
Disturbance inputs applied to the vehicles, according to (3) but without limiting the time window to \( t < T \) (see the main text), for \( N = 10 \) and for \( N = 50 \) vehicles, respectively. The figure is showing \( d_{k,1}(t) \) as the corresponding \( d_{k,2}(t) \) are just the same multiplied by \( \Delta t \).

\[
\begin{align*}
\Delta t = 1.0 & \quad \text{are just the same multiplied by } d_1, \text{see the main text}, \text{for } t < 50.2 N_d. \\
\Delta t = 5.0 & \quad \text{at time } t = 5.0, \text{for a vehicle } t = 2, \text{and for } t = 0.2, \text{as illus-}
\end{align*}
\]

Fig. 1. Disturbance inputs applied to the vehicles, according to (3) but without limiting the time window to \( t < T \) (see the main text), for \( N = 10 \) and for \( N = 50 \) vehicles, respectively. The figure is showing \( d_{k,1}(t) \) as the corresponding \( d_{k,2}(t) \) are just the same multiplied by \( \Delta t \).

\[
\begin{align*}
e(t) & = 0 & \quad \text{for } t < T \text{as } e(t) = 0 & \quad \text{at time } in independent of } N, \text{for a vehicle } t = 2, \text{and for } t = 0.2, \text{as illus-}
\end{align*}
\]

Fig. 2. Evolution of the distance errors \( e_k(t) \) for \( t > T \), for a vehicle chain (see the main text for details) subject to the input disturbances shown in Fig. 1, and for \( N = 10 \) and \( N = 50 \), respectively. Note the different scales on both axes. In agreement with our analysis, we have taken \( T = N \Delta t/5 \). For this time interval, a significant number of \( e_k(t) \) is supposed to follow the solution described by Lemma 1; the latter is plotted as black dots, which indeed superimpose with a number of simulated curves.

\[
\begin{align*}
\Delta t = 1.0 & \quad \text{in continuous-time. After exact integration, this}
\end{align*}
\]

The simulation takes arbitrary values \( a = 1, b_1 = 2, b_2 = 0.5, \text{and } \Delta t = 0.1; \text{for the first and last vehicle, we just drop from the feedback law the term associated to the missing neighbor.}

The precise scaling of input disturbances to apply and of output signals to monitor, depends on the definition of string stability that one wishes to consider. We will illustrate the BIBO type scaling of Definition 4, with \( \alpha = 1 \) independent of \( N \). As the illustrated controller is linear, it is just a matter of scaling to translate the simulation to other definitions.

Fig. 3. Evolution of \( e_k(t) \) for larger times \( t \); i.e., the plots in Fig. 2 are, in fact, zooms on the beginning of the present plots. For \( O(t) > T \) (marked with a black dot), the result of Lemma 1 no longer holds and the system behavior does depend on the particular controller choice. We observe that, the chosen controller: (i) for fixed \( N \) indeed each error eventually stabilizes to a bounded value; and (ii) the analysis of Lemma 1, i.e., with errors taken into account only up to time \( t = T \), is very optimistic.

In Fig. 2, we show the simulated spacing error between consecutive vehicles under this model, up to \( t = T \) and for two different chain lengths \( N = 10 \) and \( N = 50 \). The black squares are the solution given by Lemma 1. One indeed observes that many (central) vehicles follow this solution, while others are progressively affected by the boundary effects and behave differently. Note that as \( N \) increases, in accordance with Lemma 1, the error at a given time becomes lower; this is due to the consecutive vehicles’ disturbance inputs becoming more similar as \( N \) increases. However, in return, the solution of Lemma 1 remains valid for a longer time \( T \) and, therefore, the overall the error at time \( T \), which we can easily compute, keeps increasing unboundedly with \( N \).

We can also have a look at the behavior of the chain for \( t > T \); see Fig. 3. A striking observation is that the errors keep increasing way beyond the point covered by our analysis (black dot very close to the origin). However, as this depends on the chosen controller, we must be careful about further conclusions. The main conclusion might, thus, just be that for a fixed \( N \), the chosen controller indeed stabilizes the errors to bounded values; i.e., it does effectively stabilize the system. Only, it does not so uniformly in \( N \), and this is what string instability essentially means.

IV. CONCLUDING DISCUSSION

This paper significantly extends the scope of an impossibility result regarding the string stability toward input disturbances acting on all subsystems. Indeed, while the existing results have centered on LTI systems, we here allow controllers to be nonlinear, \( N \)-dependent, and time varying, thus possibly modulated and digitally quantized—as well as using any type of perfect local communication at finite speed. The analysis involves no complicated elements once the setting and example are identified, but as the search for alternative controllers had remained open so far, it appears to give a definite answer, clearly narrowing down the options toward achieving string stability. The following are the essential features for the impossibility.

1) Second-order integrator model for individual subsystems: If the dynamics was first order, our counterexample would not work.

2) Relative measurements: Variations that do solve string stability by adding an absolute velocity term are known; see, e.g., time-headway policies [8]–[10]. With respect to this criterion, academ-
ically, string instability appears more than ever as a property of
distributed sensing. In practice, using absolute velocity in the feed-
back controller or damping becomes a question of hardware and
application tradeoff. We must mention that the string instability
issue is not directly linked to the low observability for long-range
modes in distributed systems with relative measurements [16]. In-
deed, here the target variables are not the absolute displacements
\( x_k \), for which, indeed, there would be an observability issue, but
rather the relative displacements \( e_l \), which are directly measured.
Also, see the previous point.

3) Homogeneous controller, i.e., the same logic with the same param-
eter values at all the vehicles: Technically, the possibility remains
that heterogeneous controllers, i.e., letting the different vehicles
react differently to the same signals, could solve the issue. How-
ever, we currently have no clue how to design this heterogeneity—
unless one would allow parameters increasing unboundedly with
chain length \( N \), which, however, would pose other obvious prob-
lems. Controllers periodic in vehicle number do not seem to work.

4) Discrete-time controller: This should be representative in practice
of a realistic digital controller. Rigorously, our counterexample
analysis would break down when reducing the discretization step
\( \Delta t \) with \( N \). However, a property that only holds with infinitely large
bandwidth \( 1/\Delta t \) for communication and/or control, is usually not
robust in practice; this suggests that any “reasonable” continuous-
time controller would fail too. Note that the standard string stability
model here includes neither measurement nor communication im-
perfections, while with extreme continuous-time controllers that
are badly modeled by finite \( \Delta t \), those can be expected to become
important.

Also, note that we have only identified one particular, badly rejected
disturbance input. There may be disturbance inputs for which the sys-
tem reaction is even worse.

With this, we believe to have given at least a much more comprehen-
sive picture of what can be done on the standard academic property
of string stability. If this string stability property appears critical in some
key applications, those results should help guide a possible search for
very particular controllers to achieve it, if it is feasible at all. A point
that we did not study is string stability with respect to disturbances on
the initial state, instead of on input signals; a similar analysis might be
possible.

A different option for the future is to acknowledge that the academic
definition of string stability is too strong to be useful, thus even an
extended framework with nonlinear controllers and so on. In that sense,
we can think of two reasonable variations on string stability.

1) One option is to consider the tradeoff in a more integrated pic-
ture for finite \( N \): To have a given acceptable error, what are the
best possible combinations of absolute-velocity-dependence \( h \), control-communication bandwidths \( 1/\Delta t \), possibly nonlinear
effects, and associated gains in the presence of other noises, as a
function of chain length \( N \)? This study would be carried out while
knowing that the limit for infinite \( N \) will not work, but will also
not be essential for most applications.

2) Another approach would be, as already mentioned in Remark 2, to
acknowledge that the worst-case formulation of string stability is
too strong: As the worst-case disturbance could become more and
more unlikely with increasing \( N \), it might be more telling to take
the limit \( N \to \infty \) with a probability distribution over disturbances.
In [17], precisely this approach is taken for the behavior of a lattice
of simple linearly coupled systems.

As a final word, we may reflect upon the more profound implications
of our impossibility result. The investigation of [17], for instance, is
motivated by the stability of physical matter, which, after all, appears
to be governed by forces depending on relative states. Implications are
also expected for the numerical simulation of related PDEs. It may be
an important theoretical aim to pin down what essential element in the
system structure leads to this impossibility.

A first point in this direction is that the analysis toward our Theorem
2 can be easily extended to other spatial interconnection structures,
e.g., a \( D \)-dimensional lattice of \( N \) possibly nonlinear systems. One has the following.

1) We can keep our counterexample with \( d_k \) increasing along one
dimension of the lattice from 0 to \( \alpha \) with steps \( \frac{\alpha}{N^{1/D}} \), and constant
along the other dimensions.

2) Computing the acceptable \( T \) and \( \alpha \) for each case, we get the
relevant error growing like \( N^{\beta} \), where \( \beta \geq 0 \) depends on the lattice
dimension and on the choice of definition, but it is always \( > 0 \) for
Definitions 2–4.

Compared to [17], we thus generalize the setting by allowing any
nonlinear, time-varying local interactions toward improving the situa-
tion, but we obtain a more negative result by considering the worst
bounded disturbance, over time and over subsystem indices. If the aim
is to break a system into parts, this particular disturbance may be useful
insight. In contrast, to understand the stability of lattices in a natural
environment, one may has to acknowledge that bad disturbances, in
fact, become negligibly probable with increasing \( N \). This sets a maybe
unexpected link between distributed systems and error correcting
codes, where scaling to larger codes must essentially rely on the in-
creasing unlikelihood of uncorrectable errors [23].

References

[1] D. Swaroop and J. K. Hedrick, “String stability of interconnected
systems,” IEEE Trans. Autom. Control, vol. 41, pp. 349-357, Mar.
1996.

[2] D. Swaroop, “String stability of interconnected systems: An applica-
tion to platooning in automated highway systems,” Ph.D. dissertation, Dept.
Mech. Eng., Inst. Transp. Stud., Univ. California, Berkeley, Berkeley, CA,
USA, 1994.

[3] K. Yamamoto and M. C. Smith, “Bounded disturbance amplification for
mass chains with passive interconnection,” IEEE Trans. Autom. Control,
vol. 61, no. 6, pp. 1565-1574, Jun. 2016.

[4] Y. Yamamoto and M. C. Smith, “Design of passive interconnections in
tall buildings subject to earthquake disturbances to suppress inter-storey
drifts,” J. Phys.; Conf. Ser., vol. 744, 2016, Art. no. 012063.

[5] S. Sheikhholeslam and C. Desoer, “Longitudinal control of a platoon of
vehicles,” in Proc. Amer. Control Conf., 1990, pp. 291-297.

[6] P. Seiler, A. Pant, and K. Hedrick, “Disturbance propagation in vehicle
strings,” IEEE Trans. Autom. Control, vol. 37, no. 10, pp. 1835-1842,
Oct. 2004.

[7] P. Barooah and J. P. Hespanha, “Error amplification and disturbance prop-
agation in vehicle strings with decentralized linear control,” in Proc.
IEEE Conf. Decis. Control, 2005, pp. 4964-4969.

[8] S. Klinge and R. H. Middleton, “Time headway requirements for string
stability of homogenous linear unidirectionally connected systems,” in
Proc. IEEE Conf. Decis. Control, 2009, pp. 1992–1997.

[9] J. A. Rogge and D. Aeyels, “Vehicle platoons through ring cou-
ping,” IEEE Trans. Autom. Control, vol. 53, no. 6, pp. 1370-1377, Jul.
2008.

[10] S. Knorn, A. Donaire, J. C. Aguero, and R. H. Middleton, “Passivity-
based control for multi-vehicle systems subject to string constraints,”
Automatica, vol. 50, pp. 3224–3230, 2014.

[11] J. Ploeg, D. P. Shukla, N. van de Wouw, and H. Nijmeijer, “Controller syn-
thesis for string stability of vehicle platoons,” IEEE Trans. Intell. Transp.
Syst., vol. 15, no. 2, pp. 854–865, Apr. 2014.

[12] J. Ploeg, N. van de Wouw, and H. Nijmeijer, “l_p string stability of cas-
caded systems: Application to vehicle platooning,” IEEE Trans. Control
Syst. Technol., vol. 22, no. 2, pp. 786–793, Mar. 2014.

[13] B. Besseling and K. H. Johansson, “String stability and a delay-based
spacing policy for vehicle platoons subject to disturbances,” IEEE Trans.
Autom. Control, vol. 62, no. 9, pp. 4376–4391, Sep. 2017.
[14] S. Stüdli, M. M. Seron, and R. H. Middleton, “From vehicular platoons to general networked systems: String stability and related concepts,” *Annu. Rev. Control*, vol. 44, pp. 157–172, 2017.

[15] P. Barooah, P. G. Mehta, and J. P. Hespanha, “Mistuning-based control design to improve closed-loop stability of vehicular platoons,” *IEEE Trans. Autom. Control*, vol. 54, no. 9, pp. 2100–2113, Sep. 2009.

[16] A. Sarlette and R. Sepulchre, “Control limitations from distributed sensing: Theory and extremely large telescope application,” *Automatica*, vol. 50, pp. 421–430, 2014.

[17] B. Bamieh, M. R. Jovanovic, P. Mitra, and S. Patterson, “Coherence in large-scale networks: dimension dependent limitations of local feedback,” *IEEE Trans. Autom. Control*, vol. 57, pp. 2235–2249, Sep. 2012.

[18] I. Herman, S. Knorn, and A. Ahlen, “Disturbance scaling in bidirectional vehicle platoons with different asymmetry in position and velocity coupling,” *Automatica*, vol. 82, pp. 13–20, 2017.

[19] I. Herman, D. Martinec, Z. Hurak, and M. Sebek, “Nonzero bound on Fiedler eigenvalue causes exponential growth of H-infinity norm of vehicular platoon,” *IEEE Trans. Autom. Control*, vol. 60, no. 8, pp. 2248–2253, Aug. 2015.

[20] A. Farnam and A. Sarlette, “String stability towards leader thanks to asymmetric bidirectional controller,” in *Proc. 20th IFAC World Congr.*, 2017, pp. 10335–10341.

[21] W. J. O’Connor, “Wave-based analysis and control of lump-modeled flexible robots,” *IEEE Trans. Robot.*, vol. 23, no. 2, pp. 342–352, Apr. 2007.

[22] A. Farnam, “Towards impossibility and possibility results for string stability of platoon of vehicles,” Ph.D. dissertation, Dept. Electron. Inf. Syst., Ghent Univ., Ghent, Belgium, 2018.

[23] F. G. MacWilliams and N. J. A. Sloane, *The Theory of Error-correcting Codes*. Amsterdam, The Netherlands: Elsevier, 1977.