INTRODUCTION

Let us consider the problem of optimal design of a telecommunications network. In such a network, digital data are to be sent between pairs of terminal nodes. The communicating terminals are, of course, not connected directly to each other but only indirectly via a network of switching nodes. In operating and maintaining a telecommunications network, several optimization problems need to be solved, each on a different time scale.

For the routing problem, the network is considered fixed, and certain demands are given. The direct connections (links) between the switching nodes of a given network have certain given transmission capacities, usually measured in units of GiBit/s. The objective of the routing problem is to find a way to send the data along paths of the network, so that the transmission capacities on the individual links are not exceeded. This makes the routing problem a classical problem of combinatorial optimization, namely a multicommodity flow problem. However, the routing problem usually is complicated by further requirements, like quality-of-service (QoS) guarantees. For instance, for voice and multimedia communications, it is essential to bound the delays, i.e., the amount of time between sending and receiving a data packet, along the paths of a routing for the data of a terminal pair. In these situations, it is often also requested to send the data of a terminal pair along a unique path of the network, i.e., the flow cannot be split.

On a larger time scale, of course, the capacities of the links are not fixed. The capacities are determined both by the characteristics of the cables and by the switching nodes. The cables are replaced very rarely because this is extremely expensive; however the capacities of most links can be increased by replacing the transmitter–receiver technology installed in the switching nodes. We note that, in fiber-optic technology, the installed capacities are usually symmetric. In the capacitated network design problem, or more precisely the capacity expansion problem, a
set of (projected) demands are given, which the current network cannot satisfy; the objective is to find a set of links and new capacities, such that the demands can be satisfied, and such that the installation costs are minimal. On an even larger time scale, also the addition of new links, changing the topology of the network, can be planned and implemented, to increase the capability of the network. Network design problems have been considered in a large body of literature, see for instance 

Bienstock and Günlük (1996), Bienstock et al. (1998), Atamtürk (2002) and the references within.

In network design problems, one is often concerned with survivability requirements. When one link or several links fail, there should still be a routing that satisfies all demands; this requirement is usually called arc survivability. Similarly, when a node fails, there should still be a routing that satisfies all demands. However, the demands where the failing node is the source or the destination need not be considered. This requirement is called node survivability. Network design problems under node and arc survivability constraints have been studied in the literature, see for instance Grötschel and Monma (1990), Grötschel et al. (1992), Stoer (1992), Stoer and Dahl (1994), Lissner et al. (1995), Bienstock and Muratore (2001).

In the following we will discuss some of the models and solution approaches regarding network design problems under survivability constraints. To this end, let us fix some notation. Let $c$ denote a vector of installed capacities, and let $d$ denote a vector of demands; we shall make the spaces where these vectors live precise in the following sections. We shall denote by $F_{c,d}$ the set of feasible routings. Again, the precise space that $F_{c,d}$ is a subset of will depend on the choice of the formulation; we will discuss it later.

Stoer and Dahl (1994) seem to have been the first to address survivability questions in capacitated networks; before that paper, research had focused on modeling graph-theoretic connectivity constraints. They based their model on the characterization of capacities that allow a fractional routing of the demands by so-called metric inequalities (Iri, 1971, Onaga and Kakusho, 1971). Therefore their model does not contain variables for the no-fault routing and the fault routings; such a model is known as a projected formulation.

There are two shortcomings in this approach. First, allowing fractional routings often is too weak a relaxation of the problem. For the existence of an integer multicommodity flow, however, there is no known polyhedral description in general. Therefore, no projected formulation for the case of integral routings is known in the literature. Second, more complicated constraints like those coming from QoS guarantees cannot be incorporated into the model of this approach. For instance, bounds on the admissible delays along paths of the routing cannot be modeled, since there are no path variables present. Stoer and Dahl (1994) and various other authors consider the model of complete rerouting, i.e., in the case of a fault the new routing can be completely unrelated to the no-fault routing.

Other models of survivability require that in the case of a fault only the interrupted flow needs to be rerouted. There are two main models here. In the model of local rerouting, the failure of a link $e = (u, v)$ carrying aggregated flow $x_{e}^{agg}$ in the no-fault routing creates a new demand to route $x_{e}^{agg}$ units of flow from node $u$ to node $v$ in an auxiliary network where the failed link $e$ is not present. Local rerouting is also called link-based re-establishment in parts of the literature (Pièro and Medhi, 2004). In the model of global rerouting, first the flow along paths that use the failed link is removed from the no-fault routing; in this way, some capacities on links along these paths are released. Then the removed parts of the demands are routed through the auxiliary network. Clearly global rerouting is more economical than local rerouting, since the released capacities can be used...
for the fault routings. Global rerouting is also called path-based re-establishment (Pióro and Medhi, 2004).

**Contribution of this paper.** In this paper we investigate the structure of the family of feasible-routing sets \( F_{c,d} \) when the installed capacities \( c \) and the required demands \( d \) vary. In our mathematical model, both the capacities and the demands can take arbitrary (non-negative) integer values. Thus the family of feasible-routing sets is infinite. The key observation in order to be able to deal with this infinite family is that Minkowski sums of these sets play an important rôle. We will show that in general the feasible-routing sets satisfy

\[
F_{c^1,d^1} + F_{c^2,d^2} \subseteq F_{c^1+c^2,d^1+d^2};
\]

then we consider the pairs \((c^1, d^1)\) and \((c^2, d^2)\) where the inclusion is in fact an equality.

First, it is possible to address the question whether a capacity installation \( c \) is feasible for a demand vector \( d \). Whenever there is a pair \((c^1, d^1)\) and \((c^2, d^2)\) such that \( c = c^1 + c^2 \) and \( d = d^1 + d^2 \) and in (1) equality holds, then \( c \) is feasible for \( d \) if and only if \( c^1 \) is feasible for \( d^1 \) and \( c^2 \) is feasible for \( d^2 \).

Second, it is possible to address survivability questions as well. As a first example, let \( a \) be an arbitrary link (arc) of the network. We will define a real-valued functional \( g_a \) on the family of feasible-routing sets by defining \( g_a(F_{c,d}) \) to be the minimum amount of flow carried on \( a \) in any routing of \( F_{c,d} \). Then the network with capacity installation \( c \) and demand \( d \) is arc-survivable (in the model of complete rerouting) if and only if \( g_a(F_{c,d}) = 0 \) for all arcs \( a \). It turns out the functional \( g_a \) is Minkowski-additive. This implies that, if equality holds in (1), then \((c, d)\) defines a survivable network if and only if \((c^1, d^1)\) and \((c^2, d^2)\) do.

**Outline.** The outline of the paper is as follows. In section 2 we introduce a class of mathematical optimization problems called (mixed-)integer Minkowski programs as our general model; their formulation includes constraints involving non-linear functionals like \( g_a \). We prove a general reformulation theorem that asserts that mixed-integer Minkowski programs can be reformulated as integer linear programs, whenever there exists a suitable finite generating set.

In section 3 and section 4, we show that in two important settings, related to fractional and integral routing, there exists a finite generating set for the families of feasible-routing sets \( F_{c,d} \). In section 5 we make the formulations of the network design problems precise and discuss the structure of the generating sets that arise. This enables us to reformulate the network design problem as an integer linear program.

In section 6 we address questions of survivable network design in the model of complete rerouting both for fractional and integral flows. Again the survivability conditions can be modeled using Minkowski-additive functionals in the framework of integer Minkowski programs. The general reformulation theorem, applied to this formulation, then gives a new formulation of the survivable network design problem as an integer linear program. The resulting formulation as an integer linear program does not contain variables for the no-fault routing and the routings in the individual failure scenarios. Hence, the formulation can be regarded as a projected formulation. This is particularly relevant in the case of integral routings, where no projected formulation was previously known.

A prospective advantage of the proposed formulation of the survivable network design problem is that the formulation only grows moderately with the number of failure scenarios. Indeed, only one linear inequality and no extra variable is needed for each failure scenario. The number of variables only depends on the topology
of the network. Moreover, the reformulation only needs to be computed once for a given network topology; it can then be used for solving the survivable network design problem for arbitrary demands. On the other hand, the usual non-projected formulations contain one set of routing variables for every failure scenario.

2. Integer Minkowski programs

2.1. Definitions and first results. In this section we define (mixed-)integer Minkowski programs and show that under certain conditions there exist reformulations as integer linear programs. (Mixed-)integer Minkowski programs consist of integer variables, a linear objective function and inequality constraints involving a certain class of nonlinear functions. We first remind the reader of the definition of Minkowski sums.

Definition 2.1.

(i) Let \( A_1, A_2 \subseteq \mathbb{R}^d \) be sets. The Minkowski sum (pointwise sum) of \( A_1 \) and \( A_2 \) is defined as

\[
A_1 + A_2 = \{ x^1 + x^2 : x^1 \in A_1, x^2 \in A_2 \}.
\]

(ii) Let \( A \subseteq \mathbb{R}^d \) be a set and let \( k \in \mathbb{Z}_+ \) be a non-negative integer. We will denote by \( kA \) the set obtained by taking the Minkowski sum of \( k \) copies of the set \( A \),

\[
kA = \{ x^1 + \cdots + x^k : x^1, \ldots, x^k \in A \}.
\]

In the same way, we will use the notation \( \sum_{i \in I} k_i A_i \).

Note that when one of the summands is the empty set, then the Minkowski sum is empty too.

Remark 2.2. Frequently, Minkowski sums are considered in the context of convex sets only. For convex sets, the notation \( kA \) is usually defined for arbitrary scalars \( k \in \mathbb{R} \) as

\[
\{ kx : x \in A \};
\]

for \( k \in \mathbb{Z}_+ \) this gives the same set as in (2). However, when \( A \) is not a convex set, the set \( kA \) as defined in (2) is not necessarily the same as the set \( \{ kx : x \in A \} \).

We will consider a class of nonlinear functions that are obtained in a two-stage process. In the first stage, we define a set-valued mapping \( \mathcal{A} : \mathbb{Z}^n \rightarrow 2^{\mathbb{R}^d_+} \) that is superadditive:

Definition 2.3. A set-valued mapping \( \mathcal{A} : \mathbb{Z}^n \rightarrow 2^{\mathbb{R}^d_+} \) is called superadditive if it satisfies

\[
\mathcal{A}(z^1 + z^2) \supseteq \mathcal{A}(z^1) + \mathcal{A}(z^2) \quad \text{for } z_1, z_2 \in \mathbb{Z}^n,
\]

where + in the right-hand side denotes a Minkowski sum.

In the second stage, we define a functional \( \kappa \) on the range of \( \mathcal{A} \) with the following properties:

Definition 2.4. Let \( \kappa : 2^{\mathbb{R}^d_+} \supseteq \mathcal{A}(\mathbb{Z}^n) \rightarrow \mathbb{R} = \mathbb{R} \cup \{ +\infty \} \) be a functional with \( \kappa(\emptyset) = +\infty \) and \( A, B, C \in \mathcal{A}(\mathbb{Z}^n) \).

(i) \( \kappa \) is called Minkowski-additive if \( C = A + B \) implies \( \kappa(C) = \kappa(A) + \kappa(B) \).

(ii) \( \kappa \) is called antitone if \( A \subseteq B \) implies \( \kappa(B) \leq \kappa(A) \).

With these preparations, we are now in the position to define (mixed-)integer Minkowski programs.
Definition 2.5. A (mixed-)integer Minkowski program is an optimization problem of the following structure:

\[
\begin{align*}
\min & \quad W^T z \\
\text{s.t.} & \quad A(z) \neq \emptyset \\
& \quad B \cdot z = v \\
& \quad \kappa^i(A(z)) + h_i^T z \leq \delta_i \quad \text{for } i = 1, \ldots, k \\
& \quad z \in \mathbb{Z}^n,
\end{align*}
\]

where \( W \in \mathbb{R}^n \) is a cost vector, \( A : \mathbb{Z}^n \to 2^\mathbb{R}^d \) is a superadditive set-valued mapping, \( B \in \mathbb{Z}^{m \times n} \) a matrix, \( v \in \mathbb{Z}^m \), \( \kappa : \mathbb{A}(\mathbb{Z}^n) \to \mathbb{R} \) for \( i = 1, \ldots, k \) a Minkowski-additive and antitone functional, and \( h_i \in \mathbb{R}^n \) for \( i = 1, \ldots, k \).

In the mixed-integer case, we have \( A(z) \subseteq \mathbb{R}^d_+ \) for all \( z \in \mathbb{Z}^n \), whereas in the integer case \( A(z) \subseteq \mathbb{Z}^d_+ \) for all \( z \in \mathbb{Z}^n \).

The following theorem will concretize the conditions under which we can reformulate a (mixed-)integer Minkowski program as an integer linear program.

Theorem 2.6. Let \( \{\bar{z}^j\}_j \subseteq \mathbb{Z}^n \) with \( A(\bar{z}^j) \neq \emptyset \) for all \( j \in J \) be a finite generating set of the family of sets \( A(z) \), i.e., for \( z \in \mathbb{Z}^n \) there is a representation

\[
z = \sum_{j \in J} \lambda_j \bar{z}^j \quad \text{where} \quad \lambda_j \in \mathbb{Z}_+,
\]

such that

\[
A(z) = A\left(\sum_{j \in J} \lambda_j \bar{z}^j\right) = \sum_{j \in J} \lambda_j A(\bar{z}^j).
\]

Then the (mixed-)integer Minkowski program \( 4 \) can be reformulated as an integer linear program:

\[
\begin{align*}
\min & \quad \sum_{j \in J} (W^T \bar{z}^j) \lambda_j \\
\text{s.t.} & \quad \sum_{j \in J} (B \bar{z}^j) \lambda_j = v \\
& \quad \sum_{j \in J} \left[ \kappa^i(A(\bar{z}^j)) + h_i^T \bar{z}^j \right] \lambda_j \leq \delta_i \quad \text{for } i = 1, \ldots, k \\
& \quad \lambda_j \in \mathbb{Z}_+ \\
& \quad j \in J
\end{align*}
\]

Proof. Let \( \lambda_j^* \), \( j \in J \) be a feasible solution of the integer program \( 4 \). We need to prove that \( z^* = \sum_{j \in J} \lambda_j^* \bar{z}^j \) is a feasible solution of the corresponding (mixed-)integer Minkowski program \( 4 \). Relation \( 5 \) yields:

\[
A(z^*) = A\left(\sum_{j \in J} \lambda_j^* \bar{z}^j\right) \subseteq \sum_{j \in J} \lambda_j^* A(\bar{z}^j),
\]

and as \( A(\bar{z}^j) \neq \emptyset \) this implies \( A(z^*) \neq \emptyset \). We also have \( B \cdot z^* = v \). It remains to prove that the linear constraints of the (mixed-)integer Minkowski program are satisfied. Let \( i \in \{1, \ldots, k\} \). As \( \kappa^i \) is an antitone functional for all \( i = 1, \ldots, k \) and as \( 6 \) holds, we have:

\[
\kappa^i(A(z^*)) + h_i^T z^* \leq \sum_{j \in J} \lambda_j^* \kappa^i(A(\bar{z}^j)) + \sum_{j \in J} \lambda_j^* h_i^T \bar{z}^j \leq \delta_i.
\]

This proves the feasibility of \( z^* = \sum_{j \in J} \lambda_j^* \bar{z}^j \) for \( 4 \).
On the other hand, let \( \tilde{z} \) be a feasible solution of the (mixed-)integer Minkowski program \([4]\). We know that there is a decomposition \( \tilde{z} = \sum_{j \in J} \beta_j \tilde{z}^j \) such that

\[
\mathcal{A}(\tilde{z}) = \sum_{j \in J} \beta_j \mathcal{A}(\tilde{z}^j),
\]

where \( \beta_j \in \mathbb{Z}_+ \) for all \( j \in J \). We shall prove that \( \beta_j, j \in J, \) is a feasible solution of the integer linear program \([4]\). Surely we have \( B \cdot \tilde{z} = \sum_{j \in J} (B \tilde{z}^j) \beta_j = v \). Let \( i \in \{1, \ldots, k\} \) be fixed. As \( \kappa^i \) is a Minkowski-additive functional for all \( i = 1, \ldots, k \) we have

\[
\delta_i \geq \kappa^i(\mathcal{A}(\tilde{z})) + h_i^T \tilde{z} = \sum_{j \in J} \beta_j [\kappa^i(\mathcal{A}(\tilde{z}^j)) + h_i^T \tilde{z}^j].
\]

Therefore, \( \beta_j, j \in J \) is a feasible solution of the integer linear program \([4]\).

Clearly also the objective value of the integer linear program \([4]\) is equal to the objective value of the corresponding (mixed-)integer Minkowski program \([4]\) for every cost vector \( W \in \mathbb{R}^n \).

2.2. Families of truncated sets and truncated integer Minkowski programs. The results of the previous subsection may be improved in the sense that it is possible to formulate integer Minkowski programs on families of truncated sets as integer linear program. This result is important for our application in network design because there are usually lots of circulations of flow as feasible solutions for the multicommodity flow problem which are not used for a “regular” objective function and therefore should be cut away. We give the definition of a truncated set first and continue giving the results for truncated (mixed-)integer Minkowski programs.

**Definition 2.7.** Let \( u, v \in \mathbb{R}^n \) be vectors. We say that \( u \) reduces \( v \) and denote \( u \preceq v \) if \( u^{(i)} v^{(i)} \geq 0 \) and \( |u^{(i)}| \leq |v^{(i)}| \) for all components \( i = 1, \ldots, n \).

**Definition 2.8.** Let \( \mathcal{C} \subseteq \mathbb{R}^n \) (or \( \mathcal{C} \subseteq \mathbb{Z}^n \)) be a set of vectors. Let \( \mathcal{A} : \mathbb{Z}^n \to 2^{\mathbb{R}^d} \) be a set-valued mapping which is super-additive and \( z \in \mathbb{Z}^n \). We call

\[
\text{tr}_C(\mathcal{A}(z)) = \{ x \in \mathcal{A}(z) : \exists c \in C \text{ with } c \subseteq x \}
\]

the truncated set corresponding to \( z \) and \( \mathcal{A} \) with respect to \( \mathcal{C} \).

The definition of truncated sets is illustrated in Figure 1 for a discrete set of vectors and a polytope. Both are truncated by a fixed set \( \mathcal{C} \) of two vectors.

**Definition 2.9.** A truncated set \( \text{tr}_C(\mathcal{A}(z)) \) is called indecomposable if there is no \( 0 \neq z_1, z_2 \in \mathbb{Z}^n \) with

\[
\text{tr}_C(\mathcal{A}(z)) \subseteq \text{tr}_C(\mathcal{A}(z_1)) + \text{tr}_C(\mathcal{A}(z_2)).
\]

**Lemma 2.10.** If \( \text{tr}_C(\mathcal{A}(z)) \) is indecomposable then \( \mathcal{A}(z) \) is indecomposable, too.

**Proof.** We have \( \text{tr}_C(\mathcal{A}(z)) \subseteq \mathcal{A}(z) \). Suppose there is a decomposition \( \mathcal{A}(z) = \mathcal{A}(z_1) + \mathcal{A}(z_2) \). Let \( x \in \text{tr}_C(\mathcal{A}(z)) \subseteq \mathcal{A}(z) \); then there is a decomposition \( x = x_1 + x_2 \) with \( x_i \in \mathcal{A}(z_i), i = 1, 2 \). In particular: \( x_i \subseteq x \) (because \( x_i \in \mathbb{R}^d \)). Suppose there is \( c \in C \) with (w.l.o.g.) \( c \subseteq z_1 \), implying \( c \subseteq z \), which is a contradiction. Thus \( z_i \in \text{tr}_C(\mathcal{A}(z_i)) \) and we have \( \text{tr}_C(\mathcal{A}(z)) \subseteq \text{tr}_C(\mathcal{A}(z_1)) + \text{tr}_C(\mathcal{A}(z_2)) \), contradicting the indecomposability of \( \text{tr}_C(\mathcal{A}(z)) \).

The previous lemma shows that there is a finite generating system of families of truncated sets provided there is such a finite generating system for the corresponding family of sets. We go on showing that under certain conditions it is possible to reformulate integer Minkowski programs on truncated sets as integer linear programs.
Furthermore, let $W \in \mathbb{R}^n$ denote a vector of costs, $B \in \mathbb{Z}^{n \times n}$ a matrix, $v \in \mathbb{Z}^n$, $\kappa^i: \mathcal{A}(\mathbb{Z}^n) \to \mathbb{R}$ for $i = 1, \ldots, k$ is a Minkowski-additive and antitone functional, and $h_i \in \mathbb{R}^n$ for $i = 1, \ldots, k$. We consider the following truncated integer Minkowski program:

$$\begin{align*}
\min & \quad W^T z \\
\text{s.t.} & \quad \text{tr}_C(A(z)) \neq \emptyset \\
& \quad B \cdot z = v \\
& \quad \kappa^i(\text{tr}_C(A(z))) + h_i^T z \leq \delta_i \quad \text{for } i = 1, \ldots, k \\
& \quad z \in \mathbb{Z}^n.
\end{align*} \quad (7)$$

**Lemma 2.11.** Let $\{\bar{z}_j\}_{j \in \mathcal{J}}$ denote the finite generating system of a family of truncated sets, i.e., for $z \in \mathbb{Z}^n$ there are $\lambda_j \in \mathbb{Z}_+$, $j \in \mathcal{J}$, such that $z = \sum_{j \in \mathcal{J}} \lambda_j \bar{z}_j$ and

$$\text{tr}_C(A(z)) \subseteq \sum_{j \in \mathcal{J}} \lambda_j \text{tr}_C(A(\bar{z}_j))$$

If $\kappa^i(\text{tr}_C(A(z))) = \kappa^i(A(z))$ for all $z \in \mathbb{Z}^n$ and $i = 1, \ldots, k$ then we can reformulate the truncated integer Minkowski program (7) as integer linear program:

$$\begin{align*}
\min & \quad \sum_{j \in \mathcal{J}} (W^T \bar{z}_j) \lambda_j \\
\text{s.t.} & \quad \sum_{j \in \mathcal{J}} (B \bar{z}_j) \lambda_j = v \\
& \quad \sum_{j \in \mathcal{J}} [\kappa^i(\text{tr}_C(A(\bar{z}_j))) + h_i^T \bar{z}_j] \lambda_j \leq \delta_i \quad \text{for } i = 1, \ldots, k \\
& \quad \lambda_j \in \mathbb{Z}_+ \\
& \quad \lambda_j \in \mathbb{Z}_+ \quad \text{for } j \in \mathcal{J}
\end{align*} \quad (8)$$

**Proof.** Let $z$ be a feasible solution of the truncated integer Minkowski program (7). Then there are $\beta_j \in \mathbb{Z}_+$ such that $z = \sum_{j \in \mathcal{J}} \beta_j \bar{z}_j$ and

$$\text{tr}_C(A(z)) \subseteq \sum_{j \in \mathcal{J}} \beta_j \text{tr}_C(A(\bar{z}_j)).$$
It is clear that \( \sum_{j \in J} (B \bar{z}^j) \lambda_j = v \) holds and as the functions \( \kappa^i \) are antitone for \( i = 1, \ldots, k \) we have:

\[
\delta_i \geq \kappa^i(\text{tr}(A(z))) + h_i^T z \geq \sum_{j \in J} [\kappa^i(\text{tr}(A(\bar{z}^j))) + h_i^T \bar{z}^j] \beta_j
\]

for \( i = 1, \ldots, k \). This means that \( \beta_j \in \mathbb{Z}_+ \) for \( j \in J \) is a feasible solution of the integer linear program \( \mathcal{I} \).

It remains to show the other direction. Let \( \lambda_j \in \mathbb{Z}_+ \) for \( j \in J \) be a feasible solution of the integer linear program \( \mathcal{I} \). We set \( z := \sum_{j \in J} \lambda_j \bar{z}^j \) and show that \( z \) is a feasible solution of the truncated integer Minkowski program \( \mathcal{I} \). We have:

\[
v = \sum_{j \in J} (B \bar{z}^j) \lambda_j = B \cdot z.
\]

On the other hand, as \( z = \sum_{j \in J} \lambda_j \bar{z}^j \) we have \( A(z) \geq \sum_{j \in J} \lambda_j A(\bar{z}^j) \). This implies:

\[
\delta_i \geq \sum_{j \in J} [\kappa^i(\text{tr}(A(\bar{z}^j))) + h_i^T \bar{z}^j] \lambda_j
\]

\[
= \sum_{j \in J} [\kappa^i(A(\bar{z}^j)) + h_i^T \bar{z}^j] \lambda_j
\]

\[
\geq \kappa^i(A(z)) + h_i^T z
\]

\[
= \kappa^i(\text{tr}(A(z))) + h_i^T z,
\]

for \( i = 1, \ldots, k \). This means that \( z \) is a feasible solution of the truncated integer Minkowski program \( \mathcal{I} \). \( \square \)

In the following sections, we will investigate some finitely generated families of sets \( A(z) \) and the structures of the associated truncated (mixed-)integer Minkowski programs.

### 3. The integer case: Atomic fibers

In this section we will investigate the situation when the mapping \( A \) generates a family of discrete sets. In our application, the discrete sets will be sets of feasible integral routings. To be more precise, let \( A \in \mathbb{Z}^{d \times n} \) be a matrix, and let the mapping \( A: \mathbb{Z}^d \rightarrow 2^{\mathbb{R}_+^n} \) be defined by

\[
A(b) = P_{A,b} = \{ z : Az = b, \ z \in \mathbb{Z}_+^n \}. \tag{9}
\]

The set \( P_{A,b} \) is known as the fiber of \( b \) under the linear map \( f_A: \mathbb{Z}_+^n \rightarrow \mathbb{Z}^d, \ x \mapsto Ax \).

\[\text{Eisenschmidt et al.} \ (2006)\] considered the family of fibers of a fixed matrix \( A \in \mathbb{Z}^{d \times n} \), when the right-hand side vector \( b \in \mathbb{Z}^d \) varies. They established a theory of Minkowski decomposition for fibers; in this context, the non-decomposable fibers are called atomic:

**Definition 3.1.** We call a fiber \( P_{A,b} \) atomic, if \( P_{A,b} = P_{A,b_1} + P_{A,b_2} \) implies \( b = b_1 \) or \( b = b_2 \).

For our purposes, we will need to consider a slight generalization of the notion of atomic fibers, where we restrict the set of right-hand sides from \( \mathbb{Z}^n \) to some subset. As a motivating example, consider flow-conservation constraints that appear in standard node-arc formulations. These constraints are linear equations \( A_i z = b_i \) with a right-hand side \( b_i = 0 \). It is desirable that the flow-conservation constraints also hold in the Minkowski summands \( P_{A,b_1} \), \( P_{A,b_2} \) of a set \( P_{A,b} \) of feasible integral
routings. Therefore, we wish to restrict the decomposition into atomic fibers such that \( b_i = 0 \) for all components corresponding to the flow-conservation constraints.

This example suggests to restrict the right-hand sides to a sublattice of \( \mathbb{Z}^d \), but later we will see that it is also useful to consider restriction to a submonoid of \( \mathbb{Z}^d \).

**Definition 3.2.** Let \((M, +)\) with \( M \subseteq \mathbb{Z}^d \) be a monoid, \((\Lambda, +)\) a lattice with \( \Lambda \subseteq \mathbb{Z}^d \), and let \( A \in \mathbb{Z}^{d \times n} \) be a matrix.

1. A fiber \( P_{A,b}^I \) with \( b \in \Lambda \) is called atomic w.r.t. \( \Lambda \) if there is no decomposition \( b = b_1 + b_2 \) with \( P_{A,b}^I = P_{A,b_1}^I + P_{A,b_2}^I \), where \( b_1, b_2 \in \Lambda \).

2. A fiber \( P_{A,b}^I \) with \( b \in M \) is called atomic w.r.t. \( M \) if there is no decomposition \( b = b_1 + b_2 \) with \( P_{A,b}^I = P_{A,b_1}^I + P_{A,b_2}^I \), where \( b_1, b_2 \in M \).

Once again we refer to the theory of indecomposable fibers w.r.t. monoids and lattices developed by Eisenschmidt et al. (2006a) and recall a result which is important for our analysis in this paper:

**Lemma 3.3.** Let \( M \subseteq \mathbb{Z}^n \) be a finitely generated monoid and let \( A \in \mathbb{Z}^{d \times n} \) be a matrix. There are only finitely many fibers of the matrix \( A \) that are atomic w.r.t. \( M \).

Note that for \( M = \langle \pm A_1, \ldots, \pm A_n \rangle \), where \( A_i, i = 1, \ldots, n \), denote the columns of matrix \( A \), the above lemma shows that there are only finitely many atomic fibers of a matrix \( A \).

### 3.1. Truncated fibers

The usual formulations of the routing problem (multi-commodity flow problem) include routings that contain certain flow circulations as feasible solutions. When a useful (“regular”) objective function is chosen, an optimal solution to the routing problem will never contain a flow circulation, when we consider the routing commodity by commodity. The reason is that a routing with circulations will always be dominated by the routing where the circulations have been removed. For our purposes, however, it makes sense to cut away circulations explicitly; we will see later that it can significantly simplify the computations.

This application gives rise to the notion of truncated fibers.

**Definition 3.4.** Let \( \mathcal{C} = \{c_1, \ldots, c_s\} \) be a finite set of vectors. Let \( P_{A,b}^I \) be a fiber of a matrix \( A \in \mathbb{Z}^{d \times n} \). We call

\[
\text{tr}_C(P_{A,b}^I) = \{ z \in P_{A,b}^I : \exists c \in \mathcal{C} \text{ with } c \subseteq z \}
\]

the truncated fiber of \( A \) with right-hand side \( b \) with respect to \( \mathcal{C} \).

**Definition 3.5.** A truncated fiber \( \text{tr}_C(P_{A,b}^I) \) is called indecomposable (w.r.t. a monoid \( M \)) if there is no \( b_1, b_2 \neq 0 \) (\( b_1, b_2 \in M \)) with \( b = b_1 + b_2 \) and

\[
\text{tr}_C(P_{A,b}^I) = \text{tr}_C(P_{A,b_1}^I) + \text{tr}_C(P_{A,b_2}^I).
\]

**Corollary 3.6.** If \( \text{tr}_C(P_{A,b}^I) \) is indecomposable (w.r.t. a monoid \( M \)), then \( P_{A,b}^I \) is atomic (w.r.t. the monoid), too.

**Proof.** This is a direct consequence of Lemma 3.10. \( \square \)

This leads to the main result of this section: We can reformulate the integer Minkowski program on truncated fibers as an integer linear program. Consider the integer Minkowski program:

\[
\begin{align*}
\min & \quad W^T b \\
\text{s.t.} & \quad \text{tr}_C(P_{A,b}^I) \neq \emptyset \\
& \quad B \cdot b = u' \\
& \quad r_i(\text{tr}_C(P_{A,b}^I)) + h_i^T b \leq \delta_i \quad \text{for } i = 1, \ldots, k \\
& \quad b \in \mathbb{Z}^d,
\end{align*}
\]
where \( W \in \mathbb{R}^d \) is a vector of costs, \( B \in \mathbb{Z}^{d \times d} \), \( u \in \mathbb{Z}^d \), \( \kappa^i \) antitone and Minkowski-additive functionals, \( h_i \in \mathbb{R}^d \) and \( \delta_i \in \mathbb{R} \) for \( i = 1, \ldots, k \). Then we have:

**Corollary 3.7.** If \( \kappa^i(\operatorname{tr}_C(P^I_{A,b_i})) = \kappa^i(P^I_{A,b_i}) \) for all \( b \in \mathbb{Z}^d \) and \( i = 1, \ldots, k \) then we can reformulate the integer Minkowski program (10) as an integer linear program:

\[
\min \sum_{j \in J} (W^T \tilde{b}^j) \lambda_j
\]

s.t.

\[
\sum_{j \in J} (B \tilde{b}^j) \lambda_j = u
\]

\[
\sum_{j \in J} \left[ \kappa^i(\operatorname{tr}_C(P^I_{A,b_i})) + h_i^T \tilde{b}^j \right] \lambda_j \leq \delta_i \quad \text{for } i = 1, \ldots, k
\]

\[
\lambda_j \in \mathbb{Z}_+ \quad \text{for } j \in J
\]

*Proof.* This is a direct consequence of Lemma 3.8 \( \square \)

### 3.2. Projections of fibers

The two basic formulations for the routing problem are the node-arc formulation and the path formulation. It turns out that the node-arc formulation essentially is a projection of the path formulation. By the term projection we mean linear, integral transformations of problems from a high dimensional space of variables to a space of variables of lower dimension.

Because of this relation between the node-arc and path formulations, we also need to consider projections of (atomic) fibers. We consider the fibers of the integral matrix \( A \in \mathbb{Z}^{d \times n} \) under a projection described by a matrix \( P \in \mathbb{Z}^{m \times n} \). We remark that projections of fibers are not necessarily fibers of some matrix \( \tilde{A} \); they can have a more complicated structure, see Williams (1992). In our application, however, the following setting is general enough. We suppose that there is an integral transformation matrix \( N \in \mathbb{Z}^{c \times d} \) for the set of right-hand sides \( b \in \mathbb{Z}^d \), with the columns of \( N \) being linearly independent. Furthermore we suppose that there is an integral matrix \( A \in \mathbb{Z}^{c \times m} \) such that \( NA = A\Pi \). Then we have the following property:

\[
\Pi \cdot P^I_{A,b} = \Pi \cdot \{ z \in \mathbb{Z}_+^n : Az = b \} = \Pi \cdot \{ z \in \mathbb{Z}_+^n : NAz = Nb \}
\]

\[
= \Pi \cdot \{ z \in \mathbb{Z}_+^n : \tilde{A}z = \tilde{b} \} = \Pi \cdot \{ y \in \mathbb{Z}_+^m : \tilde{A}y = \tilde{b} \} = P^I_{\tilde{A},\tilde{b}},
\]

where \( \tilde{b} = Nb \). Equation (11) gives the following lemma.

**Lemma 3.8.** If \( P^I_{A,b} \) is decomposable then \( \Pi \cdot P^I_{A,b} = P^I_{\tilde{A},\tilde{b}} \) is decomposable w.r.t. the monoid \( B = \{ b = N \cdot b : b \in A\mathbb{Z}_+^n \} \).

*Proof.* This is a direct consequence of the linearity the projection \( \Pi \):

\[
P^I_{A,b} = \Pi \cdot P^I_{A,b} = \Pi \cdot (P^I_{A,b_1} + P^I_{A,b_2}) = \Pi \cdot P^I_{A,b_1} + \Pi \cdot P^I_{A,b_2}
\]

\[
= P^I_{\tilde{A},\tilde{b}_1} + P^I_{\tilde{A},\tilde{b}_2}.
\]

An analogous assertion is true for truncated fibers.

**Lemma 3.9.** Let \( C = \{ c_1, \ldots, c_l \} \subseteq \mathbb{Z}_+^n \) a finite set of vectors and let \( D \subseteq \mathbb{Z}_+^n \) such that \( \tilde{A} \cdot d \in B \) for all \( d \in D \), where \( B \) is the monoid from Lemma 3.8. For all \( c \in C \) let there exist \( d \in D \) with \( d \subseteq \pi(c) \). Then we have: If \( \operatorname{tr}_C(P^I_{A,b}) \) is decomposable, then \( \operatorname{tr}_D(P^I_{A,b}) \) is decomposable w.r.t. \( B \).
Proof. As a first step we will show that \( \text{tr}_D(P^I_{\bar{A},\bar{b}}) = \text{tr}_D(\Pi \cdot P^I_{\bar{A},\bar{b}}) \subseteq \Pi \cdot \text{tr}_C(P^I_{\bar{A},\bar{b}}) \). To this aim let \( x \in \text{tr}_D(\Pi \cdot P^I_{\bar{A},\bar{b}}) \). Then there is \( y \in P^I_{\bar{A},\bar{b}} \) with \( \Pi \cdot y = x \). Suppose there is \( c \in C \) with \( c \subseteq y \). Then there is \( d \in D \) with \( d \subseteq \Pi \cdot c \subseteq \Pi \cdot y = x \) which contradicts the fact that \( x \in \text{tr}_D(P^I_{\bar{A},\bar{b}}) \). Thus \( y \in \text{tr}_C(P^I_{\bar{A},\bar{b}}) \) and consequently \( x = \Pi \cdot y \in \Pi \cdot \text{tr}_C(P^I_{\bar{A},\bar{b}}) \). This proves our first claim. Now let \( \text{tr}_C(P^I_{\bar{A},\bar{b}}) \) be decomposable, i.e., there are vectors \( b_1, b_2 \) with \( \text{tr}_C(P^I_{\bar{A},\bar{b}}) \subseteq \text{tr}_C(P^I_{\bar{A},b_1}) + \text{tr}_C(P^I_{\bar{A},b_2}) \). By linearity of projections we have

\[
\Pi \cdot \text{tr}_C(P^I_{\bar{A},\bar{b}}) \subseteq \Pi \cdot \text{tr}_C(P^I_{\bar{A},b_1}) + \Pi \cdot \text{tr}_C(P^I_{\bar{A},b_2}).
\]

We will show that \( \text{tr}_D(P^I_{\bar{A},\bar{b}}) \subseteq \text{tr}_D(P^I_{\bar{A},b_1}) + \text{tr}_D(P^I_{\bar{A},b_2}) \). Let \( x \in \text{tr}_D(P^I_{\bar{A},\bar{b}}) \) be a representation \( x = x_1 + x_2 \) with \( x_i = \Pi \cdot \text{tr}_C(P^I_{\bar{A},b_i}) \). We will show that \( x_i \in \text{tr}_D(P^I_{\bar{A},b_i}) \). Let \( w \in W \) and consider \( x \in \Pi \cdot \text{tr}_D(P^I_{\bar{A},b_i}) \) for \( i = 1, 2 \). Suppose not and let w.l.o.g. \( x_1 \notin \text{tr}_D(P^I_{\bar{A},b_i}) \). Then there is \( d \in D \) with \( d \subseteq x_1 \). But then \( d \subseteq x_1 + x_2 = x \) because \( x_i \in \mathbb{Z}^n_+ \). This contradicts the fact that \( x \in \text{tr}_D(P^I_{\bar{A},\bar{b}}) \) and concludes the proof. \( \Box \)

This implies: If we have a finite generating system for the fibers \( P^I_{\bar{A},\bar{b}} \) (or \( \text{tr}_C(P^I_{\bar{A},\bar{b}}) \)) then we have a finite generating system for the projected fibers \( P^I_{\bar{A},\bar{b}} \) (or \( \text{tr}_D(P^I_{\bar{A},\bar{b}}) \)) w.r.t. the monoid \( \mathcal{B} \). Therefore we may reformulate the integer Minkowski program on the projected fibers as an integer linear program, because we may do so for the original fibers.

4. The mixed-integer case: Indecomposable Polytopes

4.1. The general case. In this section we will consider finitely generated families of polyhedral sets depending on integral vectors. We refer to the notation of [Henk et al. 2003, Köppe 2002]. Let \( W \in \mathbb{Z}^{m \times n} \) be a fixed but arbitrary integral matrix with row vectors \( w^i \in \mathbb{Z}^n, 1 \leq i \leq m \). We assume that

\[
\text{pos}\{w^1, \ldots, w^m\} = \mathbb{R}^n,
\]

where \( \text{pos} \) denotes the positive hull. Thus for every \( u \in \mathbb{R}^m \) the set \( P_u = \{ y \in \mathbb{R}^n : Wy \leq u \} \) is a polytope. We are interested in the set of all nonempty polytopes with integral right-hand side arising in this way. We set:

\[
\mathcal{U}(W) = \{ u \in \mathbb{R}^m : P_u \neq \emptyset \},
\]

and consider \( \mathcal{U}(W) \cap \mathbb{Z}^m \).

Definition 4.1. A polytope \( P_z, z \in \mathcal{U}(W) \cap \mathbb{Z}^m \) is called integrally decomposable if there exist \( P_{z_1}, P_{z_2} \) not homothetic to \( P_z \) such that \( P_z = P_{z_1} + P_{z_2} \) and \( z = z_1 + z_2 \), \( z^i \in \mathcal{U}(W) \cap \mathbb{Z}^m \). \( P_z \) is called integrally decomposable otherwise.

We have the following result in [Henk et al. 2003]:

Theorem 4.2. There exist finitely many vectors \( h^1, \ldots, h^k \in \mathcal{U}(W) \cap \mathbb{Z}^m \) such that for every polytope \( P_z, z \in \mathcal{U}(W) \cap \mathbb{Z}^m \), there exist \( h^{j_1}, \ldots, h^{j_{2m-2-n}} \) and non-negative integers \( \lambda_{j_1}, \ldots, \lambda_{j_{2m-2-n}} \) such that

\[
P_z = \sum_{i=1}^{2m-2-n} \lambda_{j_i} P_{h^{j_i}} \quad \text{and} \quad z = \sum_{i=1}^{2m-2-n} \lambda_{j_i} h^{j_i}.
\]

This means in particular, that it is possible to model a mixed-integer Minkowski program on polytopes with integral right-hand side as an integer linear program according to Theorem 2.6.
4.2. Restricted right-hand sides. In our application to network design the polytopes \( P_z \) introduced in the previous subsection will represent the sets of feasible vectors of flow of particular networks. Therefore it makes sense to claim non-negativity of the points in a polytope \( P_z \). We assume that our matrix \( W \) is of a special structure: \( W = (W, -\text{Id})^T \in \mathbb{Z}^{(m+n)\times n} \) and we restrict our attention to the following lattice of right-hand sides:

\[
\Psi = \{ u = (\bar{u}, 0)^T \in \mathbb{Z}^{m+n} : \bar{u} \in \mathbb{Z}^m \}.
\]

For \( u \in \Psi \) we have:

\[
P_u = \{ y \in \mathbb{R}^n : Wy \leq u \} = \{ y \in \mathbb{R}^n_+ : \bar{W}y \leq \bar{u} \}.
\]

As in the integral case of atomic fibers we are interested in sublattices of \( \Psi \), because of flow-conservation constraints or non-negativity constraints for demands and capacities for example.

**Definition 4.3.** Let \( \Lambda \subseteq \Psi \) denote a lattice and \( M \subseteq \Psi \) denote a monoid.

(i) A polytope \( P_u \) with \( u \in \Lambda \) is called integrally indecomposable w.r.t. \( \Lambda \) if there are no vectors \( 0 \neq u_1, u_2 \in \Lambda \) with \( P_u = P_{u_1} + P_{u_2} \) and \( u = u_1 + u_2 \).

(ii) A polytope \( P_u \) with \( u \in M \) is called integrally indecomposable w.r.t. \( M \) if there are no vectors \( 0 \neq u_1, u_2 \in M \) with \( P_u = P_{u_1} + P_{u_2} \) and \( u = u_1 + u_2 \).

It was already proved in Henk et al. (2003), Remark 3.1, that there are only finitely many integrally indecomposable polytopes w.r.t. a sublattice \( \Lambda \) of \( \mathbb{Z}^m \). We may extend this result to monoids under certain conditions. Let \( M \) be a monoid which is finitely generated, i.e., which is generated by \( m_1, \ldots, m_t \in M \) and let \( \Lambda_M \) the lattice generated by \( m_1, \ldots, m_t \).

**Lemma 4.4.** Let \( M \) be a monoid and let \( W \mathbb{Z}^n_+ \cap \Lambda_M \subseteq M \). Then there are only finitely many integrally indecomposable polytopes w.r.t. \( M \).

**Proof.** We know that there are only finitely many integrally indecomposable polytopes \( P_z \) w.r.t. \( \Lambda_M \). As \( M \subseteq \Lambda_M \) we know, that if \( z \in M \) and \( P_z \) integrally indecomposable w.r.t. \( \Lambda_M \) then \( P_z \) is integrally indecomposable w.r.t. \( M \). On the other hand, let \( P_z = P_{z_1} + P_{z_2} \) with \( z = z_1 + z_2 \), \( z, z_1, z_2 \in \Lambda_M \) and \( P_{z_1}, P_{z_2} \neq \emptyset \). Then \( z, z_i \in W \mathbb{Z}^n_+ \), \( i = 1, 2 \). As \( W \mathbb{Z}^n_+ \cap \Lambda_M = W \mathbb{Z}^n_+ \cap \Lambda_M \subseteq M \), we have \( z, z_1, z_2 \in M \). This means that the polytopes which are integrally indecomposable w.r.t. \( \Lambda_M \) are integrally indecomposable w.r.t. \( M \) and vice-versa. This implies the finiteness of the number of integrally indecomposable polytopes w.r.t. \( M \). \( \square \)

**Corollary 4.5.** Let \( C \) be a set of vectors (not necessarily finite). Consider the truncated polytopes \( \emptyset \neq \text{tr}_C(P_u) = \{ y \in \mathbb{R}^n : Wy \leq u, \exists c \in C \text{ with } c \subseteq y \} \). There is a finite generating set for these truncated polytopes.

**Proof.** This is a direct consequence of Lemma 4.4. \( \square \)

**Corollary 4.6.** One may reformulate integer Minkowski programs on (truncated) polytopes with integral right-hand side as integer linear programs.

**Proof.** This comes from Theorem 2.11 and Theorem 2.6. \( \square \)

5. Network design problems and atomic fibers

In this section, we will treat the network design problem in terms of the analysis of atomic fibers. The outline of this section is as follows: The first two subsections will introduce two formulations of the network design problem, the node-arc and the path-cycle formulation. The following subsection will give the connection between the path-cycle and the node-arc formulation and introduce the notion of irreducible networks. The following two subsections will give the connection between
the irreducible networks and the atomic fibers for both our formulations. The last subsection will finally present the network design problem as an integer Minkowski program and its reformulations as an integer linear program.

The reason for studying both formulations is the following. The path formulation is easier to study than the node-arc formulation: In the path formulation, ordinary atomic fibers appear, whereas in the node-arc formulation, we need to consider atomic fibers with respect to a monoid of feasible right-hand side vectors. On the other hand, the node-arc formulation is essentially a projection of the path-cycle formulation. Therefore, the set of atomic fibers is in general much smaller in the node-arc formulation. Consequently, the computation of the atomic fibers is more efficient, and also the reformulation as an integer linear program is more compact when we start with the node-arc formulation.

Let \( G = (V, A) \) be the supply digraph, which is connected. Each arc \( a \in A \) of the graph has a capacity \( c_a \in \mathbb{Z}_+ \) such that \( c \in \mathbb{Z}_+^{[A]} \) and we are given a demand vector \( d = (d_1, \ldots, d_k) \), \( d_i \in \mathbb{Z}_+ \forall i \in \{1, \ldots, k\} \). For each commodity \( l = 1, \ldots, k \) let \( s_l, t_l \in V \) be its source, \( t_l \in V \) its sink and \( \Pi_l \) the set of all paths from \( s_l \) to \( t_l \). Such a setting is called a multicommodity network. We will denote it by \( N = (V, A, d, c) \).

### 5.1. Node-arc formulation

In this subsection, we will give an exact definition of the problems we consider in this paper. Let \( f_a^l \in \mathbb{Z}_+ \) be the part of the flow of commodity \( l \) which uses arc \( a \). The problem of finding a feasible flow w.r.t. the capacity vector \( c \) and the demand vector \( d \) can be formulated as follows:

\[
\begin{align*}
\sum_{l=1}^{k} f_a^l + s_a &= c_a & \forall a \in A \\
\sum_{a \in \delta^+(s_i)} f_a^l - \sum_{a \in \delta^-(s_i)} f_a^l &= d_i & \forall l = 1, \ldots, k \\
\sum_{a \in \delta^+(x)} f_a^l - \sum_{a \in \delta^-(x)} f_a^l &= 0 & \forall x \in V \setminus \{s_l, t_l\}, \forall l = 1, \ldots, k \\
f_a^l, s_a &\in \mathbb{Z}_+ & \forall a \in A, \forall l = 1, \ldots, k,
\end{align*}
\]

where \( s_a, a \in A \), denote the slack variables. We denote by \( C \) the matrix corresponding to the system of equations \((13)\).

**Definition 5.1.** Let \( G = (V, A) \) be a digraph, \( d = (d_1, \ldots, d_k) \in \mathbb{Z}_+^k \) a given demand vectors and \( c \in \mathbb{Z}_+^{[A]} \) a given capacity vector. Let \( K_a^l \in \mathbb{R}_+ \) denote the costs of routing one unit of commodity \( l \) through arc \( a \). The program

\[
\begin{align*}
\min \quad & K^T f \\
\text{s.t.} \quad & C \cdot (f, s)^T = (c, d, 0)^T \\
& f_a^l, s_a \in \mathbb{Z}_+ & \forall a \in A, \forall l = 1, \ldots, k
\end{align*}
\]

is called the multicommodity flow problem for integer flows with respect to the node-arc formulation.

In the multicommodity flow problem, we are given a demand vector and a capacity vector. We want to find a routing of the demands, which is minimal w.r.t. the costs of routing flow. In the network design problem, we are given a vector of demand and want to find a capacity vector minimizing installation costs, such that we can find a routing for the demands.

**Definition 5.2.** Let \( G = (V, A) \) be a digraph, \( d = (d_1, \ldots, d_k) \in \mathbb{Z}_+^k \) a given demand vector. Let additionally \( W_a, a \in A \), denote the costs of installing one unit
of capacity on arc a. The program
\[
\min \ W^T c \\
\text{s.t.} \quad C \cdot (f, s)^T = (c, d, 0)^T \\
f_a, s_a \in \mathbb{Z}_+ \\
c_a \in \mathbb{Z}_+ \\
\forall a \in A, \forall l = 1, \ldots, k \\
(15)
\]
is called the network design problem for integer flows with respect to the node-arc formulation.

5.2. Path-cycle formulation. Let \( W \) be the set of directed cycles in \( G = (V, A) \). 
\( y_p^l \) denotes the part of commodity \( l \) routed on path \( p, p \in \Pi_l \). \( y_w^l \) denotes the part of commodity \( l \), which circulates on cycle \( w \in W \). The problem of finding a feasible flow w.r.t. the capacity vector \( c \) and the demand vector \( d \) can be formulated as follows:
\[
\sum_{l=1}^{k} \left( \sum_{p \in \Pi_l} y_p^l + \sum_{w \in W} y_w^l \right) + s_a^l = c_a \quad \forall a \in A \\
(16a)
\]
\[
\sum_{p \in \Pi_l} y_p^l = d_l \quad \forall l \in \{1, \ldots, k\} \\
(16b)
\]
\[
y_p^l, y_w^l \in \mathbb{Z}_+ \\
\forall p \in \Pi_l, \forall l \in \{1, \ldots, k\}, \\
(16c)
\]
where \( s_a, a \in A \) denote the slack variables.

As in the previous subsection we define the network design problem and the multicommodity flow problem for integer flows although this time w.r.t. the path-cycle formulation. With the notation of the previous subsection, we have the following formulation of the multicommodity flow problem:
\[
\min \sum_{a \in A} \sum_{l=1}^{k} K_a^l \left( \sum_{p \in \Pi_l} y_p^l + \sum_{w \in W} y_w^l \right) \\
\text{s.t.} \quad (17)
\]
and the following formulation of the network design problem:
\[
\min \ W^T c \\
\text{s.t.} \quad (18)
\]
\[
c_a \in \mathbb{Z}_+, \forall a \in A. \\
(18a)
\]
Now we look at the multicommodity flow problem: As we have non-negative costs associated with the arcs \( (K \in \mathbb{R}_+^{A \times k}) \), there will be no cycle flow in the optimal solution, i.e., \( y_w^l = 0 \ \forall w \in W, \forall l \in \{1, \ldots, k\} \). So we can eliminate the cycle-flow variables \( y_w^l \). For the network design problem, too, we can eliminate the cycle-flow variables \( y_w^l \). We obtain the following formulation:
\[
\sum_{p \in \Pi_l} y_p^l = d_l \quad \forall l \in \{1, \ldots, k\} \\
(19a)
\]
\[
\sum_{l=1}^{k} \sum_{p \in \Pi_l} y_p^l + s_a \leq c_a \quad \forall a \in A \\
(19b)
\]
\[
y_p^l \in \mathbb{Z}_+ \\
\forall p \in \Pi_l, \forall l \in \{1, \ldots, k\} \\
(19c)
\]
\[
s_a \in \mathbb{Z}_+ \\
\forall a \in A, \\
(19d)
\]
where \( s_a, a \in A \) denote the slack variables.
Let $D$ denote the matrix corresponding to the left-hand side of formulation (19). We obtain a new formulation of the multicommodity flow problem

$$
\begin{align*}
\min & \quad \sum_{a \in A} \sum_{l=1}^{k} K^l_a (\sum_{p \in \Pi_l} y^l_p) \\
\text{s.t.} & \quad D \cdot (y, s)^T = (d, c)^T \\
& \quad y^l_p \in \mathbb{Z}_+ \quad \forall p \in \Pi_l, \forall l = 1, \ldots, k \\
& \quad s_a \in \mathbb{Z}_+ \quad \forall a \in A
\end{align*}
$$

and a new formulation of the network design problem

$$
\begin{align*}
\min & \quad W^T c \\
\text{s.t.} & \quad D \cdot (y, s)^T = (d, c) \\
& \quad y^l_p \in \mathbb{Z}_+ \quad \forall p \in \Pi_l, \forall l = 1, \ldots, k \\
& \quad c_a, s_a \in \mathbb{Z}_+ \quad \forall a \in A
\end{align*}
$$

This formulation is called the path formulation.

5.3. The node-arc formulation as a projection of the path-cycle formulation. It is clear that one can convert feasible solutions of the multicommodity flow problem in the path-cycle formulation to feasible solutions of the node-arc formulation via a projection. Let $y$ be a feasible integer solution of the multicommodity flow problem with respect to the path-cycle formulation. We set:

$$
f^l_a := \sum_{p \in \Pi_l} y^l_p + \sum_{w \in W} \sum_{a \in w} y^l_w \quad \forall a \in A, \forall l \in \{1, \ldots, k\}.
$$

Then, $f$ is a feasible integer solution of the multicommodity flow problem with respect to the node-arc formulation. Indeed, we have integrality of the components of $f$ because $y$ is integer. The capacity constraints are respected because:

$$
\sum_{l=1}^{k} f^l_a + s_a = \sum_{l=1}^{k} \left( \sum_{p \in \Pi_l} y^l_p + \sum_{w \in W} \sum_{a \in w} y^l_w \right) + s_a = c_a \quad \forall a \in A.
$$

It remains to check whether the flow-conservation constraints are respected. To this aim, we look at a path $p = (s_l, x_1, \ldots, x_n, t_l)$ with $y^l_p > 0$. The flow on path $p$ respects the flow-conservation constraints for every $x_i \in p$. This observation is true for every path $p \in \Pi$. The same assertion is valid for all cycles $w$ with $y^l_w > 0$. Now we look at some node $x_i \in V \setminus \{s_l, t_l\}$. We have:

$$
\sum_{a \in \delta^+(x_i)} f^l_a - \sum_{a \in \delta^-(x_i)} f^l_a = 0 \quad \forall l \in \{1, \ldots, k\}.
$$

Analogous arguments yield:

$$
\sum_{a \in \delta^+(s_l)} f^l_a - \sum_{a \in \delta^-(s_l)} f^l_a = d_l \quad \forall l \in \{1, \ldots, k\}.
$$

Therefore the path-cycle flow $y$ determines the node-arc flow $f$ uniquely.

Also, one can convert feasible solutions of the multicommodity flow problem in node-arc formulation to feasible solutions in path-cycle formulation.

**Lemma 5.3** (see Ahuja et al. 1993, Theorem 3.5). Every non-negative arc flow $f$ can be represented as a path and cycle flow $y$ (though not necessarily uniquely).

However, the solution is not uniquely determined in general.
Example 5.4. To see an example of this non-uniqueness, consider the one-commodity-digraph in Figure 2. There are 4 paths from the source $s$ to the sink $t$: $p_1 = \{1, 2\}$, $p_2 = \{1, 4\}$, $p_3 = \{3, 2\}$ and $p_4 = \{3, 4\}$. We consider the arc-flow vector $f = (1, 1, 1, 1)$, i.e., one unit of flow on all arcs. This flow can be represented in two ways as path-flow: $y^1 = (y^1_{p_1}, y^1_{p_2}, y^1_{p_3}, y^1_{p_4}) = (1, 0, 0, 1)$ or $y^2 = (y^2_{p_1}, y^2_{p_2}, y^2_{p_3}, y^2_{p_4}) = (0, 1, 1, 0)$.

5.4. Irreducible networks in the node-arc and path formulation. We will now introduce the notion of irreducibility of networks. To this aim, we consider the set of feasible solutions of the multicommodity flow problem in the node-arc and in the path formulation.

Definition 5.5. Let $G = (V, A)$ be a digraph, $d \in \mathbb{Z}_+^k$ the demands and $c \in \mathbb{Z}_+^{|A|}$ the capacities. We denote by $F_P$ the set of feasible integer solutions of the multicommodity flow problem on $N = (V, A, d, c)$ w.r.t. the path formulation. Analogously, we denote by $F_{NA}$ the set of feasible integer solutions w.r.t. the node-arc formulation.

Definition 5.6. Let $G = (V, A)$ be a digraph, $d \in \mathbb{Z}_+^k$ the demands and $c \in \mathbb{Z}_+^{|A|}$ the capacities. We denote by $F_{NP}$ the set of all non-cyclic feasible integer solutions of the multicommodity flow problem on $N = (V, A, d, c)$. This means: If $y \in F_{NP}$, then $y$ contains no circulation of flow. $F_{NP}$ is defined analogously. Of course we have the following relations:

$$F_{NP} \subseteq F_P \quad F_{NA} \subseteq F_{NP}.$$

To illustrate these definitions, we consider the following example.

Example 5.7. Let $G = (V, A)$ the digraph in Figure 3.

We have 4 paths in this example: $p_1 = \{1, 4\}$, $p_2 = \{1, 2\}$, $p_3 = \{5, 6\}$ and $p_4 = \{5, 3, 4\}$. Let $d = 2$ and $c = (1, 1, 1, 1, 1, 1)$. Then $y = (y_{p_1}, y_{p_2}, y_{p_3}, y_{p_4}) = (0, 1, 0, 1)$ is a feasible integer solution of the multicommodity flow problem w.r.t. the path formulation. This means $y \in F_P$. But $y$ contains a circulation of flow on arcs 2 and 3. Therefore it is
not contained in $\mathcal{F}^{\mathcal{P}}$. The arc flow $f$ which is determined by $y$ is contained in $\mathcal{F}^{\mathcal{NA}}$ but not in $\mathcal{F}^{\mathcal{NA}}$.

Now we can give the definition of irreducibility of networks.

**Definition 5.8.** Let $G = (V, A)$ be a digraph, $d \in \mathbb{Z}_{+}^{k}$ the demand and $c \in \mathbb{Z}_{+}^{|A|}$ the arc-capacity of a network. A decomposition w.r.t. the path formulation (w.r.t. the node-arc formulation) of the network $N = (V, A, d, c)$ is given by a decomposition of the capacity vector and the demand vector $c = c_1 + c_2$, $c_1, c_2 \in \mathbb{Z}_{+}^{|A|}$, $d = d_1 + d_2$, $d_1, d_2 \in \mathbb{Z}_{+}^{k}$, such that the set of non-cyclic feasible integer solutions of the multicommodity flow problem can be obtained as the Minkowski-sum: $\mathcal{F}^{\mathcal{P}} \subseteq \mathcal{F}^{\mathcal{P}}_1 + \mathcal{F}^{\mathcal{P}}_2$ ($\mathcal{F}^{\mathcal{NA}} = \mathcal{F}^{\mathcal{NA}}_1 + \mathcal{F}^{\mathcal{NA}}_2$), where $\mathcal{F}^{\mathcal{P}}_i$ ($\mathcal{F}^{\mathcal{NA}}_i$) is the set of non-cyclic feasible integer solutions of the multicommodity flow problem on $N_i = (V, A, d_i, c_i)$, $i = 1, 2$. If the network cannot be decomposed in this way, it is called irreducible or indecomposable w.r.t. the path formulation (w.r.t. the node-arc formulation).

The notion of irreducibility depends on the formulation of the multicommodity flow problem, as the following example illustrates.

**Example 5.9.** Let us consider the network in Figure 4, with all the arc capacities equal to 1.

![Figure 4](image)

**Figure 4.** A network with demand $d = 2$

We have already considered this example in Figure 2. The set of solutions of the multicommodity flow problem w.r.t. the node-arc formulation is a singleton: $\mathcal{F}^{\mathcal{NA}} = \mathcal{F}^{\mathcal{NA}} = \{(1, 1, 1, 1)\}$. The network is decomposable w.r.t. the node-arc formulation. Its possible decompositions are shown in Figure 5.

![Figure 5](image)

**Figure 5.** Decompositions of Figure 4 with demands $d_i = 1$, $i = 1, 2$

In contrast to this, the network in Figure 4 is irreducible w.r.t. the path formulation. With the notation of Figure 2 the set of solutions of the multicommodity flow problem w.r.t. the path formulation is: $\mathcal{F}^{\mathcal{P}} = \mathcal{F}^{\mathcal{P}} = \{(1, 0, 0, 1), (0, 1, 1, 0)\}$. There is no non-trivial decomposition of the network with the set of solutions of the multicommodity flow problem decomposing according to it.

Nevertheless we have a connection between irreducibility w.r.t. the node-arc formulation and irreducibility w.r.t. the path formulation. This connection will be formulated and proved in Theorem 5.14.
5.5. **Irreducible networks and atomic fibers in the path formulation.** Now we want to explore the connection between irreducible networks and atomic fibers. We recall that a network is referred to be decomposable w.r.t. the path formulation for a given vector $d$ of demand and a given vector $c$ of capacity, if there is a decomposition $d_1, d_2$ and $c_1, c_2$ of these vectors, such that the feasible non-cyclic solutions of the multicommodity flow problem decompose according to it. The network is irreducible w.r.t. the path formulation otherwise. In fact we would like to prove that the irreducible networks form a certain subset of the atomic fibers, i.e., we want to show that they are the truncated fibers w.r.t. flow-circulations. We look at the multicommodity flow problem as defined in formulation (20). Let $c \in \mathbb{Z}_+^{|A|}$ be a given capacity vector and $d \in \mathbb{Z}_+^k$ a given demand vector.

\[
\begin{align*}
\min & \sum_{a \in A} \sum_{l=1}^k K_{al}^l (\sum_{p \in \Pi_l} y_{lp}^l) \\
\text{s.t.} & \quad D \cdot (y, s)^T = (d, c)^T \\
& \quad y_{lp}^l \in \mathbb{Z}_+ \quad \forall p \in \Pi_l, \; \forall l = 1, \ldots, k \\
& \quad s_a \in \mathbb{Z}_+ \quad \forall a \in A
\end{align*}
\]

Let

\[P_{D,b} := \{(y, s) \in \mathbb{Z}_+^{|A||\Pi|} \times \mathbb{Z}_+^{|A|} : D \cdot (y, s)^T = b\},\]

where $b = (d, c)^T$ is the right-hand side vector of (20), denote the set of feasible solutions of the multicommodity flow problem. Obviously, $P_{D,b}$ is a fiber and the elements $(y, s) \in P_{D,b}$ are in bijection with the feasible integer routings of the multicommodity flow problem on $N = (V, A, d, c)$. This means: If $y \in \mathcal{F}^P$ is a feasible integer solution of the multicommodity flow problem on $N = (V, A, d, c)$ then $(y, s) \in P_{D,b}$, where

\[s_a := c_a - \sum_{l=1}^k \sum_{p \in \Pi_l} y_{lp}^l,\]

We have the other direction, too. If $(y, s) \in P_{D,b}$, then $y$ is a feasible integer routing on $N = (V, A, d, c)$, i.e., $y \in \mathcal{F}^P$.

Having this bijection between the set of feasible integer solutions of the multicommodity flow problem $\mathcal{F}^P$ and the elements in the fiber $P_{D,b}$ we will now establish the connection between the non-cyclic solutions of the multicommodity flow problem and truncated fibers. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ denote the generating set of flow circulations on digraph $G = (V, A)$ w.r.t. the path formulation. Now, we consider $\text{tr}_\Gamma(P_{D,b}).$

**Lemma 5.10.** The elements of the truncated fiber $\text{tr}_\Gamma(P_{D,b})$ are in bijection with the non-cyclic solutions of the multicommodity flow problem $\mathcal{F}^P$.

**Proof.** Suppose $y \in \mathcal{F}^P$. Then $(y, s) \in \text{tr}_\Gamma(P_{D,b})$ with $s$ defined as above. The other direction is clear as well. \hfill \Box

**Corollary 5.11.** If $N = (V, A, d, c)$ is an irreducible network w.r.t. the path formulation, then $P_{D,b}$ with $b = (c, d)$ is an atomic fiber.

**Proof.** This is a direct consequence of Lemma 5.10 and Lemma 5.10. \hfill \Box

5.6. **Irreducible networks and atomic fibers in the node-arc formulation.** Up to now we have looked at the atomic fibers of the multicommodity flow problem with respect to the path formulation. In this subsection we will treat the atomic fibers of the multicommodity flow problem with respect to the node-arc formulation.
Consider the multicommodity flow problem w.r.t. the node-arc formulation as defined in (14):

\[
\begin{align*}
\min \quad & K^T f \\
\text{s.t.} \quad & C \cdot (f, s)^T = (c, d, 0)^T \\
& f^l_a, s_a \in \mathbb{Z}_+ \\
& \forall a \in A, \forall l = 1, \ldots, k
\end{align*}
\]

We look at the fibers of the following form:

\[
P_{C,b} := \{ (f, s) \in \mathbb{Z}_+^{k|A|} \times \mathbb{Z}_+^{|A|} : C \cdot (f, s)^T = b^T \}.
\]

If \( N = (V, A, d, c) \) is a network, then there is a corresponding non-empty fiber \( P_{C,b} \) with \( b = (c, d, 0) \). As for the path formulation the elements in the fiber correspond to the solutions of the multicommodity flow problem on \( N \) and vice-versa.

But contrarily to the path formulation not all fibers of the matrix \( C \) correspond to networks on the digraph \( G = (V, A) \). Only a subset of the fibers correspond to networks on the digraph \( G \). This subset consists of those fibers with right hand sides \( b \) of the form \( b = (c, d, 0) \) where \( c \in \mathbb{Z}_+^{|A|} \), \( d \in \mathbb{Z}_+^k \) and 0 the zero-vector. On the other hand all fibers with right hand sides of this form correspond to networks of the digraph \( G \). This means that we are dealing with a monoid \( M \) of right-hand sides \( b \), which can be defined as

\[
M := \{ (c, d, 0) : c \in \mathbb{Z}_+^{|A|}, d \in \mathbb{Z}_+^k, 0 \in \mathbb{Z}_+^{k(|V|-2)} \}.
\]

As in the previous subsection, we want to install a connection between the fibers which are atomic w.r.t. \( M \) and the irreducible networks. Therefore, let \( \Delta = \{ \delta_1, \ldots, \delta_m \} \) be the set of generators of flow-circulations on the digraph \( G = (V, A) \) w.r.t. the node-arc formulation. Then, we are able to install the connection between irreducible networks and the fibers which are atomic w.r.t. \( M \).

**Lemma 5.12.** Let \( G = (V, A) \) be a digraph, \( c \in \mathbb{Z}_+^{|A|} \) the capacity and \( d \in \mathbb{Z}_+^k \) the demand. \( N = (V, A, d, c) \) is an irreducible network if and only if the corresponding truncated fiber \( \text{tr}_\Delta(P_{C,b}) \) with \( b = (c, d, 0) \) is indecomposable w.r.t. the monoid \( M \) in (22).

**Proof.** The proof is analogous to the proof of Lemma 5.11. \( \square \)

**Corollary 5.13.** If \( N = (V, A, d, c) \) is an irreducible network w.r.t. the node-arc formulation, the \( P_{C,b} \) is an atomic fiber w.r.t. \( M \), where \( b = (c, d, 0) \).

**Proof.** This is a consequence of Lemma 2.10 and of Lemma 5.12. \( \square \)

It remains to show the connection between irreducible networks in the node-arc and irreducible networks in the path formulation.

**Theorem 5.14.** A network which is irreducible w.r.t. the node-arc formulation is irreducible w.r.t. the path formulation.

**Proof.** The assertion of the theorem is equivalent to the following assertion: A network which is decomposable w.r.t. the path formulation is decomposable w.r.t. the node-arc formulation. But this assertion is a consequence of Lemma 5.11. We have seen in section 5.4 that the node-arc formulation is a projection of the path-cycle formulation. If we denote \( \Lambda = \{ \lambda_1, \ldots, \lambda_t \} \) the generators of flow-circulations w.r.t. the path-cycle formulation, then we have: \( \Gamma \subseteq \Lambda \) and we obtain:

A network which is irreducible w.r.t. the node-arc formulation is irreducible w.r.t. the path-cycle formulation. But the non-cyclic solutions of the multicommodity flow problem w.r.t. the path-cycle formulation are in bijection with the non-cyclic solutions of the multicommodity flow problem w.r.t. the path formulation. In particular: The irreducible networks in both formulations are in bijection and the assertion of the theorem follows. \( \square \)
5.7. Formulating the network design problem as an integer linear program: An example. We have seen in the previous subsections that the network design problem for integer flows fits in our framework of the integer Minkowski programs because there we want to ensure that the set of solutions of the multicommodity flow problem for the optimal capacity \( c \) is not empty. We have also seen that there is a finite generating system for the family of feasible solutions sets of the multicommodity flow problem. Even though there are no Minkowski-additive and antitone functionals involved in the integer Minkowski programs corresponding to the network design problem we may reformulate it as an integer linear program. As this reformulation introduces an integer variable for each irreducible network the reformulation process generates an “extended version” of the path formulation. Note that a vector \( b = (c, d, 0)^T \), where \( d = e_i, i \in \{1, \ldots, k\} \), and \( c = \chi \) the incidence-vector of a path routing \( d = e_i \) is necessarily contained in the set of irreducible networks. Therefore, all path variables \( y_p \) are contained in the reformulation as an integer linear program.

We will illustrate the reformulation process by an example.

Example 5.15. Consider the digraph \( G = (V, A) \) with three arcs and three nodes in Figure 6. For given \( d \in \mathbb{Z}_+^3 \) and given \( c \in \mathbb{Z}_+^3 \) the node-arc formulation for the digraph in Figure 6 looks as follows:

\[
\begin{align*}
  f_1 + s_1 &= c_1 \\
  f_2 + s_2 &= c_2 \\
  f_3 + s_3 &= c_3 \\
  f_1 + f_2 &= d \\
  -f_2 + f_3 &= 0
\end{align*}
\]

(23)

Let \( C \) denote the matrix corresponding to the left-hand side of the multicommodity flow problem (23). Then the condition of finding a feasible integer flow translates to the following condition:

\[ P_{C,b} = \{ (f,s) \in \mathbb{Z}_+^6 : C \cdot (f,s)^T = b \} \neq \emptyset, \]

where \( b = (c,d,0)^T \). Now the network design problem for the digraph \( G = (V, A) \) for given demand \( \bar{d} \in \mathbb{Z}_+^3 \) and vector of costs \( w \in \mathbb{R}_+^3 \) may be formulated as an integer Minkowski program:

\[
\min w^T c \\
\text{s.t.} \quad d = \bar{d} \\
P_{C,b} \neq \emptyset \\
c \in \mathbb{Z}_+^6, d \in \mathbb{Z}_+^3
\]

(24)

Figure 7 shows the fibers of the matrix \( C \) which are atomic w.r.t. the monoid \( M = \{(c,d,0) \in \mathbb{Z}_+^6\} \). As there are no circulations possible on the digraph \( G \), these atomic fibers correspond to the irreducible networks with underlying digraph \( G \).

Now we will reformulate the network design problem with respect to the digraph \( G \) and prescribed demand \( \bar{d} \in \mathbb{Z}_+^3 \) as an integer linear program. Therefore, we introduce a variable \( \lambda_i \in \mathbb{Z}_+ \) for all irreducible networks with underlying digraph \( G \). Then the

\[ \text{Figure 6. The digraph } G = (V, A) \text{ for one commodity} \]
The irreducible networks with underlying digraph $G = (V, A)$

reformulated network design problem is the following one:

$$\min w_1 \lambda_1 + w_2 \lambda_{II} + w_3 \lambda_{III} + w_1 \lambda_{IV} + (w_2 + w_3)\lambda_V + (w_1 + w_2 + w_3)\lambda_V$$

s.t.

$$\lambda_1, \ldots, \lambda_{VI} \in \mathbb{Z}_+.$$ 

6. DESIGN OF SURVIVABLE NETWORKS

An optimal solution of the basic network design problem might give a capacity installation that admits only one feasible routing. Clearly, if a link or a switching node fails, not all demand can be routed through the remaining network. In this section, we will consider various notions of survivability of networks. The two basic notions are the arc survivability and the node survivability. Ensuring arc survivability means to compute a capacity installation such that there exists a feasible routing of the demand even if an arbitrary link of the network fails. If a node $v$ of the network fails, then all arcs incident with node $v$ fail simultaneously and the commodities incident with node $v$, i.e., having sink or source equal to $v$, fail simultaneously. Ensuring node survivability means to compute a capacity installation such that there exists a feasible routing of the remaining demand even if an arbitrary node of the network fails.

The two notions can be studied in a unified and generalized failure model. We refer to the failure model in Pióro and Medhi (2004):

**Definition 6.1.** A failure model is defined as a finite set $\Sigma$ of failure states of the network. Each failure state $\sigma \in \Sigma$ is characterized by a vector of arc-availability coefficients $\alpha_\sigma = (\alpha_{1\sigma}, \ldots, \alpha_{|A|\sigma})$, $0 \leq \alpha_{i\sigma} \leq 1$, and a vector of demand coefficients $\chi_\sigma = (\chi_{1\sigma}, \ldots, \chi_{k\sigma})$, $0 \leq \chi_{j\sigma} \leq 1$. The arc-availability coefficients $\alpha_{i\sigma}$ represent the proportion of the capacity of arc $i$ which is available in failure state $\sigma \in \Sigma$. The demand coefficients $\chi_{j\sigma}$ denote the demand of commodity $j$ which must be satisfied in failure state $\sigma \in \Sigma$.

It is obvious that an arc-availability coefficient $\alpha_{i\sigma}$ is equal to 0 corresponds to a total failure of arc $i$, whereas $0 < \alpha_{i\sigma} < 1$ corresponds to a partial failure of this arc and $\alpha_{i\sigma} = 1$ means that there is no failure of arc $i$.

In this section we will consider several particular cases for the arc-availability and the demand coefficients:
This yields immediately that the solutions of the multicommodity flow problem are antitone. Its non-negativity comes from the non-negativity of feasible flows. We have:

This implies finally the Minkowski-additivity of the arc-survivability functionals $g_a$. W.l.o.g. we may assume, that $\chi_\sigma = 1$ and $\alpha_\sigma \in \{0, 1\}^{|A|}$ 

This situation corresponds to a partial failure of arc $i$ and either no failure or total failure of arcs $j \neq i$.

With the help of these “pure” scenarios one may also model mixes of them, e.g., simultaneous failures of several nodes, total failures of several arcs and a partial failure of a single arc. Note that within our consideration of arc survivability of networks we always mean arc survivability w.r.t. complete rerouting. This means that there are no routing restrictions imposed in the failure case: The no-fault and the fault routing may be completely unrelated.

6.1. Arc survivability w.r.t. complete rerouting. In this section we will concentrate on the survivability of networks w.r.t. total or partial failures of networks. The property of a network to be arc-survivable in case of failure of an arc will be modeled via Minkowski-additive and antitone functionals depending on the set of feasible solutions of the multicommodity flow problem for fixed capacity, demand and a particular arc of the network. These functionals will be called arc-survivability functionals. Consider the functionals below: here we define them for the node-arc formulation of the network design problem. But it is possible to define them in an analogous way for the path formulation. In the following, the vector $b$ will always denote $b = (c, d, 0)$.

\[ g_a(P_{C,b}) := \min_{(f,s) \in P_{C,b}} \left( \sum_{i=1}^{k} f_i^a \right). \]  

A network $N = (V, A, d, c)$ is survivable w.r.t. complete rerouting in case of total failure of arc $a \in A$ if and only if $g_a(P_{C,b}) = 0$.

**Lemma 6.2.** Let $a \in A$ be fixed. The functional $g_a(P_{C,b})$ is Minkowski-additive, i.e., if $N = (V, A, d, c)$ is decomposable into $N^1 = (V, A, d^1, c^1)$ and $N^2 = (V, A, d^2, c^2)$, then we have $g_a(P_{C,b}) = g_a(P_{C,b^1}) + g_a(P_{C,b^2})$. Furthermore, $g_a(P_{C,b})$ is an antitone and non-negative functional.

**Proof.** Let $N = (V, A, d, c)$ be a network and $a \in A$ be an arc. Let $f^*$ be the solution of the multicommodity flow problem \(\text{(20)}\) on $N$ with minimum aggregated flow on arc $a$. Then we have:

\[ g_a(P_{C,b}) = \sum_{i=1}^{k} (f^*)_a^i. \]

W.l.o.g. we may assume, that $f^* \in F_N^{N^A}$, i.e., $f^*$ admits no circulation of flow.

Now let $N = N_1 + N_2$, implying that $f^* = f_1 + f_2$ with $f_i \in F_N^{N^A}$, $i = 1, 2$. This yields immediately that the solutions of the multicommodity flow problem \(\text{(20)}\) on $N_i$, $f_i$, $i = 1, 2$, admit each minimum aggregated flow on arc $a$. Therefore we have:

\[ g_a(P_{C,b}) = g_a(P_{C,b^1}) + g_a(P_{C,b^2}). \]

This implies finally the Minkowski-additivity of the arc-survivability functionals $g_a(P_{C,b})$.

The arc-survivability functional $g_a(P_{C,b})$ is antitone as the minimum-function is antitone. Its non-negativity comes from the non-negativity of feasible flows. \(\square\)
Since \( g_a(P_{C,b}) \) is Minkowski-additive and antitone we can apply Theorem 2.6. We set \( \delta := 0 \) and add the inequalities

\[
\sum_{i=1}^{t} \lambda_i g_a(P_{C,b_i}) \leq 0 \quad \forall a \in A
\]

to the reformulation of the network design problem as integer linear program to ensure survivability w.r.t. complete rerouting in case of failure of an arc. As the functionals \( g_a(P_{C,b}) \) are non-negative we can replace these inequalities by the following equalities:

\[
\sum_{i=1}^{t} \lambda_i g_a(P_{C,b_i}) = 0 \quad \forall a \in A
\]

and arrive at the following reformulation of the network design problem under arc survivability constraints as integer linear program:

\[
\begin{align*}
\text{min} & \quad W^T \left( \sum_{i=1}^{t} \lambda_i c_i \right) \\
\text{s.t.} & \quad \sum_{i=1}^{t} \lambda_i d_i = d \\
& \quad \sum_{i=1}^{t} \lambda_i g_a(P_{C,b_i}) = 0 \quad \forall a \in A \\
& \quad (c_i, d_i, 0)^T = b_i \in F(C) \quad \forall i = 1, \ldots, t \\
& \quad \lambda_i \in \mathbb{Z}_+ \quad \forall i = 1, \ldots, t
\end{align*}
\]

(26)

**Remark 6.3.** We remark that this integer linear program is infeasible if the topology of the network does not support a survivable installation of capacities. This case will occur if there is a commodity pair that does not have two arc-disjoint paths.

We will illustrate this reformulation process with the help of an example.

**Example 6.4.** We resume Example 5.15 and incorporate the arc-survivability functionals in our reformulation. Consider Figure 8. The values of the arc-survivability functionals on the arcs of the irreducible networks are marked in bold.

**Figure 8.** The values of the arc-survivability functionals
Now we may consider the reformulation of the network design problem under arc-survivability constraints. For our example this reformulation is the following one:

\[
\begin{array}{llll}
\text{min } w_1 \lambda_1 + w_2 \lambda_2 + w_3 \lambda_3 + w_4 \lambda_4 + (w_5 + w_6) \lambda_5 & \text{s.t.} & \lambda_1 + \lambda_2 + \lambda_3 + (w_5 + w_6) \lambda_5 = \bar{d} \\
0 \lambda_1 + 0 \lambda_2 + 0 \lambda_3 + 1 \lambda_5 + 0 \lambda_4 & = 0 \\
0 \lambda_1 + 0 \lambda_2 + 0 \lambda_3 + 0 \lambda_4 + 1 \lambda_5 & = 0 \\
0 \lambda_1 + 0 \lambda_2 + 0 \lambda_3 + 0 \lambda_4 + 1 \lambda_5 & = 0 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{Z}_+.
\end{array}
\]

Remark 6.5. Note that if a network is arc-survivable w.r.t. the failure of an arc it will decompose into survivable, irreducible summands. This implies that all variables \( \lambda_i \) for non-survivable, irreducible networks \( N_i \) are fixed to 0 in the reformulation as integer linear program of the survivable network design problem.

6.2. Partial failures. Up to now, we have considered total failures of arcs. But it is possible to model so-called partial failures of arcs. We want to clarify this with the help of the failure model of Pióro and Medhi [2004]. The above discussion has enabled us to model the failure of a network in each of these failure states. But we may model partial failures of arcs in an analogous way. To this aim let \( \Sigma \) with \( |\Sigma| = |A| \) be the set of failure states with \( \chi_{\sigma_a} = 1 \), \( \alpha_{\sigma_a} = 0 \) and \( \alpha_{\sigma_a} = 1 \) for all \( a \neq e \in A \). The above model guarantees survivability of the network in each of these failure states. But we may model partial failures of arcs in \( \Sigma \) in a similar way. To this aim let \( \Sigma \) with \( |\Sigma| = |A| \) be the set of failure states with \( \chi_{\sigma_a} = 1 \), \( 0 < \alpha_{\sigma_a} < 1 \) and \( \alpha_{\sigma_a} = 1 \) for all \( a \neq e \in A \). Then incorporating the following inequalities to our model guarantees survivability of the network in each of these failure states:

\[
g_a(c, d) \leq \alpha_{\sigma_a} e_a.
\]

These constraints fit in our framework of integer Minkowski programs: in section 2.1 we have seen that we may incorporate non-linear constraints

\[
\kappa(A(z)) + h^T z \leq \delta
\]

with \( \kappa \) Minkowski-additive and antitone. We already know that the arc-survivability functionals \( g_a(P_{C,b}) \) are Minkowski-additive, antitone and non-negative. Now let \( h_a := -\alpha_{\sigma_a} e_a \) with \( e_a \in \mathbb{Z}_+^{[A]+(k-1)|V|} \) the unit-vector admitting a 1 in the \( a \)-th position and let \( \tau := 0 \). Then the condition

\[
g_a(P_{C,b}) + h^T b \leq \tau
\]

is equivalent to inequality (27).

6.3. Simultaneous failures. Up to now, we have considered the case of survivability w.r.t. complete rerouting in case of failure of only one arc. Our approach to model survivability of a network in case of failure of an arc can easily be modified to model survivability w.r.t. complete rerouting in case of simultaneous failure of up to \( q \) arcs. In the model of Pióro and Medhi [2004] this means: Let \( \Sigma \) be an index set of failure states with \( |\Sigma| < \infty \). \( \chi_{\sigma} = 1 \) for all \( \sigma \in \Sigma \) and \( \alpha_{\sigma} \in \{0, 1\}^{|A|} \). Now let \( \sigma \in \Sigma \) be fixed and let \( a_1, \ldots, a_q \) be the set of arcs with \( \alpha_{a_1, \sigma} = 0 \). Consider the Minkowski-additive and antitone functionals presented below:

\[
g_{a_1, \ldots, a_q}(P_{C,b}) := \min_{(f,s) \in P_{C,b}} \left( \sum_{l=1}^k (f_{a_1}^l + f_{a_2}^l + \ldots + f_{a_q}^l) \right).
\]

We will call functions of the above type \textit{arc-survivability functionals for simultaneous failure of arcs} \( a_1, \ldots, a_q \). The Minkowski-additivity of these functionals can be shown analogously to the proof of Lemma 6.2. Antitonicity and non-negativity of the function follow as well. As before, a network \( N = (V, A, c, d) \) is survivable w.r.t. complete rerouting in case of simultaneous failure of arcs \( a_1, \ldots, a_q \) if and only if \( g_{a_1, \ldots, a_q}(P_{C,b}) = 0 \). Applying Theorem 2.6 and adding these equalities to
the formulation of the network design problem as integer Minkowski program for all failure states $\sigma \in \Sigma$ ensures arc survivability of the network. As before survivability can be guaranteed by adding linear equalities to the formulation as integer linear program.

6.4. Node survivability w.r.t. complete rerouting. The arc-survivability functionals for simultaneous failure of $q$ arcs will allow us to model node survivability of a network. If a node $v \in V$ fails then all arcs incident with this node fail simultaneously. Additionally the demand of all commodities incident with $v$ fail as well. This implies in particular that the load of the network is lower in the failure case! Therefore we have two different cases: Either node $v$ is the source or sink of a commodity or it is not. We consider a transformation of our digraph (see Figure 9) which enables us to model node survivability of a network via arc-survivability functionals. For each commodity $l$ we introduce an auxiliary arc $\bar{a}_l = (s_l, t_l)$. Our digraph now consists of the set of nodes $V$ and the disjoint union of ordinary arcs $A$ and auxiliary arcs $\bar{A}$, i.e., $G = (V, A \cup \bar{A})$. Each arc $\bar{a}_l \in \bar{A}$ has a capacity of $c_{\bar{a}_l} = d_l$. If a node $v$ fails we want the commodities incident with this node to be routed via the auxiliary arcs $\bar{a} \in \delta(v)$.

![Figure 9. Transforming the digraph $G = (V, A)$ to $G' = (V, A \cup \bar{A})$]

Now we will model node survivability of a network. To this aim let $v \in V$ be fixed. We want to model survivability of the network in case of failure of node $v$. In the model of Pióro and Medhi (2004) the corresponding failure state would be the following one: $\chi_{\sigma} \in \{0, 1\}^k$ with $\chi_{\sigma} = 0$ if and only if $v \in \{s_l, t_l\}$ and $\alpha_{\sigma} \in \{0, 1\}^{|A|}$ with $\alpha_{\sigma} = 0$ if and only if $a \in \delta(v)$. Now we want to model survivability of the network in this failure state. First of all we have to ensure the existence of a no-fault routing which does not use any of the auxiliary arcs and satisfies all the demands. This means that the minimum simultaneous flow on all auxiliary arcs is equal to 0:

$$g_{\bar{a}_1, \ldots, \bar{a}_k}(P_{C,b}) = 0. \quad (28)$$

Additionally one has to ensure that there exists a feasible vector of flow in case of failure of a node $v$. Then the following constraint ensures survivability of the network in case of failure of arc $v$.

$$g_{a \in A \cup \delta(v)}(P_{C,b}) = 0 \quad (29)$$

With the help of these auxiliary arcs and the arc-survivability functional for simultaneous failure of a set of arcs it is possible to model node survivability of a network by adding the above constraints to our formulation of the network design problem as integer Minkowski program. As shown in the previous section it is possible to guarantee node survivability by adding linear constraints to the reformulation as integer linear program.

It is clear that one may model more complex failure situations by combining and adapting the above scenarios.
Conclusions

In a forthcoming paper (Eisenschmidt, Köppe, and Laugier, 2006b), we will present algorithms to compute atomic fibers for network design problems and computational results. We remark that we expect that the integer linear programs arising from the reformulation method will be extremely large. It is therefore desirable to devise a method for computing atomic fibers on the fly, and to use it within a specialized branch-cut-and-price algorithm. However, algorithmic questions like this are beyond the scope of this paper; they will be the topic of future research.

References

Ravindra K. Ahuja, Thomas L. Magnanti, and James B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1993.

Alper Atamtürk. On capacitated network design cut-set polyhedra. *Mathematical Programming*, 92:425–437, 2002.

Daniel Bienstock and Oktay Günlük. Capacitated network design – polyhedral structure and computation. *INFORMS Journal on Computing*, 8:243–259, 1996.

Daniel Bienstock and Gabriella Muratore. Strong inequalities for capacitated survivable network design problems. *Math. Programming*, 89:127–147, 2001.

Daniel Bienstock, Sunil Chopra, Oktay Günlük, and Chih-Yang Tsai. Minimum cost capacity installation for multicommodity network flows. *Mathematical Programming*, 81:177–199, 1998.

Elke Eisenschmidt, Raymond Hemmecke, and Matthias Köppe. Computation of atomic fibers of $\mathbb{Z}$-linear maps. Manuscript, 2006a.

Elke Eisenschmidt, Matthias Köppe, and Alexandre Laugier. On the computation of atomic fibers for the design of survivable networks. Manuscript, 2006b.

Martin Grötschel and Clyde L. Monma. Integer polyhedra arising from certain network design problems with connectivity constraints. *SIAM J. Discret. Math.*, 3(4):502–523, 1990. ISSN 0895-4801. doi: http://dx.doi.org/10.1137/0403043.

Martin Grötschel, Clyde L. Monma, and Mechthild Stoer. Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints. *Oper. Res.*, 40(2):309–330, 1992a. ISSN 0030-364X.

Martin Grötschel, Clyde L. Monma, and Mechthild Stoer. Facets for polyhedra arising in the design of communication networks with low-connectivity constraints. *SIAM Journal on Optimization*, 2(3):474–504, 1992b.

Martin Henk, Matthias Köppe, and Robert Weismantel. Integral decomposition of polyhedra and some applications in mixed integer programming. *Mathematical Programming, Series B*, 94(2–3):193–206, 2003. doi: 10.1007/s10107-002-0315-0.

M. Iri. On an extension of the maximum-flow minimum-cut theorem to multicommodity flows. *Journal of the Operations Research Society of Japan*, 13:129–135, 1971.

Matthias Köppe. *Exact Primal Algorithms for General Integer and Mixed-Integer Linear Programs*. Dissertation, Otto-von-Guericke-Universität Magdeburg, 2002. Published by Shaker Verlag, Aachen, 2003.

Abdel Lisser, Robert Sarkissian, and Jean-Philippe Vial. Optimal joint synthesis of base and reserve telecommunications networks. Research report NT/PAA/ATR/ORI4491, Centre National d’Études des Télécommunications, Issy-les-Moulineaux, France, 1995a.

Abdel Lisser, Robert Sarkissian, and Jean-Philippe Vial. Survivability in telecommunication networks. Technical report, Logilab, University of Geneva, 102 Bd Carl-Vogt, CH-1211, 1995b.
K. Onaga and O. Kakusho. On feasibility conditions of multicommodity flows in networks. *IEEE Transactions on Circuit Theory*, CT-18(4):425–429, 1971.

Michal Pióro and Deepankar Medhi. *Routing, Flow, and Capacity Design in Communication and Computer Networks*. Elsevier, 2004.

Mechthild Stoer. *Design of Survivable Networks*, volume 1531 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.

Mechthild Stoer and Geir Dahl. A polyhedral approach to multicommodity survivable network design. *Numerische Mathematik*, 68:149–167, 1994.

Mechthild Stoer and Geir Dahl. A cutting plane algorithm for multicommodity survivable network design problems. *INFORMS Journal on Computing*, 10:1–11, 1998.

H. Paul Williams. The elimination of integer variables. *J. Opl. Res. Soc.*, 43(5):387–393, 1992.

Elke Eisenschmidt, Otto-von-Guericke-Universität Magdeburg, Department of Mathematics, Institute for Mathematical Optimization (IMO), Universitätsplatz 2, 39106 Magdeburg, Germany

E-mail address: eisensch@imo.math.uni-magdeburg.de

Matthias Köppe, Otto-von-Guericke-Universität Magdeburg, Department of Mathematics, Institute for Mathematical Optimization (IMO), Universitätsplatz 2, 39106 Magdeburg, Germany

E-mail address: mkoeppe@imo.math.uni-magdeburg.de

Alexandre Laugier, France Télécom R&D/BIZZ/DIAM, 905 rue Albert Einstein, 06921 Sophia Antipolis Cedex, France

E-mail address: alexandre.laugier@orange-ft.com