Finite-gap solutions of the modified Novikov–Veselov equations: their spectral properties and applications

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1 Introduction

In this article we construct finite-gap solutions of the modified Novikov–Veselov equations and discuss the algebro-geometric properties of the corresponding spectral problem and its connection with the solutions of the modified Korteweg–de Vries equations. The article adheres to [1] wherein finite-gap potentials were written down (with a short sketch of the proof) for a two-dimensional Dirac operator.

The modified Veselov–Novikov equations were introduced by Bogdanov in [2] and have the shape of a Manakov “L,A,B”-triple [3]:

\[ \frac{\partial L}{\partial t_n} = L A_n + B_n L, \] (1)

where

\[ L = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \] (2)

and \( A_n \) and \( B_n \) are matrix differential operators; moreover, the order of \( A_n \) equals \( 2n + 1 \). Furthermore, the highest term \( A_n \) equals

\[ A_n = \begin{pmatrix} \partial^{2n+1} + \bar{\partial}^{2n+1} & 0 \\ 0 & \partial^{2n+1} + \bar{\partial}^{2n+1} \end{pmatrix} + \cdots \] (3)

This hierarchy is adjoined to the operator \( L \) and preserves the zero part of its spectrum. Indeed, if

\[ L \psi = 0 \] (4)

then the equation

\[ \frac{\partial \psi}{\partial t_n} + A_n \psi = 0 \] (5)
deforms the eigenfunctions with the zero eigenvalue; i.e., if (5) holds for all \( t \) and if \( \psi \) satisfies (4) for some \( t_0 \) then \( \psi \) satisfies (4) for all \( t \).

The first equation \((n = 1)\) looks like

\[
U_t = \left( U_{zzz} + 3U_z V + \frac{3}{2} UV_z \right) + \left( U\bar{z}\bar{z} + 3U\bar{z} \bar{V} + \frac{3}{2} U\bar{V} \right),
\]

where

\[
V\bar{z} = (U^2)\bar{z}
\]

and \( z = x + iy \in \mathbb{C} \). This equation preserves realness of the potentials \( U \) and, for real potentials depending on a single spatial variable \( x \), it is reduced to the modified Korteweg–de Vries equation

\[
U_t = \frac{1}{4} U_{xxx} + 6U_x U^2
\]

(here \( V = U^2 \)).

The second equation of the modified Novikov–Veselov hierarchy was written down in [4].

2 Finite-gap potentials of a two-dimensional Dirac operator and finite-gap solutions of the modified Novikov–Veselov equations

Like in [1], here we restrict exposition to the case in which the spectral surface \( \Gamma \) of the operator \( L \) is smooth. Even for the operator corresponding to the Clifford torus, this surface is a sphere with a double point [5]. Explicit formulas for the surfaces \( \Gamma \) with double points could also be constructed, but the general construction would be much more cumbersome. In the most important case when the normalized surface remains connected, such solutions appear in the limit of the solutions described below.

We begin with recalling the necessary results of [1]: Propositions 1 and 2.

Consider the more general operator

\[
\tilde{L} = \left( \begin{array}{cc} 0 & \partial \\ -\bar{\partial} & 0 \end{array} \right) + \left( \begin{array}{cc} U & 0 \\ 0 & V \end{array} \right).
\]

The operator \( \tilde{L} \) is said to be finite-gap if it has the following shape.

**Proposition 1** (1) Let \( \Gamma \) be a compact Riemann surface of genus \( g \), let \( \infty \pm \) be a pair of distinct points in \( \Gamma \); let \( k_{1\pm}^{-1} \) be local parameters in neighborhoods about these points; moreover, \( k_{1\pm}^{-1}(\infty \pm) = 0 \); and let \( D \) a nonspecial effective divisor of degree \( g + 1 \) on \( \Gamma \setminus \{ \infty \pm \} \); i.e., \( D = P_1 + \ldots + P_{g+1} \), where \( P_i \in \Gamma \setminus \{ \infty \pm \} \).

Then
(1) There exists a unique vector-function \( \psi(z, z, P) = (\psi_1, \psi_2) \), where \( z \in \mathbb{C} \), such that \( \psi \) is meromorphic in \( P \) on \( \Gamma \setminus \{ \infty \pm \} \), has poles only at the points of \( D \), and has the following asymptotics:

\[
\psi = \exp(k_+z) \begin{pmatrix} 1 & \xi_{11}^+/k_+ + O(k_+^-) \\ 0 & 1 \end{pmatrix} \quad \text{as } P \to \infty_+,
\]

\[
\psi = \exp(k_-\bar{z}) \begin{pmatrix} 1 & \xi_{11}^-/k_- + O(k_-^-) \\ 0 & 1 \end{pmatrix} \quad \text{as } P \to \infty_-;
\]

(2) there exists a unique operator \( \tilde{L} \) of the shape (7) such that \( \tilde{L}\psi = 0 \).

The potentials of \( \tilde{L} \) have the shape

\[
U = -\xi_{21}^+, \quad V = \xi_{11}^-.
\]

The proof, implied in [1], bases on the general properties of the Baker–Akhiezer functions [6,7].

Consider the space \( \mathcal{E}_z = \mathcal{E}(\Gamma, \infty_\pm, k_{\pm}, D, z) \) formed by the functions \( \varphi \) on \( \Gamma \) satisfying the following conditions:

1. \( \varphi \) is meromorphic outside \( \infty_\pm \) and has poles only at the points of \( D \);
2. \( \varphi^+ = \exp(-k_+z) \) is holomorphic in a neighborhood of \( \infty_+ \) and \( \varphi^- = \exp(-k_-z) \) is holomorphic in a neighborhood of \( \infty_- \).

If \( \varphi_1, \varphi_2 \in \mathcal{E}_z \), then the function \( \varphi_1/\varphi_2 \) is meromorphic on the whole surface \( \Gamma \) and, for \( z \) in general position, the divisor of its poles is nonspecial. This implies for a general \( z \) that

1. by the Riemann–Roch theorem, \( \dim \mathcal{E}_z = 2 \);
2. \( \varphi \) is uniquely determined by the values \( \varphi^+(\infty_+ \pm) \) and \( \varphi^-(\infty_- \pm) \).

Now, take a basis \( \psi_1, \psi_2 \) for \( \mathcal{E}_z \) normalized by the conditions

\[
\psi_1 \exp(-k_+z) = 1, \quad \psi_2 \exp(-k_+z) = 0 \quad \text{at } \infty_+,
\]

\[
\psi_2 \exp(-k_-\bar{z}) = 0, \quad \psi_2 \exp(-k_-\bar{z}) = 1 \quad \text{at } \infty_-.
\]

These basis functions have analytic continuations in \( z \) on the whole complex plane \( \mathbb{C} \) to functions defining the components of a Baker–Akhiezer function \( \psi = (\psi_1, \psi_2) \) satisfying the conditions of the proposition.

Furthermore, the components of \( \psi \) give normalized bases for \( \mathcal{E}_z \) which are unique. This yields uniqueness of \( \psi \) and proves the first claim of the proposition.

If the potentials of \( L \) are defined by (8) then \( L\psi \in \mathcal{E}_z \) for every \( z, \exp(-k_+z) L\psi = 0 \) at \( \infty_+ \), and \( \exp(-k_-\bar{z})L\psi = 0 \) at \( \infty_- \). The last two equalities mean that \( L\psi = 0 \) everywhere.

The proof of Proposition 1 is over.

Remark 1. In the case when \( \Gamma \) has \( m \) double points exhausting all singularities, the construction of \( \psi \) reduces to the following: Let \( \Gamma \) be the normalization of \( \Gamma \) obtained by “unsticking” the double points, and under the projection \( \Gamma \to \Gamma \) the pairs of points \( (Q_1^+, Q_1^-), \ldots, (Q_m^+, Q_m^-) \) go into the \( m \) double points. Then \( \psi \) is a Baker–Akhiezer function with the same asymptotics but it has the divisor of poles \( D \) of degree \( g+m+1 \) and satisfies the normalization conditions \( \psi(Q_j^+) = \psi(Q_j^-), \ j = 1, \ldots, m \).
Proposition 2 Suppose that the spectral data \((\Gamma, \infty, k, D)\) of a finite-gap operator \(L\) satisfy the following conditions:

1. there is a holomorphic involution \(\sigma : \Gamma \to \Gamma\) such that \(\sigma(\infty) = \infty\), \(\sigma(k) = -k\), and a meromorphic differential \(\omega\) on \(\Gamma\) with zeros in \(D + \sigma(D)\), two poles at \(\infty\), and principal parts \(\pm k^2 + O(k^{-1})\) \(dk^{-1}\);

2. there is an antiholomorphic involution \(\tau : \Gamma \to \Gamma\) such that \(\tau(\infty) = \infty\), \(\tau(k) = -\bar{k}\), and a meromorphic differential \(\tilde{\omega}\) on \(\Gamma\) with zeros in \(D + \tau(D)\), two poles at \(\infty\), and principal parts \(\pm k^2 + O(k^{-1})\) \(dk^{-1}\).

Then the operator \(L\) has the shape (2) with a real potential \(U\).

Now, we are in a position to formulate a theorem about finite-gap solutions of the modified Novikov–Veselov equations.

Theorem 1 Suppose that the spectral data \((\Gamma, \infty, k, D)\) satisfy the conditions of Propositions 1 and 2. Then

1. there is a unique vector-function \(\psi(z, \bar{z}, t_1, \ldots, P)\) such that
   \(\psi\) depends on \(z, P \in \Gamma\), time variables \(t_1, \ldots\), only finitely many of which may differ from zero, \(P \in \Gamma\);
   
   \(\psi\) is meromorphic in \(P\) on \(\Gamma \setminus \{\infty\}\) and has poles only at the points of \(D\);
   
   the following asymptotics hold:

   \[
   \psi \sim \exp \left( k^3 t_1 + \ldots + k^{2n+1} t_n + \ldots \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(k^{-1}) \quad \text{as } P \to \infty,
   \]

   \[
   \psi \sim \exp \left( k^3 t_1 + \ldots + k^{2n+1} t_n + \ldots \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(k^{-1}) \quad \text{as } P \to -\infty;
   \]

2. there are a unique operator \(L\) of the shape (2) such that \(L\psi = 0\) and unique operators \(A_n\) with principal parts (3) such that equations (5) are satisfied;

3. the potential \(U = U(z, \bar{z}, t_1, \ldots)\) of the operator \(L\) satisfies the modified Novikov–Veselov equations (1).

Existence and uniqueness of \(\psi, L,\) and \(A_n\) are proved in the same way as in Proposition 1. This implies that

\[
\frac{\partial(L\psi)}{\partial t_n} = \frac{\partial L}{\partial t_n} \psi + L \frac{\partial \psi}{\partial t_n} = \left( \frac{\partial L}{\partial t_n} - L A_n \right) \psi = 0.
\]

Proceeding as in the derivation of the Novikov–Veselov equations [8], we now calculate some operators \(B_n\) such that \(L A_n + B_n L\) are the operators of multiplication by a matrix, i.e., matrix differential operators of zero order. For example, for \(n = 1\)

\[
B_1 = 3 \begin{pmatrix} 0 & U z + U \bar{z} \\ -U z - U \bar{z} & 0 \end{pmatrix} +

+ 3 \begin{pmatrix} 0 & U(\bar{V} - V) \\ -U z + U(\bar{V} - V) & 0 \end{pmatrix}.
\]
Define $E_{z,t}$ as the space of functions $\varphi$ that are meromorphic on $\Gamma \setminus \{\infty_{\pm}\}$, have poles only at the points of $D$, and are such that $\varphi^+ = \varphi \exp(-k_{\pm}z - k_{\pm}^3t_1 - \ldots)$ is holomorphic in a neighborhood of $\infty_+$ and $\varphi^- = \varphi \exp(-k_{-}\bar{z} - k_{-}^3t_1 - \ldots)$ is holomorphic in a neighborhood of $\infty_-$. For $z, t_1, \ldots$ in general position, these spaces are two-dimensional and their elements are uniquely determined by the values of $\varphi^{\pm}$ at $\infty_{\pm}$.

The multiplication

$$(L_{t_1} - L A_n - B_n L) : E_{z,t} \to E_{z,t}$$

by a matrix independent of $P \in \Gamma$ carries $\psi$ to the vector-function $\psi(x, P)$ whose components $\psi_{j,n}$ belong to $E_{z,t}$ for arbitrary $z, t_1, \ldots$. By the definition of $L$, $A_n$, and $B_n$, we have

$$\psi_{j,n}^{\pm}(\infty_{\pm}) = \psi_{\bar{j},n}^{\mp}(\infty_{\mp}) = 0$$

for $j = 1, 2$. It follows that $\psi_{(n)} = 0$ and (9) is the multiplication by the zero $(2 \times 2)$-matrix. Hence, equations (1) are satisfied, which completes the proof of Theorem 1.

**Theorem 2** Suppose that the spectral data satisfy the conditions of Theorem 1; moreover, there exists a meromorphic function

$$\lambda : \Gamma \to \mathbb{C} P^1 = \hat{\mathbb{C}},$$

having exactly two poles at the points $\infty_{\pm}$ with the Laurent parts

$$\lambda = \pm ik_{\pm} + O(1) \text{ as } k_{\pm} \to \infty.$$

Then the function $\psi(z, \bar{z}, P)$ has the shape

$$\psi(z, \bar{z}, P) = \tilde{\psi}(x, P) \exp(\lambda(P)y),$$

the potential of the operator $L$ depends only on $x$ and its dynamics in times $t_n$ is described by the equations of the modified Korteweg–de Vries hierarchy.

Proof of the theorem reduces to the following. According to the theory of Baker–Akhiezer functions, there exists a unique function $\tilde{\psi}(x, P)$ with the prescribed spectral data $(\Gamma, \infty_{\pm}, k_{\pm}, D)$ and the asymptotics

$$\tilde{\psi} \approx \exp(k_+ x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ as } k_+ \to \infty, \quad \tilde{\psi} \approx \exp(k_- x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ as } k_- \to \infty.$$

Afterwards, it is easy to see that the function $\tilde{\psi}(x, P) \exp(\lambda(P)y)$ satisfies the conditions of Theorem 1 and so from uniqueness we infer its coincidence with $\psi(z, \bar{z}, P)$.

The potential of the operator $L$ is determined by the function $\tilde{\psi}$; in consequence, it depends only on $x$. Moreover, its deformations in $t_n$ constitute the hierarchy of equations associated with the one-dimensional Dirac operator

$$\tilde{L} = \frac{1}{2} \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix},$$
and the spectral problem (4) is reduced to the Zakharov–Shabat problem
\[
\left[ \begin{array}{cc}
0 & \partial_x \\
-\partial_x & 0
\end{array} \right] + \left[ \begin{array}{cc}
2U & 0 \\
0 & 2U
\end{array} \right] \psi = \left[ \begin{array}{cc}
0 & i\lambda \\
i\lambda & 0
\end{array} \right] \psi.
\]
It is easy to see that the arising equations constitute the modified Korteweg–de Vries hierarchy.

**Remark 2.** Under the conditions of Theorem 2, the Riemann surface \( \Gamma \) is hyperelliptic, since on it there is a meromorphic function with exactly two poles.

Theorem 2, together with the formulas of § 3, provides derivation of explicit formulas for finite-gap solutions of the modified Korteweg–de Vries hierarchy. In another way (using the Miura transformation), this was made in [9].

### 3 Explicit formulas for potentials and solutions

Assume given spectral data \((\Gamma, \infty_\pm, k_\pm, D)\) satisfying the conditions of Theorem 1.

On the Riemann surface \( \Gamma \) of genus \( g \), choose a canonical basis of 1-cycles: \( \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \). By definition, its intersection form is as follows:

\[
\alpha_j \circ \beta_k = \delta_{jk}, \quad \alpha_j \circ \alpha_k = \beta_j \circ \beta_k = 0.
\]

Given the basis, we construct

1. the normalized basis of holomorphic 1-forms \( \omega_1, \ldots, \omega_g \):

   \[
   \int_{\alpha_k} \omega_j = \delta_{jk};
   \]

2. the matrix of \( \beta \)-periods of holomorphic 1-forms:

   \[
   \Omega_{jk} = \int_{\beta_k} \omega_j;
   \]

3. the theta function of the surface \( \Gamma \):

   \[
   \vartheta(u) = \sum_{N \in \mathbb{Z}^g} \exp \pi i ((\Omega N, N) + 2(N, u)),
   \]

where \( u \in \mathbb{C}^g \).

The complex torus \( J(\Gamma) = \mathbb{C}^g / \{ M + \Omega N : M, N \in \mathbb{Z}^g \} \) is called the Jacobian variety of \( \Gamma \) and the mapping

\[
P \to A(P) = \left( \int_{P_0}^P \alpha_1, \ldots, \int_{P_0}^P \alpha_g \right)
\]

from \( \Gamma \) into \( J(\Gamma) \) is called the Abelian mapping. Here \( P_0 \) is a fixed point in \( \Gamma \). By linearity, the Abelian mapping extends to divisors, and the expression
\( A(D_1) - A(D_2) \) is soundly defined for effective divisors of the same degree and is independent of the choice of \( P_0 \) in the definition of \( A \).

Denote by \( Q \) and \( R \) effective divisors of degree \( g \) such that the following relations hold:

\[
A(Q) + A(\infty_-) - A(D) = A(R) + A(\infty_+) - A(D) = 0.
\]

This amounts to fulfillment of the linear equivalences

\[
D = P_1 + \ldots + P_{g+1} \sim Q_1 + \ldots + Q_g + \infty_-,
\]
\[
D = P_1 + \ldots + P_{g+1} \sim R_1 + \ldots + R_g + \infty_+,
\]

where \( Q = Q_1 + \ldots + Q_g \) and \( R = R_1 + \ldots + R_g \).

Denote by \( \eta^\pm_l \) the meromorphic 1-forms that are uniquely determined by the following conditions:

1. \( \eta^\pm_l \) has the only pole at \( \infty^\pm \) with the Laurent part \( dk^\pm_l \); 
2. the integrals of \( \eta^\pm_l \) vanish over the \( \alpha \)-cycles.

With each form \( \eta^\pm_l \), associate the vector of \( \beta \)-periods

\[
(U^\pm_l)^j = \frac{1}{2\pi i} \int_{\beta_j} \eta^\pm_l
\]

and the constants \( a^\pm_l \) and \( b^\pm_l \) defined by the conditions

\[
\int_{P_0}^{P} \eta^\pm_l - a^\pm_l = k^\pm_l + O(k^{-1}) \text{ near } \infty^\pm, \quad \int_{P_0}^{P} \eta^\pm_l - b^\pm_l = O(k^{-1}) \text{ near } \infty_. \tag{11}
\]

Here we consider the same paths from \( P_0 \) to \( P \) for all \( l \); i.e., defining asymptotics, we fix some homotopic class of paths from \( P_0 \) to small neighborhoods of the infinities.

Denote by \( \delta \) the vector of Riemann constants which is defined as follows: for a point \( u \in J(\Gamma) \) in general position, the function \( \vartheta(A(P) - u) \) has zeros at exactly \( g \) points \( S_1, \ldots, S_g \); moreover, \( u + \delta = A(S_1) + \ldots + A(S_g) \) (this function is multiple valued and its values are determined by the choice of the integration path in the definition of the Abelian mapping and differ by nonzero multipliers).

Also, choose an odd half-period \( \varepsilon \in J(\Gamma) \); i.e., \( \vartheta(\varepsilon) = 0 \) and \( 2\varepsilon \equiv 0 \) on \( J(\Gamma) \).

Existence and uniqueness, if any, of all the above-indicated objects are well known from the theory of Riemann surfaces (see, for instance, [10]).

Define the following functions:

\[
\Phi_1(z, t) = z \left( \int_{P_0}^{P} \eta^+_1 - a^+_1 \right) + \bar{z} \left( \int_{P_0}^{P} \eta^-_1 - b^-_1 \right) + \sum_{l \geq 1} t_l \left( \int_{P_0}^{P} (\eta^+_l + \eta^-_{2l+1}) - (a^+_l + b^-_{2l+1}) \right),
\]
\[ \Phi_2(z, t) = z \left( \int_{\mathcal{P}_0} \eta^+ - b^+_1 \right) + \bar{z} \left( \int_{\mathcal{P}_0} \eta^- - a^-_1 \right) + \sum_{l \geq 1} t_l \left( \int_{\mathcal{P}_0} (\eta^+_{2l+1} + \eta^-_{2l+1}) - (a^+_{2l+1} + b^+_{2l+1}) \right), \]

\[ \Psi(z, t) = z(a^+_1 - b^-_1) + \bar{z}(b^-_1 - a^-_1) + \sum_{l \geq 1} t_l (a^+_{2l+1} - a^-_{2l+1} + b^+_{2l+1} - b^-_{2l+1}), \]

\[ F_1(z, t) = U^+_1 + U^-_1 + \sum_{l \geq 1} (U^+_{2l+1} + U^-_{2l+1}) t_l + \delta - A(Q), \]

\[ F_2(z, t) = U^+_1 + U^-_1 + \sum_{l \geq 1} (U^+_{2l+1} + U^-_{2l+1}) t_l + \delta - A(R). \]

To simplify the notation of the arguments of theta functions, denote the values of the Abelian mapping on some divisor (in particular, at a point) \( S \) by \( S \) rather than \( A(S) \).

**Theorem 3** The function \( \psi \) of Proposition 1 has the following shape:

\[ \psi_1(z, t, P) = \exp(\Phi_1(z, t)) \cdot \frac{\vartheta(P + F_1(z, t))}{\vartheta(P + \delta - Q)} \cdot \frac{\vartheta(\infty + \delta - Q)}{\vartheta(\infty + F_1(z, t))} \times \]

\[ \times \frac{\vartheta(\varepsilon + P - \infty_-)}{\vartheta(\varepsilon + \infty_+ - \infty_-)} \cdot \prod_{j}^{g+1} \vartheta(\varepsilon + \infty_+ - P_j) \cdot \prod_{j}^{g} \vartheta(\varepsilon + P - Q_j), \]

\[ \psi_2(z, t, P) = \exp(\Phi_2(z, t)) \cdot \frac{\vartheta(P + F_2(z, t))}{\vartheta(P + \delta - R)} \cdot \frac{\vartheta(\infty_+ - \delta - R)}{\vartheta(\infty_+ + F_2(z, t))} \times \]

\[ \times \frac{\vartheta(\varepsilon + P - \infty_-)}{\vartheta(\varepsilon + \infty_- - \infty_-)} \cdot \prod_{j}^{g+1} \vartheta(\varepsilon + \infty_- - P_j) \cdot \prod_{j}^{g} \vartheta(\varepsilon + P - R_j), \]

The potential \( U \) has the shape

\[ U(z, t) = -C \exp(\Psi(z, t)) \cdot \frac{\vartheta(\infty_+ + F_2(z, t))}{\vartheta(\infty_- + F_2(z, t))}, \]

where

\[ C = \frac{\prod_{j}^{g+1} \vartheta(\varepsilon + \infty_- - P_j) \prod_{j}^{g} \vartheta(\varepsilon + \infty_+ - R_j)}{\prod_{j}^{g+1} \vartheta(\varepsilon + \infty_+ - P_j) \prod_{j}^{g} \vartheta(\varepsilon + \infty_- - R_j)} \times \]

\[ \times \frac{1}{\vartheta(\varepsilon + \infty_- \varepsilon_+)} \cdot \sum_{l \geq 1} (\varepsilon + b^-_1)^l \frac{\vartheta(\varepsilon)}{\vartheta(\varepsilon + \varepsilon_+)} \]

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Here we assume that in the definition of the Abelian mapping at a point $P$ we use the same paths from $P_0$ to $P$ (because $P$ has several occurrences in the same formula) and the paths joining $P_0$ with small neighborhoods of the points $\infty_{\pm}$ coincide with those in the definition of the constants (11).

The formulas for $\psi$ are verified directly by using the properties of theta functions [10, 11]. For the one-dimensional reduction (when the potential $U$ depends only on a single spatial variable) these formulas were derived in [12] in detail. To derive these formulas for $U$, it suffices to use the fact that $\partial A(P)/\partial k_+^{-1} = U_1^+$ at $\infty_+$.

**Remark 3.** The formulas for $\psi$ and $U$ can be simplified if we account for the fact that all differentials $\eta_{2m+1}^\pm$ for $m \geq 0$ are anti-invariant under the involution $\sigma$. This implies that $\psi$ and $U$ can be written down via the theta functions of the Prym variety of the covering $\Gamma \to \Gamma/\sigma$. However, as compared with the case of a two-dimensional Schrödinger operator, which is finite-gap at a single energy level [8, 13], here we cannot control the topological type of the involution (in the case of a Schrödinger operator the analogous involution has only two smooth fixed points and the dimension of the Prym variety is half of the dimension of $J(\Gamma)$). We merely recall a simple inequality for the dimensions of the Jacobian and Prym varieties:

$$\text{genus}(\Gamma) - \text{genus}(\Gamma/\sigma) = \dim \text{Prym}(\Gamma, \sigma) \geq \left\lfloor \frac{\text{genus}(\Gamma)}{2} \right\rfloor. \quad (12)$$

### 4 The Floquet spectrum

The Riemann surface $\Gamma$ appears naturally for the operators with periodic coefficients; namely, it is the Floquet spectrum “at the zero energy level $E$.”

Assume that the potentials $U$ and $V$ of the operator (7) are periodic with respect to a lattice $\Lambda \subset \mathbb{C}$ of rank 2. A function $\psi : \mathbb{C} \to \mathbb{C}$ is said to be a Floquet function of $L$ with eigenvalue $E$ and quasi-impulses $(k_1, k_2)$ if

$$L\psi = E\psi, \quad \psi(z + \gamma) = \exp(2\pi i (\Re \gamma \cdot k_1 + \Im \gamma \cdot k_2))\psi(z)$$

for $\gamma \in \Lambda$. Each such function admits a representation of the shape

$$\psi(z) = \exp(2\pi i (xk_1 + yk_2))\varphi(z),$$

where $\varphi(z)$ is periodic with respect to $\Lambda$ and satisfies the equation

$$L_k \varphi = E\varphi, \quad (13)$$

with

$$L_k = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & \pi(k_2 + ik_1) \\ \pi(k_2 - ik_1) & V \end{pmatrix},$$

and $\varphi$ can be regarded as a function on the two-dimensional torus $\mathbb{C}/\Lambda$. Choosing a constant $C$ so that the operator

$$\mathcal{A} = \begin{pmatrix} C & \partial \\ -\bar{\partial} & C \end{pmatrix}$$

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be invertible on \( L_2(\mathbb{C}/\Lambda) \), rewrite (13) for \( \xi = A \psi \) and obtain

\[
\left[ 1 + \begin{pmatrix} U - (C + E) & \pi(k_2 + ik_1) \\ \pi(k_2 - ik_1) & V - (C + E) \end{pmatrix} A^{-1} \right] \xi = 0.
\]

The last equation has the shape

\[
(1 + A(k_1, k_2, E))\xi = 0,
\]  

where \( A(k_1, k_2, E) \) is a polynomial pencil in \( k_1, k_2, E \) of compact operators from \( L_2(\mathbb{C}/\Lambda) \) into \( L_2(\mathbb{C}/\Lambda) \). Now, the polynomial Fredholm alternative, first established by Keldysh [14], implies that equation (14) is solvable if and only if \( (k_1, k_2, E) \) belongs to some complex-analytic submanifold of positive codimension in \( \mathbb{C}^3 \). The same alternative implies that if \( E = 0 \) then equation (14) is solvable if and only if \( (k_1, k_2) \) belongs to the complex-analytic submanifold \( \hat{\Gamma} \) of codimension 1 in \( \mathbb{C}^2 \) (see also [1]).

As it is easy to see, the manifold \( \hat{\Gamma} \) is invariant under the action of the dual lattice \( \Lambda^* \):

\[
k_1 \rightarrow k_1 + \text{Re} \gamma^*, \quad k_2 \rightarrow k_2 + \text{Im} \gamma^*, \quad \gamma^* \in \Lambda^*,
\]

where \( \Lambda^* \) consists of the vectors \( \gamma^* \in \mathbb{C} \) such that \( (\gamma, \gamma^*) = \text{Re} \gamma \cdot \text{Re} \gamma^* + \text{Im} \gamma \cdot \text{Im} \gamma^* \in \mathbb{Z} \) for all \( \gamma \in \Lambda \).

The factor-manifold \( \Gamma = \hat{\Gamma} / \Lambda^* \) is called the Floquet spectrum of the operator \( L \) “at the zero energy level.”

For the twice periodic operators \( L \) described in Proposition 1, it is the Riemann surface \( \Gamma \) that represents the Floquet spectrum. In this case the function \( \psi \) defines an analytic family of Floquet functions which has poles arising when we fix asymptotics. A rigorous proof of these facts can be carried out by the methods of perturbation theory and was implemented for important two-dimensional scalar operators in [15]. We should consider perturbations of the zero potentials (in our case \( U = V = 0 \)) for which the structure of the Floquet spectrum is rather simple.

**Example 1.** \( U = V = 0 \).

We may assume that \( \Lambda = \mathbb{Z} + i\mathbb{Z} \). The Floquet functions are parametrized by two families \( \psi^1 = (e^{\lambda z}, 0) \) and \( \psi^2 = (0, e^{\mu z}) \), the surface \( \Gamma \) splits into a \( \lambda \)-plane and a \( \mu \)-plane, the \( \lambda \)-plane is compactified by the point \( \infty_+ \), and the \( \mu \)-plane is compactified by the point \( \infty_- \).

**Example 2.** \( U = V = d \neq 0 \).

We again assume that \( \Lambda = \mathbb{Z} + i\mathbb{Z} \).

The Floquet functions are linear combinations of the functions

\[
\tilde{\psi}(z, \bar{z}, \lambda) = \left( \exp \left( \lambda z - \frac{d^2}{\lambda} \bar{z} \right), \frac{-d}{\lambda} \exp \left( \lambda z - \frac{d^2}{\lambda} \bar{z} \right) \right),
\]

where \( \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). The surface \( \Gamma \) is the complex sphere, the \( \lambda \)-plane, compactified by the point at infinity. Two “infinities” are distinguished on \( \Gamma \): \( \infty_+ \), where \( \lambda = \infty \), and \( \infty_- \), where \( \lambda = 0 \). The parameters \( k_\pm \) have the shape

\[
k_+ = \lambda, \quad k_- = -\frac{d^2}{\lambda},
\]
the divisor $D$ consists of the point $d \in \mathbb{C}^*$, and the Baker–Akhiezer function $\psi$ has the shape

$$
\psi = \frac{\lambda}{\lambda - d} \left( \exp \left( \frac{\lambda z - d^2}{\lambda^2} \right), -\frac{d}{\lambda} \exp \left( \frac{\lambda z - d^2}{\lambda^2} \right) \right).
$$

**Example 3.** $U = V$ is a function in one variable.

Let $U = U(x)$ and $U(x + T) = U(x)$, where $T$ is a minimal period. Equation (4) is rewritten as the Zakharov–Shabat system (10) which in terms of $\eta_1 = \psi_1 + i\psi_2$ and $\eta_2 = \psi_1 - i\psi_2$ takes the shape

$$(\partial_x + 2iU)\eta_1 = -i\lambda\eta_2, \quad (\partial_x - 2iU)\eta_2 = -i\lambda\eta_1,$$

implying

$$(\partial_x^2 + 4U^2 + 2iU_x)\eta_1 = -\lambda^2\eta_1, \quad (\partial_x^2 + 4U^2 - 2iU_x)\eta_2 = -\lambda^2\eta_2.$$

The passage from the operator $(\partial_x^2 + 4U^2 + 2iU_x) = (\partial_x - 2iU)(\partial_x + 2iU)$ to the operator $(\partial_x^2 + 4U^2 - 2iU_x) = (\partial_x + 2iU)(\partial_x - 2iU)$ is referred to as the Miura transformation.

According to the theory of a periodic one-dimensional Schrödinger operator [6], for an operator of the shape $-\partial^2 + V(x)$ there is a Floquet–Bloch function $\tilde{\psi}(x, P)$ satisfying the equation

$$(-\partial^2 + V(x))\tilde{\psi} = E\tilde{\psi}$$

and defined on a two-fold covering of the complex $E$-plane of the shape

$$\tilde{\Gamma} = \{(\mu, E) : \mu^2 = P(E)\}.$$

Moreover, $P(E)$ is an entire function which has only simple zeros and is a polynomial of odd degree when the number of these zeros is finite (i.e., when the operator is finite-gap), and $\mu$ is a quasi-impulse in $x$.

Let $L$ be a finite-gap Dirac operator with a periodic one-dimensional potential $U(x)$. Then the Floquet spectrum of $L$ projects onto the Floquet–Bloch spectra of the operators $(\partial_x^2 + 4U^2 + 2iU_x)$ and $(\partial_x^2 + 4U^2 - 2iU_x)$; moreover, this projection has the simple shape $(k_1, k_2) \rightarrow (k_1, k_2^2)$. Therefore, the Floquet–Bloch spectra of two one-dimensional operators connected by the Miura transformation coincide (this was first proved in [16]) and are the factor-spaces $\Gamma/\omega$ with respect to the involution $\omega : (k_1, k_2) \rightarrow (k_1, -k_2)$. The functions $\eta_1$ and $\eta_2$ are, up to multiplication by meromorphic functions on $\Gamma$, the lifts to the covering of the Floquet–Bloch functions of the corresponding one-dimensional operators.

In this language, Proposition 2 is interpreted as follows (similar explanations of the presence of involutions of spectral surfaces were given in [17] for many other matrix operators).

**Proposition 3** Let $L$ be a twice periodic operator of the shape (7).
1. If $U = \bar{V}$ then the Floquet spectrum is preserved by the anti-involution
   $$(k_1, k_2) \rightarrow (-\bar{k}_1, -\bar{k}_2);$$

2. If $U = \bar{U}$ and $V = \bar{V}$ then the Floquet spectrum is preserved by the anti-involution
   $$(k_1, k_2) \rightarrow (\bar{k}_1, \bar{k}_2).$$

Proof. 1. Assume $U = \bar{V}$ and $L\psi = 0$, where $\psi = (\psi_1, \psi_2)$ is a Floquet function with quasi-impulses $(k_1, k_2)$. By direct substitution we check that the function $\hat{\psi} = (\bar{\psi}_2, -\bar{\psi}_1)$ is a Floquet function with quasi-impulses $(-\bar{k}_1, -\bar{k}_2)$.

2. If $k = (k_1, k_2)$ belongs to the Floquet spectrum then $\dim \ker L_k > 0$. From the explicit form of the operators we see that $L^*k = L\bar{k}$. The index of an elliptic operator on the compact manifold $\mathbb{C}/\Lambda$ is determined by its principal part; in the case of $L_k$ we have
   $$\text{ind} L_k = \dim \ker L_k - \dim \ker L_k^* = \text{ind} \left( \begin{array}{cc} 0 & \partial \\ -\bar{\partial} & 0 \end{array} \right) = 0,$$
   since the principal part is selfadjoint. Therefore, the inequality $\dim \ker L_k > 0$ implies $\dim \text{Coker} L_k \neq 0$ and $\dim L_{\bar{k}} > 0$, which completes the proof of Proposition 3.

In terms of Proposition 2, the anti-involution $(k_1, k_2) \rightarrow (\bar{k}_1, \bar{k}_2)$ is $\tau$, and the composite of two anti-involutions of Proposition 3 is the involution $\sigma$.

5 On soliton deformations of tori

According to [1], each two-dimensional torus $\Sigma$, immersed in $\mathbb{R}^3$ and smooth of the class $C^3$, is represented by the formulas

$$X^1(z, \bar{z}) = X_0^1 + \frac{i}{2} \int_0^z \left( (\bar{\psi}_2^2 + \psi_1^2) dz' - (\bar{\psi}_1^1 + \psi_2^2) d\bar{z}' \right),$$

$$X^2(z, \bar{z}) = X_0^2 + \frac{1}{2} \int_0^z \left( (\bar{\psi}_2^2 - \psi_1^2) dz' - (\bar{\psi}_1^2 + \psi_2^2) d\bar{z}' \right), \quad (15)$$

$$X^3(z, \bar{z}) = X_0^3 + \int_0^z (\psi_1 \bar{\psi}_2 d\bar{z}' + \bar{\psi}_1 \psi_2 d\bar{z}'),$$

where $X_0 \in \mathbb{R}^3$ is a point lying on the torus and $\psi$ is a Floquet function of the operator (2) defined on $\mathbb{C}$ and periodic with respect to the lattice $\Lambda$. Moreover, the function $\psi$ is multiplied by $\pm 1$ under shifts by periods and the torus $\Sigma$ is conformally equivalent to the flat torus $\mathbb{C}/\Lambda$.

For surfaces locally defined by formulas (15), B. G. Konopelchenko introduced deformations that are described by the modified Novikov–Veselov equations (see [18]): one should deform $U$ in accordance with the modified Novikov–Veselov equations, at that deforming the function $\psi$ in accordance with (5). The deformation of $\psi$ gives rise to a local deformation of the surface.
It is these deformations that stimulated us to present formulas for $\psi$ and its soliton deformations (see §3).

It was shown in [4] that if a torus is given in advance then this deformation, corresponding to (6), gives rise to a global deformation of the torus; moreover, the Willmore functional (the integral of squared mean curvature) is a first integral

$$ W(\Sigma) = 4 \int_{\mathcal{C}/\Lambda} U^2(z, \bar{z}) \, dx dy. $$

The well-known Willmore conjecture asserts that the minimum of the Willmore functional on immersed tori equals $4\pi^2$. The Willmore functional is invariant under the conformal transformations of $\mathbb{R}^3$ that do not send the points of the torus to the point at infinity.

Basing on the fact that a minimum of such variational problem must be nondegenerate (after factorization by the action of the conformal group), it was conjectured in [4] that the minima of this functional are stationary under the deformations generated by equation (6). We can extend this as follows: for all equations of the modified Novikov–Veselov hierarchy, the minima of $W$ for fixed conformal classes are stationary under the induced deformations.

According to this conjecture, for the minima of the Willmore functional, the Prym variety of the covering $\Gamma \to \Gamma/\sigma$ must be one-dimensional and the deformations generated by the modified Novikov–Veselov equations reduce to translations of tori along themselves. According to (12), this implies $\text{genus}(\Gamma) \leq 3$. For the surfaces of genus 3, the dimension of the Prym variety equals 3 if $\sigma$ is a hyperelliptic involution and 2 if $\sigma$ has 4 fixed points (the case in which the dimension equals 1 corresponds to an involution without fixed points). Consequently, the last conjecture implies that for the minima we must have $\text{genus}(\Gamma) \leq 2$.

References

[1] Taimanov, I.A. The Weierstrass representation of closed surfaces in $\mathbb{R}^3$. Functional Anal. Appl. 32:4 (1998), 49–62

[2] Bogdanov, L.V. The Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg–de Vries equation. Theor. Math. Phys. 70 (1987), 309–314.

[3] Manakov, S.V. The inverse problem method and two-dimensional soliton equations. (Russian) Uspekhi Mat. Nauk 31:5 (1976), 245–246.

[4] Taimanov, I.A. Modified Novikov–Veselov equation and differential geometry of surfaces. Transl. Amer. Math. Soc. Ser. 2, 179 (1997), 133–151.

[5] Taimanov, I.A. Surfaces of revolution in terms of solitons. Ann. Global Anal. Geom., 15 (1997), 419–435.
[6] Dubrovin, B.A., Krichever, I.M., and Novikov, S.P. Integrable systems. I. Dynamical systems, IV, 177–332, Encyclopaedia Math. Sci., 4, Springer, Berlin, 2001.

[7] Krichever, I.M. Methods of algebraic geometry in the theory of nonlinear equations. Russian Math. Surveys 32:6 (1977), 185–213.

[8] Veselov, A.P., and Novikov, S.P. Finite-zone two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations. Soviet Math. Dokl. 30 (1984), 588–591.

[9] Gesztesy, F., Quasi-periodic, finite-gap solutions of the modified Korteweg–de Vries equation. in: Ideas and Methods in Mathematical Analysis, Stochastics, and Applications, Cambridge University Press, 1992, 1, pp. 428–471.

[10] Dubrovin, B.A. Theta-functions and nonlinear equations. Russian Math. Surveys 36:2 (1982), 11–92.

[11] Fay, J. Theta Functions on Riemann Surfaces. Lecture Notes in Math., 52. Springer-Verlag, Berlin (1973).

[12] Previato, E. Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation. Duke Math. J. 52 (1985), 329–377.

[13] Veselov, A.P., and Novikov, S.P. Finite-zone two-dimensional Schrödinger operators. Potential operators. Soviet Math. Dokl. 30 (1984), 705–708.

[14] Keldysh, M.V. On the eigenvalues and eigenfunctions of certain classes of nonselfadjoint equations. (Russian) Dokl. Akad. Nauk SSSR, 77:1 (1951), 11–14.

[15] Krichever, I.M. Spectral theory of two-dimensional periodic operators and its applications. Russian Math. Surveys 44:2 (1989), 145–225.

[16] Ehlers, F., and Knörrer, H., An algebro-geometric interpretation of the Bäcklund transformation for the Korteweg–de Vries equation. Comment. Math. Helv. 57 (1982), 1–10.

[17] Dubrovin, B.A., Matrix finite-zone operators. J. Sov. Math. 28 (1985), 20–50.

[18] Konopelchenko, B.G., Induced surfaces and their integrable dynamics. Stud. Appl. Math. 96 (1996), 9–52.