Control Engineering Analysis of Mechanical Pitch Systems

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Abstract. With the help of a local stability analysis the coefficient range of a discrete damper, used for centrifugal forced, mechanical pitch system of small wind turbines (SWT), is gained for equilibrium points. By a global stability analysis the gained coefficient range can be validated. An appropriate approach by Takagi-Sugeno is presented in the paper.

1. Introduction
Within the last few years there has been a positive market development for small wind turbines (SWT) with less than 100kW. To enhance this development further cost reductions have to be achieved, e.g. by using simple mechanical components like mechanical pitch systems.

In [1] different types of mechanical pitch systems are described with advantages and disadvantages.

In this paper the necessary coefficient of a discrete damper is achieved for equilibrium points by analysing the pitch angle stability using the example of a variable speed, direct driven 5kW SWT.

First the local stability is analysed for equilibrium points of a linearised model with the help of the Hurwitz criteria. Secondly, an approach to verify the necessary damping coefficient by analysing the global stability of a non-linear model of the wind turbine with the help of a Takagi-Sugeno transformation is introduced.

2. The analysed mechanical pitch system
The analysed centrifugal forced, mechanical pitch system is characterised by blades, moving outwards along their blade axis with increasing rotation speed due to centrifugal forces. The movement is bounded to a range of 10mm. A spring is loaded with the outwards movement and pushes the blade inwards to the starting position with decreasing rotation speed.

The pitching motion (rotation) is caused by a pin mounted to the blade and guided in a helical ascending notch in the blade sliding bearing, integrated in the hub body (see Figure 1). Therefore the blade movement along the blade axis is correlated to the blade rotation and guarantees a restricted power input at wind speed above rated wind speed.

At the blade starting position the spring is loaded with the pretension $F_{S0}$, which acts against the centrifugal force $F_C$. With the help of the pretension $F_{S0}$ the rated rotation speed $\omega_r$ can be defined: If the centrifugal force $F_C$ exceeds the pretension $F_{S0}$ the blade movement starts.
Furthermore a discrete damper - inline with the blade axis - is integrated in the system to influence the blade movement (cf. Figure 1).

3. Mechanical, mathematical and control engineering modelling

3.1. The mechanical and mathematical model

To analyse the influence of the discrete damper on the dynamic pitch behaviour the real system is reduced to a rigid body model of the drive train and rotor and doesn’t take blade, drive train and tower deformations as well as grid feedback influences into account.

So the mechanical model is characterised just by three degrees of freedom (see Figure 1): Rotor rotation $\phi$ along the generator axis, blade pitching rotation $\theta$ along blade axis and blade movement $z$ along the blade axis.

The coordinate system is located in the blade root and defined acc. Germanischer Lloyd Guideline [3], rotating along the generator axis.

The rotor rotation is characterised by the accelerating rotor torque $T_R$ and the decelerating generator (air gap) torque $T_G$. Taking the rotor and drive train mass moment of inertia $J$ into account (the inertia for rotations along the blade axis is considered to be negligible) the rotation along the generator axis can be derived from the following torque balance (1):

$$ J \cdot \dot{\phi} = T_R - T_G $$

(b) Legend

Figure 1. Mechanical model of the SWT pitch system
With the definitions for the accelerating rotor torque $T_R$ and the decelerating generator torque $T_{G,r}$ given in [1] the following equation (2) can be derived from the torque balance (1):

$$\ddot{\varphi} = \frac{1}{J} \cdot \left( \frac{1}{2} \cdot \rho \cdot v^2 \cdot R^3 \cdot c_Q \left( \lambda (\omega, v, \theta) - \frac{P_{G,r}}{\omega(v)} \right) \right) = \ddot{\varphi} (\omega, v, \theta)$$

(2)

The blade movement along the blade axis is resulting from the centrifugal force $F_C$, spring force $F_S$ and damper force $F_D$. Taking the blade inertia $m_B$ into account the blade movement can be derived from the following force balance (3):

$$m_B \cdot \ddot{z} = F_C - F_S - F_D$$

(3)

According to the definitions for the centrifugal $F_C$, spring $F_S$, damper force $F_D$ and the ratio $i_{z\theta}$ between blade movement $z$ and blade rotation $\theta$ given in [1] the following force balance (4) can be derived from (3):

$$\ddot{\theta} = \frac{1}{m_B \cdot i_{z\theta}} \cdot \left( m_B \cdot r_{CG} \cdot \omega^2 - F_{S,0} - k_S \cdot i_{z\theta} \cdot \theta - d \cdot i_{z\theta} \cdot \dot{\theta} \right) = \ddot{\theta} (\omega)$$

(4)

So the coupling between drive train rotation and blade pitching rotation is represented in the differential equations (2) and (4) by the rotation speed $\omega$ (with $\omega = \dot{\varphi}$), being a parameter of both differential equations of motion.

3.2. Control engineering model and classification

The comparison of a general command response block diagram with the specific block diagram of the analysed pitch system reveals, that the blocks of (4) represent the controller $G_C$ and the blocks of (2) represent the controlled system transfer function $G_S$. Therefore the spring coefficient $k_S$ and damper coefficient $d$ can be considered as the controller variables of that kind of mechanical pitch system.

4. Local stability analysis

Developing a new mechanical SWT pitch system the pitch angle stability for varying wind speeds $\Delta v$ has to be guaranteed by the right specification of the discrete damper coefficient $d$. Therefore the actuation input, described by the transfer function $G_{uz}$ between changing pitch angle $\Delta \theta$ as actuation signal $u$ and varying wind speed $\Delta v$ as disturbance $z$ has to be analysed. From this analysis the range for the controller variables - spring coefficient $k_S$ and damping coefficient $d$ (cf. chapter 3.2) - can be derived. For a mechanical pitch systems the spring coefficient depends on the desired pretension $F_{S,0}$, as described in chapter 2, so just the damping coefficient $d$ needs to be derived from the local stability analysis. The stability analysis can be performed analytically and validated e.g. with appropriate SWT pitch system test benches, like operated at Reiner Lemoine Institut.

4.1. Linearisation and transformation

To solve the non linear differential equations (2) and (4) by Laplace transformation, both equations have to be linearised with a Taylor polynomial for equilibrium points resulting in

$$\dot{\omega} = \frac{1}{J} \cdot \left( k_v \cdot v + k_\omega \cdot \omega + k_\theta \cdot \theta - k_G \cdot \omega \right)$$

(5)

and

$$\dot{\theta} = \frac{1}{m_B \cdot i_{z\theta}} \cdot \left( k_C \cdot \omega - k_S \cdot i_{z\theta} \cdot \theta - d \cdot i_{z\theta} \cdot \dot{\theta} \right)$$

(6)
with the linearisation coefficients

\[ k_v = \frac{\partial T_R}{\partial v} |_c \quad k_\omega = \frac{\partial T_R}{\partial \omega} |_c \quad k_\theta = \frac{\partial T_R}{\partial \theta} |_c \quad k_G = \frac{\partial M_G}{\partial \omega} |_c \quad k_C = \frac{\partial F_C}{\partial \omega} |_c. \]  

(7)

Now solving the linear differential equations (5) and (6) by Laplace transformation results in

\[ \omega \cdot s = \frac{1}{J} (k_v \cdot v + k_\omega \cdot \omega + k_\theta \cdot \theta - k_G \cdot \omega) \]

\[ \iff \omega = \frac{1/J}{s - k_\omega / J + k_G / J} \cdot (k_v \cdot v + k_\theta \cdot \theta) \]  

(8)

and

\[ \theta \cdot s^2 = \frac{1}{m_B \cdot i_{z\theta}} \cdot (k_C \cdot \omega - k_S \cdot i_{z\theta} \cdot \theta - d \cdot i_{z\theta} \cdot \theta \cdot s) \]

\[ \iff \theta = \frac{1}{s^2 + d/m_B \cdot s + k_S/m_B} \cdot k_C \cdot \omega. \]  

(9)

4.2. Equation of motion & derivation of the actuation transfer function \( G_{uz} \)

Inserting (8) in (9) the dynamic of the whole system can be described with just one equation in the complex s-domain:

\[ \theta = \frac{1}{s^2 + d/m_B \cdot s + k_S/m_B} \cdot \left( \frac{1}{(m_B \cdot i_{z\theta})} \cdot \frac{1/J}{s - k_\omega / J + k_G / J} \cdot k_v \cdot v \right) \]  

(10)

The linearised model (10) for equilibrium points is verified by a comparison with the non linear model, described by (2) and (4).

From (10) the desired actuation transfer function \( G_{uz} \) can be transposed easily:

\[ G_{uz} \equiv \frac{u}{z} = \frac{\theta}{v} = \frac{1/J \cdot k_C \cdot k_v}{(m_B \cdot i_{z\theta}) \cdot (s^2 + d/m_B \cdot s + k_S/m_B) \cdot (s - k_\omega / J + k_G / J) - k_\theta \cdot k_C \cdot 1/J} \]

\[ \cdot \frac{1}{(m_B \cdot i_{z\theta}) / J \cdot k_C \cdot k_v} \]  

(11)

4.3. Stability analysis by Hurwitz criteria

The local stability at equilibrium points can be analysed with the Hurwitz criteria for the characteristic denominator polynomial of \( G_{uz} \). According to Hurwitz all coefficients \( a_i \) and determinants

\[ D_i = \det \begin{pmatrix} a_1 & a_3 & a_5 & \cdots \\ a_2 & a_4 & \cdots & \cdots \\ a_0 & 0 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]  

(12)
of the so called Hurwitz matrix $H$ need to be positive. In case of the presented mechanical pitch system

$$G_{a_2} = \theta = \frac{b_0 \cdot s^0}{a_3 \cdot s^3 + a_2 \cdot s^2 + a_1 \cdot s + a_0}$$

(13)

this leads to:

$$a_3 = 1 > 0$$

$$a_2 = \frac{d}{m_B} \frac{k_\omega}{J} + \frac{k_G}{J} > 0$$

$$a_1 = -\frac{d}{m_B} \frac{k_\omega}{J} + \frac{d}{m_B} \frac{k_G}{J} + \frac{k_S}{m_B} > 0$$

$$a_0 = 1/\left(\frac{m_B}{J}\right) \left( -k_S \cdot k_\omega + k_S \cdot k_G - k_\theta \cdot k_C \cdot m_B^2 \cdot i_{z\theta} \right) \geq 0$$

$$D_2 = a_1 \cdot a_2 - a_0 \cdot a_3 \geq 0$$

(14)

4.4. Damper coefficient range derived from the stability analysis

From the Hurwitz criteria (14) the necessary minimum damper coefficient $d_{\text{min}}$ - resulting in pitch angle stability for each equilibrium point - can be derived. Therefor the coefficients $a_i$ and determinate $D_2$ of ten discrete wind speeds $v$ within the SWT pitch system operation range (from $v_{\text{PitchStart}} = 9.5 \text{m/s}$ to $v_{\text{extreme}} = 59.5 \text{m/s}$) are calculated for a varying damping coefficient $0 \leq d \leq 10000$. For each damping coefficient $d$ all coefficients $a_i$ are positive, but with an increasing damping coefficient $d$ the determinate $D_2$ rises linearly from negative minimum to positive maximum. The necessary minimum damping coefficient $d_{\text{min}}(v)$ of a discrete wind speed $v$ is determined by the smallest positive value of the determinate $D_2$ and illustrated in Figure 2.

![Figure 2. Minimum damping coefficients $d_{\text{min}}(v)$ of different wind speeds $v$](RLI_PivotPitch_SimulationResults-2014_05_31.xlsx)

To guarantee the pitch system stability of the presented mechanical pitch system for all wind speeds the damping coefficient of its discrete damper needs to be equal of greater than 1410 kg/s.
5. **Approach for the global stability analysis**

5.1. **Reformulation the motion equation into Takagi Sugeno form**

Introducing the state vector

$$\mathbf{x} = [x_1 \ x_2 \ x_3]^T = [\omega \ \theta \ \dot{\theta}]^T$$

and the input signal $u = v$, the equations (2) and (4) can be written in nonlinear state-space form as

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{P_G \omega}{J} & \frac{1}{x_1^2} & 0 & 0 \\ \frac{m_B \omega}{J} & 0 & 0 & 1 \\ \frac{F_{S,0}}{l_{z \theta}} x_1 & -\frac{k_s}{m_B} - \frac{d}{m_B} & \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} \end{bmatrix} u$$

\hspace{1cm} (16)

with $\lambda(x_1, u) = \frac{R \omega}{x_1}$.

In the following, the nonlinear state-space model (16) is transformed into an equivalent Takagi-Sugeno structure. This structure, originally introduced in the context of fuzzy systems, is a weighted combination of linear models

$$\dot{\mathbf{x}} = \sum_{i=1}^{N_r} h_i(\mathbf{z}) \left( \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \mathbf{u} \right), \quad \sum_{i=1}^{N_r} h_i(\mathbf{z}) = 1$$

(17)

where $\mathbf{A}_i$ and $\mathbf{B}_i$ are constant matrices and $h_i$ are nonlinear functions of the premise variables $\mathbf{z}$, which can depend on the system states, inputs and external variables. $N_r$ denotes the number of linear submodels. The stability of TS models can be shown by solving linear matrix inequalities (LMIs). The linear submodels will now be derived by applying the sector nonlinearity approach [4],[5], which allows us to obtain an exact representation of the nonlinear model.

The matrix entries $a_{31} = f_1(x_1)$ and $b_1 = f_2(x_1, x_2, u)$ in (16) are bounded functions, if the states and the input are bounded and $x_1 > 0$. This is fulfilled in the operating range of interest. Thus, these functions can be written as

$$f_1(x_1) = w_{11}(x_1) \bar{f}_1 + w_{12}(x_1) \underline{f}_1$$

(18)

$$f_2(x_1, x_2, u) = w_{21}(x_1, x_2, u) \bar{f}_2 + w_{22}(x_1, x_2, u) \underline{f}_2$$

(19)

where

$$w_{11}(x_1) := \frac{f_1(x_1) - \underline{f}_1}{\bar{f}_1 - \underline{f}_1}, \quad w_{12}(x_1) := \frac{\bar{f}_1 - f_1(x_1)}{\bar{f}_1 - \underline{f}_1}$$

$$w_{21}(x_1, x_2, u) := \frac{f_2(x_1, x_2, u) - \underline{f}_2}{\bar{f}_2 - \underline{f}_2}, \quad w_{22}(x_1, x_2, u) := \frac{\bar{f}_2 - f_2(x_1, x_2, u)}{\bar{f}_2 - \underline{f}_2}$$

(20)

$\bar{f}_i$ and $\underline{f}_i$ denote the maximum and minimum values of the function $f_{1,2}$, i.e. the sector boundaries. From (20) it follows that the weighting functions fulfill the convexity condition.
\( w_{j1} + w_{j2} = 1 \). Using the convexity condition and defining the membership functions by all possible permutations of

\[ \{w_{11}, w_{12}\} \times \{w_{21}, w_{22}\} \]

with

\[ h_1 := w_{11} w_{21}, \quad h_2 := w_{11} w_{22}, \quad h_3 := w_{12} w_{21}, \quad h_4 := w_{12} w_{22} \quad (21) \]

the functions \( f_1 \) and \( f_2 \) can also be written as weighted combination of sector boundaries \([2]\]

\[ f_1 = (w_{11} f_1 + w_{12} f_1) (w_{21} + w_{22}) = h_1 f_1 + h_2 f_1 + h_3 f_1 + h_4 f_1, \quad (22) \]

\[ f_2 = (w_{21} f_2 + w_{22} f_2) (w_{11} + w_{12}) = h_1 f_2 + h_2 f_2 + h_3 f_2 + h_4 f_2. \quad (23) \]

Using the expressions (22) and (23) instead of \( a_{31} = f_1(x_1) \) and \( b_1 = f_2(x_1, x_2, u) \) in \( A(x) \) and \( B(x, u) \) (16) and multiplying all other constant matrix entries by

\[ \frac{N_r}{\sum_{i=1}^{N_r} h_i(z)} = 1, \quad \text{where} \quad z = [x_1, x_2, u]^T \quad (24) \]

the nonlinearities has thus been extracted and concentrated into the membership functions \( h_i \). Thus the state-space model (16) can now be written in Takagi-Sugeno form (17) as

\[ \dot{x} = \sum_{i=1}^{N_r} h_i(z) \ (A_i x + B_i u), \quad N_r = 4. \quad (25) \]

5.2. LMI criteria for stability of the pitch system

The global asymptotic stability (GAS) of the pitch system at equilibrium points is investigated using the direct Lyapunov method. If a continuously differentiable function \( V(x) \) satisfying

\[ V(x) > 0, \quad \dot{V}(x) < 0 \quad (26) \]

then it is a Lyapunov function and the mechanical pitch system or more precisely, the model of the pitch system, is GAS at equilibrium points. A common Lyapunov function candidate is the quadratic one

\[ V(x) = x^T P x \quad (27) \]

with \( P = P^T \). The derivative of \( V \) along the trajectories of the state-space model
(16) respectively (17) is given by
\[
\dot{V}(x) = x^T P x + x^T P x
\] (28)

Using (17) the condition is equivalent to
\[
\dot{V}(x) = \sum_{i=1}^{N_r} h_i(z) (A_i x + B_i u)^T P x + x^T P \sum_{i=1}^{N_r} h_i(z) (A_i x + B_i u)
\]
\[
= x^T \left( \sum_{i=1}^{N_r} h_i(z) A_i^T P + P \sum_{i=1}^{N_r} h_i(z) A_i \right) x + 2 x^T P \sum_{i=1}^{N_r} h_i(z) B_i u
\]
\[
= x^T \sum_{i=1}^{N_r} h_i(z) \left( A_i^T P + P A_i \right) x + 2 x^T P \sum_{i=1}^{N_r} h_i(z) B_i u
\] (29)

The GAS is guaranteed, if the stability of the autonomous system (with zero-input response) and the input-to-state stability is proven.

So first the stability of the autonomous system is investigated, which is guaranteed, if a Lyapunov function candidate according (27) satisfies the condition

\[
\dot{V}(x) = x^T \sum_{i=1}^{N_r} h_i(z) (A_i^T P + P A_i) x < 0
\] (30)

To satisfy equation (30) the condition

\[
(A_i^T P + P A_i) < 0
\] (31)

needs to be fulfilled.

Combining the linear inequality (31) with

\[
P < 0,
\] (32)

resulting from (26) for a quadratic Lyapunov function candidate according (27), leads to a linear matrix inequality (LMI). If a solution $P$ of the LMI exists, the autonomous system with its state-space vector $x$ is GAS.

Secondly the state-to-space stability is investigated: To estimate the maximum value of the autonomous system (30),

\[-\lambda \|x\|^2 < 0 \quad \text{with} \quad \lambda > 0
\] (33)

is introduced\(^1\) with

\(^1\) $\lambda$ is an arbitrary, positive value not be mistaken for the tip speed ratio in equation (2)
Now the description of the zero-input response of the autonomous system in (29) can be replaced with its estimated maximum value according to (33).

Furthermore the maximum value of the input-to-space description in (29) (with $0 \leq h_i(z) \leq 1$)

$$2 x^T P \sum_{i=1}^{N_r} h_i(z) B_i u = 2 \sum_{i=1}^{N_r} h_i(z) x^T P B_i u$$

is estimated introducing

$$2 \max_i \|P B_i\| \|x\| \|u\|.$$  (36)

Equation (36) fulfills the condition

$$2 \sum_{i=1}^{N_r} h_i(z) x^T P B_i u < 2 \max_i \|P B_i\| \|x\| \|u\|.$$  (37)

With the abbreviation

$$\delta := \max_i \|P B_i\|$$  (38)

the input-to-state description (35) is replaced with its estimated maximum value

$$2 \sum_{i=1}^{N_r} h_i(z) x^T P B_i u < 2 \delta \|x\| \|u\|.$$  (39)

Combing both estimated maximum values (33) and (39) to the derivative of the Lyapunov function leads to

$$\dot{V}(x) = -\lambda \|x\|^2 + 2 \delta \|x\| \|u\|.$$  (40)

If equation (40) is expanded

$$\dot{V}(x) = -\lambda \|x\|^2 + 2 \delta \|x\| \|u\|$$

$$= -\frac{\lambda}{2} \|x\|^2 - \left( \sqrt{\frac{\lambda}{2}} \|x\|^2 + 2 \sqrt{\frac{\lambda}{2}} \|x\| \delta \sqrt{\frac{2}{\lambda}} \|u\| - \delta^2 \left( \sqrt{\frac{2}{\lambda}} \|u\|^2 + \frac{\delta^2}{\lambda} \|u\|^2 \right) \right)$$

$$= -\frac{\lambda}{2} \|x\|^2 - \left( \sqrt{\frac{\lambda}{2}} \|x\| - \delta \sqrt{\frac{2}{\lambda}} \|u\| \right)^2 + \delta^2 \frac{2}{\lambda} \|u\|^2$$  (41)
and its maximum value estimated (by neglecting the second term in (41))

\[
\dot{V}(x) < -\frac{\lambda}{2} \|x\|^2 + \delta^2 \frac{2}{\lambda} \|u\|^2 ,
\]  

(42)

then input \(u\) and state vector \(x\) are separated in different terms, resulting in an input-to-state criteria

\[
\|u\| > \frac{\lambda}{2\delta} \|x\| .
\]  

(43)

With respect to the preceding considerations the following theorem is obtained

**Theorem** Consider a model of the mechanical pitch system in TS form (25). Suppose there exists a common matrix \(P = P^T > 0\), such that the following linear matrix inequation is satisfied

\[
\forall i \in \{1, ..., N_r\}, \quad A_i^T P + P A_i < 0 \quad (44)
\]

and further suppose that the wind speed as system input \(u\) is bounded, fulfilling the input-to-state-stability criteria \(|u|_{\text{max}} > \frac{\lambda}{2\delta} \|x\|_{\text{max}}\), then the mechanical pitch system (25) is globally asymptotically stable.

To evaluate the input-to-state criteria (43), the maximum value of equation (31) is estimated with the negative product of the Lyapunov function candidate \(V\) and a decay rate \(\epsilon\). Once the decay rate \(\epsilon\) is determined, the value of \(\lambda\) in equation (43) can be calculated from the maximum eigenvalue \(\lambda_{\text{max}}(P)\) like described in [6] and the GAS is verified according to the given theorem.

### 6. Conclusion and future works

In this paper the necessary minimum coefficient of a discrete damper used for an exemplary mechanical pitch system of a small wind turbine is gained for equilibrium points. Therefor the pitch angle stability is analysed with the help of the Hurwitz criteria of a linearised mathematical model.

For the global asymptotic stability analysis, using the direct Lyapunov method, the non linear state-space model is transformed into Takagi-Sugeno form and the stability criteria is derived. Within further investigations on exemplarily SWT configurations the global stability will be verified by this approach. So it will be possible to specify the damper coefficients within the development process of new SWT mechanical pitch systems.

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