LIPSHITZ MAPS FROM SURFACES

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Abstract. We give a simple procedure to estimate the smallest Lipschitz constant of a degree 1 map from a Riemannian 2-sphere to the unit 2-sphere, up to a factor of 10. Using this procedure, we are able to prove several inequalities involving this Lipschitz constant. For instance, if the smallest Lipschitz constant is at least 1, then the Riemannian 2-sphere has Uryson 1-Width less than 12 and contains a closed geodesic of length less than 160. Similarly, if a closed oriented Riemannian surface does not admit a degree 1 map to the unit 2-sphere with Lipschitz constant 1, then it contains a closed homologically non-trivial curve of length less than 4\pi. On the other hand, we give examples of high genus surfaces with arbitrarily large Uryson 1-Width which do not admit a map of non-zero degree to the unit sphere with Lipschitz constant 1.

This paper is about estimating the best Lipschitz constant of a degree 1 map from a closed oriented Riemannian surface to the unit 2-sphere. The difficulty of this problem depends on the genus of the surface. In case the surface is a topological 2-sphere, we give a simple procedure for estimating the best Lipschitz constant within a factor of twelve.

Pick a point p on the Riemannian 2-sphere \((S^2, g)\). Consider the distance spheres around p. Let D be the largest diameter of any connected component of any of the distance spheres around p.

Theorem 0.1. The best Lipschitz constant of a degree 1 map from \((S^2, g)\) to the unit 2-sphere is more than \(1/D\) and less than \(12/D\).

We now recall some vocabulary, which we will use all through the paper, designed to describe how large and how wide Riemannian manifolds are. The hypersphericity of a Riemannian n-manifold M is the supremal R so that there is a contracting map of non-zero degree from M to the n-sphere of radius R. In this paper, we will mostly focus on degree 1 maps, so we define the degree 1 hypersphericity of a Riemannian n-manifold M as the supremal R so that there is a contracting map of degree 1 from M to the n-sphere of radius R. The Uryson k-Width of a metric space X is the infimal W so that there is a continuous map from X to a k-dimensional polyhedron whose fibers have diameter less than W. The main theorem about hypersphericity and Uryson Width is an estimate by Gromov in [6] that the hypersphericity of an n-manifold is less than its Uryson (n-1)-width.

Rephrased in this vocabulary, our first theorem states that the degree 1 hypersphericity of \((S^2, g)\) is between D/12 and D. Using Gromov’s result, it follows that the Uryson 1-Width of \((S^2, g)\) is also between D/12 and D. In particular, we see that the hypersphericity and the Uryson 1-Width of a 2-sphere agree up to a factor of twelve. Since it is not hard to write an efficient algorithm to estimate D to any desired accuracy, we can efficiently estimate the hypersphericity and Uryson 1-width of a Riemannian 2-sphere up to a factor of twelve.
We prove Theorem 0.1 in section 1. In the next three sections, we apply the first theorem to give estimates relating the degree 1 hypersphericity of a surface to other geometric quantities. While the first theorem is quite easy, some of these applications are more difficult. In the last section of the paper, we show that the first theorem fails for surfaces of high genus.

In section 2, we bound the length of the shortest closed geodesic on a 2-sphere by its degree 1 hypersphericity.

**Theorem 0.2.** If $(S^2, g)$ has degree 1 hypersphericity less than 1, then it contains a closed geodesic of length less than 160.

This theorem extends a result of Croke, who proved in [4] that if the area of a sphere is less than $4\pi$, then it contains a closed geodesic of length less than 100. The main step in the proof is to produce a sufficiently nice family of short curves on $(S^2, g)$. In particular, we prove along the way the following theorem.

**Theorem 0.3.** If $(S^2, g)$ has degree 1 hypersphericity less than 1, then it admits a map to a tree whose fibers have length less than 120.

By our first theorem, we already know that $(S^2, g)$ admits a map to a tree whose fibers have diameter less than 12. It may happen, however, that the fibers are extremely long curves with many wiggles. Roughly speaking, the above theorem means that it is possible to straighten out all of these wiggles (possibly after replacing the target tree by a tree with much more branching).

In section 3, we apply our estimates to surfaces in Euclidean space. Our first estimate says that if we slice the unit ball into non-singular surfaces, then one of these surfaces has a large hypersphericity.

**Theorem 0.4.** Let $f$ be an embedding of the unit $n$-ball into $\mathbb{R}^n$, and let $f_0$ be the first $n-2$ coordinates of $f$. The level sets of $f_0$ are planar surfaces in the unit $n$-ball. One of the level sets of $f_0$ has a connected component with degree 1 hypersphericity at least $1/24$.

Our second estimate is a variation of the isoperimetric inequality. For a 2-sphere embedded in $\mathbb{R}^3$, our inequality controls the volume inside the 2-sphere in terms of the hypersphericities of pieces of the boundary.

**Theorem 0.5.** Let $U$ be a bounded open set in $\mathbb{R}^3$, with boundary diffeomorphic to a 2-sphere. Then the boundary of $U$ contains disjoint subsets $S_i$, so that the following inequality holds.

$$\text{Vol}(U) < 10^6 \sum \text{HS}(S_i)^3.$$  

In this equation, $\text{HS}(S_i)$ stands for the degree 1 hypersphericity of $S_i$.

Our third estimate gives a lower bound for the hypersphericity of the boundary of $U$.

**Theorem 0.6.** Let $U$ be a bounded open set in $\mathbb{R}^3$ with boundary diffeomorphic to a 2-sphere, and suppose that $U$ admits an area-contracting diffeomorphism to the unit ball. Then the degree 1 hypersphericity of the boundary of $U$ is at least $1/200$.

The constants in the previous two theorems are probably rather poor. In particular, it seems possible that if $U$ admits an area-contracting diffeomorphism to the unit ball, then the hypersphericity of the boundary of $U$ is at least 1.
Section 4 proves some estimates for surfaces of higher genus. We first prove two theorems bounding the lengths of homologically interesting curves in terms of the hypersphericity.

**Theorem 0.7.** Let $(\Sigma, g)$ be a closed, oriented surface of genus at least 1 and degree 1 hypersphericity less than 1. Then it contains a homologically non-trivial closed curve of length less than $4\pi$.

**Theorem 0.8.** Let $(\Sigma, g)$ be a closed oriented surface of genus $G$ and degree 1 hypersphericity less than 1. Then it contains homologically independent curves $C_1, ..., C_G$ with the length of $C_k$ bounded by $200k$.

By a similar method, we bound the Uryson 1-Width of a surface in terms of its degree 1 hypersphericity and its genus.

**Theorem 0.9.** Let $(\Sigma, g)$ be a closed oriented surface of genus $G$ and degree 1 hypersphericity less than 1. Then the Uryson 1-Width of $(\Sigma, g)$ is less than $200G + 12$.

The power of $G$ in this estimate may not be sharp, but in section 5 we will construct examples of surfaces with degree 1 hypersphericity 1 and Uryson 1-Width on the order of $G^{1/2}$, and surfaces with hypersphericity 1 and Uryson 1-Width on the order of $G^{3/4}$.

In spite of the results in section 4, I don’t really understand what a high genus surface with small hypersphericity is like. For example, I am very far from being able to write an algorithm that estimates the hypersphericity of a surface within a constant factor. In section 5, we will give examples of metrics on surfaces of very high genus with arbitrarily small hypersphericity and Uryson 1-Width at least 1.

**Theorem 0.10.** There exist closed oriented Riemannian surfaces with arbitrarily small hypersphericity and Uryson 1-Width at least 1.

We will give three different examples of surfaces proving this theorem. The three different examples have small hypersphericity for three different “reasons”, and I include them to suggest that the problem of estimating the best Lipschitz constant of a degree 1 map from a surface to the unit sphere is a complicated problem, which can involve several different pieces of geometry and topology. Our three examples will use a little homotopy theory, a little lattice theory, and a little bit of the theory of systoles.

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1. Hypersphericity of 2-Spheres

In this section, we estimate the best Lipshitz constant of a degree 1 map from a Riemannian 2-sphere \((S^2, g)\) to the unit 2-sphere.

Pick any point \(p\) in \((S^2, g)\). Let \(S(p, R)\) be the metric sphere around \(p\) of radius \(R\) - that is, the set of points \(x\) in \((S^2, g)\) whose distance from \(p\) is \(R\. Let \(D\) be the the largest diameter of any connected component of the distance sphere \(S(p, R)\) for any radius \(R\. We will prove that the smallest Lipshitz constant of any degree 1 map from \((S^2, g)\) to the unit sphere is approximately \(\frac{5}{D}\).

**Theorem 1.1.** The smallest Lipshitz constant of a degree 1 map from \((S^2, g)\) to the unit 2-sphere is at least \(\frac{\pi}{2D}\) and at most \((2 + \sqrt{2})\frac{\pi}{D}\).

**Proof.** To prove the upper bound, we will construct a Lipshitz degree 1 map. For some radius \(R\), one component of \(S(p, R)\) has diameter \(D\). Choose two points \(q\) and \(r\) in that component, so that the distance from \(q\) to \(r\) is \(D\. We define \(F_0(x) = (\text{dist}(p, x), \text{dist}(q, x))\). The function \(F_0\) maps \((S^2, g)\) to \(\mathbb{R}^2\), with Lipshitz constant \(\sqrt{2}\).

We will consider the restriction of \(F_0\) to three curves in \(S^2\): a minimal geodesic \(g_{pq}\) from \(p\) to \(q\), a minimal geodesic \(g_{pr}\) from \(p\) to \(r\), and a curve \(\gamma_{qr}\) joining \(q\) to \(r\). For now, we will assume that the component of \(S(p, R)\) containing \(q\) and \(r\) is path connected, and we take \(\gamma_{qr}\) to lie in this component.

The image under \(F_0\) of the geodesic \(g_{pq}\) is the straight line from \((0, R)\) to \((R, 0)\). Because \(\gamma_{qr}\) lies in \(S(p, R)\), the image of \(\gamma_{qr}\) under \(F_0\) is contained in the line \(x = R\). The image of the geodesic \(g_{pr}\) cannot be determined exactly, but we will prove that it lies above a certain line. Let \(g_{pr}(t)\) parametrize \(g_{pr}\) by arclength with \(g_{pr}(0) = p\) and \(g_{pr}(R) = r\). By definition, \(\text{dist}(p, g_{pr}(t)) = t\). Then \(F_0 \circ g_{pr}(t) = (t, \text{dist}(q, g_{pr}(t)))\), and the image \(F_0(g_{pr})\) is the graph of the function \(f(t) = \text{dist}(q, g_{pr}(t))\). The function \(f\) has Lipshitz norm less than or equal to 1. Since the distance from \(q\) to \(r\) is \(D\), \(f(R) = D\). Therefore, the graph of \(f\) lies above the line from \((R - D/2, D/2)\) to \((R, D)\).

Let \(T\) be the open triangle in \(\mathbb{R}^2\) with vertices \((R, 0)\), \((R, D)\), and \((R - D/2, D/2)\). We have proved that \(F_0(g_{pr})\) does not intersect \(T\), and it is easy to check that \(F_0(g_{pq})\) and \(F_0(\gamma_{qr})\) don’t intersect \(T\) either. Let \(c\) be the embedded circle in \((S^2, g)\) formed by joining \(g_{pq}\), \(\gamma_{qr}\), and \(g_{pr}\). After choosing an appropriate orientation of \(c\), \(F_0: c \to \mathbb{R}^2 - T\) has degree one.
The triangle $T$ contains an embedded disk $U$ of radius $\frac{1}{\sqrt{2+2}}D$. Using the exponential map, we construct a homeomorphism $\pi_1$ from the disk $U$ to the Northern hemisphere of the unit sphere. We define a map $\pi_2$ from the disk $U$ to the Southern hemisphere of the unit sphere by composing $\pi_1$ with a reflection through the plane of the equator. We extend $\pi_1$ and $\pi_2$ to all of $\mathbb{R}^2$ as follows. For any point $x$ outside of $U$, let $p(x)$ be the nearest point to $x$ on the boundary of $U$. Now, define $\pi_1(x) = \pi_1(p(x))$ and $\pi_2(x) = \pi_2(p(x))$. The maps $\pi_1$ and $\pi_2$ agree on the boundary of $U$ and hence on all the points outside of $U$. The maps $\pi_1$ and $\pi_2$ have Lipschitz constant $(\sqrt{2} + 1)\pi/D$.

Finally we define our degree 1 map. The curve $c$ divides $(S^2, g)$ into two disks, $D_1$ and $D_2$, which we name so that the orientation on $c$ is compatible with that of $D_1$ and incompatible with that of $D_2$. We define $F$ on each disk as follows:

\[
F(x) = \begin{cases} 
\pi_1 \circ F_0 & \text{if } x \text{ is in } D_1, \\
\pi_2 \circ F_0 & \text{if } x \text{ is in } D_2.
\end{cases}
\]

$F$ is continuous because $F_0(c)$ lies outside of $U$, and $\pi_1 = \pi_2$ outside $U$. $F$ is degree 1 because $F_0 : c \to \mathbb{R}^2 - T$ is degree 1. $F$ has Lipschitz constant $(2 + \sqrt{2})\pi/D$.

If the connected component of $S(p, R)$ containing $q$ and $r$ is not path connected, it is still true that its $\delta$-neighborhood is path connected for every $\delta > 0$, and we can take $\gamma_{qr}$ to be a path lying as close as we like to $S(p, R)$. This curve $\gamma_{qr}$ suffices for our construction, producing a map with Lipschitz constant $\epsilon + (2 + \sqrt{2})\pi/D$ for an arbitrary positive $\epsilon$. Since the space of maps with a fixed Lipschitz constant between compact spaces is compact, there is a limit of these maps which is Lipschitz with constant $(2 + \sqrt{2})\pi/D$.

To prove the lower bound, we recall a theorem of Gromov about the Uryson width. In section F of [6], Gromov proves that if a Riemannian manifold $(M, g)$ has Uryson $1$-width less than $\frac{\pi}{2}$, then any contracting map from $(M, g)$ to the unit 2-sphere is null-homotopic. (In section 5, we discuss this result of Gromov in detail.) The quotient map from $(S^2, g)$ to the space of connected components of the distance spheres around $p$ has fibers of diameter no more than $D$. If this space of connected components were a 1-polyhedron, then the Uryson 1-Width of $(S^2, g)$ would be no more than $D$. Applying Gromov’s theorem, we would see that the Lipschitz constant of a degree 1 map from $(S^2, g)$ to the unit 2-sphere is at least $\frac{\pi}{(2D + \epsilon)}$.

We deal with the general case by perturbing $\text{dist}(\cdot, p)$ to a Morse function. The space of connected components of the fibers of a Morse function is an honest 1-polyhedron. It is an exercise in point-set topology to check that for each $\epsilon$ there is a $\delta$ so that any function $g$ obeying $|g(x) - \text{dist}(x, p)| < \delta$ has fibers whose connected components have diameter less than $D + \epsilon$. We take a Morse function $g$ obeying this inequality, which proves that the Uryson 1-Width of $(S^2, g) < D + \epsilon$ for every $\epsilon$. \hfill \Box

As a corollary, we are able to estimate the degree 1 hypersphericity, the hypersphericity, and the Uryson 1-Width of $(S^2, g)$ in terms of the number $D$.

$$\frac{D}{(\pi(2 + \sqrt{2}))} \leq \text{degree 1 HS}(S^2, g) \leq \text{HS}(S^2, g) \leq 2\text{UW}_1(S^2, g)/\pi \leq 2D/\pi.$$
In this formula, degree 1 HS stands for the degree 1 hypersphericity, HS stands for the hypersphericity, and UW\textsubscript{1} stands for the Uryson 1-Width. In particular, we see that the degree 1 hypersphericity, the hypersphericity, and the Uryson 1-Width of a 2-sphere all agree up to a bounded factor. By contrast, we will prove in section 5 that for surfaces of high genus, any two of these invariants can disagree by an arbitrarily large factor.

2. Families of Short Curves and Closed Geodesics

In this section we will bound the length of the shortest closed geodesic on \((S^2, g)\) by a multiple of the degree 1 hypersphericity. The existence of a short closed geodesic follows from fairly standard methods once we have a sufficiently nice family of short curves on \((S^2, g)\). We begin by proving the existence of such a family.

**Theorem 2.1.** Suppose \((S^2, g)\) has degree 1 hypersphericity less than 1. Then there is a map \(F\) from \((S^2, g)\) to a trivalent tree \(T\), so that each fiber of the map has length less than 120. The fibers of the map have controlled topology: the preimage of any point in an edge of \(T\) is a circle, the preimage of a terminal vertex of \(T\) is a point, and the preimage of a trivalent vertex of \(T\) is homeomorphic to the greek letter \(\theta\).

Here is an outline of the proof. The first step is to find a collection of short disjoint circles in \((S^2, g)\), cutting it into components of bounded diameter. We begin by taking the connected components of distance spheres around some point \(p\) in \((S^2, g)\). These disjoint circles have bounded diameter, but they may be very long. We replace each distance sphere by a homologous curve which is a union of short circles. These curves are short, but they may not be disjoint. By applying an appropriate minimization procedure to this set of circles, we make them disjoint. These short disjoint circles will be fibers of the map that we are going to construct. The second step of the proof is to fill in the map on each region between the short disjoint circles. Since these regions have small diameter, all the geodesic segments in them are short. We construct a map on each region whose fibers consist of two or three geodesic segments plus some curves of negligible length. As a result, all of the fibers have bounded length.

**Proof.** Pick a point \(p\) in \((S^2, g)\), and consider the distance spheres around \(p\). (We first smooth the distance function so that almost every distance sphere is a union of circles.) Let \(S_i\) be the distance sphere around \(p\) of radius \(12i\), and let \(T_i\) be the distance sphere around \(p\) of radius \(12i + 6\). Of course \(T_i\) separates \(S_i\) from \(S_{i+1}\). Equivalently, each curve from \(S_i\) to \(S_{i+1}\) has non-zero (topological) intersection number with one of the components of \(T_i\). By Theorem 1.1, each component \(T_{i,j}\) of \(T_i\) has diameter less than 12.

We now use a standard trick to replace each \(T_{i,j}\) with a union of shorter curves, which still separate \(S_i\) from \(S_{i+1}\). For each component \(T_{i,j}\), take a sequence of points \(q_{i,j,k}\) going around it, with consecutive points very close together. Let \(\gamma_{i,j,k}\) be a minimal geodesic from \(q_{i,j,k}\) to \(q_{i,j,0}\). Note that each \(\gamma_{i,j,k}\) is disjoint from \(S_i\) and from \(S_{i+1}\). Let \(T_{i,j,k}\) be the closed curve formed by composing \(\gamma_{i,j,k}\), \(\gamma_{i,j,k+1}\), and the portion of \(T_{i,j}\) from \(q_{i,j,k}\) to \(q_{i,j,k+1}\). Now each \(T_{i,j,k}\) has length less than 24, each \(T_{i,j,k}\) is disjoint from \(S_i\) and \(S_{i+1}\), and each curve joining \(S_i\) to \(S_{i+1}\) has non-zero topological intersection number with one of the \(T_{i,j,k}\). Unfortunately, the
$T_{i,j,k}$ may be far from disjoint. One of the main steps of this proof is to modify them to make them disjoint.

We will modify the $T_{i,j,k}$ using a minimization procedure. First, we define a minimal basis. Let $(U, g)$ be a compact surface with a Riemannian metric and convex boundary, and suppose $H_1(U, \mathbb{Q})$ has dimension $n$. We consider all sets of $n$ circles which form a rational basis for the first homology group of $U$. We define a partial ordering on these bases as follows. Let $S$ be a set of $n$ oriented curves in $U$, which form a rational basis for $H_1(U)$, and let the lengths of the curves be $s_1 \leq \ldots \leq s_n$. For two bases $S$ and $T$, we say that $T$ is smaller than $S$ if $t_i \leq s_i$ for every $i$ and $t_i < s_i$ for some $i$. We call $S$ a minimal basis if there is no smaller basis.

Because $U$ is compact, there is a minimal basis. Each curve in a minimal basis minimizes length in its homology class. Because the boundary of $U$ is convex, each curve in a minimal basis is a closed geodesic. Also, no curve is a multiple covering of a closed geodesic, because we could replace it with a single covering of that geodesic, making the basis smaller. These geodesics must be distinct, else they would not form a basis, and so they intersect transversally.

Following Gromov, we define a curve $\gamma$ to be straight if for any two points $x$ and $y$ on $\gamma$, the distance in $U$ from $x$ to $y$ is equal to the distance in $\gamma$ from $x$ to $y$. In section 5 of [7], Gromov showed that a minimal rational basis for the first homology group consists of straight circles. We will prove a slightly stronger statement. We define a curve $\gamma$ to be strictly straight if, for any two points $x$ and $y$ on $\gamma$ and any minimal segment $p$ from $x$ to $y$, $p$ lies in $\gamma$. A strictly straight circle is always an embedded closed geodesic.

**Lemma 2.2.** If $(U, g)$ is a compact surface with convex boundary and $S$ is a minimal rational basis for the first homology of $U$, then every curve in $S$ is strictly straight. Any two curves in $S$ intersect at most once. If $U$ is planar, the curves in $S$ are disjoint.

**Proof.** Suppose that one curve $\gamma$ in $S$ is not strictly straight. Then there are points $x$ and $y$ in $\gamma$ and a minimal segment $p$ from $x$ to $y$ which does not lie in $\gamma$. Since $p$ and $\gamma$ are both geodesics, $p$ must be transverse to $\gamma$ at $x$ and $y$. Now, we define $\gamma_1$ and $\gamma_2$ to be the closed curves formed by connecting $p$ to either path in $\gamma$ from $x$ to $y$. Since $p$ is minimal, each of $\gamma_1$ and $\gamma_2$ is no longer than $\gamma$. Moreover, $\gamma_1$ and $\gamma_2$ have corners at $x$ and $y$, so they can be homotoped to curves which are strictly shorter than $\gamma$. Now either $\gamma_1$ or $\gamma_2$ is linearly independent from the other curves in the basis $S$. Replacing $\gamma$ in $S$ by one of these curves gives a strictly smaller rational basis. Therefore, every curve in a minimal basis is strictly straight.

Suppose that two geodesics $\gamma_1$ and $\gamma_2$ in $S$ intersect in two points $x$ and $y$. Let $p$ be any minimal segment from $x$ to $y$. Since $\gamma_1$ and $\gamma_2$ are strictly straight, $p$ lies in both $\gamma_1$ and $\gamma_2$, and so $\gamma_1$ and $\gamma_2$ coincide. Since $\gamma_1$ and $\gamma_2$ are distinct, they do not intersect in two points.

Now suppose that $U$ is a planar domain. Suppose that two of the curves in $S$ intersect in only one point. Since they are distinct closed geodesics, they intersect transversally, and their intersection number must be 1 or -1. But since $U$ is a planar domain, the intersection number of any two closed curves in $U$ is zero. Therefore, the curves in $S$ are disjoint. □

With this lemma, we can modify the curves $T_{i,j,k}$ to make them disjoint. Let $U_i$ be the portion of $(S^2, g)$ between $S_i$ and $S_{i+1}$. By making an arbitrarily small
bilinear change of the metric, we can assume that each $U_i$ has convex boundary. To see this, for instance, we can use the normal exponential map to give a metric with a geodesic boundary and then slightly increase the metric along the boundary. It is not difficult. Then, in each $U_i$, we consider a minimal rational basis of the first homology. Notice that each $U_i$ is a planar domain. By Lemma 2.2, a minimal basis is realized by disjoint curves. We keep only the curves in the basis with length less than 24, and label these $\gamma_{i,j}$. Since each $T_{i,j,k}$ has length less than 24, the homology class of each $T_{i,j,k}$ lies in the rational span of the $\gamma_{i,j}$. Now any path from $S_i$ to $S_{i+1}$ has non-zero intersection number with some $T_{i,j,k}$, and so it must have non-zero intersection number with some $\gamma_{i,j}$. Therefore, the $\gamma_{i,j}$ separate $S_i$ from $S_{i+1}$.

Cut $(S^2, g)$ along the curves $\gamma_{i,j}$. We claim that each component of the complement has radius less than 36. Let $X$ be a component of the complement, not containing $p$, and $B$ the boundary component of $X$ closest to $p$. Any point in $X$ is within 24 from $B$, using the distance function of $(S^2, g)$. This estimate remains true for the intrinsic distance in $X$. To see this, pick any point $x$ in $X$, and consider a minimal geodesic $g$ from $x$ to $p$. We break $g$ into pieces $g_i = g \cap U_i$. Each curve $g_i$ is a single minimal geodesic in $U_i$. Then we replace each $g_i$ with the minimal geodesic $\tilde{g}_i$ in the slightly modified metric on $U_i$ between the endpoints of $g_i$. We define $\tilde{g}$ to be the union of all $\tilde{g}_i$. The curve $\tilde{g}$ still goes from $x$ to $p$, and it has practically the same length as $g$. Since each $\gamma_{i,j}$ is strictly straight in the modified metric on $U_i$, $\tilde{g}_i$ crosses $\gamma_{i,j}$ no more than once. Therefore, $\tilde{g}$ crosses each $\gamma_{i,j}$ no more than once. In particular, $\tilde{g}$ stays in $X$ until it hits $B$, and we see that the intrinsic distance from $x$ to $B$ is less than 24. Now pick any point $b$ in $B$. The distance from any point in $X$ to $b$ is less than 36. On the other hand, if $X$ contains $p$, and $x$ is any point in $X$, then the curve $\tilde{g}$ from $x$ to $p$ stays in $X$ and has length less than 24.

Let us summarize what we have done so far. We have found a set of disjoint embedded closed curves $\gamma_i$ on $(S^2, g)$, each of length less than 24, so that each component of their complement has radius less than 36. The curves $\gamma_i$ which we have just constructed will be fibers of the map we are going to build. It remains for us to define the map on each component of their complement. We call the components $U_k$. For each $U_k$, we will construct a map $F_k$ from $U_k$ to a tree $T_k$, with the property that each boundary component of $U_k$ is the fiber of a terminal vertex of $T_k$. Putting together all of these maps, we get a single map from $(S^2, g)$ to a tree $T$ whose fibers are just the fibers of the maps $F_k$. Therefore, our theorem reduces to the following lemma.

**Lemma 2.3.** Let $U$ be a planar domain, compact with boundary. Suppose each boundary component of $U$ has length less than 24, and that the radius of $U$ is less than 36. Then $U$ admits a map to a trivalent tree $T$, so that each fiber has length less than 120. Each boundary component of $U$ is the fiber of a terminal vertex of $T$. The fibers of the map have controlled topology: the preimage of a point on an edge is a circle, the preimage of a terminal vertex is either a point or a boundary component, and the preimage of a trivalent vertex is homeomorphic to the letter $\theta$.

**Proof.** We pick a point $p$ so that every point in $U$ lies in the ball around $p$ of radius 36. After a small bilinear change of the metric, we can assume that the boundary is convex. Therefore, every minimal path from an interior point to $p$ avoids the boundary and is a geodesic. The proof will focus on the set of all minimal geodesics.
in \( U \) with one endpoint at \( p \). Away from the point \( p \) and from the cut locus of the exponential map, this set of minimal geodesics forms a foliation, and we will view the set of minimal geodesics on the whole space as a foliation with some singular leaves. The plan of the proof is to modify this foliation until it consists of disjoint closed leaves (with mild singularities) which will be the fibers of the desired map.

For an arbitrary metric, the foliation by minimal geodesics may have very complicated singularities. Morally, we should be able to reduce the singularities to standard types by making a \( C^\infty \)-small perturbation of the metric. For technical reasons, it is more convenient to perturb the geodesics in a different way. Let \( S \) be the circle around \( p \) of radius \( \epsilon \), where \( \epsilon \) is much less than the injectivity radius of the exponential map at \( p \). The normal exponential map is a mapping from the normal bundle of \( S \) to \( U \), and the images of the fibers of the normal bundle are exactly the geodesics through \( p \). Instead of perturbing the metric of \( U \), we let \( \tilde{S} \) be a \( C^\infty \)-small perturbation of the circle \( S \) and use the normal exponential map of \( \tilde{S} \) in place of the normal exponential map of \( S \).

We now define our perturbed foliation precisely. We parametrize \( \tilde{S} \) by the coordinate \( \theta \). For \( r > 0 \), we define the normal exponential map \( F(\theta, r) \) to be the endpoint of the geodesic segment of length \( r \) leaving \( \tilde{S} \) at \( \theta \) in the outward normal direction. Our perturbed geodesics are given by \( F(\theta, r) \) for fixed \( \theta \) and varying \( r \). The minimal geodesics of our perturbed family extend only until the infimal value of \( r \) at which \( F(\theta, r) = F(\theta', r') \) with \( r' < r \). This defines a foliation with singular leaves on the exterior of \( \tilde{S} \).

For each value of \( r \), the function \( F(\theta, r) \) defines a Legendrian curve in \( U \), and so we can view \( F \) as a 1-parameter family of Legendrian curves. The singularities of the foliation by geodesics normal to \( \tilde{S} \) correspond exactly to the singularities of this family of Legendrian curves. Because \( \tilde{S} \) was generic, this family contains only standard types of singularities, as described in [1]. A generic Legendrian curve has only two kinds of singularities: self-intersections and cusps. A generic 1-parameter family of Legendrian curves has several more kinds of singularities, which occur at discrete values of the parameter. These singularities are self-tangencies, triple-intersections, cusp formations and cancellations, and cusp crossings.

The possible singularities of the foliation are reduced further by some metric considerations. The Legendrian curve \( F(\theta, r) \) includes the set of all points at distance \( r \) from \( \tilde{S} \), and it may also include some points which are closer to \( \tilde{S} \). We call the set of points at distance \( r \) from \( \tilde{S} \) the metric front of the Legendrian curve. Because we are taking the foliation by minimal geodesics, only the singularities on the metric front of the Legendrian curve correspond to singularities of the foliation. Cusps, cusp cancellations, and cusp crossings cannot occur on the metric front of the Legendrian curves \( F(\theta, r) \). Therefore, the foliation by minimal geodesics normal to \( \tilde{S} \) only has singularities corresponding to self-intersections, self-tangencies, cusp formations, and triple-intersections.

A self-intersection of the Legendrian curve corresponds to a double point of our foliation where two equal rays meet and terminate. Because self-intersections occur generically on Legendrian curves, the double points form a 1-dimensional manifold. All other singularities occur at isolated points. A self-tangency of the Legendrian curve corresponds to a special double point, where two equal rays meet head-on (making an angle of \( \pi \)). We will not need to distinguish these self-tangency points from ordinary double points. A cusp formation of the Legendrian curve corresponds
to a singular point of the foliation where a single ray ends at a point. A triple-intersection of the Legendrian curve corresponds to a triple point of the foliation where three equal rays meet and terminate.

For now, we extend the foliation to the interior of the curve $\tilde{S}$ by taking minimal geodesics to $p$. The resulting foliation on the interior of $\tilde{S}$ is non-singular except at $p$, where it has a serious singularity which we will have to deal with later. By putting the boundary in general position, we can assume that the only singular points which occur on the boundary are double points. There are a finite number of boundary double points. Since the boundary is convex, the foliation is transverse to the boundary away from the boundary double points.

To summarize, we have constructed a foliation of $U$ by perturbed geodesics. Because the perturbation was very slight, we may assume that each perturbed geodesic has length less than 36. The foliation has five kinds of singular points: the point $p$, double points, cusp formation points, triple points, and boundary double points. We include a diagram illustrating the five types of singular points.

We will do three surgeries to the foliation. The first surgery fixes the foliation so that each boundary component of $U$ is a leaf of the foliation. It takes place in a small neighborhood of the boundary of $U$. The second surgery fixes the foliation so that it extends over $p$. It takes place in a small neighborhood of $p$. The second surgery uses the fact that $U$ is a planar domain. The third surgery, which is only cosmetic, replaces each singular leaf ending in a cusp formation point by a singular leaf consisting of a single point.

We describe the first surgery. Let $c$ be a boundary component of $U$. The goal of the first surgery is to modify the foliation near $c$ so that $c$ becomes a leaf.

Step 1. Modify the foliation in a neighborhood of each boundary double point so that it is non-singular and transverse to $c$. (This creates a new triple point of the foliation in the interior of $U$ near the old boundary double point.)
Step 2. Parametrize a small neighborhood $N$ of $c$ by $S^1 \times [-\epsilon, \epsilon]$, so that each curve $\{\theta\} \times [-\epsilon, \epsilon]$ is a leaf of the foliation and each circle $S^1 \times \{r\}$ has length less than 24. (The circle with $r$-coordinate $-\epsilon$ is the boundary $c$.) Let all the circles with $r$-coordinate less than $0$ be leaves of the modified foliation. In particular, $c$ is a leaf of the modified foliation. Next, make a singular leaf of the modified foliation by joining the circle with $r$-coordinate $0$ to two rays from that circle to $p$. The two rays should be chosen so that they bisect the circle.

This singular leaf cuts the remainder of $N$ into two components, which are each parametrized by $(0, \pi) \times (0, \epsilon]$. On each component we will replace the radial foliation whose leaves are $\{\theta\} \times (0, \epsilon)$ by a rectangular foliation.

For each number $a$ strictly between $0$ and $1/2$, we make a leaf of the rectangular foliation consisting of two radial line segments and one circular arc. The radial line segments have $\theta$-coordinate equal to $\pi a$ and $\pi(1 - a)$ and $r$-coordinate greater than or equal to $a\epsilon$. The circular arc has $r$-coordinate equal to $a\epsilon$ and $\theta$-coordinate ranging from $\pi a$ to $\pi(1 - a)$. Finally, there is an exceptional leaf consisting of a radial line with $\theta$-coordinate $\pi/2$ and $r$-coordinate greater than or equal to $\epsilon/2$. The foliation has a singularity at $(\pi/2, \epsilon/2)$, where it is homeomorphic to a cusp formation point. This finishes the first surgery.

At the end of the first surgery, all but finitely many leaves of the foliation are curves that start and end at $p$ and have length less than 84. There are three kinds of singular leaves. First, there are leaves with one triple point, which we will call triple leaves. The first surgery turned each boundary double point into a triple point. Each new triple point lies on a leaf consisting of three rays and an arc parallel to the boundary component $c$. Since this arc has length less than 12, the length of each new triple leaf is less than 120. In addition, the original triple points
are still present. Each original triple point lies on a leaf consisting of three rays, with total length less than 108. Second, there are leaves with two rays joining a circle which is close to a boundary circle, which we will call boundary leaves. Since each boundary circle has length less than 24, each boundary leaf has length less than 96. Finally, there are rays ending at cusp formation points, which we will call cusp leaves. Every leaf has length less than 120.

We describe the second surgery. The goal of the second surgery is to modify the foliation near p so that it extends over p.

Step 1. Draw a small circle c around p, and foliate the disk bounded by c by small concentric circles, with a point at p. Pick two points on c whose rays meet at a double point, and add the rays to the circle to make a singular leaf L.

Step 1: Adding concentric circles around p

Because U is a planar domain, the leaf L divides the remainder of U into two components, which we will repair one at a time. Each component is a planar domain bounded by an arc a of the circle c and a double leaf L joining the two endpoints of a. We have a foliation on the interior of this planar domain which extends smoothly to L but which is transverse to a, and we need to modify it to make it tangent to a.

Step 2. We define S to be the set of all points where a singular leaf meets the arc a. If the set S were empty, then each point of a would lie on a double leaf. The map taking each point to the point on the other side of its double leaf would be a continuous involution of the arc a with no fixed points, which is impossible. If there is exactly one point in S, then V is a disk containing one cusp leaf and no other singular leaves. In this case, we are in a situation similar to the last part of the first surgery. We parametrize a by (0, 1), so that the cusp leaf crosses a at 1/2, and so that for each number x between 0 and 1, the leaf crossing a at x meets the leaf crossing a at 1 − x in a double point. We then replace the foliation in a neighborhood of a by a rectangular foliation, as in the first surgery.
If $S$ has more than one point, then we consider the two outermost points of $S$. By a continuity argument, these two points must belong to the same leaf, which we call $L'$.

The foliation between $L$ and $L'$ does not contain any singular leaves and so it must have a very simple form. Between $L$ and $L'$, there are two intervals of $a$. We parametrize the first interval by $(0, 1/2)$ and the second interval by $(3/2, 2)$ so that the leaf crossing $a$ at $x$ meets the leaf crossing $a$ at $2-x$ in a double point. We let $a'$ be a curve very close to $a$ but slightly on the interior of the planar domain (slightly farther from $p$). We then replace the foliation between $a$ and $a'$ by a rectangular foliation as in the first surgery.

We have now extended our foliation a bit further. We have not yet repaired the region between $L'$ and $a'$. If $L'$ is a triple leaf, this region has two components, and if $L'$ is a boundary leaf, this region has one component. Each component is a planar domain bounded by an arc $a'$ and a double leaf $L'$ joining the two endpoints of $a'$. We have a foliation on the interior of this planar domain which extends smoothly to $L'$ but which is transverse to $a'$, and we need to modify it to make it tangent to $a'$. Since these are exactly the assumptions we made at the beginning of Step 2, we can continue our construction inductively, finding the outermost singular leaf of each new component, and extending the foliation up to that leaf. Since there are only finitely many singular leaves, this process terminates.

At the end of the surgery, the foliation is no longer singular at the point $p$. Since the circles and arcs in the surgery can be taken arbitrarily small, the surgery has a negligible effect on the lengths of leaves. There are now only two kinds of singular leaves. The triple leaves and boundary leaves are both homeomorphic to the letter
θ, and the cusp leaves are homeomorphic to line segments with one cusp formation singularity at each end.

We describe the third surgery. The goal of the third surgery is to modify the foliation in a small neighborhood of a cusp leaf to replace the cusp leaf by a point.

In the neighborhood of a cusp leaf, the foliation is homeomorphic to the level sets of the distance function from a line segment in $\mathbb{R}^2$. The distance function admits a compactly supported modification so that it has only one non-degenerate minimum and no other critical points. The level sets of the modified function give a new foliation with a singular point instead of a singular line segment. This operation increases the lengths of leaves by an arbitrarily small amount.

Finally, we define the target tree to be the space of leaves of our foliation and the map to be the quotient map. All the claims of the lemma have been satisfied. □

This finishes the proof of Theorem 2.1. Using this theorem, we can bound the length of a closed geodesic.

**Theorem 2.4.** Suppose $(S^2, g)$ has degree 1 hypersphericity less than 1. Then it contains a closed geodesic of length less than 160.

**Proof.** We will use two well-known existence theorems for closed geodesics, which go back to Birkhoff. The first result is the curve shortening theorem. Starting with any closed curve $c$ in $(S^2, g)$ which is not a closed geodesic, there is a canonical homotopy which decreases the length of the curve over time and which either converges to a closed geodesic or contracts the curve to a point. The second result concerns families of short curves. If there is a family of curves in $(S^2, g)$ of length less than $L$ which sweep out a degree non-zero map from $S^2$ to $(S^2, g)$, then there is a closed geodesic in $(S^2, g)$ of length less than $L$. This theorem can be proved by using Morse theory on the space of maps from the circle to $(S^2, g)$. Both results are explained in Croke’s paper [4].

We will use the family of short curves guaranteed by Theorem 2.1. In the special case that our tree has no triple points, then the fibers of our map sweep out $(S^2, g)$ by circles of length less than 120, and by Birkhoff’s theorem there is a closed geodesic of length less than 120. The main point of this proof is to deal effectively with the singular fibers.

It turns out that singular fibers homeomorphic to the wedge of two circles are more convenient for our proof than singular fibers homeomorphic to the letter $\theta$. By modifying our map near the singular fibers, we can replace the singular fibers by fibers homeomorphic to the wedge of two circles.
an edge of the tree is a circle, and the preimage of each terminal vertex is a point. Each fiber has length less than 157.

We will use a curve shortening lemma for wedges of two circles, which we now prove. Let \( f \) be a map from the wedge of two circles to \((S^2, g)\) with length less than 157. We will prove that \( f \) can be continuously deformed within the space of Lipshitz maps until it reaches a local minimum of the length functional, and so that every map on the way has length less than 160. We can choose a constant \( \epsilon \) sufficiently small that if \( f_0 \) and \( f_1 \) are two maps from the wedge of two circles to \((S^2, g)\), and if the \( C^0 \) distance between them is less than \( \epsilon \), then there is a homotopy of maps \( f_t \) between them so that the length of each \( f_t \) is less than 160. Consider all the 1-parameter families of maps starting with \( f \) and with length less than 160. Let \( l \) be the infimal length of all maps in such families. Let \( f_i \) be a sequence of maps in such families with the length of \( f_i \) equal to \( l_i \rightarrow l \). We can assume that each \( f_i \) has length less than 157. After reparametrization, we can assume that each \( f_i \) has Lipshitz constant less than 1000. The space of maps with Lipshitz constant less than 1000 is precompact in the \( C^0 \) topology, so the \( f_i \) have a \( C^0 \) sublimit \( f_\infty \), and the length of \( f_\infty \) is less than or equal to \( l \). For large \( i \), the \( C^0 \) distance from \( f_i \) to \( f_\infty \) is less than \( \epsilon \), and so we can deform \( f_i \) to \( f_\infty \) without increasing its length over 160. Thus we can deform \( f \) to \( f_\infty \) without increasing its length over 160, and the length of \( f_\infty \) must be equal to \( l \). If any function \( g \) was within \( \epsilon \) of \( f_\infty \) in the \( C^0 \) metric and had length less than \( l \), we could deform \( f_\infty \) to \( g \) without increasing its length over 160, contradicting the definition of \( l \). Therefore, \( f_\infty \) minimizes length in a \( C^0 \) neighborhood.

We now describe the possible maps of a wedge of two circles into a surface which are local minima for the length function. We let \( p \) be the image under a map \( f \) of the base point of the wedge of two circles. Clearly, each circle in the wedge is mapped to a geodesic starting and ending at \( p \), which might or might not be the constant map. If exactly one of these circles is constant, then the other one must be a closed geodesic. If neither circle is constant, then there are four geodesic segments leaving \( p \), and the length-minimizing condition implies that the sum of the four corresponding unit vectors is zero. Because the tangent space at \( p \) is 2-dimensional, this condition guarantees that the 4 vectors consist of two pairs of opposite vectors. Therefore, the image of the wedge of two circles consists of either 1 or 2 closed geodesics. To summarize, either there is a closed geodesic of length less than 160, or each singular fiber can be contracted to a point through a family of wedges of two circles with length less than 160.

From now on, we will assume that there is no closed geodesic of length less than 160 and work up to a contradiction. Because of the standard curve-shortening theorem, we know that each non-singular fiber contracts to a point through curves of length less than 160. Each family of curves contracting a non-singular fiber to a point gives rise to a map from a disk to \((S^2, g)\), the boundary of the disk mapping to the fiber. We will refer to a family of curves of length less than 160 which contracts a non-singular fiber to a point as a contracting disk for that fiber. Any two contracting disks for the same fiber can be glued together to get a map from the 2-sphere to \((S^2, g)\), and if the degree of this map is non-zero then Birkhoff’s theorem guarantees a closed geodesic of length less than 160. Therefore, for each non-singular fiber, all the contracting disks are homologous.
Pick one terminal vertex $t$ of $T$ as a root and let $p$ be the preimage of this vertex in $(S^2, g)$. Each non-singular fiber divides $(S^2, g)$ into two disks, and we will refer to the disk that does not contain $p$ as the far disk of that non-singular fiber. We will now prove that the contracting disk of every non-singular fiber is homologous to its far disk. We will work inductively from the outermost branches of the tree towards the root. For fibers on the same edge as a terminal vertex other than $t$, the fibering itself gives a contraction of the fiber to a point whose contracting disk is the far disk of the fiber. Notice that once we have proved our claim for one fiber, then the claim follows for all other fibers over the same edge of $T$. Therefore, it suffices to prove our claim inductively one edge at a time.

We consider an edge $E$. Let $u$ be the endpoint of $E$ farthest from $t$, and let $E_1$ and $E_2$ be the two edges meeting $E$ at $u$. By induction, we may assume that our claim holds for both $E_1$ and $E_2$, and we will prove that it holds for $E$. Let $F$ be the fiber over a point in $E$ very close to $u$, and let $F_1$ and $F_2$ be fibers over points in $E_1$ and $E_2$ very close to $u$. Orient the fibers so that $F$ is homologous to the sum of $F_1$ and $F_2$ within the inverse image of a small neighborhood of $u$. We know that the singular fiber over $u$ contracts to a point through a family of wedges of circles of length less than 160. By restricting in various ways, this one contraction produces a contraction of $F$, $F_1$, and $F_2$. We let $D$, $D_1$, and $D_2$ be the contracting disks for these fibers produced by the contraction of the singular fiber. Since these disks all come from the same contraction of the singular fiber, $D$ is homologous to $D_1 + D_2$. By induction, we may assume that $D_1$ is homologous to the far disk of $F_1$ and that $D_2$ is homologous to the far disk of $F_2$. Therefore $D$ is homologous to the far disk of $F_1$ plus the far disk of $F_2$, which is the far disk of $F$.

On the other hand, for fibers over the edge containing the root $t$, the disk containing $p$ is obviously a contracting disk. Since the far disk is also a contracting disk, Birkhoff’s theorem guarantees a closed geodesic of length less than 160.

3. Applications to Surfaces in Euclidean Space

In this section, we will prove three estimates for the hypersphericity of a spherical or planar surface in Euclidean space. Recall that for a surface with boundary, $(\Sigma, \partial \Sigma)$, the degree 1 hypersphericity is defined using degree 1 maps from the pair $(\Sigma, \partial \Sigma)$ to the pair $(S^2, \ast)$. The first estimate that we will prove is a slicing inequality.

**Theorem 3.1.** (Slicing Inequality) Let $f$ be an embedding of the unit $n$-ball in $\mathbb{R}^n$, and let $f_0$ be the first $n-2$ coordinates of $f$. Then one of the level sets of $f_0$ contains a connected component with degree 1 hypersphericity greater than $1/24$.

The proof of this theorem is based on the following lemma.

**Lemma 3.2.** Let $\pi$ be the projection from $S^n$ to the first $n-2$ coordinates of $\mathbb{R}^{n+1}$. Suppose that $g$ is a Riemannian metric on $S^n$, and that the degree 1 hypersphericity of every fiber of $\pi$ is less than 1. Then the Uryson $(n-1)$-width of $(S^n, g)$ is less than 12.

**Proof.** Let $s$ be a continuous section of the projection $\pi$. We define the function $F(x)$ to be the distance, in the intrinsic metric of the fiber through $x$, between $x$ and the point $s(\pi(x))$. The function $F$ is clearly continuous away from the singular fibers of $\pi$. Since the metric $g$ is bounded by a constant multiple of the standard metric
on $S^n$, the function $F(x)$ goes to zero as $x$ approaches any of the singular fibers. Therefore, $F$ is continuous. The function $(\pi, F)$ maps $S^n$ to $\mathbb{R}^{n-1}$. By Theorem 1.1, each connected component of the fibers of the map $(\pi, F)$ has diameter less than 12. We consider the induced map from $(\pi, F)$ to the space of connected components of level sets of $(\pi, F)$. By definition, the fibers of this map have diameter less than 12. If the space of connected components of level sets of $(\pi, F)$ were an $(n-1)$-polyhedron, we would be done.

The space of connected components of level sets of $(\pi, F)$ may be very complicated. We explain how to approximate it by $(n-1)$-dimensional polyhedra, by a method that goes back to Alexandrov. (For more information on polyhedral approximation, see the book *Dimension Theory* by Hurewicz and Wallman, [9], especially section 9 of chapter 5.) We cover $\mathbb{R}^{n-1}$ by very small open sets $B_i$. We can arrange these open sets so that each point lies in $B_i$ for at most $n$ values of $i$. The connected components of the sets $(\pi, F)^{-1}(B_i)$ form an open cover of $(S^n, g)$. There may be infinitely many open sets in this cover, but since $S^n$ is compact we may take a finite subcover. Each point of $(S^n, g)$ lies in at most $n$ of the open sets in the open cover. We now consider the map from $(S^n, g)$ to the nerve of this cover, which is an honest $(n-1)$-dimensional polyhedron. Each fiber of this map lies in a connected component of the preimage of one of the sets $B_i$. By taking a very fine open cover, we can arrange that the diameter of each connected component of the preimage of each $B_i$ is less than 12.

Let us apply the above lemma to the pullback of the standard metric on $S^n$ by an arbitrary diffeomorphism. By Gromov’s estimate in [6], the Uryson $(n-1)$-width of the unit $n$-sphere is at least $\pi/2$. Therefore, one of the level sets of $\pi$ has hypersphericity at least $\pi/24$. Thus we have proven a slicing inequality for the unit $n$-sphere.

With these lemmas we give the proof of Theorem 3.1. First, rescale the function $f$ so that the image lies in the unit ball of $\mathbb{R}^n$. This rescaling does not affect the fibers of $f_0$. Next, we embed the unit ball as a hemisphere of the unit $n$-sphere, again without changing the first $n-2$ coordinate functions. Now, we put a metric $g$ on the $n$-sphere as follows. On the image of the map $f$, we use the pushforward of the metric on the unit ball. On the complement of the image of $f$, we make the Riemannian metric very small. We are now in the situation of Lemma 3.2. Since the unit ball with the boundary collapsed to a point admits a degree 1 map to the unit sphere with Lipschitz constant $\pi$, the Uryson $(n-1)$-width of the metric $g$ is at least $1/2$. Therefore, one of the fibers of the map $\pi$ has degree 1 hypersphericity at least $1/24$. But the fibers of the map $\pi$ are practically the level sets of $f_0$ with the boundary collapsed to a point. Therefore, one of the connected components of a level set of $f_0$ has degree 1 hypersphericity at least $1/24$. This finishes the proof of Theorem 3.1.

Our next estimate is a variation of the isoperimetric inequality in $\mathbb{R}^3$.

**Theorem 3.3.** *(Isoperimetric Inequality)* Let $U$ be a bounded open set in $\mathbb{R}^3$ with boundary diffeomorphic to a 2-sphere. Then the boundary of $U$ contains disjoint open sets $S_i$, so that the following inequality holds.

$$\text{Vol}(U) < 10^6 \sum \text{HS}(S_i)^3.$$  

In this equation, $\text{HS}(S_i)$ stands for the degree 1 hypersphericity of $S_i$. 
To get a sense of the estimate involved in this theorem, consider how it applies to a rectangular solid \( R \), with side lengths \( R_1 \leq R_2 \leq R_3 \). The classical isoperimetric inequality gives a bound for the volume of \( R \) on the order of \( R^{3/2} \). To apply our isoperimetric inequality, note that the boundary of \( R \) has Urysohn 1-width on the order of \( R_2 \). Also, the degree 1 hypersphericity of \( S_i \) is bounded by a multiple of the square root of the area of \( S_i \). Putting these bounds together, we see that \( \sum HS(S_i)^3 \) is bounded by a multiple of \( R_2^2 R_3 \). On the other hand, taking the sets \( S_i \) as disks of radius \( R_2 \), we get \( \sum HS(S_i)^3 \) on the order of \( R_2^2 R_3 \). Therefore, our isoperimetric inequality gives a bound for the volume of \( R \) on the order of \( R_2^2 R_3 \). The constant in our isoperimetric inequality is much worse than that in the standard inequality, but the dependence on \( R_2 \) and \( R_3 \) is better. Our inequality gives the best dependence on \( R_2 \) and \( R_3 \) that can be expected, because the boundary of the rectangle \( R \) can be isotoped without stretching to enclose a volume on the order of \( R_2^2 R_3 \).

Now we turn to the proof of Theorem 3.3. Fix a point \( p \) in the interior of \( U \). For each number \( R \), let \( B(R) \) be the ball of radius \( R \) around \( p \), and let \( S(R) \) be the intersection of the boundary of \( U \) with \( B(R) \). We let \( \phi_R \) be a diffeomorphism of \( \mathbb{R}^3 \) with the unit 3-sphere minus a point \( Q \), with the following properties. The diffeomorphism \( \phi_R \) maps \( B(R) \) to almost the entire unit 3-sphere using the exponential map at the conjugate point to \( Q \). The diffeomorphism \( \phi_R \) maps the complement of \( B(R) \) into a tiny ball in the 3-sphere around the point \( Q \). The Lipschitz constant of \( \phi_R \) is \( \pi/2 \). Using the maps \( \phi_R \), we define a map \( F \) from \( \partial U \times (0, \infty) \) to the unit 3-sphere, by taking \( F(x, R) = \phi_R(x) \). For very large values of \( R \), \( \phi_R \) maps the boundary of \( U \) into a tiny neighborhood of the conjugate point of \( Q \), and for very small values of \( R \), \( \phi_R \) maps the boundary of \( U \) into a tiny neighborhood of \( Q \). Therefore, we can easily complete \( F \) to give a degree 1 map from \( \partial U \times [0, \infty) \) to the unit 3-sphere, mapping \( \partial U \times \infty \) to the conjugate point of \( Q \) and \( \partial U \times 0 \) to \( Q \).

Now we assign a metric \( g \) to \( \partial U \times [0, \infty) \), so that for each \( R \), the restriction of \( g \) to the fiber \( \partial U \) is given by taking the original metric on \( S(R) \), rescaling it by \( \pi/2 \), and making the metric on the rest of the boundary of \( U \) very small. Then we make the metric \( g \) very large in the directions transverse to the fibers. In this way, we arrange that the map \( F \) has Lipschitz constant 1, which shows that the metric \( g \) has hypersphericity 1 and hence Urysohn 2-width at least \( \pi/2 \). By Lemma 3.2., one of the fibers of \( \partial U \times (0, \infty) \) has degree 1 hypersphericity at least \( \pi/24 \). But each fiber is practically a rescaling of \( S(R) \) with the boundary contracted to a point. Therefore, for some value of \( R \), \( S(R) \) has degree 1 hypersphericity at least \( R/24 \).

For each point \( p \) in \( U \), call \( B(p, R) \) good if \( S(R) \) has degree 1 hypersphericity at least \( R/24 \). By the Vitali covering lemma, there are disjoint good balls \( B(p_i, R_i) \) with total volume at least one eighth the volume of \( U \). Now an easy computation shows that \( \sum HS(S(p_i, R_i))^3 > \text{Vol}(U)/10^6 \). \( \square \)

The theorem that we just proved did not impose any lower bound on the hypersphericity of the boundary of \( U \). In order to prove such a bound, we have to make a stronger assumption about the geometry of \( U \). In particular, we will prove the following variation of the isoperimetric inequality in \( \mathbb{R}^3 \).

Theorem 3.4. (Isoperimetric Inequality) Let \( U \) be a bounded open set in \( \mathbb{R}^3 \) with boundary diffeomorphic to a 2-sphere. Suppose that there is an area-contracting...
map of non-zero degree from the pair \((U, \partial U)\) to the unit 3-sphere \((S^3, \ast)\). Then the degree 1 hypersphericity of the boundary of \(U\) is at least \(1/60\).

Proof. The main ingredient of the proof is Theorem 2.1. According to Theorem 2.1, if the boundary of \(U\) has degree 1 hypersphericity less than \(1/60\), then it admits a map \(\pi\) to a trivalent tree \(T\) whose fibers have length less than 2. The fibers also have controlled topology, and in particular the fiber of each point on an open edge of \(T\) is a circle.

Each of these fibers bounds a disk with bounded area in \(\mathbb{R}^3\). These small area disks are useful in controlling the degree of area-contracting maps. Unfortunately, the small disks need not lie in \(U\) and so they can intersect in a complicated way. To circumvent this problem, we will construct a Riemannian metric \(g\) on the 3-ball, whose restriction to the boundary is isometric to \(\partial U\). The metric on the interior will be sufficiently large that for any two points \(x\) and \(y\) on the boundary, the distance in \((B^3, g)\) between \(x\) and \(y\) is equal to the distance in \(\partial U\) between \(x\) and \(y\). At the same time, the metric on the interior will be sufficiently small that it can be cut by disjoint disks of bounded area into components of bounded volume.

We first choose a very large but finite collection of the fibers of the map \(\pi\). Each fiber in our collection will be a circle. By choosing sufficiently many fibers, we may assume that each component of their complement has very small area and has at most three boundary components. We call the fibers in our collection \(\{F_i\}\). At each fiber \(F_i\), we attach a 2-cell \(D_i\) to \(\partial U\) with an attaching map taking the boundary of the 2-cell diffeomorphically to \(F_i\). We give the 2-cell \(D_i\) the Riemannian metric of a hemisphere of radius \(R_i\), where \(2\pi R_i\) is the length of the fiber \(F_i\), in such a way that the attaching map is an isometry. Since the length of \(F_i\) is less than 2, the radius of \(D_i\) is less than \(1/\pi\) and the area of \(D_i\) is less than \(2/\pi\).

For each component \(A_j\) of the complement of the curves \(F_i\) in \(\partial U\), we consider the sphere \(S_j\) consisting of the union of \(A_j\) and the disks filling each component of the boundary of \(A_j\). At each 2-sphere \(S_j\), we attach a 3-cell \(B_j\) to our 2-complex with an attaching map taking the boundary of the 3-cell diffeomorphically to \(S_j\). We give the 3-cell \(B_j\) a Riemannian metric consisting of a cylinder \(S_j \times [0, 1]\) capped by a very small metric on a ball joined to the end \(S_j \times \{1\}\). The attaching map is the identity from \(S_j \times \{0\}\) to \(S_j\). The resulting 3-complex is homeomorphic to a 3-ball with boundary \(\partial U\), and we have equipped it with a Riemannian metric which we will call \(g\).

Because the area of \(A_j\) is negligibly small and because the boundary of \(A_j\) consists of at most three fibers, the sphere \(S_j\) has area less than \(6/\pi\). The volume of \(B_j\) is negligibly more than the area of \(S_j\) times 1, so the volume of \(B_j\) is less than \(6/\pi\). Next we show that the diameter of \(S_j\) is less than 2. Let \(x\) and \(y\) be any two points in \(S_j\). Suppose for now that \(x\) is in \(D_1\) and \(y\) is in \(D_2\). Because we have chosen the fibers \(F_i\) extremely densely, there is a point \(z\) in \(A_j\) which is extremely close to both \(F_1\) and \(F_2\). The distance in \(S_j\) from \(z\) to either \(x\) or \(y\) is less than 1, and so the distance from \(x\) to \(y\) is less than 2. If \(x\) and \(y\) are both in the same disk, then the distance between them is less than 1. If one or both of \(x\) and \(y\) lies in \(A_j\), the distance between \(x\) and \(y\) is still less than 2 because any point in \(A_j\) can be moved a negligibly small distance into one of the disks \(D_i\) in \(S_j\).

We check that the distance in \((B^3, g)\) between two points \(x\) and \(y\) in \(\partial U\) is equal to the distance between \(x\) and \(y\) in \(\partial U\). Let \(p\) be any path in \((B^3, g)\) between \(x\) and \(y\). For any 3-cell \(B_j\), let \(p_0\) be a connected component of \(p \cap B_j\), with endpoints
Let \( t_1 \) and \( t_2 \) in \( S_j \). We replace \( p_0 \) by a minimal geodesic in \( S_j \) from \( t_1 \) to \( t_2 \), which we call \( p_0' \). The length of \( p_0' \) is no more than the diameter of \( S_j \) which is 2. If \( p_0 \) intersects \( S_j \times \{1\} \subset B_j \), then it has length at least 2, and so it is at least as long as \( p_0' \). On the other hand, if \( p_0 \) does not intersect \( S_j \times \{1\} \subset B_j \), then \( p_0 \) lies in the product \( S_j \times [0,1] \) and is at least as long as \( p_0' \). Repeating this argument for each component of \( p \) meeting each ball \( B_j \), we replace \( p \) by a path \( p' \) of equal or smaller length lying on the union of \( \partial U \) and the disks \( D_i \). For any 2-cell \( D_i \), let \( p_0' \) be a connected component of \( p' \cap D_i \) with endpoints \( u_1 \) and \( u_2 \) in \( F_i \). We replace \( p_0' \) by the minimal path in \( F_i \) from \( u_1 \) to \( u_2 \), which we call \( p_0'' \). Since this minimal path is a portion of the equator of the hemisphere \( D_i \), \( p_0'' \) is at least as long as \( p_0' \). Repeating this argument for each component of \( p' \) meeting each disk \( D_i \), we replace \( p' \) by a path \( p'' \) of equal or smaller length lying in \( \partial U \). This proves the estimate for the distance.

Since \( \partial U \) is embedded in \( \mathbb{R}^3 \), we have an isometric embedding map \( i \) from \( \partial U \) to \( \mathbb{R}^3 \). We view the map \( i \) as a map from the boundary of \((B^3, g)\) to \( \mathbb{R}^3 \). If we consider \( \partial U \) as a metric space using the distance function of \((B^3, g)\), the map \( i \) has Lipschitz constant 1. We now use a simple extension result for Lipschitz maps, which goes back to McShane in \[12\]. Any Lipschitz map from a subspace of some metric space to the real numbers admits an extension to the whole metric space with the same Lipschitz constant. We extend each coordinate function of the embedding \( i \) to all of \((B^3, g)\), giving a \( \sqrt{3} \)-Lipschitz map from \((B^3, g)\) to \( \mathbb{R}^3 \).

Now let \( F \) be any area-contracting map from \((U, \partial U)\) to the unit 3-sphere, taking the boundary of \( U \) to the base point of \( S^3 \). Our aim is to prove that \( F \) has degree 0. We first extend \( F \) to all of \( \mathbb{R}^3 \) by mapping the complement of \( U \) to the basepoint of \( S^3 \). We will consider \( F \circ i \), which is a map from \((B^3, g)\) to the unit 3-sphere, taking the boundary of \( B^3 \) to the basepoint of \( S^3 \), and with the same degree as \( F \). We now use our decomposition of \( B^3 \). Each 3-cell \( B_j \) in \((B^3, g)\) has volume less than \( 6/\pi \), and so the image of \( B_j \) has volume less than \( 18\sqrt{3}/\pi < 12 \). The boundary of \( B_j \) consists of \( A_j \) together with at most three disks \( D_i \). Each disk \( D_i \) has area less than \( 2/\pi \), and so the image of each disk \( D_i \) has area less than \( 6/\pi \). Since \( F \circ i \) maps \( A_j \) to the basepoint of \( S^3 \), the image of each disk is a 2-cycle. Each 2-cycle in the unit 3-sphere of area \( A < 4\pi \) bounds a 3-chain of volume less than \( \pi^2(A/4\pi)^{3/2} \), so the image of each disk \( D_i \) bounds a 3-chain of volume less than \( \pi^2((6/\pi)/(4\pi))^{3/2} < 1 \). Let \( C_j \) be the 3-cycle in \( S^3 \) which is the union of the image of \( B_j \) and the 3-chains bounded by each \( D_i \) in the boundary of \( B_j \). Each \( C_j \) has volume less than 15, and since the volume of the unit 3-sphere is \( 2\pi^2 \), each \( C_j \) is null-homologous. But the image of the fundamental homology class of \((B^3, \partial B^3)\) is homologous to the union of the cycles \( C_j \) with appropriate orientations. Therefore, the map \( F \) has degree 0. \( \square \)

4. Estimates for Surfaces of Non-Zero Genus

In this section, we prove some estimates on the geometry of surfaces of non-zero genus with bounded hypersphericity.

We first apply the methods of section 1 to estimate the systoles of closed orientable surfaces. In this paper, the systole of a Riemannian surface will be the shortest length of a homologically non-trivial closed curve in the surface. (Sometimes systole refers to the shortest homotopically non-trivial closed curve.) To give some context, we recall the main estimates for the systoles of surfaces. Besicovitch
proved that the systole of a surface of area $A$ is bounded by $\sqrt{A}/2$, and Gromov proved that for surfaces of high genus $G$ the systole is bounded by $c(G)\sqrt{A}$ where $c(G)$ falls off almost as fast as $1/\sqrt{G}$. (See [7] for Gromov’s theorem and chapter 4 of [8] for some history and many other results.) We will bound the systole of a closed orientable surface by its degree 1 hypersphericity.

**Theorem 4.1.** Suppose that $(\Sigma, g)$ is a closed oriented surface of genus at least 1, with degree 1 hypersphericity less than 1. Then $(\Sigma, g)$ contains a homologically non-trivial closed curve of length less than $4\pi$.

**Proof.** Let $\gamma$ be a shortest homologically non-trivial curve in $\Sigma$, and let $L$ be the length of $\gamma$. If $L$ is less than $4\pi$, then we are done, so we may assume that $L$ is at least $4\pi$. Pick two points on $\gamma$, $p$ and $q$, with $\text{dist}(p, q) = L/4$, and consider the function $F_0(x) = (\text{dist}(p, x), \text{dist}(q, x))$. As Gromov proved in section 5 of [7], the distance between two points of $\gamma$ is the length of the shorter of the two paths in $\gamma$ joining the two points. (This result of Gromov is easy, and a stronger easy result is proved in Lemma 2.2 of this paper.) Therefore, we can exactly determine the restriction of $F_0$ to $\gamma$. The image of $F_0$ is the square $S_1$ with vertices $(0, L/4), (L/4, 0), (L/2, L/4)$, and $(L/4, L/2)$.

We will prove that any curve homologous to $\gamma$ includes a point within $2\pi$ of $q$. Let $\tau$ be any curve homologous to $\gamma$, and suppose that the distance from $\tau$ to $q$ is at least $2\pi$. In that case, $F_0(\tau)$ lies above the line $y = 2\pi$ in $\mathbb{R}^2$, and so avoids the square $S_2$, with vertices $(L/4 - \pi, \pi), (L/4, 0), (L/4 + \pi, \pi)$, and $(L/4, 2\pi)$. Since $S_2$ is a square of side-length $\sqrt{2}\pi$, it contains a disk $D$ of radius $\pi/\sqrt{2}$. Since we assume $L > 4\pi$, $S_2$ is contained in $S_1$, and therefore $F_0(\gamma)$ also avoids the interior of $S_2$. Let $\Sigma_i$ be the connected components of the complement of $\gamma$ and $\tau$ in $\Sigma$. (After putting $\tau$ in general position, we can assume there are only finitely many connected components.) Let $B_i$ be the boundary of $\Sigma_i$. Each $B_i$ consists of a subset of $\gamma$ and a subset of $\tau$, so the map $F_0$ takes $B_i$ to $\mathbb{R}^2 - D$. Moreover, the portion of $F_0(B_i)$ below the line $y = 2\pi$ must be either the portion of $F_0(\gamma)$ below that line or else empty, and so the image of $B_i$ has winding number $-1$, $0$, or $1$ around $D$. Because $\gamma$ is homologous to $\tau$, an appropriate sum of integer multiples of the $B_i$ is equal to $\gamma - \tau$ (as 1-chains). Since the image of $\gamma - \tau$ has winding number 1 around $D$, at least one of the $B_i$ has non-zero winding number around $D$. Say that $B_1$ has winding number equal to 1 or -1.

We now define a contracting degree 1 map $F$ from $\Sigma$ to the unit sphere. Using the exponential map, we construct a homeomorphism $\pi_1$ from the disk $D$ to the Northern hemisphere of the unit sphere. We define a map $\pi_2$ from the disk $D$ to the Southern hemisphere of the unit sphere by composing $\pi_1$ with a reflection through the plane of the equator. We extend $\pi_1$ and $\pi_2$ to all of $\mathbb{R}^2$ as follows. For any point $x$ outside of $D$, let $p(x)$ be the nearest point to $x$ on the boundary of $D$. Now, define $\pi_1(x) = \pi_1(p(x))$ and $\pi_2(x) = \pi_2(p(x))$. The maps $\pi_1$ and $\pi_2$ agree on the boundary of $D$ and hence on all the points outside of $D$. The maps $\pi_1$ and $\pi_2$ have Lipschitz constant $1/\sqrt{2}$. 


Finally we define our degree 1 map. Recall that $\Sigma_1$ is an open set in $\Sigma$ with boundary $B_1$. Let $\Sigma_2$ be the complement of $\Sigma_1$. We define $F$ on each component as follows:

\begin{equation}
F(x) = \begin{cases} 
\pi_1 \circ F_0 & \text{if } x \text{ is in } \Sigma_1, \\
\pi_2 \circ F_0 & \text{if } x \text{ is in } \Sigma_2.
\end{cases}
\end{equation}

$F$ is continuous because $F_0(B_1)$ lies outside of $D$, and $\pi_1 = \pi_2$ outside $D$. $F$ is degree 1 or -1 because $F_0 : B_1 \to \mathbb{R}^2 - D$ is degree 1 or -1. After composing with a reflection, we may assume that $F$ is degree 1. $F$ has Lipschitz constant 1. Since we assumed that the degree 1 hypersphericity of $(\Sigma, g)$ is less than 1, there is no such map, and we conclude that the distance from $\tau$ to $q$ is less than $2\pi$.

Since this estimate holds for any curve $\tau$ homologous to $\gamma$, there is no curve homologous to $\gamma$ in the open manifold $\Sigma - B(q, 2\pi)$. We claim that $B(q, 2\pi)$ contains a curve with non-trivial homology class in $H_1(\Sigma)$. To see this, consider the Mayer-Vietoris sequence associated to the covering $\Sigma = U \cup V$, where $U$ is $B(q, 2\pi)$ and $V$ is a very small neighborhood of the complement of $B(q, 2\pi)$: $H_1(U) \oplus H_1(V) \to H_1(\Sigma) \to H_0(U \cap V)$.

We know that the homology class of $\gamma$ is not in the image of $H_1(V)$ in $H_1(\Sigma)$. If the homology class of $\gamma$ is in the image of $H_1(U)$, we are done. If not, then it has a non-zero image $\alpha$ in $H_0(U \cap V) = H_0(\partial U)$. If $\partial U$ has $k$ components, $c_1, \ldots, c_k$, then $H_0(\partial U)$ has one generator $[c_i]$ for each component $c_i$, and $\alpha = \sum \langle \gamma, c_i \rangle [c_i]$, where $\langle \gamma, c_i \rangle$ is the intersection number of $\gamma$ and $c_i$ in $\Sigma$. If $\alpha$ is not zero, then one of the intersection numbers is not zero, and so one of the boundary components is homologically non-trivial in $\Sigma$. In either case, we have found a homologically non-trivial curve in $B(q, 2\pi)$. We can now factor that curve, by adding many geodesics to and from $q$, into a sequence of curves with length bounded by a number as close as we like to $4\pi$. One of these curves must have a non-trivial homology class. □

We now turn to two more difficult estimates on the geometry of a surface of small hypersphericity. The first estimate bounds the lengths of $G$ homologically independent curves on $\Sigma$.

**Theorem 4.2.** Suppose that $(\Sigma, g)$ is a closed oriented surface of genus $G$ with degree 1 hypersphericity less than 1. Then $\Sigma$ contains homologically independent curves $C_1, \ldots, C_G$ with the length of $C_k$ bounded by $200k$.

I don’t know whether this theorem can be improved to bound the length of $C_k$ by a constant independent of $k$ and $G$. By the same method, we will prove a bound on the Uryson Width of a surface of small hypersphericity in a given genus.

**Theorem 4.3.** If $(\Sigma, g)$ is a closed oriented surface of genus $G$ and degree 1 hypersphericity less than 1, then $(\Sigma, g)$ has Uryson 1-Width less than $200G + 12$.

In this theorem, the important parameter is the dependence of the Uryson 1-Width on the genus. Given our systolic inequality, it is rather easy to bound the Uryson 1-Width by a multiple of $4^G$. To get the linear bound above, we will have to work harder. On the other hand, in section 5 we will give examples of surfaces with degree 1 hypersphericity less than 1 and Uryson 1-Width on the order of $\sqrt{G}$.

Both theorems follow from a lemma about surfaces in which each large ball is a planar domain. We define the planarity radius of a surface $(\Sigma, g)$ as the supremal...
number $R$ so that each metric ball of radius $R$ is a planar domain. When the planarity radius of a surface is large compared to its hypersphericity, an analogue of Theorem 1.1 holds in any genus.

**Lemma 4.4.** Let $(\Sigma, g)$ be a closed surface with planarity radius 100 and hypersphericity less than 1. Then the Uryson width of $(\Sigma, g)$ is bounded by 50. Also, there is a set of disjoint curves on $\Sigma$, each of length less than 24, whose complement is a planar domain.

**Proof.** Since this proof is rather long, I have divided it into six steps.

Claim 1. Every ball in $\Sigma$ of radius 80 has Uryson Width less than 12.

pf. Let $\Sigma_1$ be the metric space formed from a ball of radius 90 around $p$ by collapsing each component of the boundary to a point. The manifold $\Sigma_1$ is a topological 2-sphere. Any contracting map from $\Sigma_1$ to the unit sphere extends to a contracting map of the same degree from all of $\Sigma$. To see this, let $A$ be the annulus consisting of those points whose distance from $p$ is between 90 and 100. The boundary of $A$ consists of two parts: the boundary of the ball of radius 90, which we will call $B$, and the boundary of the ball of radius 100, which we will call $C$. Each component $A_i$ of $A$ is a planar domain, with one boundary component in $B$ and some number of boundary components in $C$. A contracting map $f$ from $\Sigma_1$ to the unit sphere is a map from the ball of radius 90, taking each component $B_i$ of $B$ to a single point $x_i$ in the unit sphere. Let $x$ be a point in the unit sphere and $g_i$ a minimal geodesic from $x_i$ to $x$. We extend $f$ to $A$ by mapping each component $A_i$ of the annulus to the geodesic $g_i$, parametrizing $g_i$ linearly according to the distance from $p$. This map is a contracting map from the ball of radius 100 to the unit sphere, taking the entire boundary of the ball to $x$. We now extend our map to all of $\Sigma$ by mapping the complement of $B(p, 100)$ to $x$. The new map is contracting and has the same degree as $f$. Therefore, the hypersphericity of $\Sigma_1$ is less than 1. By Theorem 1.1, each connected component of a distance sphere around $p$ in $\Sigma_1$ has diameter less than 12 in $\Sigma_1$. It follows that each connected component of a distance sphere around $p$ of radius less than 84 has diameter less than 12 in $\Sigma$. This finishes the proof of claim 1.

We define $\Gamma$ to be the subgroup of the fundamental group of $\Sigma$ generated by lasso-shaped curves consisting of an arbitrary neck followed by a loop of length less than 24. It is easy to check that this subgroup is normal, and we call the quotient group $Q$. Let $\hat{\Sigma}$ be the cover of $\Sigma$ corresponding to $\Gamma$, and note that $Q$ acts on $\hat{\Sigma}$ with quotient $\Sigma$. The fundamental group of $\hat{\Sigma}$ is $\Gamma$ and is generated by lassos in $\hat{\Sigma}$ with loops of length less than 24.

Claim 2. $\hat{\Sigma}$ is a planar domain.

pf. Let $B$ be any ball of radius 80 in $\hat{\Sigma}$. The fundamental group of $B$ is generated by lassos with loops equal to the boundary circles of $B$. By Claim 1, each of these boundary circles has diameter less than 12 in $\Sigma$, so each lasso can be factored into a composition of lassos with loops of length less than 24. Therefore, the inverse image of $B$ in $\hat{\Sigma}$ consists of isometric copies of $B$. Every ball of radius 80 in $\hat{\Sigma}$ is an inverse image of a ball of radius 80 in $\Sigma$, and is a planar domain. Any two curves in $\hat{\Sigma}$ of length less than 24 have intersection number zero, because if they intersect at all, then they both lie in a ball of radius 80, and that ball is a planar domain. On the other hand, any curve in $\hat{\Sigma}$ is homologous to a union of curves of length
less than 24. Therefore, the intersection number of any two curves in $\Sigma$ is zero. It follows that $\Sigma$ is a planar domain.

Claim 3. $\Sigma$ has Uryson width less than 24.

pf. In order to prove this claim, we extend Theorem 1.1 to complete metrics on planar domains, with a somewhat worse constant. In the non-compact case, we define the hypersphericity in terms of compactly supported maps - maps taking the complement of a compact set in the domain to a base point on the range. (Compactly supported maps have a well-defined degree.) Our proof of Theorem 1.1 does not quite extend to this case, because the map we construct takes the complement of a compact set to the equator of the target sphere. We can repair the problem easily by composing our map with a degree 1 map of the 2-sphere that takes the equator to a point. Using the exponential map, we can construct such a map with Lipshitz constant 2, so we have proved that a planar domain with hypersphericity 1 has Uryson 1-Width less than 24. Moreover, our construction yields a map which is supported inside a ball of radius 24. Therefore, we have proven that if every ball of radius 24 in a planar domain has hypersphericity less than 1, then the whole surface has Uryson width less than 24. Since $\Sigma$ is a planar domain, and every ball of radius 80 in $\Sigma$ has hypersphericity less than 1, Claim 3 follows.

Recall from section 2 that a curve in a Riemannian manifold is called strictly straight if any minimal segment joining two points on the curve is contained in the curve. Now let $S$ be the set of all strictly straight circles in $\Sigma$, of length no more than 24. Since $\Sigma$ is planar, the strictly straight circles in $\Sigma$ are disjoint by Lemma 2.2. (The union of all the strictly straight circles in $S$ is a closed set, which we also call $S$.)

Claim 4. Any curve $\gamma$ from one end of $\Sigma$ to another intersects a circle in $S$.

pf. We do this proof in two cases. First we deal with the special case that the genus of $\Sigma$ is 1. By Theorem 4.1, the shortest homologically non-trivial geodesic in $\Sigma$ has length less than $4\pi$. We pick a point $p$ on this geodesic and consider the ball of radius 80 around a lift of $p$. We see that $\Sigma$ is diffeomorphic to $R \times S^1$, and that the strictly straight geodesics include a circle in the class $[S^1]$.

Second, we deal with the main case that the genus of $\Sigma$ is at least 2. We let $c$ be a curve of length less than 24 with non-zero intersecton number with $\gamma$. On a compact manifold, we know that there is a curve homologous to $c$ which consists of a union of shorter strictly straight circles. This is not always true on a non-compact manifold, but we will prove that it is true on $\Sigma$. We first homotope the circle $c$ to a minimal geodesic in its free homotopy class. Because the genus of $\Sigma$ is at least 2, the circle $c$ cannot be moved more than a bounded distance without stretching, and so there is an actual minimizer. If this minimizer is not strictly straight, we surger it along a minimal segment into two circles. Then we repeat the procedure, homotoping each circle to a minimal geodesic in its free homotopy class, and so on. Because of bounded geometry, each time we do a surgery, splitting a closed geodesic of length less than 24 into two pieces, neither piece null-homotopic, and then homotoping the pieces to minimal geodesics, the length of each new piece is less than $(1-\epsilon)$ times the length of the old circle. Because the injectivity radius of $\Sigma$ is bounded below, this procedure must terminate after a finite number of surgeries, with a union of strictly straight circles each of length less than 24. Since the union is homologous to $c$, one of the curves intersects $\gamma$. 
Claim 5. Q has no torsion.

pf. Suppose Q had a torsion element q of order n. Pick a point x in ̂Σ and let γ be a minimal geodesic from x to q · x. The union of γ, q · γ, q^2 · γ, ..., and q^{n-1} · γ is a circle c in ̂Σ. If we parametrize γ by [0, 1], then there is a corresponding parametrization of c by [0, n], so that two points in c differing by an integer are related by a power of q. By Claim 3, there is a map π from ̂Σ to a tree T, whose fibers have diameter less than 24. Applying π to the parametrization of c, we get a map π_0 from [0, n] to the tree T. We will modify this map π_0 so that each interval [k, k + 1] (for k an integer) meets each vertex of T in a connected segment, whose image in ̂Σ has length no more than 24. If the preimage of a vertex t of T intersected with the interval [k, k + 1] consists of several numbers, we define a_{min} to be the smallest of the numbers and a_{max} to be the largest of the numbers. Then we modify π_0 so that it maps the interval [a_{min}, a_{max}] to the vertex t. Since the image of [k, k + 1] is a minimal geodesic in ̂Σ and the diameter of each fiber of π is less than 24, the length of the image of the interval [a_{min}, a_{max}] is less than 24. We repeat this procedure for each vertex t in T. We refer to the resulting map from [0, n] to T as ̂π.

The map ̂π obeys two estimates. First, the inverse image of any vertex intersected with any interval [k, k + 1] consists of a closed sub-interval. Second, ̂π agrees with π, except on the union of finitely many disjoint open intervals, each of which is mapped to a single vertex of t and each of which parametrizes a segment of length less than 24. Now, it is a topological fact that two numbers in [0, n] differing by an integer are mapped to the same point of T by ̂π. Call the two numbers a and b, and let p and q be the points parametrized by a and b. It may not be the case that p and q are mapped to the same point by π, because we have modified π along some disjoint intervals. Let a_0 and b_0 be the closest endpoints of the intervals containing a and b, and let p_0 and q_0 be the points in c parametrized by a_0 and b_0. The distance from p to p_0 is less than 12, as is the distance from q to q_0. Also, p_0 and q_0 are mapped to the same point of T by the original π, so the distance between them is no more than 24. Therefore, the distance between p and q is no more than 48. On the other hand, the distance between any two different points in the same Q-orbit is at least 80. This contradiction establishes the claim.

Claim 6. Any two points in the same Q orbit are in different components of ̂Σ − S.

pf. Let γ be any curve connecting x to q · x, avoiding all the circles in S. Since there is no torsion in Q, the union of all the curves q^n · γ, for all integers n, is a single curve running from one end of ̂Σ to another. By Claim 4, this curve must intersect a circle in S. But since S is invariant under the action of Q (or under any isometry), γ must intersect a circle in S.

Because S is Q-invariant, the curves in S descend to disjoint curves in ̂Σ of length less than 24, dividing ̂Σ into planar domains. This establishes the last statement of the lemma. We now bound the Uryson width of ̂Σ. We select some union of components of ̂Σ − S that form a fundamental domain for the action of Q on ̂Σ − S, and on each one we construct a map to a tree for which each boundary circle is a fiber, and with fibers of diameter less than 50. On each component, we pick a point p and begin by taking the distance spheres around p as our fibers (or more properly, their intersection with the component we have selected). These have diameter less than 24, but they do not extend to the boundary with each boundary component as
a fiber. We surger the fibers near each boundary circle, so that the boundary circle is a fiber. This increases their diameter by no more than 24, and so the Uryson Width of $\Sigma$ is less than 50.

Now we turn to the applications of Lemma 4.4. If a surface has planarity radius less than 100, then it contains two smooth embedded curves, each of length less than 200, intersecting transversely at one point. To see this, we find a non-planar ball of radius less than 100 in our surface. We slightly change the metric on the ball to make the boundary convex and then consider a minimal rational basis of the first homology group of the ball. Each curve in the ball is homologous to a sum of curves of length less than 200, and so every curve in the minimal basis will have length less than 200. According to Lemma 2.2, each curve will be strictly straight, and so any pair of curves will intersect no more than once, and if they do intersect once the intersection will be transverse. Since the domain is not planar, two curves in the basis must intersect, and this proves the claim.

Therefore, an arbitrary surface has either two short embedded curves intersecting transversally at one point or a large planarity radius. Exploiting these two structures, we can quickly prove Theorems 4.2 and 4.3.

Proof of Theorem 4.2: If the planarity radius of $\Sigma$ is less than 100, then there are two curves of length less than 200 with intersection number one, intersecting in a single point. We call these curves $C_1$ and $C_2$. Removing these curves from $\Sigma$ leaves a manifold $U$ whose boundary is a single circle (with corners). We can embed $U$ into a closed manifold $\Sigma'$ of genus $G-1$, so that the boundary circle bounds a disk in $\Sigma'$. Now we extend the metric on $U$ to give a Riemannian metric on $\Sigma'$, but make this metric very small on the complement of $U$. If we make the metric sufficiently small on the complement of $U$, then there is a degree 1 map from $\Sigma$ to $\Sigma'$ with Lipschitz constant arbitrarily close to 1, and so the degree 1 hypersphericity of $\Sigma'$ is less than 1.

By induction on the genus, $\Sigma'$ contains $G-1$ homologically independent curves $C_1', ..., C_{G-1}'$ with the length of $C_k'$ bounded by 200$k$. By taking a minimal basis of homology of $\Sigma'$, we can assume that each curve is strictly straight, and by altering them very slightly we can assume that each curve passes through the complement of $U$ at most once. Now we define a curve $C_{k+2}$ in $\Sigma$ as follows. Begin with the portion of the curve $C_k'$ lying in $U$ and lift it to $\Sigma$. If this curve is closed, we are done. If it is not closed, then it is a single segment with endpoints on the union of $C_1$ and $C_2$. Since the diameter of this union is 200, we can close the curve by adding a portion of $C_1$ and a portion of $C_2$ of total length no more than 200. Therefore, the length of $C_{k+2}$ is less than $200(k+1)$. We have now defined curves $C_1, ..., C_{G+1}$ in $\Sigma$ with the length of $C_k$ bounded by 200$k$.

We now show that the curves $C_k$ are homologically independent. Let $S$ be the rational span of their homology classes in $H_1(\Sigma)$. The map from $\Sigma$ to $\Sigma'$ induces a map from $S$ to $H_1(\Sigma')$. The kernel of this map is at least two-dimensional, generated by $C_1$ and $C_2$. On the other hand, the curve $C_{k+2}$ is mapped to a curve homologous to $C_k'$, and so the image of this map has dimension at least $G-1$. Therefore, the space $S$ has dimension $G+1$ and the curves $C_k$ are homologically independent.

On the other hand, if the planarity radius of $(\Sigma, g)$ is more than 100, then the lemma guarantees $G$ homologically independent curves of length less than 24.
Proof of Theorem 4.3: If the planarity radius of \((\Sigma, g)\) is less than 100, then we again find two curves of length less than 200 intersecting transversally in a point, define \(U\) to be the manifold with boundary formed by removing these two curves, embed \(U\) in a surface \(\Sigma'\) of genus \(G-1\), and extend the metric on \(U\) to a metric on \(\Sigma'\) which is very small away from \(U\). We can then define a degree 1 map \(F\) from \(\Sigma\) to \(\Sigma'\) with Lipschitz constant only slightly more than 1, and with the property that the preimage of a set of diameter \(D\) has diameter less than \(D + 200\). (The map \(F\) is essentially the map which collapses the two curves to a point.) The degree 1 hypersphericity of \(\Sigma'\) is less than 1, and so by induction on the genus we may assume that there is a map \(\pi'\) from \(\Sigma'\) to a graph whose fibers have diameter less than \(200(G-1) + 12\). (The base for the induction, \(G=0\), is just Theorem 1.1.) We define \(\pi\) to be the composition of \(\pi'\) with \(F\). The diameters of the the fibers of \(\pi\) are less than \(200G + 12\), and in this case the theorem holds. On the other hand, if \((\Sigma, g)\) has planarity radius more than 100, then by the lemma it has Uryson 1-width less than 50, and the theorem holds in this case too.

5. Strange Metrics on Surfaces of High Genus

In this section we will construct surfaces of arbitrarily small hypersphericity and with Uryson 1-Width at least 1.

**Theorem 5.1.** There exists a sequence of closed oriented surfaces \((\Sigma_i, g_i)\) with hypersphericity tending to 0 and Uryson 1-Width at least 1.

The construction is roughly as follows. First, we pick a space \((M, g)\) along with a non-zero homology class \(h\) in \(H_2(M, \mathbb{Z})\), which can be realized by an oriented surface. We pick \(\Sigma_1\) to be any surface in \(M\) in the homology class \(h\). The surface \(\Sigma_i\) is \(\Sigma_1\) joined to a very dense mesh of thin tubes filling \(M\), and the metric \(g_i\) is the induced metric from \(M\). The resulting surfaces can be chosen to converge to \(M\) in the Gromov-Hausdorff metric. The details of this construction are carried out by Cassorla in [3].

Using the geometry of \(M\), we will prove inequalities about the hypersphericity and Uryson 1-Width of the surfaces \(\Sigma_i\). Our main tool to relate the geometry of \(\Sigma_i\) to the geometry of \(M\) will be Gromov’s extension theorem for maps of small width, and some minor generalizations. Recall that the width of a map is defined to be the supremal diameter of its fibers. We now recall Gromov’s extension theorem for maps of small width, which appeared in [3].

**Theorem (Gromov).** If \(p\) is a surjective map from \(X\) to \(Y\) with width less than \(\pi/2\), and if \(f\) is a contracting map from \(X\) to a unit sphere (of any dimension), then \(f\) is homotopic to a map that factors through \(p\).

**Corollary (Gromov).** If \((M^n, g)\) has Uryson \((k-1)\) width less than \(\pi/2\), then every contracting map from \(M\) to the unit \(k\)-sphere is null-homotopic.

As a special case of the corollary, we see that the hypersphericity of a Riemannian \(n\)-manifold is bounded by \((\pi/2)\) times its Uryson \((n-1)\)-Width, which is an inequality we used several times in this paper.

Using Gromov’s extension theorem, we can construct surfaces with arbitrarily small degree 1 hypersphericity but with hypersphericity at least 1. For example, let \((\Sigma, g)\) be a degree 2 branched covering of the unit 2-sphere, branched over an \(\epsilon\)-dense set of points, with the induced metric (strictly speaking the induced metric
is singular, but we can add a tiny positive symmetric form to make a Riemannian metric. The branched covering map has width less than $3\varepsilon$, and it’s certainly surjective. By Gromov’s theorem, any contracting map from $\Sigma$ to the sphere of radius $3\varepsilon$ factors through the branched covering and has even degree. Therefore, the degree 1 hypersphericity of $\Sigma$ is less than $3\varepsilon$. On the other hand, the branched covering is itself a contracting degree 2 map to the unit sphere, and so the hypersphericity of $\Sigma$ is at least 1.

We will define a class of maps called maps of small girth which generalize the surjective maps of small width, and extend Gromov’s theorem to the maps of small girth. This generalization is not very serious, but it will be convenient for our purposes, and we will include more details of the proof than appear in [6]. To arrive at the definition of the girth of a map, we make a slight shift in point of view. Instead of a polyhedron $Y$, the target of our map is a polyhedron $Y$ equipped with a cover by open sets, $Y = \bigcup U_i$. Instead of taking a surjective map from $X$ to $Y$, we take a map $p$ whose image intersects each open set $U_i$ in the cover. For any point $y$ in $Y$, let $U_y$ be the union of all $U_i$ containing $y$. Instead of considering the fiber $p^{-1}(y)$, we consider the fattened fiber $p^{-1}(U_y)$. We say that the map $p$ has girth less than $w$ if $p^{-1}(U_y)$ has radius less than $w$.

We will say that a Riemannian manifold $M$ has convexity radius at least $w$ if each ball of radius at most $w$ is convex with respect to minimal geodesics. A set $U$ in $M$ is convex with respect to minimal geodesics if for any two points $x$ and $y$ in $U$, there is a unique minimal geodesic in $M$ joining $x$ to $y$, and this minimal geodesic lies in $U$. For instance, any complete Riemannian manifold with sectional curvature less than 1 and injectivity radius at least $\pi$ has convexity radius at least $\pi/2$.

Now we can prove one version of the extension lemma.

**Lemma 5.2.** If $p$ is a map from $X$ to $Y$ with girth less than $w$, $f$ is a contracting map from $X$ to $M$, and $M$ is a Riemannian manifold with convexity radius at least $w$, then there is a map $g$ from $Y$ to $M$ so that $f$ is homotopic to $g \circ p$. Moreover, the homotopy moves each point of $X$ by no more than $2w$.

**Proof.** First we replace the $U_i$ by a locally finite sub-cover, and we redefine $U_y$ accordingly. Since the new $U_y$ is contained in the old $U_i$, the fattened fibers $p^{-1}(U_y)$ still have radius strictly less than $w$. We pick a number $\varepsilon$ so that the girth of $p$ is less than $w - \varepsilon$.

We define a map $G$ from $Y$ to the set of convex sets in $M$. For each point $y$ in $Y$, $G(y)$ is defined to be the convex hull of the $\varepsilon$-neighborhood of $f(p^{-1}(U(y)))$. By the definition of girth, $p^{-1}(U(y))$ lies inside a ball of radius $w - \varepsilon$ in $X$. Since $f$ is contracting, $f(p^{-1}(U(y)))$ lies inside a ball of radius $w - \varepsilon$ in $M$, and its $\varepsilon$ neighborhood lies in a ball of radius $w$. Since each ball of radius $w$ in $M$ is convex, $G(y)$ is contained in a ball of radius $w$. The purpose of the $\varepsilon$ neighborhood is to ensure that $G(y)$ is open.

The main point of the proof is to construct a “section” of the map $G$: that is, a continuous map $g$ from $Y$ to $M$ so that $g(y)$ lies in $G(y)$ for each $y$. This step is a bit technical. I think the moral reason why we can find a section is just that each set $G(y)$ is contractible, but there are technicalities because the sets $G(y)$ may not vary continuously with $y$.

We divide $Y$ into simplices so small that each simplex lies in one of the $U_i$. Then we construct the map $g$ one skeleton at a time, so that at each stage it is continuous and $g(y)$ lies in $G(y)$. The map from the zero skeleton is trivial to construct. By
induction, it suffices to extend our section from the boundary of a simplex to its interior. We pick a simplex $\Delta$ and assume that $g$ is defined on its boundary so that $g(y)$ lies in $G(y)$ for each $y$ on the boundary. We choose a homeomorphism of $\Delta$ with the ball of radius 1. Since the simplex lies in one of the $U_i$, there is a point $m$ in $M$ which lies in $G(y)$ for every $y$ in $\Delta$. We define $g(y)$ to be $m$ for each $y$ in a ball of radius $1 - \delta$, for some small number $\delta$ which we will choose later. Then, we extend $g$ to the annulus $S^{k-1} \times (1 - \delta, 1)$ by mapping each ray $\theta \times (1 - \delta, 1)$ to the minimal geodesic between $g(\theta, 1)$ and $m$. Our extension $g$ is clearly continuous, and it only remains to check that $g(y)$ lies in $G(y)$ for every $y$.

The boundary of $\Delta$ is compact, so it meets only finitely many of the $U_i$. We number these $U_1, ..., U_n$. For any subset $i$ of the numbers from 1 to $n$, we define $U_i$ to be the set of points in the boundary of the simplex which lie in $U_i$ exactly when $i$ is a member of $I$. We define $G_I$ to be the convex hull of the epsilon-neighborhood of $f(p^{-1}(\cup_{i \in I} U_i))$. The condition that $g(y)$ lies in $G(y)$ means exactly that $g$ maps $U_i$ into $G_I$ for each $I$. Now the sets $U_i$ are not closed, but each point in the closure of $U_i$ lies in a $U_J$ where $J$ is a subset of $I$, and so we see that $g$ maps the closure of $U_i$ into $G_I$ for each $I$. Since the sets $G_I$ are open, $g$ maps a small neighborhood of the closure of each $U_i$ into $G_I$. Therefore, we may choose closed sets $V_i \subset U_i$, so that when we define $V_i$ as the set of points $y$ which lie in $V_i$ exactly when $i$ is a member of $I$, $g(y)$ lies in $G_I$ for each $y$ in $V_i$. This tiny improvement gives us the space we need to prove that $g(y)$ lies in $G(y)$. We define $W_i \subset S^{n-1}$ to be the intersection of $V_i$ with $S^{n-1}$, and we choose a number $\delta$ so small that $W_i \times [0, \delta]$ lies in $U_i$ for each $i$. For each point $\theta$ in $S^{n-1}$ and each $t$ in $(1 - \delta, 1)$, $V(\theta, 1)$ is a subset of $U(\theta, 1)$ and $g(\theta, 1)$ lies in $G(\theta, t)$. Since $m$ also lies in $G(\theta, t)$ for each $t$, we see that the minimal geodesic from $m$ to $g(\theta, 1)$ lies in $G(\theta, t)$, and therefore $g(y)$ lies in $G(y)$ for each $y$.

Both $f(x)$ and $g(p(x))$ lie in $G(p(x))$ which lies in a ball of radius $w$. Since the convexity radius of $M$ is at least $w$, $f(x)$ and $g(p(x))$ are joined by a unique minimal geodesic, and this minimal geodesic varies continuously with $x$. The family of geodesics, parametrized proportionally to length, is a homotopy from $f$ to $g \circ p$. This homotopy moves each point of $X$ less than $2w$. \hfill $\Box$

With this version of the extension lemma, we can prove estimates relating the Uryson 1-widths and hypersphericities of the surfaces $\Sigma_i$ to the geometry of $M$.

We prove that the Uryson 1-Width of each surface $\Sigma_i$ is at least the convexity radius of $M$. To see this, suppose that $\Sigma_i$ admitted a map $p$ to a 1-polyhedron $Y$ whose fibers had diameter less than the convexity radius of $M$. After changing the target, we can assume that $p$ is surjective. Therefore, the girth of $p$ is less than the convexity radius of $M$. The inclusion of $\Sigma_i$ in $M$ is a contracting map (because $\Sigma_i$ has the induced metric from $M$). Therefore, the extension lemma applies, proving that the inclusion of $\Sigma_i$ in $M$ factors through $p$ up to homotopy. But the map $p$ kills the fundamental homology class of $\Sigma_i$, and the inclusion of $\Sigma_i$ does not. Therefore, the Uryson 1-Width of $\Sigma_i$ is at least the convexity radius of $M$.

Next we bound the hypersphericity of $\Sigma_i$ for large values of $i$. Because the surfaces $\Sigma_i$ are converging to $M$ in the Gromov-Hausdorff metric, for large values of $i$, the inclusion of $\Sigma_i$ in $M$ has girth $\epsilon_i$ tending to 0. Now, by the extension lemma, any contracting map from $\Sigma_i$ to the sphere of radius $\epsilon_i$ factors up to homotopy through the inclusion of $\Sigma_i$ in $M$. If every map from $M$ to the 2-sphere kills the homology class $h$, then the hypersphericity of $\Sigma_i$ is less than $\epsilon_i$. For example, if $M$
is the complex projective plane and \( h \) is any non-zero homology class in \( H_2(M, \mathbb{Z}) \),
then every map from \( M \) to the 2-sphere kills \( h \).

If we take \( M \) to be the complex projective plane with its standard metric, these
two estimates prove Theorem 5.1.

The constructions that we have made for surfaces of high genus also apply to
manifolds of higher dimension. The Gromov-Hausdorff convergence is provided by
a theorem of Ferry and Okun from [5]. We state a weak version of their theorem,
which suffices for our purposes.

**Theorem** (Ferry and Okun). *Let \( A \) be a connected manifold of dimension at least
3 and \( B \) be a connected Riemannian manifold, and let \( f \) be a map from \( A \) to \( B \)
which is surjective on \( \pi_1 \). Then there is a sequence of metrics \( g_i \) on \( A \) and of maps
\( f_i \) from \( A \) to \( B \) homotopic to \( f \), so that \( (A, g_i) \) converges in the Gromov-Hausdorff
metric to \( B \) by the maps \( f_i \).

The construction is roughly as follows. First, homotope \( f \) so that the image
contains a very dense set of points in \( B \). Then, homotope \( f \) along various curves
joining the preimages of this dense set, so that each nearby pair of points in the
dense set is joined by a minimal geodesic lying in the image of \( f \). This last step
requires us to find curves in the domain which map to the right homotopy class in
the range, which requires surjectivity on \( \pi_1 \).

In particular, this theorem implies that the maps \( f_i \) have girth tending to 0 and
Lipschitz constant tending to 1. Using this theorem, we can construct metrics on
the 4-sphere with arbitrarily small hypersphericity and Uryson 3-width at least 1.
To see this, take \( A \) to be the 4-sphere, \( B \) to the quaternionic projective plane, and
\( f \) to be the hyperplane embedding. Applying the theorem of Ferry and Okun, we
get metrics \( g_i \) on \( A \) and maps \( f_i \) with girth tending to 0. Applying the estimates
just as for surfaces, we conclude that the Uryson 3-width of \((S^4, g_i)\) is at least the
convexity radius of the quaternionic projective plane and that its hypersphericity
tends to 0.

We will give two other sequences of surfaces with hypersphericity tending to 0
and Uryson 1-width at least 1. These examples are included to show that estimating
the hypersphericity of a surface is rather complicated. In each example, a different
piece of geometry or topology is used to rule out a non-zero degree contracting map
to a small sphere. In our first example, the obstruction came from homotopy theory.
In the second example, it will come from lattice theory. In the third example, it will
come from some considerations related to the theory of systoles. These examples
require more work to construct than our first example, but they also have slightly
stronger properties. For instance, we will construct surfaces \( \Sigma_i \) so that the smallest
Lipschitz constant of a homotopically non-trivial map to \( CP^\infty \) tends to infinity while
the Uryson 1-Width remains bounded below.

The estimates in these examples depend on a modification of the extension
lemma, giving a Lipschitz extension of a Lipschitz function. In order to state the
modified lemma, we make some definitions. We say that a map \( p \) is dense on
the scale \( S \) if the image of \( p \) meets every ball of radius \( S \). We say that a map
\( p \) is expanding on the scale \( S \) if, for any two points \( x_1 \) and \( x_2 \) in the domain,
\( \text{dist}(p(x_1), p(x_2)) > \text{dist}(x_1, x_2) - S \). For points in the domain closer together than
\( S \), this inequality is vacuous, but for points in the domain much farther apart than
\( S \), the map \( p \) is nearly expanding. We say that a Riemannian manifold has bounded
geometry on a scale \( S \) if its sectional curvature is pinched between \( 1/S^2 \) and \(-1/S^2 \),
and if its injectivity radius is at least $\pi S$. For instance, the unit sphere has bounded geometry on the scale 1.

**Lemma 5.3.** Let $X$ and $Y$ be locally compact path metric spaces. Let $p$ be a map from $X$ to $Y$ which is dense on the scale $W$ and expanding on the scale $W$, for some number $W$ less than $1/200$. Let $M$ be a complete Riemannian $n$-manifold with bounded geometry on the scale 1. Let $f$ be a contracting map from $X$ to $M$. Then there is a map $h$ from $Y$ to $M$, with Lipshitz constant $50n$, so that $f$ is homotopic to $h \circ p$.

**Proof.** First we prove that the girth of $p$ is bounded by $4W$. To see this, we let $U_i$ be the cover of $Y$ by $W$-balls. The set $U_y$ is defined as the union of all $W$-balls meeting $y$, which is the $2W$-ball centered at $y$. Since the image of $p$ is $W$-dense, the set $U_y$ is contained in a $3W$-ball around a point in the image of $p$, say $p(x)$. Since $p$ is expanding on the scale $W$, the preimage of $U_y$ is contained in the $4W$ ball around $x$.

Because the girth of $p$ is bounded by $4W$, which is less than $1/50$, and the convexity radius of $M$ is at least $\pi/2$, we can apply Lemma 5.2, which constructs a map $g$ from $Y$ to $M$ obeying the estimate $\text{dist}(f(x), g(p(x))) < 8W$.

We will prove that the map $g$ which we constructed in Lemma 5.2 obeys a quasi-Lipshitz inequality, bounding the distance between $g(y_1)$ and $g(y_2)$ by $\text{dist}(y_1, y_2) + 19W$. Since $p$ is dense on the scale $W$, there are points $p(x_1)$ and $p(x_2)$ within $W$ of $y_1$ and $y_2$, respectively. Recall from the proof of Lemma 5.2 that the set $G(y)$ is the convex hull of a small neighborhood of $f(p^{-1}(U_y))$, and that $g(y)$ lies in $G(y)$. Since $g(y_1) \in U_{y_1}$, the point $f(x_1)$ lies in $G(y_1)$. Since the radius of $G(y)$ is less than $4W$, the distance between $f(x_1)$ and $g(y_1)$ is less than $8W$. The distance from $g(y_1)$ to $g(y_2)$ is less than $\text{dist}(g(y_1), f(x_1)) + \text{dist}(f(x_1), f(x_2)) + \text{dist}(f(x_2), g(y_2))$, which is less than $\text{dist}(f(x_1), f(x_2)) + 16W$. Since $f$ is contracting, the distance from $f(x_1)$ to $f(x_2)$ is less than the distance from $x_1$ to $x_2$. Since $p$ is expanding on the scale $W$, this distance is less than $\text{dist}(p(x_1), p(x_2)) + W$. But the point $p(x_1)$ lies within $W$ of $y_1$, and the point $p(x_2)$ lies within $W$ of $y_2$. Putting all these estimates together, we get the quasi-Lipshitz inequality.

We now explain how to build a Lipshitz map $h$ on the scaffold of the quasi-Lipshitz map $g$. This construction is adapted from the paper [11] of Lang, Pavlovic, and Schroeder, in which they construct Lipshitz extensions of Lipshitz maps into negatively curved spaces. For each $y$ in $Y$, we define a set $H(y)$ in $M$ as the intersection of the closed balls $B(g(y'), 2\text{dist}(y, y') + 20W)$ for every $y'$ in $Y$. Because of our quasi-Lipshitz inequality, the ball of radius $W$ around $g(y)$ lies in $H(y)$, and so $H(y)$ is not empty. On the other hand, $H(y)$ lies within the ball of radius $20W$ around $g(y)$, a ball of radius less than $1/10$. We can think of $H(y)$ as showing the rough location of $h(y)$. Notice that if the distance from $y$ to $y'$ is at least $20W$, then the ball $B(g(y'), 2\text{dist}(y, y') + 20W)$ contains the ball around $g(y)$ of radius $20W$, which in turn contains $H(y)$. Therefore, $H(y)$ is an intersection of balls of radius at most $60W$. Because of the bounded geometry of $M$, each of these balls is convex, and so $H(y)$ is a convex set.

Unlike the sets $G(y)$, the sets $H(y)$ vary nicely with $y$. It is easy to see that $H(y)$ varies continuously with $y$. Because each set $H(y)$ contains a ball of radius $W$, $H(y)$ is a convex set with interior. We define the function $d(m)$ to be the infimum of the expression $2\text{dist}(y, y') + 20W - \text{dist}(m, g(y'))$ as $y'$ varies over $Y$. The function $d(m)$ is continuous. It is positive on the interior of $H(y)$, zero on the boundary, and negative on the complement of $H(y)$. The set $H(y_0)$ contains the set where $d(m)$
is greater than $2\text{dist}(y, y_0)$ and is contained in the set where $d(m)$ is greater than $-2\text{dist}(y, y_0)$. Since $d$ is continuous, we see that $H(y)$ varies continuously with $y$, using the Hausdorff topology on closed subsets of $M$. We will show something much stronger, namely that the Hausdorff distance between $H(y)$ and $H(y')$ is bounded by $5\text{dist}(y, y')$. This proof is very closely modeled on a similar proof in [11], in a slightly different situation. Let $m'$ be a point in $H(y')$ and let $m$ be the closest point to $m'$ in $H(y)$. We will bound the distance from $m$ to $m'$ by $5\text{dist}(y, y')$. Since the argument applies to each point $m'$ in $H(y')$ and by symmetry to each point in $H(y)$, this estimate will bound the Hausdorff distance between $H(y)$ and $H(y')$ by $5\text{dist}(y, y')$.

Since $m$ lies on the edge of $H(y)$, it must lie on the edge of some of the balls defining $H(y)$. We will call a point $y_1$ in $Y$ taut if $m$ lies on the edge of $B(g(y_1), 2\text{dist}(y_1, y) + 20W)$. If $y_1$ is a taut point, then we will call the ray from $m$ to $g(y_1)$ a taut ray. We proved above that if the distance from $y_1$ to $y$ is more than $20W$, then $H(y)$ lies inside $B(g(y_1), 2\text{dist}(y_1, y) + 20W)$. Therefore, every taut point lies within $20W$ of $y$, and every taut ray has length less than $60W$, which is less than $1/3$. We will prove that the angle between any two taut rays is no more than $(3/4)\pi$.

Let $y_1$ and $y_2$ be two taut points. We define the vectors $a$ and $b$ to be the inverse images of $g(y_1)$ and $g(y_2)$ under the exponential map of $M$ at $m$. The absolute value of $a$ equals the distance from $m$ to $g(y_1)$, which equals $2\text{dist}(y_1, y) + 20W$ by the definition of a taut ray. Similarly, the absolute value of $b$ equals the distance from $m$ to $g(y_2)$, which equals $2\text{dist}(y_2, y) + 20W$. By the quasi-lipshitz estimate for $g$, the distance from $g(y_1)$ to $g(y_2)$ is less than $\text{dist}(y_1, y_2) + 20$, which is less than half of $|a| + |b|$. Since taut rays have length less than $1/3$, $g(y_1)$ and $g(y_2)$ lie inside the ball of radius $1/3$ around $m$, and since this ball is convex, the minimal geodesic between them lies in the ball as well. By the Rauch comparison theorem, within the ball of radius $1/3$, the exponential map does not contract lengths by more than the factor $\frac{1/3}{\sin(1/3)}$. Therefore, the distance from $a$ to $b$ is no more than $\frac{1/3}{\sin(1/3)}$ times the distance from $g(y_1)$ to $g(y_2)$, and $|a - b| < \frac{54/53}{(27/53)(|a| + |b|)}$. By trigonometry, the angle between $a$ and $b$ is less than $(3/4)\pi$.

Because $m$ is the closest point to $m'$ in $H(y)$, the taut rays from $m$ must contain a ray pointing directly away from $m'$ in their convex hull. Since the taut rays cluster together, none of them can lie too far from this ray pointing directly away from $m'$. In particular, the angle between a taut ray and the ray from $m$ to $m'$ must be greater than $(3/4)\pi$.

It suffices to prove our Lipshitz inequality for $y$ very close to $y'$, and since $H(y)$ varies continuously with $y$, we can assume that $m$ is very close to $m'$. Let $y_1$ be a taut point in $Y$. Because $m'$ is very close to $m$, the distance between them is bounded by $(1/\cos(3/4)\pi)(\text{dist}(g(y_1), m') - \text{dist}(g(y_1), m))$. By the definition of a taut point, $\text{dist}(g(y_1), m) = 2\text{dist}(y_1, y) + 20W$. On the other hand, since $m'$ lies in $H(y')$, $\text{dist}(g(y_1), m') \leq 2\text{dist}(y_1, y') + 20W$. Therefore, the distance from $m$ to $m'$ is less than $2(2/\cos(3/4)\pi)\text{dist}(y_1, y')$, which is less than $5\text{dist}(y_1, y')$.

We now define $h(y)$ to be the center of mass of the $1/10$-neighborhood of $H(y)$. Centers of mass for measures in Riemannian manifolds are explained in section 4 of [11]. We prove some estimates very closely following theirs, but in a slightly different situation. First, we define center of mass. For a probability measure $\mu$ on a Riemannian manifold, we define a function $D_\mu(p) = \int \text{dist}(p, q)^2 d\mu(q)$. If the
manifold has sectional curvature less than 1 and injectivity radius more than \( \pi \), and if the distance from \( p \) to \( q \) is less than \( 2/5 \), then the function \( d(\cdot, q)^2 \) is strictly convex at \( p \), with Hessian greater than 1. Therefore, if \( p \) lies in a ball of radius \( 1/5 \) and \( \mu \) is supported in the same ball, \( D_\mu \) is also strictly convex at \( p \) with Hessian greater than 1. Since the ball of radius \( 1/5 \) is convex with respect to minimal geodesics, the gradient of the function \( D_\mu \) points out at every point on the edge of the ball. Therefore, the function \( D_\mu \) has a unique minimum in the ball of radius \( 1/5 \). The point where this minimum is attained is called the center of mass of \( \mu \), and we will denote it \( c_\mu \).

Because the Hessian of \( D_\mu \) is greater than 1 on the ball of radius \( 1/5 \), \( D_\mu(p) > D_\mu(c_\mu) + (1/2)dist(p, c_\mu)^2 \) for each \( p \) in the ball. Using this estimate, we can bound the distance between the centers of mass of two measures, supported on the same ball of radius \( 1/5 \).

\[
\text{dist}(c_\mu, c_{\mu'})^2 < D_\mu(c_{\mu'}) - D_\mu(c_\mu) + D_{\mu'}(c_\mu) - D_{\mu'}(c_{\mu'}). 
\]

Plugging in the definition of \( D_\mu \) we get the following expression.

\[
= \int (\text{dist}(c_\mu, q)^2 - \text{dist}(c_{\mu'}, q)^2)(d\mu' - d\mu).
\]

Factoring the difference of squares, we can bound this expression.

\[
< (4/5)\text{dist}(c_\mu, c_{\mu'}) \int |d\mu' - d\mu|.
\]

Canceling one factor of \( \text{dist}(c_\mu, c_{\mu'}) \) from each side leaves us with the following bound.

\[
\text{dist}(c_\mu, c_{\mu'}) < (4/5) \int |d\mu' - d\mu|.
\]

The probability measures we will use are the renormalized volume measures of the 1/10 neighborhood of \( H(y) \), which we call \( \mu(y) \). If \( y' \) is sufficiently close to \( y \), then \( \mu(y) \) and \( \mu(y') \) will be supported on a ball of radius \( 1/5 \). Again, if they are sufficiently close, then \( \int |d\mu(y') - d\mu(y)| \) will be bounded by the surface area to volume ratio of the 1/10 neighborhood of \( H(y) \) times the Hausdorff distance between the 1/10 neighborhood of \( H(y) \) and the 1/10 neighborhood of \( H(y') \). The surface area to volume ratio of a 1/10 neighborhood of any set in an n-manifold with \( \text{Ric} > -(n-1) \) is at most that of the 1/10 ball in hyperbolic n-space, which is less than 11n. Finally, the Hausdorff distance between the 1/10 neighborhoods of \( H(y) \) and \( H(y') \) is no more than the Hausdorff distance between \( H(y) \) and \( H(y') \), which is less than 5\( \text{dist}(y, y') \). Putting the bounds together, we see that the Lipshitz constant of \( h \) is less than \( 44n \).

We define the hypersphericity of a homology class \( h \) in \( H_2(M) \) for a Riemannian manifold \( (M, g) \) as the supremal \( R \) so that there is a contracting map from \( (M, g) \) to the 2-sphere of radius \( R \) which does not kill the homology class \( h \). If \( M \) is a closed oriented surface and \( h \) is the fundamental class of \( M \), then we recover the usual definition of hypersphericity. By Lemma 5.3, if the hypersphericity of \( h \) in \( M \) is less than \( \epsilon \), then for sufficiently large \( i \), the hypersphericity of \( \Sigma_i \) is less than \( 100\epsilon \).
We will now construct some sequences of higher dimensional manifolds \((M_n, g_n)\) with bounded geometry on the scale 1 and with hypersphericity of \(h_n\) in \(M_n\) tending to 0, where \(h_n\) is non-zero and can be realized by an oriented surface. For each manifold \(M_n\), we have constructed a sequence of surfaces \(\Sigma_i\) in the homology class \(h_n\) converging to \((M_n, g_n)\). By Gromov’s extension lemma, we know that the Uryson 1-Width of \(\Sigma_i\) is greater than 1 for every \(i\). On the other hand, by the Lipshitz extension lemma, we know that for sufficiently large \(n\) and sufficiently large \(i\), the hypersphericity of \(\Sigma_i\) is as small as we like.

Our first examples of manifolds with bounded geometry carrying homology classes of small hypersphericity are high-dimensional flat tori. Minkowski discovered that in high dimensions a random flat torus behaves differently from a rectangular torus. A rectangular torus of volume 1 has injectivity radius at most a half, but a random flat \(N\)-torus of volume 1 has injectivity radius on the order of \(\sqrt{N}\). (See the Minkowski-Hlawka theorem in a book on the geometry of numbers, such as [2].)

One reason for the difference in behavior is that in high dimensions a ball of volume 1 has radius on the order of \(\sqrt{c}\) for some constant \(c\), and the volume of the unit \(n\)-sphere is greater than 1 for every \(i\). Pick \(\omega\) such that \(\omega \geq 1\) for every \(i\). On the other hand, by the Lipshitz extension lemma, we know that for sufficiently large \(n\) and sufficiently large \(i\), the hypersphericity of \(\Sigma_i\) is as small as we like.

Let \(T\) be a flat torus with volume 1 and injectivity radius on the order of \(\sqrt{N}\). Any homologically non-trivial curve in \(T\) has length on the order of \(1/\sqrt{N}\). Therefore \(F\) is not surjective and has degree 0.

Next, we extend our lemma to homology classes of dimension 2 or higher.

**Lemma 5.4.** Let \(T\) be a flat \(N\)-torus with injectivity radius 1 and volume less than \((c\sqrt{N})^{-N}\). Then \(T\) has a one dimensional homology class \(h\), so that any map to the unit circle which does not kill \(h\) has Lipshitz constant at least \(2\pi c\sqrt{N}\).

**Proof.** Suppose the conclusion does not hold. Let \(h_1\) be any 1-dimensional homology class, and let \(f_1\) be a map to \(S^1\) not killing \(h_1\), with Lipshitz constant less than \(2\pi c\sqrt{N}\). Then let \(h_2\) be a one dimensional homology class killed by \(f_1\), and let \(f_2\) be a map to \(S^1\) not killing \(h_2\), with Lipshitz constant less than \(2\pi c\sqrt{N}\). Let \(h_3\) be a 1-dimensional homology class killed by \(f_1\) and \(f_2\), and let \(f_3\) be a map to \(S^1\) not killing \(h_3\), with Lipshitz constant less than \(2\pi c\sqrt{N}\). Continuing in this way, we produce a map \(F\) of non-zero degree from \(T\) to the direct product of \(N\) circles. Since each coordinate of \(F\) has Lipshitz constant less than \(2\pi c\sqrt{N}\), the volume dilation of \(F\) is less than \((2\pi)^N(c\sqrt{N})^N\), and the volume of the image of \(F\) is less than \((2\pi)^N\). Therefore \(F\) is not surjective and has degree 0.

Next, we extend our lemma to homology classes of dimension 2 or higher.

**Lemma 5.5.** Let \(T_0\) be a flat \(N\)-torus with injectivity radius 1 and volume less than \((c\sqrt{N})^{-N}\), let \(T_1\) be the unit cube \((n-1)\)-torus, and let \(T\) be their Cartesian product. Then \(T\) has an \(n\)-dimensional homology class \(h\) so that any map from \(T\) to the unit \(n\)-sphere which does not kill \(h\) has \(n\)-dilation at least \(\omega_n c\sqrt{N}\), where \(\omega_n\) is the volume of the unit \(n\)-sphere. In particular, the map has Lipshitz constant at least \((\omega_n c\sqrt{N})^{1/n}\).

**Proof.** Again, we suppose the conclusion is false. Let \(a\) be the fundamental homology class of \(T_1\) in \(T\). Let \(h_1\) be a one dimensional homology class in \(T_0\), and let \(f_1\) be a map from \(T\) to the unit \(n\)-sphere not killing \(h_1 \times a\), with \(n\)-dilation less than \(\omega_n c\sqrt{N}\). Pick \(h_2\), a one-dimensional homology class in \(T_0\), so that \(f_1\) kills \(h_2 \times a\), and let \(f_2\) be a map from \(T\) to the unit \(n\)-sphere not killing \(h_2 \times a\), with \(n\)-dilation
less than $\omega_n c \sqrt{N}$. Proceeding in this way, construct $f_i$ for $i$ from 1 to $N$. Now let $T'$ be $T_0 \times T_1^N$, and let $\pi_i$ be the projection of $T'$ onto a copy of $T$ provided by the product of $T_0$ with the $i$th copy of $T_1$. Now we define a map $F$ from $T'$ to the Cartesian product of $N$ unit $n$-spheres. The map to the $i$th unit $n$-sphere is given by $f_i \circ \pi_i$. It is easy to check that $F$ has non-zero degree. On the other hand, since each component of $F$ has $n$-dilation less than $\omega_n c \sqrt{N}$, the volume dilation of $F$ is less than $(\omega_n)^N (c \sqrt{N})^N$, and the volume of the image of $F$ is less than $\omega_n^N$. Therefore $F$ is not surjective and has degree zero. This contradiction finishes the proof of the lemma.

Taking $n$ to be 2, Lemma 5.5 gives us a sequence of flat tori $T^N$ with injectivity radius 1 and 2-dimensional homology classes $h_N$, with the hypersphericity of $h_N$ in $T^N$ tending to 0. Taking surfaces $\Sigma_i$ in the class $h_N$ converging to $T^N$, we get more examples of surfaces with Uryson 1-Width at least 1 and hypersphericity tending to 0.

Arguments using maps of small girth and homotopy theoretic obstructions cannot prove the small hypersphericity of these surfaces $\Sigma_i$. The reason is that, by the first extension lemma, any map of small girth from $\Sigma_i$ to $X$ extends to a map from $X$ to $T^N$ homotopic to the inclusion of $\Sigma_i$ in $T^N$. Since there is a continuous map from $T^N$ to $S^2$ not killing the homology class of $\Sigma_i$, we see that there is a continuous map from $X$ to $S^2$ not killing the homology class of $\Sigma_i$.

Slightly stronger results are available using the Conway-Thomson lattices. Conway and Thomson proved that there are self-dual lattices in $\mathbb{R}^N$ with shortest vector on the order of $\sqrt{N}$. (The theorem of Conway and Thomson is given as Theorem 9.5 in [13].) If we take $T(N)$ to be the quotient of $\mathbb{R}^N$ by a Conway-Thomson lattice, we get a torus with systole on the order of $\sqrt{N}$, and with the property that every homotopically non-trivial map to the unit circle has Lipschitz constant at least on the order of $\sqrt{N}$. This property shows that the Conway-Thomson tori obey a stronger version of Lemma 5.2: the conclusion of the lemma holds for every homology class $h$ in $H_1(T(N))$.

Using the Conway-Thomson tori, we can produce metrics on high genus surfaces with arbitrarily large homology systole and no homotopically non-trivial contracting map to the unit circle. In particular, such a surface admits no degree 1 contracting map to the unit square torus. (Recall that, by Theorem 4.1, any surface with homology systole at least $4\pi$ admits a degree 1 contracting map to the unit 2-sphere.) We take a surface $\Sigma$ of genus $G$ embedded in a Conway-Thomson torus $T(2G)$ as its Jacobian torus. The first homology group of $\Sigma$ is identified with the first homology group of $T(2G)$. We isotope $\Sigma$ so that, for many homology classes $h_i$ in the first homology group of $\Sigma$, there is a closed loop in the class $h_i$ which lies very near to the straight circle in $T^N$ in the corresponding homology class. We put the induced metric on $\Sigma$. It is not hard to check that $\Sigma$ has (homology) systole on the order of $\sqrt{N}$, and that every homotopically non-trivial map from $\Sigma$ to the unit circle has Lipschitz constant on the order of $\sqrt{N}$. Taking $N$ large proves our claim.

The Conway-Thomson lattices were suggested to me by Mikhail Katz, who was the first to apply them to systolic problems in his paper [10].

We now turn to our last example of a manifold of bounded geometry carrying a homology class of small hypersphericity. This example is based on some considerations arising in the theory of systoles. Let us define the length of an integral homology class to be the smallest length of a union of curves realizing that homology
class. On manifolds of dimension at least 3, it can happen that the norm of a class \(\alpha\) is arbitrarily large and at the same time the norm of the class 100\(\alpha\) is arbitrarily small. We give a simple construction of a Riemannian manifold demonstrating this phenomenon.

Begin with the direct product \(S^1 \times S^2\), with the standard metric. Let \(T\) be a small tubular neighborhood of an embedded curve homologous to \(100[S^1]\). On the interior of \(T\), we modify the metric by a conformal factor of \(\epsilon\), and on the exterior of \(T\) we modify the metric by a conformal factor of \(R\), where \(R\) is some enormous number depending on \(\epsilon\). (Near the edge of \(T\), we can smooth the jump so that the Riemannian metric remains smooth.) The length of 100\(\epsilon\) is around \(200\pi\). By taking \(R\) sufficiently large, we can make the length of \([S^1]\) as large as we like.

This phenomenon can occur on manifolds with bounded geometry as well. We will construct a metric \(g_N\) on \(S^1 \times S^2\) with bounded geometry on the scale 1, and with the property that the homology class \(N[S^1]\) has length only \(2\pi\). We let \(T\) be a small tubular neighborhood of an embedded curve homologous to \(N[S^1]\). We put a Riemannian metric on \(S^1 \times S^2\) which, restricted to \(T\), is just the product metric of the disk of radius 1 with the circle of radius 1. Then we rescale this metric by a large factor \(R\), so that it has bounded geometry on the scale 1. Next, we will change the metric on the interior of \(T\) so that the homology class \(N[S^1]\) is realized by a curve of length \(2\pi\). We choose coordinates on \(T\) so that the disk of radius \(R\) has standard polar coordinates \(r\) and \(\theta\), and the circle of radius \(R\) has a coordinate \(z\) varying from 0 to \(2\pi\). Our rescaled metric on \(T\) is given in these coordinates as \(g = dr^2 + r^2d\theta^2 + R^2dz^2\). We will replace this metric by a metric \(g_N = dr^2 + r^2d\theta^2 + f(r)^2dz^2\), where \(f(r) = R\) for \(R/2 \leq r \leq R\), and where \(f(r) = 1\) for \(0 \leq r \leq 1\). By a routine computation, it follows that the curvature of this metric is bounded by 1 as long as the first and second derivatives of \(\log f\) are bounded by \(1/2\). For large values of \(R\), we can easily find a (non-strictly) monotonic function \(f\), satisfying the equations above and with the first and second derivatives of \(\log f\) as small as we like. Finally, since \(f\) is monotonic, any curve in \(T\) contracts to a curve which runs along the core circle defined by the equation \(r = 0\). Since this core circle has length \(2\pi\), we see that every closed geodesic in \((T, g_N)\) has length at least \(2\pi\). Given that the curvature of \(g_n\) is bounded by 1, the bound on the lengths of geodesics proves that the part of \((T, g_N)\) a distance at least \(\pi\) from the boundary of \(T\) has injectivity radius at least \(\pi\). On the other hand, \(g_N\) has injectivity radius at least \(\pi\) on the rest of \(S^1 \times S^2\), where it is equal to our original rescaled metric. Therefore, \(g_N\) has bounded geometry on the scale 1, while \(N[S^1]\) is realized by a circle of length \(2\pi\).

Any homotopically non-trivial map from \((S^1 \times S^2, g_N)\) to the unit circle must have Lipshitz constant at least \(N\). Therefore, the hypersphericity of the class \([S^1]\) is at most \(1/N\). If we take the product of this Riemannian manifold with itself, we get a metric on \((S^1 \times S^1 \times S^2 \times S^2)\) with bounded geometry on the scale 1, and the hypersphericity of the homology class \([S^1 \times S^1]\) is bounded by \(\sqrt{\pi}/N\), because \((S^1 \times S^1 \times S^2 \times S^2, g_N \times g_N)\) contains a torus of area \(4\pi^2\) in the homology class \(N^2[S^1 \times S^1]\). Any map to a sphere which does not kill \([S^1 \times S^1]\) induces a map from this torus of degree at least \(N^2\). If the map is contracting, then the area of the image sphere is bounded by \(4\pi^2/N^2\), and so the hypersphericity of the class \([S^1 \times S^1]\) is bounded by \(\sqrt{\pi}/N\). As in previous examples, we can construct a sequence of surfaces \(\Sigma_n\) in \((S^1 \times S^1 \times S^2 \times S^2)\), in the homology class \([S^1 \times S^1]\),
and converging in the Gromov-Hausdorff sense to \((S^1 \times S^1 \times S^2 \times S^2, g_N \times g_N)\). Using Lemma 5.3, we see that these surfaces have Uryson 1-Width at least 1 and arbitrarily small hypersphericity.

Finally, I would like to mention that the Lipshitz constant of a non-zero degree map from \(\Sigma_i\) to \(CP^\infty\) is arbitrarily large. Unfortunately, Lemma 5.3 does not provide such a bound, because the dimension of \(CP^\infty\) is infinite. We can still prove the bound by an ad hoc trick. First triangulate the manifold \((S^1 \times S^1 \times S^2 \times S^2, g_N \times g_N)\) into very small, approximately Euclidean simplices. Say that the simplices are all larger than some small number \(\delta\). Then take a surface \(\Sigma_i\) which is Gromov-Hausdorff \(\delta\)-close to the total space, for some \(\delta\) much smaller than \(\epsilon\). Let \(f\) be a Lipshitz map from \(\Sigma_i\) to \(CP^\infty\), with Lipshitz constant \(L\). By Lemma 5.2, we get a map from the 0-skeleton of the triangulation to \(CP^\infty\) with Lipshitz constant very close to 1. We then extend this map to all of \((S^1 \times S^1 \times S^2 \times S^2, g_N \times g_N)\) by straightening each simplex. It is not hard to check that we get a map with Lipshitz constant less than \(cL\), where \(c\) is a constant depending on the dimension of our domain manifold (which is 6), but not depending on the metric \(g_N\) (because each extension occurs on an approximately Euclidean simplex). Suppose that \(f\) has degree non-zero. Then, we restrict the extension of \(f\) to the small area torus in the homology class \(N^2[S^1 \times S^1]\). The restriction is a \(cL\)-Lipshitz map from a torus of area \(4\pi^2\) to \(CP^\infty\) with degree at least \(N^2\). Therefore, \(L\) is greater than a multiple of \(N\), which proves our claim.

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