LOCAL GALOIS REPRESENTATIONS AND COHOMOLOGY

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ABSTRACT. The aim of this article is to investigate a curious dichotomy between the condition on the local Galois representations and the appearance of the corresponding p-adic local Langlands in the supersingular or the ordinary parts of the cohomologies. We give a positive answer to a question raised by Chojecki [Cho18].

1. INTRODUCTION

Fix a prime $p$ and $E$ a finite extension of $\mathbb{Q}_p$. Let $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(E)$ be a global Galois representation (continuous, irreducible representation that is unramified outside a finite set of primes) with $\rho_p: G_p \to \text{GL}_2(E)$ local Galois representation obtained by the restriction of the Galois representation to the decomposition group $G_p := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ satisfying the hypothesis of Fontaine-Mazur conjecture (i.e $\rho$ is odd and $\rho_p$ potentially semi-stable). By [SW99] (see also [Pan22]), $\rho$ comes from elliptic modular form $f$. The aim of this article is to investigate under what condition of $\rho_p$, they appear in the cohomologies of the supersingular and ordinary parts of the modular curves.

For $F = \mathbb{Q}$ and $\rho_p$ absolutely irreducible, a similar theorem was proved by Chojecki in [Cho15, Theorem 6.3] for mod $p$ situation and [Cho18] for $p$-adic situation. In [Cho18, p. 469], Chojecki asked if the above theorem can be generalized to the situation when $\rho_p$ is reducible, non-split. Our theorem answers the question raised by Chojecki. For totally real fields, we generalize Chojecki’s theorem [BR22] in the mod $p$ situation again under the assumption that $\rho_p$ is absolutely irreducible. Our theorem substantiates that ordinary representations will appear in the ordinary part of the cohomology similar to Chojecki whose theorem tells that supersingular representations appear in the supersingular part of the cohomology.

Recall that for any schemes $X$ and any ring $A$, $\mathcal{H}^1(X)_A := \Pi^1_{\text{et}}(X, A)$ and $\mathcal{H}^1_{\text{ord}}$ denotes the cohomology of the ordinary part of the modular curve (see §2.4). We also denote by $\mathcal{H}^1_{\text{ord}}$, the completed cohomology of the ordinary locus. Let $B(\rho_p)$ be the automorphic representation.

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associated by the \( p \)-adic Langlands correspondences to \( \rho_p \) (cf \( \S 3 \)). Let \( \epsilon \) be the \( p \)-adic cyclotomic character and \( \eta_1, \eta_2 \) be two unitary characters of \( \mathbb{Q}_p^\times \) with \( \mathcal{B}_{st} \) the semi-stable period ring of Fontaine. Note that this period ring \( \mathcal{B}_{st} \) contains \( \mathbb{C}_p := \widehat{\mathbb{Q}}_p \).

**Theorem 1.1.** Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(E) \) be a global Galois representation as above and \( \rho_p \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2\epsilon^{-1} \end{pmatrix} \) with \( \eta_1 \cdot \eta_2^{-1} \neq \epsilon^{\pm 1} \) be a potentially crystalline reducible non-split local \( p \)-adic Galois representation with distinct Hodge-Tate between \((0, k - 1)\) and \( B(\rho_p) \) be the automorphic representation associated by the \( p \)-adic local Langlands correspondence. Then, there exists a principal series sub representation \( \pi_1 \) of \( B(\rho_p) \) such that

\[
\rho_p \otimes_{\mathbb{C}_p} \pi_1 \subset \hat{H}^1_{\text{B}_{st},\text{ord}}.
\]

We also have the inclusion:

\[
\rho \subset \text{Hom}_G(B(\rho_p), \hat{H}^1_{\text{B}_{st},\text{ord}}).
\]

Here, we take the principal series representations in the category of locally analytic representations. Note that the representation \( H^1_{\text{ord}} \) is smooth but not admissible as \( \text{GL}_2(\mathbb{Q}_p) \) representation. We prove the theorem in \( \S 4 \) and the main ingredient is a recent result due to Colmez-Niziol-Dospinescu [CDN20, Theorem 5.8].

2. **MODULAR CURVES AND DRINFELD TOWERS**

### 2.1. Modular curves of infinite level.

Recall some notation about modular curve at infinity following [Cho18, p. 460]. Let \( \mathbb{A}_f \) be the set of all finite adeles over \( \mathbb{Q} \). Let \( W_p \) (respectively \( WD_p \)) be the Weil group (respectively the Weil-Deligne group). For a compact open subgroup \( K = K_p \times K^p \subset \text{GL}_2(\mathbb{A}_f) \), denote by

\[
Y(K) := \text{GL}_2(\mathbb{Q}) \setminus (\mathbb{C} - \mathbb{R}) \times \text{GL}_2(\mathbb{A}_F)/K.
\]

This is a canonical model defined over \( \mathbb{Q} \). Following Scholze, we consider these curves as adic spaces over \( \text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p}) \). Consider the compactification \( X(K) \) of \( Y(K) \) as an adic space over \( \text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p}) \). Recall the notion of equivalence \( \sim \) following [SW13, Definition 2.4.1]. For sufficiently small compact prime to \( p \) level \( K^p \subset \text{GL}_2(\mathbb{A}_f) \), by now famous theorem due to Scholze [Sch15, Theorem III.1.2] there exist adic space \( Y(K^p) \) and \( X(K^p) \) such that

\[
Y(K^p) \sim \lim_{K_p} Y(K_p K^p); X(K^p) \sim \lim_{K_p} X(K_p K^p).
\]

Now, we have a notion of supersingular \( Y(K)^{ss} \) and ordinary part \( Y(K)^{ord} \) (respectively \( X(K)^{ss} \) and \( X(K)^{ord} \)) on the special fiber of these modular curves (respectively on the compatified curves).
First assume $K_p = \text{GL}_2(\mathbb{Z}_p)$, we define the ordinary (respectively supersingular) part of the modular surface $Y(\text{GL}_2(\mathbb{Z}_p)K^p)_{\text{ord}}$ (respectively $Y(\text{GL}_2(\mathbb{Z}_p)K^p)^{\text{ss}}$) to be the inverse image of the ordinary (respectively supersingular) part of the special fiber of $Y(\text{GL}_2(\mathbb{Z}_p)K^p)$.

By [SW13, Proposition 2.4.3], there exist adic spaces $Y^{ss}$, $Y^{\text{ord}}$ and $X^{ss}$, $X^{\text{ord}}$ over $\text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p})$ such that

$$Y^{ss} \sim \lim_{\leftarrow K_p} Y(K_p)_{K_p}^{\text{ss}}, X(K_p)^{ss} \sim \lim_{\leftarrow K_p} X(K_p)_{K_p}^{\text{ss}}, Y^{\text{ord}} \sim \lim_{\leftarrow K_p} Y(K_p)_{K_p}^{\text{ord}}, X(K_p)^{\text{ord}} \sim \lim_{\leftarrow K_p} X(K_p)_{K_p}^{\text{ord}}.$$ 

For an arbitrary compact open $K_p \subset \text{GL}_2(\mathbb{Z}_p)$, we define $Y(K_p)^{\text{ord}}$ (respectively $Y(K_p)^{\text{ss}}$) as the pullback of $Y(\text{GL}_2(\mathbb{Z}_p)K^p)_{\text{ord}}$ (respectively $Y(\text{GL}_2(\mathbb{Z}_p)K^p)^{\text{ss}}$).

For a fixed tame level $K^p$, write $Y = Y(K^p)$ and $X = X(K^p)$. Let $\mathbb{P}^{1, \text{ad}}$ be the adic projective space of dimension 1. Recall that Scholze [Sch15, p. 1012, Theorem 3.3.18] defined a $\text{GL}_2(\mathbb{Q}_p)$-equivariant Hodge Tate period map

$$\pi_{HT} : X \to \mathbb{P}^{1, \text{ad}}$$

and $X^{\text{ord}} = \pi_{HT}^{-1}(\mathbb{P}^{1, \text{ad}})$ and $Y^{ss} = \pi_{HT}^{-1}(\mathbb{P}^{1, \text{ad}} - \mathbb{P}^{1}(\mathbb{Q}_p))$.

Following [SW13, Chapter 6], denote by $L\text{T}_{K^p}$ the Lubin-Tate space for $\text{GL}_2(\mathbb{Q}_p)$ at the level $K_p$ with $K_p$ a compact open subgroup of $\text{GL}_2(\mathbb{Q}_p)$. These are the local analogues of the global objects like modular curves. By [SW13, Theorem 6.3.4], there exists a perfectoid space $L\text{T}_{\infty}$ over $\text{Spa}(\mathbb{C}_p, O_{\mathbb{C}_p})$ such that

$$L\text{T}_{\infty} \sim \lim_{\leftarrow K_p} L\text{T}_{K^p}.$$ 

Now these two spaces are connected by the $p$-adic uniformization theorem [Sch15, p. 972]:

$$X^{ss} \simeq L\text{T}_{\infty}.$$ 

Following [CDN22], we replace the group $G = \text{GL}_2(\mathbb{Q}_p)$ by the group $G' = \text{GL}_2(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix})$. Let $X(K^p)^p$, $Y(K^p)^p$ (respectively $L\text{T}_{\infty}^p$) be the quotient of the curves $X(K^p)^p$, $Y(K^p)^p$ and $L\text{T}_{\infty}^p$ by the the matrix $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$.

2.2. **Drinfeld tower for $F = \mathbb{Q}$**. Recall now the construction of the Drinfeld tower [CDN20 §0.1]. For $l \neq p$, by the work of Faltings, Fargues, Harris, Taylor, the étale cohomology groups of the Drinfeld tower encode the classical Langlands and classical Jacquet Langlands for $\text{GL}_2(\mathbb{Q}_p)$. It is expected that the $p$-adic étale cohomology groups also encode the hypothetical $p$-adic local Langlands. Let $G = \text{GL}_2(\mathbb{Q}_p)$ and $\tilde{G}$ be the group of invertible elements of the quaternion algebra $D$ with center $\mathbb{Q}_p$. Let $\Omega_{D,p} := \mathbb{P}^{1, \text{ad}} - \mathbb{P}^{1}(\mathbb{Q}_p)$ the Drinfeld’s $p$-adic upper half plane. In [Dri76], Drinfeld defined certain covering $\tilde{M}_n$ of $\Omega_{D,p}$. This covering is
defined over \( \hat{\mathbb{Q}}_p := \hat{\mathbb{Q}}_p^{nr} \). Note that the action of \( W_p \) is compatible with the natural action of \( \hat{\mathbb{Q}}_p \). There is a natural covering map \( \mathcal{M}_{n+1} \to \mathcal{M}_n \to \Omega_{Dr,p} \) compatible with the natural action of \( G \) and \( \hat{G} \). Denote by \( \mathcal{M}_n := C \times_{\hat{\mathbb{Q}}_p} \hat{\mathcal{M}}_n \) and \( \mathcal{M}_\infty \) is the projective limit of all \( \mathcal{M}_n \). Denote by \( \hat{\mathcal{M}}_\infty \) the \( p \)-adic completion of \( \mathcal{M}_\infty \). This is now a perfectoid space in the sense of Scholze. By [CDN20, p. 316], \( \mathcal{M}_n \) possess a \( G \times \hat{G} \times W_p \) equivariant semi-stable model over \( O_K \) with \( K \) a finite extension of \( \hat{\mathbb{Q}}_p \). Again by Scholze [Sch15, p. 972], \( LT_\infty \simeq \mathcal{M}_\infty \). These towers realize both Jacquet Langlands and classical Langlands even for equal characteristics (for details see [Str05], [Str08]).

2.3. Étale sheaves and exact sequences. Let \( X \) be a scheme and \( j : U \hookrightarrow X \) be an open immersion. Let \( Z = X \setminus U \) and \( i : Z \to X \) be the inclusion map. Let \( \mathcal{F} \) be an étale sheaf on \( X \), we get a following exact sequence of sheaves on \( X \)

\[
0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0,
\]

which gives the following long exact sequence of cohomology groups

\[
\cdots \to H^0(X, i_*i^*\mathcal{F}) \to H^1(X, j_!j^*\mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(X, i_*i^*\mathcal{F}) \to \cdots
\]

The cohomology with compact support is defined by \( H^r_c(U, \mathcal{G}) := H^r(X, j_*\mathcal{G}) \), for \( r \geq 0 \) and any étale sheaf \( \mathcal{G} \) on \( U \). Also, by definition of \( i^* \) we have \( H^i(X, i_*i^*\mathcal{F}) = H^i(Z, i^*\mathcal{F}) \) for \( i = 0, 1 \). Therefore we get

\[
\cdots \to H^0(Z, i^*\mathcal{F}) \to H^1_c(U, j^*\mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(Z, i^*\mathcal{F}) \to \cdots
\]

We will also consider the cohomology with support on \( Z \). For this let \( Z \) be a closed subvariety (or subscheme) of \( X \). For any étale sheaf \( \mathcal{F} \) on \( X \) we have the following long exact sequence of cohomology groups

\[
\cdots \to H^r_Z(X, \mathcal{F}) \to H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F}) \to H^{r+1}(X, \mathcal{F}) \to \cdots
\]

By the general formalism of six operations for Berkovich spaces (see [Ber93]) and by the comparison results of étale cohomology of schemes and its analytification (see [Ber95]) we can use the above two long exact sequence (with compact support and with support on \( Z \)) for the modular curves settings. We follow the notation of [Cho15, Section 2.1] in this section and write down the above exact sequence for the elliptic modular curves. Let \( \mathcal{X}_1(Np^m) \) be the Katz-Mazur compactification of the modular curve associated to the moduli problem \( \Gamma(p^m) \cap \Gamma_1(N) \) with \( (N, p) = 1 \). This the model defined over \( \mathbb{Q} \) (see also [BE10, §3] for adelic definition of level that are same because the class number is 1).

In the case, we consider \( X = \mathcal{X}_1(Np^m)_{an} \), \( U = \mathcal{X}_1(Np^m)_{ss} \) and \( Z = \mathcal{X}_1(Np^m)_{ord} \). There is a following long exact sequence with \( \mathcal{F} \) constant sheaves \( \mathbb{Q}_p \) and \( \mathbb{F}_p \).

\[
\cdots \to H^0(\mathcal{X}_1(Np^m)_{ss}) \to H^1_{\mathcal{X}_{an}}(\mathcal{X}_1(Np^m)_{an}) \to H^1(\mathcal{X}_1(Np^m)_{an}) \to H^1(\mathcal{X}_1(Np^m)_{ss}) \to \cdots
\]
Define, the *ordinary* and *supersingular* part of the cohomology groups as
\[ H^1_{\text{ord}} := H^1_{\text{ét}}(\mathcal{X}_1(Np^m)_{\text{ord}}, \mathbb{Q}_p), \quad H^1_{\text{ss}} := H^1_{\text{ét}}(\mathcal{X}_1(Np^m)_{\text{ss}}, \mathbb{Q}_p). \]

Using [Cho18, p. 462], we have an explicit description
\[ H^1_{\text{ord}} = \text{Ind}_{B(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)} \mathcal{H}^0(\{\infty\}, R^1\pi_{HT, \ast}(\mathbb{Q}_p)). \]

For an irreducible principal series representation \( \pi_p \), Scholze [Sch18] defined \( S^1(\pi_p) \) and there is a close relation between \( S^1(\pi_p) \) and \( H^1_{\text{ord}}[\pi_p] \) [CDN22]. Recall that Judith Ludwig [Lud17] proved that \( S^2(\pi_p) = 0 \) and \( S^1(\pi_p) \neq 0 \). Let \( A \) be a \( \mathbb{Q}_p \) algebra. Using [Sch12, Theorem 7.17] and \( X \sim \varprojlim_{K_p} X(K_pK^p) \), we define the \( p \)-adic completed cohomology of \( X \) to be
\[ (2.1) \quad \widehat{H}^1(K^p)_A := \varprojlim_n \lim_{K_p} H^1_{\text{ét}}(X(K_pK^p), \mathbb{Z}/p^n\mathbb{Z}) \otimes A. \]

### 3. \( p \)-adic and mod \( p \) Local Langlands for \( \text{GL}_2(\mathbb{Q}_p) \)

Following [BB10] and [Bre10a], we recall some basic facts about \( p \)-adic and mod \( p \) local Langlands. Explicit construction of the Banach space \( B(\rho_p) \) can also be found in [CDP14], [BE10], [Eme06b, Conj. 3.3.1, p. 297]. Fix a finite extension \( E \) of \( \mathbb{Q}_p \) and a vector space \( V \) over \( E \). According to the \( p \)-adic local Langlands correspondence, for every \( \rho : G_p \to \text{GL}(V) \), we can associate an admissible unitary Banach space representation \( B(V) \) that satisfies the following properties:

1. For two representations \( V, V' \) of \( G_p \), we have \( V \simeq V' \) if and only if as a \( \text{GL}_2(\mathbb{Q}_p) \) representation, we have topological isomorphism (\( \text{GL}_2(\mathbb{Q}_p) \) equivariant) between \( B(V) \) and \( B(V') \).
2. If \( V \) has a determinant \( \chi \) then \( B(V) \) has central character \( \chi \cdot \epsilon \).
3. For any continuous character \( \chi : G_p \to E^\times \), there is a topological isomorphism of vector spaces:
\[ B(V \otimes \chi) \simeq B(\otimes) \otimes (\chi \circ \text{det}). \]
4. The map \( V \to B(V) \) is compatible with the extension of scalars.
5. If \( V \) is irreducible then \( B(V) \) is topologically irreducible.
(6) Recall that mod $p$ and $p$-adic local Langlands are compatible. In other words, there is a commutative diagram \([CG15]\):

\[
\begin{array}{ccc}
V & \longrightarrow & B(V) \\
\downarrow & & \downarrow \\
B(V) & \longrightarrow & 
\end{array}
\]

Under the above maps,
- If $B(V)$ is a principal series representation, then $\overline{B(V)}$ will again be a principal series representation.
- If $B(V)$ is special then $\overline{B(V)}$ is a special representation.
- If $B(V)$ is supercuspidal then $\overline{B(V)}$ is a supersingular representation.

Let $(\rho, V)$ be a two dimensional crystalline representation of the local Galois group $G_p$ with distinct Hodge Tate weights between $(0, k - 1)$.

Recall that there are few possibilities for $B(\rho_{f,p})$ ([BE10] [Eme06b] §6) (see also [Col14]):

**Proposition 3.1.**

1. (absolutely reducible) Let $\rho_{f,p} \simeq \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$ with $\eta_1 \cdot \eta_2^{-1} \neq \epsilon \pm 1$. In this case,

   \[ B(\rho_{f,p}) \simeq \text{Ind}^\mathbb{G}_m^\mathbb{Z}_p(\eta_1 \otimes \eta_2 \epsilon^{-1}) \bigoplus \text{Ind}^\mathbb{G}_m^\mathbb{Z}_p(\eta_2 \otimes \eta_1 \epsilon^{-1}) \]

2. (reducible non split, case I) If $\rho_{f,p} \simeq \begin{pmatrix} \eta_1 & \star \\ 0 & \eta_2 \end{pmatrix}$ with $\star \neq 0$ and $\eta_1 \cdot \eta_2^{-1} \neq \epsilon \pm 1$, then the corresponding automorphic representation $B(\rho_{f,p})$ satisfies the exact sequence:

   \[ 0 \to \pi_1 \to B(\rho_{f,p}) \to \pi_2 \to 0; \]

   with $\pi_1 := \text{Ind}^{\mathbb{G}_m^\mathbb{Z}_p}(\eta_2 \otimes \eta_1) \epsilon_0$ and $\pi_2 := \text{Ind}^{\mathbb{G}_m^\mathbb{Z}_p}(\eta_1 \otimes \eta_2) \epsilon_0$.

3. (reducible non split, case II) If $\rho_{f,p} \simeq \begin{pmatrix} \eta_1 & \star \\ 0 & \eta_2 \end{pmatrix}$ with $\star \neq 0$ and $\eta_1 \cdot \eta_2^{-1} = \epsilon \pm 1$ then the corresponding automorphic representation $B(\rho_{f,p})$ has a Jordan-Hölder filtration $0 \subset \pi_1 \subset \pi_2 \subset \pi$ with $\pi_1 \simeq (\chi \circ \det) \circ \text{St}$ and $\frac{\pi}{\pi_2} \simeq \text{Ind}^{\mathbb{G}_m^\mathbb{Z}_p}(\eta_2 \otimes \eta_1 \epsilon^{-1})$.

4. If $\rho_{f,p}$ is absolutely irreducible the $B(\rho_{f,p})$ is irreducible.

Note that reducible non-split case I is the analogue of principle series representation, while case II is the analogue of the twists of Steinberg or special representations of the classical local Langlands correspondences.
4. Local Galois representations and cohomologies of modular curves

In this section we prove Theorem [8]. Recall that by assumption \( \rho_{f,p} \) reducible and non-split and hence \( \rho_p \simeq \begin{pmatrix} \eta_1 & \ast \\ 0 & \eta_2 \end{pmatrix} \) with \( \eta_1 \cdot \eta_2^{-1} \neq \epsilon \pm 1 \) and \( \ast \neq 0 \).

For any de Rham representation \( V \), we can associate a two dimensional filtered \((\phi, N, G_p)\) module \( M \) to \( V \). Now with this \( M \) thanks to recent development due to Breuil, Berger, Colmez, Paskunas, we can associate \( p \)-adic local Langlands correspondence \( B(\rho_p) := LL(M) \). We defined modular curves at infinity in Section [21]. For a representation \( \pi \) of \( GL_2(\mathbb{Q}_p) \), we denote by \( \pi^* \) the dual representation of \( \pi \) and let us recall \( \eta_{r_{H},n} \) as in [CDN20, p. 346].

Let \( P \) be a principal series representation (not necessarily irreducible) and \( Q \) be any admissible representation of \( G := GL_2 \) with diagonal torus \( T = \mathbb{G}_m \times \mathbb{G}_m \).

For \( i \in \{1, 2\} \), let \( \chi_i : \mathbb{Q}_p^\times \to \mathbb{O}_p^\times \) be integral characters. For a character \( \chi = \chi_1 \otimes \chi_2 \) of the torus \( T(\mathbb{Q}_p) \), denote by \( (\text{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)})(\chi)^{et} \) the set of all continuous (equivalently locally analytic) functions \( f : GL_2(\mathbb{Q}_p) \to \mathbb{E} \) such that \( h{\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}} g = \chi_1(a)\chi_2(d) h(g) \). On these Banach spaces, the group \( GL_2(\mathbb{Q}_p) \) acts by right translation and makes them unitary \( GL_2(\mathbb{Q}_p) \) Banach spaces. Each parabolic \( P \) determines a modulus character \( \delta_P \) on the torus \( T \) with values in \( \mathbb{Q}_p^\times \).

Consider the group \( G = GL_2(\mathbb{Q}_p) \) and recall that we define \( I_P^G(\chi) := (\text{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)})(\chi)^{et} \).

In the next proposition, we consider \( \text{Hom} \) in the category of continuous (equivalently locally analytic) representation of the group \( G = GL_2(\mathbb{Q}_p) \).

**Proposition 4.1.** Let us assume that \( I_P^G(\chi) \) is not a Steinberg representation. Then, we have

\[
\text{Hom}_{C_p[G]}(I_P^G(\chi), \mathbb{B}_{st} \otimes \mathbb{Q}_p \ H^1_{et}(LT_{\infty}, \mathbb{Q}_p)) = 0.
\]

**Proof.** First, we show that \( \text{Hom}_{C_p[G]}(I_P^G(\chi), \mathbb{B}_{st} \otimes \mathbb{Q}_p \ H^1_{et}(LT_{\infty}, \mathbb{Q}_p)) = 0 \). If possible, there exists a non-zero \( \phi : I_P^G(\chi) \to C_p \otimes \mathbb{Q}_p \ H^1_{et}(LT_{\infty}, \mathbb{Q}_p) \) for some \( n \). By [Nic21] and [Ben22] Theorem 13.4.10, p. 177, there exists a period isomorphism \( \alpha_{st} : \mathbb{B}_{st} \otimes \mathbb{Q}_p \ H^1_{et}(LT_{\nu}, \mathbb{Q}_p) \simeq H^1_{dR}(LT_{\nu}) \otimes \mathbb{Q}_p^\ast \mathbb{B}_{st} \). Hence, we get a non-zero homomorphism \( I_P^G(\chi) \to H^1_{dR,c}(LT_{\nu}) \otimes \mathbb{Q}_p^\ast \mathbb{B}_{st} \).

We use [CDN20] p. 352 and follow the notation of loc. cit. that says that

\[
H^1_{dR,c}(LT_{\nu}) = \bigoplus_{M \in \Phi_{N=}} JL^1(M) \otimes_{C} WD^1(M) \otimes_{C}(LL^1(M))^\ast.
\]

In Equation [4.1] the direct summation is over modules \( M \) which are indecomposable as a Weil Deligne representations and most importantly of rank 2. In particular, there will not be any contribution in equation [4.1] from irreducible principal series representations. By our
assumption for all modules $M$, there exists a homomorphism
\[
I_G^G(\chi) \to (LL^1(M))^* \otimes_{Q_p} B_{st};
\]
with $LL^1(M)$ not an irreducible principal series representation. We now recall some basic facts from §3 (see also [Eme06b, Conj. 3.3.1, p. 297]). Note that $p$-adic Langlands are established due to work of Brueil, Berger, Colmez, Paskunas, Dospinescu and several others. There doesn’t exist a homomorphism as above. Note that dual of a principal series representation is again a principal series representation.

Finally, we use [CDN20, p. 346] given by $\phi \to \phi \otimes nr_{a,G}$ with inverse given by $\psi \to \psi \otimes nr^{-1}_{a,G}$ to get
\[
\text{Hom}_G(I_G^G(\chi), H^1_{dR,c}(LT_\infty) \otimes_{Q_p} B_{st}) = \text{Hom}_G(I_G^G(\chi) \otimes nr_{a,G}, H^1_{dR,c}(LT^p_\infty) \otimes_{Q_p} B_{st}) \otimes nr^{-1}_{a,G} \otimes nr^{-1}_{a,W_p}.
\]
That completes the proof of the proposition.

The integral version of the above comparison isomorphism theorem is proved by a recent work due to Scholze-Bhatt-Morrow [BMS18].

5. EMERTON’S ADJOINT FUNCTORS OF INDUCED REPRESENTATIONS

Consider the parabolic subgroup $P$ of $G$ with Levi decomposition $P = M \cdot N$. Let $\mathcal{G}$ (respectively $\mathcal{M}$) be the category of smooth $G$ representations (respectively smooth $M$ representations). The Jacquet module [Cas93] of $V$ is the set of all $N$ co-invariants $V_N$. The Jacquet functor $J_P : \mathcal{G} \to \mathcal{M}$ is the functor $J_P(V) = V_N$. This functor has the following important properties:

- The functor $J_P$ is exact (both left and right).
- If $V \neq 0$ and irreducible then the Jacquet module $J_P(V) \neq 0$ if and only if $V$ appears as sub representations of the parabolically induced representations.

Emerton generalized this functor to a slightly general (locally analytic or equivalently continuous) classes of representations in [Eme06a] and [Eme07]. Soon after, Emerton invented the functor $\text{Ord}_P$ [Eme10a, Eme10b] that works for representations over Artinian rings and that is again an adjoint of induced representations. Now, the Jacquet functor and $\text{Ord}_P$ functors are closely related [Sor17].

We have left exact additive functors $\mathcal{S} : \mathcal{G} \to \mathcal{M}$ given by $V \to \mathcal{S}(V) := \text{Ord}_P(V)$ (Ordinary modules of $V$). We have another exact sequence from the category of $M$ representations to the $M$ representations given by $\mathcal{S}(V) := \text{Hom}_M(U, V)$ this is again and left exact. Note that
\( \mathcal{F}(V) \) takes injective objects to \( G \) acyclic objects. By the Grothendieck spectral sequence and Frobenius reciprocity, we have [Eme10b, p. 429]

\[
E_2^{i,j} := \text{Ext}^i_M(U, R^j \text{Ord}_P(V)) \Rightarrow \text{Ext}^{i+j}_G(I^G_P(U), V).
\]

In particular, it gives rise to the exact sequence:

(5.1) \[
0 \to \text{Ext}^1_M(U, \text{Ord}_P(V)) \to \text{Ext}^1_G(I^G_P(U), V) \to \text{Hom}_M(U, R^1(\text{Ord}_P(V))) \to \ldots.
\]

Recall that the cohomology group \( H^1_c(X^{ss}) \) is a smooth representation of \( GL_2(\mathbb{Q}_p) \) [DLB17, Proposition 3.6] but not admissible.

**Lemma 5.1.** Let \( I^G_P(\chi) \) be not a twist of Steinberg. For any sub-representation \( W \) of \( H^1_c(X^{ss}) \), we have

\[
\text{Ext}^1_G(I^G_P(\chi), W) = 0
\]

**Proof.** Let \( V = I^G_P(\chi) \) and \( W = \ker(f_1) \). By [Hau16, Proposition B.2, p. 267], we have

\[
\dim_E(\text{Ext}^1_G(V, W)) \leq \dim_{k_E}(\text{Ext}^1_G(V_1, W_1)
\]

with \( k_E := O_E/m_E \) the residue field of \( E \). We show that \( \text{Ext}^1_G(I^G_P(\chi), W_1) = 0 \) with \( W = \ker(f_1) \).

We show that \( \text{Ext}^1_M(\chi, \text{Ord}_P(V)) = \text{Hom}_M(\chi, \text{R}^1(\text{Ord}_P(V))) = 0 \). Recall that [Hu17, Proposition 2.9, p. 553] with the notation of the loc. cit:

1. \( R^1\text{Ord}_P(\text{Ind}_B^G U) \simeq U \otimes \alpha^{-1}; \text{Ord}_P(I^G_P(U)) \simeq U^s. \)
2. For supersingular representations \( V \), we have \( R^1\text{Ord}_P(V) = \text{Ord}_P(V) = 0 \).

Recall that as \( G \) representation, we have

\[
H^1_c(X^{ss}) \simeq \bigoplus U_i
\]

with \( U_i \) either supercuspidal and Stenberg. Hence, \( V_1 = H^1_c(X^{ss}) \simeq \bigoplus V_i \) with \( V_i \)'s Stenberg or supersingular. We deduce that \( \text{Hom}_M(\chi, R^1(\text{Ord}_P(V)) = 0 \) since \( I^G_P(\chi) \) is not a Stenberg representation.

From the above we have \( \text{Hom}_M(\chi, \text{Ord}_P(V)) = 0 \) with \( V_1 = H^1_c(X^{ss}). \) Note that \( W_1 \subset V_1 \) and \( \text{Hom}_G(\chi, -) \) is a left exact functor, hence \( \text{Hom}_M(\chi, \text{Ord}_P(W)) = 0 \).

Since \( \chi \) is an irreducible representations, we use the isomorphism:

\[
\chi \otimes \text{Hom}_M(\chi, \text{Ord}_P(W)) \simeq \text{Ord}_P(W).
\]

Hence, we deduce that \( \text{Ext}^1_M(\chi, \text{Ord}_P(W)) = 0 \). From the Grothendieck spectral sequence [5.1] we deduce that

\[
\text{Ext}^1_G(I^G_P(\chi), V_1) = 0.
\]
We now proceed to prove our main theorem. Consider the following subgroups of the cohomology groups \[BE10\]. Let \(c \in G_{\mathbb{Q}}\) be the complex conjugation. For any cohomology group \(H\), denote by \(H^{\pm}\) the \(\pm\) eigenspace of \(c\). Let \(f\) be a \(p\)-adic cusp form of weight \(k \geq 2\), level \(N = Mp^m\) with \(m \geq 1\) and \((M, p) = 1\) and character \(\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times\) defined over a finite extension \(E\) of \(\mathbb{Q}_p\) contained in \(\overline{\mathbb{Q}}_p\). Assume that \(f\) is a Hecke eigenform with \(T_l f = a_l f\) for all \(l \in \mathbb{N}\). For any Hecke module \(X\), denote by \(X^f := \{x | x \in X; T_l x = a_l x\}\).

We now prove Theorem 1.1.

**Proof.** Let \(\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(E)\) be a global Galois representation (continuous, irreducible representations associated with a modular form that is unramified outside a finite set of primes) with \(\rho_p : G_p \to \text{GL}_2(E)\) local Galois representation obtained by the restriction of the Galois representation to the decomposition group \(G_p := \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)\). By \[SW99\], with the assumption as in the Theorem \(\rho = \rho_f\) for some elliptic Hecke eigenform \(f\). Denote by the modular curve over \(\mathbb{Q}\) by \(X = X_1(Np^m)\).

Since \(\rho_{f, p}\) is reducible and non-split with the corresponding characters satisfying the assumption of the theorem, we have an inclusion (cf. Proposition 3.1),

\[\pi_1 \hookrightarrow B(\rho_{f, p});\]

with \(\pi_1 = I_{G_{\mathbb{Q}}}^G(\chi_1)\). Recall that \(\text{Hom}_G(I_{p}^G(\chi), C_p \otimes_{\mathbb{Q}_p} H^1(LT_\infty)) = 0\) by Proposition 4.1 and observe that \(X_{ss} \simeq LT_\infty\) (cf. §2.3). By the long exact sequence in § 2.3 we have

\[0 \to H^0_c(X_{ss}^{f, \pm} \otimes_{C_p} B_{st}) \to H^0(X)^{f, \pm} \otimes_{C_p} B_{st} \to \cdots \to H^1(X)^{f, \pm} \otimes_{C_p} B_{st} \to H^1(X_{ss}^{f, \pm} \otimes_{C_p} B_{st}) \to \cdots .\]

From the long exact sequence, we get a short exact sequence

\[0 \to \text{Ker}(f_1) \to H^1_c(X_{ss}^{f, \pm} \otimes_{C_p} B_{st}) \to \text{Im}(f_1) \to 0.\]

Applying the left exact Hom functor, we get

\[0 \to \text{Hom}_G(I_{p}^G(\chi), \text{Ker}(f_1)) \to \text{Hom}_G(I_{p}^G(\chi), B_{st} \otimes H^1_c(X_{ss}^{f, \pm})) \to \text{Hom}_G(I_{p}^G(\chi), \text{Im}(f_1)) \to \text{Ext}^1_G(I_{p}^G(\chi), \text{Ker}(f_1)) \to \text{Ext}^1_G(I_{p}^G(\chi), H^1_c(X_{ss}^{f, \pm} \otimes_{C_p} B_{st})) \cdots .\]
By [CDN20, Theorem 0.6, p. 317], the cohomology group $H^1_c(LT_{\infty}^{\mathbb{Q}_F}, \mathbb{C}_p)$ is admissible. Since admissible representations form a Serre subcategory of the category of smooth representation, hence subrepresentation of admissible representations are admissible. Note that

$$\text{Hom}_G(I_c^G(\chi), H^1_c(X^{\text{ss}})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st}) = 0.$$ 

Since $\ker(f_1) \subset H^1_c(X^{\text{ss}})^{f,\pm} \otimes_{\mathbb{C}_p} \mathbb{B}_{st}$, we have $\text{Hom}_G(I_c^G(\chi), \ker(f_1)) = 0$ and $\text{Hom}_G(I_c^G(\chi), \text{Im}(f_1)) \to \text{Ext}^1_G(I_c^G(\chi), \ker(f_1))$. By Lemma 5.1, we have $\text{Ext}^1_G(I_c^G(\chi), \ker(f_1)) = 0$ and hence

$$\text{Hom}_G(I_c^G(\chi), \text{Im}(f_1)) = 0.$$

We also have an exact sequence

$$0 \to \text{Im}(f_1) \to H^1(X)^{f,\pm} \otimes \mathbb{B}_{st} \to \text{Im}(f_2) \to 0.$$ 

By applying Hom functor [DF04, Theorem 10, p. 785, Chapter 17], we get

$$0 \to \text{Hom}_G(I_c^G(\chi), \text{Im}(f_1)) \to \text{Hom}_G(I_c^G(\chi), \mathbb{B}_{st} \otimes_{\mathbb{C}_p} H^1(X)^{f,\pm})$$

$$\to \text{Hom}_G(I_c^G(\chi), \text{Im}(f_2)) \to \text{Ext}^1_{C_p[G]}(I_c^G(\chi), \text{Im}(f_1))$$

$$\to \text{Ext}^1_{C_p[G]}(I_c^G(\chi), \mathbb{B}_{st} \otimes H^1(X)^{f,\pm}) \to \text{Ext}^1_{C_p[G]}(I_c^G(\chi), \text{Im}(f_2))...$$

Hence, we have inclusions of cohomology groups:

$$\text{Hom}_G(I_c^G(\chi), \mathbb{B}_{st} \otimes_{\mathbb{C}_p} H^1(X)^{f,\pm}) \hookrightarrow \text{Hom}_G(I_c^G(\chi), \text{Im}(f_2)) \hookrightarrow \text{Hom}_G(I_c^G(\chi), \mathbb{B}_{st} \otimes H^1(X^{\text{ord}})).$$

As a consequence, we have

$$\pi_1 \otimes \text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{C}_p} H^1(X)^{f,\pm}) \hookrightarrow \pi_1 \otimes \text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} H^1(X^{\text{ord}})).$$

Now, $\pi_1$ is irreducible as $G$ representation (cf. [3]). We deduce that

$$\pi_1 \otimes \text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes H^1(X)^{f,\pm}) \subset \mathbb{B}_{st} \otimes H^1(X^{\text{ord}}).$$

As $f$ is a cusp form, we have [Bre10b, Lemma 2.1.4]

$$H^1_c(X)^{f,\pm} \simeq H^1(X)^{f,\pm}.$$ 

On the other hand by [BE10, Proof of Theorem 5.7.3], we have

1. $\text{Hom}_{C_p[G]}(\rho_f, \text{Hom}_{GL_2(\mathbb{Q}_p)}(\pi_1, \text{Im}(f_2))) = \text{Hom}_{C_p[G]}(\rho_f, \text{Hom}_{GL_2(\mathbb{Q}_p)}(\pi_1, H^1_c(X)^{f,\pm}) \simeq L.$
2. $\text{Hom}_{C_p[G]}(\rho_f, H^1_c(X)^{f,\pm}) \simeq \text{Hom}_{C_p[G]}(\rho_f, H^1_c(X)^{f,\pm}).$

From the above, we have an inclusion $\rho_f \subset \text{Hom}_G(\pi_1, H^1_{et}(X)^{f,\pm})$. By 5.2, we deduce that (as the field $C_p$ is flat)

$$\rho_f \otimes_{\mathbb{C}_p} \pi_1 \subset \text{Hom}_G(\pi_1, \mathbb{B}_{st} \otimes H^1(X)^{f,\pm}) \otimes \pi_1 \subset \mathbb{B}_{st} \otimes_{\mathbb{C}_p} H^1(X^{\text{ord}})^{f,\pm} \subset \mathbb{B}_{st} \otimes_{\mathbb{C}_p} H^1(X^{\text{ord}});$$
proving the first part of our theorem by Proposition 2.1.

We now proceed to prove the second part. Since $\text{Im}(f_2) \subset H^1(X_{\text{ord}})^{f, \pm}$, we have

$$\rho_f \hookrightarrow \text{Hom}_G(\pi, \mathbb{B}_{st} \otimes \mathbb{Q}_p \otimes H^1(X)^{f, \pm}) \hookrightarrow \text{Hom}_G(\pi, \text{Im}(f_2)) \hookrightarrow \text{Hom}_G(\pi, \mathbb{B}_{st} \otimes \mathbb{Q}_p H^1(X_{\text{ord}})^{f, \pm}) \hookrightarrow \text{Hom}_G(\pi, \mathbb{B}_{st} \otimes \mathbb{Q}_p H^1(X_{\text{ord}})).$$

Again, we complete the proof by Proposition 2.1. □

Main theorem of Chojecki [Cho18] and our Theorem 1.1 together give a nice dichotomy:

**Corollary 1.**

- If $\rho_p$ is absolutely irreducible then $\rho \otimes_{\mathbb{C}_p} B(\rho_{f,p}) \hookrightarrow \widehat{H}^1_{\text{B}, \text{ss}}$.

- If $\rho_p$ is reducible and non-split then there exist a principal series subrepresentation $\pi_1$ of $B(\rho_{f,p})$ such that $\rho_p \otimes_{\mathbb{C}_p} \pi_1 \subset \widehat{H}^1_{\text{B}, \text{ord}}$.

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