Hamilton-Jacobi method for a simple resonance

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Abstract

It is well known that a generic small perturbation of a Liouville-integrable Hamiltonian system causes breakup of resonant and near-resonant invariant tori. A general approach to the simple resonance case in the convex real-analytic setting is developed, based on a new technique for solving the Hamilton-Jacobi equation. It is shown that a generic perturbation creates in the core of a resonance a partially hyperbolic lower-dimensional invariant torus, whose Lagrangian stable and unstable manifolds, described as global solutions of the Hamilton-Jacobi equation, split away from this torus at exponentially small angles. Optimal upper bounds with best constants are obtained for exponentially small splitting in the general case.

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1 Introduction

The notion of Arnold diffusion refers to a generic instability of Hamiltonian systems with three and higher degrees of freedom \([11, 3]\). A notable exception are Liouville-integrable systems allowing the construction of global action-angle variables \([2]\). Small perturbations of such systems provide a natural set-up to study the instability. Recently Mather \([24]\) using methods of analysis in the large (which to a great extent had been created by himself) \([22, 23]\) announced the proof of the existence of Arnold diffusion in the three degrees of freedom (convex, real-analytic) case.

Consider a Hamiltonian system of \(n + 1, n \geq 2\) degrees of freedom in the cotangent bundle \(T^*T^{n+1} \cong \mathbb{R}^{n+1} \times \mathbb{T}^{n+1}\) of a torus \(T^{n+1} \equiv (\mathbb{R}/2\pi\mathbb{Z})^{n+1}\). Take an open convex domain \(\Omega \subseteq \mathbb{R}^{n+1}\). The phase space \(\mathcal{M} = \Omega \times \mathbb{T}^{n+1}\) has a natural exact symplectic structure \(\omega\). Consider a Hamiltonian function

\[
H : \mathcal{M} \to \mathbb{R}, \text{ such that } |H - H_0|_{\mathcal{M}} = \varepsilon \ll 1, \text{ for some } H_0 : \Omega \to \mathbb{R}. \tag{1}
\]

Suppose \(H\) is real-analytic, i.e. it can be extended holomorphically into a neighborhood of \(\mathcal{M}\) in \(\mathbb{C}^{2n+2}\), let \(\cdot |_{\mathcal{M}}\) above be the supremum-norm. Also suppose that \(H_0\) is strictly convex. Then one can simply take \(\Omega = \{ p \in \mathbb{R}^{n+1} : H_0(p) < E_0 \}\) for some \(E_0 > 1\).

If \((p, q)\) are (global) canonical coordinates on \(\mathcal{M}\), or the action-angle variables \([2]\) with \(w = dq \wedge dp\), the Hamiltonian \((1)\) has an expression

\[
H = H(p, q, \varepsilon) = H_0(p) + \varepsilon H_1(p, q, \varepsilon), \quad (p, q, \varepsilon) \in \Omega \times \mathbb{T}^{n+1} \times \mathbb{R}_+.
\tag{(1')}
\]

The Hessian matrix \(D^2H_0(p)\) is positive definite for every \(p \in \Omega\); \(0 \leq \varepsilon \ll 1\) is a small parameter. The perturbation \(H_1\) is \(2\pi\)-periodic in each angle \(q_j, j = 0, \ldots, n\). The system \(H1\) is autonomous\(^1\). If \(\varepsilon = 0\), it is Liouville integrable. Its phase space is foliated by invariant tori, whereupon \(p(t) = \text{const.},\) and \(q(t) = \text{const.} + \omega(p)t\), where \(\omega(p) = DH_0(p)\) is a frequency. Each torus is a Lagrangian manifold.

The central question of local analysis of system \((1)\) is what geometric objects replace the invariant tori when \(\varepsilon \neq 0\). The KAM theorem \([17, 11, 22, 23]\) asserts that as \(\varepsilon \to 0_+\), an asymptotically full measure set of these tori is stable.

Resonant unperturbed tori are foliated by tori of lower dimension. The property of a torus being resonant or non-resonant is clearly intrinsic, as well as the notion of the multiplicity \(m\) of a resonance, i.e. the difference in the dimensions of the original resonant torus and the minimum foliation torus. In the above coordinate representation the resonances correspond to the values of the action \(p\), when the components of the frequency vector \(\omega(p) \in \mathbb{R}^{n+1}\) are linearly dependent over the integers \(\mathbb{Z}\), \(m\) being the dimension of the kernel of a linear map \(\omega[k] = (k, \omega)\), for \(k \in \mathbb{Z}^{n+1}\). Resonant tori as well as the non-resonant ones sufficiently close to the former, typically get destroyed for any \(\varepsilon \neq 0\). The set of destroyed tori is residual on the unperturbed energy surface \(H_0^{-1}(E)\) for a regular value of \(E\). It is known \([9]\) that given a specific resonance, the majority (in the sense of the Lebesgue measure in \(\mathbb{R}^{n+1-m}\)) of the corresponding resonant tori result in particular in the appearance of partially hyperbolic, or whiskered tori of dimension less by \(m = 1, \ldots, n\). Characteristic exponents of these tori are typically \(O(\sqrt{\varepsilon})\). Singular perturbation theory for manifolds asymptotic to these tori has a number of subtleties \([8, 7, 13, 15, 36, 8]\) which would not be there, were the above characteristic exponents \(O(1)\), see also \([7, 6, 11]\).

Non-resonant tori, sufficiently close to resonances experience a complicated topological perestroika. If \(n = 1\), the result is a cantorus \([22]\) supporting an invariant action-minimizing measure \([23]\). Higher-dimensional relatives of cantori are not so well understood, unless the local analysis can be in a sense reduced to the \(n = 1\) case \([12]\).

The purpose of this paper is to develop from scratch the local theory for a simple resonance, \(m = 1\). A resonance is identified by an integer lattice point \(k_0 \in \mathbb{Z}^{n+1} \setminus \{0\}\). An unperturbed torus, marked by \(p = p_0 \in \Omega\) is resonant with respect to \(k_0\) iff the corresponding frequency \(\omega_0 = \omega(p_0)\) lies on the

\(^1\)The case when \(H_1\) depends on time periodically can be treated in the usual way \([2]\) whereupon the convexity assumption about \(H_0\) should be substituted by quasi-convexity \([14]\) and the non-degeneracy assumption in Theorem \([3]\) by isenergetic non-degeneracy \([4]\). Convexity is far the easiest non-degeneracy assumption to deal with; for more subtle non-degeneracy settings in the KAM theory see e.g. \([36]\).
“resonant hyperplane” \( \{ \omega \in \mathbb{R}^{n+1} : \langle k_0, \omega \rangle = 0 \} \). As \( H_0 \) is smooth and strictly convex, the “frequency map” \( p \to \omega(p) \) is a global diffeomorphism. The values of the action \( p \) satisfying the above resonance condition lie on a smooth hypersurface in \( \mathbb{R}^{n+1} \), which intersects each regular level set of \( H_0 \) transversely (for otherwise \( \langle k_0, D^2H_0(p_0)k_0 \rangle = 0 \) and is a graph over the hyperplane \( \langle k_0, p \rangle = 0 \). Thus metric and topological properties of sets on the resonant hypersurface can be described in terms of the images in the resonant hyperplane, via the frequency map.

Given \( k_0 \), one chooses a value \( p_0 \) on the intersection of a regular level set of \( H_0 \) with the resonant hypersurface, such that the corresponding frequency \( \omega_0 \) is non-resonant over \( \mathbb{Z}^{n+1} \) modulo one-dimensional sub-lattice generated by \( k_0 \). If one denotes the corresponding unperturbed simple resonance \((n+1)\)-torus as \( T_0 \), the latter is foliated by a one-parameter family of \( n \)-tori, which can be parameterized by some \( x \in \mathbb{T} \):

\[
T_0 = \bigcup_{x \in \mathbb{T}} T_x. \tag{2}
\]

It is assumed that \( \omega_0 \) is “far enough” from higher multiplicity resonances. To express the latter property, Kolmogorov’s Diophantine condition \( \mathbb{L}_n \) over the quotient lattice is used. The set of all such frequencies \( \omega_0 \) has a positive Lebesgue measure on the resonant hyperplane \( \mathbb{L}_n \).

Study of simple resonances and their role in global dynamics for the general system \( \mathbb{L}_n \) had begun at least as early as Poincaré [27]. Arnold [1] used a simple resonance model to suggest a local mechanism for universal instability, or diffusion, based on the existence of intersections of Lagrangian manifolds, asymptotic to whiskered tori, alias the splitting of separatrices phenomenon. Splitting in a more general context was studied by Chirikov [8] emphasizing its role in the general diffusion scenario and conjecturing a number of generic asymmetric exponentially small bounds apropos of the splitting and the diffusion speed. For the latter, the theorem of Nekhoroshev [26, 19, 28] gives the upper bound \( \exp \left( \epsilon^{-\frac{\alpha}{\log \alpha}} \right) \).

More recently models for simple resonances and splitting of separatrices have been investigated in a great number of works, see [7, 14, 13, 15, 35, 9, 21] among others. For a more extensive bibliography list see the treatise [21] by Lochak et al, to which one can add some 30 more titles which have become available since the year 2000. The latter work [21] among other things develops a normal form theory for local near-resonance dynamics, see also [19, 28]). However, the underlying multiple step averaging procedure is rather general and does not allow to study the splitting in all the detail. As an alternative Lochak et al advocate the Hamilton-Jacobi method, which they illustrate for a particular Hamiltonian from the Arnold example [11] (also published separately as [37]) and draft formulations of a number of theorems, which are proved herein.

A fundamental question apropos of exponentially small splitting (to which the Nekhoroshev-like normal form theory fails to provide an answer) is one of the best constants for the upper estimates involved. Such constants have been obtained for various cuts of a specific model, coupling a pendulum-like one degree of freedom Hamiltonian system with a bunch of rotators [7, 14, 10, 15, 36, 9]. Ideally, the upper bounds would be supported by lower bounds, which constitute a very delicate issue and are available only for a few particular examples [10, 33, 15, 21]. The issue is not addressed in this paper.

An important result concerning the splitting problem in the general simple resonance context is due to Eliasson [13] (see also [9]) who proved the estimate \( 2n + 2 \) on the minimum number of homoclinic orbits to a whiskered torus of dimension \( n \) at the resonance core, but not the exponentially small splitting estimate. The main building blocks for the splitting theory near a simple resonance are presented in [21] although many are without proofs, apparently due to a variety of technical difficulties. This paper attempts to do it, as it turns out that most of these difficulties can be bypassed owing to a technique, rather different from those used in the above listed references (for the exception of [37]) and appears to be more “natural” for the problem involved. The technique certainly applies to the above mentioned model, for which it gives the (known) best constant \( \frac{2}{5} \) and also shows that the latter is the largest value that the best constant in question can assume in principle.

In essence, our technique is the Hamilton-Jacobi approach prompted by Poincaré [27] cast as a “hyperbolic KAM theorem”. However it is developed in an entirely different geometric context than the traditional one founded by Graft [16]. The present geometric scenario was founded in [34]; this paper shapes it into “KAM theory on semi-infinite bi-cylinders over tori”.

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The paper is organized as follows. Section 2 starts out with the preliminaries in order to describe the standard normal form near a chosen simple resonance (2.6) and Lemma 2.1. A non-degeneracy Assumption 1 is made concerning the “hyperbolic part” of the truncated (integrable) normal form, whereupon the splitting problem is set up somewhat heuristically for the localization (2.14) of the normal form Hamiltonian near the truncated normal form separatrix. The set-up emphasizes what is called a “sputnik” property (2.16) thereof, combining the $2\pi$-periodicity of the foliation (2.1) and reversibility of the truncation (2.7) of the normal form Hamiltonian (2.6). At that point one of the main results, Theorem 1 of the paper is formulated. The formulation is still somewhat heuristic, due to the necessity of developing a certain amount of machinery.

Section 3 develops this machinery, underlying the aforesaid version of KAM theory. It is based on a simple holomorphic map introducing “energy-time” coordinates (3.1, 3.5) in which the base space is not compact. Section 3.1 is almost entirely dedicated to the relevant formalism. The hyperbolic KAM theorem, Theorem 2 follows, providing global generating functions for perturbed separatrices as the solutions of the Hamilton-Jacobi equation, Corollary 2.1. The proof of Theorem 2 incorporates a two-parameter trick, yielding an optimal (in the sense of the parameter dependence) smallness condition (3.33).

Section 4 presents the theory for the splitting, based on application of Theorem 2 and a global sputnik property. Assumption 3. The role of the sputnik is to ensure that the one-form giving the splitting distance be exact, which allows one immediately to give a lower bound for a number of homoclinic orbits [13]. This is inherent in the homoclinic splitting problems, being expressed by relations (2.10) (2.2) (1.3). The main theorem of the section, Theorem 3 claims the principal exponentially small estimate (4.13) which is adapted as (2.19) to the simple resonance normal form in Theorem 1, concluding its proof. The estimate (2.19) contains a pair of best constants ($\rho, \sigma_2$), well defined for a specific Hamiltonian with a given analyticity domain, see (3.7, 3.8).

2 Normal form near separatrix at a simple resonance

This section prepares the Hamiltonian (1) for the set-up of the theory developed in the sequel. It consists in choosing a simple resonance action value and restricting the Hamiltonian to its small neighborhood, where a suitable normal form can be produced. Under generic assumptions, this normal form can be viewed as a perturbation of an integrable reversible system containing a separatrix. These steps are standard, see e.g. [39, 13]. However the further analysis is essentially different from that one traditionally encounters in the literature. The Hamiltonian gets localized near one branch of the separatrix of the truncated normal form Hamiltonian. Localization near the other branch can be seen from the former one as a symmetry, referred to as a sputnik.

2.1 Preliminaries

Canonical transformations of the phase space

Throughout the paper, a number of canonical transformations is introduced. These transformations belong to an “affine” class, corresponding to the phase space bundle structure.

For the system (1) the phase space is a subset of $T^\ast T^{n+1}$, with canonical coordinates $(p, q) \in R^{n+1} \times T^{n+1}$. An automorphism $a$ of the base space induces a family of canonical transformations

$$\Xi = \Xi(a, S) : \begin{cases} q &= a(q'), \\ p &= t(d^2a)^{-1}p' + dS, \end{cases}$$

parameterized by a closed one-form $dS$ on the base space. As the latter is $T^{n+1}$, the one-form $dS$ is described by the generating function $S(q) = \langle \xi, q \rangle + \hat{S}(q)$, with some $\xi \in R^{n+1} \equiv [dS]$ specifying the $H^1(T^{n+1}, R)$ cohomology class of $dS$, and some function $\hat{S}(q)$, which is a zero-form on $T^{n+1}$, i.e. is $2\pi$-periodic in each component $q_j$, $j = 0, \ldots, n$ of $q$.

The notation $\langle ., . \rangle$ stands for the canonical coupling between $R^{n+1}$ (and later $R^n$) and its dual space and is identified with the Euclidean scalar product; $t(d^2a)^{-1}$ denotes the transpose inverse of the Jacobi matrix $da$. 4
The case when the map $a$ is linear, so $a$ and $da$ can be identified with a matrix from $SL(n+1, \mathbb{Z})$ is referred to as a symplectic rotation. The one-form $dS$ effects a shift of the origin within each fiber. This shift is fiber-independent if $S(q) = (p_0, q)$, given $p_0 \in \mathbb{R}^{n+1}$, which then becomes the origin in each fiber.

**Simple resonance Diophantine condition**

A resonance is identified by a minimal integer lattice point $k_0 \in \mathbb{Z}^{n+1} \setminus \{0\}$. I.e. there is no $k \in \mathbb{Z}^{n+1}$, such that $k_0 = jk$, $j = 2, 3, \ldots$. Then in any lattice basis $e_0, \ldots, e_n$, the components of $k_0$ are relatively prime. The choice of the basis $\{e_j\}_{j=0,\ldots,n}$ determines a coordinate chart $(p, q)$: take $q_j \in \mathbb{R}e_j/2\pi Ze_j$, $j = 0, \ldots, n$, let $p_j$ be a momentum canonically conjugate to $q_j$.

Consider a one-dimensional lattice $\mathbb{Z}k_0$ and a direct sum decomposition $\mathbb{Z}^{n+1} = \mathbb{Z}k_0 \oplus \mathbb{Z}^{n+1}/\mathbb{Z}k_0$. Choose a lattice basis in the quotient lattice $\mathbb{Z}^{n+1}/\mathbb{Z}k_0$ and let $k_1, \ldots, k_n \in \mathbb{Z}^{n+1}$ represent it in $\mathbb{Z}^{n+1}$, so $\{k_j\}_{j=0,\ldots,n}$ is a lattice basis in $\mathbb{Z}^{n+1}$. For a moment, let us call it a *direct sum decomposition basis*, generated by $k_0$. One can expand each $k_j$ over the “old” basis $\{e_i\}_{i=0,\ldots,n}$. As in the first equation in (2.1) let us write it as $k = a(e)$, where a linear operator $a$ is identified with a matrix from $SL(n+1, \mathbb{Z})$. The first row of this matrix simply gives the coordinates of $k_0$ in the basis $\{e_j\}_{j=0,\ldots,n}$; the rest of the rows depend on a particular choice of the basis in the quotient lattice $\mathbb{Z}^{n+1}/\mathbb{Z}k_0$. Clearly $k_0$ defines a direct sum decomposition basis modulo $SL(n, \mathbb{Z})$.

**Definition 1** A vector $\omega \in \mathbb{R}^{n+1}$ is Diophantine modulo $k_0$ with an exponent $\tau \geq n-1$ and a constant $\gamma > 0$ (one writes $\omega \in \mathcal{W}_{\tau, \gamma}^n(k_0)$) if $\langle k_0, \omega \rangle = 0$ and there exists a direct sum decomposition basis $\{k_j\}_{j=0,\ldots,n}$ generated by $k_0$, such that for all $k \in \mathbb{Z}^{n+1}$, represented as $k = (k_0, \ldots, k_n) \in \mathbb{Z}^{n+1}$ in this basis, one has

$$|\langle k, \omega \rangle| \geq \gamma |k|_0^{-\tau}, \quad |k|_0 = \sum_{j=1}^{n} |k_j|.$$  \hspace{1cm} (2.2)

Note that given $k \in \mathbb{Z}^{n+1}$ and $\omega \in \mathbb{R}^{n+1}$, the “small divisor” $\langle k, \omega \rangle$ does not depend on the choice of the lattice basis $\{e_j\}_{j=0,\ldots,n}$. However, the quantity $|k|_0$ in the right hand side of (2.2) is not $SL(n, \mathbb{Z})$-invariant and can attain any positive integer value for a given $k$.

**Definition 1** is rather unwieldy, and in order to describe metric properties of Diophantine vectors one is forced to fix the lattice basis $\{e_j\}_{j=0,\ldots,n}$. As the lattice element $k_0$ is considered fixed throughout the paper, one should naturally render $\{e_j\}_{j=0,\ldots,n}$ and $\{k_j\}_{j=0,\ldots,n}$ the same basis. Then $k_0 = (1, 0, \ldots, 0)$ and for any $\omega$ such that $\langle k_0, \omega \rangle = 0$, clearly $\omega = (0, \omega)$ for some $\omega \in \mathbb{R}^n$. Given the pair $(\tau, \gamma)$, let us further use the notation $\mathcal{W}_{\tau, \gamma}^n$ for a set of $\omega \in \mathbb{R}^n$, satisfying a stronger definition than Definition 1

$$\mathcal{W}_{\tau, \gamma}^n = \left\{ \omega \in \mathbb{R}^n : \forall k \in \mathbb{Z}^n \setminus \{0\}, \ |\langle k, \omega \rangle| \geq \gamma |k|^{-\tau}, \ |k| = \sum_{i=1}^{n} |k_i| \right\}. \hspace{1cm} (2.3)$$

The above is the standard Kolmogorov Diophantine condition [17]. Some well known metric properties of the set $\mathcal{W}_{\tau, \gamma}^n$ (with a sufficiently small $\gamma$, see e.g. [30, 4], also [12] for complementary results) are that it is non-empty if $\tau \geq n-1$, and for $\tau > n-1$ has a full Lebesgue measure as $\gamma \to 0$.

**2.2 Localization near a simple resonance**

For the Hamiltonian $H_0$ given a resonance $k_0$, let $p_0 \in \Omega$ lie on a regular level set $H_0^{-1}(E)$ and $\omega_0 = DH_0(p_0)$ satisfy Definition 1. Then one can choose the lattice basis $\{e_j\}_{j=0,\ldots,n}$ with $e_0 = k_0$, such that $\omega_0 = (0, \omega)$, with $\omega_0 \in \mathcal{W}_{\tau, \gamma}^n$, with some $\tau$, which is fixed throughout the paper and $\gamma = \gamma_0$. Moreover, the origin for the action variables can be set at $p_0$, i.e. $p_0 = (0, \ldots, 0)$. As $H$ is defined modulo a constant, let $E = H_0(0) = 0$. 

5
This fixes the choice of the action-angle variables \((p, q)\) and denoting \(p = (y, I) \in \mathbb{T} \times \mathbb{T}^n\), \(q = (x, \varphi) \in \mathbb{T} \times \mathbb{T}^n\), one can write down the following representation for the Hamiltonian \(H\):\footnote{If \(k_0 \neq (1, 0, \ldots, 0)\) in the given lattice basis and the Hamiltonian \(H\) is given in the form \(\Pi\), the representation \(\Pi\) can certainly be achieved by means of a canonical transformation in the form \(\Pi\) combining a symplectic rotation and a shift of the action origin to \(p_0\).\(\Pi\). With a specific \(H\) and \(k_0\) in mind, the adaptation of the parameters in the main estimate \(\Pi\) in Theorem \(\Pi\) is straightforward, similar to how it will have embraced the parameter \(\theta\) in the sequel.}

\[
H(p, q, \varepsilon) = \langle \omega_0, I \rangle + \frac{1}{2} \langle Q_0 p, p \rangle + O_3(p) + \varepsilon H_1(p, q, \varepsilon),
\]

where \(Q_0\) is a constant positive definite matrix.

**Notation:**

N.1. The set of non-negative or positive, integer or real numbers is denoted as \(\mathbb{Z}_+\) or \(\mathbb{Z}_{++}\), \(\mathbb{R}_+\) or \(\mathbb{R}_{++}\), respectively.

N.2. Bold lowercase symbols usually denote \((n + 1)\)-vector quantities. Uppercase symbols often but not always denote \(n\)-vector quantities. E.g. above \(p = (y, I)\), \(q = (x, \varphi)\); further \(g(q) = (g(q), G(q))\).

N.3. The symbol notation \(u(z, \cdot) = O_\alpha(z; \cdot)\), with \(\alpha \in \mathbb{R}\), implies that \(\lim_{z \to 0} \frac{|u(z, \cdot)|}{\|z\|^\alpha}\), where \(\|\cdot\|\) is the Euclidean norm, exists and is uniformly bounded from above for the whole range of the variables \((\cdot)\) (which may be omitted in the notation, as well as \(\alpha = 1\)) by some constant \(C\) which may depend on \((n, \tau)\) and perhaps other quantities fundamental for the problem, to be specified. \(C\) will be “as large as necessary” and may increase without notice. To suppress \(C\) (or \(C^{-1}\)) in estimates, the \(\lesssim\) sign is often used instead of \(\leq\).

N.4. For real \(\kappa, \sigma > 0\) and \(j \in \mathbb{Z}_{++}\) \((j = 1 \text{ usually being omitted})\) let

\[
\mathbb{B}_j^\kappa \overset{\text{def}}{=} \{ z \in \mathbb{C}^j : \|z\| \leq \kappa \},
\]

\[
\mathbb{T}_j^\sigma \overset{\text{def}}{=} \{ z \in \mathbb{C}^j : \Re z \in \mathbb{T}^j, |\Im z| \leq \sigma \}
\]

be complex extensions of a disk and a torus.

It will always be assumed by default that \(\kappa > 1\), and it will not enter the estimates. All the analyticity and non-degeneracy parameters are by default positive, as well as \(\varepsilon\). In addition, if \(\delta\) for instance denotes the analyticity loss in the variable \(\varphi \in \mathbb{T}^n\), it will be assumed by default that \(0 < \delta < \sigma\).

N.5. Scalar functions \(2\pi\)-periodic in each variable, holomorphic and uniformly bounded inside \(\mathbb{T}^j\), whose restrictions on \(\mathbb{T}^j\) are real-analytic form Banach spaces \(\mathfrak{B}_\sigma(\mathbb{T}^j)\), with topology induced by the supremum norm \(\|\cdot\|_\sigma\). The space of all Taylor series with coefficients in \(\mathfrak{B}_\sigma(\mathbb{T}^j)\), uniformly convergent inside \(\mathbb{B}_j^\kappa\), with the supremum norm \(\|\cdot\|_{\kappa, \sigma}\) is denoted as \(\mathfrak{B}_{\kappa, \sigma}(\mathbb{T}^n)\). Referring to real-analytic functions on complex domain in the sequel means referring to their holomorphic extensions.

The same notation stands for the supremum norms of vector functions. At places the subscripts in the norm notation can be omitted. If a function, whose norm is evaluated depends on additional parameters, omitting these dependencies in the estimates implies their uniformity.

Thus, in the convex real-analytic set-up, there exists a set of parameters \(\{\gamma_0, \kappa_0, \sigma_0, R_0, M_0, \varepsilon_0\}\), with \(\kappa_0 > 1\) and \(\varepsilon_0 \ll 1\), such that the Hamiltonian \(\Pi\) satisfies the following.

**Model statement:**

1. For all \(\varepsilon \in [0, \varepsilon_0]\), \(H(p, q, \varepsilon) \in \mathfrak{B}_{\kappa_0, \sigma_0}(T^* \mathbb{T}^{n+1})\) and \(|H_0|_{\kappa_0} \leq M_0\), \(|H_1|_{\kappa_0, \sigma_0} \leq 1\).

2. The frequency \(\omega_0 \in \mathfrak{B}_{\gamma_0, \tau_0}\).

3. The constant matrix \(Q_0\) is positive definite, \(\|Q_0^{-1}\| \leq R_0^{-1}\).
For a specific Hamiltonian \([\mathcal{H}]\) the analyticity considerations may somehow single out the choice of the original coordinates \((p, q)\), see e.g. [10]. As a result, the parameters above as well as the bounding constants may also depend on \(k_0\).

\section*{2.3 Normal form near a resonance}

It is well known that one can come up with a normal form near a resonance [4]. Such a normal form for the Hamiltonian \([2.4]\) was used as a motivation for the results of [13] among others. Note that the normal form transformation belongs to the class \([2.1]\) and cannot be iterated in this form.

As \(\kappa_0 > 1\), one can accept \(M_0\) and \(1\) respectively as bounds for several orders of derivatives of \(H_0\) and \(H_1\) in \(p\). Assuming \(R_0 < 1\), let us get rid of this parameter as far as the quadratic part of \(H_0\) is concerned. Rewrite \([2.4]\) scaling the time and actions by factor \(\sqrt{\varepsilon/R_0}\), i.e. \(p \to \sqrt{\varepsilon/R_0} p, H \to \sqrt{R_0/\varepsilon} H\), and then divide the Hamiltonian by \(\sqrt{R_0}\) (tantamount to yet another time scaling). Then \([2.4]\) changes to

\[
H(p, q, \varepsilon) = \omega_1, I + \frac{1}{2} (Q_1 p, p) + \varepsilon^{-1} O_3(\sqrt{\varepsilon/R_0} p) + H_1(\sqrt{\varepsilon/R_0} p, q, \varepsilon),
\]

with the notations

\[
\omega_1 = \frac{\omega_0}{\sqrt{R_0}}, \quad Q_1 = R_0^{-1} Q_0.
\]

The analyticity domain of the scaled Hamiltonian \(H\) in the scaled action \(p\) is now a complex ball of radius \(O(\sqrt{R_0}/\varepsilon)\). The matrix \(Q_1\) is such that its spectrum is contained in \([1, C(n)M_0/R_0]\).

Decompose the quantity \(H_1(0, q, \varepsilon) = H_1(0, x, \varphi, \varepsilon)\) into a \(\varphi\)-mean \(U(x, \varepsilon)\) and an oscillatory part:

\[
H_1(0, x, \varphi, \varepsilon) = \int_{\pi n} H_1(0, x, \varphi, \varepsilon) d\varphi + \left( H_1(0, x, \varphi, \varepsilon) - \int_{\pi n} H_1(0, x, \varphi, \varepsilon) d\varphi \right)
\]

\[
= U(x, \varepsilon) + \{ H_1(0, x, \varphi, \varepsilon) \}.
\]

To get rid of the \(\varphi\)-oscillatory term \(\{ H_1(0, x, \varphi, \varepsilon) \}\) consider a canonical transformation \(\Xi_\nu\), which is tantamount to the shift \(p \to p + dS_\nu(q, \varepsilon)\). The 1-form \(dS_\nu\) is exact and is given by a \(2\pi\)-periodic in each component of \(q = (x, \varphi)\) function \(S_\nu\), satisfying a PDE

\[
D_{\omega_1} S_\nu(x, \varphi, \varepsilon) = -\{ H_1(0, x, \varphi, \varepsilon) \},
\]

with the general notation for \(\omega \in \mathbb{R}^n\)

\[
D_\omega \overset{\text{def}}{=} (\omega, D_\varphi).
\]

Note that \(x, \varepsilon\) enter the above equation as parameters, in particular \(S_\nu\) is defined modulo a function of \(x\). The solution of this equation exists in a somewhat larger space than that for the right hand side. The following result is well known [41].

\textbf{Proposition 2.3.1} Let \(\omega \in \mathbb{W}_{\kappa, \gamma}\). For a function \(v \in \mathcal{B}_{\kappa'}(\mathbb{T}^n)\) with zero average on \(\mathbb{T}^n\), the solution of the equation \(D_{\omega_1} u = v\) exists in the space \(\mathcal{B}_{\kappa'}(\mathbb{T}^n)\) for any \(\sigma' < \sigma\). If \(\sigma - \sigma' = \delta, \zeta = \gamma \delta^\rho\), then

\[
|u|_{\sigma'} \lesssim \zeta^{-1} |v|_{\sigma}, \quad |du|_{\sigma'} \lesssim (\zeta \delta)^{-1} |v|_{\sigma}.
\]

Then given \(\varepsilon\) small enough to ensure that the transformation \(\Xi_\nu\) be near identity, i.e. \(|dS_\nu| \ll 1\), suppressing \(\varepsilon\) in the notation one gets for \(H_\nu \equiv H \circ \Xi_\nu\):

\[
H_\nu(p, q) = (\omega_1, I) + \frac{1}{2} (Q_1 p, p) + U(x) + \left[ f_1(q) + \langle g_1(p, q), p \rangle \right],
\]

where

\[
|f_1|_{\sigma_1} \lesssim M_0 R_0^{-1} |dS_\nu|_{\sigma_1}^2, \quad |g_1|_{\sigma_1} \lesssim M_0 \sup (R_0^{-1} |dS_\nu|_{\sigma_1}, \sqrt{\varepsilon R_0^{-3}}),
\]
Assumption 1 (Perturbation of general position) the following assumption.

\[ \theta \in \mathbb{R}^n \] with a constant vector \( \theta \), uniformly non-degenerate absolute maximum on \( \mathbb{R}^n \). Without loss of generality one can assume

Let us take a closer look at the truncated normal form (2.7). The second clause of Lemma 2.1 implies that \( U(x) = O(1) \), i.e.

\[ \varepsilon \lesssim M_0^{-2}R_0[\inf(\varsigma_0 \delta_0), R_0]^2, \] \hspace{1cm} (2.8)

with the notation \( \varsigma_0 = \gamma_0 \delta_0^\gamma \). The results so far are summarized as follows.

**Lemma 2.1 (Normal form lemma)** Let \( p_0 \in \Omega \) lie on a regular level set of \( H_0 \) and for some \( k_0 \in \mathbb{Z}^{n+1} \), let \( \omega_0 = DH_0(p_0) \) be Diophantine modulo \( (k_0) \). Suppose, the localization (2.3) of Hamiltonian (1) near \( p = p_0 \) satisfies the Model statement with the frequency \( \omega_0 \in \mathbb{M}^\gamma_{\tau, \gamma_0} \) and the set of parameters \( \{ \gamma_0, \kappa_0, \sigma_0, M_0, \varepsilon_0 \} \). For \( \sigma_1 < \sigma_0 \), let \( \delta_0 = \sigma_0 - \sigma_1, \varsigma_0 = \gamma_0 \delta_0^\gamma \), and suppose

\[ \varepsilon_0 \lesssim (\varsigma_0 \delta_0)^2. \]

For any \( \varepsilon \in (0, \varepsilon_0) \), the Hamiltonian (1) can be cast into the normal form (2.6) where:

1. \( H_\nu \in \mathfrak{B}_{\kappa_1, \sigma_1}(T^* \mathbb{T}^{n+1}) \), with \( \kappa_1 = O(R_0/\sqrt{\varepsilon}) \);

2. The constant matrix \( Q_1 \) is positive definite, with \( \| Q_1^{-1} \| \lesssim 1, \| Q_1 \| \lesssim M_0/R_0 \);

3. One has

\[ |f_1|_{\sigma_1} \lesssim \varepsilon M_0(\varsigma_0 \delta_0)^{-2}, \quad |g_1|_{\sigma_1, \kappa_1} \lesssim \sqrt{\varepsilon}(M_0/\sqrt{R_0})[\inf(\varsigma_0 \delta_0), R_0]^{-1}. \] \hspace{1cm} (2.9)

**2.4 Localization near separatrix**

Let us take a closer look at the truncated normal form (2.7). The second clause of Lemma 2.1 implies that without loss of generality one can assume

\[ \langle Q_1p, p \rangle = y^2 + 2y(\theta, I) + \langle \Theta I, I \rangle, \]

with a constant vector \( \theta \in \mathbb{R}^n \), and a constant positive definite matrix \( \Theta \in \mathbb{R}^{n^2} \), whose smallest eigenvalue is at least one (a greater than one coefficient multiplying \( y^2 \) being favorable). In order to proceed one needs the following assumption.

**Assumption 1 (Perturbation of general position)** The function \( U = U(x, \varepsilon) \) possesses a unique uniformly non-degenerate absolute maximum on \( \mathbb{T} \) for all \( \varepsilon \in [0, \varepsilon_0] \), with a characteristic exponent \( \lambda \in \mathbb{R}_{++} \).

Without loss of generality let the maximizer be \( x = 0 \) for each \( \varepsilon \) (which as far as the above assumption is concerned is non-essential and will be further omitted in the notation) with \( U(0) = 0 \). Assumption(1) then is tantamount to the claim

\[ U(0) = 0, \quad U_x(0) = 0, \quad U_{xx}(0) = -\lambda^2, \quad \lambda^{-1} = O(1); \]

\[ \forall x_c \in \mathbb{T} \setminus \{0\} : \quad U_x(x_c) = 0, \quad U(x_c) < 0. \] \hspace{1cm} (2.10)

For the truncated normal form Hamiltonian (2.7) the action \( I \) is an integral of motion. For \( I = 0 \) one can single out a one-dimensional natural integrable system, whose Hamiltonian is \( g^2/2 + U(x) \). This is a reversible
Hamiltonian system in $T^*\mathbb{T} \cong \mathbb{R} \times \mathbb{T}$. Near a zero energy level, its phase portrait is reminiscent of the classical pendulum, Fig. 1. There is a saddle $(x, y) = (0, 0)$ connected to itself by a pair of simple non-contractable curves $y = \pm \sqrt{-2U(x)}$, forming a single $\infty$-shaped curve, further referred to as the separatrix.

Let $U(x) = \lambda^2U_1(x)$. Conditions (2.10) allow one to define a $4\pi$-periodic separatrix function $\psi(x)$, determined in general as well as the constant $\lambda$ by the pair $(k_0, H_1)$ and possibly depending on $\varepsilon$:

$$\psi(x) = \begin{cases} -\sqrt{-2U_1(x)}, & x \in [-2\pi, 0), \\ -\sqrt{2U_1(x)}, & x \in [0, 2\pi) \end{cases}$$

(having chosen the branch of the square root where $\sqrt{1} = 1$). Thus $\psi(0) = 0, \psi_x(0) = 1, \psi_{xx}(0) = O(1)$ and

$$\psi \circ l_{2\pi} = -\psi, \quad l_{2\pi} : x \to x + 2\pi.$$  

By (2.10) the function $\psi$ has no other zeroes on the real axis, but even multiples of $\pi$. For instance in the classical pendulum case $U(x) = \cos x - 1, \psi(x) = 2\sin x/2$. In the general case one can write $U_1(x) = (\cos x - 1)V(x)$, with some real-analytic $2\pi$-periodic function $V$, which has no zeroes on the real axis and $V(0) = 1$. Therefore $\psi(x) = 2\sin(x/2)\psi_1(x)$, where the function $\psi_1$ is real-analytic, $2\pi$-periodic, has no zeroes on the real axis, and $\psi_1(0) = 1$. Thus the function $\psi$ is real-analytic and has no zeroes in some neighborhood of the real axis, except even multiples of $\pi$. In particular, this property will be valid for $|3x| \leq \sigma_2 \leq \sigma_1$, for some $\sigma_2$.

In the full phase space $T^*\mathbb{T}^{n+1}$ the separatrix is represented by a Lagrangian manifold

$$W_\varepsilon = \{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{T}^{n+1} : p = dS_\varepsilon(q) = (\lambda\psi(x), 0)\},$$  

where the function $\psi$ is viewed as a double-valued function on $\mathbb{T}$, corresponding to an exact double-valued one-form $dS_\varepsilon$ on $\mathbb{T}^{n+1}$, given by a $\varphi$-independent generating function $S_\varepsilon(x, \varphi) = \lambda \int \psi(x)dx$, modulo a constant. The separatrix forms a coinciding unstable-stable manifold to an invariant torus $T_\varepsilon$ at $x = 0$, see Fig. 1.

One can localize (2.6) near the manifold $W_\varepsilon$. Let us make a formal change\(^3\) $y \to y + \lambda\psi(x)$ and denote $L_\psi$ the corresponding canonical transformation, acting as the identity on the pair $(I, \varphi)$. The transformation $L_\psi^{-1}$ acts on the base space variable $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ as a period doubling map $\mathbb{T} \to \mathbb{T}'$, where

$$\mathbb{T}' = \mathbb{R}/4\pi\mathbb{Z}.$$  

The transformation $L_\psi$ changes the phase space to $T^*(\mathbb{T}' \times \mathbb{T}^n)$ and incurs a topological change on the separatrix $W_\varepsilon$, doubling the point $(x, y) = (0, 0)$ on its projection on the $(x, y)$-plane, see the following Fig. 1. The manifold $W_\varepsilon$ now corresponds to the zero section of the bundle $T^*(\mathbb{T}' \times \mathbb{T}^n)$, with the identical zero generating function.

Let $H_\psi \equiv H_\nu \circ H_\psi$, now a $4\pi$-periodic function of $x$:

$$H_\psi(p, q) = \lambda\psi(x)y + (\omega_1 + \lambda\psi(x)\theta, I) + \frac{1}{2}\langle Q_1 p, p \rangle + [f_\psi(q) + \langle g_\psi(p, q), p \rangle].$$  

(2.14)

In particular $f_\psi = f_1 + \psi\tilde{f}_\psi$, where the function $\tilde{f}_\psi$ is determined by $g_1$ in (2.9). At the first glance after the transformation $L_\psi$, the bound for both $f_\psi$ and $g_\psi$ will be that for $g_1$ in (2.9). However this is not quite the case, as one may recall that the generating function $S_\nu$ of the canonical transformation $\Xi_\nu$ in Lemma (2.21) is defined modulo a function of $x$. Thus one can combine the two transformations $\Xi_\nu \circ L_\psi$ into one with the generating function $S_\nu + \lambda \int \psi(x)dx$, which enables one to improve the above estimates as follows:

$$|f_\psi|_\sigma \lesssim \sqrt{\varepsilon}M_0 \sup[\sqrt{\varepsilon}(s_0\delta_0)^{-2}, R_0^{-3/2}],$$

$$|g_\psi|_\kappa, \sigma \lesssim \sqrt{\varepsilon}M_0[\sqrt{\varepsilon}\inf(s_0\delta_0, R_0)]^{-1}.$$  

(2.15)

\(^3\)Note that one should not worry here about the analyticity domains, as $\kappa_1$ in Lemma (2.4) is large enough.
Remark 2.1: Without loss of generality, \( \lambda \leq 1 \). As far as the power of the parameter \( R_0 \) is concerned (the second entry in the sup and inf above) the estimates depend on whether or not the original unperturbed Hamiltonian \( H_0 \) in (11) contains super-quadratic terms.\(^4\)

Along with \( L_{\psi} \) let us denote \( L_{-\psi} \) a canonical transformation, effecting the shift \( y \to y - \lambda \psi(x) \), corresponding to the “lower” separatrix on the phase portrait of the truncated normal form Hamiltonian, Fig. 1. Studying the Hamiltonian \( H_{-\psi} \equiv H_\psi \circ L_{-\psi} \), further referred to as the spunktik of \( H_\psi \) in essence adds nothing new, as \( L_{-\psi} \) is tantamount to \( L_{\psi} \) followed by a shift \( l_{2\pi} \) of the \( x \)-variable, in view of \( 2\pi \)-antiperiodicity of the function \( \psi(x) \) and \( 2\pi \)-periodicity in \( x \) of the normal form Hamiltonian \( H_\psi \). On the other, the spunktik \( H_{-\psi} \) turns out to be a convenient way to describe \( H_\psi \) transformation, corresponding to the extension \( l_{2\pi} \) : \( (x, \varphi) \to (x + 2\pi, \varphi) \) of the shift \( l_{2\pi} \) in the base space (be it \( \mathbb{T}^{n+1}, \mathbb{T}' \times \mathbb{T}^n \) or further \( \mathbb{R} \times \mathbb{T}^n \); also let \( l_{-2\pi} = l_{2\pi}^{-1} \) ) one has

\[
L_{-\psi} \circ l_{2\pi} = L_{2\pi} \circ L_{\psi} \quad \text{and} \quad H_{-\psi} \circ l_{2\pi} = H_\psi,
\]

after applying \( H_\psi \) to the first relation. Using the same symbols \( L_{\pm\psi} \) for the transformations effecting the change \( y \to y \pm \lambda \psi(x) \) on \( T^*(\mathbb{T}' \times \mathbb{T}^n) \), one also has

\[
H_\psi = H_{\psi} \circ l_{2\pi} = H_{-\psi} \circ l_{-\psi},
\]

and a useful identity follows:

\[
H_{\psi} = H_{\psi} \circ l_{2\pi} \circ L_{\psi}^2 = H_{\psi} \circ L_{-\psi}^{-2} \circ l_{2\pi}.
\]

For the flow of the Hamiltonian \( H_{\psi, t} \equiv H_{\psi, t} \circ L_{\psi} \) on \( T^*(\mathbb{T}' \times \mathbb{T}^n) \), the invariant manifold \( W_t \) contains a pair of invariant whiskered tori \( T_{u,t} \) and \( T_{s,t} \) such \( x = 0 \) on the former torus and \( x = 2\pi \) on the latter one. \( W_t \) is the unstable manifold for \( T_{u,t} \) and the stable manifold for \( T_{s,t} \). The two tori can be identified via the transformation \( L_{\psi}^{-1} \).

The transformation \( L_{\psi} \) not only “doubles” the base space but also the separatrix. Indeed, the two branches thereof in the truncated normal form \( H_{\psi, t} \) are not only graphs over the base space, but also one over the other. Thus the tori \( T_{u,t} \) and \( T_{s,t} \) not only possess an unstable/stable manifold respectively, which is \( W_t \), the zero section of \( T^*(\mathbb{T}' \times \mathbb{T}^n) \), but also its spunktik

\[
W'_t = \{(p,q) \in \mathbb{R}^{n+1} \times (\mathbb{T}' \times \mathbb{T}^n) : p = -2dS_t(q) = (-2\lambda \psi(x), 0)\},
\]

which is the stable manifold for \( T_{u,t} \) and the unstable manifold for \( T_{s,t} \). This fact is in essence reflected by (2.10) being this the symmetry in the Hamiltonian \( H_\psi \) which enables one to identify\(^5\) the manifolds \( W_t \) and \( W'_t \). On the other hand, \( W'_t \) is clearly a flow-invariant zero section for the Hamiltonian \( H_{-\psi, t} = H_{-\psi, t} \circ L_{-\psi} \). Further a spunktik Hamiltonian will be marked by a prime, e.g. \( H_{-\psi} = H_{-\psi}' \).

**Splitting problem**

As the perturbation in (2.10) is not identically zero (more precisely the zero order term thereof in the Taylor expansion in \( p \)) the manifold \( W_t \) no longer lies inside the energy surface of \( H_\psi \) (for which it is the zero section). One can expect the following scenario, Fig. 1.

A perturbation of general position causes the manifold \( W_t \) to bifurcate, or split into a pair of distinct Lagrangian manifolds, denoted as \( W_u \) and \( W_s \). If the perturbation is small enough, the two manifolds can be described as graphs of closed one-forms \( dS_u \) and \( dS_s \). The forms \( dS_{u,s} \) are well defined over the cylinders

---

\(^4\)The perturbation in (2.10) is evaluated not only at \( y = 0 \), but also quite far away from it on the separatrix. The characteristic size of the separatrix in the original action variables of the Hamiltonian (2.11) is \( \Delta \sim \sqrt{\varepsilon R_1} \), and in order that the super-quadratic term be considered as a perturbation of the (quadratic in momenta) resonant normal form, one should have \( \varepsilon \gg \Delta^3 \), thus \( \varepsilon \ll R_3^3 \). It is nevertheless irrelevant apropos of the estimates, regarding the preservation of the invariant torus at \( x = 0 \), in particular because \( \psi(0) = 0 \) (one would have to look still closer at the structure of the acquired term \( f_3 \psi \) to see that).

\(^5\)By looking locally at the Hamiltonian vector field generated by \( H_\psi \), an observer won’t be able to tell whether it is applied at a point \( (y, I, x, \varphi) \) or \( (y - 2 \lambda \psi(x), I, x + 2\pi, \varphi) \). Thus they won’t be able to tell \( W_t \) and \( W'_t \) as well as \( T_{u,t} \) and \( T_{s,t} \) apart.
\( \mathcal{I}_{u,s} \times \mathbb{T}^n \) respectively, where \( \mathcal{I}_u = [-2\pi + r, 2\pi - r] \), for some \( r < 1 \) (characteristic of the quantity \( \psi \)) and \( \mathcal{I}_s = 1 - 2\pi (\mathcal{I}_u) \). Each manifold \( W_{u,s} \) contains an invariant torus \( \mathcal{T}_{u,s} \), being the unstable manifold for \( \mathcal{T}_u \) and the stable one for \( \mathcal{T}_s \).

One should be able to identify the tori \( \mathcal{T}_{u,s} \) via the transformation \( L^{-1}_\psi \). This in particular requires that both one-forms \( dS_u \) belong to the same cohomology class:

\[
[dS_u] = [dS_s] = \xi \in \mathbb{R}^n \cong H^1(\mathcal{I}_{u,s} \times \mathbb{T}^n, \mathbb{R}).
\]

This fact is easy to establish due to the fact that the sputnik manifold \( W'_u \) will split just the same, to which there will correspond a pair of closed one-forms \( dS'_u \) and \( dS'_s \). It will be easy to see that say \( dS_u \) and \( dS'_u \) have the same cohomology class, as they in particular describe the same torus \( \mathcal{T}_u \). On the other hand \( (2.16) \) claims a congruency between the graphs of the forms \( dS_u, dS'_s \).

To measure the distance between the two manifolds \( W_u \) and \( W_s \) and to study their intersections, one naturally uses an exact 1-form \( d\mathcal{S} = dS_u - dS_s \), well defined on the union of two disjoint cylinders \( (\mathcal{I}_- \cup \mathcal{I}_+) \times \mathbb{T}^n \) where \( \mathcal{I}_- = [-2\pi + r, -r] \) and \( \mathcal{I}_+ = -\mathcal{I}_- \). Let \( \beta \in \mathbb{Z}_2 \equiv \{+, -\} \) be the sign of \( x \), then the notation \( d\mathcal{S}_\beta \) stands for the restriction of \( d\mathcal{S} \) on \( \mathcal{I}_\beta \times \mathbb{T}^n \). The values of \( \beta = +, - \) correspond to the splitting of the “upper” and “lower” separatrices of the truncated normal form Hamiltonian \( H_{\tau, \pi} \), see Fig. 1. The splitting of the sputnik manifold \( W'_u \), alias the zero section for \( H' = H_{-\psi} \) is described by \( -d\mathcal{S}_- \), by \( (2.16) \).

The generating function \( \mathcal{G}_\beta \) is real-analytic on \( \mathcal{I}_\beta \times \mathbb{T}^n \), where it satisfies a linear homogeneous Hamilton-Jacobi equation, whose coefficients can be made constant via a change of variables. Analyzing the result of the latter change, one has the following theorem.

**Theorem 1** Let \( p_0 \in \Omega \) lie on a regular level set of \( H_0 \) and for some \( k_0 \in \mathbb{Z}^{n+1} \), let \( \omega_0 = DH_0(p_0) \) be in \( \mathfrak{S}^{\tau, \rho, \epsilon}(k_0) \), corresponding to a simple resonance torus \( \mathcal{T}_0 \) with the foliation \( (3) \) in terms of \( x \in \mathbb{T} \). Suppose the localization \( (2.4) \) of Hamiltonian \( (1) \) near \( p = p_0 \) satisfies the Model statement with the frequency \( \omega_0 = \mathfrak{S}_{\tau, \rho, \epsilon} \) and parameters \( \{k_0, \sigma_0, R_0, M_0, \varepsilon_0\} \). Suppose the perturbation \( H_1 \) satisfies Assumption \( (1) \) with the characteristic exponent \( \lambda \) and the separatrix function \( \psi \). For \( \sigma_1 < \sigma_0 \), let \( \delta_0 = \sigma_0 - \sigma_1, \gamma_0 = \gamma_0 \delta_0 \); suppose

\[
\varepsilon_0 \leq (C M_0)^{-2} R_0 |\lambda^2 \inf(\omega_0 \delta_0, R_0))|^2 \equiv \eta^2,
\]

for some large enough \( C = C(n, \tau, k_0, \sigma_0, \psi) \).

For any \( \varepsilon \in (0, \varepsilon_0) \), continuously if \( H_1(\cdot, \varepsilon) \) is continuous, there exists a pair of analytic Lagrangian manifolds \( W_{u,s} \), intersecting at an invariant \( n \)-torus \( \mathcal{T} \), on which the flow of \( (1) \) is conjugate to a rotation by \( \omega_0 \). Each manifold is locally a graph over \( \mathbb{T} \times \mathbb{T}^n \). Away from \( \mathcal{T} \), the distance \( \mathcal{S} \) between \( W_u \) and \( W_s \) is bounded by

\[
|\mathcal{S}| \leq \sqrt{\varepsilon \eta^{-1}} \sum_{k \in \mathbb{Z}^n \backslash \{0\}} \exp \left(-|k, \rho R_0 \omega_0 \frac{\lambda^2}{\varepsilon} + \sigma_2 \theta| - |k| \sigma_1 \right),
\]

where \( \rho < \frac{\pi}{2}, \sigma_2 \leq \sigma_1 \), and the quantities \( \rho, \sigma_2, \sigma_1 \) are well defined for \( H \). The manifolds \( W_u \) and \( W_s \) also intersect along at least \( 2n + 2 \) orbits, biasymptotic to \( \mathcal{T} \).

**Remark 2.2:** In view of the smallness condition \( (2.18) \) and the fact that \( \theta \leq M_0 \) in \( (2.19) \), the contribution of the quantity \( \sigma_2 \theta \) in the estimate \( (2.19) \) becomes important for a Diophantine \( \omega_0 \) when \( |k| \sim \varepsilon^{-\frac{1}{\pi M_0}} \) and will play an extra role if one attempts to estimate \( |\mathcal{S}| \) from below \( (33) \). Where exactly the quantities \( \rho \) and \( \sigma_2 \) arise is explained further, see in particular \( (3.7, 3.9) \) and Fig. 2.

It is possible to simplify \( (2.14) \) further by eliminating the constant \( \theta \in \mathbb{R}^n \) therein, letting

\[
\Xi_\theta: \begin{cases} 
  x = x', \\
  \varphi = \varphi' + \theta x', \\
  y = y' - \langle \theta, I \rangle, \\
  I = I'.
\end{cases}
\]

(2.20)
The base space transformation, corresponding to $\Xi_\theta$ will be denoted as $a_\theta$. Unless $\theta \in \mathbb{Q}^n$, the pre-image of the base space $T' \times T^n$ under the transformation $a_\theta$ is a bi-infinite cylinder $\mathbb{R} \times T^n$. In other words, the transformation $\Xi_\theta$ almost surely results in the loss of $4\pi$-periodicity in the “hyperbolic coordinate” $x$.

If $H_\theta \equiv H_\psi \circ \Xi_\theta$ then

$$H_\theta(p, q) = \lambda \psi(x)y + \langle \omega_1, I \rangle + \frac{1}{2} \langle Q_2 p, p \rangle + [f_\theta(q) + \langle g_\theta(p, q), p \rangle],$$

(2.21)

where the matrix $Q_2$ arises from $Q_1$ as a result of Gaussian elimination of off-diagonal elements in the first row and the first column. So $Q_2$ is non-degenerate with the determinant at least one and the eigenvalue of largest absolute value being bounded in terms of $M_0 R_0^{-1}$.

An apparent change of the analyticity domain of Hamiltonian $\Xi_\theta$ as far as the variables $\varphi$ are concerned is easy to take into account; this will be done in Section 4. For now let us assume that $H_\theta(\cdot, \varphi)$ is real-analytic for $\varphi \in T^n$ for some $\sigma$. The only inevitable analyticity loss so far has been $\delta_0$ in the application of Lemma 2.1. The actions $p$ live in a complex ball around the origin, whose radius is “as large as necessary”, provided (2.8) is satisfied.

It is easy to see that

$$\Xi_\theta \circ L_\psi = L_\psi \circ \Xi_\theta, \quad L_{2n}^j \circ \Xi_\theta = \Xi_\theta \circ L_{2\pi}^{2j} \circ L_{2n}^j,$$

where for $j \in \mathbb{Z}$, $L_{2\pi}^j$ is a canonical transformation, corresponding to the base space diffeomorphism $L_{2\pi}^j : (x, \varphi) \rightarrow (x, \varphi + 2\pi j \vartheta)$. Then (2.21) gets modified to

$$H_\theta = H_\psi \circ L_{2\pi}^{2j} \circ L_{\psi}^{2\alpha(j)} \circ L_{2\pi}^{j} = H_\varphi \circ L_{2\pi}^{2\alpha(j)} \circ L_{\psi}^{2j} \circ L_{2\pi}^{j},$$

(2.22)

where $\alpha(j) \equiv \begin{cases} 0, & j \text{ even}, \\ 1, & j \text{ odd} \end{cases}$ is the parity of $j$. In other words (2.22) reads

$$H_\theta(y, I, x, \varphi) = H_\psi(y + 2\alpha(j)\lambda \psi(x), I, x - 2\pi j, \varphi + 2\pi j)$$

$$= H_\psi(y - 2\alpha(j)\lambda \psi(x - 2\pi j), I, x - 2\pi j, \varphi + 2\pi j).$$

The manifold $W_\epsilon$ is now represented by the zero section of the bundle $T^*(\mathbb{R} \times T^n)$, which contains unstable or stable tori $\mathcal{T}_{j, \epsilon}$ for respectively even or odd values of $j$. All the unstable [stable] tori can be identified with one another via the transformation $\Xi_\theta^{-1}$. In Fig. 1 the tori $\mathcal{T}_{0, \epsilon}, \mathcal{T}_{1, \epsilon}$ correspond to $j = 0, \pm 1$ respectively.

Clearly, as it was the case with $H_\varphi$, it suffices still suffices knowing $H_\theta$ on the interval $x \in [0, 2\pi)$ only. The relation (2.22) applied to the truncated Hamiltonian

$$H_{\theta, \epsilon} = \lambda \psi(x)y + \frac{y^2}{2} + \langle \omega_1, I \rangle + \frac{1}{2} \langle \Theta I, I \rangle$$

simply implies that if one writes

$$\lambda \psi(x)y + \frac{y^2}{2} = \frac{y}{2}(2\lambda \psi(x) + y),$$

one sees the sputnik manifold $W'_\epsilon$, where $y = -2\lambda \psi(x)$. Naturally this manifold is the zero section for the Hamiltonian $H'_\theta \equiv H_{-\psi} \circ \Xi_\theta$. This will transform the latter expression to

$$-\lambda \psi(x)y + \frac{y^2}{2},$$

which is tantamount to the shift $x \rightarrow x + 2\pi$ in view of $2\pi$-antiperiodicity of $\psi$. Similarly changing $y \rightarrow -2\lambda \psi(x) + y$ in the perturbation $\lambda \psi(x) + \frac{y^2}{2}$ is tantamount to changing $x \rightarrow x + 2\pi$ and $\varphi \rightarrow \varphi - 2\pi \theta$.

It is convenient to treat $H_\theta$ as a multi-valued real-analytic function on $T^*(\mathbb{R} \times T^n)$, whose branch is specified by fixing an even value of $j$ in (2.22) and $j = 0$ suffices for consideration. The branches differ by the shift of the angle $\varphi$ by an integer multiple of $4\pi \theta$. The splitting problem is well-posed for a chosen branch of $H_\theta$ and the magnitude of splitting is clearly the same on each branch. Technically, first one restricts $x$ to an interval $I_\theta = I_\psi$. Theorem 2 furnishes a Lagrangian manifold $W_0$ as a graph over $I_\theta \times T^n$, containing an
invariant torus $T_0$ near $x = 0$, for which it is the unstable manifold. $W_0$ is described by a generating function $S_0$. Theorem 2 also results in the stable sputnik manifold $W_0'$ of the torus $T_0$, described by a generating function $S_0'$. Theorem 2 also results in the stable sputnik manifold $W_0'$ of the torus $T_0$, described by a generating function $S_0'$. Then $x$ is restricted to an interval $I_- = I_s$ and one gets the Lagrangian manifold $W_-$ as a graph over $I_- \times \mathbb{T}^n$ containing an invariant torus $T_-$ near $x = -2\pi$, described by the generating function $S_-$. By $S_+ = S_0' \circ l_{2\pi} \circ l_{-2\pi\theta}$. Moreover starting from the pair $(S_0, S_-)$ (corresponding to $j = 0, -1$) using (2.22) one can define pairs of manifolds $(W_j, W_j')$ for all $j$, containing invariant tori $T_j$ ($W_j$ or $W_j'$ being respectively unstable or stable manifolds for even or odd values of $j$ respectively) as graphs over $I_{2\pi j}(I_0) \times \mathbb{T}^n$, with generating functions $S_j = S_0 - S_j \circ l_{4\pi} \circ l_{4\pi\theta}$, all characterized by the same $\xi$. In particular, $S_+ = S_- \circ l_{-4\pi} \circ l_{4\pi\theta}$ corresponds to $j = 1$.

All the manifolds $W_j$ for even or odd $j$ are identified respectively with $W_u$ or $W_s$ via the transformation $\Xi_\theta^{-1}$, it suffices to introduce the splitting function in the same way as it was described above, identifying $\beta = \pm$ with $j = \pm 1$ respectively. With the same notation one has $S_\beta = S_0 - S_\beta$, well defined on $I_\beta \times \mathbb{T}^n$.

This completes the construction of the normal form, and calls for a structural stability theory for Hamiltonians like $H_\theta$, which underlies the proof of Theorem 1. Another goal is to make this theory amenable to the presence of the sputnik symmetry, in order to be able to conclude that $\xi = [dS_j]$, $\forall j$. Both issues are studied at length in the next section.

3 KAM theory on semi-infinite bi-cylinders over tori

As the forthcoming theory is self-contained, the notation in this section may be occasionally different from the preceding sections. E.g. the function $\psi(x)$ is introduced axiomatically, rather than by (2.11). For structural stability no symmetry properties of $\psi$ are required, which on the other hand are essential for the splitting...
problem. The details of the set-up arising in connection with the 2π-antiperiodicity of \( \psi \) are not addressed until Section 4.

### 3.1 Energy-time coordinates

**Time-map**

Consider a real-analytic function \( \psi(x) \) for \( x \) in a closed real interval \( \mathcal{I} \), for definitly containing the points \( \pm \pi \) in the interior. Suppose \( \psi(0) = 0, \psi_x(0) = 1, \psi(x) \neq 0 \) on \( \mathcal{I} \setminus \{0\} \) and is uniformly bounded with its first two derivatives. Then \( \psi(x) \) allows a holomorphic extension into some closed rectangular domain \( \mathcal{D} \subseteq \mathbb{C} \), symmetric with respect to the real axis and containing \( \mathcal{I} \) together with a ball \( \mathbb{B}_r \) at the origin for some \( r \); such that \( \psi \) and its first two derivatives are bounded inside \( \mathcal{D} \) and apart from that \( |\psi(x)| \geq r/2 \) in \( \mathcal{D} \setminus \mathbb{B}_r \). The pair \( (\psi, \mathcal{D}) \) is regarded as fixed.

For \( x \in \mathcal{D} \setminus \{0\} \) consider a map \( s \) from \( \mathcal{D} \) into a Riemannian surface \( \mathcal{S} \) of the logarithmic type and its inverse \( x \) as follows:

\[
s : x \rightarrow \int_{-\pi}^{\pi} \frac{d\zeta}{\psi(\zeta)}, \quad x = s^{-1},
\]

as well as

\[
\chi : \mathcal{S} \rightarrow \mathcal{D}, \quad \chi = \psi \circ x.
\]

The maps \( x, \chi \) can be represented by homonymous functions of a complex variable \( s \in \mathbb{C} \), which are 2\( \pi \)-periodic and well defined in a family of semi-infinite strips about the rays \( \Re s \in (-\infty, T], \Im s = \pi j, j \in \mathbb{Z} \) for some \( T \), see Fig. 2. They are real-valued on the above rays and vanish exponentially as \( \Re s \rightarrow -\infty \). Without loss of generality \( T > 1 \).

Let us further consider only such values of the parameters \( r, T \) that \( 0 < r_\psi < r < 2r_\psi < 1, 1 < T_\psi < T < 2T_\psi \), for some fixed pair \( (r_\psi, T_\psi) \) defined in terms of \( (\psi, \mathcal{D}) \) and such that \( \mathbb{B}_{2r_\psi} \subset \mathcal{D} \) and \( T_\psi > -2\log r_\psi \). The functions \( x(s), \chi(s) \) are holomorphic in the half-plane \( \Re s \leq -2T_\psi \). Further estimates will ignore constants depending on the pair \( (\psi, \mathcal{D}) \) as well as constants \( n, \tau \) in the Diophantine condition \( \text{[23]} \) by using the \( \lesssim, \gtrsim \) and \( \sim \) symbols in an obvious way.

For \( \rho > 0 \) and \( s \in \mathbb{C} \) let

\[
\Lambda_{T,\rho} = \{ \Re s \leq T, |\Im s| \leq \rho \} \cup \{ \Re s \leq -2T_\psi \} \cup \{ \Re s \leq T, |\Im s - \pi| \leq \rho \},
\]

be further referred to as complex bi-strips. Their projections on the union of the real axis and the line \( \Im s = \pi \) will be denoted by omission of the index \( \rho \). The index \( T \) may also be omitted in qualitative argument. On the other hand, \( \Lambda_\infty \) stands for a pair of lines \( \mathbb{R} \cup \mathbb{R} + i \) (the + or - sign henceforth having a priority over \( \cup, \cap \)). The difference between the case of a finite \( T \) and \( T = \infty \) will be emphasized. Also define a bounded one-strip (rectangle)

\[
\Pi_{T,\rho} = \{ s \in \mathbb{C} : |\Re s| \leq T, |\Im s| \leq \rho \},
\]

with the same index drop rules. Clearly \( \Lambda_{T,\rho} = \Pi_{T,\rho} \cup \Pi_{T,\rho} + i\pi \).

For any real-analytic function \( \tilde{u} \) on \( \mathcal{D} \), the composition \( u = \tilde{u} \circ s \) returns real values for \( \Im s = 0, \pi \), let’s coin the term “bi-real-analytic” for that. With the above notations for the domains, the functions \( x(s), \chi(s) \) will be referred to as bi-real-analytic for \( s \in \Lambda_{T,\rho} \), for some \( (T, \rho) \). For a function \( \psi \), given by \( \text{[23]} \) one will naturally have \( \rho < \frac{\pi}{2} \) in Section 4.

Suppose \( s \in \Lambda \) and \( h \) is a canonically conjugate momentum to \( x \). Consider a canonical transformation \( \Xi_s \) from \( T^* \Lambda \) into \( T^* \mathcal{I} \) as follows

\[
\Xi_s : \begin{cases} 
  x = x(s), \\
  y = \frac{h}{\chi(s)}
\end{cases},
\]

where the maps \( x, \chi \) have been defined by \( \text{[3.1 3.2]} \).
Let us extend the maps \( s, x \) to maps \( s, x \) between \( \Lambda \times \mathbb{T}^n \) and \( \mathcal{I} \times \mathbb{T}^n \) acting as the identity on \( \varphi \in \mathbb{T}^n \), in accordance with the general convention of using bold symbols referring to the whole base space. Extend accordingly the transformation \( \Xi_s \) to \( \Xi_s \), incorporating the pair \( (I, \varphi) \in T^* \mathbb{T}^n \). Let

\[
C_T = \Lambda_T \times \mathbb{T}^n, \quad C_T = \Lambda_T \times \mathbb{T}^n, \quad \tilde{C}_T = C_T \cap C_T^\ast \tag{3.6}
\]

be referred to as bi-cylinders over tori, further just “bi-cylinders”. In particular, \( C_\infty = \Lambda_\infty \times \mathbb{T}^n \). In this section only semi-infinite bi-cylinders \( C_T \) will be dealt with. Bi-infinite and bounded bi-cylinders \( C_\infty \) and \( \tilde{C}_T \) will come into play in Section 4. In qualitative argument, the index \( T \), if finite (unlike \( T = \infty \)) is often omitted further.

**Analyticity domains**

Let us describe more precisely the analyticity domains for the map \( s \). Let us extend the maps \( s, x \) to maps \( s, x \) between \( \Lambda \times \mathbb{T}^n \) and \( \mathcal{I} \times \mathbb{T}^n \) acting as the identity on \( \varphi \in \mathbb{T}^n \), in accordance with the general convention of using bold symbols referring to the whole base space. Extend accordingly the transformation \( \Xi_s \) to \( \Xi_s \), incorporating the pair \( (I, \varphi) \in T^* \mathbb{T}^n \). Let

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Let us describe more precisely the analyticity domains for the map \( s \) in order to further define the necessary function spaces on them. Technical difficulties will arise from the fact that the bi-cylinders \( C_T, C_\infty \) are not compact. E.g. a “near-identity” transformation \( a \) of Alexandroff compactification of \( C_T \) should not necessarily preserve \( \{ s = -\infty \} \), i.e. the differential \( da \) may be unbounded. Similarly a Hamiltonian of general position on \( T^* \mathcal{C} \) may be unbounded as \( s \to -\infty \), unless the momentum \( h = 0 \). The reason is clearly because \( ds(0) \) does not exist. In a series of papers [7, 14, 15], etc. these difficulties were overcome via improper integration techniques.

Analyticity domains of functions involved will be characterized by positive parameter vectors \( p \) as follows. Let \( p = (r, T, \rho, \sigma) \in \mathbb{R}^4_+ \). Introduce partial order \( p' = (r', T', \rho', \sigma') \preceq p \) if \( r' \leq r, T' \leq T, \rho' \leq \rho, \sigma' \leq \sigma \). If \( p' \preceq p \) and \( |p - p'| = \inf(r - r', T - T', \rho - \rho', \sigma - \sigma') > 0 \), write \( p' < p \). Addition of parameter vectors, as well as multiplication by positive real numbers is defined component-wise, as well as the difference \( p - p' \) for \( p' < p \). For \( \Delta \in \mathbb{R}_+, \Delta < |p| \) the notation \( p' = p - \Delta \) means subtracting \( \Delta \) component-wise. In the sequel the components and dimension of the parameter vectors \( p \) may vary; \( p \) can incorporate \( T = \infty \).

Given \( T \) such that both points \( x : \Re s(x) = T \) (of opposite signs) are in the interior of \( \mathcal{I} \), one may want to be able to describe the widest complex strip \( \Lambda_T, \rho \) for some \( \rho \) such that the image \( x(\Lambda_T, \rho) \) be contained in \( \mathcal{D} \). This can be done as follows, see Fig. 2.

![Figure 2: The map x. The figure illustrates how the quantities \( \rho, \sigma \) can be determined relative to the pair \( (\psi, \mathcal{D}) \). The shaded region on the left, including the circle around the origin is the domain \( \mathcal{I}_{T, \rho} \).](image)

Let the level set \( \Re s(x) = T \) for \( x \in \mathcal{D} \) intersect the real axis transversely at a pair of points \( P_0 \in (\pi, \infty) \) and \( P_\pi \in (-\infty, 0) \). Let \( \gamma_{0,T} \) be a connected component of the above level set containing the former point and \( \gamma_{\pi,T} \) - containing the latter point (the two \( \gamma \)'s may coincide). The points \( P_0, P_\pi \) are connected to the origin by the level curves \( \Im s(x) = 0, \pi \) respectively. Let \( P_\zeta \in \gamma_{0,T} \) or \( \gamma_{\pi,T} : \Im (P_\zeta) = \zeta \). For the level set
\( \Im(s(x) = \zeta, x \in \mathcal{D}, \) let \( \gamma^*_\zeta \) be the connected component, whose closure contains the origin. Let

\[
\begin{align*}
\rho_+ &= \sup \{ \zeta: 0 < \zeta \leq \pi, P_\zeta \in \gamma^*_\zeta, P_{-\zeta} \in \gamma^*_{-\zeta} \}, \\
\rho_- &= \sup \{ \zeta: 0 < \zeta \leq \pi, P_{\pi+\zeta} \in \gamma^*_{\pi+\zeta}, P_{\pi-\zeta} \in \gamma^*_{\pi-\zeta} \}, \\
\rho &= \inf(\rho_-, \rho_+).
\end{align*}
\]

In other words, the quantity \( \rho \) simply shows for how long the points \( P_{0, \pi} \) can be moved along the connected components of the level curves \( \Re s = T \), to which they belong, so that the whole segment of the level curve of \( \Im(s(x) \) connecting them to the origin remains contained in \( \mathcal{D} \). Either \( \rho \in (0, \pi/2] \) or \( \rho = \pi \), which corresponds to the case when \( \Re s = T \) is a simple closed curve contained in \( \mathcal{D} \), which together with its interior forms the image of the half-plane \( \Re s \leq T \) in \( \mathcal{D} \). If \( \rho < \frac{\pi}{2} \) (the only case of interest for the splitting problem) the equality \( |\Im(s(x)| = \rho \) or \( |\Im(s(x) - \pi| = \rho \) can be achieved on four different level curve segments \( \gamma^*_\zeta \), where \( \zeta = \pm \rho \) or \( \zeta - \pi = \pm \rho \) in Fig. 2. Marking these curve segments simply as \( \gamma^*_j, j = 1, 2, 3, 4 \), define

\[
\sigma_2 = \inf_{j=1,2,3,4} \sup_{x \in \gamma^*_j} |\Im(x)|.
\]

In essence, these are the parameters \( \rho, \sigma_2 \) entering the main estimate (2.19) of Theorem II the quantity \( \sigma_2 \) will not reappear until the end of Section 4. If one is willing to go into more detail, one should consider the above quantities as four-vectors to account for each \( \gamma^*_j \).

For \( p = (r, T, \rho) \) denote

\[
\mathcal{I}_p = \mathbb{B}_r \cup x(\Lambda_{\rho, T}).
\]

For complex extensions of the bi-cylinders defined by (3.9) introduce the notations

\[
\mathcal{C}_{T, \rho, \sigma} = \Lambda_{T, \rho} \times \mathbb{T}^n,
\]

as well as \( \mathcal{C}^-_{T, \rho, \sigma}, \mathcal{C}^+_{T, \rho, \sigma}, \hat{\mathcal{C}}_{T, \rho, \sigma} \) analogous to (3.10).

**Function spaces**

Defined below are the necessary spaces of bi-real-analytic functions on the bi-cylinders \( \mathcal{C} \) as well as their maps. This is done simply via the composition of real-analytic functions on or diffeomorphisms of \( \mathcal{I} \times \mathbb{T}^n \) with the bi-real-analytic map \( x \). One needs the following formalism in order to proceed toward an implicit function theorem for structural stability of vector fields or Hamiltonians on \( \mathcal{C} \) or \( T^* \mathcal{C} \) to be further used for exponentially small splitting estimates. However, the theorem in question is interesting in its own right as a "non-compact" version of KAM theory.

Let \( \mathcal{B}^j(\mathcal{D}), j \in \mathbb{Z}_+ \) be spaces of functions real-analytic and uniformly bounded in \( \mathcal{D} \), whose Taylor series at \( x = 0 \) starts at order \( j; j = 0 \) will be further omitted. With topology induced by the supremum norm, \( \mathcal{B}^j(\mathcal{D}) \) are Banach spaces. Any \( u \in \mathcal{B}^j(\mathcal{D}) \) can be represented as \( u(x) = x^j v(x) \), where \( v \in \mathcal{B}(\mathcal{D}) \), or alternatively as \( u(x) = \psi^j(x) w(x) \), where \( w \in \mathcal{B}(\mathcal{D}) \). One can take the supremum of \( |v| \) or \( |w| \) for an equivalent norm of \( u \), the comparison constants depending on the pair \( (\psi, \mathcal{D}) \) only. For \( p = (r, T, \rho) \) define the spaces \( \mathcal{B}^j(\mathcal{I}_p) \) in the same way as \( \mathcal{B}^j(\mathcal{D}) \). As \( \mathcal{I}_p \subseteq \mathcal{D} \), clearly \( \mathcal{B}^j(\mathcal{D}) \subseteq \mathcal{B}^j(\mathcal{I}_p) \subseteq \mathcal{B}^j(\mathcal{I}_{p'}) \) for \( p' < p \). For coherence with the forthcoming notation, let us write \( \mathcal{B}^j(\mathcal{I}) \) instead of \( \mathcal{B}^j(\mathcal{I}_p) \).

If a function \( u \in \mathcal{B}_p(\mathcal{I}) \) has an extra analytic dependence in \( \varphi \in \mathbb{T}^n, 2\pi \)-periodic in each component of \( \varphi \), one adds an extra component \( \sigma \) in the above parameter vector \( p \) and writes \( u \in \mathcal{B}_p^j(\mathcal{I} \times \mathbb{T}^n) \), \( u = u(x, \varphi) \). Define the set \( \mathcal{B}^j_{p}(\mathcal{C}) \) of all holomorphic functions \( u \) on \( \mathcal{C}_{p, \rho, \sigma} \), such that \( u = \tilde{u} \times x \) for some \( \tilde{u} \in \mathcal{B}^j_{p}(\mathcal{I} \times \mathbb{T}^n) \). E.g. consider a graph \( y = D_y s(x) \) of a one-form in the variables \( (y, x) \in T^* \mathcal{I} \). If \( S(s) = \tilde{S}[x(s)] \), then the corresponding graph in the variables \( (h, s) \) obtained via \( D_x \) is \( h = dS(s) \), so \( y = \chi^{-1}(s) dS(s) = O(1) \). I.e. a \( \varphi \)-independent one-form \( ds(s) \) over \( \mathcal{C} \) vanishes exponentially as \( s \to \infty \).
\( \mathfrak{B}_p^2(\mathcal{C}) \) is a closed subspace in the Banach space of all bounded holomorphic functions on \( \mathcal{C}_{r,T,\rho,\sigma} \) and thus a Banach space itself, with the supremum norm \( \| \cdot \|_p \). Note that if \( u \in \mathfrak{B}_p^2(\mathcal{C}) \), then the function \( u[s(x), \varphi] \) allows analytic continuation into the neighborhood \( \mathfrak{B}_p \) of \( x = 0 \), vanishing at \( x = 0 \) to the \( j \)th order. This can be taken for an independent definition of the spaces \( \mathfrak{B}_p^2(\mathcal{C}) \). Moreover, if \( u(s, \varphi) \in \mathfrak{B}_p^2(\mathcal{C}) \), a multiplier \( \chi_j(s) \) can be factored out, i.e.

\[
u(s, \varphi) = \chi_j(s) v(s, \varphi), \quad v \in \mathfrak{B}_p^2(\mathcal{C}), \quad |v|_p \leq |u|_p.
\]

Section 4 will deal with bi-real-analytic functions on bi-infinite and bounded bi-cylinders. To this effect, if \( u \in \mathfrak{B}_{r,T,\rho,\sigma}(\mathfrak{C}) \) allows a uniformly bounded analytic continuation as \( T \to \infty \), then write \( u \in \mathfrak{B}_{r,T,\rho,\sigma}(\mathfrak{C}) \) or \( u \in \mathfrak{B}_{r,\rho,\sigma}(\mathfrak{C}_{\infty}) \).

Besides \( \mathfrak{B}_{r,T,\rho,\sigma}(\mathfrak{C}) \) stands for the space of bi-real-analytic functions \( u(s, \varphi) \), which are \( 2\pi \)-periodic in each component of \( \varphi \) and uniformly bounded in the bounded bi-cylinder \( \mathfrak{C}_{r,T,\rho,\sigma} \). Also let \( \mathfrak{B}_{r,T,\rho,\sigma}(\Pi \times \mathbb{T}^n) \) be the space of real-analytic functions \( u(s, \varphi) \), which are \( 2\pi \)-periodic in each component of \( \varphi \) and uniformly bounded in the bounded one-cylinder \( \Pi \times \mathbb{T}^n \), defined by \( \mathfrak{B}_p^2(\mathfrak{C}) = [\mathfrak{B}_{r,T,\rho,\sigma}(\Pi \times \mathbb{T}^n)]^2 \). With the supremum-norm \( \| \cdot \|_p \), each of the above spaces is a Banach space. Component-wise supremum norm \( \| \cdot \|_p \) or the equivalent Euclidean norm \( \| \cdot \|_p \) will be used for vector functions.

For any \( u \in \mathfrak{B}_p(\mathcal{C}) \), there exists a unique decomposition

\[
u(s, \varphi) = u_0(\varphi) + u_1(s, \varphi), \quad \text{where} \quad u_0 \in \mathfrak{B}_p(\mathbb{T}^n), \quad u_1 \in \mathfrak{B}_p^2(\mathcal{C}).
\]

Using it, define the average \( \langle u \rangle \) “at infinity” as

\[
\langle u \rangle \overset{\text{def}}{=} \int_{\mathbb{T}^n} u_0(\varphi)d\varphi.
\]

For \( u \in \mathfrak{B}_p(\mathcal{C}) \), its component \( u_1 \) satisfies an obvious exponential estimate\(^6\) in \( \mathcal{C}_p \):

\[
|u_1(s, \varphi)| \lesssim e^s|u_1|_p.
\]

Let us further describe the maps of the bi-cylinder \( \mathfrak{C} \) induced by real-analytic diffeomorphisms of \( \mathcal{I} \times \mathbb{T}^n \) after a change \( x = x(s) \). Given \( p = (r, T, \rho, \sigma) \), a sufficiently small \( \Delta \in \mathbb{R}_{++} \) and \( p' = p - \Delta \), let

\[
\tilde{a} = \mathbf{id} + \tilde{b} : q \to q + \tilde{b}(q), \quad q = (x, \varphi), \quad \mathcal{D} = \mathbb{D} \times \mathbb{T}^n, \quad \tilde{b} = (\tilde{b}, \tilde{B}) \in \mathfrak{B}_p(\mathcal{I} \times \mathbb{T}^n)^{n+1}, \quad |\tilde{b}|_p \lesssim \Delta
\]

be a smooth map of \( \mathcal{I}_{r',T',\rho',\sigma'} \times \mathbb{T}^n_{\sigma'} \) into \( \mathcal{I}_{r,T,\rho,\sigma} \times \mathbb{T}^n_{\sigma} \), well defined for a small enough constant in the above estimate for \( |\tilde{b}|_p \). It will always be assumed that \( \Delta < \Delta_\psi \), where the latter is “small enough” in terms of the pair \((\psi, \mathcal{D})\). The natural norm for \( \tilde{b} \) is the \( C^1 \)-norm in \( \mathfrak{B}_p(\mathcal{I} \times \mathbb{T}^n) \), which is easy to estimate knowing the \( C^0 \)-norm on some intermediate space \( \mathfrak{B}_{p'}(\mathcal{I} \times \mathbb{T}^n) \) with \( p' < p'' \leq p \). Details regarding intermediate parameter values will be mostly bypassed.

Let \( \mathfrak{D}_{p,\Delta}(\mathcal{I} \times \mathbb{T}^n) \) be the set of all such diffeomorphisms and define \( \mathfrak{D}_{p,\Delta}(\mathfrak{C}) \) as the set of all maps

\[
\mathfrak{a} : \mathcal{C}_{p'} \to \mathcal{C}_p, \quad \exists \mathfrak{a} \in \mathfrak{D}_{p,\Delta}(\mathcal{I} \times \mathbb{T}^n) : \mathfrak{a} = s \circ \tilde{a} \circ x.
\]

The analyticity indices can be dropped in the qualitative argument. For \( \tilde{a} \) one can come up with a unique representation \( \tilde{a} = \tilde{a}_1 \circ \tilde{a}_0 \), where \( \tilde{a}_0 : x \to x + b_0(\varphi) \) acts on \( \varphi \) as the identity, while \( \tilde{a}_1 = \mathbf{id} + \tilde{b} \) preserves \( x = 0 \), i.e. \( \tilde{b} = (\tilde{b}, \tilde{B}) \in \mathfrak{B}_p(\mathcal{I} \times \mathbb{T}^n) \times \mathfrak{B}_p(\mathcal{I} \times \mathbb{T}^n)^n \). Then \( \mathfrak{a} = \tilde{a}_1 \circ \tilde{a}_0 \), where the transformation \( \tilde{a}_1 \) preserves \( \{s = -\infty\} \). Naturally one can write \( \tilde{a}_1 = \mathbf{id} + \tilde{b} \), where \( \tilde{b} = (b, B) \in [\mathfrak{B}_p(\mathcal{C})]^n \). Indeed for the change \( \varphi \to \varphi + B(x, \varphi) \) all one has to do is to define \( B(s, \varphi) = \tilde{B}[x(s), \varphi] \).

\(^6\)Clearly not any real-analytic function of \( s \) vanishing at infinity at an exponential rate will be a member of one of the above spaces. E.g. for \( u(s) = se^s \), the function \( \tilde{u} = u[s(x)] \) is not analytic at \( x = 0 \).
change of the \( x \)-variable corresponding to \( \tilde{a}_1 \) can be written as \( x \to x + \tilde{b}(x, \varphi) \) for \( \tilde{b} \in \mathfrak{B}_p(\mathbb{I} \times \mathbb{T}^n) \), i.e. one can write \( \tilde{b} = \psi(x)v(x, \varphi) \) with \( v \in \mathfrak{B}_p(\mathbb{I} \times \mathbb{T}^n) \). Thus given \( x = x(s) \) one gets

\[
s \to s + \int_{x}^{x+\psi(x)v(x, \varphi)} \frac{d\zeta}{\psi(\zeta)} = s + \sum_{j=0}^{\infty} \left[ \psi(x)v(x, \varphi) \right]^{j+1} \frac{D_j}{(j+1)!} \frac{1}{\psi(x)} \equiv s + b(s, \varphi),
\]

for some \( b \in \mathfrak{B}_p(\mathbb{C}) \), with the norm \( |b|_p \approx |v|_p \approx |\tilde{b}|_p \). In particular the change of \( s \) under \( a_1 \) is asymptotically an identity as \( s \to -\infty \). The above expression can be viewed as a \( C^1 \) functional from \( \mathfrak{B}^1_1(\mathbb{I} \times \mathbb{T}^n) \) into \( \mathfrak{B}_p(\mathbb{C}) \), mapping zero into zero and whose differential is bounded away from zero in some neighborhood of zero.

Then if \( \eta \) is as far as one can get, as the series expansion analogous to the preceding formula will not converge uniformly with \( v \). Continuity can be achieved by extending it to \( s \geq 0 \).

Remark 3.1: Unless \( b_0 \equiv 0 \), the quantities \( a_0[s, b_0(\varphi)] \) are neither real-valued, nor continuous for real \( s \leq -2T_\psi \). Continuity can be achieved by extending it to \( \{ s = -\infty \} \), then the defining component \( a_0 \) or \( a_0 \) maps \( \{ s \leq -2T_\psi \}, \exists s \geq 0, \pi \} \cup \{ s = -\infty \} \) into \( \{ s \leq -2T_\psi \}, \exists s = 0, \pi \} \cup \{ s = -\infty \} \) and the differential \( da_0 = ds \circ da_0 \circ dx \) is unbounded. Further calculations will use the expressions

\[
d a_0^{-1} = \left[ \frac{\psi[x(s)+b_0(\varphi)]}{\chi(s)} - \frac{db_0(\varphi)}{\chi(s) 1 d\alpha} \right], \quad \psi[x(s)+b_0(\varphi)] = \chi(s) + [1+\chi(s)\eta_1(s)]b_0(\varphi) + \eta_2[s, b_0(\varphi)]b_0^{(\varphi)}, \quad (3.16)
\]

where the quantities \( \eta_1, \eta_2 \) viewed as functions of \( (s, \varphi) \) are in \( \mathfrak{B}_p(\mathbb{C}) \) by the assumptions on \( \psi \).

As one is interested in the coordinate changes \( a \in \mathfrak{D}_p, \Delta(\mathbb{C}) \) only as far as their action on functions from \( \mathfrak{B}_p(\mathbb{C}) \) is concerned, they are naturally represented by an element \( \hat{b} = (b_0, b) \) of \( \mathfrak{B}_p(\mathbb{T}^n) \times \mathfrak{B}_p(\mathbb{C}) \) into \( \mathfrak{B}_{p(1)}^0(\mathbb{C}) \), with the product topology and vector supremum norm \( | \cdot |_p \), the origin corresponding to the identity transformation.

Then if \( u \in \mathfrak{B}_p(\mathbb{C}) \) and \( |\tilde{b}|_p \lesssim \Delta \), \( u' = u \circ a(\tilde{b}) \in \mathfrak{B}_p(\mathbb{C}) \) with \( p' = p - \frac{1}{\Delta} \). Moreover with \( p'' = p - \frac{2}{\Delta} \) one can write

\[
|u - u \circ a|_p \lesssim |du|^p \|\tilde{b}|_p' \lesssim \frac{1}{\Delta} |u|_p \|\tilde{b}|_p',
\]

by the Cauchy inequality.

Apart from \( a = a_1 \circ a_0 \), the general form for the transformation \( a \) can be also taken as

\[
a(\tilde{b}) = a_0(b_0) + b : \begin{cases} s \to a_0[s, b_0(\varphi)] + b(s, \varphi), \\
\varphi \to \varphi + B(s, \varphi). \end{cases} \quad (3.18)
\]

In order to deal with functions of \( s \), which are unbounded at infinity, let us introduce a function space \( \mathfrak{B}_p^+(\mathbb{C}) \cong \mathfrak{B}_p(\mathbb{T}^n) \times \mathfrak{B}_p(\mathbb{C}) \) as follows, see (3.11):

\[
u(s, \varphi) \in \mathfrak{B}_p^+(\mathbb{C}) \text{ iff } u(s, \varphi) = \frac{v(s, \varphi)}{\chi(s)}, \quad v(s, \varphi) \in \mathfrak{B}_p(\mathbb{C}).
\]

The norm on \( \mathfrak{B}_p^+(\mathbb{C}) \) is simply \( |v|_p \). Since one can write in the spirit of (3.12) \( v(s, \varphi) = v_0(\varphi) + \chi(s)v_1(s, \varphi) \), with \( v_1 \in \mathfrak{B}_p(\mathbb{C}) \), then

\[
u(s, \varphi) = \frac{v_0(\varphi)}{\chi(s)} + v_1(s, \varphi),
\]

\[
u_0(s, \varphi) = \frac{v_0(\varphi)}{\chi(s)} + v_1(s, \varphi),
\]

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and sup(|v₀|, |v₁|) can be taken for the norm |u|₁ as well. Also let \( \mathfrak{B}_p^{(-1)}(C) \equiv \mathfrak{B}_p^−(C) \times |\mathfrak{B}_p(C)|^n \). An element of this space describes a bi-real-analytic vector field on the bi-cylinder \( C \). Clearly \( \mathfrak{B}_p^{(-1)}(C) \cong \mathfrak{B}_p^{(0,1)}(C) \), the elements of the latter space representing the maps of \( C \). If \( g \in \mathfrak{B}_p^{(-1)}(C) \) is a vector field and \( a \in \mathfrak{D}_{p,\Delta}(C) \) then \( g \circ a \) is not in \( \mathfrak{B}_p^{(-1)}(C) \), however. Indeed, as the result of the transformation \( a_0 \) the quantity \( \frac{v₀(∂φ)}{ψ|x(s) + b₀(φ)|} \) which can blow up for a finite \( s \). However \( da^{-1}g \circ a \) corresponding to the “new” vector field does belong to \( \mathfrak{B}_p^{(-1)}(C) \), see (3.16). Also, a simple calculation shows that in order to estimate the norm for partial derivatives of a function \( u \in \mathfrak{B}_p^{−}(C) \) one can still use the Cauchy formula \( |du|₁ \lesssim \Delta^{-1}|u|₁ \).

As far as Hamiltonian functions on \( T^∗C \) are concerned, consider the Banach space \( \mathfrak{B}_{κ, p}[T^∗(I × \mathbb{T}^n)] \) (with the sup-norm) of bounded real-analytic Hamiltonian functions on \( T^∗(I × \mathbb{T}^n) \), given by Taylor series with coefficients in \( \mathfrak{B}_p(I × \mathbb{T}^n) \), uniformly convergent for the momenta \( \tilde{p} = (y, I) \) inside \( \mathbb{R}_κ^{n+1} \), \( κ > 1 \). Define the space \( \mathfrak{B}_{κ, p}(T^∗C) \) of Hamiltonians on \( T^∗C \) as the subset of holomorphic functions on \( T^∗C \), such that

\[
H \in \mathfrak{B}_{κ, p}(T^∗C) \text{ iff } \exists \hat{H} \in \mathfrak{B}_{κ, p}[T^∗(I × \mathbb{T}^n)]: H = \hat{H} \circ Ξ.
\]

Thus the members of \( \mathfrak{B}_{κ, p}(T^∗C) \) are given by power series in \( \frac{H}{χ(s)}, I \), with coefficients in \( \mathfrak{B}_p(C) \).

The final remarks on the notation are that sometimes, if it is clear to which of the above spaces a function \( u \) belongs, the norm of \( u \) may be referred to simply as \( |u| \) rather than \( |u|₁ \). If \( u \in \mathfrak{B}_p(C) \) and has bounded partial derivatives, the notation \( |u|₁ \) will stand for the \( C¹ \)-norm. The notation \( |u|₁ \) will stand for the supremum norm of \( u \), restricted to the real values of all its variables, except \( s \) which is either real or \( \exists s = π \).

**Conjugacy problem**

In the formal framework developed above one can set up a conjugacy problem for a class of bi-real-analytic perturbations of a constant vector field

\[
x₀ = λ \frac{∂}{∂s} + ω \frac{∂}{∂φ} \tag{3.19}
\]

on the semi-infinite bi-cylinder \( C \), with \( λ \in \mathbb{R}_{++} \) and a Diophantine \( ω \in \mathfrak{W}_{r, r} \). In the same way as (3.20), the unperturbed vector field can be taken slightly more general, i.e.

\[
x₀ = λ \frac{∂}{∂s} + ω + θ(s), \frac{∂}{∂φ},
\]

with an angle-independent \( n \)-vector function \( θ(s) \), whose each component is a member of the space \( \mathfrak{B}_p^{−}(C) \).

This case is reducible to (3.19) after a change

\[
s = s', \ θ = θ' + λ⁻¹ \int_{−∞}^0 (θ(s + t)dt).
\]

The question of structural stability of the vector field \( x₀ \) under the group \( \mathfrak{D}(C) \) of bi-real-analytic maps of \( C \) is roughly as follows: given a vector field \( x = x₀ + g \), where \( g \in \mathfrak{B}_p^{−}(C) \) (with the parameter vector \( p = (r, T, ρ, σ) \)) and \( |g|₁ \) is small enough, does there exist a coordinate change \( a \in \mathfrak{D}_{p,\Delta}(C) \), i.e. \( a(\tilde{b}) = a₀(b₀) + b \) with \( a₀ = (a₀₀, i d) \) and \( b = (b₀, b) \in \mathfrak{B}_p^{(0,1)}(C) \), \( p' = p - \Delta \), such that

\[
da⁻¹ \circ a = x₀ \tag{3.20}
\]

The general answer to this question is no, as it is for the torus, for one can take \( x = x₀ + ξ \), with a constant \( ξ \in \mathbb{R}^{n+1} \) (\( x₀ \) being further identified with a constant vector \( (λ, ω) \in \mathbb{R}^{n+1} \)). Note that within the map class \( \mathfrak{D}_{p,\Delta}(C) \), the answer is no even if only the “longitudinal” component of the vector \( ξ \) is nonzero, as the
scalings of the variable $s$ are outside this class. Hence, conjugacy should be sought modulo $\xi \in \mathbb{R}^{n+1}$, asking for a pair $(a(b), \xi)$, such that
\[ da^{-1}(x_0 + g + \xi) \circ a = x_0. \] (3.21)

The problem can be relatively easily shown to satisfy the input of an implicit function theorem of Nash-Moser type, following the papers of Zehnder [40, 41], who made further generalizations in the abstract set-up to embrace the KAM theory with its small divisors. One essential modification is that here one should deal with the differential operator
\[ D_{\lambda, \omega} \overset{\text{def}}{=} \lambda D_s + (\omega, D_{\varphi}) = \lambda D_s + D_{\varphi}, \] (3.22)
rather than just $D_{\varphi}$, defined by (2.4). Auxiliary results apropos of solvability of linear PDEs involving the operator $D_{\lambda, \omega}$ in the set-up of various function spaces introduced earlier are presented in Appendix A.

However, estimates resulting from an application of the abstract theorem are unsatisfactory for the analysis of the normal form near a simple resonance (2.6) with its hierarchy of orders of magnitude and parameter dependencies. With extra scruple one can benefit by quasi-linearity of underlying equations, intermittent use of $C^1$ and $C^0$ estimates, similarly to the classical KAM case [31]. This is done in Section 3.2, resulting in particular in Corollary 3.2.

**Application to $H_\theta$**

Let the function $\psi$ be given by (2.11). Then the domain $D$ can be taken as a closed rectangle of some semi-width, bounded from above by $\sigma_1$, symmetric with respect to the interval $I_0 = [-2\pi + r, 2\pi - r]$ of the real axis for some $r < \inf(\sigma_1, 1)$. The pair $(\psi, D)$ as well as the parameter bounds $r$, $T$, $\psi$, $\Delta\psi$ are well defined, in particular one can ensure $T_\psi > 1$, because the mean value of $\psi$ over $\mathbb{T}'$ is zero. Then given $T \in (T_\psi, 2T_\psi)$ one can use (3.19) see Fig. 2, to determine the constants $\rho$ and $\sigma_2$ as well as the domain $I_\rho$.

The application of the transformation $\Xi_s$ to the Hamiltonian (2.21), whose $x$-variable is restricted on $I_\rho$ with the notation $(p; q) = (h, I; s, \varphi)$ results in the “new” Hamiltonian $H_s = H_\theta \circ \Xi_s \in \mathcal{B}_{s \rho}(T^* \mathbb{C})$ as follows:
\[
H_s(p, q) = \lambda h + \langle \omega_1, I \rangle + \langle Q_2 \tilde{p}, \tilde{p} \rangle + f_0(x(q)) + \langle g_0(\tilde{p}, x(q)), \tilde{p} \rangle
\] (3.23)

where $\tilde{p} = (\frac{\psi}{\chi(s)}, I) = (y, I)$, and the function $\chi(s)$ is defined by (3.21) [32]. Or, including the momentum-dependent part of $g_\theta$ into the “unperturbed” Hamiltonian:
\[
H_s(p, q) = \lambda h + \langle \omega_1, I \rangle + O_2(\tilde{p}; q) + [f(q) + \langle g(q), p \rangle]
\] (3.24)

where $f(s, \varphi) = f_0(x(s), \varphi)$, $g(s, \varphi) = \text{diag}(\chi^{-1}(s), 1, \chi_0)$, and this difference between $V$ and $V_s$ is that the latter may contain super-linear terms in $p$. For the sputnik Hamiltonian $H'_s = H'_\theta \circ \Xi_s \in \mathcal{B}_{s \rho}(T^* \mathbb{C})$ one gets
\[
H'_s = H_s \circ L^{-2}_\chi, \text{ with } L_{\chi_1} \equiv I_{\psi} \circ \Xi_s, \quad L_{\chi} : (h, I, s, \varphi) \to (h + \lambda \chi^2(s), I, s, \varphi).
\] (3.25)

Then similarly to the two previous formulas
\[
H'_s(p, q) = H_{-\lambda, \tau}(p, q) + V'_s(p, q)
\] (3.26)
i.e. $H_{-\lambda, \tau}$ is the same as $H_{\lambda, \tau}$, but for the sign of the first term and $V'_s = V_s \circ L^{-2}_\chi$, whereupon the terms $f' \in \mathcal{B}_{p}(C)$ and $g' \in \mathcal{B}_{p}^{(1)}(C)$ correspond to the zero and first order terms of the Taylor expansion in $\tilde{p}$; the
rest of the expansion is absorbed by the term $O_2'(\hat{p}; q)$. Just the same, $V'$ will denote the expression in the square brackets in (3.20) above.

By Lemma 2.4 one can take the radius $\kappa$ of convergence of both Taylor series in $\hat{p}$ as large as necessary, for instance $\kappa > 2$ and for all $\hat{p} \in \mathcal{B}_\kappa$ assume that uniformly over the (complexified) base space, the absolute value of each eigenvalue of $D^2_{pp}H_s$ and $D^2_{pp}H'_{s}$ is uniformly bounded from below by $1/2$ and from above by a constant times $M_0R_0^{-1}$.

### 3.2 Hyperbolic KAM theorem

This section develops a KAM-type approach to Hamiltonian systems in $T^*(\mathcal{C})$, such as the transformed simple resonance Hamiltonian (3.24). The section contains the statement of Theorem 2, the principal part of its proof and a number of corollaries, one of each is the solution to the conjugacy problem (3.21). Yet the prototype of Theorem 2 can be found in [34] the theory applicable to a generic simple resonance normal form and for the purpose of studying the lower technical improvements, allowing for a supposedly optimal parameter dependence, necessary in order to make the theory applicable to a generic simple resonance normal form and for the purpose of studying the lower bounds for exponentially small splitting [10], [33].

Theorem 2 per se represents an alternative to the traditional approach to whiskered tori, largely due to Graff [16]. The present approach appears to be more natural for describing the whiskers as semi-infinite cylinders over tori globally\footnote{Graff’s theorems apply to partially hyperbolic tori of all dimensions $\leq n$, but are in essence local near a hyperbolic equilibrium, where the whiskers, being very “short” cylinders over tori, can be naturally described by naive generating functions. The present theory takes advantage of the fact that in the simple “one-hyperbolic” case discussed, the truncated normal form Hamiltonian (2.8) is integrable and the generating function $S_t$ (2.9) is defined globally over a “long” cylinder; this necessitates a considerably different analysis. The present approach seems to be extendable to the case of lower-dimensional tori, despite in the latter case one certainly cannot hope to have a simplistic global description for the whiskers. Nevertheless, one may try to consider only narrow strips thereof in tubular neighborhoods of transverse homoclinic orbits, generically existing in the phase space of the non-integrable “hyperbolic” sub-system, corresponding to the truncated normal form if one lets $\dim z > 1$ in (2.9).}, as it avoids compactification of the base space and elucidates the connection with other methods of study of manifolds asymptotic to invariant tori, developed for instance in [17], [13], [37], [9]. The theorem contains significant
application of the transformation $\Psi_j$, the principal part of the Hamiltonian (modulo a constant) picks up a term equal to a small constant times $h$, thus slightly changing the value of $\lambda$.

As $j \to \infty$, the Hamiltonian vector field in question, restricted to the manifold $p = dS$ thus becomes conjugate to (3.19) (with a slightly changed value of $\lambda$) via a canonical transformation $\Psi = \Xi_1 \circ \Xi_2 \circ \ldots$. Note that the classical Kolmogorov’s theorem [17] allows a similar geometric interpretation [11] with the base space, of course being a torus, rather than a bi-cylinder.

Set-up and statements of Theorem 2 and Iterative lemma

The unperturbed Hamiltonian in (3.24) belongs to a certain class. Fix a Diophantine $\omega \in \mathbb{R}^{s,\varphi}$ and define a class $\mathcal{R}_\omega$ as follows.

Definition 2 (Unperturbed Hamiltonian) A function $H_\lambda$ on $T^*\mathcal{C}$ belongs to the class $\mathcal{R}_\omega$ if modulo a constant, it can be represented as follows:

$$H_\lambda(h, I, s, \varphi) = \lambda h + \langle \omega, I \rangle + O_2(p, q),$$

(3.28)

where $\lambda \neq 0$ (further assumed positive) and

1. $O_2(p, q) = O_2(\frac{1}{\lambda h}, I; q) \in \mathcal{B}_{\kappa, p}(T^*\mathcal{C})$, for some parameter vector $(\kappa, p)$;

2. $\exists R, M \in \mathbb{R}_+, \text{such that } \forall (\tilde{p}, q) \in \mathbb{B}_p(\mathcal{C})^\omega \times \mathcal{C}_p$, $\|D^2_{\tilde{p}q}O_2(\tilde{p}; q)\| \leq R^{-1}$ and $\|D^2_{\tilde{p}q}O_2(\tilde{p}; q)\| \leq M$.

Given $H_\lambda \in \mathcal{R}_\omega$, consider its small perturbation

$$H = H_\lambda + V, \quad V(p, q) = f(q) + \langle g(q), p \rangle,$$

(3.29)

where

$$f \in \mathcal{B}_p(\mathcal{C}), \quad g \in \mathcal{B}_p^{(-1)}(\mathcal{C}), \quad |f|_p \leq \mu, \quad |g|_p \leq \mu \nu^{-1}, \quad \text{for some } 0 \leq \mu < \nu \leq 1.$$  

(3.30)

The parameter $\nu$ is further used to obtain the desired smallness condition (3.33) generally indicating that the above described iterative procedure allows larger upper bounds for the norm of $g$ than the norm of $f$ in the perturbation $V$.

Theorem 2 (Hyperbolic KAM theorem) If $\mu$ is small enough, there exists a canonical transformation

$$\Psi = \Psi(a, S) : \begin{cases} q = a(q'), \\ p = t(da)^{-1}p' + dS, \end{cases}$$

(3.31)

such that for any $\kappa' < \kappa$, $p' < p$ and some parameter values $N', R', M'$, different from $\lambda, R, M$ respectively by $O(\mu)$, one has $H \circ \Psi \in \mathcal{R}_\omega$, with the new parameter set $\{\kappa', p', N', R', M'\}$ and $\Delta = |p - p'|$:

1. The transformation $a = a(b) \in \mathcal{D}_{p, \Delta}(\mathcal{C})$, with $|b|_{1, p'} = O(\mu)$.

2. The one-form $dS$ is defined by the generating function $S(q) = \langle \xi, \varphi \rangle + \tilde{S}(s, \varphi)$, with $\xi \in \mathbb{R}^n$, $\tilde{S} \in \mathcal{B}_p(\mathcal{C})$, and $|\xi|, |\tilde{S}|_{1, p'} = O(\mu)$.

In the above non-technical formulation of Theorem 2 “$\mu$ small enough” means that it satisfies the following smallness condition (3.33). The symbols $O(\mu)$ depend on the parameter values from both the old and the new parameter sets. Further without loss of generality, one can assume that the quantities $\delta = \sigma - \sigma', \Delta = |p - p'|, \lambda, R, M^{-1}, |\omega|^{-1} \leq 1$. The exact estimates are summarized below. In applications, one or more of them can turn out to be functions of a small parameter $\varepsilon$, and the magnitudes of the analyticity loss in the variables $(s, \varphi)$ can differ considerably. Then the following estimates can be adjusted if necessary, see e.g. footnote 11 below.
Parameter statement of Theorem 2

Let
\[ \kappa' = \kappa - 1, \quad \varsigma = \inf(\gamma \delta^\tau, \lambda), \quad \eta = R \inf(M^{-1} \varsigma \Delta, \nu). \] (3.32)

There exists a constant \( C \), depending only on \( n, \tau, \psi, D \), such that if
\[ \mu \leq C^{-2} \eta^2 \lesssim (R/M)^2 \Delta^2 \inf(\varsigma, \nu)^2, \] (3.33)
the following estimates hold:
\[ |\hat{S}|_{p'} \leq C \mu \varsigma^{-1}, \quad |d\hat{S}|_{p'} \leq C \mu \varsigma \Delta^{-1}, \quad |dS|_{p'} \leq C \mu \eta^{-1}, \]
\[ |\hat{b}|_{p'} \leq C \mu (\eta \varsigma)^{-1}, \quad |d\hat{b}|_{p'} \leq C \mu (\eta \varsigma \Delta)^{-1}, \]
\[ \lambda^{-1}|\lambda' - \lambda| \leq C \mu (\eta \varsigma)^{-1}, \]
\[ M^{-1}|M' - M|, R^{-1}|R' - R| \leq C \mu (\eta \varsigma \Delta)^{-1}. \] (3.34)

Theorem 2 can be cast into the abstract generalized Newton method framework [40], [41]. However, in order to obtain the desired parameter dependencies, a direct proof is given. The main tool is furnished by the following lemma, fulfilling a single Newton’s iteration.

Lemma 3.1 (Iterative lemma) For a Hamiltonian (3.28-3.30) with a parameter set \( \{ \kappa, p, \lambda, R, M, \mu, \nu \} \), if \( \mu \) is small enough (condition (3.36) below), there exists a canonical transformation
\[ \Psi = \Psi(a, S) : \begin{cases} q = a(q'), \\ p = t(da)^{-1}p' + dS, \end{cases} \] (3.35)
such that for any \( \kappa' < \kappa, \ p' < p \), and some new parameter values \( \lambda', R', M' \), different from \( \lambda, R, M \) respectively by \( O(\mu) \), one has for \( \Delta = |p - p'| \):

1. The transformation \( a = a(b) \in D_{p, \Delta}(C) \) is such that \( |\hat{b}|_{1, p'} = O(\mu) \).
2. The one-form \( dS \) is defined by the generating function \( S(q) = \langle \xi, \varphi \rangle + \hat{S}(s, \varphi) \), with \( \xi \in \mathbb{R}^n, \hat{S} \in B_{p'}(C) \), and \( |\xi|, |S|_{1, p'} = O(\mu) \).
3. The Hamiltonian \( H \circ \Psi = H_{\lambda'} + V' \), with \( H_{\lambda'} \in \mathcal{H}_{\omega} \), satisfies Definition 2 and (3.30) with a parameter set \( \{ \kappa', p', \lambda', R', M', \mu', \nu' \} \), where \( \mu' = O(\mu^2) \) and \( \nu' \) is independent of \( \mu \).

Clearly, the quantities \( \Psi, a, S, \) etc. in the Iterative lemma are not the same as their homonyms in Theorem 2; it should not cause confusion. In the non-technical formulation above, the quantities \( \mu', \nu' \) and the symbols \( O(\mu) \) depend on the parameter values from both the old and the new parameter sets. They are further specified as follows.

Parameter statement of Iterative lemma

Let \( \kappa - \kappa' < 1 \) and \( \varsigma, \eta \) be computed via formulae (3.32). There exists a constant \( C \), depending only on \( n, \tau, \psi, D \), such that if
\[ \mu \leq C^{-2} \eta \varsigma \Delta(\kappa - \kappa'), \] (3.36)
the following relations hold:
\[
|\hat{S}|_{p'} \leq C\mu\xi^{-1}, \quad |d\hat{S}|_{p'} \leq C\mu(\xi\Delta)^{-1}, \quad |\xi| \leq C\mu\eta^{-1},
\]
\[
|\hat{b}|_{p'} \leq C\mu(\eta\kappa)^{-1}, \quad |d\hat{b}|_{p'} \leq C\mu(\eta\kappa\Delta)^{-1},
\]
\[
\lambda^{-1}|\lambda' - \lambda| \leq C\mu(\eta\lambda)^{-1},
\]
\[
M^{-1}|M' - M|, R^{-1}|R' - R| \leq C\mu(\eta\kappa\Delta)^{-1},
\]
\[
\nu' = \inf(M^{-1}\xi\Delta, \nu), \quad \mu' \leq C^2\mu^2\eta^{-2}.
\]

The proof of the Iterative lemma is given in the next section. Detail of the proof is important in order to justify the following remarks, concerning Theorem 2. Once one accepts the Iterative lemma, the rest of the proof of Theorem 2 becomes a routine iteration scheme, given in Appendix B.

Corollaries and remarks

Theorem 2 essentially states that the zero section \( p' = 0 \) is an invariant Lagrangian manifold for the Hamiltonian \( H \circ \Psi \in \mathcal{N}_0 \). Due to the “affineness” of the transformation \( \Psi(a, S) \), in the “old” coordinates \( (p, q) \) this manifold is a section of \( \mathcal{C} \), given by
\[
W(s, \varphi) = \{(h, I, s, \varphi) \in \mathbb{R}^{n+1} \times \mathcal{C} : h = D_s\hat{S}_1(s, \varphi), I = \xi + D_\varphi \hat{S}(s, \varphi)\},
\]
where \( \hat{S} \in \mathcal{B}_p'(\mathcal{C}) \), i.e. \( \hat{S}(s, \varphi) = \hat{S}_0(\varphi) + \hat{S}_1(s, \varphi) \) in the sense of (3.12). Recall that \( \mathcal{C} = \Lambda \cup \mathbb{T}^n \) is a bi-cylinder, i.e. \( \Lambda = 0 \) or \( \pi \).

The manifold \( W(s, \varphi) \) has been compactified by incorporating an invariant torus \( \mathcal{T} \) corresponding to \( \{s' = -\infty\} \). The flow on \( \mathcal{T} \) is conjugate to rotation with the frequency \( \omega \). With \( S' = S \circ a \) and \( q' = (s', \varphi') \), another parameterization for \( W \) is
\[
W(q') = \{(p, q) : q = a(q'), p = t[d\hat{a}(q')]^{-1}dS'(q'), q' \in \mathcal{C}\},
\]
and get \( \mathcal{T} \) as
\[
\mathcal{T} = \bigcap_{T \leq 0} W([s', \varphi') \in \Lambda_T \times \mathbb{T}^n].
\]

Remark 3.2: In the sequel \( \mathcal{T} \) will be often described as located “near” \( s = -\infty \); this verbiage refers precisely to the representations 3.38 and the underlying estimates.

The representations 3.38 constitute the basis for the splitting analysis in Section 4. For easier cross-reference let us recap the above as a corollary.

**Corollary 2.1 (Hamilton-Jacobi equation)** Let \( H_0(0, q) = 0 \). The function \( S(q) = \langle \xi, \varphi \rangle + \hat{S}(q) \) satisfies the Hamilton-Jacobi equation on \( \mathcal{B}_p'(\mathcal{C}) \):
\[
H(\partial_q S(q), q) = c_0,
\]
with a real \( c_0 \) bounded by 3.31.

Let us now look back at the conjugacy problem 3.21. It corresponds precisely to the case \( f \equiv 0 \) in the perturbation 3.29. Then the parameters \( M, R, \nu \) are redundant (can be all set to 1 in the estimates) and the unperturbed Hamiltonian \( H_\lambda \) can be thought momentum-linear. Then given the perturbation \( q \in \mathcal{B}_p(\mathcal{C}) \), the aim of the conjugacy problem is to find a transformation \( \Psi(a) = \Psi(a, 0) \) to conform with 3.31 that is with \( S \equiv 0 \), as well as a constant \( \xi \in \mathbb{R}^{n+1} \) (which is not unrelated to the vector \( (\lambda - \lambda', \xi) \in \mathbb{R}^{n+1} \) such that \( (H_\lambda + (g + \xi, p)) \circ \Psi = H_{\lambda'} \)). Then the proof of Lemma 3.11 and Theorem 2 can be straightforwardly adjusted to yield the following corollary.
Corollary 2.2 There exists a constant $C(n, \tau, \psi, D)$, such that if $g \in B_p^2(C)$, with

$$|g|_p \leq C^{-2\zeta_\Delta},$$

there exists a pair $(a, \xi) \in (D_p, \Delta \times \mathbb{R}^{n+1})$, such that the map $a = a(\hat{b})$ effects (3.21) and

$$||\xi||, |\hat{b}|_{1,p'} \leq C|g|_p(\lambda \Delta)^{-1}. \quad (3.42)$$

Besides, if $H = \hat{H} \circ \Xi_\mu$, then the transformation $\Xi_\nu \circ \Psi \circ \Xi_\mu^{-1}$ effects the structural stability of the normal form

$$\hat{H}_\lambda(y, I, x, \varphi) = \lambda \psi(x)y + \langle \omega, I \rangle + O_2(\dot{p}; x, \varphi)$$

under small perturbations, proving the existence of an invariant manifold which is a graph over the variables $(x, \varphi) \in [-2\pi + r, 2\pi - r] \times \mathbb{T}^n$, containing an invariant torus near $x = 0$ [31].

The next corollary is quite obvious with respect to $\hat{H}$. It claims that if the latter is perturbed by a pair $(\hat{f}, \hat{g})$ such that $\hat{f}, d\hat{f}, \hat{g}$ all vanish at $x = 0$, then the invariant torus at $x = 0$ satisfies the Hamilton equations and does not move: in particular the constants $\xi, \xi_0$ in Theorem 2 and Corollary 2.4 should be zero. For instance, this is the case in Arnold’s example [1] (the Hamilton-Jacobi formalism for such a degenerate perturbation was developed in [37]). This fact can also be established by going through the proof of Lemma 3.1 and will be used further to claim Corollary 2.4 essential for the splitting problem.

Corollary 2.3 (Degenerate perturbation) Suppose $f \in B_p^2(C)$ and $g = (g, G) \in B_p(C) \times [B_p^1(C)]^n$. Then apropos of the transformation (3.21) one has $\xi = 0$, $S \in B_p^2(C)$, $b_0 \equiv 0$, $b = (b, B) \in B_p(C) \times [B_p^1(C)]^n$ and $c_0 = 0$ in Corollary 2.4.

The following remarks address yet more technical issues.

Remark 3.3: (Estimates) The smallness condition (3.33) for $\mu$ appears to be optimal as far as the parameter dependencies are concerned: if $\lambda, \nu = 1, \Delta = \delta$ it reproduces the standard KAM theorem optimal smallness condition, see e.g. [29]. The use of an extra parameter $\nu$ has been essential here to express that the order of magnitude of $g$ is inherently somewhat greater than that of $f$, as far as the perturbation is concerned. This fact can obstruct accessibility of the condition (3.33) if one pursues Kolmogorov’s approach to KAM theory, see e.g. [26]. Similar (standard KAM) estimates resulting from the general abstract implicit function theorem machinery are also worse [40], [41]. Under the assumptions of Corollary 2.3 the estimates of the Parameter statement of Theorem A.1 shall be modified as follows. Apart from $\xi = 0$, one should use $\zeta = \lambda$ and formally set $R = 1$ in all the estimates. Finally, if $\lambda = 0$ in Definition 2 then obviously $|\lambda|$ should substitute $\lambda$ in the estimates.

Remark 3.4: (Local uniqueness and parameter dependence) Given $H_\lambda$ and a small perturbation $(f, g)$ obeying the smallness condition (3.33), the pair $(a, \dot{S})$ is unique (as it is in standard analytic KAM theorem). Indeed, the unique solution $u(v)$ of the PDE in Proposition A.1 provides the right inverse of the operator $D_{\lambda, v}$, which is also its left inverse, guaranteeing uniqueness, see [40]. In other words, local uniqueness follows from the uniqueness of PDE solutions (modulo a constant) in Appendix A. Similarly, in the case of a continuous (e.g. on the pair $(f, g)$) or real-analytic dependence of $H$ in an extra parameter (e.g. $\mu$), the pair $(a, S)$ retains the same type of dependence in the parameter. Local uniqueness is indispensable for the splitting problem to be well-posed.

Remark 3.5: (Other settings) It is known in KAM theory that the non-degeneracy assumption in Definition 2 allows many variations [32]. Theorem 2 can be adapted to these settings in the same way as the standard KAM theorem. For instance $H_\lambda$ in (2.29) can be only linear in the actions $I$, provided that the perturbation $V$ does not depend on $I$ either, the so-called “isochronous” case. In this case the transformation

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8This is certainly not true for “large” perturbations: each hyperbolic manifold (unstable or stable) should have a counterpart (stable or unstable) or sputnik, see the coming Corollary 2.4.
Proof: Let \( T(3.40) \), i.e. \( a \) and an invariant manifold \( W(3.30) \), suppose the assumptions of Corollary 2.3. As \( W \), pair \((H_1 \circ \Psi) \) can be identified. As \( p \lambda \) than one characteristic exponent \( L \) normal form Hamiltonian (2.7). To this effect, the action of \( a \) note that as \( \varphi \) in terms of \( \psi \) according to (3.31) Theorem 2. Then the closed one-forms \( dS \) and \( dS^1 \) belong to the same cohomology class \( \xi \in H^1(C, R) \). Both \( dS \) and \( dS^1 \) satisfy the Hamilton-Jacobi equation (3.41) of Corollary 2.1 for \( H \), on the same energy level \( c_0 \). The closure \( W' \) of the graph of the form \( dS \) intersects the manifold \( W \) defined by (3.38) for the form \( dS \) at the torus \( T' \) defined by (3.40), i.e \( T' = T \).

Remark 3.6: Regarding the normal form Hamiltonian \( \text{(3.20)} \), alias \( \text{(3.23)} \), the sputnik transformation \( L_s = L_s(a^*, S^*) \) arises from the fact that the lower separatix is a graph of over the upper one for the truncated normal form Hamiltonian \( \text{(3.24)} \). To this effect, the action of \( L_s \) is \( H_{\lambda, t} \circ L_s = H_{-\lambda, t} \), natural for symplectic flows. So in this case \( a^* = 1d \). However further in Section 4, the diffeomorphism \( a^* \) will incorporate a shift of the variable \( \varphi \) by \( 2\beta \pi \theta \), \( \beta \in \{ + , - \} \) (and of \( s \) by \( -\beta \pi \)), see (3.22) and (4.14) to come. Assumption 2 is a generalization, which may be useful for instance of higher multiplicity resonances when there is more than one characteristic exponent \( \lambda \), see also footnote 4.

Note that as \( a^* \) acts as the identity on the set \( \{ s = -\infty \} \) (possibly causing reparameterization of the angles \( \varphi \) in terms of \( \varphi_s \)) it is not only \( S^* \) but also \( S^* \circ a^* \) \( \in \mathcal{B}_p(C) \), and the unperturbed invariant tori for \( H \) and \( H' \) can be identified. As \( H' \) satisfies the input of Theorem 2 there exist a canonical transformation \( \Psi'(a^1, S^1) \) and an invariant manifold \( W' \) containing an invariant torus \( T' \), versus the transformation \( \Psi(a, S) \) and the pair \((W, T) \) for \( H \), described by (3.36) and (4.10). Note that the claim \( \Psi' = L_{-1} \circ \Psi \) would be false, as the latter transformation would still make the manifold \( W \) the zero section, rather than \( W' \).

Corollary 2.4 (Sputnik) Under Assumption 2 let \( \Psi(a, S) : H \circ \Psi \in \mathcal{N}_{\omega}, \Psi'(a^1, S^1) : H' \circ \Psi' \in \mathcal{N}_{\omega} \) according to \( \text{(3.37)} \). Then the closed one-forms \( dS \) and \( dS^1 \) belong to the same cohomology class \( \xi \in H^1(C, \mathbb{R}) \). Both \( dS \) and \( dS^1 \) belong to the same cohomology class and satisfy the Hamilton-Jacobi equation (3.41) of Corollary 2.1 for \( H \), on the same energy level \( c_0 \). The closure \( W' \) of the graph of the form \( dS \) intersects the manifold \( W \) defined by (3.38) for the form \( dS \) at the torus \( T \) defined by (3.40), i.e \( T' = T \).

Proof: Let \( H_2 = H \circ \Psi \circ L_s \). Since \( H \circ \Psi \in \mathcal{N}_{\omega} \), by the explicit form (3.43) of \( L_s \), under the assumptions on the pair \((a^*, S^*) \), the Hamiltonian \( H_2 = H' \circ (L_{-1} \circ \Psi \circ L_s) \) satisfies the conditions of Corollary 2.3 for \( \mu \) small enough.

9Equivalent here is meant in the same sense as the equivalence for orders of magnitude or norms, that is up to a constant factor. Cleariy \( \lambda, R, M \) in Theorem 4 can be just bounds, rather than the actual values of non-degeneracy parameters. I.e. a change \( \lambda \to -\lambda \) or \( M \to 2M \) is inconsequential. Besides the transformation \( L_s \) may in principle entail some extra analyticity loss of the order \( \Delta \). Necessary amendments are easy to make for a concrete example.
Indeed, \((H \circ \Psi) \circ L_s = (H_s + [c_1 h + O_2(p; q)]) \circ L_s\), where \(c_1\) is a constant and the term in square brackets is \(O(\mu)\). Then the assumption on \(L_s\) (in particular its preservation of \(\{ s = -\infty \}\)) implies the previously made statement, namely

\[ H_2(p_s, q_s) = \lambda_s h_s + \langle \omega, I_s \rangle + O_2(p_s^*; q_s^*) + V_2(p_s, q_s), \]

where \(\lambda_s = \lambda' + O(\mu)\) and \(\hat{p}_s = \left(\frac{\hbar}{\chi(s)}, I_s\right)\).

In fact, the pair \((f_2, g_2)\), comprising the perturbation \(V_2\) above, which has arisen as the result of substitution of \((3.43)\) into the Hamiltonian \(W\) is amenably to Corollary 2.3. However, one should not worry about the precise smallness condition such that \(\lambda_s = \chi(s)\). As \(S'\) is proportional to \(\chi^2(s_s)\), the quantities \(f_2(q_s), g_2(q_s)\) behave as \(s_s \to -\infty\) amenable to Corollary 2.3.

Then for \(\mu\) small enough\(^{10}\) there exists a canonical transformation \(\Psi_2 = \Psi_2(a_2, S_2)\) in the form \((3.31)\), such that \(H_2 \circ \Psi_2 \in \mathfrak{H}_s\). The flow of the Hamiltonian \(H_2(p_s, q_s) = \lambda_s h_s + \langle \omega, I_s \rangle + \ldots\) contains an invariant Lagrangian manifold \(W_s = \{p_s = dS_2(q_s), q_s \in C\}\) near the zero section \(\{p_s = 0, q_s \in C\}\), with the generating function \(S_2 \in \mathfrak{H}_s^p(C)\), containing an invariant torus \(T\), corresponding to the limit as \(s_s \to -\infty\) and \(p_s = 0\). The same flow also contains the invariant manifold \(W = \{p_s = -dS^* (\alpha^* (q_s)), q_s \in C\}\), i.e. the closure of the zero section for \(H \circ \Psi\) (set \(p = 0\) in \((3.32)\)), containing the same torus. Note that both \([dS^*] = [dS_2] = 0\).

The claim now is that \(W_s = W'\), by local uniqueness. Indeed, \(H \circ \Psi \circ L_s \circ \Psi_2 = H \circ L_s \circ (L_2^{-1} \circ \Psi \circ L_s \circ \Psi_2) \in \mathfrak{H}_s\) and the transformation in the parentheses is near identity. On the other hand, \((H \circ L_s) \circ \Psi_2 \in \mathfrak{H}_s\), so one must have \(\Psi' = L_s^{-1} \circ \Psi \circ L_s \circ \Psi_2, \) for \(\mu\) small enough. Both the left and the right hand side of the latter identity must have the (unique) form \((3.31)\) in terms of the pair \((a_1, S_1)\), which by Theorem 2 is well defined as long as \(S_1\) satisfies \((3.3)\). Then \(W_s\) and \(W'\) clearly belong to the same energy level.

This essentially completes the proof. One may notice that there is a natural semidirect product structure that the canonical transformation composition induces on pairs \((a, S)\). Namely in the “old” coordinates \((p, q)\) the manifold \(W\) is represented in terms of the generating function \(S' = S' + S' \circ (a')^{-1}\), on the other hand equal \(S + S' \circ a^{-1} \circ S \circ (a \circ a'^{-1})^{-1} \), which (as \([dS^*] = [dS_2] = 0\) clearly implies \([dS] = [dS']\), i.e \(S(s, \varphi) = \langle \xi, \varphi \rangle + \hat{S}(s, \varphi)\) and \(S'(s, \varphi) = \langle \xi, \varphi \rangle + \hat{S}'(s, \varphi), \) with the same \(\xi\). However, despite \(S\) and \(S'\) are both elements of \(\mathfrak{H}_s^p(C)\), one cannot claim that necessarily \(S_0(\varphi) = S'_0(\varphi)\) in the sense of the decomposition \((3.12)\). □

Remark 3.7: (Notation) The final remark in this section is that the theory developed above can obviously be applied to the restriction of the Hamiltonian \(H(\cdot, s)\) to “one strips” \(s \in \{Rs \leq T, |3s| \leq \rho\} \cup \{Rs \leq -2T\}\) or \(s \in \{Rs \leq T, 3s - \pi \leq \rho\} \cup \{Rs \leq -2T\}\). Let us reserve the notation \(\Lambda_{\beta, T, \rho}\) with \(\beta = +\) or \(-\) respectively for the lower or the upper one of the strips above. The indices \((T, \rho)\) can be omitted in accordance with the notational convention introduced in Section 3.7.2, also \(T = \infty\) can be used. In the same fashion the subscript \(\beta\) may be added to the notations \(C, H, F, S, a, \) etc. Clearly, as far as the application of Theorem 2 to the restrictions \(H_{\beta}\) over \(C_{\beta}\) is concerned, it results in the same constants \(c_0, \xi\) as well as the \(s\)-independent components \(S_0(\varphi)\) of the generating functions \(\hat{S}_\beta\), which then represent analytic continuations of one another for \(\beta \in \{+, -\}\). These \(\beta\)-notations will be used in Section 4.

3.3 Proof of Iterative lemma

The proof of Lemma 3.1 consists of several steps inherent in KAM-type theorems [17, 41]. In the classical case, Proposition 3.3.1 plays the key role. Here these are Propositions 3.3.1 and 3.3.2. The notations \(\Psi, a, S, \) etc. in this section pertain to the formulation of the Iterative lemma and its parameter statement, rather then Theorem 2.\(^{10}\)

\(^{10}\)Note from the remark on estimates that the smallness condition for the norm of \((f_2, g_2)\) to guarantee the transformation \(\Psi_2\) is in fact more relaxed than the right hand side of \((3.33)\). However, one should not worry about the precise smallness condition here. Theorem 2 warrants the existence of the transformations \(\Psi, \Psi'\) as long as \((3.33)\) is satisfied, and this is all one needs, plus local uniqueness.
First, notice that any “affine” transformation Ψ[\(\mathbf{a}(\hat{\mathbf{b}}), S\)] described by (3.35) (where \(\mathbf{a} \in \mathcal{D}_{p,\Delta}(C)\), with the norms \(|\mathbf{b}|_{p'}\) and \(|S|_{1,p'}\) small enough, say \(O(\mu)\)) will act in such a way that \(H \circ \Psi \in \mathcal{B}_{p'}(T^* C)\) for \(H \in \mathcal{B}_{p,\Delta}(T^* C)\), despite the unboundedness as \(s \to -\infty\). This follows from the definitions of the spaces \(\mathcal{B}_{p}(C)\), \(\mathcal{D}_{p,\Delta}(C)\), \(\mathcal{B}_{n,\Delta}(T^* C)\) in Section 3.1 and can be verified directly. In particular, if \(S \in \mathcal{B}_{p'}(C)\) and has bounded partial derivatives, the quantity \(\frac{D_s S(s,x)}{\chi(s)}\) is bounded. Thus a substitution \((h, I) \to (\hat{h}, \hat{I}) + dS\) into (3.29) will not affect its structure as a Taylor series in \(\hat{\mathbf{b}}\) (3.35).

Also note that \(\frac{1}{\chi(s)} \circ \Psi[\mathbf{a}(\hat{\mathbf{b}}), 0] = \frac{b[h(v(s,\omega))]}{\chi(s)}\), where \(v \in \mathcal{B}_{p'}(C)\) and is \(O(\mu)\). This is equivalent to an earlier made statement that \(d\mathbf{a}^{-1} g \circ \mathbf{a} \in \mathcal{B}_{p}^{(-1)}(C)\) for \(g \in \mathcal{B}_{p}^{(-1)}(C)\). Combining it with the fact that by (3.17) a single Taylor coefficient, member of \(\mathcal{B}_{p}(C)\) in the Hamiltonian will only change by \(O(\mu)\) as a result of the action of the transformation \(\mathbf{a}\), one can see that \(H \circ \Psi \in \mathcal{B}_{p'}(T^* C)\) if \(\mu\) is small enough.

**Homological equation**

The quantities \((\mathbf{a}, S)\) are to eliminate the perturbation \(V\) to the leading order. In order to do so they should solve approximately the “homological equation”:

\[
H_\lambda \circ \Psi(\mathbf{a}, S) - H_\lambda + V - c_0 - c_1 h = O_2(\hat{\mathbf{p}}) + O(\mu^2),
\]

which is only possible for some specific values of the constants \(c_0, c_1\) to be found. Writing (3.44) out in essence requires only a direct substitution of (3.35) into (3.29) followed by an estimate for the “remainder” \(O(\mu^2)\). In order to get the equation for \(S\), it is enough to plug \(p \to p + dS = p + (0, \xi) + d\hat{S}\) into the Hamiltonian assuming \(S = O(\mu)\).

Furthermore if \(\mathbf{a} = \mathbf{a}_0(b_0) + \mathbf{b}\) as in (3.15), with \(\hat{\mathbf{b}} = (b_0, \mathbf{b}) = O(\mu)\), a calculation using (3.16) and \(d\mathbf{a}^{-1} = (\text{id} + d\mathbf{a}_0^{-1} \mathbf{b})^{-1}(d\mathbf{a}_0^{-1})^{-1}\) yields

\[
d\mathbf{a}^{-1} \begin{bmatrix} \lambda \\ \omega \end{bmatrix} = \begin{bmatrix} \lambda \\ \omega \end{bmatrix} - \frac{1}{\chi} \begin{bmatrix} (-\lambda + D\omega)b_0 \\ 0 \end{bmatrix} - \begin{bmatrix} D_{\lambda,\omega} b - \eta b_0 \\ D_{\lambda,\omega} B \end{bmatrix} + O(|\hat{\mathbf{b}}|^2_{p'}) \tag{3.45}
\]

where the function \(\eta(s) \in \mathcal{B}_{p}(C)\) is determined solely by \(\psi\). As the constant vector \((\lambda, \omega)\) formally belongs to \(\mathcal{B}_{p}^{(-1)}(C)\), the result of having multiplied it from the left by \(d\mathbf{a}^{-1}\) is a vector-function from \(\mathcal{B}_{p}^{(-1)}(C)\), whose leading order is given above.

It is convenient to think of \(\mathbf{b} \in \mathcal{B}_{p}^{0}(C) \cong \mathcal{B}_{p}^{(-1)}(C)\) as an element of the latter space, formally writing

\[
\hat{\mathbf{b}} = \begin{bmatrix} \frac{b_0}{\chi} + \mathbf{b} \\ B \end{bmatrix},
\]

\[
D_{\lambda,\omega} \hat{\mathbf{b}} = \frac{1}{\chi} \begin{bmatrix} (-\lambda + D\omega)b_0 \\ 0 \end{bmatrix} + \begin{bmatrix} D_{\lambda,\omega} b - \eta b_0 \\ D_{\lambda,\omega} B \end{bmatrix},
\]

as the latter expression appears in (3.45). In other words, in order to do the estimates throughout the rest of the proof, one can set \(d\mathbf{a}^{-1} = \text{id} - d\mathbf{b}\), although \(s \to s + \frac{b_0(\phi)}{\chi(s)}\) would not be a legitimate transformation for the \(s\)-variable, defined by (3.14).

Then one ends up having a pair of first order linear PDEs: one for \(\hat{S}\) and one for \(\hat{\mathbf{b}}\). The first PDE is amenable to Proposition A.1, the second one to Propositions A.1 and A.2 together, alias Proposition A.4. It requires the appropriate choice of constants \(c_0\) and \(c_1\) respectively, as well as the one-form \(dS\) cohomology class representative \(\xi \in \mathbb{R}^n\) in (3.35). The latter quantity \(\xi\) is chosen to ensure that the right hand side in the equation for \(B\) has a zero average in the sense of (3.46) enabling the choice of the constant \(c \in \mathbb{R}^{n+1}\) in Proposition A.4 simply as \(c = (c_1, 0)\) (owing to the non-degeneracy assumption on the quadratic part of \(H_\lambda\), see Definition 2). Thus (3.44) is equivalent to the following system of equations
\[ c_0 = \langle f \rangle + \langle \xi, \omega \rangle, \quad D_{\lambda, \omega} \hat{S} = \langle f \rangle - f, \quad \langle \hat{G} \rangle = 0, \quad D_{\lambda, \omega} \hat{b} = g_1 + D_{pp}^2 H(p, q)\|p = 0 \xi - c, \]

where
\[ \xi = (0, \xi), \quad g_1 = g + D_{pp}^2 H(p, q)\|p = 0 \hat{S}; \quad g_1 = (g_1, G_1). \]

Following Kolmogorov [17] one starts solving (3.47) with the equation for \( \hat{S} \), then finds \( \xi \) to satisfy the penultimate one, then solves the last equation, the constants \( c_{0,1} \) being determined along the way.

The equations for the quantities \( \hat{S}, \hat{b} \) are clearly amenable to Propositions [A.1 A.4]. The norm of the solution, according to these propositions, is simply estimated by \( \zeta^{-1} \) times the norm of the right-hand side. Applying the propositions results in analyticity loss. As a matter of fact, one encounters the analyticity loss six times along the way: solving the equation for \( \hat{S} \), evaluating the derivatives, solving the equation for \( \hat{b} \), evaluating the derivatives, making sure that not only \( a \in \mathcal{D}_{p, \Delta}(C) \), but the \( C^1 \) estimate for the norm of \( \hat{b} \) is valid throughout the maximum range of \( a \), and finally inverting it. Hence, strictly speaking one should introduce five intermediate spaces between say \( \mathcal{B}_p(C) \) and \( \mathcal{B}_{p'}(C) \), and the parameters \( \delta, \Delta \) should be scaled by factor 6. These standard steps are bypassed, and all the estimates for \( \hat{S} \) and \( \hat{b} \), as well as their derivatives, no matter that they may also be valid in some intermediate (smaller) spaces, are all written in the target spaces \( \mathcal{B}_{p'}(C) \) and \( \mathcal{B}_{p'}^{(-1)}(C) \equiv \mathcal{B}_{p'}^{(0,1)}(C) \) right away. The scaling of the analyticity loss parameters is absorbed into the constant \( C \) in (3.30). Here are the details.

First, by Proposition [A.1] and the Cauchy inequality, \( \hat{S} \) and its partial derivatives are in \( \mathcal{B}_{p'}(C) \), with the estimates\(^{11}\)
\[ |\hat{S}|_{p'} \lesssim \mu \kappa^{-1}, \quad |d\hat{S}|_{p'} \lesssim \mu (\kappa \Delta)^{-1}. \tag{3.49} \]

Then the quantity \( g_1 \) in (3.48) belongs to the space \( \mathcal{B}_{p'}^{(-1)}(C) \) (in fact, any intermediate space between \( \mathcal{B}_{p'}^{(-1)}(C) \) and \( \mathcal{B}_{p}^{(0,1)}(C) \)) with the estimate
\[ |g_1|_{p'} \lesssim \mu |M (\kappa \Delta)^{-1} + \nu^{-1}|. \]

So the following expression is well defined:
\[ \hat{S} = -(D_{II}^2 H(p, q)|p = 0)^{-1}(G_1). \]

In order to estimate it, note that \( \xi \in \mathbb{R}^n \) depends only on the right-hand side as a real function of \( q \). Thus,
\[ |\xi| \lesssim \mu R^{-1} \left(M[\text{inf}(\gamma, \lambda)]^{-1} + \nu^{-1}\right), \quad |d\hat{S}|_{p'} \lesssim \mu \eta^{-1}, \tag{3.50} \]

where in the first estimate the bounding constant depends on \( (\sigma, \rho) \); this is not the case in the second, rougher estimate, see (3.32) for the formula for the quantity \( \eta \). This gives the first three of the estimates (3.32), as well as
\[ |c_0| \lesssim \mu |\omega| R^{-1} \left(M[\text{inf}(\gamma, \lambda)]^{-1} + \nu^{-1}\right) \leq \mu |\omega| \eta^{-1}, \tag{3.51} \]

the bounding constant in the first estimate depending on \( (\sigma, \rho) \), but not in the second estimate. Being finite, the constant \( c_0 \) is further dropped.

\(^{11}\)One can go slightly more subtle estimating the derivative \( D_{\lambda} \hat{S} \), as it does not depend on \( \hat{S}_0 \), where \( \hat{S} = \hat{S}_0 + \hat{S}_1 \) in the sense of the decomposition (3.24). The norm of \( \hat{S}_1 \) however, is estimated by Proposition [A.1] without any small divisors as \( \mu \lambda^{-1} \). Then one can take a minimum of the following two estimates. One is to apply the Cauchy inequality, acquiring a factor \( \Delta^{-1} \). The other is to deduce from the equation itself that \( |D_{\lambda} \hat{S}| \leq \lambda^{-1} (|\omega| |D_{\varphi} \hat{S}| + \mu) \). The same thing can be done further estimating the \( C^1 \)-norm of \( \hat{b} \). So a \( C^1 \)-estimate for the solution of the equation \( D_{\lambda, \omega} u = v \) on \( C \) can be obtained by dividing the norm of \( v \) by \( \text{inf}[\varsigma, \lambda \sup(\Delta, \lambda |\omega|^{-1} \delta)] \) rather than \( \Delta \), causing a straightforward modification of (3.37) and (3.34).
The upper bound on the value of $c_1$ is then obtained from the last equation in \ref{9.37}, after using Proposition \ref{7.4}, with the norm of the right hand side in $\mathfrak{B}_p^{(-1)}(C)$ bounded by $\mu \eta^{-1}$:

$$
|c_1| \lesssim \mu \eta^{-1}.
$$

(3.52)

Then the $C^0$ norm of $\hat{b}$ (as an element of $\mathfrak{B}_p^{(0,1)}(C)$ or of its representation \ref{8.46} as an element of $\mathfrak{B}_p^{(-1)}(C)$) is bounded in terms of $\mu(\eta \kappa)^{-1}$ and the $C^1$ norm - in terms of $\mu(\eta \kappa \Delta)^{-1}$, which also ensures $|\hat{b}|_{p'} \lesssim \Delta$, i.e. $a \in \mathfrak{D}_p, \Delta(C)$, as well as its inverse if one takes a big enough constant in \ref{8.37}.

Finally, the presence of the additional multiplier $(\kappa - \kappa')$, in the smallness condition \ref{8.30}, is to ensure that the image of $\mathbb{E}^{n+1}_{\kappa'}$ in (non-canonical) momenta $\tilde{p} = (\frac{\hbar}{\chi(s)}, I)$ under the map $\Psi(a, S)$ is contained in $\mathbb{E}^n_{\kappa'}$, which is tantamount to requiring $|\hat{b}|_{1,p'} \lesssim \kappa - \kappa'$. The smallness condition \ref{8.30} by itself guarantees that $|c_1| < \lambda$.

Analysis of transformed Hamiltonian

It remains to estimate the term $O(\mu^2)$ in \ref{9.44}. Let us assume the smallness condition \ref{8.30} for $\mu$ and use the above obtained bounds for $S$ and $\hat{b}$. Let $C$ be large enough, say $100(n + 1)$ times the bounding constant for all the inequalities in the preceding section.

1. From \ref{8.45} it’s easy to see that $1d + d\hat{b}$ is essentially a “stretch factor” for non-canonical momenta $\tilde{p}$. Thus for the growth and non-degeneracy parameters $M', R'$ of the Taylor series $H \circ \Psi \in \mathfrak{B}_{\kappa', p'}(T^* C)$ one has $M^{-1}|M' - M|, R^{-1}|R' - R| \lesssim |\hat{b}|_{1,p'}$. Together with \ref{8.52} it gives the corresponding estimates of \ref{8.37}. The contribution from the “shift” by $d\mathfrak{S}$ in \ref{8.38} is negligible, as the non-degeneracy assumptions in Definition \ref{9.23} are global in $\tilde{p} \in \mathbb{E}_{\kappa}^{n+1}$.

2. The “new” perturbative momentum-zero-order term $f'$ is formed by several contributions. The first one comes from the momentum-super-linear part of $H$ and is bounded by $M|S|_{p'}^{1}$. The contribution from the linear terms is bounded by $|c_1||\hat{S}|_{1,p'} + |g|_{p'}|S|_{1,p'}$. Note that as $D_\mu \hat{S} \in \mathfrak{B}_p^{1}(C)$, the “new” momentum-independent term $f'$ is in $\mathfrak{B}_p^{1}(C)$ indeed. The final contribution is $f \circ a - f$, with a bound $\mu|\hat{b}|_{1,p'}$, by \ref{9.17.}. Combining it with \ref{8.50} and \ref{8.52} yields

$$
|f'|_{p'} \leq C^2 \mu^2 \eta^{-2}.
$$

3. The ”new” perturbative term $g'$ in the first order in the momentum has a component, coming from the momentum-super-linear part of $H$, bounded by $M|\hat{b}|_{1,p'}|S|_{1,p'}$. Another contribution comes from the acquired term $c_1h$: its norm can be bounded by $|c_1||\hat{b}|_{1,p'}$. Finally, the remainder $|d a^{-1} (g \circ a - g)$ must be included into account, with the bound $\mu \nu^{-1}|\hat{b}|_{1,p'}$, by \ref{9.17}. The first contribution clearly dominates the second one, and one can write

$$
|g'|_{p'} \leq \frac{1}{2} C^2 \mu^2 [M \eta^{-2}(\zeta \Delta)^{-1} + \nu^{-1} \eta^{-1} (\zeta \Delta)^{-1}] \leq C^2 \mu^2 \eta^{-2} \sup[M(\zeta \Delta)^{-1}, \nu^{-1}].
$$

(8.37)

The last pair formulas complete the set of estimates \ref{8.37} and the proof. The final remark to make here is that formally setting $d a^{-1} = d - d \hat{b}$ with $\hat{b} \in \mathfrak{B}_p^{(-1)}(C) \cong \mathfrak{B}_p^{(0,1)}(C)$ does not bring in extra error. E.g. it can be taken precisely for the sought quantity, to which after $\hat{b}$ has been determined one can unambiguously match a transformation $a \in \mathfrak{D}_p, \Delta(C)$ as long as $|\hat{b}|_{p'}$ is small enough. □

4 Splitting problem

This section contains the principal part of the proof of Theorem \ref{9.1}. The theorem follows from the analytic splitting theory in $T^*(C_{\infty})$, developed further on the basis of the main results of the previous section.
4.1 Preliminaries

Energy-time coordinates

Let us start out with the necessary additions to the set-up in the beginning of Section 3.1. One still uses formulae (4.1) for the energy-time coordinates \((h, s)\). However there is extra structure underlying the splitting problem. Namely, suppose the function \(\psi(x)\) introduced in Section 3.1 and determining the transformation \(s\) is \(2\pi\)-antiperiodic (as it is in (2.11) due to reversibility of the truncated normal form (2.7)). Then \(\psi(x)\) is \(4\pi\)-periodic, and the domain \(\mathcal{D}\) for the \(x\)-variable is a complex extension of \(\mathbb{T} = \mathbb{R}/4\pi\mathbb{Z}\). Technically, assume that it contains a pair of balls of radius \(r\) centered at \(x = 0\) and \(x = 2\pi\) and that outside these balls in \(\mathcal{D}\) one has \(|\psi(x)| \geq \tau/2\). In addition, without loss of generality one can assume that

\[
P.V. \int_{-\pi}^{\pi} \frac{d\zeta}{\psi(\zeta)} = 0,
\]

where P.V. indicates that the integral is taken in the principal value sense\(^{12}\).

Rewrite the \(2\pi\)-antiperiodicity property (4.1) as \(l_{2\pi} \circ \psi = \iota \circ \psi\), where \(\iota: s \to -s\) is sign inversion. Extend the latter to a diffeomorphism \(\iota: (s, \varphi) \to (-s, \varphi)\). As the image of the map \(s\) defined by (3.1) acting on \(\mathcal{D} \setminus \{0, 2\pi\}\), one can simply consider an imaginary circle \(\mathbb{C}/2\pi\mathbb{Z}\), as all the functions of \(s\) we are dealing with here are \(2\pi\)-periodic. Otherwise a branch of \(s\) can be fixed by drawing a branch cut in \(\mathcal{D}\) as a “semicircle” in \(\mathbb{T}\), connecting the points \(x = 0\) and \(2\pi\), but not containing \(x = \pi\). The inverse map \(x\) is represented by a homonymous function of a complex variable \(s\), which is \(2\pi\)-periodic and analytic in the bi-infinite bi-strip \(\Lambda_{T, \rho} \cup \Lambda_{-T, \rho}\), see (3.3). For now, \(4\pi\)-periodicity of the function \(\psi\) does not allow one to distinguish the values of \(x(s)\) modulo \(4\pi\). The half-width \(\rho\) and the parameter \(\sigma_2\) can be still defined by (4.1) with the choice of, say \(T = T_\psi \geq -2\log r_\psi\), i.e. the level set \(\Re s = T\) will be contained inside the ball of radius \(2r_\psi\) centered at \(x = 2\pi\). In other words,

\[
\rho = \sup\{\zeta > 0: \text{both curves } \Im s(x) = \pm \zeta \text{ lie in } \mathcal{D}, \text{ connecting the points } x = 0 \text{ and } x = 2\pi\},
\]

while \(\sigma_2 = \inf \sup |\Im x|\) over the above pair of curves. In this case one should definitely have

\[
\rho < \frac{\pi}{2}, \tag{4.2}
\]

for a bounded \(\mathcal{D}\). One reason, for instance is that the level curves \(\Re s = \mp T\) for \(T \geq T_\psi\) are contained inside balls of radius \(2r_\psi\) centered at \(x = 0\) and \(x = 2\pi\) respectively and are not homotopic in \(\mathcal{D} \setminus \{0, 2\pi\}\). Furthermore, in terms of the maps \(s, x\) (4.1) the \(2\pi\)-antiperiodicity of the function \(\psi(x)\) combined with (4.1) result in:

\[
s \circ l_{i\pi} = l_{i\pi} \circ \iota \circ s, \quad x \circ \iota = l_{2\pi} \circ x \circ l_{i\pi}, \tag{4.3}
\]

where \(l_{i\pi}: s \to s + i\pi\) (also defining a diffeomorphism \(l_{i\pi}: (s, \varphi) \to (s+i\pi, \varphi)\) and a canonical transformation \(l_{i\pi}\) acting on the momenta as the identity) and \(l_{\pm i\pi}\) are identified on \(\mathbb{C}/2\pi\mathbb{Z}\) as well as \(l_{\pm 2\pi}\) on \(\mathbb{T}\).

Fig. 3 provides an illustration, and the antiperiodicity property is expressed there by the fact that the area where \(\Re x \in [-2\pi, 0)\) is congruent to the area where \(\Re x \in [0, 2\pi)\) flipped about the real axis and translated left by \(2\pi\). Then as far as the definition (3.7) of the quantity \(\rho\), relevant to the pair \((\psi, \mathcal{D})\) is concerned, by continuity there must exist a level curve \(\gamma_\rho^\mathcal{D}\) of \(\Im s(x)\), emanating from \(x = 0\) with \(|\zeta| \leq \pi/2\) which will exit a bounded domain \(\mathcal{D}\) before (ever) arriving to the point \(x = 2\pi\). This necessitates the existence of singular points of the functions \(x(s)\) off the real axis for \(\Re s \leq T_\psi\), \(\rho \leq |\Im s| \leq \frac{\pi}{2}\), inside the unshaded rectangular regions in Fig. 3, as it is the case with the classical pendulum, where the singular points are \(s = \pm i\frac{\pi}{2}\).

Dealing with the classical pendulum where \(\psi(x) = 2\sin(x/2)\), the semi-width of \(\mathcal{D}\) can be taken arbitrarily large which means \(\rho = \rho(\psi, \mathcal{D})\) approaches \(\frac{\pi}{2}\) from below\(^{13}\). For the pendulum \(\psi(x)\) is also odd, so one should have \(D_x \psi(\pi) = D_x \chi(0) = 0\) and \(\chi(s) = 2\ \text{sech} \ s\), an even function of \(s\).

\(^{12}\)Otherwise the lower limit of integration \(i\) in the defining formula (4.1) should be substituted by some \(a \in (0, 2\pi)\) and the integral in (4.1) shall be taken from \(a - 2\pi\) to \(a\). Such an \(a\) always exists by continuity, \(2\pi\)-antiperiodicity and positivity on \((0, 2\pi)\) of \(\psi(x)\).

\(^{13}\)As \(\rho\) approaches \(\frac{\pi}{2}\) the functions from the space \(\mathcal{B}_i(C)\), etc. naturally grow unbounded. For optimal splitting estimates one wants to have \(\rho\) as large as possible, which results in various technical nuances in the literature, see [13], [14] etc.
However, further analysis of Hamiltonian \((2.21, 2.22)\), see also \((2.23)\), does require the ability to distinguish the values of \(x\) modulo \(4\pi\) (unless \(\theta \in \mathbb{Z}^n\)). In order to do so one can treat the function \(x(s)\) whose domain is shown in the upper left image in Fig. 3 as a multi-valued, bi-real-analytic, \(2i\pi\)-periodic function with values in \(\mathbb{C}\). Its single branch labelled by \(j \in \mathbb{Z}\) will be defined by fixing \(x(0) = 4\pi j\). This is equivalent to taking \(\mathcal{D}\) as a bi-infinite strip about the real axis, drawing branch cuts at all the translations of \([-2\pi, 0]\) by \(4\pi j\) and taking the lower limit of integration in \((3.1)\) equal to \(\pi + 4\pi j\), see the lower left image in Fig. 3. As a multi-valued map, \(x(s)\) has branch points inside the unshaded rectangular regions in Fig. 3. There are branch cuts emanating from a corner of each shaded region, whose exact appearance depends on how exactly the covering of \(\mathcal{D}\) (identification of \(x \in \mathbb{C}\) modulo \(4\pi\)) is defined. See the caption to Fig. 3.

**Application to \(H_\theta\)**

In this case the pair \((\psi, \mathcal{D})\) is well defined, see the end of Section 3.1. Recall that the Hamiltonian \(H_\theta\) \((2.21)\) is viewed as a multi-valued function on \(T^*(\mathbb{T}^n \times \mathbb{T}^n)\), where in view \((2.22)\) a branch is identified by a (real) value of \(H_\theta\) at \((y, I, x, \varphi) = (0, 0, 0, 0)\). The application of the transformation \(\Xi_s\) to a chosen branch of \(H_\theta\) results in the “new” Hamiltonian \(H_\psi \circ \Xi_s = H_\psi(h, I, s, \varphi) \in \mathfrak{H}_{k,\rho}(T^*\mathbb{C})\), with \(p = (r, \infty, \rho, \sigma)\) given by \((2.21)\). Application to any other branch of \(H_\theta\) is tantamount to the shift of the angles \(\varphi\) by a multiple of \(4\pi \theta\) and does not require extra consideration. As the values of the variable \(x\) cannot be identified modulo \(4\pi\) \((2.22)\) unless \(\theta \in \mathbb{Z}^n\), as far as the domain for the variable \(s\) of \(H_\psi\), is concerned it should contain branch cuts, e.g. as shown in the right image in Fig. 3. The splitting problem for \(H_\psi\) can be briefly described as follows.

In the absence of perturbation, the bi-infinite bi-cylinder \(C_\infty\) is an invariant manifold \(W_\epsilon\) which is the unstable manifold to an invariant torus \(T_{\beta,\epsilon}\) at \(s = -\infty\), see Figs 1, 3. As \(s \to \infty\) along the two different lines \(3s = 0\) and \(3s = \pi\), one arrives into a pair of different tori \(T_{\beta,\epsilon}\) \((\beta \in \{+, -\}\), the + sign corresponding to the former line; compare with Remark 3.7) corresponding to \(x = \pm 2\pi\) respectively in Fig. 1. \(W_\epsilon\) is a part of the stable manifold for these tori, which can be analytically continued further by choosing one of them and then flipping the branch cuts in Fig. 3 with respect to the imaginary axis, whereupon the chosen branch (above or below the branch cut along the line \(3s = \frac{\pi}{2}\) in Fig. 3) can be analytically continued into the strip, whereof it was separated by a branch cut. And so on.

The perturbed situation, with respect to Figs 1, 3 is qualitatively as follows. Let \(H_\theta\) be the restriction of \(H_\psi\) on \(T^*C_T\), for \(T < \infty\). Theorem 2 stipulates the existence of the perturbed manifold \(W_0\), defined as a
graph over the semi-infinite bi-cylinder $C_T$ in terms of a generating function $S_0$. $W_0$ contains an invariant torus $T_0$ near (in the sense of Remark 3.2) $\{s = -\infty\}$, for which it is an unstable manifold. Furthermore, in order to apply Theorem 2 (twice) on the bi-cylinder $C_T$ going into $\{s = +\infty\}$, one should use the analytical continuation of $H_s(\cdot, s)$ in $s$ from either the strip $|3s| \leq \rho$ or the strip $|3s - \pi| \leq \rho$. Such an analytic continuation is roughly tantamount to flipping the branch cuts in Fig. 3 with respect to the imaginary axis. Denote these analytic continuations as $H_\beta$ for $\beta = +, -$ respectively. By these analytic continuations should in particular differ from one another by the $4\pi\theta$-shift of the $\varphi$-variables and can be both described using the sputnik Hamiltonian $H_0^\beta = H_0 \circ L_\chi^{-2} \circ L_\pi^{-\beta} \circ H_0$, the set-up of Corollary 2.4 being guaranteed by 2.22.

An application of Theorem 2 to each Hamiltonian of the pair $H_\beta$ results in a manifold $W_\beta$ defined as a graph over the semi-infinite bi-cylinder $C_T$ via a generating function $S_\beta$ and containing an invariant torus $T_\beta$ near $\{s = +\infty\}$ (i.e. near $x = -2\pi, 2\pi$ for $H_\beta$, with $\beta = \text{sign} \, x$) where $W_\beta$ is the stable manifold.

The Hamiltonians $H_0$ and $H_\beta$ coincide for $s$ in the bounded strip around $\mathbb{R}$ and $\mathbb{R} + i\pi$, respectively for $\beta = +, -$. This enables one to define the splitting function on a finite bi-cylinder $\hat{C}$ as $S_0 - S_\beta$ for $s$ in the corresponding strip. An important issue that the cohomology classes of the one-forms $\{H_s = \cdot, s\}$ for $s \leq r, s \leq s, s \leq l, \rho \rangle$ and $\{s \leq r, s \leq s, s \leq l, \rho \rangle$ should in particular differ from one another by the $4\pi\theta$-shift of the $\varphi$-variables and can be both described using the sputnik Hamiltonian $H_0^\beta = H_0 \circ L_\chi^{-2} \circ L_\pi^{-\beta} \circ H_0$, the set-up of Corollary 2.4 being guaranteed by 2.22.

Technically, let us start out by calling $H_\beta(\cdot, s)$ the $2\pi\theta$-periodic restriction of $H_s(\cdot, s)$ into the complex one-strips $\Lambda_{\beta, \infty, \rho}$ described in Remark 3.7. Instead of dealing with $H_\beta$ near $\{s = +\infty\}$, consider the Hamiltonians $H_\beta \circ \mathcal{J}_\beta$, where

$$ \mathcal{J}_\beta : (h, I, s, \varphi) \to \begin{cases} (-h, I, -s, \varphi), & \beta = +, \\ (-h, I, -s + 2i\pi, \varphi), & \beta = -. \end{cases} $$

Addition of $2i\pi$ to $-s$ in the second line is optional inside the functional dependencies, as all the functions of $s$ involved are $2\pi\theta$-periodic. It has been done above simply to make sure that the restriction $L_{s, \rho}$ of $L_{s, \rho}$ to the base space maps the strip $|3s - \pi| < \rho$ into itself; it uses $2\pi\theta$-periodicity of $H_s(\cdot, s)$ in $s$. Then combining 2.22 with 3.5 one gets

$$ H_\beta \circ \mathcal{J}_\beta = H_{-\beta} \circ L_\chi^{-2} \circ L_{2\pi \theta}^{-\beta} \circ L_\pi^{-\beta} \equiv H_{-\beta} \circ L_{s, \beta}, \quad (4.4) $$

the transformation $L_\chi$ having been defined earlier by 3.25. Namely (the correction below $[+2i\pi]$ not appearing for $\beta = +$)

$$ H_\beta(-h, I, -s[+2i\pi], \varphi) = H_{-\beta}(h - 2\lambda \chi^2(s - \beta i\pi), I, s - \beta i\pi, \varphi + \beta 2i\pi \theta). \quad (4.5) $$

Indeed, the last two formulas follow from 3.3 regarding the presence of branch cuts for the map $(x, \varphi)$ by simply matching $\beta$ with sign $x$, for real $x$. Namely, in the second formula in 1.4, the shifts $l_{2\pi \theta}$ and $l_{+2\pi}$ have been identified. To make a choice of the sign for the formula 1.4 all one has to do is to act by $L_{-2\pi \theta}$ on $x \in (0, 2\pi)$ and by $L_{2\pi \theta}$ on $x \in (-2\pi, 0)$; in the same fashion $L_{-\pi, s}$ acts on $s$ : $3s = i$, corresponding to $x \in (-2\pi, 0)$, and $l_{2\pi \theta}$ acts on real $s$ corresponding to $x \in (0, 2\pi)$.

The two functions $H_\beta(\cdot, s) \circ \mathcal{J}_\beta$ allow analytic continuation in $s$ into the one-strip $\Lambda_{\beta, T, \rho}$, see Remark 3.7, where they are characterized by the same array of non-degeneracy and smallness parameters (in particular $\nu = 1$). The pair of transformations $L_{s, \beta}$ defined in 1.3 plays the role of sputnik transformations, as they obviously satisfy Assumption 2. On the other hand, the Hamiltonians $H_\beta(\cdot, s)$ represent the same analytic function $H_0(\cdot, s)$. The rest of the development is clear: Theorem 2 and Corollary 2.4 are satisfied by $H_0$ and a pair of its sputniks $H_\beta \circ \mathcal{J}_\beta$. We proceed with some generalization.

### 4.2 Splitting theory on $C_\infty$

Let $\beta \in \mathbb{Z}_2 \equiv \{+, -\}, \kappa \gg 1, \rho = (r, \infty, \rho, \sigma)$. Consider a Hamiltonian $H \equiv H_0 \in \mathfrak{B}_{\kappa, \rho}(T^*C)$ in the cotangent bundle $T^*C_\infty$ of the bi-infinite bi-cylinder $C_\infty$, in the form 3.29. Let $H_\beta(\cdot, s)$ be the restrictions of the function $H_0(\cdot, s)$ into bi-infinite strips $|3s| \leq \rho$ and $|3s - \pi| \leq \rho$ for $\beta = +, -$ respectively. Suppose, the Hamiltonians $H_\beta \circ \mathcal{J}_\beta = -L_{h, \omega} + \omega(\omega, I) + \ldots$ allow analytic continuation into one-strips $s \in \Lambda_{\beta, \infty, \rho}$ introduced by Remark 3.7, namely into the region $\{\Re s \leq -2T_{\psi}\}$, $2\pi\theta$-periodically. For the above analytic continuations let us still use the notations $H_\beta \circ \mathcal{J}_\beta$, the latter quantities being well defined in the cotangent bundle over
the one-cylinders $C_{\beta,p}$. Suppose, the restrictions of the quantities $H_0(\cdot,s)$ and $H_\beta \circ \mathcal{I}_\beta(\cdot,s)$ over $\mathbb{R}s \leq T < \infty$ satisfy the conditions of Theorem 2 with the same analyticity parameters $k,r,T,\rho,\sigma$ and equivalent (footnote 9) non-degeneracy and smallness parameters $\lambda,R,M,\mu,\nu$.

In particular this means that for $\mu = 0$ the by-infinite bi-cylinder $C_\infty$ would be an invariant Lagrangian manifold for $H_0$, asymptotic to a torus $T_0$ at $s = -\infty$ and a pair of tori $T_\beta$ at $s = +\infty$, see Fig. 3. In addition, assume the following.

**Assumption 3 (Sputniks on $C_\infty$)** There exists a pair of canonical transformations

$$L_{*\beta} = L_{*\beta}(\mathbf{a}_\beta^*,S_\beta^*) := \begin{cases} q = \mathbf{a}_\beta^*(q_s), \\ p = \frac{1}{(d\mathbf{a}_\beta^*)^{-1}} \mathbf{p}_s + dS_\beta^*(q), \end{cases} \quad (4.6)$$

where $\mathbf{a}_\beta^*$ are diffeomorphisms of $C_\infty$, such that $\mathbf{a}_\beta^* - 1d \in |\mathcal{B}_p(C)|^{n+1}$, as well as $S_\beta^* \in \mathcal{B}_p^2(C)$, such that

$$H_\beta \circ \mathcal{I}_\beta = H_{-\beta} \circ L_{*\beta}.$$ 

Then one has the following lemma.

**Lemma 4.1** Let $i \in \{0,\beta\}$. There exist invariant Lagrangian manifolds

- $W_0 = \{(h,I,s,\varphi) \in \mathbb{R}^{n+1} \times \mathcal{C} : h = D_\varphi S_0(s,\varphi), I = D_\varphi S_0(s,\varphi)\}$,
- $W_\beta = \{(h,I,s,\varphi) \in \mathbb{R}^{n+1} \times \mathcal{C} : h = D_\varphi S_\beta(s,\varphi), I = D_\varphi S_\beta(s,\varphi)\}$, \quad (4.7)

contained in the level set $H^{-1}(c_0)$ for some $c_0$ satisfying (3.31). One has $S_0 \in \mathcal{B}_p^2(C)$, $S_\beta \circ \mathcal{I} \in \mathcal{B}_p(C)$, they satisfy the corresponding bounds of (3.33) and $[dS_\beta] = \xi \in \mathbb{R}^n, \forall \varphi$. The manifolds $W_i$ contain invariant tori $T_i$ (whereupon the flow is conjugate to a rotation with the frequency $\omega$) near $s = -\infty$ for $i = 0$ and $s = +\infty$ for $i = \beta$.

**Proof:** The lemma is an immediate consequence of Theorem 2 and Corollary 2.4, see also Remark 3.7 following the latter. Indeed, Assumption 3 implies that Assumption 2 is satisfied in the sense that $H_\beta \circ \mathcal{I}_\beta$ is a sputnik of $H_0$ (restricted as a function of $s$ from the bi-strip $\Lambda_{T,\rho}$ to a one-strip $\Lambda_{-\beta,T,\rho}$ to yield $H_{-\beta}$) under the sputnik transformation $L_{*\beta}$.

The above lemma is central for the splitting problem, for now one can introduce the splitting distance as an exact one-form on the bounded bi-cylinder $\hat{\mathcal{C}}_\beta = \bigcup_{\beta \in \{+,-\}} \hat{\mathcal{C}}_{\beta,T}$: by defining it separately on each of the above components (corresponding to $\beta = \text{sign } x$ as far as the original simple resonance splitting problem is concerned):

$$d\hat{\mathcal{S}}(s,\varphi) = d[S_0(s,\varphi) - S_\beta(s,\varphi)], \quad (s,\varphi) \in \hat{\Lambda}_{\beta,T',\rho'} \times \mathbb{T}_\nu^n.$$

Let us call the function $\hat{\mathcal{S}} \in \mathcal{B}_p(\hat{\mathcal{C}})$ the splitting potential. Note that the (complexified) domains $\hat{\mathcal{C}}_{\beta,T',\rho',\sigma'}$ are disjoint for different $\beta$. In terms of the notations introduced in (3.34), one has $\hat{\Lambda}_{+,T,\rho} = \Pi_{T,\rho}$, $\hat{\Lambda}_{-,T,\rho} = \Pi_{T,\rho} + i$, where $\Pi_{T,\rho}$ is simply a symmetric rectangle in $\mathbb{C}$ with the half-length $T$ and half-width $\rho < \frac{T}{2}$.

To make things easier, let us simply view $\hat{\mathcal{S}}(s,\varphi)$ as a double-valued function, bounded and real-analytic for $(s,\varphi) \in \Pi_{T',\rho} \times \mathbb{T}_\nu^n$, by changing $s \rightarrow s + i \pi$ for $s \in \hat{\Lambda}_{-T',\rho'}$ (the strip about the line $\Re s = \pi$). Hence the rest of the statements and estimates will be valid for either one of the two branches of $\hat{\mathcal{S}}$ over the domain $\Pi \times \mathbb{T}_\nu^n$, the index $\beta$ being mostly omitted. The functions one is dealing with are members of the space $\mathcal{B}_{T',\rho',\sigma'}(\Pi \times \mathbb{T}_\nu^n)$, whose element $u$ can be represented as a uniformly convergent Fourier series in $\varphi$ with coefficients $u_k(s), k \in \mathbb{Z}^n$, bounded and holomorphic for $s \in \Pi_{T',\rho'}$, with $u_{-k}(s) = u_k^*(s) = u_k(s)$ (* marking the complex conjugate). Thus in the sequel parameter vectors $\mathbf{p}$ will have three components $(T,\rho,\sigma)$, and as usual $\mathbf{p}' < \mathbf{p}$ and $\Delta = |\mathbf{p'} - \mathbf{p}|$. Besides the use of a certain finite number of intermediate values $\mathbf{p''}$ of the analyticity parameters such that $\mathbf{p'} < \mathbf{p''} < \mathbf{p}$ is implied by default along the way.
Lemma 4.2 The splitting potential $\mathcal{S}$ satisfies a homogeneous quasi-linear PDE in $\Pi_{T',\rho'} \times \mathbb{T}_n$:

$$([\lambda + \lambda_\mu(s, \varphi)]D_s + (\omega + \omega_\mu(s, \varphi), D_\varphi))\mathcal{S} = 0,$$  

(4.9)

where the pair $g = (\lambda_\mu, \omega_\mu) \in \mathbb{B}_{T',\rho',\sigma'}(\Pi \times \mathbb{T}^n)^{n+1}$ satisfies the bounds

$$|\lambda_\mu, \omega_\mu|_{p'} \leq C\mu^{-1},$$

(4.10)

where $\eta$ is defined by (3.23) and $C$ is of the same order as in Theorem 2.

Proof: Follows by Corollary 2.1, each single $S$ is a solution of the Hamilton-Jacobi equation for $H$ on the energy level $c_0$, thus (4.9) is obtained by subtracting the equation for the former function from the same equation for the latter one. The estimate (4.10) follows from Definition 2 and the bound for $g$ in (3.30) as well as the bound (3.34) for the norm of $dS$ (which also turns out to be the estimate for $M|dS|$), see (3.49) for detail. As the result one may have to multiply the constant $C$ in Theorem A.1 by a factor, depending on $n$ and $\tau$ only. □

A prototype of the following lemma is due to Eliasson [13].

Lemma 4.3 A branch of the function $\mathcal{S}$ for $(s, \varphi) \in \Pi \times \mathbb{T}^n$ has at least $n + 1$ critical points $\varphi_c = \varphi_c(s)$, given $s$, i.e. where $D_\varphi \mathcal{S}(s, \varphi_c) = 0$.

Proof: This is a consequence of the fact that given $s \in \Pi$, the function $\mathcal{S}(s, \varphi)$ is $2\pi$-periodic in each component of $\varphi$, by Lemma 4.1 essentially stating that the one-form $d\mathcal{S}$ is exact. The number $n + 1$ of critical points is the Ljusternik-Schnirelmann characteristic of the torus $\mathbb{T}^n$. □

The statement of the next lemma is similar to Corollary 2.2 claiming the structural stability of the constant vector field $x_0 = \lambda \frac{\partial}{\partial x} + (\omega) \frac{\partial}{\partial x'}$ on the bounded one-cylinder $\Pi \times \mathbb{T}^n$ under small perturbations. It is crucial for the exponentially small estimate (2.19). The prototype of this result was proved by Sauzin [37] regarding the so-called characteristic vector field.

Lemma 4.4 There exists a constant $C = C(n, p)$ but independent of $\omega$, such that for $x = x_0 + g$, with $g \in \mathbb{B}_p(\Pi \times \mathbb{T}^n)$ such that

$$|g|_p \leq C^{-2}\lambda\Delta,$$

(4.11)

there exists a diffeomorphism $a = \text{id} + b$ with $b \in \mathbb{B}_{p'}(\Pi \times \mathbb{T}^n)^{n+1}$, effecting the conjugation $da^{-1} \circ x \circ a = x_0$, with

$$|b|_{1, p'} \leq C|g|_p(\lambda\Delta)^{-1}.$$  

(4.12)

Proof: This lemma is yet another implicit function theorem regarding the operator $D_{\lambda_\omega}$ introduced in (3.22). One can rewrite the conjugacy problem in question as

$$g \circ (\text{id} + b) - D_{\lambda_\omega} b = 0.$$

Or in Hamiltonian terms, one seeks a canonical transformation $\hat{\Psi} : q = a(q')$, $p = (da)^{-1} p'$, such that a linear Hamiltonian $H(p, q) = \lambda h + \langle \omega, I \rangle + \langle g(q), p \rangle$ is conjugate to $\lambda h + \langle \omega, I \rangle$.

The vector field conjugacy problem is certainly amenable to an abstract implicit function theorem [40], however the latter would not provide the optimal condition (4.11), as well as for the conjugacy problem (3.24). In order to get (4.11) one should follow the standard iterative scheme mimicking the proof of Corollary 2.2 (which in turn is a particular case of the proof of Theorem 2) basing it however on Proposition A.3 rather than Proposition A.1.

The latter proposition analyzes the possibility of finding a solution $u$ to a PDE $D_{\lambda_\omega} u = v$ on $\Pi_{T', \rho} \times \mathbb{T}_n^\sigma$, such that $|u|_{p'}$ can be bounded irrespective of $\omega$. The kernel of the operator $D_{\lambda_\omega}$ on $\mathbb{B}_p(\Pi \times \mathbb{T}^n)$ consists of all functions, which are represented by Fourier series in the variable $\phi = \varphi - \frac{\pi}{2}s$, and the norm of such a function in $\Pi_{T', \rho} \times \mathbb{T}_n^\sigma$ clearly does depend on $\omega$. To avoid it one is naturally led to solving a Cauchy problem...
set up by conditions \[A.3\] in Appendix A, as a way to determine the required inverse of the operator \(D_{\lambda, \omega}\).

A single application of Proposition \[A.5\] furnishes the approximate (first order) solution \(b_1\), satisfying \[4.12\] whereupon the perturbation \(g\) changes to \(g_1\), the norm of \(g_1\) bounded by \(|g_1|_{b_1, g_1, p}\). This fact constitutes the analogue of the Iterative lemma \[3.11\] whereupon the standard dyadic iterative procedure (see e.g. Appendix B) is run. \(\square\)

Results of the type of Lemmas \[4.2, 4.4\] as far as exponentially small splitting is concerned have been a target of a number of works of Lazutkin starting from \[18\] and followers, see e.g. \[10\]. Indeed from Lemmas \[4.2, 4.4\] one can further easily deduce the following upper bound for the infinity norm (the supremum over the real values of the variables only) for each branch of the splitting function \(S\).

**Theorem 3** Suppose, the assumptions of Section \[4.2\] are satisfied and the smallness condition \[5.3.6\] holds, with a large enough \(C = C(n, \tau, \psi, p)\). Then

\[
|S|_\infty \leq C\mu\eta^{-1} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \exp \left( -|\langle k, \frac{\omega}{\lambda} \rangle|\rho' - |k|\sigma' \right), \tag{4.13}
\]

where \(\eta\) is defined by \[5.3.1\], \(\rho' = \rho - \Delta < \pi/2\), \(\sigma' = \sigma - \delta\).

**Proof:** The estimate is clearly the same for each branch of the double-valued function \(S\) on \(\Pi \times \mathbb{T}^n\). Consider one branch. Let \(a\) be the conjugating diffeomorphism of Lemma \[1.1\] (the estimates of the lemma are uniform in \(\beta\)). Then the function \(S' = S \circ a\) is constant along the flow lines of the constant vector field \(x_0\) on the bounded one-cylinder \(\Pi \times \mathbb{T}^n\). Then it is a real-analytic function on \(\mathbb{T}^n\): one can formally write \(S' = S'(\varphi - \frac{\omega}{\lambda}s) = S'(\phi)\), where \(\phi = \varphi - \frac{\omega}{\lambda}s\) and expand it into the Fourier series

\[
S'(s, \varphi) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \mathcal{G}^k e^{-i\langle k, \frac{\omega}{\lambda} \rangle s} e^{i(k, \varphi)}. \tag{4.14}
\]

Then \(S'(s, \varphi)\) is a quasi-periodic function of \(s \in \Pi_{T', \rho'}\). Besides the standard complex analysis technique for estimating the Fourier coefficients yields (once again scaling the analyticity loss parameters by, say factor 4)

\[
|\mathcal{G}^k e^{-i\langle k, \varphi \rangle s}| \leq e^{-|k|\sigma'} |S|_{p'}, \quad \forall s \in \Pi_{T', \rho'}.
\tag{4.15}
\]

This implies, as \(|\Im s| \leq \rho'\) that

\[
|\mathcal{G}^k| \leq |S|_{p'} \exp \left( -|\langle k, \frac{\omega}{\lambda} \rangle|\rho' - |k|\sigma' \right),
\]

and consequently \[4.13\] as one can use \[3.3.1\] for \(|S|_{p'}\), while \(|S|_\infty \simeq |S'|_\infty \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |\mathcal{G}^k|\). \(\square\)

**Conclusion of the proof of Theorem 1**

The splitting problem for the Hamiltonian \(H_s\) given by \[3.24\] satisfies the conditions of Theorems \[2\] and \[3\] with \(\omega = \omega_1 = \frac{\omega}{\sqrt{\lambda}}\), \(R = 1\) and \(M = M_0/R_0\).

Assuming \[2.28\] and letting \(\mu = \sqrt{\lambda} \left(\frac{M_0}{\sqrt{R_0}}\right) \inf(\omega_0, R_0), \nu = 1\), it is easy to check that for the application of Theorem \[2\] (with \(\kappa > 1\), well-defined quantities \(r_\psi, T_\psi\), as well as \(\rho\) defined by \[3.7\], \(\sigma = \sigma_1 + \frac{1}{2} \delta_0\) and \(\Delta \lesssim \delta_0\) one has \(\eta^{-1} \lesssim \sup(M_0 R_0^{1/2} \lambda \delta_0^{-1})^{-1}) \lesssim \lambda^{-1}\). Thus \[2.28\] with the proper choice of the bounding constant, depending on \((n, \tau, \psi, D, \lambda, \sigma)\) ensures the applicability of Theorem \[2\].

Besides, the sputnik property, in order to satisfy Assumption \[3\] has come a long way: \[2.10, 2.22, 2.16\]. To establish the fact that all the three tori \(T_{0, \beta}\) claimed by Lemma \[1.1\] correspond to the same torus \(T\) (see Fig. 1) for the Hamiltonian \[2.6\] one should chase back through relations \[4.1, 4.2, 2.22, 2.16\] and notice that the manifolds \(W_{\beta}\) arise from the sputnik manifold \(W_0\) to \(W_0\), described by Corollary \[2.4\] with the sputnik
transformation \( L_{\gamma} = L_{\chi}^{-2} \), alias \( L_{\psi}^{-2} \), solely via a translation \( x \to x - \beta 2\pi \) in terms of the variables \((x, \varphi)\) of Hamiltonian \((2.24)\).

Finally, we show how the transformation \( \Xi_{\theta} (2.24) \) with the underlying base space transformation \( a_{\theta} : (x, \varphi) \to (x, \varphi + \theta x) \) affects the estimate \((4.13)\). Suppose \( \xi_{\theta} \) is the splitting potential, defined according to \((4.8)\) for the Hamiltonian \( H = H_{x} \). Theorem \( 4 \) followed by Lemmas \((2.2, 4.1)\) imply that the Hamilton-Jacobi equation for the quantity \( \xi_{\psi} = \xi_{\theta} \circ a_{\theta}^{-1} \) in the variables \((x, \varphi)\) of Hamiltonian \((2.14)\) is conjugate to

\[
\left( \lambda \psi(x) D_{x} + \langle \omega + \lambda \theta \psi(x), D_{\varphi} \rangle \right) \xi_{\psi}(x, \varphi) = 0,
\]

by a near-identity change \( a \) of variables \((x, \varphi)\) (where \( \xi_{\psi} = \xi_{\theta} \circ a \)) with the total analyticity loss of the order of \( \delta_{0} \). Therefore instead of \((4.14)\) one has

\[
\xi_{\psi}(s, \varphi) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \xi_{\psi}(k, \varphi) e^{-i(k, \frac{\varphi}{\lambda} s + \theta x(s))} e^{i(k, \varphi)},
\]

where \( \xi_{\psi} = \xi_{\theta} \circ s \). Then with \( \rho' = \rho - \Delta \) and \( \sigma'_{2} = \sigma_2 - \Delta \), by mimicking \((4.15)\) one gets the estimate

\[
|\xi_{\psi}(k)| \leq |\xi_{\psi}(\rho') \exp \frac{-|\langle k, \frac{\varphi}{\lambda} \omega_{1} + \frac{\varphi'}{\lambda} \theta \rangle|}{|k| \sigma_1},
\]

which completes the proof of Theorem \( 4 \) upon removing primes for the pair \((\rho, \sigma_2)\) in the formulation of the theorem. Note that the lower bound \( 2n + 2 \) for the number of homoclines comes from the application of Lemma \((4.3)\) to the two (upper, lower) separatrix branches, see Fig. 1. \( \square \)

5 Appendices

A First order linear PDEs on bi-cylinders

The following set of proposition addresses the issue of the existence of the right inverse for the operator \( D_{x} \) defined by \((3.22)\), with \( \omega \in \mathbb{R}_{r, \gamma} \) and \( \lambda > 0 \).

**Proposition A.1** Let \( p = (r, T, \rho, \sigma) \) and \( v \in \mathcal{B}_{p}(C) \). There exists a real \( c \), \(|c| \leq |v|_{p} \), such that the solution of the equation \( D_{x} \rho u = v - c \) exists in \( \mathcal{B}_{p}(C) \) for \( p' = (r, T, \rho, \sigma') \) with \( 0 < \sigma' < \sigma \). Let \( \sigma - \sigma' = \delta \) and \( \zeta = \inf(\gamma \delta r, \lambda) \). Then

\[
|u|_{p'} \lesssim \zeta^{-1} |v|_{p}.
\]

**Proof:** Following \((3.12)\), let \( v = v_{0} + v_{1} \), where \( v_{0} \in \mathcal{B}_{p}(T^{n}) \) and \( v_{1} \in \mathcal{B}_{p}(C) \). Seek \( u(s, \varphi) = u_{0}(\varphi) + u_{1}(s, \varphi) \), such that \( D_{x} u_{0} = v_{0} - c \) and \( D_{x} u_{1} = v_{1} \). The first equation obeys Proposition \((2.3.1)\) if one chooses \( c = (v_{0}) \). The second one is solved in \( C_{p} \) by the method of characteristics, regarding the fact that one can write \( v_{1}(s, \varphi) = \chi(s) w(s, \varphi) \), for some \( w \in \mathcal{B}_{p}(C) \):

\[
u_{1}(s, \varphi) = \int_{-\infty}^{0} v_{1}(s + \lambda t, \varphi + \omega t) dt = \lambda^{-1} \int_{-\infty}^{0} x^{\alpha}(s + t) w(s + t, \varphi + \omega t) dt
\]

\[
= \lambda^{-1} \int_{-\infty}^{s} x^{\alpha}(\zeta) w[\zeta, \varphi - \omega^{-1} (s - \zeta)] d\zeta
\]

\[
= \lambda^{-1} \int_{0}^{x(s)} w[s(x), \varphi - \omega^{-1} (s - s(x))] dx
\]

The latter integral was obtained via the substitution \( \zeta = s(x) \), see \((3.10)\), the integration fulfilled along the level curve \( 3 s(x) = 3 s \), see Fig. 2. The latter integral is bounded by a \((\psi, D)\)-depending constant times the norm of \( v_{1} \), see \((3.11)\). Thus \( u_{1} \in \mathcal{B}_{p}(C) \), with the norm \(|u_{1}|_{p} \lesssim \lambda^{-1} |v|_{p} \). Bi-real analyticity of \( u \) follows by construction. \( \square \)
Proposition A.2 Given \( v \in \mathcal{B}_\sigma(T^n) \) the solution of the equation \((-\lambda + D_\omega)u = v\) exists in \( \mathcal{B}_\sigma(T^n) \) and
\[
|u|_\sigma \lesssim \lambda^{-1}|v|_\sigma.
\]

Proof: Clearly \( u(\varphi) \) can be found explicitly as a Fourier series in \( \varphi \), whose coefficients \( u_k \) are expressed via the Fourier coefficients \( v_k \) of \( v(\varphi) \) as follows: \( u_k = \frac{v_k}{-\lambda + i(k, \omega)} \), \( k \in \mathbb{Z}^n \). If \( v_{-k} = v^*_k \) (complex conjugate), this property i.e. real analyticity is clearly obtained by \( u \). □

Proposition A.3 Let \( p = (r, T, \rho, \sigma) \) and \( v \in \mathcal{B}_p(\mathcal{C}) \). There exists a real constant \( c \), \( |c| \lesssim |v|_p \), such that the solution of the equation \( D_{\lambda, \omega}u = v - c \) exists in \( \mathcal{B}_p(\mathcal{C}) \) for \( p' = (r, T, \rho, \sigma') \) with \( 0 < \sigma' < \sigma \). With the same \( \delta \) and \( \zeta \) as in Proposition A.1, one has
\[
|u|_{p'} \lesssim \zeta^{-1}|v|_p.
\]

Proof: By Definition of the space \( \mathcal{B}_p(\mathcal{C}) \), \( v \) admits a unique decomposition \( v(s, \varphi) = \frac{v_0(\varphi)}{\chi(s)} + v_1(s, \varphi) \), where \( v_0 \in \mathcal{B}_\sigma(T^n) \) and \( v_1 \in \mathcal{B}_p(\mathcal{C}) \). Seek the solution \( u(s, \varphi) = \frac{v_0(\varphi)}{\chi(s)} + u_1(s, \varphi) \). Then
\[
1 \chi \left(-\lambda \frac{d\chi}{\chi} + D_\omega\right) u_0 + D_{\lambda, \omega} u_1 = \frac{v_0}{\chi} + v_1 - c.
\]
As \( \frac{d\chi}{\chi}(s) = 1 + \chi(s)\eta_1(s) \), see (3.16) where the function \( \eta_1(s) \in \mathcal{B}_p(\mathcal{C}) \), \( u_0 \) can be taken as the solution of the equation \((-\lambda + D_\omega)u_0 = v_0 \), which exists by Proposition A.2 while \( u_1 \) should satisfy
\[
D_{\lambda, \omega} u_1 = v_1 + \lambda \eta_1 u_0 - c.
\]
The first two terms in the right hand side are members of \( \mathcal{B}_p(\mathcal{C}) \), so Proposition A.1 does the job, with the constant \( c = \langle v_1 \rangle + \langle v_0 \rangle \psi_x(0) \). □

Combining Propositions A.1 and A.3 one gets

Proposition A.4 Let \( v \in \mathcal{B}_p^{(-1)}(\mathcal{C}) \). There exists a constant \( c \in \mathbb{R}^{n+1} \), \( |c| \lesssim |v|_p \), such that the solution of the \((n+1)\)-vector equation \( D_{\lambda, \omega} u = v - c \) exists in \( \mathcal{B}_p^{(-1)}(\mathcal{C}) \) for \( p' = (r, T, \rho, \sigma') \) with \( 0 < \sigma' < \sigma \). With the same \( \delta \) and \( \zeta \) as in Proposition A.1, one has
\[
|u|_{p'} \lesssim \zeta^{-1}|v|_p.
\]

Let \( v = (v, V) \). If for \( j = 1, \ldots, n \), \( \langle V_j \rangle = 0 \), then \( c = (c, 0) \), where \( c \) is the same as in Proposition A.3.

The proposition herein pertains to a conjugacy problem on the bounded one-cylinder \( \Pi \times T^n \). Let \( p = (T, \rho, \sigma) \). A function \( v \in \mathcal{B}_p(\Pi \times T^n) \) is given as a Fourier series
\[
v(s, \varphi) = \sum_{k \in \mathbb{Z}^n} v_k(s) e^{i(k, \varphi)},
\]
where \( v_{-k}(s) = v^*_k(s) = v_k(s^*) \); \( * \) marks the complex conjugate. Given \( \omega \in \mathbb{R}^n \) (not necessarily Diophantine) let
\[
\mathbb{Z}_\omega^n = \{ k \in \mathbb{Z}^n : \langle k, \omega \rangle > 0 \}.
\]
For \( k \in \mathbb{Z}_\omega^n \) and a fixed \( \rho' < \rho \) (let also \( p' = (T', \rho', \sigma') < p, \Delta = |p - p'| \)) denote
\[
\begin{align*}
  u_k[v] &= -\frac{i}{\lambda} \int_0^{\rho'} v_k(i\zeta) e^{-\frac{(k, \omega)}{\lambda} \zeta} d\zeta. 
\end{align*}
\]
For \( k \in -\mathbb{Z}_\omega^n \), let \( u_k[v] = u^*_{-k}[v] \), for \( k \) such that \( \langle k, \omega \rangle = 0 \), let \( u_k[v] = 0 \).
Proposition A.5 Let $\lambda > 0$. A Cauchy problem

$$D_{\lambda, \omega} u = v, \quad u(0, \varphi) = \sum_{k \in \mathbb{Z}^n} u_k[v] e^{i(k, \varphi)}$$

has a unique solution $u \in \mathcal{B}_{p'}(\Pi \times T^n)$, with $|u|_{p'} \leq C\lambda^{-1} |v|_{p'}$, $|du|_{p'} \leq C(\lambda \Delta)^{-1} |v|_{p'}$ where $C$ may depend on $n$ and $p'$ but is independent of $\omega$.

Proof: Seek $u(s, \varphi) = \sum_{k \in \mathbb{Z}^n} u_k(s) e^{i(k, \varphi)}$. Then $u_k(s)$ satisfy

$$\lambda u_k' + i(k, \omega) u_k = v_k, \quad u_k(0) = u_k[v].$$

For $k$ such that $\langle k, \omega \rangle = 0$ let

$$u_k(s) = \frac{1}{\lambda} \int_0^s v_k(t) dt.$$

For $k \in \mathbb{Z}^n$ let

$$u_k(s) = \frac{1}{\lambda} \int_0^s v_k(t) e^{i(k, \omega)(t-s)} dt,$$

where the integral can be taken along the part of the imaginary axis until $\Im t = \Im s$ and then along the horizontal line. For $k \in \mathbb{Z}^n$ take the lower limit of integration as $-ip'$. Then the integrand is always bounded by $\sup_{s \in \Pi \times T^n} |v_k(s)|$ in the absolute value. The initial conditions are satisfied: by real analyticity

$$v_k(0) = u_k[v].$$

If $v$ one has $v_k(-i\zeta) = v_k(i\zeta)$ for $\zeta \in \mathbb{R}$. In particular, the solution $u$ is also real analytic. Note that for the elements $u(s, \varphi)$ of $\mathcal{B}_{p'}(\Pi \times T^n)$, represented by Fourier series in $\varphi$ with analytic coefficients $u_k(s)$ the supremum norm in $\Pi_{T, p} \times T^n$ is equivalent to the norm defined as $\sum_{k \in \mathbb{Z}^n} \sup_{\Pi_{T, p}} |u_k(s)|$, the comparison constants depending in particular on $\sigma$. $\square$

B Conclusion of the proof of Theorem 2

First note that the smallness condition \[3.39\] has simply combined the smallness condition \[3.33\] on $\mu$ for the Iterative lemma to be valid with the lemma’s remainder estimate \[3.34\] on $\mu'$, simply to ensure that $\mu' < \mu$. The rest to ensure that actually $\mu'$ is many enough times smaller than $\mu$, so that the Iterative lemma can be applied again and again, with a smaller and smaller analyticity loss. This is achieved simply by choosing the constant $C$ in \[3.39\] small enough. Without loss of generality (for this part of the proof) assume that $\kappa - \kappa' = r - r' = \rho - \rho' = T - T' = \sigma - \sigma' = \Delta = \delta$, as well as $\delta < \lambda$ and $\nu > M^{-1} \varsigma \delta$.

Take a geometric sequence $\{\delta_j = 2^{-j}\delta\}_{j \geq 1}$. Let $\sigma_j = \sigma - \sum_{l=1}^j \delta_l$. Define sequences $\{\kappa, r, T, \rho\}_{j \geq 1}$ in the same way. Denote $p_j = (r_j, T_j, \rho_j, \sigma_j)$. Identify the parameters $\kappa, r, T, \rho, \sigma, \lambda, R, M, \nu, \mu$ with themselves, endowed with zero indices. Let $C_0 > 1$ be the constant, whose existence is stated by Lemma 3.1.

If $\mu$ satisfies the smallness condition \[3.39\] with some $C \geq 2^{2\tau+3} C_0$, the assumption \[3.33\] of Lemma 3.1 is satisfied for a single application of the lemma, with an analyticity loss $\delta_1 = 2^{-1} \delta$ and a parameter $\varsigma_1 = 2^{-\tau} \varsigma$ (playing the role of $\varsigma$ in \[3.32\]) implying that $\eta_1 \geq 2^{-(\tau+1)} \rho$. This results in a coordinate change $\Xi_1 = \Xi_1(\mathbf{a}_1, S_1)$. At the output, according to \[3.37\] one will have the perturbation parameters $\nu_1 \geq 2^{-(\tau+1)} M^{-1} \delta^{\tau+1}$ and $\mu_1 \leq 2^{2\tau+2} C^2 C^{-2} \mu \leq 2^{2\tau-4} \mu$, as well as the new quantities $\lambda_1, R_1, M_1$ such that

$$m_1 \leq \sup (\lambda_0 |\lambda_0 - \lambda_1|, R_0^{-1} |R_1 - R_0|, M_0^{-1} |M_1 - M_0|) \leq 2^{2\tau+2} C_0 C^{-2} \leq \frac{1}{8}.$$

Now let $C \geq 2^{2\tau+5} C_0$, and assume that the Iterative lemma can be applied repeatedly for $j \geq 2$, with an input parameter set $\{\kappa_{j-1}, r_{j-1}, \rho_{j-1}, T_{j-1}, \sigma_{j-1}, \lambda_{j-1}, R_{j-1}, M_{j-1}, \mu_{j-1}, \nu_{j-1}\}$ and an analyticity loss $\delta_j$, resulting in a transformation $\Psi_j = \Psi_j(\mathbf{a}_j, S_j)$ and an output parameter set $\{\kappa_j, r_j, T_j, \rho_j, \sigma_j, \lambda_j, R_j, M_j, \mu_j, \nu_j\}$, such that

$$\nu_j \geq (2M)^{-1} 2^{-j(\tau+1)} \delta^{\tau+1},$$

$$\mu_j \leq 2^{-j(2\tau+4)} \mu,$$

$$m_j = \sup (\lambda_{j-1}^{-1} |\lambda_j - \lambda_{j-1}|, R_{j-1}^{-1} |R_j - R_{j-1}|, M_{j-1}^{-1} |M_j - M_{j-1}|) \leq \frac{1}{8}.$$
This would imply that the sequence \( \{\mu_j\nu_j^{-1}\}_{j>1} \) vanishes geometrically.

Suppose, the above assumption is true for \( l = 1, \ldots, j-1 \) (the case \( j = 1 \) has been checked). Then on the \( j \)th application of the lemma, one can let \( \varsigma_j = 2^{-j\tau}\varsigma \), hence \( \eta_j \geq 2^{-j(\tau+1)-2}\eta \), because the inductive assumption implies that \( \lambda_{j-1} > \frac{1}{2} \), \( R_{j-1} > \frac{2}{2} \), \( M_{j-1} < 2M \). The condition (3.36) where each parameter involved has been endowed with an index \( j \) is satisfied; in fact, the right hand side of it majorates a vanishing geometric sequence with a ratio \( 2^{2\tau+3} \), whereas the left hand side is majorated by a vanishing geometric sequence with a ratio \( 2^{2\tau+4} \) (by the induction assumption). Another application of the Iterative lemma yields

\[
\mu_{j+1} \leq 2^{(2\tau+2)+5}C_02C^{-2}\mu_j \leq 2^{-(2\tau+4)}\mu_j.
\]

The fact that \( m_j < \frac{1}{8} \) is easy to verify, similar to the case \( j = 1 \). This justifies having \( 2M \) in the above assumption about \( \nu_j \) and completes the proof of the induction assumption.

Now the statement of Theorem 2 and its Parameter statement follow from chasing through the iterative scheme (3.27) and the estimates of the Parameter statement of Lemma 3.1. The existence of the limits \( a = a_1 \circ a_2 \circ \ldots \) and \( S = \sum_{j=1}^{\infty} S_j \) follows from the fast convergence of the estimates for their norms and completeness of the spaces \( \mathcal{B}_p'(C), \mathcal{B}_{p'}^{(0,1)}(C) \). Finally, the estimates (3.34) pretty much reproduce the corresponding estimates of the Iterative lemma. Indeed, due to the geometric convergence of the series \( \sum_{j\geq1} S_j \) and the composition \( a_1(b_1) \circ a_2(b_2) \circ \ldots \), it suffices to estimate the norms of \( S_1 \) and \( b_1 \) only. □
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