Universality of the anomalous enstrophy dissipation at the collapse of three point vortices on Euler-Poincaré models

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Abstract

Anomalous enstrophy dissipation of incompressible flows in the inviscid limit is a significant property characterizing two-dimensional turbulence. It indicates that the investigation of non-smooth incompressible and inviscid flows contributes to the theoretical understanding of turbulent phenomena. In the preceding study [10], a unique global weak solution to the Euler-α equations, which is a regularized Euler equations, for point-vortex initial data is considered, and thereby it has been shown that, as $\alpha \to 0$, the evolution of three point vortices converges to a self-similar collapsing orbit dissipating the enstrophy in the sense of distributions at the critical time. In the present paper, to elucidate whether or not this singular orbit can be constructed independently on the regularization method, we consider a functional generalization of the Euler-α equations, called the Euler-Poincaré models, in which the incompressible velocity field is dispersively regularized by a smoothing function. We provide a sufficient condition for the existence of the singular orbit, which is applicable to many smoothing functions. As examples, we confirm that the condition is satisfied with the Gaussian regularization and the vortex-blob regularization that are both utilized in the numerical scheme solving the Euler equations. Consequently, the enstrophy dissipation via the collapse of three point vortices is a generic phenomenon that is not specific to the Euler-α equations but universal within the Euler-Poincaré models.

1 Introduction

In the description of two-dimensional (2D) turbulent flows at high Reynolds number, there appears a remarkable discrepancy in flow regularity between viscous flows in their inviscid limit and non-viscous ones. That is to say, smooth solutions to the 2D incompressible Euler equations conserve both of the energy and the enstrophy, which are the $L^2$ norms of the velocity field and the scalar vorticity. On the other hand, it has been reported in [2, 15, 17] that the conservation of the energy and the dissipation of the enstrophy in the inviscid limit give rise to the inertial range of the energy density spectrum corresponding to the backward energy cascade and the forward enstrophy cascade in 2D turbulence. The discrepancy strongly insists that turbulent flows subject to the 2D Navier-Stokes equations converge to non-smooth flows governed by the 2D Euler equations as the Reynolds number gets infinitely large. Hence, the investigation of such singular flows plays a crucial role in the theoretical understanding of 2D fluid turbulence.

The first mathematical attempt to tackle this problem starts with constructing non-smooth weak solutions to the Euler equations dissipating the enstrophy. The global existence of a unique weak solution has been established for the initial vorticity distributions $\omega_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$
with $1 < p \leq \infty$ \cite{4,20,23}. However, it has been unfortunately shown in \cite{4} that weak solutions to the Euler equations for $\omega_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ with $p > 2$ cannot dissipate the enstrophy in the sense of distributions. Therefore, it is necessary to deal with vorticity distributions with a weaker regularity such as distributions in the space of finite Radon measures $\mathcal{M}(\mathbb{R}^2)$ on $\mathbb{R}^2$. In spite that the existence result has been extended to the case of $\omega_0 \in \mathcal{M}(\mathbb{R}^2)$ with a distinguished sign \cite{4,19}, it is still required that the velocity field induced by the vorticity distributions belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Consequently, if the vorticity distribution is represented by a $\delta$-measure, called a point vortex, for instance, it is difficult to construct a unique global weak solution to the 2D Euler equations for this initial data, since its inducing velocity field is no longer the element of $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. To overcome this mathematical difficulty, we regularize incompressible velocity fields by introducing a smoothing function with a parameter $\varepsilon$. If we successfully construct a unique global weak solution to the equations for the regularized velocity fields, we shall obtain a non-smooth incompressible and inviscid flow dissipating the enstrophy by taking the $\varepsilon \to 0$ limit of this weak solution.

An example of such regularized Euler equations is the Euler-$\alpha$ equations, where $\alpha > 0$ is the smoothing parameter. The Navier-Stokes-$\alpha$ and the Euler-$\alpha$ equations are originally derived as models of 2D turbulence \cite{7,13}. The existence of a unique global weak solution to the 2D Euler-$\alpha$ equations for $N$-point vortex initial data, referred to as $\alpha$-point vortices, has been shown in \cite{10}. Then the evolution of the weak solution can be described in terms of the dynamics of those $\alpha$-point vortices. It was discovered in \cite{22}, and it has recently been made mathematically rigorous in \cite{10}, that under a certain circumstance, the evolution of three $\alpha$-point vortices converges to a self-similar collapsing orbit in finite time as $\alpha \to 0$ and the variational part of the enstrophy dissipates in the sense of distributions at the event of collapse. In addition, it has also been revealed that this is a singular mechanism that gives rise to the irreversibility of time in conservative systems.

Another important regularization appears in the numerical scheme to solve the 2D Euler equations, which is known as the vortex blob method \cite{1,3,16}. In this scheme, descretizing initial smooth vorticity distributions by a set of many point vortices, we approximate the evolution of the vorticity distributions with those of the point vortices, in which the regularized velocity field induced by a point vortex at $x_0$ is given by

$$ u^\sigma(x) = \frac{1}{2\pi} \left( \frac{(x-x_0)^\perp}{|x-x_0|^2 + \sigma^2} \right). $$

Here, $\sigma$ denotes the smoothing parameter. As $\sigma \to 0$, we remark that the regularized velocity $u^\sigma(x)$ tends to a singular velocity field that does not belong to $L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

Here arises a natural question which we are concerned with in the present paper: Is the anomalous enstrophy dissipation via the triple collapse found in the Euler-$\alpha$ equations as $\alpha \to 0$ obtained similarly for the flows regularized by the vortex blob method as $\sigma \to 0$? This is not only a theoretical extension of the preceding study \cite{10}, but it should also be figured out whether or not the anomalous enstrophy dissipation can be constructed regardless of the regularization method.

As a matter of fact, these two regularizations of the incompressible velocity fields are generalized in a unified manner, which is called the Euler-Poincaré (EP) system \cite{11,12}. It is derived from an application of Hamilton’s principle to a dispersive kinetic energy action function with a smoothing parameter $\varepsilon$. The EP system is formally equivalent to the Euler equations when $\varepsilon$ is exactly zero. The existence of the unique global solutions to the Euler-Poincaré equations for initial vorticity distributions in $\mathcal{M}(\mathbb{R}^2)$ has been established in \cite{8} as to the Euler-$\alpha$ equations. Furthermore, since the Euler-Poincaré equations share common mathematical structures with the Euler-$\alpha$ equations, the evolution of the weak solution can be investigated in terms of the dynamics of $\varepsilon$-point vortices, which is introduced in this paper. Accordingly, one expects that the motion of the three $\varepsilon$-point vortices gives rise to the anomalous enstrophy dissipation via the self-similar collapse as we have shown in \cite{10}. On the other hand, in the Euler-Poincaré models, the enstrophy and the energy varying with the evolution of $\varepsilon$-point vortices are represented by Fourier transforms in terms of the
smoothing function unlike the Euler-α equations where it is represented by elementary functions, which makes the mathematical treatment difficult.

The paper is organized as follows. In Section 2.1 we introduce the Euler-Poincaré equations, and the existence and uniqueness theorem is stated. We then give a mathematical formulation of the Euler-Poincaré point-vortex (EP-PV) system in Section 2.2 and its associated enstrophy and energy variations are defined in Section 2.3. After introducing the three ε-point vortex problem in Section 3.1, we summarize the main results in Section 3.2, in which we provide a sufficient condition for the emergence of the anomalous enstrophy dissipation. Sections 3.3 and 3.4 are devoted to the proofs of the main results. In Section 4, we show that the enstrophy dissipation occurs for the flows regularized by the Gaussian kernel and by the vortex-blob method as applications of the main results. Final section is concluding remarks. Appendix A provides some properties of auxiliary functions, which play essential role in the proof of the main results.

2 The Euler-Poincaré system

2.1 The Euler-Poincaré equations

We derive a regularized 2D Euler equation, called the Euler-Poincaré equation, for incompressible velocity fields based on the framework of [7, 13]. For an incompressible velocity field $v$, let us define $u^\varepsilon$ by

$$u^\varepsilon(x) = \int_{\mathbb{R}^2} h^\varepsilon(x - y) v(y) dy,$$

where a smoothing function $h^\varepsilon$ is given by

$$h^\varepsilon(x) = \frac{1}{\varepsilon^2} h\left(\frac{x}{\varepsilon}\right).$$

for a scalar function $h(x)$ on $\mathbb{R}^2$. We assume that $h$ is an integrable function and may have a singularity at the origin. Since $u^\varepsilon$ is smoother than $v$ owing to its definition, we call $u^\varepsilon$ and $v$ a **regular velocity** and a **singular velocity**, respectively. In a similar manner, we define the **singular vorticity** $q$ and the **regular vorticity** $\omega^\varepsilon$ by $q = \text{curl} v$ and $\omega^\varepsilon = \text{curl} u^\varepsilon$. Let us remark that $\text{div} u^\varepsilon = 0$ and $\omega^\varepsilon = h^\varepsilon * q$ when the convolution commutes with the differential operator. Then the Euler-Poincaré equations for $(u^\varepsilon, v)$ in $\mathbb{R}^2$ are given by

$$\partial_t v + (u^\varepsilon \cdot \nabla) v - (\nabla v^T) \cdot u^\varepsilon - \nabla \Pi = 0, \quad \text{div} u^\varepsilon = \text{div} v = 0,$$

where $\Pi$ is a generalized pressure. The first momentum equation is derived from the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^2} v(x) \cdot u^\varepsilon(x) dx$$

subject to the divergence-free condition through Hamilton’s principle. Taking the curl of (2.3), we obtain the transport equations for the singular vorticity advected by the regular velocity:

$$\partial_t q + (u^\varepsilon \cdot \nabla) q = 0, \quad u^\varepsilon = K^\varepsilon * q, \quad K^\varepsilon = K * h^\varepsilon,$$

where $u^\varepsilon = K * \omega^\varepsilon$ owing to the Biot-Savart formula. It has been shown in [8] that the initial value problem of (2.4) has a unique global weak solution in the space of Radon measures $\mathcal{M}(\mathbb{R}^2)$ on $\mathbb{R}^2$, in which the following equation for the Lagrangian flow map $\eta^\varepsilon$ associated with the regular velocity is considered.

$$\partial_t \eta^\varepsilon(x, t) = u^\varepsilon(\eta^\varepsilon(x, t), t), \quad \eta^\varepsilon(x, 0) = x.$$
The solution of (2.5) yields that of (2.4) with the initial vorticity \( q_0 \) as follows.

\[
q(x, t) = q_0(\eta^\varepsilon(x, -t)).
\]

(2.6)

Here, the following functions are introduced to characterize singularities and decay rates of functions.

\[
\chi_{\log}^-(x) = \begin{cases} 
(1 - \log |x|)^{-1} & |x| \leq 1, \\
0 & |x| > 1,
\end{cases}
\quad \chi_{\log}^+(x) = \begin{cases} 
0 & |x| \leq 1, \\
|x|^\alpha & |x| > 1.
\end{cases}
\]

We also set \( \chi_{\alpha}(x) = |x|^\alpha \) for \( x \in \mathbb{R}^2 \). Then the following theorem holds.

**Theorem 2.1.** (3) Suppose that \( h \in C^1(\mathbb{R}^2) \cap W_1^1(\mathbb{R}^2) \) satisfies \( \chi_{\log}^+ h \in L^1(\mathbb{R}^2) \) and

\[
\chi_{\log}^- h \in L^\infty(\mathbb{R}^2), \quad \chi_1 \nabla h \in L^\infty(\mathbb{R}^2).
\]

Then, for any initial vorticity \( q_0 \in \mathcal{M}(\mathbb{R}^2) \), there exists a unique global weak solution of (2.3) such that \( \eta^\varepsilon \in C^1(\mathbb{R}; \mathcal{G}) \), \( u^\varepsilon \in C(\mathbb{R}; C(\mathbb{R}^2; \mathbb{R}^2)) \) and \( q \in C(\mathbb{R}; \mathcal{M}(\mathbb{R}^2)) \), where \( \mathcal{G} \) denotes the group of homeomorphisms on \( \mathbb{R}^2 \) that preserve the Lebesgue measure.

**Remark 2.2.** The assumptions of Theorem 2.1 are satisfied with two well-known regularization of the Euler equations; the Euler-\( \alpha \) equations for \( h(x) = K_0(|x|)/(2\pi) \) and the vortex blob method for \( h(x) = 1/(\pi(|x|^2 + 1)^{-1}) \). See [3, 13].

### 2.2 The Euler-Poincaré point vortex system

In what follows, we suppose that the smoothing function \( h \) in (2.2) satisfies the assumptions of Theorem 2.1. In addition, suppose that \( h \) is radial, namely \( h_r(|x|) = h(x) \), and it satisfies

\[
\int_{\mathbb{R}^2} h(x) dx = 2\pi \int_0^\infty r h_r(r) dr = 1.
\]

(2.8)

We first investigate the properties of \( K^\varepsilon \). As shown in [3], under the assumptions of Theorem 2.1, \( K^\varepsilon \) belongs to \( C_0(\mathbb{R}^2) \) and it is quasi-Lipschitz continuous with \( K^\varepsilon(0) = 0 \). It is also important to remark that \( K^\varepsilon \) is defined by \( K^\varepsilon = \nabla G^\varepsilon \), where \( G^\varepsilon \) is a solution to the following Poisson equation for \( h^\varepsilon \):

\[
-\Delta G^\varepsilon = h^\varepsilon.
\]

(2.9)

If \( h \) is radial, so is \( G^\varepsilon \), say \( G^\varepsilon(x) = G^\varepsilon(|x|) \) and we have the relation,

\[
G^\varepsilon(x) = G^1\left(\frac{x}{\varepsilon}\right) - \frac{1}{2\pi} \log \varepsilon.
\]

(2.10)

Then, we have

\[
K^\varepsilon(x) = \frac{x^+}{\varepsilon |x|} \frac{d G^1}{d r} \left(\frac{|x|}{\varepsilon}\right) = K(x) P_K \left(\frac{|x|}{\varepsilon}\right),
\]

(2.11)

where \( P_K(r) \) is defined by

\[
P_K(r) = -2\pi r \frac{d G^1}{d r}(r).
\]

(2.12)

Suppose now that the initial vorticity field is represented by a set of \( \delta \)-distributions,

\[
q_0(x) = \sum_{n=1}^N \Gamma_n \delta(x - x_0^n),
\]

(2.13)

where \( x_0^n = (x_0^n, y_0^n) \in \mathbb{R}^2 \) for \( n = 1, \ldots, N \) are their point supports of the \( \delta \)-singularities, called \( \varepsilon \)-point vortices. The strength \( \Gamma_n \in \mathbb{R} \) corresponds to the circulation around the \( \varepsilon \)-point vortex at \( x_0^n \). Theorem 2.1 shows that there exists a unique global weak solution to (2.3) with the initial data (2.13). More precisely, we have the following proposition.
Proposition 2.3. Suppose that \( h \) satisfies the assumptions of Theorem 2.1. Then, the solution to (2.4) with the initial data (2.13) is expressed by

\[
q(x, t) = \sum_{n=1}^{N} \Gamma_n \delta(x - \eta^\varepsilon(x^0_n, t)).
\]

Moreover, the point vortices in the Euler-Poincaré system never collapse.

**Proof.** Since Theorem 2.1 assures the existence of a unique global solution to (2.5), we have

\[
q(x, t) = q_0(\eta^\varepsilon(x, -t)) = \sum_{n=1}^{N} \Gamma_n \delta(\eta^\varepsilon(x, -t) - x^0_n).
\]

If \( \eta^\varepsilon(x, -t) = x^0_n \) then \( x = \eta^\varepsilon(x^0_n, t) \) else \( x \neq \eta^\varepsilon(x^0_n, t) \), which implies \( \delta(\eta^\varepsilon(x, -t) - x^0_n) = \delta(x - \eta^\varepsilon(x^0_n, t)) \). Moreover, it follows from the uniqueness of the flow map that \( \eta^\varepsilon(x^0_m, t) \neq \eta^\varepsilon(x^0_n, t) \) for \( m \neq n \) and an arbitrary \( t \in \mathbb{R} \). Thus, there is no collapse. \( \square \)

The evolution of \( \varepsilon \)-point vortices is described by \( x^\varepsilon_n(t) = \eta^\varepsilon(x^0_n, t) \). It follows from Proposition 2.3 and (2.11) with \( K^\varepsilon(0) = 0 \), the equations (2.5) with the initial vorticity (2.13) are equivalent to

\[
\frac{d}{dt} x^\varepsilon_n(t) = u^\varepsilon(x^\varepsilon_n(t), t) = -\frac{1}{2\pi} \sum_{m \neq n}^{N} \Gamma_m (x^\varepsilon_n - x^\varepsilon_m)^\perp \frac{l_{mn}}{(l_{mn}^\varepsilon)^2} P_K \left( \frac{l_{mn}^\varepsilon}{\varepsilon} \right), \quad n = 1, \ldots, N, \tag{2.15}
\]

where \( l_{mn}^\varepsilon(t) = |x^\varepsilon_n(t) - x^\varepsilon_m(t)| \) and \( x^\varepsilon_n(0) = x^0_n \). The evolution equation for \( \varepsilon \)-point vortices is called the Euler-Poincaré point vortex (EP-PV) system. According to Proposition 2.3 a weak solution to the 2D Euler-Poincaré equations provides a solution of the EP-PV system and vice versa. Now, let us see some properties of the EP-PV system. Considering the relation

\[
G^\varepsilon(|x|) = -\frac{1}{2\pi} \log |x| + H_G \left( \frac{|x|}{\varepsilon} \right)
\]

with

\[
H_G(r) = -\log r - 2\pi G^\varepsilon_1(r),
\]

we find that (2.15) is formulated as a Hamiltonian dynamical system. That is to say, it is equivalent to

\[
\Gamma_n \frac{dx^\varepsilon_n}{dt} = \frac{\partial \mathcal{H}^\varepsilon}{\partial y^\varepsilon_n}, \quad \Gamma_n \frac{dy^\varepsilon_n}{dt} = -\frac{\partial \mathcal{H}^\varepsilon}{\partial x^\varepsilon_n}, \quad n = 1, \ldots, N,
\]

with the Hamiltonian

\[
\mathcal{H}^\varepsilon = -\frac{1}{2\pi} \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m \left[ \log l_{mn}^\varepsilon + H_G \left( \frac{l_{mn}^\varepsilon}{\varepsilon} \right) \right]. \tag{2.17}
\]

The EP-PV system (2.15) admits four conserved quantities \( (\mathcal{H}^\varepsilon, Q^\varepsilon, P^\varepsilon, I^\varepsilon) \), where

\[
Q^\varepsilon + iP^\varepsilon = \sum_{n=1}^{N} x^\varepsilon_n + iy^\varepsilon_n(t), \quad I^\varepsilon = \sum_{n=1}^{N} \Gamma_n \left[ (x^\varepsilon_n)^2 + (y^\varepsilon_n)^2 \right].
\]

We then have the following integrability of the EP-PV system.
We are concerned with the enstrophy and the energy varying with the evolution of

2.3 Variations of energy and enstrophy

The Fourier transform of the vorticity field (2.14) is represented by

\[
\int_{\mathbb{R}^2} \mathcal{H}[\omega^\varepsilon](x, t)^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^2} |\mathcal{F}[\omega^\varepsilon](k, t)|^2 \, dk = \int_0^{2\pi} \pi s(\mathcal{F}[\omega^\varepsilon](s, t), t)^2 \, ds,
\]

where \(|f| = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta\). Here, we define the enstrophy density spectrum \(\mathcal{E}_N^\varepsilon\) by

\[
\mathcal{E}_N^\varepsilon(s, t) = \pi s(\mathcal{F}[\omega^\varepsilon](s, t), t)^2.
\]

The Fourier transform of the vorticity field (2.14) is represented by

\[
\mathcal{F}[q](k, t) = \frac{1}{2\pi} \sum_{n=1}^{N} \Gamma_n e^{-ik \cdot x^\varepsilon_n(t)}.
\]

Hence, we have

\[
|\mathcal{F}[q](k, t)|^2 = \frac{1}{4\pi^2} \left[ \sum_{n=1}^{N} \Gamma_n^2 + 2 \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m \cos (k \cdot (x^\varepsilon_n(t) - x^\varepsilon_m(t))) \right].
\]
Since $h^\varepsilon$ is radial and $\mathcal{F}[\omega^\varepsilon] = 2\pi \mathcal{F}[h^\varepsilon] \mathcal{F}[q]$ owing to $\omega^\varepsilon = h^\varepsilon * q$, we obtain

$$2\bar{\mathcal{E}}^\varepsilon(s,t) = \frac{s}{4\pi} \left[ 2\pi \hat{h}^\varepsilon(s) \right]^2 \left[ \sum_{n=1}^{N} \Gamma_n^2 + 2 \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(s l_{mn}^\varepsilon \cos \theta) \, d\theta \right]$$

$$= \frac{s}{4\pi} \left[ 2\pi \hat{h}^\varepsilon(s) \right]^2 \left[ \sum_{n=1}^{N} \Gamma_n^2 + 2 \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m J_0(s l_{mn}^\varepsilon) \right],$$

where $J_0(s)$ is a Bessel function of the first kind. Accordingly, the total enstrophy for the EP-PV system is expressed by

$$\frac{1}{2} \int_{\mathbb{R}^2} |\omega^\varepsilon(x,t)|^2 \, dx = \int_{0}^{\infty} 2\bar{\mathcal{E}}^\varepsilon(s,t) \, ds$$

$$= \frac{1}{4\pi \varepsilon^2} \sum_{n=1}^{N} \Gamma_n^2 \int_{0}^{\infty} s \left| 2\pi \hat{h}(s) \right|^2 \, ds + \frac{1}{2\pi \varepsilon^2} \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m \int_{0}^{\infty} s \left| 2\pi \hat{h}(s) \right|^2 J_0 \left( \frac{s l_{mn}^\varepsilon}{\varepsilon} \right) \, ds.$$
It is reduced to the following equation for the distance \( l \):
\[
E^\varepsilon(t) = -\frac{1}{2\pi} \sum_{n=1}^{N} \sum_{m=n+1}^{N} \frac{\Gamma_n \Gamma_m}{s} \left[ \log l_{mn}^\varepsilon(t) + \int_{0}^{\infty} \frac{1}{s} \left( 1 - \left| 2\pi \hat{h}(s) \right| \right) J_0 \left( \frac{l_{mn}^\varepsilon(t)}{s} \right) ds \right].
\] (2.22)

Note that the integrand of the second term has no singularity, since it follows from (2.8) that \( 2\pi \hat{h}(0) = 1 \).

### 3 Main results

#### 3.1 The three \( \varepsilon \)-vortex problem

We consider the three \( \varepsilon \)-point vortex problem, i.e. the EP-PV system with \( N = 3 \), whose Hamiltonian is expressed explicitly by
\[
\mathcal{H}^\varepsilon = -\frac{1}{2\pi} \left( \Gamma_2 \Gamma_3 \log l_{23}^\varepsilon + \Gamma_3 \Gamma_1 \log l_{31}^\varepsilon + \Gamma_1 \Gamma_2 \log l_{12}^\varepsilon \right)
\]
\[
-\frac{1}{2\pi} \left[ \Gamma_2 \Gamma_3 H_G \left( \frac{l_{23}^\varepsilon}{\varepsilon} \right) + \Gamma_3 \Gamma_1 H_G \left( \frac{l_{31}^\varepsilon}{\varepsilon} \right) + \Gamma_1 \Gamma_2 H_G \left( \frac{l_{12}^\varepsilon}{\varepsilon} \right) \right].
\]

It is reduced to the following equation for the distance \( l_{mn}^\varepsilon \):
\[
\frac{d}{dt} (l_{mn}^\varepsilon)^2 = \frac{2}{\pi} \sum_{k=1}^{3} \varepsilon P_K \left( \frac{l_{km}^\varepsilon}{\varepsilon} \right) - \frac{1}{(l_{mk}^\varepsilon)^2} P_K \left( \frac{l_{km}^\varepsilon}{\varepsilon} \right), \quad l_{mn}^\varepsilon(0) = |x_m^0 - x_n^0|.
\] (3.1)

where \( k, m, n \in \{1, 2, 3\} \) with \( k \neq m \neq n \) and \( A^\varepsilon \) denotes the signed area of the triangle formed by the three \( \varepsilon \)-point vortices. Its sign is positive if \( x_1^\varepsilon, x_2^\varepsilon \) and \( x_3^\varepsilon \) at the vertices of the triangle appear counterclockwise, while it is negative if they do clockwise. Remember that we have the two invariants in terms of the lengths: \( \mathcal{H}^\varepsilon \) and
\[
M^\varepsilon = \Gamma_2 \Gamma_3 (l_{23}^\varepsilon)^2 + \Gamma_3 \Gamma_1 (l_{31}^\varepsilon)^2 + \Gamma_1 \Gamma_2 (l_{12}^\varepsilon)^2.
\]

In order to take the \( \varepsilon \to 0 \) limit, we introduce the following scaled variables:
\[
X_n(t) = \frac{1}{\varepsilon} \varepsilon x_n^\varepsilon (\varepsilon^2 t + t^*), \quad L_{mn}(t) = \frac{1}{\varepsilon} \varepsilon l_{mn}^\varepsilon (\varepsilon^2 t + t^*)
\] (3.2)

for \( m, n \in \{1, 2, 3\} \) with \( m \neq n \), where \( t^* \in \mathbb{R} \) is an arbitrary constant determined later. Then, the evolution equation for \( X_n(t) \) is described by
\[
\frac{d}{dt} X_n = -\frac{1}{2\pi} \sum_{m \neq n} \Gamma_n \frac{(X_n - X_m) \perp L_{mn}}{L_{mn}^2} P_K (L_{mn}), \quad X_n(0) = \frac{x_n^\varepsilon (t^*)}{\varepsilon}.
\] (3.3)

It is also a Hamiltonian system with the Hamiltonian
\[
\mathcal{H} = \mathcal{H}^1 = -\frac{1}{2\pi} \left[ \Gamma_2 \Gamma_3 H_P (L_{23}^2) + \Gamma_3 \Gamma_1 H_P (L_{31}^2) + \Gamma_1 \Gamma_2 H_P (L_{12}^2) \right],
\] (3.4)

where
\[
H_P(r) = \log \sqrt{r} + H_G (\sqrt{r}),
\] (3.5)
and it has an invariant quantity,
\[ M = M^1 = \Gamma_2 \Gamma_3 L^2_{23} + \Gamma_3 \Gamma_1 L^2_{31} + \Gamma_1 \Gamma_2 L^2_{12}. \]  
(3.6)

The evolution of the distance \( L_{mn} \) is governed by
\[ \frac{d}{dt} L^2_{mn} = \frac{2}{\pi} \Gamma_k A \left[ \frac{1}{L_{nk}} P_K(L_{nk}) - \frac{1}{L_{km}} P_K(L_{km}) \right]. \]  
(3.7)

where \( A = A^1 \) is the signed area of the triangle formed by the three \( \varepsilon \)-point vortices at \( X_1(t), X_2(t) \) and \( X_3(t) \). We easily observe that the relative equilibria of \((3.7)\) are equilateral triangles or collinear configurations. See Proposition 1 of \([10]\) for its proof.

We remark that the solutions of \((2.15)\) and \((3.1)\) are recovered from those of the scaled systems \((3.3)\) and \((3.7)\) via
\[ x^2_n(t) = \varepsilon X_n \left( \frac{t-t^*}{\varepsilon^2} \right), \quad l^2_{mn}(t) = \varepsilon L_{mn} \left( \frac{t-t^*}{\varepsilon^2} \right). \]  
(3.8)

In terms of the scaled variables, the variation of enstrophy \( \mathcal{Z}^\varepsilon(t) \) is described by
\[ \mathcal{Z}^\varepsilon(t) = -\frac{1}{\varepsilon^2} \mathcal{Z}_0 \left( \frac{t-t^*}{\varepsilon^2} \right), \quad \mathcal{Z}_0(\tau) = -\frac{1}{2\pi} \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m Z_{mn}(\tau), \]  
(3.9)
in which
\[ Z_{mn}(\tau) = \int_0^\infty s \left| 2\pi \hat{h}(s) \right|^2 J_0 (s L_{mn}(\tau)) \, ds. \]  
(3.10)

Regarding the energy variation \( E^\varepsilon(t) \), since \( \mathcal{H}^\varepsilon \) is constant, we rewrite \((2.22)\) as follows.
\[ E^\varepsilon(t) = \mathcal{H}^\varepsilon + E_0 \left( \frac{t-t^*}{\varepsilon^2} \right), \quad E_0(\tau) = -\frac{1}{2\pi} \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m E_{mn}(\tau), \]  
(3.11)
in which
\[ E_{mn}(\tau) = -H_G(L_{mn}(\tau)) + \int_0^\infty \frac{1}{s} \left( 1 - \left| 2\pi \hat{h}(s) \right|^2 \right) J_0 (s L_{mn}(\tau)) \, ds. \]

The energy dissipation rate \( \mathcal{D}_E^\varepsilon \) is obtained by differentiating \( E^\varepsilon(t) \):
\[ \mathcal{D}_E^\varepsilon(t) = \frac{1}{\varepsilon^2} \frac{d}{dt} E_0 \left( \frac{t-t^*}{\varepsilon^2} \right). \]

**Remark 3.1.** In summary, from a given radial smoothing function \( h(r) = h_r(|x|) \), the functions \( P_K(|x|), H_G(|x|), H_P(|x|) \) and \( L_P(|x|) \) are derived as follows. For \( h^\varepsilon \) defined by \((2.2)\), the function \( G_1^\varepsilon(|x|) \) is obtained as the solution of the Poisson equation \((2.3)\). Setting \( \varepsilon = 1 \), we have \( G_1^1(|x|) \), yielding the functions \( P_K \) and \( H_G \) as \((2.12)\) and \((2.10)\), respectively. According to \((3.5)\), the function \( H_P \) is derived from \( H_G \). The function \( L_P(|x|) \) is defined by
\[ L_P(|x|) = \frac{1}{|x|} P_K \left( \sqrt{|x|} \right). \]  
(3.11)

Note that \( L_P \) also satisfies
\[ L_P(r) = 2 \frac{d}{dr} H(r), \]  
(3.12)
Table 1: Functions associated with the smoothing function $h_\varepsilon(r)$ for the EP-PV system corresponding to $K_0(r)$ for the $\alpha$PV system. Their common properties used in the analysis of the three vortex problem are also provided.

| $h_\varepsilon(r)$ | $K_0(r)$ | properties               |
|-------------------|--------|--------------------------|
| $P_K(r)$          | $B_K(r)$ | monotone increasing, upward convex, $0 \leq P_K < 1$ |
| $H_G(r)$          | $K_0(r)$ | monotone decreasing, downward convex, $(A.2)$ |
| $H_P(r)$          | $h_0(r)$ | monotone increasing, upward convex |
| $L_P(r)$          | $h_K(r)$ | positive, monotone decreasing |

Those functions play a significant role in the investigation of the dynamics of the three $\varepsilon$-point vortices shown later.

The EP-PV system is a generalization of the $\alpha$PV system derived from the Euler-$\alpha$ equations considered in [10]. We remark that for the modified Bessel function $K_0(|x|)$ as the smoothing function in the $\alpha$PV system, the functions $P_K(|x|)$, $H_G(|x|)$, $H_P(|x|)$ and $L_P(|x|)$ are correspondingly denoted by $B_K(|x|)$, $K_0(|x|)$, $h_0(|x|)$ and $h_K(|x|)$ in the paper [10]. Table 1 is a summary of the correspondence between those functions and their common properties, whose proofs are provided in Appendix A.

### 3.2 Main theorems

As in [10], the existence of the enstrophy dissipating solution of the three $\alpha$PV system is shown under the condition

$$
\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0. \quad (3.13)
$$

In view of (3.13), we may assume $\Gamma_1 \geq \Gamma_2 > 0 > \Gamma_3$ without loss of generality. Note that that (3.13) yields $H_\varepsilon = H$. We then show the existence of the evolution of the three $\varepsilon$-point vortices whose enstrophy varies and energy is conserved in the sense of distributions in the $\varepsilon \to 0$ limit. To state the theorem, we introduce the following functions that are defined only from the strengths $\Gamma_1$ and $\Gamma_2$.

$$
\psi(r) = \left(\frac{1}{1 + r}\right)^{1/\Gamma_1} \left(\frac{r}{1 + r}\right)^{1/\Gamma_2}, \quad k_\pm = \left(\frac{\Gamma_1 + \Gamma_2 \pm \sqrt{\Gamma_1^2 + \Gamma_1 \Gamma_2 + \Gamma_2^2}}{\Gamma_2}\right)^2 \quad (3.14)
$$

and $k_0$ is either $k_-$ or $k_+$ such that

$$
k_0 = \arg\min_{k \in \{k_-, k_+\}} \psi\left(\frac{\Gamma_1}{\Gamma_2} k\right). \quad (3.15)
$$

Then we have the following main theorem.

**Theorem 3.2.** Let $\varepsilon \in C^1(\mathbb{R}^2)$ be a positive radial function satisfying (2.7), (2.8), $\chi_3 + \eta \in L^\infty(\mathbb{R}^2)$ with $\eta > 0$, $\chi_1 \nabla h \in L^1(\mathbb{R}^2)$ and $h'_\varepsilon < 0$. Suppose (3.13) and the constant $H_\varepsilon$ satisfies

$$
\frac{\Gamma_1^2 \Gamma_2^2}{4\pi(\Gamma_1 + \Gamma_2)} \log \left(\psi\left(\frac{\Gamma_1}{\Gamma_2} k_0\right)^{-1}\right) < H_\varepsilon < 0. \quad (3.15)
$$

We also assume that, for any initial configuration with $H_\varepsilon = H_\varepsilon^\ast$, the corresponding solution of (3.7) does not converge to a relative equilibrium as either of $t \to \pm \infty$. Then, there exists a constant $t^*$ such that $t^*_m(t^*) \to 0$ as $\varepsilon \to 0$ and

$$
\lim_{\varepsilon \to 0} \mathcal{P}^\varepsilon = -z_0 \delta(-t^*), \quad \lim_{\varepsilon \to 0} \mathcal{P}^E = 0
$$

10
in the sense of distributions, where
\[ z_0 = \int_{-\infty}^{\infty} \mathcal{Z}_0(\tau) d\tau. \] (3.16)

Theorem 3.2 asserts that the enstrophy variation converges to the \( \delta \)-measure with the mass of \(-z_0\) as \( \varepsilon \to 0 \). In other words, the total enstrophy variation converges to the Heaviside function \( \delta \) as follows.
\[ \int_{-\infty}^{t} \mathcal{Z}^{\varepsilon}(\tau) d\tau \to -z_0 \delta(t - t^*). \] (3.17)

If \( z_0 > 0 \), the enstrophy dissipation occurs. Let us here note that the solution to the EP-PV system is time reversible, since it is a Hamiltonian system. Hence, as discussed in [10], even if the direction of time is reversed, we have the same convergence (3.17), which claims that the self-similar triple collapse always dissipates the enstrophy as long as \( z_0 > 0 \). This is the emergence of the irreversibility of time direction in the conservative dynamical system. However, it is still unknown whether or not the enstrophy always dissipates in that limit, since the sign of \( z_0 \) has not yet been determined.

The following corollary gives a sufficient condition for the enstrophy dissipation, which is described in terms of the function \( Z(r) \) coming from (3.10):
\[ Z(r) = \int_{0}^{\infty} s \left| 2\pi \hat{h}(s) \right|^2 J_0 \left( s \sqrt{r} \right) ds. \] (3.18)

**Corollary 3.3.** Suppose that \( Z(r) \) is monotone decreasing and downward-convex. Then, for any initial configuration satisfying the assumptions of Theorem 3.2 and \( M \geq 0 \), we have \( z_0 > 0 \). For the case of \( M < 0 \), if the functions \( Z(r) \) and \( H_P(r) \) satisfy the additional condition
\[ Z''(r)H_P'(r) - Z'(r)H''_P(r) > 0, \] (3.19)
then we have \( z_0 > 0 \).

**Remark 3.4.** While the EP-PV system and the \( \alpha \)PV system have the same Hamiltonian structure, the difference between them consists in the functions describing the Hamiltonian, whose correspondence are listed in Table 1. However, the following theorems and lemmas can be proven in the same way as those for the \( \alpha \)PV system in [10], since we just need to use the common properties shared with those functions in Table 1. Accordingly, the proofs are accomplished by formally replacing \( B_K, K_0 h_0, h_K \) with \( P_K, H_G, H_P, L_P \), which are not shown in this paper to avoid redundancy.

The following two theorems correspond to Theorem 2 and Theorem 3 of [10].

**Theorem 3.5.** Under the same assumptions of Theorem 3.2 in the \( \varepsilon \to 0 \) limit, the solution of (2.15) with \( N = 3 \) converges to the self-similar collapsing solution for \( t < t^* \) and the expanding solution for \( t > t^* \) with the same value of the Hamiltonian \( \mathcal{H}_c \) in the three point-vortex system.

This indicates that the three \( \varepsilon \)-vortex points collapse self-similarly at \( t = t^* \) in the \( \varepsilon \to 0 \) limit. Hence the enstrophy dissipation occurs at the event of the collapse. As a matter of fact, (3.13) is the necessary condition for the existence of the enstrophy dissipation via the triple collapse, which is stated as follows.

**Theorem 3.6.** Suppose \( L_{mn}(t) \to +\infty \) as \( t \to \pm \infty \) for \( m \neq n \). Then, (3.13) holds.

**Remark 3.7.** For \( M < 0 \), as we see in Section 3.4 and Section 4, the following lemmas help us to check the condition (3.19) whose proofs are the same as those of Lemma 5, Lemma 6 and Proposition 7 of [10].
Lemma 3.8. Suppose (3.13) and $M < 0$. Then, if $\mathcal{H} \leq 0$, then we have either $\gamma_1 > 1 > \gamma_2$ or $\gamma_2 > 1 > \gamma_1$, where $\gamma_1 = L_{23}/L_{12}$ and $\gamma_2 = L_{31}/L_{12}$, and these relations can not change throughout the evolution.

Lemma 3.9. Suppose (3.13), $M < 0$ and $\mathcal{H} \leq 0$. Then, every level curve of the Hamiltonian starting from collinear configurations is monotone increasing as a function of $L_{23}^2$ and it asymptotically approaches infinity along a straight line as $L_{23}^2 \rightarrow \infty$.

3.3 Proof of Theorem 3.2

Formally, it is easy to show the convergence of $\mathcal{X}$ and $\mathcal{E}$ in the sense of distributions. For any compactly supported smooth function $\phi(\tau)$, if $\mathcal{E}_0$ and $dE_0/d\tau$ decay rapidly enough to be integrable on $\mathbb{R}$ and $E_0$ vanishes at infinity, we have

$$
\langle \mathcal{X}, \phi \rangle = -\int_{-\infty}^{\infty} \frac{1}{\varepsilon^2} \mathcal{E}_0 \left( \frac{t - t^*}{\varepsilon^2} \right) \phi(t) dt = -\int_{-\infty}^{\infty} \mathcal{E}_0(\tau) \phi(\varepsilon^2 \tau + t^*) d\tau
$$

$$
- \phi(t^*) \int_{-\infty}^{\infty} \mathcal{E}_0(\tau) d\tau = -z_0 \phi(t^*),
$$

and

$$
\langle \mathcal{E}_0, \phi \rangle = \int_{-\infty}^{\infty} \frac{1}{\varepsilon^2} \frac{dE_0}{d\tau} \left( \frac{t - t^*}{\varepsilon^2} \right) \phi(t) dt = \int_{-\infty}^{\infty} \frac{dE_0}{d\tau}(\tau) \phi(\varepsilon^2 \tau + t^*) d\tau
$$

$$
\phi(t^*) \int_{-\infty}^{\infty} \frac{dE_0}{d\tau}(\tau) d\tau = \phi(t^*) [E_0(\tau)]_{-\infty}^{\infty} = 0,
$$

as $\varepsilon \to 0$. In order to make above argument mathematically rigorous, since both $\mathcal{E}_0$ and $dE_0/d\tau$ are continuous functions on $\mathbb{R}$, we will show that those functions decay rapidly, and $E_0$ vanishes at infinity.

We first remark that any solution of (3.3) subject to the assumptions of Theorem 3.2 satisfies

$$
L_{mn}(t) \sim \mathcal{O}(|t|^{1/2}), \quad t \to \pm \infty. 
$$

This asymptotic behavior is obtained by investigating the level curves of the Hamiltonian. Its proof proceeds in the same way as that of Section 4 in [10] as we mention in Remark 3.4.

To show that the value of $z_0$ given in (3.14) is well-defined, it is sufficient to see that $Z_{mn}(\tau)$ is integrable on $\mathbb{R}$. Considering the relation between the Fourier transform (2.18) and the Hankel transform (2.19), we find

$$
Z_{mn} = \int_0^{\infty} s \left| 2\pi \hat{h}(s) \right|^2 J_0(sL_{mn}) ds = (2\pi)^2 \mathcal{F} \left[ |\hat{h}|^2 \right] (x),
$$

in which $L_{mn} = |x|$. It follows from

$$
|\hat{f}|^2 = (\mathcal{F} [f])^2 = \frac{1}{2\pi} \mathcal{F} \left[ f \ast f \right], \quad \mathcal{F} [f] = \frac{1}{2\pi} \mathcal{F}^{-1} [f]
$$

for any radial function $f$ that we obtain

$$
Z_{mn}(\tau) = (2\pi)^2 \mathcal{F} \left[ |\hat{h}|^2 \right] (x) = \mathcal{F}^{-1} \left[ \mathcal{F} [h \ast h] \right] (x) = \int_{\mathbb{R}^2} h(x - y)h(y) dy.
$$
Since \( h_r(r) \) is positive and monotone decreasing, it follows that
\[
\int_{\mathbb{R}^2} h_r(|x - y|) h_r(|y|) dy = \int_{|y| \leq |x|/2} h_r(|x - y|) h_r(|y|) dy + \int_{|y| > |x|/2} h_r(|x - y|) h_r(|y|) dy \\
\leq h_r\left(\frac{|x|}{2}\right) \left[ \int_{|y| \leq |x|/2} h_r(|y|) dy + \int_{|y| > |x|/2} h_r(|x - y|) dy \right] \\
\leq 2\|h\|_{L^1} h_r\left(\frac{|x|}{2}\right).
\]

Then, for sufficiently large \( \tau_0 > 0 \), owing to (3.20), we have
\[
\int_{\tau_0}^{\infty} Z_{mn}(\tau) d\tau = \int_{\tau_0}^{\infty} \int_{\mathbb{R}^2} h(x - y) h(y) dy d\tau \leq 2\|h\|_{L^1} \int_{\tau_0}^{\infty} h_r\left(\frac{L_{mn}(\tau)}{2}\right) d\tau \\
= 2\|h\|_{L^1} \int_{\tau_0}^{\infty} h_r\left(c_0\tau^{1/2}\right) d\tau = c\|h\|_{L^1} \int_{\tau_0}^{\infty} h_r(r) dr \leq c\|h\|_{L^1}^2.
\]

and similarly
\[
\int_{-\infty}^{-\tau_0} Z_{mn}(\tau) d\tau = \int_{-\infty}^{-\tau_0} \int_{\mathbb{R}^2} h(x - y) h(y) dy d\tau \leq c\|h\|_{L^1}^2.
\]

It is easy to check that \( h \in L^2(\mathbb{R}^2) \) and
\[
\int_{-\tau_0}^{\tau_0} \int_{\mathbb{R}^2} h(x - y) h(y) dy d\tau \leq c\|h\|_{L^2}^2.
\]

Therefore, we conclude that \( Z_{mn}(\tau) \) is integrable on \( \mathbb{R} \) so that \( \tau_0 \) is finite.

Next, we show that \( E_0(\tau) \) vanishes at infinity and \( dE_0/d\tau \) is integrable on \( \mathbb{R} \). Similar to the enstrophy, we confirm those claims for \( E_{mn} \) instead of \( E_0 \). Let us recall the definition of \( E_{mn} \):
\[
E_{mn}(\tau) = -H_G(L_{mn}(\tau)) + \int_0^\infty \frac{1}{s} \left( 1 - \left| 2\pi \hat{h}(s) \right|^2 \right) J_0(s L_{mn}(\tau)) ds \\
\equiv -H_G(L_{mn}(\tau)) + E_J(L_{mn}(\tau)),
\]
where
\[
E_J(r) = \int_0^\infty \frac{1}{s} \left( 1 - \left| 2\pi \hat{h}(s) \right|^2 \right) J_0(s r) ds. \quad (3.21)
\]

Regarding the function \( H_G(r) \) in (2.11), we have
\[
H_G(|x|) = -\log|x| - 2\pi G_r^1(|x|) = \int_{\mathbb{R}^2} \left( \log \frac{|x - y|}{|x|} \right) h(y) dy.
\]

By dividing \( \mathbb{R}^2 \) into three domains and evaluating the integration on these domains separately, we obtain
\[
|H_G(|x|)| \leq \int_{|y| - |x| \leq \eta |x|} \left| \log \frac{|x - y|}{|x|} \right| h(y) dy \\
+ \int_{\eta |x| < |y| - |x| \leq |x|/\eta} \left| \log \frac{|x - y|}{|x|} \right| h(y) dy \\
+ \int_{|y| - |x| > |x|/\eta} \left| \log \frac{|x - y|}{|x|} \right| h(y) dy.
\]
Finally, the third term is estimated as

\[ \int_{||y|-|x||\leq \eta/x} |\log \frac{x-y}{x}| h(y) dy = \int_{|z|\leq \eta} |\log \frac{x}{|x|} - z| h(|x|z)|x|^2 dz \]

\[ \leq ||\chi_3 h||_{L^\infty} \int_{|z|\leq \eta} \frac{1}{|z|^3} |\log \frac{x}{|x|} - z| dz \]

\[ \leq \frac{||\chi_3 h||_{L^\infty}}{|x|} \int_{|z|\leq \eta} |\log \frac{x}{|x|} - z| dz \leq \frac{c_1}{|x|} ||\chi_3 h||_{L^\infty}. \]

Since \( |x| < ||y|-|x||/\eta \) yields \( \eta < |x-y|/|x| < 2 + \eta^{-1} \), we estimate the second term as

\[ \int_{|y|>|x|} \int_{|y|-|x||\leq \eta/x} |\log \frac{x-y}{x}| h(y) dy \leq c_1 \int_{|y|>|x|} \int_{|y|-|x||\leq |x|/\eta} h(y) dy \leq \frac{c_1}{|x|} ||\chi_1 h||_{L^1}. \]

Finally, the third term is estimated as

\[ \int_{|y|>|x|} \int_{|y|-|x||\leq \eta/x} |\log \frac{x-y}{x}| h(y) dy = \int_{|z|>1+1/\eta} |\log (1+|z|)h(|x|z)|x|^2 dz \]

\[ \leq \int_{|z|>1+1/\eta} |z|h(|x|z)|x|^2 dz \]

\[ \leq \frac{1}{|x|} \int_{|z|>1+1/|x|} |z|h(z) dz \leq \frac{1}{|x|} ||\chi_1 h||_{L^1}. \]

Remembering that \( h \) satisfies \( \chi_3 h \in L^\infty(\mathbb{R}^2) \) and \( \chi_1 h \in L^1(\mathbb{R}^2) \) under the assumptions of Theorem 3.2, we obtain

\[ |H_G(L_{mn})| \leq \frac{c}{L_{mn}} (||\chi_3 h||_{L^\infty} + ||\chi_1 h||_{L^1}). \]

In order to investigate the function \( E_J(r) \), let us observe the properties of the Fourier and Hankel transforms. It is easy to check that \( ||\mathcal{F}[f]||_{L^\infty} \leq (2\pi)^{-1} ||f||_{L^1} \) and

\[ |k| \frac{d}{ds} (|k|) = \frac{-1}{2\pi} \int_{\mathbb{R}^2} (2f(|x|) + |x|f'(|x|)) e^{-ixk} dx. \]

Then, we find \( ||\hat{h}||_{L^\infty} \leq (2\pi)^{-1} ||h||_{L^1} \) and \( ||\chi_1 h \hat{h}/ds||_{L^\infty} \leq (2\pi)^{-1} (2||h||_{L^1} + ||\chi_1 \nabla h||_{L^1}) \), and we also have \( ||d\hat{h}/ds||_{L^\infty} \leq (2\pi)^{-1} ||\chi_1 h||_{L^1} \). Since \( \hat{h}(0) = 1 \) owing to (2.8), the mean value theorem yields

\[ 2\pi \hat{h}(s) - 1 = 2\pi \hat{h}(s) - 2\pi \hat{h}(0) \leq 2\pi s \int_0^1 \frac{d\hat{h}}{ds}(\tau s) d\tau. \]

Hence, we have the following estimate for (3.21).

\[ |E_J(|x|)| = \left| \int_0^\infty \frac{1}{s} \left( 1 - 2\pi \hat{h}(s) \right) \left( 1 + 2\pi \hat{h}(s) \right) J_0 (s|x|) ds \right| \]

\[ \leq 2\pi \int_0^\infty \int_0^1 \frac{d\hat{h}}{ds}(\tau s) \left( 1 + 2\pi \hat{h}(s) \right) |J_0 (s|x|)| ds \]

\[ = \frac{2\pi}{|x|} \int_0^1 \frac{d\hat{h}}{ds} \left( \frac{u}{|x|} \right)^\tau \left( 1 + 2\pi \hat{h} \left( \frac{u}{|x|} \right) \right) J_0 (u) du. \]
Setting the constant $1/2 < \alpha < 1$, we obtain
\[
|E_J(|x|)| \leq \frac{2\pi}{|x|} \left| \chi_\alpha \frac{d\hat{h}}{ds} \right|_{L^\infty} \left( \int_0^1 \tau^{-\alpha} d\tau \int_0^\infty \frac{|x|^\alpha}{u^\alpha} \left( 1 + 2\pi \left| \hat{h} \left( \frac{u}{|x|} \right) \right| \right) |J_0(u)| du \right.
\]
\[
\leq \frac{c}{|x|^{1-\alpha}} \left| \chi_\alpha \frac{d\hat{h}}{ds} \right|_{L^\infty} \left( 1 + \|\hat{h}\|_{L^\infty} \right) \int_0^1 \tau^{-\alpha} d\tau \int_0^\infty \frac{1}{u^\alpha} |J_0(u)| du,
\]
in which the rightmost integral with respect $u$ is well-defined owing to $J_0(0) = 1$ and $J_0(r) \sim r^{-1/2}$ as $r \to \infty$. Note that it follows from the inequality $|x|^\alpha \leq 1 + |x|$ for $1/2 < \alpha < 1$ that
\[
\left| \chi_\alpha \frac{d\hat{h}}{ds} \right|_{L^\infty} \leq \left| \frac{d\hat{h}}{ds} \right|_{L^\infty} \leq \frac{1}{2\pi} \left( 2\|\hat{h}\|_{L^1} + \|\chi_1 h\|_{L^1} + \|\chi_1 \nabla h\|_{L^1} \right).
\]
Hence, we have
\[
|E_J(L_{mn})| \leq \frac{c}{L^{1-\alpha}_{mn}} \left( ||\hat{h}\|_{L^1} + \|\chi_1 h\|_{L^1} + \|\chi_1 \nabla h\|_{L^1} \right) \left( 1 + \|\hat{h}\|_{L^\infty} \right).
\]
Combining the estimates for $H_G(r)$ and $E_J(r)$ and considering $3.20$, we conclude that $E_{mn}(r)$ vanishes at infinity with the order $O(|r|^{-(1-\alpha)/2})$, $1/2 < \alpha < 1$ so that $E_0(r)$ decays with the same order.

We finally show that $dE_{mn}/dt$ is integrable on $\mathbb{R}$. Regarding the derivative of $H_G(r)$, it follows from $|dH_G(|x|)|/|dr| = |\nabla H_G(|x|)|$ that
\[
\left| \frac{dH_G(|x|)}{dr} \right| \leq \int_{\mathbb{R}^2} \frac{|x - y|}{|x - y|^2} - \frac{x}{|x|^2} \left| h(y)dy \right| = \frac{1}{|x|} \int_{\mathbb{R}^2} \frac{|y|}{|x - y|} h(y)dy
\]
\[
\leq \frac{1}{|x|^2} \int_{\mathbb{R}^2} \left( 1 + \frac{|y|}{|x - y|} \right) |y| h(y)dy \leq \frac{1}{|x|^2} \left( \|\chi_1 h\|_{L^1} + \int_{\mathbb{R}^2} \frac{|y|^2}{|x - y|} h(y)dy \right).
\]
The second term in the right-hand side is estimated as follows.
\[
\int_{\mathbb{R}^2} \frac{|y|^2}{|x - y|} h(y)dy \leq \|\chi_2 h\|_{L^\infty} \left[ \int_{|x - y| \leq 1} \frac{1}{|x - y|} dy + \int_{|x - y| > 1 \cap |y| \leq 1} \frac{1}{|x - y|} dy \right]
\]
\[
+ \|\chi_{3+n} h\|_{L^\infty} \left[ \int_{|x - y| > 1} \frac{1}{|y|^{2+\eta}} dy + \int_{|y| > |x - y| > 1} \frac{1}{|x - y|^{2+\eta}} dy \right]
\]
\[
\leq c \left( \|\chi_2 h\|_{L^\infty} + \|\chi_{3+n} h\|_{L^\infty} \right).
\]
We thus obtain
\[
\left| \frac{dH_G(|x|)}{dr} \right| \leq \frac{c}{|x|^2} \left( \|\chi_1 h\|_{L^1} + \|\chi_2 h\|_{L^\infty} + \|\chi_{3+n} h\|_{L^\infty} \right),
\]
and we know that right-hand side is finite owing to the assumptions of $h$. In order to estimate the derivative of $E_J(r)$, we rewrite \[3.21\] by
\[
E_J(|x|) = \int_0^\infty \frac{1}{u} \left( 1 - 2\pi \hat{h} \left( \frac{u}{|x|} \right) \right)^2 |J_0(u)| du.
\]
Its derivative is then expressed by
\[
\frac{dE_J}{dr}(|x|) = \frac{8\pi^2}{|x|^2} \int_0^\infty \hat{h} \left( \frac{u}{|x|} \right) \frac{d\hat{h}}{ds} \left( \frac{u}{|x|} \right) |J_0(u)| du.
\]
Similar to the calculation estimating \( E_J(r) \), we have
\[
\left| \frac{dE_J}{dr}(|x|) \right| = \frac{8\pi^2}{|x|^{2-\alpha}} \int_0^\infty \left| \frac{\hat{h}(u)}{|x|} \right| \left( \frac{u}{|x|} \right)^\alpha \left| \frac{d\hat{h}}{ds} \left( \frac{u}{|x|} \right) \right| \frac{1}{u^\alpha} |J_0(u)| \, du
\]
\[
\leq \frac{c}{|x|^{2-\alpha}} \|\hat{h}\|_{L^\infty} \left\| \chi_\alpha \frac{d\hat{h}}{ds} \right\|_{L^\infty} \leq \frac{c}{|x|^{2-\alpha}} (\|h\|_{L^1} + \|\chi_1 h\|_{L^1} + \|\chi_1 \nabla h\|_{L^1}) \|\hat{h}\|_{L^\infty},
\]
in which \( 1/2 < \alpha < 1 \). Summarizing the estimates and (3.20), we find
\[
\left| \frac{dE_{mn}}{d\tau}(\tau) \right| = \left| \left( -\frac{dH_G}{dr}(L_{mn}(\tau)) + \frac{dE_J}{dr}(L_{mn}(\tau)) \right) \frac{1}{2L_{mn}(\tau)} \right| \left| \frac{dL^2_{mn}}{d\tau}(\tau) \right|
\]
\[
\leq \frac{c}{(L_{mn}(\tau))^{3-\alpha}} \sim O(|\tau|^{-1-(1-\alpha)/2}), \quad \tau \to \pm \infty.
\]
Consequently, \( dE_{mn}/d\tau \) and \( dE_0/d\tau \) are integrable on \( \mathbb{R} \).

### 3.4 Proof of Corollary 3.3

In order to show \( z_0 > 0 \), it is enough to prove that \( \mathcal{Z}_0 \) is a positive function. Note that, owing to (3.13), \( \mathcal{Z}_0 \) is rewritten by
\[
\mathcal{Z}_0 = \frac{\Gamma_1 \Gamma_2}{2\pi} \left( \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} Z(L^2_{23}) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} Z(L^2_{31}) - Z(L^2_{12}) \right)
\]
and (3.6) is equivalent to
\[
\frac{\Gamma_2}{\Gamma_1 + \Gamma_2} L^2_{23} + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} L^2_{31} = L^2_{12} - \frac{M}{\Gamma_1 \Gamma_2}.
\]
Since \( Z(r) \) is monotone decreasing and downward-convex, it follows from \( M \geq 0 \) that
\[
\mathcal{Z}_0 > \frac{\Gamma_1 \Gamma_2}{2\pi} \left[ Z \left( L^2_{12} - \frac{M}{\Gamma_1 \Gamma_2} \right) - Z(L^2_{12}) \right] \geq 0.
\]
Hence, we easily obtain the result for \( M \geq 0 \) as desired. To see the case of \( M < 0 \), let us introduce the following notations,
\[
L = L^2_{12}, \quad \mu = \frac{L^2_{23}}{L^2_{12}}, \quad \hat{\mu} = \frac{L^2_{31}}{L^2_{12}}.
\]
According to Lemma 3.3, under the conditions \( M < 0 \) and \( \mathcal{J} < 0 \), either \( \mu < 1 < \hat{\mu} \) or \( \hat{\mu} < 1 < \mu \) holds true for all time. We here consider the case \( \mu < 1 < \hat{\mu} \). As for the other case, we can show the fact similarly. Note that (3.6) implies
\[
\hat{\mu} = \hat{\mu}(\mu) = -\frac{\Gamma_2}{\Gamma_1} \mu + \frac{1}{\Gamma_3 \Gamma_1 L} (M - \Gamma_1 \Gamma_2 L).
\]
For any fixed \( L > 0 \) and \( M < 0 \), we rewrite \( \mathcal{Z}_0 \) as the function of \( \mu \).
\[
\mathcal{Z}_0 = Z_{L,M}(\mu) = \frac{\Gamma_1 \Gamma_2}{2\pi} \left( \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} Z(\mu L) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} Z(\hat{\mu}(\mu) L) - Z(L) \right).
\]
Then, we have
\[
\frac{dZ_{L,M}}{d\mu}(\mu) = \frac{\Gamma_1 \Gamma^2_2 L}{2\pi(\Gamma_1 + \Gamma_2)} (Z'(\mu L) - Z'(\hat{\mu}(\mu) L)) < 0,
\]
since $Z'(r)$ is negative and monotone increasing. Similarly, since the Hamiltonian $H_{L,M}^{(3.1)}$ is expressed by
\[ H_{L,M} = \frac{\Gamma_1 \Gamma_2}{2\pi} \left[ \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} H_P(\mu L) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} H_P(\mu L) - H_P(L) \right], \]
we have
\[ \frac{dH_{L,M}}{d\mu}(\mu) = \frac{\Gamma_1 \Gamma_2^2 L}{4\pi(\Gamma_1 + \Gamma_2)} (L_P(\mu L) - L_P(\mu L)) > 0 \]
owing to $|L|$. Hence, it is sufficient to show that $Z_{L,M}(\mu_0) > 0$ for the constant $\mu_0$ satisfying $H_{L,M}(\mu_0) = 0$. Indeed, since $\mathcal{H} = H_{L,M}(\mu) < 0$ is equivalent to $\mu < \mu_0$, we obtain $Z_0 = Z_{L,M}(\mu) > Z_{L,M}(\mu_0) > 0$ for $\mu < \mu_0$, i.e. $\mathcal{Z}_0$ is always positive when $\mathcal{H}$ is negative.

Since $H_P(r)$ is continuous and monotone increasing, as shown in Appendix A.4, there exists the inverse function $H_P^{-1}(r)$ so that the function $Z_H(r) = Z(H_P^{-1}(r))$ is well-defined. Then, the first and second derivatives of $Z_H(r)$ are expressed by
\[ \frac{dZ_H}{dr}(r) = \frac{Z'(H_P^{-1}(r))}{H_P'(H_P^{-1}(r))}, \]
\[ \frac{d^2Z_H}{dr^2}(r) = \frac{Z''(H_P^{-1}(r))}{(H_P'(H_P^{-1}(r)))^2} - \frac{Z'(H_P^{-1}(r))}{(H_P'(H_P^{-1}(r)))^3} \frac{H''_P(H_P^{-1}(r))}{(H_P'(H_P^{-1}(r)))^3}, \]
where $DZ(r) = Z''(r)H_P'(r) - Z'(r)H''_P(r)$. Note that $H_P(r) = L_P(r)/2$ is a positive function. Owing to $|L|$ and positivity of $H_P'(r)$, we find $d^2Z_H/dr^2 > 0$ so that $Z_H(r)$ is downward-convex. Hence, considering $H_{L,M}(\mu_0) = 0$, namely
\[ \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} H_P(\mu_0 L) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} H_P(\mu_0 L) = H_P(L), \]
we obtain
\[ Z_{L,M}(\mu_0) = \frac{\Gamma_1 \Gamma_2}{2\pi} \left[ \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} Z_H(H_P(\mu_0 L)) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} Z_H(H_P(\mu_0 L)) - Z_H(H_P(L)) \right] \]
\[ > \frac{\Gamma_1 \Gamma_2}{2\pi} Z_H \left( \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} H_P(\mu_0 L) + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} H_P(\mu_0 L) \right) - Z_H(H_P(L)) \]
\[ = \frac{\Gamma_1 \Gamma_2}{2\pi} [Z_H(H_P(L)) - Z_H(H_P(L))] = 0. \]

Since we assume $\mathcal{H} < 0$, we achieve the conclusion.

4 Applications to various smoothing functions

We apply the main results to some smoothing functions for the Euler-Poincaré models to show the existence of the anomalous enstrophy dissipation via the triple collapse.

4.1 Gaussian kernel

The simplest smoothing function for the Euler-Poincare model is the Gaussian kernel, which is given by
\[ h(x) = \frac{1}{\pi} e^{-|x|^2}. \]
It is easy to confirm that \( h \) satisfies the assumptions of Theorem 3.2. In order to prove \( z_0 > 0 \), we show that the functions \( Z \) and \( H_P \), which are derived from \( h \) as (3.18) and (3.5) respectively, satisfy the assumptions of Corollary 3.3. The function \( Z(r) \) is monotone decreasing and downward convex, since we have
\[
Z(r) = \int_{\mathbb{R}^2} h(x - y)h(y)dy = \frac{1}{\pi^2} \int_{\mathbb{R}^2} e^{-\frac{1}{2}||x||^2} \int_{\mathbb{R}^2} e^{-2\frac{1}{2}||y-x||^2}dy = \frac{1}{2\pi} e^{-\frac{1}{2}r} = \frac{1}{2\pi} e^{-\frac{r}{2}},
\]
in which \( r = ||x||^2 \). Regarding the sufficient condition (3.19), since
\[
Z''(r)H_P'(r) - Z'(r)H_P''(r) = \frac{1}{8\pi} e^{-\frac{r}{4}} (H_P'(r) + 2H_P''(r)) \equiv \frac{1}{8\pi} e^{-\frac{r}{4}} DH_P(r),
\]
it is enough to show that \( DH_P(r) \) is a positive function. Owing to (A.1) and (A.3), \( DH_P(r) \) is expressed by
\[
DH_P(r) = \frac{1}{2r^2} ((r-2)P_K(\sqrt{r}) + 2re^{-r}) \equiv \frac{1}{2\pi^2} p_0(r).
\]
It follows from \( p_0'(r) = P_K(\sqrt{r}) - re^{-r} \) and \( p_0''(r) = re^{-r} > 0 \) with \( p_0'(0) = 0 \) that \( p_0' \) is positive and so \( p_0 \) is monotone increasing. In addition, owing to \( p_0(0) = 0 \), we find \( p_0 > 0 \) and thus \( DH_P > 0 \). Therefore, we conclude that the enstrophy dissipates in the EP-PV system with (4.1).

4.2 The vortex blob system

We consider the vortex blob regularization as another important example of the Euler-Poincaré models, in which the smoothing function \( h^\sigma \) is given by
\[
h^\sigma(x) = \frac{1}{\sigma^2} h_b \left( \frac{x}{\sigma} \right), \quad h_b(x) = \frac{1}{\pi(||x||^2 + 1)^2}.
\]
Since \( h_b \) satisfies the assumptions of Theorem 3.2, the enstrophy variation in the three point vortex problem of the \( \sigma \)PV system converges to the \( \delta \)-measure with the mass of \( -z_0 \) in the \( \sigma \rightarrow 0 \) limit. In what follows, we confirm numerically that the constant \( z_0 \) is strictly positive. For \( r = ||x||^2 \), the function \( Z(r) \) is given by
\[
Z(r) = \int_{\mathbb{R}^2} h_b(x - y)h_b(y)dy = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \frac{1}{||x-y||^2} \frac{1}{(||y||^2 + 1)^2}dy
\]
\[
= \frac{1}{\pi^2} \int_0^\infty \frac{s}{(s^2 + 1)^2} \int_0^{2\pi} \frac{1}{(r + s^2 + 1 - 2r^{1/2}s\cos\theta)^2} r d\theta ds
\]
\[
= \frac{2}{\pi} \int_0^\infty \frac{s}{(s^2 + 1)^2} \int_0^{2\pi} \frac{r + s^2 + 1}{(r + s^2 + 1 - 4rs\cos\theta)^{3/2}} r^{3/2} d\theta ds
\]
\[
= \frac{1}{\pi} \int_1^\infty \frac{1}{s^2} \frac{r + s}{(r - s)^2 + 4r}^{3/2} ds.
\]
As shown in Figure 11 since
\[
Z'(r) = -\frac{2}{\pi} \int_1^\infty \frac{1}{s^2} \frac{r^2 + (s + 1)r - 2s^2 + 3s}{(r - s)^2 + 4r}^{5/2} ds < 0,
\]
\[
Z''(r) = \frac{6}{\pi} \int_1^\infty \frac{1}{s^2} \frac{r^3 + (s + 2)r^2 + (5s^2 + 6s + 2)r + 3s^3 - 12s^2 + 10s}{(r - s)^2 + 4r}^{7/2} ds > 0,
\]

18
Figure 1: Plots of the functions $Z(r)$, $Z'(r)$, $Z''(r)$ and $DZ(r)$ for the vortex-blob regularization.

$Z(r)$ is monotone decreasing and downward-convex. Hence, we conclude that $z_0$ is positive for $M \geq 0$.

Next, we see the case of $M < 0$. In the vortex blob method, the Hamiltonian is expressed by

$$H^\sigma = -\frac{1}{2\pi} \sum_{n=1}^{N} \sum_{m=n+1}^{N} \Gamma_n \Gamma_m \log \sqrt{(l_{mn}^\sigma)^2 + 1},$$

and thus $H_P(r) = \log \sqrt{r + 1}$. The condition (3.19) is then equivalent to

$$DZ(r) \equiv Z''(r)(r + 1) + Z'(r) > 0.$$

Figure 1 also shows the graph of $DZ(r)$, where there exists $r_0 > 0$ such that $DZ(r_0) = 0$ and $DZ(r) > 0$ for $r > r_0$. On the other hand, according to Lemma 3.9, when three $\sigma$-point vortices with any $M < 0$ and $H^\sigma < 0$ starts from any collinear configuration at the initial moment, the distance $L_{mn}$ achieves its minimal value. That is to say, if we consider the initial data satisfying $L_{mn}^2 = r_0$ for any $m \neq n$, then $DZ(L_{mn})$ is always positive throughout the time evolution. Thus, owing to Corollary 3.3 we obtain $z_0 > 0$.

5 Concluding Remarks

We have introduced the EP-PV system describing the evolution of $\varepsilon$-point vortices in the Euler-Poincaré models and proven the existence of the evolution of the three $\varepsilon$-point vortices whose enstrophy varies at the triple collapse in the sense of distributions in the $\varepsilon \to 0$ limit. Moreover, we give a sufficient condition for the existence of the anomalous enstrophy dissipation via the triple collapse. All conditions are described in terms of the radial smoothing function $h(x) = h_r(|x|) \in C^1(\mathbb{R}^2) \cap W^1_1(\mathbb{R}^2)$; There exists a self-similar collapsing orbit of three $\varepsilon$-point vortices with the distributional enstrophy variation in the limit of $\varepsilon \to 0$, if $h$ is monotone decreasing ($h'_r < 0$) and
it satisfies the logarithmic singularity condition in the neighborhood of the origin \( \chi^+_\log h \in L^\infty(\mathbb{R}^2) \) and the decay rate conditions at infinity \( \chi_1 \nabla h \in L^\infty(\mathbb{R}^2) \) and \( \chi_{3+\eta}h \in L^\infty(\mathbb{R}^2) \) with \( \eta > 0 \). In addition, the sufficient condition for the enstrophy variation being dissipative is described in terms of \( Z(r) \) and \( H_P(r) \) that are defined from the smoothing function \( h \). Those conditions are applicable to many smoothing functions including the Euler-\( \alpha \) model, the Gaussian model and the vortex-blob model as confirmed in [10] and Section 4. Hence, we conclude that the anomalous enstrophy dissipation via the collapse of three point vortices is universally constructed within the framework of the Euler-Poincaré models.

Let us finally mention the future direction. It is interesting to investigate the enstrophy variation together with the evolution of many \( \varepsilon \)-point vortices. According to [13], the \( N \) point vortices in the PV system can collapse self-similarly in finite time under certain circumstances. Thus, there is a possibility of obtaining the enstrophy dissipation by considering the collapse of the \( N \) vortex problem in the EP-PV system. As a matter of fact, for the \( \alpha \)PV system, the enstrophy dissipation has been observed numerically in [9] via a quadruple self-similar collapse as \( \alpha \to 0 \). However, since the EP-PV system is not integrable for \( N \geq 4 \) in general, it is not an easy task to prove this. Further mathematical analysis is required.

## A Properties of auxiliary functions

We introduce some functions associated with a given smoothing function \( h_r(x) \) that is positive and monotone decreasing. Here, we show the properties of those functions that are essentially used in the proofs of the main results in the same way as in [10]. See also in Table 1.

1. **The function** \( P_K(r) \) The function \( P_K(r) \) defined by (2.12) is monotone increasing and upward-convex. Note that the derivative of \( P_K(r) \) is expressed by

\[
\frac{d}{dr} P_K(r) = 2\pi \frac{d}{dr} \left( -r \frac{dG^1_r}{dr}(r) \right) = 2\pi r h_r(r),
\]

since \( G^1_r \) is a radial function and satisfies \( -\Delta G^1_r(|x|) = h_r(|x|) \). Hence, it follows that

\[
\frac{d}{dr} P_K(\sqrt{r}) = \frac{1}{2\sqrt{r}} \frac{dP_K}{dr}(\sqrt{r}) = \pi h_r(\sqrt{r}) > 0, \quad \frac{d^2}{dr^2} P_K(\sqrt{r}) = \frac{\pi}{2\sqrt{r}} h'_r(\sqrt{r}) < 0. \quad (A.1)
\]

Moreover, as we see in [3], \( P_K(r) \) satisfies

\[
P_K(0) = 0, \quad \lim_{r \to \infty} P_K(r) = \int_{\mathbb{R}^2} h(x) dx = 1,
\]

and we thus have \( 0 \leq P_K(r) < 1 \).

2. **The function** \( L_P(r) \) The function \( L_P(r) \) defined by (3.11) is monotone decreasing. Indeed, its derivative is given by

\[
\frac{d}{dr} L_P(r) = -\frac{1}{r^2} P_K(\sqrt{r}) + \frac{\pi}{r} h_r(\sqrt{r}) \equiv -\frac{1}{r^2} l_0(r),
\]

where \( l_0(r) = P_K(\sqrt{r}) - \pi r h_r(\sqrt{r}) \). Since it follows that

\[
\frac{d}{dr} l_0(r) = -\frac{\pi}{2} \sqrt{r} h'_r(\sqrt{r}) > 0, \quad l_0(0) = P_K(0) = 0,
\]

we find that \( l_0(r) \) is a positive function and thus \( L_P(r) \) is monotone decreasing.
3. **The function** $H_G(r)$  The function $H_G(r)$ defined by (2.13) is monotone decreasing and downward-convex. Owing to $0 \leq P_K(r) < 1$, the first and second derivatives of $H_G(\sqrt{r})$ are given by

\[
\frac{d}{dr}H_G(\sqrt{r}) = -\frac{1}{2\sqrt{r}} \left( \frac{1}{\sqrt{r}} + 2\pi \frac{dG_1^1}{dr}(\sqrt{r}) \right) = -\frac{1}{2r} (1 - P_K(\sqrt{r})) < 0
\]

and

\[
\frac{d^2}{dr^2}H_G(\sqrt{r}) = \frac{1}{2r^2} \left( 1 - P_K(\sqrt{r}) \right) + \frac{\pi}{2r} h_r(\sqrt{r}) > 0.
\]

Note that

\[
H_G(r) \sim -\log r - 2\pi G_1^1(0), \quad r \to 0,
\]

since $G_1^1(0)$ is finite.

4. **The function** $H_P(r)$  Its definition is given by (3.5). It is monotone increasing and upward-convex, since we have

\[
\frac{d}{dr}H_P(r) = \frac{1}{2r} P_K(\sqrt{r}) = \frac{1}{2} L_P(r) > 0, \quad \frac{d^2}{dr^2}H_P(r) = \frac{1}{2} \frac{d}{dr} L_P(r) < 0. \tag{A.3}
\]

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