CORRECTIONS OF ORDER \((Z\alpha)^6 \frac{m_e^2}{m_\mu}\) IN THE 
MUONIUM FINE STRUCTURE, INDUCED 
BY THREE PHOTON EXCHANGE DIAGRAMS

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Abstract

In the framework of the quasipotential method we calculate the 
contributions of the kind \(\alpha^6 \ln \alpha\), \(\alpha^6\) of three-photon exchange dia-
grams to the energy spectrum of muonium \(n^3S_1\) state in the leading 
order on \(m_e/m_\mu\). Analytical expression of obtained correction for arbi-
trary principal quantum number is equal to \((Z\alpha)^6(m_e/m_\mu)m_e(6 \ln(2) - 
35/48)/n^3\).
The investigation of muonium and positronium fine structure represents one of the basic tests of quantum electrodynamics, which is sensitive to radiative corrections of higher order on $\alpha$ [1]. Many papers [2, 3, 4, 5] are devoted to the calculation of different contributions to the fine structure of hydrogen-like atom energy levels. The interest to this problem remains unchanged [6, 7, 8]. The progress, achieved in the last years during the calculation of logarithmic contributions of order $\alpha^6 \ln \alpha$ in the positronium fine structure intervals $(2^3S_1 \div 1^3S_1, 2^3S_1 \div 2^3P_J)$ [9, 10, 11], doesn’t abolish the necessity of calculation of higher order corrections $O(\alpha^6)$ [12]. The development of experimental methods, based on Doppler-free two-photon spectroscopy, allows the "large" structure intervals to be measured for the muonium and the positronium [13, 14, 15].

$$\Delta E_{exp}^{Ps}(2^3S_1 \div 1^3S_1) = \left\{ \begin{array}{l} 1233607218, 9 \pm 10, 7 \text{ MHz} \\ 1233607216, 4 \pm 3, 2 \text{ MHz} \end{array} \right.$$ (1)

$$\Delta E_{exp}^{Mu}(2^3S_1 \div 1^3S_1) = 2455527936 \pm 120 \pm 140 \text{ MHz}. \quad (2)$$

The increase of the experimental accuracy of muonium fine structure interval measurements (just as for positronium), planned in the near future, makes very actual the calculation of radiative corrections of higher order on $\alpha$. In this paper we have performed studies of the contributions of order $O(\alpha^6 \ln \alpha)$, $O(\alpha^6)$ from three-photon exchange diagrams to the muonium fine structure. The contribution of these diagrams to the muonium hyperfine structure was obtained in [16]. Our calculations are based on the Schrodinger-type local quasipotential equation [17]

$$\left( \frac{b^2}{2\mu_R} - \frac{\mu_R^2}{2\mu_R} \right) \psi_M(\vec{p}) = \int \frac{d\vec{q}}{(2\pi)^3} V(\vec{p}, \vec{q}, M) \psi_M(\vec{q}), \quad (3)$$

where $b^2 = E_1^2 - m_1^2 = E_2^2 - m_2^2$, $\mu_R = E_1 E_2 / M$ is relativistic reduced mass, $M = E_1 + E_2$ is the bound state mass, $m_1, m_2$ are the masses of the electron and the muon. As an initial approximation of quasipotential $V(\vec{p}, \vec{q}, M)$ for the bound state system ($e^- \mu^+$) we choose the ordinary Coulomb potential. On the basis of equation (3) in [18] we have obtained some relativistic corrections $m\alpha^6$ in the positronium fine structure from the one-photon, two-photon interaction and the second order perturbation theory. There are six diagrams, shown on Fig.1, which determine the three-photon exchange interaction in the muonium.

Let consider the first diagram of Fig.1. The corresponding amplitude already has the factor $\alpha^6$, which appears due to electromagnetic vertices and
Coulomb wave function. So, in the first stage of our calculations we have neglected by the electron and muon vector momentum of relative motion in the initial and final states, taking into account that the necessary accuracy is already achieved. Then the first diagram amplitude (Fig.1) takes the form:

\[ T_1^{3\gamma} = -\frac{(Z\alpha)^3}{4\pi^5} \int d^4p \int d^4p' \frac{\gamma_\lambda^3(\hat{q}_1 - \hat{p}' + m_1)\gamma_\nu^3(\hat{p}_1 - \hat{p} + m_1)\gamma_\mu^3}{(p^2 - w^2 + i\epsilon)(p'^2 - w^2 + i\epsilon)((p - p')^2 + i\epsilon)} \]

\[ <\gamma_\mu^2(\hat{p}_2 + \hat{p} + m_2)\gamma_\nu^2(\hat{q}_2 + \hat{p}' + m_2)\gamma_\lambda^2 > \]

\[ \frac{1}{D_e(p)D_e(p')D_\mu(-p)D_\mu(-p')} \]

where \( D_{e,\mu}(p) \) are the denominators of electron and muon propagators:

\[ D(\pm p) = p^2 - w^2 \pm 2mp^0 + i\epsilon, \quad w^2 = -b^2; \]

Figure 1: **Feynman diagrams of three-photon exchange interaction in the system** \((e^-\mu^+)\).
using of Coulomb gauge is the most natural for exchange photons, because the Coulomb interaction dominates in the system \( (e^{-\mu^+}) \). Nevertheless, the equivalence of Coulomb and Feynman gauges in the scattering approximation for the three-photon diagrams calculations was shown in [16]. To construct the quasipotential of the system \( (e^{-\mu^+}) \) with \( L=0 \) and \( J=1 \), that corresponds to \( T_{1}^{3\gamma} \), let introduce the projector operator for initial and final states of the following kind [18, 19]:

\[
\hat{\pi} = \frac{1}{2\sqrt{2}} \frac{\hat{P} + M}{M} \hat{\varepsilon},
\]

where \( P = p_1 + p_2 = q_1 + q_2 \) is full four-momentum of the system, and \( \varepsilon^\mu \) is the vector of \(^3S_1\) muonium state. Projecting the particles on \(^3S_1\) state by means of (6), we avoid cumbersome matrix multiplication in the bispinor averages and immediately pass on to calculation of total trace in (4). As a result, the quasipotential of first diagram may be written in the form:

\[
V_{1}^{3\gamma} = -\frac{(Z\alpha)^3}{\pi^5} \int d^4p \int d^4p' \frac{F_1(p, p')}{D_\gamma(p)D_\gamma(p')D_\gamma(p-p')D_\gamma(p)D_\gamma(p')D_\mu(-p)D_\mu(-p')},
\]

\[
D_\gamma(p) = p^2 - w^2 + i\varepsilon,
\]

\[
F_1(p, p') = f_{12}(p, p')m_2^2 + \frac{1}{3}f_{11}m_2, \quad f_{12} = pp' - 4m_1^2 - 2m_1p_0 - 2m_1p'_0 - 2p_0p'_0,
\]

\[
f_{11}(p, p') = 2m_1p^2 + p_0p'^2 + 10m_1pp' + 2p_0pp' + 2p'_0pp' + 2m_1p^2 + p'_0p'^2 + 6m_1p_0 + 6m_1p'_0 + 4m_1p_0^2 + 4m_1p'_0^2 - 4m_1p_0p'_0.
\]

We keep in (7) only the terms proportional to \( m_2^2 \) and \( m_2 \), taking in mind the determination of contribution to muonium fine structure in the leading order on parameter \( m_1/m_2 \). As will soon become evident, we can’t restrict in \( F(p, p') \) only by terms \( \sim m_2^2 \). The quasipotentials of the rest amplitudes of Fig.1 may be constructed in a similar way. They differ from each other due to momentum dependence in muonic denominators and to the kind of functions \( f_{i2} \) (\( i=1, \ldots, 6 \)).

The parts of \( F_i(p, p') \), proportional to \( m_2^2 \), coincide in all six amplitudes. Let remark, that when substitute \( \hat{\varepsilon} \to \gamma_5 \) in projector operator (6) (\(^1S_0\) state), we obtain the same function \( f_{12}(p, p') \), as for the \(^3S_1\) muonium. This means, that the muonium hyperfine splitting appears as effect of higher order on \( m_1/m_2 \). Functions \( f_{i1} \) are equal to:

\[
f_{21} = -10m_1p^2 - 5p_0p'^2 + 10m_1pp' + 4p_0pp' - 4p'0pp' + 2m_1p^2 + 4p'_0p'^2 + 12p_0m_1^2 -
\]

\[
(9)
\]
\[
-6m_1^2p_0 + 4m_1p_0^2 + 8m_1p_0p_0' - 8m_1p_0'^2 + 4p_0p_0',
\]
\[
f_{31} = 2m_1^2p_0^2 + 4p_0p_0'^2 + 10m_1pp' - 4p_0pp' + 4p_0'pp' - 10m_1p^2 - 5p_0p^2 - (10)
\]
\[
-6m_1^2p_0 + 12m_1p_0^2 - 8m_1p_0'^2 + 8m_1p_0p_0' + 4m_1p_0^2 + 4p_0p_0'^2,
\]
\[
f_{41} = 2m_1^2p_0^2 + p_0p_0'^2 - 2m_1pp' + 4p_0pp' + 2p_0'pp' - 10m_1p^2 - 8p_0p^2 - 12m_1p_0 + (11)
\]
\[
+6m_1p_0' + 4m_1p_0^2 - 4m_1p_0p_0' + 4m_1p_0^2 - 8p_0p_0'.
\]
\[
f_{51} = -10m_1p_0^2 + 2m_1pp' + 4p_0pp' + 2m_1p^2 + p_0'p_0^2 + 6m_1p_0 - (12)
\]
\[
-12m_1p_0' + 4m_1p_0^2 - 4m_1p_0p_0' + 4m_1p_0^2 - 8p_0p_0'.
\]
\[
f_{61} = -10m_1p_0^2 + 5p_0p_0'^2 - 2m_1pp' + 4p_0pp' + 4p_0'pp' - 10m_1p^2 - 5p_0p^2 - (13)
\]
\[
-6m_1p_0 - 6m_1p_0' - 8m_1p_0^2 - 4m_1p_0p_0' - 8m_1p_0^2.
\]

Integral function in (4) has simple poles on loop energies \( p_0, p_0' \) in electron, muon and photon propagators. So, the most natural way of integration (4) consists in the calculation of integrals on \( p_0, p_0' \) at the first step, using the method of residues. But such an approach of calculation leads, nevertheless, to rather complicated intermediate expressions, what makes highly questionable its subsequent analytical integration on spatial momenta \( \vec{p}, \vec{p}' \). So, we have used different approach of integration in (4), connected with transformation of muonic denominators, accounting the necessary calculational accuracy on parameter \( m_1/m_2 \). Considering that the spatial momentum of muonic motion in the intermediate state \( |\vec{p}| < m_2 \), we obtain:

\[
D_\mu(p) = p^2 - w^2 + 2m_2p_0 \approx 2m_2 \left( p_0 - \frac{p_0^2 + w^2}{2m_2} + i\epsilon \right) \approx 2m_2(p_0 + i\epsilon), \quad (14)
\]

where the second approximate equality means that we neglect by the muon kinetic energy in the intermediate state. Doing so, we suppose that the integration contour on variable \( p_0 \) must be closed in the lower halfplane. Considering the terms, proportional to \( m_2^2 \) in the numerators of all six diagrams (function \( f_{12}(p, p') \)), we have arrived to the need of expression transformation, which includes the sum of muonic denominators (5). Using the second approximate equality from (14), we obtain:

\[
\frac{1}{D_\mu(-p)D_\mu(-p')} + \frac{1}{D_\mu(-p)D_\mu(p' - p)} + \frac{1}{D_\mu(-p')D_\mu(p - p')} + \\
+ \frac{1}{D_\mu(p)D_\mu(p' - p')} + \frac{1}{D_\mu(p')D_\mu(p' - p)} + \frac{1}{D_\mu(p')D_\mu(p)'} \approx
\]
\[
\approx \frac{(-2\pi i)\delta(p_0)}{2m_2} \frac{(-2\pi i)\delta(p'_0)}{2m_2}.
\tag{15}
\]

In the energy spectrum the expression (15) will cause the corrections of order \(O(\alpha^4)\), which are canceled by the similar terms from the iteration part of the quasipotential. Consequently, to find the necessary contribution of order of \(\alpha^6\), we must use first approximate equality in (14). Taking the difference

\[
\frac{1}{2m_2 \left(p_0 - \frac{p^2+w^2}{2m_2} + i\epsilon\right)} - \frac{1}{2m_2 (p_0 + i\epsilon)} \approx \frac{(p^2 + w^2)}{4m_2^2 (p_0 + i\epsilon)^2}.
\tag{16}
\]

let represent the quantity \(1/D_\mu(p)\) in the form:

\[
\frac{1}{D_\mu(p)} \approx \frac{1}{2m_2 (p_0 + i\epsilon)} + \frac{(p^2 + w^2)}{4m_2^2 (p_0 + i\epsilon)^2}.
\tag{17}
\]

Second addendum of (17) is of higher order on \(m_1/m_2\) in comparison with the first. But it leads to the necessary order correction on the other parameter \(\alpha\). Using the splitting (17), in the sum (15), let extract the terms, which generate the correction \(O(\alpha^6)\) and \(O(\alpha^6 \ln \alpha)\) in the energy spectrum. We may write them in the following manner:

\[
\frac{(p^2 + w^2)}{8m_2^3} \left[ \frac{2\pi i\delta(p'_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p'_0 - p_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p_0)}{(p_0 + i\epsilon)^2} \right] +
\tag{18}
\]

\[
+ \frac{(p^2 + w^2)}{8m_2^3} \left[ \frac{-2\pi i\delta(p'_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p'_0 - p_0)}{(p_0 + i\epsilon)^2} + \frac{2\pi i\delta(p_0)}{(p_0 + i\epsilon)^2} \right] +
\]

\[
+ \frac{(\vec{p} - \vec{p}')^2 + w^2}{8m_2^3} \left[ - \frac{2\pi i\delta(p'_0)}{(p_0 + i\epsilon)^2} + \frac{2\pi i\delta(p'_0 - p_0)}{(p_0 + i\epsilon)^2} - \frac{2\pi i\delta(p_0)}{(p_0 + i\epsilon)^2} \right] +
\]

It is evident from three-photon interaction amplitude of the type (6), that the parts of (18) give the necessary order corrections on \(\alpha\) in the studied fine structure intervals. The same order corrections \(O(m_1/m_2)\), as well as (18), will arise from the quasipotential terms containing the functions \(f_{i1}(p, p')\), when we use the second approximation of (14) for muonic denominators. To do definite conclusion about the order of appearing terms in energy spectrum, which are determined by these quasipotential addenda, let transform them for greater simplification. Let consider for definiteness massless terms in the function \(f_{i1}(p, p')\), proportional to \(\sim p^2, p'^2, pp'\):

\[
3p^2 \left[ \frac{1}{D_\mu(p)} + \frac{1}{D_\mu(-p)} - \frac{1}{D_\mu(p - p')} - \frac{1}{D_\mu(p' - p)} \right] +
\tag{19}
\]
The transformation of other terms in functions $f_i(p,p')$ may be carried out by analogy. The next period of calculation consists in the integration over four-momenta in expressions (18)-(19). The typical two-loop integral, that results on this way has the following structure [16]:

$$K_i = (4\pi)^2 \int \frac{d^4p' d^4p}{-(2\pi)^8 (p'^2 - w^2 + i\epsilon)(p^2 - p'^2 + i\epsilon)D_i(p')D_i(p)}$$

(20)

where $G_i(p_0, p_0, m_1)$ contains one $\delta$-function, and $P(p', \vec{p}, w)$ is a polynom. To calculate fundamental integrals (20) we have used Feynman parameterization in order to combine the denominators of the particle propagators, and the symmetry properties of the integral with the replacement $p \leftrightarrow p'$. There are the next set of functions $G_i(p_0, p_0, m_1)$, which appear in this paper:

$$G_1 = -\frac{2\pi i\delta(p_0 - p'_0)2m_1}{(p_0 + i\epsilon)^2}, \quad G_2 = -\frac{2\pi i\delta(p_0)2m_1}{(p'_0 + i\epsilon)^2}, \quad G_3 = -2\pi i\delta(p_0)2m_1,$$

$$G_4 = -2\pi i\delta(p'_0)2m_1, \quad G_5 = -2\pi i\delta(p_0 - p'_0)2m_1.$$ 

(21)

The results of the integrations for $K_i$ (20) have presented in the table [16].

### Table of the integrations $K_i$ (20), appearing in the muonium fine structure calculations

|     | $p'^2(\vec{p}\vec{p}')$ | $(\vec{p}\vec{p}')^2$ | $p'^2(\vec{p}\vec{p}')$ | $w^2(\vec{p}\vec{p}')$ |
|-----|-------------------------|----------------------|-------------------------|------------------------|
| $K_1$ | $2 \ln 2 - \frac{1}{3}$ | $2 \ln 2 - \frac{1}{3}$ | $2 \ln 2 - \frac{1}{3}$ | $0$ |
| $K_2$ | $\frac{1}{2} \ln \frac{m^2}{2w} - \frac{1}{32}$ | $\frac{1}{4} \ln \frac{m^2}{4w} - \frac{1}{32}$ | $\ln \frac{m^2}{4w} - \frac{1}{8}$ | $\frac{1}{4}$ |
| $K_3$ | $-\frac{1}{2} \ln \frac{m^2}{2w} - \frac{1}{32}$ | $\frac{1}{4} \ln \frac{m^2}{4w} - \frac{1}{8}$ | $\ln \frac{m^2}{2w} - \frac{1}{8}$ | $\frac{1}{8}$ |
| $K_4$ | $\frac{1}{2} \ln \frac{m^2}{2w} - \frac{1}{8}$ | $\frac{1}{4} \ln \frac{m^2}{4w} - \frac{1}{3}$ | $\ln \frac{m^2}{2w} - \frac{1}{8}$ | $\frac{1}{8}$ |
| $K_5$ | $\ln 2$ | $\ln 2 - \frac{1}{2}$ | $\ln 2$ | $0$ |
Then the contributions, defined by expressions (8-13) and (18) are correspondingly equal:

\[ \Delta B_1 = -\frac{1}{2} (Z\alpha)^6 \frac{m_1^2}{m_2} \]  

\[ \Delta B_2 = (Z\alpha)^6 \frac{m_1^2}{m_2} (6 \ln 2 - \frac{35}{48}) \]  

We have introduced in (20) the photon mass \( w \) to avoid ”infrared” singularities. The ”infrared” logarithms \( \ln w \), containing this photon mass (see table of integrals \( K_i \)), and appearing at intermediate expressions, are mutually cancelled in the corrections \( \Delta B_1, \Delta B_2 \).

Let consider now the quasipotential addenda, containing the momenta of particle relative motion in the initial and final states. We denote them by \( \vec{r}_1 \) and \( \vec{r}_2 \) correspondingly. Their consideration leads to modification of \( f_{i1} \), which acquire the following additional terms:

\[ \Delta f_{21} = 10m_1p'r_2 + 5p_0'p'r_2 + m_1pr_2 + 3p_0'p_0r_2, \]  

\[ \Delta f_{31} = -m_1p'r_1 - 3p_0p'r_1 - 10m_1pr_1 - 5p_0'p_0r_1, \]  

\[ \Delta f_{41} = m_1(7p'r_1 + 5p'r_2 + 10pr_1 + 5pr_2) + 6p_0p'r_1 + 5p_0p'r_2 + 8p_0'pr_1 + 5p_0'pr_2, \]  

\[ \Delta f_{51} = m_1(-5p'r_1 - 10p'r_2 - 5pr_1 - 7pr_2) - 5p_0p'r_1 - 8p_0p'r_2 - 5p_0'pr_1 - 6p_0'pr_2, \]  

\[ \Delta f_{61} = m_1(-11p'r_1 - 11p'r_2 - 11pr_1 - 11pr_2) - 8p_0p'r_1 - 8p_0p'r_2 - 8p_0'pr_1 - 8p_0'pr_2. \]  

Using again the symmetry properties of appearing integrals under simultaneous variable replacement \( p \leftrightarrow p', r_1 \leftrightarrow r_2 \), we obtain cancellation of all integrations in (24)-(28). So, the contribution the particle relative motion in the fine structure with the accuracy \( O(m_1/m_2) \) is equal to zero. Thus the full value of the calculated correction \( (Z\alpha)^6 m_1^2 / m_2 \) for hydrogen-like system S-states is defined as a sum of expressions (22) and (23):

\[ \Delta B = (Z\alpha)^6 \frac{1}{n^3} \frac{\mu^3}{m_1m_2} \left( 6 \ln 2 - \frac{35}{48} \right). \]  

Numerical value of obtained contribution (29) for the muonium fine structure interval \( ^2S_1 \div ^1S_1 \) takes the value 0.271 Hz. We have used in this paper the diagrammatic approach to the calculation of the corrections of order \( (Z\alpha)^6 m_1^2 / m_2 \). We have made the most complicated part of calculations, connected with the two-loop integrations. In order to obtain the total value
of necessary order contribution it is important to complete these results by similar corrections from one-photon, two-photon interactions as well as by the results from the second order perturbation theory [18]. It is the aim of future studies.

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