THE BOLTZMANN EQUATION FOR PLANE COUETTE FLOW

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Abstract. In the paper, we study the plane Couette flow of a rarefied gas between two parallel infinite plates at \( y = \pm L \) moving relative to each other with opposite velocities \((\pm \alpha L, 0, 0)\) along the \( x \)-direction. Assuming that the stationary state takes the specific form of \( F(y, v_x - \alpha y, v_y, v_z) \) with the \( x \)-component of the molecular velocity sheared linearly along the \( y \)-direction, such steady flow is governed by a boundary value problem on a steady nonlinear Boltzmann equation driven by an external shear force under the homogeneous non-moving diffuse reflection boundary condition. In case of the Maxwell molecule collisions, we establish the existence of spatially inhomogeneous non-equilibrium stationary solutions to the steady problem for any small enough shear rate \( \alpha > 0 \) via an elaborate perturbation approach using Caflisch’s decomposition together with Guo’s \( L^\infty \cap L^2 \) theory. The result indicates the polynomial tail at large velocities for the stationary distribution. Moreover, the large time asymptotic stability of the stationary solution with an exponential convergence is also obtained and as a consequence the nonnegativity of the steady profile is justified.

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1. Introduction

The steady state of a rarefied gas between two parallel plates with the same temperatures and opposite velocities is one of the most fundamental boundary-value problems in kinetic theory, see the books of Kogan [28], Cercignani [11], Garzó-Santos [22], and Sone [33]. In particular, numerical analysis of the plane Couette flow of rarefied gas on the basis of the nonlinear Boltzmann equation has been extensively conducted in the physical literatures, cf. [29–31, 34, 35]. On the other hand, the mathematical study on this problem, even in the case when there is a temperature gap between two plates and a constant external force parallel to the boundaries, has been carried out by Esposito-Lebowitz-Marra [18, 19] for the hydrodynamic description of the steady rarefied gas flow via the approximation of the corresponding compressible Navier-Stokes equations with no-slip boundary

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condition. The result in [19] for hard sphere model was later extended in [13] to the case of hard intermolecular potentials with Grad’s angular cutoff as well as the Maxwell molecule case for which only the polynomial decay of the stationary solution for large velocities is obtained compared to the exponential decay for hard sphere model. In addition, closely related to the plane Couette flow, the stationary Boltzmann equation for rarefied gas in a Couette flow setting between two coaxial rotating cylinders was also studied extensively by Arkeryd-Nouri [2,3] in the fluid dynamic regime, see also a recent work [1] for further investigation of ghost effect induced by curvature.

The current study of the plane Couette flow with boundaries is motivated by the previous work [14] by the first two authors for uniform shear flow via the Boltzmann equation without boundaries. We refer readers to [9,12,21,27,36,37] and references therein for more details of the topic on uniform shear flow. In particular, in a recent significant progress [9], Bobylev-Notavelázquez studied the self-similar asymptotics of solutions in large time for the Boltzmann equation with a general deformation of small strength and they also showed that the self-similar profile can have the finite polynomial moments of higher order as long as the deformation strength is smaller. In this paper, we will take into account the effect of shear force induced by the relative motion of the boundaries. We hope that the current study can shed some light on the relation between the Couette flow with boundary and the uniform shear flow without boundary. A rigorous justification of the behavior of solutions in the limit \( L \to \infty \) is left for future research.

To specify the problem, we consider the rarefied gas between two parallel infinite plates with the same uniform temperature \( T_0 > 0 \), one at \( y = +L \) is moving with velocity \((U_+, 0, 0)\) and \( U_+ = \alpha L \) and the other at \( y = -L \) is moving with velocity \((U_-, 0, 0)\) and \( U_- = -\alpha L \), where \( \alpha > 0 \) is a parameter for the shear rate, see Figure 1 below. Moreover, we assume that the gas molecules are of the Maxwellian type and reflected diffusively on the plates \( y = \pm L \).

\[ F(y, v) = \frac{1}{\kappa n} Q(F, F) \] (1.1)

subject to the diffuse reflection boundary conditions at \( y = \pm L \) respectively, i.e.,

\[ F(\pm L, v) = M T_0 (v_x - U_{\pm}, v_y, v_z) \int_{v_y \leq 0} F(\pm L, v) |v_2| dv \quad \text{for } v_y \geq 0, \] (1.2)

as well as a given total mass

\[ \frac{1}{2L} \int_{-L}^{L} \int_{\mathbb{R}^3} F(y, v) dv dy = M \] (1.3)

for some positive constant \( M > 0 \). Here, the non-dimensional parameter \( \kappa n > 0 \) is the Knudsen number given by the ratio of the mean free path to the typical length and \( M T_0 = M_{T_0}(v) \).
associated with the uniform wall temperature $T_0$ at $y = \pm L$ is a global Maxwellian of the form
\[ M_{T_0}(v) = \frac{1}{2\pi T_0^2} e^{-\frac{|v_x|^2 + |v_y|^2 + |v_z|^2}{2T_0}}, \quad v = (v_x, v_y, v_z) \in \mathbb{R}^3. \]

For the Maxwell molecule model, the collision operator operator $Q$, which is bilinear and acts only on velocity variable, takes the form of
\[ Q(F_1, F_2)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B_0(\cos \theta)\left[ F_1(v_1')F_2(v') - F_1(v_1)F_2(v) \right] d\omega dv_1, \quad (1.4) \]
where the velocity pairs $(v_1, v)$ and $(v_1', v')$ satisfy the relation
\[ v_1' = v_1 - [(v_1 - v) \cdot \omega]\omega, \quad v' = v + [(v_1 - v) \cdot \omega]\omega, \quad (1.5) \]
denoting the $\omega$-representation according to conservation of momentum and energy in the elastic collision, i.e., $v_1 + v = v_1' + v'$ and $|v_1|^2 + |v|^2 = |v_1'|^2 + |v'|^2$, respectively. Throughout the paper, we assume that the collision kernel $B_0(\cos \theta)$ with $\cos \theta = (v - v_1) \cdot |v - v_1|$, depending only on the angle $\theta$ between the relative velocity $v - v_1$ and $\omega$, satisfies the Grad’s angular cutoff assumption
\[ 0 \leq B_0(\cos \theta) \leq C|\cos \theta|, \quad (1.6) \]
for a generic constant $C > 0$.

In the paper, for the boundary-value problem (1.1), (1.2) and (1.3) with finite Knudsen number, we look for stationary solutions of the following specific form
\[ F_{st}(y, v_x - \alpha y, v_y, v_z), \quad (1.7) \]
where the horizontal molecular velocity $v_x - \alpha y$ is sheared linearly along the $y$-direction. After plugging (1.7) into (1.1), (1.2), (1.3) and normalizing $L$, $M$ and $T_0$ to be one for brevity, the stationary distribution function $F_{st}$ is determined by the following boundary-value problem
\[ \begin{aligned}
&\begin{cases}
  v_y \partial_y F_{st} - \alpha v_y \partial_c F_{st} = Q(F_{st}, F_{st}), \quad y \in (-1, 1), \quad v = (v_x, v_y, v_z) \in \mathbb{R}^3,
\end{cases} \\
&F_{st}(\pm 1, v)|_{v_y \leq 0} = \sqrt{2\pi \mu} \int_{v_y \geq 0} F_{st}(\pm 1, v)|v_y| dv, \quad v \in \mathbb{R}^3, \\
&\frac{1}{2} \int_{-1}^{1} \int_{\mathbb{R}^3} F_{st}(y, v) dv dy = 1,
\end{aligned} \quad (1.8) \]
with the global Maxwellian $\mu = (2\pi)^{-3/2} e^{-|v|^2/2}$. This paper aims to establish the existence of solutions to the above boundary value problem (1.8) for any small enough shear rate $\alpha > 0$, as well as its large time asymptotic stability.

To solve (1.8), we will apply the perturbation approach by taking the shear rate as a small parameter. If $\alpha = 0$, $F_{st} = \mu$ is the unique equilibrium solution to the boundary value problem (1.8). However, for $\alpha > 0$, the external shear force drives the rarefied gas far from the equilibrium. Precisely, we set
\[ F_{st} = \mu + \sqrt{\mu}\{\alpha G_1 + \alpha^2 G_R\}, \quad (1.9) \]
with
\[ \int_{-1}^{1} \int_{\mathbb{R}^3} \sqrt{\mu} G_1 dv dy = \int_{-1}^{1} \int_{\mathbb{R}^3} \sqrt{\mu} G_R dv dy = 0. \quad (1.10) \]
By plugging (1.9) into (1.8) and comparing coefficients of the equation in the order of $\alpha$, we obtain the equation for $G_1$
\[ v_y \partial_y G_1 + LG_1 = -v_x v_y \sqrt{\mu}, \quad (1.11) \]
with boundary condition
\[ G_1(\pm 1, v)|_{v_y \leq 0} = \sqrt{2\pi \mu} \int_{v_y \geq 0} \sqrt{\mu} G_1(\pm 1, v)|v_y| dv, \quad (1.12) \]
and the equation for the remainder $G_R$

$$v_y \partial_y G_R - \alpha v_y \partial_{v_y} G_R + \frac{\alpha}{2} v_x v_y G_R + L G_R = v_y \partial_{v_y} G_1 - \frac{1}{2} v_x v_y G_1 + \Gamma(G_1, G_1) + \alpha \{ \Gamma(G_R, G_R) + \Gamma(G_1, G_R) \} + \alpha^2 \Gamma(G_R, G_R),$$

(1.13)

with boundary condition

$$G_R(\pm 1, v)_{| v_y \leq 0} = \sqrt{2\pi} \mu \int_{v_y \geq 0} \sqrt{\mu} G_R(\pm 1, v) |v_y| \, dv.$$  

(1.14)

Here, the linear and nonlinear collision operator $L$ and $\Gamma$ are given by

$$L f = -\mu^{-\frac{1}{2}} \{ Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu) \},$$

and

$$\Gamma(f, g) = \mu^{-\frac{1}{2}} \{ Q(\sqrt{\mu} f, \sqrt{\mu} g) + Q(\sqrt{\mu} g, \sqrt{\mu} f) \},$$

respectively. Properties of these two operators will be presented in Section 2. Note that to solve $G_1$, both (1.11) and (1.12) with the restriction $\int_{-1}^{1} \int_{\mathbb{R}^3} \sqrt{\mu} G_1 \, dv \, dy = 0$ are invariant under the transformation $G_1(y, v) \rightarrow -G_1(y, -v_x, v_y, v_z)$. Thus, if the solution is unique, $G_1$ is odd in $v_x$, namely,

$$G_1(y, v) = -G_1(y, -v_x, v_y, v_z), \quad -1 \leq y \leq 1, \quad v = (v_x, v_y, v_z) \in \mathbb{R}^3.$$  

(1.15)

Hence, the diffuse reflection boundary condition (1.12) for $G_1$ can be reduced to the homogeneous inflow boundary condition

$$G_1(\pm 1, v)_{| v_y \leq 0} = 0.$$  

(1.16)

The first result in the paper for the existence of the Couette flow problem is stated as follows. To the end, we use a velocity weight function

$$w_q = w_q(v) := (1 + |v|^2)^q$$

(1.17)

with an integer $q > 0$.

**Theorem 1.1.** Assume that the Boltzmann collision kernel is of the Maxwell molecule type (1.6). Then, the boundary value problem (1.8) admits a unique steady solution $F_{st} = F_{st}(y, v) \geq 0$ of the form (1.9) satisfying (1.10) and the following estimates on $G_1$ and $G_R$, respectively.

(i) The first-order correction $G_1 = G_1(y, v)$, uniquely solved by the boundary value problem (1.11) and (1.16), satisfies (1.15) and for any integers $m \geq 0$ and $q \geq 0$,

$$\| w_q \partial_{v_x}^{m} G_1 \|_{L^\infty} \leq \tilde{C}_1,$$

(1.18)

where $\tilde{C}_1 > 0$ is a constant depending only on $m$ and $q$.

(ii) The remainder $G_R = G_R(y, v)$, uniquely solved by the boundary value problem (1.13) and (1.14), satisfies that there is an integer $q_0 > 0$ such that for any integer $q \geq q_0$, there is $\alpha_0 = \alpha_0(q) > 0$ depending on $q$ such that for any $\alpha \in (0, \alpha_0)$ and any integer $m \geq 0$

$$\| w_q \partial_{v_x}^{m} G_R \|_{L^\infty} \leq \tilde{C}_{m, q},$$

(1.19)

where $\tilde{C}_{m, q} > 0$ is a constant depending only on $m$ and $q$ but independent of $\alpha$.

Some remarks on Theorem 1.1 are given as follows.

**Remark 1.1.** The steady solution $F_{st}$ to the boundary value problem (1.8) is essentially constructed in the regime where the collision is dominated and the shearing effect is weak. By (1.18) and (1.19), the steady solution takes the form of

$$F_{st} = \mu + \alpha \sqrt{\mu} G_1 + O(1) \alpha^2$$

(1.20)
with the remainder of the second order decay in large velocities only polynomially. The order of the polynomial decay can be arbitrarily large as long as the shear rate is sufficiently small. It generally holds \( \alpha_0(q) \to 0 \) as \( q \to \infty \), and in particular, one may take

\[
\alpha_0(q) = \frac{\nu_0}{8q}
\]

as shown in the proof. The result is consistent with the one in [14] for uniform shear flow without boundaries in the spatially homogeneous setting.

Remark 1.2. Without using the odd-in-\( v_x \) property as in (1.15), the existence of \( G_1(y, v) \) to the BVP (1.11) under the diffuse reflection boundary condition (1.12) also can be established by the same approach as for treating the remainder \( G_R \). Here, we take this formulation only for brevity of presentation because the proof for the homogeneous inflow boundary is relatively easier than that for the diffuse reflection boundary.

Remark 1.3. We notice that it is necessary to deal with the \( v_x \)-derivative estimates due to the appearance of the shear force term \( v_y \partial_{v_y} F_{st} \), in particular, the term \( v_y \partial_{v_y} G_1 \) becomes a source term in the equation (1.13) for \( G_R \). We emphasize that although one can obtain the derivative estimates as in (1.18) and (1.19) in \( v_x \), it is impossible to obtain similar estimate on derivative in \( v_y \) because \( G_1(y, v) \) is discontinuous at \( v_y = 0 \), see (4.25) for an explicit form of \( G_1 \) when the non-local collision term is omitted.

To establish the nonnegativity of the stationary profile \( F_{st}(y, v) \), we further study the following initial boundary value problem of the Boltzmann equation with a shear force

\[
\begin{aligned}
\partial_t F + v_y \partial_y F - \alpha v_y \partial_{v_y} F &= Q(F, F), \quad t > 0, \quad y \in (-1, 1), \quad v = (v_x, v_y, v_z) \in \mathbb{R}^3, \\
F(0, y, v) &= F_0(y, v), \quad y \in (-1, 1), \quad v \in \mathbb{R}^3, \\
F(t, \pm 1, v) |_{v_y \leq 0} &= \sqrt{2\pi \mu} \int_{v_y \geq 0} F(t, \pm 1, v) |v_y| dv, \quad t \geq 0, \quad v \in \mathbb{R}^3.
\end{aligned}
\]  

(1.21)

One may expect that the solution of the time-dependent problem (1.21) tends in large time toward that of the steady problem (1.8). For this, the second result is concerned with the large time asymptotic stability of the stationary solution \( F_{st} \) which gives the nonnegativity of \( F_{st} \).

Theorem 1.2. Let \( F_{st}(y, v) \) be the steady state obtained in Theorem 1.1 corresponding to a shear rate \( \alpha \in (0, \alpha_0) \). There are constants \( \varepsilon_0 > 0, \lambda_0 > 0 \) and \( C > 0 \), independent of \( \alpha \), such that if initial data \( F_0(y, v) \geq 0 \) satisfy

\[
\| w_q [F_0(y, v) - F_{st}(y, v)] \|_{L^\infty} \leq \varepsilon_0
\]

with

\[
\int_{-1}^1 \int_{\mathbb{R}^2} [F_0(y, v) - F_{st}(y, v)] dv dy = 0,
\]  

(1.22)

then the initial boundary value problem (1.21) admits a unique solution \( F(t, y, v) \geq 0 \) satisfying the following decay estimate:

\[
\| w_q [F(t, y, v) - F_{st}(y, v)] \|_{L^\infty} \leq Ce^{-\lambda_0 t} \| w_q [F_0(y, v) - F_{st}(y, v)] \|_{L^\infty},
\]  

(1.23)

for any \( t \geq 0 \).

Remark 1.4. Thanks to Theorem 1.1, the expansion (1.20) for the steady state \( F_{st}(y, v) \) is uniform in all \( \alpha \in (0, \alpha_0) \) when the large enough integer \( q \) is chosen and hence \( \alpha_0 = \alpha_0(q) > 0 \) is fixed. Thus, the exponential time decay estimate (1.23) also holds uniformly for any \( \alpha \in (0, \alpha_0) \), in particular, \( C \) and \( \lambda_0 \) are independent of \( \alpha \). As \( \alpha \to 0 \), we are able to recover the exponential convergence of the solution \( F(t, y, v) \) to the global Maxwellian \( \mu \) in \( L^\infty \)-norm weighted by the polynomial velocity weight \( w_q(v) \).
In what follows we present key points and strategy in the proof of the main results stated above. As pointed out in a recent nice survey by Esposito-Marra [20], stationary non-equilibrium solutions to the Boltzmann equation, despite their relevance in applications, are much less studied than time-dependent solutions, and no general existence theory is available, due to technical difficulties. Readers may refer to [20] and references therein for a thorough review on this subject. As for the Boltzmann equation on the plane Couette flow, [19] and [13] mentioned before seem to be the only mathematical works on the fluid dynamic approximation solutions in the steady case for small Knudsen number. But it remains unsolved how to justify the large time asymptotics toward the stationary solution for the time-dependent problem in the same setting of the fluid limit. In this paper, motivated by [14], instead of constructing the fluid dynamic approximation solutions, we focus on the existence and dynamical stability of the plane Couette flow with the finite Knudsen number for both the steady and unsteady problems.

First of all, for the original Couette flow problem (1.1), (1.2) and (1.3), we note that a direct perturbation approach by linearization of the boundary condition in \( \alpha \) in terms of the techniques in [13,18,19] or [16] can be applied to prove the existence of stationary solutions, because the inhomogeneous data appear only on the tangent \((x,z)\)-plane. The solution thus obtained has the structure around global Maxwellians of the form

\[
F(y,v) = \mu(v) + \sqrt{\mu(v)}(\alpha g_1 + \alpha^2 g_2 + \cdots)
\]

corresponding to the linearization of the wall Maxwellians at \( y = \pm L \)

\[
\mu(v_x \pm \alpha L, v_y, v_z) = \mu(v) + (\pm \alpha L)\mu_1(v) + (\pm \alpha L)^2\mu_2(v) + \cdots.
\]

On the other hand, in the formulation used in this paper, we rather look for the solution of the specific structure (1.7), and hence the problem can be reduced to solve (1.8) for the Boltzmann equation driven by an external shear force under the homogeneous non-moving diffuse reflection boundary condition. This means that the solution to the Couette flow problem (1.1), (1.2) and (1.3) is established around the local Maxwellian \( \mu(v_x - \alpha y, v_y, v_z) \) instead of the global Maxwellian \( \mu \) such that the kinetic diffusive reflection boundary condition (1.2) is satisfied for the background solution \( \mu(v_x - \alpha y, v_y, v_z) \). In addition, as mentioned before, it seems more convenient to use the formulation with shear forces than the original one driven by the relative motion of boundaries in order to understand the asymptotic behavior of solutions in the limit \( L \to \infty \), that is, how the Couette flow with boundaries converges to a shear flow without boundary that is closely related to what has been studied in the previous works [14] for uniform shear flow in the spatially homogeneous setting.

We also comment on the boundary value problem (1.11) and (1.12) for determining the first order correction term \( G_1(y, v) \). Notice that the inhomogeneous source term \( -v_x v_y \sqrt{\mu} \) in (1.11) does not satisfy the boundary condition (1.12), so a space-dependent non-trivial solution is induced. If the boundary condition is omitted and only the spatially homogeneous equation is considered, the corresponding solution can be written as

\[
L^{-1}(-v_x v_y \sqrt{\mu}) = -\frac{1}{2b_0} v_x v_y \sqrt{\mu}
\]

(1.24)

with the positive constant \( b_0 := 3\pi \int_{-1}^{1} B_0(z)z^2(1 - z^2)\,dz \). The form (1.24) is then consistent with the uniform shear flow in [14]. To solve the boundary value problem (1.11) and (1.12), the same approach as for treating the remainder \( G_R \) can be applied. However, in order to simplify the proof, we have made use of an additional property (1.15) to reduce the diffusive reflection boundary condition (1.12) to the homogeneous inflow boundary condition (1.16). To treat (1.11) and (1.16), we develop a direct \( L^\infty - L^2 \) method without using the stochastic cycles as in [24]. In particular, thanks to the splitting \( L = \nu_0 - K \), if the non-local term \( KG_1 \) is omitted, the solution to the boundary value problem

\[
v_y \partial_y G_1 + \nu_0 G_1 = \mathfrak{F}, \quad G_1(\pm 1, v)|_{v_x = 0} = 0,
\]
can be explicitly expressed as

\[ G_1(y, v) = 1_{\nu_y > 0} \int_y^1 e^{-\frac{\nu_y (y-y')}{v'}} v_y^{-1} \tilde{G}(y', v) dy' + 1_{\nu_y < 0} \int_y^1 e^{-\frac{\nu_y (y-y')}{v'}} v_y^{-1} \tilde{G}(y', v) dy'. \]

Moreover, we use the bootstrap argument as in [15] to treat the following problem with a parameter \( \sigma \in [0, 1] \):

\[ v_y \partial_y G_1 + \nu_y G_1 = \sigma KG_1 + \tilde{G}, \quad G_1(\pm 1, v)|_{v_y \leq 0} = 0. \]

With the solvability starting from \( \sigma = 0 \), we are able to iteratively solve the above boundary value problem for \( \sigma \) over the intervals \([0, \sigma_1], [\sigma_1, 2\sigma_1] \), and so on, where \( \sigma_1 > 0 \) is small enough such that \( \sigma_1 KG_1 \) can be regarded as a source term in the \( L^\infty \) estimation. Therefore, in the end, the original problem corresponding to \( \sigma = 1 \) can be solved. In this procedure, the uniform \( L^\infty \) estimate can be obtained through the interplay with the \( L^2 \) estimates in terms of the Guo’s technique in [24].

Here, we have omitted the discussions on the mass conservation (1.10) for \( G_1 \). In fact, inspired by [24], an extra damping term \( \epsilon G_1 \) with the vanishing parameter \( \epsilon > 0 \) has to be used, cf. Section 4 for details.

We now discuss some key points about estimating the remainder \( G_R \) for the existence of solutions to the boundary value problem (1.13) and (1.14). The direct \( L^\infty - L^2 \) approach is no longer available because the linear term \( \frac{1}{2} \alpha v_x v_y G_R \) cannot be controlled in the large velocity regime. Notice that this term arises from the action of the shear force on the exponential weight function \( \sqrt{\mu} \) in the perturbation. To overcome it, as in [14], we apply the Caflisch’s decomposition

\[ \sqrt{\mu} G_R = G_{R,1} + \sqrt{\mu} G_{R,2} \]

and \( G_{R,1} \) and \( G_{R,2} \) satisfy the coupled boundary value problems

\[
\begin{aligned}
& v_y \partial_y G_{R,1} - \alpha v_y \partial v_y G_{R,1} + \nu_y G_{R,1} = \chi_M KG_{R,1} - \frac{1}{2} \alpha \sqrt{\mu} v_x v_y G_{R,2} + F_1, \\
& G_{R,1}(\pm 1, v)|_{v_y \leq 0} = 0,
\end{aligned}
\]

(1.25)

and

\[
\begin{aligned}
& v_y \partial_y G_{R,2} - \alpha v_y \partial v_y G_{R,2} + LG_{R,2} = (1 - \chi_M) \mu^{-\frac{\chi_M}{2}} KG_{R,1} + F_2, \\
& G_{R,2}(\pm 1, v)|_{v_y \leq 0} = \sqrt{2\pi \mu} \int_{v_y \geq 0} \sqrt{\mu} G_R(\pm 1, v)|_{v_y} dv,
\end{aligned}
\]

(1.26)

respectively. Then, in (1.25), the term \(-\frac{1}{2} \alpha \sqrt{\mu} v_x v_y G_{R,2} \) can be controlled due to the appearance of \( \sqrt{\mu} \). Here, since the operator norm of \( K \) may not be small, the term \( \chi_M KG_{R,1} \) over the large velocity regime can be viewed as a source in (1.25) for \( G_{R,1} \), while the complementary term \((1 - \chi_M) \mu^{-\frac{\chi_M}{2}} KG_{R,1} \) is taken as a source in (1.26) for \( G_{R,2} \). A crucial observation inspired by [4] in estimating \( G_{R,1} \) is that the norm of the weighted operator \( w_q \chi_M K \) on \( L^\infty_\nu \) with the polynomial velocity weight \( w_q = (1 + |v|^2)^q \) can be arbitrarily small as long as \( M \) and \( q \) are chosen sufficiently large, see Lemma 2.4. Notice that Lemma 2.4 holds only for the Maxwell molecule potential as shown in the proof. Compared to the previous work [14] for uniform shear flow, it is more complicated to solve the coupling steady boundary value problems (1.25) and (1.26) in a bounded domain. We now list the main steps in the proof:

• **Step 1.** We first modify the coupled boundary value problems with two parameters \( \epsilon > 0 \) being small enough and \( 0 \leq \sigma \leq 1 \), see (5.10), and obtain the a priori estimates uniform in \( \epsilon \) and \( \sigma \) in the \( L^\infty \) framework, see Lemma 5.1 and the proof for Proposition 5.1. For the proof of Lemma 5.1, we apply Guo’s approach in [24] to the shear flow problem in a slab. In particular, we introduce the mild formulation (5.23) to treat the diffuse boundary condition with the help of Lemma 8.1, and reprove Ukai’s trace theorem in Lemma 3.1 for the \( L^2 \) estimates.

• **Step 2.** Similar to solving the first order correction term \( G_1 \), we design an explicit procedure to solve the parametrized boundary value problem (5.10) iteratively for \( \sigma \in [0, 1] \) from \( \sigma = 0 \) to \( \sigma = 1 \) for any fixed \( \epsilon > 0 \), see Lemma 5.2. Notice that the problem for \( \sigma = 0 \) is reduced to the one without the nonlocal collision terms under the homogeneous inflow boundary condition so that the characteristic method can be directly applied.
• Step 3. We study the limit $\epsilon \to 0$ to obtain the desired solution, see Subsection 5.4 for details. The key point is to obtain the macroscopic estimates in order to bound the $L^2$ norm of $G_{R,2}$ in terms of the $L^\infty$ norm of $G_{R,1}$. We apply the dual argument developed first in [16]. Note that it is delicate to deduce these estimates to be uniform for any small parameter $\epsilon > 0$.

With the existence of stationary solution $F_{st}$, the asymptotic stability of the perturbation $F = F_{st} + \sqrt{\mu}f$ as (6.1) is considered in the reformulated IBVP as (6.2). Technically, we follow the same strategy as for treating the steady problem. Precisely, we also use the decomposition
\[ \sqrt{\mu}f = f_1 + \sqrt{\mu}f_2 \]
with $f_1, f_2$ satisfying the coupled IBVPs (6.4), (6.5) and (6.6), (6.7), respectively. In order to treat initial data with only the polynomial velocity weight, we set $f_2(0, y, v) \equiv 0$ and the boundary conditions on $f_1$ and $f_2$ both as diffuse reflections which are slightly different from (1.25) and (1.26) in the steady problem. Moreover, in contrast with the steady case, we need to construct suitable temporal energy functionals so as to close the a priori estimates. In particular, the energy functional for the second component $f_2$ in the Caflisch’s decomposition is complicated, because there is a subtle interplay with $f_1$. For this, we make use of the linear combination of estimates for the two functionals, where the smallness of the shear rate $\alpha$ and finiteness of the domain play an important role. Specifically, we obtain estimates (7.2) and (7.3) for the weighted $L^\infty$ norms. To treat $L^2$ estimates on the right hand side of (7.3), we construct another functional $E_{int}(t)$ in Lemma 7.2, see (7.29), to capture the macroscopic dissipation, and conclude the desired estimates (7.33) and hence (7.36).

Finally, we remark that there have been extensive studies on the stability of shear flow in the multi-dimensional space domain in the context of fluid dynamic equations, cf. [32], in particular, we mention important contributions to the mathematical theories in [6–8] by Bedrossian et. al. for either an infinite 2D channel domain $\mathbb{T}_x \times \mathbb{R}_y$, or an infinite 3D channel domain $\mathbb{T}_x \times \mathbb{R}_y \times \mathbb{T}_z$, and an interesting work by Ionescu-Jia [26] for the asymptotic stability of the Couette flow for the 2D Euler equations in the 2D finite channel domain $\mathbb{T}_x \times [0, 1]$ with the zero normal velocities at two boundary planes $y = 0, 1$, see also the nice survey [5] and references therein. In fact, in comparison with the 1D problem (1.8) under consideration, it would be more interesting to study the existence and asymptotic stability of stationary solutions in the multi-dimensional setting corresponding to those works on fluid dynamic equations. Moreover, it is also challenging to study the fluid dynamic limit for these problems as in [13,18,19] when the vanishing Knudsen number is taken into account. We expect that this paper together with [14] can shed some light on the future investigation on the above problems.

The rest of this paper is organized as follows. In Section 2, we give some basic estimates on the linearized and nonlinear collision operators. In particular, we obtain Lemma 2.4 which is crucially used to obtain the smallness of the nonlocal operator $\mathcal{K}$ for large velocity. In Section 3, we revisit Ukai’s trace theorem in both the steady and time-dependent cases for the transport operator with the shear force in the 1D setting under consideration. In Section 4 and Section 5, we establish estimates on the first order correction $G_1$ and the remainder $G_{R}$, respectively, and hence complete the proof of Theorem 1.1 without showing nonnegativity of the stationary solution. Then we study the time-dependent problem for the local in time existence in Section 6 and the exponential time asymptotic stability of the stationary solution in Section 7 so that the nonnegativity of stationary solution follows. The appendix Section 8 includes some estimates on the boundary product measure when there are multiple bounces induced by the diffuse boundary condition.

Notations. We list some notations and norms used in the paper. Throughout this paper, $C$ denotes some generic positive (generally large) constant and $\lambda$ denote some generic positive (generally small) constant. $D \lesssim E$ means that there is a generic constant $C > 0$ such that $D \leq CE$. $D \sim E$ means $D \lesssim E$ and $E \lesssim D$. $\lambda$ indicates the characteristic function on the set $A$. We denote $\| \cdot \|$ the $L^2((-1,1) \times \mathbb{R}^3)$–norm or the $L^2((-1,1), \mu)$–norm or $L^2(\mathbb{R}^3)$–norm. Sometimes without any confusion, we use $\| \cdot \|_{L^\infty}$ to denote either the $L^\infty([-1,1] \times \mathbb{R}^3)$–norm or the $L^\infty(\mathbb{R}^3)$–norm. Moreover, $(\cdot, \cdot)$ denotes the $L^2$ inner product in $(-1,1) \times \mathbb{R}^3$ with the $L^2$ norm $\| \cdot \|$. And $(\cdot, \cdot)$ denotes the $L^2$ inner product in $\mathbb{R}^3$. We denote by $\gamma_+ = \{(1,v) | v \in \mathbb{R}^3, v_y > 0\} \cup \{(-1,v) | v \in \mathbb{R}^3, v_y < 0\}$
the outgoing set, by $\gamma_- = \{(1, v)|v \in \mathbb{R}^3, v_y < 0\} \cup \{(-1, v)|v \in \mathbb{R}^3, v_y > 0\}$ the incoming set, and by $\gamma_0 = \{(\pm 1, v)|v \in \mathbb{R}^3, v_y = 0\}$ the grazing set. Furthermore $|f|_{2, \pm} = |f1_{\gamma_{\pm}}|_2$ represent the $L^2$ norm of $f(y, v)$ at the boundary $y = \pm 1$. Finally, we define
\[
P_{\gamma}f(\pm 1, v) = \sqrt{\mu(v)} \int_{n(\pm 1) \cdot v' > 0} f(x, v') \sqrt{\mu(v')} (n(\pm 1) \cdot v') dv',
\]
where $n(\pm 1) = (0, \pm 1, 0)$. One sees that $P_{\gamma}f$ defined on $\{\pm 1\} \times \mathbb{R}^3$, is an $L^2$-projection with respect to the measure $|v_y| \sqrt{\mu(v)} dv$ for any function $f$ defined on $\gamma_+$. 

2. Basic estimates

In this section we summarize some basic estimates to be used in the following sections. Let us first give some elementary estimates for the linearized collision operator $L$ and nonlinear collision operator $\Gamma$, defined by
\[
Lg = -\mu^{-1/2} \{Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu)\},
\]
and
\[
\Gamma(f, g) = \mu^{-1/2} Q(\sqrt{\mu}f, \sqrt{\mu}g) = \int_{\mathbb{R}^3} \int_{S^2} B_0 \mu^{1/2}(v_\ast)[f(v')g(v') - f(v)g(v)] d\omega dv_\ast,
\]
respectively. It is known that
\[
Lf = \nu f - Kf
\]
with
\[
\begin{cases}
\nu = \int_{\mathbb{R}^3} \int_{S^2} B_0(\cos \theta) \mu(v_\ast) d\omega dv_\ast = \nu_0, \\
Kf = -\frac{1}{2} \left\{ Q(\mu^{1/2}f, \mu) + Q_{\text{gain}}(\mu, \mu^{1/2}f) \right\},
\end{cases}
\]
where $Q_{\text{gain}}$ denotes the positive part of $Q$ in (1.4). Note that $\nu_0$ is a positive constant in the case of Maxwell molecule collision. The kernel of $L$, denoted as $\ker L$, is a five-dimensional space spanned by
\[
\{1, v, |v|^2 - 3\} \sqrt{\mu} := \{\phi_i\}_{i=1}^5.
\]
Define a projection from $L^2$ to $\ker L$ by
\[
P_0 g = \{a_g + b_g \cdot v + (|v|^2 - 3)c_g\} \sqrt{\mu}
\]
for $g \in L^2$, and correspondingly denote the operator $P_1$ by $P_1 g = g - P_0 g$, which is orthogonal to $P_0$ in $L^2$.

It is also convenient to define
\[
L f = -\{Q(f, \mu) + Q(\mu, f)\} = \nu f - Kf,
\]
with
\[
\nu f = \nu_0 f, \quad Kf = Q(f, \mu) + Q_{\text{gain}}(\mu, f) = \sqrt{\mu}K(\frac{f}{\sqrt{\mu}}),
\]
according to (2.3).

The following lemma is concerned with the integral operator $K$ given by (2.3), and its proof in case of the hard sphere model was given by [24, Lemma 3, pp.727]. Recall (1.17) for the polynomial velocity weight $w_q$.

**Lemma 2.1.** Let $K$ be defined as in (2.3), then it holds
\[
Kf(v) = \int_{\mathbb{R}^3} k(v, v_\ast) f(v_\ast) dv_\ast
\]
with
\[
|k(v, v_\ast)| \leq C\{1 + |v - v_\ast|^{-2}\} e^{-\frac{1}{2}|v - v_\ast|^2 - \frac{1}{2}|v_\ast|^2} e^{-\frac{1}{2}v^2 - \frac{1}{2}v_\ast^2}.
\]
Moreover, let
\[
k_w(v, v_\ast) = w_q(v)k(v, v_\ast)w_{-q}(v_\ast)
\]
(2.5)
with \( q \geq 0 \), then it also holds
\[
\int_{\mathbb{R}^3} k_\varepsilon(v, v_*) e^{\frac{|v-v_*|^2}{s}} dv_* \leq \frac{C}{1 + |v|},
\]
for any \( \varepsilon \geq 0 \) small enough.

For the weighted derivative-in-\( v_* \) estimates on the nonlinear operator \( \Gamma \), we have the following lemma.

**Lemma 2.2.** In the Maxwell molecular case, it holds that
\[
\| w_q \partial_v^m \Gamma(f, g) \|_{L^2} \leq C \sum_{m' \leq m} \| w_q \partial_v^{m'} f \|_{L^2} \| w_q \partial_v^{m-m'} g \|_{L^2},
\]
and
\[
\| w_q \partial_v^m \Gamma(f, g) \|_{L^\infty} \leq C \sum_{m' \leq m} \| w_q \partial_v^{m'} f \|_{L^\infty} \| w_q \partial_v^{m-m'} g \|_{L^\infty},
\]
for any integers \( m \geq 0 \) and \( q \geq 0 \).

**Proof.** We prove (2.7) only, since the proof for (2.6) is similar and it follows from the proof of [25, Lemma 2.3, pp.611]. By definition (2.2), we have
\[
\partial_v^m \Gamma(f, g) = \partial_v^m \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B_0 \mu^{1/2}(v_* f(v_* g(v)) \omega dv_* - \partial_v^m \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B_0 \mu^{1/2}(v_* f(v_* g(v)) \omega dv_*
\]
\[
= \partial_v^m \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B_0 \mu^{1/2}(v_* f(v_* g(v)) \omega dv_* = c_0 \partial_v^m g(v) \int_{\mathbb{R}^3} \mu^{1/2}(v_* f(v_* g(v)) \omega dv_*
\]
where we have used \( \int_{\mathbb{R}^2} B_0 \omega = c_0 \) for a constant \( c_0 > 0 \). Recalling (1.5), by a change of variable \( \tilde{u} = v_* - v \), we then have
\[
\partial_v^m \Gamma(f, g) = \partial_v^m \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B_0 \mu^{1/2}(\tilde{u} + v) f(v + \tilde{u}) g(v + \tilde{u}) \omega d\tilde{u}
\]
\[
= \sum_{m_1 + m_2 \leq m} C_{m_1, m_2}^{m} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B_0 (\partial_v^{m_1-m_2} \mu^{1/2}(\tilde{u} + v) (\partial_v^{m_1} f)(v + \tilde{u}) (\partial_v^{m_2} g)(v + \tilde{u}) \omega d\tilde{u}
\]
where \( \tilde{u} = (\tilde{u} \cdot \omega) \omega \) and \( \tilde{u} = \tilde{u} - \tilde{u} \). Then, by taking directly the \( L^\infty \) norm, (2.7) holds because
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B_0 (\partial_v^{m} \mu^{1/2}(v_* f) \omega dv_* < \infty,
\]
for any integer \( m \geq 0 \). This completes the proof of Lemma 2.2. \( \square \)

The following lemma can be found in [23, Lemmas 3.2 and 3.3, pp.638-639], where the case for the hard sphere model was considered.

**Lemma 2.3.** In the Maxwell molecular case, there is a constant \( \delta_0 > 0 \) such that
\[
\langle Lf, f \rangle = \langle LP_1 f, P_1 f \rangle \geq \delta_0 \| P_1 f \|^2.
\]
Moreover, for any integer \( m > 0 \), there are constants \( \delta_1 > 0 \) and \( C > 0 \) such that
\[
\langle \partial_v^m Lf, \partial_v^m f \rangle \geq \delta_1 \| \partial_v^m f \|^2 - C \| f \|^2.
\]

**Proof.** Since (2.8) is quite elementary, we only show (2.9). As in Lemma 2.2, the key point here is to show that the action of the derivatives \( \partial_v^m \) on the nonlocal operator \( L \) does not involve any other partial derivatives such as \( \partial_v \) or \( \partial_v^2 \). By (2.1) and (2.8), we have
\[
\langle \partial_v^m Lf, \partial_v^m f \rangle = \langle L\partial_v^m f, \partial_v^m f \rangle + \sum_{m_1 \leq m} C_{m_1}^{m} \langle \partial_v^{m_1} L\partial_v^{m_1} f, \partial_v^{m_1} f \rangle \geq \delta_0 \| P_1 \| \partial_v^m f \|^2 - \sum_{m_1 \leq m} C_{m_1}^{m} \| \partial_v^{m_1} \| \partial_v^{m_1} f \|^2 - \sum_{m_1 \leq m} C_{m_1}^{m} | \langle \partial_v^{m_1} L\partial_v^{m_1} f, \partial_v^{m_1} f \rangle |.
\]

(2.10)
with
\[ I_{m_1 < m} \partial^{m-m_1}_{v_x} L \partial^{m_1}_{v_x} f \]
\[ = - \sum_{m_1 + m_2 < m} C_{m_1, m_2}^{m_1} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\partial^{m-m_1-m_2}_{v_x} \mu^{1/2})(\tilde{u} + v)(\partial^{m_1}_{v_x} f)(v + \tilde{u}_\perp) (\partial^{m_2}_{v_x} \mu^{1/2})(v + \tilde{u}_\parallel) \, d\omega \tilde{d}u \]
\[ + \partial^{m}_{v_x} \mu^{1/2}(v) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0 \mu^{1/2}(v_*) (v_*) \, d\omega dv_* \]
\[ - \sum_{m_1 + m_2 < m} C_{m_1, m_2}^{m_1} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\partial^{m-m_1-m_2}_{v_x} \mu^{1/2})(\tilde{u} + v)(\partial^{m_1}_{v_x} f)(v + \tilde{u}_\perp) (\partial^{m_2}_{v_x} \mu^{1/2})(v + \tilde{u}_\parallel) \, d\omega \tilde{d}u, \]
where we have used the change of variable $\tilde{u} = v_* - v$ again. Consequently, as for (2.6), it follows
\[ \sum_{m_1 < m} C_{m_1}^{m_1} |(\partial^{m-m_1}_{v_x} L \partial^{m_1}_{v_x} f, \partial^{m_1}_{v_x} f)| \leq \eta \|\partial^{m_1}_{v_x} f\|^2 + C_\eta \sum_{m_1 < m} \|\partial^{m_1}_{v_x} f\|^2 \]
\[ \leq \eta \|\partial^{m_1}_{v_x} f\|^2 + C_\eta \eta_1 \|\partial^{m_1}_{v_x} f\|^2 + C_\eta_1 \|f\|^2, \quad (2.11) \]
for some enough constants $\eta > 0$ and $\eta_1 > 0$, where Sobolev’s interpolation inequality $\|\partial^{m_1}_{v_x} f\|^2 \leq \eta_1 \|\partial^{m_1}_{v_x} f\|^2 + C_\eta_1 \|f\|^2$ has been used.

One the other hand, it can be easily checked that
\[ \|P_1[[\partial^{m}_{v_x} f]]| \geq \|\partial^{m}_{v_x} f| - \|P_0[\partial^{m}_{v_x} f]| \geq \|\partial^{m}_{v_x} f| - \|f\|. \quad (2.12) \]
Finally, plugging (2.11) and (2.12) into (2.10) gives (2.9). This completes the proof of Lemma 2.3.

Next, the following lemma which was proved in [14, Proposition 3.1, pp.13] plays a significant role in obtaining the $L^\infty$ estimates of the first component in the Caflisch’s decomposition of solutions.

**Lemma 2.4.** Let $K$ be given by (2.4), then for any nonnegative integer $m \geq 0$, there is $C > 0$ such that for any arbitrarily large $q > 0$ we have
\[ \sup_{|v| \geq M} w_q |\partial^m v_x K f| \leq \frac{C}{q} \sum_{0 < m' \leq m} \|w_q \partial^{m'} v_x f\|_{L^\infty}, \quad (13.2) \]
for some $M = M(q) > 0$. In particular, one can choose $M = q^2$.

**Proof.** Since the general case
\[ \sup_{|v| \geq M} w_q |\partial^m v_x K f| \leq \frac{C}{q} \sum_{0 < m' \leq m} \|w_q \partial^{m'} v_x f\|_{L^\infty} \]
was proved in [14, Proposition 3.1, pp.13], as in Lemma 2.3 we only point out that the derivative $\partial^m v_x$ acting on the nonlocal operator $K$ does not involve other derivatives such $\partial v_y$ or $\partial v_z$. Indeed, in view of (2.4), similar to the proof of Lemma 2.3, we have
\[ \partial^m v_x K f = \sum_{m_1 \leq m} C_{m_1}^{m} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\partial^{m_1}_{v_x} f)(v + \tilde{u}_\perp) (\partial^{m-m_1}_{v_x} \mu)(v + \tilde{u}_\parallel) \, d\omega \tilde{d}u - \partial^{m_1}_{v_x} \mu(v) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0 f(v_*) \, d\omega dv_* \]
\[ + \sum_{m_1 \leq m} C_{m_1}^{m} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\partial^{m_1}_{v_x} f)(v + \tilde{u}_\perp) (\partial^{m-m_1}_{v_x} \mu)(v + \tilde{u}_\parallel) \, d\omega \tilde{d}u \]
\[ = \sum_{m_1 \leq m} C_{m_1}^{m} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\partial^{m_1}_{v_x} f)(v') (\partial^{m-m_1}_{v_x} \mu)(v') \, d\omega dv_* - (\partial^{m_1}_{v_x} \mu)(v) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0 f(v_*) \, d\omega dv_* \]
\[ + \sum_{m_1 \leq m} C_{m_1}^{m} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_0(\partial^{m_1}_{v_x} f)(v') (\partial^{m-m_1}_{v_x} \mu)(v') \, d\omega dv_* \]
Then, by the similar calculation for estimating $I_1$ and $I_2$ in [14, Proposition 3.1, pp.13] yields (2.13). This completes the proof of Lemma 2.4.
3. A Trace Theorem

In this section, we present the following Ukai’s trace theorem, see also Lemma 2.3 in [17, Page 22 of 119] and Lemma 3.2 in [17, Page 56 of 119], respectively.

**Lemma 3.1.** Let $\varepsilon > 0$ and $y \in [-h, h]$ with $0 < h < +\infty$, and denote the near-grazing set of $\gamma_+$ or $\gamma_-$ as

$$
\gamma_{\pm} \equiv \{(y, v) \in \gamma_{\pm} : |y| \leq \varepsilon \text{ or } |v| \geq \frac{1}{\varepsilon}, \ v = (v_x, v_y, v_z)\}.
$$

Then, there exists a constant $C_{\varepsilon, h} > 0$ depending on $\varepsilon$ and $h$ such that

$$
|f 1_{\gamma_{\pm}}(t)|_{L^1} \leq C_{\varepsilon, h} \left\{ \|f\|_{L^1} + \|\{v_y \partial_y - \alpha v_y \partial_{v_z}\}f\|_{L^1} \right\}.
$$

Moreover, it also holds

$$
\int_0^T |f 1_{\gamma_{\pm}}(t)|_{L^1} dt \leq C_{\varepsilon, h} \left\{ \|f(0)\|_{L^1} + \int_0^T \left[ \|f(t)\|_{L^1} + \|\{\partial_t + v_y \partial_y - \alpha v_y \partial_{v_z}\}f(t)\|_{L^1} \right] dt \right\},
$$

for any $T \geq 0$.

**Proof.** To prove (3.1), we only consider the case that the boundary phase is outgoing, because the incoming case can be treated similarly. We introduce a parameter $t \in \mathbb{R}$ and treat $(y, v)$ as functions of $t$. Define the characteristic line $[s, Y(s, t, y, v), V(s, t, y, v)]$ passing through $(y, v) = (t, y(t, v(t)))$ such that

$$
dY = v_y dt, \quad dV = -\alpha v_y dt.
$$

Then it follows

$$
Y(s, t, y, v) = y - (t - s)v_y, \quad V(s, t, y, v) = (v_x + \alpha(t - s)v_y, v_y, v_z),
$$

for $(y, v) \in \gamma_{+\gamma}$. Along this trajectory, one has the identity

$$
f(y, v) = f(Y(s, t, y, v), V(s, t, y, v)) + \int_s^t \frac{d}{d\tau} f(Y(\tau; t, y, v), V(\tau; t, y, v)) d\tau.
$$

On the other hand, $(y, v) \in \gamma_{+\gamma}$ also implies $h \varepsilon \leq t_b(y, v) \leq \frac{1}{\varepsilon}$, where $t_b$ is given as (5.17). Therefore, by taking $s \in [t - t_b(y, v), t]$, we get from (3.5) that

$$
\int_{\gamma_{+\gamma}} |f(y, v)||v_y| dv \leq C_{\varepsilon, h} \int_{\gamma_{+\gamma}} \int_{t - t_b(y, v)}^t |f(Y(s, t, y, v), V(s, t, y, v))||v_y|dv
$$

$$
+ C_{\varepsilon, h} \int_{\gamma_{+\gamma}} \int_{t - t_b(y, v)}^t |\frac{d}{ds} f(Y(s, t, y, v), V(s, t, y, v))||v_y| ds dv
$$

$$
= C_{\varepsilon, h} \int_{\gamma_{+\gamma}} \int_{t - t_b(y, v)}^t |f(Y(s, t, y, v), V(s, t, y, v))||v_y| dv
$$

$$
+ C_{\varepsilon, h} \int_{\gamma_{+\gamma}} \int_{t - t_b(y, v)}^t |[v_y \partial_Y - \alpha v_y \partial_{V_z}]f(Y(s, t, y, v), V(s, t, y, v))||v_y| dv
$$

Next, in light of the Jacobian

$$
\partial(Y(s), V(s)) = \frac{\partial(Y(s), V(s))}{\partial(s, v)} = \begin{pmatrix}
  v_y & 0 & s & 0 \\
  -\alpha v_y & 1 & -\alpha s & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix} = v_y,
$$

(3.7)
and by a change of variable

\[ \tilde{y}, u = [Y(s; t, y, v), V(s; t, y, v)] = [y - (t - s)v_y, v_x + \alpha(t - s)v_y, v_y, v_z], \]

one gets

\[ \int_{\gamma_+}^{\gamma_+} \int_{t - ts}^t \left| f(Y(s; t, y, v), V(s; t, y, v)) \right| dv dy ds \leq \int_{\mathbb{R}^3} \int_{-h}^h \left| f(\tilde{y}, u) \right| d\tilde{y} du. \]  

(3.8)

Similarly, by noticing that \( \partial_Y f(Y, V) = \partial_y f(\tilde{y}, u), \partial_V f(Y, V) = \partial_u f(\tilde{y}, u) \) and \( v_y = u_y \), one has

\[ \int_{\gamma_+}^{\gamma_+} \int_{t - ts}^t \left[ |v_y \partial_Y - \alpha v_y \partial_V| f(Y(s; t, y, v), V(s; t, y, v)) \right] dv dy ds \leq \int_{\mathbb{R}^3} \int_{-h}^h \left[ |u_y \partial_y - \alpha u_y \partial_u| f(\tilde{y}, u) \right] d\tilde{y} du. \]  

(3.9)

Consequently, the desired estimate (3.1) in the case of outgoing boundary follows from (3.8), (3.9) and (3.6).

We now turn to prove (3.2). For \( f \in L^1([T_1, T] \times [-h, h] \times \mathbb{R}^3) \), we first show that

\[ \int_{T_1}^T \int_{v \cdot n(y_f) > 0} \int_0^{t_{s, y_f, u}} \left| f(\tilde{t} + s, Y(\tilde{t} + s; \tilde{y}, y_f, u), V(\tilde{t} + s; \tilde{y}, y_f, u)) \right| u_y dv ds dt \leq \int_{T_1}^T \int_{-h}^h \int_{\mathbb{R}^3} \left| f(\tilde{t}, \tilde{y}, y_f, u) \right| d\tilde{y} dv dt. \]  

(3.10)

where \( y_f = \pm h, T \geq T_1 \geq 0 \) and

\[ Y(\tilde{t} + s; \tilde{y}, y_f, u) = y_f + su_y, \quad V(\tilde{t} + s; \tilde{y}, y_f, u) = (u_x - \alpha su_y, u_y, u_z), \]

with

\[ Y(\tilde{t}; \tilde{y}, y_f, u) = y_f, \quad V(\tilde{t}; \tilde{y}, y_f, u) = u = (u_x, u_y, u_z). \]

Actually, given \( (t, y, u) \in [T_1, T] \times [-h, h] \times \mathbb{R}^3 \), let us define \( y_f = y + t b(y, u)u_y = \pm h \), and denote

\[ y = Y(t; s, y_f, u) = y_f + su_y, \quad v = V(t; s, y_f, u) = (u_x - \alpha su_y, u_y, u_z), \]

for \( u \cdot n(y_f) > 0 \). It is easy to see that \( 0 \geq s \geq -t b(y_f, u) \), and it is natural to require that \( t - s \leq T \). By a change of variable \( (y, v) \to (s, u) \) and using (3.7), one has

\[ \int_{T_1}^T \int_{u \cdot n(y_f) > 0} \int_0^{t_{s, y_f, u}} \left| f(t, Y(t; s, y_f, u), V(t; s, y_f, u)) \right| u_y dv ds dt \leq \int_{T_1}^T \int_{-h}^h \int_{\mathbb{R}^3} \left| f(t, y, v) \right| dv dt. \]  

(3.11)

On the other hand, if we denote \( \tilde{t} = t - s \), then it follows \( s \geq T_1 - \tilde{t} \) due to \( t \geq T_1 \). In summary, one has

\[ s \geq \max\{ -t b(y_f, u), T_1 - \tilde{t} \}, \quad T_1 \leq \tilde{t} \leq T. \]

Therefore, we have by change of variable \( t \to \tilde{t} \) that

\[ \int_{T_1}^T \int_{u \cdot n(y_f) > 0} \int_0^{t_{s, y_f, u}} \left| f(t, Y(t; s, y_f, u), V(t; s, y_f, u)) \right| u_y dv ds dt = \int_{T_1}^T \int_{T_1}^{T_1} \int_{u \cdot n(y_f) > 0} \int_0^{t_{s, y_f, u}} \left| f(\tilde{t} + s, Y(\tilde{t} + s; \tilde{y}, y_f, u), V(\tilde{t} + s; \tilde{y}, y_f, u)) \right| u_y dv ds d\tilde{t}. \]  

(3.12)
Consequently, (3.11) and (3.12) imply (3.10). In addition, it follows that
\[
\begin{align*}
  f(t, yf, u) &= f(t + s, Y(t + s; t, yf, u), V(t + s; t, yf, u)) \\
  &+ \int_0^s \frac{d}{d\tau} f(t + \tau, Y(t + \tau; t, yf, u), V(t + \tau; t, yf, u)) d\tau \\
  &= f(t + s, Y(t + s; t, yf, u), V(t + s; t, yf, u)) \\
  &+ \int_s^0 [\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t + \tau, Y(t + \tau; t, yf, u), V(t + \tau; t, yf, u)) d\tau.
\end{align*}
\] (3.13)

For any \((t, yf, u) \in [\varepsilon_1, T] \times \gamma_+ \setminus \gamma^\varepsilon_+\) with \(\varepsilon_1 > 0\) to be determined later and for \(0 \leq s \leq \max\{t_b(yf, u), \varepsilon_1 - t\}\), we then get from (3.13) and (3.10) that
\[
\begin{align*}
  \min\{h\varepsilon, \varepsilon_1\} \int_{\varepsilon_1}^T \int_{u-n(yf)>0} |f(t, yf, u)||u_y| dt du dt \\
  &\leq \int_{\varepsilon_1}^T \int_{u-n(yf)>0} \int_{\max\{-t_b(yf, u), -t\}}^0 |f(t + s, Y(t + s; t, yf, u), V(t + s; t, yf, u))||u_y| dtds du \\
  &+ \int_{\varepsilon_1}^T \int_{\max\{-t_b(yf, u), -t\}}^0 \int_{u-n(yf)>0} \int_s^0 |[\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t + \tau, Y(t + \tau; t, yf, u), V(t + \tau))||u_y| d\tau dt du dt \\
  &\leq \int_0^T \int_{u-n(yf)>0} \int_{\max\{-t_b(yf, u), -t\}}^0 |f(t + s, Y(t + s; t, yf, u), V(t + s; t, yf, u))||u_y| dtds du \\
  &+ \int_0^T \int_{\max\{-t_b(yf, u), -t\}}^0 \int_{u-n(yf)>0} \int_s^0 |[\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t + \tau, Y(t + \tau; t, yf, u), V(t + \tau))||u_y| d\tau dt du dt \\
  &\leq \int_0^T \int_{-h}^h \int_{\mathbb{R}^3} |f(t, y, u)| dtdy du \\
  &+ \int_0^T \int_{\max\{-t_b(yf, u), -t\}}^0 \int_{u-n(yf)>0} \int_s^0 |[\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t + \tau, Y(t + \tau; t, yf, u), V(t + \tau))||u_y| d\tau dt du dt,
\end{align*}
\] (3.14)

where we have used the fact that \(h\varepsilon \leq t_b(yf, u) \leq \frac{h}{2}\) due to \((yf, u) \in \gamma_+ \setminus \gamma^\varepsilon_+\).

Next, applying Fubini’s Theorem and using (3.10) again, one also has
\[
\begin{align*}
  \int_0^T \int_{u-n(yf)>0} \int_{\max\{-t_b(yf, u), -t\}}^0 \int_s^0 |[\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t + \tau, Y(t + \tau; t, yf, u), V(t + \tau))||u_y| d\tau dt du dt ds \\
  &= \int_0^T dt \int_{u-n(yf)>0} \int_{\max\{-t_b(yf, u), -t\}}^0 ds \int_{\max\{-t_b(yf, u), -t\}}^0 d\tau |[\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t + \tau)||u_y| \\
  &\leq \max\{h\varepsilon, \varepsilon_1\} \int_0^T dt \int_{u-n(yf)>0} \int_{\max\{-t_b(yf, u), -t\}}^0 d\tau |[\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t + \tau)||u_y| \\
  &\leq \max\{h\varepsilon, \varepsilon_1\} \int_0^T dt \int_{-h}^h dy \int_{\mathbb{R}^2} du |[\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}] f(t, y, u)|.
\end{align*}
\] (3.15)

Once (3.14) and (3.15) are obtained, it remains now to compute
\[
\int_0^{\varepsilon_1} \int_{u-n(yf)>0} |f(t, yf, u)||u_y| dt du dt.
\]

In fact, if we choose \(\varepsilon_1\) to be small enough so that \(\varepsilon_1 \leq h\varepsilon\), at this stage, the backward trajectory hits the initial plane first. Therefore, for \((t, yf, u) \in [0, \varepsilon_1] \times \gamma_+ \setminus \gamma^\varepsilon_+\), by directly using (3.7) and
applying (3.10) once again, it follows
\[
\int_0^{\varepsilon_1} \int_{u-n(y_f)>0} |f(t, y_f, u)||u_y|dudt
\leq \int_0^{\varepsilon_1} \int_{u-n(y_f)>0} |f(0, Y(t, y_f, u), V(t, y_f, u))||u_y|dudt
+ \int_0^{\varepsilon_1} \int_{u-n(y_f)>0} \int_{-\tau}^0 |\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}|f(t + \tau)||u_y|d\tau dudt
\leq C \int_{-h}^h \int_{\mathbb{R}^3} |f(0, y, u)|dydu + C \int_0^{\varepsilon_1} \int_{-h}^h \int_{\mathbb{R}^3} |\partial_t + u_y \partial_y - \alpha u_y \partial_{u_y}|f(t)|dydudt.
\]

The proof of Lemma 3.1 is then completed. 

4. Steady problem: the first order correction

In this and the next sections, we are going to show Theorem 1.1 for the existence of solutions to the steady problem (1.8). Recall (1.9) and (1.10). For the purpose, we will first study in this section the first order correction term \( G_1 \) determined by the boundary value problem (1.11) and (1.16). Notice that (1.15) and (1.12) are satisfied. Existence of the remainder \( G_R \) for the boundary value problem (1.13) and (1.14) will be considered in the next section. Indeed, we have the following proposition.

**Proposition 4.1.** The boundary value problem (1.11) and (1.16) admits a unique solution \( G_1 = G_1(y, v) \) satisfying
\[
G_1(-v_x) = -G_1(v_x), \quad \int_{-1}^1 \int_{\mathbb{R}^3} G_1(y, v)dvdy = 0, 
\]
and
\[
\|w_q \partial v_x^m G_1\|_{L^\infty} \leq \tilde{C}_1, 
\]
for any integers \( m \geq 0 \) and \( q \geq 0 \), where \( \tilde{C}_1 > 0 \) is a constant depending only on \( m \) and \( q \).

To prove this proposition, let \( 0 < \epsilon < 1 \) and \( 0 \leq \sigma \leq 1 \), then we consider the following general approximation equations
\[
\epsilon G_1 + v_y \partial_y G_1 + v_0 G_1 = \sigma KG_1 + \tilde{\mathcal{F}}, 
\]
and
\[
G_1(\pm 1, v)|_{v_y \leq 0} = 0, 
\]
where the source term \( \tilde{\mathcal{F}} = \tilde{\mathcal{F}}(y, v) \) is given and satisfies \( \tilde{\mathcal{F}}(-v_x) = -\tilde{\mathcal{F}}(v_x) \). Recall that \( v_0 \) and \( K \) are defined by (2.3). The above boundary value problem can be formally reduced to
\[
v_y \partial_y G_1 + LG_1 = \tilde{\mathcal{F}},
\]
and
\[
G_1(\pm 1, v)|_{v_y \leq 0} = 0,
\]
as \( \sigma \to 1^- \) and \( \epsilon \to 0^+ \). To prove this rigorously, we deduce the following a priori estimate.

**Lemma 4.1** (a priori estimate). The solution to the boundary value problem (4.3) and (4.4) satisfies the following uniform estimate with respect to both \( \sigma \) and \( \epsilon \):
\[
\sum_{0 \leq m \leq N_0} \|w_q \partial v_x^m G_1\|_{L^\infty} \leq \mathcal{C}_0 \sum_{0 \leq m \leq N_0} \|w_q \partial v_x^m \tilde{\mathcal{F}}\|_{L^\infty}, 
\]
where \( N_0 \) is an arbitrary non-negative integer and the constant \( \mathcal{C}_0 > 0 \) is independent of \( \epsilon \) and \( \sigma \).
Proof. The proof of (4.5) is divided into two steps.

$L^\infty$ estimates. Let $\mathcal{G}_m = w_q \partial^{m}_{v_{x}} G_1$ for $m \geq 0$ and $q \geq 0$, then $\mathcal{G}_m$ satisfies

$$
\mathcal{G}_m = w_q \partial^{m}_{v_{x}} G_1 + \sigma 1_{m > 0} \sum_{m' < m} C^{m'}_m w_q (\partial^{m-m'}_{v_{x}} K)(\partial^{m'}_{v_{x}} G_1) - w_q \partial^{m}_{v_{x}} \mathcal{F},
$$

(4.6)

and

$$
\mathcal{G}_m(\pm 1, v)|_{v_{x} \leq 0} = 0.
$$

(4.7)

We write the solution of (4.6) and (4.7) in the following mild form

$$
\mathcal{G}_m(y, v) = \sigma \int_{-1}^{y} e^{-\frac{m+1}{Q}(y-y')} \frac{w_q}{v_y} K(w_q \mathcal{G}_m)(y')dy'
$$

$$
+ \sigma 1_{m > 0} \sum_{m' < m} C^{m'}_m \int_{-1}^{y} e^{-\frac{m+1}{Q}(y-y')} \frac{w_q}{v_y} (\partial^{m-m'}_{v_{x}} K)(\partial^{m'}_{v_{x}} G_1)(y')dy'
$$

$$
- \int_{-1}^{y} e^{-\frac{m+1}{Q}(y-y')} \frac{w_q}{v_y} \partial^{m}_{v_{x}} \mathcal{F} dy' := \sum_{i=1}^{6} \mathcal{I}_i, \quad \text{for } v_y > 0,
$$

(4.8)

and

$$
\mathcal{G}_m(y, v) = -\sigma \int_{y}^{1} e^{-\frac{m+1}{Q}(y-y')} \frac{w_q}{v_y} K(w_q \mathcal{G}_m)(y')dy'
$$

$$
- \sigma 1_{m > 0} \sum_{m' < m} C^{m'}_m \int_{y}^{1} e^{-\frac{m+1}{Q}(y-y')} \frac{w_q}{v_y} (\partial^{m-m'}_{v_{x}} K)(\partial^{m'}_{v_{x}} G_1)(y')dy'
$$

$$
+ \int_{y}^{1} e^{-\frac{m+1}{Q}(y-y')} \frac{w_q}{v_y} \partial^{m}_{v_{x}} \mathcal{F} dy' := \sum_{i=1}^{6} \mathcal{I}_i, \quad \text{for } v_y < 0.
$$

(4.9)

We next compute $\mathcal{I}_i$ ($1 \leq i \leq 6$) term by term. Since

$$
\begin{cases}
1_{v_y > 0} \int_{-1}^{y} e^{-\frac{m+1}{Q}(y-y')} v_y^{-1} dy' \leq \frac{1}{\nu_0 + \epsilon}(1 - e^{-\frac{2(\nu_0+\epsilon)}{Q^2}}) < \frac{1}{\nu_0 + \epsilon}, \\
1_{v_y < 0} \int_{y}^{1} e^{-\frac{m+1}{Q}(y-y')} v_y^{-1} dy' \leq \frac{1}{\nu_0 + \epsilon},
\end{cases}
$$

(4.9)

we see that

$$
||\mathcal{I}_3||, ||\mathcal{I}_6|| \leq C ||w_q \partial^{m}_{v_{x}} \mathcal{F}||_{L^{\infty}}.
$$

In view of definition (2.3) and Lemma 2.2, it follows

$$
||\mathcal{I}_2||, ||\mathcal{I}_5|| \leq C \sum_{m > 0} ||w_q (\partial^{m-m'}_{v_{x}} K)(\partial^{m'}_{v_{x}} G_1)||_{L^{\infty}} \leq C \sum_{m > 0} ||w_q \partial^{m}_{v_{x}} G_1||_{L^{\infty}}.
$$

Consequently, we have

$$
||\mathcal{G}_m(y, v)|| \leq \mathbf{1}_{v_y > 0} \int_{-1}^{y} e^{-\frac{m+1}{Q}(y-y')} v_y^{-1} \int_{\mathbb{R}^3} k_{w}(v, v')(||\mathcal{G}_m(v', y')||)dv'dy'
$$

$$
+ \mathbf{1}_{v_y < 0} \int_{y}^{1} e^{-\frac{m+1}{Q}(y-y')} v_y^{-1} \int_{\mathbb{R}^3} k_{w}(v, v')(||\mathcal{G}_m(v', y')||)dv'dy'
$$

$$
+ C \sum_{m > 0} ||w_q \partial^{m}_{v_{x}} G_1||_{L^{\infty}} + C ||w_q \partial^{m}_{v_{x}} \mathcal{F}||_{L^{\infty}},
$$

(4.10)

where $k_{w}$ is given in Lemma 2.1. Then we iterate (4.10) once more to obtain

$$
||\mathcal{G}_m(y, v)|| \leq \sum_{i=1}^{6} \mathcal{I}_{1,i},
$$

(4.11)
In this case, we have either

\begin{equation}
J_{1,1} = 1_{v_y > 0}\sigma^2 \int_{y'} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y^{-1} \int_{R^3} k_w(v, v') 1_{v_y' > 0} \int_{y'} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y'^{-1} \times \int_{R^3} k_w(v', v'')|\mathcal{G}_m(v'', y'')|dv'' dy'' dv' dy',
\end{equation}

\begin{equation}
J_{1,2} = 1_{v_y < 0}\sigma^2 \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y^{-1} \int_{R^3} k_w(v, v') 1_{v_y' < 0} \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y'^{-1} \times \int_{R^3} k_w(v', v'')|\mathcal{G}_m(v'', y'')|dv'' dy'' dv' dy',
\end{equation}

\begin{equation}
J_{1,3} = 1_{v_y < 0}\sigma^2 \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y^{-1} \int_{R^3} k_w(v, v') 1_{v_y' > 0} \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y'^{-1} \times \int_{R^3} k_w(v', v'')|\mathcal{G}_m(v'', y'')|dv'' dy'' dv' dy',
\end{equation}

\begin{equation}
J_{1,4} = 1_{v_y < 0}\sigma^2 \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y^{-1} \int_{R^3} k_w(v, v') 1_{v_y' < 0} \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y'^{-1} \times \int_{R^3} k_w(v', v'')|\mathcal{G}_m(v'', y'')|dv'' dy'' dv' dy',
\end{equation}

\begin{equation}
J_{1,5} = 1_{v_y > 0}\sigma \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y^{-1} \int_{R^3} k_w(v, v') \left(C 1_{m>0} \sum_{m' < m} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathcal{F}\|_{L^\infty}\right) dv' dy',
\end{equation}

\begin{equation}
J_{1,6} = 1_{v_y < 0}\sigma \int_{y} e^{-\frac{\nu v_y}{v_y}(y-y')} v_y^{-1} \int_{R^3} k_w(v, v') \left(C 1_{m>0} \sum_{m' < m} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathcal{F}\|_{L^\infty}\right) dv' dy'.
\end{equation}

By using (4.9) and Lemma 2.1, we see that the last two terms can be bounded as

\begin{equation}
|J_{1,5}|, |J_{1,6}| \leq C 1_{m>0} \sum_{m' < m} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathcal{F}\|_{L^\infty} = C 1_{m>0} \sum_{m' < m} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathcal{F}\|_{L^\infty}.
\end{equation}

For the other four terms, we only compute \(J_{1,2}\) because the other three terms can be treated similarly. The estimates are divided into three cases. First of all, we take \(M > 0\) large enough.

**Case 1.** \(|v| > M\). In this case, Lemma 2.1 and (4.9) directly give

\begin{equation}
J_{1,2} \leq \frac{C}{1 + M\|\mathcal{G}_m\|_{L^\infty}}.
\end{equation}

**Case 2.** \(|v| \leq M\) and \(|v'| > 2M\), or \(|v'| \leq 2M\) and \(|v''| > 3M\). In this case, we have either \(|v - v'| > M\) or \(|v' - v''| > M\) so that one of the following two estimates holds correspondingly

\begin{equation}
k_w(v, v') \leq C e^{-\frac{\nu v}{v'}^2} k_w(v, v') e^{\frac{\nu v}{v'}^2}, \quad k_w(v', v'') \leq C e^{-\frac{\nu v}{v''}^2} k_w(v', v'') e^{\frac{\nu v}{v''}^2}.
\end{equation}

This together with Lemma 2.1 and (4.9) gives

\begin{equation}
J_{1,2} \leq C e^{-\frac{\nu v^2}{v'}^2} \|\mathcal{G}_m\|_{L^\infty}.
\end{equation}

**Case 3.** \(|v| \leq M\), \(|v'| \leq 2M\) and \(|v''| \leq 3M\). In this situation, we make use of the boundedness of the operator \(K\) on the complement of a singular set. For any large \(N > 0\), we choose a number \(M(N)\) to define

\begin{equation}
k_{w,M}(v, v') \equiv 1_{|v - v'| \geq \frac{1}{N}, |v'| \leq 2M} k_w(v, v'), k_{w,M}(v', v'') \equiv 1_{|v' - v''| \geq \frac{1}{N}, |v''| \leq 3M} k_w(v', v''),
\end{equation}

such that

\begin{equation}
\sup_v \int_{R^3} |k_{w,M}(v, v') - k_w(v, v')|dv' \leq \frac{1}{N}.
\end{equation}
and
\[ \sup_{v'} \int_{R^3} \left| k_{w,M}(v', v'') - k_w(v', v'') \right| dv'' \leq \frac{1}{N}. \]

Moreover, note that \( k_{w,M}(v, v'), k_{w,M}(v', v'') \leq C_M \). We further rewrite
\[ k_w(v, v')k_w(v', v'') = [k_w(v, v') - k_{w,M}(v, v')]k_w(v', v'') + k_{w,M}(v, v')k_{w,M}(v, v'). \]

The first two difference terms lead to the small contribution of \( \mathcal{I}_{1,2} \) as
\[ \frac{C}{N} \| \mathcal{G}_m \|_{L^\infty}. \]

For the last term, we use the following decomposition
\[ \mathbf{1}_{v_y > 0} \sigma^2 \int_1^y e^{-\frac{\alpha_0}{\alpha_0} (y-y')} v_y^{-1} \int_{|v'| \leq 2M, |v''| \leq 3M} k_{w,M}(v, v')k_{w,M}(v', v'') \times \mathbf{1}_{v_y < 0} \left[ \int_{y'}^{y+\eta_0} + \int_{y'}^{y+\eta_0} \right] e^{-\frac{\alpha_0}{\alpha_0} (y-y')} |v_y|^{-1} \mathcal{G}_m(v'', y'') dv'' dy'' dy'. \]

where \( \eta_0 > 0 \) is suitably small. For \( \mathcal{I}_{1,2} \), since \( y'' - y' \geq \eta_0 \), it follows that
\[ \mathbf{1}_{v_y < 0} e^{-\frac{\alpha_0}{\alpha_0} (y-y')} |v_y|^{-1} \leq \frac{C}{\eta_0}, \]

which together with Lemma 2.1 as well as (4.9) implies
\[ \mathcal{I}_{1,2} \leq \left\{ \int_{|v''| \leq 3M} \int_{-1}^1 \left| \partial_{v_y}^m G_1(v', y'') \right|^2 dv'' dy'' \right\}^{\frac{1}{2}}. \]

As to \( \mathcal{I}_{1,2} \), since \( y'' - y' \leq \eta_0 \), we have that for \( \beta \in (0, 1) \),
\[ \int_{|v'| \leq 2M} \mathbf{1}_{v_y < 0} \int_{y'}^{y+\eta_0} e^{-\frac{\alpha_0}{\alpha_0} (y-y')} |v_y|^{-1} dy'' dv' \]
\[ = \int_{|v'| \leq 2M} \mathbf{1}_{v_y < 0} \int_{y'}^{y+\eta_0} e^{-\frac{\alpha_0}{\alpha_0} (y-y')} \left| \frac{y' - y''}{v_y} \right|^{\beta} |y' - y''|^{1-\beta} dy'' dv' \]
\[ \leq C \int_{|v'| \leq 2M} |v_y|^{-1+\beta} dv' \int_{y'}^{y+\eta_0} |y' - y''|^{1-\beta} dy'' \leq C_M \eta_0^{1-\beta}, \]

where we have used the fact that
\[ e^{-\frac{\alpha_0}{\alpha_0} |y-y'|} \left| \frac{y' - y''}{v_y} \right|^{\beta} < +\infty. \]

Plugging (4.13) into \( \mathcal{I}_{1,2} \), we get
\[ \mathcal{I}_{1,2} \leq C_M \eta_0^{1-\beta} \| \mathcal{G}_m \|_{L^\infty}. \]

As a consequence, one has
\[ \mathcal{I}_{1,2} \leq \left\{ \frac{C}{N} + C_M \eta_0^{1-\beta} + C e^{-\frac{\alpha_0^2}{\alpha_0}} \right\} \| \mathcal{G}_m \|_{L^\infty} + C_M \| \partial_{v_y}^m G_1 \|. \]

Substituting the above estimates into (4.11), we conclude
\[ \| \mathcal{G}_m \|_{L^\infty} \leq C_{1_m > 0} \sum_{m < m} \| w_q \partial_{v_y}^m G_1 \|_{L^\infty} + C \| \partial_{v_y}^m G_1 \| + C \| w_q \partial_{v_y}^m \mathcal{F} \|_{L^\infty}. \]

(4.14)

A linear combination of (4.14) from \( m = 0 \) to \( m = N_0 \) gives the following a priori estimate
\[ \sum_{0 \leq m \leq N_0} \| \mathcal{G}_m \|_{L^\infty} \leq C \sum_{0 \leq m \leq N_0} \| \partial_{v_y}^m G_1 \| + C \sum_{0 \leq m \leq N_0} \| w_q \partial_{v_y}^m \mathcal{F} \|_{L^\infty}, \]

where \( C > 0 \) depends on \( N_0 \) and \( q \). This concludes the \( L^\infty \) estimate.
$L^2$ estimates. To close the $L^\infty$ estimate (4.15), we need to derive the $L^2$ estimate for $G_1$. For this, we first consider the zero-th order $L^2$ estimate $|G_1|$. Notice that $G_1 = P_0 G_1 + P_1 G_1$ and $P_0 G_1 = [a_1 + b_1 \cdot v + c_1 (|v|^2 - 3)] \sqrt{\nu}$ with $b_1 = [b_{1,1}, b_{1,2}, b_{1,3}]$. Moreover, it holds

$$a_1 = \langle G_1, \sqrt{\nu} \rangle, \quad b_1 = \langle G_1, v \sqrt{\nu} \rangle, \quad c_1 = \frac{1}{6} \langle G_1, |v|^2 \sqrt{\nu} \rangle.$$ 

On the other hand, from (4.8) with $m = 0$, it holds $G_1(y, -v_x, v_y, v_z) = -G_1(y, v_x, v_y, v_z)$, namely $G_1$ is odd in $v_x$. This implies

$$a_1 = b_{1,2} = b_{1,3} = c_1 = 0. \quad (4.16)$$

To obtain the $L^2$ estimate of the macroscopic component, it remains now to deduce the $L^2$ estimate of $b_{1,1}$. Actually, one can show that

$$\|b_{1,1}\|^2 \leq C \|P_1 G_1\|^2 + C \int_{v_y \geq 0} |v_y| G_1^2(\pm 1) dv + C \|w_q \mathcal{F}\|_{L^\infty}, \quad (4.17)$$

where $C > 0$ is a constant independent of $\epsilon$ and $\sigma$. For this, we define

$$\Psi = \Psi_{b,1} = v_y v_x \frac{d}{dy} \phi_{b,1}(y) \sqrt{\nu},$$

where

$$-\phi''_{b,1} = b_{1,1}, \quad \phi_{b,1}(\pm 1) = 0.$$ 

For the above boundary value problem on $b_{1,1}$, it follows

$$\|\phi_{b,1}\|_{H^2} \leq C \|b_{1,1}\|, \quad |\phi'_{b,1}(\pm 1)| \leq C \|b_{1,1}\|. \quad (4.18)$$

Taking the inner product of (4.3) and $\Psi_{b,1}$ over $(-1, 1) \times \mathbb{R}^3$, we get

$$\epsilon \langle G_1, \Psi_{b,1} \rangle - (v_y G_1, \partial_y \Psi_{b,1}) + \langle v_y G_1(1), \Psi_{b,1}(1) \rangle - \langle v_y G_1(-1), \Psi_{b,1}(-1) \rangle + (1 - \sigma) \nu_0 \langle G_1, \Psi_{b,1} \rangle + \sigma \langle LG_1, \Psi_{b,1} \rangle = \langle \mathcal{F}, \Psi_{b,1} \rangle. \quad (4.19)$$

We now compute the terms in (4.19) one by one. By Cauchy-Schwarz’s inequality and (4.18), one has

$$[\epsilon + (1 - \sigma) \nu_0] |\langle G_1, \Psi_{b,1} \rangle| \leq [\epsilon + (1 - \sigma) \nu_0] \|P_0 G_1, \Psi_{b,1}\| + 2 \nu \|P_1 G_1, \Psi_{b,1}\|$$

$$\leq \eta [\epsilon + (1 - \sigma) \nu_0] \|b_{1,1}\|^2 + C_{\eta} \|P_1 G_1\|^2,$$

$$-(v_y G_1, \partial_y \Psi_{b,1}) = - (v_y P_0 G_1, \partial_y \Psi_{b,1}) - (v_y P_1 G_1, \partial_y \Psi_{b,1}) \geq \|b_{1,1}\|^2 - \eta \|b_{1,1}\|^2 - C_{\eta} \|P_1 G_1\|^2,$$

$$|\langle \mathcal{F}, \Psi_{b,1} \rangle| \leq \eta \|b_{1,1}\|^2 + C_{\eta} \|w_q \mathcal{F}\|_{L^\infty}.$$

And by Lemma 2.2, it follows

$$\sigma \|LG_1, \Psi_{b,1}\| \leq \eta \|b_{1,1}\|^2 + C_{\eta} \|P_1 G_1\|^2.$$ 

For the boundary term, one has from (4.4) and (4.18) that

$$\langle v_y G_1(1), \Psi_{b,1}(1) \rangle - \langle v_y G_1(-1), \Psi_{b,1}(-1) \rangle$$

$$= \int_{v_y > 0} v_y G_1(1) \Psi_{b,1}(1) dv - \int_{v_y < 0} v_y G_1(-1) \Psi_{b,1}(-1) dv$$

$$\leq \eta \|b_{1,1}\|^2 + C_{\eta} \int_{v_y \geq 0} |v_y| G_1^2(\pm 1) dv.$$

Combining the above estimates for the terms in (4.19), we have (4.17).

We now deduce the $L^2$ estimate on the microscopic component $P_1 G_1$. Direct energy estimate on (4.3) gives

$$[\epsilon + (1 - \sigma) \nu_0] \|G_1\|^2 + \delta_0 \sigma \|P_1 G_1\|^2 + \frac{1}{2} \int_{v_y \geq 0} |v_y| G_1^2(\pm 1) dv \leq \eta \|G_1\|^2 + C_{\eta} \|w_q \mathcal{F}\|_{L^\infty}^2. \quad (4.20)$$
Thus, (4.17) and (4.20) as well as (4.16) yield
\[ \|G_1\|^2 + \int_{v_y \geq 0} |v_y|G_1^2(\pm 1)dv \leq C\|w_y \tilde{\mathcal{F}}\|_{L^\infty}. \] (4.21)

Furthermore, for the higher order $L^2$ estimates on $G_1$, we have from $(\partial_{v_x}^m G_1, \partial_{v_x}^m (4.3))$ with $m \geq 1$ that
\[ \|\epsilon + (1 - \sigma)\nu_0\|\partial_{v_x}^m G_1\|^2 + \delta_0 \sigma\|\partial_{v_x}^m G_1\|^2 \]
\[ + \frac{1}{2} \int_{v_y \geq 0} |v_y|\partial_{v_x}^m G_1^2(\pm 1)dv \leq C\|G_1\|^2 + C\|w_y \partial_{v_x}^m \tilde{\mathcal{F}}\|_{L^\infty}, \] (4.22)

where Lemma 2.3 has been used for $\sigma(\partial_{v_x}^m L G_1, \partial_{v_x}^m G_1)$.

Finally, the a priori estimate (4.5) follows from (4.15), (4.21) and (4.22). This completes the proof of Lemma 4.1.

With the a priori estimate (4.5), we now prove the following existence result for general linear equations (4.3) and (4.4). Before doing this, we first define the following function space
\[ \mathcal{X}_{N_0} = \{g = g(y, v) | \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m g\|_{L^\infty} < +\infty, \ g(-v_x) = -g(v_x)\}, \]
associated with the norm
\[ \|g\|_{\mathcal{X}_{N_0}} = \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m g\|_{L^\infty}. \]

And for convenience, we also define a linear operator $\mathcal{L}_\sigma$ by
\[ \mathcal{L}_\sigma g = [\epsilon + v_y \partial_y + v_0 - \sigma K]g. \]

**Lemma 4.2.** Assume $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(y, v)$ satisfies
\[ \tilde{\mathcal{F}}(-v_x) = -\tilde{\mathcal{F}}(v_x), \ \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m \tilde{\mathcal{F}}\|_{L^\infty} < +\infty, \] (4.23)
then there exists a unique solution $G_1 = G_1(y, v)$ to (4.3) and (4.4) with $\sigma = 1$ satisfying
\[ G_1(-v_x) = -G_1(v_x), \]
and
\[ \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} \leq \mathcal{C}_0 \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m \tilde{\mathcal{F}}\|_{L^\infty}, \] (4.24)

where $\mathcal{C}_0 > 0$ is a constant depending only on $N_0$ and $q$.

**Proof.** The proof is based on a bootstrap argument in the following three steps.

**Step 1. Existence for $\sigma = 0$.** If $\sigma = 0$, then (4.3) and (4.4) are reduced to
\[ \epsilon G_1 + v_y \partial_y G_1 + v_0 G_1 = \tilde{\mathcal{F}}, \]
and
\[ G_1(\pm 1, v)|_{v_y \geq 0} = 0, \]

which has a unique explicit solution
\[ G_1(y, v) = 1_{v_y > 0} \int_{-1}^{y} e^{-\frac{(y_0 + v_0 y')}{v_y}} v_y^{-1} \tilde{\mathcal{F}}(y', v)dy' + 1_{v_y < 0} \int_{y}^{1} e^{-\frac{(v_0 + y_0' + y - v_y')}{v_y}} v_y^{-1} \tilde{\mathcal{F}}(y', v)dy'. \] (4.25)

Moreover, one sees that $G_1(-v_x) = -G_1(v_x)$ according to (4.23), and a direct calculation implies
\[ \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m G_1\|_{L^\infty} \leq \mathcal{C}_0 \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x}^m \tilde{\mathcal{F}}\|_{L^\infty}. \]

**Step 2. Existence for any $\sigma \in [0, \sigma_*]$ with some $\sigma_* > 0$.** Suppose $\sigma \in (0, 1]$, we consider a more general equation
\[ \epsilon G_1 + v_y \partial_y G_1 + v_0 G_1 = \sigma K G_1 + \tilde{\mathcal{F}}, \] (4.26)
with
\[ G_1(\pm 1, v)|_{v=0} = 0. \]  

(4.27)

To solve this boundary value problem, we design the following approximation equations
\[ \epsilon G_1^{n+1} + v_y \partial_y G_1^{n+1} + \nu_0 G_1^{n+1} = \sigma KG_1^n + \tilde{\mathcal{F}}, \]

with
\[ G_1^{n+1}(\pm 1, v)|_{v=0} = 0, \]

starting from \( G_1^0 = 0 \). Once \( G_1^n \) is given, \( G_1^{n+1} \) is well defined by step 1 and satisfies the estimate
\[
\sum_{0 \leq m \leq N_1} \| w_q \partial_{v_x} G_1^{n+1} \|_{L^{\infty}} \leq C_0 \sum_{0 \leq m \leq N_1} \| w_q \partial_{v_x} KG_1^n \|_{L^{\infty}} + \mathcal{C}_0 \sum_{0 \leq m \leq N_0} \| w_q \partial_{v_x} \tilde{F} \|_{L^{\infty}},
\]

(4.28)

where \( \mathcal{C}_1 > 0 \) depends only on \( K \). If we choose \( \sigma_* > 0 \) such that \( \mathcal{C}_0 \mathcal{C}_1 \sigma_* \leq \frac{1}{2} \), then (4.28) implies
\[
\sum_{0 \leq m \leq N_0} \| w_q \partial_{v_x} G_1^n \|_{L^{\infty}} \leq 2 \mathcal{C}_0 \sum_{0 \leq m \leq N_0} \| w_q \partial_{v_x} \tilde{F} \|_{L^{\infty}},
\]

(4.29)

for any \( n \geq 0 \). Furthermore, one can also show that for \( \sigma \in [0, \sigma_*] \),
\[
\|[G_1^{n+1} - G_1^n]|_{X_N} \leq \mathcal{C}_0 \mathcal{C}_1 \sigma [G_1^n - G_1^{n-1}] \|_{X_N} \leq \frac{1}{2} \|[G_1^n - G_1^{n-1}]\|_{X_N},
\]

(4.30)

which implies that \( G_1^n \to G_1 \) strongly in \( X_N \). In addition, it is easy to see that \( G_1^{n+1}(v_x) = -G_1^{n+1}(-v_x) \) if \( G_1^n(v_x) = -G_1^n(-v_x) \) holds. Thus, for \( \sigma \in [0, \sigma_*] \), there exists a unique solution \( G_1 \in X_N \) to the problem (4.26) and (4.27). Actually, the \( a \) priori estimate (4.5) implies that we still have the bound
\[
\sum_{0 \leq m \leq N_0} \| w_q \partial_{v_x} G_1 \|_{L^{\infty}} \leq \mathcal{C}_0 \sum_{0 \leq m \leq N_0} \| w_q \partial_{v_x} \tilde{F} \|_{L^{\infty}}.
\]

In other words, it follows
\[
\|[\Sigma_1^{-1} \tilde{F}]|_{X_N} \leq \mathcal{C}_0 \| \tilde{F} \|_{X_N}.
\]

(4.31)

**Step 3. Existence for \( \sigma \in [0, 0.2 \sigma_*] \).** By using (4.31) and performing similar calculations as for obtaining (4.29) and (4.30), one can see that there exists a unique solution \( G_1 \in X_N \) to the lifted equation
\[
\epsilon G_1 + v_y \partial_y G_1 + \nu_0 G_1 - \sigma_* KG_1 = \sigma KG_1 + \tilde{F}, \quad G_1(\pm 1, v)|_{v=0} = 0,
\]

with \( \sigma \in [0, \sigma_*] \). Therefore, the solution mapping \( \Sigma_1^{-1} \) is also well-defined on \( X_N \) and the estimate (4.24) holds for \( \sigma = 2 \sigma_* \).

Finally, repeating the above procedure step by step, one can reach \( \sigma = 1 \) so that \( \Sigma_1^{-1} \) exists and (4.24) also follows simultaneously. This completes the proof of Lemma 4.2. \( \square \)

**Proof of Proposition 4.1:** By setting \( \tilde{F} = -v_x v_y \sqrt{\mu} \) in Lemma 4.2, we see that for any \( \epsilon > 0 \) there exists a unique solution \( G_1^\epsilon \in X_N \) to the boundary value problem
\[
\epsilon G_1^\epsilon + v_y \partial_y G_1^\epsilon + LG_1^\epsilon = -v_x v_y \sqrt{\mu}, \quad G_1^\epsilon(\pm 1, v)|_{v=0} = 0.
\]

Notice that \( G_1^\epsilon \) satisfies (4.1) and the estimate
\[
\|G_1^\epsilon\|_{X_N} \leq \hat{C}_1,
\]

where \( \hat{C}_1 > 0 \) is independent of \( \epsilon \). Furthermore, we define a positive sequence \( \{\epsilon_n\}_{n=1}^\infty \) such that \( |\epsilon_{n+1} - \epsilon_n| \leq 2^{-n} \), then \( \epsilon_n \to 0^+ \) as \( n \to +\infty \). We consider the following approximation equations
\[
\epsilon_n G_1^{\epsilon_n} + v_y \partial_y G_1^{\epsilon_n} + LG_1^{\epsilon_n} = -v_x v_y \sqrt{\mu},
\]

where \( \epsilon > 0 \) is independent of \( \epsilon \). Furthermore, we define a positive sequence \( \{\epsilon_n\}_{n=1}^\infty \) such that \( |\epsilon_{n+1} - \epsilon_n| \leq 2^{-n} \), then \( \epsilon_n \to 0^+ \) as \( n \to +\infty \). We consider the following approximation equations
\[
\epsilon_n G_1^{\epsilon_n} + v_y \partial_y G_1^{\epsilon_n} + LG_1^{\epsilon_n} = -v_x v_y \sqrt{\mu},
\]

where \( \epsilon > 0 \) is independent of \( \epsilon \). Furthermore, we define a positive sequence \( \{\epsilon_n\}_{n=1}^\infty \) such that \( |\epsilon_{n+1} - \epsilon_n| \leq 2^{-n} \), then \( \epsilon_n \to 0^+ \) as \( n \to +\infty \). We consider the following approximation equations
\[
\epsilon_n G_1^{\epsilon_n} + v_y \partial_y G_1^{\epsilon_n} + LG_1^{\epsilon_n} = -v_x v_y \sqrt{\mu},
\]
with
\[ G_1^{e_n}(\pm 1, v)|_{v_y \leq 0} = 0. \]
Then letting \( \tilde{\Theta}_{n+1} = G_1^{e_{n+1}} - G_1^{e_n} \), one sees that \( \tilde{\Theta}_{n+1} \) satisfies
\[
\epsilon^{n+1}\tilde{\Theta}_{n+1} + v_y \partial_y \tilde{\Theta}_{n+1} + LG\tilde{\Theta}_{n+1} = -(\epsilon^{n+1} - \epsilon^n)G_1^{e_n},
\]
with
\[ \tilde{\Theta}_{n+1}|_{v_y \leq 0} = 0. \]
Thanks to Lemma 4.2, it follows that
\[
\|\tilde{\Theta}_{n+1}\|_{X_{N_0}} \leq C_0|\epsilon^{n+1} - \epsilon^n|\|G_1^{e_n}\|_{X_{N_0}} \leq C|\epsilon^{n+1} - \epsilon^n|.
\]
This means that \( \{G_1^{e_n}\}_{n=1}^{\infty} \) is a Cauchy sequence in \( X_{N_0} \). Thus, letting \( n \to \infty \), the limit function denoted by \( G_1 \) is the unique solution of (1.11) and (1.16). Moreover, \( G_1 \) satisfies (4.1) and the bound (4.2). The proof of Proposition 4.1 is then completed. \( \square \)

5. Steady problem: remainder

Based on Proposition 4.1, one can further show the following existence result for the remainder \( G_R \). Recall the steady problem (1.8) as well as (1.9) and (1.10).

**Proposition 5.1.** The boundary value problem (1.13) and (1.14) admits a unique solution \( G_R = G_R(y, v) \) with \( \sqrt{\mu}G_R \) satisfying
\[
\int_{-1}^{1} \int_{\mathbb{R}^3} G_R(y, v) \, dv \, dy = 0.
\]
And there is an integer \( q_0 > 0 \) such that for any integer \( q \geq q_0 \), there is \( \alpha_0 = \alpha_0(q) > 0 \) depending on \( q \) such that for any \( \alpha \in (0, \alpha_0) \) and any integer \( m \geq 0 \), it holds that
\[
\|w_q \partial_v^m G_R\|_{L^\infty} \leq C, \]
where \( C > 0 \) is a constant depending only on \( m \) and \( q \) but independent of \( \alpha \).

5.1. Caflisch’s decomposition. To prove Proposition 5.1, we follow the strategy of the proof in [14] for treating the shear force term in the framework of perturbation. In fact, notice that there is a growth term \( \frac{1}{2}v_x v_y G_R \) in the equation (1.13). To treat this growth in velocity, the key point is to use the Caflisch’s decomposition [10] and an algebraic weighted estimate introduced originally by Arkeryd-Esposito-Pulvirenti [4]. For the purpose, we first decompose the remainder \( G_R \) as
\[
\sqrt{\mu}G_R = G_{R,1} + \sqrt{\mu}G_{R,2}, \tag{5.1}
\]
where \( G_{R,1} \) and \( G_{R,2} \) satisfy the following two boundary value problems, respectively,
\[
v_y \partial_y G_{R,1} - \alpha v_y \partial_v G_{R,1} + v_0 G_{R,1} = \chi_M K G_{R,1} - \frac{\alpha}{2} \sqrt{\mu} v_x v_y G_{R,2} - \frac{1}{2} v_y v_y G_{R,1} + \sqrt{\mu} v_x v_y G_{1} + \sqrt{\mu} v_y \partial_v G_{1} + \sqrt{\mu} G_{1} + Q(\sqrt{\mu} G_{1}, v_y G_{1}) + \alpha Q(\sqrt{\mu} G_{1}, \sqrt{\mu} G_{R}) + \alpha^2 Q(\sqrt{\mu} G_{R}, \sqrt{\mu} G_{R}), \ y \in (-1, 1), \ v \in \mathbb{R}^3, \tag{5.2}
\]
\[ G_{R,1}(\pm 1, v)|_{v_y \leq 0} = 0, \ v \in \mathbb{R}^3, \tag{5.3}\]
and
\[
v_y \partial_y G_{R,2} - \alpha v_y \partial_v G_{R,2} + LG_{R,2} = (1 - \chi_M) \mu^{-\frac{1}{2}} K G_{R,1}, \ y \in (-1, 1), \ v \in \mathbb{R}^3, \tag{5.4}\]
\[ G_{R,2}(\pm 1, v)|_{v_y \leq 0} = \sqrt{2\pi \mu} \int_{v_y \geq 0} \sqrt{\mu} G_R(\pm 1, v)|_{v_y} \, dv, \ v \in \mathbb{R}^3. \tag{5.5}\]
Here $\chi_M(v)$ is a non-negative smooth cutoff function such that
\[
\chi_M(v) = \begin{cases} 
1, & |v| \geq M + 1, \\
0, & |v| \leq M,
\end{cases}
\]
and $K$ is defined by (2.4). The existence of (5.2), (5.3), (5.4) and (5.5) can be constructed via the approximation sequence by iteratively solving the following system
\[
\begin{align*}
\epsilon G_{R,1}^{n+1} + v_y \partial_y G_{R,1}^{n+1} & - \alpha v_y \partial_{v_x} G_{R,1}^{n+1} + v_0 G_{R,1}^{n+1} \\
\quad = \chi_M K G_{R,1}^{n+1} - \frac{\alpha}{2} \sqrt{\pi} v_y v_y G_{R,1}^{n+1} - \frac{1}{2} \sqrt{\pi} v_y \partial_{v_x} G_{R,1}^{n+1} + Q(\sqrt{\mu} G_1, \sqrt{\mu} G_1) \\
& + \alpha(Q(\sqrt{\mu} G_1, G_1) + Q(\sqrt{\mu} G_1, \sqrt{\mu} G_1)) + \alpha^2 Q(\sqrt{\mu} G_1, \sqrt{\mu} G_1), \\ 
G_{R,1}^{n+1}(\pm 1, v)|_{v_y \leq 0} &= 0, \quad v \in \mathbb{R}^3,
\end{align*}
\]
and
\[
\begin{align*}
\epsilon G_{R,2}^{n+1} + v_y \partial_y G_{R,2}^{n+1} & - \alpha v_y \partial_{v_x} G_{R,2}^{n+1} + L G_{R,2}^{n+1} \\
\quad = (1 - \chi_M) \mu^{-\frac{1}{2}} K G_{R,1}^{n+1}, \quad y \in (-1,1), \quad v \in \mathbb{R}^3,
\end{align*}
\]
for a small parameter $\epsilon > 0$, where we have set $[G_0^{0,1}, G_0^{0,2}] = [0, 0]$ for $n = 0$.

The proof of Proposition 5.1 follows by three steps. First, similarly for treating the existence of $G_1$, we introduce a modified coupled boundary value problem with two parameters $\epsilon > 0$ and $0 \leq \sigma \leq 1$. This boundary value problem is directly solvable via the characteristic method in case of $\sigma = 0$ corresponding to the homogeneous inflow data, and we then lift the value of $\sigma$ from $\sigma = 0$ for the zero inflow data to $\sigma = 1$ for the full diffuse reflection boundary condition by a bootstrap argument. Second, we establish the limit $n \to +\infty$ for any fixed parameter $\epsilon > 0$. Third, we pass the limit $\epsilon \to 0^+$ to obtain the desired solution which satisfies (5.2), (5.3), (5.4) and (5.5). As a result, with the help of (5.1), we get the solution to the original boundary value problem (1.13) and (1.14).

5.2. A priori estimates with parameters $\epsilon$ and $\sigma$. Let us first show that $[G_{R,1}^{n+1}, G_{R,2}^{n+1}]$ is well-defined once $[G_{0,1}^{0,1}, G_{0,2}^{0,2}]$ is given. To do this, we apply the method of contraction mapping. We define the linear vector operator parameterized by $\sigma \in [0, 1]$ as follows:
\[
\mathcal{L}_\sigma = \begin{pmatrix} 
\mathcal{L}_\sigma^1 & \mathcal{L}_\sigma^2 
\end{pmatrix} = \begin{pmatrix} 
\mathcal{L}_\sigma^1 \mathcal{L}_\sigma^2 
\end{pmatrix} = [\mathcal{L}_\sigma^1, \mathcal{L}_\sigma^2][G_1, G_2],
\]
where
\[
\mathcal{L}_\sigma^1[G_1, G_2] = \begin{cases} 
\epsilon G_1 + v_y \partial_y G_1 - \alpha v_y \partial_{v_x} G_1 + v_0 G_1 - \sigma \chi_M K G_1 + \frac{\alpha v_y v_y}{2} \sqrt{\mu} G_2, \quad y \in (-1,1), \\
G_1(\pm 1, v) 1_{\{|v_y| \leq 0\}},
\end{cases}
\]
and
\[
\mathcal{L}_\sigma^2[G_1, G_2] = \begin{cases} 
\epsilon G_2 + v_y \partial_y G_2 - \alpha v_y \partial_{v_x} G_2 + v_0 G_2 - \sigma K G_2 - \sigma (1 - \chi_M) \mu^{-\frac{1}{2}} K G_1, \quad y \in (-1,1), \\
G_2(\pm 1, v) 1_{\{|v_y| \leq 0\}} - \sigma \sqrt{2 \pi \mu} \int_{v_y \geq 0} (G_1 + \sqrt{\mu} G_2)(\pm 1, v)|v_y| dv.
\end{cases}
\]
We then consider the solvability of the following coupled linear system
\[
\begin{cases} 
\mathcal{L}_\sigma^1[G_1, G_2] = F_1, \quad \mathcal{L}_\sigma^2[G_1, G_2] = F_2, \quad y \in (-1,1), \\
\mathcal{L}_\sigma^1[G_1, G_2] = 0, \quad \mathcal{L}_\sigma^2[G_1, G_2] = F_{2,b}, \quad y = \pm 1,
\end{cases}
\]
where $F_1$, $F_2$, and $F_{2,b}$ are given, and $\langle F_1, 1 \rangle + \langle F_2, \sqrt{\mu} \rangle = 0$. In the rest of the proof, for brevity, we denote
\[
\mathcal{F}_1 = \begin{cases} 
F_1, \quad y \in (-1,1), \\
0, \quad y = \pm 1,
\end{cases}
\]
and
\[
\mathcal{F}_2 = \begin{cases} 
F_2, \quad y \in (-1,1), \\
F_{2,b}(\pm 1, v), \quad y = \pm 1.
\end{cases}
\]
In what follows, we look for solutions to the system (5.10) in the Banach space
\[ X_{α,N_0} = \left\{ [G_1, G_2] \left| \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_y}^m [G_1, G_2]\|_{L^\infty} < +\infty \right. \right\}, \]  
(5.11)
supplemented with the norm
\[ \|G_1, G_2\|_{X_{α,N_0}} = \sum_{0 \leq m \leq N_0} \{ \|w_q \partial_{v_y}^m G_1\|_{L^\infty} + \|w_q \partial_{v_y}^m G_2\|_{L^\infty} \}. \]
(5.12)

Let us now deduce the a priori estimate for the parameterized linear system (5.10).

**Lemma 5.1 (a priori estimate).** Let \([G_1, G_2] \in X_{α,N_0}\) with \(α > 0\) and \(N_0 \geq 0\) be a solution to (5.10) with \(ε > 0\) suitably small and \(σ \in [0, 1]\), and let \((\mathcal{F}_1, \mathcal{F}_2) \in X_{α,N_0}\) with \(\langle \mathcal{F}_1, 1 \rangle + \langle \mathcal{F}_2, \sqrt{μ} \rangle = 0\). There is \(q_0 > 0\) such that for any \(q \geq q_0\) arbitrarily large, there are \(α_0 = α_0(q) > 0\) and large \(M = M(q) > 0\) such that for any \(0 < α < α_0\), the solution \([G_1, G_2]\) satisfies the following estimate
\[ \|G_1, G_2\|_{X_{α,N_0}} = \|L^{-1}_σ[\mathcal{F}_1, \mathcal{F}_2]\|_{X_{α,N_0}} \leq C_σ \sum_{0 \leq m \leq N_0} \{ \|w_q \partial_{v_y}^m \mathcal{F}_1\|_{L^\infty} + \|w_q \partial_{v_y}^m \mathcal{F}_2\|_{L^\infty} \}, \]  
(5.13)
where the constant \(C_σ > 0\) may depend on \(ε\) but not on \(σ\) and \(α\).

**Proof.** The proof is divided into two steps.

**Step 1.** \(L^\infty\) estimates. Let \(0 \leq m \leq N_0\) and \(q > 0\), we denote
\[ [H_{1,m}, H_{2,m}] = [w_q \partial_{v_y}^m G_1, w_q \partial_{v_y}^m G_2], \]
then we see that \([H_{1,m}, H_{2,m}]\) satisfies
\[ εH_{1,m} + v_y \partial_y H_{1,m} - αv_y \partial_x H_{1,m} + 2qα \frac{v_y v_x}{1 + |v|^2} H_{1,m} + ν_0 H_{1,m} - σχ_M w_q K \left( \frac{H_{1,m}}{w_q} \right) \]
\[ - σ1_{m>0} \sum_{1 \leq m' \leq m} C_{m'} m_w \partial_{v_y}^{m'} (χ_M K) \partial_{v_y}^{m-m'} G_1 \]
\[ + α \sum_{0 \leq m' \leq m} C_{m'} m_w \partial_{v_y}^{m'} \left( \frac{v_x v_y}{2} \sqrt{μ} \right) \partial_{v_y}^{m-m'} G_2 = w_q \partial_{v_y}^m \mathcal{F}_1, \]
(5.14)
and
\[ εH_{2,m} + v_y \partial_y H_{2,m} - αv_y \partial_x H_{2,m} + 2qα \frac{v_y v_x}{1 + |v|^2} H_{2,m} + ν_0 H_{2,m} - σw_q K \left( \frac{H_{2,m}}{w_q} \right) \]
\[ - σ1_{m>0} \sum_{1 \leq m' \leq m} C_{m'} m_w \partial_{v_y}^{m'} K \partial_{v_y}^{m-m'} G_2 \]
\[ - σ \sum_{0 \leq m' \leq m} C_{m'} m_w \partial_{v_y}^{m'} \left( (1 - χ_M) μ^{-\frac{1}{2}} \right) \partial_{v_y}^{m-m'} K G_1 = w_q \partial_{v_y}^m \mathcal{F}_2, \]
(5.15)
\[ H_{2,m}(±1, v)_1_{\{v_y = 0\}} - σw_q \partial_{v_y} m \left( \sqrt{2πμ} \right) \int_{v_y > 0} (G_1 + \sqrt{μ} G_2)(±1, v)|v_y| dv = w_q \partial_{v_y} \mathcal{F}_{2, b}(±1). \]
(5.16)
Recall the trajectory \([Y(s; t, y, v), V(s; t, y, v)]\) defined as (3.4). In addition, for \((y, v) \in [-1, 1] × \mathbb{R}^3\), we define the backward exit time \(t_b(y, v)\) as
\[ t_b(y, v) = \inf \left\{ s : y - sv_y \notin (-1, 1), s > 0 \right\}, \]
(5.17)
which is the first time at which the backward characteristic line \([Y(s; t, y, v), V(s; t, y, v)]\) emerges from \((-1, 1)\). Note that at the boundary \(y = ±1\), \(t_b(y, v)\) is well-defined if \(±v_y > 0\). For any \((y, v)\), we use \(t_b(y, v)\) when it is well-defined. Furthermore, we denote \(y_b(y, v) = y - t_b(y, v)v_y \in \{-1, 1\}\), and for random variable \(v_k\), we define the backward time cycle
\[ (t_0, y_0, v_0) = (t, y, v), (t_{k+1}, y_{k+1}, v_{k+1}) = (t_k - t_b(y_k, v_k), y_b(y_k, v_k), v_{k+1}), k \geq 0. \]
(5.18)
We also set
\[ Y^{l}_e(s; t, y, v) = 1_{[t_{i+1}, t_i)}(s)\{y_l + (s - t_i)v_{l_y}\}, \]
\[ V^{l}_e(s; t, y, v) = 1_{[t_{i+1}, t_i)}(s)(v_{lx} + \alpha(t_l - s)v_{l_y}, v_y, v_{lz}). \]

Note that \([Y^{0}_e(s), V^{0}_e(s)] = [Y(s), V(s)]\) and \(y_l = \pm 1\) for \(l \geq 1\). Moreover, \(t_l\) can be negative.

Define \(V_j = \{v_j \in \mathbb{R}^3 \mid v_j \cdot n(y_j) > 0\}\), where \(n(y_j) = (0, 1, 0)\) if \(y_j = 1\) and \(n(y_j) = (0, -1, 0)\) if \(y_j = -1\).

Let the iterated integral for \(k \geq 2\) be defined as
\[
\int_{V_{k-1}} \int_{V_{k-2}} \cdots \int_{V_1} \prod_{i=1}^{k-1} d\sigma_l = \int_{V_{k-1}} \int_{V_{k-2}} \cdots \int_{V_1} d\sigma_{k-1} \int_{V_1} d\sigma_1, \tag{5.19}
\]

where \(d\sigma_l = \sqrt{2\pi \mu_l}|v_{l_y}| dv_l\) is a probability measure.

Without loss of generality, we assume \(\lim_{|t| \to +\infty} |H_{1, m}, H_{2, m}|(t) = 0\). Along the characteristic line (3.3), for \((y, v) \in [-1, 1] \times \mathbb{R}^3 \setminus (\gamma_- \cup \gamma_0)\), we write the solution of the system (5.13) and (5.14) in the mild form as follows:

\[ H_{1, m}(t) = H_{1, m}(y(t), v(t)) \]
\[ = \sigma \int_{t_1}^t e^{-\int_{t_1}^s \mathcal{A}'(r, V(r))dr} \left\{ \chi_M w_q \mathcal{K} \left( \frac{H_{1, m}}{w_q} \right) \right\} (Y(s), V(s)) ds \]
\[ + \sigma 1_{m > 0} \int_{t_1}^t e^{-\int_{t_1}^s \mathcal{A}'(r, V(r))dr} \sum_{1 \leq m' \leq m} C_{m'} \left\{ w_q \partial^{m'}_{v_x} (\chi_M \mathcal{K}) \partial^{m'-m}_{v_x} G_1 \right\} (Y(s), V(s)) ds \]
\[ - \alpha \int_{t_1}^t e^{-\int_{t_1}^s \mathcal{A}'(r, V(r))dr} \sum_{0 \leq m' \leq m} C_{m'} \left\{ w_q \partial^{m'}_{v_x} \left( \frac{v_x v_y}{2} \sqrt{\mu} \right) \partial^{m'-m}_{v_x} G_2 \right\} (Y(s), V(s)) ds \]
\[ + \int_{t_1}^t e^{-\int_{t_1}^s \mathcal{A}'(r, V(r))dr} (w_q \partial^{m}_{v_x} F_1) (Y(s), V(s)) ds, \]

where
\[ \mathcal{A}'(r, V(r)) = v_0 + \epsilon + 2q \alpha V_q(\nu_0) V_x(\tau) \geq \nu_0 / 2, \tag{5.20} \]

provided that \(\epsilon > 0\) and \(2q \alpha > 0\) are suitably small. By Lemma 2.4, it is straightforward to see

\[
\sup_{-\infty < t < +\infty} \|H_{1, m}(t)\|_{L^\infty} \leq C \frac{q}{\alpha} \sum_{m' \leq m} \sup_{-\infty < t < +\infty} \|H_{1, m'}(t)\|_{L^\infty} + C \alpha \sum_{m' \leq m} \sup_{-\infty < t < +\infty} \|H_{2, m'}(t)\|_{L^\infty} \]
\[ + \sup_{-\infty < t < +\infty} \|w_q \partial^{m}_{v_x} F_1(t)\|_{L^\infty}. \tag{5.21} \]

By taking \(q\) sufficiently large, (5.21) further gives

\[
\sum_{0 \leq m \leq N_0} \|H_{1, m}\|_{L^\infty} \leq C \alpha \sum_{0 \leq m \leq N_0} \|H_{2, m}\|_{L^\infty} + \sum_{0 \leq m \leq N_0} \|w_q \partial^{m}_{v_x} F_1\|_{L^\infty}. \tag{5.22} \]
Moreover, in the above expressions we have used the following notations

\[ (\gamma, \nu) \in [-1, 1] \times \mathbb{R}^3 \setminus (\gamma_- \cup \gamma_0), \text{ and for } k \geq 2, \]

\[ I_5 = \sigma^{k-1} \int_{t_1}^{t} e^{-\int_{t}^{t'} A'(\tau, V(\tau)) d\tau} \left[ w_q \mathcal{H}_2(t, y, V) \right] (Y(t_1)) \int_{t_i}^{t} \left( w_q \mathcal{G}_2(t, y, V_{cl}^{-1}(t)) \right) d\Sigma_{k-1}(t), \]

\[ I_6 = \sum_{l=2}^{k-1} \sigma^{l-1} \mathcal{W} \int_{\Pi_{j=1}^{l-1} \nu_j} \left( w_q \mathcal{F}_2(t, y, V_{cl}^{-1}(t)) \right) d\Sigma_{l}(t), \]

\[ I_7 = \sigma^{l} \sum_{l=1}^{k-1} \mathcal{W} \int_{\Pi_{j=1}^{l-1} \nu_j} \int_{t_{l+1}}^{t_l} w_q \mathcal{F}_2(Y_{cl}^{l}, V_{cl}^{l})(s) d\Sigma_{l}(s) ds, \]

\[ I_8 = \sigma^{l} \sum_{l=1}^{k-1} \mathcal{W} \int_{\Pi_{j=1}^{l-1} \nu_j} \int_{t_{l+1}}^{t_l} \left( w_q K \left( \frac{H_2,0}{w_q} \right) \right) (Y_{cl}^{l}, V_{cl}^{l})(s) d\Sigma_{l}(s) ds, \]

\[ I_9 = \sigma^{l} \sum_{l=1}^{k-1} \mathcal{W} \int_{\Pi_{j=1}^{l-1} \nu_j} \int_{t_{l+1}}^{t_l} \left( (1 - \chi_M) w_q \mu^{-\frac{3}{2}} \mathcal{K} \left( \frac{H_4,0}{w_q} \right) \right) (Y_{cl}^{l}, V_{cl}^{l})(s) d\Sigma_{l}(s) ds, \]

\[ I_{10} = \sigma^{l} \mathcal{W} \sum_{l=1}^{k-1} \int_{\Pi_{j=1}^{l-1} \nu_j} \left( \frac{w_q}{\sqrt{\mu}} \mathcal{G}_1 \right) (t_i, y_i, V_{cl}^{-1}(t_i)) d\Sigma_{l}(t_i). \]

Moreover, in the above expressions we have used the following notations

\[ \Sigma_{l}(s) = \prod_{j=l+1}^{k-1} ds_j e^{-\int_{t_j}^{t_{j+1}} A'(\tau, V_{cl}(\tau)) d\tau} \tilde{w}_2(v_{l}) ds_l \prod_{j=1}^{l-1} \frac{\tilde{w}_2(v_{j})}{\tilde{w}_2(V_{cl}(t_{j+1}))} \prod_{j=1}^{l-1} e^{-\int_{t_j}^{t_{j+1}} A'(\tau, V_{cl}(\tau)) d\tau} ds_j, \]

and

\[ \tilde{w}_2(v) = (\sqrt{2\pi} w_q \sqrt{\mu})^{-1}. \]
The $L^\infty$ estimates for $H_{2,m}$ is more complicated because $K$ has no smallness property. To overcome this, we have to iterate (5.23) twice. Let us first compute $I_n$ ($1 \leq n \leq 10$) term by term. Recalling the definition (2.5) for $k_w$, one directly has by (5.20)

$$|I_1| \leq \int_{t}^{t'} e^{-\frac{\nu}{q}(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v')|H_{2,m}(s, Y(s), v')| dv'ds.$$ 

By Lemma 2.2, it follows

$$|I_2| \leq C1_{m>0} \sum_{m' \leq m-1} \|w_q \partial_{v_z}^m G_2\|_{L^\infty} \int_{t}^{t'} e^{-\frac{\nu}{q}(t-s)} ds \leq C1_{m>0} \sum_{m' \leq m-1} \|w_q \partial_{v_z}^m G_2\|_{L^\infty},$$

and similarly,

$$|I_3| \leq C \sum_{m' \leq m} \|w_q \partial_{v_z}^m G_1\|_{L^\infty}.$$

It is straightforward to see

$$|I_4| \leq C\|w_q \partial_{v_z}^m F_2\|_{L^\infty} + C\|w_q \partial_{v_z}^m F_{2,b}\|_{L^\infty}.$$

Next, notice that

$$|W| \leq Cm!q!e^{-\frac{\nu(t-t_0)}{2}}.$$ 

In the sequel, for simplicity, we denote $C_{m,q}$ for the constant $m!q!$. By Lemma 8.1, it follows

$$|I_5| \leq CC_{m,q}2^{-C_2T_0^\frac{3}{2}}e^{-\frac{\nu}{q}(t-t_1)}\|H_{2,0}\|_{L^\infty}, \quad |I_6| + |I_7| \leq CkC_{m,q}e^{-\frac{\nu}{q}(t-t_1)} \{\|w_q F_2\|_{L^\infty} + \|w_q F_{2,b}\|_{L^\infty}\},$$

and

$$|I_9|, \quad |I_{10}| \leq CC_{m,q}k e^{-\frac{\nu}{q}(t-t_1)}\|H_{1,0}\|_{L^\infty},$$

where we have taken $T_0 = t - t_k$ with $k = C_1T_0^\frac{3}{2}$, and both $C_1 > 0$ and $C_2 > 0$ are given in Lemma 8.1.

Putting all the estimates for $I_n$ ($1 \leq n \leq 10$) above together and adjusting the constants, we have

$$|H_{2,m}(t)| \leq CC_{m,q}e^{-\frac{\nu}{q}(t-t_1)} \int_{t_1}^{t} e^{-\frac{\nu}{q}(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v')|H_{2,m}(s, Y(s; t, y, v), v')| dv'ds$$

$$+ CC_{m,q}e^{-\frac{\nu}{q}(t-t_1)} \sum_{l=1}^{k-1} \int_{t_1-l}^{t_1} \int_{\mathbb{R}^3} k_w(V_l(s), v')|H_{2,0}(s, Y_l(s; t, y, v), v')| dv'ds \, d\Sigma(l)ds,$$

where

$$Q(t) = C1_{m>0} \sum_{m' \leq m-1} \sup_{-\infty<s\leq t} \|H_{2,m'}(s)\|_{L^\infty} + C \sum_{m' \leq m} \sup_{-\infty<s\leq t} \|H_{1,m'}(s)\|_{L^\infty}$$

$$+ C \sup_{-\infty<s\leq t} \|w_q \partial_{v_z}^m F_2(s)\|_{L^\infty} + C \sup_{-\infty<s\leq t} \|w_q \partial_{v_z}^m F_{2,b}(s)\|_{L^\infty}$$

$$+ CC_{m,q}2^{-C_2T_0^\frac{3}{2}} \sup_{-\infty<s\leq t} \|H_{2,0}(s)\|_{L^\infty} + CC_{m,q}k \sup_{-\infty<s\leq t} \|H_{1,0}(s)\|_{L^\infty}$$

$$+ CC_{m,q}k \sup_{-\infty<s\leq t} \|w_q F_2(s)\|_{L^\infty} + CC_{m,q}k \sup_{-\infty<s\leq t} \|w_q F_{2,b}(s)\|_{L^\infty}.$$ 

Then let us define a new backward time cycle as

$$(t'_{l+1}, y'_{l+1}, v'_{l+1}) = (t'_l - t_b(y'_l, v'_l), y_b(y'_l, v'_l), v'_{l+1}),$$
and the starting point
\[(t_0', y_0', v_0') = (s, y', v') := (s, Y(s), v') or (s, Y_{cl}^t(s), v'),\]
for some \(s \in \mathbb{R}\) and \(l \in \mathbb{Z}^+\). Furthermore, for \(\ell \in \mathbb{Z}^+\), we also denote
\[
\begin{align*}
\bar{V}_{cl}^q(s'; s, y', v') &= 1_{[t_{\ell+1}, t_{\ell}]}(s') (y' + (s - t_{\ell}' \nu_{\ell})), \\
\tilde{V}_{cl}^q(s'; s, y', v') &= 1_{[t_{\ell+1}, t_{\ell}]}(s') (v' + \alpha(t_{\ell}' - s)\nu_{\ell} y', v_{\ell}', v_{\ell}').
\end{align*}
\]
To be consistent, we set \([\bar{Y}_{cl}^0(s'), \tilde{Y}_{cl}^0(s')] := [\bar{V}(s'), \tilde{V}(s')]\).

Iterating (5.26) again, one has
\[
|H_{2,m}(l)| \leq CC_{m,q}^2 \int_{t_1}^t e^{-2\varphi(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v') \int_{t_1}^s e^{-2\varphi(s-s')} \int_{\mathbb{R}^3} k_w(\bar{V}(s'; Y(s), v'), v'') \\
	imes |H_{2,m}(s', \bar{V}(s'; Y(s), v'), v'')| dv' ds' dv' ds \\
+ CC_{m,q}^2 \int_{t_1}^t e^{-2\varphi(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v') e^{-2\varphi(s-t_{\ell}')} \sum_{\ell=1}^{t_{\ell}'-1} \int_{\mathbb{R}^3} k_w(\bar{V}_{cl}^q(s'; s, y', v'), v'') \\
	imes |H_{2,0}(s', \bar{V}_{cl}^q(s'; Y_{cl}^t(s), v'), v'')| dv' ds' dv' ds + CC_{m,q}^2 \int_{t_1}^t e^{-2\varphi(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v') \mathcal{Q}(s) dv' ds \\
+ CC_{m,q}^2 \int_{t_1}^t e^{-2\varphi(t-s)} \int_{\mathbb{R}^3} k_w(\bar{V}_{cl}^q(s'; s, y', v'), \mathcal{Q}(s)) dv' ds,
\]
where according to Lemma 8.1 and Remark 8.1, we choose \(\iota \in \mathbb{Z}^+\) such that \(\iota \sim (\tilde{T}_0)^{\tilde{\eta}}\) with \(\tilde{T}_0 = s - t_{\ell}'\) being suitably large. We claim that
\[
\|H_{2,m}\|_{L^\infty} \leq \eta \{\|H_{2,m}\|_{L^\infty} + \|H_{2,0}\|_{L^\infty}\} + C(\tilde{T}_0) \left\{\|\partial_{x_v}^m \mathcal{G}_2\| + \|\mathcal{G}_2\|\right\} + C \sup_{s \leq t_{\ell}'} \mathcal{Q}(s),
\]
where \(\eta > 0\) is suitably small. To prove (5.28), we only estimate the fourth term on the right hand side of (5.27), because the other terms can be estimated similarly. For any sufficiently small \(\eta_0 > 0\), we first divide \([t_{\ell+1}', t_{\ell}']\) as \([t_{\ell+1}', t_{\ell}' - \eta_0] \cup (t_{\ell}' - \eta_0, t_{\ell}']\), then rewrite the fourth term on the right hand side of (5.27) as
\[
\begin{align*}
\mathcal{J} := CC_{m,q}^2 \sum_{l=1}^{k-1} \int_{t_{l+1}}^{t_l} \int_{\mathbb{R}^3} k_w(V_{cl}^t(s'; s, y', v')) e^{-2\varphi(s-t_{\ell}')} \sum_{\ell=1}^{t_{\ell}'-1} \int_{\mathbb{R}^3} k_w\left(\bar{V}_{cl}^q(s'; s, y', v'), \mathcal{Q}(s)\right) dv' ds' dv' ds + \mathcal{J}_1 + \mathcal{J}_2.
\end{align*}
\]
By Lemma 8.1, it is easy to see
\[
\mathcal{J}_2 \leq CC_{m,q}^2 \eta_0 \sup_{s \leq t_{\ell}'} \|H_{2,0}\|_{L^\infty}.
\]
For \(\mathcal{J}_1\), the computation is divided into the following three cases.
Case 1. \(|V_{cl}^l(s; v)| > M\) or \(|\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v')| > M\). In this case, by Lemma 2.1, it follows that

\[
\int_{\mathbb{R}^3} |k_w(V_{cl}^l(s; v), v')| dv' \leq \int_{\mathbb{R}^3} |k_w(\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v'), v'')| dv'' \leq \frac{C(q)}{1 + M},
\]

where \(C(q) > 0\) and depends on \(q\). Therefore, one has by using Lemma 8.1 again

\[
|J_1| \leq \frac{CC_{m,q}C_0,qC^2(q)}{1 + M} \|H_{2,0}\|_{L^\infty}.
\]

Note that here and in the sequel, \(l\) and \(\ell\) run over \([1, k - 1]\) and \([1, \ell - 1]\), respectively.

Case 2. \(|V_{cl}^l(s; v)| \leq M\) and \(|v'| > 2M\), or \(|\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v')| \leq M\) and \(|v''| > 2M\). In this regime, we have either \(|V_{cl}^l(s; v) - v'| > M\) or \(|\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v') - v''| > M\). Then either of the following two estimates holds correspondingly

\[
k_w(V_{cl}^l(s; v), v') \leq Ce^{-\frac{k_v^2}{4}} k_w(V_{cl}^l(s; v), v') e^{-\frac{|V_{cl}^l(s; v) - v'|^2}{4}},
\]

\[
k_w(\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v'), v'') \leq Ce^{-\frac{k_v^2}{4}} k_w(\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v'), v'') e^{-\frac{|\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v') - v''|^2}{4}}.
\]

This together with Lemma 2.1 imply

\[
|J_1| \leq \frac{CC_{m,q}C_0,qC^2(q)e^{-\frac{k_v^2}{4}}}{N} \|H_{2,0}\|_{L^\infty}.
\]

Case 3. \(|V_{cl}^l(s; v)| \leq M\), \(|v'| \leq 2M\), \(|\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v')| \leq M\) and \(|v''| \leq 2M\). The key point here is to convert the \(L^1\) integral with respect to the double \(v\) variables into the \(L^1\) norm with respect to the variables \(y\) and \(v\). To do so, for any large \(N > 0\), we choose a number \(M(N)\) to define \(k_{w,M}(u, v')\) as (4.12), then decompose

\[
k_w(V_{cl}^l, v') k_w(\bar{V}_{cl}^l, v'') = (k_w(V_{cl}^l, v') - k_{w,M}(V_{cl}^l, v')) k_w(\bar{V}_{cl}^l, v'') + (k_{w,M}(V_{cl}^l, v') k_{w,M}(V_{cl}^l, v') k_w(\bar{V}_{cl}^l, v'').
\]

From Lemma 8.1, the first two difference terms lead to a small contribution in \(J_1\) as

\[
\frac{CC_{m,q}C_0,qC^2(q)}{N} \|H_{2,0}\|_{L^\infty}.
\]

For the remaining main contribution of the bounded product \(k_{w,M}(V_{cl}^l, v') k_{w,M}(\bar{V}_{cl}^l, v'')\), we denote \(\tilde{y} = \bar{Y}_{cl}^l(s'; Y_{cl}^l(s), v'), \tilde{y} = y - (t' - s')v'_{ty}\) and apply a change of variable \(v'_{ty} \to \tilde{y}\). Then one has

\[
\left|\frac{\partial \tilde{y}}{\partial v'_{ty}}\right| = \left|\frac{\partial(y' - (t' - s')v'_{ty})}{\partial v'_{ty}}\right| = |t' - s'| \geq \eta_0.
\]

We now estimate this part as follows

\[
CC_{m,q}C_0,q \sum_{l=1}^{k-1} \int_{\mathbb{R}^3} \int_{|v'| \leq 2M} k_w,M(V_{cl}^l(s; v), v') e^{-\frac{k_v^2}{4}(s-t')_l} \int_{|v'_{ty}| \leq 2M} k_{w,M}(V_{cl}^l, v') \int_{|v''| \leq 2M} k_{w,M}(\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v'), v'') \int_{|v''| \leq 2M} k_{w,M}(\bar{V}_{cl}^l(s'; Y_{cl}^l(s), v'), v'') |d\Sigma(s') ds' d\Sigma(s) ds |
\]

\[
\leq C(M, m, q) \sup_{s, s' \leq t} \|G_2(s)\|_{L^\infty} \sup_{v, v'} \left\{\int_{t_k}^{t_1} e^{-\frac{m(t_j - s)}{2}} e^{-\frac{k_v^2}{4}(s-t')_l} \int_{t_k}^{t_1} e^{-\frac{m(t_j - s)}{2}} ds' ds \right\}
\]

\[
\leq C(M, m, q) \sup_{s, s' \leq t} \|G_2(s)\|_{L^\infty}.
\]
Putting all the estimates for \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) together, we now obtain

\[
\mathcal{J} \leq CC_{m,q}C_0\eta_0 t \|H_{2,0}\|_{L^\infty} + \frac{CC_{m,q}C_0C^2(q)}{1 + M} \|H_{2,0}\|_{L^\infty} + CC_{m,q}C_0C^2(q)e^{-\frac{\epsilon M^2}{2}} \|H_{2,0}\|_{L^\infty} + C \sum_{m,q} \|G_2(s)\|.
\]

As mentioned before, by performing the similar calculations for the other terms on the right hand side of (5.27), one has

\[
\|H_{2,m}\|_{L^\infty} \leq \frac{CC_{m,q}C_0C^2(q)}{1 + M} \|H_{2,m}\|_{L^\infty} + CC_{m,q}C_0C^2(q)e^{-\frac{\epsilon M^2}{2}} \|H_{2,m}\|_{L^\infty} + C \sum_{m,q} \|G_2(s)\| + C \sum_{s \leq t} \|\partial_{v_s}^m G_2(s)\| + C \sup_{s \leq t} Q(s).
\]

Since \( \epsilon \sim (T_0)^{\frac{1}{2}} \), by taking \( M \) and \( N \) large enough and \( \eta_0 = (T_0)^{-\frac{3}{4}} \), small enough, (5.28) further yields (5.29). Finally, taking a linear combination of (5.28) with \( m = 0, 1, \ldots, N_0 \), we conclude

\[
\sum_{0 \leq m \leq N_0} \|H_{2,m}\|_{L^\infty} \leq C(N_0, q, \tilde{T}_0) \sum_{0 \leq m \leq N_0} \|\partial_{v_s}^m G_2\| + C(N_0, q, T_0) \sum_{0 \leq m \leq N_0} \|H_{1,m}\|_{L^\infty} + C(N_0, q, T_0) \sum_{0 \leq m \leq N_0} \left\{ \|w_q \partial_{v_s}^m F_2\|_{L^\infty} + \|w_q \partial_{v_s}^m F_2, b\|_{L^\infty} \right\}.
\]

**Remark 5.1.** We point out that the estimates (5.22) and (5.30) obtained above are independent of \( \epsilon \). Moreover, both \( T_0 \) and \( T_0 \) are independent of \( t \), because starting from any \( t \in (-\infty, +\infty) \) we can trace back \( k \) some to some \( t_k \) which can be negative.

**Step 2.** \( L^2 \) estimate. To close the final estimate, we turn to obtain the \( L^2 \) estimate of \( \partial_{v_s}^m G_2 \) with \( 0 \leq m \leq N_0 \). The goal is to prove that for given \( \epsilon > 0 \) there exists \( C(\epsilon) > 0 \) depending on \( \epsilon \) such that

\[
\sum_{0 \leq m \leq N_0} \|\partial_{v_s}^m G_2\|^2 + \sum_{0 \leq m \leq N_0} \|\partial_{v_s}^m G_2\|_{L^2}^2 \leq C(\epsilon) \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_s}^m G_1\|_{L^\infty}^2 + C(\epsilon) \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_s}^m F_2\|_{L^\infty}^2 + \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_s}^m F_2, b\|_{L^\infty}^2.
\]

For this, we begin with the following equations for \( G_2 \)

\[
\begin{cases}
\epsilon G_2 + v_y \partial_y G_2 - \alpha v_y \partial_v G_2 + \nu_0 G_2 - \sigma K G_2 - \sigma (1 - \chi_M) \mu^{-\frac{1}{2}} K G_1 = F_2, \ y \in (-1, 1), \\
G_2(\pm 1, v) 1_{\{v_s \geq 0\}} - \sigma \sqrt{2\pi \mu} \int_{v_s \geq 0} \sqrt{\mu} G(\pm 1, v) |v_y| dv = F_{2,b}.
\end{cases}
\]

Taking the inner product of (5.32) and \( G_2 \) over \( (y, v) \in (-1, 1) \times \mathbb{R}^3 \), we have, for \( \eta > 0 \)

\[
(\epsilon + (1 - \sigma) \nu_0) \|G_2\|^2 + \sigma \delta_0 \|P_1 G_2\|^2 + \frac{1}{2} \|I - P_2\| G_2^2 + \frac{1}{2} (1 - \sigma) \|P_2 G_2\|^2 \leq \|G_2, F_2\| + \|((1 - \chi_M) \mu^{-\frac{1}{2}} K G_1, G_2)\| + \eta \|P_2 G_2\|^2 + C_\eta \|w_q G_1\|_{L^\infty}^2 + C_\eta \|w_q F_{2, b}\|_{L^\infty}^2.
\]
where the following estimate on the boundary term has been used

\[
\int_{\mathbb{R}^3} v_y G_2^2(1) dv - \int_{\mathbb{R}^3} v_y G_2^2(-1) dv \\
= \int_{v_y > 0} v_y G_2^2(1) dv - \int_{v_y < 0} v_y (\sigma P_y G_2 + \sigma \tilde{P}_y G_1 + F_{2,b})^2(1) dv \\
- \int_{v_y < 0} v_y G_2^2(-1) dv - \int_{v_y > 0} v_y (\sigma P_y G_2 + \sigma \tilde{P}_y G_1 + F_{2,b})^2(-1) dv \\
\geq (1 - \sigma^2) \int_{v_y > 0} v_y (P_y G_2)^2(1) dv + \int_{v_y > 0} v_y (\{I - P_y\} G_2)^2(1) dv \\
+ (1 - \sigma^2) \int_{v_y < 0} |v_y|(P_y G_2)^2(-1) dv + \int_{v_y < 0} |v_y|(\{I - P_y\} G_2)^2(-1) dv \\
- \eta \int_{v_y < 0} v_y (P_y G_2)^2(1) dv - C_\eta \|w_y F_{2,b}(1)\|_{L^\infty}^2 - \eta \int_{v_y > 0} |v_y|(P_y G_2)^2(-1) dv \\
- C_\eta \|w_y F_{2,b}(-1)\|_{L^\infty}^2 - C_\eta \int_{v_y < 0} |v_y|\tilde{P}_y G_1(1)^2 dv - C_\eta \int_{v_y > 0} |v_y|\tilde{P}_y G_1(-1)^2 dv \\
\geq |\{I - P_y\} G_2|_{L^2}^2 + (1 - \sigma)|P_y G_2|_{L^2}^2 + \eta|P_y G_2|_{L^2}^2 - C_\eta \|w_y F_{2,b}(\pm 1)\|_{L^\infty}^2 - C_\eta \|w_y G_1\|_{L^\infty}^2.
\]

Here we have used the notation

\[
\tilde{P}_y G_1(\pm 1) = \sqrt{2\pi \mu} \int_{v_y \geq 0} G_1(\pm 1)|v_y| dv,
\]

and the estimate

\[
\int_{v_y \geq 0} G_1(\pm 1)|v_y| dv \leq C \|w_y G_1\|_{L^\infty}, \text{ for } q > 5/2.
\]

Next, since

\[
|P_y G_2(\pm 1)|_{L^2}^2 = \int_{v_y \geq 0} \left( \int_{v_y \geq 0} G_2(\pm 1)\sqrt{\mu} |v_y| dv \right)^2 2\pi \mu(v)|v_y| dv = \left[ \int_{v_y \geq 0} G_2(\pm 1)\sqrt{\mu} |v_y| dv \right]^2,
\]

by dividing the domain for integration as

\[
\{v \in \mathbb{R}^3 : v_y > 0\} = \{v \in \mathbb{R}^3 : 0 < v_y < \varepsilon \text{ or } v_y > 1/\varepsilon\} \cup \{v \in \mathbb{R}^3 : \varepsilon \leq v_y \leq 1/\varepsilon\},
\]

one sees that the grazing part of \(|P_y G_2(1)|_{L^2}^2|\) is bounded by the Hölder inequality as

\[
\left( \int_{v^*} \mu(v)|v_y| dv \right) \int_{v_y > 0} |G_2(1)|^2 v_y dv \lesssim \varepsilon \int_{v_y > 0} |G_2(1)|^2 v_y dv.
\]

For non-grazing region, we have by using the trace Lemma 3.1 that

\[
\int_{\{v \in \mathbb{R}^3 : v_y > 0\} \backslash v^*} |G_2(1)|^2 v_y dv \leq C\|G_2\|^2 + C\|v_y \partial_v G_2\|^2 - \sigma v_y \partial_v G_2|_{L^1} \\
\leq C\|G_2\|^2 + C\{(L G_2, G_2)\} + C\{(1 - \chi_M) \mu^{-1/2} \mathcal{K} G_1, G_2\} + \{(F_2, G_2)\} \\
\leq C\|G_2\|^2 + C\|w_y G_1\|_{L^\infty}^2 + C\|F_2\|^2.
\]

Putting (5.34) and (5.35) together, one has

\[
|P_y G_2|_{L^2}^2 \leq \varepsilon \{(I - P_y) G_2|_{L^2}^2 + C\|G_2\|^2 + C\|w_y G_1\|_{L^\infty}^2 + C\|F_2\|^2.
\]

Consequently, (5.33) and (5.36) give

\[
\|G_2\|^2 + \|G_2|_{L^2}^2 \leq C(\|w_y F_2|_{L^\infty}^2 + \|w_y F_{2,b}||_{L^\infty}^2 + \|w_y G_1\|_{L^\infty}^2).
\]

**Remark 5.2.** Note that the constant \(C(\varepsilon)\) in (5.37) is independent of the parameter \(\sigma\).
It remains to deduce the $L^2$ estimate of the higher order velocity derivatives. For this, applying $\partial_{v_x}^m (m \geq 1)$ to (5.32) to have
\[
\begin{cases}
\epsilon \partial_{v_x}^m \mathcal{G}_2 + v_y \partial_y \partial_{v_x}^m \mathcal{G}_2 - \alpha v_y \partial_{v_x}^m \mathcal{G}_2 + \nu_0 \partial_{v_x}^m \mathcal{G}_2 - \sigma \partial_{v_x}^m K \mathcal{G}_2 \\
- \sigma \partial_{v_x}^m [(1 - \chi_M)\mu^{-\frac{1}{2}}K\mathcal{G}_2] = \partial_{v_x}^m \mathcal{F}_2, \quad y \in (-1, 1),
\end{cases}
\tag{5.38}
\]

Taking the inner product of (5.38) and $\partial_{v_x}^m \mathcal{G}_2$, we deduce
\[
(\epsilon + (1 - \sigma)\nu_0)\|\partial_{v_x}^m \mathcal{G}_2\|^2 + \sigma \delta_0 \|\partial_{v_x}^m \mathcal{G}_2\|^2 + \frac{1}{2}\|\partial_{v_x}^m \mathcal{G}_2\|^2, + C\sum_{m' \leq m} \|w_q \partial_{v_x}^{m'} \mathcal{G}_1\|_{L^\infty} + C(m)\|P_\mathcal{G}_2\|_{L^2, +} + C\|\partial_{v_x}^m \mathcal{F}_2\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathcal{F}_{2, b}\|_{L^\infty},
\tag{5.39}
\]
where we have used the following estimates for the incoming boundary term by (5.38)
\[
\int_{v_x \leq 0} |v_y| |\partial_{v_x}^m \mathcal{G}_2(\pm 1, v)\mathbf{1}_{\{v_y \geq 0\}}| dv \leq C\|\partial_{v_x}^m \mathcal{G}_2\|_{L^2, +} + C\|w_q \mathcal{G}_1\|_{L^\infty} + C\|w_q \partial_{v_x}^m \mathcal{F}_{2, b}\|_{L^\infty}.
\]

Then, (5.39) and (5.37) give (5.31). With (5.31), (5.12) follows from (5.22), (5.30) and (5.31). This completes the proof of the lemma. \(\square\)

5.3. Existence for the linear problem with $\sigma = 1$ and $\epsilon > 0$. With Lemma 5.1, we now turn to prove the existence of solution to (5.10) for a fixed parameter $\epsilon > 0$ in $L^\infty$ framework by the contraction mapping argument.

**Lemma 5.2.** Under the same assumption of Lemma 5.1, there exists a unique solution $[\mathcal{G}_1, \mathcal{G}_2] \in \mathbf{X}_{\alpha, N_0}$ to (5.10) with $\sigma = 1$ satisfying
\[
\sum_{0 \leq m \leq N_0} \left\{ \|w_q \partial_{v_x}^m \mathcal{G}_1\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{G}_2\|_{L^\infty} \right\} 
\leq C \sum_{0 \leq m \leq N_0} \left\{ \|w_q \partial_{v_x}^m \mathcal{F}_1\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_2\|_{L^\infty} + \|w_q \partial_{v_x}^m \mathcal{F}_{2, b}\|_{L^\infty} \right\}.
\tag{5.40}
\]

**Proof.** The proof is based on the a priori estimate (5.12) established in Lemma 5.1 and a bootstrap argument. As for Lemma 4.2, the proof is also divided into the following three steps.

**Step 1.** Existence for $\sigma = 0$. If $\sigma = 0$, then (5.10) is reduced to
\[
\epsilon \mathcal{G}_1 + v_y \partial_y \mathcal{G}_1 - \alpha v_y \partial_{v_x} \mathcal{G}_1 + \nu_0 \mathcal{G}_1 + \alpha \frac{v_x v_y}{2} \sqrt{\mathcal{G}_2} = \mathcal{F}_1, \quad y \in (-1, 1),
\]
\[
\mathcal{G}_1(\pm 1, v)\mathbf{1}_{\{v_y \leq 0\}} = 0,
\]
and
\[
\epsilon \mathcal{G}_2 + v_y \partial_y \mathcal{G}_2 - \alpha v_y \partial_{v_x} \mathcal{G}_2 + \nu_0 \mathcal{G}_2 = \mathcal{F}_2, \quad y \in (-1, 1),
\]
\[
\mathcal{G}_2(\pm 1, v)\mathbf{1}_{\{v_y \leq 0\}} = \mathcal{F}_{2, b},
\]
respectively. Then, in this simple case, the existence of $L^\infty$-solutions can be directly proved by the characteristic method to have
\[
\|\mathcal{Z}_{\epsilon}^{-1}[\mathcal{F}_1, \mathcal{F}_2]\|_{\mathbf{X}_{\alpha, N_0}} \leq C \mathcal{Z} \|\mathcal{Z}^{-1}[\mathcal{F}_1, \mathcal{F}_2]\|_{\mathbf{X}_{\alpha, N_0}}.
\tag{5.41}
\]

**Step 2.** Existence for $\sigma \in [0, \sigma_*)$ for some $\sigma_* > 0$. Let $\sigma \in (0, \sigma_*)$, we now consider
\[
\epsilon \mathcal{G}_1 + v_y \partial_y \mathcal{G}_1 - \alpha v_y \partial_{v_x} \mathcal{G}_1 + \nu_0 \mathcal{G}_1 + \alpha \frac{v_x v_y}{2} \sqrt{\mathcal{G}_2} = \sigma \chi_M K \mathcal{G}_1 + \mathcal{F}_1, \quad y \in (-1, 1),
\tag{5.42}
\]
\[
\mathcal{G}_1(\pm 1, v)\mathbf{1}_{\{v_y \leq 0\}} = 0,
\tag{5.43}
\]
and
\[
\epsilon \mathcal{G}_2 + v_y \partial_y \mathcal{G}_2 - \alpha v_y \partial_{v_x} \mathcal{G}_2 + \nu_0 \mathcal{G}_2 = \sigma K \mathcal{G}_2 + \sigma (1 - \chi_M)\mu^{-\frac{1}{2}}K \mathcal{G}_1 + \mathcal{F}_2, \quad y \in (-1, 1),
\tag{5.44}
\]
For the above system, we design the following approximation scheme

\[ \epsilon G_1^{n+1} + v_y \partial_y G_1^{n+1} - \alpha v_y \partial_y G_1^{n+1} + \nu_0 G_1^{n+1} + \alpha \frac{v_x v_y}{2} \sqrt{n} G_2^{n+1} = \sigma \chi M K G_1^n + F_1 := F_1^{(1)}, \]

and

\[ \epsilon G_2^{n+1} + v_y \partial_y G_2^{n+1} - \alpha v_y \partial_y G_2^{n+1} + \nu_0 G_2^{n+1} = \sigma K G_2^n + \sigma (1 - \chi M) \mu^{-\frac{1}{2}} K G_1^n + F_2 := F_2^{(1)}, \]

with \([G_1^0, G_2^0] = [0, 0]\). The goal in the following proof has twofold: (i) \([G_1^n, G_2^n]_{n=0}^\infty\) is uniformly bounded in \(X_{\alpha, N_0}\), and (ii) \([G_1^n, G_2^n]_{n=0}^\infty\) is a Cauchy sequence in \(X_{\alpha, N_0}\). By \eqref{5.41}, it follows

\[
\|G_1^{n+1}, G_2^{n+1}\|_{X_{\alpha, N_0}} \leq C_{XF} \{ \|F_1^{(1)}, F_2^{(1)}\|_{X_{\alpha, N_0}} + \sum_{0 \leq m \leq N_0} \|w_q \partial^m_{y, x} F_2, b\|_{L^\infty} \}
\]

where \(C_{1} > 0\) is independent of \(\sigma\) and \(n\). Choosing \(0 < \sigma_* < 1\) suitably small such that

\[
C_{XF} \sigma_* C_1 \leq \frac{1}{2},
\]

\eqref{5.50} implies that

\[
\|G_1^n, G_2^n\|_{X_{\alpha, N_0}} \leq 2 M_0,
\]

for all \(n \geq 0\). Moreover, by \eqref{5.46}, \eqref{5.47}, \eqref{5.48}, and \eqref{5.49} and applying \eqref{5.41}, one has

\[
\|G_1^{n+1}, G_2^{n+1}\|_{X_{\alpha, N_0}} \leq C_{XF} \sigma C_1 \|G_1^n, G_2^n\|_{X_{\alpha, N_0}} \leq \frac{1}{2} \|G_1^n, G_2^n\|_{X_{\alpha, N_0}}
\]

with the condition \eqref{5.51}. Consequently, \eqref{5.53} and \eqref{5.52} imply that the systems \eqref{5.42}-

\[ \text{and} \quad (5.44)-(5.45) \]

has a unique solution \([G_1, G_2] \in X_{\alpha, N_0}\) for any \(\sigma \in [0, \sigma_*]\). Moreover, by Lemma 5.1, we have the following uniform estimate

\[
\|G_1, G_2\|_{X_{\alpha, N_0}} \leq C_{XF} \sum_{0 \leq m \leq N_0} \left\{ \|w_q \partial^m_{y, x} F_1\|_{L^\infty} + \|w_q \partial^m_{y, x} F_2\|_{L^\infty} + \|w_q \partial^m_{y, x} F_2, b\|_{L^\infty} \right\},
\]

which is equivalent to

\[
\|F_1, F_2\|_{X_{\alpha, N_0}} \leq C_{XF} \|F_1, F_2\|_{X_{\alpha, N_0}}.
\]

**Step 3. Existence for \(\sigma \in [0, 2 \sigma_*]\) for some \(\sigma_* > 0\).** By using \eqref{5.54} and performing the similar calculations as for obtaining \eqref{5.52} and \eqref{5.53}, for \(\sigma \in [0, \sigma_*]\), one can see that there exists a unique solution \([G_1, G_2] \in X_{\alpha, N_0}\) to the lifted system

\[
\epsilon G_1 + v_y \partial_y G_1 - \alpha v_y \partial_y G_1 + \nu_0 G_1 + \alpha \frac{v_x v_y}{2} \sqrt{n} G_2 - \sigma_* \chi M K G_1 = \sigma \chi M K G_1 + F_1, \quad y \in (-1, 1),
\]

\[
G_1(\pm 1, v)1_{\{v_y \leq 0\}} = 0,
\]

and

\[
\epsilon G_2 + v_y \partial_y G_2 - \alpha v_y \partial_y G_2 + \nu_0 G_2 - \sigma_* K G_2 - \sigma_* (1 - \chi M) \mu^{-\frac{1}{2}} K G_1
\]

\[
= \sigma K G_2 + \sigma (1 - \chi M) \mu^{-\frac{1}{2}} K G_1 + F_2, \quad y \in (-1, 1),
\]
\[ G_2(\pm 1, v)1_{\{v_y \leq 0\}} - \sigma \sqrt{2\pi \mu} \int_{v_y \geq 0} (G_1 + \sqrt{\mu}G_2)(\pm 1, v)|v_y|dv \]
\[ = \sigma \sqrt{2\pi \mu} \int_{v_y \geq 0} (G_1 + \sqrt{\mu}G_2)(\pm 1, v)|v_y|dv + \mathcal{F}_{2,b}. \]

In other words, we have shown the existence of \( \mathcal{L}_{\sigma_*}^{-1} \) on \( X_{\alpha, n_0} \) and (5.12) holds true for \( \sigma = 2 \sigma_* \).

Therefore, by repeating this procedure in finite time, one can see that \( \mathcal{L}_{\sigma_*}^{-1} \) exists when \( \sigma = 1 \) and (5.40) follows correspondingly. This completes the proof of the lemma. 

\[ \square \]

5.4. Estimates on remainder. We are ready to complete the proof of Proposition 5.1.

Proof of Proposition 5.1. We now prove the existence of the coupled system (5.2) and (5.4) under the diffuse boundary conditions (5.3) and (5.5), respectively.

Let us first go back to the approximation system (5.6), (5.7), (5.8) and (5.9). By applying Lemma 5.2, for fixed \( \epsilon > 0 \), we see that \( \{G_n^{R_1}, G_n^{R_2}\} \) is well defined when \( \{G_n^{R_1}, G_n^{R_2}\} \) is given and the solution belongs to \( X_{\alpha, n_0} \) defined in (5.11) for \( N_0 \geq 0 \).

We now show that \( \{G_n^{R_1}, G_n^{R_2}\}_{n=0}^{\infty} \) is a Cauchy sequence in \( X_{\alpha, n_0} \), which hence implies that its limit denoted by \( \{G^{R_1}, G^{R_2}\} \) is the unique solution of the following system

\[ \epsilon G^{R_1}_1 + v_y \partial_y G^{R_1}_1 - \alpha v_y \partial v_y G^{R_1}_1 + v_0 G^{R_1}_1 - \chi M K G^{R_1}_1 \]
\[ = -\frac{\alpha}{2} \sqrt{\mu}v_x v_y G^{R_2}_2 - \frac{1}{2} \sqrt{\mu}v_x v_y G_1 + \sqrt{\mu}v_y \partial v_y G_1 + Q(\sqrt{\mu}G_1, \sqrt{\mu}G_1) \]
\[ + \alpha \{Q(\sqrt{\mu}G^{R_2}_1, \sqrt{\mu}G_1) + Q(\sqrt{\mu}G_1, \sqrt{\mu}G^{R_2}_1)\} + \alpha^2 Q(\sqrt{\mu}G^{R_2}_1, \sqrt{\mu}G^{R_2}_1) \]
\[ := N, \ y \in (-1, 1), \ v \in \mathbb{R}^3, \]  
(5.55)

and

\[ \epsilon G^{R_2}_1 + v_y \partial_y G^{R_2}_1 - \alpha v_y \partial v_y G^{R_2}_1 + \chi M K G^{R_2}_1 \]
\[ = \left(1 - \chi M\right) \mu^{-\frac{1}{2}} K G^{R_1}_1, \ y \in (-1, 1), \ v \in \mathbb{R}^3, \]  
(5.56)

Furthermore, we will show that the convergence of the sequence \( \{G_n^{R_1}, G_n^{R_2}\}_{n=0}^{\infty} \) is independent of \( \epsilon \). For this, we first prove that

\[ ||\{G_n^{R_1}, G_n^{R_2}\}||_{X_{\alpha, n_0}} \leq 2C_0, \]  
(5.59)

where \( C_0 > 0 \) is independent of \( \epsilon \) and \( n \) for all \( n \geq 0 \). We apply induction in \( n \). Notice \( \{G_0^{R_1}, G_0^{R_2}\} = [0, 0] \). If \( n = 1 \), then the system (5.6), (5.7), (5.8) and (5.9) reads

\[ \epsilon G^{R_1}_1 + v_y \partial_y G^{R_1}_1 - \alpha v_y \partial v_y G^{R_1}_1 + v_0 G^{R_1}_1 - \chi M K G^{R_1}_1 + \frac{\alpha}{2} \sqrt{\mu}v_x v_y G^{R_2}_2 \]
\[ = -\frac{1}{2} \sqrt{\mu}v_x v_y G_1 + \sqrt{\mu}v_y \partial v_y G_1 + Q(G_1, G_1) := S^0, \ y \in (-1, 1), \ v \in \mathbb{R}^3, \]  
(5.60)

and

\[ \epsilon G^{R_2}_1 + v_y \partial_y G^{R_2}_1 - \alpha v_y \partial v_y G^{R_2}_1 + \chi M K G^{R_1}_1 \]
\[ = \left(1 - \chi M\right) \mu^{-\frac{1}{2}} K G^{R_1}_1, \ y \in (-1, 1), \ v \in \mathbb{R}^3, \]  
(5.61)

and

\[ \epsilon G^{R_2}_2 + v_y \partial_y G^{R_2}_2 - \alpha v_y \partial v_y G^{R_2}_2 + L G^{R_2}_2 = \left(1 - \chi M\right) \mu^{-\frac{1}{2}} K G^{R_2}_1, \ y \in (-1, 1), \ v \in \mathbb{R}^3, \]  
(5.62)

and

\[ G^{R_1}_1(\pm 1, v)|_{v_y \leq 0} = 0, \ v \in \mathbb{R}^3, \]  
(5.63)
Performing similar calculations as for deriving (5.22) and (5.30), one has
\[
\sum_{0 \leq m \leq N_0} \| w_q \partial^n v_{x_1} G_{R,1} \|_{L^\infty} \leq C \alpha \sum_{0 \leq m \leq N_0} \| w_q \partial^n v_{x_2} G_{R,2} \|_{L^\infty} + C \sum_{0 \leq m \leq N_0} \| w_q \partial^m S^0 \|_{L^\infty} \\
\leq C \alpha \sum_{0 \leq m \leq N_0} \| w_q \partial^n v_{x_2} G_{R,2} \|_{L^\infty} + C,
\]
and
\[
\sum_{0 \leq m \leq N_0} \| w_q \partial^n v_{x_2} G_{R,2} \|_{L^\infty} \leq C \sum_{0 \leq m \leq N_0} \| \partial^n v_{x_2} G_{R,2} \| + C \sum_{0 \leq m \leq N_0} \| w_q \partial^m G_{R,1} \|_{L^\infty},
\]
where the constant $C > 0$ is independent of $\epsilon$, see also Remark 5.1.

Since the mass of $G_{R,2}$ is not conserved, in order to estimate the macroscopic component of $G_{R,2}$ we instead turn to obtain the $L^2$ estimate of $G_{R,1}$. Recall $\sqrt{\mu}G_{R,1} = G_{R,1}^1 + \sqrt{\mu}G_{R,2}^1$. By (5.60), (5.61), (5.62) and (5.63), it is easy to see that $G_{R,1}^1$ satisfies
\[
\epsilon G_{R,1}^1 + v_y \partial_y G_{R,1}^1 - \alpha v_y \partial_y G_{R,1}^1 + \frac{\alpha}{2} v_x v_y G_{R,1}^1 + LG_{R,1}^1 = \mu^{-\frac{1}{2}} S^0,
\]
and
\[
G_{R,1}^1(\pm 1, v) |_{v_y \leq N_0} = \sqrt{2\pi \mu} \int_{v_y \geq 0} \sqrt{\mu}G_{R,1}^1(\pm 1, v) | v_y | dv.
\]
Next, for $n \geq 1$, denote
\[
P_0 G_{R}^n = (a^n + b^n \cdot v + c^n (|v| - 3)) \sqrt{\mu}, \quad P_0 G_{R,2}^n = (a^n_2 + b^n_2 \cdot v + c^n_2 (|v| - 3)) \sqrt{\mu},
\]
and define the projection $P_0$, from $L^2$ to ker $\mathcal{L}$, as
\[
P_0 G_{R,1}^n = (a_1^n + b_1^n \cdot v + c_1^n (|v| - 3)) \mu.
\]
In addition, we will also use the following notations
\[
b_i^n = [b_{i,1}^n, b_{i,2}^n, b_{i,3}^n], \quad i = 1, 2, \quad b^n = [b_1^n, b_2^n, b_3^n].
\]
Note that
\[
a^n = a_1^n + a_2^n, \quad b^n = b_1^n + b_2^n, \quad c^n = c_1^n + c_2^n, \quad \int_{-1}^1 a^n(y) dy = 0.
\]
Since
\[
\|[a_1^n, b_1^n, c_1^n]\| \leq C \|[P_0 G_{R,1}^n]\| \leq C \|w_q G_{R,1}^n\|_{L^\infty},
\]
for $q > 5/2$, to obtain the estimate of $\|[a_1^n, b_2^n, c_2^n]\|$, it suffices to derive the $L^2$ estimates of $[a^n, b^n, c^n]$. In what follows, we will show that the $L^2$ norm of the macroscopic part of $G_{R,1}^1$ can be indeed dominated by its microscopic component and other known terms. We estimate $[a_1^n, b_1^n, c_1^n]$ by the dual argument. First of all, we let $\Psi(y, v) \in C^\infty([-1, 1] \times \mathbb{R}^3)$, and take the inner product of (5.66) and $\Psi$ over $(-1, 1) \times \mathbb{R}^3$, to obtain
\[
\epsilon(G_{R,1}, \Psi) - (v_y G_{R,1}, \partial_y \Psi) + (v_y G_{R,1}(1), \Psi(1)) - (v_y G_{R,1}(-1), \Psi(-1)) + \alpha(v_y G_{R,1}, \partial_y \Psi)
\]
\[
+ \frac{\alpha}{2} (v_x v_y G_{R,1}, \Psi) + (LG_{R,1}, \Psi) = (\mu^{-\frac{1}{2}} S^0, \Psi).
\]
Estimate on $a_1^n$. Let
\[
\Psi = \Psi a_1 = v_y \frac{d}{dy} \phi_{a_1}(y)(|v| - 3) \sqrt{\mu},
\]
where
\[
\phi_{a_1}'' = a_1, \quad \phi_{a_1}'(1) = 0.
\]
Thus
\[
\| \phi_{a_1} \|_{H^2} \leq C \|a_1\|.
\]
Plugging \( \Psi = \Psi_{a^1} \) into (5.71), we now compute the equation term by term. First of all, by Cauchy-Schwarz inequality with \( \eta > 0 \) and using (5.73), one has

\[
\|a^1\|^2 + C\eta \|G_{1,R}\|^2 + \|G_{1,R,1}\|_\infty^2 \}
\]

\[
-\langle v_y G_{1,R}^1, \partial_y \Psi_{a^1} \rangle
\]

\[
\ge 5\|a^1\|^2 - \eta\|a^1\|^2 - C\eta \{\|P_{1,G_{1,R,2}}\|^2 + \|w_y G_{1,R,1}\|_\infty^2 \}
\]

\[
(\sigma G_{1,R,1} \mu^{-1} \tilde{\Psi}), \Psi_{a^1} \rangle
\]

\[
\alpha (\|v_y G_{1,R}^1, \partial_y \Psi_{a^1}) | + \alpha^2 (\|v_y G_{1,R,1}^1, \Psi_{a^1} | \le C\alpha \|b_1\|^2 + \|w_y G_{1,R,1}\|_\infty^2 + \|G_{1,R,2}\|_\infty^2
\]

Then by Lemmas 4.1 and 2.2, it follows

\[
(\mu^{-1} \tilde{\Psi}^0, \Psi_{a^1}) = |(v_y \partial_{v_x} G_1 - \frac{1}{2} v_y G_1 + \Gamma(G_1, G_1), \Psi_{a^1} | \le \eta\|a^2\|^2 + C\eta \|G_{1,R}\|^2 + C\eta \|G_{1,R,1}\| _\infty^2 \}
\]

\[
\le \eta\|a^1\|^2 + C\eta \|G_{1,R}\|^2 + C\eta \|G_{1,R,1}\| _\infty^2 \}
\]

\[
\le \eta\|a^1\|^2 + C\eta \|G_{1,R}\|^2 + C\eta \|G_{1,R,1}\| _\infty^2 \}
\]

provided that \( q > 3/2 \).

Next, noting that \( LG_{1,R} \) = \(-\{\Gamma(G_{1,R}, \sqrt{\mu}) + \Gamma(\sqrt{\mu}, G_{1,R}) \} \), one has by a similar argument as above that

\[
|\langle LG_{1,R}, \Psi_{a^1} \rangle | \le |\langle L (G_{1,R,1} \mu^{-1} \tilde{\Psi}), \Psi_{a^1} \rangle | + |\langle LG_{1,R,2,1}, \Psi_{a^1} \rangle |
\]

\[
\le \eta\|a^1\|^2 + C\eta \|G_{1,R,1}\| _\infty^2 \}
\]

\[
(\mu^{-1} \tilde{\Psi}^0, \Psi_{a^1}) = |(v_y G_{1,R}^1, \Psi_{a^1}) - (v_y G_{1,R}^1, \Psi_{a^1}) | \}
\]

Putting all the estimates above together, we have

\[
\|a^1\|^2 \le C\|P_{1,G_{1,R,2}}\|^2 + C\|w_y G_{1,R,1}\| _\infty^2 \}
\]

\[
+ C\eta \{\|G_{1,R}\|^2 + \|b_1\|^2 \} C\|b_1\|^2 + C.
\]

**Estimate on \( b_1 \).** Let

\[
\Psi = \Psi_{b_1} = \left\{ \begin{array}{ll}
{v_y v_x \frac{d}{dy} \phi_{b_1}} (y) \sqrt{\mu}, & i = 1, \\
{v_y v_z \frac{d}{dy} \phi_{b_3}} (y) \sqrt{\mu}, & i = 3, \\
{v_y^2 (|v|^2 - 5) \frac{d}{dy} \phi_{b_2}} (y) \sqrt{\mu}, & i = 2,
\end{array} \right.
\]

where

\[
-\phi_{b_1}'' = b_1^1, \quad \phi_{b_1}(\pm 1) = 0.
\]

Then it holds

\[
\|\phi_{b_1}\|_{H^2} \le C\|b_1\|, \quad |\phi_{b_1}'(\pm 1)| \le C\|b_1\|.
\]

We now compute each term in (5.71) with \( \Psi = \Psi_{b_1} \). By Cauchy-Schwarz inequality and (5.77), one has

\[
\|a^1\|^2 \le C\|P_{1,G_{1,R,2}}\|^2 + C\|w_y G_{1,R,1}\| _\infty^2 \}
\]

\[
+ C\eta \{\|G_{1,R}\|^2 + \|b_1\|^2 \} C\|b_1\|^2 + C.
\]
\[\alpha|\langle v_y G^1_{R_1}, \partial_y \Psi_{b_1} \rangle| + \frac{\alpha}{2}|\langle v_x v_y G^1_{R_1}, \Psi_{b_1} \rangle| \leq C \alpha(\|a^1, c^1\|^2 + \|w_q G^1_{R,1}\|_{L^\infty}^2 + \|G^1_{R,2}\|^2).
\]

Similar to (5.74) and (5.75), it follows
\[\langle \mu - \frac{1}{2} S^0, \Psi_{b_1} \rangle \leq \eta\|b^1\|^2 + C,\]

and
\[|\langle LG^1_{R_1}, \Psi_{b_1} \rangle| \leq \left|\langle L \left( G^1_{R,1} \mu - \frac{1}{2} \right), \Psi_{b_1} \rangle \right| + \|\langle LG_{R,2}, \Psi_{b_1} \rangle\|_{L^\infty} \leq \eta\|b^1\|^2 + C.\]

For the boundary term, noting that
\[G^1_{R}(\pm 1)|_{v_y \neq 0} = P_{\gamma} G^1_{R}(\pm 1) + \{I - P_{\gamma}\} G^1_{R}(\pm 1)|_{v_y = 0},\]

we have
\[\langle v_y G^1_{R}(1), \Psi_{b_1}(1) \rangle - \langle v_y G(-1), \Psi_{b_1}(-1) \rangle = \langle v_y P_{\gamma} G^1_{R}(1), \Psi_{b_1}(1) \rangle - \langle v_y P_{\gamma} G^1_{R}(-1), \Psi_{b_1}(-1) \rangle - \langle v_y \{I - P_{\gamma}\} G^1_{R}(1), \Psi_{b_1}(1) \rangle + \langle v_y \{I - P_{\gamma}\} G^1_{R}(-1), \Psi_{b_1}(-1) \rangle \leq \eta\|b^1\|^2 + C \eta(\|w_q G^1_{R,1}\|_{L^\infty}^2 + \|P_{1} G^1_{R,2}\|^2).
\]

where the fact that \(\langle v_y P_{\gamma} G^1_{R}(\pm 1), \Psi_{b_1}(\pm 1) \rangle = 0\) has been used.

We now conclude from the above estimates for \(b^1\) with \(1 \leq i \leq 3\) that
\[
\|b^1\|^2 \leq C\|P_{1} G^1_{R,2}\|^2 + C\|w_q G^1_{R,1}\|_{L^\infty}^2 + C \alpha(\|G^1_{R,2}\|^2 + \|a^1, c^1\|^2) + C\|I - P_{\gamma}\|_{L^2}^2 + C\|c^1\|^2 + C.
\]

Estimate on \(c^1\). Let
\[\Psi = \Psi_{c_1} = v_y(\|v\|^2 - 5) \frac{d}{dy} \phi_{c_1}(y) \sqrt{n},\]

where
\[-\phi_{c_1}'' = c^1, \ \phi_{c_1}(\pm 1) = 0.\]

One has
\[\|\phi_{c_1}\|_{H^2} \leq C\|c^1\|, \ \|\phi_{c_1}'(\pm 1)\| \leq C\|c^1\|.
\]

By Cauchy-Schwarz inequality and (5.80), it follows
\[\|G^1_{R}(\Psi_{c_1})\| \leq \epsilon(\|P_{0} G^1_{R}, \Psi_{c_1}\| + \epsilon(\|P_{1} G^1_{R}, \Psi_{c_1}\|) \leq C \epsilon\|c^1\|^2 + C \epsilon(\|P_{1} G^1_{R,2}\|^2 + \|w_q G^1_{R,1}\|_{L^\infty}^2)\},\]

\[-\langle v_y G^1_{R}, \partial_y \Psi_{c_1} \rangle = -\langle v_y P_{0} G^1_{R}, \partial_y \Psi_{c_1} \rangle - \langle v_y P_{1} G^1_{R}, \partial_y \Psi_{c_1} \rangle \geq 30\|c^1\|^2 - \eta\|c^1\|^2 - C \eta(\|P_{1} G^1_{R,2}\|^2 + \|w_q G^1_{R,1}\|^2_{L^\infty}),\]

\[\alpha|\langle v_y G^1_{R}, \partial_y \Psi_{c_1} \rangle| + \frac{\alpha}{2}|\langle v_x v_y G^1_{R}, \Psi_{c_1} \rangle| \leq C \alpha(\|b^1\|^2 + \|w_q G^1_{R,1}\|^2_{L^\infty} + \|G^1_{R,2}\|^2).
\]

Also similar to derive 5.74 and (5.75), one has
\[\langle \mu - \frac{1}{2} S^0, \Psi_{c_1} \rangle \leq \eta\|c^1\|^2 + C,\]

and
\[|\langle LG^1_{R}(\Psi_{c_1})\rangle \leq \left|\langle L \left( G^1_{R,1} \mu - \frac{1}{2} \right), \Psi_{c_1} \rangle \right| + \|\langle LG_{R,2}, \Psi_{c_1} \rangle\|_{L^\infty} \leq \eta\|c^1\|^2 + C \eta(\|w_q G^1_{R,1}\|_{L^\infty}^2 + \|P_{1} G^1_{R,2}\|^2).
\]

For the boundary term, by applying (5.78) and using
\[\langle v_y P_{\gamma} G^1_{R}(\pm 1), \Psi_{c_1}(\pm 1) \rangle = 0,\]
we have
\[ \langle v_y G_R^{(1)}(1), \Psi_{\epsilon_1}(1) \rangle - \langle v_y G_{R}^{(1)}(-1), \Psi_{\epsilon_1}(-1) \rangle \leq \eta \| c_1 \|^2 + C \eta \| I - P_\gamma \| G_{R,2}^1 \|_{L^\infty}^2 + C \eta \| w_y G_{R,1}^1 \|_{L^\infty}^2. \]

Combining the above estimates on \( c_1 \) give
\[ \| c_1 \|^2 \leq C \| \mathbf{P}_1 G_{R,2}^1 \|^2 + C \| w_y G_{R,1}^1 \|_{L^\infty}^2 + C \eta \| G_{R,2}^1 \|^2 \]
\[ + C \| I - P_\gamma \| G_{R,2}^1 \|_{L^\infty}^2 + C \eta. \]  

(5.81)

Finally, a linear combination of (5.76), (5.79) and (5.81) gives
\[ \| a^1, b^1, c_1 \|^2 \leq C \| \mathbf{P}_1 G_{R,2}^1 \|^2 + C \| w_y G_{R,1}^1 \|_{L^\infty}^2 + C \eta \| G_{R,2}^1 \|^2 \]
\[ + C \| I - P_\gamma \| G_{R,2}^1 \|_{L^\infty}^2 + C \eta. \]  

(5.82)

This together with (5.69) and (5.70) implies that \([a_2, b_2, c_2] \) satisfies
\[ \| a_2, b_2, c_2 \|^2 \leq C \| \mathbf{P}_1 G_{R,2}^1 \|^2 + C \| w_y G_{R,1}^1 \|_{L^\infty}^2 + C \| I - P_\gamma \| G_{R,2}^1 \|_{L^\infty}^2 + C \eta. \]  

(5.83)

In order to obtain the estimates for \( \| \mathbf{P}_1 G_{R,2}^1 \| \), we have to further consider the BVP for \( G_{R,2}^1 \) as follows:
\[ \begin{align*}
\epsilon G_{R,2}^1 + v_y \partial_y G_{R,2}^1 - \alpha v_y \partial_{v_y} G_{R,2}^1 + L G_{R,2}^1 - (1 - \chi_M) \mu^{-\frac{1}{2}} K G_{R,1}^1 &= 0, \\
G_{R,2}^1(\pm 1, v) 1_{\{ v_y \leq 0 \}} &= \sqrt{2\pi} \mu \int_{v_y \leq 0} \sqrt{\mu} G_{R}^1(\pm 1, v) |v_y| dv.
\end{align*} \]

Applying the estimates (5.33), (5.36) and (5.39) with \( \sigma = 1 \), \( F_2 = 0 \) and \( F_{2,R} = 0 \), one has
\[ \epsilon \| G_{R,2}^1 \|^2 + \delta_0 \| \mathbf{P}_1 G_{R,2}^1 \|^2 + \frac{1}{2} \| I - P_\gamma \| G_{R,2}^1 \|_{L^\infty}^2 + \eta \| P_\gamma G_{R,2}^1 \|_{L^\infty}^2 + C \| G_{R,2}^1 \|^2 \]
\[ + C \| I - P_\gamma \| G_{R,2}^1 \|_{L^\infty}^2 + C \eta \| G_{R,2}^1 \|_{L^\infty}^2, \]  

(5.84)

and
\[ \epsilon \| \partial_{v_y} G_{R,2}^1 \|^2 + \delta_0 \| \partial_{v_y} G_{R,2}^1 \|^2 + \frac{1}{2} \| \partial_{v_y} G_{R,2}^1 \|^2 \]
\[ \leq C \| G_{R,2}^1 \|^2 + C \sum_{m \leq m} \| w_y \partial_{v_y} G_{R,2}^1 \|_{L^\infty}^2 + C(m) \| P_\gamma G_{R,2}^1 \|_{L^\infty}^2, \]  

(5.85)

where all the constants on the right hand side are independent of \( \epsilon \). Then (5.82), (5.84), (5.85) and (5.86) give
\[ \begin{align*}
\sum_{0 \leq m \leq N_0} \| \partial_{v_y} G_{R,2}^1 \|^2 + \sum_{0 \leq m \leq N_0} \| \partial_{v_y} G_{R,2}^1 \|_{L^\infty}^2 &\leq C \sum_{0 \leq m \leq N_0} \| w_y \partial_{v_y} G_{R,2}^1 \|_{L^\infty}^2 + C \| w_y \partial_{v_y} G_{R,1}^1 \|_{L^\infty}^2 + C. \end{align*} \]

(5.87)

Consequently, a linear combination of (5.64), (5.65) and (5.87) gives
\[ \begin{align*}
\sum_{0 \leq m \leq N_0} \{ \| w_y \partial_{v_y} G_{R,2}^1 \|_{L^\infty}^2 + \| w_y \partial_{v_y} G_{R,1}^1 \|_{L^\infty}^2 \} &\leq C_0, \end{align*} \]

for some suitably large \( C_0 > 0 \). Therefore (5.59) holds for \( n = 1 \).

We now assume that (5.59) is valid for \( n = k \geq 1 \) and then prove that it holds for \( n = k + 1 \). In fact, applying the estimates (5.22) and (5.30) to the system (5.6)-(5.7) and (5.8)-(5.9) with \( n = k \), one has
\[ \begin{align*}
\sum_{0 \leq m \leq N_0} \| w_y \partial_{v_y} G_{R,1}^{k+1} \|_{L^\infty} &\leq C \sum_{0 \leq m \leq N_0} \| w_y \partial_{v_y} G_{R,2}^1 \|_{L^\infty} + C \sum_{0 \leq m \leq N_0} \| w_y \partial_{v_y} G_{R,1}^1 \|_{L^\infty}, \end{align*} \]

(5.88)

and
\[ \begin{align*}
\sum_{0 \leq m \leq N_0} \| w_y \partial_{v_y} G_{R,2}^{k+1} \|_{L^\infty} &\leq C \sum_{0 \leq m \leq N_0} \| w_y \partial_{v_y} G_{R,2}^1 \|_{L^\infty} + C \sum_{0 \leq m \leq N_0} \| w_y \partial_{v_y} G_{R,1}^1 \|_{L^\infty}, \end{align*} \]

(5.89)
where

\[ S^k = -\frac{1}{2} \sqrt{\nu_x v_y} G_1 + \sqrt{\nu_y \partial_v} G_1 + Q(\sqrt{\nu_x}, \sqrt{\nu_y} G_1) + \alpha Q(\sqrt{\nu_x} G^k, \sqrt{\nu_y} G^k) + \alpha^2 Q(\sqrt{\nu_x} G^k, \sqrt{\nu_y} G^k). \]

By Lemma 4.1, Lemma 2.2 and the induction assumption, we have

\[
\sum_{0 \leq m \leq N_0} \| w_q \partial_v^m S^k \|_{L^\infty} \leq C + C\alpha \sum_{0 \leq m \leq N_0} \left\{ \| w_q \partial_v^m G_{R,1}^{k-1} \|_{L^\infty} + \| w_q \partial_v^m G_{R,1}^{k} \|_{L^\infty} \right\}
+ C\alpha^2 \sum_{0 \leq m \leq N_0} \left\{ \| w_q \partial_v^m G_{R,1}^{k-1} \|_{L^\infty}^2 + \| w_q \partial_v^m G_{R,1}^{k} \|_{L^\infty}^2 \right\}. \tag{5.90}
\]

For the \( L^2 \) estimate, by performing a parallel calculation as for (5.83), one has

\[
\| [a_1^{k+1}, b_1^{k+1}, c_1^{k+1}] \|_{L^2}^2 \leq C \| P_1 G_{R,2}^{k+1} \|_{L^2}^2 + C \| w_q G_{R,1}^{k+1} \|_{L^\infty}^2 + C \| I - P_\gamma \| G_{R,2}^{k+1} \|_{L^2}^2 \]
\[ + C \sum_{j=1}^3 \| (\mu - \frac{1}{2} S^k, \Psi_j) \|. \tag{5.91}
\]

Here \( \Psi_j \) \((1 \leq j \leq 3)\) are chosen as \( \Psi_{a^{k+1}} \), \( \Psi_{b^{k+1}} \) and \( \Psi_{c^{k+1}} \) in the same way as for \( \Psi_{a^{k}} \), \( \Psi_{b^{k}} \) and \( \Psi_{c^{k}} \), respectively. Hence, (5.91) also gives

\[
\| [a_2^{k+1}, b_2^{k+1}, c_2^{k+1}] \|_{L^2}^2 \leq C \| P_1 G_{R,2}^{k+1} \|_{L^2}^2 + C \| w_q G_{R,1}^{k+1} \|_{L^\infty}^2 + C \| I - P_\gamma \| G_{R,2}^{k+1} \|_{L^2}^2 \]
\[ + C\alpha^2 \left\{ \| w_q G_{R,1}^{k+1} \|_{L^\infty}^2 + \| w_q G_{R,2}^{k+1} \|_{L^\infty} \right\}
+ C\alpha^4 \left\{ \| w_q G_{R,1}^{k+1} \|_{L^\infty} + \| w_q G_{R,2}^{k+1} \|_{L^\infty}^4 \right\}, \tag{5.92}
\]

by applying Lemma 2.2 and the relation (5.69).

On the other hand, similar to the estimates (5.84), (5.85) and (5.86), it also follows

\[
e \| G_{R,1}^{k+1} \|_{L^2}^2 + \delta_0 \| P_1 G_{R,2}^{k+1} \|_{L^2}^2 + \frac{1}{2} \| I - P_\gamma \| G_{R,2}^{k+1} \|_{L^2}^2 \]
\[ \leq \eta \| P_\gamma \| G_{R,2}^{k+1} \|_{L^2}^2 + \eta \| G_{R,1}^{k+1} \|_{L^\infty}^2 + C_{\eta} \| w_q G_{R,1}^{k+1} \|_{L^\infty}, \tag{5.93}
\]

\[
\| P_\gamma \| G_{R,2}^{k+1} \|_{L^2}^2 + \frac{1}{2} \| I - P_\gamma \| G_{R,2}^{k+1} \|_{L^2}^2 + C \| G_{R,1}^{k+1} \|_{L^\infty}^2 + \| w_q G_{R,1}^{k+1} \|_{L^\infty}, \tag{5.94}
\]

and

\[
e \| \partial_v G_{R,2}^{k+1} \|_{L^2}^2 + \delta_0 \| \partial_v G_{R,2}^{k+1} \|_{L^2}^2 + \frac{1}{2} \| \partial_v G_{R,2}^{k+1} \|_{L^2}^2 \]
\[ \leq C \| G_{R,2}^{k+1} \|_{L^2}^2 + C \sum_{m \leq m} \| w_q G_{R,1}^{m+1} \|_{L^\infty}^2 + \| w_q \partial_v G_{R,1}^{m+1} \|_{L^\infty} \tag{5.95}
\]

As a consequence, combining estimates (5.92), (5.93), (5.94) and (5.95) gives

\[
\sum_{0 \leq m \leq N_0} \| \partial_v G_{R,2}^{k+1} \|_{L^2}^2 + \sum_{0 \leq m \leq N_0} \| \partial_v G_{R,2}^{k+1} \|_{L^2}^2 \]
\[ \leq C \sum_{0 \leq m \leq N_0} \left\{ \| w_q \partial_v G_{R,1}^{m+1} \|_{L^\infty}^2 + C\alpha^2 \left\{ \| w_q G_{R,1}^{k} \|_{L^\infty} + \| w_q G_{R,1}^{k} \|_{L^\infty}^2 \right\}
+ C\alpha^4 \left\{ \| w_q G_{R,1}^{k} \|_{L^\infty} + \| w_q G_{R,2}^{k} \|_{L^\infty}^4 \right\}. \tag{5.96}
\]

Finally, by taking \( C_1 > 0 \) suitably large, we have from (5.88), (5.89), (5.90) and (5.96) that

\[
\| [G_{R,1}^{k+1}, G_{R,2}^{k+1}] \|_{X_{\alpha, N_0}} \leq C_0 + C_1 \alpha \| [G_{R,1}^{k+1}, G_{R,2}^{k+1}] \|_{X_{\alpha, N_0}} + C_1 \alpha^2 \| [G_{R,1}^{k+1}, G_{R,2}^{k+1}] \|_{X_{\alpha, N_0}}^2 \]
\[ \leq C_0 \{ 1 + 2C_0 C_1 + 4C_1 C_0^2 \alpha^2 \} \leq \frac{5}{4} C_0,
\]

provided that \( \alpha \) is chosen to be sufficiently small. Thus (5.59) holds for \( n = k + 1 \). Therefore, (5.59) holds for all \( n \geq 0 \).

We now turn to prove that \( [G_{R,1}^{n}, G_{R,2}^{n}]_{n=0}^{\infty} \) is a Cauchy sequence in \( X_{\alpha, N_0} \). For this, denote

\[
\mu_{1/2} \hat{G}_{n+1} = \hat{G}_{R,1}^{n+1} + \mu_{1/2}^{1/2} \hat{G}_{R,2}^{n+1}.
\]
with
\[ [\tilde{G}_{R,1}^{n+1}, \tilde{G}_{R,2}^{n+1}] = [G_{R,1}^{n+1} - G_{R,1}^n, G_{R,2}^{n+1} - G_{R,2}^n]. \]

Then \([\tilde{G}_{R,1}^{n+1}, \tilde{G}_{R,2}^{n+1}]\) satisfies
\[
eq \alpha \{ Q(\sqrt{\tilde{G}_{R,1}^{n+1}}, \sqrt{\tilde{G}_{R,1}^{n+1}}) + Q(\sqrt{\tilde{G}_{R,2}^{n+1}}, \sqrt{\tilde{G}_{R,2}^{n+1}}) + \alpha^2 \{ Q(\sqrt{\mu \tilde{G}_{R,1}^{n+1}}, \sqrt{\mu \tilde{G}_{R,1}^{n+1}}) + Q(\sqrt{\mu \tilde{G}_{R,2}^{n+1}}, \sqrt{\mu \tilde{G}_{R,2}^{n+1}}) + Q(\sqrt{\mu \tilde{G}_{R,1}^{n+1}}, \sqrt{\mu \tilde{G}_{R,2}^{n+1}}) \} \]
=: N, y \in (-1,1), v \in \mathbb{R}^3,
\]
and
\[
eq \sqrt{2\pi} \mu \int_{v \geq 0} \sqrt{\mu \tilde{G}_{R,1}^{n+1}(\pm 1, v)} |v|dv, \ v \in \mathbb{R}^3.
\]

We claim that
\[
\|[\tilde{G}_{R,1}^{n+1}, \tilde{G}_{R,2}^{n+1}]\|_{X_{\alpha,0}} \leq C_m \|[\tilde{G}_{R,1}^{n}, \tilde{G}_{R,2}^{n}]\|_{X_{\alpha,0}}, \tag{5.97}
\]
under the condition (5.59). In fact, on the one hand, by performing a similar calculation as for obtaining (5.88), (5.89) and (5.96), one has
\[
\|[\tilde{G}_{R,1}^{n+1}, \tilde{G}_{R,2}^{n+1}]\|_{X_{\alpha,0}} \leq C \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x} N(\tilde{G}_{R,1}^{n}, \tilde{G}_{R,2}^{n})\|_{L^\infty}.
\]

On the other hand, we have from Lemma 2.2 that
\[
\sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x} N(\tilde{G}_{R,1}^{n}, \tilde{G}_{R,2}^{n})\|_{L^\infty} \leq C \sum_{0 \leq m \leq N_0} \left\{ \|w_q \partial_{v_x} \tilde{G}_{R,1}^{n}\|_{L^\infty} + \|w_q \partial_{v_x} \tilde{G}_{R,2}^{n}\|_{L^\infty} \right\}
+ C \alpha^2 \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x} \tilde{G}_{R,1}^{n}\|_{L^\infty} \sum_{0 \leq m \leq N_0} \|w_q \partial_{v_x} \tilde{G}_{R,2}^{n}\|_{L^\infty},
\]
which is further bounded by
\[
C \alpha \sum_{0 \leq m \leq N_0} \left\{ \|w_q \partial_{v_x} \tilde{G}_{R,1}^{n}\|_{L^\infty} + \|w_q \partial_{v_x} \tilde{G}_{R,2}^{n}\|_{L^\infty} \right\},
\]
according to (5.59). Thus, the claim (5.97) holds. In other words, let \(\alpha > 0\) be suitably small, then \([G_{R,1}^n, G_{R,2}^n]\) is a Cauchy sequence in \(X_{\alpha,0}\). Hence,
\[
[G_{R,1}^n, G_{R,2}^n] \rightarrow [G_{R,1}^\epsilon, G_{R,2}^\epsilon]
\]
strongly in \(X_{\alpha,0}\) as \(n \rightarrow +\infty\). Moreover, the convergence is uniform with respect to \(\epsilon\), and the limit \([G_{R,1}^\epsilon, G_{R,2}^\epsilon]\) is a unique solution to (5.55)-(5.56) and (5.57)-(5.58). In addition, \([G_{R,1}^\epsilon, G_{R,2}^\epsilon]\) also satisfies
\[
\|[G_{R,1}^\epsilon, G_{R,2}^\epsilon]\|_{X_{\alpha,0}} \leq C, \tag{5.98}
\]
where \(C > 0\) is independent of \(\epsilon\).

Furthermore, by taking the limit \(\epsilon \rightarrow 0\), we can repeat the same procedure like letting \(n \rightarrow \infty\) so that the limit function \([G_{R,1}, G_{R,2}]\) in \(X_{\alpha,0}\) is the unique solution of (5.2)-(5.3) and (5.4)-(5.5) with the same bound as (5.98). Then the proof of Proposition 5.1 is completed.

Finally, Theorem 1.1 is an immediate consequence of Proposition 4.1 and Proposition 5.1, except for the non-negativity of the solution \(F_{\alpha}(y, v)\) that will be proved from the dynamical stability of \(F_{\alpha}(y, v)\) in Theorem 1.2.
6. Unsteady problem: local existence

We now turn to the time-dependent situation. To solve the initial boundary value problem (1.21), we set the perturbation as

\[ F(t, y, v) = F_{st}(y, v) + \sqrt{\mu}f(t, y, v), \quad (6.1) \]

then \( f = f(t, y, v) \) satisfies

\[
\begin{aligned}
\partial_t f + v_y \partial_y f - \alpha v_y \partial_{v_y} f + \frac{\alpha}{2} \partial_{v_x} v_y f + Lf &= \Gamma(f, f) + \alpha \{ \Gamma(G_1 + \alpha G_R, f) + \Gamma(f, G_1 + \alpha G_R) \}, \\
t > 0, \quad y \in (-1, 1), \quad v = (v_x, v_y, v_z) \in \mathbb{R}^3, \\
\sqrt{\mu} f(0, y, v) \overset{\text{def}}{=} f_0(y, v) = F(0, y, v) - F_{st}(y, v), \quad y \in (-1, 1), \quad v \in \mathbb{R}^3, \\
(f(t, \pm1, v)|_{v_y \leq 0} &= \sqrt{2\pi\mu} \int_{v_y \geq 0} f(t, \pm1, v) \sqrt{\mu} v_y |dv|, \quad t \geq 0, \quad v \in \mathbb{R}^3.
\end{aligned}
\]

The proof of the global existence of solutions as well as the large time behavior will be left to the next section. To resolve the difficulty caused by the growth term \( \frac{\alpha}{2} \partial_{v_x} v_y f \), it is still necessary to introduce the decomposition

\[ \sqrt{\mu} f = f_1 + \sqrt{\mu} f_2, \quad (6.3) \]

where \( f_1 \) and \( f_2 \) satisfy the following initial boundary value problem

\[
\begin{aligned}
\partial_t f_1 + v_y \partial_y f_1 - \alpha v_y \partial_{v_y} f_1 + v_0 f_1 &= \chi_M \mathcal{K} f_1 - \frac{\alpha}{2} \sqrt{\mu} v_y f_2 + \alpha \{ Q(\sqrt{\mu} f, \sqrt{\mu} G_1 + \alpha G_R) + Q(\sqrt{\mu} G_1 + \alpha G_R, \sqrt{\mu} f) \} \\
&\quad + Q(\sqrt{\mu} f, \sqrt{\mu} f), \\
f_1(0, y, v) = f_0(y, v) = F_0 - F_{st}, \quad f_1(\pm1, v)|_{v_y \leq 0} = \sqrt{2\pi\mu} \int_{v_y \geq 0} f_1(\pm1, v) v_y |dv|, \quad (6.4)
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_t f_2 + v_y \partial_y f_2 - \alpha v_y \partial_{v_y} f_2 + Lf_2 &= (1 - \chi_M) \mu^{-\frac{1}{2}} \mathcal{K} f_1, \\
f_2(0, y, v) = 0, \quad f_2(\pm1, v)|_{v_y \leq 0} = \sqrt{2\pi\mu} \int_{v_y \geq 0} \sqrt{\mu} f_2(\pm1, v) v_y |dv|, \quad (6.5)
\end{aligned}
\]

respectively. Note that initial data for \( f_2 \) is set to be zero.

We will look for solutions to (6.4)-(6.5) and (6.6)-(6.7) in the following function space

\[ Y_{\alpha, T} = \left\{ (\mathcal{G}_1, \mathcal{G}_2) : \sup_{0 \leq t \leq T} \{ \| w_q \mathcal{G}_1(t) \|_{L^\infty} + \| w_q \mathcal{G}_2(t) \|_{L^\infty} \} < +\infty \right\}, \]

associated with the norm

\[ \| [\mathcal{G}_1, \mathcal{G}_2] \|_{Y_{\alpha, T}} = \sup_{0 \leq t \leq T} \{ \| w_q \mathcal{G}_1(t) \|_{L^\infty} + \| w_q \mathcal{G}_2(t) \|_{L^\infty} \}. \]

**Theorem 6.1** (Local existence). Under the conditions in Theorem 1.2, there exits \( T_* > 0 \) depending on \( \alpha \) such that the coupled system (6.4)-(6.5) and (6.6)-(6.7) admits a unique local in time solution \([f_1(t, y, v), f_2(t, y, v)]\) satisfying

\[ \| [f_1, f_2] \|_{Y_{\alpha, T_*}} \leq C_0 \varepsilon_0, \]

for some \( C_0 > 0 \).
Proof. We first consider the following system for approximation solutions

\[
\begin{align*}
p_t f_1^{n+1} + v_y \partial_y f_1^{n+1} - \alpha v_y \partial_v f_1^{n+1} + \nu_0 f_1^{n+1} &= \chi_M K f_1^n - \frac{\alpha}{2} \sqrt{\mu v_y} f_2^n + H(f_1^n, f_2^n), \\
f_1^{n+1}(0, y, v) &= f_0(y, v) = F_0 - F_{st}, \quad f_1^{n+1}(\pm 1, v)|_{v_y \leq 0} = \sqrt{2\pi\mu} \int_{v_y \geq 0} f_1^n(\pm 1, v)|_{v_y} dv,
\end{align*}
\]

(6.8)

and

\[
\begin{align*}
p_t f_2^{n+1} + v_y \partial_y f_2^{n+1} - \alpha v_y \partial_v f_2^{n+1} + \nu_0 f_2^{n+1} &= K f_2^n + (1 - \chi_M) \mu^{\frac{1}{2}} K f_1^n, \\
f_2^{n+1}(0, y, v) &= 0, \quad f_2^{n+1}(\pm 1, v)|_{v_y \leq 0} = \sqrt{2\pi\mu} \int_{v_y \geq 0} \sqrt{\mu} f_2^n(\pm 1, v)|_{v_y} dv,
\end{align*}
\]

(6.9)

where

\[
H(f_1^n, f_2^n) = \alpha \{Q(\sqrt{\mu} f^n), \sqrt{\mu}(G_1 + \alpha G_R)) + Q(\sqrt{\mu}(G_1 + \alpha G_R), \sqrt{\mu} f^n)\} + Q(\sqrt{\mu} f^n, \sqrt{\mu} f^n),
\]

and \(\sqrt{\mu} f^n = f_1^n + \sqrt{\mu} f_2^n\). Set \([f_0^0, f_0^n] = [f_0, 0]\).

Next, one can show inductively that there exists a finite \(T_* > 0\) such that

\[
\sup_{0 \leq t \leq T_*} \|w_q[f_1^n, f_2^n](t)\|_{L^\infty} \leq C_0 \varepsilon_0,
\]

(6.10)

for any \(m \geq 0\), provided that

\[
\|w_q[f_1^n, f_2^n]\|_{L^\infty} = \|w_q[F_0(y, v) - F_{st}(y, v)]\|_{L^\infty} \leq \varepsilon_0.
\]

This also implies that \([f_1^{n+1}, f_2^{n+1}]\) is well-defined by (6.8)-(6.9) and (6.10)-(6.11) if \([f_1^n, f_2^n]\) is bounded as in (6.12). Denote

\[
[\mathfrak{G}_1^n, \mathfrak{G}_2^n] = [f_1^n, f_2^n], \quad \sqrt{\mu} \mathfrak{G}^n = \mathfrak{G}_1^n + \sqrt{\mu} \mathfrak{G}_2^n,
\]

then \(\mathfrak{G}_1^n\) and \(\mathfrak{G}_2^n\) satisfy

\[
\begin{align*}
p_t \mathfrak{G}_1^{n+1} + v_y \partial_y \mathfrak{G}_1^{n+1} - \alpha v_y \partial_v \mathfrak{G}_1^{n+1} + 2 \mu \frac{v_x v_y}{1 + |v|^2} \mathfrak{G}_1^{n+1} + \nu_0 \mathfrak{G}_1^{n+1} &= \chi_M w_q K \left( \frac{\mathfrak{G}_1^n}{w_q} \right) - \frac{\alpha}{2} \sqrt{\mu v_y} \mathfrak{G}_2^n + w_q H(f_1^n, f_2^n), \\
\mathfrak{G}_1^{n+1}(0, y, v) &= w_q f_0(y, v), \quad \mathfrak{G}_1^{n+1}(\pm 1, v)|_{v_y \leq 0} = \tilde{w}_1^{-1} \int_{v_y \geq 0} \tilde{w}_1 \sqrt{2\pi \mu \mathfrak{G}_1^n(\pm 1, v)|_{v_y} dv},
\end{align*}
\]

(6.13)

and

\[
\begin{align*}
p_t \mathfrak{G}_2^{n+1} + v_y \partial_y \mathfrak{G}_2^{n+1} - \alpha v_y \partial_v \mathfrak{G}_2^{n+1} + 2 \mu \frac{v_x v_y}{1 + |v|^2} \mathfrak{G}_2^{n+1} + \nu_0 \mathfrak{G}_2^{n+1} &= w_q K \left( \frac{\mathfrak{G}_2^n}{w_q} \right) + (1 - \chi_M) w_q \mu^{\frac{1}{2}} K f_1^n, \\
\mathfrak{G}_2^{n+1}(0, y, v) &= 0, \quad \mathfrak{G}_2^{n+1}(\pm 1, v)|_{v_y \leq 0} = \tilde{w}_2^{-1} \int_{v_y \geq 0} \tilde{w}_2 \sqrt{2\pi \mu \mathfrak{G}_2^n(\pm 1, v)|_{v_y} dv},
\end{align*}
\]

(6.14)

with \([\mathfrak{G}_1^n, \mathfrak{G}_2^n] = w_q[f_1^n, f_2^n] = w_q[f_0, 0]\). Here,

\[
\tilde{w}_1 = \hat{w}_1(v) = (\sqrt{2\pi} w_q \mu)^{-1},
\]

and \(\tilde{w}_2\) is given by (5.25).
Along the same characteristic line \( (3.3) \) by noting that \( s \) is no longer a parameter and it is non-negative, \((6.13)-(6.14)\) and \((6.15)-(6.16)\) are equivalent to

\[
\Phi_1^{n+1}(t, y, v) = 1_{t_1 \leq 0} e^{-\int_{0}^{t_1} A(\tau, V(\tau)) d\tau} (w_q f_0)(V(0), V(0)) \\
+ 1_{t_1 > 0} e^{-\int_{t_1}^{t} A(\tau, V(\tau)) d\tau} \left( \frac{C_1}{w_q} \right)^{t_1} \int_{n(y_1), v_1 > 0} \tilde{w}_1 \sqrt{2\pi\mu \Phi_1^{n}}(t_1, y_1, v_1) \left| v_1 y_1 \right| dv_1 \\
\]

\[
+ \int_{\max(0, t_1)}^{t} e^{-\int_{t_1}^{\tau} A(\tau, V(\tau)) d\tau} \left( \frac{C_1}{w_q} \right)^{\tau_1} \int_{n(y_1), v_1 > 0} \tilde{w}_1 \sqrt{2\pi\mu \Phi_1^{n}}(t_1, y_1, v_1) \left| v_1 y_1 \right| dv_1 \\
\]

\[
- \alpha \int_{\max(0, t_1)}^{t} e^{-\int_{t_1}^{\tau} A(\tau, V(\tau)) d\tau} \frac{V_x(s) V_y(s)}{2} \sqrt{\mu} (V(s)) \Phi_2^{n}(V(s)) ds \\
+ \int_{\max(0, t_1)}^{t} e^{-\int_{t_1}^{\tau} A(\tau, V(\tau)) d\tau} \left( w_q H(f_1^{n}, f_2^{n}) \right)(V(s)) ds,
\]

(6.17)

and

\[
\Phi_2^{n+1}(t, y, v) = 1_{t_1 > 0} e^{-\int_{t_1}^{t} A(\tau, V(\tau)) d\tau} \left( \frac{C_1}{w_q} \right)^{t_1} \int_{n(y_1), v_1 > 0} \tilde{w}_1 \sqrt{2\pi\mu \Phi_2^{n}}(t_1, y_1, v_1) \left| v_1 y_1 \right| dv_1 \\
\]

\[
+ \int_{\max(0, t_1)}^{t} e^{-\int_{t_1}^{\tau} A(\tau, V(\tau)) d\tau} \left( \frac{C_1}{w_q} \right)^{\tau_1} \int_{n(y_1), v_1 > 0} \tilde{w}_1 \sqrt{2\pi\mu \Phi_2^{n}}(t_1, y_1, v_1) \left| v_1 y_1 \right| dv_1 \\
\]

\[
- \alpha \int_{\max(0, t_1)}^{t} e^{-\int_{t_1}^{\tau} A(\tau, V(\tau)) d\tau} \left[ w_q K \left( \frac{C_1}{w_q} \right) \right] (V(s)) ds \\
+ \int_{\max(0, t_1)}^{t} e^{-\int_{t_1}^{\tau} A(\tau, V(\tau)) d\tau} \left[ \left. 1 - \left( \frac{\sqrt{\mu}}{w_q} \right)^{t_1} \right] \right] (V(s)) ds,
\]

(6.18)

where

\[
A(\tau, V(\tau)) = \nu_0 + 2q\alpha \frac{V_x(\tau)V_y(\tau)}{1 + \left| V(\tau) \right|^2} \geq \nu_0 / 2,
\]

provided that \( q\alpha \) is suitably small.

For the boundary terms \( I_b^{(1)} \) and \( I_b^{(2)} \), we use the equations \((6.17)\) and \((6.18)\) recursively to obtain

\[
I_b^{(1)} = \sum_{j=1}^{5} I_j^{(1)}, \quad I_b^{(2)} = \sum_{j=1}^{3} I_j^{(2)}
\]

with

\[
I_1^{(1)} = 1_{t_1 > 0} e^{-\int_{0}^{t_1} A(\tau, V(\tau)) d\tau} \tilde{w}_1 \left( V(t_1) \right) \int_{n(y_1), v_1 > 0} \tilde{w}_1 \sqrt{2\pi\mu \Phi_1^{n+1}}(t_1, y_1, v_1) \left| v_1 y_1 \right| dv_1
\]

\[
I_2^{(1)} = \gamma_0^{(1)} \sum_{k=1}^{k-1} \int_{n(y_1), v_1 > 0} 1_{t_1 \leq 0} (w_q f_0)(V^{(1)}(0), V^{(1)}(0)) d\tilde{\Sigma}_1^{(1)}(t_1)
\]

\[
I_3^{(1)} = \gamma_0^{(1)} \int_{n(y_1), v_1 > 0} \left\{ \int_{l=1}^{l-1} \int_{l=1}^{l_1} \left[ \chi M w_q K \left( \frac{C_1}{w_q} \right) \right] (Y(l)_{V^{(1)}}, V^{(1)}_{V^{(1)}}) d\tilde{\Sigma}_1^{(1)}(t_1)
\]

\[
I_4^{(1)} = -\alpha \gamma_0^{(1)} \int_{n(y_1), v_1 > 0} \left\{ \int_{l=1}^{l-1} \int_{l=1}^{l_1} \frac{V_x V_y}{2} \sqrt{\mu} \left( \frac{C_1}{w_q} \right)^{t_1} \right\} (Y(l)_{V^{(1)}}, V^{(1)}_{V^{(1)}}) d\tilde{\Sigma}_1^{(1)}(t_1)
\]

\[
I_5^{(1)} = \gamma_0^{(1)} \sum_{k=1}^{k-1} \int_{n(y_1), v_1 > 0} \left\{ \int_{l=1}^{l-1} \int_{l=1}^{l_1} \left( w_q H(f^{n-1}, f^{n-1}_2) \right) (Y(l)_{V^{(1)}}, V^{(1)}_{V^{(1)}}) d\tilde{\Sigma}_1^{(1)}(t_1)
\]
\[ I_1^{(2)} = \frac{1}{W_0^{(2)}} \int_{t_1 > 0} e^{-f_1^{t_1}} A(T, V(t))\, dt \sum_{i \leq t} \int_{t_{i+1} \leq t_i} \int_{t_{i+1} > t_i} \left\{ w_1 K \left( \frac{\Theta_{i-1}^n}{w_q} \right) \right\} (Y_{e_i}^l, V_{e_l}) (s) dS_{i-1}^{(2)} (s) ds, \]

where \( k \geq 2 \). Here, similar to \((5.24), \Sigma_i^{(1)} (s) (i = 1, 2) \) is given as

\[ \Sigma_i^{(1)} (s) = \prod_{j = i+1}^{k-1} \int_{t_j \leq t_i} \int_{t_{i+1} > t_i} \left\{ w_1 K \left( \frac{\Theta_{i-1}^n}{w_q} \right) \right\} (Y_{e_i}^l, V_{e_l}) (s) dS_{i-1}^{(2)} (s) ds, \]

To obtain \((6.12)\), one can first prove that for fixed finite \( k > 0 \) and any \( t \geq 0 \),

\[ \sup_{0 \leq t \leq T_*} \sup_{0 \leq s \leq t} \| \Theta_1^n + \Theta_2^n \|_{L^\infty} \leq C(k) \| w_q f_0 \|_{L^\infty} \leq \frac{1}{2} C_0 \varepsilon_0, \quad (6.19) \]

by choosing \( C_0 > 0 \) suitably large. Note that \((6.19)\) can be easily obtained by using \((6.17)\) and \((6.18)\) recursively because \( k \) is finite.

In the following, we prove \((6.12)\) for \( m = n + 1 \) under the assumption that it holds for \( m \leq n \). By letting \( t \leq T_* \) with \( T_* > 0 \) being suitably small and applying Lemma 8.1, we have

\[ \sup_{0 \leq t \leq T_*} \| \Theta_1^n + \Theta_2^n \|_{L^\infty} \leq \left( \frac{C}{q} + C \varepsilon \right) \sup_{1 \leq t \leq T_*} \| \Theta_1^{n+1-l} \|_{L^\infty} + C \sup_{0 \leq t \leq T_*} \| \Theta_2^{n+1-l} \|_{L^\infty} \]

\[ + C \sup_{1 \leq t \leq T_*} \| \Theta_1^{n+1-l} \|_{L^\infty} \| \Theta_2^{n+1-l} \|_{L^\infty} + C \| w_q f_0 \|_{L^\infty} \leq C \| w_q f_0 \|_{L^\infty} \left( \frac{C}{q} + C + C \varepsilon \right) C_0 \varepsilon_0 + C \| w_q f_0 \|_{L^\infty} \]

\[ \leq C \| w_q f_0 \|_{L^\infty} \leq C \left( \frac{C}{q} + C + C \varepsilon \right) C_0 \varepsilon_0 + C \| w_q f_0 \|_{L^\infty}, \quad (6.20) \]

and

\[ \sup_{0 \leq t \leq T_*} \| \Theta_2^n \|_{L^\infty} \leq C T_* \sup_{1 \leq t \leq T_*} \| \Theta_1^{n+1-l} \|_{L^\infty} \]

\[ + C(T_* + \varepsilon) \sup_{1 \leq t \leq T_*} \| \Theta_2^{n+1-l} \|_{L^\infty}, \quad (6.21) \]

where Lemma 2.4 has been used to have the factor \( \frac{1}{7} \) in \((6.20)\), and the coefficient \( T_* \) on the right hand side of \((6.21)\) comes from the last two terms in \((6.18)\) as well as \( I_2^{(2)} \) and \( I_3^{(2)} \). Choosing \( T_* \) and \( \varepsilon \) suitably small so that \( C(T_* + \varepsilon) \leq \frac{1}{8} \), and using the induction argument, we have from \((6.21)\) and \((6.20)\) that

\[ \| \Theta_2^n \|_{L^\infty} \leq \frac{1}{8} \sup_{0 \leq t \leq k} \| \Theta_2^n \|_{L^\infty} \]

\[ \leq \frac{1}{8} \left( \frac{C}{q} + C \varepsilon \right) C_0 \varepsilon_0 + C \| w_q f_0 \|_{L^\infty} \leq \frac{1}{8} \left( \frac{C}{q} + C \varepsilon \right) C_0 \varepsilon_0 + C \| w_q f_0 \|_{L^\infty}, \]

where \( \left[ \frac{n}{k} \right] \) stands for the largest integer no more than \( \frac{n}{k} \). Therefore,

\[ \| \Theta_1^n \|_{L^\infty} + \| \Theta_2^n \|_{L^\infty} \leq \frac{1}{8} \sup_{0 \leq t \leq k} \| \Theta_2^n \|_{L^\infty} \]

\[ \leq \frac{1}{8} \left( \frac{C}{q} + C \varepsilon \right) C_0 \varepsilon_0 + C \| w_q f_0 \|_{L^\infty} + \left( \frac{C}{q} + C \varepsilon \right) C_0 \varepsilon_0 + C \| w_q f_0 \|_{L^\infty}, \quad n \geq k. \]
This together with (6.19) implies that (6.12) holds for \( m = n+1 \) because \( q > 0 \) can be sufficiently large and \( \varepsilon_0 > 0 \) as well as \( \alpha > 0 \) can be suitably small.

Let us now show that \( \{[f^n_1, f^n_2]\}_{n=1}^\infty \) converges strongly in the space \( Y_{a,T} \). We denote \([\tilde{G}^n_1, \tilde{G}^n_2] = [\tilde{G}^n_1 - \tilde{G}^{n-1}_1, \tilde{G}^n_2 - \tilde{G}^{n-1}_2] \) with \( n \geq 1 \). Then \([\tilde{G}^n_1, \tilde{G}^n_2] \) satisfies
\[
\partial_t \tilde{G}^{n+1}_1 + v_y \partial_y \tilde{G}^{n+1}_1 - \alpha v_y \partial_y \tilde{G}^{n+1}_1 + 2q\alpha \frac{v_x v_y}{1 + |v|^2} \tilde{G}^{n+1}_1 + \nu_0 \tilde{G}^{n+1}_1 = \chi_M w_q K \left( \frac{\tilde{G}^n_1}{w_q} \right) - \frac{\alpha}{2} \sqrt{\nu} v_y \tilde{G}^n_2 + w_q [H(f^n_1, f^n_2) - H(f^{n-1}_1, f^{n-1}_2)],
\]
and
\[
\partial_t \tilde{G}^{n+1}_2 + v_y \partial_y \tilde{G}^{n+1}_2 - \alpha v_y \partial_y \tilde{G}^{n+1}_2 + 2q\alpha \frac{v_x v_y}{1 + |v|^2} \tilde{G}^{n+1}_2 + \nu_0 \tilde{G}^{n+1}_2 = w_q K \left( \frac{\tilde{G}^n_2}{w_q} \right) + (1 - \chi_M) w_q \mu^\frac{1}{2} \mathcal{K} \tilde{f}^n_1,
\]
for \( \tilde{G}^{n+1}_1(0, y, v) = 0 \), \( \tilde{G}^{n+1}_1(\pm 1, v) |_{v_y = 0} = \tilde{w}_1^{-1} \int \tilde{w}_1 \sqrt{2\pi} \mu \tilde{G}^n_1(\pm 1, v) |_{v_y} dv \), and
\[
\tilde{G}^{n+1}_2(0, y, v) = 0, \quad \tilde{G}^{n+1}_2(\pm 1, v) |_{v_y = 0} = \tilde{w}_2^{-1} \int \tilde{w}_2 \sqrt{2\pi} \mu \tilde{G}^n_2(\pm 1, v) |_{v_y} dv,
\]
where \( \tilde{f}^n = f^n_1 - f^{n-1}_1 \), and \( \sqrt{\nu} \tilde{G}^n = \tilde{G}^n_1 + \sqrt{\nu} \tilde{G}^n_2 \). Then similar to (6.20) and (6.21), one has
\[
\sup_{0 \leq t \leq T^*} \| \tilde{G}^{n+1}_1 \|_{L^\infty} \leq \left( \frac{C}{q} + C \varepsilon \right) \sup_{1 \leq l \leq k} \sup_{0 \leq t \leq T^*} \| \tilde{G}^{n-l}_1 \|_{L^\infty} + C \alpha \sup_{1 \leq l \leq k} \sup_{0 \leq t \leq T^*} \| \tilde{G}^{n-l}_2 \|_{L^\infty} + C \varepsilon_0 \sup_{1 \leq l \leq k} \sup_{0 \leq t \leq T} \| \tilde{G}^{n-l}_1 \|_{L^\infty} + \| \tilde{G}^{n-l}_2 \|_{L^\infty}, \tag{6.22}
\]
and
\[
\sup_{0 \leq t \leq T} \| \tilde{G}^{n+1}_2 \|_{L^\infty} \leq CT^* \sup_{1 \leq l \leq k} \sup_{0 \leq t \leq T^*} \| \tilde{G}^{n-l}_1 \|_{L^\infty} + C(T^* + \varepsilon) \sup_{1 \leq l \leq k} \sup_{0 \leq t \leq T^*} \| \tilde{G}^{n-l+1}_2 \|_{L^\infty}. \tag{6.23}
\]
Plugging (6.22) into (6.23) gives
\[
\sup_{0 \leq t \leq T} \| \tilde{G}^{n+1}_2 \|_{L^\infty} \leq C \left( \frac{1}{q} + \alpha + \varepsilon_0 + T^* + \varepsilon \right) \sup_{1 \leq l \leq k} \sup_{0 \leq t \leq T} \| \tilde{G}^{n-l+1}_1, \tilde{G}^{n-l+1}_2 \|_{L^\infty}. \tag{6.24}
\]
By taking \( q > 0 \) sufficiently large and \( \alpha > 0 \), \( \varepsilon_0 > 0 \), as well as \( T^* \), \( \varepsilon \) suitably small, we have from (6.24) and (6.22) that
\[
\sup_{0 \leq t \leq T^*} \| \tilde{G}^{n+1}_1 \|_{L^\infty} + \| \tilde{G}^{n+1}_2 \|_{L^\infty} \leq \frac{1}{8(k-1)} \sup_{0 \leq t \leq T^*} \| \tilde{G}^n_1, \tilde{G}^n_2 \|_{L^\infty}, n \geq k.
\]
On the other hand, \( \sup_{0 \leq t \leq T^*} \| \tilde{G}^n_1, \tilde{G}^n_2 \|_{L^\infty} \) is bounded due to (6.12). Hence it follows that \( \{[f^n_1, f^n_2]\}_{n=1}^\infty \) is a Cauchy sequence in the space \( Y_{a,T} \), and there is a unique \([f_1, f_2] \in Y_{a,T} \) such that \( [f^n_1, f^n_2] \) converges strongly to \([f_1, f_2] \) as \( n \to +\infty \). Hence \([f_1, f_2] \) is the desired local in time solution to the coupled system (6.4)-(6.5) and (6.6)-(6.7). This completes the proof of Theorem 6.1. \( \square \)

7. Unsteady Problem: Asymptotic Stability and Positivity

This section is about the global existence and large time behavior of solution to the initial boundary value problem (6.2). Recall the decomposition (6.3) with \( f_1 \) and \( f_2 \) satisfying the coupled system (6.4)-(6.5) and (6.6)-(6.7). Firstly, we focus on the uniform \( L^\infty \cap L^2 \) estimates under the \textit{a priori} assumption
\[
\sup_{s \geq 0} \{ e^{\lambda_0 s} \| w_q f_1(s, y, v) \|_{L^\infty} + e^{\lambda_0 s} \| w_q f_2(s, y, v) \|_{L^\infty} \} \leq \bar{\varepsilon}, \tag{7.1}
\]
for a constant $\varepsilon > 0$ suitably small, where $\lambda_0 > 0$ independent of $\alpha$ is to be determined later. And then we will give the proof of Theorem 1.2.

7.1. $L^\infty$ estimates. As in the proof of Theorem 6.1, the $L^\infty$ estimates of $f$ follows from the uniform $L^\infty$ estimates on $f_1$ and $f_2$.

**Lemma 7.1.** Let $0 < \lambda_0 \leq \frac{\nu_0}{q}$, then under the assumption (7.1), it holds that

$$\sup_{0 \leq s \leq t} e^{\lambda_0 s} \|w_q f_1(s)\|_{L^\infty} \leq C_q \|w_q f_0\|_{L^\infty} + C(\alpha + \varepsilon) \sup_{0 \leq s \leq t} e^{\lambda_0 s} \|w_q f_2(s)\|_{L^\infty},$$

and

$$\sup_{0 \leq s \leq t} e^{\lambda_0 s} \|w_q f_2(s)\|_{L^\infty} \leq C\|w_q f_0\|_{L^\infty} + C \sup_{0 \leq s \leq t} e^{\lambda_0 s} \|w_q f_2(s)\|,$$

for any $t \geq 0$.

**Proof.** For brevity, set

$$[g_1, g_2](t, y, v) = e^{\lambda_0 t} w_q(v) [f_1, f_2](t, y, v)$$

with $\lambda_0 > 0$ to be chosen. Then, the IBVP for $[g_1, g_2]$ is given as follows:

$$\partial_t g_1 + v_y \partial_y g_1 - \alpha v_y \partial_v g_1 + 2q\alpha \frac{v_x v_y}{1 + |v|^2} g_1 + (\nu_0 - \lambda_0) g_1$$

$$= -\chi_M w_q K \left( \frac{g_1}{w_q} \right) - \frac{\alpha}{2} \sqrt{\mu} v_x v_y g_2 + e^{\lambda_0 t} w_q H(f_1, f_2),$$

$$g_1(0, y, v) = w_q f_0(x, v), \quad g_1(\pm 1, v)|_{v_y \geq 0} = \tilde{w}_1^{-1} \int_{v_y \geq 0} \tilde{w}_1 \sqrt{2\pi \mu} g_1(\pm 1, v) |v_y| dv,$$

and

$$\partial_t g_2 + v_y \partial_y g_2 - \alpha v_y \partial_v g_2 + 2q\alpha \frac{v_x v_y}{1 + |v|^2} g_2 + (\nu_0 - \lambda_0) g_2$$

$$= w_q K \left( \frac{g_2}{w_q} \right) + (1 - \chi_M) w_q \mu^{-\frac{1}{2}} K \left( \frac{g_1}{w_q} \right),$$

$$g_2(0, y, v) = 0, \quad g_2(\pm 1, v)|_{v_y \geq 0} = \tilde{w}_2^{-1} \int_{v_y \geq 0} \tilde{w}_2 \sqrt{2\pi \mu} g_2(\pm 1, v) |v_y| dv.$$

Along the characteristic line (3.3) the solution to the above problem can be written in the mild form:

$$g_1(t, y, v) = 1_{t_1 \leq t} e^{-\int_{t_1}^t A_1(\tau, V(\tau)) \, d\tau} (w_q f_0)(Y(0), V(0))$$

$$+ \int_{t_1}^t e^{-\int_s^t A_1(\tau, V(\tau)) \, d\tau} \left\{ \chi_M w_q K \left( \frac{g_1}{w_q} \right) \right\} (V(s)) \, ds$$

$$- \alpha \int_{t_1}^t e^{-\int_s^t A_1(\tau, V(\tau)) \, d\tau} V_1(s) V_2(s) \sqrt{\mu} (V(s)) g_2(V(s)) \, ds$$

$$+ \int_{t_1}^t e^{-\int_s^t A_1(\tau, V(\tau)) \, d\tau} e^{\frac{\nu_0}{q} s} (w_q H(f_1, f_2)) (V(s)) \, ds + \sum_{n=1}^5 J_n^{(1)},$$

and

$$g_2(t, y, v) = \int_{t_1}^t e^{-\int_s^t A_1(\tau, V(\tau)) \, d\tau} \left\{ (1 - \chi_M) \mu^{-\frac{1}{2}} w_q K \left( \frac{g_1}{w_q} \right) \right\} (V(s)) \, ds$$

$$+ \int_{t_1}^t e^{-\int_s^t A_1(\tau, V(\tau)) \, d\tau} \left[ w_q K \left( \frac{g_2}{w_q} \right) \right] (V(s)) \, ds + \sum_{n=1}^3 J_n^{(2)},$$

where

$$A_1(\tau, V(\tau)) = \nu_0 - \lambda_0 + 2q\alpha \frac{V_1(\tau)V_2(\tau)}{1 + |V(\tau)|^2}. $$
We will take \(0 < \lambda_0 \leq \frac{N}{2}\) and let \(2q \alpha \leq \frac{N}{2}\) such that \(A_1(\tau, V(\tau)) \geq \frac{N}{2}\). Moreover, for an integer \(k \geq 2\), the terms \(J_n^{(1)} (1 \leq n \leq 5)\) in (7.5) are given by

\[
J_1^{(1)} = \sum_{n=1}^{k-1} \int_{t_j}^{t_{j+1}} \mathbf{1}_{t_1 > 0} e^{-f_{t_j}^{(1)} A_1(\tau, V(\tau))d\tau} \frac{1}{w_1}(V(t_1)) \prod_{j=1}^{l-1} \mathbf{1}_{t_j > 0} g_{n+1-k}^{(k-1)} (t_k, v_k, V^{k-1}_c(t_k)) d\hat{\Sigma}_k^{(1)} (t_k),
\]

\[
J_2^{(1)} = \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{t_1 > 0} (w_q f_0) (V^{(1)}(0), V^{(1)}(0)) d\hat{\Sigma}_l^{(1)} (0),
\]

\[
J_3^{(1)} = \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{t_1 > 0} (w_q f_0) (V^{(1)}(0), V^{(1)}(0)) d\hat{\Sigma}_l^{(1)} (0),
\]

\[
J_4^{(1)} = \alpha \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{t_1 > 0} (w_q f_0) (V^{(1)}(0), V^{(1)}(0)) d\hat{\Sigma}_l^{(1)} (0),
\]

\[
J_5^{(1)} = \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{t_1 > 0} (w_q f_0) (V^{(1)}(0), V^{(1)}(0)) d\hat{\Sigma}_l^{(1)} (0),
\]

And the terms \(J_n^{(2)} (1 \leq n \leq 3)\) in (7.6) are

\[
J_1^{(2)} = \sum_{n=1}^{k-1} \int_{t_j}^{t_{j+1}} \mathbf{1}_{t_1 > 0} e^{-f_{t_j}^{(1)} A_1(\tau, V(\tau))d\tau} \frac{1}{w_2}(V(t_1)) \prod_{j=1}^{l-1} \mathbf{1}_{t_j > 0} g_2 (t_k, v_k, V^{k-1}_c(t_k)) d\hat{\Sigma}_k^{(2)} (t_k),
\]

\[
J_2^{(2)} = \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{t_1 > 0} (w_q f_0) (V^{(1)}(0), V^{(1)}(0)) d\hat{\Sigma}_l^{(2)} (0),
\]

\[
J_3^{(2)} = \sum_{l=1}^{k-1} \int_{t_l}^{t_{l+1}} \mathbf{1}_{t_1 > 0} (w_q f_0) (V^{(1)}(0), V^{(1)}(0)) d\hat{\Sigma}_l^{(2)} (0),
\]

where

\[
\hat{\Sigma}_l^{(1)} (s) = \prod_{j=1}^{l-1} ds e^{-f_{t_j}^{(1)} A_1(\tau, V(\tau))d\tau} \tilde{\omega}_i (v_i) ds = \prod_{j=1}^{l-1} \frac{w_i (v_i)}{w_i (V^{(1)}(t_j))} \prod_{j=1}^{l-1} e^{-f_{t_j}^{(1)} A_1(\tau, V(\tau))d\tau} ds, \quad i = 1, 2.
\]

Consequently, for any \(t \geq 0\), by applying Lemmas 2.2, 2.4 and 8.1 as well as the \(a \text{ priori}\) assumption (7.1), we get from (7.5) that

\[
\sup_{0 \leq s \leq t} ||g_1(s, y, v)||_{L^\infty} \leq C_q ||w_q f_0||_{L^\infty} + \left( C_q + C_\varepsilon \right) \sup_{0 \leq s \leq t} ||g_1(s, y, v)||_{L^\infty} + C(\alpha + \varepsilon) \sup_{0 \leq s \leq t} \{ ||g_1(s, y, v)||_{L^\infty} + ||g_2(s, y, v)||_{L^\infty} \},
\]

that gives (7.2).

For \(g_2\), similar to (5.26), one has

\[
|g_2(t, y, v)| \leq C_q e^{-\frac{N}{2}(t-t_1)} \int_{\max(t_1,0)}^{t} e^{-\frac{N}{2}(t-s)} \int_{\mathbb{R}^3} k_w (V(s), v') |g_2(s, Y(s; t, y, v), v')| dv' ds + C_q e^{-\frac{N}{2}(t-t_1)} \sum_{l=1}^{k-1} \int_{t_l}^{t} \int_{\mathbb{R}^3} k_w (V^{(1)}(s), v') \times |g_2(s, Y^{(1)}(s; t, y, v), v')| dv' d\hat{\Sigma}_l^{(2)} (s) ds + \mathcal{P}(t),
\]
where
\[ \mathcal{P}(t) = C_q \sup_{0 \leq s \leq t} \| g_1(s) \|_{L^\infty} + \tilde{C}_q \sup_{0 \leq s \leq t} \| g_2(s) \|_{L^\infty}. \] (7.8)

We now have by iterating (7.7) that
\[
\begin{align*}
|g_2(t, y, v)| & \leq C_q \int_{\max \{t_1, 0\}}^{t} e^{-\frac{\alpha}{2}(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v') \int_{\max \{t_1', 0\}}^{s} e^{-\frac{\alpha}{2}(s-s')} \int_{\mathbb{R}^3} k_w(\tilde{V}(s'; Y(s), v'), v'') \\
& \quad \times |g_2(s', \tilde{Y}(s'; Y(s), v'), v'')| \, dv'' \, ds' \, dv' \, ds \\
& + C_q \int_{\max \{t_1, 0\}}^{t} e^{-\frac{\alpha}{2}(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v') e^{-\frac{\alpha}{2}(s-t_1')} \sum_{\ell=1}^{\infty} \int_{\max \{t_{\ell+1}', 0\}}^{t_{\ell}} \int_{\mathbb{R}^3} k_w(V_{\ell}(s'; v), v') \int_{\max \{t_{\ell}' , 0\}}^{s} e^{-\frac{\alpha}{2}(s-s')} \int_{\mathbb{R}^3} k_w(\tilde{V}(s'; Y_{\ell}(s'; v), v'), v'') \\
& \quad \times |g_2(s', \tilde{Y}(s'; Y_{\ell}(s; v), v'), v'')| \, dv'' \, ds' \, dv' \, ds' \, dv'' \, ds' \, dv' \, ds' \, dv' \, ds. \\
& + C_q \int_{\max \{t_1, 0\}}^{t} \int_{\mathbb{R}^3} k_w(V_{\ell}(s; v), v') \int_{\max \{t_{\ell} + 1, 0\}}^{t_{\ell}} \int_{\mathbb{R}^3} k_w(\tilde{V}_{\ell}(s; v), v') e^{-\frac{\alpha}{2}(s-t_{\ell}')} \sum_{\ell=1}^{\infty} \int_{\max \{t_{\ell+1}' , 0\}}^{t_{\ell}} \int_{\mathbb{R}^3} k_w(V_{\ell}(s; v), v') \int_{\max \{t_{\ell}' , 0\}}^{s} e^{-\frac{\alpha}{2}(s-s')} \int_{\mathbb{R}^3} k_w(\tilde{V}(s; Y_{\ell}(s; v), v'), v'') \\
& \quad \times |g_2(s', \tilde{Y}(s'; Y_{\ell}(s; v), v'), v'')| \, dv'' \, ds' \, dv' \, ds' \, dv'' \, ds' \, dv' \, ds' \, dv' \, ds'. \\
& + C_q \int_{\max \{t_1, 0\}}^{t} e^{-\frac{\alpha}{2}(t-s)} \int_{\mathbb{R}^3} k_w(V(s), v') \mathcal{P}(s) \, dv' \, ds \\
& + C_q \int_{\max \{t_1, 0\}}^{t} \int_{\mathbb{R}^3} k_w(V_{\ell}(s; v), v') \mathcal{P}(s) \, dv' \, ds. \\
& \leq C_q \sup_{0 \leq s \leq t_0} \| g_2(s) \|_{L^\infty} \leq C \sup_{0 \leq s \leq t_0} \| g_1(s) \|_{L^\infty} + C(T_0) \sup_{0 \leq s \leq t_0} \| f_2(s) \| + C \sup_{0 \leq s \leq t_0} \mathcal{P}(s), \\
\end{align*}
\] (7.9)

With (7.9), similar to (5.28), for sufficiently large $T_0 > 0$, we have
\[
\sup_{0 \leq s \leq T_0} \| g_2(s) \|_{L^\infty} \leq C \tilde{E} \sup_{0 \leq s \leq T_0} \| g_2(s) \|_{L^\infty} + C(T_0) \sup_{0 \leq s \leq T_0} \| f_2(s) \| + C \sup_{0 \leq s \leq T_0} \mathcal{P}(s),
\]
which together with (7.8) gives
\[
\sup_{0 \leq s \leq T_0} \| g_2(s) \|_{L^\infty} \leq C \sup_{0 \leq s \leq T_0} \| g_1(s) \|_{L^\infty} + C(T_0) \sup_{0 \leq s \leq T_0} \| f_2(s) \|. 
\] (7.10)

Next, combining (7.2) at $t = T_0$ and (7.10), one has
\[
\sup_{0 \leq s \leq T_0} \| [g_1, g_2](s) \|_{L^\infty} \leq C \| w_q(f_1(0, y, v), f_2(0, y, v)) \|_{L^\infty} + C(T_0) \sup_{0 \leq s \leq T_0} \| f_2(s) \| \\
\leq C \| w_q f_0 \|_{L^\infty} + C(T_0) \sup_{0 \leq s \leq T_0} \| f_2(s) \|. 
\]

Then it follows that for any $t \in [0, T_0]$,
\[
\| w_q(f_1, f_2)(t) \|_{L^\infty} \leq C e^{-\lambda_0 t} \| w_q f_0 \|_{L^\infty} + C(T_0) e^{-\lambda_0 t} \sup_{0 \leq s \leq T_0} \| f_2(s) \|. 
\] (7.11)

In particular, we have
\[
\| w_q(f_1, f_2)(T_0) \|_{L^\infty} \leq C e^{-\lambda_0 T_0} \| w_q(f_1(0, y, v), f_2(0, y, v)) \|_{L^\infty} + C(T_0) e^{-\lambda_0 T_0} \sup_{0 \leq s \leq T_0} \| f_2(s) \| \\
\leq C e^{-\lambda_0 T_0} \| w_q f_0 \|_{L^\infty} + C(T_0) e^{-\lambda_0 T_0} \sup_{0 \leq s \leq T_0} \| f_2(s) \|. 
\] (7.12)

Moreover, (7.11) can be extended to
\[
\| w_q(f_1, f_2)(t) \|_{L^\infty} \leq C e^{-\lambda_0 (t-s)} \| w_q(f_1, f_2)(s) \|_{L^\infty} + C(T_0) e^{-\lambda_0 (t-s)} \sup_{s \leq \tau \leq t} \| f_2(\tau) \|, 
\] (7.13)
for any $t \in [s, s + T_0]$ with $s \geq 0$. 
Next, for any integer \( m \geq 1 \), we can repeat the estimate (7.12) in finite times so that the functions \([f_1, f_2](t_0 + s)\) for \( l = m - 1, m - 2, \ldots, 0 \) satisfy
\[
\|w_q[f_1, f_2](mT_0)\|_{L^\infty} \leq C e^{-\lambda_0 T_0} \|w_q[f_1, f_2](\{m - 1\} T_0)\|_{L^\infty} + C(T_0) e^{-\lambda_0 T_0} \sup_{\{m - 1\} T_0 \leq s \leq mT_0} \|f_2(s)\|
\]
\[
\leq C e^{-\lambda_0 T_0} \|w_q[f_1, f_2](\{m - 1\} T_0)\|_{L^\infty} + C(T_0) e^{-\lambda_0 (m-1) T_0} \sup_{\{m - 1\} T_0 \leq s \leq mT_0} \|e^{\lambda s} f_2(s)\|
\]
\[
\leq C e^{-\lambda_0 mT_0} \|w_q[f_1, f_2](0)\|_{L^\infty} + C(T_0) \sum_{l=0}^{m-1} e^{-m\lambda_0 T_0} \sup_{\{m-l-1\} T_0 \leq s \leq \{m-l\} T_0} \|e^{\lambda s} f_2(s)\|
\]
\[
\leq C e^{-\lambda_0 mT_0} \|w_q[f_1, f_2](0)\|_{L^\infty} + C(T_0) e^{-m\lambda_0 T_0} \sup_{0 \leq s \leq mT_0} \|e^{\lambda s} f_2(s)\|. \tag{7.14}
\]
Furthermore, for any \( t \geq T_0 \), we can find an integer \( m \geq 0 \) such that \( t = mT_0 + s \) with \( 0 \leq s \leq T_0 \). Then we have, on one hand, by (7.14), that
\[
\|w_q[g_1, g_2](mT_0)\|_{L^\infty} \leq C \|w_q f_0\|_{L^\infty} + C(T_0) \sup_{0 \leq s \leq mT_0} \|e^{\lambda s} f_2(s)\|. \tag{7.15}
\]
On the other hand, (7.13) implies that
\[
\|w_q[f_1, f_2](t)\|_{L^\infty} = \|w_q[f_1, f_2](mT_0 + s)\|_{L^\infty}
\]
\[
\leq C e^{-\lambda s} \|w_q[f_1, f_2](mT_0)\|_{L^\infty} + C(T_0) e^{-\lambda s} \sup_{mT_0 \leq t \leq mT_0 + s} \|e^{\lambda \tau} f_2(\tau)\|
\]
which is equivalent to
\[
\|w_q[g_1, g_2](t)\|_{L^\infty} = \|w_q[g_1, g_2](mT_0 + s)\|_{L^\infty}
\]
\[
\leq C \|w_q[g_1, g_2](mT_0)\|_{L^\infty} + C(T_0) \sup_{mT_0 \leq t \leq mT_0 + s} \|e^{\lambda \tau} f_2(\tau)\|. \tag{7.16}
\]
Consequently, applying (7.15) to (7.16) gives the second estimate (7.3). This together with (7.2) concludes the \( L^\infty \) estimate on \( f_1 \) and \( f_2 \), and then it completes the proof of Lemma 7.1. \( \square \)

### 7.2. \( L^2 \) estimates

In order to close the \( L^\infty \) estimate in terms of (7.2) and (7.3), we need to deduce the \( L^2 \) estimate on \( e^{\lambda t} f_2(t, y, v) \). As pointed out in Section 4, the key is to obtain the dissipation estimate of the macroscopic component of \( f_2 \) as well as \( f_1 \) through the conservation of mass. Therefore, we need resort to the original perturbation \( \sqrt{\mu} g_\lambda := g_1 + \sqrt{\mu} g_2 \) with some abuse of notations
\[
[g_1, g_2](t, y, v) := e^{\lambda t} [f_1, f_2](t, y, v) \tag{7.17}
\]
compared to (7.4) in the previous subsection. Note that the velocity weight is no longer needed for the \( L^2 \) estimates. Indeed, the only time-weighted function \( g_\lambda \) satisfies the IBVP
\[
\begin{aligned}
& \partial_t g_\lambda + v_y \partial_y g_\lambda - \alpha v_y \partial_v g_\lambda + \frac{4}{3} v_x v_y g_\lambda + L g_\lambda - \lambda_0 g_\lambda \\
& \quad = e^{\lambda t} \Gamma(f, f) + \alpha e^{\lambda t} \left( \Gamma(G_1 + \alpha G_R, f) + \Gamma(f, G_1 + \alpha G_R) \right),
\end{aligned}
\tag{7.18}
\]
\[
t > 0, \, y \in (-1, 1), \, v = (v_x, v_y, v_z) \in \mathbb{R}^3,
\]
\[
\sqrt{\mu} g_\lambda(0, y, v) = f_0(y, v) - F(0, y, v) - F_{st}(y, v), \, y \in (-1, 1), \, v \in \mathbb{R}^3,
\]
\[
g_\lambda(t, \pm 1, v)|_{v_y = 0} = - \sqrt{2\pi \mu} \int_{v_y = 0} \sqrt{\mu} g_\lambda(t, \pm 1, v)|v_y| dv, \, t \geq 0, \, v \in \mathbb{R}^3.
\]
Note that since \( \int_{\mathbb{R}^3} f(t, y, v) \sqrt{\mu} dv dy = 0 \) holds according to (1.22) and (6.2), it is direct to see that
\[
\int_{-1}^{1} \int_{\mathbb{R}^3} g_\lambda(t, y, v) \sqrt{\mu} dv dy = 0, \quad \forall \, t \geq 0.
\]
Next, as in (5.67) and (5.68), we define
\[ P_0g_\lambda = (a_\lambda + b_\lambda \cdot v + c_\lambda (|v|^2 - 3))\sqrt{\mu}, \]
and
\[ P_0g_1 = (a_\lambda,1 + b_\lambda,1 \cdot v + c_\lambda,1 (|v|^2 - 3))\mu. \]
We also use the notation \( b_\lambda = (b_\lambda^1, b_\lambda^2, b_\lambda^3). \) Obviously,
\[ a_\lambda = a_\lambda,1 + a_\lambda,2, \quad b_\lambda = b_\lambda,1 + b_\lambda,2, \quad c_\lambda = c_\lambda,1 + c_\lambda,2; \quad \int_{-1}^{1} a_\lambda(t, y) dy = 0, \quad \forall \ t \geq 0. \] (7.19)
As in Section 4, we are able to prove the following result in order to capture the macroscopic dissipation of \( g_\lambda. \)

**Lemma 7.2.** Under the assumption (7.1), there exists an instant functional \( \mathcal{E}_{int}(t) \) satisfying
\[ |\mathcal{E}_{int}(t)| \leq \|g_2\|^2 + \|w_q g_1\|_{L^\infty}^2 \] (7.20)
such that for any \( t \geq 0, \)
\[ \frac{d}{dt} \mathcal{E}_{int}(t) + \lambda ||a_\lambda, b_\lambda, c_\lambda||^2 \leq C||P_1 g_2||^2 + C\|w_q g_1\|_{L^\infty}^2 + \]
\[ + C(\alpha + \tilde{\epsilon})\|g_2\|^2 + C\|s_1 - P\| g_2 \|_{L^2}^2. \] (7.21)

**Proof.** The proof of (7.21) is similar to that of (5.82) in Section 4. For brevity, we only show how to derive the \( L^2 \) estimate on \( a_\lambda. \) By letting \( \Psi = \Psi(t, y, v) \in C^\infty([0, \infty) \times [-1, 1] \times \mathbb{R}^3) \) be a test function and taking the inner product of (7.18) and \( \Psi, \) one has
\[ \frac{d}{dt} (g_\lambda, \Psi) - (g_\lambda, \partial_t \Psi) - (v_y g_\lambda, \partial_y \Psi) + (v_y, g_\lambda \Psi)(1) - (v_y, (g_\lambda \Psi)(-1)) + \alpha (v_y g_\lambda, \partial_y \Psi) + \frac{\alpha}{2} (v_x v_y g_\lambda, \Psi) + ((-\lambda_0 + L) g_\lambda, \Psi) = (H, \Psi). \] (7.22)
Choose
\[ \Psi = \Psi_{a_\lambda} = v_y \partial_y \phi_{a_\lambda}(t, y) (|v|^2 - 10)\sqrt{\mu}, \]
where
\[ \partial^2 \phi_{a_\lambda} = a_\lambda, \quad \partial_y \phi_{a_\lambda}(\pm 1) = 0, \quad \int_{-1}^{1} a_\lambda(y) dy = 0. \] (7.23)
It follows
\[ \|\phi_{a_\lambda}\|_{H^2} \leq C\|a_\lambda\|. \] (7.24)
We now compute the terms in (7.22) one by one. The Cauchy-Schwarz inequality and (7.24) directly give
\[ |(g_\lambda, \Psi_{a_\lambda})| \leq C\|g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2, \]
\[ \alpha |(v_y g_\lambda, \partial_y \Psi_{a_\lambda})| \leq C\alpha\|g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2, \]
\[ \frac{\alpha}{2} |(v_x v_y g_\lambda, \Psi_{a_\lambda})| \leq C\alpha\|g_2\|^2 + C\|w_q g_1\|_{L^\infty}^2, \]
\[ |(Lg_\lambda, \Psi_{a_\lambda})| \leq \eta ||a_\lambda,2|^2 + C\eta \|g_2\|^2 + C\eta \|w_q g_1\|_{L^\infty}^2. \]
And we have from Lemma 2.2 and the a priori assumption (7.1) that
\[ |(H, \Psi_{a_\lambda})| \leq C(\alpha + \tilde{\epsilon} + \eta) ||a_\lambda||^2 + C\eta(\alpha + \tilde{\epsilon})\{\|w_q g_2\|_{L^\infty}^2 + \|w_q g_1\|_{L^\infty}^2 \}. \]
where we have used
\[
|\langle e^{\lambda t} \Gamma(f, f), \Psi_{a\lambda} \rangle| \leq \eta \|a\lambda\|^2 + C_\eta \int_0^t \int_{\mathbb{R}^3} \left[ |v_y||v|^2 | - 10)^2 \sqrt{\mu} \right]^2 \, dv \, dy
\]
\[
\leq \eta \|a\lambda\|^2 + C_\eta \|w_y Q(\mu^2 g_\lambda, \mu^2 g_\lambda) \|_L^2 \int_0^t \left( \int_{\mathbb{R}^3} w_y \left[ |v_y||v|^2 | - 10)^2 \right] \, dv \right)^2 \, dy
\]
\[
\leq \eta \|a\lambda\|^2 + C_\eta \|w_y \mu^2 g_\lambda \|_L^2
\]
\[
\leq \eta \|a\lambda\|^2 + C_\eta \varepsilon^2 \{ \|w_y g_\lambda \|_L^2 + \|w_y g_2 \|_L^2 \}.
\]
For the second term on the left hand side of (7.22), from the inner product \(\langle (7.31), \sqrt{\mu} \rangle\), we have in the weak sense that
\[
\partial_t a\lambda + \partial_y b_\lambda^2 = 0,
\]
which yields
\[
|\langle g\lambda, \partial_t \Psi_{a\lambda} \rangle| \leq C \|b_\lambda\|^2 + C \|P_1 g_2\|^2 + C \|w_y g_1\|_L^2.
\]
In particular, the third term on the left hand side of (7.22) gives the following main contribution
\[
-(v_y g\lambda, \partial_y \Psi_{a\lambda}) = -(v_y P_0 g\lambda, \partial_y \Psi_{a,2}) - (v_y P_1 g\lambda, \partial_y \Psi_{a,1})
\]
\[
\geq 5\|a\lambda\|^2 - \eta \|a\lambda\|^2 - C_\eta \|P_1 g\lambda\|^2.
\]
The boundary term \(\langle v_y, (g\lambda \Psi_{a\lambda})(1) \rangle - \langle v_y, (g\lambda \Psi_{a\lambda})(-1) \rangle\) vanishes due to the boundary condition in (7.23). Putting all the above estimates for \(a\lambda\) together, we have
\[
\frac{d}{dt} \langle g\lambda, \Psi_{a\lambda} \rangle + \kappa \|a\lambda\|^2 \leq C \|b_\lambda\|^2 + C \|P_1 g_2\|^2 + C \|w_y g_1\|_L^2 + C(\alpha + \varepsilon) \|w_y g_2\|_L^2. \tag{7.25}
\]
Next, let
\[
\Psi = \Psi_{b\lambda} = \left\{
\begin{array}{l}
v_y v_x \frac{d}{dy} \phi_{b\lambda,1}(y) \sqrt{\mu}, \quad i = 1, \\
v_y v_x \frac{d}{dy} \phi_{b\lambda,3}(y) \sqrt{\mu}, \quad i = 3, \\
v_y^2 (|v|^2 - 5) \frac{d}{dy} \phi_{b\lambda,2}(y) \sqrt{\mu}, \quad i = 2,
\end{array}
\right.
\]
where
\[-\phi''_{b\lambda} = b_i, \quad \phi_{b\lambda}(\pm 1) = 0,
\]
and
\[
\Psi = \Psi_{c\lambda} = v_y (|v|^2 - 5) \frac{d}{dy} \phi_{c\lambda}(y) \sqrt{\mu},
\]
where
\[-\phi''_{c\lambda} = c_i, \quad \phi_{c\lambda}(\pm 1) = 0.
\]
Similar to (5.79) and (5.81), one can show that
\[
\frac{d}{dt} \langle g\lambda, \Psi_{b\lambda} \rangle + \kappa \|b\lambda\|^2 \leq C \|c\lambda\|^2 + C \|P_1 g_2\|^2 + C \|w_y g_1\|_L^2 + C(\alpha + \varepsilon) \|w_y g_2\|_L^2
\]
\[
+ C \|\{I - P_\gamma\} g_2\|_{L,+,2}^2, \tag{7.26}
\]
and
\[
\frac{d}{dt} \langle g\lambda, \Psi_{c\lambda} \rangle + \kappa \|c\lambda\|^2 \leq C \|P_1 g_2\|^2 + C \|w_y g_1\|_L^2 + C(\alpha + \varepsilon) \|w_y g_2\|_L^2
\]
\[
+ C \|\{I - P_\gamma\} g_2\|_{L,+,2}^2, \tag{7.27}
\]
respectively. Note that the decomposition \(\sqrt{\mu} \chi = g_1 + \sqrt{\mu} g_2\) has been also used to handle the terms involving \(\langle v_y, (\{I - P_\gamma\} g_\lambda \Psi)(1) \rangle - \langle v_y, (\{I - P_\gamma\} g_\lambda \Psi)(-1) \rangle\).
Consequently, by choosing $0 < \kappa_1 \ll \kappa_2 \ll 1$, we have from $\kappa_1 \times (7.25) + \kappa_2 \times (7.26) + (7.27)$ that
\[
\frac{d}{dt}\{\kappa_1(g_\lambda, \Psi_{b_\lambda}) + \kappa_2(g_{\lambda}, \Psi_{b_\lambda}) + (g_{\lambda}, \Psi_{c_\lambda})\} + \kappa\|a_\lambda, b_\lambda, c_\lambda\|^2 \\
\leq C\|P_1g_2\|^2 + C\|\mathcal{P}_1g_2\|_{L^\infty}^2 + C(\alpha + \bar{\varepsilon})\|\mathcal{P}_1g_2\|_{L^\infty}^2 + C\|\{I - P_\gamma\}g_2\|_{2,+,L}^2. \tag{7.28}
\]
Finally, (7.21) follows from (7.28) by defining
\[
E_{\text{int}}(t) = \kappa_1(g_\lambda, \Psi_{b_\lambda}) + \kappa_2(g_{\lambda}, \Psi_{b_\lambda}) + (g_{\lambda}, \Psi_{c_\lambda}). \tag{7.29}
\]
Note that (7.20) is satisfied. Thus the proof of Lemma 7.2 is completed. \hfill \Box

Now, with Lemma 7.2 and Lemma 7.1, we are ready to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** The global existence of solution to the problem (6.2) follows from the local existence constructed in Section 6 and the *a priori* estimates in the weighted $L^\infty$ space by the continuity argument. Therefore, to prove Theorem 1.2, it remains to show the uniform estimate (1.23) under the *a priori* assumption (7.1). Indeed, by (7.19), we can rewrite (7.21) as
\[
\frac{d}{dt}E_{\text{int}}(t) + \lambda\|P_0g_2\|^2 \leq C\|P_1g_2\|^2 + C\|\mathcal{P}_1g_2\|_{L^\infty}^2 + C(\alpha + \bar{\varepsilon})\|\mathcal{P}_1g_2\|_{L^\infty}^2 + C\|\{I - P_\gamma\}g_2\|_{2,+,L}^2, \tag{7.30}
\]
where $[g_1, g_2]$ is defined in (7.17). For the $L^2$ estimate on $P_1g_2$, note that $g_2$ satisfies
\[
\partial_t g_2 + v_y \partial_y g_2 - \alpha v_y \partial_v g_2 + (-\lambda_0 + L)g_2 = (1 - \chi_M)\mu^{-\frac{1}{2}}Kg_1, \tag{7.31}
\]
and
\[
g_2(0, y, v) = 0, \quad g_2(\pm 1, v)|_{y = 0} = \sqrt{2\pi \mu} \int_{v_y \geq 0} g_2(\pm 1, v)|_{v_y} dv.
\]
By taking the inner product of (7.31) and $g_2$ with respect to $y$ and $v$ over $(-1, 1) \times \mathbb{R}^3$, one has
\[
\frac{d}{dt}\|g_2\|^2 + \|\{I - P_\gamma\}g_2\|_{2,+,L}^2 + \delta_0\|P_1g_2\|^2 \leq C(\eta + \lambda_0)\|\mathcal{P}_1g_2\|^2. \tag{7.32}
\]
Let $\tilde{C} > 0$ be a constant sufficiently large. By taking the summation of $\tilde{C} \times (7.32)$ and (7.30) we have
\[
\frac{d}{dt}[\tilde{C}\|g_2(t)\|^2 + E_{\text{int}}(t)] + \lambda\|g_2\|^2 + \lambda\|\{I - P_\gamma\}g_2\|_{2,+,L}^2 \leq C\|\mathcal{P}_1g_2\|_{L^\infty}^2 + C(\alpha + \bar{\varepsilon})\|\mathcal{P}_1g_2\|_{L^\infty}^2. \tag{7.33}
\]
Denote
\[
E(t) = \tilde{C}\|g_2(t)\|^2 + E_{\text{int}}(t).
\]
For $\tilde{C} > 0$ being large enough, from (7.20) there exist constants $C_1 > 0$ and $C_2 > 0$ such that for any $t \geq 0$,
\[
2\tilde{C}\|g_2(t)\|^2 + C_2\|\mathcal{P}_1g_2(t)\|_{L^\infty}^2 \geq E(t) \geq \frac{C}{2}\|g_2(t)\|^2 - C_1\|\mathcal{P}_1g_2(t)\|_{L^\infty}^2. \tag{7.34}
\]
Then, from (7.33) and (7.34), it follows
\[
\frac{d}{dt}E(t) + \frac{2\lambda}{C}E(t) + \lambda\|\{I - P_\gamma\}g_2\|_{2,+,L}^2 \leq C\|\mathcal{P}_1g_2\|_{L^\infty}^2 + C(\alpha + \bar{\varepsilon})\|\mathcal{P}_1g_2\|_{L^\infty}^2.
\]
Hence
\[
E(t) + \lambda \int_0^t e^{-\frac{2\lambda}{C}(t-s)}\|\{I - P_\gamma\}g_2(s)\|_{2,+,L}^2 ds \\
\leq E(0) e^{-\frac{2\lambda}{C}t} + C \int_0^t e^{-\frac{2\lambda}{C}(t-s)}\|\mathcal{P}_1g_2(s)\|_{L^\infty}^2 ds + C(\alpha + \bar{\varepsilon}) \int_0^t e^{-\frac{2\lambda}{C}(t-s)}\|\mathcal{P}_1g_2(s)\|_{L^\infty}^2 ds \\
\leq E(0) + C \sup_{0 \leq s \leq t} \|\mathcal{P}_1g_2(s)\|_{L^\infty}^2 + C(\alpha + \bar{\varepsilon}) \sup_{0 \leq s \leq t} \|\mathcal{P}_1g_2(s)\|_{L^\infty}^2, \tag{7.35}
\]
for any $t \geq 0$. Therefore, by using (7.34) and (7.17), it follows from (7.35) that
\[
\sup_{0 \leq s \leq t} e^{\lambda_0 s}\|f_2(s)\| \leq C \sup_{0 \leq s \leq t} e^{\lambda_0 s}\|\mathcal{P}_1f_1(s)\|_{L^\infty} + C(\alpha + \bar{\varepsilon}) \sup_{0 \leq s \leq t} e^{\lambda_0 s}\|\mathcal{P}_1f_2(s)\|_{L^\infty}.
\]
By putting the above estimate back to (7.3) and using the smallness of $\alpha$ and $\varepsilon$, one has
\[ \sup_{0 \leq s \leq t} e^{\lambda_{st}} \| w_q f_2(s) \|_{L^\infty} \leq C \| w_q f_0 \|_{L^\infty} + C \sup_{0 \leq s \leq t} e^{\lambda_{st}} \| w_q f_1(s) \|_{L^\infty}. \] (7.36)

Moreover, by plugging (7.36) to (7.2) and using the smallness of $\alpha$ and $\varepsilon$ as well as (7.36), it holds that
\[ \sup_{0 \leq s \leq t} e^{\lambda_{st}} \| w_q [f_1, f_2](s) \|_{L^\infty} \leq C \| w_q f_0 \|_{L^\infty}, \]
which gives (1.23). Since $\| w_q f_0 \|_{L^\infty}$ is sufficiently small, the a priori assumption (7.1) is closed.

Finally, the non-negativity of the global solution constructed above can be proved similar to [14] so that the proof of Theorem 1.2 is completed.

8. Appendix

Recall the backward time cycle starting at $(t_0, y_0, v_0) = (t, y, v)$ in (5.18), the boundary probability measure $d\sigma_l$ on $\mathcal{V}_l$ in (5.19) and the product measure $d\Sigma_l(s)$ over $\prod_{j=1}^{k-1} \mathcal{V}_j$ in (5.24). The following lemma gives an estimate on the measure of the phase space $\Pi_{j=1}^{k-1} \mathcal{V}_j$ when there are $k$ times bounce.

Lemma 8.1. For any $\varepsilon > 0$ and any $T_0 > 0$, there exists an integer $k_0 = k_0(\varepsilon, T_0)$ such that for any integer $k \geq k_0$ and for all $(t, y, v) \in [0, T_0] \times [-1, 1] \times \mathbb{R}^3$, it holds
\[ \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_j(t, y, v, v_1, v_2, \ldots, v_{k-1}) > 0\}} \Pi_{l=1}^{k-1} d\sigma_l \leq \varepsilon. \] (8.1)
In particular, let $T_0 > 0$ large enough, there exist constants $C_1$ and $C_2 > 0$ independent of $T_0$ such that for $k = C_1 T_0^{5/4}$ with a suitable choice of $C_1$ such that $k$ is an integer and for all $(t, y, v) \in [0, \infty) \times [-1, 1] \times \mathbb{R}^3$, it holds
\[ \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k(t, y, v, v_1, v_2, \ldots, v_{k-1}) > 0\}} \Pi_{l=1}^{k-1} d\sigma_l \leq \left( \frac{1}{2} \right)^{C_2 T_0^{5/4}}. \] (8.2)
Furthermore, for any $q > 0$ in the weight function $w_q(v)$, there exist constants $C_3$ and $C_4 > 0$ independent of $k$ and $T_0$ such that
\[ \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq t_l\}} \int_0^{t_l} d\Sigma_l(s) ds \leq C_3, \] (8.3)
and
\[ \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} > 0\}} \int_0^{t_{l+1}} d\Sigma_l(s) ds \leq C_4. \] (8.4)

Proof. We only give the proof for (8.3), since (8.1), (8.2), and (8.4) can be proved similarly by using Lemma 23 in [24, pp. 781]. Recall the definition (5.24). We have
\[
\int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq t_l\}} \int_0^{t_l} d\Sigma_l(s) ds
\]
\[= \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq t_l\}} \int_0^{t_l} \prod_{j=l+1}^{k-1} d\sigma_j e^{-\int_0^{t_l} \mathcal{A}^\tau(\tau, V_j^0(\tau)) d\tau} \tilde{w}(v_j) d\sigma_l \prod_{j=1}^{l-1} \frac{\tilde{w}(v_j)}{\tilde{w}(V_j^0(t_{j+1}))} \times \prod_{j=1}^{l-1} e^{-\int_0^{t_j} \mathcal{A}^\tau(\tau, V_j^0(\tau)) d\tau} d\sigma_j ds.\]
which using direct calculations, can be bounded by

$$\int_{t_0}^{t_f} \int_{j=1}^{j_k-1} \int_{V_j} 1 d\sigma_j e^{-\frac{\nu}{\nu_j^2} (t_i-s)} \tilde{w}(v_i) d\sigma_i d\sigma_j ds$$

$$\leq C \int_{t_0}^{t_f} \int_{j=1}^{j_k-1} e^{-\frac{\nu}{\nu_j^2} (t_i-s)} \tilde{w}(v_i) d\sigma_i d\sigma_j ds \leq C.$$

Here we have used

$$\int_{V_i} \tilde{w}_2(v_i) d\sigma_i < +\infty,$$

and

$$\frac{\tilde{w}(v_i)}{\tilde{w}(V_{ij}^2(t_j+1))} = \frac{w_q(V_{ij}^2(t_j+1))w_j^2(V_{ij}^2(t_j+1))}{w_q(v_j)^2w_j^2(v_j)} = \frac{(1 + |V_{ij}^2(t_j+1)|^2)^q}{(1 + |v_j|^2)^q} e^{-\frac{|v_j|^2 - |V_{ij}^2(t_j+1)|^2}{4}} \leq (1 + 4\alpha^2)^q e^{\alpha^2},$$

by the Peetre’s inequality and the fact that $|V_{ij}^2(t_j+1) - v_j| = \alpha|t_b(v_j)v_{jj}| \leq 2\alpha$. Then the proof of lemma is completed.

\[ \square \]

Remark 8.1. The time interval $[0, T_0]$ in Lemma 8.1 can be replaced by any interval $[s, t]$ with the length $t - s = T_0$. In addition, since

$$\int_{V_1} \tilde{w}_1(v_1) d\sigma_i < +\infty, \quad q > 3/2,$$

and $\mathcal{A}', A$ and $A_1$ have the same lower bound $\nu_0/2$, the statement in Lemma 8.1 is also valid if $\Sigma_i(s)$ is replaced by either $\Sigma_i(s)$ or $\Sigma^{(i)}_i(s)$ ($i = 1, 2$).

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