New Class of Gravitational Wave Templates for Inspiralling Compact Binaries

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(Dated: February 2, 2008)

Compact binaries inspiralling along quasi-circular orbits are the most plausible gravitational wave (GW) sources for the operational, planned and proposed laser interferometers. We provide new class of restricted post-Newtonian accurate GW templates for non-spinning compact binaries inspiralling along PN accurate quasi-circular orbits. Arguments based on data analysis, theoretical and astrophysical considerations are invoked to show why these time-domain Taylor approximants should be interesting to various GW data analysis communities. [7 ]

PACS numbers: 04.30.Db, 04.25.Nx

Introduction.— Inspiralling comparable mass compact binaries are the most plausible sources of gravitational radiation for the operational, planned and proposed laser interferometric GW interferometers. GW data analysts, analyzing noisy data from the interferometers, require accurate and efficient temporally evolving GW polarizations, $h_+(t)$ and $h_\times(t)$, the so-called GW search templates. It is expected that weak GW signals, buried in the noisy interferometric data, should be extracted by employing the technique of ‘matched filtering’. This is an optimal technique if and only if one can construct search templates that accurately model expected GW signals from astrophysical sources, especially in their phase evolution. Till the latest stages of binary inspiral, GW signals may be accurately modeled using the post-Newtonian (PN) approximation to general relativity. The PN approximation to the dynamics of inspiralling compact binaries, usually modeled to consist of point masses, provides, for example, the equations of motion as corrections to the Newtonian one in terms of $(v/c)^2 \sim Gm/c^2 r$, where $v$, $m$, and $r$ are the characteristic orbital velocity, the total mass, and the typical orbital separation, respectively. In PN computations, it is customary to treat a non-spinning inspiralling compact binary to consist of two point masses moving in quasi-circular orbits. These PN computations, to date provided four quantities that are required to do astrophysics with GW interferometers. For inspiralling compact binaries, the relevant four quantities are the 3PN accurate dynamical (orbital) energy $E(x)$, expressed as a PN series in terms of $x = (Gm\omega_{3\text{PN}}/c^3)^{2/3}$, $\omega_{3\text{PN}}(t)$ being the 3PN accurate orbital angular frequency, the 3.5PN accurate expression for GW energy luminosity $\mathcal{L}(x)$ and the 2.5PN amplitude corrected expressions for $h_+(t)$ and $h_\times(t)$, written in terms of the orbital phase $\phi$ and $x$.

GW data analysts employ these inputs to construct various types of search templates and let us take a closer look at the so-called TaylorT1 and TaylorT2 waveforms implemented in the LSC Algorithms Library (LAL) [2]. These two template families employ the following expression for the so-called restricted PN waveform

$$ h(t) \propto \left( \frac{Gm\omega(t)}{c^3} \right)^{2/3} \cos 2\phi(t), \quad (1) $$

where the proportionality constant may be set to unity for non-spinning compact binaries. At a given PN order, the above mentioned two families provide two slightly different ways to compute $\omega(t)$ and $\phi(t)$. The TaylorT1 family numerically solves the following two differential equations:

$$ \frac{d\phi(t)}{dt} = \omega(t); \quad \frac{d\omega(t)}{dt} = -\mathcal{L}(\omega) \left/ \frac{dE}{d\omega} \right., \quad (2) $$

where, for example, $\mathcal{L}(\omega)$ and $E$ are respectively the 3.5PN accurate GW energy luminosity and the 3PN accurate orbital energy for TaylorT1 3.5PN waveforms. In other words, for a given PN member of the TaylorT1 family, $\omega(t)$ and $\phi(t)$ are computed by numerically solving the related approximants in Eq. [2]. To construct a member of TaylorT2 family, say TaylorT2 3.5PN, we require 3.5PN (Taylor expanded) accurate version of $d\omega(t)/dt$, appearing in Eq. [2]. The differential equations that define $\omega(t)$ and $\phi(t)$ for TaylorT2 3.5PN waveforms can be symbolically displayed as

$$ \frac{d\phi(t)}{dt} = \omega(t); \quad \frac{d\omega(t)}{dt} = \frac{96}{5} \left( \frac{GM\omega}{c^3} \right)^{5/3} \omega^2 \left\{ 1 + \mathcal{O}(\nu) + \mathcal{O}(\nu^{3/2}) + \mathcal{O}(\nu^2) + \mathcal{O}(\nu^{5/2}) + \mathcal{O}(\nu^3) + \mathcal{O}(\nu^{7/2}) \right\}, \quad (3) $$

where $\nu = 1/c^2$ is a PN ordering parameter and the explicit expressions for these PN contributions may be extracted from Refs. [1]. In the above equation, the chirp mass $M = m\eta^{3/5}$, where $\eta$ is the usual symmetric mass ratio and $m$ being the total mass of the binary.
In this paper, we provide prescriptions to compute three new types of time-domain Taylor approximants that should be, in our opinion, interesting to various GW data analysis communities. Let us first list the salient features of these new templates that also employ an expression similar to Eq. (1) to generate waveforms. The three important features of our Taylor approximants are the following. The first point is that, in comparison with TaylorT1 and Taylor T2 waveforms, for a given GW frequency window and at a given PN order, our prescriptions will provide more accumulated GW cycles. Further, our approaches to compute $h(t)$ are numerically as cheap (expensive) as TaylorT1 and Taylor T2 waveforms. Let us consider the second point. It is desirable to construct GW templates using the mathematical formulation employed to construct the (heavily employed) PN accurate relativistic Damour-Deruelle timing formula for binary pulsars [3]. This is because formally GW phasing for inspiralling compact binaries and timing of relativistic binary pulsars are quite similar. Our construction of these new Taylor approximants are indeed influenced by the GW phasing formalism, available in Ref. [4], that provided a method to construct GW templates for compact binaries of arbitrary mass ratio moving in inspiralling eccentric orbits. We recall that the techniques adapted in Ref. [4] were influenced by the mathematical formulation, developed in Ref. [5], to compute the Damour-Deruelle timing formula. Finally, a recent preliminary investigation indicates that our new Taylor approximants, at the dominant radiation reaction order, should be very efficient in capturing GWs from compact binaries inspiralling along PN accurate and mildly eccentric orbits [6]. This is, in our opinion, a very attractive feature for GW data analysists as GWs from inspiralling (astrophysical) compact binaries should have some tiny eccentricities, when their GWs enter the bandwidth of laser interferometers.

Let us describe how we construct these new types of PN accurate time-domain Taylor approximant GW search templates. GWs from inspiralling (astrophysical) compact binaries will have some tiny eccentricities around orbital frequencies of 20Hz. For example, using Ref. [1], it is not that difficult to show that the orbital eccentricity of the Hulse-Taylor binary pulsar when its GWs enter the bandwidth of laser interferometers indicates that our new Taylor approximants, at PN accurate circular orbits, the dominant radiation reaction order, should be very efficient in capturing GWs from compact binaries inspiralling along PN accurate and mildly eccentric orbits [6]. This is, in our opinion, a very attractive feature for GW data analysts as GWs from inspiralling (astrophysical) compact binaries should have some tiny eccentricities, when their GWs enter the bandwidth of laser interferometers.

\begin{align}
\omega_{3\text{PN}} &= \left(1 + 3 \xi^{2/3} + \left(\frac{39}{2} - 7 \eta\right) \xi^{4/3} + \left[\frac{315}{2}\right] \right) \\
&+ 7 \eta^2 + \left(-\frac{817}{4} + \frac{123}{32} \pi^2 \right) \eta \xi^2, \quad (4)
\end{align}

where $\xi = G m n/c^3$. Let us now compute employing PN accurate expressions for $\mathcal{E}(x)$ and $\mathcal{L}(x)$, available in Ref. [1], the following 3PN accurate expression for the orbital energy $\mathcal{E}$ and 3.5PN accurate GW energy luminosity $\mathcal{L}$, in terms of $\xi$, as

\begin{align}
\tilde{\mathcal{E}}(\xi) &= \xi^{2/3} \left\{1 + \left[\frac{5}{4} - \frac{11}{12} \right] \xi^{2/3} + \left[\frac{45}{8} - \frac{21}{8} \eta\right] \xi^{4/3} \\
&+ \left(-\frac{1}{24} \eta^2\right) \xi^{5/3} + \left[\frac{7975}{192} - \frac{35}{5184} \eta^3 + \frac{1031}{288} \eta^2\right] \xi^2 \right\}, \quad (5a)
\end{align}

\begin{align}
\mathcal{L}(\xi) &= \frac{32}{5} \eta^2 \xi^{10/3} \left\{1 + \left[\frac{35}{12} - \frac{10413}{336}\right] \xi^{2/3} \\
&+ 4 \pi \xi + \left[\frac{458461}{20129} - \frac{204}{504} \eta + \frac{65}{18} \eta^2\right] \xi^{4/3} \\
&+ \left[\frac{583}{24} + \frac{26753}{672} \pi \xi^{5/3} + \left[\frac{16}{3} \right] \right] \xi^{7/3} \\
&+ \left[\frac{41}{3} \right] \eta^2 + \left[\frac{1310663537}{2284800} - \frac{6881951}{7776}\right] \eta \\
&+ \left[\frac{375997}{3024} \eta^2 - \frac{775}{324} \eta^3 + \frac{1712}{105} \gamma + \log(4 \xi^{1/3})\right] \xi^{2} \\
&+ \left[\frac{771833}{2026} - \frac{624559}{1728} \eta + \frac{193855}{3024} \eta^2\right] \pi \xi^{7/3} \right\}, \quad (5b)
\end{align}

where $\tilde{\mathcal{E}} = -2 E$, $E$ being the dimensionless non-relativistic energy per unit reduced mass $\tilde{m}$ and $\gamma$ being the Euler’s gamma. We are now in a position to construct, in our terminology, TaylorK1 3.5PN and TaylorK2 3.5PN restricted PN waveforms. In our approach, the form of the restricted PN waveform, Eq. (11), becomes $h(t) \propto \left(\frac{2 \gamma n (n \xi)^{2/3}}{\cos 2 \phi(t)}\right)^{2/3}$. This is allowed because at Newtonian order $\omega = n$ and the amplitude is indeed Newtonian accurate in Eq. (11). For TaylorK1 3.5PN accurate waveform, $(n(t)$ and $\phi(t)$ are numerically obtained.
using the following two differential equations
\[
\frac{d\phi}{dt} = \omega_{3\text{PN}}, \tag{6a}
\]
\[
\frac{dn}{dt} = -\xi(\xi) \left/ \frac{dE}{dn} \right. \tag{6b}
\]

To construct our TaylorK2 3.5PN waveforms, as expected, we Taylor expand, in terms of \( \xi \), the RHS of Eq. (6b) and this leads to
\[
\frac{d\phi}{dt} = \omega_{3\text{PN}}, \tag{7a}
\]
\[
\frac{dn}{dt} = \frac{96}{5} \eta n^3 \xi^{5/3} \left\{ 1 + \left( \frac{1273}{336} - \frac{11}{4} \eta \right) \xi^{2/3} + 4 \pi \xi \right.
\]
\[
+ \left( \frac{438887}{18144} \frac{59}{18} \eta^2 - \frac{49507}{2016} \eta \right) \xi^{4/3} + \left( \frac{20033}{672} \right.
\]
\[
- \left( \frac{189}{8} \eta \right) \pi \xi^{5/3} + \left[ \frac{16}{3} + \frac{287}{24} \eta \right] \pi^2
\]
\[
- \frac{5605}{5092} \eta^3 + \frac{617285}{8064} \eta^2 + \frac{16554367}{31104} \eta
\]
\[
+ \left[ \frac{38047038863}{139708800} - \frac{1712}{105} \left( \gamma + \log 4 \xi^{1/3} \right) \right] \xi^2
\]
\[
+ \left( \frac{91495}{1512} \eta^2 - \frac{1608185}{6048} \eta + \frac{9710111}{4032} \right) \pi \xi^{7/3} \right\} . \tag{7b}
\]

Let us now specify, for example, the limits of integration for \( n \) to construct TaylorK1 3.5PN and TaylorK2 3.5PN waveforms. For initial LIGO, it is customary to use \( \omega_i \) and \( \omega_f \), the initial and final values of \( \omega \), to be 40 \( \pi \) Hz and \((6^{3/2} m)^{-1}\) Hz, where \( \omega_f \) is twice the conventional orbital angular frequency of the innermost stable circular orbit for a test particle around a Schwarzschild black hole. With these inputs, the initial and final values of \( n \), denoted by \( n_i \) and \( n_f \), are numerically computed using Eq. (4). This is justified because of the observation in Ref. [10] that the quadrupolar GW frequency from a compact binary, having PN accurate orbital motion, appears at \((1 + k) n/\pi \). At 3PN order, for a compact binary having \( m = 11.4 M_\odot \) and \( \eta = 0.108 \) we have \( n_i \sim 111.32 \) Hz and \( n_f \sim 679.3 \) Hz. In our approaches to construct, for example, TaylorK1 2PN and TaylorK2 2PN waveforms, we use only the 2PN accurate relation connecting \( \omega \) and \( n \).

Let us now compute in the time domain the accumulated number of GW cycles, \( N_{GW} \), in a given GW frequency window, by numerically integrating Eqs. (5) and (7) representing temporal evolutions for TaylorK1 and TaylorK2 waveforms at four different PN orders, namely 2PN, 2.5PN, 3PN and 3.5PN orders, for three canonical compact binaries usually considered in the GW literature [we restrict these orbital evolutions such that emitted GWs are in the GW frequency window defined by 40 Hz and \((6^{3/2} \pi m)^{-1}\) Hz]. Let us also compare these \( N_{GW} \) with what is expected from TaylorT1 and TaylorT2 waveforms at these four different PN orders. The numbers, relevant for initial LIGO, are listed in Table I where we compare \( N_{GW} \) resulting from TaylorK2 and TaylorT2 prescriptions [results are similar while comparing TaylorK1 and TaylorT1]...

**TABLE I:** Accumulated number of GW cycles, relevant for initial LIGO, for three types of canonical binaries at four different PN orders using TaylorK2 and TaylorT2 waveforms. The values of \( N_{GW} \) arising from TaylorT2 waveforms are given in parentheses. We note that TaylorK2 waveforms provide more \( N_{GW} \) compared to TaylorT2 waveforms. For high mass binaries, the convergence of \( N_{GW} \) is not that pronounced for TaylorK2 waveforms compared to TaylorT2 waveforms.

| \( m_1/M_\odot : m_2/M_\odot \) | 1.4 : 1.4 | 1.4 : 10 | 10 : 10 |
|------------------|-------|--------|--------|
| 2PN              | 1616.4 (1613.5) | 345.6 (333.8) | 57 (52.6) |
| 2.5PN            | 1613.5 (1605.8) | 333.8 (333.1) | 53.8 (52.6) |
| 3PN              | 1623.4 (1614) | 347.3 (330.9) | 57.6 (52.9) |
| 3.5PN            | 1620.6 (1615.4) | 342.4 (330.5) | 56.2 (52.5) |

We are aware that LAL also provides routines to create TaylorT3 waveforms. In this prescription, both \( \phi(t) \) and \( \omega(t) \), appearing in Eq. (1), are given as explicit PN accurate functions of time. These explicit time dependencies are usually expressed in terms of the so-called ‘adimensional’ time variable \( \theta = \frac{c^3}{\pi G m} (t_c - t) \), where \( t_c \) is the PN accurate coalescence time. It is indeed possible for us to compute \( n(t) \), using Eqs. (7), as a PN series in terms of \( \theta \). However, we are reluctant to repeat what is done in TaylorT3 waveforms to get \( \phi(t) \) with the help of Eqs. (7). Observe that radiation reaction and hence temporal evolution of \( n \) first appears at 2.5PN order and therefore, in our opinion, it is better to keep \( d\phi/dt \) to at least 2PN order in Eqs. (7) to be consistent in a PN way [see Refs. [4, 6] where similar approaches are employed].

It is important to note, while constructing these time-domain Taylor waveforms, that we employed the following two arguments. The first one is the standard argument that equates the rate of decrease of the conserved orbital energy of a compact binary to the opposite of GW luminosity. However, for constructing TaylorT1, TaylorT2, TaylorK1 and TaylorK2 waveforms, one requires additional PN accurate relations relating \( \omega \) (or \( n \) as the case may be) to the conserved orbital energy. Further, we speculate that the two different ways of computing \( d\omega/dt \), enforced in TaylorT1 and TaylorT2 waveforms, may be based on the fact that observationally \( d\omega/dt \) (or the above mentioned standard argument) is only tested to the Newtonian radiation reaction order by the accurate timing of binary pulsars. Therefore, it is natural to ask if we can construct \( h(t) \) employing only the energy balance argument. This is indeed possible as demonstrated below...
cumulated GW cycles in a given GW frequency window and may be useful in detecting GWs from inspiraling compact binaries that should have ‘teeny-weeny’ orbital eccentricities. Further, our approaches are influenced by the way PN accurate Damour-Deruelle timing formula was constructed. Therefore, we feel that our TaylorK1, TaylorK2 and TaylorEt waveforms should be of certain interest to the practitioners of LAL. Further, we feel that our restricted PN waveforms should be useful for the the recently initiated mock LISA data challenge task force.

The data analysis implications of these templates, relevant for both ground and space based GW detectors, are under active investigations in collaborations with Stas Babak, Sukanta Bose, Christian Röver and Manuel Tessmer. The GW phase evolution under our prescription is also being compared with its counterpart in numerical relativity based binary black inspiral.

I am indebted to Gerhard Schäfer for illuminating discussions and persistent encouragements. Lively discussions with Manuel Tessmer are warmly acknowledged. This work is supported in part by the DFG (Deutsche Forschungsgemeinschaft) through SFB/TR7 “Gravitationswellenastronomie” and the DLR (Deutsches Zentrum für Luft- und Raumfahrt) through “LISA Germany”.

\[ h(t) \propto \tilde{E}(t) \cos 2 \phi(t), \]

\[
\frac{d\phi}{dt} = \frac{c^3}{2\hbar}\left\{ 1 + \frac{1}{8}\left( 9 + \eta \right) \zeta + \left[ \frac{891}{128} - \frac{201}{64} \eta + \frac{11}{12} \eta^2 \right] \zeta^2 
+ \left[ \frac{24861497}{24861496} \right] \eta \pi^5/2 + \left[ \frac{577}{576} \right] \pi^2 \eta^2 + \left[ \frac{12096}{2304} \right] \eta \pi^7/2 \right\},
\]

where \( \phi = t e^3 / G m \) and \( \zeta = \tilde{E} \). We call the resulting \( h(t) \) as TaylorEt waveforms. The values of \( \zeta \) corresponding to \( \omega_t \) and \( \omega_f \) can numerically evaluated using the RHS of Eq. (8a) for \( d\phi/dt = \tilde{\omega} \).

We evaluated \( N_{GW} \) associated with TaylorEt 3.5PN waveforms for the three canonical compact binaries and the numbers are the following. For neutron star binaries, \( m = 2.8M_\odot \) and \( \eta = 0.25 \), \( N_{GW} = 1617.4 \) and for the usual black hole-neutron star binaries, \( m = 11.4M_\odot \) and \( \eta = 0.108 \), we have \( N_{GW} = 335.4 \). For typical stellar mass black hole binaries, \( m = 20M_\odot \) and \( \eta = 0.25 \), one gets \( N_{GW} = 54.0 \). It is interesting to note that we get larger \( N_{GW} \), compared to TaylorT 3.5PN waveforms and lower values compared to TaylorK 3.5PN waveforms. It should be related to the fact that it takes more time for TaylorEt prescription to reach \( \omega_f \) form \( \omega_t \) compared to TaylorT1 (or TaylorT2) approach and the opposite is true for the cases of TaylorK1 (or TaylorK2). The observation that TaylorEt waveforms also provide more number of GW cycles in a given GW frequency window, in our opinion, makes it our third prescription to compute \( h(t) \).

**Conclusions.**— We provided new ways of constructing restricted time-domain PN accurate waveforms for non-spinning compact binaries inspiralling along PN accurate quasi-circular orbits. Our prescriptions employed PN accurate expressions for the conserved orbital energy and GW luminosity, available in Refs. [4, 5] in a democratic manner and heavily depended on certain PN accurate gauge invariant quantities, first introduced in Ref. [6]. These template waveforms provide more number of ac-

\[ \frac{d\mathcal{L}}{dt} = \frac{c^3}{5\hbar} \left\{ \left[ \frac{12096}{2304} \right] \eta \pi^7/2 \right\}, \]

\[ \text{where } i = t e^3 / G m \text{ and } \zeta = \tilde{E}. \text{ We call the resulting } h(t) \text{ as TaylorEt waveforms. The values of } \zeta \text{ corresponding to } \omega_t \text{ and } \omega_f \text{ can numerically evaluated using the RHS of Eq. (8a) for } d\phi/dt = \tilde{\omega}. \]

\[ \frac{d\phi}{dt} = \zeta^3/2 \left\{ 1 + \frac{1}{8} \left[ 9 + \eta \right] \zeta + \left[ \frac{891}{128} - \frac{201}{64} \eta + \frac{11}{12} \eta^2 \right] \zeta^2 
+ \left[ \frac{3072}{3072} + \frac{205}{64} \pi^2 \right] \eta + \frac{1215}{1024} \eta^2 
+ \frac{45}{1024} \right\}, \]

\[ \zeta^3/2 \left\{ 1 + \frac{1}{8} \left[ 9 + \eta \right] \zeta + \left[ \frac{891}{128} - \frac{201}{64} \eta + \frac{11}{12} \eta^2 \right] \zeta^2 
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