Bures-Wasserstein Geometry

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Abstract The Bures-Wasserstein distance is a Riemannian distance on the space of positive definite Hermitian matrices and is given by: \( d(\Sigma, T) = \left[ \text{tr} \Sigma + \text{tr} T - 2 \text{tr}(\Sigma^{1/2} T \Sigma^{1/2})^{1/2} \right]^{1/2} \). This distance function appears in the fields of optimal transport, quantum information, and optimisation theory. In this paper, the geometrical properties of this distance are studied using Riemannian submersions and quotient manifolds. The Riemannian metric and geodesics are derived on both the whole space and the subspace of trace-one matrices. In the first part of the paper a general framework is provided, including different representations of the tangent bundle for the SLD Fisher metric. The last part of the paper unifies till now independent arguments and results from quantum information theory and optimal transport. The Bures-Wasserstein geometry is related to the Fubini-Study metric and the Wigner-Yanase information.

Keywords Information geometry · positive definite matrices · Bures distance · Wasserstein metric · Optimal transport · Quantum information

1 Introduction

In this paper we investigate the geometrical properties of the Bures-Wasserstein (BW) distance on the space of positive definite symmetric matrices, \( \mathbb{P}(n) \). For \( \Sigma, T \in \mathbb{P}(n) \), this distance function is given by:

\[
d_{\mathbb{P}(n)}^{BW}(\Sigma, T) = \left[ \text{tr}(\Sigma) + \text{tr}(T) - 2 \text{tr}\left(\left(\Sigma^{1/2} T \Sigma^{1/2}\right)^{1/2}\right)\right]^{1/2}
\]

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This function appears in optimal transport as a distance measure on the space of mean-zero Gaussian densities, where it is called the Wasserstein distance. In quantum information theory, this is a distance measure between quantum states or density matrices, called the Bures distance.

The development of this subject started when Rao realised that the Fisher information defines a Riemannian metric on the space of probability measures \[19\]. He obtained this metric by mapping the positive orthant of the unit sphere, equipped with the Euclidian metric, to the probability simplex using the square map. Later, the study of the geometrical properties of the probability simplex was extended to the quantum realm with notable contributions of Nagaoka and Petz. An overview of this field can be found in \[10\] and \[3\]. The distance measure defined in \([1]\) was introduced by Helstrom \([11]\) and Bures \([6]\) as a measure of similarity between quantum states. In \([23]\), Uhlmann derives the geometrical properties of this distance measure using a generalisation of the argument of Rao. This derivation is described in more detail in section \([5]\).

In the context of optimal transport, this distance measure was derived to be the \(L^2\)-Wasserstein distance on the space of covariance matrices for mean zero Gaussian distributions \([17]\). Its geometrical properties were first studied in this context by Takatsu \([21]\). It is interesting to note that the argument used in this paper is similar to but independent of the argument by Uhlmann fifteen years prior. Recently \([14]\) and \([5]\) built upon the work of Takatsu. Bhatia discusses both the quantum and the optimal transport interpretation of the distance and introduces the name Bures-Wasserstein distance. Furthermore, the argument of Takatsu is refined using facts on Riemannian submersions and quotient manifolds. The current paper adapts the construction of Bhatia in order to obtain the geometrical structure for the submanifold of trace-one matrices, which is of particular importance in quantum information. The aim is to both unify work from optimal transport and quantum information and simplify the original argument by Uhlmann.

More recently, the geometrical structure discussed in this paper is of interest in the field of optimisation. It turns out that for this choice of geometry the exponential and logarithmic map are cheap to evaluate, which makes it particularly suitable for numerical computations \([15]\).

The first section of the paper discusses preliminary facts needed to put the main results into perspective. A definition of the \((m)\) -and \((e)\)-representations is introduced which is easily compatible with the existing definitions from both classical and quantum information geometry. These representations are worked out explicitly for the SLD Fisher metric and the Bogoliubov metric. The main results of this paper can be found in the second section of the paper, where the geometrical structure of the BW distance is investigated first on \(P(n)\) and this is then restricted to the trace-one subset, \(D(n)\). The last section of the paper compares the geometrical structure obtained in the foregoing to similar
results in the field. An overview of the notation used in the paper can be found on page 18.

1.1 Preliminaries

Differential geometry

Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds and $F : \mathcal{M} \to \mathcal{N}$ a smooth map. We will denote the differential of $F$ at $p \in \mathcal{M}$ by: $dF_p : T_p \mathcal{M} \to T_{F(p)} \mathcal{N}$. For $g$ a Riemannian metric on $\mathcal{M}$, the length of a tangent vector $v \in T_p \mathcal{M}$ is given by: $||v||_g = (g(v, v))^{1/2}$, and the length of a curve $\gamma : [a, b] \to \mathcal{M}$ is given by:

$$L(\gamma) = \int_a^b ||\gamma'(t)||_g dt$$

where $\gamma'(t) = \frac{d\gamma}{dt}(t)$. The Riemannian distance between $p, q \in \mathcal{M}$ is defined to be:

$$d_{\mathcal{M}}(p, q) = \inf \{ L(\gamma) : \gamma : [a, b] \to \mathcal{M}, \gamma(a) = p, \gamma(b) = q \}. \quad (3)$$

See chapter 2 of [13] for details.

**Definition 1** Let $V$ be a real or complex vector space. An affine subspace of $V$ is a subset $A \subset V$ together with a vector subspace $\tilde{V} \subset V$ such that:

- $\forall a, b \in A, \exists v \in \tilde{V}$ such that $a + v = b$
- $\forall v \in \tilde{V} \text{ and } a \in A \text{ we have: } a + v \in A$

Now let $\mathcal{M}$ be an open convex subset of an affine subspace $A$ of $V$ with associated vector space $\tilde{V}$ and $p \in \mathcal{M}$ fixed. We have that the following map is a vector space isomorphism [12]:

$$\text{id}_1 : \tilde{V} \to T_p \mathcal{M} \quad (4)$$

$$\tilde{v} \mapsto v$$

$$\text{id}_1(\tilde{v})(f) = \frac{d}{dt}f(p + t\tilde{v})|_{t=0}. \quad (5)$$

where $f$ is any smooth function on $\mathcal{M}$. We can therefore identify every tangent vector in $T_p \mathcal{M}$ with an element of $\tilde{V}$ through $\text{id}_1$. Given a basis $(e_1, ..., e_m)$ for $V$, we define the Euclidean inner product on $\tilde{V}$ to be such that $\langle e_i, e_j \rangle_{\text{Eucl}} = \delta_{ij}$. The Euclidian metric $\tilde{g}$ on $\mathcal{M}$ is defined such that for $\tilde{v}, \tilde{w} \in \tilde{V}$, we have $\tilde{g}_p(\text{id}_1(\tilde{v}), \text{id}_1(\tilde{w})) = \text{Re} \left( \langle \tilde{v}, \tilde{w} \rangle_{\text{Eucl}} \right)$. The Riemannian distance associated to $\tilde{g}$ is denoted $\tilde{d}$.

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and $(e_1, ..., e_m)$ a fixed basis for $V$. This basis induces a coordinate map $k : V \to \mathbb{K}^m$ such that for $k(v) = x$ we have $v = \sum_i x_i e_i$. 


We define the $(m)$-representation of an element in $T_p\mathcal{M}$ to be the coordinate representation of its $id_1$-associated element in $\tilde{V}$. Or in symbols,

\begin{equation}
(m) : T_p\mathcal{M} \xrightarrow{id_1} \tilde{V} \overset{k}{\longrightarrow} \mathbb{K}^m
\end{equation}

Using Riesz’ representation theorem we let $id_2$ be the identification between $T_p\mathcal{M}$ with its dual $T_p^*\mathcal{M}$ such that: $v \leftrightarrow v'(\cdot) = \hat{g}_p(v,\cdot)$. We define the $(m^*)$-representation of an element of $T_p^*\mathcal{M}$ to be the $(m)$-representation of its $id_2$-associated element in $T_p\mathcal{M}$. In symbols,

\begin{equation}
(m^*) : T_p^*\mathcal{M} \xrightarrow{id_2} T_p\mathcal{M} \xrightarrow{(m)} \mathbb{K}^m.
\end{equation}

A general Riemannian metric $g$ on $\mathcal{M}$ gives a final identification, $id_3(g)$, between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$ in the same way as above: $v \leftrightarrow v' = \hat{g}_p(v,\cdot)$. Given a metric $g$, we define the $(e)$-representation of an element of $T_p\mathcal{M}$ as the $(m^*)$-representation of its $id_3(g)$-associated element in $T_p^*\mathcal{M}$. In symbols,

\begin{equation}
(e) : T_p\mathcal{M} \xrightarrow{id_3(g)} T_p^*\mathcal{M} \xrightarrow{(m^*)} \mathbb{K}^m.
\end{equation}

Note that the definition of the $(m)$-and $(e)$-representation implies:

\begin{equation}
\hat{g}_p(v, w) = \text{Re} \left( \langle u(e), w(m^*) \rangle \right)
\end{equation}

with on the right the standard inner product on $\mathbb{K}^m$.

\textbf{Remark 1} The definitions above are inspired by Chapter 2 of [2]. We will see in section 4.2 and 4.3 that the definitions correspond to the ones given in e.g. [1], [10].

\textbf{Matrix identities}

Let $\text{GL}(n)$ be the space of invertible complex matrices and $U(n)$ the unitary matrices. Every $M \in \text{GL}(n)$ can be written as $M = \Sigma U$ where $\Sigma = (MM^*)^{1/2} \in \mathbb{P}(n)$ and $U = \Sigma^{-1} M \in U(n)$. This is called the polar decomposition of $M$ and $U$ is called the unitary polar factor. In the following theorem, sometimes referred to as Uhlmann’s theorem, the unitary polar factor shows up in a maximisation problem.

\textbf{Theorem 1} Consider the following maximisation problem:

\begin{equation}
\sup_{V \in U(n)} \text{Re} \text{tr} (\Sigma VT)
\end{equation}

Then the supremum is attained for $V = U^*$, with $U$ the unitary polar factor of $T\Sigma$.

\textbf{Proof} [5], [3].

The solution for $X$ to the Lyapunov equation: $\Sigma X + X \Sigma = H$ with $\Sigma \in \mathbb{P}(n), H \in \mathbb{H}(n)$ will be denoted $L_\Sigma(H)$. It turns out that this solution exists and is unique [4].
1.2 Classical information geometry

In this section we will study $\mathcal{M}_+(\Omega)$ and $\mathcal{P}_+(\Omega)$, the space of strictly positive (resp. probability) measures on $\Omega = \{\omega_1, \ldots, \omega_n\}$. These spaces are open subsets of affine subspaces of the vector space of signed measures $\mathcal{S}(\Omega)$. The canonical basis of $\mathcal{S}(\Omega)$ is given by the set of Dirac delta measures $(\delta_1, \ldots, \delta_n)$ such that $\delta_i(\omega_j) = \delta_{ij}$. This basis gives us the $(m)$-representation as described in the preliminaries. Now we let the metric on $\mathcal{M}_+(\Omega)$ and $\mathcal{P}_+(\Omega)$ be the Fisher information metric, given by:

$$g^F_\mu(a, b) = \sum_{i=1}^n \frac{a_i^{(m)} b_i^{(m)}}{\mu(\omega_i)}$$

(12)

with $a, b$ in the tangent space of $\mathcal{M}_+(\Omega)$ or $\mathcal{P}_+(\Omega)$ at $\mu$. The $(e)$-representation is given as follows:

$$a_i^{(e)} = \frac{a_i^{(m)}}{\mu_i}$$

(13)

Another way of obtaining the $(e)$-representation for this case is by applying the $(m)$-representation to the pushforward of $a$ under the logarithm map. If $\log : \mathcal{M}_+(\Omega) \ni \mu \mapsto \log(\mu) \in \mathcal{S}(\Omega)$ then:

$$a_i^{(e)} = (d\log_\mu(a))^{(m)}.$$

(14)

We will see however that this expression of the $(e)$-representation is not general enough for the quantum case.

**Hellinger distance, Fisher metric and Fisher distance**

In this section we derive the Riemannian metric corresponding to the Hellinger distance on $\mathcal{M}_+(\Omega)$. Then we find the Riemannian distance corresponding to the restriction of this metric to $\mathcal{P}_+(\Omega)$. These will turn out to be the Fisher metric and Fisher distance respectively. This derivation was first due to Rao [19] and can be seen as a special case of the derivation given in the second part of the paper.

Restricted to the subset of diagonal matrices, it is easy to see that the BW distance on $\mathbb{P}(n)$ given in [11] has the following form:

$$d_{\mathbb{P}(n)}^{BW}(D_1, D_2) = \left[ \operatorname{tr} \left( D_1^{1/2} - D_1^{1/2} \right)^2 \right]^{1/2}.$$  

(15)

Interpreting these diagonal matrices as elements of $\mathcal{M}_+(\Omega)$, we note that this distance corresponds to the Hellinger distance, given by: $d^H(\mu, \nu) = \sqrt{\sum_{i=1}^n (\mu(\omega_i)^{1/2} - \nu(\omega_i)^{1/2})^2}$. The Hellinger distance can be obtained as the pushforward of the Euclidian distance under the square map:

$$(\mathcal{M}_+(\Omega), \tilde{d}) \ni \mu \mapsto \mu^2 \in (\mathcal{M}_+(\Omega), d^H)$$

(16)
Extending the structure on the left from a distance function to its corresponding Riemannian metric $\bar{g}$, we have the following isometry:

\[
(M_+(\Omega), \bar{g}) \ni \mu \mapsto \mu^2 \in (M_+(\Omega), g^F) \quad (17)
\]

From (16) and (17) it follows that the Hellinger distance is the geodesic distance for the Fisher metric in $M_+(\Omega)$.

We now aim to find the Riemannian distance for the Fisher metric restricted to $P_+(\Omega)$. It turns out that for this subset the geodesic distance is no longer the Hellinger distance. In order to find the right geodesic distance, we can use the fact that the following restriction of (17) remains an isometry:

\[
(S_{M_+(\Omega)}, \bar{g}) \ni \mu \mapsto \mu^2 \in (P_+(\Omega), g^F) \quad (18)
\]

where $S_{M_+(\Omega)} \equiv \{ \mu \in M_+(\Omega) : \sum \mu(\omega_i)^2 = 1 \}$, the unit sphere in $(M_+(\Omega), \bar{g})$.

We know that on this space the geodesics are given by great circles and therefore we can also compute the Riemannian distance. Using the fact that this distance is carried over by the isometry we obtain the Riemannian distance for the space of probability measures with the Fisher metric, called the Fisher distance. This is given by:

\[
d_F(p, q) = \arccos \left( \sum_{i=1}^{n} (p(\omega_i)q(\omega_i))^{1/2} \right). \quad (19)
\]

1.3 Quantum information geometry

Let $\mathbb{H}(n)$ be the set of Hermitian matrices and $\mathcal{D}(n)$ be the subset of positive definite Hermitian matrices with trace one. Within the context of section 1.1 we have $M = \mathcal{D}(n)$, $V = \mathbb{C}^{n \times n}$ and $\tilde{V} = \{ H \in \mathbb{H}(n) : \text{tr}(H) = 0 \}$. The basis vectors for $\mathbb{C}^{n \times n}$ are simply given by $(A_{11}, A_{12}, ..., A_{nn})$, where the $ij$-th entry of $A_{ij}$ is one and the rest zero. From this we get the $(m)$-representation for $T^*_\rho \mathcal{D}(n)$. For the submanifold of diagonal matrices (probability measures) Chentsov showed that the Fisher metric is the unique metric satisfying certain (statistically) natural conditions on the metric [16]. Petz proved that for $\mathcal{D}(n)$, this uniqueness no longer exists [18]. One of the suggested generalisations is the symmetrised logarithmic derivative (SLD) Fisher metric. See e.g. [11], [10].

For $H, K \in T^*_\rho \mathcal{D}(n)$ this Hermitian metric is given explicitly by:

\[
g^\text{SLD}_\rho(H, K) = 2 \text{tr} \left( L_\rho \left( H^{(m)} \right) K^{(m)} \right) \quad (20)
\]

We will derive in the second part of this paper that that the Riemannian metric corresponding to the BW distance on $P(n)$ is given by:

\[
g^\text{BW}_\Sigma(H, K) = \frac{1}{2} \text{Re} \text{tr} \left( L_\Sigma \left( H^{(m)} \right) K^{(m)} \right). \quad (21)
\]

\footnote{The isometries in this section are defined up to a constant.}
Furthermore we will prove that the Riemannian distance on $D(n)$ for this metric is given by:

$$d_{BW}^{D(n)}(\rho_1,\rho_2) = \arccos \left( \text{Re tr} \left( \left( \rho_2^{1/2} \rho_1 \rho_2^{1/2} \right)^{1/2} \right) \right)$$  \hspace{1cm} (22)

Because the real parts of $g^{SLD}$ and $g^{BW}$ are equal on $D(n)$, we can conclude that $d_{BW}^{D(n)}$ is the distance function for $g^{SLD}$ (up to a constant).

\textit{(e)-representations in quantum information geometry}

The SLD Fisher metric for $H, K \in T_\rho D(n)$ is given up to a constant by:

$$g^{SLD}_\rho(H, K) = \text{tr} \left( L_\rho \left( H^{(m)} \right) \right) K^{(m)}.$$  \hspace{1cm} (23)

From the preliminaries it follows that for this choice of metric and $H \in T_\rho D(n)$ we have the following relation between the $(m)$- and $(e)$-representation:

$$H^{(e)} = L_\rho \left( H^{(m)} \right)$$  \hspace{1cm} (24)

$$H^{(m)} = H^{(e)} \rho + \rho H^{(e)}$$  \hspace{1cm} (25)

Expressing the SLD Fisher metric in terms of the $(e)$-representation therefore gives the potentially more familiar form:

$$g^{SLD}_\rho(H, K) = \text{tr} \left( H^{(e)} \left( K^{(e)} \rho + \rho K^{(e)} \right) \right)$$  \hspace{1cm} (26)

Another common metric is the Bogoliubov metric. In the $(m)$-representation this is given by:

$$g^{Bo}_\rho(H, K) = \text{tr} \left( (d \log_\rho(H))^{(m)} K^{(m)} \right).$$  \hspace{1cm} (27)

The relation of the $(m)$- and $(e)$-representation is given by:

$$H^{(e)} = (d \log_\rho(H))^{(m)}$$  \hspace{1cm} (28)

$$H^{(m)} = \int_0^1 \rho^\lambda H^{(e)} \rho^{1-\lambda} d\lambda$$  \hspace{1cm} (29)

The Bogoliubov metric in $(e)$-representation is therefore given by:

$$g^{Bo}_\rho(H, K) = \text{tr} \left( H^{(e)} \int_0^1 \rho^\lambda K^{(e)} \rho^{1-\lambda} d\lambda \right).$$  \hspace{1cm} (30)

\textit{Remark 2} In the rest of the paper we will exclusively and implicitly use the $(m)$-representation for the elements of the tangent bundle of $P(n)$ and $D(n)$.
2 Bures-Wasserstein Geometry

In this section we explore the geometry induced by the Bures-Wasserstein distance. We start by finding the metric and geodesics corresponding to the BW distance on $\mathcal{P}(n)$. Subsequently, we restrict the obtained metric to $\mathcal{D}(n)$ and derive the corresponding distance function and geodesics. The flow of the argument is analogous to Section 1.2, where we start from the Hellinger distance on $\mathcal{M}_+(\Omega)$, derive the Fisher metric and subsequently find the Riemannian distance and geodesics for this metric restricted to the submanifold $\mathcal{P}_+(\Omega)$. We start by discussing some general results from Riemannian geometry.

Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be Riemannian manifolds and $\pi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ a smooth submersion. We can make the following orthogonal decomposition of the tangent space at $p \in \mathcal{M}$:

$$ T_p \mathcal{M} = \mathcal{V}(\pi, p) \oplus \mathcal{H}(\pi, p, g) \quad (31) $$

where $\mathcal{V}(\pi, p)$ is the kernel of $d\pi_p$ and $\mathcal{H}(\pi, p, g)$ is its orthogonal complement with respect to the metric at $p$. We will refer to these subspaces as vertical and horizontal respectively. A curve $\gamma$ in $\mathcal{M}$ is said to be horizontal if $\gamma'(t)$ is horizontal for all $t$. We say that a submersion $\pi$ is Riemannian if for all $p \in \mathcal{M}$ and $v, w \in \mathcal{H}(\pi, p, g)$ the following holds:

$$ g_p(v, w) = h_{\pi(p)}(d\pi_p v, d\pi_p w). \quad (32) $$

That is, $d\pi_p|_{\mathcal{H}(\pi, p, g)}$ is a vector space isometry.

**Theorem 2** If $(\mathcal{M}, g)$ is a Riemannian manifold and $G$ compact Lie group of isometries of $(\mathcal{M}, g)$ acting freely on $\mathcal{M}$, then there exists a unique $h$ such that the quotient map $\pi : (\mathcal{M}, g) \to (\mathcal{M}/G, h)$ is a Riemannian submersion.

**Proof** Corollary 2.29 in [13].

**Theorem 3** Let $\pi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be a Riemannian submersion. For every geodesic $\gamma$ in $(\mathcal{M}, g)$ such that $\gamma'(0)$ is horizontal we have:

- $\gamma'(t)$ is horizontal for all $t$.
- $\pi \circ \gamma$ is a geodesic in $(\mathcal{N}, h)$ of the same length as $\gamma$.

For every curve $\tilde{\gamma}$ in $(\mathcal{N}, h)$ we have that:

- there exists a unique horizontal curve in $\mathcal{M}$, denoted $\text{lift}(\tilde{\gamma})$, such that $\pi \circ \text{lift}(\tilde{\gamma}) = \tilde{\gamma}$.

**Proof** [5], [13].

**Theorem 4** Let $\pi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be a Riemannian submersion and $d_\mathcal{M}$ the Riemannian distance function on $\mathcal{M}$. The Riemannian distance function on $\mathcal{N}$, $d_\mathcal{N}$, is equal to:

$$ d_\mathcal{N}(p, q) = \inf_{\bar{p} \in \pi^{-1}(p), \bar{q} \in \pi^{-1}(q)} d_\mathcal{M}(\bar{p}, \bar{q}). \quad (33) $$
We will call \( d'_N \) the pushforward distance.

**Proof** Recall from equation (3) in the preliminaries that the Riemannian distance function on \( N \) is given by

\[
d_N(p, q) = \inf \{ L(\tilde{\gamma}) : \tilde{\gamma} : [0, 1] \to N, \tilde{\gamma}(0) = p, \tilde{\gamma}(1) = q \}
\]  

(34)

For every \( \tilde{\gamma} \) on the RHS we can find a curve \( \gamma \) in \( M \), namely \( \text{lift}(\tilde{\gamma}) \), such that \( L(\gamma) = L(\tilde{\gamma}) \) and \( \gamma(0) \in \pi^{-1}(p) \) and \( \gamma(1) \in \pi^{-1}(q) \). Therefore we have \( d_N(p, q) \geq d'_N(p, q) \).

For the reverse, we note that for every curve \( \gamma \) in \( M \) we have

\[
L(\gamma) = \int_0^1 ||\gamma'(t)||_g \ dt
\]  

(35)

\[
\geq \int_0^1 ||d\pi_\gamma(\gamma'(t))||_h \ dt
\]  

(36)

\[
= \int_0^1 ||(\pi \circ \gamma)'(t)||_h \ dt
\]  

(37)

\[
= L(\pi \circ \gamma)
\]  

(38)

where in the second line we use that \( \pi \) is a Riemannian submersion. From this it follows immediately that \( d_N(p, q) \leq d'_N(p, q) \).

\( \square \)

2.1 Geometry on the space \( \mathbb{P}(n) \)

**Riemannian metric and distance function**

In this section we will prove the following theorem.

**Theorem 5** The Bures-Wasserstein distance on \( \mathbb{P}(n) \) given in (11) is a Riemannian distance. The corresponding metric at \( \Sigma \) is given by:

\[
g_{\Sigma}^{BW}(H, K) = \text{Re} \ \text{tr}(L_{\Sigma}(H)\Sigma L_{\Sigma}(K)) = \frac{1}{2} \text{Re} \ \text{tr}(L_{\Sigma}(H)K)
\]  

(39)

In order to prove theorem 5 we need some preliminary results. The final proof can be found on page 12.

From Theorem 2 we know that there exists a metric \( \tilde{h} \) such that the quotient map \( (GL(n), \tilde{g}) \to (GL(n)/U(n), \tilde{h}) \) is a Riemannian submersion. We make the following identification: \( GL(n)/U(n) \ni M \cdot U(n) \leftrightarrow MM^* \in \mathbb{P}(n) \). This gives us the following map:

\[
\pi : (GL(n), \tilde{g}) \to (\mathbb{P}(n), h)
\]  

(40)

\[
M \mapsto MM^*
\]  

(41)
We will derive the preliminary results in the following order. First, we find the horizontal and vertical subspaces for \( \pi \) (proposition 1), which we use to show that \( h \) is given by (39) (proposition 2). Next, we show that the BW distance on \( \mathbb{P}(n) \) is equal to the pushforward distance given in (33) for \( \pi \) (proposition 3). Then we can use theorem 4 to conclude that the BW distance is actually the Riemannian distance for \( \mathfrak{g}^{BW} \).

**Proposition 1** Let \( \mathbb{H}(n) \) and \( \mathbb{H}^\perp(n) \) be the set of Hermitian and skew-Hermitian matrices, respectively. The vertical and horizontal space of \( \pi \) at \( M \in (GL(n), \bar{g}) \) are given by:

\[
\mathcal{V}(\pi, M) = \{ K (M^{-1})^* : K \in \mathbb{H}^\perp(n) \} \tag{42}
\]

\[
\mathcal{H}(\pi, M, \bar{g}) = \{ HM : H \in \mathbb{H}(n) \} \tag{43}
\]

**Proof** We have:

\[
d_{\pi M}(A) = AM^* + MA^*. \tag{44}
\]

Therefore \( d_{\pi M}(A) = 0 \iff A \in \mathcal{V}(\pi, M) \). Furthermore, we have that \( \bar{g}_M(K(M^{-1})^*, A) = 0 \forall K \in \mathbb{H}^\perp(n) \iff A \in \mathcal{H}(\pi, M, \bar{g}). \)

**Proposition 2** The metric \( h \) on \( \mathbb{P}(n) \) is given by (39).

**Proof** Because \( \pi \) is a Riemannian submersion we know from (32), that for \( A, B \in \mathcal{H}(\pi, M, \bar{g}) \), \( h \) needs to satisfy:

\[
\bar{g}_M(A, B) = h_{MM^*}(d\pi_M A, d\pi_M B) \tag{45}
\]

working this out gives:

\[
\Re \text{ tr}(AB^*) = h_{MM^*}(MA^* + AM^*, MB^* + BM^*). \tag{46}
\]

Now we plug in \( A = \tilde{H}M, B = \tilde{K}M \) for \( \tilde{H}, \tilde{K} \in \mathbb{H}(n) \). Then (46) becomes:

\[
\Re \text{ tr}(\tilde{H}MM^* \tilde{K}) = h_{MM^*}(MM^* \tilde{H} + \tilde{H}MM^*, MM^* \tilde{K} + \tilde{K}AA^*). \tag{47}
\]

If we set \( M = \Sigma^{1/2} \) and \( \tilde{H} = \mathcal{L}_\Sigma(H), \tilde{K} = \mathcal{L}_\Sigma(K) \), we get for general \( \Sigma \in \mathbb{P}(n) \) and \( H, K \in \mathbb{H}(n) \):

\[
h_{\Sigma}(H, K) = \Re \text{ tr}(\mathcal{L}_\Sigma(H) \Sigma \mathcal{L}_\Sigma(K)). \tag{48}
\]

Using the properties of the trace we have:

\[
\Re \text{ tr}(\mathcal{L}_\Sigma(H) \Sigma \mathcal{L}_\Sigma(K)) = \Re \text{ tr}(\mathcal{L}_\Sigma(K) \Sigma \mathcal{L}_\Sigma(H)) = \Re \text{ tr}(\mathcal{L}_\Sigma(H) \mathcal{L}_\Sigma(K) \Sigma) \tag{49}
\]

Adding the first and last expression gives:

\[
2h_{\Sigma}(H, K) = \text{ tr} \left[ \mathcal{L}_\Sigma(H) \left( \Sigma \mathcal{L}_\Sigma(K) + \mathcal{L}_\Sigma(K) \Sigma \right) \right] = \Re \text{ tr}(\mathcal{L}_\Sigma(H)K). \tag{50}
\]

Dividing both sides by two gives the final result. \( \square \)
In order to show that the BW distance on $P(n)$ is equal to the pushforward distance of $\pi$, we first have to investigate the distance on $(GL(n), \bar{g})$. We know that on $(C^{n} \times n, \bar{g})$ the distance is given by: $\bar{d}_{C^{n} \times n}(A, B) = ||A - B||_{2} = |\text{tr}((A - B)(A - B)^{*})|^{1/2}$. Because $GL(n) \subset C^{n} \times n$ we have $d_{GL(n)} \geq \bar{d}_{C^{n} \times n}$.

However we can show, using the following lemmata, that for some choices of $A$ and $B$ the curve $\gamma(t) = (1 - t)A + tB$ stays in $GL(n)$ and thus the two distances are equal.

**Lemma 1** For $\Sigma, T \in P(n)$ and $U = T\Sigma(T\Sigma^{2}T)^{-1/2}$, the unitary polar factor of $T\Sigma$, we have that $TU\Sigma^{-1} \in P(n)$.

**Proof** [5]. \(\square\)

**Lemma 2** For $A = \Sigma$ and $B = TU$ with $\Sigma, T \in P(n)$ and $U$ as in lemma [7] we have that $\gamma(t) = (1 - t)A + tB$ is in $GL(n)$ for $t \in [0, 1]$.

**Proof** We can write:

$$\gamma(t) = ((1 - t)I + tTU\Sigma^{-1}) \Sigma.$$ 

By the previous lemma know that $TU\Sigma^{-1}$ is positive definite. Therefore we have that $((1 - t)I + tTU\Sigma^{-1})$ is positive definite for $t \in [0, 1]$ and thus in $GL(n)$. Since $GL(n)$ is closed under multiplication we have that $\gamma(t) \in GL(n)$.

\(\square\)

Now we are in position to study the pushforward distance [53] for $\pi$. We have that $(M, g) = (GL(n), \bar{g})$ and $N = P(n)$. Plugging this in gives the following distance function:

$$d'_{P(n)}(\Sigma_1, \Sigma_2) = \inf\{d_{GL(n)}(M_1, M_2) : M_i \in \pi^{-1}(\Sigma_i)\}$$

$$= \inf_{U, V \in U(n)} d_{GL(n)}(\Sigma_1^{1/2}V, \Sigma_2^{1/2}U)$$

**Proposition 3** The BW distance on $P(n)$ is equal to the pushforward distance $d'_{P(n)}$. That is,

$$\left[\text{tr}(\Sigma_1) + \text{tr}(\Sigma_2) - 2\text{tr}\left(\left(\Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2}\right)^{1/2}\right)\right]^{1/2} = \inf_{U, V \in U(n)} d_{GL(n)}(\Sigma_1^{1/2}V, \Sigma_2^{1/2}U)$$

Moreover, the infimum on the right is attained when $V = I$ and $U$ the unitary polar factor of $\Sigma_2^{1/2}\Sigma_1^{1/2}$ given by $\Sigma_2^{1/2}\Sigma_1^{1/2}\left(\Sigma_2^{1/2}\Sigma_1^{1/2}\right)^{-1/2}$. 

Proof From the discussion above we know:

\[
d_{P(n)}^2(\Sigma_1, \Sigma_2) = \inf_{U, V \in U(n)} d^2_{GL(n)}(\Sigma_1^{1/2} V, \Sigma_2^{1/2} U) \quad (55)
\]

\[
\geq \inf_{U, V \in U(n)} d^2_{\mathbb{C}^{n \times n}}(\Sigma_1^{1/2} V, \Sigma_2^{1/2} U) \quad (56)
\]

\[
= \inf_{U, V \in U(n)} \|\Sigma_1^{1/2} V - \Sigma_2^{1/2} U\|^2. \quad (57)
\]

\[
= \inf_{U, V \in U(n)} \text{tr} \left( \left( \Sigma_1^{1/2} V - \Sigma_2^{1/2} U \right) \left( \Sigma_1^{1/2} V - \Sigma_2^{1/2} U \right)^* \right) \quad (58)
\]

\[
= \text{tr}(\Sigma_1) + \text{tr}(\Sigma_2) - 2 \sup_{U, V \in U(n)} \text{Re tr} \left( \Sigma_1^{1/2} V^* U \Sigma_2^{1/2} \right) \quad (59)
\]

We saw in theorem 1 of the preliminaries that the supremum on the right is obtained for \( V \) and \( U \) as in the proposition. Moreover, by lemma 2 we have that for this choice \((1 - t) \Sigma_1^{1/2} V + t \Sigma_2^{1/2} U \) stays in \( GL(n) \) and thus we have \( d_{GL(n)} = d_{\mathbb{C}^{n \times n}}. \) Therefore we get equality in (56) and conclude:

\[
d_{P(n)}^2(\Sigma_1, \Sigma_2) = \left[ \text{tr}(\Sigma_1) + \text{tr}(\Sigma_2) - 2 \text{tr} \left( \left( \Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2} \right)^{1/2} \right) \right]^{1/2} \quad (60)
\]

\[
= d_{P(n)}^{BW}(\Sigma_1, \Sigma_2) \quad (61)
\]

\[\square\]

Proof (of theorem 5) By theorem 2 we know that there exists a unique metric \( h \) such that \( \pi \) as defined in equation (40) is a Riemannian submersion. In lemma 2 we saw that this metric is given by \( g^{BW} \), as defined in (39) in the statement of the theorem. From theorem 4 we know that the Riemannian distance for this metric is given by the pushforward distance for \( \pi \). In proposition 3 we saw that this distance is equal to the BW-distance on \( P(n) \). This is what we set out to proof.

\[\square\]

Geodesics

In order to find a geodesic between \( \Sigma_1 \) and \( \Sigma_2 \) in \( P(n) \), according to theorem 3 we need to find a geodesic \( \gamma \) in \( GL(n) \) between points in \( \pi^{-1}(\Sigma_1) \) and \( \pi^{-1}(\Sigma_2) \) such that \( \gamma'(0) \) is horizontal.

**Theorem 6** A geodesic between \( \Sigma_1 \) and \( \Sigma_2 \) in \( (P(n), g^{BW}) \) is given by \( \pi \circ \gamma \), where

\[
\gamma(t) = (1 - t) \Sigma_1^{1/2} + t \Sigma_2^{1/2} U \quad (62)
\]

and \( U \) is again the unitary polar factor of \( \Sigma_2^{1/2} \Sigma_1^{1/2} \).
**Proof** It is clear that $\gamma$ is a geodesic in $(\mathbb{C}^{n\times n}, \bar{g})$. We saw in lemma 2 that $\gamma$ stays in $GL(n)$. It remains to show $\gamma'(0) \in H(\pi, \Sigma_1^{1/2}, \bar{g})$. We have:

$$\gamma'(0) = \Sigma_2^{1/2}U - \Sigma_1^{1/2}$$

(63)

$$= \left(\Sigma_2^{1/2}U \Sigma_1^{1/2}^{-1} - I\right) \Sigma_1^{1/2}$$

(64)

By lemma 1 we have that $\Sigma_2^{1/2}U \Sigma_1^{1/2}^{-1}$ is Hermitian and thus the same holds for $\Sigma_2^{1/2}U \Sigma_1^{1/2} - I$. Using proposition 1 we conclude $\gamma'(0)$ is horizontal. The statement of the theorem now follows from theorem 3. \qed

2.2 Geometry on the space $\mathcal{D}(n)$

**Inner product and distance function**

Let us denote the unit sphere in $(\mathbb{C}^{n\times n}, \bar{g})$ by:

$$S_{\mathbb{C}^{n\times n}} \equiv \{A \in \mathbb{C}^{n\times n} : \text{tr}(AA^*) = 1\}$$

(65)

and $S_{GL(n)} \equiv S_{\mathbb{C}^{n\times n}} \cap GL(n)$, its restriction to $GL(n)$. Note that:

$$\pi^{-1}(\mathcal{D}(n)) = S_{GL(n)},$$

(66)

We now apply theorem 2 to this submanifold of $GL(n)$. We choose the metric $g$ to be the restriction of $\bar{g}$ to $S_{GL(n)}$ and the Lie group $G$ again $U(n)$. Since both the metric and the quotient map are just restrictions of the ones in theorem 5 the resulting metric on $S_{GL(n)}/U(n) \cong \mathcal{D}(n)$ will also be the restriction of $\bar{g}^{BW}$. From theorem 4 we therefore know that the Riemannian distance function on $\mathcal{D}(n)$ corresponding to this restricted metric is given by the pushforward distance defined in (33). Just as in the classical case (section 1.2) where the Fisher distance on $\mathcal{P}_+(\Omega)$ is different from the Hellinger distance, it will turn out that the Riemannian distance on $\mathcal{D}(n)$ is different from the BW distance on $\mathcal{P}(n)$. In order to compute the distance on $\mathcal{D}(n)$, we first investigate the geometry on $S_{GL(n)}$.

Geodesics on a Euclidian sphere are obtained by intersecting the sphere with (hyper)planes through the origin. If $M$ and $N$ are two non-antipodal point on $S_{\mathbb{C}^{n\times n}}$ we can obtain the unnormalised geodesic by projecting the geodesic in $\mathbb{C}^{n\times n}$ onto $S_{\mathbb{C}^{n\times n}}$. More specifically, if $\gamma(t) = (1 - t)M + tN$ is the geodesic in $\mathbb{C}^{n\times n}$, then

$$\tilde{\gamma}(t) = \frac{\gamma(t)}{||\gamma(t)||_2}$$

(67)

is the unnormalised geodesic in $S_{\mathbb{C}^{n\times n}}$. Moreover, we have that $\gamma(t) \in GL(n) \implies \tilde{\gamma}(t) \in GL(n)$ since they are scalar multiples of each other. The distance on $S_{\mathbb{C}^{n\times n}}$ is given by:

$$d_{S_{\mathbb{C}^{n\times n}}}(M, N) = \arccos(\Re \text{tr}(MN^*))$$

(68)
Just as before, we have that in general \( d_{\mathbb{S}^{n \times n}} \leq d_{\mathbb{S}^{GL(n)}} \), but when \( \tilde{\gamma} \) stays in \( GL(n) \), we have that the two distances are equal. We are now in position to deduce the Riemannian distance function on \( D(n) \).

**Theorem 7** On \( D(n) \), the Riemannian distance for the Bures-Wasserstein metric is given by:

\[
d_{BW}^{D(n)}(\rho_1, \rho_2) = \arccos \left( \text{Re tr} \left( \left( \rho_2^{1/2} \rho_1^{1/2} \right)^2 \right) \right)
\]

(69)

**Proof** From the definition of the quotient distance we have:

\[
d_{BW}^{D(n)}(\rho_1, \rho_2) = \inf_{U, V \in U(n)} d_{\mathbb{S}^{GL(n)}} \left( \rho_1^{1/2} V, \rho_2^{1/2} U \right) \geq \inf_{U, V \in U(n)} d_{\mathbb{S}^{n \times n}} \left( \rho_1^{1/2} V, \rho_2^{1/2} U \right) = \inf_{U, V \in U(n)} \arccos \left( \text{Re tr} \left( \rho_1^{1/2} U^* \rho_2^{1/2} \right) \right)
\]

(70)

As before, the infimum is attained for \( V = I \) and \( U \) the unitary polar factor of \( \rho_2^{1/2} \rho_1^{1/2} \). From lemma 2 and the above discussion we know that for this choice of \( U \) and \( V \) we have \((1 - t)\rho_1^{1/2} V + t\rho_2^{1/2} U \in GL(n)\). Therefore we have equality in (71) and conclude the proof. \( \square \)

**Geodesics**

Analogues to theorem 6, we get the following result for the geodesics in \( D(n) \):

**Theorem 8** An unnormalised geodesic between \( \rho_1 \) and \( \rho_2 \) in \( (D(n), g^{BW}) \) is given by \( \pi \circ \tilde{\gamma} \), where

\[
\tilde{\gamma}(t) = \frac{\gamma(t)}{||\gamma(t)||_2},
\]

(73)

with

\[
\gamma(t) = (1 - t)\rho_1^{1/2} + t\rho_2^{1/2} U
\]

(74)

and \( U \) is again the unitary polar factor of \( \rho_2^{1/2} \rho_1^{1/2} \).

**Proof** We saw above that \( \tilde{\gamma}(t) \in GL(n) \) and therefore by theorem 6 it is enough to show \( \tilde{\gamma}'(0) \) is horizontal. The direction of \( \tilde{\gamma}'(0) \) can be obtained by projecting \( \gamma'(0) \in T_{\rho_1^{1/2} GL(n)} \) onto the subspace \( T_{\rho_1^{1/2} S_{GL(n)}} \). Since span\( \{\rho_1^{1/2}\} \) is the orthogonal complement of \( T_{\rho_1^{1/2} S_{GL(n)}} \) within \( T_{\rho_1^{1/2} GL(n)} \), this projection is given by:

\[
\tilde{\gamma}'(0) \propto \gamma'(0) - \text{Re tr} \left( \rho_1^{1/2} \gamma'(0)^* \right) \rho_1^{1/2}
\]

(75)

From the proof of theorem 6 it follows that \( \gamma'(0) \) is horizontal. Since \( \text{Re tr} \left( \rho_1^{1/2} \gamma'(0)^* \right) \) is just a scalar and \( \rho_1^{1/2} \), viewed as a tangent vector, is horizontal, it follows that \( \tilde{\gamma}'(0) \) is also horizontal. \( \square \)
3 BW geometry and quantum information

In quantum information theory, a quantum state or density operator is a mathematical formulation of the state of a system and is represented by a trace-one positive semi-definite matrix $\rho$. We will denote the set of these matrices by $D_0(n)$. From the spectral theorem it follows that any $\rho \in D_0(n)$ can be written as $\rho = UDU^*$ with $U \in U(n)$ and $D$ a real positive diagonal matrix with trace one. Since this latter matrix can interpreted as a probability distribution on $\{1, 2, ..., n\}$, $D(n)$ can be viewed as a generalisation of the space of probability distributions. See [24] for a more detailed account of this statement. The $\rho$'s for which $D$ is a Dirac delta function are called pure states and their set is denoted $D_p(n)$.

**Fubini-Study metric**

Note that a pure state $\rho \in D_p(n)$ can also be written as follows: $\rho = \phi\phi^T$ where $\phi \in S_{C^n} = \{\psi \in C^n : ||\psi||_2 = 1\}$. The set of pure states can be identified with the complex projective space. This space is obtained by identifying two the elements in $S_{C^n}$ that differ a complex factor, that is the quotient space $S_{C^n}/U(1)$. The identification between the pure states and $S_{C^n}/U(1)$ is given explicitly as follows: $D_p(n) \ni \rho = \phi\phi^T \leftrightarrow [\phi] \in S_{C^n}/U(1)$. Combining the above, we get the following quotient map:

$$S_{C^n} \ni \phi \mapsto \phi\phi^T \in D_p(n).$$

(76)

If we equip the unit sphere with the Euclidian metric, the resulting quotient metric on the set of pure states will be the Fubini-Study metric with corresponding distance measure $d^{FS}([\phi], [\psi]) = |\langle \phi, \psi \rangle|$. See [3] for a more comprehensive description.

From section 2 we know that the BW metric is obtained in a similar way. If we let $S_{GL} = \{A \in GL(n) : \text{tr}(AA^*) = 1\}$, the unit sphere in $GL(n)$, then $D(n)$ can be identified with $S_{GL}/U(n)$ such that $\rho \leftrightarrow [\rho^{1/2}]$. It turns out that when we equip $S_{GL}$ with the Euclidian metric, the resulting quotient metric on $D(n)$ is the BW metric. This shows that the BW metric can be viewed as a generalization of the Fubini-Study metric for mixed states, with (22) corresponding to $d^{FS}$ for pure states.

**Uhlmann and Takatsu**

In 1992 the German theoretical physicist Armin Uhlmann gave a lecture at the Symposium of mathematical physics in Toru titled "Density operators as an arena for geometry" [23]. In this talk he considers density operators as reductions of elements of a larger Hilbert space $H^{ext}$ called the purification space. If the reduction of a vector $M \in H^{ext}$ is equal to a density operator $\rho$
we call this vector a purification of $\rho$. This modus operandi is often referred to as: "Going to the church of the larger Hilbert space" [7]. $H^{ext}$ is given by the space of $n$-dimensional operators with the Hilbert-Schmidt (Euclidian) inner product and the reduction map is given by: $M \mapsto MM^* = \rho$. Note that the fiber of $\rho$ is given by: $\{\rho^{1/2}U : U \in U(n)\}$. The fact that multiple vectors in $H^{ext}$ correspond to the same density operators is referred to as gauge freedom.

Minimizing the distance between purifications of $\rho_1$ and $\rho_2$ using this freedom gives the BW distance and computing the pushforward of the Hilbert-Schmidt inner product by the reduction map gives the BW metric.

As one can see, the approach taken by Uhlmann is similar as the argument in section 2. The purification space takes the role of $GL(n)$, the reduction map is the quotient map $\pi$ and the gauge freedom is justified by theorem 4.

In 2008 an argument in a similar language to ours was given independently by Takatsu, this time for the Wasserstein distance between two mean-zero Gaussian distributions [20]. Since this distance is identical to the Bures distance, results from both fields can be carried over.

Wigner-Yanase information

In section 1.2 it was described that the Fisher metric is the pushforward metric of the Euclidian metric under the square map. That is, the following map is an isometry:

$$(\mathcal{M}_+(\Omega), \mathfrak{g}) \ni \mu \mapsto \mu^2 \in (\mathcal{M}_+(\Omega), \mathfrak{g}^F) .$$

(77)

If one would replace $\mathcal{M}_+(\Omega)$ by $\mathbb{P}(n)$, the pushforward of $\mathfrak{g}$ will be the Wigner-Yanase metric as described in [9].

We will now describe what the metric on the left should be so that its pushforward becomes the BW metric. We write $(\cdot)_{H(M)} : T_MGL(n) \to H(\pi, M, \mathfrak{g})$ for the orthogonal projection on the horizontal subspace of $T_MGL(n)$ where $\pi$ is still as defined in (40).

**Lemma 3** For $M \in GL(n)$ and $A \in H(\pi, M, \mathfrak{g})$, we have:

$$\mathcal{L}_{MM^*}(d\pi_M A) = AM^{-1}$$

(78)

**Proof** Since $A \in H(\pi, M, \mathfrak{g})$ we have that $AM^{-1} \in \mathbb{H}(n)$. We check:

$$MM^*(AM^{-1}) + (AM^{-1}) MM^* = MM^*(AM^{-1})^* + (AM^{-1}) MM^*$$

$$= MA^* + AM^*$$

$$= d\pi_M A$$

(80)

(81)

We define the following metric on $\mathbb{P}(n)$ for $H, K \in T_{\Sigma} \mathbb{P}(n)$:

$$\mathfrak{g}^H_{\Sigma}(H, K) \equiv \text{Re tr} \left( H_{\Sigma} \left( K_{\Sigma} \right)^* \right).$$

(82)
Note that the projection \((\cdot)_{H(\Sigma)}\) happens in the ambient space \(T_{\Sigma}GL(n)\) and therefore in general \(H(\Sigma), \mathcal{K}_{H(\Sigma)} \not\in T_{\Sigma}F(n)\).

**Theorem 9** The pushforward metric of \(g^H\) under the square map is equal to the BW metric \(g^{BW}\). That is, the following map is an isometry:

\[
\pi : (\mathbb{P}(n), g^H) \to (\mathbb{P}(n), g^{BW})
\]

\[
\Sigma \mapsto \Sigma^2
\]

**Proof** We have:

\[
g^{BW}_{\Sigma^2} (d\pi \Sigma H, d\pi \Sigma K) = g^H_{\Sigma^2} (d\pi \Sigma H_{\Sigma(\Sigma)}, d\pi \Sigma K_{\Sigma(\Sigma)})
\]

\[
= \frac{1}{2} \text{Re tr} \left( \mathcal{L}_{\Sigma^2} (d\pi \Sigma H_{\Sigma(\Sigma)}) d\pi \Sigma K_{\Sigma(\Sigma)} \right)
\]

\[
= \frac{1}{2} \text{Re tr} \left( H_{\Sigma(\Sigma)} \Sigma^{-1} (\Sigma (K_{\Sigma(\Sigma)})^* + K_{\Sigma(\Sigma)} \Sigma) \right)
\]

\[
= \text{Re tr} \left( (H_{\Sigma(\Sigma)} \Sigma)^* \right)
\]

\[
= g^H_{\Sigma^2} (H, K)
\]

\[
\square
\]

**Conclusion**

In this paper we have presented the derivation of the Riemannian metric corresponding to the BW distance on \(\mathbb{P}(n)\), following the exposition of [5]. Subsequently, we have adapted this argument so that the Riemannian distance and geodesics could be found for this metric on the subset \(D(n)\). This part can be considered original work of the author. In the last part we have compared the geometrical structure to similar structures within quantum information.

**Further questions**

The geometrical structure derived in this paper lives on the space of positive definite matrices. In quantum information, the larger set allowing for eigenvalues to be zero is studied. It would be interesting to investigate whether the argument to obtain this geometrical structure can be generalised to the set of positive semi-definite matrices.

The Talagrand inequality in its original form [22] gives a bound on the 2-Wasserstein distance between two Gaussian distributions in terms of their relative entropy:

\[
W^2(\mu, \nu) \leq D(\mu || \nu).
\]

The Wasserstein distance can be written in terms of the covariance matrices of its arguments. The current exposition shows that written in this form, this distance measure appears in quantum information. A related distance measure
on the space of covariance matrices is the Von Neumann relative entropy. A further study of the relation between these two distances could result in a distance measure between two (classical) distributions in terms of the Von Neumann relative entropy of their covariance matrices, similar to [8], and could potentially lead to a Talagrand-type inequality.

**Notation**

\[
\begin{align*}
\Omega &= \{\omega_1, \ldots, \omega_n\} \\
S(\Omega) &= \{\mu : \Omega \rightarrow \mathbb{R}\} \\
\mathcal{M}_+(\Omega) &= \{\mu \in S(\Omega) : \mu(\omega_i) > 0, \forall i \in \{1, \ldots, n\}\} \\
\mathcal{P}_+(\Omega) &= \{p \in \mathcal{M}_+(\Omega) : \sum_i p(\omega_i) = 1\} \\
GL(n) &= \{M \in \mathbb{C}^{n \times n} : \det(M) \neq 0\} \\
U(n) &= \{U \in GL(n) : U^*U = I\} \\
\mathbb{H}(n) &= \{H \in \mathbb{C}^{n \times n} : H = H^*\} \\
\mathbb{H}^-(n) &= \{K \in \mathbb{C}^{n \times n} : K = -K^*\} \\
\mathbb{P}(n) &= \{\Sigma \in \mathbb{H}(n) : \phi^*\Sigma\phi \geq 0, \forall \phi \in \mathbb{C}^n\} \\
\mathcal{D}(n) &= \{p \in \mathbb{P}(n) : \text{tr}(p) = 1\} \\
\mathcal{L}_\Sigma(H) &= X \text{ such that } X\Sigma + \Sigma X = H
\end{align*}
\]

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