Turbulent plane Poiseuille flow

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Abstract The understanding of fully developed turbulence remains a major unsolved problem of statistical physics. A challenge there is how to use what we understand of this problem to build-up a closure method, that is to express the time-averaged turbulent stress tensor as a function of the time-averaged velocity field \( u(x) \). We have shown that the closure problem is strongly restricted due to constraints on the time-averaged quantities, and to scaling laws derived from the idea that dissipation in fully developed turbulence is by singular events resulting from an evolution described by the Euler equations. It implies that the turbulent stress is a non-local function in space of the time-averaged velocity \( u(x) \), involving an integral kernel, an extension of classical Boussinesq theory of turbulent viscosity. We treat one of the simplest possible physical situation, the turbulent Poiseuille flow between two parallel plates. In this case, the integral kernel takes a simple form leading to full analysis of the time-averaged turbulent flow. In the limit of a very large Reynolds number, one has to match a viscous boundary layer near the walls bounding the flow and an outer solution in the bulk of the flow, a non-trivial asymptotic analysis because of logarithms. Besides the boundary layers close to the walls, there is another “inner” boundary layer near the center plane of the flow. Our expression for the turbulent stress tensor yields ultimately the complete structure of the boundary layer, including in locations where viscosity becomes important.

In recent works, we introduced a new way of modeling turbulent flows in “real situations,” that is, where turbulence is due to an incompressible flow at large Reynolds number \( Re \) in given geometries. This was partly for the purpose to show how to use “concretely” the idea that, in such a turbulent flow, the Reynolds stress tensor (RST), or turbulent tensor, can be described by explicit expressions of the global (in the sense of space dependent and time-averaged) velocity field. These expressions are strongly constrained by the fact that the dissipation in such a turbulent flow does depend only on the parameters of the time-average of the velocity field. As shown in previous papers [1,2], this leads quite naturally to write the turbulence stress as a non-local quantity, depending quadratically on the time average velocity field \( u(x) \) and its space derivatives, with non-diagonal components of the RST, of the form

\[
\sigma_{ij}^{Re}(x) = \gamma \rho |\nabla \times u(x)|^{1-\alpha} \int \nabla' |\nabla \times u(x')|^{\alpha} K(x, x') (u_{i,j} + u_{j,i})(x')
\] (1)
where the exponent $\alpha$ is such that $0 < \alpha < 1$, $\gamma$ is a dimensionless constant, $\rho$ is the mass density and $K(x, x')$ is the integral kernel which has the dimension of the inverse of a length.

The expression (1) for the RST, together with the balance of momentum, allowed a detailed analysis of the turbulent mixing layer [2], that led to an almost fully explicit solution in the case of a small velocity difference between the two parallel flows merging in the wake of the splitting plate. In the mixing layer setup, we proposed to take a simple expression for the kernel, based on dimension analysis, whereas it is irrespective of the boundary conditions. This choice is acceptable because the solid boundaries play a minor role in this problem, contrary to many real turbulent flows. Therefore, we thought it of interest to use the same theory in a turbulent flow where solid boundaries play instead a fundamental role. Here, we study a classical example of such a flow, namely the Poiseuille flow between two parallel planes. In such a flow, one has first to find explicitly the integral kernel that enters into the expression of the RST. Assuming non-slip boundary conditions, we get a turbulent stress decaying smoothly to zero as one approaches the boundary, that constitutes one of the fundamental assumptions in Prandtl boundary layer theory leading to a log-dependence law for the profile. However, compared to this well-known theory, the present work deals with equations not limited to the neighborhood of the wall. Therefore, it is possible to do the full matching between the viscous sublayer and the flow far from it. This is no trivial matter because of the occurrence of another logarithm in the boundary layer solution.

Our detailed analysis has shown that the Dirichlet condition on the walls allows to fix the value of the exponent $\alpha$ that makes significant advance in comparison with our previous works where the value of $\alpha$ remained free. Moreover, we found a rather unexpected phenomenon, namely the existence of a pair of boundary layers on either side of the central plane of the channel, far from the walls. In the transverse plane, along the $z$ coordinate, the velocity is maximum in the center, as expected, but we observe that the profile of $u(z)$ displays a small roof-like behavior of short range. This range is about the same as the width of boundary layers close to the walls, proportional to the inverse of the Reynolds number $Re$.

The closure equation (1) is explained in Sect. 1. The geometry of the problem is defined in Sect. 2, where we give the equations to be solved together with their boundary conditions. In Sect. 3, we write first the equation for the balance of momentum in the simplest possible form and solve it near the wall to get the log-law of the wall, and then we carry explicitly the matching of this boundary layer solution with the solution in the bulk, namely outside the close vicinity of the wall, and we study the abrupt but continuous slope behavior observed numerically in the center of the fluid layer. Finally, we use the mixing length model (including the value of the von Karman constant derived from observations) to get the constant $\gamma$, the only free parameter remaining in our theory.

Section 4 is more speculative than the previous ones. We try to draw some conclusions from what is done in the previous sections concerning the general question of the turbulent drag on blunt bodies, a question with an history going back to the Principia. Our point is that there are two kinds of drag, one due to the “skin friction” with a friction coefficient decaying (slowly) as $Re$ increases, and another drag, named turbulent drag here, formed when the flow has to round an obstacle (and therefore absent in Poiseuille flow with perfectly smooth walls). This turbulent drag is the one found by Newton, and it is proportional to the square of the velocity with a coefficient tending to a constant of proportionality remaining finite as $Re$ tends to infinity. So to speak we explain what is the physical origin of this Newton drag, related to the separation of the flow in a potential and non-potential domain, a point considered in details by Landau but without relating it to the quadratic Newton’s drag. Before the conclusion, we consider, in sec. 5, two other examples of turbulent flows behind an obstacle leading to different scalings for the boundary layers.
1 Equation for the closure

The aim of this section is to explain the derivation of Eq. (1) expressing the RST as a function of the average velocity field. Since it is an impossible task yet to derive the RST from a direct solution of the time-dependent equations for the fluid velocity, we found it appropriate to derive this expression from various physical constraints with as few uncontrolled assumptions as possible. Our starting point is that the RST represents actually the effect of finite time singularities of solutions of the time-dependent Euler equations, as written in the abstract. Dissipation by such singularities has been highlighted in a previous paper [3] by the analysis of the structure factors

\[ S_n(\tau) = \langle (a(t) a(t+\tau))^n \rangle \]

where \( a(t) \) is the Eulerian acceleration (time derivative of the velocity at a given spatial point), compared with data taken in Modane’s wind tunnel at high Reynolds number. The structure functions display a striking change of form at short distance, for \( \tau \) about the Kolmogorov’s time, if \( n \) is larger than a critical value.

The role of singularities has in particular the non-trivial consequence that the RST depends in a non-analytical way of the velocity field, reflected in Eq. (1) by the absolute value of the vorticity \( |\nabla \times u(x)| \). Indeed, the presence of an absolute value in the definition of the RST is not new, even though it does not seem to have been linked to singular events breaking the smoothness of the solution and then breaking the analyticity of the relationship between RST and average velocity. The same can be said about the non-analyticity of the drag force on a fast moving object, which is proportional\(^1\) to \( |U| U \) where \( U \) is the velocity of this object with respect to the fluid at rest at infinity, see Eq. (79). Thanks to the absolute value, the RST changes sign as velocity is reversed, a way to represent the loss of time reversal symmetry and so the dissipation occurring in singular events. In particular, this yields, as expected, a change of sign of the drag force as the velocity is reversed.

Now let us return to \( \sigma_{ij}^{Re}/\rho \), which has the dimension of a velocity square, since it is defined by the relation

\[ \sigma_{ij}^{Re}(x)/\rho = \langle u'_i(x) u'_j(x) \rangle \]

where \( u'_i, u'_j \) are the components of \( u' \), the fluctuating part of the velocity field. The velocity is present in Eq. (1) through the vorticity terms and the strain tensor. As we do not consider homogeneous turbulence, we define the stress tensor in the turbulent part of the flow, the one where eddies are formed. In this domain, vorticity is amplified by instabilities and vortex stretching. As stated by Landau,\(^2\) turbulence mainly exists in the part of the fluid where the vorticity is nonzero in average, which he called ” region of turbulent flow or turbulent region.” This explains the occurrence of the vorticity in the definition of the RST which cancels outside the turbulent region, by definition. Let us notice that other models describing the RST as proportional to a turbulent viscosity do not satisfy the condition \( \sigma^{Re} = 0 \) outside the turbulent domain where the flow is potential (as in front of an obstacle, but not in its wake for instance).

The term \( (u_{i,j} + u_{j,i})(x') \) is there to yield a rank two symmetrical tensor in a geometrically coherent way, as given for instance by the viscous stress, see the discussion in [1]. Note first that the RST, as a product of these two terms, vorticity and strain, depends only on the spatial derivatives of \( u \). Because of that the RST is Galilean invariant, since adding a constant velocity to \( u(r) \) does not change at all \( \sigma^{Re} \) although turbulence is not Galilean invariant, even by the role of the Reynolds number. Secondly let us note that the RST defined by Eq.

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1 This is the classical drag law of Newton, quadratic with respect to the velocity. Notice that Newton’s drag law was also derived by Maxwell (without the absolute value) who did not quote Newton.

2 In Sect. (35) of [4], Landau wrote ” the dissipation occurs mainly in the region of rotational turbulent flows and hardly at all outside this region.”
(1) is not aligned with the strain or viscous tensor at the same place, \( \sigma_{ij}^v = -\mu (u_{j,i} + u_{i,j}) \), where \( \mu \) is the viscosity coefficient, as it is in almost all models starting with the empirical Boussinesq hypothesis which have been proved [5] to be not adapted to describe wakes behind an obstacle, or in the vicinity of frontiers like the bottom of a river or the boundary of a pipe.

Let us consider now the kernel \( K(x, x') \) and look at its physical dimension in 3D geometry (when the velocity field depends on three variables). Our model Eq. (1) has the physical dimension \([U]^2[K][L] \) where \([L] \), \([U] \) and \([K] \) are, respectively, the physical dimensions of a length, a velocity and of the kernel \( K(x, x') \). Comparing with Eq. (2), we get

\[
[K] = 1/[L],
\]

as written in Introduction. Far from the boundaries, because of the symmetry under translation and rotation, the kernel in Eq. (1) is a function of \( x-x' \), then the integral is a spatial convolution on two functions, \( K(x) \) and a function depending of the velocity. It follows that the kernel, as a function of \( x \), can be seen as the Green function, or fundamental solution, associated with a differential operator. Taking account of Eq. (3), this differential operator is of order two, and then it obeys the Laplace equation

\[
\nabla^2 K(x, x') = \delta(x - x')
\]

where \( \delta(\cdot) \) is the 3D Dirac distribution. The solution of Eq. (4) is

\[
K(x, x') = \frac{1}{|x - x'|}.
\]

which satisfies Eq. (3).

Equation (5) was used in our previous papers [1,2], because we assumed that the solid boundaries play a minor role in respect to the study of the turbulent domain.\(^3\) Strictly speaking this choice is valid in the case of the simplest (and unrealistic) geometrical setup, where no obstacle stops the flow, whereas if there is some solid surface, one has to change the integral kernel in such a way that the Prandtl boundary condition\(^4\) on the solid surface is satisfied.

Equation (5) is clearly not compatible in general with the existence of an obstacle, because all the RST components must vanish on the surface of the solid if the no slip condition is imposed.\(^5\) Therefore, we propose to define the kernel \( K(x, x') \), as a function of \( x \), by Eq. (4) with Dirichlet boundary conditions on solid surfaces limiting the flow, the solid surfaces or obstacle being immobile in the frame where the RST is written. As shown below, the Dirichlet Green’s function leads to rather simple calculation in the case of a plane Poiseuille flow. Note that this model generalizes previous attempts by Taylor [7] and others to extend the simple mixing length theory of Prandtl, to geometries more complex than the one of a flow parallel to a simple plane.

There remain two points concerning the expression of the RST in Eq. (1) in the case of a plane Poiseuille flow. First, we have to note that the geometry of this setup reduces the triple

\(^3\) In the cases of circular pipes and mixing layer set up considered in our previous papers [1,2], we have chosen to take a full 3D propagator, although the velocity fields depend on two variables only. Following the present argument, in the 2D case the kernel would be of the form \( \log \sqrt{x^2 + z^2} \).

\(^4\) Prandtl boundary conditions include the conditions on the surface itself, taking into account the possible roughness, and the Prandtl boundary layers equations. Let us notice that the classical Prandtl equations were shown to be still valid for smooth (no slip boundary conditions) curved surfaces when the viscosity goes to zero [6].

\(^5\) In the more general case of non-smooth surfaces, the RST components \( \sigma_{ij} \) including the velocity fluctuations normal to the solid surface vanish, but not the longitudinal ones.
integral to a single one, because $u$ depends on a single variable, therefore the Green’s function is not given by Eq. (5), as detailed in next section. Secondly, the value of the exponent $\alpha$ needs to be known, as well as the one of the dimensionless prefactor $\gamma$. The value of $\alpha$ is determined in a rather unexpected way by the solution of the equation for the plane Poiseuille flow including the boundary conditions on the walls where $\sigma_{xz} = 0$. It turns out that this condition imposes $\alpha = 0$. Even though the viscosity plays a major role close to the plane surfaces where it dictates the width of the viscous boundary layer, it does not enter into play to impose the value of $\alpha$, see the derivation of Eq. (78) below.

The other quantity introduced in our expression of the RST is the prefactor $\gamma$. As shown below, this prefactor enters in the solution for the log-law of the wall. By comparing our result with the well-known formula of Prandtl–von Karman, we have deduced

$$\gamma \approx 0.03$$

Note that this procedure amounts to fit $\gamma$ with experiments since the Prandtl–von Karman model includes the so-called von Karman constant $\kappa$ which has been determined over the years by analysis of experimental results. This closes our expression of the RST with no freedom leftover.

2 Statement of the problem

The geometry under consideration is a turbulent plane Poiseuille flow in between two parallel (and infinite) plates located, respectively, at the elevation $z = -h/2$ and $z = h/2$, see Fig. 1. Because of the symmetry with respect to the middle-plane located at $z = 0$, the time average velocity $u$ has a single component, $u$, along the $x$ axis, which depends on $z$, the relevant variable in our problem. The third dimension associated with the $y$ coordinate will play no role at all. The fluid has a kinematic viscosity $\nu$ and a mass density $\rho$. It is driven by a constant uniform pressure gradient $g$ along $x$,

$$g = \frac{d\rho}{dx}. \quad (6)$$

The only one component of the RST we shall deal with is

$$\sigma_{xz} = \rho \langle u'_x u'_z \rangle \quad (7)$$

where $u'_i$ is the fluctuating part of the velocity component along the $i$-coordinate. The time-averaged stress balance along $x$ yields the equation

$$\frac{1}{\rho} \frac{d\sigma_{xz}}{dz} - \nu \frac{d^2 u(z)}{dz^2} = -\frac{g}{\rho} \quad (8)$$

Fig. 1 Plane Poiseuille setup
or, after integration over the variable $z$,

$$
\frac{\sigma_{xz}}{\rho} - \nu \frac{du(z)}{dz} = \frac{|g|}{\rho} z \tag{9}
$$

where $|g| = -g$ because we assume that the pressure gradient is negative.

Note that we use straight derivatives because all functions involved depend on the variable $z$ only, as written above. Without the contribution of the turbulent stress, this equation reduces to the one of the laminar plane Poiseuille flow when the boundary conditions $u(z) = 0$ are imposed for $z = \pm h/2$. What we shall do now is to look at the solution of this equation in the opposite limit where $\nu$ is small, which is equivalent to the limit of a very large Reynolds number. This requires to express the turbulent stress $\sigma_{xz}$ as a function of the mean velocity field $u(z)$ and then to solve the equation with the boundary condition, an implicit problem.

As stated in Introduction, our model (1) for the RST satisfies various symmetries based on the fact that dissipation is due to singular events described by solutions of the Euler fluid equations. In the geometry under consideration, $\sigma_{xz}$ depends on $z$ only, and the kernel $K$ is the Green function of the 1D Laplacian. Equation (1) becomes

$$
\frac{\sigma_{xz}(z)}{\rho} = \gamma \frac{du(z)}{dz} \left|^{1-\alpha} \int_{-h/2}^{h/2} dz' \left|^{\alpha} \frac{du(z')}{dz'} \right| K(z, z') \frac{du(z')}{dz'} \quad \tag{10}
$$

without integration constant because we assume that the velocity field $u(z)$ is an even function. In the expression (10) $\gamma$ is a numerical constant independent of the Reynolds number and of order unity with respect to $Re$. Various simplifications appear compared to the expressions given in [1,2]. First, the vorticity has been written here as the derivative $\frac{du(z)}{dz}$, as it follows from the geometry. Furthermore the equation for the kernel $K(z, z')$ is

$$
\frac{d^2 K(z, z')}{dz^2} = \delta(z - z'). \tag{11}
$$

where $\delta(.)$ is Dirac’s delta function. Note that one may also use the 3D expression $K(x, x') = \delta(x - x')\delta(y - y')K(z, z')$ because the integration of Dirac’s delta functions gives unity, that reduces the triple integral in Eq. (1) to a single one.

Because Eq. (10) is posed with $z$ as variable, the solution satisfies the Dirichlet boundary condition

$$
K \left( \pm \frac{h}{2}, z' \right) = 0 \tag{12}
$$

and so yields a stress tensor $\sigma_{xz}(z)$ vanishing for $z = \pm h/2$, an important constraint. Note that whatever the boundary, all the RST components with velocity fluctuation $u'_i$ normal to the solid boundary are zero, because both the average velocity and its fluctuations vanish on the surface. Here, $u'_z = 0$ for $z = \pm h/2$. The solution of Eq. (11) with these boundary conditions is

$$
K(z, z') = \frac{1}{2} \left( |z - z'| + \frac{2}{h} zz' - \frac{h}{2} \right) \tag{13}
$$

The function $K(z, z')$ has two properties that will play a role later, first it is an even function under the joint change of sign of $z$ and $z'$,

$$
K(z, z') = K(-z, -z'), \tag{14}
$$
Furthermore $K(z, z')$ is a continuous function of $z$ and $z'$ and its derivative with respect to $z$ has a finite jump of 1 at $z = z'$. The integral over $z'$ in (10) can be written as

$$L(z) = \int_{-h/2}^{h/2} dz' K(z, z') f(z')$$

where

$$f(z') = \left| \frac{du(z')}{dz'} \right| \frac{du(z')}{dz'}.$$  

We note that the last term in $K$, namely $-\frac{h}{2}$, doesn’t contribute to the odd integrant. The remaining part of the integral gives

$$L(z) = \frac{1}{2} \int_{-h/2}^{h/2} dz' f(z') \left( |z - z'| + 2 \frac{zz'}{h} \right)$$

From Eq. (8), $u(z)$ is an even function of $z$, and $\sigma_{xz}(z)$ an odd function of $z$. Setting abruptly $\nu = 0$ in Eq. (8), the solution of this equation, which is odd with respect to $z$, is

$$\sigma_{xz}(z) = |g| z$$

Obviously $\sigma_{xz}(z)$, as given by this expression, does not vanish for $z = \pm h/2$, so that some (non-trivial) changes must be made to get a solution satisfying the boundary condition $\sigma_{xz}(\pm h/2) = 0$.

The scaling laws to be derived from Eq. (8) have to relate the order of magnitude of the quantities with a physical meaning, which is here specifically the average velocity $u$, to the parameters of the problem, namely $g$, $h$ and $\nu$. Taking $h/2$ as length scale, one gets rid of any quantity with a physical dimension in the problem by taking

$$u_* = \left( \frac{|g|h}{2\rho} \right)^{1/2}$$

as unit velocity.\(^6\)

Taking this scaling law one makes appear, instead of $\nu$, the inverse of a Reynolds number

$$\frac{1}{Re} \approx \frac{2\nu}{u_* h}.$$  \(^20\)

which is usually named ”Reynolds friction” and denoted $Re_\tau$ in the fluid mechanics literature.\(^7\) However, we find below that $Re$ is better estimated by the relation (44). We are looking for the solution of Eq. (8) in the limit of a very large Reynolds number, namely for a small viscosity. Therefore, it is natural in this limit to assume first that the term of viscous stress in Eq. (8) is negligible. Neglecting this term leads to a parameterless equation where all terms have formally the same order of magnitude. The solution $u(z)$ we are looking for is an even function of $z$, as we shall see. However, it has no reason to satisfy the imposed boundary condition $u = 0$ for $z = \pm h/2$. This situation is fairly common in fluid mechanics where the boundary condition for the velocity is different in perfect fluids and in viscous fluids. At high Reynolds number, this makes a priori the viscosity significant only in a thin layer near

\(^6\) This is an idea going back to the second half of the eighteenth century and due to the French engineer Chézy [8] with $g/2\rho$ replaced by the slope of the bottom of a river times the acceleration of gravity and $h$ a length depending on the depth and width of the flowing river.

\(^7\) See for example Eq. (17.6) p. 518 in Ref. [9], 8th revised and enlarged edition. Note that $u_\tau$ in this Eq. (17.6) corresponds to our $u_*$.\(^
the solid surface. For smooth surfaces (no slip condition), the tangential velocity drops from a finite value far from the layer to zero on the solid surface. There is an added complexity in this problem, because there is another boundary layer at the center of the cell, at $z = 0$, as discussed in Sect. 3.3.

We can derive a simpler expression of the integral in Eq. (17), by using various symmetries of the integrant. Let us introduce the function $F(z)$

$$F(z) = \int_0^z dz' f(z')$$

(21)

which is an even function of $z$. Integrations by part yield

$$L(z) = \int_{-h/2}^{h/2} dz' K(z, z') f(z') = \frac{1}{2} \int_{-h/2}^{h/2} dz' F(z') - \frac{z}{h} d$$

(22)

where

$$d = \int_{-h/2}^{h/2} dz' F(z')$$

(23)

3 Solution

This question of the boundary layer has been discussed for a long time and remains tricky because of the occurrence of logarithms in the solution. The present approach is in principle more straightforward than some others, because it relies on a fully explicit equation for the stress valid all the way in the turbulent flow, from the wall until the bulk.

The difficulty in this problem is twofold. First one has to solve the equation for $u(z)$ in the turbulent domain without the viscosity term and then to match this solution with a boundary layer where viscosity plays a role. The first problem (average velocity in the turbulent domain far from the wall in a sense to be made precise) is already non-trivial because it relies on the solution of a parameterless nonlinear integro-differential equation. Even though one can only hope to get a numerical solution, one needs at least to have some information on this solution, particularly near $|z| = h/2$ where it has to be matched with the one in the boundary layer, this one depending on the viscosity.

Below we solve firstly the problem close to the wall, then away from it, and finally we match the two solutions. We show that the full solution agrees with the observation that the friction coefficient (namely the dimensionless coefficient in the Chézy formula for the average speed as a function of $g$) tends to zero logarithmically as the Reynolds number increases. This non-trivial property is special to pipe flows with infinitely smooth boundaries. As discussed in Sect. 4, there is no such a logarithmic decrease of the friction (with respect to the Reynolds number) in the $C_x$ coefficient of Newton’s quadratic law for the drag of blunt bodies at large speed. We explain why the $C_x$ coefficient tends to a constant at infinite Reynolds number.

3.1 Boundary layer solution

We shall prove that the local linearity of $L(.)$ with respect to the distance to the wall will ultimately yield the “log-law of the wall,” but contrary to the standard derivation of this law, we do not have to assume that the turbulent stress is proportional to the distance to the wall, but we derive this from an explicit expression for the stress $\sigma_{xz}(z)$. The latter expression is a priori valid near the wall and far from it, and satisfies the boundary condition that $\sigma_{xz}(z) = 0$
on the wall, as stated above. We emphasize that the boundary layer exists because the solution of Eq. (9) has to satisfy this condition.

Near \(|z| = h/2\), the first term in the Taylor expansion of \(L(z)\) is

\[
L(z)_{z \to \pm \frac{h}{2}} \approx \pm \left( \frac{h}{2} - |z| \right) \left( \frac{d}{h} - F \left( \frac{h}{2} \right) \right)
\]  

(24)

which is linear with respect to the distance to the wall, and vanishes at \(|z| = h/2\). We notice that it depends on the whole solution via \(d\), defined in Eq. (23). This is a physical consequence of the underlying theory which does not assume the existence of any scaled length besides the width of the channel, outside the close neighborhood of the wall. It follows that the Reynolds stress tensor depends on turbulent fluctuations existing in the whole fluid domain and cannot be seen as depending on local quantities only. Said otherwise, the implicit expression (24) is a consequence of our assumption that there is no “small” length scale in the turbulent velocity field, only the length scales of the flow imposed by the geometry.

From Eq. (10), we have

\[
\frac{\sigma_{xz}(z)}{\rho} = \gamma | \frac{d u(z)}{d z} |^{-\alpha} L(z)
\]  

(25)

which vanishes at \(|z| = h/2\), as expected. However, since the vanishing of \(\sigma_{xz}(z)\) close to the wall is not satisfied by the solution (18) obtained by neglecting the viscous stress, we must include the viscous stress close to the wall. Later, we prove that we have also to include the viscous stress close to \(z = 0\).

Setting \(\tilde{z} = h/2 - |z|\) as a local variable (distance to the wall) much smaller than \(h\), and dividing the two members of Eq. (9) by \(\frac{|g| h}{2 \rho}\) (the r.h.s value for \(z\) close to \(h/2\)), we get the following equation for \(\frac{d u(\tilde{z})}{d \tilde{z}}\)

\[
\gamma_1 | \frac{d u(\tilde{z})}{d \tilde{z}} |^{1-\alpha} \tilde{z} + \gamma_2 \frac{d u(\tilde{z})}{d \tilde{z}} = 1
\]  

(26)

where \(\gamma_1 = 2 \gamma (d/h - F(h/2)) \rho/(gh)\) and \(\gamma_2 = 2 \rho v/(|g|h)\) is proportional to the viscosity. Introducing the typical velocity \(u_*\) defined in Eq. (19), the scaled velocity

\[
\hat{u} = u/u_*
\]  

(27)

is of order unity, and its derivative is of order

\[
d \hat{u} / d \tilde{z} \sim 1/\tilde{z}.
\]  

(28)

Equation (26) becomes

\[
\hat{\gamma}_1 | \frac{d \hat{u}(\tilde{z})}{d \tilde{z}} |^{1-\alpha} \hat{\tilde{z}} + \frac{h}{Re} \frac{d \hat{u}(\tilde{z})}{d \tilde{z}} = 1
\]  

(29)

where \(\hat{\gamma}_1 = 2 \gamma (\hat{d}/h - \hat{F}(1/2))\) is a dimensionless quantity of order unity, with \(\hat{F}\) and \(\hat{d}\) given in Eqs. (23) and (21), respectively, but with \(\hat{u}\) instead of \(u\). Let us now consider the matching domain between the wall and the bulk, where \(\tilde{z}\) is much smaller than \(h\). In this domain, the three terms of Eq. (29) must have the same order of magnitude. Using Eq. (28), the condition for the first term to be of order unity is \(|\tilde{z}|^\alpha \sim 1\), or

\[
\alpha = 0
\]  

(30)

independently of the value of the Reynolds number (supposed to be large). We emphasize that the viscosity plays no role to set the exponent value which derives from the condition \(\sigma_{xz}(\pm h/2) = 0\). Using Eq. (30), one get \(f(z) = du/dz\), and Eq. (21) becomes

\[F(z) = \hat{F}(\hat{z}) = \hat{d} / h - \hat{F}(\hat{z})
\]
\(u(z) - u(0)\). However, the Galilean invariance under the addition of a constant velocity along \(x\) allows to set

\[ F(z) = u(z) \]  

which implies \(F(h/2) = 0\). In summary, we find that the exponent \(\alpha\), which was free initially, is actually determined by the geometry of the Poiseuille setup, namely the plane boundaries. For \(\alpha = 0\), the Reynolds stress is given by the expression

\[ \sigma_{xz}(z)/\rho = \frac{\nu}{\ell_\nu} \frac{du(z)}{dz} L(z) \]  

where \(L(z)\) is given by Eq. (22). Moreover, \(L(z)\) is a linear function of \(u\),

\[ L(z) = \frac{1}{2} \int_{-z}^{z} dz' u(z') - \frac{z}{h} d \]  

with \(d\) simply given by the velocity profile integrated over \(z\),

\[ d = \int_{-h/2}^{h/2} dz' u(z'). \]  

Finally, Eq. (9) takes the implicit form

\[ \gamma \frac{du(z)}{dz} L(z) - \nu \frac{du(z)}{dz} = \left| \frac{g}{\rho} \right| \frac{h}{2} \]  

which is the one we shall study from now. Close to the walls, one has

\[ L(z)_{z \to \pm h/2} \approx \pm \left( \frac{1}{2} - \frac{|z|}{h} \right) d \]  

Using the variable \(\tilde{z} = h/2 - z\), Eq. (35) becomes

\[ \frac{d\tilde{u}(\tilde{z})}{d\tilde{z}} (\gamma \frac{d}{h} \tilde{z} + \nu) = \left| \frac{g}{\rho} \right| \frac{h}{2} , \]  

which is finally the equation we have to solve close to the wall.

Taking into account that \(u = 0\) for \(\tilde{z} = 0\) (on the wall), the solution of Eq. (37) is

\[ u(\tilde{z})_{\text{inner}} = \frac{|g|h^2}{2\nu \gamma d} \ln \left( \frac{\tilde{z} + \frac{v h}{\gamma d}}{\frac{v h}{\gamma d}} \right) \]  

where \(d\) has the dimension of a kinematic viscosity, \(\gamma\) is dimensionless and

\[ \ell_\nu = \frac{v h}{\gamma d} \]  

is the width of the boundary layer, which is much smaller than \(h\) at large Reynolds, see Eq. (44). Note that the solution (38) has some resemblance with the one derived within the standard von Karman–Prandtl boundary layer theory, as presented by Landau [4] in Sect. (42) for instance. There are, however, some significant differences. The main difference comes from the fact that Eq. (38) is derived from our non-local model (1) for the Reynolds stress, which reduces to Eqs. (35)–(36), that directly leads to the log-law (38) valid close to the wall and another one valid farther away in what we call the outer region, as explained below. Let us recall that our integral expression (1) was set in order to be free of any additional
length scale besides those of the setup. Differently the Prandtl–von Karman model defines the following expression for Reynolds stress tensor,

$$\sigma_{xy}^{Pr}(\tilde{z})/\rho = \ell_m^2 \frac{du(\tilde{z})}{d\tilde{z}} | \frac{du(\tilde{z})}{d\tilde{z}} |,$$

(40)

which is local. In Eq. (40), the mixing length $\ell_m = \kappa \tilde{z}$ is proportional to the distance $\tilde{z}$ from the wall. The constant $\kappa$ was deduced from experimental observations, $\kappa = 0.41$ for parallel flows. By analogy with the concept of mean free path in thermodynamics, $\ell_m$ can been seen as the mean free path of small eddies close to the wall. Note that Prandtl himself described his model as “only a rough approximation” which has been improved since its original version, see for example [10]. Nevertheless, we found interesting to compare our model with the mixing length one which is easier to handle analytically and numerically, because it offers an opportunity to deduce an approximate value of the parameter $\gamma$ which remains free in our model, although $\kappa$ is known. Such a comparison is presented in Sect. 3.5.

3.2 Inner–outer matching near the wall

From the boundary layer solution given in Eq. (38), one can already guess that the matching of this “inner solution” (i.e., solution near the wall) with the outer solution (solution far from the wall) requires some care. The goal is to obtain same solutions in the matching domain. The outer solution in the domain $l_v \ll \tilde{z} \ll h$ is solution of Eq. (37) without the viscous stress, namely with $\nu = 0$. It is of the form

$$u(\tilde{z}) = \frac{|g|h^2}{2\rho \gamma d} (\ln \tilde{z}/\ell + C)$$

(41)

where $C$ is a constant and $\ell \leq h$. Therefore, the solution of this “outer” problem diverges logarithmically near the wall and all the same must match the “inner solution.” Both solutions have a logarithmic behavior, but the arguments of the two logarithms differ by a factor of order $\ell_v \approx 1/Re$ that represents two widely different length scales, $h$ for the outer solution and $h/Re$ for the inner solution. The ratio of the two length scales is, however, constant. Therefore, the corresponding difference between the two expressions for the velocity can be compensated by adding a constant to the outer solution. This is possible because the equation to be satisfied by this outer solution is formally invariant under the addition of a uniform velocity since Eq. (10) involves derivatives of $u(z)$ only. This is, after all, only a reflection of the Galilean invariance (in the $x$ direction) of the fluid equations for an inviscid fluid and perfectly smooth walls.

Said otherwise, the matching between Eqs. (38) and (41) is obtained using the constant $C$. Writing the log-term in Eq. (38) as $(\ln((\tilde{z} + \ell_v)/\ell) - \ln(\ell_v/\ell))$, where $\ell$ is any length, makes clear that the right choice for the outer solution is to write the parenthesis of Eq. (41) as

$$\ln(\tilde{z}/\ell) + C = (\ln(\tilde{z}/\ell) - \ln(\ell_v/\ell)) .$$

(42)

This leaves open the choice of $\ell$. Because this is related to the outer solution, this length scale has to be proportional to the length scale involved in this outer solution, namely $h$. The following choice $C \approx -\ln(\ell_v/h)$ or

$$C \approx \ln(\gamma Re)$$

(43)

should match the outer solution with the inner solution, by including the large positive constant $C$ in the two expressions. In Eq. (43) and in the following, the Reynolds number is defined by the relation
Fig. 2 Profile of the scaled velocity \((2\rho \gamma d/|g|h^2) u(z/h)\), versus \(\hat{z} = z/h\), along the direction \(\hat{z}\) perpendicular to the two plates, \(a\) for \(\gamma \ell_v/h = 10^{-2}\), or \(\gamma Re = 10^2\), \(b\) for \(\gamma Re = 10^3\). In each figure, the solid red curve displays the full numerical solution, and the two portions of curves, solid-blue and dashed-purple, display the inner and outer solution, respectively, (the outer solution (46) with \(a\) \(c = 0.35\), \(b\) \(c = 0.38\)). The inset in \(b\) shows a zoom close to \(z = h/2\).

\[ Re = d_u/v, \]  

(44)

where \(d_u = d\) defined in Eq. (34) that replaces the one estimated above in Eq. (20). We emphasize that the solution (38) is valid in the whole inner domain close to the wall and at the edge of the outer domain where it matches the solution at finite distance from the wall.\(^8\)

To summarize the above study, we have decomposed the whole solution between the walls into three parts belonging to three different domains. The first domain is the one far from the walls, named ”bulk” domain, where the solution may be written as

\[ u(z/h)_{\text{bulk}} = \frac{|g|h^2}{2\rho \gamma d} (U(z/h) + \text{Cst}). \]  

(45)

The function \(U(z/h)\) is a numerical function of the dimensionless ratio \(z/h\), independent of \(Re\). This function tends to \(\ln(1/2 - |z|/h)\), as \(|z|\) tends to \(h/2\), in order to allow the matching of the full solution at the edge of the viscous boundary layer. The constant in Eq. (45) is such that \(u(z)\) vanishes at the wall. The merging of the bulk solution with the outer one

\[ u(z/h)_{\text{outer}} = \frac{|g|h^2}{2\rho \gamma d} \left( \ln \left( \frac{1}{2} - \frac{|z|}{h} \right) + \ln \gamma Re + c \right). \]  

(46)

is obtained by choosing the constant \(c << \ln \gamma Re\) in such a way that the three portions join successively. The bulk-outer matching has to be performed numerically; it completes the matching analytically derived (up to the small constant \(c\)) of the outer solution with the third part of the solution, the inner one (38) vanishing at the walls. The matching of the three domains is illustrated in Fig. 2.

Outside of the boundary layer, the curve \(u(z)\) is expected to become flatter and flatter, the added uniform velocity field (proportional to \(\ln \gamma Re + c\)) making presumably the dominant

\(^8\) There has been attempts to compare accurately the log law of the wall with experimental data. This meets several challenges. First, the fitting of the experimental data and the theory requires to know accurately the velocity parameter \(u_\ast\), a parameter which can be inferred only indirectly from measurements of the local stress and/or from fit of the velocity data. Secondly, as we show, this log-law of the wall is only valid in a range of distances to the wall that is intermediate between the thickness of the viscous sublayer and the far wall dependence of the velocity profile that depends in a non-trivial way of the Reynolds number (and so of the average flow speed). All this, added to the difficulty of making a clear-cut difference between a logarithm and a power law with a small exponent, makes it hard to pinpoint the range of values of the distance to the wall where the log law applies, see Ref. [11].
part of this velocity field. To see this effect, one should take large values of $\ln(\gamma Re)$ that can be difficult to achieve in real situations. Nevertheless, we notice that the friction coefficient has been measured on a large range of $Re$ (ten decades), in the case of pipes with very smooth surfaces and also rough ones. In the case of smooth surfaces, the friction was also found to decay slowly as a function of the logarithm of the Reynolds number, as shown by Moody’s diagram [12]. We discuss below the implications of those remarks for the general problem of turbulent drag on objects of an arbitrary shape.

3.2.1 Numerical solution

The goal is to solve the integro-differential Eq. (35) with boundary conditions at $z = h/2$,

$$ (u)_{z=h/2} = 0 \quad \text{and} \quad \left( \frac{du}{dz} \right)_{z=h/2} = -\frac{|g|h}{2\rho \nu} = -\frac{u^2_s}{\nu} \quad (47) $$

First, we note that we can get rid of the parameters in the r.h.s. of Eq. (35) by scaling $z$, $\nu$ and the velocity $u$. Defining the dimensionless quantities $\hat{u} = u/u_s$, $\hat{v} = \nu / (hu_s)$, $\hat{z} = z/h$, and $d_{\hat{u}} = \int_{-1/2}^{+1/2} \hat{u}(z')dz'$, Eq. (35) becomes

$$ \gamma \left| \frac{d\hat{u}}{d\hat{z}} \right| L(\hat{u}, \hat{z}) - \hat{v} \frac{d\hat{u}}{d\hat{z}} = 2\hat{z} \quad (48) $$

where

$$ L(\hat{u}, \hat{z}) = \int_{-\hat{z}}^{\hat{z}} \hat{u}(z')dz' - \hat{z}d_{\hat{u}} \quad (49) $$

is still a linear function of $\hat{u}$. The latter property allows to notice that Eqs. (35) and (48) present a dilatation invariance, which can be stated as follows: if $u(z)$ is solution of Eq. (35) for a given set of parameters $(\gamma, \nu, u_s)$, then $v(z) = \lambda u(z)$ is also solution of Eq. (35) but with the parameters $\gamma \rightarrow \gamma/\lambda^2$ and $\nu \rightarrow \nu/\lambda$, although $u_s$ keeps unchanged, whatever $\lambda$. Similarly, if $\hat{u}(\hat{z})$ is solution of Eq. (48) for the given set of parameters $(\gamma, \hat{v})$, then any dimensionless function

$$ \hat{v}(\hat{z}) = \lambda \hat{u}(\hat{z}) \quad (50) $$

is also solution of Eq. (48) but with the parameters $\gamma \rightarrow \gamma/\lambda^2$ and $\hat{v} \rightarrow \hat{v}/\lambda$. The latter property has been found advisable for the iterative numerical method we have used. From now, we introduce the notation $d_{\hat{u}}$ instead of the integrated velocity profile $d$, and we write $d_{\hat{v}}$ for the integrated profile of the variable $v$. Setting

$$ \lambda = \sqrt{\gamma d_{\hat{v}}} \quad (51) $$

with $d_{\hat{v}} = \int_{-1/2}^{+1/2} \hat{v}(z')dz'$, one get the relation between the scaled quantities $\hat{v}(\hat{z})$ and $\hat{u}(\hat{z})$ and their averaged value (over $z$),

$$ \hat{v}(\hat{z}) = \gamma d_{\hat{u}} \hat{u}(\hat{z}) \quad d_{\hat{v}} = \gamma d_{\hat{u}}^2 \quad (52) $$

and between the corresponding velocities $u = u_s \hat{u}$, and $v = u_s \hat{v}$

$$ v(\hat{z}) = \gamma d_{\hat{u}} u(\hat{z}) \quad d_v = \gamma d_{\hat{u}}^2 \quad (53) $$

The integro-differential Eq. (48) written for the dimensionless function $\hat{v}(\hat{z})$ becomes,

$$ \frac{d\hat{v}}{d\hat{z}} \left( \frac{1}{d_{\hat{v}}} |L(\hat{v}, \hat{z}) + v_{\hat{v}}| \right) = -2\hat{z} \quad (54) $$
where \( L(.) \) is defined in Eq. (49) and \( \nu^* = \hat{\nu}/\sqrt{\gamma d^*_u} \). Using the definition (44) for the Reynolds number,

the scaled viscosity writes

\[
\nu^* = \hat{\nu}/\sqrt{\gamma d^*_u} = \frac{1}{\gamma Re} \tag{55}
\]

Using the change of variables \( u \rightarrow v = \lambda u \), the iteration method has shown up very efficient. We start from an even log-profile which fulfills the boundary conditions (56) at the wall. For the scaled velocity \( \hat{v} \), these conditions are

\[
(\hat{v})_{z=1/2} = 0 \quad \text{and} \quad \left( \frac{d\hat{v}}{dz} \right)_{z=1/2} = -\frac{1}{\nu^*} \tag{56}
\]

We have chosen the simple test function for the initial profile

\[
\hat{v}_{\text{test}}(\hat{z}) = \beta \ln \left( 1 + k \left( \frac{1}{4} - \hat{z}^2 \right) \right) \tag{57}
\]

which vanishes at the wall. Close to the boundary \( \hat{z} = 1/2 \) the parenthesis in Eq. (57) can be approximated by its first order expansion in terms of the scaled variable \( \tilde{z} = 1/2 - \hat{z} \) (as above but in scaled form),

\[
\hat{v}_{\text{test}}(\tilde{z}) \approx \beta \ln(1 + k \tilde{z}) \tag{58}
\]

and the condition (56) for the derivative yields

\[
\beta k = \frac{1}{\nu^*} \tag{59}
\]

A second relation between the two parameters \( \beta \) and \( k \) can be deduced by looking at the mid-plane velocity. From Eq. (57), we have \( \hat{v}_{\text{test}}(0) = \beta \ln(1 + k/4) \). The outer solution for \( \hat{v}(\tilde{z}) \) is the solution of Eq. (54) without the viscous term and for \( \tilde{z} \) small, as performed above for the outer solution of \( u(\tilde{z}) \), see Eqs. (41) and (43). Taking into account the successive rescaling, we get the outer solution \( \hat{v}(\tilde{z}) = (\ln(\tilde{z}/\ell) - \ln(\ell^*_u/\ell^*_v)) \). Extending this solution until the center of the flow, one may fit the outer solution with the test function by setting \( \ell = h/4 \), and \( \hat{v}(\tilde{z} = 1/2) = 1 \), or \( \beta = 1 \). Finally, using Eq. (59), the profile of the scaled velocity \( \hat{v}(\tilde{z}) \) is obtained by iterating the solutions of Eq. (54) and using the test function

\[
\hat{v}_{\text{test}}(\tilde{z}) = \ln \left( 1 + \left( \frac{1}{4} - \tilde{z}^2 \right)/\nu^* \right) \tag{60}
\]

as beginning term in the iteration process. The convergence of the solution occurs very rapidly for \( \nu^* \) values of order \( 10^{-1} - 10^{-2} \), but needs more precision machine and more steps as \( \nu \) decreases. This is due to the very strong heterogeneity of the velocity field \( u \), which get a transverse profile becoming stiffer and stiffer near the walls as \( \nu \) decreases.

The profile of the velocity \( u(\hat{z})/u(0) \), or \( \hat{v}(\tilde{z})/\hat{v}(0) \), is shown in Fig. 3 for \( \nu^* \) decreasing from \( 10^{-2} \) to \( 10^{-8} \). For \( \nu^* \) smaller than \( 10^{-2} \) the profile displays a wedge-like form in the bulk associated with a strong gradient near the walls. As \( Re \) increases (or \( \nu^* \) decreases), the bulk profile enlarges and the slope at the wall increases in agreement with Eq. (56).

In the same range of Reynolds number, we plot in Fig. 4a the inverse of \( d^*_v = \gamma d^2_u \) which is proportional to the friction coefficient. The curve \( 1/d^*_v \) decreases with respect to \( \ln(Re) \), as expected. Figure 4b, c completes the study of the friction behavior, see the discussion in Sect. 4.
Fig. 3  a Profile of the scaled velocity $\hat{v}(\hat{z})/\hat{v}(0) = u(\hat{z})/u(0)$, versus $\hat{z} = z/h$, along the direction $z$ perpendicular to the two plates, for $\nu_{\text{c}} = 10^{-1}$, (red) $10^{-3}$ (brown) and $10^{-8}$ (blue) or $\gamma Re = 10$, $10^3$ and $10^8$, see Eq. (55). The round profile at small Reynolds number gets stiff wings and flattens as $Re$ increases, as illustrated in (b) where $(u(0) - d)/d$ is plotted versus $\log_{10}(\gamma Re)$, with $d = (1/h) \int_{-h/2}^{h/2} dz u(z)$.

Fig. 4  Friction coefficient defined here as $f = 1/d\hat{v}$, given in Eq. (52), is the mean value of $\hat{v}$ (averaged over the whole profile), and is equal to $\gamma d^2_{\text{c}}$. a $1/d\hat{v}$ is plotted versus $|\log_{10}(\nu_{\text{c}})|$ (or $\log_{10}(\gamma Re)$). It reflects the behavior of $f$, defined in (80), as $Re$ increases. b, c $1/\sqrt{f}$ versus $\log_{10}(\gamma Re \sqrt{f})$ and versus $\log_{10}(\gamma Re)$, respectively. The blue lines are linear fits. Note that both curves (b, c) seem to increase linearly, although the abscissa slightly differs, this is because the difference is small, it is $\sim \log(\log(\gamma Re))$. The von Karman law in (b) is expected to be valid for $Re$ larger than $10^5$.

The profile of the stress tensor $\sigma_{xz}$ is drawn in Fig. 5 for $\nu_{\text{c}}$ decreasing from $10^{-1}$ to $10^{-3}$. As $Re$ increases, the Reynolds stress tends to a linear function of $z$ in the bulk, which falls down more and more abruptly at the boundaries.

In summary, the above study of the solution near the walls shows a big change of behavior with respect to the physical parameters that reflects very strong velocity field heterogeneity.

We have two remarks; the first one concerns the Reynolds number. In Eq. (20), $Re$ was estimated by using $u_*$ defined in Eq. (19) as the order of magnitude of the velocity in the bulk, namely without the effect of the boundary layer. However, this value $u_*$ is noticeably underestimated when viscosity effects are included, especially in the limit of $v$ tending to zero. This is due to the effect of the added constant $C$, see Eq. (43). The actual mean velocity in the flow is much better represented by $d_u/h = u_* d_u$ which increases approximately as
Fig. 5 Profile of the Reynolds stress component $\sigma_{xz}$, versus $\hat{z} = z/h$, along the direction $z$ perpendicular to the two plates, for $\nu \hat{v} = 10^{-1}$ (red, dashed), $10^{-2}$ (brown, dotted) and $10^{-3}$ (blue, solid), or $\gamma \text{Re} = 10^{-1}$, $10^{2}$ and $10^{3}$. As $\text{Re}$ increases, the profile becomes linear in the bulk, except on the edges (boundary layers) where it falls down to zero, more and more abruptly.

$(\ln(1/\nu \hat{v}))^{1/2}$ at large $\text{Re}$, that justifies the definition of the Reynolds number as $\text{Re} = d_u/\nu$ leading to Eq. (55).

A second remark concerns the link between $d_\hat{u}$ and $d_\hat{v}$ which are, respectively, the values of $\hat{u}$ and $\hat{v}$ averaged over the whole profile. We showed above that $d_\hat{u}$ is equal to the square of $d_\hat{u}$ times $\gamma$, see Eq. (52). Therefore, one finds that the decrease of the friction factor $f$ (defined as $f = 1/\gamma d_\hat{u}^2$) with respect to $\ln(\text{Re})$ agrees qualitatively with the observations in the case of smooth walls [12].

3.3 Inner–outer matching near the middle of the flow

As we discovered when looking for a numerical solution of the equation of momentum balance, there is another boundary layer near the center of the flow $z = 0$. We do not believe this is an artefact of our modeling of the turbulent stress. The occurrence of this boundary layer can be explained as follows. Because the average velocity $u(z)$ is an even function of $z$, it reaches a maximum at $z = 0$ where $du/dz$ and the RST vanish. Supposing this is a smooth maximum, it is also a place where the fluid velocity is almost uniform. Therefore, this is a place where, physically, the source of turbulence is absent: no turbulence is created on a uniform flow. Therefore the balance of momentum can become dominated by viscous friction. This explanation is obviously quite approximate, nevertheless it is supported by a detailed analysis of the solution of Eq. (9) near $z = 0$ in the limit $\nu \to 0$. Close to $z = 0$ the analysis of the solution in this limit uses the same equation as in Sect. 3.2, but with different boundary conditions.

The Reynolds stress $\sigma_{xz}$, given by Eq. (32), is an odd function of $z$, $|du(z)/dz|$ is an even function of $z$, and $L(z)$ is an odd function of $z$. Near $z = 0$ one can approximate $L(z)$ by the first non-vanishing term in its Taylor expansion, $L(z) \approx \beta_u \gamma z$, with

$$\beta_u = u(0) - d_u/h.$$  (61)

Notice that the parameter $\beta_u$ depends on the full velocity field, not on the local velocity only. Plugging the latter approximation of $L(z)$ in Eq. (32) one finds the following ordinary differential equation for $u(z)$ near $|z| = 0$

$$\beta_u \gamma z \frac{du(z)}{dz} - \nu \frac{du(z)}{dz} = \frac{|g| z}{\rho}$$  (62)

This equation can be integrated at once, with the result

$$u(0) - u(z) = \frac{|g|}{\rho (\beta_u \gamma z)} \left( \beta \gamma |z| - \nu \ln \left( \frac{\nu + \beta_u \gamma |z|}{\nu} \right) \right)$$  (63)
from a positive to a negative derivative. In the small domain of width of order thickness of the central plane, the two terms in the l.h.s. of Eq. (62) enter into play successively. First the viscous contribution of the Reynolds stress tensor, a quantity which could depend on the viscosity via the gradient of the velocity, $\nu (\partial u / \partial z)$, for $\nu = 10^{-2}$ (green dashed line), $10^{-4}$ (blue, dotted-dashed line), $10^{-6}$ (orange dotted line) and $10^{-8}$ (red solid line), or $\gamma Re = 10^2, 10^4, 10^6$ and $10^8$, respectively. Close to $z = 0$, the slope of the derivative increases with the Reynolds number, then evolves smoothly in the bulk. 

The above study of the turbulent plane Poiseuille flow was restricted to the role played by the diagonal components of the Reynolds stress. From the relationship (7) between turbulent stress and velocity fluctuations, we infer that, besides $\sigma_{xx}$, other components of this stress do not cancel and so should be considered for possible effects on the average velocity. For instance the jump of the derivative $u_z$ at the wall, that gives the constant slope visible in Fig. 6a. On the distance $\delta z$, the jump of the derivative $u_z \equiv du/dz$, labeled $\delta u_{z,z}$, is

$$
\delta u_{z,z} = |g|/ (\rho \gamma \beta_u)
$$

(65)
a quantity which could depend on the viscosity via $\beta_u$. Nevertheless, Fig. 6 shows that the jump of $du/dz$ seems to be constant in the domain of large $Re$ values numerically investigated. Differently the jump of $dv/dz$, $\delta v_{z,z} = \lambda \delta u_{z,z}$ increases with $Re$ (upper curve with blue points) because the coefficient $\lambda$ increases with $Re$. It is interesting to notice that the jump of the slope reflects the formation of two adjoining boundary layers aside the central plane.

3.4 $\sigma_{ii}$ components of the Reynolds stress tensor

The above study of the turbulent plane Poiseuille flow was restricted to the role played by the component $\sigma_{xx}$ of the Reynolds stress. From the relationship (7) between turbulent stress and velocity fluctuations, we infer that, besides $\sigma_{xx}$, other components of this stress do not cancel and so should be considered for possible effects on the average velocity. For instance the diagonal components, $\sigma_{xx} = \rho(u_x'^2)$, $\sigma_{yy} = \rho(u_y'^2)$ and $\sigma_{zz} = \rho(u_z'^2)$ are generally nonzero because they are averages of squares. The model of the turbulent Reynolds stress described in Ref. [2] includes, in addition to the term (1) studied above, the following diagonal tensor,

$$
\sigma_{ii}(x) = \delta_{ij} \gamma_i \rho |\nabla \times u(x)|^{1-\alpha} \int d x' K(x, x') |\nabla \times u(x')|^{\alpha+1}
$$

(66)
where $\delta_{ij}$ is the Kronecker symbol and the constants $\gamma_i$ are introduced to fulfill the Schwarz inequalities for the correlation functions $\langle u'_i u'_j \rangle$, named realizability conditions, see Ref. [13]. For the case $\alpha = 0$ treated here, the latter expression becomes $\sigma_{ii}(x) = \gamma_i \rho |\nabla \times u(x)| \int d x' K(x, x') |\nabla \times u(x')|$, which gives for our setup,

$$\sigma_{zz}(z)/\rho = \gamma_z |\frac{du(z)}{dz}| L(z),$$

(67)

Finally, let us notice that the diagonal component $\sigma_{zz}$ of the RST is proportional to the non-diagonal one, Eq. (32), although they are defined by different expressions, respectively, Eqs. (66) and (1). We are going to show that the diagonal components in Eq. (67) do not contribute to forces driving the Poiseuille flow. For such flow the diagonal tensor $\sigma_{ii}$ can be seen as a time-averaged pressure depending on the spatial coordinates, and both quantities, pressure and diagonal tensor, are impulse carrier. As written in Ref. [2], the couple $(u, p)$ is not unique in dynamical incompressible systems, since $p$ is a scalar gauge field defined up to an additive scalar function. In other words, the pressure is not an independent variable, but a Lagrangian multiplier necessary to ensure the incompressibility, since it fulfills the relation $\Delta p = u_{i,j}u_{j,i}$ with summation over the same indices.

In order to prove that the diagonal components in Eq. (67) do not contribute to forces driving the Poiseuille flow, let consider, for instance, the balance of forces in the $z$ direction,

$$\frac{d}{dz}(\rho u_z^2 + \sigma_{zz} + p(z)) = 0$$

(68)

where $u_z$ is a hypothetical component of the average velocity in the $z$ direction and $p(z)$ is a scalar function which could also depend on $z$.

One may set $p(z) = -\sigma_{zz}(z)$, that cancels the contribution of the diagonal term $\sigma_{zz}$ to the forces along $z$, and leads to $u_z = 0$, so that the balance of forces in the $z$ direction is realized without flow velocity in this direction. Note however that all this relies on the validity of the assumption that all quantities (like the Reynolds stress tensor) depend on $z$ only, which is relevant in the case of pipe flows of uniform cross section along $x$ (those with a translation invariance in the direction perpendicular to this cross section). It does not imply, however, that solutions depending on $z$-only are the only possible ones. Consider, for instance, a cross section made of two circles of different diameters connected through a finite segment near the axis of their centers. There shall be less friction on the wall for the largest circle and so, for the same pressure head, the speed is bigger in the biggest circle. This will yield a shear layer at the juncture of the two circles and therefore an instability generating a dependence of the average velocity with respect to the variables $y$ and $x$. Such an undulation will likely give a fairly complex average velocity field.

3.5 Comparison with the Prandtl model

The above study display results depending on a single unknown parameter, the dimensionless constant $\gamma$, in factor the right-hand side of Eq. (1). We hope that our model will be tested by experiments, but it would be yet interesting to get an order of magnitude of $\gamma$. This can be obtained by comparing the numerical solutions of our model with the solutions of the Prandtl model, as follows.

Using the definition (40) for the Reynolds stress component $\sigma_{xz}$, setting $X = \frac{du(z)}{dz}$ and $\ell_m = \kappa (h/2 - z)(h/2 + z)/h$, the differential equation for the velocity field $u(z)$ becomes

$$\ell_m^2 X^2 - \nu X - |z| \frac{|g|}{\rho} = 0$$

(69)

\[ \text{Springer} \]
This simple polynomial equation has a positive determinant \( \Delta = v^2 + 4|z| \ell_m^2/\rho \). The solution which is finite at the wall is

\[
X(z) = (v - \Delta^{1/2})/(2\ell_m^2),
\]

which gives \( X(\pm h/2) = \mp|g|/(2\rho v) \) at the solid boundaries, and \( X(0) = 0 \) in the center of the flow. After integration of Eq. (70) over the transverse variable \( z \), and taking account of the parity of the velocity \( u \), we get the profile of the velocity scaled to \( u_\ast \). As \( v \) increases, we observe that the profile flattens, as shown in Fig. 7a, that makes a joined property with our model.

Another result which is common with both models is that the viscous stress drives the solution in the close vicinity of the center. This can be proved by looking to Eq. (69) in the limit of \( z \) small. For \( v = 0 \), the solution of this equation is \( X = a\sqrt{|z|} \) with \( a = 4(|g|/\rho)^{1/2}/(\kappa h) \). It follows that without the viscous term, the slope of the derivative is infinite at \( z = 0 \). Including the viscous term, and setting \( \hat{u} = u/u_\ast \), as above, \( \hat{z} = z/h, \hat{X} = d\hat{u}/d\hat{z}, \) and \( \hat{v} = v/(u_\ast h) \), Eq. (71) becomes

\[
\frac{\kappa^2}{16} \hat{X}^2 - \hat{v} \hat{X} - \frac{1}{2} |\hat{z}|^2 = 0.
\]

where we set \( \ell_m = \kappa h/4 \) at first order in \( z \), Close to the center, more precisely for \( \hat{z} \ll 32(\hat{v}/\kappa)^2 \), the first term of Eq. (71) is much smaller than the second one, and the derivative is

\[
\hat{X} = \frac{|\hat{z}|}{2\hat{v}}
\]

In this domain, the viscous stress plays a dominant role; in particular, it allows that \( X(z) \), the derivative of the velocity, has a finite slope in the central plane although the slope is infinite for \( v = 0 \), see the inflexion point in the dashed curve of Fig. 7b. From Eq. (71), one may estimate the width of the central boundary layers, \( \delta \hat{z} \), by imposing that the three terms are of the same order of magnitude. It gives

\[
\delta \hat{z} = \frac{32}{\kappa^2 \hat{v}^2}
\]

which is of order \( 100/Re^2 \).

Just beyond the frontier of these boundary layers, the derivative of the velocity, \( \hat{X} \), is still a solution of Eq. (71) without the viscous stress, we get

\[
\hat{X} = \frac{3}{2} / \kappa |\hat{z}|^{1/2}
\]

which has a large slope \( d\hat{X}/d\hat{z} = \sqrt{\frac{2}{\kappa}} |\hat{z}|^{-1/2} \), decreasing with \( |z| \), as shown in Fig. 7b. The inset in this figure displays no jump, contrary to what has been observed with the integral model (compare with Fig. 6). In the mixing length model, the slope of the derivative does not abruptly fall to zero; it keeps a large value in this domain where \( \hat{z} \) is small. It is given by the relation \( d\hat{X}/d\hat{z} = \sqrt{\frac{2}{\kappa}} |\hat{z}|^{-1/2} \).

In summary, the central boundary layers also exist with the Prandtl model. For large \( Re \), they have a width of order \( 100 h/Re^2 \), that is smaller than the width of order \( 1/Re \) found in the frame of our model. Moreover, these layers do not manifest by an abrupt change of the slope of \( du/dz \) at the frontier of the central boundary layer and the bulk, because the turbulent stress is quadratic with respect to \( du/dz \) in Eq. (40), whereas it is locally proportional to \( zdu/dz \) within the frame of Eq. (8).
Let us now look at the solution close to the wall. First, we note that the inner solution (in the domain where the viscosity dominates) is linear,

\[
u_{\text{inner}}(\tilde{z})/u_*= \frac{1}{\nu} \tilde{z} \ll \frac{\nu}{\kappa u_*} \tag{75}\]

where \(u_*\) is defined in Eq. (19), see the green dashed lines in Fig. 7c. Following our previous analysis, one could try to define an outer solution (in the domain where the viscosity is much smaller than \(\ell_m | \frac{du(\tilde{z})}{d\tilde{z}} |\)), which takes a log behavior shown in Fig. 7c, dotted blue curves,

\[
u_{\text{out}}(\tilde{z})/u_* = \frac{1}{\kappa} \ln(\tilde{z}/h) + C \quad \frac{\nu}{\kappa u_*} \ll \tilde{z} \ll 1 \tag{76}\]

We must notice that the matching of the inner solution with the outer one is less satisfactory in the case of the Prandtl model than in the case of our non-local model. The Prandtl model requires to match a linear function with a log-one. The linear function is valid on a very short range, see the inset of Fig. 7c, and the logarithmic outer solution (defined up to a constant \(C\)) must merge at \(z \ll 1/2\) with the integral of the solution (70). This makes a difference with our model where the inner–outer merging is easy, that allows to perform an adequate matching, and lastly to make a complete analysis of the full solution between the walls, including the inner–outer matching near the wall, as shown above, see Fig. 2. We emphasize that the Prandtl solution in the bulk,

\[
u_{\text{bulk}}(\tilde{z})/u_* = \frac{1}{\kappa} \int_{\tilde{z}}^{1/2} d\tilde{z}' \sqrt{1 - 2\tilde{z}'^2} \tag{77}\]

is flatter than a log-function, due to the presence of the decreasing numerator in the integrand, as illustrated in Fig. 7c where the solid red curve is flatter than the dotted blue curve corresponding to the log-law (76).

The positive aspect of this Prandtl model is that it is analytically tractable and contains a parameter \(\kappa\) already fitted with observations, so we can use this advantage to derive an approximate value of the dimensionless parameter \(\gamma\) of our model, which was yet unknown. This can be deduced by comparing the behavior of \(d_u\), the amplitude of the velocity (averaged over \(z\)), as a function of the Reynolds number \(Re = d_u/\nu\). We note that both models display a linear behavior with respect to \(\log(Re)\). An approximate fit of the two curves has been obtained for

\[\gamma = 0.028, \tag{78}\]

as shown in Fig. 7d.

### 4 Skin friction versus turbulent drag

The turbulent Poiseuille flow makes a standard example of turbulent flow with skin friction due to boundary layers along the solid surface and shear-generated turbulence away of it. Such shear turbulence was also considered in Reference [2] in the mixing layer behind a splitter plate, a situation quite different from the Poiseuille flow, specifically because the splitter plate undergoes a lift force due to the turbulent drag behind it. Therefore, it is of interest to look for general conclusions to be drawn in situations where lift forces exist or not. Of course, we think about the case already considered by Newton in the Principia, namely the one of a blunt body moving quickly in a fluid immobile far from it. Newton showed that the drag felt
Fig. 7 Mixing length model. a Profile of the scaled velocity \( u(z/h)/u(0) \), from Eq. (71), for \( \nu/h u_* = 10^{-2} \) (lower curve) and \( 10^{-4} \) (upper curve). b Profile of the derivative \( d\hat{u}/d\hat{z} \) versus \( \hat{z} = z/h \). In the main figure, the two profiles, for \( \nu = 0 \) (red-dashed) and \( \hat{\nu} = 10^{-4} \) (blue), are undistinguishable. The inset is a zoom close to the center, for \( -3 \delta < \hat{z} < 3 \delta \), showing an infinite slope in the center for \( \nu = 0 \), but a finite slope for \( \hat{\nu} = 10^{-4} \) (blue curve), see text. c Matching of the three portions of the solution: the dashed green line represents the inner solution (75), the blue dotted curve is for the outer solution (76) and the red curve displays the full numerical solution, for \( \nu/h u_* = 10^{-4} \). The inset is a zoom close to \( z/h = 1/2 \), the vertical segment is placed at \( \tilde{z} = \nu/\kappa u_* \). d Scaled velocity \( d\hat{u}/h u_* \) (averaged over \( z \)) versus \( \log_{10}(Re) \). Our model (red points) with \( \gamma = 0.028 \) approximately fits the Prandtl model (blue curve).

by this object grows like the square of its velocity with respect to the fluid at rest at infinity. Neglecting vector indices, Newton’s law of drag behind an obstacle is

\[
F_N = -C_x \rho S |U|U
\]

where \( S \) is the cross section of the object moving at speed \( U \), and \( C_x \) is a numerical constant depending only on the shape of the moving body. The absolute value is to recall that the drag force \( F_N \) changes sign as the velocity is reversed (a non-trivial property of turbulence related to dissipation). Newton’s formula is remarkable in many ways. Some comments would deserve to be made.

The experiments show that the coefficient \( C_x \) may have a complex behavior as the Reynolds number increases, depending on the shape of the body. An example is the “drag crisis” studied for spheres, cylinders and more generally for sufficiently profiled bodies,9 for which the drag coefficient drops off suddenly as Reynolds number increases, when the flow begins to become turbulent in the boundary layer. Increasing the Reynolds number, the drag crisis is followed by an asymptotic constant behavior, toward a value practically insensitive to the roughness of the surface of the blunt body.

9 Finally explained by Prandtl.
On the other hand, for Poiseuille-typed flows, and more generally for pipe flows of arbitrary cross section, the “skin” drag force per unit length is set as,

$$D_r = -f \rho |U| U h$$  \hspace{1cm} (80)

where $U$ is the turbulent velocity, $h$ the pipe of diameter, and $f$ is the skin friction coefficient, or Chézy coefficient, which depends on the Reynolds number, and displays a sensitivity to the smoothness of the surfaces. Experimentally $\sqrt{f}$ decays approximately like $1/\ln(Re)$ at large $Re$, for flows in pipes with very smooth surfaces, as it follows from our study of turbulent Poiseuille flow, see Fig. 4.

A significant question posed by real pipe flows for a long time is the effect of the roughness of the solid surfaces. This is related to the behavior of the skin friction coefficient at very large values of the Reynolds number, because the thickness of the viscous boundary layer is of order $h/Re$ and so can become in real flows of the order of magnitude of the irregularities of the surface $\ell_{ru}$, defining the length scale of the roughness of the wall. A simple argument shows that the effect of the length $\ell_{ru}$ amounts to increase the thickness of the viscous sublayer, $\ell_v$, or to increase the skin friction, as observed. Let us assume that the velocity vanishes in a small domain close to the surface, $0 < \tilde{z} < \ell_s$, where $\ell_s = k \ell_{ru}$ is a fraction of the roughness length, the coefficient $k$ depending on various parameters like the orientation of the asperities with respect to the main flow. The integration of Eq. (37) between $\ell_s$ where the solution is zero, and $\tilde{z}$, leads to the solution

$$u(\tilde{z}) = \frac{|g|h^2}{2\rho \gamma d} \ln \left( \frac{\tilde{z} + \ell_v}{\ell_s + \ell_v} \right)$$  \hspace{1cm} (81)

with $\ell_v = \frac{wh}{\gamma d}$, as in (39). It follows that the constant $C$ appearing in the outer solution in Eq. (43) becomes

$$C = -\ln(\ell_s + \ell_v),$$  \hspace{1cm} (82)

This relation shows that the effective width of the boundary layer is higher for rough surfaces than for smooth ones. The difference between $\ell_s$ and $\ell_v$ lies in the fact that $\ell_s$ does not depend on the Reynolds number. This leads to the conclusion that the velocity of the plug flow to be added to match the boundary layer due to the roughness of the pipe surface, is of order $(gh/\rho)^{1/2} \ln(\ell_s/h)$ in the limit of very large Reynolds number, so that the friction factor, instead of tending to zero as $\sqrt{f} \sim 1/\ln(Re)$ (in the case of smooth surface), tends now to a small constant

$$\sqrt{f}_{Re \to \infty} \sim \frac{1}{\ln(\ell_s/h)}. $$  \hspace{1cm} (83)

The relation (83) agrees with the Colebrook relation [12], $1/\sqrt{f} = -2 \log_{10}(a \ell_s/h + b/Re \sqrt{f})$ where $a$ and $b$ are constant factors depending on the geometry of the flow, extending the von Karman–Prandtl expression (same with $a = 0$), to the case of rough surfaces.

Because the skin friction is sensitive to the structure of the wall at the scale of the viscous boundary layer thickness, it is possible to change the contribution of the exchange of momentum between the fluid and the wall by tuning the structure of the roughness of the wall [14], for instance, by undulations of this wall in the range of length scales of order of the viscous sublayer. This introduces anisotropy in the balance of momentum at this scale and so at bigger scales of the flow, beyond the boundary layer. This process was used, for instance, in the form of riblets on Olympic-class rowing shells and on the hull of the winner of the 1987 America’s Cup, and studied for several decades [15].

Without claim of a rigorous theory for those different behaviors of $C_v$ and $f$ at large Reynolds number, we outline below an explanation based partly on the present study. In
turbulent Poiseuille flow, the drag comes only from the viscous friction on the wall (leaving aside the central jet in the middle of the bulk, see Sect. 3.3), related itself to the component \( \sigma_{xz}(z) \) of the Reynolds stress there. But this turbulent stress is null on the wall, then it does not act directly on it, it acts only as a boundary value for the viscous layer. As shown above, the abrupt drop of \( \sigma_{xz}(z) \) close to the wall is possible using the viscosity, which enters into play in the interconnection domain between the inner/outer solution which behaves as \( \log(Re) \), see Eq. (43), that allows to transfer the \( \sigma_{xz} \) component across the boundary layer where it becomes viscous. This explains why the logarithm of the Reynolds number appears in the friction factor which decays to zero at large \( Re \). This remains true (the presence of the \( \log Re \) term in the solution close to the wall), whenever all non-diagonal components \( <u'_i u'_j> \) vanish at the boundaries, which occurs in channel and pipe flows, and more generally whenever \( i \) and/or \( j \) is a local Cartesian component of the velocity fluctuation in the direction normal to the wall.

On the other hand, in flows with a general structure, like around a blunt body at large speed, there is another contribution to the stress, besides turbulent Reynolds stress, which is the isotropic pressure. Let us recall that D’Alembert paradox states that an inviscid incompressible flow around an obstacle exerts no drag because of the exact balance between the upstream and the downstream contribution to the force coming from the pressure that contradicts the observation of significant drag on bodies moving relative to a fluid.

The explanation of this paradox is that the time-dependent Euler equations in 3D do not make in general a well-posed problem, which needs some regularization of small scales, either by viscosity or by other effects like radiation of sound waves because of the compressibility. Once the solution is regularized, the singularities are transformed into local dissipation events, something coherent with our representation of the RST by an integral in space of a quantity quadratic with respect to the average velocity field.

When the turbulent drag is due to the pressure difference between the two sides of the obstacle, of order of magnitude \( \rho u^2 \), the drag force takes the form given by the Newton’s relation (79), without dependence on \( Re \) (asymptotically at large \( Re \)) because the \( Re \) dependence concerns the skin force, an auxiliary boundary layer problem with viscosity. Note that the absolute value in Eq. (10) which breaks the symmetry under velocity reversal, is necessary for the description of both phenomena, the turbulent drag and the skin friction.

Where the upstream/downstream pressure difference comes from remains to be explained. Flows around bluff bodies are split into a potential flow upstream,\(^{10}\) and a vortical flow in the so-called Kirchhoff bubble downstream where vorticity is sustained by shear instability. We point out that Kelvin theorem is not valid for the flow lines of the time-averaged velocity field. To understand what happens to the time-averaged flow lines, let us consider the case of the mixing layer formed behind a plate separating two flows with different velocities, and parallel to the plate: the upstream domain is potential, and the downstream domain is filled with vorticity. The Kirchhoff bubble downstream is a wedge, where typical Kelvin–Helmholtz instability grows. This was enlightened in Ref. [2] where our integral model ensures a continuous distribution of vorticity in the wedge, without adding an artificial turbulent viscosity. Averaging over time, the wedge is clearly separated from the upstream potential flow by a surface or, at the minimum, by a thin layer, and time average flow lines may be connected to the upstream ones. On time average the separating surface must satisfy constraints of conservation of the mechanical invariants of mass and momentum. Concerning the momentum this implies that pressure (and only the pressure) on the side of the potential flow must be equal

\(^{10}\)The flow is potential upstream because the Kelvin theorem states that circulation is convected along flow lines.
to the sum of the pressure and of the diagonal part of the Reynolds stress on the side of the Kirchhoff bubble. Therefore, the pressure is less on the turbulent side than on the potential side. Because of the boundary condition of vanishing of the normal turbulent stress on any material surface, the Reynolds stress does not yield any force on the blunt body. There the normal force comes only from the pressure (without contribution of the Reynolds stress). Because of the continuity of the normal stress across the boundary, be it sharp or not, of the Kirchhoff bubble, the pressure is lessened in the Kirchhoff bubble that yields the lift and drag force on the blunt body. This explains where the pressure drag comes from, and lastly where Newton’s drag originates.

One may draw some conclusions from the way the inner–outer matching is done near the wall in the case of the turbulent Poiseuille flow. As we have just shown, the matching links the coefficient $C = \ln(\gamma Re)$ of the logarithmic dependence of the solution, with the amplitude of a uniform mean velocity $du/dh$, solution of the equation in the turbulent domain. The addition of the constant solution $C$ arising from the boundary condition is possible in our model where the turbulent stress depends on the vorticity and therefore is not affected by the addition of a constant velocity. Another point of interest raised by our approach of the structure of the average velocity field of turbulent flows is the possibility of bifurcations of the solution in the limit of a very large Reynolds number. This kind of bifurcation is known to occur in such flow, for instance, in the wake of fast moving cars, where this wake loses its left/right symmetry and so yields an unwanted torque on the car [16]. Because our equation for the averaged velocity field is nonlinear, its solutions may break symmetries like the left/right symmetry of some wakes.

5 Other simple examples of interaction of a fast flow with an obstacle

This section is to use results of our study of the plane Poiseuille flow to situations with an obvious resemblance to it, although with some differences. Some examples are presented and solved only from the point of view of scaling laws to show the generality of the theory outlined in this paper, and they will be more detailed in a future work.

The first problem we consider is somehow common in the framework of boundary layer theory. This is the one of a parallel flow propagating along an infinite half-plane at zero incidence. Let $x$ be the direction of the flow, $z$ the direction perpendicular to the solid surface of Cartesian equation $z = 0$ for $x > 0$, and $h(x)$ the thickness of the boundary layer growing near the plane, along $x$. Actually, as $x$ gets big enough, the flow becomes turbulent in this boundary layer, so that the knowledge of its average thickness $h(x)$ requires some turbulence modeling.

Our starting point is the writing of the fundamental law of dynamics written in the frame of the half plane where the quantities are steady in average. Basically we follow the same line of reasoning as Newton when he derived the square law of drag at large speeds. The fundamental law of dynamics is

$$\frac{dP}{dt} = F$$

(84)

where $P = mu$ is the momentum of a mass $m$ of fluid moving with a velocity $u$ in the $x$-direction, $d/dt$ is a total derivative and $F$ the force undergone by this mass due to viscous friction on the plane. In our frame, the rate of change $dP/dt$ is due to the convective part. Therefore $d/dt = ud/dx$ where $u$ is the convective velocity in the $x$-direction, approximately equal to the velocity at infinity, except in the viscous sublayer close to the plate. The variation
\[
\frac{d}{dx} \text{ is assumed to be mainly described by the change of height reflecting the change of carried mass. Therefore, if one considers a slice } dx \text{ of fluid of height } h(x) \text{ and surface unity, its mass is } m = \rho dx h(x), \text{ its momentum is } P = \rho dx h(x) u \text{ and the convective derivative of this momentum is } \frac{dP}{dt} = \rho dx u^2 \frac{dh(x)}{dx}. \text{ This must be equal to the viscous force exerted by the solid. This force is } dx \eta \frac{d(\tilde{u})}{dz} \text{ where } \eta = \rho \nu \text{ is the shear viscosity of the fluid and } \tilde{u} \text{ the order of magnitude of the velocity along } x \text{ in the viscous sublayer. Equating the two quantities, viscous friction on the solid and loss of linear momentum of the fluid one finds}
\]
\[
\frac{u^2}{\epsilon} \frac{dh}{dx} = \frac{\tilde{u}}{\epsilon} = \nu \frac{d}{dx}
\]
where \( \tilde{u} / \epsilon \) is an approximate expression for the normal derivative \( d\tilde{u}/dz \) on the solid surface. This leaves unsettled the determination of two quantities, \( \tilde{u} \) and \( \epsilon \). To estimate this viscous sublayer thickness \( \epsilon \), we assume that it is given by the same order of magnitude as for the plane Poiseuille flow, Eq. (39), by taking for \( h(x) \) the local thickness of the turbulent boundary layer. This yields
\[
\epsilon \sim \frac{h(x)}{Re_h}
\]
where \( Re_h \) is the (local) Reynolds number in the boundary layer,
\[
Re_h(x) = \frac{\tilde{u} h(x)}{\nu}.
\]
This yields
\[
\frac{dh}{dx} \sim \left( \frac{\tilde{u}}{u} \right)^2
\]
If one takes \( \tilde{u} \sim u \) this yields \( dh/dx \sim 1 \), but this is wrong because the velocity close to the wall is smaller than far away in the bulk. From our estimate for the turbulent Poiseuille flow, one may infer that \( u^2 \sim \tilde{u}^2 \ln(Re_h) \), see Eqs. (19) and (46). Therefore, our equation for the thickness of the turbulent boundary layer is
\[
\frac{dh}{dx} \sim \frac{1}{\ln(\tilde{u} h(x)/\nu)}.
\]
At first order in \( 1/\ln(x) \), this estimate agrees with the result given in Ref. [9] where the thickness of the turbulent boundary layer was found to increase as \( x/\ln x \) for large \( x \).11 In summary, our developments point to an almost linear expansion of the boundary layer when taking into account the viscosity, in agreement with a thickness increasing with the distance as \( x/\ln x \) at zero incidence. This result is consistent with Landau’s remark in Ref. [4] based on similarity considerations in the event of a flow expanding at infinity in all directions.12 Furthermore, one may infer that at nonzero incidence the turbulent boundary layer should grow with the distance \( x \) as
\[
h(x) \sim \frac{x}{\ln(\tilde{u} h(x)/\nu)} + x k(\theta_i)
\]
11 Along a half-plate at zero incidence, observations made a hundred years ago display a great increase of the boundary layer thickness at the transition from laminar (close to \( x=0 \) where the thickness behaves as \( h(x) \sim 5(\nu x/\mu)^{1/2} \)) to turbulent behavior, occurring at the critical value \( u x/\nu \approx 5 \cdot 10^5 \).
12 See section 36 of [4] where it is stated that the turbulent domain formed behind a plate displays an angular dependence, because there are no constant at our disposal having the dimension of a length. Therefore, the average velocity in the turbulent boundary layer is expected to depend on the ratio \( z/x \), as studied in our recent mixing layer paper [2].
where \( k() \) depends on the incidence angle \( \theta_i \) and \( k(0) = 0 \).

The other example of flow we shall consider is the one of a parallel flow impinging at \( x = 0 \) on a cylinder of circular cross section of radius \( r_0 \) with its generatrices along the \( x \) direction. The equation for the balance of momentum along \( x \) is derived by using the same general idea as for the fast flow past a half plane but by estimating the flux of momentum across a disc of radius \( r(x) \). Let \( r(x) \) be the radius of the turbulent layer generated by the viscous friction, and let us assume that \( x \) is large enough to ensure \( r(x) \gg r_0 \). The viscous stress on the surface of the cylinder is \( \rho \nu \tilde{u} \epsilon r_0 \) with \( \epsilon \sim r/Re_r \) where \( Re_r \) is the Reynolds number of order \( \tilde{u}r(x)/\nu \). Putting together those estimates, one obtains

\[
u^2 r \frac{dr}{dx} \sim r_0 \tilde{u}^2
\]

or

\[
\frac{dr}{dx} \sim \frac{r_0}{\ln(\tilde{u}r(x)/\nu)}.
\]

At first order with respect to \( 1/\ln(x) \) and large distance from the edge of the cylinder, this estimate points to an increase of the boundary layer thickness with the distance as

\[
r(x) \sim \sqrt{r_0 \frac{x}{\ln x}}
\]

This ends our study of examples of boundary layers near obstacles.

6 Conclusions and perspectives

In this paper, we wrote fully explicitly the integral equation for the balance of momentum including the closure of the turbulent stress introduced in ref \[1\] on the basic assumption that dissipation is caused by singular events described by solutions of Euler equations. Because of the fully explicit character of this closure, associated with an integral expression of the Reynolds stress tensor, we have shown that it is possible to obtain a clear matching between the inner boundary layer solution, valid close to the wall, and the outer solution far from the wall (but beyond the bulk). Particularly, we obtain the log-law close to the wall, in a so to say rational way, that is by handling explicit equations from the beginning.

Roughly speaking, our integral representation of the turbulent stress is a way to get rid of an impossible definition of Prandtl mixing length far from a boundary layer in an arbitrary geometry. Thanks to this solution of an unique equation, we have the full velocity profile everywhere between the two walls. This allows to understand how the profile of the velocity, written in Eq. (45), flattens and gets stiff wings as \( Re \) increases, becoming the sum of an almost uniform velocity, proportional to \( \ln(\gamma Re) + c \), plus an added velocity proportional to \( U(z) \) which looks as a wedge of increasing angle with round shoulders, as shown in Fig. 3.

The occurrence of logarithms in the skin friction explains that the friction factor, in pipes and channels with smooth surfaces, tends to decay to zero qualitatively as a function of the logarithm of the Reynolds number, and also that the drag coefficient of blunt bodies is insensitive to \( Re \) when their surfaces are perfectly smooth, but are sensitive to \( Re \), including at large values of it, when their surfaces are rough.

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