**Ion sound instability driven by ion flow in plasma.**

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Ion sound waves driven by the ion flow in a system of finite length is considered via analytical and numerical methods. The ion sound waves in a plasma with finite electron temperature are modified by the presence of stationary ion flow resulting in negative and positive energy modes. The instability develops due to coupling of negative and positive energy modes mediated by reflections from boundaries. It is shown that the length of the system measured in units of the Debye length is an important parameter that determines the stability properties, $L/\lambda_d$. In the short system, $L/\lambda_d \leq 1$, corresponding to the case of strong dispersion, the fluctuations dynamics is described the equations similar to the Pierce plasma diode case. In the opposite limit, $L/\lambda_d > 1$, the wave dispersion is weak and instability sets up as a result of weak deviation from quasineutrality. Analytical theory is developed for both cases and is compared with results of direct initial value numerical simulations.

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I. INTRODUCTION

Many natural settings of space and laboratory plasmas often include equilibrium flows of ions and/or electrons. Such situations occur in various plasma devices for electric propulsion, plasma diodes, plasma accelerators, plasma processing devices, and emissive probe diagnostics. Plasmas permeated by energetic beams also typical situations in space and astrophysics. Such plasmas represent a typical example of a non-equilibrium system prone to instabilities due to presence of free energy reservoir from stationary flows. Typical example of such systems is a Pierce diode. It has a configuration with two electrodes with high energetic electron beam between them, while ions play role of positively charged background. The numerical simulation, nonlinear effects, experimental application and other development of Pierce-like systems are widely discussed in literature.

There is a wide class of problems which have ion flows unlike Pierce diode. For example such systems could appear due to ion heating in double layer, or ions are accelerated artificially by electric fields (electric propulsion systems). Systems with ion flows are unstable when beam velocity is bigger than ion sound velocity \(v_0 > c_s\) because of reverse Landau damping; this instability is called ion acoustic beam-plasma instability (IABPI).

In this paper we analyze similar to IABPI example - ion sound instability induced by ion flow. However, in our case we consider instability inducement by boundaries condition effects. As we will see later this instability arises in case different to IABPI which is \(v_0 < c_s\). In our example, unlike IABPI, we consider only hydrodynamic effects. In general, the considered fluctuations are of the ion sound type albeit modified by the stationary ion flow. To take this into account, all electrons are assumed to be in Boltzmann equilibrium (electron inertia effects are neglected), ion are assumed cold (to avoid Landau damping) and have an uniform velocity with respect to the electron component.

The excitation of ion sound waves by ion flows is important for a number of situations such as in presheath region of the plasma-material boundary. More generally, an unmagnetized plasma with an ion flow is a specific example of plasma diode (Pierce diode). In an infinite plasma, the stationary ion flow results in the Doppler shift of the frequency of the ion sound waves freq, \(\omega \rightarrow \omega - k_x v_0\). It is shown here that in a finite length systems, the ion sound waves can be destabilized by the ion flow. This instability is different from the plasma-beam (two-stream type) instabilities, where the presence of the gradients in velocity space is important.
We employ analytical and numerical methods to analyze the structure of unstable eigen-modes, determine the dispersion relations and conditions for the instability, and find the frequencies and growth rates of the unstable modes.

II. BASIC EQUATION

The modes stability is studied by using the linearized hydrodynamic equations for cold ion beam:

\[
\begin{align*}
\frac{\partial n_i}{\partial t} + v_0 \frac{\partial n_i}{\partial z} + n_0 \frac{\partial v_i}{\partial z} &= 0, \\
\frac{\partial v_i}{\partial t} + v_0 \frac{\partial v_i}{\partial z} + e \frac{\partial \phi}{m_i \partial z} &= 0.
\end{align*}
\]

The electrons are assumed to be adiabatic and follow Boltzmann relation, for low frequencies \(\omega < kv_{Te}\).

\[n_e = \frac{n_0 e}{T_e} \phi,\]

The full system is closed by the Poisson equation

\[\frac{\partial^2 \phi}{\partial z^2} = -4\pi e (n_i - n_e),\]

where \(n_i, n_e, \phi\) - perturbation of ion, electron density and electrostatic potential respectively, \(n_0\) - equilibrium density, \(e, m_i\) - charge and mass of ions, \(T_e\) - electron temperature, \(v_0\) - speed of ion flow, \(v_{Te}^2 = \frac{T_e}{m_e}\) - electron thermal velocity.

For the ion injection from the left boundary, the following boundary condition are used

\[\phi(z = 0) = \phi(z = L) = n_i(z = 0) = v_i(z = 0) = 0,\]

where \(L\) - length of the system.

III. EIGEN-MODES

To study the system analytically we seek solution in the form \(\sim e^{-i\omega t}\). Then, the equations can be reduced to one equation in the form:

\[v_0^2 \phi''' - 2i\omega v_0 \phi'' + \left[ \frac{c_s^2}{d_e^2} - \omega^2 - \frac{v_0^2}{d_e^2} \right] \phi'' + \frac{2i\omega v_0}{d_e^2} \phi' + \frac{\omega^2}{d_e^2} \phi = 0,\]

where

\[c_s = \sqrt{\frac{T_e}{m_i}}, d_e = \sqrt{\frac{T_e}{m_e}}, v_0 = \frac{v_i}{D}, D = \frac{T_e}{m_e} \ln(\frac{L}{a}), \]

\[L = \text{length of the system}, a = \text{characteristic length}.\]
where prime is a derivative in respect to $z$, $c_s^2 = T_e/m_i$ - the ion sound velocity, $d_e^2 = T_e/4\pi e^2 n_0$ - is the Debye length.

In the limit of $v_0 \to 0$, for perturbations of the form $\sim e^{ikz}$ one obtains the dispersion equation for standard ion sound waves

$$\omega^2 = \frac{k^2 c_s^2}{1 + k^2 d_e^2}. \tag{6}$$

General solution of (5) can be sought in the form $\phi \sim e^{\lambda z}$ resulting in the equation for $\lambda$

$$v_0^2 \lambda^4 - 2i\omega v_0 \lambda^3 + \omega^2 \left[ \frac{c_s^2}{d_e^2} - \omega^2 - \frac{v_0^2}{d_e^2} \right] + \frac{2i\omega v_0}{d_e^2} \lambda + \frac{\omega^2}{d_e^2} = 0,$$

or in more convenient form:

$$d_e^2 \left( \lambda - \frac{i\omega}{v_0} \right)^2 \left( \lambda^2 - \frac{1}{d_e^2} \right) + \frac{c_s^2}{v_0^2} \lambda^2 = 0. \tag{7}$$

A. Full quasineutrality case

The simplest approximation of the full system (1) could be obtained by neglecting the effects of dispersion, so that $n_i = n_e$. Thus, we consider the length scales much longer than the Debye length - the charge separation scale.

In this limits, the solution of system (1) can be obtained in the form:

$$\phi(z) = C_1 \exp \left( \frac{i\omega z}{v_0 + c_s} \right) + C_2 \exp \left( \frac{i\omega z}{v_0 - c_s} \right). \tag{8}$$

By imposing boundary conditions (4), we obtain the stable eigen-modes with the frequencies $\omega$

$$\omega_n = \frac{\pi n v_0^2 - c_s^2}{L c_s}, n \in \mathbb{Z}. \tag{9}$$

This frequency is real that is why quasi-neutral case is stable. It will be shown that dispersion cases are unstable, so the dispersion plays important role in instability mechanism.

It is worth to note, that the formula (9) is not valid in the zero electron temperature limit ($T_e \to 0, c_s \to 0$) because in this case the solution for electrostatic potential will differ from (8) and will be:

$$\phi(z) = (C_1 + C_2 z)e^{\frac{i\omega}{v_0} z}, \tag{10}$$

while boundary conditions will give us the frequencies:

$$\omega_n = \frac{2\pi n}{L} v_0, n \in \mathbb{Z}. \tag{11}$$
B. Week dispersion case

The nature of dispersion in this case is charge separation, so the conditions for week dispersion could be written as

\[ kd_e \ll 1 \text{ or } d_e \ll L, \]  \hspace{1cm} (12)

where wave number \( k \sim \frac{1}{L} \). Using dispersion equation for plasma without flows (6) in this approximation we can estimate frequencies.

\[ \omega \sim k c_s \text{ or } \omega \sim \frac{d_e}{L} \omega_{pi}. \]  \hspace{1cm} (13)

where \( \omega_{pi} = \frac{c_s}{d_e} \) is a plasma frequency.

Formally we can solve (7) treating Deby length as a small parameter. However, some roots are tend to infinity with zero Deby length limit, so to find all roots we must seek them in the form \( \lambda = \frac{\xi}{d_e} \). In this case equation (7) could be expressed as:

\[ (\xi - \frac{i\omega}{v_0}d_e)^2(\xi^2 - 1) + \frac{c_s^2}{v_0^2} \xi^2 = 0. \]  \hspace{1cm} (14)

First couple of solutions coincide with ones in quasi neutral case:

\[ \lambda_{1,2} = \frac{i\omega}{v_0} \pm c_s + O(d_e^2). \]  \hspace{1cm} (15)

Second pair of solutions are:

\[ \lambda_{3,4} = \pm \frac{i}{v_0 d_e} \sqrt{c_s^2 - v_0^2} + \frac{i \omega c_s^2}{v_0} \frac{1}{c_s^2 - v_0^2} + O(d_e). \]  \hspace{1cm} (16)

Since all roots are different we can write the general solution of (5) in this form:

\[ \phi(z) = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} + C_3 e^{\lambda_3 z} + C_4 e^{\lambda_4 z}. \]  \hspace{1cm} (17)

Expressing ion velocity and density from the full system (1):

\[ 4\pi e n_i = \frac{\phi}{d_e^2} - \phi'', \]  \hspace{1cm} (18)

\[ 4\pi e n_0 v_i = \frac{v_0}{d_e^2} \phi + \frac{c_s^2 - v_0^2}{i \omega d_e^2} \phi' - v_0 \phi'' + \frac{v_0^2}{i \omega} \phi'''. \]  \hspace{1cm} (19)

We can write dispersion equation as a condition for existence of nontrivial solution for \( C_1, C_2, C_3, C_4 \) in the linear system of equations (4) as:

\[ D = det \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} & e^{\lambda_4 L} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} \right) = 0, \]  \hspace{1cm} (20)
where

\[ \mu_k = \left( \frac{c_s^2}{v_0^2} - 1 \right) \lambda_k + d^2 \lambda_k^3. \]  

(21)

The dispersion equation (20) is difficult to solve analytically as there numerous solutions in the whole complex plane. However we are interested only on those which has bigger imaginary part, since those unstable mode will dominate. The numerical solution for length of the system bigger than Debay length \( (L = 10d_e) \) is shown on Figure 1. On that graph we can see that no-flow approximation \([13]\) gives reasonable results.

![Figure 1](image)

**FIG. 1:** The solution of (20) for \( L = 10d_e \) illustrate existence of oscillatory \((\Re(\omega) \neq 0)\) and aperiodic \((\Re(\omega) = 0)\) instability zones. The frequency is measured in ion plasma frequency units.

The shape of the Figure 1 implies that for fixed system length \( L \), instability growth rate depends only on dimensionless ion flow velocity \( v_0/c_s \), moreover those dependence has clear oscillatory \((\Re(\omega) \neq 0)\) and aperiodic \((\Re(\omega) = 0)\) zones. The border of those zones could be found analytically using the fact the on the border wave frequency is zero. Expanding (20) in Taylor series in terms of frequencies:

\[ D(\omega) = D(0) + \frac{\partial D(0)}{\partial \omega} \omega + O(\omega^2) = 0. \]  

(22)
Using that $D(0) \equiv 0$, $\frac{\partial D(0)}{\partial \omega} = 0$ yields:

$$\frac{v_0}{c_s} = \frac{1}{\sqrt{1 + \pi^2 n^2 \frac{d_e^2}{L^2}}} \text{ where } n = 1, 2, 3, \ldots$$

(23)

C. Strong dispersion case

On the contrary to quasi-neutral case in small wave length limit charge separation play important role. The conditions for strong dispersion could be written as

$$kd_e \gg 1 \text{ or } d_e \gg L,$$

(24)

where wave number $k \sim \frac{1}{L}$. Using dispersion equation for plasma without flows (6) we can estimate frequencies.

$$\omega \sim \omega_{pi}.$$  

(25)

In this case we treat reciprocal value of Debye length $(1/d_e)$ as a small parameter. In this case roots to the (7) are:

$$\lambda_{1,2} = 0 \text{ and } \lambda_{3,4} = i\frac{\omega \pm \omega_{pi}}{v_0},$$

(26)

the general solution:

$$\phi(z) = C_1 \exp \left( i\frac{\omega + \omega_{pi}}{v_0} z \right) + C_2 \exp \left( i\frac{\omega - \omega_{pi}}{v_0} z \right) + C_3 z + C_4.$$  

(27)

This situation is mathematically equivalent to the Pierce instability\textsuperscript{10} of the electron beam between cathode and anode. Imposing boundary conditions (4), one obtains an homogeneous linear system, which has nontrivial solutions when the following dispersion equation is satisfied:

$$2\xi\alpha(1 - e^{i\xi} \cos \alpha) + i(\xi^2 + \alpha^2) \sin \alpha e^{i\xi} + i\frac{\xi^2}{\alpha}(\xi^2 - \alpha^2) = 0,$$

(28)

where $\xi = \frac{L\omega}{v_0}$ and $\alpha = \frac{L\omega_{pi}}{v_0}$.

It was shown [by Pierce\textsuperscript{11} and others\textsuperscript{15}], that the dispersion equation (28) for:

$$\alpha < \pi \quad - \text{ has stable solution,}$$

$$\pi < \alpha < 2\pi \quad - \text{ has aperiodic instability,}$$

$$2\pi < \alpha < (2N - 1)\pi \quad - \text{ has oscillatory instability,}$$

(29a)

(29b)

(29c)
where \( N = 1, 2, 3, \ldots \), with maximum growth rate:

\[
\gamma \sim \frac{v_0}{L}.
\]  

(30)

The are a lot of different solutions of the dispersion equation (28) on whole complex plane, however we are interested only on those which has bigger imaginary part, since those unstable mode will dominate. The solution which meets that criteria is shown on Figure 2. It is also confirms the existence of aperiodic and oscillatory instability zone from (29). Which is similar to week dispersion example.

\[ \text{FIG. 2: Solution of (28) and simulation benchmark shows Pirce oscillatory (\( \Re(\omega) \neq 0 \)) and aperiodic (\( \Re(\omega) = 0 \)) instability zones.} \]

IV. NUMERICAL SOLUTION (METHOD)

To confirm our analytical results we solve the system (1) numerically. Equations among that system have different structure and we employ the following strategy.

The first two equations of (1) are considered as an explicit initial value problem (IVP), and the third and fourth equations of (1) are solved as a boundary value problem (BVP).
Those subsystems are solved numerically in time. To obtain the time dependent evolution of (IVP) and (BVP) they are solved iteratively. The Poisson equation in (BVP) is solved at the beginning of each time step. The system (BVP) uses given ion density profile (either from the initial condition or from previous time step) to produce the electrostatic potential profile. The known potential distribution allows us to solve problem (IVP) in time. As the final step, we update, the ion density and velocity profiles obtained from IVP.

Common ways to solve a BVP like problem are family of shooting methods and finite difference schemes. We considered shooting methods due to their simplicity. Therefore, we chose on multiple shooting method (MSM) because it is easy to parallelize, it has no disadvantages of simple shooting methods (e.g., limitations on a system length) and since our system is linear, MSM is reduces to bunch of IVP and linear system of equations.

Our IVP is a system of hyperbolic PDEs, which can be expressed in a conservation form, because of the nature of our task (our IVP consists of continuity and Euler equations, which are conservative). This forces us to treat our system with a class of finite volume methods. The simplest finite volume method is an upwind scheme, however we cannot use this scheme for all cases because in our case it is possible to obtain waves, which are spreading in both directions. Those waves will make upwind unconditionally unstable. For the solution to this issue we used Lax-Friedrichs and Harten, Lax, Van Leer (HLL) which belongs to Godunov family methods. Lax-Friedrichs shows quite nice results, and it is much simpler than Godunov methods, since you don’t have to solve Riemann problem on each time step, but it requires high discretization for correct results due to its numerical viscosity. Another drawback is that when solution contains big gradients (sharp edges), it could manifest Gibbs phenomenon - oscillatory behaviour. We also tried MacCormack method, but it shows very oscillatory behavior, since it has no artificial viscosity as Lax-Friedrichs method. In such situation Godunov methods can give big advantage. Such schemes can be characterized by the solution of Riemann problem with computational cells in order to obtain numerical fluxes. These methods one can divide on approximate Riemann solvers and exact Riemann solvers. We used one of the kind of approximate Riemann solves - the HLL method.
A. Results of numerical simulations

For convenience all further results will be expressed in dimensionless units:

\[
\frac{n}{n_0} \rightarrow n, \quad \frac{z}{d_e} \rightarrow z, \quad \frac{e\phi}{T_e} \rightarrow \phi, \quad t\omega_{pi} \rightarrow t, \quad \frac{v}{c_s} \rightarrow v, \quad \frac{L}{d_e} \rightarrow L, \quad \frac{v_0}{c_s} \rightarrow v_0. \tag{31}
\]

The results of numerical simulations are compared with analytical results for weak and strong dispersion. We start our simulations with initial conditions of an uniformly distributed random noise and observing the evolution of the following quantities

\[
N^2 = \int_0^L n^2(z)dz, \quad \Phi^2 = \int_0^L \phi^2(z)dz, \quad V^2 = \int_0^L v^2(z)dz. \tag{32}
\]

Depending on the value of input parameters \((L, v_0)\) dumped (stable) or growing (unstable) solutions were observed. Unstable solution were fitted to the following curves

\[
N^2, V^2, \Phi^2 \sim \cos(2\Re(\omega)t + \theta)e^{2\gamma t}, \tag{33}
\]

to recover the real frequency and growth rates.

When the length of the system exceeds the Debye length \((L \sim 10d_e)\), the week dispersion approximation results are recovered. Example of frequency and growth rate dependence over ion flow velocity \(v_0\) is shown on Figures 3,4. Those graphs are very similar to the analytical results shown on Figure 1. In fact the difference of the results are in order of small parameter of analytical theory \((d_eL)\), however due to increasing density of instability zones, velocity space require unreasonable resolution to repeat transcendent part \((v_0 \sim 0)\) of the analytical solution.

In the regime, when the length of the system is much smaller than the Debye length \((L \sim 0.1d_e)\) the difference between analytical solution of strong dispersion approximation and numerical solution was less than few percent. The simulation checkpoints are shown on Figure 2.

The number of zeros of unstable spatial eigenfunctions of density, velocity and electrostatic potential which arise in Pierce diode correlates with a number of zone \((29)\) and fully defined by Pierce parameter \(\alpha\). In more general case stability of the system is governed by two parameters \((v_0, L)\) however analogy could be made. We define a zone for a fixed system length for ranges of ion velocity as shown on Figure 3. It was found that in general case the number of zeros correlates with a number of zone as well, examples of eigenfunctions are shown on Figure 5.

In aperiodic zones (where real part of frequency is zero) the number of zeros holds the same all time, but in oscillatory zones the number of zeros could decrease for short times example of that behaviour is shown on Figure 6.
FIG. 3: Results of numerical simulation for $L = 10$ confirms solution to (20), Figure 11 and existence of oscillatory ($\Re(\omega) \neq 0$) and aperiodic ($\Re(\omega) = 0$) instability zones.

B. Behaviour of week dispersion waves

If we add Doppler shift due to ion flow velocity to the dispersion equation (6), and consider dispersion effect to be week ($kd_e \ll 1$) we can obtain waves velocities:

$$v_{1,2} = v_0 \pm c_s$$  \hspace{1cm} (34)

It is worth to note that one of the couple of roots of (7) correspond to those velocities. Other couple describes slow dispersion effect. To show wave travelling very long system was chosen ($L = 1000d_e$) because in such system dispersion is small enough to see separate wave packets. Initial condition was chosen to be Gaussian pike in the middle of the system. Those evolution is shown on Figures 7. First of them shows initial condition then we see that the Gaussian peak separates into two wave packets which are moving with $v_{1,2}$ from (34). When the fastest packet meets the right wall (which has no boundary conditions except the one for electrostatic potential) it passes through the wall barely reflected. Instability arise when slowest packet meets the left wall (with Dirichlet boundary conditions for all variables) and is reflected. Later the reflected wave and dispersion tail starts to form an unstable eigenfunction to the according instability zone (23).
FIG. 4: Results of numerical simulation for $L = 5$ confirms solution to (20) and existence of oscillatory ($\Re(\omega) \neq 0$) and aperiodic ($\Re(\omega) = 0$) instability zones.

FIG. 5: Unstable spatial eigenfunctions of density, velocity and electrostatic potential for $L = 10$, for different instability zones from Figures 1, 3, 4.

C. Behaviour of strong dispersion waves

In strong dispersion case the equation (6), oscillation with ion plasma frequency $T_0$ show wave travelling short system was chosen ($L = 0.1d_e$). Initial condition was chosen to be Gaussian pike in the middle of the system. Those evolution is shown on Figures 8. First picture is a initial
FIG. 6: Unstable spatial eigenfunctions of density, velocity and electrostatic potential at oscillatory zone #2 $v_0 = 0.78$ for $L = 10$

Gaussian pike which starts to travel with velocity of ion flow ($v_0$) and to decrease in size, in the same time another pike arise from the left border and starts to travel with same velocity. When the initial Gaussian pike meets the left boundary (which has no boundary conditions except the one for electrostatic potential) it passes through, while another pike starts to transform to unstable eigenfunction.

V. CONCLUSION

In this paper was discussed the model of instability induced by ion flow in closed systems such as various plasma devices for electric propulsion, plasma diodes, plasma accelerators, plasma processing devices and emissive probe diagnostics. It was shown that the length of the system measured in units of the Debye length and ion flow velocity measured in units of the ion sound velocity are an important parameters which controls the instability regime.

For the long systems ($d_e \ll L$) dispersion equation was solved. It demonstrates the existence of aperiodic and oscillatory instability zones borders of which is described by formula (23). This
equation could be used as stability criteria for system length and ion velocity:

\[
\frac{1}{1 + \pi^2 \frac{d_e^2}{L^2}} < \frac{v_0^2}{c_s^2}.
\]  

(35)

It is worth to know that this condition reduces to Pirce criteria \((\alpha < \pi)\) in the limit of strong dispersion \((d_e \ll L)\).

It is interesting that the ion acoustic beam plasma instability (IABPI) has almost complimentary to (35) condition. The dispersion equation for IABPI in our approximations could be written as:

\[
1 + \frac{\omega_{pi}^2}{k^2 c_s^2} - \frac{\omega_{pi}^2}{\omega - k v_0} + i \sqrt{\frac{\pi}{2}} \frac{m_e \omega_{pi}^2}{m_i k^3 c_s^3} = 0,
\]  

(36)

where \(m_e\) - is electron velocity. Treating \(\epsilon = \sqrt{\frac{\pi}{2}} \frac{m_e}{m_i}\) as a small parameter, we can obtain grows rate:

\[
\gamma = \frac{\epsilon k c_s}{2(1 + k^2 d_e^2)^2}\left(1 \pm \frac{v_0}{c_s} \sqrt{1 + k^2 d_e^2}\right),
\]

(37)

which is stable if

\[
1 + k^2 d_e^2 > \frac{v_0^2}{c_s^2}.
\]  

(38)

It also was shown that for the short systems \((d_e \gg L)\) mathematical model is equivalent to the Pirce instability case.

Analytical theory for both cases was compared to results of direct initial value numerical simulation.

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FIG. 7: Evolution of Gaussian initial condition for the case of week dispersion. In Figure 7a—initial conditions (IC); Figure 7b—IC is splinting into two traveling wave packets, the fast one to the right with velocity $v_0 + c_s = 1.9$ and the slow one to the left with velocity $v_0 - c_s = -0.1$; Figures 7c, 7d—the fast wave packet is passing through the right wall barely reflecting; Figures 7e, 7f—the slow one starting to reflect from the left wall and to form an unstable eigen function.
FIG. 8: Evolution of Gaussian initial condition for the case of strong dispersion. In Figure 8a - initial conditions; Figure 8b - the initial Gauss travels with velocity $v_0 = 0.025$ to the right and decreasing in size, at the same time another pike start to grow and travel to the right with the same velocity; Figures 8c, 8d - the initial pike passes through the left boundary while new pike forms the unstable eigen function.