A vector equilibrium problem for the two-matrix model in the quartic/quadratic case

Maurice Duits\(^1\), Dries Geudens\(^2\) and Arno B J Kuijlaars\(^2\)

\(^1\) California Institute of Technology, Mathematics 253-37, Pasadena, CA 91125, USA
\(^2\) Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, B-3001 Leuven, Belgium

E-mail: mduits@caltech.edu, dries.geudens@wis.kuleuven.be and arno.kuijlaars@wis.kuleuven.be

Received 6 July 2010
Published 11 February 2011
Online at stacks.iop.org/Non/24/951

Recommended by A R Its

Abstract

We consider the two sequences of biorthogonal polynomials \((p_{k,n})_{k=0}^{\infty}\) and \((q_{k,n})_{k=0}^{\infty}\) related to the Hermitian two-matrix model with potentials \(V(x) = x^2/2\) and \(W(y) = y^4/4 + ty^2\). From an asymptotic analysis of the coefficients in the recurrence relation satisfied by these polynomials, we obtain the limiting distribution of the zeros of the polynomials \(p_{n,n}\) as \(n \to \infty\). The limiting zero distribution is characterized as the first measure of the minimizer in a vector equilibrium problem involving three measures which for the case \(t = 0\) reduces to the vector equilibrium problem that was given recently by two of us. A novel feature is that for \(t < 0\) an external field is active on the third measure which introduces a new type of critical behaviour for a certain negative value of \(t\).

We also prove a general result about the interlacing of zeros of biorthogonal polynomials.

Mathematics Subject Classification: 31A05, 42C05, 60B20

1. Introduction

1.1. Two-matrix model and biorthogonal polynomials

The Hermitian two-matrix model is a probability measure

\[
\frac{1}{Z_n} e^{-n\text{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} \, dM_1 \, dM_2
\]

(1.1)
defined on the space of pairs of \(n \times n\) Hermitian matrices \((M_1, M_2)\). Here, \(dM_1 \, dM_2\) is the standard Lebesgue measure on pairs of Hermitian matrices, \(V\) and \(W\) are the two potentials in
the model which are typically polynomials of even degree with a positive leading coefficient, \( \tau \neq 0 \) is a coupling constant and \( Z_n \) is the normalizing constant

\[
Z_n = \int \int e^{-n \text{Tr}(V(M_1) + W(M_2) - \tau M_1 M_2)} \, dM_1 \, dM_2.
\]

With the two-matrix model (1.1), we associate two sequences of monic polynomials \((p_k,n)_{k=0}^\infty\) and \((q_k,n)_{k=0}^\infty\), where \( \deg p_k,n = \deg q_k,n = k \), called biorthogonal polynomials, defined by the property

\[
\int \int p_k,n(x)q_{j,n}(y) e^{-n(V(x) + W(y) - \tau xy)} \, dx \, dy = 0, \quad k \neq j. \tag{1.2}
\]

Existence and uniqueness of these polynomials was proved by Ercolani and McLaughlin [10]. They also showed that the zeros of these polynomials are real and simple. It will be one of our results that the zeros interlace, see theorem 2.1.

It is well known that the eigenvalue correlations of the matrices \( M_1 \) and \( M_2 \) from (1.1) are determinantal with correlation kernels that can be expressed in terms of the biorthogonal polynomials and their transforms, see [2, 3, 10, 11, 15]. As an example of these relations we have that

\[
\mathbb{E} [\det(xI_n - M_1)] = p_{n,n}(x),
\]
\[
\mathbb{E} [\det(yI_n - M_2)] = q_{n,n}(y),
\]

which show that the diagonal biorthogonal polynomials \( p_{n,n} \) and \( q_{n,n} \) can be considered as ‘typical’ characteristic polynomials for \( M_1 \) and \( M_2 \), respectively. In this sense the zeros of \( p_{n,n} \) are typical eigenvalues of the matrix \( M_1 \). This explains our interest in these zeros as \( n \to \infty \).

1.2. A vector equilibrium problem

In [8] two of us studied the limiting eigenvalue behaviour of the matrix \( M_1 \) in the two-matrix model (1.1) for the case of an even polynomial \( V \) with positive leading coefficient and

\[
W(y) = \frac{1}{4} y^4.
\]

The biorthogonal polynomial \( p_{n,n} \) associated with this model can be characterized by a Riemann–Hilbert problem of size 4 \( \times \) 4 [3, 13]. The Deift–Zhou steepest descent method was successfully applied to the Riemann–Hilbert problem from [13]. A crucial ingredient in [8] is the introduction of a vector equilibrium problem with external field and upper constraint that describes the limiting mean eigenvalue distribution.

To state the vector equilibrium problem we use notions from logarithmic potential theory [17]. We define the logarithmic energy of a finite positive measure \( \nu \) on \( \mathbb{C} \) as

\[
I(\nu) = \iint \log \frac{1}{|x - y|} \, d\nu(x) \, d\nu(y),
\]

and the logarithmic potential as

\[
U^\nu(z) = \int \log \frac{1}{|z - x|} \, d\nu(x). \tag{1.3}
\]

If \( \nu_1 \) and \( \nu_2 \) are positive measures on \( \mathbb{C} \) with \( I(\nu_1), I(\nu_2) < \infty \), we also define their mutual logarithmic energy as

\[
I(\nu_1, \nu_2) = \iint \log \frac{1}{|x - y|} \, d\nu_1(x) \, d\nu_2(y).
\]
The vector equilibrium problem from [8] asks to minimize the energy functional

\[ I(\rho_1) - I(\rho_1, \rho_2) + I(\rho_2) - I(\rho_2, \rho_3) + I(\rho_3) + \int \left( V(x) - \frac{3}{4}|\tau x|^{4/3} \right) d\rho_1(x), \]

among all measures \(\rho_1, \rho_2\) and \(\rho_3\) with finite logarithmic energy that satisfy

(a) \(\rho_1\) is supported on \(\mathbb{R}\) and \(\rho_1(\mathbb{R}) = 1\);
(b) \(\rho_2\) is supported on \(i\mathbb{R}\) and \(\rho_2(i\mathbb{R}) = 2/3\);
(c) \(\rho_3\) is supported on \(\mathbb{R}\) and \(\rho_3(\mathbb{R}) = 1/3\);
(d) \(\rho_2\) satisfies the constraint \(\rho_2 \leq \sigma\) where \(\sigma\) is the unbounded measure on \(i\mathbb{R}\) defined as

\[ d\sigma(z) = \frac{\sqrt{3}}{2\pi^1} e^{4/3|z|^{1/3}} dz, \quad z \in i\mathbb{R}. \]

Here, \(dz\) is the complex line element on \(i\mathbb{R}\).

In [8] it is shown that there is a unique minimizer \((\nu_1, \nu_2, \nu_3)\) of the vector equilibrium problem. The measure \(\nu_1\) is supported on a finite union of disjoint intervals. For the case of one interval (one-cut case) it was shown in [8] that the density of \(\nu_1\) is the limiting mean density of the eigenvalues of \(M_1\). See [16] for the extension to the multi-cut case.

1.3. Aim of this paper

It is the aim of this paper to give a new perspective on the nature of the above vector equilibrium problem. In [8] the vector equilibrium problem was simply posed out of the blue, while in this work it arises after certain calculations.

In addition, we derive a similar vector equilibrium problem for the case that

\[ V(x) = \frac{x^2}{2} \quad \text{and} \quad W(y) = \frac{y^4}{4} + ty^2, \quad (1.4) \]

where \(t \in \mathbb{R}\) is a real parameter. When \(t < 0\) this vector equilibrium problem has the novel feature that an external field is acting on the third measure as well.

Due to the fact that \(V(x)\) is quadratic in (1.4) the second sequence \((q_{k,n})\) of biorthogonal polynomials is actually a sequence of orthogonal polynomials that satisfy a three term recurrence relation. The biorthogonal polynomials \(p_{k,n}\) also satisfy a recurrence relation with recurrence coefficients that can be expressed in terms of the recurrence coefficients for \(q_{k,n}\). We are going to analyse these recurrence coefficients as \(n \to \infty\) and obtain the vector equilibrium problem from this analysis. This approach is similar in spirit to the analysis of [14] for the recurrence coefficients of multiple orthogonal polynomials in a model of non-intersecting squared Bessel paths.

Furthermore, we prove that the first component \(\nu_1\) of the vector of measures \((\nu_1, \nu_2, \nu_3)\) minimizing this vector equilibrium problem is equal to the limiting zero distribution of the diagonal polynomials \(p_{n,n}\) as \(n \to \infty\). The measure \(\nu_1\) is also the limiting mean eigenvalue distribution of the matrices \(M_1\) in the two-matrix model. This is proved in a forthcoming paper [9].

In the next section we give a more detailed description of the main results in this paper.

2. Statement of main results

2.1. Interlacing zeros

Our first result deals with biorthogonal polynomials defined by (1.2) with general potentials \(V\) and \(W\). Our result is that the zeros of consecutive biorthogonal \(p_{k,n}\) and \(p_{k+1,n}\) are interlacing.
This was proved by Woerdeman [19] for a special case. Two ordered sequences of real numbers \( \alpha_1, \ldots, \alpha_k \) and \( \beta_1, \ldots, \beta_{k+1} \) are said to interlace if

\[ \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_{k-1} < \beta_k < \alpha_k < \beta_{k+1}. \]

**Theorem 2.1.** Take \( \tau \neq 0 \) and suppose that \( V \) and \( W \) are functions for which the integrals

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^j e^{-n(V(x)+W(y)-\tau xy)} \, dx \, dy, \quad k = 0, 1, 2, \ldots, \quad j = 0, 1, 2, \ldots \]

converge. Define the two sequences of monic biorthogonal polynomials \( (p_k,n)_{k=0}^{\infty} \) and \( (q_k,n)_{k=0}^{\infty} \) as in (1.2). Then the following statements hold for every \( k = 1, 2, \ldots \)

(a) The zeros of \( p_k,n \) and \( p_{k+1,n} \) interlace, and similarly, the zeros of \( q_k,n \) and \( q_{k+1,n} \) interlace.
(b) If the potentials \( V \) and \( W \) are even, then the positive zeros of \( p_k,n \) and \( p_{k+2,n} \) interlace, and similarly, the positive zeros of \( q_k,n \) and \( q_{k+2,n} \) interlace.

Theorem 2.1 is proved in section 3.

2.2. Limit of zero counting measures

In the rest of the paper we restrict ourselves to the quadratic and quartic potentials (1.4). The biorthogonal polynomials associated with this model are thus defined by

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{k,n}(x)q_{j,n}(y)e^{-n(x^2/2+y^4/4+xy^2/2-\tau xy)} \, dx \, dy = 0, \quad k \neq j. \]

We also assume without loss of generality that \( \tau > 0 \).

As a second result, we will show that in this case the limiting zero distribution of the diagonal polynomials \( p_{n,n} \) exists. For a polynomial \( P \) of degree \( n \), we introduce the normalized zero counting measure

\[ \nu(P) = \frac{1}{n} \sum_{P(x) = 0} \delta_x, \]

where the sum is taken over all zeros of \( P \) counted with multiplicity. We say that a sequence of measures \( \nu_n \) converges weakly to the measure \( \nu \) if

\[ \int f \, d\nu_n \to \int f \, d\nu, \]

for every bounded continuous function \( f \).

**Theorem 2.2.** There exists a Borel probability measure \( \nu_1 \) with \( \text{supp}(\nu_1) \subset \mathbb{R} \) such that \( \nu_1 \) is the limiting zero distribution of the diagonal polynomials \( p_{n,n} \), i.e.

\[ \lim_{n \to \infty} \nu(p_{n,n}) = \nu_1, \]

where the limit is in the sense of weak convergence of measures.

The proof of theorem 2.2 is given in section 7. It is constructive. For an explicit formula of the limiting zero distribution \( \nu_1 \), we refer to (7.1).

Note that theorem 2.2 is about the limiting distribution of the zeros of the biorthogonal polynomial \( p_{n,n} \). The measure \( \nu_1 \) is also the limiting mean distribution of the eigenvalues of the matrix \( M_1 \) from the two-matrix model, but this is not proved in this paper. This will follow from an analysis of the Riemann–Hilbert problem as in [8], which is presented in [9].
2.3. Vector equilibrium problem

Our final result is that the limiting zero distribution $\nu_1$ can be characterized by a vector equilibrium problem depending on external fields $V_1$ and $V_3$, and on a constraint $\sigma$. These objects will be described next in terms of the solutions of the equation

$$\omega^3 + t\omega = \tau z.$$  \hspace{1cm} (2.2)

For $t = 0$, the vector equilibrium problem reduces to the vector equilibrium problem described in section 1.2.

**External field $V_1$ on $\mathbb{R}$**. For $z = x \in \mathbb{R}$, equation (2.2) has either one or three real solutions. We use $\omega_1(x)$ to denote the real solution with the largest absolute value. This is the only real solution for $t \geq 0$ and also for $t < 0$ and $z = x$ with $|x| > x^*$ where

$$x^* = x^*(t) = \frac{2(-t)^{3/2}}{3\sqrt{3}\tau}, \quad t \leq 0.$$  \hspace{1cm} (2.3)

For $t < 0$ and $-x^* \leq x \leq x^*$ there are three real solutions of (2.2) which we denote by $\omega_j(x)$, $j = 1, 2, 3$, and which we number such that

$$|\omega_1(x)| \geq |\omega_2(x)| \geq |\omega_3(x)|.$$  \hspace{1cm} (2.4)

This means in fact that $\omega_2(x) < \omega_3(x) < 0 < \omega_1(x)$ if $x \in (0, x^*)$ and $\omega_1(x) < 0 < \omega_3(x) < \omega_2(x)$ if $x \in (-x^*, 0)$.

In both cases, the external field $V_1$ is defined by

$$V_1(x) = \begin{cases} \frac{x^2}{2} + \min_{y \in \mathbb{R}} (W(y) - \tau xy) \\ \frac{x^2}{2} - \frac{3}{4} \omega_1(x)^4 - \frac{1}{2} t\omega_1(x)^2, \quad x \in \mathbb{R}. \end{cases}$$  \hspace{1cm} (2.5)

The second identity in (2.5) comes from the fact that the minimum of $W(y) - \tau xy$ is taken at $y = \omega_1(x)$ and $\tau x = \omega_1(x)^3 + t\omega_1(x)$ by (2.2).

**External field $V_3$ on $\mathbb{R}$**. The external field $V_3$ vanishes identically for $t \geq 0$

$$V_3(x) \equiv 0, \quad \text{for } x \in \mathbb{R}, \text{ if } t \geq 0.$$  \hspace{1cm} (2.6)

For $t < 0$ and $x \in \mathbb{R}$, we define

$$V_3(x) = \begin{cases} \frac{3}{2} \omega_2(x)^4 + \frac{1}{2} t\omega_2(x)^2 - \frac{3}{4} \omega_3(x)^4 - \frac{1}{2} t\omega_3(x)^2, \quad \text{for } |x| < x^*, \\ 0, \quad \text{for } |x| \geq x^*. \end{cases}$$  \hspace{1cm} (2.7)

where $x^*$ is given in (2.3), and $\omega_2(x)$ and $\omega_3(x)$ are the solutions of (2.2) that satisfy (2.4).

While $V_1(x)$ is related to the global minimum of the function

$$W(y) - \tau xy = \frac{y^4}{4} + \frac{t y^2}{2} - \tau xy, \quad y \in \mathbb{R},$$  \hspace{1cm} (2.8)

$V_3(x)$ can be interpreted as the positive difference between the local maximum and the other local minimum of (2.8) on $\mathbb{R}$, which indeed exist if and only if $t < 0$ and $|x| < x^*$. 
Upper constraint $\sigma$ on $i\mathbb{R}$. For $z = iy \in i\mathbb{R}$, equation (2.2) has either one or three purely imaginary solutions. There are three purely imaginary solutions if and only if $t \geq 0$ and $|y| \leq y^*$ where

$$y^* = y^*(t) = \frac{2t^{1/2}}{3\sqrt{3}t}, \quad t > 0.$$  \hspace{1cm} (2.9)

Otherwise there is only one purely imaginary solution and the two other solutions are located symmetrically with respect to the imaginary axis. We then let $\omega_1(z)$ be the solution of (2.2) with positive real part. For convenience we put

$$y^* = 0, \quad \text{if } t < 0.$$  \hspace{1cm} (2.9)

The upper constraint $\sigma$ is defined as follows. The support of $\sigma$ is

$$\text{supp}(\sigma) = (-\infty, -iy^*] \cup [iy^*, i\infty),$$  \hspace{1cm} (2.10)

and $\sigma$ has the density on $\text{supp}(\sigma)$ given by

$$\frac{d\sigma(z)}{|dz|} = \frac{\tau}{\pi} \text{Re} \omega_1(z), \quad z \in \text{supp}(\sigma),$$  \hspace{1cm} (2.11)

for every fixed $t \in \mathbb{R}$.

Now we state our final main result.

**Theorem 2.3.** The measure $\nu_1$ from theorem 2.2 is the first component of the unique vector of measures $(\nu_1, \nu_2, \nu_3)$ minimizing the energy functional

$$E(\rho_1, \rho_2, \rho_3) = I(\rho_1) - I(\rho_1, \rho_2) + I(\rho_2) - I(\rho_2, \rho_3) + I(\rho_3)$$

$$+ \int V_1(x) \, d\rho_1(x) + \int V_3(x) \, d\rho_3(x),$$

among all vectors $(\rho_1, \rho_2, \rho_3)$ of measures with finite logarithmic energy satisfying

(a) $\rho_1$ is supported on $\mathbb{R}$ and $\rho_1(\mathbb{R}) = 1$,
(b) $\rho_2$ is supported on $i\mathbb{R}$ and $\rho_2(i\mathbb{R}) = 2/3$,
(c) $\rho_3$ is supported on $\mathbb{R}$ and $\rho_3(\mathbb{R}) = 1/3$,
(d) $\rho_2$ satisfies the constraint $\rho_2 \leq \sigma$.

Here $V_1$ and $V_3$ are defined in (2.5) and (2.6)–(2.7), respectively, and $\sigma$ is defined by (2.10)–(2.11).

Theorem 2.3 is proved in section 8.

### 2.4. Phase diagram and critical behaviour

The proof of theorem 2.3 is constructive, and we find fairly explicit formulae for the minimizing measures $\nu_j$. Indeed, we obtain $\nu_j$ as an average

$$\nu_j = \int_0^1 \mu_j^\xi \, d\xi$$

of measures $\mu_j^\xi$ depending on a parameter $\xi$ and these measures are given by formulae (5.15) and (6.15).

It follows from the analysis leading to these formulae that the supports of the measures $\nu_1, \sigma - \nu_2$ and $\nu_3$ have the following form:

$$\text{supp}(\nu_1) = [-\alpha, -\beta] \cup [\beta, \alpha],$$

$$\text{supp}(\sigma - \nu_2) = i\mathbb{R} \setminus (-iy, iy),$$

$$\text{supp}(\nu_3) = \mathbb{R} \setminus (-\delta, \delta),$$

for some $\alpha > \beta \geq 0, \gamma, \delta \geq 0$ depending on $t \in \mathbb{R}$ and $\tau > 0$. 

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Figure 1. The phase diagram in the $tt$-plane: the critical curves $\tau = \sqrt{t + 2}$ and $\tau = \sqrt{-1/t}$ separate the four cases. The cases are distinguished by the fact whether 0 is in the support of the measures $\nu_1, \sigma - \nu_2$ and $\nu_3$, or not.

We may distinguish a number of cases, depending on whether $\beta, \gamma$, or $\delta$ are equal to zero, or not. At least one of these is zero, and generically, no two consecutive ones are zero. Our analysis leads to the phase diagram in the $tt$-plane shown in figure 1.

Case I. $\beta = 0, \gamma > 0$ and $\delta = 0$. Thus in this case there are no gaps in the supports of the measures $\nu_1$ and $\nu_3$ on the real line. The constraint $\sigma$ is active along an interval $[-iy, iy]$ on the imaginary axis.

Case II. $\beta > 0, \gamma > 0$ and $\delta = 0$. In case II there is a gap in the support of $\nu_1$, but there is no gap in the support of $\nu_3$, which is the full real line. The constraint is active along an interval along the imaginary axis.

Case III. $\beta > 0, \gamma = 0$ and $\delta > 0$. In case III there is a gap in the supports of $\nu_1$ and $\nu_3$, but the constraint on the imaginary axis is not active.

Case IV. $\beta = 0, \gamma > 0$ and $\delta > 0$. In this case the measure $\nu_1$ is supported on one interval. However, there is a gap $(-\delta, \delta)$ in the support of $\nu_3$. The constraint $\sigma$ is active along an interval $[-iy, iy]$ on the imaginary axis.

Cases III and IV are new in the sense that they do not appear in [8]. The opening of a gap in the support of $\nu_3$ is due to the external field $V_3$ that acts on the third measure in the vector equilibrium problem. As $V_3$ is identically zero for $t \geq 0$, cases III and IV do not appear if $t \geq 0$, as can be seen in figure 1.

Critical behaviour occurs at the curves that separate the different cases from each other. These critical curves are given by the equations

$$\tau = \sqrt{t + 2}, \quad -2 \leq t < \infty \quad \text{and} \quad \tau = \sqrt{-1/t}, \quad -\infty < t < 0.$$  

On the critical curves two of the numbers $\beta, \gamma$ and $\delta$ are equal to zero. For example, on the curve between case III and case IV, we have $\beta = \gamma = 0$, while $\delta > 0$. Finally, note the multi-critical point

$$t = -1, \quad \tau = 1$$  

in the phase diagram, where $\beta = \gamma = \delta = 0$. All four cases come together at this point in the $tt$-plane.

We do not discuss the critical and multi-critical behaviour any further in this paper. However, it would be particularly interesting to analyse the nature of the multi-critical point.
2.5. Overview of the rest of the paper

The rest of the paper is devoted to the proofs of the three main results. Theorem 2.1 is proved in section 3. The other sections deal with the specific model (1.4) and lead to the proofs of theorems 2.2 and 2.3 in sections 7 and 8. The intermediate sections contain auxiliary results that will be essential to these proofs.

In section 4 it is shown that the biorthogonal polynomials satisfy recurrence relations with recurrence coefficients that have certain asymptotic behaviours. We introduce a new parameter $\xi$ and consider the asymptotic behaviour in the $t\xi$-phase space. Two different types of asymptotic behaviour will lead to a separation of the phase space into two regions $C_1$ and $C_2$, where $C_1$ is the one-cut case region and $C_2$ the two-cut case region.

With every $\xi$-value, we associate a vector equilibrium problem for three measures. These equilibrium problems serve as building blocks for the vector equilibrium problem of theorem 2.3. We introduce and analyse the one-cut case in section 5 and the two-cut case in section 6. Due to the different asymptotic behaviour of recurrence coefficients, the analysis in both sections is significantly different.

3. Proof of theorem 2.1

In this section we prove theorem 2.1. The proof of this theorem is inspired by [1] and uses the following theorem from [10].

**Theorem 3.1.** Suppose that $\tau \neq 0$. Let $V$ and $W$ be functions for which the integrals
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^j e^{-n(V(x)+W(y) - \tau xy)} \, dx \, dy, \quad k = 0, 1, 2, \ldots, \quad j = 0, 1, 2, \ldots
\]
converge. Then, for each $k$, there is a unique monic polynomial $p_{k,n}$ and a unique monic polynomial $q_{k,n}$ such that the families of polynomials $(p_{k,n})_{k,n=0}^{\infty}$ and $(q_{k,n})_{k,n=0}^{\infty}$ satisfy (1.2). In addition, the zeros of these polynomials are real and simple.

**Proof.** This is [10, theorem 1]. \[\square\]

To establish theorem 2.1 (a) it is clearly enough to prove the statements about the zeros of $p_{k,n}$, since the results about the zeros of $q_{k,n}$ follow by symmetry.

**Proof of theorem 2.1 (a).** Fix an integer $k \geq 1$ and consider the linear combination $A p_{k,n} + B p_{k+1,n}$ with $(A, B) \neq (0, 0)$. We claim that this polynomial has no real multiple zeros.

To see this, assume that $x_0$ is a real zero of multiplicity at least two. Then we can write
\[
A p_{k,n}(x) + B p_{k+1,n}(x) = (x - x_0)^2 r(x),
\]
where $r$ is polynomial of degree $\leq k - 1$. From the biorthogonality (1.2) it follows that
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(x) y^l (x - x_0)^2 e^{-n(V(x)+W(y) - \tau xy)} \, dx \, dy
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(x) y^l e^{-n(V(x)) - \frac{2}{\tau} \log |x-x_0| + W(y) - \tau xy)} \, dx \, dy = 0,
\]
for $l = 0, 1, 2, \ldots, k - 1$. Applying the uniqueness part of theorem 3.1 to the modified potential $V(x) - (2/n) \log |x - x_0|$, we obtain $r \equiv 0$. Thus, $A = B = 0$, which yields
a contradiction and, therefore, proves the claim that \(Apk,n + Bpk+1,n\) has no real multiple zeros if \((A, B) \neq (0, 0)\). It follows that the linear system of equations
\[
\begin{pmatrix}
    pk,n(x) & pk+1,n(x) \\
   (pk,n(x) & pk+1,n(x)
\end{pmatrix}
\begin{pmatrix}
    A \\
    B
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0
\end{pmatrix}
\]
has only the trivial solution \(A = B = 0\), for every \(x \in \mathbb{R}\). Therefore, the matrix has a non-zero determinant, and thus
\[
pk,n(x)p'_{k+1,n}(x) - p'_{k,n}(x)pk+1,n(x) \neq 0, \quad x \in \mathbb{R}.
\]
By continuity and the behaviour as \(x \to \infty\), we conclude from this that
\[
pk,n(x)p'_{k+1,n}(x) - p'_{k,n}(x)pk+1,n(x) > 0, \quad x \in \mathbb{R}.
\]
Now consider two consecutive zeros \(x_l\) and \(x_{l+1}\) of \(pk_{k+1,n}\). Because these zeros are simple, we have that
\[
pk,n(x_l)p'_{k+1,n}(x_{l+1}) < 0.
\]
From (3.1) we find \(pk,n(x_l)p'_{k+1,n}(x_l) > 0\) and \(pk,n(x_{l+1})p'_{k+1,n}(x_{l+1}) > 0\). Hence, we obtain
\[
pk,n(x_l)p_{k,n}(x_{l+1}) < 0.
\]
Therefore, \(pk,n\) must have a zero between \(x_l\) and \(x_{l+1}\). Hence, in between any two consecutive zeros of \(pk_{k+1,n}\), there is a zero of \(pk,n\), which implies that the zeros of \(pk,n\) and \(pk_{k+1,n}\) interlace.
\]
**Proof of theorem 2.1 (b).** The proof of (b) follows the same strategy. Let \(k \geq 1\) and assume that the linear combination \(Apk,n + Bpk+2,n\) has a positive multiple root, say \(x_0 > 0\). Because \(V\) and \(W\) are even potentials, \(pk,n\) and \(pk+2,n\) are either both even or both odd. Therefore, also \(-x_0\) is a double zero and we can write
\[
Apk,n(x) + Bpk+2,n(x) = (x^2 - x_0^2)^2 r(x),
\]
where \(r\) is polynomial of degree \(\leq k - 2\). Then, \(r \equiv 0\) and, thus, \(A = B = 0\) as in the proof of part (a).

Hence \(Apk,n + Bpk+2,n\) with \((A, B) \neq (0, 0)\) has no positive multiple zeros. Therefore, the linear system of equations
\[
\begin{pmatrix}
    pk,n(x) & pk+2,n(x) \\
pk,n(x) & pk+2,n(x)
\end{pmatrix}
\begin{pmatrix}
    A \\
    B
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0
\end{pmatrix}
\]
has only the trivial solution for every \(x > 0\). Thus, as in the proof of part (a),
\[
pk,n(x)p'_{k+2,n}(x) - p'_{k,n}(x)pk+2,n(x) > 0, \quad x > 0.\]
(3.2)
The proof of interlacing of the positive zeros follows from (3.2) in the same way as before. □

**Remark.** It was shown in [10] that the zeros of biorthogonal polynomials are real and simple. We can prove this result in an alternative way as follows.

First assume that \(pk,n\) has a non-real zero \(x_0 = a + bi\), \(a, b \in \mathbb{R}, b \neq 0\). Then also \(\overline{x}_0 = a - bi\) is a zero of \(pk,n\). Thus, \(pk,n\) can be written as
\[
pk,n(x) = ((x - a)^2 + b^2)r(x),
\]
where \(r\) is a polynomial of degree \(k - 2\). Now observe that
\[
\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} y^l ((x - a)^2 + b^2) r(x) e^{-n(V(x)+W(y)-\tau xy)} dx dy
\right)
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(x) y^l e^{-n(V(x)+\frac{1}{2}\log((x-a)^2+b^2)+W(y)-\tau xy)} dx dy = 0,
\]
for \( l = 0, 1, 2, \ldots, k - 1 \). Note that the second equality follows from (1.2). Applying the uniqueness part of theorem 3.1 to \( W(y) \) and the modified potential \( V(x) - (1/n) \log((x - a)^2 + b^2) \) we obtain \( r \equiv 0 \) which is a contradiction. Therefore, the zeros of \( p_{k,n} \) are real. Moreover, by putting \( b = 0 \) in the above argument we also obtain that the zeros are simple.

Of course, a similar reasoning shows that the zeros of \( q_{k,n} \) are real and simple.

4. Preliminaries on recurrence coefficients

4.1. Relations between recurrence coefficients

In the rest of the paper we consider the model with quartic and quadratic potentials (1.4). In that case both sequences \((p_{k,n})_k\) and \((q_{k,n})_k\) of biorthogonal polynomials defined by (2.1) satisfy a recurrence relation with recurrence coefficients that are related to each other as described in the following lemma.

Lemma 4.1. For \( n \geq 1 \), the sequence of polynomials \((q_{k,n})_{k=0}^\infty\) is orthogonal with respect to the weight

\[
w(y) = e^{-ny^4/4 + ty^2/2 - \tau^2 y^2/2}, \quad y \in \mathbb{R},
\]

and, therefore, satisfies a recurrence relation of the form

\[
yq_{k,n}(y) = q_{k+1,n}(y) + a_{k,n}q_{k-1,n}(y), \quad k = 0, 1, 2, \ldots,
\]

where \( q_{-1,n} \equiv 0, a_{0,n} = 0 \) and \( a_{k,n} > 0, k = 1, 2, \ldots \).

In addition, the sequence of polynomials \((p_{k,n})_{k=0}^\infty\) satisfies a recurrence relation of the form

\[
xp_{k,n}(x) = p_{k+1,n}(x) + b_{k,n}p_{k-1,n}(x) + c_{k,n}p_{k-3,n}(x), \quad k = 0, 1, 2, \ldots,
\]

where \( p_{-1,n} \equiv p_{-2,n} \equiv p_{-3,n} \equiv 0 \). The recurrence coefficients are related as follows

\[
b_{k,n} = a_{k,n}(a_{k-1,n} + a_{k,n} + a_{k+1,n} + t) = \tau^2 a_{k,n} + \frac{k}{n}, \quad k \geq 1,
\]

\[
c_{k,n} = \tau^2 a_{k-2,n}a_{k-1,n}a_{k,n}, \quad k \geq 3.
\]

Proof. The lemma is known in much greater generality for general polynomial potentials \( V \) and \( W \), see [2]. Explicit calculations that lead to (4.2), (4.3) and (4.4) for the case \( t = 0 \) are given in [19]. These calculations extend to general \( t \in \mathbb{R} \) in a straightforward way.

4.2. Asymptotic behaviour of recurrence coefficients

Our next result concerns the asymptotic behaviour of the recurrence coefficients \( a_{k,n}, b_{k,n} \) and \( c_{k,n} \) as \( k, n \to \infty \) such that \( k/n \to \xi > 0 \). Let us introduce some notation. We write

\[
\lim_{k/n \to \xi} X_{k,n} = X,
\]

if \( \lim_{k/n \to \xi} X_{k,n} = X \) holds for every two sequences of positive integers \((k_j)\) and \((n_j)\) that satisfy \( k_j/n_j \to \infty \) and \( k_j/n_j \to \xi \) as \( j \to \infty \). In the same spirit we write

\[
\lim_{k/n \to \xi} X_{k,n} = X
\]

if \( k \) even.
if $\lim_{j \to \infty} X_{k_j,n_j} = X$ holds for every sequence of positive even integers $(k_j)$ and every sequence of positive integers $(n_j)$ that satisfy $k_j, n_j \to \infty$ and $k_j/n_j \to \xi$ if $j \to \infty$. The limit with the subscript 'odd' is defined similarly.

The limiting behaviour of the recurrence coefficients $a_{k,n}, b_{k,n}, c_{k,n}$ as $k, n \to \infty, k/n \to \xi$ depends on the values of $t \in \mathbb{R}$ and $\xi > 0$. We consider the coupling constant $\tau > 0$ fixed. We define the critical $\xi$-values

$\xi_{cr} = \begin{cases} 
\frac{1}{4} (\tau^2 - t)^2, & \text{if } t < \tau^2, \\
0, & \text{if } t \geq \tau^2.
\end{cases}$

In the $t\xi$-plane the equation $\xi = \xi_{cr}$, $t < \tau^2$, defines a semi-parabola that separates the upper half of the $t\xi$-plane into two regions

$C_1: \xi > \xi_{cr}$, $-\infty < t < \tau^2$,

$C_2: 0 < \xi < \xi_{cr}$,

see figure 2. We refer to $C_1$ as the one-cut case region, since the zeros of the orthogonal polynomials $q_{k,n}$ accumulate on one interval as $k, n \to \infty$ and $k/n \to \xi > \xi_{cr}$. If $t < \tau^2$ and $\xi \in (0, \xi_{cr})$ the zeros of $q_{k,n}$ accumulate on two disjoint intervals and therefore we call $C_2$ the two-cut case.

We now state the following theorem.

**Theorem 4.2.**

(a) If $\xi > \xi_{cr}$ then the limits of $a_{k,n}, b_{k,n}, c_{k,n}$ as $k/n \to \xi$ exist and we have

$$
\lim_{k/n \to \xi} a_{k,n} = a(\xi) := \frac{\tau^2 - t + \sqrt{(\tau^2 - t)^2 + 12\xi}}{6},
$$

(4.5)

$$
\lim_{k/n \to \xi} b_{k,n} = b(\xi) := a(\xi)(3a(\xi) + t) = \tau^2 a(\xi) + \xi,
$$

(4.6)

$$
\lim_{k/n \to \xi} c_{k,n} = c(\xi) := \tau^2 a^3(\xi).
$$

(4.7)

(b) If $t < \tau^2$ and $0 < \xi < \xi_{cr}$, then the recurrence coefficients $a_{k,n}, b_{k,n}, c_{k,n}$ exhibit 2-periodic behaviour as $k/n \to \xi$ and we have

$$
\lim_{k/n \to \xi, k \text{ even}} a_{k,n} = a_0(\xi) := \frac{\tau^2 - t - \sqrt{(\tau^2 - t)^2 - 4\xi}}{2},
$$

(4.8)

$$
\lim_{k/n \to \xi, k \text{ odd}} a_{k,n} = a_1(\xi) := \frac{\tau^2 - t + \sqrt{(\tau^2 - t)^2 - 4\xi}}{2},
$$

(4.9)

$$
\lim_{k/n \to \xi, k \text{ even}} b_{k,n} = b_0(\xi) := a_0(\xi)(a_0(\xi) + 2a_1(\xi) + t),
$$

(4.10)

$$
\lim_{k/n \to \xi, k \text{ odd}} b_{k,n} = b_1(\xi) := a_1(\xi)(2a_0(\xi) + a_1(\xi) + t),
$$

(4.11)

$$
\lim_{k/n \to \xi, k \text{ even}} c_{k,n} = c_0(\xi) := \tau^2 a_0^2(\xi)a_1(\xi),
$$

(4.12)

$$
\lim_{k/n \to \xi, k \text{ odd}} c_{k,n} = c_1(\xi) := \tau^2 a_0(\xi)a_1^2(\xi).
$$

(4.13)
Figure 2. $tξ$-phase diagram: the semi-parabola separates the one-cut case region $C_1$ from the two-cut case region $C_2$.

(c) If $t < τ^2$ and $ξ = ξ_{cr}$, then

\[ a(ξ) = a_0(ξ) = a_1(ξ), \quad b(ξ) = b_0(ξ) = b_1(ξ), \quad c(ξ) = c_0(ξ) = c_1(ξ), \]

and all of the above limit relations continue to hold for $ξ = ξ_{cr}$.

**Proof.** The recurrence coefficients $a_{k,n}$ appear in a recurrence relation for the orthogonal polynomials $p_{k,n}$. These polynomials are orthogonal with respect to the weight (4.1). For this type of orthogonal polynomials the asymptotic behaviour of recurrence coefficients was studied by Bleher and Its in two papers. The paper [4] deals with the two-cut case $0 < ξ < ξ_{cr}$. In [5] the critical case $ξ = ξ_{cr}$ is studied and the results for the one-cut case $ξ > ξ_{cr}$ are given as well.

Having (4.5), (4.8) and (4.9), the limits, (4.6), (4.7) and (4.10)–(4.13) follow directly from lemma 4.1. □

5. Asymptotic analysis in one-cut case

5.1. Results from the literature

In what follows we will associate with each $ξ > 0$ a function of the form

\[ s(w) = w + d^{(0)} + \frac{d^{(1)}}{w} + \frac{d^{(2)}}{w^2} + \frac{d^{(3)}}{w^3}, \quad d^{(3)} \neq 0. \]

Such functions appear as symbols of banded Toeplitz matrices [6], and we need certain results [7, 14] that were derived in that context. Although we will not use Toeplitz matrices in this paper, we still refer to $s$ as the symbol.

We denote the solutions of the algebraic equation $s(w) = z$ by $w_j(z)$, $j = 1, \ldots, 4$ and order them by their absolute value, such that

\[ |w_1(z)| \geq |w_2(z)| \geq |w_3(z)| \geq |w_4(z)| > 0. \]

Typically, there is strict inequality in (5.2). If for certain $z \in \mathbb{C}$ two solutions have the same absolute value, then we pick an arbitrary numbering that satisfies (5.2). Furthermore, we define

\[ \Gamma_j = \{ z \in \mathbb{C} \mid |w_j(z)| = |w_{j+1}(z)| \}, \quad j = 1, 2, 3, \]

which are finite unions of analytic arcs and exceptional points, see [6, 7]. A point $z \in \mathbb{C}$ for which the algebraic equation $s(w) = z$ has a multiple solution is called a branch point.
We use the solutions $w_j(z)$ to the algebraic equation to define three Borel measures

\[ d\mu_j(z) = \frac{1}{2\pi i} \sum_{k=1}^{j} \left( \frac{w'_k(z)}{w_k(z)} - \frac{w'_k(z)}{w_k(z)} + \frac{w'_k(z)}{w_k(z)} + \frac{w'_k(z)}{w_k(z)} \right) dz, \tag{5.4} \]

for $z \in \Gamma_j$, $j = 1, 2, 3$. Here, it is assumed that every analytic arc of $\Gamma_j$ is provided with an orientation and that $dz$ denotes the complex line element on $\Gamma_j$ according to this orientation. Furthermore, $w_k(\pm z)$ is the limiting value of $w_k(\tilde{z})$ as $\tilde{z} \to z$ from the $\pm$ side on each of the arcs in $\Gamma_j$. The $+$ side ($-$ side) is on the left (right) if one traverses $\Gamma_j$ according to the orientation.

The vector of measures $(\mu_1, \mu_2, \mu_3)$ is characterized as the unique minimizer of a vector equilibrium problem.

**Theorem 5.1.** Define the energy functional $E_0$ as

\[ E_0(\rho_1, \rho_2, \rho_3) = I(\rho_1) - I(\rho_1, \rho_2) + I(\rho_2) - I(\rho_2, \rho_3) + I(\rho_3), \]

where $\rho_1, \rho_2, \rho_3$ are positive measures on $\mathbb{C}$ with finite logarithmic energy. Then the following statements hold.

(a) The vector of measures $(\mu_1, \mu_2, \mu_3)$ given by (5.4) is the unique minimizer for the functional $E_0$ among all vectors $(\rho_1, \rho_2, \rho_3)$ of positive measures with finite logarithmic energy, satisfying

\begin{align*}
&\text{(i)} \supp(\rho_j) \subset \Gamma_j, \text{ for } j = 1, 2, 3 \text{ and } \\
&\text{(ii)} \rho_1(\Gamma_1) = 1, \rho_2(\Gamma_2) = 2/3 \text{ and } \rho_3(\Gamma_3) = 1/3.
\end{align*}

(b) The measures $\mu_1, \mu_2, \mu_3$ satisfy for some constant $\ell$

\begin{align*}
\ell &= -2U^{\mu_1}(z) + U^{\mu_3}(z) = \log \left| \frac{w_1(z)}{w_2(z)} \right|, \tag{5.5} \\
U^{\mu_1}(z) - 2U^{\mu_2}(z) + U^{\mu_3}(z) &= \log \left| \frac{w_2(z)}{w_3(z)} \right|, \tag{5.6} \\
U^{\mu_2}(z) - 2U^{\mu_3}(z) &= \log \left| \frac{w_3(z)}{w_4(z)} \right|, \tag{5.7}
\end{align*}

for every $z \in \mathbb{C}$.

**Proof.** The proof of theorem 5.1 can be found in [7]. The conditions (5.5)–(5.7) are the Euler–Lagrange variational conditions for the vector equilibrium problem. Note that the right-hand side of the $j$th variational condition vanishes if $z \in \Gamma_j$. In [7] there also appear constants $\ell_2$ and $\ell_3$ in (5.6) and (5.7). However, these constants vanish because $\Gamma_2$ and $\Gamma_3$ are unbounded and $U^{\mu_j}(z) = -\mu_j(\Gamma_j) \log |z| + o(1),$ as $z \to \infty$. \hfill $\square$

If the symbol (5.1) depends on a parameter $\xi > 0$, say

\[ s(w; \xi) = w + d^{(0)}(\xi) + \frac{d^{(1)}(\xi)}{w} + \frac{d^{(2)}(\xi)}{w^2} + \frac{d^{(3)}(\xi)}{w^3}, \tag{5.8} \]

then we use $w_j(z; \xi)$, $\Gamma_j(\xi)$ and $\mu_j^\xi$ to indicate the dependence of the notions from (5.2), (5.3) and (5.4), respectively, on the parameter $\xi$.

Next, we state a result of Kuijlaars and Román [14] on polynomials satisfying certain recurrence relations. It will be the key ingredient of the proof of theorem 2.2 given in section 7.
Theorem 5.2. Let for each \( n \in \mathbb{N} \) a sequence of monic polynomials \((p_{k,n})_{k=0}^{\infty}\) be given where \( \deg p_{k,n} = k \). Furthermore, suppose that

(a) these polynomials satisfy the recurrence relations
\[
x p_{k,n}(x) = p_{k+1,n}(x) + d^{(0)}_{k,n} p_{k,n} + d^{(1)}_{k,n} p_{k-1,n} + d^{(2)}_{k,n} p_{k-2,n} + d^{(3)}_{k,n} p_{k-3,n},
\]
for certain real recurrence coefficients \( d^{(j)}_{k,n} \), \( j = 0, 1, 2, 3 \);

(b) the polynomials \( p_{k,n} \) have real and simple zeros \( x_{1,n}^{k,n} < \cdots < x_{k,n}^{k,n} \) satisfying for each \( k \) and \( n \) the interlacing property
\[
x_{j+1}^{k+1,n} < x_{j,n}^{k,n} < x_{j+1}^{k+1,n}, \quad \text{for } j = 1, \ldots, k;
\]

(c) for each \( j = 0, 1, 2, 3 \) the set of recurrence coefficients
\[
\{d^{(j)}_{k,n} \mid k + 1 \leq n\}
\]
is bounded;

(d) there exist continuous functions \( d^{(j)} : (0, +\infty) \to \mathbb{R} \), \( j = 0, 1, 2, 3 \), such that for each \( \xi > 0 \)
\[
\lim_{k/n \to \xi} d^{(j)}_{k,n} = d^{(j)}(\xi),
\]
and \( d^{(3)}(\xi) \neq 0 \); 

(e) we have
\[
\Gamma_1(\xi) \subset \mathbb{R}, \quad \text{for every } \xi > 0
\]
where \( \Gamma_1(\xi) \) is the set defined as in (5.3) corresponding to the \( \xi \)-dependent function (5.8) with \( d^{(1)}(\xi) \) coming from (5.10).

Then, the normalized zero counting measures \( \nu(p_{k,n}) \) have a weak limit as \( k, n \to \infty \) with \( k/n \to \lambda > 0 \) given by
\[
\lim_{k/n \to \lambda} \nu(p_{k,n}) = \frac{1}{\lambda} \int_0^\lambda \mu_{\xi}^\lambda d\xi,
\]
where for each \( \xi > 0 \) the measure \( \mu_{\xi}^\lambda \) is given by (5.4) corresponding to the function (5.8).

Proof. This is [14, theorem 1.2] for the case of a five term recurrence (5.9). \( \square \)

The intuition behind theorem 5.2 is that the zeros of \( p_{k,n} \) are eigenvalues of a \( k \times k \) matrix with five non-zero diagonals
\[
\begin{pmatrix}
d^{(0)}_{0,n} & 1 & 0 & \cdots & \cdots & 0 \\
d^{(1)}_{1,n} & d^{(0)}_{1,n} & 1 & 0 & \cdots & \\
d^{(2)}_{2,n} & d^{(1)}_{2,n} & d^{(0)}_{2,n} & 1 & 0 & \cdots \\
d^{(3)}_{3,n} & d^{(2)}_{3,n} & d^{(1)}_{3,n} & d^{(0)}_{3,n} & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} \quad (5.12)
\]

Under the assumption (5.10) the entries are slowly varying along the diagonals if \( k \) and \( n \) are large, so that locally the matrix (5.12) looks like a five-diagonal Toeplitz matrix. Then for each \( \xi \) one considers the exact Toeplitz matrices with the entries \( d^{(3)}(\xi), d^{(2)}(\xi), d^{(1)}(\xi), d^{(0)}(\xi), 1 \).
along the diagonals for which it is known, see [6, 7, 12, 18], that the eigenvalues accumulate on \( \Gamma_j(\xi) \) as the size grows, with \( \mu_j^\xi \) as limiting normalized eigenvalue counting measure. The distribution of the eigenvalues of (5.12) is then obtained by averaging of the measures \( \mu_j^\xi \) as in (5.11).

5.2. Analysis of the symbol \( s_1 \) in the one-cut case

It will be our goal to apply theorem 5.2 to the biorthogonal polynomials \( p_{k,n} \) that have the recurrence relation (4.2). The recurrence coefficients in (4.2) have the appropriate limits only in the one-cut case. We discuss this case first.

We therefore assume that \( \xi > \xi_{cr} \). In that case we have by theorem 4.2 that the recurrence coefficients \( b_{k,n}, c_{k,n} \) have limits \( b(\xi), c(\xi) \) given by (4.6) and (4.7) as \( k, n \to \infty \) and \( k/n \to \xi \). We therefore associate with \( \xi > \xi_{cr} \) the symbol

\[
 s_1(w; \xi) = w + \frac{b(\xi)}{w} + \frac{c(\xi)}{w^3}. \tag{5.13}
\]

As already noted before, we use \( w_j(z; \xi) \) for \( j = 1, 2, 3, 4 \) and \( \Gamma_j(\xi), \mu_j^\xi \) for \( j = 1, 2, 3 \) to denote the quantities related to the symbol (5.13).

In order to apply theorem 5.2 we need to know that \( \Gamma_j(\xi) \subset \mathbb{R} \), see assumption (e) in theorem 5.2. Then the proof of theorem 2.3 follows the approach outlined in [14, section 7].

So, to obtain an external field \( V_j \) acting on \( \nu_j \) and an upper constraint \( \sigma \) acting on \( \nu_2 \) we will need that \( \Gamma_j(\xi) \) is an increasing and \( \Gamma_2(\xi) \) a decreasing set as a function of \( \xi \). These features are contained in the following theorem.

**Theorem 5.3.** Let \( \tau > 0 \) and \( t \in \mathbb{R} \). Then for every \( \xi > \xi_{cr} \) we have that

\[
 \Gamma_1(\xi) \subset \mathbb{R}, \quad \Gamma_2(\xi) \subset i\mathbb{R} \quad \text{and} \quad \Gamma_3(\xi) \subset \mathbb{R}.
\]

For \( \xi > \xi_{cr} \) the set \( \Gamma_1(\xi) \) is increasing as a function of \( \xi \), while the set \( \Gamma_2(\xi) \) is decreasing.

More precisely, there exist \( \alpha(\xi), \gamma(\xi) > 0 \) such that

\[
 \Gamma_1(\xi) = [-\alpha(\xi), \alpha(\xi)],
\]

\[
 \Gamma_2(\xi) = i\mathbb{R} \setminus (-i\gamma(\xi), i\gamma(\xi)),
\]

\[
 \Gamma_3(\xi) = \mathbb{R}. \tag{5.14}
\]

In addition, we have that

(a) \( \xi \mapsto \alpha(\xi) \) is strictly increasing for \( \xi > \xi_{cr} \) with

\[
 \lim_{\xi \to +\infty} \alpha(\xi) = +\infty \quad \text{and} \quad \lim_{\xi \to 0^+} \alpha(\xi) = 0 \quad \text{if} \quad t \geq \tau^2,
\]

(b) \( \xi \mapsto \gamma(\xi) \) is strictly increasing for \( \xi > \xi_{cr} \) with

\[
 \lim_{\xi \to +\infty} \gamma(\xi) = +\infty \quad \text{and} \quad \lim_{\xi \to 0^+} \gamma(\xi) = y^* \quad \text{if} \quad t \geq \tau^2,
\]

see (2.9) for the definition of \( y^* \).

Figure 3 shows the sets \( \Gamma_1(\xi), \Gamma_2(\xi) \) and \( \Gamma_3(\xi) \) in the complex plane in the one-cut case.

The proof of theorem 5.3 is given in the next subsection. Here we note that as a consequence of theorem 5.3 two consecutive sets among \( \Gamma_1(\xi), \Gamma_2(\xi) \) and \( \Gamma_3(\xi) \) are not overlapping, which implies that (5.4) can be written more simply as

\[
 d\mu_j^\xi(z) = \frac{1}{2\pi i} \left( \frac{w'_j(z; \xi)}{w_j(z; \xi)} - \frac{w'_{j*}(z; \xi)}{w_{j*}(z; \xi)} \right) dz, \quad z \in \Gamma_j(\xi), \quad j = 1, 2, 3. \tag{5.15}
\]
Figure 3. The sets $\Gamma_1(\xi)$ (plain), $\Gamma_2(\xi)$ (dashed) and $\Gamma_3(\xi)$ (dotted) in the one-cut case. We have that $\Gamma_1(\xi) = [-\alpha(\xi), \alpha(\xi)]$, $\Gamma_2(\xi) = (-i\infty, -i\gamma(\xi)] \cup [i\gamma(\xi), i\infty)$ and $\Gamma_3(\xi) = \mathbb{R}$.

5.3. Proof of theorem 5.3

5.3.1. Branch points. As a first step to the proof of theorem 5.3 we calculate the branch points for the algebraic equation $s_1(w; \xi) = z$.

Lemma 5.4. Let $\xi > \xi_{cr}$. Define $u(\xi), v(\xi) > 0$ such that

\begin{align*}
u(\xi)^2 &= \frac{\tau^2 a(\xi) + \xi + \sqrt{(\tau^2 a(\xi) + \xi)^2 + 12\tau^2 a_1(\xi)^2}}{2} > 0, \quad (5.16) \\
v(\xi)^2 &= \frac{\tau^2 a(\xi) + \xi - \sqrt{(\tau^2 a(\xi) + \xi)^2 + 12\tau^2 a_1(\xi)^2}}{2} < 0. \quad (5.17)
\end{align*}

Then, the branch points are $\pm \alpha(\xi), \pm i\gamma(\xi)$ where

\begin{align*}
\alpha(\xi) &= s_1(u(\xi), \xi) = 2u(\xi) - \frac{2v(\xi)^2}{3u(\xi)} > 0 \\
-i\gamma(\xi) &= s_1(i v(\xi), \xi) = i \left( 2v(\xi) - \frac{2u(\xi)^2}{3v(\xi)} \right), \quad (5.18)
\end{align*}

and $\alpha(\xi), \gamma(\xi) > 0$. Moreover, we can rewrite the symbol as

$$s_1(w; \xi) = w + \frac{u(\xi)^2 - v(\xi)^2}{w} + \frac{u(\xi)^2 v(\xi)^2}{3w^3}.$$ 

Proof. The proof is straightforward. Note that $\pm u(\xi)$ and $\pm iv(\xi)$ are the zeros of the derivative of $s_1(w; \xi)$ with respect to $w$. From (5.16) and (5.17) it can be shown that $3u(\xi)^2 - v(\xi)^2 > u(\xi)^2 - 3v(\xi)^2 > 0$ if $\xi > \xi_{cr}$. The positivity of $\alpha(\xi)$ and $\gamma(\xi)$ follows from these inequalities.

5.3.2. The restriction of $s_1$ to $\mathbb{R}$ and $i\mathbb{R}$. Consider the algebraic equation

$$s_1(x; \xi) = x + \frac{u(\xi)^2 - v(\xi)^2}{x} + \frac{u(\xi)^2 v(\xi)^2}{3x^3} = z, \quad (5.19)$$

for real values of $z$. Figure 4(a) shows a sketch of the graph of the symbol $s_1(x; \xi)$ for real values of $x$. The solutions to (5.19) for $z \in \mathbb{R}$ are real or come in pairs of complex conjugate numbers. By lemma 5.4, $\pm u(\xi)$ is a double solution of the equation if $z = \pm \alpha(\xi)$. Then, as is clear from the graph in figure 4(a), the other two solutions are complex conjugate and their modulus is smaller than $u(\xi)$. 
Now consider the restriction of the symbol $s_1$ to the imaginary axis
$$s_1(iy; \xi) = i\left(y - \frac{u(\xi)^2 - v(\xi)^2}{y} + \frac{u(\xi)^2 v(\xi)^2}{3y^3}\right).$$
We claim that $s_1$ has four purely imaginary zeros: $\pm iy_1, \pm iy_2$, where $y_1 > y_2 > 0$. To see this, recall that $u(\xi)^2 - 3v(\xi)^2 > 0$, so that $(u(\xi) - \sqrt{3}v(\xi))(u(\xi) + \sqrt{3}v(\xi))/3 > 0$. This can be rewritten as
$$u(\xi)^2 - v(\xi)^2 > 2u(\xi)v(\xi)/\sqrt{3}.$$ (5.20)
Now consider the biquadratic equation
$$y^4 - (u(\xi)^2 - v(\xi)^2)y^2 + \frac{u(\xi)^2 v(\xi)^2}{3y^3} = 0.$$ By (5.20) the discriminant is positive and less than $(u(\xi)^2 - v(\xi)^2)^2 > 0$. Then the claim follows. Figure 4(b) shows the graph of the restriction of $s_1$ to the imaginary axis.

Consider the algebraic equation
$$y - \frac{u(\xi)^2 - v(\xi)^2}{y} + \frac{u(\xi)^2 v(\xi)^2}{3y^3} = z.$$ (5.21)
for $z \in \mathbb{R}$. For $z = \pm \gamma(\xi)$ the equation has four real solutions: the double solution $\mp v(\xi)$ and two strictly positive/negative solutions. The latter are simple and $v(\xi)$ lies between their moduli.

5.3.3. Auxiliary lemmas. We start by proving two lemmas.

**Lemma 5.5.** Let $\xi > \xi_{cr}$. Assume that $w_a, w_b \in \mathbb{C}$ are such that $w_a \neq w_b$, $|w_a| = |w_b|$ and $s_1(w_a; \xi) = s_1(w_b; \xi) = z$. Then $z \in \mathbb{R} \cup i\mathbb{R}$.

**Proof.** The complex numbers $w_a^2$ and $w_b^2$ lie on a circle of radius $\rho = |w_a^2| = |w_b^2|$ centred at the origin of the complex plane. We can factorize $s_1(w; \xi)$ as
$$s_1(w; \xi) = \frac{(w^2 + y_1^2)(w^2 + y_2^2)}{w^3},$$
where \( \pm iy_1, \pm iy_2 \) are the zeros of the symbol \( s_1(w; \xi) \). Thus

\[
|s_1(w, \xi)| = \frac{\text{dist}(w^2, -y_1^2)\text{dist}(w^2, -y_2^2)}{\rho^{3/2}}, \quad \text{if } |w^2| = \rho.
\]

Since \(-y_1^2, -y_2^2 < 0\), it follows that

\[
[-\pi, \pi] \to \mathbb{R} : \theta \mapsto \frac{\text{dist}(\rho e^{i\theta}, -y_1^2)\text{dist}(\rho e^{i\theta}, -y_2^2)}{\rho^{3/2}}
\]
is an even function that is strictly decreasing as \( \theta \) increases from 0 to \( \pi \). Thus, equality

\[
\text{dist}(\rho e^{i\theta}, -y_1^2)\text{dist}(\rho e^{i\theta}, -y_2^2) = \text{dist}(\rho e^{i\theta}, -y_1^2)\text{dist}(\rho e^{i\theta}, -y_2^2),
\]
with \( \theta_0, \theta_b \in [-\pi, \pi] \), can only occur if \( \theta_b = \theta_0 \). Then it follows from the assumptions of the lemma that \( w_a^2 = w_b^2 \) or \( w_a^2 = -w_b^2 \). This gives rise to three possible cases: \( w_a = -w_b \) and \( w_a = \pm w_b \). Substituting these results into the algebraic equation yields

\[
z = s_1(w_a; \xi) = s_1(-w_b; \xi) = -s_1(w_b; \xi) = -z,
\]
\[
z = s_1(w_a; \xi) = s_1(w_b; \xi) = s_1(-w_b; \xi) = \bar{z}, \quad \text{or}
\]
\[
z = s_1(w_a; \xi) = s_1(-w_b; \xi) = -s_1(w_b; \xi) = -\bar{z}.
\]

In all three cases \( z \in \mathbb{R} \cup i\mathbb{R} \). \( \square \)

Lemma 5.6. Let \( \xi > \xi_{cr} \). Then \( \Gamma_j(\xi) \subset \mathbb{R} \cup i\mathbb{R} \) for \( j = 1, 2, 3 \). Moreover,

\[
\Gamma_1(\xi) \cap \Gamma_2(\xi) = \Gamma_2(\xi) \cap \Gamma_3(\xi) = \emptyset.
\]

Proof. Fix \( j \in \{1, 2, 3\} \). If \( z \in \Gamma_j(\xi) \) and \( s_1(w, \xi) = z \) has a double solution, then \( z \) is one of the branch points \( \pm a(\xi) \) or \( \pm i\gamma(\xi) \). So \( z \in \mathbb{R} \cup i\mathbb{R} \). If \( z \in \Gamma_j(\xi) \) and \( s_1(w, \xi) = z \) does not have a double solution, then \( w_j(\xi) \neq w_{j+1}(\xi) \) and \( |w_j(\xi)| = |w_{j+1}(\xi)| \). Then it follows from lemma 5.5 that \( z \in \mathbb{R} \cup i\mathbb{R} \). This proves that \( \Gamma_j(\xi) \subset \mathbb{R} \cup i\mathbb{R} \).

Now assume that for a certain value of \( z \in \mathbb{R} \cup i\mathbb{R} \) and \( j \in \{1, 2\} \) \( z \in \Gamma_j(\xi) \cap \Gamma_{j+1}(\xi) \). Then \( |w_j(z; \xi)| = |w_{j+1}(z; \xi)| = |w_{j+2}(z; \xi)| \). We distinguish four cases. First consider the case \( z = 0 \). Recall that the algebraic equation \( s_1(w, \xi) = 0 \) has four different imaginary solutions: \( \pm iy_1 \) and \( \pm iy_2 \). This contradicts the assumption. \( z \) cannot be one of the branch points \( \{\pm a(\xi), \pm i\gamma(\xi)\} \). Next assume that \( z \in \mathbb{R} \setminus \{0, \pm a(\xi)\} \). From the proof of lemma 5.5 it follows that \( w_j(z; \xi) = \gamma, \) so that two roots coincide and \( z \) is a branch point. This possibility was already excluded. If \( z \in i\mathbb{R} \setminus \{0, \pm iy(\xi)\} \) we analogously obtain \( w_j(z; \xi) = -\gamma \) and \( w_{j+2}(z; \xi) = -\gamma \). Then again \( z \) is a branch point. Since we could exclude all four possible cases, we can conclude that \( \Gamma_1(\xi) \cap \Gamma_2(\xi) = \Gamma_2(\xi) \cap \Gamma_3(\xi) = \emptyset \). \( \square \)

5.3.4. Transformed symbol. Lemma 5.6 is used in the proof of theorem 5.3. Another ingredient of that proof is the transformed symbol \( S_1 \), defined as

\[
S_1(W; \xi) = s_1(a(\xi)W; \xi) = a(\xi) \left( W + \frac{3}{W} + t \frac{1}{W} + t^2 \frac{1}{W^3} \right), \quad (5.22)
\]
see (5.13), (4.6) and (4.7). The advantage of the transformed symbol \( S_1 \) over the symbol \( s_1 \) is that it depends on \( \xi \) in a much easier way. This will significantly simplify the calculations.

We denote the zeros of \( (dS_1/dW)(W; \xi) \) by \( \pm U(\xi) \) and \( \pm iV(\xi) \), where \( U(\xi), V(\xi) > 0 \). One can check that \( \pm U(\xi) = \pm u(\xi)/a(\xi) \) and \( \pm iV(\xi) = \pm v(\xi)/a(\xi) \), where \( u(\xi) \) and \( v(\xi) \)
are defined as in (5.16) and (5.17). Moreover, $S_1$ gives rise to the same branch points as $s_1$ does, namely
\[
S_1(\pm U(\xi) ; \xi) = s_1(\pm u(\xi) ; \xi) = \pm \alpha(\xi)
\]
and
\[
S_1(\pm i V(\xi) ; \xi) = s_1(\pm iv(\xi) ; \xi) = \mp iy(\xi).
\]
(5.23)

### 5.3.5. Proof of theorem 5.3

**Proof.** We will use the restriction of the symbol to the real axis to determine the sets $\Gamma_j(\xi) \cap \mathbb{R}$. Figure 4(a) shows the typical form of the graph of this restriction.

Recall the algebraic equation (5.19). For $z \in (-\alpha(\xi), \alpha(\xi))$ we find two pairs of complex conjugate solutions. Therefore, $(-\alpha(\xi), \alpha(\xi)) \subset \Gamma_1(\xi) \cap \Gamma_3(\xi)$. If $z = \pm \alpha(\xi)$ the equation has the double solution $\pm u(\xi)$ and one pair of complex conjugate solutions with smaller modulus. We conclude that $[-\alpha(\xi), \alpha(\xi)) \subset \Gamma_1(\xi) \cap \Gamma_3(\xi)$. Next, take $z > \alpha(\xi)$. The equation then has two real solutions and one pair of complex conjugate solutions. The largest real solution is denoted by $x_1(z)$, the smallest real solution by $x_2(z)$, and the complex conjugate solutions by $x_3(z) = x_4(z)$. From lemma 5.6 it follows that the equation does not admit three solutions with equal moduli. Thus, the situation $|x_2(z)| = |x_3(z)| = |x_4(z)|$ cannot occur if $z > \alpha(\xi)$. Because $|x_2(\alpha(\xi))| > |x_3(\alpha(\xi))| = |x_4(\alpha(\xi))|$ and the roots of a polynomial equation are continuous with respect to the coefficients of the equation, we have that $(\alpha(\xi), +\infty) \subset \Gamma_3(\xi) \cap (\alpha(\xi), +\infty) \cap \Gamma_1(\xi) = (\alpha(\xi), +\infty) \cap \Gamma_2(\xi) = \emptyset$. We obtain similar results if $z \in (-\infty, -\alpha(\xi))$. At this moment we conclude
\[
\Gamma_1(\xi) \cap \mathbb{R} = [-\alpha(\xi), \alpha(\xi)], \quad \Gamma_2(\xi) \cap \mathbb{R} = \emptyset \quad \text{and} \quad \mathbb{R} \subset \Gamma_3(\xi).
\]

Let us now restrict the symbol $s_1$ to the imaginary axis to determine the sets $\Gamma_j(\xi) \cap i\mathbb{R}$. Figure 4(b) shows a typical graph of $\gamma \mapsto -i\gamma_1(i\gamma; \xi)$. We proceed in a similar way. Consider for real values of $z$ the algebraic equation (5.21). For $z \in (0, \gamma(\xi))$ this equation has four real solutions with different moduli. Therefore, $(0, i\gamma(\xi)) \cap \Gamma_j(\xi) = \emptyset$ for $j = 1, 2, 3, 4$. For $z = 0$ we obtain the solutions $\pm y_1, \pm y_2$. Since $y_1 > y_2$, 0 belongs to $\Gamma_1(\xi)$ and $\Gamma_3(\xi)$, but not to $\Gamma_2(\xi)$. This is consistent with (5.24). If $z = \gamma(\xi)$, the equation has the double solution $-v(\xi)$, a real solution with modulus less than $v(\xi)$ and a real solution with modulus greater than $v(\xi)$. Thus, $i\gamma(\xi)$ belongs to $\Gamma_2(\xi)$. Next take $z > \gamma(\xi)$. The equation then has two different real solutions and one pair of complex conjugate solutions. Using a similar continuity argument as before we obtain $(i\gamma(\xi), +\infty) \subset \Gamma_2(\xi)$ and $(i\gamma(\xi), +\infty) \cap \Gamma_1(\xi) = (i\gamma(\xi), +\infty) \cap \Gamma_3(\xi) = \emptyset$. The same procedure works for $z < 0$. Summarized this is
\[
\Gamma_1(\xi) \cap i\mathbb{R} = \Gamma_3(\xi) \cap i\mathbb{R} = \{0\} \quad \text{and} \quad \Gamma_2(\xi) \cap i\mathbb{R} = i\mathbb{R} \setminus (-i\gamma(\xi), i\gamma(\xi)).
\]
(5.25)

Combining (5.24) and (5.25) proves (5.14).

To prove (a) and (b) the transformed symbol $S_1$ will be useful. First, we prove that $\xi \mapsto \alpha(\xi)$ is an increasing function. Take $\xi > \xi_c$. It follows from (5.23) that
\[
\frac{d\alpha(\xi)}{d\xi} = \frac{dS_1}{d\xi}(U(\xi); \xi) = \frac{\partial S_1}{\partial W}(U(\xi); \xi) \frac{dU(\xi)}{d\xi} + \frac{\partial S_1}{\partial U}(U(\xi); \xi).
\]

Since $U(\xi)$ is a zero of the derivative of $S_1$, the first term on the right-hand side vanishes. Taking the partial derivative with respect to $\xi$ in (5.22) yields
\[
\frac{d\alpha(\xi)}{d\xi} = \left( \frac{U(\xi) + 3}{U(\xi)} \right) \frac{d\alpha(\xi)}{d\xi}.
\]

Observe that the function $\xi \mapsto a(\xi)$ is increasing, see (4.5). Because $U(\xi) > 0$ we conclude that $\xi \mapsto a(\xi)$ is an increasing function. It can be proved similarly that $\xi \mapsto \gamma(\xi)$ is an increasing function for $\xi > \xi_{cr}$.

Next, we show that $\lim_{\xi \to +\infty} a(\xi) = +\infty$. From (4.5) it follows that $a(\xi) \sim \sqrt{\xi}$ as $\xi \to \infty$.

Using (5.16)–(5.18) we compute

$$
\alpha(\xi) = \frac{2}{3u(\xi)}(3u(\xi)^2 - v(\xi)^2)
= \frac{2}{3u(\xi)} \left(2(\tau^2 a(\xi) + \xi) + \sqrt{(\tau^2 a(\xi) + \xi)^2 + 12\tau^2 a(\xi)^3}\right) \sim \sqrt{\xi}.
$$

Therefore, $\xi \mapsto \alpha(\xi)$ is unbounded. The limit $\lim_{\xi \to +\infty} \gamma(\xi) = +\infty$ can be proved analogously.

Our final task is to calculate the limits of $\alpha(\xi)$ and $\gamma(\xi)$ as $\xi \to 0^+$ for $t \geq \tau^2$. In the limit $\xi = 0$ the transformed symbol is

$$
S_1(W; 0) = tW + \frac{\tau^2}{W^3},
$$

because $\lim_{\xi \to 0^+} a(\xi) = 0$. Its derivative

$$
\frac{dS_1}{dW}(W; 0) = -\frac{t}{W^2} - 3\frac{\tau^2}{W^4},
$$

has only two zeros, denoted by

$$
\pm iV(0) = \pm i\sqrt{\frac{3\tau^2}{t}}.
$$

It follows that

$$
\lim_{\xi \to 0^+} V(\xi) = V(0) = \sqrt{\frac{3\tau^2}{t}} \quad \text{and} \quad \lim_{\xi \to 0^+} U(\xi) = +\infty.
$$

Substituting these results into (5.26) yields

$$
\lim_{\xi \to 0^+} \alpha(\xi) = \lim_{\xi \to 0^+} S_1(U(\xi); \xi) = \lim_{\xi \to 0^+} \frac{t}{W} + \frac{\tau^2}{W^3} = 0
$$

and

$$
\lim_{\xi \to 0^+} i\gamma(\xi) = \lim_{\xi \to 0^+} S_1(-iV(\xi); \xi) = \frac{t}{-iV(0)} + \frac{\tau^2}{(-iV(0))^3} = iy^*.
$$

see (2.9) for $y^*$.

6. Asymptotic analysis in two-cut case

6.1. Doubling the recurrence relation

In section 5 we introduced and analysed the symbol $s_1(w; \xi)$ in the one-cut case.

Here we want to do something similar for the two-cut case. Note, however, that the recurrence coefficients $b_{k,n}$ and $c_{k,n}$ in (4.2) do not have limits as $k/n \to \xi \in (0, \xi_{cr})$. Instead, there is two-periodic limiting behaviour given by (4.10)–(4.13). This is a fundamental difference with the one-cut case and, therefore, the construction from the previous section does not apply to the two-cut case.
We analyse the two-cut case by doubling the recurrence relation (4.2). This yields a new recurrence relation in which the coefficients have limits. Indeed, we obtain
\begin{equation}
x^2 p_{k,n}(x) = p_{k+2,n}(x) + A_{k,n} p_{k,n}(x) + B_{k,n} p_{k-2,n}(x) + C_{k,n} p_{k-4,n}(x) + D_{k,n} p_{k-6,n}(x). \quad (6.1)
\end{equation}
where
\begin{align*}
A_{k,n} &= b_{k,n} + b_{k+1,n}, \\
B_{k,n} &= c_{k,n} + c_{k+1,n} + b_{k,n}b_{k-1,n}, \\
C_{k,n} &= b_{k,n}c_{k-1,n} + c_{k,n}b_{k-3,n}, \\
D_{k,n} &= c_{k,n}c_{k-3,n}.
\end{align*}
The limits of these coefficients as \( k/n \to \infty \) such that \( k/n \to \xi \in (0, \xi_{cr}) \) exist and are denoted by
\begin{align*}
A(\xi) &= \lim_{k/n \to \xi} A_{k,n} = b_0(\xi) + b_1(\xi), \quad (6.2) \\
B(\xi) &= \lim_{k/n \to \xi} B_{k,n} = c_0(\xi) + c_1(\xi) + b_0(\xi)b_1(\xi), \quad (6.3) \\
C(\xi) &= \lim_{k/n \to \xi} C_{k,n} = b_0(\xi)c_1(\xi) + c_0(\xi)b_1(\xi), \quad (6.4) \\
D(\xi) &= \lim_{k/n \to \xi} D_{k,n} = c_0(\xi)c_1(\xi). \quad (6.5)
\end{align*}
see also (4.10)–(4.13).

In analogy with (5.13) we define the symbol
\begin{equation}
\hat{s}_2(w; \xi) = w + A(\xi) + \frac{B(\xi)}{w} + \frac{C(\xi)}{w^2} + \frac{D(\xi)}{w^3}. \quad (6.6)
\end{equation}
We use the subscript 2 to recall that we are in the two-cut case. The hat refers to the fact that the recurrence relation was doubled to obtain this symbol. Also the quantities that are associated with the symbol (6.6) will be equipped with a hat. Thus we use \( \hat{w}_j(z; \xi) \) to denote the solutions of \( \hat{s}_2(w; \xi) = z \) with the usual ordering
\begin{equation*}
|\hat{w}_1(z; \xi)| \geq |\hat{w}_2(z; \xi)| \geq |\hat{w}_3(z; \xi)| \geq |\hat{w}_4(z; \xi)|.
\end{equation*}
Furthermore, we have \( \hat{\Gamma}_j(\xi) \) and \( \hat{\mu}_j^\pm \) for \( j = 1, 2, 3 \).

It is remarkable that \( \hat{s}_2 \) has the factorization
\begin{equation}
\hat{s}_2(w; \xi) = \left( \frac{w + \xi}{w^3} \right)^2 \left( w^2 - t \tau^2 w + \tau^4 w + \tau^4 \xi \right). \quad (6.7)
\end{equation}
This follows from (6.2)–(6.5) and the explicit expressions for \( b_0(\xi), b_1(\xi), c_0(\xi) \) and \( c_1(\xi) \) from theorem 4.2. Note that \( w = -\xi \) is always a double zero of (6.7). Thus 0 is always a branch point and in fact an endpoint of one of the sets \( \hat{\Gamma}_j(\xi) \), as will follow from the analysis in the next subsection.

6.2. Analysis of the symbol \( \hat{s}_2 \) in the two-cut case

The rest of this section is devoted to the proof of the following theorem, which is the two-cut case version of theorem 5.3.

**Theorem 6.1.** Fix \( t < \tau^2 \) and \( 0 < \xi < \xi_{cr} \). Then we have that
\begin{equation*}
\hat{\Gamma}_1(\xi) \subset \mathbb{R}^+, \quad \hat{\Gamma}_2(\xi) \subset \mathbb{R}^- \quad \text{and} \quad \hat{\Gamma}_3(\xi) \subset \mathbb{R}^+.
\end{equation*}
More precisely, there exist $\tilde{\alpha}(\xi) > \tilde{\beta}(\xi) \geq 0$, $\tilde{\gamma}(\xi) \leq 0$ and $\tilde{\delta}(\xi) \geq 0$ such that

\begin{align*}
\tilde{\Gamma}_1(\xi) &= [\tilde{\beta}(\xi), -\infty, \tilde{\alpha}(\xi)], \\
\tilde{\Gamma}_2(\xi) &= (-\infty, \tilde{\gamma}(\xi)], \\
\tilde{\Gamma}_3(\xi) &= [\tilde{\delta}(\xi), +\infty].
\end{align*}

(6.8)

In addition, we have for every fixed $t < \tau^2$

(a) $\xi \mapsto \tilde{\alpha}(\xi)$ is strictly increasing for $0 < \xi < \xi_{cr}$ with

$$
\lim_{\xi \to 0^+} \tilde{\alpha}(\xi) = \tau^2(\tau^2 - t) \quad \text{and} \quad \lim_{\xi \to \xi_{cr}^-} \tilde{\alpha}(\xi) = \lim_{\xi \to \xi_{cr}^+} \alpha(\xi)^2;
$$

(b) $\tilde{\beta}(\xi) = 0$ if and only if $t < -\tau^2$ and $-\tau^2 \leq \xi < \xi_{cr}$. Otherwise $\xi \mapsto \tilde{\beta}(\xi)$ is positive and strictly decreasing with

$$
\lim_{\xi \to 0^+} \tilde{\beta}(\xi) = \tau^2(\tau^2 - t),
\lim_{\xi \to \xi_{cr}^-} \tilde{\beta}(\xi) = 0, \quad \text{for } -\tau^2 \leq t < \tau^2,
\lim_{\xi \to \xi_{cr}^-} \tilde{\beta}(\xi) = 0, \quad \text{for } t \leq -\tau^2;
$$

(c) $\tilde{\gamma}(\xi) = 0$ if and only if $t < 0$ and $\xi \leq -\tau^2$. Otherwise $\xi \mapsto \tilde{\gamma}(\xi)$ is negative and strictly decreasing with

$$
\lim_{\xi \to 0^+} \tilde{\gamma}(\xi) = -\frac{4t^3}{27\tau^2} = -(\gamma^*)^2, \quad \text{for } 0 < t < \tau^2,
\lim_{\xi \to \xi_{cr}^-} \tilde{\gamma}(\xi) = 0, \quad \text{for } t < 0,
\lim_{\xi \to \xi_{cr}^+} \tilde{\gamma}(\xi) = -\lim_{\xi \to \xi_{cr}^-} \gamma(\xi)^2;
$$

(d) $\tilde{\delta}(\xi) = 0$ if and only if $-\tau^2 < t < \tau^2$ and $-\tau^2 \leq \xi < \xi_{cr}$. Otherwise $\xi \mapsto \tilde{\delta}(\xi)$ is positive and strictly decreasing with

$$
\lim_{\xi \to 0^+} \tilde{\delta}(\xi) = \frac{4(-t)^3}{27} = (x^*)^2, \quad \text{for } t < 0,
\lim_{\xi \to \xi_{cr}^-} \tilde{\delta}(\xi) = 0, \quad \text{for } t \leq -\tau^2,
\lim_{\xi \to \xi_{cr}^-} \tilde{\delta}(\xi) = 0, \quad \text{for } -\tau^2 \leq t < 0.
$$

Theorem 6.1 indicates a refinement of the $t\xi$-phase diagram. We divide the two-cut case region $C_2$ into three subregions $C_{2a}$, $C_{2b}$ and $C_{2c}$, depending on whether $\tilde{\beta}(\xi)$, $\tilde{\gamma}(\xi)$ and $\tilde{\delta}(\xi)$ are zero, or not. The regions are separated by the critical ray

$$
\xi = -\tau^2, \quad t < 0,
$$

which is tangent to the critical semi-parabola. The three regions are

| Region          | Conditions                     |
|-----------------|--------------------------------|
| $C_{2a}$        | $t < -\tau^2$, $-\tau^2 < \xi < \xi_{cr}$ |
| $C_{2b}$        | $t < 0$, $0 < \xi < -\tau^2$    |
| $C_{2c}$        | $-\tau^2 < t < \tau^2$, $\max(-\tau^2, 0) < \xi < \xi_{cr}$ |

see figure 5. Then, according to theorem 6.1, we have the following for $\xi < \xi_{cr}$.

- if $(t, \xi) \in C_{2a}$ then $\tilde{\beta}(\xi) = 0$, $\tilde{\gamma}(\xi) < 0$ and $\tilde{\delta}(\xi) > 0$;
- if $(t, \xi) \in C_{2b}$ then $\tilde{\beta}(\xi) > 0$, $\tilde{\gamma}(\xi) = 0$ and $\tilde{\delta}(\xi) > 0$;
- if $(t, \xi) \in C_{2c}$ then $\tilde{\beta}(\xi) > 0$, $\tilde{\gamma}(\xi) < 0$ and $\tilde{\delta}(\xi) = 0$.

The proof of theorem 6.1 is in the following subsection. The approach is similar to the one used throughout the previous section, but there are certain complications.
6.3. Proof of theorem 6.1

6.3.1. The zeros of \( \hat{\mathcal{S}}_2 \) and graphs. The symbol \( \hat{\mathcal{S}}_2 \) has a double zero in \(-\xi\), see (6.7). The two remaining zeros are also negative. We order them such that \( x_1 \leq x_2 < 0 \). The location of \((t, \xi)\) in the phase diagram determines the way the zeros are ordered, see figure 5.

Lemma 6.2. The zeros of \( \hat{\mathcal{S}}_2(w; \xi) \) are ordered as follows:

\[
\begin{align*}
-\xi &< x_1 < x_2, & \text{if } (t, \xi) \in C_{2a}, \\
x_1 &< -\xi < x_2, & \text{if } (t, \xi) \in C_{2b}, \\
x_1 &< x_2 < -\xi, & \text{if } (t, \xi) \in C_{2c}, \\
x_1 = -\xi < x_2, & \text{if } \xi = -t\tau^2 \text{ and } t < -\tau^2, \\
x_1 &< x_2 = -\xi, & \text{if } \xi = -t\tau^2 \text{ and } -\tau^2 < t < 0, \\
x_1 = x_2 = -\xi, & \text{if } (t, \xi) = (-\tau^2, \tau^4).
\end{align*}
\]

Proof. The proof of this lemma is straightforward. □

The three figures 6, 7 and 8 show sketches of the graph of \( \hat{\mathcal{S}}_2 \) for \((t, \xi)\) belonging to \(C_{2a}\), \(C_{2b}\) and \(C_{2c}\), respectively. The graphs have the following properties:

- \( \hat{\mathcal{S}}_2 \) has four negative zeros, counted with multiplicity. \(-\xi\) is a double zero.
- \( \hat{\mathcal{S}}_2 \) attains a local minimum \( \hat{\alpha}(\xi) > 0 \) in a point \( w^*_1 > 0 \).
- \( \hat{\mathcal{S}}_2 \) attains three local maxima \( \hat{\beta}(\xi) \geq 0, \hat{\gamma}(\xi) \leq 0 \) and \( \hat{\delta}(\xi) \geq 0 \) in the respective points \( w^*_1 < w^*_2 < w^*_3 < 0 \). \( \hat{\beta}(\xi) \) and \( \hat{\delta}(\xi) \) are local maxima. \( \hat{\gamma}(\xi) \) is a local minimum. Only the extremum attained at the double zero \(-\xi\) is zero.
- If \((t, \xi) \in C_{2a}\), then \( w^*_1 = -\xi \), so that \( \hat{\beta}(\xi) = 0, \hat{\gamma}(\xi) < 0 \) and \( \hat{\delta}(\xi) > 0 \).
- If \((t, \xi) \in C_{2b}\), then \( w^*_2 = -\xi \), so that \( \hat{\beta}(\xi) > 0, \hat{\gamma}(\xi) = 0 \) and \( \hat{\delta}(\xi) > 0 \).
- If \((t, \xi) \in C_{2c}\), then \( w^*_3 = -\xi \), so that \( \hat{\beta}(\xi) > 0, \hat{\gamma}(\xi) < 0 \) and \( \hat{\delta}(\xi) = 0 \).
6.3.2. Auxiliary lemmas. The following two lemmas are the analogues of lemmas 5.5 and 5.6.

**Lemma 6.3.** Let $\xi \in (0, \xi_{cr})$. Assume that $w_a, w_b \in \mathbb{C}$ are such that $w_a \neq w_b$, $|w_a| = |w_b|$ and $\hat{s}_2(w_a; \xi) = \hat{s}_2(w_b; \xi) = z$. Then, $z \in \mathbb{R}$.

**Proof.** The complex numbers $w_a$ and $w_b$ lie on a circle of radius $\rho = |w_a| = |w_b|$ centred at the origin of the complex plane. We can factorize $\hat{s}_2(w; \xi)$ as

$$\hat{s}_2(w; \xi) = \frac{(w + \xi)^2(w - x_1)(w - x_2)}{w^3},$$

where $x_1, x_2 < 0$. Thus,

$$|\hat{s}_2(w; \xi)| = \frac{\text{dist}(w, -\xi)^2\text{dist}(w, x_1)\text{dist}(w, x_2)}{\rho^3}, \quad \text{if } |w| = \rho.$$

Since $x_1, x_2, -\xi < 0$, it follows that

$$[-\pi, \pi] \to \mathbb{R} : \theta \mapsto \frac{\text{dist}(\rho e^{i\theta}, -\xi)^2\text{dist}(\rho e^{i\theta}, x_1)\text{dist}(\rho e^{i\theta}, x_2)}{\rho^3}$$

is an even function that is strictly decreasing as $\theta$ increases from $0$ to $\pi$. Thus the equality for $\theta = \theta_a$ and $\theta = \theta_b$ can only occur if $\theta_b = \pm \theta_a$. Then, it follows from the assumptions of the lemma that $w_b = \overline{w_a}$ and

$$z = \hat{s}_2(w_b; \xi) = \hat{s}_2(w_a, \overline{\xi}) = \overline{\hat{s}_2(w_a, \xi)} = \overline{z}.$$

Therefore, $z \in \mathbb{R}$. \qed

**Lemma 6.4.** Let $\xi \in (0, \xi_{cr})$. Then, $\hat{\Gamma}_j(\xi) \subset \mathbb{R}$ for $j = 1, 2, 3$. Moreover, $\hat{\Gamma}_1(\xi) \cap \hat{\Gamma}_2(\xi) = \hat{\Gamma}_3(\xi) \cap \hat{\Gamma}_3(\xi) = \emptyset$.

**Proof.** The proof is analogous to that of lemma 5.6. \qed

6.3.3. Transformed symbol. For convenience, let us again introduce a transformed symbol $\hat{\Sigma}_2$:

$$\hat{\Sigma}_2(W; \xi) = \hat{s}_2(\xi W; \xi) = \xi \left(\frac{W + 1)^2}{W} - t^2 \left(\frac{W + 1)^2}{W^2} + \xi^4 \frac{(W + 1)^3}{W^3}\right) \right). \quad (6.9)$$
see (6.7). Note that \( \hat{S}_2 \) depends on \( \xi \) in a simple way. The branch points of the transformed symbol \( \hat{S}_2 \) coincide with the branch points of the symbol \( \hat{s}_2 \). To see this, define
\[
W^*_j (\xi) = \frac{w^*_j (\xi)}{\xi} \quad \text{for} \quad j = 0, 1, 2, 3.
\]
Then,
\[
\frac{\partial}{\partial W} \hat{S}_2 (W^*_j (\xi); \xi) = 0, \quad \text{for} \quad j = 0, 1, 2, 3,
\]
and
\[
\hat{\alpha}(\xi) = \hat{S}_2 (W^*_0 (\xi); \xi), \quad \hat{\beta}(\xi) = \hat{S}_2 (W^*_1 (\xi); \xi), \quad \hat{\gamma}(\xi) = \hat{S}_2 (W^*_2 (\xi); \xi), \quad \hat{\delta}(\xi) = \hat{S}_2 (W^*_3 (\xi); \xi).
\] (6.10)

6.3.4. Proof of theorem 6.1.

**Proof.** We prove the first part of the theorem only for the case that \( (t, \xi) \in C_{2a} \). The other cases can be treated similarly. Figure 6 shows the graph of the symbol \( \hat{S}_2 \) for the case \( (t, \xi) \in C_{2a} \).

For \( z \in (\hat{\gamma}(\xi), 0) \), the equation \( \hat{S}_2 (x; \xi) = z \) has four different negative solutions. Therefore, \( (\hat{\gamma}(\xi), 0) \cap \hat{\Gamma}_1 (\xi) = \emptyset, j = 1, 2, 3 \). If \( z = \hat{\gamma}(\xi) \) the equation has the double solution \( w^*_2 \), a negative solution with a greater modulus and a negative solution with a smaller modulus. We conclude that \( \hat{\gamma}(\xi) \in \hat{\Gamma}_2 (\xi) \). Next, take \( z < \hat{\gamma}(\xi) \). Then, the equation has two negative solutions and one pair of complex conjugate solutions. Using the continuity argument (which was also used in the proof of theorem 5.3) we obtain that the modulus of the complex conjugate solutions lies between the moduli of the negative solutions. Therefore, \( (-\infty, \hat{\gamma}(\xi)) \subset \hat{\Gamma}_2 (\xi) \).

Now focus on \( z \geq 0 \) with the extra assumption that \( \hat{\gamma}(\xi) > \hat{\delta}(\xi) \). We can make a similar reasoning if \( \hat{\alpha}(\xi) \leq \hat{\delta}(\xi) \). If \( z = 0 \) the equation has a double solution \( -\xi \) and two different negative solutions with a smaller modulus. Therefore, \( 0 \) belongs to \( \hat{\Gamma}_1 (\xi) \). If \( z \in (0, \delta(\xi)) \) we find two different negative solutions and one pair of complex conjugate solutions. The continuity argument guarantees that the modulus of the complex solutions is the greatest, so that \( [0, \delta(\xi)) \subset \hat{\Gamma}_1 (\xi) \). Proceeding in the same way, we obtain \( [\delta(\xi), \hat{\alpha}(\xi)] \subset \hat{\Gamma}_1 (\xi) \cap \hat{\Gamma}_2 (\xi) \).
For $z > \tilde{\alpha}(\xi)$ the equation has two different positive solutions and one pair of complex conjugate solutions. By the continuity argument we obtain $(\tilde{\alpha}(\xi), \infty) \subset \tilde{\Gamma}_1(\xi)$ or $(\tilde{\alpha}(\xi), \infty) \subset \tilde{\Gamma}_3(\xi)$. Since $\tilde{\Gamma}_1(\xi)$ is a compact set, see [7], only the second inclusion holds.

Collecting this information we obtain $\tilde{\Gamma}_1(\xi) = [0, \tilde{\alpha}(\xi)]$, $\tilde{\Gamma}_2(\xi) = (-\infty, \tilde{\gamma}(\xi)]$ and $\tilde{\Gamma}_3(\xi) = [\tilde{\delta}(\xi), \infty)$. so that (6.8) is proved under the assumption that $(t, \xi) \in C_{2a}$ and $\tilde{\alpha}(\xi) > \tilde{\delta}(\xi)$. One can prove all other cases in a similar way.

Next, let us study the behaviour of $\tilde{\alpha}(\xi)$, $\tilde{\beta}(\xi)$, $\tilde{\gamma}(\xi)$ and $\tilde{\delta}(\xi)$ as $\xi$ increases. Since all branch points are of the form $\tilde{S}_2(W^*_{j}(\xi); \xi)$ for some $j = 0, 1, 2, 3$, see (6.9) and (6.10), we are interested in the sign of

$$\frac{d}{d\xi} \tilde{S}_2(W^*_j(\xi); \xi) = \frac{\partial \tilde{S}_2}{\partial W}(W^*_j(\xi); \xi) \frac{dW^*_j(\xi)}{d\xi} + \frac{\partial \tilde{S}_2}{\partial \xi}(W^*_j(\xi); \xi)$$

$$= \frac{(W^*_j(\xi) + 1)^2}{W^*_j(\xi)},$$

(6.11)

where the last equality holds because of (6.9).

Since $W^*_0(\xi) = w^*_0(\xi)/\xi > 0$, we obtain from (6.11) that $\xi \mapsto \tilde{\alpha}(\xi)$ is a strictly increasing function. Putting $\xi = -1$ in (6.11), we also get that that $0 = \tilde{S}_2(-1; \xi)\) remains zero for fixed $t$ and $(t, \xi)$ in a fixed subregion $C_{2a}$, $C_{2b}$ or $C_{2c}$. The other two branch points can be written as $\tilde{S}_2(w^*_j(\xi); \xi)$ with $w^*_j(\xi) < 0$ and $w^*_j(\xi) \neq -\xi$. In terms of the transformed symbol they are $\tilde{S}_2(W^*_j(\xi); \xi)$ with $W^*_j(\xi) < 0$ and $W^*_j(\xi) \neq -1$. Equation (6.11) then implies that these branch points are strictly decreasing functions of $\xi$.

Let us now concentrate on the behaviour of the branch points as $\xi \to \xi_{ca}$—for fixed $t$, so that $(t, \xi)$ approaches the critical semi-parabola. This behaviour is a straightforward corollary of the following claim:

$$\tilde{S}_2(w^2; \xi_{ca}) = s_1(w; \xi_{ca})^2, \quad w \in \mathbb{C} \setminus \{0\}.$$  

(6.12)
Let us prove this claim. Using (5.13), (6.6) and (4.14), we compute
\[
\lim_{\xi \to \xi_0^+} \hat{\Gamma}(w^2; \xi) = w^2 \left( 1 + \frac{b_0(\xi_0)}{w^2} + \frac{c_0(\xi_0)}{w^4} \right) \left( 1 + \frac{b_1(\xi_0)}{w^2} + \frac{c_1(\xi_0)}{w^4} \right) \\
= \left( w + \frac{b(\xi_0)}{w} + \frac{c(\xi_0)}{w^3} \right)^2 \\
= \lim_{\xi \to \xi_0^+} s_1(w; \xi)^2.
\]

The calculation of the limits of the branch points if \( \xi \to 0^+ \) can be done in the same way as in the proof of theorem 5.3 and is left to the reader. \( \square \)

6.4. Reformulation of results in the two-cut case

In what follows it will be convenient to undo the doubling of the recurrence and consider instead of the functions \( \hat{\mu}_j(z; \xi) \), the sets \( \Gamma_j(\xi) \), and the measures \( \hat{\mu}_j^\xi \) also for \( \xi \in (0, \xi_0) \),
\[
\begin{align*}
\hat{\mu}_j(z; \xi) &= \hat{\mu}_j(z^2; \xi), \\
\Gamma_j(\xi) &= \{ z \in \mathbb{C} \mid z^2 \in \Gamma_j(\xi) \}, \\
&= \{ z \in \mathbb{C} \mid |z|^2 \in \hat{\Gamma}_j(\xi) \}
\end{align*}
\tag{6.13}
\]
and
\[
\begin{align*}
\frac{d\mu_1^\xi(x)}{dx} &= |x| \frac{d\hat{\mu}_1^\xi(x^2)}{dx}, \quad x \in \Gamma_1(\xi), \\
\frac{d\mu_2^\xi(z)}{dz} &= |z| \frac{d\hat{\mu}_2^\xi(z^2)}{dz}, \quad z \in \Gamma_2(\xi), \\
\frac{d\mu_3^\xi(x)}{dx} &= |x| \frac{d\hat{\mu}_3^\xi(x^2)}{dx}, \quad x \in \Gamma_3(\xi).
\end{align*}
\tag{6.14}
\]
Then, with these definitions
\[
\begin{align*}
\Gamma_j(\xi) &= \{ z \in \mathbb{C} \mid |w_j(z; \xi)| = |w_{j+1}(z; \xi)| \}, \quad j = 1, 2, 3 \\
\end{align*}
\tag{6.15}
\]
and
\[
\frac{d\mu_j^\xi(z)}{dz} = \frac{1}{2} \frac{1}{2\pi i} \frac{w_j'(z; \xi)}{w_j(z; \xi)} d z, \quad z \in \Gamma_j(\xi), \quad j = 1, 2, 3.
\]

One can also check that the \( \mu_j^\xi \) are measures on \( \Gamma_j(\xi) \) with total masses
\[
\int d\mu_1^\xi = 1, \quad \int d\mu_2^\xi = \frac{2}{3} \quad \text{and} \quad \int d\mu_3^\xi = \frac{1}{3}.
\]

By theorem 6.1 we have
\[
\begin{align*}
\Gamma_1(\xi) &= [-\alpha(\xi), -\beta(\xi)] \cup [\beta(\xi), \alpha(\xi)], \\
\Gamma_2(\xi) &= (-i\infty, -i\gamma(\xi)] \cup [i\gamma(\xi), +i\infty), \\
\Gamma_3(\xi) &= (-\infty, -i\delta(\xi)] \cup [i\delta(\xi), +\infty).
\end{align*}
\]
with
\[
\alpha(\xi) = \sqrt{-\alpha(\xi)}, \quad \beta(\xi) = \sqrt{-\beta(\xi)}, \quad \gamma(\xi) = \sqrt{-\gamma(\xi)} \quad \text{and} \quad \delta(\xi) = \sqrt{-\delta(\xi)}.
\]

Theorem 6.1 also shows that for every fixed \( t < \tau^2 \)
(a) \( \xi \mapsto \alpha(\xi) \) is strictly increasing for \( 0 < \xi < \xi_0 \), with
\[
\lim_{\xi \to 0^+} \alpha(\xi) = \tau \sqrt{\tau^2 - t} \quad \text{and} \quad \lim_{\xi \to \xi_0^-} \alpha(\xi) = \lim_{\xi \to \xi_0^+} \alpha(\xi),
\]
and
\[
\lim_{\xi \to 0^+} \beta(\xi) = \tau \sqrt{\tau^2 - t} \quad \text{and} \quad \lim_{\xi \to \xi_0^-} \beta(\xi) = \lim_{\xi \to \xi_0^+} \beta(\xi).
\]
(b) \( \beta(\xi) = 0 \) if and only if \( t < -\tau^2 \) and \( -t\tau^2 \leq \xi < \xi_{cr} \). Otherwise \( \xi \mapsto \beta(\xi) \) is positive and strictly decreasing with
\[
\lim_{\xi \to 0^+} \beta(\xi) = \tau \sqrt{\tau^2 - 1},
\lim_{\xi \to \xi_{cr}^-} \beta(\xi) = 0, \quad \text{for } -\tau^2 \leq t < \tau^2,
\lim_{\xi \to -\tau^2^-} \beta(\xi) = 0, \quad \text{for } t \leq -\tau^2.
\]

(c) \( \gamma(\xi) = 0 \) if and only if \( t < 0 \) and \( 0 < \xi \leq -t\tau^2 \). Otherwise \( \xi \mapsto \gamma(\xi) \) is positive and strictly increasing with
\[
\lim_{\xi \to 0^+} \gamma(\xi) = \frac{2t^{3/2}}{3\sqrt{3}} = y^*, \quad \text{for } 0 \leq t < \tau^2,
\lim_{\xi \to -\tau^2^-} \gamma(\xi) = 0, \quad \text{for } t \leq 0,
\lim_{\xi \to \xi_{cr}^-} \gamma(\xi) = \lim_{\xi \to \xi_{cr}^+} \gamma(\xi)
\]

(d) \( \delta(\xi) = 0 \) if and only if \(-\tau^2 < t < \tau^2 \) and \(-t\tau^2 \leq \xi < \xi_{cr} \). Otherwise \( \xi \mapsto \delta(\xi) \) is positive and strictly decreasing with
\[
\lim_{\xi \to 0^+} \delta(\xi) = \frac{2(-t)^{3/2}}{3\sqrt{3}} = x^*, \quad \text{for } t \leq 0,
\lim_{\xi \to \xi_{cr}^-} \delta(\xi) = 0, \quad \text{for } t \leq -\tau^2,
\lim_{\xi \to -\tau^2^-} \delta(\xi) = 0, \quad \text{for } -\tau^2 \leq t < 0.
\]

Figure 9 shows sketches of the sets \( \Gamma_j(\xi) \), \( j = 1, 2, 3 \), for \((t, \xi)\) belonging to \( C_{2a}, C_{2b} \) and \( C_{2c} \).

By theorem 5.1 we know that the measures \( (\hat{\mu}^\xi_1, \hat{\mu}^\xi_2, \hat{\mu}^\xi_3) \) are characterized by a vector equilibrium problem. The transformed measures \( (\mu^\xi_1, \mu^\xi_2, \mu^\xi_3) \) from (6.14) then also are characterized by a vector equilibrium problem, as stated in the following theorem.

**Theorem 6.5.**

(a) The vector of measures \( (\mu^\xi_1, \mu^\xi_2, \mu^\xi_3) \) given in (6.14) is the unique minimizer for the functional
\[
E_0(\rho_1, \rho_2, \rho_3) = I(\rho_1) - I(\rho_1, \rho_2) + I(\rho_2) - I(\rho_2, \rho_3) + I(\rho_3),
\]
for all vectors \((\rho_1, \rho_2, \rho_3)\) of positive measures with finite logarithmic energy \( I(\rho_1) < \infty \), satisfying

(i) \( \text{supp}(\rho_j) \subset \Gamma_j(\xi) \), for \( j = 1, 2, 3 \) and
(ii) \( \rho_1(\Gamma_1(\xi)) = 1, \rho_2(\Gamma_2(\xi)) = 2/3 \) and \( \rho_3(\Gamma_3(\xi)) = 1/3 \).

(b) The measures \( \mu^\xi_1, \mu^\xi_2, \mu^\xi_3 \) satisfy for some constant \( \ell^\xi \)
\[
\ell^\xi - 2U^{\mu^\xi_1}(z) + U^{\mu^\xi_2}(z) = \frac{1}{2} \log \frac{|w_1(z; \xi)|}{|w_2(z; \xi)|}, \quad (6.16)
\]
\[
U^{\mu^\xi_1}(z) - 2U^{\mu^\xi_2}(z) + U^{\mu^\xi_3}(z) = \frac{1}{2} \log \frac{|w_2(z; \xi)|}{|w_3(z; \xi)|}, \quad (6.17)
\]
\[
U^{\mu^\xi_2}(z) - 2U^{\mu^\xi_3}(z) = \frac{1}{2} \log \frac{|w_3(z; \xi)|}{|w_4(z; \xi)|}, \quad (6.18)
\]
A vector equilibrium problem for the two-matrix model in the quartic/quadratic case

Figure 9. The sets $\Gamma_1(\xi)$ (plain), $\Gamma_2(\xi)$ (dashed) and $\Gamma_3(\xi)$ (dotted) for $(t, \xi) \in C_{2a}$ (a), $(t, \xi) \in C_{2b}$ (b) and $(t, \xi) \in C_{2c}$ (c).

**Proof.** It follows from (1.3) and (6.14) that

$$U^{\mu_1}(z) = \frac{1}{2} U^{\tilde{\rho}_1}(z^2).$$

Applying this to the variational conditions (5.5)–(5.7) in the context of the symbol $\tilde{s}_2$ and using (6.13) establishes (6.16)–(6.18).

7. **Proof of theorem 2.2**

We now give the proof of theorem 2.2, which is based on theorem 5.2. For fixed $t$, we define the measure $\nu_1$ as an average of the measures $\mu_1^\xi$

$$\nu_1 = \int_0^{1} \mu_1^\xi \, d\xi. \quad (7.1)$$

This measure will be the limiting normalized zero distribution of the polynomials $p_{n,n}$ as $n \to \infty$. Note that the measure $\mu_1^\xi$ is defined by (5.4) in the one-cut case (i.e. $\xi > \xi_{\text{cr}}$), and by (6.14) in the two-cut case.

7.1. **Proof of theorem 2.2 for $t \geq \tau^2$**

We first prove theorem 2.2 for $t \geq \tau^2$. This is the simplest case, since we deal with the one-cut case for every $\xi > 0$ and we can apply theorem 5.2 almost immediately.

**Proof.** Let $t \geq \tau^2$. Conditions (a) and (d) of theorem 5.2 follow from lemma 4.1 and theorem 4.2. The interlacing condition (b) follows from theorem 2.1 (a), and condition (e) is
contained in theorem 5.3. So in order to be able to apply theorem 5.2 it remains to establish condition (c). This will be done in the following lemma.

Having verified all conditions, theorem 5.2 can be applied and theorem 2.2 follows for $t \geq \tau^2$. 

To complete the preceding proof we still have to establish the following lemma. It states that the recurrence coefficients $b_{k,n}$ and $c_{k,n}$ remain bounded in case $k/n$ is bounded.

**Lemma 7.1.** Let $t \in \mathbb{R}$. Then the two sets of recurrence coefficients $\{b_{k,n} \mid k + 1 \leq n\}$ and $\{c_{k,n} \mid k + 1 \leq n\}$ are bounded.

**Proof.** By (4.3) and (4.4) it is sufficient to prove that the set

$$\{a_{k,n} \mid k + 1 \leq n\}$$

is bounded. From the asymptotics of the orthogonal polynomials $q_{n,n}$, see [4, 5], it follows that there exists $M > 0$ such that the zeros of the diagonal polynomials $q_{n,n}$, $n = 1, 2, 3, \ldots$, belong to $[-M, M]$. Then, by interlacing, see theorem 2.1, the zeros of all $q_{k,n}$ with $k \leq n$ belong to $[-M, M]$. Note that the zeros of $q_{k,n}$ coincide with the eigenvalues of the tridiagonal Jacobi matrix

$$J_{k,n} = \begin{pmatrix}
0 & \sqrt{a_{1,n}} & 0 & \cdots & 0 \\
\sqrt{a_{1,n}} & 0 & \sqrt{a_{2,n}} & \cdots & 0 \\
0 & \sqrt{a_{2,n}} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \sqrt{a_{k-1,n}} \end{pmatrix}.$$

It is well known that the eigenvalues of a real symmetric matrix $X$ interlace with the eigenvalues of the matrix obtained from $X$ by deleting the first row and column. Applying this $k - 2$ times, we find that the eigenvalues of

$$\begin{pmatrix}
0 & \sqrt{a_{k-1,n}} \\
\sqrt{a_{k-1,n}} & 0
\end{pmatrix}$$

are in $[-M, M]$, which implies that $a_{k-1,n} \leq M^2$ for every $k \leq n$. This proves the lemma. 

**7.2. Proof of theorem 2.2 for $t < \tau^2$**

The situation for $t < \tau^2$ is more complicated. For small values of $\xi$, namely $\xi < \xi_{cr}$, we are in the two-cut case, while for $\xi > \xi_{cr}$ we deal with the one-cut case.

The two-cut case was handled in section 6 by doubling the recurrence relation. This led to the symbol $\tilde{s}_2$ and related notions. It will be convenient to double the recurrence relation also in the one-cut case. The doubled recurrence relation is (6.1), and so for $\xi > \xi_{cr}$, we define

$$A(\xi) = \lim_{k/n \to \xi} A_{k,n} = 2b(\xi),$$

$$B(\xi) = \lim_{k/n \to \xi} B_{k,n} = 2c(\xi) + b(\xi)^2,$$

$$C(\xi) = \lim_{k/n \to \xi} C_{k,n} = 2b(\xi)c(\xi),$$

$$D(\xi) = \lim_{k/n \to \xi} D_{k,n} = c(\xi)^2.$$
and
\[ \hat{s}_1(w; \xi) = w + A(\xi) + \frac{B(\xi)}{w} + \frac{C(\xi)}{w^2} + \frac{D(\xi)}{w^3}. \]

The subscript 1 is added to recall that we are in the one-cut case. The hat refers to the fact that the recurrence relation was doubled to obtain this symbol.

As before, we consider the algebraic equation \( \hat{s}_1(w; \xi) = z \) for complex \( z \) and denote its solutions by \( \hat{w}_j(z; \xi) \), \( j = 1, 2, 3, 4 \), ordered such that
\[ |\hat{w}_1(z; \xi)| \geq |\hat{w}_2(z; \xi)| \geq |\hat{w}_3(z; \xi)| \geq |\hat{w}_4(z; \xi)|. \]

Furthermore, we define for \( \xi > \xi_{cr} \)
\[ \hat{\Gamma}_1(\xi) = \{ z \in \mathbb{C} | |\hat{w}_1(z; \xi)| = |\hat{w}_2(z; \xi)| \}, \]
and
\[ d\mu^\xi_1(x) = \frac{1}{2\pi i} \left( \frac{\hat{w}_1'(z)}{\hat{w}_1(z)} - \frac{\hat{w}_2'(z)}{\hat{w}_2(z)} \right) dz, \]
for \( z \in \hat{\Gamma}_1(\xi) \). The following lemma establishes the link between these notions and the corresponding ones related to the original symbol \( s_1 \).

**Lemma 7.2.** Let \( \xi > \xi_{cr} \) Then, \( z \in \Gamma_1(\xi) \) if and only if \( z^2 \in \hat{\Gamma}_1(\xi) \). Moreover,
\[ d\mu^\xi_1(x) = |x| d\hat{\mu}^\xi_1(x^2) dx, \quad x \in \Gamma_1(\xi). \]  

**Proof.** The proof of this lemma is straightforward. We omit the details. \( \square \)

We now prove theorem 2.2 if \( t < \tau^2 \).

**Proof of theorem 2.2.** for \( t < \tau^2 \). Let \( t < \tau^2 \).

For even \( k = 2l \), we have that \( p_{k,n} \) is an even polynomial and we can write
\[ p_{k,n}(x) = r_{l,n}(x^2), \]  
where \( r_{l,n} \) is a monic polynomial of degree \( l \). Rewriting the doubled recurrence relation 6.1 in terms of the polynomials \( r_{l,n} \) yields
\[ x r_{l+1,n}(x) = r_{l+1,n}(x) + \hat{A}_{l,n} r_{l,n}(x) + \hat{B}_{l,n} r_{l-1,n}(x) + \hat{C}_{l,n} r_{l-2,n}(x) + \hat{D}_{l,n} r_{l-3,n}(x), \]
where
\[ \hat{A}_{l,n} = A_{2l,n}, \quad \hat{B}_{l,n} = B_{2l,n}, \quad \hat{C}_{l,n} = C_{2l,n} \quad \text{and} \quad \hat{D}_{l,n} = D_{2l,n}. \]

Therefore, the polynomials \( r_{l,n} \) satisfy condition (a) of theorem 5.2.

We have for \( \xi > 0 \)
\[ \lim_{k/n \to \xi} A_{k,n} = A(\xi), \quad \lim_{k/n \to \xi} B_{k,n} = B(\xi), \quad \lim_{k/n \to \xi} C_{k,n} = C(\xi), \quad \lim_{k/n \to \xi} D_{k,n} = D(\xi), \]
where \( A(\xi), B(\xi), C(\xi) \) and \( D(\xi) \) are defined by (7.2)–(7.5) if \( \xi > \xi_{cr} \) and by (6.2)–(6.5) if \( \xi < \xi_{cr} \). It follows that
\[ \lim_{l/n \to \xi} \hat{A}_{l,n} = A(2\xi), \quad \lim_{l/n \to \xi} \hat{B}_{l,n} = B(2\xi), \quad \lim_{l/n \to \xi} \hat{C}_{l,n} = C(2\xi), \quad \lim_{l/n \to \xi} \hat{D}_{l,n} = D(2\xi). \]

This establishes condition (d) of theorem 5.2. Condition (c) follows from lemma 7.1 and (b) from theorem 2.1(b). For condition (e) we need that \( \hat{\Gamma}_1(2\xi) \subset \mathbb{R} \). If \( 2\xi > \xi_{cr} \), this is
guaranteed by lemma 7.2 and theorem 5.3. If \(2\xi \in (0, \xi_{cr})\), this follows from theorem 6.1. Applying theorem 5.2, we obtain
\[
\lim_{n \to 1/2} \nu_{n} = 2 \int_{0}^{1/2} \hat{\mu}_{1}^{2\xi} d\xi = \int_{0}^{1} \hat{\mu}_{1}^{\xi} d\xi.
\]
Then by (7.7), (6.14) and (7.6), we obtain
\[
\lim_{n \to \infty} \nu_{n} = \int_{0}^{1} \hat{\mu}_{1}^{\xi} d\xi = \int_{0}^{1} \hat{\mu}_{1}^{\xi} d\xi.
\]
by the definition of \(\nu_{1}\). In a similar way one finds the same limit \(\nu_{1}\) for the subsequence of odd \(n\), completing the proof of theorem 2.2. Alternatively, this follows from the interlacing of zeros. □

8. Proof of theorem 2.3

8.1. Averaging the vector equilibrium problems

In this section we prove that the limiting zero distribution \(\nu_{1}\) can be characterized by a vector equilibrium problem for three measures with external fields acting on the first and the third measure, and a constraint acting on the second measure, see theorem 8.1. Recall that \(\nu_{1}\) is given by (7.1). Similarly we define the measures \(\nu_{2}\) and \(\nu_{3}\) by
\[
\nu_{j} = \int_{0}^{1} \mu_{j}^{\xi} d\xi, \quad j = 2, 3. \tag{8.1}
\]
It follows from theorem 5.3 and theorem 6.1 that the supports of \(\mu_{j}^{\xi}\) as a function of \(\xi\) are increasing if \(j = 1, 3\) and decreasing if \(j = 2\). Hence,
\[
supp(\nu_{j}) = \bigcup_{0 < \xi \leq 1} supp(\mu_{j}^{\xi}) = \begin{cases} 
\Gamma_{1}(1) \subset \mathbb{R}, & \text{for } j = 1, \\
\Gamma_{2}(0+) \subset i\mathbb{R}, & \text{for } j = 2, \\
\Gamma_{3}(1) \subset \mathbb{R}, & \text{for } j = 3.
\end{cases}
\]
We also know that \(\nu_{1}(\mathbb{R}) = 1, \nu_{2}(i\mathbb{R}) = 2/3, \nu_{3}(\mathbb{R}) = 1/3\).

The vector of measures \((\nu_{1}, \nu_{2}, \nu_{3})\) is characterized by a vector equilibrium problem.

**Theorem 8.1.** Define
\[
\tilde{V}_{1}(x) = \frac{1}{2} \int_{0}^{\xi_{cr}} \log \left| \frac{w_{1}(x; \xi)}{w_{2}(x; \xi)} \right| d\xi + \int_{\xi_{cr}}^{\infty} \log \left| \frac{w_{1}(x; \xi)}{w_{2}(x; \xi)} \right| d\xi, \quad x \in \mathbb{R}, \tag{8.2}
\]
\[
\tilde{\sigma} = \int_{0}^{\infty} \mu_{2}^{\xi} d\xi, \tag{8.3}
\]
\[
\tilde{V}_{3}(x) = \frac{1}{2} \int_{0}^{\xi_{cr}} \log \left| \frac{w_{3}(x; \xi)}{w_{4}(x; \xi)} \right| d\xi + \int_{\xi_{cr}}^{\infty} \log \left| \frac{w_{3}(x; \xi)}{w_{4}(x; \xi)} \right| d\xi, \quad x \in \mathbb{R}. \tag{8.4}
\]
Then, \((\nu_{1}, \nu_{2}, \nu_{3})\) is the unique vector of measures minimizing the energy functional
\[
E(\rho_{1}, \rho_{2}, \rho_{3}) = I(\rho_{1}) - I(\rho_{1}, \rho_{2}) + I(\rho_{2}) - I(\rho_{2}, \rho_{3}) + I(\rho_{3}) + \int \tilde{V}_{1}(x) d\rho_{1}(x) + \int \tilde{V}_{3}(x) d\rho_{3}(x),
\]
among all measures $\rho_1$, $\rho_2$ and $\rho_3$ satisfying
(a) $\rho_1$ is supported on $\mathbb{R}$ and $\rho_1(\mathbb{R}) = 1$,
(b) $\rho_2$ is supported on $i\mathbb{R}$ and $\rho_2(i\mathbb{R}) = 2/3$,
(c) $\rho_3$ is supported on $\mathbb{R}$ and $\rho_3(\mathbb{R}) = 1/3$,
(d) $\rho_2$ satisfies the constraint $\rho_2 \leq \tilde{\sigma}$.

**Proof.** We have already shown that $(v_1, v_2, v_3)$ satisfies the conditions (a), (b) and (c). Also (d) holds, because of the definitions (8.1) and (8.3). Since $\Gamma_2(\xi)$ is a set that decreases as $\xi$ increases, we have

$$\text{supp}(\tilde{\sigma} - v_2) = \Gamma_2(1).$$

Since the energy functional is strictly convex it is sufficient to show that the Euler–Lagrange variational conditions associated with the minimization problem are satisfied, namely

$$2U^{v_1}(x) - U^{v_2}(x) + \tilde{V}_1(x) \begin{cases} = \ell & \text{for } x \in \text{supp}(v_1), \\ > \ell & \text{for } x \in \mathbb{R} \setminus \text{supp}(v_1), \end{cases} \quad (8.5)$$

for some $\ell$,

$$- U^{v_1}(z) + 2U^{v_2}(z) - U^{v_3}(z) \begin{cases} = 0 & \text{for } z \in \text{supp}(\tilde{\sigma} - v_2), \\ < 0 & \text{for } z \in i\mathbb{R} \setminus \text{supp}(\tilde{\sigma} - v_2), \end{cases} \quad (8.6)$$

and

$$- U^{v_2}(x) + 2U^{v_3}(x) + \tilde{V}_3(x) \begin{cases} = 0 & \text{for } x \in \text{supp}(v_3), \\ > 0 & \text{for } x \in \mathbb{R} \setminus \text{supp}(v_3). \end{cases} \quad (8.7)$$

We establish (8.5)–(8.7) by integrating the Euler–Lagrange variational conditions (5.5)–(5.7)

$$\ell^\xi - 2U^{\mu_1}(z) + U^{\mu_2}(z) = \log \left| \frac{w_1(z; \xi)}{w_2(z; \xi)} \right|, \quad (8.8)$$

$$U^{\mu_1}(z) - 2U^{\mu_2}(z) + U^{\mu_3}(z) = \log \left| \frac{w_2(z; \xi)}{w_3(z; \xi)} \right|, \quad (8.9)$$

$$U^{\mu_2}(z) - 2U^{\mu_1}(z) = \log \left| \frac{w_3(z; \xi)}{w_4(z; \xi)} \right|, \quad (8.10)$$

if $\xi > \xi_{\text{cr}}$ and (6.16)–(6.18)

$$\ell^\xi - 2U^{\mu_1}(z) + U^{\mu_2}(z) = \frac{1}{2} \log \left| \frac{w_1(z; \xi)}{w_2(z; \xi)} \right|, \quad (8.11)$$

$$U^{\mu_1}(z) - 2U^{\mu_2}(z) + U^{\mu_3}(z) = \frac{1}{2} \log \left| \frac{w_2(z; \xi)}{w_3(z; \xi)} \right|, \quad (8.12)$$

$$U^{\mu_2}(z) - 2U^{\mu_1}(z) = \frac{1}{2} \log \left| \frac{w_3(z; \xi)}{w_4(z; \xi)} \right|, \quad (8.13)$$

for $\xi \in (0, \xi_{\text{cr}})$ with respect to $\xi$.

Integrating (8.11) from 0 to $\min(\xi_{\text{cr}}, 1)$ and (8.8) from $\min(\xi_{\text{cr}}, 1)$ to 1 yields

$$\ell - 2U^{v_1}(z) + U^{v_3}(z) = \frac{1}{2} \int_0^{\min(\xi_{\text{cr}}, 1)} \log \left| \frac{w_1(z; \xi)}{w_2(z; \xi)} \right| d\xi + \int_{\min(\xi_{\text{cr}}, 1)}^1 \log \left| \frac{w_1(z; \xi)}{w_2(z; \xi)} \right| d\xi,$$

for some constant $\ell \in \mathbb{R}$.

Let $x \in \mathbb{R}$. Since $|w_1(x; \xi)| \geq |w_2(x; \xi)|$ for every $\xi > 0$, we can extend the integration to infinity and obtain an inequality

$$\ell - 2U^{v_1}(x) + U^{v_3}(x) \leq \tilde{V}_3(x), \quad (8.14)$$
since \( \tilde{V}_1 \) is given by (8.2). Equality holds in (8.14) if and only if \(|w_1(x; \xi)| = |w_2(x; \xi)|\) for every \( \xi > 1 \). That is, if and only if \( x \in \bigcap_{\xi > 1} \Gamma_1(\xi) = \Gamma_1(1) = \text{supp}(v_1) \).

The first equality holds since the sets \( \Gamma_j(\xi) \) are increasing as \( \xi \) increases. This proves (8.5).

The proof of (8.7) is similar.

Integrating (8.12) from 0 to \( \min(\xi_{cr}, 1) \) and (8.9) from \( \min(\xi_{cr}, 1) \) to 1 yields

\[
U^{w_1}(z) - 2U^{w_2}(z) + U^{w_3}(z) = \frac{1}{2} \int_{0}^{\min(\xi_{cr}, 1)} \log \left| \frac{w_2(z; \xi)}{w_3(z; \xi)} \right| \, d\xi + \int_{\min(\xi_{cr}, 1)}^{1} \log \left| \frac{w_2(z; \xi)}{w_3(z; \xi)} \right| \, d\xi.
\]

Since \(|w_2(z; \xi)| \geq |w_3(z; \xi)|\) for every \( \xi > 0 \) it follows that

\[
U^{w_1}(z) - 2U^{w_2}(z) + U^{w_3}(z) \geq 0, \quad z \in \mathbb{C}. \tag{8.15}
\]

Equality holds in (8.15) if and only if \(|w_2(z; \xi)| = |w_3(z; \xi)|\) for every \( \xi \in (0, 1) \). That is, if and only if

\[
z \in \bigcap_{0 < \xi < 1} \Gamma_2(\xi) = \Gamma_2(1) = \text{supp}(\tilde{\sigma} - v_2).
\]

The first equality holds since the sets \( \Gamma_2(\xi) \) are decreasing as \( \xi \) increases. This proves (8.6).

This completes the proof of theorem 8.1

\[ \square \]

8.2. Auxiliary lemmas

From theorem 8.1 we know that \((v_1, v_2, v_3)\) is the minimizer for a vector equilibrium problem with external fields and an upper constraint. To complete the proof of theorem 2.3 we evaluate the external fields \( \tilde{V}_1 \) and \( \tilde{V}_3 \) and the measure \( \tilde{\sigma} \) and show that they are equal to the external fields \( V_1 \) (up to a constant) and \( V_3 \) and the constraint \( \sigma \) that appear in theorem 2.3.

Although the calculations involved may not look too elegant, we think it is remarkable that they can be performed at all. We start by defining and investigating two functions. We define for \( j = 1, 2, 3, 4 \),

\[
F_j(z, \xi) = z - \tau^2 \frac{a(\xi)}{w_j(z; \xi)} - w_j(z; \xi), \quad \xi > \xi_{cr}, \quad z \in \mathbb{C}, \tag{8.16}
\]

where \( w_j(z; \xi) \) is defined as in section 5.2. If \( t < \tau^2 \) we also introduce for \( j = 1, 2, 3, 4 \),

\[
G_j(z, \xi) = \frac{z\xi}{w_j(z; \xi)} + \xi, \quad 0 < \xi < \xi_{cr}, \quad z \in \mathbb{C}, \tag{8.17}
\]

where \( w_j(z; \xi) \) is defined by (6.13). We will establish three lemmas concerning these functions. The first lemma states that \( F_j \) and \( G_j \) are antiderivatives of the integrands in (8.2) and (8.4). This explains our interest in these functions. The second lemma states that \( F_j \) and \( G_j \) continuously connect to each other at the boundary point \( \xi_{cr} \) of their domains of definition. The third lemma gives the limiting behaviour of \( F_j(z, \xi) \) and \( G_j(z, \xi) \) as \( \xi \to 0 \).

**Lemma 8.2.** Let \( \cdot \) denote the partial derivative with respect to \( z \).

(a) For every \( t \in \mathbb{R} \) and \( z \in \mathbb{C} \) the equality

\[
\frac{\partial F_j(z, \xi)}{\partial \xi} = \frac{w_j'(z; \xi)}{w_j(z; \xi)}, \quad \xi > \xi_{cr} \tag{8.18}
\]

holds. Here, \( w_j(z; \xi) \) is defined as in section 5.2.
A vector equilibrium problem for the two-matrix model in the quartic/quadratic case

(b) For every \( t < \tau^2 \) and \( z \in \mathbb{C} \) the equality

\[
\frac{\partial G_j}{\partial \xi}(z, \xi) = \frac{1}{2} \frac{w_j'(z; \xi)}{w_j(z; \xi)}, \quad 0 < \xi < \xi_{cr},
\]  

holds. Here, \( w_j(z; \xi) \) is defined by (6.13).

**Proof.** (a) Let \( \xi > \xi_{cr}, z \in \mathbb{C} \) and define \( W_j(z; \xi) \) for \( j = 1, 2, 3, 4 \) as

\[
W_j(z; \xi) = \frac{w_j(z; \xi)}{a(\xi)}. \tag{8.20}
\]

Since \( s_1(w_j(z; \xi)) = z \) we find by (5.13) and the explicit expressions (4.6) and (4.7) for \( b(\xi) \) and \( c(\xi) \) that

\[
S_1(W_j(z; \xi); \xi) = a(\xi) \left( W_j(z; \xi) + 3 W_j(z; \xi)^2 + t \frac{\tau^2}{W_j(z; \xi)} \right)^2 = z, \tag{8.21}
\]

see (5.22) for the definition of \( S_1(W; \xi) \). By (8.17) and (8.20), we can write

\[
F_j(z, \xi) = z - \frac{\tau^2}{W_j(z; \xi)} - a(\xi)W_j(z; \xi), \tag{8.22}
\]

so that

\[
\frac{\partial F_j}{\partial \xi}(z, \xi) = \frac{\partial W_j}{\partial \xi}(z; \xi) \tau^2 - t - \frac{6}{W_j(z; \xi)^2} + \frac{\tau^2}{W_j(z; \xi)} \frac{\partial W_j}{\partial \xi}(z; \xi). \tag{8.23}
\]

Taking the derivative of (8.21) with respect to \( \xi \), we eliminate \( a'(\xi) \) from (8.23) to obtain after some calculations

\[
\frac{\partial F_j}{\partial \xi}(z, \xi) = \frac{\partial W_j}{\partial \xi}(z; \xi) \frac{\tau^2 - t - 6a(\xi)}{W_j(z; \xi)^2 + 3} - \frac{1}{a(\xi)} \frac{\partial W_j}{\partial \xi}(z; \xi), \tag{8.24}
\]

where the second equality follows from (4.5). Calculating partial derivatives of (8.21) with respect to \( z \) and \( \xi \) yields

\[
\frac{\partial S_1}{\partial W} \frac{\partial W_j}{\partial z} = 1, \quad \frac{\partial S_1}{\partial W} \frac{\partial W_j}{\partial \xi} + \frac{\partial S_1}{\partial \xi} = 0,
\]

from which we obtain

\[
\frac{\partial W_j}{\partial \xi} = \frac{\partial W_j}{\partial z} \frac{\partial S_1}{\partial \xi} = \frac{\partial W_j}{\partial z} \frac{a'(\xi)}{W_j(z; \xi)} + \frac{W_j^2 + 3}{W_j},
\]

where the last equality follows from (5.22). Now, combine this with (8.24) to obtain

\[
\frac{\partial F_j}{\partial \xi}(z, \xi) = \frac{W_j'(z; \xi)}{W_j(z; \xi)}. \tag{8.25}
\]

Recalling (8.20) completes the proof of part (a).

(b) The proof of part (b) follows along the same lines. Let \( 0 < \xi < \xi_{cr} \) and \( z \in \mathbb{C} \). Define \( \tilde{W}_j(z; \xi) \) for \( j = 1, 2, 3, 4 \) by

\[
\tilde{W}_j(z; \xi) = \frac{\tilde{w}_j(z; \xi)}{\xi}, \tag{8.25}
\]

so that \( \tilde{W}_j(z^2; \xi) = \frac{w_j(z; \xi)}{\xi} \), see (6.13). Then by (8.17) we can write

\[
G_j(z, \xi) = \frac{z}{W_j(z^2; \xi) + 1}. \tag{8.26}
\]
\[ \frac{\partial G_j}{\partial \xi}(z, \xi) = -\frac{\partial \hat{W}_j}{\partial \xi}(z^2; \xi) \frac{z}{(\hat{W}_j(z^2; \xi) + 1)^2}. \quad (8.27) \]

Recall the definition (6.9) of the transformed symbol \( \hat{S}_2 \) and note that by the definition (8.25)

\[ \hat{S}_2(\hat{W}_j(z; \xi); \xi) = \xi (\hat{W}_j(z; \xi) + 1) \hat{W}_j(z^2; \xi) - t \tau - \frac{\xi}{2} \hat{W}_j(z^2; \xi)^2, \quad (8.28) \]

see (6.9). Calculating partial derivatives of (8.28) with respect to \( z \) and \( \xi \) yields

\[ \frac{\partial \hat{S}_2}{\partial W} \frac{\partial \hat{W}_j}{\partial z} = 1, \quad \frac{\partial \hat{S}_2}{\partial W} \frac{\partial \hat{W}_j}{\partial \xi} + \frac{\partial \hat{S}_2}{\partial \xi} = 0. \]

Therefore,

\[ -\frac{\partial \hat{W}_j}{\partial \xi} = \frac{\partial \hat{W}_j}{\partial z} \frac{\partial \hat{S}_2}{\partial W} = \frac{\partial \hat{W}_j}{\partial Z} \hat{W}_j + 1 \]

where the last equality follows from (6.9). If we combine this with (8.27) we obtain

\[ \frac{\partial G_j}{\partial \xi}(z, \xi) = z \hat{W}_j'(z^2; \xi) \frac{\hat{W}_j(z^2; \xi)^2}{\hat{W}_j(z^2; \xi)}. \]

This completes the proof of part (b) because of (8.25). \( \square \)

**Lemma 8.3.** Assume \( t < \tau^2 \). Then for \( z \in \mathbb{C} \) and \( j = 1, 2, 3, 4 \) the equality of the limits

\[ \lim_{\xi \to \xi_\tau^+} G_j(z, \xi) = \lim_{\xi \to \xi_\tau^-} F_j(z, \xi) \]

holds. Here, \( F_j \) and \( G_j \) are defined as in (8.16) and (8.17).

**Proof.** We claim that

\[ \lim_{\xi \to \xi_\tau^+} W_j(z; \xi)^2 = \lim_{\xi \to \xi_\tau^-} \hat{W}_j(z^2; \xi). \]

To prove the claim, observe that, as a corollary of (6.12),

\[ \lim_{\xi \to \xi_\tau^+} w_j(z; \xi)^2 = \lim_{\xi \to \xi_\tau^-} \hat{w}_j(z^2; \xi). \]

Then, from (4.5) it follows that

\[ \lim_{\xi \to \xi_\tau} a(\xi) = \frac{\tau^2 - t}{2}, \quad (8.30) \]

so that also

\[ \lim_{\xi \to \xi_\tau^+} W_j(z; \xi)^2 = \left( \frac{\tau^2 - t}{2} \right)^{-2} \lim_{\xi \to \xi_\tau^-} w_j(z; \xi)^2 = \lim_{\xi \to \xi_\tau^-} \frac{\hat{w}_j(z^2; \xi)}{\xi^{\alpha}} = \lim_{\xi \to \xi_\tau^-} \hat{W}_j(z^2; \xi), \]

which proves the claim.

Note also that (8.30) and (8.21) yield

\[ z = \lim_{\xi \to \xi_\tau^+} \left( \frac{\tau^2 - t}{2} W_j(z; \xi) + \frac{3 \tau^2 - t}{2} W_j(z; \xi) + \frac{\tau^2}{W_j(z; \xi)^3} \right). \quad (8.31) \]
Given this, the proof of (8.29) comes down to a calculation. Using (8.22) and (8.31), we can rewrite the right-hand side of (8.29) as
\[
\lim_{\xi \to \xi_0^+} F_j(z, \xi) = \lim_{\xi \to \xi_0^+} \left( \frac{\tau^2 - t}{2} \frac{1}{W_j(z; \xi)} + \frac{\tau^3}{W_j(z; \xi)^3} \right) = \lim_{\xi \to \xi_0^+} \frac{z}{W_j(z; \xi)^2 + 1},
\]
where also the second equality follows from (8.31). Lemma 8.3 then follows from the claim and equation (8.26).

We also need the limiting behaviour of the \(F_j\) and \(G_j\) functions as \(\xi \to 0^+\). Here we make a connection with the function \(\omega_1\) that is defined in section 2.3, and the functions \(\omega_2\) and \(\omega_3\) that are defined on the interval \([-x^*, x^*]\) in case \(t < 0\).

**Lemma 8.4.**

(a) If \(t \geq \tau^2\), then
\[
\lim_{\xi \to 0^+} F_1(x, \xi) = 0, \quad \text{for } x \in \mathbb{R},
\]
\[
\lim_{\xi \to 0^+} F_2(x, \xi) = x - \tau \omega_1(x), \quad \text{for } x \in \mathbb{R}.
\]
(b) If \(t < \tau^2\), then
\[
\lim_{\xi \to 0^+} G_1(x, \xi) = 0, \quad \text{for } x \in \mathbb{R},
\]
\[
\lim_{\xi \to 0^+} G_2(x, \xi) = x - \tau \omega_1(x), \quad \text{for } x \in \mathbb{R}.
\]

If \(t < 0\), then also for \(j = 3, 4\),
\[
\lim_{\xi \to 0^+} G_j(x, \xi) = x - \tau \omega_{j-1}(x), \quad \text{for } x \in [-x^*, x^*].
\]

(c) If \(t \geq \tau^2\) and \(z \in (-i\infty, -iy^*] \cup [iy^*, +i\infty)\), then
\[
\lim_{\xi \to 0^+} (F_2(z^+, \xi) - F_2(z^-, \xi)) = 2\tau \text{Re} \omega_1(z),
\]
where \(F_2(z^+, \xi) = \lim_{h \to 0^+} F_2(z^+ h, \xi)\).

(d) If \(t < \tau^2\) and \(z \in (-i\infty, -iy^*] \cup [iy^*, +i\infty)\), then
\[
\lim_{\xi \to 0^+} (G_2(z^+, \xi) - G_2(z^-, \xi)) = 2\tau \text{Re} \omega_1(z),
\]
where \(G_2(z^+, \xi) = \lim_{h \to 0^+} G_2(z^+ h, \xi)\).

**Proof.**

(a) Let \(t \geq \tau^2\). For \(\xi > 0\), we define
\[
\omega_j(z; \xi) = \frac{\tau a(\xi)}{w_{j+1}(z; \xi)}, \quad j = 0, 1, 2, 3.
\]
Since \(s_j(w_j(z; \xi); \xi) = z, j = 1, 2, 3, 4\), see (5.13), we obtain that (8.39) are the four solutions of the equation
\[
a(\xi) \left( \frac{\tau^2}{\omega^2} + 3\omega \right) + t\omega + \omega^3 = \tau z,
\]
ordered such that
\[
|\omega_0(z; \xi)| \leq |\omega_1(z; \xi)| \leq |\omega_2(z; \xi)| \leq |\omega_3(z; \xi)|.
\]
As $\xi \to 0^+$, we have that $a(\xi) \to 0$. Then, (8.40) has one solution that tends to 0 as well, hence
\[
\lim_{\xi \to 0^+} \omega_0(z; \xi) = 0,
\]
while the other solutions tend to the solutions of the cubic equation
\[
\omega^3 + t\omega = \tau z,
\]
that we already encountered in (2.2).

In terms of (8.39), $F_j(z, \xi)$ can be rewritten as
\[
F_j(z, \xi) = z - \tau a_{j-1}(z; \xi) - \frac{\tau a(\xi)}{a_{j-1}(z; \xi)}, \quad j = 1, 2, 3, 4.
\]
Moreover, using (8.40) we get
\[
F_j(z, \xi) = \frac{\omega_0(z; \xi)}{\tau} \left(3a(\xi) + t - \tau^2 + \omega_0(z; \xi)^2\right),
\]
from which (8.32) follows by letting $\xi \to 0$.

Let $x \in \mathbb{R}$ with $x \neq 0$. Then by theorem 5.3 (a) we have $x \notin \Gamma_1(\xi)$ for sufficiently small $\xi$. Thus $w_1(x; \xi)$ and $w_2(x; \xi)$ are real, while $w_3(x; \xi)$ and $w_4(x; \xi)$ are non-real and complex conjugate, see figure 4(a). Hence, by (8.39), $\omega_1(x; \xi)$ is real for small $\xi$ and, therefore, converges as $\xi \to 0$ to a real solution of (2.2). In the present situation there is only one real solution, which was previously defined as $\omega_1(x)$. Thus $\omega_1(x; \xi) \to \omega_1(x)$ as $\xi \to 0^+$ for $x \in \mathbb{R}$ with $x \neq 0$, but in view of continuity it also holds for $x = 0$. Then, (8.33) is obtained by taking the limit $\xi \to 0$ in (8.41) with $j = 1$ and noting that $a(\xi) \to 0$.

(b) We prove part (b) only for $t < 0$ and $x \in [0, x^+]$. The proof for the other cases follows from similar considerations.

For $t < 0$, we recall that
\[
\tilde{\delta}_j(w_j(z; \xi); \xi) = \tau^2, \quad j = 1, 2, 3, 4,
\]
see section 6.1 and (6.13). By figure 5 we are in the case $C_{2b}$ for small enough $\xi$. See figure 7 for the graph of $\tilde{\delta}_2$ in this case.

Let $x \in [0, x^+]$. Then $x^2 < \tilde{\delta}(\xi)$ for sufficiently small $\xi$ by part (d) of theorem 6.1. It then follows that $w_3(x; \xi)$ and $w_4(x; \xi)$ are real. We now distinguish two cases depending on whether $x^2$ is smaller than or not.

- If $x^2 < \tau^2(\tau^2 - t)$ then it follows from part (b) of theorem 6.1 that $x^2 < \tilde{\delta}(\xi)$ for small $\xi$.

Then by (8.42) and figure 7 we have
\[
w_1(x; \xi) < w_2(x; \xi) < -\xi < w_3(x; \xi) < w_4(x; \xi) < 0.
\]
By the definition (8.17) this implies
\[
G_2(x; \xi) < G_1(x; \xi) < 0 < x < G_4(x; \xi) < G_3(x; \xi).
\]
- If $x^2 > \tau^2(\tau^2 - t)$ then by part (a) of theorem 6.1 we have that $x^2 > \tilde{\delta}(\xi)$ for small enough $\xi$. In this case in figure 7 the local minimum $\tilde{\delta}(\xi)$ at $w_0^*$ is smaller than the local maximum $\tilde{\delta}(\xi)$ at $w_1^*$. We then get by (8.42) and figure 7 that
\[-\xi < w_3(x; \xi) < w_4(x; \xi) < 0 < w_2(x; \xi) < w_1(x; \xi).
\]
Thus by (8.17)
\[
0 < G_1(x; \xi) < G_2(x; \xi) < x < G_4(x; \xi) < G_3(x; \xi).
\]
Next, we get from (8.42) and (8.17) that the \( G_j(x; \xi) \), \( j = 1, \ldots, 4 \), are the four solutions of
\[
G^4 - 3xG^3 + (3x^2 + i\tau^2 + \xi)G^2 + (-x^3 + \tau^4x - i\tau^2x - 2\xi x)G + \xi x^2 = 0.
\]
Letting \( \xi \to 0 \), we see that one of the \( G_j \)'s tends to 0 while the other three tend to the solutions of
\[
G^3 - 3xG^2 + (3x^2 + i\tau^2)G - x^3 + \tau^4 (\tau^2 - i) x = 0. \tag{8.45}
\]
In view of (8.43) and (8.44) it must be \( G_j(x; \xi) \) that tends to 0 and so we proved (8.34).

Now let \( \omega_j(x; \xi) = \frac{1}{\tau}(x - G_{j+1}(x; \xi)) \), so that
\[
G_j(x; \xi) = x - \tau \omega_{j-1}(x; \xi), \quad j = 2, 3, 4. \tag{8.46}
\]
It then follows from (8.43) and (8.44) that in both cases
\[
\omega_1(x; \xi) > 0 > \omega_2(x; \xi) > \omega_3(x; \xi). \tag{8.47}
\]
As \( \xi \to 0 \), we have that \( G_j(x; \xi) \), \( j = 2, 3, 4 \) tend to the three solutions of (8.45). Then from (8.46) we get that \( \omega_j(x; \xi) \), \( j = 1, 2, 3 \), tend to the solutions of
\[
\omega^3 + t \omega = \tau x. \tag{8.48}
\]
In view of (8.47) and the earlier definitions of \( \omega_j(x) \), \( j = 1, 2, 3 \), it is then easy to check that
\[
\lim_{\xi \to +0} \omega_j(x; \xi) = \omega_j(x).
\]
This proves (8.35) and (8.36) because of (8.46).

(c) Let \( t \geq \tau^2 \) and \( z \in (-i\infty, -i\Im^*) \cup (i\Im^*, i\infty) \). Recall that \( \omega_1(z) \) is defined as the solution of \( \omega^3 + t \omega = \tau z \) with positive real part. As in the proof of part (a) we use again the definition (8.39) for \( \omega_j(z; \xi) \). These functions are defined with possible cuts on parts of the real and imaginary axis. We prove that
\[
\lim_{\xi \to +0} \omega_{1, -}(z; \xi) = \omega_1(z). \tag{8.49}
\]
where \( \omega_{1, -}(z; \xi) \) denotes the limiting value on the imaginary axis taken from the right half plane.

To prove (8.49), we note that, by theorem 5.3(b), \( z \in \Gamma_2(\xi) \) for sufficiently small \( \xi \). Then, \( w_{2, +}(z; \xi) \) and \( w_{2, -}(z; \xi) \) have the same imaginary part, but opposite real part. Note that \( w_{2}(z; \xi) \) is positive for sufficiently large \( x > 0 \) and that \( w_{2}(z; \xi) \) does not take purely imaginary values for \( z \) in the right half plane because it solves the algebraic equation (5.19). It then follows from the continuity of \( w_{2}(z; \xi) \) that \( w_{2, -}(z; \xi) \) has positive real part. Then, by (8.39), also \( \omega_{1, -}(z, \xi) \) has positive real part for small enough \( \xi \). This proves (8.49).

Since \( w_{2, +}(z; \xi) \) and \( w_{2, -}(z; \xi) \) have opposite real part, we know that
\[
F_2(z+, \xi) - F_2(z-, \xi) = -2\Re F_2(z-, \xi) = 2\tau \Re \left( \frac{\omega_{1, -}(z; \xi) - \tau a(\xi)}{\omega_{1, -}(z; \xi)} \right).
\]
Then, (8.37) follows by letting \( \xi \to 0 \) and using (8.49) and \( a(\xi) \to 0 \).

(d) Part (d) can be proved in a similar way. We do not give details. \( \square \)
8.3. Proof of theorem 2.3

Theorem 2.3 follows immediately from theorems 2.2 and 8.1 and the following result that connects $\tilde{V}_1$, $\tilde{V}_3$ and $\tilde{\sigma}$ as defined in theorem 8.1, with $V_1$, $V_3$ and $\sigma$ that are defined in section 2.3.

**Theorem 8.5.** There is a constant $C$, depending on $t$ and $\tau$ such that

$$\tilde{V}_1(x) = V_1(x) + C, \quad x \in \mathbb{R}. \quad (8.50)$$

Furthermore we have

$$\tilde{V}_3(x) = V_3(x), \quad x \in \mathbb{R}, \quad (8.51)$$

and

$$\tilde{\sigma} = \sigma. \quad (8.52)$$

**Proof of (8.50).** We distinguish two cases.

**Case 1.** $t \geq \tau^2$. In this case (8.2) takes the simple form

$$\tilde{V}_1(x) = \int_0^{\infty} \log \left| \frac{w_1(x; \xi)}{w_2(x; \xi)} \right| d\xi.$$  

Fix $x \in \mathbb{R}$. As $\Gamma_1(\xi)$ is an unboundedly increasing set, there exists $\xi^*(x) \geq 0$ such that $x \in \Gamma_1(\xi)$ if $\xi \geq \xi^*(x)$ and $x \notin \Gamma_1(\xi)$ if $\xi < \xi^*(x)$. If $x \in \Gamma_1(\xi)$, we have that $|w_1(x; \xi)| = |w_2(x; \xi)|$, so that the upper bound in the integral can be replaced by $\xi^*(x)$. The derivative of $\tilde{V}_1$ can be written as

$$\tilde{V}_1'(x) = \lim_{\xi \to 0} \int_{\xi^*(x)}^{\xi} \left( \frac{w_1'(x; \xi)}{w_1(x; \xi)} - \frac{w_2'(x; \xi)}{w_2(x; \xi)} \right) d\xi.$$  

Part (a) of lemma 8.2 now yields

$$\tilde{V}_1'(x) = \lim_{\xi \to 0} \left( F_2(x, \xi) - F_1(x, \xi) \right) + \left( F_1(x, \xi^*(x)) - F_2(x, \xi^*(x)) \right). \quad (8.53)$$

By definition of $\xi^*(x)$, $x$ is on the boundary of $\Gamma_1(\xi^*(x))$. Therefore, $x$ is one of the branch points $\pm \alpha(\xi^*(x))$. The algebraic equation $s_1(w; \xi^*(x)) = x$ then has the double solution $w_1(x; \xi^*(x)) = w_2(x; \xi^*(x))$, so that the last terms of (8.53) vanish, see (8.16). The first terms can be handled by (8.32) and (8.33). We obtain

$$\tilde{V}_1'(x) = x - \tau \omega_1(x).$$

Using (2.2), we can integrate this equation with respect to $x$

$$\tilde{V}_1(x) = \frac{x^2}{2} - \int \left( 3\omega_1(x)^2 + t \omega_1(x) \right) d\omega_1(x) = \frac{x^2}{2} - \frac{3}{4} \omega_1(x)^4 - \frac{1}{2} t \omega_1(x)^2 + C,$$

where $C$ is a constant of integration, which proves (8.50) in view of (2.5).

**Case 2.** $t < \tau^2$. $\tilde{V}_1$ is given by

$$\tilde{V}_1(x) = \frac{1}{2} \int_{\xi^*}^{\infty} \log \left| \frac{w_1(x; \xi)}{w_2(x; \xi)} \right| d\xi + \int_{\xi^*}^{\infty} \log \left| \frac{w_1(x; \xi)}{w_2(x; \xi)} \right| d\xi.$$  

Fix $x \in \mathbb{R}$. The set $\Gamma_1(\xi)$ is unboundedly increasing if $\xi$ increases. Therefore, there exists $\xi^*(x) \geq 0$ such that $x \in \Gamma_1(\xi)$ if $\xi \geq \xi^*(x)$ and $x \notin \Gamma_1(\xi)$ if $\xi < \xi^*(x)$.
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Assume that $\xi^*(x) > \xi_{cr}$. The other possibility, $\xi^*(x) \leq \xi_{cr}$, is simpler and will be left to the reader. We obtain for the derivative of $\tilde{V}_1$

$$\tilde{V}'_1(x) = \frac{1}{2} \lim_{\xi_1 \to 0^+} \lim_{\xi_2 \to \xi_{cr}^-} \int_{\xi_1}^{\xi_2} \left( \frac{w_1'(x; \xi)}{w_1(x; \xi)} - \frac{w_2'(x; \xi)}{w_2(x; \xi)} \right) \, d\xi$$

By lemma 8.2 we obtain

$$\tilde{V}'_1(x) = \lim_{\xi_1 \to 0^+} (G_2(x, \xi_1) - G_1(x, \xi_1))$$

As in the previous case, it follows from the definition of $\xi^*(x)$ that $F_1(x, \xi^*(x)) = F_2(x, \xi^*(x))$. The limit at $\xi_{cr}$ vanishes as a result of lemma 8.3. So what remains is the limit for $\xi_1 \to 0^+$, which by (8.34) and (8.35), yields

$$\tilde{V}'_1(x) = x - \tau \omega_2(x).$$

This leads to (8.50) in the same way as in the other case. \qed

Proof of (8.51). From theorem 5.3 it follows that $\Gamma_3(\xi) = \mathbb{R}$ if $\xi > \xi_{cr}$. Then, for every $x \in \mathbb{R}$, we have that $|w_3(x, \xi)| = |w_2(x; \xi)|$. Therefore, the second integral of (8.4) vanishes. We now distinguish two cases.

Case 1. $t \geq 0$ or $t < 0$ and $\xi \geq \xi^*$. If $t \geq 0$, then $\Gamma_3(\xi) = \mathbb{R}$ for every $\xi > 0$. If $t < 0$ and $\xi > \xi^*$, then it follows from theorem 6.1 that $x \in \Gamma_3(\xi)$ for all $\xi \in (0, \xi_{cr})$. So in both situations, we have that $|w_3(x, \xi)| = |w_2(x; \xi)|$ for every $x \in \mathbb{R}$ and $\xi \in (0, \xi_{cr})$. Then, (8.4) ensures us that $\tilde{V}_2(x) = 0$. As $\tilde{V}_2(x) = 0$ in this case, by (2.6) and (2.7).

Case 2. $t < 0$ and $|\xi| < \xi^*$. If $|\xi| < \xi^*$, there exists $\xi^*(x) < \xi_{cr}$ such that $x \in \Gamma_3(\xi)$ if $\xi^*(x) \leq \xi$ and $x \not\in \Gamma_3(\xi)$ if $\xi < \xi^*(x)$, because $\Gamma_3(\xi)$ is an increasing set. Then we obtain

$$\tilde{V}'_2(x) = \frac{1}{2} \lim_{\xi_1 \to 0^+} \int_{\xi_1}^{\xi_2} \left( \frac{w_3'(x; \xi)}{w_3(x; \xi)} - \frac{w_2'(x; \xi)}{w_2(x; \xi)} \right) \, d\xi.$$ 

Applying part (b) of lemma 8.2 yields

$$\tilde{V}'_2(x) = \lim_{\xi_1 \to 0^+} (G_4(x, \xi_1) - G_3(x, \xi_1)) = (G_3(x, \xi^*(x)) - G_4(x, \xi^*(x))).$$

The last two of terms cancel each other because $x = \pm \delta(\xi^*(x))$ and $G_3(x, \xi^*(x)) = G_4(x, \xi^*(x))$ by definition of $\xi^*$. The limit for $\xi_1 \to 0^+$ is calculated using (8.36) and it follows that

$$\tilde{V}'_2(x) = \tau (\omega_2(x) - \omega_3(x)).$$

Integrating with respect to $x$ yields

$$\tilde{V}_2(x) = \frac{1}{2} \omega_2(x)^2 - \frac{1}{2} \omega_3(x)^2 - \frac{1}{4} t \omega_2(x)^2 + C,$$

where $C$ is a constant of integration. Substituting $x = \pm x^*$ one can check that $C = 0$. This proves (8.51) in view of (2.7). \qed
Proof of (8.52). By (8.3) we have
\[ \text{supp}(\tilde{\sigma}) = \bigcup_{\xi > 0} \text{supp}(\mu_{\xi}^2) = \bigcup_{\xi > 0} \Gamma_2(\xi) \]
and this is either \( i\mathbb{R} \) in case \( t \leq 0 \), or \( (-\infty, -iy^*) \cup [iy^*, i\infty) \) in case \( t > 0 \). This coincides with the support of \( \sigma \).

Let \( z = iy \) with \( y > y^* \). The sets \( \Gamma_2(\xi) \) are decreasing as \( \xi \) increases, and there is a \( \xi^*(z) > 0 \) such that \( z \in \Gamma_2(\xi) \) if and only if \( \xi \leq \xi^*(z) \). Then by (8.3) we have
\[ \frac{d\tilde{\sigma}(z)}{dz} = \frac{d\mu_{\xi}^2(z)}{dz} \frac{d\xi}{d\nu}, \quad (8.54) \]
since there is no contribution to the integral for \( \xi > \xi^* \).

The form of \( d\mu_{\xi}^2(z)/dz \) depends on whether we are in the one-cut or two-cut case. Let us assume that \( t < \tau^2 \) and \( \xi^*(z) > \xi^* \), so that both cases appear in (8.54). We will not give details about the other cases, which are simpler.

Since \( \xi^*(z) > \xi^* \), we split up the integral (8.54) and use (5.15) and (6.15) to obtain
\[ \frac{d\tilde{\sigma}(z)}{dz} = \frac{1}{4\pi i} \lim_{\xi_1 \to 0^+} \left( G_2(z+; \xi_1) - G_2(z-; \xi_1) \right) \]
\[ + \frac{1}{2\pi i} \int_{\xi_1}^{\xi^*} \left( \frac{w_{2-}(z; \xi)}{w_{2+}(z; \xi)} - \frac{w_{2+}(z; \xi)}{w_{2+}(z; \xi)} \right) d\xi. \]

Applying lemma 8.2 yields
\[ \frac{d\tilde{\sigma}(z)}{dz} = \frac{1}{2\pi i} \lim_{\xi \to 0^+} \left( G_2(z+; \xi) - G_2(z-; \xi) \right) \]
\[ + \lim_{\xi \to \xi^*} \left( G_2(z-; \xi) - F_2(z, -\xi^*) + F_2(z+, \xi^*) - G_2(z+; \xi^*) \right) \]
\[ + F_2(z-, \xi^*(z)) - F_2(z+; \xi^*(z)). \quad (8.55) \]

By definition of \( \xi^*(z) \), \( z \) is on the boundary of \( \Gamma_2(\xi^*(z)) \). Therefore, \( z \) is the branch point \( iy(\xi^*(z)) \). The algebraic equation \( s_1(w; \xi) = z \) then has the double solution \( w_{2}(z; \xi^*(z)) = w_{3}(z; \xi^*(z)) \), so that the last two terms of (8.55) cancel each other. Also the limit at \( \xi^* \) vanishes as a result of lemma 8.3. The limit as \( \xi_1 \to 0^+ \) is calculated using (8.38) which gives
\[ \frac{d\tilde{\sigma}(z)}{dz} = \frac{\tau}{\pi i} \text{Re} \omega_1(z). \]

By symmetry, the same formula is valid for \( z = -iy \) with \( y > y^* \). This proves the equality (8.52) in view of the definition (2.11) of \( \sigma \). \( \square \)

Acknowledgments

Dries Geudens is Research Assistant of the Fund for Scientific Research, Flanders (Belgium).

Arno Kuijlaars is supported in part by FWO-Flanders projects G.0427.09 and G.0641.11, by K U Leuven research grant OT/08/33, by the Belgian Interuniversity Attraction Pole P06/02, and by a grant from the Ministry of Education and Science of Spain, project code MTM2005-08648-C02-01.

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