UNIVALENT FUNCTIONS AND INTEGRABLE SYSTEMS

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Abstract. We study one-parameter expanding evolution families of simply connected domains in the complex plane described by infinite systems of evolution parameters. These evolution parameters in some cases admit Hamiltonian formulation and lead to integrable systems. One example of such parameters is complex moments for the Laplacian growth that form a Whitham-Toda integrable hierarchy. Another example we deal with is related to expanding coefficient bodies for conformal maps given by Löwner subordination chains. The coefficients bodies are proved to form a Liouville partially integrable Hamiltonian system for each fixed index and the first integrals are obtained. We also discuss the contact structure of this system.

1. Introduction

Complex dynamical systems, iterations and construction of Lie semigroups with respect to the composition operation lead to the study of expanding systems of plane domains. These systems can be thought of as one-parameter families of domains (typically simply connected and univalent) $\Omega(t)$ that form subordination chains in the Riemann sphere $\hat{\mathbb{C}}$, which are defined for $0 \leq t < T$ where $T$ may be $\infty$. This means that $\Omega(t) \subset \Omega(s)$, $\Omega(t) \neq \Omega(s)$, whenever $t < s$.

Kufarev [16] studied a one-parameter family of domains $\Omega(t)$, and regular functions $g(z,t)$, $g : \Omega(t) \times [0,T) \to U$, where $U = \{\zeta : |\zeta| < 1\}$. He proved differentiability of $g(z,t)$ with respect to $t$ for $z$ from the Carathéodory kernel $\Omega(0)$ of $\Omega(t)$.

Sometimes, the evolution of $\Omega(t)$ can be characterized by infinite sets of evolution parameters given as a result of integration of certain dynamical systems. Hamiltonian interpretation of such systems yields their integrability. The interest to integrable systems is caused first of all by Hamiltonian systems of mathematical physics where the Hamiltonian is thought of as an energy functional and its critical points correspond to minimal energy of particle systems. However, other areas of mathematics also use Hamiltonian formalism, e.g., optimal control, classical mechanics, etc.

One of the examples of integrable systems of evolution parameters corresponding to one-parameter families of expanding domains is the problem of Laplacian growth (Hele-Shaw problem). The Laplacian growth means the Dirichlet problem for a harmonic potential where the boundary of the phase domain $\Omega(t)$ is unknown a-priori (free boundary), and in fact, is defined by the normality of its motion. The classical (strong) solvability of this problem suggests that phase domains $\Omega(t)$ are bounded by analytic curves. It turns out, that given an initial domain

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Ω(0) the set of Richardson’s moments (see the definition in [24]) solves completely the inverse potential problem, thus describing the evolution of phase domains as long as the classical solution exists. As it has been shown in [1], [15], [18], [27], the Laplacian growth problem can be embedded into a larger hierarchy of domain variations (Whitham-Toda hierarchy) for which all the complex moments are treated as independent variables (generalized time variables), and form an integrable system.

Another example of evolution parameters is the set of coefficients of the Riemann maps of $U$ onto the family of the phase domains $\Omega(t)$ that has attracted attention of the complex analysts for a more than eighty year period of the last century. The Bieberbach conjecture [8] has proved to be the most intriguing problem that forced investigations in geometric function theory during this period. It finally has been solved in 1984 by de Branges [9], who proved that the $n$-th Taylor coefficient $b_n$ of a conformal homeomorphism $f(\zeta)$, normalized by $f(0) = 0$, $f'(0) = 1$, does not exceed its index: $|b_n| \leq n$. This class of functions we denote by $S$. The inequality is sharp and the equality is attained only for rotations of the Koebe function $k(z) = z(1-z)^{-2}$. De Branges’ proof has put an end to the numerous attempts to prove or disprove the Bieberbach conjecture. However, a much more general and difficult problem is to describe the range of first coefficients as a set of values $V_n = \{(b_2, \ldots, b_n), f \in S\}$. In the trivial case $V_2$ is the disk of radius 2. Only the first non-trivial coefficient body $V_3 = (b_2, b_3)$ has been described completely by Schaeffer and Spenser in 1951 in their famous monograph [25]. A qualitative description of $V_n$, $n \geq 3$, has been partially given in [5]. Apart from these two monographs there are only few works where a progress in such a problem has been made (see, e.g., [23]).

The key stages of our presentation are as follows. Firstly, a boundary point of $V_n$ is given by a unique function whose first coefficients $b_2, \ldots, b_n$ define the rest of coefficients. Secondly, this function unlike the Laplacian growth, presents a full mapping (the complement of the image of $U$ is of measure zero) of the unit disk onto the complex plane minus an analytic graph with the root at infinity. Thirdly, erasing this graph we obtain a family of expanding domains completely described by the initial domain, and in its turn by the point of $\partial V_n$. Finally, we obtain a $(2n-3)$ dimensional family of evolution dynamics completely described as a flow generated by a certain vector field, which admits a Hamiltonian formulation. It is proved to be partially integrable in the Liouville sense and the first integrals are obtained. Moreover, it turns out that the first integrals that generate additional directions form a horizontal space so this Hamiltonian system possesses a contact structure. It is important that the evolution exists all the time $0 \leq t < \infty$.

Our paper is devoted to a detail realization of this plan and the description of this system of evolution parameters, its Hamiltonian formulation, and finally, its integration. The central idea of the Hamiltonian formulation is forced by the Löwner evolution (see, e.g., [2], [17], [20]) that is based on evolution equations in partial and ordinary derivatives mutually linked. The controllable dynamical system for evolution parameters turns out to be Hamiltonian under the assumption that the necessary optimality conditions hold. So, we are especially interested in subordination chains of domains generated by Riemann maps corresponding to the critical
controls, and therefore, to the trajectories, that in particular, give all boundary points to the coefficient body for univalent maps.

From one side our main results describe evolution coefficient systems for certain type of Löwner’s dynamics of domains and maps that admit the Hamiltonian formulation and are integrable. The coefficient bodies will form a certain type of hierarchy (dependent on the coefficient indices) and the first integrals will be derived. From the other side, we give all boundary points of the coefficient bodies $V_n$.

2. THE LÖWNER AND LÖWNER-KUFAREV EVOLUTION

There are many papers and monographs devoted to the Löwner and Löwner-Kufarev equations. However we revisit this theory from the point of view of correspondence between Löwner equations in partial and ordinary derivatives and univalence of solutions to the partial derivative Löwner equation.

Let us consider a subordination chain of simply connected hyperbolic domains $\Omega(t)$ in the complex plane $\mathbb{C}$, which is defined for $0 \leq t < T$, where $T$ may be infinity. We suppose that the origin is an interior point of the kernel of $\{\Omega(t)\}_{t=0}^{T}$. Let us normalize the growth of the evolution of this subordination chain by the conformal radius of $\Omega(t)$ with respect to the origin to be $e^t$. By the Riemann Mapping Theorem we construct a subordination chain of mappings $f(\zeta,t)$, $\zeta \in U$, where each function $f(\zeta,t) = e^{it} + b_2(t)\zeta^2 + \ldots$ is a holomorphic univalent map of $U$ onto $\Omega(t)$ for every fixed $t$. Pommerenke $[20, 21]$ introduced such chains in order to generalize Löwner’s equation. His result says that given a subordination chain of domains $\Omega(t)$ defined for $t \in [0,T)$, there exists an analytic regular function $p(\zeta,t) = 1 + p_1(t)\zeta + p_2(t)\zeta^2 + \ldots$, $\zeta \in U$, such that $\text{Re } p(\zeta,t) > 0$ and

\begin{equation}
\frac{\partial f(\zeta,t)}{\partial t} = \zeta \frac{\partial f(\zeta,t)}{\partial \zeta} p(\zeta,t),
\end{equation}

for $\zeta \in U$ and for almost all $t \in [0,T)$. The class of holomorphic functions $p$ defined as above we denote by $C$. The equation (1) is called the Löwner-Kufarev equation due to two seminal papers by Löwner $[17]$ with

\begin{equation}
p(\zeta,t) = \frac{e^{iu(t)} + \zeta}{e^{iu(t)} - \zeta},
\end{equation}

where $u(t)$ is a continuous function regarding to $t \in [0,T)$, and by Kufarev $[16]$ in general case, where this equation appeared for the first time.

The equation (1) represents a growing evolution of simply connected domains. Let us consider the reverse process. Given an initial domain $\Omega(0) \equiv \Omega_0$ (and therefore, the initial mapping $f(\zeta,0) \equiv f_0(\zeta)$), and a function $p(\zeta,t)$ from the class $C$, we solve the equation (1) and ask whether the solution $f(\zeta,t)$ represents a subordination chain of simply connected domains. The initial condition $f(\zeta,0) = f_0(\zeta)$ is not given on the characteristics of the partial derivative
equation (1), hence the solution exists and is unique. Assuming $s$ as a parameter along the characteristics we have
\[ \frac{dt}{ds} = 1, \quad \frac{d\zeta}{ds} = -\zeta p(\zeta, t), \quad \frac{df}{ds} = 0, \]
with the initial conditions $t(0) = 0$, $\zeta(0) = z$, $f(\zeta, 0) = f_0(\zeta)$, where $z$ is in $U$. This leads us to the Cauchy problem for the L"owner-Kufarev equation in ordinary derivatives for a function $\zeta = w(z,t)$
\[ \frac{dw}{dt} = -wp(w, t), \]
with the initial condition $w(z, 0) = z$.

We see that the equation (3) is exactly the characteristic equation for (1). Unfortunately, this approach requires the extension of $f_0(w^{-1}(\zeta, t))$ into $U$ because the solution to (1) is the function $f(\zeta, t)$ given as $f_0(w^{-1}(\zeta, t))$, where $\zeta = w(z, s)$ is a solution of the initial value problem for the characteristic equation (3) that maps $U$ into $U$. Therefore, the solution of the initial value problem for the equation (1) may be non-univalent.

Let $S$ stand for the usual class of all univalent holomorphic functions $f(z) = z + a_2 z^2 + \ldots$ in the unit disk. Solutions to the equation (3) are regular univalent functions $w(z, t) = e^{-t}z + a_2(t)z^2 + \ldots$ in the unit disk that map $U$ into itself. Conversely, every function from the class $S$ can be represented by the limit
\[ f(z) = \lim_{t \to \infty} e^t w(z, t), \]
where there exists a function $p(z, t)$ from the class $C$ for almost all $t \geq 0$, such that $w(z, t)$ is a solution to the equation (3) (see [21, pages 159–163]). Each function $p(z, t)$ generates a unique function from the class $S$. The reciprocal statement is not true. In general, a function $f \in S$ can be determined by different functions $p \in C$.

From [21, page 163] it follows that we can guarantee the univalence of the solutions to the L"owner-Kufarev equation in partial derivatives (1) assuming the initial condition $f_0(\zeta)$ given by the limit (4) with the function $p(\cdot, t)$ chosen to be the same in the equations (1) and (3).

In his original paper [17] L"owner dealt with functions that map $U$ onto domains each of which was obtained by slitting $\mathbb{C}$ along a unique Jordan curve. This led him to a subclass $S'$ of the class $S$ of one-slit maps which is dense in $S$. Each function $f \in S'$ is obtained as the limit (4) where $w(z, t)$ is a solution to the Cauchy problem for the L"owner equation
\[ \frac{dw}{dt} = -u \frac{e^{iu(t)} + w}{e^{iu(t)} - w}, \]
with the initial condition $w(z, 0) = z$ where $u(t)$ is a continuous control function. The corresponding L"owner equation in partial derivatives is just equation (1) with the function $p(\zeta, t)$ given by (2). The solution to this equation is univalent all the time if and only if the initial condition is given as the limit (4) for the same $p$ given by (2). It maps the unit disk onto a growing family of one-slit domains. The class of such domains lies dense in the space of all simply connected domains in the sense of Carathéodory’s kernel convergence.
We proceed with generalized Löwner chains of domains with several slits. Let \( \Omega(t), 0 \in \Omega(t), t \in [0,T) \), be a subordination chain of simply connected domains each of which is obtained by slitting \( \mathbb{C} \) along a finite number of Jordan curves with \( m+1 \) distinct endpoints. The \( m+1 \)-th endpoint is at infinity. The one-parameter family \( f(\zeta,t) \) stands for the corresponding subordination chain of normalized conformal maps \( f(\zeta,t) = e^{t\zeta} + b_2(t)\zeta^2 + \ldots, \zeta \in \mathbb{U}, t \in [0,T) \). These maps satisfy the equation (1) with the function \( p(\zeta,t) \) given by (2) where \( u(t) \) is a piecewise continuous control function regarding to \( t \in [0,T) \) (see, e.g., Goluzin [12]). However, this representation is not unique and another approach has been suggested originally by Kufarev and described in [2]. It was based on the function \( p(\zeta,t) \) in (1) given by the following formula

\[
p(\zeta,t) = \sum_{k=1}^{m} \lambda_k(t) \frac{e^{iu_k(t)} + \zeta}{e^{iu_k(t)} - \zeta},
\]

where \( u_k(t) \) are continuous control functions regarding to \( t \in [0,T) \) and \( \lambda_k(t) \) are measurable non-negative control functions, \( \sum_{k=1}^{m} \lambda_k(t) = 1 \).

Constructing the characteristic equation (3) we come to the Löwner-Kufarev equation in ordinary derivatives and the limit (4) gives a subclass \( S^{''} \) of the class \( S \) of multi-slit maps which is also dense in \( S \). The function \( w(z,t) \) in (4) is a solution to the Cauchy problem for the Löwner-Kufarev equation

\[
\frac{dw}{dt} = -w \sum_{k=1}^{m} \lambda_k(t) \frac{e^{iu_k(t)} + w}{e^{iu_k(t)} - w},
\]

with the initial condition \( w(z,0) = z \) where \( u_k(t) \) and \( \lambda_k(t) \) are defined as in (6).

Similarly to the one-slit evolution, the equation (1) with the function \( p \) given by (6) possesses univalent solutions all the time if and only if the initial condition is given as the limit (4) for the same \( p \) in (7) given by (6).

Although the representation (1, 6) or (7) is not unique neither, it allows to elaborate a unique one. Given a subordination chain \( \Omega(t) \) of multi-slit domains there exists a unique set of positive numbers \( \{ \lambda_k \}_{k=1}^{m}, \sum_{k=1}^{m} \lambda_k = 1 \), and a unique set of continuous control functions \( \{ u_k(t) \}_{k=1}^{m} \), such that \( \Omega(t) = f(U,t) \), where \( f \) is a solution to the equation (1) with the function \( p \) given by (6), and \( \lambda_k(t) \equiv \lambda_k \). This statement follows from the results of [23]. An analogous uniqueness statement for the representation (7) is true too.

3. Evolution parameters and hierarchies

As it was stated in Introduction, one of the examples of integrable systems of evolution parameters corresponding to one-parameter families of domains is the problem of Laplacian growth. In its classical formulation it deals with the smooth family \( \{ \Omega(t) \}, t \in [0,T) \) of phase domains, i.e., when \( \partial \Omega(t) \) are smooth (\( C^\infty \)) interfaces for each \( t \), and the normal velocity \( v_n \) at the boundary continuously depends on \( t \) at any point of \( \partial \Omega(t) \). Each \( \Omega(t) \) is supposed to be simply connected in \( \mathbb{C} \) for any \( t \in [0,T) \) fixed. Starting from the initial moment \( t = 0 \) the boundary becomes even analytic smooth for \( t \in (0,T) \). Richardson’s complex moments completely describe...
this evolution, say given an initial phase domain $\Omega(0)$ there exists a unique infinite set of the moments for $\Omega(t)$ at each $t \in (0, T)$ and vice versa. One of the reasons for a slit dynamics to appear in this process is extending the solution to exist beyond the blowup time $T$. One way of the regularization of the problem is to introduce a surface tension parameter instead of zero boundary condition in the classical Hele-Shaw formulation (see, e.g., [14]). This can produce a crack morphology in which a part of the boundary develops a thin finger penetrating the phase domain. Another proposition (see [13]) is to continue smooth dynamics by slit dynamics beyond the blowup time when a possible cusp is developed at the boundary. Of course, in this case one relaxes smoothness of the boundary permitting the corresponding Riemann map to have singularities at the unit circle.

Another reason to consider slit dynamics is the determinate evolution, duality, and univalence of the solutions to the Löwner and Löwner-Kufarev equations (in ordinary and partial derivatives) stated in the preceding section.

In view of the involved Riemann maps the most natural evolution parameters are the Taylor coefficients of these maps. Of course, the infinite set of coefficients completely describes the evolution given by the maps $f : U \to \Omega(t)$. However in general, the differential equations for these coefficients obtained from the Löwner or Löwner-Kufarev equations do not admit any reasonable Hamiltonian formulation that leads to integration of this (infinitely dimensional) system.

Instead we propose to construct hierarchies defined by the coefficient bodies as an object to study. Let $f \in S$. We construct the coefficient bodies by setting

$$V_n = \{(b_2, \ldots, b_n) : f(\zeta) = \zeta + \sum_{k=2}^{\infty} b_k \zeta^k \in S\}, \quad n \geq 2,$$

in the $(2n - 2)$-dimensional Euclidean space. Every point of the boundary of $V_n$ is given by a unique function $f \in S$ that maps the unit disk onto the plane minus piecewise analytic Jordan arcs forming a tree with a root at infinity having at most $n - 1$ tips. Each $V_n$ defines a class $\{\Omega_0\}$ of initial domains for multi-slit subordination evolutions $\{\Omega(t)\}$, and the corresponding systems of differential equations for coefficients form an integrable system $D_n$ for each $n$, moreover, its dual system $D^*_n$ obtained by the characteristic equation admits Hamiltonian formulation. The latter systems form hierarchies $\{D_n\}_{n=2}^\infty$ and $\{D^*_n\}_{n=2}^\infty$. This idea up to some extent will be proved in the next sections.

4. COEFFICIENT BODIES

By the coefficient problem for univalent functions we mean the problem of precise finding the regions $V_n$ defined above. These compact sets have been investigated by a great number of authors, but the most remarkable source is a famous monograph [25] written by Schaeffer and Spencer in 1950. Among other contributions to the coefficient problem we distinct a monograph by Babenko [5] that contains a good collection of qualitative results on the coefficient bodies $V_n$. The results concerning the structure and properties of $V_n$ include
(i) \( V_n \) is homeomorphic to a \((2n-2)\)-dimensional ball and its boundary \( \partial V_n \) is homeomorphic to a \((2n-3)\)-dimensional sphere;

(ii) every point \( x \in \partial V_n \) corresponds to exactly one function \( f \in S \) which will be called a boundary function for \( V_n \);

(iii) with the exception for a set of smaller dimension, at every point \( x \in \partial V_n \) there exists a normal vector satisfying the Lipschitz condition;

(iv) there exists a connected open set \( X_1 \) on \( \partial V_n \), such that the boundary \( \partial V_n \) is an analytic hypersurface at every point of \( X_1 \). The points of \( \partial V_n \) corresponding to the functions that give the extremum to a linear functional belong to the closure of \( X_1 \).

It is worth to note that all boundary functions have a similar structure. They map the unit disk \( U \) onto the complex plane \( \mathbb{C} \) minus piecewise analytic Jordan arcs forming a tree with a root at infinity and having at most \( n-1 \) tips. This assertion underlines the importance of multi-slit maps in the coefficient problem for univalent functions.

Löwner’s approach is based on the following idea: a function mapping \( U \) onto the complement of a single slit admits univalent dynamics for \( t \in [0, \infty) \) represented by the Löwner equation (1-2) forming a subordination chain \( f(\zeta, t) = \sum_{n=1}^{\infty} b_n(t)\zeta^n \). Let us deduce a system of differential equations for the coefficients of \( f(\zeta, t) \) substituting the expansion of \( f(\zeta, t) \) into (1). This gives

\[
\dot{b}_k = kb_k + 2 \sum_{j=1}^{k-1} jb_j e^{-i(k-j)t}, \quad b_k(0) = a_k, \quad t \geq 0,
\]

\( k = 1, 2, \ldots \), where \( f(\zeta, 0) = f_0(\zeta) = \sum_{n=1}^{\infty} a_n(1)\zeta^n \). In particular, \( b_1(t) = e^t \).

Going to the Löwner equation in characteristics (5) we write \( w(z, t) = \sum_{n=1}^{\infty} a_n(t)z^n \) and substitute this function in (1). To formulate the result we introduce the following notations

\[
a(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ a_1(t) & 0 & \ldots & 0 & 0 \\ a_2(t) & a_1(t) & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \ldots & \ldots \\ a_{n-1}(t) & a_{n-2}(t) & \ldots & a_1(t) & 0 \end{pmatrix}.
\]

Then the differential equation for \( a(t) \) is of the form

(8) \[
\dot{a} = -a - 2 \sum_{s=1}^{n-1} e^{-isu} A^s a,
\]

\[
a(0) \equiv a^0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

In particular, \( a_1(t) = e^{-t} \) and \( \lim_{t \to \infty} e^t a_k(t) = a_k \).
5. Hamiltonian formulation for the coefficient system. Integrability

5.1. Hamiltonian dynamics and integrability. Let us recall briefly the Hamiltonian and symplectic definitions and concepts that will be used in the sequel. There exists a vast amount of modern literature dedicated to different approaches to and definitions of integrable systems (see, e.g., [3], [4], [7], [28]).

The classical definition of a completely integrable system in the sense of Liouville applies to a Hamiltonian system. If we can find independent conserved integrals which are pairwise involutory (vanishing Poisson bracket), this system is completely integrable (see e.g., [3], [4], [7]). That is each first integral allows us to reduce the order of the system not just by one, but by two. We formulate this definition in a slightly adopted form as follows.

A dynamical system in $C^{2n}$ is called Hamiltonian if it is of the form

$$\dot{x} = \nabla_s H(x),$$

where $\nabla_s$ denotes the symplectic gradient given by

$$\nabla_s = \left( \frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_{2n}}, -\frac{\partial}{\partial x_1}, \ldots, -\frac{\partial}{\partial x_n} \right).$$

The function $H$ in (9) is called the Hamiltonian function of the system. It is convenient to redefine the coordinates $(x_{n+1}, \ldots, x_{2n}) = (\psi_1, \ldots, \psi_n)$, and rewrite the system (9) as

$$\dot{x}_k = \frac{\partial H}{\partial \psi_k}, \quad \dot{\psi}_k = -\frac{\partial H}{\partial x_k}, \quad k = 1, 2, \ldots, n.$$

The system has $n$ degrees of freedom. The two-form $\omega = \sum_{k=1}^{n} dx \wedge d\bar{\psi}$ admits the standard Poisson bracket $\{\cdot, \cdot\}$

$$\{f, g\} = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial \psi_k} - \frac{\partial f}{\partial \psi_k} \frac{\partial g}{\partial x_k} \right)$$

associated with $\omega$. The symplectic pair $(C^{2n}, \omega)$ defines the Poisson manifold $(C^{2n}, \{\cdot, \cdot\})$. These notations may be generalized for a symplectic manifold and a Hamiltonian dynamical system on it.

The system (10) may be rewritten as

$$\dot{x}_k = \{x_k, H\}, \quad \dot{\psi}_k = \{\psi_k, H\}, \quad k = 1, 2, \ldots, n,$$

and the first integrals $\Phi$ of the system are characterized by

$$\{\Phi, H\} = 0.$$

In particular, $\{H, H\} = 0$, and the Hamiltonian function $H$ is an integral of the system (9). If the system (11) has $n$ functionally independent integrals $\Phi_1, \ldots, \Phi_n$, which are pairwise involutory $\{\Phi_k, \Phi_j\} = 0, k, j = 1, \ldots, n$, then it is called completely integrable in the sense of Liouville. The function $H$ is included in the set of the first integrals. The classical theorem of Liouville and Arnold [3] gives a complete description of the motion generated by the completely integrable system (10). It states that such a system admits action-angle coordinates around a connected
regular compact invariant manifold. One can work with real Poisson manifolds instead, making use of the real Hamiltonian function $2\text{Re } H$ keeping all other formulas changeless.

If the Hamiltonian system admits only $1 \leq k < n$ independent involutory integrals, then it is called partially integrable. The case $k = 1$ is known as the Poincaré–Lyapunov theorem which states that a periodic orbit of an autonomous Hamiltonian system can be included in a one-parameter family of such orbits under a nondegeneracy assumption. A bridge between these two extremal cases $k = 1$ and $k = n$ has been proposed by Nekhoroshev [19] and proved later in [6], [10], [11]. The result states the existence of $k$-parameter families of tori under suitable nondegeneracy conditions.

5.2. Coefficient system. We will show that the system (8) becomes an integrable system when treated as a description of the boundary hypersurface $\partial V_n$. Solutions $a(t)$ to (8) for different control function $u(t)$ (piecewise continuous, in general) being multiplied by the factor $e^{t}$ represent all points of $\partial V_n$ as $t \to \infty$. The trajectories $e^{t}a(t)$, $0 \leq t < \infty$, fill $V_n$ so that every point of $V_n$ belongs to a certain trajectory $e^{t}a(t)$. The endpoints of these trajectories can be interior or else boundary points of $V_n$. In this way, we set $V_n$ as the closure of the reachable set for the control system (8).

According to property (ii) of $V_n$ given in the previous section, every point $x \in \partial V_n$ is attained by exactly one trajectory $e^{t}a(t)$ which is determined by a choice of the piecewise continuous control function $u(t)$. The function $f \in S$ corresponding to $x$ is a multi-slit map of $U$. If the boundary tree of $f$ has only one tip, then there is a unique continuous control function $u(t)$ in $t \in [0, \infty)$ that corresponds to $f$. The case of multi-slit maps we will consider in the next section.

To reach a boundary point $x \in \partial V_n$ corresponding to a one-slit map, the trajectory $e^{t}a(t)$ has to obey extremal properties, i.e., to be an optimal trajectory. The continuous control function $u(t)$ must be optimal, and hence, it satisfies a necessary condition of optimality. The Maximum Principle is a powerful tool to be used that provides a joint interpretation of two classical necessary variational conditions: the Euler equations and the Weierstrass inequalities (see, e.g., [22]).

To realize the maximum principle we consider an adjoint vector

$$
\psi(t) = \begin{pmatrix}
\psi_1(t) \\
\vdots \\
\psi_k(t) \\
\psi_n(t)
\end{pmatrix},
$$

with complex valued coordinates $\psi_1, \ldots, \psi_n$, and the real Hamiltonian function

$$
H(a, \psi, u) = 2\text{Re } \left[ -a - 2 \sum_{s=1}^{n-1} e^{-isu(t)} A^s a^T \bar{\psi} \right],
$$
where \( \tilde{\psi} \) means the vector with complex conjugate coordinates. To come to the Hamiltonian formulation for the coefficient system we require that \( \tilde{\psi} \) satisfies the adjoint system of differential equations

\[
\frac{d\tilde{\psi}}{dt} = -\frac{\partial H}{\partial a}, \quad 0 \leq t < \infty.
\]

Taking into account (8) we rewrite (13) as

\[
\frac{d\tilde{\psi}}{dt} = \left( E + 2 \sum_{s=1}^{n-1} e^{-isu(t)}(s + 1)(A^T)^s \right) \tilde{\psi},
\]

where \( E \) is the unit matrix.

The maximum principle states that any optimal control function \( u^*(t) \) possesses a maximizing property for the Hamiltonian function along the corresponding trajectory, i.e.,

\[
\max_u H(a^*(t), \psi^*(t), u) = H(a^*(t), \tilde{\psi}^*(t), u^*), \quad t \geq 0,
\]

where \( a^* \) and \( \psi^* \) are solutions to the system (8, 14) with \( u = u^*(t) \).

The maximum principle (15) yields that

\[
\frac{\partial H(a^*(t), \psi^*(t), u^*)}{\partial u} \bigg|_{u=u^*(t)} = 0.
\]

Evidently, (8), (13), and (16) imply that

\[
\frac{dH(a^*(t), \psi^*(t), u^*(t))}{dt} = 0,
\]

for an optimal differentiable control function \( u^*(t) \).

The Hamiltonian formalism for system (8, 14) will lead to integrability. First, we show how \( \psi \) can be expressed in terms of the phase variable \( a \). In the following theorem causing no confusion we will use \( t(z) \) to denote the matrix transposition.

**Theorem 1.** Let \( a(t) \) and \( \psi(t) \), \( \psi(T) = (v_1, \ldots, v_n) \) obey the system (8, 14), \( T \geq 0 \). Then \( \psi_k(t) = c_{n-k+1}, k = 1, \ldots, n \), where \( c_1, \ldots, c_n \) are the Taylor coefficients of the expansion

\[
\frac{(\tilde{v}_n z + \cdots + \tilde{v}_1 z^n)w'(z, T)}{w'(z, t)} = \sum_{k=1}^{\infty} c_k(t) z^k.
\]

**Proof.** Let \( w(z, t) = \sum_{k=1}^{\infty} a_k(t) z^k \) be a solution to the Löwner differential equation (5). Differentiating (5) with respect to \( z \) we immediately have

\[
\frac{d}{dt} \left( \frac{z}{w'(z, t)} \right) = \frac{z}{w'(z, t)} \left( \frac{e^{iu} + w}{e^{iu} - w} + \frac{2e^{iu}w}{(e^{iu} - w)^2} \right).
\]

Considering the expansion

\[
\frac{z}{w'(z, t)} = \sum_{k=1}^{\infty} q_k(t) z^k,
\]
we obtain

\[ \frac{dq(t)}{dt} = \left( E + 2 \sum_{s=1}^{n-1} e^{-isu(t)}(s + 1)A^s \right) q, \]

where \( q(t) = (q_1(t), \ldots, q_n(t))^T \). We observe that system (19) differs from system (14) only by the transposition sign.

In order to satisfy the condition \( q(T) = (\bar{v}_1, \ldots, \bar{v}_n)^T \) we denote by

\[ g(z, t) = \frac{(\bar{v}_n z + \cdots + \bar{v}_1 z^n)w'(z, T)}{w'(z, t)} = \sum_{k=1}^{\infty} c_k(t)z^k, \]

and observe that \( g(z, t) \) obeys the same equation (18) where \( z/w'(z, t) \) is substituted by \( g(z, t) \).

Hence,

\[ c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ \vdots \\ c_n(t) \end{pmatrix} \]

obeys the system (19) substituting \( q(t) \) by \( c(t) \). It is easily seen that

\[ c(T) = \begin{pmatrix} \bar{v}_n \\ \vdots \\ \vdots \\ \bar{v}_1 \end{pmatrix}. \]

The difference in the transposition sign implies that \( \bar{\psi}_k = c_{n-k+1} \), \( k = 1, \ldots, n \). This completes the proof. \( \square \)

Putting \( T = 0 \) in this theorem we come to the following corollary.

**Corollary 1.** Let \( a(t) \) and \( \tilde{\psi}(t), \psi(0) = (v_1, \ldots, v_n)^T \) obey the system (8), (18), \( t \geq 0 \). Then \( \tilde{\psi}_k(t) = c_{n-k+1}, k = 1, \ldots, n \), where \( c_1, \ldots, c_n \) are the Taylor coefficients of the expansion

\[ \frac{(\bar{v}_n z + \cdots + \bar{v}_1 z^n)}{w'(z, t)} = \sum_{k=1}^{\infty} c_k(t)z^k. \]

**Remark 1.** Since \( \text{Im} \ a_1(t) = 0 \), the Hamiltonian function \( H(a, \psi, u) \) does not depend on \( \text{Im} \ \psi_1 \). Therefore, without loss of generality, we can put \( \psi_1(0) = v_1 \) to be real. This assumption leaves \( a, \psi_2, \ldots, \psi_n \) changeless.

Now we are able to apply the maximum principle. An optimal continuous control function satisfies the maximizing property (15) and obeys equation (16). The Hamiltonian \( H(a, \psi, u) \) is a trigonometric polynomial with respect to \( u \) of degree \( n-1 \) if \( \psi_n \neq 0 \). Let \( \psi(0) = (v_1, \ldots, v_n)^T \).
Note that \( \psi_n(t) = e^t \). We assume that \( v_n \neq 0 \), and diminish \( n \) to \( n - 1 \), otherwise. At \( t = 0 \), we have
\[
H(a^0, (v_1, \ldots, v_n)^T, u) = -v_1 e^{-t} - 2 \sum_{s=1}^{n-1} (\text{Re} \, v_{s+1} \cos(su) + \text{Im} \, v_{s+1} \sin(su)).
\]

For the optimal \( u^* \),
\[
H_u(a^0, (v_1, \ldots, v_n)^T, u^*) = 2 \sum_{s=1}^{n-1} (s \, \text{Re} \, v_{s+1} \sin(su^*) - s \, \text{Im} \, v_{s+1} \cos(su^*)) = 0.
\]
Suppose that \( v_2, \ldots, v_n \) are such that
\[
H_{uu}(a^0, (v_1, \ldots, v_n)^T, u^*) = 2 \sum_{s=1}^{n-1} (s^2 \, \text{Re} \, v_{s+1} \cos(su^*) + s^2 \, \text{Im} \, v_{s+1} \sin(su^*)) \neq 0,
\]
for the optimal \( u^* \). As we see, the term with \( v_1 \) does not influence the process of maximizing, and we will assume it zero later in course. The inequality holds for all \( (v_2, \ldots, v_n) \in \mathbb{R}^{2n-2} \) except for a set \( A \) of dimension at most \( 2n - 4 \). Indeed, due to the maximizing property of the optimal control function, this condition breaks down if
\[
H_{uu}(a^0, (v_1, \ldots, v_n)^T, u^*) = H_{uuu}(a^0, (v_1, \ldots, v_n)^T, u^*) = 0.
\]
Thus, the set \( A \) is determined by two linearly independent equations
\[
\sum_{s=1}^{n-1} (s^2 \, \text{Re} \, v_{s+1} \cos(su^*) + s^2 \, \text{Im} \, v_{s+1} \sin(su^*)) = 0,
\]
\[
\sum_{s=1}^{n-1} (s^3 \, \text{Re} \, v_{s+1} \sin(su^*) - s^3 \, \text{Im} \, v_{s+1} \cos(su^*)) = 0,
\]
with the fixed optimal \( u^* \), varying \( v_2, \ldots, v_n \). Therefore, \( A \) is a linear space of dimension \( 2n - 4 \) in \( \mathbb{R}^{2n-2} \).

**Definition 1.** We say that a vector \( (v_2, \ldots, v_n) \) satisfies the regularity condition on \([0, T]\) if
\[
H_{uu}(a^*(t), \psi^*(t), u^*) \neq 0, \quad t \in [0, T],
\]
for the optimal \( u^* \) and \( \psi^*(0) = (v_1, \ldots, v_n)^T \) along the optimal trajectory \( (e^t a^*(t), \psi^*(t)) \) corresponding to \( u^* \).

Evidently, if \( (v_2, \ldots, v_n) \in \mathbb{R}^{2n-2} \setminus A \), then \( (v_2, \ldots, v_n) \) satisfies the regularity condition on \([0, T]\) for \( T > 0 \) and small enough. Let us denote by \( Y(T) \) the set of all \( (v_2, \ldots, v_n) \) satisfying the regularity condition on \([0, T]\).

Let \( (v_2, \ldots, v_n) \in Y(T) \). Then
\[
H_{uu}(a, \psi, u) \neq 0, \quad t \in [0, T],
\]
for \( (a, \psi, u) \) from a neighborhood of \( (a^*, \psi^*, u^*) \) and this neighborhood depends on \( t \). The equation
\[
H_u(a, \psi, u) = 0,
\]
together with the regularity condition on \([0,T]\) determines an analytic implicit function \(u = u(a, \psi)\) in a neighborhood of \((a^*, \psi^*)\), and \(u(a, \psi)\) lies in a neighborhood of \(u^* \equiv u(a^*, \psi^*)\).

We substitute \(u = u(a, \psi)\) in the phase system \(\mathcal{S}\) and the adjoint system \(\mathcal{O}\) and solve the Cauchy problem with the initial data \(a(0) = a^0\), \(\psi(0) = (v_1, \ldots, v_n)^T\). Since \(u = u^*\) along \((a^*, \psi^*)\), the Hamiltonian \(H(a, \psi, u)\) satisfies the maximum principle. Note that the neighborhood of \((a^*+\psi^*+t)\) can be determined by a neighborhood of the corresponding vector \((v_2, \ldots, v_n) = (\psi_2(0), \ldots, \psi_n(0))\). According to Corollary 1 the function \(\psi(t)\) can be represented as a function of the phase variable \(a\) and of the initial conditions \((v_1, \ldots, v_n)\), say \(\psi(t) = \varphi(a(t), v_1, \ldots, v_n)\).

**Theorem 2.** Let \((v_2, \ldots, v_n)\) satisfy the regularity condition on \([0,T]\). Then the system \(\mathcal{S} [\mathcal{O}]\) with \(u = u(a, \psi) = u(a, \varphi(0, v_2, \ldots, v_n))\) is partially Liouville integrable. Moreover, the first integrals form a contact structure.

**Proof.** We substitute \(\psi(t) = \varphi(a(t), v_1, \ldots, v_n)\) in the Hamiltonian and obtain
\[
H(a(t), \psi(t), u) = H(a(t), \varphi(a(t), v_1, \ldots, v_n), u) = H(a(t), v_1, \ldots, v_n, u).
\]
To each point \((v_1, \ldots, v_n)\) from a neighborhood of a vector satisfying the regularity condition, there corresponds an optimal continuous control function. The maximum principle implies that
\[
\mathcal{H}_u(a(t), v_1, \ldots, v_n, u) \bigg|_{u = u(a, \varphi(a, v_1, \ldots, v_n))} = 0,
\]
that, together with \(\mathcal{S} \mathcal{O}\), gives
\[
(21) \quad \mathcal{H}(a(t), v_1, \ldots, v_n, u(a, \varphi(a, v_1, \ldots, v_n))) = \text{const}.
\]

In order to prove the partial integrability we will find the first complex integrals \(\Phi_1, \ldots, \Phi_n\). We already have one real integral \(\Phi_1\).

By the corollary from Theorem 1 we deduce that
\[
(22) \quad \sum_{k=1}^{n} \psi_{n-k+1}^k z^k = w'(z, t) \sum_{k=1}^{n} \psi_{n-k+1}^k z^k + w' z^k \sum_{k=n+1}^{\infty} c_k z^k.
\]
We denote by \(\Phi(a_1, \ldots, a_n)\) the vector of the first integrals of the Hamiltonian system \(\mathcal{S} [\mathcal{O}]\) given by
\[
\left( \begin{array}{cccc} 
\Phi_1 \\
\Phi_2 \\
\Phi_n \\
\end{array} \right) = \left( \begin{array}{cccc}
a_1 & 2a_2 & \ldots & (n-1)a_n \\
0 & a_1 & \ldots & (n-2)a_{n-1} \\
0 & 0 & \ldots & (n-3)a_{n-2} \\
0 & 0 & \ldots & 0 \\
\end{array} \right) \left( \begin{array}{c}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_n \\
\end{array} \right),
\]
with the control \(u\) given in the statement of the theorem. Indeed, the equality \(\mathcal{S} \mathcal{O}\) implies that \(\Phi_k = \psi_k\) are constants for all \(t\) and \(k = 1, \ldots, n\). The first integral \(\mathcal{S} \mathcal{O}\) allows us to conclude that \(\{\Phi_k, \mathcal{H}\} = 0\). One checks that \(\{\Phi_s, \Phi_k\} = -s \Phi_{s+k-1}\) for all \(1 \leq s < k\) and \(s + k - 1 \leq n\). Otherwise, \(\{\Phi_s, \Phi_k\} = 0\) for \(s + k - 1 > n\). This implies that
• \([n/2]\) first integrals \(\{\Phi_{[n/2]+1}, \ldots, \Phi_n\}\) are pairwise involutory;
• the integrals \(\{\Phi_1, \ldots, \Phi_{[n/2]}\}\) are not pairwise involutory but their Poisson brackets give all the rest of integrals. This structure is said to be contact (which is not uncoupled because, for example, \(\{\Phi_1, \ldots, \Phi_n\} = -\Phi_n \neq 0\)).

This completes the proof. \qed

Remark 2. We observe that

\[
H(a_0, (v_1, \ldots, v_n)^T, u^*) = -v_1 - 2 \sum_{s=1}^{n-1} (\Re v_{s+1} \cos(su^*) + \Im v_{s+1} \sin(su^*))
\]

at \(t = 0\), which is the constant in the right-hand side of (21). It contains an additive term \((-v_1)\). The optimal control function \(u^*\) does not depend on \(v_1\). Hence, we can put \(v_1 = 0\) without loss of generality. This assumption does not influence \(a, \psi_2, \ldots, \psi_n\), and defines uniquely the constant in the right-hand side of (21). Thus, the first integral (21) can be rewritten as

\[
(23) \quad H(a(t), v_2, \ldots, v_n, u(a, \varphi(a, v_2, \ldots, v_n))) = \text{const}.
\]

The integral (23) gives a local parametric description of the boundary of the coefficient body for bounded univalent functions from \(S\) (such that \(f \in S, |f(z)| < M, 1 < M \leq e^T\)). The points of this part of the boundary correspond to the functions that map \(U\) onto the disk \(|w| < M\) slit along an analytic curve with exactly one tip at its interior point. The boundary hypersurface is homeomorphic to an open set on the 2\(n\) - 3-dimensional sphere in \(\mathbb{R}^{2n-2}\) and must be parameterized by 2\(n\) - 3 free parameters. We should select 2\(n\) - 3 independent parameters among 2\(n\) - 2 variables \(v_2, \ldots, v_n\). Observe that the Hamiltonian \(H(a, \psi, u)\) and the solution \(\psi\) to the adjoint system (13) are linear with respect to \(\psi\). The implicit function \(u = u(a, \psi)\) is invariant upon multiplying \(\psi\) by a positive number. Therefore, both \(H\) and \(\psi\) may be determined up to a positive multiplier without loss of generality. Putting, e.g., \(|v_n| = 1\) we reduce the number of free parameters in (23) to 2\(n\) - 3 that gives a local parametric representation of the boundary.

The integral (21) is continued from \([0, T]\) to a bigger interval \([0, T + \varepsilon]\) as long as the regularity conditions are preserved. Let \(T > 0\) be the minimal positive number such that the regularity conditions break up at \(T\) for \(v_2, \ldots, v_n\), i.e.,

\[
\mathcal{H}_{uu}(a(T), v_2, \ldots, v_n, u(a(T), \varphi(a(T), v_2, \ldots, v_n))) = 0,
\]

\[
\mathcal{H}_{uuu}(a(T), v_2, \ldots, v_n, u(a(T), \varphi(a(T), v_2, \ldots, v_n))) = 0.
\]

Denote by

\[
G_1(t, v_2, \ldots, v_n) := \mathcal{H}_{uu}(a(t), v_2, \ldots, v_n, u(a(t), \varphi(a(t), v_2, \ldots, v_n))),
\]

\[
G_2(t, v_2, \ldots, v_n) := \mathcal{H}_{uuu}(a(t), v_2, \ldots, v_n, u(a(t), \varphi(a(t), v_2, \ldots, v_n))).
\]

In these formulas we admit that \(a(t)\) is determined by \(v_2, \ldots, v_n\) according to the phase system (5). The system \(G_1(t, v_2, \ldots, v_n) = G_2(t, v_2, \ldots, v_n) = 0\) of linearly independent equations defines a (2\(n\) - 4)-dimensional manifold in the (2\(n\) - 3)-dimensional space of free parameters \(v_2, \ldots, v_n, |v_n| = 1\).
Thus, all integrals \(23\) are continued on \(t \in [0, \infty)\) for all \(v_2, \dots, v_n\) except for a \((2n - 4)\)-dimensional set. Using continuity of integrals \(23\) with respect to \(v_2, \dots, v_n\) we conclude that the relation \(23\) holds for all \((v_2, \dots, v_n) \in \mathbb{R}^{2n-3}\) and for all \(t \geq 0\).

6. Multi-slit evolution

Now we generalize the results of the preceding section to the multi-slit evolution. We do this for the maps with the boundary tree having exactly two finite tips because other cases can be easily obtained by analogy. The L"owner-Kufarev equation \(7\) has the form

\[
\frac{dw}{dt} = -w \left( \lambda e^{iu_1(t)} + w \right) + \frac{1 - \lambda}{e^{iu_2(t)} - w},
\]

where \(u_1\) and \(u_2\) are continuous control functions and \(\lambda \in (0, 1)\). The differential equation for phase variables \(8\) becomes

\[
\dot{a} = -a - 2 \sum_{s=1}^{n-1} \left( \lambda e^{-isu_1} + (1 - \lambda) e^{-isu_2} \right) A^s a,
\]

The Hamiltonian is written as

\[
\tilde{H}(a, \psi, u_1, u_2, \lambda) = \text{Re} \left[ \left( -a - 2 \sum_{s=1}^{n-1} \left( \lambda e^{-isu_1} + (1 - \lambda) e^{-isu_2} \right) A^s a \right) \bar{\psi} \right],
\]

and the adjoint system \(14\) as

\[
\frac{d\bar{\psi}}{dt} = \left( E + 2 \sum_{s=1}^{n-1} \left( \lambda e^{-isu_1} + (1 - \lambda) e^{-isu_2} \right) (s + 1)(A^T)^s \right) \bar{\psi}.
\]

The maximizing condition \(15\) now splits into two bits

\[
\max_{u_1, u_2, \lambda} \tilde{H}(a^*(t), \psi^*(t), u_1, u_2, \lambda) = \tilde{H}(a^*(t), \psi^*(t), u_1^*, u_2^*, \lambda), \quad t \geq 0,
\]

where \(a^*\) and \(\psi^*\) are solutions to the system \(24, 25\) with \(u_1 = u_1^*(t), u_2 = u_2^*(t)\), and moreover,

\[
\text{Re} \left[ \left( \sum_{s=1}^{n-1} \left( e^{-isu_1^*} - e^{-isu_2^*} \right) A^s a \right) \bar{\psi} \right] = 0.
\]

Indeed, if condition \(27\) were not true the value of the optimal \(\lambda\) would be 0 or 1, and the corresponding boundary function would be a one-slit map that contradicts our supposition. The maximum principle \(26\) implies that

\[
\frac{\partial \tilde{H}(a^*(t), \psi^*(t), u_1, u_2, \lambda)}{\partial u_j} \bigg|_{u_j = u_j^*(t)} = 0, \quad j = 1, 2.
\]
Evidently, (24), (25), and (28) imply that

\[ \frac{d\tilde{H}(a^*(t), \psi^*(t), u^*_1, u^*_2, \lambda)}{dt} = 0, \]

for optimal differentiable control functions \( u^*_1(t) \) and \( u^*_2(t) \).

Obviously, Theorem 1 and the corollary thereafter still true in this case and \( \psi \) can be expressed in terms of the phase variable \( a \). Continuous optimal control functions \( u^*_1, u^*_2 \) satisfy the maximizing property (26) and obey equations (28). The Hamiltonian \( \tilde{H}(a, \psi, u_1, u_2, \lambda) \) is a trigonometric polynomial with respect to \( u_1 \) and \( u_2 \) of degree \( n - 1 \) if \( \psi_n \neq 0 \). Let again \( \psi(t) = (v_1, \ldots, v_n)^T \), and we assume that \( v_n \neq 0 \), and diminish \( n \) to \( n - 1 \), otherwise. We have \( \psi_n(t) = e^t \). An additional parameter is \( \lambda \in (0,1) \), which is constant. At \( t = 0 \), we have

\[ \tilde{H}(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda) = \tilde{H}_u(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda) = 0. \]

Suppose that \( \lambda, v_2, \ldots, v_n \) are such that

\[ \tilde{H}_{u_1}u_1(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda), \quad \text{or} \quad \tilde{H}_{u_2}u_2(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda), \]

do not vanish. This holds for all \( (\lambda, v_2, \ldots, v_n) \in B \subset (0,1) \times \mathbb{R}^{2n-2} \) except for a set \( \tilde{A} \) of dimension at most \( 2n - 4 \). The linear space \( B \) is obtained by restriction \( \tilde{A} \) which, being rewritten at \( t = 0 \) for the optimal control functions, is equivalent to

\[ H(a^0, (v_1, \ldots, v_n)^T, u^*_1) - H(a^0, (v_1, \ldots, v_n)^T, u^*_2) = 0. \]

So, the dimension of \( B \) is \( 2n - 2 \). The set \( \tilde{A} \) is determined by two linearly independent equations

\[ \tilde{H}_{u_1}u_1(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda) = \tilde{H}_{u_2}u_2(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda) = 0, \]

or

\[ \tilde{H}_{u_2}u_1(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda) = \tilde{H}_{u_1}u_2(a^0, (v_1, \ldots, v_n)^T, u^*_1, u^*_2, \lambda) = 0, \]

with the fixed optimal \( u^*_1, u^*_2 \), varying \( (\lambda, v_2, \ldots, v_n) \) in \( B \). Therefore, \( \tilde{A} \) is a linear space of dimension \( 2n - 4 \) in \( B \subset \mathbb{R}^{2n-2} \).

**Definition 2.** We say that a vector \( (\lambda, v_2, \ldots, v_n) \) satisfies the regularity condition on \( [0, T] \) for multi-slit evolution if

\[ \tilde{H}_{u_1}u_1(a^*, \psi^*, u^*_1, u^*_2, \lambda) \neq 0, \quad \text{or} \quad \tilde{H}_{u_2}u_2(a^*, \psi^*, u^*_1, u^*_2, \lambda) \neq 0, \quad t \in [0, T], \]

for the optimal \( u^*_1, u^*_2 \) and \( \psi^0 = (v_1, \ldots, v_n)^T \) along the optimal trajectory \( (e^t a^*(t), \psi^*(t)) \) corresponding to \( u^*_1, u^*_2 \).
If \((\lambda, v_2, \ldots, v_n) \in B \setminus \hat{A}\), then \((\lambda, v_2, \ldots, v_n)\) satisfies the regularity condition on \([0, T]\) for \(T > 0\) and small enough. Let us denote by \(\tilde{Y}(T)\) the set of all \((\lambda, v_2, \ldots, v_n)\) satisfying the regularity condition on \([0, T]\).

Let \((v_2, \ldots, v_n) \in \tilde{Y}(T)\). Then
\[
\tilde{H}_{u_1u_1}(a, \psi, u_1, u_2, \lambda) \neq 0, \quad \text{or} \quad \tilde{H}_{u_2u_2}(a, \psi, u_1, u_2, \lambda) \neq 0, \quad t \in [0, T],
\]
for \((a, \psi, u_1, u_2, \lambda)\) from a neighborhood of \((a^*, \psi^*, u_1^*, u_2^*, \lambda)\) and this neighborhood depends on \(t\). The equations
\[
H_{u_1}(a, \psi, u_1, u_2, \lambda) = H_{u_2}(a, \psi, u_1, u_2, \lambda) = 0,
\]

together with the regularity condition on \([0, T]\) determine analytic implicit functions \(u_1 = u_1(a, \psi, \lambda)\) \(u_2 = u_2(a, \psi, \lambda)\) in a neighborhood of \((a^*, \psi^*, \lambda)\), and each of \(u_1 = u_1(a, \psi, \lambda)\) \(u_2 = u_2(a, \psi, \lambda)\) lies in a neighborhood of \(u_1^* \equiv u_1(a^*, \psi^*, \lambda), u_2^* \equiv u_2(a, \psi, \lambda)\) respectively.

We substitute \(u_1 = u_1(a, \psi, \lambda), u_2 = u_2(a, \psi, \lambda)\) in the phase system (24) and the adjoint system (25) and solve the Cauchy problem with the initial data \(a(0) = a^0, \psi(0) = (v_1, \ldots, v_n)^T, \lambda, (v_2, \ldots, v_n) \in B\). According to Corollary 1 the function \(\psi(t)\) can be represented as a function of the phase variable \(a\) and the initial conditions \((v_1, \ldots, v_n)\), say \(\psi(t) = \varphi(a(t), v_1, \ldots, v_n, \lambda)\). As in the preceding section, both \(\tilde{H}\) and \(\psi\) may be determined up to a positive multiplier without loss of generality. Putting, e.g., \(|v_n| = 1\) we reduce the number of free parameters to \(2n - 3\) that gives a local parametric representation of the boundary. Repeating all further steps of preceding section we prove the following theorem.

**Theorem 3.** Let \((\lambda, v_2, \ldots, v_n)\) satisfy the regularity condition on \([0, T]\). Then the system (24) with
\[
\begin{align*}
  u_1 &= u_1(a, \psi, \lambda) = u_1(a, \varphi(a(t), v_1, \ldots, v_n), \lambda), \\
  u_2 &= u_2(a, \psi, \lambda) = u_2(a, \varphi(a(t), v_1, \ldots, v_n), \lambda),
\end{align*}
\]

is partially Liouville integrable. Moreover, this statement is continued for all \((\lambda, v_2, \ldots, v_n) \in B\) and for all \(t \geq 0\).

Theorems 2,3 also solve the problem of integrability for the corresponding coefficient systems in Section 4 substituting the optimal control functions in the Cauchy problem. One can find exact solutions in explicit form for \(n = 3\) in, e.g., [2] and [26].

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