Universal Densities Exist for Every Finite Reference Measure

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Abstract—As it is known, universal codes, which estimate the entropy rate consistently, exist for stationary ergodic sources over finite alphabets but not over countably infinite ones. We generalize universal coding as the problem of universal densities with respect to a fixed reference measure on a countably generated measurable space. We show that universal densities, which estimate the differential entropy rate consistently, exist for finite reference measures. Thus finite alphabets are not necessary in some sense. To exhibit a universal density, we adapt the nonparametric differential (NPD) entropy rate estimator by Feutrill and Roughan. Our modification is analogous to Ryabko’s modification of prediction by partial matching (PPM) by Cleary and Witten. Whereas Ryabko considered a mixture over Markov orders, we consider a mixture over quantization levels. Moreover, we demonstrate that any universal density induces a strongly consistent Cesáro mean estimator of conditional density given an infinite past. This yields a universal predictor with the $0-1$ loss for a countable alphabet. Finally, we specialize universal densities to processes over natural numbers and on the real line. We derive sufficient conditions for consistent estimation of the entropy rate with respect to infinite reference measures in these domains.

Index Terms—Universal coding, prediction by partial matching, quantization, density estimation, universal prediction.

I. INTRODUCTION

Consider the family of stationary ergodic measures over a given countable alphabet. It is known that universal measures, i.e., those consistently estimating the entropy rate in the almost sure sense and in expectation, exist for any finite alphabet. A simple example thereof is the PPM (prediction by partial matching) measure, also called the $R$-measure, constructed gradually by Cleary and Witten [1] and by Ryabko [2], [3].

Universal measures are important for many reasons. They matter not only in practical data compression but also in various problems of statistical inference, as advocated in [4]. Here, we name a few examples of their applications:

- The Shannon-Fano code taken with respect to a universal measure is an instance of a lossless universal code for data compression. Other important instances of universal codes were discovered in [5], [6], [7]. These other codes do not necessarily induce a universal measure due to the strict Kraft inequality.
- Universal measures are an important building block also in estimation of the Markov order [8] and of the hidden Markov order [9]. Indirectly, they are also connected to upper bounds for mutual information and showing disjointness of classes of finite-state and perigraphic processes, discussed in statistical language modeling [10], [11].
- As shown in [12], having a measure that is universal in expectation, we can construct a strongly consistent Cesáro mean estimator of the marginal measure for memoryless sources. Since this convergence holds in the Kullback-Leibler divergence, it also holds in the total variation—by the Pinsker inequality [13], [14].
- Moreover, universal measures induce universal predictors with the $0-1$ loss under mild conditions [15], see also [16] for other related results. In particular, if there is an estimator of the conditional density given an infinite past that is strongly consistent in the total variation, it also induces a universal predictor with the $0-1$ loss, see [17] and [18].

It is known, alas, that universal measures or codes do not exist for a countably infinite alphabet [12], [19], [20]. It may seem that the assumption of a finite alphabet is necessary in general. In this paper, we disprove this hypothesis by casting the problem of universal measures into universal densities, i.e., Radon-Nikodym derivatives with respect to a given reference measure.

The direct inspiration of the following constructions comes from a recent paper by Feutrill and Roughan [21]. They considered a problem of estimating the differential entropy rate $h_\lambda$ (with respect to the Lebesgue measure $\lambda$) of Gaussian processes with long memory, such as the fractional Gaussian noise (FGN) or ARFIMA processes. They observed that the differential entropy rate can be roughly estimated via the simple nonparametric differential (NPD) entropy rate estimator, which reads

$$\hat{h}_{\text{NPD}}(x_1, \ldots, x_n) = \hat{H}(\left\{\frac{x_1}{\Delta}, \ldots, \frac{x_n}{\Delta}\right\}) + \log \Delta,$$

(1)

where $\Delta > 0$ is a fixed bin width and $\hat{H}(y_1, \ldots, y_n)$ is a consistent estimator of the entropy rate for a countably infinite alphabet. A reasonable choice of $\hat{H}(y_1, \ldots, y_n)$ can be the estimator by Kontoyiannis et al. [22], which is consistent under the Doeblin condition, whereas its simple modification discussed in [23] is consistent for any stationary ergodic process but over a finite alphabet. What is interesting, for the estimator by [22], the NPD estimates are empirically quite close to the true differential entropy rate $h_\lambda$ of FGN and ARFIMA processes even for $\Delta = 1$. Feutrill and Roughan [21, Theorem VII.1] tried to argue that the NPD estimator tends to the differential entropy rate for $\Delta \to 0$ and $n \to \infty$ but their treatment of the joint limit was not rigorous.

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Having learned about this result, we thought that there must be some solid mathematical idea underneath. Suppose that the observations are bounded in the unit interval, $x_i \in [0,1]$. Let us take the bin width $\Delta = 2^{-l}$ where $l = 0, 1, 2, \ldots$ and apply a universal measure $R^l$ for a finite alphabet $\{1, 2, \ldots, 2^l\}$ as a special case of estimator $\hat{H}(y_1, \ldots, y_n)$. Then we obtain $\hat{h}_{NPD}(x_1, \ldots, x_n) = -\frac{1}{n} \log R^l_\lambda(x_1, \ldots, x_n)$, where function

$$R^l_\lambda(x_1, \ldots, x_n) := 2^{-l} \cdot R^l(\lfloor 2^l x_1 \rfloor, \ldots, \lfloor 2^l x_n \rfloor)$$

(2)
is a probability density. Making one step further, let weights $w_i > 0$ with $\sum_{l=0}^{\infty} w_l = 1$. Then we may consider the mixture over all quantization levels,

$$R_\lambda(x_1, \ldots, x_n) := \sum_{l=0}^{\infty} w_l \cdot 2^{-l} \cdot R^l(\lfloor 2^l x_1 \rfloor, \ldots, \lfloor 2^l x_n \rfloor),$$

(3)

which is also a probability density. The transition from (2) to (3) is analogous to Ryabko’s [2], [3] derivation of the universal $R$-measure over a finite alphabet by taking a mixture of PPM measures by [1] over all Markov orders. Hence, for any stationary ergodic process $(X_i)_{i \in \mathbb{Z}}$ with $X_i \in \{0, 1\}$, we have

$$\lim_{n \to \infty} \sup_{A \subseteq \{0,1\}^n} \left[ -\frac{1}{n} \log R_\lambda(X_1, \ldots, X_n) \right] \leq \inf_{\lambda \geq 0} h^l_\lambda \text{ a.s.},$$

(4)

where quantized entropy rates $h^l_\lambda$ are the limits of $-\frac{1}{n} \log R^l_\lambda(X_1, \ldots, X_n)$ by universality of measures $R^l$. Moreover, since $R_\lambda$ is a probability density then by the asymptotic equipartition [24] and by [25, Theorem 3.1], we obtain

$$\lim_{n \to \infty} \inf_{\lambda \geq 0} \left[ -\frac{1}{n} \log R_\lambda(X_1, \ldots, X_n) \right] \geq h_\lambda \text{ a.s.}$$

(5)

Thus density $R_\lambda$ estimates entropy rate $h_\lambda$ consistently if $\inf_{l \geq 0} h^l_\lambda = h_\lambda$.

In this paper, we prove that this holds true in a more general setting. Density (3) is an instance of a more general construction, which we call the NPD density to honor the idea by Feurtrill and Roughan. The NPD density can be defined with respect to an arbitrary finite reference measure on a countably generated measurable space. Moreover, if the reference measure is finite then the respective NPD density is universal, i.e., it estimates consistently the differential entropy rate with respect to this reference measure. This result can be proved using the asymptotic equipartition for densities [24], Barron’s inequality [25, Theorem 3.1], and the monotone convergence of $f$-divergences for filtrations [11, Chapter 22, Problem 4], a by-product of our earlier investigations [26, Lemma 2]. The last one yields $\inf_{l \geq 0} h^l_\lambda = h_\lambda$ indeed.

Thus, it is not finiteness of the alphabet but rather finiteness of the reference measure that allows for universal densities. Nonexistence of universal codes for a countably infinite alphabet is tightly connected to nonexistence of a uniform probability measure over the set of natural numbers. The relevance of finite reference measures may be known to experts in density estimation, see [27, Lemma 2], although that particular result concerns memoryless sources. In this context, we also recall that nonexistence of universal measures for countably infinite alphabets rests on nonexistence of a consistent density estimator for memoryless sources over such alphabets [12]. Thus, there are interactions between density estimation and universal coding, worth further exploration.

Having constructed the NPD density and proven its universality, we will also discuss some applications. These are as follows.

- Inspired by work [12], we show that if a universal density exists (for a given subclass of processes) then it induces a strongly consistent Cesàro mean estimator of the conditional density given an infinite past, in the total variation, as it follows by the Pinsker inequality. Any such conditional density estimator solves also the problem of universal prediction with the $0 - 1$ loss for a countable alphabet. This strengthens our earlier results from work [15], which dealt with a finite alphabet.

- As some examples, we specialize NPD densities to the classes of processes over a countably infinite alphabet and on the real line. We easily name some sufficient conditions that allow for consistent estimation of the entropy rate with respect to infinite reference measures in these domains, see also [20]. In particular, as we show, there exists a strongly consistent entropy rate estimator with respect to the Lebesgue measure in the class of stationary ergodic Gaussian processes.

The organization of the paper is as follows. In Section II, we amply recall preliminaries to keep the paper relatively self-contained. Section III contains the main result, i.e., the construction of the universal NPD density with respect to a finite reference measure. Having an instance of a universal density, in Section IV, we construct a consistent Cesàro mean estimator of the conditional density given an infinite past. Consequently, this estimator of conditional density is turned into a universal predictor in the countable alphabet setting. As some further examples, in Section V, we present applications of the total NPD density to processes over a countably infinite alphabet and on the real line. The paper is concluded in Section VI, where we sketch open problems.

II. PRELIMINARIES

In this section we establish our setting and report the received knowledge. We discuss the introductory material such as the entropy rate, the asymptotic equipartition, the definition of universal measures, and the universal PPM measure for a finite alphabet.

A. General Setting

Let $(\mathcal{X}, \mathcal{C}, \mu)$ be a countably generated measurable space with a $\sigma$-finite measure $\mu$ on it. Measure $\mu$ will be called the reference measure. Simple familiar examples are the counting measure $\mu(\mathcal{A}) = \gamma(\mathcal{A}) := \text{card } \mathcal{A}$ for a countable alphabet $\mathcal{X}$ or the Lebesgue measure $\mu([a, b]) = \lambda([a, b]) := b - a$ for $\mathcal{X} = \mathbb{R}$. Consider the product space $(\mathcal{X}^\mathbb{Z}, \mathcal{C}^\mathbb{Z})$ and put random variables $X_k : \mathcal{X}^\mathbb{Z} \ni (x_i)_{i \in \mathbb{Z}} \mapsto x_k \in \mathcal{X}$. We write the tuples of points as $x_{j:k} := (x_j, x_{j+1}, \ldots, x_k)$. For a probability measure $R$ on $(\mathcal{X}^\mathbb{Z}, \mathcal{C}^\mathbb{Z})$, we denote its finite-dimensional restrictions $R_n(A) := R(X_{1:n} \in A)$ and if $R_n \ll \mu^n$ then
we write the densities
\[ R_\mu(x_{1:n}) := \frac{dR_n}{d\mu^n}(x_{1:n}). \tag{6} \]

The space of stationary ergodic measures on \((\mathcal{X}, \mathcal{X}^\mathbb{N})\) with respect to the shift operation will be denoted as \(\mathcal{E}\). A measure \(P \in \mathcal{E}\) where \(P_n \ll \mu^n\) is called a memoryless source if \(P_n(x_{1:n}) = \prod_{i=1}^n P_n(x_i)\).

Fix a measure \(P \in \mathcal{E}\) where \(P_n \ll \mu^n\). Convergence in probability with respect to measure \(P\) is denoted
\[ \lim_{n \to \infty} Y_n = Y \text{ i.p.} \iff \forall \epsilon > 0 \lim_{n \to \infty} P(|Y_n - Y| > \epsilon) = 0. \tag{7} \]

The expectation operator \(\mathbb{E}\) and the quantifier “almost surely” (a.s.) will be also taken throughout with respect to \(P\). Throughout the paper, symbol log denotes the logarithm to a fixed underspecified base. We define the block entropy
\[
\begin{align*}
\mu(n) &= \mathbb{E}[-\log P_\mu(x_{1:n})] \\
&= -\int P_\mu(x_{1:n}) \log P_\mu(x_{1:n}) d\mu^n(x_{1:n}). \tag{8}
\end{align*}
\]

A short notice, if the reference measure \(\mu\) is the counting measure then \(\mu(n) \geq 0\) since \(P_\mu(x_{1:n}) \leq 1\). By contrast, if \(\mu\) is a probability measure then \(\mu(n) \leq 0\) since \(-\mu(n)\) is the Kullback-Leibler divergence between \(P_\mu\) and \(\mu^n\).

In any case, by stationarity and by the Jensen inequality, the block entropy is subadditive, \(\mu(n + m) \leq \mu(n) + \mu(m)\). Hence, by the Fekete lemma for subadditive sequences [28], see also [11, Theorem 5.11], sequence \(\mu(n)/n\) is decreasing and there exists the entropy rate
\[
\mu := \lim_{n \to \infty} \frac{\mu(n)}{n} = \inf_{n \geq 1} \frac{\mu(n)}{n}. \tag{9}
\]

Moreover, by the Shannon-McMillan-Breiman (SMB) theorem, in many cases, we have the asymptotic equipartition property, namely,
\[
\lim_{n \to \infty} [-\log P_\mu(x_{1:n})]/n = \mu \text{ a.s.,} \tag{10}
\]

as noticed by Shannon [29] for memoryless sources and consecutively generalized in [24], [30], [31], and [32] to other settings.

In particular, let us consider the class of stationary ergodic measures with a finite entropy rate,
\[
\mathbb{E}(\mu) := \{P \in \mathcal{E} : P_n \ll \mu^n \text{ and } |\mu| < \infty\}. \tag{11}
\]

Let us also denote the conditional density
\[
P_\mu(x_{n+1:n+m}|x_{1:n}) := \frac{P_\mu(x_{n+1:n+m})}{P_\mu(x_{1:n})}. \tag{12}
\]

As shown by Barron [24, Theorem 1], we have the asymptotic equipartition (10) for an arbitrary \(P \in \mathbb{E}(\mu)\). This fact is a consequence of the Breiman ergodic theorem [30], [33, Theorem 12(c)] since, as shown by Barron [24, Proof of Theorem 1], for \(P \in \mathbb{E}(\mu)\) there exists the limit of conditional densities
\[
P_\mu(X_0|X_{-\infty:-1}) := \lim_{n \to \infty} P_\mu(X_0|X_{-n:-1}) \text{ a.s.} \tag{13}
\]

and \(\mathbb{E}\sup_{n \in \mathbb{N}} |\log P_\mu(X_0|X_{-n:-1})| < \infty\), whereas
\[
\mu = \mathbb{E}[-\log P_\mu(X_0|X_{-\infty:-1})]. \tag{14}
\]

Obviously, the asymptotic equipartition is an interesting property from a statistical perspective since it suggests that the entropy rate can be estimated, which is known to be the case for a finite counting reference measure. However, the problem of applying statement (10) for estimating the unknown entropy rate is that we need to estimate the unknown probability measure.

B. Universal Measures

To approach the problem of entropy estimation in a reasonable way, let us consider another probability measure \(R\) where \(R_n \ll \mu^n\), which need not be stationary or ergodic. As it is a part of an older information-theoretic folklore, for any \(m > 1\) we have a particular case of Markov’s inequality, called sometimes the Barron inequality,
\[
P \left( \log \frac{dP_n}{dR_n}(X_{1:n}) \leq -\log m \right) \leq \frac{1}{m}, \tag{15}
\]

shown for example in [25, Theorem 3.1] or applied implicitly much earlier in [34]. Since \(dP_n/dR_n(x_{1:n}) = P_\mu(x_{1:n})/R_\mu(x_{1:n})\), as a result, by an easy application of the Borel-Cantelli lemma, we obtain
\[
\liminf_{n \to \infty} \left(-\log R_\mu(X_{1:n})/n \geq \mu \right. \text{ a.s.} \tag{16}
\]

Since the Kullback-Leibler divergence is nonnegative in general,
\[
D(P_n||R_n) := \mathbb{E} \left[ \log \frac{dP_n}{dR_n}(X_{1:n}) \right]
\]
\[
= \mathbb{E} \left[ \log \frac{P_\mu(X_{1:n})}{R_\mu(X_{1:n})} \right] \geq 0, \tag{17}
\]

we also have a similar result in expectation:
\[
\liminf_{n \to \infty} \mathbb{E} \left(-\log R_\mu(X_{1:n})/n \geq \mu \right. \text{ a.s.} \tag{18}
\]

Consequently, we will consider three definitions of a universal measure. By these definitions, a universal measure can be used for estimating the entropy rate of an unknown stationary ergodic process.

Definition 1: A probability measure \(R\) where \(R_n \ll \mu^n\) is called strongly universal with respect to a measure \(\mu\) if for every measure \(P \in \mathbb{E}(\mu)\),
\[
\lim_{n \to \infty} [-\log R_\mu(X_{1:n})]/n = \mu \text{ a.s.} \tag{19}
\]

Definition 2: A probability measure \(R\) where \(R_n \ll \mu^n\) is called universal in expectation with respect to a measure \(\mu\) if for every measure \(P \in \mathbb{E}(\mu)\),
\[
\lim_{n \to \infty} \mathbb{E} [-\log R_\mu(X_{1:n})]/n = \mu. \tag{20}
\]

Definition 3: A probability measure \(R\) where \(R_n \ll \mu^n\) is called universal in probability with respect to a measure \(\mu\) if for every measure \(P \in \mathbb{E}(\mu)\),
\[
\lim_{n \to \infty} [-\log R_\mu(X_{1:n})]/n = \mu \text{ i.p.} \tag{21}
\]

Remark: A measure \(R\) that satisfies (19), (20), and (21) is simply called universal with respect to \(\mu\). If the reference
measure is the counting measure over a countable alphabet \( \mathbb{X} \), i.e., \( \mu(A) = \gamma(A) := \text{card } A \) for \( A \subset \mathbb{X} \), then we speak of measures that are universal with respect to alphabet \( \mathbb{X} \), respectively. In this case, we drop the subscript \( \gamma \) and write \( P_n(x) \to P(x) \), \( R_n(x) \to R(x) \), and \( h_n \to h \).

In fact, implications (19) \( \implies \) (20) \( \implies \) (21) hold usually. Obviously, (19) \( \implies \) (21) since the almost sure convergence implies the convergence in probability by the Riesz theorem. By contrast, implications (19) \( \implies \) (20) and (20) \( \implies \) (21) need a more explicit information-theoretic proof. In the next two propositions we generalize facts that are well known in the case of the counting reference measure.

**Proposition 1:** If measure \( \mu \) is strongly universal with respect to alphabet \( \mathbb{X} \) and \( R_\mu(x_{1:n}) \geq K \epsilon^n \) holds uniformly for some \( 0 < K < 1 \) and \( c > 0 \) then measure \( \mu \) is universal in expectation with respect to measure \( \mu \).

**Proof:** Since \( R_\mu(x_{1:n}) \geq K \epsilon^n \), we have \( -\log R_\mu(x_{1:n})/n \leq -\log c - \log K \). Hence by the Fatou lemma with the minus sign and by the strong universality of measure \( \mu \), we obtain

\[
\lim_{n \to \infty} \frac{\mathbb{E}[n \log R_\mu(X_{1:n})]}{n} \leq \mathbb{E} \lim_{n \to \infty} \frac{-\log R_\mu(X_{1:n})}{n} = h_\mu. \tag{22}
\]

Combining this with inequality (18) yields the claim. \( \square \)

A careful reader may have noticed that the above proof applies the Fatou lemma rather than the dominated convergence. The latter is usually invoked for proving universality with respect to the counting reference measure.

**Proposition 2:** A measure universal in expectation is universal in probability.

**Proof:** Let measure \( \mu \) be universal in expectation. Let an \( \epsilon > 0 \). In view of the asymptotic equipartition (10) and inequality (16), it is sufficient to show

\[
\lim_{n \to \infty} P \left( -\frac{\log R_\mu(X_{1:n})}{n} + \frac{\log P_\mu(X_{1:n})}{n} \geq \epsilon \right) = 0. \tag{23}
\]

Hence, by the Markov inequality, \( P(X \geq \epsilon) \leq \mathbb{E} X_+ / \epsilon \), it suffices to prove

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \log \frac{P_\mu(X_{1:n})}{R_\mu(X_{1:n})} \right] > 0. \tag{24}
\]

But this follows from equality

\[
\frac{1}{n} \mathbb{E} \left[ \log \frac{P_\mu(X_{1:n})}{R_\mu(X_{1:n})} \right] = -\frac{1}{n} \mathbb{E} \left[ \log \frac{P_\mu(X_{1:n})}{R_\mu(X_{1:n})} \right] + \frac{1}{n} \mathbb{E} \left[ \log \frac{P_\mu(X_{1:n})}{R_\mu(X_{1:n})} \right]. \tag{25}
\]

The first term on the right hand side tends to 0 by the universality in expectation whereas the second term tends to 0 since

\[
\mathbb{E} \left[ \log \frac{P_\mu(X_{1:n})}{R_\mu(X_{1:n})} \right] = \int_0^\infty P \left( \log \frac{P_\mu(X_{1:n})}{R_\mu(X_{1:n})} \leq -u \right) du \leq \int_0^\infty \frac{d\log m}{m} = \int_1^\infty \frac{m \log e}{m^2} = \log e \tag{26}
\]

is uniformly bounded by the Barron inequality (15).

Thus it is sufficient usually to demonstrate strong universality and a uniform lower bound on a given density. Other flavors of universality follow hence automatically.

**C. Finite Alphabet**

For a finite alphabet, universal measures are known to exist. We recall an example of a universal measure for this case, called the PPM (prediction by partial matching). The PPM measure was discovered gradually. Cleary and Witten [1] considered roughly Markov approximations PPM\(_k\) defined in (27) and coined name PPM, which we prefer as more distinctive. Ryabko [2], [3] considered the infinite series PPM defined in (29), called it the R-measure, and proved that it is universal. To be precise, Ryabko used the Krichevsky-Trofimov smoothing (+1/2) rather than the Laplace smoothing (+1) applied in (27). This difference is minor and does not affect universality.

The definition that we use is as follows.

**Definition 4 (PPM Density):** Let the alphabet be \( \mathbb{X} = \{a_1, \ldots, a_D\} \). Adapting the definitions from works [1], [2], [3], [35], the PPM density of order \( k \geq 0 \) is defined as

\[
\text{PPM}_k^D(x_{1:n}) := \begin{cases} 
D^{-k-1} \prod_{i=k+2}^n N(x_{i-k-1} x_{i+1} x_{i+2}) + 1, & k \leq n-2, \\
D^{-n}, & k \geq n-1,
\end{cases} \tag{27}
\]

where the frequency of a substring \( w_{1:k} \) in a string \( x_{1:n} \) is

\[
N(w_{1:k}|x_{1:n}) := \sum_{i=1}^{n-k+1} \mathbb{1}\{x_{i:i+k-1} = w_{1:k}\}. \tag{28}
\]

Subsequently, we define the (total) PPM density as

\[
\text{PPM}_k^D(x_{1:n}) := \sum_{k=0}^\infty w_k \text{PPM}_k^D(x_{1:n}), \tag{29}
\]

\[
w_k := \frac{1}{k+1} - \frac{1}{k+2}. \tag{30}
\]

Infinite series (29) is a sum over finitely many distinct terms and it is computable in the sense of computability theory, see [15], since we have

\[
\text{PPM}_k^D(x_{1:n}) = D^{-n} \text{ for } k \geq L(x_{1:n}), \tag{31}
\]

where \( L(x_{1:n}) \) is the maximal length of a repetition in \( x_{1:n} \),

\[
L(x_{1:n}) := \max \{k \geq 0 : x_{i+1:i+k} = x_{j+1:j+k} \text{ for some } 0 \leq i < j \leq n-k\}, \tag{32}
\]

an important information-theoretic statistic in its own right [11, Chapter 9].

Let us show that the total PPM density yields a probability measure.

**Theorem 1:** There exists a measure \( R \) such that \( R(x_{1:n}) = \text{PPM}_0^D(x_{1:n}) \).

**Remark:** This measure \( R \) will be denoted PPM\(_D^0\).
Proof: By the Kolmogorov process theorem, it suffices to show that
\[ \sum_{x_{n+1} \in X} \text{PPM}^D(x_{1:n+1}) = \text{PPM}^D(x_{1:n}). \] (33)
But this follows by the monotone convergence from
\[ \sum_{x_{n+1} \in X} \text{PPM}^D_k(x_{1:n+1}) = \text{PPM}^D_k(x_{1:n}), \] (34)
which in turn follows by the definition of \( \text{PPM}^D_k(x_{1:n}) \). □

The strong universality of the total PPM measure follows by the Stirling approximation and by the Birkhoff ergodic theorem. Moreover, we have the uniform lower bound
\[ \text{PPM}^D(x_{1:n}) \geq w_{n-1} \text{PPM}^D_{n-1}(x_{1:n}) \geq \frac{D^{-n}}{(n+1)^2} > \frac{1}{4} (2D)^{-n}. \] (35)

Hence by Propositions 1 and 2 follows also the universality in expectation and in probability.

Theorem 2 ([2], [3], [35]): Measure \( \text{PPM}^O \) is universal with respect to alphabet \( \mathbb{X} = \{a_1, \ldots, a_D\} \).

III. MAIN RESULTS

In this section, we exhibit the main result. We show that universal measures exist if the reference measure is an arbitrary finite measure. We do it by an effective construction. Our constructive example of a universal measure is called the PPM (nonparametric differential) measure to honor the quantization idea by Feurtrill and Roughan [21]. They carried out the construction of the NPD estimator half-way—as detailed in Section I.

Let us proceed to the construction of the NPD measure. Let notation \( X_l \uparrow X \) denote a filtration of a \( \sigma \)-field \( X \), i.e., a sequence of nested \( \sigma \)-fields \( X_l \subset X_{l+1} \) where \( l = 0, 1, 2, \ldots \) and \( \sigma(\bigcup_{l \geq 0} X_l) = X \). Assuming that the reference measure \( \mu \) is a finite measure on a countably generated measurable space, we will demonstrate universality of the following constructive object.

Definition 5 (NPD Density): Let \( (\mathbb{X}, X, \mu) \) be a countably generated finite measure space. Let \( X_l \uparrow X \) where \( l = 0, 1, 2, \ldots \) be a filtration where the \( \sigma \)-fields \( X_l \) are finite with \( X_0 = \{\mathbb{X}, \emptyset\} \). Such a filtration exists since \( X \) is countably generated. Let \( \chi_l \) be the finite partitions that generate \( \sigma \)-fields \( X_l \), respectively. We treat classes \( \chi_l \) as finite alphabets of symbols \( A \in \chi_l \). We introduce quantizations of points \( x \in \mathbb{X} \) as symbols \( x^l := A \) for \( x \in A \in \chi_l \). Moreover, for \( l = 0, 1, 2, \ldots \) let \( R_l \) be certain measures that are universal for alphabets \( \chi_l \). We define the NPD density of order \( l \geq 0 \) as
\[ \text{NPD}^l_{\mu}(x_{1:n}) := \frac{R_l(x_{1:n})}{\prod_{i=1}^{\text{card} \chi_l}(x_i^l)}. \] (36)
Subsequently, we define the (total) NPD density as
\[ \text{NPD}_\mu(x_{1:n}) := \sum_{l=0}^{\infty} w_l \text{NPD}^l_{\mu}(x_{1:n}), \] (37)
\[ w_l := \frac{1}{l+1} - \frac{1}{l+2}. \] (38)

Let us note that the NPD measure depends implicitly on filtration \( X_l \uparrow X \), universal measures \( R_l \) for finite alphabets, and reference measure \( \mu \). Actually, for the universality of the NPD density, it does not matter which universal measures \( R_l \) we use in definition (36). There is some analogy between the PPM series (29) and the NPD series (37). Weights \( w_l \) in series (29) weigh different Markov approximations, whereas weights \( w_l \) in series (37) weigh different quantization levels. Thus, we may say that our development of the quantization idea by Feurtrill and Roughan [21] is analogous to Ryabko’s [2], [3] development of the PPM measures by Cleary and Witten [1]. Whereas Cleary and Witten [1] and Feurtrill and Roughan [21] considered approximations of a fixed order, the order meaning the Markov order or the quantization level respectively, the idea of Ryabko [2], [3] and of us is to apply a mixture of infinitely many orders. As we have seen, this guarantees that the total PPM measure is universal and it is reasonable to expect that so is the total NPD density.

Before we demonstrate universality, let us take a closer look at the NPD densities. The total NPD density is measurable and finite \( \mu \)-almost everywhere, as it follows by the monotone convergence. Just an explicit proof for a sanity check.

Theorem 3: We have \( \text{NPD}_\mu(x_{1:n}) < \infty \) for \( \mu \)-almost all \( x_{1:n} \).

Proof: For each \( l \geq 0 \), we have
\[ \int \text{NPD}^l_{\mu}(x_{1:n}) d\mu^n(x_{1:n}) = \sum_{x_{1:n}} R^l(x_{1:n}) = 1. \] (39)
Since \( \text{NPD}^l_{\mu}(x_{1:n}) \geq 0 \), hence by the monotone convergence, we obtain
\[ \int \text{NPD}^l_{\mu}(x_{1:n}) d\mu^n(x_{1:n}) = \sum_{l=0}^{\infty} w_l \int \text{NPD}^l_{\mu}(x_{1:n}) d\mu^n(x_{1:n}) = 1. \] (40)
Since the integral is finite, the integrand is finite almost everywhere. □

Although the total NPD density can be divergent for particular tuples \( x_{1:n} \), we can control its finiteness pretty well in some important cases.

Example 1 (Dyadic Partitions): Let the universal measures in (36) be the PPM measures, \( R_l \equiv \text{PPM}^{\text{card} \chi_l} \). Then
\[ \text{NPD}^l_{\mu}(x_{1:n}) = \frac{(\text{card} \chi_l)^{-n}}{\prod_{i=1}^{\text{card} \chi_l}(x_i^l)} \text{ for } l \geq M(x_{1:n}), \] (41)
where \( M(x_{1:n}) \) is the minimal quantization level that puts points \( x_1, \ldots, x_n \) into different bins,
\[ M(x_{1:n}) := \min \{ l \geq 0 : L(x_{1:n}) = 0 \}. \] (42)
A particularly regular case arises for uniformly dyadic partitions:
\[ \text{card} \chi_l = 2^l \text{ and } \mu(x_i^l) = 2^{-l}. \] (43)
Such partitioning is feasible if the reference measure \( \mu \) is a nonatomic probability measure, such as the normal distribution \( N(m, \sigma^2) \) to be discussed in Section V-B. Then \( \text{NPD}^l_{\mu}(x_{1:n}) = 1 \) for \( l \geq M(x_{1:n}) \) and series \( \text{NPD}_\mu(x_{1:n}) \).
is finite if \( M(x_{1:n}) \) is finite, whereas statistic \( M(X_{1:n}) \) is finite \( \mu^2 \)-almost surely.

Universality was stated in Definitions 1–3 as a property of measures rather than their densities. Thus, let us see the following statement.

**Theorem 4:** There exists a measure \( R \) such that \( R_{\mu}(x_{1:n}) = \text{NPD}_\mu(x_{1:n}) \).

**Remark:** This measure \( R \) will be denoted NPD.

**Proof:** We may construct measures

\[
R_n(A) := \int_{x_{1:n} \in A} \text{NPD}_\mu(x_{1:n}) d\mu^n(x_{1:n}).
\]

To show that measures \( R_n \) induce measure \( R \) on infinite sequences, by the Kolmogorov process theorem, it suffices to show that \( R_{n+1}(A \times X) = R_n(A) \). In turn, using the Fubini theorem this is implied by condition

\[
\int_{x_{n+1} \in X} \text{NPD}_\mu(x_{1:n+1}) d\mu(x_{n+1}) = \text{NPD}_\mu(x_{1:n}),
\]

The above follows by the monotone convergence from

\[
\int_{x_{n+1} \in X} \text{NPD}_\mu(x_{1:n+1}) d\mu(x_{n+1}) = \text{NPD}_\mu(x_{1:n}),
\]

which is true since each \( R^i \) in (36) is a measure. \( \square \)

In order to prove universality of the total NPD measure, we will apply a lemma that concerns convergence of \( f \)-divergences for filtrations:

**Lemma 1 ([11, Chapter 3, Problem 4]):** For an interval \( A \) let \( f : A \rightarrow [0, \infty] \) be a nonnegative, continuous, and convex measurable function, let \( \nu \ll \rho \) be two finite measures on a measurable space, and let \( G_n \uparrow G \) be a filtration. We have

\[
\lim_{n \to \infty} \int f \left( \frac{d\nu}{d\rho} \right)^n d\rho = \int f \left( \frac{d\nu}{d\rho} \right) d\rho,
\]

where the sequence on the left hand side is increasing.

**Remark:** Lemma 1 follows, via the martingale convergence, by a synergy of the Fatou lemma and the Jensen inequality. The Fatou lemma yields that the left hand side is larger than the right hand side, whereas the Jensen inequality yields the reversed inequality. The idea is the same as the proof of [26, Lemma 2], which concerns continuity of conditional mutual information for \( \sigma \)-fields.

Now we will derive the main result of this section.

**Theorem 5:** Measure NPD is universal with respect to the finite measure \( \mu \).

**Proof:** It suffices to show that for \( P \in \mathbb{E}(\mu) \), we have

\[
\limsup_{n \to \infty} \frac{- \log \text{NPD}_\mu(X_{1:n})}{n} \leq h_\mu \text{ a.s.}
\]

The strong universality follows hence by the converse bound (16). By contrast, the universality in expectation and in probability follows by Proposition 1 and 2 and inequality

\[
\text{NPD}_\mu(x_{1:n}) \geq \frac{1}{2} \text{NPD}_\mu^0(x_{1:n}) = \frac{1}{2} \mu(X)^n.
\]

So as to demonstrate (48), we first observe that by the strong universality of measures \( R^i \), we have

\[
\lim_{n \to \infty} \frac{- \log R_i(x_{1:n})}{n} = \inf_{n \geq 1} \frac{- \log P_n(x_{1:n})}{n} \text{ a.s.}
\]

On the other hand, by the Birkhoff ergodic theorem, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left[ - \log \mu(X_i) \right] = C_l := \mathbb{E} \left[ - \log \mu(X_1) \right] \text{ a.s.}
\]

Moreover, each cross entropy \( C_l \) is finite since each \( C_l \) is a sum over finitely many finite elements—by \( P_1 \ll \mu \). Denote quantities

\[
h_\mu(n) := \mathbb{E} \left[ - \log P_n(X_1^n) \right] = \mathbb{E} \left[ - \log P_n(X_1^n) \right] - n C_l.
\]

Since cross entropies \( C_l \) are finite, equations (50) and (51) imply

\[
\lim_{n \to \infty} \frac{- \log \text{NPD}_\mu(X_{1:n})}{n} \leq \inf_{n \geq 1} \frac{h_\mu(n)}{n} \text{ a.s.}
\]

Since \( \text{NPD}_\mu(x_{1:n}) \geq w_l \text{NPD}_\mu^l(x_{1:n}) \) then for any \( l \geq 0 \), we obtain

\[
\limsup_{n \to \infty} \frac{- \log \text{NPD}_\mu(X_{1:n})}{n} \leq \limsup_{n \to \infty} \frac{- \log w_l \text{NPD}_\mu^l(X_{1:n})}{n} = h_\mu \text{ a.s.}
\]

It remains to show that \( \inf_{l \geq 0} h_\mu^l = h_\mu \). For this goal we observe that

\[
\frac{P_n(x_{1:n})}{\prod_{i=1}^{n} \mu(x_i)} = dP_n | x_n \rangle \langle x_n |_{\mu(x_{1:n})}.
\]

Hence we have

\[
h_\mu^l = \inf_{n \geq 1} \frac{h_\mu(n)}{n} = \inf_{n \geq 1} \frac{1}{n} \int \eta \left( \frac{dP_n | x_n \rangle \langle x_n |_{\mu(x_{1:n})}}{d\mu^n | x_n \rangle \langle x_n |_{\mu(x_{1:n})}} \right) d\mu^n,
\]

where \( \eta(x) := -x \log x \). We switch the order of infimums,

\[
\inf_{l \geq 0} h_\mu^l = \inf_{n \geq 1} \frac{1}{n} \int \eta \left( \frac{dP_n | x_n \rangle \langle x_n |_{\mu(x_{1:n})}}{d\mu^n | x_n \rangle \langle x_n |_{\mu(x_{1:n})}} \right) d\mu^n
\]

and we apply Lemma 1 to function \( f(x) = \log 2 - \eta(x) \). Hence

\[
\inf_{l \geq 0} h_\mu^l = \inf_{n \geq 1} \frac{1}{n} \int \eta \left( \frac{dP_n | x_n \rangle \langle x_n |_{\mu(x_{1:n})}}{d\mu^n | x_n \rangle \langle x_n |_{\mu(x_{1:n})}} \right) d\mu^n = \inf_{n \geq 1} \frac{h_\mu(n)}{n} = h_\mu.
\]

The proof is complete. \( \square \)

**IV. APPLICATIONS**

In this section, we show that if a universal measure exists then we may construct a strongly consistent Cesàro mean estimator for the limiting conditional density given an infinite past. Subsequently, we show that any strongly consistent estimator of this conditional density yields a universal predictor for a countable alphabet.
A. Conditional Density Estimation

Here, we will show that a universal density, if it exists, induces a strongly consistent Cesàro mean estimator of the conditional density. For this goal, we consider a general reference measure \( \mu \) as in Section II-A.

A stochastic process \( (R^{(n)}_\mu)_{n \geq 1} \) is called a conditional density estimator if each random variable \( R^{(n)}_\mu(x) \) is a (marginal) probability density with respect to measure \( \mu \) and each function \( R^{(n)}_\mu(x) \) is a measurable function of random variables \( X_{n-1} \). We also denote the respective random measure

\[
P^{(\infty)}(x) := \lim_{n \to \infty} P^{(n)}(x),
\]

where

\[
P^{(n)}(x) := P_\mu(x|X_{n-1}).
\]

We denote the respective random measure \( P^{(\infty)}(A) := \int_A P^{(\infty)}(x) \, d\mu(x) \). Obviously \( P^{(\infty)} = P_1 \) if \( P \) is a memoryless source.

Inspired by the construction of [12], we will consider the following object:

**Definition 6:** Consider a probability measure \( R \) where \( R_n \ll \mu^n \). The Cesàro mean measure \( \bar{R} \) is defined via conditional densities

\[
\bar{R}_\mu(x_n|x_{1:n-1}) := \frac{1}{n} \sum_{i=0}^{n-1} R_\mu(x_n|x_{n-i};n-1).
\]

Let us observe that we may introduce a conditional density estimator

\[
\bar{R}^{(n)}_\mu(x) := \bar{R}_\mu(x|X_{n-1}).
\]

We will call it the Cesàro mean density estimator. We denote the respective random measure \( \bar{R}^{(n)}(A) := \int_A \bar{R}^{(n)}(x) \, d\mu(x) \). In work [12], a similar conditional density estimator was considered, albeit with a reflected time arrow. The Cesàro mean measure \( \bar{R} \) reminds also of linear interpolation models used for statistical language modeling in the 1990’s [36].

For the Cesàro mean density estimator, we have the following result which generalizes [12, Theorem 1] for memoryless sources.

**Theorem 6:** Consider a measure \( P \in \mathbb{E}(\mu) \), a probability measure \( R \) where \( R_n \ll \mu^n \), and the Cesàro mean density estimator \( (\bar{R}^{(n)}_\mu)_{n \geq 1} \). We have

\[
\mathbb{E}[-\log R_\mu(X_{1:n})]/n - h_\mu \geq \mathbb{E} D(P^{(\infty)}||\bar{R}^{(n)}).
\]

**Proof:** We essentially apply the proof idea of [12, Theorem 1], which is a restriction of the present claim to \( \mu \) being the counting measure and \( P \) being a memoryless source. In the reasoning rewritten in a more transparent notation we apply the stationarity and the Jensen inequality,

\[
\mathbb{E}[-\log R_\mu(X_{1:n})]/n - h_\mu = \mathbb{E}\left[-\frac{1}{n} \sum_{i=1}^{n} \log R_\mu(X_i|X_{1:i-1})\right] - h_\mu
\]

\[
= \mathbb{E}\left[-\frac{1}{n} \sum_{i=0}^{n-1} \log R_\mu(X_0|X_{i};n-1)\right] - h_\mu
\]

\[
\geq \mathbb{E}\left[-\log \frac{1}{n} \sum_{i=0}^{n-1} R_\mu(X_0|X_{i};1)\right] - h_\mu
\]

\[
= \mathbb{E} D(P^{(\infty)}||\bar{R}^{(n)}),
\]

where the last transition is due to equality

\[
h_\mu = \mathbb{E}[-\log P^{(\infty)}(X_0)]
\]

shown by Barron [24, Proof of Theorem 1].

We recall the total variation distance of probability measures \( P_1 \) and \( R_1 \) on a measurable space \( (X, \mathcal{X}) \), defined as

\[
\delta(P_1, R_1) := \sup_{A \in \mathcal{X}} |P_1(A) - R_1(A)|.
\]

If \( P_1, R_1 \ll \mu \) then \( \delta(P_1, R_1) = \frac{1}{n} \int |P_1(x) - R_1(x)| \, d\mu(x) \). We also recall the Pinsker inequality [14], which reads

\[
\delta(P_1, R_1) \leq \sqrt{\frac{D(P_1||R_1)}{2 \log e}}.
\]

In consequence, if a universal density exists and the entropy rate is finite then the Cesàro mean density estimator is strongly consistent in the total variation.

**Theorem 7:** Suppose that measure \( R \) is universal in expectation with respect to a reference measure \( \mu \). Then for every measure \( P \in \mathbb{E}(\mu) \), we have

\[
\lim_{n \to \infty} \delta(P^{(\infty)}, \bar{R}^{(n)}) = \lim_{n \to \infty} \delta(P^{(n)}, \bar{R}^{(n)}) = 0 \text{ a.s.}
\]

**Proof:** If \( R \) is universal in expectation then for every measure \( P \in \mathbb{E}(\mu) \), we have

\[
0 = \lim_{n \to \infty} \mathbb{E}[-\log R_\mu(X_{1:n})]/n - h_\mu
\]

\[
\geq \lim_{n \to \infty} \mathbb{E} D(P^{(\infty)}||\bar{R}^{(n)}) \geq 0.
\]

Now by the Pinsker inequality (66) and by the dominated convergence, we obtain

\[
0 = \lim_{n \to \infty} \mathbb{E}\left[\delta(P^{(\infty)}, \bar{R}^{(n)})\right]^2 = \mathbb{E}\left[\lim_{n \to \infty} \delta(P^{(\infty)}, \bar{R}^{(n)})\right]^2,
\]

which implies the almost sure convergence for \( \delta(P^{(\infty)}, \bar{R}^{(n)}) \).

Since

\[
h_\mu = \lim_{n \to \infty} \mathbb{E}[-\log P^{(n)}_\mu(X_0)]
\]

for a stationary \( P \) then we also have

\[
\lim_{n \to \infty} \mathbb{E} D(P^{(n)}||\bar{R}^{(n)}) = \lim_{n \to \infty} \mathbb{E} D(P^{(\infty)}||\bar{R}^{(n)}) = 0.
\]

Thus, analogously we derive the almost sure convergence for \( \delta(P^{(n)}, \bar{R}^{(n)}) \).
B. Universal Prediction

Theorem 7 strengthens and generalizes the celebrated Ornstein theorem, originally stated for binary stationary ergodic processes [18]. Ornstein’s theorem plays an important role in the theory of universal prediction of binary processes with the 0 − 1 loss [16]. Analogously, we can apply Theorem 7 to develop a theory of universal prediction for processes over an arbitrary countable alphabet, strengthening the recent result of [15] by the way.

The exact development, recalling the basic facts from [15], as follows. For a countable alphabet \( \mathcal{X} \), we will consider densities with respect to the counting measure \( \gamma \). Is as follows. For a countable alphabet \( \mathcal{X} \), denoted without subscript \( \gamma \) according to our earlier convention. A predictor is an arbitrary function \( f : \mathcal{X}^* \rightarrow \mathcal{X} \). The predictor \( f_P \) induced by a probability measure \( P \) is defined as a maximizer of conditional probability,

\[
    f_P(x_{1:n-1}) = \arg \max_{x_n \in \mathcal{X}} P(x_n|x_{1:n-1}).
\]  

(72)

In the problem of universal prediction we seek for a measure-independent predictor that minimizes the relative frequency of prediction mistakes. That is, we apply the logarithmic loss encountered in the problem of universal coding.

By the Azuma-Hoeffding inequality [37], for any probability measure \( P \), the rate of mistakes is equal to the rate of their conditional probabilities, namely,

\[
    \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \{1 \{ X_i \neq f(X_{1:i-1}) \} \} - P(X_i \neq f(X_{1:i-1})|X_{1:i-1}) = 0 \text{ a.s.,}
\]  

(73)

see [15, Theorem 3.5] for the derivation. Therefore, as shown in [33] and [15, Theorem 3.5] using the Breiman ergodic theorem [30], [33, Theorem 12(c)], for any measure \( P \in \mathcal{E} \) and any predictor \( f \), we have

\[
    \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1 \{ X_i \neq f(X_{1:i-1}) \} \geq u \text{ a.s.,}
\]  

(74)

where we define the unpredictability rate

\[
    u := E \left[ 1 - \max_{x \in \mathcal{X}} P(x|X_{1:n-1}) \right].
\]  

(75)

By contrast, for any measure \( P \in \mathcal{E} \) and its induced predictor \( f_P \), we have

\[
    \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1 \{ X_i \neq f_P(X_{1:i-1}) \} = u \text{ a.s.}
\]  

(76)

Thus by an analogy to universal measures, we propose universal predictors.

Definition 7: A predictor \( f \) is called strongly universal with respect to a reference measure \( \mu \) if for any measure \( P \in \mathcal{E}(\mu) \),

\[
    \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1 \{ X_i \neq f(X_{1:i-1}) \} = u \text{ a.s.}
\]  

(77)

Definition 8: A predictor \( f \) is called strongly universal with respect to a reference measure \( \mu \) if for any measure \( P \in \mathcal{E}(\mu) \),

\[
    \lim_{n \to \infty} E \left[ \frac{1}{n} \sum_{i=1}^{n} 1 \{ X_i \neq f(X_{1:i-1}) \} \right] = u.
\]  

(78)

Definition 9: A predictor \( f \) is called strongly universal with respect to a reference measure \( \mu \) if for any measure \( P \in \mathcal{E}(\mu) \),

\[
    \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1 \{ X_i \neq f(X_{1:i-1}) \} = u \text{ i.p.}
\]  

(79)

Remark: By analogy, a predictor that satisfies (77), (78), and (79) is simply called universal with respect to \( \mu \). But we have implications (77) \( \Rightarrow \) (78) \( \Rightarrow \) (79) always. Implication (77) \( \Rightarrow \) (78) follows by the dominated convergence. Implication (78) \( \Rightarrow \) (79) follows by the Markov inequality and the almost sure lower bound (74). The exact proofs resemble proofs of Propositions 1 and 2. Thus each strongly universal predictor is universal.

In work [15, Theorems 3.12 and 3.18], it was shown that for a finite alphabet \( \mathcal{X} \) and a strongly universal measure \( R \), the induced predictor \( f_R \) is universal if we have a uniform bound for conditional probabilities of form

\[
    - \log R(x_{n+1}|x_n^i) \leq \epsilon_n \sqrt{n/\log n}, \quad \lim_{n \to \infty} \epsilon_n = 0.
\]  

(80)

Being uniform in symbols, condition (80) can be satisfied only if the alphabet is finite. Subsequently, we will show that condition (80) can be dropped if measure \( R \) is universal in expectation and if we consider predictor \( f_R \), induced by the Cesàro mean measure \( \bar{R} \), rather than predictor \( f_R \), induced by the original measure \( R \). Hence we have a universal predictor also for a countably infinite alphabet.

Theorem 8: Consider a countable alphabet \( \mathcal{X} \). Suppose that measure \( R \) is universal in expectation with respect to a reference measure \( \mu \). The Cesàro mean predictor \( \bar{f}_R \) is universal with respect to measure \( \mu \).

Proof: Consider a measure \( P \in \mathcal{E}(\mu) \). By Theorem 7, we have a generalization of the Ornstein theorem [18], namely,

\[
    \lim_{n \to \infty} 2\epsilon(P^{(n)}, \bar{R}^{(n)}) = \lim_{n \to \infty} \sum_{x \in \mathcal{X}} |P(x|X_{1:n-1}) - \bar{R}(x|X_{1:n-1})| = 0 \text{ a.s.}
\]  

(81)

since \( P^{(n)}, \bar{R}^{(n)} \ll \gamma \). As it follows from a simple application of the Breiman ergodic theorem [30], [33, Theorem 12(c)], statement (81) implies a generalization of the Bailey theorem [17], namely,

\[
    \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{x \in \mathcal{X}} |P(x|X_{1:i-1}) - \bar{R}(x|X_{1:i-1})| = 0 \text{ a.s.}
\]  

(82)

Since we have a so called prediction inequality

\[
    P(X_i = f_P(X_{1:i-1})|X_{1:i-1}) - P(X_i = f_R(X_{1:i-1})|X_{1:i-1}) \leq \sum_{x \in \mathcal{X}} |P(x|X_{1:i-1}) - \bar{R}(x|X_{1:i-1})|
\]  

(83)
noticed in work [15, Proposition 3.11] then by corollary (73) of the Azuma-Hoeffding inequality, we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \{X_i \neq f_R(X_{1:i-1})\} = 0 \quad \text{a.s.} \tag{84}
\]
That is, the Cesàro mean predictor \( f_R \) is universal. \( \square \)

The Cesàro mean predictor \( f_R \) for \( R = \text{NPD} \) that applies the PPM measures is quite complex. It contains five nested maximizations, summations, and products that may contribute to a pessimistic time complexity \( O(n^5) \), where \( n \) is the length of the sample. In the future research, it would be advisable to seek for a universal predictor with a smaller time complexity.

V. EXAMPLES

As some supplementary examples, in this section, we scale down the NPD measure to two cases where consistent estimation of the entropy rate is not possible in general. These are the countably infinite alphabet and the real line. The general infeasibility of consistent estimation in these cases becomes intuitive by virtue of additional assumptions that we are bound to make. Once these conditions are met, the corrected NPD estimator is strongly consistent.

A. Countably Infinite Alphabet

Subsequently, let us consider a countably infinite alphabet \( \mathbb{X} \). We recall that we have earlier adopted notation \( P_1: \mathbb{X} \ni x \mapsto P(x) \) for the marginal density of measure \( P \) on \( (\mathbb{X}^\mathbb{Z}, \mathcal{X}^\mathbb{Z}) \) with respect to the counting measure \( \gamma(A) = \text{card } A \) on \( (\mathbb{X}, \mathcal{A}) \). Thus for probability measures \( P \) and \( R \), we denote the Shannon entropy and the Kullback-Leibler divergence taken with respect to their marginal densities:
\[
\begin{align*}
H(P_1) &:= -\sum_{x \in \mathbb{X}} P(x) \log P(x), \\
D(P_1||R_1) &:= \sum_{x \in \mathbb{X}} P(x) \log \frac{P(x)}{R(x)}.
\end{align*}
\]
It may appear a bit surprising that we have \( P_1 \) on the left hand side and \( P \) on the right hand side but we prefer not to multiply the notational conventions, which are sufficiently overloaded to deal with quantization levels.

We recall that there is no consistent conditional density estimator for a countably infinite alphabet, in general.

**Theorem 9 ([12, Theorem 2]):** Let the alphabet \( \mathbb{X} \) be countably infinite. Let \( (R^{(n)})_{n \geq 1} \) be an arbitrary conditional density estimator. Then there is a memoryless source \( P \) such that \( H(P_1) < \infty \) and \( D(P_1||R^{(n)}) = \infty \) a.s. for all \( n \).

In view of Theorems 6 and 9, for a countably infinite alphabet \( \mathbb{X} \), for each measure \( R \) there exists a memoryless source \( P \) with \( H(P_1) < \infty \) such that
\[
\lim_{n \to \infty} \mathbb{E}[-\log R(X_{1:n})]/n = \infty. \tag{87}
\]
Since each memoryless source \( P \) is stationary ergodic with entropy rate \( H(P_1) \) we obtain the known result.

**Theorem 10 ([19], [12, Theorem 3]):** There is no measure universal in expectation with respect to a countably infinite alphabet.

For a countably infinite alphabet \( \mathbb{X} \), consider now a reference measure \( \mu \) to be contrasted with the counting measure \( \gamma \). We suppose that \( \mu \ll \gamma \) and \( \mu \gg \gamma \). Entropies of \( P \) with respect to \( \mu \) are written as \( h_\mu(n) \) or \( h_\mu \), whereas entropies of \( P \) with respect to \( \gamma \) are written as \( h(n) \) or \( h \). We have
\[
P(x_{1:n}) = P_\mu(x_{1:n}) \frac{d\mu}{d\gamma}(x_{1:n}) = P_\mu(x_{1:n}) \prod_{i=1}^{n} \mu(x_i). \tag{88}
\]

Let us write the marginal cross entropy
\[
H(P_1|R_1) := H(P_1) + D(P_1||R_1) = -\sum_{x \in \mathbb{X}} P(x) \log R(x). \tag{89}
\]
If \( H(P_1||\mu) < \infty \) then from equation (88), applying stationarity, we obtain
\[
h(n) = h_\mu(n) - n \sum_{x \in \mathbb{X}} P(x) \log \mu(x) = h_\mu(n) + nH(P_1||\mu). \tag{90}
\]
Hence \( h = h_\mu + H(P_1||\mu) \geq 0 \). If \( \mu \) is a probability measure then \( h_\mu \leq 0 \). Consequently, we have a sufficient condition
\[
P \in \mathbb{E}, \mu(x) > 0 \text{ for all } x \in \mathbb{X}, H(P_1||\mu) < \infty \quad \implies \quad P \in \mathbb{E}(\mu). \tag{91}
\]
In particular, we may estimate the entropy rate in the following way, compare it with more complicated characterizations in [20]:

**Theorem 11:** Consider a countably infinite alphabet \( \mathbb{X} \) and a probability measure \( \mu \) such that \( \mu(x) > 0 \) for all \( x \in \mathbb{X} \). Let \( P \in \mathbb{E} \) with \( H(P_1||\mu) < \infty \). Then
\[
\lim_{n \to \infty} \frac{1}{n} \left[ -\log \text{NPD}_\mu(X_{1:n}) - \sum_{i=1}^{n} \log \mu(X_i) \right] = h \quad \text{a.s.} \tag{92}
\]

**Remark:** In particular, estimator (92) is strongly consistent for any source over a finite but unknown alphabet.

**Proof:** Quantity \( -n^{-1} \log \text{NPD}_\mu(X_{1:n}) \) is a strongly consistent estimator of \( h_\mu \), whereas \( -n^{-1} \sum_{i=1}^{n} \log \mu(X_i) \) is a strongly consistent estimator of \( H(P_1||\mu) \) by the Birkhoff ergodic theorem. Since \( H(P_1||\mu) < \infty \) then the sum of these two estimators converges to \( h \). \( \square \)

Let us consider a particular quantization of the set of natural numbers.

**Example 2 (Incremental Partitions):** Let the alphabet be the set of natural numbers, \( \mathbb{X} = \mathbb{N} \). Suppose that \( \mu(m) > 0 \) for all \( m \in \mathbb{N} \). Let \( P \in \mathbb{E} \) with cross entropy \( H(P_1||\mu) < \infty \). Let the universal measures in (36) be \( R^f = \text{PPM}^{\text{card } \mathbb{X}} \).

In this case, estimator (92) differs from the PPM measure since \( \mu(X_i) \leq \mu(X_i) \). To observe simplifications of estimator (92), let us take partitions
\[
\chi_l := \{ \{1\}, \{2\}, \ldots, \{l\}, \{l+1, l+2, \ldots\} \}, \quad l \geq 0. \tag{93}
\]
Let us denote the optimal quantization level \( Q_n \) and the optimal Markov order \( R_n \) for sample \( X_{1:n} \) as some elements
\[
(Q_n, R_n) \in \arg\max_{q, r \geq 0} \frac{\text{PPM}^{q+1}(X_{1:n})}{\mu(q)(X_{1:n})}. \tag{94}
\]
Since we have $\text{PPM}^r_{R_n}(x_{1:n}) < \text{PPM}^q(x_{1:n})$ in general and $\text{PPM}^q(x_{1:n}) = q^{-n}$ for $r \geq n - 1$, we may fix pair $(Q_n, R_n)$ so that

$$Q_n \leq \max X_{1:n}, \quad R_n \leq n - 1. \quad (95)$$

Using the well known idea for the PPM measure, we may also bound

$$0 \leq -\log \text{NPD}_n(X_{1:n}) + \log \frac{\text{PPM}^{Q_{n+1}}_{R_n}(X^Q_{1:n})}{\mu^n(X^Q_{1:n})} \leq -\log w_n - \log w_n. \quad (96)$$

Thus we may specialize Theorem 11 as the following proposition.

**Proposition 3:** Consider the setting of Example 2. Suppose that

$$\lim_{n \to \infty} \frac{\log \max X_{1:n}}{n} = 0 \text{ a.s.} \quad (97)$$

Then we have

$$\lim_{n \to \infty} \frac{1}{n} \left[ -\log \text{PPM}^{Q_{n+1}}_{R_n}(X^Q_{1:n}) + C_n(Q_n) \right] = h \text{ a.s.}, \quad (98)$$

where we define

$$C_n(q) := -\sum_{i=1}^n \log \frac{\mu(X_i)}{\mu(X'_i)} = -\sum_{i=1}^n \mathbb{1}\{X_i > q\} \log \mu(X_i|X_i > q) \geq 0. \quad (99)$$

**Proof:** Convergence (98) follows from convergence (92) and the sandwich bound (96). \(\square\)

Term $C_n(Q_n)$ is a nonnegative correction of the finite-alphabet entropy estimator taken for the optimal quantization level and the optimal Markov order. In the following, we will show that this correction vanishes ultimately almost surely if the process is upper bounded. To shed some light on this issue, let us introduce the minimal sufficient quantization level of a probability measure $P \in \mathcal{E}(\mu)$ relative to measure $\mu$, which is defined as

$$Q := \inf \left\{ q \geq 0 : \frac{P(X|X_{-\infty:-1})}{\mu(X_Q)} = \frac{P(X|X_{-\infty:-1})}{\mu(X_0)} \text{ a.s.} \right\}. \quad (100)$$

In particular, the minimal sufficient quantization level is $Q = 0$ for the memoryless source $P = \mu^x$, which seems somewhat counterintuitive. What is more intuitive, we have $Q \leq M$ if $X_i \leq M$ since $X_i^M = \{X_i\}$ holds almost surely.

As an auxiliary result, we will show that for any stationary ergodic measure $P$ with entropy $h_\mu < \infty$, statistic $Q_n$ does not underestimate parameter $Q$.

**Proposition 4:** Consider the setting of Example 2. We have

$$\lim_{n \to \infty} Q_n \geq Q \text{ a.s.} \quad (101)$$

**Proof:** We apply the proof idea of [11, Theorem 6.12] for the inconsistent estimator of the Markov order given by the PPM measure, the inconsistency result due to Csiszar and Shields [38]. By contradiction, let us assume that

$$\liminf_{n \to \infty} Q_n = q \text{ holds with a positive probability for some } q < Q. \text{ Then on the respective random points, we obtain}$$

$$h_\mu = \lim_{n \to \infty} \frac{1}{n} \left[ -\log \text{NPD}_n(X_{1:n}) \right] \text{ by universality of NPD} \quad$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[ -\log \frac{\text{PPM}^{Q_{n+1}}_{R_n}(X^Q_{1:n})}{\mu^n(X^Q_{1:n})} \right] \text{ (by (96))} \quad$$

$$\geq \lim_{n \to \infty} \frac{1}{n} \left[ -\log \frac{\text{PPM}^{Q_{n+1}}_{R_n}(X^q_{1:n})}{\mu^n(X^q_{1:n})} \right] \text{ (since } Q_n = q \text{ infinitely often)} \quad$$

$$\geq \lim_{n \to \infty} \frac{1}{n} \left[ -\log \frac{P(X^q | X_{-\infty:-1})}{\mu(X^q_0)} \right] \text{ (by the Barron inequality)} \quad$$

$$= \mathbb{E} \left[ -\log \frac{P(X^q | X_{-\infty:-1})}{\mu(X^q_0)} \right] \quad$$

$$\geq \mathbb{E} \left[ -\log \frac{P(X^q | X_{-\infty:-1})}{\mu(X_0)} \right] \quad$$

$$(by q < Q \text{ and the definition of } Q) \quad = h - H(P_1 || \mu). \quad (102)$$

Inequality $h_\mu > h - H(P_1 || \mu)$ cannot be true so our assumption is false. Thus $\liminf_{n \to \infty} Q_n \geq Q$ almost surely. \(\square\)

Now we prove the desired claim about vanishing of $C_n(Q_n)$.

**Proposition 5:** Consider the setting of Example 2 with $X_i \leq M < \infty$. Then

$$\lim_{n \to \infty} C_n(Q_n) = 0 \text{ a.s.} \quad (103)$$

**Proof:** Without loss of generality, suppose that $M := \text{ess sup} X_i < \infty$ almost surely. We have $Q \leq M$. Observe that

$$Q = \inf \{ q \geq 0 : \forall m > q P(X_0 = m|X_{-\infty:-1}) = K \mu(m) \text{ a.s.} \}, \quad (104)$$

where $K \geq 0$ is a certain random variable. Since $\mu(m) > 0$ and $P(X_0 = m|X_{-\infty:-1}) = 0$ for $m > M$, hence $K = 0$ almost surely and $P(X_0 = m|X_{-\infty:-1}) = 0$ for all $m > Q$. In other words, $Q \geq \text{ess sup} X_i = M \geq \max X_{1:n} \geq Q_n$. Consequently, $\lim_{n \to \infty} Q_n = M$ holds almost surely in view of Proposition 4. Since also $X_i^M = \{X_i\}$ almost surely, we have $C_n(M) = 0$ and $\lim_{n \to \infty} C_n(Q_n) = 0$ almost surely in this case. \(\square\)

**B. Real Line and Gaussian Processes**

Consider alphabet $X = \mathbb{R}$ and the reference measure $\mu$ being the normal distribution $N(m, \sigma^2)$ to be contrasted with the Lebesgue measure $\lambda$. Entropies of $P$ with respect to $\mu$ are written as $h_\mu(n)$ or $h_\mu$, whereas entropies of $P$ with respect to $\lambda$ are written as $h_\lambda(n)$ or $h_\lambda$. We have

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x-m)^2}{2\sigma^2} \right) \quad (105)$$

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Let us write the rescaled second moment of the marginal distribution
\[ M_2 := \int_{-\infty}^{\infty} \frac{(x - m)^2}{2\sigma^2} P_\lambda(x) dx = \frac{\text{Var} X_i + (E X_i - m)^2}{2\sigma^2}, \]  
\[ (106) \]
denoting variance \( \text{Var} X_i := E(X_i - E X_i)^2 \).

If \( M_2 < \infty \), we obtain like in Section V-A that
\[ h_\lambda(n) = h_\mu(n) - n \int_{-\infty}^{\infty} P_\lambda(x) \log \frac{d\mu}{d\lambda}(x) dx. \]
\[ = h_\mu(n) + n \left[ M_2 \log e + \log \sigma \sqrt{2\pi} \right]. \]  
\[ (107) \]
Hence \( h_\lambda = h_\mu + M_2 \log e + \log \sigma \sqrt{2\pi} \). Since \( \mu \) is a probability measure then \( h_\mu \leq 0 \). Consequently, we have a sufficient condition
\[ P \in \mathbb{E}, \ P_n \ll \lambda^0, \ |E X_i| < \infty, \ \text{Var} X_i < \infty, \ |h_\lambda| < \infty \]
\[ \implies P \in \mathbb{E}(\mu). \]  
\[ (108) \]

In this case, we may estimate the entropy rate in the following way.

**Theorem 12:** Consider \( \mathcal{X} = \mathbb{R} \). Let \( \mu \) be the normal distribution \( N(m, \sigma^2) \), whereas \( \lambda \) be the Lebesgue measure. Suppose that \( P \in \mathbb{E} \) with \( P_n \ll \lambda^0, \ |E X_i| < \infty, \ \text{Var} X_i < \infty, \ |h_\lambda| < \infty \). Then
\[ \lim_{n \to \infty} \frac{1}{n} \left[ -\log \text{NPD}_\mu(X_{1:n}) + \sum_{i=1}^{n} \frac{(X_i - m)^2}{2\sigma^2} \right] \log e \]
\[ + \log \sigma \sqrt{2\pi} = h_\lambda \text{ a.s.} \]  
\[ (109) \]

**Proof:** Quantity \( -n^{-1} \log \text{NPD}_\mu(X_{1:n}) \) is a strongly consistent estimator of the entropy rate \( h_\mu \), whereas \( -n^{-1} \sum_{i=1}^{n} (X_i - m)^2/2\sigma^2 \) is a strongly consistent estimator of the rescaled second moment \( M_2 \). Since \( M_2 < \infty \) then the linear combination of these estimators tends to \( h_\lambda \). \( \square \)

Thus, the corrected NPD estimator (109) is a strongly consistent estimator of the entropy rate for all nondeterministic stationary ergodic Gaussian processes since \( |E X_i| < \infty, \ \text{Var} X_i < \infty, \ |h_\lambda| < \infty \) holds in this case. As we have mentioned in Section I, Feutrill and Roughan [21] supposed that the NPD estimator can estimate the entropy rate of a Gaussian process but they did not carry out this idea rigorously enough.

Consider the dyadic filtration (43) from Example 1. In this case, computing quantizations \( X_i \) requires computing quantiles of the normal distribution. An interesting question is whether such a filtration is optimal. Moreover, we may suppose that the corrected NPD estimator (109) improves if we take parameters \( m \) and \( \sigma \) close to the expectation and variance of \( X_i \) with respect to \( P \). Obviously, we can specialize Theorem 12 as the following proposition.

**Proposition 6:** Consider \( \mathcal{X} = \mathbb{R} \). Let \( \mu \) be the normal distribution \( N(m, \sigma^2) \), whereas \( \lambda \) be the Lebesgue measure. Suppose that \( P \in \mathbb{E} \) with \( P_n \ll \lambda^0, \ E X_i = m, \ \text{Var} X_i = \sigma^2, \) and \( |h_\lambda| < \infty \). Then
\[ \lim_{n \to \infty} \frac{1}{n} \left[ -\log \text{NPD}_\mu(X_{1:n}) + \log \sigma \sqrt{2\pi e} \right] = h_\lambda \text{ a.s.} \]  
\[ (110) \]

The problem with the above estimator is that we need to know the exact expectation and the variance of \( X_i \). Can we estimate them from sample \( X_{1:n} \) and plug the result into the NPD estimator? Consider random measures \( \mu_n \sim N(m_n, \sigma^2_n) \), where
\[ m_n := \frac{1}{n} \sum_{i=1}^{n} X_i, \ \sigma^2_n := \frac{1}{n} \sum_{i=1}^{n} (X_i - m_n)^2. \]  
\[ (111) \]
We may ask whether there holds still convergence
\[ \lim_{n \to \infty} \frac{1}{n} \left[ -\log \text{NPD}_{\mu_n}(X_{1:n}) \right] + \log \sigma \sqrt{2\pi e} = h_\lambda \text{ a.s.} \]  
\[ (112) \]
However, this question is not stated precisely enough. Let us note that measure NPD depends implicitly on the quantization path \( X_i \rightarrow X \) and measure \( \mu \). When we variate the reference measure \( \mu \rightarrow \mu_n \), it is not clear whether we have also to vary \( \sigma \)-fields \( X_i \rightarrow X_i \). In any case, the proof technique of Theorem 5 works no longer and we have no simple guarantee of consistency.

**VI. CONCLUSION**

Drawing an inspiration from the nonparametric differential (NPD) entropy rate estimator by Feutrill and Roughan [21], we have constructed a universal NPD density that works for stationary ergodic sources on any countably generated measurable space—as long as we agree to define the entropy relative to a finite reference measure. Our theoretical development of the NPD estimator is analogous to Ryabko’s [2], [3] development of the PPM measure by Cleary and Witten [1]. Whereas Ryabko considered a countably infinite mixture of source estimates of distinct Markov orders, we have considered an analogous mixture of source estimates of distinct quantization levels.

As we have shown, the NPD density solves the problem of consistent entropy rate estimation. Moreover, it can be used to obtain a strongly consistent Cesáro mean estimator of the conditional density given an infinite past, in the total variation, using the idea of [12]. This in turn solves the problem of universal prediction with the \( 0 - 1 \) loss for a countable alphabet, cf. [15]. The NPD density can also shed light on sufficient conditions for consistent estimation of the entropy rate with respect to infinite reference measures, cf. [20].

There is one hanging gun that has not shot in this play, however. Among the applications of universal coding, in Section I, we have mentioned estimation of the (hidden) Markov order, see [8] and [9] for classical references. Analogously, we may ask whether a similar approach can be developed for estimation of the minimal sufficient quantization level for real-valued stochastic processes. The general method of Markov order estimation requires comparing universal measures or codes with the maximum likelihood. However, we are not sure whether this method can be translated to quantization level estimation and whether the concept of the minimal sufficient quantization level of an arbitrary process can be reasonably defined, see definition (100) which is somewhat counterintuitive. Solving this issue is deferred to another work. We recall that we have also stated some related open questions in Section V-B that concern the optimal quantization of Gaussian processes.
Another important open topic is the computational complexity of universal densities. The infinite series (37) can be truncated or approximated in some cases, as we have shown in Examples 1 and 2. However, we have not cared about the time complexity or the speed of convergence of universal densities, being satisfied by their general computability or consistency. In the future research, one should develop explicit bounds for time and memory complexity of the total NPD density and to find other universal densities that can be computed faster. Yet another idea for future research is to investigate the rate of convergence of the NPD entropy rate estimates and of the Cesàro mean estimator of conditional density given an infinite past. We think that both mathematical statistics and information theory can benefit from such analyses.

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