Novel Effect Induced by Spacetime Curvature in Quantum Hydrodynamics

T. Koide
Instituto de Física, Universidade Federal do Rio de Janeiro,
C.P. 68528, 21941-972, Rio de Janeiro, RJ, Brazil

T. Kodama
Instituto de Física, Universidade Federal do Rio de Janeiro,
C.P. 68528, 21941-972, Rio de Janeiro, RJ, Brazil and
Instituto de Física, Universidade Federal Fluminense, 24210-346, Niterói, RJ, Brazil

The interplay between quantum fluctuation and spacetime curvature is shown to induce an additional term in the energy-momentum tensor of fluid using the generalized framework of the stochastic variational method (SVM). To illustrate the effect of this quantum-curvature (QC) term, the Friedmann-Robertson-Walker (FRW) metric is applied. We show that this effect contributes to the cosmological acceleration although it is too small in the present non-relativistic toy model. Our result suggests the possibility that the vector space represented by the wave function may not exist for quantum mechanics in the curved spacetime.

I. INTRODUCTION

There is no established formulation of quantum mechanics in the curved spacetime. Normally the formulation in the Euclidean spacetime is assumed to be held without a major modification. For example, the existence of the Hilbert space is required from the beginning. This is fairly natural but not trivial a priori.

The stochastic variational method (SVM) is one of quantization schemes [1–3]. Quantization is then formulated as the stochastic optimization of classical actions: the Schrödinger equation is derived by applying the stochastic variation to the action which leads to the Newton equation under the standard classical variation. As a remarkable feature, the existence of the Hilbert space is not necessarily required. Instead, the optimized dynamics is represented in the Madelung-Bohm-type hydrodynamic form (quantum hydrodynamics) [4], and it can be cast into the Schrödinger equation only when the fluid velocity is expressed by the gradient of a velocity potential [36]. If such a hydrodynamics in the curved spacetime has an additional term, the velocity potential is not introduced in general and then the quantized dynamics is not expressed with the wave function (See below Eq. (21)). Thus it is interesting to study the existence of the vector space formed by the wave function in the curved spacetime in the framework of SVM.

The purpose of this paper is, assuming that the variational principle is a fundamental requirement in the quantization, to develop the generalized SVM applicable to curved spacetime systems. For the sake of simplicity, we consider non-relativistic systems of particle and continuum medium [5].

Then we show that the interplay between quantum fluctuation and spacetime curvature induces the quantum-curvature (QC) term which prevents us from introducing the wave function but is necessary to find a conserved energy-momentum tensor. To investigate the effect, we apply the Friedmann-Robertson-Walker (FRW) metric and discuss that the effect of the QC term contributes to the cosmological acceleration.

In this paper, h, c and G denote the Planck constant, the speed of light and the gravitational constant, respectively. The stochastic quantity is denoted by ( ˆ ). The difference dA(t) = A(t + dt) − A(t), independently of the sign of dt. The Einstein notation of the summation is used.

II. FORMULATION

For a curved spacetime characterized by the metric gµν, it is possible to find a local Minkowskian system with the metric ηab (= diag(−1, 1, 1, 1)). The spacetime position in the former coordinate is denoted by xμ while that in the latter by yσ. The Greek indices α, β, · · · are used to label the general coordinate. The Latin indices a, b, · · · are for the local Minkowskian coordinate but i, j, k are reserved to denote the spatial components of xμ. Then the tetrads [6] are eiμ(X) = ∂xμ/∂yσ|X=x and eνσ(X) = ∂yσ/∂xμ|X=x, satisfying gμνeiνeλχ = ηab and ηabeiμeνσ = gμν. Note that g = det(gμν).

In the SVM quantization, quantum effects are introduced through stochastic motions. We consider a non-relativistic Brownian motion in the curved spacetime where the time component is given by dx0/dt = cdt, and a limited spacetime geometry satisfying

\[ g_{00} = 0, \quad \partial_t g_{00} = 0. \]  (1)

The stochastic differential equation (SDE) of such a Brownian motion is already known for the curved space [4, 5]. The same equation is applied to the curved spacetime using the tetrad. Then the forward SDE with dt > 0

\[ dA = A(t + dt) − A(t) = \sqrt{g} \frac{dW}{\sqrt{dt}} \]  (2)

where the time component is given by dx0/dt = cdt, and a limited spacetime geometry satisfying

\[ g_{00} = 0, \quad \partial_t g_{00} = 0. \]  (1)
is given by
\[ d\hat{x}^i = u^i_+(\hat{x}_t)dt + \frac{\hbar}{M} \mathcal{L}^i_+(\hat{x}_t) dW^a_t, \tag{2} \]
where \( \hat{x}_t \) represents \((x^0_t, \hat{x}_t) \) and the Stratonovich definition of the product is given by
\[ f(\hat{x}_t) \circ_s d\hat{W}_t^a = f(\hat{x}_{t+dt/2}) d\hat{W}_t^a, \tag{3} \]
for an arbitrary smooth function \( f(x) \). It should be noted that Leibniz’s rule of differential for stochastic quantities is formally held when the Stratonovich definition is applied \[8\]. The standard Wiener process \( W_t^a \) has only spatial components \((W_t^0 = 0) \) which satisfy
\[ E[d\hat{W}_t^a] = 0, \]
\[ E[(d\hat{W}_t^a)(d\hat{W}_t^b)] = |dt| \delta^{ab} \delta_{\Delta t} \] (\( a, b \neq 0 \)),
where \( E[ ] \) represents the stochastic ensemble average. The purpose of SVM is to find the unknown smooth function \( u^i_+(x) \) by the optimization.

The change of the tetrad is determined by the Levi-Civita-Ito stochastic parallel transport \[7\],
\[ de_{\alpha}^\mu(\hat{x}_t) = -\Gamma_{\alpha \beta}^\mu(\hat{x}_t) \delta^\beta_\gamma(\hat{x}_t) \circ_s d\hat{x}_t^\gamma, \tag{6} \]
where \( \Gamma_{\alpha \beta}^\mu(x) \) is the Christoffel symbol. The length of the transported vector is conserved in this definition. Note however that a different stochastic transport is considered in Nelson’s stochastic mechanics of the curved space \[2\] where, thus, the length is not conserved.

In the formulation of the variational principle, we should fix not only an initial condition but also a final condition. This implies that the forward SDE alone is not sufficient \[2\]. We further introduce the backward SDE for \( dt < 0 \) as
\[ d\hat{x}^i = u^i_- (\hat{x}_t)dt + \frac{\hbar}{M} \mathcal{L}^i_- (\hat{x}_t) \circ_s d\hat{W}_t^a, \tag{7} \]
where \( u^i_-(x) \) is another unknown function and \( \hat{W}_t^a \) is another Wiener process which satisfies the same correlations as \( W_t^a \). The backward SDE should correspond to the time-reversed process of the forward SDE. Thus there exists a condition associating \( u^i_-(x) \) with \( u^i_+(x) \). To find it, we derive the two Fokker–Planck equations for the probability density defined by \[37\],
\[ \rho(x) = \int \frac{cdt}{\sqrt{\pi}} E[\delta^{(4)}(x^\mu - \hat{x}_t^\mu)], \tag{8} \]
from the forward and backward SDEs, independently. Since these equations are equivalent and the condition \[11\] is considered, we obtain the consistency condition,
\[ u^i_+(x) = u^i_-(x) + \frac{\hbar}{M} g^{ij} \partial_j \ln \rho(x). \tag{9} \]
See also Refs. \[2\]. Using this, the two Fokker–Planck equations are reduced to the equation of continuity \[2\],
\[ \nabla_\mu (\rho(x) u^\mu(x)) = 0, \tag{10} \]
where \( \nabla_\mu \) represents the covariant derivative and
\[ u^\mu(x) = (u^0, u^i(x)) = \left( c, \frac{u^i_+(x) + u^i_-(x)}{2} \right). \tag{11} \]

The stochastic trajectories are zigzag and thus the standard definition of the particle velocity is not applicable. The possible time differential is studied by Nelson \[12\], finding two quantities: the mean forward derivative,
\[ D_t f(\hat{x}_t) = \lim_{dt \to 0^+} E \left[ \frac{f(\hat{x}_{t+dt}) - f(\hat{x}_t)}{dt} \right] | \mathcal{P}_t \], \tag{12} \]
and the mean backward derivative,
\[ D^- f(\hat{x}_t) = \lim_{dt \to 0^-} E \left[ \frac{f(\hat{x}_{t+dt}) - f(\hat{x}_t)}{dt} \right] | \mathcal{F}_t. \tag{13} \]
These expectation values are conditional averages, where \( \mathcal{P}_t (\mathcal{F}_t) \) indicates to fix values of \( \hat{x}_t \) for \( t' < t \) \(( t' > t \)). For the \( \sigma \)-algebra of all measurable events of \( \hat{x}_t, \mathcal{P}_t \) and \( \mathcal{F}_t \) represent an increasing and a decreasing family of sub-\( \sigma \)-algebras, respectively. These derivatives are connected through the stochastic partial integration \[2\],
\[ \int_a^b ds E[\dot{Y}_s D_+ \dot{X}_s] = - \int_a^b ds E[\dot{X}_s D_- Y_s] \]
\[ + \int_a^b ds \frac{d}{ds} E[\dot{X}_s Y_s]. \tag{14} \]

As another new property which appears in the formulation for the curved spacetime, \( dy^\mu = dx^\mu \tau^\mu_\nu(x) \) is generalized to the stochastic trajectories. Although this is not unique, we adapt the generalization using the Stratonovich definition,
\[ dy^\mu(\hat{x}_t) = d\hat{x}^\mu \circ_s \tau^\mu_\nu(\hat{x}_t), \tag{15} \]
leading to
\[ \mathcal{L}^\mu_-(\hat{x}_t) D_\pm y^\nu(\hat{x}_t) = u^\nu_\pm(\hat{x}_t). \tag{16} \]
Similarly, for an arbitrary smooth vector function \( A^\mu(x) \), we find
\[ \mathcal{L}^\mu_-(\hat{x}_t) D_\pm (A^\mu(\hat{x}_t) \tau^\mu_\nu(\hat{x}_t)) = \mathcal{L}^\mu_-(\hat{x}_t) E \left[ \frac{dA^\mu(\hat{x}_t)}{dt} \circ_s \tau^\mu_\nu(\hat{x}_t) + A^\mu(\hat{x}_t) \circ_s \frac{d\tau^\mu_\nu(\hat{x}_t)}{dt} \right] | \mathcal{P}_t(\mathcal{F}_t) \]
\[ = \left( c \nabla_\mu + u^i_- (\hat{x}_t) \nabla_j \pm \frac{\hbar}{2M} g^{jk} (\hat{x}_t) \nabla_j \nabla_k \right) A^\mu(\hat{x}_t). \tag{17} \]
Here we used \( \frac{d}{dt} \circ_s \tau^\mu_\nu + \tau^\mu_\nu \circ_s \frac{d}{dt} \tau^\mu_\nu = 0 \).

Let us apply these definitions to the variation of the stochastic action,
\[ I = \int_{t_i}^{t_f} dt E[L], \tag{18} \]
where the stochastic Lagrangian of a single-particle system of the mass $M$ is

$$L = \frac{M}{4} \sum_{z=\pm} (D_z \dot{y}^a) \eta_{ab} (D_z \dot{y}^b) - V(\dot{x}_t),$$  \hspace{1cm} (19)$$

with $\dot{y}^a = y^a(\dot{x}_t)$ and $V(x)$ being a potential energy. In the stochastic systems, the tetrad is considered to be more fundamental than the metric and the Lagrangian is expressed by the tetrad. The kinetic term is replaced by the average of the contributions from the mean forward and backward derivatives. See [3, 15, 16] for other choices. This reduces to the standard classical Lagrangian in the vanishing limit of $\hbar$.

The variation of the trajectory is defined by

$$\dot{x}_t' \rightarrow \dot{x}_t + \delta f^i(\dot{x}_t, t),$$  \hspace{1cm} (20)$$

where the infinitesimal smooth function satisfies $\delta f^i(\dot{x}_t, t_i) = \delta f^i(\dot{x}_t, t_f) = 0$. We find the optimized solution for any choice of $\delta f^i(\dot{x}, t)$ and also for any stochastic distribution of $\dot{x}_t$. Then $v^i(x)$ is given by the solution of the following equation,

$$(v^0 \nabla_0 + v^j \nabla_j) v^i + \frac{g^{ij}}{M} \partial_j V = \frac{g^{ij} R^k}{2M^2} \left( \partial_j \left( \frac{1}{\sqrt{\rho}} \Delta_{LB} \sqrt{\rho} - \frac{1}{2} R^h_k \partial_h \ln \rho \right) \right).$$  \hspace{1cm} (21)$$

Note that $\Delta_{LB} = g^{ij} \nabla_i \nabla_j$ is the Laplace-Bertrami operator but the sum runs only for $i, j = 1, 2, 3$ because of the correlation defined by Eq. [3]. The first term on the right hand side is known as the quantum potential which exists even in the Euclidean space and produces various non-trivial quantum behaviors [4, 13]. The second term with the Ricci tensor $R_{ij} = g^{kl} R_{kl} = g^{kl} R^\mu_{j\mu}$ [14] is the new term induced by the interplay between quantum fluctuation and spacetime curvature. This is the term which we have called quantum-curvature (QC) term.

To cast Eq. (21) into the form of the Schrödinger equation, $v^i$ should be expressed in terms of the velocity potential which becomes the phase of the wave function. See also Eq. (30) in Ref. [3]. However, because of the possible $x$ dependence in the Ricci tensor, Eq. (21) is not represented by the velocity potential and hence the stochastic optimized dynamics is not necessarily expressed with the wave function.

The above formulation for particle systems can be extended to continuum media. The behavior of the non-relativistic simple fluid is described by the mass density and the velocity field. The corresponding stochastic Lagrangian density is obtained from the classical Lagrangian density for the ideal fluid. In the Lagrangian coordinates, it is expressed as [15, 17]

$$\mathcal{L} = \rho M_0(\xi) \times \left[ \frac{1}{4} \sum_{z=\pm} (D_z \dot{y}^a) \eta_{ab} (D_z \dot{y}^b) - \frac{\varepsilon_m(\rho M_0(\xi) / J(\partial_\xi))}{\rho M_0(\xi) / J(\partial_\xi)} \right].$$  \hspace{1cm} (22)$$

where $\rho M_0(\xi)$ is the initial mass distribution, $\varepsilon_m$ is the internal energy density and $J(\partial_\xi) = \det[\partial y(\dot{x}_t) / \partial y(\xi)]$. Note that the particle trajectory in the previous discussion is replaced by that of the fluid element, $\dot{x}_t' \rightarrow \dot{x}_t(\xi)$ where $\xi$ denotes the initial position of the fluid element. The mass density of the simple fluid is given by $\rho M(x) = M(\rho(x))$ and then the consistency condition [14] is still maintained by replacing $\rho(x)$ with $\rho M(x)$. Note that the Gross-Pitaevskii equation is derived from this when the Euclidean SVM is applied [3, 10].

For the sake of later convenience, we express the optimized result with the fluid momentum density as

$$\nabla_0(\rho_M v^i v^0) = -\nabla_j T^{ij}_m,$$  \hspace{1cm} (23)$$

where the fluid stress tensor is defined by

$$T^{ij}_m = \rho_M v^i v^j + g^{ij} P_m - \frac{\hbar^2}{4M^2} g^{\alpha \beta} g_{\mu \nu}^{\prime} \rho_M \nabla_\beta \partial_\alpha \ln \rho_M.$$  \hspace{1cm} (24)$$

The adiabatic pressure $P_m$ is [15]

$$P_m = -\frac{d}{d \rho_M} \left[ \frac{\varepsilon_m(\rho M)}{\rho_M} \right],$$  \hspace{1cm} (25)$$

and the QC term and the quantum potential are unified to the last term. The suffix $m$ indicates the contribution from matter. Note that $\rho_M v^i v^0$ and $T^{ij}_m$ form a part of the energy-momentum tensor. The quantum potential is not sufficient and the QC term is necessary to express the conserved energy-momentum tensor.

### III. EFFECT OF QC TERM

To investigate the effect of the QC term in cosmology, we apply the FRW metric [14], $g_{\mu \nu} = \text{diag}(-1, a^2/(1-Kr^2), a^2r^2, a^2 r^2 \sin^2 \theta)$, where $a$ is the FRW scale factor and $K$ is a parameter associated with the geometry, which takes the value 1 (spherical), 0 (Euclidean) or $-1$ (hyperspherical). We further consider that the distribution of matter with large scales is homogeneous and isotropic. Then, dropping the velocity field and the spatial derivative terms in Eq. (24), the stress tensor becomes

$$T^{ij}_m = g^{ij} \left( P_m - \frac{\mathcal{A}}{8\pi G} \right),$$  \hspace{1cm} (26)$$

$$\Lambda_{QC} = -\frac{2\pi \hbar^2 G}{M^2 c^5} H_0 \partial_0 \rho_M,$$  \hspace{1cm} (27)$$

where $H_0 = a^{-1}(da/dt)$ is the Hubble constant and the dimension of $\Lambda_{QC}$ is the same as that of the cosmological constant $\Lambda$. The second term in $T^{ij}_m$ gives the negative contribution to the pressure [22]. To see this, we use the energy conservation [14] as

$$\partial_0 \rho_M \approx c^{-2} \Omega_m \partial_0 \varepsilon = -3c^{-3} \Omega_m H_0 (\varepsilon + P),$$  \hspace{1cm} (28)$$

where $\varepsilon$ and $P$ are the total energy density of the universe and the corresponding pressure, respectively. The energy
ratio $\Omega_m$ is defined by $\rho_MC^2/\varepsilon$. We assumed that the time dependence of $\Omega_m$ is small. Then $\Lambda_{QC}$ becomes a positive function,

$$\Lambda_{QC} = \frac{6\pi G}{c^4} \alpha \Omega_m(\varepsilon + P),$$

(29)

where the adimensional parameter is defined by

$$\alpha = \left( \frac{\hbar H_0}{Mc^2} \right)^2 \approx 10^{-82},$$

(30)

with $M$ being chosen to be, for example, the mass of the hydrogen atom $M = 1.67 \times 10^{-27}$ kg. That is, $\Lambda_{QC}$ induces the effect analogous to $\Lambda$ which contributes to the accelerating expansion of the universe.

Whereas the above qualitative behavior is consistent, the magnitude is far from the observed value. Substituting the critical density to $\varepsilon + P$, the order is estimated by $\Lambda_{QC} \approx 10^{-135}$ m$^{-2}$ with $\Omega_m = 0.1$, which is too small compared to the accepted value, $\Lambda \sim 10^{-52}$ m$^{-2}$ [27]. It should be however noted that $\Lambda_{QC}$ in Eq. (27) can be inhomogeneous and the above estimation is not appropriate for the quantitative discussion of the accelerating expansion. See also the later discussion.

On the other hand, we observe that, applying Eq. (27) to a localized mass distribution, the second term in $T_{\mu\nu}^m$ gives a negative contribution for $\partial_\mu \rho_M < 0$ and a positive constitution for $\partial_\mu \rho_M > 0$, respectively. Therefore the QC term suppresses the diffusion of the mass distribution and then the gravitational force seems to be effectively enhanced. Such a behavior is analogous to a part of the effects expected by the dark matter [21, 22].

IV. CONCLUDING REMARKS

In this paper, the stochastic variational method was generalized to the non-relativistic curved spacetime systems for the first time. This is applicable not only to a non-relativistic particle system but also to a continuum medium. We further showed that the interplay between quantum fluctuation and spacetime curvature induces the quantum-curvature term, which prevents us from deriving the Schrödinger equation by introducing the wave function but is necessary to find the conserved energy-momentum tensor. To investigate its effect, the Friedmann-Robertson-Walker metric was applied. The effect of the QC term contributes to the cosmological acceleration although its quantitative influence is too small.

However the explanation of cosmology is not our purpose. We intended to find a possibility for the large modification of quantum mechanics and the present result gives a positive sign under the assumption that the variational principle is a fundamental requirement in the quantization. It will be important to pursue the possible modifications of quantum mechanics from a wide perspective [16, 23, 24].

There exist various effects which have not yet been discussed and may contribute to the quantitative estimation. For example, the QC term can exist in QED. The relativistic effect may modify the QC term and then the Wiener process in the present work may be substituted by the Poisson process [27]. Moreover, to have the consistent back reaction between the quantized matter and the classical curved spacetime, the quantum-classical hybrids should be considered [28, 29]. It is also interesting to study the vacuum energy and the particle creation in the field theory [30, 33]. These are challenges for the future.

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[12] E. Nelson, “Derivation of the Schrödinger Equation from Newtonian Mechanics”, Phys. Rev. 150, 1079 (1966).

[13] A. S. Sanz, “Investigation Puzzling Aspects of the Quantum Theory by Means of Its Hydrodynamic Formulation”, Found. Phys. 45, 1152 (2015).

[14] S. Weinberg, Cosmology, (Oxford, New York, 2008).

[15] T. Koide and T. Kodama, “Navier-Stokes, Gross-Pitaevskii and generalized diffusion equations using the stochastic variational method”, J. Phys. A: Math. Theor. 45, 255204 (2012).

[16] T. Koide and T. Kodama, “Generalization of uncertainty relation for quantum and stochastic systems”, Phys. Lett. A382, 1472 (2018).

[17] J. H. Gaspar Elsas, T. Koide and T. Kodama “Noether’s Theorem of Relativistic-Electromagnetic Ideal Hydrodynamics”, Braz. J. Phys. 45, 334 (2015).

[18] D. Huterer and D. L. Shafer, “Dark energy two decades after: observables, probes, consistency tests”, Rep. Prog. Phys. 81, 016901 (2018).

[19] P. Brax, “What makes the Universe accelerate? A review on what dark energy could be and how to test it”, Rep. Prog. Phys. 81, 016902 (2018).

[20] D. Vasak, et al., “Covariant Canonical Gauge theory of Gravitation resolves the Cosmological Constant Problem“, arXiv:1802.07137.

[21] J. L. Feng, “Dark Matter Candidates from Particle Physics and Methods of Detection” Annu. Rev. Astron. Astrophys. 48, 495 (2010).

[22] K. M. Zurek, “Asymmetric Dark Matter: Theories, signatures, and constraints”, Phys. Rep. 537, 91 (2014).

[23] T. Koide, “Perturbative expansion of irreversible work in Fokker-Planck equation à la quantum mechanics”, J. Phys. A: Math. Theor. 50, 325001 (2017).

[24] T. Koide, “Nonequilibrium work relation from Schrödinger’s unrecognized probability theory”, J. Phys. Commun. 2, 021001 (2018).

[25] J.-P. Gazeau, “From Classical to Quantum Models: The Regularising Role of Integrals, Symmetry and Probabilities”, Found. Phys. https://doi.org/10.1007/s10701-018-0210-3 (2018).

[26] H. Bergeron, et al., “A baby Majorana quantum formalism”, arXiv:1701.04026v3.

[27] T. Kudo, I. Ohba and H. Nitta, “A derivation of the Dirac equation in an external field based on the Poisson process”, Phys. Lett. A286, 227 (2001) and references therein.

[28] T. Koide, “Classicalization of quantum variables and quantum-classical hybrids”, Phys. Lett. A379, 2007 (2015).

[29] H.-T. Elze, “Quantum-classical hybrid dynamics – a summary”, H.-T. Elze, J. Phys. Conf. Ser. 442, 012007 (2013).

[30] T. Koide and T. Kodama, “Stochastic variational method as quantization scheme: Field quantization of the complex Klein-Gordon equation”, Prog. Theor. Exp. Phys. 093A03 (2015).

[31] R. O. Ramos, M. V. dos Santos and I. Waga “Matter creation and cosmic acceleration”, Phys. Rev. D89, 083524 (2014).

[32] M. V. dos Santos, I. Waga and R. O. Ramos, “Degeneracy between CCDM and ΛCDM cosmologies”, Phys. Rev. D90, 127301 (2014).

[33] B. L. Hu and E. Verdaguer, “Stochastic Gravity: Theory and Applications” Living Rev. Relativity, 11, 3 (2008).

[34] T. Takabayasi, “On the Formulation of Quantum Mechanics associated with Classical Pictures” Prog. Theor. Phys. 8, 143 (1952).

[35] T. C. Wallstrom, “On the derivation of the Schrödinger equation from stochastic mechanics” Found. Phys. Lett 2, 113 (1989).

[36] See however the criticism by Takabayasi [34] and Wallstrom [35].

[37] For the sake of simplicity, we omitted the initial distribution of the particle here but it does not affect our formulation.

[38] This term contains also the contribution from the quantum potential.