The DSR-deformed relativistic symmetries and the relative locality of 3D quantum gravity

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Abstract
Over the last decade there were significant advances in the understanding of quantum gravity coupled to point particles in 3D ((2+1)-dimensional) spacetime. Most notably it is emerging that the theory can be effectively described as a theory of free particles on a momentum space with anti-deSitter geometry and with noncommutative spacetime coordinates of the type $[x^\mu, x^\nu] = i\hbar g^{\mu\nu} x^\rho$. We here show that the recently proposed relative-locality curved-momentum-space framework is ideally suited for accommodating these structures’ characteristics of 3D quantum gravity. Through this we obtain an intuitive characterization of the DSR-deformed Poincaré symmetries of 3D quantum gravity, and find that the associated relative spacetime locality is of the type producing dual-gravity lensing.

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(Some figures may appear in colour only in the online journal)

1. Introduction

We here report a study which is relevant for two of the most active areas of quantum-gravity research over the last decade. Some aspects of our analysis contribute to the ongoing development of DSR-deformed relativistic symmetries at the Planck scale, while other aspects of our analysis are inspired by previous studies of quantum-gravity in 3D ((2+1)-dimensional) spacetime.

The first studies of DSR-deformed relativistic symmetries intended [1, 2] to provide an alternative interpretation of results on quantum-gravity modifications of special relativistic laws, such as modifications of the on-shell relation of the type $p^2 = E^2 - m^2 + \Delta_{QG}(\ell, E)$, with $\Delta_{QG}(\ell, E)$ some quantum-gravity correction and $\ell$ expected to be given roughly by the inverse of the Planck scale. At first these quantum-gravity-research results producing laws not
compatible with special relativity were interpreted as inevitably associated with the presence of a non-relativistic preferred-frame picture. Starting with [1, 2] (and now finding support in a rather sizable literature, see, e.g., [3–7]) it was understood that some of these modifications of special-relativistic laws could be accommodated in scenarios (‘DSR scenarios’) that are still fully relativistic, preserving the principle of equivalence of inertial frames, if one allows for ℓ-deformed laws of transformation between observers [1, 2]. These relativistic proposals then have as invariant characteristic scales of the transformation rules not only the speed-of-light scale $c$ (here mute because of our choice of units $c = 1$) but also the Planck scale $\ell^{-1}$.

Interest in the 3D quantum-gravity problem started to pick up during the 1980s [8–11]. Our study is primarily connected to more recent work coupling 3D gravity to point particles [12–19], showing that several potentially different approaches agree on some results, which at this point should then be viewed as robust. In particular, it is found that the momenta of the particles are described by elements of the isometry group of the ‘model space-time’ which provides gluing data for the non-trivial topology describing them. The first intuition of this can be found in studies from the 1990s [20–22] in a metric formalism. More recent refined descriptions [12–17] established results such that the momentum of particles coupled to Chern–Simons gravity is given by holonomies of the gauge group of the theory along non-contractible loops containing the puncture describing the particle. The connection between metric descriptions and Chern–Simons descriptions was investigated in [14, 15]. For reasons that shall be clarified also by our line of analysis (see later) the momentum-space features of this characterization can be deduced already at the level of the classical theory, and they persist when the theory is quantized [16–19]. And for the quantum theory the counterpart of this non-trivial geometry of momentum space turns out to be noncommutativity of the spacetime coordinates.

We are interested in the case of 3D gravity without a cosmological constant, where one ends up with a momentum space with anti-deSitter geometry, and noncommutativity of the spacetime coordinates of the type [23]

$$[x^\mu, x^\nu] = i\hbar \ell \varepsilon_{\mu \nu} x^\rho$$

which is the case we here label3 ‘spinning noncommutative spacetime’.

Crucial for our analysis is the observation that these features of curvature of momentum space and noncommutativity of spacetime coordinates have already provided the starting point for some DSR-relativistic scenarios. In particular, there was a considerable amount of work on a DSR scenario centered on [24–26] a momentum space with deSitter geometry and spacetime noncommutativity of $\kappa$-Minkowski4 type [28, 29]. We here show that the techniques and approaches developed in those contexts can indeed be adapted to the scenario inspired by the 3D quantum gravity. In particular, the ‘relative-locality curved-momentum-space framework’ [30, 31], which had been valuably applied to the $\kappa$-Minkowski-based picture [24, 26], is here found to be also applicable to the 3D-gravity-inspired picture. The relative-locality framework can be applied to an even wider class of theories, but specifically in the context of DSR-relativistic theories it empowers us to properly implement within a spacetime picture the deformations of translation transformations that are typically encountered. This will also play a key role in our analysis.

The most significant results we obtain establish that, as preliminarily suggested by some previous studies (see, e.g., [18, 19, 32–34]), 3D quantum gravity is a theory with DSR-deformed

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3 This is inspired by the analogy between (1) and the angular-momentum algebra (which could also suggest the name ‘spin spacetime’ [23]).

4 Interestingly, also the possibility of $\kappa$-Minkowski noncommutativity can arise in the 3D context of Chern–Simons theories, but only at the cost of renouncing [27] to some aspects of the relevance for the Einstein–Hilbert action.
relativistic symmetries. And we show that a characteristic aspect of our 3D-gravity-inspired analysis is ‘dual-gravity lensing’, one of the least studied among possible features of a scenario with deformation of relativistic symmetries (previously considered explicitly only in [35, 36]).

We feel that there are rather general benefits in performing studies such as ours at the interface between research on DSR-deformed relativistic symmetries and research on 3D quantum gravity. On the DSR side one should note that the construction of 4D models with DSR-deformed relativistic symmetries is at present at an advanced but incomplete stage, and the well-understood 3D-quantum-gravity context can be ideally suited for giving guidance toward uncovering other significant implications of these deformations. The debate on DSR often revolves around whether these relativistic deformations should at all be considered in relation to the quantum-gravity problem, and the fact that they necessarily arise in the 3D-quantum-gravity context surely provides a strong element of support for advocates of the study of DSR-deformed relativistic symmetries. And the well-understood 3D-quantum-gravity context is also ideally suited for giving guidance on the conceptual side: features like relative locality and noncommutativity of momentum-composition laws may appear puzzling when introduced by hand in a DSR picture, so the fact that we expose here their inevitability in the 3D-gravity context can change the balance of intuitions on such features. Moreover, 3D quantum gravity also provides an explicit example of the sort of mechanisms which are expected to produce DSR-deformed laws of kinematics: for 3D quantum gravity we can actually integrate out gravity [18, 19] and verify that its effects are reabsorbed into novel relativistic properties for the gravity-free propagation of particles. It is not unnatural to conjecture that also in 4D quantum gravity there would be some regime of observation such that the only quantum-gravity effects there tangible can be reabsorbed into novel relativistic properties for the gravity-free propagation of particles, but in the 4D case providing explicit examples where this intuition applies is beyond our present technical abilities.

On the 3D-gravity side we stress how reliance on expertise gained in previous studies of DSR deformations might amplify the potentialities for 3D results to inspire phenomenological programs for real 4D quantum gravity. Surely 4D quantum gravity will be very different from its 3D version, but it is legitimate to speculate that some of the features uncovered in the much simpler 3D context might also apply to the 4D context we are really interested in. But such a legitimate speculation could be valuable only if it can be scrutinized experimentally, whereas most results on 3D quantum gravity so far have been of rather formal nature. By uncovering a role for DSR-deformed relativistic symmetries in the 3D-gravity context we here open the way for using 3D-gravity as guidance for proposals of ‘DSR phenomenology’, some of which are already at rather advanced stage of development (see, e.g., [37–39]).

As mentioned, we adopt units such that the speed-of-light scale is set to 1, and we denote by \( \ell \) the inverse of 3D-quantum-gravity Planck scale. It turns out to be sufficient for our purposes to assume that \( \ell \) is very small, and therefore several of our results are shown only at leading (linear) order in \( \ell \). Also note that for the antisymmetric tensor \( \varepsilon_{\mu\nu\rho} \), we adopt conventions such that \( \varepsilon_{012} = -1 \) and indices are raised and lowered with the metric \( \eta_{\mu\nu} = (-1, 1, 1) \). This in particular implies that the defining commutation relations for the spinning spacetime could also be written as \([x^1, x^2] = -i\hbar \ell \hat{\chi}^0, [x^1, x^0] = -i\hbar \ell \hat{\chi}^2, [x^2, x^0] = i\hbar \ell \hat{\chi}^1\).

### 2. Anti-deSitter momentum space and spinning spacetime

Before getting started with our analysis it is useful for our purposes to characterize quantitatively, in this section, the known facts about 3D gravity and quantum gravity that were described qualitatively in our opening remarks. None of the points made in this section is original, since they can all be found here and there in the literature: the main points are
actually textbook material on the Lie group $SL(2, R)$ (see, e.g., [40]), while their applications to 3D-gravity and the usefulness in that context of the parametrization we discuss had already been stressed in previous studies (see in particular [18, 22]). In subsection 4.1 we will show how these results can be revisited adopting the perspective of the relative-locality framework.

We focus on the case without a cosmological constant, and we are mainly interested in the connection between geometry of momentum space and spacetime noncommutativity. We already mentioned that the momentum space has anti-deSitter geometry, but more precisely the momentum space is the Lie group $SL(2, R)$, group of linear transformations acting on $\mathbb{R}^2$, with determinant equal to 1. And our first task is to expose the anti-deSitter geometry of this momentum space. To do this, we note that it is possible to write the generic element $p$ of $SL(2, R)$ as a combination of the identity matrix and of the elements of a basis of $sl(2, R)$, the Lie algebra of $SL(2, R)$

$$p = u I - 2 \xi_\mu X^\mu. \quad (2)$$

Here $I$ is the identity $2 \times 2$ matrix and the $X^\mu$ are

$$\begin{align*}
X^0 &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & X^1 &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & X^2 &= \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\end{align*} \quad (3)$$

which constitute a basis of $sl(2, R)$, and the requirement of having determinant equal to 1 (det $p = 1$) implies that the parameters $u, \xi_\mu$ must be constrained to satisfy

$$u^2 - \xi_\mu \xi^\mu = 1. \quad (4)$$

This constraint provides, as announced, the definition of a three-dimensional anti-deSitter geometry.

And note that (3) implies that the $X^\mu$ satisfy by construction (up to a dimensionful constant) the commutation relations of the spinning spacetime

$$[X^\mu, X^\nu] = \varepsilon^{\mu\nu\rho} X^\rho. \quad (5)$$

Among the choices of coordinates for this momentum-space geometry used in the 3D-gravity literature, particularly convenient for our purposes is the choice of coordinates $p^\mu$ such that

$$p^\mu = \sqrt{1 + \ell^2 p^\mu p_\mu} = 2 \ell p^\mu X^\mu, \quad (6)$$

since we shall find that this choice of coordinates allows one to describe the metric very compactly and to formulate the law of composition of momenta explicitly in terms of the algebraic properties of the $X^\mu$ matrices.

The on-shell condition satisfied by these momenta turns out [18, 22, 23] to be

$$\ell^{-2} (\arcsin (\sqrt{1 - \ell^2 p^\mu p_\mu}))^2 = m^2, \quad (7)$$

where $m$ is the mass of the particle. We do not revisit the arguments leading to this form of the on-shell relation, since we shall anyway rederive it from the relative-locality perspective in section 4.

Next we must note that the group structure of our momentum space implies that the law of composition of momenta is nonlinear. In fact, if we multiply two group elements

$$\begin{align*}
p &= \sqrt{1 + \ell^2 p^\mu p_\mu} = 2 \ell p^\mu X^\mu, \\
q &= \sqrt{1 + \ell^2 q^\mu q_\mu} = 2 \ell q^\mu X^\mu
\end{align*} \quad (8)$$

An alternative coordinatization which is also frequently used (see, e.g., [22]) adopts coordinates $P_\alpha, P_\beta, P_\gamma$ which are essentially Euler angles and are connected to the coordinates of our equation (6) by the relations

$$p_0 = \ell^{-1} \sin(P_\alpha \ell) \cosh(P_\beta \ell), \quad p_1 = \ell^{-1} \cos(P_\alpha \ell) \sinh(P_\beta \ell), \quad p_2 = \ell^{-1} \sin(P_\alpha \ell) \sinh(P_\beta \ell).$$

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we obtain a new element $pq$.

$$pq = (\sqrt{1 + \ell^2 p^\mu p_\mu} \sqrt{1 + \ell^2 q^\nu q_\nu} + \ell^2 p^\mu q_\mu)\mathbb{I} - 2\ell(\sqrt{1 + \ell^2 q^\nu q_\nu} + \sqrt{1 + \ell^2 p^\mu q_\mu} - \ell\epsilon_{\mu}^{\nu\rho} p_\nu q_\rho)X^\mu.$$

(9)

where we made use of the identity

$$X^\mu X^\nu = \frac{1}{2} \eta^{\mu\nu} \mathbb{I} + \frac{1}{2} \epsilon_{\mu}^{\nu\rho} X^\rho.$$

(10)

We also observe, and this will play an important role in what follows, that there is a simple but nonlinear relation between the coordinates $(p \oplus q)_\mu$ of $pq$ and the coordinates $p_\mu$, $q_\mu$ of $p,q$ which can be easily inferred from (9):

$$(p \oplus q)_\mu = \sqrt{1 + \ell^2 q^\nu q_\nu} p_\mu + \sqrt{1 + \ell^2 p^\mu q_\mu} - \ell\epsilon_{\mu}^{\nu\rho} p_\nu q_\rho.$$

(11)

### 3. Relativistic kinematics on the spinning spacetime

We are now ready for performing the first task of our analysis, which is the one of exposing the DSR-relativistic symmetries of the emerging framework. We shall be satisfied analyzing the classical limit of the construction described in the previous section, characterized by spacetime coordinates with Poisson brackets given by

$$\{x^\mu, x^\nu\} = \epsilon^{\mu\nu\rho} x^\rho,$$

(12)

and by a momentum space with coordinates $p_\mu$, constrained on mass shells governed by

$$\ell^{-2}(\arcsin(\sqrt{1 - \ell^2 p^\mu p_\mu}))^2 = m^2,$$

(13)

obeying the composition law

$$(p \oplus q)_\mu = \sqrt{1 + \ell^2 q^\nu q_\nu} p_\mu + \sqrt{1 + \ell^2 p^\mu q_\mu} - \ell\epsilon_{\mu}^{\nu\rho} p_\nu q_\rho.$$

(14)

The relevant DSR-deformed relativistic symmetries are particularly simple (with respect to other much-studied examples [25]) since the action of Lorentz-sector generators on momenta remains undeformed. Indeed the mass shell (13) is found to be invariant and the composition law (14) is found to be covariant when adopting the standard

$$[R, p_0] = 0 \quad [N_1, p_0] = p_1 \quad [N_2, p_0] = p_2$$

(15)

$$[R, p_1] = -p_2 \quad [N_1, p_1] = p_0 \quad [N_2, p_1] = 0$$

(16)

$$[R, p_2] = p_1 \quad [N_1, p_2] = 0 \quad [N_2, p_2] = p_0,$$

(17)

where $R$ denotes the generator of rotations, while $N_1$ and $N_2$ denote the generators of boosts.

The action of symmetry generators on coordinates is affected by the deformation (12), but only in rather localized manner. We want to show that compatibility with (12) is achieved by leaving undeformed the Poisson brackets among translation generators

$$\{p_\mu, p_\nu\} = 0$$

(18)

and the action on spacetime coordinates of rotations and boosts

$$[R, x^0] = 0 \quad [N_1, x^0] = -x^1 \quad [N_2, x^0] = -x^2$$

(19)

$$[R, x^1] = -x^2 \quad [N_1, x^1] = -x^0 \quad [N_2, x^1] = 0$$

(20)

$$[R, x^2] = x^1 \quad [N_1, x^2] = 0 \quad [N_2, x^2] = -x^0.$$

(21)
The only deformation we shall need concerns translations acting on spacetime coordinates, a deformation which is required for compatibility with \( \{x^\mu, x^\nu\} = \ell \epsilon^{\mu\nu\rho} x^\rho \); one could not possibly adopt the standard \( \{p^\mu, x^\nu\} = -\delta^\nu_\mu \) since then the Jacobi identities would not be satisfied (in particular \( \{p^\mu, \{x^\nu, x^\rho\}\} + \{x^\rho, \{p^\mu, x^\nu\}\} + \{x^\nu, \{x^\rho, p^\mu\}\} \neq 0 \)).

For the main results of our study, reported in the following sections, we shall only make use of leading-\( \ell \)-order formulas. And we can here show that at leading order in \( \ell \) there is a unique specification of \( \{p^\mu, x^\nu\} \), given by

\[
\{p^\mu, x^\nu\} \simeq -\delta^\nu_\mu + \frac{\ell}{2} \epsilon^{\nu\rho\mu} p^\rho,
\]

which satisfies all the Jacobi identities, taking into account (15)–(21).

We show this uniqueness property of \( \{p^\mu, x^\nu\} = -\delta^\nu_\mu + \frac{\ell}{2} \epsilon^{\nu\rho\mu} p^\rho \), by starting with the following parametrization:

\[
\{p^\mu, x^\nu\} = -\delta^\nu_\mu + \ell f_{\mu \nu \rho} p^\rho,
\]

which gives the most general possibility at leading order, compatibly with the absence of deformation in the \( \ell \to 0 \) limit.

We get started determining the parameters \( f_{\mu \nu \rho} \) by considering the Jacobi identity

\[
\{p^\mu, \{x^\nu, x^\rho\}\} + \{x^\rho, \{p^\mu, x^\nu\}\} + \{x^\nu, \{x^\rho, p^\mu\}\} = 0
\]

which implies

\[
f_{\mu \nu \rho} - f_{\mu \rho \nu} = \epsilon^{\nu \rho \mu}.
\]

Evidently this fixes the antisymmetric part of \( f_{\mu \nu \rho} \) in the indices \( \nu \rho \) and allows us to proceed now assuming

\[
f_{\mu \nu \rho} = t_{\mu \nu \rho} + \frac{1}{2} \epsilon^{\nu \rho \mu},
\]

where \( t_{\mu \nu \rho} \) must be symmetric in the indices \( \nu \rho \).

Next we consider the Jacobi identities for \( R_{\mu \nu \rho \sigma}, p_{\mu \nu}, x^\rho \), for \( N_1, p_{\mu \nu}, x^\rho \), and for \( N_2, p_{\mu \nu}, x^\rho \). It is easy to show that these amount to the following requirement

\[
f_{\mu \nu \rho} \epsilon^{\lambda \gamma \rho} + f_{\rho \nu \lambda} \epsilon^{\rho \gamma \mu} + f_{\rho \lambda \mu} \epsilon^{\nu \rho \gamma} = 0.
\]

Rewriting this requirement making use of (26) one then easily finds that the corresponding requirement for \( t_{\mu \nu \rho} \) must hold:

\[
t_{\mu \nu \rho} \epsilon^{\lambda \gamma \rho} + t_{\rho \nu \lambda} \epsilon^{\rho \gamma \mu} + t_{\rho \lambda \mu} \epsilon^{\nu \rho \gamma} = 0.
\]

And, using the fact that \( t_{\mu \nu \rho} \) is symmetric for exchange of indices \( \nu \rho \), one can easily show [41] that the only possibility is

\[
t_{\mu \nu \rho} = 0.
\]

Considering that we had already established (26) this result of \( t_{\mu \nu \rho} = 0 \) confirms our thesis that the choice \( f_{\mu \nu \rho} = \frac{1}{2} \epsilon^{\nu \rho \mu} \) is unique.

Because of our focus on leading-order results it is not necessary for us to go beyond (22), whose uniqueness we have established at leading order. But we just note down the observation that the following exact formula:

\[
\{p^\mu, x^\nu\} = -\delta^\nu_\mu \sqrt{1 + \frac{\ell^2}{4} p^\rho p_\rho + \frac{\ell}{2} \epsilon^{\nu \rho \mu} p^\rho}
\]

satisfies all the Jacobi identities exactly (no leading-order truncation), taking into account \( \{x^\mu, x^\nu\} = \ell \epsilon^{\mu\nu\rho} x^\rho \) and (15)–(21).

In summary we are here confronted with a DSR-relativistic setup such that the deformation is confined in translations. We have highlighted (22) (generalizable to (30)). And this should
be combined with the fact that the momentum charges are composed following the nonlinear law (14), which in turn produces, as we shall discuss in greater detail later, a deformation of the action of translation transformations on multiparticle systems. Take for example a system composed of two particles, respectively with phase-space coordinates $p_\mu, x^n$ and $q_\mu, y^n$: then a translation parametrized by $b^\rho$, and generated by the total-momentum charge $(p \oplus q)_\rho$, acts for example on the particle with phase-space coordinates $p_\mu, x^n$ as follows:

$$b^\rho \{ (p \oplus q)_\rho, x^n \} \simeq b^\rho \{ p_\rho, x^n \} - \ell b^\rho \varepsilon_\rho^{\sigma \gamma} q_\gamma \{ p_\sigma, x^n \},$$

where on the right-hand side we were again satisfied to show the leading-order Planck-scale modification.

4. Describing dynamics within the relative-locality framework

Up to this point we focused on a description of the momentum-space and spacetime structures of the theory here of interest, with emphasis on relativistic implications. Our next task is to describe particle dynamics governed by these structures. We shall be here satisfied with a description of dynamics in the classical limit, but this is already a severe challenge, particularly because of the implications of momentum-space curvature. A formalism suitable for our purposes was only recently developed: this is the relative-locality curved-momentum-space framework of [30, 31]. The main objective of this section is to formulate a relative-locality curved-momentum-space theory which incorporates the 3D-gravity structures we discussed.

4.1. On-shell relation and relative-locality geometry of momentum space

In section 2 we established that the metric of the momentum space of 3D gravity is the anti-deSitter-space metric. And we also commented briefly on previous results indicating that the momenta are governed by an on-shell condition of the form

$$\ell^{-2} \left( \arcsin \left( \sqrt{-\ell^2 p^\mu p_\mu} \right) \right)^2 = m^2.$$  (32)

In the relative-locality framework, which we shall use for formulating dynamics, the momentum-space metric and the on-shell relation must be linked [30, 31] by the requirement that the on-shell relation be describable in terms the geodesic distance $D(0, p_\mu)$ of the momentum $p_\mu$ from the origin:

$$D^2(0, p_\mu) = m^2.$$  (33)

In this subsection we verify that this requirement enforced by the relative-locality framework reproduces the expected result (32). For this purpose we exploit the fact that our anti-deSitter momentum space can be embedded very easily in $\mathbb{R}^{2,2}$,

$$ds^2 = -du^2 - (d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2.$$  (34)

By embedding our anti-deSitter momentum space in $\mathbb{R}^{2,2}$ we can then describe the metric on the anti-deSitter momentum space as a metric induced by the $\mathbb{R}^{2,2}$ metric. Our embedding

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6 Further details on this prescription are given in [30, 31]. The distance $D(0, p_\mu) = \int_0^1 \sqrt{g^{\mu\nu} \dot{k}_\mu(s) \dot{k}_\nu(s)} ds$ of the momentum $p_\mu$ from the origin must be computed along a geodesic of the Levi-Civita connection of the momentum space metric $g^{\mu\nu}$, with endpoints of the geodesic such that $k_\mu(0) = 0$ and $k_\mu(1) = p_\mu$. We find convenient to use embedding coordinates in the analysis of this geodesic distance. (The result (35) can also be obtained by exploiting the fact that our momentum space is a Lie group and then defining the metric over this space using the Killing form of its Lie algebra.)
coordinates are \( Y_I = \sqrt{1 + \ell^2 p^\mu p_\mu, \ell p_\mu} \) so we can evidently describe the pull-back of the metric (34) to our \( SL(2, R) \) as follows:

\[
d s^2 = - (d p_0)^2 + (d p_1)^2 + (d p_2)^2 - \frac{\ell^2 p^\mu p_\nu d p_\mu d p_\nu}{1 + \ell^2 p^\mu p_\mu}. \tag{35}
\]

In order to characterize the geodesic distance \( D(0, p_\mu) \) it is convenient to adopt the description of the geodesics from the viewpoint of the embedding space \( \mathbb{R}^{2,2} \). We start by noting that a geodesic on the anti-deSitter hypersurface (which is the image of our embedding), equipped with the Levi-Civita connection associated with the metric (35), can be described by the Lagrangian

\[
L = \dot{Y}^I \dot{Y}_I + \lambda (Y^I Y_I + 1). \tag{36}
\]

The kinetic term describes the free motion in \( \mathbb{R}^{2,2} \) and \( \lambda \) is a Lagrange multiplier imposing that the motion should be on the anti-deSitter hypersurface. The equations of motion one derives from the Lagrangian (36) are simply

\[
\ddot{Y}_I = \lambda Y_I, \quad Y^I Y_I + 1 = 0. \tag{37}
\]

The first equation is a simple second-order differential equation, while the second one defines the anti-deSitter hypersurface.

The geodesics going out from the origin and arriving at a point \( Y_I = (\sqrt{1 + \ell^2 p^\mu p_\mu, \ell p_\mu}) \) are characterized by the value of \( p^\mu p_\mu \). If \( p^\mu p_\mu > 0 \), then the geodesic is space-like and we have (taking the absolute value of \( \dot{Y}^I \dot{Y}_I \) when computing the geodesic distance)

\[
\ell^2 p^\mu p_\mu = \sinh^2 (D(0, Y_I)). \tag{38}
\]

If \( p^\mu p_\mu = 0 \), then the geodesics is light-like and \( D(0, Y_I) = 0 \). Finally, if \( p^\mu p_\mu < 0 \), then the geodesic is time-like and we have

\[
\ell^2 p^\mu p_\mu = - \sin^2 (D(0, Y_I)). \tag{39}
\]

Using that \( D(0, Y_I) = \ell D(0, p_\mu) = \ell m \) we can rewrite the previous equations as

\[
\ell^2 p^\mu p_\mu = \sinh^2 (\ell m) \quad p^\mu p_\mu > 0 \tag{40}
\]

\[
p^\mu p_\mu = 0 \quad p^\mu p_\mu = 0 \tag{41}
\]

\[
\ell^2 p^\mu p_\mu = - \sin^2 (\ell m) \quad p^\mu p_\mu < 0. \tag{42}
\]

Since our mass-shell condition should be a perturbation of the special relativistic one, the physically relevant cases are the last two, that can be written together as

\[
\ell^2 p^\mu p_\mu = - \sin^2 (\ell m), \tag{43}
\]

where \( m \) is now allowed to be 0.

We note that our physical momentum space is then described by the condition

\[
- \ell^2 \leq p^\mu p_\mu \leq 0, \tag{44}
\]

where the first inequality comes from the anti-deSitter nature of our momentum space and the second one comes from the requirement for the mass-shell condition to have the right special relativistic limit.

Rewriting (43) in the spirit of (33), we have

\[
\ell^2 \left( \arcsin \left( \sqrt{-\ell^2 p^\mu p_\mu} \right) \right)^2 = m^2, \tag{45}
\]

which, as announced, reproduces the prediction (32) based on previous 3D-gravity results.
4.2. Spinning affine connection on momentum space

In the characterization of the geometry of momentum space adopted in the relative-locality framework one combines information on the metric of momentum space (in the sense discussed in the previous subsection) with a specification of the connection coefficients on momentum space, which must be based [30, 31] on the form of the law of composition of momenta near the origin of momentum space. Near the origin of momentum space the composition law relevant for our spinning-spacetime case takes the form

\[(p \oplus q)_{\mu} \simeq p_{\mu} + q_{\mu} - \ell \epsilon^{\nu \rho}_{\mu} p_{\nu} q_{\rho}.\] (46)

Following [30, 31] one interprets as momentum-space connection coefficients \(\Gamma_{\mu}^{\nu \rho}(0)\) the coefficients of the leading-order term of the expansion of the composition law near the origin of momentum space:

\[(p \oplus q)_{\mu} \simeq p_{\mu} + q_{\mu} - \ell \Gamma_{\mu}^{\nu \rho}(0) p_{\nu} q_{\rho} + \cdots\] (47)

(adopting conventions [31] such that the connection coefficients are dimensionless).

Therefore in our case the connection coefficients are

\[\Gamma_{\mu}^{\nu \rho}(0) = \epsilon_{\mu}^{\nu \rho}.\] (48)

Also part of the notion of ‘relative-locality momentum-space geometry’ introduced in [30, 31] are definitions for torsion, curvature of the connection and nonmetricity, given in terms of properties of the law of composition of momenta. This geometric interpretation shall play no direct role in our analysis, since we only need the condition of on-shellness and the law of conservation of momenta (and our quantitative findings of course do not depend on whether or not one adopts such a geometric interpretation). However, some authors (most notably [35]) have argued that work on the relative-locality framework should gradually uncover a connection between certain qualitative predictions of the framework and certain corresponding aspects of the geometry of momentum space, including such geometric aspects of the law of composition of momenta. We therefore note down for completeness the relevant observations, even though we shall here not dwell any further on such speculations.

Following the definitions given in [30, 31] our momentum space has torsion

\[T_{\mu}^{\nu \rho}(0) = - \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\rho} \left( (p \oplus q)_\mu - (q \oplus p)_\mu \right)_{p=q=0} = 2\ell \Gamma_{\mu}^{\nu \rho}(0) = \ell \Gamma_{\mu}^{\nu \rho}(0) - \Gamma_{\mu}^{\nu \rho}(0)\]

\[= 2\ell \epsilon_{\mu}^{\nu \rho},\] (49)

the curvature of the connection (evaluated in the origin) vanishes:

\[R_{\mu}^{\nu \rho \sigma}(0) = 2 \frac{\partial}{\partial p_\nu} \frac{\partial}{\partial q_\rho} \frac{\partial}{\partial k_\sigma} \left( (p \oplus q) \oplus k - p \oplus (q \oplus k) \right)_\mu |_{p=q=k=0} = 0,\] (50)

and we determine the value in the origin of the nonmetricity tensor [30, 31] in terms of the affine connection here discussed and the metric of anti-deSitter momentum space discussed in section 2, finding that it also vanishes:

\[N_{\mu}^{\nu \sigma}(0) = \nabla_\mu g^{\nu \sigma}(0) = g^{\mu \nu \rho}(0) + \ell \Gamma_{\rho}^{\nu \sigma}(0) g^{\rho \mu}(0) + \ell \Gamma_{\sigma}^{\nu \rho}(0) g^{\rho \mu}(0) = 0.\]

4.3. Classical regime within the relative-locality framework

As announced we shall focus on dynamics in the classical limit, as formalized within the relative-locality framework of [30, 31]. We work at leading order in \(\ell\) and we follow the prescriptions of [30, 31] for formulating interactions among particles through boundary terms, enforcing momentum conservation, at endpoints of worldlines. For example the case of a single
two-body-particle-decay process (see figure 1) is then described in terms of the following action \[26, 30, 31\]:

\[
S = \int_{-\infty}^{\infty} \left( \left( \delta_{\mu}^{\nu} - \frac{\ell}{2} \epsilon^{\mu\alpha\nu} p_{\alpha} \right) x^\nu p_{\mu} + \hat{N}_p (p^\mu p_{\mu} - m'^2) \right) ds
\]

\[
+ \int_{s_0}^{\infty} \left( \left( \delta_{\mu}^{\nu} - \frac{\ell}{2} \epsilon^{\mu\alpha\nu} q_{\alpha} \right) y^\nu q_{\mu} + \hat{N}_q (q^\mu q_{\mu} - m''^2) \right) ds - \xi^{\mu}_{(0)} K_{\mu}^{(0)}(s_0). \tag{51}
\]

Here the Lagrange multipliers \(\hat{N}_k, \hat{N}_p, \hat{N}_q\) enforce in a standard way the on-shell relation of particles, denoting with \(m\) the mass of the incoming particle and denoting with \(m'\) and \(m''\) the masses of the outgoing particles. Also note that the symplectic form was specialized to the case of spacetime coordinates such that \(\{x^\mu, x^\nu\} = \ell \epsilon_{\mu\nu}\), so that then (see (30)) one has

\[
\{p_{\mu}, x^\nu\} = -\delta_{\mu}^{\nu} + \frac{\ell}{2} \epsilon^{\nu\alpha\mu} p_{\alpha}.
\tag{52}
\]

And the most innovative part of the formalization introduced in \[30, 31\] is the presence of boundary terms at endpoints of worldlines enforcing momentum conservation, such as the one characterized in (51) by \(K_{\mu}^{(0)}(s_0)\). In particular, on the basis of what we established in the previous section we can specify the boundary term in (51) as follows:

\[
K_{\mu}^{(0)}(s_0) = (k_\mu - (p \oplus q)_\mu) = k_\mu - p_\mu - q_\mu + \ell \epsilon_{\mu\nu} p_{\nu} q_{\nu}.
\tag{53}
\]

Many significant implications of the momentum-space geometry are a consequence of the way in which translational invariance manifests itself, as stressed already in section 3. Translations are still generated by the total-momentum charges, but these are obtained from single-particle charges via the deformed \(\oplus\) composition law. And we shall find that, as in similar analyses of the relative-locality framework [26, 30, 31, 35], this produces relativity of spacetime locality. Actually, relativity of spacetime locality appears to be a generic consequence of a nontrivial geometry of momentum space, which already affects such theories for free particles [42], when distantly boosted observers are considered. It amounts to the possibility that pairs of events established to be coincident by nearby observers may be described as events that are not exactly coincident in the coordinatizations of those events by distant observers [30, 31, 42]. Because of these coordinate artifacts one cannot trust the description of a given observer Alice of events distant from her: one must in such cases replace Alice description with the coordinatization of the events by some observer Bob near to them. The same physical content one usually (i.e. with trivial translation transformations) produces by simply deriving...
the equations of motion, here requires us to handle both the equations of motion and the laws of transformation among distant observers. We shall see an example of this mechanism explicitly in the following section.

5. Dual-gravity lensing

Our last task is to expose one aspect of the relativity of spacetime locality present in our 3D-gravity-inspired theory. For this it will suffice to consider an example of causally connected interactions, such as the one here shown in figure 2.

The situation in figure 2 is described within the relative-locality framework by an action of the type

$$S = \int_{s_0}^{s_1} \left( \left( \delta_{\mu}^{\nu} - \frac{\ell}{2} \epsilon_{\mu\sigma}^{\alpha\beta} \right) x^\nu p_{\mu} + \mathcal{N}_{\mu} (p^\mu p_{\mu} - m^2) \right) \, ds$$

$$+ \int_{s_0}^{+\infty} \left( \left( \delta_{\nu}^{\mu} - \frac{\ell}{2} \epsilon_{\nu\sigma}^{\alpha\beta} k_{\sigma} \right) z^\nu k_{\mu} + \mathcal{N}_{\nu} (k^\nu k_{\mu} - m^2) \right) \, ds$$

$$+ \int_{s_0}^{s_1} \left( \left( \delta_{\nu}^{\mu} - \frac{\ell}{2} \epsilon_{\nu\sigma}^{\alpha\beta} q_{\sigma} \right) y^\nu q_{\mu} + \mathcal{N}_{\nu} (q^\nu q_{\mu} - \mu^2) \right) \, ds$$

$$+ \int_{s_1}^{+\infty} \left( \left( \delta_{\nu}^{\mu} - \frac{\ell}{2} \epsilon_{\nu\sigma}^{\alpha\beta} p_{\sigma} \right) y^\nu p_{\mu} + \mathcal{N}_{\nu} (p^\nu p_{\mu} - \mu^2) \right) \, ds$$

$$+ \int_{s_1}^{+\infty} \left( \left( \delta_{\nu}^{\mu} - \frac{\ell}{2} \epsilon_{\nu\sigma}^{\alpha\beta} q_{\sigma} \right) y^\nu q_{\mu} + \mathcal{N}_{\nu} (q^\nu q_{\mu} - \mu^2) \right) \, ds$$

$$- \xi_{\nu}^{\mu} K_{\nu}^{[0]} (s_0) - \xi_{\nu}^{\mu} K_{\nu}^{[1]} (s_1)$$

(54)

where we give each particle possibly a different mass \((m, m', m'', \mu, \mu', \mu'')\). Concerning the conservation laws, which are to be codified in the boundary terms \(K_{\nu}^{[0]} (s_0), K_{\nu}^{[1]} (s_1)\), we follow the prescription given in [26] for having causally connected interactions preserving translational invariance, so we adopt as boundary terms

$$K_{\nu}^{[0]} (s_0) = (q \oplus p)_{\mu} - (q \oplus p')_{\mu}$$

$$= p_{\mu} - p'_{\mu} - k_{\mu} - \ell \epsilon_{\mu}^{\alpha\beta} (q_{\alpha} p_{\beta} - q_{\alpha} p'_{\beta} - q_{\alpha} k_{\beta} - p'_{\alpha} k_{\beta})$$

(55)
and
\[ K_{\mu}^{[1]}(s_1) = (q \otimes p' \otimes k)_{\mu} - (p'' \otimes q' \otimes k)_{\mu} = q_{\mu}' + p_{\mu}' - q_{\mu}' - \xi_{\mu}^{\nu} (q_{\nu} p_{\rho} + q_{\nu} k_{\rho} + p_{\nu} k_{\rho} - p_{\nu} q_{\rho} - q_{\nu} k_{\rho}). \]  

(56)

We are now all set to derive equations of motion and boundary conditions, by varying \((54)\) keeping momenta fixed \([30, 31]\) at \(\pm \infty\). For the equations of motion one easily finds
\[ p_{\mu} = 0, \quad q_{\mu} = 0, \quad \dot{q}_{\mu} = 0, \quad \dot{p}_{\mu} = 0, \]  

(57)

\[ C_{\rho} = 0, \quad C_{q} = 0, \quad C_{\dot{q}} = 0, \quad C_{\dot{p}} = 0, \quad C_{p} = 0, \quad C_{\dot{p}} = 0. \]  

(58)

\[ \dot{x}^\mu = 2 N_{\rho} p^\mu, \quad \dot{y}^\mu = 2 N_{\rho} q^\mu, \quad \dot{y}_v^\mu = 2 N_{\rho} v^\mu, \]  

(59)

and the boundary conditions at endpoints of worldlines are
\[ z^\mu(0) = -\xi^{[0]}_{[0]} \delta K^{[0]}_{\mu} \delta k_{[0]} \left( \delta \mu^{[0]} + \frac{\epsilon}{2} \mu_{[0]} \rho \right) = \xi^{[0]}_{[0]} - \frac{\epsilon}{2} e_{[0]}^{\mu} (k_{[0]} - 2 q_{[0]} - 2 p_{[0]}), \]  

(60)

\[ \xi^{[0]}_{[0]} \delta K^{[0]}_{\mu} \delta p_{[0]} \left( \delta \mu^{[0]} + \frac{\epsilon}{2} \mu_{[0]} \rho \right) = \xi^{[0]}_{[0]} - \frac{\epsilon}{2} e_{[0]}^{\mu} (p_{[0]} - 2 k_{[0]} + 2 q_{[0]}), \]  

(61)

Evidently (and unsurprisingly) when only ‘soft interactions’ \([26]\) are involved, i.e. all particles involved have energies small enough that the \(\epsilon\)-deformation can be ignored, a standard special relativistic situation is recovered. We are going to focus in particular on what the theory predicts for the particle exchanged between the two interactions, assuming it is a massless particle. And we shall make reference to an observer Alice located where the interaction with conservation law \(K_{\mu}^{[0]} = 0\) takes place, and an observer Bob located where the conservation law \(K_{\mu}^{[0]} = 0\) takes place.

If both interactions in figure 2 are soft then Alice describes the exchanged massless particle according to
\[ x_{A^{(1)}}^{1} = x_{A^{(0)}}^{0}, \quad x_{A^{(1)}}^{2} = 0, \]  

(62)

where we assumed that the first interaction occurs exactly in Alice’s origin, and we further specialized conventions so that the massless particles exchanged between two soft interactions
propagate along a common $x^1$ axis of Alice and Bob. The index $(s)$ introduced in (62) will be here consistently used to identify equations written for a soft particle (a particle only taking part in interactions for which the $\ell$-deformation is negligible).

Bob is distant and at rest with respect to Alice, and we want that the massless particle exchanged between two soft interactions is detected, through the second interaction, in Bob’s origin. So Bob must be connected to Alice by a translation of parameters $b^\nu = (b^0, b^1, 0)$ with $b^0 = 0$. Evidently the worldline described as in (62) according to Alice’s coordinatization maintains that description in Bob’s coordinatization:

$$x^1_B = x^0_B + b^0 - b^1 = x^0_B, \quad x^2_B = 0.$$  \hspace{1cm} (63)

Of course, what we are interested in understanding is how this situation changes if the processes are ‘hard’ enough for the $\ell$-deformation to be tangible. As those familiar with relative locality will be expecting (and newcomers will see here below), for such ‘hard’ particles it is not even obvious at the onset of the analysis which of them will reach Bob in his origin. We shall find that some hard particles emitted in Alice’s origin do reach Bob’s origin, but this will come about only upon allowing ourselves to consider hard particles not necessarily emitted along Alice’s $x^1$ axis. We chose the above notation such that the particle exchanged between the interactions has phase-space coordinates $p'^\mu, x'^\mu$ and we shall keep using consistently this notation. And we characterize the direction of propagation of this exchanged particle according to Alice via an angle $\theta$:

$$p'^1 = p^0 \cos \theta$$
$$p'^2 = -p^0 \sin \theta.$$  \hspace{1cm} (64)

The case of one such particle emitted from Alice’s origin would then be described according to

$$x'^1_A = x^0_A \cos \theta$$
$$x'^2_A = -x^0_A \sin \theta.$$  \hspace{1cm} (65)

In order to establish if any of these particles (i.e. if for any value of energy $p'^0$ and emission angle $\theta$ with respect to Alice’s $x^1$ axis) manages to reach Bob’s origin we evidently need to determine Bob’s description of the worldlines described by Alice according to (65). The crucial step for this is for us to establish the relationship between Alice’s and Bob’s description of the coordinates of the particle exchanged between the two interactions in figure 2, i.e. we need to perform a translation transformation properly taking into account the deformed law of composition of momentum charges.

Following previous results on translation symmetry in the relative-locality framework [26, 30, 31] (see also [43]), we implement the relevant translation transformations through the action of the total-momentum charge:

$$x'^\mu_B = x'^\mu_A + b^\nu (q \oplus p'^\nu \otimes k_v, x'^\mu) = x'^\mu_A - b^\nu \frac{\partial (q \oplus p'^\nu \otimes k_v)}{\partial p'^\nu} \left( \delta^\mu_\sigma + \frac{\ell}{2} \epsilon^\mu_\sigma^\rho \rho' \right)$$
$$= x'^\mu_A + b^\nu (-\delta^\nu_\nu + \epsilon^{\sigma \gamma}_\nu (k_\gamma - q_\gamma)) \left( \delta^\mu_\sigma + \frac{\ell}{2} \epsilon^\mu_\sigma^\rho \rho' \right)$$
$$= x'^\mu_A - b^\nu + b^\nu \frac{\ell}{2} \epsilon^{\mu}_\nu \rho' \left( p'^\nu + 2k_\nu - 2q_\nu \right) \equiv x'^\mu_A - \Delta^\mu.$$  \hspace{1cm} (66)
Analogously we find the translations for the other coordinates:

\[
x_B^\mu = x_A^\mu + b^\nu (q \oplus p^\nu) = x_A^\mu - b^\nu + \frac{\ell}{2} \epsilon^\mu_{\nu\rho} (p_\rho - 2q_\rho) b^\nu
\]

\[
y_B^\mu = y_A^\mu + b^\nu (q \oplus p' \oplus k) = y_A^\mu - b^\nu + \frac{\ell}{2} \epsilon^\mu_{\nu\rho} (p_\rho + 2p'_\rho + 2k_\rho) b^\nu
\]

\[
z_B^\mu = z_A^\mu + b^\nu (q \oplus p' \oplus k) = z_A^\mu - b^\nu + \frac{\ell}{2} \epsilon^\mu_{\nu\rho} (k_\rho - 2q_\rho - 2p'_\rho) b^\nu
\]

\[
x_B^\mu = x_A^\mu + b^\nu (p'' \oplus q' \oplus k) = x_A^\mu - b^\nu + \frac{\ell}{2} \epsilon^\mu_{\nu\rho} (p''_\rho + 2q'_\rho + 2k_\rho) b^\nu
\]

\[
y_B^\mu = y_A^\mu + b^\nu (p'' \oplus q' \oplus k) = y_A^\mu - b^\nu + \frac{\ell}{2} \epsilon^\mu_{\nu\rho} (q_\rho - 2k_\rho - 2p''_\rho) b^\nu.
\]

Consistently with what had been previously established [26, 30, 31] in the literature on translation symmetry in the relative-locality framework, one can easily verify that these translation transformations produced by the total-momentum charge leave the equations of motion (57)–(60) and the boundary conditions (61) unchanged. As announced above, our main focus is going to be on the case of the particle with coordinates \( x^\mu \) which is exchanged between the two interactions in figure 2. On the basis of (66) one finds that for the worldline of this particle, described by Alice according to (65), Bob’s description is

\[
x_B^1 = \cos \theta x_A^0 - \cos \theta \Delta^0 - \Delta^1
\]

\[
x_B^2 = -\sin \theta x_A^0 - \sin \theta \Delta^0 - \Delta^2,
\]

where \( \Delta^\mu \) was introduced in (66): \( \Delta^\mu \equiv b^\nu - \frac{\ell}{2} \epsilon^\mu_{\nu\rho} (p''_\rho + 2k_\rho - 2q_\rho) b^\nu \).

We cannot \textit{a priori} insist on having the hard photon reach Bob in his spacetime origin, but we can enforce (at least for some choices of \( \theta \) and of the translation parameters \( b^\mu \)) that the hard photon goes through Bob’s spatial origin \( x_B^1 = x_B^2 = 0 \). We observe that the equation of motion can be easily rearranged as follows:

\[
x_B^1 = -\tan \theta x_B^0 - \tan \theta \Delta^1 - \Delta^2
\]

\[
x_B^0 = \frac{x_B^1 - \cos \theta \Delta^0 + \Delta^1}{\cos \theta}.
\]

Enforcing that the particle goes through \( x_B^1 = x_B^2 = 0 \) we have that

\[
\tan \theta = -\frac{\Delta^2}{\Delta^1}
\]

\[
x_B^0 = \frac{\Delta^1}{\cos \theta} - \Delta^0.
\]

The first of these two equations leads us to the conclusion that

\[
\theta \simeq \tan \theta = \frac{\Delta^2}{\Delta^1} = \frac{b \ell (k_1 - q_1 + k_0 - q_0) + b \xi (p''_0 + p'_1)}{-b - b \ell (k_2 - q_2) - b \xi^2 p''_1^2} \approx \frac{b \ell (k_1 - q_1 + k_0 - q_0)}{b + b \ell (k_2 - q_2)}
\]

\[
\simeq \ell (k_1 - q_1 + k_0 - q_0),
\]

where we used\(^7\) the fact that \( \theta \) vanishes at zeroth order in \( \ell \) and we are working at leading order in \( \ell \).

\(^7\) Note in particular that a term \( \ell p''_1 \) should only be taken into account at next-to-leading order, since from (69) one sees that \( \ell p''_1 \simeq \ell b p''_0 \) and \( \theta \) vanishes at zeroth order in \( \ell \). Similarly from (68) one sees that \( p''_0 + p'_1 \simeq (1 - \cos \theta) p''_0 \simeq \theta^2 p''_0 \).
Figure 3. Schematic of the findings of our analysis of massless particles exchanged between distant observers Alice and Bob. The $x^4$ axis in the figure is determined by the direction of the soft (low-energy) massless particle emitted at Alice that reaches Bob. Hard (high-energy) particles emitted at Alice that also reach Bob are the ones that Alice describes as going along a direction forming an angle $\theta$ (whose value is determined by our equation (72)) with the $x^4$ axis. We drew a macroscopic angle $\theta$ for better visibility, but actually this angle is extremely small (even if the particle energies involved are as high as, say, 1 TeV the angle $\theta$ still only is of the order of $10^{-16}$).

Our result (72) is evidently noteworthy: we found a non-zero result for $\theta$, i.e. the worldlines of hard particles that reach Bob from Alice must not be parallel to the worldlines of the soft particles that reach Bob from Alice. This is the feature known as ‘dual-gravity lensing’ in the relative-locality literature [35, 36]. Let us postpone further comments on this until after we have established the time at which such hard particles cross Bob’s spatial origin. This is easily done by substituting our result for $\theta$ in the second part of equations (71):

$$x_B^0 = \frac{\Delta^1}{\cos \theta} - \Delta^0 \approx \Delta^1 - \Delta^0 \approx b + b(k_2 - q_2) - b(k_2 - q_2) = 0. \quad (73)$$

So we have that the relevant hard particles do cross the origin of Bob’s reference frame. In setting up the analysis we only had enough handles to specialize to the case of hard particles going through Bob’s spatial origin, but then the result is such that those actually go through Bob’s spacetime origin.

Figure 3 summarizes the findings of our analysis of massless particles exchanged between distant observers Alice and Bob.

Evidently, the aspects of ‘dual-gravity lensing’ shown in figure 3 can deserve a few extra comments. Like previous cases in which ‘dual-gravity lensing’ was encountered [35, 36], we observe that relative locality plays a key role. Let us focus for example on the event where the soft (red) worldline crosses Bob’s worldline and the event where the hard (blue) worldline crosses the line $x_B^2 = x_B^0 = 0$. These two events are coincident, as manifested in the coordinatization by the nearby observer Bob, but Alice’s inferences about these two events, which are distant from Alice, would describe them as noncoincident.

In previous related studies [35, 36] one also finds an implicit invitation to study the energy dependence of dual-gravity lensing. In our case the angle $\theta$ that governs the magnitude of the lensing is of order $\ell E_*$, where $E_*$ is a characteristic energy scale of the process involved. And it is interesting to compare what is expected as difference between a case with some $E_*$ and a case with some $E'_*$ bigger than $E_*$. The study of dual-gravity lensing reported in [35] found that essentially the relative angle of lensing, $\theta' - \theta$, would have to be proportional to the sum of the energy scales involved, $E'_* + E_*$. Instead, the study reported in [36] would predict in such cases a relative lensing effect going like difference of the energy scales involved, $E'_* - E_*$. 


Evidently, the case we here uncovered is in the same class as the one of [36], since indeed in our case the angle $\theta$ goes like $\ell E_\star$, and therefore $\theta' - \theta \approx \ell E_\star' - \ell E_\star$.

6. Outlook

We feel that the study we reported here should be viewed as confirming the usefulness of techniques developed by research on DSR-deformed relativistic symmetries for the analysis of scenarios inspired by research on 3D gravity coupled to point particles.

And surely reference to results derived within an actual gravity model (in spite of being only a 3D model) gives poignancy to DSR studies. In this respect, our result for dual-gravity lensing in the 3D-gravity-inspired scenario has significance from a broader DSR perspective. Previous studies of dual-gravity lensing [35, 36] had provided possible alternative pictures of dual-gravity lensing, and it is valuable to see which one results from a theory which had been of interest independently of relative-locality research.

The most significant limitation of our analysis comes from neglecting quantum aspects of the DSR and relative-locality features. But the first example of analysis of such quantum effects is only very recent, reported in [44], and relies heavily on the fact basis developed in nearly 20 years of investigations of $\kappa$-Minkowski noncommutativity. The much younger literature on the spinning spacetime here of interest still does not offer some of the ingredients used in [44] for the description of quantum effects. However, we feel that particularly the spinning-spacetime study reported in [23] should provide a valuable starting point for such future analyses.

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