INTEGRAL POINTS ON SUBVARIETIES OF SEMIABELIAN VARIETIES, II

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Abstract. This paper proves a finiteness result for families of integral points on a semiabelian variety minus a divisor, generalizing the corresponding result of Faltings for abelian varieties. Combined with the main theorem of the first part of this paper, this gives a finiteness statement for integral points on a closed subvariety of a semiabelian variety, minus a divisor.

In addition, the last two sections generalize some standard results on closed subvarieties of semiabelian varieties to the context of closed subvarieties minus divisors.

Recall that a semiabelian variety is a group variety $A$ such that, after suitable base change, there exists an abelian variety $A_0$ and an exact sequence

$$(0.1) \quad 0 \rightarrow \mathbb{G}_m^\mu \rightarrow A \xrightarrow{\rho} A_0 \rightarrow 0.$$  

(In this paper a variety is an integral separated scheme of finite type over a field. Since a group variety has a rational point, the base field is algebraically closed in the function field. In characteristic zero, this implies that the variety is geometrically integral.)

Let $k$ be a number field, and let $S$ be a finite set of places of $k$ containing all archimedean places. Let $R$ be the ring of integers of $k$ and let $R_S$ be the localization of $R$ away from places in $S$. Let $X$ be a quasi-projective variety over $k$. A model for $X$ over $R_S$ is an integral scheme, surjective and quasi-projective over $\text{Spec} R_S$, together with an isomorphism of the generic fiber over $k$ with $X$. An integral point of $X$ (or, loosely speaking, an integral point of $X$) is an element of $X(R_S)$.

The first part [V 3] of this paper proved a finiteness statement (Theorem 0.3) for families of integral points on closed subvarieties $X$ of a semiabelian variety $A$ over $k$. This second and final part proves a similar result (Theorem 0.2) for certain open subvarieties of $A$.

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Specifically, the open subvarieties under consideration are those that can be written as the complement of a divisor. These two finiteness results then combine very easily to give a finiteness statement (Theorem 0.4) for a closed subvariety minus a divisor.

The main result of this paper is the following:

**Theorem 0.2.** Let $D$ be an effective divisor on $A$, let $V = A\setminus \text{Supp} D$, and let $\mathcal{V}$ be a model for $V$ over $\text{Spec} R$. Then the set $\mathcal{V}(R)$ of integral points on $V$ equals a finite union $\bigcup_i \mathcal{B}_i(R)$, where each $\mathcal{B}_i$ is a subscheme of $\mathcal{V}$ whose generic fiber $\mathcal{B}_i$ is a translated semiabelian subvariety of $A$.

In ([F], Thm. 2), Faltings proved a corresponding statement for integral points on abelian varieties: if $D$ is an ample effective divisor on an abelian variety $A$, then (with notations as above) the set $\mathcal{V}(R)$ is finite. In the semiabelian case this is no longer true (see Examples 1.1 and 1.3); however it is true that Theorem 0.2 implies Faltings’ result. Indeed, since an ample divisor $D$ on an abelian variety $A$ is still ample when restricted to a nontrivial translated abelian subvariety, the result follows by induction on the dimension of $A$.

As with Faltings’ result, the proof of Theorem 0.2 proceeds by reducing to a statement on diophantine approximation (Theorem 3.6); in addition, this paper relies heavily on results from [V 3].

In [V 3] we proved the following result.

**Theorem 0.3.** Let $X$ be a closed subvariety of a semiabelian variety $A$, and let $\mathcal{X}$ be a model for $X$. Then $\mathcal{X}(R)$ equals a finite union $\bigcup_i \mathcal{B}_i(R)$, where each $\mathcal{B}_i$ is a subscheme of $\mathcal{X}$ whose generic fiber $\mathcal{B}_i$ is a translated semiabelian subvariety of $A$.

Since the conclusion of this theorem is the same as for Theorem 0.2, the two theorems can be combined and generalized:

**Theorem 0.4.** Let $X$ be a closed subvariety of a semiabelian variety $A$, let $D$ be an effective divisor on $X$, and let $\mathcal{V}$ be a model for $X \setminus D$. Then $\mathcal{V}(R)$ equals a finite union $\bigcup_i \mathcal{B}_i(R)$, where each $\mathcal{B}_i$ is a subscheme of $\mathcal{V}$ whose generic fiber $\mathcal{B}_i$ is a translated semiabelian subvariety of $A$.

**Proof.** Let $\mathcal{X}$ be a model for $X$ such that $\mathcal{V} \subseteq \mathcal{X}$ (this can be accomplished, if necessary, by enlarging $S$). By Theorem 0.2,

$$\mathcal{V}(R) \subseteq \mathcal{X}(R) = \bigcup_i \mathcal{B}_i(R) ;$$

then Theorem 0.3 implies that

$$\mathcal{V}(R) = \bigcup_i (\mathcal{B}_i \cap \mathcal{V})(R) = \bigcup_i \bigcup_j \mathcal{B}_{ij}(R).$$

The situation regarding integral points on complements of sets of codimension $\geq 2$ is not as clean; in this case most of the rational points are also integral.
I do not believe that Theorem 0.2 has been conjectured by anyone, except that it follows from the general conjectures of [V 1] (see Theorem 5.16).

The first section of the paper gives some examples showing that a stronger conclusion in Theorem 0.2 is impossible. The second section begins the proof proper by showing some results on completions of semiabelian varieties; it is this section that contains most of what is new. Section 3 proves the main approximation lemma via extensions of Thue’s method from [F] and [V 3]. Section 4 then combines these results to conclude the proof of Theorem 0.2. The last two sections prove some theorems suggested by the similarities between Theorems 0.2 and 0.3.

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§1. Some examples

This section gives some examples showing that one cannot hope to get finiteness of the set of integral points in Theorem 0.2.

Example 1.1. Let $A = \mathbb{G}_m^2$ and let $D$ be the diagonal on $A$. Completing $A$ in the obvious way to $(\mathbb{P}^1)^2$, it follows that the closure of $D$ is ample. Yet

$$A \setminus D \cong (\mathbb{G}_m \setminus \{1\}) \times \mathbb{G}_m,$$

so it may have infinitely many integral points (depending on the model).

Of course in this case there is a nontrivial group action on $A$. The following definition, which will be useful throughout this paper, formalizes this idea.

Definition 1.2. Let $X$ be a variety on which a semiabelian variety $A$ acts, and let $Y$ be a subvariety of $X$.

(a). Let $B(A,Y)$ denote the identity component of the subgroup

$$\{a \in A(\overline{\mathbb{Q}}) \mid a + Y = Y\}$$

in $A$.

(b). If $X = A$, acting on itself by translation, then we write $B(Y) = B(A,Y)$. The restriction to $Y$ of the quotient map $A \to A/B(Y)$ exhibits $Y$ as a fiber bundle with fiber $B(Y)$. This map is called the Ueno fibration associated to $Y$. It is trivial when $B(Y)$ is.

Classically, the Ueno fibration is defined for closed subvarieties. The image of the Ueno fibration has trivial Ueno fibration.

Example 1.3. Let $E$ be an elliptic curve and let $A = \mathbb{G}_m \times E$. Let $f$ be a nonzero rational function on $E$ with a pole at a rational point $P$. Let $U$ be the largest subset of $E$ on which $f$ is defined and nonzero, and let $D \subseteq \mathbb{G}_m \times U \subseteq A$ be its graph. Then $A \setminus D$ has trivial Ueno fibration, yet it contains the nontrivial translated subgroup $\mathbb{G}_m \times \{P\}$.

Both of these examples illustrate that the non-completeness of semiabelian varieties introduces issues not present in the case of abelian varieties.
§2. Completions of semiabelian varieties

This section collects some results about completions of semiabelian varieties with various desirable properties.

Throughout this section, all varieties are over a field of characteristic zero, although it may well be true that everything holds over fields of arbitrary characteristic.

Definition 2.1. Let $G$ be a variety. A completion of $G$ is a complete variety $X$ with an open immersion $G \hookrightarrow X$. We often identify $G$ with its image in $X$. Given two completions $X_1$ and $X_2$ of $G$, we say that $X_1$ dominates $X_2$ if there exists a morphism $X_1 \to X_2$ compatible with the immersions $G \hookrightarrow X_1$ and $G \hookrightarrow X_2$. A completion $X$ of a group variety $G$ is equivariant if the group law $G \times G \to G$ extends to a morphism $G \times X \to X$.

As noted in ([V 3], Sect. 2), a semiabelian variety $A$ is isomorphic to

$$\mathbb{P}((\mathcal{O}_{A_0} + \mathcal{M}_1) \times_{A_0} \cdots \times_{A_0} \mathbb{P}((\mathcal{O}_{A_0} + \mathcal{M}_\mu))$$

for some $\mathcal{M}_1, \ldots, \mathcal{M}_\mu \in \text{Pic}^0 A_0$. Here the notation $\mathbb{P}(\mathcal{L}_1 + \mathcal{L}_2)$ means $\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$ minus the sections corresponding to the canonical projections to $\mathcal{L}_1$ and $\mathcal{L}_2$. This paper needs a slightly more general situation in which $V$ is a projective variety, $\mathcal{M}_1, \ldots, \mathcal{M}_n \in \text{Pic}^0 V$, and $W = \mathbb{P}(\mathcal{O}_V + \mathcal{M}_1) \times_V \cdots \times_V \mathbb{P}(\mathcal{O}_V + \mathcal{M}_\mu)$. The group $\mathbb{G}_m^n$ still acts on $W$, and $W$ is a fiber bundle over $V$ with fiber $\mathbb{G}_m^\mu$. As usual, let $\rho: W \to V$ denote the canonical projection. In practice $V$ will be birational to $A_0$.

Throughout this paper, all fiber bundles will have fiber equal to the variety underlying a group variety. The structure group of such bundles will always be the group of translations.

Lemma 2.2. Any equivariant completion $\overline{G}$ of $\mathbb{G}_m^n$ determines a completion $\overline{W}$ of $W$ for which $\rho$ extends to a fiber bundle $\overline{\rho}: \overline{W} \to V$ with fiber $\overline{G}$.

Proof. Cover $V$ by open subsets $U_i$ on which $\rho^{-1}(U_i)$ is isomorphic to a product $\mathbb{G}_m^n \times U_i$. We will form $\overline{W}$ by glueing completions $\overline{G} \times U_i$ of $\rho^{-1}(U_i)$. This is possible since the glueing isomorphisms on $\rho^{-1}(U_i \cap U_j)$ are translations by sections of $\rho$, and $\overline{G}$ is an equivariant completion. See also ([Se 2], Sect. 1.3).

If $V = A_0$ then $W = A$ and the resulting completion $\overline{W} =: \overline{A}$ is then invariant under translation by elements of $A$.

Proposition 2.3. Let $A$ and $\rho: A \to A_0$ be as in (0.1), let $A'$ be another semiabelian variety, and let $\theta: A \to A'$ be a group homomorphism. Let $X$ and $X'$ be equivariant completions of $A$ and $A'$, respectively, as in Lemma 2.2. Let $L$ and $L'$ be ample line sheaves on $X$ and $X'$, respectively, and let $h_L$ and $h_{L'}$ be associated height functions. Then for all $P \in A(\overline{Q})$,

$$h_{L'}(\theta(P)) \ll h_L(P) + O(1).$$

(2.3.1)
Proof. Let $\Gamma$ be the closure of the graph of $\theta$ in $X \times X'$, let $M$ be an ample line sheaf on $\Gamma$, and let $h_M$ be an associated height function. Then basic properties of heights (functoriality and ([V 1], 1.2.9f)) imply that

$$h_L(\theta(P)) \ll h_M((P, \theta(P))) + O(1).$$

Thus, by replacing $X'$ with $\Gamma$ and $A'$ with $A$, we reduce to the case where $A' = A$ and $X'$ dominates $X$ (since $\Gamma$ is also an equivariant completion of $A$). Let $\phi : X' \to X$ be the morphism. By Kodaira’s lemma ([V 1], 1.2.7), we have $m\phi^*L \sim L' + E$ for some $m > 0$ and some effective divisor $E$ on $X'$. Thus (2.3.1) holds outside of the base locus $B$ of $E$. Let $\tau : A \to A$ denote translation by an element $a \in A$. Then for $P \notin \tau^{-1}(B)$,

$$h_{L'}(\theta(P)) \ll h_{\tau_*L'}(\theta(P)) + O(1)$$

$$= h_{L'}(\tau(\theta(P))) + O(1)$$

$$\ll h_{L'}(\tau(P)) + O(1)$$

$$= h_{\tau_*L}(P) + O(1)$$

$$\ll h_L(P) + O(1).$$

Here the first and last steps follow from ([V 1], 1.2.9d), since $\tau^*L$ is algebraically equivalent to $L$ on $X$. The proposition then follows by applying (2.3.2) to elements $a_1, \ldots, a_r \in A$ chosen such that the corresponding translations $\tau_1, \ldots, \tau_r$ satisfy

$$\bigcap_{i=1}^r \tau_i^{-1}(B \cap A) = \emptyset.$$  

\[ \]

Theorem 2.4. Let $A_0$ be a smooth projective variety, let $\mathcal{M}_1, \ldots, \mathcal{M}_\mu \in \text{Pic}^0 A_0$, and let

$$A = \mathbb{P}'(\mathcal{O}_{A_0} \oplus \mathcal{M}_1) \times_{A_0} \cdots \times_{A_0} \mathbb{P}'(\mathcal{O}_{A_0} \oplus \mathcal{M}_\mu).$$

Let $D$ be an effective divisor on $A$. Assume that $B(\mathbb{G}_m^\mu, D) = 0$ (where $\mathbb{G}_m^\mu$ acts on fibers of $A$ over $A_0$ in the obvious manner). Then there exists an equivariant completion $\overline{G}$ of $\mathbb{G}_m^\mu$ with corresponding completion $\overline{A}$ of $A$ (as in Lemma 2.2) and a projective birational morphism $\pi_0 : \overline{A}_0 \to A_0$, satisfying the following conditions.

1. Let $\overline{A} = \overline{A} \times_{A_0} \overline{A}_0$ and let $\pi : \overline{A} \to \overline{A}$ be the canonical projection onto the first factor. Then there exists a Cartier divisor $\overline{D}$ on $\overline{A}$ such that $(\pi_*\overline{D}) \cap A = D$, where $\pi_*$ refers to $\overline{D}$ as a Weil divisor. Moreover, $\overline{D}$ is ample on fibers of the map $\overline{\rho} : \overline{A} \to A_0$.

2. The support of $\overline{D}$ does not contain $\pi^{-1}(T)$ for any subset $T \subseteq \overline{A}$ corresponding to a $G$-orbit of $\overline{G}$.

3. Let $E$ be the exceptional set for $\pi_0 : \overline{A}_0 \to A_0$ and let $\overline{\rho} : \overline{A} \to \overline{A}_0$ be the map corresponding to $\overline{\rho} : \overline{A} \to A_0$. Then there exists a Cartier divisor $\overline{D}$ on
\( \tilde{A} \) and a divisor \( F \) on \( \tilde{A} \) supported only on \( \tilde{\rho}^{-1}(E) \) satisfying the numerical equivalence

\[
\pi^* \mathcal{D} \equiv \tilde{D} + F.
\]

(4). The set \( \rho^{-1}(\pi_0(E)) \) is contained in \( \text{Supp} \ D \).

(5). Finally, \( \mathcal{G} \) has only finitely many \( G \)-orbits.

Moreover, if \( A_0 \) is abelian (and \( A \) is semiabelian) and if \( D \) has trivial Ueno fibration, then \( \mathcal{D} \) is ample.

Proof. We first consider the case \( A = \mathbb{G}_m^\mu \). In this case \( \pi \) is an isomorphism, (4) is vacuous, and conditions (1) and (3) are equivalent to the closure \( \overline{\mathcal{D}} \) of \( D \) in \( \overline{A} \) being Cartier and ample.

Let \( f \in k[x_1, x_1^{-1}, \ldots, x_\mu, x_\mu^{-1}] \) be a Laurent polynomial such that \( D = (f) \).

Let \( M = \mathbb{Z}^\mu \), and for \( m = (m_1, \ldots, m_\mu) \in M \) let \( x^m \) denote \( x_1^{m_1} \cdots x_\mu^{m_\mu} \). Write \( f = \sum c_m x^m \) and let \( \Delta = \Delta_f \) be the convex hull of \( \{ m \in M \mid c_m \neq 0 \} \) in \( M_\mathbb{R} := M \otimes_{\mathbb{Z}} \mathbb{R} \). This is a polyhedron called the Newton polyhedron. Its vertices are lattice points. Let \( m^{(0)}, \ldots, m^{(\ell)} \) be the lattice points in \( \Delta \). For vertices \( m \) of \( \Delta \), let \( \sigma_m \subseteq M_\mathbb{R} \) be the cone

\[
\left\{ \sum_{i=0}^\ell \lambda_i (m^{(i)} - m) \mid \lambda_i \geq 0 \ \forall \ i \right\}.
\]

After replacing \( \Delta \) with a positive integral multiple \( n\Delta \), we may assume that for all vertices \( m \) of \( \Delta \), the set \( \{ m^{(i)} - m \mid i = 0, \ldots, \ell \} \) generates the monoid \( \sigma_m \cap M \) (see Gordan’s lemma, ([TE], p. 7)). This multiple \( n\Delta \) corresponds to \( f^n \), which corresponds to \( nD \).

Following ([O], Sect. 1), we have a morphism \( \phi: \mathbb{G}_m^\mu \rightarrow \mathbb{P}^\ell \) defined by

\[
(t_1, \ldots, t_\mu) \mapsto [t_1^{m^{(0)}} : \cdots : t_\mu^{m^{(\ell)}}].
\]

Since \( D \) has trivial Ueno fibration, the Newton polyhedron does not lie in any hyperplane of \( M_\mathbb{R} \); therefore this map is actually an embedding. Let \( \overline{\mathcal{G}} \) be the closure of the image of \( \phi \). Then \( \overline{\mathcal{G}} \) is a toric variety (for definitions see [TE] or [D]). In particular it is an equivariant completion of \( \mathbb{G}_m^\mu \) with only finitely many orbits. These orbits are in incidence-preserving one-to-one correspondence with the faces of \( \Delta \). Finally, by ([TE], p. 6 Thm. 1), \( \overline{\mathcal{G}} \) is normal.
Lemma 2.4.2. Let $D$ be a divisor on $\mathbb{G}_m^n$ with trivial Ueno fibration. Let $\overline{G}$ be the toric variety corresponding to a suitably large multiple of the Newton polyhedron $\Delta$ of the defining equation $f$ of $D$. Then the closure $\overline{D}$ of $D$ in $\overline{G}$ is Cartier and ample, and its support does not contain any $\mathbb{G}_m^n$-orbit of $\overline{G}$.

Proof. The polynomial $f$ defines a global section of $\mathcal{O}(1)$ on $\mathbb{P}^\ell$ by $s = c_0x_0 + \cdots + c_\ell x_\ell$, where $f = \sum c_i x_m^{(i)}$. Given a face $\delta$ of $\Delta$, the closure of the associated orbit in $\overline{G}$ is determined by the vanishing of all $x_i$ for which $m^{(i)} \notin \delta$. By definition of $\delta$, there is an index $i$ for which $c_i \neq 0$ and $m_i \in \delta$; therefore $s$ does not vanish identically on the orbit associated to $\delta$. Thus $(s) = (\overline{D})$, which is therefore ample. □

The lemma implies conditions (1)–(4), thus proving the case $A = \mathbb{G}_m^n$. Before proving the general case, some more lemmas are needed.

Lemma 2.4.3. Let $\overline{G}$ be a completion of $\mathbb{G}_m^n$ as described above, and let $V$, $W$, and $\overline{W}$ be as in Lemma 2.2. Recall that $\text{Num} X$ denotes the numerical equivalence class group of a variety $X$. Let $i_1: \overline{G} \to \overline{W}$ be a closed fiber of $\rho: \overline{W} \to V$ (equivariant under the action of $\mathbb{G}_m^n$), and let $i_2: V \to \overline{W}$ be a section of $\rho$ associated to a zero-dimensional orbit of $\overline{G}$. Then

(a). the map

$$(i_1^*, i_2^*): \text{Num} \overline{W} \to \text{Num} \overline{G} \times \text{Num} V$$

is an isomorphism, and is independent of the choice of $i_1$ and $i_2$;

(b). every closed integral curve on $\overline{W}$ is numerically equivalent to an effective sum of curves in the images of $i_1$ and $i_2$; and

(c). a divisor $D$ on $\overline{W}$ is ample if and only if the divisors $i_1^* D \in \text{Pic} \overline{G}$ and $i_2^* D \in \text{Pic} V$ are ample.

Proof. First consider part (b). We start by claiming that every closed integral curve on $\overline{W}$ is numerically equivalent to an effective sum of curves in fibers of $\rho$ and curves in sections of $\rho$ associated to zero-dimensional orbits of $\overline{G}$. Let $C$ be a closed integral curve in $\overline{W}$ and let $\chi: \mathbb{G}_m \to \mathbb{G}_m^n$ be a nontrivial one-parameter subgroup. Translations of $C$ by $\chi(a)$ for $a \in \mathbb{G}_m$ define a surface $Y \subseteq \overline{W} \times \mathbb{G}_m$ with $Y \cap (\overline{W} \times \{1\}) = C$. Let $\overline{Y}$ be the closure of $Y$ in $\overline{W} \times \mathbb{P}^1$, and let $Y_0 = \overline{Y} \cap (\overline{W} \times \{0\})$. Then $Y_0$ is a sum of curves in $\overline{W}$ which is numerically equivalent to $C$ and is invariant under translations by $\chi$. Thus each irreducible component either lies in $\overline{W} \setminus W$ or lies in a fiber of $\rho: \overline{W} \to V$. The claim then follows by induction on dimension.

Given any two fibers of $\rho$, an effective sum of curves in one fiber is numerically equivalent to an effective sum of curves in the other. Hence, in the above claim, the curves in fibers of $\rho$ can be assumed to lie in the image of $i_1$.

To prove (b), it remains to show that the curves in sections of $\rho$, as above, can be taken to lie in the image of $i_2$. To show this, it suffices to show that for any two sections $\sigma_1$ and $\sigma_2$ of $\rho$ as above and any closed integral curve $C \subseteq V$, $\sigma_1(C)$ is algebraically equivalent to $\sigma_2(C)$. Let $\Gamma$ be the graph whose vertices are zero-dimensional
orbits of $\overline{G}$ and whose edges are one-dimensional orbits. Since $\Gamma$ is connected, it suffices to consider sections $\sigma_1$, $\sigma_2$ contained in the subset of $\overline{W}$ corresponding to the closure of a one-dimensional orbit in $\overline{G}$. In this case it is easy to show explicitly that $\sigma_1(C)$ and $\sigma_2(C)$ are algebraically equivalent, since this subset of $\overline{W}$ is isomorphic to $\mathbb{P}(\mathcal{O}_V \oplus \mathcal{M})$ for some $\mathcal{M} \in \text{Pic}^0 V$. Thus (b) holds. Moreover, $i_2^*: \text{Num} \overline{W} \to \text{Num} V$ is independent of the choice of $i_2$.

Next consider the map $i_1^*: \text{Pic} \overline{W} \to \text{Pic} \overline{G}$.

Since $\overline{G}$ has trivial Albanese, this map is independent of the fiber chosen. Since divisors on $\mathbb{G}_m^\mu$ are all principal, it follows that $\text{Pic} \overline{G}$ is generated by the closures of the orbits of codimension one; hence $i_1^*: \text{Pic} \overline{W} \to \text{Pic} \overline{G}$ is surjective. By the Seesaw theorem ([Mi], Thm. 5.1), the kernel is $\text{Pic} V$. Thus there is an exact sequence

$$0 \to \text{Pic} V \to \text{Pic} \overline{W} \to \text{Pic} \overline{G} \to 0.$$ 

The map $i_1^*: \text{Pic} \overline{W} \to \text{Pic} V$ splits this sequence. (This splitting depends on the choice of $i_2$.)

By part (b), an element in $\text{Pic} \overline{W}$ is numerically equivalent to zero if and only if its components in $\text{Pic} V$ and $\text{Pic} \overline{G}$ are both numerically equivalent to zero. Hence there is an exact sequence

$$0 \to \text{Num} V \to \text{Num} \overline{W} \to \text{Num} \overline{G} \to 0,$$

which is again split. In this case, though, the splitting is independent of the choice of zero-dimensional orbit, by the argument using $\Gamma$.

Since $i_1$ and $i_2$ are closed immersions, it follows that if $D$ is an ample divisor on $\overline{W}$ then its components in $\text{Num} \overline{G}$ and $\text{Num} V$ must also be ample. The converse follows from part (b) and from Kleiman’s criterion for ampleness. \hfill \Box

**Lemma 2.4.4.** Let $A$ be a semiabelian variety with $\mu = 1$, let $\overline{A}$ be the completion of $A$ associated to the (unique) completion $\mathbb{G}_m^\mu \subseteq \mathbb{P}^1$, and let $\overline{\rho}: \overline{A} \to A_0$ be the extension of $\rho: A \to A_0$ as in (0.1). Let $\sigma: A_0 \to \overline{A}$ and $\sigma': A_0 \to \overline{A}$ be the sections of $\overline{\rho}$ corresponding to $0 \in \mathbb{P}^1$ and $\infty \in \mathbb{P}^1$, respectively. Let $D$ be a closed subset of $\overline{A}$ of pure codimension one. Assume that $D$ meets the generic fiber of $\rho$, but that it does not contain the image of $\sigma$ or $\sigma'$. Then (in the notation of Definition 1.2) there exists an abelian subvariety $C$ of $A$ such that $\rho(C) = B(\sigma^{-1}(D))$.

**Proof.** By replacing $A$ with a translate of $\rho^{-1}(B(\sigma^{-1}(D)))$, we may assume that $B(\sigma^{-1}(D)) = A_0$. We may also assume that $D$ is irreducible (discard all but one suitable irreducible component). Then $D$ does not meet the image of $\sigma$; since the images of $\sigma$ and $\sigma'$ are algebraically equivalent, it does not meet the image of $\sigma'$.\hfill \Box
either. Thus $D \subseteq A$. Recall that $\overline{A} \cong \mathbb{P}(\mathcal{O}_{A_0} \oplus \mathcal{M})$ for some line sheaf $\mathcal{M} \in \text{Pic}^0 A_0$. Let $Y$ be some normal projective variety admitting a birational morphism $i: Y \to D$. Let $E = \overline{A} \times_{A_0} Y$ and $\mathcal{N} = i^* \rho^* \mathcal{M}$, so that $E = \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{N})$. This contains a divisor $D'$ (equal to the image of the map $i \times_{A_0} \text{Id}_Y$) which has degree 1 over $Y$ and does not meet the divisors on $E$ corresponding to the projections of $\mathcal{O}_Y \oplus \mathcal{N}$ onto either of its direct summands. Therefore the surjection $\mathcal{O}_Y \oplus \mathcal{N} \to \mathcal{L}$ corresponding to $D'$ is an isomorphism on each direct summand; thus $\mathcal{N} \cong \mathcal{O}_Y$. Therefore $\mathcal{M}$ lies in the kernel of $\text{Pic}^0 A_0 \to \text{Pic}^0 Y$.

But since the Albanese of $Y$ maps onto $A_0$, the above map on $\text{Pic}^0$ must be finite; hence $\mathcal{M}$ is torsion in $\text{Pic}^0 A_0$, of order dividing the degree of $\rho_{\mid D}$. It then follows that $D$ is a translated subgroup of $A$. Let $C$ be that subgroup. □

Lemma 2.4.5. Let $\overline{G}$ be the completion of $G^\mu_m$ associated to some polyhedron $\Delta$ and let $\overline{A}$ be the corresponding completion of $A$ (Lemma 2.2). Let $\sigma: A_0 \to \overline{A}$ be the section of $\overline{\rho}: \overline{A} \to A_0$ corresponding to some zero-dimensional orbit of $\overline{G}$. Let $D$ be a closed subset of $\overline{A}$ of pure codimension 1. Assume that $D$ meets the generic fiber of $\overline{\rho}$ on all subsets of $\overline{A}$ corresponding to closures of positive dimensional orbits of $\overline{G}$, but that $D$ does not contain any subset of $\overline{A}$ corresponding to a zero dimensional orbit. Then

$$\rho(B(D \cap A)) = B(\sigma^{-1}(D)) .$$

Proof. Let $B = B(\sigma^{-1}(D))$. The section $\sigma$ corresponds to a vertex $m$ of $\Delta$. Let $\{Y_i \mid i \in I\}$ be the set of $A$-orbits of $\overline{A}$ corresponding to edges (i.e., one-dimensional faces) of $\Delta$ incident to $m$. The closures $\overline{Y}_i$ contain the image of $\sigma$. Applying Lemma 2.4.4 to $D \cap \overline{Y}_i$ on each $Y_i$ gives an abelian subvariety mapping onto $B$. Since the edges of $\Delta$ incident to $m$ do not lie in any hyperplane of $M$, this implies that the line sheaves $\mathcal{M}_1, \ldots, \mathcal{M}_\mu$ used in defining $A$ ([V 3], Sect. 2) are torsion when restricted to $B$. Thus there exists an abelian subvariety $C \subseteq A$ for which $\rho(C) = B$.

We now show that $B(D \cap A)$ contains $C$. After replacing $A_0$ with an isogenous abelian variety, we may assume that $C$ has degree 1 over $B$, and that $\mathcal{M}_1, \ldots, \mathcal{M}_\mu$ are trivial on $B$. Then $\mathcal{M}_1, \ldots, \mathcal{M}_\mu$ all descend to line sheaves on $A_0/B$; this then defines a fiber bundle $\overline{A} \to \overline{A}'$ with fiber $B$. Since this morphism has at least one fiber which does not meet $D$, it follows that $B(D \cap A) \supseteq C$. Thus $\rho(B(D \cap A)) \supseteq B$. The opposite inclusion is trivial. □

We now finish the proof of Theorem 2.4. The divisor $D$ on the generic fiber of $\overline{\rho}$ defines a hyperplane in $\mathbb{P}^\ell$ defined over $K(A_0)$, and therefore a rational section $\phi: A_0 \to \mathbb{P}(\mathcal{E}^\vee)$ for some appropriate vector sheaf $\mathcal{E}$ of rank $\ell + 1$ on $A_0$. Let $\tilde{A}_0$ be the closure of the graph of this rational map, and let $\pi_0: \tilde{A}_0 \to A_0$ be the canonical projection. Then $\phi$ extends to a morphism $\tilde{\phi}: \tilde{A}_0 \to \mathbb{P}(\mathcal{E}^\vee)$. Let $\overline{G}$ be the completion of $G^\mu_m$ associated to $D$ on the generic fiber of $\rho$, as above. Let $\overline{A}$ be the associated completion of $A$; this can be naturally identified with a subscheme of $\mathbb{P}(\mathcal{E})$, and therefore $\tilde{\phi}$ defines a Cartier divisor $\tilde{D}_0$ on $\tilde{A} := \overline{A} \times_{A_0} A_0$. Moreover, by
construction no fiber of $\tilde{\rho}$ is contained in $\tilde{D}_0$, so since $\tilde{D}_0$ is of pure codimension one, it follows that all components of $\tilde{D}_0$ map onto $\tilde{A}_0$. Thus $\tilde{D}_0$ is the strict transform of the horizontal (over $A_0$) part of $D$. Add to this the pull-back of the vertical part of $D$ to get $\tilde{D}$. This gives (1), (2), and (5).

Let $\sigma: A_0 \to \overline{A}$ be a section of $\tilde{\rho}$ as in Lemma 2.4.5 and let $\tilde{\sigma}: \tilde{A}_0 \to \tilde{A}$ be the corresponding section of $\tilde{\rho}$. Then $\tilde{D}$ corresponds via Lemma 2.4.3 to the divisor $\tilde{\sigma}^*\tilde{D}$ on $\tilde{A}_0$ and some ample divisor $D_1$ on $\overline{G}$. Form $\pi_0*\tilde{\sigma}^*\tilde{D}$ as a Weil divisor; it is also Cartier since $A_0$ is smooth. Let $\overline{D}$ be a divisor on $\overline{A}$ corresponding to $\pi_0*\tilde{\sigma}^*\tilde{D}$ and $D_1$. By construction, $\pi_0^*(\pi_0*\tilde{\sigma}^*\tilde{D}) - \tilde{\sigma}^*\overline{D}$ is a divisor supported only on $E$. Pulling back to $\overline{A}$ then gives (3).

Moreover, the support of $\pi_0*\tilde{\sigma}^*\tilde{D}$ is just $\sigma^{-1}(\text{Supp }\overline{D})$, so if $A_0$ is an abelian variety and if $D$ has trivial Ueno fibration, then Lemma 2.4.5 implies that $\pi_0*\tilde{\sigma}^*\tilde{D}$ has trivial Ueno fibration. Thus by ([Mu], §6, Application 1, p. 60), $\pi_0*\tilde{\sigma}^*\tilde{D}$ is ample and therefore Lemma 2.4.3 implies that $\overline{D}$ is ample.

We now show (4). Zariski’s Main Theorem ([Ha], III 11.4) and its proof imply that there is a Zariski-open subset $U$ of $A_0$ over which $\pi_0$ is an isomorphism, and the fibers over all $P \notin U$ are positive dimensional. For those $P$, it follows that $\overline{D}$ maps onto $\rho^{-1}(P)$ . This gives (4).

**Proposition 2.5.** Let $\overline{A}$ be the equivariant completion of a semiabelian variety $A$ associated to some Newton polyhedron $\Delta$, and let $T$ be an orbit of $\overline{A}$. Then there exists an open subset $U$ of $\overline{A}$ containing $A$ and $T$, and an equivariant projection $p: U \to T$ whose restriction to $T$ is the identity.

**Proof.** We continue using the notation of the proof of Theorem 2.4.

It suffices to prove the proposition in the case where $A = \mathbb{G}_m^\nu$. Let $\delta$ be the face of $\Delta$ corresponding to $T$. Let $\sigma_\delta$ be the cone in $M_\mathbb{R}$ generated by elements $m'-m$ with $m' \in \Delta$ and $m \in \delta$. Then the set

$$U := \text{Spec } k[\sigma_\delta \cap M]$$

is an open affine subset of $\overline{A}$. (Here $k[\sigma_\delta \cap M]$ is a monoid ring.) Let $\tau_\delta$ be the largest subgroup of $\sigma_\delta$. Then $\text{Spec } k[\tau_\delta \cap M] = T$, and the injection $T \hookrightarrow U$ corresponds to the surjection $k[\sigma_\delta \cap M] \twoheadrightarrow k[\tau_\delta \cap M]$ defined by $x^m \mapsto 0$ for all $m \notin \tau_\delta$.

We then let $p$ be the morphism corresponding to the canonical injection $k[\tau_\delta \cap M] \hookrightarrow k[\sigma_\delta \cap M]$. □

§3. The main approximation result

This section proves the main approximation result used in the proof of Theorem 0.2. First we recall a standard definition and a definition from ([V 3], 7.1).

**Definition 3.1.** A line sheaf or Cartier divisor $L$ on a complete variety $X$ is nef (“numerically effective”) if $(L \cdot C) \geq 0$ for all integral curves $C$ on $X$.

**Definition 3.2.** Let $X/k$ be a variety. A generalized Weil function is a function $g: \coprod U(k_v) \to \mathbb{R}$, where $U$ is a nonempty Zariski-open subset of $X$, such that
there exists a proper birational morphism $\Phi: X^* \to X$ such that $g \circ \Phi$ extends to a Weil function for some divisor $D^*$ on $X^*$. It is called effective if $D^*$ is an effective divisor. The support of $g$, written $\text{Supp}_g$, is defined to be the set $\Phi(\text{Supp} D^*)$. N.B.: This is not the set where $g \neq 0$ (the analysts’ definition of support).

**Definition 3.3.** Let $Y$ be a proper closed subset of a variety $X/k$. Then a logarithmic distance function for $Y$ is an effective generalized Weil function $g$ on $X$ such that, if $\Phi: X^* \to X$ is as in Definition 3.2, then the divisor $D^*$ (as above) has $\text{Supp} D^* = \Phi^{-1}(Y)$. Note that this is not really minus the logarithm of the distance to $Y$, especially near singularities, but we do have the following easy fact.

**Lemma 3.4.** Let $Y$ be a proper closed subset of a complete variety $X/k$, let $\phi: X' \to X$ be a proper birational morphism, and let $Y' = \phi^{-1}(Y)$. Let $g$ and $g'$ be logarithmic distance functions for $Y$ and $Y'$, respectively. Then

$$g + O(1) \gg \ll g' + O(1)$$

Moreover, $O(1)$ refers to $M_k$-constants, as in ([L], Ch. 10, §1).

**Proof.** We may assume that $X^*$ is the same for $g$ and $g'$; let $D$ and $D'$ be the divisors on $X^*$ associated to $g$ and $g'$, respectively. Then since $\text{Supp} D = \text{Supp} D'$, it follows that $nD - D'$ and $n'D' - D$ are effective for sufficiently large $n$ and $n'$. This implies the lemma. \qed

For future reference, we also note that if $X$ is an equivariant completion of $A$, if $\lambda_\infty$ is a logarithmic distance function for $X \setminus A$, if $X \setminus A = \bigcup T_i$, where $T_i$ are finitely many subsets of $X$ (e.g., orbits), and if $\lambda_i$ are logarithmic distance functions for $\overline{T_i}$ on $X$ for all $i$, then $\sum \lambda_i$ is a logarithmic distance function for $X \setminus A$ and therefore

$$\lambda_\infty + O(1) \gg \ll \sum \lambda_i + O(1).$$

Here again $O(1)$ refers to $M_k$-constants.

**Theorem 3.6.** Let $X$ be an equivariant completion of a semiabelian variety $A$. Let $h_L$ be a height function on $X$ with respect to an ample line sheaf $L$. Let $\lambda_w$ be a generalized Weil function on $X$ at a place $w \in S$, and let $\lambda_\infty,w$ be a logarithmic distance function for $X \setminus A$ on $X$. Then there do not exist a real number $\kappa > 0$ and a subset $\mathcal{S} \subseteq A(R_S)$ satisfying the conditions (1)

$$\lambda_w(P) \geq \kappa h_L(P)$$

for all $P \in \mathcal{S}$; and (2) for all $\eta > 0$ the inequality

$$\lambda_\infty,w(P) \leq \eta h_L(P)$$
holds for infinitely many \( P \in \mathcal{I} \).

\textbf{Proof.} First we claim that the theorem is independent of the choice of completion of \( A \). Indeed, suppose \( X' \) is another completion, with ample line sheaf \( L' \) and height function \( h_{L'} \). Without loss of generality we may assume that \( X' \) dominates \( X \) via \( \phi: X' \to X \). By Proposition 2.3,

\begin{equation}
(3.6.3) \quad h_L(P) \gg h_{L'}(P)
\end{equation}

for almost all \( P \in \mathcal{I} \). Moreover \( \lambda_w \) is also a generalized Weil function on \( X' \). Thus (3.6.1) holds for \( X \) if and only if it holds for \( X' \) (with a different \( \kappa \)). Also, Lemma 3.4 and (3.6.3) imply that (3.6.2) holds for \( X \) if and only if it holds for \( X' \) (with a different \( \eta \)).

Thus we may assume that \( X \) is the equivariant completion associated to the injection \( \mathbb{G}_m^\mu \hookrightarrow (\mathbb{P}^1)^\mu \). Following [V 3], we will denote \( X \) by \( \overline{A} \) from now on. We may assume that \( \lambda_w \) is effective. Every generalized Weil function is dominated by a Weil function (of a divisor on \( \overline{A} \)), so we may also assume that \( \lambda_w = \lambda_{D,w} \) is a Weil function for an effective divisor \( D \) on \( \overline{A} \).

The basic idea of the remainder of the proof is to incorporate the extra machinery of [V 3] into the argument of ([F], Sect. 6). As in [V 3], we let \( L_0 \) be an ample symmetric divisor on \( A_0 \), and let \( L_1 = \overline{A} \setminus \overline{A} \) (taking all components with multiplicity one). Then, by basic properties of height functions, we may assume that

\[ L = \rho^*L_0 + L_1 \]

Unlike [F], it is not necessary here to assume that \( D \) is ample; instead, let \( \ell \) be a positive integer such that \( \ell L - D \) is ample.

Let \( \delta > 0 \) be a rational number, and choose a positive rational \( \epsilon < 1 \) and a positive integer \( n \) satisfying

\begin{equation}
(3.6.4) \quad n\epsilon < \frac{\kappa \delta}{[k: \mathbb{Q}]}
\end{equation}

and

\begin{equation}
(3.6.5) \quad \frac{2\delta^n}{n!} < \frac{\epsilon^{\dim A}}{(5\dim A \ell \dim A)^n}.
\end{equation}

As in ([V 3], Sect. 3), let \( s = (s_1, \ldots, s_n) \) be a tuple of positive integers. For integers \( i \) and \( j \) in \( \{1, \ldots, n\} \) let \( s_i \cdot \text{pr}_i - s_j \cdot \text{pr}_j \) denote the morphism from \( A^n \) to \( A \) defined using the group law. Also as in [V 3], given any product, \( \text{pr}_i \) denotes the projection morphism from that product to its \( i^{\text{th}} \) factor.

For closed subvarieties \( X_1, \ldots, X_n \) of \( A \), let \( \overline{X}_1, \ldots, \overline{X}_n \) denote their respective closures in \( \overline{A} \). Let \( \psi_s: \prod \overline{X}_i \to \overline{A}^{n-1} \) be the rational map whose components are the
restrictions of \( s_i \cdot \text{pr}_i - s_{i+1} \cdot \text{pr}_{i+1} \) as \( i \) varies from 1 to \( n-1 \). Let \( W_s \) be the closure of the graph of this rational map, and let \( \pi_s: W_s \to \prod X_i \) be the natural projection.

For \( n \)-tuples \( s \) of positive integers and for rational \( \epsilon \) we define
\[
L_{\epsilon,s} = \sum_{i=2}^{n} (s_{i-1} \cdot \text{pr}_{i-1} - s_i \cdot \text{pr}_i)^* \rho^* L_0 + \sum_{i=2}^{n} (s_{i-1}^2 \cdot \text{pr}_{i-1} - s_i^2 \cdot \text{pr}_i)^* L_1 + \epsilon \sum_{i=1}^{n} s_i^2 \text{pr}_i^* L
\]
as a \( \mathbb{Q} \)-divisor class on \( W_s \) and
\[
M_{\epsilon,s} = \sum_{i=2}^{n} (s_{i-1} \cdot \text{pr}_{i-1} - s_i \cdot \text{pr}_i)^* \rho^* L_0
\]
\[+ s_i^2 \text{pr}_i^* L_1 + 2 \sum_{i=2}^{n-1} s_i^2 \text{pr}_i^* L_1 + s_n^2 \text{pr}_n^* L_1 + \epsilon \sum_{i=1}^{n} s_i^2 \text{pr}_i^* L \]
as a \( \mathbb{Q} \)-divisor class on \( \prod X_i \). Note that these differ from their counterparts in \([V \; 3]\): the first two sums are taken over a smaller set of pairs of indices, in line with \([F]\). Adding all pairs of indices is possible, but more complicated.

As in \([V \; 3]\), these definitions extend by homogeneity to tuples \( s \) of positive rational numbers: let \( a \) be the lowest common denominator and let \( W_s = W_{as}, \pi_s = \pi_{as}, L_{\epsilon,s} = a^{-2} L_{\epsilon,as} \), and \( M_{\epsilon,s} = a^{-2} M_{\epsilon,as} \).

The natural injection of \([V \; 3], 3.6)\) carries over to this case: for any positive integer \( d \) canceling all denominators of \( s \) and \( \epsilon \), we have
\[
\mathcal{O}(dL_{\epsilon,s}) \hookrightarrow \mathcal{O}(dM_{\epsilon,s}).
\]

With these choices, the overall strategy is the same as in \([V \; 3], \text{Sect. 4}\), except that there is no set \( Z \). As in \textit{op. cit.}, we let \( h(X_i) \) denote the height of the closed subvariety \( X_i \) of \( A \), taken relative to the ample line sheaf \( L \). We omit the subscript \( L \) since heights of subvarieties will not be taken relative to any other line sheaf (for points, however, we will retain the subscript since heights of points will be taken relative to other line sheaves).

The strategy is to choose \( P_1, \ldots, P_n \in \mathcal{S} \) satisfying the following conditions:

\[(3.6.9.1)\] \( h_L(P_1) \geq c_1 \).
\[(3.6.9.2)\] \( h_L(P_{i+1})/h_L(P_i) \geq c_2 \geq 1 \) for all \( i = 1, \ldots, n-1 \).
\[(3.6.9.3)\] \( P_1, \ldots, P_n \) all point in roughly the same direction in \( A(R_S) \otimes \mathbb{Z} \mathbb{R} \), up to a factor \( 1 - \epsilon_1 \): see \([V \; 3], 13.2 \) and \( 13.3 \).

The main part of the proof involves closed subvarieties \( X_1, \ldots, X_n \) of \( A \). We start with \( X_1 = \cdots = X_n = A \) and successively find collections with \( \sum \dim X_i \) strictly smaller. At each stage, \( X_1, \ldots, X_n \) are assumed to satisfy the following conditions:

\[(3.6.10.1)\] Each \( X_i \) contains \( P_i \).
Each $X_i$ is geometrically irreducible and defined over $k$.

The degrees $\deg X_i$ satisfy $\deg X_i \leq c_3$.

The heights $h(X_i)$ will be bounded by the formula

$$\sum_{i=1}^{n} \frac{h(X_i)}{h_L(P_i)} \leq c_4 \sum_{i=1}^{n} \frac{1}{h_L(P_i)}.$$ 

Eventually, this inductive process reaches the point where some $X_j$ is zero dimensional; i.e., $X_j = P_j$. When that happens, by (3.6.10.4),

$$1 = \frac{h(X_j)}{h_L(P_j)} \leq \frac{\sum_{i=1}^{n} h(X_i)}{\sum_{i=1}^{n} h_L(P_i)} \leq c_4 \frac{\sum_{i=1}^{n} 1}{h_L(P_1)} \leq c_4 n.$$ 

This contradicts (3.6.9.1) if $c_1 > c_4 n$.

For $i = 1,\ldots,n$ let $s_i$ be rational numbers close to $1/\sqrt{h_L(P_i)}$ and let $d$ be a large sufficiently divisible integer. Let $\mathcal{F}$ be the model for $\prod X_i$ constructed in ([V 3], Sect. 10); briefly, this is a model just large enough to extend the definition of $M_{\epsilon,s}$. Let $\Gamma_d(\prod X_i, dM_{\epsilon,s})$ denote the submodule of $\Gamma(\prod X_i, dM_{\epsilon,s})$ consisting of sections having index $\geq \delta$ along $D \times \cdots \times D$, relative to multiplicities $ds_1^2,\ldots,ds_n^2$. Also let $h^0_d(\prod X_i, dM_{\epsilon,s})$ denote the rank of this module. We start by obtaining two estimates for some ranks. Except for notation, this follows [F].

**Lemma 3.6.11.** If $d$ is sufficiently divisible, then

$$(3.6.11.1) \quad h^0(W_s, dL_{\epsilon,s}) \geq \epsilon^{\dim X_1} \prod_{i} (ds_i^2)^{\dim X_i} \frac{\deg X_i}{(\dim X_i)!} - o(d^{\Sigma \dim X_i}),$$

where the implicit constant in $o(\cdot)$ is independent of $d$.
Proof. By homogeneity, we may assume that $s_1, \ldots, s_n$ are all integers. We have

$$(3.6.11.2) \left( s_1^2 \rho^* L \right)^{\dim X_1}$$

\[
\cdot \prod_{i=2}^{n} \left( (s_{i-1} \cdot \rho^* L_0 + (s_1^2 \cdot \rho^* L_1) \right)^{\dim X_i}
\]

$$= s_1^2 \dim X_1 \left( \left( s_2 \rho^* L \right)^{\dim X_2} \right)$$

\[
\cdot \prod_{i=3}^{n} \left( (s_{i-1} \cdot \rho^* L_0 + (s_1^2 \cdot \rho^* L_1) \right)^{\dim X_i}
\]

$$= \cdots = \prod_{i=1}^{n} s_i^{2 \dim X_i} \left( \left( \dim X_i \right) \right).$$

Since each of the terms $(s_{i-1} \cdot \rho^* L_0 + (s_1^2 \cdot \rho^* L_1)$ and $s_1^2 \rho^* L$ in the definition (3.6.12) of $L_{c,s}$ is nef, (3.6.11.2) gives

$$(3.6.11.3) \left( L_{\sum_i \dim X_i} \right) \geq \epsilon^{\dim X_1} \left( \sum_{i=1}^{n} \dim X_i \right) \prod_{i=1}^{n} s_i^{2 \dim X_i} \left( \left( \dim X_i \right) \right).$$

Here the symbol in parentheses on the right denotes a multinomial coefficient. By ([V 3], Lemma 6.1), $L_{c,s}$ is ample. Hence, if $d$ is sufficiently large then all the higher cohomology groups vanish, giving

$$h^0(W_s, dL_{c,s}) = \frac{(dL_{c,s})^{\sum_i \dim X_i}}{(\sum_i \dim X_i)!} + o(d^{\sum_i \dim X_i}).$$

Combining this with (3.6.11.3) gives (3.6.11.1). □

Lemma 3.6.12. Let $\Gamma(\mathcal{Y}, dM_{c,s})$, $\Gamma(W_s, dL_{c,s})$, and $\Gamma_0(\prod_i X_i, dM_{c,s})$ be identified with submodules of $\Gamma(W_s, d\pi^*_s M_{c,s})$ via $\pi^*_s$ and (3.6.8). Then there is a constant $c > 0$, depending only on $A$, $D$, $L$, $\ell$, $\epsilon$, $\dim X_1, \ldots, \dim X_n$, and the bounds on $\deg X_1, \ldots, \deg X_n$, such that if $d$ is sufficiently large and divisible then the rank of the $R$-module

$$(3.6.12.1) \Gamma(\mathcal{Y}, dM_{c,s}) \cap \Gamma(W_s, dL_{c,s}) \cap \Gamma_0(\prod_i X_i, dM_{c,s})$$

is bounded from below by $c h^0(W_s, dL_{c,s})$.

Proof. Let $Y_i = D \cap X_i$. We first show that the upper bound

$$(3.6.12.2) h^0(\prod_i X_i, dM_{c,s}) - h^0(\prod_i X_i, dM_{c,s}) \leq \frac{\delta^n}{n!} \prod_i ds_i^2 \cdot \prod_i (5ds_i^2)^{\dim Y_i} \cdot \frac{\deg Y_i}{(\dim Y_i)!}$$


Theorem 3.6.14. For all tuples \( (s_1, \ldots, s_n) \) of positive rational numbers and for all sufficiently large (and divisible) \( d \) (depending on \( s \)), there exists a section \( \gamma \in \Gamma(Y, dM_{\varepsilon, s}) \) such that \( \|\gamma\|' \) and \( \|\gamma\|'' \) are bounded and such that the inequality

\[
\prod_{v \mid \infty} \|\gamma\|_{\sup, v} \leq \exp\left( c d \sum_{i=1}^{n} s_i^2 \right)
\]

holds. Let \( L' = (4 + \epsilon)\rho^*L_0 + (2 + \epsilon)L_1 \leq 5L \). As noted in [F], it suffices to prove the inequality

\[
h^0 \left( \prod Y_i, \sum ds_i^2 pr_i^* L' - \sum e_i pr_i^* D \right) \leq h^0 \left( \prod Y_i, \sum ds_i^2 pr_i^* L' \right)
\]

for all tuples \( (e_1, \ldots, e_n) \in \mathbb{N}^n \) satisfying \( e_1/ds_1^2 + \cdots + e_n/ds_n^2 \leq \delta \). (Here \( \mathbb{N} = \{0, 1, 2, \ldots\} \)) This follows from the facts that a translate of \( D \) is algebraically equivalent to an effective divisor on \( Y_i \), and that \( L' \) is ample.

Then, (3.6.5), (3.6.11.1), (3.6.12.2), and the inequality \( \deg Y_i \leq \ell \deg \mathcal{X}_i \) imply that the rank of the module (3.6.12.1) is bounded from below by \( ch^0(W_s, dL_{\varepsilon, s}) \) for some \( c > 0 \).

Next we put three metrics on \( \mathcal{O}(M_{\varepsilon, s}) \). First, fix a metric on \( \mathcal{O}(L_0) \) whose curvature is translation invariant. For \( m = 1, \ldots, \mu \) let \([0]_m \) and \([\infty]_m \) be the divisors on \( \mathcal{A} \) corresponding to the divisors \( pr_m^*[0] \) and \( pr_m^*[\infty] \), respectively, on \( (\mathbb{P}^1)^\mu \). Then \( L_1 = \sum_{m=1}^{\mu} ([0]_m + [\infty]_m) \). By (IV 3), Prop. 2.6) at all places \( v \) there are Weil functions \( \lambda_{m,v} \) for \([0]_m - [\infty]_m \) satisfying

\[
\lambda_{m,v}(P + Q) = \lambda_{m,v}(P) + \lambda_{m,v}(Q) \quad \text{for all } P, Q \in A(C_v).
\]

(Here \( C_v \) denotes the completion of the algebraic closure of the completion \( k_v \) of \( k \) at \( v \); it is algebraically closed.) These can be used as in (IV 3, 2.8) to define smooth metrics on \( \mathcal{O}(L_1) \). This defines a smooth metric \( \|\cdot\|' \) on \( \mathcal{O}(M_{\varepsilon, s}) \) via the expression (3.6.7). Next, for each \( v \in S \) let \( \|\cdot\|'' \) be the singular metric constructed in (IV 3, 10.5–10.6), so that \( \|\cdot\|'' \) is equivalent to the metric on \( \mathcal{O}(dL_{\varepsilon, s}) \) via the embedding (3.6.8). Finally, let

\[
\|\gamma\|''' = \frac{\|\gamma\|''}{\sum \exp(-\delta ds_i^2 pr_i^* \lambda_{D,v})}.
\]

If a section has index \( \geq \delta \) at \( D \times \cdots \times D \), then this singular metric remains bounded; one may take this as a definition of the index at \( D \times \cdots \times D \). Note that the singular metrics \( \|\cdot\|' \) and \( \|\cdot\|''' \) are of the form (IV 3, 10.5); therefore (IV 3, Lemma 11.2) applies.

Note also that we take the intersection in (3.6.12.1) instead of combining \( \|\cdot\|' \) and \( \|\cdot\|''' \); this is because of problems at infinity as illustrated in the examples in Sect. 1. This is why (3.6.2) is necessary.

As in (IV 3, Thm. 12.4 and Remark 12.6), we obtain a small section:
holds. Here the constant $c$ is independent of $s$ and $d$.

\textbf{Proof.} This proof is a matter of obtaining bounds for volumes of various lattices in the diagram

\[ 0 \to \Gamma(\mathcal{V}, dM_{\epsilon, s}) \longrightarrow \Gamma\left(\mathcal{V}, d\sum s_i^2 \text{pr}_i^* L'\right)^a \longrightarrow \Gamma\left(\mathcal{V}, d\sum s_i^2 \text{pr}_i^* L''\right)^b \]

\[ \cup \]

\[ \Gamma'(\mathcal{V}, dM_{\epsilon, s}) \longrightarrow \Gamma'(\mathcal{V}, d\sum s_i^2 \text{pr}_i^* L')^a \cup \]

\[ \Gamma''(\mathcal{V}, dM_{\epsilon, s}) \longrightarrow \Gamma''(\mathcal{V}, d\sum s_i^2 \text{pr}_i^* L')^a . \]

This is as in [V 3]: the top row is the Faltings complex, with

\[ L' = 4\rho^*L_0 + 2L_1 + \epsilon L \]

and

\[ L'' = 8\rho^*L_0 + 2L_1 + \epsilon L ; \]

the symbols $\Gamma'$ in the middle row denote the submodules of sections $\gamma$ for which $\|\gamma\|'$ is bounded; and the symbols $\Gamma''$ in the bottom row denote the submodules of sections $\gamma$ for which both $\|\gamma\|$ and $\|\gamma\|''$ are bounded.

The proof is exactly the same as in [V 3], except that the fifth paragraph is repeated because of the extra row in the above diagram. \hfill \Box

By (3.6.13), ([V 3], Prop. 10.10), (3.6.10.4), (3.6.1), and the choice of the $s_i$, we have

\[ \|\gamma(P_1, \ldots, P_n)\|_w = \|\gamma(P_1, \ldots, P_n)\|''_w \sum_{i=1}^n \exp(-\delta ds_i^2 \lambda_{D,w}(P_i)) \]

\[ \leq \|\gamma\|''_{\sup,w} \cdot n \max_{1 \leq i \leq n} \exp(-\delta ds_i^2 \lambda_{D,w}(P_i)) \]

\[ \leq \|\gamma\|_{\sup,w} \exp(-\delta d\kappa) \cdot \exp\left(cd \sum_{i=1}^n s_i^2\right) \]

and therefore

\[ (3.6.15) \quad -\log \|\gamma(P_1, \ldots, P_n)\|_w \geq -\log \|\gamma\|_{\sup,w} + d\kappa \delta - cd \sum_{i=1}^n s_i^2 . \]
Likewise, letting $\alpha_{vmi} = \exp(-pr^*_i \lambda_{m,v})$ and applying \([V 3]\), (10.6) and Prop. 10.10) to $\| \cdot \|$ gives

\[ (3.6.16) \]
\[- \log \| \gamma(\Pi_1, \ldots, \Pi_n) \|_v \geq - \log \| \gamma \|_{\sup,v} + \sum_{m=1}^{\mu} \sum_{i=2}^{n} \log \frac{(ds^2_{m,i-1} + \alpha_{vmi})^2 (1 + \alpha_{vmi})}{(1 + \alpha_{vm,i-1})^2} - cd \sum_{i=1}^{n} s^2_i \]

for all $v \in S$; cf. \([V 3]\), Prop. 12.5). If the $\Pi_i$ are chosen so that \([V 3]\), 13.5) holds, then by \([V 3]\), 13.7) we have

\[ (3.6.17) \]
\[ ds^2_1 h_{L_1}(\Pi_1) + 2 \sum_{i=2}^{n-1} ds^2_i h_{L_1}(\Pi_i) + ds^2_n h_{L_1}(\Pi_n) \leq \frac{1}{[k : \mathbb{Q}]} \sum_{\mu} \sum_{m=1}^{n} \sum_{i=1}^{2} \log \frac{(ds^2_{m,i-1} + \alpha_{vmi})^2}{(1 + \alpha_{vm,i-1})^2 (1 + \alpha_{vmi})^2} \]
\[ + d(n - 1)\epsilon_1 + cd \sum_{i=1}^{n} s^2_i. \]

But also, choosing $\Pi_1, \ldots, \Pi_n$ so that (3.6.2) holds for sufficiently small $\eta$, we have

\[ (3.6.18) \]
\[ \frac{1}{[k : \mathbb{Q}]} \sum_{m=1}^{\mu} \sum_{i=2}^{n} \log \frac{(ds^2_{m,i-1} + \alpha_{wm}^2)^2}{(1 + \alpha_{wm,i-1})^2 (1 + \alpha_{wm}^2)^2} \leq d\epsilon_2 \]

for some $\epsilon_2 > 0$ depending only on $\eta$, $\mu$, $n$, and $[k : \mathbb{Q}]$. Adding (3.6.16) for
\( v \in S \setminus \{ w \} \) to (3.6.15) and applying (3.6.14.1), (3.6.18), and (3.6.17) then gives

\[
\frac{1}{[k : Q]} \sum_{v \in S} - \log \| \gamma(P_1, \ldots, P_n) \|_v
\]

\[
\geq \frac{1}{[k : Q]} \left( \sum_{v \in S \setminus \{ w \}} \sum_{m=1}^{\mu} \sum_{i=2}^{n} - \log \left( \frac{d_{s_{m-1}^2 - 1}^2 \alpha_{wm,i-1} + d_{s_{m1}^2}^2 \alpha_{wm1}}{1 + \alpha_{wm,i-1}^2} \right)^2 + d\kappa \delta \right)
\]

\[
+ \frac{1}{[k : Q]} \sum_{v \in S} - \log \| \gamma \|_{\sup,v} - cd \sum_{i=1}^{n} s_i^2
\]

\[
\geq \frac{1}{[k : Q]} \left( \sum_{v \in S} \sum_{m=1}^{\mu} \sum_{i=2}^{n} - \log \left( \frac{d_{s_{m-1}^2 - 1}^2 \alpha_{wm,i-1} + d_{s_{mi}^2}}{1 + \alpha_{wm,i-1}^2} \right)^2 + d\kappa \delta \right)
\]

\[
- d\epsilon_2 - cd \sum_{i=1}^{n} s_i^2
\]

\[
\geq d \left( \frac{\kappa \delta}{[k : Q]} - (n - 1)\epsilon_1 - \epsilon_2 + s_1^2 h_{L_1}(P_1) + 2 \sum_{i=2}^{n-1} s_i^2 h_{L_1}(P_i) + s_n^2 h_{L_1}(P_n) \right)
\]

\[
- cd \sum_{i=1}^{n} s_i^2
\]

On the other hand, as in ([V 3], 13.6), for suitably chosen \( P_1, \ldots, P_n \), we have

\[
\frac{1}{[k : Q]} \deg M_{\tau,s} \mid E \leq (n - 1)\epsilon_1 + n\epsilon + s_1^2 h_{L_1}(P_1) + 2 \sum_{i=2}^{n-1} s_i^2 h_{L_1}(P_i) + s_n^2 h_{L_1}(P_n) + c \sum_{i=1}^{n} s_i^2
\]

on the arithmetic curve \( E \) corresponding to \( (P_1, \ldots, P_n) \).

Combining these two inequalities gives

\[
\frac{1}{[k : Q]} \sum_{v \not\in S} - \log \| \gamma \|_v \leq d \left( n\epsilon - \frac{\kappa \delta}{[k : Q]} + 2(n - 1)\epsilon_1 + \epsilon_2 \right) + cd \sum_{i=1}^{n} s_i^2
\]

By (3.6.4) this gives a negative upper bound if \( \epsilon_1 \) and \( \epsilon_2 \) are sufficiently small, leading to a positive lower bound for the index of \( \gamma \) at \( (P_1, \ldots, P_n) \) (with multiplicities \( ds_1^2, \ldots, ds_n^2 \)). The argument then concludes by applying the product theorem in the usual way, as in ([V 2], Sect. 18).

\[\square\]

§4. Proof of Theorem 0.2

This section uses the notation given in the introduction of the paper.

First, we may assume that the Ueno fibration is trivial. This is because the theorem is preserved under pulling back by quotient morphisms. We may enlarge \( k \) so that the
toric part of $A$ splits; i.e., the exact sequence (0.1) holds already over $k$. Finally, it will suffice to assume that $\mathcal{V}(R_S)$ is Zariski-dense and obtain a contradiction. To see this, apply Theorem 0.3 to any irreducible component of the closure of $\mathcal{V}(R_S)$ and then proceed by induction on the dimension.

Let $\tilde{A}, \pi: \tilde{A} \to A, \tilde{D}$, and $\tilde{D}$ be as in Theorem 2.4. For $v \in S$ let $\lambda_{\tilde{D}, w}$ be a Weil function for $\tilde{D}$ on $\tilde{A}$. Then there exist $w \in S$ and $\kappa_0 > 0$ such that

$$\lambda_{\tilde{D}, w}(\tilde{P}) \geq \kappa_0 h_{\tilde{D}}(\tilde{P})$$

for all $\tilde{P} \in \tilde{A}$ corresponding to elements $P \in \mathcal{S}_0$, where $\mathcal{S}_0 \subseteq \mathcal{V}(R_S)$ is Zariski-dense in $A$. We now claim that, after shrinking to a possibly smaller $S \subseteq S_0$, the inequality

$$(4.1) \quad \lambda_{\tilde{D}, w}(\tilde{P}) \geq \kappa h_{\mathcal{T}}(P)$$

holds as well. If not, then by (2.4.1) $h_{F}(\tilde{P}) \geq \kappa h_{\mathcal{T}}(P)$ for some $\kappa > 0$. But since $\pi(\text{Supp } F)$ is contained in $\text{Supp } D$, we have $h_{F}(\tilde{P}) = \sum_{v \in S} \lambda_{F,v}(\tilde{P}) + O(1)$ and therefore

$$\lambda_{F,w}(\tilde{P}) \geq \kappa h_{\mathcal{T}}(P)$$

for some $\kappa > 0$ and some $w \in S$ (after shrinking $\mathcal{S}$). Pushing down to $A_0$ then gives an inequality which contradicts Theorem 3.6 in the case where $\mu = 0$ (which is also ([F], Thm. 2)). Thus (4.1) holds.

For $A$-orbits $T \neq A$ of $A$ let $\tilde{T}$ be the closure and let $\lambda_{\mathcal{T}, w}$ be some logarithmic distance function for $\mathcal{T}$. Choose an orbit $T$ of minimal dimension such that

$$\lambda_{\mathcal{T}, w}(P) \geq \eta h_{\mathcal{T}}(P)$$

for some $\eta > 0$ and for all $P$ in some Zariski-dense subset $\mathcal{S}'$ of $\mathcal{S}$. If there is no such orbit then let $T = A$.

Let $p: A \to T$ be the restriction to $A$ of the equivariant projection defined in Proposition 2.5. Let $\tilde{T} = \pi^{-1}(T)$; then $p$ lifts to a morphism $\tilde{p}: \pi^{-1}(A) \to \tilde{T}$. Points $\tilde{p}(\tilde{P})$ for $\tilde{P} \in \tilde{A}$ lying over $P \in \mathcal{S}'$ approach $\tilde{D} \cap \tilde{T}$ as in (3.6.1); for a suitable $\kappa' > 0$, we have

$$(4.2) \quad \lambda_{\tilde{D}, w}(\tilde{p}(\tilde{P})) \geq \kappa' h_{\mathcal{T}}(P).$$

But also, by Proposition 2.3, there is a constant $c_1 > 0$ such that

$$h_{\mathcal{T}}(P) \geq c_1 h_{\mathcal{T}}(p(P)) + O(1)$$

for all $P \in \mathcal{S}'$. We may therefore replace $h_{\mathcal{T}}(P)$ in (4.2) with $h_{\mathcal{T}}(p(P))$. This gives (3.6.1) for $p(\mathcal{S}')$ on $T$, since $\lambda_{\tilde{D}, w}$ is a generalized Weil function on $T$. The condition (3.6.2) also holds, by minimality of the choice of $T$, and by (3.5). This leads to a contradiction, by Theorem 3.6 applied to $\tilde{T}$. \qed
§5. Some additional geometry

The similarity between the conclusions of Theorems 0.2 and 0.3 suggests that some of the results traditionally proved for closed subvarieties of semiabelian varieties could be proved for closed subvarieties minus divisors, too. This section generalizes results of Ueno and Fujita on the logarithmic Kodaira dimension of such varieties.

Unless otherwise specified, all varieties are over an algebraically closed field $k$ of characteristic zero.

Many of these results probably extend to positive characteristic, but additional work would be needed due to the unavailability of resolution of singularities in positive characteristic.

For general references on group varieties, see Rosenlicht \([R]\); for other references on closed subvarieties of semiabelian varieties, see Abramovich \([A]\).

Theorem 5.1. Let $k$ be any field, let $X$ be a complete nonsingular variety over $k$, and let $D$ be a divisor on $X$. Suppose that $H^0(X,mD) \neq 0$ for some $m \in \mathbb{Z}_{>0}$. Let $m_0$ be the index of the subgroup of $\mathbb{Z}$ generated by all such $m$. Then there exist constants $c_2 \geq c_1 > 0$ and an integer $\kappa$ with $0 \leq \kappa \leq \dim X$ such that

$$c_1 m^\kappa \leq h^0(X,mm_0D) \leq c_2 m^\kappa$$

for all sufficiently large $m$.

Proof. See ([I 3], Thm. 10.2). $\square$

Definition 5.2. Let $k$ be any field and let $X$ be a complete nonsingular variety over $k$.

(a). Let $D$ be a divisor on $X$. The $D$-dimension of $X$, denoted $\kappa(X,D)$, is the number $\kappa$ in Theorem 5.1 if $D$ satisfies the conditions of that theorem; otherwise it is $-\infty$.

(b). Let $\mathcal{L}$ be a line sheaf on $X$. Then the $\mathcal{L}$-dimension of $X$, denoted $\kappa(X,\mathcal{L})$, is defined to be $\kappa(X,D)$ for any divisor $D$ on $X$ such that $\mathcal{L} \cong \mathcal{O}(D)$.

(c). A divisor $D$ (resp. line sheaf $\mathcal{L}$) on $X$ is big if $\kappa(X,D)$ (resp. $\kappa(X,\mathcal{L})$) equals $\dim X$.

A divisor $D$ on $X$ is big if and only if $h^0(X,mD) \gg m^{\dim X}$ for all sufficiently large and divisible integers $m$.

Definition 5.3. Let $V$ be a nonsingular quasi-projective variety.

(a). A smooth completion of $V$ is an open immersion $V \hookrightarrow X$ into a nonsingular projective variety $X$ such that $D := X \setminus V$ is a normal crossings divisor (taken with all multiplicities equal to one). Such a smooth completion will often be denoted by a pair $(X,D)$.

(b). The logarithmic canonical divisor of $V$ on $X$ is $K_{(X,D)} := K_X + D$. If $(X,D)$ is clear from the context, then it may be denoted $K_V$ and called simply the logarithmic canonical divisor of $V$. 
The logarithmic Kodaira dimension of $V$ is the number $\kappa(X,K_{(X,D)})$; by ([I3], Thm. 11.2) it depends only on $V$. It is denoted $\overline{\kappa}(V)$.

We say that $V$ is of logarithmic general type if $\overline{\kappa}(V) = \dim V$; i.e., if $K_V$ is big.

Let $\pi : V \to W$ be a morphism to a nonsingular quasi-projective variety $W$, let $V \hookrightarrow X$ and $W \hookrightarrow Y$ be smooth completions such that $\pi$ extends to a morphism $\overline{\pi}: X \to Y$, and let $K_V$ and $K_W$ be the logarithmic canonical divisors of $V$ and $W$, respectively. Then the relative logarithmic canonical divisor of $V$ over $W$ with respect to $\overline{\pi}$ is $K_{V/W} := K_V - \overline{\pi}^*K_W$. Again, mention of $\overline{\pi}$ may be omitted if it is clear from the context.

The first result of this section is that if a closed subvariety of a semiabelian variety, minus a divisor, has trivial Ueno fibration, then it is of logarithmic general type. For closed subvarieties this was proved by Iitaka ([I1] and [I2]); the proof here is an easy adaptation of that proof. Since Iitaka’s exposition often leaves out details, however, we provide a more complete proof here. This also provides the opportunity to change the proof a little, by replacing a cardinality argument with an argument on the field of definition of a group subvariety.

The first step in the proof consists of proving it in the special case when the closed subvariety is the whole semiabelian variety; in other words, a semiabelian minus a divisor $D$ with $B(D) = 0$ has logarithmic general type. In the end of the paper [I2], Iitaka remarks that this was proved by T. Fujita, but I have been unable to find a reference. Therefore, we give a proof here, adapting a proof of Mumford ([Mu], §6, pp. 60–61) for part of the way.

**Lemma 5.4.**

(a). Let $\overline{G}$ be a toric smooth completion of $G^\mu_m$. Then the logarithmic canonical divisor of $G^\mu_m$ on $\overline{G}$ is trivial, and the isomorphism between the canonical line sheaf and the trivial line sheaf on $\overline{G}$ is translation invariant.

(b). Let $D$ be an effective divisor on $G^\mu_m$ with trivial Ueno fibration. Then there exists a toric smooth completion $\overline{G}$ of $G^\mu_m$, such that the closure $\overline{D}$ of $D$ in $\overline{G}$ does not contain any $G^\mu_m$-orbit of $\overline{G}$.

**Proof.** A nonsingular toric variety $\overline{G}$ with principal orbit $G^\mu_m$ can be described by giving an open cover in which each open subset is isomorphic to $A^\mu$, and the action of $G^\mu_m$ is by monomials. Therefore, the divisor $\overline{G} \setminus G^\mu_m$ is a normal crossings divisor (and hence $\overline{G}$ is a smooth completion of $G^\mu_m$). Also, the differential $\mu$-form $dx_1/x_1 \wedge \cdots \wedge dx_\mu/x_\mu$ is a generator on this open subset for the canonical line sheaf of $G^\mu_m$ on $\overline{G}$. This generator is translation invariant, and is a nonzero constant multiple of the corresponding generator over any other such open subset. This proves part (a).

For the proof of (b) we first recall some definitions from the theory of toric varieties. A **fan** is a finite set $\Sigma$ of polyhedral cones in $\mathbb{R}^\mu$ such that every $\sigma \in \Sigma$ is a closed, rational, polyhedral cone not containing any nontrivial linear subspace; all faces of all $\sigma \in \Sigma$ also lie in $\Sigma$; and for all $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of $\sigma$ and of $\sigma'$. A
barycentric subdivision of $\Sigma$ associated to a rational ray $\lambda$ is the fan $\Sigma'$ consisting of all cones in $\Sigma$ not containing $\lambda$, plus the convex hull of $\lambda$ with each face of each $\sigma \in \Sigma$ containing $\lambda$.

By Lemma 2.4.2 there exists an equivariant completion $G_1$ of $\mathbb{G}_m$ such that the closure $D_1$ of $D$ in $G_1$ is Cartier and ample. This is a toric variety. Moreover, $D_1$ does not contain any orbit of $G_1$.

By ($[\mathcal{D}]$, 8.1), $G_1$ can be desingularized by applying a finite sequence of barycentric subdivisions to the corresponding fan $\Sigma$. Consider one such barycentric subdivision: let $\lambda$ be a rational ray, and let $\phi: X' \to X$ be the corresponding morphism of toric varieties. Orbits in $X'$ map onto orbits in $X$, so if the closure $\overline{D}_X$ of $D$ in $X$ does not contain any orbit in its support, then the same is true of the closure $\overline{D}_{X'}$ of $D$ in $X'$. In particular, this applies to the exceptional set of $\phi$, which is the closure of the orbit corresponding to the ray $\lambda$. Thus $\overline{D}_X$ and $\overline{D}_{X'}$ are related by pull-back of Cartier divisors: $\overline{D}_{X'} = \phi^* \overline{D}_X$.

Let $\phi: \overline{G} \to G_1$ be the desingularization corresponding to the composite of these barycentric subdivisions. By induction, the closure $\overline{D}$ of $D$ in $\overline{G}$ does not contain any orbit under the action of $\mathbb{G}_m$. Moreover, $\overline{G}$ is a toric nonsingular equivariant completion of $\mathbb{G}_m$. Thus part (b) holds. □

**Corollary 5.5.** Let $\rho: X \to Y$ be a fiber bundle with fiber $\mathbb{G}_m^\mu$, where $X$ and $Y$ are varieties, and let $D$ be an effective divisor on $X$ with $B(\mathbb{G}_m^\mu, D) = 0$. Then there exists a toric smooth completion $\overline{G}$ of $\mathbb{G}_m^\mu$ such that the closure $\overline{D}$ of $D$ in the corresponding (relative) completion $\overline{X}$ of $X$ does not contain any subset corresponding to a $\mathbb{G}_m^\mu$-orbit of $\overline{G}$.

**Proof.** Applying Lemma 5.4b to the generic fiber of $\rho$ gives a toric smooth completion $\overline{G}_\eta$ of $\mathbb{G}_m^\mu$ over the generic point $\eta$ of $Y$. Since toric varieties are defined by discrete data, $\overline{G}_\eta$ is of the form $\overline{G} \times_k \eta$ for some toric smooth completion $\overline{G}$ of $\mathbb{G}_m^\mu$ over $k$. This is the desired $\overline{G}$. □

**Lemma 5.6.** Let $\pi: X \to Y$ be a fiber bundle with fiber $B$, where $X$ and $Y$ are nonsingular varieties and $B$ is a semiabelian variety. Assume that there exists a smooth completion $(\overline{Y}, E)$ of $Y$ such that $\pi$ extends to a fiber bundle over $\overline{Y}$. Then there exists a smooth completion $(\overline{X}, D)$ of $X$ such that $\pi$ extends to a fiber bundle $\overline{\pi}: \overline{X} \to \overline{Y}$, and such that $K_{\overline{X}} + D = \overline{\pi}^*(K_{\overline{Y}} + E)$. Consequently, $\overline{\kappa}(X) = \overline{\kappa}(Y)$. Moreover, $\overline{\pi}$ can be constructed so that its fiber is an equivariant completion of $B$ corresponding to a toric smooth completion of the toric part of $B$.

**Proof.** We first claim that if $A$ is a semiabelian variety, if $\overline{\mathcal{A}}$ is an equivariant completion as in Lemma 2.2 with $\overline{G}$ a toric smooth completion of $\mathbb{G}_m^\mu$, and if $D = \overline{\mathcal{A}} \setminus A$, then $K_{(\overline{\mathcal{A}}, D)}$ is trivial, and the isomorphism $\mathcal{O}(K_{(\overline{\mathcal{A}}, D)}) \cong \mathcal{O}_{\overline{\mathcal{A}}}$ commutes with translation. If $A$ is an abelian variety, then this is classical, since in fact $\Omega^1_A/k \cong \mathcal{O}_{\overline{A}}^{\dim A}$ and that isomorphism commutes with translation. If $A = \mathbb{G}_m^\mu$, then this follows from Lemma 5.4a. In general, we have $\rho^{-1}(U) \cong U \times \mathbb{G}_m^\mu$, so $K_{\rho^{-1}(U)/U} = 0$ for open $U \subseteq A_0$. In particular, this applies to the exceptional set of $\overline{\pi}$, which is the closure of the orbit corresponding to the ray $\lambda$. Thus $\overline{D}_X$ and $\overline{D}_{X'}$ are related by pull-back of Cartier divisors: $\overline{D}_{X'} = \phi^* \overline{D}_X$.

Let $\phi: \overline{G} \to G_1$ be the desingularization corresponding to the composite of these barycentric subdivisions. By induction, the closure $\overline{D}$ of $D$ in $\overline{G}$ does not contain any orbit under the action of $\mathbb{G}_m$. Moreover, $\overline{G}$ is a toric nonsingular equivariant completion of $\mathbb{G}_m$. Thus part (b) holds. □
in an open covering of $A_0$, and the transition functions between these isomorphisms consist of translations on $\mathbb{G}_m^\mu$, so the isomorphisms $K_{\rho^{-1}(U)/U} = 0$ patch together to give us $K_{A/A_0} = 0$. This isomorphism is invariant under translations by $A$, since the same is true on suitable open subsets of the sets $\rho^{-1}(U)$.

To prove the lemma itself, let $(\mathcal{Y}, E)$ be as assumed, and let $\mathcal{B}$ be an equivariant completion of $B$ as above. This determines a smooth $B$-equivariant completion $(\mathcal{X}, D)$ of $X$, as in Lemma 2.2. As before, a patching argument then gives $K_{\mathcal{X}} + D = \bar{\pi}^*(K_{\mathcal{Y}} + E)$. The assertion $\bar{\kappa}(X) = \bar{\kappa}(Y)$ follows immediately from this.

\begin{lemma}
Let $A$ be a semiabelian variety and let $B$ be a semiabelian subvariety.

Then there exist equivariant completions of $A$ and $A/B$ such that the canonical map $A \to A/B$ extends to a morphism between the completions.
\end{lemma}

\begin{proof}
As usual, let $\rho: A \to A_0$ be the maximal abelian quotient of $A_0$. Choose an isomorphism $\text{Ker } \rho \cong \mathbb{G}_m^\mu$ such that the first $\mu(B)$ factors correspond to the maximal torus in $B$. One can then use the completion $A$ corresponding to the completion $\mathbb{G}_m^\mu \hookrightarrow (\mathbb{P}^1)^\mu$.
\end{proof}

\begin{lemma}[Theorem of the Square]
Let $A$ be a semiabelian variety and let $\mathcal{A}$ be an equivariant completion of $A$ corresponding to a nonsingular toric equivariant completion of $\mathbb{G}_m^\mu$ (in the notation of (0.4)). For $x \in A$ let $T_x: \mathcal{A} \to \mathcal{A}$ denote translation by $x$. Then for all line sheaves $\mathcal{L}$ on $\mathcal{A}$ and all $x, y \in A(k)$, $\mathcal{L} \otimes T_x^* \mathcal{L} \cong T_x^* \mathcal{L} \otimes T_y^* \mathcal{L}$.
\end{lemma}

\begin{proof}
Let $\mathcal{G}$ be the equivariant completion of $\mathbb{G}_m^\mu$ mentioned above. Then (since all divisors on $\mathbb{G}_m^\mu$ are principal) every divisor on $\mathcal{G}$ is linearly equivalent to a divisor supported on closures of orbits. Therefore $\mathcal{L} \cong \mathcal{O}(D_1 + \rho^*D_2)$, where $D_1$ is a divisor supported only on subsets of $\mathcal{A}$ corresponding to orbits of $\mathcal{G}$, and $D_2$ is a divisor on $A_0$. The lemma then follows, since $D_1$ is invariant under translation, and since the theorem of the square holds on $A_0$.
\end{proof}

The following is an adaptation of a result proved by Mumford for abelian varieties; cf. ([Mu], §6, pp. 60–61).

\begin{lemma}
Let $A$ be a semiabelian variety, let $D$ be an effective divisor on $A$, and let $\mathcal{A}$ be an equivariant completion of $A$ corresponding to a completion $\mathcal{G}$ of $\mathbb{G}_m^\mu$ satisfying the conditions of Corollary 5.5. Let $\bar{D}$ be the closure of $D$ in $\mathcal{A}$. Then

(a). the linear system $|2\bar{D}|$ is base-point free;
(b). if $B(D) = 0$, then the morphism $\mathcal{A} \to \mathbb{P}^N$ induced by $|2\bar{D}|$ is generically finite; and
(c). if $B(D) = 0$, then $\bar{D}$ is big.
\end{lemma}

\begin{proof}
Lemma 5.8 implies that $T_x^* \bar{D} + T_x^* \bar{D} \sim 2\bar{D}$ for all $x \in A(k)$, so given any $P \in \mathcal{A}$ it suffices to find $x \in A(k)$ such that $P \pm x \notin \text{Supp } \bar{D}$. For suitably generic
choices of $x$, $\text{Supp} \overline{D}$ does not contain any orbit of $\mathbb{G}_m^n$ in the fiber of $\rho$ containing $P+x$ or $P-x$. This condition depends only on $\rho(x)$. The condition $P \pm x \notin \text{Supp} \overline{D}$ is then satisfied for a generic choice of $x$ within such a fiber. This proves (a).

Now suppose $B(D) = 0$, and let $\phi$ be a morphism $\overline{A} \to \mathbb{P}^N$ induced by $|2\overline{D}|$. If $\phi$ is not generically finite, then there is an integral curve $C$ in $\overline{A}$, meeting $A$, such that $\phi(C)$ is a point. Then for all $x$, $\text{Supp}(T_x^*\overline{D} + T_x^*\overline{D})$ either contains $C$, or is disjoint from $C$. In particular, it is disjoint from $C$ for almost all $x$. Hence the same is true of $\text{Supp} T_x^*\overline{D}$, and of $T_x^*D_0$ for all irreducible components $D_0$ of $\overline{D}$.

We now claim that all such components $D_0$ are invariant under translation by $x_2 - x_1$ for all $x_1, x_2 \in C \cap A$. Indeed, since all divisors $T_x^*D_0$ are algebraically equivalent, their restrictions to $C$ must have the same degree, which must be zero since $C$ is usually disjoint from $T_x^*D_0$. Let $x_1, x_2 \in C \cap A$ and $y \in D_0$. Then $x_1 \in T_y^{*} D_0$, so also $x_2 \in T_{y-x_1}^{*} D_0$, and therefore $y \in T_{x_2-x_1}^{*} D_0$. This holds for all $y \in D_0$, so $D_0 \subseteq T_{x_2-x_1}^{*} D_0$. By symmetry they are equal, thus proving the claim.

But now it follows that $\overline{D}$ is invariant under translation by $x_2 - x_1$ for all $x_1, x_2 \in C \cap A$, contradicting the assumption that $B(D) = 0$. Thus $\phi$ is generically finite.

Finally, $\overline{D}$ is big, because $2\overline{D}$ is the pull-back of a hyperplane via the generically finite morphism $\phi$. $\square$

**Lemma 5.10.** Let $V$ be a nonsingular quasi-projective variety and let $X$ be a smooth completion of $V$. Let $D$ be an effective divisor on $X$, all of whose irreducible components meet $V$. Assume also that $\text{Supp} D$ does not contain any irreducible local intersection of components of $X \setminus V$. Then there exists a smooth completion $X'$ of $V \setminus D$ admitting a morphism $\pi: X' \to X$ such that the relative logarithmic canonical divisor of $\pi^{-1}(V \setminus D)$ over $V$ is effective, with support equal to $\pi^{-1}(\text{Supp} D)$.

**Proof.** Let $F$ be the divisor $X \setminus V$ (with all multiplicities equal to one). By Hironaka’s resolution of singularities ([Hi], pp. 142–143, Main Theorem II) there exists a sequence $X_r \to X_{r-1} \to \cdots \to X_0 = X$ of blowings-up such that

1. $\pi_{i+1}: X_{i+1} \to X_i$ is the blowing-up of an irreducible nonsingular subvariety $C_i$ which has normal crossings with $E_i \cup F_i$, where $F_i$ is the inverse image of $F$ in $X_i$, and $E_i$ is the exceptional set of the morphism $\pi_1 \circ \cdots \circ \pi_i$;
2. $C_i$ is contained in the strict transform of $D$ for all $i$; and
3. The strict transform of $D$ in $X_r$ is nonsingular and has normal crossings with $E_r \cup F_r$.

We claim that for all $i$ the relative logarithmic canonical divisor of $X_i \setminus (E_i \cup F_i)$ over $X \setminus F$ is an effective divisor whose support equals $E_i$. This will be proved by induction. It is trivial if $i = 0$. Assume it is true for $i$. We may assume that $C_i$ is not a divisor. If $C_i$ is not locally an intersection of components of $E_i \cup F_i$, then the relative logarithmic canonical divisor of $X_{i+1} \setminus (E_{i+1} \cup F_{i+1})$ over $X_i \setminus (E_i \cup F_i)$ is effective, with support equal to the exceptional divisor of $\pi_{i+1}$, so the inductive hypothesis is true for $i + 1$. If, on the other hand, $C_i$ is locally an intersection of components of
Lemma 5.11. Let $A$ be a semiabelian variety, and let $D$ be an effective divisor on $A$ with $B(D) = 0$. Then $A \setminus D$ is of logarithmic general type.

Proof. By Lemmas 5.4 and 5.9, there exists a toric smooth completion $\overline{A}$ of $A$ such that the closure $\overline{D}$ of $D$ is big, and such that $\overline{D}$ does not contain any local intersection of components of $\overline{A} \setminus A$. By Lemma 5.10, there exists a proper birational morphism $\pi: \overline{A} \to \overline{A}$ such that $\overline{A}$ is a smooth completion of $A \setminus D$ and such that the relative logarithmic canonical divisor of $\pi^{-1}(A \setminus D)$ (in $\overline{A}$) over $A$ is effective, with support equal to $\pi^{-1}(\text{Supp} \overline{D})$. In particular, it is big. But, by Lemma 5.6 with $Y$ equal to a point, the logarithmic canonical divisor of $A \setminus D$ (on $\overline{A}$) is zero; hence the logarithmic canonical divisor of $A \setminus D$ (on $\overline{A}$) is big. Thus $A \setminus D$ is of logarithmic general type.

Corollary 5.12 (Fujita). Let $D$ be an effective divisor on a semiabelian variety $A$. Then $\kappa(A \setminus D) \geq 0$, with equality if and only if $D = 0$.

Proof. Let $A' = A/B(D)$ and $D' = D/B(D)$. By Lemma 5.11, $\kappa(A' \setminus D') = \dim A'$. By Lemmas 5.6 and 5.7, $\kappa(A \setminus D) = \kappa(A' \setminus D')$. Thus $\kappa(A \setminus D) \geq 0$, with equality if and only if $B(D) = A$. The latter condition holds if and only if $D = 0$.

Lemma 5.13. Let $k$ be any field (of any characteristic, not necessarily algebraically closed). Let $A$ be a semiabelian variety over $k$, with maximal abelian quotient $\rho: A \to A_0$. Let $S$ be a geometrically integral scheme over $k$ with $S(k) \neq \emptyset$, and let $B$ be a reduced closed subscheme of $A \times_k S$. Assume that $B$ is a group subscheme of the group scheme $A \times_k S$ over $S$, that $B$ is smooth over $S$, that the restriction of the map $\rho(S): A \times_k S \to A_0 \times_k S$ to $B$ is smooth, and that $B$ is geometrically connected over $k$. Then there is a group subvariety $B$ of $A$ such that $B = B \times_k S$. (Cf. ([Mi], Prop. 20.3).)

Proof. Pick $s \in S(k)$, and let $B = B_s$. Since $B$ is reduced, it is sufficient to show that $B \times_k k = B \times_k S \times_k k$, set-theoretically. Hence we may assume that $k$ is algebraically closed.
Let $K$ be the function field of $S$. The image of $\rho_{(S)}|_{B_K}: B_K \to A_0 \times_k K$ is a connected group subvariety; hence an abelian subvariety; by ([Mi], Cor. 20.4) it is of the form $B_0 \times_k K$ for some abelian subvariety $B_0$ of $A_0$. By smoothness and dimensionality considerations, $\rho_{(S)}$ maps $B$ onto $B_0 \times_k S$. By shrinking $A$, we may therefore assume that $B_0 = A_0$.

Now consider the group $C_K := \text{Ker} \rho_{(S)} \cap B_K$. It is a subgroup of an algebraic torus; hence by ([B], 8.5 and 8.4 Corollary), it is a diagonalizable group. Let $K'$ be a finite extension of $K$ over which $C_K$ splits, and let $S'$ be a corresponding generically finite cover of $S$. Then, by ([B], 8.7), $C_K \times_K K'$ is of the form $F \times \mathbb{G}_m^\mu$, where $F$ is a finite group and $\mu \in \mathbb{N}$. Hence there exists a diagonalizable group $C$ over $k$ and a nonempty open subgroup $U$ of $S'$ such that the closure of $B_K \times K'$ in $B \times S U$ is $U$-isomorphic to $C \times_k U$. By rigidity ([B], 8.10), it follows that the induced map $C \times_k U \to A$ factors through the projection onto the first factor; hence we may regard $C$ as a subgroup of $A$. Since $B$ is closed, it follows that $B \supseteq C \times_k S$.

After replacing $A$ with $A/C$ and $B$ with its image in $(A/C) \times_k S$, we may assume that $B$ is generically finite over $A_0 \times_k S$, of degree 1. Since $B$ is also reduced, it corresponds to a (reduced) rational point on the generic fiber of $\rho_{(S)}|_{B}$, hence $B$ is the closure of the image of a rational section $\sigma: U \to A \times_k S$, where $U$ is an open dense subset of $A_0 \times_k S$. Translating by closed points of $A_0$ and using the fact that $B$ is a group subscheme, we see that $U$ is of the form $A_0 \times_k V$ for some open dense subset $V \subseteq S$ (and that $\sigma$ is a homomorphism of $V$-group schemes). Thus $B$ is a family of regular sections of $\rho: A \to A_0$, parametrized by $V$. But $\rho: A \to A_0$ admits at most one regular section passing through the group identity of $A$, since the ratio of any two such sections is a regular map $A_0 \to \mathbb{G}_m^{\mu(A)}$. Thus $B \cap (A \times_k V) = B' \times_k V$ for some $B' \subseteq A$. Since $B$ is closed, it follows that $B \supseteq B' \times_k S$. Since all irreducible components of $B$ dominate $A_0 \times_k S$, we have $B = B' \times_k S$. Finally, $B' = B_k = B$, and we are done. \hfill \square

Recall that a regular field extension is a field extension $K/k$ such that $K$ is linearly disjoint from $k$ over $k$; equivalently (in characteristic zero), $k$ is algebraically closed in $K$.

**Lemma 5.14.** Let $k$ be any field, let $K/k$ be a regular field extension, let $A$ be a geometrically integral semiabelian variety over $k$, and let $B$ be a geometrically integral group subvariety of $A_K := A \times_k K$. Then there exists a group subvariety $B_k$ of $A$ such that $B = B_k \times A A_K$.

**Proof.** This lemma is already known when $A$ is an abelian variety: see ([Mi], Cor. 20.4). The proof here is essentially the same proof, using the stronger Lemma 5.13.

Let $S$ be a variety over $k$ with $K(S) = K$. Since $K/k$ is regular, $S$ is geometrically integral. Let $B$ be the closure of $B$ in $A \times_k S$. After replacing $S$ with an open subvariety, we may assume that $B$ is smooth over $S$, that $\rho_{(S)}|_{B}: B \to A_0 \times_k S$ is smooth, and that the morphisms $\text{Spec} K \to B$, $B \to B$, and $B \times_K B \to B$ expressing the group structure of $B$ extend to morphisms $S \to B$, etc. This latter condition ensures that $B$ is a group subscheme of $A \times_k S$.
By ([R], p. 412), there is a finite separable extension $k'$ of $k$ such that $S(k') \neq \emptyset$. We may assume that $k'$ is Galois over $k$. After base change to $k'$, Lemma 5.13 implies the existence of $B_{k'} \subseteq A \times_k k'$ such that $B \times_K Kk' = B_{k'} \times_{k'} Kk'$ (as subvarieties of $A \times_k Kk'$). This equality determines $B_{k'}$ uniquely; hence it is invariant under $\operatorname{Gal}(k'/k)$. By descent (see, for example, ([Se 1], V.20)), $B_{k'}$ is of the form $B_k \times_k k'$ for some semiabelian subvariety $B_k$ of $A$.

**Lemma 5.15.** Let $X$ be a closed subvariety of a semiabelian variety $A$. Then $\bar{\kappa}(X) \geq 0$, with equality if and only if $X$ is a translated semiabelian subvariety of $A$.

**Proof.** See ([I 2], Thm. 4 and Thm. 2).

**Theorem 5.16.** Let $A$ be a semiabelian variety, let $X$ be a closed subvariety of $A$, and let $D$ be an effective Weil divisor on $X$. Then

(a) $\bar{\kappa}(X \setminus D) \geq 0$, with equality if and only if $X$ is a translated semiabelian subvariety of $A$ and $D = 0$;

(b) $\bar{\kappa}(X \setminus D) + \dim B(X \setminus D) = \dim X$; and

(c) $B(X \setminus D) = 0$ if and only if $X \setminus D$ is of logarithmic general type.

**Proof.** The inequality $\bar{\kappa}(X \setminus D) \geq 0$ is immediate from Lemma 5.15 and the inequality $\bar{\kappa}(X \setminus D) \geq \bar{\kappa}(X)$. The remainder of part (a) follows by Lemmas 5.15 and 5.12.

Next consider (b). By Hironaka’s resolution of singularities and by ([I 3], Thm. 10.3), there exists a complete nonsingular variety $X'$, a normal crossings divisor $D'$ on $X'$, a proper birational morphism $\pi: X' \setminus D' \to X \setminus D$, a complete nonsingular variety $W$, and a dominant morphism $\Phi: X' \to W$ such that

(i) $\dim W = \bar{\kappa}(X \setminus D)$,

(ii) $K(X')$ is regular over $K(W)$, and

(iii) if $X'_m$ denotes the generic fiber of $\pi$, then $h^0(X'_m, mK_{(X', D')}) \leq 1$ for all $m \in \mathbb{N}$, and is not always zero.

The canonical divisor of the generic fiber of $\pi$ is $K_{X'}|_{X'_m}$; this follows by looking at the First Exact Sequence for differentials, which is exact on the left in this case, and taking highest exterior powers. Therefore the logarithmic canonical divisor of the generic fiber of $\pi|_{X' \setminus D'}$ is $(K_{X'} + D')|_{X'_m}$. Thus, by (iii), the logarithmic Kodaira dimension of the generic fiber of $\pi|_{(X' \setminus D')}$ is zero. We may regard this as a subvariety of $A_\eta := A \times_k K(W)$; after base change to $\overline{K(W)}$, it is a translated semiabelian subvariety of $A \times_k \overline{K(W)}$. By Lemma 5.14, that subvariety comes from a subvariety $B$ of $A$. Since the generic fiber is invariant under translation by $B$, the same holds for $X \setminus D$. Thus $B(X \setminus D) \supseteq B$, and therefore by (i),

$$\dim B(X \setminus D) \geq \dim B = \dim X - \bar{\kappa}(X \setminus D).$$

But also, by Lemmas 5.6 and 5.7,

$$\bar{\kappa}(X \setminus D) = \bar{\kappa}((X \setminus D)/B(X \setminus D)) \leq \dim X - \dim B(X \setminus D).$$
Combining this with (5.16.1) gives part (b).

Part (c) follows immediately from (b).

§6. The Kawamata Structure Theorem

This section generalizes the Kawamata Structure Theorem to the context of a closed subvariety of a semiabelian variety, minus a divisor.

Theorem 6.1. Let \( A \) be a semiabelian variety over an algebraically closed field of characteristic zero, let \( X \) be a closed subvariety of \( A \), and let \( D \) be an effective divisor on \( X \). Let \( Z = Z(X \setminus D) \) be the union of all positive dimensional translated semiabelian subvarieties of \( A \) contained in \( X \setminus D \). Then \( Z \) is a Zariski-closed subset of \( X \setminus D \), and each irreducible component has nontrivial Ueno fibration.

Proof. We may assume that \( X \setminus D \) has trivial Ueno fibration, for otherwise the theorem is trivial (with \( Z = X \setminus D \)). By noetherian induction it then suffices to show that \( Z \) is not Zariski-dense. Let \( B = B(X) \) and \( X' = X/B \); then \( X' \) has trivial Ueno fibration and there is a fiber bundle \( \theta : X \to X' \) with fiber \( B \).

Let \( Z' \) (resp. \( Z'' \)) be the union of all nontrivial translated semiabelian subvarieties of \( A \) which are contained in \( X \setminus D \) and which lie (resp. do not lie) in fibers of \( \theta \). Then \( Z = Z' \cup Z'' \). But \( Z'' \subseteq \theta^{-1}(Z(X')) \), which is not Zariski-dense by the Kawamata Structure Theorem for closed subvarieties of semiabelian varieties ([N], Lemma 4.1). Thus it suffices to show that \( Z' \) is not Zariski-dense.

Let \( Z_0 \) be the union of all nontrivial translated abelian subvarieties making up \( Z' \), and let \( Z_1 \) be the union of all nontrivial images of \( \mathbb{G}_m \) lying in \( X \setminus D \) and contained in fibers of \( \theta \). Then

\[
Z' = Z_0 \cup Z_1 .
\]

(Note that a nontrivial image of \( \mathbb{G}_m \) is a translated semiabelian subvariety, by ([I 2], Thm. 2.).)

First consider \( Z_0 \). Suppose \( Z_0 \neq \emptyset \) and let \( C \) be a translated abelian subvariety of \( A \) lying in \( X \setminus D \) and lying in a fiber of \( \theta \). Then the closure of \( D \) in \( \overline{A} \) does not meet \( C \); hence this is true of all translates of \( C \) unless they lie in the closure of \( D \). Thus \( B(X \setminus D) \) contains the abelian subvariety corresponding to \( C \). This contradicts the assumption that \( B(X \setminus D) \) is trivial, so \( Z_0 = \emptyset \).

This leaves \( Z_1 \). The remainder of this proof is motivated by the proof of Theorem 0.2.

Let \( B_0 \) be the abelian quotient of \( B \) and let \( B_1 \) be the toric part, so that there exists an exact sequence \( 0 \to B_1 \to B \to B_0 \to 0 \). Then \( \theta : X \to X' \) factors as

\[
X \xrightarrow{\phi} X_0 \xrightarrow{\psi} X' ,
\]

where \( \phi \) and \( \psi \) are fiber bundles with fibers \( B_1 \) and \( B_0 \), respectively.

Let \( \pi_0 : \overline{X_0} \to X_0 \), \( X \subseteq \overline{X} \), \( \pi : \overline{X} \to \overline{X} \), and \( \overline{D} \) be as in Theorem 2.4. Let \( \overline{B}_1 \) be the equivariant completion of \( B_1 \) corresponding to \( \overline{X} \). Let \( \chi : \mathbb{G}_m \to X \setminus D \) be
a nontrivial morphism whose image is contained in a fiber of $\theta$. Then its image must lie in a fiber of $\phi$, say, $\text{Im } \chi \subseteq \phi^{-1}(x)$, $x \in X_0$. Let $\tilde{x} \in \tilde{X}_0$ be a point lying over $x \in X_0$; then $\chi$ lifts to $\tilde{\chi} : \mathbb{G}_m \to \tilde{X}$. Since $\tilde{D}$ is ample on fibers of $\tilde{\phi} : \tilde{X} \to \tilde{X}_0$, the closure of the image of $\tilde{\chi}$ must meet $\tilde{D}$. Let $P$ be a point where they meet, let $T$ be the $B_1$-orbit of $\tilde{B}_1$ corresponding to the $B_1$-orbit of $\tilde{\phi}^{-1}(\tilde{x})$ containing $P$, and let $p : U \to T$ be the projection defined in Proposition 2.5; $U \supseteq B_1$. This $p$ defines a projection $q$ from an open subset of $\tilde{X}$ to the subset $\tilde{T}$ of $\tilde{X}$ corresponding to $T$. Then $q \circ \tilde{\chi}$ defines a morphism $\mathbb{A}^1 \to T$. Since $T$ is isomorphic to a product of $\mathbb{G}_m$’s, it follows that this morphism must be trivial. Thus the image of $\tilde{\chi}$ must lie in the proper Zariski-closed subset $q^{-1}(\text{Supp } \tilde{D} \cap \tilde{T})$, so the image of $\chi$ must lie in the corresponding proper Zariski-closed subset of $X$.

Since there are only finitely many such $\tilde{T}$, it follows that $Z_1$ cannot be Zariski-dense. □

References

[A] D. Abramovich, Subvarieties of semiamelab varieties, Compos. Math. 90 (1994), 37–52.
[B] A. Borel, Linear algebraic groups, second enlarged edition, Grad. Texts in Math. 126, Springer, 1991.
[D] V.I. Danilov, The geometry of toric varieties, Russian Math. Surveys 33:2 (1978), 97–154.
[F] G. Faltings, Diophantine approximation on abelian varieties, Ann. of Math. (2) 133 (1991), 549–576.
[Ha] R. Hartshorne, Algebraic geometry, Grad. Texts in Math. 52, Springer, 1977.
[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. (2) 79 (1964), 109–203.
[I 1] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, Complex analysis and algebraic geometry (W. L. Baily, Jr. and T. Shioda, eds.), Iwanami, Tokyo, 1977, pp. 175–189.
[I 2] S. Iitaka, Logarithmic forms of algebraic varieties, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 23 (1976), 525–544.
[I 3] S. Lang, Fundamentals of diophantine geometry, Springer, 1983.
[Mi] J.S. Milne, Abelian Varieties, Arithmetic geometry (G. Cornell and J.H. Silverman, eds.), Springer, 1986, pp. 103–150.
[Mu] D. Mumford, Abelian Varieties, Oxford University Press, 1970.
[N] J. Noguchi, Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties, Nagoya Math. J. 83 (1981), 213–233.
[O] T. Oda, Convex bodies and algebraic geometry—toric varieties and applications, I, Proceedings of the algebraic geometry seminar (Singapore, 1987) (M. Nagata and T.A. Peng, eds.), World Scientific, Singapore, 1988, pp. 89–94.
[R] M. Rosenlicht, Some basic theorems on algebraic groups, Am. J. Math 78 (1956), 401–443.
[Se 1] J.-P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1975; English translation, Algebraic groups and class fields, Graduate texts in mathematics 117, Springer, 1988.
[Se 2] J.-P. Serre, Quelques propriétés des groupes algébriques commutatifs, Nombres transcendents et groupes algébriques (M. Waldschmidt, ed.), Astérisque 69–70, 1979, pp. 191–203.
[TE] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal Embeddings I, Lecture Notes in Math. 339, Springer, 1973.
[V 1] P. Vojta, Diophantine approximations and value distribution theory, Lecture Notes in Math. 1239, Springer, 1987.
[V 2] , Applications of arithmetic algebraic geometry to diophantine approximations, Arithmetic Algebraic Geometry, Trento, 1991, Lecture Notes in Math. 1553, Springer, 1993.

[V 3] , Integral points on subvarieties of semiabelian varieties, I, Invent. Math. 126 (1996), 133–181.

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