Magnetic orbital motion and $0.5e^2/h$
conductance of quantum-anomalous-Hall
hybrid strips

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Abstract

The magnetic-induced orbital motion of quasi-particles affects the conductance properties of a hybrid strip of a quantum-anomalous-Hall topological material with induced superconductivity. We elucidate the scenario of topological NSN ideal junctions in presence of orbital magnetic motion, showing how it leads to a halved quantized conductance $0.5e^2/h$ even in absence of Majorana modes. The magnetic orbital effect favours Fermionic charged modes with finite wave numbers, in contradistinction to Majorana zero modes which are chargeless zero-energy modes with vanishing wave number. The bias sensitivity of the 0.5 conductance plateau allows to distinguish the two cases. Conductance oscillations due to backscattering interference are absent in the charged Fermion case.

Keywords

Quantum anomalous Hall, topological insulators, Majorana modes.

Introduction

Quantum transport with devices of robust tunable properties is a very active field in Condensed Matter physics. The reasons are twofold: on the one hand, fundamental understanding of the often bizarre behaviors of a quantum system; on the other hand, practical applications allowing to overcome present technological bottlenecks. Specifically, we refer in this Letter to topological-transport systems, able to host current-carrying states along the edges or boundaries of specific arrangements of different materials. The so-called Majorana zero modes in strips of quantum-anomalous-Hall insulator (QAHI) material, with hybrid superconductivity, clearly manifest the above dual interest: they are electron-hole quantum superpositions that may allow quantum computation through braiding.

Majorana zero modes have also been studied in hybrid semiconductor-superconductor nanowires, where they remain attached to the nanowire ends and yield electrical zero-bias (anomalous) conductance peaks. In contrast, Majorana modes in QAHI strips propagate along the lateral edges and they are characterized by a $0.5e^2/h$ conductance plateau when a single Majorana mode is active. The corresponding energy-momentum dependence $\varepsilon(k)$ is linear, with the shape of a diagonal cross centered around $k = 0$ and $\varepsilon = 0$. These modes are chiral, with right- and left-movers localizing on opposite edges of the strip. Interferometry with Majorana beam splitters was consid-
In QAHI’s the material magnetization is fundamental for the characterization of the topological phases; each phase having a topological invariant given by the number of zero-energy edge modes present. The internal magnetization is controlled by an external magnetic field. Here, the maximum (saturation) magnetizations in opposite perpendicular orientations are typically, sweeping $B_z$ from negative to positive all the system phases between magnetization reversal; usually, a perpendicularly oriented magnetic field acting like a trigger for the magnetization. The external field acting like a trigger for the magnetization is controlled by an external magnetic field. Besides being a trigger for the internal magnetization, a perpendicularly oriented magnetic field also has a direct action on the charged electron and hole quasiparticles. The direct coupling of the field with the quasiparticles’ spin can be neglected or absorbed in the similar coupling given by the internal magnetization. In addition, however, there is a direct influence of the external field on the orbital quasiparticle motion. This is the focus of our present study. We will show how the orbital effect modifies the scenario of the Majorana modes, leading to charged Fermionic modes with nonvanishing wavenumbers (as opposed to chargeless Majorana zero modes with $k\approx 0$).

For a 2D strip, infinite along $x$ and with lateral extension $L_y$, the minimal substitution $\vec{p} \rightarrow \vec{p} \pm h y / \ell_z^2$, where $\ell_z^{-2} = eB_z / \hbar c$ is the magnetic length, describes the orbital field effect. The ± signs yield the well known different motion of opposite charges in a given $B_z$ field. In a topological QAHI the $p^2$ term is rather small but, nevertheless, the above minimal substitution cannot be neglected in the strong $p$-linear contribution of Rashba spin-orbit type. Notice also that in a field of, say, $B_z \gtrsim 1$ mT it is $\ell_z \lesssim 1$ μm and, thus, it makes orbital effects again relevant for strips of $L_y \gtrsim 1$ μm and fields in the millitesla range.

Remarkably, although the orbital effect alters the character of the zero modes, the conductance still remains $0.5e^2 / h$ in cases when a single charged mode propagates through the device. We refer to a configuration such that the bias drop is symmetrical in the two terminals ($\pm V/2$) and the central superconductor is effectively grounded ($\mu = 0$), with no current flowing from it to the leads. We conclude that a halved conductance plateau is not a unique signature of a Majorana mode, but it is also possible with Fermionic modes having different electron and hole transmissions such that one mode is transmitted and the other is reflected.

An essential difference between the above two scenarios is that the transport by a single charged Fermion requires a nonvanishing energy. As shown below, this causes the small-bias limit of the 0.5 conductance plateau to disappear in the charged Fermion case while it survives in the Majorana one. Besides, for reduced transverse widths $L_y$, the Majorana transmission is characterized by conductance oscillations. These oscillations are absent if transmission occurs through a charged Fermion mode in a perfect way. It is worth noting here that 0.5 conductance plateaus in absence of Majorana modes have also been suggested in Refs. 21,22 as a result of percolation through disordered QAHI strips. In this sense, we assume that disorder is small or moderate, such that the chiral edge character of the modes is still well defined throughout the device.

**Model and transport**

We use the model of double-layer 2D hybrid QAHI strip of Refs. 4–10. More specifically, with the same notation and parameter definitions of Ref. 20 the Hamiltonian reads

$$
\mathcal{H} = \left[ m_0 + m_1 \left( p_x^2 + p_y^2 \right) \right] \tau_z \lambda_x + \Delta_Z \sigma_z - \frac{\alpha}{\hbar} \left( p_x \sigma_y - p_y \sigma_x \right) \tau_z \lambda_z + \Delta_p \tau_x + \Delta_m \tau_x \lambda_z. \tag{1}
$$

Usual spin, electron-hole isospin and layer pseudospin are represented by $\sigma$, $\tau$ and $\lambda$ Pauli matrices, respectively. The superconductor energy-gap parameter in the two layers $\Delta_{1,2}$...
define the above $\Delta_{p,m}$ parameters by $\Delta_{p,m} = (\Delta_1 \pm \Delta_2)/2$. In an NSN double junction these parameters are nonzero only in the central S section of length $L_x$, as sketched in Fig. 1a. $\Delta_Z$ represents the Zeeman-like term due to the magnetization of the material and $\alpha$ the spin-orbit coupling. Typical values of the Hamiltonian parameters are discussed in Ref. 20. In particular, the energy and length scales are meV and $\mu$m, respectively.

With the minimal substitution, three new orbital terms are added to Eq. (1),

$$\mathcal{H}_{orb} = m_1 \left( \frac{\hbar^2}{\ell_z^2} y^2 \tau_z - 2 \frac{\hbar^2}{\ell_z^2} y p_x \right) \lambda_x + \frac{\alpha}{\ell_z^2} y \sigma_y \lambda_z . \quad (2)$$

The first two contributions in Eq. (2) originate in the $m_1$ quadratic-in-momentum term and they are familiar from the kinetic contributions of the standard quantum Hall effect; the last one originates in the spin-orbit-like $\alpha$ term. We shall keep all three terms of Eq. (2), although, in the present context of QAHI parameters the smallness of $m_1$ leads to a dominant magnetic $\alpha$-term modification. Note also that with our energy (meV) and length ($\mu$m) units the conversion between $\ell_z^{-2}$ and $B_z$ in milliteslas is $B_z/mT \approx 0.7 \ell_z^{-2}/\mu$m$^{-2}$; that is, $\ell_z^{-2} = 1 \mu$m$^{-2}$ corresponds to $B_z = 0.7$ mT.

The formalism of phase-coherent transport in hybrid superconducting nanostructures was reviewed in Ref. 23. In particular, for our present purposes, the current-voltage relations in the two terminals $i = 1, 2$ read

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} , \quad (3)$$

where the conductance matrix is given in terms of the number of propagating modes in each terminal $N_i^\alpha(E)$ and the transmission probabilities between terminals $P_{ij}^{\alpha\beta}(E)$ (quasiparticle e/h type is indicated by $\alpha, \beta = \pm$) as

$$a_{ij} = \delta_{ij}N_i^+(E_i) + P_{ij}^{++}(E_j) - P_{ij}^{+-}(E_j) . \quad (4)$$

The energies $E_i$ in Eq. (4) are related to the terminal-superconductor biases $E_i = eV_i - \mu$.

In a setup with a grounded superconductor and a symmetrical bias drop $V_{1,2} = \pm V/2$ the conductance is given by

$$G(E) = \frac{e^2}{2\hbar} \left[ P_{11}^{++}(E) + P_{11}^{+-}(E) + P_{12}^{+-}(E) + P_{12}^{++}(E) \right] , \quad (5)$$

where all probabilities are evaluated at the same energy $E = eV/2$. Equation (5) is easily derived from Eq. (3) using particle-hole symmetry $P_{ij}^{\alpha\beta}(E) = P_{ij}^{\beta\alpha}(-E)$ and flux conservation $N_i^\alpha(E) = \sum_{j\beta} P_{ij}^{\alpha\beta}(E)$.

In the Majorana scenario, a 0.5e$^2$/h conductance results from all transmission probabilities in Eq. (5) taking a value $\approx 0.25$. This is understood as an incident Fermionic quasiparticle that splits and reflects in equal amounts as an electron and as a hole due to the peculiarities of the Majorana mode. In presence of the magnetic orbital effect, a 0.5e$^2$/h conductance also results when one Fermion is fully transmitted, say $P_{12}^{++} = 1$, while all the rest $P_{ij}^{\beta\alpha}$ vanish. This single-charged-mode scenario is made possible by the breaking of electron-hole degeneracy, which results in different transmissions for conjugate quasiparticles at the same energy $P_{ij}^{\alpha\beta}(E) \neq P_{ij}^{\beta\alpha}(E)$. Nevertheless, we stress again that electron-hole symmetry is still preserved by the orbital effect with the relations $P_{ij}^{\beta\alpha}(E) = P_{ij}^{\alpha\beta}(-E)$.

The specific calculations of NSN topological junctions discussed below are performed with the complex-$k$ method used previously by us in Refs. 25-28 to solve the Bogoliubov-deGennes equation. More specifically, the reader is addressed to Ref. 20 for the details in the present context of hybrid QAHI strips. Our approach allows to take fully into account the finite size influence connected with specific values of $L_x$ and $L_y$ (Fig. 1h).

Results

Figure 1 shows how the above two scenarios of 0.5e$^2$/h conductance can be easily distinguished by looking at the bias dependence $G(E)$ of an NSN double junction. In presence of a Majo-
the two Fermion modes of opposite charge $E$ decreases from positive values to zero the in-

equality that evolves, in the small bias limit, to an $e^2/h$ conductance plateau. The reason of this
difference at small bias is clear from Fig. 1: as $E$ decreases from positive values to zero the in-
verted band is activated, allowing the transport by the two Fermion modes of opposite charge
to $P_{12}^{++} \approx P_{12}^{--} \approx 1$ and a value $G \approx e^2/h$ from Eq. (5).

For intermediate energies the halved conductance is found in both scenarios of Fig. 1 and,
therefore, it may become complicate to ascer-
tain the $0.5e^2/h$ origin looking only at a single bias, or at a reduced window of biases. In this
respect, Fig. 1 suggests that the conductance oscillations around $0.5e^2/h$ are a differentiating
mechanism in strips of $L_y$ smaller than a few microns. Indeed, the conductance oscillations are
due to the interference of the back-scattered chiral Majoranas, a mechanism that is absent in the transport by charged Fermion modes.

The phase diagrams with the number of propagating edge modes at low energies ($\mathcal{N}$) are shown in Fig. 2 for selected values of the energy. In the limit of vanishing energy $\mathcal{N}$ is the characteristic Chern number of each topologi-

cal phase. At nonzero (sizeable) energies the phases with $\mathcal{N} = 1$ become prominent, extend-
ing up to relatively large magnetic fields (Fig. 2c,d). For $E \to 0$, however, the $\mathcal{N} = 1$ phase is severely quenched, surviving only in a small region at low fields and with nonvanishing superconductivity (Fig. 2a).

It is remarkable that the $\mathcal{N} = 1$ phase of an edge charged Fermion can be seen even with
vanishing superconductivity (Fig. 2d), although it requires the presence of magnetic field ($\ell_z^{-2} > 0$) and nonvanishing energy (bias). This is showing that the essential requirements for this phase are particle-hole symmetry and breaking of particle-hole degeneracy as discussed above, rather than a strong superconducting gap $\Delta_{1,2}$. The superconducting gap is nevertheless essential for the single Majorana phase, restricted
to $\ell_z^{-2} \approx 0$. Dotted lines in Fig. 2 mark the position of the $k = 0$ energy gap closing. Ide-
ally, this line separates the energy-band behavior from a gapped band structure to a double cross at non zero $k$’s, i.e., a transition $\mathcal{N} = 0 \to \mathcal{N} = 2$. The separation in $k$ values between the two crosses increases with the magnetic field, $\Delta k \propto L_y \ell_z^{-2}$ and, only for $\ell_z^{-2} \approx 0$ a single cross at the origin is effectively present. This agrees with Fig. 2b and with Fig. 2a for $\ell_z^{-2} > 3 \mu m^{-2}$. With nonzero $E$’s and $\ell_z^{-2}$’s the intermediate phase $\mathcal{N} = 1$ may still emerge either because $E$ exceeds the $k = 0$ maximum (Fig. 2c,d), or because a small gap is still present at finite $k$ before the double cross is fully formed (Fig. 2a). The latter, however, is a marginal finite-$L_y$ effect yielding the severely quenched $\mathcal{N} = 1$ phase in Fig. 2.

We have explored above the model dependence on $\Delta_Z$ and $\ell_z^{-2}$ assumed to be inde-
pendent parameters. In experiment however, the material magnetization ($\Delta_Z$) is triggered by the magnetic field ($B_z$ or, equivalently, $\ell_z^{-2}$) in be-
tween saturation values. Noting the symmetry properties

$$G(E, \Delta_Z, B_z) = G(E, -\Delta_Z, B_z)$$

$$= G(E, \Delta_Z, -B_z)$$

$$= G(E, -\Delta_Z, -B_z) ,$$ (6)

it is always possible to represent a general tra-
jectory in the $\Delta_Z - B_z$ plane by its correspond-
ing image in the first quadrant of the plane. In this respect, the thin diagonal lines of Fig. 2 are a possible linear dependence $\Delta_Z - \ell_z^{-2}$ mod-
ing a sweep of magnetic field from negative to positive values in a magnetization reversal. The corresponding conductances can be seen in Fig. 3 for different values of the energy (bias).

The conductance traces of Fig. 3 show that the $0.5e^2/h$ plateau becomes conspicuous with increasing bias as a result of the magnetic or-

bital effect. In our model this bias dependence is to be expected whenever the conductance transition and its associated plateau occurs at magnetic fields $\ell_z^{-2} \gtrsim 1 \mu m^{-2}$, or equivalently $B_z \gtrsim 0.7 \mu T$. As argued above, in this sce-
nario the $0.5e^2/h$ plateau reflects the breaking of electron-hole degeneracy by the magnetic or-
bital motion with a given nonvanishing energy. It is not to be excluded, however, that the $0.5e^2/h$ plateau due to a genuine Majorana mode can be observed, although our above analysis indicates that this would require rather small magnetic fields such that the field sweep in Fig. 2 moves towards the horizontal axis. We also stress that in the Majorana scenario the $0.5$ plateau should still survive in the limit of vanishing bias $E \to 0$; therefore, observing a clear bias dependence as in Fig. 3 would discern the validity of the charged Fermion scenario suggested here.

Conclusions

A $0.5e^2/h$ conductance plateau in ideal quantum-anomalous-Hall hybrid strips can be attributed to a chiral Majorana mode at very small magnetic fields, or to a charged Fermionic mode at sizeable fields ($B_z > 1 \text{ mT}$). For the latter, the magnetic orbital motion has to break the electron-hole degeneracy at a given finite energy, while still keeping electron-hole symmetry in the transmissions for opposite-sign energies. Both situations can be realized in a two terminal setup with a grounded superconductor region and a symmetrical bias drop in the two terminals. The vanishing-bias dependence of the $0.5$ plateau, disappearing for the charged Fermion and surviving for the chiral Majorana, has been suggested as a means to discern the two scenarios of a halved conductance. In addition, while the $0.5$ plateau due to a Majorana is predicted to display energy oscillations when the transverse width is small $L_y \lesssim 3 \mu \text{m}$, the plateau due to a charged Fermion is predicted to be much more $L_y$-independent and flat. The latter is characterized by an almost perfect transmission with magnetic motion along the edge, largely insensitive to $L_y$.

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Figure 1: Panels b), d): band structure of a hybrid strip of $L_y = 3 \mu m$ with b) a Majorana zero mode at $k = 0$ and d) two charged Fermionic modes at finite $k$’s. The charge of each state is given by the symbol color scale. The Zeeman and magnetic length parameters are $\Delta Z = 1.2$ meV and $\ell_z^{-2} = 0.1 \mu m^{-2}$ for b); and $\Delta Z = 1.5$ meV and $\ell_z^{-2} = 4 \mu m^{-2}$ for d). The right panels c) and e) show the bias-dependent conductances of an NSN double junction with $L_x = 10 \mu m$ (a) such that the central region is described by the band structure shown on its corresponding left panel. Other parameters: $m_0 = 1$ meV, $m_1 = 10^{-3}$ meV$\mu m^2/\hbar^2$, $\alpha = 0.26$ meV$\mu m$, $\Delta_{1,2} = (1,0.1)$ meV.
Figure 2: Colorscale plots of the number of propagating modes \( N \) at a fixed energy \( E \). We used \( E = 0.01 \text{ meV} \) (a,b) and \( E = 0.3 \text{ meV} \) (c,d). Left panels (a,c) are for the superconducting strip with \( \Delta_1 = 1 \text{ meV} \) and \( \Delta_2 = 0.1 \text{ meV} \); right ones (b,d) are for \( \Delta_{1,2} = 0 \). Other parameters as in Fig. 1. Dotted lines show the position of the \( k = 0 \) gap closings while thin straight lines represent a typical evolution in a sweep of magnetic fields.

Figure 3: Conductance traces for a magnetic field sweep along the straight lines shown in Fig. 2. The bias (energy) is kept fixed in each trace to the indicated value. For clarity, each trace has been shifted with respect to the preceding one, in increasing energy order, by an increment of 0.5 horizontally and 0.05 vertically. The \( 0.5e^2/h \) plateau is greatly enhanced with increasing bias.