CONJECTURAL LARGE GENUS ASYMPTOTICS OF
MASUR–VEECH VOLUMES AND OF AREA SIEGEL–VEECH
CONSTANTS OF STRATA OF QUADRATIC DIFFERENTIALS

AMOL AGGARWAL, VINCENT DELECROIX, ÉLISE GOUJARD, PETER ZOGRAF,
AND ANTON ZORICH

Abstract. We state conjectures on the asymptotic behavior of the Masur–
Veech volumes of strata in the moduli spaces of meromorphic quadratic dif-
ferentials and on the asymptotics of their area Siegel–Veech constants as the
genus tends to infinity.

Brief history of the subject and state of the art

The Masur–Veech volumes of strata in the moduli spaces of Abelian differentials
and in the moduli spaces of meromorphic quadratic differentials with at most simple
poles were introduced in fundamental papers [Ma] and [V]. These papers proved
that the Teichmüller geodesic flow is ergodic on each connected component of each
stratum with respect to the Masur–Veech measure introduced in these papers, and
that the total measure of each stratum is finite.

Masur–Veech volumes of strata of Abelian differentials. The first efficient
evaluation of volumes of the strata in the moduli spaces of Abelian differentials was
performed by A. Eskin and A. Okounkov [EO] twenty years later. The Masur–Veech
volumes of several low-dimensional strata of Abelian differentials were computed
just before that and by different methods in [Zo1]. The algorithm of A. Eskin and
A. Okounkov was implemented by A. Eskin in a rather efficient computer code
which already at this time allowed to compute volumes of all strata of Abelian
differentials up to genus 10, and volumes of some strata, like the principal one, up
to genus 60 (or more).

Direct computations of volumes of the strata of Abelian differentials in genera
accessible at this time combined with numerical experiments with the Lyapunov ex-
ponents providing an approximate value of the Siegel–Veech constant $c_\text{area}$ allowed
A. Eskin and A. Zorich to state conjectures on asymptotic behavior of volumes
of the strata in the moduli spaces of Abelian differentials and on the large genus
asymptotics of the associated Siegel–Veech constants. These conjectures were stated
at the end of 2003, but published in [EZo] only after publication of [EKZo]. The
latter paper proved the relation between the sum of the Lyapunov exponents and
the Siegel–Veech constant, and, thus, justified the prior experimental evidence.

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The study of the Masur–Veech volumes of the strata of Abelian differentials was tremendously advanced in recent years. D. Chen, M. Möller and D. Zagier proved in [CMöZa] the large genus asymptotic formula for the Masur–Veech volume of the principal stratum of Abelian differentials, confirming the conjectural formula from [EZo] for this particular case. A. Sauvaget proved in [S1] the asymptotics of Masur–Veech volume conjectured in [EZo] for the minimal stratum of Abelian differentials. A. Aggarwal proved in [Ag1] the conjectural volume asymptotics for all strata by combinatorial methods. Finally, D. Chen, M. Möller, A. Sauvaget and D. Zagier proved in [CMöSZa] the conjecture in maximal generality for all connected components of all strata of Abelian differentials combining combinatorics, geometry, and intersection theory. A. Sauvaget found in [S2] the asymptotic expansion in inverse powers of $g$ for these volumes. The same authors proved in [CMöSZa] very efficient recursive formula for the Masur–Veech volumes of the strata of Abelian differentials, which allows to find their exact values up to very large genera. Also, A. Aggarwal in [Ag2] and D. Chen, M. Möller, A. Sauvaget and D. Zagier in [CMöSZa] proved all conjectures from [EZo] on large genus asymptotics of the Siegel–Veech constants. Masur–Veech volumes and Siegel–Veech constants of hyperelliptic connected components were computed by J. Athreya, A. Eskin and A. Zorich in [AEZ2]. Due to these series of papers we now have comprehensive information on the Masur–Veech volumes and on Siegel–Veech constants for all strata of Abelian differentials.

**Masur–Veech volumes of strata of quadratic differentials.** Our knowledge of the Masur–Veech volumes and of the Siegel–Veech constants for the strata of meromorphic quadratic differentials is still relatively poor. Paper [AEZ2] proved exact closed formula (conjectured by M. Kontsevich) for the Masur–Veech volumes and for the Siegel–Veech constants in genus zero. The first reasonable table of volumes of strata of quadratic differentials was computed by E. Goujard in [G2], where she applied the algorithm of Eskin and Okounkov [EO] to compute volumes of all strata of dimension up to 11.

In recent paper [DGZZ] the authors found a formula for the Masur–Veech volumes of the strata with simple zeroes and simple poles through intersection numbers of $\psi$-classes on the Deligne–Mumford compactification $\overline{M}_{g,n}$ of the moduli space of complex curves and stated the conjecture on the large genus asymptotics of the Masur–Veech volume of the principal stratum of holomorphic quadratic differentials. The recent paper [CMöS] suggested an alternative formula for the Masur–Veech volumes of the same strata through certain very special linear Hodge integrals. The very recent paper of [Kaz] provided extremely efficient recursive formula for these Hodge integrals, which allows to compute volumes of strata with simple zeroes and several simple poles in genera about 100 in split seconds. This computation corroborates the conjectural formula of the authors for these strata. According to paper [CMöS], work in progress [YZZ] also corroborates and develops this conjecture. Moreover, as it will be explained in Section 2 formula of Chen–Möller–Sauvaget combined with efficient recursive algorithm of Kazarian, together allow to extend the conjecture to general strata in the moduli space of meromorphic quadratic differentials with at most simple poles.
1. Conjectural asymptotic formula

Let \( d = (d_1, \ldots, d_n) \) be an unordered partition of a positive integer number \( 4g - 4 \) divisible by 4 into a sum \(|d| = d_1 + \cdots + d_n = 4g - 4\), where \( d_i \in \{-1, 0, 1, 2, \ldots\} \) for \( i = 1, \ldots, n \). Denote by \( \Pi_{4g-4} \) the set of those partitions as above, which satisfy the additional requirement that the number of entries \( d_i = -1 \) in \( d \) is at most \( \log(g) \).

**Conjecture 1.** For any \( d \in \hat{\Pi}_{4g-4} \) one has

\[
\text{Vol} \ Q(d_1, \ldots, d_n) = \frac{4}{\pi} \prod_{i=1}^{n} \frac{2^{d_i+2}}{d_i+2} \cdot (1 + \varepsilon_1(d)),
\]

where

\[
\lim_{g \to \infty} \max_{d \in \Pi_{4g-4}} |\varepsilon_1(d)| = 0.
\]

**Remark 1.** We use the normalization of volumes as in \([AEZ2], [G2], [DGZZ]\) and \([CMoS]\). In particular, all the zeroes are labeled.

**Conjecture 2.** For non-hyperelliptic components \( Q \) of all strata \( Q(d) \) of meromorphic quadratic differentials with at most simple poles, where \( d \in \hat{\Pi}_{4g-4} \) and \( g \geq 6 \) one has

\[
c_{\text{area}}(Q) = \frac{1}{4} \cdot (1 + \varepsilon_2(d)),
\]

where

\[
\lim_{g \to \infty} \max_{d \in \Pi_{4g-4}} |\varepsilon_2(d)| = 0.
\]

**Corollary 1.** Applying the formula from \([EKZo]\) for the sum of the Lyapunov exponents of the Hodge bundle over the Teichmüller geodesic flow in any non-hyperelliptic connected component of a stratum \( Q(d) \) of meromorphic quadratic differentials with \( d \in \hat{\Pi}(4g-4) \) we get

\[
\lambda_1^+ + \cdots + \lambda_g^+ = \frac{1}{24} \cdot \sum_{d_i \in d} \frac{d_i(d_i + 4)}{d_i + 2} + \frac{\pi^2}{12} + \frac{\pi^2}{3} \cdot \varepsilon_2(d).
\]

**Remark 2.** Paper \([CMoS]\) proved formulae for the Masur–Veech volume of the principal strata of meromorphic quadratic differentials and for the corresponding Siegel–Veech constant \( c_{\text{area}} \) in terms of intersection numbers. This paper also suggested analogous conjectural formula for other strata. We hope that the algebro-geometric conjectures from \([CMoS]\) and the numerical conjectures as above might be mutually useful.

2. Numerical evidence.

**Siegel–Veech constant.** Numerical evaluation of Lyapunov exponents allows to compute approximate values of Siegel–Veech constants \( c_{\text{area}}(Q) \) for connected components \( Q \) of strata \( Q(d) \) in the moduli space of meromorphic quadratic differentials with at most simple poles through the following formula from \([EKZo]\):

\[
\lambda_1^- + \cdots + \lambda_{g-2}^- = \frac{1}{24} \cdot \sum_{d_i \in d} \frac{d_i(d_i + 4)}{d_i + 2} + \frac{1}{4} \cdot \sum_{d_j \in d, d_j \equiv 1 \mod 2} \frac{1}{d_j + 2} + \frac{\pi^2}{3} \cdot c_{\text{area}}(Q).
\]
Here the effective genus $g_{\text{eff}}$ is defined as $g_{\text{eff}} = \hat{g} - g$, where $\hat{g}$ is the genus of the canonical double cover $p : \hat{C} \to C$ (ramified at simple poles and at zeroes of odd orders of the quadratic differential $q$) such that $p^*q = \hat{\omega}$ is a square of a globally defined Abelian differential $\hat{\omega}$ on $\hat{C}$.

Simulations performed for numerous strata in large genera provide strong numerical evidence for conjectural asymptotics (2) from Conjecture 2. For example, experiments performed for about 20 random strata with $g = 0, 1, 2, 3$ simple poles in genera in the range from 20 to 30 give approximate values of $c_{\text{area}}(Q)$ varying from 0.247 to 0.259 for $10^6$ iterations of fast Rauzy induction. Analogous experiments with the principal stratum $Q_{g,0} = Q(1^g q^4)$ give approximate values of $c_{\text{area}}(Q_{g,0})$ varying from 0.251 to 0.253 for $10^7$ iterations of fast Rauzy induction for genera from 20 to 30 and approximate values of $c_{\text{area}}(Q_{g,0})$ varying from 0.255 to 0.258 for $10^8$ iterations of fast Rauzy induction for genera about 40. Similar simulations for the principal strata were independently performed by Ch. Fougeron.

As we explained in the introduction, combining very recent results [CM6S] and [Kaz] we can compute exact values of volumes of strata $Q_{g,n} = Q(1^g q^{4+n}, -1^n)$ and then compute exact values of the Siegel–Veech constants $c_{\text{area}}(Q_{g,n})$ due to the following result.

**Theorem (G1).** Let $g$ be a strictly positive integer, and $n$ nonnegative integer. When $g = 1$ we assume that $n \geq 2$. Under the above conventions the following formula is valid:

\[
(4) \quad c_{\text{area}}(Q_{g,n}) = \frac{1}{\text{Vol}(Q_{g,n})} \cdot \left( \frac{1}{8} \sum_{g_1 + g_2 = g} \frac{\ell!}{\ell_1! \ell_2!} \frac{n!}{(n_1 - 1)!(n_2 - 1)!} \cdot \frac{(d_1 - 1)! (d_2 - 1)!}{(d - 1)!} \text{Vol}(Q_{g_1,n_1}) \times \text{Vol}(Q_{g_2,n_2}) + \right.
\]

\[+ \left. \frac{1}{16} \frac{(4g - 4 + n) n(n - 1)}{(6g - 7 + 2n)(6g - 8 + 2n)} \text{Vol}(Q_{0,3}) \times \text{Vol}(Q_{g,n - 1}) + \right. \]

\[+ \left. \frac{\ell!}{(\ell - 2)!} \frac{(d - 3)!}{(d - 1)!} \text{Vol}(Q_{g-1,n+2}) \right) . \]

Here $d = \dim_C Q_{g,n} = 6g - 6 + 2n$, $d_i = 6g_i - 6 + 2n_i$, $\ell = 4g - 4 + n$, $\ell_i = 4g_i - 4 + n_i$.

The above formula gives exact values of $c_{\text{area}}(Q_{g,n})$ for pairs $g, n$ up to $g = 250$ and larger. The analysis of the resulting data provides serious numerical evidence towards validity of Conjecture 2. In particular, for $g = 3, 4, \ldots, 250$, the Siegel–Veech constant $c_{\text{area}}(Q_{g,0})$ is monotonously decreasing from 0.284275 to 0.250285. For a fixed genus $g$ and small values $n = 0, 1, 2, \ldots$ of $n$, the Siegel–Veech constant $c_{\text{area}}(Q_{g,n})$ does not fluctuate much. For example, the approximate values of $c_{\text{area}}(Q_{250,n})$ for $n = 0, 1, \ldots, 6$ are given by the following list:

\{0.250285, 0.250118, 0.249992, 0.249909, 0.249867, 0.249867, 0.249909\}.

**Volume asymptotics.** The arguments towards Conjecture 4 are indirect. We start by recalling the scheme which was used to formulate analogous Conjecture in [EZo] for large genus asymptotics of Masur–Veech volumes of strata of Abelian differentials. Certain phenomena which had a status of conjectures at the early stage of the project [EZo] are proved in the recent paper [Ag2] of A. Aggarwal. We
conjecture that strata in the moduli spaces of meromorphic quadratic differentials have analogous geometric properties.

Having explained the scheme in the case of Abelian differentials (where everything is proved by now) we describe necessary adjustments which we use for the case of quadratic differentials. In this latter case, part of results are still conjectural.

Stating volume asymptotics conjecture for Abelian differentials. Paper [EMaZo] provided a formula for the Siegel–Veech constant \( c_{\text{area}}(H(m)) \) of any (connected component of any) stratum \( H(m) \) in the moduli space of Abelian differentials. This formula expressed \( c_{\text{area}}(H(m)) \) through Masur–Veech volumes of the associated principal boundary strata normalized by \( \text{Vol} \, H(m) \). It was conjectured that in large genera a dominant part of the contribution to \( c_{\text{area}}(H(m)) \) comes from the boundary strata corresponding to simplest degenerations (represented by so-called configurations of multiplicity one). Neglecting contributions of more sophisticated boundary strata one obtains particularly simple approximate formula for \( c_{\text{area}}(H(m)) \) as an explicit weighted sum of ratios of volumes of simplest boundary strata over the volume of the stratum \( H(m) \). The only way to get the constant asymptotic value \( c_{\text{area}}(H(m)) \approx \frac{1}{2} \) in this expression leads to the formula

\[
(5) \quad \text{Vol} \, H(m_1, \ldots, m_n) = \text{const} \cdot \prod_{i=1}^{n} \frac{1}{m_i + 1} \cdot (1 + \varepsilon_3(m)),
\]

where \( \varepsilon_3(m) \to 0 \) as \( g \to +\infty \) uniformly for all partitions \( m \) of \( 2g - 2 \) into a sum of positive integers.

Evaluation of the universal constant \( \text{const} \) in the above formula is a separate nontrivial problem. Since the Masur–Veech volumes of strata appear in the approximate formula for \( c_{\text{area}}(H(m)) \) only in ratios of the volume of some smaller stratum over the volume of the original stratum, the global normalization constant cancels out in every ratio. In the case of Abelian differentials, the constant \( \text{const} \) was guessed from numerics. Namely, in the particular case of the principal stratum, the original formula of A. Eskin and A. Okounkov from [EO] allows to compute \( \text{Vol} \, H(1^{2g-2}) \) for genus \( g = 60 \) and higher, which allowed to guess the correct value \( \text{const} = 4 \). Asymptotics (4) was rigorously proved for particular cases in [CMZo1] and in [S1] and then in [Ag1] and in [CMZo2] in general case.

We address the reader interested in more details to [Zo2], where the notions of configuration of multiplicity one is explained in details, and where the contributions of such configurations to all possible Siegel–Veech constants, including \( c_{\text{area}}(H(m)) \), are computed in full details. The conjecture that the total contribution to \( c_{\text{area}}(H(m)) \) coming from more complicated configurations becomes negligible in large genera is recently proved by A. Aggarwal in [Ag2].

Stating volume asymptotics conjecture for quadratic differentials. Developing technique from [EMaZo] and using the topological description of the principal boundary strata in the moduli spaces of quadratic differentials given in [MaZo], E. Goujard obtained in [G1] an expression for the Siegel–Veech constant \( c_{\text{area}}(Q(d)) \) through Masur–Veech volumes of the stratum \( Q(d) \) and of its principal boundary. Formula (4) is a particular case of this more general formula.

Analogously to the situation with Abelian differentials, we expect that the total contribution to \( c_{\text{area}}(Q(d)) \) coming from the principal boundary strata different
from the simplest ones becomes negligible in high genera. We describe this conjecture in more details in Section 3. Extracting from the formula of Goujard for $c_{\text{area}}(Q(d))$ the contribution coming from these simplest degenerations we get an approximate formula for $c_{\text{area}}(Q(d))$ through a weighted sum of ratios of volumes of the boundary strata divided by the volume of the stratum under consideration. In the particular case of the strata $Q_{g,n}$ this corresponds to removing from (4) the summands containing products of the volumes, see Section 3 for more details.

Similarly to the case of Abelian differentials, (up to a global normalization factor) expression (4) is the unique expression, such that for any $d \in \hat{\Pi}$ the resulting weighted sum tends to the asymptotic value $c_{\text{area}}(Q(d)) \approx \frac{4}{\pi}$. We provide this weighted sum for $c_{\text{area}}(Q(d))$ in Section 3 below.

As in the case of Abelian differentials, this approach does not allow to find the global constant factor in the asymptotic formula (1). The corresponding factor $\frac{4}{\pi}$ was originally conjectured in [DGZZ] for the large genus asymptotics of the Masur–Veech volume of the principal stratum of holomorphic quadratic differentials. This conjecture is based on fine geometric considerations combined with elaborate computations performed in [DGZZ].

The formula from [CMos] for the same volume combined with a very efficient recursion from [Kaz] for the corresponding linear Hodge integrals, involved in this formula, provide together very serious numerical evidence for validity of our conjecture for the strata with only simple zeroes and with simple poles, when the number of simple poles is small enough.

For any pair $g, n$ of nonnegative integers define

$$\text{Vol}^\text{appr} Q_{g,n} = \frac{4}{\pi} \cdot 2^n \cdot \left(\frac{8}{3}\right)^{4g-4+n}.$$ 

This value corresponds to the right-hand side of expression (4) applied to the partition $d = (1^{4g-4+n}, -1^n)$, where the factor $(1 + \varepsilon_1(d))$ is omitted. The sequence of ratios $\frac{\text{Vol}^\text{appr} Q_{g,0}}{\text{Vol} Q_{g,0}}$ is monotonously decreasing for the range $g = 3, 4, \ldots, 250$, from $\frac{\text{Vol} Q_{2,0}^\text{appr}}{\text{Vol} Q_{2,0}} \approx 1.01892$ to $\frac{\text{Vol} Q_{250,0}^\text{appr}}{\text{Vol} Q_{250,0}} \approx 1.00027$. For any fixed genus $g$ and small values $n = 0, 1, 2, \ldots$ of $n$, the volumes do not fluctuate much. For example, the approximate values of $\frac{\text{Vol} Q_{250,n}^\text{appr}}{\text{Vol} Q_{250,n}}$ for $n = 0, 1, \ldots, 5$ are given by the following list:

\{1.00027406, 1.00027477, 1.00027493, 1.00027481, 1.00027469, 1.00027484\}.

In particular, these numerical data corroborates our conjectural value $\frac{4}{\pi}$ of the universal normalizing constant.

Remark 3. Analogous analysis of numerical data was independently performed by D. Chen, M. Möller and A. Sauvaget who used the algorithm from [YZZ].

Range of validity of the conjecture. E. Goujard computed exact values of the Masur–Veech volumes of strata in the moduli space of quadratic differentials for strata of dimension at most 11. Consider expression (4) with omitted factor $(1 + \varepsilon_1(d))$ and divide it by the exact value of the volume $\text{Vol} Q(d)$. The resulting ratio evaluated for strata of genera 5 and 6 of dimension 11 varies from 1.037
to 1.135, which suggests that formula (1) gives reasonable approximation of the Masur–Veech volume already for strata of relatively small genus.

3. Contribution of configurations of multiplicity one to \( c_{\text{area}}(Q(d)) \).

We refer the reader to [MaZo] and to [G2] for the notions “configurations of homologous saddle connections” and “principal boundary strata”.

**Conjecture 3.** For all strata \( Q(d_1, \ldots, d_n) \) with \( (d_1, \ldots, d_n) \in \tilde{\Pi}(4g-4) \), the Siegel–Veech constant \( c_{\text{area}}(Q(d_1, \ldots, d_n)) \) is asymptotically supported on the following set of configurations:

\[
C_{b,1}(d_i, d_j)
\]

where \( d_i, d_j, a_1, a_2 \) are such that \( d_i, d_j \geq 1, \{d_i, d_j\} \subset \{d_1, \ldots, d_n\}, \) where \( i \neq j \), \( a_1, a_2 \geq 0 \), \( a_1 + a_2 \geq 3 \) and \( a_1 + a_2 + 2 \in \{d_1, \ldots, d_n\} \).

**Corollary 2.** For \( n \leq \log(g-1) - 2 \) a combination of Conjecture 1 and Conjecture 3 implies Conjecture 4

**Proof.** First note that our normalization of the Masur–Veech volume implies that

\[
\text{Vol } Q(0, d_1, \ldots, d_n) = 2 \text{Vol } Q(d_1, \ldots, d_n),
\]

see [AEZ2]. Hence,

\[
\varepsilon_1(0, d_1, \ldots, d_n) = \varepsilon_1(d_1, \ldots, d_n),
\]

where the quantity \( \varepsilon_1(d) \) is defined in [G1]. Thus, it is sufficient to prove Corollary 2 for partitions which do not contain entries 0 (representing the marked points).

Choose any pair of indices \( 1 \leq i < j \leq n \) such that \( d_i, d_j \geq 1 \). Applying Theorem 1 from [G1], we obtain the following expression for the contribution \( c_{\text{area}}(C_{b,1}(d_i, d_j)) \) of the configuration \( C_{b,1}(d_i, d_j) \) to \( c_{\text{area}}(Q(d)) \):

\[
c_{\text{area}}(C_{b,1}(d_i, d_j)) = \frac{(d - 3)!}{(d - 1)!} \cdot 2d_i d_j \cdot \frac{\text{Vol } Q_{g-1}(d_1, \ldots, d_i - 2, \ldots, d_j - 2, \ldots, d_n)}{\text{Vol } Q_g(d_1, \ldots, d_n)},
\]

where \( d = \dim_{\mathbb{C}} Q_g(d_1, \ldots, d_n) = 2g + n - 2 \). Note that the symbol \( d \) in bold denotes a partition in \( \tilde{\Pi}(3g - 3 + n) \), while the symbol \( d \) denotes the complex dimension of the stratum \( Q(d) \).

Assuming that \( g \gg 1 \) and using Conjecture 1 we obtain

\[
(6) \quad c_{\text{area}}(C_{b,1}(d_i, d_j)) \sim \frac{(d_i + 2)(d_j + 2)}{2^3 d^2}.
\]

Choose any index \( i \) in \( \{1, \ldots, n\} \) such that \( d_i \geq 3 \) (if such index exists). By Theorem 1 from [G1], for any pair of nonnegative integers \( a_1, a_2 \) satisfying \( a_1 + a_2 = d_i - 2 \), the contribution \( c_{\text{area}}(C_{b,11}(a_1, a_2)) \) of the configuration \( C_{b,11}(a_1, a_2) \) to \( c_{\text{area}}(Q(d)) \) has the following form:

\[
c_{\text{area}}(C_{b,11}(a_1, a_2)) = \frac{(d - 3)!}{(d - 1)!} \cdot 2(a_1 + a_2) \cdot \frac{\text{Vol } Q_{g-1}(d_1, \ldots, d_i - 4, \ldots, d_n)}{\text{Vol } Q_g(d_1, \ldots, d_n)},
\]
so, assuming that $g \gg 1$ and using Conjecture \[1\] we obtain the following contribution to $\mathrm{area}(\mathcal{Q}(d))$ of all pairs $a_1,a_2$ corresponding to the fixed $d_i$ as above:

\[
\begin{align*}
(7) \quad \frac{1}{2} \sum_{a_1=0}^{d_i-2} c_{\text{area}}(\mathcal{C}_{b,II}(a_1,a_2)) &= \frac{1}{24d^2} (d_i - 1)(d_i + 2) = \\
&= \frac{1}{24d^2} \left( (d_i + 2)^2 - 3(d_i + 2) \right).
\end{align*}
\]

We use notation $\mathcal{Q}(d_1,\ldots,d_n) = \mathcal{Q}(-1^{\mu-1},1^{\mu_1},2^{\mu_2},\ldots)$ to record the multiplicities of entries $-1,1,2,\ldots$ in the partition $d$.

Equation \[6\] implies that

\[
\begin{align*}
(8) \quad 2^3d^2 \sum_{1 \leq i,j \leq n \atop d_i,d_j \geq 1} c_{\text{area}}(\mathcal{C}_{b,1}(d_i,d_j)) &\sim \sum_{1 \leq i,j \leq n} (d_i + 2)(d_j + 2) - \\
&- \mu_1 \sum_{i=1}^{n} (d_i + 2) + \frac{\mu_1(\mu_1 + 1)}{2}.
\end{align*}
\]

Equation \[7\] implies that

\[
\begin{align*}
(9) \quad 2^4d^2 \sum_{1 \leq i,j \leq n \atop d_i \geq 3 \atop a_1,a_2 \geq 0} c_{\text{area}}(\mathcal{C}_{b,II}(a_1,a_2)) &\sim \sum_{i=1}^{n} (d_i + 2)^2 - 3 \sum_{i=1}^{n} (d_i + 2) - \\
&- (\mu_1 + 9\mu_1 + 16\mu_2) + 3(\mu_1 + 3\mu_1 + 4\mu_2).
\end{align*}
\]

Note that

\[
(10) \quad \sum_{i=1}^{n} (d_i + 2) = \left( \sum_{i=1}^{n} d_i \right) + 2n = 4g - 4 + 2n = 2d,
\]

and, in particular, $\mu_1 + \mu_1 + \mu_2 + \cdots = n < d$, which implies that

\[
(11) \quad \frac{2\mu_1 - 6\mu_1 - 12\mu_2}{d^2} \to 0 \quad \text{as } d \to +\infty.
\]

Note also, that by assumption the number of simple poles in a stratum is much smaller than the genus $g$. Since $g < d$ we get

\[
\frac{\mu_1}{d} \to 0 \quad \text{as } d \to +\infty.
\]

We conclude that for partitions $d$ as in the statement of Corollary \[2\] with $\mu_0 = 0$ we get the following asymptotics as $g \to +\infty$:

\[
\begin{align*}
c_{\text{area}}(\mathcal{Q}(d)) &\sim \sum_{1 \leq i,j \leq n \atop d_i,d_j \geq 1} c_{\text{area}}(\mathcal{C}_{b,1}(d_i,d_j)) + \sum_{d_i \geq 3 \atop a_1,a_2 \geq 0} c_{\text{area}}(\mathcal{C}_{b,II}(a_1,a_2)) \sim \\
&\sim \frac{1}{4} \left(\frac{1}{(2d)^2} \left( \sum_{i=1}^{n} (d_i + 2) \right)^2 - 3 \sum_{i=1}^{n} (d_i + 2) \right) = \frac{1}{4} \left(\frac{1}{(2d)^2} \left( (2d)^2 - 3 \cdot 2d \right) \sim \frac{1}{4},
\end{align*}
\]

where the first equivalence is the statement of Conjecture \[3\] and the second equivalence is the combination of \[8\]–\[11\].
Remark 4. Note that in this proof we used much weaker restriction on the growth rate of the number of simple poles, namely it was sufficient to have \( \mu - 1 = o(g) \) as \( g \to +\infty \).

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Harvard University, Department of Mathematics, 1 Oxford Street, Cambridge, MA 02138, USA
E-mail address: agg_a@math.harvard.edu

LaBRI, Domaine universitaire, 351 cours de la Libération, 33405 Talence, FRANCE
E-mail address: 20100.delecroix@gmail.com

Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351 cours de la Libération, 33405 Talence, FRANCE
E-mail address: elise.goujard@gmail.com

St. Petersburg Department, Steklov Math. Institute, Fontanka 27, St. Petersburg 191023, and Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O. 29B, St.Petersburg 199178 Russia
E-mail address: zograf@pdmi.ras.ru

Center for Advanced Studies, Skoltech; Institut de Mathématiques de Jussieu – Paris Rive Gauche, Case 7012, 8 Place Aurélie Nemours, 75205 PARIS Cedex 13, France
E-mail address: anton.zorich@gmail.com