Canonical Formulation of Gravitational Teleparallelism in 2+1 Dimensions in Schwinger’s Time Gauge

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Abstract

We consider the most general class of teleparallel gravitational theories quadratic in the torsion tensor, in three space-time dimensions, and carry out a detailed investigation of its Hamiltonian formulation in Schwinger’s time gauge. This general class is given by a family of three-parameter theories. A consistent implementation of the Legendre transform reduces the original theory to a one-parameter family of theories. By calculating Poisson brackets we show explicitly that the constraints of the theory constitute a first-class set. Therefore the resulting theory is well defined with regard to time evolution. The structure of the Hamiltonian theory rules out the existence of the Newtonian limit.

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§1. Introduction

Gravitational theories in three space-time dimensions have attracted considerable attention in the last years[1]. In particular quantum effects in this simplified geometrical context were investigated (see, for instance, references[2, 3]). The hope is that lower dimensional theories would provide hints as to the quantization of four-dimensional general relativity. It is known that vacuum Einstein’s theory in 2+1 dimensions does not yield a suitable description of the gravitational field[4, 5]. Since in 2+1 dimensions the Riemann and Ricci tensors have the same number of components, the vanishing of Einstein’s equations imply that the full curvature tensor vanishes as well. Therefore the source-free space-time is flat, and thus the existence of black holes is prevented. It would be interesting to find a theoretical formulation of 2+1 general relativity that display two important features: the Newtonian limit and a black hole solution.

General relativity can be described in the alternative framework of the teleparallel geometry. A four-dimensional formulation of gravitational teleparallel theories, quadratic in the torsion tensor, and formulated with arbitrary parameters was proposed by Hayashi and Shirafuji[6]. This formulation was successful in that the desirable features above were obtained for a large class of theories. A canonical formulation of the teleparallel equivalent of general relativity (TEGR) was developed in [7].

In this paper we consider an arbitrary teleparallel theory of gravity in 2+1 dimensions, expressed by a three-parameter family of theories, and show that by means of a consistent Legendre transform this arbitrary theory reduces to a one-parameter family of theories. The 2+1 decomposition is carried out in Schwinger’s time gauge[8]. Moreover, we calculate all relevant Poisson brackets and conclude that the constraints of the theory are first class. As we will show, the 2+1 constraint algebra differs slightly from the previously evaluated algebra in 3+1 dimensions[7]. The resulting theory shares similarities with the Hamiltonian formulation of the four-dimensional teleparallel equivalent of general relativity[7].

The main motivation for considering the TEGR is that the energy and momentum of the gravitational field can be interpreted as arising from the constraint equations of the theory[9]. All analysis carried out so far indicate that the gravitational energy is consistently defined by means of the expression that arises in the realm of the TEGR. The most relevant and successful
application amounts to the evaluation of the irreducible mass of rotating black holes\[10\]. Recently the loss of mass by means of gravitational waves in the context of Bondi’s radiating metric has been investigated\[11\].

A family of three-parameter teleparallel theories in 2+1 dimensions, in the Lagrangian formulation, was proposed and investigated by Kawai\[12\]. By establishing conditions on the parameters, black hole solutions were obtained\[12, 13\]. Such black hole solutions are quite different from the Schwarzschild and Kerr black holes in 3+1 dimensions. However, Kawai did not consider the cosmological constant in the theory.

The arbitrary teleparallel theory we address corresponds precisely to Kawai’s formulation. We have constructed the canonical formulation of the latter by applying Dirac’s formalism for constrained Hamiltonian systems\[14\]. Therefore we arrive at a one-parameter class of theories by only requiring it to have a well defined Hamiltonian formulation.

The paper is divided as follows. In §2 we introduce the Lagrangian formulation. The 2+1 space-time decomposition and the canonical formalism is developed in §3. In this section we provide the details of the Legendre transform. The constraint algebra is presented in §4. The calculations that yield the constraint algebra are too intricate, and therefore we have omitted it here. In §5 we show that the conditions obtained on the parameters of the theory rule out the existence of the Newtonian limit. Finally in §6 we present our conclusions.

Our notation is the following: space-time indices \(\mu, \nu, \ldots\) and global SO(2,1) indices \(a, b, \ldots\) run from 0 to 2. In the 2+1 canonical decomposition Latin indices from the middle of the alphabet indicate space indices according to \(\mu = 0, i, a = (0), (i)\). The flat space-time metric is fixed by \(\eta_{(0)(0)} = -1\).

§2. Lagrangian formulation of teleparallelism in (2+1) dimensions

We begin by stating the four basic postulates that the Lagrangian density for the gravitational field in the empty space, in the teleparallel geometry, must satisfy. It must be invariant under (i) coordinate transformations, (ii) global Lorentz (SO(2,1)) transformations, (iii) parity transformations, and (iv) must be quadratic in the torsion tensor. The most general Lagrangian density is constructed out of triads \(e^a\mu\), and is given by\[12\]
\[ L_0 = e \left( c_1 t_{abc} + c_2 v^a v_a + c_3 a_{abc} a^{abc} \right) \]  

(2.1)

where \( c_1, c_2 \) and \( c_3 \) are constants and

\[ t_{abc} = \frac{1}{2} (T_{abc} + T_{bac}) + \frac{1}{4} (\eta_{ca} v_b + \eta_{cb} v_a) - \frac{1}{2} \eta_{ab} v_c , \]  

(2.2)

\[ v_a = T^b_{ba} \equiv T_a , \]  

(2.3)

\[ a_{abc} = \frac{1}{3} (T_{abc} + T_{cab} + T_{bca}) , \]  

(2.4)

\[ T_{abc} = e^\mu_b e^\nu_c T_{a\mu\nu} = e^\mu_b e^\nu_c (\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) . \]  

(2.5)

where \( e = \det(e^a_\mu) \).

The Lagrangian density (2.1) is constructed out of the anti-symmetric part of the connection \( \Gamma^\lambda_{\mu\nu} = e^a_\lambda \partial_\mu e_{a\nu} \), whose curvature tensor vanishes identically. Such connection defines a space with absolute parallelism, or teleparallelism.

The definitions above correspond to the irreducible components of the torsion tensor. In order to construct the Hamiltonian formulation we need to rewrite \( L_0 \) such that the torsion tensor appears as a multiplicative quantity. It can be shown that \( L_0 \) can be rewritten as

\[ L_0 = e \left( c_1 X^{abc} T_{abc} + c_2 Y^{abc} T_{abc} + c_3 Z^{abc} T_{abc} \right) , \]  

(2.6)

where

\[ X^{abc} = \frac{1}{2} T^{abc} + \frac{1}{4} T^{bac} - \frac{1}{4} T^{cab} + \frac{3}{8} (\eta^{ca} v^b - \eta^{ba} v^c) , \]  

(2.7)

\[ Y^{abc} = \frac{1}{2} (\eta^{ab} v^c - \eta^{ac} v^b) , \]  

(2.8)

\[ Z^{abc} = \frac{1}{3} (T^{abc} + T^{bca} + T^{cab}) . \]  

(2.9)

The definitions above satisfy

\[ X^{abc} = -X^{acb}, Y^{abc} = -Y^{acb}, Z^{abc} = -Z^{acb} , \]  

(2.10)
The quantities $X^{abc}$, $Y^{abc}$, $Z^{abc}$ have altogether the same number of components of $T^{abc}$. It can be verified that

$$X^{abc} + X^{bca} + X^{cab} \equiv 0 \quad . \quad (2.11)$$

Let us define $\Sigma^{abc}$ by

$$\Sigma^{abc} = c_1 X^{abc} + c_2 Y^{abc} + c_3 Z^{abc} \quad . \quad (2.12)$$

In terms of $\Sigma^{abc}$, $L_0$ can be simply written as

$$L_0 = e \Sigma^{abc} T_{abc} \quad . \quad (2.13)$$

In order to carry out the Hamiltonian formulation we need to write the Lagrangian density in a Palatini-type Lagrangian density. The latter is achieved by introducing the field variable $\Delta^{abc} = -\Delta_{acb}$, that will be ultimately identified with the torsion tensor by means of the field equations. By following the procedure of [7] we write the first order differential form of $L_0$ as

$$L (e_{\mu \nu}, \Delta^{abc}) = -e \left( c_1 \Theta^{abc} + c_2 \Omega^{abc} + c_3 \Gamma^{abc} \right) \left( \Delta^{abc} - 2T_{abc} \right) \quad , \quad (2.14)$$

where $\Theta^{abc}$, $\Omega^{abc}$ and $\Gamma^{abc}$ are defined in similarity to $X^{abc}$, $Y^{abc}$ and $Z^{abc}$, respectively:

$$\Theta^{abc} = \frac{1}{2} \Delta^{abc} + \frac{1}{4} \Delta^{bac} - \frac{1}{4} \Delta^{cab} + \frac{3}{8} \left( \eta^{ca} \Delta^b - \eta^{ba} \Delta^c \right) \quad , \quad (2.15)$$

$$\Omega^{abc} = \frac{1}{2} \left( \eta^{ab} \Delta^c - \eta^{ac} \Delta^b \right) \quad , \quad (2.16)$$

$$\Gamma^{abc} = \frac{1}{3} \left( \Delta^{abc} + \Delta^{bca} + \Delta^{cab} \right) \quad . \quad (2.17)$$

The three quantities above are anti-symmetric in the last two indices. The quantity $\Delta^b$ is defined by $\Delta^b = \Delta^a_a b$.

The field equations are most easily obtained by making use of the following three identities:

$$X^{abc} \Delta_{abc} = \Theta^{abc} T_{abc} \quad , \quad (2.18)$$
\[ Y^{abc} \Delta_{abc} = \Omega^{abc} T_{abc} \quad , \quad (2.19) \]
\[ Z^{abc} \Delta_{abc} = \Gamma^{abc} T_{abc} \quad . \quad (2.20) \]

By taking into account the identities above the Lagrangian density in first order formalism is identically rewritten as

\[ L = -ec_1 \Theta^{abc} \Delta_{abc} + 2ec_1 X^{abc} \Delta_{abc} - ec_2 \Omega^{abc} \Delta_{abc} + 2ec_2 Y^{abc} \Delta_{abc} \]

\[ -ec_3 \Gamma^{abc} \Delta_{abc} + 2ec_3 Z^{abc} \Delta_{abc} \quad . \quad (2.21) \]

The variation of \( L \) with respect to \( \Delta_{def} \) is given by

\[ \frac{\delta L}{\delta \Delta_{def}} = -ec_1 \frac{\delta (\Theta^{abc} \Delta_{abc})}{\delta \Delta_{def}} + 2ec_1 X^{abc} \frac{\delta (\Delta_{abc})}{\delta \Delta_{def}} \]

\[ -ec_2 \frac{\delta (\Omega^{abc} \Delta_{abc})}{\delta \Delta_{def}} + 2ec_2 Y^{abc} \frac{\delta (\Delta_{abc})}{\delta \Delta_{def}} \]

\[ -ec_3 \frac{\delta (\Gamma^{abc} \Delta_{abc})}{\delta \Delta_{def}} + 2ec_3 Z^{abc} \frac{\delta (\Delta_{abc})}{\delta \Delta_{def}} \quad . \quad (2.22) \]

\[ \frac{\delta L}{\delta \Delta_{def}} = -2ec_1 \Theta^{def} + 2ec_1 X^{def} - 2ec_2 \Omega^{def} \]

\[ +2ec_2 Y^{def} - 2ec_3 \Gamma^{def} + 2ec_3 Z^{def} \quad , \quad (2.23) \]

where we have used

\[ \frac{\delta (\Theta^{abc} \Delta_{abc})}{\delta \Delta_{def}} = 2\Theta^{abc} \frac{\delta \Delta_{abc}}{\delta \Delta_{def}} \quad , \quad (2.24) \]

\[ \frac{\delta (\Omega^{abc} \Delta_{abc})}{\delta \Delta_{def}} = 2\Omega^{abc} \frac{\delta \Delta_{abc}}{\delta \Delta_{def}} \quad , \quad (2.25) \]
since the quantities between parentheses in the left hand side of the expressions above are quadratic in $\Delta_{abc}$. This result can be verified by explicit calculations.

By considering the action integral

$$I = \int Ld^3x,$$  

(2.27)

and imposing the variation

$$\frac{\delta I}{\delta \Delta_{def}} = \int \frac{\delta L}{\delta \Delta_{def}} d^3x = 0,$$  

(2.28)

we arrive at

$$c_1 \left( X^{def} - \Theta^{def} \right) + c_2 \left( Y^{def} - \Omega^{def} \right) + c_3 \left( Z^{def} - \Gamma^{def} \right) = 0.$$

(2.29)

By taking into account equations (2.7) $\sim$ (2.9) and (2.15) $\sim$ (2.17), the only solution to equation (2.29), for arbitrary values of $c_i$, is given by

$$\Delta_{abc} = T_{abc} = e_b^\mu e_c^\nu T_{\mu\nu},$$

(2.30)

that implies

$$X^{abc} = \Theta^{abc},$$

(2.31)

$$Y^{abc} = \Omega^{abc},$$

(2.32)

$$Z^{abc} = \Gamma^{abc}.$$  

(2.33)

Note that the equation (2.29) represents nine equations for nine unknown quantities $\Delta_{abc}$.

§3. The Hamiltonian formulation
In this section it will be necessary to make a change of notation. The three-dimensional space-time triads of the last section will be denoted here as $3^{a}_\mu$, whereas diads restricted to the two-dimensional spacelike surface will be represented simply by $e^a_\mu$. This distinction is mandatory in the 2+1 decomposition of the triads. The latter is similar to the 3+1 decomposition of tetrad fields.

The 2+1 decomposition of triads is given by

$$3^{a}_k = e^a_k$$

$$3^{a'i} = e^{a'i} + \frac{N^i}{N} \eta^a$$

$$e^{a'i} = \overline{F}^k e^a_k \quad \eta^a = -N \quad 3^{a'0}$$

$$3^{a}_0 = N^i e^a_i + N \eta^a$$

$$\eta^a e_{ak} = 0 \quad \eta_a \eta^a = -1 \quad 3 e = N e$$ \hspace{1cm} (3.1)

where $N$ and $N^i$ are the lapse and shift functions, respectively, and

$$3 e = \det \left(3^{a}_\mu \right)$$

$$g_{ij} = e^{a'i} e_{aj}$$

$$\overline{g}^{ij} g_{jk} = \delta^i_k .$$ \hspace{1cm} (3.2)

Therefore,

$$\left( \overline{F}^k \right) \sim \left( g_{ij} \right)^{-1} .$$ \hspace{1cm} (3.3)

It follows that

$$e^{b'k} e_{b'j} = \delta^k_j ,$$

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\[ e^a_i e^b_i = \eta^{ab} + \eta^a \eta^b. \] (3.4)

The components \( e^{ai} \) and \( e^a_k \) are now restricted to the two-dimensional space-like surface.

The Hamiltonian formulation is achieved by rewriting the Lagrangian density (2.14) in the form \( L = p_q - H \). For this purpose we define, in analogy with (2.12), the field quantity \( \Lambda^{abc} \) according to

\[ \Lambda^{abc} = c_1 \Theta^{abc} + c_2 \Omega^{abc} + c_3 \Gamma^{abc}, \] (3.5)

By means of (3.5) we define \( P^{ai} \), the momentum canonically conjugated to \( e_{ai} \). It is given by

\[ P^{ai} = 4^3 e \Lambda^{a0i} = 4 e e^b_i \eta_c \Lambda^{abc}. \] (3.6)

In terms of \( P^{ai} \) we write the Lagrangian density as

\[ L = P^{ai} \dot{e}^{ai} + 3 e a_0 \partial_i P^{ai} \]
\[ +2 Ne \Lambda^{aij} T_{aij} + N^k P^{ai} T_{aik} \]
\[ -Ne \Lambda^{abc} \Delta_{abc} - \partial_i \left[ 3 e a_0 P^{ai} \right]. \] (3.7)

We note that there is no time derivative of \( 3 e a_0 \). Thus the momentum \( P^{a0} \) canonically conjugated to \( e_{a0} \) is taken to vanish from the outset.

At this point we adopt Schwinger’s time gauge [8],

\[ \eta^a = \delta^a_{(0)}, \quad \eta_a = -\delta_a^{(0)}. \] (3.8)

Conditions above imply

\[ 3 e_{(0)} = e^{(0)} = 0. \]

The time gauge is taken to hold before varying the action, since the fixing of this gauge is not a consequence of any local symmetry of the theory. The imposition of time gauge actually corresponds to a reduction in the configuration space of the theory. Whereas the fixation of this gauge in the
Lagrangian formulation of a teleparallel theory involves some intricacies, in the Hamiltonian formulation it amounts to a straightforward procedure.

By making use of the Lagrangian field equations we identify

\[ \Delta_{aij} = T_{aij} \]  

in \( L \), since these equations do not involve time derivatives.

Following \[7\], we wish to establish a 2+1 decomposition for \( \Lambda^{abc} \) that distinguishes the components of the latter that are projected (restricted) to the two-dimensional spacelike surface from those that define the canonical momenta. Assuming Schwinger’s time gauge we write

\[ \Lambda^{abc} = \frac{1}{4e} \left( \eta^b e^c_i P^{ai} - \eta^c e^b_i P^{ai} \right) + e^b_i e^c_j \Lambda^{aij}, \]  

(3.10)

where

\[ \Lambda^{aij} = e^a_k \Lambda^{kij}. \]  

(3.11)

In the expression above \( \Lambda^{kij} \) is a tensor on the two-dimensional spacelike surface.

The Legendre transform would be straightforward if \( \Lambda^{aij} \) would depend only on \( e^{(k)j} \) and its spatial derivatives. However in general this is not the case. The main issue is that after the Legendre transform has been performed, the Hamiltonian density cannot depend on the “velocities” \( \Delta_{a0j} = T_{a0j} \), which contain terms of the type \( \dot{e}_{ai} \). The quantities \( \Lambda^{aij} \) in (3.10) in general contain terms like \( \Delta_{a0j} \). Such terms cannot be present in the final form of the Hamiltonian density obtained via \( H = P^{ai}(x) \dot{e}_{ai}(x) - L \). This goal will be achieved by posing restrictions on the constants \( c_i \), as we will see, and by invoking the time gauge condition.

Since the momenta is defined by (3.6), and since \( P^{ai} \) is an irreducible component of \( \Lambda^{abc} \) as given by (3.10), we expect the contribution of (3.11) to the Lagrangian density not to yield velocities terms of the type \( \Delta_{a0j} \). Thus \( \Lambda^{aij} \) in (3.10) must not depend on the momenta, and therefore it cannot lead to the emergence of velocities in the final form of \( L \).

As we mentioned earlier, the time gauge condition reduces the configuration space of the theory from the SO(2,1) (in \( L \)) to the SO(2) group (in \( H \)). As a consequence of \( \dot{e}_{(0)i} = 0 \) the teleparallel geometry is restricted to the
two-dimensional spacelike surface.

We will now express the several components of $L$ in (3.7) by means of the 2+1 decomposition of the triads and of $\Lambda^{abc}$. Considering definitions (3.5) and (3.6) we can obtain by explicit calculations the expression of $P^{(0)k}$. It is given by

$$P^{(0)k} = -2eT^{(0)}_{(i)k} \left( \frac{3}{4}c_1 + c_2 \right) + eT^k\left( \frac{3}{2}c_1 - 2c_2 \right).$$  \hspace{1cm} (3.12)

where $T^k = T^{(i)}_{(i)k}$.

Considering first $3e_{a0}\partial_i P^{ai}$ we find

$$3e_{a0}\partial_i P^{ai} = N^k e_{(j)k}\partial_i P^{(j)i} - N\partial_k P^{(0)k}.$$  \hspace{1cm} (3.13)

As to the surfaced term $-\partial_i(3e_{a0}P^{ai})$ we have

$$-\partial_i \left[ P^{ai} \left( N^k e_{ak} + N\eta_a \right) \right] = -\partial_i \left[ N_k P^{ki} \right] + \partial_i \left[ NP^{(0)i} \right].$$  \hspace{1cm} (3.14)

Let us consider the term $-Ne\Lambda^{abc}\Delta_{abc}$. By means of (3.1), (3.8) and (3.10) this term can be rewritten after a long calculation as

$$-Ne\Lambda^{abc}\Delta_{abc} = +N \left( \frac{c_1}{4} - \frac{c_3}{3} \right)^{-1} \times$$

$$\times \left\{ \frac{1}{16e} \left( P^{ij}P_{ji} - P^{(0)l}P^{(0)l} \right) + \frac{1}{2} \left( e_{(m)i}P^{(m)j}\Lambda^{(0)ij} \right) - ee_{(m)i}e_{(n)k}\Lambda^{(n)ij}\Lambda_{(m)kj} \right.$$  

$$+ \left( \frac{3c_1}{8c_2} - \frac{1}{2} \right) \left\{ \frac{1}{16e} \left[ P^2 - P^{(0)l}P^{(0)l} \right] + \frac{1}{2} P^{(0)k}e_{(m)j}\Lambda^{(m)jk} \right.$$
\[-e e_{(m)} e^{(m) ik} e_{(n)} e^{(n)j} \Lambda^{(n)j}_k \] 

\[+ \left( \frac{c_1}{4} + \frac{2c_3}{3} \right) \left\{ \frac{1}{4} \Delta_{ij(0)} P^{ji} - e \Delta_{i(0)j} \Lambda^{(0)ij} \right. \]

\[+ \left. e \Delta_{ikj} \Lambda^{kij} + \Delta_{(0)(0)j} P^{(0)i} + \Delta_{(0)ij} P^{ij} \right\} \] 

(3.15)

Space indices are raised and lowered by means of $e^{(i)j}$ and $e^{(k)l}$. The quantity $P$ is defined by $P = P^{(ijkl)} e^{(ijkl)}$. We observe that $\Lambda^{(0)ij}$ contains time derivatives $\dot{e}_{(i)j}$. However, we note that the third term on the right hand side of the expression above can be rewritten as

\[\frac{1}{2} (e_{(m)} e^{(m) j} \Lambda^{(0)ij}) = \frac{1}{2} (P_{[ij]} \Lambda^{(0)ij}), \] 

(3.16)

where $[..]$ denotes anti-symmetrization. It is known that in tetrad type theories of gravity the anti-symmetric part of the canonical momenta vanishes weakly. Let us obtain here the full expression of $P_{[ij]}$. Making use of (3.5) and (3.6) we find

\[P_{[ij]} + e T_{(0)ij} \left( c_1 - \frac{4}{3} c_3 \right) + e T_{[i(0)j]} \left( c_1 + \frac{8}{3} c_3 \right) = 0 \] 

(3.17)

Note that in the time gauge the term $T_{(0)ij}$ vanishes.

We consider finally the term $2N e \Lambda^{ai} T_{aij}$. In the 2+1 decomposition it reads

\[2N e \Lambda^{aij} T_{aij} = 2N e (\Lambda^{(0)ij} T_{(0)ij} + \Lambda^{(k)ij} T_{(k)ij}) \] 

(3.18)

The time gauge simplifies the expression above in two aspects. Because of it, the first term on the right hand side vanishes. Moreover, it can be verified by explicit calculations that the second term does not contain velocity terms $\Delta_{a0j} = T_{a0j}$.

By carefully inspecting expressions (3.12) and (3.15) we arrive at the conditions that allow a well defined Legendre transform. From (3.12) we observe that we must demand

\[c_2 + \frac{3c_1}{4} = 0 \.

(3.19)

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Furthermore by requiring
\[ c_1 + \frac{8c_3}{3} = 0, \]  
the last five terms of expression (3.15) drop out. Four of these terms are velocity dependent. In view of the argument presented above, according to which the momentum \( P^{ai} \) is an irreducible component of \( \Lambda^{abc} \), these terms must be eliminated in the Legendre transform. As a consequence, expression (3.17) also becomes exempt of velocity terms. In the time gauge equation (3.17) reads
\[ P_{[ij]} = 0, \]  
and therefore it must appear in the Hamiltonian formulation as constraint equations.

We remark that the imposition of (3.19) implies the elimination of the “velocity” \( T_{(0)0k} \) from the expression of \( P^{(0)i} \). However, in view of the time gauge condition such term does not contain any time derivative. It contains only a spatial derivative of the lapse function. We know that the variation of the Hamiltonian with respect to the lapse function yields the Hamiltonian constraint. It turns out that we still have to require (3.19), otherwise the presence of such derivative of the lapse function would render a very complicated expression for the Hamiltonian constraint, which does not satisfy the constraint algebra to be presented in the next section. Moreover the lapse function would no longer play the role of a genuine Lagrange multiplier.

The actual necessity of demanding condition (3.19) will be demonstrated in §4. It will be shown that without this condition the Hamiltonian formulation cannot be consistently established.

We can write the final form of \( L \) by collecting the remaining terms. We choose to write the Lagrangian density in terms of \( c_1 \) only. By factorizing the lapse and shift functions we obtain
\[ L = P^{(j)i} \dot{e}_{(j)i} + N^k C_k + NC - \partial_i \left[ N_k P^{ki} + N(3c_1 e T^{j}) \right] + \lambda^{ij} P_{[ij]} \]  
where \( \lambda^{ij} \) are Lagrange multipliers. The Hamiltonian and vector constraints are given respectively by
\[ C = \frac{1}{6c_1} \left( P^{ij} P_{ji} - P^2 \right) + e T^{ikj} \Sigma_{ikj} - \partial_k \left[ 3c_1 e T^k \right], \quad (3.23) \]

\[ C_k = e_{(j)k} \partial_i P^{(j)i} + P^{(j)i} T_{(j)ik}. \quad (3.24) \]

We remark that \( \Sigma_{kij} \) that appears in \( C \) is a function of \( e_{(ij)} \) and its spatial derivatives only. The Lagrange multipliers \( \lambda^{ij} \) are ultimately determined by evaluating the field equation \( \dot{e}_{(ij)}(x) = \{ e_{(ij)}(x), \mathcal{H} \} \), where \( \mathcal{H} \) is the total Hamiltonian, and by imposing \( P_{[ij]} = 0 \).

Therefore we conclude this section by observing that the imposition of a well defined Legendre transform has reduced the three-parameter to a one-parameter family of theories.

§4. The constraint algebra

A consistent implementation of the constraint algebra is a necessary condition for the Hamiltonian formulation. However, it is not a sufficient condition. It remains to be verified whether the constraint structure of the theory is consistently implemented in the sense of Dirac’s formulation of constrained Hamiltonian systems. We recall that the Hamiltonian formulation of the TEGR is determined by a set of first class constraints[7].

The Hamiltonian formulation determined by (3.21) \( \sim \) (3.24) is very much similar to the 3+1 Hamiltonian formulation of the TEGR, the difference residing in the presence of the constant \( c_1 \) in (3.22) \( \sim \) (3.24) and in the numerical coefficient of the \( P^2 \) term in the Hamiltonian constraint \( C \).

The calculations that lead to the constraint algebra between (3.21), (3.23) and (3.24) are extremely long and intricate. Here we just provide the final expressions. Regardless of the value of \( c_1 \) the constraint algebra “closes”, and therefore (3.21), (3.23) and (3.24) constitute a first class set. Except for the numerical value of the contraction of the metric tensor with itself, the calculations in 2+1 are almost identical to the calculations in the 3+1 formulation.

The constraint algebra is given by

\[ \{ C(x), C(y) \} = \left[ -g^{ik}(x) C_i(x) - 2 \partial_j P^{[jk]}(x) \right] \]
\[
+ P_{[mn]} \left( -\frac{1}{2} T^{kmn} + T^{mnk} \right) \frac{\partial}{\partial x^k} \delta(x - y) - (x \leftrightarrow y), \quad (4.1)
\]

\[
\{ C(x), C_k(y) \} = C(y) \frac{\partial}{\partial y^k} \delta(x - y), \quad (4.2)
\]

\[
\{ C_j(x), C_k(y) \} = -C_k(x) \frac{\partial}{\partial x^j} \delta(x - y) + C_j(y) \frac{\partial}{\partial y^k} \delta(x - y). \quad (4.3)
\]

Moreover we have

\[
\{ C(x), P^{[m(n)]}(y) \} = \{ C_k(x), P^{[m(n)]}(y) \} = 0 \quad (4.4)
\]

\[
\{ P^{[m(n)]}(x), P^{[i(j)]}(y) \} = \left( \eta^{(n)(j)} P^{[m(i)]}(x) - \eta^{(n)(i)} P^{[m(j)]}(x) + \eta^{(m)(i)} P^{[n(j)]}(x) \right) \delta(x - y) \quad (4.5)
\]

Note that by making \( P^{[m(n)]} = 0 \), we obtain the constraint algebra of the 2+1 ADM formulation. The value of \( c_1 \) remains arbitrary.

We are now in a position to discuss the necessity of \( (3.19) \). By not requiring \( (3.19) \) the expression of \( P^{(0)k} \), given by \( (3.12) \), acquires an extra term given by

\[
- 2e T^{(0)}_{(0)} \left( \frac{3}{4} c_1 + c_2 \right) = 2e \left( \frac{3}{4} c_1 + c_2 \right) g^{kl} \frac{1}{N} \partial_l N .
\]

As a consequence the Hamiltonian density acquires an extra term as well. Considering equation \( (3.13) \) this extra term reads

\[
W = -2e N \left( \frac{3}{4} c_1 + c_2 \right) \partial_k \left( e g^{kl} \frac{1}{N} \partial_l N \right).
\]

It is easy to see that this extra term spoils the closure of the constraint algebra. The simplest way of observing the emergence of troublesome terms in the constraint algebra of the theory is by verifying the consistency of the vector constraint \( C_k \). We remark that the expression of \( C_k \) does not depend on the imposition of condition \( (3.19) \). Therefore this condition is not \textit{a priori}
assumed. By calculating the time evolution \( \dot{C}_k(x) = \{C_k(x), \mathcal{H}\} \), where \( \mathcal{H} \) is the total Hamiltonian, we must evaluate

\[
\int d^3y \{C_k(x), W(y)\} =
\]

\[-2e \left( c_2 + \frac{3}{4} c_1 \right) \frac{1}{N} (\partial_i N)(\partial_j N)(\epsilon^a g^{ij} - 2\epsilon^{il} e^{aj})(\partial_k e_{ak} + \partial_k e_{al}) .
\]

We cannot take the right hand side of the expression above as a constraint (either first class or second class) in conjunction with the additional terms in the expression of \( \dot{C}_k(x) \), otherwise several other constraints would emerge by means of consistency conditions, and eventually all degrees of freedom would be exhausted. Therefore condition (3.19) is mandatory.

§5. The absence of a Newtonian limit

The existence of the Newtonian limit was investigated in ref. [12] by considering static fields with circular symmetry, in the absence of spinorial particles and without cosmological constant. It was found a relation that leads to the Newtonian limit and that in our notation reads

\[
3c_1 + 4c_2 = -6c_1c_2 ,
\]

\[
c_1c_2 \neq 0 . \quad (5.1)
\]

Conditions (3.19) and (3.20) violate conditions above for the existence of a Newtonian limit. Therefore a well defined theory from the point of view of the initial value problem cannot display such limit.

The field equations for the Lagrangian density (2.1) were obtained in ref. [12]. It was noticed that the field equations are equivalent to Einstein’s three-dimensional field equations if the parameters satisfy

\[
c_1 + \frac{2}{3} = 0 ,
\]

\[
c_2 - \frac{1}{2} = 0 ,
\]

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By comparing the equations above with (3.19) and (3.20) we conclude that the theory defined by (3.22) \( \sim \) (3.24) is equivalent to the source free Einstein’s general relativity in 2+1 dimensions provided we fix \( c_1 = -\frac{2}{3} \). However, it must be noted that the interaction type between matter (spin \( \frac{1}{2} \)) fields and the gravitational field in teleparallel theories is different from that in Einstein’s theory (see equation (3.20’) of [14]).

§6. Conclusions

In this paper we have investigated the existence of a viable theory of 2+1 dimensional gravity by only requiring it to have a well defined Hamiltonian formulation. A consistent implementation of the Legendre transform reduced the original three-parameter to a one-parameter theory. The resulting theory corresponds to a constrained Hamiltonian system with first class constraints, with total Hamiltonian given by

\[
\mathcal{H} = -\int d^3 x \left( NC + N^i C_i + \lambda^{ij} P_{ij} - \partial_i \left[ N_k P^{ki} + N (3c_1 e^T) \right] \right).
\]

The final form of the theory shares similarities with the 3+1 canonical formulation of the teleparallel equivalent of general relativity. The successful applicability of the Hamiltonian formalism to lower dimensional formalisms is a positive feature of Dirac’s formulation of Hamiltonian constrained systems. Furthermore it supports the conjecture that teleparallel theories may acquire a prominent status in the investigation of gravity theories.

Finally we mention that a theory obtainable from the Lagrangian density (3.22) by adding a negative cosmological constant has the well known BTZ black hole solution[15]. The BTZ black hole solution is found by ascribing the free parameter \( c_1 \) the value \( c_1 = -\frac{2}{3} \). This investigation will be presented elsewhere.

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