Thermodynamic properties of generalized exclusion statistics

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Abstract

We analytically calculate some thermodynamic quantities of ideal $g$-on gas obeying generalized exclusion statistics. We show that the specific heat of $g$-on gas ($g \neq 0$) vanishes linearly in any dimension as $T \to 0$ when the particle number is conserved and exhibits an interesting dual symmetry that relates the particle-statistics at $g$ to the hole-statistics at $1/g$ at low temperatures. We derive the complete solution on cluster coefficients $b_l(g)$ as a function of Haldane’s statistical interaction $g$ in $D$ dimensions. We also find that the cluster coefficients $b_l(g)$ and the virial coefficients $a_l(g)$ are exactly mirror symmetric ($l=$odd) or antisymmetric ($l=$even) about $g = 1/2$. In two dimensions, we completely determine the closed forms about the cluster and the virial coefficients of the generalized exclusion statistics, which exactly agree with the virial coefficients of anyon gas of linear energies. We show that the $g$-on gas with zero chemical potential shows thermodynamic properties similar to the photon statistics. We discuss some physical implications of our results.

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I. INTRODUCTION

Recently there has been extensive interest on generalized exclusion statistics (GES) [1–14] initiated by Haldane. It was realized [2,5,7–12] that there are several models obeying GES where interparticle interactions can be regarded as purely statistical interactions by generalized exclusion principle. The GES defined by Haldane generalizes the Pauli’s exclusion principle through the linear differential relation

\[ \Delta d_{\alpha} = - \sum_{\beta} g_{\alpha\beta} \Delta N_{\beta}, \] (1.1)

where \( \Delta d_{\alpha} \) is the change of available one-particle Hilbert space dimension when the number of added particles amounts to \( \Delta N_{\beta} \) and \( \alpha \) and \( \beta \) indicate different particle species. This definition of statistics is independent of space dimension and obviously interpolates between boson \( (g_{\alpha\beta} = 0) \) and fermion \( (g_{\alpha\beta} = \delta_{\alpha\beta}) \) continuously. Note \( g_{\alpha\beta} \) can have arbitrary nonnegative values.

An earlier form of fractional statistics-anyons [15] is related to braiding properties of particle trajectories, which is peculiar property of two dimensions unlike GES. The GES superficially seems to have little to do with the braid-group notion of 2D statistics. But several people showed [1,2,7–9] that anyon gas in the lowest Landau level (LLL) satisfies GES given by the statistical interaction \( g = \alpha \) (braiding statistics parameter). Moreover, Murthy and Shankar argued [5] that anyons obey the GES by relating the statistical interaction \( g \) to the high-temperature limit of the second virial coefficient (VC) of the anyon gas which shows a nontrivial dependence on 2D braid statistics [16].

As discussed by Nayak and Wilczek [3], free anyons are not ideal \( g \)-ons but interacting \( g \)-ons. It will be an interesting question whether the ideal GES provides good approximations of anyon model where Chern-Simons flux is attached to charged particles [15]. In this respect, it will be shown that ideal \( g \)-on statistics provide a good leading approximation of anyon statistics, i.e., the anyon statistics has a correct limit of GES, in first-order perturbation-cluster expansion [17,18] and in the linear energies in a regulating harmonic potential [19].
In this paper we analytically calculate some thermodynamic quantities of ideal $g$-on gas with $g_{\alpha\beta} = g\delta_{\alpha\beta}$. For the single-particle energy given by $\epsilon(p) = ap^n$, we verify the exclusion statistics satisfy the third law of thermodynamics as conjectured by Nayak and Wilczek [3], i.e., when the particle number is conserved, the specific heat of $g$-on gas ($g \neq 0$) vanishes linearly in any dimension as $T \to 0$ and show that the specific heat at low temperature exhibits an interesting dual symmetry that relates the particle-statistics at $g$ to the hole-statistics at $1/g$. We derive the complete solution on cluster coefficients $b_l(g)$ and thus determine the VCs $a_l(g)$ as a function of Haldane’s statistical interaction $g$ in $D$ dimensions. We also demonstrate that the cluster coefficient (CC) $b_l(g)$ and the VC $a_l(g)$ are exactly mirror symmetric ($l=\text{odd}$) or antisymmetric ($l=\text{even}$) about $g = 1/2$, so that the semion ($g = 1/2$) which is the case of 1D spinon as discussed by Haldane [1,21] must exhibit a peculiar property, $a_{2l}(1/2) = b_{2l}(1/2) = 0$ for $\forall l \in \mathbb{N}$. In the case of $D = n$, we completely determine the cluster and the virial coefficients of the GES in closed forms, which exactly agree with the VCs of anyon gas obtained by Sen [17] and Dasnieres de Veigy [19]. This result implies that the free anyon gas with a small $\alpha$ or with linear energies can be well described by the ideal $g$-on gas. We also show that the $g$-on gas with zero chemical potential shows thermodynamic properties similar to the photon statistics and all the thermodynamic quantities for a given volume depend only on the temperature and their dependences can be determined by the simple dimensional arguments. Finally we discuss some physical implications of our results.

II. THERMODYNAMICS OF GENERALIZED EXCLUSION STATISTICS

Consider the one-particle energy spectrum divided into a large number of energy cells $\epsilon_\alpha$, each of which contains $G_\alpha$ independent levels and $N_\alpha$ identical particles. In the limit of a discrete energy spectrum, $G_\alpha$ would be the degeneracy factor. If $d_{N_\alpha}$ is the dimension of the one-particle Hilbert space in the $\alpha$th cell with the coordinates of $N_\alpha - 1$ particles held fixed, the Haldane’s definition of exclusion principle, Eq.(1.1), leads to
\[ d_{N_\alpha} = G_\alpha - g(N_\alpha - 1). \]

Let \( W(\{N_\alpha\}) \) be the number of configurations of the system corresponding to the set of occupation numbers \( \{N_\alpha\} \). Then an elementary combinatorial argument gives

\[
W(\{N_\alpha\}) = \prod_\alpha \frac{(d_{N_\alpha} + N_\alpha - 1)!}{N_\alpha!(d_{N_\alpha} - 1)!}.
\]  

(2.1)

Under the constraints of fixed particle number and energy

\[
\begin{align*}
N &= \sum_\alpha N_\alpha, \\
E &= \sum_\alpha \epsilon_\alpha N_\alpha,
\end{align*}
\]  

(2.2)

the grand partition function \( Z \) is determined by Haldane-Wu state-counting rule (2.1) as follows

\[
Z = \sum_{\{N_\alpha\}} W(\{N_\alpha\}) \exp\{\sum_\alpha N_\alpha(\mu - \epsilon_\alpha)/kT\},
\]

(2.3)

where \( \mu \) and \( T \) are the Lagrange multipliers incorporating the constraints (2.2) of fixed particle number and energy respectively.

The stationary condition of the grand partition function \( Z \) with respect to \( N_\alpha \) gives the statistical distribution \( n_\alpha \equiv \bar{N}_\alpha/G_\alpha \) of ideal \( g \)-on gas with chemical potential \( \mu \) and temperature \( T \) as derived by Wu

\[
n_\alpha = \frac{1}{w(\zeta_\alpha) + g},
\]

(2.4)

where the function \( w(\zeta_\alpha) \) satisfies the functional equation

\[
w(\zeta_\alpha)^g[1 + w(\zeta_\alpha)]^{1-g} = \zeta_\alpha \equiv e^{(\epsilon_\alpha - \mu)/kT}.
\]

(2.5)

As usual statistics, all the thermodynamic functions of the GES can be evaluated around the set \( \{\bar{N}_\alpha\} \) of the most probable occupation numbers in thermodynamic limit.

The Eqs.(2.4) and (2.5) give correct solutions for the familiar Bose \((g = 0)\) and Fermi \((g = 1)\) distributions since \( w(\zeta_\alpha) = \zeta_\alpha - 1 \) for \( g = 0 \) and \( w(\zeta_\alpha) = \zeta_\alpha \) for \( g = 1 \). Since the function \( F_g(w) \equiv w^g(1 + w)^{1-g} - \zeta_\alpha \) is a monotonically increasing function for \( w > 0 \),
the algebraic equation (2.5) has only one positive root for \( g \neq 0 \) and \( T \neq 0 \) and thus the statistics is unique for a given \( g \). Also, \( n_\alpha \) decreases monotonically as \( g \) increases. In other wards, as \( g \) is increased from boson \(( g = 0)\) toward fermion \(( g = 1)\), the occupation number \( n_\alpha \) decreases for a fixed \( \zeta_\alpha \).

At \( T = 0 \) the ditribution (2.4) reveals a quite surprising phenomenon such as formation of Fermi surface for \( g \neq 0 \) [2,3]

\[
\begin{align*}
n_\alpha &= \begin{cases} 
1/g, & \text{if } \epsilon_\alpha < \mu = \epsilon_F, \\
0, & \text{if } \epsilon_\alpha > \mu = \epsilon_F,
\end{cases} 
\end{align*}
\]  

(2.6)

where the Fermi surface \( \epsilon_\alpha = \epsilon_F \) is determined by the condition \( \sum_{\epsilon_\alpha < \epsilon_F} G_\alpha n_\alpha = N \). This property has important effects on the low-temperature thermodynamics of the \( g \)-on gas to be demonstrated later. For the Bose-Einstein statistics \(( g = 0)\), \( n_\alpha \rightarrow \infty \) when \( \epsilon_\alpha \rightarrow \mu \) and the well-known Bose-Einstein condensation can occur.

As observed by Nayak and Wilczek [3] and Rajagopal [4], the equation (2.5) displays a duality property that relates the particle-statistics at \( g \) to the hole-statistics at \( 1/g \) with a rescaled \( \zeta_\alpha^{-1/g} \)

\[
w(\zeta_\alpha^{-1/g}; \frac{1}{g}) = 1/w(\zeta_\alpha; g).
\]  

(2.7)

In terms of the distribution \( n_\alpha \) in Eq.(2.4), the thermodynamic potential \( \Omega = -kT \ln Z \) is given by

\[
\Omega \equiv -PV = -kT \sum_\alpha G_\alpha \ln \frac{1 + (1-g)n_\alpha}{1 - gn_\alpha}.
\]  

(2.8)

From the Eq.(2.5), one can see the following properties on \( w(\zeta_\alpha) \);

If \( \epsilon_\alpha \rightarrow \infty \), then \( w(\zeta_\alpha) \rightarrow \zeta_\alpha = e^{(\epsilon_\alpha - \mu)/kT} = \frac{1}{z} e^{\epsilon_\alpha/kT} \),

(2.9)

and,

if \( \epsilon_\alpha \rightarrow 0 \) and \( T \rightarrow 0 \), then \( w(\zeta_\alpha) \rightarrow w_0 \equiv \begin{cases} 
e^{-\epsilon_F/gkT}, & \text{if } g \neq 0, \\
\frac{1-z}{z}, & \text{if } g = 0,
\end{cases} 
\]  

(2.10)
where the fugacity $z \equiv e^{\mu/kT}$.

In our calculations, we will use periodic boundary conditions over the system's volume $V \equiv L^D$ and thus the $D$-dimensional momentum eigenvalue $p$ of single-particle is given by

$$p = \frac{2\pi\hbar}{L}n$$

where the $D$-dimensional vector $n$ takes only integer values. Then, in thermodynamic limit as $V \to \infty$, a sum over $p$ can be replaced by an integration

$$\sum_p \to \frac{V}{(2\pi\hbar)^D} \int d^Dp.$$

A state of an ideal $g$-on gas can be specified by specifying a set of occupation numbers $\{n_p\}$ so defined that there are $n_p$ particles having the momentum $p$ in the state under consideration. Of course, we must take the cell size of the single-particle energy spectrum to be sufficiently large, i.e. $G_\alpha \gg 1$, in order to safely ignore the specific complication of GES coming from negative probabilities as pointed out by Nayak and Wilczek [3]. Then the occupation number $N_\alpha$ of the $\alpha$th cell will be the sum of $n_p$ over all the levels in the $\alpha$th cell. With this in mind, the total energy $E$ and the total number of particles $N$ in Eq.(2.2) can be represented as the sum over the momentum $p$

$$N = \sum_p \bar{n}_p(g), \quad (2.11)$$
$$E = \sum_p \epsilon_p \bar{n}_p(g) \quad (2.12)$$

with $\bar{n}_p(g) = \bar{N}_\alpha/G_\alpha$ determined by the Eqs.(2.4) and (2.5). In the following, we will denote the statistical distribution $\bar{n}_p(g)$ at statistics $g$ as $n_g$ for simplicity. We assume the single-particle energy of the ideal $g$-on gas is given by

$$\epsilon_p = ap^n, \quad p = |p|.$$

Using the spherical coordinates in $D$ dimensions, the pressure $P$ given by Eq.(2.8) can be represented as the following integration in terms of the statistical distribution $n_g$ at $g$. 

6
\[
\frac{P}{kT} = \frac{2\pi^{D/2} a^{-D/n}}{(2\pi \hbar)^{D/2} n} \int_0^\infty d\epsilon \frac{\epsilon^{D/n} - 1}{\ln \left( \frac{1+(1-g)n_0}{1-gn_0} \right)},
\]
(2.14)

where the last step is obtained through a partial integration and the Eq.(2.4). In the same way, the particle density \( \rho \equiv N/V \) is given by

\[
\rho = \frac{2\pi^{D/2} a^{-D/n}}{(2\pi \hbar)^{D/2} n} \int_0^\infty d\epsilon \frac{\epsilon^{D/n} n_0}{\ln \left( \frac{1+(1-g)n_0}{1-gn_0} \right)}.
\]
(2.15)

where the second step is obtained through a partial integration and using the relation

\[
kT \frac{dn_0}{d\epsilon} = -z \frac{dn_0}{dz}.
\]

The equations (2.14) and (2.15) can be written in the following standard form [21]

\[
\frac{P}{kT} = \frac{1}{\lambda_D f_D^{(g)}(\frac{D}{n}+1)(z)}
\]
(2.16)

\[
\rho = \frac{1}{\lambda_D f_D^{(g)}(\frac{D}{n})(z)},
\]
(2.17)

where

\[
f_D^{(g)}(\nu)(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty dy y^{\nu-1} n_0 (z^{-1} e^y), \quad y = \frac{\epsilon}{kT}
\]
(2.18)

and the thermal wavelength \( \lambda_D \equiv \frac{n^{-1/2} (2\pi \hbar)^{D/2} \Gamma(D/2)}{2\pi^{D/2} \Gamma(D/n)} \). One can determine the Fermi energy \( \epsilon_F \) in Eq.(2.6) in terms of the density \( \rho \) by the Eq.(2.17),

\[
\epsilon_F = [g(4\pi)^{D/2} \Gamma(\frac{D}{2} + 1)] \frac{\hbar^2}{\pi a(\hbar^2)} n.
\]
(2.19)

From the definition in the Eq.(2.18), we obtain the following relation

\[
f_D^{(g)}(\frac{D}{n})(z) = z \frac{\partial}{\partial z} f_D^{(g)}(\frac{D}{n}+1)(z).
\]
(2.20)

From the Eq.(2.12) the energy \( E \) can be also expressed as

\[
\frac{E}{V} = \frac{D kT}{n} \lambda_D f_D^{(g)}(\frac{D}{n}+1)(z),
\]
(2.21)

which obviously exhibits the following statistics independent relation

\[
P = \frac{n E}{D V}.
\]
(2.22)
According to the fundamental principles of statistical mechanics, the entropy $S$ of the
system is given by
\[
\frac{S}{k} = \sum_{p} \left\{ n_{g} \frac{\epsilon_{p} - \mu}{kT} + \ln \frac{1 + (1 - g)n_{g}}{1 - gn_{g}} \right\}
\]
\[
= D + n \frac{PV}{kT} - N \ln z,
\] (2.23)
where the second expression is obtained by using the relation (2.22). The density fluctuations
in the grand partition function $Z$ can be expressed as the following relation
\[
<N^{2}> - <N>^{2} = z \frac{\partial}{\partial z} z \frac{\partial}{\partial z} \ln Z = \frac{V}{kT} z \frac{\partial}{\partial z} z \frac{\partial}{\partial z} P
\]
\[
= \tilde{N} kT \rho \kappa_{T},
\] (2.24)
where the isothermal compressibility $\kappa_{T} \equiv -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_{T} = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial P} \right)_{T}$ and the Eq.(2.15) is used.
It can be easily shown that the isothermal compressibility in the case of $\mu \neq 0$ is given by
\[
\kappa_{T} = \frac{1}{kT} \frac{1}{\lambda^{D} \rho^{2}} z \frac{\partial}{\partial z} f^{(g)}_{\mu}(z).
\] (2.25)
The equation (2.24) shows that the density fluctuations of the $g$-on gas are also vanishingly
small in the thermodynamic limit provided that $\kappa_{T}$ is finite.

III. GENERAL PROPERTIES OF GENERALIZED EXCLUSION STATISTICS

A. General cases

First we will calculate the thermodynamic quantities such as the total energy $E$, the
specific heat $C_{V} \equiv (\partial E/\partial T)_{V}$ at constant volume, the entropy $S$, and the isothermal com-
pressibility $\kappa_{T}$ for the single-particle energies given by $\epsilon(p) = ap^{n}$ at low temperature and
high density, i.e., $\lambda^{D} \rho \gg 1$.

At low $T$ and high $\rho$ limit $(g \neq 0)$, the Eq.(2.18) can be evaluated by a method so called
Sommerfeld expansion [21] after partial integrations with respect to $y$
\[
f^{(g)}_{\nu}(z) = -\frac{1}{\Gamma(\nu + 1)} \left( \frac{\mu}{kT} \right)^{\nu} \sum_{j=0}^{\nu} \binom{\nu}{j} \left( \frac{\nu}{\mu} \right)^{j} C_{j}^{g},
\] (3.1)
where the numerical numbers $C^g_j$ at statistics $g$ is defined by

$$C^g_j = \int_{-\infty}^{\infty} dx \, x^j \frac{dn_g(x)}{dx}, \quad x = (\epsilon - \mu)/kT$$  \hspace{1cm} (3.2)

and $C^g_0 = -\frac{1}{g}$. Then the Eqs.(2.17) and (2.21) is given by

$$N/V = -\frac{1}{\lambda^D(kT)\frac{\pi}{n} \Gamma(D/n)} \frac{n}{D} \sum_{j=0}^{D/n} \left( \frac{D/n}{j} \right) \left( \frac{kT}{\mu} \right)^j C^g_j,$$  \hspace{1cm} (3.3)

$$E/V = -\frac{1}{\lambda^D(kT)\frac{\pi}{n} \Gamma(D/n)} \frac{n}{D+n} \sum_{j=0}^{1+n} \left( \frac{1+n}{j} \right) \left( \frac{kT}{\mu} \right)^j C^g_j.$$  \hspace{1cm} (3.4)

Let us rewrite the Eqs.(3.3) and (3.4) in the following forms [3]

$$\epsilon_F^{D/n} = \mu^{D/n} \left[ 1 - g \sum_{j=1}^{D} \left( \frac{kT}{\mu} \right)^j \left( \frac{D/n}{j} \right) C^g_j \right],$$  \hspace{1cm} (3.5)

$$E = E_0 \left( \frac{\mu}{\epsilon_F} \right)^{1+n} \left[ 1 - g \sum_{j=1}^{D} \left( \frac{kT}{\mu} \right)^j \left( 1 + \frac{n}{j} \right) C^g_j \right],$$  \hspace{1cm} (3.6)

where the Fermi energy $\epsilon_F$ is given by the Eq.(2.19) and $E_0 = \frac{D}{D+n} N \epsilon_F$ which is the ground state energy of the $g$-on gas at the given density.

Here we will convert an $x$-integration into a $w$-integration using the Eqs.(2.4) and (2.5)

$$C^g_j = \int_{-\infty}^{\infty} dx \, x^j \frac{dn_g(x)}{dx} = -\int_{0}^{\infty} \frac{1}{(w+g)^2} \left( g \ln w + (1-g) \ln(1+w) \right)^j$$

$$+ \int_{0}^{w_0} \frac{1}{(w+g)^2} \left( g \ln w + (1-g) \ln(1+w) \right)^j$$

$$= -\int_{0}^{\infty} \frac{1}{(w+g)^2} \left( g \ln w + (1-g) \ln(1+w) \right)^j + O(e^{-\epsilon_F/gkT}).$$  \hspace{1cm} (3.7)

Since $w_0 \approx e^{-\epsilon_F/gkT} \ll 1$, the second integration can be certainly neglected, which is the reason we use the partial-integrated forms on $N/V$ and $E$. The dual symmetry given by Eq.(2.7) relates the coefficients $C^g_j$ at statistics $g$ to the coefficients $C^1_j$ at $\frac{1}{g}$ up to $O(e^{-\epsilon_F/gkT})$ as follows
\[ C_j^g = (-1)^j g^{-2} C_j^g, \] (3.8)

which can be proved by the substitution, \( w \to \frac{1}{w} \), from Eq. (3.7). Note that only fermions \((g = 1)\) have the exact particle-hole dual symmetry: \( C_j^1 = 0 \) for all odd \( j \) [24]. After a partial integration, \( C_j^g (j \geq 1) \) becomes

\[ C_j^g = -g^{j-1} \{ \ln w \}^j \big|_{w=0} - j \int_0^\infty \frac{dw}{w(1+w)} \{ g \ln w + (1 - g) \ln(1 + w) \}^{j-1}. \] (3.9)

Using the following integral formulas

\[
\int dw \frac{\ln w}{w(1+w)} = \frac{1}{2} (\ln \frac{w}{1+w})^2 + \int dw \frac{\ln(1+w)}{w(1+w)} \tag{3.10}
\]
\[
\int_0^\infty dw \frac{\ln(1+w)}{w(1+w)} = \zeta(2) = \frac{\pi^2}{6}, \tag{3.11}
\]

we can easily calculate \( C_1^g \) and \( C_2^g \). The results are

\[ C_1^g = 0, \]
\[ C_2^g = -\frac{\pi^2}{3}. \] (3.12)

The higher coefficients of \( C_j^g \) can be obtained numerically

\[
C_3^\frac{1}{3} = -3.6062, \quad C_3^2 = 7.2123, \quad C_4^\frac{1}{4} = -29.2227, \quad C_4^2 = -116.891 \]
\[
C_3^\frac{1}{3} = -4.8082, \quad C_3^3 = 14.4247, \quad C_4^\frac{1}{4} = -26.6973, \quad C_4^3 = -240.276 \] (3.13)
\[
C_3^\frac{1}{3} = -5.4093, \quad C_3^4 = 21.637, \quad C_4^\frac{1}{4} = -25.9758, \quad C_4^4 = -415.612, \text{ etc.}
\]

The above results clearly display the dual symmetry (3.8). The chemical potential \( \mu \), the total energy \( E \), and the specific heat \( C_V \) are given by

\[ \mu = \epsilon_F \left[ 1 - g \frac{\pi^2}{6} (\frac{D}{n} - 1) \frac{kT}{\epsilon_F} \right] + O(T^3) \] (3.14)
\[ E = E_0 \left[ 1 + g \frac{\pi^2}{6} (1 + \frac{D}{n}) \frac{kT}{\epsilon_F} \right] + O(T^3) \] (3.15)
\[ C_V/k = g \frac{\pi^2}{3} \left( 1 + \frac{D}{n} \right) \frac{E_0}{\epsilon_F} kT + O(T^2). \] (3.16)

Note that, at low temperatures, the chemical potential \( \mu \) increases in the case of \( D/n < 1 \), remains stable in \( D/n = 1 \), and decreases in the case of \( D/n > 1 \) when the temperature
increases. This is consistent with the well-known results of the fermion gas when $n = 2$ [22]. Using the Eq.(3.14) and Eq.(3.15), one can easily calculate the entropy $S$ defined by Eq.(2.23)

$$
\frac{S}{k} = \frac{1}{kT} \left( \frac{D + n}{D} E - \mu N \right)
= g^{\frac{\pi^2}{3}} (1 + \frac{D}{n}) \frac{E_0}{\epsilon_F} kT + O(T^2)
\simeq C_V/k. \tag{3.17}
$$

When the particle number is conserved, the specific heat of the $g$-on gas ($g \neq 0$) vanishes linearly in any dimension as $T \to 0$. Therefore the GES obviously satisfies the third law of thermodynamics as conjectured by Nayak and Wilczek [3], which expresses that the ground state of the many-body $g$-on system is nondegenerate. Our above results also agree with those of Nayak and Wilczek in the case of $g = \frac{1}{2}$ and, in the case of $g = 1$ and $D/n = 3/2$, reproduce the well-known results of fermion gas [21]. The linear dependences on $T$ and $g$ of the specific heat of the $g$-on gas can be roughly understood as the same way as ordinary fermion gas. Of course, the key point is the existence of Fermi surface at low temperature. (The energy of particles excited over $\epsilon_F$ is of order $gkT$ and the number of excited particles is of order $(kT/\epsilon_F)N$. Thus the total excitation energy amounts to $g(kT/\epsilon_F)NkT$, so that $C_V \approx g(kT/\epsilon_F)Nk$.) Note that the linear dependences on $T$ and $g$ of specific heat are satisfied even in D dimensions with $\epsilon(p) = ap^n$.

Also the isothermal compressibility $\kappa_T$ given by Eq.(2.23) can be calculated by using the Eq.(3.13) and Eq.(3.14)

$$
\kappa_T = g \frac{D}{n} \frac{E_0}{\epsilon_F} \frac{a}{\epsilon_F} \frac{D}{2} \left[ 1 - g \frac{\pi^2}{6} \left( \frac{D}{n} - 1 \right) \left( \frac{kT}{\epsilon_F} \right)^2 + O(T^3) \right]. \tag{3.18}
$$

Note that the leading behavior is linear in $g$ and the correction of order $T^2$ vanishes when $D = n$.

Since the statistics at $g = 0$ is a well-known bosonic one, we would not present a detailed analysis. But, for comparison, we want to present only the results
\[ N = \frac{1}{\lambda D} \sum_{l=1}^{\infty} \frac{z^l}{l^\nu} + \frac{1}{\lambda D} \sum_{l=1}^{\infty} \frac{z^l}{l^1} \]
\[ E = \frac{D kT}{n} \sum_{l=1}^{\infty} \frac{z^l}{l^{1+\nu}}. \]

When \( T \to 0 \) or \( z \to 1 \), the above results show the characteristic properties of Bose-Einstein condensation [21] unlike the statistics at \( g \neq 0 \). The important point is that only the Bose-Einstein statistics \( (g = 0) \) exhibits the Bose-Einstein condensation as mentioned in Sec.II.

Now we will investigate the thermodynamic properties at high \( T \) and low \( \rho \) in \( D \) dimensions. This region corresponds to \( \lambda D \rho \ll 1 \) or \( z \ll 1 \). For this purpose we will take the following expansion of the function \( f^{(g)}_\nu(z) \) in Eq.(2.18)
\[ f^{(g)}_\nu(z) = \sum_{l=1}^{\infty} b^{(\nu)}_l(g)z^l. \]

According to the statistical relation (2.20), the coefficients \( b^{(\nu)}_l(g) \) satisfy the following relation
\[ b^{(\nu-1)}_l(g) = l \cdot b^{(\nu)}_l(g). \]

The following results can be obtained from the Eq.(3.21)
\[ \frac{P}{kT} = \frac{1}{\lambda D^\nu} \sum_{l=1}^{\infty} b^{(\nu)}_l(g) z^l \]
\[ \rho = \frac{1}{\lambda D} \sum_{l=1}^{\infty} l b^{(\nu)}_l(g) z^l, \]
where \( b_l(g) \equiv b^{(\nu)}_{l+1}(g) \) is the \( l \)th cluster coefficient. The virial expansion [21] of the equation of state is defined to be
\[ \frac{P}{kT} = \rho \sum_{l=1}^{\infty} a_l(g)(\lambda D \rho)^l, \]
where \( a_l(g) \) is called the \( l \)th VC. By substituting Eq.(3.25) into Eqs.(3.23) and (3.24) and requiring that the resulting equation to be satisfied for every \( z \) we obtain
\[a_1 = b_1 = 1\]
\[a_2 = -b_2\]
\[a_3 = 4b_2^2 - 2b_3\]
\[a_4 = -20b_2^3 + 18b_2b_3 - 3b_4\]
\[a_5 = 2(56b_2^4 - 72b_2^2b_3 + 9b_3^2 + 16b_2b_4 - 2b_5)\]
\[a_6 = -672b_2^5 + 112b_2^3b_3 - 315b_2b_3^2 - 280b_2^2b_4 + 60b_3b_4 + 50b_2b_5 - 5b_6\]
\[\ldots\]

(3.26)

To get the \(l\)th CCs \(b_l(g)\), let us expand the distribution \(n(\eta)\) as follows

\[n(\eta) = \sum_{i=1}^{\infty} c_i(g)\eta^i,\]  

where \(\eta = \zeta^{-1} = ze^{-\frac{kT}{\lambda}}\). Then the coefficients \(b^{(\nu)}_l(g)\) in Eq.(3.21) are

\[b^{(\nu)}_l(g) = c_l(g)\frac{1}{\nu^l}.\]  

(3.28)

It can be easily calculated that for bosons, \(c_l(0) = 1\) and for fermions, \(c_l(1) = (-1)^{l+1}\).

Although the coefficients \(b^{(\nu)}_l(g)\) or \(c_l(g)\) for general \(g\) can be obtained by expanding the Eq.(2.4) and Eq.(2.5) in terms of \(z\), we will show in the next section the CC \(b^{(\nu)}_l(g)\) can be derived as the following closed form

\[b^{(\nu)}_l(g) = \frac{1}{\nu^l} \prod_{m=1}^{l-1} \frac{m - gl}{m}.\]  

(3.29)

All the VCs \(a_l(g)\) can be obtained by the Eq.(3.26) since we have already known all the CCs \(b_l(g)\) from Eq.(3.29).

The first few terms of the pressure \(P\) and the particle density \(\rho\) in Eq.(3.23) and Eq.(3.24) are given by

\[P = \frac{kT}{\lambda^D} \left(z - \frac{2g - 1}{2^{n+1}} z^{2^\nu} + \frac{9g^2 - 9g + 2}{2 \cdot 3^{n+1}} z^{3^\nu} + \ldots\right),\]  

(3.30)

\[\lambda^D \rho = z - \frac{2g - 1}{2^n} z^{2^\nu} + \frac{9g^2 - 9g + 2}{2 \cdot 3^n} z^{3^\nu} + \ldots.\]  

(3.31)
The virial expansion (3.25) becomes

\[ PV = NkT \left[ 1 + \frac{2g - 1}{2^{1 + \frac{D}{n}}} \lambda^D \rho + \left\{ \frac{(2g - 1)^2}{4^D/n} - \frac{9g^2 - 9g + 2}{3^{1 + \frac{D}{n}}} \right\} (\lambda^D \rho)^2 + \ldots \right] \]. (3.32)

In other words the second VC and the third VC are as follows

\[ a_2 = \frac{2g - 1}{2^{1 + \frac{D}{n}}} \] (3.33)
\[ a_3 = \frac{(2g - 1)^2}{4^D/n} - \frac{9g^2 - 9g + 2}{3^{1 + \frac{D}{n}}} \]. (3.34)

Note that \( a_3 \) as well as \( a_2 \) is statistics-dependent in \( D \) dimensions but \( a_3 \) is statistics-independent only in \( 2D \) or more accurately in the case of \( D/n = 1 \). Of course, when \( g = 0 \) or 1, our above results reproduce the standard results of statistical mechanics.

Using the results given by Eq.(3.31) and (3.30), the entropy \( S \) in Eq.(2.23) and the compressibility \( \kappa_T \) in Eq.(2.25) in the Boltzmann limit can be obtained

\[ \frac{S}{k} \simeq N\left[ \frac{D + n}{n} - \ln \lambda^D \rho \right] \]
\[ + N\left[ \frac{D + n}{n} \frac{2g - 1}{2^{1 + \frac{D}{n}}} \lambda^D \rho - \ln(1 + \frac{2g - 1}{2^{\frac{D}{n}}} \lambda^D \rho) \right] \] (3.35)
\[ \kappa_T \simeq \frac{1}{P} \left[ 1 + \left\{ \frac{9g^2 - 9g + 2}{2 \cdot 3^{\frac{D}{2}}} - \frac{(2g - 1)^2}{4^D/n} \right\} \lambda^{2D} (kT)^2 \right]. \] (3.36)

The first part in Eq.(3.35) is just the Sackur-Tetrode equation [21] and the second part is a correction due to quantum statistics. Note that this quantum correction has no linear dependence on \( \rho \) when \( D = n \) and there exists a sign change at \( g = 1/2 \).

We will prove a very interesting mirror symmetry of the CC \( b_l(g) \) and the VC \( a_l(g) \) about \( g = 1/2 \). Let us start with the following relation derived by the Eqs.(2.4) and (2.5)

\[ (1 - gn_g)^g[1 + (1 - g)n_g]^{1-g} = n_g \frac{1}{z} e^{\frac{\epsilon}{kT}}. \] (3.37)

Rewrite the above Eq.(3.37) as

\[ [1 - (\frac{1}{2} + \tilde{g})n_{\tilde{g}}]^{\frac{1}{2} + \tilde{g}}[1 + (\frac{1}{2} - \tilde{g})n_{\tilde{g}}]^{\frac{1}{2} - \tilde{g}} = n_{\tilde{g}} \frac{1}{z} e^{\frac{\epsilon}{kT}}, \] (3.38)

where \( \tilde{g} = g - \frac{1}{2} \). Obviously one can see that the form of the Eq.(3.38) is invariant under the mirror transformation
\[ \tilde{g} \rightarrow -\tilde{g} \]
\[ z \rightarrow -z \quad (3.39) \]
\[ n_\tilde{g} \rightarrow -n_{-\tilde{g}}. \]

The above mirror symmetry implies that, if \( n_\tilde{g}(z) \) is a solution of Eq. (3.38), then \( -n_{-\tilde{g}}(-z) \) is also a solution. Since the algebraic equation (3.38) must have a unique nonnegative solution for a given \( g \) as remarked at the beginning of Sec. II, we can obtain the following interesting symmetry property:

\[ n_{-\tilde{g}}(-z) = -n_{\tilde{g}}(z). \quad (3.40) \]

Note that the above mirror symmetry about the distribution \( n_g \) is generally satisfied for all temperature and density.

The following mirror symmetry is the direct consequence of the Eq. (3.40) derived from Eq. (3.27)

\[ c_{2l+1}(-\tilde{g}) = c_{2l+1}(\tilde{g}) \]
\[ c_{2l}(-\tilde{g}) = -c_{2l}(\tilde{g}). \quad (3.41) \]

The above mirror symmetry immediately leads to the mirror symmetry about the pressure \( P \) and the particle density \( \rho \)

\[ f^{(-\tilde{g})}_{D/n}(-z) = -f^{(\tilde{g})}_{D/n}(z) \quad \text{or} \quad \rho(-z, -\tilde{g}) = -\rho(z, \tilde{g}) \]
\[ f^{(-\tilde{g})}_{D/n+1}(-z) = -f^{(\tilde{g})}_{D/n+1}(z) \quad \text{or} \quad P(-z, -\tilde{g}) = -P(z, \tilde{g}). \quad (3.42) \]

From the Eq. (3.42), one can easily verify the following mirror symmetry about \( g = 1/2 \) of the CC \( b_l(g) \) and the VC \( a_l(g) \) in the Eqs. (3.23)-(3.25)

\[ b_{2l+1}(-\tilde{g}) = b_{2l+1}(\tilde{g}) \quad a_{2l+1}(-\tilde{g}) = a_{2l+1}(\tilde{g}) \]
\[ b_{2l}(-\tilde{g}) = -b_{2l}(\tilde{g}) \quad a_{2l}(-\tilde{g}) = -a_{2l}(\tilde{g}). \quad (3.43) \]

In particular, semions with \( g = 1/2 \) exhibit a surprising property, \( a_{2l}(\frac{1}{2}) = b_{2l}(\frac{1}{2}) = 0 \) for \( \forall l \in \mathbb{N} \). We can observe that our previous explicit results show mirror symmetries given by Eqs. (3.40)-(3.43) consistently.
B. D=n

In this section we will consider the special case of $D = n$, which allows more analytic results. This includes the usual non-relativistic system with $\epsilon_p = p^2/2m$ in two dimensions as a special case. In the case of $D = n$, we have an explicit solution \[2\] directly obtained from Eq.(2.17) and Eq.(2.5):

$$
\rho = \frac{1}{\lambda^D} \ln \frac{1 + w(z^{-1})}{w(z^{-1})} \Rightarrow \rho e^{-\lambda^D \rho} + e^{-\lambda^D \rho} = 1. \tag{3.44}
$$

This result will be very useful in our calculations of the CCs and the VC s.

When $D = n$, there is no need to perform the Sommerfeld expansion used in the previous subsection. From the Eq.(2.21), the total energy of the system can be evaluated to be

$$
E = \frac{V}{\lambda^D} \int_{w(z^{-1})}^{\infty} dw \frac{1}{w(1+w)} \{ \mu + gkT \ln w + (1-g)kT \ln(1+w) \}
= \frac{V}{\lambda^D} \left[ \mu \ln \frac{1 + w(z^{-1})}{w(z^{-1})} - \frac{gkT}{2} (\ln \frac{1 + w(z^{-1})}{w(z^{-1})})^2 + kT \int_{w(z^{-1})}^{\infty} dw \ln(1+w) \right]. \tag{3.45}
$$

The above expression can be rewritten as more elegant form using the Eq.(3.44)

$$
\frac{E}{V} = \frac{1}{2} gkT \lambda^D \rho^2 + \frac{kT}{\lambda^D} \int_0^{\lambda^D \rho} du \frac{u}{e^u - 1}, \tag{3.46}
$$

where $e^u - 1 = 1/w$.

First consider the low temperature and high density regions, i.e., $\lambda^D \rho \gg 1$. Since $\lambda^D \rho \gg 1$ and thus $w(z^{-1}) \simeq e^{-\lambda^D \rho} \ll 1$, the integral in Eq.(3.45) or Eq.(3.46) can be evaluated up to $O(e^{-\lambda^D \rho})$

$$
E = \mu N - \frac{gkT}{2} \frac{\lambda^D}{V} N^2 + \frac{\pi^2 kT}{6} \frac{V}{\lambda^D} + O(e^{-\lambda^D \rho}). \tag{3.47}
$$

For $g = 0$, the Eq.(3.47) becomes

$$
E = \frac{\pi^2 kT}{6} \frac{V}{\lambda^D} + O(e^{-\lambda^D \rho}) \tag{3.48}
$$

since $\mu \simeq -kT \exp(-\lambda^D \rho) \to 0^{-}$ as $T \to 0$. For $g \neq 0$, the Eq.(3.47) becomes

$$
E = \frac{1}{3} N \epsilon_F \left[ 1 + \frac{g \pi^2}{3} \left( \frac{kT}{\epsilon_F} \right)^2 + O(e^{-\lambda^D \rho}) \right]. \tag{3.49}
$$
Now the specific heat \( C_V \) at constant volume is given by

\[
C_V/k = \begin{cases} 
\frac{2}{3} \frac{2\pi^2}{(2\pi h)^2 T(D/2)} \frac{NkT}{\rho} + O(e^{-\lambda P \rho}), & \text{if } g = 0, \\
g \frac{2\pi^2 E_0 \omega_k}{\varepsilon_F} kT + O(e^{-\lambda P \rho}), & \text{if } g \neq 0.
\end{cases}
\] (3.50)

The above results, of course, coincide with those of the general case when \( D/n = 1 \). But note that, as \( T \to 0 \), the chemical potential \( \mu \) remains constant, i.e., equals to \( \epsilon_F \) and the specific heat has only a linear correction on \( T \) up to exponentially small corrections. This is a peculiar property of the energy spectrum with constant density of states [22].

We will now investigate a systematic way of calculating the CCs and VCs of the GES in the Boltzmann limit using the explicit result, Eq.(3.44). In this region the equation (3.46) can be explicitly calculated in terms of the density \( \rho \) and thus all the VCs can be determined. A straightforward calculation leads to

\[
\frac{P}{kT} = \rho + \frac{2g - 1}{4} \lambda P \rho^2 + \rho \sum_{l=1}^{\infty} \frac{B_{l+1}}{(l+2)!} (\lambda P \rho)^{l+1},
\] (3.51)

where \( B_l \) is the \( l \)th Bernoulli number (\( B_1 = -\frac{1}{2} \), \( B_2 = \frac{1}{6} \), \( B_4 = -\frac{1}{30} \), etc.) and \( B_{2l+1} = 0 \) for \( l \geq 1 \) [23]. From the above equation, the VCs in the Eq.(3.25) can be found

\[
\begin{align*}
 a_1 &= 1, \\
 a_2 &= \frac{2g - 1}{4}, \\
 a_{2l+1} &= \frac{B_{2l}}{(2l+1)!}, \\
 a_{2l+2} &= 0, \quad \text{for } \forall l \geq 1.
\end{align*}
\] (3.52)

Remarkably the VCs except the second VC are \( g \)-independent and moreover all the even VCs except \( a_2(g) \) vanish. Of course, these properties on the VCs are consistent with the general properties depicted in the previous subsection.

In order to calculate the CCs \( b_l(g) \), we will again use the Eq.(3.44). From the Eq.(3.44), we can deduce the following relation:

\[
\frac{\tilde{\rho}}{z} = \frac{\tilde{\rho} e^{\gamma \tilde{\rho}}}{e^{\tilde{\rho}} - 1} = \sum_{l=0}^{\infty} \frac{B_l(\gamma)}{l!} \tilde{\rho}^l,
\] (3.53)

where \( \tilde{\rho} \equiv \lambda P \rho \), \( \gamma \equiv 1 - g \) and \( B_l(\gamma) \) are the Bernoulli polynomials [23]. Since \( z(\tilde{\rho}) \) can be expressed in closed form in terms of Eq.(3.53), the series expansion \( \tilde{\rho}(z) \) in the Eq.(3.24) is
an inversion of the power series $z(\hat{\rho})$ [23]. Using the method of the calculus of residues, the CCs $b_l^{(2)}$ can be expressed as

$$b_l^{(2)} = \frac{1}{l!} \left. \frac{d^{l-1}}{d\hat{\rho}^{l-1}} \left( \frac{\hat{\rho}}{z} \right) \right|_{z=0}$$

$$= \frac{1}{l!} B_{l-1}^{(l)}(\gamma l), \quad (3.54)$$

where $B_{n}^{(l)}(x)$ are generalized Bernoulli polynomials [24] of order $l$:

$$\frac{t^{l}e^{xt}}{(e^{t}-1)^{l}} = \sum_{n=0}^{\infty} B_{n}^{(l)}(x) \frac{t^{n}}{n!},$$

$$B_{n}^{(l+1)}(x) = \frac{n!}{l!} \frac{d^{l-n}}{dx^{l-n}} (x-1)(x-2) \cdots (x-n), \quad l \geq n. \quad (3.55)$$

Thus the CCs $b_l^{(2)}$ are determined as

$$b_l^{(2)}(g) = \frac{1}{l^2} \prod_{m=1}^{l-1} \frac{\gamma l - m}{m}. \quad (3.56)$$

Notice that the coefficients $c_l(g) = b_l^{(0)}(g)$ in Eq.(3.28) are independent of order $\nu$. Hence we can obtain remarkably simple result about the coefficients $b_l^{(\nu)}(g)$ for general $\nu$ from the Eq.(3.56)

$$b_l^{(\nu)}(g) = \frac{1}{l^\nu} \prod_{m=1}^{l-1} \frac{\gamma l - m}{m}. \quad (3.57)$$

From the above equation and Eq.(3.26), the mirror symmetry of the CCs and the VCs in the Eq.(3.43) can be also directly proved. Interestingly the similar results have been noticed in the Calogero-Sutherland model [25], anyon gas in the LLL [7], and the anyon model with linear energies [19] in a regulating harmonic potential.

When $D = n$, we have completely determined all the CCs and the VCs and some lowest terms among them are

$$b_1 = 1, \quad a_1 = 1$$

$$b_2 = \frac{1-2g}{4}, \quad a_2 = \frac{2g-1}{4}$$

$$b_3 = \frac{1}{3^2} \prod_{m=1}^{3} (1 - \frac{3}{m}g), \quad a_3 = \frac{1}{36}$$

$$b_4 = \frac{1}{4^2} \prod_{m=1}^{4} (1 - \frac{4}{m}g), \quad a_4 = 0$$

$$b_5 = \frac{1}{5^2} \prod_{m=1}^{5} (1 - \frac{5}{m}g), \quad a_5 = -\frac{1}{3600}\quad (3.58)$$
and $a_6 = 0$, etc. The above results exactly coincide with the VCs (CCs) of anyon gas obtained by Sen \[17\] and Dasnières de Veigy \[19\]. This result clearly shows that the anyon gas with a small $\alpha$ or with linear energies ($g = 1 - \alpha^2$) can be well approximated by the ideal $g$-on gas. Thus the anyon gas confined in the LLL and the anyon gas with linear energies in a regulating harmonic potential can be regarded as the ideal $g$-on gas. Although the CCs show complicated statistics-dependences, the VCs are extremely simple and statistics-independent except the second VC. This implies that there are miraculous cancellations in the virial expansion.

C. $\mu = 0$

Let us consider the case the particle number is not conserved, $\mu = 0$. In this case, the number of particles $N$ itself is not a given constant but must be determined from the conditions of thermal equilibrium, $(\partial \Omega/\partial N)_{T,V} = \mu = 0$ \[21\]. The particle density $\rho$ and the pressure $P$ given by Eq.(2.17) and Eq.(2.16) respectively become

$$\rho = \frac{1}{\lambda^D} f_{\frac{D}{n}}^{(g)}(1) \equiv \frac{1}{\lambda^D} I_{\frac{D}{n}}^g,$$
$$\frac{P}{kT} = \frac{1}{\lambda^D} f_{1+\frac{D}{n}}^{(g)}(1) \equiv \frac{1}{\lambda^D} I_{1+\frac{D}{n}}^g,$$

(3.59)

where the definite integrals $I_{\frac{D}{n}}^g$ and $I_{1+\frac{D}{n}}^g$ are pure numbers. This shows that the pressure $P$ and the particle density $\rho$ depend only on the temperature $T$. Actually, for a given volume, all the thermodynamic quantities should be determined only by the temperature and their temperature dependences can be obtained by simple dimensional arguments. For example, the energy density $E/V$ and the pressure are proportional to $(kT)^{1+\frac{D}{n}}$ and the particle density and the entropy density $S/V$ to $(kT)^{\frac{D}{n}}$.

In the adiabatic expansion (or compression), the above results imply that the volume and temperature are related by $V T^{\frac{D}{n}} = $constant and the pressure and volume by $P V^{\frac{D}{n}+1} = $constant as the photon statistics or an ordinary extreme relativistic gas.

When $D/n = 1$, using the $w$-integration, the integrals $I_1^g$ and $I_2^g$ in Eq.(3.59) can be expressed as follows
\[
I_1^g = \frac{\xi}{g}, \quad I_2^g = -\frac{\xi^2}{2g} + \sum_{l=1}^{\infty} \frac{l+\xi}{l^2} e^{-\xi l},
\]

where \( \xi = \ln(1 + p) \) and \( p \) is a positive root of the algebraic equation, \( p^g(1 + p)^{1-g} = 1 \). The numerical evaluation of some of the coefficients are as follows:

\[
\begin{align*}
I_1^1 & = 0.9624, \quad I_2^1 = 0.4812, \quad I_3^1 = 0.9870, \quad I_2^2 = 0.6580 \\
I_1^2 & = 1.1467, \quad I_3^2 = 0.3822, \quad I_2^2 = 1.0785, \quad I_3^3 = 0.5664 \\
I_1^3 & = 1.2891, \quad I_1^4 = 0.3223, \quad I_2^3 = 1.1400, \quad I_1^4 = 0.5050, \quad \text{etc.}
\end{align*}
\]

**IV. CONCLUSION**

We have analytically calculated some thermodynamic quantities of ideal \( g \)-on gas obeying generalized exclusion statistics. We have shown that the specific heat of \( g \)-on gas \((g \neq 0)\) vanishes linearly in any dimension as \( T \to 0 \) when the particle number is conserved and exhibits an interesting dual symmetry that relates the particle-statistics at \( g \) to the hole-statistics at \( 1/g \) at low temperatures. In the low temperature the thermodynamic properties at \( g \neq 0 \) are similar to those of fermion due to the existence of Fermi surface unlike the case of \( g = 0 \) (bosons) where the Bose-Einstein condensation can occur.

We have found the closed form about the CCs \( b_l(g) \) as a function of Haldane’s statistical interaction \( g \) in \( D \) dimensions. Unlike the low temperature case, high temperature behaviors are bifurcated at \( g = 1/2 \). The statistical interactions at \( g < 1/2 \) are attractive (bosonic) while those at \( g > 1/2 \) are repulsive (fermionic). The CCs \( b_l(g) \) and the VCs \( a_l(g) \) are exactly mirror symmetric \((l=\text{odd})\) or antisymmetric \((l=\text{even})\) about \( g = 1/2 \). The case of \( g = 1/2 \) (semions) has a very peculiar property, \( a_{2l}(1/2) = b_{2l}(1/2) = 0 \) for \( \forall l \in \mathbb{N} \). Spinon excitations in the Heisenberg spin chain with inverse-square exchange can be described by the semionic statistics \((g = 1/2)\) \([1,20]\), which can be also related to half-filling Laughlin’s boson fractional quantum Hall states \([26]\). Thus the high temperature behaviors in these models may exhibit very interesting properties.
In the case of the energy spectrum with constant density of states, i.e. $D = n$, we have obtained full exact results and calculated all the CCs and the VCs of the $g$-on gas and we have confirmed that they exactly agree with the virial coefficients of anyon gas of linear energies obtained by Dasnières de Veigy [19]. Although this result implies that the anyon gas with linear energies can be well approximated by the ideal $g$-on gas and so the anyon gas of linear energy can be regarded as the ideal $g$-on gas, the free anyon gas cannot be generally regarded as the ideal $g$-on gas, e.g., in the case with nonlinear energy spectrum or strong coupling [18] where it is shown the statistical dependence of third VC appears in second order correction. Even so, there may still remain the nonperturbative similarities such as the mirror symmetry about semions between anyon statistics and GES. Note that it was reported by Sen [27] that the third VC of anyons exhibits similar mirror symmetry about semions. It will be very interesting if the higher virial coefficients of the anyon gas turn out to be also mirror symmetric or antisymmetric about semions.

In the case of $\mu = 0$, the number of particles $N$ is not a given constant as in a degenerate gas but the quantity being determined from the conditions of thermal equilibrium. As the photon statistics, for a given volume, all the thermodynamic quantities of the $g$-on gas depend only on the temperature and their dependences can be determined by the simple dimensional arguments.

In their paper [3], Nayak and Wilczek suggested that the electrons in Mott insulators would behave as 2-ons and thus the Mott insulators have a Fermi surface of anomalous size and anomalous values of the specific heat. Our results show that the specific heat and the volume of the Fermi surface of 2-ons are twice as large as ordinary fermions in the unit of Fermi energy. Also, Ihm suggested [28] that the reduction of the number of allowed configurations of the crystalline ice can be recognized as arising from the fractional exclusion due to so-called ice rule and this reduction can be describable as the GES with the statistical interaction $g = 0.867$. But it still remains open to construct an appropriate three-dimensional model describing the GES unlike the one-dimensional soluble models nicely fitting with the notion of the GES such as Calogero-Sutherland model [10], antiferromagnetic
spin chain with inverse-square exchange \cite{20}, and twisted Haldane-Shastry model \cite{12}. So it will be very interesting although very difficult to construct higher dimensional models nicely fitting with the notion of the GES.

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