Gauge dependence in topological gauge theories

Carlo M. Becchi\footnote{E-mail: becchi@genova.infn.it} 
and 
Stefano Giusto\footnote{E-mail: giusto@genova.infn.it} 

Dipartimento di Fisica dell’ Università di Genova 
Via Dodecaneso 33, I-16146, Genoa, Italy

Camillo Imbimbo\footnote{E-mail: imbimbo@infnge.ge.infn.it} 

INFN, Sezione di Genova, 
Via Dodecaneso 33, I-16146, Genoa, Italy

ABSTRACT

We parametrize the gauge-fixing freedom in choosing the Lagrangian of a topological gauge theory. We compute the gauge-fixing dependence of correlators of equivariant operators when the compactified moduli space has a non-empty boundary and verify that only a subset of these has a gauge independent meaning. We analyze in detail a simple example of such anomalous topological theories, 4D topological Yang-Mills on the four-sphere and instanton number $k = 1$. 
1. The problem

Topological gauge theories [1] are characterized by finite-dimensional gauge orbit spaces — for example, the moduli space of Riemann surfaces or the moduli space of (anti)self-dual four-dimensional euclidean instantons. Equivariant cohomology classes of the BRS operator are associated with closed forms on the moduli space $\mathcal{M}$ which depend in general on the gauge-fixing choice, i.e. on the choice of the Lagrangian: different gauge-fixings lead to closed forms which differ by exact terms. The evaluation of correlators reduces to finite-dimensional integration of closed forms over (cycles of) the moduli space: mathematically this corresponds to compute intersection numbers on $\mathcal{M}$. Thus, only if the integrals of exact forms over $\mathcal{M}$ vanish, one is guaranteed a priori that the gauge-fixing ambiguity does not affect integrated correlators.

However, in practice, $\mathcal{M}$ is hardly ever compact. The prototypical example of the present paper will be topological Yang-Mills theory on a four-dimensional variety $X$. In this case the non-compactness of $\mathcal{M}_k$ — the moduli space of instantons of Pontryagin number $k$ — is due to instantons of arbitrarily small size. Because of this, the proper definition of intersection theory on $\mathcal{M}_k$ [2] involves the integration over the compactification $\overline{\mathcal{M}}_k$ of $\mathcal{M}_k$ which includes instantons of zero-size. One can show that $\overline{\mathcal{M}}_k$ has no boundary whenever the “strata” $\mathcal{M}_{k,l}$ — whose points are the $k$–instantons given by the superposition of a $k – l$-instanton and $l$ zero-size one-instantons — are of codimension greater than one, for $l > 0$. This happens when $k$ is in the so-called “stable range”, which means that $k$ is sufficiently large ($4k > 3b^+(X) + 4$, when $G = SU(2)$). If $k$ belongs to the stable range, integrals of exact forms over $\overline{\mathcal{M}}_k$ vanish and thus the gauge-fixing ambiguity is immaterial at the level of physical correlators.

In this article we study topological gauge theories whose compactified moduli spaces have non-empty boundaries. A simple example of this situation is topological Yang-Mills on the four-dimensional sphere $S_4$, gauge group $SU(2)$ and instanton number $k = 1$, not in the stable range: the compactified moduli space is a 5-dimensional closed ball $\overline{B}_5$, whose boundary, the four-dimensional sphere $S_4$, has codimension one. This is also the case when explicit expressions for the instanton fields are known, so that a detailed analysis of the gauge-fixing ambiguity is possible.

We first show that a generic choice for the Lagrangian is parametrized by a bosonic gauge background $\tilde{A}(x; m)$ and a fermionic ghost background $\tilde{c}(x; m)$. If the transition functions for $\tilde{A}$ and $\tilde{c}$ are appropriately chosen, $\tilde{A} + \tilde{c}$ defines a connection on a certain
infinite-dimensional bundle on $X \times \mathcal{M}_k$; then correlators of equivariant operators are globally defined closed forms on the moduli space — i.e. elements of the De Rham cohomology. If a more general choice for $\tilde{A}$ and $\tilde{c}$ is made, correlators are not globally defined on $\mathcal{M}_k$. They rather have to be interpreted — together with their “descendents” — as cocycles of the Čech-De Rham sheaf on $\mathcal{M}_k$ identified by a set of local Ward identities \cite{3}. Well-known results in cohomology theory ensure that these cocycles are associated with globally defined De Rham classes.

Next, we derive the Ward identity that captures the gauge-fixing dependence of integrated correlators when the moduli space boundary is non-empty. This will allow us to identify a subset of non-trivial operators whose correlators have a gauge independent meaning. We shall see that, for $k = 1$ and $X = S_4$, gauge invariant correlators capture the cohomology of the boundary of the moduli space or, equivalently, the cohomology of the (compactified) moduli space relative to its boundary. We suspect that this is a phenomenon occurring whenever the compactified moduli space has a boundary, though we do not prove this in general.

Topological Yang-Mills on $R_4$ for $k = 1$ has been considered in a paper by D. Anselmi \cite{4}. This paper studies the whole set of correlators of equivariant closed forms without discussing their gauge dependence. We shall consider the $R_4$ theory elsewhere. Here we simply notice that the moduli space of this theory admits a natural partial compactification into $S_4 \times R_+$, which contains a non-trivial 4-cycle. Of course, if one restricts oneself to integration of correlators over this non-trivial 4-cycle, one does not encounter the boundary-related gauge ambiguities that we discuss here.

2. The Lagrangian

The BRS transformation laws characterizing the theory are \cite{5}:

\[
\begin{align*}
    s A &= -D c + \psi \\
    s \psi &= -[c, \psi] - D \phi \\
    s c &= -c^2 + \phi \\
    s \phi &= -[c, \phi].
\end{align*}
\]

Here, $s$ is the nilpotent BRS operator and all the fields are forms on $S_4$ with values in the Lie algebra of the gauge group $SU(2)$. The one-forms $A = A_\mu dx^\mu$ and $\psi = \psi_\mu dx^\mu$ are the gauge connection and the gaugino field respectively. The zero-forms $c$ and $\phi$ are the ghost and superghost fields. $A$, $\psi$, $c$, $\phi$ have ghost number 0, 1, 1 and 2 respectively. $D$
is the covariant exterior differential and commutators are taken in the Lie algebra of the
gauge group.

The generators of the equivariant cohomology of $s$ satisfy the descent equations

\begin{align*}
s \frac{1}{2} \text{Tr} F^2 &= -d \text{Tr} F \psi \\
s \text{Tr} F \psi &= -d \text{Tr} (\phi F + \frac{1}{2} \psi^2) \\
s \text{Tr} (\phi F + \frac{1}{2} \psi^2) &= -d \text{Tr} \phi \psi \\
s \text{Tr} \phi \psi &= -\frac{1}{2} d \text{Tr} \phi^2 \\
s \frac{1}{2} \text{Tr} \phi^2 &= 0,
\end{align*}

where Tr denotes the invariant Killing form on the Lie algebra of $SU(2)$.

The Lagrangian of the theory is $s$–trivial, reflecting the topological character of the
theory. We choose it in the following form:

\begin{equation}
L = s \text{Tr} \left[ \tilde{\Gamma} F_+ + \tilde{c} \tilde{D} \ast (A - \tilde{A}) + \tilde{\phi} \tilde{D} \ast (\psi - \tilde{\psi}) \\
+ x_i W^{(i)} \ast (A - \tilde{A}) \right].
\end{equation}

We have introduced the anti-fields $\Gamma$, $\tilde{c}$, $\tilde{\phi}$ with values in the gauge group Lie algebra, ghost
number -1, -1, and -2 and form degree 2, 0, and 0, respectively. Their BRS transformation
properties are:

\begin{align*}
s \tilde{\Gamma} &= -[c, \tilde{\Gamma}] + \Lambda & s \Lambda &= -[c, \Lambda] - [\phi, \tilde{\Gamma}] \\
s \tilde{c} &= -[c, \tilde{c}] + \Sigma & s \Sigma &= -[c, \Sigma] - [\phi, \tilde{c}] \\
s \tilde{\phi} &= -[c, \tilde{\phi}] + \Delta & s \Delta &= -[c, \Delta] - [\phi, \tilde{\phi}].
\end{align*}

$\Lambda$, $\Sigma$ and $\Delta$ are Lagrangian multipliers which have the same form degree as $\tilde{\Gamma}$, $\tilde{c}$ and $\tilde{\phi}$,
respectively.

The Hodge-star duality operator $\ast$ acting on forms on $X$ is defined via a space-time
background metric $g$. $F_+ = \frac{1}{2} (F + \ast F)$ is the self-dual part of the curvature two-form $F = dA + A^2$.

$\tilde{A} = \tilde{A}(x; m)$ is the gauge field background and $\tilde{D}$ is the corresponding covariant
derivative. $m \equiv (m^i)$ with $i = 1, \ldots, \dim \mathcal{M}_k$ labels a point in the moduli space. $\tilde{A}(x; m)$
is a gauge connection which belongs in the gauge orbit associated with $m$.

The moduli $m$ are dynamical variables, therefore we extend the action of the BRS
operator on the moduli space identifying it with the exterior derivative $\delta$:

\begin{equation}
sm^i = dm^i.
\end{equation}
The last term in Eq. (2.3) fixes the zero modes of $A$ and $\psi$ associated with the tangent vectors to the moduli space. \{$W^{(i)}$\} must be a system of one-forms identifying a basis of the cotangent space to $\mathcal{M}_k$. $x_i$, $y_i$ are supermultiplets of global Lagrange multipliers:

$$s x_i = y_i . \quad (2.6)$$

The basic property of the Lagrangian in Eq. (2.3) is that the corresponding functional measure, at fixed moduli, localizes operators in the algebra generated by $A$, $\psi$, $c$ and $\phi$ to certain master values. To prove this, consider first the following terms in the Lagrangian:

$$\text{Tr} \left[ \Lambda F_+ + \Sigma \tilde{D} \star (A - \tilde{A}) + y_i W^{(i)} \star (A - \tilde{A}) \right] . \quad (2.7)$$

Integrating out the Lagrange multipliers $\Lambda$, $\Sigma$ and $y_i$ localizes $A$ to the background $\tilde{A}$. In fact, the first term restricts the functional integration to the anti self-dual connections and the second term implements the projection to a local gauge slice which intersects the gauge orbit labelled by $m$. The integration over $y_i$ projects out connections which do not belong to the orbit $m$.

Next, turn to the terms containing the multipliers $\bar{\Gamma}$, $\bar{c}$ and $x_i$:

$$\text{Tr} \left[ \bar{\Gamma} (\tilde{D} \psi)_+ + \bar{c} \tilde{D} \star (\psi - \tilde{D}c - s \tilde{A}) - x_i W^{(i)} \star (\psi - \tilde{D}c - s \tilde{A}) \right] . \quad (2.8)$$

The first term puts $(\tilde{D} \psi)_+ = (\tilde{D}(\psi - \tilde{D}c - s \tilde{A}))_+ = 0$. The second and third terms give the constraint $\psi = \tilde{D}c + s \tilde{A}$. Then, the $\Delta$-dependent term

$$\text{Tr} \left[ \Delta \tilde{D} \star (\psi - \tilde{\psi}) \right] = \text{Tr} \left[ \Delta \tilde{D} \star (\tilde{D}c + s \tilde{A} - \tilde{\psi}) \right] , \quad (2.9)$$

leads to the equation

$$\tilde{D} \star (\psi - \tilde{\psi}) = \tilde{D} \star (s \tilde{A} + \tilde{D}c - \tilde{\psi}) = 0 , \quad (2.10)$$

which determines the ghost master field $\tilde{c}$:

$$\tilde{c} = \frac{1}{\Delta} \tilde{D} \star (\tilde{\psi} - s \tilde{A}) , \quad (2.11)$$

where $\tilde{\Delta} \equiv \star \tilde{D} \star \tilde{D} + \tilde{D} \star \tilde{D} \star$ is the Laplacian relative to the background $\tilde{A}$. $\tilde{\Delta}$ is not degenerate because of the absence of reducible connections on $\mathcal{M}_k$. The same equation (2.10) constrains the background field $\tilde{\psi}$ to be equal to the master field for $\psi$ — which is $s \tilde{A} + \tilde{D}c$ — up to the addition of a one-form $\delta$, with $\tilde{D} \star \delta = 0$. However, the background
field $\tilde{\psi}$ appears in the Lagrangian (2.3) exclusively via the covariant divergence $\tilde{D} \star \tilde{\psi}$. Therefore we can identify the master field for $\psi$ with $\tilde{\psi}$ with no loss of generality.

The remaining part of the Lagrangian is written, after some algebra, as:

$$\text{Tr} \left[ \tilde{\bar{\psi}} \tilde{D} \star \tilde{D}(\phi - s\tilde{c} - \tilde{c}^2) \right].$$

(2.12)

Thus the superghost master field $\tilde{\phi}$ equals $s\tilde{c} + \tilde{c}^2$.

Summarizing, we proved the equation

$$\langle X(A, c, \psi, \phi) \rangle = X(\tilde{A}, \tilde{c}, \tilde{\psi}, \tilde{\phi}),$$

(2.13)

with the master fields $\tilde{A}$, $\tilde{\psi}$, $\tilde{c}$ and $\tilde{\phi}$ obeying BRS transformation rules identical to (2.1):

$$s \tilde{A} = -\tilde{D} \tilde{c} + \tilde{\psi} \quad s \tilde{\psi} = -[\tilde{c}, \tilde{\psi}] - \tilde{D} \tilde{\phi}$$

$$s \tilde{c} = -\tilde{c}^2 + \tilde{\phi} \quad s \tilde{\phi} = -[\tilde{c}, \tilde{\phi}].$$

(2.14)

3. The gauge-fixing dependence of topological field theories

It follows from the previous analysis that the gauge-fixing freedom in our Lagrangian corresponds to the choice of $\tilde{A}(x; m)$ and $\tilde{c}(x; m)$. We now investigate the dependence of vacuum averages of gauge invariant and equivariant operators on the backgrounds $\tilde{A}$ and $\tilde{c}$. If $\tilde{A}'$ is another gauge background, there exists a gauge transformation $U(x; m) \in SU(2)$, depending in general on $m$, which relates it to $\tilde{A}$:

$$\tilde{A} \to \tilde{A}' = U^{-1}(\tilde{A} + v) U$$

(3.1)

where $v \equiv dUU^{-1}$ is a space-time one-form with values in the Lie algebra of the gauge group. Suppose for a moment that we simultaneously change the ghost background as follows

$$\tilde{c} \to \tilde{c}' = U^{-1}(\tilde{c} + \hat{v}) U,$$

(3.2)

where $\hat{v} \equiv sUU^{-1}$ is a one-form on moduli space with values in the gauge Lie algebra. It then follows from Eqs. (2.14) that the backgrounds $\tilde{\psi}$ and $\tilde{\phi}$ transform covariantly, i.e. $\tilde{\psi} \to U^{-1}\tilde{\psi} U$ and $\tilde{\phi} \to U^{-1}\tilde{\phi} U$. The BRS operator in the definition of the Lagrangian (2.3) transform covariantly under gauge transformations of the quantum fields acting on the ghost $c$ as in Eq. (3.2). Note that these quantum gauge transformations do not coincide with the classical gauge transformations, which act on the ghost $c$ homogeneously. Thus, if
$X$ is a classically gauge invariant and \textit{equivariant} operator, its vacuum average is invariant under the simultaneous variation of the backgrounds in Eqs. (3.1) and (3.2):

$$\langle X (A, \psi, \phi) \rangle_{\tilde{A}, \tilde{c}} \to \langle X (A, \psi, \phi) \rangle_{\tilde{A}', \tilde{c}'},$$  \hspace{1cm} (3.3)

Eq. (3.3) shows than an \textit{arbitrary} variation of the bosonic background $\tilde{A} \to U^{-1}(\tilde{A} + v) \ U$ is equivalent to a shift of the ghost background, $\tilde{c} \to U^{-1}(\tilde{c} - \tilde{v}) \ U$. Therefore the dependence of the vacuum averages on the gauge-fixing can be computed by considering their dependence on arbitrary variations of the ghost background $\tilde{c}$, keeping $\tilde{A}$ fixed. Essentially, this has already been done in the context of 2D topological gravity in Ref. [3]. If

$$\tilde{c} \to \tilde{c}' \equiv \tilde{c} + \eta,$$  \hspace{1cm} (3.4)

then $\bar{\psi} \to \bar{\psi} + \bar{D} \eta$. Thus the variation of the Lagrangian is

$$\mathcal{L} \to \mathcal{L} - s (\bar{\phi} D \ast \bar{D} (\tilde{c}' - \tilde{c})) = \mathcal{L} - s I_\eta \mathcal{L},$$  \hspace{1cm} (3.5)

where $I_\eta$ is the operator which shifts the superghost, $I_\eta \phi \equiv \eta$. Therefore:

$$\Delta_\eta \langle X \rangle \equiv \langle X \rangle_{\tilde{A}, \tilde{c}'} - \langle X \rangle_{\tilde{A}, \tilde{c}} = \int_0^1 dt \ [s \langle I_\eta X \rangle_{\tilde{A}, \tilde{c}(t)} + \langle I_\eta s X \rangle_{\tilde{A}, \tilde{c}(t)}],$$  \hspace{1cm} (3.6)

where $\tilde{c}(t) = t \tilde{c}' + (1 - t) \tilde{c}$ interpolates between $\tilde{c}$ and $\tilde{c}'$.

Consider the principal bundle $\mathcal{P}$ over the moduli space whose fiber is the group of local gauge transformations and whose total space is the space of (anti)-selfdual connections with instanton number $k$. The choice of the bosonic background $\tilde{A}$ corresponds to a choice of a section of this bundle. In general $\mathcal{P}$ is non-trivial. Then, the section $\tilde{A}$ is only locally defined and one needs to compare vacuum averages taken with different $\tilde{A}$ and $\tilde{A}'$ related by a gauge transformation $U$ as in Eq. (3.1). If the corresponding ghost backgrounds $\tilde{c}$ and $\tilde{c}'$ are related by Eq. (3.2) with precisely the \textit{same} $U$ as in (3.1), they define a \textit{connection} on the principal bundle $\mathcal{P}$. Eq. (3.3) shows that with such a choice of gauge-fixing on the various patches of moduli space, vacuum averages of equivariant and gauge invariant operators are globally defined forms on moduli space. A collection of backgrounds $(\tilde{A}, \tilde{c})$ satisfying Eqs. (3.1) and (3.2) is said to define a (global) gauge of De Rham type.

However different gauge choices are possible and may be computationally convenient in certain circumstances. In particular, one can choose the gauge-fixing to be of the form $(\tilde{A}, 0)$ on each patch of the moduli space. We call this choice of the (global) gauge of Čech
type for the following reason. With this gauge choice functional averages of equivariant observables are not globally defined forms; however Eq. (3.6) shows that in this situation averages of equivariant observables jump by exact terms when going from one patch to another. Starting from Eq. (3.6) it is possible to derive a descent of Ward identities whose solution is a cocycle of the Čech-De Rham sheaf over the moduli space, equivalent in cohomology to the global form defined by the De Rham gauge [3].

In our example, \( \overline{\mathcal{M}}_1 \) is contractible, \( P \) is trivial and global sections \( \tilde{A} \) exist. Any choice of the ghost background \( \tilde{c} \) defines a good connection on \( P \) and produces averages of gauge invariant and equivariant observables \( X \) which are globally defined. If \( X \) is \( s \)-closed, averages \( \langle X \rangle_{\tilde{A},\tilde{c}} \) computed with different \( \tilde{c} \) differ by exact terms, as implied by the Ward identity (3.6). When \( X \) has ghost number five, we can integrate it over \( \mathcal{M}_1 = B_5 \), the closed five-dimensional ball, but the result of the integration depends in general on the choice of \( \tilde{c} \):

\[
\Delta_\eta \int_{\mathcal{M}_1} \langle X \rangle = \int_{\mathcal{M}_1} \int_0^1 dt \int_0^1 \eta \langle I_\eta X \rangle_{\tilde{A},\tilde{c}(t)} = \int_{\partial\mathcal{M}_1} \int_0^1 dt \int_0^1 \eta \langle I_\eta X \rangle_{\tilde{A},\tilde{c}(t)}. \tag{3.7}
\]

This equation implies, nonetheless, that \( s \)-closed operators \( X \) which are independent of the superghost field \( \phi \) have vacuum expectation values which are independent of the gauge-fixing choice. From Eq. (2.2) one sees that such \( X \) are in the algebra generated by \( \int_{C_3} \text{Tr} F\psi \), for all the space-time 3-cycles \( C_3 \). A top form on \( \overline{\mathcal{M}}_1 \) is obtained by considering

\[
\Omega (C_3^{(i)}) = \prod_{i=1}^5 \int_{C_3^{(i)}} \text{Tr} F\psi. \tag{3.8}
\]

Note that \( \Omega \) is \( s \)-trivial because the third homology of \( S_4 \) is empty: since \( C_3^{(i)} = \partial B_4^{(i)} \), with \( B_4^{(i)} \) 4-chains in \( S_4 \), the descent equations (2.2) imply that

\[
\int_{C_3^{(i)}} \text{Tr} F\psi = s \int_{B_4^{(i)}} \frac{1}{2} \text{Tr} F^2. \tag{3.9}
\]

The integral of \( \Omega \) over \( \overline{\mathcal{M}}_1 \) is not necessarily zero because of the non-trivial boundary \( \partial\overline{\mathcal{M}}_1 \equiv S_4 \). Since the pull-back of \( \Omega \) on the boundary \( \partial\overline{\mathcal{M}}_1 \) vanishes, \( \Omega \) defines an element of the cohomology of \( \overline{\mathcal{M}}_1 \) relative to its boundary, which, thanks to the triviality of the cohomology of \( \overline{\mathcal{M}}_1 \), coincides with the cohomology of \( \partial\overline{\mathcal{M}}_1 \). Let us compute this class explicitly. Consider the bosonic background [3]:

\[
\tilde{A} = U^{-1} dU, \tag{3.10}
\]
where
\[ U = \frac{1}{\sqrt{\rho^2 + (x - m)^2}} \begin{pmatrix} \rho \\ x - m \end{pmatrix}, \] (3.11)
is \((2 \times 1)\)-matrix of quaternions. The quaternions \( x = x^\mu \sigma_\mu \) and \( m = m^\mu \sigma_\mu \) (with \( \sigma_\mu = (1, i \sigma_i) \) where \( \sigma_i, i = 1, 2, 3 \) are the Pauli matrices) correspond to the points \((x^\mu)\) and \((m^\mu)\) of \( R^4 \).

The coordinates \( m \) and \( \rho \) appearing in (3.11) label instantons on \( R^4 \) centered in \( m \) with size \( \rho \). By means of the stereographic projection of the four-sphere Eq. (3.11) defines as well an instanton solution on \( S^4 \), taken with a conformally flat metric. Thus, \( \rho \) and \( m \) are also coordinates on the moduli space of \( k = 1 \) instantons on \( S^4 \) but only local ones: they are not to be identified with size and position of the instantons on \( S^4 \). To see this, let \((y^i) = (y^0, y^\mu)\), with \( i = 1, \ldots, 5 \) and \( \mu = 1, \ldots, 4 \), be cartesian coordinates of \( R^5 \). If one thinks of \( \overline{M}_1 = \overline{B}_5 \) as the unit 5-ball centered in the origin of \( R^5 \), \((y^i)\) are global coordinates on it which are related to \( \rho \) and \( m \) by means of the equations
\[ \rho = \frac{\sqrt{1 - y^2}}{1 - y^0} \equiv \frac{\lambda}{1 - y^0} \] (3.12)
\[ m^\mu = \frac{y^\mu}{1 - y^0}, \]
where \( y^2 \equiv \sum_i (y^i)^2 \). From Eq. (3.12) it is clear that \( \rho \) and \( m \) are good coordinates for \( \overline{M}_1 \) only on a patch with \( y^0 \neq 1 \). There is a natural action of the rotation group \( O(5) \) on the cartesian coordinates \((y^i)\), which induces, via the coordinates transformations (3.12), an action on \( \rho \) and \( m \). This is precisely the \( O(5) \) action induced on the moduli space of instantons by the group of isometries of the space-time variety \( S^4 \). Therefore, the \( O(5) \)-invariant \( \lambda = \sqrt{1 - y^2} \) in Eq. (3.12) is to be identified with the size of the instanton and the angular coordinates of \( R^5 \) determine the position of its center in \( S^4 \). The center \( y^i = 0 \) of \( \overline{M}_1 \) represents the \( O(5) \)-invariant instanton of maximal size while the points on the four-sphere with \( \lambda = 0 \), which is the boundary of \( \overline{M}_1 \), are zero-size instantons.

The \((2 \times 1)\)-matrix of quaternions in Eq. (3.11), rewritten in terms of the coordinates \((y^i)\), is
\[ U = \frac{1}{\sqrt{\lambda^2 + X^2}} \begin{pmatrix} \lambda \\ X \end{pmatrix}, \] (3.13)
where \( X \equiv (1 - y^0)x - y \).

8
For the evaluation of $\Omega$ the choice of $\tilde{c}$ is irrelevant, as shown above. However to illustrate the gauge dependence of the generic equivariant operators it is useful to consider the following ghost background

$$\tilde{c} = U^{-1} s U,$$

(3.14)

with the same $U$ as in Eq. (3.10). $\tilde{A} + \tilde{c}$ is a connection on the principal bundle over $X \times \overline{M}_1 = S_4 \times \overline{M}_1$ which is the product of the $SU(2)$ bundle over $S_4$ associated with the instanton and the bundle $P$ over $\overline{M}_1$. The curvature of this connection is [5]:

$$F \equiv \tilde{F} + \tilde{\psi} + \tilde{\phi} = (d + s)(\tilde{A} + \tilde{c}) + (\tilde{A} + \tilde{c})^2.$$

(3.15)

The vacuum averages of the operators in the descent equation s (2.2) are encoded in the Pontryagin form associated to $F$:

$$\frac{1}{2} \text{Tr} F^2 = \frac{1}{2} \text{Tr} \tilde{\phi}^2 + \text{Tr} \tilde{\phi} \tilde{\psi} + \text{Tr} (\phi \tilde{F} + \frac{1}{2} \tilde{\psi}^2) + \text{Tr} \tilde{F} \tilde{\psi} + \frac{1}{2} \text{Tr} \tilde{F}^2,$$

(3.16)

which is a four-form on $S_4 \times \overline{M}_1$. The boundary of $S_4 \times \overline{M}_1$ is $S_4 \times \partial \overline{M}_1$. $\partial \overline{M}_1$ is naturally identified with the space-time 4-sphere, and we always imply this identification in the following.

The computation of the Pontryagin form associated to $F$ parallels the evaluation of the Pontryagin form associated to $\tilde{F} = d\tilde{A} + \tilde{A}^2$. The superconnection $\tilde{A} + \tilde{c}$ evaluated at constant $\lambda$ is given by

$$\tilde{A} + \tilde{c} = U^{-1}(d + s) U = \frac{1}{\lambda^2 + X^2} [X^\nu (d + s) X\nu - X^\mu (d + s) X\mu],$$

(3.17)

an expression which is formally identical to the familiar formula for $\tilde{A} = U^{-1} d U$. Analogously, the Pontryagin form associated to $F$ pulled-back on the boundary, $\lambda = 0$, is:

$$\frac{1}{2} \text{Tr} F^2|_{S_4 \times \partial \overline{M}_1} = \delta(\Sigma) \epsilon_{\mu \nu \rho \sigma} (d + s) X\mu (d + s) X\nu (d + s) X\rho (d + s) X\sigma

= \delta_{C_4} - \delta_{C_4(\bar{P})}.$$  

(3.18)

The two four-cycles

$$C_4 = \{ (x,y) \in S_4 \times \partial \overline{M}_1 \mid x = y \}$$  

$$C_4(\bar{P}) = \{ (x,y) \in S_4 \times \partial \overline{M}_1 \mid (x,y) = (x,\bar{P}) \},$$

(3.19)

are the solutions in $S_4 \times \partial \overline{M}_1$ of the equation $X = 0$. The point $\bar{P} \equiv (y^0 = 1, y^\mu = 0)$ is the image in $\partial \overline{M}_1$ of the point in space-time defining the stereographic projection. $\delta_C$ is
the delta-function with support on the cycle $C$, a form of degree equal to the codimension of $C$.

From Eq. (3.18) one obtains the pull-back on $\partial M_1$ of the vacuum averages of the (non-trivial) operators in the descent, computed in the backgrounds (3.10) and (3.14):

$$
\langle \frac{1}{2} \text{Tr} \phi^2(C_0) \rangle_{\tilde{A}, \tilde{c}} = \delta C_0 - \delta \bar{P} \quad \langle \int_{C_1} \text{Tr} \phi \psi \rangle_{\tilde{A}, \tilde{c}} = \delta C_1
$$

$$
\langle \int_{C_2} \text{Tr} (F \phi + \frac{1}{2} \psi^2) \rangle_{\tilde{A}, \tilde{c}} = \delta C_2 
\langle \int_{C_3} \text{Tr} (F \psi) \rangle_{\tilde{A}, \tilde{c}} = \delta C_3 .
$$

The vacuum averages of the same operators computed in the gauge $(\tilde{A}, 0)$ change according to the Ward identity (3.7):

$$
\langle \frac{1}{2} \text{Tr} \phi^2(C_0) \rangle_{\tilde{A}, 0} = \langle \int_{C_1} \text{Tr} \phi \psi \rangle_{\tilde{A}, 0} = 0
$$

$$
\langle \int_{C_2} \text{Tr} (F \phi + \frac{1}{2} \psi^2) \rangle_{\tilde{A}, 0} = \delta C_2 - s \int_{C_2} \tilde{F} \tilde{c}
$$

$$
\langle \int_{C_3} \text{Tr} (F \psi) \rangle_{\tilde{A}, 0} = \delta C_3 .
$$

We see that the class on the boundary captured by $\Omega (C^{(i)}_3)$ depends on the linking number of the five cycles $C^{(i)}_3$ in $S_4$:

$$
\int_{\partial M_1} \Omega (C^{(i)}_3) = L(C^{(i)}_3) \equiv B^{(1)}_4 \cup C^{(2)}_3 \cdots \cup C^{(5)}_3 .
$$

Before concluding, let us remark that the dependence of the vacuum average of $\frac{1}{2} \text{Tr} \phi^2$ on $\bar{P}$ reflects its dependence on the choice of $U$. In fact, consider a rotation of $O(5)$ acting simultaneously on the space-time coordinates $x^\mu$ and on the moduli $(y^i)$. It amounts, up to a gauge transformation, to multiplying $U$ on the left by a moduli-dependent matrix $R$ in $\text{Sp}(2) \approx O(5)$, which does not change $\tilde{A}$ but shifts $\tilde{c}$ by $U^{-1} R^{-1} (s \ R) \ U$. The Ward identity (3.7) then predicts a variation of $\langle \frac{1}{2} \text{Tr} \phi^2 \rangle$ which turns out to be equivalent precisely to the corresponding $O(5)$ rotation acting on $\bar{P}$. The term $\delta \bar{P}$ is absent in the same correlator computed in the theory on $R_4$, but it is crucial in our context to guarantee that

$$
\int_{\partial M_1} \langle \frac{1}{2} \text{Tr} \phi^2(C_0) \rangle = 0 .
$$

Acknowledgements

It is a pleasure to thank R. Stora for interesting discussions. This work is partially supported by the ECPR, contract SC1-CT92-0789.
References

[1] E. Witten, Nucl. Phys. B 340 (1990) 281.
[2] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four-Manifolds*, (Clarendon Press, Oxford, 1995).
[3] C. Becchi and C. Imbimbo, hep-th/9510003, Nucl. Phys. B462 (1996) 571.
[4] D. Anselmi, hep-th/9411049, Nucl. Phys. B 439 (1995) 617.
[5] L. Baulieu and I.M. Singer, Nucl. Phys. Proc. Suppl. B5 (1988) 12.
[6] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, Phys. Lett. 65A, (1978) 185.