A connection with parallel totally skew-symmetric torsion on a class of almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics

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Abstract
The subject of investigations are the almost hypercomplex manifolds with Hermitian and anti-Hermitian (Norden) metrics. A linear connection $D$ is introduced such that the structure of these manifolds is parallel with respect to $D$ and its torsion is totally skew-symmetric. The class of the nearly Kähler manifolds with respect to the first almost complex structure is of special interest. It is proved that $D$ has a $D$-parallel torsion and is weak if it is not flat. Some curvature properties of these manifolds are studied.

Keywords: almost hypercomplex manifold, pseudo-Riemannian metric, anti-Hermitian metric, Norden metric, indefinite metric, neutral metric, natural connection, parallel structure, totally skew-symmetric torsion, strong HKT-connection, weak HKT-connection.

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Introduction

The linear connections whose torsion is a 3-form, i.e. connections with totally skew-symmetric torsion, are known as Kähler with torsion (shortly, KT-) connections or Bismut connections on an almost Hermitian manifold \[39, 40\]. There exists, on any Hermitian manifold, a unique KT-connection whose torsion tensor is a 3-form \[14, 10\]. In \[10\] all almost contact metric, almost Hermitian and $G_2$-structures admitting a connection with totally skew-symmetric torsion tensor are described. Such a connection on almost complex manifolds with Norden metric is introduced and investigated in \[31, 32, 33\]. A connection of this type on almost contact manifolds with B-metric (which is the corresponding metric of the Norden metric in the odd dimensional case) is introduced in \[29\].
A good quaternionic analog of Kähler geometry is given by the *hyper-Kähler with torsion* (shortly, HKT-) geometry. An HKT-manifold is a hyper-Hermitian manifold admitting a hyper-Hermitian connection with totally skew-symmetric torsion, i.e. for which the three KT-connections associated to the three Hermitian structures coincide. This connection is said to be an *HKT-connection*. This geometry is introduced in [22] and later studied for instance in [16, 1, 2, 3, 39]. A case of particular interest is when the torsion 3-form of such HKT-connection is closed. In this case the HKT-manifold is called *strong* otherwise it is called *weak*. The HKT-geometry is a natural generalization of the hyper-Kähler geometry, since when the torsion is zero the HKT-connection coincides with the Levi-Civita connection. The HKT-connections have applications in some branches of theoretical and mathematical physics. For instance, these connections appear on supersymmetric sigma models with Wess-Zumino term [11, 23, 24] and in supergravity theory [15, 36]. Some of the applications of KT- and HKT-geometries in physics are as follows: Strong KT- and HKT-geometries have applications in type II string theory and in two-dimensional supersymmetric sigma models [38, 11, 23]. Weak KT- and HKT-geometries have applications in supersymmetric quantum mechanics [15, 7]. Strong and weak HKT-geometries have applications in the moduli spaces of gravitational solitons and black holes [15, 34, 22].

In this work we continue the investigations on an almost hypercomplex manifold \((M, H)\) with a metric structure \(G\), generated by a pseudo-Riemannian metric \(g\) of neutral signature [20, 21].

As it is known, if \(g\) is a Hermitian metric on \((M, H)\), the derived metric structure \(G\) is the known hyper-Hermitian structure. It consists of the given Hermitian metric \(g\) with respect to the three almost complex structures of \(H\) and the three Kähler forms associated with \(g\) by \(H\) (see, e.g. [1]).

Here, the considered metric structure \(G\) has a different type of compatibility with \(H\). The structure \(G\) is generated by a neutral metric \(g\) such that the first (resp., the other two) of the almost complex structures of \(H\) acts as an isometry (resp., act as anti-isometries) with respect to \(g\) in each tangent fibre. Suppose that the almost complex structures of \(H\) act as isometries or anti-isometries with respect to the metric, then the existence of an anti-isometry generates exactly the existence of one more anti-isometry and an isometry. Thus, \(G\) contains the metric \(g\) and three \((0,2)\)-tensors associated by \(H\) – a Kähler form and two metrics of the same type. The existence of such bilinear forms on an almost hypercomplex manifold is proved in [20]. The neutral metric \(g\) is Hermitian with respect to the first almost complex structure of \(H\) and \(g\) is an anti-Hermitian (i.e. Norden) metric regarding the other two almost complex structures of \(H\). For this reason we call the derived manifold \((M, H, G)\) an *almost hypercomplex manifold with Hermitian and anti-Hermitian metrics* or shortly an *almost \((H, G)\)-manifold*.

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The geometry of an arbitrary almost $(H, G)$-manifold is the geometry of the hypercomplex structure $H = \{J_1, J_2, J_3\}$ and the neutral metric $g$ or equivalently – the geometry of the metric structure $G = \{g, g_1, g_2, g_3\}$, $g_\alpha(\cdot, \cdot) := g(J_\alpha \cdot, \cdot)$, $\alpha = 1, 2, 3$. In this geometry, there are important so-called natural connections for the $(H, G)$-structure (briefly, the $(H, G)$-connections), i.e. $H$ and $g$ are parallel with respect to such a connection.

If the three almost complex structures of $H$ are parallel with respect to the Levi-Civita connection $\nabla$ of $g$, then we call such $(H, G)$-manifolds of Kähler type pseudo-hyper-Kähler manifolds and we denote their class by $\mathcal{K}$. Therefore, outside of the class $\mathcal{K}$, the Levi-Civita connection $\nabla$ is no longer an $(H, G)$-connection. There exist countless natural connections on an almost $(H, G)$-manifold in the general case.

In this paper we construct a natural connection $D$ with totally skew-symmetric torsion tensor on almost hypercomplex manifolds with Hermitian and Norden metrics. The presence of almost complex structures with Hermitian and Norden metrics gives us a reason to restrict our consideration to those classes where the corresponding natural connections with totally skew-symmetric torsion exist. Namely, these are the class $\mathcal{H}_1$ of almost Hermitian manifolds and the class of $\mathcal{W}_3$ of quasi-Kähler manifolds with Norden metric. A central place in our investigations takes the $\mathcal{H}_1$’s subclass $\mathcal{W}_1$ of nearly Kähler manifolds – an important part in the theory of geometric structures of non-integrable type. The study of these manifolds was initiated by A. Gray in 1970’s as the concept of weak holonomy and they have been investigated since then by many authors.

The present paper is organized as follows. In Section 1 we present some necessary facts about the considered manifolds. Section 2 is devoted to the study of the properties of the considered manifolds of a special class. Namely, the manifolds which belong to $\mathcal{H}_1$ with respect to $J_1$ in the Gray-Hervella classification and to $\mathcal{W}_3$ with respect to $J_2$ and $J_3$ in the Ganchev-Borisov classification. After that, we characterize the more specialized subclass (denoted by $\mathcal{W}_{133}$) of the above class, where the manifolds are nearly Kählerian with respect to $J_1$. In Section 3 we introduce the notion of a pseudo-HKT-connection (shortly, pHKT-connection) on an almost $(H, G)$-manifold and determine the class of the considered manifold, where such a connection exists. It is this class considered in the first part of Section 2. Then, we construct the pHKT-connection $D$ in a special case when the almost $(H, G)$-manifold belongs to $\mathcal{W}_{133}$. At the end of Section 3 we show that this connection has a $D$-parallel torsion and prove that any non-$D$-flat $(M, H, G)$ is a weak pHKT-manifold.

The point in question in this work is the existence and the geometric characteristics of the considered manifolds with totally skew-symmetric torsion. The main result of this paper is that the known KT-connection on a nearly Kähler manifold plays the role of the pseudo-HKT-connection on the corresponding almost $(H, G)$-manifold.
1. Almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics

1.1. The almost \((H, G)\)-manifolds

Let \((M, H)\) be an almost hypercomplex manifold, i.e. \(M\) is a \(4n\)-dimensional differentiable manifold and \(H = (J_1, J_2, J_3)\) is a triple of almost complex structures with the properties:

\[
J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \quad J_\alpha^2 = -I
\]

for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\) and \(I\) denotes the identity \([6], [1]\).

The standard structure of \(H\) on a \(4n\)-dimensional vector space with a basis \(\{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}\}\) \(k=0,1,...,n-1\) has the form \([37]\):

\[
\begin{align*}
J_1 X_{4k+1} &= X_{4k+2}, & J_2 X_{4k+1} &= X_{4k+3}, & J_3 X_{4k+1} &= -X_{4k+4}, \\
J_1 X_{4k+2} &= -X_{4k+1}, & J_2 X_{4k+2} &= X_{4k+4}, & J_3 X_{4k+2} &= X_{4k+3}, \\
J_1 X_{4k+3} &= -X_{4k+4}, & J_2 X_{4k+3} &= -X_{4k+1}, & J_3 X_{4k+3} &= -X_{4k+2}, \\
J_1 X_{4k+4} &= X_{4k+3}, & J_2 X_{4k+4} &= -X_{4k+2}, & J_3 X_{4k+4} &= X_{4k+1}.
\end{align*}
\]

Further, \(x, y, z, w\) will stand for arbitrary differentiable vector fields on \(M\).

Let \(g\) be a pseudo-Riemannian metric on \((M, H)\) with the properties

\[
g(x, y) = \varepsilon_\alpha g(J_\alpha x, J_\alpha y),
\]

where

\[
\varepsilon_\alpha = \begin{cases} 
1, & \alpha = 1; \\
-1, & \alpha = 2; 3.
\end{cases}
\]

In other words, for \(\alpha = 1\), the metric \(g\) is Hermitian with respect to \(J_1\), and in the case \(\alpha = 2\) or \(\alpha = 3\), the metric \(g\) is an anti-Hermitian (i.e. Norden) metric with respect to \(J_\alpha\) (\(\alpha = 2\) or \(\alpha = 3\), respectively) \([12]\). Moreover, the associated bilinear forms \(g_1, g_2, g_3\) are determined by

\[
g_\alpha(x, y) = g(J_\alpha x, y) = -\varepsilon_\alpha g(x, J_\alpha y), \quad \alpha = 1, 2, 3.
\]

Following \([1.3]\) and \([1.5]\), the metric \(g\) and the associated bilinear forms \(g_2\) and \(g_3\) are necessarily pseudo-Riemanian metrics of neutral signature \((2n, 2n)\). The associated bilinear form \(g_1\) is the associated (Kähler) 2-form.

A structure \((H, G) = (J_1, J_2, J_3, g_1, g_2, g_3)\) is introduced and investigated in \([20], [21], [31], [27]\) and \([28]\). The cases when the original metric \(g\) is Hermitian or anti-Hermitian with respect to the almost complex structures of \(H\) are considered. In \([20]\) it is proved that the unique possibility for an anti-Hermitian metric to be considered on an almost hypercomplex manifold is the case when the given metric is Hermitian with respect to the first and moreover it is an anti-Hermitian metric with respect to other two structures of \(H\). Therefore, we call \((H, G)\) an almost hypercomplex structure with Hermitian and anti-Hermitian
metrics on \( M \) (or, in short, an almost \((H, G)\)-structure on \( M \)). Then, we call a
manifold with such a structure briefly an almost \((H, G)\)-manifold.

The structural tensors of an almost \((H, G)\)-manifold are the three \((0,3)\)-tensors determined by

\[
F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z), \quad \alpha = 1, 2, 3, \tag{1.6}
\]

where \( \nabla \) is the Levi-Civita connection generated by \( g \).

The tensors \( F_\alpha \) have the following fundamental identities:

\[
F_\alpha(x, y, z) = -\varepsilon_\alpha F_\alpha(x, z, y) = -\varepsilon_\alpha F_\alpha(x, J_\alpha y, J_\alpha z),
\]

\[
F_\alpha(x, J_\alpha y, z) = \varepsilon_\alpha F_\alpha(x, y, J_\alpha z);
\]

\[
F_\alpha(x, y, z) = F_\beta(x, J_\gamma y, z) = \varepsilon_\beta F_\gamma(x, y, J_\beta z)
\]

\[
= -F_\gamma(x, J_\beta y, z) + \varepsilon_\gamma F_\beta(x, y, J_\gamma z) \tag{1.8}
\]

for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\).

In \([21]\) we study a special class \( K \) of the \((H, G)\)-manifolds – the so-called
there pseudo-hyper-Kähler manifolds. The manifolds from the class \( K \) are the
\((H, G)\)-manifolds for which the complex structures \( J_\alpha \) are parallel with respect
to the Levi-Civita connection \( \nabla \), generated by \( g \), for all \( \alpha = 1, 2, 3 \).

As \( g \) is an indefinite metric, there exist isotropic vectors on \( M \), i.e. \( g(x, x) = 0 \) for a nonzero vector \( x \). In \([20]\) we define the invariant square norm

\[
\|\nabla J_\alpha\|^2 = g^{ij} g^{kl} g((\nabla_i J_\alpha) e_k, (\nabla_j J_\alpha) e_l), \tag{1.9}
\]

where \( \{e_i\}_{i=1}^n \) is an arbitrary basis of the tangent space \( T_p M \) at an arbitrary
point \( p \in M \). We say that an almost \((H, G)\)-manifold is an isotropic pseudo-hyper-Kähler manifold if \( \|\nabla J_\alpha\|^2 = 0 \) for all \( J_\alpha \) of \( H \). Clearly, if \((M, H, G)\)
is a pseudo-hyper-Kähler manifold, then it is an isotropic pseudo-hyper-Kähler manifold. The inverse statement does not hold. For instance, in \([20]\) we have
constructed an almost \((H, G)\)-manifold on a Lie group, which is an isotropic pseudo-hyper-Kähler manifold but it is not a pseudo-hyper-Kähler manifold.

1.2. Properties of the Kähler-like tensors

A tensor \( L \) of type \((0,4)\) with the properties:

\[
L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),
\]

\[
L(x, y, z, w) + L(y, z, x, w) + L(z, x, y, w) = 0 \tag{1.10}
\]

is called a curvature-like tensor. The last equality of \((1.10)\) is known as the first
Bianchi identity of a curvature-like tensor \( L \).

We say that a curvature-like tensor \( L \) is a Kähler-like tensor on an almost
\((H, G)\)-manifold when \( L \) satisfies the properties:

\[
L(x, y, z, w) = \varepsilon_\alpha L(x, y, J_\alpha z, J_\alpha w) = \varepsilon_\alpha L(J_\alpha x, J_\alpha y, z, w), \tag{1.11}
\]
where $\varepsilon_\alpha$ is determined by (1.4).

Let the curvature tensor $R$ of the Levi-Civita connection $\nabla$, generated by $g$, be defined, as usual, by $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z$. The corresponding $(0, 4)$-tensor, denoted by the same letter, is determined by $R(x, y, z, w) = g(R(x, y)z, w)$. Obviously, $R$ is a Kähler-like tensor on an arbitrary pseudo-hyper-Kähler manifold.

A Kähler-like tensor $L$ on an arbitrary almost $(H, G)$-manifold has the same properties (1.10) and (1.11) of $R$ on a pseudo-hyper-Kähler manifold. Thus, we obtain the following geometric characteristic of the Kähler-like tensors on an almost $(H, G)$-manifold, similarly to Theorem 2.3 in [21], where it is proved that the hyper-Kähler $(H, G)$-manifolds are flat, i.e. $R = 0$ in $\mathcal{K}$. Using the same idea, we establish the truthfulness of the following

**Proposition 1.1** ([28]). Every Kähler-like tensor on an almost $(H, G)$-manifold is zero. □

2. Properties of the $(H, G)$-manifolds of a certain class

2.1. Class $\mathcal{G}_1(J_1) \cap \mathcal{W}_3(J_2) \cap \mathcal{W}_3(J_3)$

For the sake of our further purposes, we restrict the class of $(M, H, G)$ to the class $\mathcal{G}_1(J_1) \cap \mathcal{W}_3(J_2) \cap \mathcal{W}_3(J_3)$. In other words, $(M, J_1, g)$ is an almost Hermitian manifold with neutral metric of the class $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ according to the classification in [19], and $(M, J_\alpha, g)$, $\alpha = 2, 3$, are almost complex manifolds with Norden metric of the class $\mathcal{W}_3$ (the so-called quasi Kähler manifolds with Norden metric), according to the classification in [12]. There these classes are defined as follows:

$$\mathcal{G}_1(J_1) : \quad F_1(x, x, z) = F_1(J_1x, J_1x, z), \quad (2.1)$$

$$\mathcal{W}_3(J_\alpha) : \quad \bigoplus_{x, y, z} F_\alpha(x, y, z) = 0, \quad \alpha = 2, 3. \quad (2.2)$$

**Theorem 2.1.** Let $M$ be an almost $(H, G)$-manifold which is a quasi-Kähler manifold with Norden metric regarding $J_2$ and $J_3$. Then it belongs to the class $\mathcal{G}_1$ with respect to $J_1$.

**Proof.** Suppose $(M, J_\alpha, g)$, $\alpha = 2, 3$, belongs to the class $\mathcal{W}_3$ of the quasi-Kähler manifolds with Norden metric. Then, according to (2.2), we have respectively

$$F_2(x, y, z) + F_2(y, x, z) = -F_2(z, x, y), \quad (2.3)$$

$$F_3(x, y, z) + F_3(y, x, z) = -F_3(z, x, y). \quad (2.4)$$

The relation $J_2 = -J_1 \circ J_3$ implies

$$F_2(x, y, z) = F_3(x, y, J_1z) - F_1(x, J_3y, z). \quad (2.5)$$

Next, using consecutively (2.3), (2.5) and (2.4), we obtain

$$F_2(x, y, z) = -F_2(x, y, z) - F_2(y, x, z)$$

$$= -F_3(x, y, J_1z) + F_1(x, J_3y, z) - F_3(y, x, J_1z) + F_1(y, J_3x, z),$$

6
\[ F_2(z, x, y) = F_3(J_1z, x, y) + F_1(x, J_3y, z) + F_1(y, J_3x, z). \]  

(2.6)

Having in mind \( J_3 = -J_2 \circ J_1 \), we express the term \( F_3(J_1z, x, y) \) from the line above as follows

\[ F_3(J_1z, x, y) = -F_2(J_1z, J_1x, y) - F_1(J_1z, x, J_2y). \]  

(2.7)

Then, from (2.6) and (2.7) we have

\[ F_2(z, x, y) + F_2(J_1z, J_1x, y) = F_1(x, J_3y, z) + F_1(y, J_3x, z) - F_1(J_1z, x, J_2y). \]  

(2.8)

We replace the substitutions \( z \rightarrow J_1z \) and \( x \rightarrow J_1x \) in (2.8) and applying properties (1.7) and (1.1), we obtain

\[ F_2(J_1z, J_1x, y) + F_2(z, x, y) = -F_1(J_1x, J_2y, z) + F_1(y, J_3x, z) + F_1(z, J_1x, J_2y). \]  

(2.9)

The difference of (2.8) and (2.9) is the following

\[ F_1(x, J_3y, z) + F_1(J_1x, J_2y, z) - F_1(J_1z, x, J_2y) - F_1(z, J_1x, J_2y) = 0. \]

Next, we substitute \( J_3y \) for \( y \) in the last equality and applying (1.7) we obtain

\[ F_1(x, z, y) - F_1(J_1x, J_1z, y) - F_1(J_1z, J_1x, y) + F_1(z, x, y) = 0, \]  

(2.10)

which is equivalent to (2.11), i.e. \( (M, J_1, g) \) belongs to the class \( G_1 \).

\[ \textbf{Theorem 2.2.} \textit{Let } M \textit{ be an almost } (H, G) \text{-manifold from the class } G_1(J_1) \cap W_3(J_2) \cap W_3(J_3). \textit{If } (M, J_1, g) \textit{ belongs to the subclass } W_0(J_1) : F_1 = 0 \textit{ of the Kähler manifolds with neutral metric then } (M, H, G) \textit{ is a pseudo-hyper-Kähler manifold.} \]

\textit{Proof.} Suppose \( F_1 = 0 \) holds. By virtue of (1.8) we have

\[ F_2(x, y, z) = F_3(x, J_1y, z) = F_3(x, y, J_1z) \]  

(2.11)

and by substitutions \( y \rightarrow J_1y, z \rightarrow J_1z \) the following relations are valid

\[ F_2(x, J_1y, J_1z) = -F_3(x, y, J_1z) = -F_3(x, J_1y, z). \]  

(2.12)

Comparing (2.11) and (2.12), we obtain

\[ F_2(x, J_1y, J_1z) = -F_2(x, y, z). \]  

(2.13)

Having in mind the property \( F_2(x, J_2y, J_2z) = F_2(x, y, z) \) from (1.7), equality (2.13) implies

\[ F_2(x, J_3y, J_3z) = -F_2(x, y, z). \]  

(2.14)
Now, from (2.11) we have
\[ F_2(x, y, J_2 z) = F_3(x, y, J_3 z). \] (2.15)

According to the conditions \( S_{x, y, z} F_2(x, y, z) = S_{x, y, z} F_3(x, y, z) = 0 \) for a manifold in \( W_3(J_2) \cap W_3(J_3) \) and since \( F_2 \) and \( F_3 \) are symmetric by the second and the third arguments, then by virtue of (2.15) the following equality is valid
\[ F_2(J_2 z, x, y) = F_3(J_3 z, x, y). \] (2.16)

Using \( F_3(x, J_3 y, J_3 z) = F_3(x, y, z) \) from (1.7), we obtain
\[ F_2(z, x, y) = F_2(z, J_3 x, J_3 y). \] (2.17)

Equalities (2.14) and (2.17) imply \( F_2 = 0 \) and therefore \( F_3 = 0 \) is also valid, i.e. \( (M, H, G) \in \mathcal{K} \).

2.2. Class \( W_{133} \)

In this subsection we consider a more specialized manifold \( (M, H, G) \) with non-integrable structures \( J_\alpha (\alpha = 1, 2, 3) \), namely a manifold which is nearly Kählerian with neutral metric regarding \( J_1 \) and quasi Kählerian with Norden metric regarding \( J_2 \) and \( J_3 \). In other words, \( (M, H, G) \) belongs to the class \( W_1(J_1) \cap W_3(J_2) \cap W_3(J_3) \) (in short, \( W_{133} \)) according to the corresponding classifications in [17] and [12], where the basic class \( W_1(J_1) \) is defined by
\[ W_1(J_1) : F_1(x, y, z) = -F_1(y, x, z) \] (2.18)
or equivalently \( F_1(x, y, z) = \frac{1}{3} \sum_{x, y, z} F_1(x, y, z) = \frac{1}{3} d g_1(x, y, z) \). Furthermore, the property \( F_1(J_1 x, J_1 y, z) = -F_1(x, y, z) \) holds for a \( W_1(J_1) \)-manifold since \( W_1(J_1) \subset \mathcal{S}_{1}(J_1) \) and \( 2.24 \).

It is known ([19]) that the class \( \mathcal{S}_1 = W_1 \oplus W_3 \oplus W_4 \) of almost Hermitian manifolds \( (M, J_1, g) \) exists in general form when the dimension of \( M \) is at least 6. At dimension 4, \( \mathcal{S}_1 \) is restricted to its subclass \( W_4 \). Thus, the manifold \( (M, H, G) \) belonging to the class \( W_{133} \) exists when \( \text{dim} \ M \geq 8 \).

2.2.1. Properties of the structural \((0,3)\)-tensors in \( W_{133} \)

According to (1.7) for \( \alpha = 1 \) and (2.18), we have
\[ F_1(x, y, z) = -F_1(y, x, z) = -F_1(x, z, y), \] (2.19)
i.e. \( F_1 \) is a 3-form, and moreover
\[ F_1(x, y, z) = -F_1(J_1 x, J_3 y, z) = -F_1(J_3 x, y, J_1 z) = -F_1(x, J_1 y, J_3 z), \] (2.20)
\[ F_1(J_1 x, y, z) = F_1(x, J_3 y, z) = F_1(x, y, J_1 z). \] (2.21)
Lemma 2.3. The structural tensors \( F_\alpha \) \( (\alpha = 1, 2, 3) \) of \( (M, H, G) \in W_{133} \) has the following properties:

\[
F_1(J_2x, J_y z) = -F_1(J_3x, J_2y, z), \\
F_1(J_2x, J_y J_2z) = -F_1(J_3x, J_y J_3z), \\
F_1(x, J_2y, J_2z) = -F_1(x, J_3y, J_3z),
\]

(2.22)

\[
F_2(x, y, J_1z) = F_2(x, J_1y, J_1z) = F_2(x, J_3y, J_3z), \\
F_3(x, y, J_1z) = F_3(x, J_1y, J_1z) = F_3(x, J_2y, J_2z).
\]

(2.23)

Proof. The first equality of (2.22) follows from the first equality of (2.20) applying \( J_2 \) to \( x \) and \( y \). In a similar way, we obtain the other two properties in (2.22).

We apply \( J_3 \) to \( x \) and \( y \) in (2.20). Then, using (1.7) and (2.19), we obtain \( F_2(z, J_3x, J_3y) = F_2(z, x, y) \). The remaining part of (2.23) follows from the latter equality by the substitutions \( x \to J_2x \), \( y \to J_2y \) and (1.7).

The proof of (2.24) is analogous to the previous one. □

The last lemma imply immediately the following

Lemma 2.4. The structural tensors \( F_\alpha \) \( (\alpha = 1, 2, 3) \) of \( (M, H, G) \in W_{133} \) has the following properties:

\[
F_1(J_2x, J_3y, z) = F_1(J_3x, J_2y, z), \\
F_1(J_2x, y, J_3z) = F_1(J_2x, y, J_3z), \\
F_1(x, J_2y, J_3z) = F_1(x, J_2y, J_3z),
\]

(2.25)

\[
F_2(x, y, J_1z) = -F_2(x, J_1y, z), \\
F_2(x, y, J_3z) = -F_2(x, J_3y, z), \\
F_3(x, y, J_1z) = -F_3(x, J_1y, z), \\
F_3(x, y, J_2z) = -F_3(x, J_2y, z).
\]

(2.26)

(2.27)

The latter two lemmas imply the following

Proposition 2.5. The structural tensors \( F_\alpha \) \( (\alpha = 1, 2, 3) \) of a \( W_{133} \)-manifold \( (M, H, G) \) have the following properties:

\[
2F_2(x, y, z) = F_1(x, y, J_3z) - F_1(x, J_3y, z), \\
2F_3(x, y, z) = F_1(x, J_2y, z) - F_1(x, J_2y, z),
\]

(2.28)

\[
F_2(x, y, z) = -F_3(J_1x, y, z), \\
F_1(x, y, J_1z) + F_2(x, y, J_2z) + F_3(x, y, J_3z) = 0.
\]

(2.29)

(2.30)

(2.31)

Proof. From (1.8) we have

\[
F_2(x, y, z) = F_3(x, J_1y, z) + F_1(x, y, J_3z), \\
F_2(x, y, z) = -F_1(x, J_3y, z) + F_3(x, y, J_1z)
\]
which we sum up, apply (2.27) so the result is (2.28). Similarly, we obtain (2.29).

Next, we replace $z$ with $J_2 z$ and $J_3 z$ in (2.28) and (2.29), respectively. Then, the addition of the resultant equalities yields (2.31), because of (2.21).

**Proposition 2.6.** Let $M$ be an almost $(H, G)$-manifold from the class $W_{133}$. If $(M, J_\alpha, g)$ (for some $\alpha = 1, 2, 3$) belongs to the subclass $W_0(J_\alpha) : F_\alpha = 0$ of the corresponding manifolds of Kähler type then $(M, H, G)$ is a pseudo-hyper-Kähler manifold.

**Proof.** It follows from the interconnections between $F_1$, $F_2$ and $F_3$ in Proposition 2.5. □

Having in mind Proposition 2.6, we can restrict our attention to the class of the so-called strict $W_{133}$-manifolds $(M, H, G)$, i.e. $(M, J_\alpha, g)$ does not belong to $W_0(J_\alpha)$ for any $\alpha = 1, 2, 3$.

### 2.2.2. Curvature properties in $W_{133}$

Let us recall the following curvature properties on a nearly Kähler manifold $(M, J_1, g)$ known from [17]:

$$R(x, y, J_1 z, J_1 w) - R(x, y, z, w) = A_1(x, y, z, w),$$

where $A_1(x, y, z, w) = g((\nabla_x J_1) y, (\nabla_z J_1) w) = g(T(x, y), T(z, w))$ and

$$R(J_1 x, J_1 y, J_1 z, J_1 w) = R(x, y, z, w).$$

(2.33)

**Theorem 2.7.** The curvature tensor $R$ on $(M, H, G) \in W_{133}$ has the following property with respect to the almost hypercomplex structure $H$:

$$R(x, y, z, w) + \sum_{\alpha=1}^3 R(x, y, J_\alpha z, J_\alpha w)$$

$$= \sum_{\alpha=1}^3 \left\{ A_\alpha(x, y, z, w) - A_\alpha(y, z, x, w) \right\},$$

(2.34)

where $A_\alpha(x, y, z, w) = g((\nabla_x J_\alpha) y, (\nabla_z J_\alpha) w)$, $\alpha = 1, 2, 3$.

**Proof.** We apply the covariant derivative by $\nabla$ to (2.31) and obtain

$$\sum_{\alpha=1}^3 \left\{ (\nabla_x F_\alpha)(y, z, J_\alpha w) + A_\alpha(y, z, x, w) \right\} = 0.$$

We alternate the last equality with respect to $x$ and $y$. Further, we apply the following corollary of the Ricci identity on an $(H, G)$-manifold known from [28]:

$$(\nabla_x F_\alpha)(y, z, J_\alpha w) - (\nabla_y F_\alpha)(x, z, J_\alpha w)$$

$$= R(x, y, J_\alpha z, J_\alpha w) - \varepsilon_\alpha R(x, y, z, w).$$

(2.35)

The result is (2.34). □
Corollary 2.8. The curvature tensor $R$ on $(M, H, G) \in W_{133}$ has the following properties with respect to the almost complex structures $J_2$ and $J_3$, respectively:

$$2R(x, y, z, w) + 2R(x, y, J_2 z, J_2 w) = \sum_{\alpha=1}^{3} \left\{ A_\alpha(x, z, y, w) - A_\alpha(y, z, x, w) \right\} - A_1(x, y, z, w),$$

(2.36)

and

$$2R(x, y, J_3 z, J_3 w) = \sum_{\alpha=1}^{3} \left\{ A_\alpha(x, z, y, w) - A_\alpha(y, z, x, w) \right\} - A_1(x, y, J_3 z, J_3 w),$$

(2.37)

Proof. From (2.32) we have

$$R(x, y, J_1 z, J_1 w) = R(x, y, z, w) + A_1(x, y, z, w).$$

After the substitutions $z \rightarrow J_2 z$ and $w \rightarrow J_2 w$ in the last equality we obtain

$$R(x, y, J_2 z, J_2 w) = R(x, y, J_2 z, J_2 w) + A_1(x, y, J_2 z, J_2 w).$$

We replace the left-hand sides of the last two equalities in (2.31) and get (2.36).

In a similar way we obtain (2.37). Moreover, we establish the property $A_1(x, y, J_2 z, J_2 w) = -A_1(x, y, J_3 z, J_3 w)$ because of the first line of (2.22).

As an immediate consequence of Corollary 2.8 we obtain for the scalar curvature and its associated quantities by $J_\alpha$ the following relations:

$$\tau - \tau_1^{**} = \|\nabla J_1\|^2, \quad \tau + \tau_\alpha^{**} = -\frac{1}{2} \|\nabla J_\alpha\|^2, \quad \alpha = 2, 3.$$  

(2.38)

where $\tau_\alpha^{**} = g^{ij} g^{ks} R(e_i, e_k, J_\alpha e_s, J_\alpha e_j)$, $\alpha = 1, 2, 3$.

According to (2.30) we get directly

$$\|\nabla J_2\|^2 = \|\nabla J_3\|^2.$$  

(2.39)

Using (2.28) and (2.29) we obtain

$$\|\nabla J_2\|^2 = -\frac{1}{2} \|\nabla J_1\|^2 - \frac{1}{2} g^{ij} g^{ks} g^{pq} F_1(e_i, e_k, e_p) F_1(e_j, J_3 e_s, J_3 e_q),$$

$$\|\nabla J_3\|^2 = -\frac{1}{2} \|\nabla J_1\|^2 - \frac{1}{2} g^{ij} g^{ks} g^{pq} F_1(e_i, e_k, e_p) F_1(e_j, J_2 e_s, J_2 e_q).$$

Since the third property in (2.22) holds, then summing up the latter two equalities we obtain

$$\|\nabla J_1\|^2 + \|\nabla J_2\|^2 + \|\nabla J_3\|^2 = 0.$$
Because of \((2.39)\) the last equality implies
\[
\|\nabla J_1\|^2 = -2 \|\nabla J_2\|^2 = -2 \|\nabla J_3\|^2.
\] (2.40)

From \((2.38)\) and \((2.40)\) we obtain the following relation
\[
3\tau + \tau_1^{**} = -4\tau_2^{**} = -4\tau_3^{**}.
\] (2.41)

3. Pseudo-HKT-connection on an \((H, G)\)-manifold which is a nearly Kähler manifold with respect to \(J_1\)

3.1. Constructing the pHKT-connection

As it is known from [13], a linear connection \(D\) is called a natural connection on an almost complex manifold \((M, J)\) with Norden metric \(g\) if the almost complex structure \(J\) and the metric \(g\) are parallel with respect to \(D\), i.e. \(DJ = Dg = 0\).

The notion of the hyper-Hermitian connection in the hyper-Hermitian geometry is known. This is a linear connection such that the three almost complex structures and the Hermitian metric are parallel regarding this connection.

Following this idea we give

**Definition 3.1.** A linear connection \(D\) is called a natural connection on an almost hypercomplex manifold \((M, H)\) with a pseudo-Riemannian metric \(g\) if the almost hypercomplex structure \(H = (J_1, J_2, J_3)\) and the metric \(g\) are parallel with respect to \(D\), i.e. \(DJ_1 = DJ_2 = DJ_3 = Dg = 0\).

As a corollary for a natural connection \(D\) on \((M, H, G)\) we have also \(Dg_1 = Dg_2 = Dg_3 = 0\), having in mind \((1.5)\).

If \(T\) is the torsion tensor of \(D\), i.e. \(T(x, y) = D_x y - D_y x - [x, y]\), then the corresponding tensor field of type \((0,3)\) is determined by \(T(x, y, z) = g(T(x, y), z)\).

In a similar way to KT- and HKT-connections we introduce

**Definition 3.2.** A natural connection \(D\) is called a pseudo-HKT-connection on an almost \((H, G)\)-manifold (briefly, pHKT-connection) if the torsion tensor \(T\) of \(D\) is totally skew-symmetric.

For an almost complex manifold with Hermitian metric \((M, J, g)\), in [10] it is proved that there exists a unique KT-connection if and only if the Nijenhuis tensor \(N_J(x, y, z) := g(N_J(x, y), z)\) is a 3-form, i.e. the manifold belongs to the class of cocalibrated structures \(S_1 = W_1 \oplus W_3 \oplus W_4\). KT-connections on nearly Kähler manifolds are investigated in [35] for instance.

Then, there exists a unique KT-connection \(D^1\) for the almost Hermitian manifold \((M, J_1, g) \in S_1\) on the considered almost \((H, G)\)-manifold such that
\[
g\left(D^{1}_{xy} z\right) = g\left(\nabla_{xy} z\right) + Q_1(x, y, z),
\] (3.1)

where
\[
Q_1(x, y, z) = \frac{1}{2} \left\{ F_1(x, y, J_1 z) + F_1(y, z, J_1 x) - F_1(z, x, J_1 y) \right\}.
\] (3.2)
The difference $Q_1$ of $D^1$ and $\nabla$ is a totally skew-symmetric tensor because of the relation $T_1 = 2Q_1$ with the corresponding torsion $T_1$ of the KT-connection $D^1$. Then, in the case when $\mathcal{G}_1$ is restricted to $\mathcal{W}_1$ for $J_1$, because of (2.18), expression (3.2) assumes the form

$$Q_1(x, y, z) = \frac{1}{2} F_1(x, y, J_1 z).$$

In [31] it is proved that there exists a unique KT-connection on an almost complex manifold with Norden metric $(M, J, g)$ if and only if the manifold is quasi-Kählerian with Norden metric or $(M, J, g)$ belongs to the class $\mathcal{W}_3$.

Then, there exists a unique KT-connection $D^\alpha$ ($\alpha = 2, 3$, respectively) for the almost complex manifold with Norden metric $(M, J_\alpha, g) \in \mathcal{W}_3(J_\alpha)$ on the considered almost $(H, G)$-manifold such that

$$g(D^\alpha_x y, z) = g(\nabla_x y, z) + Q_\alpha(x, y, z),$$

where

$$Q_\alpha(x, y, z) = -\frac{1}{4} \sum_{x, y, z} F_\alpha(x, y, J_\alpha z).$$

Let us construct a linear connection $D$, using the KT-connections $D^1$, $D^2$ and $D^3$, on an almost $(H, G)$-manifold from the class $\mathcal{G}_1(J_1) \cap \mathcal{W}_3(J_2) \cap \mathcal{W}_3(J_3)$ as follows

$$g(D_x y, z) = g(\nabla_x y, z) + Q(x, y, z),$$

where

$$Q(x, y, z) = -\frac{1}{2} Q_1(x, y, z) + Q_2(x, y, z) + Q_3(x, y, z).$$

We apply (2.31), (1.7), (2.18) and (2.21) in (3.7) and obtain

$$Q(x, y, z) = \frac{1}{2} F_1(x, y, J_1 z).$$

Then, the linear connection $D_1$, introduced by (3.6) and (3.7), has the following definitional equality

$$g(D_x y, z) = g(\nabla_x y, z) + \frac{1}{2} F_1(x, y, J_1 z)$$

or equivalently

$$D_x y = \nabla_x y - \frac{1}{2} J_1 (\nabla_x J_1) y.$$ (3.9)

We verify directly that $D$ is a natural connection on $(M, H, G)$ and therefore, because of the properties of $F_1$ and the uniqueness of $D^1$, $D^2$ and $D^3$, we obtain

**Proposition 3.1.** The linear connection $D$ defined by (3.9) is a unique pHKT-connection on an almost $(H, G)$-manifold from the class $\mathcal{W}_{133}$. □

Let us remark that the pHKT-connection $D$ on an almost $(H, G)$-manifold coincides with the known KT-connection $D^1$ on nearly Kähler manifolds studied in [10], [35] for the Riemannian case and [8] for the pseudo-Riemannian case.
3.2. Characteristics of the constructed pHKT-connection

3.2.1. $D$-parallel torsion of $D$

Since $D$ coincides with $D^1$, then the pHKT-connection $D$ defined by (3.9) on an almost $(H, G)$-manifold from $W_{133}$ is $D$-parallel, $dT = 0$, and henceforth $T$ is coclosed, $dT = 0$ (10), (26), (4).

3.2.2. Curvature tensor of $D$

Let us consider the curvature tensor $K$ of the connection $D$, i.e. $K = [Dx, Dy]z - D[Dx, y]z$ and $K(x, y, z, w) = g(K(x, y)z, w)$. The relation between the connections $D$ and $\nabla$ generates the corresponding relation between their curvature tensors $K$ and $R$.

**Proposition 3.2.** Let $(M, H, G)$ be an almost $(H, G)$-manifold from $W_{133}$ and $D$ be the pHKT-connection defined by (3.9). Then the curvature tensors $K$ of $D$ and $R$ of $\nabla$ has the following relation

$$K(x, y, z, w) = R(x, y, z, w) + \frac{1}{4} A_1(x, y, z, w) + \frac{1}{4} S_{x, y, z} A_1(x, y, z, w). \ (3.10)$$

Let us remark that such a relation is given in [18], [32] and [10].

**Proposition** implies directly that $r^D$ is symmetric and $J_1$-invariant, which is also known from [10] and [32] for the Riemannian case. Moreover, for the scalar curvatures we have

$$r^D = r^\nabla - \frac{1}{4} \|T\|^2, \ (3.11)$$

where $\|T\|^2 = \|\nabla J_1\|^2$.

Further, we give some necessary and sufficient conditions for flatness of $D$ in the following

**Proposition 3.3.** Let $(M, H, G)$ be an almost $(H, G)$-manifold from $W_{133}$ and $D$ be the parallel pHKT-connection defined by (3.9). Then the following characteristics of this connection are equivalent:

(i) $D$ is strong;

(ii) $D$ has a $\nabla$-parallel torsion;

(iii) $D$ is flat.

**Proof.** Using $DT = 0$ we obtain

$$(\nabla x T)(y, z, w) = \frac{1}{2} S_{x, y, z} A_1(x, y, z, w). \ (3.12)$$

Then we compute the exterior derivative of the 3-form $T$ as

$$dT(x, y, z, w) = 2 S_{x, y, z} A_1(x, y, z, w). \ (3.13)$$
Therefore, the pHKT-connection is strong, i.e. $dT = 0$, if and only if the identity 
$$\mathbf{S} A_1(x, y, z, w) = 0$$ is valid, which is equivalent to $\nabla T = 0$, according to (3.12). Thus, (i) is equivalent to (ii).

The curvature tensor $K$ satisfies the properties of the first line of (1.10) and the Kähler-like property (1.11), since $DJ_\alpha = 0$, $\alpha = 1, 2, 3$. Then, $K$ is a Kähler-like tensor if and only if the identity 
$$\mathbf{S} K(x, y, z, w) = 0$$ holds.

On the other hand, the first Bianchi identity for $K$ with torsion $T$
$$\mathbf{S} \ K(x, y, z, w) = \mathbf{S} \ \{T(T(x, y), z, w) + (D_x T)(y, z, w)\}$$
takes the following form, since $T$ is $D$-parallel:

$$\mathbf{S} \ K(x, y, z, w) = \mathbf{S} \ A_1(x, y, z, w). \quad (3.14)$$

Therefore, according to (3.13) and (3.14), we have that $D$ is a strong pHKT-connection if and only if the curvature tensor $K$ of $D$ is a Kähler-like tensor. Then, because of Proposition 1.1 $K$ is zero, i.e. the equivalence (i) ⇔ (iii) is valid. This completes the proof.

**Proposition 3.4.** Let $(M, H, G)$ be an almost $(H, G)$-manifold from $\mathcal{W}_{133}$ and $D$ be the parallel pHKT-connection defined by (3.9). If $D$ is flat or strong then $(M, H, G)$ is $\nabla$-flat, isotropic-hyper-Kählerian and the torsion of $D$ is isotropic.

**Proof.** According to Proposition 3.3 if $(M, H, G)$ is $D$-flat then we obtain the following form of the curvature tensor $R$ of the Levi-Civita connection $\nabla$, applying (3.10):

$$R(x, y, z, w) = -\frac{1}{4} A_1(x, y, z, w). \quad (3.15)$$

Substituting $J_1 z$ and $J_1 w$ for $z$ and $w$ in (3.15), respectively, and using the properties $A_1(x, y, J_1 z, J_1 w) = -A_1(x, y, z, w)$ and (2.32) for a nearly Kähler manifold $(M, J_1, g)$, we obtain $A_1 = 0$. Therefore, according to (3.14), $R = 0$ and then the connections $\nabla$ and $D$ are both flat. Moreover, having in mind (2.40) and (3.11), $(M, H, G)$ is isotropic-hyper-Kählerian and $T$ is isotropic.

**Corollary 3.5.** Let $(M, H, G)$ be an almost $(H, G)$-manifold from $\mathcal{W}_{133}$. If it is not $D$-flat then it does not admit a strong pHKT-connection, i.e. any non-$D$-flat $(M, H, G)$ is a weak pHKT-manifold.

It is known from [8] that there are no strict flat nearly pseudo-Kähler manifolds of dimension less than 12.

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