Hypothesis Elimination in Kleene Semirings
(Extended Abstract)

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Abstract—A Kleene Semiring (KS) is an algebraic structure satisfying the axioms of Kleene algebra, minus the annihilation axioms \( x.0 = 0 = 0.x \). We show that, like Kleene algebra (KA), KS admits efficient elimination of various kinds of equational hypotheses, in particular Hoare formulas (\( x = 0 \)). Our method is purely proof-theoretic, and can be used to eliminate Horn hypotheses in any suitable Horn-equational theory. Moreover, it gives a simple condition under which hypotheses eliminations can be combined.

I. INTRODUCTION

Kleene algebra (KA) [7] and its descendants, such as Kleene algebra with tests (KAT) [8] and omega algebra [2] have proved useful in reasoning about programs and program transformations, e.g. theorems about concurrency control [2], static analysis [9], and compiler optimization [10]. These algebras well-suited to such applications for several reasons. The operators of Kleene algebra (the regular expression operators \( 0,1,+,\ldots, \) and \( * \)) correspond naturally to program operators (the miracle, skip, nondeterministic choice, sequential composition, and finite repetition). KA is easy to teach and to use; its equational theory is just the equational theory of regular expressions, and its formulation of induction is particularly simple (e.g., no well-founded sets). KA is particularly well-suited to program reasoning requiring commutativity arguments, because arbitrary terms (not just tests) can be used as inductive hypotheses [2]. Using tests, KA can faithfully encode most things one wants to do in PSPACE (e.g., automata constructions), including arguments that are awkward in alternative formalisms (e.g., a 10 page TLA proof of the reduction theorem in [11] shrinks to half a page [2]). Finally, the equational theory of these algebras is computationally tractable (PSPACE-complete [7], [3]), in contrast to alternatives such as relational algebra.

Most interesting applications of these algebras require reasoning in the presence of additional equational hypotheses giving the required properties of the program fragments. For example, if \( p \) and \( q \) are tests, the equation \( p.x.q = 0 \) represents the Hoare triple \( \{ p \} x \{ \neg q \} \), and in omega algebra, \( (p.x)^* = 0 \) expresses termination of the program “\( \text{while } p \text{ do } x \)”. Following Kozen, we call such equations Hoare formulas.

An important property of these algebras is that Hoare hypotheses can be efficiently eliminated using the following theorem, first proved in [1]: for any ground terms \( x, y, \) and \( z \),

\[
(x = 0 \vdash y = z) \iff (\langle f(y) = f(z) \rangle)
\]

where \( f(u) = T.x.T + u \), “\( \langle \rangle \)” is the product operator (i.e., sequential composition), and \( T \) is the maximal element of the algebra (or \( \Sigma^* \), where \( \Sigma \) is the sum of all letters appearing in \( x, y, \) or \( z \)). This shows that these algebras remain PSPACE-complete even of we allow Hoare hypotheses. This flavor of hypothesis elimination has been extended to other kinds of hypotheses, such as \( x \leq 1 \) where \( x \) doesn’t contain the product operator [1], \( p.a = p \) where \( p \) is a test and \( a \) is atomic [5], and elimination of combinations of hypotheses [6].

One of the limitations of these algebras is that the annihilation axioms \( 0.x = 0 = x.0 \) are problematic when we want to reason about total correctness or specifications. For example, if \( x \) is a nonterminating program, we would expect \( x.0 \) to be equal to \( x \) rather than 0. Because of this, several KA-like algebras keep \( 0.x = 0 \) but omit \( x.0 = 0 \), e.g. [13], [4], [12]. The axiom \( 0.x = 0 \) is problematic if \( x \) represents a specification of a function with precondition \( \text{false} \) and some \text{non-\text{false}} \) postcondition. We would like to extend hypothesis elimination techniques to these weaker algebras.

In this paper, we give a general technique for eliminating a set of Horn-equational hypotheses from a Horn-equational theory. In contrast to previous hypothesis elimination techniques, which required constructing explicit algebras, our method is purely proof-theoretic, and so can be used to eliminate appropriate classes of hypotheses from any theory under suitable conditions. It also yields a simple condition under which we can combine such eliminations.

Our main result is that Hoare hypotheses can be eliminated from Kleene Semirings (KS), whose axioms are Kozen’s axioms for KA without the annihilation axioms. Thus, we can replace the annihilation axioms with arbitrary Hoare axioms, while keeping the decision procedure for equality in PSPACE.

A. Notation

In this paper, an “algebra” is given by an operator signature (including the set of constants) and a set of (ground) Horn-equational axioms on this signature. Thus, the usual presentation of an algebra as a set of axioms, with the variables of each universally quantified over elements of the algebra, will be tacitly treated instead as a set of axiom schemas, where these variables are actually metavariables ranging over terms; we can take this liberty because we are concerned in this paper only with the ground theory. We will also tacitly take the equality rules as an axiom scheme, and so consider them explicit axioms of the algebra, i.e. for every operator op of the algebra, and tuples of terms \( u \) and \( v \), we have an implicit axiom \( u = v \Rightarrow \text{op}(u) = \text{op}(v) \) (and analogously for the reflexivity and transitivity axioms of equality). This will allow us to talk about proofs without having to special-case equality.
The algebra under consideration will be determined by the context. The word “constant” means a nullary function of the algebra, “operator” means a non-nullary function of the algebra “term” means a term of the algebra, “equation” means an equation between terms, and “formula” means a Horn formula whose literals are equations. Except when indicated otherwise, \(a, b, c, x, y, \) and \(z\) are metavariables ranging over terms, \(u\) and \(v\) are metavariables ranging over tuples of terms, \(H \vdash F\) (where \(H\) is a set of formulas and \(F\) is a formula) means that the conclusion of \(F\) is provable in the algebra whose axioms are those of the algebra along with the formulas of \(H\) and the hypotheses of \(F\), and \(H \vdash C\) (where \(C\) is a set of formulas) means \(H \vdash F\) for every formula \(F\) in \(C\).

Except when explicitly indicated, all identifiers are single letters, and are either metavariables representing terms or (meta)functions from terms to terms. Juxtaposition of identifiers always denotes function application (right associative, \(p(a)\) is the function applied first).

II. HYPOTHESIS ELIMINATION

Here we give a general method for hypothesis elimination in Horn-equational theories. Fix an algebra, let \(H\) be a set formulas, and let \(F\) range over sets of formulas, and \(E\) range over equations. To eliminate \(H\), we define a suitable elimination function \(f\), for which we establish

\[(\forall E, F : (H, F \vdash E) \leftrightarrow (f F \vdash f E))\]

We prove (1) as follows:

\[H, F \vdash E \quad \Rightarrow \quad \{(2)\}\]
\[fF \vdash f E \quad \Rightarrow \quad \{\vdash\}\]
\[H, f F \vdash f E \quad \Rightarrow \quad \{(3)\}\]
\[H, F \vdash E\]

We are left with the obligations

\[(2) \quad (\forall E, F : (H, F \vdash E) \Rightarrow (f F \vdash f E))
\]

\[(3) \quad (\forall x : H \vdash f x = x)\]

We prove (2) by induction on the proof of \(H, F \vdash E\). Each step of the proof is an axiom of the algebra or a formula in \(H\) or \(E\). The case of a formula of \(F\) is trivial, since we have \(fF\) as a hypothesis. For the cases where the step is an axiom \(A\) of the algebra or a formula of \(H\), we need to show

\[(4) \quad \vdash f (A)
\]
\[(5) \quad \vdash f (H)\]

This leaves as proof obligations (3), (4), and (5). For each case of \(H\), we will define a suitable \(f\) and show that it satisfies these obligations.

A. Eliminating multiple hypotheses

The approach above already allows simultaneous elimination of multiple hypotheses (since \(H\) can contain any number of formulas), but requires sharing a single elimination function \(f\). There are two ways to eliminate hypotheses in stages.

Suppose we want to eliminate two sets of hypotheses, \(H\) and \(H'\), that we can eliminate in some algebra using elimination functions \(f\) and \(f'\) respectively. One possibility is to use \(f\) to reduce \(H, H' \vdash F\) to \(f H' \vdash f F\). However, in general we might not be able to eliminate \(f H'\), even though we could eliminate \(H'\).

As an alternative, we can first add \(H'\) to the algebra (without changing \(F\)), eliminate \(H\) (in the new algebra), and finally eliminate \(H'\) from \(f(F)\) in the original algebra. The addition of \(H'\) to the algebra in the first step introduces a new proof obligation

\[(6) \quad H' \vdash f(H')\]

and the resulting elimination function is \(f' \circ f\) (where \(f\) is the function applied first).

As an example of this, suppose that the formulas of \(H'\) are all equations (without hypotheses), and that \(f\) is a function of the form \(f x = x + t\) for some term \(t\); this was the form of the elimination function for hypotheses of the form \(a = 0\) in Kleene algebra, where \(t = \top, a \top [1]\). Then the proof obligation for an equation \(e1 = e2\) in \(H'\) reduces to \(e1 = e2 + e1 + t = e2 + t\), which follows immediately from equality reasoning. This gives a trivial proof that elimination of \(a = 0\) in Kleene algebra can be combined with the elimination of other equational hypotheses, which was previously proved in [6] using a more complex argument.

An obvious generalization is that in any theory, if a set of hypotheses can be eliminated with an elimination function that is polymorphic in its argument, then the hypotheses can be eliminated alongside any set of eliminable equational hypotheses.

III. KLEENE SEMIRINGS

For the rest of the paper, we work in the theory of Kleene semirings, the axioms of which are simply Kozen’s axioms for Kleene algebra [7] with the annihilation axioms removed; as usual, \(x \leq y\) abbreviates \(x + y = y\):

\[(x + y) + z = x + (y + z)\]
\[x + y = y + x\]
\[x + x = x\]
\[0 + x = x\]
\[x(y,z) = (x,y)z\]
\[1.x = x.1 = x\]
\[x(y + z) = x.y + x.z\]
\[(x + y).z = x.z + y.z\]
\[x* = 1 + x + x*.x*\]
\[x.y \leq x \Rightarrow x.y* = x\] (* ind)
\[x.y \leq y \Rightarrow x*.y = y\] (* ind)

The equational theory of KS is, like KA, PSPACE-complete. Moreover, the equational theory of KS, restricted to terms not
mentioning 0, is the same as the similarly restricted equational theory of KA.

An initial model of KS can be constructed as follows. (We give an informal construction here; a more precise construction is given later as a corollary of the hypothesis elimination theorem for hypotheses of the form \( a = 0 \).) Define a closed language over an alphabet that includes the symbol 0 to be a language \( L \) such that (1) \( 0 \in L \), and (2) for all strings \( r,s,t \) (each possibly empty) such that \( r.s.t \in L \), \( s.0.t \in L \). Define the closure of a language to be the smallest closed language that contains it. Interpret each operator of KS as in KA, but operating on closed languages and closing the result. For example, if \( b \) and \( c \) are symbols, the the language denoted by the expression \( b.c \) is the the set of strings generated (treating 0 as an ordinary symbol) from the regular expression \( 0^*.(0 + 0.c + b.0 + b.0^*.c).0^* \).

An example of a relational model of KS is the following. Terms denote binary relations on a set (representing states), where each relation maps relates input \( t \) to an output iff that output is \( \perp \). The operators are interpreted as in the relational model of KA, as are the constants, except for 0 which is the identity relation restricted to \( \perp \). Note that this model satisfies \( 0.x = 0 \), but not \( x.0 = 0 \).

IV. ELIMINATING \( a = 0 \)

In this section, we present our main result, the elimination of equations of the form \( a = 0 \).

The absence of the annihilation axioms makes the definition of a suitable elimination function much more complex than that of the elimination function used for Kleene algebra [1]. With the annihilation axioms, the assumption \( a = 0 \) can be viewed as saying that we are in a modified language model where terms denote sets of strings not containing superstrings of strings of \( a \). (Operators in this model behave as usual, then remove superstrings of \( a \) from the result.) This model is isomorphic to one where terms represent languages that include all superstrings of \( a \), hence the definition \( f(x) = T.a.T + x \), where \( T = \Sigma^* \), where \( \Sigma \) is the sum of all symbols in the alphabet. But in the absence of the annihilation axioms, this construction does not work, because we would be unable to prove (5).

Without the annihilation axioms, we can no longer imagine working in a simple string model. Instead, we imagine working in an ordered string model, where string \( s \) "refines" string \( t \) iff \( t \) can be transformed into \( s \) by a sequence of improvement steps, where a string is improved by replacing an arbitrary substring with an arbitrary string of \( a \) (or the string 0). Terms now denote nonempty sets of strings closed under refinement (i.e. if a set contains \( t \) and \( s \) refines \( t \), then the set contains \( s \)).

In this model, we can think of \( f \) as the closure operator (i.e., \( f.x \) computes the language of all strings that refine strings of \( x \)). The problem is how to define \( f \) as a function from terms to terms. Because substrings of strings from \( a \) added in improvement steps can themselves be rewritten by later improvement steps, the key is to define a term \( m \) that gives an explicit formula for \( f.a \). \( f.x \) itself can then be defined as the set of strings obtainable by breaking up a string of \( x \) into a finite set of substrings (some of which might be empty) and replacing some of these substrings with strings from \( m \).

Formally, for any term \( x \), define \( px \) (the “prefixes” of \( x \)) and \( sx \) (the “suffixes” of \( x \)) as follows (\( c \) ranges over all constant symbols, including 0 and 1):

\[
\begin{align*}
     pc &= 1 + c \\
     p(x+y) &= px + py \\
     s(x+y) &= sx + sy \\
     p(x.y) &= px + x.py \\
     s(x.y) &= sy + sx.y \\
     p(x^*) &= x^*.px \\
     s(x^*) &= sx.x^*
\end{align*}
\]

The theorems we need regarding these functions are proved in the appendix. Note that we cannot simply claim obvious properties like these because they hold for ordinary languages, since we are effectively in an ordered language model.

We next define the terms \( l \) and \( m \) as follows:

\[
\begin{align*}
     l &= pa^*.a.sa^* \\
     m &= l.(psa.l)^*
\end{align*}
\]

Intuitively, \( m \) consists of all of the strings obtainable by starting with a string of \( a \) and repeatedly replacing an arbitrary substring with a string of \( a \) (or 0).

Finally, we define the function \( f \) as follows, by induction on the term structure of its argument:

\[
\begin{align*}
     fc &= (1 + m).(c + m).(1 + m) \quad \text{for constant } c \\
     f(x+y) &= fx + fy \\
     f(x.y) &= fx.fy + px.m.sf y \\
     f(x^*) &= gx^* \\
     gx &= fx + px.m.(psf.x.m)^*.sf x
\end{align*}
\]

Intuitively, the strings of \( fx \) are strings of \( x \), chopped into substrings (some empty), with \( m \) substituted for some of the substrings. The following proofs show that this \( f \) satisfies the formulas (3) ((26) and (27) below), (4) ((25) below), and (5) ((21) below).

The following facts are proved about \( p \) and \( s \) (all by induction on \( x \), except for the last which is proved by induction on the derivation of \( \vdash x = y \)).

\[
\begin{align*}
     (7) & \quad 1 + x \leq px \\
     (8) & \quad ppx = px \\
     (9) & \quad psx = spx \\
     (10) & \quad px.0 = x.0 \\
     (12) & \quad (\vdash x = y) \Rightarrow (\vdash px = py)
\end{align*}
\]

The following properties are proved by direct calculation, using the definitions of \( l \) and \( m \) and the above properties of \( p \) and \( s \):

\[
\begin{align*}
     (13) & \quad m.psa.m \leq m \\
     (14) & \quad pl = pa^* + l.psa \\
     (15) & \quad pl.m = m \\
     (16) & \quad pm.m = m \\
     (17) & \quad m.psm.m \leq m
\end{align*}
\]
The following properties are proved by induction on $x$:

(18) $x + m \leq f.x$
(19) $x \leq ps.m \land f.x \leq x + px.m.sx \Rightarrow pf.x.m \leq px.m$
(20) $x \leq ps.m \Rightarrow f.x \leq x + px.m.sx$

The remaining properties are proved by direct calculation:

(21) $x \leq m \Rightarrow f.x = m$
(22) $pgx = pf.x.(pm + (m.ps.fx)^*)$
(23) $pgx.m.sgx \leq gx$

V. PROPERTIES OF $p$ AND $s$

(7) $1 + x \leq px$

Proof by induction on $x$:

$x = c :$

$p.y.z = \{\text{def } x\}$
$p.c = \{\text{def } p\}$
$1 + c = \{\text{def } x\}$

$x = y + z :$

$p.y + p.z = \{\text{def } p\}$
$p(y + z) = \{\text{def } x\}$

$(1 + y) + (1 + z) = \{\text{def } x\}$

$x = y^* :$

$p.y + p.z = \{\text{def } p\}$
$p(y + z) = \{\text{def } x\}$

$(1 + y) + (1 + z) = \{\text{def } x\}$

$x = y^*.$

$p.y + p.z = \{\text{def } p\}$
$p(y + z) = \{\text{def } x\}$

$x = psx = spx$

Proof by induction on $x$:

$x = c :$

$p.sx = \{\text{def } x\}$
$p.s = \{\text{def } s\}$
$p(1 + c) = \{\text{def } p\}$
$1 + c = \{\text{def } s\}$

$x = y + z :$

$p.sx = \{\text{def } x\}$
$p.s = \{\text{def } s\}$

$(1 + y) + (1 + z) = \{\text{def } x\}$

$x = y^*.$

$p.sx = \{\text{def } x\}$
$p.s = \{\text{def } s\}$

$(1 + y) + (1 + z) = \{\text{def } x\}$
Since the first and last terms are equal, the first and sixth terms are equal.

\[ p(x\cdot y) = x\cdot(p(x + py)) \]

\[ p(px\cdot y) = px\cdot py \]

\[ px\cdot px \leq \{ KS \} \]

\[ px\cdot px \leq \{ 1 \leq px (7) \} \]

\[ p(px\cdot y) = \{ proof above \} \]

\[ px\cdot (px + py) = \{ ppx = px (8) \} \]

\[ px\cdot (px + py) = \{ px\cdot px = px\cdot (proof above) \} \]

\[ px\cdot (1 + py) = \{ 1 \leq py (7) \} \]

\[ px\cdot py \]

\[ p(px\cdot y) = \{ def p \} \]

\[ px + px\cdot py = \{ (8) \} \]

\[ px + px\cdot py = \{ 1 \leq py (7), \; so \; px \leq px\cdot py \} \]

\[ px\cdot py \]

(10) \( px\cdot 0 = x\cdot 0 \)

Proof by induction on \( x \):

\[ x = c : \]

\[ px\cdot 0 = \{ \text{def } x \} \]

\[ pc\cdot 0 = \{ \text{def } p \} \]

\[ (1 + c)\cdot 0 = \{ \text{KS} \} \]

\[ 1.0 + c.0 = \{ 1.0 = 0; \; 0 + c.0 = c.0 \} \]

\[ c.0 \]

(9) \( px\cdot 0 = x\cdot 0 \)

Proof by induction on \( x \):

\[ x = y + z : \]

\[ px\cdot 0 = \{ \text{def } x \} \]

\[ (py + pz)\cdot 0 = \{ \text{ KS} \} \]

\[ py\cdot 0 + pz\cdot 0 = \{ py\cdot 0 = y\cdot 0, \; pz\cdot 0 = z\cdot 0 \; \text{ (ind hyp)} \} \]

\[ y\cdot 0 + z + 0 = \{ \text{ KS} \} \]

\[ (y + z)\cdot 0 = \{ \text{def } x \} \]

\[ x.0 \]

(12) \( \vdash x = y \Rightarrow (\vdash px = py) \)

Proof by induction on the proof of \( x = y \).

\[ (x + y) + z = x + (y + z) : \]

\[ p((x + y) + z) = \{ \text{def } p \} \]

\[ (px + py) + pz = \{ \text{KS} \} \]

\[ px + (py + pz) = \{ \text{def } p \} \]

\[ p(x + (y + z)) \]

\[ x + y = y + x : \]

\[ p(x + y) = \{ \text{def } p \} \]

\[ px + py = \{ \text{KS} \} \]

\[ py + px = \{ \text{def } p \} \]

\[ p(y + x) \]

\[ x + x = x : \]

\[ p(x + x) = \{ \text{def } p \} \]

\[ px + px = \{ \text{KS} \} \]

\[ px \]

\[ 0 + x = x : \]

\[ p(0 + x) = \{ \text{def } p \} \]

\[ p(0 + px) = \{ \text{def } p \} \]

\[ 1 + 0 + px = \{ 1 \leq px (7) \} \]

\[ px \]
x.(y,z) = (x.y).z:
p(x.(y,z)) = \{ \text{def } p \}
px + x.(py + y.pz) = \{ \text{KS } \}
px + x.py + x.y.pz = \{ \text{def } p \}
p(x.y).z = \{ \text{def } p \}
x.1 = x:
p(x.1) = \{ \text{def } p \}
px + x.p1 = \{ \text{def } p \}
px + x = \{ x \leq px \} \quad (7)
px
x.(y+z) = x.y + x.z:
p(x.(y+z)) = \{ \text{def } p \}
px + x.(py + pz) = \{ \text{KS } \}
px + x.py + x.pz = \{ \text{KS } \}
(px + x.py + px + pz) = \{ \text{def } p \}
p(x.y) + p(x,z) = \{ \text{def } p \}
p(x.y + x.z)

x^* = 1 + x + x^*.x^*:
p(1 + x + x^*.x^*) = \{ \text{def } p \}
1 + px + p(x^*.x^*) = \{ (11) \}
1 + px + x^*.x^*.px = \{ px \leq x^*.px \quad \text{KS} \}
1 + px + x.x^*.px = \{ x^*.x^* = x^* \quad \text{KS} \}
1 + px + x^*.px = \{ 1 + px \leq x^*.px \quad \text{KS} \}
px^* \quad \text{KS}
p(x^*) = \{ \text{def } p \}

x.y \leq x \Rightarrow x.y^* = x:
Suppose x.y \leq x.

Then by the induction hypothesis, p(x.y) \leq px, so

px + x.y^* = \{ x.y^* = x \}
px + x.y = \{ \text{def } p \}
p(x.y) \quad \text{KS}
px \quad \text{KS}
px^* \quad \text{KS}

For the induction axiom x.y \leq y \Rightarrow x^*.y = y, suppose x.y \leq y. Then by the induction hypothesis, p(x.y) = px + x.py \leq py, so

p(x^*.y) = \{ (11) \}
x^*. (px + py) = \{ px \leq py \quad \text{hyp} \}
x^*.py = \{ x.py \leq py \quad \text{ind hyp} \}
so x^*.py = py(* \text{ind})}
py

\square

A. Properties of m

m.psa.m \leq m

m.psa.m = \{ \text{def } m \}
l.(psa.l)*.psa.l.(psa.l)* \leq \{ (psa.l).(psa.l)* \leq (psa.l)* \}
l.(psa.l)*.(psa.l)* = \{ (psa.l)*.(psa.l)* = (psa.l)* \}
m

\square

(14) pl = pa^* + l.psa

pl = \{ \text{def } l \}
p(pa^*.a.sa*) = \{ (11) \}
pa^*.p(a.sa*) = \{ \text{def } p \}
pa^*.p(a + a.sa*.psa) = \{ pa^*.pa = pa^*(11) \}
pa^* + pa^*.a.sa*.psa = \{ pa^*.a.sa* = l \quad \text{def } l \}
pl.psa

\square

(15) pl.m = m

pl.m = \{ pl = pa^* + l.psa \quad (14) \}
(pl + l.psa).m = \{ \text{def } m \}
(pa^* + l.psa).l.(psa.l)* \leq \{ psa.l.(psa.l)* \leq (psa.l)* \}
(pl^* + 1).l.(psa.l)* \leq \{ 1 \leq pa^* \}
pa^*.l.(psa.l)* = \{ \text{def } l \}
pa^*.p.(psa.l)* = \{ pa^*.pa = pa^* \quad \text{KS} \}
pa^*.psa.(a.sa*.l)* = \{ \text{def } l \}
l.(psa.l)* \leq \{ 1 \leq pl(7) \}
pl.m

\square

(16) pm.m = m

pm.m = \{ \text{def } m \}
p(l.(psa.l)*).m = \{ \text{def } p \}
(pl + l.(psa.l)*.p(psa.l)).m = \{ \text{def } m \}
(pl + m.p(psa.l)).m = \{ pm.m = psa.pl \quad (11) \}
(pl + m.p.psa).m = \{ \text{def } m \}
m + m.psa.m = \{ m.psa.m \leq m \quad (13) \}
m
B. Properties of $f$ and $g$

(18) $x + m \leq fx$

Proof by induction on $x$:

$x = c$ :

\[
fx = \{\text{def } x\}
\]

\[
fc = \{\text{def } f\}
\]

\[
(1 + m)\cdot (c + m) \cdot (1 + m) \geq \{\text{KS}\}
\]

\[
c + m = \{\text{def } x\}
\]

\[
x + m \geq \{\text{def } x\}
\]

$x = y + z$ :

\[
fx = \{\text{def } x\}
\]

\[
f(y + z) = \{\text{def } f\}
\]

\[
fy + fz \geq \{y + m < fy \text{ (ind hyp)}\}
\]

\[
y + m + fz \geq \{z + m < fz \text{ (ind hyp)}\}
\]

\[
y + m + z + m = \{\text{KS}\}
\]

\[
(y + z) + m = \{\text{def } x\}
\]

\[
x + m \geq \{\text{def } x\}
\]

$x = y.z$ :

\[
f(y.z) = \{\text{def } f\}
\]

\[
fy.fx + pfy.m.sfz \geq \{1 \leq pfy, 1 \leq sfz \text{ (7)}\}
\]

\[
fy.fx + m \geq \{y < fy, z < fz \text{ (ind hyp)}\}
\]

\[
y + m + m = \{\text{def } x\}
\]

\[
x + m \geq \{\text{def } x\}
\]

$x = y^*$ :

\[
f(y^*) = \{\text{def } f, g\}
\]

\[
(fy + pfy.m.(psfy.m)^* sfy)^* \geq \{1 \leq (psfy.m)^*\}
\]

\[
(fy + pfy.m.sfy)^* \geq \{1 \leq pfy \text{ (7)}\}
\]

\[
(fy + m.sfy)^* \geq \{1 \leq sfy \text{ (7)}\}
\]

\[
(fy + m)^* \geq \{y < fy \text{ (ind hyp)}\}
\]

\[
y + m^* \geq \{\text{KS}\}
\]

\[
y + m = \{\text{def } x\}
\]

\[
x + m \geq \{\text{def } x\}
\]

$x \leq psm \land fx \leq x + px.m sx \Rightarrow pfx.m \leq px.m$

Proof by induction on $x$: assuming $x \leq psm$,

$x = c$ :

\[
x = \{\text{def } x\}
\]

\[
(1 + m).c + m.(1 + m) \geq \{\text{KS}\}
\]

\[
c + m = \{\text{def } x\}
\]

\[
x + px.m + m.sx + psm.m \leq \{\text{def } x\}
\]

\[
x + px.m + m.sx + m \leq \{\text{def } x\}
\]

\[
x + px.m.sx \leq \{\text{def } x\}
\]

\[
x + px.m.sx \leq \{\text{def } x\}
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x + px.m.sx \leq \{\text{def } x\}
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x + px.m.sx \leq \{\text{def } x\}
\]

\[
x + px.m.sx \leq \{\text{def } x\}
\]
\[ x = y^*; \text{ let } z = px.m.sy; \text{ then} \]
\[ fx = \begin{cases} \text{def } f & \text{if } \text{psm}_m \leq m \text{ (20)} \\ \text{def } f & \text{if } \text{monotonicity of } p \text{ (16)} \\ \text{def } f & \text{if } \text{fx} \text{ (18)} \end{cases} \]

\[ (y^* + px.m.sy.y^*) \leq \begin{cases} y^* + px.m.sy.y^* & \text{def } z \\ y^* + px.m.sy.y^* & \text{def } s \end{cases} \]

Proof by induction on the structure of \( x \):

\[ x = y + z: \]
\[ fx = \begin{cases} \text{def } x & \text{if } \text{fx} \\ \text{def } f & \text{if } \text{fx} \end{cases} \]

\[ x = y^*; \text{ let } z = px.m.sy; \text{ then} \]
\[ pgx.m.sgx = \begin{cases} \text{def } g & \text{if } \text{pgx.m.sgx} \text{ (22)} \\ \text{def } g & \text{if } \text{pgx.m.sgx} \text{ (23)} \end{cases} \]

Let \( r = psfx.m \) and \( t = psfx.pm \); then
\[ pgx = \begin{cases} \text{def } g & \text{if } \text{pgx} \text{ (11)} \\ \text{def } p & \text{if } \text{pgx} \text{ (11)} \end{cases} \]
\[ x = y : \]
\[
pfxm = \{ \text{def } x \} \]
\[
pfy.m = \{ \text{def } f \} \]
\[
pgym = \{ \text{def } p \} \]
\[
gym.gym.m = \{ 1 \leq sgy \ (7) \} \]
\[
gym.gym.m.sgy \leq \{ (23) \} \]
\[
gym.gym = \{ \text{KS} \} \]
\[
gym = \{ \text{def } f \} \]
\[
f(y) = \{ \text{def } f \} \]
\[
fx
\]
\[ x = y : \]
\[
f(x + x) = \{ \text{def } f \} \]
\[
fx + fx = \{ \text{KS} \} \]
\[
fx
\]
\[ a = 0 : \]
\[
fa = \{ (21) \} \]
\[
m = \{ (21) \} \]
\[
f0
\]
\[
(x.y).z = x.(y.z) : \text{ let } r = pf.x.m \text{; then }
\]
\[
f((x.y).z) = \{ \text{def } f \} \]
\[
f(x.y).z + pf(x.y).m.sfx = \{ \text{def } f \} \]
\[
(fx.fy + r.sfy).fz + ( pf + fx.pfy + pfx.pm + r.psfy).m.sfx = \{ pm.m = m \} \]
\[
(fx.fy + r.sfy).fz + ( pf + fx.pfy + r.psfy).m.sfx = \{ \text{KS} \} \]
\[
fx.fy + r.sfy.m.sfx = \{ m = sm \} \]
\[
x^* = 1 + x + x^*.x^* \]
\[
\text{let } r = p(gx^*.m.s(gx^*); \text{ then }
\]
\[
f(1 + x + x^*.x^*) = \{ \text{def f} \}
\]
\[
f1 + fx + (gx^*).(gx^* + r) = \{ \text{def } f, p, s \}
\]
\[
m + 1 + fx + gx^*.gx^* + r = \{ gx^*.gx^* = gx^* \}
\]
\[
1 + m + fx \leq gx^* \]
\[
gx^* + gx^*.pgx.m.sgx.gx^* \leq \{ pgx.m.sgx \leq gx \ (23) \}
\]
\[
fx^* = \{ \text{def } f \}
\]
\[ x.y \leq x \Rightarrow x.y^* = x:
\]
\[
\text{assume } f(x.y) < fx .
\]
\[
\text{then } fx.fy + pfx.m.sfy = f(x.y) < fx, \text{ so }
\]
\[
f(x.y) = \{ g \}
\]
\[
fx.gy + pf.x.m.sgy.gy \leq \{ pf.x.m.sgy \leq fx \text{ (below)} \}
\]
\[
fx.gy = \{ fx.gy \leq fx \text{ (below); (* ind)} \}
\]
\[
fx \leq \{ \text{KS} \}
\]
\[
fx.gy + pf.x.m.sgy = \{ \text{def } f \}
\]
\[
fx(y) = \{ \text{def } f \}
\]
\[
fx.gy = \{ gy \leq fy + px.pfy.m.sgy \}
\]
\[
fx.fy + px.pfy.m.sgy = \{ \text{def } gy \}
\]
\[
fx.fy + pfx.m.sgy = \{ pfx.m.sgy \leq fx \text{ (below)} \}
\]
\[
fx.fy + fx = \{ \text{def } f \}
\]
\[
fx + fx = \{ \text{def } f \}
\]
\[ \]
\[ 1.x = x : \]
\[
f.lx = \{ \text{def } f \}
\]
\[
1.fx + pf1.m.sfx = \{ \text{def } f \}
\]
\[
(1 + m).fx + (1 + pm).m.sfx = \{ pm.m = m = m (16) \}
\]
\[
fx + m.fx + m.sfx = \{ fx \leq sfx \ (7) \}
\]
\[
\{ \text{so } m.fx \leq m.sfx \}
\]
\[
fx + m.sfx = \{ m.sfx \leq fx \ (24) \}
\]
\[
f
\]
\[ C. f \text{ preserves axioms }
\]
\[
(25) \vdash f(A)
\]
\[ \]
\[ \]
\[ \]
\[
pfx.m.sfy \\
pfx.m.(sm + (psfy.m)^*) sfy \\
pfx.m.(psfy.m)^*.sfy \\
pfx.m.sfy \\
pfx.m.psfy.m \\
pfox.m.sfy \\
pfx.m.psfy.m \leq \{ \text{def } p \} \\
pf(\text{fox.m.sfy}) \cdot m \leq \{ \text{fox.m.sfy} \leq f(x) \text{ (hyp)} \} \Box \\
pfx.m \\
\]

(26) \( a = 0 \vdash f(x) = x \)

Proof: induction on \( x \) (using \( a = 0 \vdash m = 0 \)):

\[
x = c : \\
f x = \{ \text{def } x \} \\
f c = \{ \text{def } f \} \\
(1 + 0).(0 + c).(1 + 0) = \{ \text{KS } \} \\
c \\
\]

\[
x = y + z : \\
f x = \{ \text{def } x \} \\
f y + z = \{ \text{def } f \} \\
f y + f z = \{ f y = y, f z = z \text{ (ind hyp)} \} \\
y + z \\
\]

\[
x = y.z : \\
f x = \{ \text{def } x \} \\
f y.z = \{ \text{def } f \} \\
f y.f z + p f y.0 sf z = \{ 0 \leq 1 \} \\
f y.f z = \{ f y = y, f z = z \text{ (ind hyp)} \} \\
y.z = x \\
\]

\[
x = y^*: \\
f x = \{ \text{def } x \} \\
f y^* = \{ \text{def } f, g \} \\
(f y + f y.0.(psfy.0)^*.sf y)^* = \{ pf y.0 = f y.0 \} \quad \text{(10)} \\
(f y + f y.0.(psfy.0)^*.sf y)^* = \{ \text{KS } \} \\
(f y + f y.(0.psfy.0)^*.sf y)^* = \{ 0.sfy = 0.f y \} \quad \text{(10)} \\
(f y + f y.(0.f y)^*.0.f y)^* = \{ f y = y \text{ (ind hyp)} \} \\
(0 + y.(0.g)^*.y)^* = \{ \text{def } x \} \\
y^* = \{ \text{def } x \} \\
x \Box \\
\]

(27) \( (\vdash f(u) = f(v)) \Rightarrow (\vdash f(\text{op}(u) = f(\text{op}(v)))) \)

Case analysis on \( \text{op} \) (using (12)):

\[
\text{op} = + : \\
f (u_0 + u_1) = \{ \text{def } f \} \\
f u_0 + f u_1 = \{ f u_0 = f v_0, f u_1 = f v_1 \text{ (hyp)} \} \\
f v_0 + f v_1 = \{ \text{def } f \} \\
f (v_0 + v_1) = \{ \text{def } f \} \\
\]

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