Spanners in randomly weighted graphs: independent edge lengths

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Abstract

Given a connected graph $G = (V, E)$ and a length function $\ell : E \to \mathbb{R}$ we let $d_{v,w}$ denote the shortest distance between vertex $v$ and vertex $w$. A $t$-spanner is a subset $E' \subseteq E$ such that if $d'_{v,w}$ denotes shortest distances in the subgraph $G' = (V, E')$ then $d'_{v,w} \leq td_{v,w}$ for all $v, w \in V$. We show that for a large class of graphs with suitable degree and expansion properties with independent exponential mean one edge lengths, there is w.h.p. a 1-spanner that uses $\approx \frac{1}{2} n \log n$ edges and that this is best possible. In particular, our result applies to the random graphs $G_{n,p}$ for $np \gg \log n$.

1 Introduction

Given a connected graph $G = (V, E)$ and a length function $\ell : E \to \mathbb{R}$ we let $d_{v,w}$ denote the shortest distance between vertex $v$ and vertex $w$. A $t$-spanner is a subset $E' \subseteq E$ such that if $d'_{v,w}$ denotes shortest distances in the subgraph $G' = (V, E')$ then $d'_{v,w} \leq td_{v,w}$ for all $v, w \in V$. In general, the closer $t$ is to one, the larger we need $E'$ to be relative to $E$. Spanners have theoretical and practical applications in various network design problems. For a recent survey on this topic see Ahmed et al [1]. Work in this area has in the main been restricted to the analysis of the worst-case properties of spanners. In this note, we assume that edge lengths are random variables and do a probabilistic analysis.

Suppose that $G = ([n], E)$ is almost regular in that

\[(1 - \theta)dn \leq \delta(G) \leq \Delta(G) \leq (1 + \theta)dn\]  

where $1 \geq d \gg \frac{\log \log n}{\log^{1/2} n}$ and $\theta = \frac{1}{\log^{1/2} n}$. Here $\delta, \Delta$ refer to minimum and maximum degree respectively.

We will also assume either that $d > 1/2$ or

\[|E(S, T)| \geq \psi |S| |T|\]  

for all $|S|, |T| \geq \theta n$.  

(2)

Here $\psi = \frac{\omega \log \log n}{\log^{1/2} n} \leq d$ where $\omega = \omega(n) \to \infty$ as $n \to \infty$ and $E(S, T)$ denotes the set of edges of $G$ with one end in $S \subseteq [n]$ and the other end in $T \subseteq [n]$, $S \cap T = \emptyset$.

Let $\mathcal{G}(d)$ denote the set of graphs satisfying the stated conditions, (1) and (2). We observe that $K_n \in \mathcal{G}(1)$ and that w.h.p. $G_{n,p} \in \mathcal{G}(p)$, as long as $np \gg \log n$. The weighted perturbed model of Frieze [5] where randomly weighted edges are added to a randomly weighted $dn$-regular graph also lies in $\mathcal{G}(d).

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Suppose that the edges \( \{i, j\} \) of \( G \) are given independent lengths \( \ell_{i,j}, 1 \leq i < j \leq n \) that are distributed as the exponential mean one random variable, denoted by \( E(1) \). In general we let \( E(\lambda) \) denote the exponential random variable with mean \( 1/\lambda \).

When \( G = K_n \), Janson \[9\] proved the following: W.h.p. and in expectation

\[
d_{1,2} \approx \frac{\log n}{n}; \quad \max_{j>1} d_{1,j} \approx \frac{2 \log n}{n}; \quad \max_{i,j} d_{i,j} \approx \frac{3 \log n}{n}. \tag{3}
\]

Here (i) \( A_n \approx B_n \) if \( A_n = (1 + o(1))B_n \) and (ii) \( A_n \gg B_n \) if \( A_n/B_n \to \infty \), as \( n \to \infty \).

It follows that w.h.p. the length of the longest edge in any shortest path is at most \( L = (3 + o(1)) \log n \). It follows further that w.h.p. if we let \( E' \) denote the set of edges of length at most \( L \) then this is a 1-spanner of size \( O(n \log n) \).

We tighten this and extend it to graphs in the class \( G(d) \).

**Theorem 1.** Let \( G \in G(d) \) or let \( G \) be a \( dn \)-regular graph with \( d > 1/2 \) where the lengths of edges are independent exponential mean one. The following holds w.h.p.

(a) The minimum size of a 1-spanner is asymptotically equal to \( \frac{1}{2} n \log n \).

(b) If \( 2 \leq \lambda = O(1) \) then a \( \lambda \)-spanner requires at least \( n \log n \) edges.

A companion paper deals with \((1 + \varepsilon)\)-spanners in embeddings of \( G_{n,p} \) in \([0, 1]^2\) as studied by Frieze and Pegden \[7\]. Here we choose \( n \) random points \( X = \{X_1, X_2, \ldots, X_n\} \) in \([0, 1]^2\) and connect a pair \( X_i, X_j \) with probability \( p \) by an edge of length \( |X_i - X_j| \).

## 2 Proof of Theorem 1

The proof of Theorem 1 uses a few parameters. We will list some of them here for easy reference:

\[
\theta = \frac{1}{\log^{1/2} n}; \quad k_0 = \log n; \quad k_1 = \theta n; \quad \alpha = 1 - 2\theta.
\]

\[
\ell_0 = \frac{(1 + \sqrt{\theta}) \log n}{dn}; \quad \ell_1 = \frac{5 \log n}{dn}; \quad \ell_2 = \ell_0 - \frac{(\log \log n)^2}{dn}; \quad \ell_3 = \frac{\log n}{200\lambda dn}.
\]

We also use the Chernoff bounds for the binomial \( B(n, p) \): for \( 0 \leq \varepsilon \leq 1 \),

\[
P(B(n, p) \leq (1 - \varepsilon)np) \leq e^{-\varepsilon^2 np/2},
\]

\[
P(B(n, p) \geq (1 + \varepsilon)np) \leq e^{-\varepsilon^2 np/3},
\]

\[
P(B(n, p) \geq \alpha np) \leq \left( \frac{e}{\alpha} \right)^{\alpha np}.
\]

It will only be in Section 2.2 that we will need to use condition [2].

### 2.1 Lower bound for part (a)

We identify sets \( X_v \) (defined below) of size \( \approx \log n \) such that w.h.p. a 1-spanner must contain \( X_v \) for \( n - o(n) \) vertices \( v \). The sets \( X_v \) are the edges from \( v \) to its nearest neighbors. If an edge \( \{v, x\} \) is missing from a set \( S \subseteq E(K_n) \) then a path from \( v \) to \( x \) must go to a neighbor \( y \) of \( v \) and then traverse \( K_n - v \) to reach \( x \). Such a path is likely to have length at least the distance promised by \( \[3\] \), scaled by \( d^{-1} \).

We first prove the following:
Lemma 2. Fix \( v, w_1, w_2, \ldots, w_\ell \) for \( \ell = O(\log n) \) and let \( \alpha = 1 - 2\theta \). Then,

\[
\Pr\left( \exists 1 \leq i \leq \ell : d_{v, w_i} \leq \frac{\alpha \log n}{dn} \right) = o(1).
\]

Proof. There are at most \( ((1 + \theta)dn)^{k-1} \) paths using \( k \) edges that go from vertex \( v \) to vertex \( w_i, 1 \leq i \leq \ell \). The random variable \( E(1) \) dominates the uniform \([0, 1]\) random variable \( U_1 \). We write this as \( E(1) \succ U_1 \). As such we can couple each edge weight with a lower bound given by a copy of \( U_1 \). The length of one of these \( k \)-edge paths is then at least the sum of \( k \) independent copies of \( U_1 \). The fraction \( x^k/k! \) is an upper bound on the probability that this sum is at most \( x \) (tight if \( x \leq 1 \)). Therefore,

\[
\Pr\left( \exists 1 \leq i \leq \ell : d_{v, w_i} \leq \frac{\alpha \log n}{dn} \right) \leq \ell \sum_{k=1}^{n-1} \left( (1 + \theta)dn \right)^{\frac{k-1}{k}} \leq \frac{\ell}{dn} \sum_{k=1}^{10\log n} \left( \frac{e^{1+\theta} \alpha \log n}{k} \right)^k + O(n^{-10}) \leq \frac{10\ell \log n}{dn^{1-\alpha e^\sigma}} + o(1) = o(1).
\]

For a vertex \( v \in [n] \), let

\[
A_v = \left\{ w \neq v : \ell_{v, w} \leq \frac{\log n}{dn} \right\}.
\]

Lemma 3. W.h.p. \( |A_v| \leq 4 \log n \) for all \( v \in [n] \).

Proof. We have, from the Chernoff bounds and \( E(1) \succ U_1 \) that

\[
\Pr(|A_v| \geq 4 \log n) \leq \Pr\left( \text{Bin} \left( (1 + \theta)dn, \frac{\log n}{dn} \right) \geq 4 \log n \right) \leq \left( \frac{e(1 + \theta)}{4} \right)^{4\log n} = o(n^{-1}).
\]

The lemma follows from the union bound, after multiplying the RHS of (5) by \( n \).

For \( v \in [n] \), let \( \delta_v \) be the distance from \( v \) to its nearest neighbor. Let

\[
B = \left\{ v : \delta_v \geq \frac{\log^{1/2} n}{dn} \right\}.
\]

Lemma 4. \( |B| \leq ne^{-\log^{1/3} n} \) w.h.p.

Proof. We have

\[
\mathbb{E}(|B|) \leq n \left( \exp \left\{ - \frac{\log^{1/2} n}{dn} \right\} \right)^{(1-\theta)dn} = ne^{-(1-\theta)\log^{1/2} n}.
\]

The lemma follows from the Markov inequality.

Let

\[
X_v = \left\{ e = \{v, x\} : \ell(e) \leq \delta_v + \frac{\alpha \log n}{dn} \right\}.
\]
Lemma 5. Let \( S \subseteq E(K_n) \) define a 1-spanner. Then w.h.p. \( S \supseteq X_v \) for all but \( o(n) \) vertices \( v \).

Proof. Let \( G_S = ([n], S) \) and suppose that \( v \notin B \). Then

\[
\delta_v + \frac{\alpha \log n}{dn} < \frac{\log^{1/2} n}{dn} + \frac{\alpha \log n}{dn} < \frac{\log n}{dn}
\]

(6)

and so \( X_v \subseteq \{v\} \times A_v \) and in particular \( |X_v| \leq 4 \log n \) w.h.p. by Lemma 3.

If \( G_S \) does not contain an edge \( e = \{v, x\} \in X_v \), then the \( G_S \)-distance from \( v \) to \( x \) is then w.h.p. at least

\[
\delta_v + \frac{\alpha \log n}{dn} > d_{v,x}.
\]

(7)

To obtain (7) we have used Lemma 2 applied to \( K_n - v \) with \( x \) replacing \( v \) and \( w_1, w_2, \ldots, w_{\ell} \) being the remaining neighbors of \( v \) in \( K_n \).

So, if

\[
C = \{v \notin B : \exists 1\text{-spanner } S \supseteq X_v\},
\]

then \( \mathbb{E}(|C|) = o(n) \).

Any 1-spanner must contain \( X_v, v \in [n] \setminus (B \cup C) \) and the lemma follows from the Markov inequality.

Now \( |X_v| \) dominates \( Bin \left((1 - \theta)dn, 1 - \exp \left\{-\frac{\alpha \log n}{dn}\right\}\right) \) and so by the Chernoff bounds

\[
\mathbb{P} \left(|X_v| \leq (1 - \varepsilon)\alpha \log n + O \left(\frac{\log^2 n}{n}\right)\right) \leq e^{-\varepsilon^2 \alpha \log n/(2 + o(1))} = o(1) \text{ for } \varepsilon = \log^{-1/3} n.
\]

Applying Lemma 5 we see that w.h.p. a 1-spanner contains at least \( \frac{1 - o(1)}{2} \log n \) edges. The factor 2 comes from the fact that \( \{v, w\} \) can be in \( X_v \cap X_w \). (In this case the edge \( \{v, w\} \) contributes twice to the sum of the \( |A_v| \)’s.) Note that we do not need (2) to prove the lower bound.

2.2 Upper bound for part (a)

Let \( \ell_0 = \frac{(1 + \sqrt{\theta}) \log n}{dn} \) and \( \ell_1 = \frac{5 \log n}{dn} \) and \( E_0 = \{e : \ell(e) \leq \ell_0\} \). Now \( |E(G)| \in (1 \pm \theta)dn^2/2 \) and so the Chernoff bounds imply that w.h.p. \( |E_0| \approx \frac{1}{2} n \log n \) and our task is to show that adding \( o(n \log n) \) edges to \( E_0 \) gives us a 1-spanner w.h.p. We will do this by showing that w.h.p. there are only \( o(n \log n) \) edges \( e \) with \( \ell(e) > \ell_0 \) that are the shortest path between their endpoints. Adding these \( o(n \log n) \) edges to \( E_0 \) creates a 1-spanner, since every edge on a shortest path in a graph is itself a shortest path between its endpoints.

Janson [9] analysed the performance of Dijkstra’s algorithm on the complete graph \( K_n \) with exponential edge-weights; we will adapt his argument to our setting on a graph \( G \) satisfying conditions (1) and (2).

In particular, we analyze Dijkstra’s algorithm for shortest paths from vertex 1 where edges have exponential weights. Recall that after \( i \) steps of the algorithm we have a tree \( T_i \) and a set of values \( d_v, v \in [n] \) such that for \( u \in T_i \), \( d_u \) is the length of the shortest path from 1 to \( u \). For \( v \notin T_i \), \( d_v \) is the length of the shortest path from 1 to \( v \) that follows a path from 1 to \( u \in T_i \) and then uses the edge \( \{u, v\} \). Let \( \delta_i = \max \{v \in T_i : d_v\} \).

The constraints on the length \( l(u, v) \) of the edge \( \{u, v\} \) for \( u \in T_i, v \notin T_i \) are that \( d_u + l(u, v) \geq \delta_i \) or equivalently that \( l(u, v) \geq \delta_i - d_u \). Fixing \( T_i \) and the lengths of edges within \( T_i \) or its complement, every set of lengths \( \{l(u, v)\}_{u \in T_i, v \notin T_i} \) satisfying these constraints would give the same history of the algorithm to this point.

Due to the memoryless property of the exponential distribution we then have that \( l(u, v) = \delta_i - d_u + E_{u,v} \) where \( E_{u,v} \) is a mean-1 exponential, independent of all other \( E(u', v') \).

Thus the Dijkstra algorithm is equivalent in distribution to the following discrete-time process:
• Set \( v_1 = 1, T_1 = \{1\} \).

• Having defined \( T_i \), associate a mean-1 exponential \( E_{u,v} \) to each edge \( \{u,v\} \in E(T_i, \bar{T}_i) \) that is independent of the process to this point. Define \( e_{i+1} \) to be the edge \( \{u,v\} \in E(T_i, \bar{T}_i) \) minimizing \( \delta_i + E_{u,v} \), and define \( v_{i+1} \) to be the vertex for which \( e_{i+1} = \{v_j, v_{i+1}\} \) for some \( v_j \in T_i \). Finally define \( d_{v_{i+1}} \) by \( \delta_i + E_{v_j, v_{i+1}} \).

Finally, note that, as the minimum of \( r \) rate-1 exponentials is an exponential of rate \( r \), this is equivalent in distribution to the following process:

• Set \( v_1 = 1, T_1 = \{1\} \).

• Having defined \( v_i, T_i \), define a vertex \( v_{i+1} \) by choosing an edge \( e_{i+1} = \{v_j, v_{i+1}\} (j \leq i) \) uniformly at random from \( E(T_i, \bar{T}_i) \), set \( T_{i+1} = T_i \cup \{v_{i+1}\} \), and define \( d_{1, v_{i+1}} = d_{1, v_i} + E_i \gamma_i \) where \( E_i \gamma_i \) is an (independent) exponential random variable of rate \( \gamma_i = E(T_i, \bar{T}_i) \).

It follows that

\[
\mathbb{E}(d_{1,m}) = S_m := \sum_{i=1}^{m-1} \mathbb{E} \left( \frac{1}{\gamma_i} \right) \quad \text{and} \quad \mathbb{V}ar(d_{1,m}) = \sum_{i=1}^{m-1} \mathbb{E} \left( \frac{1}{\gamma_i^2} \right).
\]

Observe that we have

\[
(1 - \theta)i(dn - i) \leq \gamma_i \leq (1 + \theta)i(dn) \quad \text{w.h.p.}
\]

and so for \( 1 \leq i \leq \theta n \) we have

\[
\gamma_i = idn(1 + \zeta_i) \quad \text{where} \quad |\zeta_i| = O(\theta) \quad \text{w.h.p.}
\]

Also, we have

\[
\gamma_i = (n - i)dn(1 + \zeta_i) \quad \text{where} \quad |\zeta_i| = O(\theta) \quad \text{w.h.p.}
\]

for \( n - \theta n \leq i \leq n \).

It follows that

\[
S_{\theta n} = (1 + O(\theta)) \sum_{i=1}^{\theta n} \frac{1}{dn i} = \frac{\log n}{dn} + O \left( \frac{\log^{1/2} n}{n} \right) \quad \text{w.h.p.} \quad (8)
\]

**Lemma 6.** W.h.p. \( \max_{i,j} d_{i,j} \leq \ell_1 = \frac{5\log n}{dn} \).

**Proof.** Following [9], let \( k_1 = \theta n \) and \( Y_i = E_i^{\gamma_i}, 1 \leq i < n \) so that \( Z_1 = d_{1,k_1} = Y_1 + Y_2 + \cdots + Y_{k_1} \). For \( t < 1 - \frac{1 + o(1)}{dn} \) we have implies that w.h.p. for \( m = k_1 - 1 \),

\[
\mathbb{E}(e^{tdnZ_1}) = \mathbb{E} \left( \prod_{i=1}^{m} e^{tdnY_i} \right) = \sum_{x} \mathbb{E} \left( \prod_{i=1}^{m} e^{tdnY_i} \mid \gamma_m = x \right) \mathbb{P}(\gamma_m = x) = \mathbb{E} \left( \prod_{i=1}^{m-1} e^{tdnY_i} \right) \sum_{x} \mathbb{E}(e^{tdY_m} \mid \gamma_m = x) \mathbb{P}(\gamma_m = x) = \mathbb{E} \left( \prod_{i=1}^{m-1} e^{tdnY_i} \right) \sum_{x} \frac{x}{x - tdn} \mathbb{P}(\gamma_m = x) = \mathbb{E} \left( \prod_{i=1}^{m-1} e^{tdnY_i} \right) \left( 1 - \frac{(1 + o(1))t}{i} \right)^{-1}.
\]

Here the term in (9) stems from the fact that given \( \gamma_m, Y_m \) is independent of \( Y_1, Y_2, \ldots, Y_{m-1} \).

Then for any \( \beta > 0 \) we have
\[ P \left( Z_1 \geq \frac{\beta \log n}{dn} \right) \leq E(e^{t dn Z_1 - t \beta \log n}) \leq e^{-t \beta \log n} \prod_{i=1}^{k_1-1} \left( 1 - \frac{(1 + o(1))t}{i} \right)^{-1} = e^{-t \beta \log n} \exp \left\{ \sum_{i=1}^{k_1-1} \left( \frac{(1 + o(1))t}{i} + O \left( \frac{1}{i^2} \right) \right) \right\} = \exp \{ (1 + o(1) - \beta) t \log n \}. \]

It follows, on taking \( \beta = 2 + o(1) \) that w.h.p.

\[ d_{j,k_1} \leq \frac{(2 + o(1)) \log n}{dn} \text{ for all } j \in [n]. \]

Letting \( \hat{T}_{k_1} \) be the set corresponding to \( T_{k_1} \) when we execute Dijkstra’s algorithm starting at vertex 2. First consider the case where \( d \leq 1/2 \) and (2) holds. Then, using (2), we have that either \( T_{k_1} \cap \hat{T}_{k_1} \neq \emptyset \) or,

\[ P \left( \exists e \in T_{k_1} : \hat{T}_{k_1} : X(e) \leq \frac{1}{n} \right) \leq \exp \left\{ -\frac{\psi \theta^2 n^2}{n} \right\} = o(n^{-2}) \] (10)

This shows that we fail to find a path of length \( \leq \frac{(4 + o(1)) \log n}{dn} + \frac{1}{n} \) between a fixed pair of vertices with probability \( o(n^2) \). In particular, taking a union bound over all pairs of vertices, we obtain that w.h.p.

\[ \max_{i,j} d_{i,j} \leq \left( \frac{4 + o(1)}{dn} \right) \log n + \frac{1}{n}. \]

If \( G \) has \( \delta(G) \geq (1 - \tau)dn \) with \( d = 1/2 + \varepsilon, \varepsilon > 0 \) constant, then any pair of vertices has at least \( (2\varepsilon - 2\theta)n \) common neighbors. We pair up the vertices of \( T_{k_1}, \hat{T}_{k_1} \) and bound the probability that we cannot find a path of length 2 whose endpoints consist of one of our pairs, and which uses only edges of length at most \( \frac{\log n}{n \log n} \), as

\[ \left( e^{-\left( \frac{\log n}{n \log n} \right)^2} \right)^{-\theta n(2\varepsilon n - 2\theta n)} = o(n^{-2}). \]

Again we are done by a union bound over possible pairs.

We now consider the probability that a fixed edge \( e \) satisfies that \( \ell(e) > \ell_0 \) and that \( e \) is a shortest path from 1 to \( n \).

**Lemma 7.** Let \( E(e) \) denote the event that \( \ell(e) > \ell_0 \) and \( e \) is a shortest path from 1 to \( n \).

\[ P \left( E \left| \max_{j} d_{1,j} \leq \ell_1 \right. \right) = o \left( \frac{\log n}{n} \right). \]

**Proof.** Without loss of generality we write \( e = \{1, n\} \). If \( E = E(e) \) occurs then we have the occurrence of the event \( F \) where

\[ F = \{ d_{1,m} + \ell(f_m) \geq \ell(e), m = 2, 3, \ldots, n - 1 \} \]

and \( f_m \) denotes the edge joining vertex \( n \) to the vertex whose shortest distance from vertex 1 (in \( G \setminus \{n\} \)) is the \( m \)th smallest. (If the edge does not exist then \( \ell(f_m) = \infty \) in the calculation below.) Indeed this follows from Dijkstra’s algorithm; the event \( F \) indicates that at every step of the algorithm, no path shorter than the edge \( \{1, n\} \) is found.

Let \( n_0 = n(1 - d/2) \). We need \( \ell(f_m) + d_m \geq \xi = \ell(e) \) for all \( m \) in order that \( F \) occurs. If \( d_{1,n_0} = x \) then this is implied by \( \bigcap_{m=1}^{n_0} \{ \ell(f_m) \geq \xi - x \} \). Using the independence of the \( \ell(f_m) \) and \( d_{1,i}, i = 2, \ldots, n_0 \), we bound

\[ P(F \mid \max_{i,j} d_{1,j} \leq \ell_1) \leq \frac{1}{P(\max_{j} d_{1,j} \leq \ell_1)} \int_{\xi = \ell_0}^{\ell_1} e^{-\xi} \int_{x = 0}^{\infty} P \left( \bigcap_{m=1}^{n_0} \{ \ell(f_m) \geq \xi - x \} \right) dP \{ d_{1,n_0} = x \} d\xi \] (11)

and using the fact that there are at least \( dn/2 - 1 \) indices \( m \) for which \( \ell(f_m) < \infty \) we bound
Lemma 8. Together with Lemma 6, Lemma 7 implies that w.h.p. the number of edges $e$ for which $\mathcal{E}(e)$ occurs is $o(n \log n)$. Adding these to $E_0$ gives us a 1-spanner of size $\approx \frac{1}{2}n \log n$.

2.3 Lower bound for part (b)

Lemma 8. Fix a set $A$ such that $|A| \leq a_0 = O(\log n)$. Let $\mathcal{P}$ be the event that there exists a path $P$ of length at most $\ell_4 = \frac{\log n}{200d^2 n}$ joining two distinct vertices of $A$. Then $\mathbb{P}(\mathcal{P}) = O(n^{o(1)-199/200})$. 
There are a number of related questions one can tackle:  

**Lemma 9.** Let $B_1$ denote the set of vertices whose incident edges of length smaller than $\ell_3 = \ell_4/\lambda$ do not number in the range $I = [\frac{\log n}{300d\lambda}, \frac{\log n}{100d\lambda}]$. Then, w.h.p. $|B_1| \leq n^{1-1/5000\lambda}$. (Recall that we are bounding the size of a $\lambda$-spanner from below.)

**Proof.** The Chernoff bounds imply that

$$
\mathbb{P}(v \in B_1) \leq \mathbb{P}\left( Bin\left((1 + \theta)dn, 1 - \exp\left\{-\frac{\log n}{200\lambda dn}\right\}\right) \notin I \right) = \\
\mathbb{P}\left( Bin\left((1 + \theta)dn, \frac{\log n}{200\lambda dn} + O\left(\frac{\log^2 n}{n^2}\right)\right) \notin I \right) \leq 2 \exp\left\{-\frac{(1 + o(1)) \log n}{2 \times 9 \times 200\lambda}\right\} \leq n^{-1/4000\lambda}.
$$

The result follows from the Markov inequality.

**Lemma 10.** Let $B_2$ denote the set of vertices $v$ for which $|\{w : \ell_{v,w} \leq \ell_4\}| \geq \log n$. Then $B_2 = \emptyset$ w.h.p.

**Proof.** The Chernoff bounds imply that

$$
\mathbb{P}(B_2 \neq \emptyset) \leq n \mathbb{P}\left( Bin\left((1 + \theta)dn, 1 - \exp\left\{-\frac{\log n}{200dn}\right\}\right) \geq \log n\right) = o(1).
$$

Let $B_3$ denote the set of vertices $v$ for which there is a path of length at most $\ell_4$ joining neighbors $w_1, w_2$ such that $\ell_{v,w_i} \leq \ell_3, i = 1, 2$. Lemma 8 with $A$ equal to the set of neighbors $w$ of vertex $v$ such that $\ell_{v,w} \leq \ell_3$ shows that $|B_3| = o(n)$ w.h.p. (The fact that we can take $|A| = O(\log n)$ follows from Lemma 3) Lemmas 9 and 10 then imply that if $v \notin B_1 \cup B_3$ then a $\lambda$-spanner has to include the at least $\log n/(300d\lambda)$ edges incident to $v$ that are of length at most $\ell_3$. This completes the proof of part (b) of Theorem 1.

### 3 Summary and open questions

We have determined the asymptotic size of the smallest 1-spanner when the edges of a dense (asymptotically) regular graph $G$ are given independent lengths distributed as $E_2$, modulo the truth of (2) or the degree being $dn, d > 1/2$.

There are a number of related questions one can tackle:

1. We could replace edge lengths by $E_2^s$ where $s < 1$. This would allow us to generalise edge lengths to distributions with a density $f$ for which $f(x) \approx x^{1/s}$ as $x \to 0$. This is a more difficult case than $s = 1$ and it was considered by Bahmidi and van der Hofstad [3]. They prove that w.h.p. $d_{1,2}$ grows like $\frac{n^s}{\Gamma(1+1/s)}$, where $\Gamma$ denotes Euler’s Gamma function. The analysis is more complex than that of [9] and it is not clear that our proof ideas can be generalised to handle this situation.

2. The results of Theorem 1 apply to $G_{n,p}$. It would be of some interest to consider other models of random or quasi-random graphs.
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