Lower Bounds for Adversarially Robust PAC Learning

Abstract

In this work, we initiate a formal study of probably approximately correct (PAC) learning under evasion attacks, where the adversary’s goal is to misclassify the adversarially perturbed sample point \( \tilde{x} \), i.e., \( h(\tilde{x}) \neq c(\tilde{x}) \), where \( c \) is the ground truth concept and \( h \) is the learned hypothesis. Previous work on PAC learning of adversarial examples have all modeled adversarial examples as corrupted inputs in which the goal of the adversary is to achieve \( h(\tilde{x}) \neq c(x) \), where \( x \) is the original untampered instance. These two definitions of adversarial risk coincide for many natural distributions, such as images, but are incomparable in general.

We first prove that for many theoretically natural input spaces of high dimension \( n \) (e.g., isotropic Gaussian in dimension \( n \) under \( \ell_2 \) perturbations), if the adversary is allowed to apply up to a sublinear \( o(\|x\|) \) amount of perturbations on the test instances, PAC learning requires sample complexity that is exponential in \( n \). This is in contrast with results proved using the corrupted-input framework, in which the sample complexity of robust learning is only polynomially more.

We then formalize hybrid attacks in which the evasion attack is preceded by a poisoning attack. This is perhaps reminiscent of “trapdoor attacks” in which a poisoning phase is involved as well, but the evasion phase here uses the error-region definition of risk that aims at misclassifying the perturbed instances. In this case, we show PAC learning is sometimes impossible all together, even when it is possible without the attack (e.g., due to the bounded VC dimension).

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1 Introduction

Learning predictors is the task of outputting a hypothesis \( h \) using a training set \( S \) in such a way that \( h \) can predict the correct label \( c(x) \) of unseen instances such as \( x \) with high probability. A normal successful learner, however, could be vulnerable to adversarial perturbations. In particular, it was shown (Szegedy et al., 2014; Biggio et al., 2013; Goodfellow et al., 2015) that deep neural nets (DNNs) are vulnerable to so-called adversarial examples that are the result of small (even imperceptible to human eyes) perturbations on the original input \( x \). Since the introduction of such attacks, many works have studied defenses against them and more attacks are introduced afterwards (Biggio et al., 2013, 2014; Goodfellow et al., 2015; Papernot et al., 2016b; Carlini & Wagner, 2017; Xu et al., 2017; Madry et al., 2017).

A fundamental question in robust learning is whether one can design learning algorithms that achieve “generalization” even under such adversarial perturbations. Namely, we want to know when we can learn a robust classifier \( h \) that still correctly classifies its inputs even if they are adversarially perturbed in a limited way. Indeed, one can ask when the \((\varepsilon, \delta)\) PAC (probably approximately correct) learning (Valiant, 1984) is possible in adversarial settings. More formally, the goal here is to learn a robust \( h \) from the data set \( S \) consisting of \( m \) independently sampled labeled (non-adversarial) instances in such a way that, with probability \( 1 - \delta \) over the learning process, the produced \( h \) has error at most \( \varepsilon \) even under “limited” adversarial perturbations of the input. This limitation is carefully defined by some metric \( d \) defined over the input space \( \mathcal{X} \) and some upper bound “budget” \( b \) on the amount of perturbations that the adversary can introduce. I.e., we would like to minimize

\[
\text{AdvRisk}(h) = \Pr_{x \sim D} \left[ \exists \tilde{x} : d(x, \tilde{x}) \leq b, h(\tilde{x}) \neq c(\tilde{x}) \right] \leq \varepsilon
\]

where AdvRisk is the “adversarial” risk, and \( c(\cdot) \) is the ground truth (i.e., the concept function).

**Error-region adversarial risk.** The above notion of adversarial risk has been used implicitly or explicitly in previous work (Gilmer et al., 2018; Diochnos et al., 2018; Bubeck et al., 2018; Degwekar & Vaikuntanathan, 2019; Ford et al., 2019) and was formalized by Diochnos et al. (2018) as the “error-region” adversarial risk, because adversary’s goal here is to push \( x \) into the error region

\[
\mathcal{E} = \{ x | h(x) \neq c(x) \}
\]

**Corrupted-input adversarial risk.** Another notion of adversarial risk (that is similar, but still different from the error-region adversarial risk explained above) has been used in many works such as (Feige et al., 2015; Madry et al., 2017) in which the perturbed \( \tilde{x} \) is interpreted as a “corrupted input”. Namely, here the goal of the learner is to find the label of the original untampered point \( x \) by only having its corrupted version \( \tilde{x} \), and thus adversary’s success criterion is to reach \( d(x, \tilde{x}) \leq b, h(\tilde{x}) \neq c(x) \). Hence, in that setting, the goal of the learner is to find an \( h \) that minimizes

\[
\Pr_{x \sim D} \left[ \exists \tilde{x} : d(x, \tilde{x}) \leq b, h(\tilde{x}) \neq c(x) \right].
\]

It is easy to see that, if the ground truth \( c(x) \) does not change under \( b \)-perturbations, \( c(x) = c(\tilde{x}) \), the two notions of error-region and corrupted-input adversarial risk will be equal. In particular, this is the case for practical distributions of interest, such as images or voice, where sufficiently-small perturbations usually do not change human’s judgment about the true label. However, if \( b \)-perturbations can change the ground truth, \( c(x) \neq c(\tilde{x}) \), the two definitions are incomparable.

Several works have already studied PAC learning with provable guarantees under adversarial perturbations (Bubeck et al., 2018; Cullina et al., 2018; Feige et al., 2018; Attias et al., 2018; Khim & Loh, 2018; Yin et al., 2018; Montasser et al., 2019). However, all these works use the corrupted-input notion of adversarial risk. In particular, it is proved by Attias et al. (2018) that robust learning might require more data, but it was also shown by Attias et al. (2018) and Bubeck et al. (2018) that in natural settings, if robust classification is feasible, robust classifiers could be found with a sample complexity that is only polynomially larger than that of normal learning. This leads us to our central question:

What problems are PAC learnable under evasion attacks that perturb instances into the error region? If PAC learnable, what is their sample complexity?
1.1 Our Contribution

In this work, we initiate a formal study of PAC learning under adversarial perturbations, where the goal of the adversary is to increase the error-region adversarial risk using small (sublinear $o(||x||)$) perturbations of the inputs $x$. Therefore, in what follows, whenever we refer to adversarial risk, by default it means the error-region variant.

**Result 1: exponential lower bound on sample complexity.** Suppose the instances of a learning problem come from a metric probability space $(\mathcal{X}, D, d)$ where $D$ is a distribution and $d$ is a metric defining some norm $||\cdot||$. Suppose the input instances have norms $||x|| \approx n$ where $n$ is a parameter related (or in fact equal) to the data dimension. One natural setting of study for PAC learning is to study attackers that can only perturb $x$ by a sublinear amount $o(||x||) = o(n)$ (e.g., $\sqrt{n}$).

Our first result is to prove a strong lower bound for the sample complexity of PAC learning in this setting. We prove that for many theoretically natural input spaces of high dimension $n$ (e.g., isotropic Gaussian in dimension $n$ under $\ell_2$ perturbations), PAC learning of certain problems under sublinear perturbations of the test instances requires exponentially many samples in $n$, even though the problem in the no-attack setting is PAC learnable using polynomially many samples. This holds e.g., when we want to learn half spaces in dimension $n$ under such distributions (which is possible in the no-attack setting). We note that even though PAC learning is defined for all distributions, proving such lower bound for a specific input distribution $D$ over $\mathcal{X}$ only makes the negative result stronger.

**Remark 1.1** (Approximation error in error-region robust learning). If a learning problem is realizable in the no-attack setting, i.e., there is a hypothesis $h$ that has risk zero over the test instances, it means that the same hypothesis $h$ will have adversarial (true) risk zero over the test instances as well, because any perturbed point is still going to be correctly classified. This is in contrast with corrupted-input notion of adversarial risk that even in realizable problems, the smallest corrupted-input (true) adversarial risk could still be large, and even at odds with correctness (Tsipras et al., 2018). This means that our results rule out (efficient) PAC learning even in the agnostic setting as well, because in the realizable setting there is at least one hypothesis with error-region adversarial risk zero while (as we prove), in some settings learning a model with adversarial risk (under sublinear perturbations) close to zero requires exponentially many samples.

**Result 2: ruling out PAC learning under hybrid attacks.** We then study PAC learning under adversarial perturbations that happen during both training and testing phases. We formalize hy-
brid attacks in which the final evasion attack is preceded by a poisoning attack [Biggio et al., 2012; Papernot et al., 2016a]. This attack model bears similarities to “trapdoor attacks” [Gu et al., 2017] in which a poisoning phase is involved before the evasion attack, and here we give a formal definition for PAC learning under such attacks. Our definition of hybrid attacks is general and can incorporate any notion of adversarial risk, but our results for hybrid attacks use the error-region adversarial risk.

Under hybrid attacks, we show that PAC learning is sometimes impossible all together, even though it is possible without such attacks. For example, even if the VC dimension of the concept class is bounded by $n$, if the adversary is allowed to poison only $1/n^{10}$ fraction of the $m$ training examples, then it can do so in such a way that a subsequent evasion attack could then increase the adversarial risk to $\approx 1$. This means that PAC learning is in fact impossible under such hybrid attacks.

We also note that classical results about malicious noise (Valiant, 1985; Kearns & Li, 1993) and nasty noise (Bshouty et al., 2002) could be interpreted as ruling out PAC learning under poisoning attacks. However, there are two differences: (I) The adversary in these previous works needs to change a constant fraction of the training examples, while our attacker changes only an arbitrarily small inverse polynomial fraction of the training set, and hence it does not add any misclassified examples to the pool. Thus the poisoning attack used here is a clean/correct label attack (Mahloujifar et al., 2018a; Shafahi et al., 2018).

2 Defining Adversarially Robust PAC Learning

Notation. By $\tilde{O}(f(n))$ we refer to the set of all functions of the form $O(f(n) \log(f(n)))^{O(1)}$. We use capital calligraphic letters (e.g., $\mathcal{D}$) for sets and capital non-calligraphic letters (e.g., $D$) for distributions. $x \leftarrow \mathcal{D}$ denotes sampling $x$ from $\mathcal{D}$. For an event $\mathcal{S}$, we let $\mathcal{D}(\mathcal{S}) = \Pr_{x \sim \mathcal{D}}[x \in \mathcal{S}]$.

A classification problem $\mathcal{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H})$ is specified by the following components. The set $\mathcal{X}$ is the set of possible instances, $\mathcal{Y}$ is the set of possible labels, $\mathcal{D}$ is a class of distributions over instances $\mathcal{X}$. In the standard setting of PAC learning, $\mathcal{D}$ includes all distributions, but since we deal with negative results, we sometimes work with fixed $\mathcal{D} = \{D\}$ distributions, and show that even distribution-dependent robust PAC learning is sometimes hard. In that case, we represent the problem as $\mathcal{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{C}, \mathcal{D}, \mathcal{H})$. The set $\mathcal{C} \subseteq \mathcal{Y}^\mathcal{X}$ is the concept class and $\mathcal{H} \subseteq \mathcal{Y}^\mathcal{X}$ is the hypothesis class. In general, we can allow randomized concept and hypothesis functions to model, in order, label uncertainly (usually modeled by a joint distribution over instances and labels) and randomized predictions. All of our results extend to randomized learners and randomized hypothesis functions, but for simplicity of presentation, we treat them as deterministic mappings. By default, we consider 0-1 loss functions where $\text{loss}(y', y) = 1[y' = y]$. For a given distribution $D \in \mathcal{D} \text{ and a concept function } c \in \mathcal{C}$, the risk of a hypothesis $h \in \mathcal{H}$ is the expected loss of $h$ with respect to $D$, namely $\text{Risk}(D, c, h) = \Pr_{x \sim D}[\text{loss}(h(x), c(x))]$. An example $z$ is a pair $z = (x, y)$ where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. An example is usually sampled by first sampling $x \leftarrow D$ for some $D \in \mathcal{D}$ followed by letting $y = c(x)$ for some $c \in \mathcal{C}$. A sample sequence $S = (z_1, \ldots, z_m)$ is a sequence of $m$ examples. As is usual, sometimes we might refer to a sample sequence as the training set. By $S \leftarrow (D, c(D))^m$ we denote the process of obtaining $S$ by sampling $m$ iid samples from $D$ and labeling them by $c$.

Our learning problems $\mathcal{P}_n = (\mathcal{X}_n, \mathcal{Y}_n, \mathcal{C}_n, \mathcal{D}_n, \mathcal{H}_n)$ are usually parameterized by $n$ where $n$ denotes the “data dimension” or (closely) capture the bit length of the instances. Thus, the “efficiency” of the algorithms could depend on $n$. Even in this case, for simplicity of notation, we might simply write $\mathcal{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{C}, \mathcal{D}, \mathcal{H})$. By default, we will have $\mathcal{C} \subseteq \mathcal{H}$, in which case we call $\mathcal{P}$ realizable. This means that for any training set for $c \in \mathcal{C}, D \in \mathcal{D}$, there is a hypothesis that has empirical and true risk zero; though finding such $h$ might be challenging.

Evasion attacks. An evasion attacker $A$ is one that changes the test instance $x$, denoted as $\tilde{x} \leftarrow A(x)$. The behavior and actions taken by $A$ could, in general, depend on the choices of $D \in \mathcal{D}, c \in \mathcal{C}$, and $h \in \mathcal{H}$. As a result, in our notation, we provide $A$ with access to $D, c, h$ by giving them as special inputs to $A^{D,c,h}$ denoting the process as $\tilde{x} \leftarrow A[D, c, h](x)$. We use calligraphic font $\mathcal{A}$ to denote a class/set of attacks. For example, $\mathcal{A}$ could contain all attackers who could change test instance $x$ by at most $b$ perturbations under a metric defined over $\mathcal{X}$.

\footnote{This dependence is information theoretic, and for example, $A$ might want to find $\tilde{x}$ that is misclassified, in which case its success is defined as $h(\tilde{x}) \neq c(\tilde{x})$ which depends on both $h, c$.}
Poisoning attacks. A poisoning attacker $A$ is one that changes the training sequence as $\tilde{S} \leftarrow A(S)$. Such attacks, in general, might add examples to $S$, remove examples from $S$, or do both. The behavior and actions taken by $A$ could, in general, depend on the choices of $D \in D, c \in C$ (but not on $h \in H$, as it is not produced by the learner at the time of the poisoning attack). As a result, we provide implicit access to $D, c$ with sample complexity $n$. We use calligraphic font $\mathcal{A}$ to denote a class/set of attacks. For example, $\mathcal{A}$ could contain attacks that change $1/n$ fraction of $S$ only using clean labels (Mahloujifar et al. 2018b; Shafahi et al. 2018).

Hybrid attacks. A hybrid attack $A = (A_1, A_2)$ is a two phase attack in which $A_1$ is a poisoning attacker and $A_2$ is an evasion attacker. One subtle point is that $A_2$ is also aware of the internal state of $A_1$, as they are a pair of coordinating attacks. More formally, $A_1$ outputs an extra “state” information $\hat{x}$ which will be given as an extra input to $A_2$. As discussed above, $A_1$ can depend on $D, c$, and $A_2$ can depend on $D, c, h$ as defined for evasion and poisoning attacks.

We now define PAC learning under adversarial perturbation attacks. To do so, we need to first define our notion of adversarial risk. We will do so by employing the error-region notion adversarial risk as formalized in Diochnos et al. (2018) adversary aims to misclassify the perturbed instance $\tilde{x}$.

**Definition 2.1** (Error-region (adversarial) risk). Suppose $A$ is an evasion adversary and let $D, c, h$ be fixed. The error-region (adversarial) risk is defined as follows.

$$\text{AdvRisk}_A(D, c, h) = \Pr_{x \leftarrow D, \tilde{x} \leftarrow A(D, c, h)(x)}[h(\tilde{x}) \neq c(\tilde{x})].$$

For randomized $h$, the above probability is also over the randomness of $h$ chosen after $\tilde{x}$ is selected.

We now define PAC learning under hybrid attacks, from which one can derive also the definition of PAC learning under evasion attacks and under poisoning attacks.

**Definition 2.2** (PAC learning under hybrid attacks). Suppose $\mathcal{P}_n = (X_n, Y_n, \mathcal{C}_n, D_n, \mathcal{H}_n)$ is a realizable classification problem, and suppose $A$ is a class of hybrid attacks for $\mathcal{P}_n$. $\mathcal{P}_n$ is PAC learnable with sample complexity $m(\varepsilon, \delta, n)$ under hybrid attacks of $A$, if there is a learning algorithm $L$ such that for every $n, 0 < \varepsilon, \delta < 1, c \in \mathcal{C}, D \in D$, and $(A_1, A_2) \in A$, if $m = m(\varepsilon, \delta, n)$, then

$$\Pr_{S \leftarrow (D, c(D))^m, (\tilde{S}, \tilde{t}) \leftarrow A(D, c, \tilde{t})(S), h \leftarrow L(\tilde{S})} \left[ \text{AdvRisk}_{A_2}[D, c, h](\tilde{h}, c, D) > \varepsilon \right] \leq \delta.$$

PAC learning under (pure) poisoning attacks or evasion attacks could be derived from Definition 2.2 by letting either of $A_1$ or $A_2$ be a trivial attack that does no tampering at all.

We also note that one can obtain other definitions of PAC learning under evasion or hybrid attacks in Definition 2.2 by using other forms of adversarial risk, e.g., corrupted-input adversarial risk (Feige et al. 2015, 2018; Madry et al. 2017; Schmidt et al. 2018; Attias et al. 2018).

### 3 Lower Bounds for PAC Learning under Evasion and Hybrid Attacks

Before proving our main results, we need to recall the notion of Normal Lévy families, and define a desired and common property of set of concept functions with respect to the distribution of inputs.

**Notation.** Let $(\mathcal{X}, d)$ be a metric space. For $S \subseteq \mathcal{X}$, by $d(x, S) = \inf \{d(x, y) \mid y \in S\}$ we denote the distance of a point $x$ from $S$. We also let $S_b = \{y \mid d(x, y) \leq b, x \in S\}$ be the $b$-expansion of $S$. When there is also a measure $D$ defined over the metric space $(\mathcal{X}, d)$, the concentration function is defined and denoted as $\alpha(b) = 1 - \inf \{\Pr_D[|E|] \mid \Pr_D[|E|] \geq 1/2\}$.

**Definition 3.1** (Normal Lévy families). A family of metric probability spaces $(X_n, d_n, D_n)_{n \in \mathbb{N}}$ with concentration function $\alpha_n(\cdot)$ is called a normal Lévy family if there are $k_1, k_2$, such that

$$\alpha_n(b) \leq k_1 \cdot e^{-k_2 b^2/n}$$

$^4$For example, an attack model might require $A$ to choose its perturbed instances still using correct/clean labels, in which case the attack is restricted based on the choice of $c$.

$^5$Another common formulation of Normal Lévy families uses $\alpha_n(b) \leq k_1 \cdot e^{-k_2 b^2 n}$, but here we scale the distances up by $n$ to achieve “typical norms” to be $\approx n$, which is the dimension.
Examples. Many natural metric probability spaces are Normal Lévy families. For example, all the following examples under normalized distance (to make the typical norms $\approx n$) are normal Lévy families as stated in Definition 3.1: the unit $n$-sphere with uniform distribution under the Euclidean or geodesic distance, $\mathbb{R}^n$ under Gaussian distribution and Euclidean distance, $\mathbb{R}^n$ under Gaussian distribution and Euclidean distance, the unit $n$-cube and unit $n$-ball under the uniform distribution and Euclidean distance, any product distribution of dimension $n$ under the Hamming distance. See (Ledoux, 2001; Giannopoulos & Milman, 2001; Milman & Schechtman, 1986) for more examples.

The following lemma was proved in Mahloujifar et al. (2018b) when Normal Lévy input spaces.

Theorem 3.4 (Limits of adversarially robust PAC learning).

As we will see, Part 1a and Part 1b of Theorem 3.4 are special cases of the following more quantitative variant afterwards (Lemma 3.5).

 Examples. The set of homogeneous half spaces in $\mathbb{R}^n$ are $\alpha$-close for all $\alpha \in (0, 1]$ under any of the following natural distributions: uniform over the unit sphere, uniform inside the unit ball, and isotropic Gaussian. This can be proved by picking two half spaces that their disagreement region under the mentioned distributions is exactly $\alpha$. The set of (monotone, or not necessarily monotone) conjunctions are $\alpha$-close for $\alpha = 2^{-k}$ for all $k \in \{2, \ldots, n\}$ under the uniform distribution over $\{0, 1\}^n$. This can be proved by looking at $c_1 = x_1 \wedge \ldots \wedge x_{k-1}$ and $c_2 = x_1 \wedge \ldots \wedge x_{k-1} \wedge x_k = c_1 \wedge x_k$. Since all the variables that appear in $c_1$ also appear in $c_2$, we have that $\Pr_{x \sim \mathcal{D}}[c_1(x) \neq c_2(x)]$ is equal to $\Pr_{x \sim \{0,1\}^n}[c_1(x) = 1 \land (c_2(x) = 0)]$, and as a consequence this is equal to $2^{-k} - 2^{-k} = 2^{-k}$.

We now state and prove our main results. Theorem 3.4 is stated in the asymptotic form considering attack families that attack the problem for sufficiently large index $n \in \mathbb{N}$ of the problem. We describe a quantitative variant afterwards (Lemma 3.5).

Theorem 3.4 (Limits of adversarially robust PAC learning). Suppose $\mathcal{P}_n = (\mathcal{X}, \mathcal{Y}, \mathcal{C}, \mathcal{D}, \mathcal{H})$ is a realizable classification problem and that $\mathcal{X}$ is a Normal Lévy Family (Definition 3.1) over $\mathcal{D}$ and a metric $d$, and that $C$ is $\Theta(\alpha)$-close with respect to $D$ for all $\alpha \in [2^{-\Theta(n)}, 1]$. Then, the following hold even for $\mathcal{P}_n$ learning with parameters $\varepsilon = 0.9, \delta = 0.49$.

1. Sample complexity of PAC learning robust to evasion attacks:
   (a) Exponential lower bound: Any PAC learning algorithm that is robust against all attacks with a sublinear tampering $b = o(n)$ budget under the metric $d$ requires exponential sample complexity $m \geq 2^{\Omega(n)}$.
   (b) Super-polynomial lower bound: PAC learning that is robust against all tampering attacks with budget $b = \tilde{O}(\sqrt{n})$, requires at least $m \geq n^{\omega(1)}$ many samples.

2. Ruling out PAC learning robust to hybrid attacks:

Suppose the tampering budget of the evasion adversary can be any $b = \tilde{O}(\sqrt{n})$, and let $B_{\lambda}$ be any class of poisoning attacks that can remove $\lambda = \lambda(n)$ fraction of the training examples for an (arbitrary small) inverse polynomial $\lambda(n) \geq 1/\text{poly}(n)$. Let $R$ be the class of hybrid attacks that first do a poisoning by some $B \in B_{\lambda}$ and then an evasion by some adversary of budget $b = \tilde{O}(\sqrt{n})$. Then, $\mathcal{P}_n$ is not PAC learnable (regardless of sample complexity) under hybrid attacks in $R$.

As we will see, Part 1a and Part 1b of Theorem 3.4 are special cases of the following more quantitative lower bound that might be of independent interest.

Lemma 3.5. For the setting of Theorem 3.4, if the tampering budget is $b = \rho \cdot n$, for a fixed function $\rho = \rho(n) = o(1)$, then any PAC learning algorithm for $\mathcal{P}_n$ under evasion attacks of tampering budget $b = b(n)$, even for parameters $\varepsilon = 0.9, \delta = 0.49$ requires sample complexity at least $m(n) \geq 2^{\Omega(\rho^2 \cdot n)}$.
Examples. Here we list some natural scenarios that fall into the conditions of Theorem 3.4. All examples of Normal Lévy families listed after Definition 3.1, together with the concept class of half spaces satisfy the conditions of Theorem 3.4 and hence cannot be PAC learned using a poly(n) number of samples. The reason is that one can always find two half spaces whose symmetric difference has measure exactly $\epsilon$. Moreover, as discussed in examples following Definition 3.3, even discrete problems such as learning monotone-conjunctions under the uniform distribution (and Hamming distance as perturbation metric) fall into the conditions of Theorem 3.4, for which a lower bound on their sample complexity (or even impossibility) of robust PAC learning could be obtained.

Remark 3.6 (Evasion-robust PAC learning in the RAM computing model with real numbers). We remark that if we allow (truly) real numbers represent the concept and hypothesis classes, one can even rule out PAC learning (not just lower bounds on sample complexity) under similar perturbations describe in Part I. Indeed, by inspecting the same proof of Theorem 3.4 for Part I one can get such results, e.g., for learning half-spaces in dimension $n$ when inputs come from isotropic Gaussian. However, we emphasize that such (seemingly) stronger lower bounds are not realistic, as in real settings, we eventually work with finite precision to represent the concept functions (of half spaces). This makes the set of concept functions finite, in which case the test error eventually reaches zero, using perhaps exponentially many samples. Theorem 3.4, however, has the useful feature that it applies even in those settings, as long as the concept functions are rich enough to allow the sufficiently close (but not too close) pairs under the distribution $D$ according to Definition 3.5.

In what follows, we will first prove Lemma 3.5. We will then use Lemma 3.5 to prove Theorem 3.4.

Proof of Lemma 3.5. Let $m = m(0.9, 0.49, n)$ be the sample complexity of the (presumed) learner $L$ that achieves $(\epsilon, \delta)$-PAC learning for $\epsilon = 0.9, \delta = 0.49$. If $m = 2^{\Omega(n)}$ already, we are done, as it is even larger than what Lemma 3.5 states, so let $m = 2^{\omega(n)}$, and we will derive a contradiction. Since the distribution $D$ is fixed, in the discussion below, we simply denote $\text{Risk}(D, h, c)$ as $\text{Risk}(h, c)$.

Recall that, by assumption, for all $\epsilon \in [2^{-\Theta(n)}, 1]$, there are $c_1, c_2 \in C$ that are $\Theta(\epsilon)$-close under the distribution $D$. Because $m = 2^{\omega(n)}$, it holds that $1/m \geq \omega(2^{-\Theta(n)})$, and so there are $c_1, c_2 \in C$ such that for $\Delta(c_1, c_2) = \{x \in X \mid c_1(x) \neq c_2(x)\}$ we have

$$\Omega\left(\frac{1}{m}\right) \leq \Pr_{x \leftarrow D}[x \in \Delta(c_1, c_2)] \leq \frac{1}{100m}.$$ 

Now, consider $m$ i.i.d. samples that are given to the learner $L$ as a training set $S$. With probability at least 0.99 of the sampling of $S$, all $x \in S$ would be outside $\Delta(c_1, c_2)$, in which case $L$ would have no way to distinguish $c_1$ from $c_2$. So, if we pick $c \leftarrow \{c_1, c_2\}$ at random and pick test instance $x \leftarrow (D \mid \Delta(c_1, c_2))$, the hypothesis $h = L(S)$ fails with probability at least 0.99/2. Thus, we can fix the choice of $c \in \{c_1, c_2\}$, such that with probability 0.99/2 > 0.49 we get a $h \leftarrow L(S)$ where

$$\text{Risk}(h, c) = \Pr_{x \leftarrow D}[h(x) \neq c(x)] \geq \frac{1}{2} \cdot \Pr_{x \leftarrow D}[x \in \Delta(c_1, c_2)] \geq \Omega\left(\frac{1}{m}\right).$$

For this fixed $c$ and any such learned hypothesis $h$ with $\text{Risk}(h, c) = \Omega(1)/m$, by Lemma 3.2, the adversarial risk reaches $\text{AdvRisk}_{A_0}(h, c) \geq 0.99$ by an attack $A \in A_0$ that has tampering budget:

$$b = O(\sqrt{n}) \cdot (\sqrt{\ln(O(m))} + \sqrt{O(1)}) \leq t \cdot (\sqrt{n \cdot \ln m})$$

for universal constant $t$. But, we said at the beginning that the tampering budget of the adversary is $\rho(n) \cdot n$. Therefore, it should be that

$$\rho(n) \cdot n < t \cdot (\sqrt{n \cdot \ln m}),$$

as otherwise the evasion-robust PAC learner is not actually robust as stated. Thus, we get

$$m \geq e^{\rho(n)^2 \cdot n / t} = 2^{\Omega(\rho(n)^2 \cdot n)}$$

which finishes the proof of Lemma 3.5. $\square$

We now prove Theorem 3.4 using Lemma 3.5.

Proof of Theorem 3.4. Using Lemma 3.5, we will first prove Part 1a, then Part 1b, and then Part 2. Throughout, $\epsilon = 0.9, \delta = 0.49$ are fixed, so the sample complexity $m = m(n)$ is a function of $n$. 7
Proving Part 1a. We claim that PAC learning resisting all \( b = o(n) \)-tampering attacks requires sample complexity \( m \geq 2\Omega(n) \). The reason is that, otherwise, there will be an infinite sequence of values \( n_1 < n_2 < \ldots \) for which \( m = m(n_i) \leq 2^\gamma(n_i) \) for \( \gamma(n) = o(1) \). However, in that case, if we let \( \rho(n) = \gamma(n)^{1/3} \), because \( \rho(n) = o(n) \), by Lemma 3.5, the sample complexity is 
\[
m(n_i) \geq 2^{\Omega(\rho(n_i)^2 \cdot m_i)} = \omega(2^{\gamma(n_i) \cdot m_i}).
\]
However, this is a contradiction as we previously assumed \( m(n_i) \leq 2^{\gamma(n_i) \cdot (n_i)} \).

Proving Part 1b. Suppose the adversary can tamper instances with budget \( b(n) = \kappa(n) \cdot \sqrt[n]{n} \) for \( \kappa(n) \in \text{polylog}(n) \). Since we can rewrite \( b(n) = \rho(n) \cdot n \) for \( \rho(n) = \kappa(n) / \sqrt[n]{n} \), then by Lemma 3.5, the sample complexity of \( L \) should be at least 
\[
m(n) \geq 2^{\Omega(\rho(n)^2 \cdot n)} = 2^{\Omega(\kappa(n)^2)}. 
\]
Therefore, if we choose \( \kappa(n) = \log(n)^2 \), the sample complexity of \( L \) becomes \( m \geq n^{\log n} \geq n^{\omega(1)} \).

Proving Part 2. Let be \( c_1, c_2 \in C \) be such that for \( \Delta(c_1, c_2) = \{ x \in \mathcal{X} \mid c_1(x) \neq c_2(x) \} \) we have 
\[
\Omega(\lambda) \leq \Pr_{x \leftarrow D(c_1, c_2)}[x \in \Delta(c_1, c_2)] \leq \lambda. 
\]
Consider a poisoning attacker \( A_1 \) that given a data set \( S \), it removes any \( (x, y) \) from \( S \) such that \( x \in \Delta(c_1, c_2) \). Note that the (expected) number of such examples is \( \Pr[x \in \Delta(c_1, c_2)] \leq \lambda \). Let \( S' \) be the modified training set. The learner \( L(S') \) now has no way to distinguish between \( c_1 \) and \( c_2 \). Thus, like in Lemma 3.5, we can fix \( c \in \{ c_1, c_2 \} \), such that \( L(S') \) always produces \( h \) where 
\[
\text{Risk}(h, c) = \Pr_{x \leftarrow D'}[h(x) \neq c(x)] \geq \frac{1}{2} \cdot \Pr_{x \leftarrow D'}[x \in \Delta(c_1, c_2)] \geq \Omega(\lambda). 
\]
For this fixed \( c \) and any such learned hypothesis \( h \) with \( \text{Risk}(h, c) = \Omega(\lambda) \), by Lemma 3.2, the adversarial risk (under attacks) reaches \( \text{AdvRisk}_{A_\lambda}(h, c) \geq 0.99 \) by an attack \( A \in A_\lambda \) that changes test instances \( x \) by at most \( b \) for 
\[
b = O(\sqrt{n} \cdot (\sqrt{\ln(O(1/\lambda))} + \sqrt{O(1)}) \leq O(\sqrt{n} \cdot \ln(1/\lambda)). 
\]
Since \( \lambda = 1 / \text{poly}(n) \), it holds that \( b = \tilde{O}(\sqrt{n}) \). □

4 Extensions

In this section, we describe some extensions to Theorem 3.4 in various directions.

Extension to randomized predictors. In Theorem 3.4 we ruled out PAC learning (or its small sample complexity) even for very large values \( \varepsilon = 0.9, \delta = 0.49 \). One might argue that proving such lower bound could not be impossible because a trivial hypothesis (for the setting where \( Y = \{0, 1\} \)) can achieve \( \varepsilon = 0.5 \) by outputting random bits. However, this trivial predictor is randomized, while Theorem 3.4 is proved for deterministic hypotheses. For the case of randomized hypotheses, one can adjust the proof of Theorem 3.4 to get similar lower bounds for \( \varepsilon = 0.49, \delta = 0.49 \) as follows.

In the proof of Theorem 3.4 we first showed that small sample complexity implies the existence of \( c \) that with probability \( > 0.49 \) it will have an error region with a non-negligible measure. When the hypothesis is randomized, however, we cannot work with the traditional notion of error region, because on every point \( x \in \mathcal{X} \), the hypothesis could be wrong \( h(x) \neq c(x) \) with some probability in \([0, 1]\). We can, however, work with the relaxed notion of “approximate error” region, defined as \( \mathcal{AE}(h, c) = \{ x \mid \Pr[h(x) \neq c(x)] \geq 1/2 \} \), where the probability is over the randomness of \( h \).

In proofs of both Lemma 3.5 and Theorem 3.4 we deal with two close concept functions \( c_1, c_2 \) that are “indistinguishable” for the hypothesis \( h \) and then conclude that for each point \( x \in \Delta(c_1, c_2) \), \( h \) makes a mistake on at least one of \( c_1, c_2 \). If \( h \) is randomized, we cannot say this anymore, but we can still say that for each such point \( x \in \Delta(c_1, c_2) \), for at least one of \( c_1, c_2 \), \( h(x) \) is wrong with probability at least \( 0.5 \). Therefore, we get the same lower bound on the size of the \( \mathcal{AE} \) as we got in Lemma 3.5 and Theorem 3.4. However, expanding the set \( \mathcal{AE} \) instead of an actual error-region, implies that the adversarially perturbed points \( \tilde{x} \) that fall into \( \mathcal{AE} \) are now misclassified with probability \( 0.5 \). Thus, at least 0.99 fraction of inputs can be perturbed into \( \mathcal{AE} \) to be misclassified with probability \( > 0.49 \).
Lower bound for PAC learning of a “typical” concept function. Theorem 3.4 only proves the existence of at least one concept function \( c \in \mathcal{C} \) for which the (presumed) robust PAC learner will either fail (to PAC learn) or will need large sample complexity. Now, suppose concept functions themselves come from a (natural) distribution and we only want to robustly PAC learn most of them. Indeed, we can extend the proof of Theorem 3.4 to show that for natural settings, the impossibility result extends to at least half of the concept functions, not just a few pathological cases.

To extend Theorem 3.4 to the more general “typical” failure over \( c \leftarrow \mathcal{C} \) (stated as Claim 4.2 below) we need the following definition as an extension to Definition 3.3.

**Definition 4.1** (Uniformly \( \alpha \)-close function families). Suppose \( D \) is a distribution over \( \mathcal{X} \), and let \( \mathcal{C} \) be a set of functions from \( \mathcal{X} \) to some set \( \mathcal{Y} \). We call \( \mathcal{C} \) uniformly \( \alpha \)-close with respect to \( D \), if there is a joint distribution \((c_1, c_2)\) where both coordinates are uniformly distributed over \( \mathcal{C} \), and that for all \((c_1, c_2) \leftarrow (c_1, c_2)\), it both holds that \( c_1, c_2 \in \mathcal{C} \) and that \( \Pr_{x \leftarrow D}[c_1(x) \neq c_2(x)] = \alpha \).

**Claim 4.2.** In Theorem 3.4 and Lemma 3.3 make the only change in the setting as follows. The concept class \( \mathcal{C} \) now satisfies the stronger condition of being uniform \( \alpha \)-close with respect to \( D \). Then, the same limitations of PAC learning hold for at least measure half of \( c \leftarrow \mathcal{C} \).

Here we sketch why Claim 4.2 holds. The difference is that now, instead of knowing the existence of an \( \alpha \)-close pair \((c_1, c_2)\), we have distribution \((c_1, c_2)\) samples from which satisfy the \( \alpha \)-close property. Therefore, for all samples \((c_1, c_2) \leftarrow (c_1, c_2)\), at least one of \( c_1 \) or \( c_2 \) is “bad” for the (presumed) PAC learner \( L \) (with the same proof before). But, since each of the coordinates in \((c_1, c_2)\) is marginally uniform, therefore, at least measure 1/2 of \( c \leftarrow \mathcal{C} \) is bad for \( L \).

**Example.** Consider the uniform measure over homogeneous half spaces in dimension \( n \) as the set of concept functions \( \mathcal{C} \): choose a point \( w \) in the unit sphere and select the half space \( \{ x \mid \langle x, w \rangle \geq 0 \} \). It is easy to see that \( \mathcal{C} \) with such measure is uniformly \( \alpha \)-close with respect to the isotropic Gaussian distribution (or uniform distribution over the unit sphere). Thus, Claim 4.2 applies to this case.

### 5 Conclusion and Open Questions

We examined evasion attacks, where the adversary can perturb instances during test time, as well as hybrid attacks where the adversary can perturb instances during both training and test time. For evasion attacks we gave an exponential lower bound on the sample complexity even when the adversary can perturb instances by an amount of \( o(n) \), where \( n \) is the data dimension capturing the “typical” norm of an input. For hybrid attacks, PAC learning is ruled out altogether when the adversary can poison a small fraction of the training examples and still perturb the test instance by a sublinear amount \( o(n) \) (or even \( \tilde{O}(\sqrt{n}) \)).

Our result shows a different behavior when it comes to PAC learning for error-region adversarial risk compared to previously used notions of adversarial robustness based on corrupted inputs. In particular, in the error-region variant of adversarial risk, realizable problems stay realizable, as normal risk zero for a hypothesis \( h \) also implies (error-region) adversarial risk zero for the same \( h \). This makes our results more striking, as they apply to agnostic learning as well.

**Open questions.** Our Theorem 3.4 relies on a level of tampering to be at least \( \tilde{O}(\sqrt{n}) \) to imply the super-polynomial lower bounds. One natural question is to find the exact threshold of perturbations needed that triggers super-polynomial lower bounds on sample complexity.

Another important direction is to study the sample complexity of PAC learning (with concrete parameters \( \varepsilon, \delta \)) for practical distributions such as images or voice. Our lower bounds of this work are only proved for theoretically natural distributions that are provably concentrated in high dimension. Mahloujifar et al. (2019) presents a method for empirically approximating the concentration of such distributions given i.i.d. samples from them.

Finally, we ask if similar results could be proved for corrupted-input adversarial risk. Note that previous work studying learning under corrupted-input adversarial risk (Bubeck et al., 2018b; Cullina et al., 2018; Feige et al., 2018; Attias et al., 2018; Khim & Loh, 2018; Yin et al., 2018; Montasser et al., 2019) focus on agnostic learning, by aiming to get close to the “best” robust classifier. However, it is not clear how good the best classifier is. It remains open to find out when we can learn robust classifiers (under corrupted-input risk) in which the total adversarial risk is small.
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