THE DEGREE OF THE GENERATORS OF THE CANONICAL RING

OF SURFACES OF GENERAL TYPE WITH \( p_g = 0 \)

by

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Introduction.

Upper bounds for the degree of the generators of the canonical rings of surfaces of general type were found by Ciliberto [C]. In particular it was established that the canonical ring of a minimal surface of general type with \( p_g = 0 \) is generated by its elements of degree lesser or equal to 6, ([C], th. (3.6)). This was the best bound possible to obtain at the time, since Reider’s results, [R], were not yet available. In this note, this bound is improved in some cases (theorems (3.1), (3.2)).

In particular it is shown that if \( K^2 \geq 5 \), or if \( K^2 \geq 2 \) and \( |2K_S| \) is base point free this bound can be lowered to 4. This result is proved by showing first that, under the same hypothesis, the degree of the bicanonical map is lesser or equal to 4 if \( K^2 \geq 3 \), (theorem (2.1)), implying that the hyperplane sections of the bicanonical image have not arithmetic genus 0. The result on the generation of the canonical ring then follows by the techniques utilized in [C].

Notation and conventions.

We will denote by \( S \) a projective algebraic surface over the complex field. Usually \( S \) will be smooth, minimal, of general type.

We denote by \( K_S \), or simply by \( K \) if there is no possibility of confusion, a canonical divisor on \( S \). As usual, for any sheaf \( \mathcal{F} \) on \( S \), we denote by \( h^i(S, \mathcal{F}) \) the dimension of the cohomology space \( H^i(S, \mathcal{F}) \), and by \( p_g \) and \( q \) the geometric genus and the irregularity of \( S \).

By a curve on \( S \) we mean an effective, non zero divisor on \( S \). We will denote the intersection number of the divisors \( C, D \) on \( S \) by \( C \cdot D \) and by \( C^2 \) the self-intersection of the divisor \( C \). We denote by \( \equiv \) the linear equivalence for divisors on \( S \). \( |D| \) will be the complete linear system of the effective divisors \( D' \equiv D \), and \( \phi_D : S \to \mathbb{P}(H^0(S, O_S(D)^\vee)) = |D|^\vee \) the natural rational map defined by \( |D| \).

We will denote by \( \Sigma_d \) the rational ruled surface \( \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(d)) \), for \( d \geq 0 \). \( \Delta_\infty \) will denote the section of \( \Sigma_d \) with minimum self-intersection \( -d \) and \( \Gamma \) will be a fibre of the projection to \( \mathbb{P}^1 \).

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1. Auxiliary results.

In this section various results that will be used in the sequel are listed.

(1.1) (Xiao [X], Reider [R]) Let \( S \) be a minimal surface of general type with \( p_g = 0 \). If \( K^2 \geq 2 \), \( \phi_{2K} \) is generically finite and if \( K^2 \geq 5 \), \( \phi_{2K} \) is a morphism.

(1.2) (Catanese [Ca], Reider [R], Bombieri-Catanese [BC]) Let \( X \) be the canonical model of a minimal surface \( S \) of general type with \( p_g = 0 \). If \( K^2_S \geq 3 \), \( \phi_{3K_X} \) is an embedding and if \( K^2_S = 2 \), \( \phi_{3K_X} \) is a birational morphism.

(1.3) (Xiao [X], p.37) Let \( S \) be a minimal surface of general type with \( p_g = 0 \). If \( S \) contains a genus 2 pencil, then \( K^2 \leq 2 \).

(1.4) (Ciliberto, [C], th. (3.5)) The canonical ring of a minimal surface of general type with \( p_g = q = 0 \) is generated by its elements of degree \( \leq 6 \).

(1.5) (Ciliberto, [C], theorem (1.11), observation (1.3) ) Let \( |D| \) be a complete, base point free linear series on a smooth irreducible curve \( C \). If \( |D| \) is not composed with a rational involution, or if \( \dim|D| = 1 \), then the natural map \( H^0(C, \mathcal{O}_C(D)) \otimes H^0(C, \omega_C) \to H^0(C, \mathcal{O}_C(D) \otimes \omega_C) \) is surjective.

(1.6) (De Franchis, [DF], see also Catanese, Ciliberto, [CC]) Let \( X, Y \) be two smooth surfaces and \( f : X \to Y \) be a finite double cover. If \( p_g(Y) = q(Y) = 0 \) and \( q(Y) \geq 0 \), then the Albanese image of \( X \) is a curve \( C \), the involution determined by \( f \) acts as \((-1)\) on \( \text{Alb}(X) \) and there is a morphism \( h : Y \to \mathbb{P}^1 \) such that \( f : X \to Y \) is the fiber product of \( p : C \to \mathbb{P}^1, \mathbb{P}^1 \) and \( h \).

(1.7) (Arakelov, see [Be], pg. 343) Let \( S \) be a minimal surface of general type and let \( f : S \to B \) be a genus \( b \) pencil of curves of genus \( g \). Then \( K^2 \geq 8(g-1)(b-1) \).

(1.8) (Ciliberto, Francia, Mendes Lopes, [CFM]) Let \( S \) be an algebraic surface and \( \mu \in \text{Pic}^0(S) \). If \( C \) is a curve on \( S \) such that \( C^2 > 0 \), then \( \mu|_C \) is non-trivial.

2. Degree of the bicanonical map

(2.1) **Theorem.** Let \( S \) be a minimal surface of general type with \( p_g = 0 \), \( K^2 \geq 3 \), such that \( \phi_{2K} \) is a morphism. Then degree \( \phi_{2K} \leq 4 \) and \( X := \text{Im} \phi_{2K} \) is a surface of degree bigger or equal to \( n \) in \( \mathbb{P}^n \).

**Proof.** By (1.1) \( \phi_{2K} \) is generically finite. Let \( d \), \( m \) be respectively the degrees of \( \phi_{2K} \) and \( X \). Since, by assumption, \( |2K| \) is basepoint free, one has \( d \cdot m = (2K)^2 = 4K^2 \).

Because \( p_2 = K^2 + 1 \) and \( X \) is not contained in any hyperplane of \( \mathbb{P}^{K^2} \), \( \deg X \geq K^2 - 1 \). An easy calculation yields then that \( d > 4 \) can hold only in the following two cases:

(a) \( K^2 = 5 \) and \( d = 5 \) or
(b) \( K^2 = 3 \) and \( d = 6 \).

Notice that in these cases (and exactly in these) we have \( \deg X = K^2 - 1 \). Now to prove the theorem we are going to show that neither case can occur.

(a) Suppose that \( K^2 = 5 \) and \( d = 5 \). It is well known that a surface of degree 4 in \( \mathbb{P}^5 \) is one of the following surfaces:
(i) $\Sigma_0$ embedded by $|\Delta_\infty + 2\Gamma|$ or $\Sigma_2$ embedded by $|\Delta_\infty + 3\Gamma|$;
(ii) $\Sigma_4$ mapped by $|\Delta_\infty + 4\Gamma|$ (i.e. a cone over the rational curve of degree 4);
(iii) the Veronese surface.

Since $X$ is the bicanonical image of $S$, $X$ must be the Veronese surface. In fact in both cases (i) and (ii), the pull-back of the image of $|\Gamma|$ would give a pencil $D$ on $S$ such that $2K \cdot D = 5$, an odd number, which is impossible. Hence if degree $\phi_{2K} = 5$, $X$ is the Veronese surface. Denote by $L$ the image on $X$ of a line of $P^2$ and let $F$ be the inverse image of $L$ on $S$. Notice that, because $h^0(S, \mathcal{O}_S(F)) \geq 3$ and $\chi(\mathcal{O}_S) = 1$, $h^1(S, \mathcal{O}_S(F)) \geq 2$. Since $2F$ is linearly equivalent to $2K$, $\eta := K - F$ is a 2-torsion element of $Pic(S)$.

Let $g : \tilde{S} \to S$ be the étale double cover associated to $\eta$. One has $K_S^2 = 10$, $\chi(\mathcal{O}_S) = 2$ and $q(\tilde{S}) = q(S) + h^1(S, \mathcal{O}_S(F)) \geq 2$. Therefore we can apply (1.6) to the double cover $g : \tilde{S} \to S$, obtaining the existence of a fibration $f : \tilde{S} \to B$, with $B$ a smooth curve of genus $b \geq 2$. Because $K_S^2 = 10$, by (1.7) the genus of the general fibre of $f$ must be 2. This implies in turn that also $\tilde{S}$ has a fibration in genus 2 curves, contradicting (1.3). Therefore the case $K^2 = 5$, $d = 5$ does not occur.

(b) If $K^2 = 3$ and $\deg \phi_{2K} = 6$, the image of $\phi_{2K}$ is necessarily the quadric cone in $P^3$. Otherwise the image of $\phi_{2K}$ would be the non-singular quadric in $P^3$ and thus $2K \equiv D_1 + D_2$, with $K \cdot D_i = 3$, which is not congruent to $D_i^2 = 0$ (mod 2). So the image of $\phi_{2K}$ is a quadric cone and therefore we have $2K \equiv 2D + G$, where $|D|$ is a rational pencil without fixed part, satisfying $K \cdot D = 3$ and $G$ is an effective divisor, possibly 0, such that $K \cdot G = 0$. Notice that, if $G \neq 0$, every irreducible component $\theta$ of $G$ is a curve such that $\theta^2 = -2$, $K \cdot \theta = 0$. We can write $G$ uniquely as $G = 2G' + \Gamma$, where either $\Gamma = 0$ or $\Gamma = \theta_1 + \ldots + \theta_r$, with $\theta_1, \ldots, \theta_r$ distinct irreducible curves. If $\Gamma \neq 0$, following the same argument as in [CDe], prop.1.5, one can show that the curves $\theta_1, \ldots, \theta_r$ are disjoint, and thus the curve $\Gamma := \theta_1 + \ldots + \theta_r$ is smooth.

In either case we have $\Gamma \equiv 2\gamma$, where $\gamma \equiv K - D - G'$ and so we can consider the smooth double cover $\pi : S' \to S$, branched over $\Gamma$ and determined by $\gamma$. By the double cover formulas one has (with $r=0$, if $\Gamma = 0$)

\[
\begin{align*}
\chi(S') &= 2\chi(S) + \frac{1}{2}\gamma \cdot (K_S + \gamma) = 2 - \frac{r}{4} ; \\
K_{S'}^2 &= 2(K_S + \gamma)^2 = 2(3 - \frac{1}{2}r) = 6 - r ; \\
p_g(S') &= h^0(S, \mathcal{O}_S(K_S + \gamma)) + h^0(S, \mathcal{O}_S(K_S)) = h^0(S, \mathcal{O}_S(D + G' + \Gamma)) + 0 \geq 2.
\end{align*}
\]

Suppose that $\Gamma \neq 0$. Because $\chi(S') \geq 1$, we must have $r = 4$, yielding $\chi(S') = 1$ and $q(S') \geq 2$. In turn, by (1.6), this implies that the Albanese image of $S'$ is a curve of genus $b = q(S')$. But then we have a contradiction to the inequality (1.7). In fact $S'$ contains exactly four exceptional divisors, lying over the curves $Z_i$ and thus the minimal model $\tilde{S}$ of $S'$ has $K_{\tilde{S}}^2 = 6 < 8(q - 1)$.

Therefore $\Gamma = 0$, $\gamma$ is a 2-torsion divisor and $\pi$ is étale. $S'$ is thus a minimal surface with $\chi(S') = 2$ and $K_{S'}^2 = 6$. Since $p_g \geq 2$, $S'$ is irregular and in fact $q = 1$ ( $q \geq 2$ leads to the same contradiction as in the previous paragraph).

Let us consider now the Albanese fibration of $S'$, $\alpha : S' \to E = Alb(S')$. By (1.6), the fixed point free involution $\iota$ induced by $\pi : S' \to S$ acts as $(-1)$ on $E$ and therefore there are 4 fibres of the Albanese pencil which are stable under $\iota$. Since $\pi$ is étale, this implies that the pencil induced on $S$ by the Albanese pencil has 4 double fibres $2F_1, ..., 2F_4$. The existence of these 4 double fibres implies in turn the existence in $Pic(S)$ of seven distinct non-zero
2-torsion divisors, namely \( \eta_{ij} = F_i - F_j \), \( i, j \in \{1, 2, 3, 4\}, i < j \) and \( \eta = F_1 + F_2 - F_3 - F_4 \). Notice that, given a non-trivial 2-torsion divisor \( \mu \) on \( S \), by the Riemann-Roch theorem one has always \( h^0(S, \mathcal{O}_S(K + \mu)) \geq 1 \).

Consider now the pencil \(|D|\). Since \( K \cdot D = 3 \), \( D^2 \) is an odd number bigger than 0 and either \( G \neq 0 \), \( D^2 = 1 \) and \( g(D) = 3 \), or \( G = 0 \), \( D^2 = 3 \) and \( g(D) = 4 \). In either case a general curve \( D' \) in \(|D|\) is smooth. Furthermore, by (1.8), the seven linear systems \(|K + \eta_{ij}|\), \( i, j \in \{1, 2, 3, 4\}, i < j \) and \(|K + \eta|\) cut on \( D' \) seven distinct effective divisors \( N_{ij} \) and \( N \) of degree 3.

Consider \( \text{Im} \ r : H^0(S, \mathcal{O}_S(2K)) \to H^0(D', \mathcal{O}_{D'}(2K)) \). Since \( \phi_{2K}(D') \) is a line and \(|2K|\) is basepoint free, \( \text{Im} \ r \) is a \( g^6_3 \) without base points. Now \( 2N_{ij} \), for \( i, j \in \{1, 2, 3, 4\}, i < j \) and \( 2N \) belong to \( \text{Im} \ r \) and each of these divisors gives a contribution of at least 3 for the degree of the ramification divisor \( R \) of the 6:1 morphism \( D' \to \mathbb{P}^1 \). Therefore \( \deg R \geq 21 \), and so by the Hurwitz formula one has \( 2g(D') - 2 \geq 6(-2) - 21 = 9 \), i.e. \( g(D') \geq 6 \), which contradicts \( g(D') \leq 4 \). Thus also the case \( K^2 = 3, d = 6 \) cannot occur. \( \diamond \)

(2.2) **Observations.**

1. If \( K^2 = 2 \) and \(|2K|\) is base-point free, then the degree of the bicanonical map is 8.

2. The degree of the bicanonical map of the Burniat surface with \( K^2 = 3 \) (cf. [P]) is 4 and therefore, at least for \( K^2 = 3 \), the bound established in theorem (2.1) is sharp.

3. The degree of the generators of the canonical ring

Let us fix some notation first. \( C \) will be a general element in \(|2K_S|\), whilst \( H_m := H^0(C, \mathcal{O}_C(mK_S)) \) and \( R_m := H^0(S, \mathcal{O}_S(mK_S)) \). We will denote by \( R \) the canonical ring of \( S \) and by \( H := \bigoplus_{m=1}^{\infty} H_m \).

Notice that, because \( q = 0 \), the restriction maps \( r_m : R_m \to H_m \) are surjective for every \( m \in \mathbb{N} \), and so the natural graded morphism \( r : R \to H \) induced by the restriction maps is also surjective. Furthermore \( \ker r \) is a principal ideal of \( R \), generated by one element of degree 2 (cf.[C], §2). By (1.4), \( R \) is generated by its elements of degree lesser or equal to 6. To lower this bound to 5 it will be enough to show that the map \( H_3 \otimes H_3 \to H_5 \) is surjective, and similarly to lower the bound to 4 it will be enough to show that the map \( H_2 \otimes H_3 \to H_5 \) is surjective (cf.[C], §2).

(3.1) **Proposition.** Let \( S \) be a minimal surface of general type with \( p_g = 0 \), \( K^2 \geq 2 \) such that \(|2K|\) has no fixed components. Then the canonical ring of \( S \) is generated by its elements of degree \( \leq 5 \).

**Proof.** By (1.1) \( \text{Im} \phi_{2K} \) is a surface and so by our hypothesis the general curve \( C \in |2K| \) is irreducible. Since, by (1.2), \( \phi_{3K} \) is birational, \( C \) is not hyperelliptic and therefore the map

\[
H^0(C, \omega_C) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})
\]

is surjective. Since, by adjunction, \( \omega_C \simeq \mathcal{O}_C(3K_S) \), the map \( H_3 \otimes H_3 \to H_6 \) is surjective. Hence, by (1.4) and the previous considerations we have the assertion. \( \diamond \)

(3.2) **Proposition.** Let \( S \) be a minimal surface of general type with \( p_g = q = 0 \). The canonical ring of \( S \) is generated by its elements of degree \( \leq 4 \) if \( S \) satisfies one of the following conditions:
i) \( K^2 \geq 5 \); or

ii) \(|2K|\) is base point free and \( K^2 \geq 2 \).

**Proof.** By (1.4) and proposition (3.1), it suffices to show that the map \( H_2 \otimes H_3 \to H_5 \) is surjective. Let \( C \) be a general bicanonical curve. Since \(|2K|\) is base point free (in case (i) by (1.1)), the linear series \(|\mathcal{O}_C(2K_S)|\) is base point free.

If \( K^2 \geq 3 \), by theorem (2.1), \( \text{Im}\phi_{2K} \) is not a surface of minimal degree \( n - 1 \) in \( \mathbb{P}^n \). Therefore \(|\mathcal{O}_C(2K_S)|\) is not composed with a rational involution and so, by (1.5), we have the surjectivity of the above map.

If \( K^2 = 2 \), one has \( h^0(S, \mathcal{O}_S(2K)) = 3 \) and thus \( \dim|\mathcal{O}_C(2K_S)| = 1 \), implying, again by (1.5), that the map \( H_2 \otimes H_3 \to H_5 \) is surjective. 

(3.3) **Observation.**

To my knowledge, all the known examples of minimal surfaces of general type with \( p_g = q = 0 \) and \( K^2 = 2, 3, 4 \) have bicanonical system free from base points.

**References.**

[BC] E. Bombieri and F. Catanese, *The tricanonical map of a surface with \( K^2 = 2, p_g = 0 \)*, C.P. Ramanujam - A tribute, Bombay, 1978.

[Be] A. Beauville, *L’inégalité \( p_g \geq 2q - 4 \) pour les surfaces de type générale*, Bull. Soc. Math. de France, 110 (1982), 343-346

[C] C. Ciliberto, *Sul grado dei generatori dell’anello canonico di una superficie di tipo generale*, Rend. Sem. Mat. Univers. Politecn. Torino, 41, 3 (1983), 83-111

[Ca] F. Catanese, *Footnotes to a theorem of I. Reider*, Algebraic Geometry, Lecture Notes in Math, 1417, Springer, 1990.

[CC] F. Catanese, C. Ciliberto, *On the irregularity of cyclic coverings of algebraic surfaces*, in Proceedings of the conference ”Geometry of complex projective varieties”, Cetraro, 1990, Mediterranean Press.

[CDe] F. Catanese, O.Debarre, *Surfaces with \( K^2 = 2, p_g = 1, q = 0 \)*, J.Reine angew. Math, 395 (1989), 1-55

[CFM] C. Ciliberto, P.Francia, M. Mendes Lopes, *Remarks on the bicanonical map for surfaces of general type*, to be published in Math.Zeit.

[DF] M. De Franchis, *Sugl’integrali di Picard relativi a una superficie doppia*, Rend. Circ. Mat. Palermo XX (1905), 331-334

[P] C.A.M. Peters, *On certain examples of surfaces with \( p_g = 0 \) due to Burniat*, Nagoya Math. J., 66 (1977), 109-119

[R] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. 127 (1988), 309-316

[X] G. Xiao, *Finitude de l’application canonique des surfaces de type général*, Boll. Soc. Math. France, 113 (1985), 23-51

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