On stationary metrics in five dimensions

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ABSTRACT: It is well-known that the Kerr-metric (rotating black hole in four dimensions) has Petrov type D. We prove a similar property in five dimensions. The Myers-Perry metric (rotating black hole in five dimensions) with one non-zero angular momentum has Petrov type $D_2$. Conversely, we show that the Myers-Perry solution is unique within a certain restricted class of metrics of Petrov type $D_2$.

KEYWORDS: Classical Theories of Gravity, Black Holes.
1. Introduction

The metric of a four-dimensional rotating mass is described by the Kerr-metric. This metric is algebraically special: it has Petrov type D. As far as we know, there is no easy mathematical or physical reason for this surprising result. The Schwarzschild metric also has Petrov type D. However, this already follows from the isometries of the metric, without using the detailed form of it. A rotating black hole has very few isometries and it has taken physicists a long time to find its metric. Kerr found it by looking within the class of metrics of Petrov type D; he was lucky to find it there.

In this paper we point out that something similar holds in five dimensions. The five-dimensional Schwarzschild metric has Petrov type $22$ (see ref [1] for a discussion of the five-dimensional Petrov classification). This is not surprising – it follows again from the isometries. Things are more interesting for rotating black holes. In Section 2, we prove that the Myers-Perry solution [2] with one non-zero angular momentum has Petrov type $22$ as well. In Section 3, we prove a converse statement: the Myers-Perry solution is the only five-dimensional metric satisfying the following conditions.

1. The metric has 3 commuting Killing vectors and is invariant under 2 discrete $\mathbb{Z}_2$ transformations.

2. The metric has Petrov type $22$.

3. The metric is asymptotically flat at infinity.

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2. The Petrov type of the Myers-Perry solution

In this section, we derive the Petrov type of the Myers-Perry metric [2]

\[
\begin{align*}
 ds^2 &= -\frac{R^2}{\rho^2} \left( dt - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\rho^2} \left( (r^2 + a^2) d\phi - adt \right)^2 + \\
 &\quad \frac{\rho^2}{R^2} dr^2 + \rho^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2,
\end{align*}
\]

with \( \rho^2 = r^2 + a^2 \cos^2 \theta \) and \( R^2 = r^2 - 2M + a^2 \). The Petrov classification in five dimensions [1] is basically an invariant classification of the Weyl spinor

\[
\psi_{abcd} = (\gamma^{ij})_{ab}(\gamma^{kl})_{cd}C_{ijkl},
\]

where \( C_{ijkl} \) is the Weyl tensor and \( \gamma_i \) are the five-dimensional gamma matrices\(^1\). We choose the following dual tetrad

\[
\begin{align*}
 e^t &= \frac{R}{\rho} \left( dt - a \sin^2 \theta d\phi \right), & e^\phi &= \frac{\sin \theta}{\rho} \left( (r^2 + a^2) d\phi - adt \right), \\
 e^r &= \frac{\rho}{R} dr, & e^\theta &= \rho d\theta, & e^\psi &= r \cos \theta d\psi.
\end{align*}
\]

A calculation then gives the Weyl polynomial

\[
\Psi = \psi_{abcd}x^a x^b x^c x^d = \frac{48M}{\rho^2} \left( \frac{(x^1)^2}{z} - \frac{(x^2)^2}{\bar{z}} - \frac{(x^3)^2}{z} + \frac{(x^4)^2}{\bar{z}} \right)^2,
\]

with \( z = r + ia \cos \theta \). This polynomial is the square of a polynomial of degree 2, hence, the Myers-Perry solution has Petrov type 22 in the notation of ref. [1].

3. The Myers-Perry solution is unique

We now prove that the Myers-Perry solution is unique within a certain restricted class of metrics of Petrov type 22. The outline of the proof is as follows. We make ansätze for the tetrad, the connection \( \omega \) and the Weyl spinor \( \Psi \) and solve the equations

\[
\begin{align*}
 \Omega &= d\omega + \omega \wedge \omega, \\
 d\Omega + [\omega, \Omega] &= 0.
\end{align*}
\]

In these equations, the curvature \( \Omega \) is completely determined by the Weyl spinor \( \Psi \) because the Ricci tensor vanishes. The condition on the Petrov type simplifies the Weyl spinor \( \Psi \) drastically. This makes it possible to find all solutions. The ansätze can be found in section 3.1, the simplification of the Weyl spinor is discussed in section 3.2 and the solutions of the field equations are given in section 3.3. It can be seen that only the Myers-Perry solution is flat at infinity, which proves its uniqueness. Although the Petrov condition simplifies the field equations very much, the derivation of the solutions is still rather tedious. Some details can be found in appendix A.

\(^1\)We use the following convention for the gamma matrices: \( \gamma_1 = i \sigma_1 \otimes 1, \gamma_2 = \sigma_2 \otimes 1, \gamma_3 = \sigma_3 \otimes \sigma_1, \gamma_4 = \sigma_3 \otimes \sigma_2, \gamma_5 = \sigma_3 \otimes \sigma_3 \).
3.1 The ansätze

Our coordinates are $t, \phi, r, \theta, \psi$. We assume that the metric has 3 commuting Killing vectors which we choose as $\partial_t$, $\partial_\phi$ and $\partial_\psi$. In addition, we assume that the metric is invariant under two $\mathbb{Z}_2$ transformations:

- $(t, \phi) \rightarrow (-t, -\phi)$, this reflects that the metric is invariant if the rotation reverses and time flows backwards
- $\psi \rightarrow -\psi$.

The most general ansatz for the tetrad is then as follows. The vectors $e_t$ and $e_\phi$ are linear combinations of $\partial_t$ and $\partial_\phi$, the vectors $e_r$ and $e_\theta$ are linear combinations of $\partial_r$ and $\partial_\theta$ and $e_\psi$ is proportional to $\partial_\psi$. The coefficients in these linear combinations depend only on the variables $r$ and $\theta$. The commutators then read

$$
\begin{align*}
[e_t, e_r] &= \mu e_t + \alpha e_\phi \\
[e_t, e_\theta] &= \nu e_t + \beta e_\phi \\
[e_\phi, e_r] &= \kappa e_t + \zeta e_\phi \\
[e_\phi, e_\theta] &= \lambda e_t + \eta e_\phi \\
[e_r, e_\phi] &= \sigma e_\psi \\
[e_r, e_\theta] &= \delta e_\psi + \varepsilon e_\theta \\
[e_\theta, e_\phi] &= \tau e_\psi
\end{align*}
$$

(3.3)

It can be shown that the most general ansatz for the Weyl spinor consistent with the above tetrad, is

$$
\begin{align*}
\psi_{1111} &= \bar{\psi}_{3333} = (T_{11} - T_{21}) + i(T_{12} - T_{22}) \\
\psi_{1122} &= \bar{\psi}_{3344} = W_1 + iW_2 \\
\psi_{1133} &= V_1 - V_2 \\
\psi_{1144} &= \bar{\psi}_{2233} = I_2 + iI_3 \\
\psi_{2222} &= \bar{\psi}_{4444} = (T_{11} + T_{21}) + i(T_{12} + T_{22}) \\
\psi_{2244} &= V_1 + V_2 \\
\psi_{1234} &= \frac{1}{2}I_1
\end{align*}
$$

(3.4)

The ansatz for the tetrad breaks the local Lorentz group $O(1, 4)$ to $O(1, 1)_{t\phi} \times O(2)_{r\theta}$. Under this group the ansatz for the Weyl spinor transforms as

$$\begin{align*}
(2, 2) \oplus (2, 1) \oplus (1, 2) \oplus 2 \cdot (1, 1) \oplus (1', 1')
\end{align*}$$

3.2 The condition on the Petrov type

The Weyl polynomial associated with the Weyl spinor (3.4) is $\Psi = \psi_{abcd} x^a x^b x^c x^d$. It can be written as a square of a polynomial of degree 2 if and only if one of the following four cases holds.
Case A: \( I_1 = 0, W_1^2 + W_2^2 = V_1^2 - V_2^2 = I_2^2 + I_3^2 \) and an extra condition on \( T_{ij} \):
- If \( V_i \) is light-like, this extra condition is \( T_{ij} = 0 \).
- If \( V_i \) is time-like, it is difficult to write this extra condition in a manifestly \( O(1, 1) \times O(2) \) invariant way. If we choose the particular frame \( W_2 = V_2 = 0 \) and \( W_2 = V_1 \), it reads \( T_{11} = 3I_2, T_{22} = -3I_3 \) and \( T_{12} = T_{21} = 0 \).

Case B: \( I_2 = I_3 = T_{ij} = V_i = 0 \) and \( W_1^2 + W_2^2 = I_1^2 \).

Case C: \( I_2 = I_3 = T_{ij} = W_i = 0 \) and \( V_1^2 - V_2^2 = I_1^2 \).

Case D: \( T_{ij} = W_i = V_i = 0 \) and \( I_2^2 + I_3^2 = I_1^2 \).

### 3.3 Solutions of the field equations

The classification contains the following metrics:

1. The Kasner metric with exponents \((-1/2, 1/2, 1/2, 1/2)\)

   \[
   ds^2 = -dt^2 + \frac{1}{t}dx^2 + t(dy^2 + dz^2 + du^2) \tag{3.5}
   \]

   This metric is not asymptotically flat.

2. \( \frac{1}{r^2} \left( dt + 2af(\theta)d\phi \right)^2 + r^2f'(\theta)^2d\phi^2 + \left( \frac{b}{r^2} - \frac{a^2}{r^4} \right)^{-1}dr^2 + r^2d\theta^2 + (br^2 - a^2)d\psi^2, \) \tag{3.6}

   with \( f(\theta) \) a polynomial of degree 2. If one does a Wick transformation \( r \rightarrow ir \), the metric can be interpreted as a five-dimensional spatially homogeneous cosmological model \(^2\). The isometry algebra is \( A_{4,10} \oplus \mathbb{R} \) in the notation of ref. [4]. The metric reduces to the Kasner metric if \( a = 0 \). This metric is not asymptotically flat.

3. \( \frac{R^2}{\rho^2} \left( dt - a \sin^2 \theta d\phi \right)^2 + \frac{k_2 - k_1 \cos^2 \theta}{\rho^2} \left( (r^2 + a^2) d\phi - adt \right)^2 + \frac{\rho^2}{R^2} dr^2 + \frac{\sin^2 \theta}{k_2 - k_1 \cos^2 \theta} \rho^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2, \) \tag{3.7}

   where \( \rho^2 = r^2 + a^2 \cos^2 \theta \) and \( R^2 = k_1 r^2 - 2M + k_2 a^2 \). This metric has Lorentz signature only if \( k_2 \geq k_1 \geq 0 \). If \( k_1 \neq 0 \), the metric is actually the Myers-Perry solution (2.1). If \( k_1 = 0 \), it is not asymptotically flat.

Hence, we see that the Myers-Perry solution is the unique metric that (i) has three commuting Killing vectors \( \partial_t, \partial_\phi \) and \( \partial_\psi \), (ii) is invariant under \( (t, \phi) \rightarrow (-t, -\phi) \) and \( \psi \rightarrow -\psi \), (iii) is asymptotically flat and (iv) has Petrov type 22.

\(^2\)I would like to thank S. Hervik for pointing this out.
4. Conclusions

The well-known “no-hair” theorems of four-dimensional gravity do not hold in five dimensions for the general stationary case. For example, in [5] a rotating ring was found with the same mass and angular momentum as the Myers-Perry solution, but with a different horizon topology. In the static case on the other hand, one can prove that asymptotically flat solutions are unique [6, 7]. See also ref. [8] for a speculation that uniqueness properties hold in higher dimensions if one (i) specifies the horizon topology and (ii) only considers stable solutions. The uniqueness theorems can also be saved by adding supersymmetry: in ref. [9] it is proven that the BMPV solution [10] is the only supersymmetric black hole of minimal $N = 1$, $D = 5$ supergravity. It is nice that the Myers-Perry solution is unique within the class of metrics of Petrov type $22$. This result could shed some light on the structure of solutions of five-dimensional gravity.

It would be interesting to generalize the above analysis. Some possibilities are the following.

- We have restricted our study of the Myers-Perry solution by putting one of the angular momenta to zero. The above analysis can probably be generalized to the case with two angular momenta.

- Preliminary investigations indicate that the rotating ring found by Reall [5] has Petrov type $22$. The generalization of this rotating ring to two non-zero angular momenta is not known. It is possible that it can be found in the class of metrics of Petrov type $22$.

- The charged generalization of the Myers-Perry solutions is not known. The following reasons suggest that it might be of Petrov type $22$.
  
  - As shown in this paper, the uncharged limit has Petrov type $22$.
  
  - The non-rotating limit (five-dimensional Reissner-Nordstrøm metric) has Petrov type $22$ as well.
  
  - The charged rotating black hole in four dimensions (the Kerr-Newmann metric) has the same Petrov type as the uncharged rotating black hole, i.e. Petrov type $D$.

It seems worthwhile to look for the charged generalization within the class of metrics of Petrov type $22$.

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A. More details

In this appendix, we solve the Einstein equations with the ansätze (3.3) and (3.4). As discussed in section 3.2, the Petrov condition leads to four different cases. These are discussed separately.

Case A

We only discuss the case where \(V_i\) is time-like. \(V_i\) light-like is a special case of Case C, which is discussed below. We assume \(V_1 \neq 0\), otherwise, it follows from (3.2) that the space is flat. From the Bianchi equations (3.2), we obtain

\[
\begin{align*}
\tau I_3 &= aV_1 \\
\tau I_2 &= -(2\nu + \tau)V_1 \\
\sigma I_3 &= \lambda V_1 \\
\sigma I_2 &= (2\zeta + \sigma)V_1
\end{align*}
\]

Combining the above equations with the Petrov condition (3.2) leads to

\[
\begin{align*}
\nu \sigma + \tau \sigma + \tau \zeta &= 0 \quad (A.1) \\
\alpha^2 + 4\nu^2 + 4\nu\tau &= 0 \quad (A.2) \\
\lambda^2 + 4\zeta^2 + 4\zeta\sigma &= 0 \quad (A.3)
\end{align*}
\]

Furthermore, combining (A.1) with (3.1) gives

\[
\lambda \alpha = 4\zeta \nu \quad (A.4)
\]

At this point, we want to choose coordinates such that \(e_r = A\zeta \partial_r\) and \(e_\theta = B\nu \partial_\theta\), where \(A\) and \(B\) are two functions. However, we can only make this choice if \(\zeta\) and \(\nu\) are both non-zero. Therefore, we have to treat the cases \(\zeta = 0\) or \(\nu = 0\) separately. This is a bit unfortunate because it will turn out that the solutions in the latter cases can be obtained as limits of solutions in the case \(\zeta \neq 0 \neq \nu\). Anyway, we break up case A into 5 subcases.

Case A.1: \(\nu \neq 0\) and \(\zeta \neq 0\)

From (3.1) and (A.4), one obtains \(e_r = -3\zeta \nu\) and \(e_\theta = -3\zeta \nu\). Using these equations, one can show that there are coordinates such that

\[
e_r = \frac{\rho^2}{r} \zeta \partial_r \quad \text{and} \quad e_\theta = -\frac{\rho^2}{a^2 \sin \theta \cos \theta} \nu \partial_\theta,
\]

where \(\rho^2 = r^2 + a^2 \cos^2 \theta\). The solution of (3.1) and (3.2) is the metric (3.7).

Case A.2: \(\nu = 0\) and \(\tau = 0\)

From the Bianchi identities, one obtains \(\alpha = 0\) and \(\mu = -\zeta\). The resulting field equations are then easily solved. The solution is the metric (3.6).
Case A.3: $\nu = 0$ and $\tau \neq 0$
From the Bianchi identities, one obtains $\lambda = \alpha = 0$, $I_3 = 0$, $I_2 = -V_1$ and $\zeta = -\sigma$. The solutions in this case are the Schwarzschild metric and the Kasner metric (3.5).

Case A.4: $\zeta = 0$ and $\sigma = 0$
This case leads again to the metric (3.6).

Case A.5: $\zeta = 0$ and $\sigma \neq 0$
This case leads again to the Schwarzschild metric or the Kasner metric.

Case B
We can choose $W_2 = 0$ and $W_1 = I_1$ by an $O(2)_{r\theta}$ transformation. Furthermore, we can use an $O(1,1)$ boost to put $\alpha = \beta = 0$. The Bianchi equations yield $\kappa = 0$, $\varepsilon = -\zeta$, $\mu = \zeta$, $\delta = 0$ and $\tau = 0$. Using the commutation relations (3.3), we can choose coordinates in which the metric has the form

$$ds^2 = A(r)d\psi^2 + B(r)dr^2 + r^2d\bar{s}^2(t,\theta,\phi).$$

From Einstein’s equations, it directly follows that the three-dimensional metric $d\bar{s}^2$ is Einstein, hence has constant curvature. If $d\bar{s}^2$ is Euclidean, we obtain the Kasner metric (3.5). If $d\bar{s}^2$ is a sphere, we obtain the Schwarzschild metric.

Case C
If $I_1 = 0$, we can choose $V_1 = V_2$ by an $O(1,1)_{t\phi}$ transformation. If $I_1 \neq 0$, we make $V_2 = 0$ and $V_1 = I_1$ by a boost. In both cases, Einstein’s equations lead to flat space.

Case D
We put $\alpha = \beta = \nu = 0$ by an $O(1,1)_{t\phi} \times O(2)_{r\theta}$ transformation. From the Bianchi identities, one can rather easily see that there are only solutions if $I_3 = 0$ and $I_2 = I_1$. The Bianchi identities then give $\eta = \kappa = \lambda = 0$ and $\zeta = \mu = -\sigma$. The solution is the Kasner metric (3.5).

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