On the Scalar Manifold of Exceptional Supergravity

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Abstract

We construct two parametrizations of the non compact exceptional Lie group $G = E_7(-25)$, based on a fibration which has the maximal compact subgroup $K = E_6 \times U(1)/Z_3$ as a fiber. It is well known that $G$ plays an important role in the $\mathcal{N} = 2$ $d = 4$ magic exceptional supergravity, where it describes the U-duality of the theory and where the symmetric space $\mathcal{M} = G/K$ gives the vector multiplets’ scalar manifold.

First, by making use of the exponential map, we compute a realization of $G/K$, that is based on the $E_6$ invariant $d$-tensor, and hence exhibits the maximal possible manifest $[(E_6 \times U(1))/Z_3]$-covariance. This provides a basis for the corresponding supergravity theory, which is the analogue of the Calabi-Vesentini coordinates.

Then we study the Iwasawa decomposition. Its main feature is that it is $SO(8)$-covariant and therefore it highlights the role of triality. Along the way we analyze the relevant chain of maximal embeddings which leads to $SO(8)$.

It is worth noticing that being based on the properties of a “mixed” Freudenthal-Tits magic square, the whole procedure can be generalized to a broader class of groups of type $E_7$.

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1 The “mixed” magic square and the 56 of the Lie algebra \( \mathfrak{e}_7(-25) \)

Exceptional Lie groups act as symmetries in many physical systems. In particular, non compact forms of the group \( E_7 \) enter as U-duality of \( d = 3 \) and \( d = 4 \) supergravity theories. Here we focus on the \( \mathcal{N} = 2 \) \( d = 4 \) magic exceptional supergravity, where the relevant real form is \( G = E_7(-25) \).

As the first step we need to construct the Lie algebra \( \mathfrak{e}_7(-25) \). To this aim, we are going to follow the technique outlined in Sec. 7 of [1], which is based on the non-symmetric “mixed” magic square [2][3][4]:

| Table 1: The “mixed” magic square |
|----------------------------------|
| \( \mathbb{R} \) | \( \mathbb{C} \) | \( \mathbb{H} \) | \( \mathbb{O} \) |
| \( SO(3) \) | \( SU(3) \) | \( USp(6) \) | \( F_{4(-52)} \) |
| \( SU(3) \) | \( SU(3) \oplus SU(3) \) | \( SU(6) \) | \( E_{6(-78)} \) |
| \( Sp(6, \mathbb{R}) \) | \( SU(3, 3) \) | \( SO^*(12) \) | \( E_{7(-25)} \) |
| \( F_{4(4)} \) | \( E_{6(2)} \) | \( E_{7(-5)} \) | \( E_{8(-24)} \) |

The rows and columns contain the division algebras of the real numbers \( \mathbb{R} \), the complex numbers \( \mathbb{C} \), the quaternions \( \mathbb{H} \), the octonions \( \mathbb{O} \) and their split forms \( \mathbb{H}_S \) and \( \mathbb{O}_S \).

Then the Tits formula gives the Lie algebra \( \mathcal{L} \) corresponding to row \( \mathbb{A} \) and column \( \mathbb{B} \) as [4]:

\[
\mathcal{L} (\mathbb{A}, \mathbb{B}) = \text{Der} (\mathbb{A}) \oplus \text{Der} (\mathfrak{J}_3 (\mathbb{B})) + (\mathbb{A} \times \mathfrak{J}_3' (\mathbb{B})).
\]

Here, the symbol \( \oplus \) denotes direct sum of algebras, whereas \( \times \) stands for direct sum of vector spaces. Furthermore, \( \text{Der} \) means the linear derivations, \( \mathfrak{J}_3 (\mathbb{B}) \) denotes the rank-3 Jordan algebra on \( \mathbb{B} \), and the priming amounts to considering only traceless elements. One of the main ingredients entering in the last term is the Lie product, which extends the multiplication to \( \mathbb{A} \times \mathfrak{J}_3' (\mathbb{B}) \). Its explicit expression for \( \mathbb{A} = \mathbb{H}_S \) and \( \mathbb{B} = \mathbb{O} \) can be found e.g. in [5].

For the Lie algebra of \( E_7(-25) \) the Tits formula [1] yields:

\[
\mathfrak{e}_{7(-25)} = \mathcal{L} (\mathbb{H}_S, \mathbb{O}) = \text{Der} (\mathbb{H}_S \oplus \text{Der} (\mathfrak{J}_3 (\mathbb{O}))) \oplus (\mathbb{H}_S \times \mathfrak{J}_3' (\mathbb{O})) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{f}_4 + (\mathbb{H}_S \times \mathfrak{J}_3' (\mathbb{O})).
\]

The second step is to identify the subalgebra \( \mathfrak{R} \) generating the maximal compact subgroup \( K := (E_{6(-78)} \times U(1))/\mathbb{Z}_3 \) of \( E_{7(-25)} \). This can be achieved by using the Tits formula [1] once more to compute the manifestly \( \mathfrak{f}_4 \)-covariant expression for \( \mathfrak{e}_{6(-78)} \):

\[
\mathfrak{e}_{6(-78)} = \mathcal{L} (\mathbb{C}, \mathbb{O}) = \mathcal{L} (\mathbb{R}, \mathbb{O}) + (\mathbb{I} \times \mathfrak{J}_3' (\mathbb{O})) = \text{Der} (\mathfrak{J}_3 (\mathbb{O})) + (\mathbb{I} \times \mathfrak{J}_3' (\mathbb{O})) = \mathfrak{f}_4 + (\mathbb{I} \times \mathfrak{J}_3' (\mathbb{O})),
\]

where we are picking the only imaginary unit \( i \in \mathbb{H}_S \) which satisfies \( i^2 = -1 \). Thus, we obtain:

\[
\mathfrak{R} = \mathfrak{ad}_i + \text{Der} (\mathfrak{J}_3 (\mathbb{O})) + (\mathbb{I} \times \mathfrak{J}_3' (\mathbb{O})),
\]

with \( \mathfrak{ad}_i \in \mathbb{H}_S \) the adjoint action of \( i \), generating the maximal compact subgroup \( U(1) \) of the group \( SL(2, \mathbb{R}) \) appearing in [2].

An explicit construction of the matrices \( \phi_I, I = 1, \ldots, 78 \), realizing the \( \mathfrak{e}_{6(-78)} \) subalgebra in its irreducible representation \( \text{Fund} = 27 \) has been performed e.g. in Sec. 2.1 of [5] by making use of [3] and of the explicit expression of the \( f_4(-52) \) in its irrep. \( \text{Fund} = 26 \) previously computed in [7].

Finally, by putting together all these algebraic objects, we find that an explicit symplectic realization of the Lie algebra \( \mathfrak{e}_{7(-25)} \) in its irreducible representation \( \text{Fund} = 56 \) is as follows [8].

The generators of the maximal compact subgroup \( K \) (antihermitian matrices):

\[
\mathfrak{e}_{6(-78)} : \quad Y_I = \begin{pmatrix}
\phi_I & 0 & 0 & 0 \\
\overrightarrow{\mathcal{T}} & 0 & 0 & 0 \\
0 & \overrightarrow{\mathcal{T}} & -\phi_I & 0 \\
0 & 0 & \mathcal{T} & 0 \\
\end{pmatrix}, \quad I = 1, \ldots, 78
\]
The generators of the coset $M = G/K$ (hermitian matrices):

$$u(1): \quad Y_{79} = \begin{pmatrix}
0_{27} & 0_{27} & 2iA_{\alpha} & i\sqrt{2} \bar{\varphi}_{\alpha} \\
-i\sqrt{2} & 0_{27} & 0 & 0 \\
i\sqrt{2} & 0_{27} & -i\sqrt{2} & 0 \\
0_{27} & 0_{27} & i\sqrt{2} & 0 \\
0_{27} & 0_{27} & 0 & i\sqrt{2} \\
0_{27} & 0_{27} & 0 & 0 \end{pmatrix};$$

(6)

The generators of the coset $M = G/K$ (hermitian matrices):

$$Y_{\alpha +79} = \frac{1}{2} \begin{pmatrix}
0_{27} & 0_{27} & 2iA_{\alpha} & i\sqrt{2} \bar{\varphi}_{\alpha} \\
0_{27} & 0_{27} & 0 & 0 \\
0_{27} & 0_{27} & 0 & 0 \\
0_{27} & 0_{27} & 0 & 0 \\
0_{27} & 0_{27} & 0 & 0 \\
0_{27} & 0_{27} & 0 & 0 \end{pmatrix}, \quad \alpha = 1, \ldots, 27;$$

(7)

$$Y_{\alpha +106} = \frac{1}{2} \begin{pmatrix}
0_{27} & 0_{27} & -2A_{\alpha} & \sqrt{2} \bar{\varphi}_{\alpha} \\
0_{27} & 0_{27} & 0 & 0 \\
0_{27} & 0_{27} & 0 & 0 \\
0_{27} & 0_{27} & 0 & 0 \\
0_{27} & 0_{27} & 0 & 0 \end{pmatrix}, \quad \alpha = 1, \ldots, 27.$$  

(8)

Here $I_n$ is the $n \times n$ identity matrix, $0_{27}$ is the $27 \times 27$ null matrix, $\mathbf{0}_n$ is the zero vector in $\mathbb{R}^n$, and $\bar{\varphi}_{\alpha}, \alpha = 1, \ldots, 27$, is the canonical basis of $\mathbb{R}^{27}$.

The matrices $A_{\alpha}$ are defined in terms of the $d$-tensor of the 27 of $E_6(-78)$. There is a cubic form, which is defined for any $j_1, j_2, j_3 \in \mathfrak{f} \mathfrak{d}(\mathbb{O})$ as $[9, 10, 11]$:

$$\text{Det}(j_1, j_2, j_3) := \frac{1}{3!} \text{Tr}(j_1 \circ j_2 \circ j_3) + \frac{1}{6!} \text{Tr}(j_1) \text{Tr}(j_2) \text{Tr}(j_3) - \frac{1}{6} \left( \text{Tr}(j_1) \text{Tr}(j_2 \circ j_3) + \text{cyclic perm.} \right);$$

(9)

where $\circ$ is the product in $\mathfrak{f} \mathfrak{d}(\mathbb{O})$. By choosing a basis $\{j_{\alpha}\}_{\alpha=1,\ldots,26}$ of $\mathfrak{f} \mathfrak{d}(\mathbb{O})$ normalized as $\langle j_{\alpha}, j_{\beta} \rangle := \text{Tr}(j_{\alpha} \circ j_{\beta}) = 2\delta_{\alpha\beta}$, a completion to a basis for $\mathfrak{f} \mathfrak{d}(\mathbb{O})$ can be obtained by adding $j_{27} = \sqrt{\frac{2}{3}} I_3$. Then the matrices $A_{\alpha}$'s are $27 \times 27$ symmetric matrices, whose components, explicitly computed in $[5]$, satisfy the following relation $[9]$

$$(A_{\alpha})^3 = \frac{3}{2} \text{Det}(j_{\alpha}, j_\gamma, j_\beta) =: \frac{1}{\sqrt{2}} d_{\alpha\gamma\beta},$$

(10)

where $d_{\alpha\gamma\beta} = d_{(\alpha\gamma\beta)}$ is the totally symmetric rank-3 invariant $d$-tensor of the 27 of of $E_6(-78)$, with a normalization suitable to match $\text{Det}(j_{\alpha}, j_\gamma, j_\beta)$ given by $[9]$. Whenever the choice of the basis $\{j_{\alpha}\}$ is exploited in order to distinguish the identity matrix from the traceless ones, the $d_{\alpha\beta\gamma}$ of $E_6$ has a maximal manifestly $F_4(-52)$-invariance only. However, it is crucial to point out that, being expressed only in terms of the invariant $d$-tensor, the result (10) does not depend on the particular choice of the basis $\{j_{\alpha}\}$. Thus, the expressions of $Y_{\alpha +79}$ (7) and of $Y_{\alpha +106}$ (8) exhibit the maximal manifest compact $[(E_6 \times U(1))/\mathbb{Z}_3]$-covariance.

A couple of remarks on the properties of the matrices $Y_{\alpha}$’s are in order. The first is that they satisfy:

$$Y_\alpha \in \text{usp}(28, 28), \quad A = 1, \ldots, 133.$$  

Moreover, in order to guarantee that the period of the maximal torus in the $E_6$ subgroup equals $4\pi$, the standard choice for the period of the spin representations of the orthogonal subgroups $[7, 6]$, the matrices
Y’s are orthonormalized as \( \langle Y, Y' \rangle_{56} := \frac{1}{12} \text{Tr}(Y Y') \) with signature \((-79, +54)\). As a consequence, the components \( (A_\alpha)_{\beta\gamma} \) are normalized as \( A_{\alpha\beta\gamma} A^{\alpha'\beta'\gamma'} = 5 \delta_{\alpha\alpha'} \).

This is consistent with the normalization of the \(d\)-tensor of \((E_6(-26))\) adopted e.g. in [12], which is dictated by the expression \( f(z) := \frac{1}{3!} \sum A^{\alpha\beta\gamma} z^\alpha z^\beta z^\gamma \) for the Kähler-invariant \((X^0)^2\)-rescaled holomorphic prepotential function characterizing special Kähler geometry (see e.g. [13][14][15], and Refs. therein).

2 Manifestly \([(E_6 \times U(1))/Z_3]\)-covariant Construction of the Coset \( \mathcal{M} \)

In this Section we construct a manifestly \([(E_6 \times U(1))/Z_3]\)-covariant parametrization of the symmetric space \( \mathcal{M} = \frac{E_7(-25)}{E_6(-78) \times U(1)} / Z_3 \).

As we have seen in the previous Sec. 1, it is generated by the matrices \( Y_{79+i}, (7) \) and \( I = 1, \ldots, 54 \). Through the exponential mapping, it can be defined as follows:

\[
\mathcal{M} := \exp \left( \sum_{\alpha=1}^{27} x_\alpha Y_{106+\alpha} + y_\alpha Y_{79+\alpha} \right), \quad \text{with } x_\alpha \in \mathbb{R}, \ y_\alpha \in \mathbb{R}, \text{ for } \alpha = 1, \ldots, 27. \tag{12}
\]

In order to make the complex structure of \( \mathcal{M} \) manifest, it is convenient to introduce the following complex linear combinations of the matrices:

\[
\zeta_\alpha := \frac{1}{\sqrt{2}} (Y_{79+\alpha} + i Y_{106+\alpha}), \quad \bar{\zeta}_\alpha := \frac{1}{\sqrt{2}} (Y_{79+\alpha} - i Y_{106+\alpha}) \tag{13}
\]

together with the corresponding complex linear combinations of the parameters:

\[
z_\alpha := \frac{1}{\sqrt{2}} (y_\alpha + i x_\alpha), \quad \bar{z}_\alpha := \frac{1}{\sqrt{2}} (y_\alpha - i x_\alpha), \tag{14}
\]

which allows to rewrite (12) as

\[
\mathcal{M} := \exp \left( \sum_{\alpha=1}^{27} \bar{z}_\alpha \zeta_\alpha + z_\alpha \bar{\zeta}_\alpha \right). \tag{15}
\]

By introducing the 27 dimensional complex vector \( z := \sum_{\alpha=1}^{27} z_\alpha \bar{e}_\alpha \), describing the scalar fields, and the 28 \times 28 matrix \( A := \begin{pmatrix} -\sqrt{2} \sum_{\alpha=1}^{27} \bar{z}_\alpha A_\alpha & z \\ z^T \end{pmatrix} \), the expression for \( \mathcal{M} \) (15) enjoys the simple form:

\[
\mathcal{M} := \exp \left( \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \right) = \begin{pmatrix} \text{Ch} (\sqrt{AA^\dagger}) & A \text{Sh} (\sqrt{AA^\dagger}A) \\ A^\dagger \text{Sh} (\sqrt{AA^\dagger}) & \text{Ch} (\sqrt{AA^\dagger}) \end{pmatrix}. \tag{16}
\]

This is a Hermitian matrix, of the same form as the finite coset representative worked out [16] for the split (i.e. maximally non-compact) counterpart \( \mathcal{M}_{N=8} = \frac{E_7(-2)}{SU(8)/Z_2} \), which is the scalar manifold of maximal \( \mathcal{N} = 8, D = 4 \) supergravity, associated to \( j_3 (\mathbb{O}_S) \). However, while \( \mathcal{M}_{N=8} \) is real, because of (11) \( \mathcal{M} \) is an element of \( USp(28, 28) \).
By using the machinery of special Kähler geometry (see e.g. [13][14][15], and Refs. therein), the symplectic sections defining the symplectic frame associated to the coset parametrization introduced above can be directly read from (16):

\[
\mathcal{M} = \begin{pmatrix} u^\Lambda (z, \overline{z}) & v_A (z, \overline{z}) \\ v^\Lambda (z, \overline{z}) & u_A (z, \overline{z}) \end{pmatrix},
\]

where the symplectic index $\Lambda = 0, 1, \ldots, 27$ (with 0 pertaining to the $\mathcal{N} = 2, D = 4$ graviphoton), and $i = \pi, 28$. Thus, the symplectic sections read (see e.g. [17][15] and Refs. therein; subscript “28” omitted):

\[
f_i^\Lambda : = \frac{1}{\sqrt{2}} (u + v)_i^\Lambda = \left( \mathcal{T}_{\pi i}^\Lambda, f^\Lambda \right) = \exp \left( \frac{1}{2} K \right) \left( \mathcal{T}_{\pi i}^\Lambda X^\Lambda, X^\Lambda \right); \tag{18}
\]

\[
h_{i\Lambda} : = -i \frac{1}{\sqrt{2}} (u - v)_{i\Lambda} = \left( \mathcal{T}_{\pi i}^\Lambda h^\Lambda, h_{\Lambda} \right) = \exp \left( \frac{1}{2} K \right) \left( \mathcal{T}_{\pi i}^\Lambda M^\Lambda, M^\Lambda \right), \tag{19}
\]

where $\mathcal{D}$ is the Kähler-covariant differential operator,

\[
\mathcal{V} := \left( L^\Lambda, M^\Lambda \right)^T = \exp \left( \frac{1}{2} K \right) \left( X^\Lambda, F^\Lambda \right)^T \tag{20}
\]

is the symplectic vector of Kähler-covariantly holomorphic sections, and

\[
K := -\ln \left[ i \left( X^\Lambda F^\Lambda - X^\Lambda F^\Lambda \right) \right] \tag{21}
\]

is the Kähler potential determining the corresponding geometry. A more explicit expression for (16) would be needed in order to check that the prepotential $F$ does not exist (i.e., $2F = X^\Lambda F^\Lambda = 0$) in the symplectic frame we have just introduced, which can be considered the analogue of the Calabi-Vesentini basis [12][13], whose manifest covariance is the maximal one.

### 3 The Iwasawa Decomposition and the role of triality

Now we are going to find another parametrization for the coset $\mathcal{M}$, provided by the Iwasawa decomposition. In this case the maximal manifest covariance is broken down to a subgroup $SO(8)$, thus providing a manifestly triality-symmetric description.

The manifold $\mathcal{M}$ has rank 3, which means that the maximal dimension of the intersection between a Cartan subalgebra of $E_{7(-25)}$ and the generators of $\mathcal{M}$ is 3. In particular, we can pick 3 such generators to be the diagonal generators of the Jordan algebra $\mathfrak{J}_3 (\mathbb{O})$ itself, namely $h_1 = Y_{123}, h_2 = Y_{132}$ and $h_3 = Y_{133}$.

The following step is to determine a basis $\mathcal{W}_+$ of $54 - 3 = 51$ positive roots $\lambda^+_i$, $i = 1, \ldots, 51$ with respect to $\mathfrak{h}_3$. Then the Iwasawa decomposition of the coset $\mathcal{M}$ is defined as:

\[
\mathcal{M} := \exp (x_1 h_1 + x_2 h_2 + x_3 h_3) \exp \left( \sum_{i=1}^{51} y_i \lambda^+_i \right). \tag{22}
\]

As anticipated, one of its main features is that since the elements $h_1, h_2, h_3 \in \mathfrak{h}_3$ commute with a 28-dimensional subalgebra $\mathfrak{so}(8)$, the Iwasawa parametrization of $\mathcal{M}$ exhibits a manifest maximal covariance given by $SO(8)$. Therefore, the 51-dimensional linear space $\Lambda_+$ generated by the positive roots $\mathcal{W}_+$ is invariant under the (adjoint) action of $SO(8)$, and it decomposes into irreps. of $SO(8)$ as:

\[
\Lambda_+ = 1^3 + 8^2 c + 8^2 c + 8^2 s,
\]

which is a manifestly triality-symmetric decomposition. In particular, at the level of algebras $\mathfrak{so}(8) = \text{tri}(\mathbb{O})$ with the automorphism group $\text{Aut}(\mathfrak{t} (\mathbb{O})) = Spin(8)$ of the normed triality over the octonions $\mathbb{O}$ [20].
It is worth remarking that the appearance of the square for the three $8$ irreps. in (23) is a consequence of the complex (special Kähler) structure of the coset $\mathcal{M}$. Moreover, it should be observed that the $SO(8)$ entering in (23) can be identified as:

$$SO(8) \subset [(SO(10) \times U(1)) \cap E_{4(-52)}].$$

(24)

This can be understood by noticing that it can be obtained from both the following chains of maximal symmetric embeddings [21]:

$$E_7(-25) \supset E_6(-78) \times U(1) \supset SO(10) \times U(1)' \supset SO(8) \times U(1)'' \supset U(1)''(25)$$

and

$$E_7(-25) \supset E_6(-78) \times U(1) \supset F_4(-52) \times U(1) \supset SO(9) \times U(1) \supset SO(8) \times U(1).$$

(26)

In the last line of (25) the first two $U(1)$ factors have the physical meaning of “extra” $T$-dualities generated by the Kaluza-Klein reductions, respectively $D = 5 \rightarrow D = 4$, and $D = 6 \rightarrow D = 5$.

Denoting with subscripts $U(1)$-charges, the adjoint irrep. $133$ of $E_7(-25)$ branches according to (25) as (see e.g. [21]):

$$133 = 78_0 + 1_0 + 27_{-2} + 27_{+2}$$

(27)

$$= 1_{0,0} + 16_{0,-3} + 16'_{0,+3} + 45_{0,0} + 1_{0,0}$$

$$+ 1_{-2,+4} + 10_{-2,-2} + 16_{-2,+1}$$

$$+ 1_{+2,-4} + 10_{+2,+2} + 16'_{+2,-1}$$

$$= 1_{0,0,0} + 8_{c,0,-3,1} + 8_{s,0,-3,-1} + 8_{c,0,+3,-1} + 8_{s,0,+3,1}$$

$$+ 1_{0,0,0} + 8_{c,0,0,+2} + 8_{c,0,0,-2} + 28_{0,0,0} + 1_{0,0,0}$$

$$+ 1_{-2,+4,0} + 1_{-2,-2,+2} + 1_{-2,-2,-2} + 8_{c,-2,-2,0} + 8_{c,-2,1,+1} + 8_{c,-2,+1,-1}$$

$$+ 1_{+2,-4,0} + 1_{+2,-2,+2} + 1_{+2,-2,-2} + 8_{c,+2,2,0} + 8_{c,+2,-1,-1} + 8_{s,+2,-1,1}.$$

4 Final Remarks

It is very interesting to remark that being based only on the algebraic properties of the “mixed” Freudenthal-Tits magic square in Table 1 the construction of the basis with the maximal possible covariance [16] and the computation of the Iwasawa decomposition [22] described here can be both generalized [8] at least to a broader class of minimally non-compact, simple groups of type $E_7$ [23]. Moreover, it also turns out that, like for $\mathcal{M}$, in all these cases the maximal covariance (at least at the Lie algebra level) of the Iwasawa decomposition is given by the automorphism algebra of the corresponding normed triality [20].

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References

[1] I. Yokota, Exceptional Lie Groups, arXiv:0902.0431 [math.DG].
[2] C. H. Barton and A. Sudbery, *Magic squares and matrix models of Lie algebras*, Adv. in Math. **180**, 596 (2003), math/0203010 [math.RA].

[3] M. Güngaydin, G. Sierra, P. K. Townsend, *Exceptional Supergravity Theories and the Magic Square*, Phys. Lett. **B133B**, 72 (1983).

[4] J. Tits, *Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles. I. Construction*, (French), Nederl. Akad. Wetensch. Proc Ser. A **69**, 223 (1966).

[5] S. L. Cacciatori, F. D. Piazza and A. Scotti, *E7 groups from octonionic magic square*, arXiv:1007.4758 [math-ph].

[6] F. Bernardoni, S. L. Cacciatori, B. L. Cerchiai and A. Scotti, *Mapping the geometry of the E6 group*, J. Math. Phys. **49**, 012107 (2008), arXiv:0710.0356 [math-ph].

[7] F. Bernardoni, S. L. Cacciatori, B. L. Cerchiai and A. Scotti, *Mapping the geometry of the F4 group*, Adv. Theor. Math. Phys. Vol. 12, Number 4, 889 (2008), arXiv:0705.3978 [math-ph].

[8] S. L. Cacciatori, B. L. Cerchiai, A. Marrani, *Magic Coset Decompositions*, preprint CERN-PH-TH/2012-020, arXiv:1201.6314 [hep-th].

[9] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Geom. Dedicata **19**, 7 (1985).

[10] L. Borsten, M. J. Duff, S. Ferrara, A. Marrani and W. Rubens, *Small Orbits*, arXiv:1108.0424 [hep-th].

[11] L. Borsten, M. J. Duff, S. Ferrara, A. Marrani and W. Rubens, *Explicit Orbit Classification of Reducible Jordan Algebras and Freudenthal Triple Systems*, arXiv:1108.0908 [math.RA].

[12] L. Andrianopoli, R. D’Auria, S. Ferrara and M. A. Lledó, *Gauging of Flat Groups in Four Dimensional Supergravity*, JHEP **0207**, 010 (2002), hep-th/0203206.

[13] A. Strominger, *Special Geometry*, Commun. Math. Phys. **133**, 163 (1990).

[14] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry Structure of Special Geometries*, Nucl. Phys. **B400**, 463 (1993), hep-th/9210068.

[15] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and T. Magri, *N= 2 Supergravity and N= 2 superYang-Mills Theory on General Scalar Manifolds : Symplectic Covariance, Gaugings and the Momentum Map*, J. Geom. Phys. **23**, 111 (1997), hep-th/9605032.

[16] B. de Wit and H. Nicolai, *N= 8 Supergravity*, Nucl. Phys. **B208**, 323 (1982).

[17] A. Ceresole, R. D’Auria and S. Ferrara, *The Symplectic Structure of N= 2 Supergravity and its Central Extension*, Nucl. Proc. Suppl. **46**, 67 (1996), hep-th/9509160.

[18] A. Ceresole, R. D’Auria, S. Ferrara and A. Van Proeyen, *Duality Transformations in Supersymmetric Yang-Mills Theories coupled to Supergravity*, Nucl. Phys. **B444**, 92 (1995), hep-th/9502072.

[19] E. Calabi and E. Vesentini, *On Compact, Locally Symmetric Kähler Manifolds*, Ann. Math. **71**, 472 (1960).

[20] J. C. Baez, *The Octonions*, Bull. Am. Math. Soc. **39**, 145 (2002), math/0105155 [math-ra].

[21] R. Gilmore : “*Lie Groups, Lie Algebras, and Some of Their Applications*”, Dover, New York, 2006.

[22] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rept. **79**, 1 (1981).

[23] R. B. Brown, *Groups of Type E7*, J. Reine Angew. Math. **236**, 79 (1969).