Green-function method in the theory of ultraslow electromagnetic waves in an ideal gas with Bose-Einstein condensates

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(Dated: January 16, 2009)

We propose a microscopic approach describing the interaction of an ideal gas of hydrogenlike atoms with a weak electromagnetic field. This approach is based on the Green-function formalism and an approximate formulation of the method of second quantization for quantum many-particle systems in the presence of bound states of particles. The dependencies of the propagation velocity and damping rate of electromagnetic pulses on the microscopic characteristics of the system are studied for a gas of hydrogenlike atoms. For a Bose-Einstein condensate of alkali-metal atoms we find the conditions when the electromagnetic waves of both the optical and microwave regions are slowed. In the framework of the proposed approach, the influence of an external homogeneous and static magnetic field on the slowing phenomenon is studied.

PACS numbers: 03.75.Hh, 05.30.-d, 42.25.Bs
Keywords: atom-photon collisions, Bose-Einstein condensation, bound states, Green’s function methods, quantization (quantum theory)

I. INTRODUCTION

The possibility of observation of ultraslow light pulses in vapors of alkali-metal atoms at extremely low temperatures was shown in 1999 [1]. To the present moment, there are quite a lot of theoretical works devoted to the description of this phenomenon (probably the earliest of which is represented in Ref. [2]). Naturally, for the most complete description of electromagnetic waves slowing in a Bose-Einstein condensate (BEC), the development of an appropriate microscopic approach is needed.

But, while constructing this approach, some problems appear. First of all, one needs to use the method of second quantization for a microscopic description of the response of the BEC to the perturbation by the external electromagnetic field. The second-quantization method uses the creation and annihilation operators of atoms. This fact requires consideration of atoms as elementary objects, which do not have internal structure. However, atoms are compound objects with a rather complicated internal structure. Moreover, in the processes of interaction of neutral atoms with an electromagnetic field, this structure plays a crucial role. In particular, it leads also to a sufficient reduction of the group velocity of propagation of electromagnetic waves in a BEC.

Thus, a problem appears concerned with the introduction of the creation and annihilation operators, which must preserve the information about the internal structure and, first of all, about the energy state of the atoms. In other words, we discuss the formulation of the second-quantization method for a many-particle system in the presence of bound states of particles (atoms). This problem is solved in Ref. [3]. In that work, an approximate formulation of the second quantization method for such a system was constructed in the case when the binding energy of compound particles is much larger than the kinetic energy. The possibility of development of such a theory is demonstrated by considering a system that consists of fermions of two different kinds and bound states (atoms) that can be formed only by two fermions of different kinds (ions and electrons). The choice of this model in Ref. [3] was dictated not by methodological difficulties, but to get more visual results. In that work, the Hamiltonians of interaction for particles of different kinds (including bound states) were constructed. The Hamiltonian of interaction of all particles in the system (including neutral compound particles) with the electromagnetic field was found. Because of this, the authors constructed a nonrelativistic quantum electrodynamics of systems which may consist of bound states of particles. Within this theory, the Maxwell-Lorentz equations were obtained. These allow study of the response of a system consisting of charged “elementary” particles and bound states to perturbation by an external electromagnetic field. For solving this problem it is natural to use the Green-function formalism (see, e.g., Refs. [4, 5]).

In Ref. [3] the Green-function method was generalized to the case of description of this response of the systems mentioned above. In terms of Green functions, the expressions for such macroscopic characteristics as conductivity, permittivity, and magnetic permeability of a hydrogenlike plasma were found.

In the present paper, the formulas that were found in Refs. [3, 5] are used to study the phenomenon of electromagnetic pulses slowing in a BEC. According to the main results from Refs. [3, 6], let us briefly summarize some basic principles of the proposed approach.
II. HAMILTONIAN OF HYDROGENLIKE ATOMIC GAS IN AN EXTERNAL ELECTROMAGNETIC FIELD

Usually, BEC is studied in gases of alkali-metal atoms. It is known that the internal structure of alkali metals is similar to the internal structure of a hydrogen atom (hydrogenlike atoms). Thus, such atoms may be considered as bound states of two elementary particles (the valence electron and the atomic core). In this case, the results of Refs. 3, 4 become suitable for a description of the interaction of a weak electromagnetic field with a BEC. We use it to find the limits of applicability for it in a low-energy approximation is given in the Appendix. We use it to find the limits of applicability for the model of the ideal gas. Taking into account this fact, the system Hamiltonian may be written as follows:

\[ \hat{H}(t) = \hat{H}_0 + \hat{H}_{\text{int}} + \hat{V}(t), \quad \hat{H}_0 = \hat{H}_{\text{ph}} + \hat{H}_{p}, \quad (1) \]

where \( \hat{H}_{\text{ph}} \) is the Hamiltonian for free photons and \( \hat{H}_{\text{int}} \) is the Hamiltonian of interaction between atoms. Note that we neglect the Hamiltonian \( \hat{H}_{\text{ph}} \) below. We also consider the system of hydrogenlike atoms as an ideal gas. Thus, we do not take into account the presence of the interaction Hamiltonian \( \hat{H}_{\text{int}} \) in the next calculations. Its microscopic expression is found in Ref. 3 in the framework of the developed formulation of the second-quantization method. This Hamiltonian is constructed on the basis of the Coulomb interaction between particles that can form a bound state (hydrogenlike atom). The explicit form for it in a low-energy approximation is given in the Appendix. We use it to find the limits of applicability for the model of the ideal gas.

The operator \( \hat{H}_{p} \) in Eq. (1) is the Hamiltonian for free particles (atoms),

\[ \hat{H}_{p} = \sum_{\alpha} \int dx \left( \frac{1}{2m} \frac{\partial \tilde{n}_{\alpha}^\dagger(x)}{\partial x} \frac{\partial \tilde{n}_{\alpha}(x)}{\partial x} + \varepsilon_{\alpha} \tilde{n}_{\alpha}^\dagger(x) \tilde{n}_{\alpha}(x) \right), \quad (2) \]

where \( \tilde{n}_{\alpha}^\dagger(x) \) and \( \tilde{n}_{\alpha}(x) \) are the creation and annihilation operators of hydrogenlike (alkali-metal) atoms with the set of quantum numbers \( \alpha \) at the point \( x \), respectively; \( \varepsilon_{\alpha} \) is the atomic energy corresponding to this state; and \( m \) is the atomic mass.

The operator \( \hat{V}(t) \) in Eq. (1) represents the Hamiltonian that describes the interaction of atoms with an electromagnetic field,

\[ \hat{V}(t) = -\frac{1}{e} \int dx \mathbf{A}(x, t) \hat{j}(x) + \int dx \varphi(x, t) \hat{\sigma}(x), \quad (3) \]

where \( \mathbf{A}(x, t) \) and \( \varphi(x, t) \) are the vector and scalar potentials of the external electromagnetic field, and \( \hat{j}(x) \) and \( \hat{\sigma}(x) \) are the operators of the current and charge densities for neutral atoms, respectively (see Ref. 3 for details),

\[ \hat{\sigma}(x) = \frac{1}{\sqrt{V}} \sum_{p-p'} \sum_{\alpha\beta} e^{i(x(p'-p))} \sigma_{\alpha\beta}(p - p') \tilde{n}_{\alpha}^\dagger(p) \tilde{n}_{\beta}(p'), \quad (4) \]

\[ \hat{j}(x) = \frac{1}{\sqrt{V}} \sum_{p-p'} \sum_{\alpha\beta} e^{i(x(p'-p))} \left( I_{\alpha\beta}(p - p') \right) \]

\[ + \frac{(p + p')}{2m} \sigma_{\alpha\beta}(p - p') \tilde{n}_{\alpha}^\dagger(p) \tilde{n}_{\beta}(p'), \quad (5) \]

where \( V \) is the volume of the system. The matrix elements of the charge and current densities in Eqs. (4) and (5) can be expressed in terms of the atomic wave functions \( \varphi_{\alpha}(x) \):

\[ \sigma_{\alpha\beta}(k) = e \int dy \varphi_{\alpha}^*(y) \varphi_{\beta}(y) \]

\[ \times \left[ \exp \left( \frac{i m_e}{m} ky \right) - \exp \left( -\frac{i m_e}{m} ky \right) \right], \quad (6) \]

\[ I_{\alpha\beta}(k) = \frac{ie}{2m_e} \int dy \left( \frac{\partial \varphi_{\alpha}^*(y)}{\partial y} \varphi_{\beta}(y) - \varphi_{\alpha}^*(y) \frac{\partial \varphi_{\beta}(y)}{\partial y} \right) \]

\[ \times \left[ \exp \left( -\frac{i m_e}{m} ky \right) + \frac{m_e}{m_p} \exp \left( \frac{i m_e}{m} ky \right) \right], \quad (7) \]

where \( e \) is the electron charge absolute value, and \( m_p \) and \( m_e \) are the masses of the atomic core and electron, respectively \( (m = m_p + m_e) \).

III. RESPONSE OF A GAS OF HYDROGENLIKE ATOMS TO PERTURBATION BY AN EXTERNAL ELECTROMAGNETIC FIELD: GREEN FUNCTIONS

Here we find the expressions that describe the influence of a weak external electromagnetic field on the system under consideration. To this end, following Ref. 4, let us recall some main principles of the Green-function method.

Consider a system that in some moment of time \( t \) is characterized by a statistical operator \( \hat{\rho}(t) \). Then the system Hamiltonian may be written as

\[ \hat{H}(t) = \hat{H}_0 + \hat{V}(t), \quad \hat{V}(t) = \int dx \mathbf{F}(x, t) \hat{\xi}(x), \]

where \( \hat{H}_0 \) is the Hamiltonian of an ideal atomic gas and \( \mathbf{F}(x, t) \) is the Hamiltonian of interaction of atoms with the external electromagnetic field, \( \mathbf{F}(x, t) \) are the quantities, which define the external field, and \( \hat{\xi}(x) \) are the quasilocals operators, which refer only to the studied system (they are not associated with the external field; see Ref. 3). Note also that the summation convention over the repeated index \( i \) is meant.
Let us consider below that the influence of the external field on the system is weak. Due to this fact, we can develop perturbation theory over \( V(t) \). Thus, for the mean value of the arbitrary quasilocal operator \( \hat{a}(x) \) in the linear order in \( V(t) \), one gets

\[
\text{Sp} \hat{\rho}(t) \hat{a}(x) = \text{Sp} w \hat{a}(0) + a^F(x, t),
\]

(8)

\[
a^F(x, t) = \int_{-\infty}^{\infty} dt' \int d\mathbf{x}' G_{\alpha \xi} (\mathbf{x} - \mathbf{x}', t - t') F_i(\mathbf{x}', t'),
\]

where \( w \) is the equilibrium statistical operator (the Gibbs operator) for the system under consideration,

\[
w = \exp \left[ \Omega - \beta \left( \hat{\mathcal{H}}_0 - \sum_{\alpha} \mu_{\alpha} \hat{N}_\alpha \right) \right].
\]

(9)

Here \( \beta = 1/T \) is the reciprocal temperature in energy units, \( \mu_{\alpha} \) is the chemical potential of atoms in the state \( \alpha \), and \( \hat{N}_\alpha = \int d\mathbf{x} \hat{\eta}_0^\dagger (\mathbf{x}) \hat{n}_0(\mathbf{x}) \) is the operator of the total number of atoms in the state \( \alpha \). The temperature and the chemical potential are found from the following equations:

\[
\text{Sp} w \hat{\mathcal{H}}_0 = \mathcal{H}_0, \quad \text{Sp} w \hat{N}_\alpha = N_\alpha.
\]

The dependence of the thermodynamic potential \( \Omega = \Omega(T, \mu_{\alpha}) \) is defined by \( \text{Sp} w = 1 \).

The function \( G_{\alpha \xi} (\mathbf{x} - \mathbf{x}', t - t') \) in (8),

\[
G_{\alpha \xi} (\mathbf{x} - \mathbf{x}', t - t') = -i \theta(t - t') \text{Sp} w [\hat{a}(x, t), \hat{\xi}(\mathbf{x}', t')],
\]

(10)

is the two-time retarded Green function of the operators \( \hat{a}(x, t), \hat{\xi}(\mathbf{x}', t') \), which are taken in the Heisenberg representation,

\[
\hat{a}(x, t) = e^{i\hat{\mathcal{H}}_0 t} \hat{a}(x) e^{-i\hat{\mathcal{H}}_0 t}.
\]

The quantity \( \theta(t) \) in Eq. (10) is the Heaviside function.

Taking the Fourier transforms of the quantities \( a^F \) and \( F_i \),

\[
a^F(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \, d\omega \, e^{-i(t\omega - \mathbf{k}\mathbf{x})} a^F(\mathbf{k}, \omega),
\]

\[
F_i(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \, d\omega \, e^{-i(t\omega - \mathbf{k}\mathbf{x})} F_i(\mathbf{k}, \omega),
\]

one gets

\[
a^F(\mathbf{k}, \omega) = G_{\alpha \xi} (\mathbf{k}, \omega) F_i(\mathbf{k}, \omega)
\]

(11)

with

\[
G_{\alpha \xi} (\mathbf{k}, \omega) = \int_{-\infty}^{\infty} dt \int d\mathbf{x} \, e^{i(t\omega - \mathbf{k}\mathbf{x})} G_{\alpha \xi} (\mathbf{x}, t).
\]

(12)

Let us use the derived expressions for studying the response of a gas of alkali-metal atoms to the external electromagnetic field. To this end, we must replace the quantities \( F_i(x, t) \) in Eq. (9) by the vector \( A(x, t) \) and scalar \( \varphi(x, t) \) potentials of the external electromagnetic field. The operators \( \hat{a}(x, t) \) and \( \hat{\xi}(\mathbf{x}', t') \) must be replaced in this case by the operators of the current density \( \hat{j}(x) \) and charge density \( \hat{\sigma}(x) \) (see Eqs. (31)–(33)). By the Hamiltonian \( \hat{\mathcal{H}}_p \) in Eq. (9) one must take the Hamiltonian \( \hat{\mathcal{H}}_p \) (see Eqs. (11) and (2)). In next calculations we use also the Maxwell-Lorentz equations for the average fields in the system (see in that case Refs. [3, 5])

\[
\text{rot} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \quad \text{div} \mathbf{E} = 4\pi \sigma,
\]

\[
\text{rot} \mathbf{H} = -\frac{4\pi}{c} \mathbf{j}, \quad \text{div} \mathbf{H} = 0,
\]

(13)

where the quantities \( \sigma(x, t) \) and \( \mathbf{j}(x, t) \) are the mean values of the charge and current densities (see (3)–(5)),

\[
\mathbf{j}(x, t) = \text{Sp} \hat{\rho}(t) \hat{j}(x), \quad \sigma(x, t) = \text{Sp} \hat{\rho}(t) \hat{\sigma}(x),
\]

(14)

and the quantities \( \mathbf{E}(x, t) \) and \( \mathbf{H}(x, t) \) are the electric and magnetic field intensities, which can be expressed in terms of the potentials of an external electromagnetic field (see Refs. [4, 5]). Note that the mean values of the quantities \( \sigma(x, t) \) and \( \mathbf{j}(x, t) \) should be calculated in accordance with Eq. (5). For the equilibrium state, characterized by the statistical operator \( \hat{\mathcal{H}}_0 \), the mean values of these quantities vanish (see below).

Omitting rather cumbersome calculations, we come to the following formulas for the mean values of the charge and current densities that are induced by the weak external electromagnetic field (see Ref. [5] for more details):

\[
\sigma(x, t) = \int_{-\infty}^{\infty} dt' \int d^3 x' \left( -G_{ix}(x - x', t - t') \right.
\]

\[
\times \left. \frac{1}{c} A_i(x', t') + G(x - x', t - t') \varphi(x', t') \right), \]

(15)

\[
\mathbf{j}(x, t) = \int_{-\infty}^{\infty} dt' \int d^3 x' \left( -G_{kl}(x - x', t - t') \right.
\]

\[
\times \left. \frac{1}{c} A_l(x', t') + G_k(x - x', t - t') \varphi(x', t') \right). \]

(16)

The Green functions, in terms of which the expressions (15) and (16) are written, are defined by

\[
G(x, t) = -i \theta(t) \text{Sp} w [\hat{\sigma}(x, t), \hat{\sigma}(0)],
\]

\[
G_{kl}(x, t) = -i \theta(t) \text{Sp} w [\hat{j}_{kl}(x, t), \hat{\sigma}(0)],
\]

\[
G_{kl}(x, t) = -i \theta(t) \text{Sp} w [\hat{j}_{kl}(x, t), \hat{j}_{kl}(0)],
\]

(17)

where the charge and current density operators (see Eqs. (34) and (35)) are taken in the Heisenberg representation.

Let us recall that we consider the response of an ideal gas of hydrogenlike atoms to a perturbation by an external electromagnetic field. As mentioned above, this consideration is reasonable in the case of a description...
of a BEC in dilute vapors of alkali-metal atoms. Taking into account this fact, the expressions for the Fourier transforms of the Green functions (see Eq. (12)) can be written as

\[ G(k, \omega) = \frac{1}{V} \sum_{\alpha, \beta} \sigma_{\alpha \beta}(k) \sigma_{\beta \alpha}(-k) \times \frac{f_\alpha(p - k) - f_\beta(p)}{\varepsilon_\alpha(p) - \varepsilon_\beta(p - k) + \omega + i0}, \quad (18a) \]

\[ G_l(k, \omega) = \frac{1}{V} \sum_{\alpha, \beta} \left( \frac{2p - k}{2M} \sigma_{\alpha \beta}(k) + I_{\alpha \beta}(k) \right)_l \times \frac{\sigma_{\beta \alpha}(-k) [f_\alpha(p - k) - f_\beta(p)]}{\varepsilon_\alpha(p) - \varepsilon_\beta(p - k) + \omega + i0}, \quad (18b) \]

\[ G_{lj}(k, \omega) = \frac{1}{V} \sum_{\alpha, \beta} \left( \frac{2p - k}{2M} \sigma_{\alpha \beta}(k) + I_{\alpha \beta}(k) \right)_l \times \left( \frac{2p - k}{2M} \sigma_{\beta \alpha}(-k) + I_{\beta \alpha}(-k) \right)_j \times \frac{f_\alpha(p - k) - f_\beta(p)}{\varepsilon_\alpha(p) - \varepsilon_\beta(p - k) + \omega + i0}. \quad (18c) \]

Here \( \varepsilon_\alpha(p) = \varepsilon_\alpha + p^2/2m \) is the total energy of an atom, and \( f_\alpha(p) \) is the distribution function of the ideal gas of hydrogenlike atoms in equilibrium, which is defined by the relation

\[ S_{\alpha \alpha'}(p)\tilde{\eta}_{\alpha'}(p') = f_\alpha(p)\delta_{\alpha \alpha'}(\mathbf{p} - \mathbf{p}'), \]

where \( \delta_{\alpha \alpha'} \) and \( \Delta(\mathbf{p} - \mathbf{p}') \) denote the Kronecker symbols. Bearing in mind that the unperturbed state of a gas is characterized by the Hamiltonian \( \hat{H} \), and using Eq. (9), it is easy to come to the explicit form for the distribution function \( f_\alpha(p) \),

\[ f_\alpha(p) = (\exp\{[\varepsilon_\alpha(p) - \mu_\alpha]/T\} - 1)^{-1}. \quad (19) \]

Employing the last expression and Eqs. (1) - (7), one may ensure that the mean values of \( \hat{\sigma}(\mathbf{x}) \) and \( \hat{\mathbf{j}}(\mathbf{x}) \) with the statistical operator \( \omega \) (see Eqs. (8) and (9)) vanish. We recall that such mean values are not accounted in the derivation of Eqs. (13) and (16). Let us also note that the total set of the Green functions (see Eqs. (18)) is given for a completeness of the description. Below we use only the expression for the scalar Green function (18a).

As to Eqs. (18), it is necessary to note the following. Strictly speaking, these formulas are correct only in the case when the system is considered in a state of total statistical equilibrium. Thus, to produce excited atoms with the set of quantum numbers \( \alpha \) in this state, one needs to keep matter in equilibrium with radiation. For that, in turn, one needs to preserve the number of photons in the volume in which the gas of hydrogenlike atoms is contained (or to return the photons by the special set of “mirrors”).

In the opposite case, the statistical operator (9) describes a certain quasistationary state of the system, but not the state of total statistical equilibrium. By the term “quasistationary equilibrium” we mean a state of the system in which atoms can be found only in quantum states with lifetimes much longer than other characteristic times of system relaxation (e.g., the relaxation due to collision processes). For example, as these states we can consider the hyperfine ground states of alkali-metal atoms (see Ref. (6)) and also states whose existence is stimulated by the external field (e.g., resonant laser radiation). Therefore, in the case of absence of equilibrium between radiation and matter, the sum over all quantum states \( \alpha, \beta \) in Eqs. (18) must be replaced by the sum over these characteristic quantum states. Exactly these states are considered below in this paper.

In terms of the derived Green function (18a), one can express the permittivity of the studied system (see in this case also Refs. (4, 5)) as

\[ \epsilon^{-1}(k, \omega) = 1 + \frac{4\pi k^{-2} G(k, \omega)}{1 + 4\pi k^{-2} G(k, \omega)}. \quad (20) \]

Taking into account the explicit form (18a), one finds

\[ \epsilon^{-1}(k, \omega) = 1 + \frac{4\pi k^{-2} \sum_{\alpha, \beta} \sigma_{\alpha \beta}(k) \sigma_{\beta \alpha}(-k) \times \frac{f_\alpha(p - k) - f_\beta(p)}{\varepsilon_\alpha(p) - \varepsilon_\beta(p - k) + \omega + i0}}{1 + 4\pi k^{-2} G(k, \omega)}. \quad (21) \]

We must note that, in accordance with Ref. (4), the formulas (18), (20), and (21) can be used only in the region of relatively large \( k \). Regarding the present paper, this criterion of applicability can be written as \( \nu \lambda^3 \ll 1 \) (\( \lambda \) is the laser wavelength and \( \nu \) is the density of atoms). In the case \( \nu \lambda^3 \gtrsim 1 \), the mentioned expressions become incorrect. This results from the fact that in this region the interaction between particles may play a significant role. Note that we neglect it in deriving Eqs. (18), (20), and (21). To take into account the interaction effects, one needs to derive the equations for the Green functions and to find a hierarchy for the correlation functions (see, e.g., Refs. (4, 7, 8)). In fact, the use of these procedures results in a redefinition of the quantity \( G(k, \omega)k^{-2} \) in Eq. (20) because of the account of correlation corrections (8, 9). But, in the case when this quantity is small, Eqs. (18), (20), and (21) and the expressions that result from them remain correct in the first order of perturbation theory even if one takes the correlations into account. Hence, we consider that the mentioned quantity in Eq. (20) is small in the next calculations,

\[ |4\pi k^{-2} G(k, \omega)| \ll 1. \quad (22) \]

This inequality makes it possible to ignore the correlation corrections to the mentioned term in the first order of
where $\nu$ over $T$  

Taking into account this fact, after integration of (21) e.g., Ref. [4]):

$\delta$ is set proportional to the Dirac delta function this fact, the distribution function should consider all atoms in a BEC state. Because of static magnetic field (see Fig. 1, and also Ref. [6]).

that form the BEC (see Refs. [6, 10]),

equals the energy of the lowest level of the atomic states critical temperature in particular, results in rather strong dependence of the numbers of the ground state. Let us note that this fact, where by the index $\alpha$ is the density of condensed atoms in the $\alpha$ state. We return to the discussion of this approximation below.

As is known, at extremely low temperatures a Bose-Einstein condensate of alkali-metal atoms can be formed. The main peculiarity of an ideal Bose gas is that at temperatures lower than the critical temperature, $T \leq T_c$, the chemical potential $\mu_\alpha$ does not depend on $T$ and equals the energy of the lowest level of the atomic states that form the BEC (see Refs. [6, 10]),

$$\mu_\alpha = \varepsilon_{\alpha_0},$$

where by the index $\alpha_0$ we denote the set of quantum numbers of the ground state. Let us note that this fact, in particular, results in rather strong dependence of the critical temperature $T_c$ on the intensity of an external static magnetic field (see Fig. 1 and also Ref. [6]).

At temperatures much lower than $T_c$, $T \ll T_c$, we should consider all atoms in a BEC state. Because of this fact, the distribution function $f_\alpha(p)$ (see Eq. (19)) is set proportional to the Dirac delta function $\delta(p)$ (see, e.g., Ref. [4]):

$$f_\alpha(p) = (2\pi)^3 \nu_\alpha \delta(p),$$

where $\nu_\alpha$ is the density of condensed atoms in the $\alpha$ state. Taking into account this fact, after integration of (21) over $p$, the formula for the permittivity of a gas in the BEC state ($T \to 0$) takes the form

$$\varepsilon^{-1}(\mathbf{k}, \omega) \approx 1 + \frac{4\pi}{k^2} \sum_{\alpha,\beta} \sigma_{\alpha\beta}(\mathbf{k}) \sigma_{\beta\alpha}(-\mathbf{k}) \times$$

$$\times \left( \frac{\nu_\alpha}{\omega + \Delta \varepsilon_{\alpha\beta} - \varepsilon_\alpha + i\gamma_{\alpha\beta}} \right. + \left. \frac{\nu_\beta}{\omega + \Delta \varepsilon_{\alpha\beta} + \varepsilon_\alpha + i\gamma_{\alpha\beta}} \right),$$

(24)

where $\varepsilon_\alpha = k^2/2m$ is the recoil energy of an atom, and the quantity $\sigma_{\alpha\beta}(\mathbf{k})$ is still defined by Eq. (1). Here we also introduce the natural linewidth $\gamma_{\alpha\beta}$, which is related to the probability of a spontaneous transition between the $\alpha$ and $\beta$ states. Note that, in our calculations, we use the system of units, in which the Planck constant $\hbar$ equals unity. Thus, the frequency is measured in energy units.

As one can conclude from the relation (24), in the case of the BEC regime, the response of the gas increases proportionally to the density of condensed atoms. This effect takes place due to the coherent behavior of the atoms in the condensate. Thus, one may say that the atoms respond to the perturbation cooperatively. As for the mathematical form, this fact corresponds to the substitution of the wave functions of a single atom for the condensate wave functions (see Eqs. (6), and (21)–(24)). In the limit of zero temperatures, the mentioned phenomenon results in increase of collective effects, while the value of the corresponding linewidth (which is defined by the imaginary summand $i\gamma_{\alpha\beta}$) remains the same as for the isolated atom.

It is easy to see that at frequencies close to the energy intervals $\Delta \varepsilon_{\alpha\beta}$ ($\Delta \varepsilon_{\alpha\beta} \equiv \varepsilon_\alpha - \varepsilon_\beta$), some peculiarities appear in Eq. (24). Because of the small values of the recoil energy and the natural linewidth, these peculiarities are similar to the resonance ones. The mentioned behavior of the permittivity must have a strong impact on the dispersion characteristics of a gas and, as known, becomes the governing condition for slowing of the electromagnetic pulses.

To study the propagation of electromagnetic waves in a BEC we must supplement Eq. (24) with the dispersion relation for free waves, which can spread in the system

$$\omega^2 \varepsilon(k, \omega) \mu(k, \omega) - k^2 = 0,$$

(25)

where $\mu(k, \omega)$ denotes the magnetic permeability of the gas. This quantity may also be found in the framework of the Green-function formalism (see also Ref. [3]).

**IV. DISPERSION CHARACTERISTICS OF A TWO-LEVEL SYSTEM IN THE BEC STATE**

In this section we try to simplify the description of the interaction of electromagnetic waves with a BEC. To this end, let us consider a case when the frequency of the external field is close to the transition frequency between
In the Appendix.

Estimates of the applicability of this approach are given in Eq. (A1). Approximate estimates of the applicability of this approach are given in the Appendix.

Let us denote by indices 1 and 2 the quantities related to the lower and the upper states of the two-level system and omit the terms that are related to other states. Then, Eq. (21) takes the form

\[ \epsilon^{-1}(k, \omega) \approx 1 + \frac{a(k)}{\delta \omega + i \gamma} \]  

(26)

with

\[ a(k) = 4\pi(\nu_1 - \nu_2) \frac{g_1 g_2 |\sigma_{12}(k)|^2}{k^2} \]  

(27)

Here \( g_j \) is the degeneracy order of the \( j \) level (\( j = 1, 2 \)) with respect to total momentum, \( \delta \omega = (\omega - \Delta \varepsilon_{21}) \) is the detuning of the external field (\( |\delta \omega| \ll \Delta \varepsilon_{21} \)), and \( \gamma = \gamma_{12} \) is the linewidth related to the transition probability from the upper to lower state. We also neglect the so-called recoil energy \( \varepsilon_r \) (see Eq. (22)) because of the fact that in the cases studied below it is rather small in comparison with the characteristic quantities that give the leading contribution to the slowing phenomenon.

The expression (27) can be written also in the more usual form (cf. Refs. 8, 9)

\[ \epsilon(k, \omega) \approx 1 - \frac{a(k)}{\delta \omega + i \gamma} \frac{1}{1 + C}, \]  

(28)

where \( C = a(k)/(\delta \omega + i \gamma) \). Note that the derived expression, accurate to within a numerical factor 1/3 before \( C \), coincides with the Lorentz-Lorenz relation (see, e.g., Refs. 12, 13). Let us emphasize that we derive it in the framework of the microscopic approach. The formula (28) results directly from Eqs. (21)–(23). Thus, it contains the microscopic parameters of the studied system, which are calculated in the microscopic theory. In accordance with Eq. (23), the expression (28) carries information about the response of the ideal gas in the pure Bose-condensed state. Note that in a rather dense condensate there also can appear additional collective effects related to the interaction between atoms. Depending on the value of parameter \( (\nu \lambda^3) \), one should consider such effects (see Refs. 8, 9, 13, 14). As it is mentioned above, the account of interaction effects in the region \( \nu \lambda^3 \gg 1 \) results in redefining the shift \( C \). But let us recall that we consider the case of small \( C \) (see Eq. (22)),

\[ |a(k)/(\delta \omega + i \gamma)| \ll 1. \]  

(29)

In some sense, this inequality results from the fact that we describe the system of alkali-metal atoms as an ideal gas. This consideration means that we neglect the interaction between atoms (see Eq. (A1)). Approximate estimates of the applicability of this approach are given in the Appendix.

To analyze the propagation properties of electromagnetic waves in a BEC, let us introduce the refractive index and damping factor. We also set the magnetic permeability equal to unity in the dispersion relation (26), \( \mu(k, \omega) \approx 1 \). It is easy to verify that this approximation is valid in this case. The refractive index \( n(k, \omega) \) and the damping factor \( n''(k, \omega) \) are expressed in terms of the real \( \epsilon' \) and imaginary \( \epsilon'' \) parts of the permittivity \( \epsilon \), \( \epsilon = \epsilon' + i \epsilon'' \), as follows:

\[ n' = \sqrt{\frac{\epsilon'^2 + \epsilon''^2 + \epsilon'}{2}}, \quad n'' = \sqrt{\frac{\epsilon'^2 + \epsilon''^2 - \epsilon'}{2}} \]  

(30)

The quantities \( \epsilon' \) and \( \epsilon'' \), in turn, are found from Eq. (26):

\[ \epsilon' = \frac{\delta \omega(\delta \omega + a) + \gamma^2}{(\delta \omega + a)^2 + \gamma^2}, \quad \epsilon'' = \frac{\gamma a}{(\delta \omega + a)^2 + \gamma^2}. \]  

(31)

It is known that the group velocity of the electromagnetic pulse is defined by the relation

\[ v_g = \frac{c}{n' + \omega(\partial n'/\partial \omega)}, \]  

(32)

where \( c \) denotes the speed of light in vacuum. Hence, in the case of small energy losses and strong dispersion, one finds from Eq. (31)

\[ v_g \approx 2c \frac{((\delta \omega + a)^2 + \gamma^2)^2}{\delta \omega \left((\delta \omega + a)^2 - \gamma^2\right)}. \]  

(33)

This expression gives the possibility of studying the dependence of the group velocity not only on the frequency detuning \( \delta \omega \) and the linewidth \( \gamma \), but also on the characteristic parameters of the system, which give a contribution to the value of the parameter \( a \) (see Eq. (27)).

Now we define the conditions when significant slowing of electromagnetic pulses in a two-level system with a BEC is possible without essential absorption. Considering the limit \( \delta \omega \to 0 \) and employing Eq. (31), one gets the limits for the real and imaginary parts of the permittivity:

\[ \lim_{\delta \omega \to 0} \epsilon' = \frac{\gamma^2}{\gamma^2 + a^2}, \quad \lim_{\delta \omega \to 0} \epsilon'' = \frac{\gamma a}{\gamma^2 + a^2}. \]

According to these relations, the condition of small dissipation \( (|\epsilon''| \ll \epsilon') \) may be written as

\[ \frac{|a|}{\gamma} \ll 1. \]

Note that this inequality defines not only the limits of the transparency region for electromagnetic waves, it also defines the upper limits of applicability of our approach (see Eqs. (22) and (23)). Really, as it is easy to see, in the case \( |a| \sim \gamma \), the second term in the denominator of Eq. (23) (the so-called Lorentz-Lorenz shift \( C \)) can become of the order of unity. But, as mentioned above, we assume that this term is small \( (|C| \ll 1) \).
We need also to use the condition defining sufficient pulse slowing (the strong dispersion condition), \(v_g \ll c\), which, in accordance with (33), take the form
\[
\frac{c}{v_g} \equiv \frac{\Delta \varepsilon_{21} |a|}{2\gamma^2} \gg 1.
\]
(34)
In a system with defined energy structure of atoms \((\Delta \varepsilon, \gamma = \text{const})\) the only parameter that may be varied is the level occupation difference \((\nu_1 - \nu_2)\), which is included in the parameter \(a\) (see Eq. (27)). Taking into account this fact and the above inequalities, one gets
\[
\frac{\gamma}{\Delta \varepsilon_{21}} \ll \frac{|a|}{\gamma} \ll 1.
\]
(35)
This inequality characterizes the region of densities (parameter \(a\)) in which the phenomenon of significant slowing of electromagnetic waves in a BEC may be observed.

Next, on the basis of the derived formulas, we consider two cases of slowing of weak electromagnetic waves in a BEC. The first one is concerned with electromagnetic pulses in the optical region of the spectrum (resonance D lines of alkali-metal atoms), and the second one is related to microwaves (hyperfine transitions of the ground states of alkali metals).

V. SLOWING OF ELECTROMAGNETIC WAVES IN THE OPTICAL RANGE IN A BEC

As was mentioned above, a significant slowing of electromagnetic pulses was observed in a condensed gas of sodium atoms in Ref. [1]. In this experiment, for the observation of ultraslow light pulses the effect of electromagnetically induced transparency (EIT; see Ref. [15] for details) was used. For realization of this effect one needs to use an additional laser of large intensity besides the laser pulse, which is slowed in the experiment. But we should note that the description of this experiment in the framework of the Green-function formalism is impossible. The main reason is that for EIT support it is necessary to use a high-power (coupling) laser. Thus, the influence of such a laser on a BEC cannot be described correctly within the linear response theory. But let us emphasize that the use of a high-power laser is not a necessary condition for the existence of the slowing phenomenon.

Therefore, we consider the above microscopic theory in the case of slowing of weak light pulses in a sodium BEC at temperatures \(T \ll T_c\) in the absence of EIT. In particular, in the framework of the developed approach we study the slowing phenomenon, when the laser frequency is close to the transition from the ground state \((l = 0, \, 3P_{1/2} \text{ state, } F = 2 \text{ for sodium})\) to the excited one \((l = 1, \, 3P_{3/2} \text{ state, } F' = 1)\). Note, that the dipole transition \(1 \rightarrow 2\) between these states is allowed.

First of all, we emphasize the following. The matrix element of the charge density \(\sigma_{12}\) (see Eq. (4)) contains the wave functions of a hydrogenlike atom \(\varphi_{1,2} (y)\), which have a sharp maximum at the point \(y\). Due to this fact we can expand the exponents \(\exp \left[ i (m_p/m) \nu y \right]\) and \(\exp \left[ -i (m_e/m) \nu y \right]\) into series of perturbation theory over \((ky) \ll 1\). Then, in the first order of perturbation theory, one gets
\[
\sigma_{12}^{(1)} (k) \approx i k d_{12},
\]
where \(d_{12}\) is the atomic dipole moment, which is related to the dipole transition \(1 \rightarrow 2\):
\[
d_{12} = e r_{12}, \quad r_{12} = \int dy \, \varphi_1^*(y) \varphi_2 (y) y.
\]
(36)
For the quantitative estimates we take for the sodium atom \(|\sigma_{12} (k)|^2 \approx k^2 d^2 / 3\) that results in (see Eq. (21)) \(a = (4\pi/3)(\nu_1 - \nu_2) d^2\). Taking \(d^2 \approx (3.52 e r_0)^2 S_{FF}\) [10], where \(r_0\) is the Bohr radius and \(S_{FF}\) is the relative strength of the \(F \rightarrow F'\) transition, \(\Delta \varepsilon_{21} \approx 2.1 \text{ eV}\), and the linewidth \(\gamma = S_{FF}/T_c = 2 \left( T_c = 4.1 \times 10^{-8} \text{ eV is the natural linewidth} \right)\), below we consider \(F = 2 \rightarrow F' = 2\) transition, \(S_{22} = 1/4\). Since we consider the propagation of weak electromagnetic waves through the condensate, we assume that the density of atoms in the excited state is negligibly small. In other words, we treat the occupation of the excited level is small due to low intensity of the propagating pulse \((\nu_2 \ll \nu_1)\). Next, using Eq. (35), one can obtain the following inequality, which defines the conditions of slowing in condensed sodium vapor at \(T \ll T_c\):
\[
3 \times 10^5 \ll \nu \ll 10^{14} \text{ cm}^{-3}.
\]
(37)
From this inequality it is easy to see that the density value of condensed atoms in the case of real experiments with BECs \((\nu = 5 \times 10^{12} \text{ cm}^{-3}; \text{ see Ref. [1]}\) lies in the limits of the derived interval.

Note that the main parameter that can be measured in experimental conditions is the relative intensity of the transmitted light. This quantity for a homogeneous medium is defined by the expression \(I/I_0 = \exp \left[ -e''(\omega) k L \right]\), where \(L\) is the transversal size of the atomic sample. Taking the density of condensed atoms in the ground state \(\nu = 5 \times 10^{11} \text{ cm}^{-3}, k = 1.066 \text{ cm}^{-1}, L = 0.004 \text{ cm},\) one finds \(I/I_0 \approx e^{-2.2} \approx 0.11\). This allows to state that in the limit of zero temperatures, \(T \ll T_c\), ultraslow light pulses of weak intensity may be directly observed for these parameters (according to Eq. (36), \(v_g/c \approx 10^{-6}\)). However, a significant part (up to 89%) of the pulse is absorbed by the medium.

It is easy to see that the intensity of the transmitted light has a rather strong dependence on the system parameters, which make a contribution to the exponent. Thus, the situation with the slowing phenomenon may completely change, when temperature effects are taken into account. These effects may increase the value of \(e'' (\omega)\) resulting in the strong reduction of the relative intensity of the transmitted light. For example, with increase of \(e'' (\omega)\) by several times the intensity reduces to
less than 1%. It is possible to take into account the influence of temperature effects on the propagation properties in the framework of the Green-function formalism. However, this is a separate problem, which is not considered in the present paper.

Note also that in the case described above the group velocity of the pulse is negative (anomalous dispersion in the case of a normal population, \( \nu_1 > \nu_2 \)). We return to the discussion of this fact in Sec. IV.

Next, let us consider light slowing in a multicomponent BEC employing the developed theory. Nowadays, the realization of a multicomponent Bose condensate in experimental conditions is rather achievable (see, e.g., Ref. [13]). The formulation of this problem is interesting for the following reasons. As is known, in an external magnetic field the Zeeman splitting of levels occurs, and the spacing between different components of the multiplet \( \Delta \varepsilon_{\text{mag}} \) is proportional to the magnetic field intensity. Thus, in a weak external magnetic field this energy difference is much smaller than the difference between the nonsplit levels of the hyperfine transition \( \Delta \varepsilon_{\text{hy}} \). If the atoms can occupy different magnetic states of the multiplet, we can talk about the existence of a multicomponent BEC. In such a condensate the effects related to the resonant absorption for some defined frequencies may be significantly smaller than in the case described above.

As an example, we consider a two-component condensate of sodium atoms in the quantum states \( 3S_{1/2}, F = 2, m_F = -2 \) and \( 3S_{1/2}, F = 2, m_F = -1 \) (below we denote them with subscripts 1 and 2, respectively). Now, let us study the response of the system to laser radiation, which is detuned relative to the transition between these two states and the state \( 3P_{3/2} \) (denoted below by the subscript 3), as shown in Fig. 2.

By using Eq. (24), we study the dependence of the dispersion characteristics of an ideal gas of sodium atoms in a BEC state on the frequency of external radiation. For simplicity, let us treat as equal the transition probabilities from the upper state to one of the lower states, \( \gamma_{31} = \gamma_{32} \). We also consider the densities of atoms in two lower states \( \nu_1 = \nu_2 = 5 \times 10^{12} \text{ cm}^{-3} \) and \( \sigma_{13} = \sigma_{23} \).

In fact, these assumptions are made only for convenience, but they are not necessary in principle for numerical evaluations. Let the energy of splitting be \( \Delta \varepsilon_{\text{mag}} = 8 \gamma_{31} \). Taking all other parameters the same as in the above estimates, one finds the dependencies of the dispersion characteristics, as illustrated in Fig. 3. From these graphs we conclude that in such a system the signal may slow down approximately to 264 m/s.

At first sight, the last result contradicts a well-known statement from classical optics (see, e.g., Ref. [18]) that in a two-level gas a large index of refraction is usually related to a high absorption ratio. Hence, on atomic resonances the signal is mostly absorbed. However, this contradiction is only apparent. Let us emphasize one more time that in the case described above the real part of the susceptibility \( \chi' = (\epsilon' - 1) \), which gives the main contribution in the change of the refractive index, is rather small in magnitude (as one can conclude from Fig. 3 \( |\chi'| \lesssim 10^{-2} \)). The imaginary part \( \chi'' = \epsilon'' \), which defines the absorption ratio, is rather small too, \( \chi'' \lesssim 10^{-2} \) (in the central part \( \chi'' \approx 10^{-3} \)). In other words, in the considered case, the matter is not dense for laser radiation (the refractive index is close to unity), but has a strong

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**FIG. 2:** Scheme of energy levels of a sodium atom. For the sake of simplicity the levels are drawn not to scale, and the hyperfine structure of the excited state is not shown.

**FIG. 3:** Dependencies of the refractive index, damping factor, and transmitted light intensity for the two-component Bose condensate.
dispersion due to closely situated resonances. This is the reason why the group velocity of the pulse may be sufficiently reduced, while its absorption ratio remains rather low.

Let us note that in the regions of anomalous dispersion in Fig. 5 (where the derivative of the refractive index is negative) there is good agreement with the results obtained in Ref. [12], in which the authors studied the dependence of the refractive index on the laser intensity. The agreement takes place in the limit of low light intensity. In fact, we consider only this limiting case in the present paper.

As for the obtained dependencies as whole, they are similar to the dependencies that are observed in the EIT regime [15]. One can, by analogy with Ref. [1], state that the phenomenon of simultaneous manifestation of low group velocity value and low absorption ratio in the central part of Fig. 5 is the result of destructive interference between two absorption-emission paths, which correspond to the transitions $1 \rightarrow 3$ and $2 \rightarrow 3$, respectively. The difference is that the spacing between these transitions equals not the Rabi frequency value $\Omega_R$, but the energy of splitting of two occupied magnetic sublevels $\Delta \varepsilon_{\text{mag}}$. If we compare the dependence of light absorption (the lowest graph in Fig. 5) with the analogous one in the EIT scheme [15], we can see that the intensities of the transmitted light in the central part are nearly the same ($I/I_0 \approx 0.3$). However, in the EIT case, the maximum is sharper due to the nonlinear nature of the effect.

Let us emphasize that in our case the energy spacing between magnetic states must be larger than the linewidth of the transitions. In other words, these sublevels cannot be situated arbitrarily near each other. Therefore, the intensity of the transmitted light depends directly on the energy spacing $\Delta \varepsilon_{\text{mag}}$ and on the density of the condensed states in the condensate $\nu_{1,2}$. At the same time, the group velocity of the pulse also strongly depends on both these characteristics. Thus, if one tries to change the characteristics of the system for much stronger reduction of the group velocity value (increase in the density of atoms or decrease in the energy spacing $\Delta \varepsilon_{\text{mag}}$), the ratio of absorption will correspondingly increase (see also Ref. [18]). In our case, the absorption ratio is the main limiting factor for the achievement of lower values of the group velocity.

As for the effect of electromagnetically induced transparency, in the experiment the Rabi frequency (which is analogous to the energy $\Delta \varepsilon_{\text{mag}}$ in our case) is usually set to be of the same order as the linewidth. Thus, the effect of slowing due to the Zeeman splitting is somewhat weaker ($v_g \approx 264$ m/s in comparison with $v_g \approx 32$ m/s in Ref. [1]), but the authors hope that the observation of this kind of slowing is rather possible in the present experimental conditions.

VI. SLOWING OF MICROWAVES BY HYPERFINE GROUND STATES OF ALKALI-METAL ATOMS

As mentioned above, the proposed microscopic theory is correct when the occupation of the excited atomic states is stimulated by an external electromagnetic field (the case that is studied in Sec. VI, or when the lifetimes of the excited states are much larger than the relaxation times. Note that for alkali-metal atoms all sublevels of the hyperfine ground state can be treated as such long-living states. This follows from the fact that the dipole transitions are forbidden for these states. Waves whose frequency corresponds to the energy difference between the levels of hyperfine structure are usually called microwaves. Since there are many trapping-related experiments with ultracold atoms prepared in different hyperfine states, the problem of the possibility of slowing microwaves becomes rather significant.

By considering the example of cesium atoms in the BEC state, we try to find the conditions in which, in our opinion, the slowing phenomenon may be observed experimentally. To this end, according to the developed approach, let us show that the region defined by Eq. (35) exists. Note that a brief description of the application of the microscopic approach for the description of microwave slowing may be found in Ref. [19].

It is known that alkali-metals ($^{133}$Cs, in particular) do not have a dipole moment in the ground state. This agrees with Eq. (39), from which one can find that the dipole moment $d$ equals zero for the wave functions of the ground state. Thus, to describe the slowing of microwaves on the levels of hyperfine structure with forbidden dipole transitions, we should expand the charge density matrix element $\sigma_{12}$ (see Eq. 4) to the second order in $(ky) \ll 1$. As a result, one gets

$$\sigma_{12}^{(2)}(k) \approx \frac{e}{3}(kr_0)^2,$$

where $r_0$ is the atomic radius (for the cesium ground state $r_0 \approx 2.6 \times 10^{-8}$ cm [20]). Let us consider that the levels are degenerate, $g_1 = 7$, $g_2 = 9$ ($g_j = 2F_j + 1$, where $F_j$ is the total spin of an atom in the $j$ hyperfine state, $j = 1, 2$,) and also set $k = (\Delta \varepsilon_{21}/c)$ with $\Delta \varepsilon_{21} \approx 3.8 \times 10^{-5}$ eV (microwaves with frequency 9.2 GHz). We take the natural linewidth $\gamma$ of the upper hyperfine level equal to the value that correspond to the anticipated accuracy ($10^{-16}$) in “cesium fountain clock” experiments [21], $\gamma \approx 3.8 \times 10^{-21}$ eV. Then, on the basis of Eqs. (27) and (35), one may get the following region where the slowing phenomenon may be observed:

$$10^{-1} \ll |\nu_1 - \nu_2| \ll 10^{15} \text{ cm}^{-3}.$$

It is easy to see that the effect becomes greater when the density difference increases. When it reaches the upper limit of (39), damping effects prevail in the system and the signal is mostly absorbed by the medium. We stress that the region of densities (39), in our opinion, looks
Let us assume that the magnetic sublevels of the upper level of hyperfine structure are kept away from each other (the linewidth of the transition is much smaller than the energy difference between sublevels $\Delta \varepsilon_{\text{mag}}$, as shown in Fig. 3). Then, a signal tuned exactly to the transition between the occupied lower state and one of the upper states (narrow black arrowed lines in Fig. 3) may be slowed down, as described above. The only difference from the previous situation is that the states are no longer degenerate ($g_1 = g_2 = 1$) and the linewidth in Eq. (33) corresponds to the transition probability from the chosen sublevel with the projection $j = (−F', ..., F')$ to the occupied lower state. Taking all other values the same when deriving the inequality (39), one can find the inequality in the presence of the external magnetic field

$$10 \ll |\nu_1 - \nu_2| \ll 10^{17} \text{ cm}^{-3}. \quad (40)$$

In the experiment $^{22}$, condensed cesium atoms with a peak density of $1.3 \times 10^{13}$ cm$^{-3}$ were kept in a trap. Thus, according to Eq. (40), one finds that in such a system ultraslow microwaves may be observed. Moreover, direct calculations show that the group velocity of the signal (see Eq. (34)) propagating in the mentioned system is close to $3 \times 10^{-4}$ cm/s.

Note that here we must make important remarks relating to the possibility of experimental observation of ultraslow microwaves in a BEC. In fact, it is impossible to make a direct measurement of the group velocity value in the conditions of a real experiment. It is usually calculated by measuring the time of pulse delay. The characteristic dimensions of the pulse (and, specifically, the wavelength) must be of the order of the atomic sample size, in order for it to propagate in the system. In the present experimental conditions this relation is not always satisfied. Usually, the characteristic sizes of a condensate are of the order of a millimeter (or even submillimeter), whereas the wavelength, related to the hyperfine transition, is of the order of centimeters (e.g., for cesium in the absence of an external field $\lambda_{hf} = 3.3$ cm).

In view of the aforesaid, there are two possible ways to avoid this problem. The first one, which is rather difficult at this moment, is to use other configurations of magneto-optical traps (and, probably, other cooling techniques) that may allow production of condensed clouds with centimeter sizes. The second one, and probably more realistic, is to use a rather strong bias field, which results (according to the Paschen-Back effect) in increase of the spacing between levels and, consequently, in the reduction of the resonance wavelength. At the present moment, such fields are completely accessible, although they must be rather strong. For example, to reduce the clock transition wavelength to $\lambda_{hf} = 2$ mm in cesium, the field intensity must be of the order of 45 kG, which is quite achievable from the standpoint of the trapping-related experiments (see Ref. $^{22}$). Let us note also that, corresponding to the results $^{6}$ (see also Fig. 1), the presence of the additional magnetic field may sufficiently raise the value of the critical temperature.

![FIG. 4: Level scheme of cesium atom in an external magnetic field. The first sign in brackets corresponds to the total spin $F$, the second one to its projection $m_F$. For the sake of visibility, the figure is not drawn to scale and only the most relevant levels are shown.](image-url)
Note that, in order to observe the effect for the limiting case considered while deriving Eq. 35, one needs to use a microwave signal with detuning $\delta \omega_s$ much smaller than the level linewidth:

$$\delta \omega_s \ll \gamma_j.$$  \hfill (41)

In a real experiment it is rather difficult to to satisfy such a condition. This is due to the fact that the natural linewidth $\gamma_j$ of the hyperfine structure levels is rather small (dipole transitions are forbidden and the transitions come from the higher order effects).

In this case, the detuning of a maser between close-lying magnetic sublevels of the hyperfine ground state structure may save the situation. At the same time, we must satisfy the condition

$$\gamma_j \ll \delta \omega_s \ll \Delta \varepsilon_{mag},$$

where $\Delta \varepsilon_{mag}$ is the energy difference between these sublevels. For example, a microwave signal that is detuned relative to the transitions between two neighboring states of the upper hyperfine multiplet and the occupied lower state (wide gray arrowed line in Fig. 4) may satisfy this condition at certain intensities of the magnetic fields.

Let all atoms be condensed in the lower (occupied) state with the density $\nu$ and energy $\varepsilon_0$. Then, it is not difficult to come to the following expression for the permittivity (see Eqs. 24 and 27):

$$\varepsilon^{-1}(k, \omega) \approx 1 + a \sum_{j=-F'}^{F'} \left( \frac{1}{\omega - \Delta \varepsilon_j(B) + i \gamma_j} \right), \hfill (42)$$

where $\Delta \varepsilon_j = (\varepsilon_{F', m_{F'}}) - \varepsilon_0$. The derived expression defines the dependence of the refractive index on the frequency of the microwave radiation. Therefore, it also defines the conditions for pulse slowing (see Eqs. 30 and 32). It is easy to see that in this case the condition of wave slowing (cf. Eq. 33) takes the form

$$\frac{\Delta \varepsilon_{mag}}{\Delta \varepsilon_{hf}} \ll \frac{4a}{\Delta \varepsilon_{mag}} \ll \frac{\Delta \varepsilon_{mag}}{2 \gamma}. \hfill (43)$$

The structure of the $^{133}\text{Cs}$ upper hyperfine state in the presence of magnetic field is schematically shown (not to scale) in Fig. 5. From this graph, one may conclude that the energy difference $\Delta \varepsilon_{mag}$ is small either in the region of weak magnetic fields or in the region of levels intersection (strong magnetic fields).

In the presence of an external magnetic field the dependence of the refractive index has a rather complicated form due to the large number of magnetic states (nine sublevels of the upper $F = 4$ state; see Fig. 3). Hence, it is more convenient to show such dependence in a figure.

The dependencies of the refractive index and damping factor on the frequency are shown in Fig. 6. Notice that in the central part ($\delta \omega_s = 0$, corresponding to the wide gray arrowed line in Fig. 4) the refractive index has a slope with steepness dependent on the interval $\Delta \varepsilon_{mag} = \varepsilon_{(4,4)} - \varepsilon_{(4,3)}$, which, in turn, depends on the magnetic field intensity. This means that a microwave signal that is detuned in this way may be sufficiently slowed. Moreover, its group velocity depends directly on the magnetic field intensity.

The direct calculations show that, e.g., for an ideal gas of condensed cesium atoms ($\nu = 1.3 \times 10^{13} \text{ cm}^{-3}, \gamma_j = 3.8 \times 10^{-21} \text{ eV}$) with the energy of splitting $\Delta \varepsilon_{mag} = 5.7 \times 10^{-20} \text{ eV} (\Delta \varepsilon_{mag}/\gamma_j = 15$, such as shown in Fig. 5), the pulse can be slowed down to $0.01 \text{ cm/s}$. But we emphasize that in this case (weak bias field) the pulses with wavelengths of the order of centimeters may be slowed in a BEC. Thus, one must use condensed clouds of the same size, which is, as mentioned above, hardly achievable from the standpoint of present experimental conditions.

In the case of strong magnetic fields (close to the region of intersection of levels of the Paschen-Back splitting; see Fig. 5), the graphs of the refractive index and damping factor have qualitatively the same dependence, as shown in Fig. 6. Moreover, the wavelength of the corresponding radiation in these conditions is of the order of millimeters, which is quite acceptable from the standpoint of a real experiment. But, as follows from the estimates that are based on Eqs. 32 and 13, the conditions for slowing of electromagnetic waves are satisfied only in a region of magnetic field intensity $B_{cr}$ that approximately equals 583 kG. In this region of bias field one can effectively control the ultraslow microwave phenomenon in a BEC. Present scientific papers note the achievement of ultrastrong intensities of magnetic fields (e.g., of the order of 600 kG; see in that case Ref. 28), but the authors of the present work, being theorists, cannot state whether it is possible to use such fields in BEC-related experiments.
FIG. 6: Refractive index and damping factor dependencies for cesium atoms in the BEC state in the region of frequencies close to the transitions between magnetic sublevels of hyperfine splitting. Left and right peaks correspond to the transitions $|3,3⟩ → |4,−4⟩$ and $−|3,3⟩ → |4,−3⟩$, respectively. The parameter used for the Zeeman splitting energy $Δε_{mag}/γ_j = 15$.

VII. CONCLUSION

Thus, we propose a microscopic approach for the description of the linear response of an ideal gas of hydrogenlike atoms in the BEC state to perturbation by an external electromagnetic field. The use of this approach allows us to find the permittivity of dilute gases of alkali-metal atoms in a BEC state as a function of the microscopic characteristics of the system at temperatures $T ≪ T_c$.

The main advantage of the considered approach is that it comes from a microscopic description, which is based on the approximate formulation of the second quantization method for systems that consist of bound states of particles. Another advantage is its convenience for studying the dispersion characteristics of gases regardless of the type and number of quantum states in the system. As a possible drawback one can refer the fact that in the form in which it is formulated in this work it cannot be correctly used for the description of the response to a strong perturbation (as may probably be done in the approaches) owing to the use of linear theory. In other words, this approach cannot be used without considerable modification to systems in which strong laser radiation is used (e.g., the coupling field for the provision of EIT). Note that we neglected also the interaction between atoms (in view of the consideration of rather dilute gases) and the influence of temperature effects. Temperature effects in the theory developed may be taken into account, but this represents a separate problem.

The developed approach is demonstrated on the example of the reduction of the speed of light in a BEC of sodium vapor with parameters close to the experimental ones. It is shown that in some conditions the signal may be sufficiently slowed in the absence of an additional strong (coupling) field. A strong field is used only to provide electromagnetically induced transparency leading to the reduction of the damping factor value. Thus, it is not a necessary condition for the observation of ultrashort pulses.

The following cases are considered: the optical signal is tuned to the resonance frequency and the signal is detuned relative to close transitions in the two-component condensate. It is shown that in these situations ultrashort pulses may be observed. The conditions for slowing, which depend on the macroscopic characteristics of the system, are found. The dispersion dependencies are studied also.

The approach is applied for studying the propagation of microwaves in a BEC. It is shown that the microwave signal, which is tuned to the transition between different hyperfine levels of the ground state, can propagate in the system with rather small energy loss. It is pointed out that the sign of the group velocity of the propagating pulse depends directly on the occupation difference of the states.

The influence of an external static and homogeneous magnetic field (taking into account the Zeeman and Paschen-Back effects) on the slowing parameters is studied. Considering cesium atoms, it is shown that the pulse can be slowed down to extremely small values in some conditions.

APPENDIX A: ON THE CORRECTNESS OF THE IDEAL GAS APPROXIMATION

Let us now say some words relating to the limits of applicability of the developed approach. Note that we consider the condensed gas of alkali-metal atoms as ideal. In other words, we assume that the interaction between atoms is small in comparison with the binding energy and in comparison with the interaction of particles with the external field.

To estimate the influence of effects related to the interatomic interaction, let us consider the Hamiltonian $\hat{H}_{\text{int}}$, which is introduced in Eq. (1). As mentioned above, the microscopic expressions for it are found in Ref. [3] in the framework of the developed formulation of the second-quantization method. It is also shown that in the low-energy approximation (or in the case of low temperatures) this operator can be written as

$$\hat{H}_{\text{int}} = \int dx \, dy \, v_{αβγδ}(x − y)\hat{n}_α^\dagger(x)\hat{n}_β^\dagger(y)\hat{n}_γ(y)\hat{n}_δ(x), \quad (A1)$$
where the Einstein summation convention over indices \(\alpha, \beta, \gamma, \delta\) is meant, and the kernel \(v_{\alpha\beta;\gamma\delta}(z)\) has the form
\[
v_{\alpha\beta;\gamma\delta}(z) = \frac{1}{25} \left[ z^2 (d_{\alpha\delta} d_{\beta\gamma}) - 3 (zd_{\alpha\delta}) (zd_{\beta\gamma}) \right], \tag{A2}\]
in which the quantities \(d_{\alpha\beta}\) are the matrix elements of the dipole moment operator of a single atom. Such matrix elements can be expressed in terms of the atomic wave functions \(\varphi_{\alpha}(y)\) in different quantum states \(\alpha, \beta, \gamma, \delta\),
\[
d_{\alpha\beta} = e \int dy \, \varphi_{\alpha}^{*}(y) \varphi_{\beta}(y) y. \tag{A3}\]
Thus, one can see that in this case the Hamiltonian (A1) corresponds to the dipole-dipole interaction of atoms.

It is easy to verify that in the case of the interaction of atoms in the ground state, the expression (A1) is inapplicable. It results from the fact that the dipole moment of the hydrogen-like atom in the ground state is equal to zero. One may find the Hamiltonian of interaction \(H_{\text{int}}^{(0)}\) of the atoms in the ground state from the second-order terms of the perturbation theory over the interaction (A1). If we assume that this interaction is small in comparison with the binding energy (2), we get (see Ref. [29]),
\[
H_{\text{int}}^{(0)} = \frac{1}{2} \int dx \, dy \, v^{(0)}(x - y) \hat{\eta}_{\alpha_0}^{\dagger}(x) \hat{\eta}_{\alpha_0}^{\dagger}(y) \hat{\eta}_{\alpha_0}(y) \hat{\eta}_{\alpha_0}(x), \tag{A4}
\]
where \(\hat{\eta}_{\alpha_0}(x)\) and \(\hat{\eta}_{\alpha_0}(x)\) are the creation and annihilation operators of hydrogen-like (alkali-metal) atoms in the ground state with the set of quantum numbers \(\alpha_0\) at the point \(x\), respectively. The quantity \(v^{(0)}(z)\) can be expressed by the relation
\[
v^{(0)}(z) = \sum_{\beta,\lambda}^{'} |v_{\beta\lambda;\alpha_0\alpha_0}(z)|^2 \frac{2 \varepsilon_{\alpha_0} - \varepsilon_{\beta} - \varepsilon_{\lambda}}{2 \varepsilon_{\alpha_0} - \varepsilon_{\beta} - \varepsilon_{\lambda}}, \tag{A5}\]
where \(v_{\beta\lambda;\alpha_0\alpha_0}(z)\) is defined by Eq. (A2), and \(\varepsilon_{\alpha_0}\) is the ionization energy of an atom. The prime in the sum means that the terms with \(\beta = \alpha_0\) and \(\lambda = \alpha_0\) are omitted. Hence, one can conclude that the Hamiltonian (A1) corresponds to the well-known Hamiltonian of the van der Waals interaction (see, e.g., Ref. [29]).

We must note that it is convenient to replace the operators (A1) and (A4) by effective Hamiltonians (see in this case Refs. [29, 30]). Thus, the effective Hamiltonian of the dipole-dipole interaction takes the form
\[
H_{\text{int}}^{(\text{eff})} = U_{\alpha\beta;\gamma\delta} \int dx \, \hat{\eta}_{\alpha_0}^{\dagger}(x) \hat{\eta}_{\gamma}(x) \hat{\eta}_{\gamma}(x) \hat{\eta}_{\alpha_0}(x), \tag{A6}
\]
with (see also Ref. [31])
\[
U_{\alpha\beta;\gamma\delta} = \frac{4\pi}{3} d_{\alpha\delta} d_{\beta\gamma}. \tag{A7}
\]
In the case of interaction of atoms in the ground state, the Hamiltonian (A1) can be replaced by
\[
\hat{H}_{\text{int}}^{(0)} = \frac{1}{2} U_0 \int dx \, \hat{\eta}_{\alpha_0}^{\dagger}(x) \hat{\eta}_{\alpha_0}^{\dagger}(x) \hat{\eta}_{\alpha_0}(x) \hat{\eta}_{\alpha_0}(x), \tag{A8}
\]
where
\[
U_0 = \frac{4\pi \hbar^2 a_s}{m} \tag{A9}
\]
and \(a_s\) is the s-wave scattering length (see, e.g., Ref. [30]). Note that the quantities \(U_{\alpha\beta;\gamma\delta}\) and \(U_0\) may be defined by Eqs. (A7) and (A9) only in the case when the quantum fluctuations, which are related to the elementary excitations in the system, are negligibly small [30].

To find the approximate criteria of applicability of our theory, let us consider a dilute weakly nonideal Bose gas, which interacts with a weak external electromagnetic field. It is known that the standard thermodynamic perturbation theory is inapplicable to such a gas in the region below the critical point \(T_c\). It results from the fact that there appear diverging terms in the region of small momenta (see in this case, e.g., Refs. [4, 30]). Therefore, one needs to use a special perturbation theory. Such a theory was constructed by Bogoliubov in [32]. Within this theory it is shown (and rigorously proven in Ref. [33]) that the creation and annihilation operators may be considered as \(c\) numbers,
\[
\hat{\eta}_{\alpha_0}^{\dagger}(x) \to \nu_{\alpha} e^{-\phi}, \quad \hat{\eta}_{\alpha_0}(x) \to \nu_{\alpha} e^{\phi}, \tag{A10}
\]
where \(\phi\) is some phase, which is irrelevant in our consideration. To account for the existence of the overcondensate particles (the density of which is much smaller than the density of the condensate particles) one needs to introduce a gas of elementary excitations, or quasiparticles with an energy spectrum, which is now called the Bogoliubov spectrum. The special perturbation theory, which was proposed by Bogoliubov, removes the divergences of the correlation functions in the region of small wave vectors [30]. In addition to the mentioned spectrum, this theory allows one also to find the ground state energy of the weakly nonideal Bose gas, its chemical potential (which has a nonzero value in case of interaction), and the corrections to these values, which result from the existence of quantum fluctuations. For example, the contribution of interaction effects to the ground state energy \(\varepsilon_0\) can be written as (see Ref. [30])
\[
\varepsilon_0 = U \frac{N^2}{2V} \left(1 + \frac{128}{15\pi^2} (\nu a_s^3)^{1/2}\right), \tag{A10}
\]
where \(U\) is the physical coupling constant (see Eqs. (A7) and (A9)), \(N\) is the overall number of atoms in the weakly nonideal Bose gas, \(V\) is the system volume, \(\nu\) is the density of particles, and \(a_s\) is the scattering length (see Eq. (A9)). The second summand in brackets in Eq. (A10) characterizes the quantum fluctuation contribution to the ground state energy of the gas. The formula (A10) is correct, when the so-called gas parameter \(\nu a_s^3\) is small,
\[
\nu a_s^3 \ll 1. \tag{A11}
\]
Below we assume that the relation (A11) is satisfied. Therefore, in the following considerations we do not take
into account the contribution of the quantum correlations to the characteristics of the weakly nonideal Bose gas interacting with the external radiation.

We note that the formalism developed in the present paper can be applicable to the system under consideration only in the case when the interatomic interaction is small compared to the interaction with the external field. Thus, let us compare the energy related to the interaction with the weak external radiation and the contribution of the interaction between atoms to the ground state energy $E_0$. If we neglect the existence of the quantum fluctuations in the system, this contribution, in accordance with Eq. (A10), takes the form

$$E_0 = V \sum_\alpha U_\alpha \nu_\alpha \nu_\alpha^0,$$  
(A12)

where $U_\alpha$ is the coupling constant, which characterizes the effective Hamiltonian of interaction between atoms with the set of quantum numbers $\alpha$ (see Eqs. (A6)–(A9)), and $\nu_\alpha$ is the density of such atoms in the condensate (by the index $\alpha_0$ we denote the set of quantum numbers of the ground state). The energy related to the interaction with the external electromagnetic field is characterized by the Hamiltonian $V (t)$ (see Eq. (3)). Its mean value $E_E$ can be expressed in the framework of the Green-function formalism (see, e.g., Refs. 4, 5). But, one can estimate it also in a different way. This energy must be of the same order of magnitude as the energy which is pumped into the system by the external field [6]. As mentioned above, the expression for the energy that is absorbed by the system from the external field can be written as (see Eq. (31))

$$E_E \sim V \frac{E^2}{8\pi} \xi,$$  
(A13)

where $E$ is the amplitude of the electric field in the pulse, $\xi = \{1 - \exp(-\epsilon^\prime(\omega)kL)\}$ is the part of the energy that is absorbed by the system in the case of propagation of a single pulse, and $L$ is the characteristic linear size of the atomic cloud.

The ideal gas approximation is correct when the energy $E_0$ is small in comparison with $E_E$,

$$\sum_\alpha U_\alpha \nu_\alpha \nu_\alpha^0 \ll \frac{E^2}{8\pi} \xi.$$  
(A14)

In Sec. IV we study the dispersion characteristics of a two-level system in the BEC state. There we consider that the density of atoms in the excited state (denoted by the set of quantum numbers $\alpha_1$) is much less than the density of atoms in the ground state (denoted by the set of quantum numbers $\alpha_0$), $\nu_\alpha \ll \nu_\alpha^0$. The number of atoms in the excited states can be estimated with the assumption that all energy (A13) that is absorbed by the system is spent on the excitation of certain states $\alpha_1$. If the intensity of the electromagnetic field is rather small, one gets (see, e.g., Ref. 34)

$$\nu_\alpha \sim \frac{E^2}{h\omega} \xi.$$  
(A15)

From the inequality $\nu_\alpha \ll \nu_\alpha^0$ we have

$$\frac{E^2}{h\omega} \xi \ll \nu_\alpha^0.$$  
(A16)

Therefore, using Eq. (A15) we come to the following relation for the contribution of the interaction (A12) in the ground state energy (see also Eqs. (A7), (A9)):

$$E_0 = \frac{4\pi \hbar^2 a_s}{m} \nu_\alpha^2 + \frac{4\pi}{3} |d_{\alpha_0 \alpha_1}|^2 \nu_\alpha \frac{E^2}{h\omega} \xi.$$  
(A17)

Thus, the inequality (A14) can be written as

$$\frac{4\pi \hbar^2 a_s \nu_\alpha^2}{m} + \frac{4\pi}{3} |d_{\alpha_0 \alpha_1}|^2 \nu_\alpha \frac{E^2}{h\omega} \xi \ll \frac{E^2}{8\pi} \xi.$$  

This relation defines the limits for the intensity of the external electromagnetic field and the density of condensate particles in the ground state.

To find numerical estimates, we take the value of the field intensity from the experiment [1], in which a laser with power density 5 mW/cm$^2$ and $\lambda = 589$ nm was used. The scattering length $a_s$ for a number of alkali-metals in the ground state is of the order of a few nanometers [30]. Taking the density $\nu_\alpha \sim 10^{13}$ cm$^{-3}$ and $\xi \approx 0.5$, one easily obtains

$$\frac{4\pi \hbar^2 a_s \nu_\alpha^2}{m} \sim 10^{-12} \text{ erg cm}^{-3},$$

$$\frac{4\pi}{3} |d_{\alpha_0 \alpha_1}|^2 \nu_\alpha \frac{E^2}{h\omega} \xi \sim 10^{-14} \text{ erg cm}^{-3},$$

$$\frac{E^2}{8\pi} \xi \sim 10^{-6} \text{ erg cm}^{-3}.$$  

It is easy to see that in this case the inequality (A17) is correct. Let us note that the relation (A16) is satisfied also with a rather large "reserve".

Now we can conclude that the relations (22) and (29) (see also Eqs. (37) and (40)) and the inequalities (A14), (A16), and (A17) define the limits of applicability of our theory for the description of the response of a condensed gas of alkali-metals to a perturbation by a weak external field. For completeness of the description, one should also add the condition

$$\frac{E^2}{8\pi} \xi \ll \varepsilon_{\alpha \alpha_1} \nu_\alpha^2,$$

which characterizes the smallness of the energy related to the interaction with the external field in comparison with the characteristic energy that may be enough to ionize atoms in the condensate.

Note also that here we do not consider the interaction of closely located radiating dipoles. This can be explained by the fact that the effects produced by this interaction are quadratic in the density of atoms in the excited state (dipoles), and the density of dipoles in dilute vapors of alkali-metals is low (the average distance between dipoles is much larger than the characteristic
wavelength). First, the experimentalists \[1\] come to such a conclusion. Second, one can come to it after the following consideration. As for excited states produced by external pumping, the statement about low density of dipoles can be proven by estimates based on Eq. (A15). Note also that some part of the excited atoms can be produced by collisions in the system. But for a steady-state system the density of such atoms can be comparable only in the case when there is an equilibrium between the radiation (photons) and matter (see Ref. \[4\]). For a BEC in dilute vapors of alkali-metals, in which the ultraslow light phenomenon is observed, such equilibrium can be achieved, e.g., by a special set of mirrors that returns photons to the system. In this case, the interaction of closely located radiating dipoles must be taken into account because it can have a strong impact on the phenomenon under study. The mentioned effect should be taken into account also for rather dense gases in the BEC state.

Of course, all of the estimates introduced in this section are rather approximate. Probably more explicit estimates can be made only from an approach related to the Green-function formalism (see, e.g., Refs. \[7\], \[8\]).

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