TEMPORAL INTERPRETATION OF INTUITIONISTIC QUANTIFIERS:
MONADIC CASE

GURAM BEZHANISHVILI AND LUCA CARAI

ABSTRACT. In a recent paper we showed that intuitionistic quantifiers admit the following temporal interpretation: “always in the future” (for ∀) and “sometime in the past” (for ∃). In this paper we study this interpretation for the monadic fragment MIPC of the intuitionistic predicate logic. It is well known that MIPC is translated fully and faithfully into the monadic fragment MS4 of the predicate S4 (Gödel translation). We introduce a new tense extension of S4, denoted by TS4, and provide an alternative full and faithful translation of MIPC into TS4, which yields the temporal interpretation of monadic intuitionistic quantifiers mentioned above. We compare this new translation with the Gödel translation by showing that both MS4 and TS4 can be translated fully and faithfully into a tense extension of MS4, which we denote by MS4t. This is done by utilizing the algebraic and relational semantics for the new logics introduced. As a byproduct, we prove the finite model property (fmp) for MS4t and show that the fmp for the other logics involved can be derived as a consequence of the fullness and faithfulness of the translations considered.

1. INTRODUCTION

It is well known that, unlike classical quantifiers, the interpretation of intuitionistic quantifiers is non-symmetric in that ∀xA is true at a world w iff A is true at every object a in the domain Dw of every world v accessible from w, and ∃xA is true at w iff A is true at some object a in the domain Dw of w. This non-symmetry is also evident in the Gödel translation of the intuitionistic predicate logic IQC into the predicate S4, denoted QS4, since ∀xA is translated as □∀xA' and ∃xA as ∃xA', where A' is the translation of A. Because of this, it is common to give a temporal interpretation of the intuitionistic universal quantifier as “always in the future.” In [5] we showed that it is also possible to give a temporal interpretation of the intuitionistic existential quantifier as “sometime in the past.”

In this paper we concentrate on the monadic (one-variable) fragment of IQC. It is well known that this fragment is axiomatized by Prior’s monadic intuitionistic propositional calculus MIPC [7, 25]. The monadic fragment of QS4 was studied by Fischer-Servi [13] who showed that the Gödel translation of IQC into QS4 restricts to the monadic case. We denote this monadic fragment by MS4, introduce a tense counterpart of it, which we denote by TS4, modify the Gödel translation, and prove that it embeds MIPC into TS4 fully and faithfully. This allows us to give the desired temporal interpretation of intuitionistic monadic quantifiers as “always in the future” (for ∀) and “sometime in the past” (for ∃).

While MS4 and TS4 are not comparable, we introduce a common extension, which we denote by MS4t. The system MS4t can be thought of as a tense extension of MS4. We prove that there exist full and faithful translations of MIPC, MS4, and TS4 into MS4t, yielding the following diagram, which commutes up to logical equivalence. In the diagram, the Gödel translation is denoted by ( )t.
our new translation by ( )♯, and the three translations into MS4.t by ( )♭, ( )# and ( )†, respectively.

We prove these results by utilizing the algebraic and relational semantics, and by showing that each of these systems is canonical. In addition, we prove that MS4.t has the fmp. It is then an easy consequence of the fullness and faithfulness of the translations considered that the other systems also have the fmp. That MIPC has the fmp was first proved by Bull [6], and an error in the proof was corrected independently by Fischer-Servi [14] and Ono [23]. To the best of our knowledge, the proof of the fmp for TS4 (and possibly also for MS4) is new. We conclude the paper by comparing the above translations with the translation of IQC into a version of predicate S4.t studied in [5].

2. Translation of MIPC into MS4

In this preliminary section we briefly recall the syntax and semantics of MIPC and MS4, and give an alternate proof that the Gödel translation of MIPC into MS4 is full and faithful.

2.1. MIPC. We start by recalling the definition of Prior’s monadic intuitionistic propositional calculus MIPC. Let \( L \) be a propositional language and let \( L_{∀∃} \) be an extension of \( L \) with two modalities \( ∀ \) and \( ∃ \).

**Definition 2.1.** The monadic intuitionistic propositional calculus MIPC is the intuitionistic modal logic in the propositional modal language \( L_{∀∃} \) containing

1. all theorems of the intuitionistic propositional calculus IPC;
2. the S4-axioms for \( ∀ \):
   (a) \( ∀(p ∧ q) ↔ (∀p ∧ ∀q) \),
   (b) \( ∀p → p \),
   (c) \( ∀p → ∀∀p \);
3. the S5-axioms for \( ∃ \):
   (a) \( ∃(p ∨ q) ↔ (∃p ∨ ∃q) \),
   (b) \( p → ∃p \),
   (c) \( ∃∃p → ∃p \),
   (d) \( (∃p ∧ ∃q) → ∃(∃p ∧ q) \);
4. the axioms connecting \( ∀ \) and \( ∃ \):
   (a) \( ∃∀p ↔ ∀p \),
   (b) \( ∃p ↔ ∀∃p \);

and closed under the rules of modus ponens, substitution, and necessitation \((ϕ/∀ϕ)\).

**Remark 2.2.** There are a number of axioms that are equivalent to the axiom (3d) (see, e.g., [2, Lem. 2(d)]).

The algebraic semantics for MIPC is given by monadic Heyting algebras. These algebras were first introduced by Monteiro and Varsavsky [22] as a generalization of monadic (boolean) algebras of Halmos [17]. For a detailed study of monadic Heyting algebras we refer to [2, 3, 4].
Definition 2.3. Let $H$ be a Heyting algebra.
   
   (1) A unary function $i : H \to H$ is an interior operator on $H$ if
       
       (a) $i(a \land b) = ia \land ib$,
       
       (b) $i1 = 1$,
       
       (c) $ia \leq a$,
       
       (d) $ia \leq iia$.

   (2) A unary function $c : H \to H$ is a closure operator on $H$ if
       
       (a) $c(a \lor b) = ca \lor cb$,
       
       (b) $c0 = 0$,
       
       (c) $a \leq ca$,
       
       (d) $cca \leq ca$.

Definition 2.4. A monadic Heyting algebra is a triple $\mathfrak{A} = (H, \forall, \exists)$ where $H$ is a Heyting algebra, $\forall$ is an interior operator on $H$, and $\exists$ is a closure operator on $H$ satisfying:

   (1) $\exists(\exists a \land b) = \exists a \land \exists b$,
   
   (2) $\forall \exists a = \exists a$,
   
   (3) $\exists \forall a = \forall a$.

Let $\text{MHA}$ be the class of all monadic Heyting algebras.

Remark 2.5. Let $(H, \forall, \exists)$ be a monadic Heyting algebra.

   (1) There are a number of equivalent conditions to Definition 2.4(1) (see, e.g., [2, Lem. 2(d)]). These together with the conditions connecting $\forall$ and $\exists$ yield that the fixpoints of $\forall$ form a subalgebra $H_0$ of $H$ which coincides with the subalgebra of the fixpoints of $\exists$. Moreover, $\forall$ and $\exists$ are the right and left adjoints of the embedding $H_0 \to H$, and up to isomorphism each monadic Heyting algebra arises this way (see, e.g., [2, Sec. 3]).

   (2) The non-symmetry of $\forall$ and $\exists$ is manifested by the fact that the $\forall$-analogue $\forall(\exists a \lor b) = \exists a \lor \forall b$ of Definition 2.4(1) does not hold in general.

The standard Lindenbaum-Tarski construction (see, e.g., [26]) yields that monadic Heyting algebras provide a sound and complete algebraic semantics for $\text{MIPC}$.

We next turn to the relational semantics for $\text{MIPC}$. There are several such (see, e.g., [3]), but we concentrate on the one introduced by Ono [23].

Definition 2.6. An $\text{MIPC}$-frame is a triple $\mathfrak{F} = (X, R, Q)$ where $X$ is a set, $R$ is a partial order, $Q$ is a quasi-order (reflexive and transitive), and the following two conditions are satisfied:

   (O1) $R \subseteq Q$,
   
   (O2) $xQy \Rightarrow (\exists z)(xRz \& zEQy)$.

Here $E_Q$ is the equivalence relation defined by $xE_Qy$ iff $xQy$ and $yQx$.

Let $\mathfrak{F} = (X, R, Q)$ be an $\text{MIPC}$-frame. As usual, for $x \in X$, we write

   $R[x] = \{y \in X \mid xRy\}$ and $R^{-1}[x] = \{y \in X \mid yRx\}$,

and for $U \subseteq X$, we write

   $R[U] = \bigcup\{R[u] \mid u \in U\}$ and $R^{-1}[U] = \bigcup\{R^{-1}[u] \mid u \in U\}$.

We use the same notation for $Q$ and $E_Q$. Since $E_Q$ is an equivalence relation, we have that $E_Q[x] = (E_Q)^{-1}[x]$ and $E_Q[U] = (E_Q)^{-1}[U]$. 
We call a subset $U$ of $X$ an $R$-upset provided $U = R[U]$ (i.e., $x \in U$ and $xRy$ imply $y \in U$). Let $\text{Up}(X)$ be the set of all $R$-upsets of $\mathcal{F}$. It is well known that $\text{Up}(X)$ is a Heyting algebra, where the lattice operations are set-theoretic union and intersection, and $U \rightarrow V$ is calculated by

$$U \rightarrow V = \{ x \in X \mid R[x] \cap U \subseteq V \} = X \setminus R^{-1}[U \setminus V].$$

In addition, for $U \in \text{Up}(X)$, define

$$\forall Q(U) = X \setminus Q^{-1}[X \setminus U] \text{ and } \exists Q(U) = E_Q[U].$$

Then $\mathcal{F}^+ = (\text{Up}(X), \forall, \exists)$ is a monadic Heyting algebra (see, e.g., [3, Sec. 6]).

**Remark 2.7.** If $U \in \text{Up}(X)$, then Definition 2.6(O2) implies that $E_Q[U]$ equals $Q[U]$. That $\exists Q(U)$ equals $Q[U]$ motivates our interpretation of $\exists$ as “sometime in the past.” Indeed, taking $Q[U]$ is the standard way to associate an operator on $\wp(X)$ to the tense modality “sometime in the past” (see, e.g., [28, p. 151]). As a consequence of this, $\mathcal{F}^+_0$ is the set of $Q$-upsets of $\mathcal{F}$.

Each monadic Heyting algebra $\mathfrak{A} = (H, \forall, \exists)$ can be represented as a subalgebra of $\mathcal{F}^+$ for some $\text{MIPC}$-frame $\mathfrak{F}$. For this we recall the definition of the canonical frame of $\mathfrak{A}$.

**Definition 2.8.** Let $\mathfrak{A} = (H, \forall, \exists)$ be a monadic Heyting algebra. The canonical frame of $\mathfrak{A}$ is the frame $\mathfrak{A}^+_+ = (X_\mathfrak{A}, R_\mathfrak{A}, Q_\mathfrak{A})$ where $X_\mathfrak{A}$ is the set of prime filters of $H$, $R_\mathfrak{A}$ is the inclusion relation, and $xQ_\mathfrak{A}y$ if $x \cap H_0 \subseteq y$ (equivalently, $x \cap H_0 \subseteq y \cap H_0$).

By [3] Sec. 6], $\mathfrak{A}^+_+$ is a $\text{MIPC}$-frame.

**Definition 2.9.** We call an $\text{MIPC}$-frame $\mathfrak{F}$ canonical if it is isomorphic to $\mathfrak{A}^+_+$. For some monadic Heyting algebra $\mathfrak{A}$, we define the Stone map $\beta : \mathfrak{A} \rightarrow \text{Up}(X_\mathfrak{A})$ by

$$\beta(a) = \{ x \in X_\mathfrak{A} \mid a \in x \}.$$ 

By [3] Sec. 6], $\beta : \mathfrak{A} \rightarrow (\mathfrak{A}^+_+)^+$ is a one-to-one homomorphism of monadic Heyting algebras. Thus, we arrive at the following representation theorem for monadic Heyting algebras.

**Proposition 2.10.** Each monadic Heyting algebra $\mathfrak{A}$ is isomorphic to a subalgebra of $(\mathfrak{A}^+_+)^+$.

**Remark 2.11.**

1. The image of $\mathfrak{A}$ inside $(\mathfrak{A}^+_+)^+$ can be recovered by introducing a Priestley topology on $X_\mathfrak{A}$. This leads to the notion of perfect $\text{MIPC}$-frames and a duality between the category of monadic Heyting algebras and the category of perfect $\text{MIPC}$-frames; see [3, Thm. 17].

2. When $\mathfrak{A}$ is finite, its embedding into $(\mathfrak{A}^+_+)^+$ is an isomorphism, and hence the categories of finite monadic Heyting algebras and finite $\text{MIPC}$-frames are dually equivalent.
The next corollary is an immediate consequence of the above considerations.

**Corollary 2.12.** MIPC is canonical; that is,

\[ \forall \in \text{MHA} \Rightarrow (\forall_+) \in \text{MHA}. \]

A valuation on an MIPC-frame \( \mathcal{F} = (X, R, Q) \) is a map \( v \) associating an \( R \)-upset of \( X \) to any propositional letter of \( L_{\forall\exists} \). The connectives \( \land, \lor, \rightarrow, \neg \) are then interpreted as in intuitionistic Kripke frames, and \( \forall, \exists \) are interpreted by

\[
\begin{align*}
    x \vDash_v \forall \varphi & \iff (\forall y \in X)(xQy \Rightarrow y \vDash_v \varphi), \\
    x \vDash_v \exists \varphi & \iff (\exists y \in X)(xEQy \& y \vDash_v \varphi).
\end{align*}
\]

As usual, we say that \( \varphi \) is valid in \( \mathcal{F} \), and write \( \mathcal{F} \vDash \varphi \), if \( x \vDash_v \varphi \) for every valuation \( v \) and every \( x \in X \).

Soundness of MIPC with respect to this semantics is straightforward to prove. For completeness, it is sufficient to utilize the algebraic completeness and the representation theorem for monadic Heyting algebras. As a result, we arrive at the following:

**Theorem 2.13.** MIPC \( \vdash \varphi \) iff \( \mathcal{F} \vDash \varphi \) for every MIPC-frame \( \mathcal{F} \).

We conclude this section by recalling that MIPC has the fmp. This was first established by Bull [7] using algebraic semantics. His proof contained a gap, which was corrected independently by Fischer-Servi [14] and Ono [23]. A semantic proof is given in [15], which is based on the technique developed by Grefe [16]. We will give yet another proof of this result in Section 5.

**2.2. MS4.** Let \( L_{\Box\forall} \) be a propositional bimodal language with two modal operators \( \Box \) and \( \forall \).

**Definition 2.14.** The monadic S4, denoted MS4, is the smallest classical bimodal logic containing the S4-axioms for \( \Box \), the S5-axioms for \( \forall \), the left commutativity axiom

\[ \Box \forall \varphi \rightarrow \forall \Box \varphi, \]

and closed under modus ponens, substitution, \( \Box \)-necessitation, and \( \forall \)-necessitation.

As usual, \( \Diamond \) is an abbreviation for \( \neg \Box \neg \) and \( \exists \) is an abbreviation for \( \neg \forall \neg \).

**Remark 2.15.** Recalling the definition of fusion of two logics (see [15]), MS4 is obtained from the fusion S4 \( \otimes \) S5 by adding the left commutativity axiom \( \Box \forall \varphi \rightarrow \forall \Box \varphi \) which is the monadic version of the converse Barcan formula. The monadic version of the Barcan formula is the right commutativity axiom \( \forall \Box \varphi \rightarrow \Box \forall \varphi \). Adding it to MS4 yields the product logic S4 \( \times \) S5; see [15, Ch. 5] for details.

The algebraic semantics for MS4 is given by monadic S4-algebras. To define these algebras, we first recall the definition of S4-algebras and S5-algebras.

**Definition 2.16.**

1. An S4-algebra, or an interior algebra, is a pair \( \mathfrak{B} = (B, \Box) \) where \( B \) is a boolean algebra and \( \Box \) is an interior operator on \( B \) (see Definition 2.3(1)).

2. An S5-algebra, or a monadic algebra, is an S4-algebra \( \mathfrak{B} = (B, \forall) \) that in addition satisfies \( a \leq \forall \exists a \) for all \( a \in B \).

We are ready to define monadic S4-algebras.

**Definition 2.17.** A monadic S4-algebra, or an MS4-algebra for short, is a tuple \( \mathfrak{B} = (B, \Box, \forall) \) where
(1) \((B, \Box)\) is an S4-algebra,
(2) \((B, \forall)\) is an S5-algebra,
(3) \(\Box \forall a \leq \forall \Box a\) for each \(a \in B\).

**Lemma 2.18.** The axiom \(\Box \forall a \leq \forall \Box a\) in Definition 2.17 can be replaced by any of the following:

1. \(\Box \forall a = \forall \Box a\).
2. \(\forall \Box \exists a = \Box \exists a\).
3. \(\exists \exists a = \exists \Box a\).
4. \(\Box \exists a = \exists \Box a\).
5. \(\exists \Box a \leq \exists \exists a\).

**Proof.** Showing that (1) and (2) are equivalent to \(\Box \forall a \leq \forall \Box a\) is straightforward. That (3) and (4) are equivalent to (5) can be proved similarly (see [8] for details). We show that (2) and (3) are equivalent. Suppose (2) holds. Then for each \(a \in B\), we have

\[\forall \Box \exists a = \forall \Box \exists a = \Box \forall a = \Box \exists a.\]

Using \(\forall \Box \exists a = \Box \exists a\) twice, we obtain

\[\exists \exists a = \exists \forall \exists a = \forall \exists a = \Box \exists a,\]

yielding (3). Proving (2) from (3) is analogous. \(\square\)

**Remark 2.19.** As noted above, the inequality \(\Box \forall a \leq \forall \Box a\) is equivalent to the equality \(\forall \Box \forall a = \Box \forall a\). This yields that the set \(B_0\) of \(\forall\)-fixpoints of an MS4-algebra \((B, \Box, \forall)\) forms an S4-subalgebra of \((B, \Box)\) such that \(\forall\) is the right adjoint to the embedding \(B_0 \to B\). Moreover, up to isomorphism each MS4-algebra arises this way. This is similar to the case of monadic Heyting algebras (see Remark 2.5).

The Lindenbaum-Tarski construction yields that MS4-algebras provide a sound and complete algebraic semantics for MS4.

The relational semantics for MS4 was first introduced by Esakia [12].

**Definition 2.20.** An **MS4-frame** is a triple \(\mathfrak{F} = (X, R, E)\) where \(X\) is a set, \(R\) is a quasi-order, \(E\) is an equivalence relation, and the following commutativity condition is satisfied:

\((\forall x, y, z \in X)(x E y \& y R z) \Rightarrow (\exists u \in X)(x R u \& u E z).\)

\(\begin{array}{c}
\text{Figure 2. Condition (E).}
\end{array}\)

For an MS4-frame \(\mathfrak{F} = (X, R, E)\), let \(\wp(X)\) be the powerset of \(X\) and for \(U \in \wp(X)\) let

\[\Box_R(U) = X \setminus R^{-1}[X \setminus U]\]

and \(\forall_E(U) = X \setminus E[X \setminus U].\)
Since $R$ is a quasi-order, $(\varphi(X), \square_R)$ is an $S4$-algebra; and since $E$ is an equivalence relation, $(\varphi(X), \forall_E)$ is an $S5$-algebra (see [18] Thm. 3.5). In addition, the commutativity condition yields that $3^+ := (\varphi(X), \square_R, \forall_E)$ is an $MS4$-algebra.

In fact, as in the case of monadic Heyting algebras, each $MS4$-algebra $\mathfrak{B} = (B, \square, \forall)$ is isomorphic to a subalgebra of $3^+$ for some $MS4$-frame $\mathfrak{F}$. We can take $\mathfrak{F}$ to be the canonical frame of $\mathfrak{B}$. Let $H$ be the set of $\square$-fixpoints and $B_0$ the set of $\forall$-fixpoints. Then $H$ is a Heyting algebra which is a bounded sublattice of $B$, and $B_0$ is an $S4$-subalgebra of $(B, \square)$.

**Remark 2.21.** If $\mathfrak{B} = 3^+$, then the elements of $H$ are the $R$-upsets of $3$ and the elements of $B_0$ are the $E$-saturated subsets of $3$ (that is, unions of $E$-equivalence classes).

**Definition 2.22.** Let $\mathfrak{B} = (B, \square, \forall)$ be an $MS4$-algebra. The canonical frame of $\mathfrak{B}$ is the frame $\mathfrak{B}_+ = (X_\mathfrak{B}, R_\mathfrak{B}, E_\mathfrak{B})$ where $X_\mathfrak{B}$ is the set of ultrafilters of $B$, $xR_\mathfrak{B}y$ iff $x \cap H \subseteq y \cap H$, and $xE_\mathfrak{B}y$ iff $x \cap B_0 = y \cap B_0$.

**Lemma 2.23.** If $\mathfrak{B}$ is an $MS4$-algebra, then $\mathfrak{B}_+$ is an $MS4$-frame.

**Proof.** Since $(B, \square)$ is an $S4$-algebra, $R_\mathfrak{B}$ is a quasi-order (see [18] Thm. 3.14)); and since $(B, \forall)$ is an $S5$-algebra, $E_\mathfrak{B}$ is an equivalence relation (see [18] Thm. 3.18)). It remains to show that Definition 2.20(E) is satisfied. Let $x, y, z \in X_\mathfrak{B}$ be such that $xE_\mathfrak{B}y$ and $yR_\mathfrak{B}z$. This means that $x \cap B_0 = y \cap B_0$ and $y \cap H \subseteq z$. Let $F$ be the filter of $\mathfrak{B}$ generated by $(x \cap H) \cup (z \cap B_0)$. We show that $F$ is proper. Otherwise, since $x \cap H$ and $z \cap B_0$ are closed under meets, there are $a \in x \cap H$ and $b \in z \cap B_0$ such that $a \wedge b = 0$. Therefore, $a \leq b$. Thus, $a = \square a \leq \square b$, so $\square b \in x$. Since $B_0$ is an $S4$-subalgebra of $(B, \square)$ and $b \in B_0$, we have $\square b \in B_0$. This yields $\square b \in x \cap B_0 = y \cap B_0$, which implies $\square b \in y \cap H \subseteq z$. Therefore, $b \in z$ which contradicts $b \in z$. Thus, $F$ is proper, and so there is an ultrafilter $u$ of $B$ such that $F \subseteq u$. Consequently, $x \cap H \subseteq u$ and $z \cap B_0 \subseteq u \cap B_0$. Since $z \cap B_0$ and $u \cap B_0$ are both ultrafilters of $B_0$, we conclude that $z \cap B_0 = u \cap B_0$. Thus, there is $u \in X_\mathfrak{B}$ with $xR_\mathfrak{B}u$ and $uE_\mathfrak{B}z$. \hfill \Box

**Definition 2.24.** We call an $MS4$-frame canonical if it is isomorphic to $\mathfrak{B}_+$ for some $MS4$-algebra $\mathfrak{B}$.

For an $MS4$-algebra $\mathfrak{B}$, it follows from [18] Thm. 3.14] that the Stone map $\beta : B \rightarrow \varphi(X_\mathfrak{B})$ is a one-to-one homomorphism of $MS4$-algebras. Thus, we arrive at the following representation theorem.

**Proposition 2.25.** Each $MS4$-algebra $\mathfrak{B}$ is isomorphic to a subalgebra of $(\mathfrak{B}_+)^+$.

**Remark 2.26.** To recover the image of $\mathfrak{B}$ in $\varphi(X_\mathfrak{B})$ we need to endow $X_\mathfrak{B}$ with a Stone topology. This leads to the notion of perfect $MS4$-frames and a duality between the category of $MS4$-algebras and the category of perfect $MS4$-frames (see [8] for details). When $\mathfrak{B}$ is finite, its embedding into $(\mathfrak{B}_+)^+$ is an isomorphism, and hence the categories of finite $MS4$-algebras and finite $MS4$-frames are dually equivalent.

As an immediate consequence of the above considerations, we obtain that if $\mathfrak{B}$ is an $MS4$-algebra, then so is $(\mathfrak{B}_+)^+$. Thus, we have:

**Corollary 2.27.** $MS4$ is canonical.

A valuation on an $MS4$-frame $\mathfrak{F} = (X, R, E)$ is a map $v$ associating a subset of $X$ to each propositional letter of $L_{\square \forall}$. Then the boolean connectives are interpreted as usual,

$$
x \vDash v \square \varphi \iff (\forall y \in X)(xRy \Rightarrow y \vDash v \varphi),$$

$$x \vDash v \forall \varphi \iff (\forall y \in X)(xEy \Rightarrow y \vDash v \varphi).$$
As usual, we say that \( \varphi \) is valid in \( \mathcal{F} \), in symbols \( \mathcal{F} \models \varphi \), if \( x \models_{v} \varphi \) for every valuation \( v \) and \( x \in X \).

Soundness of MS4 with respect to this semantics is straightforward to prove, and completeness follows from the algebraic completeness and the representation theorem for MS4-algebras proved above.

**Theorem 2.28.** MS4 \( \vdash \varphi \) iff \( \mathcal{F} \models \varphi \) for every MS4-frame \( \mathcal{F} \).

In addition, MS4 has the fmp. While this can be proved directly using algebraic technique, we will derive it as a consequence of the fmp of a stronger multimodal system in Section 5.

### 2.3. Gödel translation.

We recall that the Gödel translation of MIPC into MS4 is defined by:

\[
\begin{align*}
\bot^t &= \bot \\
p^t &= \Box p \\
(\varphi \land \psi)^t &= \varphi^t \land \psi^t \\
(\varphi \lor \psi)^t &= \varphi^t \lor \psi^t \\
(\varphi \rightarrow \psi)^t &= \Box(\neg \varphi^t \lor \psi^t) \\
(\forall \varphi)^t &= \forall \varphi^t \\
(\exists \varphi)^t &= \exists \varphi^t
\end{align*}
\]

It was shown by Fischer-Servi [13] that this translation is full and faithful, meaning that

\[
\text{MIPC} \vdash \varphi \text{ iff } \text{MS4} \vdash \varphi^t.
\]

Fischer-Servi used the translations of MIPC and MS4 into IQC and QS4 respectively, and the predicate version of the Gödel translation. In [14] she gave a different proof of this result using the fmp for MIPC. We give yet another proof utilizing relational semantics for MIPC and MS4. Our proof generalizes the semantic proof that the Gödel translation of IPC into S4 is full and faithful (see, e.g., [9, Sec. 3.9]). We require the following lemma.

**Lemma 2.29.** For any formula \( \chi \) of \( \mathcal{L}_{\forall \exists} \), we have

\[
\text{MS4} \vdash \chi^t \rightarrow \Box \chi^t.
\]

**Proof.** We first show that MS4 \( \vdash \exists \Box \varphi \rightarrow \Box \exists \varphi \) for any formula \( \varphi \) of \( \mathcal{L}_{\Box \forall} \). For this, by algebraic completeness, it is sufficient to prove that the inequality \( \exists \Box a \leq \Box \exists a \) holds in every MS4-algebra \( (B, \Box, \forall) \). Let \( a \in B \). We have

\[
\exists \Box a \leq \Box \exists a = \Box \forall \exists a \leq \forall \forall \exists a = \forall \exists a \leq \Box \exists a.
\]

We are now ready to prove that MS4 \( \vdash \chi^t \rightarrow \Box \chi^t \) by induction on the complexity of \( \chi \). This is obvious when \( \chi = \bot \). The cases when \( \chi \) is \( p \), \( \varphi \rightarrow \psi \), or \( \forall \varphi \) follow from the axiom \( \Box \varphi \rightarrow \Box \Box \varphi \). We next consider the cases when \( \chi \) is \( \varphi \land \psi \) or \( \varphi \lor \psi \). Suppose that the claim is true for \( \varphi \) and \( \psi \), so \( \varphi^t \rightarrow \Box \varphi^t \) and \( \psi^t \rightarrow \Box \psi^t \) are theorems of MS4. Then \( \varphi^t \land \psi^t \rightarrow \Box(\varphi^t \land \psi^t) \) and \( \varphi^t \lor \psi^t \rightarrow \Box(\varphi^t \lor \psi^t) \) are also theorems of MS4. Finally, if \( \chi \) is \( \exists \varphi \) and MS4 \( \vdash \varphi^t \rightarrow \Box \varphi^t \), then MS4 \( \vdash \exists \varphi^t \rightarrow \Box \exists \varphi^t \). Therefore, since MS4 \( \vdash \exists \Box \varphi^t \rightarrow \exists \Box \varphi^t \), we conclude that MS4 \( \vdash \Box \exists \varphi^t \rightarrow \Box \exists \varphi^t \). \( \square \)

In the next definition we generalize to MS4-frames the well-known definition of skeleton (see, e.g., [9, Sec. 3.9]).

**Definition 2.30.** Let \( \mathcal{F} = (X, R, E) \) be an MS4-frame. Define the relation \( Q_E \) on \( X \) by setting \( xQ_Ey \) iff \( (\exists z \in X)(xRz \& zEy) \). Then the skeleton \( \mathcal{F}' = (X', R', Q') \) of \( \mathcal{F} \) is defined as follows. Let \( \sim \) be the equivalence relation on \( X \) given by \( x \sim y \) iff \( xRy \) and \( yRx \). We let \( X' \) be the set of equivalence classes of \( \sim \), and define \( R' \) and \( Q' \) on \( X' \) by \( [x]R'[y] \iff xRy \) and \( [x]Q'[y] \iff xQEy \).
On the other hand,

(1) If \( \mathfrak{F} \) is an MS4-frame, then \( \mathfrak{F}' \) is an MIPC-frame.

(2) For each valuation \( v \) on \( \mathfrak{F} \) there is a valuation \( v' \) on \( \mathfrak{F}' \) such that for each \( x \in \mathfrak{F} \) and \( \mathcal{L}_{\forall\exists} \)-formula \( \varphi \), we have

\[
\mathfrak{F}', [x] \models_{v'} \varphi \text{ iff } \mathfrak{F}, [x] \models_{v} \varphi^t.
\]

(3) For each \( \mathcal{L}_{\forall\exists} \)-formula \( \varphi \), we have

\[
\mathfrak{F}' \models \varphi \text{ iff } \mathfrak{F} \models \varphi^t.
\]

(4) For each MIPC-frame \( \mathfrak{G} \) there is an MS4-frame \( \mathfrak{F} \) such that \( \mathfrak{G} \) is isomorphic to \( \mathfrak{F}' \).

**Proof.** (1). It is well known that \( (X', R') \) is an intuitionistic Kripke frame. That \( Q' \) is well defined follows from Condition (E). Showing that \( Q' \) is a quasi-order, and that (O1) and (O2) hold in \( \mathfrak{F}' \) is straightforward.

(2). Define \( v' \) on \( \mathfrak{F}' \) by \( v'(p) = \{ [x] \in X' \mid R[x] \subseteq v(p) \} \). We show that \( \mathfrak{F}', [x] \models_{v'} \varphi \text{ iff } \mathfrak{F}, [x] \models_{v} \varphi^t \) by induction on the complexity of \( \varphi \). Since \( v'(p) = \{ [x] \mid \mathfrak{F}, x \models_{v} \Box p \} \), the claim is obvious when \( \varphi \) is a propositional letter. We prove the claim for \( \varphi \) of the form \( \forall \psi \) and \( \exists \psi \) since the other cases are well known. Suppose \( \varphi = \forall \psi \). By the definition of \( Q' \) and induction hypothesis, we have

\[
\mathfrak{F}', [x] \models_{v'} \forall \psi \text{ iff } \mathfrak{F}, [x] \models_{v} \psi^t \text{ iff } (\forall y \in X')(xQ_Ey \Rightarrow \mathfrak{F}', [y] \models_{v'} \psi)
\]

On the other hand,

\[
\mathfrak{F}, x \models_{v} (\forall \psi)^t \text{ iff } \mathfrak{F}, x \models_{v} \Box \forall \psi^t
\]

Thus, \( \mathfrak{F}', [x] \models_{v'} \exists \psi \text{ iff } \mathfrak{F}, [x] \models_{v} \psi^t \).

Suppose \( \varphi = \exists \psi \). As noted in Remark 2.7, \( Q' \) and \( E_{Q'} \) coincide on \( R' \)-upsets, and it is straightforward to see by induction that the set \( \{ [y] \mid \mathfrak{F}', [y] \models_{v'} \psi \} \) is an \( R' \)-upset. Therefore, by the induction hypothesis,

\[
\mathfrak{F}', [x] \models_{v'} \exists \psi \text{ iff } (\exists y \in X')(xQ_E[y] \& \mathfrak{F}', [y] \models_{v'} \psi)
\]

On the other hand,

\[
\mathfrak{F}, x \models_{v} (\exists \psi)^t \text{ iff } \mathfrak{F}, x \models_{v} \exists \psi^t
\]

since, by Lemma 2.29, the set \( \{ y \mid \mathfrak{F}, y \models_{v} \psi^t \} \) is an \( R \)-upset, and \( E \) and \( Q_E \) coincide on \( R \)-upsets. Thus, \( \mathfrak{F}', [x] \models_{v'} \exists \psi \text{ iff } \mathfrak{F}, [x] \models_{v} (\exists \psi)^t \).
(3). If \( \mathfrak{F} \models \varphi^t \), then there is a valuation \( v \) on \( \mathfrak{F} \) such that \( \mathfrak{F}, x \not\models \varphi^t \) for some \( x \in X \). By (2), \( v' \) is a valuation on \( \mathfrak{F}' \) such that \( \mathfrak{F}', [x] \not\models \varphi \). Therefore, \( \mathfrak{F}' \not\models \varphi \). If \( \mathfrak{F}' \not\models \varphi \), then there is a valuation \( w \) on \( \mathfrak{F}' \) and \( [x] \in X' \) such that \( \mathfrak{F}', [x] \not\models_w \varphi \). Let \( v \) be the valuation on \( \mathfrak{F} \) given by \( v(p) = \{ [x] | [x] \in w(p) \} \).

Since \( \mathfrak{F}' \) is an MIPC-frame, \( w(p) \) is an \( R' \)-upset of \( \mathfrak{F}' \) for each \( p \). So \( v(p) \) is an \( R \)-upset of \( \mathfrak{F} \) for each \( p \). Therefore, \( w = v' \) because

\[
v'(p) = \{ [x] \in X' | R[x] \subseteq v(p) \} = \{ [x] \in X' | x \in v(p) \} = w(p).
\]

Thus, \( \mathfrak{F}', [x] \not\models'_w \varphi \). By (2), \( \mathfrak{F}, x \not\models \varphi^t \). Consequently, \( \mathfrak{F} \not\models \varphi^t \).

(4). Let \( \mathfrak{G} = (X, R, Q) \) be an MIPC-frame. We show that \( \mathfrak{F} = (X, R, E_Q) \) is an MS4-frame. If \( xE_Qy \) and \( yRz \), then by definition of \( E_Q \) and condition (O1) of MIPC-frames, \( xQy \) and \( yQz \). Since \( Q \) is transitive, \( xQz \). Condition (O2) then implies that there is \( u \in X \) with \( xRu \) and \( uEQz \). Thus, \( \mathfrak{F} \) is an MS4-frame. Since \( R \) is a partial order, \( \sim \) is the identity relation. It then follows from condition (O2) that \( Q = E_{Q_E} \), and hence \( \mathfrak{G} \) is isomorphic to \( \mathfrak{F}' \). □

**Remark 2.32.** In general, we cannot recover an MS4-frame \( \mathfrak{F} = (X, R, E) \) from its skeleton \( \mathfrak{F}' \) even if \( R \) is a partial order. Indeed, it is not always the case that \( E = E_{Q_E} \). However, if \( \mathfrak{F} \) is canonical (and in particular finite), then \( E = E_{Q_E} \); see [3, Sec. 2] for details.

**Theorem 2.33.** The Gödel translation of MIPC into MS4 is full and faithful; that is,

\[
\text{MIPC} \vdash \varphi \quad \text{iff} \quad \text{MS4} \vdash \varphi^t.
\]

**Proof.** To prove faithfulness, suppose that \( \text{MS4} \not\vdash \varphi^t \). By Theorem 2.28, there is an MS4-frame \( \mathfrak{F} \) such that \( \mathfrak{F} \not\models \varphi^t \). By Proposition 2.31, \( \mathfrak{F}' \) is an MIPC-frame and \( \mathfrak{F}' \not\models \varphi \). Thus, by Theorem 2.13, MIPC \( \not\models \varphi \). For fullness, let MIPC \( \not\models \varphi \). Then there is an MIPC frame \( \mathfrak{G} \) such that \( \mathfrak{G} \not\models \varphi \). By Proposition 2.31(4), there is an MS4-frame such that \( \mathfrak{G} \) isomorphic to \( \mathfrak{F}' \). Therefore, \( \mathfrak{F}' \not\models \varphi^t \). Proposition 2.31(3) implies that \( \mathfrak{F} \not\models \varphi^t \). Thus, MS4 \( \not\models \varphi^t \). □

**Remark 2.34.** The original proof of McKinsey and Tarski [20, 21] that the Gödel translation of IPC into S4 is full and faithful was algebraic. They proved that the □-fixpoints of each S4-algebra form a Heyting algebra, and that each Heyting algebra arises this way. In the monadic setting, while we still have that the □-fixpoints of each MS4-algebra form a monadic Heyting algebra, it is not the case that each monadic Heyting algebra arises this way (see [8] for details). Nevertheless, Fischer-Servi [14] proved that each finite monadic Heyting algebra does. Thus, while we can prove faithfulness in the same fashion as McKinsey and Tarski, proving fullness requires to first establish the finite model property for MIPC.

### 3. Translation of MIPC into TS4

In this section we introduce a new multimodal tense system TS4 in which, as we will show in the next section, MIPC embeds fully and faithfully by a modified Gödel translation. For this we require to recall the well-known tense system S4.t.

#### 3.1. S4.t

The tense logic S4.t is the extension of the least tense logic K.t in which both tense modalities satisfy the S4-axioms. This system was studied by several authors. In particular, Esakia [10] showed that an extension of the Gödel translation embeds the Heyting-Brouwer logic HB of Rauszer [27] into S4.t fully and faithfully. The language of HB is obtained by enriching the language of IPC by an additional connective of coimplication, and the logic HB is the extension of IPC by the axioms for coimplication, which are dual to the axioms for implication. Wolter [29] extended the celebrated Blok-Esakia Theorem to this setting.
Let $\mathcal{L}_T$ be the propositional tense language with two modalities $\Box_F$ and $\Box_P$. As usual, $\Box_F$ is interpreted as “always in the future” and $\Box_P$ as “always in the past.” We use the following standard abbreviations: $\Diamond_F$ for $\neg\Box_F\neg$ and $\Diamond_P$ for $\neg\Box_P\neg$. Then $\Diamond_F$ is interpreted as “sometime in the future” and $\Diamond_P$ as “sometime in the past.”

**Definition 3.1.** Let $S_4.t$ be the smallest classical bimodal logic containing the $S_4$-axioms for $\Box_F$ and $\Box_P$, the tense axioms
\[
\begin{align*}
p & \rightarrow \Box_P\Diamond Fp \\
p & \rightarrow \Box_F\Diamond Pp
\end{align*}
\]
and closed under modus ponens, substitution, $\Box_F$-necessitation, and $\Box_P$-necessitation.

Algebraic semantics for $S_4.t$ was studied by Esakia [10, 11], where the duality theory for $S_4$-algebras was generalized to $S_4.t$-algebras.

**Definition 3.2.** An $S_4.t$-algebra is a triple $(B, [\Box_F, \Box_P])$ where $(B, [\Box_F])$, $(B, [\Box_P])$ are $S_4$-algebras and for each $a \in B$ we have
\[
\begin{align*}
(\text{PF}) & \quad a \leq \Box_P\Diamond Fa \\
(\text{FP}) & \quad a \leq \Box_F\Diamond Pa
\end{align*}
\]

The Lindenbaum-Tarski construction yields that $S_4.t$-algebras provide a sound and complete algebraic semantics for $S_4.t$. Relational semantics for $S_4.t$ is given by $S_4.t$-frames.

**Definition 3.3.** An $S_4.t$-frame is a pair $\mathfrak{F} = (X, Q)$ where $X$ is a set and $Q$ is a quasi-order on $X$.

Let $Q^-$ be the converse of $Q$. For $U \in \wp(X)$ let
\[
[Q^-(U)] = X \setminus Q^-[X \setminus U]
\]
Since $Q$ is a quasi-order, so is $Q^-$, so $(\wp(X), [Q^-])$ and $(\wp(X), [Q^-])$ are $S_4$-algebras. A standard argument (see [18, Thm. 3.6]) gives that $\mathfrak{F}^+ := (\wp(X), [Q, [Q^-]])$ satisfies $\text{(PF)}$ and $\text{(FP)}$. Therefore, $\mathfrak{F}^+$ is an $S_4.t$-algebra.

In fact, each $S_4.t$-algebra $\mathcal{B} = (B, [\Box_F, \Box_P])$ is isomorphic to a subalgebra of $\mathfrak{F}^+$ for some $S_4.t$-frame $\mathfrak{F}$. As usual, we can take $\mathfrak{F}$ to be the canonical frame of $\mathcal{B}$. Let $H_F$ and $H_P$ be the sets of $\Box_F$-fixpoints and $\Box_P$-fixpoints, respectively. Since $\Box_F$ and $\Box_P$ are $S_4$-operators, $H_F$ and $H_P$ are Heyting algebras.

**Remark 3.4.** Let $(B, [\Box_F, \Box_P])$ be an $S_4.t$-algebra. It follows from Definition 3.2 that $H_F$ coincides with the set of $\Diamond_P$-fixpoints and $H_P$ with the set of $\Diamond_F$-fixpoints. Moreover, $\neg$ maps $H_F$ to $H_P$ and vice versa. Indeed, if $a \in H_F$, then $a = \Box_P a$. By $\text{(PF)}$, $\Diamond_P a = \Diamond_P \Box_F a \leq a$, so $\Diamond_P a = a$, and hence $\Box_P a = \neg a = \neg \Diamond_P a = \neg a$. Therefore, $\neg a \in H_P$. Similarly, if $a \in H_P$, then $\neg a \in H_F$. Thus, $\neg$ is a dual isomorphism between $H_F$ and $H_P$.

**Definition 3.5.** Let $\mathcal{B} = (B, [\Box_F, \Box_P])$ be an $S_4.t$-algebra. The canonical frame of $\mathcal{B}$ is the frame $\mathcal{B}_+ = (X_\mathcal{B}, Q_\mathcal{B})$ where $X_\mathcal{B}$ is the set of ultrafilters of $B$ and $xQ_\mathcal{B}y$ iff $x \cap H_F \subseteq y$; equivalently, $y \cap H_P \subseteq x$.

By a standard argument, if $\mathcal{B}$ is an $S_4.t$-algebra, then $\mathcal{B}_+$ is an $S_4.t$-frame.

**Definition 3.6.** We call an $S_4.t$-frame canonical if it is isomorphic to $\mathcal{B}_+$ for some $S_4.t$-algebra $\mathcal{B}$. 
A standard argument now yields the following representation theorem.

**Proposition 3.7.** If \( \mathfrak{B} \) is an \( \mathsf{S}4.t \)-algebra, then \( \mathfrak{B} \) is isomorphic to a subalgebra of \( (\mathfrak{B}_+)^+ \).

**Remark 3.8.** To recover the image of \( \mathfrak{B} \) in \( \varphi(X_{\mathfrak{B}}) \) we need to endow \( X_{\mathfrak{B}} \) with a Stone topology. This leads to the notion of perfect \( \mathsf{S}4.t \)-frames and a duality between the category of \( \mathsf{S}4.t \)-algebras and the category of perfect \( \mathsf{S}4.t \)-frames (see [11]). When \( \mathfrak{B} \) is finite, its embedding into \( (\mathfrak{B}_+)^+ \) is an isomorphism, and hence the categories of finite \( \mathsf{S}4.t \)-algebras and finite \( \mathsf{S}4.t \)-frames are dually equivalent.

As an immediate consequence, we obtain:

**Corollary 3.9.** \( \mathsf{S}4.t \) is canonical.

While \( \mathsf{S}4.t \)-frames coincide with \( \mathsf{S}4 \)-frames, the difference is in the interpretation of the modalities as we use \( Q \) to interpret \( \Box_F \) and \( Q^- \) to interpret \( \Box_P \).

A *valuation* on an \( \mathsf{S}4.t \)-frame \( \mathfrak{F} = (X,Q) \) is a map \( v \) associating a subset of \( X \) to each propositional letter of \( \mathcal{L}_T \). The classical connectives are interpreted as usual, and the tense modalities are interpreted as

\[
\begin{align*}
x \vDash_v \Box_F \varphi & \iff (\forall y \in X)(xQy \Rightarrow y \vDash_v \varphi), \\
x \vDash_v \Box_P \varphi & \iff (\forall y \in X)(yQx \Rightarrow y \vDash_v \varphi).
\end{align*}
\]

As usual, we say that \( \varphi \) is *valid* in \( \mathfrak{F} \), in symbols \( \mathfrak{F} \vDash \varphi \), if \( x \vDash_v \varphi \) for every valuation \( v \) and \( x \in X \).

Soundness of \( \mathsf{S}4.t \) with respect to this semantics is straightforward to prove. Completeness follows from the algebraic completeness and the representation of \( \mathsf{S}4.t \)-algebras.

**Theorem 3.10.** \( \mathsf{S}4.t \vDash \varphi \text{ iff } \mathfrak{F} \vDash \varphi \text{ for every } \mathsf{S}4.t \text{-frame } \mathfrak{F} \).

That \( \mathsf{S}4.t \) has the fmp belongs to folklore. We were unable to find it stated explicitly in the literature. It will follow from our results in Section 5.

3.2. \( \mathsf{TS4} \). The tense logic \( \mathsf{TS4} \) will combine \( \mathsf{S}4 \) with \( \mathsf{S}4.t \). We will use \( \mathsf{S}4 \) to interpret intuitionistic connectives, and \( \mathsf{S}4.t \) to interpret monadic intuitionistic quantifiers. Let \( \mathcal{ML} \) be the multimodal propositional language with three modalities \( \Box, \Box_F, \) and \( \Box_P \). We use \( \diamond, \Box_F, \) and \( \Box_P \) as usual abbreviations.

**Definition 3.11.** The logic \( \mathsf{TS4} \) is the least classical multimodal logic containing the \( \mathsf{S}4 \)-axioms for \( \Box, \Box_F, \) and \( \Box_P \), the tense axioms for \( \Box_F \) and \( \Box_P \), the connecting axioms

\[
\begin{align*}
diamond p & \rightarrow \Box_F p \\
\Box_F p & \rightarrow \diamond(\diamond_F p \land \Box_P p)
\end{align*}
\]

and closed under modus ponens, substitution, and three necessitation rules (for \( \Box, \Box_F, \) and \( \Box_P \)).

Algebraic semantics for \( \mathsf{TS4} \) is given by \( \mathsf{TS4} \)-algebras.

**Definition 3.12.** A \( \mathsf{TS4} \)-algebra is a quadruple \( \mathfrak{B} = (B, \Box, \Box_F, \Box_P) \) where \( (B, \Box) \) is an \( \mathsf{S}4 \)-algebra, \( (B, \Box_F, \Box_P) \) is an \( \mathsf{S}4.t \)-algebra, and for each \( a \in B \) we have:

\[
\begin{align*}
&T1 \quad \diamond a \leq \Box_F a \\
&T2 \quad \Box_F a \leq \diamond(\diamond_F a \land \Box_P a)
\end{align*}
\]

The Lindenbaum-Tarski construction then yields that \( \mathsf{TS4} \)-algebras provide a sound and complete algebraic semantic for \( \mathsf{TS4} \).
Lemma 3.17. A TS4-frame is a triple \(\mathfrak{F} = (X, R, Q)\) where \(X\) is a set and \(R, Q\) are quasi-orders on \(X\) such that \(R \subseteq Q\) and \(xQy\) implies that there is \(z \in X\) such that \(xRz\) and \(zE_Qy\).

Remark 3.14.

1. The only difference between TS4-frames and MiPC-frames is that in TS4-frames the relation \(R\) is a quasi-order, while in MiPC-frames it is a partial order.

2. It is straightforward to check that if \((X, R, Q)\) is a TS4-frame, then \((X, R, E_Q)\) is an MS4-frame, and that if \((X, R, E)\) is an MS4-frame, then \((X, R, Q_E)\) is a TS4-frame. (We recall that, as in Definition 3.30, \(Q_E\) is defined by \(xQ_Ey\) iff \(\exists z \in X)(xRz \& zE_Qy)\). If \((X, R, Q)\) is a TS4-frame, by definition we have that \(xQy\) iff \(\exists z \in X)(xRz \& zE_Qy)\). Thus, \(Q = Q_{E_Q}\).

Remark 3.13. Since \(\mathfrak{F}\) is a TS4-frame, \(\mathfrak{F}^+ = (\varphi(X), \Box_R, \Box_Q, \Box_{Q^-})\) is a TS4-algebra.

Proof. Since \(R\) and \(Q\) are quasi-orders, \((\varphi(X), \Box_R)\) is an S4-frame and \((\varphi(X), \Box_Q, \Box_{Q^-})\) is a S4-frame. It remains to show that \(\mathfrak{F}^+\) satisfies \((T1)\) and \((T2)\).

\((T1)\) Since \(R \subseteq Q\), we have \(\Box_R(U) = R^{-1}[U] \subseteq Q^{-1}[U] = \Diamond_Q(U)\).

\((T2)\) Let \(x \in \Diamond_Q(U) = Q^{-1}[U]\), so there is \(y \in U\) with \(xQy\). Then there is \(z \in X\) with \(xRz\) and \(zE_Qy\). Therefore, \(z \in Q^{-1}[y] \subseteq Q^{-1}[U] = \Diamond_Q(U)\) and \(z \in Q[y] \subseteq Q[U] = \Diamond_{Q^-}(U)\). Thus, \(x \in R^{-1}[z] \subseteq R^{-1}[\Diamond_Q(U) \cap \Diamond_{Q^-}(U)] = \Diamond_R(\Diamond_Q(U) \cap \Diamond_{Q^-}(U))\). This shows that \(\Diamond_Q(U) \subseteq \Diamond_R(\Diamond_Q(U) \cap \Diamond_{Q^-}(U))\).

\(\Box\)

We next prove that each TS4-algebra is represented as a subalgebra of \(\mathfrak{F}^+\) for some TS4-frame \(\mathfrak{F}\). For a TS4-algebra \((B, \Box, \Box_F, \Box_P)\) let \(H, H_F,\) and \(H_P\) be the Heyting algebras of the \(\Box\)-fixpoints, \(\Box_F\)-fixpoints, and \(\Box_P\)-fixpoints, respectively.

Definition 3.16. Let \(\mathfrak{B} = (B, \Box, \Box_F, \Box_P)\) be a TS4-algebra. The canonical frame of \(\mathfrak{B}\) is the frame \(\mathfrak{B}^+ = (X_B, R_B, Q_B)\) where \(X_B\) is the set of ultrafilters of \(B\), \(xR_By\) if \(x \cap H \subseteq y\), and \(xQ_By\) if \(x \cap H_F \subseteq y\), which happens iff \(y \cap H_P \subseteq x\).

Lemma 3.17. If \(\mathfrak{B}\) is a TS4-algebra, then \(\mathfrak{B}^+\) is a TS4-frame.

Proof. Clearly \(R_B\) and \(Q_B\) are quasi-orders. To prove that \(R_B \subseteq Q_B\) we first show that \(H_F \subseteq H\). Let \(a \in H_F\). Then \(a = \Box_Fa = -\Diamond_F\neg a = \neg \Diamond_F\neg a\). By \((T1)\),

\(-\Diamond_F\neg a = -\Diamond_F\neg a = \Box_F\neg a \leq \Box a\).

Therefore, \(a = \Box a\), and so \(a \in H\). Now suppose that \(xR_By\), so \(x \cap H \subseteq y\). Let \(a \in x \cap H_F\). Then \(a \in x \cap H \subseteq y\). Thus, \(a \in y\), and hence \(xQ_By\).

To prove the other condition, let \(xQ_By\), so \(x \cap H_F \subseteq y\). We show that \((x \cap H) \cup (y \cap H_F) \cup (y \cap H_P)\) generates a proper filter of \(B\). Otherwise, since \(H, H_F, H_P\) are closed under meets, there are \(a \in x \cap H, b \in y \cap H_F,\) and \(c \in y \cap H_P\) such that \(a \wedge b \wedge c = 0\). By Remark 3.14, \(H_F\) coincides with the set of \(\Diamond_P\)-fixpoints and \(H_P\) with the set of \(\Diamond_F\)-fixpoints. Therefore, since \(b \in H_F\) and \(c \in H_P\), we have \(\Diamond_P(b \wedge c) \wedge \Diamond_F(b \wedge c) = b \wedge c\). Thus, \(a \wedge \Diamond_P(b \wedge c) \wedge \Diamond_F(b \wedge c) \leq a \wedge b \wedge c = 0\), yielding \(a \leq -\Diamond_P(b \wedge c) \wedge \Diamond_F(b \wedge c)\). Since \(a \in H\), we have

\(a = \Box a \leq -\Diamond_P(b \wedge c) \wedge \Diamond_F(b \wedge c) = -\Diamond_P(b \wedge c) \wedge \Diamond_F(b \wedge c)\).
Consequently, \( a \land \Box (\Diamond_P (b \land c) \land \Diamond_F (b \land c)) = 0 \). By (1.2),
\[
a \land \Diamond_F (b \land c) \leq a \land \Box (\Diamond_P (b \land c) \land \Diamond_F (b \land c)) = 0.
\]
Because \( b \land c \leq \Diamond_F (b \land c) \), \( b \land c \in y \), and \( y \) is a filter, we have \( \Diamond_F (b \land c) \in y \). Since \( x \cap H_F \subseteq y \), we have \( y \cap H_F \subseteq x \). Therefore, \( \Diamond_F (b \land c) \in y \cap H_F \subseteq x \) and \( a \in x \). Thus, \( 0 = a \land \Diamond_F (b \land c) \in x \), a contradiction. Consequently, there is an ultrafilter \( z \) such that \( (x \cap H) \cup (y \cap H_F) \cup (y \cap H_P) \subseteq z \). But then \( x \cap H \subseteq z \), \( y \cap H_F \subseteq z \), and \( y \cap H_P \subseteq z \). This gives that \( xR_{Bz} z \), \( yQ_{Bz} z \), and \( zQ_{Bz} y \), as desired.

**Definition 3.18.** We call a TS4-frame *canonical* if it is isomorphic to \( \mathfrak{B}_+ \) for some TS4-algebra \( \mathfrak{B} \).

Let \( \mathfrak{B} \) be a TS4-algebra. Since \( \beta : B \rightarrow \wp(X_{\mathfrak{B}}) \) is an embedding of TS4-algebras, we obtain the following representation theorem for TS4-algebras.

**Proposition 3.19.** Each TS4-algebra \( \mathfrak{B} \) is isomorphic to a subalgebra of \( (\mathfrak{B}_+)^+ \).

**Remark 3.20.** To recover the image of \( \mathfrak{B} \) in \( \wp(X_{\mathfrak{B}}) \) we need to endow \( X_{\mathfrak{B}} \) with a Stone topology. This leads to the notion of perfect TS4-frames and a duality between the categories of TS4-algebras and perfect TS4-frames (see [8] for details). When \( \mathfrak{B} \) is finite, its embedding into \( (\mathfrak{B}_+)^+ \) is an isomorphism, and hence the categories of finite TS4-algebras and finite TS4-frames are dually equivalent.

Since \( (\mathfrak{B}_+)^+ \) is a TS4-algebra, as an immediate consequence we obtain:

**Corollary 3.21.** TS4 is canonical.

Let \( \mathfrak{F} = (X, R, Q) \) be a TS4-frame. A *valuation* of \( \mathcal{ML} \) into \( \mathfrak{F} \) associates with each propositional letter a subset of \( X \). The classical connectives are interpreted as usual, \( \Box \) is interpreted using the relation \( R \), and \( \Box_F, \Box_P \) are interpreted using the relation \( Q \):
\[
\begin{align*}
  x \vDash v \Box \varphi & \iff (\forall y \in X)(xR_y \Rightarrow y \vDash_v \varphi), \\
  x \vDash v \Box_F \varphi & \iff (\forall y \in X)(xQ_y \Rightarrow y \vDash_v \varphi), \\
  x \vDash v \Box_P \varphi & \iff (\forall y \in X)(yQ_x \Rightarrow y \vDash_v \varphi).
\end{align*}
\]
Consequently,
\[
\begin{align*}
  x \vDash v \Diamond \varphi & \iff (\exists y \in X)(xR_y \land y \vDash_v \varphi), \\
  x \vDash v \Diamond_F \varphi & \iff (\exists y \in X)(xQ_y \land y \vDash_v \varphi), \\
  x \vDash v \Diamond_P \varphi & \iff (\exists y \in X)(yQ_x \land y \vDash_v \varphi).
\end{align*}
\]

**Theorem 3.22.** TS4 \( \vdash \varphi \iff \mathfrak{F} \vDash \varphi \) for every TS4-frame \( \mathfrak{F} \).

**Proof.** Soundness is straightforward to prove, and completeness follows from the algebraic completeness and the representation of TS4-algebras (Proposition 3.19).

In Section 5 we will prove that TS4 has the fmp.

### 3.3. Gödel translation adjusted

In this section we modify the Gödel translation to embed MIPC into TS4 fully and faithfully.

**Definition 3.23.** The translation \( (-)^\sharp : \text{MIPC} \rightarrow \text{TS4} \) is defined as \( (-)^\sharp \) on propositional letters, \( \bot, \land, \lor, \) and \( \rightarrow; \) and for \( \forall \) and \( \exists \) we set:
\[
\begin{align*}
  (\forall \varphi)^\sharp & = \Box_F \varphi^\sharp, \\
  (\exists \varphi)^\sharp & = \Diamond_P \varphi^\sharp.
\end{align*}
\]
Thus, $\forall$ is interpreted as “always in the future” and $\exists$ as “sometime in the past.”

We adapt Definition 2.30 to the setting of $TS4$-frames by utilizing the correspondence between $TS4$-frames and $MS4$-frames described in Remark 3.14.

**Definition 3.24.** Let $\mathfrak{F} = (X, R, Q)$ be a $TS4$-frame, and let $\sim$ be the equivalence relation given by $x \sim y$ iff $xRy$ and $yRx$. We set $X'$ to be the set of equivalence classes of $\sim$, and define $R'$ and $Q'$ on $X'$ by $[x]R'[y]$ iff $xRy$ and $[x]Q'[y]$ iff $xQy$. We call $\mathfrak{F}' = (X', R', Q')$ the skeleton of $\mathfrak{F}$.

**Proposition 3.25.**

1. If $\mathfrak{F}$ is a $TS4$-frame, then $\mathfrak{F}'$ is an $MIPC$-frame.
2. For each valuation $v$ on $\mathfrak{F}$ there is a valuation $v'$ on $\mathfrak{F}'$ such that for each $x \in \mathfrak{F}$ and $\mathcal{L}_{\forall\exists}$-formula $\varphi$, we have

$$\mathfrak{F}', [x] \models v' \varphi \iff \mathfrak{F}, x \models v \varphi^\sharp.$$  

3. For each $\mathcal{L}_{\forall\exists}$-formula $\varphi$, we have

$$\mathfrak{F}' \models \varphi \iff \mathfrak{F} \models \varphi^\sharp.$$  

4. Any $MIPC$-frame $\mathfrak{G}$ is also a $TS4$-frame and $\mathfrak{G}'$ is isomorphic to $\mathfrak{G}$.

**Proof.**

1. It is well known that $(X', R')$ is an intuitionistic Kripke frame. $Q'$ is well defined on $X'$ because $R \subseteq Q$ in $\mathfrak{F}$. Showing that $Q'$ is a quasi-order, and that (O1) and (O2) hold in $\mathfrak{F}'$ is straightforward.

2. As in Proposition 2.31(2), we define $v'$ by $v'(p) = \{[x] \in X' \mid R[x] \subseteq v(p)\}$, and show that $\mathfrak{F}', [x] \models v' \varphi \iff \mathfrak{F}, x \models v \varphi^\sharp$ by induction on the complexity of $\varphi$. It is sufficient to only consider the cases when $\varphi$ is of the form $\forall \psi$ or $\exists \psi$. Suppose $\varphi = \forall \psi$. Then by the definition of $Q'$ and induction hypothesis,

$$\mathfrak{F}', [x] \models v' \forall \psi \iff (\forall[y] \in X')([x]Q'[y] \Rightarrow \mathfrak{F}', [y] \models v' \psi)$$

$$\iff (\forall y \in X)(xQy \Rightarrow \mathfrak{F}', y \models v \psi^\sharp)$$

$$\iff \mathfrak{F}, x \models v \mathbf{\Box} \psi^\sharp$$

$$\iff \mathfrak{F}, x \models v (\forall \psi)^\sharp.$$  

Suppose $\varphi = \exists \psi$. As noted in Remark 2.7, $Q'$ and $E_{Q'}$ coincide on $R'$-upsets. Since the set $\{[y] \mid \mathfrak{F}', [y] \models v' \psi\}$ is an $R'$-upset, by the induction hypothesis, we have

$$\mathfrak{F}', [x] \models v' \exists \psi \iff (\exists[y] \in X')([x]E_{Q'}[y] \& \mathfrak{F}', [y] \models v' \psi)$$

$$\iff [x] \in E_{Q'}([y] \mid \mathfrak{F}', [y] \models v' \psi])$$

$$\iff [x] \in Q'([y] \mid \mathfrak{F}', [y] \models v' \psi])$$

$$\iff x \in Q([y] \mid \mathfrak{F}', [y] \models v' \psi])$$

$$\iff x \in Q([y] \mid \mathfrak{F}, y \models v \psi^\sharp)$$

$$\iff \mathfrak{F}, x \models v \mathbf{\Diamond} \psi^\sharp$$

$$\iff \mathfrak{F}, x \models v (\exists \psi)^\sharp.$$  

3. The proof is analogous to that of Proposition 2.31(3).
(4). Let \( G = (X, R, Q) \) be an MIPC-frame. It is clear from the definition of TS4-frames that \( G \) is also a TS4-frame. Since \( R \) is a partial order, \( \sim \) is the identity relation. Therefore, \( G \) is isomorphic to \( G^\sim \). □

**Theorem 3.26.** The translation \((-)^\sim\) of MIPC into TS4 is full and faithful; that is,

\[ MIPC \vdash \varphi \text{ iff } TS4 \vdash \varphi^\sim. \]

**Proof.** To prove faithfulness, suppose that \( TS4 \not\vdash \varphi^\sim \). By Theorem 3.22, there is a TS4-frame \( \mathcal{F} \) such that \( \mathcal{F} \not\models \varphi^\sim \). By Proposition 3.25, \( \mathcal{F}^\sim \) is an MIPC-frame and \( \mathcal{F}^\sim \not\models \varphi \). Thus, by Theorem 2.13, \( MIPC \not\vdash \varphi \). For fullness, if \( MIPC \not\vdash \varphi \), then there is an MIPC-frame \( G \) such that \( G \not\models \varphi \).

By Proposition 3.25(4), \( G \) is also a TS4-frame and it is isomorphic to \( G^\sim \). Therefore, \( G^\sim \not\models \varphi \). Proposition 3.25(3) then yields that \( G \not\models \varphi^\sim \). Thus, \( TS4 \not\vdash \varphi^\sim \). □

4. **Translations into MS4.t**

In Sections 2 and 3 we described full and faithful translations of MIPC into MS4 and TS4, respectively. This yields the following diagram.

![Diagram](https://example.com/diagram.png)

There does not appear to be a natural way to translate MS4 into TS4 or vice versa (see [8] for details). The aim of this section is to define a new tense system and show that both MS4 and TS4 embed fully and faithfully into it, thus completing the above diagram.

4.1. **MS4.t.** Let \( \mathcal{L}_{TV} \) be the propositional language with the tense modalities \( \Box_F \) and \( \Box_P \), and the monadic modality \( \forall \). In order to stress that the language \( \mathcal{L}_{TV} \) is different from \( \mathcal{ML} \) and TS4, we use different symbols for the tense modalities.

**Definition 4.1.** The tense MS4, denoted MS4.t, is the least classical multimodal logic containing the S4.t-axioms for \( \Box_F \) and \( \Box_P \), the S5-axioms for \( \forall \), the left commutativity axiom

\[ \Box_F \forall p \rightarrow \forall \Box_F p, \]

and closed under modus ponens, substitution, and the necessitation rules (for \( \Box_F, \Box_P \), and \( \forall \)).

**Remark 4.2.** We can think of MS4.t as the tense extension of MS4. It is worth stressing that MS4.t is not the monadic fragment of the standard predicate extension QS4.t of S4.t. To see this, it is well known that the Barcan formula \( \forall x \Box_F \varphi \rightarrow \Box_F \forall x \varphi \) and the converse Barcan formula \( \Box_F \forall x \varphi \rightarrow \forall x \Box_F \varphi \) are both theorems of any tense predicate logic, hence of QS4.t as well. Thus, the monadic fragment of QS4.t contains both the left commutativity axiom \( \Box_F \forall p \rightarrow \forall \Box_F p \) and the right commutativity axiom \( \forall \Box_F p \rightarrow \Box_F \forall p \). On the other hand, it is easy to see (e.g., using the Kripke semantics for MS4.t which we will define shortly) that, while MS4.t contains the left commutativity axiom, the right commutativity axiom is not provable in MS4.t.

Algebraic semantics for MS4.t is given by MS4.t-algebras.
Definition 4.3. An MS4.t-algebra is a tuple \( \mathfrak{B} = (B, \Box_F, \Box_P, \vee) \) where \( (B, \Box_F, \Box_P) \) is an S4.t-algebra and \( (B, \Box_F, \vee) \) is an MS4-algebra.

As usual, the Lindenbaum-Tarski construction yields that MS4.t is sound and complete with respect to MS4.t-algebras.

As with S4 and S4.t, we have that MS4.t-frames are simply MS4-frames, the difference is in interpreting tense modalities. Thus, the following lemma is straightforward.

Lemma 4.4. If \( \mathfrak{F} = (X, R, E) \) is an MS4.t-frame, then \( \mathfrak{F}^+ := (\varphi(X), \Box_R, \Box_{R^*}, \vee_E) \) is an MS4.t-algebra.

We next prove that each MS4.t-algebra is represented as a subalgebra of \( \mathfrak{F}^+ \) for some MS4.t-frame \( \mathfrak{F} \). For an MS4.t-algebra \( (B, \Box_F, \Box_P, \vee) \) let \( H_F, H_P \) and \( B_0 \) be the \( \Box_F \)-fixpoints, \( \Box_P \)-fixpoints, and \( \vee \)-fixpoints, respectively. Clearly \( H_F \) and \( H_P \) are Heyting algebras and \( B_0 \) is a boolean subalgebra of \( B \).

Definition 4.5. Let \( \mathfrak{B} = (B, \Box_F, \Box_P, \vee) \) be an MS4.t-algebra. The canonical frame of \( \mathfrak{B} \) is the frame \( \mathfrak{B}^+ = (X_{\mathfrak{B}}, R_{\mathfrak{B}}, E_{\mathfrak{B}}) \) where \( X_{\mathfrak{B}} \) is the set of ultrafilters of \( B \), \( xR_{\mathfrak{B}}y \) iff \( x \cap H_F \subseteq y \) iff \( y \cap H_P \subseteq x \), and \( xE_{\mathfrak{B}}y \) iff \( x \cap B_0 = y \cap B_0 \).

Since MS4.t-frames are MS4-frames, the next lemma is obvious.

Lemma 4.6. If \( \mathfrak{B} \) is an MS4.t-algebra, then \( \mathfrak{B}^+ \) is an MS4.t-frame.

Thus, since \( \beta : B \rightarrow \varphi(X_{\mathfrak{B}}) \) is an embedding of S4.t-algebras and MS4-algebras, we obtain the following representation theorem for MS4.t-algebras.

Proposition 4.7. Each MS4.t-algebra \( \mathfrak{B} \) is isomorphic to a subalgebra of \( (\mathfrak{B}^+) \).  

Remark 4.8. To recover the image of the embedding of \( \mathfrak{B} \) into \( (\mathfrak{B}^+) \) we need to endow \( \mathfrak{B}^+ \) with a Stone topology. This leads to the notion of perfect MS4.t-frames and a duality between the categories of MS4.t-algebras and perfect MS4.t-frames (see [8] for details). When \( \mathfrak{B} \) is finite, its embedding into \( (\mathfrak{B}^+) \) is an isomorphism, and hence the categories of finite MS4.t-algebras and finite MS4.t-frames are dually equivalent.

By Lemmas [4.4] and [4.6] if \( \mathfrak{B} \) is an MS4.t-algebra, then so is \( (\mathfrak{B}^+) \). As an immediate consequence, we obtain:

Corollary 4.9. MS4.t is canonical.

A valuation on an MS4.t-frame \( \mathfrak{F} = (X, R, E) \) is a map \( v \) associating to each propositional letter of \( L_{T\forall} \) a subset of \( \mathfrak{F} \). The boolean connectives are interpreted as usual, and

\[
\begin{align*}
\mathfrak{F}, x &\models \Box_F \varphi \quad \text{iff} \quad (\forall y \in X)(xR y \Rightarrow y \models \varphi), \\
\mathfrak{F}, x &\models \Box_P \varphi \quad \text{iff} \quad (\forall y \in X)(yR x \Rightarrow y \models \varphi), \\
\mathfrak{F}, x &\models \forall \varphi \quad \text{iff} \quad (\forall y \in X)(xEy \Rightarrow y \models \varphi).
\end{align*}
\]

Theorem 4.10. MS4.t \models \varphi \text{ iff } \mathfrak{F} \models \varphi \text{ for every MS4.t-frame } \mathfrak{F}.

Proof. Soundness is a consequence of the soundness of the relational semantics for MS4 and S4.t. Completeness follows from the algebraic completeness and the representation of MS4.t-algebras (see Proposition 4.7). \( \square \)

In Section 5 we will prove that MS4.t has the fmp.
4.2. Translations of TS4 and MS4 into MS4_t. We next define two full and faithful translations 
$(-)^\# : MS4 \rightarrow MS4_t$ and $(-)^\uparrow : TS4 \rightarrow MS4_t$. The translation of MS4 into MS4_t will reflect 
that MS4_t is the tense extension of MS4.

Definition 4.11. We define the translation $(-)^\# : MS4 \rightarrow MS4_t$ by replacing in each formula $\varphi$ 
of $L_{\Box^\neg}$ every occurrence of $\Box$ with $\Box_F$.

Theorem 4.12. The translation $(-)^\#$ of MS4 into MS4_t is full and faithful; that is,

$$MS4 \vdash \varphi \iff MS4_t \vdash \varphi^\#.$$ 

Proof. By definition, MS4_t-frames are MS4-frames and valuations on MS4-frames and MS4_t-frames coincide. 
The boolean connectives and monadic modality $\forall$ are interpreted the same way in MS4-frames and MS4_t-frames. 
Also, the interpretation of $\Box$ in MS4-frames coincides with the interpretation of $\Box_F$ in MS4_t-frames. 
This implies that for each frame $\mathfrak{F} = (X, R, E)$, valuation $v$, and $x \in X$, we have $\mathfrak{F}, x \models \varphi$ iff $\mathfrak{F}, x \models \varphi^\#$ for every $L_{\Box^\neg}$-formula $\varphi$. The result then follows 
from the soundness and completeness of MS4 and MS4_t with respect to their relational semantics (see Theorems 2.28 and 4.10).

Definition 4.13. Define the translation $(-)^\uparrow : TS4 \rightarrow MS4_t$ by

$$p^\uparrow = p \quad \text{for each propositional letter } p$$

$(-)^\uparrow$ commutes with the boolean connectives

$$(\Box \varphi)^\uparrow = \Box_F \varphi^\uparrow$$

$$(\Box_F \varphi)^\uparrow = \Box_F \forall \varphi^\uparrow$$

$$(\Box_F \varphi)^\uparrow = \forall \Box_F \varphi^\uparrow.$$

Definition 4.14. For an MS4_t-frame $\mathfrak{F} = (X, R, E)$ we define $\mathfrak{F}^\uparrow = (X, R, Q_E)$.

Proposition 4.15. 
(1) If $\mathfrak{F}$ is an MS4_t-frame, then $\mathfrak{F}^\uparrow$ is a TS4-frame.

(2) Each valuation $v$ on $\mathfrak{F}$ is also a valuation on $\mathfrak{F}^\uparrow$ such that for each $x \in \mathfrak{F}$ and ML-formula $\varphi$, we have

$$\mathfrak{F}^\uparrow, x \models v \varphi \iff \mathfrak{F}, x \models v \varphi^\uparrow.$$ 

(3) For each ML-formula $\varphi$, we have

$$\mathfrak{F}^\uparrow \models v \varphi \iff \mathfrak{F} \models v \varphi^\uparrow.$$ 

(4) For any TS4-frame $\mathfrak{G}$ there is an MS4_t-frame $\mathfrak{F}$ such that $\mathfrak{G} = \mathfrak{F}^\uparrow$.

Proof. (1). Since MS4_t-frames coincide with MS4-frames, we already observed in Remark 3.11(2) that $\mathfrak{F}^\uparrow$ is a TS4-frame.

(2). It is clear that if $v$ is a valuation on $\mathfrak{F}$, then $v$ is also a valuation on $\mathfrak{F}^\uparrow$. We show that $\mathfrak{F}^\uparrow, x \models v \varphi$ iff $\mathfrak{F}, x \models v \varphi^\uparrow$ by induction on the complexity of $\varphi$. The only nontrivial cases are when $\varphi$ is of the form $\Box \psi$, $\Box_F \psi$ and $\forall \Box_F \psi$. Suppose $\varphi = \Box \psi$. Then, by the induction hypothesis,

$$\mathfrak{F}^\uparrow, x \models v \Box \psi \iff (\forall y \in X)(xRy \Rightarrow \mathfrak{F}^\uparrow, y \models v \psi)$$

$$\iff (\forall y \in X)(xRy \Rightarrow \mathfrak{F}, y \models v \psi^\uparrow)$$

$$\iff \mathfrak{F}, x \models v \Box_F \psi^\uparrow$$

$$\iff \mathfrak{F}, x \models (\Box \psi)^\uparrow.$$
Suppose $\varphi = \Box_F \psi$. Then, by the induction hypothesis,

\[
\mathfrak{F}^\dagger, x \models_F \Box_F \psi \iff (\forall y \in X)(yQ_E y \Rightarrow \mathfrak{F}^\dagger, y \models_v \psi)
\]

iff (\forall z \in X)(zRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}^\dagger, y \models_v \psi))

iff (\forall z \in X)(zRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}, y \models \psi^\dagger))

iff $\mathfrak{F}, x \models_v \Box_F \psi^\dagger$

iff $\mathfrak{F}, x \models_v (\Box_F \psi)^\dagger$.

Suppose $\varphi = \Box_P \psi$. Then, by the induction hypothesis,

\[
\mathfrak{F}^\dagger, x \models_P \Box_P \psi \iff (\forall y \in X)(yQ_E x \Rightarrow \mathfrak{F}^\dagger, y \models_v \psi)
\]

iff (\forall y, z \in X)(yRz \& zEx \Rightarrow \mathfrak{F}^\dagger, y \models_v \psi)

iff (\forall y \in X)(zRz \Rightarrow (\forall y \in X)(yRz \Rightarrow \mathfrak{F}^\dagger, y \models_v \psi))

iff (\forall y \in X)(zRz \Rightarrow (\forall y \in X)(yRz \Rightarrow \mathfrak{F}, y \models \psi^\dagger))

iff $\mathfrak{F}, x \models_v \Box_P \psi^\dagger$

iff $\mathfrak{F}, x \models_v (\Box_P \psi)^\dagger$.

(3). The proof that $\mathfrak{F}^\dagger \models \varphi$ iff $\mathfrak{F} \models \varphi^\dagger$ is analogous to that of Proposition 2.31(3).

(4). Let $\mathfrak{G} = (X, R, Q)$ be a TS4-frame. As we observed in Remark 3.14, $\mathfrak{G} = (X, R, E_Q)$ is an MS4-frame, and so an MS4.t-frame. By definition of TS4-frames we have that $Q = Q_{E_Q}$, and hence $\mathfrak{G} = \mathfrak{F}^\dagger$.

\[\text{Theorem 4.16.} \quad \text{The translation } (-)^\dagger \text{ of TS4 into MS4.t is full and faithful; that is,}
\]

TS4 $\vdash \varphi \quad \iff \quad$ MS4.t $\vdash \varphi^\dagger$.

\[\text{Proof.} \quad \text{To prove faithfulness, suppose that MS4.t } \not \models \varphi^\dagger. \quad \text{By Theorem 3.10, there is an MS4.t-frame } \mathfrak{F} \quad \text{such that } \mathfrak{F} \not \models \varphi^\dagger. \quad \text{By Proposition 4.15, } \mathfrak{F}^\dagger \text{ is a TS4-frame and } \mathfrak{F}^\dagger \not \models \varphi. \quad \text{Thus, TS4 } \not \models \varphi \text{ by Theorem 3.22.} \quad \text{For fullness, if TS4 } \not \models \varphi, \text{ then there is a TS4-frame } \mathfrak{G} \text{ such that } \mathfrak{G} \not \models \varphi. \quad \text{By Proposition 4.15(4), there is an MS4.t-frame } \mathfrak{F} \text{ such that } \mathfrak{G} \text{ is isomorphic to } \mathfrak{F}^\dagger. \quad \text{Therefore, } \mathfrak{F}^\dagger \not \models \varphi. \quad \text{Proposition 4.15(3) then implies that } \mathfrak{F} \not \models \varphi^\dagger. \quad \text{Thus, MS4.t } \not \models \varphi^\dagger. \quad \square
\]

\[\text{Remark 4.17.}
\]

(1) The definition of the translation $(-)^\dagger : \text{TS4 } \rightarrow \text{MS4.t}$ is suggested by the correspondence between TS4-frames and MS4.t-frames. Indeed, given an MS4.t-frame $\mathfrak{F}$, the relation $Q_E$ in $\mathfrak{F}^\dagger$ is the composition of $R$ and $E$, and the inverse relation $Q_E^*$ is the composition of $E$ and $R^{-1}$. Therefore, the modalities $\Box_F$ and $\Box_P$ are translated as $\Box_P \forall$ and $\forall \Box_P$, respectively.

(2) It is natural to consider a modification of $(-)^\dagger$ where $\Box_P$ is translated as $\Box_P \forall$. However, such a modification is neither full nor faithful. Nevertheless, its composition with $(-)^\natural : \text{MIPC } \rightarrow \text{TS4}$ is full and faithful, as we will see at the end of Section 4.3.
4.3. **Translations of MIPC into MS4.t.** We denote the composition of (−)# and (−)† by (−)†#, and the composition of (−)† and (−)♭ by (−)♭†. Since we proved that all these four translations are full and faithful, we also have that (−)♭# and (−)♭† are full and faithful translations of MIPC into MS4.t. We have thus obtained the following diagram of full and faithful translations. We next show that this diagram is commutative up to logical equivalence in MS4.t.

![Diagram of full and faithful translations]

**Lemma 4.18.** For any formula ϕ of $L_{\forall \exists}$, we have

$$\text{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \Box P \varphi^{\#}.$$  

**Proof.** By Lemma 2.29 and Theorem 4.12, MS4.t ⊨ $\varphi^{t\#} \rightarrow \Box P \varphi^{\#}$. Therefore, MS4.t ⊨ $\Diamond P \varphi^{t\#} \rightarrow \Diamond P \Box P \varphi^{\#}$. The tense axiom then gives MS4.t ⊨ $\Diamond P \varphi^{t\#} \rightarrow \varphi^{\#}$. Thus, MS4.t ⊨ $\varphi^{t\#} \leftrightarrow \Diamond P \varphi^{\#}$. □

**Theorem 4.19.** For any $L_{\forall \exists}$-formula χ we have

$$\text{MS4.t} \vdash \chi^{t\#} \leftrightarrow \chi^{\#}.$$  

**Proof.** The two compositions compare as follows:

\[
\begin{align*}
\bot^{t\#} &= \bot & \bot^{\#} &= \bot \\
\varphi^{t\#} &= \Box F P & \varphi^{\#} &= \Box F P \\
(\varphi \land \psi)^{t\#} &= \varphi^{t\#} \land \psi^{t\#} & (\varphi \land \psi)^{\#} &= \varphi^{\#} \land \psi^{\#} \\
(\varphi \lor \psi)^{t\#} &= \varphi^{t\#} \lor \psi^{t\#} & (\varphi \lor \psi)^{\#} &= \varphi^{\#} \lor \psi^{\#} \\
(\varphi \rightarrow \psi)^{t\#} &= \Box F (\neg \varphi^{t\#} \lor \psi^{t\#}) & (\varphi \rightarrow \psi)^{\#} &= \Box F (\neg \varphi^{\#} \lor \psi^{\#}) \\
(\forall \varphi)^{t\#} &= \Box F \forall \varphi^{t\#} & (\forall \varphi)^{\#} &= \Box F \forall \varphi^{\#} \\
(\exists \varphi)^{t\#} &= \exists \varphi^{t\#} & (\exists \varphi)^{\#} &= (\Diamond P \varphi)^{\#} = (\neg \Box P \neg \varphi^{\#})^{\#} \\
& & &= \neg \Box P \neg \varphi^{\#} \\
\end{align*}
\]

Thus, they are identical except the $\exists$-clause. Therefore, to prove that MS4.t ⊨ $\chi^{t\#} \leftrightarrow \chi^{\#}$ it is sufficient to prove that MS4.t ⊨ $\varphi^{t\#} \leftrightarrow \varphi^{\#}$ implies MS4.t ⊨ $\exists \varphi^{t\#} \leftrightarrow \exists \Box P \neg \varphi^{\#}$. Since MS4.t ⊨ $\neg \Box P \neg \varphi^{\#}$, it is enough to prove that MS4.t ⊨ $\exists \varphi^{t\#} \leftrightarrow \exists \Box P \varphi^{\#}$. From the assumption MS4.t ⊨ $\varphi^{t\#} \leftrightarrow \varphi^{\#}$ it follows that MS4.t ⊨ $\exists \Box P \varphi^{t\#} \leftrightarrow \exists \Box P \varphi^{\#}$. By Lemma 4.18, MS4.t ⊨ $\varphi^{t\#} \leftrightarrow \Diamond P \varphi^{t\#}$ and hence MS4.t ⊨ $\exists \varphi^{t\#} \leftrightarrow \exists \Box P \varphi^{\#}$. Thus, MS4.t ⊨ $\exists \varphi^{t\#} \leftrightarrow \exists \Box P \varphi^{\#}$. □

As we pointed out in Remark 4.17.2, there is another natural translation of MIPC into MS4.t.

**Definition 4.20.** Let (−)♭ : MIPC → MS4.t be the translation that differs from (−)†# and (−)♭† only in the $\exists$-clause:

$$(\exists \varphi)^♭ = \Diamond P \exists \varphi.$$  

The translation (−)♭ provides a temporal interpretation of intuitionistic monadic quantifiers that is similar to the translation (−)♭† (see also Section 6).
**Theorem 4.21.** For any $\mathcal{L}_{\forall_3}$-formula $\chi$ we have

$$\text{MS4}.t \vdash \chi^b \leftrightarrow \chi^{t\#}.$$  

Consequently, the translation $(-)^b$ of $\text{MIPC}$ into $\text{MS4}.t$ is full and faithful.

**Proof.** The translations $(-)^b$ and $(-)^{t\#}$ are identical except the $\exists$-clause. Therefore, to prove that $\text{MS4}.t \vdash \chi^b \leftrightarrow \chi^{t\#}$ it is sufficient to prove that $\text{MS4}.t \vdash \varphi^b \leftrightarrow \varphi^{t\#}$ implies $\text{MS4}.t \vdash \diamond_p \exists \varphi^b \leftrightarrow \exists \varphi^{t\#}$.

By Lemma 4.18, $\text{MS4}.t \vdash (\exists \varphi)^{t\#} \leftrightarrow \diamond_p (\exists \varphi)^{t\#}$ which means $\text{MS4}.t \vdash \exists \varphi^{t\#} \leftrightarrow \diamond_p \exists \varphi^{t\#}$. From the assumption $\text{MS4}.t \vdash \varphi^b \leftrightarrow \varphi^{t\#}$ it follows that $\text{MS4}.t \vdash \diamond_p \exists \varphi^b \leftrightarrow \diamond_p \exists \varphi^{t\#}$. Thus, $\text{MS4}.t \vdash \diamond_p \exists \varphi \leftrightarrow \exists \varphi^{t\#}$. Since $(-)^{t\#}$ is full and faithful, it follows that $(-)^b$ is full and faithful as well. 

As a result, we obtain the following diagram of full and faithful translations that is commutative up to logical equivalence in $\text{MS4}.t$.

$$\begin{array}{ccc}
\text{MIPC} & \xrightarrow{(-)^b} & \text{MS4} \\
\downarrow & & \downarrow \ \\
\text{TS4} & \xrightarrow{(-)^t} & \text{MS4}.t
\end{array}$$

5. **Finite Model Property**

In this section we prove that the logics studied in this paper all have the fmp. Our strategy is to first establish the fmp for $\text{MS4}.t$, and then use the full and faithful translations to conclude that all the logics we have considered have the fmp.

Let $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ be an $\text{MS4}.t$-algebra and $S \subseteq B$ a finite subset. Then $(B, \forall)$ is an $S_5$-algebra. Let $(B', \forall')$ be the $S_5$-subalgebra of $(B, \forall)$ generated by $S$. It is well known (see [1]) that $(B', \forall')$ is finite. Define $\Box'_F$ and $\Box'_P$ on $B'$ by

$$\Box'_F a = \bigvee \{ b \in B' \cap H_F \mid b \leq a \}$$

$$\Box'_P a = \bigvee \{ b \in B' \cap H_P \mid b \leq a \}.$$  

We denote $(B', \Box'_F, \Box'_P, \forall')$ by $\mathfrak{B}_S$.

**Lemma 5.1.** $\mathfrak{B}_S$ is an $\text{MS4}.t$-algebra.

**Proof.** By definition, $(B', \forall')$ is an $S_5$-algebra. Since $(B, \Box_F)$ and $(B, \Box_P)$ are both $S_4$-algebras, a standard argument (see [19, Lem. 4.14]) shows that $(B', \Box'_F)$ and $(B', \Box'_P)$ are also $S_4$-algebras. We show that $(B', \Box'_F, \Box'_P)$ is an $S_4$-t-algebra. Let $H_F$ be the algebra of $\Box'_F$-fixpoints and $H_P$ the algebra of $\Box'_P$-fixpoints of $\mathfrak{B}$. As noted in Remark 3.14, $-\vdash$ is a dual isomorphism between $H_F$ and $H_P$. Therefore,

$$\Box'_F a := -\Box'_F -a = -\bigvee \{ b \in B' \cap H_F \mid b \leq -a \}$$

$$= -\bigvee \{ b \in B' \cap H_F \mid a \leq -b \}$$

$$= \bigwedge \{-b \mid b \in B' \cap H_F, a \leq -b \}$$

$$= \bigwedge \{ c \in B' \cap H_P \mid a \leq c \}.$$
Since this meet is finite and □_P commutes with finite meets, we obtain

\[ \square_P \Diamond_P' a = \square_P \left( \bigwedge \{ c \in B' \cap H_P \mid a \leq c \} \right) \]
\[ = \bigwedge \{ \square_P c \mid c \in B' \cap H_P, \ a \leq c \} \]
\[ = \bigwedge \{ c \in B' \cap H_P \mid a \leq c \} \]
\[ = \Diamond_P' a. \]

Thus, \( \Diamond_P' a \in B' \cap H_P \) which yields

\[ \square_P \Diamond_P' a = \bigvee \{ b \in B' \cap H_P \mid b \leq \Diamond_P' a \} = \Diamond_P' a. \]

Similarly, we have that \( \Diamond_P' a = \bigwedge \{ c \in B' \cap H_F \mid a \leq c \} \) from which we deduce that \( \square_P \Diamond_P' a = \Diamond_P' a. \) This implies that \( a \leq \square_P \Diamond_P' a \) and \( a \leq \square_P \Diamond_P' a. \) Consequently, \( (B, \square_P, \Diamond_P) \) is an S4-t-algebra.

It remains to show that \( \square_P \forall^P a \leq \forall^P a \) holds in \( \mathcal{B}_S. \) For this it is sufficient to show that the set \( B'_0 := B' \cap B_0 \) of the \( \forall'-\)fixpoints of \( B' \) is an S4-subalgebra of \( (B', \square_P') \) because then \( \square_P \forall^P a = \forall^P \square_P' a \leq \forall^P a. \) Suppose that \( d \in B'_0. \) Then \( \square_P' d = \bigvee \{ b \in B' \cap H_F \mid b \leq d \}. \) Let \( b \in B' \cap H_F. \) By Lemma 2.18, \( \exists b = \exists \square_P b = \square_P \exists \exists_P b = \square_P \exists b. \) Therefore, \( \exists b \in B' \cap H_F. \) Moreover, \( b \leq \exists b \) and \( b \leq d \) implies \( \exists b \leq d. \) Thus, \( \square_P' d = \bigvee \{ \exists b \mid b \in B' \cap H_F, \ b \leq d \}. \) Since \( (B', \forall') \) is an S5-algebra, \( B'_0 \) is the set of \( \exists'-\)fixpoints of \( B' \) and is closed under finite joins. Consequently, \( \square_P' d \in B'_0 \).

**Theorem 5.2.** MS4.t has the fmp.

**Proof.** It is sufficient to prove that each \( L_{TV} \)-formula \( \varphi \) refuted on some MS4.t-algebra is also refuted on a finite MS4.t-algebra. Let \( \overline{t}(x_1, \ldots, x_n) \) be the term in the language of MS4.t-algebras that corresponds to \( \varphi \), and suppose there is an MS4.t-algebra \( \mathcal{B} = (B, \square_F, \square_P, \forall) \) and \( a_1, \ldots, a_n \in B \) such that \( \overline{t}(a_1, \ldots, a_n) \neq 1 \) in \( \mathcal{B}. \) Let

\[ S = \{ t'(a_1, \ldots, a_n) \mid t' \text{ is a subterm of } t \}. \]

Then \( S \) is a finite subset of \( B. \) Therefore, by Lemma 5.1 \( \mathcal{B}_S = (B', \square_F', \square_P', \forall) \) is a finite MS4.t-algebra. It follows from the definition of \( \square_F' \) that, for each \( b \in B' \), if \( \square_F b \in B' \), then \( \square_F' b = \square_F b. \) Similarly, if \( \square_P b \in B \), then \( \square_P b = \square_P b. \) Thus, for each subterm \( t' \) of \( t \), the computation of \( t' \) in \( \mathcal{B}_S \) is the same as that in \( \mathcal{B}. \) Consequently, \( t(a_1, \ldots, a_n) \neq 1 \) in \( \mathcal{B}_S \), and we have found a finite MS4.t-algebra refuting \( \varphi. \)

**Remark 5.3.** Lemma 5.1 in particular proves that \( \mathcal{B}_S \) is an S4.t-algebra. Thus, the proof of the fmp for MS4.t contains the proof of the fmp for S4.t. In fact, MS4.t is a conservative extension of S4.t.

We conclude this section by showing that the fmp for TS4, MS4, and MIPC is a consequence of Theorem 5.2.

**Theorem 5.4.**

1. TS4 has the fmp.
2. MS4 has the fmp.
3. MIPC has the fmp.

**Proof.** (1). Suppose that TS4 \( \nvdash \varphi. \) By Theorem 4.16, MS4.t \( \nvdash \varphi^+. \) Since MS4.t has the fmp, there is a finite MS4.t-algebra \( \mathcal{B} \) such that \( \mathcal{B} \nmodels \varphi^+. \) As noted in Remark 4.8, \( \mathcal{B} \) is isomorphic to \( (\mathcal{B}_+)^+. \)
This yields that \( \mathcal{B}_+ \not\models \varphi^\dag \). By Proposition \([1.15](2)\), \( (\mathcal{B}_+)\dag \not\models \varphi \). We have thus obtained a finite \(TS4\)-frame \( (\mathcal{B}_+)\dag \) refuting \( \varphi \). So \( ((\mathcal{B}_+)\dag)^+ \) is a finite \(TS4\)-algebra such that \( ((\mathcal{B}_+)\dag)^+ \not\models \varphi \).

(2). Similar to the proof of (1) but uses the translation \( (-)^\# : MS4 \rightarrow MS4.t \) instead of \( (-)^\dag \).

(3). Similar to the proof of (1) but uses the composition \( (-)^\dag \# : MIPC \rightarrow MS4.t \) instead of \( (-)^\dag \).

Alternatively, we can use the other translations \( (-)^\ddagger \) and \( (-)^\circ \) of \( MIPC \) into \( MS4.t \).

\[ \square \]

6. Connection with the full predicate case

In \([5]\) we studied a temporal translation of the predicate intuitionistic logic \( IQC \) that is the predicate analogue of the translation \( (-)^\circ \) of Definition \([4.20]\). We proved that this translation embeds \( IQC \) fully and faithfully into a weakening of the tense predicate logic \( QS4.t \). This weakening is necessary since \( QS4.t \) proves the Barcan formula for both \( \square_F \) and \( \square_P \), so Kripke frames of \( QS4.t \) have constant domains, and hence they validate the translation of the constant domain axiom \( \forall x(A \lor B) \rightarrow (A \lor \forall xB) \), where \( x \) is not free in \( A \). Since this is not provable in \( IQC \), the translation cannot be full. Instead we considered the tense predicate logic \( QS^*S4.t \) in which the universal instantiation axiom \( \forall xA \rightarrow A(y/x) \) is replaced by its weakened version \( \forall y(\forall xA \rightarrow A(y/x)) \). The main result of \([5]\) proves that \( IQC \) translates fully and faithfully into \( QS^*S4.t \) (provided the translation is restricted to sentences).

It is natural to investigate the relationship between \( MS4.t \) and predicate extensions of \( S4.t \). As we already pointed out in Remark \([1.2]\), \( MS4.t \) is not the monadic fragment of \( QS4.t \). In addition, \( MS4.t \) cannot be the monadic fragment of \( QS^*S4.t \) either since the formula \( \forall xA \rightarrow A \) is not in general provable in \( QS^*S4.t \), whereas \( \forall \varphi \rightarrow \varphi \) is provable in \( MS4.t \). On the other hand, call a formula \( \varphi \) (in the language of \( MS4.t \)) \emph{bounded} if each occurrence of a propositional letter in \( \varphi \) is under the scope of \( \forall \). Bounded formulas play the same role as sentences of \( QS^*S4.t \) containing only one fixed variable. It is quite plausible that for a bounded formula \( \varphi \) we have \( MS4.t \vdash \varphi \) iff \( QS^*S4.t \) proves the translation of \( \varphi \) where each occurrence of a propositional letter \( p \) is replaced with the unary predicate \( P(x) \) and \( \forall \) is replaced with \( \forall x \) (for a similar translation of \( MIPC \) and its extensions into \( IQC \) and its extensions, see \([21]\)). If true, this would yield that the monadic sentences provable in \( QS^*S4.t \) are exactly the bounded formulas \( \varphi \) provable in \( MS4.t \). It would also yield that restricting the translation \( IQC \rightarrow QS^*S4.t \) of \([5]\) to the monadic setting gives the translation \( (-)^\circ : MIPC \rightarrow MS4.t \) for bounded formulas.

It is natural to seek an axiomatization of the full monadic fragment of \( QS^*S4.t \). Note that in this fragment \( \forall \) does not behave like an \( S5 \)-modality. For example, \( \forall \varphi \rightarrow \varphi \) is not in general a theorem of this fragment.

Finally, the translation \( (-)^\# : MS4 \rightarrow MS4.t \) suggests a translation of \( QS4 \) into \( QS^*S4.t \) which replaces each occurrence of \( \square \) with \( \square_P \). It is easy to see that for sentences this translation is full and faithful. Composing it with the standard Gödel translation of \( IQC \) into \( QS4 \) yields a translation \( IQC \rightarrow QS^*S4.t \) which is different from the translation of \([5]\). This translation restricts to the translation \( (-)^\dag \# : MIPC \rightarrow MS4.t \) for bounded formulas. Thus, the upper part of the diagram of Section 4.3 extends to the predicate case.

On the other hand, we do not see a natural way to interpret the tense modalities of \( TS4 \) as monadic quantifiers, and hence we cannot think of a natural predicate logic which could take the role of \( TS4 \) in the diagram of Section 4.3. Thus, the lower part of the diagram does not seem to have a natural extension to the predicate case. Nevertheless, we can consider the predicate analogue of the translation \( (-)^\ddagger : MIPC \rightarrow MS4.t \). Arguing as in Theorems \([1.19] \) and \([4.21]\) yields a translation...
of IQC into $Q^\circ S4.t$ that is full and faithful on sentences and coincides, up to logical equivalence in $Q^\circ S4.t$, with the other two predicate translations described in this section.

We thus obtain the following diagram in the predicate case which is commutative up to logical equivalence in $Q^\circ S4.t$.

\[ \begin{array}{ccc}
IQC & \cong & Q^\circ S4.t \\
\downarrow & & \downarrow \\
QS4 & \cong & Q^\circ S4.t \\
\end{array} \]

**References**

[1] H. Bass. Finite monadic algebras. *Proc. Amer. Math. Soc.*, 9:258–268, 1958.
[2] G. Bezhanishvili. Varieties of monadic Heyting algebras. I. *Studia Logica*, 61(3):367–402, 1998.
[3] G. Bezhanishvili. Varieties of monadic Heyting algebras. II. Duality theory. *Studia Logica*, 62(1):21–48, 1999.
[4] G. Bezhanishvili. Varieties of monadic Heyting algebras. III. *Studia Logica*, 64(2):215–256, 2000.
[5] G. Bezhanishvili and L. Carai. Temporal interpretation of intuitionistic quantifiers. AiML 2020, to appear, 2020.
[6] R. A. Bull. A modal extension of intuitionist logic. *Notre Dame J. Formal Logic*, 6(2):142–146, 1965.
[7] R. A. Bull. MIPC as the formalisation of an intuitionist concept of modality. *J. Symbolic Logic*, 31(4):609–616, 12 1966.
[8] L. Carai. *New directions in duality theory for modal logic*. PhD thesis, New Mexico State University, forthcoming 2021.
[9] A. Chagrov and M. Zakharyaschev. *Modal logic*. Oxford University Press, New York, 1997.
[10] L. Esakia. The problem of duality in the intuitionistic logic and Browerian lattices. In *V Inter. Congress of Logic, Methodology and Philosophy of Science*, pages 7–8. Canada, 1975.
[11] L. Esakia. Semantical analysis of bimodal (tense) systems. In *Logic, Semantics and Methodology*, pages 87–99 (Russian). Metsniereba Press, Tbilisi, 1978.
[12] L. Esakia. Provability logic with quantifier modalities. *Intensional Logics and Logical Structure of Theories*, pages 4–9 (Russian), 1988.
[13] G. Fischer-Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36(3):141–149, 1977.
[14] G. Fischer-Servi. The finite model property for MIPQ and some consequences. *Notre Dame Journal of Formal Logic*, 19(4):687–692, 1978.
[15] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*. North-Holland Publishing Co., Amsterdam, 2003.
[16] C. Grefe. Fischer Servi’s intuitionistic modal logic has the finite model property. In *Advances in modal logic, Vol. 1 (Berlin, 1996)*, volume 87 of *CSLI Lecture Notes*, pages 85–98. CSLI Publ., Stanford, CA, 1998.
[17] P. R. Halmos. Algebraic logic. I. Monadic Boolean algebras. *Compositio Math.*, 12:217–249, 1956.
[18] B. Jónsson and A. Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.
[19] J. C. C. McKinsey and A. Tarski. The algebra of topology. *Ann. of Math.*, 45:141–191, 1944.
[20] J. C. C. McKinsey and A. Tarski. On closed elements in closure algebras. *Ann. of Math.*, 47:122–162, 1946.
[21] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *J. Symbolic Logic*, 13:1–15, 1948.
[22] A. Monteiro and O. Varsavsky. Álgebras de Heyting monádicas. *Actas de las X Jornadas de la Unión Matemática Argentina, Bahía Blanca*, pages 52–62, 1957.
[23] H. Ono. On some intuitionistic modal logics. *Publications of the Research Institute for Mathematical Sciences*, 13(3):687–722, 1977.
[24] H. Ono. Some problems in intermediate predicate logics. *Reports on Mathematical Logic*, 21:55–67, 1987.
[25] H. Ono and N.-Y. Suzuki. Relations between intuitionistic modal logics and intermediate predicate logics. *Rep. Math. Logic*, (22):65–87 (1989), 1988.
[26] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*. Państwowe Wydawnictwo Naukowe, Warsaw, 1963.
[27] C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations. *Fund. Math.*, 83(3):219–249, 1973/74.
[28] S. K. Thomason. Semantic analysis of tense logics. *J. Symbolic Logic*, 37:150–158, 1972.
[29] F. Wolter. On logics with coimplication. *J. Philos. Logic*, 27(4):353–387, 1998.

Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, USA
*E-mail address*: guram@nmsu.edu

Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, USA
*E-mail address*: lcarai@nmsu.edu