**Universality at Breakdown of Quantum Transport on Complex Networks**

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We consider single-particle quantum transport on parametrized complex networks. Based on general arguments regarding the spectrum of the corresponding Hamiltonian, we derive bounds for a measure of the global transport efficiency defined by the time-averaged return probability. For tree-like networks, we show analytically that a transition from efficient to inefficient transport occurs depending on the (average) functionality of the nodes of the network. In the infinite system size limit, this transition can be characterized by an exponent which is universal for all tree-like networks. Our findings are corroborated by analytic results for specific deterministic networks, dendrimers and Viseck fractals, and by Monte Carlo simulations of iteratively built scale-free trees.

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for $\chi \ll O(1)$ there is no strict implication on the exact value of the infinite time limit of $\bar{\pi}(t)$. However, previous results suggest that in particular the maxima of $\bar{\pi}(t)$ are well reproduced by the lower bound $|\bar{\alpha}(t)|^2$, therefore, also indicating that the values of the RHS of Eq. (2) lie close to the values of $\chi$ [8].

Combining Eqs. (1) and (2) allows to estimate $\chi$ from below, knowing only the spectral density $\rho(E_\ast)$ of one arbitrary eigenvalue $E_\ast$,

$$\chi = \sum_E \rho^2(E) \geq \rho^2(E_\ast) + \frac{1}{N} \left[ 1 - \rho(E_\ast) \right] = \chi. \quad (3)$$

Here, we assume a completely flat density $\rho(E)$ on its support aside from $E_\ast$. $\chi$ allows for a rather accurate estimation of $\chi$ if $E_\ast$ is a single highly degenerate eigenvalue compared to all other eigenvalues, i.e., if $\rho(E_\ast) \gg \rho(E \neq E_\ast)$, or if all eigenvalues are non-degenerate, i.e., $\rho(E) = 1/N$ for all $E$. These two limits correspond to vastly different networks. For instance, chain-like networks with $H$ belonging to the above mentioned class have eigenvalues with (mostly) the same degeneracy, i.e., $\rho(E) = \text{const.}$ for all $E$, which yields $\chi = \chi = 1/N$ [8], suggesting very efficient transport. In contrast to this, stars of the same size typically have a single highly degenerate eigenvalue yielding $\chi = \chi = 1 - (4N - 6)/N^2$ [8], thus, rendering transport inefficient. Obviously, in the infinite system size limit one has

$$\chi_\infty \equiv \lim_{N \to \infty} \chi = \begin{cases} 0 & \text{for chains} \\ 1 & \text{for stars} \end{cases}. \quad (4)$$

Breakdown of Quantum Transport – For networks whose spectra dependent on a tunable parameter $\sigma \in \mathbb{R}$ such that for large values of $\sigma$ the spectral density of the selected eigenvalue $E_\ast$ is of order $O(1/N)$ and for small values of $\sigma$ of order $O(1)$, one might observe a transition from efficient to inefficient transport. Let this transition occur at a given parameter value $\sigma_c$. At this value the transport breaks down which is reflected by a change of $\chi(\sigma)$ from values of $O(1/N)$ for $\sigma > \sigma_c$ to values of $O(1)$ for $\sigma < \sigma_c$. This is reminiscent of a phase transition where, in our case, the quantity $1 - \chi_\infty(\sigma)$ represents the order parameter. (We use the usual terminology of phase transitions in order to stress the similarities and to avoid to overload the paper with new terminology.) Consequently, we associate with this transition a critical exponent defined by

$$\kappa \equiv \lim_{\sigma \to \sigma_c} \frac{\log |1 - \chi_\infty(\sigma)|}{\log |\sigma - \sigma_c|}. \quad (5)$$

Since $\chi_\infty(\sigma) \leq \chi_\infty(\sigma)$, one has $\kappa \geq \kappa_c$, where $\kappa$ is the exponent associated with the transition for $\chi_\infty(\sigma)$. As we will show below, tree-like networks which have a parametrized transition from chain-like topologies to star-like topologies yield the same exponent $\kappa$. This allows us to group networks, depending to their asymptotic (global) quantum transport efficiency, into universal classes defined by $\kappa$.

![Tree-like Networks](image)

Tree-like Networks – Figure 1 shows examples of three tree-like networks: (a) a scale-free tree (SFT), where the functionalities are drawn from the probability distribution $P(f_j = x) \propto x^{-s}$ (here $s = 2.5$), (b) a Dendrimer (D) of generation 5 with functionality $f = 3$, and (c) a Viscek fractal (VF) of generation 3 with functionality $f = 4$. Three different types of nodes are marked: leaf nodes (green triangles), parents (brown circles) of leave nodes, and all other (open circles).

As is easily shown, if a parent node $j$ has two different leaves $l$ and $k$, then the superposition state $(|l\rangle - |k\rangle)/\sqrt{2}$ is a normalized eigenstate of the network. The total amount of these superposition states - all belonging to the same eigenvalue $E_\ast = H(f_j = 1) = 1$, regardless
based on Eq. (3), we will employ Eq. (8) of leaves $N$ of leaves $N_L$. As shown in the limit when $\delta \to 0$ one obtains as a lower bound the average over the set of all parents. Given the tree-like topology, we can estimate $N_L$ based on $N$ and the average functionality of those nodes which are not leaves, $\langle f \rangle_{N \setminus N_L}$. For trees there are $N - 1$ bonds connecting the $N$ nodes. In the two extreme cases of chains ($\langle f \rangle_{N \setminus N_L} = 2$) and stars ($\langle f \rangle_{N \setminus N_L} = N - 1$), one has 2 and $N - 1$ leaves, respectively. For other trees the number of leaves

\[
N_L = N - \frac{N - 2}{\langle f \rangle_{N \setminus N_L} - 1},
\]

which lies in the interval $[2, N - 1]$. We are now in the position of expressing $g(E)$ in terms of $N$ and the averaged functionalities $\langle f \rangle_{N_P}$ and $\langle f \rangle_{N \setminus N_L}$. Inserting into the RHS of Eq. (3), one obtains as a lower bound the final result up to order $1/N$:

\[
\chi \geq \left( 1 - \frac{1}{\langle f \rangle_{N \setminus N_L} - 1} \right)^2 \left( 1 - \frac{1}{\langle f - \delta \rangle_{N_P} - 1} \right)^2 + \frac{1}{N} \left[ 1 - \left( \frac{\langle f \rangle_{N \setminus N_L} - 2}{\langle f \rangle_{N \setminus N_L} - 1} \right)^2 \langle f - \delta \rangle_{N_P} - 1 \right] + 4 \left( \frac{\langle f \rangle_{N \setminus N_L} - 2}{\langle f \rangle_{N \setminus N_L} - 1} \right)^2 \left( \frac{\langle f - \delta \rangle_{N_P} - 2}{\langle f - \delta \rangle_{N_P} - 1} \right)^2.
\]

In the limit $\langle \delta \rangle_{N_P} / \langle f \rangle_{N_P} \ll 1$, i.e., when a parent node is only rarely coupled to more than one other parent node, one has $\langle f - \delta \rangle_{N_P} \approx \langle f \rangle_{N_P}$, where $\langle f \rangle_{N_P}$ can often be written as a function of $\langle f \rangle_{N \setminus N_L}$. If the functionality of non-leaf nodes does not systematically depend on the position in the network, one has $\langle f \rangle_{N_P} = \langle f \rangle_{N \setminus N_L} \equiv \langle f \rangle$. We note, that there are exceptions, see also the VF below. Equation (8) defines only a lower bound to $\chi$ because we have neglected those eigenstates which are not simple superpositions of two states localized at leaves belonging to the same parent. In the infinite system size limit, these states are negligible close to the transition point, such that the equality holds for $\chi_{\infty}$.

Considering the inverse of the average functionality as the network’s adjustable parameter, i.e., $\sigma = 1 / \langle f \rangle$, we can deduce a universal behavior at the breakdown of quantum transport. In the limit $N \to \infty$ we obtain

\[
\chi_{\infty} = \left( 1 - \frac{1}{\langle f \rangle - 1} \right)^4,
\]

which results to first order in $1 / \langle f \rangle$ in $1 - \chi_{\infty} = 4 / \langle f \rangle$ and we get

\[
\kappa = \lim_{1/\langle f \rangle \to 0} \frac{\log [1 - \chi_{\infty}]}{\log [1 / \langle f \rangle]} = 1,
\]

regardless of the original underlying network, be it deterministic or random. Therefore, all tree-like networks will yield the same universal exponent $\kappa$. We note, that the presence of loops could eventually lead to different exponents.

Examples – In order to corroborate our general findings, we consider the three examples of tree-like networks depicted in Fig. 1. All these networks allow for a parametrized transition from chain-like to star-like topologies, depending on the (average) functionality. While D and VF allow for a direct computation of $\chi$ and $\xi$ based on Eq. (3), we will employ Eq. (8) for SFT, which, depending on the average functionality, can have many leaves. For normalisation purpose it is inevitable for finite systems to impose a maximal functionality $f_{\text{max}} \leq N - 1$ such that the average becomes $\langle f \rangle = \frac{\sum_{j=1, i \neq j} f_{i,j}}{\sum_{j=1}^{f_{\text{max}}} f_i}$. Inserting this average in Eq. (8) yields $\Delta_{\text{SFT}}$. We note that $\langle f \rangle$ depends on the scaling parameter $s$ and that $1 / \langle f \rangle \to 0$ when $s \downarrow 2$. In limit $N \to \infty$ the averages are related to the Riemann zeta function $\zeta(s) = \sum_{j=1}^{\infty} f^{-s}$ allowing to write the leading terms for values of $s \gtrsim 2$ as

\[
\Delta_{\text{SFT}} = 1 - 4 \frac{\zeta(s) - 1}{\zeta(s - 1) - \zeta(s)}.
\]

Therefore, the critical exponent follows as

\[
\kappa = \lim_{s \downarrow 2} \frac{\log (1 - \Delta_{\text{SFT}})}{\log (s - 2)} = 1,
\]

which again confirms our general statement about the universal behavior of tree-like networks. Figure 2 shows the dependence of $1 - \chi_{\text{SFT}}$ on the scale-free parameter $s$ and, as inset, on $\langle f \rangle$ for different sizes $N$ and also
for $N \to \infty$. For finite SFT, we have compared our analytic estimation given by Eq. (8) (lines) with Monte-Carlo simulations (symbols) for SFT grown iteratively by the algorithm given in [14] with the connectivity matrix defining $H$. In the numerical computations we have considered ensemble averages of $\chi_{\text{SFT}}$ with ensembles sizes of $R = 10^9/N$. All curves show the expected scaling close to the transition point where $1/\langle f \rangle \to 0$ and $s \searrow 2$, respectively. One notes from the inset of Fig. 2 that with increasing $N$ the finite-size effects become less pronounced, leading eventually to a sharp transition for $s \searrow 2$.

FIG. 2. (Color online) Numerical (symbols) and analytical (lines) results for $1 - \chi_{\text{SFT}}$: For finite $N = 10^2, 10^3$, and $10^4$, the ensemble averages for $1 - \chi_{\text{SFT}}$ with the connectivity matrix defining $H$ and with $R = 10^9/N$ realizations are shown (in linear scale) as functions of the scale free parameter $s$. The analytical estimates are obtained from Eq. (8) with $\delta = 0$ for finite SFT and, in the limit $N \to \infty$, from Eq. (11). The inset shows the same curves (in log-log scale) as functions of $1/\langle f \rangle$. In both plots, one notices the clear sign of the breakdown of the quantum transport, with the expected linear behavior.

In both cases we find the breakdown of transport in the limit $1/\langle f \rangle \to 0$. In order to be comparable to the SFT, we express both, $\chi_{\text{SFT}}^D$ and $\chi_{\text{SFT}}^\infty$, as functions of $\langle f \rangle$. For $\langle f \rangle_{N_{\text{SFT}}} = f$, which is not the case for VF, with (13), we obtain for $1/\langle f \rangle \to 0$ that $1 - \chi_{\text{SFT}}^D \sim 1/\langle f \rangle$ and that $1 - \chi_{\text{SFT}}^\infty \sim 1/\langle f \rangle$. Thus, also here we find the breakdown leading to the (exact) exponent $\kappa = \kappa = 1$.

We finally want to stress the differences between D and VF on the one hand and SFT on the other hand: For all structures, the transition happens when $1/\langle f \rangle \to 0$ or $1/\langle f \rangle \to 0$. However, for D and VF, having fixed deterministic functionalities $f$, there is no parameter allowing to study the behavior of $\chi_{\infty}$ beyond the critical point. Moreover, the limit $f \to \infty$ seems rather artificial in the $N \to \infty$ limit since no real system can ever reach this. The situation is different for SFT: Even though $\langle f \rangle$ diverges for scaling parameters $s \leq 2$, it is possible to study the behavior of $\chi_{\text{SFT}}$ in this parameter region. We further note that for finite SFT, one observes the maximal values of $\chi_{\text{SFT}} < 1$ only in the limit when $s \to 0$.

Conclusions – We have shown, that the breakdown of quantum transport on complex tree-like networks shows a universal behavior characterized by a global transport efficiency measure based on the time-averaged return probability. Parametrizing the corresponding Hamiltonian by the (average) functionality of the nodes of the network allows to derive bounds for this measure, leading in the infinite system size limit to a characteristic universal exponent for all tree-like networks. We anticipate that a similar treatment might also be feasible for other networks, also including loops.

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