ONE-DIMENSIONAL PROBLEM OF RHEOLOGICAL LAW OF MOLECULAR AND MOLAR TRANSFER IN FLUIDS

Abstract: The fluid flow is considered in the paper according to the rheological law of molecular and molar transfer in a flow. The obtained differential equation of the third order was solved analytically for the one-dimensional problem of fluid flow in a round pipe. The flow pattern obtained for the selected model according to the analytical solution was given.

Key words: fluid motion, differential equation, analytical solution.

Language: English

Citation: Khudjaev, M. (2020). One-dimensional problem of rheological law of molecular and molar transfer in fluids. ISJ Theoretical & Applied Science, 07 (87), 279-285.

Soi: http://s-o-i.org/1.1/TAS-07-87-57 Doi: https://dx.doi.org/10.15863/TAS.2020.07.87.57

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\[
\begin{cases}
\frac{\partial v_1}{\partial t} = -\frac{dp}{dx_1} + \mu \left( \frac{\partial^2 v_1}{\partial x_2^2} + \frac{1}{x_2} \frac{\partial v_1}{\partial x_2} \right) + m_1 \left[ \frac{\partial^3 v_1}{\partial t \partial x_2^2} + \frac{1}{x_2} \frac{\partial^2 v_1}{\partial t \partial x_2} \right] \\
+ v_1 \left( \frac{\partial^3 v_1}{\partial x_1 \partial x_2^2} + \frac{1}{x_2} \frac{\partial^2 v_1}{\partial x_1 \partial x_2} \right) + v_2 \left( \frac{\partial^3 v_1}{\partial x_2^3} + \frac{1}{x_2} \frac{\partial^2 v_1}{\partial x_2^2} \right) + \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_2} + \frac{\partial^2 v_1}{\partial x_2^2} \\
\frac{\partial v_1}{\partial x_1} + \frac{1}{x_2} \frac{\partial (x_2 v_2)}{\partial x_2} = 0.
\end{cases}
\] (3)

A model one-dimensional problem is formulated.

The equations of motion under the assumptions $v_1 = v_1(x_2, t), \quad v_2 = \text{const}$,

\[
m_1 \left( \frac{\partial^3 v_1}{\partial t \partial x_2^2} + \frac{1}{x_2} \frac{\partial^2 v_1}{\partial t \partial x_2} \right) + m_1 v_2 \left( \frac{\partial^3 v_1}{\partial x_2^3} + \frac{1}{x_2} \frac{\partial^2 v_1}{\partial x_2^2} \right) + \mu \left( \frac{\partial^3 v_1}{\partial x_2^3} + \frac{1}{x_2} \frac{\partial^2 v_1}{\partial x_2^2} \right) = N. \] (4)

Equation (4) is solved under the following initial and boundary conditions

\[
\begin{align*}
&v_1 = 0, \quad \frac{\partial v_1}{\partial x_2} = 0, \quad \frac{\partial^2 v_1}{\partial x_2^2} = 0 \quad t = 0, \\
&\frac{\partial v_1}{\partial x_2} = 0, \quad v_1 < \infty \quad x_1 = 0, \\
&v_1 = 0 \quad x_1 = R.
\end{align*}
\] (5)

The boundary conditions of equation (4) are the cohesion conditions and axial symmetry. At initial time, the fluid in the infinitely long round pipe is at rest, and at time $t = 0$, a pressure drop $dp/dx_1$ occurs, that later remains constant in time.

To solve the posed problem, modifying (4) we obtain the equation convenient for integration

\[
m_1 \frac{1}{x_2} \frac{\partial}{\partial x_2} \left( x_2 \frac{\partial^2 v_1}{\partial t \partial x_2} \right) + m_1 v_2 \frac{1}{x_2} \frac{\partial}{\partial x_2} \left( x_2 \frac{\partial^2 v_1}{\partial x_2^2} \right) + \mu \frac{1}{x_2} \frac{\partial}{\partial x_2} \left( x_2 \frac{\partial v_1}{\partial x_2} \right) = N. \] (6)

Multiplying both sides of this equation by $x_2$ and integrating the obtained values by $x_2$, we have:

\[
m_1 x_2 \frac{\partial^2 v_1}{\partial t \partial x_2} + m_1 v_2 x_2 \frac{\partial^2 v_1}{\partial x_2^2} + \mu x_2 \frac{\partial v_1}{\partial x_2} = \frac{N x_2^2}{2} + c_1. \] (7)

Applying this procedure for the second time, we arrive at the equation...
After Laplace transform with respect to $t$, the sequential equation is written in the form

$$\frac{\partial \bar{v}_1}{\partial x_2} + \frac{1}{x_2} \frac{\partial \bar{v}_1}{\partial t} + \frac{\mu}{m_1\nu_2} v_1 = \frac{N x_2^2}{4 m_1\nu_2} + \frac{c_1}{m_1\nu_2} \ln x_2 + \frac{c_2}{m_1\nu_2},$$  \hspace{1cm} (8)

where $s$ is the transform parameter.

The conditions for $t = 0$ from (5) were taken into account in passing to the images. The obtained inhomogeneous equation (9) was solved by the method of variation of constants. The homogeneous part of the equation has the following solution

$$\bar{v}_1 = c \ e^{-b x_2},$$

where $b = \frac{s}{v_2} + \frac{\mu}{m_1\nu_2}$.

If to consider $c$ not as an arbitrary constant, but as some function of $x_2$, i.e. $c = c\left(x_2\right)$, then we can choose the function $c\left(x_2\right)$ so that function (8) becomes a solution to the inhomogeneous equation (9).

$$c\left(x_2\right) = \frac{N}{4 m_1\nu_2 s} e^{b x_2} \left(\frac{x_2^2}{b} - \frac{2 x_2}{b^2} - \frac{2}{b^3}\right) + \frac{c_1}{m_1\nu_2 s} \left[\frac{e^{b x_2} \ln x_2}{b} - \left(\ln x_2 + \sum_{n=1}^{\infty} \frac{(b x_2)^n}{n \cdot n!}\right)\right].$$ \hspace{1cm} (12)

Substituting the found expression $c\left(x_2\right)$ into equality (8), we obtain the sought for solution to the inhomogeneous equation (9) in the form:

$$\bar{v}_1\left(x_2\right) = \frac{N}{4 m_1\nu_2 s} \left(\frac{x_2^2}{b} - \frac{2 x_2}{b^2} - \frac{2}{b^3}\right) + \frac{c_1}{m_1\nu_2 s} \left[-\frac{1}{b} e^{-b x_2} \ln x_2 - e^{-b x_2} \sum_{n=1}^{\infty} \frac{(b x_2)^n}{n \cdot n!}\right] + \frac{c_2}{m_1\nu_2 s} + \frac{c_3}{m_1\nu_2 s} e^{-b x_2}.$$ \hspace{1cm} (13)

To determine the integration constants $c_1$, $c_2$, $c_3$, we use the boundary conditions from (5). Since for $x_2 \to \infty$ the following is appropriate
\[
\frac{c_1}{m_1v_2}s\left[\frac{1}{b}e^{-bx_2} \ln x_2 - e^{-bx_2} \sum_{n=1}^{\infty} \frac{(bx_2)^n}{n!}\right] \to \infty,
\]

then from the boundedness condition of axial velocity \(c_1 = 0\) is determined. From the condition of cohesion, i.e. \(v_1 = 0\), at \(x_2 = R\) the following relation is obtained
\[
c_2 = \frac{Nb}{4} \left(\frac{R^2 - 2R}{b^2} - \frac{2}{b^3}\right) - c_3 be^{-br}. \quad (14)
\]

The value of the constant \(c_3\) is determined from the condition of axial symmetry, i.e. \(d v_1 / dx_2 = 0\) at \(x_2 = 0\):
\[
\frac{-v_1(x_2)}{4m_1v_2} = \frac{N}{s(s + \mu/m_1)} \left[\frac{x_2^2 - R}{s(s + \mu/m_1)} - 2(x_2 - R)\frac{v_2}{s(s + \mu/m_1)^2} + 2\frac{v_2^3}{s(s + \mu/m_1)^3}\left(e^{\frac{s + \mu}{m_1}} - e^{\frac{s + \mu}{m_1}}\right)\right]. \quad (15)
\]

In (15) we turn now to the original. From the table of originals and images [10] for the first two terms we have
\[
\frac{1}{s(s + \mu/m_1)} = \frac{m_1}{\mu} \left(1 - e^{-\mu/m_1}\right),
\]
\[
\frac{1}{s(s + \mu/m_1)} = \left(\frac{m_1}{\mu}\right)^2 \left(1 - e^{-\mu/m_1} - \frac{\mu}{m_1} e^{\mu/m_1}\right). \quad (16)
\]

For the last term of equation (15) the Duhamel integral [10] is used:
\[
sG(s)F(s) = f(0)g(t) + \int_0^t f(\tau)g(t - \tau)d\tau, \quad (17)
\]
in this case
\[
c_3 = -\frac{N}{2b^3}.
\]

Substituting the value of \(c_3\) into (14), \(c_2\) is determined in the final form
\[
c_2 = -\frac{N}{4} \left(R^2 - 2R \frac{R}{b} - \frac{2}{b^3}\right) + \frac{N}{2b^2} e^{-br}. \quad (18)
\]

After simple modifications, we obtain an expression for the velocity in the images.
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\[
\begin{align*}
  v_1(t,x_2) &= \frac{dp}{dx_1} \frac{1}{4\mu} \left( x_2^2 - R \right) \left( 1 - e^{-\frac{\mu t}{m_l}} \right) - \\
  &- 2(x_2 - R) v_2 \frac{m_l}{\mu} \left( 1 - e^{-\frac{\mu t}{m_l}} - t \frac{\mu}{m_l} e^{-\frac{\mu t}{m_l}} \right) + \\
  &+ 2v_2^2 e^{-\frac{\mu x_2}{m_2v_2}} \left[ - \frac{\mu}{m_1} \left( \frac{t^2}{2} + \frac{t^2}{2} \frac{m_l x_2}{m_2 v_2} + t \frac{m_l}{\mu} + \frac{m_l^2}{\mu^2} \right) - \frac{m_l^2}{\mu^2} \right] - \\
  &- 2v_2^2 e^{-\frac{\mu x_2}{m_2v_2}} \left[ - \frac{\mu}{m_1} \left( \frac{t^2}{2} + \frac{t^2}{2} \frac{\mu R}{m_1 v_2} + t \frac{m_l}{\mu} + \frac{m_l^2}{\mu^2} \right) - \frac{m_l^2}{\mu^2} \right].
\end{align*}
\]  (19)

**Discussion of obtained solution (19).** Function (19) is a general solution of equation (4) and describes the flow rate distribution in a round pipe. Here the expressions in square brackets for \( t > \frac{x_2}{v_2} \) are equal to zero, and as \( t \to \infty \) the expression for the stationary distribution of the fluid velocity in the pipe is:

\[
  v_1 = \frac{dp}{dx_1} \frac{1}{4\mu} \left( x_2^2 - R \right) - 2(x_2 - R) v_2 \frac{m_l}{\mu} + 2v_2^2 \frac{m_l^2}{\mu^2} \left( e^{\frac{\mu R}{m_2 v_2}} - e^{-\frac{\mu x_2}{m_2 v_2}} \right). 
\]  (20)

In the absence of molar transfer in motion, i.e. for \( m_l v_2 = 0 \), it is possible to derive a formula from (20) for the velocity distribution of a viscous fluid in a pipe [11].

**Results of computational experiments.** Analysis of numerical calculations carried out for the stationary case based on the obtained analytical solution of the model problem of fluid flow in a round tube according to the selected rheological law showed that a set of parameters \( a_1 = \frac{m_l v_2}{\mu} \) plays a characterizing role in the flow motion.

Figure 1 shows the velocity profiles in dimensionless coordinates for various values of \( a_1 \).

Curve 1 corresponds to a zero value of \( a_1 \), i.e. to the velocity distribution of viscous Newtonian fluid (the Poiseuille solution).

Curves 2-5 are obtained at values of \( a_1 = 0.36; 0.73; 1.45; 2.18 \), respectively.
As follows from the presented velocity distribution curves, as the molar transfer coefficient of the momentum increases, the region of maximum velocities moves from the middle of the pipe to the peripheral region. Curve 2 corresponds to the fluid flow with the formation of a flow core. This nature of the flow is manifested at values of $a_1$ in the range from 0.01 to 0.6. Starting from $a_1 > 0.6$, the velocity along the flow axis decreases, the region of maximum velocity moves toward the wall. The decrease in velocity along the flow axis, according to the author, is associated with an increase in the molar force of internal friction.

**Conclusions**

Thus, the calculations based on equation (20) showed:

- the consistency of the selected rheological model for describing the distribution of flow velocity;
- the consistency of the model for determining other hydrodynamic flow parameters depending on the physicomechanical properties of fluids.

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August 2020, 113129
https://doi.org/10.1016/j.cma.2020.113129

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