Measuring Congruence on High Dimensional Time Series

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Abstract

A time series is a sequence of data items; typical examples are videos, stock ticker data, or streams of temperature measurements. Quite some research has been devoted to comparing and indexing simple time series, i.e., time series where the data items are real numbers or integers. However, for many application scenarios, the data items of a time series are not simple, but high-dimensional data points. E.g., in video streams each pixel can be considered as one dimension, leading to $k$-dimensional data items with $k = 12 \times 288$ already for low resolution videos with $128 \times 96$ pixels per frame.

Motivated by an application scenario dealing with motion gesture recognition, we develop a distance measure (which we call congruence distance) that serves as a model for the approximate congruency of two complex time series. This distance measure generalizes the classical notion of congruence from point sets to complex time series.

We show that, given two input time series $S$ and $T$, computing the congruence distance of $S$ and $T$ is NP-hard. Afterwards, we present two algorithms with quadratic and quasi-linear runtime, respectively, that compute an approximation of the congruence distance. We provide theoretical bounds that relate these approximations with the exact congruence distance, as well as experimental results, which indicate that our approach yields accurate approximations of the congruence distance.

1 Introduction

Similarity search or nearest neighbour search is a common problem in computer science and has a wide range of applications (see Section 1.1 for examples). Given a dataset (in our case, a set of time series), a query (in our case, a time series), the problem is to find nearest neighbours to the query in the dataset, regarding a certain distance or similarity function. The difference between distance and similarity functions is that a distance function returns 0 for exact matches and a higher value otherwise, whereas similarity functions return greater values for more similar input data. In this paper, we consider distance functions only. There are two main variations of the nearest neighbour search problem. The first variation is called the $\varepsilon$-nearest neighbour search ($\varepsilon$-NN search), where the search returns all elements from the dataset having a distance of at most $\varepsilon$ to the query. The second variation is called Top-$k$ nearest neighbour search, where those $k$ elements having the smallest distance to the query will be returned. In each case, a requirement in practical systems is the fast computation of the distance function.

The datasets considered in this paper are time series, i.e., sequences of elements in $M$, for a metric space $(M, d)$. Examples of time series include simple time series where $M = \mathbb{R}$ (e.g. temperature measurements or stock data) and multi variate time series where $M = \mathbb{R}^k$ (e.g. motion trackings in three dimensional space or videos).

The distance functions defined and analyzed in this paper measure the (approximate) congruence of two time series. Thereby, the distance between two time series $S$ and $T$ shall be 0 iff two $S$ can be transformed into $T$ by rotation, translation, and mirroring; in this case, $S$ and $T$ are said to be congruent. A value greater than 0 shall correlate to the amount of transformation needed to turn the time series into congruent ones.

1.1 Motivation and Related Work

Simple time series are finite sequences holding one number per time step. There is a vast field of applications for simple time series in likely all scientific areas, including geo science (temperature measurements, earthquake prediction), medicine (heart rate measurements), and finance (stock ticker data). Depending on the application, different similarity measurements of time series are used (e.g. Landmarks [21], Dynamic Time Warping [20], and Longest Common Subsequence [20]). Different techniques evolved to speed up nearest neighbour searches [9,25]. Esling and Agon published a survey on simple time series [13].

Let us continue with a few examples highlighting the role of multi-dimensional time series.

Motion Gesture Recognition  The interest in motion gesture recognition has drastically increased over the last decade, especially in combination with augmented reality systems, as for example the Oculus Rift [3]. Recent products, like the LeapMotion [1] or Microsoft Kinect [2], are able to recognize the posture of the hands and body, respectively. These applications belong to appearance based approaches of motion gesture recognition, since they use cameras to recognize the posture at each time. A second category of posture recognition systems include gloves [11], which is more than 30 years old. The area of their applications has grown more and more from medicine and health care up to recent applications as, for example, controlling a Smartphone [13, 17]. Approaches using systems like these gloves are called skeletal based. The main difference is, that the gesture recognition software retrieves the key information, i.e. the trajectory of the body parts, instead of one or multiple video streams of that person.

Our interest, and the application of our work for motion gesture recognition, is the classification of gestures rather than the capturing itself. There are various different approaches to treat this problem, e.g. Computer Vision based techniques [26], trajectory based techniques [24], approaches based on State Machines [16], etc.

Considering the motion of a finger tip and its direction as a time series in \( \mathbb{R}^k \), our approach contributes to the skeletal based algorithms. From our point of view, the problem of motion gesture recognition narrows down to the problem of finding the most similar time series. Hereby, similarity of two time series means the measurement of their congruence. Since motion gestures usually are not performed exactly as stored in a database, we need a fine granular or approximative congruence measurement. For example, a circle can be drawn more like an ellipse, but is more congruent to a circle than to a square or a line (see Figure 1 and Figure 2). To the best of our knowledge, in the literature already existing techniques.

Content Based Video Copy Detection Nowadays, a vast amount of video data is uploaded and shared on community sites such as YouTube or Facebook. This leads to various tasks such as copyright protection, duplicate detection, analysing statistics of particular broadcast advertisements, or searching for large videos containing certain scenes or clips. Two basic approaches exist to address these challenges, namely watermarking and content based copy detection (CBCD). Watermarking suffers from being vulnerable to transformations frequently performed during copy creation of a video (e.g. resizing or reencoding). Furthermore, watermarking cannot be used on videos unmarked before distribution. In contrast, CBCD is about finding copies of an original video by specifically comparing the contents and is thus more robust against transformations done during copy creation. These transformations include resolution, format, and encoding changes, addition of noise, blurring, flipping, (color) negation, and gray-scaling. Hence, copies are near-duplicates and it is natural to use a distance or similarity function to discover them.

Many approaches compare features created per image [8, 27]. Global features include mean color values and color histograms. In contrast to global features, local features (e.g. Harris Corners, SIFT, or SURF) are more robust against transformations when searching for similar images [12, 22, 23]. However, these techniques suffer from weak robustness against transformations as for example flipping or negation.

Considering a video with \( k \) pixels per image as a time series in \( \mathbb{R}^k \), the transformations flipping, negation, and gray-scaling correspond to mirroring, rotating, and translating the time series and thus do not change the congruence distance to another video. Furthermore, a global or local feature could be stored per image and regarded as state per time step. Hence, the congruence distance function introduced in the present paper seems to be a good basis for video distance functions in combination with already existing techniques.

Congruence Calculation The classical CONGRUENCE problem basically determines whether two point sets \( A, B \subseteq \mathbb{R}^k \) are congruent considering isometric transformations (i.e., rotation, translation, and mirroring) [6, 15]. For two and three dimensional spaces, there are results providing algorithms with runtime \( \mathcal{O}(n \cdot \log n) \) [6]. For larger dimensionalities, they provide an algorithm with runtime \( \mathcal{O}(n^{k-2} \cdot \log n) \). For various reasons (e.g. bounded floating point precision, physical measurement errors), the approximated CONGRUENCE problem is of much more interest in practical applica-
tions. Different variations of the approximated CONGRUENCE problem have been studied (e.g. what types of transformations are used, is the assignment of points from $A$ to $B$ known, what metric is used)\

The CONGRUENCE problem is related to our work, since the problem is concerned with the existence of isometric functions such that a point set maps to another point set. The main difference is, that we consider ordered lists of points (i.e. time series) rather than pure sets.

1.2 Main Contributions

In this paper, we use a model for complex time series covering models of time series known from the literature as well as high dimensional time series. Focusing on high dimensional time series, our main contributions are as follows:

1. We define and analyze an intuitive congruence measurement (congruence distance) which can be computed by solving an optimization problem with highly nonlinear constraints.

2. We show that the calculation of the congruence distance is an NP-hard problem. This is done by constructing a technically involved polynomial time reduction from the NP-hard 1-IN-3-SAT problem.

3. We provide two approximations to the congruence distance (delta distance, and reduced delta distance) that can be computed in polynomial time. Studying their approximative, we obtain:

- The approximations yield lower bounds on the congruence distance.
- There exist pathetic examples revealing that the relative error can grow arbitrarily.
- Our experimental results suggest a stable behaviour of the approximations in practical applications.

1.3 Organization

The rest of this paper is structured as follows. In Section 2, we provide basic notation used throughout the paper. In Section 3, we fix the notion of time series, and we present distance measures that turn the set of ordered lists of points (i.e. time series) rather than pure sets.

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2 Preliminaries

Basic notation By $N$, $R$, $R_{\geq c}$ we denote the set of non-negative integers, the set of reals, and the set of all reals $\geq c$, for some $c \in R$, respectively. For integers $x, y$ we write $[x, y]$ for the interval consisting of all integers $z$ with $x \leq z \leq y$, and we write $[x, y)$ for $[x, y] \setminus \{y\}$.

By $R^k$ and $R^{k \times k}$, for $k \in N$, we denote the set of all vectors of length $k$, resp., all $(k \times k)$-matrices with entries in $R$. For a vector $v \in R^k$ we write $v_i$ for the entry in position $i$.

Similarly, for a matrix $M \in R^{k \times k}$ we write $m_{i,j}$ for the entry in row $i$ and column $j$. By $\epsilon_i$, we denote the $i$-th unit vector in $R^k$, i.e., the vector with entry 1 in the $i$-th position and entry 0 in all other positions.

We write $Mv$ for the product of the matrix $M \in R^{k \times k}$ and the vector $v \in R^k$. We write $\lambda v$ and $\lambda M$ for the product of the number $\lambda \in R$ with the vector $v$ and the matrix $M$, respectively (i.e., for all $i, j \in [1, k]$, the $i$-th entry of $\lambda v$ is $\lambda v_i$, and the entry in row $i$ and column $j$ of $\lambda M$ is $\lambda m_{i,j}$).

By $\|\|_p$, for $p \in R_{\geq 1}$, we denote the usual $p$-norm on $R^k$; i.e., $\|v\|_p = \left(\sum_{i=1}^{k} |v_i|^p\right)^{1/p}$ for all $v \in R^k$.

By $\langle \cdot, \cdot \rangle$ we denote the usual scalar product on $R^k$: i.e., for $u, v \in R^k$ we have $\langle u, v \rangle = \sum_{i=1}^{k} u_i v_i$. In particular, $\|v\|_2 = \sqrt{\langle v, v \rangle}$ for all $v \in R^k$. Recall that two vectors $u, v \in R^k$ are orthogonal iff $\langle u, v \rangle = 0$.

A matrix $M \in R^{k \times k}$ is called orthogonal if the absolute value of its determinant is 1. Equivalently, $M$ is orthogonal iff $\langle m_i, m_j \rangle = 1$ and $\langle m_i, m_j \rangle = 0$ for all $i, j \in [1, k]$ with $i \neq j$, where $m_i$ denotes the vector in the $i$-th column of $M$. We write $MO(k)$ to denote the set of all orthogonal matrices in $R^{k \times k}$. Recall that angles and lengths are invariant under multiplication with orthogonal matrices, i.e.:

$$\forall u, v \in R^k : M \in MO(k) : \langle Mu, Mv \rangle = \langle u, v \rangle.$$

$$\forall u \in R^k, M \in MO(k) : \|Mu\|_2 = \|u\|_2.$$

In general, a vector norm is an arbitrary mapping $\|\| : R^k \rightarrow R_{\geq 0}$ that satisfies the following axioms:

$$\forall v \in R^k : \|v\| = 0 \implies v = 0.$$

$$\forall \lambda \in R, v \in R^k : \|\lambda v\| = |\lambda| \cdot \|v\|.$$

$$\forall u, v \in R^k : \|u + v\| \leq \|u\| + \|v\|.$$  

Clearly, $\|\|_p$ is a vector norm ($\ell_p$ norm) for any $p \in R_{\geq 1}$.

A matrix norm is a mapping $\|\| : R^{k \times k} \rightarrow R_{\geq 0}$ satisfying the following axioms:

$$\forall M \in R^{k \times k} : \|M\| = 0 \implies M = 0.$$

$$\forall \lambda \in R, M \in R^{k \times k} : \|\lambda M\| = |\lambda| \cdot \|M\|.$$

$$\forall M, M' \in R^{k \times k} : \|M + M'\| \leq \|M\| + \|M'\|.$$  

The particular matrix norms considered in this paper are the max column norm $\|\|_m$ and the $p$-norm $\|\|_p$, for $p \in R_{\geq 1}$, which are defined as follows: For all $M \in R^{k \times k}$:

$$\|M\|_m = \max_{1 \leq j \leq k} \max_{1 \leq i \leq k} |M_{i,j}|.$$  

$$\|M\|_p = \left(\sum_{i=1}^{k} \sum_{j=1}^{k} |M_{i,j}|^p\right)^{1/p}.$$
Let \( R \) be a finite sequence of elements in \( M \). By \( \|\cdot\| \) we denote the set of all time series over \( M \) of arbitrary length, i.e., \( \mathcal{T}_M = \bigcup_{n \in \mathbb{N}} M^n \). If \( M \) is clear from the context, we will omit the subscript \( M \) and simply write \( T \) instead of \( \mathcal{T}_M \). For \( n \in \mathbb{N} \) we then write \( \mathcal{T}_n \) to denote the set \( M^n \) of all time series of length \( n \) over \( M \). It is straightforward to verify the following.

**Proposition 3.1.** \((\mathcal{T}_n, \|\cdot\|_p)\) is a metric space.

Next, we want to extend \( \|\cdot\| \) to a distance measure on time series of arbitrary length, i.e., we want to extend \( \|\cdot\| \) to a mapping \( \mathcal{T}_M \times \mathcal{T}_M \to \mathbb{R}_{\geq 0} \). For this, the following notation is convenient.

**Definition 3.2.** Let \( T = (t_0, \ldots, t_{n-1}) \in \mathcal{T} \) be a time series, let \( b \in [0, n] \), and let \( \ell \in [1, n-b] \). Then \( T_b^\ell := (t_b, \ldots, t_{b+\ell-1}) \) is the subseries of \( T \) of length \( \ell \) starting at index \( b \).

If \( S = (s_0, \ldots, s_{m-1}) \in M^m \) and \( T = (t_0, \ldots, t_{n-1}) \in M^n \) are two time series of lengths \( m \leq n \), then we let

\[
\|d\| (T, S) := \|d\| (S, T) := \min_{b \in [0, n-m]} \|d\| (S, T_b^m).
\]

I.e., the distance between \( S \) and \( T \) is computed by finding the best match of the shorter time series regarded as a window over the longer time series. We will write

\[
d_p(\cdot, \cdot)
\]

instead of \( \|d\| (\cdot, \cdot) \) for the special case where \( M = \mathbb{R}^k \), \( \|\cdot\|_p \) for some \( p \in \mathbb{R}_{\geq 1} \), and \( d \) is the Euclidean distance \( d_2 \) defined via \( d_2(x, y) = \|x - y\|_2 \) for all \( x, y \in \mathbb{R}^k \).

It is easy to see that many other distance functions (e.g. DTW and LCSS [7,10,20]) that have been considered in the literature for time series over \( \mathbb{R} \) or \( \mathbb{R}^k \) can be adopted to time series over \( M \) for a metric space \((M, d)\) accordingly.

To avoid confusion between \( d \), \( \|d\|_p \), \( d_p \), and further distance functions considered in this paper, we will henceforth write \( d \) (or variants thereof) to denote distance functions for relating time series (i.e., \( d \) will be a function from \( \mathcal{T}_M \times \mathcal{T}_M \) to \( \mathbb{R}_{\geq 0} \)), and we will write \( d \) (or variants thereof) to denote distance functions for relating individual states in the time series (i.e., \( d \) will be a function from \( M \times M \) to \( \mathbb{R}_{\geq 0} \)). The latter will be called state distance function.

We will speak of metric time series whenever considering time series over \( M \) for a metric space \((M, d)\). For a given vector norm \( \|\cdot\|_p \), the associated function \( \|d\| \) will serve as a distance measure for time series over \( M \).

Let us conclude this section with a few examples that illustrate the generality of metric time series.
Examples 3.3. As already explained above, simple time series are a special case of time series where $M = \mathbb{R}$, $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ is defined via $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$, and $|||| = ||||_p$ for some $p \in \mathbb{R}_{\geq 1}$.

Complex time series, i.e., time series where the states are elements in $\mathbb{R}^k$ for some fixed $k$, are the special case where $M = \mathbb{R}^k$, $d: \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}_{\geq 0}$ is the Euclidean distance $d_2$, $||||_p$ for some $p \in \mathbb{R}_{\geq 1}$, and hence $||d|| = d_2$.

For an arbitrary undirected connected graph $G = (V, E)$, we can consider the mapping $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$ where $d(u, v)$ is the length of a shortest path between nodes $u$ and $v$ of $G$. Note that $(V, d)$ is a metric space. Given an arbitrary vector norm $||||$, we can view sequences of nodes of $G$ as time series over $M = V$, and $||d||$ as a distance measure between such time series.

In the remainder of this paper we restrict attention to time series over $M = \mathbb{R}^k$ and state distance functions $d_p$.

4 Time Series Congruence

Let $M := \mathbb{R}^k$ and let $T := T_M$. If $T = (t_0, \ldots, t_{n-1}) \in T$ is a time series, $M \in \mathbb{R}^{k \times k}$ is a matrix, and $v \in \mathbb{R}^k$ is a vector, we write $M \cdot T + v$ for the time series $(t'_0, \ldots, t'_{n-1})$ where $t'_i = M t_i + v$ for each $i \in [0, n]$.

We say that two time series $S, T \in T$ are congruent, if $S$ can be transformed into $T$ by rotation, mirroring, or translation. This is formalized in the following definition.

Definition 4.1. Consider the metric space $(\mathbb{R}^k, d)$ for $d := d_2$. Two time series $S$ and $T$ of the same length $n$ are called congruent (for short: $S \cong C T$) if there is a matrix $M \in \mathcal{M}(k)$ and a vector $v \in \mathbb{R}^k$ such that $T = M \cdot S + v$.

It is easy to see that for each $n \in \mathbb{N}$, the congruence relation $\cong C$ is an equivalence relation on the class of all time series over $\mathbb{R}^k$ of length $n$.

According to the motivation provided in Section [ ] we aim at a distance measure that regarding two time series $S$ and $T$ as very similar if $T$ is obtained from $S$ via rotation, mirroring, or translation, i.e., which satisfies the following congruence requirement.

Definition 4.2 (Congruence Requirement). Let $k \in \mathbb{N}$, let $M = \mathbb{R}^k$, and let $T = T_M$. A function $d: T \times T \rightarrow \mathbb{R}_{\geq 0}$ satisfies the congruence requirement iff for all time series $S, T \in T$ the following is true:

$$d(S, T) = 0 \iff S \cong C T.$$  

The following example highlights some intuition for the congruence distance function that is provided in Definition [4.4].

Example 4.3. Consider the time series

$$S := \left(\begin{array}{ccc} -4 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad T := \left(\begin{array}{ccc} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Obviously, $d_1(S, T) = 5$. Now, let us rotate $T$ by 90 degrees counterclockwise, i.e., let us compute $M \cdot T$ for the matrix

$$M := \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right). \quad \text{Then, } \quad M \cdot T = \left(\begin{array}{ccc} -3 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

and $d_1(S, M \cdot T) = 1 + \sqrt{2} < 5$.

Thus, without rotation, we need to add a vector of Euclidean length 5 to the first state of $T$ in order to transform $T$ into $S$. But after rotating $T$ by 90 degrees counterclockwise, we only need to add a vector of length 1 to the first state and a vector of length $\sqrt{2}$ to the third state of $M \cdot T$ to obtain the time series $S$.

Adding vectors to certain states can be interpreted as investing energy to make both time series having the same structure, i.e., being “congruent”. Hence, the congruence distance defined below can be viewed as a measure for the minimum amount of energy needed to make both time series congruent.

Definition 4.4 (Congruence Distance). Let $k \in \mathbb{N}$, $M = \mathbb{R}^k$, $T = T_M$, and $p \in \mathbb{R}_{\geq 1}$. The congruence distance $d_p^C(S, T)$ between two time series $S, T \in T$ is defined via

$$d_p^C(S, T) := \min_{M \in \mathcal{MO}(k), v \in \mathbb{R}^k} d_p(S, M \cdot T + v).$$

Note that, although $\mathcal{MO}(k)$ and $\mathbb{R}^k$ are infinite sets, it can be shown that the “min” used in the definition of $d_p^C(S, T)$ does exist, and that for given $S, T$ there are $M \in \mathcal{MO}(k)$ and $v \in \mathbb{R}^k$ such that $d_p^C(S, T) = d_p(S, M \cdot T + v)$; a proof can be found in the appendix. It is not difficult to see that the following holds for $M = \mathbb{R}^k$ and $T_n = M^p$ for each $n \in \mathbb{N}$:

Proposition 4.5. $(T_n, d_p^C)$ is a pseudo metric space.

The proof is given in the appendix. Obviously, calculating $d_p^C(S, T)$ for arbitrary $S, T \in T$ is a nonlinear optimization problem that can be solved using numeric solvers. However the problem is computationally difficult: As we show in the next subsection, already the calculation of $d_1^C$ is NP-hard.

4.1 NP-Hardness

In this subsection we restrict attention to $d_1 = ||d_2||_1$ and the according congruence distance $d_1^C$. Consider the following problem:

$d_1^C$-COMPUTATION

Input: A number $k \in \mathbb{N}$ and two time series $S$ and $T$ of equal length over $\mathbb{R}^k$.

Task: Compute (a suitable representation of) the number $d_1^C(S, T)$.

This subsection’s main result is:

Theorem 4.6. If $P \neq \text{NP}$, then $d_1^C$-COMPUTATION cannot be solved in polynomial time.
The remainder of Subsection 4.1 is devoted to the proof of Theorem 4.6 which constructs a reduction from the NP-complete problem 1-in-3-SAT. Recall that 1-in-3-SAT is the problem where the input consists of a propositional formula $\Phi$ in 3-cnf, i.e., in conjunctive normal form where each clause is a disjunction of literals over 3 distinct variables. The task is to decide whether there is an assignment $\alpha$ that maps the variables occurring in $\Phi$ to the truth values 0 or 1, such that in each disjunctive clause of $\Phi$ exactly one literal is satisfied by $\alpha$; we will call such an assignment $\alpha$ a 1-in-3 model of $\Phi$.

Our reduction from 1-in-3-SAT to $d_1^n$-COMPUTATION$_\nu$ will proceed as follows: A given 3-cnf formula $\Phi$ with $k$ variables $V_1, \ldots, V_k$ will be mapped to two time series $S_\Phi$ and $T_\Phi$ over $\mathbb{R}^k$, which represent the formula $\Phi$ and its variables, respectively. Our construction of $S_\Phi$ and $T_\Phi$ will ensure that for a certain number $c(\Phi)$ the following is true: $d_1^n(S_\Phi, T_\Phi) = c(\Phi) \iff$ there is a 1-in-3 model of $\Phi$.

The basic idea for our choice of $S_\Phi$ and $T_\Phi$ is that each dimension of $\mathbb{R}^k$ represents one variable. An orthogonal matrix, mirroring the $i$-th dimension then will correspond to negating the $i$-th variable $V_i$.

To formulate the proof, the following notation will be convenient. For a propositional formula $\Phi$ with $k$ variables, we write $V_1, \ldots, V_k$ to denote the variables occurring in $\Phi$. A literal over a variable $V_i$ is a formula $L_i \in \{V_i, \neg V_i\}$. A disjunctive (conjunctive) 3-clause is a formula $\Psi = \bigvee_{i \in I} L_i$ ($\Psi = \bigwedge_{i \in I} L_i$) with $L_i \in \{V_i, \neg V_i\}$, $|I| = 3$, and $I \subseteq [1, k]$. A 3-cnf formula is a formula $\Phi = \bigwedge_{j=1}^m \Psi_j$, where $m \geq 1$ and each $\Psi_j$ is a disjunctive 3-clause.

Furthermore, we will use the following notation for concatenating time series. Let $\ell \geq 1$, and let $S_j = (s^j_0, \ldots, s^j_{n_j-1})$ be a time series over $\mathbb{R}^k$ for each $j \in [1, \ell]$. Then, by

$$S_1 \times \cdots \times S_\ell$$

we denote the time series

$$(s^1_0, \ldots, s^{n_1-1}_0, \ldots, s^n_0, \ldots, s^n_{n_\ell-1}).$$

If $i_1 < \cdots < i_\ell$ is an increasing sequence of integers and $S_{i_j}$ is a time series over $\mathbb{R}^k$, for each $j \in [1, \ell]$, then for $I = \{i_1, \ldots, i_\ell\}$ we let

$$\otimes_{i \in I} S_i := S_{i_1} \times \cdots \times S_{i_\ell}.$$  

From a 3-cnf formula $\Phi$ to time series $S_\Phi$ and $T_\Phi$. For a given 3-cnf formula $\Phi$ let $k$ be the number of variables occurring in $\Phi$. Let $\Phi$ be of the form $\bigwedge_{j=1}^m \Psi_j$, where $m \geq 1$ and each $\Psi_j$ is a disjunctive 3-clause of the form $\bigvee_{i \in I_j} L_i$, where $L_i \in \{V_i, \neg V_i\}$, $|I_j| = 3$, and $I_j \subseteq [1, k]$.

For a disjunctive 3-clause $\Psi = \bigvee_{i \in I} L_i$ let

$$\Psi' := \bigvee_{j \in I} \left( L_j \land \bigwedge_{i \in I_{(j)}} \neg T_i \right),$$

where $T_i := \neg V_i$ and $\neg V_i := V_i$. Clearly, an assignment $\alpha$ satisfies $\Psi'$ iff it is a 1-in-3 model of $\Psi$. And $\alpha$ satisfies $\Psi' := \bigwedge_{j=1}^m \Psi'_j$ iff it is a 1-in-3 model of $\Phi = \bigwedge_{j=1}^m \Psi_j$.

The formulas $\Gamma_j$ for $j \in I$ are called the conjunctive 3-clauses implicit in $\Psi$.

We define an embedding $\theta$ of variables, literals, and conjunctive 3-clauses into $\mathbb{R}^k$ as follows: For each $i \in [1, k]$ let

$$\theta(V_i) := e_i \quad \text{and} \quad \theta(\neg V_i) := -e_i.$$  

For a literal $L_i$ we let $l_i := \theta(L_i)$. For a conjunctive 3-clause $\Gamma = \bigwedge_{i \in I} L_i$, we let

$$\gamma := \theta(\Gamma) := \sum_{i \in I} \theta(L_i) = \sum_{i \in I} l_i.$$  

In particular, for $\Gamma_j$ as defined above, we obtain that

$$\gamma_j := \theta(\Gamma_j) = l_j - \sum_{i \in I_{(j)}} l_i.$$  

For each disjunctive 3-clause $\Psi = \bigvee_{i \in I} L_i$ we let

$$e_i := \sum_{i \in I} e_i$$

and define the following time series over $\mathbb{R}^k$:

$$S_\Psi := \bigotimes_{i \in I} (6e_i, -6e_i), \quad T_\Psi := \bigotimes_{i \in I} (6e_i, 6e_i),$$

$$S_\Phi := \bigotimes_{j=1}^m ((\gamma_j), \quad T_\Phi := (e_1, e_1, e_1)$$

$$\tilde{S}_\Phi := S_\Psi \times S_\Psi, \quad \tilde{T}_\Psi := T_\Phi \times T_\Phi.$$  

(1)

For a 3-cnf formula $\Phi = \bigwedge_{j=1}^m \Psi_j$ all these time series will be concatenated to the two time series

$$S_\Phi := \bigotimes_{j=1}^m \left( \tilde{S}_{\Psi_j} \right), \quad T_\Phi := \bigotimes_{j=1}^m \left( \tilde{T}_{\Psi_j} \right).$$  

Finally, to be able to handle translations, we concatenate the time series with their mirrored duplicates:

$$\hat{S}_\Phi := S_\Phi \times -S_\Phi, \quad \hat{T}_\Phi := T_\Phi \times -T_\Phi.$$  

Our aim is to compute a number $c(\Phi)$ such that the following is true: $c_1^n(S_\Phi, T_\Phi) = c(\Phi)$ iff $\Phi$ has a 1-in-3 model. For obtaining this, we will proceed in several steps, the first of which is to compute a number $c^{M\text{O}}(\Phi)$ such that $\Phi$ has a 1-in-3 model iff $d_1^n(S_\Phi, T_\Phi) = c^{M\text{O}}(\Phi)$, for

$$d_1^{M\text{O}}(S, T) := \min_{M \in M\text{O}(k)} d_1(S, M \cdot T).$$  

(2)

The idea behind our choice of the time series $S_\Phi$ and $T_\Phi$ is as follows: $S_\Phi$ and $T_\Phi$ force the orthogonal matrix $M$ to have a suitable shape when leading to the minimal distance, i.e. to have all the $e_i$'s as Eigenvectors with Eigenvalues of 1 or $-1$ — in other words: each vector $e_i$ will either be left untouched or will be negated. The time series $S_\Phi$ represents the disjunctive
3-clause $\Psi$, while $T_\Psi$ holds the vector representing the variables used in $\Psi$. The minimum of $\tilde{S}_\Psi = S_\Psi \times S_\Psi$ to $\tilde{T}_\Psi = T_\Psi \times T_\Psi$ will then be reached if the vector $\sum_{i \in I} e_i$ is rotated in such a way that it matches one of the vectors of $S_\Psi$. Hence, assigning a propositional variable $V_i$ the value 0 corresponds to negating the $i$-th dimension, and assigning $V_i$ the value 1 leaves that dimension untouched.

Relating $d_{1}^{\mathcal{MO}}(S_\Psi, T_\Psi)$ with 1-in-3 models of $\Phi$ The next observation will be helpful for our proofs.

Lemma 4.7. Let $M \in \mathcal{MO}(k)$, $i \in [1,k]$, $a_i := d_2(e_i, M e_i)$, and $b_i := d_2(-e_i, M e_i)$. Then,

$$b_i = \sqrt{4 - a_i^2} \quad \text{and} \quad a_i = \sqrt{4 - b_i^2}.$$ 

Proof. We let $e_i := d_2(e_i, -e_i)$. For the special case where $k = 2$, Thales' Theorem tells us that $a_i^2 + b_i^2 = c_i^2$. The same holds true for arbitrary $k$, as the following computation shows.

Clearly, $c_i = d_2(e_i, -e_i) = \|2e_i\|_2 = 2$, and thus $c_i^2 = 4$. Furthermore, $a_i^2 = d_2(e_i, M e_i)^2 = \|e_i - M e_i\|_2^2 = (e_i - M e_i, e_i - M e_i) = (e_i, e_i) + (M e_i, M e_i) - 2(e_i, M e_i) = 2 - 2(e_i, M e_i)$. And $b_i^2 = d_2(-e_i, M e_i)^2 = \|-e_i - M e_i\|_2^2 = (e_i + M e_i, e_i + M e_i) = (e_i, e_i) + (M e_i, M e_i) + 2(e_i, M e_i) = 2(e_i, M e_i) + 2(e_i, e_i) = 4 = c_i^2$.

Thus, $b_i = \sqrt{4 - a_i^2}$ and $a_i = \sqrt{4 - b_i^2}$. □

From now on, whenever given a matrix $M \in \mathcal{MO}(k)$, we will always use the following notation: $a_i := d_2(e_i, M e_i)$, and $b_i := d_2(-e_i, M e_i)$. From Lemma 4.7 we know that $b_i = \sqrt{4 - a_i^2}$ and $a_i = \sqrt{4 - b_i^2}$.

For a disjunctive 3-clause $\psi$ and a matrix $M \in \mathcal{MO}(k)$ we let

$$d_\psi(M) := d_1(\tilde{S}_\Psi, M \cdot \tilde{T}_\Psi) = d_1(S_\Psi, M \cdot T_\Psi) + d_1(S_\Psi, M \cdot T_\Psi).$$

In the next lemmas, we will gather information on the size of $d_\psi(M)$ (cf. the appendix for proof of Lemma 4.8 and Lemma 4.10).

Lemma 4.8. Let $\Psi = \bigvee_{i \in I} L_i$ be a disjunctive 3-clause, let $M \in \mathcal{MO}(k)$. Then,

$$d_1(S_\Psi, M \cdot T_\Psi) = 6 \cdot \sum_{i \in I} (a_i + b_i).$$

Lemma 4.9. Let $\Psi = \bigvee_{i \in I} L_i$ be a disjunctive 3-clause.

(a) For each $M \in \mathcal{MO}(k)$ we have

$$d_1(S_\Psi, M \cdot T_\Psi) \geq 4\sqrt{2} - 3 \cdot \sum_{i \in I} \min(a_i, b_i).$$

(b) Let $M$ be an element in $\mathcal{MO}(k)$ such that $Me_1 = \gamma$, where $\gamma = \theta(\Gamma)$ for some conjunctive 3-clause $\Gamma$ implicit in $\Psi$. Then $d_1(S_\Psi, M \cdot T_\Psi) = 4\sqrt{2}$. 

(c) Let $M$ be an element in $\mathcal{MO}(k)$ such that $Me_i \in \{e_i, -e_i\}$ for all $i \in I$, and $d_1(S_\Psi, M \cdot T_\Psi) = 4\sqrt{2}$. Then $Me_1 = \gamma$, where $\gamma = \theta(\Gamma)$ for some conjunctive 3-clause $\Gamma$ implicit in $\Psi$.

Proof. Let $\Gamma_j$, for $j \in I$, be the conjunctive 3-clauses implicit in $\Psi$, and let $\gamma_j = \theta(\Gamma_j)$.

For proving (a), let $M$ be an arbitrary element in $\mathcal{MO}(k)$. Note that by definition of $S_\Psi$ and $T_\Psi$, we have

$$d_1(S_\Psi, M \cdot T_\Psi) = \sum_{j \in I} d_2(\gamma_j, Me_1).$$

By the triangle inequality and the symmetry we know that $d(x, z) \geq d(x, y) - d(y, z)$ is true for all pseudo metric spaces $(M,d)$ and all $x, y, z \in M$. Thus, for any vector $v \in \mathbb{R}^k$ and for any $j \in I$ we have

$$d_2(\gamma_j, Me_1) \geq d_2(\gamma_j, v) - d_2(v, Me_1),$$

and hence

$$d_1(S_\Psi, M \cdot T_\Psi) \geq \sum_{j \in I} \left( d_2(\gamma_j, v) - d_2(v, Me_1) \right) = \left( \sum_{j \in I} d_2(\gamma_j, v) \right) - 3 \cdot d_2(v, Me_1).$$

Let us choose $v \in \mathbb{R}^k$ as follows: We let $v := \sum_{i \in I} s_i e_i$ where $s_i := 1$ if $a_i \leq b_i$, and $s_i := -1$ otherwise. Then,

$$d_2(v, Me_1) = \|v - Me_1\|_2 = \left\| \sum_{i \in I} (s_i e_i - M e_i) \right\|_2 \leq \sum_{i \in I} \|s_i e_i - M e_i\|_2 = \sum_{i \in I} d_2(s_i e_i, M e_i).$$

Note that $d_2(s_i e_i, M e_i)$ is equal to $a_i$ if $s_i = 1$, and it is equal to $b_i$ if $s_i = -1$. Thus, due to our choice of $s_i$, we know that $d_2(s_i e_i, M e_i) = \min(a_i, b_i)$, and hence

$$3 \cdot d_2(v, Me_1) \leq 3 \cdot \min(a_i, b_i).$$

Our next goal is to show that $\sum_{j \in I} d_2(\gamma_j, v) \geq 4 \cdot \sqrt{2}$. For simplicity let us consider w.l.o.g. the case where $I = \{1, 2, 3\}$. For $i \in I$ let $l_i = \theta(\Gamma_i)$ (thus, $l_i \in \{e_i, -e_i\}$). Then, w.l.o.g. we have

$$\gamma_1 = l_1 - l_2 - l_3, \quad \gamma_2 = -l_1 + l_2 - l_3, \quad \gamma_3 = -l_1 - l_2 + l_3.$$

For showing that $\sum_{j \in I} d_2(\gamma_j, v) \geq 4 \cdot \sqrt{2}$, we make a case distinction according to $v$.

Case 1: $v = \gamma_i$ for some $i \in I$. In this case, $d_2(\gamma_i, v) = 0$, and for each $j \in I \setminus \{i\}$, it is straightforward to see that $d_2(\gamma_j, v) = \sqrt{8} = 2\sqrt{2}$. Thus, $\sum_{j \in I} d_2(\gamma_j, v) = 4\sqrt{2} \approx 5.665$.

Case 2: $v = -\gamma_i$ for some $i \in I$. In this case, $d_2(\gamma_i, v) = 2 \cdot \|\gamma_i\|_2 = 2\sqrt{3}$. Furthermore, for each $j \in I \setminus \{i\}$, it is straightforward to see that $d_2(\gamma_j, v) = \sqrt{4} = 2$. Thus, $\sum_{j \in I} d_2(\gamma_j, v) = 4 + 2\sqrt{3} > 4\sqrt{2}$.

Case 3: $v = l_1 + l_2 + l_3$. Then, for each $j \in I$ we have $d_2(\gamma_j, v) = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$. Thus, $\sum_{j \in I} d_2(\gamma_j, v) = 3 \cdot 2\sqrt{2} = 6\sqrt{2} > 4\sqrt{2}$. 

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Case 4: \( v = -l_1 - l_2 - l_3 \). Then, for each \( j \in I \) we have \( d_2(\gamma_j, v) = 3 \cdot 2 = 6 > 4 \sqrt{2} \).

Note that Cases 1–4 comprise all possible cases for \( v \), and in all these cases, \( \sum_{j \in I} d_2(\gamma_j, v) \geq 4 \sqrt{2} \). Together, with (8) and (7) we obtain that equation (4) is correct. This completes the proof of (a).

For the proof of (b), let \( M \) be an element in \( \mathcal{MO}(k) \) such that \( M e_j = \gamma_i \), for some \( i \in I \). From equation (5) we then obtain

\[
d_1(S_\Psi, M \cdot T_\Psi) = \sum_{j \in I} d_2(\gamma_j, \gamma_i).
\]

According to Case 1 above, \( \sum_{j \in I} d_2(\gamma_j, \gamma_i) = 4 \sqrt{2} \). This completes the proof of (b).

For the proof of (c), let \( M \) be an element in \( \mathcal{MO}(k) \) such that \( d_1(S_\Psi, M \cdot T_\Psi) = 4 \sqrt{2} \) and \( M e_i \in \{e_i, -e_i\} \) for all \( i \in I \). Thus, \( M e_i \) is equal to a vector \( v' = \sum_{i \in I} s'_i e_i \) where \( s'_i \in \{1, -1\} \) for each \( i \in I \). The above case distinction (for \( v' \) rather than \( v \)) tells us that \( \sum_{j \in I} d_2(\gamma_j, v') = 4 \sqrt{2} \) iff \( v' = \gamma_i \) for some \( i \in I \). This completes the proof of (c), since \( \sum_{j \in I} d_2(\gamma_j, v') = d_1(S_\Psi, M \cdot T_\Psi) \).

From the two previous lemmas, we easily obtain the following Lemma (cf. the appendix for a proof):

**Lemma 4.10.** Let \( \Psi = \bigvee_{i \in I} L_i \) be a disjunctive 3-clause, let \( \Gamma \) be one of the conjunctive 3-clauses implicit in \( \Psi \), and let \( \gamma := \theta(\Gamma) \). For each \( i \in I \) let \( l_i \in \{e_i, -e_i\} \) be such that \( \gamma = \sum_{i \in I} l_i \). Then, there exists an \( M \in \mathcal{MO}(k) \) with \( M e_i = l_i \) for each \( i \in I \).

**Lemma 4.11.** Let \( \Psi = \bigvee_{i \in I} L_i \) be a disjunctive 3-clause. Then,

\[
\min_{M \in \mathcal{MO}(k)} d_\Psi(M) = 36 + 4 \cdot \sqrt{2}.
\]

Furthermore, every \( M \in \mathcal{MO}(k) \) with \( d_\Psi(M) = 36 + 4 \sqrt{2} \) has the following properties:

(*) \( M e_i \in \{e_i, -e_i\} \), for every \( i \in I \).

(**) \( M e_\Gamma = \gamma \), where \( \gamma = \theta(\Gamma) \) for a conjunctive 3-clause \( \Gamma \) implicit in \( \Psi \).

**Proof.** We first show that \( d_\Psi(M) \geq 36 + 4 \sqrt{2} \) is true for all \( M \in \mathcal{MO}(k) \). To this end, let \( M \) be an arbitrary matrix in \( \mathcal{MO}(k) \). By the Lemmas 4.8 and 4.9 we know that \( d_\Psi(M) = d_1(S_\Psi, M \cdot T_\Psi) + d_1(S'_\Psi, M' \cdot T'_\Psi) \geq 6 \cdot \sum_{i \in I} (a_i + b_i) + 4 \sqrt{2} - 3 \cdot \sum_{i \in I} \min(a_i, b_i) \)

\[
= 4 \sqrt{2} + \sum_{i \in I} (6a_i + 6b_i - 3 \min(a_i, b_i))
\]

\[
= 4 \sqrt{2} + 3 \cdot \sum_{i \in I} (2a_i + 2b_i - \min(a_i, b_i))
\]

What is the smallest value possible for \( N_i \)? Recall that \( a_i = \sqrt{4 - b_i^2} \) and \( b_i = \sqrt{4 - a_i^2} \). Thus, in case that \( a_i \leq b_i \) we have

\[
N_i = a_i + 2 \cdot \sqrt{4 - a_i^2},
\]

and in case that \( a_i > b_i \) we have

\[
N_i = b_i + 2 \cdot \sqrt{4 - b_i^2}.
\]

Furthermore, if \( a_i \leq \sqrt{2} \), then \( b_i \geq \sqrt{2} \), and hence \( a_i \leq b_i \). If \( a_i > \sqrt{2} \), then \( b_i < \sqrt{2} \), and hence \( b_i < a_i \).

Therefore, \( N_i \) is the minimum value of the function

\[
f : \{x \in \mathbb{R} : 0 \leq x \leq \sqrt{2}\} \rightarrow \mathbb{R}
\]

defined via

\[
f(x) := x + 2 \cdot \sqrt{4 - x^2}.
\]

It is not difficult to verify that \( f(0) = 4 \), and \( f(x) > 4 \) for all \( x \in \mathbb{R} \) with \( 0 < x \leq \sqrt{2} \).

Thus, \( N_i \geq 4 \) for true for every \( i \in I \). This leads to

\[
d_\Psi(M) \geq 4 \sqrt{2} + 3 \cdot \sum_{i \in I} 4 = 4 \sqrt{2} + 3 \cdot 4 = 4 \sqrt{2} + 36.
\]

Combining this with Lemma 4.10, we obtain that

\[
\min_{M \in \mathcal{MO}(k)} d_\Psi(M) = 36 + 4 \sqrt{2}.
\]

Now let us consider an arbitrary \( M \in \mathcal{MO}(k) \) for which \( d_\Psi(M) = 36 + 4 \cdot \sqrt{2} \). From the computations above we know that for each \( i \in I \) it must be true that \( N_i = 4 \) and hence \( a_i = 0 \) or \( b_i = 0 \). Since \( a_i = d_2(e_i, M e_i) \) and \( b_i = d_2(-e_i, M e_i) \), this implies that \( M e_i \in \{e_i, -e_i\} \). Hence, the lemma’s statement (**) holds.

Furthermore, since \( a_i = \sqrt{4 - b_i^2} \), we know that \( \{a_i, b_i\} = \{0, 2\} \). Hence, Lemma 4.8 implies that \( d_1(S'_\Psi, M \cdot T'_\Psi) = 6 \cdot 3 \cdot 2 = 36 \). By definition, we have

\[
d_\Psi(M) = d_1(S'_\Psi, M \cdot T'_\Psi) + d_1(S_\Psi, M \cdot T_\Psi);
\]

and by assumption we have \( d_\Psi(M) = 36 + 4 \sqrt{2} \). Thus, \( d_1(S'_\Psi, M \cdot T'_\Psi) = 4 \sqrt{2} \). From Lemma 4.9(c) we therefore obtain that the lemma’s statement (**) is correct.

The previous lemma tells us, in particular, that each \( M \in \mathcal{MO}(k) \), for which \( d_\Psi(M) \) is minimal, belongs to the set

\[
\mathcal{MO}(k)_C := \{ M \in \mathcal{MO}(k) : m_i \in \{e_i, -e_i\} \}
\]

where \( m_i \) denotes the vector in the \( i \)-th column of \( M \). Henceforth, the elements in \( \mathcal{MO}(k)_C \) will be called boolean matrices.

For each boolean matrix \( M \in \mathcal{MO}(k)_C \), we let \( A(M) \) be the assignment \( a \) with \( a(V_i) = 1 \) if \( m_{i,i} = 1 \), and \( a(V_i) = 0 \) if \( m_{i,i} = -1 \). Obviously, \( A \) is a bijection between \( \mathcal{MO}(k)_C \) and the set of all assignments to the propositional variables \( V_1, \ldots, V_k \).
Lemma 4.12. Let $\Phi = \bigvee_{j=1}^{m} \Psi_j$ be a 3-cnf formula with $m$ disjunctive clauses. Then, $\Phi$ has a 1-in-3 model iff
\[
\min_{M \in \mathcal{MO}(k)} d_1(S_{\Phi}, M \cdot T_{\Phi}) = m \cdot (36 + 4\sqrt{2}).
\]

Proof. According to our definition of $S_{\Phi}$ and $T_{\Phi}$, the following is true for every $M \in \mathcal{MO}(k)$:
\[
d_1(S_{\Phi}, M \cdot T_{\Phi}) = \sum_{j=1}^{m} d_1(S_{\Phi_j}, M \cdot T_{\Phi_j}).
\]
Furthermore, by Lemma 4.11 we know for each $j \in [1, m]$ that $d_1(S_{\Phi_j}, M \cdot T_{\Phi_j}) \geq 36 + 4\sqrt{2}$. Thus,
\[
\min_{M \in \mathcal{MO}(k)} d_1(S_{\Phi}, M \cdot T_{\Phi}) \geq m \cdot (36 + 4\sqrt{2}).
\]

To prove the lemma’s “if direction”, assume that there is an $M \in \mathcal{MO}(k)$ such that $d_1(S_{\Phi}, M \cdot T_{\Phi}) = m \cdot (36 + 4\sqrt{2})$.

Then, for each $j \in [1, m]$ we have
\[
d_{\Phi_j}(M) = d_1(S_{\Phi_j}, M \cdot T_{\Phi_j}) = 36 + 4\sqrt{2}.
\]

Thus, according to Lemma 4.11 $M$ has the properties $(\ast)$ and $(\ast\ast)$. In particular, $M$ is a boolean matrix in $
\mathcal{MO}(k)$. Let $\alpha := A(M)$ be the variable assignment associated with $M$. In the following, we show that $\alpha$ is a 1-in-3 model of $\Psi_j$, for each $j \in [1, m]$.

Fix an arbitrary $j \in [1, m]$ and let $\Psi := \Psi_j$. Let $I = \{i_1, i_2, i_3\} \subseteq [1, k]$ such that $\Psi = (L_{i_1} \lor L_{i_2} \lor L_{i_3})$, where $L_i$ is a literal over the variable $V_i$, for each $i \in I$.

From $(\ast)$ and $(\ast\ast)$ we know that $M e_i \in \{e_i, \neg e_i\}$ for every $i \in I$, and $M e_I = \gamma$, where $\gamma = \theta(\Gamma)$ for a conjunctive 3-clause $\Gamma$ implicit in $\Psi$. W.l.o.g., $\Gamma = (L_{i_1} \land \neg L_{i_2} \land \neg L_{i_3})$. Thus, $\gamma = l_{i_1} - l_{i_2} - l_{i_3}$, where $l_i = \theta(L_i)$ for each $i \in I$.

For each $i \in I$ let $m_i$ be the vector in the $i$-th column of $M$. Then, the following is true:
\[
m_{i_1} + m_{i_2} + m_{i_3} = M e_I = \gamma = l_{i_1} - l_{i_2} - l_{i_3}.
\]

Hence, $m_{i_1} = l_{i_1}$, $m_{i_2} = -l_{i_2}$, and $m_{i_3} = -l_{i_3}$. Therefore, the associated variable assignment $\alpha := A(M)$ satisfies the literal $L_{i_1}$ but not the literals $L_{i_2}$, $L_{i_3}$. Hence, $\alpha$ is a 1-in-3 model of $\Psi_j$.

In summary, we have shown that $\alpha$ is a 1-in-3 model of $\Psi_j$, for each $j \in [1, m]$. Therefore, $\alpha$ also is a 1-in-3 model of $\Phi$. This completes the proof of the “if direction”.

For the proof of the “only-if direction”, let us consider the case whether $\Phi$ has a 1-in-3 model. I.e., there exists a variable assignment $\alpha$ which, for each $j \in [1, m]$, satisfies exactly one literal in the disjunctive clause $\Psi_j$. Let $M$ be the boolean matrix with $A(M) = \alpha$. It suffices to prove that
\[
d_{\Phi_j}(M) = 36 + 4\sqrt{2}
\]
is true for every $j \in [1, m]$. To this end, fix an arbitrary $j \in [1, m]$ and let $\Psi := \Psi_j$. Let $I = \{i_1, i_2, i_3\} \subseteq [1, k]$, and for each $i \in I$ let $L_i$ be a literal over $V_i$, such that $\Psi = (L_{i_1} \lor L_{i_2} \lor L_{i_3})$.

Since $\alpha$ is a 1-in-3 model of $\Psi_j$, it satisfies a conjunctive 3-clause $\Gamma$ that is implicit in $\Psi$. W.l.o.g.,
\[
\Gamma = (L_{i_1} \land \neg L_{i_2} \land \neg L_{i_3}).
\]

Let $\gamma := \theta(\Gamma)$, and for each $i \in I$ let $l_i \in \{e_i, \neg e_i\}$ be such that $\gamma = \sum_{i \in I} l_i$.

Since $\alpha$ satisfies $\Gamma$, and since $M = A^{-1}(\alpha)$, it is straightforward to verify along the definition of the mappings $A$ and $\theta$ that $M e_i = l_i$ is true for each $i \in I$. From Lemma 4.10 we therefore obtain that $d_{\Phi}(M) = 36 + 4\sqrt{2}$. \hfill \Box

Note that Lemma 4.12 establishes the goal formulated directly before equation (2): When choosing $c^{\mathcal{MO}}(\Phi) := m \cdot (36 + 4\sqrt{2})$ whenever $\Phi$ is a 3-cnf formula consisting of $m$ disjunctive 3-clauses, Lemma 4.12 tells us that $\Phi$ has a 1-in-3 model if, and only if, $d_1^{\mathcal{MO}}(S_{\Phi}, T_{\Phi}) = c^{\mathcal{MO}}(\Phi)$.

Relating $d_1^{\mathcal{C}}(S_{\Phi}, T_{\Phi})$ with 1-in-3 models of $\Phi$ Until now, we only considered transformations using orthogonal matrices. However, the congruence distance $d_1^{\mathcal{C}}$ allows distance minimization also by translating with an arbitrary vector. The following lemma considers these transformations of time series too (cf. the appendix for a proof).

Lemma 4.13. Let $S, T$ be two time series of the same length over $\mathbb{R}^k$, let $\bar{S} := S \times -S$ and $\bar{T} := T \times -T$. The following is true for every $M \in \mathcal{MO}(k)$ and every $v \in \mathbb{R}^k$:
\[
d_1(S, M \cdot T) \leq d_1(S, M \cdot T + v)
\]

As a consequence of Lemma 4.12, Lemma 4.13, and the definition of $d_1^{\mathcal{C}}$, we immediately obtain the following (cf. the appendix for a proof).

Theorem 4.14. Let $\Phi = \bigvee_{j=1}^{m} \Psi_j$ be a 3-cnf formula with $m$ disjunctive clauses. Then, $\Phi$ has a 1-in-3 model iff
\[
d_1^{\mathcal{C}}(S_{\Phi}, T_{\Phi}) = m \cdot (72 + 8\sqrt{2}).
\]

An algorithm solving 1-in-3-Sat

of Theorem 4.14

Assume that $A$ is an algorithm which, on input of two time series $S$ and $T$ of equal length, computes $d_1^{\mathcal{C}}(S, T)$. Using this algorithm, the problem 3-IN-1-SAT can be solved as follows.

Upon input of a 3-cnf formula $\Phi$, construct the time series $S_{\Phi}$ and $T_{\Phi}$. Clearly, this can be done in time polynomial in the size of $\Phi$. Letting $k$ be the number of variables occurring in $\Phi$, run algorithm $A$ with input $k, S_{\Phi}, T_{\Phi}$. After a number of steps polynomial in the size of $\Phi$, $A$ will output (a suitable representation of) the number $d_1^{\mathcal{C}}(S, T)$. Now, check if this number is equal to (a suitable representation of) the number $m \cdot$
(72+8√2), where \( m \) is the number of disjunctive clauses of \( \Phi \). If so, output “yes”; otherwise output “no”.

From Theorem 4.14 we know that the algorithm’s output is “yes” if, and only if, \( \Phi \) has a 1-in-3 model. Thus, we have constructed a polynomial-time algorithm solving the NP-complete problem 1-in-3-SAT. In case that \( P \neq \text{NP} \), such an algorithm cannot exist. □

Note that according to the above proof, already the restriction of \( d_1^\mathcal{S} \)-COMPUTATION to input time series over \{0, 1, −1, 6, −6\} cannot be accomplished in polynomial time, unless \( P = \text{NP} \).

4.2 The structure \( \Delta S \) of a time series \( S \)

In this subsection we consider the well-known self-similarity matrix of a time series. Usually, the self-similarity matrix is used to analyze a time series for patterns (e.g., using Recurrence Plots [12]).

The important property that makes the self-similarity matrix useful for approximating the congruence distance, is its invariance under transformations considered for the congruence distance, i.e., rotation, translation, and mirroring.

Considering an arbitrary time series \( T = (t_0, \ldots, t_{n-1}) \in \mathcal{T}_M \) over a metric space \((\mathbb{M}, d)\), the self-similarity matrix

\[
\Delta T := \begin{pmatrix} d(t_i, t_{i+j}) \end{pmatrix}_{i \in [0, n-1], j \in [1, n-i]}
\]

describes the inner structure of the time series. Thus, we also call the self-similarity matrix \( \Delta T \) the structure of the time series \( T \).

Throughout the remainder of this subsection, we will restrict attention to time series over \((\mathbb{R}^k, d_2)\).

The next theorem shows that for such time series, the structure \( \Delta T \) completely describes the sequence \( T \) up to congruence, i.e., up to rotation, translation, and mirroring of the whole sequence in \( \mathbb{R}^k \).

**Theorem 4.15.** Consider the metric space \((\mathbb{R}^k, d)\) for \( d := d_2 \), and let \( S, T \) be two time series of length \( n \) over \( \mathbb{R}^k \). Then, \( S \) and \( T \) are congruent if they have the same structure, i.e.:

\[
S \cong C T \iff \Delta S = \Delta T.
\]

Basically, this theorem holds because the Euclidean Distance is invariant under isometric functions (cf. the appendix for a detailed proof).

4.3 The Delta Distance

Our approach for approximating the congruence distance between two time series \( S \) and \( T \) is to compare the self-similarity matrices of \( S \) and \( T \) via a suitable matrix norm. This is formalized in the following definition.

**Definition 4.16** (Delta Distance).

Let \( \mathcal{T} \) be the class of all time series over \( \mathbb{R}^k \), and let \( \|\cdot\| \) be a matrix norm. Let \( S, T \in \mathcal{T} \) be two time series of length \( m \) and \( n \) (\( m \leq n \)), respectively. The delta distance \( d_{\|\cdot\|}^\Delta(S, T) \) is defined as follows:

\[
d_{\|\cdot\|}^\Delta(T, S) := d_{\|\cdot\|}^\Delta(S, T) := \min_{b \in [0, n-m]} \left( d_{\|\cdot\|}^\Delta(S, T^b_m) \right).
\]

We will consider the cases where \( \|\cdot\| \) is the max column norm \( \|\cdot\|_\text{m} \) or the \( p \)-Norm \( \|\cdot\|_p \) for some \( p \in \mathbb{R}_{>1} \).

In these cases we will write \( d_m^\Delta \) and \( d_p^\Delta \), respectively, to denote \( d_{\|\cdot\|}^\Delta \).

Obviously, for time series of the same length, the complexity of computing the delta distance \( d_m^\Delta \) grows quadratically with the length of the time series. In particular, for time series \( S \) and \( T \) of equal length, \( d_m^\Delta(S, T) \) and \( d_m^\Delta(S, T) \) can be computed in time quadratic in the length of \( S \) and \( T \).

Our next aim is to show that the the delta distance \( d_m^\Delta \) provides a lower bound on the congruence distance \( d_1^\Delta \), as formulated in the following theorem.

**Theorem 4.17.** For all time series \( S \) and \( T \) over \( \mathbb{R}^k \), the following holds:

\[
d_m^\Delta(S, T) \leq 2 \cdot d_1^\Delta(S, T).
\]

For proving Theorem 4.17, we will employ the following two lemmas (cf. the appendix for their proofs).

**Lemma 4.18.** Let \( \mathcal{T} \) be the set of all time series over \( \mathbb{R}^k \). Let \( \|\cdot\| \) be a matrix norm. Let \( p \in \mathbb{R}_{>1} \), and let \( C \) be a function from \( \mathbb{N} \) to \( \mathbb{R} \). If for all \( n \in \mathbb{N} \) and all time series \( S, T \in \mathcal{T} \) of length \( n \) we have

\[
d_m^\Delta(S, T) \leq C(n) \cdot d_p(S, T),
\]

then

\[
d_m^\Delta(S, T) \leq C(\min \{\#S, \#T\}) \cdot d_p(S, T)
\]

holds for all time series \( S, T \in \mathcal{T} \) (i.e., also for time series of different lengths).

**Lemma 4.19.** The following holds for the max column norm \( \|\cdot\|_\text{m} \) and for all time series \( S \) and \( T \) over \( \mathbb{R}^k \):

\[
d_m^\Delta(S, T) \leq 2 \cdot d_1^\Delta(S, T).
\]

of Theorem 4.17

W. l. o. g., let \( \#S = n \leq \#T \). For arbitrary \( M \in \mathcal{MO}(k) \) and \( v_0 \in \mathbb{R}^k \), Theorem 4.15 tells us that

\[
\Delta T = \Delta(M \cdot T + v_0).
\]

Hence, rotation, translation, and mirroring of \( T \) does not affect \( d_m^\Delta(S, T) \). Applying Lemma 4.19, we obtain

\[
d_m^\Delta(S, T) = d_m^\Delta(S, M \cdot T + v_0) \leq 2 \cdot d_1(S, M \cdot T + v_0)
\]

Thus, the desired inequality holds:

\[
d_m^\Delta(S, T) \leq \inf_{M \in \mathcal{MO}(k), v_0 \in \mathbb{R}^k} \left( 2 \cdot d_1(S, M \cdot T + v_0) \right)
\]

\[
= 2 \cdot d_1^\Delta(S, T)
\]

□
Figure 4: Two time series $S$ and $T_ε$ in $\mathbb{R}^2$, such that $\frac{d^2_{\delta}(S,T_ε)}{d_1^2(S,T_ε)} \xrightarrow{ε \to 0} +\infty$.

Similarly to Lemma 4.19, we can also prove the following (cf. the appendix for a proof).

**Lemma 4.20.** The following holds for the matrix norm $\|\cdot\|_1$ and for all time series $S,T$ over $\mathbb{R}^k$:

$$d^2_{\delta}(S,T) \leq (\min\{\#S,\#T\} - 1) \cdot d_1^2(S,T).$$

Combining this lemma with the proof of Theorem 4.17, we obtain the following.

**Theorem 4.21.** For all time series $S$ and $T$ over $\mathbb{R}^k$, the following holds: Then the following inequality holds:

$$d^2_{\delta}(S,T) \leq (\min\{\#S,\#T\} - 1) \cdot d_1^2(S,T).$$

The Theorems 4.21 and 4.17 show that the delta distances $d^2_{\delta}$ and $d^2_{\alpha}$ provide lower bounds for the congruence distance $d^2_{\delta}$. On the other hand, the ratio of the congruence distance and the delta distance can grow arbitrarily as shown with the following example.

**Example 4.22.** Consider $M = \mathbb{R}^2$ and the euclidean distance $d = d_2$. We show that for each $C > 0$ time series $S,T$ exist such that $\frac{d^2_{\delta}(S,T)}{d^2_1(S,T)} \geq C$ (cf. Figure 6). Let $ε > 0$, $a := \sqrt{1-ε^2}$, and consider $S = (s_0, s_1, s_2), T_ε = (t_0, t_1, t_2) \in T_3$ with

$$S = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}, \quad T_ε = \begin{pmatrix} -a & 0 \\ 0 & ε \end{pmatrix}.$$

Then $\delta^2(S,T_ε) = 2(1-a) = 2(1-\sqrt{1-ε^2})$. We claim that $d^2_1(S,T_ε) \geq \frac{1}{2} ε$ (cf. the Appendix A, Claim 2 for a proof). Then,

$$\frac{d^2_{\delta}(S,T_ε)}{d^2_1(S,T_ε)} \geq \frac{1}{4} \cdot \frac{ε}{1-\sqrt{1-ε^2}} \xrightarrow{ε \to 0} +\infty.$$

5 The Reduced Delta Distance

While computing the congruence distance $d^2_{\delta}$ of two given time series $S$ and $T$ is an NP-hard problem, the computation of the delta distances $d^2_{\alpha}$ and $d^2_{\delta}$ can be accomplished in time quadratic in the lengths of $S$ and $T$. For practical usage, however, a distance measure that can be computed in linear or quasi-linear time, would be highly desirable.

In this section, we propose the $d^\delta$ distance function, for which the distance between two time series of length $n$ can be computed in time $O(n \log n)$. The idea underlying the definition of $d^\delta$ is the same as for the delta distance function $d^2_{\alpha}$, except that only log $n$ columns of the matrix $\Delta S - \Delta T$ are computed.

For giving the precise definition of the reduced delta distance $d^\delta$, we need the following notation.

We write $\text{Pow}_2$ for the set $\{2^n : n \in \mathbb{N}\}$ of all powers of 2. Let $T = (t_0, \cdots, t_{n-1}) \in T$ be a time series over the metric space $(\mathbb{R}^k, d)$. The reduced structure of $T$ is the matrix

$$δT := \left( d(t_i, t_{i+j}) \right)_{i \in [0,n-1], j \in [1,n-i] \cap \text{Pow}_2}.$$

It contains the subset of the entries of $\Delta T$ which compare two states $t_i$ and $t_{i+j}$ having a time distance $j$ that is a power of 2.

**Definition 5.1** (Reduced Delta Distance).

Let $T$ be the class of all time series over $\mathbb{R}^k$, and let $\|\cdot\|$ be a matrix norm. Let $S,T \in T$ be two time series of lengths $m$ and $n$ ($m \leq n$), respectively. The reduced delta distance $d^\delta_{\|\cdot\|}(S,T)$ is defined as follows:

$$d^\delta_{\|\cdot\|}(T,S) := d^\delta_{\|\cdot\|}(S,T) := \min_{b \in [0,n-m]} \left( d^\delta_{\|\cdot\|}(S,T_{b,m}) \right).$$

In case that $\|\cdot\|$ is the max column norm $\|\cdot\|_\infty$ or the p-Norm $\|\cdot\|_p$ for some $p \in \mathbb{R}_{>1}$, we will write $d^\delta_{\infty}$ and $d^\delta_{p}$, respectively, to denote $d^\delta_{\|\cdot\|}$.

In particular, since $δT$ has $O(n \log n)$ entries, the values $d^\delta_{\infty}(S,T)$ and $d^\delta_{\|\cdot\|}(S,T)$ can be computed in time $O(n \log n)$, if $S$ and $T$ are two time series of length $n$. Thus, using $d^\delta$ (instead of $d^\alpha$ or $d^\delta_{\|\cdot\|}$) has the benefit that the distance between two time series of equal length can be computed in quasi-linear time.

On the other hand, using $d^\delta$ instead of $\Delta T$ has the drawback that Theorem 4.15 (i.e., the congruence requirement) does not hold for $d^\delta T$. The following example shows that there are time series $S,T \in T$ with $\Delta S \neq \Delta T$ but $δS = δT$.

**Example 5.2.** Consider the following time series $S, T \in T$ over the metric space $(\mathbb{R}^2, d_2)$:

$$S = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then

$$\Delta S = \begin{pmatrix} 4 & 5 & 0 \\ 3 & 4 & 5 \end{pmatrix}, \quad \Delta T = \begin{pmatrix} 4 & 5 & 8 \\ 3 & 4 & 5 \end{pmatrix}.$$

but

$$\delta S = \begin{pmatrix} 4 & 5 & 0 \\ 3 & 4 & 5 \end{pmatrix} = δT.$$

**Cheap Lower Bound for Congruence**

In Section 4.3 we showed that the congruence distance yields an upper bound for the delta distance. Analogously, in this section we show that the congruence distance yields an upper bound for the reduced delta distance.
distance. Viewed from the other side, the reduced delta distance function can thus be regarded as a computationally cheap approximation of the congruence distance, which provides a lower bound for the congruence distance.

**Theorem 5.3.** For all time series \( S \) and \( T \) over \( \mathbb{R}^k \), the following holds:

\[
d_m^\delta(S, T) \leq 2 \cdot d_1^\delta(S, T).
\]

**Proof.** Obviously, \( d_m^\delta(S, T) \leq d_m^\gamma(S, T) \) holds for all time series \( S, T \) over \( \mathbb{R}^k \). Hence, the desired inequality follows with Theorem 4.17. \( \square \)

The proof of the following theorem is much more algebraic and can be found in the appendix.

**Theorem 5.4.** For all time series \( S \) and \( T \) over \( \mathbb{R}^k \), the following holds:

\[
d_1^\delta(S, T) \leq \left[2 \cdot \log(\min\{\#S, \#T\} - 1)\right] \cdot d_1^\gamma(S, T).
\]

### 6 Experimental Results

Although we have boundaries for the delta distance function, the ratio of \( d_1^\delta \) to \( d_m^\delta \) could be arbitrarily bad. With the experiments below, we show that the ratio has a stable average. We consider the delta distance functions using the max column matrix norm only (i.e. we focus on \( d_m^\delta \) and \( d_m^\gamma \)) since we achieved best results with it.

**Algorithm 1** The algorithm \texttt{gen} generates new time series from a given time series such that it is likely to be aligned to its origin. The parameters \( \eta \) and \( E \) are considered to be clear from the context.

```
Algorithm: gen

Input: \( S = (s_0, \ldots, s_{n-1}) \), \( \eta > 1 \), \( E \in \mathbb{N} \),
Output: \texttt{gen}(S)
repeat \( E \) times:
Choose a random subset \( I \subseteq \{0, \ldots, n-1\} \)
Calculate the barycenter \( b := |I|^{-1} \sum_{i \in I} s_i \)
Choose a random value \( \mu \in [\eta^{-1}, \eta] \)
for each \( i \in I \):
\( s_i := b + \mu \cdot (s_i - b) \)
end
return \( S \)
```

We performed the experiments using the TRECVID benchmark dataset [4] which consists of a set of videos, denoted by \( \mathcal{TR} \), and consider them as high-dimensional time series. The dataset consists of around 8000 videos from 13 sec to 2 h. We downsampled them to grayscale videos with a resolution of 128 × 96 Pixels and a framerate of two images per second, i.e. we consider the metric space \( (M = \mathbb{R}^{12,288}, d) \) with \( |M| = 8000 \). Each video \( T = (t_0, \ldots, t_{|T|}) \in \mathcal{TR} \) is considered as a time series with \( 26 \leq \#T \leq 14,400 \), and with each image \( t_i \in \mathbb{R}^{12,288} \) considered as a 12,288 dimensional vector with each pixel corresponding to one dimension.

Since it is too complex to compute the exact value of \( d_1^\delta \), we used a transformation function \( g : \mathcal{TR} \rightarrow \mathcal{T} \) to generate new time series for each time series \( S \in \mathcal{TR} \), and computed the distance \( d_1(S, \texttt{gen}(S)) \geq d_1^\gamma(S, \texttt{gen}(S)) \). Intuitively, \( g \) creates random time series such that for a certain time series, the time series distance to the random time series is close to their congruence distance. Technically, \( g \) explodes or implodes random states of the time series around their barycenter. The exact algorithm is shown in Algorithm 1. Hence, we can provide an upper bound for the relative error of \( d_m^\delta \) and \( d_m^\gamma \):

\[
e_m^\gamma(S, \texttt{gen}(S)) := \frac{d_1(S, \texttt{gen}(S))}{2 \cdot d_m^\gamma(S, \texttt{gen}(S))} \geq \frac{d_1^\gamma(S, \texttt{gen}(S))}{2 \cdot d_m^\gamma(S, \texttt{gen}(S))}
\]

\[
e_m^\delta(S, \texttt{gen}(S)) := \frac{d_1(S, \texttt{gen}(S))}{2 \cdot d_m^\delta(S, \texttt{gen}(S))} \geq \frac{d_1^\delta(S, \texttt{gen}(S))}{2 \cdot d_m^\delta(S, \texttt{gen}(S))}
\]

Table 1 shows the average ratios \( E[e_m^\gamma(S, \texttt{gen}(S))] \) and \( E[e_m^\delta(S, \texttt{gen}(S))] \) for all time series and their generated time series for 12 sets of parameters of the algorithm \texttt{gen}. A closer inspection of the algorithm \texttt{gen} shows that the congruence distance increases with increasing parameter \( \eta \). Thus, the results from Table 1 suggest that the boundary for the relative error decreases with increasing congruence distance. Note, that we only approximate the relative error from top. We did not investigate the variation of the error for different numbers of explosions \( E \), but it could probably vary as a result of the deviation of \( d_1(S, \texttt{gen}(S)) \) from \( d_1^\gamma(S, \texttt{gen}(S)) \).

Figure 6 shows the results in more detail for fixed parameters \( \eta = 1.5 \) and \( E = 5 \). The experimental results give evidence for the presumption that the congruence distance function is more accurate on time series having a large congruence distance. On the other hand,

| \( E \setminus \eta \) | 1.1 | 2 | 10 |
|----------------|-----|---|----|
| 1              | 1.62 | 1.53 | 1.32 |
| 5              | 1.82 | 1.62 | 1.22 |
| 10             | 1.78 | 1.62 | 1.22 |
| 50             | 1.85 | 1.35 | 1.13 |

Table 1: Experimental results: The average \( E[e_m^\gamma(S, \texttt{gen}(S))] \) (left) and \( E[e_m^\delta(S, \texttt{gen}(S))] \) (right) for different parameter settings of \texttt{gen}. 
the relative error increases for time series having a low congruence distance.

7 Conclusion and Future Work

In this paper, we introduced and analyzed the problem of measuring the congruence between two time series. After having proved that its computation is NP-hard, we provided two measures for approximating the congruence distance in polynomial time. The first (namely, the delta distance) measures the congruence in a way that the distance of two time series is 0 iff they are congruent. The second loses this benefit, but can be computed in quasi-linear instead of quadratic time. Furthermore, we showed that all provided distances fulfill the triangle inequality on time series of the same length.

We simplified the problem of motion gesture recognition to measure the congruence of two time series in 3-dimensional space. In practical applications, multiple time series at the same time need to be considered, which is one of our next steps to investigate. Furthermore, the congruence distance is not robust against scaling of the time series, which is an important future work too. Also, we are currently carrying out experiments on motion gesture recognition and content-based video copy detection to evaluate the utility of the congruence distance. To achieve more robust distance functions, local time shifting techniques like Dynamic Time Warping need to be adapted to the congruence distance provided in this paper.

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A Details Omitted in Section 4

Lemma A.1. Definition 4.4 is well-defined.

Proof. Consider arbitrary but fixed $S, T \in T_n$, $M \in \mathcal{MO}(k)$, and $v \in \mathbb{R}^k$. We write $\bar{v}$ and $\bar{0}$ for the time series $(y_0, \ldots, y_{n-1})$ and $(z_0, \ldots, z_{n-1})$ in $T_n$, where $y_i = v$ and $z_i = 0 \in \mathbb{R}^k$, for each $i \in [0, n]$.

It is not difficult to verify that the following is true:

$$d_p(S, M \cdot T + v) \geq d_p(M \cdot T + v, \bar{0}) - d_p(S, \bar{0})$$

$$\geq d_p(\bar{v}, \bar{0}) - d_p(M \cdot T, \bar{0}) - d_p(S, \bar{0})$$

Note that $d_p(\bar{v}, \bar{0}) = \|d_2(\bar{v}, \bar{0})\|_p = \left( \|v\|_2^p \right)^{1/p} = n^{1/p} \cdot \|v\|_2$. Thus, we have

$$d_p(S, M \cdot T + v) \geq n^{1/p} \cdot \|v\|_2 - d_p(T, \bar{0}) - d_p(S, \bar{0}).$$

Let

$$I := \inf_{M' \in \mathcal{MO}(k), v' \in \mathbb{R}^k} d_p(S, M' \cdot T + v').$$

Clearly, $I \leq d_p(S, T)$ (for this, consider $v' = 0 \in \mathbb{R}^k$ and $M'$ the identity matrix). Now, let $\varepsilon > 0$, and let

$$r := \frac{d_p(S, T) + d_p(T, \bar{0}) + d_p(S, \bar{0}) + \varepsilon}{n^{1/p}}.$$ 

Then, for any $v \in \mathbb{R}^k$ with $\|v\|_2 > r$, we have

$$d_p(S, M \cdot T + v) \geq I + \varepsilon, \quad \text{for every } M \in \mathcal{MO}(k).$$

Thus, for computing $I$ it suffices to restrict attention to $v' \in \mathbb{R}^k$ with $\|v'\|_2 \leq r$. I.e.,

$$I = \inf_{M' \in \mathcal{MO}(k), v' \in V} d_p(S, M' \cdot T + v'),$$

for $V := \{ v \in \mathbb{R}^k : \|v\|_2 \leq r \}.$

We let $X := \mathcal{MO}(k) \times V$ and $Y := \mathbb{R}_{\geq 0}$ and let $f : X \to Y$ be defined via

$$f(M, v) := d_p(S, M \cdot T + v).$$

Furthermore, view each element $(M, v) \in X$ as a vector in $\mathbb{R}^{k^2+k}$, and choose the Euclidean distance $d_2$ as a metric on $X$. It is straightforward to see that, with respect to this metric,

1. $X$ is a bounded set, and
2. $f$ is a continuous function.

Now consider an arbitrary sequence $\xi := (M_i, v_i)_{i \in \mathbb{N}}$ with $M_i \in \mathcal{MO}(k)$ and $v_i \in V$ for all $i \in \mathbb{N}$ such that

$$I = \lim_{i \to \infty} d_p(S, M_i \cdot T + v_i) = \lim_{i \to \infty} f(M_i, v_i).$$

Since $X = \mathcal{MO}(k) \times V$ is a bounded set (w.r.t. $d_2$), the sequence $\xi$ is bounded, and thus contains a convergent (w.r.t. $d_2$) subsequence $\xi' = (M_{i_j}, v_{i_j})_{j \in \mathbb{N}}$, with $i_1 < i_2 < \cdots$ (Bolzano-Weierstrass Theorem). Let $(M^*, v^*) \in \mathcal{MO}(k) \times \mathbb{R}^k$ be the limit, i.e.,

$$(M^*, v^*) := \lim_{j \to \infty} (M_{i_j}, v_{i_j}).$$

Since $f$ is a continuous function, we obtain that

$$\lim_{j \to \infty} f(M_{i_j}, v_{i_j}) = f(M^*, v^*).$$

Thus,

$$I = \lim_{i \to \infty} f(M_i, v_i) = \lim_{j \to \infty} f(M_{i_j}, v_{i_j}) = f(M^*, v^*) = d_p(S, M^* \cdot T + v^*).$$

Hence, Definition 4.4 is well-defined. \qed
of Proposition 4.5
It is easy to see that $d_p$ and thus $d_p^C$ are symmetric functions, i.e., $d_p^C(S, T) = d_p^C(T, S)$. Furthermore, the triangle inequality for $d_p$ follows from the axioms of a norm.

To prove the triangle inequality of $d_p^C$, let $M_1, M_2 \in \mathcal{MO}(k)$ and $v_1, v_2 \in \mathbb{R}^k$ such that $d_p^C(S, T) = d_p(M_1 \cdot S + v_1, T)$ and $d_p^C(T, U) = d_p(T, M_2 \cdot U + v_2)$. Then, the triangle inequality follows:

$$d_p^C(S, U) \leq d_p(M_1 \cdot S + v_1, M_2 \cdot U + v_2) \leq d_p(M_1 \cdot S + v_1, T) + d_p(T, M_2 \cdot U + v_2) = d_p^C(S, T) + d_p^C(T, U).$$

\[\square\]

of Lemma 4.8
Recalling that $d_1 = \|d_2\|_1$, for the particular choice of $S'_\Psi$ and $T'_\Psi$, we obtain that

$$d_1(S'_\Psi, M \cdot T'_\Psi) = 6 \cdot \sum_{i \in I} (d_2(e_i, Me_i) + d_2(-e_i, Me_i)). \tag{8}$$

Hence, $d_1(S'_\Psi, M \cdot T'_\Psi) = 6 \cdot \sum_{i \in I} (a_i + b_i)$.

\[\square\]

of Lemma 4.10
Let $M$ be the $(k \times k)$-matrix with columns $m_1, \ldots, m_k$ such that $m_i = l_i$ for each $i \in I$, and $m_j = e_j$ for each $j \in [1, k] \setminus I$. It is straightforward to verify that $M \in \mathcal{MO}(k)$ and $Me_i = l_i$ for each $i \in I$.

Now let $M$ be an arbitrary element in $\mathcal{MO}(k)$ such that $Me_i = l_i$ for all $i \in I$. Using the Lemmas 4.8 and 4.9 we obtain that

$$d_\Psi(M) = d_1(S'_\Psi, M \cdot T'_\Psi) + d_1(S'_\Psi, M \cdot T'_\Psi) = 6 \cdot \sum_{i \in I} (a_i + b_i) + 4\sqrt{2}.$$

Recall that $a_i = d_2(e_i, Me_i)$ and $b_i = d_2(-e_i, Me_i)$. Since $Me_i = l_i \in \{e_i, -e_i\}$, we obtain that

$$a_i + b_i = d_2(e_i, -e_i) = 2 \cdot \|e_i\|_2 = 2.$$

Hence, $6 \cdot \sum_{i \in I} (a_i + b_i) = 6 \cdot 3 \cdot 2 = 36$, and $d_\Psi(M) = 36 + 4\sqrt{2}$.

\[\square\]

of Lemma 4.13
Let $S, T, v$ be as in the lemma’s assumption. Let us fix an arbitrary $M \in \mathcal{MO}(k)$. Let $T' := M \cdot T$. Let $S = (s_0, \ldots, s_{n-1})$ and let $T' = (t_0, \ldots, t_{n-1})$. Then,

$$d_1(\bar{S}, M \cdot \bar{T}) = \left( \sum_{i=0}^{n-1} d_2(s_i, t_i) \right) + \left( \sum_{i=0}^{n-1} d_2(-s_i, -t_i) \right) = \sum_{i=0}^{n-1} 2 \cdot \|s_i - t_i\|_2,$$

and

$$d_1(\bar{S}, M \cdot \bar{T} + v) = \sum_{i=0}^{n-1} \left( \|s_i - t_i - v\|_2 + \|s_i - t_i + v\|_2 \right).$$
Letting $u_i := s_i - t_i$, we obtain that

$$d_1(\tilde{S}, M \cdot \tilde{T}) = \sum_{i=0}^{n-1} 2 \|u_i\|_2,$$

and

$$d_1(\tilde{S}, M \cdot \tilde{T} + v) = \sum_{i=0}^{n-1} (\|u_i + v\|_2 + \|u_i - v\|_2).$$

For proving the lemma, it therefore suffices to show that

$$2 \|u_i\|_2 \leq \|u_i + v\|_2 + \|u_i - v\|_2$$

(9) is true for every $i \in [0, n]$. In what follows, we prove that the inequality (9) is in fact true for every vector $u_i \in \mathbb{R}^k$.

For achieving this, the following claim will be useful.

Claim 1. $|a + b| + |a - b| \geq 2|a|$ is true for all $a, b \in \mathbb{R}$.

Proof. Let us first restrict attention to the case where $a \geq 0$. In this case, the following is true:

$$|a + b| + |a - b| = \begin{cases} a + b + a - b = 2a, & \text{if } a \geq |b| \\ a + |b| - a + |b| = 2|b|, & \text{if } a < |b| \end{cases}$$

In both cases, $|a + b| + |a - b| \geq 2|a|$, and we are done.

Now let us consider the case where $a < 0$. Then, $a' := -a > 0$, and $|a + b| + |a - b| = |a' - b| + |a' + b|$. For the latter, we already know that it is $\geq 2|a'| = 2|a|$. This completes the proof of Claim 1. \qed

Now, let $u$ be an arbitrary vector in $\mathbb{R}^k$. Clearly, there exists an element $M' \in \mathcal{MO}(k)$ such that $M' u = a e_1$ for some $a \in \mathbb{R}$. For this $M'$ let us write $b_i$ for the entry in the $i$-th component of the vector $M' v$, for every $i \in [1, k]$. Then, the following is true:

$$2 \|u\|_2 = 2 \|M' u\|_2 = 2|a|$$

and

$$\|u + v\|_2 + \|u - v\|_2 = \|M'(u + v)\|_2 + \|M'(u - v)\|_2$$

$$= \|M' u + M' v\|_2 + \|M' u - M' v\|_2$$

$$= \|ae_1 + M' v\|_2 + \|ae_1 - M' v\|_2$$

$$\geq |a + b_1| + |a - b_1|.$$  

By Claim 1, the latter is $\geq 2|a|$. In summary, we obtain that

$$2 \|u\|_2 \leq \|u + v\|_2 + \|u - v\|_2$$

is true for all $u, v \in \mathbb{R}^k$. This completes the proof of Lemma 4.13. \qed

of Theorem 4.11

From Lemma 4.13 we obtain that

$$d_1^c(\tilde{S}_\Phi, \tilde{T}_\Phi) = \min_{M \in \mathcal{MO}(k)} d_1(\tilde{S}_\Phi, M \cdot \tilde{T}_\Phi).$$

Furthermore, by the choice of $\tilde{S}_\Phi$ and $\tilde{T}_\Phi$ we have for every $M \in \mathcal{MO}(k)$ that

$$d_1(\tilde{S}_\Phi, M \cdot \tilde{T}_\Phi) = 2 \cdot d_1(S_\Phi, M \cdot T_\Phi).$$

Hence, $d_1^c(\tilde{S}_\Phi, \tilde{T}_\Phi) = 2 \cdot \min_{M \in \mathcal{MO}(k)} d_1(S_\Phi, M \cdot T_\Phi)$ \textit{Lemma 4.12} \quad 2 \cdot m \cdot (36 + 4\sqrt{2}). \qed
of Theorem 4.15

Let \( S = (s_0, \ldots, s_{n-1}) \) and \( T = (t_0, \ldots, t_{n-1}) \). For the direction “\( \Rightarrow \)” assume that \( T = M \cdot S + v_0 \) for some orthogonal matrix \( M \in \mathcal{MO}(k) \) and some vector \( v_0 \in \mathbb{R}^k \). Then \( \Delta S = \Delta T \), since the following equation holds for all \( i, j \in [0, n) \):

\[
d(t_i, t_j)^2 = d(Ms_i + v_0, Ms_j + v_0)^2
\]

\[
= \|Ms_i + v_0 - Ms_j - v_0\|_2^2
\]

\[
= \|M(s_i - s_j)\|_2^2
\]

\[
= \|s_i - s_j\|_2^2
\]

\[
= d(s_i, s_j)^2
\]

For the opposite direction, assume \( \Delta S = \Delta T \). Let \( S' = (0, s'_1, \ldots, s'_{n-1}) = S - s_0 \) and \( T' = (0, t'_1, \ldots, t'_{n-1}) = T - t_0 \). Obviously, \( \Delta S' = \Delta S \) and \( \Delta T' = \Delta T \). Hence, \( \Delta S' = \Delta T' \), and we thus obtain the equality of scalar products:

\[
(s'_i - s'_j, s'_i - s'_j) = d(s'_i, s'_j)^2 = d(t'_i, t'_j)^2 = (t'_i - t'_j, t'_i - t'_j) .
\]

Hence,

\[
\|s'_i\|_2^2 + \|s'_j\|_2^2 - 2(s'_i, s'_j) = \|t'_i\|_2^2 + \|t'_j\|_2^2 - 2(t'_i, t'_j).
\]

Using \( \|s'_i\|_2^2 = d(0, s'_i)^2 = d(0, t'_i) = \|t'_i\|_2^2 \), we obtain that

\[
(s'_i, s'_j) = (t'_i, t'_j) .
\]

Let \( \mathcal{B} := \{s'_1, \ldots, s'_{n-1}\} \) be a basis of the vector space

\[
V := \text{span}(s'_1, \ldots, s'_{n-1}) ,
\]

let \( W := \text{span}(t'_1, \ldots, t'_{n-1}) \), and consider the linear function \( F : V \rightarrow W \) defined via \( F(s'_i) := t'_i \) for each \( j \in [1, m] \). For this function \( F \), the following holds for all \( a, b \in [1, m] \):

\[
\langle F(s'_a), F(s'_b) \rangle = \langle t'_a, t'_b \rangle = \langle s'_a, s'_b \rangle .
\]

Thus, \( F \) is orthogonal and the family \( \{t'_1, \ldots, t'_{n-1}\} \) is a basis of \( \text{span}(t'_1, \ldots, t'_{n-1}) \).

To see that \( F(s'_i) = t'_i \) holds for all \( i \in [0, n) \) consider the gramian matrix

\[
G := \begin{pmatrix} \langle s'_i, s'_j \rangle \end{pmatrix} \in \mathbb{R}^{n \times n},
\]

\[
= \begin{pmatrix} \langle t'_i, t'_j \rangle \end{pmatrix} \in \mathbb{R}^{n \times n} .
\]

Note that the gramian matrix \( G \) is nonsingular because \( \mathcal{B} \) is a basis. Now, for \( i \in [0, n) \) and \( a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{R}^m \) such that

\[
s'_i = a_1s'_1 + \cdots + a_ms'_m \text{ and } t'_i = b_1t'_1 + \cdots + b_mt'_m
\]

we have

\[
G \cdot a = \sum_{h=1}^{m} a_h \langle s'_h, s'_i \rangle = \sum_{h=1}^{m} a_h s'_h s'_i = \langle s'_i, s'_j \rangle = \langle t'_i, t'_j \rangle = \sum_{h=1}^{m} b_h \langle t'_h, t'_i \rangle = G \cdot b
\]

and thus \( a = b \). Hence

\[
F(s'_i) = \sum_{h=1}^{m} a_h F(s'_h) = \sum_{h=1}^{m} a_h t'_h = \sum_{h=1}^{m} b_h t'_h = t'_i
\]

i.e., \( F(s'_i) = t'_i \) holds for all \( i \in [0, n) \).

From linear algebra we know that \( F \) can be extended to an orthogonal function \( F' : \mathbb{R}^k \rightarrow \mathbb{R}^k \). Consider the matrix \( M' \in \mathcal{MO}(k) \) such that

\[
F'(v) = w \quad \iff \quad M'v = w
\]

is true for all \( v \in \mathbb{R}^k \). In particular, \( M's'_i = t'_i \) is true for all \( i \in [0, n) \), and thus \( M' \cdot S' = T' \). Hence,

\[
T - t_0 = T' = M' \cdot S' = M' \cdot (S - s_0) = M' \cdot S - s_0,
\]

and therefore \( T = M' \cdot S + (t_0 - s_0) \).
of Lemma 4.18
Let $S, T \in \mathcal{T}$ with $\#S = m \leq n = \#T$ and choose $i \in [0, n-m]$ such that $d_p(S, T_i^m) = d_p(S, T)$. Then, the desired inequality follows:

$$d_\| \|_p^\Delta (S, T) \leq d_\| \|_p^\Delta (S, T_i^m) \leq C(m) \cdot d_p(S, T_i^m) = C(m) \cdot d_p(S, T)$$

\[ \Box \]

of Lemma 4.19
First consider the case where $\#S = \#T = n$. Choose $j^*$ such that the $j^*$-th column of $\Delta S - \Delta T$ has the maximum sum, i.e.,

$$\sum_{i=0}^{n-1-j^*} |d_2(s_i, s_{i+j^*}) - d_2(t_i, t_{i+j^*})| = d_m^\Delta(S, T).$$

Then, for $d := d_2$ we have

$$d_m^\Delta(S, T) = \sum_{i=0}^{n-1-j^*} |d(s_i, s_{i+j^*}) - d(t_i, t_{i+j^*})| \leq \sum_{i=0}^{n-1-j^*} (d(s_i, t_i) + d(s_{i+j^*}, t_{i+j^*})) \leq 2 \cdot \sum_{i=0}^{n-1} d(s_i, t_i) = 2 \cdot d_1(S, T)$$

For $S, T \in \mathcal{T}$ of different lengths, the inequality follows using Lemma 4.18 and $C(n) := 2$. \[ \Box \]

of Lemma 4.20
First, assume $\#S = \#T = n$. Using $\|u + v\|_1 = \|u\|_1 + \|v\|$ for vectors $u, v \in \mathbb{R}^k$ with non-negative entries only we get the inequality:

$$d_1^\Delta(S, T) \leq \left\| (d(s_i, t_i) + d(s_{i+j}, t_{i+j}))_{0 \leq i < n-1, 1 \leq j < n-1} \right\|_1$$

$$= \left\| (n-i-1) \cdot (d(s_i, t_i))_{0 \leq i < n-1} \right\|_1 + \left\| (d(s_{i+j}, t_{i+j}))_{0 \leq i < n-1, 1 \leq j < n-1} \right\|_1$$

$$= \left\| ((n-i-1) \cdot d(s_i, t_i))_{0 \leq i < n-1} \right\|_1 + \left\| (i \cdot d(s_i, t_i))_{0 \leq i < n-1} \right\|_1$$

$$= \left\| ((n-1) \cdot d(s_i, t_i))_{0 \leq i < n} \right\|_1 = (n-1) \cdot d_1(S, T)$$

For the first inequation we used the triangle inequality and symmetry of $d$:

$$d(s_i, s_{i+j}) \leq d(s_i, t_i) + d(t_i, t_{i+j}) + d(t_{i+j}, s_{i+j})$$

$$\implies |d(s_i, s_{i+j}) - d(t_i, t_{i+j})| \leq d(s_i, t_i) + d(s_{i+j}, t_{i+j})$$

Considering $S, T \in \mathcal{T}$ with $\#S = m > n = \#T$, the inequality follows using Lemma 4.18 and $C(n) := n-1$. \[ \Box \]
Figure 6: Two time series $S$ and $T_{\varepsilon}$ in $\mathbb{R}^2$, such that $\frac{d_C(S,T_{\varepsilon})}{dP(S,T_{\varepsilon})} \xrightarrow{\varepsilon \to 0} +\infty$.

**Claim 2.** Consider $\mathcal{M} = \mathbb{R}^2$, $d = d_2$, and let $\varepsilon > 0$, $a := \sqrt{1 - \varepsilon^2}$. Then for $S, T_{\varepsilon} \in T_3$ with

$$S := \begin{pmatrix} -a \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

and $T_{\varepsilon} := \begin{pmatrix} -a \\ \varepsilon \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$

the following inequality holds:

$$\delta_C(S,T_{\varepsilon}) \geq \frac{\varepsilon}{2}.$$

**Proof.**

Denote $S = (s_0, s_1, s_2)$ and $T_{\varepsilon} = (t_0, t_1, t_2)$.

Assume that $M \in \mathcal{MO}(k)$ and $v_0 \in \mathcal{M}$ exist such that

$$\delta(S, M \cdot T_{\varepsilon} + v_0) < \frac{\varepsilon}{2}.$$

Then, considering the linear function

$$\iota : \mathcal{M} \longrightarrow \mathcal{M},
\quad v \longmapsto M \cdot v + v_0,$$

the following inequality must hold for each $i \in [0, 2]$:

$$d(s_i, \iota(t_i)) < \frac{\varepsilon}{2}.$$

Since $\iota$ is an isometric function and $\frac{1}{2}s_0 + \frac{1}{2}s_2 - s_1 = 0$, we have

$$d\left(s_1, \frac{1}{2}(t_0) + \frac{1}{2}t_2\right) = \left\| \frac{1}{2}\iota(t_0) + \frac{1}{2}\iota(t_2) - s_1 \right\|_2
\leq \frac{1}{2}d(\iota(t_0), s_0) + \frac{1}{2}d(\iota(t_2), s_2)
< \frac{\varepsilon}{2},$$

i.e., we obtain a contradiction with

$$d(s_1, \iota(t_1)) \geq d\left(\iota(t_1), \iota\left(\frac{1}{2}(t_0 + t_2)\right)\right) - d\left(s_1, \iota\left(\frac{1}{2}(t_0 + t_2)\right)\right) \geq \varepsilon - \frac{\varepsilon}{2} > \frac{\varepsilon}{2}.$$

Hence, $\delta_C(S, T_{\varepsilon}) = \inf_{M \in \mathcal{MO}(k), v_0 \in \mathcal{M}} \delta(S, M \cdot T_{\varepsilon} + v_0) \geq \frac{\varepsilon}{2}$. \qed

**B Details Omitted in Section 5**

*of Theorem 5.4*

The fiddly part of the calculation are the following inequalities for constant $k \in \mathbb{N}$:

$$|\{j \in [1,k] | j \in 2^N\}| = \left|\left\{2^0, 2^1, \ldots, 2^{\lfloor \log(k-1) \rfloor}\right\}\right| = 1 + |\log(k-1)|$$
For the second inequation in (10), the numbers $i_l$ have to be chosen properly.

\[
\left| \{ i + j = k \mid j \in 2^N, i \geq 0 \} \right| = \left| \bigcup_{l=0}^{\log k} \{ i_l + 2^l \} \right|
\]
\[
= 1 + \lfloor \log k \rfloor
\]

First, consider $S,T \in \mathcal{T}$ of the same length $\#S = n = \#T$. Then, the desired inequality follows with the following calculation and subsequently applying Lemma 4.18:

\[
d_1^\delta(S,T)
= \| \delta S - \delta T \|_1
= \left\| (d(s_i, s_{i+j}) - d(t_i, t_{i+j}))_{i \in [0,n-1], j \in [1,n-i], j \in 2^N} \right\|
\leq \left\| (d(s_i, t_i))_{i \in [0,n-1], j \in [1,n-i], j \in 2^N} \right\|
+ \left\| (d(s_{i+j}, t_{i+j}))_{i \in [0,n-1], j \in [1,n-i], j \in 2^N} \right\|
= \left\| ([\log(n - i - 1)] \cdot d(s_i, t_i))_{i \in [0,n-1]} \right\|
+ \left\| ((1 + \lfloor \log k \rfloor) \cdot d(s_k, t_k))_{k \in [1,n]} \right\|
= \lfloor \log(n - 1) \rfloor \cdot d(s_0, t_0)
+ \lfloor (1 + \lfloor \log i \rfloor + \lfloor \log(n - i - 1) \rfloor) \cdot d(s_i, t_i) \rfloor_{i \in [1,n-1]}
\leq (1 + \lfloor \log(n - 1) \rfloor) \cdot d(s_{n-1}, t_{n-1})
\leq [2 \cdot \log(n - 1)] \cdot d_1(S,T)
\]

Using Lemma 4.18 we can allow $S,T$ to be of different lengths. Since the set of entries in $\delta T$ is a subset of the entries in $\Delta T$, the desired inequality follows analogously as in the proof of Theorem 4.15 by minimizing $d_1(S, M \cdot T + v_0)$. \qed