Chapter 1

Heterotic (0,2) Gepner Models and Related Geometries

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On the sad occasion of contributing to the memorial volume “Fundamental Interactions” for my teacher Wolfgang Kummer I decided to recollect and extend some unpublished notes from the mid 90s when I started to build up a string theory group in Vienna under Wolfgang as head of the particle physics group. His extremely supportive attitude was best expressed by his saying that one should let all flowers flourish. I hope that these notes will be useful in particular in view of the current renewed interest in heterotic model building.

The content of this contribution is based on the bridge between exact CFT and geometric techniques that is provided by the orbifold interpretation of simple current modular invariants. After reformulating the Gepner construction in this language I describe the generalization to heterotic (0,2) models and its application to the Geometry/CFT equivalence between Gepner-type and Distler-Kachru models that was proposed by Blumenhagen, Schimmrigk and Wisskirchen. We analyze a series of solutions to the anomaly equations, discuss the issue of mirror symmetry, and use the extended Poincaré polynomial to extend the construction to Landau-Ginzburg models beyond the realm of rational CFTs.

In the appendix we discuss Gepner points in torus orbifolds, which provide further relations to free bosons and free fermions, as well as simple currents in $N = 2$ SCFTs and minimal models.

1.1. Introduction

When a number of different constructions for heterotic string compactifications were developed in the late 1980s it soon became clear from the coincidence of spectra that Gepner models\(^1\) and Calabi-Yau hypersurfaces in weighted projective spaces\(^2\) should be closely related. The connection\(^3\)

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was found to be provided by Landau-Ginzburg models, whose superpotential \( W(\phi_i) \) can be identified with the hypersurface equation \( W(z_i) = 0 \). A Fermat-type potential of the form \( W = \sum \phi_i^{K_i} \) then corresponds to a Gepner model with levels \( k_i = K_i - 2 \). The precise relation was later derived by Witten by virtue of his \( N = 2 \) supersymmetric gauged linear sigma model (GLSM), which – in addition to the shape parameters (complex structure moduli) in the superpotential \( W \) – contains the size parameters (Kähler moduli) of the Calabi-Yau as D-terms.

The Gepner point thus turns out to be located at small values of the Kähler moduli, way outside the range of validity of sigma model perturbation theory, so that Gepner models provide an exactly solvable CFT stronghold in the realm where strong quantum corrections invalidate any naive geometrical picture. This proved to be useful in many contexts like closed string mirror symmetry, as well as homological mirror symmetry, where, for example, the transport of exact CFT boundary states to D-branes at large volume can be studies.

In the context of perturbative heterotic strings the phenomenological condition of space-time supersymmetry in the RNS formalism implies that the \((0,1)\) superconformal invariance that is left over from the gauge-fixed world-sheet supergravity is extended to a \((0,2)\) superconformal invariance plus quantization of the U(1) charges. This is, in fact, an equivalence, because quantization of the \( N=2 \) superconformal U(1) charge implies locality of the spectral flow operator, which implements the space-time SUSY transformations on the internal CFT part of vertex operators.

In the geometric context \((0,2)\) models correspond to stable holomorphic vector bundles \( V_1 \times V_2 \subset E_8 \times E_8 \) on a Calabi-Yau manifold \( X \) with vanishing first Chern classes satisfying the anomaly cancellation condition \( ch_2(V_1) + ch_2(V_2) = ch_2(TX) \). The notion of a \((2,2)\) model then refers to the choice \( V_1 = TX \) with trivial \( V_2 \), called standard embedding, so that the structure group \( SU(3) \) of \( TX \) breaks \( E_8 \times E_8 \) to the gauge group \( E_6 \times E_8 \) in 4 dimensions. The name \((2,2)\) originates from the CFT analog of this situation where we replace the compactification manifold by an abstract \( N = (2,2) \) left-right symmetric superconformal field theory with central charge \( c = 9 \). This “internal sector” is combined with the 4 space-time coordinates \( X^\mu \) and their right-moving superpartners \( \bar{\psi}^\mu(\bar{z}) \), augmented by a left-moving \( SO(10) \times E_8 \) current algebra, whose central charge 13 adds up with 4 non-compact dimensions and the internal \( c = 9 \) to the critical

\( a \) The \((0,1)\) superconformal invariance is hence required for a consistent coupling to the superghosts in BRST quantization.
dimension 26 of the bosonic string.\(^b\) The same spectral flow mechanism that generates space-time SUSY in the right-moving sector then extends the manifest $SO(10)$ times the $U(1)$ of the $N = 2$ superconformal algebra to the low energy $E_6$ gauge symmetry of the standard embedding.\(^c\) In the geometric context this amounts to the GSO projection. For a general internal $N = 2$ SCFT with fractional charges it has to be augmented by charge quantization and is then referred to as “generalized GSO projection”.

While the general $(0,2)$ models have better phenomenological prospects, like featuring the more realistic GUT gauge groups $SO(10)$ and $SU(5)$, the $(2,2)$ case has been studied much more systematically. In the realm of $\sigma$ models one reason for this was the discovery of world sheet instanton corrections,\(^9,10\) which were believed to destabilize the vacua. A criterion for avoiding this problem was soon found by Distler and Greene;\(^11\) see also.\(^12–14\) The technical difficulty of checking the ‘splitting type’ of the stable vector bundles, however, provided a powerful deterrent for further progress. The situation became much more secure with Witten’s gauged linear sigma models,\(^4\) the $(0,2)$ version of which was used by Distler and Kachru\(^15,16\) to generalize the construction introduced by Distler and Greene.\(^11\) The resulting class of models is now believed to define honest $(0,2)$ SCFTs at the infrared fixed point.\(^17\) Somewhat ironically, with the recognition of the importance of moduli stabilization for model building, world-sheet instantons can turn from an obstacle into a virtue, and one now has to work quite hard\(^18\) to circumvent the cancellation mechanism that has been established for toric Calabi-Yau complete intersections by Beasley and Witten.\(^19\) There is also much recent work on generalizations like heterotic M-theory,\(^20\) and heterotic compactification with H-field background flux,\(^21\) but this is beyond the scope of the present note.

In the realm of exact methods a powerful generalization of Gepner’s construction\(^1\) was found by Schellekens and Yankielowicz,\(^22\) who used simple currents\(^23\) to produce a telephone book of $(1,2)$ models\(^24\) from tensor products of minimal models. For the $(0,2)$ case their huge list of models apparently was so far from complete that it never was published. At the same time closely related methods were used by Font et al.\(^25,26\) to construct pseudo-realistic models. On the CFT side the main problem is the arbitrariness in the selection of a reasonable subset from the huge set of

\(^b\) The value $c = 9$ corresponds to 6 compact dimensions $X^i$ plus the contribution from their right-moving fermionic superpartners $\psi^i$.

\(^c\) More precisely, the mechanisms are mapped to one another by the bosonic string map and its inverse, the Gepner map, respectively (see below).
available models. A landmark in this effort was Schellekens’ theorem on
the conditions for the possibility of avoiding fraction electric charges.$^{27,28}$

An interesting question is, of course, to what extent the geometric and
the CFT approaches to $(0,2)$ models overlap. The identification of models
that are accessible to both constructions would provide further evidence
for the stability of the $\sigma$ model constructions, but most importantly allows
to explore deformations of the rational models, which only live at certain
points in moduli space. Originally based on a stochastic computer search for
matching particle spectra Blumenhagen et al.$^{29,30}$ proposed a set of gauge
bundle data on a complete intersection that is conjectured to describe the
moduli space of a rational superconformal $(0,2)$ cousin of the Gepner model
on the quintic. Using the classification of simple current modular invar-
nants$^{31}$ the product invariant that these authors employ can be translated
into the canonical form$^{32}$ that exhibits its relation to orbifold twists and
discrete torsion.$^{33}$ It turns out that the breaking of the gauge group from $E_6$ to $SO(10)$ is due to a certain twist of order 4 that acts on a minimal
model factor of the internal conformal field theory (at odd level) and on
an $SO(2)$ that is part of the linearly realized $SO(2) \times SO(8) \subset SO(10)$
 gauge symmetry.$^{32}$ Assuming that the $\mathbb{Z}_4$ breaking mechanism does not
care about the rest of the conformal field theory and only acts on a Fermat
factor of a non-degenerate potential we analysed the anomaly matching
conditions and proposed a whole series of identifications$^{32}$ that provides us
with 3219 models, based on the list of 7555 weights for transverse hyper-
surfaces in weighted projective spaces,$^{34,35}$ and many more if we combine
this with other constructions like orbifolding and discrete torsion.$^{36-38}$

The purpose of this note is to collect the necessary ingredients for these
constructions, where the concept of the extended Poincaré polynomial$^{39}$ is
used to generalize the CFT approach to Landau-Ginzburg models beyond
the exactly solvable case. In section 1.2 we discuss simple current modular
invariants (SCMI)$^{23}$ and their geometric interpretation.$^{31}$ To set up the
concepts we begin with recalling the geometric orbifolding idea and use it
to motivate and interpret the formula for the most general invariant. In
section 1.3 we use simple current techniques for the implementation of the
generalized GSO projection and show how the Gepner construction gen-
eralizes to $(0,2)$ models in general and, in particular, for gauge symmetry
breaking in the proposed $\sigma$ model connection. We discuss the counting of
non-singlet spectra in terms of the information encoded in the extended
Poincaré polynomial, thus extending the scope of the construction to arbi-
trary Landau-Ginzburg orbifolds. Since the charge conjugation of $N = 2$
minimal models is a simple current modular invariant, our discussion explains the observed (0,2) mirror symmetry\(^{40,41}\) along the lines of Greene-Plesser orbifolds and their generalization due to Berglund and Hübsch,\(^{42}\) which applies to the large class of transversal potentials that are minimal in a certain sense.\(^{43}\) Section 1.4 briefly recollects the geometric side of the proposed identifications. Here we start with an ansatz for the base manifold and vector bundle data that are conjectured to describe the moduli spaces of the (0,2) models and find a unique solution to the anomaly equations. In section 1.5 we conclude with a number of topics for generalizations and further studies.

Some technical points are discussed in appendices. In appendix A we discuss Gepner points of torus orbifolds and exact CFT realizations for the extensions of \(\mathbb{Z}_2 \times \mathbb{Z}_2\) orbifolds recently classified by Donagi and Wendl-\(\ldots\)\(^{44}\) Appendix B discusses simple currents in \(N = 2\) SCFTs and their use for explaining labels and field identifications of \(N = 2\) minimal models.

1.2. Orbifolds and simple currents

The concept of an orbifold CFT originates from the geometric picture of closed strings on orbit spaces \(X/G\) where \(X\) is a smooth manifold with a discrete group action \(G\), with or without fixed points. The modding out of \(G\) has two consequences: String states on the orbifold need to be invariant under the symmetry on the covering space \(X\), which leads to a projection of the Hilbert space \(\mathcal{H}_X\) on the covering space to \(G\)-invariant states. On the other hand, new closed string states emerge, whose \(2\pi\)-periodicity on \(X/G\) corresponds to periodicity up to a group transformation \(g \in G\) on \(X\). The Hilbert space has hence to be augmented by twisted sectors \(\mathcal{H}_X^{(g)}\).

1.2.1. Orbifold CFT and modular invariance

For abstract conformal field theories \(\mathcal{C}\) that are invariant under a group \(G\) of symmetry transformations the same result can be derived from modular invariance and factorization constraints on the partition function without relying on a geometric interpretation. Depicting the one-loop partition function by a torus that indicates the double-periodic boundary conditions imposed in the path integral, \(Z_\mathcal{C} = \square\), the orbifold partition function can be obtained as a linear combination of partition functions with boundary
conditions twisted by group transformations \(g\) and \(h\),

\[
Z_C(g, h) = g \begin{array}{c} \hline \end{array} h
\]  

(1.1)
in the vertical and horizontal direction, respectively. If we interpret the horizontal direction as the spacial extension of a closed string and the vertical direction as Euclidean time then \(h\) amounts to twisted boundary conditions in the Hilbert space, while a normalized sum over twisted boundary conditions in periodic Euclidean time can be shown to be equivalent to a projector

\[
\Pi_G = \frac{1}{|G|} \sum_{g \in G} g \begin{array}{c} \hline \end{array} g^* 
\]  

(1.2)
onto \(G\)-invariant states. Under modular transformations

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})
\]  

(1.3)
boundary conditions are recombined: For the generators

\[
S : \tau \rightarrow -1/\tau, \quad T : \tau \rightarrow \tau + 1
\]  

(1.4)
of \text{PSL}(2, \mathbb{Z}) we observe

\[
S : g \begin{array}{c} \hline \end{array} h \rightarrow h^{-1} \begin{array}{c} \hline \end{array} g, \quad T : g \begin{array}{c} \hline \end{array} h \rightarrow gh \begin{array}{c} \hline \end{array} h
\]  

(1.5)
where \(T\) maps the double-periodicity \((1, \tau)\) to \((1, \tau + 1)\) and the action of \(S\) has been chosen as \((1, \tau) \rightarrow (\tau, -1)\).\(^d\) The double-periodicities are consistently defined only if \(g\) and \(h\) commute so that we need to restrict to twists obeying \(gh = hg\) in the case of non-abelian groups.

Since modular transformations mix up all twists of the periodicities along the homology cycles we expect an invariant to contain contributions from all combinations and it is easy to see that the simplest invariant solution is

\[
Z_{C/G} \equiv \frac{1}{|G|} \sum_{gh = hg} g \begin{array}{c} \hline \end{array} h
\]  

(1.6)
In the abelian case the sum over \(h\) corresponds to a sum over all twisted sectors. The sum over \(g\) then implements the projection onto invariant states; in accord with (1.2) the normalization ensures that the (invariant)

\(^d\) While \(S^2 = (ST)^3 = 1 \in \text{PSL}(2, \mathbb{Z})\) for modular group elements, the action of \(S^2 = (ST)^3 : (1, \tau) \rightarrow (-1, -\tau)\) on the world sheet amounts to parity plus time reversal. Due to CPT invariance the action of \(S\) on the Hilbert space thus squares to a charge conjugation \(S^2 = (ST)^4 = C\) of the conformal fields.
ground state contributes to the partition function with multiplicity one. Our CFT result thus coincides with what we expect for closed strings on orbifolds $X/G$. But there might be further solutions.

1.2.2. Discrete torsion and quantum symmetries

Let us start with the more general ansatz

$$Z_{C/G}^\varepsilon = \frac{1}{|G|} \sum_{gh=hg} \varepsilon(g, h) \, g \, h$$

(1.7)

with weight $\varepsilon(g, h)$ for the $(g, h)$–twisted contribution. This modification can also be motivated from geometry and is called “discrete torsion” because it is related to phase factors $\varepsilon(g, h)$ due to $B$-field flux with only “discrete” values allowed by $G$-invariance (the field strength $H = dB$ of the 2-form $B$ determines the “torsion” of the corresponding sigma model). With an analysis of the modular invariance and factorization constraints Vafa has shown that

$$\varepsilon(g, g) = \varepsilon(g, h) \varepsilon(h, g) = 1, \quad \varepsilon(g_1 g_2, h) = \varepsilon(g_1, h) \varepsilon(g_2, h).$$

(1.8)

Mathematically discrete torsion corresponds to an element of the group cohomology $H^2(G, U(1))$. For abelian groups $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$ with generators $g_i$ the most general solution is parametrized by an arbitrary choice of the phases $\varepsilon(g_i, g_j)$ for $i < j$ obeying $\varepsilon(g_i, g_j) \gcd(n_i, n_j) = 1$.

The ambiguity of the orbifold CFT that is due to discrete torsion is quite easy to understand in the operator picture because the group action is originally defined only in the untwisted sector. For the twisted sectors we do know the group action on (untwisted) operators but the action on the twisted ground states (and on the corresponding twist fields) is a priori subject to a choice. We can thus think of $\varepsilon(g, h)$ as an extra phase of the group action of $g$ in the $h$-twisted sector.

While the symmetry of the original CFT is lost after orbifolding because of the projection to invariant states, a new symmetry emerges due to selection rules for operator products of twist fields $\Sigma_{h_1}(z) \Sigma_{h_2}(w)$, to which

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$^6$ With the restriction to $gh = hg$ the formula also applies to the non-abelian case, where the sum can be interpreted to extend over conjugacy classes of twists followed by a projection onto states that are invariant under the respective normalizers.

$^7$ On a genus $n$ surface the partition function depends on $2n$ twists along homology cycles, with a corresponding prefactor $\varepsilon(g_1, g_2; \ldots; g_{2n-1}, g_{2n})$ that has to factorize into $\varepsilon(g_1, g_2) \ldots \varepsilon(g_{2n-1}, g_{2n})$. The only condition in the analysis that has to be used beyond the torus is a Dehn twist at genus 2.
we only expect contributions of fields twisted by $h_1 h_2$. The corresponding symmetry of the orbifold has been called quantum symmetry.\footnote{More generally, we can consider arbitrary \( N=2 \) SCFTs for which mirror symmetry, i.e. right-moving charge conjugation, is equivalent to an orbifold.\footnote{This is the case for the large class of Landau-Ginzburg models for which a transversal potential exists whose number of monomials is equal to the number of fields,\footnote{as was discovered by Berglund and Hübsch.}} In the abelian case the quantum symmetry is dual of the original symmetry. Modding out the quantum symmetry of a $\mathbb{Z}_n$-orbifold just gives us back the original CFT.\footnote{If we mod out two commuting group actions $\langle g_1, g_2 \rangle$ in two steps then the freedom due to discrete torsion $\varepsilon(g_2, g_1)$ can be recovered by combining the group action $g_2$ of the second orbifolding with an appropriate power of the quantum symmetry $q_1$ that emerges from the $g_1$-twist in the first orbifold. These ideas can be used to extend the Green-Plesser mirror construction of Gepner models to arbitrary orbifolds with discrete torsion.\footnote{These ideas can be used to extend the Green-Plesser mirror construction of Gepner models to arbitrary orbifolds with discrete torsion.}} If we mod out two commuting group actions $\langle g_1, g_2 \rangle$ in two steps then the freedom due to discrete torsion $\varepsilon(g_2, g_1)$ can be recovered by combining the group action $g_2$ of the second orbifolding with an appropriate power of the quantum symmetry $q_1$ that emerges from the $g_1$-twist in the first orbifold. These ideas can be used to extend the Green-Plesser mirror construction of Gepner models to arbitrary orbifolds with discrete torsion.\footnote{These ideas can be used to extend the Green-Plesser mirror construction of Gepner models to arbitrary orbifolds with discrete torsion.}

\subsection*{1.2.3. Simple currents}

Simple currents are, in a sense, generalized free fields in rational conformal field theories. For free bosons there is a shift symmetry. When it is used for orbifolding the twisted sectors correspond to winding states. For free fermions a $\mathbb{Z}_2$ symmetry is provided by the fermion number. In this case the twisted sector is the Ramond sector, with a cut in the punctured complex plane, and the projection to invariant states is the GSO projection. Simple currents, as we will see, also come with discrete symmetries. Accordingly, they can be used to construct new conformal field theories, which turn out to be given in terms of the original characters but with a certain type of non-diagonal modular invariants.

We consider a rational conformal field theory, i.e. a CFT with left- and right-moving chiral algebras $A_L$ and $A_R$ such that the conformal fields are combined into a finite number of representations $\phi_{i\ell k}$ of $A_L \otimes A_R$, where $i$ labels the representation of $A_L$. The chiral algebras contain the Virasoro algebra and possibly more. We may use the highest weight state, or primary field, in a conformal family as its representative. It is important, however, to keep in mind that the conformal weight $h_i$ is well-defined for a primary field, but only defined modulo 1 for the conformal family.

The fusion algebra $\phi_i \times \phi_j = \mathcal{N}_{ij}^k \phi_k$ of a rational CFT is the commutative associative algebra whose non-negative integral structure constants $\mathcal{N}_{ij}^k$ encode the fusion rules, i.e. the information of which representations
of the chiral algebra appear in operator product expansions $\phi_i(z)\phi_j(w)$. A simple current $J$ of a conformal field theory is a primary field that has a unique fusion product with all other primary fields, i.e.

$$J \times \phi_j = \phi(Jj), \quad j \to Jj \to J^2j \to J^3j \ldots,$$

where we use the notation $Jj$ for the label of the fusion product of $J$ and $\phi_j$. A simple current thus decomposes the field content of the CFT into orbits, which have finite length for a rational theory.

Since the OPE $J(z)\phi_j(w)$ contains only fields from a single conformal family, whose conformal weights can only differ by integers, all expansion coefficients $(z - w)^{h_{Jj} - h_J - h_j}$ have the same monodromy $e^{-2\pi i Q_J(\phi)}$ with

$$Q_J(\phi) \equiv h_J + h_j - h_{Jj} \mod 1 \quad (1.10)$$

about the expansion point $w$. The monodromy of $J(z)$ for a big circle about the positions of $\phi_j(w_j)$ and $\phi_k(w_k)$ is the product of the two respective monodromies. Thus the phase transformation $e^{-2\pi i Q_J}$ is compatible with operator products and defines a symmetry of the CFT. Before we come to the resulting orbifold CFTs, which correspond to the simple current modular invariants, we need to collect some basic definitions and facts for simple currents.

The order $N_J$ of a simple current $J$ is the length of the orbit of the identity $J^{N_J} = 1$. Because of associativity and commutativity of the fusion product the simple currents of a CFT form an abelian group $C$, which is called the center. The definition of the monodromy charge implies

$$Q_{J \times K}(\phi) \equiv Q_J(K \phi) - Q_J(K) + Q_K(\phi) \mod 1,$$

so that

$$Q_{J \times K}(\phi) \equiv Q_J(\phi) + Q_K(\phi), \quad Q_{J^m}(\phi) \equiv nQ_J(\phi). \quad (1.11)$$

$Q_J(\phi)$ is hence a multiple of $1/N_J$. It can be shown that the charge quantum of $Q_J$ is indeed $1/N_J$, so that a simple current $J$ always comes with a discrete $\mathbb{Z}_{N_J}$ phase symmetry of the CFT (not every cyclic symmetry is generated by a simple current, though). The symbol $\equiv$ henceforth denotes equality modulo integers.

For the orbifolding of a CFT we may choose to mod out some subgroup of its full symmetry group. Similarly, we now choose some fixed subgroup $G$ of the center $C$ of a CFT that is generated by independent simple currents $J_i$ of orders $N_i$. We use the notation $[\alpha] = \prod J_i^{\alpha_i}$ and $Q_i = Q_{J_i}$, where $\alpha_i$ are integers that are defined modulo $N_i$. Then we can parametrize the

\[ N_{ij}^k \geq 1 \] indicate contributions from descendents in OPEs beyond the coefficients that are implied by the Ward identities of the chiral algebra.
conformal weights and monodromy charges of all simple currents in $\mathcal{G}$ in terms of a matrix $R_{ij}$,

$$R_{ij} = \frac{r_{ij}}{N_i} \equiv Q_i(J_j) = Q_j(J_i), \quad h_{[\alpha]} = \frac{1}{2} \sum_i r_{ii} \alpha^i - \frac{1}{2} \sum_{ij} \alpha^i R_{ij} \alpha^j \quad (1.12)$$

with $r_{ij} \in \mathbb{Z}$. If $N_i$ is odd we can always choose $r_{ii}$ to be even. With this convention all diagonal elements $R_{ii}$ are defined modulo 2 for both, even and odd $N_i$.

Using the definitions of $Q$ and $R$ we obtain

$$h_{[\alpha]} \phi = h_{\phi} + h_{[\alpha]} - \alpha^i Q_i(\phi), \quad Q_i([\alpha] \phi) \equiv Q_i(\phi) + R_{ij} \alpha^j. \quad (1.13)$$

It can be shown that $S$ matrix elements for fields on the same orbits are related by phases,

$$S_{[\alpha]|\beta] \Psi = S_{\phi, \Psi} e^{2\pi i (\alpha^i Q_i(\Psi) + \beta^i Q_i(\phi) + \alpha^k R_{ki} \beta^i)}. \quad (1.14)$$

$T$-matrix elements only depend on conformal weights and, according to eq. (1.13), are related by phases $2\pi i (h_{[\alpha]} - \alpha^i Q_i(\phi) - h_{[\beta]} + \beta^i Q_i(\Psi))$.

### 1.2.4. Simple current modular invariants and chiral algebras

The partition function of a rational CFT can be written as

$$Z(\tau) = \text{Tr} e^{2\pi i L_0} e^{-2\pi i \bar{L}_0} = \sum_{ij} M_{ij} \chi_i(\tau) \bar{\chi}_j(\tau) \quad (1.15)$$

with a non-negative integral matrix $M_{ij}$ that is called a modular invariant if

$$[M, S] = [M, T] = 0 \quad \text{and} \quad M_{11} = 1 \quad (1.16)$$

since under modular transformations $\chi_i(-1/\tau) = S_{ij} \chi_j(\tau)$ and $\chi_i(\tau + 1) = T_{ij} \chi_j(\tau)$ so that $M \to S^T MS^*$ and $M \to T^T MT^*$ with symmetric unitary matrices $S$ and $T$, respectively. Modular invariants of automorphism type are permutation matrices that uniquely map representation labels of the left movers to right movers, where the permutation is an automorphism of the fusion rules. Extension-type invariants, on the other hand, combine contributions of several characters to characters of extended chiral algebras while other representations of the original chiral algebra are projected out.

Simple current modular invariants (SCMIs) are modular invariants for which $M_{jk} \neq 0$ only if $\phi_j$ and $\phi_k$ are on the same orbit, i.e. if $k = J_j$ for some simple current $J \in \mathcal{C}$. $T$-invariance requires that $h_j - h_k \in \mathbb{Z}$, and is hence also called “level matching”. Using eq. (1.13), with the above notation $[\alpha] = \prod J_i^{\alpha_i} \in \mathcal{G} \subseteq \mathcal{C}$, we thus find the condition that

$$h_j - h_{[\alpha]} \equiv \alpha^i Q_i(\phi_j) - h_{[\alpha]} \in \mathbb{Z} \quad (1.17)$$

It is easiest to first compute $R_{ij} \equiv Q_i(J_j)$ modulo 1 and then fix $R_{ij}$ modulo 2 for the diagonal elements with even $N_i$ by imposing that formula (1.12) for $h(J_i)$ has to hold.
must be an integer. If the order $N_i$ of $J_i$ is even then eq. (1.17) implies that the twist $J_i$ (like any odd power of $J_i$) can contribute to a modular invariant only if $r_{ii} = N_i R_{ii} \in 2\mathbb{Z}$. We henceforth assume that all generators of $\mathcal{G}$ satisfy this condition.

If we think of $[\alpha]$ as the twist in the orbifolding procedure, which is in accord with the number $|\mathcal{G}|$ of twisted sectors as well as with the expected quantum symmetry due to twist selection rules, it is not difficult to guess that the SCMI should impose a projection $\delta_Z(Q_i + X_{ij}\alpha^j)$ where $\delta_Z$ is one for integers and zero otherwise. The linear ansatz $X_{ij} \alpha^j$ for the phase shift in the projection is suggested by comparing eq. (1.17) with $h_{[\alpha]} \equiv -\frac{1}{2} \alpha^i R_{ij} \alpha^j$ and by the expected quantum symmetry. Using regularity assumptions it can be shown that the most general SCMI reads

$$M_{\phi,[\alpha]\phi} = \mu(\phi) \prod_i \delta_Z \left( Q_i(\phi) + X_{ij}\alpha^j \right), \quad (1.18)$$

where $T$-invariance implies $X + X^T \equiv R$ modulo 1 for off-diagonal and modulo 2 for diagonal matrix elements, $X$ is quantized by $	ext{gcd}(N_i, N_j) X_{ij} \in \mathbb{Z}$, and $\mu(\phi)$ denotes the multiplicity of the primary field $\phi$ on its orbit, i.e. $\mu(\phi) = |\mathcal{G}|/|\mathcal{G}_\phi|$ where $|\mathcal{G}_\phi|$ is the size of the orbit of the action of $\mathcal{G}$ on $\phi$. While the symmetric part $X_{(ij)} \equiv \frac{1}{2} R_{ij} \equiv X_{ij} - \frac{1}{2} X_{ij}$ is fixed by level matching, the ambiguity due to the choice of a properly quantized antisymmetric part $E_{ij} \equiv X_{ij} - \frac{1}{2} X_{ij}$ corresponds to the discrete torsion of the orbifolding procedure.

We can now briefly discuss different types of invariants. If $X = 0$ we have a pure extension invariant because all fields with non-integral charges are projected out while all fields on a simple current orbit are combined to new conformal families. $X = 0$ is only possible if the conformal weights of all simple currents $J \in \mathcal{G}$ are integral and since these currents are in the orbit of the identity they extend the chiral algebras $\mathcal{A}_L$ and $\mathcal{A}_R$ so that we obtain a new rational symmetric and diagonal CFT.

Let us define the kernel $\text{Ker}_X$ as the set of integral solutions $[\alpha]$ of $X_{ij} \alpha^j \in \mathbb{Z}$ with $\alpha_j$ defined modulo $N_j$. If this kernel is trivial then $(Q_i(\phi) + X_{ij} \alpha^j) \in \mathbb{Z}$ has a unique solution $[\alpha]$ for each charge, which defines a unique position $[\alpha] \phi$ on the orbit that only depends on the charge $Q_i(\phi)$ of $\phi$. We then obtain an automorphism invariant. In general, the extension of the right-moving chiral algebra $\mathcal{A}_R$ is give by the kernel $\text{Ker}_X$.

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1 The maximal subgroup of $\mathcal{C}$ that can contribute to a SCMI is called “effective center”.

2 'Regularity' requires that $M_{\phi,[\alpha]\phi}$ only depends on $Q_i(\phi)$, $^{47}$ Discrete Fourier sum and 2-loop modular invariance imply that the ‘phases’ are bilinear and antisymmetric. $^{31}$
and, since
\[ M_{[\alpha|\phi,\phi]} = \mu(\phi) \prod_i \delta_\mathbb{Z} \left( Q_i(\phi) + \alpha^j X_{ji} \right), \] (1.19)
the extension of the left-moving chiral algebra \( A_L \) is given by the kernel \( \text{Ker}_\mathbb{Z} X^T \) of the transposed matrix. While the extensions are of the same size, they need not be isomorphic. For example, an extension of \( A_R \) by \( \mathbb{Z}_9 \) can occur together with an extension of \( A_L \) by \( \mathbb{Z}_3 \times \mathbb{Z}_3 \).

1.3. Gepner-type (0,2) models

The right-moving sector of a heterotic string consists of four space-time coordinates and their superpartners \((X^\mu, \psi^\mu)\), a ghost plus superghost system \((b, c, \beta, \gamma)\), and a supersymmetric sigma model on a Calabi-Yau, whose abstract version is an \( N = 2, c = 9 \) SCFT \( C_{\text{int}} \). Equivalently, we can use lightcone gauge, which amounts to ignoring the ghosts and restricting space-time indices to transverse directions. The left-moving sector is a bosonic string with space-time plus ghost part \((X^\mu, b, c)\) and the same internal sector \( C_{\text{int}} \) with \( c = 9 \), whose central charges add up to \( 4 + 9 - 26 = -13 \) so that we need to add a left-moving CFT with central charge 13 for criticality. Modular invariance requires this CFT to be either an \( E_8 \times SO(10) \) or \( SO(26) \) level 1 affine Lie algebra (we will henceforth ignore the \( SO(26) \) case because it is phenomenologically less attractive). In the geometric context of a sigma model on a Calabi-Yau the superstring vacuum is then obtained by aligning space-time spinors and tensors with internal Ramond and Neveu-Schwarz sectors, respectively, and performing the GSO projection. For abstract \( N = 2 \) SCFTs \( U(1) \) charges may be quantized in fractional units so that, in addition, a projection to integral charges (generalized GSO) is required for space-time supersymmetry.

All of these operations can be understood as SCMIAs of extension type.\(^{22,39}\) To see this let us first discuss the simple currents in the relevant CFTs. For the \( D_n \cong SO(2n) \) current algebra the center \( C_n \) has order 4 and consists of the spinor representation \( s \), its conjugate \( \bar{s} \), and the vector \( v \) with \( sv = \bar{s}, \ s^2 = \bar{s}^2 = v^n, \ v^2 = 1 \Rightarrow C_n \cong \begin{cases} \mathbb{Z}_4 & \text{for } n \not\in 2\mathbb{Z} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } n \in 2\mathbb{Z} \end{cases} \) (1.20)

The conformal weights and monodromies are
\[ h_s = \frac{n}{8}, \ h_v = \frac{1}{2}, \ R_{s v} = 1, \ R_{s s} = 1/2, \ R_{v v} = \begin{cases} 3n/4 & \text{for } n \not\in 2\mathbb{Z} \\ n/4 & \text{for } n \in 2\mathbb{Z} \end{cases} \] (1.21)
since \( s^2 = v^n \) so that \( N_s = 4 \) for \( n \) odd and \( N_s = 2 \) for \( n \) even.
For the internal $N = 2$ SCFT $C_{\text{int}}$ the center always contains the super-current $J_v$ with $h = 3/2$ and $J_v^2 = 1$ and the spectral flow current $J_s$ with $h = c/24$ and $J_s^2 = J_s^h$ where $c = 3k/M$ and $1/M$ is the charge quantum in the NS sector (see appendix B). The monodromy charge $Q_v$ is 0 in the NS sector and 1/2 in the Ramond sector. $J_s = e^{i\sqrt{c/12}X}$ is the Ramond ground state of maximal $U(1)$ charge $c/6$ and can be written as a vertex operator in terms of the bosonized $U(1)$ current $J(v) = \sqrt{c/3} \partial X(z)$ so that $Q_{J_s} \equiv -\frac{1}{2}Q$ and $Q_{J_s}(J_s) \equiv -c/12$ modulo 1.

1.3.1. The $(2,2)$ case and the generalized GSO projection

In order to apply simple current techniques it is convenient to start from a left-right symmetric theory. This can be achieved by applying the bosonic string map to the right-movers,

$$SO(2)_{LC} \rightarrow D_5 \times E_8, \quad (0, v) \rightarrow (v, 0), \quad (s, \bar{s}) \rightarrow -(\bar{s}, s), \quad (1.22)$$

which maps modular invariant partition functions of heterotic strings to modular invariant partition functions of bosonic strings. The inverse map will be called Gepner map. For simplicity we discuss the spectrum in terms of light-cone space-time $SO(2)_{LC}$ representations rather than using the equivalent $SO(4) \otimes (b, c, \beta, \gamma)$, which would necessitate superghost contributions with the benefit of manifest Lorentz invariance.

Consistent quantization of the gauge fixed $N=1$ supergravity theory requires that the Ramond and NS sectors of the space-time and internal sectors are aligned. After the bosonic string map this implies that $SO(10)$ spinor representations are aligned with the Ramond sector of the internal SCFT. This can be implemented by a SCMI that extends the chiral algebra by the current $J_{RNS} = J_v \otimes v$ (which has conformal weight $h_{RNS} = 2$) because $Q_{J_v} \equiv 1/2$ for Ramond fields and $Q_v \equiv 1/2$ for $SO(10)$ spinors. Similarly, in the case of a Gepner model, where $C_{\text{int}} = C_{k_1} \otimes \ldots \otimes C_{k_l}$ is a tensor product of $N = 2$ SCFTs, the alignment can be implemented as a SCMI extending the chiral algebra by all bilinears of the respective supercurrents $J_{ij} = J_{v_i} J_{v_j}$, where $h_{ij} = 3$. Rather then defining a “superconformal tensor product” with an implicit alignment we keep the alignment procedure explicit because we will later be interested in $(0,2)$ models for which the chiral algebra extension that implements the alignment only takes place in the right-moving sector, where it is needed for consistency.

Space-time supersymmetry now requires that the spectral flow in the internal sector is combined with an $SO(10)$ spin field $s$ after the bosonic string map so that space-time bosons/fermions in the heterotic string have
NS/R contributions from the internal N=2 SCFT. This is implemented by the simple current $J_{GSO} = J_s \otimes s$, which has integral conformal weight $h_{GSO} = c/24 + n/8 = 3/8 + 5/8 = 1$ and hence can be used for a SCMI of extension type. Inspection of the massless spectrum (see below) shows that the $2 \times 16$ states in $(J_{GSO})^{\pm 1}$ together with the $U(1)$ current of the $N = 2$ SCFT lead to the 33 massless vector bosons that extend the $45_{adj}$ of $D_5$ to the $78_{adj}$ of the gauge group $E_8$ that is familiar from the standard embedding $SU(3) \subseteq E_8$. The mechanism that implements space-time SUSY in the fermionic string is hence related by the bosonic string map to the mechanism that extends $E_8 \times SO(10)$ to the gauge group $E_8 \times E_6$ of a (2,2) compactification. Since $Q_{GSO} = -\frac{1}{2}Q$ this “generalized GSO projection” implies a projection to even $U(1)$ charges in the bosonic string and, according to eq. (1.22), to odd $U(1)$ charges in the Gepner construction of the superstring when the space-time contribution is taken into account.

For sigma models on CY manifolds the charges are already quantized in (half)integral units in the (R)NS sector. The standard GSO projection can hence be regarded as a generalized GSO projection with $M = 1$. In order to simplify the comparison between abstract and geometrical constructions of $N = 2$ SCFTs it has been suggested to define an intermediate projection which extends the chiral algebra only by simple currents that have no contributions from the spacetime/gauge sector.$^{48}$ The corresponding subgroup $G_{CY}$ of the center contains all alignment currents of the building blocks of the internal SCFT plus the current $J_{CY} = J_{GSO}^R = J_{GSO}^R B_{E_8}^3$. In order to set up the enumeration of massless states of the heterotic string we recall the relevant vertex operators. On the bosonic side, where the NS vacuum has $h = -1$, there are the universal operators

$$\left( \partial X^\mu \times I_{E_8 \times D_6} + I_{st} \times J^{(E_8 \times D_6)}_{-1/2} \right) \times I_{int} \quad (1.23)$$

and the model-dependent contributions

$$I_{st} \times I_{E_8} \times \sum_{0 \leq r < 4} \left( s \right)^r \times \phi_{int} \quad (1.24)$$

For the right-movers the NS vacuum has $h = -1/2$ and the relevant vertex operators are

$$\sum_{0 \leq r < 4} \left( s \right)^r \times \phi_{int} \quad (1.25)$$

$^{1}$ The discussion in ref. $^{48}$ attempts independence of the space-time dimension $2n = 10 - 2c/3$. Note, however, that standard compactifications on K3’s have internal $N = 4$ SCFTs so that the bosonic analog of $N=2$ space-time SUSY in 6-dimensional (4,4) models is the extension of the gauge group $E_8 \times D_6$ to $E_8 \times F_4$, where the $3 = 133 - 66 - 2 \cdot 32$ $D_6$-singlet gauge bosons come from the $SU(2)$ R-symmetry currents of the $N = 4$ SCFT.
The enumeration of the non-universal states can therefore be organized according to the following data,

\[
\begin{array}{|c|c|c|}
\hline
D_5^{(P)} & h_{\text{int}} & Q_{\text{int}} \\
\hline
0 \rightarrow \Psi^\mu = v & 0 & 0 \\
s \rightarrow \Sigma = \bar{s} & \frac{3}{8} & \frac{3}{2} - \frac{1}{2} \\
v \rightarrow \Pi_a = 0 & \frac{1}{2} & \pm 1 \\
\bar{s} \rightarrow \Sigma = s & \frac{3}{8} & \frac{1}{2} - \frac{3}{2} \\
\hline
\end{array}
\]

where the entries of the “Hodge diamond” are multiplicities of internal fields with (left,right) charges \((Q, \bar{Q})\).

Since spectral flow relates (anti)chiral primary states to Ramond ground states the counting can be performed in any of these sectors, with an appropriate shift of charges. For CY compactifications Hodge duality further implies \(x = y\) where \(y = 1\) corresponds to extended \(N = 2\) space-time SUSY and \(y = 3\) yields \(N = 4\). The bosonic (left-moving) analogs of these extensions are gauge groups \(E_7\) and \(E_8\), respectively. For orbifolds with discrete torsion \(x \neq y\), i.e. any combination of \(E_6, 7, 8\) with \(N = 1, 2, 4\), is possible. The \(h_{12}\) is a complex structure deformations (we call them anti-generations of charged particles) correspond to chiral primary fields with symmetric charges \(Q = \bar{Q} = 1\) while the \(h_{11}\) is \(g\) generations count Kähler moduli, i.e. the CY Hodge diamond is rotated by \(\pi/2\) as compared to the diamond of left/right charge multiplicities of the \(N = 2\) SCFT.

1.3.2. The extended Poincaré polynomial

The aim of the extended Poincaré polynomial (EPP) is to encode all information about an \(N = 2\) superconformal theory that is necessary for computing the (charged) massless spectrum of any tensor product containing this model as one factor. It takes advantage of the fact that the generalized GSO-projection corresponds to an extension invariant so that we may, in a first step, disregard the projection to integral charge in the expression (1.18) and consider the ‘unprojected orbifold’. Eventually, to obtain the projected orbifold, we just have to omit the contributions with non-integral monodromy charges.

The Poincaré polynomial encodes charge degeneracies for \(N = 2\) SCFTs,

\[
P(t, \bar{t}) = \text{tr}_{(c, c)} t^Q \bar{Q} = (t\bar{t})^{g/6} \text{tr}_{R_{gs}} t^Q \bar{Q}, \tag{1.26}
\]
where we assume locality of symmetric spectral flow. In order to be able to combine the information of the factors of a tensor product we need to encode, in addition, information on the twists. We thus define the ‘full extended’ Poincaré polynomial as

\[ P(t, \overline{t}, x, \sigma) = \sum_{l \geq 0} \sum_{k=0}^{x^l \sigma^k} P_{l,k}(t, \overline{t}), \]

where \( P_{l,k}(t, \overline{t}) \) is the Poincaré polynomial of the unprojected sector twisted by \( J^2 l^s J^k v \), i.e. \( P_{l,k} \) is obtained by looking for all Ramond ground states \( \phi_{ij} \) with \( j = J^2 l^s J^k v i \) and the \( U(1) \) charges of \( i \) and \( j \) are encoded by the exponents of \( t \) and \( \overline{t} \), respectively.

For a tensor product with alignment of Ramond/NS sectors we obtain

\[ P(t, \overline{t}, x, \sigma) = \sum_{l \geq 0} x^l \left( \sum_{k=0}^{t^l} P_{l,k}^{(1)}(t, \overline{t}) P_{l,k}^{(2)}(t, \overline{t}) + \sigma \sum_{k=0}^{1} P_{l,k}^{(1)}(t, \overline{t}) P_{l,1-k}^{(2)}(t, \overline{t}) \right) \]

By iteration of this formula we conclude that (1.27) indeed encodes all information from the factor theories of a Gepner model that enters the computation of the charged massless spectrum. In fact, this information is still redundant: Consider a R ground state \( \phi_{ij} \) whose contribution to \( P_{l,k} \) is \( t^{Q+\frac{c}{3}} \overline{t}^{Q+\frac{c}{3}} \). Then eqs. (1.13) and (B.6) imply for the \( U(1) \) charges

\[ \overline{Q} \equiv Q + l c/3 - k \quad \Rightarrow \quad k \equiv Q + l c/3 - \overline{Q} \mod 2. \]

As the exponent of \( \sigma \) is fixed in terms of the other exponents we can set

\[ \sigma \to -1 \quad \Rightarrow \quad P(t, \overline{t}, x) = P(t, \overline{t}, x, -1). \]

The negative sign is convenient for index computations since it implies opposite signs for contributions to generations and anti-generations.\(^{39}\) For minimal models at level \( k = K - 2 \) one finds

\[ P^{(MM)}(t^K, \overline{t}^K, x) = \frac{K-1}{1+(-x)^K-2} (1-(-x)^{K-1}) \]

where the ordinary Poincaré polynomial is \( P(t^K) = \frac{1-t^{K-1}}{1-t} \).

Since the numbers of (anti)chiral primaries and of Ramond ground states are finite also in non-rational SCFTs extended Poincaré polynomials can be defined in a more general context and explicit formulas have been given for Landau-Ginzburg orbifolds.\(^{39}\)

\(^{39}\) In the original definition of the extended Poincaré polynomial\(^{49}\) Schellekens, in addition, puts \( \overline{T} = 1 \). For diagonal theories we have shown\(^{39}\) that, for a given \( Q \), all states contribute with the same sign, so that it is indeed sufficient to drop the \( \overline{Q} \)-dependence in applications to heterotic \((2,2)\) string vacua built from diagonal theories, but not necessarily for orbifolds thereof.
1.3.3. Gauge/SUSY breaking and (0,2) models

While the chiral algebra extension of a SCMI based on $J_{\text{GSO}}$ and alignment currents can be reduced by switching on discrete torsion $X \neq X^T$ this would not only break the left-moving $E_6$ but also the right-moving space-time SUSY of the heterotic string. We hence need to increase the twist group $\mathcal{G}$ at least by one additional generator of even order. While there are many possibilities for this type of models we would always end up with at least $SO(10)$. For smaller gauge groups, like the “exceptional” series $E_5 = D_5 = SO(10)$, $E_4 = A_4 = SU(5)$ and $E_3 = SU(3) \times SU(2)$ that is familiar from geometric/sigma model constructions, we have to start with smaller building blocks and use asymmetric extensions that rebuild the $D_5 \times E_8$ needed for the Gepner map only in the right-moving sector.

A natural implementation of this idea can be motivated by the free fermion construction of $D_n = SO(2n)$ in terms of $2n$ Majorana fermions with aligned spin structures. The extension of $SO(2m) \otimes SO(2n)$ to $SO(2m + 2n)$ is achieved by alignment of all spin structures and can be implemented by a SCMI of extension type with the current $J = v_{D_n} \otimes v_{D_m}$, in complete analogy to the alignment of spin structures for a tensor product of SCFTs. The exceptional series is thus obtained by starting with a gauge sector $SO(2l) \otimes SO(2)^{5-l} \otimes E_8$ and a generalized GSO projection

$$J_{\text{GSO}} = J_s \otimes s_{SO(2l)} \otimes (s_{SO(2)})^{5-l} \quad (1.31)$$

as is illustrated in the following table:

| $l$ | $E_{l+1}$ | $D_l \times D_1^{5-l}$ | $|E_{l+1}| - |D_l| - |U(1)|$ | currents $(J_{\text{GSO}})^{\pm 1}$ |
|-----|-----------|---------------------|-----------------|----------------------|
| 5   | $E_6$     | $SO_{10}$           | 32 = 78 - 45 - 1 | $|s| = 16 \quad h = \frac{2}{5} + \frac{5}{1}$ |
| 4   | $E_5 = SO_{10}$ | $SO_8 \times SO_2$ | 16 = 45 - 28 - 1 | $|s| = 8 \quad h = \frac{2}{5} + \frac{5}{1}$ |
| 3   | $E_4 = SU_5$ | $SO_6 \times (SO_2)^2$ | 8 = 24 - 15 - 1 | $|s| = 4 \quad h = \frac{2}{5} + \frac{5}{1} + \frac{1}{3}$ |
| 2   | $SU_3 \times SU_2$ | $SO_4 \times (SO_2)^3$ | 4 = 11 - 6 - 1 | $|s| = 2 \quad h = \frac{2}{5} + \frac{5}{1} + \frac{1}{3}$ |

For the rest of this paper we restrict to the case $l = 4$, i.e. to $SO(10)$ models based on a CFT of the form $\mathcal{C}_{\text{int}} \times SO(8) \times SO(2) \times E_8$ with $c = 26 - 4$.

Blumenhagen and A. Wißkirchen performed a computer search for spectra of heterotic models of this type that agree with Distler-Kachru models and came up with a small list, the most promising candidate of which is an $SO(10)$ model with 80 generations. They used the original approach of Schellekens and Yankielowicz constructing SCMIs as products of invariants for cyclic subgroups of the center. Translating their data into our language we find, in addition to $J_{\text{GSO}}$ and the alignment currents,
a $\mathbb{Z}_4$ twist whose simple current generator $J_B = (J_s^{k=3})^5 \times s_{SO(2)}$ is the product of the spinor of $SO(2)$ times the $5^{th}$ power of the spectral flow of one of the minimal model factors of the quintic.

We call $J_B$, which squares to the alignment current $J_B^2 = J_v^{k=3} \otimes v_{SO(2)}$, a Bonn twist. Since only one minimal model enters this construction it appears natural to generalize the discussion to an internal SCFT of the form

$$C_{int} = C' \otimes F_K,$$

where $F_K$ is a minimal model whose level $k = K - 2$ needs to be odd in order that $J_2^K = J_v$. In the Landau-Ginzburg description $F_K$ has a Fermat-type potential $W = \phi^K$ and is hence referred to as Fermat factor. The Bonn twist thus generalizes to $J_B = (J_F^s)^K \times s_{SO(2)}$, $N_B = 4$, $J_B^2 = J_v^F \otimes v_{SO(2)}$ (1.32)

so that the resulting (0,2) model can be defined by a SCMI based on the generators $J_B$, $J_{GSO}$ and two more alignment currents

$$J_A = v_{SO(8)} \otimes v_{SO(2)}, \quad J_C = J_v^C \otimes v_{SO(8)}. \quad (1.33)$$

The nonvanishing monodromies are $R_{BB} \equiv \frac{K-1}{2} \mod 2$, $R_{AB} \equiv \frac{1}{2} \mod 1$ and $R_{B,GSO} \equiv \frac{K-1}{4} \mod 1$. We need $J_{GSO}$ and the alignment currents $J_A$, $J_B$ and $J_C$ in the chiral algebra on the right-moving side, i.e. in the kernel of $X$, so that the corresponding columns of the matrix $X$ must be 0 mod 1, or 0 mod 1/2 in the case of $J_B$. This fixes all discrete torsions and implies

|     | $J_B$ | $J_C$ | $J_A$ | $J_{GSO}$ |
|-----|------|------|------|----------|
| $J_{GSO}$ | 0 0 | $\frac{K-1}{2}$ 0 | 0 0 | 0 0 |
| $J_A$ | 0 0 | 0 0 | 0 0 | 0 0 |
| $J_B$ | $\frac{K-1}{2}$ 0 | 0 0 | 0 0 | 0 0 |
| $J_C$ | 0 0 | 0 0 | 0 0 | 0 0 |

For a field $\phi_a, J_a$ that is twisted by

$$J = J^{2\alpha}_{GSO} J^{2\beta}_{A} J^{2\gamma}_{B} J^{2\rho}_{C}, \quad \alpha, \beta, \gamma, \rho = 0, 1 \quad (1.34)$$

this leads to the following charge projections for the monodromy charges

$$Q_{GSO} \equiv -\frac{1}{2} Q_{U(1)} \equiv 0, \quad Q_A \equiv \frac{1}{2} \rho, \quad Q_B \equiv \frac{K-1}{4} \rho, \quad Q_C \equiv 0, \quad (1.35)$$

or, equivalently, $Q_{GSO} \equiv \overline{Q}_A \equiv \overline{Q}_C \equiv 0$ and $\overline{Q}_B \equiv \frac{1}{2} \alpha + \frac{K-1}{4} \rho$ modulo 1.

The massless matter representations (chiral superfields) as well as possible gauge group extensions (vector superfields) can now be enumerated straightforwardly. Space-time quantum numbers come from representations
of the right-moving chiral algebra while the gauge group representations follow from left-moving CFT quantum numbers. The correspondences have been worked out for $E_5 = SO(10)$, $E_4 = SU(5)$ and $E_3 = SU(3) \times SU(2)$ by Blumenhagen and Wisskirchen\(^29\) (cf. their tables in section 6). For the case $SO(8) \times U(1) \subseteq E_5$ the massless matter representations are assembled by the orbits of $J_{GSO}$ as follows,

$$16 = 8^{\bar{s}}_{-1} + 8^s_1, \quad 16 = 8^s_{-1} + 8^{\bar{s}}_1, \quad 10 = 1_{-2} + 8^s_0 + 1_2,$$

where the subscripts denote the $U(1)$ charges.

Only gauge-singlet representations can depend on non-topological information, i.e. uncharged fields with $r = 0$ and $h_{int} = 1$ in eq. (1.24). All charged matter fields and non-abelian gauge group extensions can hence be determined in terms of the data encoded in the extended Poincaré polynomial of $C^\prime$. Our construction can thus be used for all Landau-Ginzburg orbifolds based on $N = 2$ SCFTs of the form $C^\prime \otimes F$ with a Fermat factor $F \sim \phi^K$ with $K \in 2\mathbb{Z} + 1$.

### 1.4. Geometry and vector bundle data

Witten’s gauged linear sigma model\(^4\) made it possible to construct a large class of $(0, 2)$ string vacua.\(^15\) The starting point is a supersymmetric abelian gauge theory that leads in the Calabi-Yau phase to a $\sigma$ model described by an exact sequence (monad)

$$0 \rightarrow V \rightarrow \bigoplus_{i=1}^{r+1} \mathcal{O}(n_i) \xrightarrow{F_i} \mathcal{O}(m) \rightarrow 0 \quad (1.37)$$

defining a bundle $V$ of rank $r$ over a complete intersection Calabi-Yau $X$. $F_i$ are homogeneous polynomials of degrees $m - n_i$ not vanishing simultaneously on $X$. For weighted projective ambient spaces we can write this data as

$$V_{n_1, \ldots, n_r+1}[m] \rightarrow \mathbb{P}_{w_1, \ldots, w_{N+4}}[d_1, \ldots, d_N], \quad (1.38)$$

where $r = 4, 5$ corresponds to unbroken gauge groups $SO(10)$ and $SU(5)$, respectively. The Calabi-Yau condition $c_1(V) = 0$ and the condition $c_1(V) = 0$, which guarantees the existence of spinors, read

$$\sum d_i - \sum w_j = m - \sum n_i = 0 \quad (1.39)$$

and the cancellation of gauge anomalies $ch_2(V) = ch_2(\mathcal{O}^2)$ with $ch_2 = \frac{1}{2}c_1^2 - c_2$ implies the quadratic diophantine constraint

$$\sum d_i^2 - \sum w_j^2 = m^2 - \sum n_i^2. \quad (1.40)$$
For a Calabi-Yau hypersurface $W = 0$ the choice of $m = d = \sum w_j$ with $n_i = w_i$ solves these equations and $F_i = \partial_i W$ corresponds to the $(2,2)$ case.

The suggested CFT/geometry correspondence$^{29}$ associates the vector bundle $V_{1,1,1,1,1}[5]$ over $\mathbb{P}_{1,1,1,1,2,2}[4,4]$ to the $(0,2)$ cousin of the Gepner model$^3$. Since the twist $J_B$ that defines the $(0,2)$ model only acts on one of the Fermat factors we expect that this is part of a larger picture, where the Gepner model data directly translate into vector bundle data $V_{n_1,...,n_5}[m]$ with $k_i = m/n_i - 2$. For the base manifold the doubling of the respective weight seems to correspond to the doubling of the order of the twist group by the Bonn twist $J_B$ (as compared to the standard construction). We hence make the ansatz

$$V_{n_1,...,n_5}[m] \rightarrow \mathbb{P}_{n_1,...,n_4,2n_5,w_6}[d_1,d_2],$$

i.e. $w_i = n_i$ for $i < 5$ and $w_5 = 2n_5$, and impose (1.39) and (1.40) or

$$d_1 + d_2 = m + n_5 + w_6, \quad d_1^2 + d_2^2 = m^2 + 3n_5^2 + w_6^2.$$  \hfill (1.42)

It is quite non-trivial and encouraging that this non-linear system has a general solution $w_6 = (m - n_5)/2 = d_1/2$ and $d_2 = (m + 3n_5)/2$. We hence conjecture a correspondence between the $(0,2)$ models defined in the previous section with the Distler-Kachru models defined by the data$^{39}$

$$V_{n_1,...,n_5}[m] \rightarrow \mathbb{P}_{n_1,...,n_4,2n_5,w_6}[m - n_5, (m + 3n_5)/2].$$ \hfill (1.43)

The increase of the codimension of the Calabi-Yau may be interpreted as providing an additional field of degree $w_6 = d_1/2$ that generates the twisted sectors for the $\mathbb{Z}_2$ orbifolding due to $J_B$.

In the Calabi–Yau phase a toric approach to the resolution of singularities appears to be most natural.$^{50}$ For the $(2,2)$ model the Newton polytope $\Delta$ of a generic transversal degree $m$ polynomial is reflexive and its polar polytope $\Delta^*$ provides a desingularization of the hypersurface in the weighted projective space $\mathbb{P}_{n_1,...,n_5}$. For the complete intersection (1.43) the Batyrev-Borisov construction$^{52}$ suggests to consider the Minkowski sum $\Delta = \Delta_1 + \Delta_2$ of the Newton polytopes $\Delta_l$ of degree $d_l$ polynomials w.r.t. the weights $w_j$. If $\Delta$ is reflexive then a natural resolution of singularities can again be based on a triangulation of the fan over $\Delta^*$. A useful collection of tools and formulas for further studies of this class of models can be found in a paper by Blumenhagen.$^{53}$

1.5. Conclusion

We discussed the construction of a large class of heterotic $(0,2)$ Gepner-type models in terms of simple current techniques and their generalization...
to Landau-Ginzburg models based on the topological information encoded by the extended Poincaré polynomial. Already without orbifolding the 7555 transversal potentials lead to 3219 models, 220 of which are of Fermat type.

For a large subclass of the potentials the mirrors of the (2,2) models can be constructed as orbifolds.\textsuperscript{42,43} In this case our analysis provides the ingredients for an orbifold mirror construction also for the (0,2) version, thus explaining the mirror symmetry that has been observed in orbifold spectra.\textsuperscript{40,41} While an algorithm for the construction of the mirror orbifold is known also in the presence of discrete torsions,\textsuperscript{46} it would be interesting to find an explicit formula for the mirror orbifold in group theoretical terms.

In addition to the phenomenological interest of heterotic models it would be interesting to test the proposed identifications by comparing spectra in geometrical phases\textsuperscript{53} and Yukawa couplings at the Landau-Ginzburg points,\textsuperscript{54} and to study generalizations with smaller gauge groups.

Acknowledgements. I would like to thank Ron Donagi and Emanuel Scheidegger for helpful discussions. This work is supported in part by the Austrian Research Funds FWF under grant Nr. P18679.

Appendix A. Gepner models, torus orbifolds & mirror symmetry

In accord with the three weighted projective spaces $\mathbb{P}_{111}[3]$, $\mathbb{W}_{112}[4]$ and $\mathbb{W}_{123}[6]$ that admit a transversal CY equation of degree $d = 3, 4, 6$, there are three Gepner models with levels $k = (1, 1, 1)$, $k = (2, 2, 0)$ and $k = (4, 1, 0)$, and superpotentials $W = X^3 + Y^3 + Z^3$, $W = X^4 + Y^4 + Z^2$ and $W = X^6 + Y^3 + Z^2$, respectively, that describe 2d tori. While the Kähler modulus is fixed at the Landau-Ginzburg point at a value that is consistent with the $\mathbb{Z}_d$ quantum symmetry originating in the GSO projection, the complex structure deformation corresponds to a deformation of $W$ by $\lambda XYZ$. At the Gepner point $\lambda = 0$ the complex structure moduli are $\tau = e^{2\pi i/d}$, where $e^{2\pi i/3}$ and $e^{2\pi i/6}$ are related by $\tau \rightarrow \tau + 1$.

We focus on $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, whose abelian extensions were recently classified and compared to free fermion models by Donagi and Wendland.\textsuperscript{44} Since we want to realize the $\mathbb{Z}_2$’s as symmetries of Gepner models we consider $\mathbb{W}_{112}[4]$ and $\mathbb{W}_{123}[6]$, for which a phase rotation of the first homogeneous coordinate corresponds to a phase rotation by $2\pi/d$ of the flat double-periodic torus coordinate $z \in T^2$ (this can be checked by counting fixed points and orders of stabilizers). The $\mathbb{Z}_2$ orbifold $z \rightarrow -z$ hence corresponds to the phase symmetry $\rho = \mathbb{Z}_2 : 1 0 0$ in both cases.

With the notation of\textsuperscript{44} as subscript and the Hodge numbers as super-
spript, the four inequivalent orbifolds by a \(Z_2 \times Z_2\) twist group \(G_T\) are \(X_0^{1,3}\), \(X_{19,19}\), \(X_{11,11}\), and \(X_{0,-3}\). They differ by the number of shifts \(z \rightarrow z + \frac{1}{2}\) that are included and we can choose the following generators,\(^{44}\)

\[
X_0^{1,3} : \begin{align*}
\theta^{(1)}(z_1, z_2, z_3) &= (-z_1, z_2, -z_3) \\
\theta^{(2)}(z_1, z_2, z_3) &= (z_1, -z_2, -z_3) 
\end{align*}
\]

(A.1)

\[
X_{19,19} : \begin{align*}
\theta^{(1)}(z_1, z_2, z_3) &= (-z_1, z_2, -z_3) \\
\theta^{(2)}(z_1, z_2, z_3) &= (z_1, -z_2, \frac{1}{2} - z_3) 
\end{align*}
\]

(A.2)

\[
X_{11,11} : \begin{align*}
\theta^{(1)}(z_1, z_2, z_3) &= (-z_1, z_2 + \frac{1}{2}, -z_3) \\
\theta^{(2)}(z_1, z_2, z_3) &= (z_1, -z_2, -z_3) 
\end{align*}
\]

(A.3)

\[
X_{0,-3} : \begin{align*}
\theta^{(1)}(z_1, z_2, z_3) &= (z_1, z_2, -z_3) \\
\theta^{(2)}(z_1, z_2, z_3) &= (-z_1, z_2 + \frac{1}{2}, \frac{1}{2} - z_3) 
\end{align*}
\]

(A.4)

Only \(\mathbb{P}_{112}[4]\) admits a second independent \(Z_2\) action, namely \(\sigma = Z_2 : 1 0 1\), which has no fixed points and hence corresponds to a shift \(z \rightarrow z + \frac{1}{2}\) of order 2. The product \(\rho \circ \sigma = Z_2 : 0 0 1\) also has 4 fixed points and corresponds to the rotation \(z \rightarrow \frac{1}{2} - z\) about \(z = \frac{1}{4}\), which is equivalent to \(\rho\). For the realization of \(X_{0-n}\) in terms of Gepner models we hence need at least \(n - 1\) factors of \(\mathbb{P}_{112}[4]\). This can be confirmed by computing the Hodge numbers with the program package PALP.\(^{55}\) In a UNIX shell environment the required input data can be assembled as follows,

```
Weight1="6 1 2 3 1 2 3 1 2 3 "
TorusQ1="/Z6: 1 2 3 0 0 0 0 0 /Z6: 0 0 0 1 2 3 0 0 0"
Weight2="12 2 4 6 2 4 6 3 3 6 "
TorusQ2="/Z6: 1 2 3 0 0 0 0 0 /Z6: 0 0 0 1 2 3 0 0 0"
Weight3="12 2 4 6 3 3 6 3 3 6 "
TorusQ3="/Z6: 1 2 3 0 0 0 0 0 /Z6: 0 0 0 1 1 2 0 0 0"
Weight4="14 1 1 2 1 2 1 1 2 1 2 "
TorusQ4="/Z4: 1 1 2 0 0 0 0 0 /Z4: 0 0 0 1 1 2 0 0 0"
X01="$Weight1 $TorusQ1 /Z2: 1 0 0 0 0 0 1 0 0 /Z2: 0 0 0 1 0 0 1 0 0"
X02="$Weight2 $TorusQ2 /Z2: 1 0 0 0 0 0 1 0 0 /Z2: 0 0 0 1 0 0 0 0 1"
X03="$Weight3 $TorusQ3 /Z2: 1 0 0 1 0 1 1 0 0 /Z2: 0 0 0 1 0 0 0 1 1"
X04="$Weight4 $TorusQ4 /Z2: 1 0 1 1 0 0 1 0 0 /Z2: 1 0 0 1 0 1 0 0 1"
```

echo –e "$X01 \n$X02 \n$X03 \n$X04" | poly.x -lf
```

where “\texttt{Weight*}” includes a sufficient number of \(\mathbb{P}_{112}[4]\) factors for the shift symmetries, “\texttt{TorusQ*}” provides two GSO projections for torus factors (the overall GSO is automatic) and “\texttt{X0*}” completes the input line for the respective \(Z_2 \times Z_2\) orbifold \(X_{0-1, \ldots, 0, -4}\). The last line pipes the input into the executable \texttt{poly.x} contained in PALP,\(^{55}\) with flags “–l” and “–f” for “Landau-Ginzburg” and “filter” (i.e. read input from pipe), respectively.

The mirror models can now be constructed using the Green-Plesser orbifold construction. In\(^{44}\) it was observed that discrete torsions often provide the mirrors. This is special to \(Z_2\)-torsions, however, for which a discrete
torsion between two phase symmetries of even order of the LG superpotential can be switched on/off by redefinition of the action on massive fields $Z^2$, as has been discussed in detail in.\textsuperscript{46} For general orders of the generators, the mirror models of orbifolds with discrete torsion again have discrete torsion\textsuperscript{46} and we do not know of any indications that mirror symmetry and discrete torsion are related for $Z_n$ twists with $n \neq 2$.\textsuperscript{37,38}

In the classification of extensions $G_S \to G \to G_T$ of the twist group,\textsuperscript{44} $G_S$ is the subgroup of shifts. Only $\mathbb{Z}_{112}[4]$ admits a symmetry that corresponds to a second independent shift $\sigma'$ of order 2, which however cannot be diagonalized simultaneously with $\sigma$. It exchanges $X$ and $Y$ and reverses the sign of $Z$. The mirror construction in this case proceeds by first taking the Green-Plesser mirror for the diagonal subgroup and then performing the mirror moddings of the remaining twists on the mirror CFT, which may involve quantum symmetries. It would be interesting to use examples from\textsuperscript{44} with non-trivial fundamental groups to further test the conjecture that mirror symmetry exchanges torsion in $H^2(X,\mathbb{Z})$ with torsion in $H^3(X,\mathbb{Z})$.\textsuperscript{56}

Appendix B. N=2 SCFT, simple currents & minimal models

The $N=2$ superconformal algebra\textsuperscript{57} is generated by the Fourier modes of $T(z)$, of its fermionic superpartners $G^\pm(z)$, and of a $U(1)$ current $J(z)$

\begin{align}
\{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)J_{r+s} + \frac{c}{12} (r^2 - \frac{1}{4}) \delta_{r+s}, \quad \text{(B.1)} \\
[L_n, G_r^\pm] &= (\frac{c}{12} - r)G_{n+r}^\pm, \quad [J_n, G_r^\pm] = \pm G_{n+r}^\pm, \quad \text{(B.2)} \\
[L_n, J_m] &= -mJ_{m+n}, \quad [J_m, J_n] = \frac{c}{6} m \delta_{m+n}, \quad \text{(B.3)}
\end{align}

where $r,s \in \mathbb{Z} + \frac{1}{2}$ in the NS sector. According to (B.1) the Ramond ground states $G_0|\alpha\rangle_R = 0$ have $h_\alpha = \frac{c}{24}$. The analogous unitarity bound in the NS sector is saturated by the chiral primary fields $G_{\pm\frac{1}{2}}|\phi\rangle = 0$, which obey $\{G_{\pm\frac{1}{2}}, G_{-\frac{1}{2}}\}|\phi\rangle = (2L_0 - J_0)|\phi\rangle = 0$ and hence $h = Q/2$. Their conjugate anti-chiral states saturate the BPS bound $h = -Q/2$.

The N=2 algebra admits the continuous spectral flow

\begin{align}
L_n \xrightarrow{t \theta} L_n + \theta J_n + \frac{c}{12} \theta^2 \delta_n, \quad J_n \xrightarrow{t \theta} J_n + \theta \delta_n, \quad G_r \xrightarrow{t \theta} G_{r+\theta} \quad \text{(B.4)}
\end{align}

which for $\theta = \pm \frac{1}{2}$ maps Ramond ground states into chiral and antichiral primary fields, respectively. Spectral flow is best understood by bosonization of the $U(1)$ current $J(z) = i \sqrt{c/3} \partial X(z)$ in terms of a free field $X$. A charged operator $O_q$ can thus be written as a normal ordered product of a vertex operator with a neutral operator $O_0$,

\begin{align}
O_q = e^{i\sqrt{c/3} qX} O_0(\partial X, \ldots, \psi, \ldots) \quad \text{(B.5)}
\end{align}
The contribution of the vertex operator to $h$ is $\frac{3q^2}{2}$ so that in unitary theories the maximal charges of Ramond ground states and chiral primary states are $c/6$ and $c/3$, respectively. In particular, the Ramond ground state $J_s = e^{i\sqrt{c/12}X}$ with maximal charge $c/6$ is a simple current. A short calculation shows that its monodromy charge is $Q_s = -\frac{1}{2}Q$. If the $U(1)$ charges $Q$ are quantized in units of $1/M$ in the NS sector then $c = 3k/M$ for some integer $k$. Since the $U(1)$ charges are shifted by $-c/6 = -k/2M$ in the Ramond sector the order $N_s$ of $J_s$ is $2M$ if $k \in 2\mathbb{Z}$ and $4M$ if $k \notin 2\mathbb{Z}$.

Already for $N = 1$ SCFTs the supercurrent $G$ is a universal simple current, which we denote by $J_v = G$. Its monodromy charge is $Q_v = 0$ for NS fields and $Q_v = 1/2$ for Ramond fields since $h_v = 3/2$ and the conformal weights of superpartners differ by integers in the Ramond sector and by half-integers for NS states. Putting the pieces together we find the matrix of monodromies

$$R_{v,v} = 0, \quad R_{v,s} = 1/2, \quad R_{s,s} = n - c/12 \quad \text{with} \quad n = \begin{cases} 0 & k \in 4\mathbb{Z} \\ 1 & k \notin 4\mathbb{Z} \end{cases} \quad (B.6)$$

where we used $h_v = c/24$ and $Q_s(J_s) = -c/12$. Note that $J_{sM}^2 = J_v^k$ (since the monodromy charges agree) so that the universal center is $\mathbb{Z}_{2M} \times \mathbb{Z}_2$ for $k \in 2\mathbb{Z}$ and $\mathbb{Z}_{4M}$ for $k \notin 2\mathbb{Z}$.

B.1. $N = 2$ minimal models

Minimal models have a number of different realizations. Here we use the coset construction for the $N = 2$ superconformal series $\mathcal{C}_k$

$$(SU(2)_k \times U(1)_1)/U(1)_{2K}, \quad c = 3k/K \quad \text{with} \quad K = k + 2 \quad (B.7)$$

as a quotient of $SU(2)$ level $k$ for $k \in \mathbb{N}$ times $U(1)_4 \cong SO(2)_1$ by $U(1)_{2K}$. Primary fields $\phi^m_l$ are labelled accordingly by $0 \leq l \leq k$, $s \mod 4$ and $m \mod 2K$ with the branching rule $l + m + s \in 2\mathbb{Z}$. The fusion rules are

$$\phi^l_{m_1} \times \phi^s_{m_2} = \sum_{|l|=|l_1-l_2|} \phi^{l_1+l_2,x_{l_1-l_2}}_{m_1+m_2} \quad (B.8)$$

so that $\phi^0_m$ and $\phi^{k+2}_m$ are simple currents. The conformal weights and the $U(1)$ charges obey

$$h \equiv \frac{l(l+2)-m^2}{4K} + \frac{s^2}{2} \mod 1, \quad Q \equiv s - \frac{m}{4} \mod 2 \quad \text{exact for} \quad \frac{|m-s|}{1-\delta_0 \leq s \leq 1} \quad (B.9)$$

where the NS and R sectors correspond to even and odd $s$, respectively. The formulas (B.9) are exact in the standard range $|m-s| \leq l$, $-1 \leq s \leq 1$ and otherwise sufficient to determine the monodromy charges of simple currents. In particular, the selection rule $l + m + s \in 2\mathbb{Z}$ is implemented by integrality of the monodromy charge $Q^2_K$ of the simple current $\phi^2_K$.  

which has integral conformal weight. According to the rules for modular invariance the branching rule thus necessitates the field identification
\[ \phi_{m}^{l} \sim \phi_{m+K}^{k-l,s+2} \quad \text{with} \quad J_{id} = \phi_{K}^{k}, \quad Q_{id} \equiv (l + m + s)/2 \quad (\text{B.10}) \]
due to an extension of the chiral algebra by the “identification current” \( J_{id} \).

The center of the minimal model at level \( k \) is hence of order \( 4K \) and generated by the spectral flow current \( J_{s} := \phi_{1}^{01} \sim \phi_{1-K}^{k3} \) and the supercurrent \( J_{s}^{0} := \phi_{0}^{02} \sim \phi_{0}^{k0} \) with \( J_{s}^{2K} = J_{s}^{R} \); more generally all above formulas for \( N = 2 \) SCFTs apply with \( M = K \). Ramond ground states and (anti)chiral primary fields are now easily identified as follows,

| anti-chiral primary | Ramond ground states | chiral primary |
|---------------------|----------------------|---------------|
| \( \phi_{l}^{0} \sim \phi_{k+1}^{k-l,2} \rightarrow |l\rangle_{a} \) | \( \phi_{l+1}^{l+1} \sim \phi_{k-l+1}^{k-l,1} \rightarrow |l\rangle_{R} \) | \( Q = -\frac{l}{k}, \quad h = -\frac{2}{k} \) |
| \( Q = \pm \left( \frac{l}{k} - \frac{2}{K} \right), \quad h = \frac{2}{k} \) | \( \phi_{-l}^{0} \sim \phi_{-k-l}^{k-l,2} \rightarrow |l\rangle_{c} \) | \( Q = \frac{l}{K}, \quad h = \frac{2}{k} \) |

The Landau-Ginzburg description of the minimal model with the diagonal modular invariant has superpotential \( W = X^{K} \) with \( X \sim \phi_{-1}^{1,0} \).

In order to determine the conformal weights and multiplicities of all fields relevant for massless string spectra we follow the discussion in ref.\(^{22} \) and first note that the supercurrent \( J_{c} \) acts as \( J_{c} \phi_{m}^{l} = \phi_{m+2}^{l} \sim \phi_{m+K}^{k-l,s} \). Choosing \( m \) such that \( -K < m \leq K \) we find that \( m \rightarrow m - K \) for \( m > 0 \) and \( m \rightarrow m + K \) for \( m \leq 0 \). It is then straightforward to check that \( l + 1 - |m| \rightarrow -(l + 1 - |m|) \), i.e. the fields inside the cone \( |m| \leq l + 1 \) are mapped to the outside and vice versa.

In the NS sector we choose \( s = 0 \). Then (B.9) gives the correct value of \( h \) inside the cone, i.e. for \( |m| \leq l \). The conformal weight of the respective superpartner is \( h + \frac{1}{2} \) and its multiplicity is 2 unless \( G_{1/2}^{+} \) or \( G_{-1/2}^{-} \) vanishes. This happens for \( |m| = l \), for which the multiplicity of the superpartner is 1 for \( l > 0 \). For \( l = m = 0 \), i.e. the superpartner \( J_{c} \) of the identity, the lowest states have \( h = 3/2 \) with multiplicity 2.

In the R sector highest weight states are annihilated by \( G_{0}^{+} \) or \( G_{0}^{-} \). They thus come in pairs \( \phi_{m}^{l+1} \) that are related by the action of \( G_{0}^{\pm} \). Usually we can fulfill \( |m| < l \) by field identification, in which case \( h \) is degenerate and given correctly by (B.9). The only exception is \( |m| = l + 1 \) where \( G_{0}^{+} = G_{0}^{-} = 0 \). In that case one has to make a choice of chirality: The Ramond ground states have \( h = c/24 \) in accordance with (B.9), and their superpartners have \( h = 1 + c/24 \). The choice \( m = l + 1 \) and \( s = 1 \) leads to the standard range given in (B.9). The only descendent that plays a role for the massless spectrum of strings is the descendent \( J_{-1}|0\rangle \) of the vacuum.
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