Enumerating Restricted Dyck Paths with Context-Free Grammars

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Abstract

The number of Dyck paths of semilength $n$ is famously $C_n$, the $n$th Catalan number. This fact follows after noticing that every Dyck path can be uniquely parsed according to a context-free grammar. In a recent paper, Zeilberger showed that many restricted sets of Dyck paths satisfy different, more complicated grammars, and from this derived various generating function identities. We take this further, highlighting some combinatorial results about Dyck paths obtained via grammatical proof and generalizing some of Zeilberger’s grammars to infinite families.

1 Introduction

As Flajolet and Sedgewick masterfully demonstrate in their seminal text, Analytic Combinatorics [3], mathematicians have occasionally borrowed the study of formal languages from computer science and linguistics for combinatorial reasons. Many combinatorial classes can be reinterpreted as languages generated by certain grammars, and these grammars often make writing down generating functions, another favorite combinatorial tool, routine.

For example, consider the well-known Dyck paths. A Dyck path is a finite list of +1’s and −1’s whose partial sums are nonnegative, and whose sum is 0. We will write $U$ (up) for +1 and $D$ (down) for −1. Thus, the following are all Dyck paths:

\[
\begin{align*}
U U D D \\
U D U D \\
U U U D U D D \\
\end{align*}
\]

A Dyck path must have even length, since “number of $U$’s” equals “number of $D$’s.” For this reason, we often refer to Dyck paths of semilength $n$ (length $2n$).

It is a famous result that the number of Dyck paths of semilength $n$ equals the $n$th Catalan number:

\[
C_n = \frac{1}{n+1} \binom{2n}{n}.
\]

There are many proofs of this fact, but here is a grammatical proof.

Let $\mathcal{P}$ denote the set of all Dyck paths. Then, $\mathcal{P}$ is generated by the unambiguous, context-free grammar

\[
\mathcal{P} = \epsilon \cup U \mathcal{P} D \mathcal{P},
\]

where $\epsilon$ denotes the empty string. In words, a path is either empty or begins with a $U$, is followed by a Dyck path (shifted to height 1), a $D$, then another Dyck path. This is a unique parsing of all Dyck paths.
Given a set of objects \( E \) each with a nonnegative integer size, let \( GF(E) = \sum_{k \geq 0} |E(k)| z^k \) be a formal generating function, where \( |E(k)| \) is the number of objects of size \( k \) in \( E \). The main result about formal grammars is that, in an unambiguous context free grammar,

\[
GF(A \cup B) = GF(A) + GF(B)
\]

and

\[
GF(AB) = GF(A)GF(B),
\]

where the “sizes” of the grammar are the lengths of the words it generates.

In our case, if \( P(z) \) is the generating function for the number of Dyck paths of semilength \( n \), then this grammar implies

\[
P(z) = GF(\epsilon) + GF(U\bar{P}D\bar{P})
\]

\[
= 1 + zP(z)^2.
\]

(There is exactly one empty Dyck path [which has semilength 0], and the presence of \( U \) and \( D \) increases the semilength by 1.) The generating function \( C(z) \) for the Catalan numbers also satisfies

\[
C(z) = 1 + zC^2(z),
\]

and since there are only two possible solutions, it is not hard to see that \( P(z) = C(z) \).

The grammatical technique offers a unifying framework: Devise a grammar and you get an equation. Sometimes the equations turn out to be well-known. Other times they are complicated messes. The enumeration of all Dyck paths is one application of this framework, and here we want to demonstrate others. In particular, we will give grammatical proofs of several combinatorial facts about restricted Dyck paths, and also establish several infinite families of grammars in closed form.

First, let us define the restrictions we shall consider.

**Definition 1.** Given a Dyck path, the **height** of the path at position \( k \) is the partial sum of the path after its \( k \)th term. A **peak** of a Dyck path at height \( h \) (or simply “at \( h \)” ) is the bigram \( UD \) where the height of the path after the \( U \) is \( h \). Similarly, a **valley** occurs at the bigram \( DU \), and its height is analogously defined. The empty path has, by convention, a peak at 0 but no valley.

Given a sequence of steps \( L \), define \( L^n \) to be the repetition of \( L \) \( n \) times. (For example, \( U^2 = UU \) and \( (UD)^3 = UDUDUD \).)

A Dyck path has an **up-run of length** \( n \) provided that it contains at least one \( U^n \) that is not preceded nor followed by \( U \). Similarly, it contains a **down-run of length** \( n \) provided that it contains at least one \( D^n \) that is neither preceded nor followed by \( D \).

We are generally considered with Dyck paths whose peaks and valley heights avoid certain sets, and whose up-run and down-run lengths avoid certain sets, and combinations of the four conditions. We will, for example, discuss the set of all Dyck paths whose peak heights avoid \( \{2, 4, 6, \ldots \} \) and have no up-run of length greater than 2.

When a set \( \mathcal{P} \) of Dyck paths has been specified and used in an expression, such as \( \mathcal{P} = U\bar{P}D\bar{P} \), it is shorthand for “any (possibly vertically shifted) Dyck path from \( \mathcal{P} \).”

**Definition 2.** For arbitrary sets of positive integers \( A, B, C, \) and \( D \), let \( P(A, B, C, D) \) be the set of Dyck Dyck paths whose peaks heights avoid \( A \), whose valleys avoid \( B \), whose up-run lengths avoid \( C \), and whose down-run lengths avoid \( D \). Let \( P_{A,B,C,D}(z) \) be be the generating function for the number of Dyck paths of semilength \( n \) in \( P(A, B, C, D) \).
Some of these sets have been studied. In [5], Peart and Woan provide a continued-fraction recurrence for the generating functions $P_{\{k\},\emptyset,\emptyset,\emptyset}(z)$. In [2], where Eu, Liu, and Yeh take this idea further and express $P_{A,\emptyset,\emptyset,\emptyset}(z)$ as a finite continued fraction whenever $A$ is finite or an arithmetic progression. In [6], Zeilberger presents a rigorous experimental method to derive equations for $P_{A,B,C,D}(z)$ when the sets involved are finite or arithmetic progressions. Proving “by hand” some of Zeilberger’s interesting discoveries ex post facto was a motivation for the present work. We generalize some of Zeilberger’s results to infinite families which are likely out of reach for symbolic methods.

Our results include several explicit grammars (and therefore generating function equations) for infinite families of the sets $A$ and $B$, and also grammatical proofs of several interesting special cases suggested in [6]. Many of these—any grammars referencing restrictions on up- or down-runs—are not in [2]. Some of our results are suggested in the OEIS [4]; see, for example, A1006 (Motzkin numbers) and A004148 (generalized Catalan numbers).

The remainder of the paper is organized as follows. Section 2 presents some results discovered by experimentation with software from [6] and proven with grammatical methods. Section 3 presents some infinite families of explicit grammars. Section 4 offers some concluding remarks about the limitations of grammars.

2 Combinatorial results

In this section we will present a number of results with grammatical proofs. We will often abuse notation and use one symbol—$P$, for example—to simultaneously denote a set of Dyck paths, a generating function, and a non-terminal symbol in a formal grammar.

**Proposition 3.** The number of Dyck paths of semilength $n$ whose peak heights avoid $\{2r+3 \mid r \geq 0\}$ and whose up-runs are no longer than 2 is 1 when $n = 0$, and $2^{n-1}$ when $n \geq 1$.

**Proof.** Let $P$ be the set of all such Dyck paths, and $Q$ the set of all Dyck paths which avoid peaks in $\{2r+2\}$ and up-runs longer than 2. Note that $P$ and $Q$ satisfy the following grammar:

$$P = \epsilon \cup UDP \cup UUDQDP$$

$$Q = \epsilon \cup UDQ.$$

This implies the following system of equations:

$$P = 1 + zP + z^2QP$$

$$Q = 1 + zQ.$$

Thus $Q(z) = (1 - z)^{-1}$ (the only path in $Q$ of semilength $n$ is $(UD)^n$) and

$$P(z) = \frac{1 - z}{1 - 2z}.$$

Therefore $[z^0]P(z) = 1$ and $[z^n]P(z) = 2^{n-1}$.

**Proposition 4.** The number of Dyck paths of semilength $n$ whose peak heights avoid $\{2r+3 \mid r \geq 0\}$ and whose up-runs are no longer than 3 equals the $(n + 1)$th generalized Catalan number $G_{n+1}$, defined by

$$G_0 = 1$$

$$G_1 = 1$$

$$G_{n+2} = G_{n+1} + \sum_{1 \leq k < n+1} G_k G_{n-k}.$$
Proof. Let $P$, $O$, and $E$ be the set of all Dyck paths with up-runs no longer than 3, and whose peak heights avoid \{2r + 3 \mid r \geq 0\}, \{2r + 2 \mid r \geq 0\}, and \{2r + 1 \mid r \geq 0\}, respectively. Observe that $P$, $O$, and $E$ satisfy the following grammar:

$$
P = \epsilon \cup UDP \cup UUDODP
$$

$$
O = \epsilon \cup UDO \cup UUUDODEO
$$

$$
E = \epsilon \cup UUDODE
$$

This grammar implies the following equations:

$$
P = 1 + zP + z^2OP
$$

$$
O = 1 + zO + z^3EO^2
$$

$$
E = 1 + z^2OE.
$$

This system has two possible solutions for $P$, but only one is holomorphic near the origin, namely

$$
P(z) = \frac{2}{1 - z - z^2 + (z^4 - 2z^3 - z^2 - 2z + 1)^{1/2}}.
$$

The generating function $G(z)$ for the generalized Catalan numbers is well-known to be

$$
G(z) = \frac{1 - z + z^2 - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}}{2z^2},
$$

and it is routine to verify that $G(z) = zP(z) + 1$. Therefore $G_{n+1} = [z^n]P(z)$ for $n \geq 0$.

The following proposition is concerned with Motzkin numbers (see A1006 in the OEIS and [1]). A Motzkin path is like a Dyck path, but includes a “sideways” step $S$ which does not change the height. The $n$th Motzkin number $M_n$ is the number of Motzkin paths of length $n$. The generating function $M = M(z)$ for $M_n$ satisfies the quadratic equation

$$
M = 1 + zM + z^2M^2.
$$

There are numerous bijections between Motzkin paths and various restricted classes of Dyck paths. Such bijections are often variations of the “folding” map

$$
UD \mapsto S
$$

$$
DU \mapsto S
$$

$$
UU \mapsto U
$$

$$
DD \mapsto D,
$$

which in general is not injective, but many restrictions on Dyck paths make it injective. For example, this idea shows that the Dyck paths of semilength $n$ with no up-runs longer than 2 are in bijection with the Motzkin paths of length $n$. We offer a grammatical proof of this fact.

**Proposition 5.** The number of Dyck paths of semilength $n$ which avoid up-runs of length 3 or more equals the $n$th Motzkin number $M_n$. 


Proof. Let \( P \) be the set of such paths. A grammar for \( P \) is
\[
P = \varepsilon \cup UUDPDP \cup UDP.
\]
Our grammar implies that
\[
P = 1 + zP + z^2P^2.
\]
This is the same equation satisfied by the Motzkin generating function, and it is easy to check that \( P(z) = M(z) \).

Proposition 6. Consider the set of Dyck paths such that no peak or valley has positive, even height. The numbers of such paths of semilength \( 2n \) and \( 2n + 1 \) are \((\frac{2n-1}{n})\) and \((\frac{2n}{n})\), respectively.

1. For any Dyck path, the first step must be up and the last step must be down to avoid negative height.

2. The parity of the height at the \( k \)th step clearly equals the parity of \( k \).
   a. if the \( 2k \)th step is up, the \((2k+1)\)th step must also be up to avoid a peak with even height.
   b. If the \( 2k \)th step is down and the height is not zero, then the next step must also be down to avoid a valley with positive even height.
   c. If the \( 2k \)th step is down and the height is 0, then the next step must be up to avoid a negative height.

Proof.

Let \( D \) be the set of Dyck Paths with semi-length \( 2n + 1 \) such that no peak-height and no valley-height is a positive even number. By our note above, it is clear that each \( d \in D \) is defined by the direction of its even numbered steps and its height at each of these steps, excluding its final step which must be down. Since \( d \) has length \( 4n + 2 \), there are \( 2n \) of these steps.

Let \( W \) be the set of all walks of length \( 2n \) with steps up and down such that the starting and ending height are both 0. Note that \( W \) consists of all permutations of \( n \) up steps and \( n \) down-steps, and therefore \( |W| = \binom{2n}{n} \). We define a bijection between \( D \) and \( W \) as follows.

Working our way through \( k = 1 \ldots 2n \): Given a walk \( w \in W \), the height going from \(-1\) to 0 or from 0 to \(-1\) at the \( k \)th step corresponds to a Dyck path whose height is zero at the \( 2k \)th step. (So the \( 2k \)th step is down and the \((2k+1)\)th step is up). Otherwise, an increase in the absolute value of the height at the \( k \)th step corresponds to a Dyck path whose \( 2k \)th and \((2k+1)\)th steps are both up, and a decrease corresponds to a Dyck path whose \( 2k \)th and \((2k+1)\)th steps are both down.

Note that the height can only go from \(-1\) to 0 at even steps and from 0 to \(-1\) at odd steps, so this is clearly injective. Its inverse is also injective since, when determining the \( k \)th step of the walk, given a Dyck path, we already know all the preceding steps in that walk.

Now, let \( D \) be the set of Dyck Paths with semi-length \( 2n \) such that no peak-height and no valley-height is a positive even number. As in the previous proof, \( d \in D \) is defined by the direction of its even numbered steps and its height at each of these steps, excluding its final step which must be down. Since \( d \) has length \( 4n \), there are \( 2n - 1 \) of these steps.

Let \( W \) be the set of all walks of length \( 2n - 1 \) with steps up and down such that the starting height is 0 and ending height is \(-1\). Note that \( W \) consists of all permutations of \( n - 1 \) up steps and \( n \) down-steps, and therefore \( |W| = \binom{2n-1}{n-1} \).

Here, we can define a bijection between \( D \) and \( W \) the same way that we defined it in the previous proof. Note that for \( w \in W \), we start at 0 and end at \(-1\), so the number of steps away
from the line between heights 0 and $-1$ is still the same as the number of steps towards it, so its image ends at height 0. Moreover, there can never be more steps towards zero than away from zero (i.e. absolute value can never decrease more than it increases), so the image will never have negative height and thus is a Dyck path. As before, the image will never have a peak or valley height that is a positive even number.

In the other direction, $d \in D$ obviously maps to a walk of length $2n - 1$ that starts at height 0. Since $d$ starts and ends at height 0, if we remove the first and last step of $d$ and split the remaining path into sub-paths of length 2, then $[1,1]$ appears the same number of times as $[-1,-1]$. Thus, the ending height if the image of $d$ will either be 0 or $-1$ (since the number of steps away from the line between 0 and $-1$ equals the number of steps toward that line). Since $d$ has semi-length $2n - 1$, $[-1,1]$ will occur an odd number of times in $d$. Therefore the image of $d$ will cross the line between 0 and $-1$ an odd number of times, starting at 0, and thus will end at $-1$.

Both maps are injective for the same reasons as before.

3 Grammatical families

In this section we provide some explicit grammars for infinite families of restricted Dyck paths. In many cases, such grammars are guaranteed to exist. The reasoning in [6] shows that, for every set of Dyck paths whose peaks, valleys, and up- and down-runs avoid specific arithmetic progressions, we may construct a finite, context-free grammar which generates them. The method implied in [6] to compute these grammars gives no hint as to their form, and this is what we try to provide here.

Our first two results are about Dyck paths whose up-run lengths avoid a fixed arithmetic progression $\{Ar + B \mid r \geq 0\}$. It turns out that when $B < A$, there is a simple context-free grammar for such paths. When $B \geq A$ the situation is more complicated, but we can derive a “grammatical equation” which again leads to a generating function.

**Proposition 7.** Let $B < A$ be non-negative integers. The set $\mathcal{P}$ of Dyck paths whose up-run lengths avoid $\{Ar + B \mid r \geq 0\}$ has the unambiguous grammar

$$\mathcal{P} = \bigcup_{0 \leq k < A \atop k \neq B} U^k(D\mathcal{P})^k \cup U^A(\mathcal{PD})^A\mathcal{P},$$

and therefore

$$P(z) = \sum_{0 \leq k < A \atop k \neq B} z^k P^k(z) + z^A P^{A+1}(z),$$

where $P(z)$ is the weightumerator of $\mathcal{P}$.

**Proof.** The grammar clearly uniquely parses the empty path, so suppose that $P \in \mathcal{P}$ has length $n > 0$. Then $P$ starts with a up-run of length $k > 0$ for some $k \neq B \mod A$. If $k < A$, then write $P = U^k DW$, where $W$ is a walk from height $k - 1$ to height 0 with the same restrictions on up-runs as $P$. For $0 \leq i < k - 1$, let $D_i$ indicate the down-step in $W$ which hits the height $i$ for the first time. Then

$$W = P_k^{-1}D_{k-2}P_{k-2}D_{k-3}...P_1D_0P_0,$$

where $P_i$ is a Dyck path shifted to height $i$ with the same restrictions on up-runs as $P$. This uniquely parses $P$ into the case $U^k(D\mathcal{P})^k$ in the grammar.
If the initial up-run has length \( k \geq A \), then write \( P = U^A W \), where \( W \) is a walk from height \( A \) to height 0 whose up-run lengths avoid \( \{ Ar + B \mid r \geq 0 \} \). By argument analogous to the previous paragraph, we can decompose \( W \) as

\[
W = P_AD_{A-1}P_{A-1}D_{A-2}...P_1D_0P_0,
\]

where \( P_i \in \mathcal{P} \). Thus \( W \) is of the form \((\mathcal{PD})^A\mathcal{P}\), and this uniquely parses \( P \) into the final case of the grammar.

We have shown that \( \mathcal{P} \) is contained in the language generated by this grammar, and it is easy to see that the first \( k \) cases of the grammar are contained in \( \mathcal{P} \). The final case, \( U^A(\mathcal{PD})^A\mathcal{P} \) is also contained in the grammar, because concatenating \( U^A \) to the beginning of a path does not change the length any of the up-runs modulo \( A \). The different cases are clearly disjoint, so the grammar is also unambiguous.

**Proposition 8.** Let \( A \leq B \) be nonnegative integers. The set \( \mathcal{P} \) of Dyck paths avoiding up-run lengths in \( \{ Ar + B \mid r \geq 0 \} \) satisfies the “grammatical equation”

\[
\mathcal{P} \cup U^B(\mathcal{DP})^B = \bigcup_{0 \leq k < A} U^k(\mathcal{DP})^k \cup U^A(\mathcal{PD})^A\mathcal{P},
\]

and therefore

\[
P(z) + zB P(z) = \sum_{0 \leq k < A} z^kP^k(z) + z^AP^{A+1}(z),
\]

where \( P(z) \) is the weight-enumerator of \( \mathcal{P} \).

Note that the right-hand side is nearly identical to the previous claim; the difference being that we can get paths in \( U^B(\mathcal{DP})^B \), which we will show below.

**Proof.** If \( P \) is a path in \( \mathcal{P} \), then we can uniquely parse \( P \) into a case of the right-hand side by the same argument given in the previous proposition.

\[
U^B(\mathcal{DP})^B = U^A U^{B-A}(\mathcal{DP})^B = U^A \{ U^{B-A}(\mathcal{DP})^{B-A} \}^{D(\mathcal{PD})^{A-1}}\mathcal{P}.
\]

The expression in brackets, \( U^{B-A}(\mathcal{DP})^{B-A} \), is in \( \mathcal{P} \), which shows that \( U^B(\mathcal{DP})^B \) is contained in \( U^A(\mathcal{PD})^A\mathcal{P} \).

Conversely, it remains to show that the left-hand side is all that the right-hand side can generate. \( \bigcup_{0 \leq k < A} U^k(\mathcal{DP})^k \) is contained in \( \mathcal{P} \) as in the previous proposition. For \( W \in U^A(\mathcal{PD})^A\mathcal{P}, \) write

\[
W = U^A P_1 D \ldots P_A D P_{A+1}.
\]

Let \( \ell \) be the length of the initial up-run in \( P_1 \). If \( \ell \neq B \) (mod \( A \)), then \( W \) contains no up-runs of lengths in \( \{ Ar + B \mid r \geq 0 \} \) and is a path in \( \mathcal{P} \). If \( \ell \equiv B \) (mod \( A \)), then \( \ell \leq B - A \). If \( \ell < B - A \) then the initial run of \( W \) has length less than \( B \). Thus, \( W \) contains no up-runs of lengths in \( \{ Ar + B \mid r \geq 0 \} \). For \( \ell = B - A \), let \( D_i \) denote the first time \( W \) steps down to height \( i \) for \( A < i < B \) and write

\[
W = U^A P_1 D \ldots P_A D P_{A+1}
= U^A U^{B-A} D_{B-1} W_{B-1} \ldots D_A W_A D P_2 D \ldots P_A D P_{A+1}
= U^B D_{B-1} W_{B-1} \ldots D_A W_A D P_2 D \ldots P_A D P_{A+1}.
\]

\( W_i \) is Dyck path shifted to height \( i \) by the definition of \( D_i \). Hence, \( W \in U^B(\mathcal{DP})^B \). \( \square \)
Proposition 9. Let $A, B \in \mathbb{Z}_{\geq 0}$ such that $B < A$. The set $\mathcal{P}$ of Dyck paths avoiding down-run lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq 0}\}$ has the unambiguous grammar

$$\mathcal{P} = \{\text{EmptyPath}\} \cup \bigcup_{1 \leq k < A, k \neq B} (U \mathcal{P})^{k-1}UD^k \mathcal{P} \cup (U \mathcal{P})^A D^A \mathcal{P},$$

and therefore

$$P(z) = 1 + \sum_{0 \leq k < A, k \neq B} z^k p^k(z) + z^A P^{A+1}(z),$$

where $P(z)$ is the weight-enumerator of $\mathcal{P}$.

Proof. It is obvious that the grammar uniquely parses the empty path, so let $P \in \mathcal{P}$ have length $n > 0$. Let $D_0$ denote the first time $P$ returns to height 0 and let $k$ be the length of the descending run in $P$ ending with the step $D_0$. Let $D_{k-1}...D_0$ denote this descending run.

If $k < A$, then write $P = UWD_{k-1}...D_0P_0$. It is clear that $W$ is a walk from height 1 to height $k - 1$ and $P_0$ is a Dyck path, where both $W$ and $P_0$ have the same restrictions on descending runs as $P$. Thus, $P_0 \in \mathcal{P}$ and, letting $U_i$ indicate the last up-step from height $i$ in $W$, we have

$$W = P_1U_1P_2U_2...P_{k-2}U_{k-2}P_{k-1}.$$  

By the definition of $U_i$, $P_j$ is a Dyck path shifted to height $j$ with the same restrictions on descending runs as $P$. This uniquely parses $P$ into the case $(U \mathcal{P})^{k-1}UD^k \mathcal{P}$.

If the first descending run in $P$ that hits height zero has length $k \geq A$, then write

$$P = WD_{A-1}...D_0P_0.$$  

It is obvious that $P_0 \in \mathcal{P}$, and $W$ is a walk from height 0 to height $A$ which never returns to height 0. By an argument analogous to the previous paragraph, we can decompose $W$ as

$$W = U_0P_1U_1P_2U_2P_3...U_{A-2}P_A,$$

where $P_i \in \mathcal{P}$. Thus, $W$ is of the form $(U \mathcal{P})^A$ and $P$ is uniquely parsed into the case $(U \mathcal{P})^A D^A \mathcal{P}$.

This proves that $\mathcal{P}$ can be generated by the given grammar. It is clear that $(U \mathcal{P})^{k-1}UD^k \mathcal{P}$ is contained in $\mathcal{P}$ for $1 \leq k < A$ and $k \neq B$. $(U \mathcal{P})^A D^A \mathcal{P}$ is also contained in $\mathcal{P}$, since concatenating $D^A$ to a Dyck path in $\mathcal{P}$ does not change the length of any down-runs modulo $A$. The different cases defined on the right-hand side are clearly disjoint, so the grammar is unambiguous.

Proposition 10. Let $A, B \in \mathbb{Z}_{\geq 0}$ such that $B \geq A$. The set $\mathcal{P}$ of Dyck paths avoiding down-run lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq 0}\}$ satisfies the grammatical equation

$$\mathcal{P} \cup (U \mathcal{P})^{B-1}UD^B \mathcal{P} = \{\text{EmptyPath}\} \cup \bigcup_{1 \leq k < A} (U \mathcal{P})^{k-1}UD^k \mathcal{P} \cup (U \mathcal{P})^A D^A \mathcal{P},$$

and therefore

$$P(z) + z^BP^B(z) = 1 + \sum_{0 \leq k < A} z^k p^k(z) + z^A P^{A+1}(z).$$

where $P(z)$ is the weight-enumerator of $\mathcal{P}$.  

Note that the right-hand side is nearly identical to that of the previous claim – the difference being that we can get paths in $(U\mathcal{P})^{B-1}UD^B\mathcal{P}$, which we will show below.

**Proof.** If $P$ is a path in $\mathcal{P}$, then we can uniquely parse $P$ into a case of the right hand side following the same argument given in the proof of Proposition 9. Note that

$$(U\mathcal{P})^{B-1}UD^B\mathcal{P} = (U\mathcal{P})^A(U\mathcal{P})^{B-A-1}UD^B-AD^A\mathcal{P} = (U\mathcal{P})^{A-1}U\{\mathcal{P}(U\mathcal{P})^{B-A-1}UD^B-A\}D^A\mathcal{P}$$

and the expression in brackets, $\mathcal{P}(U\mathcal{P})^{B-A-1}UD^B-A$, is contained in $\mathcal{P}$. Thus, any path in $(U\mathcal{P})^{B-1}UD^B\mathcal{P}$ is uniquely parsed into the case $(U\mathcal{P})^AD^A\mathcal{P}$.

Thus, the left-hand side of the equation is generated by the right-hand side. The different cases defined on the right-hand side are also clearly disjoint. It remains to show that all paths generated by the right-hand side are contained in the left-hand side. It is clear that $(U\mathcal{P})^{k-1}UD^k\mathcal{P}$ is contained in $\mathcal{P}$ for $1 \leq k < A$. For $W \in (U\mathcal{P})^AD^A\mathcal{P}$,

$$W = U_P U_{P_2} ... U_{P_A} D^A P_0.$$  

Let $\ell$ be the length of the last down-run in $P_A$. If $\ell \neq B (\bmod A)$, then $W$ contains no down-runs of lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq 0}\}$ and $W \in \mathcal{P}$. If $\ell \equiv B (\bmod A)$, then $\ell \leq B - A$. When $\ell < B - A$, the corresponding down-run in $W$ has length $< B$, and again $W$ contains no down-runs of lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq 0}\}$. For $\ell = B - A$, write

$$P_A = W_A U_A W_{A+1} U_{A+1} W_{A+2} ... W_B U_B D^{B-A},$$

where $U_i$ is the last up-step from height $i$ in $P$, and thus $W_i$ is a Dyck path shifted to height $i$.

$$W = U_P U_{P_2} ... U_{P_{A-1}} U_P A D^A P_0 = U_P U_{P_2} ... U_{P_{A-1}} U(W_A U_A ... W_B U_B D^{B-A})D^A P_0 = U_P U_{P_2} ... U_{P_{A-1}} U W_A U_A ... W_B U_B D^B P_0$$

and, hence, $W \in (U\mathcal{P})^B D^B \mathcal{P}$. \hfill $\Box$

**Proposition 11.** Let $r \in \mathbb{Z}^+$. The set $\mathcal{P}$ of Dyck paths avoiding ascending and descending runs of lengths $L \in \{1, ..., r\}$ satisfies the grammatical equation

$$\mathcal{P} \cup UD\mathcal{P} = \{\text{EmptyPath}\} \cup U^{r+1}D^{r+1}\mathcal{P} \cup U\mathcal{P} D\mathcal{P}.$$ 

and therefore

$$P(z) + zP(z) = 1 + z^{r+1}P(z) + zP^2(z),$$

where $P(z)$ is the weightenumerator of $\mathcal{P}$.

**Proof.** If $P \in \mathcal{P}$ is the empty path, then the grammar uniquely parses $P$. Otherwise, $P \in \mathcal{P}$ must begin with an ascending run of length $\ell > r$. If $\ell = r + 1$, then clearly $U^{r+1}$ must be immediately followed by the descending run $D^{r+1}$, and $P$ is uniquely parsed into the case $U^{r+1}D^{r+1}\mathcal{P}$.

If $\ell > r + 1$, then let $D_0$ denote the step where $P$ returns to height 0 for the first time and write

$$P = U_P D_0 P_2.$$ 

It is obvious that $P_2 \in \mathcal{P}$ and $P_1$ is a Dyck path shifted to height 1. By restrictions on $P$, the final descending run in $P_1$ must have length $L \geq r$. If $L = r$ then the preceding ascending run ends at
height \( r + 1 \). But the ascending runs in \( P \) must have length of at least \( r + 1 \), and hence \( P_1 \) hits height 0, contradicting the definition of \( D_n \). From here, it is clear that \( P_1 \) has the same restrictions on ascending and descending runs as \( P \). Thus, \( P \) is uniquely parsed into the case \( UPD\).

Since it is trivial that \( UDP \) is contained in \( UPD\), we have shown that the left-hand side of the given equation is generated by the right-hand side. It is also obvious that the cases defined on the right-hand side are disjoint and that \( \{\text{EmptyPath}\} \cup U^{r+1}D^{r+1}P \) is contained in \( \mathcal{P} \). A path \( UP_1DP_2 \in UPD\) is contained in \( UDP \) if \( P_1 \) is the empty path and \( \mathcal{P} \) otherwise. Thus, \( \mathcal{P} \) satisfies the given grammatical equation.

\[ \square \]

**Proposition 12.** Let \( m, n \in \mathbb{Z}^+ \). The set \( \mathcal{P} \) of Dyck paths avoiding ascending runs of lengths in \( \{1, \ldots, m\} \) and descending runs of lengths in \( \{1, \ldots, n\} \) satisfies the "grammatical equation"

\[ \mathcal{P} \cup UDP = \{\text{EmptyPath}\} \cup UPD \cup U^{n+1}D^{n+1}(PD)^{m-n}\mathcal{P}, \text{ if } m \geq n \]  

(1)

\[ \mathcal{P} \cup UDP = \{\text{EmptyPath}\} \cup UPD \cup (U\mathcal{P})^{n-m}U^{m+1}D^{n+1}\mathcal{P}, \text{ if } m \leq n. \]  

(2)

**Proof.** We have already shown that this statement is true for \( m = n \). Suppose \( m > n \). If \( P \in \mathcal{P} \) is the empty path, then the grammar uniquely parses \( P \). Otherwise, \( P \) must begin with an ascending run of length \( \ell > m \). If \( \ell = m + 1 \) then \( U^{m+1} \) is followed by a descending chain of length of at least \( n + 1 \). Let \( D_i \) denote the first time \( P \) returns to height \( i \) for \( 0 \leq i \leq m - n - 1 \), and write

\[ P = U^{m+1}D^{n+1}P_{m-n}D_{m-n-1} \ldots P_1D_0P_0. \]

It is obvious that \( P_1 \) is a Dyck path, shifted to height \( i \), that has the same restrictions on ascending runs and descending runs (with the exception of the final descending run) as \( P \). Since \( P_1 \) is a Dyck path, its final descending run must be at least as long as the ascending run preceding it. Thus, \( P_i \) is either the empty path or ends with a descending run of length \( L > m > n \). Thus, \( P \) is uniquely parsed into the case \( U^{m+1}D^{n+1}(PD)^{m-n}\mathcal{P} \).

If \( \ell > m + 1 \) then, letting \( D_0 \) denote the first time \( P \) returns to height 0, write

\[ P = UP_1DP_0. \]

Clearly, \( P_0 \in \mathcal{P} \), and \( P_1 \) is a Dyck path shifted to height 1 and has the same restrictions on ascending runs as \( P \). Using the same argument as for \( P_1 \) in the previous case, the descending runs in \( P_1 \) also have the same restrictions as \( P \). This uniquely parses \( P \) into the case \( UPD\). Finally, it is obvious that \( UDP \) is contained in \( UPD \), so the left-hand side of (1) is generated by the right-hand side.

Now, suppose \( m < n \) and let \( P \) be an element in \( \mathcal{P} \). Let \( L \) denote the length of the descending run where \( P \) returns to height 0 for the first time. If \( L = n + 1 \), then write

\[ P = WU^{m+1}D^{n+1}P_0, \]

where \( W \) is a walk from height 0 to \( n - m \) with the same restrictions on ascending and descending runs as \( P \) and \( P_0 \in \mathcal{P} \). Decomposing \( W \) and letting \( U_i \) denote the last time \( W \) leaves height \( i \), write

\[ W = U_0P_1U_1P_2 \ldots U_{m-n-1}P_{m-n}. \]

Then, for all \( i, P_i \) is clearly a Dyck path with the same restrictions on descending runs and ascending runs (with the exception of the first run) as \( P \). The first ascending run in \( P_i \) must be longer than
the descending run that follows it, which has length of at least \(n + 1 > n\). Thus, \(P_i \in \mathcal{P}\) and \(P\) has the grammar \((U\mathcal{P})^{n-m}U^{m+1}D^{n+1}\mathcal{P}\).

If \(L > n + 1\), then write

\[
P = UP_1DP_2
\]

, where \(D\) denotes the first time \(P\) returns to height 0. Then, using the same argument as we gave when \(\ell > m + 1\), \(P\) is parsed into the case \(UPDP\). Since \(UPDP\) is obviously contained in \(UPDP\), we have shown that the left-hand side of (2) is generated by the right-hand side.

In both (1) and (2), it is clear that the cases on the right-hand side are disjoint and the empty path is an element of \(\mathcal{P}\). Also, \(UP_1DP_2 \in UPDP\) is contained in \(\mathcal{P}\) if \(P_1\) is not the empty path, and is contained in \(UPDP\) otherwise. In (1), \(U^{m+1}D^{n+1}(DP)^{m-n}\mathcal{P}\) is contained in \(\mathcal{P}\), since all ascending runs clearly avoid restrictions on \(\mathcal{P}\) and the descending runs are formed by concatenating down-steps to descending runs of length of at least \(n - 1\). Similarly in (2), we have that \((UP)^{n-m}U^{m+1}D^{n+1}\mathcal{P}\) is contained in \(\mathcal{P}\). Thus, \(\mathcal{P}\) satisfies the given grammatical equation in both cases.

\(\square\)

**Proposition 13.** Let \(r, k \in \mathbb{Z}^+\) and let \(\mathcal{P}\) be the set of Dyck paths avoiding ascending runs of length \(\{1, ..., r\}\) and descending runs of length \(\{k + 1, ..., r\}\). Then the 'grammar' of \(\mathcal{P}\) is

\[
\mathcal{P} \cup UDP \cup U^{r+1}D^k(DP)^{r+1-k} = \{\text{EmptyPath}\} \cup U\mathcal{P}DP \cup U^{r+1}D^{r+1}\mathcal{P} \cup U^{r+1}(DP)^{r+1}
\]

**Proof.** If \(P \in \mathcal{P}\) is the empty path, then the grammar uniquely parses \(P\). Otherwise, \(P\) begins an ascending run of length \(\ell > r\), and we can deduce that it also ends with a descending run of length \(L > r\). If \(\ell > r + 1\), then let \(D_0\) denote the first time that \(P\) returns to the \(x\)-axis and write

\[
P = UP_1DP_2D_0P_0.
\]

It is easy to see that \(P_0\) is a path in \(\mathcal{P}\) and \(P_1\) is a Dyck path shifted to height 1. The initial ascending run in \(P_1\) has length \(\ell - 1 > r\). Thus, all ascending runs in \(P_1\) have length of at least \(r + 1\) and, since \(P_1\) is a shifted Dyck path, the final descending run in \(P_1\) must also have length of at least \(r + 1\). From here, it is easy to see that \(P_1\) has the same restrictions on ascending and descending runs as \(P\). \(P\) is therefore uniquely parsed into the case \(UPDP\).

Suppose \(\ell = r + 1\). Let \(D_i\) be the step where \(P\) returns to height \(i\) for the first time and write

\[
P = U^{r+1}D_rP_r...D_0P_0.
\]

\(P_i\) is a Dyck path for all \(i\) and, if \(P_i\) is not the empty path, it must end with a descending run of length \(r + 1\) by restrictions on ascending runs. Thus \(P_i\) is a path in \(\mathcal{P}\), and \(P\) is parsed into the case \(U^{r+1}(DP)^{r+1}\).

It is trivial that \(UPDP\) is contained \(U\mathcal{P}DP\) and \(U^{r+1}D^k(DP)^{r+1-k}\) is contained in \(U^{r+1}(DP)^{r+1}\). Thus, the left-hand side is generated by the right-hand side. Note that, on the left-hand side,

\[
UDP \cap \mathcal{P} = UDP \cap U^{r+1}D^k(DP)^{r+1-k} = \emptyset,
\]

however

\[
\mathcal{P} \cap U^{r+1}D^k(DP)^{r+1-k} = U^{r+1}D^{r+1}\mathcal{P}.
\]

Looking at the right-hand side, it is clear that \(\{\text{EmptyPath}\}, U\mathcal{P}DP, \text{ and } U^{r+1}(DP)^{r+1}\) are disjoint, and \(U^{r+1}D^{r+1}\mathcal{P}\) is contained in \(U^{r+1}(DP)^{r+1}\). Note that this resolves the issue of double counting paths in \(U^{r+1}D^{r+1}\mathcal{P}\) on the left-hand side. Thus, all that remains to show is that all the paths generated by the right-hand side are contained in the left-hand side.
The path \( UP_1DP_0 \) in \( UPD\mathcal{P} \) is clearly in \( \mathcal{P} \) if \( P_1 \) is not the empty path and in \( UDP \) otherwise. For \( W \) in \( U^{r+1}(D\mathcal{P})^{r+1} \), write
\[
W = U^{r+1}D_1P_1...D_1P_1D_0P_0.
\]
Choose the smallest \( i \) such that \( P_{r-i} \) is not the empty path or, if no such \( i \) exists, set \( i = r \). Then the first descending run in \( W \) has length \( i + 1 \). If \( i \geq k \) then \( W \) is an element of \( U^{r+1}D^k(D\mathcal{P})^{r+1-k} \). Otherwise, we claim that \( W \) is a path in \( \mathcal{P} \). It is clear that \( W \) is a Dyck path and we have seen that nonempty \( P_j \in \mathcal{P} \) must end in a descending run of length of at least \( r+1 \). Thus, we only need to show that the first descending run in \( W \) follows the restrictions in \( \mathcal{P} \). This is clearly true since \( i < k \). Hence \( W \in \mathcal{P} \), and \( \mathcal{P} \) satisfies the grammatical equation as desired. \( \square \)

4 Conclusion

We have given several grammatical proofs of various combinatorial results (some lifted from [6]) and established some infinite families of grammars. Our methods work because we are able to derive context-free grammars describing certain restricted classes Dyck paths, namely when our restrictions involved sets of arithmetic progressions. It is natural to ask if context-free grammars exist for other types of restrictions, but this is beyond our current scope.

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