Perturbative Part of the Non-Singlet Structure Function $F_2$ in the Large-$N_F$ Limit

L. Mankiewicz
Institute for Theoretical Physics
TU-München, D-85747 Garching, Germany

and

M. Maul, E. Stein
Institute for Theoretical Physics, Johann Wolfgang Goethe-Universität,
D-69054 Frankfurt am Main, Germany

We have calculated $\overline{MS}$ Wilson coefficients and anomalous dimensions for the non-singlet part of the structure function $F_2$ in the large-$N_F$ limit. Our result agrees with exact two and three loop calculations and gives the leading $N_F$ dependence of the perturbative non-singlet Wilson coefficients to all orders in $\alpha_S$.

PACS: 12.38.Cy, 13.60.Hb, 12.38.-t
Keywords: deep inelastic scattering, structure function $F_2$, perturbative coefficients, anomalous dimensions
High precision of current experimental data on deep-inelastic lepton-nucleon scattering makes it necessary to extend perturbative calculations to higher and higher orders. Although in principle QCD perturbation theory is well understood, in practice its technical complexity has limited the existing state-of-the-art next to next to leading order (NNL) calculations only to low Mellin moments $M_N, N = 2, 4, 6, 8$ of the structure functions $F_2$ and $F_L$.

Recently, it has been realized that in the large-$N_F$ limit of QCD corresponding calculations can be performed exactly to all orders [3, 4]. Although it is perhaps not easy to argue about the importance of results obtained in this limit for the real world, they are certainly useful either as an independent check of complicated computer algebra algorithms used in the exact calculations, or as a starting point for further approximation procedures, such as ‘Naive Nonabelianization’ (NNA) [5] or ‘Asymptotic Pade Approximation’ [6].

The goal of the present paper is to analyze moments $M_{2,N}(Q^2)$ of the flavor non-singlet (NS) component of twist-2 part of structure function $F_2(x, Q^2)$

$$M_{2,N}(Q^2) = \int_0^1 dx x^{N-2} F_{2NS}(x, Q^2) \quad N = 2, 4, \ldots ,$$

defined through the well known decomposition [8] of the hadronic scattering tensor of unpolarized deep inelastic lepton nucleon scattering in terms of the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$.

$$W_{\mu\nu}(p, q) = \frac{1}{2\pi} \int d^4z e^{iqz} \langle p| J_{\mu}(z) J_{\nu}(0)|p\rangle = \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) \frac{1}{2x} F_L(x, Q^2)$$

$$- \left( g_{\mu\nu} + p_{\mu} p_{\nu} \frac{q^2}{(p \cdot q)^2} - \frac{p_{\mu} q_{\nu} + p_{\nu} q_{\mu}}{p \cdot q} \right) \frac{1}{2x} F_2(x, Q^2).$$

According to the Operator Product Expansion [8] one can arrange $M_{2,N}(Q^2)$ as a product

$$M_{2,N}(Q^2) = C_{2,N}(\alpha_S(Q^2)) A_N(Q^2)$$

of the Wilson coefficient $C_{2,N}(\alpha_S(Q^2))$ and the spin-averaged matrix element $A_N(Q^2)$ of a spin-$N$, twist-2 operator

$$\langle p| \bar{\psi} \gamma^{\{\mu_1} i D^{\mu_2} \ldots i D^{\mu_N}\} \psi |p\rangle_{\mu-n} = p^{\{\mu_1} \ldots p^{\mu_N\}} A_N(\mu^2).$$

In the above equation $\{\mu \ldots \nu\}$ indicates twist-2 projection i.e., symmetric and traceless combination. Validity of equation (3) requires that both renormalization and factorization scales have been set equal to the virtuality of the external photon $Q^2$. In addition, it has been implicitly understood that squares of quark charges are properly included on the right-hand side of Eq.(4), and that flavors of quark-operators $\psi$ have been combined to yield the difference between u- and d-quark matrix elements.
The $Q^2$-dependence of matrix elements $A_N(Q^2)$, written as a solution of the renormalisation group equation,

$$A_N(Q^2) = A_N(\mu^2) \cdot \exp \left( \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha_s' \frac{\gamma_N^{(NS)}(\alpha_s')}{\beta(\alpha_s')} \right).$$

is determined by a set of anomalous dimensions $\gamma_N(\alpha_S(Q^2))$. The perturbative expansion of $C_{2,N}$ and $\gamma_N$ results in

$$C_{2,N}(\alpha_S(Q^2)) = 1 + \sum_{j=0}^{\infty} C_{2,N}^{(j)} a_s^{j+1},$$

$$\gamma_N(\alpha_S(Q^2)) = \sum_{j=0}^{\infty} \gamma_N^{(j)} a_s^{j+1},$$

where we have introduced a shorthand notation $a_s = \frac{\alpha_S(Q^2)}{4\pi}$. In the following we present results for the Wilson coefficients $C_{2,N}(\alpha_S(Q^2))$ and anomalous dimensions $\gamma_N(\alpha_S(Q^2))$ obtained in the large-$N_F$ limit, $N_F \to \infty$, keeping $a_s \cdot N_F = \text{const}$.

Technically, we have found it advantageous to follow the approach developed in [4, 9] and calculate the generating function $G(u; N)$

$$C_{2,N}^{(j)} = \left( -\frac{2}{3} N_F \right)^j \frac{d^j}{du^j} \bigg|_{u=0} G(u; N) + \mathcal{O}(N_F^{j-1}),$$

for the Wilson coefficients $C_{2,N}$ in the large-$N_F$ limit. As explained in details e.g. in [3, 4], the dominant contribution in this limit arises from an arbitrary number of fermionic loop insertions into the gluon propagator. As a consequence, the problem can be reduced to calculation of the first-order radiative corrections using modified Landau-gauge gluon propagator

$$D_{\mu\nu}(k; u) = \frac{i}{(-k^2)^{1+u}} \left( g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right).$$

In calculations of gauge invariant quantities, such as in the present case, the longitudinal part does not contribute and can be dropped. The only complication arises due to renormalization - the ‘bare’ function $G_B(u; N)$ acquires a singularity at $u = 0$ and has to be renormalized by an appropriate counterterm. As discussed in [4, 9], in minimal subtraction schemes such as $\overline{\text{MS}}$, the counterterm can be obtained in a compact form. As a result the renormalized generating function $G_R(u; N)$ can be written as

$$G_R(u; N) = G_B(u; N) + \tilde{G}_0(u; N)/u,$$

where $\tilde{G}_0(u; N)$ is determined in terms of expansion coefficients $g_k(N)$ of another function $G_0(u; N)$ in the variable $u$ around the origin

$$G_0(u; N) = \sum_{k=0}^{\infty} g_k(N) u^k,$$

$$\tilde{G}_0(u; N) = \sum_{k=0}^{\infty} \frac{1}{k!} g_k(N) u^k.$$
The merit of this complicated and apparently indirect construction lies in the fact that
the function $G_0(u; N)$ can be found directly from the calculation of one-loop radiative corrections computed in $d$-dimensions using the modified propagator $[8]$, while the anomalous dimension follows from the corresponding counterterm in a standard way. An interested reader should consult Refs. [4, 9], in particular Appendix A of Ref. [4], for details. To compute $G_R(u; N)$ and $\gamma_N$ we have calculated the Feynman diagrams depicted on Figure [1], arriving at

$$G_R(u; N) = G^A_R(u; N) + G^B_R(u; N) + G^C_R(u; N) + G^D_R(u; N)$$

$$G^A_R(u; N) = G^B_R(u; N) = 2C_F[\exp(-C)]^u \frac{\Gamma(1-u)}{u\Gamma(3-u)(1+u)}$$

$$+ \sum_{k=2}^{N} 2C_F[\exp(-C)]^u \frac{\Gamma(1-u)\Gamma(u+k-1)}{\Gamma(1+u)\Gamma(3-u)\Gamma(k)(u+k)}$$

$$- 2C_F[\exp(-C)]^u \frac{\Gamma(1-u)\Gamma(1+u)\Gamma(N)(u+N)}{\Gamma(3-u)\Gamma(1+u)\Gamma(N)(u+N)}$$

$$- \sum_{k=1}^{N} 2C_F[\exp(-C)]^u \frac{\Gamma(1-u)\Gamma(k)}{\Gamma(3-u)\Gamma(1+u)\Gamma(N)(u+N)}$$

$$+ \sum_{k=1}^{N-1} 4C_F[\exp(-C)]^u \frac{\Gamma(1-u)\Gamma(u+k)}{u\Gamma(3-u)\Gamma(1+u)\Gamma(N)(u+N)}$$

$$+ \bar{G}^A_0(u; N)$$

$$G^A_0(u; N) = - \sum_{k=2}^{N} \frac{C_F}{3} \frac{\Gamma(4+2u)}{\Gamma(1-u)\Gamma^2(2+u)\Gamma(1+u)(u+k)}$$

$$+ \bar{G}^A_0(u; N)$$

$$G^B_R(u; N) = 12C_F[\exp(-C)]^u \frac{\Gamma(1-u)\Gamma(N+u)}{\Gamma(3-u)(1+u+N)\Gamma(N)\Gamma(1+u)}$$

$$- 2C_F[\exp(-C)]^u \frac{\Gamma(1-u)(u^2-3u+2)\Gamma^2(N+u)}{u\Gamma(3-u)\Gamma(1+u)\Gamma(N+u+2)\Gamma(N)}$$

$$+ \bar{G}^B_0(u; N)$$

$$G^C_0(u; N) = \frac{C_F}{3} \frac{\Gamma(4+2u)\Gamma(N+u)}{\Gamma(1-u)\Gamma^3(1+u)\Gamma(N+u+2)}$$

$$+ \bar{G}^C_0(u; N)$$

$$G^D_R(u; N) = - 2C_F[\exp(-C)]^u \frac{\Gamma(1-u)\Gamma(N+u)}{u\Gamma(3-u)\Gamma(1+u)\Gamma(N)}$$

$$+ \bar{G}^D_0(u; N)$$

$$G^D_0(u; N) = \frac{C_F}{3} \frac{\Gamma(4+2u)}{\Gamma(1-u)\Gamma^2(1+u)\Gamma(3+u)}$$

(11)
for the generating function $G_R(u;n)$. In the above formula $C_F = 4/3$, and $C$ denotes the finite part of the quark loop insertion into the gluon propagator, $C_{MS} = -5/3$. The relation between $\tilde{G}_0(u;N)$ and $G_0(u;N)$ is given by equation (10). For the case of the anomalous dimension $\gamma_N(a_s)$ our result reads

$$
\gamma_N(a_s) = \gamma_A^N(a_s) + \gamma_B^N(a_s) + \gamma_C^N(a_s) + \gamma_D^N(a_s)
$$

$$
\gamma_A^N(a_s) = a_s C_F \frac{1}{3} \frac{\Gamma(4 + 2s)}{\Gamma(1 - s)\Gamma^2(1 + s)\Gamma(3 + s)}
$$

$$
\gamma_B^N(a_s) = -a_s C_F \sum_{k=2}^{N} \frac{1}{3} \frac{\Gamma(1 - s)\Gamma^2(2 + s)\Gamma(1 + s)(s + k)}{\Gamma(4 + 2s)\Gamma(N + 2)}
$$

$$
\gamma_C^N(a_s) = a_s C_F \frac{1}{3} \frac{\Gamma(4 + 2s)\Gamma(N + s)}{\Gamma(1 - s)\Gamma^3(1 + s)\Gamma(N + s + 2)}
$$

$$
\gamma_D^N(a_s) = a_s C_F \frac{1}{3} \frac{\Gamma(4 + 2s)}{\Gamma(1 - s)\Gamma^2(1 + s)\Gamma(3 + s)}
$$

(12)

where $s = \left(-\frac{2}{3}N_F\right) a_s$. Terms labeled as $A, B, C$ and $D$ in Eqs. (11) and (12) correspond to contributions of graphs depicted in Figure 1. We quote results obtained in the Feynman gauge i.e. contributions from the longitudinal part of the propagator (8), which cancel in the sum of all graphs, have been neglected. In the case of the anomalous dimension $\gamma_N(a_s)$ one can put together various terms in (12) and obtain a more compact expression

$$
\gamma_N(a_s) = a_s C_F \frac{1}{3} \frac{\Gamma(4 + 2s)}{\Gamma(1 - s)\Gamma(1 + s)^3}
$$

$$
\times \left[ \frac{1}{(s + 2)(s + 1)} - \frac{1}{(s + N + 1)(s + N)} + \frac{2}{(1 + s)^2} \sum_{k=2}^{N} \frac{1}{k + s} \right],
$$

(13)

$s = \left(-\frac{2}{3}N_F\right) a_s$, which agrees with result obtained earlier by Gracey [10].

Now, expanding the right-hand side of (13) in $a_s$

$$
\gamma_N(a_s) = b_1 C_F a_s + C_F \sum_{j=2}^{\infty} \left[ b_j \left(-\frac{2}{3}N_F\right)^{j-1} a_s^j \right]
$$

(14)

one can find explicit results for the anomalous dimensions in any fixed order of the perturbative expansion in the large-$N_F$-limit, see Ref. [10]. Similarly, using (7) one can derive from Eq. (14) fixed order predictions for the Wilson coefficients for the structure function $F_2$ in this limit. Note that large-$N_F$ generating function for the Wilson coefficients for the non-singlet part of the structure function $F_L(x,Q^2)$ is known from Refs. [10, 11].
In Ref. [2] Wilson coefficients for $F_L$ and $F_2$ have been calculated for the even moments $N = 2 \ldots 8$ exactly up to the third order in $\alpha_s$. The leading $N_F$ terms at the third and the fourth orders for an arbitrary $N$-th moment read

\[
C_{2,N}^{(3)} = C_F \left( -\frac{2}{3} N_F \right)^2 \left[ - \frac{9517}{216} - 2 \zeta(3) + \frac{1}{81} \frac{5659 + 108 \zeta(3)}{1 + N} - \frac{368}{9} \frac{1}{(1 + N)^2} + \frac{146}{9} \frac{1}{(1 + N)^3} \right]
\]
\[
- \frac{10}{3} \frac{1}{(1 + N)^4} + \frac{1}{81} \frac{1}{N} - \frac{2041 + 108 \zeta(3)}{N^2} + 30 \frac{134}{N^3} - \frac{30}{3} \frac{1}{N^4} + \frac{4955}{3} \frac{1}{N^5} + \frac{8}{3} \zeta(3) + \frac{4357}{162} \frac{32}{3} \frac{1}{N^2} - \frac{164}{9} \frac{1}{N} + \frac{4}{3} \frac{56}{(1 + N)^2} - \frac{1}{3} \frac{1}{(1 + N)^3} + \frac{392}{9} (1 + N) S_1(N-1) + (1 + \frac{2}{N} - \frac{2}{N}) S_1(N-1) S_2(N-1)
\]
\[
+ (\frac{-19}{6} + 2 \frac{16}{N^2} - \frac{2}{3} \frac{1}{N}) S_1(N-1)^3 + \frac{535}{18} \frac{1}{N^2} + 4 \frac{1}{3} \frac{1}{N} - \frac{2}{(1 + N)^2} - \frac{28}{3} \frac{1}{1 + N} S_2(N-1)
\]
\[
+ \frac{166}{9} \frac{4}{3} \frac{1}{N} + \frac{4}{3} \frac{1}{1 + N}) S_3(N-1) - \frac{20}{3} S_4(N-1)
\]
\[
+ \sum_{k=1}^{N-1} \left( -\frac{2}{9} \frac{146 k^4 - 139 k^3 - 8 k^2 + 3 k - 18}{(k+1)^3 k^2} S_1(k) \right)
\]
\[
+ \sum_{k=1}^{N-1} \left( \frac{2}{9} \frac{18 k^4 + 36 k^3 + 18 k^2}{(k+1)^3 k^2} S_1(k) S_2(k) \right)
\]
\[
+ \sum_{k=1}^{N-1} \frac{2}{3} \frac{(16 k^2 + 7 k - 3) S_1(k)^2}{k (k+1)^2} + \sum_{k=1}^{N-1} \frac{14}{3} \frac{S_1(k)^3}{k+1}
\]
\[
- \sum_{k=1}^{N-1} \frac{2}{3} \frac{(16 k^2 + 7 k - 3) S_2(k)}{k (k+1)^2} + \sum_{k=1}^{N-1} \frac{8}{3} \frac{S_3(k)}{k+1} \right] + O(N_F),
\]

\[
C_{2,N}^{(4)} = C_F \left( -\frac{2}{3} N_F \right)^3 \left[ \frac{386}{3} \frac{1}{(1 + N)^3} - \frac{325}{6} \frac{1}{(1 + N)^4} + \left( \sum_{k=1}^{N-1} \frac{-6}{k+1} \right) \frac{-1}{4} \zeta(3)
\]
\[
+ \frac{1}{27} \frac{27 \zeta(3) - 2068}{N^2} - \frac{1}{27} \frac{1571 + 27 \zeta(3)}{(1 + N)^2} - \frac{4955}{27} \frac{1}{N^3} + \frac{83}{6} \frac{1}{N^4} + \frac{34883}{162} \frac{1}{N^5} + \frac{1}{1620} \frac{(452200 + 5940 \zeta(3) - 2430 \zeta(4))}{1 + N} - \frac{1}{1620} \frac{540 \zeta(3) + 154360 + 2430 \zeta(4)}{N} + \frac{9}{4} \zeta(4)
\]
\[
+ \frac{23}{2} \frac{1}{N^5} + \frac{23}{2} \frac{1}{(1 + N)^5} + (-3 \zeta(4) + \frac{10}{3} \zeta(3) + \frac{25279}{324} + \frac{12}{N^4} - \frac{32}{N^5} + \frac{164}{3} \frac{1}{N^2}
\]
\[
- \frac{2077}{27} \frac{1}{N} \frac{(1 + N)^4 + (1 + N)^3 - 3}{3} \frac{1}{(1 + N)^2} + \frac{5731}{27} \frac{1}{1 + N} S_1(N-1)
\]
\[
+ \left( \frac{19}{2} - \frac{6}{N^2} + \frac{16}{(1 + N)^2} - \frac{28}{1 + N} \right) S_1(N-1) S_2(N-1)
\]

6
\begin{align}
\sum_{k=1}^{N-1} \left( -\frac{1}{3} \frac{(146 k^4 + 139 k^3 + 8 k^2 - 3 k + 18) S_2(k)}{k^2 (k+1)^3} \right) + \sum_{k=1}^{N-1} \left( \frac{3}{3} \frac{S_2(k)^2}{k+1} \right) \\
\sum_{k=1}^{N-1} \left( \frac{1}{3} \frac{(146 k^4 + 139 k^3 + 8 k^2 - 3 k + 18) S_1(k)^2}{k^2 (k+1)^3} \right) \\
\sum_{k=1}^{N-1} \left( -\frac{1}{27} \frac{(324 - 3128 k^6 - 4929 k^5 - 2427 k^4 - 257 k^3 - 333 k^2 + 270 k) S_1(k)}{(k+1)^4 k^3} \right) \\
\sum_{k=1}^{N-1} \left( \frac{2}{3} \frac{(16 k^2 + 7 k - 3) S_1(k)^3}{k (k+1)^2} \right) + \sum_{k=1}^{N-1} \frac{S_1(k)^4}{k+1} - \frac{281971}{1728} \\
\sum_{k=1}^{N-1} \left( -\frac{1}{27} \frac{(216 k^6 - 648 k^5 - 648 k^4 - 216 k^3) S_1(k) S_3(k)}{(k+1)^4 k^3} \right) \\
\sum_{k=1}^{N-1} \left( \frac{4}{3} \frac{(16 k^2 + 7 k - 3) S_3(k)}{k (k+1)^2} \right) \\
\sum_{k=1}^{N-1} \left( -\frac{1}{3} \frac{(18 k^4 + 36 k^3 + 18 k^2) S_1(k)^2 S_2(k)}{(k+1)^3 k^2} \right) \\
\sum_{k=1}^{N-1} \left( -\frac{1}{27} \frac{(864 k^6 + 2106 k^5 + 1458 k^4 + 54 k^3 - 162 k^2) S_1(k) S_2(k)}{(k+1)^4 k^3} \right) \right] + O(N_F^2),
\end{align}

where $S_a(N) = \sum_{k=1}^{N} \frac{1}{k^a}$. The corresponding results for the longitudinal structure function $F_L$ read

\begin{align}
C_{L,N}^{(3)} &= \left( -\frac{2}{3} N_F \right)^2 \frac{4 C_F}{1 + N} \left[ 203 \frac{19}{18} \frac{19}{3(1 + N)} + \frac{2}{(1 + N)^2} + \left( \frac{19}{3} - \frac{2}{(1 + N)} \right) \right] S_1(N - 1) \\
&\quad + S_1^2(N - 1) - S_2(N - 1) \right] + O(N_F),
\end{align}
\[
C_{L,N}^{(4)} = \left(-\frac{2}{3}N_F\right)^3 \frac{C_F}{1 + N} \left[\frac{4955}{27} - \frac{406}{3(1 + N)^2} + \frac{76}{(1 + N)^3} - \frac{24}{27(1 + N)^3} \right] \\
+ \left(\frac{406}{3} - \frac{76}{(1 + N)^2} + \frac{24}{(1 + N)^3} \right) S_1(N - 1) + \left(38 - \frac{12}{1 + N}\right) S_2^2(N - 1) \\
+ 4S_1^3(N - 1) - \left(38 - \frac{12}{1 + N}\right) + 12S_1(N - 1) \right) S_2(N - 1) \\
+ 8S_3(N - 1) \right] + \mathcal{O}(N_F^2),
\]

\[
C_{L,N}^{(5)} = \left(-\frac{2}{3}N_F\right)^4 \frac{C_F}{1 + N} \left[\frac{69766}{81} - \frac{19820}{27(1 + N)} + \frac{1624}{3(1 + N)^2} - \frac{304}{(1 + N)^3} + \frac{96}{1 + N} \right] \\
+ \left(\frac{19820}{27} - \frac{1624}{3(1 + N)^2} + \frac{304}{(1 + N)^3} - \frac{96}{1 + N} \right) \\
-(152 - \frac{48}{1 + N}) S_2(N - 1) + 32S_3(N - 1) \right) S_1(N - 1) \\
+ \left(\frac{812}{3} - \frac{152}{1 + N} + \frac{48}{(1 + N)^2} - 24S_2(N - 1) \right) S_2^2(N - 1) \\
+ \left(\frac{152}{3} - \frac{16}{1 + N}\right) S_3^3(N - 1) + 4S_4^4(N - 1) \\
- \left(\frac{812}{3} - \frac{152}{1 + N} + \frac{48}{(1 + N)^2} \right) S_2(N - 1) + 12S_2^2(N - 1) \\
+ \left(\frac{304}{3} - \frac{32}{1 + N}\right) S_3(N - 1) - 24S_4(N - 1) \right] + \mathcal{O}(N_F^3),
\]

respectively. We have checked that the first, second (not quoted here) and the third order coefficients agree with the leading-$N_F$ terms extracted from Ref. [4]. Results for higher order terms are too long to be written down explicitly. Instead, we have prepared a Maple program for the numerical evaluation of Wilson coefficients in the large-$N_F$ limit, available on request, which allows to compute them using results of [10, 11] and of the present paper.

Finally, in Tables [1] and [2] we have compared the NNA approximants for the coefficient functions $C_{2,N}(a_s)$ with the exact results obtained in Ref. [4], and evaluate the NNA prediction for the $\mathcal{O}(a_s^4)$ terms. It is seen that although typically the NNA procedure predicts correctly the magnitude of the perturbative coefficients, with increasing precision as $N$ becomes larger, the numerical accuracy is not optimal. We have also checked that a similar procedure applied to the anomalous dimension $\gamma_N(a_s)$ gives much worse results which is probably connected to the fact that in the latter case the perturbative expansion is not dominated by renormalons.

**Acknowledgment:** We are grateful to V. Braun and M. Beneke for useful discussions. This work has been supported by BMBF, DFG (G. Hess Programm), DESY, and German-Polish exchange program X081.91. L. M. was supported in part by the KBN grant 2 P03B 065 10.
Table 1: Comparison of the NNA approximants to the exact results of the coefficient function $C_{2,N}(a_s)$ obtained in [2] up to order $O(a_s^3)$ for $N_F = 3$. The last column contains also the prediction for the $O(a_s^4)$ terms. The $O(a_s)$ corrections agree of course exactly.

| $N_F = 3$ | Exact results [2] | NNA approximants |
|-----------|-------------------|------------------|
| $N = 2$   | $1.69377 a_s^2 + 1.42209 a_s^3$ | $71.99999 a_s^2 + 1099.02 a_s^3 + 26193.11796 a_s^4$ |
| $N = 4$   | $91.3797 a_s^2 + 1675.76 a_s^3$ | $229.3368 a_s^2 + 4256.46 a_s^3 + 103655.6199 a_s^4$ |
| $N = 6$   | $218.3596 a_s^2 + 5004.63 a_s^3$ | $378.1762 a_s^2 + 7775.28 a_s^3 + 209938.7405 a_s^4$ |
| $N = 8$   | $357.0330 a_s^2 + 9357.69 a_s^3$ | $511.9851 a_s^2 + 11283.4 a_s^3 + 306602.1048 a_s^4$ |

Table 2: Comparison of the NNA approximants to the exact results of the coefficient function $C_{2,N}(a_s)$ obtained in [2] up to order $O(a_s^3)$ for $N_F = 4$. The last column contains also the prediction for the $O(a_s^4)$ terms. The $O(a_s)$ corrections agree of course exactly.

| $N_F = 4$ | Exact results [2] | NNA approximants |
|-----------|-------------------|------------------|
| $N = 2$   | $-3.63957 a_s^2 - 169.747. a_s^3$ | $66.66666 a_s^2 + 942.230 a_s^3 + 20792.94153 a_s^4$ |
| $N = 4$   | $74.3918 a_s^2 + 901.570 a_s^3$ | $212.3489 a_s^2 + 3649.22 a_s^3 + 82285.17302 a_s^4$ |
| $N = 6$   | $190.3465 a_s^2 + 3454.66 a_s^3$ | $350.1631 a_s^2 + 6666.05 a_s^3 + 159511.6507 a_s^4$ |
| $N = 8$   | $319.1081 a_s^2 + 6973.59 a_s^3$ | $474.0603 a_s^2 + 9673.72 a_s^3 + 243390.6360 a_s^4$ |
References

[1] S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, Nucl. Phys. B 427 (1994) 41.

[2] By S.A. Larin, P. Nogueira, T. van Ritbergen, J.A.M. Vermaseren, The three loop QCD calculation of the moments of deep inelastic structure functions. NIKHEF-96-010, hep-ph/9605317

[3] D. J. Broadhurst, Zeit. Phys. C 58 (1993) 339.

[4] M. Beneke, V. M. Braun Nucl. Phys. B 426 (1994) 301.

[5] D. J. Broadhurst and A. G. Grozin, Phys. Rev. D 52 (1995) 4082; M. Beneke and V. M. Braun Phys. Lett. B 348 (1995) 513; P. Ball, M. Beneke, V. M. Braun Nucl. Phys. B 452 (1995) 563; M. Beneke, Nucl. Phys. B 405 (1993) 424.

[6] J. Ellis, M. Karliner, M. A. Samuel, A prediction for the four loop beta function in QCD, CERN-TH-96-327, hep-ph/9612202; J. Ellis, E. Gardi, M. Karliner, M. A. Samuel, Phys. Rev. D 54 (1996) 6986.

[7] R. G. Roberts, The structure of the proton. Cambridge University Press, Cambridge (1990).

[8] F. J. Yndurain, The theory of quark and gluon interactions, Springer Verlag, (1993).

[9] P. Ball, M. Beneke, V. M. Braun Nucl. Phys. B 452 (1995) 563.

[10] J. A. Gracey Phys. Lett. B 322 (1994) 141; J. A. Gracey, Large-Nf methods for computing the perturbative structure of deep inelastic scattering, talk presented at Fourth International Workshop on Software Engineering and Artificial Intelligence for High Energy and Nuclear Physics, Pisa, Italy, 3-8 April 1995, hep-ph/9509276.

[11] E. Stein, M. Meyer-Hermann, L. Mankiewicz, and A. Schäfer, Phys. Lett. B 376 (1996) 177.
Figure 1: Graphs $A, B, C$ and $D$ which contribute to the calculation of the perturbative part of $F_2(x, Q^2)$ in the large-$N_F$ limit.