Non-equilibrium almost-stationary states and linear response for gapped quantum systems

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Abstract

We prove the validity of linear response theory at zero temperature for perturbations of gapped Hamiltonians describing interacting fermions on a lattice. As an essential innovation, our result requires the spectral gap assumption only for the unperturbed Hamiltonian and applies to a large class of perturbations that close the spectral gap. Moreover, we prove formulas also for higher order response coefficients.

Our justification of linear response theory is based on a novel extension of the adiabatic theorem to situations where a time-dependent perturbation closes the gap. According to the standard version of the adiabatic theorem, when the perturbation is switched on adiabatically and as long as the gap does not close, the initial ground state evolves into the ground state of the perturbed operator. The new adiabatic theorem states that for perturbations that are either slowly varying potentials or small quasi-local operators, once the perturbation closes the gap, the adiabatic evolution follows non-equilibrium almost-stationary states (NEASS) that we construct explicitly.

Keywords. Linear response theory, adiabatic theorem, non-equilibrium stationary state, space-adiabatic perturbation theory, Kubo formula.

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1 Introduction

The simplicity and the empirical success of linear response theory [15] make it a formalism widely used in physics to calculate the response of systems in thermal equilibrium to external perturbations. However, its validity for extended systems is based on properties of the microscopic dynamics, which are often difficult to establish. It is therefore

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not surprising that the rigorous justification of linear response theory based on first
principles in specific models is a constant theme in mathematical physics, which was
prominently advertised for example by Simon [24] already in 1984.

In this work we prove the validity of linear and also higher order response theory for
perturbations of gapped interacting quantum Hamiltonians on the lattice and at zero
temperature. This framework is relevant, for example, for (topological) insulators in
solid state physics such as quantum Hall systems.

More specifically, we consider a family of quasi-local Hamiltonians for systems of
interacting fermions on finite cubes $\Lambda \subset \mathbb{Z}^d$ with a spectral gap above the ground state,
whose size is bounded below by a positive constant uniformly in the volume $|\Lambda|$. Then,
according to equilibrium statistical mechanics, the equilibrium state of the system at
sufficiently low temperature is very close to its ground state. A question of fundamental
physical importance is to understand the “response” of such systems to static pertur-
bations as, for example, a weak external electric field. Here “response” refers to the
change of expectation values of physical quantities which are induced by adiabatically
switching on the perturbation. Linear response theory proceeds by applying first or-
der time-dependent perturbation theory in an uncontrolled way, cf. Section 4. Thus,
any justification of linear response theory in the present context starts necessarily from
the analysis of solutions of the time-dependent Schrödinger equation in the adiabatic
limit. However, the standard adiabatic theorem, which provides asymptotic expansions
of these solutions to any order in the adiabatic parameter, falls short for extended in-
teracting systems for two reasons. Firstly, it yields norm-estimates that are not and
cannot be uniform in the volume $|\Lambda|$. Secondly, it must be assumed that a spectral gap
remains open uniformly in the volume $|\Lambda|$ even if the perturbation is fully turned on.

Recently, Bachmann et al. [2] were able to prove an adiabatic theorem for expec-
tation values of local observables in interacting spin systems with error estimates that
are uniform in the volume $|\Lambda|$. Their result was slightly extended and translated to the
setting of lattice fermions in [18]. While the result of Bachmann et al. is a technical
and conceptual breakthrough, it is still an extremely difficult open problem to prove
their main assumption, namely the stability of the gap of a generic gapped many-body
Hamiltonian under small perturbations. More importantly, the linear response formal-
ism is expected to be applicable also in situations where the perturbation closes the
spectral gap and the system is driven into an (almost) stationary state that need not
be an eigenstate.

In this article we formulate and prove a novel adiabatic theorem with a gap as-
sumption only on the unperturbed Hamiltonian. The class of allowed perturbations
contains slowly varying but not necessarily small potentials and small quasi-local oper-
ators. It is shown that, once the spectral gap closes, the adiabatic evolution no longer
follows the ground state of the system—which is an invariant state for the instanta-
aneous Hamiltonian—but instead a certain almost-invariant state for the now gapless
Hamiltonian. As these almost-invariant states are neither eigenstates nor functions of
the Hamiltonian, we call them non-equilibrium almost-stationary states (NEASS). Rig-
orous and uniform asymptotic expansions of the NEASS, into which the system evolves
when adiabatically turning on a perturbation, then allow for a straightforward proof of linear and higher order response theory.

Since the formulation of precise statements requires the implementation of a decent amount of non-standard mathematical concepts, we refrain from stating theorems in the introduction and instead briefly explain the main conceptual ideas behind our proof. As realised and worked out in [2], the key concepts for adiabatic approximations that hold uniformly in the volume are locality and finite speed of propagation in lattice systems. By assumption, all operators appearing (the Hamiltonian, the perturbation, the observables) are quasi-local, i.e. they are sums of local terms. While the number of summands in these operators increases with increasing volume, in any bounded region only finitely many summands make a sizeable contribution. Moreover, thanks to Lieb-Robinson bounds [17] no long range correlations are induced by the dynamics and, as a consequence, the spectral flow is generated by a quasi-local operator [12, 3]. This allowed Bachmann et al. [2] to control adiabatic approximation errors for expectations of local observables uniformly in the volume.

To dispose of the gap assumption for the perturbed Hamiltonian, in the present article we use that small quasi-local perturbations and slowly varying potentials both leave intact a local gap structure, even though the full perturbed Hamiltonian might not be gapped anymore. For small quasi-local perturbations this is expected, because for any fixed volume stability of the gap follows from standard perturbation theory. Slowly varying potentials, on the other hand, are locally almost constant and thus only shift the local terms in the Hamiltonian by a multiple of the identity. As a consequence, the NEASS of the perturbed system can be constructed by applying a unitary transformation that is generated by a sum of local terms to the ground state of the unperturbed Hamiltonian. The resulting state is almost-invariant under the dynamics generated by the perturbed Hamiltonian, because transitions out of this state either require particles to overcome the local energy gap or to tunnel long spatial distances. For a heuristic sketch of the situation see Figure 1. To implement these ideas mathematically, we heavily use and partly extend a technical machinery that has experienced important new developments during recent years. This includes Lieb-Robinson bounds for interacting fermions [19, 6], the quasi-local inverse of the Liouvillian introduced in the context of the so called spectral-flow or quasi-adiabatic evolution [12, 3], as well as ideas and technical lemmas from [2] and [18]. The mathematical justification of linear response theory in similar situations for non-interacting systems of fermions was studied e.g. in [4, 5, 7, 9, 14]. Note that while the results in [5, 14, 7] require instead of a spectral gap only a mobility gap, because of the order of limits they are not yet fully satisfactory, cf. Section 4. Clearly linear response in the presence of localisation is of great physical relevance also in the interacting case, however, the mathematical understanding of many-body localisation is still in its infancy and we are not aware of any work on justifying linear response in this setting.

For non-interacting systems NEASS were constructed e.g. in [21, 22, 23, 25] using the formalism of space-adiabatic perturbation theory. In this context a different terminology was adopted, and instead of almost-stationary states one speaks about
Figure 1: On the left panel the eigenvalues of a gapped Hamiltonian $H_0$ are sketched: The ground state is separated by a gap $g$ from the rest of the spectrum uniformly in the volume, while the eigenvalue spacing between the other eigenvalues typically goes to zero when the volume grows. On the right panel the local energy landscape of $H = H_0 + V$ with $V = \varepsilon x$, $\varepsilon \ll 1$, is sketched and the lowest solid line represents the NEASS. Local transitions out of the NEASS are strongly suppressed, since either the spectral gap of size $g$ (vertical arrow) or a spatial distance of size $\varepsilon^{-1}$ (horizontal arrow) needs to be overcome.

almost-invariant subspaces, a notion going back to [20]. The results of the present paper could thus be viewed as a generalisation of space-adiabatic perturbation theory to interacting systems.

The structure of the paper is as follows. In Section 2 we introduce the mathematical framework for quasi-local operators and extend it to partly localised quasi-local operators. While the concept of slowly varying potentials is not new, its definition for systems of varying size requires some care. Moreover, we formulate a crucial lemma, Lemma 2.1, that states that commutators of arbitrary quasi-local operators with slowly varying potentials are small and quasi-local. In Section 3 we state those results concerning the NEASS that are required for the proof of linear response theory in Section 4. Section 3 contains the general adiabatic theorem, Theorem 5.1, which we call a space-time adiabatic theorem, since it exploits the slow variation of the Hamiltonian both in space and as a function of time. Its proof is divided into two parts. The proof of the space-time adiabatic expansion is the content of Section 6.1, the proof of the adiabatic theorem itself is given in Section 6.2. The asymptotic expansion of the NEASS is stated in Proposition 5.2 and proved in Section 6.3. All statements of Section 3 follow as corollaries of the more general space-time adiabatic theory of Section 5. In Appendix A we prove Lemma 2.1 about commutators with slowly varying potentials, while in the Appendices B we collect without proofs a number of technical lemmas from other sources that need to be slightly adapted. Finally, Appendix C briefly discusses the local inverse of the Liouvillian and how to extend its mapping properties to slowly
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2 The mathematical framework

In this section we explain the precise setup necessary to formulate our main results. In a nutshell, we consider systems of interacting fermions on a subset \( \Lambda \subset \mathbb{Z}^d \) of linear size \( M \) of the \( d \)-dimensional square lattice \( \mathbb{Z}^d \). In some directions \( \Lambda \) can be closed in order to allow for cylinder or torus geometries. The Hamiltonian generating the dynamics is a quasi-local operator, that is, roughly speaking, an extensive sum of local operators. As we aim at statements that hold uniformly in the system size \( M \), we consider actually families of operators indexed by \( \Lambda \). This requires a certain amount of technical definitions, in particular, one needs norms that control families of quasi-local operators. Since the particle number depends on the size of the system, it is most convenient to work on Fock space. Many of the following concepts are standard and only slightly adapted from [3, 19, 2].

2.1 The lattice and the Hilbert space

Let \( \Gamma = \mathbb{Z}^d \) be the infinite square lattice and \( \Lambda = \Lambda(M) := \{-\frac{M}{2} + 1, \ldots, \frac{M}{2}\}^d \subset \Gamma \) the centred box of size \( M \), with \( M \in 2\mathbb{N} \). For many applications, in particular those concerning currents, it is useful to consider \( \Lambda \) being closed in some directions, say in the first \( d_c \) directions, \( 0 \leq d_c \leq d \). In particular, for \( d_c = d \) this means that \( \Lambda \) has a discrete torus geometry and for \( d_c = 0 \) it is a \( d \)-dimensional discrete cube. In order to define the corresponding metric on \( \Lambda \), let \( a \wedge b \) be the representative of \( [a-b] \in \mathbb{Z}/M \) in \( \{-\frac{M}{2} + 1, \ldots, \frac{M}{2}\} \) and define

\[
\Lambda: \Lambda \times \Lambda \to \Gamma, \quad (x, y) \mapsto x \wedge y \quad \text{with} \quad (x \wedge y)_j = \begin{cases} x_j - \frac{M}{2}y_j & \text{if } j \leq d_c \\ x_j - y_j & \text{if } j > d_c, \end{cases}
\]

Figure 2.1: Here \( d = 2 \), and \( \Lambda \) is closed in the 1-direction and open in the 2-direction.
the difference vector of two points in \( \Lambda \). With \( d : \Gamma \times \Gamma \to \mathbb{N}_0 \) denoting the \( \ell^1 \)-distance on \( \Gamma \), the \( \ell^1 \)-distance on the “cylinder” \( \Lambda \) with the first \( d_c \) directions closed is

\[
d^\Lambda : \Lambda \times \Lambda \to \mathbb{N}_0, \quad d^\Lambda(x, y) := d(0, y \Delta x).
\]

Let the one-particle Hilbert space be \( h_\Lambda := \ell^2(\Lambda, \mathbb{C}^s) \), where \( \mathbb{C}^s \) describes spin and the internal structure of the unit cell. The corresponding \( N \)-particle Hilbert space is its \( N \)-fold anti-symmetric tensor product

\[
H^\Lambda, N := \bigwedge^N_{j=1} h^\Lambda, \quad H^\Lambda, 0 := \mathbb{C}.
\]

All these Hilbert spaces are finite-dimensional and thus all linear operators on them are bounded. Let \( a_{i,x} \) and \( a^*_{i,x} \), \( i = 1, \ldots, s \), \( x \in \Gamma \), be the standard fermionic annihilation and creation operators satisfying the canonical anti-commutation relations

\[
\{a_{i,x}, a^*_{j,y}\} = \delta_{i,j} \delta_{x,y} 1_{\mathfrak{N}_\Lambda} \quad \text{and} \quad \{a_{i,x}, a_{j,y}\} = 0 = \{a^*_{i,x}, a^*_{j,y}\},
\]

where \( \{A, B\} = AB + BA \). For a subset \( X \subset \Lambda \) we denote by \( \mathcal{A}_X \subset \mathfrak{L}(\mathfrak{N}_\Lambda) \) the algebra of operators generated by the set \( \{a_{i,x}, a^*_{i,x} \mid x \in X, i = 1, \ldots, s\} \). Those elements of \( \mathcal{A}_X \) commuting with the number operator

\[
\mathfrak{N}_X := \sum_{x \in X} a^*_{x} a_{x} := \sum_{x \in X} \sum_{j=1}^s a^*_{j,x} a_{j,x}
\]

form a sub-algebra \( \mathcal{A}_X^\mathfrak{R} \) of \( \mathcal{A}_X \) contained in the sub-algebra \( \mathcal{A}_X^+ \) of even elements, i.e. \( \mathcal{A}_X^\mathfrak{R} \subset \mathcal{A}_X^+ \subset \mathcal{A}_X \). We will use the vector notation \( a_x = (a_{1,x}, \ldots, a_{s,x}) \) without further notice in the following.

### 2.2 Interactions and associated operator-families

An interaction \( \Phi = \{\Phi^{\varepsilon, \Lambda}\}_{\varepsilon \in (0,1], \Lambda = \Lambda(M), M \in 2\mathbb{N}} \) is a family of maps

\[
\Phi^{\varepsilon, \Lambda} : \{X \subset \Lambda\} \to \bigcup_{X \subset \Lambda} \mathcal{A}_X^\mathfrak{R}, \quad X \mapsto \Phi^{\varepsilon, \Lambda}(X) \in \mathcal{A}_X^\mathfrak{R},
\]

from subsets of \( \Lambda \) into the set of operators commuting with the number operator \( \mathfrak{N}_X \). The operator-family \( A = \{A^{\varepsilon, \Lambda}\}_{\varepsilon \in (0,1], \Lambda} \) associated with the interaction \( \Phi \) is the family of operators

\[
A^{\varepsilon, \Lambda} := A^{\varepsilon, \Lambda}(\Phi) := \sum_{X \subset \Lambda} \Phi^{\varepsilon, \Lambda}(X) \in \mathcal{A}_\Lambda^\mathfrak{R}. \tag{1}
\]

In the case that an interaction or the associated operator-family does not depend on the parameter \( \varepsilon \), we will drop the superscript \( \varepsilon \) in the notation. Note also that “interaction” is used here as a mathematical term for the above kind of object and should not be confused with the physics notion of interaction.
In order to turn the vector space of interactions into a normed space, it is useful to introduce the following functions that will serve to control the range of an interaction (cf. e.g. [19] and references therein). Let

\[ F(r) := \frac{1}{(1 + r)^{d+1}} \quad \text{and} \quad F_\zeta(r) := \frac{\zeta(r)}{(1 + r)^{d+1}}, \]

where

\[ \zeta \in \mathcal{S} := \{ \zeta : [0, \infty) \to (0, \infty) \mid \zeta \text{ is bounded, non-increasing, satisfies} \]
\[ \zeta(r + s) \geq \zeta(r) \zeta(s) \text{ for all } r, s \in [0, \infty) \text{ and} \]
\[ \sup \{ r^n \zeta(r) \mid r \in [0, \infty) \} < \infty \text{ for all } n \in \mathbb{N} \}. \]

For any \( \zeta \in \mathcal{S} \) and \( n \in \mathbb{N}_0 \), a corresponding norm on the vector space of interactions is defined by

\[ \| \Phi \|_{\zeta,n} := \sup_{\varepsilon \in (0,1]} \sup_{\Lambda} \sup_{x,y \in \Lambda} \sum_{X \subseteq \Lambda : x,y \in X} \Lambda\text{-diam}(X)^n \frac{\| \Phi^{\varepsilon,\Lambda}(X) \|}{F_\zeta(d^{\Lambda}(x,y))}. \]

Here \( \Lambda\text{-diam}(X) \) denotes the diameter of the set \( X \subseteq \Lambda \) with respect to the metric \( d^{\Lambda} \). The prime example for a function \( \zeta \in \mathcal{S} \) is \( \zeta(r) = e^{-ar} \) for some \( a > 0 \). For this specific choice of \( \zeta \) we write \( F_a \) and \( \| \Phi \|_{a,n} \) for the corresponding norm. However, for technical reasons the use of the more general decay functions \( \zeta \) in \( \mathcal{S} \) seems unavoidable, see also the remark after Lemma C.2.

It is important for applications to consider also interactions that are localised in certain directions around certain locations. To this end we introduce the space of localisation vectors

\[ \text{Loc} := \{0,1\}^d \times \prod_{M=2}^{\infty} \Lambda(M). \]

Note that \( L =: (\ell, l^{\Lambda(2)}, l^{\Lambda(4)}, \ldots) \in \text{Loc} \) defines for each \( \Lambda \) a \( (d - |\ell|) \)-dimensional hyperplane through the point \( l^{\Lambda} \in \Lambda \) which is parallel to the one given by \( \{x_j = 0 \text{ if } \ell_j = 1\} \). Here \( |\ell| := |\{\ell_j = 1\}| \) is the number of constrained directions. The distance of a point \( x \in \Lambda \) to this hyperplane is

\[ \text{dist}(x, L) := \sum_{j=1}^d |(x \wedge l^{\Lambda})_j| \ell_j \quad (2) \]
and we define for $\alpha > 0$ a new “distance” on $\Lambda$ by

$$d_{\Lambda}^\alpha(x, y) := d(\Lambda, x, y) + \alpha (\text{dist}(x, L) + \text{dist}(y, L)).$$

Note that $d_{\Lambda}^\alpha$ is no longer a metric on $\Lambda$ but obviously still satisfies the triangle inequality. Moreover, if $L \in \text{Loc}$ is trivial, i.e. $|\ell| = 0$, then $d_{\Lambda}^0 = d$. The corresponding norms are denoted by

$$\|\Phi\|_{\zeta, n, L, \alpha} := \sup_{\varepsilon \in (0, 1]} \Lambda \sup_{x, y \in \Lambda} \sum_{X \subset \Lambda} \Lambda-\text{diam}(X)^n \frac{\|\Phi^{\varepsilon, \Lambda}(X)\|}{F_\zeta(d_{\Lambda}^\alpha(x, y))}.$$

In the following we will either have $\alpha = 1$ or $\alpha = \varepsilon$.

An operator-family $A$ with interaction $\Phi_A$ such that $\|\Phi_A\|_{\zeta, 0, L, \alpha} < \infty$ for some $\zeta \in S$ is called quasi-local and $L_\alpha$-localised. One crucial property of quasi-local $L_\alpha$-localised operator-families is that the norm of the finite-size operator $A^{\varepsilon, \Lambda}$ grows at most as the volume of its “support” (cf. Lemma B.2)

$$\|A^{\varepsilon, \Lambda}\| \leq C \|\Phi_A\|_{\zeta, 0, L, \alpha} \alpha^{-|\ell|} M^d-|\ell|.$$

Let $B_{\zeta, n, L, \alpha}$ be the Banach space of interactions with finite $\|\cdot\|_{\zeta, n, L, \alpha}$-norm, and put

$$B_{S, n, L, \alpha} := \bigcup_{\zeta \in S} B_{\zeta, n, L, \alpha}, \quad B_{E, n, L, \alpha} := \bigcup_{a > 0} B_{a, n, L, \alpha},$$

$$B_{S, \infty, L, \alpha} := \bigcap_{n \in \mathbb{N}_0} B_{S, n, L, \alpha}, \quad B_{E, \infty, L, \alpha} := \bigcap_{n \in \mathbb{N}_0} B_{E, n, L, \alpha}.$$

Note that $\Phi \in B_{S, \infty, L, \alpha}$ merely means that there exists a sequence $\zeta_\alpha \in S$ such that $\Phi \in B_{\zeta_\alpha, n, L, \alpha}$ for all $n \in \mathbb{N}_0$. The corresponding spaces of operator-families are denoted by $L_{\zeta, n, L, \alpha}$, $L_{E, n, L, \alpha}$, $L_{E, \infty, L, \alpha}$, $L_{S, n, L, \alpha}$, and $L_{E, \infty, L, \alpha}$ respectively: that is, an operator-family $A$ belongs to $L_{\zeta, n, L, \alpha}$ if it can be written in the form (1) with an interaction in $B_{\zeta, n, L, \alpha}$, and similarly for the other spaces. Lemma A.1 in [18] shows that the spaces $B_{S, n, L, \alpha}$ and therefore also $B_{S, \infty, L, \alpha}$ are indeed vector spaces. Note that they are not algebras of operators, i.e. the product of two quasi-local $L_\alpha$-localised operators need neither be quasi-local nor $L_\alpha$-localised, but they are closed under taking commutators, see Lemmas B.3 and B.4 in Appendix B. When we don’t write the index $L_\alpha$, this means that $L = 0 := (0, 0, 0, \ldots)$ and the interaction (respectively the operator-family) is quasi-local but not localised in any direction.

### 2.3 Slowly varying potentials

Roughly speaking, a slowly varying potential is a function $v$ on $\Lambda$ with Lipschitz-constant of order $\varepsilon$. 
Definition 2.1 (Slowly varying potentials). A slowly varying potential is a family of functions \( \{ v^{\varepsilon, \Lambda} : \Lambda \to \mathbb{R} \}_{\varepsilon \in (0,1]} \) such that

\[
C_v := \sup_{\varepsilon \in (0,1]} \sup_{\Lambda} \sup_{x,y \in \Lambda} \frac{|v^{\varepsilon, \Lambda}(x) - v^{\varepsilon, \Lambda}(y)|}{\varepsilon \cdot d^\Lambda(x,y)} < \infty.
\]

The space of slowly varying potentials is denoted by \( V \).

Let us give some typical examples. If \( \Lambda \) is open in the \( j \)-direction, then for any \( v_1 \in C^1(\mathbb{R}, \mathbb{R}) \) (not necessarily bounded!) with \( \| v'_1 \|_{\infty} < \infty \) the functions

\[
v_1^{\varepsilon, \Lambda} : \Lambda \to \mathbb{R}, \quad x \mapsto v_1^{\varepsilon, \Lambda}(x) := v_1(\varepsilon x_j)
\]

both satisfy, by the mean-value theorem, for any \( x, y \in \Lambda \)

\[
|v_1^{\varepsilon, \Lambda}(x) - v_1^{\varepsilon, \Lambda}(y)| \leq \varepsilon \cdot d^\Lambda(x,y) \| v'_1 \|_{\infty} \quad \text{and} \quad |	ilde{v}_1^{\varepsilon, \Lambda}(x) - \tilde{v}_1^{\varepsilon, \Lambda}(y)| \leq \varepsilon \cdot d^\Lambda(x,y) \| v'_1 \|_{\infty}.
\]

Hence, \( v_1, \tilde{v}_1 \in V \) with \( C_{v_1} = C_{\tilde{v}_1} = \| v'_1 \|_{\infty} \) in this case. In particular, \( v_1(x) = x_j \) would be a viable option. In the case that \( \Lambda \) is closed in the \( j \)-direction, any function \( v_2 : (-1/2, 1/2] \to \mathbb{R} \) that is \( C^1 \) with periodic boundary conditions defines, by the same reasoning, a slowly varying potential

\[
v_2^{\varepsilon, \Lambda}(x) := \varepsilon M v_2 \left( \frac{x_j}{M} \right) \quad \text{with} \quad C_{v_2} = \| v'_2 \|_{\infty}.
\]

With a slowly varying potential \( v^{\varepsilon, \Lambda} \) we associate a corresponding operator-family \( V_v \) defined by

\[
V_v^{\varepsilon, \Lambda} := \sum_{x \in \Lambda} v^{\varepsilon, \Lambda}(x) a_x^* a_x.
\]

The key property of a slowly varying potential is that its commutator with any quasi-local operator-family is a quasi-local operator-family of order \( \varepsilon \), as stated more precisely in the following lemma.

Lemma 2.1. Let \( V_v \) be the operator-family of a slowly varying potential \( v \in V \) and \( A \in \mathcal{L}_{S,\infty,L_\alpha}, \) i.e. \( A \in \mathcal{L}_{\zeta_k,k+d+1,L_\alpha} \) for some sequence \( \zeta_k \) in \( S \). Then there exists an operator-family \( A_v \in \mathcal{L}_{S,\infty,L_\alpha} \) with interaction \( \Phi_{A_v} \in \mathcal{B}_{S,\infty,L_\alpha} \) satisfying

\[
\| \Phi_{A_v} \|_{\zeta_k,k,L_\alpha} \leq \frac{\varepsilon}{2} C_v \| \Phi_A^\Lambda \|_{\zeta_k,k+d+1,L_\alpha}
\]

such that

\[
[A, V_v] = \varepsilon A_v.
\]

The proof of Lemma 2.1 is given in Appendix A.
3 Non-equilibrium almost-stationary states

We have now all the tools to formulate our main result about non-equilibrium almost-stationary states (NEASS) for time-independent Hamiltonians of the form

\[ H = H_0 + V_v + \varepsilon H_1. \]

The results of this section will follow as corollaries of the space-time adiabatic theorem, Theorem 5.1 of Section 5 and the underlying space-time adiabatic expansion, Proposition 5.1. However, we consider these special cases conceptually most important and it might be difficult to read them directly off the rather technical general statement. Moreover, they are at the basis of the proof of linear response theory given in Section 4.

(A1) Assumptions on \( H_0 \).

Let \( H_0 \in L_a,\mathbb{N} \) for all \( n \in \mathbb{N}_0 \) and some \( a > 0 \) such that all \( H_0^\Lambda \) are self-adjoint.

We assume that there exists \( M_0 \in \mathbb{N} \) such that for all \( M \geq M_0 \) and corresponding \( \Lambda = \Lambda(M) \) the ground state \( E_0^\Lambda \) of the operator \( H_0^\Lambda \), with associated spectral projection \( P_0^\Lambda \), has the following properties: The degeneracy \( \kappa^\Lambda \) of \( E_0^\Lambda \) and the spectral gap are uniform in the system size, i.e. there exist \( \kappa \in \mathbb{N} \) and \( g > 0 \) such that \( \kappa^\Lambda \leq \kappa \) and \( \operatorname{dist}(E_0^\Lambda, \sigma(H_0^\Lambda) \setminus E_0^\Lambda) \geq g > 0 \) for all \( M \geq M_0 \).

A typical example of a physically relevant Hamiltonian \( H_0 \) to which our results apply is the family of operators

\[ H_{T\phi W}^\Lambda = \sum_{(x,y)\in \Lambda^2} a_x^* T(x - y) a_y + \sum_{x\in \Lambda} a_x^* \phi(x) a_x + \sum_{\{x,y\}\subset \Lambda} a_x^* a_y W(d^\Lambda(x,y)) a_y^* a_y - \mu \mathcal{N} \Lambda. \]  

For example, if the kinetic term \( T : \Gamma \to \mathcal{L}(\mathbb{C}^s) \) is a compactly supported function with \( T(-x) = T(x)^* \), the potential term \( \phi : \Gamma \to \mathcal{L}(\mathbb{C}^s) \) is a bounded function taking values in the self-adjoint matrices, the two-body interaction \( W : \mathbb{N}_0 \to \mathcal{L}(\mathbb{C}^s) \) is compactly supported and also takes values in the self-adjoint matrices, then \( H_{T\phi W}^\Lambda \in L_{a,\infty} \) for any \( a > 0 \). For non-interacting systems, i.e. \( W \equiv 0 \), on a torus and \( V \) sufficiently small, the gap condition can be checked rather directly and one typically finds it to be satisfied for values of the chemical potential \( \mu \in \mathbb{R} \) lying in specific intervals. It was recently shown in [11, 8] that for sufficiently small \( W \neq 0 \) the gap remains open.

(A2) Assumptions on the perturbations.

Let \( H_1 \in L_{S,\infty,L_{11}^H} \) be self-adjoint and let \( v \in \mathcal{V} \) be a slowly varying potential and \( V_v \) the corresponding operator-family. If \( L^H \) is nontrivial we assume that \( V_v \) is \( L^H_{\varepsilon^\gamma} \)-localised for \( \gamma \in \{0, 1\} \) in the sense that

\[ [H_0, \frac{1}{\varepsilon} V_v] \in L_{S,\infty,L_{11}^H}. \]
Remark 3.1. (a) The parameter $\gamma \in \{0, 1\}$ will appear in the statements of our results for the following reason. While a perturbation of the form $H_1$ can be localised in a fixed neighbourhood of a $(d - |\ell_H|)$-dimensional hyperplane and thus on a volume of order $M^{d-|\ell_H|}$, a slowly varying potential is typically only localised in an $\varepsilon^{-1}$-neighbourhood of such a $(d - |\ell_H|)$-dimensional hyperplane\footnote{Think for example of a smooth step function along a codimension one hyperplane with finite step size but order $\varepsilon$ slope.} and thus on a volume of order $\varepsilon^{-|\ell_H|} M^{d-|\ell_H|}$. Hence, the proper normalisation in the statements will then contain a factor $\varepsilon^{-|\ell_H|\gamma} M^{d-|\ell_H|}$.

(b) It is not known whether perturbing a generic gapped $H_0$ by a small local perturbation $\varepsilon H_1$ leaves the spectral gap open for $\varepsilon$ sufficiently small, see e.g. the discussion in Section 1.5 in [1]. However, perturbing by a slowly varying potential $V_v$ such that $\|V_v^A\|_\infty \sim M$ clearly closes the gap of $H_0$ for all values of $\varepsilon$.

In a nutshell, the following theorem about non-equilibrium almost-stationary states says that for any $n \in \mathbb{N}$ there exists a state $\Pi_n$, that is obtained from the ground state $P_n$ of $H_0$ by a unitary transformation with small quasi-local generator, such that $[\Pi_n, H] = \mathcal{O}(\varepsilon^{n+1})$. As a consequence, $\Pi_n$ is almost-invariant under the dynamics generated by $H$. In cases where the perturbation is so small that the gap of $H_0$ remains open, $\Pi_n$ is $\varepsilon^{n+1}$-close to the ground state projection of $H$ in the sense that the Taylor polynomials of both operators agree up to order $\varepsilon^n$. In general, however, $\Pi_n$ can differ greatly from the ground state or any other thermal equilibrium state of $H$. This is why we call $\Pi_n$ a non-equilibrium almost-stationary state.

Theorem 3.1 (Non-equilibrium almost-stationary states). Let the Hamiltonian $H = H_0 + V_v + \varepsilon H_1$ satisfy (A1) and (A2)$_{L^H, \gamma}$ for some $L^H \in \text{Loc}$ and $\gamma \in \{0, 1\}$. Then there is a sequence of self-adjoint operator-families $(A_\mu)$ with $A_\mu \in \mathcal{L}_{S,\infty,L^H}$, for all $\mu \in \mathbb{N}$, such that for any $n \in \mathbb{N}$ it holds that the projector

$$
\Pi_n^{\varepsilon,A} := e^{i\varepsilon S_n^A} P_n^A e^{-i\varepsilon S_n^A} \quad \text{with} \quad S_n^{\varepsilon,A} := \sum_{\mu=1}^{n} \varepsilon^{\mu-1} A_\mu^{\varepsilon,A}
$$

satisfies

$$
[\Pi_n^{\varepsilon,A}, H^{\varepsilon,A}] = \varepsilon^{n+1} \left[\Pi_n^{\varepsilon,A}, P_n^{\varepsilon,A}\right] \tag{4}
$$

for some $R_n \in \mathcal{L}_{S,\infty,L^H}$.

The state $\Pi_n^{\varepsilon,A}$ is almost-stationary for the dynamics generated by $H^{\varepsilon,A}$ in the following sense: Let $\rho^{\varepsilon,A}(t)$ be the solution of the Schrödinger equation

$$
i \frac{d}{dt} \rho^{\varepsilon,A}(t) = [H^{\varepsilon,A}, \rho^{\varepsilon,A}(t)] \quad \text{with} \quad \rho^{\varepsilon,A}(0) = \Pi_n^{\varepsilon,A}.
$$

Then for any $\zeta \in S$, $L \in \text{Loc}$ with $\ell \cdot \ell_H = 0$, and $m \geq 1$ there exists a constant $C$ such that for any $B \in \mathcal{L}_{\zeta,2d,L_1}$

$$
sup_{A(M): M \geq M_0} \frac{\varepsilon^{|\ell_H|\gamma}}{M^{d-|\ell_H|\mu}} \left| \text{tr}(\rho^{\varepsilon,A}(t) B^A) - \text{tr}(\Pi_n^{\varepsilon,A} B^A) \right| \leq C \varepsilon^{n+1} (1 + |\varepsilon^m t|^{d+1}) \|\Phi_B\|_{\zeta,2d,L_1}. \tag{5}
$$
While (4) is an immediate consequence of Proposition 5.1 for $\delta = 0$, (5) is not strictly speaking a corollary of the results of Section 5. But it can easily be concluded by combining Proposition 3.1 with a simple Duhamel argument for $\Pi_{\tilde{n}}^{\varepsilon,\Lambda}$ with $\tilde{n} = n + m(d + 1)$ and $\delta = \varepsilon^{m-1}$ as in the proof of Theorem 5.1.

Remark 3.2. (a) The trace in (5) is normalised by the volume of the region in $\Lambda$ where the perturbation “acts” and the observable $B$ “tests”. This region is the intersection of the neighbourhoods of two transversal hyperplanes, one of co-dimension $|\ell_H|$ and thickness of order $\varepsilon^{-1}$ (resp. of order one if $\gamma = 0$), and one of co-dimension $|\ell|$ and thickness of order one. Hence, the relevant volume is of order $\varepsilon^{-|\ell_H| - |\ell|}$. If the perturbation acts everywhere ($|\ell_H| = 0$) and the observable $B$ tests everywhere ($|\ell| = 0$), then the trace in (5) is the usual trace per unit volume.

(b) Note that the first $n$ terms in the asymptotic expansion of the NEASS $\Pi_{\tilde{n}}^{\varepsilon,\Lambda}$ are uniquely determined by the requirements that $(\Pi_{\tilde{n}}^{\varepsilon,\Lambda})^2 - \Pi_{\tilde{n}}^{\varepsilon,\Lambda} = O(\varepsilon^{n+1})$ and $[H^{\varepsilon,\Lambda}, \Pi_{\tilde{n}}^{\varepsilon,\Lambda}] = O(\varepsilon^{n+1})$, cf. e.g. [22]. Hence, all terms in this expansion could be equally well obtained by just applying standard regular perturbation theory, e.g. [13]. However, the unitary $e^{i\varepsilon S_{\varepsilon,\Lambda}}$ is not uniquely determined and one key feature of the above result is that the latter can be chosen quasi-local. Otherwise the almost invariance in (5) uniformly in the volume could not be concluded.

(c) The NEASS $\Pi_{\tilde{n}}^{\varepsilon,\Lambda}$ agrees with the state obtained by the quasi-adiabatic evolution [12, 3] up to terms of order $\varepsilon^{n+1}$ in the following sense: The quasi-adiabatic evolution of $Q_{0,\Lambda} := P_{\Lambda}^*$ is the solution to the evolution equation

$$i \frac{d}{d\varepsilon} Q^{\varepsilon,\Lambda} = [I_{H^{\varepsilon,\Lambda}}(\frac{d}{d\varepsilon} H^{\varepsilon,\Lambda}), Q^{\varepsilon,\Lambda}] \quad \text{with} \quad Q^{0,\Lambda} = P_{\Lambda}^*.$$ (6)

Here $I_{H^{\varepsilon,\Lambda}}$ is the quasi-local inverse of the Liouvillian discussed in Appendix C. It is straightforward to check that if $H^{\varepsilon,\Lambda}$ has a gapped ground state $P_{\Lambda}^*$ uniformly for all $\varepsilon \in [0, \varepsilon_0]$, then the quasi-adiabatic evolution $Q^{\varepsilon,\Lambda}$ of $P_{\Lambda}^*$ actually agrees with $P_{\Lambda}^*$, i.e. $Q^{\varepsilon,\Lambda} = P_{\Lambda}^*$ for all $\varepsilon \in [0, \varepsilon_0]$. In particular, $Q^{\varepsilon,\Lambda}$ and $P_{\Lambda}^*$ have the same Taylor expansion in powers of $\varepsilon$. But also the NEASS $\Pi_{\tilde{n}}^{\varepsilon,\Lambda}$ and the eigenprojection $P_{\Lambda}^*$ have the same Taylor expansion in powers of $\varepsilon$ up to order $n$, as remarked under item (b).

For applications it is essential to have an explicit expansion of expectation values in the NEASS in powers of $\varepsilon$ with coefficients given by expectations in the unperturbed ground state, the linear term being typically most important. The following statement is a special case of Proposition 5.2 about the expansion of the time-dependent NEASS. For better readability we will use the notation

$$\langle B \rangle_{\rho^\Lambda} := \text{tr}(\rho^\Lambda B^\Lambda)$$

for the expectation value of an observable $B^\Lambda$ in a state $\rho^\Lambda$. 

12
Proposition 3.1 (Asymptotic expansion of the NEASS). Under the assumptions of Theorem 3.1 there exist maps $K_{j}^{ε,Λ} : A_{\Lambda}^{R} \rightarrow A_{\Lambda}^{R}$, $j \in \mathbb{N}$, given by nested commutators with operator-families in $\mathcal{L}_{S,\infty,L_{m}^{H}}$, such that for any $n, m \in \mathbb{N}_{0}$ with $n \geq m$, any $\zeta \in \mathcal{S}$, and any $L \in \text{Loc}$ with $\ell \cdot \ell_{H} = 0$ there is a constant $C$ such that for any $B \in \mathcal{L}_{\zeta,m+1,1}$ it holds that

$$\sup_{\Lambda(M) : M \geq M_{0}} \frac{\varepsilon|H_{\gamma}|}{M^{d-|H_{\gamma}|}} \left| \langle B \rangle_{\Pi_{n}^{\Lambda}} - \sum_{j=0}^{m} \varepsilon^{j} \langle K_{j}^{\varepsilon}[B] \rangle_{\Pi_{n}^{\Lambda}} \right| = C \varepsilon^{m+1} \| \Phi_{B} \|_{\zeta,m+1,1},$$

with

$$\sup_{\Lambda(M) : M \geq M_{0}} \sup_{\varepsilon \in [0,1]} \frac{\varepsilon|H_{\gamma}|}{M^{d-|H_{\gamma}|}} \langle K_{j}^{\varepsilon}[B] \rangle_{\Pi_{n}^{\Lambda}} < \infty$$

for all $j \in \mathbb{N}$. The first terms in the expansion are given by

$$K_{0}^{\varepsilon,\Lambda} = \text{Id}, \quad K_{1}^{\varepsilon,\Lambda} := -i[A_{1}^{\varepsilon,\Lambda}, \cdot], \quad \text{and} \quad K_{2}^{\varepsilon,\Lambda} := -i[A_{2}^{\varepsilon,\Lambda}, \cdot] - \frac{1}{2}[A_{1}^{\varepsilon,\Lambda}, [A_{1}^{\varepsilon,\Lambda}, \cdot]],$$

where $A_{1} \in \mathcal{L}_{S,\infty,L_{m}^{H}}$ and $A_{2} \in \mathcal{L}_{S,\infty,L_{m}^{H}}$ were constructed in Theorem 3.1. More explicitly, abbreviating $V^{\Lambda} := \frac{1}{\varepsilon}V_{0}^{\varepsilon,\Lambda} + H_{1}^{\varepsilon,\Lambda}$ it holds that

$$\langle K_{1}^{\varepsilon}[B] \rangle_{\Pi_{n}^{\Lambda}} = \langle [[R_{0}, V], B] \rangle_{\Pi_{n}^{\Lambda}}$$

and

$$\langle K_{2}^{\varepsilon}[B] \rangle_{\Pi_{n}^{\Lambda}} = 2 \text{Re} \langle (VR_{0}VR_{0} - VP_{\varepsilon}VR_{0}R_{0}) B \rangle_{\Pi_{n}^{\Lambda}}$$

$$+ \langle (VR_{0}BR_{0}V - VR_{0}R_{0}VP_{\varepsilon}) B \rangle_{\Pi_{n}^{\Lambda}},$$

where $R_{0}^{\Lambda} := (H_{0}^{\varepsilon} - E_{\Lambda}^{\varepsilon})^{-1}(1 - P_{\varepsilon}^{\Lambda})$ denotes the reduced resolvent of $H_{0}$.

Finally, again as a corollary of Proposition 5.1 and Theorem 5.1 we note that adiabatically switching on the perturbation drives the ground state of the unperturbed Hamiltonian into the NEASS of the perturbed Hamiltonian up to small errors in the adiabatic parameter and independently of the precise form of the switching function.

Proposition 3.2 (Adiabatic switching and the NEASS). Let the Hamiltonian $H = H_{0} + V_{\varepsilon} + \varepsilon H_{1}$ satisfy (A1) and (A2)$_{L_{H},\gamma}$ for some $L_{H} \in \text{Loc}$ and $\gamma \in \{0,1\}$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth “switching” function with $f(t) = 0$ for $t \leq -1$ and $f(t) = 1$ for $t \geq 0$, and define $H(t) := H_{0} + f(t)(V_{\varepsilon} + \varepsilon H_{1})$. Let $\rho^{\varepsilon,\Lambda,n,f}(t)$ be the solution of the adiabatic time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t} \rho^{\varepsilon,\Lambda,n,f}(t) = [H^{\varepsilon,\Lambda}(t), \rho^{\varepsilon,\Lambda,n,f}(t)]$$

with adiabatic parameter $\eta \in (0,1]$ and initial datum $\rho^{\varepsilon,\Lambda,n,f}(t) = P_{\varepsilon}^{\Lambda}$ for all $t \leq -1$.

Then for any $n > d$, $\zeta \in \mathcal{S}$ and $L \in \text{Loc}$ with $\ell \cdot \ell_{H} = 0$ there exists a constant $C$ such that for any $B \in \mathcal{L}_{\zeta,2d,L_{1}}$ and for all $t \geq 0$

$$\sup_{\Lambda(M) : M \geq M_{0}} \frac{\varepsilon|H_{\gamma}|}{M^{d-|H_{\gamma}|}} \left| \langle B \rangle_{\rho^{\varepsilon,\Lambda,n,f}(t)} - \langle B \rangle_{\Pi_{n}^{\Lambda}} \right| \leq \frac{\varepsilon^{n+1} + \eta^{n+1}}{\eta^{d+1}} \sum_{\ell_{H} = 0}^{\ell_{H} = \infty} \left( 1 + \ell_{H}^{d+1} \right) \| \Phi_{B} \|_{\zeta,2d,L_{1}},$$

where $\Pi_{n}^{\Lambda}$ is the NEASS of $H^{\varepsilon,\Lambda}$ constructed in Theorem 3.1.
Remark 3.3. Proposition 3.2 shows that, as long as the adiabatic parameter $\eta$ satisfies
\[ 1 \gg \eta \gg \varepsilon^{n+1}, \]
the initial ground state $P_*$ of $H_0$ evolves, up to a small error, into a NEASS $\Pi_{\varepsilon}^\Lambda$ that is independent of the form of the switching function $f$. Since $n \in \mathbb{N}$ can be chosen arbitrarily large, this means that whenever the adiabatic switching of the perturbation occurs on a time-scale of order $\varepsilon^{-m}$ with $m > 0$, the system will be driven into the same unique NEASS constructed in Theorem 3.1. Slower switching must be excluded, because, in general, the NEASS is an almost-invariant but not an invariant state for the instantaneous Hamiltonian. On time scales asymptotically larger than any inverse power of $\varepsilon$, the NEASS need not be stable and might deteriorate because of tunnelling. Hence, it is not surprising that the relevant time scale for the adiabatic switching process depends on the strength of the perturbation.

4 Linear response theory

To put our result on the justification of response theory, Theorem 4.1, into proper context, we briefly recall the usual derivation of linear response formulas in the context of static perturbations. Assume that a system described by the Hamiltonian $H_0^\Lambda$ is initially in its zero-temperature equilibrium state $P_{\Lambda}^*$, when a static perturbation $\varepsilon V^\Lambda$ is applied. To keep notation concise, in this section we will again write $V^\Lambda := \frac{1}{\varepsilon}V_{v}^\Lambda + H_1^\Lambda$ for the sum of the two types of perturbations we consider. The dynamical process of applying the perturbation is modelled by a time-dependent Hamiltonian $H_{\varepsilon,\Lambda}(\eta s) := H_0^\Lambda + f(\eta s)\varepsilon V^\Lambda$, where $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently regular switch function with $f(t) = 0$ for $t \leq -1$ and $f(t) = 1$ for $t \geq 0$. Thus, we have $H_{\varepsilon,\Lambda}(\eta s) = H_0^\Lambda$ for all $s \leq -\frac{1}{\eta}$ and $H_{\varepsilon,\Lambda}(\eta s) = H_0^\Lambda + \varepsilon V^\Lambda$ for all $s \geq 0$. The parameter $\eta > 0$ controls the time scale $\frac{1}{\eta}$ on which the switching process occurs. The state of the system at time $s \geq 0$, when the perturbation is fully switched on, is obtained from solving the time-dependent Schrödinger equation $i \frac{d}{ds} \tilde{\rho}_{\varepsilon,\Lambda,\eta,f}(s) = [H_{\varepsilon,\Lambda}(\eta s), \tilde{\rho}_{\varepsilon,\Lambda,\eta,f}(s)]$ with initial condition $\tilde{\rho}_{\varepsilon,\Lambda,\eta,f}(s) = P_{\Lambda}^*$ for all $s \leq -\frac{1}{\eta}$. It is convenient and common practice to rescale the time variable to $t = \eta s$, which yields the standard form of the Schrödinger equation with adiabatic parameter $\eta > 0$, namely
\[ i \eta \frac{d}{dt} \rho_{\varepsilon,\Lambda,\eta,f}(t) = [H_{\varepsilon,\Lambda}(t), \rho_{\varepsilon,\Lambda,\eta,f}(t)] \]
with $\rho_{\varepsilon,\Lambda,\eta,f}(t) = \tilde{\rho}_{\varepsilon,\Lambda,\eta,f}(t/\eta)$.

The response of the system with respect to an observable $B^\Lambda$ at time $t \geq 0$ is now defined as
\[ \sigma_{\varepsilon,\Lambda,\eta,f}(t) := \frac{1}{N(\Lambda)} \left( \langle B \rangle_{\rho_{\varepsilon,\Lambda,\eta,f}(t)} - \langle B \rangle_{P_{\Lambda}^*} \right), \]
i.e. as the difference between the expectation of $B^\Lambda$ in the state after the perturbation $V^\Lambda$ was turned on and its expectation in the initial ground state $P_{\Lambda}^*$ of the unperturbed
Hamiltonian $H^\Lambda_0$. Here $N(\Lambda)$ is a normalisation depending on the localisation properties of the perturbation $V^\Lambda$ and the observable $B^\Lambda$. E.g., in the case of extensive quantities $B^\Lambda$ and perturbations $V^\Lambda$ that act everywhere, $N(\Lambda)$ must be chosen proportional to the volume $|\Lambda|$. Since in general many-body situations the quantity $\sigma^{\varepsilon,\Lambda,\eta,f}(t)$ is neither computable nor interesting, one considers the following asymptotic regimes, that lead to explicit and practically useful formulas. First, since one is interested in macroscopic systems, to avoid finite size effects one takes the thermodynamic limit $\Lambda \to \mathbb{Z}^d$. Second, one expects that in the adiabatic limit $\eta \ll 1$ of slow switching the system settles in an (almost) stationary state that has no “memory” of the switching procedure, i.e. that the response becomes independent of the precise form of the switching function $f$ and also independent of the time $t \geq 0$. Finally, one is interested in small perturbations and thus in an expansion of the response in powers of $\varepsilon$.

The standard linear response calculation now proceeds by expanding $\sigma^{\varepsilon,\Lambda,\eta,f}(0)$ first in powers of $\varepsilon$,

$$
\sigma^{\varepsilon,\Lambda,\eta,f}(0) =: \varepsilon \sigma_1^{\Lambda,\eta,f} + o_{\Lambda,\eta,f}(\varepsilon),
$$

where $o_{\Lambda,\eta,f}(\varepsilon)$ denotes a remainder term that is $o(\varepsilon)$ when $\Lambda$, $\eta$, and $f$ are kept fixed. This expansion can be easily achieved by standard time-dependent perturbation theory and yields

$$
\sigma_1^{\Lambda,\eta,f} = -\frac{i}{N(\Lambda)} \int_{-\infty}^{0} f(\eta s) \langle [B(-s), V] \rangle_{P^\Lambda} ds.
$$

Here we use the notation $B^\Lambda(s) := e^{iH^\Lambda_0 s}B^\Lambda e^{-iH^\Lambda_0 s}$ for the Heisenberg time-evolution of an operator $B^\Lambda$.

Then one considers $\sigma_1^{\Lambda,\eta,f}$ in the adiabatic and thermodynamic limit and calls the resulting quantity

$$
\sigma_1 := \lim_{\eta \to 0} \lim_{\Lambda \to \mathbb{Z}^d} \sigma_1^{\Lambda,\eta,f} = -i \lim_{\eta \to 0} \int_{-\infty}^{0} f(\eta s) \langle [B(-s), V] \rangle_{P^\infty} ds
$$

the linear response coefficient. Typically, if the limits exist, $\sigma_1$ is independent of $f$ and one chooses $f(\eta s) = e^{\eta s}$ to simplify the explicit evaluation of (12).

However, the procedure just described is only justified when the remainder term in (10) is of lower order in $\varepsilon$ uniformly in the volume $\Lambda$ and in the adiabatic parameter $\eta$. This uniformity of time-dependent perturbation theory on long (adiabatic) time scales is anything but obvious and can only be expected if the system indeed evolves into an (almost) stationary state that looses all memory of the switching process. But the occurrence of such a behaviour cannot hold unconditionally and needs to be established under suitable additional assumptions.

Recently Bachmann et al. [2] were able to prove validity of linear response for interacting spin systems and for quasi-local perturbations that do not close the spectral gap (i.e., they assume that $H^{\varepsilon,\Lambda}(t)$ has a spectral gap above its ground state $P^\varepsilon_{\Lambda}(t)$ uniformly in $\varepsilon \in [0,\varepsilon_0)$, $\Lambda$, and $t \in \mathbb{R}$). The key ingredient is their adiabatic theorem that shows that in the adiabatic limit $\eta \to 0$ for local observables $B$ the expectation
\( \langle B \rangle_{\rho,\Lambda,\eta(0)} \) converges to \( \langle B \rangle_{P^\Lambda} \), i.e. that
\[
\sigma^{\varepsilon,\Lambda}(0) := \lim_{\eta \to 0} \sigma^{\varepsilon,\Lambda,\eta,f}(0) = \langle B \rangle_{P^\Lambda} - \langle B \rangle_{P^\Lambda}
\]
uniformly in the volume \( \Lambda \). Then, using the spectral flow, they show that the asymptotic expansion of \( \sigma^{\varepsilon,\Lambda}(0) \) in powers of \( \varepsilon \) starts with a linear term,
\[
\sigma^{\varepsilon,\Lambda}(0) = \varepsilon \sigma_1^\Lambda + o(\varepsilon),
\]
where, again, the error term is uniform in \( \Lambda \). As all limits are uniform in the volume \(|\Lambda|\), one can take the thermodynamic limit in the end and, if it exists, it defines the physically meaningful linear response coefficient
\[
\tilde{\sigma}_1 := \lim_{\Lambda \to \mathbb{Z}^d} \sigma_1^\Lambda.
\]
While Bachmann et al. [2] do not discuss this in detail, it is likely that under rather mild assumptions one can show that \( \tilde{\sigma}_1 \) agrees with \( \sigma_1 \) in (12).

We now show that, using the results presented in the previous section, a similar reasoning allows us to rigorously derive linear and also higher order response coefficients even for perturbations that close the gap. However, one important conceptual difference appears. Since the system evolves into an almost-invariant state with a life-time that depends on the strength \( \varepsilon \) of the perturbation, the adiabatic switching must occur on a time-scale not longer than the life-time, cf. Remark 3.3. This puts a lower bound of the form \( \eta \geq \varepsilon^m \) for some \( m \geq 1 \) on the adiabatic parameter \( \eta \). On the other hand, as we will show, even a relatively fast switching with \( \eta \leq \varepsilon^{1/m} \) for some \( m \geq 1 \) still leads to a response coefficient that has an asymptotic expansion in powers of \( \varepsilon \) with coefficients independent of the switching function \( f \).

**Theorem 4.1 (Response theory to all orders).** Under the same assumptions as in Proposition 3.2 let again \( \rho^{\varepsilon,\Lambda,\eta,f}(t) \) be the solution of the adiabatic time-dependent Schrödinger equation (9) with adiabatic parameter \( \eta \in (0,1] \) and initial datum \( \rho^{\varepsilon,\Lambda,\eta,f}(t) = P^\Lambda \) for all \( t \leq -1 \).

For \( L \in \text{Loc} \) with \( \ell \cdot \ell_H = 0 \) and \( B \in \mathcal{L}_{L^\infty,L^1} \) define the normalised response as
\[
\sigma^{\varepsilon,\Lambda,f}(t) := \frac{\varepsilon^{l(H)} }{M^2 - |\ell_H|^2} \left( \langle B \rangle_{\rho^{\varepsilon,\Lambda,\eta,f}(t)} - \langle B \rangle_{P^\Lambda} \right),
\]
and for \( j \in \mathbb{N} \) the \( j \)th order response coefficient as
\[
\sigma_j^\Lambda := \frac{\varepsilon^{l(H)} }{M^2 - |\ell_H|^2} \langle K^\varepsilon_1[B] \rangle_{P^\Lambda},
\]
where the \( K^\varepsilon_j \)'s were defined and explicit expressions for \( \langle K^\varepsilon_1[B] \rangle_{P^\Lambda} \) and \( \langle K^\varepsilon_2[B] \rangle_{P^\Lambda} \) were given in Proposition 3.1.

Then for any \( n \in \mathbb{N} \) and \( m \geq 1 \) there exists a constant \( C \in \mathbb{R} \) independent of \( \varepsilon \), such that for \( t \geq 0 \) and \( r = \max\{2d,n+1\} \)
\[
\sup_{\Lambda(M):M \geq M_0} \sup_{\eta \in [\varepsilon^m,\varepsilon^{1/m}]} \left| \sigma^{\varepsilon,\Lambda,f}(t) - \sum_{j=1}^n \varepsilon^j \sigma_j^\Lambda \right| \leq \varepsilon^{n+1} C (1 + t^{d+1}) \| \Phi_B \|_{\varepsilon,r,L^1}. \tag{13}
\]
Remark 4.1. (a) The response coefficients \( \sigma_j^\Lambda \) are independent of the switch function \( f \), the adiabatic parameter \( \eta \), time \( t \geq 0 \), and also of \( n \) and \( m \). For slowly varying potentials of the form \( v^\epsilon,\Lambda = \epsilon v^\Lambda \) or perturbations of the form \( \epsilon H_1 \), they are also independent of \( \epsilon \). Thus the response has an asymptotic expansion in the strength \( \epsilon \) of the perturbation that is uniform in the volume and with coefficients that are constant in time and have no memory of the switching.

(b) Looking at the first two coefficients \( \sigma_1^\Lambda \) and \( \sigma_2^\Lambda \) given by (7) and (8), one notes that they agree with what one would get from just applying regular perturbation theory for isolated eigenvalues, e.g. [13]. This is true also for all higher order terms, as follows from Remark 3.2 (b).

(c) Another formula for \( \sigma_1^\Lambda = -i \frac{\epsilon |H|}{M^{-d} |\ell| |H|} \langle [V,R_0], B \rangle \rangle_p^\Lambda \) that can be checked by direct computation is
\[
\sigma_1^\Lambda = -i \left. \frac{\epsilon |H|}{M^{d-|\ell|} |H|} \right|_{\eta \to 0} \int_{-\infty}^0 e^{\eta s} \langle [B(-s), V] \rangle \rangle_p^\Lambda \, ds,
\]
\[\text{i.e. one chooses } f = \exp \text{ in (11).}\]

Proof of Theorem 4.1 According to Proposition 3.2, for \( \tilde{n} := m(n + d + 1) \) there exists \( C_1 \in \mathbb{R} \) such that for \( t \geq 0 \) and \( \epsilon \geq \eta \geq \epsilon_m \) it holds that
\[
\sup_\Lambda \frac{\epsilon |H|}{M^{d-|\ell|} |H|} \left| \langle B \rangle_{\rho^\epsilon,\Lambda,\eta,f(t)} - \langle B \rangle_{\Pi_{\tilde{n}}^\Lambda} \right| \leq C_1 \epsilon^{\tilde{n}+1} \eta^{d+1} + (1 + \epsilon^{d+1}) \|\Phi_B\|_{\zeta_{2d,2d},L_1} \leq 2C_1 \epsilon^{n+1} (1 + \epsilon^{d+1}) \|\Phi_B\|_{\zeta_{2d,2d},L_1}.
\]

According to Proposition 3.1, there exists \( C_2 \in \mathbb{R} \) such that
\[
\sup_\Lambda \frac{\epsilon |H|}{M^{d-|\ell|} |H|} \left| \langle B \rangle_{\Pi_{\tilde{n}}^\Lambda} - \sum_{j=0}^n \epsilon^j \langle K_j^\Lambda(B) \rangle_p^\Lambda \right| = C_2 \epsilon^{n+1} \|\Phi_B\|_{\zeta_{n+1,n+1},L_1},
\]
which proves (13). ☐

While we believe that our results are of general conceptual interest, let us end this section with a concrete example, where linear response for gap closing potentials can play an important role. The understanding of the fractional quantum Hall effect [16] rests on the idea that the many-body interaction itself can produce spectral gaps at fractional fillings of bands. These gaps are much smaller than gaps between Landau levels and not much is known about their stability under external perturbations. While the stability of the spectral gap for small perturbations of free fermions is now well understood [11] (and the results of [2] can be applied at least for sufficiently small perturbations), for general many-body ground states, like those in the fractional Hall
effect, such a result seems out of reach at the moment. Moreover, for the very small gaps at fractional filling it could hold only for very small perturbations anyway. However, as our results show, the validity of linear response for small and slowly varying perturbations does not depend on the global stability of spectral gaps.

5 The space-time adiabatic theorem

In this section we consider time-dependent Hamiltonians of the form

\[ H(t) = H_0(t) + V_\varepsilon(t) + \varepsilon H_1(t), \]

where only \( H_0(t) \) is assumed to have a gapped ground state, and prove a novel adiabatic theorem of the following type. While the standard adiabatic theorem asserts that solutions of the time-dependent Schrödinger equation remain near gapped spectral subspaces of the instantaneous Hamiltonian in the adiabatic limit, our theorem states that solutions remain near non-equilibrium almost-stationary states of the instantaneous Hamiltonian. The results of this section contain the statements of Section 3 as special cases.

Definition 5.1 (Spaces of time-dependent operators). Let \( I \subseteq \mathbb{R} \) be an interval. We say that a map \( A : I \to L_{\mathcal{A},n,L}^\infty \) is smooth and bounded whenever it is given by interactions \( \Phi^A(t) \) such that the maps

\[ I \to \mathcal{A}_X^\infty, \quad t \mapsto \Phi^A(t, X), \]

are infinitely differentiable for all \( \Lambda \) and \( X \subset \Lambda \) and

\[ \sup_{t \in I} \| \Phi^{(k)}_A(t) \|_{\mathcal{A},n,L} < \infty \quad \text{for all} \quad k \in \mathbb{N}_0. \]

Here \( \Phi^{(k)}_A(t) = \{(\frac{d^k}{dt^k}\Phi^A(t))\}_\Lambda \) denotes the interaction defined by the term-wise derivatives. The corresponding spaces of smooth and bounded time-dependent interactions and operator-families are denoted by \( \mathcal{B}_{I,\mathcal{A},n,L}^\infty \) and \( \mathcal{L}_{I,\mathcal{A},n,L}^\infty \).

We say that \( A : I \to \mathcal{L}_{S,\mathcal{A},n,L}^\infty \) is smooth and bounded if for any \( n \in \mathbb{N}_0 \) there is a \( \zeta_n \in S \) such that \( A : I \to \mathcal{L}_{\zeta_n,n,L}^\infty \) is smooth and bounded and write \( \mathcal{L}_{I,S,\mathcal{A},n,L}^\infty \) for the corresponding space.

(A1)_{I,L,H} Assumptions on \( H_0 \).

Let \( I \subseteq \mathbb{R} \) be an interval and let \( H_0 : I \to \mathcal{L}_{a,\infty} \) be smooth and bounded, i.e. \( H_0 \in \mathcal{L}_{I,a,\infty} \), with values in the self-adjoint operators. If \( H^H \in \text{Loc} \) is non-trivial, i.e. \( |H^H| > 0 \), then we assume in addition that \( \frac{d}{dt}H_0 \in \mathcal{L}_{I,a,\infty,L^H}^\infty \), i.e. that its time-derivative is \( L^H_1 \)-localised.

Moreover, we assume that there exists \( M_0 \in \mathbb{N} \) such that for all \( t \in I \), \( M \geq M_0 \) and corresponding \( \Lambda = \Lambda(M) \) the operator \( H_0^\Lambda(t) \) has a gapped part \( \sigma^\Lambda(t) \subset \sigma(H_0^\Lambda(t)) \) of its
spectrum with the following properties: There exist continuous functions $f^\pm_\Lambda : I \to \mathbb{R}$ and constants $g > \tilde{g} > 0$ such that $f^\pm_\Lambda(t) \in \rho(H_0^\Lambda(t))$, $f^+_\Lambda(t) - f^-_\Lambda(t) \leq \tilde{g}$,
\[ [f^+_\Lambda(t), f^-_\Lambda(t)] \cap \sigma(H_0^\Lambda(t)) = \sigma^\Lambda_*(t), \quad \text{and} \quad \operatorname{dist}(\sigma^\Lambda_*(t), \sigma(H_0^\Lambda(t)) \setminus \sigma^\Lambda_*(t)) \geq g \]
for all $t \in I$. We denote by $P^\Lambda_*(t)$ the spectral projection of $H_0^\Lambda(t)$ corresponding to the spectrum $\sigma^\Lambda_*(t)$.

Note that now also the time dependence of $H_0(t)$ is part of the perturbation and thus, possibly, a localisation of this part of the driving can be relevant.

In contrast to the previous section, we now require merely that $H_0(t)$ has a “narrow” gapped part $\sigma_*(t)$ of the spectrum which need not consist of a single eigenvalue. Typically we have in mind almost-degenerate ground states consisting of a narrow group of finitely many eigenvalues separated by a gap from the rest of the spectrum. The condition that the width $\tilde{g}$ of the spectral patch $\sigma_*(t)$ is smaller than the gap $g$ enters in the proof through the requirement that all operators appearing in the statements are quasi-local. More technically speaking, we need the range of the quasi-local inverse $\mathcal{I}_H$ of the Liouvillian have a vanishing $P_*(\cdots)P_*$-block, cf. Appendix C.

(A2)$_{I,L^u,\gamma}$ Assumptions on the perturbations.
Let $H_1 \in \mathcal{L}_{I,S,\infty,L^u}$ be self-adjoint and let $v : I \to \mathcal{V}$ be a time-dependent slowly varying potential such that the functions $t \mapsto v^{\varepsilon,\Lambda}(t,x)$ are infinitely differentiable for all $\varepsilon \in (0,1]$, $\Lambda$, and $x \in \Lambda$. Moreover, assume that all derivatives $\frac{d^k}{dt^k} v^{\varepsilon,\Lambda}(t,x)$ define again slowly varying potentials. Denote by $V_\varepsilon$ the corresponding operator-family.

If $L^H$ is non-trivial, then assume also that $\frac{d^k}{dt^k}[H_0(t), \frac{1}{\varepsilon} V_\varepsilon(t)] \in \mathcal{L}_{S,\infty,L^u}$ uniformly in $t \in I$ for all $k \in \mathbb{N}_0$.

Recall that the time-independent NEASS of the previous section was given by conjugation of the spectral projection $P_*$ of $H_0$ by the time-independent unitary $e^{iS_\varepsilon}$. In the time-dependent setting the spectral projection $P_*(t)$, the analog $\Pi^\delta_{*,n}(t) = e^{iS_\varepsilon^\Lambda(t)} P_*(t) e^{-iS_\varepsilon^\Lambda(t)}$ of the NEASS, and the unitary $e^{iS_\varepsilon^\Lambda(t)}$ connecting them, all depend on time. A special case of our adiabatic theorem is as follows. Assume that the system starts at time $t_0$ in the state $\rho(t_0) = \Pi^\delta_{*,n}(t_0)$, i.e. the initial state is the projection onto the full superadiabatic subspace. Then the solution of the time-dependent Schrödinger equation
\[ i\varepsilon \frac{d}{dt} \rho(t) = [H(t), \rho(t)] \quad (14) \]
remains close to $\Pi^\delta_{*,n}(t)$ up to small errors in $\varepsilon$ and for long times. Note that we replaced the adiabatic parameter $\eta$ of the previous sections by $\eta = \varepsilon \delta$ with $\delta \in [0, \frac{1}{\varepsilon}]$ in order to facilitate the notation in asymptotic expansions in powers of $\varepsilon$. In particular, $\Pi^\delta_{*,n}(t)$ is the static NEASS of $H(t)$ discussed in the previous section. We will show that at times $\tilde{t} \in I$ where the first $n$ time-derivatives of $H$ vanish, we have indeed that $\Pi^\delta_{*,n}(\tilde{t})$ equals the static NEASS $\Pi^0_{*,n}(\tilde{t})$. If, in addition, also the perturbation vanishes at time $\tilde{t}$, i.e. $H(\tilde{t}) = H_0(\tilde{t})$, then $\Pi^0_{*,n}(\tilde{t}) = P_*(\tilde{t})$. 19
However, the general statement concerns an adiabatic approximation to solutions of (14) with initial condition given by an arbitrary state
\[ \Pi_n^\delta(t_0) := e^{i\varepsilon S_n^\delta(t_0)} \rho_0 e^{-i\varepsilon S_n^\delta(t_0)} \]
in the range of \( \Pi_{s,n}^\delta(t_0) \), i.e. \( \rho_0 \) is a state satisfying \( \rho_0 = P_s(t_0)\rho_0 P_s(t_0) \). Then the adiabatic approximation must also track the evolution within the spectral subspace and the statement becomes that
\[ \Pi_n^\delta(t) := e^{i\varepsilon S_n^\delta(t)} P_n^\delta(t) e^{-i\varepsilon S_n^\delta(t)} \]
is close to the solution of (14) with initial condition \( V \) satisfies the effective equation
\[ i\varepsilon \frac{d}{dt} P_n^\delta(t) = [\varepsilon \delta K(t) + \sum_{\mu=0}^n \varepsilon^\mu h_\mu^\delta(t), P_n^\delta(t)] \quad \text{with} \quad P_n^\delta(t_0) = \rho_0 . \] (15)

Here \( K(t) := \mathcal{I}_{H_0(t),0} P_s(t) \) (cf. Appendix C) is a self-adjoint quasi-local generator of the parallel transport within the vector-bundle \( \Xi_{s,I} \) over \( I \) defined by \( t \mapsto P_s(t) \), i.e. its off-diagonal terms are
\[ P_s(t)K(t)P_s(t) = -iP_s(t)\dot{P}_s(t) \quad \text{and} \quad P_s(t)K(t)P_s(t) = -i\dot{P}_s(t)P_s(t) , \]
and \( P_s(t)K(t)P_s(t) = 0 \). The coefficients \( h_\mu^\delta(t) \) of the effective Hamiltonian generating the dynamics within \( \Xi_{s,I} \) are quasi-local operators that commute with \( P_s(t) \). Thus, for \( \rho_0 = P_s(t_0) \) the unique solution of (15) is \( P_s(t) \) and the effective Hamiltonian \( \sum_{\mu=0}^n \varepsilon^\mu h_\mu^\delta(t) \) is irrelevant.

Our main results are now stated in three steps. Proposition 5.1 contains the detailed properties of the super-adiabatic state that solves the time-dependent Schrödinger equation up to a small and quasi-local residual term. The adiabatic theorem, Theorem 5.1 asserts that the super-adiabatic state and the actual solution of the time-dependent Schrödinger equation are close in the sense that they agree on expectations for quasi-local observables up to terms asymptotically smaller than any power of \( \varepsilon \). Finally, in Proposition 5.2 we determine the asymptotic expansion of the NEASS.

**Proposition 5.1 (The space-time adiabatic expansion).** For some \( L^H \in \text{Loc} \) and \( \gamma \in \{0,1\} \) let the Hamiltonian \( H_0 \) satisfy Assumption (A1) and let \( H = H_0 + V_v + \varepsilon H_1 \) with \( V_v \) and \( H_1 \) satisfying (A2) and \( \mu \in \mathbb{N} \) of quasi-local \( L^H \)-localised operator families, such that for any \( n \in \mathbb{N}_0, t_0 \in I \), and any family of initial states \( \rho_0^\lambda \) with \( \rho_0^\lambda = P_s^\lambda(t_0)\rho_0^\lambda P_s^\lambda(t_0) \), the state
\[ \Pi_n^{\varepsilon,\lambda}(t) := e^{i\varepsilon S_n^{\varepsilon,\lambda}(t)} P_n^{\varepsilon,\lambda}(t) e^{-i\varepsilon S_n^{\varepsilon,\lambda}(t)} , \]
solves
\[ i\varepsilon \frac{d}{dt} \Pi_n^{\varepsilon,\lambda}(t) = [ H^{\varepsilon,\lambda}(t), \Pi_n^{\varepsilon,\lambda}(t) ] + \varepsilon^{n+1}(1 + \delta^{n+1}) [ P_n^{\varepsilon,\lambda}(t), \Pi_n^{\varepsilon,\lambda}(t) ] , \]
where
\[ S^\varepsilon,\Lambda,\delta_n(t) := \sum_{\mu=1}^n \varepsilon^{\mu-1} A^\varepsilon,\Lambda,\delta_n(t), \]

\[ P^\varepsilon,\Lambda,\delta_n(t) \] is the solution of the effective equation
\[ i\varepsilon \frac{d}{dt} P^\varepsilon,\Lambda,\delta_n(t) = \left[ \varepsilon \delta K^\Lambda(t) + \sum_{\mu=0}^n \varepsilon^\mu h^\varepsilon,\Lambda,\delta_n(t), P^\varepsilon,\Lambda,\delta_n(t) \right] \quad \text{with} \quad P^\varepsilon,\Lambda,\delta_n(t_0) = \rho^\Lambda_0 , \] (16)

and \( R^\delta_n \in \mathcal{L}_{I,S,\infty,L^\varepsilon_H} \) uniformly for \( \delta \in [0, \frac{1}{\varepsilon}] \).

Further properties of \((h^\delta_{\mu})_{\mu \in \mathbb{N}_0}\) and \((A^\delta_{\mu})_{\mu \in \mathbb{N}}\) are:
\begin{enumerate}
  \item \([h^\delta_{\mu}, P^\Lambda_{\ast}(t)] = 0 \) for all \( t \in I, \mu \in \mathbb{N}_0, \) and \( \delta \in [0, \frac{1}{\varepsilon}] \).
  \item All \( A^\delta_{\mu} \) are polynomials of degree \( \mu \) in \( \delta \) with coefficients in \( \mathcal{L}_{I,S,\infty,L^\varepsilon_H} \) and the constant coefficients \( A^0_{\mu}(t) \) are those of the almost-stationary state \( \Pi^\delta_n,\Lambda,0(t) \) for the Hamiltonian \( H^{\varepsilon,\Lambda}(t) \).
  \item If for some \( \bar{t} \in I \) it holds that \( \frac{d^k}{dt^k} H(\bar{t}) = 0 \) for \( k = 1, \ldots, n \), then \( A^\delta_{\mu}(\bar{t}) = A^0_{\mu}(\bar{t}) \) and thus \( \Pi^\delta_n,\Lambda,\delta(\bar{t}) = \Pi^\delta_n,\Lambda,0(\bar{t}) \).
\end{enumerate}

The proof of Proposition 5.1 is given in Section 6.1. Note that the generator \( S_n(t) \) of the super-adiabatic transformation at time \( t \) turns out to depend only on the Hamiltonian \( H \) and its time-derivatives at time \( t \). As a consequence, if \( \rho_0 = P_\ast(t_0) \), i.e. also \( P^\delta_n(t) = P_\ast(t) \) depends only on \( H_0 \) at time \( t \), then the super-adiabatic projection \( \Pi^\delta_n(t) \) itself depends only on the Hamiltonian \( H \) and its time-derivatives at time \( t \). In this case the NEASS has no “memory” of the switching process.

We now state the general space-time adiabatic theorem that is proved in Section 6.2.

**Theorem 5.1 (The space-time adiabatic theorem).** For some \( L^\varepsilon_H \in \operatorname{Loc} \) and \( \gamma \in \{0,1\} \) let the Hamiltonian \( H_0 \) satisfy Assumption (A1)_{L,H^\varepsilon} \) and let \( H = H_0 + V_\varepsilon + \varepsilon H_1 \) with \( V_\varepsilon \) and \( H_1 \) satisfying (A2)_{L,H^\varepsilon}. \) Let \( \Pi^\varepsilon,\Lambda,\delta_n(t) \) be the super-adiabatic state of order \( n > d \) constructed in Proposition 5.1 and \( \rho^\varepsilon,\Lambda,\delta(t) \) the solution of the Schrödinger equation
\[ i\varepsilon \frac{d}{dt} \rho^\varepsilon,\Lambda,\delta(t) = [H^\varepsilon,\Lambda(t), \rho^\varepsilon,\Lambda,\delta(t)] \quad \text{with} \quad \rho^\varepsilon,\Lambda,\delta(t_0) = \Pi^\varepsilon,\Lambda,\delta_n(t_0) . \]

Then for any \( \zeta \in \mathcal{S} \) and \( L \in \operatorname{Loc} \) with \( \ell \cdot \ell_H = 0 \) there exists a constant \( C \) such that for any \( B \in \mathcal{L}_{\zeta,2d,L_1}, \delta \in [0, \frac{1}{\varepsilon}] \), and all \( t \in I \)
\[ \sup_{\Lambda(M):M \geq M_0} \frac{\varepsilon|H|^\gamma}{M^d|\ell|^{d+1}} |\operatorname{tr}(\rho^\varepsilon,\Lambda,\delta(t) B^\Lambda) - \operatorname{tr}(\Pi^\varepsilon,\Lambda,\delta_n(t) B^\Lambda)| \leq C \varepsilon^{n+1 + \left(\frac{\varepsilon\delta}{d+1}\right)^{n+1}} |t - t_0|(1 + |t - t_0|^d) \|\Phi_B\|_{\zeta,2d,L_1} , \]

(17)

where \( \kappa := \sup \Lambda \operatorname{tr}(\rho^\Lambda_0) \).
For simplicity, we state and prove the explicit expansion of the super-adiabatic state only for the case that \( \rho_0 = P_*(t_0) \) and thus \( P_\nu^0(t) = P_*(t) \). The general case could be handled as in Theorem 3.3 in [18]. The proof of the following statement is given in Section 6.3.

**Proposition 5.2 (Asymptotic expansion of the time-dependent NEASS).** Under the assumptions of Theorem 5.1 and if \( \rho_0 = P_*(t_0) \), there exist time-dependent linear maps \( K_j^{\varepsilon,\Lambda,\delta}: A^{\Omega}\Lambda \to A^{\Omega}\Lambda, j \in \mathbb{N} \), such that for any \( k \in \mathbb{N}_0, \zeta \in \mathcal{S}, \) and \( L \in \text{Loc} \) with \( \ell \cdot \ell_H = 0 \) there is a constant \( C \) such that for any \( B \in L_{\zeta,k+1,L_1} \) it holds that

\[
\sup_{t \in I} \sup_{\Lambda(M): M \geq M_0} \frac{e^{\ell H^\gamma}}{M^{2-|\ell| - 2\ell_H}} \left| \text{tr} \left( \Pi_k^{\varepsilon,\Lambda,\delta}(t) B^\Lambda \right) - \sum_{j=0}^k \varepsilon^j \text{tr} \left( P_*(t) K_j^{\varepsilon,\Lambda,\delta}(t) [B^\Lambda] \right) \right| \leq C \varepsilon^{k+1} (1 + \delta^{k+1}) \| \Phi_B \|_{\zeta,k+1,L_1},
\]

with

\[
\sup_{t \in I} \sup_{\Lambda(M): M \geq M_0} \sup_{\varepsilon \in (0,1]} \frac{e^{\ell H^\gamma}}{M^{2-|\ell| - 2\ell_H}} \text{tr} \left( P_*(t) K_j^{\varepsilon,\Lambda,\delta}(t) [B^\Lambda] \right) < C(1 + \delta^j)
\]

for all \( j \leq k \).

Each map \( K_j^{\varepsilon,\Lambda,\delta} \) is given by a finite sum of nested commutators with the operators \( A_1^\delta, \ldots, A_j^\delta \) constructed in Theorem 3.1. Explicitly, the first terms are

\[
K_0^{\varepsilon,\Lambda,\delta}(t) = \text{Id}, \quad K_1^{\varepsilon,\Lambda,\delta}(t)[\cdot] = -i [A_1^\varepsilon, A_j^\Lambda, \cdot],
\]

\[
K_2^{\varepsilon,\Lambda,\delta}(t)[\cdot] = -i [A_2^\varepsilon, A_j^\Lambda, \cdot] + \frac{1}{2} [A_1^\varepsilon, A_j^\Lambda, A_1^\varepsilon, A_j^\Lambda, \cdot].
\]

If \( \sigma_*(t) = \{ E_\ast(t) \} \) is a single eigenvalue, then

\[
\text{tr} \left( P_*(t) K_1^{\varepsilon,\Lambda,\delta}(t) [B^\Lambda] \right) = i \delta \text{tr} \left( P_*(t) \left[ \left( P_*(t) R_0^\Lambda(t) + R_0^\Lambda(t) P_*(t) \right), B^\Lambda \right] \right) + \text{tr} \left( P_*(t) \left[ \left( R_0^\Lambda(t), \frac{1}{2} V_{\varepsilon,\Lambda}(t) + H_1^\Lambda(t) \right), B^\Lambda \right] \right)
\]

where \( R_0^\Lambda(t) := (H_0^\Lambda(t) - E_\ast(t))^{-1} (1 - P_*(t)) \) denotes the reduced resolvent of \( H_0^\Lambda(t) \). In the time-independent case, the second order term is given by [8].

### 6 Proofs of the main results

#### 6.1 The space-time adiabatic expansion

**Proof of Proposition 5.1.** To simplify the notation and to improve readability, we drop all super- and subscripts that are not necessary to distinguish different objects, as well as the dependence on time, \( \varepsilon \), and \( \Lambda \).
Taking a time derivative of \( \Pi = U_P n U^* \), where \( U := e^{iS} \), and using (16) yields
\[
i \delta e\frac{d}{dt} \Pi = i \delta e \dot{U}_P n U^* + i \delta e U P_n \dot{U}^* + U [\delta e K + h, P_n] U^* = [H, \Pi] + U \left[ i \delta e \dot{U}^* + (\varepsilon \delta K + h) - U^* H U, P_n \right] U^*,
\]
where for the second equality we employed the identities \( U \dot{U}^* = -\hat{U} U^* \) and \([H, \Pi] = U[U^* H U, P_n] U^* \). We thus need to choose the coefficients \( A_\mu \) entering in the definition of \( U \) in such a way that the remainder term satisfies
\[
U \left[ i \delta e \dot{U}^* + \varepsilon \delta K + h - U^* H_0 U - U^*(V_v + \varepsilon H_1) U, P_n \right] U^* = \varepsilon^{n+1} \left( 1 + \delta^{n+1} \right) U[U^* R_n U, P_n] U^*. \tag{18}
\]
Expanding \( U^* H_0 U \) yields
\[
U^* H_0 U = e^{-iS} H_0 e^{iS} = \sum_{k=0}^n \frac{\varepsilon^k}{k!} \mathcal{L}_S^k(H_0) + \frac{\varepsilon^{n+1}}{(n+1)!} e^{-i\hat{S}} \mathcal{L}_S^{n+1}(H_0) e^{i\hat{S}} =: \sum_{\mu=0}^n \varepsilon^\mu H_{0,\mu} + \varepsilon^{n+1} H_{0,n+1},
\]
where \( \hat{\gamma} \in [0, \varepsilon] \), each \( H_{0,\mu} \), \( \mu = 1, \ldots, n \), is defined as the sum of those terms in the series that carry a factor \( \varepsilon^\mu \), and \( \varepsilon^{n+1} H_{0,n+1} \) is the sum of all remaining terms. Above we use the notation \( \mathcal{L}_A(B) := -i[A, B] \) and \( \mathcal{L}_S^k(H_0) \) for the nested commutator \([[-iS, \cdots , [-iS, [-iS, H_0]] \cdots ]]\), where \(-iS \) appears \( k \) times. Clearly
\[
H_{0,\mu} = -\mathcal{L}_{H_0}(A_\mu) + L_\mu,
\]
where \( L_\mu \) contains a finite number of iterated commutators of the operators \( A_\nu \), \( \nu < \mu \), with \( H_0 \). Explicitly, the first orders are
\[
H_{0,0} = H_0, \quad H_{0,1} = -\mathcal{L}_{H_0}(A_1), \quad H_{0,2} = -\mathcal{L}_{H_0}(A_2) - \frac{1}{2} [A_1, [A_1, H_0]].
\]
We obtain a similar expansion for
\[
U^* V_v U := \sum_{\mu=0}^{n-1} \varepsilon^\mu V_\mu + \varepsilon^{n+1} V_n, \quad \text{and} \quad \varepsilon U^* H_1 U := \sum_{\mu=0}^{n-1} \varepsilon^{\mu+1} H_{1,\mu} + \varepsilon^{n+1} H_{1,n},
\]
with
\[
V_0 = V_v, \quad V_1 = -\mathcal{L}_{V_v}(A_1), \quad V_2 = -\mathcal{L}_{V_v}(A_2) - \frac{1}{2} [A_1, [A_1, V_v]].
\]
As a crucial observation is now, that, according to Lemma 2.1, if the operators \( A_\nu \) for \( \nu < \mu \) are polynomials in \( \delta \) of degree \( \nu \) with coefficients in \( \mathcal{L}_{L^\infty_{\mathcal{C}_0}} \), then the operator \( V_\mu \) for \( \mu \geq 1 \) is of the form
\[
V_\mu = \varepsilon \tilde{V}_\mu \quad \text{with} \quad \tilde{V}_\mu \quad \text{a polynomial of degree} \quad \mu \quad \text{with coefficients in} \quad \mathcal{L}_{L^\infty_{\mathcal{C}_0}}.
\]
We obtain the expansion of $U^*\dot{U}$ by expanding the integrand of Duhamel’s formula

$$i\varepsilon U^*\dot{U} = -\varepsilon^2 \int_0^1 e^{-i\lambda \varepsilon S} \dot{S} e^{i\lambda \varepsilon S} d\lambda$$

as a Taylor polynomial of nested commutators and then integrating term by term,

$$i\varepsilon U^*\dot{U} = -\varepsilon^2 \sum_{k=0}^{n-2} \frac{\varepsilon^k}{(k+1)!} \mathcal{L}^k(\dot{S}) - \frac{\varepsilon^{n+1}}{(n-1)!} \int_0^1 e^{-i\lambda \gamma S} \mathcal{L}^{n-1}(\dot{S}) e^{i\lambda \gamma S} d\lambda$$

$$= \sum_{\mu=1}^n \varepsilon^\mu Q_\mu + \varepsilon^{n+1} Q_{n+1}.$$

Here, again, $Q_\mu$ collects all terms in the sum proportional to $\varepsilon^\mu$. Note that $Q_\mu$ is a finite sum of iterated commutators of the operators $A_\nu$ and $A_\nu$ for $\nu < \mu$, the first terms being

$$Q_1 = 0, \quad Q_2 = -\dot{A}_1, \quad Q_3 = -\dot{A}_2 + \frac{1}{2}[A_1, \dot{A}_1].$$

Inserting the expansions into (18) leaves us with

$$i\varepsilon \delta U^*\dot{U} + \varepsilon \delta K + h - U^*H_0U - U^*(V_\varepsilon + \varepsilon H_1)U =$$

$$= \varepsilon(\delta K + \frac{1}{\varepsilon}(h_0 - H_0 - V_\varepsilon) + h_1 - H_{0,1} - H_{1,0})$$

$$+ \sum_{\mu=2}^n \varepsilon^\mu(\delta Q_\mu + h_\mu - H_{0,\mu} - \dot{V}_{\mu-1} - H_{1,\mu-1})$$

$$+ \varepsilon^{n+1}(Q_{n+1} - H_{0,n+1} - \dot{V}_n - H_{1,n}).$$

It remains to determine $A_1, \ldots, A_n$ inductively such that

$$[\mathcal{L}_{H_0}(A_1) + \delta K + \frac{1}{\varepsilon}(h_0 - H_0 - V_\varepsilon) + h_1 - H_{1,0}, P_n] \doteq 0 \quad (19)$$

and

$$[\mathcal{L}_{H_0}(A_\mu) + \delta Q_\mu + h_\mu - L_\mu - \dot{V}_{\mu-1} - H_{1,\mu-1}, P_n] \doteq 0 \quad (20)$$

for all $\mu = 1, \ldots, n$. Equations (19) and (20) can be solved for $A_\mu$, since $L_\mu, Q_\mu, \dot{V}_{\mu-1}$, and $H_{1,\mu-1}$ depend only on $A_\nu$ for $\nu < \mu$.

First, note that in the block decomposition with respect to $P_*$ (the full spectral projection), the $P_\ast^\perp(\cdots)P_\ast^\perp$-blocks on both sides of (19) resp. (20) vanish identically, independently of the choice of $A_\mu$. Second, the off-diagonal blocks of (19) resp. (20) determine $A_\mu$ uniquely. Third, we will choose the coefficient $h_\mu$ of the effective Hamiltonian in such a way that also the $P_\ast(\cdots)P_\ast$-block of (19) resp. (20) is zero.

We first consider $\mu = 1$, which is special, as we will see, for several reasons. According to Lemma C.1, the unique solution of the off-diagonal part of (19) is given by the off-diagonal part of

$$A_1 = -\delta I_{H_0,g,\tilde{g}}(K) + I_{H_0,g,\tilde{g}}(\frac{1}{\varepsilon}V_\varepsilon) + I_{H_0,g,\tilde{g}}(H_{1,0}).$$
By assumption we have that $H_{1,0} = H_1 \in \mathcal{L}_{I,S,\infty,L_H^\infty}$. Lemma C.2 implies $K = \mathcal{I}_{H_0,\hat{g},\hat{H}}(\hat{H}) \in \mathcal{L}_{S,\infty,L_H^\infty}$ and, since, according to Lemma 2.1, $H_{0,v} := [H_0, \frac{1}{\varepsilon} V_v] \in \mathcal{L}_{I,S,\infty,L_H^\infty}$, also $H_{0,v} \in \mathcal{L}_{I,S,\infty,L_H^\infty}$. Thus, again by Lemma C.2, $A_1$ is a polynomial in $\delta$ of degree one with coefficients in $\mathcal{L}_{I,S,\infty,L_H^\infty}$ and the constant term $A^0_1 = \mathcal{I}_{H_0,\hat{g},\hat{H}}(\frac{1}{\varepsilon} V_v + H_{1,0})$ is the coefficient of the almost-stationary state.

We still need to deal with the $P_s(\cdots)P_s$-block of (19). Defining

$$P_s h_0 P_s := P_s(h_0 + V_v)P_s \quad \text{and} \quad P_s h_1 P_s := P_s(H_{1,0} + \mathcal{L}_{H_0}(A_1))P_s,$$

we have that the $P_s(\cdots)P_s$-block of (19) is

$$P_s \left[ \frac{1}{\varepsilon}(h_0 - H_0 - V_v) + (h_1 - H_{1,0} + \mathcal{L}_{H_0}(A_1)), P_s \right] P_s = 0.$$  

To ensure that $h_0$ and $h_1$ are indeed quasi-local operators, we need to add also a $P_s^\perp(\cdots)P_s^\perp$-block to both of them. Using Corollary C.1 we find that

$$h_0 := H_0 + V_v - \mathcal{I}_{H_0,\hat{g},\hat{H}}(\mathcal{L}_{H_0}(V_v))$$

and

$$h_1 := H_{1,0} - \mathcal{L}_{H_0}(A_1) - \mathcal{I}_{H_0,\hat{g},\hat{H}}(\mathcal{L}_{H_0}(H_{1,0} - \mathcal{L}_{H_0}(A_1)))$$

are indeed quasi-local and diagonal, and that this new definition is compatible with (21).

For $\mu > 1$ the off-diagonal part of (20) is again solved by

$$A_\mu = \mathcal{I}_{H_0,\hat{g},\hat{H}}(-\delta Q_\mu + L_\mu + \tilde{V}_{\mu-1} + H_{1,\mu-1}).$$

Assume as induction hypothesis that for all $\nu < \mu$ and $r \in \mathbb{N}_0$ we already showed that $\frac{d^r}{dt^r} A^\delta_\nu$ are polynomials in $\delta$ of degree $\nu$ with coefficients in $\mathcal{L}_{I,S,\infty,L_H^\infty}$. Then we can conclude from Lemma B.4 and the respective definitions of $L_\mu$, $Q_\mu$, $\tilde{V}_{\mu-1}$, and $H_{1,\mu-1}$ that also $\frac{d^r}{dt^r} L_\mu$, $\frac{d^r}{dt^r} Q_\mu$, $\frac{d^r}{dt^r} \tilde{V}_{\mu-1}$, and $\frac{d^r}{dt^r} H_{1,\mu-1}$ for $r \in \mathbb{N}_0$ are all polynomials of order at most $\mu$ in $\delta$ with coefficients in $\mathcal{L}_{I,S,\infty,L_H^\infty}$. Thus with Lemma C.2 also $\frac{d^r}{dt^r} A_\mu$ is a polynomial of order $\mu$ in $\delta$ with coefficients in $\mathcal{L}_{I,S,\infty,L_H^\infty}$. In summary we conclude that for $\delta \leq \frac{1}{\varepsilon}$ we have that $\varepsilon S = \sum_{\mu=1}^n \varepsilon^\mu A_\mu \in \mathcal{L}_{I,S,\infty,L_H^\infty}$.

Finally, the $P_s(\cdots)P_s$-block of (20) is canceled by choosing

$$h_\mu = -\delta Q_\mu + L_\mu + \tilde{V}_{\mu-1} + H_{1,\mu-1} - \mathcal{L}_{H_0}(A_\mu)$$

$$- \mathcal{I}_{H_0,\hat{g},\hat{H}}(\mathcal{L}_{H_0}(-\delta Q_\mu + L_\mu + \tilde{V}_{\mu-1} + H_{1,\mu-1} - \mathcal{L}_{H_0}(A_\mu))),$$

which is again a diagonal quasi-local operator by Corollary C.1.

For the remainder term $R_n = (1 + \delta^{n+1})^{-1} U(Q_{n+1} - H_{0,n+1} - \tilde{V}_n - H_{1,n}) U^*$ first note that the terms $Q_{n+1}$, $H_{0,n+1}$, $\tilde{V}_n$, and $H_{1,n}$ are all of the form of a sum of multi-commutators of operators that are polynomials in $\varepsilon$ and $(\varepsilon\delta)$ of total degree at least $n+1$, conjugated by unitaries of the form $e^{i\gamma S}$ with $\gamma \leq \varepsilon$. The multi-commutators can be estimated by Lemma B.4 and the conjugations by Lemma B.6. The conjugation with $U$ that leads to $R_n$ is again estimated by Lemma B.6.

Finally note that if $\frac{d^{k}}{dt^{k}} H(t_0) = 0$ for all $k = 1, \ldots, n$, then $K(t_0) = 0$ and $Q_\mu(t_0) = 0$ for all $\mu = 1, \ldots, n$, and thus $A^\delta_\mu(t_0) = A^0_\mu(t_0)$ for all $\mu = 1, \ldots, n$. \qed
6.2 Proof of the adiabatic theorem

Proof of Theorem 5.1. To show (17), we first observe that

\[
\frac{e^{i|t|\gamma}}{M^{d-|t|-|t'|\delta}} \text{tr}(\rho^{\varepsilon,\Lambda,\delta}(t)B^{\Lambda}) = \frac{e^{i|t|\gamma}}{M^{d-|t|-|t'|\delta}} \sum_{X \subseteq \Lambda} \text{tr}(\rho^{\varepsilon,\Lambda,\delta}(t)\Phi_{B}^{\Lambda}(X)).
\]

We freely use the notation from Proposition 5.1 and its proof provided in the last section. Let \( U_{n}^{\varepsilon,\Lambda,\delta}(t, s) \) and \( V_{n}^{\varepsilon,\Lambda,\delta}(t, s) \) be the solutions of

\[
i \varepsilon \delta \frac{d}{dt} U_{n}^{\varepsilon,\Lambda,\delta}(t, s) = H^{\varepsilon,\Lambda}(t) U_{n}^{\varepsilon,\Lambda,\delta}(t, s)
\]

with \( U_{n}^{\varepsilon,\Lambda,\delta}(s, s) = 1 \) for \( t, s \in \mathbb{R} \)

\[
i \varepsilon \delta \frac{d}{dt} V_{n}^{\varepsilon,\Lambda,\delta}(t, s) = \left( H^{\varepsilon,\Lambda}(t) + \varepsilon \delta (1 + \delta^{n+1}) R_{n}^{\varepsilon,\Lambda,\delta}(t) \right) V_{n}^{\varepsilon,\Lambda,\delta}(t, s)
\]

with \( V_{n}^{\varepsilon,\Lambda,\delta}(s, s) = 1 \) for \( t, s \in \mathbb{R} \), respectively, and thus

\[
\rho^{\varepsilon,\Lambda,\delta}(t) = U_{n}^{\varepsilon,\Lambda,\delta}(t, t_{0}) \Pi_{n}^{\varepsilon,\Lambda,\delta}(t_{0}) U_{n}^{\varepsilon,\Lambda,\delta}(t, t_{0})
\]

and

\[
\Pi_{n}^{\varepsilon,\Lambda,\delta}(t) = V_{n}^{\varepsilon,\Lambda,\delta}(t, t_{0}) \Pi_{n}^{\varepsilon,\Lambda,\delta}(t_{0}) V_{n}^{\varepsilon,\Lambda,\delta}(t_{0}, t)
\]

Then for any local observable \( O \in A_{X}^{+} \) we have that

\[
\text{tr}\left( (\Pi_{n}^{\varepsilon,\Lambda,\delta}(t) - \rho^{\varepsilon,\Lambda,\delta}(t)) O \right) \leq \text{tr}(\rho^{\varepsilon,\Lambda,\delta}(t_{0})) \leq \kappa, \text{ with } \sup_{t \in \mathbb{R}} \|\Phi_{R_{n}}\|_{q_{0}, L_{q}^{H}} \leq C_{R_{n}}
\]

it holds that

\[
\left| \text{tr}\left( (\rho^{\varepsilon,\Lambda,\delta}(t) - \Pi_{n}^{\varepsilon,\Lambda,\delta}(t)) O \right) \right| \leq \|O\| \|X\|^{2} (\varepsilon \gamma \text{ dist}(X, L^{H})) t - t_{0} |(1 + \delta^{n+1}) t - t_{0}|^{d} \]

\[
\leq C C_{R_{n}} \varepsilon^{n+1} (1 + \delta^{n+1}) \kappa \|O\| \|X\|^{2} (\varepsilon \gamma \text{ dist}(X, L^{H})) |t - t_{0}|(1 + |t - t_{0}|^{d}) \]

\[
\leq C C_{R_{n}} \varepsilon^{n+1} (1 + \delta^{n+1}) \kappa \|O\| \|X\|^{2} (\varepsilon \gamma \text{ dist}(X, L^{H})) |t - t_{0}|(1 + |t - t_{0}|^{d}),
\]

(22)
where the second inequality follows from Lemma B.5 (due to the adiabatic time scale we pick up a factor $(\varepsilon \delta)^{-d}$ and set dist$(X, L^H) := \min_{x \in X} \text{dist}(x, L^H)$ (compare (2)). Substituting $\Phi^B(X)$ for $O$ in (22) and abbreviating $\Delta := |t - t_0|$, we obtain

$$
\left| \text{tr} \left( \left( \rho^{\varepsilon,\Lambda}(t) - \Pi^{\varepsilon,\Lambda}(t) \right) B \right) \right| \leq \frac{\varepsilon^{n+1}(1 + \delta^{n+1})}{(\varepsilon \delta)^{d+1}} k C \sum_{X \subseteq \Lambda} |X|^2 \zeta(\varepsilon \gamma) \text{dist}(X, L^H) \| \Phi^B(X) \| \Delta(1 + \Delta^d)
$$

$$
\leq \frac{\varepsilon^{n+1}(1 + \delta^{n+1})}{(\varepsilon \delta)^{d+1}} k C \sum_{x \in \Lambda} \sum_{X \subseteq \Lambda : x \in X} |X| \zeta(\varepsilon \gamma) \text{dist}(x, L^H) \| \Phi^B(X) \| \Delta(1 + \Delta^d)
$$

$$
\leq \frac{\varepsilon^{n+1}(1 + \delta^{n+1})}{(\varepsilon \delta)^{d+1}} k C \sum_{x \in \Lambda} \zeta(\varepsilon \gamma) \text{dist}(x, L^H) \sum_{y \in \Lambda} F_\zeta(d^B_{L_1}(x, y)) \times
$$

$$
\sum_{X \subseteq \Lambda : x, y \in X} \Lambda_{\text{diam}}(X)^{2d} \| \Phi^B(X) \| F_\zeta(d^B_{L_1}(x, y)) \Delta(1 + \Delta^d)
$$

$$
\leq \frac{\varepsilon^{n+1}(1 + \delta^{n+1})}{(\varepsilon \delta)^{d+1}} k C \| \Phi^B \|_{\zeta, 2d, L_1} \| F \|_\Gamma \times
$$

$$
\sum_{x \in \Lambda} \zeta(\varepsilon \gamma) \text{dist}(x, L^H) \zeta(\text{dist}(x, L)) \Delta(1 + \Delta^d)
$$

$$
\leq \frac{\varepsilon^{n+1}(1 + \delta^{n+1})}{(\varepsilon \delta)^{d+1}} k C \| \Phi^B \|_{\zeta, 2d, L_1} \| F \|_\Gamma M^{d - |\ell| - |\ell H|} \Delta(1 + \Delta^d) .
$$

In the second inequality we used that summing over all sets $X$ for which $x$ minimises the distance to $L^H$ and then over all $x \in \Lambda$ would also include each term in the sum of the previous line at least once. In the second-to-last inequality we used Lemma B.1. Hence the statement (17) of the theorem follows.

6.3 Proof of the expansion of the NEASS

Proof of Proposition 5.2. Expanding $\Pi^{\varepsilon,\Lambda,\delta}$ in the trace yields

$$
\text{tr} \left( \Pi^{\varepsilon,\Lambda,\delta} B^\Lambda \right) = \text{tr} \left( e^{i \varepsilon S^{\varepsilon,\Lambda,\delta}} P_*^\Lambda e^{-i \varepsilon S^{\varepsilon,\Lambda,\delta}} B^\Lambda \right) = \text{tr} \left( P_*^\Lambda e^{-i \varepsilon S^{\varepsilon,\Lambda,\delta}} B^\Lambda e^{i \varepsilon S^{\varepsilon,\Lambda,\delta}} \right)
$$

$$
= \text{tr} \left( P_*^\Lambda \left( \sum_{j=0}^{k} \frac{\varepsilon^j}{j!} \mathcal{L}_{S^\varepsilon}^{\Lambda}(B)^\Lambda + \frac{1}{(k + 1)!} e^{-i \tilde{\gamma} S^{\varepsilon,\Lambda,\delta}} \mathcal{L}_{S^\varepsilon}^{k+1}(B)^\Lambda e^{i \tilde{\gamma} S^{\varepsilon,\Lambda,\delta}} \right) \right)
$$

(24)

for some $\tilde{\gamma} \in [0, \varepsilon]$. Using Lemma B.3, the argument that took us from (22) to (23) gives

$$
\left\| e^{-i \tilde{\gamma} S^{\varepsilon,\Lambda,\delta}} \mathcal{L}_{S^\varepsilon}^{k+1}(B)^\Lambda e^{i \tilde{\gamma} S^{\varepsilon,\Lambda,\delta}} \right\| = \left\| \mathcal{L}_{S^\varepsilon}^{k+1}(B)^\Lambda \right\|
$$

$$
\leq C(\varepsilon^{k+1} + (\varepsilon \delta)^{k+1}) \| \Phi^B \|_{\zeta, k+1, L} M^{d - |\ell| - |\ell H|} .
$$
Hence the remainder in (24) is of order $\varepsilon^{k+1}(1 + \delta^{k+1})M^{d-|\ell|-|\ell_H|}$ and we can truncate it. Note that in the last inequality above we used the existence of a function $\tilde{\zeta} \in \mathcal{S}$ with $\mathcal{B}_{\tilde{\zeta},k+1,L_H} \subset \mathcal{B}_{\zeta,k+1,L_H}$ such that $S(t) \in \mathcal{L}_{\tilde{\zeta},k+1,L_H}$ (compare Lemma A.1 (a) in [18]). Expanding also the sum in the second line of (24) and collecting terms with the same power of $\varepsilon$ we find that

$$
\text{tr} \left( P^{A}_s e^{-i\varepsilon S^{\Lambda}} B^{A} e^{i\varepsilon S^{\Lambda}} \right) =: \text{tr} \left( P^{A}_s \sum_{j=0}^{k} \varepsilon^j \mathcal{K}^j(B^{A}) \right) + O(\varepsilon^{k+1}(1 + \delta^{k+1})\|\Phi_B\|_{\zeta,k+1,L} M^{d-|\ell|-|\ell_H|}),
$$

where $\mathcal{K}^j_j(t) : \mathcal{A}^{\mathbb{R}}_{0} \rightarrow \mathcal{A}^{\mathbb{R}}_{0}$ are linear maps given be nested commutators with various $A_{\mu}$’s. In particular,

$$
\mathcal{K}^0_0(B^{A}) = B^{A}, \quad \mathcal{K}^1_1(B^{A}) = -i[A^1_1, B^{A}],
$$

and

$$
\mathcal{K}^2_2(B^{A}) = -i[A^2_2, B^{A}] - \frac{1}{2}[A^1_1, [A^1_1, B^{A}]].
$$

For the explicit expression of the linear term we compute

$$
\text{tr}(P_s[A_1, B]) = \text{tr}([P_s, A_1]B) = \text{tr}([P_s, A_1^{OD}]B)
$$

and note that if $\sigma_s(t) = \{E_s(t)\}$ is a single eigenvalue, then for any off-diagonal operator $O = O^{OD}$ one has according to (25)

$$
\mathcal{L}_H^{-1}(O) = i [(H_0 - E_s)^{-1} P^\perp_s, O] =: i [R, O],
$$

and thus with $K^{OD} = -i[\hat{P}_s, P_s]$, that

$$
A^{OD}_1 = \mathcal{L}_{H_0,\sigma,\delta}(-\delta K + \frac{1}{\varepsilon} V_\sigma + H_1)^{OD} = \mathcal{L}_H^{-1}((-\delta K + \frac{1}{\varepsilon} V_\sigma + H_1)^{OD}) = [R, -\delta[\hat{P}_s, P_s] + i(\frac{1}{\varepsilon} V_\sigma + H_1)^{OD}].
$$

Hence,

$$
\text{tr}(P_s[A_1, B]) = -\delta\text{tr}([P_s, [R, [\hat{P}_s, P_s]]B]) + i\text{tr}([P_s, [R, \frac{1}{\varepsilon} V_\sigma + H_1]]B)
$$

$$
= -\delta\text{tr}(P_s[[\hat{P}_s R + R \hat{P}_s], B]) - i\text{tr}(P_s[[\frac{1}{\varepsilon} V_\sigma + H_1, R], B]).
$$

We skip the longer but in principle similar computation of the explicit second order term. Note however, that the uniqueness of the expansion of $\Pi^c_0$ implies that for computing this expansion one can chose a more convenient series $A_{\mu}$ not involving the map $\mathcal{I}_{H_0,\sigma,\delta}$. For example, the series $\tilde{A}_{\mu}$ obtained by choosing $\tilde{A}_{\mu}$ purely off-diagonal in each step of the construction leads to major simplifications in computing higher order terms. While this leads to a different unitary $e^{-i\varepsilon S^c}$, which presumably does not preserve quasi-locality, they result in the same NEASS up to terms of order $\varepsilon^{n+1}$.  \[\square\]
Appendices

In the following appendices we collect the various technical results used in the preceding sections. We will make use of the notation established in the previous sections, but for the sake of readability we often drop the superscripts $\varepsilon$ and $\Lambda$.

A Proof of Lemma 2.1

Before giving the proof of Lemma 2.1, we need to briefly discuss the Hilbert space structure of $A_\Lambda$. Defining $\langle A, B \rangle_{A_\Lambda} := \text{tr} A^* B$ turns the $4^{\varepsilon s}$-dimensional vector space $A_\Lambda$ into a Hilbert space. A convenient orthonormal basis is formed by the monomials. Let $\mathcal{M}_\Lambda := \{ f : \Lambda \times (1, \ldots, s) \to \{0, 1, 2, 3\}\}$ then

$$ M_f := n_f \prod_{x \in \Lambda} \prod_{j=1}^{s} m_{j, x, f(x)} \quad \text{with} \quad \left\{ \begin{array}{l} m_{j, x, 0} = 1 \\ m_{j, x, 1} = a_{j, x} \\ m_{j, x, 2} = a_{j, x}^* \\ m_{j, x, 3} = a_{j, x}^* a_{j, x} - a_{j, x} a_{j, x}^* , \end{array} \right. $$

where the ordering with respect to different sites only contributes a sign and is supposed to be fixed in any convenient way. The normalisation factor $n_f$ is specified below. The support of a monomial is the set $\text{supp}(M_f) := \{ x \in \Lambda \mid f(x, j) \neq 0 \text{ for all } j = 1, \ldots, s \}$.

The $2^{\varepsilon s}$ vectors of the form

$$ \Psi_{X_1, \ldots, X_s} := \prod_{j=1}^{s} \prod_{x \in X_j \subset \Lambda} a_{j, x}^* \Omega $$

form an orthonormal basis of Fock space, where, again, the ordering with respect to different sites only contributes a sign that is left unspecified for the moment.

It suffices to show that the monomials form an orthonormal basis of Fock space $\mathcal{F}_{\Lambda, s=1}$ for the case $s = 1$, as for $s > 1$ we are just dealing with a tensor product of copies of $\mathcal{F}_{\Lambda, s=1}$, i.e. $\mathcal{F}_{\Lambda, s} = \otimes_{j=1}^{s} \mathcal{F}_{\Lambda, s=1}$. For two monomials $M_f$ and $M_g$ we have

$$ \text{tr}(M_f^* M_g) = \sum_{X \subset \Lambda} \langle \Psi_X, M_f^* M_g \Psi_X \rangle. $$

Assume that $f(x) \neq g(x)$ for some $x \in \Lambda$. Up to a sign, the following combinations of creation and annihilation operators on the site $x$ can appear in $M_f^* M_g$,

$$ a_{x}, a_{x}^*, a_{x} a_{x}, a_{x}^* a_{x}, a_{x}^* (a_{x} a_{x} - a_{x} a_{x}^*) , a_{x} (a_{x}^* a_{x} - a_{x} a_{x}^*) , (a_{x}^* a_{x} - a_{x} a_{x}^*). $$

If one of the first six cases appears, the inner product $\langle \Psi_X, M_f^* M_g \Psi_X \rangle$ is zero for any $\Psi_X$. In the last case, the summands with $\Psi_{X \cup \{x\}}$ and with $\Psi_{X \setminus \{x\}}$ cancel because they carry the opposite sign.
In the case $f = g$, the following combinations of creation and annihilation operators on any site $x \in \Lambda$ can appear in $M_f^* M_f$,

$$1, a_x^* a_x, a_x a_x^*, (a_x^* a_x + a_x a_x^*) .$$

Thus if $f(x) = 2$ for some $x \in X$ or $f(x) = 1$ for some $x \notin X$, then $\langle \Psi_X, M_f^* M_f \Psi_X \rangle = 0$. Otherwise $\langle \Psi_X, M_f^* M_f \Psi_X \rangle = 1$ and therefore the normalisation factor is

$$n_f := 2^{-|\{x \in \Lambda | f(x) = 0 \text{ or } f(x) = 3\}|/2 .$$

**Proof of Lemma 2.1.** Because of the simple form of $V_v$, we can define the interaction of the commutator $[A, V_v]$ as

$$\Phi^A_{[A, V_v]}(Z) := \sum_{x \in \Lambda} \sum_{j=1}^s [\Phi^A_{[A]}(Z), v^{\varepsilon, A}(x) a_{j,x}^* a_{j,x}] .$$

Next we expand $\Phi^A_{[A]}(Z)$ in the basis of monomials $(M_f)_{f \in \mathfrak{M}^\Lambda}$ as

$$\Phi^A_{[A]}(Z) = \sum_{f \in \mathfrak{M}^\Lambda} c_f M_f .$$

Since $a_{j,x}^* a_{j,x}$ has non-vanishing commutator only with $a_{j,x}$ and $a_{j,x}^*$, it holds that

$$[M_f, v^{\varepsilon, A}(x) a_{j,x}^* a_{j,x}] = \begin{cases} v^{\varepsilon, A}(x) M_f & \text{if } f(x,j) = 1 \\ -v^{\varepsilon, A}(x) M_f & \text{if } f(x,j) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now $M_f \in \mathfrak{A}_\Lambda^{\|Z\|/2}$ implies that for $Z_{f,1} := \{(z,j) \in Z \times \{1, \ldots, s\} | f(z,j) = 1\}$ and $Z_{f,2} := \{(z,j) \in Z \times \{1, \ldots, s\} | f(z,j) = 2\}$ it holds that $|Z_{f,1}| = |Z_{f,2}|$. Hence we find that

$$\|\Phi^A_{[A, V_v]}(Z)\| = \left\| \sum_{x \in Z} \sum_{j=1}^s \left( \sum_{f \in \mathfrak{M}^\Lambda} c_f M_f, v^{\varepsilon, A}(x) a_{j,x}^* a_{j,x} \right) \right\|$$

$$= \left\| \sum_{f \in \mathfrak{M}^\Lambda} \left( \sum_{(x,j) \in Z_{f,1}} v^{\varepsilon, A}(x) - \sum_{(x,j) \in Z_{f,2}} v^{\varepsilon, A}(x) \right) c_f M_f \right\|$$

$$\leq \frac{|Z| s}{2} \left( \max_{x \in Z} v^{\varepsilon, A}(x) - \min_{x \in Z} v^{\varepsilon, A}(x) \right) \left\| \sum_{f \in \mathfrak{M}^\Lambda} c_f M_f \right\|$$

$$\leq \frac{\Lambda \cdot \text{diam}(Z)^d s}{2} C_v \cdot \Lambda \cdot \text{diam}(Z) \|\Phi^A_{[A]}(Z)\| ,$$

30
where we used in the first inequality that the monomials $M_f$ form an orthonormal basis. Defining $\Phi^A_{\Lambda,v} := \frac{1}{\varepsilon} \Phi^A_{\{A,V_v\}}$, we obtain

$$
\sum_{Z \supset \{x,y\}} \Lambda\text{-diam}(Z)^k \frac{\|\Phi^A_{\Lambda,v}(Z)\|}{F_{\partial_k}(d_{L_{\alpha}}^k(x,y))} = \sum_{Z \supset \{x,y\}} \Lambda\text{-diam}(Z)^k \frac{\|\frac{1}{\varepsilon} \Phi^A_{\{A,V_v\}}(Z)\|}{F_{\partial_k}(d_{L_{\alpha}}^k(x,y))} \\
\leq \frac{s}{2} C_v \sum_{Z \supset \{x,y\}} \Lambda\text{-diam}(Z)^{k+d+1} \frac{\|\Phi^A_{\Lambda,v}(Z)\|}{F_{\partial_k}(d_{L_{\alpha}}^k(x,y))} \\
\leq \frac{s}{2} C_v \|\Phi^A\|_{\zeta,k+1,\alpha}. \quad \square
$$

## B Technicalities on quasi-local operators

In this appendix we collect several results concerning local and quasi-local operators that were used repeatedly in the construction of the almost-stationary states and in the proof of the adiabatic theorem. They are all proven in Appendix C of [18], most of them based on similar lemmas in [2].

We start with a simple lemma that is at the basis of most arguments concerning localisation near $L$. It will not be explicitly used in the following, but it is important for checking that the claimed generalisations of the following results to the case of $L_{\alpha}$-localisation hold.

**Lemma B.1.** It holds that

$$
\sum_{y \in \Lambda} F_{\zeta}(d_{L_{\alpha}}^k(x,y)) \leq \zeta(\alpha \text{ dist}(x, L)) \|F\|_{r},
$$

where

$$
\|F\|_{r} := \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) < \infty.
$$

The statement for the $d_{L_{\alpha}}^k$-distance is proved in Lemma C.1 in [18] and the statement for the $d_{L_{\alpha}}^k$-distance follows by the same proof.

The next lemma shows that the norm of a quasi-local operator-family localised near $L$ grows at most like the volume of $L$.

**Lemma B.2.** Let $A \in \mathcal{L}_{\zeta,0,L_{\alpha}}$, then there is a constant $C_{\zeta}$ depending only on $\zeta$ such that

$$
\| A^A \| \leq \alpha^{-|\ell|} M^d (-|\ell|) C_{\zeta} \| \Phi^A \|_{\zeta,0,L_{\alpha}} \|F\|_{r}.
$$

This follows by the same proof as the one of Lemma C.2 in [18] using Lemma B.1. We continue with a norm estimate on iterated commutators with quasi-local operator-families, where we use $\text{ad}_A(B) := [A, B]$ to denote the adjoint map.
Lemma B.3. There is a constant $C_k$ depending only on $k \in \mathbb{N}$ such that for any $A_1 \in \mathcal{L}_{\zeta,kd,L_\alpha}$, $A_2, \ldots, A_k \in \mathcal{L}_{\zeta,kd}$, $X \subset \Lambda$, and $O \in \mathcal{A}_X^+$ it holds that

$$
\|\text{ad}_{A_k} \circ \cdots \circ \text{ad}_{A_1}(O)\| \leq C_k \|O\| \|X\|^k \zeta (\alpha \text{dist}(X, L)) \|\Phi_{A_1}\|_{\zeta, (k-1)d,L_\alpha} \prod_{j=2}^k \|\Phi_{A_j}\|_{\zeta, (k-j)d}.
$$

This follows by by the same proof as the one of Lemma C.3 in [18] using Lemma B.1.

The next lemma shows that such an iterated commutator of quasi-local operator-families is itself a quasi-local operator-family and that if one of them is $L_\alpha$-localised, then also the iterated commutator is. For the proof see Lemma C.4 in [18] and Lemma 4.5 (ii) in [2].

Lemma B.4. Let $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, $A_0 \in \mathcal{L}_{\zeta,(n+k)d,L_\alpha}$, and $A_1, \ldots, A_k \in \mathcal{L}_{\zeta,(n+k)d}$. Then $\text{ad}_{A_k} \cdots \text{ad}_{A_1}(A_0) \in \mathcal{L}_{\zeta,n,ad,L_\alpha}$ and

$$
\|\Phi_{\text{ad}_{A_k} \cdots \text{ad}_{A_1}(A_0)}\|_{\zeta, nd,L_\alpha} \leq C_{k,n} \|\Phi_{A_0}\|_{\zeta, (n+k)d,L_\alpha} \prod_{j=0}^k \|\Phi_{A_j}\|_{\zeta, (n+k)d}
$$

with a constant $C_{k,n}$ depending only on $k$ and $n$. In particular, for $A_0 \in \mathcal{L}_{\zeta,\infty,L_\alpha}$ and $A_1, \ldots, A_k \in \mathcal{L}_{\zeta,\infty}$ also $\text{ad}_{A_k} \cdots \text{ad}_{A_1}(A_0) \in \mathcal{L}_{\zeta,\infty,L_\alpha}$.

We also need to control the norm of commutators with time-evolved local operators. This is the content of the next lemma, which is Lemma C.5 in [18], see also Lemma 4.6 in [2].

Lemma B.5. Let $H \in \mathcal{L}_{\alpha,0}$ generate the dynamics $u^{\Lambda}_{t,s}$ with Lieb–Robinson velocity $v := \frac{1}{2} 2^{d+2} \|F\| \|\Phi_H\|_{\alpha,0}$. Then there exists a constant $C > 0$ such that for any $O \in \mathcal{A}_X$ with $X \subset \Lambda$, for any $A \in \mathcal{L}_{\zeta,0,L_\alpha}$ and for any $t, s \in \mathbb{R}$ it holds that

$$
\|\{A, u^{\Lambda}_{t,s}(O)\}\| \leq C \|O\| \|\Phi_{A}\|_{\zeta,0,L_\alpha} \|X\|^2 \zeta (\alpha \text{dist}(X, L_\alpha)) (1 + |t-s|^d).
$$

If $H \in \mathcal{L}_{\tilde{\zeta},0}$ for some $\tilde{\zeta} \in \mathcal{S}$, one still has that for any $T > 0$ there exists a constant $C$ such that

$$
\sup_{t,s \in [0,T]} \|\{A, u^{\Lambda}_{t,s}(O)\}\| \leq C \|O\| \|\Phi_{A}\|_{\zeta,0,L_\alpha} \|X\|^2 \zeta (\alpha \text{dist}(X, L_\alpha))
$$

The final lemma in this appendix shows that adjoining a quasi-local $L_\alpha$-localised operator-families with a unitary that is itself the exponential of a quasi-local operator-family yields a quasi-local and $L_\alpha$-localised operator-family, cf. Lemma C.7 of [18].

Lemma B.6. Let $S \in \mathcal{L}_{\zeta,0}$ be self-adjoint and let $D \in \mathcal{L}_{S,\infty,L_\alpha}$, i.e. there is a sequence $(\zeta_n)_{n \in \mathbb{N}_0}$ in $\mathcal{S}$ such that $\|\Phi_D\|_{\zeta_n,(n+1)d,L_\alpha} < \infty$. Then the family of operators

$$
\left\{ A^\Lambda := e^{-iS^\Lambda} D^\Lambda e^{iS^\Lambda} \right\}_\Lambda
$$

defines an operator-family $A \in \mathcal{L}_{S,\infty,L_\alpha}$. More precisely, there is a constant $C_{\|\Phi_S\|_{\zeta,0}}$ depending on $\|\Phi_S\|_{\zeta,0}$, $\zeta$, $(\zeta_n)_{n \in \mathbb{N}_0}$, and $d$, and a sequence $(\xi_n)_{n \in \mathbb{N}_0}$ in $\mathcal{S}$, such that

$$
\|\Phi_A\|_{\xi_n,n,L_\alpha} \leq C_{\|\Phi_S\|_{\zeta,0}} \|\Phi_D\|_{\xi_n,(n+1)d,L_\alpha}
$$

for all $n \in \mathbb{N}_0$.  

32
C Quasi-local inverse of the Liouvillian

Let $\mathcal{H}$ be a finite dimensional Hilbert space, $H \in \mathcal{L}(\mathcal{H}) =: \mathcal{A}$ be self-adjoint, $\sigma_* \subset \sigma(H)$ a subset of eigenvalues of $H$ and $P_*$ the corresponding spectral projection. The inner product $\langle A, B \rangle := \text{tr} A^* B$ turns the algebra $\mathcal{A}$ into a Hilbert space that splits into the orthogonal sum $\mathcal{A} = \mathcal{A}_P^D \oplus \mathcal{A}_P^{OD}$ with respect to $P_*,$

$$A = (P_* A P_* + P_+^* A P_+^*) + (P_+^* A P_* + P_* A P_+^*) =: A_P^D + A_P^{OD}. $$

Then the map

$$\mathcal{L}_H : \mathcal{A} \to \mathcal{A}, \quad B \mapsto \mathcal{L}_H(B) := -i[H, B]$$

is called the Liouvillian. It maps self-adjoint operators to self-adjoint operators and its restriction $\mathcal{L}_H|_{\mathcal{A}_P^{OD}} : \mathcal{A}_P^{OD} \to \mathcal{A}_P^{OD}$ to $P_*$-off-diagonal operators is an isomorphism. If $\sigma_* = \{E_*\}$ consists of a single eigenvalue, then its inverse is explicitly given by

$$(\mathcal{L}_H|_{\mathcal{A}_P^{OD}})^{-1} : \mathcal{A}_P^{OD} \to \mathcal{A}_P^{OD}, \quad B \mapsto i \left[ (H - E_*)^{-1} P_+^*, B \right], \quad \text{(25)}$$

where $(H - E_*)^{-1} P_+^*$ is a bounded operator called the reduced resolvent. See for example the discussion in Appendix D of [18] for the simple proofs of all the claims about $\mathcal{L}_H|_{\mathcal{A}_P^{OD}}.$

In the context of the so-called quasi-adiabatic evolution (called spectral flow in [3]) in [12, 3] an extension $\mathcal{I}_H$ of $(\mathcal{L}_H|_{\mathcal{A}_P^{OD}})^{-1}$ to the full algebra $\mathcal{A}$ was constructed in a way that preserves quasi-locality. To understand this quasi-local extension of the inverse of the Liouvillian, first note that for any $\lambda > 0$ one can find a real-valued, odd function $\mathcal{W}_\lambda \in L^1(\mathbb{R})$ satisfying

$$\sup\{|s|^n |\mathcal{W}_\lambda(s)| : |s| > 1\} < \infty \quad \text{for all } n \in \mathbb{N},$$

and with Fourier transform $\widehat{\mathcal{W}}_\lambda \in C^\infty(\mathbb{R})$ satisfying

$$\widehat{\mathcal{W}}_\lambda(\omega) = \frac{-i}{\sqrt{2\pi \omega}} \quad \text{for } |\omega| \geq \lambda \quad \text{and} \quad \widehat{\mathcal{W}}_\lambda(0) = 0.$$

An explicit function $\mathcal{W}_\lambda$ having all these properties is constructed in [3]. We need a slightly modified version $\mathcal{W}_{\lambda, \bar{\lambda}}$ of this function: Let $\lambda > \bar{\lambda} > 0$ and $\chi_{\lambda, \bar{\lambda}} \in C^\infty(\mathbb{R})$ a real valued function with $\chi_{\lambda, \bar{\lambda}}(\omega) = 0$ for $\omega \in [-\bar{\lambda}, \bar{\lambda}]$ and $\chi_{\lambda, \bar{\lambda}}(\omega) = 1$ for $|\omega| \geq \lambda.$ Then $\mathcal{W}_{\lambda, \bar{\lambda}}$ defined through its Fourier transform $\widehat{\mathcal{W}}_{\lambda, \bar{\lambda}} := \chi_{\lambda, \bar{\lambda}} \widehat{\mathcal{W}}_\lambda$ satisfies, in addition to the properties mentioned above for $\mathcal{W}_\lambda,$ also $\widehat{\mathcal{W}}_{\lambda, \bar{\lambda}}(\omega) = 0$ for all $\omega \in [-\bar{\lambda}, \bar{\lambda}].$

Lemma C.1. Assume $H,$ $\sigma_*,$ and $P_*$ as above and let $g := \text{dist}(\sigma_*, \sigma(H) \setminus \sigma_*) > 0.$ Then for any $g > \tilde{g} > 0$ the map

$$\mathcal{I}_{H, g} : \mathcal{A} \to \mathcal{A}, \quad \mathcal{I}_{H, g}(A) := \int_{\mathbb{R}} \mathcal{W}_{g, \tilde{g}}(s) e^{iHs} A e^{-iHs} ds$$

33
satisfies
\[ \mathcal{I}_{H,g,\tilde{g}}|_{A_{\tilde{g}}^{SD}} = \mathcal{L}_H|_{A_{\tilde{g}}^{SD}}^{-1} \]
and, if \( \text{diam}(\sigma_s) \leq \tilde{g} \),
\[ P_* \mathcal{I}_{H,g,\tilde{g}}(A) P_* = 0 \quad \text{for all } A \in \mathcal{A}. \]

\textbf{Proof.} All claims follow immediately by inserting the spectral decomposition of \( H = \sum_n E_n P_n \) into the definition of \( \mathcal{I}_H \),
\[ \mathcal{I}_{H,g,\tilde{g}}(A) = \sqrt{2\pi} \sum_{n,m} \tilde{W}_{g,\tilde{g}}(E_m - E_n) P_n A P_m, \]
and using that for \( E_n \in \sigma_s \) and \( E_m \in \sigma(H) \setminus \sigma_s \) it holds that \( |E_m - E_n| \geq g \), i.e.
\[ \tilde{W}_{g,\tilde{g}}(E_m - E_n) = \frac{-i}{\sqrt{2\pi}(E_m - E_n)}, \]
and that for \( E_n, E_m \in \sigma_s \) it holds that \( |E_m - E_n| \leq \tilde{g} \), i.e.
\[ \tilde{W}_{g,\tilde{g}}(E_m - E_n) = 0. \]

The crucial advantage of the map \( \mathcal{I}_H \) is that if \( H \) admits Lieb-Robinson bounds, then \( \mathcal{I}_H \) maps quasi-local operator-families to quasi-local operator-families, an observation originating from [12] and worked out in detail in [3]. To cover also slowly varying potentials \( V_v \) that are not contained in any of the spaces \( \mathcal{L}_{0,1,1}^{\varphi} \), we need to slightly extend these results.

\textbf{Lemma C.2.} Let \( H \in \mathcal{L}_{a,0} \) and let \( D = \{D^{\varepsilon,\Lambda}\} \) be an operator-family such that \([H, D] \in \mathcal{L}_{S,\infty,L_v^{\gamma}} \) for some \( L^H \in \text{Loc} \) and \( \gamma \in \{0,1\} \). By Lemma B.4 this is the case, in particular, if \( D \in \mathcal{L}_{S,\infty,L_v^{\gamma}} \). Then
\[ \mathcal{I}_{H,g,\tilde{g}}(D)^{\varepsilon,\Lambda} := \{ \mathcal{I}_{H^\Lambda,g,\tilde{g}}(D^{\varepsilon,\Lambda}) \}_\Lambda \]
defines an operator-family \( \mathcal{I}_{H,g,\tilde{g}}(D) \in \mathcal{L}_{S,\infty,L_v^{\gamma}} \).

\textbf{Proof.} This statement is a slight generalisation of Theorem 4.8 in [3] or Lemma 4.8 in [2], where \( D \in \mathcal{L}_{S,\infty,L_v^{\gamma}} \) is required. Since the proof is quite subtle and lengthy, we merely explain the small change due to the relaxed assumption on \( D \).

By definition of \( \mathcal{I}_{H,g,\tilde{g}} \) and the fact that \( \mathcal{W}_{g,\tilde{g}} \in L^1(\mathbb{R}) \) is odd, we have that
\[ \mathcal{I}_{H,g,\tilde{g}}(D) = \int_{\mathbb{R}} \mathcal{W}_{g,\tilde{g}}(s) e^{iHs} D e^{-iHs} \, ds = \int_{\mathbb{R}} \mathcal{W}_{g,\tilde{g}}(s) \left( e^{iHs} D e^{-iHs} - D \right) \, ds \]
\[ = i \int_{\mathbb{R}} \mathcal{W}_{g,\tilde{g}}(s) \int_0^s e^{iHu} [H, D] e^{-iHu} \, du \, ds \]
with \([H, D] \in \mathcal{L}_{S,\infty,L_v^{\gamma}} \) by assumption. Now one proceeds exactly as in [3], where the additional integration in the variable \( u \) does not change the arguments at all, since, on the one hand, for the part of the integral where \( |s| \leq T \) the bound \( \int_0^s e^{cu} \, du = \frac{1}{c} (e^{cs} - 1) \) is as good as \( e^{cs} \) itself and, on the other hand, for the part of the integral where \( |s| \geq T \) one uses that also \(|s| \mathcal{W}_{g,\tilde{g}}(s) \) decays faster than any polynomial. \( \square \)
Combining Lemma C.1 and Lemma C.2, we obtain the following corollary.

**Corollary C.1.** Assume (A1) $I, L_{H}$ for $H_{0}$ and let $D \in L_{S, \infty, L_{0}^{H}}$. Then

\[
\tilde{D} := D - I_{H_{0}(t), \tilde{g}, \tilde{g}}(L_{H_{0}(t)}(D))
\]

satisfies $\tilde{D} \in L_{S, \infty, L_{0}^{H}}$ and

\[
P_{*}(t)\tilde{D}P_{*}(t) = P_{*}(t)DP_{*}(t), \quad P_{*}(t)\tilde{D}P_{*}(t)^{\perp} = P_{*}(t)^{\perp}\tilde{D}P_{*}(t) = 0.
\]

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