GEOMETRIC PROOF OF THE $\lambda$-LEMMA

ERIC BEDFORD AND TANYA FIRSOVA

ABSTRACT. We give a geometric approach to the proof of the $\lambda$-lemma. In particular, we point out the role pseudoconvexity plays in the proof.

I. INTRODUCTION

A holomorphic motion in dimension one is the family of injections $f_\lambda : A \to \hat{C}$ over a complex manifold $\Lambda \ni \lambda$. Holomorphic motions first appeared in [13, 14] where they were used to show that a generic rational map $f : \hat{C} \to \hat{C}$ is structurally stable. This notion has since found numerous applications in holomorphic dynamics and Teichmüller Theory. Its usefulness comes from the fact that analyticity alone forces strong extendibility and regularity properties that are referred to as the $\lambda$-lemma.

Theorem 1.1.

- **Extension $\lambda$-lemma** [13], [14] Any holomorphic motion $f : \mathbb{D} \times A \to \hat{C}$ extends to a holomorphic motion $\mathbb{D} \times \bar{A} \to \hat{C}$.
- **QC $\lambda$-lemma** [14] The map $f(\lambda, a)$ is uniformly quasisymmetric in $a$.

For many applications it is important to know that a holomorphic motion can be extended to a holomorphic motion of the entire sphere. Bers & Royden [5] and Sullivan & Thurston [16] proved that there exists a universal $\delta > 0$ such that under the circumstances of the Extension $\lambda$-lemma, the restriction of $f$ to the parameter disk $\Delta_\delta$ of radius $\delta$ can be extended to a holomorphic motion $\Delta_\delta \times \hat{C} \mapsto \hat{C}$. Słodkowski [15] proved the strongest version asserting that $\delta$ is actually equal to 1:

$\lambda$-**lemma** [Słodkowski]. Let $A \subset \hat{C}$. Any holomorphic motion $f : \Delta \times A \to \hat{C}$ extends to a holomorphic motion $\Delta \times \hat{C} \mapsto \hat{C}$.

Astala and Martin [1] gave an exposition of Słodkowski’s proof from the point of view of 1-dimensional complex analysis. Chirka [8] gave an independent proof using solution to $\bar{\partial}$-equation. (See [12] for a detailed exposition of Chirka’s proof.) The purpose of this paper is to give a more geometric approach to the proof of the $\lambda$-lemma. We take Słodkowski’s approach, but replace the major technical part in Słodkowski’s proof by a transparent geometric argument and relate pseudoconvexity to the holomorphic motions.

The strongest $\lambda$-lemma fails when the dimension of the base manifold is greater than 1, even if the base is topologically contractible. This follows from the results of Earl-Kra [9] and Hubbard [11].

We give the necessary background on holomorphic motions, pseudoconvexity and Hilbert transform in Section 2. In Section 3 we show that the $\lambda$-lemma when $A$ is finite implies the $\lambda$-lemma for arbitrary $A$. We set up the notations and terminology in Section 4. We state
the filling theorem for the torus, and explain how it implies the finite $\lambda$-lemma in Section 5.
In Section 6 we prove Hölder estimates for disks trapped inside pseudoconvex domains and construct such trapping pseudoconvex domains for “graphical tori”. We use these estimates to prove the filling theorem in Section 7.

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2. **Background**

2.1. **Holomorphic motion.** Let $\Delta$ be a unit disk. Let $A \subset \mathbb{C}$. A *holomorphic motion* of $A$ is a map $f: \Delta \times A \to \mathbb{C}$ such that

1. for fixed $a \in A$, the map $\lambda \mapsto f(\lambda, a)$ is holomorphic in $\Delta$
2. for fixed $\lambda \in \Delta$, the map $a \mapsto f(\lambda, a) =: f_\lambda(a)$ is an injection and
3. the map $f_0$ is the identity on $A$.

2.2. **Pseudoconvexity.** Below we give definitions that are sufficient for our purposes.

A $C^2$ smooth function is (strictly) plurisubharmonic (written (strictly) psh) if its restriction to every complex line is strictly subharmonic. In coordinates $z = (z_1, \ldots, z_n)$, $u(z)$ is strictly psh if the matrix $\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)$ is positive definite.

A smoothly bounded domain $\Omega \subset \mathbb{C}^2$ is strictly pseudoconvex if there is a smooth, strictly psh function $\rho$ in a neighborhood of $\Omega$ such that $\{\Omega = \rho(z) < 0\}$.

**Lemma 2.1.** Let $\Omega_s \subset \mathbb{C}^2$ be a family of pseudoconvex domains with defining functions $\rho_s$, $s \in [0, 1]$. We assume that the family $\rho_s$ is continuous in $s$. Let $\phi_s: \Delta \to \mathbb{C}^2$ be a continuous family of holomorphic non-constant functions that extend continuously to $\Delta$. Set $D_s := \phi_s(\Delta)$. Suppose $\partial D_s \subset \partial \Omega_s$, $s \in [0, 1]$. Then $D_1 \subset \Omega_1$.

**Proof.** Consider the restriction of the functions $\rho_s$ to $D_s$. The functions $\rho_s \circ \phi_s : \Delta \to \mathbb{R}$ are subharmonic functions, $\rho_1 \circ \phi_1$ is the limit of $\rho_s \circ \phi_s$. By the hypothesis of the lemma, $\rho_s \circ \phi_s \leq 0$ on $\Delta$. Therefore, $\rho_1 \circ \phi_1 \leq 0$. If the maximum value 0 is attained in the interior point, $\rho_1 \circ \phi_1 \equiv 0$. It implies that $D_1 \subset \partial \Omega_1$, which is impossible. Therefore, $\rho_1 \circ \phi_1 < 0$ on $\Delta$, and $D_1 \subset \Omega_1$. $\square$

Let $M \subset \mathbb{C}^2$ be a real two-dimensional manifold. We say that $p \in M$ is a **totally real** point if $T_p M \cap iT_p M = \{0\}$. $M$ is a totally real manifold if all its points are totally real. If the manifold $M$ is totally real, it is in fact homeomorphic to the torus (see [6] and [10]). Assume $M \subset \partial \Omega$, then one can define a characteristic field of directions on $M$.

Let $p \in M$. Let $H_p \partial \Omega := T_p \Omega \cap iT_p \Omega$ be the holomorphic tangent space. $(\xi_p) := H_p \partial \Omega \cap T_p M$ is called the *characteristic direction*. We denote by $\chi(M, \Omega)$ the characteristic field of directions (see [7], Section 16.1).

2.3. **Hilbert transform.** A function $u: S^1 \to \mathbb{C}$ is Hölder continuous with exponent $\alpha$ if there is a constant $A$ such that for all $x, y \in S^1$:

$$|u(x) - u(y)| < A|x - y|^\alpha.$$  

We will consider the space $C^{1, \alpha}(S^1)$ of differentiable functions $u$ with $\alpha$- Hölder continuous derivative. The norm on the space $C^{1, \alpha}(S^1)$ is defined by the formula:
The existence of such holomorphic motion follows from the Finite set holomorphic motion of $z$. Using the diagonal method, we get a holomorphic motion that three points are dense in $\Delta$. By construction, it coincides with the holomorphic motion $u_h^n$. Reduction of the $\lambda$-lemma to the finite $\lambda$-lemma: We normalize the holomorphic motion $f$ so that three points $a_1, a_2, a_3$ stay fixed. We can assume $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$. Let $\{a_n\}$ be a sequence of points that are dense in $\hat{A}$. Let $\{z_n\}$ be a sequence of points that are dense in $\mathbb{C} \setminus \hat{A}$. Let $f_n$ be a holomorphic motion of $a_1, \ldots, a_n, z_1, \ldots, z_n$, such that $f_n(\lambda, a_i) = f(\lambda, a_i).$ The existence of such holomorphic motion follows from the Finite $\lambda$-lemma.

For any fixed $z_n$, for $k \geq n$, maps $f_k$ are defined at the point $z_n$, and functions $f_k(*, z_n) : \Delta \to \mathbb{C} \setminus \{0, 1\}$ form a normal family. So we can choose a convergent subsequence $f_k(*, z_n)$. Using the diagonal method, we get a holomorphic motion $f$, that is well defined for all $a_n$ and $z_n$ and coincides with $f$ on $a_i$ for all $i$. By the Extension $\lambda$-lemma, we extend it uniquely to the holomorphic motion of $\mathbb{C}$. By construction, it coincides with the holomorphic motion $f$ on the set $A$. 

4. Notations and Terminology

We will use the notation $\Delta \subset \mathbb{C}$ for the unit disk. We consider $\mathbb{C}^2$ with coordinates $(\lambda, w)$. The horizontal direction is parametrized by $\lambda$, the vertical by $w$. Throughout the paper we consider disks of the form

$$w = g(\lambda)$$

that will depend on two different parameters. We will use the following notations

$$g : \Delta \times S^1 \times [0, t_0] \to \mathbb{C}^2$$

$$g^\xi(\lambda) := g^\xi(\lambda, \xi) := g(\lambda, \xi, t) := g(\lambda, \xi, t).$$
4.1. **Graphical Torus.** Let $\pi : \mathbb{C}^2 \to \mathbb{C}$, $\pi(\lambda, w) = \lambda$, be the projection to the first coordinate. We say that a torus $\Gamma \subset \{\partial \Delta\} \times \mathbb{C}$ is a **graphical torus** if for each $\lambda \in \partial \Delta$, $C_\lambda := \pi^{-1}(\lambda) \in \mathbb{C}\{0\}$ is a simple closed curve that has winding number 1 around 0.

Thus, the vertical slices $\{C_\lambda : \lambda \in S^1\}$ give a foliation of $\Gamma$. We wish to construct a transverse foliation of $\Gamma$. We will consider holomorphic functions $g_\xi : \Delta \to \mathbb{C}$, which extend continuously to $\bar{\Delta}$ and such that $\gamma_\xi := g_\xi(\partial \Delta) \subset \Gamma$. We will construct a family of holomorphic disks such that $\{\gamma_\xi : \xi \in S^1\}$ form another foliation of the torus $\Gamma$ that is transverse to the original foliation.

4.2. **Family of Graphical Tori.** Let $\{C_\lambda^t : t > 0, \lambda \in \partial \Delta\}$ be smooth curves, such that

1. $C_\lambda^t$ have winding number 1 around 0;
2. for fixed $\lambda$, $C_\lambda^t$ form a smooth foliation of $\mathbb{C}\{0\}$;
3. there exists $\epsilon > 0$, so that $C_\lambda^t = \{|w|^2 = t\}$ for $t < \epsilon$.

Let

$$\Gamma^t = \{ (\lambda, w) : \lambda \in \partial \Delta, w \in C_\lambda^t \}.$$  

We set $\Gamma^0 = \{ (\lambda, 0) : \lambda \in \partial \Delta \}$. We refer to $\Gamma^t$, $t \geq 0$ as smooth family of graphical tori, though for $t = 0$ it degenerates to a circle $\Gamma^0$. The superscript $t$ will be applied to indicate the dependence on the torus $\Gamma^t$.

4.3. **Holomorphic Transverse Foliation of a Graphical Torus.** Let $\Gamma$ be a graphical torus. Let $g : \Delta \to \mathbb{C}$ be a holomorphic function that extends continuously to the closure
We say that the function \( g : \Delta \to \mathbb{C} \) defines a **holomorphic disk** \( D := \{ (\lambda, g(\lambda)) : \lambda \in \Delta \} \subset \mathbb{C}^2 \) with a **trace** \( \gamma := \partial D \).

We will construct foliations of graphical tori by traces of holomorphic disks. To do this, we will require additional properties:

We say that a function \( g : \Delta \times S^1 \to \mathbb{C} \) defines a **holomorphic transverse foliation** of a graphical torus \( \Gamma \) if

1. \( g \) is continuous.
2. For each \( \xi \in S^1 \), we let \( \{ \gamma_\xi := g(\lambda, \xi) : \lambda \in \partial \Delta \} \). The curves \( \gamma_\xi \) are simple, pairwise disjoint and define a foliation of \( \Gamma \).
3. Let \( g_\xi(\lambda) := g(\lambda, \xi) \), \( g_\xi : \Delta \to \mathbb{C} \) is holomorphic, \( g_\xi \in C^{1,\alpha}(\Delta) \)
4. \( g_\xi(\lambda) \neq 0 \), for all \( \xi \in S^1, \lambda \in \Delta \)
5. \( g_\xi(\lambda) \neq g_\eta(\lambda) \), for \( \lambda \in \Delta \) and distinct \( \xi, \eta \in S^1 \).

We will also consider holomorphic transverse foliations of a smooth family graphical tori \( \{ \Gamma_t \} \). This refers to a smooth family of foliations of graphical tori \( \Gamma_t \) with the additional assumption that the disks from \( \Gamma_{t_1} \) are disjoint from the disks from \( \Gamma_{t_2} \) if \( t_1 \neq t_2 \).

In fact the leaves in all of our foliations will be closed, and thus they are also fibrations by curves.

**5. Holomorphic transverse foliations and the Finite \( \lambda \)-lemma**

**Filling Theorem.** Let \( \Gamma \) be a graphical torus, then there exist a function \( g : \Delta \times S^1 \to \mathbb{C} \) that defines a holomorphic transverse foliation of \( \Gamma \). Moreover, the foliation is unique in the following strong sense: if there is a function \( h : \Delta \to \mathbb{C} \) that defines a holomorphic disk with trace in \( \Gamma \), and if \( h(\lambda) \neq 0 \) for \( \lambda \in \Delta \), then there exists \( \xi \in S^1 \) so that \( h = g_\xi \).

We need the following slightly stronger statement to deduce the Finite \( \lambda \)-lemma:

**Filling Theorem’.** Let \( \Gamma_t, t \in [0, \infty) \) be a family of graphical tori. There exists a function \( g : \Delta \times S^1 \times [0, \infty) \to \mathbb{C} \) that defines a holomorphic transverse foliation of the family \( \Gamma_t \). And the foliation is unique in the above mentioned strong sense.

The reduction of the Finite \( \lambda \)-lemma to Filling Theorem’ can be found in [15].

**Reduction of the Finite \( \lambda \)-lemma to Filling Theorem’.** Let \( f \) be a holomorphic motion of the points \( a_1, \ldots, a_n \). We need to extend the motion \( f \) to one more point \( a_{n+1} \). To achieve that we construct a holomorphic motion of all of \( \mathbb{C} \) and pick a leaf that passes through the point \( a_{n+1} \).

We normalize the motion so that \( a_1 = 0, f(\lambda, 0) = 0 \) for all \( \lambda \in \Delta \). Let \( \lambda = re^{i\theta} \). For each \( r \in [0, 1), e^{i\theta} \in S^1 \) the derivative \( \frac{\partial f}{\partial r}(\lambda, a_i) \) defines a vector in \( \mathbb{C} \). We can extend it to a smooth family of vector fields \( v_\theta(r) \) on \( \mathbb{C} \). By integrating the vector field for \( r \in [0, 1) \) and taking the union of solutions over \( \xi \in S^1 \), we get a smooth motion \( g : \Delta \to \mathbb{C} \) such that \( g(\lambda, a_i) = f(\lambda, a_i) \).

Let \( C_0^\lambda \) be a smooth family of simple curves that foliate \( \mathbb{C} \setminus \{ 0 \} \). We choose the foliation so that different \( a_i \) belong to different curves \( C_0^\lambda \). Take \( r < 1 \). Let \( S_r = \{ \lambda : |\lambda| = r \} \). Let \( C_\lambda = g(C_0^\lambda) \) for \( \lambda \in S_r \).

By Corollary [5] there exists a holomorphic motion with the prescribed traces \( \Gamma_t = \{ (\lambda, C_\lambda^t) : \lambda \in S_r \} \). By the uniqueness, it coincides with \( f \) on points \( a_1, \ldots, a_n \). By taking the limit as \( r \to 1 \), we obtain a holomorphic motion of \( \mathbb{C} \) that coincides with \( f \) on \( a_1, \ldots, a_n \). \( \square \)
6. TRAPPING HOL MORPHIC DISKS INSIDE PSEUDOCONVEX DOMAINS

6.1. Estimates for holomorphic disks trapped inside strictly pseudoconvex domains. The next theorem is from [4, 3]. We do not use the result of the theorem. We provide the proof to shed light on the technique we use and put the results in a general context.

Theorem 6.1. [4, 3] Let $\Omega$ be a strictly pseudoconvex domain, and let $M$ be a totally real manifold, $M \subset \partial \Omega$. Let $g : \Delta \to \Omega$ be a holomorphic function that extends as a $C^1$ smooth function to the closure $\overline{\Delta}$. Set $D = g(\Delta)$. Assume that $\gamma := \partial D \subset M$. Then there is a constant $\alpha = \alpha(M, \Omega)$, so that the angle $\angle(T_p \gamma, \xi_p) > \alpha$ is uniformly large, independently of $D$.

Lemma 6.1. Let $\Omega$ be a strictly pseudoconvex domain and let $M \subset \partial \Omega$ be a totally real two dimensional manifold. Let $g : \Delta \to \mathbb{C}^2$ be a holomorphic injection that extends to $\overline{\Delta}$ as a $C^1$ smooth function. Let $\gamma := \{g(\lambda) : \lambda \in \partial \Delta\}$. Assume that $\gamma \subset M$ and $g(\Delta) \subset \Omega$. Then for every point $p \in \gamma$, $T_p \gamma$ is transverse to the characteristic field of directions $\chi(M, \Omega)$.

Proof. Let $\rho$ be a strictly psh function such that $\Omega = \{\rho < 0\}$. The function $\rho \circ g : \Delta \to \mathbb{R}$ is subharmonic. Let $p \in \partial \Delta$. By the Hopf Lemma, the radial derivative $\frac{\partial (\rho \circ g)}{\partial r}(p) > 0$. Let $\xi_p$ be a vector that defines the characteristic direction in a point $p$. The normal vector to the disk $g(\Delta)$ in a point $p$ is $iT_p \gamma$. It does not belong to the tangent plane to $\partial \Omega$, so $iT_p \gamma$ is transverse to $i\xi_p$. Therefore, $T_p \gamma$ is transverse to $\xi_p$. \hfill $\Box$

Let $n_p$ be the unit outward normal vector to the hypersurface $\partial \Omega$. The vectors $(\xi_p, i\xi_p, n_p, in_p)$ form an orthonormal basis in $\mathbb{C}^2 \approx \mathbb{R}^4$ with respect to Euclidean inner product $(\cdot, \cdot)$. The vectors $in_p$ and $\xi_p$ form an orthonormal basis for $T_p M$. Given $\alpha$, we define a conical neighborhood of $\xi_p$:

$$K_\alpha = \{v \in T_p M : (v, \xi_p) > \alpha(v, in_p)\} \subset T_p M.$$ 

Lemma 6.2. Let $\Omega$ be a strictly pseudoconvex domain, and let $M \subset \partial \Omega$ be totally real. There exist $\alpha > 0$, and a continuous family of strongly pseudoconvex domains $\Omega_\epsilon$ such that $M \subset \partial \Omega_\epsilon$, and the characteristic fields of directions $\chi(M, \Omega_\epsilon)$ fill the cone-fields $K_\alpha$.

Proof. The manifold $M$ separates $\partial \Omega$ into two parts $(\partial \Omega)_1$, $(\partial \Omega)_2$. Let $h$ be a smooth function such that

1. $h|_M = 0$;
2. $h|_{(\partial \Omega)_1} > 0$, $h|_{(\partial \Omega)_2} < 0$;
3. $\frac{\partial h}{\partial (in_p)} > 0$, for each $p \in M$.

Let us denote by $\vec{n}$ the normal field to the hypersurfaces $\rho = \text{const}$. Since we can identify $T_p \mathbb{C}^2$ with $\mathbb{C}^2$, we can treat the normal vector field $n$ as a function defined in a neighborhood of $\partial \Omega$. We use the same letter $n$ for this function. Let $\rho_\epsilon(z) = \rho(z + \epsilon \vec{n})$, $\Omega_\epsilon = \{\rho_\epsilon < 0\}$. Then there exists $\delta$, so that for $|\epsilon| < \delta$, $\rho_\epsilon$ are plurisubharmonic. Therefore, $\Omega_\epsilon$ are strictly pseudoconvex, and characteristic fields of directions to $\Omega_\epsilon$ fill the cone field $K_\alpha$. \hfill $\Box$

Proof of Theorem 6.1. Let $D \subset \Omega$, $\partial D \subset \partial \Omega$. Then by Lemma 6.2 there exists a continuous family of strictly pseudoconvex domains $\Omega_\epsilon$, $|\epsilon| < \delta$ so that their characteristic fields of directions fill $C_\alpha$, for some $\alpha > 0$. By Lemma 2.1 $D \subset \Omega_\epsilon$ for $|\epsilon| < \delta$. Therefore, an angle estimate follows. \hfill $\Box$
6.2. Pseudoconvex domains for Graphical Tori. We wish to obtain the angle estimates for graphical tori. Let \( \eta_p \) be a vector that is tangent to the curve \( C_\lambda \) in a point \( p \). We want to think of \( \eta_p \) as a characteristic direction. However, a priori a graphical torus \( \Gamma \) doesn’t belong to a pseudoconvex domain. It belongs to a Levy flat domain \( \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \). Our strategy is to curve this Levy flat domain to obtain a family of pseudoconvex domains whose boundaries contain the torus \( \Gamma \) and so that characteristic directions span a wedge around \( \eta_p \).

**Theorem 6.2.** Let \( \Gamma \) be a graphical torus. Assume that \( g : \Delta \to \mathbb{C} \) defines a holomorphic disk \( D \) with the trace \( \gamma \subset \Gamma, g(\lambda) \neq 0 \). Then there exists a constant \( \alpha = \alpha(\Gamma) > 0 \) (independent of \( D \)) so that the angle \( \angle(\eta_p, T_p\gamma) \) is bounded below by \( \alpha \) independently of \( \Gamma \).

We need the following lemmas to prove Theorem 6.2.

Consider a family of the graphical tori \( \Gamma' \), \( \Gamma^1 = \Gamma \). Let \( F : S^1 \times \mathbb{C} \to \mathbb{R} \) be a defining function, \( F^{-1}(t) = \Gamma^t \). Let us extend \( F \) to a smooth function \( \overline{F} : \overline{\Delta} \times \mathbb{C} \to \mathbb{R} \), so that \( \overline{F}(\lambda, w) = |w|^2 \) for all \( \lambda \in \Delta, |w| \leq \epsilon \). We can also satisfy the condition \( F'_w \neq 0 \).

**Lemma 6.3.** There exists a function \( \phi : \overline{\Delta} \times \mathbb{C} \to \mathbb{R}_{\geq 0} \) so that \( \phi \) is smooth, \( \Delta_w \phi > 0 \), and restriction of \( \phi \) to \( S^1 \times \mathbb{C} \) defines a foliation of \( S^1 \times \mathbb{C} \) by \( \Gamma' \). We also require \( |\lambda| = 1 \phi^{-1}_\lambda(1) = C_\lambda \).

**Proof.** Let \( F(\lambda, w) \) be an extension defined earlier. Let \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) be an increasing convex function, \( \rho(0) = 0 \). Then \( \phi = \rho \circ F \) is also an extension of a defining function of the foliation as well.

\[
(1) \quad \Delta_w(\rho \circ F) = \frac{1}{4} \rho''|F_w|^2 + \frac{1}{4} \rho' \Delta_w F
\]

Since \( F_w(\lambda, w) \neq 0 \), when \( w \neq 0 \), so that \( \Delta_w(\rho \circ F) > 0 \) away from a neighborhood of \( w = 0 \). In a neighborhood of \( 0, \Delta_w F = 4 \). By taking \( \rho'(0) > 0 \), one can insure that \( \Delta(\rho \circ F) > 0 \).

Let us set \( \phi = \rho \circ F \).

Choose \( c \) so that \( \phi^{-1}_\lambda(1) = C_\lambda \). \( \square \)

**Lemma 6.4.** There exists a function \( \psi : \overline{\Delta} \times \mathbb{C} \to \mathbb{R} \cup \{-\infty\}, \) so that \( \psi \) is smooth, \( \Delta_w \psi < 0 \), and restriction of \( \psi \) to \( S^1 \times \mathbb{C} \) defines a foliation of \( S^1 \times \mathbb{C} \) by \( \Gamma' \). We require that \( \psi(\lambda, 0) = -\infty \) for all \( \lambda \in \overline{\Delta} \). We also require that for \( |\lambda| = 1, \psi^{-1}_\lambda(t) = C_\lambda \).

**Proof.** Consider a function \( \psi = c\rho \circ \ln F \), where \( \rho \) is increasing, concave function, \( \rho(-\infty) = -\infty \).

\[
\Delta_w(\rho \circ \ln F) = \frac{1}{4} \rho''\left|\frac{F_w}{F}\right|^2 + \frac{1}{4} \rho' \Delta_w(\ln F)
\]

Since \( F'_w \neq 0 \) when \( w \neq 0 \), we can make \( \Delta_w(\rho \circ \ln F) < 0 \). In a neighborhood of \( w = 0, \Delta_w(\ln F) = 0 \), therefore \( \Delta_w(\rho \circ \ln F) < 0 \). By choosing a constant \( c \), we can ensure that \( \psi^{-1}_\lambda(1) = C_\lambda \). \( \square \)

Let \( TT \) be the tangent space to the torus \( \Gamma \). Let \( K_\alpha \subset TT \) be the cone field:

\[
K_\alpha := \{(p, v) : v \in T_p T, (v, \eta_p) > \alpha(v, \frac{\partial}{\partial \theta})\}.
\]
\[ K_\alpha^\circ := \{(p, v) \in K_\alpha : v \neq c\eta_p, c \in \mathbb{R}\} \]

**Lemma 6.5.** For a graphical torus \( \Gamma \), there exist a family of pseudoconvex domains \( \Omega_\epsilon, \epsilon \in [-\delta, 0) \cup (0, \delta] \) and \( \alpha > 0 \), so that \( \Gamma \subset \partial \Omega_\epsilon \) and characteristic directions \( \chi(T, \Omega_\epsilon) \) fill \( K_\alpha^\circ \).

**Proof.** Take
\[ \omega_\epsilon := \frac{1}{\epsilon}(\frac{1}{|\lambda|^2} - 1) + \phi, \]
where \( \phi \) is a function constructed in Lemma 6.3.

\[ \text{Hess} \omega_\epsilon = \left( \frac{1}{\epsilon} + \frac{\partial^2 \phi}{\partial \lambda \partial \lambda} \frac{\partial^2 \phi}{\partial w \partial \lambda} \frac{\partial^2 \phi}{\partial w \partial \lambda} \Delta_w \phi \right) \]

For small enough \( \epsilon \), the Hessian is positive definite, so the function \( \omega_\epsilon \) is plurisubharmonic. The domains
\[ \Omega_\epsilon = \{(\lambda, w) : \omega_\epsilon(\lambda, w) < 1\}. \]
are strictly pseudoconvex for small \( \epsilon \).

Let \( D \) be a holomorphic disk with the trace in \( \Gamma \). The domains \( \Omega_\epsilon \) converge to \( |\lambda| < 1 \). Therefore, by Lemma 6.1, the disk \( D \) is trapped in \( \Omega_\epsilon \) for all small enough \( \epsilon \).

For small \( \epsilon \), the function
\[ \sigma_\epsilon(\lambda, w) := \frac{1}{\epsilon}(\frac{1}{|\lambda|^2} - 1) - \psi \]
is strictly plurisubharmonic. By the same reasoning, the disks are trapped in
\[ \Sigma_\epsilon = \{(\lambda, w) : \sigma_\epsilon < -1\} \]
when \( \epsilon \) is sufficiently small. \( \square \)

**Proof of Theorem 6.2.** By Lemma 6.1 the tangent \( T_{\gamma}\gamma \) is transverse to characteristic directions. Therefore, the angle estimate follows. \( \square \)

**Corollary 6.1.** Let \( g : \Delta \to \mathbb{C} \) define a holomorphic disk with the trace in \( \Gamma \), \( g(\lambda) \neq 0 \) for \( \lambda \in \Delta \). Assume that \( = g \in C^1(\bar{\Delta}) \). Then there exists \( C \) depending only on \( \Gamma \) such that \( |g'(\lambda)| < C \) for all \( \lambda \in \Delta \). The derivative estimate stays valid for graphical tori that are small perturbations of \( \Gamma \).

**Proof.** It is enough to estimate \( g'(\lambda) \) for \( |\lambda| = 1 \). Then \( \lambda = e^{i\theta}, \) so \( |g'_\theta| = |g'_\theta| \). Let \( u, v, \theta, r \) be an orthonormal system of coordinates in a neighborhood of \( \Gamma \). We assume that \( u|_{\Gamma} \) is a coordinate along \( C_\lambda \) and \( v, u \) are coordinates in \( \lambda = const \) plane. Then \( g'_\theta = u'_\theta \) and the angle estimate implies that \( |u'_\theta| \) is uniformly bounded from below. \( \square \)

7. **PROOF OF THE FILLING THEOREM**

The proof is by continuity method. At many points we follow the treatment of [11]. For each \( \lambda \in S^1 \) we can foliate interior of \( C_\lambda^t \setminus \{0\} \) by simple smooth curves \( C_\lambda^t \) so that
(1) \( C_\lambda^t = \{|z| = t\} \) for \( t \leq \epsilon \);
(2) \( C_\lambda^t \) depend smoothly on \( \lambda \).
Let

$$\Gamma^t = \{(\lambda, w): \lambda \in S^1, w \in C^t_\lambda\}$$

$$\Gamma^0 = \{(\lambda, 0): \lambda \in S^1\}$$

\(\Gamma^t\) by definition is a smooth family of graphical tori, \(\Gamma^1 = \Gamma\). For \(t \leq \epsilon\), the tori \(\Gamma^t\) are foliated by the vertical leaves \(w = \text{const}\). We will prove that the set \(\mathcal{S}\) of parameters \(t\) such that \(\Gamma^t\) is foliated is open and closed in \([0, 1]\), so \(\mathcal{S} = [0, 1]\), and the torus \(\Gamma\) is foliated. Moreover, we will prove that the foliation is unique in the strong sense.

Let \(F: S^1 \times \mathbb{C} \rightarrow \mathbb{R}\) be a defining function of the foliations \(C^t_\lambda\). For each fixed \(\lambda\),

\[C^t_\lambda = \{(\lambda, w): F(\lambda, w) = t\}\]

The function \(F\) depends smoothly on \(\lambda\). We assume that \(F^w_0(\lambda, w) \neq 0\) for \(w \neq 0, \lambda \in S^1\).

**Lemma 7.1.** Assume that the winding number of a curve \(\{\gamma(\lambda): \lambda \in S^1\}\) around \(0\) is equal to zero. Then the winding number of the curve \(\{F^w_0(\lambda, \gamma(\lambda)): \lambda \in S^1\}\) around \(0\) is equal to zero.

**Proof.** There is a homotopy of the curve \(\gamma, G: \gamma \times [0, 1] \rightarrow \mathbb{C} \setminus \{0\}\) so that \(G(\gamma \times \{0\}) = \gamma, G(\gamma \times \{1\}) = \text{const}\). The winding number of the curves \(\{F^w_0(\lambda, \gamma(t)): \lambda \in S^1\}\) around \(0\) is well defined, so it stays constant. Hence, it is equal to zero.

7.1. **Regularity.**

**Theorem 7.1.** Let \(\Gamma\) be a graphical torus. Let \(g: \bar{\Delta} \rightarrow \mathbb{C}\) be a function that defines a holomorphic disk with the trace \(g(\partial \Delta) \in \Gamma\). Assume \(g': L^\infty(\Delta), g \neq 0 \forall \lambda \in \Delta\). Then \(g' \in C^\alpha(\bar{\Delta})\).

**Proof.** We include \(\Gamma\) into a family of graphical tori \(\Gamma^t, \Gamma^1 = \Gamma\). Let \(F: S^1 \times \mathbb{C} \rightarrow \mathbb{R}\) be a defining function for \(\Gamma^t, F^{-1}(t) = \Gamma^t\).

(2)

\[F(\lambda, g(\lambda)) = 1.\]

Let \(\lambda = e^{i\theta}\). Since \(g' \in L^\infty(\Delta), g \in C^\alpha(\bar{\Delta})\) and \(g_\theta\) exist a.e.

Differentiating equation (2) with respect to \(\theta:\)

\[\lambda i F_\lambda(\lambda, g(\lambda)) - \text{Im}(F_w(\lambda, g(\lambda))g'(\lambda)\lambda) = 0\text{ a.e.}\]

Since the winding number of \(\{g(\lambda): \lambda \in S^1\}\) around \(0\) is zero. By Lemma 7.1 the winding number of \(\{F_w(\lambda, g(\lambda)): \lambda \in S^1\}\) around \(0\) is zero as well.

\[F_w(\lambda, g(\lambda)) = e^{\alpha(\lambda) + ib(\lambda)}\]

Since \(g(\lambda)\) is \(\alpha\)-Hölder continuous, \(b(\lambda)\) is Hölder continuous function, so is its Hilbert transform \(Hb(\lambda)\).

\[\text{Im}\left(\lambda e^{Hb(\lambda) - ib(\lambda)}g'(\lambda)\right) = e^{-a(\lambda)}F'_x(\lambda, g(\lambda))\lambda, \text{a.e.}\]

\[\text{Im}\lambda e^{Hb(\lambda) - ib(\lambda)}g'(\lambda)\text{ is a.e. equal to a Hölder continuous, so it is Hölder continuous. The function } \lambda e^{Hb(\lambda) - ib(\lambda)}g'(\lambda)\text{ is holomorphic so it is in } C^\alpha\text{ class. Therefore, } g' \in C^\alpha(\bar{\Delta}).\]
7.2. Openness. In [2], the stability of foliation by holomorphic disks is proved if one starts from the standard torus.

**Theorem 7.2.** Let $\Gamma^t$ be a family of graphical tori, $t \in [0, \infty)$. Assume that a function $g^{10} : \Delta \times \mathbb{S}^1 \rightarrow \mathbb{C}$ defines a holomorphic transverse foliation of a graphical torus $\Gamma^{10}$. Then there exists $\delta$ and a function $\tilde{g} : \Delta \times \mathbb{S}^1 \times (t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{C}$ that defines a transverse holomorphic foliation of $\Gamma^t$ for $|t - t_0| < \delta$.

**Proof.** Hilbert transform

$$H : C^{1,\alpha}(\mathbb{S}^1) \rightarrow C^{1,\alpha}(\mathbb{S}^1)$$

is a bounded linear operator. We change the standard normalization $Hu(0) = 0$ to $Hu(1) = 0$. We denote by $C^{1,\alpha}_{\mathbb{R}}(\mathbb{S}^1) \subset C^{1,\alpha}(\mathbb{S}^1)$ be the subspace of real-valued functions. The curve \( \{g^{10}_\xi(\lambda), \lambda \in \mathbb{S}^1\} \) has winding number 0 around zero, since $g^{10}_\xi(\lambda) \neq 0$ for $\lambda \in \Delta$. Therefore, by Lemma 7.1, the curve \( \{F_w(\lambda, g^{10}_\xi(\lambda)) : \lambda \in \mathbb{S}^1\} \) has winding number 0 around 0:

$$F_w(\lambda, g^{10}_\xi(\lambda)) = e^{\alpha_\xi (\lambda) + i b_\xi (\lambda)},$$

where $\alpha_\xi(\lambda), b_\xi(\lambda)$ are Hölder continuous with exponent $\alpha$. Thus, $H b_\xi(\lambda)$ is Hölder continuous as well.

$X_\xi(\lambda) := e^{H b_\xi(\lambda) - i b_\xi(\lambda)}$ is a holomorphic function on $\Delta$ and is proportional to the normal vector to $C^t_{\lambda}$ in points $(\lambda, g^{10}_\xi(\lambda))$.

Functions of the form $(u(\lambda) + i Hu(\lambda))X_\xi(\lambda)$ give all holomorphic functions that are Hölder continuous up to the boundary with the condition that $(u_\xi(1) + iHu_\xi(1))X_\xi(1)$ is proportional to the normal vector to $C^t_{\lambda}$ in a point $g^{10}_\xi(1)$. There exists an $\epsilon$ such that for each point $\eta \in C^t_{\lambda}$, $|t - t_0| < \epsilon$, there is only one normal vector that intersects $C^t_{\lambda}$ in a point $\eta$.

The space $C^{0}(\mathbb{S}^1, C^{1,\alpha}(\mathbb{S}^1))$ is a Banach space with the norm

$$||u|| = \sup_{\xi \in \mathbb{S}^1, \lambda \in \mathbb{S}^1} |u_\xi(\lambda)| + \sup_{\xi \in \mathbb{S}^1, \lambda \in \mathbb{S}^1} |u'_\xi(\lambda)| + \sup_{\xi \in \mathbb{S}^1, \lambda_1 \neq \lambda_2 \in \mathbb{S}^1} \frac{|u'_\xi(\lambda_1) - u'_\xi(\lambda_2)|}{|\lambda_1 - \lambda_2|^\eta}.$$

Consider an operator $F : \mathbb{R} \times C^{0}(\mathbb{S}^1, C^{1,\alpha}_{\mathbb{R}}(\mathbb{S}^1)) \rightarrow C^{0}(\mathbb{S}^1, C^{1,\alpha}_{\mathbb{R}}(\mathbb{S}^1))$:

$$F : u_\xi \rightarrow F(\lambda, g^{10}_\xi(\lambda) + (u_\xi + i Hu_\xi)X_\xi(\lambda)) - t$$

One can check that $F$ is a continuous operator on a Banach space.

$$F_{u_\xi} = F_w(\lambda, g^{10}_\xi(\lambda))X_\xi(\lambda)u_\xi = e^{\alpha_\xi(\lambda) - H b(\xi)} u_\xi(\lambda)$$

is an invertible linear operator, the unique preimage of 0 is $u^t_\xi$. And the function $g(\lambda, \xi, t) = g^{10}_\xi(\lambda) + u^t_\xi(\lambda)$ defines a holomorphic transverse foliation and is of class $C^{1,\alpha}$ on $\Delta$.

By continuity, for $\xi \neq \eta, g^{10}_\xi(\lambda) \neq g^{10}_\eta(\lambda)$ for $\lambda \in \Delta$.

This also gives us the openness for one disk.

**Theorem 7.3.** Let $\Gamma^t, t \in I$ be a family of graphical tori. Let $g^{10} : \Delta \rightarrow \mathbb{C}$ be a function that defines a holomorphic disk with the trace in the torus $\Gamma^{10}$. Assume that $g^{10} \in C^{1,\alpha}(\Delta)$, $g^{10}(\lambda) \neq 0$ for $\lambda \in \Delta$. Then there exists $\delta$ and a continuous function $g : \Delta \times (t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{C}$ such that $g^t(\lambda) := g(\lambda, t)$ defines a holomorphic disk with the trace in $\Gamma^t$ and $g^t \in C^{1,\alpha}(\Delta)$, $g^t(\lambda) \neq 0$ for $\lambda \in \Delta$. 
7.3. Closedness.

**Theorem 7.4.** Let $\Gamma^t$, $t \in [0, \infty)$ be a family of graphical tori. Suppose that there exists $g : \bar{\Delta} \times S^1 \times [0, t_0) \to \mathbb{C}$ that defines a holomorphic transverse foliation of $\Gamma^t$. Then $g$ can be extended to $g : \bar{\Delta} \times S^1 \times [0, t_0 \to \mathbb{C}$ that defines a holomorphic transverse foliation.

**Proof.** By Corollary 6.1 there exists $C$ that depends only on $\Gamma^{t_0}$ so that $| (g^{t_0}_g)' | < C$ for $t < t_0$ close to $t_0$. Since the space of bounded holomorphic functions on $\Delta$ is compact, we can pass to the limit. Let $g^{t_0}_g$ be the limits, $| (g^{t_0}_g)' | < C$. By Regularity Theorem 7.1, $g^{t_0}_g \in C^{1,\alpha}(\bar{\Delta})$. □

This also gives us closedness for a family of disks.

**Theorem 7.5.** Let $\Gamma^t$, $t \in [0, \infty)$ be a family of graphical tori. Assume that $g : \bar{\Delta} \times [0, t_0) \to \mathbb{C}$ is a continuous function such that $g^{t_0} : g^{t_0} \in C^{1,\alpha}(\bar{\Delta})$, $g^{t_0}(\lambda) \neq 0$ for $\lambda \in \Delta$. Then $g$ can be extended to a continuous $g : \bar{\Delta} \times [0, t_0] \to \mathbb{C}$ such that $g^{t_0}$ defines a holomorphic disk with the trace in $\Gamma^{t_0}$, $g^{t_0}(\lambda) \neq 0$ for $\lambda \in \Delta$.

7.4. Uniqueness. Let $\Gamma^c = \{(\lambda, w) : |w| = c, |\lambda| = 1\}$ be standard tori. Let $g : \Delta \to \mathbb{C}$ be a function that defines a holomorphic disk with the trace of $g(\partial \Delta) \in \Gamma^c$. By the Minimum Modulus Theorem, $\min g$ is attained on the boundary. Maximum modulus is attained on the boundary as well. So $|g(\lambda)| = \text{const.}$ Therefore, $g(\lambda) = \text{const.}$

**Theorem 7.6.** Let $\Gamma$ be a graphical torus. Let $g, h : \bar{\Delta} \to \mathbb{C}$ be functions that define holomorphic disks with traces in $\Gamma$, $g(\lambda) \neq 0$, $h(\lambda) \neq 0$ for $\lambda \in \Delta$. Assume that $g(1) = h(1)$. Then there exists $\epsilon$ such that $|g(\lambda) - h(\lambda)| < \epsilon$ for $\lambda \in S^1$ implies $g(\lambda) \equiv h(\lambda)$.

Note that the same $\epsilon$ works for tori close to $\Gamma$.

**Proof.** Let $a(\lambda, s) = F(\lambda, g(\lambda) + s(h(\lambda) - g(\lambda)))$, $a(\lambda, 0) = a(\lambda, 1) = t$. Then

$$\int_0^1 a_x ds = 0 = \text{Re}(h(\lambda) - g(\lambda)) \int_0^1 F_w(g(\lambda) + s(h(\lambda) - g(\lambda)))ds.$$  \hspace{1cm} (3)

The winding number of the curve $\{g(\lambda) : \lambda \in S^1\}$ around 0 is equal to zero. Hence, by Lemma 7.1, the winding number of

$$\{F_w(\lambda, g(\lambda)) : \lambda \in S^1\}$$

around 0 is equal to zero.

Therefore, for small enough $\epsilon$, the winding number of the curve

$$\left\{ \int_0^1 F_w(g(\lambda) + s(h(\lambda) - g(\lambda))) ds : \lambda \in S^1 \right\}$$

around 0 is equal to zero, so

$$\int_0^1 F_w(g(\lambda) + s(h(\lambda) - g(\lambda))) ds = e^{a(\lambda) + ib(\lambda)}.$$  

The function $b(\lambda)$ is a bounded Hölder continuous function.

By equation (3), $\arg (g(\lambda) - h(\lambda)) = \frac{\pi}{2} - b(\lambda)$, where $b(\lambda)$ is a bounded function. It contradicts the fact that $g(1) - h(1) = 0$. □
7.5. Global Uniqueness.

**Theorem 7.7.** Let $\Gamma$ be a graphical torus. Let $g^1, h^1 : \Delta \to \mathbb{C}$ be functions that define holomorphic disks with traces in $\Gamma$, $g^1, h^1 \in C^{1,\alpha}(\Delta)$. Assume that $g^1(1) = h^1(1)$. Then $g^1(\lambda) = h^1(\lambda)$.

**Proof.** We include torus $\Gamma$ into a family of graphical tori $\Gamma^t$, $t \in [0, 1]$, $\Gamma^1 = \Gamma$. By Theorems 7.3, 7.5, there exist functions $g, h : \Delta \times [0, 1] \to \mathbb{C}$ such that $g(\lambda, 1) = g^1$, $h(\lambda, 1) = h^1$ and $g^t(\lambda) := g(\lambda, t)$ define holomorphic disks with the traces in tori $\Gamma^t$. There exists $\epsilon$ such that for $t < \epsilon$, $\Gamma^t = \{(\lambda, w) : |w| = t, \lambda \in S^1\}, t \in [0, 1]$ are standard tori with uniqueness of solutions. For $t < \epsilon$, $g^t \equiv h^t$. Let $t_0 = \sup\{t : h^t \equiv g^t\}$. If $t_0 \neq 1$, then by applying Theorem 7.6, we get a contradiction. □

At this point we have proved the Filling theorem. For Filling Theorem’ the only statement remains is to show that disks for $\Gamma^t$ are disjoint from disks for $\Gamma^{t_2}$ when $t_1 \neq t_2$. Suppose $D^t_{\xi_1}$ is a disk with boundary in $\Gamma^t$. If $D^t_{\xi_1} \cap D^{t_2} \neq \emptyset$, then since the traces are in $\Gamma^t$, $\Gamma^{t_2}$ we will have $D^t_{\xi_1} \cap D^{t_2}_{\xi_2} = \emptyset$ for all $\xi_1, \xi_2 \in S^1$. By Filling Theorem, the disks $D^t_{\xi_1}, t \in [t_1, t_2]$, which is a contradiction.

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