Isolated singularities in the heat equation behaving like fractional Brownian motions

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Abstract

We consider solutions of the linear heat equation in $\mathbb{R}^N$ with isolated singularities. It is assumed that the position of a singular point depends on time and is Hölder continuous with the exponent $\alpha \in (0, 1)$. We show that any isolated singularity is removable if it is weaker than a certain order depending on $\alpha$. We also show the optimality of the removability condition by showing the existence of a solution with a nonremovable singularity. These results are applied to the case where the singular point behaves like a fractional Brownian motion with the Hurst exponent $H \in (0, 1/2]$. It turns out that $H = 1/N$ is critical.

Key words: removability, isolated singularity, heat equation, fractional Brownian motion.

Abbreviated title: Singularities in the heat equation

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1 Introduction

In the field of partial differential equations, it is an important and interesting problem to study singularities of solutions. For instance, let us consider the Laplace equation $\Delta u = 0$ in $\Omega \setminus \{\xi_0\}$, where $\Omega$ is a domain in $\mathbb{R}^N$ and $\xi_0 \in \Omega$. We say that a singularity of $u$ at the point $x = \xi_0$ is removable if there exists a classical solution $\tilde{u}$ of the Laplace equation in $\Omega$ such that $\tilde{u} \equiv u$ in $\Omega \setminus \{\xi_0\}$. It is well known that the singularity of a solution $u$ at $x = \xi_0$ is removable if $u(x) = o(|x|^{2-N})$ for $N \geq 3$ and $u(x) = o(\log(1/|x|))$ for $N = 2$ as $x \to \xi_0$. This condition is optimal, because the fundamental solution of the Laplace equation is given by

$$\Psi(x) := \begin{cases} C_N|x|^{2-N} & \text{if } N \geq 3, \\ C_2 \log(1/|x|) & \text{if } N = 2, \end{cases}$$

where $C_N > 0$ denotes a constant depending on $N$. Similar results have been obtained for nonlinear elliptic equations as well, see, e.g., [1, 4, 13, 16, 17] and references cited therein. On the other hand, for the heat equation $u_t = \Delta u$ in $(\mathbb{R}^N \setminus \{\xi_0\}) \times (0, T)$ with $N \geq 3$ and $T > 0$, Hsu [7] and Hui [8] proved that the singular point $\xi_0$ is removable if and only if $|u(x, t)| = o(\Psi(x - \xi_0))$ as $x \to \xi_0$ locally uniformly on some time interval. This result is optimal since $\Psi$ is a stationary solution of the heat equation.

In this paper, we study the heat equation in the case where the position of a singular point depends on time, which is formulated as

$$u_t = \Delta u, \quad (x, t) \in D,$$

where

$$D := \{(x, t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\}, \ t \in (0, T)\}.$$ 

Throughout of this paper, we assume that $N \geq 3$ and the singular point $\xi : \mathbb{R} \to \mathbb{R}^N$ is continuous. Our interest for (1.1) is in the removability of a singularity and the existence of a positive singular solution, especially when the singular point behaves like a fractional Brownian motion with the Hurst exponent $H \in (0, 1/2]$. See [3, 11, 12] for related results concerning the fractional Brownian motion with the Hurst exponent $H \in (0, 1/N]$.

For a solution $u$ of (1.1), the singularity at $x = \xi(t)$ is said to be removable if there exists a function $\tilde{u}$ which satisfies the heat equation in $\mathbb{R}^N \times (0, T)$ in the classical sense and $\tilde{u} \equiv u$ on $D$. Takahashi-Yanagida [14] studied the case where $\xi(t)$ is locally $1/2$-Hölder continuous in $t \in \mathbb{R}$ and showed that if a solution $u$ of (1.1) satisfies $u(x, t) = o(\Psi(x - \xi(t)))$ as $x \to \xi(t)$ locally uniformly in $t \in (0, T)$, then the singularity of $u$ at $x = \xi(t)$ is removable. They also showed the existence of solutions with nonremovable time-dependent singularities that are asymptotically radially symmetric (see [6, 9, 15] for related results about time-dependent singularities in nonlinear parabolic equations).
Kan-Takahashi [10] studied the case where the limit
\[
\lim_{t \uparrow T} \frac{\xi(T) - \xi(t)}{(T - t)^\alpha}
\]
exists for some \( \alpha \in (0, 1/2) \), and showed the existence of a solution whose limiting profile loses the asymptotic radial symmetry.

Our first objective is to extend the removability condition to the case where \( \xi(t) \) is \( \alpha \)-Hölder continuous with \( \alpha \in (0, 1) \). Taking the result of [10] into account, we state our result by using the integral of \( u(x, t) \) on a ball
\[
B^r(t) := \{ x \in \mathbb{R}^N : |x - \xi(t)| \leq r \}.
\]

**Theorem 1.1** (Removability). Let \( N \geq 3 \) and \( \alpha \in (1/N, 1) \). Suppose that \( \xi(t) \) is locally \( \alpha \)-Hölder continuous in \( t \in (0, T) \). If a positive solution \( u \) of (1.1) satisfies
\[
\int_{B^r(t)} u(x, t) dx = \begin{cases} o(r^2) & \text{for } 1/2 \leq \alpha < 1, \\ o(r^{1/\alpha}) & \text{for } 1/N < \alpha < 1/2 \end{cases}
\]
as \( r \to 0 \) locally uniformly in \( t \in (0, T) \), then the singularity of \( u \) at \( x = \xi(t) \) is removable.

If a solution \( u \) is positive and bounded in a neighborhood of \( \xi(t) \), then the solution satisfies
\[
c_1 r^N \leq \int_{B^r(t)} u(x, t) dx \leq c_2 r^N, \quad r \in (0, 1)
\]
with some constant \( c_1, c_2 > 0 \). This implies that for \( 0 < \alpha \leq 1/N \), we cannot judge the removability from the integral on \( B^r(t) \). In fact, we will see that for every \( 0 < \alpha \leq 1/N \), there exists a solution with a singularity at \( \xi(t) \) that is bounded at \( \xi(t) \). Therefore, to study the removability, we need to examine more precise profile of the solution around the singular point.

Our idea of the proof of Theorem 1.1 is to extend the method in [14] that is based on a construction of a suitable cut-off function. In order to construct the desired function, the Hölder exponent \( \alpha = 1/2 \) is critical in some sense. Using the cut-off function, we can show under the condition in Theorem 1.1 that the solution satisfies (1.1) in a weak sense. Then the parabolic regularity implies that the solution satisfies the heat equation on \( \mathbb{R}^N \) in the classical sense.

Next, we show that the condition in Theorem 1.1 is optimal. To show the optimality, we consider the initial value problem
\[
\begin{cases}
u_t - \Delta v = \delta(x - \xi(t)), & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\]
(1.2)
where $\delta(\cdot)$ denotes a Dirac measure concentrated at the origin $0 \in \mathbb{R}^N$, and the initial value $u_0(x)$ is assumed to be continuous, positive and bounded on $\mathbb{R}^N$. By solving this problem, we show the existence of a solution of (1.2) with a nonremovable singularity at $\xi(t)$. In the case of $\alpha > 1/2$, it was shown in [14] that if $\xi(t)$ is $\alpha$-Hölder continuous, (1.2) has a solution satisfying

$$u(x, t) = C_N| x - \xi(t) |^{-N+2} + o(| x - \xi(t) |^{-N+2}) \quad (x \to \xi(t)). \quad (1.3)$$

In this paper, we are particularly interested in the case where $\xi(t)$ is Hölder continuous with the exponent $\alpha \leq 1/2$. For example, any sample path of the fractional Brownian motion with the Hurst exponent $H \in (0, 1/2]$ is $(H - \epsilon)$-Hölder continuous almost everywhere in $t$, where $\epsilon > 0$ is an arbitrarily small constant. We note that $H = 1/2$ corresponds to the ordinary Brownian motion. Therefore, it becomes an interesting question to ask what happens to the solution of (1.2) depending on $H$.

In order to investigate the behavior of solutions to (1.2) with such $\xi(t)$, we first consider the case where the singularity $\xi(t)$ satisfies more general conditions. Let $s_0 \in (0, T]$ be arbitrarily fixed. For small $r > 0$, we define a function

$$\sigma(r) := \inf \{ s \in [0, s_0] : |\xi(T-s) - \xi(T)| > r \}.\quad (1.4)$$

(We set $\sigma(r) = s_0$ if $|\xi(T-s) - \xi(T)| \leq r$ for all $s \in [0, s_0]$.) We also define a set

$$\mathcal{T}(r) := \{ s \in [0, s_0] : |\xi(T-s) - \xi(T)| \leq r \} \quad (1.4)$$

and a function

$$\tau(r) := \mu(\mathcal{T}(r)), \quad (1.5)$$

where $\mu(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}$, that is

$$\tau(r) = \int_0^{s_0} 1_{\{ |\xi(T-s) - \xi(T)| \leq r \}} ds.$$ 

Clearly, $\sigma(r) \leq \tau(r)$ for all $r > 0$. In the field of probability theory, the functions $\sigma(r)$ and $\tau(r)$ are called the first exit time and the occupation time, respectively.

The following result gives a lower bound of solutions of (1.2).

**Theorem 1.2.** Let $N \geq 3$. Assume that $\sigma(r) = O(r^2)$ as $r \to 0$. Then for any $\theta \in (0, 1)$, there exist constants $C > 0$ and $R > 0$ such that the solution of (1.2) satisfies

$$\int_{B^r(T)} u(x, T)dx \geq C \{ \sigma(\theta r) + r^N \}$$

for all $r \in (0, R)$.
We note that if \( \xi(t) \) moves not less slowly than \( (T-t)^{1/2} \) as \( t \uparrow T \), then the assumption on \( \sigma \) in Theorem 1.2 is satisfied. More precisely, \( \sigma(r) = O(r^2) \) as \( r \to 0 \) if and only if

\[
\liminf_{s \downarrow 0} s^{-1/2} \sup_{T-s < t < T} |\xi(T) - \xi(t)| \in (0, \infty].
\]

Next, we give a pointwise upper bound of solutions of (1.2).

**Theorem 1.3.** Let \( N \geq 3 \). Then there exist constants \( C > 0 \) and \( R > 0 \) such that the solution \( u \) of (1.2) satisfies

\[
\int_{B^r(T)} u(x, T) dx \leq C \left[ \min \left\{ r^N \int_r^\infty \tau(l) l^{-N-1} dl, r^2 \right\} + r^N \right]
\]

for all \( r \in (0, R] \).

We note that if \( r^{-\kappa} \tau(r) \) is nonincreasing in \( r \) with some constant \( \kappa < N \), then the condition on \( \tau \) in Theorem 1.3 is simplified as

\[
r^N \int_r^\infty \tau(l) l^{-N-1} dl = r^N \int_r^\infty l^{-\kappa} \tau(l) l^{-N-1+\kappa} dl \leq r^N r^{-\kappa} \tau(r) \int_r^\infty l^{-N-1+\kappa} dl \leq C \tau(r).
\]

We apply Theorems 1.2 and 1.3 to some special cases. We first consider the simple case where \( \xi(t) \) is \( \alpha \)--Hölder continuous at \( t = T \) with \( 0 < \alpha \leq 1/2 \).

**Corollary 1.4.** Let \( N \geq 3 \). Suppose that \( \xi \) satisfies

\[
c_1(T-t)^\alpha \leq |\xi(T) - \xi(t)| \leq c_2(T-t)^\alpha, \quad t \in [t_0, T],
\]

with some \( \alpha \in (0, 1/2] \), \( c_1, c_2 > 0 \) and \( t_0 \in [0, T) \). Then there exist constants \( C_1, C_2 > 0 \) and \( R > 0 \) such that the solution \( u \) of (1.2) satisfies the following inequalities for \( r \in (0, R] \):

(i) If \( 1/N < \alpha \leq 1/2 \), then

\[
C_1 r^{1/\alpha} \leq \int_{B^r(T)} u(x, T) dx \leq C_2 r^{1/\alpha}.
\]

(ii) If \( 0 < \alpha \leq 1/N \), then

\[
C_1 r^N \leq \int_{B^r(T)} u(x, T) dx \leq C_2 r^N.
\]

When \( \xi(\cdot) \) is a sample path of the fractional Brownian motion, then we can apply Theorems 1.2 and 1.3 to obtain the following result.
Theorem 1.5 (Fractional Brownian motion). Let $N \geq 3$. Suppose that $\xi(\cdot)$ is a sample path of the fractional Brownian motion with the Hurst exponent $H$. Then for every $t \in (0, T]$ and $R > 0$, there exist $C_1(\omega), C_2(\omega) > 0$ such that the solution $u$ of (1.2) satisfies the following inequalities for $r \in (0, R]$ with probability one:

(i) If $1/N < H \leq 1/2$, then

$$C_1 r^{1/H} \left\{ \log(1/r) \right\}^{1/(2H) - \delta} \leq \int_{B_r(T)} u(x, t) dx \leq C_2 r^{1/H} \left\{ \log(1/r) \right\}^{1+\delta}.$$ 

(ii) If $H = 1/N$, then

$$C_1 r^N \leq \int_{B_r(T)} u(x, t) dx \leq C_2 r^N (\log 1/r)^2 \left\{ \log(1/r) \right\}^{1+\delta}.$$ 

(iii) If $0 < H < 1/N$, then

$$C_1 r^N \leq \int_{B_r(T)} u(x, t) dx \leq C_2 r^N (\log 1/r) \left\{ \log(1/r) \right\}^{1+\delta}.$$

This paper is organized as follows. In Section 2 we discuss the removability of singularities. In Sections 3 we consider (1.2) and derive a lower bound and an upper bound of the solutions by using the functions $\sigma(r)$ and $\tau(r)$. In Section 4 we study the case where $\xi(t)$ is a sample path of the fractional Brownian motion.

2 Removability of singularities

In this section, we give a proof of Theorem 1.1 by using a suitable cut-off function. The following lemma was proved in [14].

Lemma 2.1. Let $N \geq 1$, $t_1, t_2 \in \mathbb{R}$ ($t_1 < t_2$) and $\alpha \in (0, 1]$. Suppose that $\xi(t)$ is $\alpha$-Hölder continuous in $t \in [t_1, t_2]$ for some $\alpha \in (0, 1]$. Then there exist $\delta_0 = \delta_0(\alpha, N, t_1, t_2) \in (0, 1)$ and $C = C(\alpha, N, t_1, t_2) > 0$ independent of $x, t$ with the following property: For every $\delta \in (0, \delta_0)$, there exists a cut-off function $\eta \in C^\infty(\mathbb{R}^N \times \mathbb{R})$ such that

$$0 \leq \eta(x, t) \leq 1,$$

$$\eta(x, t) = \begin{cases} 1 & \text{if } |x - \xi(t)| \geq \delta, \\ 0 & \text{if } |x - \xi(t)| \leq \delta/2, \end{cases}$$

$$|
abla \eta| \leq C \delta^{-1}, \quad |\Delta \eta| \leq C \delta^{-2}, \quad |(\eta)_t| \leq C \delta^{-1/\alpha}$$

for $(x, t) \in \mathbb{R}^N \times [t_1, t_2]$.

Now we prove Theorem 1.1.
Proof of Theorem 1.1. For $0 < t_1 < t_2 < T$, let $\varphi \in C^\infty_0(\mathbb{R}^N \times (0,T))$ be a test function with a compact support in $\mathbb{R}^N \times (t_1,t_2)$. Then the Weyl lemma for the heat equation (see, e.g., [4, Section 6]) implies that if $u$ satisfies

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} u(\varphi_t + \Delta \varphi) \, dx \, dt = 0 \quad (2.1)$$

for any $\varphi$, then it turns out that $u$ actually belongs to $C^{2,1}(\mathbb{R}^N \times (0,T))$. Hence, it suffices to prove (2.1) for the removability of the singularity of $u$ at $x = \xi(t)$.

Suppose that $u$ satisfies for some $k > 0$

$$\int_{B^r(t)} u(x,t) \, dx = o(r^k) \quad \text{as } r \to 0$$

locally uniformly in $t \in (0,T)$.

Let $\varepsilon \in (0,1)$ be arbitrarily given. By assumption on $u$, we can take $\delta_1 = \delta_1(t_1,t_2,\varepsilon) \in (0,1)$ such that

$$\int_{B^r(t)} u(x,t) \, dx \leq \varepsilon r^k, \quad 0 < r < \delta_1, \quad t \in [t_1,t_2] \subset (0,T).$$

Let $\delta \in (0,\min\{\delta_0,\delta_1\})$, where $\delta_0$ is defined in Lemma 2.1. For this $\delta$, we take a cut off function $\eta \in C^\infty_0(\mathbb{R}^N)$ constructed in Lemma 2.1. Multiplying (1.1) by $\eta \varphi$ and integrating it by parts, we obtain

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} u\{(\varphi \eta)_t + \Delta (\varphi \eta)\} \, dx \, dt = 0. \quad (2.2)$$

By simple calculations, we have

$$\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u\{(\varphi \eta)_t - \varphi_t\} \, dx \, dt \right| \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u|(\varphi \eta)_t - \varphi_t \, dx \, dt \leq C_1(1 + \delta^{-1/\alpha}) \int_{t_1}^{t_2} \int_{B^\delta(t)} u(x,t) \, dx \, dt \leq C_2 \varepsilon(1 + \delta^{-1/\alpha}) \delta^k$$

and

$$\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u\{\Delta (\varphi \eta) - \Delta \varphi\} \, dx \, dt \right| \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u|\Delta (\varphi \eta) - \Delta \varphi \, dx \, dt \leq C_3(1 + \delta^{-1} + \delta^{-2}) \int_{t_1}^{t_2} \int_{B^\delta(t)} u(x,t) \, dx \, dt \leq C_4 \varepsilon(1 + \delta^{-1} + \delta^{-2}) \delta^k,$$
where \( C_1, C_2, C_3, C_4 > 0 \) are constants independent of \( \varepsilon \). Hence if \( k = \max\{2, 1/\alpha\} \), we obtain

\[
\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u(\varphi_t + \Delta \varphi) \, dx \, dt \right| \leq C_5 \varepsilon,
\]

where \( C_5 > 0 \) is a constant independent of \( \varepsilon \) and \( \delta \in (0, \min\{\delta_0, \delta_1\}) \). Since \( \varepsilon > 0 \) is arbitrary, this implies (2.1). This completes the proof.

3 Nonremovable singularities

In this section, we consider the case where the singularity at \( \xi(t) \) is not removable. Let us consider the initial value problem (1.2) with \( u(x, 0) = u_0(x) \), where \( u_0(x) \) is continuous positive and bounded on \( \mathbb{R}^N \). By using the heat kernel

\[
G(x, y, t) := \frac{1}{(4\pi t)^{N/2}} \exp \left( -\frac{|x - y|^2}{4t} \right),
\]

the solution of (1.2) is expressed as

\[
u(x, t) = \int_{\mathbb{R}^N} G(x, y, t)u_0(y) \, dy + \int_0^t G(x, \xi(s), t - s)ds
\]

(see [10, 14]). Since the first term of the right hand side is smooth, positive and bounded in \( t > 0 \), we see that for \( R > 0 \), there exist constants \( C_1 > 0 \) and \( C_2 > 0 \) depending on \( R \) and \( u_0 \) such that

\[
C_1 r^N \leq \int_{B_r(T)} \int_{\mathbb{R}^N} G(x, y, T)u_0(y) \, dy \, dx \leq C_2 r^N \quad \text{for } r \in (0, R).
\]

Hence, it suffices to examine the second term

\[
F(x, T) := \int_0^T G(x, \xi(t), T - t) \, dt
\]

\[
= \int_0^T \frac{1}{(4\pi(T - t))^{N/2}} \exp \left( -\frac{|x - \xi(t)|^2}{4(T - t)} \right) \, dt.
\]

Here, by exchanging the order of integration and changing the variables \( y = x - \xi(T) \) and \( \eta(s) = \xi(T - s) - \xi(T) \), we have

\[
\int_{B_r(T)} F(x, T) \, dx = \int_{|x - \xi(T)| \leq r} \int_0^T \frac{1}{(4\pi(T - t))^{N/2}} \exp \left( -\frac{|x - \xi(t)|^2}{4(T - t)} \right) \, dt \, dx
\]

\[
= \int_0^T \frac{1}{(4\pi s)^{N/2}} \int_{B_r(s)} \exp \left( -\frac{|y - \eta(s)|^2}{4s} \right) \, dy \, ds,
\]
where $B^r_0$ is defined as
\[ B^r_0 := \{ y : |y| \leq r \} . \]

In the following, $C$ stands for generic constants whose value may change from line to line but does not depend on other variables.

Proof of Theorem 1.2. Let $0 < \theta < 1$ be arbitrarily given. By assumption, $\sigma$ satisfies $\sigma(\theta r) \leq Ar^2$ for all $r \in (0, R)$ with some constant $A > 0$ and $R > 0$. For $s \leq \sigma(\theta r) \leq Ar^2$ and a constant $\delta > 0$, we have
\[ \{ y : |y - \eta(s)| \leq \delta s^{1/2} \} \subset \{ y : |y| \leq \theta r + \delta s^{1/2} \} \subset \{ y : |y| \leq (\theta + \delta A^{1/2})r \} . \]

Hence, if we take $\delta > 0$ so small that $\theta + \delta A^{1/2} \leq 1$, then
\[ B^r_0 \supset \{ y : |y - \eta(s)| \leq \delta s^{1/2} \} . \]

Now, for $s \leq \sigma(\theta r)$, we have
\[
\int_{B^r_0} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy \geq \int_{|y - \eta(s)| \leq \delta s^{1/2}} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy \\
\geq C \int_{|y - \eta(s)| \leq \delta s^{1/2}} dy \geq C s^{N/2} .
\]

Hence we obtain
\[
\int_{B^r(T)} F(x, T) dx \geq \int_0^{\sigma(\theta r)} \frac{1}{(4\pi s)^{N/2}} \int_{B^r_0} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy ds \\
\geq \int_0^{\sigma(\theta r)} \frac{1}{(4\pi s)^{N/2}} \times C s^{N/2} ds \\
\geq C \sigma(\theta r) .
\]

This completes the proof.

Before giving a proof of Theorem 1.3 we give an upper bound of positive solutions.

Lemma 3.1. For any solution of (1.2), there exists a constant $C > 0$ such that
\[
\int_{B^r(T)} u(x, T) dx \leq Cr^2
\]
for $r > 0$. 

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Proof. Set \( B^r_1(s) := \{ y \in \mathbb{R}^N : |y - \eta(s)| \leq r \} \). We then have
\[
\int_{B^r_0} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy = \int_{B^r_0 \cap B^r_1(s)} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy + \int_{B^r_0 \setminus B^r_1(s)} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy \\
\leq \int_{B^r_0 \cap B^r_1(s)} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy + \int_{B^r_1(s) \setminus B^r_0} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy \\
= \int_{B^r_1(s)} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy \\
= \int_{B^r_0} \exp \left( - \frac{|y|^2}{4s} \right) dy.
\]

Hence, there holds
\[
\int_{B^r(T)} F(x, T) dx = \int_0^T \frac{1}{(4\pi s)^N/2} \int_{B^r_0} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy ds \\
\leq \int_0^T \int_{B^r_0} \frac{1}{(4\pi s)^N/2} \exp \left( - \frac{|y|^2}{4s} \right) dy ds \\
= C \int_0^T \int_{0}^{r} \frac{1}{(4\pi s)^N/2} \exp \left( - \frac{\rho^2}{4s} \right) \rho^{N-1} d\rho ds.
\]

Evaluating the last integral, we obtain
\[
\int_{B^r(T)} F(x, T) dx \leq Cr^2.
\]

This completes the proof.

Proof of Theorem 1.3. We choose \( R > 0 \) so small that \([0, s_0] \setminus T(2R)\) is not empty and let \( r \in (0, R) \). If \(|\eta(s)| \geq 2r\) and \(|y| \leq r\), then
\[
|y - \eta(s)| \geq |\eta(s)| - |y| \geq \frac{1}{2} |\eta(s)|.
\]

Hence we divide the integration into three parts to obtain
\[
\int_{B^r_0} F(x, T) dx \leq \int_0^T \int_{B^r_0} \frac{1}{(4\pi s)^N/2} dy ds \\
+ \int_{s \in [0, s_0] \setminus T(2r)} \int_{B^r_0} \frac{1}{(4\pi s)^N/2} \exp \left( - \frac{|\eta(s)|^2}{16s} \right) dy ds \\
+ \int_{s \in T(2r)} \int_{B^r_0} \frac{1}{(4\pi s)^N/2} \exp \left( - \frac{|y - \eta(s)|^2}{4s} \right) dy ds \\
=: I_1(r) + I_2(r) + I_3(r).
\]
It is easy to see that $I_1(r)$ is bounded by $Cr^N$. Using the integration by parts for the Lebesgue-Stieltjes measure $d\tau(l)$, we have

$$I_2(r) \leq \int_{s \in [0, s_0] \setminus T(2r)} \int_{B_r} |\eta(s)|^{-N}dyds$$

$$\leq Cr^N \int_{|\eta(s)| > 2r} |\eta(s)|^{-N}ds$$

$$\leq Cr^N \int_{2r}^{\infty} l^{-N}d\tau(l)$$

$$= -Cr^N (2r)^{-N} \tau(2r) + Cr^N \int_{2r}^{\infty} Nl^{-N-1}\tau(l)dl$$

$$\leq Cr^N \int_{r}^{\infty} l^{-N-1}\tau(l)dl.$$

Finally, since

$$\int_{B_r(T)} \exp\left(-\frac{|y - \eta(s)|^2}{4s}\right)dy \leq \int_{\mathbb{R}^N} \exp\left(-\frac{|y|^2}{4s}\right)dy \leq Cs^{N/2},$$

the integral $I_3(r)$ satisfies

$$I_3(r) \leq C \int_{s \in T(2r)} ds = C\tau(2r).$$

Combining these estimates and using

$$r^N \int_{2r}^{\infty} l^{-N-1}\tau(l)dl \geq r^N \tau(2r) \int_{2r}^{\infty} l^{-N-1}dl \geq C\tau(2r),$$

we obtain

$$\int_{B_r(T)} F(x, T)dx \leq Cr^N + Cr^N \int_{r}^{\infty} l^{-N-1}\tau(l)dl + C\tau(2r)$$

$$\leq Cr^N + Cr^N \int_{r}^{\infty} l^{-N-1}\tau(l)dl.$$ 

By Lemma 3.1, the proof is completed. 

Proof of Corollary 1.4. First, if $\xi$ satisfies

$$|\xi(T) - \xi(t)| \leq c_2(T-t)^\alpha, \quad t \in [t_0, T],$$

for some $c_2 > 0$ and $t_0 \in [0, T)$, we have $\sigma(r) \geq (r/c_2)^{1/\alpha}$. Then by Theorem 1.2 there exists $C_1 > 0$ and $R_1 > 0$ such that

$$\int_{B_r(T)} u(x, T)dx \geq C_1 \max\{r^{1/\alpha}, r^N\}, \quad r \in (0, R_1).$$
Next, if
\[ |\xi(T) - \xi(t)| \geq c_1(T - t)^\alpha, \quad t \in [t_0, T], \]
then \( \tau(r) \leq (r/c_1)^{1/\alpha} \). On the other hand, there exists \( c_0 > 0 \) such that \( \tau(l) = \tau(c_0) \) for \( l \geq c_0 \). Hence, we have
\[ r^N \int_{l}^{\infty} l^{-N-1} \tau(l) \, dl \leq C r^{1/\alpha} \quad \text{if } \alpha > 1/N, \]
\[ r^N \int_{l}^{\infty} l^{-N-1} \tau(l) \, dl \leq C r^N \int_{c_0}^{\infty} l^{-N-1} \, dl \leq C r^N \quad \text{if } \alpha < 1/N. \]

Then, by Theorem 1.3, there exist \( C_2 > 0 \) and \( R_2 > 0 \) such that
\[ \int_{B^r(T)} u(x, T) \, dx \leq C_2 r^{1/\alpha} \quad \text{if } \alpha > 1/N, \]
\[ \int_{B^r(T)} u(x, T) \, dx \leq C_2 r^N \quad \text{if } \alpha < 1/N \]
for \( r \in (0, R_2) \). By setting \( R = \min\{R_1, R_2\} \), the proof for \( \alpha \neq 1/N \) is completed.

Finally, we consider the case \( \alpha = 1/N \). In this case, we need a more precise estimate for \( I_2(r) \) than the proof of Theorem 1.3. Since \( c_1 s^{1/N} \leq |\eta(s)| \leq c_2 s^{1/N} \) and \( [0, s_0] \setminus \mc{T}(2r) \subset [cr^N, s_0] \) for some \( c > 0 \), we have
\[ I_2(r) = \int_{s \in [0, s_0] \setminus \mc{T}(2r)} \int_{B^r_r(r)} \frac{1}{(4\pi s)^{N/2}} \exp \left( -\frac{|\eta(s)|^2}{16s} \right) \, dy \, ds \]
\[ \leq \int_{s \in [0, s_0] \setminus \mc{T}(2r)} \int_{B^r_r(r)} \frac{1}{(4\pi s)^{N/2}} \exp \left( -cs^{2/N} \right) \, dy \, ds \]
\[ \leq C r^N \int_{cr^N}^{\infty} \frac{1}{(4\pi s)^{N/2}} \exp \left( -cs^{2/N} \right) \, ds \]
\[ \leq C r^N. \]
This completes the proof.

4 Fractional Brownian motion

In this section, we consider the case where \( \xi(t) \) is a sample path of the fractional Brownian motion with the Hurst exponent \( H \) with \( 0 < H \leq 1/2 \). We sometimes write \( \xi_t \) for \( \xi(t) \). We denote by \( E \) and \( P \) the expectation and the probability, respectively, of the fractional Brownian motion starting from the origin. We first show that the Lebesgue measure \( \tau(r) \) of \( \mc{T}(\rho) \) defined by (1.4) has the following properties:
Proposition 4.1. Suppose that $\xi(t)$ is a sample path of the fractional Brownian motion with the Hurst exponent $H$. Then for any $\delta > 0$, there exists a constant $r_0(\omega) > 0$ such that
\[
\sigma(r) \geq \{\log \log(1/r)\}^{-1/(2H)-\delta}r^{1/H}
\]
for $r \in (0, r_0]$ with probability one.

Proof. Let
\[
A_m := \{\sigma(e^{-m}) \leq (\log m)^{-1/(2H)-\delta/2}e^{-m/H}\}.
\]
By using (3.14) of [18], we have
\[
P(A_m) \leq C \exp(-c(\log m)^{1+\delta H}).
\]
By the Borel-Cantelli lemma, we have
\[
P(\limsup_{m \to \infty} A_m) = 0.
\]
Then for $e^{-m-1} < r \leq e^{-m}$, $\sigma$ satisfies
\[
\sigma(r) \geq \sigma(e^{-m-1}) \geq (\log(m + 1))^{-1/(2H)-\delta/2}e^{-(m+1)/H} \geq \{\log \log(1/r)\}^{-1/(2H)-\delta}r^{1/H}
\]
for all sufficiently large $m > 0$ with probability one. Therefore, we have
\[
\sigma(r) \geq \{\log \log(1/r)\}^{-1/(2H)-\delta}r^{-1/H}
\]
for all sufficiently small $r > 0$ with probability one. This proves the desired result.

Proposition 4.2. Suppose that $\xi(t)$ is a sample path of the fractional Brownian motion with the Hurst exponent $H$. Then the following holds:

(i) If $H > 1/N$, for any $\delta > 0$, there exists a constant $r_0(\omega) > 0$ such that
\[
\tau(r) \leq r^{1/H}\{\log \log(1/r)\}^{1+\delta}
\]
for $r \in (0, r_0]$ with probability one.

(ii) If $H = 1/N$, for any $\delta > 0$, there exists a constant $r_0(\omega) > 0$ such that
\[
\tau(r) \leq r^N(\log(1/r))\{\log \log(1/r)\}^{1+\delta}
\]
for $r \in (0, r_0]$ with probability one.

(iii) If $H < 1/N$, for any $\delta > 0$, there exists a constant $r_0(\omega) > 0$ such that
\[
\tau(r) \leq r^N\{\log \log(1/r)\}^{1+\delta}
\]
for $r \in (0, r_0]$ with probability one.
Before giving a proof of Proposition 4.2, we complete the proof of Theorem 1.5.

Proof of Theorem 1.5. By Theorem 1.2 and Proposition 4.1, we obtain the lower bound immediately. By Proposition 4.2 and a similar computation to the proof of Corollary 1.4, we have

\[ r^N \int_0^\infty l^{-N-1} \tau(l) dl \leq C r^{1/H} \{ \log \log(1/r) \}^{1+\delta} \quad \text{if } H > 1/N, \]

\[ r^N \int_0^\infty l^{-N-1} \tau(l) dl \leq C r^N \log(1/r) \{ \log \log(1/r) \}^{1+\delta} \quad \text{if } H = 1/N, \]

\[ r^N \int_0^\infty l^{-N-1} \tau(l) dl \leq C r^N \log(1/r) \{ \log \log(1/r) \}^{1+\delta} \quad \text{if } H < 1/N. \]

Thus the desired bounds are obtained.

To show Proposition 4.2, we need some preparations. We consider \( f(x) \) satisfying

\[ f(x) \geq 0, \quad \|f\|_1 < \infty, \quad \|f\|_\infty < \infty, \]

and define

\[ Z := \int_0^\infty f(\xi_s) ds, \quad Z_t := \int_0^t f(\xi_s) ds. \]

Proposition 4.3.

(i) If \( H > 1/N \), then

\[ E[Z^n] \leq n! C_1^n \]

for all \( n \in \mathbb{N} \), where \( C_1 := \sqrt{\frac{2}{\pi H N - 1}} \|f\|_1 + \|f\|_\infty} .\)

(ii) If \( H = 1/N \), then

\[ E[Z^n] \leq n! C_2^n (\log t)^n \]

for all \( n \in \mathbb{N} \), where \( C_2 := \sqrt{2 \pi \|f\|_1 + \|f\|_\infty} .\)

(iii) If \( 0 < H < 1/N \), then

\[ E[Z^n] \leq n! C_3^n t^{(1-HN)n} \]

for all \( n \in \mathbb{N} \), where \( C_3 := \sqrt{\frac{2}{\pi (1-HN)}} \|f\|_1} .\)

We can write

\[ E[Z^n] = \int_{0<t_1<t} \cdots \int_{0<t_n<t} E[f(\xi_{t_1}) \cdots f(\xi_{t_n})] dt_1 \cdots dt_n = n! a_n(t), \]

where

\[ a_n(t) := \int \cdots \int_{0<t_1<\cdots<t_n<t} E[f(\xi_{t_1}) \cdots f(\xi_{t_n})] dt_1 \cdots dt_n. \]
In addition, let
\[
\begin{align*}
  b_n(t) &:= \int_{t_j-t_{j-1} > 1} \cdots \int_{0 < t_1 < \cdots < t_n < t} E[f(\xi_{t_1}) \cdots f(\xi_{t_n})] dt_1 \cdots dt_n, \\
b_0(t) &:= 1, \\
t_0 := 0,
\end{align*}
\]
and \(a_n := a_n(\infty), b_n := b_n(\infty)\).

To show Proposition 4.3 we give the following lemmas.

**Lemma 4.4.** The density of \(\xi_{t_1}, \xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}}\) satisfies
\[
p_N(x_1, \ldots x_n) \leq \left( \frac{2}{\pi} \right)^{nN/2} \prod_{j=1}^{n} (t_j - t_{j-1})^{-HN}.
\]

**Proof.** Let \(L_n(t_1, \ldots, t_n)\) be the covariance matrix of \(\xi_{t_1}^1, \xi_{t_2}^1 - \xi_{t_1}^1, \ldots, \xi_{t_n}^1 - \xi_{t_{n-1}}^1\), where \(\xi^1\) is a sample path of the one-dimensional fractional Brownian motion.

By \([2, Lemma 3.3]\), we have
\[
\det L_n(t_1, \ldots, t_n) \geq \frac{1}{2^n} \prod_{j=1}^{n} (t_j - t_{j-1})^{2H},
\]
where \(t_0 = 0\). Then the density of \(\xi_{t_1}^1, \xi_{t_2}^1 - \xi_{t_1}^1, \ldots, \xi_{t_n}^1 - \xi_{t_{n-1}}^1\) satisfies
\[
p(x_1, \ldots x_n) \leq \left( \frac{2}{\pi} \right)^{n/2} \prod_{j=1}^{n} (t_j - t_{j-1})^{-H},
\]
and hence we obtain the desired result.

**Lemma 4.5.** If \(H > 1/N\), then
\[
b_n \leq n! \hat{C}_1^n
\]
for all \(n \in \mathbb{N}\), where \(\hat{C}_1 = \sqrt{\frac{2}{\pi} \frac{1}{HN-1} \|f\|_1}\).

**Proof.** First we note that
\[
E[f(\xi_{t_1}) \cdots f(\xi_{t_n})] \leq \left( \frac{2}{\pi} \right)^{nN/2} \prod_{j=1}^{n} (t_j - t_{j-1})^{-HN}
\]
\[
\times \int_{\mathbb{R}^N} \cdots \int_{\mathbb{R}^N} f(x_1) f(x_1 + x_2) \cdots f(x_1 + \cdots + x_n) dx_1 \cdots dx_n
\]
\[
= \left( \frac{2}{\pi} \right)^{nN/2} \prod_{j=1}^{n} (t_j - t_{j-1})^{-HN} \|f\|_1^n.
\]
Then we have
\[ b_n \leq \left( \frac{2}{\pi} \right)^{nN/2} \|f\|_1^n \int \cdots \int_{0<t_1<\ldots<t_n} \prod_{j=1}^n (t_j - t_{j-1})^{-HN} dt_1 \ldots dt_n \]
\[ = \left( \frac{2}{\pi} \right)^{nN/2} \|f\|_1^n (HN - 1)^{-n}. \]

Hence we obtain the desired inequality.

**Lemma 4.6.** If \( H = 1/N \), then
\[ b_n(t) \leq n! \hat{C}_2^n (\log t)^n \]
for all \( n \in \mathbb{N} \), where \( \hat{C}_2 = \sqrt{2/\pi} \|f\|_1 \).

**Proof.** It suffices to show \( a_n \leq C_1^n \). Since
\[ E[f(\xi_t) \ldots f(\xi_{tn})] \leq \left( \frac{2}{\pi} \right)^{nN/2} \prod_{j=1}^n (t_j - t_{j-1})^{-HN} \|f\|_1^n, \]
we have
\[ b_n(t) \leq \left( \frac{2}{\pi} \right)^{nN/2} \|f\|_1^n \int \cdots \int_{0<t_1<\ldots<t_n} \prod_{j=1}^n (t_j - t_{j-1})^{-HN} dt_1 \ldots dt_n \]
\[ \leq \left( \frac{2}{\pi} \right)^{nN/2} \|f\|_1^n (\log t)^n. \]

Hence we obtain the desired result.

Now, let us prove Proposition 4.3 by using the above lemmas.

**Proof of Proposition 4.3** First we show (i). When \( n = 1 \),
\[ a_1 = \int_0^\infty E[f(\xi_{t_1})] dt_1 \]
\[ = \int_0^1 E[f(\xi_{t_1})] dt_1 + \int_1^\infty E[f(\xi_{t_1})] dt_1 \]
\[ \leq \|f\|_\infty + b_1 \leq \|f\|_\infty + \hat{C}_1 \leq C_1. \]
When \( n = 2 \),
\[
a_2 = \int_0^{t_1} E[f(\xi_{t_1}) f(\xi_{t_2})] dt_1 dt_2
\]
\[
= \int_0^{t_1} E[f(\xi_{t_1}) f(\xi_{t_2})] dt_1 dt_2 + \int_{t_1}^{t_2} E[f(\xi_{t_1}) f(\xi_{t_2})] dt_1 dt_2 + \int_{t_1 < t_2 - 1} E[f(\xi_{t_1}) f(\xi_{t_2})] dt_1 dt_2 + \int_{1 < t_2 < t_1 < 1} E[f(\xi_{t_1}) f(\xi_{t_2})] dt_1 dt_2
\]
\[
\leq \|f\|_\infty^2 + 2\|f\|_\infty b_1 + b_2
\]
\[
\leq \|f\|_\infty^2 + 2\|f\|_\infty \hat{C}_1 + \hat{C}_1^2
\]
\[
\leq C_1^2.
\]

For general \( n \), by the same computation as above, we have
\[
a_n \leq \sum_{k=0}^{n} n C_k \|f\|_\infty^k \hat{C}_1^{n-k} = (\|f\|_\infty + \hat{C}_1)^n = C_1^n
\]
and hence we obtain (i).

By the same way as the proof of (i), we have (ii) with the aid of Lemma 4.6. Moreover, from
\[
E[f(\xi_{t_1}) \ldots f(\xi_{t_n})] \leq \left(\frac{2}{\pi}\right)^{nN/2} \prod_{j=1}^{n} (t_j - t_{j-1})^{-H} \|f\|_1^n,
\]
we see that
\[
a_n(t) \leq \left(\frac{2}{\pi}\right)^{nN/2} \|f\|_1^n \int_0^{t} \ldots \int_{0 < t_1 < \ldots < t_n < t} \prod_{j=1}^{n} (t_j - t_{j-1})^{-H} dt_1 \ldots dt_n
\]
\[
\leq \left(\frac{2}{\pi}\right)^{nN/2} \|f\|_1^n (1 - H N)^{-n H (1 - H N)n}.
\]
Therefore, (iii) holds.

**Lemma 4.7.**

(i) If \( H > 1/N \), then there exists \( C < \infty \) such that
\[
E[\tau(e^{-m})^n] \leq n! C^n e^{-mn/H}
\]
for all \( m, n \in \mathbb{N} \).

(ii) If \( H = 1/N \), then there exists \( C < \infty \) such that
\[
E[\tau(e^{-m})^n] \leq n! C^n e^{-mnN} m^n
\]
for all \( m, n \in \mathbb{N} \).
(iii) If $H < 1/N$, then there exists $C < \infty$ such that
\[ E[\tau(e^{-m})^n] \leq n! C^m e^{-mNn} \]
for all $m, n \in \mathbb{N}$.

**Proof.** Note that by the stationary increment property of the fractional Brownian motion,
\[ \tau(e^{-m}) = d \int_0^{s_0} 1_{\{|\xi(s)| \leq e^{-m}\}} ds, \]
where $X \overset{d}{=} Y$ means that $X$ has same distribution as $Y$. In addition, by the scaling property, we have
\[ \int_0^{s_0} 1_{\{|\xi(s)| \leq e^{-m}\}} ds = d e^{-m/H} \int_0^{s_0 \exp(m/H)} 1_{\{|\xi(s)| \leq 1\}} ds. \]
Note that $\|f\|_1 < \infty$ and $\|f\|_\infty < \infty$ hold. Thus, if we substitute $1_{\{|x| \leq 1\}}$ for $f$ in Proposition 4.3, we obtain the desired result.

Finally, we give the following lemma.

**Lemma 4.8.**
(i) If $H > 1/N$, there exist $C, c > 0$ such that
\[ P(\tau(e^{-m}) \geq Me^{-m/H}) \leq Ce^{-cM} \]
for any $M < \infty$ and $m \in \mathbb{N}$.
(ii) If $H = 1/N$, there exist $C, c > 0$ such that
\[ P(\tau(e^{-m}) \geq Me^{-mN} m) \leq Ce^{-cM} \]
for any $M < \infty$ and $m \in \mathbb{N}$.
(iii) If $H < 1/N$, there exist $C, c > 0$ such that
\[ P(\tau(e^{-m}) \geq Me^{-mN}) \leq Ce^{-cM} \]
for any $M < \infty$ and $m \in \mathbb{N}$.

**Proof.** First, we show (i). By Lemma 4.7, there exist $c > 0$ and $C < \infty$ such that for any $m \in \mathbb{N}$
\[ E[\exp(c\tau(e^{-m}e^{m/H}))] < C. \]
Thus, by Chebyshev’s inequality, we obtain
\[ P(\tau(e^{-m}) \geq Me^{-m/H}) = P(\exp(c\tau(e^{-m}e^{m/H})) \geq e^{cM}) \leq Ce^{-cM} E[\exp(c\tau(e^{-m}e^{m/H}))] \leq Ce^{-cM}. \]
Similarly, we have (ii) and (iii).
Now, we are in a position to show the main proposition in this section.

**Proof of Proposition 4.2.** First, we consider the case of $H > 1/N$. Let

$$A_m := \{ \tau(e^{-m}) \geq (\log m)^{1+\delta/2} e^{-m/H} \}.$$ 

By Lemma 4.8

$$P(A_m) \leq C \exp(-c(\log m)^{1+\delta/2}).$$

By the Borel-Cantelli lemma, we have

$$P(\limsup_{m \to \infty} A_m) = 0.$$ 

Hence if $e^{-m-1} < r \leq e^{-m}$, then

$$\tau(r) \leq \tau(e^{-m} \leq (\log m)^{1+\delta/2} e^{-m/H} \leq \{\log \log (1/r)\}^{1+\delta} r^{1/H}$$

for all sufficiently large $m$ with probability one. Thus we obtain

$$\tau(r) \leq \{\log \log (1/r)\}^{1+\delta} r^{1/H}$$

for all sufficiently small $r > 0$ with probability one. This proves (i).

The assertions (ii) and (iii) can be proved in the same way.

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