Modeling anisotropic magnetized White Dwarfs with $\gamma$ metric

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The effect of magnetic fields in the Equations of State (EoS) of compact objects is the splitting of the pressure in two components, one parallel and the other perpendicular to the magnetic field. This anisotropy suggests the necessity of using structure equations considering the axial symmetry of the magnetized system. In this work, we consider an axially symmetric metric in spherical coordinates, the $\gamma$-metric, and construct a system of equations to describe the structure of spheroidal compact objects. In addition, we connect the geometrical parameter $\gamma$ linked to the spheroid’s radii, with the source of the anisotropy. So, the model relates the shape of the compact object to the physics that determines the properties of the composing matter. To illustrate how our structure equations work, we obtain the mass-radii solutions for magnetized White Dwarfs in the weak magnetic field regime. Our results show that the main effect of the magnetic field anisotropy in White Dwarfs structure is to cause a prolate deformation of these objects. Since this effect is only relevant at low densities, it does not affect the maximum values of magnetized White Dwarf’s masses, which remain under Chandrasekhar limit.

I. INTRODUCTION

Magnetic fields are present in almost all stars during their stellar evolution, becoming huge in the final stage, when they turn into compact objects. Measurements of periods and spin down of soft-gamma repeaters (SGR) and X-ray luminosities of anomalous X-ray pulsars (AXP) [1], support the idea of the existence of magnetars: neutron stars with surface magnetic fields as large as $10^{14} - 10^{18}$ G [2]. In the case of White Dwarfs (WDs), observed surface magnetic fields range from $10^6$ G to $10^9$ G [3]. Although the inner magnetic fields can not be observed directly, their bounds can be estimated with theoretical models based on macroscopic and microscopic analysis. The maximum magnetic fields estimated for WDs are around $10^{13}$ G [4, 5] and about $5 \times 10^{18}$ G for neutron stars [6].

From a microscopic point of view, a magnetic field acting on a fermion gas breaks the spherical symmetry and produces an anisotropy in the quantum-statistical average of the energy-momentum tensor. The effect of this anisotropy is the splitting of the pressure into two components, one along the magnetic field —the parallel pressure $P_{\parallel}$— and another in the transverse direction —the perpendicular pressure $P_{\perp}$, so that $T_{\nu}^{\mu} = \text{diag}(\rho, -P_{\perp}, -P_{\perp}, -P_{\parallel})$. Consequently, a gas of fermions under the action of a constant and uniform magnetic field has an anisotropic —axially symmetric— equation of state (EoS) [4]. For this reason, when modeling the structure of magnetized compact objects, one should consider axial symmetry instead of the spherical symmetry used when solving the Tolman-Oppenheimer-Volkoff (TOV) equations.

Our first attempt addressing this issue, on Refs. [4–6], was to consider a metric in cylindrical coordinates $(t, r, \phi, z)$ to obtain Einstein’s field equations following the procedure described in Ref. [3]. This model lead us to obtain some information about the effects of the magnetic field in terms of the shape —prolateness or oblateness— of the compact object as well as upper limits for the values of the magnetic field that can sustain these stars. However, since we assumed that all the magnitudes depend only on the radial coordinate $r$, we were unable to determine the total mass.

Therefore, we return to spherical coordinates. Let us remark that anisotropies in the energy-momentum tensor are admitted in spherical symmetry as long as the tensor has the form $T_{\nu}^{\mu} = \text{diag}(\rho, -p_r, -p_r, -p_t)$, where $p_r$ is a radial pressure and $p_t$ is a tangential one [5, 10]. However, this is not compatible with the anisotropy due to magnetic fields. Thus, we are going to use an axially symmetric metric in spherical coordinates to account for the magnetic anisotropy of the system.

Hence, in this work, we start from a metric presenting a $\gamma$ parameter associated to the deformation of the stars. This metric was previously presented in [11, 12] and allows to obtain a set of structure equations that generalize the TOV equations to axially symmetric objects. The novelty of our treatment consists in computing the total mass as for a spheroidal object and proposing an ansatz...
to relate $\gamma$ with the ratio between the pressures, which connects the physics of the system with its geometry.

As an example and test case, we solve these anisotropic structure equations for magnetized WDs in the weak magnetic field regime \[13\]. We have chosen magnetized WDs because of their connection with supernovae Chandrasekhar WDs \[14\] and our previous studies of these objects in cylindrical coordinates \[4\]. Also, the weak magnetic field approximation for magnetized WDs EoS has the advantage of allowing a semi-analytical treatment of the EoS instead of a numerical interpolation.

Section II is devoted to the magnetized WDs EoS, while the anisotropic structure equations are presented in Section III. Results for magnetized WDs and their discussion can be found in Section IV and concluding remarks in Section V.

II. EOS FOR MAGNETIZED WHITE DWARFS

Typical WDs are composed by carbon or oxygen atoms. The role of the different particles conforming these atoms in the star’s physics depends on their masses. Due to its relative low mass, only the degenerated gas of relativistic electrons determine the pressure that compensates the gravitational collapse of the star. The heavier neutrons and protons behave non-relativistically, and contribute mainly to the mass and energy density.

The pressures and the energy density of the electron gas in magnetized WDs are obtained starting from the thermodynamical potential \[7\].

\[
\Omega(B, \mu, T) = -\frac{eB}{2\pi^2} \sum_{l=0}^{\infty} g_l \left[ \epsilon_l + \frac{1}{\beta} \ln \left( 1 + e^{-\beta(\epsilon_l - \mu)} \right) \right]
\]

being $\epsilon_l = \sqrt{p_{\perp}^2 + 2|eB|l + m^2}$ the electron spectrum in a magnetic field. In Eq. (1) the magnetic field $B$ is supposed uniform, constant, and in the $z$ direction, $l$ stands for the Landau levels and the factor $g_l = 2 - \delta_{l0}$ includes the spin degeneracy of the fermions for $l \neq 0$. $\beta$ is the inverse of the absolute temperature $T$, $\mu$ the chemical potential, $m$ the electron mass and $e$ its charge.

Note that, in general, the thermodynamical potential on Eq. (1) can be divided in two contributions

\[
\Omega(B, \mu, T) = \Omega^{\text{vac}}(B) + \Omega^{\text{st}}(B, \mu, T).
\]

The second term, $\Omega^{\text{st}}(B, \mu, T)$, arises from statistical considerations and reads

\[
\Omega^{\text{st}}(B, \mu, 0) = -\frac{eB}{2\pi^2} \sum_{l=0}^{\infty} g_l (\mu - \epsilon_l) \Theta(\mu - \epsilon_l),
\]

where $\Theta(\cdot)$ is the unit step function.

As pointed out in the introduction, the maximum magnetic field estimated for WDs interiors is around $10^{13}$ G, a value that is of the same order of the critical magnetic field for electrons $B_c = m^2/e = 4.4 \times 10^{13}$ G (Schwinger field \[2\]). With this upper bound, we consider magnetic field values such that $B < B_c$. Moreover, in astrophysical scenarios the energy scales are determined by the temperature and the density. Then, for WDs in the zero temperature limit, the parameter that will set the relative relevance of magnetic field effects on the system is the density. Due to the typical densities on WDs and the values of the magnetic fields used, the electron gas is characterized by the relation $eB \ll m^2 \ll \mu^2$, which is known as the weak field limit. In this regime, the distance between Landau levels ($\sim eB$) is small and we can consider the discrete spectrum as a continuum. This allows us to replace the sum over $l$ in Eq. (1) by an integral through the Euler-MacLaurin formula \[17\].

\[
\frac{eB}{2} \sum_{l=0}^{\infty} g_l f(2eBl) \approx eB \int_0^\infty f(2eBl)dl + \frac{eB}{2} f(\infty) + \sum_{k=1}^{\infty} \frac{2^{2k-1}}{(2k)!} (eB)^{2k} B_{2k} \left[ f^{(2k-1)}(\infty) - f^{(2k-1)}(0) \right],
\]

where $f(2eBl) = (\mu - \epsilon_l) \Theta(\mu - \epsilon_l)$ and the coefficients $B_n$ stand for the Bernoulli numbers ($B_2 = 1/6$). Then, we can expand Eq. (1) onto the second power on $eB$, and take the classical limit by means of the change of variables $p_{\perp}^2 = 2eBl = p_x^2 + p_y^2$, with $p_\perp dp_\perp = eBdl$ \[18\]. Hence, we get the statistical part of the thermodynamical

\[\text{(3)}\]

\[\text{(4)}\]

\[\text{(5)}\]

\[\text{(6)}\]

\[\text{(7)}\]
potential as follows

$$\Omega_{st}(B, \mu, 0) = -\frac{m^4}{12\pi^2} \left[ \mu \sqrt{\mu^2 - m^2} \left(\frac{\mu^2}{m^2} - \frac{5}{2}\right) \right] + \frac{3}{2} \ln \left( \frac{\mu + \sqrt{\mu^2 - m^2}}{m} \right) + \left( \frac{B}{B_c} \right)^2 \ln \left( \frac{\mu + \sqrt{\mu^2 - m^2}}{m} \right).$$  \(6\)

From Eq. \(6\) we note that, in the weak magnetic field limit, the statistical part of the thermodynamical potential is expressed as a sum of two non-magnetic terms at \(\mu \neq 0\) and \(T = 0\), plus a third term that depends also on the magnetic field.

On the other hand, the term \(\Omega_{\text{vac}}(B)\) in Eq. \(2\):

$$\Omega_{\text{vac}}(B) = -\frac{eB}{2\pi^2} \int_0^\infty dp_3 \sum_{l=0}^\infty g_l \varepsilon_l,$$  \(7\)

does not depend on the chemical potential nor on the temperature and corresponds to the vacuum. This contribution presents an ultraviolet divergence that must be renormalized \([19]\). In the weak field limit, the renormalization of \(\Omega_{\text{vac}}\) leads to the following expression \([20]\)

$$\Omega_{\text{vac}}(B, 0, 0) = -\frac{m^4}{90(2\pi)^2} \left( \frac{B}{B_c} \right)^4.$$  \(8\)

If we consider typical values of densities and magnetic fields present in WDs, the vacuum contribution in Eq. \(8\) can be neglected when compared to the statistical one in Eq. \(6\). Therefore, the thermodynamical potential of the electron degenerate system can be approximated to \(\Omega(B, \mu, 0) = \Omega^{st}(B, \mu, 0)\) when working below the Schwinger magnetic field.

Matter inside compact stars must be in stellar equilibrium. So, we must impose charge neutrality and baryon number conservation to the energy density and the pressures. With these considerations, the magnetized WDs EoS — the pressures as a parametric function of the energy density — becomes

$$E = \Omega + \mu N + m_N \frac{A}{Z} N + \frac{B^2}{8\pi},$$  \(9a\)

$$P_\parallel = -\Omega - \frac{B^2}{8\pi},$$  \(9b\)

$$P_\perp = -\Omega - BM + \frac{B^2}{8\pi},$$  \(9c\)

where \(\Omega\) is given by Eq. \(6\), \(N = -\partial \Omega / \partial \mu\) is the electron particle density and \(M = -\partial \Omega / \partial B\) the magnetization. The term \(Nm_N A/Z\) included in Eq. \(9a\) considers the contribution of the nucleons to the energy density \([7]\).

The last term in Eqs. \(9\) is the Maxwell contribution to the pressures and energy density, \(P_\perp = E^B = -P_\parallel = B^2/8\pi\). For an electron gas, this contribution becomes relevant when the magnetic fields are higher than \(10^{14}\) G, which from our point of view are unrealistic values for WDs \([13, 21]\). Therefore, we will neglect Maxwell contribution in our calculations.

Fig. 1 shows the semi-analytical parametric EoS given in Eqs. \(9\) for magnetized WDs with a carbon/oxygen composition for \(B = 0\) and \(B = 5 \times 10^{12}\) G. Note that at first glance, the three curves coincide, but the difference between the perpendicular and the parallel pressures can be appreciated in the inset panel for low densities, being the perpendicular pressure curve softer than the parallel one.

![Figure 1. EoS for magnetized WDs at fixed values of the magnetic field \(B = 0\) and \(B = 5 \times 10^{12}\) G (in CGS units).](image)

The EoS obtained in this section can be used to solve structure equations for magnetized WDs. In Fig. 2 we present the mass-radius curves obtained with the standard isotropic TOV equations considering the pairs \((E, P_\parallel)\) and \((E, P_\perp)\) as independent EoS, as well as the corresponding non-magnetized curve \([4]\). Despite the tiny differences between the pressures, the fact of using one or the other leads to different mass-radius relations for less dense systems. This result highlights the importance of a model properly considering the anisotropy in the system.

### III. \(\gamma\)-METRIC AND STRUCTURE EQUATIONS FOR MAGNETIZED COMPACT OBJECTS

We devote this section to construct a general model suitable to study the structure of axially deformed compact objects. Our model is based on Refs. \([11, 12, 29]\), where the authors show that a deformed compact object

\[m_N = 931.494 \text{ MeV} \sim m_{n,p} \text{ and } A/Z \text{ is the number of nucleons per electron } (A/Z = 2 \text{ for carbon/oxygen WDs})\]
with axial symmetry can be described by the metric
\[ ds^2 = -\left[1 - 2M(r)/r\right]^{\frac{\delta}{\gamma}} + \frac{dr^2}{1 - 2M(r)/r} + r^2 d\theta^2 \]  
(10)
where \( \gamma = z/r \) parametrizes the polar radius \( z \) in terms of the equatorial one \( r \).

Starting from this metric, the energy-momentum tensor of the magnetised gas and using the mass of a spheroid to compute the star’s mass, we obtain the following structure equations
\[
\frac{dM}{dr} = r^2 E\gamma, 
\]  
(11a)
\[
\frac{dP_{\parallel}}{dr} = -\frac{(E + P_{\parallel})[\frac{\delta}{2} + r^2 P_{\parallel} - \frac{\delta}{2}(1 - \frac{2M}{r})\gamma]}{r^2(1 - \frac{2M}{r})\gamma}, 
\]  
(11b)
\[
\frac{dP_{\perp}}{dz} = \frac{1}{\gamma} \frac{dP_{\perp}}{dr} = -\frac{(E + P_{\perp})[\frac{\delta}{2} + r^2 P_{\perp} - \frac{\delta}{2}(1 - \frac{2M}{r})\gamma]}{\gamma r^2(1 - \frac{2M}{r})\gamma}, 
\]  
(11c)
which describe the variation of the mass and the pressures with the spatial coordinates \( r, z \) for an anisotropic axially symmetric compact object. Note that these equations are coupled through the dependence with the energy density and the mass.

Since the parallel pressure has its maximum central value at \( \theta = 0 \) (\( z \) axis) and goes to zero at the surface, we assume it depends just on the radial coordinate. The perpendicular pressure, on the contrary, is maximum in the \( \theta = \pi/2 \) plane (\( xy \) plane) and zero at \( \theta = 0 \), therefore depending on the \( z = \gamma r \) coordinate.

In general terms, the solutions of Eqs. (11) are computed similarly to how it is done usually for the TOV equations. In this case, we start from a point in the centre with \( E_0 = E(r = 0), P_{\parallel 0} = P_{\parallel}(r = 0) \) and \( P_{\perp 0} = P_{\perp}(r = 0) \) taken from the EoS on Eq. (9). The equatorial and polar radii of the star, \( R \) and \( Z = \gamma R \), are respectively defined when \( P_{\parallel}(R) = 0 \) and \( P_{\perp}(Z) = 0 \), while the mass of the star is \( M = M(R) \).

There is also a remarkable difference with respect to the solution of standard TOV equations in the manner we compute the energy density from the EoS during the integration process. To clarify this point, let us denote as \( c_1(\mu), c_2(\mu) \) the 2D parametric curves given by
\[
c_1(\mu) = (E(\mu), P_{\parallel}(\mu)) 
\]  
(12a)
\[
c_2(\mu) = (E(\mu), P_{\perp}(\mu)) 
\]  
(12b)
with \( E(\mu), P_{\parallel}(\mu) \) and \( P_{\perp}(\mu) \) defined by Eqs. (9). Given \( P_{\parallel} \) and \( P_{\perp} \), obtained in one integration step of Eqs. (11), two parametric values \( \tilde{\mu}_{\parallel} \) and \( \tilde{\mu}_{\perp} \) are computed solving the nonlinear Eqs. (9a) and (9c) respectively. The corresponding points in the curves (12a) and (12b) are \( c_1(\tilde{\mu}_{\parallel}) = (E_{\parallel}, P_{\parallel}) \) and \( c_2(\tilde{\mu}_{\perp}) = (E_{\perp}, P_{\perp}) \), where \( E_{\parallel} = E(\tilde{\mu}_{\parallel}) \) and \( E_{\perp} = E(\tilde{\mu}_{\perp}) \). Hence, in the next integration step, we update the right hand side of Eq. (11b) using the point \( c_1(\tilde{\mu}_{\parallel}) \) with \( E = E_{\parallel} \) and \( P_{\parallel} = P_{\parallel} \). Similarly, we update Eq. (11c) with \( c_2(\tilde{\mu}_{\perp}) \) by taking \( E = E_{\perp} \) and \( P_{\perp} = P_{\perp} \).

The use of different values of the energy when integrating Eqs. (11b) and (11c) is a consequence of canceling the dependence on the angular variables and assuming that \( P_{\parallel} \) evolves in the equatorial direction and \( P_{\perp} \) in the polar one. Because of this supposition, the star’s inner points for which we are computing the two pressures at each step are different and there is no particular reason for the energies to coincide.

The existence of two energies at each integration step introduces an ambiguity in Eq. (11a). Then, we need to select which of them should be used to compute the total mass. To do so, let us note that, since we are dealing with an anisotropic object the mass density is also anisotropic. Along the equatorial direction the mass density is equal to
\[
dM = \gamma r^2 E_{\parallel} dr, 
\]  
(13)
while in the polar direction it reads
\[
dM = \frac{z^2}{\gamma^2} E_{\perp} dz. 
\]  
(14)

In Eqs. (13) and (14) we have used the parallel and the perpendicular energy density in regard of the differentiation direction. Now, taking into account that \( z = \gamma r \), Eq. (14) can be transformed into
\[
dM = \gamma r^2 E_{\parallel} dr. 
\]  
(15)
Adding Eqs. (13) and (15), we get
\[
\frac{dM}{dr} = \gamma r^2 E_{\parallel} + E_{\perp} \frac{1}{2}. 
\]  
(16)
Eq. (16) indicates that, if we don’t want to lose the information about the mass density anisotropy, we have to
update the right hand side of Eq. (11a) with the average energy density $E = (E_{\parallel} + E_{\perp})/2$. This is also a consequence of neglecting the dependence on angular variables and a warning about the fact that for a complete description of the anisotropic object one should consider a full tridimensional treatment.

Since we are interested in the effects on the structure equations coming from the magnetic field and the related anisotropy, we propose to interpret $\gamma$ as the ratio between the parallel and perpendicular central pressures, $P_{\parallel o}$ and $P_{\perp o}$ respectively

$$
\gamma = \frac{P_{\parallel o}}{P_{\perp o}}.
$$

This assumption connects the geometry and the physics in the system implying that the shape of the star is determined by the anisotropy in its center. It yields reasonable results as a first approximation to the problem (see Section IV). However, a more advanced calculation should take into account the variation of the parameter $\gamma$ throughout the star, just as if considering a nested set of shells with constant value of $\gamma$.

The combination of the structure equations in Eqs. (11) with the ansatz given by Eq. (17) allows us to describe the internal variations of the mass and the pressures of a magnetized compact object. It is important to remark that by setting $B = 0$, the model automatically yields $P_{\perp} = P_{\parallel}$ and $\gamma = 1$. This means that we recover the spherical TOV equations from Eqs. (11) and thus, the standard non-magnetized solution for the structure of compact objects.

In what follows, we study the solutions of Eqs. (11) for magnetized WDs EoS, even though these structure equations describe any anisotropic axially deformed compact object provided that it is spheroidal.

### IV. MAGNETIZED WDs NUMERICAL RESULTS AND IMPORTANT REMARKS

In this section we validate the anisotropic model proposed in Section III by integrating Eqs. (11) for the EoS of magnetized WDs. The numerical results are shown in Figs. 3 and 4.

The upper panel of Fig. 3 displays the mass versus the equatorial ($R$) and the polar ($Z$) radii—the transverse and parallel ones—for $B = 5 \times 10^{12}$ G in comparison with the non-magnetized solution. At the highest central densities and smallest radii, the masses reach values close to the Chandrasekhar limit of $1.44 M_{\odot}$ [15, 16, 22]. Also, note that in the $B = 0$ case, the relation $R = Z$ is fulfilled and the curve is identical to the corresponding one in Fig. 2 as it should be, since Eqs. (11) reduce to the isotropic TOV equations.

Moreover, analyzing the magnetized solution at biggest radii, which corresponds to the lowest central densities, we find that for a certain value of mass the polar radius is higher than the equatorial one. So, the corresponding star is a prolate object. Such behavior can be seen in the variation of $\gamma$ as a function of the central density $E_0$ in the lower panel of Fig. 3, since $\gamma > 1$ means that $Z$ is always bigger than $R$. This also illustrates the connection between the geometry and the physics of the system by means of the relation among the ansatz in Eq. (17) with the central density and the radius of the stars.

The existence of two values of mass for a given equatorial radius on the upper panel of Fig. 3 can be understood as a deformation effect for low enough densities with respect to the magnetic energy of the system. If comparing to the corresponding curve of mass as a function of energy density in the lower panel on Fig. 3 we can see that the two masses come from stars with different central densities, so that the lower density star corresponds to the lower mass. In that case, the magnetic field plays an important role, producing a higher deformation on the star. This can also be explained by the balance of the forces at stake, the magnetic, the gravitational and the one from the pressure exerted by the electron gas. For a given magnetic field, at the lowest densities, the particles
increasing the maximum masses and/or producing oblate structure of these objects might be modified differently, as it may occur for neutron stars. Then, the fields are higher and the weak magnetic field limit is not satisfied, $B < 10^{13}$ G, only for WDs but also for other types of compact objects. Precisely, the observed effect is the prolate deformation of magnetic field effects on compact objects, we have connected the parameter $\gamma$, which relates the radii of the magnetized low density WDs, at least for high densities. In consequence, the main effect is the deformation of the magnetized low density WDs, at least in the weak magnetic field limit. Relating this result with the isotropic approximation was preventing any further conclusion on this matter. Also, note that once again, the isotropic approximation was preventing any further conclusion on this matter. Also, note that once again, we do not obtain masses above the Chandrasekhar limit for $B < 10^{13}$ G.

Finally, we would like to remark that although in this case we can neglect the contribution of the Maxwell term $B^2/8\pi$, it could be relevant for other EoS where magnetic fields are higher and the weak magnetic field limit is not valid, as it may occur for neutron stars. Then, the structure of these objects might be modified differently, increasing the maximum masses and/or producing oblate shapes. Thus, a more detailed study with other EoS is compelling.

V. CONCLUSIONS

In this work, we have obtained the structure of non-isotropic compact objects starting from a $\gamma$-metric and computing the mass as for a spheroid. As a result, we get a set of equations that describe the structure of an axially symmetric deformed object, provided it is spheroidal.

In the process to obtain the structure equations, we have neglected the dependence of the quantities with the angular coordinates. This means that when integrating the equations there is a lack of information and the total mass must be computed averaging the energy densities in the polar and equatorial direction. Then, a complete description of an axially symmetric object would require to consider dependence with all coordinates, which brings the necessity of more sophisticated numerical relativity techniques. However, the structure equations we present have the advantage of providing relevant information about the axially symmetric system at a low computational cost.

As we were interested in the anisotropies coming from magnetic fields effects on compact objects, we have connected the parameter $\gamma$, which relates the radii of the magnetized WDs configurations with respect to the corresponding central densities solutions in absence of magnetic field.

In order to illustrate our model we presented the solutions for magnetized WDs structure considering magnetic field values below $10^{13}$ G and densities from $10^6 - 10^{11}$ g/cm$^3$. This allowed us to use the EoS in the weak magnetic field regime, where the vacuum and the Maxwell contributions can be neglected and $P_\perp < P_\parallel$. Our results show that the effect of the magnetic anisotropy on the EoS is relevant at low densities. Hence, it does not affect maximum values of WDs' masses. More precisely, the observed effect is the prolate deformation of stable magnetized WDs configurations with respect to the corresponding central densities solutions in absence of magnetic field.

We expect to obtain oblate objects for other EoS featuring $P_\perp > P_\parallel$, which means to consider the strong magnetic field regime $eB \gg \mu^2$. Intermediate magnetic fields regime ($m^2 \ll 2eB \ll \mu^2$), which are more realistic, should be explored carefully in subsequent works not only for WDs but also for other types of compact objects.

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