Drift Analysis*

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Abstract
Drift analysis is one of the major tools for analysing evolutionary algorithms and nature-inspired search heuristics. In this chapter we give an introduction to drift analysis and give some examples of how to use it for the analysis of evolutionary algorithms.

1 Introduction
Drift analysis goes back to the seminal paper of Hajek [Haj82], and has since become ubiquitous in the analysis of Evolutionary Algorithms (EAs). Google Scholar lists more than 100,000 hits for the phrases “Drift” and “Evolutionary Algorithm”, so a comprehensive review of all applications or even just all existing drift theorems is far beyond the scope of this chapter. Instead, the chapter serves two purposes.

Firstly, it provides a self-contained introduction into drift analysis (Section 3), which is so far missing in the literature1. This introduction is suitable for graduate students or for theory-affine researchers who have not yet encountered drift analysis. This first part will contain illustrative examples, and will discuss in detail the different requirements of the most basic drift theorems, specifically on additive drift, variable drift, and multiplicative drift. Counterexamples are given to point out when some drift theorems are not applicable, or give poor results.

Secondly, Section 4 provides an overview over the most important recent developments in drift analysis, including lower and tail bounds, weak drift, negative drift, and population drift. This section is much more concise, and may also serve as a quick reference for the expert reader.

1 This article is supposed to become a chapter in a book on Theory of Evolutionary Algorithms that will be published by Springer, edited by Benjamin Doerr and Frank Neumann.

1 A briefer introduction can be found in [LO17].
2 Basics of Drift Analysis

2.1 Motivation

To analyse the runtime of an evolutionary algorithm (or more generally, any randomised algorithm), one of the most common and successful approaches consists of the following three steps.

1. Identify a quantity $X_t$, the potential (also called drift function or distance function), that adequately measures the progress that the algorithm has made until time $t$.

2. For any value of $X_t$, understand the nature of the random variable $X_t - X_{t+1}$, the one-step change of the potential.

3. Translate the data from step 2. into information about the runtime $T$ of the algorithm. In theoretical analysis, we usually define $T$ as the number of fitness function evaluations until the algorithm evaluates an optimal solution.

Drift analysis is concerned with step 3. Generally, good drift theorems require as little information as possible about the potential $X_{t+1}$, and give as much information as possible about $T$. In the basic theorems, we only require (bounds on) the expectation $E[X_t - X_{t+1} \mid X_t = s]$ for all $s$, which is called drift, in order to derive (bounds on) the expectation $E[T]$. Drift analysis has become a successful theory because the framework above is very general, and good tools for step 3 exist which apply to a multitude of situations. In contrast, step 1 and 2 often do not generalise from one problem to another. Frequently, step 1 is the part of a runtime analysis that carries the key insight, and it usually requires much more ingenuity than the other steps. On the other hand, step 2, the analysis of $X_t - X_{t+1}$, requires arguably less insight. However, step 2 is usually the most lengthy and technical part of a runtime analysis. Therefore, the complexity of a proof can often be substantially reduced if only some basic information like the expectation $E[X_t - X_{t+1} \mid X_t = x]$ is needed in step 2.

For evolutionary algorithms, a natural candidate for the potential $X_t$ is the fitness $f(x^{(t)})$ of the best individual in the current population, especially so if the population consists only of a single individual, as for example for (1 + 1) EA’s. In a sense, this fitness measures the “progress” until time $t$ since it would exactly correspond to the quality of the output if the algorithm terminated with this generation. However, it is not necessarily the best choice to measure the progress that the algorithm has made towards finding a global optimum. For example, consider the linear fitness function\(^2\) $f : \{0, 1\}^n$ with $f(x) = (n - 1) \cdot x_1 + \sum_{i=2}^n x_i$, which puts very large emphasis

\(^2\)We follow the standard convention that for an $n$-dimensional vector $x$, we denote its components with $x_1, \ldots, x_n$. 
on the first bit. The optimum (for maximization) is the string \( x_{OPT} = (1, \ldots, 1) \), but the two strings \( x_1 = (1, 0, 0, \ldots, 0) \) and \( x_2 = (0, 1, 1, \ldots, 1) \) have the same fitness \( f(x_1) = f(x_2) = n - 1 \). However, the string \( x_2 \) is much more similar to \( x_{OPT} \) than \( x_1 \), so we should choose a potential that assigns a higher rating to \( x_2 \) than to \( x_1 \). We will see later (Example 12) good choices for the potential in this example.

### 2.1.1 General Setup

Throughout this chapter we will assume that \((X_t)_{t \geq 0}\) is a sequence of non-negative random variables with a finite state space \( \mathcal{S} \subseteq \mathbb{R}_0^+ \) such that \( 0 \in \mathcal{S} \). We will denote the minimum positive state by \( s_{\min} := \min(\mathcal{S} \setminus \{0\}) \). The stopping time or hitting time of 0 of \((X_t)_{t \geq 0}\) is defined as the smallest \( t \) such that \( X_t = 0 \). We are generally interested in the drift \( \Delta_t(s) := \mathbb{E}[X_t - X_{t+1} | X_t = s] \), where \( t \geq 0 \) and \( s \in \mathcal{S} \).

As with all conditional expectations, \( \Delta_t(s) \) is not well-defined if \( \Pr[X_t = s] = 0 \). So in other words, \( \Delta_t(s) \) is undefined for situations that never occur. Obviously, this is not a practical issue, and it is convenient (and common in the community) to be sloppy about such cases. So we will use phrases like “\( \Delta_t(s) \leq 1 \) for all \( t \geq 0 \)” as a shortcut for “\( \Delta_t(s) \leq 1 \) for all \( t \geq 0 \) for which the conditional expectation \( \Delta_t(s) \) is well-defined”.

In Section 4 we will often need to work with pointwise drift rather than just conditioning on \( X_t = s \), and we will denote the filtration associated with algorithm’s history up to time \( t \) by \( \mathcal{F}_t \), cf. the discussion on filtrations below. Moreover, tail bounds will be formulated for a fixed initial search point \( X_0 = s_0 \).

Throughout the chapter, \( f \) will denote a fitness function to be optimised, either maximised or minimised. For a \((1 + \lambda)\)-algorithm, we will use the convention that \( x(t) \) is the search point after \( t \) generations.

### 2.1.2 Variants

In the literature, terminology may vary between different authors, and there are often slightly different setups considered. We highlight some variants which occur frequently.

1. **Signs.** We consider the change \( X_t - X_{t+1} \). In the literature, the difference is sometimes considered with opposite signs, \( X_{t+1} - X_t \), which is arguably a more natural choice. However, since we consider drift towards zero, with our choice the drift is usually positive instead of negative. Moreover, our choice is more consistent with the established term “negative drift”, which refers to a drift that points away from the target.

2. **Markov Chains.** Instead of any sequence of random variables, the sequence \( X_t \) is sometimes assumed to be a Markov chain, i.e., the state

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$X_t$ should completely determine the distribution of $X_{t+1}$. While this is a mathematically appealing scenario, it usually does not apply in the context of evolutionary algorithms. For instance, in the example from Section 2.1 above, the information $X_t = n - 1$ would tell us that the current fitness is $n - 1$, but the search points of this fitness differ in nature. Thus, the subsequent trajectory of search points depends on more information than is contained in $X_t$, and so do the subsequent potentials $X_{t+1}, X_{t+2}, \ldots$. So already in this very simple example, we do not have a Markov chain.

3. **Filtrations and Pointwise Drift.** We have defined the drift as a random variable that is conditioned on the value of $X_t$, i.e., $\Delta_t(x) = E[X_t - X_{t+1} | X_t = s]$. Instead, it is also possible to condition on the whole history of $X_t$, or even on the whole history of the algorithm. (Recall that in general, the potential $X_t$ does not completely describe the state of the algorithm at time $t$). In mathematical terms, the set of such histories is described by a filtration of $\sigma$-algebras $F_0 \subseteq F_1 \subseteq \ldots$, where intuitively the $\sigma$-algebra $F_t$ contains all the information that is available after the first $t$ steps of the algorithm. For example, instead of requiring that $E[X_t - X_{t+1} | X_t = s] \leq 1$ for all $t \geq 0$, we would ask that $E[X_t - X_{t+1} | F_t] \leq 1$ for all $t \geq 0$ and all histories $F_t$ up to time $t$ such that $X_t = x$ in $F_t$. In this case, we also speak of pointwise drift, and we will write $E[X_t - X_{t+1} | F_t, X_t = s] \leq 1$ to mean that for every history $F$ of the algorithm up to time $t$ with the property $X_t = s$, we have $E[X_t - X_{t+1} | F] \leq 1$.

Obviously, pointwise drift is a much stronger condition, and requiring such a strong condition in a drift theorem gives a priori a weaker theorem. However, for most applications it does not make a big difference to consider either version. Intellectually, it is arguably easier to imagine a fixed history of the algorithm, and to think about the next step in this fixed setting. Therefore, it is not uncommon in the EA community to formulate drift theorems using filtrations. However, we will also see examples (Example 2 and 12) where the weaker condition “$X_t = s$” is beneficial.

The basic drift theorems concerned with the expected runtime $E[T]$ can be formulated with either form of conditioning, and in this chapter we choose the stronger form (i.e., with weaker requirements), conditioning

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3 Mathematically speaking, it is the coarsest $\sigma$-algebra which makes all random choices of the algorithm up to time $t$ measurable.

4 This is sometimes sloppily described by $E[X_t - X_{t+1} | X_0, \ldots, X_t]$. However, note that this is not quite correct since it only conditions on the past values of $X_t$, not on the history of the algorithm. In particular, conditioning on $X_0, \ldots, X_t$ usually does not determine the current state of the algorithm (e.g., the current search point or population).

5 By abuse of notation, for brevity
on $X_t = s$. However, once the drift theorems include tail bounds, things become more subtle, and it becomes essential to condition on every possible history. Therefore, we will switch to using filtrations and pointwise drift in the last part of the chapter.

4. **Infinite Search Spaces.** We assume in this chapter that the state space $S$ is finite. This makes sense in the context of this book since in discrete optimization the search spaces, and also the state spaces of the algorithms, tend to be finite (though they may be huge). However, there are problems, especially in continuous optimization, in which infinite state spaces are more natural. Generally, all drift theorems mentioned in this chapter still go through when the state space $S \subseteq \mathbb{R}^+_0$ is infinite, but bounded. For unbounded search spaces, things become more complicated. The upper bounds on $E[T]$ in the drift theorems still hold in these cases, while the lower bounds on $E[T]$ fail in general, as we will discuss briefly after Theorem 1.

5. **Drift Versus Expected Drift.** Unfortunately, the meaning of the term “drift” is somewhat inconsistent in the literature. We have defined it as the expected change $E[X_t - X_{t+1} \mid X_t = s]$. However, some authors also use “drift” to refer to the conditional random variable $X_t - X_{t+1} \mid X_t = s$, and our definition would be the “expected drift” in their terminology. Some authors would also call the conditional expectation $E[X_t - X_{t+1} \mid \mathcal{F}_t]$ “drift”, which is itself a random variable (by the randomness in the history of the algorithm). Again, our notion of drift would be the expected drift $E_{\mathcal{F}_t} [E[X_t - X_{t+1} \mid \mathcal{F}_t]]$ in this terminology. Yet another notion uses “drift” to refer to the conditional random variable $X_t - X_{t+1} \mid \mathcal{F}_t$. Fortunately, the heterogeneous nomenclature usually does not lead to confusion, except some minor notational irritations.

3 **Elementary Introduction to Drift Analysis**

We start with an elementary introduction to drift analysis. We will discuss the three main workhorses, The Additive Drift Theorem 1, the Variable Drift Theorem 3, and the Multiplicative Drift Theorem 10. All of them give upper bounds on the expected hitting time $E[T]$, the Additive Drift Theorem also matching lower bounds.

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6 Some statements like Theorems 3 and 10 additionally require that the infimum $s_{\min} := \inf(S \setminus \{0\})$ is strictly positive.

7 See [LS16] for an example.

8 Note that the expectation of a random variable does not always give the full truth. There are cases where the value of $E[T]$ may be misleading, because $T$ is with high probability much smaller than $E[T]$, and the latter is dominated by unlikely events in which $T$ is exceptionally large. In the context of drift analysis, this particularly happens if the drift is too weak, as we discuss in Section 4, Example 18.
3.1 Additive Drift

The simplest possible drift is additive drift, i.e., $X_{t+1}$ differs from $X_t$ in expectation by an additive constant. The theorem in its modern form dates back to He and Yao [HY01, HY04], who built on work by Hajek [Haj82], and who proved the theorem using (without explicit reference) the Optional Stopping Theorem for martingales [GS01]. Here we give an elementary proof taken from [LS16], since this proof gives some insight in the differences between upper and lower bounds.

**Theorem 1** (Additive Drift Theorem [HY04]). In the situation of Section 2.1.1,

(a) if there exists $\delta > 0$ such that for all $s \in S \setminus \{0\}$ and for all $t \geq 0$,

$$\Delta_t(s) := E[X_t - X_{t+1} \mid X_t = s] \geq \delta, \quad (1)$$

then

$$E[T] \leq \frac{E[X_0]}{\delta}. \quad (2)$$

(b) if there exists $\delta > 0$ such that for all $s \in S \setminus \{0\}$ and for all $t \geq 0$,

$$\Delta_t(s) := E[X_t - X_{t+1} \mid X_t = s] \leq \delta, \quad (3)$$

then

$$E[T] \geq \frac{E[X_0]}{\delta}. \quad (4)$$

**Proof.** (a) As we are only interested in the hitting time $T$ of zero we may assume without loss of generality that $X_{T+1} = X_{T+2} = \ldots = 0$.

We may rewrite condition (1) as $E[X_{t+1} \mid X_t = s] \leq E[X_t \mid X_t = s] - \delta$. Since this holds for all $s \in S \setminus \{0\}$, and since $T > t$ if and only if $X_t > 0$, we conclude

$$E[X_{t+1} \mid T > t] \leq E[X_t \mid T > t] - \delta. \quad (5)$$

By the law of total probability we have

$$E[X_t] = \Pr[T > t] \cdot E[X_t \mid T > t] + \Pr[T \leq t] \cdot E[X_t \mid T \leq t].$$

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9They were apparently all unaware that the result had been proven even earlier by Tweedie [Twe76 Theorem 6], and a yet earlier proof in Russian has been attributed to Menshikov [MPW16 Bibliographical Notes on Section 2.6]. The Additive Drift Theorem has been proven and rediscovered many times, and it is known under various names. For example, in stability theory it is considered a special case of Dynkin’s formula [MT12 Theorem 11.3.1], or as a generalization of Foster’s criterion [B+08 Proposition 4.5]. In these contexts, drift analysis is often called Lyapunov Function Method, e.g. [MPW16 Theorem 2.6.2]. However, the hitting time is often only a side aspect in these areas.
\[ Pr[T > t] \cdot E[X_t | T > t]. \] 

Proceeding similarly for \( X_{t+1} \) we obtain

\[ E[X_{t+1}] = Pr[T > t] \cdot E[X_{t+1} | T > t] + Pr[T \leq t] \cdot \underbrace{E[X_{t+1} | T \leq t]}_{=0} \]

\[ \leq Pr[T > t] \cdot (E[X_t | T > t] - \delta) \]

\[ = E[X_t] - \delta \cdot Pr[T > t]. \quad (6) \]

Since \( T \) is a random variable that takes values in \( \mathbb{N}_0 \), we may write \( E[T] = \sum_{t=0}^{\infty} Pr[T > t] \). Thus

\[ \delta \cdot E[T] \sum_{t=0}^{\tau} \delta Pr[T > t] \leq \sum_{t=0}^{\tau} (E[X_t] - E[X_{t+1}]) = E[X_0] - E[X_{\tau+1}] \geq 0 \]

\[ \leq E[X_0], \quad (8) \]

which proves (a).

(b) Analogously to (a), Equations (5), (6), (7), and (8) hold with reversed inequalities, except for the very last step in (8). So (8) becomes:

\[ \delta \cdot E[T] \sum_{t=0}^{\tau} \delta Pr[T > t] \geq E[X_0] - E[X_{\tau+1}]. \quad (9) \]

There are only two possible cases. Either \( Pr[T > t] \), which is a non-increasing sequence, does not converge to 0. In this case, \( E[T] = \sum_{t=0}^{\infty} Pr[T > t] = \infty \), in which case (b) holds trivially. Or \( Pr[T > t] \to 0 \), and by (6) we also have

\[ E[X_{\tau+1}] = \underbrace{Pr[T > t]}_{\to 0} \cdot \underbrace{E[X_{\tau+1} | T > t]}_{\leq \max S < \infty} \to 0. \quad (10) \]

Now (b) follows from (9) and (10).

The proof also shows what can generally go wrong for infinite search spaces. The proof of (a) goes through unmodified. For (b), Inequality (9) is generally true. Moreover, it is tight if Condition (3) is tight. The problem is that \( E[X_{\tau+1}] \) may not go to zero. For example, consider the Markov chain where \( X_{t+1} \) is either 0 or 2\( X_t \), both with probability 1/2. Here \( E[T] = 2 \), but \( E[X_t - X_{t+1}] = 0 \) for all \( t \geq 0 \). In particular, Condition (2) is satisfied with \( \delta = 1 \) (or any other \( \delta > 0 \)), but the conclusion of (b) does not hold. On the other hand, for the tight choice \( \delta = 0 \), we see that we have equality in (9) since \( E[X_{\tau+1}] = E[X_0] \).

Note that if the drift in Theorem 1 is exactly \( \delta \) in each step, then the upper and lower bounds match. In this case, Theorem 1 can be seen as
an invariance theorem, which states that the expected hitting time of 0 is independent of the exact distribution of the progress, as long as the expectation of the progress (i.e., the drift) remains fixed. In particular, if $X_0$ is an integer multiple of $\delta$, this includes the deterministic case in which $X_t$ decreases in each step by exactly $\delta$, with probability 1. Thus a process of constant drift can not be accelerated (or slowed down) by redistributing the probability mass. We will resume this point in Section 3.2 when we discuss why other drift theorem are not tight in general.

We conclude the section on additive drift with an application.

Example 2 (RLS on LeadingOnes). Consider Random Local Search (RLS) on the $n$-dimensional hypercube $\{0,1\}^n$. RLS is a (1+1)-algorithm (i.e., it has population size one and generates only one offspring in each generation). The mutation operator flips exactly one bit, which is chosen uniformly at random. RLS has elitist selection, i.e., the offspring replaces the parent if and only if its fitness is at least as high as the parent’s fitness. A pseudocode description is given in Algorithm 1.

Algorithm 1: Random Local Search (RLS) maximizing a fitness function $f : \{0,1\}^n \rightarrow \mathbb{R}$.

1. Choose $x^{(0)} \in \{0,1\}^n$ uniformly at random;
2. for $t = 0, 1, 2, \ldots$ do
3.    Pick $i \in \{1, \ldots, n\}$ uniformly at random, and create $y^{(t)}$ by flipping the $i$-th bit in $x^{(t)}$;
4.    if $f(y^{(t)}) \geq f(x^{(t)})$ then
5.        $x^{(t+1)} \leftarrow y^{(t)}$;
6.    else
7.        $x^{(t+1)} \leftarrow x^{(t)}$;

We study RLS on the LeadingOnes fitness function, which returns the number of initial one-bits before the first zero bit. Formally,

$$\text{LeadingOnes}(x) = \sum_{k=1}^{n} \prod_{i=1}^{k} x_i = \max\{i \in \{1, \ldots, n\} \mid \underbrace{11\ldots1}_{i \text{ times}} \text{ is a prefix of } x\}.$$

The LeadingOnes problem is a classical benchmark problem for evolutionary algorithms, and RLS on LeadingOnes has been studied in much greater detail than we can present here, with methods and results that go far beyond drift analysis \cite{DD16, Lad05}.

Naive potential. As potential we choose in a first step $X_t := n - f(x^{(t)})$, the distance in fitness from the optimum. The state space is $\mathcal{S} = \{0, \ldots, n\}$. We need to compute the drift $\Delta_t(s) := E[X_t - X_{t+1} \mid X_t = s]$ for every $s \in \mathcal{S} \setminus \{0\}$, so we fix such an $s$. For convenience, we write $k := n - s \in$
\{0, \ldots, n-1\} for the fitness in this case. Note that \(X_t = s\) implies that the first \(k\) bits of \(x^{(t)}\) are one-bits, but the \(k+1\)-st bit is a zero-bit. Obviously, the potential changes if and only if we flip the \(k+1\)-st bit, so let us denote this event by \(E\). Since the flipped bit is chosen uniformly, we have \(\Pr[E] = 1/n\). Hence the drift is

\[
\Delta_t(s) = \Pr[E] \cdot E[X_t - X_{t+1} \mid X_t = s \text{ and } E] = \frac{1}{n} \cdot E(s). \tag{11}
\]

So it remains to bound the conditional expectation \(E(s)\). Such conditional expectations occur quite frequently when a drift is computed. Assume that \(X_t = s\) (i.e., \(f(x^{(t)}) = k = n - s\), and that \(E\) occurs. Obviously, \(E(s) \geq 1\), since we improve at least the \(k+1\)-st bit. On the other hand, we improve the fitness by at least 2 if and only if the \(k+2\)-nd bit happens to be a one-bit. Note that since the algorithm is elitist and has fitness \(f(x_t) = k\), the \(k+2\)-nd bit has had no influence on the fitness of previous search points. Therefore, by symmetry it has probability \(1/2\) to be a one-bit, and we obtain

\[
\Pr[X_t - X_{t+1} \geq 2 \mid X_t = s \text{ and } E] = \Pr[x_{k+2}^{(t)} = 1 \mid X_t = s \text{ and } E] = 1/2.
\]

Analogously, \(X_t - X_{t+1} \geq i\) if and only if the bits with indices \(k+2, \ldots, k+i\) are all one-bits, which happens with probability \(2^{-i+1}\). Since \(X_t - X_{t+1}\) is an integer non-negative random variable, we may sandwich

\[
1 \leq E(s) = \sum_{i=1}^{s} \Pr[X_t - X_{t+1} \geq i \mid X_t = s \text{ and } E] = 1 + \sum_{i=2}^{s} 2^{-i+1} - 1 = \sum_{i=1}^{\infty} 2^{-i} = 2 \tag{12}
\]

Hence, by (11),

\[
\frac{1}{n} \leq \Delta_t(k) \leq \frac{2}{n}, \tag{13}
\]

and Theorem 4 implies that

\[
\frac{n}{2} E[X_0] \leq E[T] \leq n E[X_0]. \tag{14}
\]

To estimate \(E[X_0] = n - f(x^{(0)})\), we observe that \(f(x^{(0)}) \geq i\) happens if only if the first \(i\) bits are all one-bits, which happens with probability \(2^{-i}\). Hence, a similar calculation as before shows

\[
E[f(x^{(0)})] = \sum_{i=1}^{n} \Pr[f(x^{(0)}) \geq i] = \sum_{i=1}^{n} 2^{-i} = 1 - 2^{-n} \in [0, 1], \tag{15}
\]

\footnote{Note that such an argument would not be true if we would condition on one particular history of the algorithm, cf. the discussion on filtrations in Section 2.1.1.}
and thus \( n - 1 \leq E[X_0] \leq n \). Therefore, by (14) we get \( E[T] = \Theta(n^2) \).

**Translated potential.** The analysis so far gives the asymptotics \( E[T] \), but it is not tight up to constant factors. The problem is, as (12) shows, that the inequality \( E(k) \geq 1 \) is rather coarse except for the few exceptional cases where \( k \) is almost \( n \). In fact, in the border case \( k = n - 1 \) we have equality, \( E(k) = 1 \). Hence, we do not have a perfectly constant drift, which is a reason for the discrepancy between upper and lower bound. Such border effects can often be remedied by translating the potential function. In this case, we consider

\[
Y_t := \begin{cases} 
X_t + 1, & \text{if } X_t \geq 1; \\
0, & \text{otherwise.} 
\end{cases}
\]  

(16)

The effect is that the drift increases when there is a substantial chance to reach 0 in the next step. In our case, we get an additional term for \( i = n - k + 1 \) in (12), which equals the term for \( i = n - k \). Intuitively, the term for \( i = n - k \) counts double since in this case the potential drops from 2 to 0, rather than from 1 to 0. Consequently, we get for the potential \( Y_t = s + 1 \), which corresponds as before to fitness \( f(x^{(0)}) = k = n - s \):

\[
E[Y_t - Y_{t+1} \mid Y_t = s + 1 \text{ and } \mathcal{E}_k] = \sum_{i=1}^{n-k} \Pr[Y_t - Y_{t+1} \geq i \mid Y_t = s + 1 \text{ and } \mathcal{E}_k] 
= 1 + \sum_{i=2}^{n-k} 2^{-i+1} + 2^{n-k+1} = 2.
\]  

(17)

Hence, the drift with respect to \( Y_t \) is exactly \( 2/n \), and Theorem 7 gives a tight result:

\[
E[T] = \frac{n}{2} E[Y_0].
\]  

(18)

From (16) it is easy to compute \( E[Y_0] \) exactly as

\[
E[Y_0] = n - E[f(x^{(0)})] + 1 \cdot \Pr[Y_0 > 0] \overset{15}{=} n - (1 - 2^{-n}) + 1 - 2^{-n} = n.
\]

Together with (18), the Additive Drift Theorem 7 now implies \( E[T] = n^2/2 \).

The previous example illustrates how important it is for Theorem 1 that the drift be as uniform as possible, to get matching upper and lower bounds. The example also shows that rescaling of the potential function may be a way to smoothen out inhomogeneities. Following this approach systematically leads to the variable drift theorem that we will discuss in the next section.
3.2 Variable Drift

The Additive Drift Theorem is useful because it is tight, but it requires us to find a potential function that has constant drift. Is this even always possible? The perhaps surprising answer is Yes. By choosing the canonical potential \( X_t := E[T | F_t] - t \), where \( F_t \) is the history of the algorithm up to time \( t \), we trivially get a drift of exactly 1 \([HY04, DJW12a]\). This looks like formal nonsense, since it seemingly only helps finding the runtime if we already know the runtime. However, as outlined in the introduction, finding the right potential function requires often more ingenuity than actually computing the drift, and the canonical potential gives us a natural candidate if we have any guess on what the runtime might be. The guess may come from heuristic considerations or from simulations. In practice, a more severe issue is that the history \( F_t \) (or even the current state) is too complicated to work with, and needs to be replaced by a simpler quantity that still resemble the canonical potential.

Even if we start with a variable in the “wrong” scaling, Mitavskiy, Rowe, and Cannings \([MRC09]\), and Johannsen in his PhD thesis \([Joh10]\) developed a theorem which automatically rescales the drift in the right way.

**Theorem 3 (Variable Drift Theorem \([Joh10, RS12]\)).** In the situation of Section 2.1.1, if there is an increasing function \( h : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that for all \( s \in S \setminus \{0\} \) and all \( t \geq 0 \),

\[
\Delta_t(s) \geq h(s),
\]

then

\[
E[T] \leq \frac{s_{\min}}{h(s_{\min})} + E\left[ \int_{s_{\min}}^{X_0} \frac{1}{h(\sigma)} d\sigma \right],
\]

where the expectation on the latter term is over the random choice of \( X_0 \).

We remark that the condition that \( h \) be increasing is usually satisfied, since progress typically becomes harder as the algorithm approaches an optimum. However, variants of the theorem for non-decreasing drift function do exist \([DHK12, FK13]\).

**Proof of Theorem 3 adapted from \([Joh10]\).** The main insight of the proof lies in an appropriate rescaling of \( X_t \) by the function

\[
g(s) := \begin{cases} 
\frac{s_{\min}}{h(s_{\min})} + \int_{s_{\min}}^{s} \frac{1}{h(\sigma)} d\sigma, & s \geq s_{\min}, \\
\frac{s}{h(s_{\min})}, & 0 \leq s \leq s_{\min}.
\end{cases}
\]

\(^{11}\)The situation resembles induction, where finding the right induction hypothesis is sometimes much harder than actually proving the inductive step.

\(^{12}\)Some formulations in the literature require \( h \) to be integrable. However, since we assume \( S \) to be finite, the interval \([s_{\min}, X_0]\) is a compact interval, on which every monotone function is integrable.
The integral is well-defined since $h$ is increasing. Note that $g$ is strictly increasing. We claim that for all $s \in S \setminus \{0\}$ and all $r \geq 0$,

$$g(s) - g(r) \geq \frac{s - r}{h(s)}. \quad (22)$$

To prove the claim, we distinguish three cases. First assume $s \geq r \geq s_{\text{min}}$. Then

$$g(s) - g(r) = \int_r^s \frac{1}{h(\sigma)} d\sigma \geq \int_r^s \frac{1}{h(s)} d\sigma = \frac{s - r}{h(s)}. \quad (23)$$

Similarly, if $r \geq s \geq s_{\text{min}}$, then

$$g(r) - g(s) = \int_s^r \frac{1}{h(\sigma)} d\sigma \leq \int_s^r \frac{1}{h(s)} d\sigma = \frac{r - s}{h(s)}.$$  

and multiplication with $-1$ yields the claim. The only remaining case is $s \geq s_{\text{min}} > r \geq 0$ (since we assumed $s \in S \setminus \{0\}$), and in this case,

$$g(s) - g(r) = \frac{s_{\text{min}}}{h(s_{\text{min}})} + \int_{s_{\text{min}}}^s \frac{1}{h(\sigma)} d\sigma - \frac{r}{h(s_{\text{min}})} \geq \frac{s_{\text{min}} - r}{h(s_{\text{min}})} + \frac{s - s_{\text{min}}}{h(s)} \geq \frac{s - r}{h(s)}. \quad (25)$$

Now let us consider the rescaled random variable $Y_t := g(X_t)$. This random variable takes values of the form $g(s)$, where $s \in S$. For all $s \in S \setminus \{0\}$,

$$E[Y_t - Y_{t+1} \mid Y_t = g(s)] = E[g(X_t) - g(X_{t+1}) \mid g(X_t) = g(s)] \geq \frac{\Delta_t(s)}{h(s)} \geq 1. \quad (26)$$

Hence $Y_t$ has at least a constant drift. The theorem follows by applying the Additive Drift Theorem \[ to $Y_t$.

**Example 4** (Coupon Collector, RLS on OneMax). The most classical example for variable drift is the Coupon Collector Process (CCP): there are $n$ types of coupons, and a collector wants to have at least one coupon of each type. However, the coupons are sold in opaque wrappings, so she cannot see the type of a coupon before buying it. If each type occurs with the same frequency $1/n$, how many coupons does she need to buy before she has every type at least once?

The CCP and its variants appear in various contexts within the study of EAs. The most basic example is the runtime of RLS (Algorithm 1 on page 8) for maximising the OneMax fitness function, which counts the number of one-bits in a bitstring. Formally, for $x \in \{0, 1\}^n$,

$$\text{OneMax}(x) = \sum_{i=1}^n x_i.$$  

(27)
The one-bits correspond to the coupons in the CCP that the collector has already obtained. Since RLS flips in each round exactly one bit, and a one-bit stays a one-bit forever, a round of RLS corresponds exactly to the purchase of a coupon. Thus the number of rounds of RLS on OneMax is equivalent to the number of purchases in the CCP.

To analyse the CCP, we let $X_t$ be the number of missing coupons after $t$ purchases, and as usual we denote by $T$ the hitting time of 0. Then for $X_t = s$ the probability to obtain a new type with the next purchase is $s/n$. In this case $X_t$ decreases by one, so $X_t$ has a drift of $\Delta t(s) = s/n$. The minimal positive value of $X_t$ is $s_{\text{min}} = 1$. Hence, the Variable Drift Theorem with function $h(s) = s/n$ gives the upper bound

$$E[T] \leq \frac{1}{h(1)} + E\left[\int_1^{X_0} \frac{n}{\sigma} d\sigma\right] = n(1 + E[\ln(X_0)]) \leq n \ln n + n. \quad (28)$$

The expected runtime is indeed $E[T] = n \ln n + \Theta(n)$, both for CCP [MR95] and for RLS on OneMax [DD16]. We will discuss in the next section when we can expect the bounds from the Variable Drift Theorem to be tight, and see situations in which they are rather inaccurate. Before that, we give two other, less trivial examples.

**Example 5** ((1 + $\lambda$) EA on OneMax). In 2017, Gießen and Witt [GW17b] analysed the (1 + $\lambda$) EA (Algorithm 2) for minimising the OneMax function, cf. Equation (27).

**Algorithm 2:** The (1 + $\lambda$) EA with offspring population size $\lambda$ and mutation rate $c/n$, minimising a fitness function $f : \{0, 1\}^n \to \mathbb{R}$.

1. Choose $x^{(0)} \in \{0, 1\}^n$ uniformly at random;
2. for $t = 0, 1, 2, \ldots$ do
   3. for $i = 1, \ldots, \lambda$ do
      4. Create $y^{(t,i)}$ by flipping each bit of $x^{(t)}$ independently with probability $c/n$;
      5. $y^{(t)} \leftarrow \text{argmin}\{f(y^{(t,i)})\}$ (breaking ties randomly);
      6. if $f(y^{(t)}) \leq f(x^{(t)})$ then
         7. $x^{(t+1)} \leftarrow y^{(t)}$;
      8. else
         9. $x^{(t+1)} \leftarrow x^{(t)}$;

The potential was identical with the fitness, $X_t = \text{OneMax}(x^{(t)})$. To bound the drift $\Delta_t(s)$, the authors used order statistics of the binomial distribution except for the initial conditions: for the CCP, the collector usually starts with no coupons, while RLS starts with a random bitstring.
They could show that $\Delta_t(s) \geq h(s)$, where

$$h(s) := \begin{cases} 
(1 - o(1)) \frac{\ln \lambda}{\ln n} & \text{if } s \geq \frac{n}{\ln \lambda}, \\
(1/2 - o(1)) e^{-c} \frac{1}{\ln n} & \text{if } s \geq \frac{n}{\ln \lambda}, \\
(1 - o(1)) e^{-c} \min\{c, 1\}/2 & \text{if } s \geq \frac{n}{\ln \lambda}, \\
(1 - o(1)) e^{-c} \frac{\lambda^s}{n} & \text{if } s < \frac{n}{\ln \lambda}, \\
(1 - o(1)) e^{-c} \lambda^s/2 & \text{if } s \geq \frac{n}{\ln \lambda}. 
\end{cases}$$

(29)

Obviously, computing the drift is non-trivial, and the major contribution of the paper. However, note that despite the complexity of the formula, once we know it we can easily obtain a runtime bound by the variable drift theorem:

$$E[T] \leq \frac{1}{h(1)} + E \left[ \int_1^{X_{\text{max}}} \frac{1}{h(\sigma)} d\sigma \right].$$

(30)

The integral can now be computed by splitting it into six ranges, and evaluating it with elementary calculus. Actually, $h(\sigma)$ is constant for all ranges except for the last one, which gives one of the leading terms:

$$\int_1^{n/(\lambda \sqrt{\ln n})} (1 + o(1)) e^c n \ln \frac{n/(\lambda \sqrt{\ln n})}{c\lambda} d\sigma = (1 + o(1)) e^c n \ln \frac{n/(\lambda \sqrt{\ln n})}{c\lambda}. \quad (31)$$

Proceeding like this for all six ranges, the authors obtain the final result

$$E[T] \leq (1 + o(1)) \left( \frac{e^c}{c} \cdot \frac{n \ln n}{\lambda} + \frac{1}{2} \cdot \frac{n \ln \ln \lambda}{\ln \lambda} \right).$$

(32)

The authors also prove a matching lower bound by the techniques discussed in Section 4.1.

Example 6 (Island Model on OneMax). Doerr, Fischbeck, Frahnow, Friedrich, Kötzing, and Schirneck [DFF +17] studied island models in various topologies. For the complete graph as migration topology, the algorithm consists of $\lambda$ independent $(1 + 1)$ EA’s, except that every $\tau$ rounds all individuals are updated by the current best search point, see Algorithm 3.

For the OneMax function, the most interesting phase turns out to be the phase when the current best search point has fitness in some interval $[s_0, s_1]$, where $s_0 = \min\{n, n \ln \lambda/(2\tau)\}$ and $s_1 = n/(\tau \ln \lambda)$. The authors define $X_t$ to be the fitness after $t$ migrations, i.e., $X_t = \text{OneMax}(x^{(\tau,t)})$ holds for every $1 \leq i \leq \lambda$. To identify the end of the phase, we truncate $X_t$, i.e., we define $X_t := 0$ if OneMax$(x^{(\tau,t)}) < s_0$. Note that the minimal non-zero value of $X_t$ is thus $s_{\min} = s_0$. The drift of $X_t$ for all $t \geq 0$ and all $s \in [s_0, s_1]$ turns out to be

$$\Delta_t(s) \geq h(s) := \frac{c\ln \lambda}{\ln(n \ln \lambda/(\tau s))}. \quad (33)$$

$^{14}$for the case $\lambda = \omega(1)$. The other case is similar.

$^{15}$for some parameter regimes
Algorithm 3: Island model on $\lambda$ islands and migration interval $\tau$ for minimising $f : \{0, 1\}^n \rightarrow \mathbb{R}$.

1. Choose $x^{(0,1)}, \ldots, x^{(0,\lambda)} \in \{0, 1\}^n$ uniformly at random;
2. for $t = 0, 1, 2, \ldots$ do
   3. for $i = 1, \ldots, \lambda$ do
      4. Create $y^{(t,i)}$ by flipping each bit of $x^{(t,i)}$ independently with probability $1/n$;
      5. if $f(y^{(t,i)}) \leq f(x^{(t,i)})$ then
         6. $x^{(t+1,i)} \leftarrow y^{(t,i)}$;
      7. else
         8. $x^{(t+1,i)} \leftarrow x^{(t,i)}$;
   9. if $(t + 1 \mod \tau) = 0$ then
      10. for $i = 1, \ldots, \lambda$ do
          11. $y \leftarrow \arg \min \{f(y^{(t+1,i)})\}$ (breaking ties randomly);
          12. $x^{(t+1,i)} \leftarrow y$;

for some constant $c > 0$. Note that the function $h(s)$ is increasing. Thus, by the Variable Drift Theorem \[3\] the expected number of migrations $T_0$ before a fitness of less than $d_{0}$ is achieved is bounded by

$$E[T_0] \leq \frac{s_0}{h(s_0)} + \frac{1}{c \ln \lambda} \int_{s_0}^{s_1} \ln \left(\frac{n \ln \lambda}{\sigma \tau} \right) d\sigma,$$  \hspace{1cm} (34)

where we used $X_0 \leq d_1$. The latter integral can now be evaluated by elementary analysis, and yields

$$\int_{s_0}^{s_1} \ln \left(\frac{n \ln \lambda}{\sigma \tau} \right) d\sigma = \frac{\tau}{n \ln \lambda} \left[\sigma (1 - \ln \sigma)\right]_{\tau s_0/(n \ln \lambda)}^{\tau s_1/(n \ln \lambda)},$$ \hspace{1cm} (35)

from which the authors can derive their runtime bounds.

Tightness of the Variable Drift Theorem. In general, the bound in the Variable Drift Theorem \[3\] does not need to be tight, even if we assume that $h(s)$ is a tight lower bound for the drift (i.e., if (19) is an equality). However, in many situations the bound is tight, especially if the potential $X_t$ does not jump around too much. Let us unravel the proof of Theorem \[3\] to understand this phenomenon better.

We first note that the proof is a reduction to the Additive Drift Theorem, which is tight (cf. the discussion after Theorem \[1\]). So the only possible problem is the estimate (26) on the drift. This estimate may not be tight if (22), the inequality $g(s) - g(r) \geq \frac{s - r}{h(s)}$, is too coarse. Note that for estimating the drift, we use (22) specifically for $s = X_t$ and $r = X_{t+1}$. These
are not arbitrary values; for example, for RLS on OneMax, they differ by at most one. We have proved (22) by case distinction, so let us inspect one of the cases for illustration. For convenience, we restate the argument for $s > r > s_{\text{min}}$:

$$g(s) - g(r) = \int_{r}^{s} \frac{1}{h(\sigma)} d\sigma \geq \int_{r}^{s} \frac{1}{h(s)} d\sigma = \frac{s - r}{h(s)}. \quad (23)$$

The crucial step is to use $1/h(\sigma) \geq 1/h(s)$ for the range $r \leq \sigma \leq s$. In general, this may be a bad estimate. However, if $s = X_t$ and $r = X_{t+1}$ are close to each other then $\sigma$ runs through a small range, and $1/h(\sigma)$ may not vary too much. For example, $s$ and $r$ differ at most by one for RLS on OneMax, and the function $1/h(\sigma) = n/\sigma$ does not vary much in such a small range, especially if $r$ and $s$ are large. We will see in Section 4.1 that large jumps are still tolerable if they occur with sufficiently small probability. The following artificial example from [GW17a] illustrates how large jumps can lead to bad upper bounds.

**Example 7 (RLS with shortcuts).** Consider a $(1+1)$-algorithm that in each step creates the optimum with probability $1/n$, and with probability $1-1/n$ it does an RLS step as in Algorithm 1. For minimising OneMax, we may naively try the fitness as potential, $X_t := \text{OneMax}(x(t))$. For $X_t = s > 0$, there is a probability of $1/n$ to jump directly to the optimum, thus decreasing the potential by $s$. On the other hand, there is a probability of $(1-1/n)/n$ to decrease the potential by 1 with a normal RLS step. Together, the drift is

$$\Delta_t(s) = h(s) := \frac{1}{n} \cdot s + \left(1 - \frac{1}{n}\right) s = \frac{2s}{n} - \frac{s}{n^2} = (1 \pm o(1)) \frac{2s}{n}. \quad (36)$$

Thus, the Variable Drift Theorem 3 yields

$$E[T] \leq \frac{1}{h(1)} + E \left[\int_{1}^{X_0} (1 \pm o(1)) \frac{n}{2\sigma} d\sigma \right] = \Theta(n \log n). \quad (37)$$

However, since in each step we have probability at least $1/n$ to jump directly to the optimum, the expected runtime is at most $E[T] \leq n$, so (37) is not tight. The problem can be understood by inspecting the transformed variable $Y_t := g(X_t)$ from the proof of the Variable Drift Theorem, Equation (21). For simplicity we ignoring the factor $(1 + o(1))$ in (36), and obtain

$$Y_t := \begin{cases} \frac{n}{2} (1 + \ln X_t) & \text{if } X_t \geq 1, \\ 0 & \text{if } X_t = 0. \end{cases} \quad (38)$$

Computing the drift of $Y_t$ directly, we obtain for $X_t = s$, i.e, for $Y_t = \frac{n}{2} (1 + \ln s)$.

$$E[Y_t - Y_{t+1} \mid X_t = s] = \frac{1}{n} \cdot \frac{n}{2} (1 + \ln s) + \left(1 - \frac{1}{n}\right) \frac{s}{n} \cdot \frac{n}{2} (\ln s - \ln(s - 1))$$
Thus, we do not have constant drift in the scaled potential. However, in the proof of the Variable Drift Theorem \(3\), we bound the drift by \(1\) (see \(26\)), which is the reason for the additional \(\log n\) factor.

Fortunately, it is quite common that there are no large jumps of fitness values. Mutation-based evolutionary algorithms tend to make small steps, and other nature-based search heuristics like ant-colony optimisation or estimation of distribution algorithms tend to make rather small updates on reasonable functions. However, note that this is not necessarily true for crossover operations. Also, depending on the fitness function a small (genotypical) change may cause a large (phenotypical) jump in the fitness, as the next example shows.

**Example 8 (RLS on BinVal).** We consider RLS (Algorithm 1 on page 8) for minimising the BinVal function given by

\[
\text{BinVal}(x) = \sum_{i=1}^{n} 2^i x_i. \tag{40}
\]

If we choose the potential \(X_t := \text{BinVal}(x(t))\) identical to the fitness, then we observe [DJW12a] that each one-bit has probability \(1/n\) to be flipped. If the \(i\)-th bit is flipped from one to zero, this reduces the potential by \(2^i\). Hence, at search point \(x\) with potential \(s := \text{BinVal}(x)\) the drift is

\[
E[X_t - X_{t+1} \mid x(t) = x] = \sum_{1 \leq i \leq n, \ x_i(t) = 1} \frac{1}{n} \cdot 2^i = \frac{1}{n} \sum_{i=1}^{n} 2^i x_i = \frac{s}{n}. \tag{41}
\]

In particular, since the latter term only depends on \(s\), we can write

\[
E[X_t - X_{t+1} \mid X_t = s] = \frac{s}{n}. \tag{42}
\]

Therefore we are in the situation to apply the Variable Drift Theorem \(3\) with \(h(s) = s/n\) and \(s_{\text{min}} = 1\), and obtain

\[
E[T] \leq \frac{1}{1/n} + E \left[ \int_1^{X_0} \frac{n}{\sigma} d\sigma \right] = n + n \cdot E[\ln X_0] = \Theta(n^2), \tag{43}
\]

where the last equality follows since \(X_0 \leq 2^{n+1}\), and since with probability at least \(1/2\) the first bit in \(X_0\) is a one-bit, which implies \(E[X_0] \geq 2^{n-1}\).

However, the bound \(43\) is far from tight. In fact, if we use the OneMax potential \(\text{OneMax}(x) := \sum_{i=1}^{n} x_i\), then the drift with respect to \(\text{OneMax}\)
is still $\Delta_{t}^{\text{OneMax}}(s) = s/n$, which leads to a runtime bound of $E[T] \leq n + n \cdot E[\text{OneMax}(x^{(0)})] \leq n \ln n + n^{16}$.

The reason why (43) is not tight is that there may be some very large jumps in the potential (cf. the discussion before this example). For example, consider the situation when only a single one-bit is left. By symmetry, this one-bit is at a random position. In particular, with probability at least $1/2$, the bit is in the latter half, and thus $X_{t} \geq 2^{n/2}$. Therefore, in Equation (25) we estimate $h(\sigma) \leq h(s)$ for $\sigma$ which ranges at least between $s_{\min} = 1$ and $2^{n/2}$. Thus the estimate is off by an exponential factor. Consequently, the rescaled potential $Y_{t} = g(X_{t}) = n(1 + \ln X_{t})$ does not have constant drift.

While the drift is always at least 1 by Equation (26), if there is only a single one-bit left in the latter half of the string, the rescaled potential decreases with probability $1/n$ from $Y_{t} \geq n(1 + \ln 2^{n/2}) = \Omega(n^{2})$ to 0. Hence, the drift of $Y_{t}$ in this situation is $1/n \cdot \Omega(n^{2}) = \Omega(n)$, causing the runtime bound to be almost a factor $n$ too large.

When Rescaling Beats the Variable Drift Theorem

We have seen an examples which illustrates why the Variable Drift Theorem does not always give tight results. Unfortunately, a common reason is that the potential does not represent the progress well that the algorithm has made, in which case a truly new insight is needed. However, sometimes the problem can be solved by directly considering the rescaled potential. We illustrate this by an artificial example taken from [LS16].

**Example 9 (Random Decline).** Let $a > 0$ be a constant, let $n \in \mathbb{N}^{+}$, and consider the following Markov chain on $S = \{0, \ldots, N\}$, where $N$ is a sufficiently large integer. We start with $X_{0} = n$, and for each $t \geq 0$ we draw $X_{t+1}$ uniformly at random from $\{0, 1, 2, \ldots, \min\{\lfloor aX_{t} \rfloor, N\}\}$.

If $a < 2$, then for $S \in S \setminus \{0\}$ and all $t \geq 0$ we have a drift of

$$\Delta_{t}(s) \geq s - \frac{a}{2}s = \frac{2 - a}{2} \cdot s. \quad (44)$$

Therefore, by the Variable Drift Theorem $E[T] = O(\log n)$. However, the theorem does not make any statement for $a \geq 2$.

However, let us inspect the rescaled potential $Y_{t} := 1 + \ln(X_{t})$. Then for every $s \in S \setminus \{0\}$ that is smaller than $N/a$,

$$E[Y_{t} - Y_{t+1} \mid Y_{t} = 1 + \ln s] = 1 + \ln(s) - \frac{1}{(as + 1)} \sum_{k=1}^{[as]} (1 + \ln k) \quad (45)$$

16 Alternatively, we could observe that RLS behaves exactly the same on BinVal and on OneMax, so the runtimes are the same.

17 Compared to $n$. For this exposition we will assume that $N$ is so large that the process never hits the right border.

18 Worse: The statement could be applied for non-constant $a$ like $a = 2(1 - 1/n)$, and would lead to the misleading bound $E[T] = O(n \log n)$.

19 The full calculation including error terms can be found in [LS16].
\[ \approx \ln(s) - \frac{1}{as} \left( \int_1^{as} \ln \sigma \, d\sigma \right) \]
\[ = \ln(x) - \frac{1}{as} [\sigma \ln(\sigma) - \sigma]_{\sigma=1}^{as} \]
\[ \approx \ln(s) - (\ln(as) - 1) = 1 - \ln a. \quad (45) \]

Thus we see that if \( a < e = 2.71 \ldots \) is a constant, then the drift of \( Y_t \) is also constant. Hence, by the Additive Drift Theorem \( \text{[1]} \) we get \( E[T] = O(E[Y_0]) = O(\ln n) \). So the analysis of the rescaled random variable applies to a wider range than the Variable Drift Theorem \( \text{[3]} \). In fact, the condition \( a < e \) is tight for logarithmic runtime, since for \( a \geq e \) the expected runtime is \( \omega(\ln n) \) \([\text{LS16}]\).

### 3.3 Multiplicative Drift

A very important special case of variable drift is multiplicative drift, where the drift is proportional to the potential. Introduced in \([\text{DJW10b, DJW12a, DG13}]\), it has become the most widely used variant of drift analysis in evolutionary algorithms. In fact, all the examples \( \text{[1, 7, 8, and 9]} \) had multiplicative drift. In particular, Examples \( \text{[7, 8, and 9]} \) show that the same limitations as for variable drift apply.

**Theorem 10** (Multiplicative Drift \([\text{DJW12a}]\), special case of Theorem \( \text{[3]} \)). In the situation of Section \( \text{[2.1.1]} \), suppose there exists \( \delta > 0 \) such that for all \( s \in S \setminus \{0\} \) and all \( t \geq 0 \) the drift is

\[ \Delta_t(s) \geq \delta s. \quad (46) \]

Then

\[ E[T] \leq \frac{1 + E[\ln(X_0/s_{min})]}{\delta}. \quad (47) \]

We conclude this section by giving some applications of the multiplicative drift theorem.

**Example 11** ((1 + 1) EA on Linear Functions). One of the cornerstones in the theory of evolutionary algorithms is the analysis of linear pseudo-boolean functions \( f : \{0, 1\}^n \to \mathbb{R} \), i.e., functions of the form \( f(x) = \sum_{i=1}^{n} w_i x_i \), where the \( w_i \) are constants. To avoid trivialities, we assume that the weights are non-zero, and by symmetry of the search space we may assume that they are non-negative and sorted, \( w_1 \geq w_2 \geq \ldots \geq w_n > 0 \). We have already seen two examples of such functions: \text{OneMax} in Example \( \text{[4]} \) and \text{BinVal} in Example \( \text{[8]} \).

To analyse how the (1 + 1) EA with mutation rate \( c = 1/n \) (Algorithm \( \text{[2]} \) with offspring population size \( \lambda = 1 \)) minimises a linear function, a naive
approach is to use the fitness as potential, $X_t := f(x^{(t)})$. Similar as for RLS on BinVal, this yields a multiplicative drift of at least

$$\Delta_t(s) \geq \Omega(s/n),$$

(48)

since the $(1+1)$ EA has at least a constant probability to perform an RLS step, i.e., to flip exactly one bit. Therefore, the Multiplicative Drift Theorem gives the bound

$$E[T] \leq O \left( \frac{1 + E[\ln(X_0/w_n)]}{\delta} \right).$$

(49)

For OneMax-like functions where all weights are similar, this bound is $O(n \ln n)$, which turns out to be tight. However, for other linear function like BinVal, the bound is not tight, for the same reason as for RLS on BinVal (Example 8). Rather, the expected runtime is $\Theta(n \ln n)$, as was first shown by Droste, Jansen, and Wegener in [DJW02].

For the OneMax potential $OM_t := \text{OneMax}(x^{(t)})$ the situation is rather interesting. For functions like BinVal, there are search points (e.g., the search point $(1,0,\ldots,0)$ where only the highest-valued bit is not optimised yet) in which the drift is negative, i.e., $E[OM_t - OM_{t+1} | x^{(t)} = (1,0,\ldots,0)] < 0$. Nevertheless, Jägersküpper showed [Jäg08] by a coupling argument that bits of higher weight are more likely to be optimised, so that we still have a multiplicative drift [DJW10a] for all $t \geq 0$ and all $s \in \{1,\ldots,n\}$,

$$\Delta_t(s) = E[OM_t - OM_{t+1} | OM_t = s] = \Omega(s/n),$$

(50)

from which a runtime bound $E[T] = O(n \ln n)$ follows. So this is one of the case where it is beneficial to avoid filtrations and pointwise drift, see also the paragraph Drift Versus Expected Drift in Section 2.1.2.

The results can be tightened if one considers more carefully crafted potentials. Doerr, Johannsen, and Winzen showed [DJW10b], building on ideas from [HY04], that the drift function $\varphi(x) := \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{5}{4}x_i + \sum_{i=\lfloor n/2 \rfloor + 1}^{n} x_i$ even has pointwise multiplicative drift, i.e., for all $t \geq 0$ and all search points $x \in \{0,1\}^n$,

$$E[\varphi(x^{(t)}) - \varphi(x^{(t+1)}) | x^{(t)} = x] = \Omega(\varphi(x)/n).$$

(51)

This yields again the runtime bound $E[T] = O(n \ln n)$. Pointwise multiplicative drift giving similar runtime bounds can also be achieved by other potential functions [DJW12a].

Similar techniques can also be used to show that the $(1+1)$ EA has still runtime $\Theta(n \ln n)$ on every linear function if the mutation rate is $c/n$ for an arbitrary constant $c$ [DG13, WH13, LS16]. However, this requires a considerably more complicated potential function which must necessarily depend on the mutation rate [DJW12b].
Example 12 (Minimum Spanning Trees). Consider the following minimum spanning tree (MST) problem proposed in [NW07]. Let $G = (V, E)$ be a connected graph with $n$ vertices, $m$ edges $e_1, \ldots, e_m$, and positive integer edge weights $w_1, \ldots, w_m$. We denote by $w_{\text{max}} := \max_i w_i$ the maximum weight. A bit string $x \in \{0, 1\}^m$ represents a subgraph of $G$ with vertex set $V$, where the edge $e_i$ is present if and only if $x_i = 1$. The fitness of a bit string is given by $f(x) = \sum_{i=1}^m w_i x_i + p(x)$, where $p(x)$ is a punishment term for non-trees that ensures to find a spanning tree quickly, and to stay within the set of spanning trees afterwards.

We consider the $(1 + 1)$ EA on this problem. In [NW07] it was shown that the algorithm quickly finds a spanning tree, so we assume for simplicity that the initial search point $x^{(0)}$ represents such a tree. We consider the potential function $X_t := \sum_{i=1}^n w_i x_i^{(t)} - w_{\text{opt}}$, where $w_{\text{opt}}$ is the weight of a minimum spanning tree. Then relying on results from [NW07], in [DJW12a] it is shown that the potential function has a multiplicative drift of

$$\Delta_t(s) = E[X_t - X_{t+1} | X_t = s] \geq \frac{s}{em^2}.$$  \hspace{1cm} (52)

Hence, by the Multiplicative Drift Theorem 10 the expected runtime (starting from a spanning tree) is at most

$$E[T] \leq em^2(1 + \ln(mw_{\text{max}})),$$  \hspace{1cm} (53)

since the minimum potential of a non-optimal search point is at least $s_{\text{min}} \geq 1$, and since $mw_{\text{max}}$ is an upper bound on $X_0$. It is an open question whether (53) is tight, since the best lower bound is $\Omega(m^2 \ln m)$ [NW07], which is a tight bound for RLS [DJW12a].

There are numerous other applications of the multiplicative drift theorem, including evolutionary algorithms on other problems [DJI0, DJW12a, DK15, GKL16], ant-colony optimisation [FKKS16], island models [LW17], genetic programming [DKL17], and estimation of distribution algorithms [FKKS17].

4 Advanced Drift Theorems

In this section we will review the most important developments in drift analysis in the last years, in particular lower and tail bounds, weak drift, negative drift, and population drift. Note that other than in the previous section, many advanced theorems make assumptions on the pointwise drift, cf. Section 2.1.1.

\[^{20}\text{especially on tail bounds}\]
4.1 Lower Bounds

As discussed in Section 3.2, the Variable Drift Theorem and the Multiplicative Drift Theorem only have a chance to give tight results if we have some restriction on the probability of making large jumps. From the earlier discussion on pages 15ff, it is clear that we get a matching lower bound for the Variable Drift Theorem if we apply the estimates (23), (24), and (25) only in tight cases. In particular, this is the case if $h(X_{t+1})/h(X_t)$ is always close to 1. Following this idea, we get the following lower bound.

**Theorem 13** (Variable Drift Theorem, Lower Bound 1). In the situation of Section 2.1.1, assume there is an increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $c \geq 1$ such that $s \in S \setminus \{0\}$ and all $t \geq 0$,

1. 
   \[ \Delta_t(s) \leq h(s), \]  
   \[ (54) \]

2. 
   \[ \frac{1}{c} \leq \frac{h(\max\{X_{t+1}, s_{\min}\})}{h(X_t)} \leq c. \]  
   \[ (55) \]

Then

\[ E[T] \geq \frac{1}{c} \cdot \left( \frac{s_{\min}}{h(s_{\min})} + E \left[ \int_{s_{\min}}^{X_0} \frac{1}{h(\sigma)} d\sigma \right] \right), \]  
\[ (56) \]

where the expectation on the latter term is over the random choice of $X_0$.

Note that the theorem gives a direct comparison between upper and lower bound: it says that they differ at most by a factor $c$. Despite its arguably natural form, it seems that the lower bound has never been formulated in this version in the literature, perhaps because it usually does not give tight constants. For example, consider RLS on OneMax as in Example 3. There $X_t$ is given by the fitness, and $h(s) = s/n$. The largest jump occurs when $X_t$ decreases from 2 to 1, in which case $h(X_{t+1})/h(X_t) = 1/2$. Thus the lower bound is a factor 2 from the upper bound.

Doerr, Fouz, and Witt [DFW11] have given a variant which usually gives a tighter lower bound. In fact, it gives a matching lower bounds in many applications. Note, however, that the theorem has the rather strong condition that the sequence $X_t$ is non-increasing.

**Theorem 14** (Variable Drift Theorem, Lower Bound 2 [DFW11]). In the situation of Section 2.1.1, if there are two functions $\xi, h : \mathbb{R}_0^+ \to \mathbb{R}^+$ such that $\xi$ is monotone increasing, and such that for all $s \in S \setminus \{0\}$ and for all $t \geq 0$,

\[ 21 \text{ though Feldmann and Kötzing [FK13] give bounds following the same ideas.} \]
\[ 22 \text{ But see the discussion after Theorem 15 below.} \]
1. \[ X_{t+1} \leq X_t. \quad (57) \]

2. \[ X_{t+1} \geq \xi(X_t). \quad (58) \]

3. \[ E[X_t - X_{t+1} \mid \mathcal{F}_t, X_t = s] \leq h(\xi(s)). \quad (59) \]

Then
\[ E[T] \geq \frac{s_{\min}}{h(s_{\min})} + E \left[ \int_{s_{\min}}^{X_0} \frac{1}{h(\sigma)} \, d\sigma \right], \quad (60) \]

where the expectation on the latter term is over the random choice of \( X_0 \).

To apply Theorem 14, one should first choose \( \xi \) such that Condition 2 is satisfied, and afterwards choose \( h \) in such a way that the composition \( h \circ \xi \) is the drift\(^{23}\), cf. Example 16 below.

We remark that Gießen and Witt [GW17a] have developed a version in which the deterministic Condition 2 is replaced by a probabilistic condition. The exact formulation is rather technical. However, the theorem simplifies for multiplicative drift [Wit13]. We give here the version from [LW13], which assumes bounds on the probability that \( X_t \) drops by more than a multiplicative factor. A version in which an additive bound on \( |X_t - X_{t+1}| \) is assumed can be found in [DKLL17].

**Theorem 15** (Multiplicative Drift Theorem, Lower Bound [Wit13, LW13]).

In the situation of Section 2.1.1, if there are two constants \( 0 < \beta, \delta \leq 1 \) such that for all \( s \in S \setminus \{0\} \) and all \( t \geq 0, \)

1. \[ X_{t+1} \leq X_t. \quad (61) \]

2. \[ \Pr[X_t - X_{t+1} \geq \beta X_t \mid \mathcal{F}_t, X_t = s] \leq \frac{\beta \delta}{1 + \ln(s/s_{\min})}. \quad (62) \]

3. \[ E[X_t - X_{t+1} \mid \mathcal{F}_t, X_t = s] \leq \delta s. \quad (63) \]

\[^{23}\text{In particular, the function } h \text{ in Theorem 14 is not identical to the function } h \text{ in the upper bound version, Theorem 3.}\]
Then

$$E[T] \geq \frac{1 - \beta}{1 + \beta} \cdot \frac{1 + E[\ln(X_0/s_{\text{min}})]}{\delta}.$$  \quad (64)

Recently, Doerr, Doerr, and Kötzing \cite{DDK17} showed that the monotonicity condition \eqref{eq:monotonicity} can be completely removed if \eqref{eq:condition} is replaced by the condition that for all $s, s' \in \mathcal{S} \setminus \{0\}$ with $s' \leq s$,

$$E[(s' - X_{t+1})_+ \mid \mathcal{F}_t, X_t = s] \leq \delta s',$$  \quad (65)

where $x_+$ is the truncation function which is $x$ for $x \geq 0$ and $0$ for $x < 0$. The authors show that this condition is satisfied for very natural processes. In particular it is satisfied for processes with multiplicative drift if the jump probability $p(s) := \Pr[X_{t+1} \leq s' \mid \mathcal{F}_t, X_t = s]$ is a decreasing function in $s$, whenever $s' \leq s$. This modification extends the scope of Theorem 15 considerably, since many evolutionary algorithms are non-monotone processes. Moreover, it seems likely that the proof in \cite{DDK17} can be extended to generalise related lower bounds, in particular the lower bound for variable drift in Theorem 14.

We conclude the discussion on lower bounds with an easy example to demonstrate how to apply Theorem 14 and 15.

**Example 16 (RLS on OneMax, Lower Bound).** Consider once more RLS on OneMax as in Example 4. We want to apply Theorem 15. Since $X_t$ decreases by at most one, we choose $\xi(s) := s - 1$ to satisfy \eqref{eq:condition} as tightly as possible. Since the drift is $\Delta_t(s) = s/n$, we choose $h(s) := (s + 1)/n$ so that $h(\xi(s)) = \Delta_t(s)$. Thus we obtain the lower bound

$$E[T] \geq \frac{s_{\text{min}}}{h(s_{\text{min}})} + E\left[\int_{s_{\text{min}}}^{X_0} \frac{1}{h(\sigma)} d\sigma\right] = \frac{1}{2/n} + E\left[\int_1^{X_0} \frac{n}{\sigma + 1} d\sigma\right],$$

which is easily seen to be at least $n \ln n - O(n)$.

Note that Theorem 15 would give a less tight bound if naively applied.

To satisfy \eqref{eq:condition} for $s = 2$, it would be necessary to choose $\beta \geq 1/2$, and for $s = 1$ we even need $\beta \geq 1$, which renders the bound useless. However, this problem can be overcome by truncating the search space, see \cite{DDK17} for details.

### 4.2 Tail Bounds

For additive drift, we cannot expect sharp concentration of the runtime $T$ unless we can bound the probability that $X_t$ makes large jumps. For

\begin{footnote}{In other words, it should more likely to jump into the interval $[0, s']$ if you start closer to it.}\end{footnote}
example, consider the process on $S = \{0, n\}$ in which $X_t = n$ has probability $1/n$ to jump to zero, and stays in $n$ otherwise. Then $X_t$ has drift one towards $0$, but the hitting time $T$ is not concentrated. So we need to make some assumption on the distribution of $|X_t - X_{t+1}|$. Notably, such requirements are not necessary for the upper tail bound for multiplicative drift, as pointed out by Doerr and Goldberg [DG13]. We give the simplified formulation from [DJW12a]. We also present the proof of Doerr and Goldberg, which is remarkably short and elegant.

**Theorem 17** (Multiplicative Drift, Upper Tail Bound [DG13, DJW12a]). In the situation of Section 2.1.1, suppose that $X_0 = s_0$, and that there exists $\delta > 0$ such that for all $s \in S \setminus \{0\}$ and all $t \geq 0$,

$$E[X_t - X_{t+1} \mid X_t = s] \geq \delta s.$$  \hspace{1cm} (67)

Then, for all $r \geq 0$

$$\Pr \left[ T > \left\lceil \frac{r + \ln(s_0/s_{\min})}{\delta} \right\rceil \right] \leq e^{-r}.$$  \hspace{1cm} (68)

**Proof.** For every fixed $\rho = \lceil \frac{r + \ln(s_0/s_{\min})}{\delta} \rceil \in \mathbb{N}$, by Markov’s inequality,

$$\Pr[T > \rho] = \Pr[X_{\rho} > 0] \leq \frac{E[X_{\rho}] \rho}{s_{\min}} \leq (1 - \delta)^\rho \frac{s_0}{s_{\min}},$$  \hspace{1cm} (69)

where (*) comes from applying Equation (67) and linearity of expectation $\tau$ times. Since $(1 - x) \leq e^{-x}$ for all $x \in \mathbb{R}$, we obtain $\Pr[T > \rho] \leq e^{-\rho \delta s_0/s_{\min}} \leq e^{-r}$. \qed

For all other main drift theorems, including additive drift, variable drift, and lower tails for multiplicative drift, we need assumptions on the probability of large jumps. The easiest assumption is that large jumps do not occur at all, i.e. $|X_{t+1} - X_t| < c$ for some parameter $c$. This case occurs in various situation, for example for RLS, for some ant colony optimisation algorithms like the max-min ant system MMAS, or for the compact genetic algorithm cGA. We refer the reader to Kötzing [Kötz14] for a large collection of additive drift theorems with this assumption.

While there are situations without large jumps, there are even more cases in which large jumps may occur, but are unlikely. Thus research has focused on drift theorems with assumptions on the jump probability, usually some type of exponentially falling bounds, i.e., $\Pr[|X_{t+1} - X_t| > j] \leq c \cdot (1 + \eta)^{-j}$ for some parameters $c, \eta > 0$. In this chapter we stick with this type of condition, although generalisations are possible. Kötzing has made the point that exponentially falling jump probabilities imply a sub-Gaussian

\footnote{For example, $\Pr[T > 2E[T]] = (1 - 1/n)^{2n} \approx e^{-2}$.}
distribution of $X_t - \varepsilon t$, which is sufficient to derive most known tail bounds \[K" ot16\]. Lehre and Witt have given a very general framework for drift theorems \[LW13\, LW14\], in which only weak conditions on the exponential probability generating function $e^{\lambda(X_t-X_{t+1})}$ are needed.\[27\] Most major drift theorems, including concentration bounds, can be derived from this framework, so that it arguably renders the other drift theorems unnecessary \[LW13\]. However, researchers have continued to use specialised drift theorems, possibly because the framework by Lehre and Witt comes with a substantial technical overhead.

Even with bounds on the probability of making jumps, lower tail bounds remain rather delicate. Unfortunately, it is not true in general that the runtime is concentrated around the expectation. This problem occurs when the drift is too weak, as the following counterexample shows.

**Example 18** (Runtime is Not Concentrated Around Mean for Weak Drift). We consider the following artificial random walk on the set \{0, 1, \ldots, N\} for some (very large) constant $N$. We start in $X_0 = n$, where $n$ is much smaller than $N$. For $X_t = s$, with probability $1/n^4$ we make a step to the left, $X_{t+1} := X_t - 1$, and otherwise we flip an unbiased coin to see whether we make a step to the left or to the right. We say that we do a biased step in the first case, and an unbiased step in the second.\[28\] Effectively, this process can be summarised as

\[
X_{t+1} = \begin{cases} 
X_t - 1 & \text{with probability } \frac{1}{2}(1 + 1/n^4), \\
X_t + 1 & \text{with probability } \frac{1}{2}(1 - 1/n^4).
\end{cases}
\]  

(70)

Then the drift is easily seen to be

\[
\Delta_t(s) = \frac{1}{n^4},
\]  

(71)

so that by the Additive Drift Theorem \[4\] we obtain

\[
E[T] = n^4.
\]  

(72)

So in terms of expectations, drift analysis can handle the problem quite well. However, it turns out that the expectation is completely misleading. Consider the first $n^3$ steps of the algorithm. By a union bound, with probability $1 - O(1/n)$ all of these steps are unbiased. Hence, with high probability the first $n^3$ steps are given by an unbiased random walk, also known as a Gambler’s Ruin Process. This process is well-studied, and it is known that the

\[26\] and arguably more natural, using the Azuma-Hoeffding inequality.

\[27\] more precisely, only the expectation of this function needs to be bounded.

\[28\] We have neglected the border case $X_t = N$ in the description. However, if $N$ is large enough, e.g., $N = c^n$, then we cannot hit the right border in $o(N)$ steps, so the arguments are unaffected by the right border. For Equation (72) we need that the drift is also $1/n^4$ at the border.
probability to walk from \( n \) to 0 in at most \( \alpha n^2 \) steps is \( 1 - O(\alpha^{-1/2}) \) for all \( \alpha > 1 \) [GS01]. In particular, with \( \alpha = n \), the probability that an unbiased random walk starting in \( n \) hits 0 in at most \( n^3 \) steps is \( 1 - O(n^{-1/2}) \). Thus, with high probability the stopping time \( T \) of our process satisfies \( T = O(n^3) \). Hence, with high probability \( T \) is asymptotically much smaller than its expectation \( E[T] = n^4 \).

We remark that the same example can easily be adapted to multiplicative drift, e.g., by making the probability of an unbiased step \( X_t/n^{1/2} \). Since \( X_t \) changes in each step by at most one, by Theorem 15 the bound \( E[T] = O(n^{10} \log n) \) given by the Multiplicative Drift Theorem 10 is tight up to constants factors. However, as before the runtime is \( O(n^3) \) with high probability, so that with high probability the runtime is much smaller than the expected runtime.

Despite this problem, good tail bounds for additive drift have been developed. The following theorem follows by combining Theorems 10, 12, and 13 in [Köt16].

**Theorem 19** (Additive Drift, Tail Bounds, following [Köt16]). In the situation of Section 2.1.1, suppose that \( X_0 = s_0 \), and that there exist \( \delta, \eta, r > 0 \) such that for all \( s \in S \setminus \{0\} \), all \( j \in \mathbb{N}_0 \), and all \( t \geq 0 \),

1. \[
\Pr[|X_{t+1} - X_t| > j | F_t] \leq \frac{r}{(1 + \eta)^j}. \tag{73}
\]

2a. \[
E[X_{t+1} - X_t | F_t, X_t = s] \leq \delta. \tag{74}
\]

Then, for all \( x \geq 0 \)

\[
\Pr \left[ T \leq \frac{s_0 - x}{\delta} \right] \leq \exp \left\{ -\frac{\eta x}{8} \cdot \min \left\{ 1, \frac{\eta^2 \delta x}{32rs_0} \right\} \right\}. \tag{75}
\]

If instead of 2a. we have

2b. \[
E[X_{t+1} - X_t | F_t, X_t = s] \geq \delta, \tag{76}
\]

\footnote{In fact, being mathematically sloppy the “typical case” is \( T = \Theta(n^2) \).}

\footnote{Actually, the statement in [Köt16] is stronger since it states that at no point during the whole process \( X_t \) deviates substantially from its expectation, while we only consider \( X_t \) that are relevant for the runtime.}
then

\[
\Pr \left[ T \geq \frac{s_0 + x}{\delta} \right] \leq \exp \left\{ -\frac{\eta x}{8} \cdot \min \left\{ 1, \frac{\eta^2 \delta x}{32 r s_0} \right\} \right\}.
\]  

(77)

Note that the bounds in Theorem (75) and (77) give only concentration if the right hand side is of the form \( \exp\{-\Phi\} \) for a large term \( \Phi \). In particular, consider the case that \( \delta \) and \( r \) are constants, and that \( x = \Theta(s_0) \). Then \( \Phi = \omega(1) \) if and only if the bound \( s_0/\delta \) on the expected runtime satisfies \( s_0/\delta = o(x^2) = o(s_0^2) \). On the other hand, for \( s_0/\delta = \omega(s_0^2) \) the runtime bound from the drift is larger than the time that an unbiased random walk would need to hit 0, cf. also Example 18. So it is not surprising that Theorem 19 does not give concentration in this regime. Tight concentration bounds for the regime of weak drift can be found in [Kötzli16].

Analogous tail bounds for variable drift do exist in the general framework of Lehre and Witt [LW13]. Alternatively, one can rescale \( X_t \), as discussed in Section 3.2, to turn multiplicative drift into additive drift, and apply Theorem 19. Unfortunately, both approaches tend to be considerably technical. The most important case is to obtain tight lower tail bounds for variable drift. But even with the framework of Lehre and Witt, in order to derive lower tail bounds for the \((1 + 1)\) EA on ONE MAX it is still necessary to split the process into phases of relatively constant drift [LW13]. An easy and comprehensive lower tail bound for multiplicative drift is yet missing in the literature.

4.3 Negative Drift

If the drift does not point towards zero, but rather it points with a constant rate away from zero, then it takes exponential time to cross an interval. The first theorem of this type was proven by Oliveto and Witt [OW11, OW12], following Hajek’s classical work [Haj82]. We give a formulation close to [RS14, LS16] because it avoids \( o \)-notation for the length of the interval. Explicit constants can be found in [OW15, Kötzli16, Witt17].

**Theorem 20** (Negative Drift, following [OW11, OW12, RS14, LS16]). For all \( a, b, \delta, \eta, r > 0 \), with \( a < b \), there is \( c > 0, n_0 \in \mathbb{N} \) such that the following holds for all \( n \geq n_0 \). Suppose that we are in the situation of Section 2.1.1 with \( X_0 \geq bn \), and suppose that for all \( s \in \mathbb{S} \) with \( s > an \), for all \( j \in \mathbb{N}_0 \), and for all \( t \geq 0 \) the following conditions hold:

(a) \[
E[X_t - X_{t+1} \mid \mathcal{F}_t, X_t = s] \leq -\delta,
\]  

(78)

(b) \[
\Pr[|X_t - X_{t+1}| \geq j \mid \mathcal{F}_t, X_t = s] \leq \frac{r}{(1 + \eta)^j}.
\]  

(79)
Then for the hitting time $T_a$ of $S \cap [0, an]$, 

$$\Pr[T_a \leq e^{cn}] \leq e^{-cn}. \quad (80)$$

Negative drift is helpful for proving lower bounds [RS14, OW15, LS16], but not only so. It may also be used to show that an algorithm stays in a desired parameter regime. For example, Neumann, Sudholt, and Witt used it to show that an Ant Colony Optimisation (ACO) algorithm has good runtime because all pheromone values stay in a desirable range [NSW10]. Similarly, Kötzing and Molter [KM12], as well as subsequent work [LW15, FKKS16, LW16] used negative drift to show that ACO algorithms tend to stay close to the optimum, thus enabling the algorithm to follow the optimum in a dynamically changing environment. In a different setting, Sudholt and Witt [SW16] show that the compact Genetic Algorithm cGA is efficient on OneMax because for each position the probability to sample a one-bit never becomes too low. Similar ideas have been applied for population-based non-elitist algorithms in the Strong Selection Weak Mutation (SSWM) regime [PHST17].

### 4.4 Populations

If the algorithm uses population sizes larger than one, or if it does not work at all with populations, like Ant Colony Optimisation (ACO) or Estimation of Distribution Algorithms (EDAs), then it is often challenging to find a single potential $X_t$ which captures well the quality of the current population. In some cases, it suffices to consider the current best optimum (Example 6, [DFF17]) or some average quality [FKKS15, SW16]. A systematic approach was developed by Corus, Dang, Eremeev, and Lehre [CDEL14, CDEL17], who gave the so-called Level-Based Theorem for population-based algorithms. A population-based algorithm in their sense is any algorithm of the following form. In each round it maintains a population of size $\lambda$, and from this population it generates some probability distribution $D$. For the next round, it produces independently $\lambda$ samples from $D$, which form the next generation.

This framework of population-based algorithms applies to many situations, often with a twist to the usual algorithm description. Firstly, it does include all $(\mu + \lambda)$-evolutionary or genetic algorithms if the $\lambda$ offsprings are generated independent of each other. In this case, the offspring population determines by a distribution $D$ which subsumes selection and

---

31 in some parameter regimes
32 ACO algorithms maintain pheromone values, EDAs maintain a probability distribution, rather than a population of search points.
33 conflicting terminology exists.
34 not the parent population, since from these the next parents are not sampled independently. Rather, the parents of the next generation need to compete with each other in the selection step.
mutation/crossover, from which the new offsprings are sampled. Other population-based algorithms include, surprisingly, EDAs [DL15b]. While these algorithms conceptually maintain a probability distribution rather than a population, they do produce a sample population in each round, from which the next distribution is computed. This offspring population makes them fit into the framework of population-based algorithms.

The Level-Based Theorem assumes a partitioning of the search space into fitness levels that need to be climbed by the population. It gives an upper bound on the expected runtime if certain conditions are satisfied. The exact formulation is rather technical, so we refer the reader to [CDEL17]. Qualitatively, three ingredients are required:

(a) If part of the population has at least fitness level $i$, then the probability to sample an offspring at level $i + 1$ is sufficiently large.

(b) The fraction of the population which has fitness level at least $i$ increases in expectation.

(c) The population size is large enough.

Although it was only recently developed, the level-based Theorem has already found quite a number of applications, including the analysis of genetic algorithms with a multitude of selection mechanisms and benchmark functions [CDEL17], EDAs [DL15b, LN17], the analysis of self-adaptive algorithms [DL16b], and of algorithms in situations that are dynamic [DL16b], noisy [DL15a], or provide only partial information [DL16a].

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