Braided categories of endomorphisms as invariants for local quantum field theories

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Abstract
We want to establish the “braided action” (defined in the paper) of the DHR category on a universal environment algebra as a complete invariant for completely rational chiral conformal quantum field theories. The environment algebra can either be a single local algebra, or the quasilocal algebra, both of which are model-independent up to isomorphism. The DHR category as an abstract structure is captured by finitely many data (superselection sectors, fusion, and braiding), whereas its braided action encodes the full dynamical information that distinguishes models with isomorphic DHR categories. We show some geometric properties of the “duality pairing” between local algebras and the DHR category which are valid in general (completely rational) chiral CFTs. Under some additional assumptions whose status remains to be settled, the braided action of its DHR category completely classifies a (prime) CFT. The approach does not refer to the vacuum representation, or the knowledge of the vacuum state.

1 Introduction
In most approaches to quantum field theory (QFT) one starts from a kinematical algebra (e.g., the equal-time canonical commutation relations) and constructs the dynamics along with the ground state (the vacuum). This state is represented, e.g., by the path integral (after analytic continuation),
which is notoriously difficult to construct. It is well known that renormalization requires a change of the original algebra along the way with the construction. Once this is achieved, one extracts the (time-ordered) correlation functions and scattering amplitudes.

In a recent approach based on the operator product expansion (OPE), Holland and Hollands [HH15] construct only the full interacting quantum field algebra, whose coefficient functions turn out to be much more regular at short distance than the vacuum correlation functions. The construction of the algebra is in this approach well separated from the dynamical intricacies of the vacuum state, which must be constructed in a second step.

This is very much in the spirit of the algebraic approach to quantum field theory (AQFT) [Haa96], which emphasizes the primacy of the algebra of observables along with its local structure (its subalgebras $\mathcal{A}(O)$ of observables localized in spacetime regions $O$), and studies its many different representations of physical interest. Among them, there is the vacuum representation, distinguished by the existence of an invariant vacuum state $\Omega$. The extraordinary features of this state are reflected in the Bisognano-Wichmann property [BW75], [BGL93], [Mun01] which asserts that its restriction to the algebra $\mathcal{A}(W)$ of observables in a wedge region $W$ is a KMS state for the boosts subgroup preserving that wedge. This not only predicts remarkable “thermal features” of the well-known vacuum fluctuations, including the Unruh effect [Sew80], [BV14], it also allows to construct the boost generator and the CPT operator from just the data $(\mathcal{A}(W), \Omega)$, i.e., a single von Neumann algebra and a state. Since the CPT operator differs from the asymptotic free CPT operator by the scattering matrix [Jos65], it carries most of the dynamical content of the QFT.

The enormous amount of dynamical information encoded in the quantum vacuum state is also witnessed by the following facts, which may “explain” why the construction of this state is bound to be so difficult.

Borchers [Bor92] has shown that a full (1+1)-dimensional QFT can be constructed from a single algebra $\mathcal{A}(W)$, the vacuum state $\Omega$, and a unitary positive-energy representation $U$ of the translations subgroup, such that $U(x)\Omega = \Omega$ and $U(x)\mathcal{A}(W)U(x)^* \subset \mathcal{A}(W)$ for $x \in W$. Using a pair of algebras and the vacuum state, even the translations can be constructed [Wie93]. This idea has been extended to 3+1 dimensions in different ways, by Buchholz and Summers [BS93], and by Kähler and Wiesbrock [KW01], and to chiral conformal QFT by Guido, Longo and Wiesbrock [GLW98].

All these facts are instances of modular theory, which captures subtle functional analytic properties of faithful normal states of von Neumann algebras.
This theory is essentially trivial for commutative algebras, and therefore none of these results has a classical analogue.

In a nutshell, all local algebras $\mathcal{A}(O)$ of observables along with the covariance, and hence the entire QFT, can be constructed out of one or two given von Neumann algebras and the vacuum state.

As an attempt to “by-pass” the difficult construction of the vacuum state, we want to address the question, how far one can get without knowledge of it, just given “one or two local von Neumann algebras”, and which possibly more accessible structure might be apt to substitute it?

Our input shall be the DHR category $[\text{DHR71}]$ of the QFT to be (re-)constructed, that controls the composition (“fusion”) and permutation (“braiding”) of its positive energy representations in terms of a unitary braided tensor category $(\text{UBTC})$\footnote{It is actually even a $C^*$ braided tensor category, but the $C^*$ property is automatic for rational UBTCs that we are going to deal with, see [LR97, Lem. 3.2], [Müg00, Prop. 2.1].}

In low dimensions, the DHR category may be regarded as a “dual substitute” for global symmetries [DR89], [DR90], hence it encodes important but certainly not complete information about the model. We shall see that its braided action on a model independent algebra, formulated in Section 3 as an invariant for local nets, encodes more specific dynamical information.

As abstract structures, UBTCs are quite easily accessible, especially when they have only finitely many inequivalent irreducible objects and finite-dimensional intertwiner spaces (rational QFT). In this case it suffices to know the fusion rules of the irreducible objects (superselection sectors), and solve a finite number of algebraic relations to fix the admissible tensor structures and braidings. E.g., the well-known fusion rules of the chiral Ising model admit eight solutions, hence eight inequivalent UBTCs.

We want to explore to which extent the DHR category allows to reconstruct the underlying QFT. The answer cannot be unique because two QFTs may easily share the same DHR category up to equivalence. E.g., by tensoring a QFT with another one which has no nontrivial sectors (“holomorphic CFT”, in the context of chiral conformal QFT) does not change its DHR category. By invoking its braided action, however, the distinction is revealed, see Section 8, and we offer a sufficient criterion to exclude the presence of holomorphic factors. This criterion seems to be the right one to grasp the information about localization (left/right separation) of charges, hence dually of observables, out of the DHR braiding, in the sense of Proposition 9.3. It is also a good candidate to be a necessary condition, in view of Proposition
We shall restrict ourselves to chiral conformal QFTs, because in this case complete rationality [KLM01] implies non-degeneracy of the DHR braiding, i.e., the DHR category has the abstract structure of a unitary modular tensor category (UMTC). For our purpose, this means that the braiding of DHR endomorphisms encodes a sharp distinction between left and right. Our basic idea is to start with either the global $C^*$-algebra $\mathcal{A}$ of quasilocal observables, or a single local von Neumann algebra $\mathcal{A}(I_0)$ where $I_0$ is an arbitrarily fixed bounded interval of the line $\mathbb{R}$ (or equivalently of the circle $S^1$). The local picture is technically advantageous, but not essential, see Sections 4 and 5. Indeed neither $\mathcal{A}$, nor $\mathcal{A}(I_0)$, carry any specific information about the models, by well-known results of [Haa87], [Tak70], and thus serve as a universal environment ("blanc canvas") to let the DHR category act on.

Either locally or globally, relative commutants have a geometric interpretation both on half-intervals (strong additivity) or half-lines (relative essential duality), see Proposition 2.7. Also the structure of the two-interval subfactor can be extended verbatim to a unital $C^*$-inclusion of algebras in the real line picture, see Corollary 4.9. Moreover the action of the DHR category on the observables behaves similarly locally or globally: compare modularity with Proposition 4.5, and the duality relations between observables and endomorphisms localizable in half-lines (Proposition 4.3) or intervals (Proposition 4.7), either on $\mathbb{R}$ or confined in some fixed interval $I_0$. The latter proposition gives also an affirmative answer (in the chiral conformal setting) to a conjecture of S. Doplicher [Dop82] (in (3+1)-dimensional theories), see Remark 4.8.

Our main tool to reconstruct the local substructure of the net are abstract points of the braided action of the DHR category, see Section 6. The crucial observation is that the DHR category possesses, by its very definition based on the underlying local structure, a characteristic property: its braiding trivializes $\varepsilon_{\rho,\sigma} = 1$ whenever $\rho, \sigma$ are localizable in mutually left/right separated regions of the real line. Since points are responsible for left/right splittings of the line, this motivates our definition of abstract points as suitable pairs of subalgebras that trivialize the braiding.

Using algebraic deformation techniques, abstract points can be carried wildly far-away from the naive geometric picture of two half-interval algebras, see Section 7. We therefore need to understand what is required to identify abstract point as geometric points, up to unitary equivalence. In Section 10 we show a way of deriving the completeness of the braided action as an invariant for local nets, but on a subclass of completely rational conformal
nets which we call **prime conformal nets**, see Definition 8.5. Primality of a conformal net rules out holomorphic and tensor products cases, and relies on the notion of **prime UMTC** due to [M"ug03]. In order to state the classification result we actually need two further assumptions, see Section 10, hence the content of Proposition 10.1 is still an abstract recipe, as we do not know which examples fit into the classification. Yet the recipe is quite surprising and natural, in the sense that it is essentially based on two facts about completely rational nets: the structure of the two-interval subfactor ([KLM01, Thm. 33]) and of the fixed points of the local DHR subcategories (Proposition 4.7).

In principle our techniques apply to general rational BTCs, in particular to UMTCs, thanks to realization results of [HY00] by means of endomorphisms. Hence solving the previous trivialization constraints \( \varepsilon_{\rho,\sigma} = 1 \) and then applying our machinery, can be viewed as a possible way to realize abstract UMTCs by means of suitable, e.g., prime (see Definition 8.5), conformal nets via the DHR construction. We do not discuss this “exoticity” problem for abstract UMTCs in this work, and we refer to [Kaw15] for more explanations, and to [Bis15] for a systematic positive answer on the realization of Drinfeld doubles of subfactors with index less than 4.

## 2 Conformal nets and points on the line

The purpose of this section is to collect structure properties of QFT models that shall be used for the reconstruction of local algebras from an action of the DHR category in later sections. Although these results are well known (except Proposition 2.7), it is worthwhile to exhibit them in due context.

In this work we deal with chiral conformal field theories (chiral CFTs) “in one spacetime dimension”, referring to either of the two light-like coordinates \( x^0 \pm x^1 \) in two dimensions. By conformal covariance one can equivalently consider theories on the real line \( \mathbb{R} \), or on the unit circle \( S^1 \). The latter can be regarded as a “conformal closure” of the line \( \mathbb{S}^1 \cong \mathbb{R} = \mathbb{R} \cup \{ \infty \} \) and the points of the two sets can be put in bijective correspondence via the Cayley map \( x \in \mathbb{R} \mapsto (x + i)(x - i)^{-1} \in S^1 \setminus \{ 1 \} \).

Chiral CFTs are effectively described in the algebraic setting of AQFT [Haa96]. An abundance of models of the field-theoretic literature has been reformulated in this unifying framework, giving access to model-independent insight and structure analysis [Reh15].

In the following we adopt the **real line picture** as more natural for our purposes, in particular from a representation theoretical point of view, cf.
We describe chiral CFTs by means of local conformal nets on the line in the following sense, cf. [FJ96]. Instead of points of \( \mathbb{R} \) we have bounded intervals \( I \subset \mathbb{R} \), instead of local fields we have local algebras \( \mathcal{A}(I) \). More precisely, let \( \mathcal{I} \) be the family of non-empty open bounded intervals \( I \subset \mathbb{R} \) and notice that \( \mathcal{I} \) is partially ordered by inclusion and directed. Consider a complex separable Hilbert space \( \mathcal{H} \), the vacuum space, and to every \( I \in \mathcal{I} \) assign a von Neumann algebra \( \mathcal{A}(I) = (\mathcal{A}(I))'' \) realized on \( \mathcal{H} \). The latter correspondence forms a net of algebras, which we denote by \( \{\mathcal{A}\} = \{I \in \mathcal{I} \mapsto \mathcal{A}(I)\} \).

**Definition 2.1.** A net of von Neumann algebras \( \{\mathcal{A}\} = \{I \in \mathcal{I} \mapsto \mathcal{A}(I)\} \) realized on \( \mathcal{H} \) is a local conformal net on the line if it fulfills:

- **Isotony:** if \( I, J \in \mathcal{I} \) and \( I \subset J \) then \( \mathcal{A}(I) \subset \mathcal{A}(J) \).
- **Locality:** if \( I, J \in \mathcal{I} \) and \( I \cap J = \emptyset \) then \( \mathcal{A}(I) \) and \( \mathcal{A}(J) \) elementwise commute.
- **Möbius covariance:** there is a strongly continuous unitary representation \( U \) of the Möbius group \( \text{Möb} = \text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm 1\} \) on \( \mathcal{H} \), which acts covariantly on the net, i.e.,

\[
U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)
\]

whenever \( I \in \mathcal{I} \), \( g \in \text{Möb} \) and \( gI \in \mathcal{I} \), we ask nothing otherwise.
- **Positivity of the (conformal) Hamiltonian:** the generator \( H \) of the rotations subgroup of \( \text{Möb} \) is positive.
- **Vacuum vector:** there exists a Möbius invariant vector \( \Omega \in \mathcal{H} \), unique up to scalar multiples, and cyclic for \( \{\mathcal{A}(I), U(g): I \in \mathcal{I}, g \in \text{Möb}\} \).

A local conformal net on the line (in a vacuum sector) is then specified by a quadruple \( (\{\mathcal{A}\}, U, \Omega, \mathcal{H}) \).

The following notion says when two local conformal nets are “the same”, and is particularly useful for classification purposes.

**Definition 2.2.** Two local conformal nets on the line (in their vacuum sector) \( \{\mathcal{A}\} \) and \( \{\mathcal{B}\} \), or better \( (\{\mathcal{A}\}, U_A, \Omega_A, \mathcal{H}_A) \) and \( (\{\mathcal{B}\}, U_B, \Omega_B, \mathcal{H}_B) \), are isomorphic, or unitarily equivalent, if there exists a unitary operator \( W : \mathcal{H}_A \rightarrow \mathcal{H}_B \) which intertwines the two quadruples, i.e., \( W\mathcal{A}(I)W^* = \mathcal{B}(I) \) for all \( I \in \mathcal{I} \), \( WU_A(g)W^* = U_B(g) \) for all \( g \in \text{Möb} \) and \( W\Omega_A = \Omega_B \). We write \( \{\mathcal{A}\} \cong \{\mathcal{B}\} \) for isomorphic nets.
Now starting from the local algebras of a net \{A\} as above, one can define algebras for arbitrary regions \(S \subset \mathbb{R}\) as follows. Define \(\mathcal{A}(S)\) to be the von Neumann algebra, respectively \(C^*-\)algebra, generated by all local algebras \(A(I)\) such that \(I \subset S\), depending on whether \(S\) is a bounded, respectively unbounded, region of \(\mathbb{R}\). In the first case notice that \(\mathcal{A}(S) \subset \mathcal{A}(J)\) for a sufficiently big \(J \in \mathcal{I}\), in the second case let \(\mathcal{R}(S) := \mathcal{A}(S)'\).

In this way we get the quasilocal \(C^*-\)algebra \(\mathcal{A} := \mathcal{A}(\mathbb{R})\), the algebras of “space-like” complements of intervals \(A(I')\) where \(I' := \mathbb{R} \setminus \overline{T}\), \(I \in \mathcal{I}\), the half-line ("wedge") algebras \(A(W)\) where \(W \subset \mathbb{R}\) is a non-empty open half-line, left or right oriented.

**Remark 2.3.** The latter distinction between norm and weak closure is not just technical, it is essential to understand the structure of local nets and their DHR representation theory. Assume Haag duality on \(\mathbb{R}\) (see below) and consider for instance \(I \in J\), i.e., \(\overline{T} \subset J\) where \(I, J \in \mathcal{I}\). Then \(I' \cap J = I_1 \cup I_2\) and \(\mathcal{A}(I_1 \cup I_2) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset \mathcal{A}(I') \cap \mathcal{A}(J)\) is the two-interval subfactor considered by [KLM01], and \(\vee\) is a short-hand notation for the von Neumann algebra generated. The previous inclusion is proper in many examples, in particular DHR charge transporters from \(I_1\) to \(I_2\) do not belong to \(\mathcal{A}(I_1 \cup I_2)\).

On the other hand, take \(I' = W_1 \cup W_2\), \(I \in \mathcal{I}\) and observe that

\[
\mathcal{A}(W_1 \cup W_2) = C^*\{\mathcal{A}(W_1) \cup \mathcal{A}(W_2)\} \subset \mathcal{R}(W_1 \cup W_2) = \mathcal{A}(W_1) \vee \mathcal{A}(W_2)
\]

is by Haag duality on \(\mathbb{R}\) the inclusion \(\mathcal{A}(I') \subset \mathcal{A}(I)'\), again proper in general. In this case DHR charge transporters from \(W_1\) to \(W_2\) are again not in \(\mathcal{A}(W_1 \cup W_2)\) but they belong to the weak closure \(\mathcal{R}(W_1 \cup W_2)\). Geometrically speaking, half-lines \(W_1\) and \(W_2\) “weakly touch at infinity” and allow charge transportation.

Chiral Rational CFTs (chiral RCFTs) correspond, in the algebraic setting, to a class of local conformal nets singled out by the following additional conditions imposed on the local algebras, see [KLM01], [Mug10]. Throughout this paper we will restrict to the completely rational case whenever representation theoretical issues are concerned.

**Definition 2.4.** A local conformal net on the line \{A\}, as in Definition 2.1, is called **completely rational** if the following conditions are satisfied.

(a) **Haag duality on \(\mathbb{R}\):** \(\mathcal{A}(I')' = \mathcal{A}(I)\) for all \(I \in \mathcal{I}\).

(b) **Split property:** for every \(I, J \in \mathcal{I}\), \(I \in J\) there exists a type I factor \(\mathcal{F}\) such that \(\mathcal{A}(I) \subset \mathcal{F} \subset \mathcal{A}(J)\).
(c) Finite index two-interval subfactor: $\mathcal{A}(I_1 \cup I_2) \subset \mathcal{A}(I) \cap \mathcal{A}(J)$ has finite Jones index, where $I, J \in \mathcal{I}$, $I \subset J$ and $I' \cap J = I_1 \cup I_2$ for $I_1, I_2 \in \mathcal{I}$.

With conformal covariance, see [GLW98], condition (a) is equivalent to

(a)' Strong additivity: $\mathcal{A}(I_1 \cup I_2) = \mathcal{A}(I)$ where $I \in \mathcal{I}$, $p \in I$ and $\{p\}' \cap I = I \setminus \{p\} = I_1 \cup I_2$ for $I_1, I_2 \in \mathcal{I}$.

Remark 2.5. Conditions (a) and (b) strengthen the locality assumption on the net, they are natural and fulfilled in many models. Condition (c) is the characteristic feature of “rational” theories, i.e., those with finitely many superselection sectors.

Notice that complete rationality, in the conformal setting, is a local condition, i.e., can be checked inside one arbitrarily fixed local algebra.

By conformal covariance, local conformal nets on the line $\{\mathcal{A}\}$, as in Definition 2.1 can be uniquely extended to local conformal nets on the circle, see [Lon08] for the precise definition of the latter. This fact is well known, cf. [FJ96], [LR04], [LW11], but contains some subtleties, see [Gio16, Sec. 1.2, 4.1] for the details. In particular, denoted by $\{\tilde{\mathcal{A}}\}$ the extension, it can be shown that the two definitions one might give of weakly closed half-line algebras are the same, namely $\tilde{\mathcal{A}}(W) = \mathcal{R}(W)$, and that in the Haag dual case (assumption (a)) the extension is algebraically determined by the formula $\tilde{\mathcal{A}}(I) = \mathcal{A}(I')'$. The correspondence $\{\mathcal{A}\} \mapsto \{\tilde{\mathcal{A}}\}$ is bijective up to isomorphism of nets in the sense of Definition 2.2.

As a consequence all the known properties of chiral conformal nets hold on the line as well, see, e.g., [GF93], [GL96], [GLW98]. Notably the Reeh-Schlieder theorem, the Bisognano-Wichmann property, factoriality of the local algebras, additivity and essential duality $\mathcal{R}(W)' = \mathcal{R}(W'')$. Moreover inclusions of local algebras $\mathcal{A}(I) \subset \mathcal{A}(J)$ for $I, J \in \mathcal{I}$, $I \subset J$ are known to be normal and conormal, i.e., respectively

$$\mathcal{A}(I)'' = \mathcal{A}(I), \quad \mathcal{A}(I) \vee \mathcal{A}(I)'' = \mathcal{A}(J)$$

(1)

where $\mathcal{N} := \mathcal{N}' \cap \mathcal{M}$ denotes the relative commutant of the inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras. The normality and conormality relations above do not depend on the specific geometric position of $I$ inside $J$, nor on Haag duality (assumption (a)).

With the split property (assumption (b)) both the local algebras $\mathcal{A}(I)$ for all $I \in \mathcal{I}$ and the quasilocal algebra $\mathcal{A}$ are canonical objects, in the sense that they are universal (independent of the specific model) up to spatial
isomorphism. The first as the unique injective ("hyperfinite") type $\text{III}_1$ factor by [Haa87], the second by a general result of [Tak70]. In particular, they contain no specific information about the models. Moreover locality of the net is not needed neither in [Tak70] nor to apply the result of [Haa87]. In the first only isotony enters, for the second we know that Bisognano-Wichmann’s modular covariance holds regardless of locality [DLR01].

The entire information about the chiral CFT is then encoded in the inclusions and relative commutation relations among different local algebras, i.e., in the local algebraic structure of the net. This statement is made precise by the next proposition due to M. Weiner [Wei11], which says that the vacuum sector of a local conformal net is uniquely determined by its local algebraic structure.

Proposition 2.6. [Wei11, Thm. 5.1]. Let $\{\mathcal{A}\}$ be a local conformal net as above fulfilling the split property (assumption (b)). Then $\{\mathcal{A}\}$, or better ($\{\mathcal{A}, U, \Omega, \mathcal{H}\}$, is completely determined up to isomorphism of nets, see Definition 2.2, by the isomorphism class of the local subfactors $\{\mathcal{A}(I) \subset \mathcal{A}(I_0), I \in \mathcal{I}, I \subset I_0\}$ for any arbitrarily fixed interval $I_0 \in \mathcal{I}$.

In other words, the isomorphism class of the collection of local algebras is a complete invariant for split local conformal nets.

With Haag duality on $\mathbb{R}$ (assumption (a)), there is a geometric interpretation of the relative commutant and of the normality and conormality relations (1) for inclusions of local algebras which arise for the choice of points. Namely let $I \in \mathcal{I}$, take $p \in I$ and let $\{p\}' \cap I = I \setminus \{p\} = I_1 \cup I_2$, $I_1, I_2 \in \mathcal{I}$. The relative commutant of $\mathcal{A}(I_1) \subset \mathcal{A}(I)$ is then given by

$$\mathcal{A}(I_1)^c := \mathcal{A}(I_1)' \cap \mathcal{A}(I) = \mathcal{A}(I_2).$$

It follows from conformal covariance, cf. [GLW98], that the relations (2) are actually equivalent to assumption (a).

Now a point of an interval, $p \in I$, is uniquely determined by two intervals $I_1, I_2 \in \mathcal{I}$ as above, the relative complements of $p$ in $I$. Algebraically, $p \in I$ splits $\mathcal{A}(I)$ into a pair of commuting subalgebras $\mathcal{A}(I_1), \mathcal{A}(I_2) \subset \mathcal{A}(I)$ which in the Haag dual case are each other’s relative commutants.

Similarly a point of the line, $p \in \mathbb{R}$, is uniquely determined by two half-lines $W_1, W_2 \subset \mathbb{R}$, the relative complements of $p$ in $\mathbb{R}$, and determines
two “global” unital $C^*$-inclusions $\mathcal{A}(W_1), \mathcal{A}(W_2) \subset \mathcal{A} := \mathcal{A}(\mathbb{R})$. Our first main structure result, see Proposition 2.7, shows that the same geometric interpretation of relative commutants holds in the global case. The proof is independent of assumption (a), but as a technical tool we need to assume (b). Merging the standard terminology of “relative commutant” and “essential duality” for local algebras we can call this property relative essential duality.

**Proposition 2.7.** Let $\{\mathcal{A}\}$ be a local conformal net on the line as in Definition 2.1, which fulfills the split property (assumption (b)). Consider the inclusion of unital $C^*$-algebras $\mathcal{A}(W) \subset \mathcal{A}$, where $W \subset \mathbb{R}$ is a half-line, left or right oriented, then

$$\mathcal{A}(W)^c := \mathcal{A}(W)' \cap \mathcal{A} = \mathcal{A}(W')$$

where $W' = \mathbb{R} \setminus \overline{W}$ is the opposite half-line.

**Proof.** Observe first that $\mathcal{A}(W)' = \mathcal{R}(W')$, hence the statement is equivalent to $\mathcal{A}(W) = \mathcal{R}(W) \cap \mathcal{A}$. This does not boil down to essential duality $\mathcal{R}(W)' = \mathcal{R}(W')$, because typically $\mathcal{A}(W) \subset \mathcal{R}(W)$ is proper and $\mathcal{R}(W) \nsubseteq \mathcal{A}$, see [BGL93, Sec. 1].

By the split property we have that $\mathcal{R}(W)$ is the injective factor of type $III_1$ and the same holds for its commutant. Consider then a norm continuous conditional expectation

$$E : \mathcal{B}(\mathcal{H}) \to \mathcal{R}(W)'$$

given by averaging over the adjoint action of the unitary group $G := \mathcal{U}(\mathcal{R}(W))$ of $\mathcal{R}(W)$, equipped with the ultraweak topology or equivalently with any of the other weak operator topologies.

Now, injectivity is equivalent to amenability of the unitary group, i.e., to the existence of a left invariant state (“mean”) on the unital $C^*$-subalgebra $\mathcal{C}_{ru}(G)$ of right uniformly continuous functions in $L^\infty(G)$, see [dIH79], [Pat92]. Similar to [Arv74] one can define an integral $E(b) := \int_G \text{Ad}_u(b) \, du$ with respect to such a mean $m$, for every $b \in \mathcal{B}(\mathcal{H})$, as the unique element in $\mathcal{B}(\mathcal{H})$ such that

$$\langle \varphi, \int_G \text{Ad}_u(b) \, du \rangle = \int_G \langle \varphi, \text{Ad}_u(b) \rangle \, du \quad \forall \varphi \in \mathcal{B}(\mathcal{H})_*$$

where $\mathcal{B}(\mathcal{H})_*$ is the predual, and the r.h.s. is defined by the mean on functions

$$\int_G \langle \varphi, \text{Ad}_u(b) \rangle \, du = m(f_{\varphi,b}), \quad f_{\varphi,b}(u) := \langle \varphi, \text{Ad}_u(b) \rangle.$$
One can easily see by formal computations that \( E(b)u = uE(b) \) for all \( u \in G \) hence \( E(b) \in \mathcal{R}(W)' \), see also [dH79, Lem. 1, 2]. Moreover, \( E \) is a norm one projection onto \( \mathcal{R}(W)' \), i.e., \( \|E(b)\| \leq \|b\| \) and \( E(b) = b \) if \( b \in \mathcal{R}(W)' \), hence a conditional expectation by [Tom57]. Observe that \( E \) cannot be normal because \( \mathcal{R}(W) \) is type III, see [Tak03, Ex. IX.4].

The next step is to show that \( E \) preserves the local structure of the net, i.e., maps local algebras into local algebras and \( \mathcal{A} \) into itself. So take a bounded interval \( I \) containing the origin of \( W \), we want to show that

\[
E : \mathcal{A}(I) \to \mathcal{A}(I) \cap \mathcal{R}(W)'.
\]

First, assume in addition that Haag duality on \( \mathbb{R} \) holds. Take \( a \in \mathcal{A}(I) \) and \( \mathcal{A}(I) = \mathcal{A}(I)' = (\mathcal{R}(W_1) \vee \mathcal{R}(W_2))' \) where \( I' = W_1 \cup W_2 \) and \( W_1, W_2 \) are half-lines. If for instance \( W_2 \subset W \), then every \( x \in \mathcal{R}(W_2) \) commutes with \( E(a) \in \mathcal{R}(W)' \). Take now any \( y \in \mathcal{R}(W_1) \subset \mathcal{R}(W') \), then

\[
E(a)y = \int_G \text{Ad}_u(a)y \, du = \int_G y \text{Ad}_u(a) \, du = yE(a)
\]

because \( uy = yu, u \in \mathcal{R}(W) \) and \( ay = ya, a \in \mathcal{A}(I) \) by locality. Hence \( E(a) \) commutes with \( \mathcal{R}(W_2) \) and with \( \mathcal{R}(W_1) \), and we can conclude that \( E(a) \in \mathcal{A}(I) \).

In general, a more refined and purely algebraic argument [dH79, Lem. 2 (iii)] shows directly that \( E(a) \in \mathcal{A}(I) \vee \mathcal{R}(W) \) which coincides with \( \mathcal{R}(W_1') \) by additivity, hence \( E(a) \in \mathcal{R}(W_1' \cap W') \) where \( W_1' \cap W' = I \cap W' \in \mathcal{I} \) and

\[
E : \mathcal{A}(I) \to \mathcal{A}(I \cap W') = \mathcal{A}(I) \cap \mathcal{R}(W)').
\]

Exhausting \( \mathbb{R} \) with a sequence of intervals \( I_n \) containing the origin of \( W \), by norm continuity of \( E \) we get \( E : \mathcal{A} \to \mathcal{A} \) and

\[
C^*\left(\bigcup_n \mathcal{A}(I_n \cap W')\right) = E(\mathcal{A}) = \mathcal{A}(W)^c.
\]

But also \( C^*\left(\bigcup_n \mathcal{A}(I_n \cap W')\right) = \mathcal{A}(W') \), hence \( \mathcal{A}(W)^c = \mathcal{A}(W') \) follows. \( \square \)

Remark 2.8. The techniques employed here are similar to those used in [Dop82, Sec. 5]. There, however, local algebras \( \mathcal{A}(I) \) are considered instead of half-line algebras and one does not need additivity nor essential duality to show that conditional expectations on \( \mathcal{A}(I)' \) preserve the local substructure of \( \mathcal{A} \).
As a consequence of Proposition 2.7, assuming the split property we can take the relative commutant of the inclusion \( \mathcal{A}(W') \subset \mathcal{A}(W)^c \subset \mathcal{A}(W)' \) and obtain
\[
\mathcal{A}(W) = \mathcal{A}(W)^c = \mathcal{R}(W) \cap \mathcal{A}
\] (3)
where the relative commutants refer to the inclusions \( \mathcal{A}(W) \subset \mathcal{A} \).

This is similar to the case of local algebras \( \mathcal{A}(I) \subset \mathcal{A}, I \in \mathcal{I} \) if we assume Haag duality on \( \mathbb{R} \), indeed
\[
\mathcal{A}(I) = \mathcal{A}(I)^c
\] (4)
follows by taking relative commutants of the inclusion \( \mathcal{A}(I') \subset \mathcal{A}(I)^c \subset \mathcal{A}(I)' \), cf. [DHR69, Sec. V]. The relations (3) and (4) are a global version of the normality relations (1) encountered before.

Heuristically speaking, we regard normality as an algebraic fingerprint of connectedness in the following sense. Algebras associated to intervals \( \mathcal{A}(I) \) or half-lines \( \mathcal{A}(W) \) are “connected”, relative commutants \( \mathcal{A}(I)^c \) are also “connected” in a broader sense, e.g., on the circle, because \( \mathcal{A}(I)^c = \mathcal{A}(I)^{cc} \) always holds. On the other hand, algebras \( \mathcal{A}(S) \subset \mathcal{A} \) associated to disconnected regions, e.g., \( S = I', I \in \mathcal{I} \), need not be normal. Indeed, assuming (a), the inclusion
\[
\mathcal{A}(I') \subset \mathcal{A}(I')^{cc} = \mathcal{A}(I)^c
\] (5)
is proper in many examples, see Corollary 4.9. In the case of holomorphic nets there is no algebraic distinction (in the sense of normality relations) between “connected” and “disconnected” regions at the level of nets, cf. [RT13] for an explicit isomorphism between interval and two-interval algebras in the case of graded-local Fermi nets. Notice that the unital \( C^* \)-inclusion (5) is a “global” version of the two-interval subfactor \( \mathcal{A}(I_1 \cup I_2) \subset \mathcal{A}(I_1 \cup I_2)^{cc} \subset \mathcal{A}(I)^c \) considered by [KLM01], where relative commutants are taken in \( \mathcal{A}(J) \) for \( I \sqsubset J, I' \cap J = I_1 \cup I_2 \). Indeed \( (\mathcal{A}(I_1) \vee \mathcal{A}(I_2))' \cap \mathcal{A}(J)' \cap \mathcal{A}(J) = (\mathcal{A}(I_1)' \cap \mathcal{A}(I_1 \cup I)') \cap \mathcal{A}(J) = \mathcal{A}(I)^c \cap \mathcal{A}(J) \).

In the following we shall concentrate on local conformal nets on the line \( \{A\} \), see Definition 2.1, which are in addition completely rational, as in Definition 2.4. In this case we know by [KLM01, Cor. 37] that the category of finitely reducible DHR representations of the net, denoted by DHR\(\{A\} \), has the abstract structure of a unitary modular tensor category (UMTC). Referring to [DHR71, FRS92, BKLR13, Mug12, EGO15] for the relevant definitions and further details, we just recall that DHR representations of a local quantum field theory satisfying Haag duality can be described in terms
of DHR endomorphisms of the quasilocal algebra $\mathcal{A}$, which enjoy covariance, localizability and transportability properties. They are the objects of the $C^*$ tensor category $\text{DHR}\{\mathcal{A}\}$, and their intertwiners are the morphisms. The fusion product of representations is defined through the composition of DHR endomorphisms (the monoidal product of $\text{DHR}\{\mathcal{A}\}$), which is commutative up to unitary equivalence. The unitary equivalence between $\rho \circ \sigma$ and $\sigma \circ \rho$ is given by the DHR braiding

$$\varepsilon_{\rho,\sigma} = (v^* \times u^*) \cdot (u \times v) = \sigma(u^*)v^*u\rho(v) \in \text{Hom}(\rho \sigma, \sigma \rho)$$

where $u \in \text{Hom}(\rho, \hat{\rho})$ and $v \in \text{Hom}(\sigma, \hat{\sigma})$ are unitary charge transporters to equivalent auxiliary DHR endomorphisms $\hat{\rho}$, $\hat{\sigma}$, such that $\hat{\rho}$ is localizable to the space-like left of $\hat{\sigma}$ \footnote{In [FRS92] the opposite right/left convention is adopted for the DHR braiding; this is related to a different convention for the Cayley map given at the beginning of this section.}. The unitary braiding thus defined does not depend on the specific choice of the auxiliary endomorphisms $\hat{\rho}$, $\hat{\sigma}$, and of the charge transporters $u$ and $v$, and satisfies the naturality axiom, thus turning $\text{DHR}\{\mathcal{A}\}$ into a unitary braided tensor category (UBTC). By the definition, if $\rho$ is localizable to the space-like left of $\sigma$, one may choose $u = v = 1$, hence

$$\varepsilon_{\rho,\sigma} = 1.$$ 

UMTCs are a particular class of UBTCs having irreducible tensor unit, finitely many inequivalent irreducible objects, conjugate objects and non-degenerate braiding (modularity).

The latter is the essentially new feature of DHR categories arising in low-dimensional models. Moreover, the key ingredient in the proof of modularity is the discovery of a deep connection between the algebraic structure of the net and the structure of its representation category. More precisely, the two-interval subfactor [KLM01, Thm. 33] is a Longo-Rehren subfactor [LR95, Prop. 4.10] and is uniquely determined up to isomorphism by the tensor structure of the category (forgetting the braiding), see [KLM01, Cor. 35]. Hence the DHR braiding can be seen as an additional ingredient whose definition requires, in the low-dimensional case, the choice of a point (irrespectively of its position) in order to separate the localization of DHR endomorphisms.

We close the section by mentioning that complete rationality is realized by several models: Wess-Zumino-Witten $SU(N)$-currents [Was98], Virasoro nets with central charge $c < 1$ [Car04, KL04], lattice models [DX06, Bis12], the Moonshine vertex operator algebra [KL06]. Further candidates
come from more general loop groups [GF93] and vertex operator algebras [CKLW15]. Moreover, complete rationality passes to tensor products [KLM01], group-fixed points [Xu00], finite index extensions and finite index subnets [Lon03].

3 Braided actions of DHR categories

The motivation of our work is the following: in the variety of completely rational models, one can easily find non-isomorphic ones, see Definition 2.2 having equivalent DHR categories in the sense of abstract UBTCs, see [EGNO15, Def. 8.1.7, Rmk. 9.4.7]. Examples of this can be constructed by looking at completely rational holomorphic nets, i.e., nets with only one irreducible DHR sector: the vacuum. In this case the DHR category coincides with Vec, the category of finite-dimensional complex vector spaces, up to unitary braided tensor equivalence. Take now a completely rational conformal net \( \{A\} \) and tensor it with a nontrivial holomorphic net \( \{A_{\text{holo}}\} \), then

\[
\text{DHR}\{A \otimes A_{\text{holo}}\} \simeq \text{DHR}\{A\} \boxtimes \text{DHR}\{A_{\text{holo}}\} \simeq \text{DHR}\{A\}
\]

but \( \{A\} \not\cong \{A \otimes A_{\text{holo}}\} \), because tensoring with nontrivial holomorphic nets increases the central charge by a multiple of 8. Hence the UBTC equivalence class of the DHR category is not a complete invariant for nets, i.e., the correspondence between completely rational conformal nets (up to isomorphism) and their DHR categories (up to UBTC equivalence)

\[
\{A\} \mapsto \text{DHR}\{A\}
\]

is not one-to-one. We might replace equivalence of categories with the much stronger notion of isomorphism of categories, see [ML98], but this is not what we want to do. Instead we consider the action of the DHR category on the net as additional structure, i.e., consider its realization as a braided tensor category of endomorphisms of the net. For technical reasons, we look at the action on a local algebra rather than the “global” defining action \( \text{DHR}\{A\} \subset \text{End}(A) \) on the quasilocal algebra. Namely, fix an arbitrary interval \( I_0 \in \mathcal{I} \) and consider the “local” full subcategory \( \text{DHR}^{I_0}\{A\} \subset \text{DHR}\{A\} \) whose objects are the DHR endomorphisms \( \rho \) localizable in \( I_0 \), i.e., \( \rho|_{\mathcal{A}(I_0')} = \text{id}|_{\mathcal{A}(I_0')} \).

\footnote{Here \( \simeq \) denotes UBTC equivalence and \( \boxtimes \) is the Deligne product (the “tensor product” in the category of semisimple linear categories).}
Notice that the inclusion functor in this case is also an equivalence, i.e., essentially surjective in addition
\[ \text{DHR}^{I_0} \{ \mathcal{A} \} \simeq \text{DHR} \{ \mathcal{A} \} \] (7)
because \( I_0 \) is open and there is by definition (and by Möbius covariance) no minimal localization length. Considering the action on local algebras means considering the restriction functor \( \rho \mapsto \rho|_{\mathcal{A}(I_0)} \)
\[ \text{DHR}^{I_0} \{ \mathcal{A} \} \hookrightarrow \text{End}(\mathcal{A}(I_0)) \] (8)
which is well-defined, strict tensor and faithful by Haag duality on \( \mathbb{R} \). Recall that the arrows of the endomorphism category on the right hand side are defined as
\[ \text{Hom}_{\text{End}(\mathcal{A}(I_0))}(\hat{\rho}, \hat{\sigma}) := \{ t \in \mathcal{A}(I_0) : t\hat{\rho}(a) = \hat{\sigma}(a)t, a \in \mathcal{A}(I_0) \} \]
where \( \hat{\rho}, \hat{\sigma} \in \text{End}(\mathcal{A}(I_0)) \). With conformal symmetry \[ \text{[GL96]} \] have shown that the restriction functor is also full (i.e., local intertwiners are global), hence an embedding of categories. The restriction functor is by no means essentially surjective, i.e., not every (finite index) endomorphism of the injective type \( \text{III}_1 \) factor \( \mathcal{A}(I_0) \) is realized by DHR endomorphisms of \( \{ \mathcal{A} \} \). But it has replete image, i.e., it is closed under unitary isomorphism classes in \( \text{End}(\mathcal{A}(I_0)) \).

The first interesting point concerning the embedding (8) is the following
Remark 3.1. Forgetting the braiding, the remaining abstract structure of \( \text{DHR}^{I_0} \{ \mathcal{A} \} \) is the one of a unitary fusion tensor category (UFTC). Functors between unitary categories (or *-categories) will always be assumed to preserve the *-structure. A result of Popa \[ \text{[Pop95]} \] states that an embedding \( \mathcal{C} \hookrightarrow \text{End}(\mathcal{M}) \) as above, where \( \mathcal{C} \) is a UFTC and \( \mathcal{M} \) is the unique injective type \( \text{III}_1 \) factor, is canonical in the following sense. Take two equivalent UFTCs realized as endomorphisms of injective type \( \text{III}_1 \) factors \( \mathcal{C} \subset \text{End}(\mathcal{M}) \) and \( \mathcal{D} \subset \text{End}(\mathcal{N}) \) where we can assume \( \mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H}) \). By \[ \text{[Pop95 Cor. 6.11]} \], see also \[ \text{[KLM01 Cor. 35]} \], there exists a spatial isomorphism \( \text{Ad}_U : \mathcal{M} \rightarrow \mathcal{N} \) where \( U \) is unitary in \( \mathcal{B}(\mathcal{H}) \) which implements an equivalence \( \mathcal{C} \simeq \mathcal{D} \) as follows
\[ \hat{\rho}_i \mapsto \text{Ad}_U \circ \hat{\rho}_i \circ \text{Ad}_U^* \simeq \hat{\sigma}_i \] (9)
for all \( i = 0, \ldots, n \) where \( \{ \hat{\rho}_0, \ldots, \hat{\rho}_n \} \) and \( \{ \hat{\sigma}_0, \ldots, \hat{\sigma}_n \} \) are generating sets for \( \mathcal{C} \) and \( \mathcal{D} \) respectively and \( \simeq \) stands for unitary isomorphism in \( \text{End}(\mathcal{N}) \).
If both embeddings are replete as in [8], we can extend the equivalence to an isomorphism of categories $\mathcal{C} \cong \mathcal{D}$ and every $\hat{\sigma} \in \mathcal{D}$ can be written as

$$\hat{\sigma} = \text{Ad}_U \circ \hat{\rho} \circ \text{Ad}_U^* =: \hat{U} \hat{\rho}$$

for a unique $\hat{\rho} \in \mathcal{C}$, moreover $t \mapsto \text{Ad}_U(t) =: \hat{U} t$ gives a *-linear bijection of the Hom-spaces $\text{Ad}_U : \text{Hom}(\hat{\rho}_i, \hat{\rho}_j) \to \text{Hom}(\hat{U} \hat{\rho}_i, \hat{U} \hat{\rho}_j)$. This isomorphism is manifestly strict tensor.

Take two nets \{A\}, \{B\} and consider as in [8] the replete embeddings of the respective DHR categories

$$\text{DHR}^I_0 \{\mathcal{A}\} \hookrightarrow \text{End}(\mathcal{A}(I_0)), \quad \text{DHR}^I_0 \{\mathcal{B}\} \hookrightarrow \text{End}(\mathcal{B}(I_0))$$

for some fixed interval $I_0 \in \mathcal{I}$. As we said, it may happen that $\text{DHR}\{\mathcal{A}\} \simeq \text{DHR}\{\mathcal{B}\}$ as UBTCs, hence as UFTCs forgetting the braiding. By Remark 3.1, there is a spatial isomorphism $\text{Ad}_U : \mathcal{A}(I_0) \to \mathcal{B}(I_0)$ which implements a strict tensor isomorphism between the images of the two restrictions, hence between the respective local DHR subcategories.

However, the latter isomorphism $F_U : \text{DHR}^I_0 \{\mathcal{A}\} \to \text{DHR}^I_0 \{\mathcal{B}\}$ need not preserve the braidings

$$\varepsilon^A_{\rho_1, \rho_2} = v_2^* \times u_1^* \times u_1 \times v_2 = \rho_2(u_1^*)v_2^*u_1\rho_1(v_2) \in \text{Hom}_{\text{DHR}\{\mathcal{A}\}}(\rho_1\rho_2, \rho_2\rho_1)$$

where $\rho_1, \rho_2 \in \text{DHR}^I_0 \{\mathcal{A}\}$ and $u_1, v_2$ are unitaries in $\mathcal{A}(I_0)$ such that $\text{Ad}_u \rho_1$ is localizable left to $\text{Ad}_v \rho_2$ inside $I_0$. Indeed

$$F_U(\varepsilon^A_{\rho_1, \rho_2}) = \text{Ad}_U(\rho_2(u_1^*)v_2^*u_1\rho_1(v_2)) = F_U(v_2^*) \times F_U(u_1^*) \times F_U(u_1) \times F_U(v_2)$$

is in the correct intertwiner space

$$F_U(\varepsilon^A_{\rho_1, \rho_2}) \in \text{Hom}_{\text{DHR}\{\mathcal{B}\}}(F_U(\rho_1), F_U(\rho_2), F_U(\rho_2), F_U(\rho_1))$$

but can be $F_U(\varepsilon^A_{\rho_1, \rho_2}) \neq \varepsilon^B_{F_U(\rho_1), F_U(\rho_2)}$ because, for instance, $F_U(u_1), F_U(v_2)$ need not be charge transporters which take the respective endomorphisms one left to the other inside $I_0$.

Take now two isomorphic nets \{A\}, \{B\} (see Definition 2.2). Then there is a unitary $W$ which implements spatial isomorphisms $\text{Ad}_W : \mathcal{A}(I) \to \mathcal{B}(I)$ for every $I \in \mathcal{I}$, hence for $I_0$ and all of its subintervals. The resulting strict tensor isomorphism $F_W : \text{DHR}^I_0 \{\mathcal{A}\} \to \text{DHR}^I_0 \{\mathcal{B}\}$ defined on objects as $\rho \mapsto \text{Ad}_W \circ \rho \circ \text{Ad}_W^*$ is braided in addition. Indeed $F_W$ respects the localization regions of the DHR endomorphisms, by definition, hence $F_W(\varepsilon^A_{\rho_1, \rho_2}) = \varepsilon^B_{F_W(\rho_1), F_W(\rho_2)}$. More generally
Definition 3.2. Let \( \mathcal{C} \) be an abstract strict UMTC and \( \mathcal{M} \) a von Neumann factor. A strict tensor replete embedding
\[
G : \mathcal{C} \hookrightarrow \text{End}(\mathcal{M})
\]
will be called a **braided action** of \( \mathcal{C} \) on \( \mathcal{M} \).

Remark 3.3. The previous notion is purely tensor categorical, indeed the category \( \text{End}(\mathcal{M}) \) is an enormous object which does not have a “global” braiding. However any braided action can be promoted to an actual braided functor by endowing the (replete tensor) image \( G(\mathcal{C}) \subset \text{End}(\mathcal{M}) \) with the braiding \( \hat{\varepsilon}_{G(\rho), G(\sigma)} := G(\varepsilon_{\rho, \sigma}) \). Our terminology is motivated by the importance of the realization of \( \mathcal{C} \) as a braided tensor category of endomorphism of \( \mathcal{M} \), see Definition 3.4 below for the precise formulation of this statement. The endomorphisms in the range of the embedding have automatically finite index. Moreover if \( \mathcal{M} \) is type III, they are automatically normal and injective (unital).

In our case at hand, \( \mathcal{C} := \text{DHR}^b \{ \mathcal{A} \} \) for some fixed \( I_0 \in \mathcal{I} \) and the braided action of the DHR category, remember the equivalence (7), on \( \mathcal{M}_0 := \mathcal{A}(I_0) \) is given by the restriction functor (8).

Definition 3.4. Let \( \mathcal{C}, \mathcal{D} \) be two abstract strict UMTCs and \( \mathcal{M}, \mathcal{N} \) two von Neumann factors. Two braided actions \( G_1 : \mathcal{C} \hookrightarrow \text{End}(\mathcal{M}) \) and \( G_2 : \mathcal{D} \hookrightarrow \text{End}(\mathcal{N}) \) will be called isomorphic if there is a spatial isomorphism \( \text{Ad}_U : \mathcal{M} \rightarrow \mathcal{N} \) implementing a strict tensor isomorphism between the respective images which is also braided. Equivalently, the unique strict tensor isomorphism \( F_U : \mathcal{C} \rightarrow \mathcal{D} \) which makes the following diagram commute
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G_1} & \text{End}(\mathcal{M}) \\
F_U \downarrow & & \downarrow \text{Ad}_U \\
\mathcal{D} & \xleftarrow{G_2} & \text{End}(\mathcal{N})
\end{array}
\]
is in addition a UBTC isomorphism.

Take two nets \( \{ \mathcal{A} \}, \{ \mathcal{B} \} \), their respective DHR categories together with their braided actions respectively on \( \mathcal{A}(I_0), \mathcal{B}(I_0) \) for some fixed \( I_0 \). Clearly from the previous discussion, if \( \{ \mathcal{A} \} \) and \( \{ \mathcal{B} \} \) are isomorphic nets (see Definition 2.2) then \( \text{DHR}^b \{ \mathcal{A} \} \) and \( \text{DHR}^b \{ \mathcal{B} \} \) have isomorphic braided actions (see Definition 3.4) hence we have an invariant.

Remarkably, the situation described in Definition 3.2 is general for UMTCs, in the sense that every abstract UMTC \( \mathcal{C} \) admits a braided action on the injective type III\(_1\) factor \( \mathcal{M} \).
Remark 3.5. As in Remark 3.1 we drop the braiding on $\mathcal{C}$ and consider its UFTC structure first. Without loss of generality, i.e., up to a (non-strict) tensor equivalence, we can assume that $\mathcal{C}$ is strict. Relying on a deep result of [HY00], we know that the presence of conjugates (rigidity) and the $C^*$-structure guarantee the existence of a (non-strict) tensor embedding $G : \mathcal{C} \hookrightarrow \text{End}(\mathcal{M})$, where $\mathcal{M}$ is the unique injective type $III_1$ factor. Now the image of $\mathcal{C}$ in $\text{End}(\mathcal{M})$ can be endowed with the braiding which promotes $G$ to a braided embedding, taking care of the nontrivial multiplicativity constraints of the functors, and can be completed to a UMTC $\tilde{\mathcal{C}}$ realized and replete in $\text{End}(\mathcal{M})$, which is equivalent to $\mathcal{C}$ as an abstract UMTC. The inclusion functor gives then a braided action of $\tilde{\mathcal{C}}$ on $\mathcal{M}$ in the strong sense employed in Definition 3.2. Similarly to Remark 3.1 but in this more general context, the (non-strict) tensor embedding $G : \mathcal{C} \hookrightarrow \text{End}(\mathcal{M})$ of a UFTC $\mathcal{C}$ is also expected to be unique (in a suitable sense, cf. [HP15 Conj. 3.6]).

4 Duality relations

Motivated by [Dop82] we consider the duality pairing

$$\mathcal{A} \leftrightarrow^\perp \text{DHR}\{\mathcal{A}\}$$

between the DHR category and the algebra $\mathcal{A}$ of quasilocal observables of a given (Haag dual) local conformal net $\{\mathcal{A}\}$, defined by the action $(a, \rho) \mapsto \rho(a)$.

Definition 4.1. Given a unital $C^*$-subalgebra $\mathcal{N} \subset \mathcal{A}$ we define its dual as

$$\mathcal{N}^\perp := \{\rho \in \text{DHR}\{\mathcal{A}\} : \rho(n) = n, n \in \mathcal{N}\}$$

and $\text{Hom}_{\mathcal{N}^\perp}(\rho, \sigma) := \text{Hom}_{\text{DHR}\{\mathcal{A}\}}(\rho, \sigma)$ for every $\rho, \sigma \in \mathcal{N}^\perp$. In other words, $\mathcal{N}^\perp \subset \text{DHR}\{\mathcal{A}\}$ is a full subcategory, i.e., specified by its objects only.

$\mathcal{N}^\perp$ is automatically a unital tensor category of endomorphisms of $\mathcal{A}$. Conversely

Definition 4.2. Given a unital tensor full subcategory $\mathcal{C} \subset \text{DHR}\{\mathcal{A}\}$ we define its dual as

$$\mathcal{C}^\perp := \{a \in \mathcal{A} : \sigma(a) = a, \sigma \in \mathcal{C}\}.$$
\( C^\perp \) is automatically a unital \( C^* \)-subalgebra of \( \mathcal{A} \). We have the following

**Proposition 4.3.** Let \( \{ \mathcal{A} \} \) be a local conformal net on the line fulfilling in addition Haag duality on \( \mathbb{R} \) (assumption (a)). Take \( \mathcal{A}(W) \subset \mathcal{A} \) where \( W \subset \mathbb{R} \) is a half-line, left or right oriented, then

\[
\mathcal{A}(W)^\perp = \text{DHR}^W \{ \mathcal{A} \}
\]

where \( \text{DHR}^W \{ \mathcal{A} \} \) is the full subcategory of \( \text{DHR} \{ \mathcal{A} \} \) whose objects are the endomorphisms localizable in the half-line \( W' \), opposite to \( W \).

**Proof.** One inclusion is trivial, the other follows from the definition of DHR localizability of endomorphisms and norm continuity.

Combining Proposition 2.7 and 4.3 we obtain

**Corollary 4.4.** Let \( \{ \mathcal{A} \} \) be a local conformal net on the line fulfilling Haag duality on \( \mathbb{R} \) (assumption (a)) and the split property (assumption (b)). Then

\[
\mathcal{A}(W)^\perp = \text{DHR}^W \{ \mathcal{A} \}
\]

for every half-line \( W \subset \mathbb{R} \), left or right oriented. In particular

\[
\mathcal{A}(W)^\perp \simeq \text{DHR} \{ \mathcal{A} \} \simeq \mathcal{A}(W)^{\perp c}
\]

as UBTCs.

Also, by definition, we have trivial braiding operators

\[
\varepsilon_{\rho \sigma} = 1
\]  

(11)

whenever \( \rho \in \text{DHR}^W \{ \mathcal{A} \}, \sigma \in \text{DHR}^{W'} \{ \mathcal{A} \} \) and \( W \) is a left half-line, hence \( W' \) a right half-line. Equation (11) is the characteristic feature of the DHR braiding coming from spacetime localization of charges in QFT. An abstract UBTCs need not have this kind of trivialization property for braiding operators at all.

The situation is different for local algebras \( \mathcal{A}(I) \subset \mathcal{A} \), \( I \in \mathcal{I} \), as shown by Doplicher in [Dop82, Prop. 2.3] with the split property (assumption (b)):

**Proposition 4.5.** [Dop82]. Let \( \{ \mathcal{A} \} \) be a local conformal net on the line fulfilling in addition assumptions (a) and (b), then

\[
\mathcal{A}(I)^{\perp c} = \langle \text{Inn}^I \{ \mathcal{A} \} \rangle_\oplus
\]

for every \( I \in \mathcal{I} \), where \( \text{Inn}^I \{ \mathcal{A} \} \) is the full subcategory of \( \text{DHR} \{ \mathcal{A} \} \) whose objects are the inner automorphisms localizable in \( I \) and \( (\cdot)_\oplus \) denotes the completion under (finite) direct sums in \( \mathcal{A}(I) \), i.e., the inner endomorphisms localizable in \( I \).
In particular,
\[ \mathcal{A}(I) \perp \simeq \text{DHR}\{\mathcal{A}\}, \quad \mathcal{A}(I)^c \perp \simeq \text{Vec}. \quad (12) \]

**Remark 4.6.** The previous proposition has a deep insight in the theory of DHR superselection sectors in any spacetime dimension, see also [Bor65 Lem. III-1 (erratum)], [DHR69 Sec. V], [Rob11 Sec. 1.9] and discussions therein. Notice also that the proof in [Dop82] is formulated in 3+1 dimensions and holds in the case of Abelian gauge symmetry, i.e., DHR automorphisms only. See [Müg99 Prop. 4.2] for the adaptation to the general case, and [Dri79] for related arguments. Notice also that by definition $\text{DHR}^I\{\mathcal{A}\} = \mathcal{A}(I)^\perp$. Furthermore, using now all the assumptions of complete rationality (a), (b), (c), we can prove our second main structure result

**Proposition 4.7.** Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line, then

\[ \text{DHR}^I\{\mathcal{A}\} \perp = \mathcal{A}(I') \]

for every $I \in \mathcal{I}$.

**Proof.** ($\supset$): trivial by definition of DHR localization.

($\subset$): take $a \in \mathcal{A}$ such that $\rho(a) = a$ for all $\rho \in \text{DHR}^I\{\mathcal{A}\}$. It follows easily that $a \in \mathcal{A}(I)^c = \mathcal{A}(I') \cap \mathcal{A}$ by using inner automorphisms localizable in $I$, the task is to show that $a \in \mathcal{A}(I')$. We divide the proof into three steps.

We first assume that (i) $a \in \mathcal{A}_{\text{loc}}$, i.e., $a \in \mathcal{A}(K)$ for some sufficiently big interval $I \subset K$ and that (ii) all DHR endomorphisms have dimension $d_\rho = 1$ (pointed category case).

Then the inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)^c$ is locally the two-interval subfactor $\mathcal{A}(I_1 \cup I_2) \subset \mathcal{A}(I') \cap \mathcal{A}(K) = \mathcal{A}(I)^c$ where $I' \cap K = I_1 \cup I_2$ and $I_1, I_2 \in \mathcal{I}$. Hence $a \in \mathcal{A}(I)^c$ has a unique “harmonic” expansion \[LR95\ Eq. (4.10)]

\[ a = \sum_{i=0, \ldots, n} a_i \overline{R}_i \quad (13) \]

where $a_i \in \mathcal{A}(I_1 \cup I_2)$ are uniquely determined coefficients and $\overline{R}_i \in \mathcal{A}(I)^c$ are (fixed) generators of the extension. The computation of this extension is the core of \[KLM01\]. The extension has finite index by assumption (c) and the generators are uniquely determined, up to multiplication with elements of $\mathcal{A}(I_1 \cup I_2)$, by the DHR category of $\{\mathcal{A}\}$. Indeed

\[ \overline{R}_i \in \text{Hom}_{\text{DHR}(\mathcal{A}(I))}(\text{id}, \rho_i^1 \overline{R}_i) \]
are solutions of the conjugate equations \[LR97\] Sec. 2 for the \(i\)-th sector \([\rho_i]\) where \(\rho_i^1\) is localizable in \(I_1\) and \(\bar{\rho}_i^2\) is localizable in \(I_2\), and \(n\) is the number of DHR sectors of the theory different from the vacuum \([\rho_0] = [\text{id}]\). By Frobenius reciprocity \[LR97\] Lem. 2.1] and up to multiplication with elements of \(\mathcal{A}(I_1 \cup I_2)\), the generators \(R_i\) can be thought as unitary \([\rho_i]\)-charge transporters from \(I_2\) to \(I_1\), equivalently as unitary \([\bar{\rho}_i]\)-charge transporters from \(I_1\) to \(I_2\). By assumption, for all \(\rho \in \text{DHR}\{\mathcal{A}\}\) we have

\[
a = \sum_i a_i R_i = \rho(a) = \sum_i a_i \rho(R_i)
\]

To fix ideas, from now on we assume \(I_1\) left to \(I\) and \(I_2\) right to \(I\). By naturality and tensoriality of the braiding, see \[DHR71\] Lem. 2.6], \[FRS92\] Sec. 2.2], we have

\[
\varepsilon_{\rho^1_i, \rho} \bar{\rho}_i R_i = \rho(R_i)
\]

which reduces to

\[
\rho(R_i) = \varepsilon_{\rho^1_i, \rho} \bar{\rho}_i
\]

because of the respective localization properties of the endomorphisms. In this special case we have \(\varepsilon_{\rho^1_i, \rho} = \lambda_{\rho^1_i, \rho} I\) where \(\lambda_{\rho^1_i, \rho} \in \mathbb{T}\) is a complex phase, hence \(a_i \varepsilon_{\rho^1_i, \rho} \in \mathcal{A}(I_1 \cup I_2)\) and by uniqueness of the previous expansion, if \(a_i \neq 0\) we must have \(\varepsilon_{\rho^1_i, \rho} = I\) for all \(\rho \in \text{DHR}^I\{\mathcal{A}\}\). But also \(\varepsilon_{\rho, \rho^1_i} = I\) for all \(\rho \in \text{DHR}^I\{\mathcal{A}\}\), hence \([\bar{\rho}_i]\) is degenerate. By modularity of the category all coefficients \(a_i = 0\) for \(i = 1, \ldots, n\) and we are left with \(a = a_0\) because \(R_0 = I\) can be chosen without loss of generality. In particular, \(a \in \mathcal{A}(I_1 \cup I_2)\).

We now relax the assumption (ii) about the category and allow DHR endomorphisms of dimension \(d_\rho > 1\). As above we have

\[
a = \rho(a) = \sum_i a_i \varepsilon_{\rho^1_i, \rho} R_i
\]

for all \(\rho \in \text{DHR}^I\{\mathcal{A}\}\) but now the coefficients have different localization properties and we need a more refined argument. Then rewrite

\[
a = \sum_i a_i \rho^1_i (\varepsilon_{\rho, \rho^1_i} \varepsilon_{\rho^1_i, \rho}) R_i
\]

and consider for all \(\rho \in \text{DHR}^I\{\mathcal{A}\}\) a conjugate endomorphism \(\bar{\rho}\) again localizable in \(I\) and operators \(R_\rho \in \text{Hom}_{\text{DHR}(\mathcal{A}(I))}(\text{id}, \rho \bar{\rho})\) as before. The latter are \(R_\rho \in \mathcal{A}(I)\) and can be normalized such that \(R_\rho \bar{R}_\rho = d_\rho I\). Then we can write

\[
a = d_\rho^{-1} \bar{R}_\rho R_\rho a = d_\rho^{-1} \bar{R}_\rho a \bar{R}_\rho = d_\rho^{-1} \sum_i a_i \bar{R}_\rho \rho^1_i (\varepsilon_{\rho, \rho^1_i} \varepsilon_{\rho^1_i, \rho}) \bar{R}_i \bar{R}_\rho
\]
by locality, and using $\rho_i^P(R^\rho) = R^\rho$ we have also

$$a = \rho(a) = d^{-1}_\rho \sum_i a_i \rho_i^P(R^\rho) \rho_i^1(\varepsilon_{\rho,\rho^2_i} \varepsilon_{\rho^2_i,\rho}) \overline{R_i} R^\rho$$

where on the right hand side we have formed a “killing-ring”, after [BEK99, Sec. 3], in order to exploit modularity. Then choose one representative for each sector $\rho_j \in \text{DHR}$ where $j = 0, \ldots, n$ and consider

$$(\sum_j d^2_{\rho_j}) a = \sum_j d^2_{\rho_j} \rho_j(a) = \sum_i a_i d_{\rho_i} \rho_i^P(R^\rho_{\rho_i}) \rho_i^1(\varepsilon_{\rho,\rho^2_i} \varepsilon_{\rho^2_i,\rho}) \overline{R_i} R^\rho_{\rho_i}$$

$$= \sum_i a_i (\sum_k d^2_{\rho_k}) \delta_{|\rho_i|,|\rho|} \overline{R_i} = (\sum_k d^2_{\rho_k}) a_0 \overline{R_0}$$

by unitarity of the $S$-matrix, as shown by [Reh90] in the case of UMTCs. As before we conclude $a = a_0 \in \mathcal{A}(I_1 \cup I_2)$.

It remains the case when $a \in \mathcal{A}\setminus\mathcal{A}_{\text{loc}}$ relaxing assumption (i). By the split property (assumption (b)) we have that $\mathcal{A}(I)$ is injective hence generated by an amenable group of unitaries. Averaging over its adjoint action (cf. proof of Proposition 2.7) we get a conditional expectation $E : \mathcal{B} = \mathcal{A}(I) \lor \mathcal{A}(I)' \to \mathcal{A}(I)'$ mapping for all $I \in K, K \in \mathcal{I}$

$$E(\mathcal{A}(K)) = \mathcal{A}(K) \cap \mathcal{A}(I)', \quad E(\mathcal{A}) = \mathcal{A}(I)^c.$$

Since $E$ is norm continuous we have

$$\mathcal{A}(I)^c = C^*(\cup_{n \in \mathbb{N}} \mathcal{A}(K_n) \cap \mathcal{A}(I)'), \quad I \in \mathcal{K}_n \not\subseteq \mathbb{R}, \ K_n \in \mathcal{I}$$

hence we can write $a = \lim_n a_n$ where $a_n \in \mathcal{A}(K_n) \cap \mathcal{A}(I)'$. As in the previous steps we get

$$a_n = \sum_i a_{n,i} \overline{R_i}$$

where we can choose $\overline{R_i}$ independently of $n$ (at least for big $n$). From the assumptions and norm continuity of $\rho \in \text{DHR} \{\mathcal{A}\}$ we have

$$a = \rho(a) = \lim_n \rho(a_n) = \lim_n \sum_i a_{n,i} \varepsilon_{\rho^2_i,\rho} \overline{R_i}.$$

Now we show that for all $i$ the sequences $(a_{n,i})_n$ converge to some $b_i \in \mathcal{A}(I')$. Indeed the coefficients are explicitly given [LR95, Eq. (4.10)] as

$$a_{n,i} = \lambda E_n(a_n \overline{R_i})$$

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where $\lambda$ is the $\mu_2$-index of the two-interval subfactor and we denoted by $E_n : \mathcal{A}(K_n) \cap \mathcal{A}(I') \to \mathcal{A}(K_n \cap I')$ the minimal conditional expectations, see [KLM01, Prop. 5]. Compute

$$\|a_{n,i} - a_{m,i}\| = \lambda \|E_n(a_{n,R_i^*}) - E_m(a_{m,R_i^*})\|$$

but now it holds [KLM01, Lem. 11] that $E_m(\mathcal{A}(K_n) \cap \mathcal{A}(I')) = E_n$ if $m > n$, thus

$$\lambda \|E_m((a_n - a_m)R_i)\| \leq \lambda (d_{a_n})^{1/2} \|a_n - a_m\| \to 0$$

for $n, m \to \infty$. Then $(a_{n,i})_n$ are Cauchy sequences. Since $\mathcal{A}(I')$ is by definition norm closed, the limit points $b_i \in \mathcal{A}(I')$ exist. Hence we have shown that the (local) unique expansion formula (13) makes sense also in the quasilocal limit for the inclusion $\mathcal{A}(I') \subset \mathcal{A}(I)^c$

$$a = \sum_i b_i R_i. \quad (14)$$

With the same argument as in the (local) two-interval case we can show that $\rho(a) = a$ for all $\rho \in \text{DHR}^I \{\mathcal{A}\}$ implies $b_i = 0$ whenever $i \neq 0$, hence $a = b_0 \in \mathcal{A}(I')$ and the proof is complete. \qed

**Remark 4.8.** A statement similar to the previous proposition appears in [Dop82] as a “natural conjecture” which explains the shape of the inclusion $\mathcal{A}(O') \subset \mathcal{A}(O)^c$ where $O$ is any open double cone region in Minkowski spacetime $\mathbb{R}^{3+1}$. The generators of the extension can be interpreted in that case as local measurements of (global Abelian) superselection charges, see also [DL83]. The situation here is much different: DHR superselection charges in low dimensions have non-degenerate braided statistics (opposite to permutation group), the category is modular instead of symmetric, there is no global gauge symmetry and the generators of the extension $\mathcal{A}(I') \subset \mathcal{A}(I)^c$, where $I \in \mathcal{I}$, seem to have a purely topological nature. Surprisingly (in the light of the previous facts) the proof of the statement relies essentially on modularity. To our knowledge, by now there is no other proof of the statement in different contexts.

From the previous proof, we also get the following

**Corollary 4.9.** With the assumptions of Proposition 4.7, every element $a \in \mathcal{A}(I)^c = \mathcal{A}(I') \cap \mathcal{A}$ admits a unique “harmonic” expansion, cf. [LR95, Eq. (4.10)]

$$a = \sum_{i=0,\ldots,n} b_i R_i.$$
where \( b_i \in \mathcal{A}(I') \) are uniquely determined coefficients and \( \overline{R}_i \in \text{Hom}(\text{id}, \rho_i \rho_i^* \rho_i^*) \subset \mathcal{A}(I) \) are \((\text{fixed})\) generators of the extension of unital \( C^*\)-algebras

\[
\mathcal{A}(I') \subset \mathcal{A}(I)^c.
\]

In particular, for holomorphic conformal nets it holds (cf. Proposition 2.7)

\[
\mathcal{A}_{\text{holo}}(I') = \mathcal{A}_{\text{holo}}(I)^c.
\]

\[\text{Remark 4.10.} \] Relations analogous to Proposition 4.7 hold for half-lines \( W \subset \mathbb{R} \), namely \( \text{DHR}^W(\{A\})^\perp = \mathcal{A}(W) \) as one can easily show using Proposition 2.7. We shall see later a more general argument, see Proposition 6.5.

5 Local duality relations

We turn now to the local picture, i.e., consider as environment some local algebra \( \mathcal{A}(I_0) \) for arbitrarily fixed \( I_0 \in \mathcal{I} \) instead of the quasilocal algebra \( \mathcal{A} \). Similarly to (10) we consider the local duality pairing

\[
\mathcal{A}(I_0) \leftrightarrow \text{DHR}^{I_0}(\{A\}). \tag{15}
\]

The local version of all the statements we made in Section 4 follows analogously, thanks to strong additivity, by considering \((\text{local})\) interval algebras \( \mathcal{A}(I) \subset \mathcal{A}(I_0) \) if \( I \subset I_0, \) \( I \in \mathcal{I} \), and \((\text{local})\) half-line algebras \( \mathcal{A}(I_1) \subset \mathcal{A}(I_0) \) if \( I_1 = W \cap I_0, W \subset \mathbb{R} \) is any half-line with origin \( p \in I_0 \).

In the following the symbol \( \perp \) will refer to (15). Similarly to the notion of relative commutant for unital inclusions of algebras, i.e., \( N^c = N' \cap \mathcal{A}(I_0) \) if \( N \subset \mathcal{A}(I_0) \), we introduce relative commutants of subcategories

\[\text{Definition 5.1.} \] Let \( C \subset \text{DHR}^{I_0}(\{A\}) \) be a unital full inclusion of tensor categories, we define the relative commutant as

\[ C^c := \{ \rho \in \text{DHR}^{I_0}(\{A\}) : \rho \sigma = \sigma \rho, \sigma \in C \} \]

where the equality sign means pointwise equality as endomorphisms of \( \mathcal{A}(I_0) \), or equivalently of \( \mathcal{A} \). We define \( C^c \subset \text{DHR}^{I_0}(\{A\}) \) as a full subcategory, i.e., \( \text{Hom}_{C^c}(\rho, \sigma) := \text{Hom}_{\text{DHR}(\mathcal{A})}(\rho, \sigma) \) for every \( \rho, \sigma \in C^c \).

\( C^c \) is automatically a unital tensor category of endomorphisms of \( \mathcal{A}(I_0) \). Now combining relative commutants and duals, given a subalgebra \( \mathcal{N} \subset \mathcal{A}(I_0) \) we define a unital tensor full subcategory \( C_{\mathcal{N}} \subset \text{DHR}^{I_0}(\{A\}) \) as

\[ C_{\mathcal{N}} := \mathcal{N}^c \perp \]

where by definition \( \text{Hom}_{C_{\mathcal{N}}}(\rho, \sigma) = \text{Hom}_{\text{DHR}(\mathcal{A})}(\rho, \sigma) \) for every \( \rho, \sigma \in C_{\mathcal{N}} \).
Remark 5.2. Despite we use the term “local” for the duality pairing \([15]\) and for the respective subcategories of DHR\(I_0\)\(\{A\}\) defined as above, it should be kept in mind that both \(C_N\) and DHR\(I_0\)\(\{A\}\) are categories of globally defined endomorphisms of the quasilocal algebras \(A\), which then are “localizable” in smaller regions, e.g., \(I_0\), i.e., act trivially on every local algebra \(A(J), J \subset I_0'\) and on \(N^c\).

Summarizing the previous results, we have

**Corollary 5.3.** Let \(p \in I_0\) and \(I_0 \setminus \{p\} = I_1 \cup I_2\). Let \(\mathcal{N} := A(I_1)\), then \(N^c = A(I_2), C_N = \text{DHR}^{I_1}\{A\}, C_{N^c} = \text{DHR}^{I_2}\{A\}\). Moreover, if \(I_1\) is to the left of \(I_2\), then \(\varepsilon_{\rho,\sigma} = 1\) whenever \(\rho \in C_N, \sigma \in C_{N^c}\).

**Remark 5.4.** It is well known that a point as the localization of an observable is an over-idealization, forcing fields to be distributions, and making the intersections of local algebras corresponding to regions intersecting at a point trivial. In contrast, the proper way of “lifting” points to quantum field theory rather seems to be their role as separators between local algebras, trivializing the braiding as in Corollary 5.3.

### 6 Abstract points

Let \(\{A\}\) be a completely rational conformal net on the line (Definition 2.31). In the previous two sections we essentially used the action of the DHR category, and its abstract structure of UMTC. Now we employ the DHR braiding as well, see equation (11) and comments thereafter, hence the braided action (Definition 3.2) given by the restriction functor

\[
\mathcal{C} := \text{DHR}^{I_0}\{A\} \hookrightarrow \text{End}(\mathcal{M}_0)
\]

where \(\mathcal{M}_0 := A(I_0)\) and \(I_0 \in \mathcal{I}\) is an arbitrarily fixed interval.

**Definition 6.1.** We call **abstract point** of \(\mathcal{M}_0\) an ordered pair of algebras \((\mathcal{N}, \mathcal{N}^c)\) where \(\mathcal{N} \subset \mathcal{M}_0\) such that

(i) \(\mathcal{N}\) and \(\mathcal{N}^c\) are injective type III\(_1\) factors.
(ii) \(\mathcal{N} = \mathcal{N}^c\) and \(\mathcal{N} \vee \mathcal{N}^c = \mathcal{M}_0\).
(iii) \(C_N \simeq \mathcal{C}\) and \(C_{N^c} \simeq \mathcal{C}\) as UBTCs.
(iv) \(\varepsilon_{\rho,\sigma} = 1\) whenever \(\rho \in C_N, \sigma \in C_{N^c}\).
With abuse of notation we denote abstract points by \( p := (\mathcal{N}, \mathcal{N}^c) \), and call \( \mathcal{N}, \mathcal{N}^c \) respectively the \textit{left, right relative complement of} \( p \) \textit{in} \( \mathcal{M}_0 \).

More generally, given an “abstract” UMTC \( \mathcal{C} \) together with a braided action on the injective type \( \text{III}_1 \) factor \( \mathcal{M} \), see Definition 3.2 and Remark 3.5 we can analogously define abstract points of \( \mathcal{M} \) (with respect to the braided action \( \mathcal{C} \hookrightarrow \text{End}(\mathcal{M}) \)). In the case of a UMTC coming from a completely rational conformal net, \( \mathcal{C} = \text{DHR}^{I_0}\{A\} \) together with its \textit{canonical} braided action on \( \mathcal{M}_0 \), the existence of those is the content of the previous sections.

\textbf{Remark 6.2.} Condition (iii) is indeed equivalent to essential surjectivity of the inclusion functors \( \mathcal{C}_\mathcal{N} \subset \mathcal{C} \) and \( \mathcal{C}_{\mathcal{N}^c} \subset \mathcal{C} \). In fact \( \mathcal{C}_\mathcal{N} \subset \mathcal{C} \subset \text{DHR}\{A\} \) are full inclusions by definition, the latter also essentially surjective, and the inclusion functor is trivially unitary strict tensor and braided.

\textbf{Remark 6.3.} Condition (iv) consists a priori of uncountably many constraints on braiding operators. We shall see in Proposition 6.11 that it is indeed equivalent to a finite system of equations. This makes (iv) a more tractable (“rational”) condition.

\textbf{Remark 6.4.} From Corollary 5.3 we know that ordered pairs of local algebras \((A(I_1), A(I_2))\), associated respectively to the left and right relative complements \( I_1, I_2 \) of some \( p \in I_0 \), are also abstract points of \( \mathcal{M}_0 = A(I_0) \). We shall refer to them as \textit{honest} points of \( \mathcal{M}_0 \) (with respect to the net \{\( A \}\}). The converse is not true in general, see in Sections 7 and 8.

At the level of generality of Definition 6.1 we can show the following

\textbf{Proposition 6.5.} Let \( p = (\mathcal{N}, \mathcal{N}^c) \) be an abstract point of \( \mathcal{M}_0 \), then the quadruple \((\mathcal{N}, \mathcal{N}^c, \mathcal{C}_\mathcal{N}, \mathcal{C}_{\mathcal{N}^c})\) is uniquely determined by any one of its elements.

\textbf{Proof.} It is sufficient to show that \( \mathcal{C}_{\mathcal{N}^c} \) determines \( \mathcal{N} \). By definition \( \mathcal{C}_{\mathcal{N}^c} \perp \mathcal{N}^c \perp = \mathcal{N} \perp \perp \) holds and the inclusion \( \mathcal{N} \subset \mathcal{N} \perp \) is trivial. The opposite inclusion also holds for algebras of the form \( \mathcal{N} = \mathcal{P}^c \), where \( \mathcal{P} \subset \mathcal{M}_0 \) is any unital \( \mathcal{C}^\ast \)-subalgebra of \( \mathcal{M}_0 \), cf. [Dop82, Sec. 5], in our case \( \mathcal{P} = \mathcal{N}^c \). Let \( a \in \mathcal{N} \perp \) and consider the unitary group \( \mathcal{U}(\mathcal{P}) \), then \( \text{Ad}_u \in \mathcal{N} \perp \) for all \( u \in \mathcal{U}(\mathcal{P}) \) hence \( \text{Ad}_u(a) = a \) and we conclude \( a \in \mathcal{U}(\mathcal{P})' \). Now \( \mathcal{U}(\mathcal{P}) \) linearly spans \( \mathcal{P} \), hence \( a \in \mathcal{M}_0 \cap \mathcal{P}' = \mathcal{P}^c = \mathcal{N} \).

The gain in considering together pairs of subfactors or pairs of subcategories is that we can use the braiding operators between endomorphisms as a remnant of their localization properties (left/right separation) hence, dually, of the net. The first interesting consequence of Definition 6.1 is however the following
Proposition 6.6. Let \((N, N^c)\) be a pair of subfactors of \(M_0\) fulfilling conditions (i) and (ii) in the Definition of abstract points.

If we consider for instance \(N \subset M_0\) and the associated \(C_N \subset C\), we have

- if \(\rho \in C_N\) then \(\rho \in \text{End}(N)\).
- if \(t \in \text{Hom}_{C_N}(\rho, \sigma)\) where \(\rho, \sigma \in C_N\), then \(t \in N\).
- if \(t \in N\) and \(t\rho(n) = \sigma(n)t\) for all \(n \in N\) where \(\rho, \sigma \in C_N\), then \(t \in \text{Hom}_{C_N}(\rho, \sigma)\).

In other words, we have a well-defined, faithful and full restriction functor \(\rho \mapsto \rho\lrcorner_N\)

- if \(\rho \in C_N\) and \(u \in U(N)\) then \(\text{Ad}_u \rho \in C_N\).

Hence the restriction functor has replete image, i.e., it is specified by its sectors (unitary isomorphism classes of objects) only.

Proof. First, take \(\rho \in C_N = N^c\) and \(n \in N\), then \(\rho(n)m = \rho(nm) = m\rho(n)\) for all \(m \in N^c\) and we get \(\rho(n) \in M_0 \cap N^c = N^cc = N\).

Second, take \(t \in M_0\) such that \(t\rho(a) = \sigma(a)t\) for all \(a \in M_0\), where \(\rho, \sigma \in C_N\). Now, letting \(a \in N^c\) we have \(ta = at\) hence \(t = N^cc = N\).

Third, we have \(t \in N\) and \(t\rho(n) = \sigma(n)t\) if \(n \in N\) by definition and \(t\rho(m) = \sigma(m)t\) if \(m \in N^c\) because \(tm = mt\). Now, every \(a \in M_0 = N \vee N^c\) can be written as an ultra-weak limit of finite sums \(a = \text{uw-lim} \sum_i n_i m_i\) where \(n_i \in N\) and \(m_i \in N^c\). Also, \(\rho, \sigma\) are automatically normal on \(M_0\), see [Tak02, p. 352], being \(M_0\) non-type \(I\) and \(H\) separable. Normality on \(M_0 = \mathcal{A}(I_0)\) can also be derived by DHR transportability of the endomorphisms, but we prefer the previous argument which is intrinsic and local. From these two facts we conclude that \(t\rho(a) = \sigma(a)t\) for all \(a \in M_0\), hence as DHR endomorphisms because local intertwiners are global, i.e., \(C \hookrightarrow \text{End}(M_0)\) is full.

The last point is trivial to show, but has interesting consequences (see Proposition 6.7).

The conditions stated in Definition contain many redundancies. Out of the operator algebraic assumptions (i) and (ii) on \(N\) and \(N^c\), one can derive properties of their dual categories \(C_N\) and \(C_N^c\) which are custom assumptions in \(C^*\) tensor category theory, see, e.g., [LR97]. Nevertheless, assumptions (iii) and (iv) cannot be derived from the previous, see Proposition 4.3 and 4.5, unless the net \(\{\mathcal{A}\}\) is holomorphic.
Proposition 6.7. Let \((\mathcal{N}, \mathcal{N}^c)\) be a pair of subfactors of \(\mathcal{M}_0\) fulfilling conditions (i) and (ii) in the Definition 6.1 of abstract points. Then the subcategories \(\mathcal{C}_\mathcal{N}\) and \(\mathcal{C}_{\mathcal{N}^c}\) automatically have irreducible tensor unit, subobjects, finite direct sums and conjugate objects.

In other words, they are \(C^*\) tensor categories which are also fusion and rigid.

Proof. The restriction functor \(\mathcal{C}_\mathcal{N} \hookrightarrow \text{End}(\mathcal{N})\) is full and faithful by Proposition 6.6, hence irreducibility of the tensor unit of \(\mathcal{C}_\mathcal{N}\) is equivalent to factori-ality of \(\mathcal{N}\).

In general the existence of subobjects in DHR\{\(A\}\} follows because we have a net of type \(\text{III}\) factors, i.e., \(\mathcal{A}(I_0)\) alone being type \(\text{III}\) is not sufficient to construct DHR subendomorphisms. In our case we need again Proposition 6.6 together with \(\mathcal{N}\) being type \(\text{III}\). Let \(\rho \in \mathcal{C}_\mathcal{N}\) and \(e \in \text{Hom}_{\mathcal{C}_\mathcal{N}}(\rho, \rho) \subset \mathcal{N}\)
a non-zero orthogonal projection. Choose \(v \in \mathcal{N}\) such that \(v^*v = 1\), \(vv^* = e\) and let \(\sigma(n) := v^*\rho(n)v\), \(n \in \mathcal{N}\), then \(\sigma \in \text{End}(\mathcal{N})\) by definition. In order to show \(\sigma \lessdot \rho\) in \(\mathcal{C}_\mathcal{N}\) we need to extend \(\sigma\) to \(\mathcal{M}_0\) and then to the quasilocal algebra \(\mathcal{A}\), in such a way that the intertwining relation \(v \in \text{Hom}_{\mathcal{C}_\mathcal{N}}(\sigma, \rho)\) holds, cf. Remark 5.2. Now \(\sigma(m) := v^*\rho(m)v\), \(m \in \mathcal{N}^c\), and \(\rho\) is normal on \(\mathcal{M}_0\) hence \(\sigma\) extends to \(\text{End}(\mathcal{M}_0)\) with \(\sigma|_{\mathcal{N}^c} = \text{id}\) and \(v \in \text{Hom}_{\text{End}(\mathcal{M}_0)}(\sigma, \rho)\). On the other hand \(\rho \in \mathcal{C}\) and \(\mathcal{C}\) has subobjects, hence let \(w \in \mathcal{M}_0\) and \(\tau \in \mathcal{C}\) such that \(w^*w = 1\), \(ww^* = e\) and \(w \in \text{Hom}_{\mathcal{C}}(\tau, \rho) = \text{Hom}_{\text{End}(\mathcal{M}_0)}(\tau, \rho)\). Now \(w^*v\) is unitary in \(\text{Hom}_{\text{End}(\mathcal{M}_0)}(\sigma, \tau)\) hence we can extend \(\sigma \in \mathcal{C}\) because \(\mathcal{C} \hookrightarrow \text{End}(\mathcal{M}_0)\) is replete. Thus \(\sigma \in \mathcal{C}_\mathcal{N}\) and \(v \in \text{Hom}_{\mathcal{C}_\mathcal{N}}(\sigma, \rho)\) because \(\mathcal{C}_\mathcal{N} \hookrightarrow \text{End}(\mathcal{N})\) is full.

Along similar lines one can show the existence of direct sums in \(\mathcal{C}_\mathcal{N}\).

To show existence of conjugates in \(\mathcal{C}_\mathcal{N}\) we need, in addition, results from the theory of infinite subfactors with finite index. Let \(\rho \in \mathcal{C}_\mathcal{N}\) be an irreducible DHR endomorphism, hence with finite (minimal) index \(\text{Ind}(\rho(\mathcal{M}_0), \mathcal{M}_0) < \infty\) [KLM01 Cor. 39], i.e., finite statistical dimension \(d_\rho < \infty\) [GL96 Cor. 3.7]. Let \(\Phi\) be the unique left inverse of \(\rho\), see [GL96 Cor. 2.12], which is normal on \(\mathcal{M}_0\) and localizable in \(I_0\), hence in particular \(\Phi(\mathcal{M}_0) \subset \mathcal{M}_0\).

For every \(n \in \mathcal{N}\), \(m \in \mathcal{N}^c\) we have \(\Phi(m) = \Phi(\rho(m)) = m\) and \(\Phi(n)m = \Phi(n\rho(m)) = \Phi(nm) = m\Phi(n)\) hence \(\Phi|_{\mathcal{N}^c} = \text{id}\) and \(\Phi(\mathcal{N}) \subset \mathcal{N}^c = \mathcal{N}\).

Again by Proposition 6.6 irreducibility of \(\rho\) is equivalent to irreducibility of the subfactor \(\rho(\mathcal{N}) \subset \mathcal{N}\), then \(E_{\mathcal{N}} := \rho \circ \Phi_{\mathcal{N}}\) coincides with the unique normal faithful (minimal) conditional expectation given by [Lon89 Thm. 5.5]. After setting \(\lambda := \text{Ind}(\rho(\mathcal{M}_0), \mathcal{M}_0)^{-1}\), we have the Pimsner-Popa bound
where $\lambda$ is the best constant fulfilling equation (16). In particular, it holds for all $a \in \mathcal{N} \subset \mathcal{M}_0$ and if we let $\mu := \text{Ind}(\rho(\mathcal{N}), \mathcal{N})^{-1}$ by the same argument on $\rho(\mathcal{N}) \subset \mathcal{N}$ and by uniqueness of $E|_{\mathcal{N}}$ we get $\mu \geq \lambda$, hence $\text{Ind}(\rho(\mathcal{N}), \mathcal{N}) < \infty$.

Now we turn to the construction of the conjugate endomorphism of $\rho$ in $\mathcal{C}_\mathcal{N}$. As before we begin “locally”, i.e., by construction of the restriction of the conjugate as an object of $\text{End}(\mathcal{N})$ and then extend. Let $\rho_{\mathcal{N}} := \rho|_{\mathcal{N}} \in \text{End}(\mathcal{N})$ and $\overline{\gamma} := (\rho_{\mathcal{N}})^{-1} \circ \gamma \in \text{End}(\mathcal{N})$ where $\gamma$ is a canonical endomorphisms of $\mathcal{N}$ into $\rho(\mathcal{N})$ [Lon90, Thm. 3.1]. By finiteness of the index of $\rho_{\mathcal{N}} \subset \mathcal{N}$ [Lon90, Thm. 4.1 and 5.2] we have a solution $R \in \text{Hom}(\text{End}(\mathcal{N}), \text{End}(\mathcal{N}))$ of the conjugate equations [LR97, Sec. 2] in $\text{End}(\mathcal{N})$. First, we extend $\rho_{\mathcal{N}}$ to $\mathcal{M}_0$ by making use of another formula for the canonical endomorphism $[LR95, Eq. (2.19)]$

$$\gamma(n) = \lambda d_\rho^{-1} E(\overline{R} n \overline{R}), \quad n \in \mathcal{N}. \quad (17)$$

By (17) $\gamma$ extends normally to $\text{End}(\mathcal{M}_0)$ and to the quasilocal algebra $\mathcal{A}$. Also, for $m \in \mathcal{N}^c$ we get $\gamma(m) = \lambda d_\rho^{-1} E(\overline{R} m \overline{R}) = \lambda d_\rho^{-1} E(\overline{R} \overline{R}) m = m$ by [LR95, Eq. (4.1)]], hence $\gamma|_{\mathcal{N}^c} = \text{id}$ and $\gamma(\mathcal{M}_0) \subset \rho(\mathcal{M}_0)$. It follows that we can extend normally $\overline{\gamma} := \rho^{-1} \circ \gamma \in \text{End}(\mathcal{M}_0)$ because $\rho$ is injective hence bicontinuous onto its image in the ultraweak topology [Ped79, p. 59]. Moreover we have $\overline{\rho}_{\mathcal{N}^c} = \text{id}$ and $R \in \text{Hom}_{\text{End}(\mathcal{M}_0)}(\text{id}, \overline{\rho})$, $\overline{R} \in \text{Hom}_{\text{End}(\mathcal{M}_0)}(\text{id}, \overline{\rho})$.

On the other hand $\rho \in \mathcal{C}$ and let $\hat{\rho} \in \mathcal{C}$ be a DHR conjugate of $\rho$, hence by irreducibility and [Lon90, Thm. 3.1] we have a unitary $u \in \text{Hom}_{\text{End}(\mathcal{M}_0)}(\overline{\rho}, \hat{\rho})$. As above we extend $\overline{\gamma} \in \mathcal{C}$ by repleteness of $\mathcal{C} \hookrightarrow \text{End}(\mathcal{M}_0)$, hence $\overline{\gamma} \in \mathcal{C}_\mathcal{N}$ together with $R \in \text{Hom}_{\mathcal{C}_\mathcal{N}}(\text{id}, \overline{\rho})$, $\overline{R} \in \text{Hom}_{\mathcal{C}_\mathcal{N}}(\text{id}, \overline{\rho})$, and we have the statement in the irreducible case.

Now, $\overline{R}$ can be normalized in such a way $R^* R = \overline{R} \overline{R}$ gives the (intrinsic) dimension of $\rho$ in $\mathcal{C}_\mathcal{N}$. The latter does not depend on the choice of normalized solutions in $\mathcal{C}$, and equals the statistical dimension $d_\rho$ on one side and $\text{Ind}(\rho(\mathcal{N}), \mathcal{N})^{1/2}$ on the other by [LR97, p. 121]. In particular, it holds $\lambda = \mu$ and $d_\rho^2 = \text{Ind}(\rho(\mathcal{N}), \mathcal{N})$.

The construction of conjugates extends to finite direct sums, concluding the proof of the proposition for $\mathcal{C}_\mathcal{N}$. Similarly for $\mathcal{C}_\mathcal{N}^c$ interchanging the roles of $\mathcal{N}$ and $\mathcal{N}^c$.

Remark 6.8. See [GL92, Thm. 2.2, Cor. 2.4] for a similar discussion on the conjugation of endomorphisms of subfactors.
Going back to the duality between subalgebras and subcategories, under assumption (iii) we can lift the normality relations contained in (ii) from $\mathcal{N}, \mathcal{N}^c$ to $\mathcal{C}_N, \mathcal{C}_N^c$, in the sense of Definition 5.1.

**Proposition 6.9.** Let $(\mathcal{N}, \mathcal{N}^c)$ be a pair of subfactors of $\mathcal{M}_0$ fulfilling conditions (i), (ii) and (iii) in the Definition 6.1 of abstract points. Then

\[ (\mathcal{C}_N)^c = \mathcal{C}_N^c, \quad (\mathcal{C}_N^c)^c = \mathcal{C}_N^c \]

and the operations in the diagram

\[
\begin{array}{ccc}
\mathcal{N} & \overset{\iota}{\longrightarrow} & \mathcal{C}_N^c \\
\mathcal{N}^c & \overset{\iota}{\longrightarrow} & \mathcal{C}_N \\
\end{array}
\]

are commutative and invertible.

**Proof.** Take $\rho \in \mathcal{C}_N^c$ and first assume (iv) in addition, then $\varepsilon_{\sigma, \rho} = 1$ for all $\sigma \in \mathcal{C}_N$ gives in particular $\rho \sigma = \sigma \rho$ and we can conclude $\rho \in (\mathcal{C}_N)^c$.

But we want the statement independent of braiding operators, hence we use Proposition 6.6 to draw the same conclusion. Indeed $\rho(\sigma(m)) = \rho(m) = \sigma(\rho(m))$ for all $\sigma \in \mathcal{C}_N$ and $m \in \mathcal{N}^c$, and the same holds for $n \in N$. As before, by assumption (i) and (ii) we have $\mathcal{M}_0 = N \vee \mathcal{N}^c$ and $\rho, \sigma$ are normal on $\mathcal{M}_0$. Hence $\rho \sigma = \sigma \rho$ for all $\sigma \in \mathcal{C}_N$ and again $\rho \in (\mathcal{C}_N)^c$.

Vice versa, if $\rho \in (\mathcal{C}_N)^c$ then in particular $\rho \text{Ad}_u = \text{Ad}_u \rho$ for all $u \in \mathcal{U}(\mathcal{N})$, explicitly $\rho(ua^u*) = u\rho(a)u^*$ for all $a \in \mathcal{M}_0$. Then we have $u^*\rho(u) \in \text{Hom}_{\text{End}(\mathcal{M}_0)}(\rho, \rho) = \text{Hom}_{\mathcal{C}}(\rho, \rho)$. If $\rho$ is irreducible, then $u^*\rho(u) = \lambda_u$ where $\lambda_u \in \mathbb{T}$ is a complex phase. The map $u \in \mathcal{U}(\mathcal{N}) \mapsto \lambda_u \in \mathbb{T}$ is a norm continuous unitary character, hence trivial by [Kad52, Thm. 1] because $\mathcal{N}$ is a non-type $I$ factor by assumption (i), and we have $\rho(u) = u$ for all $u \in \mathcal{U}(\mathcal{N})$. In this case, we conclude $\rho \in \mathcal{N}^\perp = \mathcal{C}_N^c$.

In general, if $\rho \in (\mathcal{C}_N)^c$ is (finitely) reducible, we can write $\rho$ as a finite direct sum of irreducibles $\rho = \oplus_{i=1}^{\ldots, n} \rho_i$ with $\rho_i \in \mathcal{C}_N^c$ by assumption (iii). Notice that we already have the inclusion $\mathcal{C}_N^c \subset (\mathcal{C}_N)^c$. Let $\rho, \sigma \in (\mathcal{C}_N)^c$ and $t \in \text{Hom}_{(\mathcal{C}_N)^c}(\rho, \sigma)$, then one has

\[ \text{Ad}_u(t)\rho(\text{Ad}_u(a)) = \sigma(\text{Ad}_u(a)) \text{Ad}_u(t) \]

for every $u \in \mathcal{U}(\mathcal{N})$, because $\text{Ad}_u \in \mathcal{C}_N$. But every $\text{Ad}_u$ is an automorphisms of $\mathcal{M}_0$ hence we get $\text{Ad}_u(t) \in \text{Hom}_{(\mathcal{C}_N)^c}(\rho, \sigma)$ and $u \in \mathcal{U}(\mathcal{N}) \mapsto \text{Ad}_u$ is a group representation of $\mathcal{U}(\mathcal{N})$ on the finite-dimensional vector space.
\( V := \text{Hom}_{(\mathcal{C}_N)\gamma}(\rho, \sigma) \), see [LR97, Lem. 3.2]. Now, \( V^*V = \text{Hom}_{(\mathcal{C}_N)\gamma}(\rho, \rho) \) is isomorphic to a finite-dimensional block-diagonal matrix algebra, e.g., if \( n = 2 \) then \( \text{Hom}_{(\mathcal{C}_N)\gamma}(\rho_1 \oplus \rho_2, \rho_1 \oplus \rho_2) \) is either the full matrix algebra \( M_2(\mathbb{C}) \cong \mathbb{C}^4 \) if \( \rho_1 \cong \rho_2 \) or diagonal matrices \( \Lambda_2(\mathbb{C}) \cong \mathbb{C}^2 \) if \( \rho_1 \not\cong \rho_2 \). Hence we can consider the Hilbert inner product on \( V \) given by the (non-normalized) trace of \( V^*V \), i.e.

\[
(t|s) := \text{Tr}(t^*s) = \sum_{i=1,\ldots,n} t_i^*(t^*s)t_i
\]

where \( t, s \in V \) and \( \{t_1, \ldots, t_n\} \subset \mathcal{M}_0 \) is a Cuntz algebra of isometries defining \( \rho = \oplus_i \rho_i \), namely \( t_i^*t_j = \delta_{i,j} \) and \( \sum_i t_i t_i^* = 1 \) and \( t_i \in \text{Hom}_{(\mathcal{C}_N)\gamma}(\rho_i, \rho) \). The definition of trace does not depend on the choice of \( \{t_1, \ldots, t_n\} \) and that matrix units of \( V^*V \) form an orthonormal basis of \( V^*V \) with respect to the previous inner product. Now, given \( t, s \in V \) and \( u \in \mathcal{U}(\mathcal{N}) \), compute

\[
(\text{Ad}_u(t)|\text{Ad}_u(s)) = \text{Tr}(ut^*su^*) = \text{Tr}(\rho(u)\rho(u^*)ut^*su^*\rho(u)\rho(u^*))
\]

\[
= \sum_{i=1,\ldots,n} t_i^*(\rho(u)\rho(u^*)t^*su^*\rho(u)\rho(u^*))t_i = u \text{Tr}(\rho(u^*)ut^*su^*\rho(u))u^* = \text{Tr}(t^*s) = (t|s)
\]

because \( \rho_i(u) = u \), being \( \rho_i \in \mathcal{C}_N\gamma \), and \( u^*\rho(u) \in V^*V \) so we can use the trace property. Hence the representation of \( \mathcal{U}(\mathcal{N}) \) on \( V \) is unitary with respect to the previous inner product, and norm continuous, as one can easily check with respect to the induced \( C^* \)-norm of \( V \subset \mathcal{M}_0 \) and then using the equivalence of norms for finite-dimensional vector spaces. Again by [Kad52] and assumption (i) the representation must be trivial, i.e., \( \text{Ad}_u(t) = t \) for all \( u \in \mathcal{U}(\mathcal{N}) \), hence \( t \in \mathcal{N}^\gamma \cap \mathcal{M}_0 = \mathcal{N}^\circ \) and we have shown \( \text{Hom}_{(\mathcal{C}_N)\gamma}(\rho, \sigma) \subset \mathcal{N}^\circ \).

In conclusion, we get that every Cuntz algebra of isometries defining the direct sum \( \rho = \oplus_i \rho_i \) lies in \( \mathcal{N}^\circ \), hence we conclude \( \rho \in \mathcal{C}_N\gamma \). Both subcategories \( \mathcal{C}_N^\circ \) and \( (\mathcal{C}_N)\gamma \) are full by definition, hence they have the same Hom-spaces, and the proof is complete. \( \square \)

Concerning condition (iv) in Definition 6.1, the following shows that the braiding contains all the information about the subcategories \( \mathcal{C}_N, \mathcal{C}_N^\circ \) and charge transportation among them.

**Lemma 6.10.** Let \( p = (\mathcal{N}, \mathcal{N}^\circ) \) be an abstract point of \( \mathcal{M}_0 \). Let \( \rho \in \mathcal{C} \), then

- \( \rho \in \mathcal{C}_N \) if and only if \( \mathcal{E}_{\rho, \text{Ad}_u} = 1 \) for all \( u \in \mathcal{U}(\mathcal{N}^\circ) \).
Let \( \rho \in \mathcal{C} \), \( v \in \mathcal{U}(\mathcal{M}_0) \) and set \( \hat{\rho} := \text{Ad}_v \rho \). We call \( v \) an abstract \( \rho \)-charge transporter to \( \mathcal{C}_{N^c} \) if it holds \( \sigma(v) = v \varepsilon_{\sigma, \rho} \) for all \( \sigma \in \mathcal{C}_{N^c} \). The terminology is motivated by the following equivalence

- \( \hat{\rho} \in \mathcal{C}_{N^c} \) if and only if \( v \) is an abstract \( \rho \)-charge transporter to \( \mathcal{C}_{N^c} \).

Analogous statements hold interchanging \( N \) with \( N^c \) and \( \varepsilon \) with \( \varepsilon_{\text{op}} \).

**Proof.** By naturality of the braiding and using the convention \( \varepsilon_{\rho, v} = 1 \) we see that triviality of braiding operators with inner automorphisms \( \text{Ad}_u \) is triviality of the action of the endomorphism on \( u \). Hence the first statement follows.

For the second, take \( \rho \in \mathcal{C} \) and \( v \in \mathcal{U}(\mathcal{M}_0) \) an abstract \( \rho \)-charge transporter to \( \mathcal{C}_{N^c} \). For every \( \sigma \in \mathcal{C}_{N^c}, a \in \mathcal{M}_0 \) compute \( \sigma \hat{\rho}(a) = \sigma(v)\rho(a)(v^*) = v\varepsilon_{\sigma, \rho}(a)v^* = \hat{\rho}(a) \) hence \( \hat{\rho} \in (\mathcal{C}_{N^c})^c = \mathcal{C}_{N^c} \) by Proposition 6.9. Viceversa, if \( \hat{\rho} = \text{Ad}_v \rho \in \mathcal{C}_{N^c} \) for some \( v \in \mathcal{U}(\mathcal{M}_0) \) then \( \varepsilon_{\sigma, \rho} = 1 \) for every \( \sigma \in \mathcal{C}_{N^c} \) by (iv). Hence \( v\varepsilon_{\sigma, \rho}(v^*) = 1 \) and we obtain the second statement.

On the other hand, after defining \( \mathcal{C}_N \), \( \mathcal{C}_{N^c} \) by duality from \( N \), \( N^c \), condition (iv) turns out to be equivalent to a finite system of equations.

**Proposition 6.11.** Let \((N, N^c)\) be a pair of subfactors of \( \mathcal{M}_0 \) fulfilling conditions (i), (ii) and (iii) in the Definition 6.1 of abstract points. For each sector labelled by \( i = 0, \ldots, n \) choose (assumption (iii)) irreducible representatives \( \rho_i \in \mathcal{C}_N \) and \( \sigma_i \in \mathcal{C}_{N^c} \) respectively in \( \mathcal{C}_N \) and \( \mathcal{C}_{N^c} \), such that \([\rho_i] = [\sigma_i]\). Then

\[
\varepsilon_{\rho_i, \sigma_j} = 1, \quad i, j = 0, \ldots, n
\]

is equivalent to condition (iv).

**Proof.** In order to show the nontrivial implication, we first take \( \rho \in \mathcal{C}_N \) and \( \sigma \in \mathcal{C}_{N^c} \) irreducible. By Proposition 6.6 we have \( \text{Ad}_{u_i} \rho = \rho_i \) and \( \text{Ad}_{v_j} \sigma = \sigma_j \) for some \( i, j \in \{0, \ldots, n\} \) and \( u_i \in \mathcal{U}(N), v_j \in \mathcal{U}(N^c) \). Naturality of the braiding gives

\[
\varepsilon_{\rho, \sigma} = \sigma(u_i^*)v_j^*\varepsilon_{\rho_i, \sigma_j}u_i\rho(v_j)
\]

hence \( \varepsilon_{\rho, \sigma} = \sigma(u_i^*)v_j^*u_i\rho(v_j) = 1 \) because, e.g., \( u_i\rho(v_j) = u_iv_j = v_ju_i \). Hence we have shown (iv) in the irreducible case.

In the reducible case, we can write direct sums \( \rho = \sum_a s_a \rho_a s_a^* \) and \( \sigma = \sum_b t_b \sigma_b t_b^* \) where \( a, b \in \{0, \ldots, n\} \) and \( \rho_a \in \mathcal{C}_N \), \( \sigma_b \in \mathcal{C}_{N^c} \) run in our choice of

\[\text{The opposite braiding of } \mathcal{C} \text{ is defined as } \varepsilon_{\rho, \sigma}^\text{op} := \varepsilon_{\sigma, \rho}^* \text{ or equivalently by interchanging left and right localization in the DHR setting.}\]
representatives and \( \{ s_a \}_a, \{ t_b \}_b \) are Cuntz algebras of isometries respectively in \( \mathcal{N}, \mathcal{N}^c \), again by Proposition 6.6. As before

\[
\varepsilon_{\rho, \sigma} = \sum_{a,b} \sigma(s_a) t_b \varepsilon_{\rho, \sigma} s_a^* s_b^* = \sum_{a,b} s_a^* t_b^* = 1
\]

so we conclude (iv) for all \( \rho \in \mathcal{C}_N, \sigma \in \mathcal{C}_N^c \).

\[ \square \]

Remark 6.12. Thinking in terms of DHR localization properties of the endomorphisms, if we have \( \rho \in \mathcal{C}_N \), \([\rho] \neq [\mathrm{id}]\), the previous statement says that it cannot be localizable in some interval \( I_\rho \) which is to the right of some localization intervals \( I_j \) of \( \sigma_j \in \mathcal{C}_N^c \) as above, for all \( j = 0, \ldots, n \), for every choice of such \( \sigma_j \in \mathcal{C}_N^c \). This would imply degeneracy of \([\rho]\), hence contradict modularity of \( \mathcal{DHR}\{ \mathcal{A} \} \). Despite this naive left/right separation picture, and the results of the last section, we shall see next how abstract points can become wildly non-geometric or “fuzzy”. This is a typical situation in QFT where points of spacetime are replaced by (field) operators.

7 Fuzzy abstract points

Let \( \{ \mathcal{A} \} \) be a completely rational conformal net on the line, let \( I_0 \in \mathcal{I}, \mathcal{M}_0 = \mathcal{A}(I_0) \) and \( \mathcal{C} = \mathcal{DHR}^{I_0}\{ \mathcal{A} \} \). Inside \( \mathcal{M}_0 \) we can find honest points (those associated to geometric points \( p \in I_0 \), see Remark 6.4), but also uncountably many families of abstract points which are fuzzy, in the sense that they are not honest anymore (with respect to \( \{ \mathcal{A} \} \)) and do not resemble any kind of geometric interpretation. The following examples give algebraic deformations of abstract points into abstract points, and of honest points into possibly fuzzy ones.

Example 7.1. Let \( p = (\mathcal{A}(I_1), \mathcal{A}(I_2)) \) be an honest point of \( \mathcal{M}_0 \) and consider localizable unitaries \( u \in \mathcal{U}(\mathcal{M}_0) \). Then \( u p u^* = (\text{Ad}_u(\mathcal{A}(I_1)), \text{Ad}_u(\mathcal{A}(I_2))) \) is an abstract point of \( \mathcal{M}_0 \), see Definition 6.1. Indeed conditions (i) and (ii) follow because \( \text{Ad}_u : \mathcal{M}_0 \to \mathcal{M}_0 \) is a normal automorphism, in particular \( \text{Ad}_u(\mathcal{A}(I_1)^c) = \text{Ad}_u(\mathcal{A}(I_1))^c \). Now if \( \rho \in \mathcal{C}_{\mathcal{A}(I_1)} \) then \( \rho \mapsto \text{Ad}_u \circ \rho \circ \text{Ad}_u^* \) is again in \( \mathcal{C} \) because \( \text{Ad}_u \circ \rho \circ \text{Ad}_u^* = u \rho(u^*) \rho(\cdot) \rho(u) u^* \) and \( u \rho(\cdot) u^* \in \mathcal{U}(\mathcal{M}_0) \). Moreover it acts trivially on \( \text{Ad}_u(\mathcal{A}(I_1))^c \) hence \( \rho \mapsto \rho \) defines a bijection between the objects of \( \mathcal{C}_{\mathcal{A}(I_1)} \) and \( \mathcal{C}_{\text{Ad}_u(\mathcal{A}(I_1))} \), and (iii) follows. One easily checks that \( \rho \mapsto \rho \) respects the tensor structure of \( \mathcal{C} \), where the action on arrows \( s \in \text{Hom}_\mathcal{C}(\rho, \sigma), \rho, \sigma \in \mathcal{C} \) is given by \( s^\rho := \text{Ad}_u(s) \). Condition (iv) is also fulfilled because \( \rho \mapsto \rho \) respects the braiding of \( \mathcal{C} \), namely

\[
\varepsilon_{\rho, \sigma} = u \sigma(u^*) \sigma(u^*) \varepsilon_{\rho, \sigma} \rho(\sigma(u^*) u^*) \rho(u) u^* = \varepsilon_{\rho, \sigma}
\]

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by naturality, hence $\varepsilon_{\rho,\sigma} = 1$ if and only if $\varepsilon_{\rho,\sigma} = 1$. In other words $u \in \mathcal{U}(\mathcal{M}_0)$, $\rho \mapsto u\rho$ gives rise to a group of UBTC autoequivalences of $\mathcal{C}$ which are also strict tensor and automorphic.

It can happen that $upu^* = p$, e.g., if $u$ is localizable away from the cut geometric point $p \in \mathcal{I}_0$. Otherwise $u$ and $p$ need not “commute” and $upu^*$ can be viewed as a “fat” point of $\mathcal{M}_0$.

**Example 7.2.** Let $p = (\mathcal{A}(I_1), \mathcal{A}(I_2))$ as in the previous example and consider the modular group of $\mathcal{M}_0$ with respect to any faithful normal state $\varphi$, e.g., the vacuum state $\omega(\cdot) = (\Omega \cdot \Omega)$ of $\mathcal{A}$. Denote by $\Delta_\varphi$ and $\sigma^\varphi_t = \text{Ad}_{\Delta_\varphi^t}$, $t \in \mathbb{R}$ respectively the modular operator and the modular group of $(\mathcal{M}_0, \varphi)$. Then $\Delta_\varphi^t p \Delta_\varphi^{-t}$ is an abstract point of $\mathcal{M}_0$, for every $t \in \mathbb{R}$. Indeed (i) and (ii) follow as before, while (iii) is guaranteed by the existence of localizable Connes cocycles $u_{\rho,t} \in \mathcal{U}(\mathcal{M}_0)$, as shown by [Lon97, Prop. 1.1], which fulfill the intertwining relation $t^\rho = \text{Ad}_{u_{\rho,t}} \rho$ on $\mathcal{M}_0$ for $t^\rho := \sigma^\varphi_t \circ \rho \circ \sigma^\varphi_{-t}$. Hence $t^\rho$ is again DHR and $t \mapsto t^\rho$ gives a tensor autoequivalence of $\mathcal{C}$, defined on arrows as $t^s := \sigma^\varphi_s(s)$. Using more advanced technology we can show that $t \mapsto t^\rho$ respects the braiding of $\mathcal{C}$. Namely

$$\varepsilon_{t^\rho, t^\sigma} = u_{\sigma,t} \varepsilon_{\rho,t} u_{\rho,t}^* u_{\sigma,t}^* = u_{\sigma,t} \varepsilon_{\rho,t} u_{\sigma,t}^* = \sigma^\varphi_t (\varepsilon_{\rho,\sigma}) = t^\rho \varepsilon_{\rho,\sigma}$$

where the first equality follows by naturality of the braiding, the second and third by tensoriality and naturality of the Connes cocycles associated to the modular action of $\mathbb{R}$, see respectively [Lon97, Prop. 1.4, 1.3]. In particular, $\varepsilon_{\rho,\sigma} = 1$ if and only if $\varepsilon_{t^\rho, t^\sigma} = 1$, hence condition (iv) is satisfied. As before $t \in \mathbb{R}$, $\rho \mapsto t^\rho$ gives rise to a group of UBTC autoequivalences of $\mathcal{C}$ which are again strict tensor and automorphic. The point $\Delta_\varphi^t p \Delta_\varphi^{-t}$ is not honest in general, but highly fuzzy.

In the special case of the vacuum state $\varphi = \omega$, the modular action is geometric and coincides with the dilations subgroup $t \mapsto \Lambda^t_{\mathcal{I}_0}$ of \text{M"ob} which preserve $\mathcal{I}_0$ (Bisognano-Wichmann property [GL96, Prop. 1.1]), hence $\Delta_\varphi^t p \Delta_\varphi^{-t} = \Lambda^{-2\pi t}_{\mathcal{I}_0}(p)$ is just a M"obius transformed honest point (with respect to $\{\mathcal{A}\}$).

In the terminology of [Tur10, App. 5] due to M. M"uger, see also [Lon97, App. A], we have found that $\mathcal{U}(\mathcal{M}_0)$ (and all of its subgroups) and $\mathbb{R}$ (for every choice of faithful normal state on $\mathcal{M}_0$) act on $\mathcal{C}$ (as UBTC strict automorphisms), and the actions are strict. One can then define the category of “$G$-fixed points”, $\mathcal{C}^G$, where $G$ denotes one of these groups with the associated action. In our case $\mathcal{C}^G = \mathcal{C}$ because all the objects $\rho$ of $\mathcal{C}$ are “$G$-equivariant”, i.e., admit a cocycle for the $G$-action, i.e., unitary isomorphisms $v_{\rho,g} : \rho \to g \rho$, }
\( g \in G \), such that \( v_{\rho,gh} = g(v_{\rho,h}) \circ v_{\rho,g} \). In Example 7.1, the cocycle identity follows because \( \rho \) are \(*\)-homomorphisms, in Example 7.2 it coincides with the characterization of the Connes cocycles.

In our case these actions are also implemented by unitaries \( U_g \in \mathcal{U}(\mathcal{H}) \), hence we have examples of (groups of) automorphisms of the braided action \( \mathcal{C} \to \text{End}(\mathcal{M}_0) \) in the sense of Definition 3.4.

8 Prime UMTCs and prime conformal nets

There are other types of abstract points, living inside completely rational nets that factorize as tensor products, which are abstract but neither honest nor fuzzy, in the sense that they are almost geometric, or better, geometric in \(1+1\) dimensions. Ruling out these cases will lead us to the notion of prime conformal nets.

Example 8.1. Consider a completely rational conformal net on the line of the form \( \{ I \in \mathcal{I} \mapsto \mathcal{A}(I) = \mathcal{A}_1(I) \otimes \mathcal{A}_2(I) \} = \{ \mathcal{A}_1 \otimes \mathcal{A}_2 \} \), where \( \{ \mathcal{A}_1 \} \), \( \{ \mathcal{A}_2 \} \) are two nontrivial nets, then \( \text{DHR}\{ \mathcal{A} \} \simeq \text{DHR}\{ \mathcal{A}_1 \} \boxtimes \text{DHR}\{ \mathcal{A}_2 \} \) as UBTCs. An equivalence is given by \( \rho \boxtimes \sigma \mapsto \rho \otimes \sigma \), \( T \boxtimes S \mapsto T \otimes S \) where essential surjectivity follows from [KLM01, Lem. 27] and the braiding on the l.h.s. is defined as \( \varepsilon_{\rho \otimes \sigma,\tau \otimes \eta} = \varepsilon_{\rho,\tau} \boxtimes \varepsilon_{\sigma,\eta} \). We can consider as before a local algebra \( \mathcal{M}_0 := \mathcal{A}_1(I_0) \otimes \mathcal{A}_2(I_0) \) for some interval \( I_0 \in \mathcal{I} \), and take two honest points \( p_1 = (\mathcal{A}_1(I_1), \mathcal{A}_1(I_2)) \) in \( \mathcal{A}_1(I_0) \) and \( p_2 = (\mathcal{A}_2(J_1), \mathcal{A}_2(J_2)) \) in \( \mathcal{A}_2(I_0) \) respectively in the two components. Now setting \( \mathcal{N} := \mathcal{A}_1(I_1) \otimes \mathcal{A}_2(J_1) \) we have that irreducibles in \( \mathcal{C}_\mathcal{N} \) are given by \( \text{Ad}_u \rho \otimes \sigma \) for some \( \rho \in \text{DHR}^{I_1}\{ \mathcal{A}_1 \} \), \( \sigma \in \text{DHR}^{J_1}\{ \mathcal{A}_2 \} \) and \( u \in \mathcal{U}(\mathcal{N}) \). Moreover, the pair of algebras \( q = (\mathcal{N}, \mathcal{N}^c) \) is an abstract point of \( \mathcal{M}_0 \), but not honest unless \( I_1 = J_1 \). In other words, \( q = p_1 \otimes p_2 \) is an honest point of \( \mathcal{M}_0 \) if and only if \( p_1 = p_2 \) as geometric points of \( I_0 \).

We recall the following definition due to [Müg03], see also [DMNO13].

Definition 8.2. A UMTC \( \mathcal{C} \) is called a prime UMTC if \( \mathcal{C} \not\simeq \text{Vec} \) and every full unitary fusion subcategory \( \mathcal{D} \subset \mathcal{C} \) which is again a UMTC is either \( \mathcal{D} \simeq \mathcal{C} \) or \( \mathcal{D} \simeq \text{Vec} \) as UBTCs.

The terminology is motivated by the following proposition, which is among the deepest results on the structure of UMTCs. It establishes prime UMTCs as building blocks in the classification program of UMTCs, see [RSW09].
Proposition 8.3. [Müg03, DGNO10]. Let $\mathcal{C}$ be a UMTC, let $\mathcal{D} \subset \mathcal{C}$ be a unitary full fusion subcategory and consider the centralizer of $\mathcal{D}$ in $\mathcal{C}$ defined as the full subcategory of $\mathcal{C}$ with objects

$$Z_{\mathcal{C}}(\mathcal{D}) := \{ x \in \mathcal{C} : \varepsilon_{x,y} = \varepsilon_{x,y}^{\text{op}}, y \in \mathcal{D} \}.$$  

It holds

- $Z_{\mathcal{C}}(\mathcal{D})$ is a unitary (full) fusion subcategory of $\mathcal{C}$, which is also replete, and $Z_{\mathcal{C}}(Z_{\mathcal{C}}(\mathcal{D})) = \overline{\mathcal{D}}$ where $\overline{\mathcal{D}}$ denotes the repletion of $\mathcal{D}$ in $\mathcal{C}$.

If $\mathcal{D}$ is in addition a UMTC, i.e., $Z_{\mathcal{D}}(\mathcal{D}) \simeq \text{Vec}$, then

- $Z_{\mathcal{C}}(\mathcal{D})$ is also a UMTC and $\mathcal{C} \simeq \mathcal{D} \boxtimes Z_{\mathcal{C}}(\mathcal{D})$ as UBTCS.

In particular, every UMTC admits a finite prime factorization, i.e.

$$\mathcal{C} \simeq \mathcal{D}_1 \boxtimes \ldots \boxtimes \mathcal{D}_n$$

as UBTCS, where $\mathcal{D}_i$, $i = 1, \ldots, n$ are prime UMTCs, fully realized in $\mathcal{C}$.

Remark 8.4. Observe that assuming $\text{DHR}\{\mathcal{A}\}$ to be prime as an abstract UMTC rules out holomorphic nets. Moreover the examples seen in [CL] cannot arise, unless one of the two tensor factors is holomorphic, i.e., $\{\mathcal{A}\} = \{\mathcal{A}_1 \otimes \mathcal{A}_{\text{holo}}\}$. The following definition is aimed to rule out also this case.

Definition 8.5. Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line. Fix arbitrarily $I_0 \in \mathcal{I}$ and let $\mathcal{M}_0 = \mathcal{A}(I_0), \mathcal{C} = \text{DHR}^{I_0}\{\mathcal{A}\}$. We call $\{\mathcal{A}\}$ a prime conformal net if the following conditions are satisfied.

- $\mathcal{C} \simeq \text{DHR}\{\mathcal{A}\}$ is a prime UMTC.

- For every ordered pair $p = (N, N^c), q = (M, M^c)$ of abstract points of $\mathcal{M}_0$, if $N \vee M^c$ is normal in $\mathcal{M}_0$ then $M \subset N$, in particular $N \vee M^c = M_0$.

Remark 8.6. Notice that the primality assumption on $\mathcal{C} \simeq \text{DHR}\{\mathcal{A}\}$ is purely categorical, i.e., invariant under equivalence of UBTCS, hence contains no information about the actual size of the category. By definition of prime UBTCS, holomorphic nets are not prime conformal nets.

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\footnote{or braided relative commutant of $\mathcal{D} \subset \mathcal{C}$. Cf. the definition of relative commutant $\mathcal{D}^c$ we introduced in Section 4 for full inclusions of tensor categories. Cf. also the definition [HP15, Def. 2.9] of relative commutant in the sense of Drinfeld.}
Remark 8.7. If \( p, q \) mutually fulfill, e.g., \( \mathcal{R} = (\mathcal{R} \cap \mathcal{S}) \vee (\mathcal{R} \cap \mathcal{S}^c) \) for \( \mathcal{R}, \mathcal{S} \in \{\mathcal{N}, \mathcal{N}^c, \mathcal{M}, \mathcal{M}^c\} \) (resembling strong additivity), then the statements \( \mathcal{M} \subset \mathcal{N} \) and \( \mathcal{N} \vee \mathcal{M}^c = \mathcal{M}_0 \) are actually equivalent.

It is easy to see that prime conformal nets cannot factor through nontrivial holomorphic subnets.

Example 8.8. Let \( \{\mathcal{A}\} \) be a prime conformal net on the line, hence not holomorphic, but factoring through a holomorphic subnet, \( \{\mathcal{A}\} = \{\mathcal{A}_1 \otimes \mathcal{A}_{\text{holo}}\} \). Considering points \( p_1 \otimes p_2 \) of \( \mathcal{M}_0 \) like in Example 8.1, it is easy to construct \( \mathcal{N} \vee \mathcal{M}^c \) which are normal in \( \mathcal{M}_0 \) but neither exhaust \( \mathcal{M}_0 \) nor have \( \mathcal{M} \subset \mathcal{N} \), e.g., enlarging \( \mathcal{M} \) in the holomorphic component. Then \( \{\mathcal{A}\} \) cannot be prime unless \( \{\mathcal{A}_{\text{holo}}\} = \{\mathbb{C}\} \).

Remark 8.9. Both the notion of primality for completely rational conformal nets and the property of not factorizing through holomorphic subnets are invariant under isomorphism of nets.

Concerning the converse of the implication seen in Example 8.8, let \( \{\mathcal{A}\} \) be a completely rational net, not necessarily prime, take \( p, q \) as in Definition 8.5. The idea is that \( (\mathcal{N} \vee \mathcal{M}^c)^c = \mathcal{N}^c \cap \mathcal{M} \) are abstract “interval algebras” which lie in the “holomorphic part” of the net whenever \( \mathcal{N} \vee \mathcal{M}^c \) is normal in \( \mathcal{M}_0 \). More precisely, we can show that they necessarily factor out in a tensor product subalgebra of \( \mathcal{M}_0 \), and that the local subcategories associated to them à la DHR are trivial, namely \( \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \text{Vec} \).

Proposition 8.10. Let \( \{\mathcal{A}\} \) be a completely rational conformal net on the line, fix \( I_0 \in \mathcal{I} \) and let \( \mathcal{M}_0 = \mathcal{A}(I_0) \), \( \mathcal{C} = \text{DHR}^{I_0}\{\mathcal{A}\} \). Consider the family \( \mathcal{F} \) of ordered pairs of abstract points \( p = (\mathcal{N}, \mathcal{N}^c) \), \( q = (\mathcal{M}, \mathcal{M}^c) \) such that \( \mathcal{N} \vee \mathcal{M}^c \) is normal in \( \mathcal{M}_0 \), then the following holds.

- For every \( (p, q) \in \mathcal{F} \) we have \( \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \text{Vec} \).
- Consider the subalgebra of \( \mathcal{M}_0 \) defined as

\[
\mathcal{M}_0^{\text{holo}} := \bigvee_{(p, q) \in \mathcal{F}} \mathcal{N}^c \cap \mathcal{M}
\]

then \( \mathcal{M}_0^{\text{holo}} \) is either \( \mathbb{C} \) or a type III\(_1\) subfactor of \( \mathcal{M}_0 \), and the same holds for the relative commutant

\[
(\mathcal{M}_0^{\text{holo}})^c = \bigcap_{(p, q) \in \mathcal{F}} \mathcal{N} \vee \mathcal{M}^c.
\]

\footnote{We identify \( \text{Vec} \) with the full subcategory of \( \mathcal{C} \) whose objects are the inner endomorphisms, cf. Proposition 4.5.}
Moreover we have a splitting
\[ \mathcal{M}_0^{\text{holo}} \vee (\mathcal{M}_0^{\text{holo}})^c \cong \mathcal{M}_0^{\text{holo}} \otimes (\mathcal{M}_0^{\text{holo}})^c \]
as von Neumann algebras.

**Proof.** Normality of \( N \vee M^c \) in \( \mathcal{M}_0 \) means \( N \vee M^c = (N \vee M^c)^{cc} \), equivalently \( (N^c \cap M)^c = N^c \cap M^c \), but there is a more useful characterization. Without assuming normality, let \( \rho \in C_N \), \( \tilde{\rho} \in C_{M^c} \) and \( u \) a unitary charge transporter from \( \rho \) to \( \tilde{\rho} \). For every \( a \in N^c \cap M \) we have \( ua = u \rho(a) = \tilde{\rho}(a)u = au \) hence \( u \in (N^c \cap M)^c = (N \vee M^c)^{cc} \). Denoting by
\[
\mathcal{U}_C(N, M^c) := \text{vN}\{u \in \text{Hom}_C(\rho, \tilde{\rho}) \cap \mathcal{U}(\mathcal{M}_0), \rho \in C_N, \tilde{\rho} \in C_{M^c}\}
\]
the von Neumann algebra generated by the charge transporters, we have
\[
N \vee M^c \subset \mathcal{U}_C(N, M^c) \subset (N \vee M^c)^{cc}
\]
(18)
where the first inclusion holds because the unitaries in \( \mathcal{U}(N) \) and \( \mathcal{U}(M^c) \) generate inner automorphisms from the vacuum. Normality of \( N \vee M^c \) in \( \mathcal{M}_0 \) turns out to be *equivalent* to \( \mathcal{U}_C(N, M^c) = \mathcal{U}_C(N, M^c)^{cc} = N \vee M^c \). Using this we can show that \( C_{N^c} \cap C_{M^c} \subset \text{Vec} \). Let \( \rho \in C_{N^c} \cap C_{M^c} \) and observe that \( C_{N^c} \cap C_{M^c} = N^c \perp M^c \perp = (N \vee M^c)^\perp \) because endomorphisms in \( C \) are normal. Now by normality of \( N \vee M^c \) in \( \mathcal{M}_0 \) we have that \( \rho \in \mathcal{U}_C(N, M^c)^\perp \), i.e., \( \rho(u) = u \) for every unitary generator \( u \in \mathcal{U}_C(N, M^c) \). On the other hand for every \( \sigma \in C_N \) and \( \tilde{\sigma} := A_{\mu} \sigma \in C_M \) we have \( \varepsilon_{\rho, \tilde{\sigma}} = 1 \) by assumption (iv), i.e., \( \rho(u) = u \varepsilon_{\rho, \sigma} \) by naturality of the braiding, hence \( \varepsilon_{\rho, \sigma} = 1 \). Again by (iv) we have \( \varepsilon_{\sigma, \rho} = 1 \) and by (iii) \( C_N \simeq C \) from which we can conclude that \( \rho \) has vanishing monodromy with every sector, hence \( \rho \in \text{Vec} \) by modularity of \( C \), showing the first statement.

The second statement follows using modular theory on abstract points of \( \mathcal{M}_0 \), see Example 7.2 [Reh00, Prop. 2.8]. Let \( \sigma_t^\omega := A_{\Delta_t^\omega}, t \in \mathbb{R} \) be the modular group of \( \mathcal{M}_0 \) associated to the vacuum state \( \omega \) of the net, we know that if \( p \) is an abstract point of \( \mathcal{M}_0 \) then \( \sigma_t^\omega(p), t \in \mathbb{R} \) are also abstract points. Furthermore \( t \mapsto \sigma_t^\omega \) respects \( \mathcal{M}_0 \) and the normality property for subalgebras of \( \mathcal{M}_0 \), hence maps \( \mathcal{F} \) onto \( \mathcal{F} \) because \( (\sigma_t^\omega)^{-1} = \sigma_t^{-1} \omega \), and we conclude \( \sigma_t^\omega(\mathcal{M}_0^{\text{holo}}) = \mathcal{M}_0^{\text{holo}}, t \in \mathbb{R} \). By Takesaki’s theorem [Tak72] we have a faithful normal conditional expectation \( E : \mathcal{M}_0 \to \mathcal{M}_0^{\text{holo}} \) intertwining \( \sigma_t^\omega \) with \( \sigma_t^\omega \circ E, t \in \mathbb{R} \), where \( \varphi \) is the faithful normal state obtained by restricting \( \omega \) to \( \mathcal{M}_0^{\text{holo}} \) and \( \sigma_t^\omega \) is the associated modular group, see [Str81] Sec. 10]. Now the vacuum state \( \omega \) is given by the unique vector invariant under
the group of $I_0$-preserving dilations by \[GL96\] Cor. B.2. This, together with the Bisognano-Wichmann property \[GL96\] Prop. 1.1], imply that $t \mapsto \sigma_t^\omega$ is ergodic on $\mathcal{M}_0$, hence $t \mapsto \sigma_t$ is ergodic on $\mathcal{M}_0^{\text{holo}}$. In other words, $\varphi$ has trivial centralizer, then by \[Lon08\] Prop. 6.6.5 $\mathcal{M}_0^{\text{holo}}$ is a factor of type III$_1$ or trivial $\mathcal{M}_0^{\text{holo}} = \mathbb{C}$. The same holds for $(\mathcal{M}_0^{\text{holo}})^c$. In particular, $\mathcal{M}_0^{\text{holo}}$ being a subfactor of $\mathcal{M}_0$, we can apply \[Tak72\] Cor. 1 to get the splitting of $\mathcal{M}_0^{\text{holo}} \lor (\mathcal{M}_0^{\text{holo}})^c$ as von Neumann tensor product, completing the proof of the second statement.

\[\square\]

9 Comparability of abstract points

In the previous sections we analysed the braiding condition (iv) in Definition \[6.1\] on honest and abstract points of a net $\{\mathcal{A}\}$, see Eq. (11), Lemma \[6.10\] Proposition \[6.11\] and showed how it can be led far away from geometry in Section \[7\].

In this section we draw some of its consequences, as in the proof Proposition \[8.10\] and to do so we introduce comparability $p \sim q$ of abstract points, along with an order relation $p < q$ compatible with the geometric ordering of honest points. The terminology is motivated by the fact that two abstract points $p \sim q$ in a prime conformal net are necessarily $p < q$ or $q < p$ or $p = q$, see Proposition \[9.5\]. The order symbols should be intended as inclusions of relative complement algebras of $p, q$ in $\mathcal{M}_0$.

Let $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$ be two abstract points of $\mathcal{M}_0$ as in Definition \[6.1\] and $(\mathcal{R}, \mathcal{S})$ be any pair of elements from $\{\mathcal{N}, \mathcal{N}^c, \mathcal{M}, \mathcal{M}^c\}$. Similarly to Eq. (18) we have that the von Neumann algebras of unitary charge transporters

\[\mathcal{U}_c(\mathcal{R}, \mathcal{S}) := \mathcal{vN}\{u \in \text{Hom}_c(\rho, \tilde{\rho}) \cap \mathcal{U}(\mathcal{M}_0), \rho \in \mathcal{C}_R, \tilde{\rho} \in \mathcal{C}_S\} \quad (19)\]

always sit in between

\[\mathcal{R} \lor \mathcal{S} \subset \mathcal{U}_c(\mathcal{R}, \mathcal{S}) \subset (\mathcal{R} \lor \mathcal{S})^{cc},\]

in particular $\mathcal{U}_c(\mathcal{R}, \mathcal{S})^{cc} = (\mathcal{R} \lor \mathcal{S})^{cc}$. Hence asking normality of (19) in $\mathcal{M}_0$ is equivalent to asking that charge transporters generate as von Neumann algebras the relative commutants, cf. \[Mug99\] Cor. 4.3], \[KLM01\] Thm. 33], i.e., $\mathcal{U}_c(\mathcal{R}, \mathcal{S}) = (\mathcal{R} \lor \mathcal{S})^{cc} = (\mathcal{R}^c \lor \mathcal{S}^c)^c$.

Notice that, e.g., $\mathcal{U}_c(\mathcal{N}, \mathcal{N})$ and $\mathcal{U}_c(\mathcal{N}, \mathcal{N}^c)$ are always normal in $\mathcal{M}_0$ by (ii) and that $\mathcal{U}_c(\mathcal{R}, \mathcal{S}) = \mathcal{U}_c(\mathcal{S}, \mathcal{R})$ by definition.
Lemma 9.1. In the above notation, assume that $\mathcal{U}(\mathcal{R}, \mathcal{S})$ is normal in $\mathcal{M}_0$ for every pair $(\mathcal{R}, \mathcal{S})$ of elements in $\{N, N^c, M, M^c\}$, then

- $\mathcal{C}_{N \cap M} = \mathcal{C}_N \cap \mathcal{C}_M$ and $\mathcal{C}_{N^c \cap M^c} = \mathcal{C}_{N^c} \cap \mathcal{C}_{M^c}$.
- $\mathcal{C}_{N \cap M^c} \subset \mathcal{C}_N \cap \mathcal{C}_{M^c}$ and $\rho \in \mathcal{C}_{N \cap M^c}$ if and only if $\rho$ is an inner endomorphism of $\mathcal{C}$; in symbols: $\mathcal{C}_{N \cap M^c} = (\mathcal{C}_N \cap \mathcal{C}_{M^c}) \cap \text{Vec}$. Similarly $\mathcal{C}_{M \cap N^c} = (\mathcal{C}_M \cap \mathcal{C}_{N^c}) \cap \text{Vec}$.

Proof. Consider the intersection of left-left relative complements $\mathcal{C}_N \cap \mathcal{C}_M$. The inclusion $\mathcal{C}_{N \cap M} \subset \mathcal{C}_N \cap \mathcal{C}_M$ reads $(N \cap M)^{c^\perp \perp} \subset N^{c^\perp} \cap M^{c^\perp} = (N^c \vee M^c)^{c^\perp}$ hence follows easily by taking duals of $(N^c \vee M^c)^{c} = (N \cap M)^c$. The opposite inclusion follows from the braiding condition and normality assumption on charge transporters. Take $\rho \in \mathcal{C}_N \cap \mathcal{C}_M$ then by (iv) we have $\varepsilon_{\rho, \tilde{\rho}} = 1$ for every $\tilde{\rho} := \text{Ad}_u \sigma \in \mathcal{C}_{M^c}$ where $\sigma \in \mathcal{C}_{N^c}$ and $u$ is a unitary generator of $\mathcal{U}(N^c, M^c)$. Hence $\rho(u) = u \varepsilon_{\rho, \sigma}^*$ by naturality of the braiding. But also $\varepsilon_{\rho, \sigma} = 1$ by assumption (iv) and $\rho \in \mathcal{U}(N^c, M^c)^{c^\perp}$ follows, hence we have the first statement. The right-right case follows similarly.

In the left-right case the inclusion $\mathcal{C}_{N \cap M^c} \subset \mathcal{C}_N \cap \mathcal{C}_{M^c}$ can be proper, as shown by Proposition 4.5 in the honest case. Take $\rho \in \mathcal{C}_N \cap \mathcal{C}_{M^c}$, by normality $\rho \in \mathcal{C}_{N \cap M^c}$ if and only if $\rho(u) = u$ for every unitary generator $u \in \mathcal{U}(N^c, M)$. But now by (iv) we have $\varepsilon_{\rho, \sigma} = 1$ for every $\tilde{\rho} := \text{Ad}_u \sigma \in \mathcal{C}_M$ where $\sigma \in \mathcal{C}_{N^c}$, $u \in \mathcal{U}(N^c, M)$, hence $\rho(u) = u \varepsilon_{\rho, \sigma}^*$ together with $\varepsilon_{\rho, \sigma} = 1$. By assumption (iii) $\mathcal{C}_{N^c} \simeq \mathcal{C}$ and modularity of $\mathcal{C}$, we can conclude that $\rho \in \mathcal{C}_{N \cap M^c}$ if and only if $\rho \in \text{Vec}$, and the proof is complete.

As already remarked, given a pair of abstract points $p = (N, N^c)$, $q = (M, M^c)$ of $\mathcal{M}_0$, the algebras $N \cap M^c$ can be viewed as abstract “interval algebras” of $\mathcal{M}_0$ with associated “local” DHR subcategories $\mathcal{C}_N \cap \mathcal{C}_{M^c}$.

Denote by $\Delta(C)$ the spectrum of $C$ and let $\mathcal{U}_{C_{N^c \cap M^c} \cap \text{Vec}}(\mathcal{N}, \mathcal{M}^c) \subset \mathcal{U}(\mathcal{N}, \mathcal{M}^c)$ be the subalgebra generated by $\rho$-charge transporters associated to sectors $[\rho] \in \Delta(C_N \cap \mathcal{C}_{M^c})$. The vacuum $[id]$ is always in the spectrum, hence $\mathcal{U}_{C_{N^c \cap M^c} \cap \text{Vec}}(\mathcal{N}, \mathcal{M}^c)$ is also intermediate in $\mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$.

Lemma 9.2. In the above notation, assume that $\mathcal{U}_{C_{N^c \cap M^c} \cap \text{Vec}}(\mathcal{N}, \mathcal{M}^c)$ and $\mathcal{U}_{C_{M^c \cap N^c} \cap \text{Vec}}(\mathcal{M}, \mathcal{N}^c)$ are normal in $\mathcal{M}_0$, then $\mathcal{C}_{N^c} \cap \mathcal{C}_M$ and $\mathcal{C}_{M^c} \cap \mathcal{C}_N$ have “modular spectrum”, i.e.

$$Z_{C_{N^c \cap M^c} \cap \text{Vec}} \subset \mathcal{C}_{M^c \cap N^c} \cap \text{Vec}.$$  

Proof. Let $\rho \in \mathcal{C}_{N^c} \cap \mathcal{C}_M$ such that $\varepsilon_{\rho, \sigma} = \varepsilon_{\rho, \sigma}^*$ for all $\sigma \in \mathcal{C}_{N^c} \cap \mathcal{C}_M$. Inspired by [Müg99 Lem. 3.2] we can write $\varepsilon_{\rho, \sigma} = u^* \rho(u)$ and $\varepsilon_{\rho, \sigma}^* = x^* \rho(x)$ where $u$
and \( x \) are unitaries transporting \( \sigma \) respectively to \( C_{M} \) and \( C_{N} \), see Lemma \[6.10\]. Hence triviality of the monodromy \( \varepsilon_{\rho,\sigma} = \varepsilon_{\rho,\sigma}^{\text{op}} \) is triviality of the action \( \rho(ux^*) = ux^* \). Moreover every generator \( w \) of \( \mathcal{U}_{C_{N} \cap C_{M}}(N,M) \) can be written as \( w = ux^* \) with \( u \) and \( x \) as above. By normality \( \mathcal{U}_{C_{N} \cap C_{M}}(N,M) = (N \vee M)^{cc} \) hence, reversing the argument, one can drop the restriction \( \sigma \in C_{N} \cap C_{M} \). By modularity of \( \mathcal{C} \) we get \( \rho \in \text{Vec} \). Analogously interchanging \( N \) and \( M \).

Normality of \( \mathcal{U}_{C_{N} \cap C_{M}}(N,M) \) obviously implies normality of \( \mathcal{U}_{C}(N,M) \).

We are now ready to introduce the notion of comparability of two abstract points \( p, q \) mentioned in the beginning of this section.

**Definition 9.3.** Let \( \{ \mathcal{A} \} \) be a completely rational conformal net on the line. In the notation of Definition \[6.1 \] two abstract points \( p = (N,N^{c}), q = (M,M^{c}) \) of \( M_{0} \) are called comparable if they fulfill the following

- \( \mathcal{U}_{C_{R} \cap C_{S}}(R,S) = \mathcal{U}_{C_{R} \cap C_{S}}^{cc}(R,S) \).
- \( R \vee S = (R \vee S)^{\perp \perp} \).

for every pair \( (R,S) \) in \( \{ N,N^{c}, M,M^{c} \} \). In this case, we write \( p \sim q \).

Observe that \( \mathcal{U}_{C_{R} \cap C_{S}}(R,S) \) and \( (C_{R} \cap C_{S})^{\perp} = (R \vee S)^{\perp \perp} \) are both intermediate algebras in the inclusions \( R \vee S \subset (R \vee S)^{cc} \). Hence comparability means that these bounds are maximally, respectively minimally, saturated.

**Remark 9.4.** We have already motivated the normality condition on charge transporters. Concerning biduality, it easily holds for left or right local half-line algebras, see Proposition \[4.3 \] Remark \[4.10 \] and for two-interval algebras, as we have shown in Proposition \[4.7 \] Notice also that comparability is manifestly reflexive, symmetric and invariant under isomorphism of nets (but not manifestly transitive).

**Proposition 9.5.** Let \( \{ \mathcal{A} \} \) be a prime conformal net on the line (Definition \[8.3 \]) and take two abstract points \( p = (N,N^{c}), q = (M,M^{c}) \) of \( M_{0} \). If \( p \sim q \) then either \( p < q \) or \( q < p \) or \( p = q \), i.e., respectively \( N \subset M \) or \( M \subset N \) or \( N = M \).

In particular, in the case of a prime conformal net, comparability of \( p \) and \( q \) can be checked on the two pairs \( (N,M^{c}), (M,N^{c}) \).

**Proof.** The idea of the proof is that \( N^{c} \cap M \) and \( M^{c} \cap N \) are, a priori, abstract interval algebras of two different tensor factors of the net. Call for short \( C_{1} := C_{N^{c}} \cap C_{M} \) and \( C_{2} := C_{M^{c}} \cap C_{N} \) and observe that

\[
C_{1} \subset Z_{C}(C_{2}), \quad C_{2} \subset Z_{C}(C_{1})
\] (20)
because for every $\rho \in C_1$, $\sigma \in C_2$ we have $\varepsilon_{\rho,\sigma} = 1 = \varepsilon_{\sigma,\rho}$ by condition (iv), in particular $\varepsilon_{\sigma,\rho} \varepsilon_{\rho,\sigma} = 1$. We also have

$$Z_{C_1}(C_1) \subset \text{Vec}, \quad Z_{C_2}(C_2) \subset \text{Vec} \quad (21)$$

by Lemma 9.2. Notice that it can be $C_1 = C_2 = \{\text{id}\}$, e.g., if $\mathcal{N} = \mathcal{M}$. In order to invoke primality of the DHR category $C$ as a UMTC, we take the closures of $C_1, C_2 \subset C$ under conjugates, subobjects, finite direct sums, tensor products and unitary isomorphism classes. Denote them respectively by $\tilde{C}_1, \tilde{C}_2$. In other words, they are the smallest replete fusion subcategories of $C$ containing $C_1, C_2$ respectively. Thanks to [Mug03, Thm. 3.2], see also [DGNO10, Thm. 3.10], they are characterized as double braided relative commutant subcategories of $C$, i.e.

$$\tilde{C}_1 = Z_C(Z_C(\tilde{C}_1)), \quad \tilde{C}_2 = Z_C(Z_C(\tilde{C}_2)).$$

Now inclusions (20) and (21) clearly extend to subobjects, direct sums, tensor products and unitary isomorphism classes, because the vanishing of the monodromy is a condition stable under such operations, see [Mug00 Sec. 2.2], and Vec is a replete fusion subcategory of $C$. We need to check that (20) and (21) extend to conjugates because neither of the two sides of (20) nor the l.h.s. of (21) are a priori rigid. Let $\rho \in C_1, \sigma \in C_2$ and choose a conjugate $\overline{\rho} \in C$ of $\rho$, we want to show that $\varepsilon_{\sigma,\overline{\rho}} \varepsilon_{\overline{\rho},\sigma} = 1$. By condition (iii) we can assume $\overline{\rho} \in C_{\mathcal{N}^c}$ up to unitary isomorphism, equivalently we could have assumed $\overline{\rho} \in C_{\mathcal{M}}$. By Proposition 5.6 we have that every solution of the conjugate equations $R \in \text{Hom}_C(\text{id}, \overline{\rho} \rho), \overline{R} \in \text{Hom}_C(\text{id}, \rho \overline{\rho})$ for $\rho, \overline{\rho}$, see [LR97 Sec. 2], lies in $\mathcal{N}^c$, in particular $\sigma(R) = R, \sigma(\overline{R}) = \overline{R}$. Hence we get $\varepsilon_{\sigma,\overline{\rho}} = R^* \overline{\rho} (\varepsilon^*_{\overline{\rho},\sigma}) \overline{\rho} \sigma(\overline{R}) = R^* \overline{\rho} \sigma(\overline{R}) = 1$ and similarly $\varepsilon_{\sigma,\overline{\rho}} = \overline{\rho} \sigma(\overline{R}) \overline{\rho} (\varepsilon^*_{\sigma,\overline{\rho}}) R = \overline{\rho} \sigma(\overline{R}) R = 1$. In particular, $\overline{\rho}$ and $\sigma$ have vanishing monodromy.

Summing up we have $\tilde{C}_1 \subset Z_C(C_2)$ and similarly $\tilde{C}_2 \subset Z_C(C_1)$. Moreover, given $\sigma \in C_2$ choose a conjugate $\overline{\sigma} \in C$ and observe that the vanishing of the monodromy of $\overline{\sigma}$ and every $\rho$ in $\tilde{C}_1$ is equivalent to the vanishing of the monodromy of $\sigma$ and every $\rho$, by rigidity of $\tilde{C}_1$, see [Mug00, Eq. (2.17)]. Hence we have

$$\tilde{C}_1 \subset Z_C(\tilde{C}_2), \quad \tilde{C}_2 \subset Z_C(\tilde{C}_1) \quad (22)$$

and the two inclusions are equivalent by the double braided relative commutant theorem. We can extend also inclusions (21) by observing that $Z_{C_1}(\tilde{C}_1) \subset Z_{C_1}(C_1) \subset \text{Vec}$ and that, given $\rho \in C_1$ and a conjugate $\overline{\rho} \in C$, the vanishing of the monodromy of $\overline{\rho}$ and every $\sigma$ in $\tilde{C}_1$ is equivalent, as
above, to the vanishing of the monodromy of $\rho$ and every $\sigma$. Thus we have $\rho \in \text{Vec}$, hence $\bar{\rho} \in \text{Vec}$, and we conclude

$$Z_{\tilde{C}_1}(\tilde{C}_1) = \text{Vec}, \quad Z_{\tilde{C}_2}(\tilde{C}_2) = \text{Vec}$$

which means modularity for the replete fusion subcategories $\tilde{C}_1, \tilde{C}_2 \subset \mathcal{C}$. By primality of $\mathcal{C}$ as a UMTC, see Definition 8.2, the two subcategories are either $\mathcal{C}$ or $\text{Vec}$ and by the inclusions (22) we can assume $\tilde{C}_1 = \text{Vec}$, up to exchanging the roles of $\mathcal{N}$ and $\mathcal{M}$.

In particular, we obtain

$$\mathcal{C}_1 = \mathcal{C}_{N^c} \cap \mathcal{C}_M \subset \text{Vec},$$

by Lemma 9.1, i.e., $(\mathcal{N}^c \cap \mathcal{M})^c = (\mathcal{N} \cap \mathcal{M})^c$. Now by comparability we have a biduality relation $(\mathcal{N} \cap \mathcal{M})^{c\perp} = (\mathcal{N} \cap \mathcal{M})^c$, while $(\mathcal{N}^c \cap \mathcal{M})^c \perp = (\mathcal{N}^c \cap \mathcal{M})^c$ follows by the same argument as in Proposition 6.5. By taking duals we have that $\mathcal{N} \cap \mathcal{M}^c$ is normal in $\mathcal{M}_0$, hence $\mathcal{M} \subset \mathcal{N}$ by the primality assumption on the net. In particular, $\mathcal{C}_1 = \{id\}$, and the proof is complete. \[\square\]

As said before, normality of $\mathcal{U}_{\mathcal{C}_{N^c} \cap \mathcal{C}_M}(\mathcal{N}, \mathcal{M}^c)$ is equivalent to saying that the inclusion $\mathcal{N} \cap \mathcal{M}^c \subset (\mathcal{N} \cap \mathcal{M})^c$ is generated by charge transporters associated to sectors $[\rho] \in \Delta(\mathcal{C}_{N^c} \cap \mathcal{C}_M)$. We could strengthen this assumption by asking that the inclusion has the structure of a Longo-Rehren inclusion associated with $\{[\rho] \in \Delta(\mathcal{C}_{N^c} \cap \mathcal{C}_M)\}$. This amounts to specifying not only the generators of the extension, but also the algebraic relations among them [KLM01, Eq. (15), Prop. 45].

We show next that the latter can be derived, in our language of abstract points, from the fusion structure of the intersection categories. However, we don’t require, a priori, $\mathcal{N} \cap \mathcal{M}^c$ to split as a von Neumann tensor product, nor $\mathcal{N}$ and $\mathcal{M}^c$ to be commuting algebras.

**Proposition 9.6.** Let $\{\mathcal{A}\}$ be a completely rational conformal net on the line and take two abstract points $p = (\mathcal{N}, \mathcal{N}^c)$, $q = (\mathcal{M}, \mathcal{M}^c)$, in the notation of Definition 6.4. If we assume that

- $\mathcal{U}_{\mathcal{C}_{N^c} \cap \mathcal{C}_M}(\mathcal{N}, \mathcal{M}^c)$ and $\mathcal{U}_{\mathcal{C}_{M^c} \cap \mathcal{C}_N}(\mathcal{M}, \mathcal{N}^c)$ are normal in $\mathcal{M}_0$,
- $\mathcal{C}_{N^c} \cap \mathcal{C}_M$ and $\mathcal{C}_{M^c} \cap \mathcal{C}_N$ are UFTCs in $\mathcal{C}$,
- $\mathcal{C}_{N} \cap \mathcal{C}_M \simeq \mathcal{C}$ and $\mathcal{C}_{N^c} \cap \mathcal{C}_{M^c} \simeq \mathcal{C}$



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then \( \mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{\text{cc}} \) and \( \mathcal{M} \vee \mathcal{N}^c \subset (\mathcal{M} \vee \mathcal{N}^c)^{\text{cc}} \) have the structure of Longo-Rehren inclusions, in the sense that the generators of the extensions fulfill the relations \([\text{KLM01}] \) Eq. (15).

**Proof.** Consider the inclusion \( \mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{\text{cc}} \). Being \( \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \) a UFTC we can arrange its irreducible sectors \( \{[\rho] \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}\} \), in the terminology of \([\text{KLM01}]\), see also \([\text{Reh90}], [\text{BEK99}]\).

By assumption, for each \([\rho_i]\) we can choose \( \overline{p}_i \in \mathcal{N} \cap \mathcal{C}_{\mathcal{M}} \), \( \rho_i \in \mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}^c} \) and \( R_i \in \text{Hom}_\mathcal{C}(\text{id}, \overline{p}_i \rho_i) \) such that \( R_i^* R_i = d_{\rho_i} \text{id} \) and \( R_0 = \text{id} \). In particular, \( R_i a = \overline{p}_i \rho_i(a) R_i \) for all \( a \in \mathcal{N} \vee \mathcal{M}^c \) and \( R_i \in (\mathcal{N}^c \cap \mathcal{M})^c = (\mathcal{N} \vee \mathcal{M}^c)^{\text{cc}} \).

Now, \( R_i R_j \in \text{Hom}_\mathcal{C}(\text{id}, \overline{p}_i \rho_i \overline{p}_j \rho_j) = \text{Hom}_\mathcal{C}(\overline{p}_i \rho_i, \overline{p}_j \rho_j) \) because, e.g., \( \mathcal{C}_{\mathcal{N}} \) and \( \mathcal{C}_{\mathcal{N}^c} \) commute in the sense of Proposition \([\text{KLM01}]\) and

\[
R_i R_j = \sum_{k,\alpha,\beta} (w_\alpha w_\beta^* \times v_\beta v_\alpha^*) \cdot (R_i \times R_j)
\]

where \( k \) runs over irreducible components \([\rho_k] < [\rho_i][\rho_j]\) and \( \alpha, \beta \) over orthonormal bases of isometries \( w_\alpha \in \text{Hom}_\mathcal{C}(\mathcal{N}, \overline{p}_k \rho_p) \), \( v_\beta \in \text{Hom}_\mathcal{C}(\rho_k, \rho_p \rho_j) \).

Then \( \sum_{k,\alpha,\beta} w_\alpha w_\beta^* \times v_\beta v_\alpha^* \cdot R_i \times R_j = \sum_{k,\alpha,\beta} w_\alpha v_\beta \lambda_{\alpha,\beta}^k R_k \) where \( \lambda_{\alpha,\beta}^k \in \mathbb{C} \) because \([\rho_k]\) is irreducible, hence \([\text{id}] < [\mathcal{N}][\rho_k]\) with multiplicity one, and \( \overline{p}_k(v_\beta) = v_\beta \). Setting \( C_{ij}^k := \sum_{\alpha,\beta} w_\alpha v_\beta \lambda_{\alpha,\beta}^k \) we have (non-canonical) intertwiners in \( \text{Hom}_\mathcal{C}(\overline{p}_k \rho_k, \overline{p}_j \rho_j) = \text{Hom}_\mathcal{C}(\overline{p}_k \rho_k, \overline{p}_j \rho_j) \) which lie in \( \mathcal{N} \vee \mathcal{M}^c \) and fulfill

\[
R_i R_j = \sum_k C_{ij}^k R_k.
\]

In particular, we have \( C_{ii}^0 \in \text{Hom}_\mathcal{C}(\text{id}, \overline{p}_i \rho_i \overline{p}_i \rho_i) \) again in \( \mathcal{N} \vee \mathcal{M}^c \), hence \( R_i^* C_{ii}^0 \) is a multiple of \( R_i \), i.e., we get

\[
R_i^* = \lambda C_{ii}^0 R_i
\]

for some \( \lambda \in \mathbb{C} \), and we have shown up to normalization constants the algebraic relations of \([\text{KLM01}] \) Eq. (15).

On the other hand, by Frobenius reciprocity \([\text{LR97}] \) Lem. 2.1] the \( R_i \) generate the extension \( \mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{\text{cc}} \) because every unitary charge transporter \( u \in \text{Hom}_\mathcal{C}(\rho, \hat{\rho}) \), \( \rho \in \mathcal{C}_{\mathcal{N}^c} \), \( \hat{\rho} \in \mathcal{C}_{\mathcal{M}^c} \) such that \([\hat{\rho}] = [\rho_i]\) for some \( i \), can be written as \( u = \lambda v \rho_i \langle r^* \rangle R_\tau = \lambda v r^* R_\tau \) for suitable \( \lambda \in \mathbb{C} \), \( v \in \mathcal{M}^c \) unitary and \( r \in \mathcal{N} \) isometric. In particular, every \( b \in (\mathcal{N} \vee \mathcal{M}^c)^{\text{cc}} \) admits a (not necessarily unique) “harmonic” expansion

\[
b = \sum_i b_i R_i \quad (24)
\]
where $b_i \in \mathcal{N} \vee \mathcal{M}$, cf. [LR95, Eq. (4.10)], [KLM01, Prop. 45], and we are done.

**Corollary 9.7.** With the assumptions of the previous proposition, $\mathcal{N} \vee \mathcal{M}^c$ is bidual in $\mathcal{M}_0$, i.e., $(\mathcal{N} \vee \mathcal{M}^c)^{\perp \perp} = \mathcal{N} \vee \mathcal{M}^c$. Moreover $\mathcal{N} \vee \mathcal{M}^c$ is normal in $\mathcal{M}_0$ if and only if $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \text{Vec}$, and $\mathcal{N} \vee \mathcal{M}^c = \mathcal{M}_0$ if and only if $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} = \{\text{id}\}$. Analogous statements hold interchanging $\mathcal{N}$ and $\mathcal{M}$, hence in particular $p \sim q$.

**Proof.** The category $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$ is automatically modular with the braiding inherited from $\mathcal{C}$, thanks to Lemma 9.2. The first statement follows by the same argument leading to Proposition 9.4 which relies on the (not necessarily unique) harmonic expansion (24), on rigidity of $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}}$ and on unitarity of its modular $S$-matrix.

Normality of $\mathcal{N} \vee \mathcal{M}^c$ implies $\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \subset \text{Vec}$ as we have seen in Proposition 8.10, the converse follows from the normality assumption on charge transporters.

The nontrivial implication in the last statement follows from biduality.

**10 Abstract points and (Dedekind’s) completeness**

In the following we show a way of deriving completeness of the invariant introduced in Section 8, Eq. (8), on the class of prime conformal nets. This section is rather speculative, in the sense that it relies on two assumptions on the “good behaviour” of abstract point (in the prime CFT case). The first is horizontal and concerns transitivity of the comparability relation $p \sim q$, the second is vertical and asks totality of the unitary equivalence $p = UqU^*$ encountered in Section 7. Here we do not discuss about the issue of deriving them, nor strengthening Definition 6.1 or 9.3 in order to do so, nor deciding how do they constrain models. We just show how the structure of the real line (Dedekind’s completeness axiom) and of a conformal net can cooperate in the reconstruction of the latter up to isomorphism from its abstract points, thanks to Proposition 9.5.

**Proposition 10.1.** Let $\{\mathcal{A}\}$ be a prime conformal net on the line (Definition 8.3), fix arbitrarily $I_0 \in \mathcal{T}$ and assume in addition that comparability $p \sim q$ is transitive, and unitary equivalence $p = UqU^*$ is total on the abstract points

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of \( M_0 = \mathcal{A}(I_0) \). Then \( \{\mathcal{A}\} \) is uniquely determined up to isomorphism by its abstract points inside \( M_0 \).

**Proof.** Take first an honest abstract point \( p = (\mathcal{A}(I_1), \mathcal{A}(I_2)) \) of \( M_0 \) with respect to \( \{\mathcal{A}\} \), as in Remark 6.4. By Remark 9.4 all the other honest points are equivalent to \( p \). We want to show that they exhaust the comparability equivalence class. Let \( q = (\mathcal{N}, \mathcal{N}^c) \) be an abstract point of \( M_0 \) such that \( q \sim p \), hence by transitivity \( q \sim r \) for every honest point \( r = (\mathcal{A}(J_1), \mathcal{A}(J_2)) \), and by Proposition 9.5 either \( r \leq q \) or \( q < r \). Consider the maximum over the first family, i.e., the von Neumann algebra generated by the left relative complements, and the minimum over the second, i.e., the intersection of the left relative complements. The resulting algebras are again honest points because the net is additive and they coincide because the real line is Dedekind complete, thus \( q \) is also honest with respect to \( \{\mathcal{A}\} \).

Now take an arbitrary abstract point \( s = (\mathcal{M}, \mathcal{M}^c) \) of \( M_0 \). By the totality assumption there is a unitary \( U \in \mathcal{U}(\mathcal{H}) \) such that \( s = UpU^* \) where \( p = (\mathcal{A}(I_1), \mathcal{A}(I_2)) \) as above. Now every unitary is eligible as an isomorphism of local conformal nets, because positivity of the energy is preserved by unitary conjugation, hence call \( \{\tilde{\mathcal{A}}\} \) the net defined on algebras by \( \tilde{\mathcal{A}}(I) := UA(I)U^* \), \( I \in \mathcal{I} \), and observe that \( s = (\tilde{\mathcal{A}}(I_1), \tilde{\mathcal{A}}(I_2)) \) is an honest point of \( \tilde{\mathcal{A}}(I_0) = \mathcal{A}(I_0) \) with respect to the new net. As before, \( r \) determines all the other honest points (because the comparability relation and its transitivity property are invariant under isomorphisms of nets), hence all the local interval algebras \( \tilde{\mathcal{A}}(I) \subset \tilde{\mathcal{A}}(I_0) \), \( I \subset I_0 \) by taking intersections. By Proposition 2.6 the latter determine \( \{\tilde{\mathcal{A}}\} \) up to isomorphism, hence \( \{\mathcal{A}\} \) as well, and the proof is complete. \( \square \)

### 11 Conclusions

In chiral conformal QFT, the DHR category \( \mathcal{C} = \text{DHR}\{\mathcal{A}\} \) is a unitary braided tensor category corresponding to the positive-energy representations of the model. In completely rational models, the braiding is non-degenerate, hence it is a modular tensor category (UMTC). While abstract UMTCs are rigid structures and cannot distinguish the underlying CFT model uniquely, we have studied the question to which extent the braided action of this category on a single (local or global) algebra \( \mathcal{A} \) is a complete invariant of the model. The strategy is to exploit the trivialization of the braiding, which is a characteristic feature of the DHR braiding, in certain geometric constellations to identify pairs of subalgebras (called “abstract points”). They are
candidates for subalgebras of local observables associated to regions (half-intervals or half-lines) separated by a geometric point. Modularity is needed to distinguish the left from the right complement, and enters in our analysis through the stronger categorical notion of primality for UMTCs. As the main tool in this direction, we established powerful duality relations between subalgebras of $\mathcal{A}$ and subcategories of $\mathcal{C}$, and a characterization of “prime” CFT models that do not factor through nontrivial subnet, either holomorphic or not. We formulate a unitary equivalence relation and a comparability relation between abstract points. Assuming that the former is total and the latter is transitive, we showed that the action of the DHR category is a complete invariant for prime CFT models, i.e., it allows (in principle) to reconstruct the local QFT up to unitary equivalence.

We assumed throughout that the action does come from a CFT, so that we only have to decide whether two inequivalent CFT can give rise to the same action. We did not address the more ambitious question of how to characterize those actions which possibly come from a CFT, thus leaving the realization problem of braided actions of abstract UMTCs by DHR categories of some local net for future research.

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References

[Arv74] W. Arveson. On groups of automorphisms of operator algebras. J. Funct. Anal. 15, 217–243, 1974.

[BEK99] J. Böckenhauer, D. E. Evans, and Y. Kawahigashi. On $\alpha$-induction, chiral generators and modular invariants for subfactors. Comm. Math. Phys. 208, 429–487, 1999.

[BGL93] R. Brunetti, D. Guido, and R. Longo. Modular structure and duality in conformal quantum field theory. Comm. Math. Phys. 156, 201–219, 1993.
[Bis12] M. Bischoff. Models in boundary quantum field theory associated with lattices and loop group models. *Comm. Math. Phys.* 315, 827–858, 2012.

[Bis15] M. Bischoff. A remark on CFT realization of quantum doubles of subfactors. Case index < 4. *preprint arXiv:1506.02606*, 2015.

[BKLR15] M. Bischoff, Y. Kawahigashi, R. Longo, and K.-H. Rehren. *Tensor Categories and Endomorphisms of von Neumann Algebras. With applications to quantum field theory. Springer Briefs in Mathematical Physics*, Vol. 3. Springer, Cham, 2015.

[Bor65] H.-J. Borchers. Local rings and the connection of spin with statistics. *Comm. Math. Phys.* 1, 281–307, 1965.

[Bor92] H.-J. Borchers. The CPT-theorem in two-dimensional theories of local observables. *Comm. Math. Phys.* 143, 315–332, 1992.

[BS93] D. Buchholz and S. J. Summers. An algebraic characterization of vacuum states in Minkowski space. *Comm. Math. Phys.* 155, 449–458, 1993.

[BV14] D. Buchholz and R. Verch. Macroscopic aspects of the Unruh effect. *preprint arXiv:1412.5892*, 2014, to appear in *Class. Quant. Grav.*

[BW75] J. J. Bisognano and E. H. Wichmann. On the duality condition for a Hermitian scalar field. *J. Math. Phys.* 16, 985–1007, 1975.

[Car04] S. Carpi. On the representation theory of Virasoro nets. *Comm. Math. Phys.* 244, 261–284, 2004.

[CKLW15] S. Carpi, Y. Kawahigashi, R. Longo, and M. Weiner. From vertex operator algebras to conformal nets and back. *preprint arXiv:1503.01260*, 2015.

[DGNO10] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik. On braided fusion categories. I. *Selecta Math. (N.S.)* 16, 1–119, 2010.

[DHR69] S. Doplicher, R. Haag, and J. E. Roberts. Fields, observables and gauge transformations. I. *Comm. Math. Phys.* 13, 1–23, 1969.

[DHR71] S. Doplicher, R. Haag, and J. E. Roberts. Local observables and particle statistics. I. *Comm. Math. Phys.* 23, 199–230, 1971.

[DL83] S. Doplicher and R. Longo. Local aspects of superselection rules. II. *Comm. Math. Phys.* 88, 399–409, 1983.

[dlH79] P. de la Harpe. Moyennabilité du groupe unitaire et propriété $P$ de Schwartz des algèbres de von Neumann. In: *Algèbres d’Opérateurs, Lecture Notes in Math.* Vol. 725, pp. 220–227. Springer, Berlin, 1979.

[DLR01] C. D’Antoni, R. Longo, and F. Radulescu. Conformal nets, maximal temperature and models from free probability. *J. Oper. Theory* 45, 195–208, 2001.
[DMNO13] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik. The Witt group of non-degenerate braided fusion categories. *J. Reine Angew. Math.* 677, 135–177, 2013.

[Dop82] S. Doplicher. Local aspects of superselection rules. *Comm. Math. Phys.* 85, 73–86, 1982.

[DR89] S. Doplicher and J. E. Roberts. A new duality theory for compact groups. *Invent. Math.* 98, 157–218, 1989.

[DR90] S. Doplicher and J. E. Roberts. Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics. *Comm. Math. Phys.* 131, 51–107, 1990.

[Dri79] W. Driessler. Duality and absence of locally generated superselection sectors for CCR-type algebras. *Comm. Math. Phys.* 70, 213–220, 1979.

[DX06] C. Dong and F. Xu. Conformal nets associated with lattices and their orbifolds. *Adv. Math.* 206, 279–306, 2006.

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor Categories* , Mathematical Surveys and Monographs Vol. 205. American Mathematical Society, Providence, RI, 2015.

[FJ96] K. Fredenhagen and M. Jörß. Conformal Haag-Kastler nets, pointlike localized fields and the existence of operator product expansions. *Comm. Math. Phys.* 176, 541–554, 1996.

[FRS92] K. Fredenhagen, K.-H. Rehren, and B. Schroer. Superselection sectors with braid group statistics and exchange algebras. II. Geometric aspects and conformal covariance. *Rev. Math. Phys.*, SI1 (Special Issue), 113–157, 1992.

[GF93] F. Gabbiani and J. Fröhlich. Operator algebras and conformal field theory. *Comm. Math. Phys.* 155, 569–640, 1993.

[Gio16] L. Giorgetti. *Braided actions of DHR categories and reconstruction of chiral conformal field theories*. PhD thesis, Georg-August-Universität Göttingen, Institut für Theoretische Physik, 2016.

[GL92] D. Guido and R. Longo. Relativistic invariance and charge conjugation in quantum field theory. *Comm. Math. Phys.* 148, 521–551, 1992.

[GL96] D. Guido and R. Longo. The conformal spin and statistics theorem. *Comm. Math. Phys.* 181, 11–35, 1996.

[GLW98] D. Guido, R. Longo, and H.-W. Wiesbrock. Extensions of conformal nets and superselection structures. *Comm. Math. Phys.* 192, 217–244, 1998.

[Haa87] U. Haagerup. Connes’ bicentralizer problem and uniqueness of the injective factor of type III1. *Acta Math.* 158, 95–148, 1987.

[Haa96] R. Haag. *Local Quantum Physics*. Springer Berlin, 1996.
[HH15] J. Holland and S. Hollands. Recursive construction of operator product expansion coefficients. *Comm. Math. Phys.* 336, 1555–1606, 2015.

[HP15] A. Henriques and D. Penneys. Bicommutant categories from fusion categories. *preprint arXiv:1511.05226*, 2015.

[HY00] T. Hayashi and S. Yamagami. Amenable tensor categories and their realizations as AFD bimodules. *J. Funct. Anal.* 172, 19–75, 2000.

[Jos65] R. Jost. *The General Theory of Quantized Fields*. Amer. Math. Soc., Providence, R.I., 1965.

[Kad52] R. V. Kadison. Infinite unitary groups. *Trans. Amer. Math. Soc.* 72, 386–399, 1952.

[Kaw15] Y. Kawahigashi. Conformal field theory, tensor categories and operator algebras. *J. Phys. A* 48, 303001, 2015.

[KL04] Y. Kawahigashi and R. Longo. Classification of local conformal nets. Case $c < 1$. *Ann. Math.* 160, 493–522, 2004.

[KL06] Y. Kawahigashi and R. Longo. Local conformal nets arising from framed vertex operator algebras. *Adv. Math.* 206, 729–751, 2006.

[KLM01] Y. Kawahigashi, R. Longo, and M. Müger. Multi-interval subfactors and modularity of representations in conformal field theory. *Comm. Math. Phys.* 219, 631–669, 2001.

[KW01] R. Kăhler and H.-W. Wiesbrock. Modular theory and the reconstruction of four-dimensional quantum field theories. *J. Math. Phys.* 42, 74–86, 2001.

[Lon89] R. Longo. Index of subfactors and statistics of quantum fields. I. *Comm. Math. Phys.* 126, 217–247, 1989.

[Lon90] R. Longo. Index of subfactors and statistics of quantum fields. II. Correspondences, braid group statistics and Jones polynomial. *Comm. Math. Phys.* 130, 285–309, 1990.

[Lon97] R. Longo. An analogue of the Kac-Wakimoto formula and black hole conditional entropy. *Comm. Math. Phys.* 186, 451–479, 1997.

[Lon03] R. Longo. Conformal subnets and intermediate subfactors. *Comm. Math. Phys.* 237, 7–30, 2003.

[Lon08] R. Longo. Lecture Notes on Conformal Nets. Part II. Nets of von Neumann Algebras. Preliminary lecture notes available at http://www.mat.uniroma2.it/~longo/Lecture_Notes.html.

[LR95] R. Longo and K.-H. Rehren. Nets of subfactors. *Rev. Math. Phys.* 7, 567–597, 1995.

[LR97] R. Longo and J. E. Roberts. A theory of dimension. *K-Theory* 11, 103–159, 1997.
K.-H. Rehren and G. Tedesco. Multilocal fermionization. *Lett. Math. Phys.* 103, 19–36, 2013.

G.L. Sewell. Relativity of temperature and the Hawking effect. *Phys. Lett.* 79A, 23-24, 1980.

Ș. Strătilă. *Modular Theory in Operator Algebras.* Editura Academiei Republicii Socialiste România, Bucharest; Abacus Press, Tunbridge Wells, 1981.

M. Takesaki. Algebraic equivalence of locally normal representations. *Pacific J. Math.* 34, 807–816, 1970.

M. Takesaki. Conditional expectations in von Neumann algebras. *J. Funct. Anal.* 9, 306–321, 1972.

M. Takesaki. *Theory of Operator Algebras I.* Encyclopaedia of Mathematical Sciences Vol. 124. Springer-Verlag, Berlin, 2002.

M. Takesaki. *Theory of Operator Algebras II.* Encyclopaedia of Mathematical Sciences Vol. 125. Springer-Verlag, Berlin, 2003.

J. Tomiyama. On the projection of norm one in $W^*$-algebras. *Proc. Japan Acad.* 33, 608–612, 1957.

V. Turaev. *Homotopy Quantum Field Theory.* EMS Tracts in Mathematics Vol. 10. Eur. Math. Soc. Zürich, 2010. Appendix 5 by Michael Müger and Appendices 6 and 7 by Alexis Virelizier.

A. Wassermann. Operator algebras and conformal field theory III. Fusion of positive energy representations of LSU(N) using bounded operators. *Invent. Math.* 133, 467–538, 1998.

M. Weiner. An algebraic version of Haag’s theorem. *Comm. Math. Phys.* 305, 469–485, 2011.

H.-W. Wiesbrock. Half-sided modular inclusions of von-Neumann-algebras. *Comm. Math. Phys.* 157, 83–92, 1993.

F. Xu. Algebraic orbifold conformal field theories. *Proc. Nat. Acad. Sci. U.S.A.* 97, 14069, 2000.