Resonance Identities for Closed Characteristics on Compact Star-shaped Hypersurfaces in $\mathbb{R}^{2n}$

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Abstract

Resonance relations among periodic orbits on given energy hypersurfaces are very important for getting deeper understanding of the dynamics of the corresponding Hamiltonian systems. In this paper, we establish two new resonance identities for closed characteristics on every compact star-shaped hypersurface $\Sigma$ in $\mathbb{R}^{2n}$ when the number of geometrically distinct closed characteristics on $\Sigma$ is finite, which extend those identities established by C. Viterbo in 1989 for star-shaped hypersurfaces assuming in addition that all the closed characteristics and their iterates are non-degenerate, and that by W. Wang, X. Hu and Y. Long in 2007 for strictly convex hypersurfaces in $\mathbb{R}^{2n}$.

Key words: Compact star-shaped hypersurfaces, closed characteristics, Hamiltonian systems, resonance identity.

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1 Introduction and main result

Let $\Sigma$ be a $C^3$ compact hypersurface in $\mathbb{R}^{2n}$ strictly star-shaped with respect to the origin, i.e., the tangent hyperplane at any $x \in \Sigma$ does not intersect the origin. We denote the set of all such hypersurfaces by $\mathcal{H}_{st}(2n)$, and denote by $\mathcal{H}_{con}(2n)$ the subset of $\mathcal{H}_{st}(2n)$ which consists of all strictly convex hypersurfaces. We consider closed characteristics $(\tau, y)$ on $\Sigma$, which are solutions of the following problem

$$\begin{cases}
\dot{y} = J N_{\Sigma}(y), \\
y(\tau) = y(0),
\end{cases} \quad (1.1)$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix in $\mathbb{R}^n$, $\tau > 0$, $N_{\Sigma}(y)$ is the outward normal vector of $\Sigma$ at $y$ normalized by the condition $N_{\Sigma}(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbb{R}^{2n}$. A closed characteristic $(\tau, y)$ is prime, if $\tau$ is the minimal period of $y$. Two closed characteristics $(\tau, y)$ and $(\sigma, z)$ are geometrically distinct, if $y(R) \neq z(R)$. We denote by $\mathcal{T}(\Sigma)$ the set of all geometrically distinct closed characteristics on $\Sigma$. A closed characteristic $(\tau, y)$ is non-degenerate, if 1 is a Floquet multiplier of $y$ of precisely algebraic multiplicity 2.

The study on closed characteristics in the global sense started in 1978, when the existence of at least one closed characteristic was first established on any $\Sigma \in \mathcal{H}_{st}(2n)$ by P. Rabinowitz in [Rab1] and on any $\Sigma \in \mathcal{H}_{con}(2n)$ by A. Weinstein in [Wei1] independently, since then the existence of multiple closed characteristics on $\Sigma \in \mathcal{H}_{con}(2n)$ has been deeply studied by many mathematicians, for example, studies in [EkL1], [EkH1], [Szu1], [LoZ1], [WHL1], and [Wan1] for convex hypersurfaces. We refer readers to the survey paper [Lon5] and the recent [Lon6] of Y. Long for earlier works and references on this subject.

But for the star-shaped hypersurfaces, one difficulty in the study on the star-shaped hypersurfaces is that the Maslov-type index and mean index of each closed characteristic may be negative. We are only aware of a few papers about the multiplicity of closed characteristics. In [Gir1] of 1984 and [BLMR] of 1985, $\# \mathcal{T}(\Sigma) \geq n$ for $\Sigma \in \mathcal{H}_{st}(2n)$ was proved under some pinching conditions. In [Vit1] of 1989, C. Viterbo proved a generic existence result for infinitely many closed characteristics on star-shaped hypersurfaces. In [HuL1] of 2002, X. Hu and Y. Long proved that $\# \mathcal{T}(\Sigma) \geq 2$ for $\Sigma \in \mathcal{H}_{st}(2n)$ on which all the closed characteristics and their iterates are non-degenerate. Recently $\# \mathcal{T}(\Sigma) \geq 2$ was proved for every $\Sigma \in \mathcal{H}_{st}(4)$ by D. Cristofaro-Gardiner and M. Hutchings in [CGH1], and it’s different proofs can also be found in [GHHM], [LLo1] and [GiG1].

In [Eke1] of 1984, I. Ekeland first discovered some resonance relations of closed characteristics for $\Sigma \in \mathcal{H}_{con}(2n)$, but which are not explicitly given. In [Vit1], C. Viterbo established two such
identities explicitly for closed characteristics on $\Sigma \in \mathcal{H}_{st}(2n)$ under the assumption that all the closed characteristics on $\Sigma$ are non-degenerate. Such identities are important ingredients in the study in [Vit1] and [HuL1]. In [WHL1] of 2007, W. Wang, X. Hu and Y. Long proved the resonance identity for every $\Sigma \in \mathcal{H}_{con}(2n)$ which removed the non-degeneracy condition. This identity plays a crucial role in the proof of their estimate $\# \mathcal{T}(\Sigma) \geq 3$ for every $\Sigma \in \mathcal{H}_{con}(6)$. Note that in [Rad1] of 1989 and [Rad2] of 1992, a similar identity for closed geodesics on compact Finsler manifolds was established by H.-B. Rademacher. Motivated by [Vit1] and [WHL1], we establish the following new identities on closed characteristics for star-shaped hypersurface $\Sigma \in \mathcal{H}_{st}(2n)$ without the non-degeneracy conditions.

**Theorem 1.1.** Suppose that $\Sigma \in \mathcal{H}_{st}(2n)$ satisfies $\# \mathcal{T}(\Sigma) < +\infty$. Denote all the geometrically distinct prime closed characteristics by $\{(\tau_j, y_j)\}_{1 \leq j \leq k}$. Then the following identities hold

$$
\sum_{1 \leq j \leq k \atop \hat{i}(y_j) > 0} \frac{\hat{\chi}(y_j)}{\hat{i}(y_j)} = \frac{1}{2},
$$

$$
\sum_{1 \leq j \leq k \atop \hat{i}(y_j) < 0} \frac{\hat{\chi}(y_j)}{\hat{i}(y_j)} = 0,
$$

where $\hat{i}(y_j) \in \mathbb{R}$ is the mean index of $y_j$ given by Definition 4.7, $\hat{\chi}(y_j) \in \mathbb{Q}$ is the average Euler characteristic given by Definition 4.8 and Remark 4.9 below. Especially by (4.32), we have

$$
\hat{\chi}(y) = \frac{1}{K(y)} \sum_{1 \leq m \leq K(y) \atop 0 \leq l \leq 2n-2} (-1)^{i(y^m)+l} k_i(y^m),
$$

where $K(y) \in \mathbb{N}$ is the minimal period of critical modules of iterations of $y$ defined in Proposition 4.6, $i(y^m)$ is the index defined in Definition 4.7 (cf. Definition 2.9 and (2.15) below), and $k_i(y^m)$s are the critical type numbers of $y^m$ given by Definition 4.3 and Remark 4.4 below.

**Remark 1.2.** When all the closed characteristics on $\Sigma \in \mathcal{H}_{st}(2n)$ together with their iterations are non-degenerate, by Remark 4.9 our identities (1.2) and (1.3) coincide with the identities (1.3) and (1.4) of Theorem 1.2 of [Vit1]. Thus our Theorem 1.1 generalizes C. Viterbo’s result in [Vit1] to the degenerate case.

When $\Sigma \in \mathcal{H}_{con}(2n)$, we can choose $K_0 = 0$ in the proof of Case (b) of Theorem 3.3 below. Then $d(K) = 0$ in (2.15). By (3.13) and (3.15), we obtain

$$
C_{S^1,l}(F_0, S^1 \cdot \bar{x}) \cong C_{S^1,l}(\bar{F}_0, S^1 \cdot \bar{y}) \cong C_{S^1,l}(\bar{F}_K, S^1 \cdot \bar{y}).
$$

Noticing that $C_{S^1,l}(\bar{F}_0, S^1 \cdot \bar{y})$ is exactly isomorphic to $C_{S^1,l}(\Psi_0, S^1 \cdot \bar{x})$ which is defined in Definition 3.1 of [WHL1], then our identity (1.2) coincides with the identity (1.3) of Theorem 1.2 of [WHL1].
Thus our Theorem 1.1 generalizes also the resonance identity in [WHL1] for convex hypersurfaces to star-shaped hypersurfaces.

We also note that some similar resonance identities for closed Reeb orbits on closed contact manifolds were established in Theorem 3.6 of [GiG1] under the context of local contact homology after we completed this paper.

The main idea in the proof of Theorem 1.1 and the arrangement of the rest of this paper are as follows.

(1) Motivated by the works [Vit1] of C. Viterbo and [WHL1] of W. Wang, X. Hu, and Y. Long, for every $\Sigma \in \mathcal{H}_{st}(2n)$ with $\#T(\Sigma) < +\infty$, we shall construct a functional $F_{a,K}$ on the space $W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ for large $a > 0$ and $K$ satisfying the requirement (2.3)-(2.4) below and establish a Morse theory of this functional $F_{a,K}$ to study closed characteristics on $\Sigma$. By a change of variable, it is equivalent to study a functional $\Psi_{a,K}$ on $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$.

As usual we use the Clarke-Ekeland dual action principle and a modification of the Viterbo index theory. Because in general such a dual action functional is not $C^2$, motivated by the studies on closed geodesics and convex Hamiltonian systems, we follow [Vit1] to introduce a finite dimensional approximation to the space $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ to get the enough smoothness. This finite dimensional approximation allows us to apply the idea of the Splitting Lemma of D. Gromoll and W. Meyer [GrM1] to obtain the periodicity of critical modules for closed characteristics, which overcomes the first difficulty in addition to the study in [WHL1]. The second difficulty and the most important thing is that, all the critical modules at a critical orbit $S^1 \cdot x$ of $F_{a,K}$ rely on $K$, and we need to show they are isomorphic to each other and thus are independent of such $K$. This is proved by Theorem 3.3 below.

Because the functional $F_{a,K}$ is not $C^2$ on $W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$, we can not get Splitting Lemma for $F_{a,K}$ directly. But in our case, the functional $F_{a,K'}$ is uniformly concave in the direction of $D_\infty(K_0)$ by (3.7) below, where $D_\infty(K_0)$ is as in Definition 3.4, $K_0 < K'$ satisfying that $K' - K_0$ is small enough. Motivated by the method of [DHK1], we obtain a Splitting Lemma type argument (cf. Lemma 3.5) to complete the proof of Theorem 3.3.

(2) To achieve the above mentioned purposes, following Proposition 2.2 of [WHL1] and Proposition 2.7 of [Vit1], we construct a special family of Hamiltonian functions which have more flexible properties at the origin and infinity, and are homogenous in the middle and near the critical orbits.

In Section 2, fixing a hypersurface $\Sigma \in \mathcal{H}_{st}(2n)$ with $\#T(\Sigma) < +\infty$, we construct a family of Hamiltonian functions in Lemma 2.4 using auxiliary functions satisfying conditions (i)-(ii) of Lemma 2.2, together with Proposition 2.5 which yields more precise requirement on the Hamiltonian.
functions near the origin and infinity. Using such modified Hamiltonian functions, we construct a functional $F_{a,K}$ on the space $W^{1,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^{2n})$ for every $a > 0$ and $K$ satisfying $(2.3)-(2.4)$, whose critical points are precisely all the closed characteristics on $\Sigma$ with periods less than $aT$ and that the origin of $W^{1,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^{2n})$ is the only constant critical point of $F_{a,K}$. By a usual change of variables, properties of $F_{a,K}$ can be studied by a functional $\Psi_{a,K}$ on $L^{2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^{2n})$. Using the finite dimensional approximation, we get the Palais-Smale condition for $F_{a,K}$ and prove that for every fixed closed characteristic $(\tau,y)$ on $\Sigma$, the Viterbo index and nullity of all the functionals $F_{a,K}$ at its critical point corresponding to $(\tau,y)$ are independent of $a$ whenever $a > \frac{\tau}{T}$.

(3) In Section 3, we prove that for every fixed closed characteristic $(\tau,y)$ on $\Sigma$, the critical modules of all the functionals $F_{a,K}$ at its critical point corresponding to $(\tau,y)$ are independent of $a$ and $K$. Here the main difficulty part is to deal with the case when $K$ crosses values in $(2\pi/T)\mathbb{Z}$. Here the main idea is to use the Splitting Lemma type argument to obtain the independence of critical modules in $K$.

(4) In Section 4, we further require the Hamiltonian function to be homogeneous near every critical orbit so that the critical modules are periodic functions of the dimension. This homogeneity of the Hamiltonian function is realized by the condition (iii) of Lemma 2.2.

(5) In Section 5, we get a degenerate version of Theorem 7.1 of [Vit1] which shows that the origin has in fact no homological contribution to the lower order terms in the Morse series.

(6) In Section 6, we use the homological information obtained in the Sections 2-5, compute all the local critical modules of the dual action functional $F_{a,K}$ and use such information to set up a Morse theory for all the closed characteristics on $\Sigma \in \mathcal{H}_{st}(2n)$. Together with the global homological information, we establish the claimed mean index identities $(1.2)-(1.3)$ and prove Theorem 1.1.

In this paper, let $\mathbb{N}, \mathbb{N}_{0}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{R}^{+}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and positive real numbers respectively. Denote by $(a,b)$ and $|a|$ the standard inner product and norm in $\mathbb{R}^{2n}$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard $L^{2}$ inner product and $L^{2}$ norm. For an $S^{1}$-space $X$, we denote by $X_{S^{1}}$ the homotopy quotient of $X$ by $S^{1}$, i.e., $X_{S^{1}} = S^{\infty} \times_{S^{1}} X$, where $S^{\infty}$ is the unit sphere in an infinite dimensional complex Hilbert space. By $t \to a^{+}$, we mean $t > a$ and $t \to a$. In this paper we use only $\mathbb{Q}$ coefficients for all homological modules.
2 Variational structure for closed characteristics and finite dimensional reduction

In this paper, we follow the frameworks of [WHL1] and [Vit1] to transform the problem (1.1) into a fixed period problem of a Hamiltonian system with some period $T > 0$, which is fixed for the rest of the paper without further restrictions, and then study its variational structure. Here we omit most of the details and only point out differences from [WHL1] when necessary.

In the rest of this paper, we fix first a $\Sigma \in \mathcal{H}_{st}(2n)$ and assume the following condition on $T(\Sigma)$:

(F) There exist only finitely many geometrically distinct prime closed characteristics $\{(\tau_j, y_j)\}_{1 \leq j \leq k}$ on $\Sigma$.

As in [WHL1], we have the following discrete subset of $\mathbb{R}^+$:

Definition 2.1 Under the assumption (F), the set of periods of closed characteristics on $\Sigma$ is defined by

$$\text{per}(\Sigma) = \{ m\tau_j \mid m \in \mathbb{N}, 1 \leq j \leq k \}.$$ and let $\hat{\tau} = \inf\{ s \mid s \in \text{per}(\Sigma) \}$.

Motivated by Lemma 2.2, Proposition 2.7 of [Vit1], and omitting the condition (iv) of Proposition 2.2 of [WHL1] to get more flexibility, we use the following auxiliary function to further define Hamiltonian functions.

Lemma 2.2 For any sufficiently small $\vartheta \in (0, 1)$, there exists a function $\varphi \equiv \varphi_\vartheta \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ depending on $\vartheta$ which has 0 as its unique critical point in $[0, +\infty)$ such that the following hold.

(i) $\varphi(0) = 0 = \varphi'(0)$, and $\varphi''(0) = 1 = \lim_{t \to 0^+} \varphi''(t)$;

(ii) $\frac{d}{dt} \left( \frac{\varphi'(t)}{t} \right) < 0$ for $t > 0$, and $\lim_{t \to +\infty} \frac{\varphi'(t)}{t} < \vartheta$; that is, $\frac{\varphi'(t)}{t}$ is strictly decreasing for $t > 0$;

(iii) In particular, we can choose $\alpha \in (1, 2)$ sufficiently close to 2 and $c \in (0, 1)$ such that $\varphi(t) = ct^\alpha$ whenever $\frac{\varphi'(t)}{t} \in [\vartheta, 1 - \vartheta]$ and $t > 0$.

Remark 2.3. As in [WHL1], Lemma 2.2 (iii) above is used only in our study in the Section 4 to obtain the periodic property of critical modules at critical points. In the other parts of this paper we use function $\varphi$ which satisfy the properties (i)-(ii) only and defined on $[0, +\infty)$. In the proof of Lemma 2.4, given an $a > \frac{\vartheta}{c}$, we choose first the parameter $\vartheta \in (0, \frac{c}{a})$ depending on $a$. Then we choose the parameter $\alpha \in (1, 2)$ depending on $a$ so that the proof of Lemma 2.2 goes through, and choose $\varphi$ to be homogeneous of degree $\alpha$ and modify it near 0 and $+\infty$ such that (i)-(ii) of Lemma 2.2 hold. We denote such choices of $\vartheta$, $\alpha$ and $\varphi$ by $\vartheta_a$, $\alpha_a$ and $\varphi_a$ respectively to indicate their dependence on $a$. In such a way, we can obtain a connected family of $\varphi_a$ satisfying
(i)-(ii) of Lemma 2.2 such that \( \varphi_a \) and its first and second derivatives with respect to \( t \) depend continuously on \( a \).

Let \( j : \mathbb{R}^{2n} \to \mathbb{R} \) be the gauge function of \( \Sigma \), i.e., \( j(\lambda x) = \lambda \) for \( x \in \Sigma \) and \( \lambda \geq 0 \), then \( j \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R}) \) and \( \Sigma = j^{-1}(1) \). Then the following lemma was proved in Proposition 2.4 (iii) of [WHL1] (cf. also Lemmas 2.1 and 2.2 of [Vit1]).

**Lemma 2.4.** Let \( \hat{a} > \hat{\tau} T \), \( \vartheta_a \in (0, \hat{\tau} aT) \) and \( \varphi_a \) be a \( C^\infty \) function associated to \( \vartheta_a \) satisfying (i)-(ii) of Lemma 2.2 and continuously depending on the parameter \( a \) as mentioned in Remark 2.3.

Define the Hamiltonian function \( \bar{H}_a(x) = a \varphi_a(j(x)) \) and consider the fixed period system

\[
\begin{cases}
\dot{x}(t) = J\bar{H}'_a(x(t)), \\
x(0) = x(T).
\end{cases}
\]

Then solutions of (2.1) are \( x \equiv 0 \) and \( x = \rho y(\tau t/T) \) with \( \varphi'_a(\rho) = \frac{\tau}{aT} \), where \((\tau, y)\) is a solution of (1.1). In particular, non-zero solutions of (2.1) are in one to one correspondence with solutions of (1.1) with period \( \tau \leq aT \).

For technical reasons we want to further modify the Hamiltonian, more precisely, we follow Page 624 of [Vit1], and let \( \epsilon_a \) satisfy \( \epsilon_a T < 2\pi \) and \( \vartheta_a \) be small enough, we can construct a function \( H_a \), coinciding with \( \bar{H}_a \) on \( U_A = \{ x \mid \bar{H}_a(x) \leq A \} \) for some large \( A \), and with \( \frac{1}{2}\epsilon_a |x|^2 \) outside some large ball, such that \( \nabla H_a(x) \) does not vanish and \( H_a''(x) < \epsilon_a \) outside \( U_A \). As in Proposition 2.7 of [Vit1], we have the following result.

**Proposition 2.5.** For \( a > \hat{\tau} \) and small \( \epsilon_a \), we choose small enough \( \vartheta_a \) such that Lemma 2.4 holds. Then there exists a function \( H_a \) on \( \mathbb{R}^{2n} \) such that \( H_a \) is \( C^1 \) on \( \mathbb{R}^{2n} \), and \( C^3 \) on \( \mathbb{R}^{2n} \setminus \{0\} \), \( H_a = \bar{H}_a \) in \( U_A \), and \( H_a(x) = \frac{1}{2}\epsilon_a |x|^2 \) for \( |x| \) large, and the solutions of the fixed period system

\[
\begin{cases}
\dot{x}(t) = JH'_a(x(t)), \\
x(0) = x(T),
\end{cases}
\]

are the same with those of (2.1).

**Remark 2.6.** Note that here the first derivative of \( H_a(x) \) with respect to \( x \in \mathbb{R}^{2n} \) and the second derivative of \( H_a(x) \) with respect to \( x \in \mathbb{R}^{2n} \setminus \{0\} \) are continuous in the parameter \( a \). Note that under these choices, the first and second derivatives of \( \epsilon_a \) with respect to \( a \) are also continuous. Here, that \( H_a \)'s form a connected family in \( a \) is crucial in our proofs below for Lemma 2.10, and Propositions 2.11 and 3.2.

As in [BLMR] (cf. Section 3 of [Vit1]), for any \( a > \hat{\tau} \), we can choose some large constant \( K = K(a) \) such that

\[
H_{a,K}(x) = H_a(x) + \frac{1}{2}K|x|^2
\]
is a strictly convex function, that is,

\[ (\nabla H_{a,K}(x) - \nabla H_{a,K}(y), x - y) \geq \frac{\epsilon}{2} |x - y|^2, \]  

(2.4)

for all \(x, y \in \mathbb{R}^{2n}\), and some positive \(\epsilon\). Let \(H_{a,K}^*\) be the Fenchel dual of \(H_{a,K}\) defined by

\[ H_{a,K}^*(y) = \sup \{ x \cdot y - H_{a,K}(x) \mid x \in \mathbb{R}^{2n} \}. \]  

(2.5)

The dual action functional on \(X = W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^{2n})\) is defined by

\[ F_{a,K}(x) = \int_0^T \left[ \frac{1}{2} (J\dot{x} - Kx, x) + H_{a,K}^*(-J\dot{x} + Kx) \right] dt. \]  

(2.6)

Then \(F_{a,K} \in C^{1,1}(X, \mathbb{R})\) holds as proved in (3.16) of [Vit1], but \(F_{a,K}\) is not \(C^2\).

**Lemma 2.7.** (cf. Proposition 3.4 of [Vit1]) Assume \(KT \notin 2\pi \mathbb{Z}\), then \(x \) is a critical point of \(F_{a,K}\) if and only if it is a solution of (2.2).

From Lemma 2.7, we know that the critical points of \(F_{a,K}\) are independent of \(K\).

**Proposition 2.8.** For every critical point \(x_a \neq 0\) of \(F_{a,K}\), the critical value \(F_{a,K}(x_a) < 0\) holds and is independent of \(K\).

**Proof.** Since \(\nabla H_{a,K}(x_a) = -J\dot{x_a} + Kx_a\), then we have

\[ H_{a,K}^*(-J\dot{x_a} + Kx_a) = (-J\dot{x_a} + Kx_a, x_a) - H_{a,K}(x_a). \]

Thus we obtain

\[ F_{a,K}(x_a) = \int_0^T \left[ \frac{1}{2} (J\dot{x_a} - Kx_a, x_a) + H_{a,K}^*(-J\dot{x_a} + Kx_a) \right] dt \]

\[ = \int_0^T \left[ -\frac{1}{2} (J\dot{x_a} - Kx_a, x_a) - H_{a,K}(x_a) \right] dt \]

\[ = \int_0^T \left[ -\frac{1}{2} (J\dot{x_a} - x_a) - H(x_a) \right] dt \]

\[ = \int_0^T \left[ \frac{1}{2} (H'_a(x_a), x_a) - H(x_a) \right] dt. \]  

(2.7)

By Lemma 2.4 and Proposition 2.5, we have \(x_a = \rho_a y(\tau t/T)\) with \(\frac{\varphi'_{\rho_a}}{\rho_a} = \frac{T}{aT} \). Hence, we have

\[ F_{a,K}(x_a) = \frac{1}{2} a\varphi'_a(\rho_a) \rho_a T - a\varphi_a(\rho_a) T. \]  

(2.8)

Here we used the facts that \(j'(y) = N_\Sigma(y)\) and \(j'(y) \cdot y = 1\).

Let \(f(t) = \frac{1}{2} a\varphi'_a(t) t - a\varphi_a(t)\) for \(t \geq 0\). Then we have \(f(0) = 0\) and \(f'(t) = \frac{a}{2} (\varphi''_a(t) t - \varphi'_a(t)) < 0\) since \(\frac{d}{dt} (\frac{\varphi'_a(t)}{t}) < 0\) by (ii) of Lemma 2.2. Together with (2.8), it yields the proposition.
As well known, when \( KT \not\in 2\pi \mathbb{Z} \), the map \( x \mapsto -J\dot{x} + Kx \) is a Hilbert space isomorphism between \( X = W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}) \) and \( E = L^2(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^{2n}) \). We denote its inverse by \( M_K \) and the functional

\[
\Psi_{a,K}(u) = \int_0^T \left( -\frac{1}{2}(M_K u, u) + H_{a,K}^*(u) \right) dt, \quad \forall u \in E. \tag{2.9}
\]

Then \( x \in X \) is a critical point of \( F_{a,K} \) if and only if \( u = -J\dot{x} + Kx \) is a critical point of \( \Psi_{a,K} \). We have a natural \( S^1 \)-action on \( X \) or \( E \) defined by

\[
\theta \cdot u(t) = u(\theta + t), \quad \forall \theta \in S^1, \ t \in \mathbb{R}. \tag{2.10}
\]

Clearly both of \( F_{a,K} \) and \( \Psi_{a,K} \) are \( S^1 \)-invariant. For any \( \kappa \in \mathbb{R} \), we denote by

\[
\Lambda_{a,K}^\kappa = \{ u \in L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}) \mid \Psi_{a,K}(u) \leq \kappa \} \tag{2.11}
\]

and

\[
X_{a,K}^\kappa = \{ x \in W^{1,2}(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^{2n}) \mid F_{a,K}(x) \leq \kappa \}. \tag{2.12}
\]

Clearly, both level sets are \( S^1 \)-invariant.

**Definition 2.9.** (cf. p.628 of [Vit1]) Suppose \( u \) is a nonzero critical point of \( \Psi_{a,K} \). Then the formal Hessian of \( \Psi_{a,K} \) at \( u \) is defined by

\[
Q_{a,K}(v) = \int_0^T (-M_K v \cdot v + H'''_{a,K}(u)v \cdot v) dt, \tag{2.13}
\]

which defines an orthogonal splitting \( E = E_- \oplus E_0 \oplus E_+ \) of \( E \) into negative, zero and positive subspaces. The index and nullity of \( u \) are defined by \( i_K(u) = \dim E_- \) and \( \nu_K(u) = \dim E_0 \) respectively.

Similarly, we define the index and nullity of \( x = M_K u \) for \( F_{a,K} \), we denote them by \( i_K(x) \) and \( \nu_K(x) \). Then we have

\[
i_K(u) = i_K(x), \quad \nu_K(u) = \nu_K(x), \tag{2.14}
\]

which follow from the definitions (2.6) and (2.9). The following important formula was proved in Lemma 6.4 of [Vit1]:

\[
i_K(x) = 2n([KT/2\pi] + 1) + i''(x) \equiv d(K) + i''(x), \tag{2.15}
\]

where the index \( i''(x) \) does not depend on \( K \), but only on \( H_a \).

By the proof of Proposition 2 of [Vit2], we have that \( v \in E \) belongs to the null space of \( Q_{a,K} \) if and only if \( z = M_K v \) is a solution of the linearized system

\[
\dot{z}(t) = JH'''_{a}(x(t))z(t). \tag{2.16}
\]
Thus the nullity in (2.14) is independent of $K$, which we denote by $\nu^v(x) \equiv \nu_K(u) = \nu_K(x)$.

In this paper, we say that $\Psi_{a,K}$ with $a \in [a_1, a_2]$ form a continuous family of functionals in the sense of Remark 2.6, when $0 < a_1 < a_2 < +\infty$.

Motivated by Proposition 3.9 and Lemma 5.1 of [Vit1] as well as Lemma 3.4 of [WHL1], we have the following

**Lemma 2.10.** For any $0 < a_1 < a_2 < +\infty$, let $K$ be fixed so that $\Psi_{a,K}$ with $a \in [a_1, a_2]$ is a continuous family of functionals defined by (2.2) satisfying (2.3) with the same $\epsilon > 0$. Then there exist a finite dimensional $S^1$-invariant subspace $G$ of $L^2(R/TZ; R^{2n})$ and a family of $S^1$-equivariant maps $h_a : G \to G^\perp$ such that the following hold.

(i) For $g \in G$, each function $h \mapsto \Psi_{a,K}(g + h)$ has $h_a(g)$ as the unique minimum in $G^\perp$.

Let $\psi_{a,K}(g) = \Psi_{a,K}(g + h_a(g))$. Then we have

(ii) Each $\psi_{a,K}$ is $C^1$ and $S^1$-invariant on $G$. Here $g_a$ is a critical point of $\psi_{a,K}$ if and only if $g_a + h_a(g_a)$ is a critical point of $\Psi_{a,K}$.

(iii) If $g_a \in G$ and $H_a$ is $C^k$ with $k \geq 2$ in a neighborhood of the trajectory of $g_a + h_a(g_a)$, then $\psi_{a,K}$ is $C^{k-1}$ in a neighborhood of $g_a$. In particular, if $g_a$ is a nonzero critical point of $\psi_{a,K}$, then $\psi_{a,K}$ is $C^2$ in a neighborhood of the critical orbit $S^1 \cdot g_a$. The index and nullity of $\Psi_{a,K}$ at $g_a + h_a(g_a)$ defined in Definition 2.9 coincide with the Morse index and nullity of $\psi_{a,K}$ at $g_a$.

(iv) For any $\kappa \in R$, we denote by

$\tilde{\Lambda}_{a,K}^\kappa = \{g \in G \mid \psi_{a,K}(g) \leq \kappa\}$. \hspace{1cm} (2.17)

Then the natural embedding $\tilde{\Lambda}_{a,K}^\kappa \hookrightarrow \Lambda_{a,K}^\kappa$ given by $g \mapsto g + h_a(g)$ is an $S^1$-equivariant homotopy equivalence.

(v) The functionals $a \mapsto \psi_{a,K}$ is continuous in a in the $C^1$ topology. Moreover $a \mapsto \psi_{a,K}$ is continuous in a neighborhood of the critical orbit $S^1 \cdot g_a$.

**Proof.** Firstly, we consider the eigenvalues of $-M_K$. Let $x(t) = e^{-L^t}x_0$ for some $L \in \frac{2\pi}{T}Z$ and $x_0 \in R^{2n}$, then $-J\dot{x} + Lx = (L + K)x$. Thus $\{-\frac{1}{L+K} \mid L \in \frac{2\pi}{T}Z\}$ is the set of all the eigenvalues of $-M_K$.

By the convexity of $H_{a,K}^*$, we have

$$(H_{a,K}^*(u) - H_{a,K}^*(v), u - v) \geq \omega|u - v|^2, \hspace{0.5cm} \forall \ a \in [a_1, a_2], \ u, v \in R^{2n},$$ \hspace{1cm} (2.18)

for some $\omega > 0$. Hence we can use the proof of Proposition 3.9 of [Vit1] to obtain the subspace $G$ and the map $h_a$. In fact, Let $G$ be the subspace of $L^2(R/(TZ); R^{2n})$ generated by the eigenvectors of $-M_K$ whose eigenvalues are less than $-\frac{\omega}{2}$, i.e.,

$$G = \text{span}\{e^{-L^t}x_0 \mid -\frac{1}{L+K} < -\frac{\omega}{2}, L \in \frac{2\pi}{T}Z, x_0 \in R^{2n}\}.$$
and $h_a(g)$ is defined by the equation
\[
\frac{\partial}{\partial h} \Psi_{a,K}(g + h_a(g)) = 0.
\] (2.19)

Then (i)-(iii) follows from Proposition 3.9 of [Vit1], and (iv) follows from Lemma 5.1 of [Vit1].

The rest part of this proof is devoted to (v).

**Claim.** For each $a \in [a_1, a_2]$ and $\epsilon > 0$ small, we have
\[
|H^*_{a+\epsilon,K}(y) - H^*_{a,K}(y)| = O(\epsilon) + O(\epsilon)|y|^2, \quad \forall y \in \mathbb{R}^{2n},
\] (2.20)
\[
|H''_{a+\epsilon,K}(y) - H''_{a,K}(y)| = O(\epsilon) + O(\epsilon)|y|, \quad \forall y \in \mathbb{R}^{2n},
\] (2.21)

where we denote by $B = O(\epsilon)$ if $|B| \leq C|\epsilon|$ for some constant $C > 0$.

In fact, we fix an $a \in [a_1, a_2]$ and let $b \in [a - \epsilon, a + \epsilon] \cap [a_1, a_2]$. For any $y \in \mathbb{R}^{2n}$, let $H'_{b,K}(y) = z_b$, then $H'_{b,K}(z_b) = y$. By the convexity of $H_{b,K}$, we have
\[
|u_1 - u_2| \leq \alpha|H'_{b,K}(u_1) - H'_{b,K}(u_2)|, \quad \forall u_1, u_2 \in \mathbb{R}^{2n}, b \in [a - \epsilon, a + \epsilon],
\] (2.22)
for some constant $\alpha > 0$ which is independent of $b$. Thus, we obtain
\[
|H''_{a+\epsilon,K}(y) - H''_{a,K}(y)| = |z_{a+\epsilon} - z_a| \\
\leq \alpha|H'_{a+\epsilon,K}(z_{a+\epsilon}) - H'_{a+\epsilon,K}(z_a)| \\
= \alpha|H'_{a,K}(z_a) - H'_{a+\epsilon,K}(z_a)| \\
= \alpha(O(\epsilon) + O(\epsilon)|z_a|) \\
= \alpha(O(\epsilon) + O(\epsilon)\alpha|y|).
\] (2.23)

Here we have used the fact that $H_a = \frac{1}{2}\epsilon_a|x|^2$ for $|x|$ large and the derivative of $\epsilon_a$ with respect to $a$ is continuous by Remark 2.6. Hence, (2.21) holds.

For (2.20), we have
\[
H^*_{b,K}(y) = z_b \cdot y - H_{b,K}(z_b).
\] (2.24)

Then
\[
|H^*_{a+\epsilon,K}(y) - H^*_{a,K}(y)| = |(y, z_{a+\epsilon} - z_a) + H_{a+\epsilon,K}(z_{a+\epsilon}) - H_{a,K}(z_a)| \\
= O(\epsilon) + O(\epsilon)|y|^2.
\] (2.25)

Here we used (2.23) and the fact that $H_a = \frac{1}{2}\epsilon_a|x|^2$ for $|x|$ large and the derivative of $\epsilon_a$ with respect to $a$ is continuous by Remark 2.6. The claim is proved.
Now we have the following estimates:

\[
|\Psi_{a+\epsilon,K}(u) - \Psi_{a,K}(u)| \leq \int_0^T |H_{a+\epsilon,K}^*(u) - H_{a,K}^*(u)| dt = O(\epsilon) + O(\epsilon)\|u\|^2, \quad (2.26)
\]

\[
\|\Psi_{a+\epsilon,K}(u) - \Psi_{a,K}(u)\|^2 = \|H_{a+\epsilon,K}^{\prime*}(u) - H_{a,K}^{\prime*}(u)\|^2 = O(\epsilon) + O(\epsilon)\|u\|^2. \quad (2.27)
\]

As in [Vit1], (2.19) and the definition of \(G\) yield

\[
\langle \Psi_{a,K}^\prime(u) - \Psi_{a,K}^\prime(v), u - v \rangle \geq \frac{\omega}{2}\|u - v\|^2, \quad \forall u - v \in G^1, \quad a \in [a_1, a_2]. \quad (2.28)
\]

Hence we have

\[
\frac{\omega}{2}\|h_{a+\epsilon}(g) - h_a(g)\|^2 \leq \langle \Psi_{a+\epsilon,K}^\prime(g + h_{a+\epsilon}(g)) - \Psi_{a+\epsilon,K}^\prime(g + h_a(g)), h_{a+\epsilon}(g) - h_a(g) \rangle = \langle \Psi_{a,K}^\prime(g + h_a(g)) - \Psi_{a,K}^\prime(g + h_a(g)), h_{a+\epsilon}(g) - h_a(g) \rangle = (O(\epsilon) + O(\epsilon)\|g + h_a(g)\|^2)^{1/2}\|h_{a+\epsilon}(g) - h_a(g)\|,
\]

where the first equality follows from (2.19) and the last equality follows from (2.27). Hence the map \(a \mapsto h_a(g)\) is continuous.

Because \(\psi_{a,K}(g) = \Psi_{a,K}(g + h_a(g))\) by definition, \(\psi_{a,K}^\prime(g) = \frac{\partial}{\partial g}\Psi_{a,K}(g + h_a(g))\) by (2.19), hence the first statement of (v) follows from (2.26) and (2.27). The last statement of (v) follows from p.629 of [Vit1] and the implicit functional theorem with parameters.

**Proposition 2.11.** For all \(b \geq a \geq \tau\), let \(F_{b,K}\) be the functional defined by (2.6), and \(x_b\) be the critical point of \(F_{b,K}\) so that \(x_b\) corresponds to a fixed closed characteristic \((\tau, y)\) on \(\Sigma\) for all \(b \geq a\). Then the index \(\nu^v(x_b)\) and nullity \(\nu^\nu(x_b)\) are constants for all \(b \geq a\). In particular, when \(H_b\) is \(\alpha\)-homogenous for some \(\alpha \in (1, 2)\) near the image set of \(x_b\), the index and nullity coincide with those defined for the Hamiltonian \(H(x) = j(x)^\alpha\) for all \(x \in \mathbb{R}^{2n}\). Especially \(1 \leq \nu^\nu(x_b) \leq 2n - 1\) always holds.

**Proof.** Denote by \(R(t)\) the fundamental solution of (2.10) satisfying \(R(0) = I_{2n}\). Then by Lemma 1.6.11 of [Eke2], whose proof does not need the convexity of \(\Sigma\), we have

\[
R(t)T_{y(0)}\Sigma \subset T_{y(\tau t)}\Sigma. \quad (2.29)
\]

Then the completely same argument of Proposition 3.5 of [WHL1] proves that \(\nu^\nu(x_a)\) is constant for all \(H_a\) satisfying Proposition 2.5 with \(a \geq \tau\) and \(1 \leq \nu^\nu(x_a) \leq 2n - 1\).

For any \(b > a > \tau\), by (iii) of Lemma 2.2, we can construct a continuous family of \(\Psi_{c,K}\) with \(c \in [a, b]\) such that \(H_b\) is homogenous of degree \(\alpha = \alpha_b\) near the image set of \(x_b\). Now we can use Lemma 2.10 (v) to obtain a continuous family of \(\psi_{c,K}\) such that \(\psi_{c,K}^\prime(g_c)\) depends continuously on
where $c \in [a, b]$, where $g_c$ is the critical point of $\psi_{c,K}$ corresponding to $M_K^{-1}x_c$. Because $\dim \ker \psi''_{c,K}(g_c) = \nu_K(M_K^{-1}x_c) = \nu_v(x_c)$ is constant, the index of $\psi''_{c,K}(g_c) = i_K(M_K^{-1}x_c) = i_v(x_c) + d(K)$ must be constant too. Thus $i_v(x_b)$ is constant for all $b \geq a$. Note that here we used (2.14), (2.15), the definition of $i_v(x_b)$ below (2.16), and Lemma 2.10 (iii). Since the index $i_v(x_b)$ and nullity $\nu_v(x_b)$ only depend on the value of $H_b$ near the image set of $x_b$ (cf. Proposition 2 of [Vit1]), then the index and nullity coincide with those defined for the Hamiltonian $H(x) = j(x)α$, $\forall x \in \mathbb{R}^{2n}$. The proof is complete.

**Proposition 2.12.** $\Psi_{a,K}$ satisfies the Palais-Smale condition on $E$, and $F_{a,K}$ satisfies the Palais-Smale condition on $X$, when $KT \notin 2\pi \mathbb{Z}$.

**Proof.** We first prove that $\Psi_{a,K}$ satisfies the Palais-Smale condition on $E = L^2(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^{2n})$. Below we use short hand notations $\Psi$, $\psi$, $h$, $H_K^*$, $F$, and $M$ for $\Psi_{a,K}$, $\psi_{a,K}$, $h_a$, $H_K^*, F_{a,K}$, and $M_K$ respectively.

Assume that $x_j = g_j + h_j \in E$ is a sequence such that $\Psi'(x_j) \to 0$, as $j \to \infty$, where $g_j \in G$, $h_j \in G^\perp$. Then

$$
\langle \Psi'(x_j), h_j - h(g_j) \rangle = o(1)\|h_j - h(g_j)\|,
$$

(2.30)

where we denote by $B_j = o(1)$ if $B_j \to 0$ as $j \to \infty$. On the other hand, since $h_j - h(g_j) \in G^\perp$ and $\frac{\partial \psi}{\partial h}(g_j + h(g_j)) = 0$, we obtain

$$
\langle \Psi'(x_j), h_j - h(g_j) \rangle = \langle \Psi'(x_j) - \Psi'(g_j + h(g_j)), h_j - h(g_j) \rangle.
$$

(2.31)

From (2.28) which implies the convexity of $\Psi$ in the direction of $G^\perp$, we have

$$
\langle \Psi'(x_j) - \Psi'(g_j + h(g_j)), h_j - h(g_j) \rangle \geq \frac{\omega}{2}\|h_j - h(g_j)\|^2,
$$

(2.32)

for some $\omega > 0$. Combining (2.30)-(2.32), we obtain

$$
\|h_j - h(g_j)\| = o(1).
$$

(2.33)

Similar to (3.16) of [Vit1], because $\nabla H_{K}^*$ is Lipschitz, we have

$$
\|\Psi'(x_j) - \Psi'(g_j + h(g_j))\| \leq C\|h_j - h(g_j)\|.
$$

(2.34)

Now $\Psi'(x_j) \to 0$, from (2.33) and (2.34), we have $\psi'(g_j) = \Psi'(g_j + h(g_j)) = o(1)$. Together with Proposition 4.1 of [Vit1], whose proof goes through in our setting without any modifications, it shows that $g_j$ has a converging subsequence. Hence, $x_j = g_j + h_j$ must have a converging subsequence by (2.33), i.e., $\Psi$ satisfies the Palais-Smale condition on $E$. 

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For the Palais-Smale condition of $F$ on $X = W^{1,2}(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^{2n})$, we have first
\[
F(Mu) = \Psi(u), \quad \forall u \in E, \tag{2.35}
\]
where $M$ is a Hilbert space isomorphism between $E$ and $X$. Then
\[
\langle F'(Mu), Mv \rangle_X = \langle \Psi'(u), v \rangle_E, \quad \forall u, v \in E. \tag{2.36}
\]
Now we use the standard $L^2$-norm for $E$, and the norm $\|M^{-1}x\|_E$ for $x \in X$ which is equivalent to the standard one. Then $\langle F'(Mu), Mv \rangle_X = \langle M^{-1}F'(Mu), M^{-1}Mv \rangle_E$ holds. Together with (2.35) it yields the following identity on $E$:
\[
M^{-1}F'(Mu) = \Psi'(u). \tag{2.37}
\]
Because $M$ is a Hilbert space isomorphism between $E$ and $X$, by (2.35) and (2.37) the Palais-Smale condition of $F$ on $X$ follows from that of $\Psi$ on $E$.

3 Parameter independence of critical modules for closed characteristics

For a critical point $u$ of $\Psi_{a,K}$ and the corresponding $x = M_Ku$ of $F_{a,K}$, let
\[
\Lambda_{a,K}(u) = \Lambda_{a,K}(u) = \{w \in L^2(\mathbb{R}/(T\mathbb{Z}), \mathbb{R}^{2n}) \mid \Psi_{a,K}(w) \leq \Psi_{a,K}(u)\}, \tag{3.1}
\]
\[
X_{a,K}(x) = X_{a,K}(x) = \{y \in W^{1,2}(\mathbb{R}/(T\mathbb{Z}), \mathbb{R}^{2n}) \mid F_{a,K}(y) \leq F_{a,K}(x)\}. \tag{3.2}
\]
Clearly, both sets are $\mathbb{S}^1$-invariant. Denote by $\text{crit}(\Psi_{a,K})$ the set of critical points of $\Psi_{a,K}$. Because $\Psi_{a,K}$ is $\mathbb{S}^1$-invariant, $\mathbb{S}^1 \cdot u$ becomes a critical orbit if $u \in \text{crit}(\Psi_{a,K})$. Note that by the condition (F), Lemma 2.4, Proposition 2.5 and Lemma 2.7, the number of critical orbits of $\Psi_{a,K}$ is finite. Hence as usual we can make the following definition.

**Definition 3.1.** Suppose $u$ is a nonzero critical point of $\Psi_{a,K}$, and $\mathcal{N}$ is an $\mathbb{S}^1$-invariant open neighborhood of $\mathbb{S}^1 \cdot u$ such that $\text{crit}(\Psi_{a,K}) \cap (\Lambda_{a,K}(u) \cap \mathcal{N}) = \mathbb{S}^1 \cdot u$. Then the $\mathbb{S}^1$-critical modules of $\mathbb{S}^1 \cdot u$ is defined by
\[
C_{\mathbb{S}^1, q}(\Psi_{a,K}, \mathbb{S}^1 \cdot u) = H_{\mathbb{S}^1, q}(\Lambda_{a,K}(u) \cap \mathcal{N}, (\Lambda_{a,K}(u) \setminus \mathbb{S}^1 \cdot u) \cap \mathcal{N}) \equiv H_q((\Lambda_{a,K}(u) \cap \mathcal{N})_{\mathbb{S}^1}, ((\Lambda_{a,K}(u) \setminus \mathbb{S}^1 \cdot u) \cap \mathcal{N})_{\mathbb{S}^1}), \tag{3.3}
\]
where $H_{\mathbb{S}^1, *}$ is the $\mathbb{S}^1$-equivariant homology with rational coefficients in the sense of A. Borel (cf. Chapter IV of [Bor1]). Similarly, we define the $\mathbb{S}^1$-critical modules $C_{\mathbb{S}^1, q}(F_{a,K}, \mathbb{S}^1 \cdot x)$ of $\mathbb{S}^1 \cdot x$ for $F_{a,K}$. 

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As well-known, this definition is independent of the choice of $\mathcal{N}$ by the excision property of the singular homology theory (cf. Definition 1.7.5 of [Cha1]). Recall that $X_{S^1}$ is defined at the end of Section 1.

We have the following for critical modules.

**Proposition 3.2.** Let $(\tau, y)$ be a closed characteristic on $\Sigma$. For any $\frac{\tau}{\pi} < a_1 < a_2 < +\infty$, let $K$ be a fixed sufficiently large real number so that (2.4) holds for all $a \in [a_1, a_2]$. Then the critical module $C_{S^1, q}(F_{a,K}, S^1 \cdot x)$ is independent of the choice of $H_a$ defined in Proposition 2.5 for any $a \in [a_1, a_2]$ in the sense that if $x_1$ and $x_2$ correspond to the same closed characteristic $(\tau, y)$ on $\Sigma$, then we have

$$C_{S^1, q}(F_{a_1,K}, S^1 \cdot x_1) \cong C_{S^1, q}(F_{a_2,K}, S^1 \cdot x_2), \quad \forall q \in \mathbb{Z}. \quad (3.4)$$

In other words, the critical modules are independent of the choices of all $a > \frac{\tau}{\pi}$, the function $\varphi_a$ satisfying (i)-(ii) of Lemma 2.2, and $H_a$ satisfying Proposition 2.5.

**Proof.** Let $\varphi_a$ be a family of functions satisfying (i)-(ii) of Lemma 2.2 and let $H_a(x)$ satisfy Proposition 2.5 parameterized by $a \in [a_1, a_2]$. Without loss of generality we can assume $H_a$ depends continuously on $a$ in the sense of Remark 2.6. For each $a \in [a_1, a_2]$, we denote by $x_a$ the corresponding solution of (2.2) with the Hamiltonian $H_a$.

Now (2.26) and (2.27) imply that $b \mapsto \Psi_{b,K}$ is continuous in the $C^1$ topology. Then $b \mapsto F_{b,K}$ is continuous in the $C^1$ topology too. Note that the number of critical orbits of each $F_{b,K}$ is finite. Hence by the continuity of critical modules (cf. Theorem 8.8 of [MaW1] or Theorem 1.5.6 on p.53 of [Cha1], which can be easily generalized to the equivariant case), our proposition holds.

Note that a similar argument shows that the critical modules are independent of the choice of $\varphi_a$ in $H_a(x) = a\varphi_a(j(x))$ whenever $a$ is fixed, $\varphi_a$ satisfies (i)-(ii) of Lemma 2.2, and $H_a$ satisfies Proposition 2.5.

The rest of this section is devoted to the proof of independence of the critical modules for closed characteristics in the choice of $K$. In the following, we fix an $a > \frac{\tau}{\pi}$, and write $F_K$ and $H$ for $F_{a,K}$ and $H_a$ respectively. We suppose also that $K \in \mathbb{R}$ satisfy (2.4), i.e.,

$$H_K(x) = H(x) + \frac{1}{2}K|x|^2 \quad \text{is strictly convex.} \quad (3.5)$$

By Lemma 2.7, the critical points of $F_K$ which are the solutions of (2.2) are the same for any $K$ satisfying that $K \notin \frac{2\pi}{\tau}\mathbb{Z}$. Recall $d(K) = 2n([KT/2\pi] + 1)$ by (2.15).
Theorem 3.3. Suppose \( \bar{x} \) is a nonzero critical point of \( F_K \). Then the \( S^1 \)-critical module 
\[
C_{S^1, d(K)+l}(F_K, S^1 \cdot \bar{x}) \cong C_{S^1, d(K')}+l(F_{K'}, S^1 \cdot \bar{x}),
\]
(3.6)
where \( KT, K'T \notin 2\pi \mathbb{Z}, l \in \mathbb{Z} \), and both \( K \) and \( K' \) satisfy (\[3.3\]).

We carry out the proof of this theorem in the following two cases: (a) \( d(K) = d(K') \); and (b) \( K < K_0 < K' \) with \( K_0 - K, K' - K_0 \) small enough, and \( \frac{K_0 T}{2\pi} \) is an integer.

It is clear that proofs of Cases (a) and (b) imply the general case.

Proof of Case (a). By Lemma 2.7, the critical orbits of \( F_\sigma \) are independent of \( \sigma \) and the number is finite by the condition (F). Then for \( \bar{x} \in \text{crit}(F_\sigma) \) there exists an \( S^1 \)-invariant open neighborhood \( U \) of \( S^1 \cdot \bar{x} \) in \( X \) such that \( F_\sigma \) has a unique critical orbit \( S^1 \cdot \bar{x} \) in \( U \), for all \( \sigma \in [K, K'] \). By (7.12) and (7.13) of [Vit1], \( \sigma \mapsto F_\sigma \) is continuous in \( C^1(\overline{U}) \) topology for \( \sigma \notin \frac{2\pi \mathbb{Z}}{\pi} \). Hence, by the continuity of critical modules (see Theorem 8.8 of [MaW1] or Theorem 1.5.6 of [Cha1], which can be easily generalized to the equivariant sense), and the Palais-Smale condition of \( F_\sigma \) given by Proposition 2.12, the proof of Case (a) is complete.

Before we give the proof of Case (b), we give one definition and two lemmas.

Definition 3.4. Assume \( K_0 \in \frac{2\pi}{\pi} \mathbb{Z} \). As in [Vit1], let
\[
D_{\infty}(K_0) = \{\exp(-JK_0 t)x \mid x \in \mathbb{R}^{2n}\},
\]
and define \( C(K_0) \) to be the orthogonal complement of \( D_{\infty}(K_0) \) in \( X \).

Let \( K_0 < K' \) such that \( K' - K_0 \) is small enough. As pointed out by Viterbo in [Vit1], \( F_{K'} \) is strictly concave in the direction of \( D_{\infty}(K_0) \). More precisely, similar to the argument to get (7.7) of [Vit1], there exists a constant \( C > 0 \) such that for all \( x \in X \) and \( h \in D_{\infty}(K_0) \) we have
\[
\langle F'_{K'}(x + h) - F'_{K'}(x), h \rangle_X \leq -C\|h\|_X^2,
\]
(3.7)
where the equality holds if and only if \( h = 0 \).

Let \( \bar{x} \) be a nonzero critical point of \( F_{K'} \) with multiplicity \( \text{mul}(\bar{x}) = m \), i.e., it corresponds to a closed characteristic \( (m\tau, y) \subset \Sigma \) with \( (\tau, y) \) being prime. Write \( \bar{x} = \bar{y} + \bar{z} \), where \( \bar{y} \in C(K_0) \), \( \bar{z} \in D_{\infty}(K_0) \). Note that \( \bar{y} \neq 0 \) must hold, because otherwise, by (7.14) of [Vit1], we have \( \bar{z} = 0 \) and then \( \bar{x} = 0 \).

Then expanding \( \bar{x} \) into its Fourier series according to the \( C(K_0) \) and \( D_{\infty}(K_0) \) components, from the fact \( \text{mul}(\bar{x}) = m \), we obtain \( \text{mul}(\bar{y}) = m \), then \( \bar{y}(t + \frac{1}{m}) = \bar{y}(t) \) for all \( t \in \mathbb{R} \) and the orbit of \( \bar{y} \), namely, \( S^1 \cdot \bar{y} \cong S^1/Z_m \cong S^1 \). Let \( p : N(S^1 \cdot \bar{y}) \to S^1 \cdot \bar{y} \) be the normal bundle of \( S^1 \cdot \bar{y} \) in \( C(K_0) \) and
let $p^{-1}(\theta \cdot \bar{y}) = N(\theta \cdot \bar{y})$ be the fibre over $\theta \cdot \bar{y}$, where $\theta \in S^1$. Let $DN(S^1 \cdot \bar{y})$ be the $g$ disk bundle of $N(S^1 \cdot \bar{y})$ for some $g > 0$ sufficiently small, i.e., $DN(S^1 \cdot \bar{y}) = \{ \xi \in N(S^1 \cdot \bar{y}) \mid \| \xi \|_{H^1} < g \}$ which is identified by the exponential map with a subset of $C(K_0)$, and let $DN(\theta \cdot \bar{y}) = p^{-1}(\theta \cdot \bar{y}) \cap DN(S^1 \cdot \bar{y})$ be the disk over $\theta \cdot \bar{y}$. Clearly, $DN(\theta \cdot \bar{y})$ is $Z_m$-invariant and we have $DN(S^1 \cdot \bar{y}) = DN(\bar{y}) \times Z_m S^1$, where the $Z_m$ action is given by

$$(\theta, v, t) \in Z_m \times DN(\bar{y}) \times S^1 \to (\theta \cdot v, \theta^{-1}t) \in DN(\bar{y}) \times S^1.$$ Hence for an $S^1$ invariant subset $\Gamma$ of $DN(S^1 \cdot \bar{y})$, we have $\Gamma/S^1 = (\Gamma_{\bar{y}} \times Z_m S^1)/S^1 = \Gamma_{\bar{y}}/Z_m$, where $\Gamma_{\bar{y}} = \Gamma \cap DN(\bar{y})$. Obviously, we also have a bundle $\tilde{p} : \tilde{N}(S^1 \cdot \bar{x}) \to S^1 \cdot \bar{x}$ of $S^1 \cdot \bar{x}$ in $X = C(K_0) \oplus D_\infty(K_0)$, where the fibre over $\theta \cdot \bar{x}$ is $N(\theta \cdot \bar{y}) \oplus D_\infty(K_0)$, $\theta \in S^1$. Let $D\tilde{N}(S^1 \cdot \bar{x})$ be the $\tilde{g}$ disk bundle of $\tilde{N}(S^1 \cdot \bar{x})$ for some $\tilde{g} > 0$ sufficiently small, i.e., $D\tilde{N}(S^1 \cdot \bar{x}) = \{ \xi \in \tilde{N}(S^1 \cdot \bar{x}) \mid \| \xi \|_{H^1} < \tilde{g} \}$ which is identified by the exponential map with a subset of $X$, and let $D\tilde{N}(\theta \cdot \bar{x}) = \tilde{p}^{-1}(\theta \cdot \bar{x}) \cap D\tilde{N}(S^1 \cdot \bar{x})$ be the disk over $\theta \cdot \bar{x}$. Clearly, $D\tilde{N}(\theta \cdot \bar{x})$ is $Z_m$-invariant and we have $D\tilde{N}(S^1 \cdot \bar{x}) = D\tilde{N}(\bar{x}) \times Z_m S^1$ where the $Z_m$ action is given by

$$(\theta, v, t) \in Z_m \times D\tilde{N}(\bar{x}) \times S^1 \to (\theta \cdot v, \theta^{-1}t) \in D\tilde{N}(\bar{x}) \times S^1.$$}

**Lemma 3.5.** Let $\bar{x}$ be a nonzero critical point of $F_{K'}$ with $\text{mul}(\bar{x}) = m$. Then there exists an open ball $B(0, r)$ in $T_{\bar{x}}(D\tilde{N}(\bar{x}))$ centered at 0 with radius $r > 0$, a local $Z_m$-equivariant homeomorphism $\phi : B(0, r) \to \phi(B(0, r)) \subset D\tilde{N}(\bar{x})$, $\phi(0) = \bar{x}$, a $Z_m$-equivariant $C^0$ map $h : B(0, r) \cap T_{\bar{y}}(DN(\bar{y})) \to D_\infty(K_0)$ such that

$$F_{K'}(\phi(\xi)) = -\| \eta \|^2_X + F_{K'}(\bar{x} + \nu + h(\nu)), \quad (3.8)$$

where $\xi = \eta + \nu$ with $\nu \in B(0, r) \cap T_{\bar{y}}(DN(\bar{y}))$ and $\eta \in D_\infty(K_0)$.

**Proof.** By (3.7), in the direction of $D_\infty(K_0)$, $F_{K'}$ is strictly concave. Then there is a map

$h : T_{\bar{y}}(DN(\bar{y})) \to D_\infty(K_0)$

uniquely defined by the relation $\nabla F_{K'}(\bar{x} + \nu + h(\nu)) \in C(K_0)$, i.e., $h(\nu)$ achieves the strict maximum of $F_{K'}(\bar{x} + \nu + g)$ for $g \in D_\infty(K_0)$. By the same proof of Lemma 2.2 of [DHK1], and noticing that (7.14) of [Vit1], we get that $h$ is continuous and $h(0) = 0$. Note that since $F_{K'}$ is $S^1$-invariant and $\text{mul}(\bar{x}) = m$, then $h$ is $Z_m$-equivariant.

Let $H^+ = T_{\bar{y}}(DN(\bar{y}))$ and $H^- = D_\infty(K_0)$. Now we define a map $\psi : T_{\bar{x}}(D\tilde{N}(\bar{x})) = H^+ \oplus H^- \to T_{\bar{x}}(D\tilde{N}(\bar{x}))$ by

$$\psi(\nu, \mu) = \begin{cases} (\nu, \psi_1(\nu, \mu)) & \text{if } \mu \neq 0, \\ (\nu, 0), & \text{if } \mu = 0, \end{cases} \quad (3.9)$$
where \( \nu \in H^+, \mu \in H^- \). Then \( \psi \) is continuous on \( H^+ \oplus H^- \).

**Claim (A).** \( \psi \) is one-to-one on \( H^+ \oplus H^- \).

Suppose \( \psi(\nu_1, \mu_1) = \psi(\nu_2, \mu_2) \) for some \( (\nu_i, \mu_i) \in H^+ \oplus H^- \), \( i = 1, 2 \). By (3.9), we have \( \nu_1 = \nu_2 \), \( \frac{\mu_1}{\|\nu_1\|_X} = \frac{\mu_2}{\|\nu_2\|_X} \) and

\[
\sqrt{F'_{K'}(\bar{x} + \nu_1 + h(\nu_1)) - F'_{K'}(\bar{x} + \nu_1 + h(\nu_1) + \mu_1)} = \sqrt{F'_{K'}(\bar{x} + \nu_2 + h(\nu_2)) - F'_{K'}(\bar{x} + \nu_2 + h(\nu_2) + \mu_2)}.
\]

Then we have \( F'_{K'}(\bar{x} + \nu_1 + h(\nu_1) + \mu_1) = F'_{K'}(\bar{x} + \nu_1 + h(\nu_1) + \mu_2) \) and we may suppose \( \mu_2 = s\mu_1 \) for some \( s \geq 1 \). By the mean value theorem, there exists \( 1 \leq t \leq s \) such that

\[
0 = F'_{K'}(\bar{x} + \nu_1 + h(\nu_1) + s\mu_1) - F'_{K'}(\bar{x} + \nu_1 + h(\nu_1) + \mu_1)
\]

\[
= (F'_{K'}(\bar{x} + \nu_1 + h(\nu_1) + t\mu_1), (s - 1)\mu_1)_X
\]

\[
= (F'_{K'}(\bar{x} + \nu_1 + h(\nu_1) + t\mu_1) - F'_{K'}(\bar{x} + \nu_1 + h(\nu_1)), (s - 1)\mu_1)_X
\]

\[
\leq -(s - 1)tC\|\mu_1\|^2_X
\]

where \( C > 0 \) is defined in (3.7), and we have used the fact that \( \nabla F'_{K'}(\bar{x} + \nu_1 + h(\nu_1)) \in C(K_0) \) implies

\[
\langle F'_{K'}(\bar{x} + \nu_1 + h(\nu_1)), \mu_1 \rangle_X = 0.
\]

Thus we have \( s = 1 \) or \( \mu_1 = 0 \), then \( \mu_1 = \mu_2 \). Claim (A) follows.

**Claim (B).** For any \( \epsilon > 0 \), there exists a positive real number \( \delta_\epsilon > 0 \) such that

\[
B_{H^+}(0, \epsilon) \times B_{H^-}(0, \delta_\epsilon) \subseteq \psi(B_{H^+}(0, \epsilon) \times B_{H^-}(0, \epsilon)),
\]

where \( B_{H^*}(0, \epsilon) \) denotes an open ball in \( H^* \), \( * = +, - \).

In fact, by (3.7) and noticing that \( \nabla F'_{K'}(\bar{x} + \nu + h(\nu)) \in C(K_0) \) implies

\[
\langle F'_{K'}(\bar{x} + \nu + h(\nu)), \mu \rangle_X = 0, \quad \forall \mu \in D_\infty(K_0),
\]

then we have

\[
F'_{K'}(\bar{x} + \nu + h(\nu)) - F'_{K'}(\bar{x} + \nu + h(\nu) + \mu)
\]

\[
= -\int_0^1 \frac{d}{dt}F'_{K'}(\bar{x} + \nu + t\nu + h(\nu) + t\mu)dt
\]

\[
= -\int_0^1 (F'_{K'}(\bar{x} + \nu + h(\nu) + t\mu), \mu)_X dt
\]

\[
= -\int_0^1 (F'_{K'}(\bar{x} + \nu + h(\nu) + t\mu) - F'_{K'}(\bar{x} + \nu + h(\nu)), \mu)_X dt
\]

\[
\geq C\int_0^1 t\|\mu\|^2_X dt = \frac{C}{2}\|\mu\|^2_X,
\]
where \( C > 0 \) is the constant defined in (3.7). Then by the continuity of \( h \) and the definition of \( \psi_1 \) in (3.9), it follows that

\[
\{t \frac{\mu}{\|\mu\|_X}, 0 \leq t < \sqrt{\frac{C}{2}}\epsilon\} \subseteq \psi_1(\nu \times B_H^{-}(0, \epsilon)), \quad \forall \mu \in \partial B_H^{-}(0, \epsilon), \nu \in B_H^{+}(0, \epsilon).
\]

Let \( \delta_\epsilon = \sqrt{\frac{C}{2}}\epsilon \), we have

\[
B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon) \subseteq \psi(B_H^{+}(0, \epsilon) \times B_H^{-}(0, \epsilon)),
\]

which proves Claim (B).

Let \( \varphi \) be the restriction of \( \psi^{-1} \) on \( B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon) \).

**Claim (C).** \( \varphi \) is continuous on \( B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon) \).

Let \( (\nu_0, \mu_0) \in \psi^{-1}(B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon)) \) and \( \{(\nu_n, \mu_n)\} \) be a sequence in \( \psi^{-1}(B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon)) \) such that \( \{(\nu_n, \mu_n)\} \) converges to \( (\nu_0, \mu_0) \), we prove that \( \{(\nu_n, \mu_n)\} \) converges to \( (\nu_0, \mu_0) \). Firstly, by definition we have \( \nu_n \to \nu_0 \). Since \( B_H^{-}(0, \epsilon) \) is compact, we can suppose that \( \mu_n \to \mu \) in \( B_H^{+}(0, \epsilon) \). Then by the continuity of \( h \) and \( F_{K'} \), \( h(\nu_n) \to h(\nu_0) \) and

\[
\sqrt{F_{K'}(\bar{x} + \nu_n + h(\nu_n)) - F_{K'}(\bar{x} + \nu_n + h(\nu_n) + \mu_n)} \to \sqrt{F_{K'}(\bar{x} + \nu_0 + h(\nu_0)) - F_{K'}(\bar{x} + \nu_0 + h(\nu_0) + \mu)} \quad \text{as} \quad n \to +\infty.
\]

Thus we get \( \psi(\nu_0, \mu_0) = \psi(\nu_0, \mu) \) by (3.9), since \( \psi \) is one-to-one, then \( \mu = \mu_0 \) and Claim (C) follows.

Now by Claims (A), (B) and (C), \( \varphi \) is an homeomorphism from \( B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon) \) to an open neighborhood of 0 in \( T_{\bar{x}}(D\tilde{N}(\bar{x})) \). We define \( \phi = \exp_{\bar{x}} \circ \tau \circ \varphi \), where \( \exp_{\bar{x}} \) is the exponential map, \( \tau \) is defined by

\[
\tau(\nu, \mu) = (\nu, h(\nu) + \mu),
\]

which is an homeomorphism from \( T_{\bar{x}}(D\tilde{N}(\bar{x})) \) to itself satisfying \( \tau(0) = 0 \). Then for \( \epsilon > 0 \) sufficiently small, \( \phi \) is an homeomorphism from \( B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon) \) to an open neighborhood of \( \bar{x} \) in \( D\tilde{N}(\bar{x}) \). Note that by the above proof, \( \phi \) is \( Z_m \)-equivariant, and for any \((\nu, \eta) \in B_H^{+}(0, \epsilon) \times B_H^{-}(0, \delta_\epsilon)\), we can write \( \eta = \psi_1(\nu, \mu) \) for some \( \mu \in B_H^{-}(0, \epsilon) \), then

\[
F_{K'}(\phi(\nu, \eta)) = F_{K'}(\phi \circ \psi(\nu, \mu)) = F_{K'}(\bar{x} + \nu + h(\nu) + \mu) = F_{K'}(\bar{x} + \nu + h(\nu)) - \|\psi_1(\nu, \mu)\|^2_X = F_{K'}(\bar{x} + \nu + h(\nu)) - \|\eta\|^2_X.
\]

Then (3.8) holds. Let \( r = \min\{\epsilon, \delta_\epsilon\} \), we complete the proof of Lemma 3.5.

**Lemma 3.6.** \( F_{K_0} \) satisfies the Palais-Smale condition on \( C(K_0) \).
Proposition 2.12, we obtain that $F$ of \cite{Vit1}, this with $e$ of \cite{Vit1}, we obtain the Palais-Smale condition for $\Psi$ in the proof of Proposition 4.1 of \cite{Vit1} should be modified to $L$. Lines 9-10 on Page 642 of \cite{Vit1}, we have $K$ where we write $L$ is the unique critical orbit of $S$. Now for $\bar{y}$, we can choose an $K$ in our case. Here the equation $M(g_n + h_n) - \nabla H_K(g_n + h_n) = \epsilon_n$ in the proof of Proposition 4.1 of \cite{Vit1}, the functional $\Psi$ on $C(K_0)$ is strictly convex (resp. concave) in the direction of $D_\infty(K_0)$ for $L < K_0$ (resp. $L > K_0$). Thus there is an $S^1$-equivariant map $z_L : C(K_0) \to D_\infty(K_0), \ y \mapsto z_L(y)$, \begin{equation}
abla F_L((y + z_L(y)) \in C(K_0), i.e., $z_L(y)$ achieves the minimum (resp. maximum) of $F_L(y + h)$ for $h$ in $D_\infty(K_0)$. Note that the map $z_L$ is $C^{0,1}$ by Page 627 of \cite{Vit1}, and when $L = K'$ the map $z_L$ is the map $h$ defined in Lemma 3.5.

Now for $y \in C(K_0)$, set $\tilde{F}_L(y) = F_L(y + z_L(y))$ for $L \in [K, K'] \setminus \{K_0\}$. Then $\tilde{F}_L$ is $C^{1,1}$ and $S^1$-invariant. By Lines 9-10 on Page 642 of \cite{Vit1}, we have

$$|\tilde{F}_L(y) - \tilde{F}_K_0(y)| \leq C|L - K_0||y||_{H^1}, \quad (3.11)$$

$$\|\nabla \tilde{F}_L(y) - \nabla \tilde{F}_K_0(y)\|_{H^1} \leq C|L - K_0||y||_{H^1}, \quad (3.12)$$

where we write $\tilde{F}_K_0$ for $F_K_0$ restricted to $C(K_0)$. Since $\bar{x} \neq 0$ is a critical point of $F_L$, there exists a unique $\bar{y} \in C(K_0)$ such that $\bar{x} = \bar{y} + \bar{z}$, where $\bar{z} \in D_\infty(K_0)$. Then $z_L(\bar{y}) = \bar{z}$ holds, which is independent of $L$. Note that $\bar{y} \neq 0$, because otherwise, by (7.14) of \cite{Vit1}, we have $\bar{z} = 0$ and then $\bar{x} = 0$.

To continue the proof, we need the following three claims.

**Claim 1.** We can choose an $S^1$-invariant open neighborhood $U_1$ of $S^1 \cdot \bar{y}$ in $C(K_0)$ such that $S^1 \cdot \bar{y}$ is the unique critical orbit of $\tilde{F}_L$ in $U_1$ for all $L \in [K, K']$.

In fact, firstly there is an $S^1$-invariant open neighborhood $V_1$ of $S^1 \cdot \bar{y}$ in $C(K_0)$ such that $S^1 \cdot \bar{y}$ is the unique critical orbit of $\tilde{F}_L$ in $V_1$ for all $L \in [K, K'] \setminus K_0$. If not, there exists a sequence of
$S^1$ orbits $\{S^1 \cdot \bar{y}_j\}_{j \geq 1} \subset C(K_0)$ such that $\lim_{j \to \infty} \bar{y}_j = \bar{y}$ and $\bar{y}_j$ is a critical point of $\bar{F}_{L_j}$ for some $L_j \in [K, K'] \setminus K_0$. Then $\bar{y}_j + z_{L_j}(\bar{y}_j)$ is a critical point of $F_{L_j}$ for all $j \in \mathbb{N}$. Since by Lemma 2.7 the critical points of $F_L$ are the same for all $L \in [K, K'] \setminus K_0$, then $\bar{y}_j + z_{L_j}(\bar{y}_j)$ is a critical point of $F_L$ for some $L \in [K, K'] \setminus \{K_0\}$ and $z_{L_j}(\bar{y}_j) = z_L(\bar{y}_j)$. But $\lim_{j \to \infty} (\bar{y}_j + z_L(\bar{y}_j)) = \bar{y} + z_L(\bar{y})$ and the critical orbits of $F_L$ are isolated by the condition (F), which yields a contradiction.

If $\nabla \bar{F}_{K_0}(z) = 0$ for some $z \in C(K_0)$, by definition, we have $w = z - \nabla H^*_{K_0}(-J \dot{z} + K_0 z) \in D_{\infty}(K_0)$. Then $\nabla H_{K_0}(z - w) = -J \dot{z} + K_0 z = -J(\dot{z} - \dot{w}) + K_0 (z - w)$, i.e., $\nabla H(z - w) = -J(\dot{z} - \dot{w})$. Thus $z - w$ is a solution of (2.2). But on the other hand, all the solutions of (2.2) are isolated $S^1$ orbits by the condition (F), so we can choose an $S^1$-invariant open neighborhood $V_2$ of $\bar{y}$ in $C(K_0)$ such that $S^1 \cdot \bar{y}$ is the unique critical orbit of $\bar{F}_{K_0}$ in $V_2$. Hence, setting $U_1 = V_1 \cap V_2$, Claim 1 is proved.

Note that $\bar{F}_L$ satisfies the Palais-Smale condition by Proposition 2.12 and Lemma 3.6. Now combining (3.11)-(3.12) with the continuity of critical modules depending on $L$ (cf. Theorem 8.8 of [MaW1] or Theorem 1.5.6 of [Ch], which can be easily generalized to the equivariant sense), we obtain the $C^1$-continuity of $\bar{F}_L$ in $L \in [K, K']$. Together with Claim 1, we obtain

$$C_{S^1, d(K)+1}(\bar{F}_L, S^1 \cdot \bar{y}) \cong C_{S^1, d(K)+1}(\bar{F}_{K_0}, S^1 \cdot \bar{y}) \cong C_{S^1, d(K)+1}(\bar{F}_{K'}, S^1 \cdot \bar{y}).$$ \hfill (3.13)

**Claim 2.** $F_L(\bar{x})$ is independent of $L$.

In fact, since $\nabla H_L(\bar{x}) = -J \dot{\bar{x}} + L \bar{x}$, then $H^*_L(-J \dot{\bar{x}} + L \bar{x}) = (-J \dot{\bar{x}} + L \bar{x}, \bar{x}) - H_L(\bar{x})$. Thus

$$F_L(\bar{x}) = \int_0^T \left[ \frac{1}{2}(J \dot{\bar{x}} - L \bar{x}, \bar{x}) + H^*_L(-J \dot{\bar{x}} + L \bar{x}) \right] dt$$
$$= \int_0^T \left[ -\frac{1}{2}(J \dot{\bar{x}} - L \bar{x}, \bar{x}) - H_L(\bar{x}) \right] dt$$
$$= \int_0^T \left[ -\frac{1}{2}(J \dot{\bar{x}}, \bar{x}) - H(\bar{x}) \right] dt,$$ \hfill (3.14)

which is independent of $L$. Thus Claim 2 is proved.

Now let $c = F_L(\bar{x})$. We define $X^c(K) = \{y \in X \mid F_K(y) \leq c\}$ and $\bar{X}^c(K) = \{y \in C(K_0) \mid \bar{F}_K(y) \leq c\}$. Let $\bar{U}$ be an $S^1$-invariant open neighborhood of $\bar{y}$ in $C(K_0)$ such that $\bar{F}_K$ has unique critical orbit $S^1 \cdot \bar{y}$ in $\bar{U}$, then $U \equiv \bar{U} \times D_{\infty}(K_0)$ is an $S^1$-invariant open neighborhood of $\bar{x}$ such that $F_K$ has unique critical orbit $S^1 \cdot \bar{x}$ in $U$.

**Claim 3.** The natural embeddings $\bar{X}^c(K) \cap \bar{U} \to X^c(K) \cap U$ and $(\bar{X}^c(K) \setminus \{\bar{y}\}) \cap \bar{U} \to (X^c(K) \setminus \{\bar{x}\}) \cap U$ are $S^1$-equivariant homotopy equivalences.

In fact, by the strictly convexity of $F_K$ in the direction of $D_{\infty}(K_0)$ and the argument of Lemma 5.1 of [VÎ1], the claim follows.
By Claim 3, we have the following:

\[ C_{s^1,d(K)+l}(F_K, S^1 \cdot \bar{x}) \cong C_{s^1,d(K)+l}(\bar{F}_K, S^1 \cdot \bar{y}). \]  

(3.15)

Together with (3.13), this yields

\[ C_{s^1,d(K)+l}(F_K, S^1 \cdot \bar{x}) \cong C_{s^1,d(K)+l}(\bar{F}_{K'}, S^1 \cdot \bar{y}). \]  

(3.16)

In Lemma 3.5, let \( f_1(\eta) = -\|\eta\|_X^2 \) for all \( \eta \in D_\infty(K_0) \), \( f_2(\nu) = F_{K'}(\bar{x} + \nu + h(\nu)) \) for all \( \nu \in B(0,r) \cap T_\bar{y}(DN(\bar{y})) \). Then the Gromoll-Meyer pair of 0 for \( f_1 \), i.e., \((W_1, W_{1-})\), is \( \mathbb{Z}_m \)-equivariant homotopy equivalent with \((B^{2n}, S^{2n-1})\) since \( f_1 \) is \( \mathbb{Z}_m \)-invariant. Note that for (3.8) of Lemma 3.5, by the definitions of \( h \) and \( \bar{F}_{K'} \), we have

\[ \frac{\partial}{\partial z} F_{K'}(\bar{y} + \nu + z_{K'}(\bar{y} + \nu)) = 0, \]

\[ f_2(\nu) = F_{K'}(\bar{x} + \nu + h(\nu)) = F_{K'}(\bar{y} + \nu + z_{K'}(\bar{y} + \nu)) = \bar{F}_{K'}(\bar{y} + \nu) \]

for \( \nu \in T_\bar{y}(DN(\bar{y})) \). Denote by \((W_2, W_{2-})\) the Gromoll-Meyer pair of \( \bar{y} \) with respect to the negative gradient vector field of \( \bar{F}_{K'} \) in \( DN(\bar{y}) \), \((W_2, W_{2-})\) is \( \mathbb{Z}_m \)-invariant since \( f_2 \) is \( \mathbb{Z}_m \)-invariant. Thus, we obtain

\[ C_{d(K)+l}(\bar{F}_{K'}|_{DN(\bar{y})}, \bar{y}) \cong H_{d(K)+l}(W_2, W_{2-}). \]  

(3.17)

Using Lemma 1.5.1 of [Cha1] and Lemma 3.5, we have

\[ C_{d(K)+l+2n}(F_{K'}|_{DN(\bar{x})}, \bar{x}) \cong H_{d(K)+l+2n}(W_1 \times W_2, (W_1- \times W_2) \cup (W_1 \times W_{2-})). \]  

(3.18)

By Definition 3.1, we have

\[ C_{s^1, \ast}(F_{K'}, S^1 \cdot \bar{x}) \]

\[ \cong H_{s^1, \ast}(X_{a,K'}(\bar{x}) \cap D\bar{N}(S^1 \cdot \bar{x}), (X_{a,K'}(\bar{x}) \setminus (S^1 \cdot \bar{x})) \cap D\bar{N}(S^1 \cdot \bar{x})), \]  

(3.19)

where \( X_{a,K'}(\bar{x}) \) is defined as in (3.2). Since all the isotropy groups \( G_x = \{ g \in S^1 \mid g \cdot x = x \} \) for \( x \in D\bar{N}(S^1 \cdot \bar{x}) \) are finite, we can use Lemma 6.11 of [FaR1] to obtain

\[ H^S_{s^1}(X_{a,K'}(\bar{x}) \cap D\bar{N}(S^1 \cdot \bar{x}), (X_{a,K'}(\bar{x}) \setminus (S^1 \cdot \bar{x})) \cap D\bar{N}(S^1 \cdot \bar{x})) \]

\[ \cong H^S(X_{a,K'}(\bar{x}) \cap D\bar{N}(S^1 \cdot \bar{x})/S^1, (X_{a,K'}(\bar{x}) \setminus (S^1 \cdot \bar{x})) \cap D\bar{N}(S^1 \cdot \bar{x})/S^1) \]

\[ \cong H^S(X_{a,K'}(\bar{x}) \cap D\bar{N}(\bar{x})/\mathbb{Z}_m, (X_{a,K'}(\bar{x}) \setminus (\bar{x})) \cap D\bar{N}(\bar{x})/\mathbb{Z}_m). \]

By the condition (F) at the beginning of Section 2, a small perturbation on the energy functional can be applied to reduce each critical orbit to nearby non-degenerate ones. Thus similar to the
Similarly, we have

\[ H_{s'}(X_{A,K'}(x) \cap D_{\bar{N}}(S^1 \cdot x), (X_{A,K'}(x) \setminus (S^1 \cdot x)) \cap D_{\bar{N}}(S^1 \cdot x)) \]
\[ \cong H_{s}(X_{A,K'}(x) \cap D_{\bar{N}}(S^1 \cdot \bar{x})/S^1, (X_{A,K'}(x) \setminus (S^1 \cdot \bar{x})) \cap D_{\bar{N}}(S^1 \cdot \bar{x})/S^1) \]
\[ \cong H_{s}(X_{A,K'}(\bar{x}) \cap D_{\bar{N}}(\bar{x})/Z_m, (X_{A,K'}(\bar{x}) \setminus (\bar{x})) \cap D_{\bar{N}}(\bar{x})/Z_m). \quad (3.20) \]

For a \( Z_m \)-space pair \((A,B)\), let

\[ H_s(A, B)_{\pm Z_m} = \{ \sigma \in H_s(A, B) \mid L_s \sigma = \pm \sigma \}, \]

where \( L \) is a generator of the \( Z_m \)-action. Note that the same argument as in Section 6.3 of [Rad2], in particular Satz 6.6 of [Rad2], Lemma 3.6 of [BaL1] or Theorem 3.2.4 of [Bre1] yields

\[ H_{s}(X_{A,K'}(\bar{x}) \cap D_{\bar{N}}(\bar{x})/Z_m, (X_{A,K'}(\bar{x}) \setminus (\bar{x})) \cap D_{\bar{N}}(\bar{x})/Z_m) \]
\[ \cong H_{s}(X_{A,K'}(\bar{x}) \cap D_{\bar{N}}(\bar{x}), (X_{A,K'}(\bar{x}) \setminus (\bar{x})) \cap D_{\bar{N}}(\bar{x}))^{Z_m}. \quad (3.21) \]

Combining (3.19)-(3.21), we have

\[ C_{S^1, d(K)+l+2n}(F_{K'}, S^1 \cdot \bar{x}) \]
\[ \cong H_{d(K)+l+2n}(X_{A,K'}(\bar{x}) \cap D_{\bar{N}}(\bar{x}), (X_{A,K'}(\bar{x}) \setminus (\bar{x})) \cap D_{\bar{N}}(\bar{x}))^{Z_m} \]
\[ \cong C_{d(K)+l+2n}(F_{K'}|_{D_{\bar{N}}(\bar{x})}, \bar{x})^{Z_m}. \quad (3.22) \]

Similarly, we have

\[ C_{S^1, d(K)+l}(\tilde{F}_{K'}, S^1 \cdot \bar{y}) \cong C_{d(K)+l}(\tilde{F}_{K'}|_{D_{\bar{N}}(\bar{y})}, \bar{y})^{Z_m}. \quad (3.23) \]

Now by (3.17) and (3.18), as in Proposition 3.10 of [WHL1], we have

\[ C_{d(K)+l+2n}(F_{K'}|_{D_{\bar{N}}(\bar{x})}, \bar{x})^{Z_m} \cong C_{d(K)+l}(\tilde{F}_{K'}|_{D_{\bar{N}}(\bar{y})}, \bar{y})^{Z_m}. \quad (3.24) \]

In fact, let \( \theta \) be a generator of the linearized \( Z_m \)-action on \( W_1 \). Then \( \theta(\xi) = \xi \) for \( 0 \neq \xi \in T_0(W_1) \) if and only if \( m| K_{0T}/2\pi \). Thus together with (3.17), (3.18) and the fact that \( \dim W_1 \) is even, it yields (3.24).

Hence, it follows from (3.22)-(3.24) that

\[ C_{S^1, d(K)+l+2n}(F_{K'}, S^1 \cdot \bar{x}) \cong C_{S^1, d(K)+l}(\tilde{F}_{K'}, S^1 \cdot \bar{y}). \quad (3.25) \]

Combining (3.16) and (3.25), using the fact that \( d(K') = d(K) + 2n \), we obtain

\[ C_{S^1, d(K)+l}(F_{K'}, S^1 \cdot \bar{x}) \cong C_{S^1, d(K)+l+2n}(F_{K'}, S^1 \cdot \bar{x}) = C_{S^1, d(K') + l}(F_{K'}, S^1 \cdot \bar{x}). \]

The proof of Theorem 3.3 is complete.
4 Periodic property of critical modules for closed characteristics

In this section, we fix \( a \) and let \( u_K \neq 0 \) be a critical point of \( \Psi_{a,K} \) with multiplicity \( \text{mul}(u_K) = m \), that is, \( u_K \) corresponds to a closed characteristic \((\tau,y) \subset \Sigma\) with \((\tau,y)\) being \( m \)-iteration of some prime closed characteristic. Precisely, by Proposition 2.5 and Lemma 2.7, we have \( u_K = -J\dot{x} + Kx \) with \( x \) being a solution of (2.2) and \( x = \rho y(t) \) with \( \frac{\varphi'(\rho)}{\rho} = \frac{t}{\alpha} \). Moreover, \((\tau,y)\) is a closed characteristic on \( \Sigma \) with minimal period \( \frac{\tau}{m} \). Hence the isotropy group satisfies \( \{ \theta \in S^1 \mid \theta \cdot u_K = u_K \} = Z_m \) and the orbit of \( u_K \), namely, \( S^1 \cdot u_K \cong S^1/Z_m \cong S^1 \). By Lemma 2.10, we obtain a critical point \( g_K \) of \( \psi_{a,K} \) corresponding to \( u_K \), and then the isotropy group satisfies

\[
\{ \theta \in S^1 \mid \theta \cdot g_K = g_K \} = Z_m.
\]

Let \( p : N(S^1 \cdot g_K) \to S^1 \cdot g_K \) be the normal bundle of \( S^1 \cdot g_K \) in \( G \) (as defined in Lemma 2.10) and let \( p^{-1}(\theta \cdot g_K) = N(\theta \cdot g_K) \) be the fibre over \( \theta \cdot g_K \), where \( \theta \in S^1 \). Let \( DN(S^1 \cdot g_K) \) be the 
\[ \text{disk bundle of } N(S^1 \cdot g_K) \text{ for some } p > 0 \text{ sufficiently small, i.e.,} \]

\[
DN(S^1 \cdot g_K) = \{ \xi \in N(S^1 \cdot g_K) \mid \|\xi\| < \rho \}\]

which is identified by the exponential map with a subset of \( G \), and let \( DN(\theta \cdot g_K) = p^{-1}(\theta \cdot g_K) \cap DN(S^1 \cdot g_K) \) be the disk over \( \theta \cdot g_K \). Clearly, \( DN(\theta \cdot g_K) \)

is \( Z_m \)-invariant and we have \( DN(S^1 \cdot g_K) = DN(g_K) \times Z_m S^1 \) where the \( Z_m \) action is given by

\[
(\theta, v, t) \in Z_m \times DN(g_K) \times S^1 \mapsto (\theta \cdot v, \theta^{-1}t) \in DN(g_K) \times S^1.
\]

Hence for an \( S^1 \) invariant subset \( \Gamma \) of \( DN(S^1 \cdot g_K) \), we have \( \Gamma/S^1 \cong (\Gamma_{g_K} \times Z_m S^1)/S^1 = \Gamma_{g_K}/Z_m \), where \( \Gamma_{g_K} = \Gamma \cap DN(g_K) \).

For a \( Z_m \)-space pair \((A,B)\), let

\[
H_\ast(A,B)^\pm Z_m = \{ \sigma \in H_\ast(A,B) \mid L_\ast \sigma = \pm \sigma \},
\]

where \( L \) is a generator of the \( Z_m \)-action. Then as in Section 6 of \([\text{Rad2}]\), Section 3 of \([\text{BaL}]\) or Lemma 3.9 of \([\text{WHL}]\), we have

**Lemma 4.1.** Suppose \( u_K \neq 0 \) is a critical point of \( \Psi_{a,K} \) with \( \text{mul}(u_K) = m \), \( g_K \) is a critical point of \( \psi_{a,K} \) corresponding to \( u_K \). Then we have

\[
C_{S^1,\ast}(\Psi_{a,K}, S^1 \cdot u_K) \cong C_{S^1,\ast}(\psi_{a,K}, S^1 \cdot g_K)
\]

\[
\cong H_\ast((\bar{\Lambda}_{a,K}(g_K) \cap DN(g_K))/Z_m, ((\bar{\Lambda}_{a,K}(g_K) \setminus \{g_K\}) \cap DN(g_K))/Z_m)
\]

\[
\cong H_\ast((\bar{\Lambda}_{a,K}(g_K) \cap DN(g_K)), ((\bar{\Lambda}_{a,K}(g_K) \setminus \{g_K\}) \cap DN(g_K)))^Z_m.
\]

where \( \bar{\Lambda}_{a,K}(g_K) = \{ g \in G \mid \psi_{a,K}(g) \leq \psi_{a,K}(g_K) \} \).

**Proof.** For reader’s conveniences, we sketch a proof here and refer to Section 6 of \([\text{Rad2}]\), Section 3 of \([\text{BaL}]\) or Lemma 3.9 of \([\text{WHL}]\) for related details.
By Lemma 2.10 (iv), we have
\[ C_{S^1, *}(\Psi_{a,K}, S^1 \cdot u_K) \cong C_{S^1, *}(\psi_{a,K}, S^1 \cdot g_K). \]

By Definition 3.1, we have
\[ C_{S^1, *}(\psi_{a,K}, S^1 \cdot g_K) \cong H_{S^1, *}(\tilde{\Lambda}_{a,K}(g_K) \cap DN(S^1 \cdot g_K), (\tilde{\Lambda}_{a,K}(g_K) \setminus (S^1 \cdot g_K)) \cap DN(S^1 \cdot g_K)). \]

Since all the isotropy groups \( A_x = \{a \in S^1 \mid a \cdot x = x\} \) for \( x \in DN(S^1 \cdot g_K) \) are finite, we can use Lemma 6.11 of [FaR1] to obtain
\[ H_{S^1, *}(\tilde{\Lambda}_{a,K}(g_K) \cap DN(S^1 \cdot g_K), (\tilde{\Lambda}_{a,K}(g_K) \setminus (S^1 \cdot g_K)) \cap DN(S^1 \cdot g_K)) \]
\[ \cong H^*(\tilde{\Lambda}_{a,K}(g_K) \cap DN(S^1 \cdot g_K)/S^1, ((\tilde{\Lambda}_{a,K}(g_K) \setminus (S^1 \cdot g_K)) \cap DN(S^1 \cdot g_K))/S^1) \]
\[ \cong H^*(\tilde{\Lambda}_{a,K}(g_K) \cap DN(g_K)/\mathbb{Z}_m, ((\tilde{\Lambda}_{a,K}(g_K) \setminus \{g_K\}) \cap DN(g_K))/\mathbb{Z}_m). \]

By the condition (F) at the beginning of Section 2, a small perturbation on the energy functional can be applied to reduce each critical orbit to nearby non-degenerate ones. Thus similar to the proofs of Lemmas 2 and 4 of [GrM1], all the homological \( \mathbb{Q} \)-modules of each space pair in the above relations are all finitely generated. Therefore we can apply Theorem 5.5.3 and Corollary 5.5.4 on pages 243-244 of [Spa1] to obtain the same relation on homological \( \mathbb{Q} \)-modules:
\[ H_{S^1, *}(\tilde{\Lambda}_{a,K}(g_K) \cap DN(S^1 \cdot g_K), (\tilde{\Lambda}_{a,K}(g_K) \setminus (S^1 \cdot g_K)) \cap DN(S^1 \cdot g_K)) \]
\[ \cong H^*(\tilde{\Lambda}_{a,K}(g_K) \cap DN(S^1 \cdot g_K)/S^1, ((\tilde{\Lambda}_{a,K}(g_K) \setminus (S^1 \cdot g_K)) \cap DN(S^1 \cdot g_K))/S^1) \]
\[ \cong H^*(\tilde{\Lambda}_{a,K}(g_K) \cap DN(g_K)/\mathbb{Z}_m, ((\tilde{\Lambda}_{a,K}(g_K) \setminus \{g_K\}) \cap DN(g_K))/\mathbb{Z}_m). \]

Note that the same argument as in Section 6.3 of [Rad2], in particular Satz 6.6 of [Rad2], Lemma 3.6 of [BaL1] or Theorem 3.2.4 of [Brc1] yields
\[ H^*(\tilde{\Lambda}_{a,K}(g_K) \cap DN(g_K)/\mathbb{Z}_m, ((\tilde{\Lambda}_{a,K}(g_K) \setminus \{g_K\}) \cap DN(g_K))/\mathbb{Z}_m) \]
\[ \cong H^*(\tilde{\Lambda}_{a,K}(g_K) \cap DN(g_K), ((\tilde{\Lambda}_{a,K}(g_K) \setminus \{g_K\}) \cap DN(g_K))\mathbb{Z}_m). \]

The above relations together complete the proof of Lemma 4.1.

By (2.6) and (2.9), we have \( C_{S^1, *}(\Psi_{a,K}, S^1 \cdot u_K) \cong C_{S^1, q}(F_{a,K}, S^1 \cdot x) \). By Proposition 3.2, the module \( C_{S^1, q}(F_{a,K}, S^1 \cdot x) \) is independent of the choice of the Hamiltonian function \( H_a \) whenever \( H_a \) satisfies conditions in Proposition 2.5. Hence in order to compute the critical modules, we can choose \( \Psi_{a,K} \) with \( H_a \) being positively homogeneous of degree \( \alpha = \alpha_a \) near the image set of every nonzero solution \( x \) of (2.2) which corresponds to some closed characteristic \((\tau, y)\) with period \( \tau \) being strictly less than \( aT \).
In other words, for a given \( a > 0 \), we choose \( \vartheta \in (0,1) \) first such that \([aT\vartheta, aT(1-\vartheta)] \supset \text{per}(\Sigma) \cap (0, aT)\) holds by the definition of the set \( \text{per}(\Sigma) \) and the assumption (F). Then we choose \( \alpha = \alpha_a \in (1, 2) \) sufficiently close to 2 by (iii) of Lemma 2.2 such that \( \varphi_a(t) = ct^\alpha \) for some constant \( c > 0 \) and \( \alpha \in (1, 2) \) whenever \( \varphi_a'(t) \in [\vartheta, 1 - \vartheta] \). Now we suppose that \( \varphi_a \) satisfies (iii) of Lemma 2.2.

Now we consider iterations of critical points of \( \Psi_{a,K} \). Suppose \( u_K \neq 0 \) is a critical point of \( \Psi_{a,K} \) with \( \text{mul}(u_K) \equiv m \), and \( g_K \) is the critical point of \( \psi_{a,K} \) corresponding to \( u_K \). By Proposition 2.5 and Lemma 2.7, we have \( u_K = -J\dot{x} + Kx \) with \( x \) being a solution of (2.2) and \( x = \rho y(\frac{\tau}{\alpha}) \) with \( \varphi'(o) = \frac{\tau}{\alpha} \). Moreover, \( (\tau, y) \) is a closed characteristic on \( \Sigma \) with minimal period \( \frac{\tau}{m} \). For any \( p \in \mathbb{N} \) satisfying \( p\tau < aT \), we choose \( K \) such that \( pK \notin \frac{2\pi}{\tau} \mathbb{Z} \), then the \( p \)th iteration \( u_{pK}^p \) of \( u_K \) is given by \( -J\dot{x}^p + pKx^p \), where \( x^p \) is the unique solution of (2.2) corresponding to \( (p\tau, y) \) and is a critical point of \( F_{a,pK} \), that is, \( u_{pK}^p \) is the critical point of \( \Psi_{a,pK} \) corresponding to \( x^p \). Hence we have

\[
x(t) = \left( \frac{\tau}{\alpha a} \right)^{\frac{1}{\alpha}} y(\tau t), \quad x^p(t) = \left( \frac{p\tau}{\alpha a} \right)^{\frac{1}{\alpha}} y(p\tau t) = \frac{1}{p^{\frac{\alpha}{\alpha-2}}} x(pt),
\]

\[
u_K(t) = -J\dot{x}(t) + Kx(t), \quad u_{pK}^p(t) = -J\dot{x}^p(t) + pKx^p(t) = \frac{1}{p^{\frac{\alpha}{\alpha-2}}} u_K(pt).
\]

We define the \( p \)th iteration \( \varphi^p \) on \( L^2(\mathbb{R}/(T \mathbb{Z}); \mathbb{R}^{2n}) \) by

\[
\varphi^p : v_K(t) \mapsto v_{pK}^p(t) \equiv p^{\frac{\alpha-1}{\alpha-2}} v_K(pt).
\]

Then there exist a \( w \in W^{1,2}(\mathbb{R}/(T \mathbb{Z}); \mathbb{R}^{2n}) \) such that

\[
v_K(t) = -J\dot{w}(t) + Kw(t), \quad v_{pK}^p(t) = -J\dot{w}^p(t) + pKw^p(t), \quad w^p(t) = p^{\frac{\alpha-1}{\alpha-2}} w(pt).
\]

By definition, we have

\[
\Psi_{a,K}(v_K) = \int_0^T \left[ \frac{1}{2}(J\dot{w} - Kw, w) + H^*_a,K(-J\dot{w} + Kw) \right] dt
\]

\[
\Psi_{a,pK}(v_{pK}^p) = \int_0^T \left[ \frac{1}{2}(J\dot{w}^p - pKw^p, w^p) + H^*_a,pK(-J\dot{w}^p + pKw^p) \right] dt
\]

\[
= \frac{1}{2} p^{\frac{\alpha}{\alpha-2}} \int_0^T (J\dot{w}(pt) - Kw(pt), w(pt)) dt
\]

\[
+ \int_0^T H^*_a,pK(p^{\frac{\alpha}{\alpha-2}} (-J\dot{w}(pt) + Kw(pt))) dt.
\]

Let \( \xi(t) = H^*_a,K(-J\dot{w}(t) + Kw(t)) \), then \( H^*_a,K(\xi(t)) = -J\dot{w}(t) + Kw(t) \). Note that when \( v_K \) belongs to a small \( L^\infty \)-neighborhood of \( u_K \), \( \xi \) belongs to a small \( L^\infty \)-neighborhood of \( x \). In the following, we suppose that \( v_K \) belongs to a small \( L^\infty \)-neighborhood of \( u_K \).
Since $H_a$ is positively homogeneous of degree $\alpha$ near the image set of $x$, we have $H'_a(p^{\frac{1}{\alpha-2}}\xi(pt)) = p^{\frac{1}{\alpha-2}}H'_a(\xi(pt))$, $H_a(p^{\frac{1}{\alpha-2}}\xi(pt)) = p^{\frac{1}{\alpha-2}}H_a(\xi(pt))$. Thus, we get

\[
H'_{a,pK}(p^{\frac{1}{\alpha-2}}\xi(pt)) = H'_a(p^{\frac{1}{\alpha-2}}\xi(pt)) + pKp^{\frac{1}{\alpha-2}}\xi(pt) = \frac{p^{\frac{1}{\alpha-2}}}{p^{\frac{\alpha}{\alpha-2}}}H'_a(\xi(pt)) = p^{\frac{1}{\alpha-2}}(-J\dot{w}(pt) + Kw(pt)).
\]

So it follows that

\[
H'_{a,pK}(p^{\frac{\alpha}{\alpha-2}}(-J\dot{w}(pt) + Kw(pt))) = H'_{a,pK}(p^{\frac{1}{\alpha-2}}\xi(pt)) \cdot p^{\frac{1}{\alpha-2}}\xi(pt) - H_{a,pK}(p^{\frac{1}{\alpha-2}}\xi(pt))
\]

\[
= p^{\frac{\alpha}{\alpha-2}}H'_{a,K}(\xi(pt)) \cdot \xi(pt) - p^{\frac{\alpha}{\alpha-2}}H_a(\xi(pt)) - pK[p^{\frac{1}{\alpha-2}}\xi(pt)]^2
\]

\[
= p^{\frac{\alpha}{\alpha-2}}(H'_{a,K}(\xi(pt)) \cdot \xi(pt) - H_{a,K}(\xi(pt)))
\]

\[
= p^{\frac{\alpha}{\alpha-2}}H'_{a,K}(H'_{a,K}(\xi(pt)))
\]

\[
= p^{\frac{\alpha}{\alpha-2}}H'_{a,K}(H'_{a,K}(\xi(pt)) - J\dot{w}(pt) + Kw(pt))
\]

Combining (4.5)-(4.7), we obtain

\[
\Psi_{a,pK}(v_{pK}^p) = p^{\frac{\alpha}{\alpha-2}}\int_0^T \frac{1}{2}(J\dot{w}(pt) - Kw(pt), w(pt)) + H_{a,K}(\xi(pt)) \cdot \xi(pt) - H_{a,K}(\xi(pt)) - pK[p^{\frac{1}{\alpha-2}}\xi(pt)]^2 dt
\]

\[
= \frac{p^{\frac{\alpha}{\alpha-2}}}{p^{\frac{\alpha}{\alpha-2}}K} \Psi_{a,K}(v_K).
\]

By direct computation, we obtain

\[
\Psi'_{a,pK}(v_{pK}^p) = -w^p + H'_{a,pK}(v_{pK}^p) = -p^{\frac{1}{\alpha-2}}w(pt) + p^{\frac{1}{\alpha-2}}\xi(pt),
\]

specially,

\[
\Psi'_{a,K}(v_K) = -w + H'_{a,K}(v_K) = -w(t) + \xi(t).
\]

Hence, we have

\[
\Psi'_{a,pK}(\phi^p(v_K)) = \Psi_{a,pK}(v_{pK}^p) = p^{-1}\phi^p(\Psi_{a,K}(v_K)).
\]

Applying it to Lemma 2.10, for $g \in DN(S^1 \cdot g_K)$, noticing that $g$ and $h_K(g) \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$, and $h_K : G_K \rightarrow G_K^1$ is continuous in the $C^0$-topology (cf. Page 628 of [Vit1]), where we write $h_K$ and $G_K$ for $h_a$ and $G$ respectively to indicate their dependence on $K$, we have that $g + h_K(g)$ belongs to a small $L^\infty$-neighborhood of $u_K$ when the radius $\varrho > 0$ of the ball $DN(S^1 \cdot g_K)$ is small enough and

\[
\frac{\partial}{\partial h} \Psi_{a,pK}(\phi^p(g) + h_{pK}(\phi^p(g))) = 0 = \frac{\partial}{\partial h} \Psi_{a,pK}(\phi^p(g) + \phi^p(h_K(g))),
\]

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where we choose $G_{pK}$ in Lemma 2.10 for $\Psi_{a,pK}$ such that $G_{pK} \supseteq \phi^p(G_K)$, in fact, in (2.18) we can choose the same $\omega > 0$ for both $K$ and $pK$, and let $G_{pK}$ be the subspace of $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ generated by the eigenvectors of $-M_{pK}$ whose eigenvalues are less than $-\frac{\omega}{2p}$; that is,

$$G_{pK} = \text{span}\{e^{-JLt}x_0 | -\frac{1}{L + pK} < -\frac{\omega}{2p}, L \in \frac{2\pi}{T} \mathbb{Z}, x_0 \in \mathbb{R}^{2n}\},$$

and then

$$\phi^p(G_K) = \text{span}\{e^{-JpLt}x_0 | -\frac{1}{L + K} < -\frac{\omega}{2p}, L \in \frac{2\pi}{T} \mathbb{Z}, x_0 \in \mathbb{R}^{2n}\} \subseteq G_{pK}.$$ 

Hence $h_{pK}(\phi^p(g)) = \phi^p(h_K(g))$ holds. This together with (4.8) yields

$$\psi_{a,pK}(\phi^p(g)) = \Psi_{a,pK}(\phi^p(g) + h_{pK}(\phi^p(g))) = \Psi_{a,pK}(\phi^p(g) + \phi^p(h_K(g))) = p^{\frac{\omega}{2p}} \psi_{a,K}(g + h_K(g)) = p^{\frac{\omega}{2p}} \psi_{a,K}(g). \quad (4.12)$$

We define a new inner product $\langle \cdot, \cdot \rangle_p$ on $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ by

$$\langle v, w \rangle_p = p^\frac{2(\alpha - 1)}{2 - \alpha} \langle v, w \rangle. \quad (4.13)$$

Then $\phi^p : DN(g_K) \to DN(g^p_{pK})$ is an isometry from the standard inner product to the above one, where $g^p_{pK} = \phi^p(g_K)$ is the critical point of $\psi_{a,pK}$ corresponding to $u^p_{pK}$ and the radii of the two normal disk bundles are suitably chosen. Clearly $\phi^p(DN(g_K))$ consists of points in $DN(g^p_{pK})$ which are fixed by the $\mathbb{Z}_p$-action. Since the $\mathbb{Z}_p$-action on $DN(g^p_{pK})$ are isometries and $f \equiv \psi_{a,pK}|_{DN(g^p_{pK})}$ is $\mathbb{Z}_p$-invariant, we have

$$f''(g) = \left(\left[\left(f|_{\phi^p(DN(g_K))}\right)^{''}\right] 0 \atop 0 \ast\right), \quad \forall g \in \phi^p(DN(g_K)). \quad (4.14)$$

Moreover, we have

$$f'(g) = (f|_{\phi^p(DN(g_K))})', \quad \forall g \in \phi^p(DN(g_K)). \quad (4.15)$$

Now we can apply the results by D. Gromoll and W. Meyer [GrM1] to the manifold $DN(g^p_{pK})$ with $g^p_{pK}$ as its unique critical point. Then $\text{mul}(g^p_{pK}) = pm$ is the multiplicity of $g^p_{pK}$ and the isotropy group $\mathbb{Z}_{pm} \subseteq S^1$ of $g^p_{pK}$ acts on $DN(g^p_{pK})$ by isometries. According to Lemma 1 of [GrM1], we have a $\mathbb{Z}_{pm}$-invariant decomposition of $T_{g^p_{pK}}(DN(g^p_{pK}))$

$$T_{g^p_{pK}}(DN(g^p_{pK})) = V^+ \oplus V^- \oplus V^0 = \{(x_+, x_-, x_0)\} \quad (4.16)$$

with $\dim V^- = i(g^p_{pK}) = i_{pK}(u^p_{pK})$, $\dim V^0 = \nu(g^p_{pK}) - 1 = \nu_{pK}(u^p_{pK}) - 1$ (cf. Lemma 2.10(iii)), and a $\mathbb{Z}_{pm}$-invariant neighborhood $B = B_+ \times B_- \times B_0$ for 0 in $T_{g^p_{pK}}(DN(g^p_{pK}))$ together with two $\mathbb{Z}_{pm}$-invariant diffeomorphisms

$$\Phi : B = B_+ \times B_- \times B_0 \to \Phi(B_+ \times B_- \times B_0) \subset DN(g^p_{pK}),$$

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and

\[ \eta : B_0 \to W(g_{pK}^p) \equiv \eta(B_0) \subset DN(g_{pK}^p), \]

and \( \Phi(0) = \eta(0) = g_{pK}^p \), such that

\[ \psi_{a,pK} \circ \Phi(x_+, x_-, x_0) = |x_+|^2 - |x_-|^2 + \psi_{a,pK} \circ \eta(x_0), \quad (4.17) \]

with \( d(\psi_{a,pK} \circ \eta)(0) = d^2(\psi_{a,pK} \circ \eta)(0) = 0 \). As usual, we call \( W(g_{pK}^p) \) a local characteristic manifold, and \( U(g_{pK}^p) = B_- \) a local negative disk at \( g_{pK}^p \). By the proof of Lemma 1 of GrM1, \( W(g_{pK}^p) \) and \( U(g_{pK}^p) \) are \( Z_{pm} \)-invariant. It follows from (4.17) that \( g_{pK}^p \) is an isolated critical point of \( \psi_{a,pK}|_{DN(g_{pK}^p)} \). Then as in Lemma 6.4 of Rad2, we have

\[ H_s(\tilde{\Lambda}_{a,pK}(g_{pK}^p) \cap DN(g_{pK}^p), (\tilde{\Lambda}_{a,pK}(g_{pK}^p) \setminus \{g_{pK}^p\}) \cap DN(g_{pK}^p)) \]

\[ = \bigoplus_{q \in \mathbb{Z}} H_q(U(g_{pK}^p), U(g_{pK}^p) \setminus \{g_{pK}^p\}) \]

\[ \otimes H_{s-q}(W(g_{pK}^p) \cap \tilde{\Lambda}_{a,pK}(g_{pK}^p), (W(g_{pK}^p) \setminus \{g_{pK}^p\}) \cap \tilde{\Lambda}_{a,pK}(g_{pK}^p)), \quad (4.18) \]

where

\[ H_q(U(g_{pK}^p), U(g_{pK}^p) \setminus \{g_{pK}^p\}) = \begin{cases} Q, & \text{if } q = i_{pK}(u_{pK}^p), \\ 0, & \text{otherwise.} \end{cases} \quad (4.19) \]

Now we have the following proposition.

**Proposition 4.2.** For any \( p \in \mathbb{N} \), we choose \( K \) such that \( pK \notin \frac{2\pi}{p} \mathbb{Z} \). Let \( u_K \neq 0 \) be a critical point of \( \Psi_{a,K} \) with \( \text{mul}(u_K) = 1 \), \( u_K = -J\dot{x} + Kx \) with \( x \) being a critical point of \( F_{a,K} \). Then for all \( q \in \mathbb{Z} \), we have

\[ C_{S^1, q}^{S^1, q}(\Psi_{a,pK}, S^1 \cdot u_{pK}^p) \]

\[ \cong \left( H_{q-i_{pK}(u_{pK}^p)}(W(g_{pK}^p) \cap \tilde{\Lambda}_{a,pK}(g_{pK}^p), (W(g_{pK}^p) \setminus \{g_{pK}^p\}) \cap \tilde{\Lambda}_{a,pK}(g_{pK}^p)) \right)^{\beta(x^p)Z_p}, \quad (4.20) \]

where \( \beta(x^p) = (-1)^{i_{pK}(u_{pK}^p)-i_{K}(u_K)} = (-1)^{iv(x^p)-iv(x)} \). In particular, if \( u_{pK}^p \) is non-degenerate, i.e., \( v_{pK}(u_{pK}^p) = 1 \), then

\[ C_{S^1, q}^{S^1, q}(\Psi_{a,pK}, S^1 \cdot u_{pK}^p) = \begin{cases} Q, & \text{if } q = i_{pK}(u_{pK}^p) \text{ and } \beta(x^p) = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.21) \]

**Proof.** Suppose \( \theta \) is a generator of the linearized \( Z_p \)-action on \( U(g_{pK}^p) \). Then \( \theta(\xi) = \xi \) if and only if \( \xi \in T_{g_{pK}^p}(\phi^p(DN(g_{pK})) \). Hence it follows from (4.12) and (4.14) that \( \xi = (\phi^p)_*(\xi') \) for a unique \( \xi' \in T_{g_{pK}^p}(DN(g_{pK}))^- \). Hence the proof of Satz 6.11 in Rad2, Proposition 2.8 in Bal1 or Proposition 3.10 in WHL1 yield this proposition. Note that \( i_{pK}(u_{pK}^p) = 2n([pKT/2\pi]+1) + iv(x^p) \) and \( i_K(u_K) = 2n([KT/2\pi]+1) + iv(x) \) follow from (2.14) and (2.15).
**Definition 4.3.** For any \( p \in \mathbb{N} \), we choose \( K \) such that \( pK \notin \frac{\mathbb{Z}}{T} \). Let \( u_K \neq 0 \) be a critical point of \( \Psi_{a,K} \) with \( \text{mul}(u_K) = 1 \), \( u_K = -J\dot{x} + Kx \) with \( x \) being a critical point of \( F_{a,K} \). Then for all \( l \in \mathbb{Z} \), let

\[
\begin{align*}
k_l,\pm_1(u^p_{pK}) &= \dim \left( H_l(W(g^p_{pK}) \cap \tilde{\Lambda}_{a,pK}(g^p_{pK}), (W(g^p_{pK}) \setminus \{g^p_{pK}\}) \cap \tilde{\Lambda}_{a,pK}(g^p_{pK})) \right)^\pm_{\mathbb{Z}^p}, \\
k_l(u^p_{pK}) &= \dim \left( H_l(W(g^p_{pK}) \cap \tilde{\Lambda}_{a,pK}(g^p_{pK}), (W(g^p_{pK}) \setminus \{g^p_{pK}\}) \cap \tilde{\Lambda}_{a,pK}(g^p_{pK})) \right)^{\beta(x^p)}_{\mathbb{Z}^p}.
\end{align*}
\]

(4.22)

(4.23)

Here \( k_l(u^p_{pK}) \)'s are called critical type numbers of \( u^p_{pK} \).

**Remark 4.4.** (i) Since

\[
C_{S^1, l+i_{pK}(u^p_{pK})}(\Psi_{a,pK}, S^1 \cdot u^p_{pK}) \cong C_{S^1, l+i_{pK}(x^p)}(F_{a,pK}, S^1 \cdot x^p)
\]

\[
\cong C_{S^1, l+d(pK)+i_{pK}(x^p)}(F_{a,pK}, S^1 \cdot x^p),
\]

by Theorem 3.3, we obtain that \( k_l(u^p_{pK}) \) is independent of the choice of \( K \) and denote it by \( k_l(x^p) \), here \( k_l(x^p) \)'s are called critical type numbers of \( x^p \).

(ii) By Proposition 2.11, we have \( k_l,\pm_1(u^p_{pK}) = 0 \) if \( l \notin [0, 2n - 2] \).

Similar to Section 7.1 of [Rad2], Theorem 2.11 of [BaL1], or Lemma 3.12 of [WHL1], we have

**Lemma 4.5.** Let \( u_K \neq 0 \) be a critical point of \( \Psi_{a,K} \) with \( \text{mul}(u_K) = 1 \). Suppose \( \nu_{mK}(u^m_{mK}) = \nu_{pmK}(u^pm_{pmK}) \) for some \( m, p \in \mathbb{N} \). Then we have \( k_l,\pm_1(u^m_{mK}) = k_l,\pm_1(u^pm_{pmK}) \) for all \( l \in \mathbb{Z} \).

**Proof.** Let \( \phi^p : DN(g^m_{mK}) \to DN(g^pm_{pmK}) \) be the \( p \)th iteration map. By (4.13), \( \phi^p \) is an isometry under the modified metric. Hence by (4.12), we have

\[
\nu_{mK}(u^m_{mK}) - 1 = \dim \ker((\psi_{a,mK}|_{DN(g^m_{mK})})'' - I) = \dim \ker((\psi_{a,mK}|_{\phi^p(DN(g^m_{mK}))})'' - I).
\]

(4.24)

Thus by (4.14) and the assumption \( \nu_{mK}(u^m_{mK}) = \nu_{pmK}(u^pm_{pmK}) \), we have that \( T_{g^m_{pmK}}(\phi^p(DN(g^m_{mK}))) \) contains the null space of the Hessian of \( \psi_{a,pmK}|_{DN(g^pm_{pmK})} \). Now by (4.15), we can use Lemma 7 of [GrM1] to obtain that \( \phi^p(W(g^m_{mK})) \equiv W(g^pm_{pmK}) \) is a characteristic manifold of \( \psi_{a,pmK}|_{DN(g^pm_{pmK})} \), where \( W(g^m_{mK}) \) is a characteristic manifold of \( \psi_{a,mK}|_{DN(g^m_{mK})} \). By (4.12), we have

\[
\phi^p : (W(g^m_{mK}) \cap \tilde{\Lambda}_{a,mK}(g^m_{mK}), (W(g^m_{mK}) \setminus \{g^m_{mK}\}) \cap \tilde{\Lambda}_{a,mK}(g^m_{mK})) \to (W(g^pm_{pmK}) \cap \tilde{\Lambda}_{a,pmK}(g^pm_{pmK}), (W(g^pm_{pmK}) \setminus \{g^pm_{pmK}\}) \cap \tilde{\Lambda}_{a,pmK}(g^pm_{pmK}))
\]

is a homeomorphism. Suppose \( \theta \) and \( \theta_p \) generate the \( Z_m \) and \( Z_{pm} \) action on \( W(g^m_{mK}) \) and \( W(g^pm_{pmK}) \) respectively. Then clearly \( \phi^p \circ \theta = \theta_p \circ \phi^p \) holds and it implies

\[
H_s(W(g^m_{mK}) \cap \tilde{\Lambda}_{a,mK}(g^m_{mK}), (W(g^m_{mK}) \setminus \{g^m_{mK}\}) \cap \tilde{\Lambda}_{a,mK}(g^m_{mK})) \cong H_s(W(g^pm_{pmK}) \cap \tilde{\Lambda}_{a,pmK}(g^pm_{pmK}), (W(g^pm_{pmK}) \setminus \{g^pm_{pmK}\}) \cap \tilde{\Lambda}_{a,pmK}(g^pm_{pmK})) \cong Z_m.
\]

(4.25)
Therefore our lemma follows.

**Proposition 4.6.** Let \( x \neq 0 \) be a critical point of \( F_{a,K} \) with \( \text{mul}(x) = 1 \) corresponding to a critical point \( u_K \) of \( \Psi_{a,K} \). Then there exists a minimal \( K(x) \in \mathbb{N} \) such that

\[
\nu^v(x^{p+K(x)}) = \nu^v(x^p), \quad i^v(x^{p+K(x)}) - i^v(x^p) \in 2\mathbb{Z}, \quad \forall p \in \mathbb{N}, \quad (4.25)
\]

\[
k_l(x^{p+K(x)}) = k_l(x^p), \quad \forall p \in \mathbb{N}, \; l \in \mathbb{Z}. \quad (4.26)
\]

We call \( K(x) \) the minimal period of critical modules of iterations of the functional \( F_{a,K} \) at \( x \).

**Proof.** As in the proof of Proposition 2.11, we denote by \( R(t) \) the fundamental solution of \( (2.16) \).

Then by Section 2 and Theorem 2.1 of \( [\text{HuL1}] \), we have \( i^v(x^p) = i(x,p) - n \) and \( \nu^v(x^p) = \nu(x,p) \) for all \( p \in \mathbb{N} \), where \( (i(x,p), \nu(x,p)) \) are index and nullity defined by C. Conley, E. Zehnder and Y. Long (cf. \( [\text{CoZ1}], [\text{LZe1}], [\text{Lon1}] - [\text{Lon4}] \)). Hence we have \( \nu^v(x^p) = \dim \ker \left( R(1)^p - I_{2n} \right) \). Denote by \( \lambda_i = \exp(\pm 2\pi \frac{r_i}{s_i}) \) the eigenvalues of \( R(1) \) possessing rotation angles which are rational multiples of \( \pi \) with \( r_i \) and \( s_i \in \mathbb{N} \) and \( (r_i, s_i) = 1 \) for \( 1 \leq i \leq q \). Let \( K(x) \) be twice of the least common multiple of \( s_1, \ldots , s_q \). Then \( (4.25) \) holds. Note that the later conclusion in \( (4.25) \) follows from Theorem 9.3.4 of \( [\text{Lon4}] \).

In order to prove \( (4.26) \), it suffices to show

\[
k_l(x^{m+qK(x)}) = k_l(x^m), \quad \forall q \in \mathbb{N}, \; l \in \mathbb{Z}, \; 1 \leq m \leq K(x). \quad (4.27)
\]

In fact, assume that \( (4.27) \) is proved. Note that \( (4.26) \) follows from \( (4.27) \) with \( q = 1 \) directly when \( p \leq K(x) \). When \( p > K(x) \), we write \( p = m + qK(x) \) for some \( q \in \mathbb{N} \) and \( 1 \leq m \leq K(x) \). Then by \( (4.27) \) we obtain

\[
k_l(x^{p+K(x)}) = k_l(x^{m+q+1}K(x)) = k_l(x^m) = k_l(x^{m+qK(x)}) = k_l(x^p),
\]

i.e., \( (4.26) \) holds.

To prove \( (4.27) \), we fix an integer \( m \in [1, K(x)] \). Let

\[A = \{ s_i \in \{ s_1, \ldots , s_q \} \mid s_i \text{ is a factor of } m \},\]

and let \( m_1 \) be the least common multiple of elements in \( A \). Hence we have \( m = m_1m_2 \) for some \( m_2 \in \mathbb{N} \) and \( \nu_{mK}(u_{m_1K}^m) = \nu^v(x^m) = \nu^v(x^{m_1}) = \nu_{m_1K}(u_{m_1K}^{m_1}) \). Thus by Remark 4.4 (i) and Lemma 4.5, we have \( k_l(x^m) = k_{l,\beta(x^m)}(u_{m_1K}^m) = k_{l,\beta(x^m)}(u_{m_1K}^{m_1}) \). Since \( m + pK(u) = m_1m_3 \) for some \( m_3 \in \mathbb{N} \), we have by Remark 4.4 (i) and Lemma 4.5 that \( k_l(x^{m+pK(x)}) = k_{l,\beta(x^{m+pK(x)})}(u_{m_1K}^{m_1}) \). By \( (4.25) \), we obtain \( \beta(x^{m+pK(x)}) = \beta(x^m) \), and then \( (4.27) \) is proved. This completes the proof. \( \blacksquare \)
Note that the above Proposition 4.6 could be established also without forcing the Hamiltonian to be homogeneous near its critical points. In fact, by Proposition 3.2, it holds for any Hamiltonian defined by Proposition 2.5.

In the following, Let $F_{a,K}$ be the functional defined by (2.6) with $H_a$ satisfying Proposition 2.5, we do not require $\tilde{H}_a$ to be homogeneous anymore.

**Definition 4.7.** Suppose the condition (F) at the beginning of Section 2 holds. For every closed characteristic $(\tau, y)$ on $\Sigma$, let $aT > \tau$ and choose $\varphi_a$ to satisfy (i)-(ii) of Lemma 2.2. Determine $\rho$ uniquely by $\frac{\varphi_a'(\rho)}{\rho} = \frac{\tau}{aT}$. Let $x = \rho y(\frac{\tau}{aT})$. Then we define the index $i(\tau, y)$ and nullity $\nu(\tau, y)$ of $(\tau, y)$ by

$$i(\tau, y) = i^v(x), \quad \nu(\tau, y) = \nu^v(x).$$

Then the mean index of $(\tau, y)$ is defined by

$$\hat{i}(\tau, y) = \lim_{m \to \infty} \frac{i(m\tau, y)}{m}. \quad (4.28)$$

Note that by Proposition 2.11, the index and nullity are well defined and is independent of the choice of $aT > \tau$ and $\varphi_a$ satisfying (i)-(ii) of Lemma 2.2.

For a prime closed characteristic $(\tau, y)$ on $\Sigma$, we denote simply by $y^m \equiv (m\tau, y)$ for $m \in \mathbb{N}$. By Proposition 3.2, we can define the critical type numbers $k_l(y^m)$ of $y^m$ to be $k_l(x^m)$, where $x^m$ is the critical point of $F_{a,K}$ corresponding to $y^m$. We also define $K(y) = K(x)$, where $K(x) \in \mathbb{N}$ is given by Proposition 4.6. Suppose $\mathcal{N}$ is an $S^1$-invariant open neighborhood of $S^1 \cdot x^m$ such that $\text{crit}(F_{a,K}) \cap (X_{a,K}(x^m) \cap \mathcal{N}) = S^1 \cdot x^m$. Then we make the following definition

**Definition 4.8.** The Euler characteristic $\chi(y^m)$ of $y^m$ is defined by

$$\chi(y^m) \equiv \chi((X_{a,K}(x^m) \cap \mathcal{N})_{S^1}, (X_{a,K}(x^m) \setminus S^1 \cdot x^m) \cap \mathcal{N})_{S^1})$$

$$\equiv \sum_{q=0}^{\infty} (-1)^q \dim C_{S^1, q}(F_{a,K}, S^1 \cdot x^m). \quad (4.29)$$

Here $\chi(A, B)$ denotes the usual Euler characteristic of the space pair $(A, B)$. The average Euler characteristic $\hat{\chi}(y)$ of $y$ is defined by

$$\hat{\chi}(y) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq m \leq N} \chi(y^m). \quad (4.30)$$

Note that by Proposition 3.2 and Theorem 3.3, $\chi(y^m)$ is well defined and is independent of the choice of $a$ and $K$. In fact, by Remark 4.4 (i), we have

$$\chi(y^m) = \sum_{l=0}^{2n-2} (-1)^{l(y^m)+4} k_l(y^m). \quad (4.31)$$
The following remark shows that $\hat{\chi}(y)$ is well-defined and is a rational number.

**Remark 4.9.** By (4.32) and Proposition 4.6, we have

\[
\hat{\chi}(y) = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq l \leq 2n-2} (-1)^{i(y^m) + 1} k_l(y^m)
\]

\[
= \lim_{s \to \infty} \frac{1}{sK(y)} \sum_{0 \leq p < s} (-1)^{i(y^{pK(y)} + m) + 1} k_l(y^{pK(y) + m})
\]

\[
= \frac{1}{K(y)} \sum_{0 \leq l \leq 2n-2} (-1)^{i(y^m) + 1} k_l(y^m).
\]

Therefore $\hat{\chi}(y)$ is well defined and is a rational number. In particular, if all $y^m$s are non-degenerate, then $\nu(y^m) = 1$ for all $m \in \mathbb{N}$. Hence the proof of Proposition 4.6 yields $K(y) = 2$. By (4.21), we have

\[
k_l(y^m) = \begin{cases} 1, & \text{if } i(y^m) - i(y) \in 2\mathbb{Z} \quad \text{and} \quad l = 0 \\ 0, & \text{otherwise.} \end{cases}
\]

Hence (4.32) implies

\[
\hat{\chi}(y) = \begin{cases} (-1)^{i(y)}, & \text{if } i(y^2) - i(y) \in 2\mathbb{Z}, \\ \frac{(-1)^{i(y)}}{2}, & \text{otherwise.} \end{cases}
\]

**Remark 4.10.** Note that $k_l(y^m) = 0$ for $l \notin [0, \nu(y^m) - 1]$ and it can take only values 0 or 1 when $l = 0$ or $l = \nu(y^m) - 1$. Moreover, the following facts are useful (cf. Lemma 3.11 of [Bal1], Remark 3.17 of [WHL1], [Cha1] and [MaW1]):

(i) $k_0(y^m) = 1$ implies $k_l(y^m) = 0$ for $1 \leq l \leq \nu(y^m) - 1$.
(ii) $k_{\nu(y^m) - 1}(y^m) = 1$ implies $k_l(y^m) = 0$ for $0 \leq l \leq \nu(y^m) - 2$.
(iii) $k_l(y^m) \geq 1$ for some $1 \leq l \leq \nu(y^m) - 2$ implies $k_0(y^m) = k_{\nu(y^m) - 1}(y^m) = 0$.
(iv) In particular, only one of the $k_l(y^m)$s for $0 \leq l \leq \nu(y^m) - 1$ can be non-zero when $\nu(y^m) \leq 3$.

## 5 Contribution of the origin

In section 3 and 4, we studied nonzero critical points of $F_{a,K}$, now we need to study the contribution of the origin to the Morse series of the functional $F_{a,K}$ on $W^{1,2}(\mathbb{R}/\mathbb{Z}; \mathbb{R}^n)$. Theorem 7.1 of [Vit1] was given under the condition that all the closed characteristics together with their iterations are non-degenerate, however, by a modification of the proof, we obtain a degenerate version in the following.

**Theorem 5.1.** Fix an $a > 0$ such that $\operatorname{per}(\Sigma) \cap (0, aT) \neq \emptyset$. Then there exists an $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_0]$ we have

\[
H_{S^1, q + a(K)}(X^\varepsilon_{a,K}, X^{-\varepsilon}_{a,K}) = 0, \quad \forall q \in \tilde{I},
\]

(5.1)
if $I$ is an interval of $\mathbb{Z}$ such that $I \cap [i(\tau, y), i(\tau, y) + \nu(\tau, y) - 1] = \emptyset$ for all closed characteristics $(\tau, y)$ on $\Sigma$ with $\tau \geq aT$.

**Proof.** By Proposition 2.8, there exists an $\varepsilon_0 > 0$ such that there are no critical values of $F_{a,K}$ in the interval $[-\varepsilon_0, \varepsilon_0]$ except 0. Hence we have

$$H_{S_1, q+d(K)}(X^\varepsilon_{a,K}, X^{-\varepsilon}_{a,K}) \cong H_{S_1, q+d(K)}(X^0_{a,K}, X^{-2}_{a,K}), \quad \forall q \in \mathbb{Z}, \varepsilon \in (0, \varepsilon_0].$$

In the following we assume $\varepsilon \in (0, \varepsilon_0]$.

Note that by the same proof of Proposition 3.2, $H_{S_1, q+d(K)}(X^\varepsilon_{a,K}, X^{-\varepsilon}_{a,K})$ is independent of the choice of $\varphi_a$ in $\tilde{H}_a(x) = a\varphi_a(j(x))$ which satisfies (i) of Lemma 2.2. Hence we can choose $a\varphi_a \equiv \phi$, where $\phi$ is defined as in Lemma 2.2 of [Vit1]. Since the homology in (5.1) depends only on the restriction of $H_a$ to a neighborhood of the origin, as in the beginning of Section 7 of [Vit1], we assume $H_a$ to be homogeneous of degree two everywhere. Now we can make some modifications of the proof of Theorem 7.1 of [Vit1] to complete our proof.

There are only four palaces in the proof of Theorem 7.1 of [Vit1] where the non-degenerate condition is used, i.e. the use of Proposition 1 of Appendix 1 (In fact, it should be Proposition 3 of Appendix 1) in Page 640 and Page 642, the use of Proposition 2 (i) of Appendix 1 in Page 644, and the use of Proposition 3 of Appendix 1 in Page 648.

However, Proposition 2 (i), Proposition 3 (a) of Appendix 1 of [Vit1] work for the degenerate case, so we only need to modify the proof in Page 648 of [Vit1]. By assumption, $I \cap [i(\tau, y), i(\tau, y) + \nu(\tau, y) - 1] = \emptyset$ for all closed characteristics $(\tau, y)$ on $\Sigma$ with $\tau \geq aT$, then by a degenerate version of Proposition 3 (b) of Appendix 1 of [Vit1], we complete our modification. More precisely, in the proof of Proposition 3 (b) of Appendix 1 of [Vit1] (we only consider Case (i)), we let $\tilde{\tau}$ be the restriction of the map $\tau$ on $\partial X$, then

$$C_*(\tilde{\tau}, (x_0, t_0)) \cong C_*(f_{t_0}, x_0). \quad (5.2)$$

In fact, since $\partial f_t(x_0)/\partial t < 0$, by the implicit functional theorem, we have that in a small enough neighborhood $U$ of $x_0$, there is an unique continuous function $t_x$ such that $(x, t_x) \in \partial X$ and $t_{x_0} = t_0$ for $x \in U$. By the fact that $\partial f_t(x_0)/\partial t < 0$, we obtain that $t_x \leq t_0$ for $x \in U$ if and only if $f_{t_0}(x) \leq 0$. Thus $\{(x, t_x) \mid \tilde{\tau}(x, t_x) \leq t_0, x \in U\}$ is homotopy equivalent with $\{x \in U \mid f_{t_0}(x) \leq 0\}$ and $\{(x, t_x) \mid \tilde{\tau}(x, t_x) \leq t_0, x \in U\} \setminus \{(x_0, t_0)\}$ is homotopy equivalent with $\{x \in U \mid f_{t_0}(x) \leq 0\} \setminus \{x_0\}$, and then (5.2) follows by definition and the homotopy invariance of homology. (5.2) is also true in an equivariant setting, and then we apply it to the functional $F_{t,K'}|S$, where $S$ is the unit sphere of $W^{1,2}(\mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$ and $F_{t,K'}|S$ is the restriction on $S$, we have

$$C_{S_1, k}(\tilde{\tau}, S_1 \cdot (x_0, t_0)) \cong C_{S_1, k}(F_{t_0,K'}|S, S_1 \cdot x_0) = 0, \quad (5.3)$$

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where \( k \in I \) and \( t_0 \in [a, a'] \), \( x_0 \) corresponds to a \( t_0T \)-periodic solution of (1.1). Then by Theorem 1.6.1 of [Cha1] (which can be easily generalized to the equivariant sense), we obtain (7.25) of [Vit1]. The proof is complete.

### 6 Proof of Theorem 1.1

In this section, we give a proof for the Theorem 1.1 with \( \tilde{H}_a(x) = a\varphi_a(j(x)), \) where \( \varphi_a \) satisfies (i)-(ii) of Lemma 2.2.

Let \( F_{a,K} \) be a functional defined by (2.6) for some \( a, K \in \mathbb{R} \) large enough and let \( \varepsilon > 0 \) be small enough such that \( [-\varepsilon, 0] \) contains no critical values of \( F_{a,K} \). We consider the exact sequence of the triple \((X, X^{-\varepsilon}, X^{-b})\) (for \( b \) large enough)

\[
\rightarrow H_{S^1, s}(X^{-\varepsilon}, X^{-b}) \rightarrow H_{S^1, s}(X, X^{-\varepsilon}) \rightarrow H_{S^1, s-1}(X^{-\varepsilon}, X^{-b}) \rightarrow \cdots, \tag{6.1}
\]

where \( X = W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^n) \). The normalized Morse series of \( F_{a,K} \) in \( X^{-\varepsilon} \setminus X^{-b} \) is defined, as usual, by

\[
M_a(t) = \sum_{q \geq 0, 1 \leq j \leq p} \dim C_{S^1, q}(F_{a,K}, S^1 \cdot v_j)t^{q-d(K)}, \tag{6.2}
\]

where we denote by \( \{S^1 \cdot v_1, \ldots, S^1 \cdot v_p\} \) the critical orbits of \( F_{a,K} \) with critical values less than \(-\varepsilon\). We denote by \( t^{d(K)}H_a(t) \) the Poincaré series of \( H_{S^1, s}(X^{-\varepsilon}, X^{-b}) \), \( H_a(t) \) is a Laurent series, and we have the equivariant Morse inequality

\[
M_a(t) - H_a(t) = (1 + t)R_a(t), \tag{6.3}
\]

where \( R_a(t) \) is a Laurent series with nonnegative coefficients. On the other hand, the Poincaré series of \( H_{S^1, s}(X, X^{-b}) \) is, by Corollary 5.11 of [Vit1], \( t^{d(K)}(1/(1-t^2)) \). The Poincaré series of \( H_{S^1, s}(X, X^{-\varepsilon}) \) is \( t^{d(K)}Q_a(t) \), according to Theorem 5.1, if we set \( Q_a(t) = \sum_{k \in \mathbb{Z}} q_k t^k \), then

\[
q_k = 0 \quad \forall k \in \hat{I}, \tag{6.4}
\]

where \( \hat{I} \) is defined in Theorem 5.1. Now by (6.1) (cf. Proposition 1 in Appendix 2 of [Vit1]), we have

\[
H_a(t) - \frac{1}{1-t^2} + Q_a(t) = (1 + t)S_a(t), \tag{6.5}
\]

with \( S_a(t) \) a Laurent series with nonnegative coefficients. Adding up (6.3) and (6.5) yields

\[
M_a(t) - \frac{1}{1-t^2} + Q_a(t) = (1 + t)U_a(t), \tag{6.6}
\]
where \( U_a(t) = \sum_{i \in \mathbb{Z}} u_i t^i \) also has nonnegative coefficients.

Now truncate (6.6) at the degrees \( 2C \) and \( 2N \), where we set \( C \) equal to \( 2n^2 \), and \( 2N > 2C \), and write \( M_a^{2N}(2C; t), Q_a^{2N}(2C; t) \cdots \) for the truncated series. Then from (6.6) we infer

\[
M_a^{2N}(2C; t) - \sum_{h=0}^{N} t^{2h} + Q_a^{2N}(2C; t) = (1 + t)U_a^{2N-1}(2C; t) + t^{2N} u_{2N} + t^{2C} u_{2C-1}.
\]

By (6.4), and the fact that for \( a \) large enough \( \tilde{I} \) contains \([2C, 2N]\), indeed let \( \alpha > 0 \) such that any prime closed characteristic \((\tau, y)\) with \( \tilde{i}(y) \neq 0 \) has \( |\tilde{i}(y)| > \alpha \). Then if \( k \geq aT/\min \{\tau_i\} \), we have \( |i(y^k)| \sim k|\tilde{i}(y)| \geq \kappa \alpha \) for \( a \to \infty \), so \( Q_a^{2N}(2C; t) = 0 \), and (6.7) can be written

\[
M_a^{2N}(2C; t) - \sum_{h=0}^{N} t^{2h} = (1 + t)U_a^{2N-1}(2C; t) + t^{2N} u_{2N} + t^{2C} u_{2C-1}.
\]

Changing \( C \) into \(-C\), \( N \) into \(-N\), and counting terms with \(-2N \leq i \leq -2C\), we obtain

\[
M_a^{-2C}(-2N; t) = (1 + t)U_a^{-2C-1}(-2N; t) + t^{-2N} u_{-2N-1} + t^{-2C} u_{-2C}.
\]

Denote by \( \{x_1, \ldots, x_k\} \) the critical points of \( F_{a,K} \) corresponding to \( \{y_1, \ldots, y_k\} \). Note that \( v_1, \ldots, v_p \) in (6.2) are iterations of \( x_1, \ldots, x_k \). Since \( C_{S^1, q}(F_{a,K}, S_1 \cdot x^m_j) \) can be non-zero only for \( q = d(K) + i(y^m_j) + l \) with \( 0 \leq l \leq 2n - 2 \), by Propositions 2.11, 4.2 and Remark 4.4, the normalized Morse series (6.2) becomes

\[
M_a(t) = \sum_{1 \leq j \leq k, 0 \leq l \leq 2n-2} k_l(y^m_j) i^{i(y^m_j) + l} = \sum_{1 \leq j \leq k, 0 \leq l \leq 2n-2} k_l(y^m_j) i^{i(y^m_j) + l},
\]

where \( K_j = K(y^m_j) \) and \( s \in \mathbb{N}_0 \). The last equality follows from Proposition 4.6.

Write \( M(t) = \sum_{h \in \mathbb{Z}} w_h t^h \). Then we have

\[
w_h = \sum_{1 \leq j \leq k, 0 \leq l \leq 2n-2} k_l(y^m_j) \# \{s \in \mathbb{N}_0 \mid i(y^m_j K_j) + l = h\}, \quad \forall 2C \leq |h| \leq 2N.
\]

Note that the right hand side of (6.10) contains only those terms satisfying \( sK_j + m_j < \frac{4T}{r_j} \). Thus (6.11) holds for \( 2C \leq |h| \leq 2N \) by (6.10).

**Claim 1.** \( w_h \leq C_1 \) for \( 2C \leq |h| \leq 2N \) with \( C_1 \) being independent of \( a, K \).

In fact, we have

\[
\# \{s \in \mathbb{N}_0 \mid i(y^m_j K_j) + l = h\}
\]
\[\begin{align*}
\text{Claim 2. There is a real constant } C_2 > 0 \text{ independent of } a, K \text{ such that} \\
\left| M_a^{2N}(2C; -1) - \sum_{\substack{1 \leq j \leq k, 0 \leq 2n - 2 \leq m \leq K_j \text{, } 0 \leq i(y_j^m) > 0}} (-1)^{i(y_j^m) + l} k_l(i(y_j^m)) \frac{2N}{K_j i(y_j^m)} \right| & \leq C_2, \\
\left| M_a^{-2C}(-2N; -1) - \sum_{\substack{1 \leq j \leq k, 0 \leq 2n - 2 \leq m \leq K_j \text{, } 0 \leq i(y_j^m) > 0}} (-1)^{i(y_j^m) + l} k_l(i(y_j^m)) \frac{2N}{K_j i(y_j^m)} \right| & \leq C_2,
\end{align*}\]

where the sum in the left hand side of (6.16) equals to \(2N \sum_{i(y_j^m) > 0} \frac{\hat{x}(y_j)}{i(y_j)}\), the sum in the left hand side of (6.17) equals to \(2N \sum_{i(y_j^m) < 0} \frac{\hat{x}(y_j)}{i(y_j)}\) by (4.32).

In fact, we have the estimates
\[\# \{ s \in \mathbb{N}_0 \mid 2C \leq i(y_j^{sK_j+m}) + l \leq 2N\}\]
Hence (1.2) and (1.3) follow from (6.16) and (6.17).

choose \( m \) where these two estimates together with (6.14), we obtain (6.16). Similarly, we obtain (6.17).

Dividing both sides of the above two identities by 2

\[
\begin{align*}
\lim_{N \to \infty} \frac{1}{2N} M^2_a(2C; -1) - (N - C + 1) &= u_2N + u_{2C - 1}, \\
M^{-2C}_a(-2N; -1) &= u_{-2N - 1} + u_{-2C}.
\end{align*}
\]

Dividing both sides of the above two identities by 2N and letting \( N \) tend to infinity, we obtain

\[
\begin{align*}
\lim_{N \to \infty} \frac{1}{2N} M^2_a(2C; -1) &= \frac{1}{2}, \\
\lim_{N \to \infty} \frac{1}{2N} M^{-2C}_a(-2N; -1) &= 0.
\end{align*}
\]

Hence (1.2) and (1.3) follow from (6.16) and (6.17).

Let us also mention that if there is no solution with \( \hat{i} = 0 \), we do not need to cut our series at \( \pm 2C; \) we can cut at \(-2N \) and \( 2N \) only, thus obtaining

\[
M(t) - \frac{1}{1 - t^2} = (1 + t)U(t), \quad (6.18)
\]
where \( M(t) \) denotes \( M_a(t) \) as \( a \) tends to infinity.

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