Low-energy spectrum of $SU(3)$ Yang-Mills Quantum Mechanics

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Abstract

The $SU(3)$ Yang-Mills Quantum Mechanics of spatially constant gluon fields is considered in the unconstrained Hamiltonian approach using the “flux-tube gauge”. The Faddeev-Popov operator, its determinant and inverse, are rather simple, but show a highly non-trivial periodic structure of six Gribov-horizons separating six Weyl-chambers. The low-energy eigensystem of the obtained physical Hamiltonian can be calculated (in principle with arbitrary high precision) using the orthonormal basis of eigenstates of the corresponding harmonic oscillator problem with the same non-trivial Jacobian only replacing the chromomagnetic potential by the 16dimensional harmonic oscillator potential. This turns out to be integrable and its eigenstates be made out of orthogonal polynomials of the 45 components of eight irreducible symmetric tensors. The calculations in this work have been carried out in all sectors $J^{PC}$ up to spin $J = 11$, and up to polynomial order 10 for even and 11 for odd parity. The low-energy eigensystem of the physical Hamiltonian of $SU(3)$ Yang-Mills Quantum Mechanics is found to converge nicely when truncating at higher and higher polynomial order (equivalent to increasing the resolution in functional space). Our results are in good agreement with the results of Weisz and Zieman (1986) using the constrained Hamiltonian approach. We find excellent agreement in the $0^{++}$ and $2^{++}$ sectors, much more accurate values in other sectors considered by them, e.g. in the $1^{--}$ and $3^{--}$ sectors, and quite accurate ”new results” for the sectors not considered by them, e.g. $2^{--}, 4^{--}, 5^{--}, 3^{++}$.

1 Introduction

The Yang-Mills Quantum Mechanics (YM QM) of spatially constant gluon fields has been studied for a long time, as a toy model for the QCD vacuum[1]-[7], as the zeroth order of a weak coupling expansion [8]-[11], and as zeroth order of a strong coupling expansion [12]-[15]. For the case of SU(2) YM theory, the symmetric gauge turned out to exist in the strong coupling limit and to be very convenient for calculations, also including the quarks [16]. The reduced gauge fields transform as symmetric tensors under spatial rotations, and the Faddeev-Popov (FP) operator turned out to be non-trivial but manageable in a way similar to the Calogero model [17]. For the case of SU(3) YM theory, however, the symmetric gauge can also be defined [18]-[20] and leads to reduced fields transforming as tensors under spatial rotations, but the corresponding FP-operator turns out to be
very complicated. In previous work [21], a new algebraic gauge for SU(3) YM theory, the flux-tube
gauge, has been proposed, which exists in the strong coupling limit and has a simple non-trivial
FP-operator. As for the case of SU(2) YM theory in the symmetric gauge [15], the corresponding
gauge reduced SU(3) YM Hamiltonian in the flux-tube gauge can be expanded in strong coupling
\( \lambda = g^{-2/3} \) with the leading order corresponding to SU(3) YM QM of spatially constant fields. The
drawback of the flux-tube gauge is however, that the reduced fields \( A \) are color-singlets, but are
not transforming as tensors under spatial rotations. We shall show in this article how this can
be circumvented by forming certain irreducible polynomials of the reduced \( A \), symmetric tensors
with definite eigenvalues of \( J,P, \) and \( C \). We shall calculate in this work the low-energy spectrum
of SU(3) YM QM in the unconstrained Hamiltonian approach using the flux-tube gauge. The
Faddeev-Popov operator, its determinant and inverse, are rather simple, but show a highly non-
trivial periodic structure of six Gribov-horizons separating six Weyl-chambers. The low-energy
eigensystem of the obtained physical Hamiltonian can be calculated (in principle with arbitrary
high precision) using the orthonormal basis of eigenstates of the corresponding harmonic oscillator
problem with the same non-trivial Jacobian only replacing the chromomagnetic potential by the
16-dimensional harmonic oscillator potential. This turns out to be integrable and its eigenstates to
be made out of orthogonal polynomials of the 45 components of eight elementary spatial tensors.

The paper is organised as follows. In Section 2 we give a short introduction to the Hamiltonian
approach of SU(3) Yang-Mills Quantum mechanics of spatially constant gluon fields \( V_{ai} \) constrained
by non-Abelian Gauss laws. In Section 3 the flux-tube gauge is defined and shown to lead to
a rather simple but non-trivial Faddeev-Popov (FP) operator. In Section 4 the corresponding
harmonic oscillator (HO) Hamiltonian, obtained by replacing the chromomagnetic potential by
an 16-dimensional harmonic oscillator potentials, is solved analytically by polynomials in the
components of eight irreducible tensors in reduced space \( A \). In Section 5 we use the obtained
eigensystem of the HO-Hamiltonian to find the eigensystem of the Hamiltonian of SU(3)-Yang-
Mills QM in dependence of truncation at higher and higher polynomial degree. The results
are compared with those of Weisz and Zieman [11] obtained in the constrained approach, and
comparison with the low glueball spectrum obtained in [22],[23] using lattice QCD. Section 6
gives our Conclusions. Some technical details are banned to Appendices A-F.

2 Constrained SU(3) Yang-Mills QM of spatially constant
fields

The action of \( SU(3) \) Yang-Mills Quantum Mechanics of spatially constant gluon fields \( V_{\mu}(t) \equiv V_{a\mu}(t)\lambda_a/2 \) is defined as

\[
S[V] := \text{Vol} \int dt \left[ -\frac{1}{4} F_{a\mu\nu} F_{a\mu\nu}^{\text{hom}} \right] = \text{Vol} \int dt \frac{1}{2} \left[ (E_{ai}^{\text{hom}})^2 - (B_{ai}^{\text{hom}})^2 \right],
\]

with the spatially constant field strength tensor

\[
F_{a\mu\nu}^{\text{hom}} := \delta_{\mu0} \partial_t V_{a\nu} - \delta_{\nu0} \partial_t V_{a\mu} + g f_{abc} V_{b\mu} V_{c\nu}, \quad a = 1,\ldots,8,
\]

or in terms of the chromoelectric and chromomagnetic parts

\[
E_{ai}^{\text{hom}} \equiv F_{a\mu}^{\text{hom}} ; \quad B_{ai}^{\text{hom}} \equiv \frac{1}{2} \epsilon_{ijk} F_{a\mu}^{\text{hom}}.
\]

The action (1) is invariant under the spatially homogeneous \( SU(3) \) gauge transformations

\[
V_{a\mu}(t)\lambda_a/2 = U[\omega(t)] V_{a\mu}(t)\lambda_a/2 \quad U^{-1}[\omega(t)].
\]
Furthermore, the action is invariant under spatial rotations $R$

$$R : \ V_{ai} \to R_{ij} V_{aj},$$  \hspace{1cm} (5)

as well as under parity transformations and charge conjugation

$$P : \ V_{ai} \lambda_a \to -V_{ai} \lambda_a \quad \quad C : \ V_{ai} \lambda_a \to -(V_{ai} \lambda_a)^*.$$  \hspace{1cm} (6)

In terms of the momenta $\Pi_{ai} = -E_{ai}^{\text{hom}}$ canonical conjugate to the spatial $V_{ai}$ one obtains the canonical Hamiltonian

$$H_C = \text{Vol} \left[ \frac{1}{2} \Pi_{ai}^2 + \frac{1}{2} \left( B_{ai}^{\text{hom}}(V) \right)^2 - gV_{a0} (f_{abc} V_{ci} \Pi_{bi}) \right].$$  \hspace{1cm} (7)

Exploiting the time dependence of the gauge transformations t put

$$V_{a0} = 0, \quad a = 1, \ldots, 8 \quad \text{(Weyl gauge)},$$  \hspace{1cm} (8)

the dynam. vaiables $V_{ai}, \Pi_{ai}$ are quantized in the Schrödinger functional approach imposing the equal-time commutation relations $\Pi_{ai} = -i\partial/\partial V_{ai}$. The physical states $\Phi$ satisfy the coupled system of Schrödinger Equ. and eight non-abelian Gauss law constraints,

$$H_0 \Phi \equiv \text{Vol} \left[ \frac{1}{2} \Pi_{ai}^2 + \frac{1}{2} \left( B_{ai}^{\text{hom}}(V) \right)^2 \right] \Phi = E \Phi;$$  \hspace{1cm} (9)

$$G_a \Phi \equiv g f_{abc} V_{ci} \Pi_{bi} \Phi = 0, \quad a = 1, \ldots, 8.$$  \hspace{1cm} (10)

The Gauss law operators $G_a$ are the generators of the residual time independent gauge transformations, satisfying $[G_a, H] = 0$ and $[G_a, G_b] = i f_{abc} G_c$. The matrix element of an operator $O$ is given in the Cartesian form

$$\langle \Phi' | O | \Phi \rangle \propto \int dV \ \Phi'^* (V) O \Phi(V).$$  \hspace{1cm} (11)

Since $H_0$ is invariant under spatial rotations $[H_0, J_i] = 0$ with

$$J_i = \epsilon_{ijk} V_{aj} \Pi_{ak}, \quad i = 1, 2, 3, \quad [J_i, J_j] = i \epsilon_{ijk} J_k;$$  \hspace{1cm} (12)

and invariant under parity $[H_0, P] = 0$ and charge conjugation $[H_0, C] = 0$ the eigenstates can be characterised by $J^{PC}$.

In their work [11] Weisz and Ziemann used the variational approach to find the eigenvalues of the constrained Schrödinger Equ. (9) with trial functions

$$\Phi^{(J)PC} (V) = P^{(J)PC}_{\text{gauge inv.}} (V) \exp[-(\omega/2) (V_{ai})^2]$$

which are gauge invariant and hence automatically satisfy the Gauss law constraints (10). In the sectors $0^{++}$ and $2^{++}$ they find rather accurate eigenvalues, in other sectors first upper bounds. In the present work we would like to demonstrate that the above constrained system becomes integrable if one replaces

$$\left( B_{ai}^{\text{hom}}(V) \right)^2 \to \omega^2 (V_{ai})^2, \quad \omega > 0 \quad \text{free parameter}$$

Using an exact gauge reduction the energy-eigensystem can be found rather accurately and used as a Hilbert-basis for the YM QM. Truncating at higher and higher numbers of nodes, a converging low-energy eigensystem of YM QM is obtained.
3 Unconstrained Hamiltonian formulation using the flux-tube gauge

3.1 Unconstrained Hamiltonian formulation of SU(3) Yang-Mills QM

In order to obtain an unconstrained Hamiltonian formulation, one can perform a point transformation of the original 24 $V_{ai}$ to a new set of adapt coordinates,

$$V_{ai}(q, S) = O_{ab}(q) A_{bi},$$ (13)

in terms of the 8 gauge angles $q_j$ parametrising the $O_{ab}(q)$ orth. $8 \times 8$ matrix adjoint to $U(q)$

$$O_{ab}(q) = (1/8) Tr \left[ U^{-1}(q) \lambda_a U(q) \lambda_b \right].$$ (14)

and 16 reduced $A_{ai}$ satisfying some gauge conditions

$$\chi_a(A) = (\Gamma_i)_{ab} A_{bi} = 0, \quad a = 1, ..., 8.$$ (15)

Preserving the CCR → old canonical momenta in terms of the new variables

$$\Pi_{ai}(q, A, p, P) = O_{ab}(q) \left[ P_bi - (\Gamma_i)_{bl} \gamma_{ls}^{-1T}(A) \left( \frac{1}{g} \Omega^{-1}_{st}(q) p_t + T_s(A, P) \right) \right],$$ (16)

with the homogeneous part of the FP operator

$$\gamma_{ab}(A) := (\Gamma_i)_{ad} f_{abc} A_{ci},$$ (17)

and the operators

$$T_a(A, P) := f_{abc} A_{bi} P_{ci}.$$ (18)

In terms of the new coordinates, the Gauss-laws become

$$G_a \Phi \equiv O_{ai}(q) \Omega^{-1}_{ki}(q) p_k \Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i} \Phi = 0 \quad (Abelianisation)$$

The unconstrained spin operator reads

$$J_i = \epsilon_{ijk} A_{aj} E_{ak}.$$ (19)

in terms of the physical electric fields

$$E_{ai} := P_{bi} - (\Gamma_i)_{bl} \gamma_{ls}^{-1T}(A) T_s(A, P).$$ (20)

The correctly ordered unconstrained Hamiltonian of SU(3) YM-QM takes the form [24]

$$H = \frac{1}{2} \left[ \frac{1}{\gamma(A)} P_{ai} |\gamma(A)| P_{ai} + \frac{1}{\gamma(A)} T_a |\gamma(A)| \left( \gamma^{-1}(T_i^T \Gamma_i) \gamma^{-1T} \right)_{ac} T_c + \left( B_{ai}^{\text{hom}}(A) \right)^2 \right],$$

using the homogeneous part of the chromomagnetic field

$$B_{ai}^{\text{hom}}(A) := (1/2) g \epsilon_{ijk} f_{abc} A_{bj} A_{ck}.$$ (21)

The matrix element of a physical operator $O$ is given by

$$\langle \Psi'| O | \Psi \rangle \propto \int dA \, |\gamma(A)| \, \Psi'^*[A] \, O \, \Psi[A].$$
3.2 Unconstrained Hamiltonian formulation of SU(3) YM-QM in the flux-tube gauge

It is our aim to find a gauge which exists and leads to a maximally simple FP-operator. This is can be achieved by putting the "flux-tube-gauge", leading to a rather simple but non-trivial FP-operator. The drawback of the this gauge is that the reduced fields \(A_{ai}\) are not spin-eigenstates, as was the case for the SU(3) symmetric gauge [18] - [20] leading to a very complicated Fp-operator. We shall show that the disadvantage, that the reduced fields in the fluxtube-gauge are not spin-eigenstates, can be circumvented by forming certain irreducible polynomials of the reduced \(A\), symmetric tensors, which have definite eigenvalues of \(J, P,\) and \(C\).

The "flux-tube-gauge" is defined as

\[
\chi_a(A) = 0 : \quad A_{a1} = 0 \quad \forall a = 1, 2, 4, 5, 6, 7 \quad \wedge \quad A_{a2} = 0 \quad \forall a = 5, 7 .
\]  

or explicitly

\[
A = \begin{pmatrix}
0 & A_{12} & A_{13} \\
0 & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33} \\
0 & A_{42} & A_{43} \\
0 & 0 & A_{53} \\
0 & A_{62} & A_{63} \\
0 & 0 & A_{73} \\
A_{81} & A_{82} & A_{83}
\end{pmatrix} \equiv (X \ Y \ Z) .
\]  

Using finally the reparametrisation

\[
A_{31} \equiv X_3 = r \cos[\psi] \quad A_{81} \equiv X_8 = r \sin[\psi] ,
\]  

the explicit expression for the homogeneous part of the FP-operator (17) is

\[
\gamma = \begin{pmatrix}
0 & -r \cos[\psi] & 0 & 0 & 0 & 0 & 0 & 0 \\
r \cos[\psi] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-Y_6/2 & 0 & -Y_4/2 & -Y_+ & 0 & Y_1/2 & -Y_2/2 & -\sqrt{3}Y_4/2 \\
0 & 0 & 0 & r \cos[\psi + \frac{2\pi}{3}] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -r \cos[\psi + \frac{2\pi}{3}] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -r \cos[\psi + \frac{4\pi}{3}] & 0 & 0 \\
-Y_4/2 & 0 & Y_6/2 & Y_1/2 & Y_2/2 & Y_- & 0 & -\sqrt{3}Y_6/2
\end{pmatrix}
\]

using the abbreviations \(Y_\pm := -(Y_3 \pm \sqrt{3}Y_8)/2\). The FP-determinant factorises

\[
|\gamma(A)| = r^6 \cos^2[3 \psi] \ Y_3Y_6 .
\]  

The inverse \(\gamma^{-1}\) of the Faddeev-Popov operator is rather simple (shown in Appendix A) and exists in the regions of non-vanishing determinant. The matrix elements are

\[
\langle \Psi_1 | O | \Psi_2 \rangle = \int d\mu_X \int d\mu_Y \int d\mu_Z \ \Psi_1^\dagger O \ \Psi_2 ,
\]

with the completely factorised

\[
\int d\mu_X := \int_0^\infty dr \ r^7 \int_0^{2\pi} d\psi \ \cos^2[3 \psi] ,
\]

\[
\int d\mu_Y := \int_{-\infty}^\infty dY_1 \int_{-\infty}^\infty dY_2 \int_{-\infty}^\infty dY_3 \int_{-\infty}^\infty dY_5 \int_0^\infty dY_4 \int_0^\infty dY_6 \int_0^\infty dY_6Y_6 ,
\]

\[
\int d\mu_Z := \prod_{a=1}^8 \int_{-\infty}^{\infty} dZ_a .
\]
Furthermore, for the operators $T_a(A,P)$ defined in (18) we find that $T_a^X = -if_{abc} X_b \partial/\partial X_c \equiv 0$, i.e.

$$T_a(A, P) \equiv T_a^Y(Y, P_Y) + T_a^Z(Z, P_Z) ,$$

where the components of the (non-reduced) $T_a^Z = -if_{abc} Z_b \partial/\partial Z_c$ satisfy the $su(3)$ algebra

$$[T_a^Z, T_b^Z] = i f_{abc} T_c^Z ,$$

whereas the reduced $T_a^Y = -if_{abc} Y_b \partial/\partial Y_c$ do not.

The physical electric fields read in the flux-tube gauge

$$E(A, P) = \begin{pmatrix} -P_2(A, P) & P_{12} & P_{13} \\ P_1(A, P) & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \\ P_{54}(A, P) & P_{42} & P_{43} \\ -P_4(A, P) & -P_+(A, P) & P_{53} \\ -P_7(A, P) & P_{62} & P_{63} \\ P_6(A, P) & P_-(A, P) & P_{73} \\ P_8 & P_{12} & P_{83} \end{pmatrix} ,$$

with the Hermitian $P_± = P_±$

$$P_+ = \frac{1}{Y_4} \left[ T_3^Y + \left( T_3^Z + \frac{1}{\sqrt{3}} T_8^Z \right) \right] , \quad P_- = \frac{1}{Y_6} \left[ T_3^Y + \left( T_3^Z - \frac{1}{\sqrt{3}} T_8^Z \right) \right] ,$$

and

$$P_1 = \frac{1}{r \cos[\psi]} \left( \bar{T}_1^Y + \bar{T}_1^Z \right) , \quad P_2 = \frac{1}{r \cos[\psi]} \left( \bar{T}_2^Y + \bar{T}_2^Z \right) ,
$$
$$P_4 = \frac{1}{r \cos[\psi + 2\pi/3]} \left( \bar{T}_4^Y + \bar{T}_4^Z \right) , \quad P_5 = \frac{1}{r \cos[\psi + 2\pi/3]} \left( \bar{T}_5^Y + \bar{T}_5^Z \right) ,
$$
$$P_6 = \frac{1}{r \cos[\psi + 4\pi/3]} \left( \bar{T}_6^Y + \bar{T}_6^Z \right) , \quad P_7 = \frac{1}{r \cos[\psi + 4\pi/3]} \left( \bar{T}_7^Y + \bar{T}_7^Z \right) ,$$

with the definition of the tilded operators $\bar{T}_i^Y$ and $\bar{T}_i^Z$ given in Appendix B.

Hence, the components of the spin angular momentum operators in reduced space

$$J_i = \epsilon_{ijk} A_{aj} E_{ak} ,$$

read explicitly in the flux-tube gauge

$$J_1 = -i \sum_{a=1,2,3,4,6,8} \left[ Y_a \frac{\partial}{\partial Z_a} - Z_a \frac{\partial}{\partial Y_a} \right] + (Z_5 P_+ - Z_7 P_-) ,
$$
$$J_2 = -i \sum_{a=3,8} \left[ Z_a \frac{\partial}{\partial X_a} - X_a \frac{\partial}{\partial Z_a} \right] - (Z_1 P_2 - Z_2 P_1) + (Z_4 P_5 - Z_5 P_4) - (Z_6 P_7 - Z_7 P_6) ,
$$
$$J_3 = -i \sum_{a=3,8} \left[ X_a \frac{\partial}{\partial Y_a} - Y_a \frac{\partial}{\partial X_a} \right] + (Y_1 P_2 - Y_2 P_1) - Y_4 P_5 + Y_6 P_7 ,$$

and the physical Hamiltonian reads

$$H = \frac{1}{2} J_X^{-1} \sum_{a=1,2,3,4,6,8} \left( \frac{\partial}{\partial X_a} J_X \frac{\partial}{\partial X_a} \right) + \frac{1}{2} J_Y^{-1} \sum_{a=1,2,3,4,6,8} \left( \frac{\partial}{\partial Y_a} J_Y \frac{\partial}{\partial Y_a} \right) + \frac{1}{2} \sum_{a=1}^8 \left( \frac{\partial}{\partial Z_a} \frac{\partial}{\partial Z_a} \right)
$$
$$+ \frac{1}{2} J_Y^{-1} \sum_{a=1,2,3,4,6,7} P_a^i J_Y P_a + \left( \frac{P_-^2 + P_+^2}{2} \right) + \frac{1}{2} \sum_{a=1}^8 \sum_{i=1}^3 \left( B_{ai}^{\text{hom}} [X, Y, Z] \right)^2 ,$$

using the abbreviations $|\gamma(A)| = J_X J_Y$ of (26) with $J_X := r^6 \cos^2 [3 \psi]$ and $J_Y := Y_4 Y_6$. 

6
3.3 Hamiltonian of SU(3) YM-QM in the flux-tube gauge

The Hamiltonian of SU(3) YM-QM in the flux-tube gauge (38) can be written in the form

\[ H[A,P] = K_X + K_Y + K_Z + \frac{1}{2r^2} \left[ \frac{(I_{1Y}^2 + I_{2Y}^2)}{\cos^2 \psi} + \frac{(I_{4Y}^2 + I_{5Y}^2)}{\cos^2 [\psi + 2\pi/3]} + \frac{(I_{6Y}^2 + I_{7Y}^2)}{\cos^2 [\psi + 4\pi/3]} \right] \]

\[ + \frac{1}{2Y_4^2} I_{1Y}^2 + \frac{1}{2Y_6^2} I_{2Y}^2 + \frac{1}{2} \left( P_{\rm hom}^{\text{h.o.}}[X,Y,Z] \right)^2. \]  

(39)

The single-direction kinetic terms read

\[ K_X = -\frac{1}{2} \left[ \frac{\partial^2}{\partial r^2} + \frac{7}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( -6 \tan[3\psi] \frac{\partial}{\partial \psi} + \frac{\partial^2}{\partial \psi^2} \right) \right], \]

\[ K_Y = -\frac{1}{2} \sum_{a=1,2,3,8} \frac{\partial^2}{\partial Y_a^2} + \sum_{a=4,6} \left( \frac{\partial^2}{\partial Y_a^2} + \frac{1}{Y_a} \frac{\partial}{\partial Y_a} - \frac{1}{Y_a^2} \left( Y_1 \frac{\partial}{\partial Y_2} - Y_2 \frac{\partial}{\partial Y_1} \right)^2 \right), \]

\[ K_Z = -\frac{1}{2} \sum_{a=1}^8 \frac{\partial^2}{\partial Z_a^2}, \]

and the interactions \( I_{1Y} \) and \( I_{2Y} \) given in Appendix B.

It has been proven a long time ago by Simon [3], that the spectrum is discrete although the chromomagnetic potential owns three flat valleys narrowing down. Although there exist classical zero energy trajectories to infinity, the quantum fluctuations in the narrowing valleys confine the wavefunction and lead to a discret spectrum. It is therefore reasonable to replace in an intermediate step the chromomagnetic potential by a separable 16-dimensional harmonic oscillator potential, and then use the obtained eigensystem to find the eigensystem of SU(3) YM QM.

3.4 The corresponding harmonic oscillator problem \( H_{\text{h.o.}} \)

Replacing in \( H(A,P) \) the magnetic potential by the separable harmonic oscillator potential with free parameter \( \omega > 0 \)

\[ \frac{1}{2} \left( B_{\text{hom}}^{\text{h.o.}}(A) \right)^2 \rightarrow \frac{1}{2} \omega^2 (A_{\text{h.o.}})^2 \equiv \frac{1}{2} \omega^2 \left[ r^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_6^2 + Y_8^2 + Z_a^2 \right] \]  

(40)

we obtain the corresponding harmonic oscillator problem (with the same measure !!!),

\[ H_{\text{h.o.}}[A,P] = H_X + H_Y + H_Z + \frac{1}{2Y_4^2} I_{1Y}^2 + \frac{1}{2Y_6^2} I_{2Y}^2 + \]

\[ + \frac{1}{2r^2} \left[ \frac{(I_{1Y}^2 + I_{2Y}^2)}{\cos^2 \psi} + \frac{(I_{4Y}^2 + I_{5Y}^2)}{\cos^2 [\psi + 2\pi/3]} + \frac{(I_{6Y}^2 + I_{7Y}^2)}{\cos^2 [\psi + 4\pi/3]} \right] \]  

(41)

The single-direction Hamiltonions read

\[ H_X = \frac{1}{2} \left[ -\frac{\partial^2}{\partial r^2} - \frac{7}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( 6 \tan[3\psi] \frac{\partial}{\partial \psi} - \frac{\partial^2}{\partial \psi^2} \right) + \omega^2 r^2 \right], \]

\[ H_Y = \frac{1}{2} \left[ \sum_{a=1,2,3,8} \left( -\frac{\partial^2}{\partial Y_a^2} + \omega^2 Y_a^2 \right) + \sum_{a=4,6} \left( -\frac{\partial^2}{\partial Y_a^2} - \frac{1}{Y_a} \frac{\partial}{\partial Y_a} + \frac{1}{Y_a^2} \left( Y_1 \frac{\partial}{\partial Y_2} - Y_2 \frac{\partial}{\partial Y_1} \right)^2 + \omega^2 Y_a^2 \right) \right], \]

\[ H_Z = \frac{1}{2} \sum_{a=1}^8 \left[ -\frac{\partial^2}{\partial Z_a^2} + \omega^2 Z_a^2 \right], \]

As stated already in our earlier work [21], this system is integrable and can be solved analytically in terms of orthogonal polynomials. We shall demonstrate this in more detail in the following paragraphs.
4 Exact solution of the corresponding harmonic oscillator problem

The operators $T_a^Y$ and $T_a^Z$ lead to the coupling between the three spatial directions.

4.1 Solutions of the $H_{h.o.}$ Schrödinger equation separable in $X,Y,Z$.

First looking for solutions for the case where the kinetic terms decouple for all three directions:

$$
\Phi_{X|Y|Z} = \Phi_X[X] \Phi_Y[Y] \Phi_Z[Z],
$$

$$
H_{h.o.} \Phi_{X|Y|Z} = (H_X + H_Y + H_Z) \Phi_{X|Y|Z} = (\epsilon_X + \epsilon_Y + \epsilon_Z) \Phi_{X|Y|Z}.
$$

with the single-direction functionals $\Phi_X, \Phi_Y, \text{and} \Phi_Z$ satisfying the separate Schrödinger equations

$$
H_X \Phi_X = \epsilon_X \Phi_X,
$$

$$
H_Y \Phi_Y = \epsilon_Y \Phi_Y \land T^{Y}_{a} \Phi_Y = 0, \quad a = 1, ..., 8,
$$

$$
H_Z \Phi_Z = \epsilon_Z \Phi_Z \land T^{Z}_{a} \Phi_Z = 0, \quad a = 1, ..., 8.
$$

Note that the X-equation is unconstrained, whereas the Y- and Z-equations are constrained.

4.1.1 Solution of the X-equation

Consider first the X-equation (43)

$$
H_X \Phi_X \equiv -\frac{1}{2} \left[ \frac{\partial^2}{\partial r^2} + \frac{7}{r} \frac{\partial}{\partial r} - \omega^2 r^2 + \frac{1}{r^2} \left( -6 \tan[3 \psi] \frac{\partial}{\partial \psi} + \frac{\partial^2}{\partial \psi^2} \right) \right] \Phi_X = \epsilon_X \Phi_X
$$

and the matrix elements

$$
\langle \Phi'_X | O_X | \Phi_X \rangle = \int_0^\infty dr r^7 \int_0^{2\pi} d\psi \cos^2[3 \psi] \Phi'_X O_X[X] \Phi_X
$$

In terms of the new coordinates

$$
s_{11} = X_a X_a = r^2, \quad s_{111} = d_{abc} X_a X_b X_c = \frac{1}{\sqrt{3}} r^3 \sin[3 \psi]
$$

which are the $s_{11}$ and $s_{111}$ components of the 6- and 10-component symmetric tensors

$$
\begin{align*}
s_{ij}^{++}[A] &:= A_{ai} A_{aj}, \\
s_{[3]ijk}^{--}[A] &:= d_{abc} A_{ai} A_{bj} A_{ck}
\end{align*}
$$

and using the scaled

$$
\bar{y} := s_{111}/s_{11}^{3/2} = \frac{1}{\sqrt{3}} \sin[3 \psi]
$$

Equ. (46) reads

$$
-\frac{1}{2} \left[ 4s_{11} \frac{\partial^2}{\partial s_{11}^2} + 16 \frac{\partial}{\partial s_{11}} - s_{11} \omega^2 + \frac{1}{s_{11}} \left( 3(1 - 3\bar{y}^2) \frac{\partial^2}{\partial \bar{y}^2} - 27\bar{y} \frac{\partial}{\partial \bar{y}} \right) \right] \Phi_X = \epsilon_X \Phi_X
$$

which can easily be solved by separation of variables. The solutions can be written

$$
\Phi_{n_1,n_2}^{X}[s_{11}, s_{111}] = \frac{\omega^2}{\sqrt{6\pi}} p_{n_1,n_2}^{(1)} (\omega s_{11}, \omega^{3/2} s_{111}) \exp[-\omega s_{11}/2]
$$
with the energy eigenvalues
\[ \epsilon_{n_1,n_2} = (4 + 2n_1 + 3n_2) \omega =: (4 + n_X) \omega \] (53)

The lowest polynomials read
\[ p_{0,0}^{(1)}(x, y) = 1 , \quad p_{1,0}^{(1)}(x, y) = \frac{1}{2} (-4 + x) , \quad p_{0,1}^{(1)}(x, y) = \frac{1}{\sqrt{10}} y , \quad p_{2,0}^{(1)}(x, y) = \frac{1}{2 \sqrt{10}} (20 - 10x + x^2) , \quad p_{1,1}^{(1)}(x, y) = \frac{1}{\sqrt{70}} (-7 + x)y , \quad ... \] (54)

The number \( n_X := 2n_1 + 3n_2 \) is the degree of the corresponding polynomial in the components of \( X \) for each solution. After Gram-Schmidt-orthogonalisation of those degenerate in energy, they form an ONB with respect to the measure \( \omega \) and the Y-Equ.(44). We mention here, that the X-equation (43), due to the special form of the FP-Jacobian (17), can be considered also for the case of singular boundary conditions. The lowest solutions for this singular case and their energy eigenvalues are shown in Appendix C. Although very interesting, we shall limit ourselves in this article on the regular case.

### 4.1.2 Solution of the Y-equation

For functionals \( \Phi_Y[s_{22}, s_{222}] \) depending only on the components
\[ s_{22} = Y_a Y_a , \quad s_{222} = d_{abc} Y_a Y_b Y_c , \] (55)
of the symmetric tensors (49), satisfying \( T_a^Y s_{22} = T_a^Y s_{222} = 0 \), i.e. \( T_a^Y \Phi_Y[s_{22}, s_{222}] = 0 \), \( a = 1, ..., 8 \), and the Y-Equ.(44)
\[ H_Y \Phi_Y[s_{22}, s_{222}] = \epsilon_Y \Phi_Y[s_{22}, s_{222}] \] (56)
is solved by the same functionals (52), and in particular polynomials, and energy eigenvalues as for the X-Equ.(43), but with \( s_{11} \) and \( s_{111} \) replaced by \( s_{22} \) and \( s_{222} \) respectively. They form an ONB with respect to the measure
\[ \langle \Phi_Y' | O_Y | \Phi_Y \rangle = \int_0^\infty dY_4 Y_4 \int_0^\infty dY_6 Y_6 \prod_{a=1,2,3,8} \left[ \int_0^\infty dY_a \right] \Phi_Y' O_Y [Y] \Phi_Y . \] (57)

### 4.1.3 Solution of the Z-equation

Similarly for the Z-equation with \( \Phi_Z[s_{33}, s_{333}] \) depending only on the components
\[ s_{33} = Z_a Z_a , \quad s_{333} = d_{abc} Z_a Z_b Z_c , \] (58)
of the symmetric tensors (49), satisfying \( T_a^Z s_{33} = T_a^Z s_{333} = 0 \), i.e. \( T_a^Z \Phi_Z[s_{33}, s_{333}] = 0 \), \( a = 1, ..., 8 \), and the Z-Equ.(45)
\[ H_Z \Phi_Z[s_{33}, s_{333}] = \epsilon_Z \Phi_Z[s_{33}, s_{333}] \] (59)
solved by the same functionals (52), and in particular polynomials, and energy eigenvalues as for the X-Equ.(43), but with \( s_{11} \) and \( s_{111} \) replaced by \( s_{33} \) and \( s_{333} \) respectively. They form an ONB with respect to the measure
\[ \langle \Phi'_Z | O_Z | \Phi_Z \rangle = \prod_{a=1, ..., 8} \left[ \int_0^\infty dZ_a \right] \Phi'_Z O_Z [Z] \Phi_Z . \] (60)
4.1.4 Trigonal form of the one-direction Hamiltonian

The polynomials can be easily obtained using the equation

$$D^{(1)}_{p_{n_1,n_2}}(x,y) = (\epsilon/\omega) P^{(1)}_{p_{n_1,n_2}}(x,y) ,$$

with the differential operator $D^{(1)} := D^{(1)}_{-2} + D^{(1)}_0$ consisting of two parts

$$D^{(1)}_{-2} := -\frac{3}{2} x^2 \partial_y^2 - 2 (x \partial_x + 3y \partial_y + 4) \partial_x , \quad D^{(1)}_0 := (2x \partial_x + 3y \partial_y + 4) ,$$

acting in the space of monomials $x^n y^m$. The operator $D^{(1)}_0$ reproduces the given monomial with eigenvalue $(2n_1 + 3n_2 + 4)$ where $(2n_1 + 3n_2)$ is its power seen as a homogeneous polynomial in the A ("A-power"), attributing $x$ the A-power 2 and $y$ the A-power 3, as is the case e.g. for $x = \omega s_{11}(A)$ and $y = \omega^{3/2} s_{111}(A)$ in Equ.(48). On the other hand, the operator $D^{(1)}_{-2}$ transforms one $x^n y^m$ into another one, $x^{m_1} y^{m_2}$, with an by two lowered A-power $(2m_1 + 3m_2) = (2n_1 + 3n_2) - 2$. Hence the operator $D^{(1)}$ is triangular in the space of monomials $x^n y^m$ and can easily be diagonalised with eigenvalues

$$\epsilon = (2n_1 + 3n_2 + 4) \omega ,$$

and the eigenfunctions the polynomials

$$p^{(1)}_{n_1,n_2}(x,y) = N \left( \sum_{m_1,m_2} A^{(1)}_{n_1,n_2} (m_1,m_2) x^{m_1} y^{m_2} \right) ,$$

with the leading ("defining") monomial $x^n y^m$ and a "tail" of monomials of decreasing, lower powers with some definite coefficients $A^{(1)}_{n_1,n_2} (m_1,m_2)$.

We shall see in the following paragraphs, that such behaviour will also for the case of solutions separable in only one space direction and the non separable general case. Before discussing these we shall give in the following subsection some examples of solution build from those separable in all $X, Y, Z$.

4.1.5 Putting the solutions of the $X$, $Y$, and $Z$-equations together

Together, we have

$$H_{h.o.} \Phi_{X|Y|Z} = (\epsilon_X + \epsilon_Y + \epsilon_Z) = \epsilon_{h.o.} \Phi_{X|Y|Z} .$$

with the energy eigenvalues

$$\epsilon_{h.o.} = (12 + n_X + n_Y + n_Z) \omega = (12 + n) \omega$$

We find the lowest solutions $\Phi_{[n]|X|Y|Z[A]} = P_{[n]|X|Y|Z} \exp[-\omega (s_{11} + s_{22} + s_{33})/2]$ with

$$\epsilon_{h.o.} = 12 \omega : P_{[0]|X|Y|Z} \propto p_{0,0}[X] p_{0,0}[Y] p_{0,0}[Z] = 1 ,$$

$$\epsilon_{h.o.} = 14 \omega : P_{[2]|X|Y|Z} \propto p_{1,0}[X] p_{0,0}[Y] p_{0,0}[Z] = -2 + \omega s_{11}/2 , \quad \text{and perm.}$$

$$\epsilon_{h.o.} = 15 \omega : P_{[3]|X|Y|Z} \propto p_{0,1}[X] p_{0,0}[Y] p_{0,0}[Z] = \omega^{3/2} s_{111}/\sqrt{10} , \quad \text{and perm.}$$

$$\epsilon_{h.o.} = 16 \omega : P_{[4]|X|Y|Z} \propto p_{2,0}[X] p_{0,0}[Y] p_{0,0}[Z] = \left(10 - 5 \omega s_{11} + \omega^2 s_{11}^2/2\right)/\sqrt{10} , \quad \text{and perm.}$$

$$\epsilon_{h.o.} = 17 \omega : P_{[5]|X|Y|Z} \propto p_{1,1}[X] p_{0,0}[Y] p_{0,0}[Z] = 4 - \omega (s_{22} + s_{33}) + \omega^2 s_{22}s_{33}/4 , \quad \text{and perm.}$$

$$\epsilon_{h.o.} = 18 \omega : P_{[6]|X|Y|Z} \propto p_{0,2}[X] p_{0,0}[Y] p_{0,0}[Z] = -7 + \omega s_{11} \omega^{3/2} s_{111}/\sqrt{70} , \quad \text{and perm.}$$

$$\epsilon_{h.o.} = 19 \omega : P_{[7]|X|Y|Z} \propto p_{1,2}[X] p_{0,0}[Y] p_{0,0}[Z] = -2 + \omega s_{22}/2 \omega^{3/2} s_{111}/\sqrt{10} , \quad \text{and perm.}$$

...
The number \( n \) is the maximal power of the polynomial \( P_n \), seen as a polynomial in the reduced gauge field \( A \). Superposing (67), and denoting

\[
S^{(0)}_{[2]} := s_{[2]ii} = s_{11} + s_{22} + s_{33} = (A_{ai})^2
\]

we can build the \( 0^{++} \) eigenstates \( \Phi^{(0)++}_{[i]} = P^{(0)++}_{[i]}[A] \exp[-\omega (A_{ai})^2/2] \), with

\[
\begin{align*}
\epsilon_{h.o.}^{(0)++} &= 12 \omega : \quad P^{(0)++}_{[0]} \propto 1 , \\
\epsilon_{h.o.}^{(0)++} &= 14 \omega : \quad P^{(0)++}_{[2]} \propto -12 + \omega s^{(0)}_{[2]} , \\
\epsilon_{h.o.}^{(0)++} &= 16 \omega : \quad P^{(0)++}_{[4]} \propto 108 - 18 \omega s^{(0)}_{[2]} + \omega^2 (s^{(0)}_{[2]})^2 .
\end{align*}
\]

We find rotational invariant solutions although the flux-tube gauge is not rotational invariant. We mention here, that for the case of singular solutions of the the \( X \)-equation (43), discussed in Appendix C, rotational invariance is broken in one direction, here the \( x \)-direction, leaving only a cylindrical symmetry.

### 4.2 Separation of one of the three directions

#### 4.2.1 Separation of the \( Y \)-equation

Starting with the case where the \( y \)-direction decouples

\[
\Phi_{XZ|Y}[X,Y,Z] = \Phi_Y[Y] \Phi_{XZ}[X,Z] ,
\]

\[
H_{h.o.} \Phi_{Y|XZ}[X,Y,Z] = (H_Y + H_{XZ}) \Phi_Y[X,Y,Z] = (\epsilon_Y + \epsilon_{XZ}) \Phi_{Y|XZ}[X,Y,Z] .
\]

where \( \Phi_Y \) satisfies the above discussed constrained \( Y \)-equation (44) and \( \Phi_{XZ} \) is to solve the constrained \( x \)-direction Schrödinger equation

\[
\begin{bmatrix}
H_X + H_Z + \frac{1}{2} r^2 \left[ \frac{(T_2^Z)^2}{\cos^2 \psi} + \frac{(T_4^Z)^2}{\cos^2 [\psi + 2\pi/3]} + \frac{(T_5^Z)^2}{\cos^2 [\psi + 4\pi/3]} \right] \end{bmatrix} \Phi_{XZ} = \epsilon_{XZ} \Phi_{XZ} ,
\]

\[
\wedge \quad T_3^Z \Phi_{XZ} = 0 \quad \wedge \quad T_8^Z \Phi_{XZ} = 0 .
\]

In terms of the components \( s_{13}, s_{113}, \) and \( s_{133} \) of the symmetric tensor (49),

\[
s_{13} = X_a Z_a , \quad s_{113} = d_{abc} X_a X_b Z_c , \quad s_{133} = d_{abc} X_a Z_b Z_c ,
\]

we find the \( \epsilon_{XZ} = 10 \omega \) solution

\[
\Phi_{[2]}^{XZ}[X,Z] \propto [\omega s_{13}] \exp [-\omega (s_{11} + s_{33})/2] ,
\]

and the \( \epsilon_{XZ} = 11 \omega \) solutions

\[
\begin{align*}
\Phi_{[3]}^{XZ}[X,Z] &\propto [\omega^{3/2} s_{113}] \exp [-\omega (s_{11} + s_{33})/2] , \\
\Phi_{[3]}^{XZ}[X,Z] &\propto [\omega^{3/2} s_{133}] \exp [-\omega (s_{11} + s_{33})/2] ,
\end{align*}
\]

We have the \( \epsilon^{(2)} = 12 \omega \) solutions

\[
\begin{align*}
\Phi_{[4]}^{XZ}[X,Z] &\propto \left[ 2 - \frac{1}{2} \omega (s_{11} + s_{33}) + \omega^2 s_{13} \right] \exp [-\omega (s_{11} + s_{33})/2] , \\
\Phi_{[4]}^{XZ}[X,Z] &\propto \left[ 6 - \frac{3}{2} \omega (s_{11} + s_{33}) + \omega^2 b_{22} \right] \exp [-\omega (s_{11} + s_{33})/2] ,
\end{align*}
\]
using additionally the 22-component
\[ b_{22}[X, Z] = f_{abc} f_{ade} X_b Z_c X_d Z_e , \] (76)
of the symmetric tensor
\[ b_{[ij]}^{++}[A] := g^{-2} B_{a_i}^{\text{hom}}[A] B_{a_j}^{\text{hom}}[A] , \] (77)
It appears in the tail of the \( \epsilon^{(2)} = 14 \omega \) solution
\[ \Phi_{XZ}^{XZ}[X, Z] \propto \left[ \omega^2 \left( \frac{1}{2} s_{11} s_{33} - s_{13}^2 - b_{22} \right) + \omega^3 s_{111} s_{133} \right] \exp \left[ -\omega(s_{11} + s_{33})/2 \right] , \] (78)
with the product \( s_{111} s_{133} \) as leading term, and is irreducible in \( A \)-space.

In general, the \( X-Z \)-solutions are of the form
\[
\Phi_{n_1, \ldots, n_8}^{XZ}[X, Z] = \frac{2}{5} \frac{\omega^4}{\sqrt{11 \pi^{3/2}}^2} p_{n_1, \ldots, n_8}^{(2)} \left( \omega s_{11}, \omega s_{12}, \omega s_{13}, \omega s_{111}, \omega s_{112}, \omega s_{122}, \omega^2 b_{22} \right) \times \exp \left[ -\omega(s_{11} + s_{33})/2 \right]
\] (79)
with energy
\[ \epsilon_{n_1, \ldots, n_8}^{(2)} := [8 + 2(n_1 + n_2) + 3(n_3 + n_4 + n_5 + n_6 + n_7) + 4n_8] \omega \] (80)
where \( p_{n_1, \ldots, n_8}^{(2)} (x_1, \ldots, x_8) \) is a polynomial in the eight variables \( x_1, \ldots, x_8 \) where the monomial with the maximal order reads \( \prod_{i=1}^8 x_i^{n_i} \). After orthogonalisation of those degenerate in energy) they form an ONB with respect to the measure
\[ \langle \Phi'_{XY} | O | \Phi_{XY} \rangle = \int d\mu_X \int d\mu_Y \Phi'_{XY} O[X, Y] \Phi_{XY} . \] (81)

### 4.2.2 Separation of the \( Z \)-equation

Next we consider the case where the \( z \)-direction decouples
\[ \Phi_{Z|XY}[X, Y, Z] = \Phi_Z[Z] \Phi_{XY}[X, Y] , \]
\[ H_{h.o} \Phi_{Z|XY}[X, Y, Z] = (H_Z + H_{XY}) \Phi_{Z|XY}[X, Y, Z] = (\epsilon_Z + \epsilon_{XY}) \Phi_{Z|XY}[X, Y, Z] . \] (82)
where \( \Phi_Z \) satisfies the above discussed constrained \( Z \)-Schrödinger-equation (45) and functional \( \Phi_{XY} \) is to solve the unconstrained x-y-direction Schrödinger equation
\[ H_{XY} \Phi_{XY} \equiv \left[ H_X + H_Y + \frac{1}{2 r^2 (Y_4 Y_6)} \left( \frac{\bar{T}_1 Y^4 Y_6 \bar{T}_1 Y^4 Y_6 \bar{T}_2 Y^4 Y_6 \bar{T}_2 Y^4 Y_6}{\cos^2 \psi} + \frac{\bar{T}_3 Y^4 Y_6 \bar{T}_3 Y^4 Y_6 \bar{T}_4 Y^4 Y_6 \bar{T}_4 Y^4 Y_6}{\cos^2 (\psi + 2\pi/3)} \right) \right] \Phi_{XY} = \epsilon_{XY} \Phi_{XY} \] (83)
Using also the additional components
\[ s_{12} = X_a Y_a , \quad s_{112} = d_{abc} X_a X_b Y_c , \quad s_{122} = d_{abc} X_a Y_b Y_c , \quad b_{33}[X, Y] = f_{abc} f_{ade} X_b Y_c X_d Y_e , \] (84)
the x-y-solutions are of the form
\[
\Phi_{n_1, \ldots, n_8}^{XY}[X, Y] = \frac{2}{5} \frac{\omega^4}{\sqrt{11 \pi^{3/2}}^2} p_{n_1, \ldots, n_8}^{(2)} \left( \omega s_{11}, \omega s_{12}, \omega s_{13}, \omega s_{111}, \omega s_{112}, \omega s_{122}, \omega^2 b_{33} \right) \times \exp \left[ -\omega(s_{11} + s_{22})/2 \right]
\] (85)
and energy
\[ \epsilon_{n_1, \ldots, n_8}^{(2)} := [8 + 2(n_1 + n_2) + 3(n_3 + n_4 + n_5 + n_6 + n_7) + 4n_8] \omega \] (86)
where \( p_{n_1, \ldots, n_8}^{(2)} \) are the same polynomials as for the x-z case. After orthogonalisation of those degenerate in energy) they form an ONB with respect to the measure
\[ \langle \Phi'_{XY} | O | \Phi_{XY} \rangle = \int d\mu_X \int d\mu_Y \Phi'_{XY} O[X, Y] \Phi_{XY} . \] (87)
4.2.3 Separation of the X-equation

Finally, we consider the case where the x-direction decouples

\[ \Phi_{X|YZ}[X, Y, Z] = \Phi_X[X] \Phi_{YZ}[Y, Z], \]

\[ H_{h.o.} \Phi_{X|YZ}[X, Y, Z] = (H_X + H_{YZ}) \Phi_{X|YZ}[X, Y, Z] = (\epsilon_X + \epsilon_{YZ}) \Phi_{X|YZ}[X, Y, Z]. \]  

(88)

where \( \Phi_X \) satisfies the above discussed unconstrained X-equation (43) and the functionals \( \Phi_{YZ} \) have to solve the constrained Y-Z-equation

\[ H_{YZ} \Phi_{YZ} \equiv \left[ H_Y + H_Z + \frac{1}{2Y^2} \left[ \frac{1}{2} \left( T_3^Y + \frac{1}{\sqrt{3}} T_8^Y \right) - T_3^Y \right] \left( T_3^Z + \frac{1}{\sqrt{3}} T_8^Z \right) \right. \]

\[ + \frac{1}{2Y^2} \left[ \frac{1}{2} \left( T_3^Z - \frac{1}{\sqrt{3}} T_8^Z \right) - T_3^Z \right] \left( T_3^Z - \frac{1}{\sqrt{3}} T_8^Z \right) \] \( \Phi_{YZ} = \epsilon_{YZ} \Phi_{YZ} \),

\[ \wedge \left( \widetilde{T}_a^Y + \widetilde{T}_a^Z \right) \Phi_{YZ} = 0, \quad a = 1, 2, 4, 5, 6, 7. \]  

(89)

In terms of the components

\[ s_{23} = Y_a Z_a, \quad s_{223} = d_{abc} Y_a Y_b Z_c, \quad s_{233} = d_{abc} Y_a Z_b Z_c, \quad b_{11}[Y, Z] = f_{abc} f_{ade} Y_b Z_c Y_d Z_e, \]  

(90)

the y-z-solutions are of the form

\[ \Phi_{YZ}^{n_1, \ldots, n_8}(Y, Z) = \frac{2\omega^4}{5\sqrt{11\pi}^{3/2}} p_{n_1, \ldots, n_8}^{(2)} \left[ \omega s_{22}, \omega s_{33}, \omega s_{23}, \omega^{3/2}s_{222}, \omega^{3/2}s_{333}, \omega^{3/2}s_{223}, \omega^{3/2}s_{233}, \omega^2 b_{11} \right] \]

\[ \times \exp \left[ -\omega(s_{22} + s_{33})/2 \right] \]  

(91)

and energy

\[ \epsilon_{n_1, \ldots, n_8}^{(2)} := [8 + 2(n_1 + n_2) + 3(n_3 + n_4 + n_5 + n_6 + n_7) + 4n_8] \omega \]  

(92)

where \( p_{n_1, \ldots, n_8}^{(2)} \) are the same polynomials as for the x-z case. After orthogonalisation of those degenerate in energy) they form an ONB with respect to the measure

\[ \langle \Phi_{YZ}'|O|\Phi_{YZ} \rangle = \int d\mu_Y \int d\mu_Z \left[ \Phi_{YZ}' O[Y, Z] \Phi_{YZ} \right]. \]  

(93)

4.2.4 Trigonal form of the two-direction Hamiltonian

The polynomials

\[ p_{n_1, \ldots, n_8}^{(2)}(x_{11}, x_{33}, x_{13}, x_{111}, x_{333}, x_{113}, x_{133}, b_{22}) \]  

(94)

can be easily obtained using the equation

\[ \left[ D_1^{(1)} + D_3^{(1)} + D_{13}^{(2)} \right] p^{(2)} = (\epsilon_{13}/\omega) p^{(2)} \]  

(95)

with the one-direction differential operators

\[ D_1^{(1)} := D^{(1)}[x_{11}, x_{111}], \quad D_3^{(1)} := D^{(1)}[x_{33}, x_{333}], \]  

(96)

and

\[ D_{13}^{(2)} := D^{(2)}[x_{11}, x_{33}, x_{13}, x_{111}, x_{333}, x_{113}, x_{133}, b_{22}], \]  

(97)

with the two-direction operator \( D^{(2)} \) shown explicitly in Appendix D. As for the one-direction operators \( D^{(1)} \), the two-direction operator \( D^{(2)} \) contains an A-power conserving part \( D_0^{(2)} \) and a
the A-power by two lowering operator $D_{-2}^{(2)}$. Hence, the eigenvalue problem (95) is triangular in the space of monomials $M(x_1, \ldots, x_8)$ in $x_1, \ldots, x_8$ and can easily be diagonalised with eigenvalues

$$\epsilon = (8 + 2(n_1 + n_2) + 3(n_3 + n_4 + n_5 + n_6 + n_7) + 4n_8) \omega \equiv (8 + n) \omega$$  \hspace{1cm} (98)$$

The eigenfunctions are polynomials

$$p_{n_1, \ldots, n_8}^{(2)} = N \left( \sum_{m_1, \ldots, m_2}^{m < n} A_{n_1, \ldots, n_8}^{(1)}(m_1, \ldots, m_8) x_1^{m_1} \cdots x_8^{m_8} + x_1^{n_1} \cdots x_8^{n_8} \right)$$  \hspace{1cm} (99)$$

with the leading "defining" monomial $\Pi_{i=1}^{8} x_i^{n_i}$ of maximal order and a "tail" of monomials of decreasing, lower powers with some definite coefficients $A_{n_1, \ldots, n_8}^{(1)}(m_1, \ldots, m_8)$.

4.2.5 Some examples of solutions

Together, we have

$$H_{h.o.} \Phi_{X|YZ} = (\epsilon_X + \epsilon_{YZ}) \Phi_{X|YZ} = \epsilon_{h.o.} \Phi_{X|YZ}, \text{ and cycl. perm.}$$  \hspace{1cm} (100)$$

with the energy eigenvalues

$$\epsilon_{h.o.} = (12 + n_X + n_{YZ}) \omega = (12 + n) \omega \text{ and cycl. perm.}$$  \hspace{1cm} (101)$$

The lowest solutions are $\Phi_{[n]} X|YZ$ $[A] = P_{[n]} X|YZ$ $[A] \exp[-\omega (A_{ai})^2 / 2]$ and cycl. perm., with

$$\epsilon_{h.o.} = 14 \omega : P_{[2]} X|YZ \propto p_{0,0}^{(1)}[X] p_{0,0,0,0,0,0,0,0}^{(2)}[Y, Z] \propto \omega s_{23}, \text{ and cycl. perm.}$$

$$\epsilon_{h.o.} = 15 \omega : P_{[3]} X|YZ \propto p_{0,0}^{(1)}[X] p_{0,0,0,0,0,1,0,0}^{(2)}[Y, Z] \propto \omega^{3/2} s_{233}, \text{ and cycl. perm.}$$

$$\epsilon_{h.o.} = 16 \omega : P_{[4]} X|YZ \propto p_{0,0}^{(1)}[X] p_{0,0,0,0,0,0,0,0}^{(2)}[Y, Z] \propto 6 - 3 \omega(s_{22} + s_{33})/2 + \omega^2 s_{11}^2, \text{ and cycl. perm.}$$

$$\epsilon_{h.o.} = 17 \omega : P_{[5]} X|YZ \propto p_{0,0}^{(1)}[X] p_{0,0,0,0,0,0,0,0}^{(2)}[Y, Z] \propto (-2 + \omega s_{11}^2)/2 \omega s_{23}, \text{ and cycl. perm.}$$

$$\epsilon_{h.o.} = 18 \omega : P_{[6]} X|YZ \propto p_{0,0}^{(1)}[X] p_{0,0,0,0,0,0,0,0}^{(2)}[Y, Z] \propto (-2 + \omega s_{11}^2)/2 \omega^{3/2} s_{233}, \text{ and cycl. perm.}$$

...  \hspace{1cm} (102)$$

By superposition, we obtain from (102) and (67), all spin-0 solutions up to polynomial order $n = 4$, namely $\Phi_{[n]}^{(0)++} [A] = P_{[n]}^{(0)++} [A] \exp[-\omega (A_{ai})^2 / 2]$, with

$$\epsilon_{h.o.}^{(0)++} = 12 \omega : P_{[0]}^{(0)++} [A] \propto 1,$$

$$\epsilon_{h.o.}^{(0)++} = 14 \omega : P_{[2]}^{(0)++} [A] \propto [-12 + \omega s_{[2]}^{(0)}],$$

$$\epsilon_{h.o.}^{(0)++} = 16 \omega : P_{[4]}^{(0)++} [A] \propto \left[ 108 - 18 \omega s_{[2]}^{(0)} + \omega^2 (s_{[2]}^{(0)})^2 \right],$$

$$P_{[4]}^{(0)++} [A] \propto \left[ 72 - 12 \omega s_{[2]}^{(0)} + \omega^2 s_{[2]}^{(0)ij} s_{[2]}^{(0)ij} \right],$$

$$P_{[4]}^{(0)++} [A] \propto \left[ 18 - 3 \omega s_{[2]}^{(0)} + \omega^2 t_{[4]}^{(0)} \right]$$  \hspace{1cm} (103)$$
4.3.1 Inclusion of the components completing (104) up to polynomial order $n = 4$ in $A$.

Furthermore, using the notation $s_{[2]}^{(2)}_{ij} := (s_{[2]}^{(2)}_{ij} - \delta_{ij}s_{[2]}^{(2)}_{kk}/3)$, we can build first spin-2 solutions $\Phi_{ij}^{(2);+}[A] = P_{ij}^{(2);+}[A]\exp[-\omega(A_{ak})^2/2]$ with

$$
\epsilon_{h.o.} = 14\omega : \quad P_{[2]}^{(2);+}[A] \propto \left[ \omega s_{[2]}^{(2)}_{ij} \right],
$$

and first vector solutions $\Phi_{[4]}^{(1);-}[A] = P_{[4]}^{(1);-}[A]\exp[-\omega(A_{ak})^2/2]$, denoting $v_{[3]}^{[i]} := s_{[3]}^{[i]jj}$,

$$
\epsilon_{h.o.} = 15\omega : \quad P_{[3]}^{(1);-}[A] \propto \left[ \omega^{3/2} v_{[3]}^{[i]} \right],
$$

and first vector solutions $\Phi_{[4]}^{(1);-}[A] = P_{[4]}^{(1);-}[A]\exp[-\omega(A_{ak})^2/2]$, denoting $v_{[3]}^{[i]} := s_{[3]}^{[i]jj}$,

$$
\epsilon_{h.o.} = 17\omega : \quad P_{[5]}^{(1);-}[A] \propto \left[ -15\omega^{3/2} v_{[3]}^{[i]} + \omega^{5/2} s_{[2]}^{(0)} v_{[3]}^{[i]} \right].
$$

In order to obtain all solutions, we have to consider the general case of non-separable solutions, discussed in the next paragraph.

### 4.3 Solutions depending on all three directions

Finally we consider the completely non-separable case

$$
H\Phi[X, Y, Z] = \epsilon \Phi[X, Y, Z].
$$

#### 4.3.1 Inclusion of the components $s_{123}$ and $(b_{23}, b_{13}, b_{12})$

Including the component $s_{123}$ of the symmetric 3-tensor $s_{[3]}_{ijk}^{[i]}[A]$, defined in (49),

$$
s_{123} = d_{abc}X_{a}Y_{b}Z_{c},
$$

depending on all three space directions, we obtain the further $\epsilon_{h.o.} = 15\omega$ solution

$$
\Phi_{[3]}[X, Y, Z] \propto \left[ \omega^{3/2} s_{123} \right] \exp \left[ -\omega(s_{11} + s_{22} + s_{33})/2 \right].
$$

From this, (67), (102), we have the spin-3 solution $\Phi_{[3]}^{(3);-}[A] = P_{[3]}^{(3);-}[A]\exp[-\omega(A_{ak})^2/2]$ with

$$
\epsilon_{h.o.} = 15\omega : \quad P_{[3]}^{(3);-}[A] \propto \left[ s_{123}^{(3)} \right],
$$

with the spin-3 part $s_{[3]}_{ijk}^{(3)}$ of the symmetric 3-tensor $s_{[3]}_{ijk}$, see Appendix E.

Including finally the components $b_{23}, b_{13}, b_{12}$ of the symmetric 4-tensor $b_{[4]}_{ij}^{[i]}[A]$, defined in (77),

$$
b_{23} = f_{abc}f_{ade}X_{b}Z_{c}X_{d}Y_{e}, \quad \text{and cycl. perm. } b_{13}, b_{12},
$$

irreducible in A-space, we obtain the triplet of $\epsilon_{h.o.} = 16\omega$ solutions

$$
\Phi_{[4]}[X, Y, Z] \propto \left[ -15\omega^{3/2} b_{23} + \omega^{2} b_{23} \right] \exp \left[ -\omega(s_{11} + s_{22} + s_{33})/2 \right], \quad \text{and cycl. perm.}
$$

and cycl. perm.

Noting also the triplet of $\epsilon_{h.o.} = 16\omega$ solutions

$$
\Phi_{[4]}[X, Y, Z] \propto \left[ \omega s_{23} + \omega^{2} s_{123}s_{13} \right] \exp \left[ -\omega(s_{11} + s_{22} + s_{33})/2 \right], \quad \text{and cycl. perm.}
$$

we obtain all spin-2 solutions $\Phi_{[n]}^{(2);+}[A] = P_{[n]}^{(2);+}[A]\exp[-\omega(A_{ak})^2/2]$ up to $n = 4$ with

$$
\epsilon_{h.o.} = 14\omega : \quad P_{[2]}^{(2);+}[A] \propto \left[ \omega s_{[2]}^{(2)}_{ij} \right],
$$

$$
\epsilon_{h.o.} = 16\omega : \quad P_{[4]}^{(2);+}[A] \propto \left[ -14\omega s_{[2]}^{(2)}_{ij} + \omega^{2} s_{[2]}^{(0)} s_{[2]}^{(2)}_{ij} \right],
$$

$$
\Phi_{[4]}^{(2);+}[A] \propto \left[ \omega^{2} s_{[2]}^{(2)}_{ij} + \omega^{2} b_{[4]}^{(2)}_{ij} \right].
$$

completing (104) up to polynomial order $n = 4$ in $A$. 

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4.3.2 Inclusion of the further irreducible vector $b_{[5]}^{-}\[A]$.

Similarly, we have the vector solutions $\Phi_{i}^{(1)-}[A] = P_{i}^{(1)-}[A] \exp[-\omega (A_{ik})^2/2]$, with

$$
\epsilon_{h.o.} = 15 \omega : \quad P_{[3]}^{(1)-}[A] \propto [\omega^{3/2} v_{[3]}] ,
$$

$$
\epsilon_{h.o.} = 17 \omega : \quad P_{[5]}^{(1)-}[A] \propto [-15 \omega^{3/2} v_{[3]} + \omega^{5/2} s_{[2]}^{(0)} v_{[3]}] ,
$$

$$
P_{[5]}^{(1)-}[A] \propto [-8 \omega^{3/2} v_{[3]} + \omega^{5/2} s_{[2]} v_{[3]}] ,
$$

$$
P_{[5]}^{(1)-}[A] \propto [-9 \omega^{3/2} v_{[3]} + \omega^{5/2} s_{[3]} s_{[2]}] ,
$$

$$
P_{[5]}^{(1)-}[A] \propto [\omega^{5/2} b_{[5]}] ,
$$

completing (105) up to polynomial order $n = 5$ in $A$. The vector $b_{[5]}^{-}\[A]$ in (109) is defined as

$$
b_{[5]}^{-}[A] := d_{abc} B_{[a]}^{hom} B_{[b]}^{hom} A_{[c]} + \frac{1}{4} (2s_{jk}s_{123} - s_{jj}s_{ikk} - s_{kk}s_{ijj}) , \quad (i \neq j \neq k).
$$

It appears e.g. in the tail of $\epsilon_{h.o.} = 19 \omega$ solution $b_{[4]} v_{[3]}$:

$$
P_{[7]}^{(1)-}[A] \propto [- \frac{9}{2} \omega^{3/2} v_{[3]} + \omega^{5/2} (- \frac{5}{4} s_{[2]}^{(0)} v_{[3]} + \frac{3}{2} s_{[2]} v_{[3]} - \frac{1}{4} s_{[3]} s_{[2]} - 5 b_{[5]})]
$$

$$
+ \omega^{7/2} b_{[4]} v_{[3]} ,
$$

and is irreducible, i.e. not representable as the product of the components of $s_{[2]}$, $s_{[3]}$, and $b_{[4]}$. Considering the solutions of similar products of components up to $n = 10$, we find no further elementary tensors beyond the four already included, the 6 components of the symmetric 2-tensor $s_{[2]}$, the 10 components of the symmetric 3-tensor $s_{[3]}$, the 6 components of the further symmetric 2-tensor $b_{[4]}$, and the 3 components of the vector $b_{[5]}$, not considering the axial sector so far.

4.3.3 Inclusion of the axial states

Indeed, using the axial scalar

$$
a_{[3]}^{-}[A] := \frac{1}{6} \epsilon_{ij} f_{abc} A_{[a]} A_{[b]} A_{[c]} = f_{abc} X_{a} Y_{b} Z_{c} ,
$$

we obtain the $\epsilon = 15 \omega$ solution

$$
\Phi_{[3]}^{+}[A] := \frac{\omega^{9/2}}{\sqrt{6\pi}} [\omega^{3/2} a_{[3]}] \exp [-\omega (s_{11} + s_{22} + s_{33}) / 2] .
$$

Now, considering the tail of the $a_{[3]} v_{[3]}$ solution with $n = 6$,

$$
\Phi_{[6]}^{+}[A] \propto [- \frac{5}{2} \omega^{2} a_{[4]} + \omega^{3} a_{[3]} v_{[3]}] \exp [-\omega (A_{ik})^2 / 2] .
$$

we find that we have to inclued also the axial $n = 4$ vector the axial vector

$$
a_{[4]}^{-}[A] := d_{abc} B_{[a]}^{hom} A_{[b]} A_{[c]} , \quad (i = 1, 2, 3)
$$

which is an irreducible polynomial $\epsilon = 16 \omega$ solution of 4-th order in $A$. Furthermore, considering the tails of the $a_{[3]} b_{[4]}$ solutions with $n = 7$,

$$
\Phi_{[7]}^{(0)-}[A] \propto [30 \omega^{3/2} a_{[3]} + \omega^{5/2} (-4 a_{[3]} s_{[2]} + 3 a_{[5]}^{(0)}) + \omega^{7/2} a_{[3]} b_{[4]}^{(0)}] \exp [-\omega (A_{ik})^2 / 2] ,
$$

$$
\Phi_{[7]}^{(2)+}[A] \propto [\omega^{5/2} (2 a_{[3]} s_{[2]} + \frac{3}{2} a_{[5]}^{(2)}) + \omega^{7/2} a_{[3]} b_{[4]}^{(2)}] \exp [-\omega (A_{ik})^2 / 2] .
$$
Table 1: Definition of the complete set of eight elementary \(SU(3)\)-invariant spatial tensors on gauge-reduced \(A\)-space. The color indices \(a, b, c\) are summed over, but the spatial indices \(i, j, k\) are not, in all lines of the table. Note that for the case \(i = j\) in the seventh line one can choose any of the two \(k \neq i, j\), both give the same \(a(5)\). The second column shows the degree \([n]\) of the tensor (as a polynomial in \(A\)). The last column shows the spin components into which the tensor can be decomposed.

| Type of Tensor | Degree \([n]\) | Degree of Spin \([\lambda]\) |
|----------------|-------------|-------------------------------|
| Sym. 2 Tensor  | \(2\)       | \(0^+, 2^+\)                  |
| Sym. 3 Tensor  | \(3\)       | \(1^-, 3^-\)                  |
| Sym. 2 Tensor  | \(4\)       | \(0^+, 2^+\)                  |
| Vector         | \(5\)       | \(1^-\)                       |
| Axial Scalar   | \(3\)       | \(0^-\)                       |
| Axial Vector   | \(4\)       | \(1^-\)                       |
| Sym. 2 Tensor  | \(5\)       | \(0^+, 2^+\)                  |
| Sym. 3 Tensor  | \(6\)       | \(1^-, 3^-\)                  |

we find that we have also to include the symmetric axial 2-tensor

\[
a^{+-}_{[5]ij}[A] := d_{abc} B^{\hom}_{ak} A_{bk}(d_{cde} A_{di} A_{ej}) , \quad (i \leq j \land k \neq i, j)
\]  

(118)

into the list of irreducible polynomials. Finally, considering the tails of the \(a_{[3]}b_{[5]}\) and \(a_{[4]}b_{[4]}\) solutions with \(n = 8\),

\[
\Phi^{+}_{[8]i}[A] \propto \left[ \omega^2 a_{[4]i} + \omega^3 \left( -\frac{1}{4}s_{[2]}^{(0)} a_{[4]i} + \frac{1}{4}s_{[2]}^{(0)} a_{[4]j} - \frac{1}{2}a_{[6]j}^{(1)} \right) + \omega^4 a_{[3]}b_{[5]} \right] \exp \left[ -\omega (A_{ak})^2/2 \right].
\]

(119)

\[
\Phi^{[3]+}_{[8]ijk}[A] \propto \left[ \omega^3 \left( \frac{1}{2}a_{[4]i}^{(3)} s_{[3]}^{(3)ij} + \frac{7}{4} \left( a_{[4]i} s_{[2]ij}^{(3)} - a_{[4]ij}^{(3)} \right) S - a_{[6]ij}^{(3)} \right) + \omega^4 \left(a_{[4]}b_{[4]}ij\right)^{(3)} \right] \exp \left[ -\omega (A_{ak})^2/2 \right].
\]

(120)

that we have, last but not least, to include also the symmetric axial 3-tensor

\[
a^{+-}_{[6]ijk}[A] := d_{abc} B^{\hom}_{ai} B^{\hom}_{bj} B^{\hom}_{ck} , \quad (i \leq j \leq k)
\]  

(121)

containing spin-1 and spin-3 parts, into the list of irreducible polynomials. In addition to the four irreducible tensors \(s_{[2]}, s_{[3]}, b_{[4]},\) and \(b_{[5]}\), the axial scalar \(a_{[3]}\), the 3 component of the axial vector \(a_{[4]}\), the 6 components of the symmetric axial 2-tensor \(a_{[5]}\), containing another axial scalar and an axial spin-2 part, and finally the symmetric axial 3-tensor \(a_{[6]}\) have to be included into the list of all irreducible polynomials, in terms of which all polynomial solutions can be represented. The complete list is shown in Table 1. Their transformation properties under spatial rotations are summarised in Appendix E. Considering the tails of all solutions from products of components of these eight irreducible symmetric tensors, up to maximal order \(n = 12\), no further irreducible polynomials have been found.

4.3.4 Triangular from of the complete Hamiltonian

We would like to point out here, that as for the 1-dimensional and the 2-dimensional cases, the polynomial solutions

\[
P_{[n]} \left[ \omega s_{[2]}, \omega^{3/2} s_{[3]}, \omega^2 b_{[4]}, \omega^{5/2} b_{[5]}, \omega^3 a_{[3]}, \omega^2 a_{[4]}, \omega^5 a_{[5]}, \omega^3 a_{[6]} \right]
\]

(122)

of the general 3-dimensional case, seen as polynomial in the 45 components of the 8 irreducible symmetrical tensors can be also obtained using a trigonal differential equation

\[
\left[D^{(1)}_1 + D^{(1)}_2 + D^{(1)}_3 + D^{(2)}_{23} + D^{(2)}_{13} + D^{(2)}_{12} + D^{(3)}_{123}\right] P_{[n]} = (12 + n) P_{[n]}
\]

(123)
with the three-direction-differential operator $D^{(3)}_{123}$. As the 1-dim. $D^{(1)} = D^{(1)}_0 + D^{(1)}_{-2}$ given in (62) and the 2-dim. $D^{(2)} = D^{(2)}_0 + D^{(2)}_{-2}$ given in (96) and shown explicitly in Appendix D, the 3-dim. Differential operator $D^{(3)} = D^{(3)}_0 + D^{(3)}_{-2}$ has a diagonal and a by an power-of-2-lowering part and hence is diagonalisable analytically. The explicit expression of $D^{(3)}_{123}$ takes several pages and is therefore not show here explicitly.

4.3.5 Comment on reducibility of polynomials in original constrained and reduced spaces

As discussed in the work of Dittner [25], for $SU(3)$ in original constrained functional space $\{V^\alpha_i, i = 1, .., D\}$ there are two irreducible $SU(3)$-invariant tensors in $D = 1$ spatial dimension,

$$V^\alpha_1V^\alpha_1 \quad \text{and} \quad d_{abc}V^a_1V^b_1V^c_1 , \quad (D = 1) \quad (124)$$

nine irreducible $SU(3)$-invariant tensors for $D = 2$,

$$V^\alpha_1V^\alpha_1, \quad d_{abc}V^a_1V^b_1V^c_1, \quad V^\alpha_2V^\alpha_2, \quad d_{abc}V^a_2V^b_2V^c_2, \quad V^\alpha_1V^\alpha_2, \quad d_{abc}V^a_1V^b_1V^c_2, \quad d_{abc}V^a_1V^b_2V^c_2, \quad C^a[V]C^a[V] , \quad d_{abc}C^a[V]C^b[V]C^c[V] , \quad (D = 2) \quad (125)$$

where $C^a[V] := d_{abc}V^b_1V^c_2$. The nineth is independent of the first 8, in the sense, that it cannot be represented as a sum of products of them. It is, however, not primitive because it is related to them via outer products. Furthermore, Dittner proved that for $D \geq 3$ there are maximally 35 independent irreducible $SU(3)$-invariant ("primitive") tensors of maximally 6th rank. For tensors of rank higher than 6, the number of constraints due to outer products exceeds the number of irreducible tensors.

In reduced functional space $\{A^a_i, i = 1, .., D\}$, considered here, we have also two $SU(3)$-invariant irreducible tensors in $D = 1$ spatial dimension,

$$s_{11} = A^a_1A^a_1 \quad \text{and} \quad s_{111} = d_{abc}A^a_1A^b_1A^c_1 , \quad (D = 1) \quad (126)$$

For reduced functional space in $D = 2$ we have different to the constrained case, only 8 $SU(3)$-invariant irreducible tensors $^1$,

$$s_{11} , s_{111} , s_{22} , s_{222} , s_{12} , s_{112} , s_{122} , \quad \text{and} \quad B^2_3 , \quad (D = 2) \quad (127)$$

In contrast to the case of original constrained functional space, the last 9th tensor, which is of rank 6, is reducible in reduced space

$$d_{abc}C^a[A]C^b[A]C^c[A] = \frac{1}{18}s^3_{12} - \frac{1}{6}s_{12}s_{11}s_{22} - \frac{1}{12}s_{111}s_{222} + \frac{3}{4}s_{112}s_{122} + \frac{1}{6}s_{12}B^2_3 , \quad (D = 2) \quad (128)$$

For the case of 3-dimensional reduced space here we find eight irreducible tensors of maximally 6th polynomial order in $A$. Since the $SU(3)$ gauge is reduced completely, outer products in color space are absent in the reduced approach, in contrast to the constrained approach.

4.4 All solutions of the corresponding harmonic oscillator problem

As demonstrated in the preceding paragraphs, the corresponding harmonic oscillator problem

$$H_{h.o.}(A, P) \Phi^{(j)PC}_{i,M} = \epsilon^{(j)PC}[\omega] \Phi^{(j)PC}_{i,M} , \quad (129)$$

\(^1\text{the eighth is related to the eighth in (125) via the identity} f_{abc}f_{ade} = (2/3)(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}) + d_{abd}d_{ace} - d_{abe}d_{acd}\)
with the same measure as for the case of Yang-Mills Quantum mechanics (27),

$$\langle \Phi_1|O|\Phi_2\rangle = \int d\mu_X \int d\mu_Y \int d\mu_Z \Phi_1^\dagger O \Phi_2 ,$$  

(130)

only replacing the chromomagnetic potential by the 16-dim. harmonic oscillator potential with parameter $\omega$, see (40), turns out to be trigonal in the space of the monomial functionals

$$M[\omega s_{[2]},\omega^{3/2}s_{[3]},\omega^{2}b_{[4]},\omega^{5/2}b_{[5]},\omega^{3/2}a_{[3]},\omega^{2}a_{[4]},\omega^{5/2}a_{[5]},\omega^{3}a_{[6]}] \exp[-\frac{1}{2}\omega (A_{ai})^2] ,$$  

(131)

and hence integrable. The $M$ are monomials in the 45 components of eight elementary $SU(3)$-invariant spatial tensors in reduced A-space shown in Table 1. Note $(A_{ai})^2 \equiv (s_{11} + s_{22} + s_{33})$.

Organising the monomial functionals according to the degree $n$ (as a polynomial in the $A$) and the conserved quantum numbers $J,M,P,C$ and applying a Gram-Schmidt orthogonalisation with respect to the measure, we obtain all exact solutions

$$\Phi^{(J)PC}_{[n]i,M}[A] = P^{(J)PC}_{[n]i,M}[\omega s_{[2]},\omega^{3/2}s_{[3]},\omega^{2}b_{[4]},\omega^{5/2}b_{[5]},\omega^{3/2}a_{[3]},\omega^{2}a_{[4]},\omega^{5/2}a_{[5]},\omega^{3}a_{[6]}] \times \exp[-\omega (A_{ai})^2/2] ,$$  

of the corresponding harmonic oscillator problem with energies

$$\epsilon_{h,o}^{(J)PC} = (12 + n) \omega ,$$

where $n$ is the degree of $P_{[n]}$ as a polynom in the $A$.

### 4.4.1 Lowest order monomials for all symmetry sectors $J^{PC}$

The lowest order monomials for each symmetry sector $J^{PC}$ are listed in Appendix F, Tables 6a-6d. In the upper part of Table 6a one finds the monomials for the case $J^{++}$ for even $J = 0, 2, 4, ..$, the first $0^{++}_{[n]}$ and $2^{++}_{[n]}$ monomials up to $n = 4$ can be read of from Equ.(103) and (108) in the last paragraphs. Similarly, the upper half of Table 6b lists the lowest monomials for the case $J^{--}$ for odd $J = 1, 3, 5, ..$, the first $1^{--}_{[n]}$ monomials up to $n = 5$ and $3^{--}_{[n]}$ monomials up to $n = 3$ can be read of from Equ.(109) and (107) respectively.

Next we note that for the sector $2^{++}_{[6]}$ in the upper part of table 6a, there appears the symmetric part $(b_{[4]ik} s_{[2]kj})^{(2)}_{\text{sym}}$. The corresponding antisymmetric part $(\epsilon_{ist} b_{[4]js} s_{[2]jt})^{(2)}_{\text{sym}}$ also of order $n = 6$ describes the lowest spin-1 monomial $1^{++}_{[6]}$ shown in the lower part of Table 6a listing the case $J^{++}$ for odd $J = 1, 3, 5, ..$. Similarly, for the sector $3^{--}_{[5]}$ in the upper part of Table 6b, there appears the symmetric parts $(s_{[3]ijk} s_{[2]pk})^{(3)}_{\text{sym}}$ and $(v_{[3]i} s_{[2]jk})^{(3)}_{\text{sym}}$, where the symmetrisation is over the open indices $i, j, k$. The corresponding antisymmetric parts $(\epsilon_{ist} s_{[3]jsk} s_{[2]kt})^{(2)}_{\text{sym}}$ and $(\epsilon_{ist} v_{[3]js} s_{[2]jt})^{(2)}_{\text{sym}}$, also of order $n = 5$, describes the lowest spin-2 monomials $2^{--}_{[5]}$ appearing in the lower part of Table 6b listing $J^{--}$ for even $J = 0, 2, 4, ..$

In the Tables 6a and 6b the number of axial components in a monomial has to be even, e.g. in the case of the $0^{++}_{[6]}$ state $(a_{[3]}^{+} a_{[3]}^{+})$. The upper and lower parts of Tables 6c and 6d list the monomials for the case of $J^{+}$ and $J^{++}$ where an odd number of axial components appear in the monomials.

### 4.4.2 Energy spectrum of the harmonic oscillator problem $H_{h,o}$

Fig.1 shows the lowest energy eigenvalue of the harmonic oscillator problem $H_{h,o}$ in each symmetry sector $J^{PC}$ for the (maximal) polynomial order of 10/11 nodes for even/odd parity.
Figure 1: The lowest energy eigenvalue of the corresponding harmonic oscillator problem $H_{h.o.}$ in each symmetry sector $J^{PC}$ for the (maximal) polynomial order of 10/11 nodes for even/odd parity.
The spectrum reflects the gauge invariant constructability of the monomials from the components of the irreducible symmetric tensors for a given symmetry sector. Hence in sectors where the monomials are build from antisymmetric parts of products of tensors, lie relatively high in energy. For example, the lowest $2^{-+}$ state appears only at $n = 5$ and therefore energy $\epsilon = 19 \omega$, the lowest $1^{++}$ state only at $n = 6$ and therefore energy $\epsilon = 20 \omega$. Similarly, their axial colleagues, the lowest $2^{-+}$ state appears only at $n = 6$ and therefore energy $\epsilon = 20 \omega$, the lowest $1^{-+}$ states only at $n = 7$ and therefore energy $\epsilon = 21 \omega$.

Very important for the use of a orthogonal basis for the case of $SU(3)$ Yang-Mills Quantum Mechanics is the analytical construction of the eigensystem of the corresponding harmonic oscillator Hamiltonian $H_{h.o.}$. This is most effectively done using Gram-Schmidt orthogonalisation described in the next paragraph.

### 4.4.3 Eigenstates of $H_{h.o.}$ from Gram-Schmidt orthogonalisation

Enumerate all possible monomials for given $JPC$ in increasing order $n$ and multiplicity $m$

$$M^{(J)PC}_r[A] := \omega^{n/2} M^{(J)PC}_{n|m} \left( s_2[A], s_3[A], b_4[A], b_5[A], a_3[A], a_4[A], a_5[A], a_6[A] \right) ,$$

(132)

e.g. for the $0^{++}$ sector

$$M^{(0)++}_1 = M^{(0)++}_0 \equiv 1 ,$$

$$M^{(0)++}_2 = \omega M^{(0)++}_{2} \equiv \omega s_{2ii} ,$$

$$M^{(0)++}_3 = \omega^2 M^{(0)++}_{41} \equiv \omega^2 s_{2ii}s_{2jj} ,$$

$$M^{(0)++}_4 = \omega^2 M^{(0)++}_{42} \equiv \omega^2 s_{2ij}s_{2jj} ,$$

$$M^{(0)++}_5 = \omega^2 M^{(0)++}_{43} \equiv \omega^2 b_{4ij} .$$

...

Consider now the Gram matrix

$$G^{(J)PC}_{rs} := \langle \langle M^{(J)PC}_r M^{(J)PC}_s \rangle \rangle_A ,$$

(133)

with the measure

$$\langle \langle \rangle \rangle_A := \int_0^{2\pi} d\psi \cos^2[3 \psi] \int_0^\infty dr r^7 \exp[-\omega r^2] \left[ \prod_{a=1}^{8} \int_{-\infty}^{\infty} dZ_a \exp[-\omega Z_a^2] \right]$$

$$\left[ \prod_{a=1}^{1,2,3,8} \int_{-\infty}^{\infty} dY_a \cdot \exp[-\omega Y_a^2] \right] \int_0^\infty dY_4Y_4 \exp[-\omega Y_4^2] \int_0^\infty dY_6Y_6 \exp[-\omega Y_6^2] .$$

Here it is very useful that the integration completely factorises into simple 1-dimensional integrations, which is due to the choice of the flux-tube gauge. Since the functionals to be integrated, are polynomials, the integrations can be carried out as replacements.

Gram-Schmidt orthogonalisation corresponds to finding a lower-triangular matrix $T$ such that

$$TG^{(J)PC}T^T = 1 ,$$

obtaining the orthogonal polynomials

$$P^{(J)PC}_n[A] := \sum_{k=1}^{n} T_{nk} M^{(J)PC}_k[A] .$$

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Then the functionals
\[ \Phi^{(j)PC}_n[A, \omega] \equiv P^{(j)PC}_n[A] \exp \left[ -\frac{1}{2} \omega (A_{ai})^2 \right] \]
form an ONB of solutions of the corresponding harmonic oscillator problem. Note that during the orthogonalisation procedure, the linear dependent states appear as zero-eigenvalues of the Gram matrix \( G^{(j)PC} \) Equ.(133) and can therefore systematically be removed.

The matrix elements of the harmonic potential can then be easily obtained using
\[
\langle \Phi^{(j)PC}_m[A, \omega] (A_{ai})^2 \Phi^{(j)PC}_n[A, \omega] \rangle = T_{mr} T_{ns} \langle \langle M^{(j)PC}_r (A_{ai})^2 M^{(j)PC}_s \rangle \rangle_A = - \frac{1}{2} \omega^2 \frac{\partial}{\partial \omega} (\ldots) .
\]

Finally, the magnetic matrix elements can be calculated using
\[
\langle \Phi^{(j)PC}_m[A, \omega] (B^2_{ai}[A]) \Phi^{(j)PC}_n[A, \omega] \rangle = T_{mr} T_{ns} \langle \langle M^{(j)PC}_r B^2_{ai}[A] M^{(j)PC}_s \rangle \rangle_A .
\]

These will be, as shown in the next section, the main steps to obtain the low energy eigensystem of SU(3) YM QM.

5 Low-energy spectrum of SU(3) YM QM

5.1 The energy spectrum of SU(3) YM QM from the corresponding harmonic oscillator problem

Consider the basis of energy eigenstates of the corresponding unconstrained harmonic oscillator Schrödinger equation orthonormal with respect to the Yang-Mills measure
\[
H_{h.o.} \Phi_n[A, \omega] \equiv \left[ T_{\textrm{kin}} + \frac{1}{2} \omega^2 A_{ai}^2 \right] \Phi_n[A, \omega] = \epsilon_n^{h.o} \Phi_n[A, \omega] .
\]

Then the matrix elements of the unconstrained Yang-Mills Hamiltonian are given as
\[
\mathcal{M}_{mn} = \langle \Phi^\dagger_m[A, \omega] \left( T_{\textrm{kin}} + \frac{1}{2} B^2_{ai}[A] \right) \Phi_n[A, \omega] \rangle_A = \delta_{nm} \epsilon_n^{h.o} - \langle \Phi^\dagger_m[A, \omega] \left( \frac{1}{2} \omega^2 A_{ai}^2 \right) \Phi_n[A, \omega] \rangle_A + \frac{1}{2} \langle \Phi^\dagger_m[A, \omega] \left( B^2_{ai}[A] \right) \Phi_n[A, \omega] \rangle_A
\]
since the kinetic terms \( T_{\textrm{kin}} \) are the same for the Yang-Mills and the corresponding harmonic oscillator problem. These are calculated analytically using formulae (134) and (135). We treat \( \omega \) as a variational parameter, which in each symmetry sector can be choose to minimize the lowest eigenvalue of the matrix \( \mathcal{M} \), which is easily diagonalised numerically with high accuracy. The most time consuming part is the calculation of the expectation values of the chromomagnetic potential according to formula (135), it takes about a week on 36-kernel micro-supercomputers for truncations at polynomial order 10 or 11 for higher spins, and 2-3 months for truncation at polynomial order 12, which is work in progress.

5.2 Results

We have calculated the low energy spectrum of SU(3) Yang-Mills Quantum Mechanics for all symmetry sectors \( J^{PC} \) up to spin \( J = 11 \), including polynomials up to 10th order (10-node resolution) for even parity and up to 11th order (11-node resolution) for odd parity.
5.2.1 Lowest level in each symmetry sector $J^{PC}$

Fig. 2 shows the lowest energy eigenvalue in each symmetry sector $J^{PC}$ as a function of the polynomial order of truncation up to 10 for even parity and up to 11 for odd parity. The values are listed in Tables 2-5. Fig. 3 shows the lowest energy eigenvalue of SU3 YM-QM in each symmetry sector $J^{PC}$ including all states up to polynomial order of 10 for even parity and up to polynomial order 11 for odd parity.

The spectrum is purely discrete in accordance with the proof of Simon [3] and the groundstate energy is obtained to be $\epsilon_0^{++} = 12.5868$ (when truncating at 12 nodes). The lowest states correspond to the 16 components of $0^{++}, 2^{++}$ and $1^{--}, 3^{--}$ which show good convergence as a function of increasing polynomial order. These 16 states correspond to the spins of the elementary dynamical variables of the "symmetric gauge" [18]-[20].

The higher the polynomial order of truncation, the less the dependence on the variational parameter $\omega$. At 10-th or 11th order of truncation the results for the spectrum is practically independent of the arbitrarily introduced parameter $\omega$.

Also good convergence as a function of increasing polynomial order show the states $4^{++}, 0^{-+}, 5^{--}, 4^{--}, 2^{--}, 1^{+-}, 2^{++}$.

For higher states one can use the flow of the energy values with increasing resolution, indicating that at higher energies the energy levels might be almost equidistant in each symmetry sector as for the corresponding harmonic oscillator spectrum Fig.1.

Deviations of the spectra of SU(3) Yang-Mills QM Fig. 3 with spectra of the corresponding harmonic oscillator Fig.1 show the effect of the chromomagnetic potential. All aspects and constraints due to gauge invariance are already included in the corresponding harmonic oscillator spectrum.

5.2.2 The lowest few levels in each symmetry-sector $J^{PC}$

Fig. 4 shows the lowest few energy eigenvalues of SU3 YM-QM in each symmetry sector $J^{PC}$ as a function of the polynomial order of truncation up to 10/11 nodes for even/odd parity.
Figure 3: The lowest energy eigenvalue of SU3 YM-QM in each symmetry sector $J^{PC}$ including all states up to polynomial order of 10 for even parity and up to polynomial order 11 for odd parity.
Consideration of the lowest few energy eigenvalues of SU3 YM-QM in each symmetry sector \( J^{PC} \) shows, that higher excitations e.g. the first excited state of spin-2 are close to degenerate to the lowest state of spin-4 similar to the integrable harmonic oscillator problem.

In order to even improve the convergence of the lowest states and to get convergent results also at higher energy, polynomials in orders of 14/15 are necessary. Hence we need very effective computer algorithms to cope with very large polynomials. This will be subject of future work.

### 5.2.3 Comparison with the results of Weisz and Ziemann in the constrained approach

The results found in the present work are in good agreement with the results of Weisz and Ziemann [11], shown in the last column denoted by ”WZ” in Tables 2-5, using the constrained Hamiltonian approach. The agreement is excellent in the 0\(^{++}\) and 2\(^{++}\), where their results are already quite accurate, although their error estimates turned out to be too optimistic. Our results are much more accurate in other sectors considered by them wit only few trial states, e.g. in 1\(^{--}\) and 3\(^{--}\) sectors, and we give quite accurate ”new results” for the states not considered by them, as e.g. 2\(^{--}\), 4\(^{--}\), 5\(^{--}\), 3\(^{++}\).

### 5.2.4 Comparison with the results of Lattice QCD

Comparing the results of the low energy spectrum of SU(3) Yang-Mills Quantum Mechanics with those obtained in Lattice QCD using asymmetric lattices, eg. by Morningstar and Peardon [22] and Chen et al. [23], using dimensionless results obtained by dividing by the lowest (spin-0) mass, show reasonable overall agreement for the 0\(^{++}\), 2\(^{++}\), 3\(^{++}\), 0\(^{--}\), 2\(^{=}\), and 1\(^{--}\), 3\(^{--}\), 2\(^{+}\), 0\(^{+}\) glueball states considered by them. Their 1\(^{--}\), 2\(^{--}\), 3\(^{--}\) results, however, are much higher in energy then those of Yang-Mills Quantum Mechanics.
6 Conclusions

It has been shown in this work, that an unconstrained Hamiltonian formulation of SU(3) Yang-Mills Quantum Mechanics of spatially constant fields, which corresponds to the lowest order in an strong coupling expansion of SU(3) YM theory, can be carried out in a rather practical way using the flux-tube gauge. The corresponding Faddeev-Popov operator is simple but non-trivial. The drawback, that the reduced gauge fields in the fluxtube-gauge themselves are not tensors under spatial rotations, as was the case for the symmetric gauge on cost of a very complicated FP-operator, can be circumvened by forming certain irreducible polynomials of the reduced $A$, eight symmetric tensors, four of which are axial, which have definite eigenvalues of J,P, and C.

The spectrum of the Hamiltonian of SU(3) Yang-Mills QM of spatially constant fields can be determined in an effective way using the exact solutions of the corresponding harmonic oscillator problem only replacing the chromomagnetic potential by a 16-dimensional harmonic oscillator potential parametrised by one parameter $\omega$, but leaving the non-trivial FP-operator unchanged. This model has been demonstrated in this work to be integrable. The eigensystem turned out to be orthogonal polynomials of the 45 components of the eighth irreducible tensors, four of them axial, multiplied by a 16-dimensional Gaussian. Its energy spectrum depends only on the polynomial order with respect to the $A$ of the eigenstate, and is highly degenerate. Using the Gram-Schmidt orthogonalisation we could find the eigensystem of the corresponding harmonic oscillator Hamiltonian up to 10th polynomial order for even, and 11th order for odd parity states. This eigensystem could then be used to find the corresponding eigensystem of SU(3) Yang-Mills Quantum Mechanics with relatively high accuracy for the low lying states, and the dependence on the variational parameter $\omega$ became very small. Very helpful for the analytical calculations is here the fact, that using the flux-tube gauge, the integrations in functional space for calculating matrix elements completely factorise, and since the eigenstates are polynomials multiplied by a 16-dim Gaussian, the integrations can be substituted by replacements.

The results are in good agreement with the results of Weisz and Zieman (1986) using the constrained Hamiltonian approach in the $0^{++}$ and $2^{++}$ sectors, much more accurate values in other sectors considered by them, e.g. in $1^{--}$ and $3^{--}$ sectors, and give quite accurate “new results” for the states not considered by them, as e.g. $2^{--}, 3^{++}$.

By considering the corresponding harmonic oscillator problem, which includes already all effects of gauge invariance, as an intermediate step, the comparison of its energy spectrum with the final Yang-Mills spectrum shows the effect of the chromomagnetic potential. In order to further investigate the emerging structures in the spectrum, even higher accuracy results and polynomials up to order 14/15 and hence even more effective programs and algorithms are necessary. An accurate knowledge of the eigensystem of SU(3) Yang-Mills QM is also necessary for strong coupling perturbation theory in small $\lambda = g^{-2/3}$ proposed in earlier work [21] analogous to the SU(2) approach [15]. Analogous to the case of SU(2) Dirac-Yang-Mills QM [16], the calculation can also be generalised to the inclusion of quarks to study the masses of mesons.

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A  Explicit form of the inverse of the FP-operator in the flux-tube gauge

The inverse $\gamma^{-1}$ of the homogeneous part of the Faddeev-Popov operator exists in the regions of non-vanishing determinant (26), and its non-vanishing matrix elements are rather simple,

\[
(\gamma^{-1})_{12} = - (\gamma^{-1})_{21} = \frac{1}{r \cos[\psi]}, \quad (\gamma^{-1})_{32} = \frac{1}{2 r \cos[\psi]} (Y_4/Y_6 - Y_6/Y_4), \quad (\gamma^{-1})_{82} = -\frac{1}{2 \sqrt{3} r \cos[\psi]} (Y_4/Y_6 + Y_6/Y_4),
\]

\[
(\gamma^{-1})_{45} = - (\gamma^{-1})_{54} = -\frac{1}{r \cos[\psi + 2\pi/3]}, \quad (\gamma^{-1})_{34} = -\sqrt{3} (\gamma^{-1})_{84} = -\frac{1}{2 \sqrt{3} r \cos[\psi + 2\pi/3]} (Y_2/Y_6),
\]

\[
(\gamma^{-1})_{35} = \frac{1}{2 \sqrt{3} r \cos[\psi + 2\pi/3]} (2Y_+/Y_4 + Y_1/Y_6), \quad (\gamma^{-1})_{85} = \frac{1}{2 \sqrt{3} r \cos[\psi + 2\pi/3]} (2Y_+/Y_4 - Y_1/Y_6),
\]

\[
(\gamma^{-1})_{67} = - (\gamma^{-1})_{76} = \frac{1}{r \cos[\psi + 4\pi/3]}, \quad (\gamma^{-1})_{36} = \sqrt{3} (\gamma^{-1})_{86} = \frac{1}{2 \sqrt{3} r \cos[\psi + 4\pi/3]} (Y_2/Y_4),
\]

\[
(\gamma^{-1})_{37} = \frac{1}{2 \sqrt{3} r \cos[\psi + 4\pi/3]} (Y_1/Y_4 - 2Y_-/Y_6), \quad (\gamma^{-1})_{87} = \frac{1}{2 \sqrt{3} r \cos[\psi + 4\pi/3]} (Y_1/Y_4 + 2Y_-/Y_6),
\]

\[
(\gamma^{-1})_{33} = \sqrt{3} (\gamma^{-1})_{83} = Y_4^{-1}, \quad (\gamma^{-1})_{38} = -\sqrt{3} (\gamma^{-1})_{88} = Y_6^{-1},
\]

 grouped into those proportional to $\cos^{-1}[\psi]$, $\cos^{-1}[\psi + 2\pi/3]$, and $\cos^{-1}[\psi + 4\pi/3]$, and those independent of $r$ and $\psi$. Such a "Weyl-decomposition" leads to a considerable simplification of the non-local potential.

B  Explicit forms of the $I_{m}^{YZ}$ ($m = 1, 2, 4, 5, 6, 7$) and $I_{\pm}^{YZ}$

The explicit expressions of the interactions $I_{m}^{YZ}$ and $I_{\pm}^{YZ}$ in (39) read

\[
I_{m}^{YZ} := \left( \frac{1}{Y_4 Y_6} \tilde{T}^Y_m Y_4 Y_6 + \tilde{T}^Z_m \right) (\tilde{T}^Y_m + \tilde{T}^Z_m)
\]

\[
I_{\pm}^{YZ} := \left[ \left( T^Z_3 \pm \frac{1}{\sqrt{3}} T^Z_8 \right) + 2T^Y_5 \right] \left( T^Z_3 \pm \frac{1}{\sqrt{3}} T^Z_8 \right)
\]

with the shifted non-Hermitean $\tilde{T}^Y_a$ and Hermitean $\tilde{T}^Z_a$, defined as

\[
\tilde{T}^Y_1 := T^Y_1 - \frac{1}{2} \left( \frac{Y_6}{Y_4} - \frac{Y_4}{Y_6} \right) T^Y_3, \quad \tilde{T}^Z_1 := T^Z_1 - \frac{1}{2Y_4} \left( T^Z_3 + \frac{1}{\sqrt{3}} T^Z_8 \right) + \frac{Y_4}{2Y_6} \left( T^Z_3 - \frac{1}{\sqrt{3}} T^Z_8 \right),
\]

\[
\tilde{T}^Y_2 := T^Y_2, \quad \tilde{T}^Z_2 := T^Z_2,
\]

\[
\tilde{T}^Y_4 := T^Y_4 - \left( \frac{Y_1}{2Y_6} + \frac{Y_+}{Y_4} \right) T^Y_3, \quad \tilde{T}^Z_4 := T^Z_4 - \frac{1}{2Y_6} \left( T^Z_3 - \frac{1}{\sqrt{3}} T^Z_8 \right) - \frac{Y_+}{Y_4} \left( T^Z_3 + \frac{1}{\sqrt{3}} T^Z_8 \right),
\]

\[
\tilde{T}^Y_5 := T^Y_5 - \frac{Y_2}{2Y_6} T^Y_3, \quad \tilde{T}^Z_5 := T^Z_5 - \frac{Y_2}{2Y_6} \left( T^Z_3 - \frac{1}{\sqrt{3}} T^Z_8 \right),
\]

\[
\tilde{T}^Y_6 := T^Y_6 + \left( \frac{Y_1}{2Y_4} - \frac{Y_-}{Y_6} \right) T^Y_3, \quad \tilde{T}^Z_6 := T^Z_6 + \frac{Y_1}{2Y_4} \left( T^Z_3 + \frac{1}{\sqrt{3}} T^Z_8 \right) - \frac{Y_-}{Y_6} \left( T^Z_3 - \frac{1}{\sqrt{3}} T^Z_8 \right),
\]

\[
\tilde{T}^Y_7 := T^Y_7 - \frac{Y_2}{2Y_4} T^Y_3, \quad \tilde{T}^Z_7 := T^Z_7 - \frac{Y_2}{2Y_4} \left( T^Z_3 + \frac{1}{\sqrt{3}} T^Z_8 \right).
\]
Table 2: The lowest few energy eigenvalues of SU3 YM-QM in the symmetry sector $J^{++}$ as a function of the polynomial order of truncation up to 10 nodes. In brackets the number of states.

C  The case of singular solutions of the X-equation (43)

The X-equation (43) is solved also by the singular functionals

$$
\Phi^{\text{sing}}_{n_1,n_2}(r, \psi) := \frac{2\omega^2}{\sqrt{6}\pi^{3/4}(\omega r^2)^{3/4}} \sqrt{1 + \sin[3\psi]} \, p^{\text{sing}}_{n_1,n_2}(\omega r^2, \sin[3\psi]) e^{-\omega r^2/2}
$$

with the energy eigenvalues

$$
\varepsilon^{\text{sing}}_{\nu,\mu} = (5/2 + 2n_1 + 3n_2) \omega = (5/2 + n) \omega
$$

with the lowest polynomials $p^{\text{sing}}_{0,0}(x, y) = 1$, $p^{\text{sing}}_{1,0}(x, y) = \frac{1}{\sqrt{10}}(5 - 2x)$, .... Due to the $\cos^2[3\psi]$ factor in the measure (17), absent in the Y- and Z-equations, they are nevertheless finite and normalisable. Hence, when including the regular solutions in other two directions, y and z, we obtain solutions, breaking rotational invariance in one direction, the x-direction, leaving only a cylindrical symmetry.
Table 3: The lowest few energy eigenvalues of SU3 YM-QM in the symmetry sector $J^{--}$ as a function of the polynomial order of truncation up to 11 nodes. In brackets the number of states.

Table 4: The lowest few energy eigenvalues of SU3 YM-QM in the symmetry sector $J^{-+}$ as a function of the polynomial order of truncation up to 11 nodes. In brackets the number of states.

Table 5: The lowest few energy eigenvalues of SU3 YM-QM in the symmetry sector $J^{+-}$ as a function of the polynomial order of truncation up to 10 nodes. In brackets the number of states. The $1^{+-}$ estimate '18' given in Tab.3 in WZ seems to be a misprint, in their Fig.1 it looks more like '19'.
D  Some explicit expressions for the 2-dim case

D.1  Explicit form of the operator  \( D^{(2)} = D^{(2)}_0 + D^{(2)}_{-2} \)

\[
D^{(2)}_0 := 2 x_{12} \partial_{x_{12}} + 3 x_{112} \partial_{x_{112}} + 3 x_{122} \partial_{x_{122}} + 4 b_{33} \partial_{b_{33}} ,
\]
\[
D^{(2)}_{-2} := -\frac{2}{3} \left( \frac{1}{4} x_{11}^2 + x_{11} x_{22} - b_{33} \right) \partial_{x_{112}}^2 - \frac{2}{3} \left( \frac{1}{4} x_{22}^2 + x_{11} x_{22} - b_{33} \right) \partial_{x_{122}}^2
\]
\[
- \frac{2}{3} (x_{11} + x_{22}) x_{12} \partial_{x_{112}} \partial_{x_{122}} - 2 x_{12} (x_{11} \partial_{x_{112}} \partial_{x_{112}} + x_{22} \partial_{x_{222}} \partial_{x_{122}})
\]
\[
+ 2 \left( \frac{1}{2} x_{11} x_{22} - b_{33} - x_{12}^2 \right) \left( \partial_{x_{111}} \partial_{x_{122}} + \partial_{x_{222}} \partial_{x_{112}} \right) - \frac{1}{2} (x_{11} + x_{22}) \left( \partial_{x_{112}}^2 + 6 \partial_{b_{33}} \right)
\]
\[
- 3 (x_{112} \partial_{x_{111}} + x_{122} \partial_{x_{222}} - x_{112}) \partial_{x_{112}} - ((x_{111} + 2 x_{122}) \partial_{x_{112}} + (x_{222} + 2 x_{112}) \partial_{x_{122}}) \partial_{x_{112}}
\]
\[
- 3 ((x_{22} x_{111} + x_{11} x_{112} - 2 x_{12} x_{112}) \partial_{x_{111}} + (x_{11} x_{222} + x_{22} x_{112} - 2 x_{12} x_{112}) \partial_{x_{222}}) \partial_{b_{33}}
\]
\[
- ((x_{22} x_{112} + x_{11} x_{222} - 2 x_{12} x_{122}) \partial_{x_{112}} + (x_{11} x_{222} + x_{22} x_{112} - 2 x_{12} x_{112}) \partial_{x_{122}}) \partial_{b_{33}}
\]
\[
- \frac{1}{2} \left[ (x_{11} + x_{22})(-x_{11} x_{22} + x_{12}^2 + 5 b_{33}) + 3(-(x_{22} x_{112} + x_{11} x_{112}) + (x_{112}^2 + x_{112}^2)) \right] \partial_{b_{33}}
\]
\[
- (2 x_{12} \partial_{x_{112}} + 4 b_{33} \partial_{b_{33}})(\partial_{x_{111}} + \partial_{x_{222}}) - 2 x_{12} \partial_{x_{112}} (2 \partial_{x_{111}} + \partial_{x_{222}}) - 2 x_{12} \partial_{x_{122}} (\partial_{x_{111}} + 2 \partial_{x_{222}})
\]

D.2  Explicit expression for the 2-dim Jacobian in terms of gauge invariant functions

In the 2-dim case we can perform a coordinate transformation of the eight gauge fields to the eight irreducible gauge invariant polynomials

\[
(X_1, X_2, Y_1, Y_2, Y_3, Y_4, Y_6, Y_8) \rightarrow x := (x_{11}, x_{22}, x_{12}, x_{111}, x_{222}, x_{112}, x_{122}, b_{33}) .
\]

Then the matrix elements take the form

\[
\langle \Phi'_{XY} | O[X, Y] | \Phi_{XY} \rangle \propto \int d^8 x \frac{\exp[-(x_{11} + x_{22})]}{\sqrt{m(x)}} p^{(2)}_1(x) O[x] p^{(2)}_2(x) ,
\]

with the polynomial

\[
m(x_{11}, x_{22}, x_{12}, x_{111}, x_{222}, x_{112}, x_{122}, b_{33}) =
\]
\[
(x_{11}^3 - 3 x_{111}^2)(x_{22}^3 - 3 x_{222}^2) + 16 b_{33}^2 + 36 b_{33}^2 \left( x_{12}^2 - x_{11} x_{22} \right) + 12 b_{33} \left[ 2 \left( x_{12} - x_{11} x_{22} \right) \right]^2
\]
\[
+ 3 x_{11} \left( x_{112} x_{222} - x_{122}^2 \right) + 3 x_{22} \left( x_{111} x_{122} - x_{112}^2 \right) + 3 x_{12} \left( x_{112} x_{122} - x_{111} x_{222} \right) \]
\[
+ 4 \left( x_{12}^2 - x_{11} x_{22} \right)^3 - x_{11} x_{22}^3 + 9 x_{11} x_{22} \left( x_{12}^2 - x_{11} x_{22} \right) + 9 x_{11} x_{22} \left( x_{12}^2 - x_{11} x_{22} \right)
\]
\[
+ 12 x_{12} \left( 3 x_{112} x_{222} - x_{111} x_{222} \right) + 36 x_{12} \left( x_{11} x_{122} + x_{22} x_{112} \right)
\]
\[
+ 18 x_{12} \left( x_{11} x_{122} + x_{22} x_{112} \right) + x_{111} x_{222} - 54 x_{112} x_{122} x_{111} x_{222} - 27 x_{112} x_{122} + 36 x_{111} x_{122}^3 .
\]

The angular momentum operator takes the form

\[
J_3 = -i \left[ 2 x_{12} (\partial_{x_{112}} - \partial_{x_{222}}) + (x_{22} - x_{11}) \partial_{x_{12}}
\right]
\]
\[
+ 3 (x_{11} \partial_{x_{111}} - x_{122} \partial_{x_{222}} + (2 x_{122} - x_{111}) \partial_{x_{112}} + (x_{22} - 2 x_{112}) \partial_{x_{122}}) .
\]
E Spin representation of the symmetric tensors

The simultaneous eigenfunctions of $J_3$ and $J^2 \equiv J_1^2 + J_2^2 + J_3^2$, given explicitly in (37), for the symmetric 2-tensor $s^{++}_{[2]ij}$ are determined as the spin-0 component

$$s^{(0)++}_{[2]0} = \frac{1}{\sqrt{3}} (s_{11} + s_{22} + s_{33}) ,$$

and the five spin-2 components $s^{(2)++}_{[2]ij} := s^{++}_{[2]ij} - \delta_{ij}s^{++}_{[2]kk}/3$,

$$s^{(2)++}_{[2]0} = \frac{1}{\sqrt{6}} (s_{11} + s_{22} - 2 s_{33}) ,$$

$$s^{(2)++}_{[2] \pm 1} = \pm s_{13} + i s_{23} ,$$

$$s^{(2)++}_{[2] \pm 2} = - \frac{1}{2} (s_{11} - s_{22}) \mp i s_{12} .$$

For the symmetric 3-tensor $s^{--}_{[3]ijk}$ we have the three spin-1 components $v^{(1)--}_{[3]i} := s^{--}_{[3]ijj}$,

$$v^{(1)--}_{[3]0} = (s_{113} + s_{223} + s_{333}) ,$$

$$v^{(1)--}_{[3] \pm 1} = \pm \frac{1}{\sqrt{2}} ((s_{111} + s_{122} + s_{133}) \pm i(s_{112} + s_{222} + s_{233})) ,$$

as well as the seven spin-3 components $s^{(3)--}_{[3]ijk} := s^{--}_{[3]ijk} - \frac{1}{15} \left( \delta_{jk} v^{--}_{[3]i} + \delta_{ik} v^{--}_{[3]j} + \delta_{ij} v^{--}_{[3]k} \right)$,

$$s^{(3)--}_{[3]0} = \frac{1}{\sqrt{30}} (3(s_{113} + s_{223}) - 2 s_{333}) ,$$

$$s^{(3)--}_{[3] \pm 1} = \mp \frac{1}{2\sqrt{10}} ((s_{111} + s_{122} - 4 s_{133}) \pm i(s_{112} + s_{222} - 4 s_{233})) ,$$

$$s^{(3)--}_{[3] \pm 2} = - \frac{1}{2} (s_{113} - s_{223}) \mp i s_{123} ,$$

$$s^{(3)--}_{[3] \pm 3} = \pm \frac{1}{2\sqrt{6}} ((s_{111} - 3 s_{122}) \mp i(s_{222} - 3 s_{112})) .$$

Analogous expressions hold for the other irreducible symmetric tensors $b^{++}_{[4]ij}$, $b^{(1)--}_{[5]i}$, as well as the irreducible symmetric axial tensors $a^{(0)++}_{[3]i}$, $a^{(1)++}_{[4]i}$, $a^{+-}_{[5]ij}$ and $a^{-+}_{[6]ijk}$.

F Lowest monomials for all symmetry sectors $J^{PC}$

This Appendix presents the Tables 6a-6d showing the monomials of lowest polynomial order in $A$ for each symmetry sector $J^{PC}$. Here the following conventions are used. Double indices are summed over. The subscript "sym" indicates symmetrization over all open indices. Furthermore only linearly independent monomials are shown, that is those that are not expressible as sums of others in the same symmetry sector. For example, the 6th order spin-2 monomial

$$\left(s^{[2]ik}s^{[2]kl}s^{[2]ij}\right)^{(2)} \equiv s_{[2]il}\left(s^{[2]ik}s^{[2]kj}\right)^{(2)} + (1/2) \left(s^{[2]kl}s^{[2]kl} - s^{[2]kk}s^{[2]ll}\right)s^{(2)}_{[2]ij}$$

is linear dependent and not shown in Table 6a.
| $J^{++}$ | $M_i^{(J)^{++}}$ |
|---------|----------------|
| 0^{++} | 1 |
| 0^{++} | $s_{[2]}^{[0]} \equiv s_{[2]}^{[0]}$ |
| 0^{++} | \((s_{[2]}^{[0]})^2, s_{[2]}^{[0]} s_{[2]}^{[0]}\), $b_{[4]}^{(0)} \equiv b_{[4]}^{[0]}$ |
| 0^{++} | \((s_{[2]}^{[0]} s_{[2]}^{[0]}), s_{[2]}^{[0]} s_{[2]}^{[0]}\), $b_{[4]}^{(0)} s_{[2]}^{[0]} s_{[2]}^{[0]}$ |
| 0^{++} | \((s_{[2]}^{[0]} s_{[2]}^{[0]}), s_{[2]}^{[0]} s_{[2]}^{[0]}\), $b_{[4]}^{(0)} s_{[2]}^{[0]} s_{[2]}^{[0]}$ |
| 0^{++} | $s_{[2]}^{[0]} \equiv s_{[2]}^{[0]}$ |

Table 6a: Lowest order monomials for the $J^{++}$ sector, for even and odd spin $J$. 

3^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |

4^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |

5^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |

6^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |

7^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |

8^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |

9^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |

10^{++} | $s_{[2]}^{[0]} s_{[2]}^{[0]}$ |
\[
\begin{array}{c|c}
\hline
J^{-} & M^{(J)^{-}} \\
\hline
1_{[3]} & v^{[3]i} \equiv s^{[3]ij} \\
1_{[5]} & s^{[0]}_{[2]} v^{[3]i}, \ s^{[2]}_{[2]} v^{[3]i}, \ s^{[3]}_{[ij} s^{[3]}_{jk]}, \ b^{[3]}_{[i]} \\
3_{[3]} & s^{[3]}_{[ijk]} \\
3_{[5]} & s^{[0]}_{[2]} s^{[3]}_{[ijk]}, \ \left( s^{[2]}_{[i]} s^{[3]}_{[jk]} \right)^{[3]}_{\text{sym}}, \ \left( s^{[2]}_{[ij]} v^{[3]}_{[jk]} \right)^{[3]}_{\text{sym}} \\
5_{[5]} & \left( s^{[3]}_{[ijk]} s^{[2]}_{lm} \right)^{(3)}_{\text{sym}} \\
7_{[7]} & \left( s^{[3]}_{[ijk]} s^{[2]}_{lm} s^{[2]}_{ns} \right)^{(7)}_{\text{sym}} \\
9_{[9]} & \left( s^{[3]}_{[ijk]} s^{[2]}_{lm} s^{[2]}_{ns} s^{[2]}_{tu} \right)^{(9)}_{\text{sym}}, \ \left( s^{[3]}_{[ijk]} s^{[3]}_{lmn} s^{[3]}_{stu} \right)^{(9)}_{\text{sym}} \\
11_{[11]} & \left( s^{[3]}_{[ijk]} s^{[2]}_{lm} s^{[2]}_{ns} s^{[2]}_{tu} s^{[2]}_{vw} \right)^{(11)}_{\text{sym}}, \ \left( s^{[3]}_{[ijk]} s^{[3]}_{lmn} s^{[3]}_{stu} s^{[2]}_{vw} \right)^{(11)}_{\text{sym}} \\
0_{[9]} & \epsilon_{ijk} b^{[4]}_{[il} s^{[3]}_{[j]lm} s^{[2]}_{[km]}, \ \epsilon_{ijk} b^{[4]}_{[il} v^{[3]}_{[j]} s^{[2]}_{[kl]}, \ \epsilon_{ijk} s^{[3]}_{[idm} s^{[2]}_{[jm} s^{[2]}_{[kn} s^{[2]}_{[ln]} \\
2_{[5]} & \left( \epsilon_{ist} s^{[3]}_{[sjk]} s^{[2]}_{[kt]} \right)^{(2)}_{\text{sym}}, \ \left( \epsilon_{ist} v^{[3]}_{[i]} s^{[2]}_{[jt]} \right)^{(2)}_{\text{sym}} \\
4_{[5]} & \left( \epsilon_{ist} s^{[3]}_{[sjk]} s^{[2]}_{[it]} \right)^{(4)}_{\text{sym}} \\
6_{[7]} & \left( \epsilon_{ist} s^{[3]}_{[sjk]} s^{[2]}_{[lt]} s^{[2]}_{[mn]} \right)^{(6)}_{\text{sym}} \\
8_{[9]} & \left( \epsilon_{ist} s^{[3]}_{[sjk]} s^{[2]}_{[lt]} s^{[2]}_{[mn]} s^{[2]}_{[ru]} \right)^{(8)}_{\text{sym}} \\
10_{[11]} & \left( \epsilon_{ist} s^{[3]}_{[sjk]} s^{[2]}_{[lt]} s^{[2]}_{[mn]} s^{[2]}_{[ru]} s^{[2]}_{[vw]} \right)^{(10)}_{\text{sym}}, \ \left( \epsilon_{ist} s^{[3]}_{[sjk]} s^{[2]}_{[lt]} s^{[3]}_{[mn]} s^{[3]}_{[u]} s^{[3]}_{[u]} \right)^{(10)}_{\text{sym}} \\
\hline
\end{array}
\]

Table 6b: Lowest order monomials for the $J^{-}$ sector, for odd and even spin $J$.

\[
\begin{array}{c|c}
\hline
J^{+} & M^{(J)^{+}} \\
\hline
0_{[3]} & a^{[3]} \\
0_{[5]} & s^{[0]}_{[2]} a^{[3]}, \ a^{[5]}_{[i]} \equiv a^{[3]}_{[ii]} \\
2_{[5]} & a^{[3]}_{[s]} s^{[2]}_{[ij]}, \ a^{[5]}_{[ij]} \\
4_{[7]} & a^{[3]}_{[s]} s^{[2]}_{[ij]} s^{[2]}_{[kl]} \left( a^{[4]}_{[4]} s^{[3]}_{[jk]} \right)^{(4)}_{\text{sym}}, \ \left( a^{[4]}_{[3]} s^{[3]}_{[jk]} \right)^{(4)}_{\text{sym}}, \ \left( a^{[3]}_{[ij]} s^{[2]}_{[kl]} \right)^{(4)}_{\text{sym}} \\
6_{[9]} & a^{[3]} \left( s^{[2]}_{[ij]} s^{[2]}_{[kl]} s^{[2]}_{[mn]} \right)^{(6)}_{\text{sym}}, \ a^{[3]} \left( s^{[3]}_{[ijk]} s^{[3]}_{[lmn]} \right)^{(6)}_{\text{sym}}, \ \left( a^{[4]}_{[4]} s^{[3]}_{[ijk]} s^{[3]}_{[lmn]} \right)^{(6)}_{\text{sym}}, \ \left( a^{[4]}_{[3]} s^{[3]}_{[ijkl]} s^{[2]}_{[mn]} \right)^{(6)}_{\text{sym}} \\
8_{[11]} & a^{[3]} \left( s^{[2]}_{[ij]} s^{[2]}_{[kl]} s^{[2]}_{[mn]} s^{[2]}_{[st]} \right)^{(8)}_{\text{sym}}, \ a^{[3]} \left( s^{[3]}_{[ijk]} s^{[3]}_{[lmn]} s^{[2]}_{[st]} \right)^{(8)}_{\text{sym}}, \ \left( a^{[4]}_{[4]} s^{[2]}_{[ijkl]} s^{[3]}_{[lmn]} s^{[2]}_{[st]} \right)^{(8)}_{\text{sym}}, \ \left( a^{[4]}_{[3]} s^{[3]}_{[ijkl]} s^{[3]}_{[lmn]} s^{[2]}_{[st]} \right)^{(8)}_{\text{sym}}, \ \left( a^{[4]}_{[3]} s^{[3]}_{[ijkl]} s^{[3]}_{[lmn]} s^{[2]}_{[st]} \right)^{(8)}_{\text{sym}} \\
1_{[7]} & \left( \epsilon_{ist} a^{[3]}_{[i]} s^{[2]}_{[jt]} \right)^{(3)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[3]}_{[i]} v^{[3]}_{[j]} \right)^{(3)}_{\text{sym}} \\
3_{[7]} & \left( \epsilon_{ist} a^{[3]}_{[i]} s^{[2]}_{[jt]} \right)^{(3)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[3]}_{[i]} v^{[3]}_{[j]} \right)^{(3)}_{\text{sym}} \\
5_{[9]} & \left( \epsilon_{ist} a^{[4]}_{[4]} s^{[3]}_{[tjk]} s^{[2]}_{[lm]} \right)^{(5)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[4]}_{[4]} s^{[3]}_{[tjk]} s^{[2]}_{[lm]} \right)^{(5)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[5]}_{[5]} s^{[2]}_{[jkl]} s^{[2]}_{[lm]} \right)^{(5)}_{\text{sym}} \\
7_{[11]} & a^{[3]} \left( \epsilon_{ist} a^{[3]}_{[i]} s^{[3]}_{[lmn]} s^{[2]}_{r} \right)^{(7)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[4]}_{[4]} s^{[3]}_{[tjk]} s^{[2]}_{[lm]} s^{[2]}_{nr} \right)^{(7)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[4]}_{[4]} s^{[3]}_{[tjk]} s^{[2]}_{[lm]} s^{[2]}_{nr} \right)^{(7)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[5]}_{[5]} s^{[3]}_{[tjk]} s^{[3]}_{mn} s^{[2]}_{r} \right)^{(7)}_{\text{sym}}, \ \left( \epsilon_{ist} a^{[5]}_{[5]} s^{[3]}_{[tjk]} s^{[3]}_{mn} s^{[2]}_{r} \right)^{(7)}_{\text{sym}} \\
\hline
\end{array}
\]

Table 6c: Lowest order monomials for the $J^{+}$ sector, for even and odd spin $J$.  

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Table 6d: Lowest order monomials for the $J^{+-}$ sector, for odd and even spin $J$.

| $J^{+-}$ | $M^{(J)^{+-}}$ |
|----------|----------------|
| $1^{++}_{[4]}$ | $a_{[4]}i$ |
| $3^{+}_{[6]}$ | $a_{[3]} S_{ijk}^{(3)} (a_{[4]} i S_{jk}^{(3)})^{(3)}_{\text{sym}} a_{[6]} i j k$ |
| $5^{+-}_{[8]}$ | $a_{[3]} (s_{ijk}^{(3)} S_{tm}^{[2]} a_{[4]} i S_{jk}^{(3)} S_{tm}^{[2]})^{(5)}_{\text{sym}} (a_{[5]} i j S_{klm}^{[3]} a_{[6]} i j k S_{tm}^{[2]})^{(5)}_{\text{sym}}$ |
| $7^{++}_{[10]}$ | $a_{[3]} (s_{ijk}^{(3)} S_{tm}^{[2]} S_{tm}^{[2]})^{(7)}_{\text{sym}} (a_{[5]} i j S_{klm}^{[3]} a_{[6]} i j k S_{tm}^{[2]})^{(7)}_{\text{sym}}$ |
| $0^{++}_{[10]}$ | $\epsilon_{ist} a_{[3]} S_{ilm}^{[2]} S_{tm}^{[2]} ; \epsilon_{ist} a_{[5]} i j S_{ijk}^{(3)} S_{tm}^{[2]} ; \epsilon_{ist} a_{[4]} i S_{tm}^{[2] i j} ; \epsilon_{ist} b_{[4]} i S_{tm}^{[2] i j}$ |
| $2^{+-}_{[6]}$ | $\left(\epsilon_{i s t} a_{[4]} S_{tm}^{[2] i j}\right)^{(2)}_{\text{sym}}$ |
| $4^{++}_{[8]}$ | $a_{[3]} \left(\epsilon_{ist} s_{l m}^{[2] i j k} S_{tm}^{[2]} \right)^{(4)}_{\text{sym}} a_{[4]} S_{tm}^{[2]} S_{tm}^{[2]} S_{tm}^{[2] kl} \left(\epsilon_{ist} a_{[2]} S_{tm}^{[2] ij k} S_{tm}^{[2] kl}\right)^{(4)}_{\text{sym}} a_{[6]} S_{tm}^{[2] ij k} S_{tm}^{[2] kl}$ |
| $6^{+-}_{[10]}$ | $a_{[3]} \left(\epsilon_{ist} s_{l m}^{[2] i j k} S_{tm}^{[2]} S_{tm}^{[2] kl}\right)^{(6)}_{\text{sym}} a_{[4]} S_{tm}^{[2]} S_{tm}^{[2] ij k} S_{tm}^{[2] kl} \left(\epsilon_{ist} a_{[2]} S_{tm}^{[2] ij k} S_{tm}^{[2] kl} S_{tm}^{[2] mn}\right)^{(6)}_{\text{sym}} a_{[6]} S_{tm}^{[2] ij k} S_{tm}^{[2] kl} S_{tm}^{[2] mn}$ |

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