A DYNAMIC FOR PRODUCTION ECONOMIES WITH MULTIPLE EQUILIBRIA

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Abstract. In this article, we extend to private ownership production economies, the results presented by Bergstrom, Shimomura, and Yamato (2009) on the multiplicity of equilibria for the special kind of pure-exchanges economies called Shapley-Shubik economies. Furthermore, a dynamic system that represents the changes in the distribution of the firms on the production branches is introduced. For the first purpose, we introduce a particular, but large enough, production sector to the Shapley-Shubik economies, for which a simple technique to build private-ownership economies with a multiplicity of equilibria is developed. In this context, we analyze the repercussions on the behavior of the economy when the number of possible equilibria changes due to rational decisions on the production side. For the second purpose, we assume that the rational decisions on the production side provoke a change in the distribution of the firms over the set of branches of production.

1. Introduction. A priori multiplicity of equilibria is generally a not desirable property of the economies. There is a vast literature analyzing the properties that guarantee the uniqueness of the Walrasian equilibrium. Under the usual assumptions of the general equilibrium theory, the sets of economies that have a given number of equilibria form open sets, in such a way that if any of the fundamentals of the economy are perturbed (not much), the new economy will not differ too much from the previous one, in particular, the number of equilibria will not change, and at the same time, changes in allocations and equilibrium prices will not be abrupt and can generally be predicted. These economies are called regular. Although they may be relatively small, these modifications may not satisfy in the same way for all economic agents. However, even when it is not frequent, it is important to take into account the fact that from time to time, some economies undergo drastic changes when some of their fundamentals are perturbed, and these changes are followed by significant repercussions on the welfare of consumers. In the framework of the

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General Equilibrium Theory abrupt changes in the consumption and prices as a result of small perturbations in fundamentals of the economy can only occur in a neighborhood of a critical economy. These are precisely the economies that appear on the borders of the sets of regular economies with a fixed number of equilibria. We postulate that changes abrupt or not, are the consequence of rational choices of the managers of the firms.

In this paper, we analyze the possible repercussions of rational decisions in the productive side of the economy on the welfare of consumers. Although we focus on production economies for a better understanding of our analysis, we will start with pure exchange economies. As it is well known, under the usual hypothesis of a neoclassical ownership production economy, the source of the multiplicity of equilibria lies in the consumption side, and not in the productions side of the economy. This fact is result of the characteristics of the supply function \( s : \mathbb{R}^2_+ \to Y \) defined by the rule \( s(p) = s_1(p) + s_2(p) \), where

\[
s_j(p) = \{ y_j(p) \in Y_j : p \cdot y_j(p) \geq p \cdot y, \text{ for all } y \in Y_j \}, \quad j = 1, 2.
\]

\[
\partial s(p) = \partial^2 \pi_j(p) \text{ is positive definite. Thus, if the aggregate excess demand is given by } z(p) = f(p) - s(p), \text{ where } f(p) = x_1(p) + x_2(p) - \omega_1 - \omega_2 \text{ and } x_i(p), \quad i = 1, 2, \text{ is the demand of the consumers, the production effect is entirely substitution, therefore the consumption substitution effects becoming stronger than the income effects. So the contribution of the production to the index tends to make it positive. See [7].}
\]

In several cases even when an exchange economy shows multiplicity of equilibria, this fact tends to disappear when production is incorporated into the model, even when initial preferences and endowments remain the same. In our analyses, we are interested in the effects of the decisions of rational agents acting on the production side over the consumption side and over the welfare. As it is well known, drastic change occurs in an economy as result of the movements in some parameters making that the number of equilibria changes.

In a private-ownership production economy, changes in the profits of the firms give place to changes in the wealth of the shareholders of these firms and therefore in their demand for consumption goods. These changes modify the set of possible equilibria of the economy and they are not always favorable for the consumers. Even more, they can be responsible for economic crises (abrupt changes in prices and equilibrium allocations) when they provoke a change in the number of equilibria. It is precisely in this transition that the more important changes are observed. We will see that the causes that lead to changes in the number of equilibria may lie in the decisions of the managers of the firms looking to maximize their profits in a framework in which different branches of production or technologies can give rise to different profits. The rest of the work is organized as follows. In section 2 we consider a pure exchange economy with multiplicity of equilibria. In section 3 we introduce a neoclassical production economy. In section 4 we consider the definitions of regular and singular economies, and the effects on the behavior of this kind of economies as a response of changes in the main parameters of the economy are argued. In section 6 we generalize the conditions of Shapley-Shubik for multiplicity of equilibria. The family of Shapley-Shubik economies is an interesting class of economies that are computationally manageable, allowing complex outcomes such as multiple equilibria. Once in this section, we show our main contribution of this paper. In section 7 we consider an economy with several firms distributed in two
production branches and we introduce a definition of regular and singular economies depending on the characteristics of these distributions. In section 8 we introduce a dynamic for the economy driven by the decisions of the managers of the firms. In section 9 we analyze the effects of these changes in the welfare. Finally, we offer some conclusions and comments.

2. Pure exchange economies. Given that we are only interested in analyzing the conditions that guarantee the multiplicity of equilibria, and that these are generally a consequence of consumer behavior rather than of the characteristics of the technology, we begin by considering economies of pure exchange, in particular the so-called Shapley-Shubik economies. These are a special kind of exchange economies with only two consumers and two goods. For this kind of economies, the conditions ensuring multiplicity of equilibria are well known.

2.1. Notations and definitions. A pure exchange economy $E$ with 2 goods and 2 consumers is a set

$$E = \{\mathbb{R}^2_+, (u^1, \omega^1), (u^2, \omega^2)\}$$

Where $u^i$ is a smooth, strictly monotonic and strictly quasi-concave function on the commodity space $\mathbb{R}^2_+$ and $\omega^i = (\omega^i_1, \omega^i_2) \in \mathbb{R}^2_+$ is the initial endowment vector for the $i$-th consumer, $i = 1, 2$. A commodity bundle is represented by a vector $x = (x_1, x_2) \in \mathbb{R}^2_+$ and, according to a unit price system $p = (p_1, p_2) \in \mathbb{R}^2_+$ the value of the commodity bundle $x$ at price $p$ is given by the inner product $p \cdot x = p_1 x_1 + p_2 x_2$. Finally by $x^i(p, p \cdot \omega^i)$ we represent the commodity set maximizing $u^i$ subject to the budget set $B^i(p) = \{x \in \mathbb{R}^2_+ | p \cdot x \leq p \cdot \omega^i\}$, $p \cdot \omega^i$ corresponds to the value of the initial endowments or initial wealth of the $i$-th consumer. In what follows we assume that utility functions $u^i$ and the consumption set $\mathbb{R}^2_+$ are fixed. Then, each pure exchange economy can be characterized by the distribution of the initial endowments $\omega = (\omega^1, \omega^2) \in \mathbb{R}^4_+$. Therefore, we use the notation

$$E_\omega = \{\mathbb{R}^2_+, (u^1, \omega^1), (u^2, \omega^2)\}$$

to denote the economy characterized by the distribution $\omega$.

**Definition 1.** Given an economy $\omega$, we say that the vector demand $x = ((x^1_1, x^1_2), (x^2_1, x^2_2))$ and the price vector $p = (p_1, p_2)$ is a *Walrasian equilibrium* if:
1. $(x^1_i, x^2_i) = x^i(p, p \cdot \omega^i), i = 1, 2$ where $u_i(x^i(p)) \geq u_i(x)$ for all $x : p \cdot x \leq p \cdot \omega^i$, $i = 1, 2$.
2. $(x^1_1 + x^1_2, x^2_1 + x^2_2) = (\omega^1_1 + \omega^1_2, \omega^2_1 + \omega^2_2)$.

First item tells us that $(x^1_1, x^2_1)$ maximizes the utility subject to the budget constraint set $B^i(p)$ for $i$-th consumer, while the second means that aggregate demand equals aggregate supply. This fact implies the definition of *individual excess demand*, which we introduce below:

**Definition 2.** Given the economy $E_\omega$, we define the individual excess demand function $z^i : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \rightarrow \mathbb{R}^2$ as the function defined by the rule:

$$z^i(p, \omega) = (x^i_1(p, \omega^i) - \omega^i_1, x^i_2(p, \omega^i) - \omega^i_2).$$

On the other hand, the aggregate excess demand is a function $z : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \rightarrow \mathbb{R}^2$ defined by the formula

$$z(p, \omega) = z^1(p, \omega) + z^2(p, \omega) \quad (1)$$
or explicitly
\[
z(p, \omega) = \left( \sum_{i=1}^{2} x_{1i}^p(p, p \cdot \omega^i) - \omega_1^p, \sum_{i=1}^{2} x_{2i}^p(p, p \cdot \omega^i) - \omega_2^p \right).
\]

For each economy \( E_\omega \), we define the equilibrium set as follows
\[
E_\omega = \{ p \in \mathbb{R}^2_{++} : z_\omega(p) = 0 \}
\]
where \( z_\omega : \mathbb{R}^2_{++} \to \mathbb{R}^2 \) defined by \( z_\omega(p) = z(p, \omega) \).

**Remark 1.** It is easy to see that \( z_\omega : \mathbb{R}^2_{++} \to \mathbb{R}^2 \) satisfies:

a) Walras’ Law \( p \cdot z_\omega(p) = 0 \) for all \( p \in \mathbb{R}^2_{++} \),
b) Homogeneity of degree zero \( z_\omega(\alpha p) = z_\omega(p) \), for all \( \alpha > 0 \).

Note that from item b) if \( p \) is an equilibrium vector price for the economy \( E_\omega \), \( \alpha p \) will also be an equilibrium for the economy \( E_\omega \). Therefore, if \( (p_1, p_2) \) is an equilibrium price vector for the economy \( \omega \), then \( (p_1/p_2, 1) \) is an equilibrium price too. In the same way from Walras’ Law and homogeneity of degree zero, we can consider several set of normalized equilibria prices:

- the one-dimensional simplex \( \Delta_+ = \{ p \in \mathbb{R}^2_{++} : p_1 + p_2 = 1 \} \) or
- the positive quadrant of the one-dimensional sphere
  \[
  S_+ = \{ p \in \mathbb{R}^2_{++} : p_1^2 + p_2^2 = 1 \}.
  \]
- or the set of relative prices \( p = (p_1/p_2, 1) \).

**Remark 2.** From Walras’ Law we know that for any \( p \in \mathbb{R}^2_{++} \), \( p z_\omega(p) = 0 \), from equation (1) this means \( p_1 z_{11}^p(p) + p_2 z_{21}^p(p) = 0 \), therefore without of generality we can assume that second coordinate depends on the first one. Then, \( z_\omega(p) = 0 \) if and only if \( z_{11}^p(p) = 0 \). Therefore to determine an equilibrium price vector it is enough to solve the equation \( z_{11}^p(p) = 0 \) and from homogeneity of degree zero this equation becomes in \( z_{21}^p(p_1) \). In conclusion, to determine an equilibrium price vector for the global economy is enough to find a price for first good such that clean the first market.

In what follows, we will always use the normalization \((p, 1)\) and we will identify the excess demand function \( z_{11}^p(p, 1) \) with \( z_{10}^p(p) \). For the sake of convenience we write \( z^1(p) \) instead of \( z_{10}^p(p) \) or if not confusion could arise we just write \( z(p) \) or \( z_\omega(p) \) whenever we want to emphasize the importance of \( \omega \).

Under the habitual assumptions and using the normalization discussed above, as conclusion of the implicit function theorem, we have that, if \( z(\bar{p}, \bar{\omega}) = 0 \) there exist suitable neighborhoods \( V \) and \( U \) of \( \bar{\omega} \) and \( \bar{p} \) respectively, and a differentiable function \( p : V \to U \) such that \( z(p(\omega), \omega) = 0 \) for all \( \omega \in V \). Note that from the fact that \( z(\alpha p, \omega) = z(p, \omega) \) this conclusion is not longer true, if we not assume the prices to normalize.

### 2.2. Shapley-Shubik pure exchange economies

In this section, we present Shapley-Shubik pure exchange economies as a model of simple exchange economies with multiple equilibria. Many of the definition below are taken from Theodore C. Bergstrom, Ken-Ichi Shimomura and Takehiko Yamato’s work [2].
Definition 3. A pure exchange economy $E$ with 2 goods and 2 consumers is a set
$$E = \{ \mathbb{R}_+^2, (u^1, \omega^1), (u^2, \omega^2) \},$$
where
1. $\mathbb{R}_+^2$ denotes the consumption set.
2. For $i = 1, 2$, $u^i$ denotes a strictly quasi-concave utility function on the commodity space $\mathbb{R}_+^2$ and $\omega^i \in \mathbb{R}_+^2$ the initial endowment vector for the $i$- consumer.

Whenever the utilities functions have the form
$$u^1(x_1, x_2) = x_1 + f_1(x_2), \text{ and } u^2(x, y) = f_2(x_1) + x_2,$$
for some strictly concave, continuously differentiable functions $f_1(\cdot), f_2(\cdot)$ from $[0, \infty)$ to $\mathbb{R}$, such that $\lim_{z \to \infty} f_i'(z) \leq 0$, $i = 1, 2$. We call to the economy $E$ Shapley-Shubik economy.

For this kind of economies we define
$$B_i = \lim_{z \to 0} f_i'(z), \quad i = 1, 2.$$

Following the notation in [2] we write $\phi_1(p) = x_2(p)$, i.e., the demand of consumer 1 for good 2 and $\phi_2(1/p) = x_2(p)$ for the demand for good 1 of consumer 2.

Remark 3. The price $p$ is an interior competitive equilibrium price if and only if the demand of both consumers are interior at $p$ and $\phi_1(p) = \frac{1}{p} \phi_1 \left( \frac{1}{p} \right)$. See Lemma 1 in [2].

2.3. Mirror-symmetric Shapley-Shubik pure exchange economy. We define a mirror-symmetric Shapley-Shubik economy as one in which $u^1(x_1, x_2) = u^2(x_2, x_1)$ for all $(x_1, x_2) \in \mathbb{R}_+^2$. This requires that $f_1(x) = f_2(x) = f(x)$ for some function $f$, satisfying the above conditions. The consumers’ utility functions are:
$$u^1(x_1, x_2) = x_1 + f(x_2),$$
$$u^2(x_1, x_2) = f(x_1) + x_2.$$

Let $B = f'(0)$ and let $\phi_1(p) = \phi_2(p) = \phi(p)$ for all $p \in (0, B]$. In [2] is proved that if $B > 1$ and $(f')^{-1}(1) \leq \min\{\omega^1_1, \omega^2_2\}$, then there is one equilibrium price at $p = 1$. Furthermore, they prove that whenever $z_{\omega}(p) = 0$ for some $p > 0$, then $z_{\omega}(p^{-1}) = 0$ is also verified, and they give necessary conditions for the existence of multiple equilibria not only in mirror-symmetric Shapley-Shubik economy but also in the general case of Shapley-Shubik economies. These conditions are stated in the following theorem.

Theorem 1. Let $((\omega^1_1, \omega), (\omega, \omega^2_2), f)$ be a mirror symmetric Shapley-Shubik economy and set $\phi_1(p) = (f')^{-1}(p)$. Then, if $\omega^1_1 = \omega = \omega^2_1$, $\phi(1) + 2\phi'(1) > \omega$ and $\phi(1) < \min\{\omega^1_1, \omega^2_2\}$ the economy has at least three equilibria prices. One at price $p_1 = 1$, another in some $p_3 > 1$ and one more in $p_1 = p_3^{-1} < 1$.

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1. $\phi(p) = (f')^{-1}$.
2. In Lemma 1, Remark 1 and footnote 2 of [2] is shown the important role play by the values $B_1$ and $B_2$.
3. In the original Theorem is assumed that the endowments are $\omega^1 = (\omega^1_1, 0)$, and $\omega^2 = (0, \omega^2_2)$. However, a slight modification in the hypothesis of theorem $(\phi(1) + 2\phi'(1) > \omega)$ allows us to consider positive amounts of second good for the first consumer and positive amounts of first one for second consumer.
Example 1. Let \( f(x) = 5.5x - \frac{1}{2}x^2 \), \( \omega = ((5, 2), (2, 3)) \) and set \( u^1(x, y) = x + f(y) \) and \( u^2(x, y) = y + f(x) \), then after some algebra the Marshall demands at prices \( p = (p_1, p_2) \) are given by

\[
x^1(p_1, p_2) = \left( 5 + 2\frac{p_2}{p_1} - \left( 5.5 - \frac{p_2}{p_1} \right) \frac{p_2}{p_1}, 5.5 - \frac{p_2}{p_1} \right),
\]

\[
x^2(p_1, p_2) = \left( 5.5 - \frac{p_1}{p_2}, 2\frac{p_1}{p_2} p_y + 3 - \left( 5.5 - \frac{p_1}{p_2} \right) \frac{p_1}{p_2} \right).
\]

\[
z^1\omega(p_x, p_y) = 2\frac{p_2}{p_1} - \left( 5.5 - \frac{p_2}{p_1} \right) \frac{p_2}{p_1} - \frac{p_1}{p_2} + 3.5.
\]

We normalize setting \( p_1 = p \) and \( p_2 = 1 \) to obtain

\[
z\omega(p) = 2 + p^{-1} - (5.5 - p^{-1}) p^{-1} - p + 3.5.
\]

Then the equilibria prices, in the first market, are given by \( p = 1/2, 1, 2 \). These equilibria correspond with \((1/2, 1), (1, 1)\) and \((2, 1)\) in the global economy.

**Figure 1.** Example 1: \( \omega = ((5, 2), (2, 3)) \), \( f(x) = 5.5x - \frac{1}{2}x^2 \), equilibria prices at \( p_1 = 1/2, p_2 = 1, p_3 = 2 \).

3. **Neoclassical private ownership production economies.** In a private ownership production economy, each household is characterized by a consumption set described by an endowment of commodities and preferences described by a utility function. Each firm is owned by the household and is characterized by technology described by a transformation function. In this section, we restrict ourselves to consider the basic definitions and theory about private ownership economies with 2 goods, 2 consumers, and two branches of production. To save notation, we refer to this kind of economies as private ownership production economies of dimension 2, which formally we define as follows:

We understand as a 2-dimensional, neoclassical ownership production economy a set

\[
E = \{ \mathbb{R}^2_+, (u^1, \omega^1), (u^2, \omega^2), Y_1, Y_2, \theta_{ij}, i, j = 1, 2 \}
\]

where

1. \( \mathbb{R}^2 \) is the commodity space.
2. There are two consumers indexed by \( i = 1, 2 \), each one is characterized by his initial endowment \( \omega_i \) and by his preference relation \( \succeq_i \), which is represented by \( C^2 \) utility functions.

3. By \( \omega = (\omega_1, \omega_2) \) we denote the initial endowments.

4. \( Y_1 \) and \( Y_2 \) are the \textit{technological or production sets} which we assume to be closed and strictly convex subset of \( \mathbb{R}^2 \) such that \( Y_1 \cap \mathbb{R}^+_+ = \{0\} \) and \( -\mathbb{R}^+_+ \subset Y_1 \). As is general, we assume that in a feasible plan of production, each positive coordinate of \( y \in Y_j \) corresponds to an output and each negative coordinate represents an input.

5. The real number \( \theta_{ij} \in [0, 1] \) represents consumer’s \( i \) share of firm \( j \)’s profit. Moreover, we assume that \( \theta_{11} + \theta_{21} = 1 \) and \( \theta_{12} + \theta_{22} = 1 \).

6. We will denote by \( Y \) the set of feasible production plans of the economy as a whole, i.e., \( Y = Y_1 + Y_2 \). We focus in the case in which the technology is defined by a production function \( g_j : \mathbb{R} \to \mathbb{R} \) using just one good as a input.

As in the case of a pure exchange economy, consumers are characterized by preferences and endowments. In addition, since the economy is of private ownership, a profit distribution rule \( \Theta = \{\theta_{ij}\}^2_{i,j=1} \) determines how each level of profits is distributed among the consumers. Hence, once \( \Theta \) and \( \omega \) are fixed the total income of the \( i \)-th consumer is given by

\[
W^i(p) = p \cdot \omega^i + \sum_{j=1}^2 \theta_{ij} \pi_j(y_j(p)),
\]

since consumers are shareholders of the firms.

Consumers are maximizing their utilities in their budgetary set. So the demand is given by

\[
x^i(p, W^i(p)) = (x_1^i(p, W^i), x_2^i(p, W^i))
\]

whenever\( u_i(x_1^i(p, W^i), x_2^i(p, W^i)) \geq u_i(x_1, x_2) \) for all \( x \in \mathbb{R}^2_+ : p \cdot x \leq W^i(p), \ i = 1, 2 \), and firms are maximizing profits in their technological set, so the plans of production are \( y^j(p) = (y^j_1(p), y^j_2(p)) \)

\[
y^j(p) : p \cdot y^j(\cdot) \geq p \cdot \omega^j \text{ for all } y \in Y_j \ j = 1, 2.
\]

The excess aggregate demand function presents the same properties as those considered above for the case of pure exchange economies. Consequently, to denote the excess aggregate demand for privately owned production economies, we will use the same notation as for pure exchange economies. So, we have

**Definition 4.** The demand function for each good \( k = 1, 2 \) will be denoted by \( z^k : \mathbb{R}^2_+ \to \mathbb{R} \) and will be defined by the law

\[
z^k(p, W^1(p), W^2(p)) = x^1_k(p, W^1(p)) + x^2_k(p, W^2(p)) - \omega^1_k - \omega^2_k - (y^1_k(p) + y^2_k(p)). \tag{2}
\]

Hence the aggregate demand function of the economy is given by the vectorial field \( z : \mathbb{R}^2_+ \to \mathbb{R}^2 \) defined by

\[
z(p, W^1(p), W^2(p)) = (z^1_1(p, W^1(p), W^2(p)), z^2_1(p, W^1(p), W^2(p))). \tag{3}
\]

Once the initial endowments and the shares of consumers in the profits of the firms have been fixed, for each \( p = (p_1, p_2) \) we can write \( z(p, W^1(p), W^2(p)) = z(p) \).
We assume that the objective of each firm is to maximize its profit, therefore at 
given price vector \( p \in \mathbb{R}_+^2 \), the \( j \)-th firm solves the problem

\[
\max_{y \in Y_j} \pi^j(p) = \max_{y \in Y_j} p \cdot y.
\]  

(4)

If for firm \( j \), the \( \mathbb{R}^2 \) vector \( y^j(p) \) solve the maximization problem (4), \( y^j(p) \) is called the supply of firm \( j \) at given price \( p \), \( j = 1, 2 \). By \( y = (y^1(p), y^2(p)) \), we denote the supply of both firms at price \( p \). Then, we define a Walrasian equilibrium for a private ownership economy as follows.

**Definition 5.** A Walrasian equilibrium for a neoclassical private ownership production economy \( E \) is a triad \( \{y^*, x^*, p^*\} \) formed by a profile of production plans \( y^* = (y^{i1}, y^{i2}) \in \mathbb{R}^2 \times \mathbb{R}^2 \), an allocation \( x^* = (x^{i1}, x^{i2}) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) and a set of prices \( p = (p_1, p_2) \in \mathbb{R}_+^2 \) such that:

1. \( x^{i1} \in \mathbb{R}_+^2, p^* \cdot x^{i1} \leq p^* \cdot \omega^i + \sum_{j=1}^2 \theta_{ij} p^* y^{ij}, i = 1, 2, y^{ij} \in Y^i, j = 1, 2. \)
2. \( u'(x^{i1}) \geq u'(x^i) \) for all \( x^i \in \mathbb{R}_+^2 \), such that \( p^* \cdot x^i \leq p^* \cdot \omega^i + \sum_{j=1}^2 \theta_{ij} p^* y^{ij}. \)
3. \( p^* y^{ij} \geq p^* y^{ij} \) for all \( y^j \in Y^j \).
4. \( \sum_{i=1}^2 x^{i1} \leq \sum_{j=1}^2 y^{ij} + \sum_{i=1}^2 \omega^i. \)

Item (1) requires that the demands of households are in their consumption sets and the production plans in the production sets. Item (2) implies that household equilibrium choices maximize utility. Item (3) that production plans maximize profits, and item (4) that equilibrium allocation is feasible.

**Remark 4.** Note that the main properties of the excess demand function, smoothness, Walras’ Law and homogeneity of degree zero reminding valid for production economies. Then, we normalize \( p_2 = 1 \) to identify the aggregate excess demand function for good 1

\[
z(p) = z^1(p, 1),
\]

see (2) and (3) above.

4. **Regular and singular economies.** Even when the theory and results presented in this section are more general, we focus our attention on economies with only two goods, since our purpose is to obtain sufficient conditions that guaranty the existence of multiple equilibria in mirror-symmetric Shapley-Shubik economies with mirror-symmetric production, which we will be defined in section 5.

In the rest of this work, we will assume that the excess demand function is smooth, i.e., as many times differentiable as we need. Under this context, we make the following definitions and classifications of an economy. But before that, we point out that the following results hold for economies with or without production, then in what follows we will assume that all the fundamentals of the economy, either privately owned or pure exchange, are fixed less the distribution of initial endowments of the economic agents, and therefore we will identify any of this two kind of economies with \( \omega \in \mathbb{R}_+^2 \).

**Definition 6.** Given an economy \( \omega = (\omega^1, \omega^2) \in \mathbb{R}_+^2 \), if \( z : \mathbb{R}_+^2 \times \mathbb{R}_+ \rightarrow \mathbb{R} \) denotes the excess demand function, we say that the economy \( \omega \) is regular whenever \( z'(p) \neq 0 \) for any \( p \in E_\omega \).

Mathematically speaking, this means that no equilibrium point is a critical point for \( z_\omega \). When this is not the case, the economy is called singular. Finally, any equilibrium prices satisfying \( z(p) \neq 0 \) is called regular equilibrium price.
Definition 7. If \( p \) is a regular equilibrium price, we define the index of \( p \) as follows:

\[
i(p) = \text{sign}(-z'(p)).
\]

Note that an equilibrium price \( p \) has positive index if and only if that \( z'(p) < 0 \), which implies that an increase in \( z \) provoke an increase in the equilibrium price of the good one, while a negative index has the interpretation that an increase in the excess demand of good 1 results in a decrease of the equilibrium prices of good 1. Moreover, positive index for an equilibrium price \( p \) corresponds with the notion of attractor in the sense that any solution of \( z(p') = 0 \) starting near to the equilibrium \( p \) converges to it.

It is a well know result that in any regular economy the sum of the index over all \( p \in E_\omega \) is equal to +1, that is,

\[
\sum_{p \in E_\omega} i(p) = +1,
\]

see for example [3],[7] or [4] for the proof of this statement. An important fact about this statement that worth pointing out is that the result remains valid even in the case of economies with production. For a proof of this fact, we recommend to see Theorem 6 in [5].

In a two goods word, equation (5) give us information about the shape of the graphic of the excess demand function. Furthermore, it gives us a useful criterion to determine the existence of multiple equilibria in the economy. First that all note that if the economy \( \omega \) is regular, then \( \sum_{p \in E_\omega} i(p) = +1 \) take place if the cardinality of \( E_\omega \) is odd. Therefore we have (see for instance [7]) that:

Theorem 2. If the economy \( \omega \) is regular, then \( \#E_\omega \) is odd.

The next example proof that the converse is not true, that is, a economy with an odd number of equilibria not necessary is regular.

Example 2. (A singular pure exchange economy) Suppose there are two goods and two consumers with utility functions:

\[
u^1(x_1, x_2) = x_1 - \frac{1}{8}x_2^{-8},
\]

\[
u^2(x_1, x_2) = -\frac{1}{8}x_1^{-8} + x_2,
\]

both utility functions are quasi-linear. Assume the endowments are \( \omega^1 = (2, r) \) and \( \omega^2 = (r, 2) \).

The Marshallian demands at prices \( p = (p_1, p_2) \) are:

\[
x^1(p_1, p_2) = \left(2 + r \left(\frac{p_2}{p_1}\right) - \left(\frac{p_2}{p_1}\right)^{8/9} \right), \left(\frac{p_1}{p_2}\right)^{1/9},
\]

\[
x^2(p_1, p_2) = \left(\frac{p_2}{p_1}\right)^{1/9}, 2 + r \left(\frac{p_1}{p_2}\right) - \left(\frac{p_1}{p_2}\right)^{8/9} \right).
\]

\[
z_x^1(p_x, p_y) = r \left(\frac{p_y}{p_x} - 1\right) - \left(\frac{p_y}{p_x}\right)^{8/9} + \left(\frac{p_y}{p_x}\right)^{1/9}.
\]
Normalizing the price of good 2 so that $p_2 = 1$, we can write the aggregate excess demand curve for good 1 as

$$z^1_\omega(p_1, 1) = x^1_1(p_1, 1) + x^2_1(p_1, 1) - (2 + r),$$

or

$$z^1_\omega(p_1, 1) = r (1/p_1 - 1) - (1/p_1)^{8/9} + (1/p_1)^{1/9}.$$

Then it is easy to see that for $r = 7/9$, $p = 1/2$ is the only equilibrium price. Since $z^1_\omega(1/2) = 0$, we conclude that the economy $\omega$ is singular. In this case, the singularity occurs in an inflection point. See figure (2) below.

![Figure 2](image_url)

**Figure 2.** Example 2: $r = 7/9$ equilibrium price at $p_1 = 0.5$ is singular

**Theorem 3.** Let $\omega$ be a regular economy, if $p^* \in E_\omega$ is such that $i(p^*) = -1$, then the economy has multiple equilibria.

**Proof.** From the fact that $\sum_{p \in E_\omega} i(p) = +1$, it follows that if $\# E_\omega = 1$ and $p^* \in E_\omega$ were such that $i(p^*) = -1$, then $+1 = \sum_{p \in E_\omega} i(p) = i(p^*) = -1$ which is a contradiction. Therefore if a regular economy with two goods has an equilibrium with negative index, then it necessarily has multiple equilibria.

**Remark 5.** It is important to remark that, last theorem guaranties that for regular economies, whenever there is an equilibrium price $p$ for which the demand at $p$ is upward sloping, the economy necessarily has at least another two prices $p_1$ and $p_2$ for which $z'_i(p_i) < 0$.

The importance of this remark lies in the fact that it gives us a basis for the construction of economies with multiple equilibria.

**Proposition 1.** If for any $p \in E_\omega$, $i(p) = +1$, then $\# E_\omega = 1$.

5. **Shapley-Shubik production economies.** As we mention in subsections 2.2 and 2.3 the pure exchanges Shapley-Shubik economies are a well studied class of economies for which the conditions under which multiple equilibria there exist are well known. Then in this section we propose the definition of private ownership Shapley-Shubik production economy, and using the results present in section 4, and we give conditions that guaranty the existences of multiple equilibria for a particular case which we will call mirror symmetric Shapley-Shubik economies with mirror production.
Definition 8. Let \( f : [0, \infty) \to \mathbb{R} \) be a strictly concave, continuous differentiable function such that \( \lim_{z \to \infty} f'(z) \leq 0 \) and \( B = \lim_{z \to 0} f'(z) \), set \( u^1(x_1, x_2) = x_1 + f(x_2) \) and \( u^2(x_1, x_2) = f(x_1) + x_2 \) then a private ownership Shapley-Shubik production economy is given by
\[
\mathcal{E} = \{ \mathbb{R}^2, (u^1, \omega^1), (u^2, \omega^2), g_1, g_2, \theta_{ij}, i, j = 1, 2 \},
\]
where \( g_j, j = 1, 2 \), is a concave single input-output technology, such that, \( x_2 = g_1(x_1) \) and \( x_1 = g_2(x_2) \).

In particular in analogy with mirror-symmetric economies whenever \( g_1 = g_2 = g \), and \( x_2 = g(x_1) \) and \( x_1 = g(x_2) \), that is, when both branches of production use same technology to produce different goods using different inputs, we will called to the economy mirror symmetric economy with mirror production.

5.1. A simple construction of multiple equilibria Shapley-Shubik production economies. Let
\[
\mathcal{E} = \{ \mathbb{R}^2, u^1(x_1, x_2), \omega^1, u^2((x_1, x_2)), \omega^2, g_1, g_2, \theta_{i,j}, i, j = 1, 2 \}
\]
be a mirror symmetric economy with mirror production economy, such that,
- utilities are defined by
  \( u^1(x_1, x_2) = x_1 + f(x_2), \quad u^2(x_1, x_2) = f(x_1) + x_2 \),
- endowments \( \omega^1 = (\omega^1_1, \omega), \omega^2 = (\omega, \omega^2_2), \omega_k > 0, k = 1, 2 \) and \( \omega > 0 \)
- \( \theta_{ij} \) represents the profit of the \( i \)-th consumer in the benefits of the \( j \)-th firm.
  It is assume that \( 0 \leq \theta_{ij} \leq 1 \) holds for \( i = 1, 2 \) and \( \theta_{1j} + \theta_{2j} = 1, j = 1, 2 \).
- Technologies are defined by a single strictly concave function \( g_1 = g_2 = g \), verifying \( x_2 = g(x_1), x_1 = g(x_2) \),
where \( f : [0, \infty) \to \mathbb{R}, i = 1, 2 \), is a strictly concave, continuously differentiable function, \( \lim_{z \to \infty} f'(z) \leq 0 \) and \( B = \lim_{z \to 0} f'(z) \). In addition, we define \( \phi(x) = (f')^{-1}(x) \) and \( \rho(x) = (g')^{-1}(x) \).

Theorem 4. Let \( \mathcal{E} \) be a mirror symmetric economy with mirror production verifying the above conditions, if \( \theta_{11} + \theta_{12} = 1 \), then \( p = 1 \) is an equilibrium for the economy \( \mathcal{E} \). Moreover, if
\[
2\phi(1) + 2\phi'(1) + 2\rho'(1) - \theta_{11}\phi(1) + \theta_{12}g(\rho(1)) - [g(\rho(1)) - \rho(1)] > \omega,
\]
then the economy \( \mathcal{E} \) has at least three equilibria.

Proof. Solving the maximization profit problem for both firms and after some algebra setting \( p_1 = p, p_2 = 1 \) we obtain
- Firm 1
  \[
y = (-\rho(p), g(\rho(p))), \quad \pi_1(p) = g(\rho(p)) - pp(p).
\]
- Firm 2
  \[
y = (g(\rho(p^{-1})), -\rho(p^{-1}))), \quad \pi_2(p) = pg(\rho(p^{-1})) - \rho(p^{-1}).
\]
While for consumers we have
If Corollary 5.

Therefore the excess demand function is given by

\[ z(p) = \omega_1^p + \omega p^{-1} + \theta_{11} \pi_1(p) p^{-1} + \theta_{12} \pi_2(p) p^{-1} - \phi(p^{-1}) p^{-1} + \phi(p) + \rho(p) - \omega + g(\rho(p^{-1})) \].

Therefore

\[ z(p) = \omega p^{-1} + \theta_{11} [g(\rho(p)) - \rho(p) p] p^{-1} + \theta_{12} \left[ g(\rho(p^{-1})) p - \rho(p^{-1}) \right] p^{-1} - \phi(p^{-1}) p^{-1} + \phi(p) + \rho(p) - \omega + g(\rho(p^{-1})) \].

Note that for \( p = 1 \), we have

\[ z(1) = (\theta_{11} + \theta_{12} - 1) (g(\rho(1)) - \rho(1)) = 0. \tag{6} \]

Last equality follows from the fact that \( \theta_{11} + \theta_{12} = 1 \), and therefore we conclude that \( p = 1 \) is an equilibrium point.

The second part follows directly from the fact that

\[ z'(1) = -\omega - [g(\rho(1)) - \rho(1)] + 2(\phi(1) + \phi'(1) + \rho'(1)) - \theta_{11} \rho(1) + \theta_{12} g(\rho(1)), \]

which is strictly positive from the hypothesis above. Then to conclude the multiplicity of equilibria we use Theorem 3 in section 4.

Even when the hypothesis \( \theta_{11} + \theta_{12} = 1 \) is not too restrictive, this hypothesis can be removed as a direct consequence of equation (6). Then we have the following corollary:

Corollary 5. If \( \rho(1) \) is a fixed point \( g \), then for any \( \theta_{ij} \in [0,1], i, j = 1, 2, p = 1 \) is an equilibrium for the mirror-symmetric economy with mirror production

\[ \mathcal{E} = \{ \mathbb{R}^2, u^1(x_1, x_2), \omega^1 = (\omega_1^1, \omega), u^2(x_1, y_2), \omega^2 = (\omega, \omega_2^1), x_2 = g(x_1), x_1 = g(x_2), \theta_{ij} \} \],

where utilities are: \( u^1(x_1, x_2) = x_1 + f(x_2) \) and \( u^2(x_1, x_2) = f(x_2) + x_1 \), with \( f(\cdot) \) strictly concave, continuously differentiable function from \( [0, \infty) \) to \( \mathbb{R} \), such that \( \lim_{z \to \infty} f'(z) \leq 0 \) and \( B = \lim_{z \to 0} f'(z) \).
6. A generalization of a production Sahply-Shubik economy. In this section, we give sufficient conditions for the multiplicity of equilibria in a generalized Shapley-Shubik economy with production. As in the definition 8, the technologies are characterized by concave production functions, but unlike those considered in subsection 5.1, they are not necessarily equal. As we will show, in this context, whenever \( p = 1 \) is an equilibrium price, the conditions (14) is sufficient to guaranty the multiplicity of equilibria. Furthermore, in the general case whenever an equilibrium price \( p \) satisfies \( z'(p) > 0 \), the multiplicity of equilibria holds, see Theorem (6) below.

More precisely, in this section we assume that firm 1 produce good 2 using as input first good by mean of the rule \( x_2 = g_1(x_1) \), where \( g_1 \) is a strictly concave production function, while firm 2 produce good 1 using as input good 2, by means of \( x_1 = g_2(x_2) \), where \( g_2 \) is also a strictly concave function. There are two consumer characterized by their utilities function \( u \) and \( u' \), and that \( \theta \) and \( \rho \) represent the shares in firm 1 and firm 2, respectively of \( i \)-th consumer, \( i = 1,2 \), and as it is usual in a private ownership production economy we assume that \( \theta_{1j} + \theta_{2j} = 1, j = 1,2 \).

1. Setting \( \rho = (g_1')^{-1} \), \( p_x = p \) and \( p_y = 1 \), we have that
   
   - The supply of firm 1 is given by \( y_1(p) = (−ρ(p), g(ρ(p))) \).
   
   - The profits for firm 1 is \( π_1(p) = g_1(ρ(p)) − pρ(p) \).

2. In the same way for setting \( \gamma = (g_2')^{-1} \).
   
   - The supply of firm 2 is given by \( y_2(p) = (g_2(γ(p^{-1})), −γ(p^{-1})) \).
   
   - The profits of firm 2 is \( π_2(p) = p g_2(γ(p^{-1})) − γ(p^{-1}) \).

3. Setting \( φ_i = (f_i')^{-1}, i = 1,2 \), since budget constraint set for consumer 1 is
   
   \[ x_1^1 + px_2^1 = ω_1^1 + pw_1 \]

   assuming that \( p_x = p \) and \( p_y = 1 \), and solving the maximization utility problem for consumer 1, we obtain that the marginal rate of substitution between second and first good must satisfies \( f'_1(x_1^2(p)) = p^{-1} \), since \( f_1' \) is invertible, we obtain

   \[ x_1^2(p) = (f_1')^{-1}(p^{-1}) = φ_1(p^{-1}) \],

   that is, the individual demand for second good by consumer 1 is given by \( φ_1(p^{-1}) \). Taking account the budget constraint set, we have

   \[ x_1^1(p) = ω_1^1 + ω_2^1 p^{-1} + \theta_{11} [g_1 (ρ(p)) − pρ(p)] p^{-1} + \theta_{12} [pg_2 (γ(p^{-1})) − γ(p^{-1})] p^{-1} − φ_1(p^{-1}) p^{-1} \]

   it follows that the individual demand of consumer 1 is given by:

   \[ x^1(p) = ω_1^1 + ω_2^1 p^{-1} + \theta_{11} [g_1 (ρ(p)) − pρ(p)] p^{-1} + \theta_{12} [pg_2 (γ(p^{-1})) − γ(p^{-1})] p^{-1} − φ_1(p^{-1}) p^{-1}, φ_1(p^{-1})] \]
In the same way we can prove that the individual demand of consumer 2 is given by
\[ x^2(p) = (\phi_2(p), \omega_1^2 p + \omega_2^2 + (1 - \theta_{11}) [g_1 (\rho(p)) - pp\rho(p)] + \\
(1 - \theta_{12}) [pg_2 (\gamma(p^{-1})) - \gamma(p^{-1})] - \phi_2(p)p). \]
Note that for consumer 2 we have
\[ x^2_1(p) = \phi_2(p) \] (9) and
\[ x^2_2(p) = \omega_1^2 p + \omega_2^2 + (1 - \theta_{11}) [g_1 (\rho(p)) - pp\rho(p)] + \\
(1 - \theta_{12}) [pg_2 (\gamma(p^{-1})) - \gamma(p^{-1})] - \phi_2(p)p, \] (10)

therefore the excess demand function for the good 1 is given by.
\[ z^1(p, W) = \omega_1^2 p^{-1} + \theta_{11} [g_1 (\rho(p)) - pp\rho(p)] p^{-1} + \\
\theta_{12} [pg_2 (\gamma(p^{-1})) - \gamma(p^{-1})] p^{-1} - \phi_1(p^{-1}) p^{-1} + \\
\phi_2(p) + \rho(p) - \omega_1^2 - g_2 (\gamma(p^{-1})), \]
where \( W = (W^1, W^2) \).

For \( p = 1 \) to be an equilibrium it is required that
\[ 0 = z^1(1, W) = (\omega_1^2 - \omega_1^2) + \theta_{11} g_1 (\rho(1)) + (1 - \theta_{11}) \rho(1) + \\
+ (\theta_{12} - 1) g_2 (\gamma(1)) - \theta_{12} \gamma(1) + \phi_2(1) - \phi_1(1). \] (12)
To simplify the notation given that the elements of the economy \( \mathcal{E} \) are fixed we will write
\[ z^1(p, W) = z^1(p). \]
Recall that from the Walras’ Law \( p \) is an equilibrium price for the economy \( \mathcal{E} \) if and only if \( z^1(p) = 0 \). Consequently an equilibrium price is regular if and only if \( \frac{d}{dp} z^1(p) = 0 \).

In what follows, if there is no danger of confusion, we will use the expressions \( z(p) \) to indicate the excess demand for the good 1, and \( z'(p) \) to indicate the derivative of the excess demand function of good 1 with respect to \( p \), that is, \( z'(p) = \frac{d}{dp} z^1(p) \).

Now differentiating \( z(p) \) with respect to \( p \) we have
\[ z'(p) = -\omega_1^2 p^{-2} - \theta_{11} [g_1 (\rho(p)) - pp\rho(p)] p^{-2} - \theta_{11} \rho(p) p^{-1} \\
- \theta_{12} [pg_2 (\gamma(p^{-1})) - \gamma(p^{-1})] p^{-2} + \theta_{12} g_2 (\gamma(p^{-1})) p^{-1} \\
+ \phi_1 (p^{-1}) p^{-2} + \phi'_1 (p^{-1}) p^{-3} + \phi_2 (p) + \rho'(p) + p^{-3} \gamma'(p^{-1}) \] (13)
To obtain an economy with a multiplicity of equilibria it is enough that the following condition holds,
functions of the form $x$

The fundamental fact in Theorem 6 is to assume the existence of some Remark 6.

and $\alpha, \beta$

Let’s assume that firms are characterized by $x$

Example 3. Let’s assume that firms are characterized by $x_2 = g_1(x_1) = ax_1^\alpha$ and $x_1 = h(x_2) = bx_2^\beta$, for $\alpha, \beta \in (0, 1)$ and $a, b > 0$ such that $\alpha a = 1$ and $b \beta = 1$. While non linear part of the utility functions for consumers are given by quadratic functions of the form

$$f_i(x) = a_i x - \frac{1}{2} x^2$$

with $a_i > 0$, $i = 1, 2$. In what follows we assume that parameters $\theta_{ij}$, $i, j = 1, 2$ and $\alpha, \beta$ are fixed, therefore our propose is to determine conditions on $a_1, a_2$ that guaranty the multiplicity of equilibria.

Then from a direct computation we obtain:

$$z(1) = \theta_{11} (a - 1) + 1 + \theta_{12} (b - 1) + a_2 - a_1,$$

$$z'(1) = -\omega - \theta_{11} a + \theta_{12} + (a_1 - 3) + \frac{1}{\beta - 1} + \frac{1}{\alpha - 1}.$$
Assuming that \( p = 1 \) is an equilibrium point, from equation (14) we need \( z'(1) > 0 \), in order to guaranty the existence of multiple equilibria, therefore from equation (16) we obtain the following condition on \( a_1 \):

\[
\frac{a_1}{1 - \beta} + \frac{1}{1 - \alpha} + 3. \quad (17)
\]

Once we fix a parameter \( a_1 \) satisfying last equation, we use equations (15) and (12) to obtain the following condition on \( a_2 \) that guaranty that \( p = 1 \) is an equilibria price.

\[
a_2 = \theta_{11}(1 - a) + \theta_{12}(1 - b) + b - 1 + a_1 \quad (18)
\]

**Example 4.** To enlighten how to apply the last two equations we consider a Shapley-Shubik private ownership economy defined by

\[
E = \{ \mathbb{R}^2, u^1(x, y), \omega^1, u^2(x, y), \omega^2, g_1, g_2, \theta_{ij}, i, j = 1, 2 \}.
\]

For consumers we assume that

\[
u^1(x_1, x_2) = x_1 + f_1(x_2), \quad \omega^1 = (2\omega), \quad \theta^1 = (\theta_{11}, \theta_{12})
\]

and

\[
u^2(x_1, x_2) = f_2(x_1) + x_2, \quad \omega^1 = (\omega, 25), \quad \theta^1 = (\theta_{11}, \theta_{12})
\]

and for technology function

\[
x_2 = g_1(x_1) = \frac{1}{\alpha} x_1^\alpha, x_1 = g_2(x_2) = \frac{1}{\beta} x_2^\beta.
\]

Where \( f_i(z) = a_i z - \frac{1}{2} z^2 \), \( \omega = \frac{94}{35} \), \( \alpha = 0.3 \) and \( \beta = 0.35 \). With sharing \( \theta_{11} = 0.75 \) and \( \theta_{12} = 0.4 \), therefore we need to find \( a_1 \) and \( a_2 \) that guaranty the existence of multiple equilibria in the economy with production describe above. Then using equation (17), we need \( a_1 > 12 \). For the sake of convince we set \( a_1 = \frac{2189}{140} \), then from equation (18) we have \( a_2 = 15 \). Therefore the excess demand function is given by

\[
z(p) = \frac{94}{91} p^{-1} + \frac{7}{4} p^{-10/7} + \frac{26}{35} p^{35/65} - \frac{2189}{140} p^{-1} + p^{-2} + 15 - p + p^{-10/7} - \frac{94}{91} \frac{100}{35} p^{100/65}
\]
6.2. Constructing economies with production with a prefixed equilibrium. In this section, we will show that, fixing a positive price \((p,1)\) and considering initial endowments large enough, it is possible to construct Shapley-Shubik economies with production for which \((p,1)\) is a regular equilibrium price, and with at least three equilibria. To do this we will consider that the non-linear portions of consumers’ utility functions are quadratic of the form 
\[ f_i(z) = a_iz - \frac{1}{2}z^2, \ a_i > 0, \ i = 1,2, \] 
and that firms produce according to the technological function described in example 4.

Let’s assume 
\[ u_1(x_1,x_2) = x_1 + a_1x_2 - \frac{1}{2}x_2^2, \ u_2(x_1,x_2) = a_2x_1 - \frac{1}{2}x_1^2 + x_2, \] 
with \(a_1, a_2 > 0\) to be determined, consider \(x_2 = ax_1^\alpha\) and \(x_1 = bx_2^\beta\), with \(\alpha, \beta \in (0,1)\), \(a\alpha = 1\) and \(b\beta = 1\) fixed, from equations (11) and (13), and a direct computation we obtain
\[
z(p) = \omega p^{-1} + \theta_{11}(a-1)p^{1/(\alpha-1)} + \theta_{12}(b-1)p^{-\beta/(\beta-1)}
\]
\[-(a_1-p^{-1})p^{-1} + (a_2 - p) + p^{1/(\alpha-1)} - \omega - bp^{-\beta/(\beta-1)},
\]
and
\[
z'(p) = -\omega p^{-2} - \left(\theta_{11}a - \frac{1}{\alpha-1}\right)p^{(2-\alpha)/(\alpha-1)}
\]+\left(\theta_{12} - \frac{1}{\beta-1}\right)p^{(1-2\beta)/(\beta-1)} + (a_1-p^{-1})p^{-2} - p^{-3} - 1.
\]
In a similar way to the one did in the example 1, we solve \(z'(p) > 0\) for \(a_1\) and \(z(p) = 0\) for \(a_2\), then have
\[
a_1 > \omega + \theta_{11}ap^{\alpha/(\alpha-1)} - \theta_{12}p^{1/(\beta-1)} + 2p^{-1}
\]
\[+p^2 + \frac{1}{1-\beta}p^{1/(1-\beta)} + \frac{1}{1-\alpha}p^{\alpha/(\alpha-1)}.\tag{19}\]
Choosing any value of \(a_1\) satisfying this equation \(a_2\) is uniquely determine by the formula
\[
a_2 = (1-p^{-1})\omega + (\theta_{11}(1-a) - 1)p^{1/(\alpha-1)}
\]+\((\theta_{12}(1-b) + b)p^{-\beta/(\beta-1)} + (a_1-p^{-1})p^{-1} + p,\tag{20}\]
then for any election of \(a_1\) satisfying (19) we obtain different economies with production for which \(p\) is an equilibrium price (equation (20) guaranties this fact). Moreover, by construction this economies have at least three equilibria prices with \(p\) unstable.

Example 5. Let’s take \(\theta_{11} = 0.75, \ \theta_{12} = 0.4, \ \alpha = 0.3, \ \beta = 0.35\) and \(\omega = \frac{94}{30}\). In order to illustrate, we construct an economy with multiple equilibria, one of which we choose to be \(p = 2\). Using this values in 19 we obtain the condition on \(a_1, a_1 > 13.26596422679823\), then for convenience we choose \(a_1 = 14\), from equation 20 we have \(a_2 = 11.18083150184725\). Then the graphic of the excess demand function is represented in the figure below, which equilibria prices are given by 
\(p_1 \approx 0.1792472498915, \ p_2 = 2\) and \(p_3 \approx 2.6915745984313\).
From examples 1 and 5 is clear that the fundamental thing in the construction of economies with multiple equilibria lies in finding a value for the parameter $a_1$ satisfying (17) or in the general case inequality (19). Even when the construction in example 5 is more general this implies a little more of computation work, therefore since in essence both construction are equal we prefer to continue working with the particular case where the basis equilibrium is $p = 1$, in each construction we will point out how to modify it to encompass the general case.

7. Distributions on the production branches of an economy. In this section, we consider economies such as those described in section 3, but now we assume the existence of several firms distributes into two branches of production, each of which is characterized by a different technology. Fixing all the parameters except for the number of firms using each of the available technologies, we can identify the economies with a distribution of firms on the set of branches of production, what means that each economy is identified with a distribution $N = (N_1, N_2)$. Where $N_1$ represents the number of firms in branch 1, and $N_2$ represents the number of firms in branch 2. To resume this fact, we will use the symbol $E_N$ to represent a generalized Shapley-Shubik economy such that the distribution of firms is given by $E_N$.

Definition 9. GSSEPB (A generalized Shapley-Shubik production economy with a distribution on production branches) For a distribution $N = (N_1, N_2)$ where $N_2 = N - N_1$ an economy is a set
\[ E_N = \{ \mathbb{R}^2, (u^i(x_1, x_2, \omega^i), Y_1, Y_2, \theta_{i,j}, N, i, j = 1, 2) \} \]
such that:

- For consumers we have that
  \[ u^1(x_1, x_2) = x_1 + f_1(x_2), \quad \omega^1 = (w_1, \omega), \quad \theta^1 = (\theta_{11}, \theta_{12}) \]
  and
  \[ u^2(x_1, x_2) = f_2(x_1) + x_2, \quad \omega^2 = (\omega, \omega_2), \quad \theta^2 = (\theta_{21}, \theta_{22}), \]
with $f_i : [0, \infty) \to \mathbb{R}$ differentiable and strictly quasi-concave functions such that $\lim_{z \to \infty} f_i'(z) \leq 0$ and $B = \lim_{z \to 0} f_i'(z)$. 

Figure 4. Example 5: equilibria prices at $p_1 \approx 0.1792472498915$, $p_2 = 2$ and $p_3 \approx 2.6915745984313$
• The technological restriction $Y_1$ and $Y_2$ are characterized by functions $g_1$ and $g_2$, verifying to be strictly concave and smooth. Moreover, we assume that the production set takes the form

$$T_1 = \{ (x_1, x_2) \in \mathbb{R}_+^2 : x_2 \leq g_1(x_1) \} \text{ and } T_2 = \{ (x_1, x_2) \in \mathbb{R}_+^2 : x_1 \leq g_1(x_2) \}$$

• The real number $\theta_{ij} \in [0, 1]$ represent the shares in the profit of firm $j$ of the $i$-th consumer.

In the following, we will consider that all the elements of the economy, except the distribution $N$ are fixed. Even when the total of firms is fixed and equal to $N$, we will consider the possibility of continuous changes over time in the distribution of the firms, this will be represented by the function $N : \mathbb{R}_+ \rightarrow \Delta$ where $\Delta = \{ (N_1, N_2) \in \mathbb{N}^2 : N_1 + N_2 = N \}$ so, $N(t) = (N_1(t), N_2(t)) \in \Delta$ for all $t \geq 0$. The changes in the distribution of the firms give rise to a continuous economies that will be represented by $E_{N(t)}$.

So, $N_1(t)$ represents the number of firms operating in branch 1, and $N_2(t)$ firms in branch 2, in each time $t$. According to the decisions of the managers these quantities change along the time. But we assume that the total number of firms is a constant $N$. That is $N = N_1(t) + N_2(t)$ for all $t$.

We will denote by $F_k$ the set of firms in branch $k = 1, 2$. Each firm in branch $F_k$ uses the technology $Y_k, k = 1, 2$. Then a distribution on the set of production branches can be considered as a distribution on the set of available technologies. We assume that technologies are free, and that managers can change of technology without costs. Changes in the distribution of the firms impact in the behavior of consumers, this is clear form the fact that the income of each consumers is a function on $N = (N_1, N_2)$

$$W^i(p, N) = p \cdot \omega^i + \sum_{k=1}^2 N_k \theta_{ik} \pi_k(y_k(p)),$$

We assume that the supply and consequently the profits of the firms in the same branch of production are the same, i.e., $y_j(p) = y_k(p)$ for all $j \in F_1$ and $y_j(p) = y_k(p)$ for all $j \in F_2$ consequently, $\pi_j(y_j(p)) = \pi_1(y_1(p))$ for all firm $j \in F_1$, and $\pi_j(y_j(p)) = \pi_2(y_2(p))$ for all firm $j \in F_2$.

A distribution of probability of the firms over the set of production branches is given by $n(t) = (n_1(t), n_2(t)) = \left( \frac{N_1(t)}{N}, \frac{N_2(t)}{N} \right)$. Considering a distribution $N = (N_1, N_1)$ the definition of excess demand function (see equation (21)) takes the form:

$$z(p, N) = x^1(p) + x^2(p) - \omega^1 - \omega^2 + N_1y_1(p) - N_2y_2(p).$$

7.1. Characterization of equilibria prices for any given distribution. Recall that from Walras’ law $z(p, N) = 0$ if and only if $z_1(p, N) = 0$. Moreover, since $N = (N_1, N - N_1)$ we define for each $N$ the equilibrium set

$$E_N = \{ p \in \mathbb{R} : z(p, N_1) = 0 \}.$$

So, once a distribution $N^* = (N_1^*, N_2^*)$, where $N_2^* = N - N_1^*$ is fixed, and taken into account that the aggregate supply of firms in branch, $i = 1, 2$, is given by $N_i^* y_i(p)$ and following the notation introduced in section 6, the excess demand function for distribution $N^*$ can be written in the form...
\[ z(p, N_1^*) = \omega_1^2 p^{-1} + \theta_{11} N_1^* [g_1 (\rho(p)) - \rho p(p)] p^{-1} \\
+ \theta_{12} (N - N_1^*) [p g_2 (\gamma(p^{-1})) - \gamma(p^{-1})] p^{-1} - \phi_1 (p^{-1}) p^{-1} \]

\[ + \phi_2 (p) + N_1^* \rho(p) - \omega_1^2 - (N - N_1^*) g_2 (\gamma(p^{-1})) \]  

(21)

then \((p, 1) = (p(N_1^*), 1)\) is an equilibrium price for the economy if and only if

\[
\phi_2(p) = - \omega p^{-1} - \theta_{11} N_1^* [g (\rho(p)) - \rho p(p)] p^{-1} \\
- \theta_{12} (N - N_1^*) [p h (\gamma(p^{-1})) - \gamma(p^{-1})] p^{-1} \\
+ \phi_1 (p^{-1}) p^{-1} - N_1^* \rho(p) + \omega + (N - N_1^*) h(\gamma(p^{-1}))
\]

Recall that from equation (9), \(\phi_2(p) = x_1^2(p)\) and \(\phi_2(p)\) represents the individual demand of first good by consumer 2. From equation (7) we have that \(x_1^2(p) = \phi_1 (p^{-1})\). In section 6, we showed that

- The supply of firm 1 is given by \(y_1(p) = (-\rho(p), g(\rho(p)))\).
- The profits for firm 1 is \(\pi_1(p) = g(\rho(p)) - \rho p(p)\).
- The supply of firm 2 is given by \(y_2(p) = (h(\gamma(p^{-1})), -\gamma(p^{-1}))\).
- The profits of firm 2 is \(\pi_2(p) = p h(\gamma(p^{-1})) - \gamma(p^{-1})\).

Hence, equation (22) can be rewritten for productive Sahpley-Shubik economy with distribution, (see definition 9)

\[
p \left[ \phi_2(p) - \omega \right] = - \left[ \omega_1^2 p + \omega + \theta_{11} N_1^* \pi_1(p) + \theta_{12} (N - N_1^*) \pi_2(p) \right] + \\
+ \phi_1 (p^{-1}) + \omega_1^2 p + (-N_1^* \rho(p) + (N - N_1^*) h(\gamma(p^{-1}))) p.
\]

Given that \((p, 1)\) is an equilibrium for the economy \(\mathcal{E}_N\), last equation has the following economical interpretation: the value of the individual excess demand for the first good of the second consumer equals the value of the net output of the commodity 1 minus the income of the consumer 1 plus \(\phi_1(p^{-1}) + \omega_1^2 p\), i.e.,

\[
p z_1^1(p) = p \left( N_1 x_1^{f_1}(p) + (N - N_1) x_1^{f_2}(p) \right) - W^1(p) + x_2^1(p) + \omega_1^2 p
\]

Where \(x_1^{f_1}(p) < 0\) is the demand for the inputs of firm 1 and \(x_1^{f_2} > 0\) is the product of firm 2.

**Definition 10.** Given an economy \(\mathcal{E}_N\)

- \(p \in E_N\) is a regular price of the economy if and only if \(z'(p, N_1) \neq 0\), in other case we say that \(p\) is a critical equilibrium price.
- The economy \(\mathcal{E}_N\) is regular if for all \(p : z(p, N_1) = 0\) it follows that \(z'(p, N_1) \neq 0\), i.e.; if all equilibria prices are regular. In other case the economy is critical or singular.

7.2. **Multiplicity of equilibria condition.** Differentiating equation (21) we have
Corollary 9. For a given distribution \( N_1^* \), the economy. Moreover, if equations (22) and (24) hold for some \( p > 0 \), the list of equilibria prices for the GSSEPB has uniqueness of equilibrium. 

Theorem 7. Let a GSSEPB \( E_{N^*} \). If for the distribution \( (N_1^*,N - N_1^*) \) there exist a positive price \( p \) such that equation (22) holds, then \( p \) is an equilibrium price for the economy. Moreover, if equations (22) and (24) hold for some \( p > 0 \), then \( p \) is a equilibrium price and the economy has at least three equilibria prices.

The next theorem is the reciprocal of the previous one.

Theorem 8. Let \( (N_1^*,N - N_1^*) \) a fix distribution and let \( p_1,p_2,\ldots,p_{2k+1} \) a complete list of equilibria prices for the GSSEPB \( E_{N^*} \). If \( k > 0 \) and \( \phi_1(p_j^{-1}) > 0 \), \( j = 1,2,\ldots,2k+1 \), then there exist some \( j \) such that \( \phi_1(p_j^{-1}) \) satisfies (24).

Corollary 9. For a given distribution \( (N_1,N - N_1) \) and a complete list \( p_1,p_2,\ldots,p_{2k+1} \) of equilibria prices for \( N_1 \), if \( \phi_1(p_j^{-1}) > 0 \) and

\[
\phi_1(p_j^{-1}) < \omega + \theta_{11}N_1^*[g_1(\rho(p_j)) - \rho(p_j)p_j] + \theta_{11}N_1^*[p_j]\]

\[
+ \theta_{12}(N - N_1^*)[p_jg_2(\gamma(p_j^{-1})) - \gamma(p_j^{-1})] - \theta_{12}(N - N_1^*)g_2(\gamma(p_j^{-1}))p_j \]  

\[
- \phi_1(p_j^{-1})p_j^{-1} - \phi_2(p_j)p_j^2 - N_1^*\rho(p_j)p_j^2 - (N - N_1^*)p_j^{-1}\gamma'(p_j^{-1})
\]

for any \( j = 1,2,\ldots,2k+1 \), then \( p_1 = p_2 = \cdots = p_{2k+1} \). That is, the economy has a unique equilibrium for distribution \( (N_1,N - N_1) \).

Corollary 10. Let \( N^* = (\tilde{N}_1,N - \tilde{N}_1) \) and \( N^{**} = (N_1^*,N - N_1^*) \) be distributions such that the GSSEPB \( E_{N^*} \) has multiple equilibria. If the distribution \( N^{**} \) verify for all \( p > 0 \) the inequality

\[
\phi_1(p^{-1}) < \omega + \theta_{11}N_1^*[g_1(\rho(p)) - \rho(p)p] + \theta_{11}N_1^*[p]
\]

\[
+ \theta_{12}(N - N_1^*)[g_2(\gamma(p^{-1})) - \gamma(p^{-1})] - \theta_{12}(N - N_1^*)g_2(\gamma(p^{-1}))p
\]

\[
- \phi_1(p^{-1})p^{-1} - \phi_2(p)p^2 - N_1^*\rho(p)p^2 - (N - N_1^*)p^{-1}\gamma'(p^{-1})
\]

Then there exists a distribution \( N^{**} = (N_1^*,N - N_1^*) \), \( \tilde{N}_1 < N_1^* \leq N_1^* \) with the property of being the distribution with the lower number of firms in branch 1, for which the economy \( E_{N^{**}} \) has uniqueness of equilibrium.

}\[
z'(p) = -\omega p^{-2} - \theta_{11}N_1^*[g_1(\rho(p)) - \rho(p)p]p^{-2} - \theta_{11}N_1^*[\rho(p)p]
\]

\[
-\theta_{12}(N - N_1^*)[p g_1(\gamma(p^{-1})) - \gamma(p^{-1})]p^{-2} + \theta_{12}(N - N_1^*)\gamma(\gamma(p^{-1}))p^{-1}
\]

\[
+ \phi_1(p^{-1})p^{-2} + \phi_2(p)p^{-3} + \phi_3(p) + N_1^*\rho(p) + (N - N_1^*)p^{-3}\gamma'(p^{-1}).
\]
A milder condition in last corollary can be stated as follows.

**Corollary 11.** Suppose that \( \phi_1(p) > 0 \) for all \( p > 0 \) and that \( \mathcal{N} = (\bar{N}_1, N - \bar{N}_1) \) is a distribution of the firms on the set of branches of production such that the economy \( \mathcal{E}_N \) has multiplicity of equilibria prices. If there exists some other distribution \( \mathcal{N}^* = (N_1^*, N - N_1^*) \), with \( \bar{N}_1 < N_1^* \leq N \), for which inequality (26) holds for all \( p \) such that \( z(p, \mathcal{N}^*) = 0 \), then the economy change from multiple equilibria to one equilibrium for some distribution \( \mathcal{N}^u = (N_1^u, N - N_1^u) \), with \( N_1 < N_1^u \leq N_1^* \).

The corollaries above give us conditions for changes from multiplicity of equilibria to uniqueness. Next corollary describe simple conditions to change from uniqueness to multiplicity of equilibria.

**Corollary 12.** Suppose that \( \phi_1(p) > 0 \) for any \( p > 0 \) and that \( \mathcal{N} = (\bar{N}_1, N - \bar{N}_1) \) is a distribution of the firms on the set of branches of production such that the GSSPB \( \mathcal{E}_N \) has uniqueness of equilibrium. If there exists some distribution \( \mathcal{N}^* = (N_1^*, N - N_1^*) \), with \( \bar{N}_1 < N_1^* \leq N \), for which (24) holds for some \( p > 0 \) such that \( z(p) = 0 \), then the economy change from uniqueness of equilibrium to multiplicity whenever \( \mathcal{N}^* \) is the distribution with the lower number of firms in branch 1, for which (24) holds.

**Remark 7.** The corollaries 10, 11 and 12 give conditions ensuring that a GSSPB change the number of equilibria when the distribution of the firms change. They show the hypotheses that ensure that if the number of companies in branch 1 increases and we starts from an economy with multiple equilibria, the economy changes to one with a single equilibrium. This change in the number of equilibria also occurs if we start from a distribution such that the number of firms in branch 1 is high enough and the economy has a multiplicity of equilibria and there is a flow from branch 1 to branch 2.

Finally recall that a singular economy is one for which there exist some equilibrium price \( p > 0 \) for which \( z'(p) = 0 \). Then we have,

**Theorem 13.** If for a distribution \( \mathcal{N} = (N_1^*, N - N_1^*) \) there exist a positive price \( p \) such that equation (22) holds and

\[
\begin{align*}
\phi_1(p^{-1}) &= \omega + \theta_{11} N_1^* \left[ g_1(\rho(p)) - \rho(p) \right] + \theta_{11} N_1^* \rho(p)p \\
&+ \theta_{12} (N - N_1^*) \left[ pg_2(\gamma(p^{-1})) - \gamma(p^{-1}) \right] - \theta_{12} (N - N_1^*) g_2 \left( \gamma(p^{-1}) \right) p \\
&- \phi_1'(p^{-1}) p^{-1} - \phi_2'(p) p^2 - N_1^* \rho'(p)p^2 - (N - N_1^*) p^{-1} \gamma'(p^{-1}).
\end{align*}
\]

Then \( p \) is a singular equilibrium for the economy \( \mathcal{E}_N \).

**Proof.** Equation (22) is equivalent to \( z(p) = 0 \) and using (23) we can show directly that \( z'(p) = 0 \) is equivalent to

\[
\begin{align*}
\phi_1(p^{-1}) &= \omega + \theta_{11} N_1^* \left[ g_1(\rho(p)) - \rho(p) \right] + \theta_{11} N_1^* \rho(p)p \\
&+ \theta_{12} (N - N_1^*) \left[ pg_2(\gamma(p^{-1})) - \gamma(p^{-1}) \right] - \theta_{12} (N - N_1^*) h \left( \gamma(p^{-1}) \right) p \\
&- \phi_1'(p^{-1}) p^{-1} - \phi_2'(p) p^2 - N_1^* \rho'(p)p^2 - (N - N_1^*) p^{-1} \gamma'(p^{-1}).
\end{align*}
\]
7.3. General examples of GGSEPB. Let’s consider again the example 3 in which the firms are characterized by the technology functions, \( x_2 = g_1(x_1) = ax_1^\alpha \) and \( x_1 = g_2(x_2) = bx_2^\beta \), for \( \alpha, \beta \in (0, 1) \) and \( a, b > 0 \) such that \( \alpha = 1 \) and \( b\beta = 1 \). Utility functions are \( u_1(x_1, x_2) = x_1 + f_1(x_2) \) and \( u_2(x_1, x_2) = f_2(x_1) + x_2 \)

\[ f_i(x) = a_i x - \frac{1}{2} x^2 \]

with \( a_i > 0, i = 1, 2 \). In what follows we assume that parameters \( \theta_{ij}, i, j = 1, 2 \) and \( \alpha, \beta \) are fixed, therefore our propose is to determine conditions on \( a_1, a_2 \) that guaranty the multiplicity of equilibria. equations (17) and equation (18) becomes

\[ a_1 \geq \omega + \theta_{11} N_1 a - \theta_{12} (N - N_1) + \frac{N_1}{1 - \alpha} + \frac{N - N_1}{1 - \beta} + 3 \]  

(29)

and

\[ a_2 = \theta_{11} N_1 (1 - a) + \theta_{12} (N - N_1) (1 - b) - N_1 + (N - N_1) b + a_1 . \]  

(30)

Therefore, once that parameters \( \alpha, \beta, a_1, a_2, \omega, \theta_{11}, \theta_{12} \) and the number of firms \( N \) are fixed the number of possible equilibria prices will depend on the distribution \( N = (N_1, N_2) \) of the firms over the set of available technologies. If for some \( N_1 \) the inequality (29)and equality (30) are verified the economy will have three equilibria.

Theorem 14. Given the parameters \( \alpha, \beta, a_1, a_2, \omega, \theta_{11} \) and \( \theta_{12} \). Let \( N = (N_1, N_2) \) be a distribution of the firms over the branches of production, such that

1. The equality in (29) is verified, then the economy has a unique singular equilibrium at \( p = 1 \).
2. \( \omega + \theta_{11} N_1 a - \theta_{12} (N - N_1) + \frac{N_1}{1 - \alpha} + \frac{1}{1 - \beta} + 3 > a_1^{n_1} \), then the economy is regular with at least three equilibria.
3. In other cases the economy has only one regular equilibrium

Proof. 1. It follows from the fact that \( \omega + \theta_{11} N_1 a - \theta_{12} (N - N_1) + \frac{N_1}{1 - \alpha} + \frac{1}{1 - \beta} + 3 = a_1^{n_1} \) if and only if \( z'(1) = 0 \).

2. It follows from the fact that

\[ a_1 > \omega + \theta_{11} N_1 a - \theta_{12} (N - N_1) + \frac{N_1}{1 - \alpha} + \frac{1}{1 - \beta} + 3 \]

if and only if \( z'(1) > 0 \).

Remark 8. Depending on the distribution of the firms on the available technology sets, the economy may be singular or regular and with multiplicity or uniqueness of equilibrium. Given that the consumers are the shareholders of the firms, their incomes depend on the distribution, and therefore equilibria prices depend also on the distribution. So, we can consider the equilibria prices \( p \) as a correspondence \( p : (0, 1) \to E_N \). Where

\[ E_N = \{ p \in \mathbb{R}_+ : z_N(p) = 0 \} . \]

This shows that the decisions of the managers of the firms, which, in principle, would only affect the productive side of the economy, can have repercussions on the global behavior of the economies. In section 9, we will analyze how these decisions can affect the welfare of the economy.
Example 6. Consider an economy

\[ \mathcal{E} = \{ \mathbb{R}^2_+, (u^1, \omega^1), (u^2, \omega^2), h, g \} \]

where \( u^1(x_1, x_2) = x_1 + f^1(x_2) \) and \( u^2(x_1, x_2) = x_2 + g(x_1) \) are the utility functions and \( f_i(z) = a_i z - \frac{1}{2} z^2, i = 1, 2 \) and \( a_1 = 90, a_2 = \frac{2059}{14} \) endowments \( \omega = ((20/91, 16/91), (16/91, 25)) \) and \( \theta_{11} = 0.75, \theta_{12} = 0.4 \), the share in the profits of firms of consumers 1 and \( \theta_{21} = 0.25, \theta_{22} = 0.6 \), for consumer 2, and the technology function verifying

\[ x_2 = g_1(x_1) = \frac{1}{\alpha} x^\alpha, x_1 = g_2(x_2) = \frac{1}{\beta} x^\beta \]

with \( \alpha = 0.3 \) and \( \beta = 0.35 \).

Note that the number of equilibria \( \Gamma \) of the economy depends on the distribution of the firms on the set of production branches. So \( \Gamma : D \to \mathbb{N} \) where
\( \mathcal{D} = \{ N \in \mathbb{R}_+^2 : N_1 + N_2 = N \} \) and \( N \) is the set of natural numbers. If the distribution \( \mathcal{E}_N \) is a regular economy, then there is a neighborhood \( V_N \subset \mathbb{R}_+^2 \) such that \( \Gamma(N) \) is constant for all \( N \in V_N \cap \mathcal{D} \). See [1].

In example 6 the economy has three equilibria for distributions (10, 40) and (11, 39), and have just one equilibrium for any other distribution. If seeking greater benefits, the managers of the firms decide to change their branches or production technology, the structure of the economy will be modified, for example by changing the number of equilibria. Before this happens the economy must have gone through a critical economy. This could lead to an abrupt change in the distributions and equilibrium prices of the economy.

8. **A dynamics for the economy.** The replicator dynamics as an appropriate representation for an economy in which rational decisions on investment are taken by the managers of the firms were firstly considered in [1]. Moving by the desire of the best returns for their investments, managers can change the technology or the production branch of their investments. Nowadays the transformation of the photography industry Kodak becomes a laboratory for medicaments against the COVID-19 is an interesting example. The investments flux from the branches where profits are lower to the branches with higher profits can be modelled using the replicator dynamics in the following way.

\[
\dot{n}_i = n_1(1 - n_1) (\pi_1(n) - \pi_2(n)),
\]

\[
\dot{n}_2 = -\dot{n}_1.
\]

Fixed the initial state of the economy, i.e.; a distribution of the firms over the production branches \( \mathcal{N}(t_0) = \frac{N_1(t_0)}{N} \) or equivalently \( n(t_0) = (n_1(t_0), n_2(t_0)) \) the evolution of the economy is given by the solution \( n(t, t_0) \) of the dynamical system (31) where \( n(t_0, t_0) = n(t_0) \) and corresponding to a continuous of economies \( \mathcal{E}_{\mathcal{N}(t)} \), where \( \mathcal{N}(t) = N n(t, t_0) \). Note that the distribution \( n(t) \) depends on time, because managers at every time can choose to change the production branches according with the relation profits obtained in time \( t \). That is, if \( \pi_1(n(t)) > \pi_2(n(t)) \) then \( \dot{n}_1(t) > 0 \) and a positive flux from branch 2 to branch 1 will occurs, if the identity reverse then the flux also. In case where \( \pi_1(n(t)) = \pi_2(n(t)) \) the economy is in steady state. Managers have not incentives to change of the branch of production.

To simplify the notation, when there is not possibility of confusion we omit the variable \( t \).

It is convenient to have, at least for the general examples discuss in this paper (see example 3), an expression for the steady state, then from a direct computation we can show that the profits of the firms are given by

\[
\pi_1(n_1) = (a - 1)p(n_1)\alpha/\alpha-1 \quad \text{and} \quad \pi_2(n_1) = (b - 1)p(n_1)^{-1/(\beta-1)}.
\]

We assume that at the end of every period, managers choose the branch of production according to the profits observed in each branch. We will witness a flow from the branches that offer the least profits to those with the highest profits. This implies a change in the distribution of the firms, quantified by the dynamical system (31). The process continues until profits equalize. The economy will now reach a stationary state, whose stability in the sense of Liapunov will indicate the possibilities of remaining as such or not overtime, under possible exogenous disturbances of the parameters of the system.
The steady state for our system is reached for some distribution \( n^* = (n_1^*, n_2^*) \) such that \( \pi_1(n_1^*) = \pi_2(n_2^*) \), i.e.:

\[
p^* = p(n_1^*) = \left[ \frac{a-1}{b-1} \right]^{(a-1)(b-1)} \alpha^\beta \left( \alpha^{-1} \right) \left( \beta^{-1} \right) = \left( \frac{a-1}{b-1} \right)^{a_1b_2}. \tag{33}
\]

So, if \( n_1 : p(n_1) > p^* \) then \( \pi_1(n_1) < \pi_2(n_2) \) and we will observe a flux from branch 1, to branch 2. and reciprocally is for \( n_1 \) we have that \( p(n_1) < p^* \).

The corollary 11 shows that there is some \( N_1^* \) such for all \( N_1 < N_1^* \) the economy has uniqueness of equilibrium and that if \( N_1 > N_1^* \) then the economy has multiplicity of equilibria. Hence the economy \( E_{N^*} \) where \( N^* = (N_1^*, N - N_1^*) \) is a singular economy. This means that if in time \( t_0 \) the profits \( \pi_1(n(t_0)) > \pi_2(n(t_0)) \) and the distribution on the branches verify the condition \( N_1(t_0) < N_1^* \), then managers prefer to invest in branch 1. In this case we will assist to a flux from branch 2, to branch 1. If this flux is maintained over time in such a way that at some instant \( t > t_0 \) the inequality \( \pi_1(n(t)) > \pi_2(n(t)) \) holds, where \( n(t) = (N_1(t)/N, N_2(t)/N) \) and \( N_1(t) > N_1^* \) then the economy will have undergone a significant change in its behavior, with repercussions on the welfare of consumers.

Let’s see an example:

**Example 7.** Using once again example (6) and noting that:

\[
N^u = (10, 40) \quad \text{see example 6)
\]

The distribution \( N^u = (10, 40) \) (see example 6), defines an economy \( E_{N^u} \) with multiplicity of equilibria. The distribution \( N^* = (14, 36) \) satisfy the hypotheses of the corollary 11, then there must be a distribution \( N^u \) such that the number of firms in branch 1 \( N_1^u \) corresponds to a minimum of the number of firms in branch 1 for distributions that determine economies with a single equilibrium and \( 10 < N_1^u < 14 \). The distribution for this example is \( N^u = (12, 38) \). In addition to the graphs (figure 7) it follows that the equation \( (26) \) it is not true for all \( p > 0 \), thus the condition given by equation \( (26) \) is just a sufficient condition for the uniqueness of the equilibrium, but not necessary.
As we will see in the next section the fact that the production side becomes more efficient or more rentable does not mean that something similar occurs in the consumption side of the economy.

Continuing with the analysis in last example, we proceed to analyze the profits of the firms in each branch. Then from a direct computation we obtain

\[
\pi_1(p) = \frac{7}{3}p^{-3/7} \quad \text{and} \quad \pi_2(p) = \frac{65}{35}p^{100/65}
\]

then using equation (33) we obtain that the equal-benefit price is given by

\[
p^* = \left[ \frac{49}{39} \right]^{91/179}
\]

In the following table we show equilibria prices and benefits for some distributions.

Since \( p^* \approx 1.1230431 \), we can see from the table above that there is not distribution for which the equal-prfits is reached, therefore we can conclude that
the steady state for this economy is not reachable. Note that for any distribution \( N = (N_1, N - N_1) \) with \( 0 \leq N_1 \leq 10 \), managers will prefer to invest in branch 1. Then, if the initial state of the economy is given by a distribution \( n(t_0) = (N_1(t_0)/50, N_2(t_0)/50) \) such that \( N_1(t_0) \leq 10 \) then, given that \( \pi_1(n(t_0)) > \pi_2(n(t_0)) \), according with the dynamical system (31) there will be an increasing flow towards branch 1. This positive flow remain until a time \( t^* > t_0 \) when \( N_1(t^*) = 12 \) because for all distribution \( N : 12 \leq N_1 < 50 \) we have that \( \pi_1(n(t_0)) < \pi_2(n(t_0)) \). From this moment on, the inequality in profits is reversed, but since equality \( \pi_1(n^*) = \pi_2(n^+) \) corresponds to a non-integer real number \( 50n^* \), it is possible that a cyclical process of emigration of firms from one branch to another and reciprocally begins. A symmetric process would be obtained if the initial state of the economy corresponds to a distribution \( N(t_0) \) such that \( 12 < N_1(t_0) < 50 \). In this case the flow will be from branch 1 to branch 2. see table 10.

It is important to note that, in the case in which the initial state of the economy is characterized by a distribution \( N(t_0) \) we will observe, as already mentioned, a positive flow towards branch 1. Consequently, at some time \( t = t^* \) the economy will have passed through a singularity since it goes from being an economy with a uniqueness of equilibrium to being one economy with a multiplicity of equilibria. Something similar happens if the initial state of the economy in time \( t_0 \) is characterized by a distribution \( N(t_0) \) such that \( 12 < N_1(t_0) < 50 \). In this case, the positive flow will be towards branch 2 and the transition to a distribution with \( N_1 = 11 \) would imply a sudden change in the behavior of the economy that would stop from having only one equilibrium to having three equilibria.

However, these two cases are not completely similar. Observe that when losing uniqueness, the economy that was initially in the state characterized by \( N_1(t_0) < 10 \) maintains, regardless of the price that is established when losing uniqueness, the same sense of the inequality in profits. For the second case, we will observe that there are two possibilities that the sense of the inequality of profits is reversed and only one that it remains. This facts alerts us to the impossibility of foreseeing \textit{a priori} the consequences of the rational decisions of the managers of the firms in a neighborhood of a singular economy. The impact of these decisions on the number

| \( N_1 \) | \( P \) | \( \pi_1(p) \) | \( \pi_2(p) \) |
|-------|------|----------|----------|
| 5     | 0.0358654 | 9.714076 | 0.011099 |
| 6     | 0.0455704 | 8.766515 | 0.016043 |
| 7     | 0.0580256 | 7.904130 | 0.023266 |
| 8     | 0.0741872 | 7.114125 | 0.033954 |
| 9     | 0.0956613 | 6.379756 | 0.050205 |
| 10    | 0.1256235 | 5.676613 | 0.076349 |
| 11    | 1      | 2.333333 | 1.857143 |
| 12    | 0.1725278 | 4.954926 | 0.124389 |
|       | 0.6155470 | 2.877271 | 0.880302 |
|       | 1.523654  | 1.948742 | 3.545224 |

\textbf{Figure 10.} Profit table for example 7.
and quality of balances can be very different. In the next section, we will analyze precisely the repercussions of these decisions on the welfare of consumers.

**Remark 9.** We consider an economy $E_n^*$ to be singular, even in the cases in which the probability distribution given by $n^* = (n_1^*, n_2^*)$ that defines the existence of a price $p$ such that $z(p, n^*) = 0$ and $dz(p, n^*)/dp = 0$ occurs for a distribution in which $Nn^*$ does not correspond to a positive integer value. This is not detrimental to the conclusions obtained, since the sudden and unexpected changes are perceived when we go from an economy characterized by a distribution $\mathcal{N}$ with $\bar{N}_1 < n_1^* \bar{N}$ to another characterized by a distribution $\mathcal{N}^*$ such that $N_1^* > n_1^* \bar{N}$.

9. **Repercussions on welfare of rational decisions on the production side.**

In this section, we will show that, although the investment decisions of company managers are rational and tend to increase the general profits of the productive side of the economy, this fact does not necessarily imply an improvement in the welfare of all consumers, even when they are shareholders of the firms, as in the case of a privately owned neoclassical economy to show that this assertion is true let us consider the utilities of the consumers of the example 7 when the distribution of the firms change according with the decision of managers.

In the table 11 are listed the values of the utilities of consumers when the distribution of the firms change.

| $N_1$ | $p$    | $u^1(x_1, x_2)$ | $u^2(x_1, x_2)$ |
|-------|--------|-----------------|-----------------|
| 5     | 0.0358654 | 2969.97331      | 10846.88104    |
| 6     | 0.0455704 | 3205.44727      | 10846.46648    |
| 7     | 0.0580256 | 3385.79796      | 10845.32682    |
| 8     | 0.0741872 | 3525.68448      | 10843.36349    |
| 9     | 0.0956613 | 3636.21159      | 10840.36528    |
| 10    | 0.1256235 | 3726.19327      | 10835.86728    |
|       | 1      | 4027.89011      | 10744.01171    |
|       | 1.0747100 | 4036.61083      | 10743.12541    |
| 11    | 0.1725278 | 3804.20679      | 10828.58676    |
| 12    | 0.6155470 | 3975.66125      | 10768.53385    |
| 12    | 1.5223654 | 4085.00484      | 10767.28381    |

| $N_1$ | $p$    | $u^1(x_1, x_2)$ | $u^2(x_1, x_2)$ |
|-------|--------|-----------------|-----------------|
| 12    | 1.7717621 | 4110.39822      | 10803.81133    |

Figure 11. Utility table for example 7.

Note that if the initial state of the economy is characterized by a distribution $\mathcal{N}(t_0)$ such that $0 < N_1(t_0) < 12$ then the number of firms in branch 1 tends to increase as result of the managers decisions. The values of the utility of consumer increases along this process, at the same time that the utility of consumers 2, decreases.

An inverse result on the utilities of consumers happens in the initial state corresponds to a distribution where $12 < N_1(t_0) < 50$. 
The considered example, although it represents a particular case, leads us to wonder if a public policy aimed at production could avoid undesirable situations in the welfare of consumers. Certainly this would suppose the participation of a central authority in the economy with the difficulties that this implies, in particular the possible adverse results.

9.1. **On the stability of the equilibria, an open question.** Even when the auctioneer dynamics considered by L. Walras and represented mathematically by P. Samuelson (see cite Sa) in the form of a system of differential equations

$$\dot{p}_i = k_i z_i(p), \ k_i > 0, \ i = 1, ... n$$

where \( n \) is the number of different products available in the economy, is controversial in economic theory, however, let us consider the following facts associated with equilibria in the case of the example 7 for the distributions corresponding to \( N_1 = 10 \) and 11. For each of the economies corresponding to these distributions there are three equilibria prices \((p_i, 1), i = 1, 2, 3\) verifying the inequalities \( p_1 < p_2 < p_3 \). The reader can easily check that according to the auctioneer’s dynamics the equilibrium \( p_2 \) is always unstable, and that each consumer would prefer another to this one, see table 11. Is it perhaps, this situation that makes this equilibrium unstable for the Samuelson-Walras dynamic? On the contrary, the equilibria \( p_1 \) and \( p_3 \) are stable. These equilibria are the most preferred for at least one of the consumers. The question that arises is whether it is this preference that makes equilibrium stability.

10. **Conclusions.** The multiplicity of the Walrasian equilibrium is an issue that worries and questions economic theory. Given the existence of multiple equilibria, what is the mechanism that makes the economy settle in one of them in particular? And furthermore, why does a regular economy with multiple equilibria, once established one of them, remain the same in time and do not change from one moment to another? One answer may lie in the stability of the Lyapunov equilibrium of a dynamic system similar to the one Samuelson considers to represent the well-known Walras auctioneer.

In the examples presented with three equilibria for regular economies, one of them is always less desired for one of the two consumers. They both prefer one of the other two although not the same. It can easily be seen that the first one, the unwanted one, is unstable for the Walras dynamic, while the other two are stable for that dynamic.

The stability of the current equilibrium is not opposed to the existence of economic changes that occur when the income of consumers changes. Among them, precisely, that of the set of possible equilibria for the new values of the parameters that define the economy. In our work, we consider that consumers are shareholders of companies, which is a common assumption in the models of privately owned neo-classical economies, the change in income is the result of the investment decisions of company managers.

For regular economies, changes in the productive side of the economy can lead to more efficient economies without major disturbances in the general welfare of the economy, or even increase it; however, not necessarily all consumers value these changes equally. Some of them may be adversely affected in their well-being.

Although in our work we only consider two consumers, it is possible to think that in many cases there are many affected and a lot in their consumption as a result of
rational decisions on the productive side of the economy. Does this possibility give rise to the consideration of industrial public policies that temper these changes that may be negative for the well-being of some consumers?

It should be noted that, in the event that the economy is critical, the changes caused by these investment decisions will be abrupt, unexpected, and impossible to predict.

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