Spin-Dependent Antenna Splitting Functions

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ABSTRACT

We consider parton showers based on radiation from QCD dipoles or ‘antennae’. These showers are built from $2 \to 3$ parton splitting processes. The question then arises of what functions replace the Altarelli-Parisi splitting functions in this approach. We give a detailed answer to this question, applicable to antenna showers in which partons carry definite helicity, and to both initial- and final-state emissions.

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Contents

1 Introduction 1

2 Proposal for the $2 \rightarrow 3$ splitting functions 2

3 Spin-0 case 7

4 Spin-1 and spin-2 case 9

5 Spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ cases 12

6 Initial-state showers 15

7 Comparison to previous results 21
1 Introduction

In the studies that are now being done to prepare for physics at the LHC, many new approaches have been proposed to the old problem of generating parton showers. The workhorse event generators PYTHIA [1] and HERWIG [2] generate parton showers by successive radiations from individual partons. The ‘splitting functions’ that define the radiation pattern are taken to be the kernels in the Altarelli-Parisi equation [3,4]. This guarantees that the radiation pattern is correct in the region in which two partons become collinear. Marchesini and Webber pointed out that it is also important to include color interference between emissions from different partons [5]. In the workhorse generators, this is implemented by angular ordering of emissions.

The program ARIADNE, by Andersson, Gustafson, Lönnblad, and Pettersson, took a different approach, implementing color coherence by considering the QCD dipole to be the basic object that radiates a parton [6,7]. The basic branching process in a parton shower is then a splitting in which two partons forming a color dipole radiate a third parton. This approach has been taken up recently by a number of authors. It is the basis for the VINCIA shower by Giele, Kosower, and Skands [8] and the parton shower implementation in SHERPA by Krauss and Winter [9]. We are also developing a parton shower based on this approach [10]. In the years between ARIADNE and the newer works, the term ‘dipole’ has been applied in QCD to a different strategy based on \(1 \rightarrow 2\) splittings with recoil taken up by a third particle [11]. To avoid confusion, we will follow [8] in calling the initial two-parton state an ‘antenna’ and a branching process with \(2 \rightarrow 3\) splittings an ‘antenna shower’.

Central to the antenna shower is the \(2 \rightarrow 3\) splitting function, the function that gives the relative branching probabilities as a function of the final momenta. The original ARIADNE program used an \textit{ad hoc} proposal satisfying the basic consistency requirements. It would be better to have a prescription that can be directly derived from QCD. Splitting to three partons has been studied in great detail in the QCD literature, but not for this application. Collinear systems of three partons are a part of the infrared structure of QCD at next-to-next-to-leading order, and calculations that reach this level need an explicit prescription for treating this set of infrared singularities. Kosower [12] defined the ‘antenna function’ as a basic starting point for the analysis of this problem. Many authors have computed antenna functions [13,14,15]. Quite recently, Gehrmann-De Ridder, Gehrmann, and Glover have built a complete formalism of ‘antenna subtraction’ for NNLO calculations [16]. The kernel in their theory can be interpreted as a \(2 \rightarrow 3\) splitting function, and it has been used to perform \(2 \rightarrow 3\) splitting in the VINCIA shower [8].

In this paper, we will take a much more direct route to the construction of \(2 \rightarrow 3\) splitting functions. We will compute these functions by writing local operators that
create two-parton final states and computing their 3-parton matrix elements. These calculations are very straightforward. They can be used to treat individually all possible sets of polarized initial and final partons.

This paper is organized as follows: In Section 2, we will present our complete set of spin-dependent $2 \to 3$ splitting functions. In Section 3, we will give the derivation for the cases with total spin zero. In Sections 4 and 5, we will give the derivation for the cases with nonzero total spin.

All of these derivations will be done in the kinematics of final-state radiation. This is the easiest situation to visualize and understand. However, the same splitting functions can be used, after crossing, to describe parton emissions that involve initial-state particles. We will explain how to use our expressions for initial-state showers in Section 6.

The $1 \to 2$ Altarelli-Parisi splitting functions are universal in the sense that they result from a well-defined singular limit of QCD amplitudes. For $2 \to 3$ splitting functions there is no such universality. The collinear and soft limits must agree with the known universal values, but away from these limits there is no unique prescription. Earlier in this introduction, we made reference to a number of previous proposals for the spin-averaged antenna splitting functions. All of these, including the ARIADNE splitting functions, have the correct soft and collinear limits and so satisfy the basic requirements. In Section 7, we will give a detailed comparison of the $2 \to 3$ splitting functions obtained using our method to previous proposals for these splitting functions.

2 Proposal for the $2 \to 3$ splitting functions

We begin by defining variables for $2 \to 3$ splitting. There are three cases of splittings that are needed for antenna showers: the final-final (FF) splitting, in which a third particle is created by coherent radiation from a two-particle system in the final state; the initial-final (IF) splitting, in which a third particle is created by coherent radiation from an initial- and a final-state particle; and initial-initial (II) splitting, in which a third particle is created by coherent radiation from two initial-state particles. It is easiest to understand the kinematics of antenna splitting for the FF case. In this section, we will explain this kinematics and give a precise prescription for the splitting functions. In Section 6, we will extend our prescription to the IF and II cases, in such a way that the same splitting functions can be used in those cases.

Consider, then, a two-parton final-state system $(A, B)$ that splits to a 3-parton system $(a, c, b)$, conserving momentum, as shown in Fig. 1(a). Let $s_{ij} = (k_i + k_j)^2$,
Figure 1: (a) Kinematics of $2 \rightarrow 3$ splitting in the final state (FF) case. (b) Phase space for $2 \rightarrow 3$ splitting in the FF case. The six regions corresponding to different orderings of $s_{ab}$, $s_{ac}$, $s_{bc}$ are shown. The region that should be well described by an antenna splitting $AB \rightarrow acb$ is shaded.

and let $Q = k_A + k_B = k_a + k_b + k_c$.

The fractional invariant masses in the final state are

$$y_{ab} = \frac{s_{ab}}{s_{AB}} , \quad y_{ac} = \frac{s_{ac}}{s_{AB}} , \quad y_{bc} = \frac{s_{bc}}{s_{AB}} . \quad (1)$$

The momentum fractions of the three particles in the $(AB)$ frame are

$$z_a = \frac{2Q \cdot k_a}{s_{AB}} , \quad z_b = \frac{2Q \cdot k_b}{s_{AB}} , \quad z_c = \frac{2Q \cdot k_c}{s_{AB}} . \quad (2)$$

These obey the identities

$$y_{ab} = (1 - z_c) , \quad y_{ac} = (1 - z_b) , \quad y_{bc} = (1 - z_a) . \quad (3)$$

and

$$y_{ab} + y_{ac} + y_{bc} = 1 , \quad z_a + z_b + z_c = 2 . \quad (4)$$

The FF phase space covers the triangle $z_a \leq 1$, $z_b \leq 1$, $z_a + z_b \geq 1$. We can divide this phase space into six triangles, each of which has a different ordering of the three quantities $y_{ab}$, $y_{ac}$, $y_{bc}$, as shown in Fig. 1(b). An antenna shower should give an accurate description of the dynamics in the two regions $y_{ac} < y_{bc} < y_{ab}$, $y_{bc} < y_{ac} < y_{ab}$ that are shaded in the figure.

A general problem in the generation of QCD radiation is that of possible double-counting. Consider, for example, the process $e^+e^- \rightarrow qgq\bar{q}$. In some part of the phase space, the first $g$ can be considered to be radiated from the antenna of the $q$ and the
second $g$; in another, the second $g$ can be considered to be radiated from the first $g$ and the $\overline{q}$. These regions should be disjoint in the full 4-body phase space. The complete solution to the problem is beyond the scope of this paper. In simple terms, though, we can make the separation by choosing the radiated gluon to be softer than the gluon from which it radiates. This corresponds to integrating each antenna only over the shaded region in Fig. 1(b). A similar approximate solution to the double-counting problem will apply in the other kinematic regions discussed in Section 6. A more detailed discussion of this issue can be found in [8,10].

Radiation from different QCD antenna is strictly independent and non-interfering only in the limit of a large number of colors in QCD, $N_c \gg 1$. Keeping only terms leading in $N_c$ is known to be a good approximation to full QCD in many circumstances. In particular, parton shower algorithms are correct only to leading order in $N_c$. In this paper, we will explicitly work only to the leading order for large $N_c$.

In the limit of large $N_c$, the rate for a $2 \rightarrow 3$ splitting is given by a formula of the form

$$N_c \frac{\alpha_s}{4\pi} \int dz_a dz_b \cdot S(z_a, z_b, z_c)$$

For example, in $e^+ e^- \rightarrow q_- g_+ \overline{q}_+$,

$$\frac{1}{\sigma_0} \frac{d\sigma}{dz_a dz_b} = N_c \frac{\alpha_s}{4\pi} \frac{z_a^2}{(1 - z_a)(1 - z_b)},$$

where $(a, c, b)$ are the $(q, g, \overline{q})$, respectively, and $-$ and $+$ denote left- and right-handed helicity, and $\sigma_0$ is the cross section for $e^+ e^- \rightarrow q_- \overline{q}_+$.[17] Eq. (5) will be our basic formula of reference. Using this notation, we can write the various $2 \rightarrow 3$ splitting functions as

$$S = \frac{N(z_a, z_b, z_c)}{y_{ab} y_{ac} y_{bc}},$$

where the numerator is a simple function of the $z_i$. For example, for the splitting $q_- \overline{q}_+ \rightarrow q_- g_+ \overline{q}_+$ given above,

$$N = y_{ab} z_a^2 = (1 - z_c)z_a^2.$$

In Table 1, we give our proposal for the numerator functions for all possible cases of massless quark and gluon splittings. The expressions are all monomials in the $y_{ij}$ and $z_i$.

In the FF kinematics, all of the $y_{ij}$ and $z_i$ are positive and so $S(z_a, z_b, z_c)$ in (7), is always positive. In IF and II kinematics, some $y_{ij}$ and $z_i$ will become negative. In most cases, the correct prescription is to take $S(z_a, z_b, z_c)$ to be the absolute value of the expression in Table 1. However, there is a line within the IF region where $z_a$ or
Table 1: Numerator functions $N(z_a, z_b, z_c)$ for the spin-dependent $2 \rightarrow 3$ splitting functions $AB \rightarrow acb$: $S = N/(y_{ab}y_{ac}y_{bc})$. Each line gives a choice of $AB$. The labels denote the polarization of the three final particles with the radiated particle $c$ in the center: $(h_a, h_c, h_b)$. The empty columns are forbidden by quark chiral symmetry. By the P and C invariance of QCD, the same expressions apply after exchanging $− \leftrightarrow +$, $q \leftrightarrow \bar{q}$, or $ABacb \leftrightarrow BAbca$.

| $g_+g_+ \rightarrow ggg$ | + + + | + + + | + + + | + + + | + + + | + + | + + | + + | + + | + + | + + | + + | + + | + + | + + | + + | + + | + + | + + |
|--------------------------|-------|-------|-------|-------|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| $g_-g_+ \rightarrow ggg$ | 0     | 0     | $y_{ab}$ | $y_{ac}$ | $y_{bc}$ | 0    | 0    | 0    | 0    | 0    |
| $g_+g_+ \rightarrow \bar{q}gq$ | -     | -     | $y_{ab}^3y_{bc}^2$ | $y_{ab}^2y_{bc}$ | -     | 0    | 0    | 0    | -    |
| $g_-g_+ \rightarrow \bar{q}gq$ | -     | -     | $y_{ab}^3y_{bc}^2$ | $y_{ab}^2y_{bc}$ | -     | 0    | 0    | 0    | -    |
| $q_-\bar{q}_+ \rightarrow qgq$ | -     | -     | -     | $y_{ab}y_{bc}$ | $y_{ab}z_b^2$ | - | - | - | - | - |
| $q_-\bar{q}_- \rightarrow qgq$ | -     | -     | -     | -     | $y_{ab}$ | - | $y_{ab}$ | - | - | - |
| $q_-g_+ \rightarrow qgg$ | -     | -     | -     | 0 | $y_{ac}^3$ | $y_{ab}^3$ | $y_{ac}^2$ | 0 | - | - |
| $q_-g_- \rightarrow qgg$ | -     | -     | -     | $z_a^3$ | $y_{ab}z_b^2$ | $y_{ac}^2$ | 0 | - | - | - |
| $q_-g_- \rightarrow q\bar{q}q$ | -     | -     | -     | - | $y_{ab}y_{ac}^2$ | $y_{ab}y_{ac}z_b$ | - | - | - | - |
| $q_-g_+ \rightarrow q\bar{q}q$ | -     | -     | -     | - | $z_ay_{ab}y_{ac}z_b^2$ | $z_ay_{ab}y_{ac}^3$ | - | - | - | - |

The splitting functions $S$ must give the correct universal behavior in the soft and collinear limits. In the soft limit, $z_c \rightarrow 0$, the numerators must go to 1 if the flavor and helicity of the final partons $a$ and $b$ match those of the initial partons $A$ and $B$; otherwise, the numerators must go to 0. It is easy to check that this test is satisfied.

In the collinear limits, we will insist that each antenna has the collinear behavior required in QCD. One often hears the following statement about soft and collinear limits: In dipole splitting ($1 \rightarrow 2$ emission), each dipole has the correct collinear behavior but the correct soft behavior is obtained by combining neighboring dipoles. In antenna splitting ($2 \rightarrow 3$ emission), each antenna has the correct soft limit but the correct collinear behavior is obtained by combining neighboring antennae. However, in the large $N_c$ limit, which we take to guide our intuition, different antennae are independent radiators with different, non-interfering, colors flowing in them. From the viewpoint of this limit, each antenna, separately, must give both the correct pattern of soft radiation and the correct pattern of collinear radiation. This philosophy differs from that of the ARIADNE group [6,7] and of [9]. We will discuss this point further when we compare with their results in Section 7.

The collinear radiation from a given hard gluon is then the sum of two contributions, one from each of the two antennae to which that hard gluon belongs. In
the large $N_c$ limit, these correspond to radiation from the color and anticolor lines of the gluon. A single antenna, which has one of these contributions, then has $1/2$ of the standard collinear emission rate. This factor of $1/2$ enters the check we will perform in a moment. The factor comes entirely from bookkeeping and is independent of the question of double-counting discussed briefly earlier in this Section.

We now discuss the check of collinear limits. Consider the limit in which $c$ becomes collinear with $a$. In this limit,

\[ z_c \to z, \quad z_a \to (1 - z), \quad z_b \to 1, \quad y_{ac} \to 0. \]

The $2 \to 3$ splitting function must reduce to

\[ S \to \frac{1}{y_{ac}} P(z), \]

here $P(z)$ is the relevant spin-dependent Altarelli-Parisi splitting function. These were presented in the original Altarelli-Parisi paper \cite{3} and are reviewed in Table 2. The functions are normalized as in (5), and as described in the previous paragraph: We take the large $N_c$ limit and divide by 2 where necessary to give the contribution from one QCD antenna. The denominator of (7) tends to $y_{ac}z(1 - z)$ in this limit. Then it is easy to check that the numerators match correctly in all cases. The limit in which $c$ becomes collinear with $b$ can be checked in the same way.

When the collinear limits and the soft limit are all nonzero, there is a unique monomial of the $y$’s and $z$’s that gives all limits correctly. In the other cases, there is some ambiguity. In all cases, it would be desirable if the results in Table 1 could be derived directly by simple Feynman diagram computations. In the next few sections, we will present those derivations.
3 Spin-0 case

To compute the $2 \rightarrow 3$ splitting functions, we will use the following method: Write an operator that, at the leading order, creates a 2-parton state with definite helicity. Then, compute the 3-particle matrix element. This realizes in a very simple way the splitting process illustrated in Fig. 1.

To create massless quarks and antiquarks of definite helicity, we will use the appropriate chiral fermion fields. To create gluons of definite helicity, we will use the operators

$$\sigma \cdot F = \frac{1}{2} \sigma^m \sigma^n F_{mn}, \quad \overline{\sigma} \cdot F = \frac{1}{2} \sigma^m \sigma^n F_{mn},$$

(11)

where $\sigma^m, \overline{\sigma}^m$ are the $2 \times 2$ matrix entries of the Dirac matrices in a chiral basis and $F_{mn}$ is the gluon field strength tensor. At leading order, $\sigma \cdot F$ creates a $+$ helicity gluon, and $\overline{\sigma} \cdot F$ creates a $-$ helicity gluon.

The 2-parton state $g_+ g_+$ in the first line of Table 1 can be created from the spin-0 operator

$$\mathcal{O} = \frac{1}{2} \text{tr}[(\sigma \cdot F)^2].$$

(12)

We can then compute the splitting function for this polarized initial state explicitly from the definition

$$S(z_a, z_c, z_b) = Q^2 \left| \frac{\mathcal{M}(\mathcal{O} \rightarrow abc)}{\mathcal{M}(\mathcal{O} \rightarrow AB)} \right|^2$$

(13)

In the next few sections, we will compute all of the splitting functions in Table 1 using this formula, with a different choice of the operator $\mathcal{O}$ for each line of the table.

To evaluate (13), we need to compute the matrix elements of $\mathcal{O}$, with total momentum $Q$ injected, to 3-gluon final states. The result can be expressed in terms of color-ordered amplitudes. We identify the color-ordered amplitude that multiplies the color structure $\text{tr}[T^a T^b T^c]$ with the splitting function. To carry out these computations, we will use the spinor product formalism. That is, instead of working with 4-vectors, we will use as our basic objects the spinor products

$$\langle ij \rangle = \overline{\alpha}_-(i) u_+(j), \quad [ij] = \bar{\alpha}_+(i) u_-(j).$$

(14)

These objects obey

$$|\langle ij \rangle|^2 = |[ij]|^2 = s_{ij}.$$  

(15)

Methods for QCD computations with spinor products and color-ordering are explained in [18,19]. In this notation, the matrix element for $\mathcal{O}$ to create a $g_+ g_+$ final state is

$$\langle g_+ g_+ | \mathcal{O} | 0 \rangle = [AB]^2.$$  

(16)
The three-gluon matrix elements of the operator (12) are given by the diagrams in Fig. 2. These diagrams have already been analyzed by Dixon, Glover, and Khoze as a part of their analysis of the coupling of the Higgs boson to multi-gluon states [20]. They find

\[ A(O \to g^+_g + g^-) = \frac{S_{AB}^2}{\langle ac \rangle \langle cb \rangle \langle ba \rangle} \]
\[ A(O \to g^+_g + g^-) = \frac{[ac]^4}{\langle ac \rangle [cb][ba]} \]
\[ A(O \to g^-g - g^+) = \frac{[ab]^4}{\langle ac \rangle [cb][ba]} \]
\[ A(O \to g^-g - g^+) = \frac{[bc]^4}{\langle ac \rangle [cb][ba]} \]

(17)

and zero for the other four cases. After squaring, using (15), and dividing by the square of (16), we obtain the first line of Table 1.

One of the major points of [20] is that the results (17) belong to series of Maximally Helicity Violating (MHV) amplitudes that have a simple form for any number of gluons emitted. Actually, all of the amplitudes that we will compute in this paper are similarly simple and belong to MHV series. The use of MHV amplitudes to study antenna splitting is explored for higher-order processes in [15].

In principle, the initial state \( g^+g^+ \) could also have been created by an operator of spin 2, or some higher spin. This would have led to a more complicated expression for the \( 2 \to 3 \) splitting function, with, however, the same soft and collinear limits. This illustrates the ambiguity in the definitions of \( 2 \to 3 \) splitting functions referred to in the introduction. The simplest results are obtained using the operator of minimal spin, and we will make that choice in all of the examples to follow.

The diagram shown in Fig. 3 gives the splitting of the two-gluon initial state to
We find
\[ \mathcal{A}(O \rightarrow q_+ q_- g_+) = \left[ \frac{ab}{ac} \right]^2 \]
\[ \mathcal{A}(O \rightarrow q_- q_+ g_-) = \left[ \frac{cb}{ac} \right]^2 \] (18)

There is no splitting to a final $g_-$. This gives the result in the third line of the table.

The initial state $q_- q_-$ can also be created by a spin 0 operator
\[ O = \bar{q}_L q_R . \] (19)

The matrix element for this operator to create a $q_- q_-$ final state is
\[ \langle q_- q_- \mid O \mid 0 \rangle = \langle AB \rangle \] . (20)

A straightforward calculation gives
\[ \mathcal{A}(O \rightarrow q_- g_+ q_-) = \frac{\langle ab \rangle^2}{(ac)(cb)} \]
\[ \mathcal{A}(O \rightarrow q_- g_- q_-) = \frac{s_{AB}}{[ac][cb]} \] (21)

These give the results shown in the sixth line of the table.

4 Spin-1 and spin-2 case

In [6], the $2 \rightarrow 3$ splitting function for $q\bar{q} \rightarrow gg\bar{q}$ was derived from the cross section for $e^+e^- \rightarrow qg\bar{q}$. From the point of view of the previous section, this corresponds to creating the 2- and 3-parton final states using the operator
\[ O = \bar{q}_L \gamma^m q_L . \] (22)
To obtain a definite matrix element, we must contract this operator with a polarization vector. A convenient choice is to introduce two new massless vectors 1 and 2, such that \( k_1 + k_2 = k_A + k_B \), and to choose the polarization vector to be \( \epsilon^\mu = \langle 1 | \gamma^\mu | 2 \rangle \). This is effectively the procedure of decaying the massive vector that couples to the operator (22) into a pair of massless vectors to facilitate the analysis; this is a standard method in spinor product calculations [21]. We then recast

\[ O = \frac{1}{2} \bar{u}_L \gamma^m q_L \langle 1 | \gamma_m | 2 \rangle . \]

(23)

The matrix element of (23) to a \( q^- q^+ \) state is

\[ \langle q^- q^+ | O | 0 \rangle = -\langle 1 A | 2B \rangle . \]

(24)

The direction of the 1-2 system chooses the helicity of the final partons. In this case, there is only one choice, and so the amplitude vanishes when 1 is parallel to \( A \) or 2 is parallel to \( B \). This will not always be true in our later examples. But, we will always be able to choose the desired helicity of \( A \) and \( B \) by choosing 1 parallel to \( B \) and 2 parallel to \( A \).

The matrix elements for the operator (23) to create 3-parton final states are

\[ \mathcal{A}(O \rightarrow q^- g^+ q^+) = \frac{\langle 1a \rangle^2 [12]}{\langle ac \rangle \langle cb \rangle} \]

\[ \mathcal{A}(O \rightarrow q^- g^- q^+) = \frac{[2b]^2 \langle 12 \rangle}{\langle ac \rangle \langle cb \rangle} . \]

(25)

To compute the results in the fifth line of the table, we must essentially divide (25) by (24) and square the result. To do this, we need a prescription for treating the expressions \( \langle 1a \rangle \) and \( [2b] \) in the numerators. The problem of relating the vectors \( a, b, c \) to \( A \) and \( B \) in an antenna splitting was discussed at length by Kosower in [22]; that paper gives a general treatment in terms of reconstruction functions to provide expressions that can be smoothly integrated in higher-order QCD calculations. This discussion is generalized to the initial-state channels in [23]. Here, we will take a more \textit{ad hoc} approach that leads to the simplest formulae with correct singular limits.

Formulae for \( \langle 1a \rangle \) and \( [2b] \) that are simple and become exact in the collinear and soft limits are found by approximating \( a \) collinear with \( A \) and \( b \) collinear with \( B \). Then identifying 1 with \( B \) and 2 with \( A \) gives

\[ |\langle 1a \rangle|^2 = s_{Ba} \rightarrow z_a s_{AB} , \quad |\langle 1b \rangle|^2 \rightarrow 0 , \quad |\langle 2a \rangle|^2 \rightarrow 0 , \quad |\langle 2b \rangle|^2 = s_{Ab} \rightarrow z_b s_{AB} . \]

(26)
and similarly for the conjugate products. Using this prescription, one obtains the fifth line of the table. This is a more formal version of the argument for these entries already given in Section 2.

In our calculations, we will encounter two more numerator objects that require reconstruction, namely, \( \langle 1c \rangle \) and \( \langle 2c \rangle \). The prescription above gives

\[
|\langle 1c \rangle|^2 = s_{Bc} \to \left(\frac{y_{bc}}{z_b}\right)s_{AB} , \quad |\langle 2c \rangle|^2 = s_{Ac} \to \left(\frac{y_{ac}}{z_a}\right)s_{AB} .
\]

(27)

However, it is potentially dangerous to write factors of \( z_a, z_b \) in the denominator. We will see in Section 6 that such factors would create unphysical singularities when continued to the IF kinematics. These unphysical singularities are avoided in the general formalism used in [22], but at the price of introducing much more complicated formulae. Fortunately, we will see that \( \langle 1c \rangle \) arises only in situations where there is no collinear singularity with \( c \) parallel to \( b \). In such cases, the remaining universal singular terms—the collinear singularity with \( c \) parallel to \( a \) and the soft singularity—correspond to kinematic limits with \( z_b \to 1 \). A similar consideration applies to \( \langle 2c \rangle \).

Thus, we choose, instead of using (27), to evaluate these quantities as

\[
|\langle 1c \rangle|^2 = s_{Bc} \to y_{bc}s_{AB} , \quad |\langle 2c \rangle|^2 = s_{Ac} \to y_{ac}s_{AB} .
\]

(28)

This gives an incorrect shape in a region where \( a \) and \( b \) are collinear, but, hopefully, we will not use the \( AB \to acb \) splitting function to evaluate the rate to fill this region of phase space.

Another choice for evaluating \( \langle 1c \rangle \) and \( \langle 2c \rangle \) is to replace both expressions by \( z_c \). However, the spinor product \( \langle 1c \rangle \) vanishes in the \( bc \) collinear limit but not in the \( ac \) collinear limit, and conversely for \( \langle 2c \rangle \), so this choice does not give the universal singularities correctly.

We now apply this formalism to compute the second and fourth lines of Table 1 associated with the \( g_-g_+ \) antenna. This antenna is created by the spin-2 operator \( \text{tr} [\gamma^m (\sigma \cdot F) \gamma^n (\sigma \cdot F)] \). To make a definite calculation, we need a spin-2 polarization vector. An appropriate choice can be found by introducing the massless vectors 1 and 2 as above and writing

\[
\epsilon^{mn} = \langle 1 | \gamma^m | 2 \rangle \langle 1 | \gamma^n | 2 \rangle \]

(29)

This effectively decays the massive spin-2 particle into two massless spinors. This method was introduced in [24] to compute the relevant amplitudes for the emission of massive gravitons at high-energy colliders.

With this prescription, we generate the \( g_-g_+ \) antenna using the operator

\[
\mathcal{O} = \frac{1}{4} \text{tr} [\gamma^m (\sigma \cdot F) \gamma^n (\sigma \cdot F)] \langle 1 | \gamma_m | 2 \rangle \langle 1 | \gamma_n | 2 \rangle
\]

(30)
The matrix element of this operator that creates the 2-parton dipole is
\[
\langle g_+ g_+ | O | 0 \rangle = \langle 1 A \rangle^2 [2 B]^2 .
\] (31)

To obtain the correct initial polarizations, we take 1 = B, 2 = A as before. The matrix elements to the possible 3-parton final states are
\[
\mathcal{A}(O \to g_+ g_+ g_+) = 0
\]
\[
\mathcal{A}(O \to g_+ g_+ g_-) = \frac{\langle 1 b \rangle^4 [12]^2 [a c]}{\langle ab \rangle \langle ac \rangle \langle cb \rangle}
\]
\[
\mathcal{A}(O \to g_+ g_- g_+) = \frac{\langle 1 c \rangle^4 [12]^2 [a c]}{\langle ab \rangle \langle ac \rangle \langle cb \rangle}
\]
\[
\mathcal{A}(O \to g_- g_+ g_+) = \frac{\langle 1 a \rangle^4 [12]^2 [a c]}{\langle ab \rangle \langle ac \rangle \langle cb \rangle},
\] (32)

and the conjugates with 1 ↔ 2 for the other four combinations. Applying the reductions (26), (27), we find the results given in the second line of the table.

The nonzero matrix elements of this operator to \(\bar{q} q g\) final states are
\[
\mathcal{A}(O \to \bar{q}_- g_+ g_+) = \frac{\langle 1 c \rangle^2 [2 b]^2}{[a c]}
\]
\[
\mathcal{A}(O \to \bar{q}_- g_+ g_+) = \frac{\langle 1 a \rangle^2 [2 b]^2}{[a c]}. 
\] (33)

The same reduction process gives the results in the fourth line of the table.

5 Spin-\(1/2\) and spin-\(3/2\) cases

The cases of quark-gluon antennae can be treated in the same way. There is one additional subtlety. In QCD, quarks are color triplets and gluons are color octets,
so a quark-gluon operator carries net color. This means that the matrix element for gluon emission from a quark-gluon operator is not gauge-invariant unless we allow the gluon also to be emitted from the initial state. This makes it unclear how to define a quark-gluon antenna.

We resolve this problem with the following prescription: We consider the quarks to be color octet particles like the gluons. Then, as in the previous sections, we extract the color-ordered contribution corresponding to emission from the antenna. In the limit of large $N_c$, the various antennae in a process radiate independently. The diagrams contributing to a quark-gluon antenna in this prescription are shown in Fig. 4. The third diagram, with an intermediate quark line, does not appear in QCD. However, it does nicely provide the missing piece that makes this sum of diagrams gauge-invariant without radiation from the initial state.

This solution is the same as that found in the earlier work of Gehrmann-De Ridder, Gehrmann, and Glover [16]. Those authors computed the quark-gluon antennae by factorizing the amplitudes for the decay of a neutralino into a massless gluino plus $gg$ or $q\bar{q}$. In their calculation, the off-shell color octet fermion is the gluino.

With this understanding, we proceed as in the previous Section. We can generate the $q_--g_-$ antenna using the operator $\bar{q}_L(\vec{\sigma} \cdot F)$. The polarization spinor can be built by introducing massless spinors 1 and 2 as above and taking $|2\rangle$ to be this spinor. Then

$$O = -i\bar{q}_L(\vec{\sigma} \cdot F) |2\rangle .$$

(34)

The matrix element of this operator that creates the 2-parton dipole is

$$\langle q_--g_--|O|0\rangle = \langle AB|B2\rangle .$$

(35)

To obtain the correct initial polarizations, we take $1 = B, 2 = A$.

The matrix elements to the possible 3-parton final states are

$$\mathcal{A}(O \rightarrow q_--g_+g_+)=0$$

$$\mathcal{A}(O \rightarrow q_--g_-g_+)=\frac{\langle ac\rangle^3\langle 2c\rangle}{\langle ab\rangle\langle ac\rangle\langle cb\rangle}$$

$$\mathcal{A}(O \rightarrow q_--g_+g_-)=\frac{\langle ab\rangle^3\langle 2b\rangle}{\langle ab\rangle\langle ac\rangle\langle cb\rangle}$$

$$\mathcal{A}(O \rightarrow q_--g_-g_-)=s_{AB}\frac{\langle 12\rangle\langle 1a\rangle}{\langle ab\rangle\langle ac\rangle\langle cb\rangle} .$$

(36)

Applying the reductions (26), (27), we find the results given in the seventh line of the table.
The nonzero matrix elements of this operator to \( q\bar{q}q \) final states are

\[
\mathcal{A}(\mathcal{O} \rightarrow q-\bar{q}q+) = \frac{\langle ac \rangle \langle 2c \rangle}{\langle cb \rangle},
\]
\[
\mathcal{A}(\mathcal{O} \rightarrow q-\bar{q}q-) = -\frac{\langle ab \rangle \langle 2b \rangle}{\langle cb \rangle}.
\] (37)

The same reduction process gives the results in the ninth line of the table.

We generate the \( q^-g_+ \) antenna using the spin-\( \frac{3}{2} \) operator \( \bar{q}_L \gamma^m (\sigma \cdot F) \). This is essentially the supersymmetry current of the system of gluons and color octet fermions. The polarization spinor can be built by introducing massless spinors 1 and 2 as above:

\[
\mathcal{O} = i\bar{q}_L \gamma^m (\sigma \cdot F) [2|1|\gamma_m|2].
\] (38)

The matrix element of this operator that creates the 2-parton dipole is

\[
\langle q^-g_+ | \mathcal{O} | 0 \rangle = \langle 1A | 2B \rangle^2.
\] (39)

To obtain the correct initial polarizations, we again take \( 1 = B, 2 = A \).

The matrix elements to the possible 3-parton final states are

\[
\mathcal{A}(\mathcal{O} \rightarrow q-g_+g_+) = \frac{\langle 1a \rangle^3 \langle 12 \rangle^2}{\langle ab \rangle \langle ac \rangle \langle cb \rangle},
\]
\[
\mathcal{A}(\mathcal{O} \rightarrow q-g_-g_+) = \frac{\langle ab \rangle \langle 2b \rangle^3 \langle 12 \rangle}{[ab][ac][cb]},
\]
\[
\mathcal{A}(\mathcal{O} \rightarrow q-g_+g_-) = \frac{\langle ac \rangle \langle 2c \rangle^3 \langle 12 \rangle}{[ab][ac][cb]},
\]
\[
\mathcal{A}(\mathcal{O} \rightarrow q-g_-g_-) = 0.
\] (40)

Applying the reductions (26), (27), we find the results given in the eighth line of the table.

The nonzero matrix elements of this operator to \( q\bar{q}q \) final states are

\[
\mathcal{A}(\mathcal{O} \rightarrow q-\bar{q}q+) = \frac{\langle 1a \rangle \langle 2b \rangle^2}{[cb]},
\]
\[
\mathcal{A}(\mathcal{O} \rightarrow q-\bar{q}q-) = -\frac{\langle 1a \rangle \langle 2c \rangle^2}{[cb]}.
\] (41)

The same reduction process gives the results in the tenth line of the table.
6 Initial-state showers

The Feynman diagram computations that we have done to find the antenna splitting functions for FF splittings can also be applied, by crossing, to IF and II splittings. The expressions in Table II are given in terms of invariant quantities that are unchanged under crossing. Thus, we can use the expressions in this table directly in other channels. At worst, a change of the overall sign is required in some cases. In this section, we will clarify this statement by analyzing the kinematics of IF and II splittings in the same variables as those used in Section 2 for FF splittings. In all cases, the kinematics is done for all massless partons only. The kinematic discussion in this section is similar to that presented in [23].

To begin, we will formalize some of the results quoted in Section 2 for the FF region. The cross section for a process \( X \to abc \) is

\[ \sigma(X \to abc) = \frac{1}{\Phi_X} \frac{s}{128\pi^3} \int dz_\alpha dz_\beta |\mathcal{M}(X \to abc)|^2, \tag{42} \]

where \( \Phi_X \) is the flux factor. Polarization and color indices have been suppressed. The left-hand side has been integrated over the orientation of the final state system but is otherwise exact. To write an expression involving the antenna splitting function, we approximate

\[ \mathcal{M}(X \to abc) \approx \mathcal{M}(X \to AB) \cdot gT \cdot \frac{\mathcal{M}(O \to abc)}{\mathcal{M}(O \to AB)}, \tag{43} \]

where \( O \) is the operator used in Sections 3–5 to represent the state \( AB \). The factor \( gT \) is the QCD coupling and color matrix; after squaring and summing over colors, this becomes \( 4\pi\alpha_sN_c \). The splitting function is defined by (13),

\[ S(z_\alpha, z_c, z_\beta) = s_{AB} \left| \frac{\mathcal{M}(O \to abc)}{\mathcal{M}(O \to AB)} \right|^2 \tag{44} \]

Then

\[ \sigma(X \to abc) \approx \sigma(X \to AB) \cdot \frac{\alpha_sN_c}{4\pi} \int dz_\alpha dz_\beta S(z_\alpha, z_c, z_\beta). \tag{45} \]

It is important to note that, in this formula or in (43), the vectors \( k_A \) and \( k_B \) are introduced as part of the approximation. They can be defined in any way that is consistent with the requirements that \( k_A \) and \( k_B \) are lightlike, \( k_A + k_B = Q \), and \( k_A \) and \( k_B \) become parallel to \( k_\alpha \) and \( k_\beta \), respectively, in the soft and collinear limits.

The logic of this derivation extends straightforwardly to the IF and II regions. The major change is that, in these cases, we need to introduce initial hadrons from which the initial partons are extracted.
Consider first the IF case. The cross section for a proton of momentum $P$ to scatter from a color-singlet system $X$ transferring momentum $Q$ to create a 2-parton system $cb$ is

$$\sigma(pX → cb) = \int dx_a f(x_a) \frac{1}{\Phi_{aX}} \frac{1}{16\pi} \int d\cos\theta_* |M(aX → cb)|^2,$$

where $\cos\theta_*$ is the scattering angle in the $cb$ center of mass system. We will approximate this formula using the expression analogous to (43)

$$\mathcal{M}(aX → cb) \approx \mathcal{M}(AX → B) \cdot gT \cdot \frac{M(aO → cb)}{\mathcal{M}(A0 → B)}.$$  

(47)

Then the splitting function is defined by the same expression $S$ as in (44), but now analytically continued into the new kinematic region. If a fermion line is crossed from the final to the initial state, an extra factor $(-1)$ should be included. In addition, $s_{AB}$ in (44) is negative in this region, giving an extra minus sign.

The decomposition of the amplitude is illustrated in Fig. 5(a). The kinematics can be described by variables $y_{ij}$ and $z_i$ obeying the relations (1) to (4). However, the vectors $k_A, k_a$ now have negative timelike component, and the vector $Q = k_A + k_B = k_a + k_b + k_c$ is spacelike, $Q^2 = s_{AB} < 0$. The phase space for this region covers the quadrilateral shown in Fig. 5(b). The region of integration is infinite, since $z_a$ can become very large, but the integral is cut off at large $z_a$ by the parton distribution function. The line $z_a > 1, z_b = 1$ corresponds to the region of initial state radiation, $c$ parallel to $a$. The line $z_a = 1, 0 < z_b < 1$ corresponds to the region of final state radiation, $c$ parallel to $b$. The line $z_a + z_b = 1$ corresponds to $b$ parallel to $a$, that is, $b$ as initial state radiation from the primary $a$. An antenna shower should give an accurate description of the dynamics in the two regions $|y_{ac}| < |y_{bc}| < 1, |y_{bc}| < |y_{ac}| < |y_{ab}|$ that are shaded in the figure. The new constraint $|y_{bc}| < 1$ is just $s_{bc} < |Q^2|$, which is stronger than the constraint that this invariant is less than $|s_{ab}|$.

To decompose (46) into an appropriate form, we choose $p_A$ and $p_B$ and then change variables. Let $p_A$ be chosen in the direction of $p_a$, so that $p_a = z_a p_A, z_a > 1$. Then $p_B = Q - p_A$. We have

$$p_a = x_a P, \quad p_A = x_A P, \quad x_a = z_a x_A,$$

(48)

with $x_A$ having the definite value $x_A = -Q^2/2P \cdot Q$ associated with scattering a massless particle from a local current. For the reaction $aQ → bc, s + t + u = Q^2$, so $t + u = Q^2 - s = Q^2 z_a$. Then

$$t = Q^2 (1 - z_b) = \frac{1}{2} Q^2 z_a (1 - \cos \theta_*)$$

(49)
Using these formulae, we can change variables from \((x_a, \cos \theta)\) to \((z_a, z_b)\). The Jacobian of this transformation is

\[
J = \frac{\partial(x_a, \cos \theta)}{\partial(z_a, z_b)} = \frac{2x_A}{z_a}
\]

Thus,

\[
\sigma(pX \to cb) = \int \frac{dz_a}{z_a^2} \int dz_b \int dx_A f(x_a x_A) \delta(x_A + Q^2/2P \cdot Q) \cdot \frac{1}{\Phi_{AX}} \frac{1}{8\pi} |\mathcal{M}(aX \to cb)|^2.
\]

This is an exact rewriting of (46). Now apply the approximation (47) and group terms to form

\[
\sigma(AX \to B) = \frac{1}{\Phi_{AX}} 2\pi \delta(Q^2 + x_A 2P \cdot Q) |\mathcal{M}(AX \to B)|^2.
\]

Then

\[
\sigma(pX \to cb) \approx \int \frac{dz_a}{z_a^2} \int dz_b \int dx_A f(x_a x_A) \sigma(AX \to B) \cdot \frac{\alpha_s N_c}{4\pi} S(z_a, z_c, z_b).
\]

As an example, consider using this formula to describe initial-state gluon radiation in deep inelastic scattering from a quark. The total gluon emission is given by the sum of the two spin-dependent splitting functions in the fifth line of Table 1, equal to

\[
\sum S = \frac{z_a^2 + z_b^2}{y_{ac} y_{cb}},
\]
The extra minus sign comes from the sign of $s_{AB}$ in (44). In the region of initial state radiation, $z_a = 1/w$, $z_b \approx 1$, $y_{ac} = -(1 - z_b)$, $y_{bc} = (1 - 1/w)$. Then, setting

$$\int dz_b \frac{1}{1 - z_b} = \log \frac{Q^2}{\mu^2},$$

we obtain

$$\sigma(pX \to cb) \approx \int dx_A \int \frac{dw}{w} f(\frac{x_A}{w}) \sigma(AX \to B) \cdot \frac{\alpha_s N_c}{4\pi} \frac{1 + w^2}{(1 - w)} \log \frac{Q^2}{\mu^2},$$

which is correct.

This is an appropriate point to discuss again the signs of the expressions in Table I. The antenna splitting functions are probabilities; thus, they should be positive. However, we define the splitting functions in the IF and II regions as analytic continuations of the values in the FF region, so their positivity must be checked explicitly.

As we move from the FF region to the IF region with $A$ and $a$ in the initial state, $y_{bc}$ becomes negative while all other $y_{ab}$, $y_{ac}$, $z_a$, $z_b$ remain positive. The factor $z_c$ can be negative, but $z_c$ does not appear in the Table. With the minus sign from $s_{AB}$ in (44), the denominator of $S(z_a, z_b, z_c)$ is positive, and so we need only check the numerator functions in given in the Table. The numerator functions for $gg \to ggg$, $qg \to qg\bar{q}$, and $gg \to qg\bar{q}$ remain positive, while the numerator functions for $gg \to gqg$ become negative. In this last case, a fermion not present in the 2-parton system is crossed from the final to the initial state, so we must supply an extra factor $(-1)$. Then all of the expressions are positive, as required. However, if we then cross from the region $z_b > 0$ to the region $z_b < 0$, one $gg \to gqg$ and one $gg \to qg\bar{q}$ amplitude changes sign. This sign change is unphysical; presumably, it is due to the simple method of reconstruction in (26) and (28). We recommend setting these two amplitudes to zero for $z_b < 0$. The region $z_b < 0$ is outside the shaded region in Fig. 5(b) where we will generally use the parton shower approximation, so most likely this difficulty is not important in practice.

Similarly, for the FI region where $b$ and $B$ and taken to be in the initial state, $y_{ac} < 0$. Then the numerators that go negative as we cross into the region are those in the $qq \to qg\bar{q}$ cases where a fermion is crossed into the initial state. Now there are four amplitudes, one each in the $qq \to qg\bar{q}$ cases and both of those in $q-gg \to qg\bar{q}$, that become negative when $z_a < 0$. Again, we recommend that these amplitudes be set to zero in this region of unphysical behavior.

In the II region, both $y_{ac}$ and $y_{bc}$ are negative. The denominator of $S(z_a, z_b, z_c)$ is positive. The numerator terms that are negative because of the sign changes are compensated by minus signs from crossing. There are no unphysical sign changes.
Figure 6: (a) Kinematics of $2 \rightarrow 3$ splitting in the initial state (II) case. (b) Phase space for $2 \rightarrow 3$ splitting in the II case. The six regions corresponding to different orderings of $|s_{ac}|$, $|s_{bc}|$, $|Q^2|$ are shown. The region that should be well described by an antenna splitting $AB \rightarrow acb$ is shaded.

The correct result is always obtained by taking the absolute value of the numerator expression from Table 1.

We now discuss the kinematics of the II case. We begin from the formula for two protons of momentum $P_A$, $P_B$ to produce a color-singlet system of momentum $Q$ plus a massless parton $c$,

$$
\sigma(pp \rightarrow cX) = \int dx_a \int dx_b f(x_a) f(x_b) \frac{1}{2s_{ab}} \frac{1}{16\pi} \int d\cos \theta \frac{2p_\ast}{\sqrt{s_{ab}}} |\mathcal{M}(ab \rightarrow cX)|^2,
$$

where $\cos \theta_\ast$ and $p_\ast$ are the scattering angle and the momentum in the $cX$ center of mass frame.

The decomposition of the amplitude is illustrated in Fig. 6(a). The kinematics can again be described by variables $y_{ij}$ and $z_i$ obeying the relations (1) to (4). Now the vectors $k_A$, $k_a$, $k_B$, $k_b$ have negative timelike component, and the vector $Q = k_A + k_B = k_a + k_b + k_c$ is also negative timelike, with $Q^2 > 0$. The phase space for this region covers the quadrant shown in Fig. 5(b), with $z_a, z_b > 1$. Again, the region of integration is infinite, but the integral is cut off by the behavior of the parton distribution functions. The line $z_a > 1$, $z_b = 1$ corresponds to the region of initial state radiation with $c$ parallel to $a$. The line $z_a = 1$, $z_b > 1$ corresponds to the region of initial state radiation with $c$ parallel to $b$. An antenna shower should give an accurate description of the dynamics in the two regions $|y_{ac}| < |y_{bc}| < 1$, $|y_{bc}| < |y_{ac}| < 1$ that are shaded in the figure. Again, the limit 1 here corresponds to constraints $|s_{ac}|, |s_{bc}| < |Q^2|$, which are stronger than the constraints that these two invariants are less than $|s_{ab}|$. 

19
In the $ab \to cX$ process, the system $X$ must recoil with some nonzero transverse momentum. Thus, it is not possible to choose $k_A$ and $k_B$ to be parallel to $k_a$, $k_b$. The invariants for the $ab \to cX$ scattering process satisfy $s + t + u = Q^2$. Since $t = Q^2(1 - z_b)$, $u = Q^2(1 - z_a)$, this means that $s = Q^2(z_a + z_b - 1)$. Alternatively, $s = x_ax_b \cdot 2P_A \cdot P_B$. We would like to choose the longitudinal fractions of $A$ and $B$, $x_A$ and $x_B$, to satisfy the relation

$$x_Ax_B \cdot 2P_A \cdot P_B = Q^2.$$  \hspace{1cm} (58)

To make this possible, we must write

$$x_a = z_ax_AC, \quad x_b = z_bxBC,$$

with \(C^2 = \frac{z_a + z_b - 1}{z_azi} \) \hspace{1cm} (60)

The function $C(z_a, z_b)$ approaches 1 when *either* $z_a$ or $z_b$ goes to 1; that is $C \approx 1$ in both collinear regions.

Also, $t + u = Q^2(2 - z_a - z_b) = Q^2z_c$, so

$$t = Q^2(1 - z_b) = \frac{1}{2}Q^2z_c(1 - \cos \theta_s).$$ \hspace{1cm} (61)

We can now use (59) and (61) to change variables from $(x_a, x_b, \cos \theta_s)$ to $(x_A, z_a, z_b)$, holding $x_B$ fixed at the value $x_B = Q^2/x_Ax_B \cdot P_A \cdot P_B$. The Jacobian of this transformation is

$$J = \frac{\partial(x_a, x_b, \cos \theta_s)}{\partial(x_A, z_a, z_b)} = \frac{2x_B}{z_c} = \frac{x_B}{s_{ab}} \frac{Q^2 \sqrt{s_{ab}}}{P_s}.$$ \hspace{1cm} (62)

Then

$$\sigma(pp \to cX) = \int d^2z_a \frac{dz_b}{z_a^2} \frac{1}{z_b^2} \int dx_A dx_B f(z_ax_AC)f(z_bxBC) x_B \delta(x_B - Q^2/x_Ax_B \cdot P_B)$$

$$\cdot \frac{1}{s_{AB}} \frac{1}{8\pi} |M(aX \to cb)|^2.$$ \hspace{1cm} (63)

This is an exact rewriting of (57). Now apply the approximation analogous to (43) or (47) and group terms to form

$$\sigma(AB \to X) = \frac{1}{2s_{AB}} 2\pi \delta(Q^2 - x_Ax_B \cdot P_B)|M(AX \to B)|^2.$$ \hspace{1cm} (64)

This gives, finally,

$$\sigma(pp \to cX) \approx \int d^2z_a \frac{dz_b}{z_a^2} \frac{1}{z_b^2} \int dx_A dx_B f(z_ax_AC)f(z_bxBC) \sigma(AB \to X) \frac{\alpha_s N_c}{4\pi} S(z_a, z_c, z_b).$$ \hspace{1cm} (65)
To test this formula, consider the case of $q\bar{q}$ annihilation with the emission of a gluon collinear with the quark $a$. The sum of spin-dependent splitting functions for this case is again (54). In the collinear region of interest, $z_a = 1/w$, $z_b \approx 1$. Repeating the step that led to (56), we find

$$\sigma(pp \to cX) \approx \int dx_A dx_B \int \frac{dw}{w} f(x_A) f(x_B) \sigma(AB \to X) \cdot \frac{\alpha_s N_c}{4\pi} \frac{1 + w^2}{1 - w} \log \frac{Q^2}{\mu^2},$$

which is the correct limit.

7 Comparison to previous results

In the Introduction, we made reference to a number of previous definitions of the antenna splitting functions. We noted that these definitions agree, as they must, in the singular soft and collinear limits. However, these prescriptions differ widely away from the boundaries of phase space. In this section, we will compare our prescription to those of ARIADNE [6, 7] and Gehrmann-De Ridder, et al. [16].

We will make this comparison over the natural phase space discussed in the previous section—the entire $(z_a, z_b)$ plane above the line $z_a + z_b = 1$. In order to describe antenna showers for initial- as well as final-state emissions, the splitting functions should extend into the region $z_a, z_b > 1$. Depending on the details of how the shower is constructed, their use might be restricted to a polygon around $z_a = z_b = 1$, or the expressions might be used for arbitrarily large values of $z_a$ and $z_b$.

We note again that the IF regions include the lines $z_a = 0$ and $z_b = 0$. Expressions for the splitting functions that are well-behaved near $z_a = z_b = 1$ can possibly have a singularity on this line, though such a singularity in the middle of the phase space would be unphysical. We used this criterion in Section 4 to exclude factors of $1/z_a$ and $1/z_b$ from appearing in (28). The antenna functions of Duhr and Maltoni [15] are typically singular along this line and so cannot be used in parton shower models in all regions.

The ARIADNE and Gehrmann-De Ridder antenna functions give expressions summed over final polarizations. To compare our splitting functions to these, we must sum over a row in Table 1. Our summed expressions are independent of the initial polarization in the soft and collinear limits, but they depend on the polarizations of $A$ and $B$ in the interior of the $(z_a, z_b)$ space. The comparison to our expressions thus also reveals where this dependence on polarization is an important effect.

The first antenna splitting functions were put forward by the ARIADNE group [6]. Their approach started from the spin-averaged cross section for the simple splitting
process $q\bar{q} \to qg\bar{q}$ in $e^+e^-$ annihilation. They then guessed the expressions for the $qg \to qgg$ and $gg \to qgg$ splittings, so that these would have a similar form to the $q\bar{q} \to qg\bar{q}$ case,

$$S = \frac{z_a^{n_a} + z_b^{n_b}}{y_{ac}y_{bc}},$$

(67)

where $n_a, n_b = 2$ for emission from a quark and 3 for emission from a gluon.

Our philosophy, explained in Section 2, is that each individual antenna should reproduce the collinear limit predicted by QCD. These expressions are symmetric under interchange of identical particles, while (67) does not have this property, so we would obtain the complete splitting function by symmetrizing (67). This gives

$q\bar{q}$ antenna: $S = \frac{z_a^2 + z_b^2}{y_{ac}y_{bc}},$

$gg$ antenna: $S = \frac{z_a^3 + z_b^3}{y_{ac}y_{bc}} + \frac{z_a^3 + z_c^3}{y_{ab}y_{bc}} + \frac{z_b^3 + z_c^3}{y_{ab}y_{ac}},$

$qq$ antenna: $S = \frac{z_a^2 + z_b^3}{y_{ac}y_{bc}} + \frac{z_a^2 + z_c^3}{y_{ab}y_{bc}}.$

(68)

The summed terms are each positive in the FF kinematic region. To obtain the ARIADNE splitting functions in the other regions, we analytically continue these formulae into the regions where $z_a$ or $z_b$ is greater than 1.

The analytic continuation of the ARIADNE and, below, the Gehrmann-de Ritter results brings in the issue of the positivity of these expressions, similar to the positivity issue for our splitting functions discussed in Section 6. For the ARIADNE and Gehrmann-De Ridder antenna functions, the expressions given are summed over spins, and the individual pieces are not independent of one another. So, if they become negative, that is a problem for the complete, spin-summed, expression. For the Gehrmann-de Ridder functions, it can be seen that this happens only the regions $z_a < 0$ and $z_b < 0$, so this is not a serious problem. However, the ARIADNE function involve $z_c^2$, which is negative in the whole region $z_a + z_b > 2$. This problem cannot be resolved by replacing $z_c$ with $|z_c|$, since this leads to expressions that do not agree with the Altarelli-Parisi factorization along the lines separating the IF regions from the II region. Fortunately, the ARIADNE functions do not become actually become negative until $z_a$ or $z_b$ becomes very large ($z_a$ or $z_b \sim 12$). However, the idea that the ARIADNE functions are sums of positive and negative terms in the initial-state regions goes against the intuition used to propose these expressions.

We are now in a position to compare the ARIADNE function to our proposal. For the $q\bar{q}$ antenna, the expression above coincides with the sum of row 5 of Table 1. For the $gg$ and $qg$ cases, the ratio of the above ARIADNE functions to those defined in Table 1 are illustrated in Figs. 7, 8, and 9. The notation in the figures is
Figure 7: Visualization of the ratio of the ARIADNE antenna function to our antenna functions for the processes $gg \rightarrow ggg$. The figures on the left and right are the comparison of the ARIADNE antenna function to our spin-summed antenna functions from row 1 and row 2 in Table 1, respectively. The boundaries of phase space for the different kinematic regions are marked in blue. The contours are plotted at ratios of 1.2, 1.5, 2.0, 3.0, and 5.0, with + indicating a region in which the ratio is greater than 1.

The following: Each figure represents the ratio of the ARIADNE splitting function to our results for a specific initial set of polarized partons, summed over final state polarizations. The ratio goes to 1 on the lines $z_a = 1$ and $z_b = 1$, which correspond to the collinear limits. Away from these lines, the contours on which the ratios are 1.2, 1.5, 2.0, 3.0, and 5.0 (toward the + symbol), and the inverses of these numbers (toward the − symbol) are shown. The $qg$ antenna function are asymmetric between partons $a$ and $b$. The IF region in the lower right is that in which the quark is in the initial state and the gluon is in the final state. The IF region in the upper left is that in which the gluon is in the initial state and the quark remains in the final state.

The ARIADNE authors gave a different interpretation to the formulae (68). They took the philosophy that the collinear limit need not result from a single antenna but rather should be the result of summing over the possible antennae that would lead to a specific final state. A three gluon final state could result from any pair of the gluons radiating the third and so should be the sum of three antennae. Then the second line of (68) would be interpreted as the sum over these three antennae. This is a reasonable point of view for the FF kinematics considered in [6]. However, in the IF and II regions, at least one of the $z_i$ will be negative and so some of the terms
Figure 8: Visualization of the ratio of the ARIADNE antenna function to our antenna function for the process $q\bar{q}_- \to qg\bar{q}$. Our antenna function for the process $q\bar{q}_+ \to qg\bar{q}$ coincides with the ARIADNE result and so is not included. The notation is as in Fig. 7.

Figure 9: Visualization of the ratio of the ARIADNE antenna function to our antenna functions for the processes $qg \to qgg$. The figures on the left and right are the comparison of the ARIADNE antenna function to our spin-summed antenna functions from row 7 and row 8 in Table 1 respectively. The notation is as in Fig. 7.
in the last two lines of (68) will become negative. Such terms cannot be interpreted as independent radiators, each emitting a gluon with positive probability. It is tempting to revise the formula in (68) by taking the absolute values of the negative terms. However, one can readily check that no such prescription gives the correct Altarelli-Parisi limit along the lines $z_a = 1$ and $z_b = 1$ at the boundaries of the IF and II regions. Thus, we believe, the ARIADNE formulae can be used in the IF and II regions only by using the formulae (68) as written and accepting that some negative signs will appear [26].

Gehrmann-De Ridder, Gehrmann and Glover [16] studied $2 \rightarrow 3$ splitting from Feynman diagrams to develop an antenna subtraction program for NNLO calculations. In doing so, they were able to extract unpolarized antenna functions for the processes $gg \rightarrow ggg$, $qg \rightarrow qgg$ and $qg \rightarrow q\bar{q}q$. To calculate the gluon-gluon antenna function, they used the effective Higgs coupling to gluons

$$L = -\frac{\lambda}{4} h F^\mu \nu F_{\mu \nu}. \quad (69)$$

This is essentially the same procedure that we used in Section 3, and it yields the same result as the sum of row 1 in Table 1. In our language, their antenna function for the gluon-gluon dipole is [27]

$$S = \frac{y_{ac}^2 + y_{bc}^2 + y_{ab}^2 + y_{ac}^2 y_{bc}^2 + y_{ab} y_{bc}^2 + y_{ab}^2 y_{ac}^2}{y_{ab} y_{ac} y_{bc}} + 4. \quad (70)$$

The comparison of this antenna function to the sum of row 2 of Table 1 is illustrated in Fig. 10.

This splitting function for $gg \rightarrow ggg$ is, however, not precisely the form of the splitting function that is used in the VINCIA parton shower [8]. They use the ‘global’ form of the Gehrmann-De Ridder antenna function, which in our language is

$$S = \frac{1}{2} \left[\frac{2y_{ab}^2 + y_{ab} y_{bc}^2 + y_{ac} y_{bc}^2 + 8}{y_{ab} y_{ac} y_{bc}} + \frac{8}{3}\right]. \quad (71)$$

To implement this antenna function, a similar procedure is used as with the ARIADNE antenna functions. That is, emissions from overlapping antenna are summed. When the three antennae contributing to $gg \rightarrow ggg$ are summed together, one recovers the result (70). This prescription works well in the FF kinematics. However, as in the ARIADNE case, it might require negative contributions in splitting functions for some antennae in the IF and II kinematics.

To construct the antenna functions involving quarks, Gerhmann-De Ridder, et al., calculated the decay of a neutralino $\chi$ to a gluon and a gluino $\psi$ through the effective operator

$$L = i\bar{\psi}\gamma^{\mu\nu}\chi F_{\mu\nu} + \text{h.c.} \quad (72)$$

25
In principle, our results should agree for a spin $\frac{1}{2}$ initial state. However, our choices (26) and (28) for handling ambiguous momentum products, produce some differences. In our language, their antenna functions involving quarks are

\[
qg \rightarrow qgg : \quad S = \frac{2y_{ab}^2 + 2y_{ac}^2 + y_{abc}^2 + y_{abc}^2 + 2y_{abc}y_{ab}^2}{y_{abc}y_{abc}} + 2 + 2y_{ac} + 2y_{ab},
\]
\[
qg \rightarrow q\bar{q}q : \quad S = \frac{(y_{ac} + y_{ab})^2y_{abc}y_{abc} - 2y_{ac}^2y_{ab}^2}{y_{abc}y_{abc}} + y_{ab} + y_{ac}.
\]

(73)

The comparison to our antenna functions is illustrated in Figs. 11 and 12. For $qg \rightarrow qgg$, our result for the spin $\frac{1}{2}$ case is indeed very close to the above expression in the FF region. For $qg \rightarrow q\bar{q}q$, our prescription (28) gives us an extra factor of $z_a$ near $z_a = 0$.

In summary, we have shown that the antenna splitting functions represented by (7) and Table 1 give a physically sensible prescription for the construction of antenna showers. These splitting functions can be used with the formulae (45), (53), (65) to generate antenna splittings in all three relevant kinematic regions. We hope that this formalism will provide a firm foundation for the construction of new parton showers based on the antenna concept.
Figure 11: Visualization of the ratio of the Gehrmann-De Ridder antenna functions to our antenna functions for the processes $qg \rightarrow qgg$. The figures on the left and right are the comparison of the Gehrmann-De Ridder antenna function to our spin-summed antenna functions from row 7 and row 8 in Table [1] respectively. The notation is as in Fig. [7].

Figure 12: Visualization of the ratio of the Gehrmann-De Ridder antenna functions to our antenna functions for the processes $qg \rightarrow qq\bar{q}$. The figures on the left and right are the comparison of the Gehrmann-De Ridder antenna function to our spin-summed antenna functions from row 9 and row 10 in Table [1] respectively. The notation is as in Fig. [7].
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[27] This expression actually differs from the one given in [16] by a factor of 1/3, which comes from allowing any gluon to become collinear with any other gluon. Since we have identified the radiated gluon, we remove this factor.