HOLONOMY GROUPS AND SPACETIMES

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Abstract. A study is made of the possible holonomy group types of a space-time for which the energy-momentum tensor corresponds to a null or non-null electromagnetic field, a perfect fluid or a massive scalar field. The case of an Einstein space is also included. The techniques developed are also applied to vacuum and conformally flat space-times and contrasted with already known results in these two cases. Examples are given.

1. Introduction

In some recent papers attempts have been made to describe space-times according to their holonomy group. The general situation was discussed in the review [1] and the solution for a vacuum and a conformally flat space-time was given, respectively in [2] and [3]. In this paper both null and non-null Einstein-Maxwell fields together with perfect fluids and massive scalar fields will be similarly studied and the holonomy classification for them obtained. The techniques developed will also be used to give an alternative approach to the vacuum and conformally flat cases.

There are several ways of classifying space-times. Two common ones are through the Petrov type of the Weyl tensor (the Petrov classification) and through the algebraic type of the energy-momentum tensor (the Segre classification). For a review see e.g. [4]. These are, however, pointwise in the sense that the Petrov or Segre type may vary from point to point (subject to continuity requirements which can be described topologically [5]). Thus such classification systems, when imposed on the space-time as a whole, require not only a restriction as to the algebraic type, but also that type to be imposed globally (so that each point of the space-time is of the same type). In practice these are really local classifications (but rather useful nonetheless). An alternative classification which is global in the sense that it applies to the whole (or at least to the appropriate part) of the space-time can be achieved by assuming the existence of certain families (usually Lie algebras) of vector fields on the space-time and which describe certain symmetries of it. This is again useful (see, e.g. [4] [5]) but complicated because of the diverse nature of the symmetries involved. The classification through holonomy groups considered here, whilst incomplete and also suffering itself from certain global types being assumed, nevertheless considers a global property of the space-time connection, namely the holonomy type.

Let $M$ be a space-time with $M$ and all structures on $M$ assumed smooth. A standard notation will be used with a comma and a semi-colon denoting the usual partial and covariant derivative and with round and square brackets representing the usual symmetrisation and skew-symmetrisation. Let $g$ be the space-time metric with signature $(− +++)$ with associated Levi-Civita connection $\Gamma$ and curvature operator $R'$ with tensor components $R'_{\alpha\beta\gamma\delta}$. For any $p \in M$ let $C^k(p)$ denote the set of all piecewise $C^k(1 \leq k \leq \infty)$ closed curves at $p$. For each $c \in C^k(p)$ there is an obvious vector space isomorphism $f_c : T_pM \rightarrow T_pM$ (where $T_pM$ is the tangent space to $M$ at $p$) defined by parallel transport around $c$ and the set of all such maps arising from all members of $C^k(p)$ is a group under the usual
Table 1. The first column gives the labelling (following [8]) for each subalgebra of $A$ (omitting the trivial flat case $R_1$ and the case $R_3$ which is impossible for space-times [1]). The second column gives a bivector basis for the subalgebra in terms of a real null tetrad $(l, n, x, y)$ for all types except $R_{13}$ where a pseudo-orthonormal basis $(u, x, y, z)$ is used. Here $\sqrt{2}u = l - n$, $\sqrt{2}z = l + n$ and the only non-vanishing inner products are $l^an_a = x^ax_a = y^ay_a = z^az_a = -u^au_a = 1$. The wedge product is used to describe bivectors so that, for example, $l \wedge n = 2l_{[a}n_{b]}$. In the $R_{12}$ row $0 \neq c \in \mathbb{R}$. The third column lists the recurrent vector fields (up to an obvious scaling) which cannot be globally scaled so as to be covariantly constant. The fourth column lists in the $< >$ brackets a spanning set for the vector space of covariantly constant vector fields on $M$. The final column lists the subalgebra dimension.

| Type | Lie Algebra | Recurrent | Constant | Dimension |
|------|-------------|-----------|----------|-----------|
| $R_2$ | $l \wedge n$ | $l, n$ | $< x, y >$ | 1         |
| $R_3$ | $l \wedge x$ | - | $< l, y >$ | 1         |
| $R_4$ | $x \wedge y$ | - | $< l, n >$ | 1         |
| $R_5$ | $l \wedge n, l \wedge x$ | $l$ | $< y >$ | 2         |
| $R_7$ | $l \wedge n, x \wedge y$ | $l, n$ | - | 2         |
| $R_8$ | $l \wedge x, l \wedge y$ | - | $< l >$ | 2         |
| $R_9$ | $l \wedge n, l \wedge x, l \wedge y$ | $l$ | - | 3         |
| $R_{10}$ | $l \wedge n, l \wedge x, n \wedge x$ | - | $< y >$ | 3         |
| $R_{11}$ | $l \wedge x, l \wedge y, x \wedge y$ | - | $< l >$ | 3         |
| $R_{12}$ | $l \wedge x, l \wedge y, l \wedge n + e(x \wedge y)$ | $l$ | - | 3         |
| $R_{13}$ | $x \wedge y, y \wedge z, x \wedge z$ | - | $< u >$ | 3         |
| $R_{14}$ | $l \wedge n, l \wedge x, l \wedge y, x \wedge y$ | $l$ | - | 4         |
| $R_{15}$ | $A$ | - | - | 6         |

operations $f_{c_1} \circ f_{c_2} = f_{c_1 \cdot c_2}$ and $f_{c_1}^{-1} = f_{c_1^{-1}}$ for $c, c_1, c_2 \in C^k(p)$. This group is Lie-isomorphic to a Lie subgroup of the Lie group $GL(4, \mathbb{R})$ and is independent of the degree of differentiability $k$ and (up to isomorphism) of $p \in M$. This group is abstractly denoted by $\Phi$ and called the holonomy group of $M$. If one repeats the above construction this time restricting to members of $C^k(p)$ which are continuously or smoothly homotopic to zero (it matters not which) one arrives at the restricted holonomy group of $M$ denoted abstractly by $\Phi^0$. In fact $\Phi^0$ is the identity component Lie subgroup of $\Phi$. If $M$ is simply connected $\Phi^0 = \Phi$. (For further details of holonomy theory including the results of this section, [6] is an excellent text).

Throughout this paper $M$ will be assumed simply connected. It then follows that $\Phi^0 = \Phi$ and hence that $\Phi$ is a connected Lie subgroup of the identity component $L_0$ of the Lorentz group $L$. Let $\phi$ be the holonomy algebra (the Lie algebra of $\Phi$ or $\Phi^0$) and $A$ the Lorentz algebra (the Lie algebra of $L$ or $L_0$). Then $\phi$ is a subalgebra of $A$ and since $\Phi$ is now connected (since $M$ is simply connected) the possibilities for $\Phi$ correspond in one-to-one fashion to the subalgebras of $A$. The subalgebra structure for $A$ is well known and given in table 1.

Now let $p \in M$ and, in some coordinate system about $p$, consider the set of matrices of the form

$$R_{bcd}^a X^c Y^d, R_{bcd,e}^a X^c Y^d Z^e, \ldots$$
where \( X, Y, Z \in T_p M \). This set can be shown to constitute a Lie algebra under matrix commutation which (up to isomorphism) is independent of the coordinates chosen. It is denoted by \( \phi'_p \), referred to as the \textit{infinitesimal holonomy algebra} of \( M \) at \( p \), and is a subalgebra of both \( \phi \) and \( A \) for each \( p \in M \). The corresponding unique connected Lie subgroup arising from \( \phi'_p \) is referred to as the \textit{infinitesimal holonomy group} of \( M \) at \( p \) and denoted by \( \Phi'_p \). If \( \dim \Phi'_p \) is independent of \( p \) then for \( p, q \in M \) \( \Phi'_p \) and \( \Phi'_q \) are (up to isomorphism) equal to each other and also to \( \Phi^0 \) (and, since \( M \) is simply connected, also to \( \Phi \)).

An important result in holonomy theory is the \textit{Ambrose-Singer theorem} (see e.g. [6]). Let \( p \in M \) and \( c \) be a differentiable curve from \( p \) to some point \( q \in M \). Denoting parallel transport along \( c \) from \( p \) to \( q \) by \( \tau \) then if \( u, v \in T_p M \) and \( R' \) is the curvature operator one can construct a linear map \( T_p M \rightarrow T_p M \) by

\[
(2) \quad w \rightarrow \tau^{-1} [R'(\tau(u)\tau(v)), \tau(w)]
\]

for \( w \in T_p M \). With \( p \) fixed, and for all choices of \( u, v \in T_p M \), \( q \in M \) and \( c \), the set of all such linear maps spans (a representation of) the holonomy algebra \( \phi \) of \( M \).

This theorem essentially says that a representation of \( \phi \) at \( p \in M \) can be obtained by choosing some \( q \in M \), finding all bivectors in the range of the curvature tensor at \( q \) (i.e. all bivectors of the form \( R^a_{bcd} H^{cd} \) at \( q \)) and parallely transporting them along some curve \( c \) to \( p \). The bivectors which occur at \( p \) for all choices of \( q \in M \) and \( c \) span \( \phi \).

It is remarked here that a detailed study of the space-time \textit{infinitesimal holonomy group} has been given [7] following earlier work in [8] and [9]. However these studies rely upon certain (explicit or implicit) assumptions being made about the constancy of the dimension of the infinitesimal holonomy group over \( M \). In this paper no such assumptions are made and interest is focussed on the "full" \textit{holonomy group} \( \Phi \) of \( M \).

2. Holonomy Reducibility

Let \( p \in M \) and, with the notation of the previous section, define \( \Phi_p \) by

\[
(3) \quad \Phi_p = \{ f_c : c \text{ a piecewise differentiable closed curve at } p \}
\]

Then the holonomy group \( \Phi \) of \( M \) is called \textit{reducible} if for some (and hence any) \( p \in M \) and for some non-trivial proper subspace \( V \subseteq T_p M \), \( V \) is invariant under each member of \( \Phi_p \). Otherwise \( \Phi \) is called \textit{irreducible}. Such a subspace \( V \) is called \textit{holonomy invariant}. Further, \( \Phi \) is called \textit{non-degenerately reducible} if a (non-trivial proper) non-null holonomy invariant subspace of \( T_p M \) exists at some (and hence every) \( p \in M \). If \( \Phi \) is reducible, but not non-degenerately reducible, it is called \textit{degenerately reducible} [10].

Regarding the Lorentz group \( L_0 \) as a 6-dimensional Lie group of \( 4 \times 4 \) non-singular matrices preserving the Lorentz matrix \( \eta_{ab} = \text{diag}(1, 1, 1) \) one may represent the Lorentz algebra \( A \) as the vector space of \( 4 \times 4 \) skew-symmetric matrices (bivectors) in the usual way (see table 1). The holonomy algebra \( \Phi \) will then be viewed as a subalgebra of \( A \) in this representation. One then has the exponential map \( \exp : \phi \rightarrow \Phi \) defined as the usual exponential of matrices (see e.g.[11]).

A global, nowhere zero vector field \( k \) on \( M \) is called \textit{recurrent} if there exists a global covector field \( q \) on \( M \) such that in any coordinate domain \( k^a_{qh} = k^a q_h \). One can now collect together the following results (see e.g. [1] [12])

\textbf{Theorem 1.} Let \( M \) be a simply connected space-time with (connected) holonomy group \( \Phi \) and holonomy algebra \( \phi \). Then

(i) \( \Phi \) is reducible if and only if the members of \( \phi \) admit a common eigenvector.
(ii) $\Phi$ is reducible if and only if $M$ admits a global recurrent vector field.

(iii) The members of $\phi$ admit a common eigenvector with zero eigenvalue if and only if $M$ admits a non-zero global covariantly constant vector field.

(iv) The holonomy group $\Phi$ is reducible if and only if it is not of type $R_{15}$. 

It is remarked here that a nowhere zero covariantly constant vector field is recurrent but not necessarily conversely. In fact, if $k$ is a recurrent vector field on $M$ then it gives rise in an obvious way to a 1-dimensional holonomy invariant subspace at each $p \in M$ and $k$ is either everywhere timelike, everywhere spacelike or everywhere null. Also if $k$ is recurrent, so is $\alpha k$ for a nowhere zero $\alpha : M \to \mathbb{R}$. Then if $k$ is recurrent and not null, the associated normalized (to $\pm 1$) vector field obtained from $k$ is covariantly constant. This result may fail if $k$ is null in the sense that it may not be possible to globally scale $k$ so that it is covariantly constant.

Finally it is mentioned that the Lorentz group $L_0$ has the nice property that it is an exponential group, that is, the exponential map $\exp : A \to L_0$ is surjective (onto). Thus every member of $L_0$ is the exponential of some member of $A$ (see, e.g.,[13]). This property can fail for the (connected) Lie subgroups of $L_0$. However one does have the following property of any connected Lie group $G$ (and, in particular for the connected Lie subgroups of $L_0$) that any member of $G$ is a product of finitely many members of $G$ each of which is the exponential of a member of the Lie algebra of $G$.

It is convenient at this point to collect together some results that will be of use later. Let $R_{ab} = R_{a[cb]}$ be the components of the Ricci tensor of $M$, $R = R_{ab}g^{ab}$ the Ricci scalar and $C_{abcd}$ the Weyl tensor components. Then the curvature tensor can be decomposed as

$$R_{abcd} = C_{abcd} + R_{a[c}g_{d]b} + g_{a[c}R_{d]b} - \frac{1}{3}RG_{abcd}$$

$$= C_{abcd} + E_{abcd} + \frac{1}{6}RG_{abcd}$$

where

$$E_{abcd} = \tilde{R}_{a[c}g_{d]b} + \tilde{R}_{b[d}g_{c]a}$$

The statement that $M$ is an Einstein space (so that $R_{ab} = \frac{1}{4}Rg_{ab}$) is equivalent to either of the statements $\tilde{R}_{ab} = 0$ or $E_{abcd} = 0$.

The Einstein field equations are

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}$$

where $T_{ab}$ are the components of the energy-momentum tensor and $\kappa \neq 0$ is the gravitational constant. The following duality properties are also useful (where $\ast$ denotes the usual duality operator)

$$\ast C_{abcd} = C_{abcd}, \ast G_{abcd} = G_{abcd}, \ast E_{abcd} = -E_{abcd}$$

It follows that, when considered in the usual $6 \times 6$ formulation as linear maps on bivector space at any $p \in M$, the maps arising from the tensors $C_{abcd}, G_{abcd}$ and $E_{abcd}$ have even rank. For the Weyl tensor at $p$ the rank is 2 in the Petrov type $N$ case, 4 for type $III$, 6 for types $II$ and $D$ and 4 or 6 for type $I$.

3. Vacuum Space-Times

This special case has been dealt with elsewhere [2] and the following theorem proved
Theorem 2. Let $M$ be a simply connected, not flat, vacuum space-time. Then the
holonomy group $\Phi$ of $M$ is one of the types $R_8$, $R_{14}$ or $R_{15}$.

The proof in [2] essentially consists of using the first equation in (8), applied now
to the curvature tensor in vacuum, to establish that whenever a bivector belongs to
the holonomy algebra $\phi$ (in the sense of table 1) then so also does its dual. From this
it follows that the holonomy algebra is even-dimensional and the Ambrose-Singer
theorem together with a consideration of the infinitesimal holonomy completes the
proof.

A mild variation of this proof starts by assuming that $M$ is not flat (otherwise
$\Phi$ is of type $R_1$) and then supposes that $\Phi$ is not of type $R_{15}$ and is hence reducible
from theorem 1. The same theorem then shows that $M$ admits a recurrent vector
field $k$. If $k$ is covariantly constant on $M$ the Ricci identity on $k$ gives $R^a_{bcd}k^d = 0$
from which it follows that the curvature tensor (equal to the Weyl tensor in vacuo) is
of Petrov type $N$ (and hence of rank 2) and that $k$ spans the repeated principal null
direction at those points of $M$ where the curvature is non-zero. Since $M$ is not flat it
follows that $k$ is a null vector field on $M$ and the curvature tensor is constructed out
of a dual pair of null bivectors with principal null direction $k$ at those points where
it is not zero. Since $k$ is covariantly constant and hence preserved under parallel
translation, the Ambrose-Singer theorem shows that $\phi$ consists of a dual pair of
null bivectors with principal null direction $k$ and hence that $\Phi$ is of type $R_8$. If $k$
is recurrent and $M$ admits no covariantly constant vector fields then $k$ must be null
and $R^a_{bcd}k^d$ cannot be identically zero on $M$. The latter statement follows because
otherwise the previous argument using the Ambrose-Singer theorem would again
lead to $\Phi$ being of type $R_8$ and from table 1 and theorem 1(iii) one then achieves
the contradiction that $k$ is covariantly constant on $M$. Thus if $k_{ab} = k_{a}p_{b}$ for some
covector field $p$ on $M$ the Ricci identity on $k$ shows that on $M$ $R_{abc}k^d = k_a F_{ab}$
where the bivector $F$ is not identically zero on $M$ and, where non-zero, is null with
principal null direction $k$ (from the vacuum condition and the identity $R_{[abcd]} = 0$).
Hence at such a point $p$ the Petrov type is $III$ with repeated principal null direction
$k$ and, as is well known, the curvature tensor is constructed from four independent
bivectors at $p$. Thus dim $\phi_p \geq 4$ and from table 1 and the fact that $\Phi$ is reducible
$\Phi$ must be of type $R_{14}$. This completes the proof.

If $\Phi$ is of type $R_8$ the Petrov type is $N$ at those points where the curvature is
non-zero. If $\Phi$ is of type $R_{14}$ the Petrov type is $N$ where the curvature tensor is
not zero and where $R_{abc}k^d = 0$ and type $III$ where $R_{abc}k^d \neq 0$. The vacuum
pp-waves [14] are essentially the only examples of holonomy type $R_8$. Examples of
type $R_{14}$ can be found in [15] [16]. The generic vacuum metric is of type $R_{15}$ (e.g.
the Schwarzschild metric).

4. Einstein Space(-Times)

In view of theorem 2, $M$ will be assumed a proper Einstein space with $E_{abcd} = 0$
and $R = \text{constant} \neq 0$ in (4). Then (4) and (8) show that $R^a_{abcd} = R^{a}_{abcd}$ and the
argument in [2] again shows that $\phi$ is dual invariant in the sense described at the
beginning of the proof of theorem 2 and hence that dim $\Phi$ is even. Table 1 then
shows that the possibilities for the type of $\Phi$ are $R_7$, $R_8$, $R_{14}$ and $R_{15}$. However
if $\Phi$ were of type $R_8$ the curvature tensor would be constructed entirely out of a
dual pair of null bivectors at each $p \in M$ with principal null direction $k \in T_p M$.
But then, at $p$, $R^{a}_{bcd}k^d = 0$ ($\Rightarrow R_{ab}k^b = 0$) and so one finds the contradiction that
$R = 0$ at $p$ (and, incidentally, that $M$ admits no covariantly constant vector fields).
Thus one has the following result.
Theorem 3. Let $M$ be a simply connected proper Einstein space(-time). Then the holonomy group $\Phi$ of $M$ is one of the types $R_7$, $R_{14}$ or $R_{15}$.

The recurrence condition and Ricci identity on any null recurrent vector field $k$ on $M$ gives

$$k_a R^a_{bcd} = k_a C^a_{bcd} + \frac{1}{12} R (k_c g_{bd} - k_d g_{bc}) = 2k_b \delta_{[c:d]}$$

Thus $k_c R^a_{b|cd}k_a = 0$ and then

$$k_c C^a_{b|cd}k_a = \frac{1}{12} R (k_c g_{bd} - k_d g_{bc})$$

where the expressions in (i) are nowhere zero on $M$. Now if $\Phi$ is of type $R_7$ it is known that there are two recurrent (null) vector fields satisfying (10) (see table 1 or [12]) and hence, using the Bel criteria for the Petrov types, such a space-time is of Petrov type $D$ everywhere with these recurrent null directions as the principal null directions. If $\Phi$ is of type $R_{14}$ only one such recurrent (null) vector field exists (table 1) and so the Petrov type is $II$ or $D$ at each point with the recurrent null vector field as repeated principal null direction. The general case is when $\Phi$ has type $R_{15}$ and the special case when $M$ has constant curvature is of this type since then $\dim \delta_p = 6$ for each $p \in M$. An example of type $R_7$ can be obtained by adjusting the $R_7$ example in the next section so that the 2-dimensional manifolds $M_1$ and $M_2$ described there have the same constant curvature. The authors are not aware of a proper Einstein space of type $R_{14}$.

5. Conformally Flat Space-times

Here one has the decomposition (4) with the Weyl tensor identically zero. Suppose $M$ admits a non-zero covariantly constant vector field $k$. Then the Ricci identity on $k$ yields $R_{abcd}k^d = 0$ and hence $R_{ab}k^b = 0$. A contraction of (4) with $k^a$ and a further contraction with $k^c$ yields, in turn, the relations

$$k_c R^b_{d|ab} = \frac{1}{3} R k_c g_{db} \quad (ii) (k_c k^c) \left( \frac{1}{3} R g_{ab} - R_{ab} \right) = \frac{1}{3} R k_a k_b$$

Now if $M$ is flat the holonomy type is $R_1$. Otherwise there exists $p \in M$ such that the curvature tensor (and hence the Ricci tensor since $M$ is conformally flat) is not zero at $p$. Thus if $M$ is not flat, (11)(ii) shows that $k$ is null at $p$ if and only if $R = 0$ at $p$. The same equation then shows that, since $k$ is covariantly constant and nowhere zero, $k$ is null on $M$ if and only if $R \equiv 0$ on $M$.

Now if $k$ is null (11)(i) shows that $R_{ab} = \alpha k_a k_b$ on $M$ where $\alpha : M \rightarrow \mathbb{R}$ (so the Ricci tensor, if not zero, has Segre type $\{211\}$ with eigenvalue zero). But then the curvature tensor is identically equal to $E_{abcd}$ (since $R \equiv 0$ on $M$) and so, either from (4) after a short calculation or directly from the classification in [17], one sees that the curvature is either zero or constructed from a dual pair of null bivectors with principal null direction $k$ at each point of $M$. A similar argument to that in the vacuum case using the Ambrose-Singer theorem then shows that $\Phi$ is of type $R_3$.

If $k$ is not null then at some $p \in M$ $R(p) \neq 0$ and (11)(ii) gives

$$R_{ab} = \frac{1}{3} R (g_{ab} - \varepsilon k_a k_b)$$

where $k$ has been assumed globally scaled so that $k_a k^a = \varepsilon = \pm 1$. From (12) and (4) one finds

$$R_{abcd} = 4 R \left( G_{abcd} - \varepsilon g_{a[c} k_{d]} - 2 \varepsilon k_a k_{[c} g_{d]} \right)$$

Now if $F$ is any bivector at $p$ satisfying $F_{ab} k^b = 0$ (13) shows that

$$R_{abcd} F^{cd} = \frac{1}{4} R F_{ab} \neq 0$$
But such bivectors form a 3-dimensional subspace $W$ of bivector space at $p$ and the complementary subspace $W^*$ to $W$ is spanned by three independent simple bivectors whose blades contain $k$. Thus if $H \in W^*$, $R_{abcd}H^{cd} = 0$ (since $R_{abcd}k^d = 0$). It follows that, since $R(p) \neq 0$, the curvature tensor has rank equal to 3 at $p$. This shows that $\phi$ has dimension 3 and hence from table 1 that $\dim \phi = 3$ and that $\Phi$ is of type $R_{10}$ if $k$ is spacelike and of type $R_{13}$ if $k$ is timelike. It follows from (12) that $R_{ab}k^b = 0$ and that the Ricci and energy-momentum tensors have Segre type $\{(1, (11))\}$ (k timelike) and $\{(1, 11)\}$ (k spacelike).

Now suppose $M$ admits a recurrent (null) vector field $k$ such that $k_{a;b} = k_ap_b$. The Ricci identity on $k$ followed by a contraction with $g^{bd}$ gives

$$k_aR^a_{bcd} = 2k_bp_{[c,d]}$$

The first of these shows that the bivector $p_{[a;b]}$ at any $p \in M$ is amongst the bivectors contributed to $\phi'_p$, and hence to $\phi$, by the curvature tensor as in (1). Thus from theorem 1, it must have $k$ as an eigenvector and so $p_{[a;b]}k^b = \nu k_a$ for $\nu : M \to \mathbb{R}$ (or, alternatively, use the identity $R_{a[bc]} = 0$ to see that $p_{[a;b]}$ is a simple bivector whose blade contains $k$). Then a contraction of (4) with $k^a k^c$ shows that $R = 0$ on $M$. Thus $R_{abcd} = E_{abcd}$ on $M$ and then (8) and the argument in [2] shows that $\phi$ is dual invariant (as described earlier) and that $\dim \phi$ is even. Since $k$ may be assumed here to be not covariantly constant, it follows that the only possibilities for $\Phi$ are the types $R_7$ and $R_{14}$. The following theorem has been proved (c.f. [3])

**Theorem 4.** Let $M$ be a simply connected conformally flat, not flat space-time. Then the holonomy group $\Phi$ of $M$ is one of the types $R_7$, $R_8$, $R_{10}$, $R_{13}$, $R_{14}$ or $R_{15}$.

It is clear from the proof that, given the holonomy group is reducible, the condition that $R = 0$ on $M$ is equivalent to $M$ admitting a null recurrent (including covariantly constant) vector field and then $\Phi$ is one of the types $R_7$, $R_8$ or $R_{14}$. If a recurrent (not globally scalable to be covariantly constant, and hence null) vector field is admitted then the possibilities for $\Phi$ are the types $R_7$ and $R_{14}$.

A closer investigation of the curvature tensor and the tensor $E_{abcd}$ reveals further information about the Segre type of the Ricci tensor when the holonomy group is reducible [3]. In fact in such cases this Segre type at $p \in M$ (which may vary with $p$) is, for a non-zero Ricci tensor, either

(i) $\{(31)\}$ with eigenvalue zero ($\Rightarrow R = 0$)
(ii) $\{2(11)\}$ with eigenvalues differing only in sign ($\Rightarrow R = 0$)
(iii) $\{(211)\}$ with eigenvalue zero ($\Rightarrow R = 0$)
(iv) $\{1,(111)\}$ with the non-degenerate (timelike) eigenvalue zero ($\Rightarrow R \neq 0$)
(v) $\{(1,11)\}$ with the non-degenerate eigenvalue zero ($\Rightarrow R \neq 0$)
(vi) $\{(1,1)\}$ with eigenvalues differing only in sign ($\Rightarrow R = 0$)

Examples of each of these holonomy type possibilities, except one, can now be given. (The authors are unaware of an example of the $R_{14}$ case). For the $R_7$ type let $(M_1, g_1)$ and $(M_2, g_2)$ be two 2-dimensional manifolds with $M_1 = M_2 = \mathbb{R}^2$ and $g_1$ and $g_2$ metrics on $M_1$ and $M_2$ with signatures $(-1,1)$ and $(1,1)$ respectively. Suppose also that $(M_1, g_1)$ has constant curvature $k < 0$ and that that $(M_2, g_2)$ has constant curvature $-k$. Then $M = M_1 \times M_2$ is a connected and simply-connected space-time with Lorentz metric $g = g_1 \times g_2$. Further if $x^0, x^1$ are coordinates for $M_1$ and $x^2, x^3$ are coordinates for $M_2$ then the metric $g$ on $M$ is represented in the coordinate system $x^a$ on $M$ by

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{AB}dx^Adx^B$$
where \( \alpha, \beta, \gamma = 0, 1, A, B, C = 2, 3 \) and where \( g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma) \) and \( g_{AB} = g_{AB}(x^C) \) are the components of \( g_l \) and \( g_q \) in the coordinates \( x^a \) and \( x^q \), respectively. For \( p \in M \) one can construct a real null tetrad \((l, n, x, y)\) in some coordinate neighbourhood \( U \) of \( p \) such that \( l \) and \( n \) are tangent to \( M_1 \) and \( x \) and \( y \) are tangent to \( M_2 \) in \( U \). Then the curvature tensor in \( U \) can be written in terms of the bivector \( A = l \wedge n \) and its dual \( A = x \wedge y \) as

\[
R_{abcd} = a \left( A_{ab} A_{cd} + A^*_{ab} A^*_{cd} \right) \quad (0 \neq a = \text{constant})
\]

It follows that the Ricci scalar \( R = 0 \) and, from (4), that the Weyl tensor is zero. The Ricci tensor has Segre type \( \{(1, 1) (1)\} \) with eigenvalues constant and differing only in sign and \((M, g)\) represents a conformally flat non-null Einstein-Maxwell field of the Bertotti-Robinson type \( [4] \). Also \( l \) and \( n \) may be extended to null vector fields on \( M \) which are recurrent (but which cannot be scaled so as to be covariantly constant) and the bivectors \( A \) and \( A^* \) are covariantly constant. It follows that the holonomy type of \((M, g)\) is \( R_7 \) (see \([7]\)).

An example of a type \( R_8 \) holonomy group is given by the plane wave space-time given in a global coordinate system \((x, y, u, v)\) on \( \mathbb{R}^4 \) by the metric \([4]\)

\[
ds^2 = dx^2 + dy^2 - 2dudv - h(u) (x^2 + y^2) \, du^2
\]

where \( h : \mathbb{R}^4 \to \mathbb{R} \) is everywhere positive. This space-time is conformally flat and represents a null Einstein-Maxwell field. The covector field \( l_a = u_{,a} \) is global, null and covariantly constant as are the bivectors \( F_{ab} = 2l_{[a x, b]} \) and \( F_{ab}^* = 2l_{[a y, b]} \). The curvature tensor is

\[
R_{abcd} = f(u) \left( F_{ab} F_{cd} + F_{ab}^* F_{cd}^* \right)
\]

for \( f : \mathbb{R}^4 \to \mathbb{R} \) positive.

Next let \( M' = \mathbb{R}^3 \) and \( g' \) be a Lorentz metric on \( M' \) which is of non-zero constant curvature. Then the usual metric product \( M = M' \times \mathbb{R} \) yields a space-time admitting a global covariantly constant spacelike vector field. The holonomy group is easily checked to be of type \( R_{10} \) (c.f. \([7]\)) and since a 7-dimensional Lie algebra of (local) Killing vector fields is admitted by \( M \), it is conformally flat.

A simply connected region of the well-known Einstein static universe which admits a global covariantly constant timelike vector field is a conformally flat perfect fluid and an example of holonomy type \( R_{13} \). The standard (simply connected) Friedmann-Robertson-Walker space-times are conformally flat and give examples of holonomy type \( R_{15} \).

### 6. Null Einstein-Maxwell Fields

In this case let \( F \) be the nowhere zero global null Maxwell bivector with principal null direction represented by the nowhere zero global null vector field \( l \). Then in a coordinate neighbourhood of some (any) \( p \in M \) there exists a (nowhere zero) spacelike vector field \( q \) orthogonal to \( l \) such that \( F_{ab} = 2l_{[a q, b]} \) and

\[
R_{abcd} = C_{abcd} + \frac{k}{2} \left( F_{ab} F_{cd} + F_{ab}^* F_{cd}^* \right)
\]

where the Einstein-Maxwell equations are \( R_{ab} = \mu l_{ab} \) (\( \Rightarrow R = 0 \)) and where \( \mu : M \to \mathbb{R} \) is nowhere zero.

Suppose first that \( \Phi \) is reducible and, as before, let \( k \) be a recurrent vector field on \( M \). If \( k \) is covariantly constant then the Ricci identity gives \( R_{abcd} k^d = 0 \), \( R_{ab} k^b = 0 \) and hence \( k_{,a} l^a = 0 \). Thus \( k \) is everywhere null or everywhere spacelike.
If $k$ is everywhere null one may take $l = k$ and so from (16) $C_{abcd} = 0$. Thus the Petrov type at any $p \in M$ is $N$ (with the repeated principal null direction spanned by $l$) or $O$. If $M$ is conformally flat, (16) together with the Ambrose-Singer theorem and the covariant constancy of $l$ shows that $\Phi$ is of type $R_8$ (c.f. section 5). If the Petrov type is $N$ at some $p \in M$ then the Weyl tensor is also constructed out of the bivectors $F$ and $\hat{F}$ at $p$ and a further use of the Ambrose-Singer theorem and (16) shows that $\phi$ contains either just the two independent bivectors $F$ and $\hat{F}$ or (if judicious cancellation occurs) just one of these bivectors. In the former case $\Phi$ is of type $R_3$ and in the latter case of type $R_3$.

If $k$ is spacelike one may choose a real null tetrad $(l, n, x, y)$ in a coordinate neighbourhood of some (any) $p \in M$ such that $k = x$ and so $R_{abcd}x^d = 0$. It follows that the curvature tensor is constructed in the neighbourhood from the bivectors $l \wedge n$, $l \wedge y$ and $n \wedge y$ and for consistency with the expression for the Ricci tensor, one must then have in this neighbourhood

$$R_{abcd} = 4\mu [a]_{b} [c]_{d}$$

(17)

The Ambrose-Singer theorem then shows that $\phi$ is spanned by null bivectors which contract to zero with $x$ (since $x$ is covariantly constant). From table 1 this can easily be seen to eliminate the type $R_6$ but not the types $R_3$ and $R_{10}$. (Alternatively if one assumes $\Phi$ is of type $R_6$ then $l$ would be recurrent and the Ambrose-Singer theorem would then show that $\phi$ consisted only of $l \wedge y$ and give the contradiction that the type was $R_3$). The only possibilities are that $\Phi$ is of type $R_3$ or type $R_{10}$.

If the type is $R_3$ then $l$ is covariantly constant on $M$ (see previous paragraph). It also follows from (16) and (17) that for either holonomy type the Petrov type is $N$ at every point of $M$ with repeated principal null direction spanned by $l$.

Now suppose that $k$ is a recurrent vector field on $M$ but that $M$ admits no covariantly constant non-zero vector fields (so that $k$ is null on $M$). Then $k_{a}R_{bcd} = 2k_{a}p_{[c]d]}$ and by the argument given in the previous section $2\mu [a]_{b} [c]_{d]}' = l \wedge y$ for some $\lambda : M \rightarrow R$. It follows that $\lambda k_{a} = \mu (l_{a}k_{\alpha}) l_{\alpha}$ and hence that, irrespective of the possible zeros of $\lambda$, $l$ and $k$ are proportional at each $p \in M$. Thus one may take $k = l$ and then (16) and the Ricci identity for $l$ give

$$l_{a}R_{bcd} = 2l_{a}p_{[c]d]} = l_{a}C_{bcd}^a \quad \Rightarrow \quad l_{a}l_{c}C_{bcd}^a = 0$$

(18)

This shows that the Petrov type at any $p \in M$ is $III$, $N$ or $O$ with $l$ spanning the repeated null direction in the first two cases. If the Petrov type is nowhere type $III$ then (16) and the Ambrose-Singer theorem and theorem 1 yield the contradiction that $l$ is covariantly constant on $M$ and so the Petrov type must be $III$ on some (necessarily open because of the remarks on rank in section 2) subset of $M$. From the algebraic structure of a type $III$ Weyl tensor at $p \in M$ one can extend $l$ to a real null tetrad $(l, n, x, y)$ at $p$ such that, using (16), the simple bivectors $l \wedge n$, $x \wedge y$ and $l \wedge x$ (at least) are contributed to $\phi_p$ and hence to $\phi$ through (1). Thus $\dim \phi \geq 3$ and the conditions on $l$ and table 1 limit the possibilities for $\Phi$ to the types $R_9, R_{12}$ and $R_{14}$. But the case $R_9$ requires each member of $\phi$ to be simple and with $l$ in its blade whilst it is easily checked that if $\Phi$ is of type $R_{12}$ then $\phi$ does not admit three independent simple bivectors. Hence $\Phi$ is of type $R_{14}$. One can thus state the following theorem.

**Theorem 5.** Let $M$ be a simply connected space-time of the null Einstein-Maxwell type. Then the holonomy group $\Phi$ of $M$ is of the type $R_3$, $R_8$, $R_{10}$, $R_{14}$ or $R_{15}$.

It is remarked that, whatever the holonomy type, the Petrov type is algebraically special (including type $O$) at any $p \in M$ with $l$ spanning a repeated principal null direction, by virtue of the Goldberg-Sachs theorem [18].
Examples of null Einstein-Maxwell fields of holonomy type $R_3$ can be found in [9] and [19] and examples of types $R_8$, $R_{10}$ and $R_{14}$ are also displayed in [9]. Since the reducible possibilities described in theorem 5 were found at each point to be of Petrov type $O$, $N$ or $III$, any type $D$ null Einstein-Maxwell fields (which exist - see [4], p256) are of holonomy type $R_{15}$.

7. NON-NULL EINSTEIN-MAXWELL FIELDS

Here the global Maxwell bivector is a non-null bivector $F$. One assumes the existence of independent global null vector fields $l$ and $n$ and functions $\alpha$ and $\beta$ such that $\alpha^2 + \beta^2$ is nowhere zero on $M$ and such that in local coordinates

$$F_{ab} = 2\alpha l_{[an]} + 2\beta x_{[ay]}$$

The Einstein-Maxwell equations then give

$$R_{ab} = -\frac{2}{\alpha^2 + \beta^2} \left( 2l_{(an)} - x_{a}x_{b} - y_{a}y_{b} \right)$$

which has Segre type $\{(1,1) (11)\}$ with its two distinct eigenvalues differing only in sign and nowhere zero on $M$.

Suppose that $M$ admits a recurrent vector field $k$. Clearly $k$ cannot be covariantly constant since then a contraction of the Ricci identity for $k$ reveals a zero eigenvalue of the Ricci tensor at each $p \in M$ which contradicts (20). Thus $k$ can be assumed null and the Ricci identity for $k$ is then $k_{a}R_{bcd} = 2k_{b}p_{[cd]}$ and so, as before, the bivector $p_{[ab]}$ is contained in $\phi$ and has $k$ as an eigenvector at each $p \in M$. A contraction of this Ricci identity then shows that $k$ everywhere spans a null Ricci eigendirection. But (20) shows that at each $p \in M$ the only null Ricci eigendirections are spanned by $l$ and $n$. So one can assume, say, that $k = l$ on $M$. Theorem 1 then shows that the bivectors spanning $\phi$ are (some subset of) the bivectors $A_{ab} = 2l_{[an]}$, $B_{ab} = 2l_{[ax]}$ and their duals. Thus the curvature tensor must be constructed only from these bivectors and, in addition, must be consistent with (20). It follows that the local expression for the curvature tensor is

$$R_{abcd} = \frac{2}{l} \left( \alpha^2 + \beta^2 \right) \left( A_{ab}A_{cd} + A_{ab}A^{*}_{cd} \right)$$

$$+ \alpha \left( B_{ab}B_{cd} - B_{ab}B^{*}_{cd} \right) + b \left( B_{ab}B^{*}_{cd} + B_{ab}B_{cd} \right)$$

$$+ c \left( A_{ab}B_{cd} + B_{ab}A_{cd} - A_{ab}B^{*}_{cd} - B_{ab}A^{*}_{cd} \right)$$

$$+ d \left( A_{ab}B^{*}_{cd} + B_{ab}A^{*}_{cd} + B_{ab}A_{cd} + A_{ab}B_{cd} \right)$$

for functions $a$, $b$, $c$ and $d$. Since $\alpha^2 + \beta^2$ is nowhere zero it can be shown from (21) and some elementary algebra that the curvature tensor contributes at least two independent bivectors to $\phi$ (through (1)) and that one of them is (simple) spacelike and is contracted to zero by $l$. Since no covariantly constant vector fields are admitted the possibilities for $\Phi$ are $R_7$, $R_9$, $R_{12}$ and $R_{14}$. However the algebras for $R_9$ and $R_{12}$ cannot admit a spacelike bivector with the above properties (because all members of $R_9$ are simple with their blades containing $l$ and the only simple members of $R_{12}$ are linear combinations of $l \wedge x$ and $l \wedge y$ and hence null). So only $R_7$ and $R_{14}$ remain. The following result has been established.

Theorem 6. Let $M$ be a simply connected space-time of the non-null Einstein-Maxwell type. Then the holonomy group $\Phi$ of $M$ is of the type $R_7$, $R_{14}$ or $R_{15}$. 

The Ricci identity above for the recurrent null vector field \( k \) and the fact that \( k \) is an eigenvector of \( p_{[a;b]} \) together show that
\[
R_{abcd}k^a k^c = e k_b k_d \quad R_{ab}k^b = f k_a
\]
for functions \( e \) and \( f \) on \( M \). These, in turn, lead from (4) (or [17]) to \( C_{abcd}k^a k^c \propto k_b k_b \). Hence, at points where the Weyl tensor is not zero, it is algebraically special with \( k \) spanning a repeated principal null direction. If the holonomy type is \( R_7 \) two such recurrent null vector fields are admitted and the Petrov type at any \( p \in M \) is thus \( O \) or \( D \). For the \( R_{14} \) type only one such recurrent null vector field occurs and so at any \( p \in M \) the Petrov type could be any of the algebraically special types (including \( O \)).

An example of a non-null (conformally flat) Einstein-Maxwell field of holonomy type \( R_7 \) was given in section 5. An example of the holonomy type \( R_{14} \) is listed in [9]. A (simply connected) Reissner-Nordström space-time is an example of holonomy type \( R_{15} \).

8. **Perfect Fluid Space-Times**

Here one assumes a global, nowhere zero, unit timelike vector field \( u \) (the fluid flow vector) and functions \( p \) (the pressure) and \( \mu \) (the energy density) on \( M \). It is assumed that \( \mu \) and \( p \) do not simultaneously vanish over some non-empty open subset of \( M \). One then has an energy- momentum tensor with components \( T_{ab} \) given by
\[
T_{ab} = (\mu + p) u_a u_b + p g_{ab} \\
\Rightarrow R_{ab} = \kappa \left[ \frac{1}{2} (\mu - p) g_{ab} + (\mu + p) u_a u_b \right]
\]
Suppose that \( k \) is a recurrent (possibly covariantly constant) vector field on \( M \). Then, by a similar argument to that given in the previous cases, \( k \) is a Ricci eigenvector, \( R_{ab}k^b = \lambda k_a \) for some function on \( M \), and so from (23)
\[
\lambda k_a = \frac{1}{2} (\mu - p) k_a + \kappa (\mu + p) \left( k^b u_b \right) u_a
\]
Now since \( k \) and \( u \) are nowhere zero it follows that if \( k \) is null \( (k^b u_b) \) is nowhere zero and so \( (\mu + p) = 0 \) everywhere on \( M \). Thus, from (23), \( M \) is an Einstein space-time with \( \mu - p \) constant on \( M \). If \( \mu - p = 0 \) \( M \) is vacuum and the conclusions of section 3 hold whereas if \( \mu - p \neq 0 \) section 4 completes the argument.

If \( k \) is timelike (and hence one can assume \( k \) is covariantly constant) then (24) with \( \lambda \equiv 0 \) and the facts that \( u \) and \( k \) are nowhere zero on \( M \) show that at each \( p \in M \mu + p = 0 \) if and only if \( \mu - p = 0 \). Recalling the exclusion clause at the beginning of this section one sees that \( \mu - p \) and \( \mu + p \) cannot both vanish over any non-empty open subset of \( M \) and so one may take \( k = u \) on \( M \). Thus \( u \) is covariantly constant on \( M \) and the Ricci identity shows that \( R_{abcd}u^d = 0 \) (\( \Rightarrow R_{ab}u^b = 0 \)) and so \( 3\mu + \mu = 0 \) on \( M \) and
\[
R_{ab} = -2\kappa p \left( g_{ab} + u_a u_b \right)
\]
The contracted Bianchi identity then shows that \( p \) and hence \( \mu \) and \( p \) separately are (non-zero) constants on \( M \). If one introduces a pseudo-orthonormal tetrad \((u, x, y, z)\) at any \( p \in M \) one finds that, for consistency with (25), the curvature tensor takes the form
\[
R_{abcd} = -\kappa p \left( X_{ab} X_{cd} + Y_{ab} Y_{cd} + Z_{ab} Z_{cd} \right)
\]
where \( X_{ab} = 2x_{[a} y_{b]} \), \( Y_{ab} = 2x_{[a} \dot{z}_{b]} \) and \( Z_{ab} = 2y_{[a} \dot{z}_{b]} \). Clearly, the holonomy group \( \Phi \) is of type \( R_{13} \).
If $k$ is spacelike (and again assumed covariantly constant) then (24) with $\lambda \equiv 0$ shows that $k_a u^a = 0$ over an open dense subset of $M$ and hence on $M$ and then that $\mu = p$ on $M$. Thus

$$R_{ab} = 2\kappa p u^a u_b$$

A similar expression to (26) can be found for the curvature tensor at $p \in M$ in terms of the pseudo-orthonormal tetrad given in an obvious notation by $(u, k, y, z)$

$$\psi$$

$$R_{abcd} = \kappa p (C_{ab} C_{cd} + D_{ab} D_{cd} + E_{ab} E_{cd})$$

where $C_{ab} = 2u_{[a}z_{b]}$, $D_{ab} = 2u_{[a}z_{b]}$ and $E_{ab} = 2y_{[a}z_{b]}$. Clearly, the holonomy group $\Phi$ is of type $R_{10}$. Thus it is not possible for both a timelike and a spacelike covariantly constant vector field to be admitted. When $k$ is timelike it is clear from (25) and (26) that the curvature tensor and the Ricci tensor tetrad components are the same for each of a 3-parameter family of pseudo-orthonormal tetrads (that is they are unchanged under the 3-dimensional subgroup $SO(3)$ of the Lorentz group.

Hence, from (4), the Weyl tensor is similarly unchanged. Since the largest such subgroup for a non-zero Weyl tensor has dimension 2 [14] it follows that in this case, $M$ is conformally flat. The following theorem is established.

**Theorem 7.** Let $M$ be a simply connected space-time representing a perfect fluid. Then if $M$ is not an Einstein space(-time) the holonomy group $\Phi$ of $M$ is of type $R_{10}$, $R_{13}$ or $R_{15}$.

The Einstein static universe discussed in section 5 is an example with a type $R_{13}$ holonomy group whilst the Gdel metric provides an example of type $R_{10}$. The standard Friedmann- Robertson-Walker metrics give examples of type $R_{15}$.

9. **Massive Scalar Fields**

If $M$ is the space-time of a massive scalar field $\psi$ defined on $M$ such that $\psi$ is not constant over any non-empty open subset of $M$ together with a non-zero constant $m$ such that the associated energy-momentum tensor is given by [20]

$$T_{ab} = \psi_a \psi_b - \frac{1}{2} (\psi^c \psi_c + m^2 \psi^2) g_{ab}$$

where $\psi_a = \psi_a$. Then (29) and the contracted Bianchi identity give

$$R_{ab} = \kappa \psi_a \psi_b + \frac{1}{2} \kappa m^2 \psi^2 g_{ab} \quad \psi^a_a = m^2 \psi$$

Now (30) reveals that at any $p \in M$, any $v \in T_p M$ orthogonal to $\psi_a$ is a Ricci eigenvector with the same eigenvalue $\frac{1}{2} \kappa m^2 \psi^2$. Hence the Ricci tensor has a triple (real) eigenvalue degeneracy and so has Segre type either $\{(211)\}$, $\{(1, 11) 1\}$ or $\{1, (111)\}$ at $p$ [17].

Now suppose $M$ admits a recurrent vector field $k$. Then, by an argument identical to one used several times earlier, the Ricci identity on $k$ shows that $k$ is a Ricci eigenvector everywhere on $M$. If $k$ is null then one can write down, using the argument of the previous section, an expression for the curvature tensor similar to (21) but now one must impose consistency with the first equation in (30). Since $k$ is a null Ricci eigenvector the Segre type of the Ricci tensor at any $p \in M$ must be $\{(211)\}$ or $\{(1, 11) 1\}$ and this expression for the curvature tensor, on contraction, shows that the latter is impossible. Thus the Segre type is everywhere $\{(211)\}$. But the first equation in (30) cannot be of this type unless $\psi_a$ spans the null Ricci eigendirection at those points where $\psi_a$ is not zero. So by the uniqueness of the null Ricci eigendirection for this type it follows that $\psi_a = \sigma k_a$ for some function $\sigma$ on $M$. Hence, since $k_{a;b} = k_a p_b$ and $\psi_{a;b} = \psi_{b;a}$ one finds that $\psi_{a;b} = \nu \psi_a \psi_b$ for...
some function $\nu$ on $M$. The second equation in (30) then gives the contradiction that $\psi \equiv 0$ on $M$.

If $k$ is not null (and assumed globally scaled so that $k_a b = 0$ and $k^a k_a = \varepsilon = \pm 1$) the Ricci identity gives $R_{ab} k^b = 0$ everywhere on $M$. This fact and the first equation of (30) yields

$$\frac{1}{2} m^2 \psi^2 k_a = 0$$

(31)

It follows that $\psi_a k^a$ may only vanish over a subset of $M$ with no interior and so over the complement of this subset, and hence over $M$, $\psi_a = \rho k_a$ for some function $\rho : M \to \mathbb{R}$. The previous equation then gives

$$\varepsilon \rho^2 + \frac{1}{2} m^2 \psi^2 = 0$$

(32)

Now in some open neighbourhood $U$ of any $p \in M$ one may write $k_a = u_{,a}$ for some function $u$ on $U$. It then follows (since $k_a b = 0 \Rightarrow \rho = \rho (u)$) that on $U$, $\psi_{a;b} = \dot{\rho} k_a k_b$ (where a dot represents differentiation with respect to $u$). Also $\psi = \psi (u)$ and $\dot{\psi} = \rho$.

The second equation in (30) then reveals that $\varepsilon \rho^2 = m^2 \psi$ on $U$. But on differentiating (32) one finds that $2\varepsilon \dot{\rho} + m^2 \psi \dot{\psi} = 0$ on $U$. Thus $\rho = 0$ and the contradiction $m^2 \psi = 0$ on $M$ is achieved. This gives the following result.

**Theorem 8.** Let $M$ be a simply connected space-time representing a massive scalar field. Then the holonomy group $\Phi$ of $M$ is of the type $R_{15}$.

10. Conclusions

The holonomy group structure of simply connected space-times which have certain specified energy-momentum tensor (or which are conformally flat) has been studied. The allowed holonomy types are (excluding the flat space-time $R_1$ type) for vacuum space-times (types $R_8$, $R_{14}$, $R_{15}$), for Einstein space-times with $R \neq 0$ ($R_7$, $R_{14}$, $R_{15}$), for conformally flat space-times ($R_7$, $R_8$, $R_{10}$, $R_{13}$, $R_{14}$, $R_{15}$), for null Einstein-Maxwell space-times ($R_3$, $R_8$, $R_{10}$, $R_{14}$, $R_{15}$), for non-null Einstein-Maxwell space-times ($R_7$, $R_{14}$, $R_{15}$), for perfect fluid space-times ($R_{10}$, $R_{13}$, $R_{15}$) and for massive scalar fields ($R_{15}$). Some of the above reducible cases (i.e. not type $R_{15}$) are, in fact, non-degenerately reducible (see caption to table 1). These space-times are locally the metric product of lower dimensional manifolds and so allow a natural product coordinate system convenient for their exploration. Similar (but not always quite so convenient) coordinate systems are admitted in the degenerately reducible cases.

Some results which arise are perhaps worth general statements. If $M$ admits a null recurrent (not necessarily covariantly constant) vector field $k$ then $M$ is algebraically special (including type $O$) in the Petrov classification and $k$ spans a repeated principal null direction of the Weyl tensor and a null Ricci eigendirection. Suppose, on the other hand that $M$ admits a non-null covariantly constant vector field $k$. Then if $k$ is timelike the Ricci identity shows that $k$ is a timelike Ricci eigenvector and so the Ricci tensor is of the diagonalisable type (i.e. Segre type $\{1,11\}$ or one of its degeneracies (see e.g. [17]). It is then easily checked that, in the usual $6 \times 6$ formulation, the tensors $R_{abcd}$ and $E_{abcd}$ are (in an obvious algebraic sense) diagonalisable and hence so is the Weyl tensor. This last statement means that the Petrov type is $O$, $I$ or $D$. If a covariantly constant spacelike vector field is admitted then, potentially, any Petrov type might occur (see [7] and c.f. [21]).

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