Non-equilibrium thermal transport in the quantum Ising chain

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We consider two quantum Ising chains initially prepared at thermal equilibrium but with different temperatures and coupled at a given time through one of their end points. In the long-time limit the system reaches a non-equilibrium steady state. We discuss properties of this non-equilibrium steady state, and characterize the convergence to the steady regime. We compute the mean energy flux through the chain and the large deviation function for the quantum and thermal fluctuations of this energy transfer.

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I. INTRODUCTION

Recent experiments have strongly renewed the attention on non-equilibrium quantum dynamics. In particular, it has become possible to accurately measure the current flowing between two leads driven out of equilibrium through macroscopical control parameter, e.g. temperature or voltage gradient. Moreover, several proposals have been advanced in order to theoretically characterize and measure the quantum fluctuations of the current. In this framework, 1d systems are peculiar: on one side they represent an approximate description for 3d systems with strong anisotropy; on the other side, the dynamics is anomalous because of the role of purely elastic scattering processes. In particular, in some remarkable cases, the low-energy description can be given in terms of integrable field theories. In these cases, the presence of an infinite set of conserved charges may result in a non-vanishing Drude weight and a ballistic heat transport. It is then crucial to establish how universal these properties are and what the role played by dimensionality is.

In this paper, we focus on the analysis of the energy-current steady state in the simplest example of solvable lattice models, the quantum Ising chain. The protocol that we choose for constructing the out-of-equilibrium steady state is that of Hamiltonian reservoirs, defined as follows. At time \( t = 0 \), the system is prepared as two independent, finite, bounded (left and right) chains thermalized at different temperatures \( \beta_l \) and \( \beta_r \). The initial system density matrix is \( \rho_0 = \rho_l \otimes \rho_r \), where \( \rho_{r/l} = Z_{r/l}^{-1} \exp(-\beta_{r/l}H_{r/l}) \). Then the two chains are coupled at their end-points in the middle, producing a single homogeneous bounded chain twice as long with Hamiltonian \( H \), and the system is left to evolve unitarily. We will study and characterize the infinite-chain, long-time dynamics of local observables, defining the stationary density matrix \( \rho_{\text{stat}} \). Although the evolution of the whole system is unitary, as long as the evolution time remains smaller than the time an elementary excitation in the bulk needs to reach the boundary, any finite part of the chain around the connection point is effectively open and coupled to two asymptotic thermal baths. The stationary density matrix \( \rho_{\text{stat}} \) describes observables on such finite parts of the chain, not in the asymptotic baths. The stationary state density matrix \( \rho_{\text{stat}} \) (given in (19) below) shows a factorized form in terms of its chiral components. Furthermore, convergence to the steady state is non-universal and follows a \( t^{-1/2} \) behavior. We will characterize the heat transport along the chain, computing the mean current \( J \), see (24), and its large-deviation function \( F(\lambda) \) which satisfies fluctuation-dissipation relations, see (31). As already remarked in (25), the derivative of \( F(\lambda) \) can be obtained from the knowledge of the mean current at shifted values of the temperatures \( \beta_l \) and \( \beta_r \), see (30).

The density matrix \( \rho_{\text{stat}} \) was already obtained in the anisotropic XY chain (a family of models which includes the Ising chain) within the formalism of \( C^* \) algebra, where the factorized form was observed; however the approach to the steady state and the full counting statistics were not discussed. It was also obtained in 1+1-dimensional conformal field theory (CFT) and massive quantum field theory (QFT) where a similar factorized form was observed. Note that an other approach to treat 1d open quantum systems based on Lindblad dynamics has been developed.

This paper is organized as follows: in Sec. [1] we present a brief description of the diagonalization procedure and the thermodynamic limit (further details are given in the Appendix); in Sec. [II] we show that local observables have a well defined large time limit that implies a factorized form for \( \rho_{\text{stat}} \); in Sec. [IV] we derive an explicit expression for the stationary energy flow; in Sec. [V] the same formalism is applied to obtain the large deviation function.
II. EXACT SOLUTION

The Ising Hamiltonians for the right and left chains are defined as, respectively \((J > 0)\)

\[
H_r = -\frac{J}{2} \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x + h \sum_{i=1}^{N} \sigma_i^z
\]

\[
H_l = -\frac{J}{2} \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x + h \sum_{i=0}^{N-1} \sigma_i^z
\]

These spin chains can be exactly solved employing the usual Jordan-Wigner procedure. After having introduced the canonical fermionic operators

\[
c_i = -\left(\prod_{j<i} \sigma_j^x\right) \sigma_i^-
\]

and their hermitian conjugates, where \(\sigma_1^\pm = 1/2(\sigma_1^x \pm i\sigma_1^y)\), the Hamiltonians become quadratic forms that can be diagonalized by means of a Bogoliubov transformation (see the Appendix), using new canonical operators

\[
\phi_{r/l}(k) = \sum_{i=-N+1}^{N} \left[ \omega_{r/l}(k) c_i + c_{r/l}^\dagger(k) \sigma_i^x \right]
\]

with \(k = 1, \ldots, N\). The Hamiltonians take the form

\[
H_{r/l} = \sum_{k=1}^{N} \epsilon(k) \phi_{r/l}^\dagger(k) \phi_{r/l}(k);
\]

the one-particle spectrum is \(\epsilon(\theta) = J\sqrt{\theta^2 + 1 - 2h \cos \theta}\) and the angles \(\theta_k\) are solutions of the transcendental equation

\[
\frac{\sin((N+1)\theta_k)}{\sin(N\theta_k)} = h^{-1}.
\]

We restrict to the case \(h > 1\), i.e. the paramagnetic phase, where the equation has exactly \(N\) real solutions in the interval \([0, \pi]\). In an analogous way, one can treat the full chain Hamiltonian \(H = H_l + H_r + V\) where \(V = -\frac{J}{2} \sigma_0^x \sigma_1^z\).

We now consider the thermodynamic limit \(N \to \infty\). For the right / left chain with Hamiltonian \(H_{r/l}\) the result is a half-infinite chain, either extending to the right or to the left. The solutions of the equation distribute uniformly in the interval \([0, \pi]\). Therefore the variable \(\theta\) becomes continuous, and we can define properly normalized operators \(\Phi_{r/l}\) that satisfy standard anticommutation relations

\[
\{\Phi_{r/l}^\dagger(\theta), \Phi_{r/l}(\theta')\} = \delta(\theta - \theta').
\]

For \(H\) the result is an infinite chain. There a bit of care is needed. In the \(N \to \infty\) limit, the chain, already symmetric under parity, becomes also translation invariant. Hence the translation operator \(P\), defined by \(P^\dagger \sigma^{\alpha}_{k} P = \sigma^{\alpha}_{k-1}\) for \(\alpha = x, y, z\), can be diagonalized simultaneously with \(H\). Since \(P \leftrightarrow P^\dagger\) under parity, this imposes a two-fold degeneracy of the single-particle spectrum, which can be seen explicitly checking the odd and even \(k\) solutions for large \(N\) of \((6)\). We stress that, in the thermodynamic limit, \(H_0\) and \(H\) have, as expected, the same single-particle spectrum with the same degeneracy. In the single-particle degenerate eigensubspace of \(H\), it is possible to fix the two fermionic operators at each \(\theta\) by imposing that they have definite \(P\)-eigenvalues, which turn out to be \(e^{\pm i\theta}\). We then define canonical operators for the full chain

\[
\{\Phi_{R/L}^\dagger(\theta), \Phi_{R/L}(\theta')\} = \delta(\theta - \theta')
\]

with

\[
P^\dagger \Phi_{R/L}(\theta) P = e^{\mp i\theta} \Phi_{R/L}(\theta).
\]

The Hamiltonian takes the form

\[
H = \int_0^\pi d\theta \epsilon(\theta) [\Phi_{R}^\dagger(\theta) \Phi_{R}(\theta) + \Phi_{L}^\dagger(\theta) \Phi_{L}(\theta)] = H_R + H_L.
\]

For the lattice definition of \(\Psi_R\) and \(\Psi_L\) we refer to the Appendix.

III. NON-EQUILIBRIUM STEADY STATE

A. Factorization of the stationary density matrix

The stationary density matrix can be defined from the large-time expectation value of local observables

\[
\lim_{t \to \infty} \lim_{N \to \infty} \text{Tr}[O(t) \rho_0] = \text{Tr}[O \rho_{\text{stat}}].
\]

It is natural to reformulate it introducing the \(S\)-matrix, formally defined as

\[
S = \lim_{t \to \infty} e^{-iHt} e^{iH_{0t}},
\]

by which we can write \(\rho_{\text{stat}} = S \rho_0 S^\dagger\). Equivalently, one can consider the action of \(S\) over observables, that can be obtained once its action over the fermionic operators \(\Psi_{R/L}(\theta)\) is known. It is useful to observe that \(S\) intertwines between the two Hamiltonians, namely \(S H_0 S^\dagger = H\). This can be understood by considering the limit case where \(\beta_l = \beta_r\). Indeed, if the initial state is at equilibrium, it is natural to expect thermalization with the final Hamiltonian, i.e. in this case \(\rho_{\text{stat}} = Z^{-1} e^{-\beta H} = Z^{-1} S e^{-\beta H_0} S^\dagger\). This property is useful because it implies that \(S^\dagger \Psi_{R/L}(\theta) S\) must be a linear combination of \(\Phi_{r/l}(\theta)\) and can therefore be represented as a \(2 \times 2\) orthogonal matrix

\[
S^\dagger \begin{bmatrix} \Psi_R(\theta) \\ \Psi_L(\theta) \end{bmatrix} S = \begin{bmatrix} \cos a(\theta) & \sin a(\theta) \\ \sin a(\theta) & -\cos a(\theta) \end{bmatrix} \begin{bmatrix} \Phi_R(\theta) \\ \Phi_L(\theta) \end{bmatrix}.
\]

(13)
It is important to remark that the action of the $S$ matrix is well defined only on local operators. Therefore $\Psi_{R/L}(\theta)$ must be replaced by a smooth superposition of fermionic operators\cite{13}, which is local in the lattice. Such superpositions include both left and right movers:

$$\tilde{\Psi}(\theta) = \int_{-\pi}^{0} d\theta' g_{\theta}(\theta') \Psi_{L}(-\theta') + \int_{0}^{\pi} d\theta' g_{\theta}(\theta') \Psi_{R}(\theta').$$ \hspace{1cm} (14)

Here $g_{\theta}(\theta')$ is a wave-packet centered in $\theta$. Locality, more precisely the exponential vanishing of $\{ \tilde{\Psi}(\theta), c_{j}\}$, $\{ \tilde{\Psi}(\theta), c_{j}^{\dagger}\}$, at large $|j|$, imposes only that $g_{\theta}(\theta')$ be $2\pi$-periodic in $\theta'$ and analytic on a neighbourhood of the real $\theta'$-line. If the wave packet is peaked enough only one of the two terms essentially contribute in (14). At the end of the calculation, we will need to take the delocalization limit $g_{\theta}(\theta') \rightarrow \delta(\theta - \theta')$.

We first provide an heuristic argument to specify the value of $a(\theta)$, that we will prove explicitly later. This heuristic argument parallels the more precise derivations developed in CFT\cite{13,14} and in massive QFT\cite{15}. The eigenfunctions of the single-particle Hamiltonians for the left and right half chains can be thought of as stationary waves due to the boundary condition at the origin. Consider an operator which is a superposition of $\Phi_{0}$ and which creates a perturbation localised on the left, say around the site $j < 0$. The time evolution with $H_{0}$ will split it into two perturbations formed by right and left movers, whose centres move respectively towards the left and the right. However, that moving towards the right will eventually hit the right-boundary thus inverting its motion under the $H_{0}$ dynamics. Therefore, for large time $t$, the evolved operator will resemble an operator formed of right movers evolved with the Hamiltonian $H$. Thus we expect

$$e^{iH_{0} t} \Phi_{r/l}(\theta) e^{-iH_{0} t} \rightarrow \infty \quad e^{iH t} \Psi_{L/R}(\theta) e^{-iH t}$$ \hspace{1cm} (15)

which would correspond to $a(\theta) = \pi/2$.

We now give a more formal derivation. Inverting the relations between the fermionic fields and the local fermions $c_{j}$’s and $c_{j}^{\dagger}$’s, one obtains the change of basis matrices

$$\begin{bmatrix} \Psi_{R}(\theta) \\ \Psi_{L}(\theta) \end{bmatrix} = \int_{0}^{\pi} d\theta' m(\theta, \theta') \begin{bmatrix} \Phi_{r}(\theta') \\ \Phi_{l}(\theta') \end{bmatrix} + \tilde{m}(\theta, \theta') \begin{bmatrix} \Phi_{r}^{\dagger}(\theta') \\ \Phi_{l}^{\dagger}(\theta') \end{bmatrix}$$ \hspace{1cm} (16)

If we take a wave packet as in (14) and we act upon it with the $S$ matrix we must deal with the expression

$$\lim_{t \rightarrow \infty} \int_{0}^{\pi} d\theta' \int_{0}^{\pi} d\theta'' g_{\theta}(\theta') e^{-it(\varepsilon(\theta') - \varepsilon(\theta''))} m(\theta', \theta'') \begin{bmatrix} \Phi_{r}(\theta'') \\ \Phi_{l}(\theta'') \end{bmatrix}$$ \hspace{1cm} (17)

and a similar one with $g_{\theta}(\theta') \rightarrow g_{\theta}(-\theta')$, and for $\tilde{m}(\theta', \theta'')$. It is useful to change variable in the integral in $\theta'$, by introducing $z = e^{-i\theta'}$. In the $z$ variable, one has to integrate over the lower semi-circumference as shown in Fig.\cite{1} where $g_{\theta}$ is analytic. We may also assume $g_{\theta}$ to be analytic on the interval $[-1, 1]$. Then, it is possible to close the contour with a curve $\gamma$ in the lower semicircle, where the imaginary part of $\varepsilon(\theta')$ is negative; this does not affect the result in the large-$t$ limit as the integral over $\gamma$ vanishes in this limit. The integral reduces to the sum of the residues inside this region. The relevant contribution coming from the change of basis matrices is

$$m(\theta', \theta'') = \frac{1}{2\pi i} \left( \frac{1}{\theta' - \theta'' + i\delta - c.c.} \right) \sigma_{x} + \ldots$$ \hspace{1cm} (18)

where the remaining part and $\tilde{m}(\theta', \theta'')$ are not singular inside the integration domain and $\sigma_{x}$ is a Pauli matrix. The regularization $\delta \sim 1/N > 0$ is associated with the finite-volume regularization.

According to \cite{11}, the thermodynamic limit has to be taken before the large time limit. Further, the large-time limit should be taken before the delocalization limit. Sending $\delta$ to $0^{+}$ before $t \rightarrow \infty$, in the large $t$ limit all the poles inside the contour have vanishing residue, except for the one at $\theta' = \theta'' - i\delta$. After this, the wave-packet width can be safely sent to 0, i.e. $g_{\theta}(\theta') = \delta(\theta - \theta')$ and one obtains the result anticipated in (15). In particular, the non-equilibrium density matrix takes the factorized\cite{13} form

$$\rho_{\text{Stat}} = Z^{-1} e^{-\beta_{L} H_{L}} \otimes e^{-\beta_{R} H_{R}}.$$ \hspace{1cm} (19)

Notice that the opposite limit $t \rightarrow -\infty$ can be treated analogously, but the contour in Fig.\cite{1} has to be closed outside the unit circle in the lower half-plane. The relevant pole becomes $\theta' = \theta'' + i\delta$ and $a(\theta) = 0$. This reflects the fact that time-reversal symmetry is actually broken in the steady state, but $\mathcal{PT}$-symmetry is still preserved.
B. Approach to the steady-state

After having investigated the stationary density matrix, it can be useful to study how fast the convergence is. As we saw, all the poles inside the contour correspond to exponentially vanishing corrections. It remains to estimate the contribution of the integral over the curve γ and we employ the saddle-point approximation. We observe that, however, we choose γ inside the lower semicircle, the largest values of the imaginary part of ϵ(θ′) will be at the boundaries, i.e., θ′ = {0, π}. We can therefore expand all the functions close to them obtaining, as usual, gaussian integrals that can be readily performed. This provides a power-law approach \( t^{-1/2} \). This correction is due to the zero modes in the single particle spectrum, that do not move during the dynamics, giving place to a field localized in the original center of the wave packet. Even in the \( h \to 1 \) gapless limit, this contribution is still present due to the zero mode at \( \theta = \pi \), at the end of the spectrum, showing that it cannot be deduced from the low-energy physics. A \( t^{1/2} \) power law approach to the steady state was also observed in the context of quantum quenches in the Ising model\(^\text{20}\).

IV. HEAT CURRENT

In this section we will derive the stationary energy flow between the two halves of the chain. We introduce \( E = 1/2(H_r - H_l) \) whose variation is the total amount of energy transferred from the left to the right part of the chain. Its time derivative \( p = i[H, E] \) is local with support on the lattice sites \( j = 0, 1 \), and one can therefore expect its long-time limit expectation value to converge to a stationary value

\[
\lim_{t \to \infty} \text{Tr}[p(t)\rho_0] = \text{Tr}[\rho_{\text{stat}}] = \mathcal{J}. \tag{20}
\]

In the stationary state the mean energy current stays constant to the value \( \mathcal{J} \), reached in the long-time limit. In terms of local fermions \( c_j \)'s and \( c_j^\dagger \)'s, \( p = \frac{ihJ}{2}(c_0^\dagger c_1 - c_1^\dagger c_0) \). When expressed through the non-local left and right mover operators \( \Psi \) the operator \( p \) is represented by the quadratic form

\[
p = J^2 \int_0^\pi d\theta d\theta' \left[ \frac{\Psi_R^\dagger(\theta)\Psi_L(\theta)}{\Psi_R(\theta')\Psi_L^\dagger(\theta')} \right] A(\theta, \theta') \left[ \frac{\Psi_R(\theta')\Psi_L^\dagger(\theta)}{\Psi_R^\dagger(\theta)\Psi_L(\theta')} \right], \tag{21}
\]

where the explicit expression for the \( 2 \times 2 \) matrix \( A(\theta, \theta') \) is not needed here. The operator \( p \) can be seen as a momentum density at site 0. The momentum density at site \( j \) is \( p_j = \mathcal{P} / p (P^\dagger) \), and summing over \( j \in \mathbb{Z} \) one obtains the total momentum \( P \) of the chain, which satisfies \( [P, c_j] = \frac{ihJ}{2}(c_{j-1} - c_{j+1}) \). Hence\(^\text{25}\)

\[
P = hJ \int_0^\pi d\theta \sin \theta \left[ n_R(\theta) - n_L(\theta) \right] \tag{22}
\]

with \( n_{R/L}(\theta) \) the number operators for the right/left moving fermions. Applying now the definitions \( \delta \) and summing over \( j \) in \( \text{Eq. } (21) \) one concludes that the diagonal elements of the matrix \( A \) are fixed to be \( \text{diag}(\pm \frac{h}{2\pi} \sin \theta) \). It turns out that this is sufficient for our purposes. It is indeed clear that the expectation value computed with the stationary matrix \( \rho_{\text{stat}} \) of a bilinear fermionic operator satisfies

\[
\text{Tr}[\Psi_R^\dagger(\theta)\Psi_R(\theta')\rho_{\text{stat}}] = \delta_{\alpha,\beta}\delta(\theta - \theta') \frac{1}{e^{\epsilon_{\alpha,\beta}(\theta)} + 1} \tag{23}
\]

for \( \alpha, \beta = R, L \), with \( \beta_R \equiv \beta_L \equiv \beta \). Performing the integration over the variable \( \theta \) we finally derive

\[
\mathcal{J}(\beta_1, \beta_r) = \frac{1}{2\pi\beta_r^2} \left[ j(\beta_1J(h+1)) - j(\beta_1J(h-1)) \right] - \frac{1}{2\pi\beta_r^2} \left[ j(\beta_rJ(h+1)) - j(\beta_rJ(h-1)) \right], \tag{24}
\]

where \( j(x) = \text{Li}_2(-e^{-x}) - x \log(1 + e^{-x}) \). The expression \( \text{Eq. } (24) \) can be evaluated in at the critical point \( h = 1 \). At the critical point, its universal conformal behaviour as \( \beta_1/r \gg J^{-1} \) is obtained using \( \text{Li}_2(1) = -\pi^2/12 \), \( \text{Li}_2(0) = 0 \), giving \( 2\pi(\beta_1^{-2} - \beta_r^{-2}) \), in agreement with the general theory developed in\(^\text{13,21,15}\) for the value \( 1/2 \) of the CFT central charge. The universal scaling limit leading to a massive QFT can be explicitly read off \( \text{Eq. } (24) \), taking \( h \to 1 \) and \( \beta_1/r, J \to \infty \) with fixed products \( \beta_1/r, J(h-1) \).

V. LARGE DEVIATION FUNCTION

We have seen that, given the factorized form of the steady state, the average current is related to the asymmetry in the number of movers in the two directions. Once the total amount of energy transferred within a time interval \( [t_1, t_2] \) has been defined as

\[
\Delta E(t_2, t_1) \equiv \int_{t_1}^{t_2} p(s) ds \tag{25}
\]

it is interesting to compute its large deviation function

\[
F(\lambda) = \lim_{t \to \infty} \frac{1}{t} \log \text{Tr}[\rho_{\text{stat}} e^{-\lambda \Delta E(t, 0)}]. \tag{26}
\]

In principle, at the quantum level, different definitions of \( F(\lambda) \) have been considered in the literature, all related to the physical processes underneath the different measurement protocols for \( \Delta E(t, 0) \)\(^\text{3,30}\). We will briefly discuss this issue at the end of the section. The present derivation parallels that done in\(^\text{13,14}\). By taking the derivative with respect to \( \lambda \) of \( \text{Eq. } (26) \) and using time-translation invariance of the steady-state one gets

\[
- \partial_\lambda F(\lambda) = \lim_{t \to \infty} \frac{1}{t} \text{Tr}[\Delta t E e^{-\lambda \Delta t E} \rho_{\text{stat}}] \tag{27}
\]
where $\Delta t E \equiv \Delta E(t/2, -t/2)$. Substituting (19) in (25), one can take the large $t$ limit using the standard representation $\delta(x) = \lim_{t \to \infty} \sin(tx)/\pi x$, obtaining

$$\lim_{t \to \infty} \Delta t E = H_R - H_L.$$  

(28)

In order to take the $t \to \infty$ limit, (28) can be substituted in the exponent of (27). One realizes that the resulting expression is equivalent to

$$-\partial_{\beta} F(\lambda) = \lim_{t \to \infty} \frac{1}{t} \int_{-t/2}^{t/2} ds \text{Tr} [p(s) \rho_{stat}(\lambda)],$$

(29)

where $\rho_{stat}(\lambda)$ is the factorized density matrix in (11) with inverse temperatures shifted as $\beta_i \to \beta_i + \lambda$ and $\beta_r \to \beta_r - \lambda$. The trace can be now computed using (21) and (23). The integrand becomes $\lambda$-independent, thus cancelling the $1/t$ prefactor and leading to

$$-\partial_{\beta} F(\lambda) = \mathcal{J}(\beta_i + \lambda, \beta_r - \lambda).$$

(30)

Performing the integration over $\lambda$, one obtains

$$F(\lambda) = [f_{\beta_i}(\lambda) - f_{\beta_i}(0)] - [f_{\beta_r}(-\lambda) - f_{\beta_r}(0)],$$

(31)

where we have defined

$$f_{\beta}(x) = \frac{\text{Li}_2(-e^{-(\beta + x)J(h+1)}) - \text{Li}_2(-e^{-(\beta + x)J(h-1)})}{2\pi(\beta + x)}.$$  

(32)

As in (24), the conformal and the scaling limit can be readily obtained. In particular, (30) has been shown in (23) to be valid generally at the gapless point, and will be shown to follow quite generally from an analysis of $PT$-symmetry in a forthcoming work (22). Finally, we notice that this expression is indeed consistent with the fluctuation-dissipation relation (23): 

$$F(\lambda) = F(\beta_r - \beta_i - \lambda).$$

(33)

We mentioned earlier that other definitions for the large deviation function $F(\lambda)$ are possible. In particular it was suggested in (5) to consider

$$\tilde{F}(\lambda) = \lim_{t \to \infty} \frac{1}{t} \log \text{Tr} [\rho_{stat} e^{-\lambda E(t)} e^{\lambda E(0)}]$$

(34)

that corresponds to a time ordered expression of (26).

Although the equivalence of these two definitions is far from being trivial even in the simple model we are considering here, we conclude this section by providing a formal and heuristic argument for which they must coincide. From time-translation invariance inside the steady state and applying the $S$-matrix it formally follows

$$\lim_{t \to \infty} E(-t/2) = \Phi S \Phi$$

$$\lim_{t \to \infty} E(t/2) = \Phi S E \Phi = \frac{1}{2}(H_L - H_R).$$

(35)

(36)

Substituting these relations into the exponents of (34) we obtain (30). Notice however that the operator $E(t)$ is non-local and thus the limits taken in (35) (36) are formal: both sides are in fact infinite, and only local observables reach a steady regime.

VI. CONCLUSIONS

In this paper, we derived the non-equilibrium density matrix for two Ising chains at different temperatures coupled in the middle at time $t = 0$, such that the final system is homogeneous. The density matrix shows a factorized form in terms of right and left moving fermions and agrees with that derived previously using different techniques. The non-equilibrium steady state supports a time-independent current flowing along the chain, signaling a ballistic mechanism for the heat transport and in agreement with recent numerical simulations. We have also evaluated for the first time the exact large-deviation function for heat transport in this model. Although the result for the density matrix is obtained for a free system, it is likely that it can be extended to a more general class of 1d quantum chain models, like integrable models. This is because conserved charges can protect transport from back-scattering, as observed in (2). Moreover, it may be possible that as in the equilibrium case, the low-energy physics is universal and well described by a suitable CFT or more generally QFT, where the exact thermal-flow stationary states have already been described (and are similarly factorized). This would suggest that interactions which are irrelevant at equilibrium are still not relevant when $\beta_i \neq \beta_r$.

We plan to extend the analysis to the case where the two chains are coupled with an impurity or more generally when the final Hamiltonian is no more translation invariant. A CFT description of this problem is still lacking and it is not even clear whether the steady state will still be factorized or how the presence of the impurity will affect the current and its fluctuations.

Appendix

In this Appendix we give more details on the diagonalization technique of section I. We have (15)

$$\omega_{i/i}^{r/l}(k) = \frac{1}{2} [A_{i/i}^{r/l}(k) + B_{i/i}^{r/l}(k)],$$

$$\xi_{i/i}^{r/l}(k) = \frac{1}{2} [A_{i/i}^{r/l}(k) - B_{i/i}^{r/l}(k)]$$

(37)

(38)

where for the right chain $1 \leq i \leq N$

$$A_{i}^{r}(k) = N_{k}(-1)^{i-1} \{ \sin(i \theta_k) - h^{-1} \sin[(i-1) \theta_k] \},$$

$$B_{i}^{r}(k) = N_{k}(-1)^{i} \sin(i \theta_k).$$

(39)

(40)

The constants $N_k$ and $N_{k}'$ ensures normalization, e.g. $\sum \delta_{k,k'} A_{i}^{r}(k') A_{i}^{l}(k') = \delta_{k,k'}$ and $\sum A_{i}^{r}(k) A_{i}^{l}(k) = \delta_{i,j}$. Up to such normalizations, analogous functions in the left chain are given by $(-N+1 \leq i \leq 0)$

$$A_{i}^{l}(k) = B_{i}^{l-1-i}(k) \quad \text{and} \quad B_{i}^{l}(k) = A_{i}^{l-1-i}(k).$$

(41)

In order to take the thermodynamic limit, it is useful to write the finite size approximation valid for $h > 1$, for
the $k$-th solution of \[ \theta_k = x_k + \frac{f(x_k)}{N+1} + O \left( \frac{1}{N+1} \right)^2 \] (42)
where $x_k = \frac{\pi k}{N+1}$ and $f(x) = \arctan \left( \frac{\sin x}{\cos x - \xi} \right)$. Substituting (42) into (39, 40) we obtain for $\theta \in [0, \pi]$

\[ A_i^r(\theta) = \sqrt{\frac{2}{\pi}} (-1)^{i+1} \sin \left[ i\theta - f(\theta) \right], \] (43)
\[ B_i^r(\theta) = \sqrt{\frac{2}{\pi}} (-1)^i \sin(i\theta), \] (44)

that are properly normalized. For $i \leq 0$ one can use (41). The thermodynamic limit in the full chain is obtained taking (39, 40), and sending $N \to 2N$ and $i \to i - N$. However this time, using (42), we notice that the solution at the same $\theta$ of (40) splits into two degenerate cases (corresponding to even and odd $k$)

\[ A_1^r(\theta) = \sqrt{\frac{1}{\pi}} \left( -1 \right)^{i-1} \sin \left( i\theta - \frac{f(\theta) + \theta}{2} \right), \] (45)
\[ A_2^r(\theta) = \sqrt{\frac{1}{\pi}} \left( -1 \right)^{i-1} \cos \left( i\theta - \frac{f(\theta) + \theta}{2} \right), \] (46)
\[ B_1^r(\theta) = \sqrt{\frac{1}{\pi}} (-1)^i \sin \left( i\theta + \frac{f(\theta) - \theta}{2} \right), \] (47)
\[ B_2^r(\theta) = \sqrt{\frac{1}{\pi}} (-1)^i \cos \left( i\theta + \frac{f(\theta) - \theta}{2} \right). \] (48)

As expected, the set of functions (45)-(48) is apart from an irrelevant phase the same one would have obtained starting with periodic boundary conditions. Two fermionic operators $\Psi_1(\theta)$ and $\Psi_2(\theta)$ can be then introduced from $A_{1,2}^i$ and $B_{1,2}^i$, in the same way as $\Phi_i(\theta)$ and $\Phi_r(\theta)$ are expanded in terms of $A_{i/r}$ and $B_{i/r}$, see (41) and (37, 38): left and right moving fermions in (9) are linear combinations of them

\[ \begin{bmatrix} \Psi_R(\theta) \\ \Psi_L(\theta) \end{bmatrix} = \frac{e^{-i\frac{(\phi_2-\phi_1)}{2}}}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & -1 \end{bmatrix} \begin{bmatrix} \Psi_1(\theta) \\ \Psi_2(\theta) \end{bmatrix}. \] (49)
$P$ is a diagonal quadratic form on the basis of the left and right movers and we only have to identify its eigenvalues. Since the full chain is invariant under discrete translations $P$ is also diagonalized by the modes $a(\theta)$ such that
\[ c_j = \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{2\pi}} e^{ij\theta} a(\theta). \]
Substituting this expression into the form $P = \frac{1}{2} \sum_{j \in \mathbb{Z}} p_j$ leads immediately to (22).

25 P. Bernard and B. Doyon, in preparation.
26 D. Bernard and B. Doyon, in preparation.
27 B. Doyon, F. Essler and J. E. Moore, work in progress.