UNIQUENESS OF ENTIRE SOLUTIONS TO QUASILINEAR EQUATIONS OF p-LAPLACE TYPE

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ABSTRACT. We prove the uniqueness property for a class of entire solutions to the equation

\[ \begin{cases} -\text{div} A(x, \nabla u) = \sigma, & u \geq 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u = 0, \end{cases} \]

where \( \sigma \) is a nonnegative locally finite measure in \( \mathbb{R}^n \), absolutely continuous with respect to the \( p \)-capacity, and \( \text{div} A(x, \nabla u) \) is the \( A \)-Laplace operator, under standard growth and monotonicity assumptions of order \( p \) \((1 < p < \infty) \) on \( A(x, \xi) \) \((x, \xi \in \mathbb{R}^n)\); the model case \( A(x, \xi) = \xi|\xi|^{p-2} \) corresponds to the \( p \)-Laplace operator \( \Delta_p \) on \( \mathbb{R}^n \).

Our main results establish uniqueness of solutions to a similar problem,

\[ \begin{cases} -\text{div} A(x, \nabla u) = \sigma u^q + \mu, & u \geq 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u = 0, \end{cases} \]

in the sub-natural growth case \( 0 < q < p-1 \), where \( \mu, \sigma \) are nonnegative locally finite measures in \( \mathbb{R}^n \), absolutely continuous with respect to the \( p \)-capacity, and \( A(x, \xi) \) satisfies an additional homogeneity condition, which holds in particular for the \( p \)-Laplace operator.

1. Introduction

We prove the uniqueness property for a class of reachable solutions to the equation

\[ \begin{cases} -\Delta_p u = \sigma, & u \geq 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u = 0, \end{cases} \quad (1.1) \]

where \( \sigma \geq 0 \) is a locally finite Borel measure in \( \mathbb{R}^n \) absolutely continuous with respect to the \( p \)-capacity, and \( \Delta_p u = \text{div}(\nabla u|\nabla u|^{p-2}) \) \((1 < p < \infty)\) is the \( p \)-Laplace operator.

More general \( A \)-Laplace operators \( \text{div} A(x, \nabla u) \) in place of \( \Delta_p \), under standard growth and monotonicity assumptions of order \( p \) on \( A(x, \xi) \) \((x, \xi \in \mathbb{R}^n)\), are treated as well (see Sec. 2). All solutions \( u \) of (1.1) are understood to be \( A \)-superharmonic (or, equivalently, locally renormalized) solutions in \( \mathbb{R}^n \) (see [19] and Sec. 3 below).

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We often use bilateral global pointwise estimates of solutions to (1.1) obtained by Kilpeläinen and Malý [20, 21] in terms of the Havin–Maz’ya–Wolff potentials (often called Wolff potentials) $W_{1,p}^\sigma$. Criteria of existence of solutions to (1.1), which ensure that $W_{1,p}^\sigma \not\equiv \infty$, can be found in [33] (see also Sec. 3 below).

In Sec. 4 we prove uniqueness of nontrivial reachable solutions to the problem

\[
\begin{aligned}
-\Delta_p u &= \sigma u^q + \mu, \quad u \geq 0 \text{ in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u &= 0,
\end{aligned}
\]

(1.2)

in the sub-natural growth case $0 < q < p - 1$, where $\mu, \sigma$ are nonnegative locally finite measures in $\mathbb{R}^n$ absolutely continuous with respect to the $p$-capacity. We observe that such a uniqueness property generally fails in the case $q \geq p - 1$.

When we treat the uniqueness problem for solutions of equations of type (1.2) for $0 < q < p - 1$ with more general $\mathcal{A}$-Laplace operators $\text{div} \mathcal{A}(x, \nabla \cdot)$ in place of $\Delta_p$, we impose the additional homogeneity condition $\mathcal{A}(x, \lambda \xi) = \lambda^{p-1} \mathcal{A}(x, \xi)$, for all $\xi \in \mathbb{R}^n$ and $\lambda > 0$ (see Sec. 4). Our main tool in the proof of uniqueness is provided by bilateral pointwise estimates for all entire solutions obtained recently in [37].

We observe that in the case $p = 2$ all superharmonic solutions of equations (1.1) and (1.2) are reachable, and hence unique. An analogue of this fact is true for more general equations with the linear uniformly elliptic $\mathcal{A}$-Laplace operator $\text{div} \mathcal{A}(x, \nabla u)$, with bounded measurable coefficients $\mathcal{A} \in L^\infty(\mathbb{R}^n)^{n \times n}$, in place of $\Delta$. In other words, all entire $\mathcal{A}$-superharmonic solutions to such equations are unique. For similar problems in domains $\Omega \subset \mathbb{R}^n$ and linear operators with positive Green’s function satisfying some additional properties (in particular, in uniform domains) the uniqueness property was obtained recently in [38].

The uniqueness of nontrivial bounded (superharmonic) solutions for (1.2) in the case $p = 2$ was proved earlier by Brezis and Kamin [7]. For solutions $u \in C(\overline{\Omega})$ in bounded smooth domains $\Omega \subset \mathbb{R}^n$ and $\mu, \sigma \in C(\overline{\Omega})$, along with some more general equations involving monotone increasing, concave nonlinearities on the right-hand side, the uniqueness property was originally established by Krasnoselskii [23, Theorem 7.14].

As shown below, for $p \neq 2$, all $p$-superharmonic solutions $u$ to (1.1) or (1.2) are reachable, and hence unique, if, for instance, the condition $\liminf_{|x| \to \infty} u = 0$ in (1.1) or (1.2), respectively, is replaced with $\lim_{|x| \to \infty} u = 0$. See Sections 3 and 4 where we discuss this and other conditions that ensure that all solutions are reachable.

Existence and bilateral pointwise estimates for all $\mathcal{A}$-superharmonic solutions to (1.2) were obtained in [37]. (See also earlier results in [9] involving minimal solutions in the case $\mu = 0$.) In particular, it is known that the measure $\sigma$ is necessarily absolutely continuous with respect to the $p$-capacity.
provided there exists a nontrivial \( u \geq 0 \) such that \(-\Delta_p u \geq \sigma u^q \) ([9], Lemma 3.6).

We remark that the proofs of the main existence results in [9, 37] for (1.2) in the case \( \mu = 0 \) used a version of the comparison principle ([9], Lemma 5.2) that contained some inaccuracies. A corrected form of this comparison principle is provided in Lemma 4.1 below. The other parts of [9, 37] are unaffected by this correction.

With regards to the existence problem, we prove additionally that we can always construct a reachable solution to either (1.1) or (1.2), whenever a solution to the corresponding equation exists (see Theorem 3.10 and Remark 4.7 below).

2. \( \mathcal{A} \)-superharmonic functions

Let \( \Omega \subseteq \mathbb{R}^n, n \geq 2 \), be an open set. By \( \mathcal{M}^+(\Omega) \) we denote the cone of nonnegative locally finite Borel measures in \( \Omega \), and by \( \mathcal{M}^+_{\text{f}}(\Omega) \) the subcone of finite measures in \( \mathcal{M}^+(\Omega) \). For \( \mu \in \mathcal{M}^+(\Omega) \), we set \( \|\mu\|_{\mathcal{M}^+(\Omega)} = \mu(\Omega) \) even if \( \mu(\Omega) = +\infty \). The space of finite signed Borel measures in \( \Omega \) is denoted by \( \mathcal{M}^b(\Omega) \). By \( \|\mu\|_{\mathcal{M}^b(\Omega)} \) we denote the total variation of \( \mu \in \mathcal{M}^b(\Omega) \).

Let \( \mathcal{A}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) be a Carathéodory function in the sense that the map \( x \to \mathcal{A}(x, \xi) \) is measurable for all \( \xi \in \mathbb{R}^n \), and the map \( \xi \to \mathcal{A}(x, \xi) \) is continuous for a.e. \( x \in \mathbb{R}^n \). Throughout the paper, we assume that there are constants \( 0 < \alpha \leq \beta < \infty \) and \( 1 < p < n \) such that for a.e. \( x \) in \( \mathbb{R}^n \),

\[
\mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \quad |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}, \quad \forall \xi \in \mathbb{R}^n,
\]

\[
[\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)] \cdot (\xi_1 - \xi_2) > 0, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n, \quad \xi_1 \neq \xi_2.
\]

In the uniqueness results of Sec. 4, we assume additionally the homogeneity condition

\[
(2.2) \quad \mathcal{A}(x, \lambda \xi) = \lambda^{p-1} \mathcal{A}(x, \xi), \quad \forall \xi \in \mathbb{R}^n, \quad \lambda > 0.
\]

Such homogeneity conditions are often used in the literature (see [18], [21]).

For an open set \( \Omega \subset \mathbb{R}^n \), it is well known that every weak solution \( u \in W^{1,p}_{\text{loc}}(\Omega) \) to the equation

\[
(2.3) \quad -\text{div} \mathcal{A}(x, \nabla u) = 0 \quad \text{in} \ \Omega
\]

has a continuous representative. Such continuous solutions are said to be \( \mathcal{A} \)-harmonic in \( \Omega \). If \( u \in W^{1,p}_{\text{loc}}(\Omega) \) and

\[
\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0,
\]

for all nonnegative \( \varphi \in C^\infty_0(\Omega) \), i.e., \( -\text{div} \mathcal{A}(x, \nabla u) \geq 0 \) in the distributional sense, then \( u \) is called a supersolution to (2.3) in \( \Omega \).

A function \( u: \Omega \to (-\infty, \infty] \) is called \( \mathcal{A} \)-superharmonic if \( u \) is not identically infinite in each connected component of \( \Omega \), \( u \) is lower semicontinuous,
and for all open sets $D$ such that $\overline{D} \subset \Omega$, and all functions $h \in C(\overline{D})$, $A$-harmonic in $D$, it follows that $h \leq u$ on $\partial D$ implies $h \leq u$ in $D$.

A typical example of $A(x, \xi)$ is given by $A(x, \xi) = |\xi|^{p-2} \xi$, which gives rise to the $p$-Laplacian $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$. In this case, $A$-superharmonic functions will be called $p$-superharmonic functions.

We recall here the fundamental connection between supersolutions of (2.3) and $A$-superharmonic functions discussed in [18].

**Proposition 2.1** ([18]). (i) If $u \in W^{1,p}_{\text{loc}}(\Omega)$ is such that
\[
-\text{div} A(x, \nabla u) \geq 0 \quad \text{in } \Omega,
\]
then there is an $A$-superharmonic function $v$ such that $u = v$ a.e. Moreover,
\[\tag{2.4} v(x) = \text{ess lim inf}_{y \to x} v(y), \quad x \in \Omega.\]

(ii) If $v$ is $A$-superharmonic, then (2.4) holds. Moreover, if $v \in W^{1,p}_{\text{loc}}(\Omega)$, then
\[-\text{div} A(x, \nabla v) \geq 0 \quad \text{in } \Omega.
\]

(iii) If $v$ is $A$-superharmonic and locally bounded, then $v \in W^{1,p}_{\text{loc}}(\Omega)$, and
\[-\text{div} A(x, \nabla v) \geq 0 \quad \text{in } \Omega.
\]

Note that if $u$ is $A$-superharmonic, then the gradient of $u$ may not exist in the sense of distributions in the case $1 < p \leq 2 - 1/n$. On the other hand, if $u$ is an $A$-superharmonic function, then its truncation $u_k = \min\{u, k\}$ is $A$-superharmonic as well, for any $k > 0$. Moreover, by Proposition 2.1(iii) we have $u_k \in W^{1,p}_{\text{loc}}(\Omega)$. Using this we define the very weak gradient
\[Du := \lim_{k \to \infty} \nabla \min\{u, k\} \quad \text{a.e. in } \Omega.
\]

If either $u \in L^\infty(\Omega)$ or $u \in W^{1,1}_{\text{loc}}(\Omega)$, then $Du$ coincides with the regular distributional gradient of $u$. In general we have the following gradient estimates [20] (see also [18]).

**Proposition 2.2** ([20]). Suppose $u$ is $A$-superharmonic in $\Omega$ and $1 \leq q < \frac{n}{n-1}$. Then both $|Du|^{p-1}$ and $A(\cdot, Du)$ belong to $L^q_{\text{loc}}(\Omega)$. Moreover, if $p > 2 - \frac{1}{n}$, then $Du$ coincides with the distributional gradient of $u$.

Note that by Proposition 2.2 and the dominated convergence theorem, we have
\[-\text{div} A(x, \nabla u)(\varphi) := \int_{\Omega} A(x, Du) \cdot \nabla \varphi \, dx
\]
\[= \lim_{k \to \infty} \int_{\Omega} A(x, \nabla \min\{u, k\}) \cdot \nabla \varphi \, dx \geq 0,
\]
whenever $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, and $u$ is $A$-superharmonic in $\Omega$. It follows from Riesz’s representation theorem (see [18] Theorem 21.2) that there exists a unique measure $\mu[u] \in \mathcal{M}^+(\Omega)$ called the Riesz measure of $u$ such that
\[-\text{div} A(x, \nabla u) = \mu[u] \quad \text{in } \Omega.
\]
3. Quasilinear equations with locally finite measure data in the entire space

In this section, we investigate the problems of existence and uniqueness of $A$-superharmonic solutions in the entire space $\mathbb{R}^n$ to the equation

\[
\begin{aligned}
-\text{div } A(x, \nabla u) &= \sigma, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u &= 0,
\end{aligned}
\]

with measures $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$ (not necessarily finite).

There has been a lot of work addressing the existence and uniqueness problem for quasilinear equations of the form

\[
\begin{aligned}
-\text{div } A(x, \nabla u) &= \sigma \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $\sigma \in L^1(\Omega)$, or, more generally, $\sigma \in \mathcal{M}_0(\Omega)$; see, e.g., [2] [4] [5] [6] [11] [10] [12] [25] [22]. For arbitrary domains, including $\mathbb{R}^n$, we refer to the papers [2] (for $L^1$ data) and [25] (for data in $\mathcal{M}_0(\Omega)$). In these papers one can find the notions of entropy solutions (see [2, 6, 22]), SOLA (solutions obtained as limits of approximations) for $L^1$ data (see [11]), reachable solutions (see [10]), and renormalized solutions (see [12]).

The current state of the art on the uniqueness problem for (3.2) is that most results require that $\sigma \ll \text{cap}_p$, i.e., $\sigma$ is absolutely continuous with respect to the $p$-capacity in the sense that $\sigma(K) = 0$ for any compact set $K \subset \Omega$ such that $\text{cap}_p(K) = 0$. The $p$-capacity $\text{cap}_p(\cdot)$ is a natural capacity associated with the $p$-Laplacian defined by

\[
\text{cap}_p(K) := \inf \left\{ \int_{\Omega} |\nabla h|^p \, dx : h \in C_0^\infty(\Omega), \ h \geq 1 \text{ on } K \right\},
\]

for any compact set $K \subset \Omega$.

For later use, we now recall the following equivalent definitions of a (global) renormalized solution to equation (3.2) (see [12]). For our purposes, we shall restrict ourselves to the case $\sigma \in \mathcal{M}_0^+(\Omega)$ and nonnegative solutions. Recall that we may use the decomposition $\sigma = \sigma_0 + \sigma_s$, where both $\sigma_0$ and $\sigma_s$ are nonnegative measures such that $\sigma_0 \ll \text{cap}_0$, and $\sigma_s$ is concentrated on a set of zero $p$-capacity.

**Definition 3.1.** Let $\sigma \in \mathcal{M}_0^+(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Then $u \geq 0$ is said to be a renormalized solution of (3.2) if the following conditions hold:

(a) The function $u$ is measurable and finite almost everywhere, and $T_k(u)$ belongs to $W^{1,p}_0(\Omega)$ for every $k > 0$, where $T_k(s) := \min\{k, s\}$, $s \geq 0$.

(b) The gradient $Du$ of $u$ satisfies $|Du|^{p-1} \in L^q(\Omega)$ for all $q < \frac{n-1}{p-1}$.

(c) For any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and any $\varphi \in W^{1,p}(\Omega) \cap$
\[ L^\infty(\Omega) \text{ such that } h(u) \varphi \in W_0^{1,p}(\Omega), \]
\[
\int_\Omega A(x, Du) \cdot \nabla (h(u) \varphi) \, dx = \int_\Omega h(u) \varphi \, d\sigma_0,
\]
and for any \( \varphi \in C^0_b(\Omega) \) (the space of bounded and continuous functions in \( \Omega \)),
\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq u \leq 2m\}} A(x, Du) \cdot Du \varphi \, dx = \int_\Omega \varphi \, d\sigma_s.
\]

**Definition 3.2.** Let \( \sigma \in M_b^+(\Omega) \), where \( \Omega \subset \mathbb{R}^n \) is a bounded open set. Then \( u \geq 0 \) is said to be a renormalized solution of (3.2) if \( u \) satisfies (a) and (b) in Definition 3.1, and if the following condition holds:

(c) For any \( h \in W^{1,\infty}(\mathbb{R}) \) with such that \( h' \) has compact support, and any \( \varphi \in W^{1,r}(\Omega) \cap L^\infty(\Omega) \), \( r > n \), such that \( h(u) \varphi \in W_0^{1,p}(\Omega) \),
\[
\int_\Omega A(x, Du) \cdot \nabla (h(u) \varphi) \, dx = \int_\Omega h(u) \varphi \, d\sigma_0 + h(+\infty) \int_\Omega \varphi \, d\sigma_s.
\]
Here \( h(+\infty) := \lim_{s \to +\infty} h(s) \).

**Definition 3.3.** Let \( \sigma \in M_b^+(\Omega) \), where \( \Omega \subset \mathbb{R}^n \) is a bounded open set. Then \( u \geq 0 \) is said to be a renormalized solution of (3.2) if \( u \) satisfies (a) and (b) in Definition 3.1, and if the following conditions hold:

(c) For every \( k > 0 \), there exists \( \lambda_k \in M_b^+(\Omega) \) concentrated on the set \( \{u = k\} \) such that \( \lambda_k \ll \operatorname{cap}_p \), and \( \lambda_k \to \sigma_s \) in the narrow topology of measures in \( \Omega \) as \( k \to \infty \), i.e.,
\[
\lim_{k \to \infty} \int_\Omega \varphi \, d\lambda_k = \int_\Omega \varphi \, d\sigma_s, \quad \forall \varphi \in C^0_b(\Omega).
\]

(d) For every \( k > 0 \),
\[
\int_{\{u < k\}} A(x, Du) \cdot \nabla \varphi \, dx = \int_{\{u < k\}} \varphi \, d\sigma_0 + \int_\Omega \varphi \, d\lambda_k
\]
for all \( \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \).

We shall also need the notion of a local renormalized (nonnegative) solution on a general open set \( \Omega \subset \mathbb{R}^n \) (not necessarily bounded) associated with a measure \( \sigma \in M^+(\Omega) \) (not necessarily finite). We recall the following equivalent definitions (see [3]), adapted to the case of nonnegative solutions.

**Definition 3.4.** Let \( \sigma \in M^+(\Omega) \), where \( \Omega \subset \mathbb{R}^n \) is an open set. Then a nonnegative function \( u \) is said to be a local renormalized solution of the equation \(-\text{div}A(x, \nabla u) = \sigma\), if the following conditions hold:

(a) The function \( u \) is measurable and finite almost everywhere, and \( T_k(u) \) belongs to \( W_0^{1,p}(\Omega) \), for every \( k > 0 \), where \( T_k(s) := \min\{k, s\} \), \( s \geq 0 \).

(b) The gradient \( Du \) of \( u \) satisfies \( |Du|^p \in L_{\text{loc}}^q(\Omega) \) for all \( 0 < q < \frac{n}{n-1} \), and \( u^{p-1} \in L_{\text{loc}}^s(\Omega) \) for all \( 0 < s < \frac{n}{n-p} \).
(c) For any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and any $\varphi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ with compact support in $\Omega$ such that $h(u) \varphi \in W^{1,p}(\Omega)$,

$$
\int_{\Omega} \mathcal{A}(x,Du) \cdot \nabla (h(u) \varphi) \, dx = \int_{\Omega} h(u) \varphi \, d\sigma_0,
$$

and for any $\varphi \in C^0_0(\Omega)$ with compact support in $\Omega$,

$$
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq u \leq 2m\}} \mathcal{A}(x,Du) \cdot Du \, \varphi \, dx = \int_{\Omega} \varphi \, d\sigma_s.
$$

**Definition 3.5.** Let $\sigma \in \mathcal{M}^+(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set. Then a nonnegative function $u$ is said to be a local renormalized solution of the equation $-\text{div} \mathcal{A}(x,\nabla u) = \sigma$, if it satisfies (a) and (b) in Definition 3.4, and if the following conditions hold:

(c) For every $k > 0$, there exists a nonnegative measure $\lambda_k << \text{cap}_p$, concentrated on the sets $\{u = k\}$, such that $\lambda_k \to \sigma_s$ weakly as measures in $\Omega$ as $k \to \infty$, i.e.,

$$
\lim_{k \to \infty} \int_{\Omega} \varphi \, d\lambda_k = \int_{\Omega} \varphi \, d\sigma_s,
$$

for all $\varphi \in C^0_0(\Omega)$ with compact support in $\Omega$.

(d) For every $k > 0$,

$$
\int_{\{u < k\}} \mathcal{A}(x,Du) \cdot \nabla \varphi \, dx = \int_{\{u < k\}} \varphi \, d\sigma_0 + \int_{\Omega} \varphi \, d\lambda_k
$$

for all $\varphi \in W^{1,p}_0(\Omega) \cap L^{\infty}(\Omega)$ with compact support in $\Omega$.

We now discuss solutions of (3.1) for general measures $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. It is known that a necessary and sufficient condition for (3.1) to admit an $\mathcal{A}$-superharmonic solution is the finiteness condition

(3.3) $$
\int_1^{\infty} \left( \frac{\sigma(B(0,\rho))}{\rho^p} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty;
$$

(see, e.g., [33, 34]). Thus, it is possible to solve (3.1) for a wide and optimal class of measures $\sigma$ satisfying (3.3) that are not necessarily finite.

We mention that (3.3) is equivalent to the condition $W_{1,p}\sigma(x) < +\infty$ for some $x \in \mathbb{R}^n$ (or equivalently quasi-everywhere in $\mathbb{R}^n$ with respect to the $p$-capacity), where

$$
W_{1,p}\sigma(x) := \int_0^{\infty} \left( \frac{\sigma(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}
$$

is the Havin–Maz’ya–Wolff potential of $\sigma$ (often called the Wolff potential); see [17, 20].

By the fundamental result of Kilpeläinen and Malý [20, 21], any $\mathcal{A}$-superharmonic solution $u$ to equation (3.1) satisfies the following global
pointwise estimates,
\[(3.4) \quad \frac{1}{K} W_{1,p} \sigma(x) \leq u(x) \leq K W_{1,p} \sigma(x), \quad \forall x \in \mathbb{R}^n,\]
where \(K > 0\) is a constant depending only on \(n, p\) and the structural constants \(\alpha\) and \(\beta\) in (2.1).

Our main goal here is to introduce a new notion of a solution to (3.1) so that existence is obtained under the natural growth condition (3.3) for \(\sigma\), and uniqueness is guaranteed as long as \(\sigma \ll \text{cap}_p\) (see Definition 3.8 below).

We begin with the following result on the existence of a minimal solution to (3.1) in case the measure \(\sigma\) is continuous with respect to the \(p\)-capacity.

**Theorem 3.6.** Let \(\sigma \in \mathcal{M}^+ (\mathbb{R}^n)\), where \(\sigma \ll \text{cap}_p\). Suppose that (3.3) holds. Then there exists a minimal \(A\)-superharmonic solution to equation (3.1).

**Proof.** Condition (5.3) implies that
\[\int_1^\infty \left( \frac{\sigma(B(x, \rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} d\rho < +\infty\]
for all \(x \in \mathbb{R}^n\). Thus,
\[\{W_{1,p} \sigma = \infty\} = \left\{ x \in \mathbb{R}^n : W_{1,p}^1 \sigma := \int_0^1 \left( \frac{\sigma(B(x, \rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} d\rho = \infty \right\}.
\]
This yields
\[\text{cap}_p(\{W_{1,p} \sigma = \infty\}) = \lim_{j \to \infty} \text{cap}_p(\{W_{1,p} \sigma = \infty\} \cap B_j(0)) = \lim_{j \to \infty} \text{cap}_p(\{x \in B_j(0) : W_{1,p}^1(\sigma|_{B_{j+1}(0)}) = \infty\}) = 0.
\]
Here we used the fact that, if \(\mu \in \mathcal{M}^+_b (\mathbb{R}^n)\), then \(\text{cap}_p(\{W_{1,p} \mu = \infty\}) = 0\) (see [1, Proposition 6.3.12]). It follows that \(\sigma(\{W_{1,p} \sigma = \infty\}) = 0\), since \(\sigma \ll \text{cap}_p\).

Let \(\sigma_k (k = 1, 2, \ldots)\) be the restriction of \(\sigma\) to the set \(B_k(0) \cap \{W_{1,p} \sigma < k\}\). We then have that \(\sigma_k\) weakly converges to \(\sigma\), and
\[\int_{\mathbb{R}^n} W_{1,p} \sigma_k d\sigma_k \leq k \sigma(B_k(0)) < +\infty.
\]
Hence, \(\sigma_k \in W^{-1,p'}(B_k(0))\) \((1/p + 1/p' = 1)\), and for each \(k > 0\), there exists a unique nonnegative solution \(u_k \in W_{0}^{1,p}(B_k(0))\) to the problem
\[(3.5) \quad \begin{cases} -\text{div} \mathcal{A}(x, \nabla u_k) = \sigma_k \text{ in } B_k(0), \\ u_k = 0 \text{ on } \partial B_k(0). \end{cases}
\]
If we set \( u_k = 0 \) in \( \mathbb{R}^n \setminus B_k(0) \), then the sequence \( \{u_k\} \) is non-decreasing, and by \([33, \text{Theorem 2.1}]\),
\[
 u_k \leq K W_{1,p}\sigma < \infty \quad d\sigma - \text{a.e.}
\]
By \([20, \text{Theorem 1.17}]\), it follows that the function \( u := \lim_{k \to \infty} u_k \) is \( A \)-superharmonic in \( \mathbb{R}^n \). Moreover, \( u \leq K W_{1,p}\sigma \), and consequently
\[
 \liminf_{|x| \to \infty} u(x) \leq K \liminf_{|x| \to \infty} W_{1,p}\sigma(x) = 0.
\]
Thus, \( u \) is an \( A \)-superharmonic solution of (3.1).

To show the minimality of \( u \), let \( v \) be another \( A \)-superharmonic solution of (3.1). From the construction of \( u \), it is enough to show that \( u_k \leq v \) for any \( k \geq 1 \). To this end, let \( \nu_j, j = 1, 2, \ldots \), be the Riesz measure of \( \min\{v, j\} \). Since \( v \) is \( A \)-superharmonic, it is also a local renormalized solution to \( -\text{div} A(x, \nabla v) = \sigma \) in \( \mathbb{R}^n \) (see \([19]\)). Hence, by a result of \([12, 3]\) and the fact that \( \sigma << \text{cap}_p \), we obtain
\[
 \nu_j = \sigma|\{v < j\} + \alpha_j
\]
for \( \alpha_j \in \mathcal{M}^+(\mathbb{R}^n) \) concentrated in the set \( \{v = j\} \).

Using the estimate \( v \leq K W_{1,p}\sigma \), we deduce
\[
 \nu_j \geq \sigma|\{v < j\} \geq \sigma|\{K W_{1,p}\sigma < j\} \geq \sigma|\{W_{1,p}\sigma < k\} \geq \sigma_k,
\]
provided \( j/K > k \). Since \( u_k \in W_{0}^{1,p}(B_k(0)) \) and \( \min\{v, j\} \in W^{1,p}(B_k(0)) \), by the comparison principle (see \([9, \text{Lemma 5.1}]\)), we estimate
\[
 u_k \leq \min\{v, j\} \leq v,
\]
provided \( j \geq Kk \). Thus, \( u = \lim_{k \to \infty} u_k \leq v \). This completes the proof of the theorem.

The proof of the minimality of \( u \) above can be modified to obtain the following comparison principle.

**Theorem 3.7 (Comparison Principle).** Let \( \sigma, \tilde{\sigma} \in \mathcal{M}^+(\mathbb{R}^n) \), where \( \sigma \leq \tilde{\sigma} \) and \( \sigma << \text{cap}_p \), \( 1 < p < n \). Then \( u \leq \tilde{u} \), where \( u \) is the minimal \( A \)-superharmonic solution of (3.1) and \( \tilde{u} \) is any \( A \)-superharmonic solution of (3.1) with datum \( \tilde{\sigma} \) in place of \( \sigma \).

**Proof.** Let \( \sigma'_k, k = 1, 2, \ldots, \) be the restriction of \( \sigma \) to the set \( B_k(0) \cap \{W_{1,p}\tilde{\sigma} < k\} \). Since \( \sigma << \text{cap}_p \), we have that \( \sigma'_k \) weakly converges to \( \sigma \). Moreover, as \( W_{1,p}\sigma'_k \leq W_{1,p}\sigma \leq W_{1,p}\tilde{\sigma} < k \) on the set \( \{W_{1,p}\tilde{\sigma} < k\} \), it follows that
\[
 \int_{\mathbb{R}^n} W_{1,p}\sigma'_k d\sigma_k' \leq k \sigma(B_k(0)) < +\infty.
\]

Hence, \( \sigma'_k \in W^{-1,p(k)}(B_k(0)) \), and for each \( k > 0 \) there exists a unique nonnegative solution \( u'_k \in W_{0}^{1,p}(B_k(0)) \) to the problem
\[
 \begin{cases}
 -\text{div} A(x, \nabla u'_k) = \sigma'_k \quad \text{in } B_k(0), \\
 u'_k = 0 \quad \text{on } \partial B_k(0).
\end{cases}
\]
Letting $u_k' = 0$ in $\mathbb{R}^n \setminus B_k(0)$, we have that the sequence $\{u_k'\}$ is non-decreasing, and by [33 Theorem 2.1],
\[ u_k' \leq K W_{1,p} \sigma. \]

Then $u_k'$ converges pointwise to an $\mathcal{A}$-superharmonic solution $u'$ of (3.1) by [20 Theorem 1.17]. On the other hand, by the comparison principle of [9 Lemma 5.1], we have
\[ u_k' \leq u_k, \quad \forall k \geq 1, \]
where $u_k$ is defined in (3.5). Hence, letting $k \to \infty$, we get $u' \leq u$, which yields $u' = u$ by the minimality of $u$.

We now let $\tilde{\sigma}_j$ ($j = 1, 2, \ldots$) be the Riesz measure of $\min\{\tilde{u}, j\}$. Recall that the Riesz measure $\tilde{\sigma}$ of $\tilde{u}$ can be decomposed as
\[ \tilde{\sigma} = \tilde{\sigma}_0 + \tilde{\sigma}_s, \]
where $\tilde{\sigma}_0 \in \mathcal{M}^+(\mathbb{R}^n)$, $\tilde{\sigma}_0 \ll \text{cap}_p$, and $\tilde{\sigma}_s \in \mathcal{M}^+(\mathbb{R}^n)$ is concentrated on a set of zero $p$-capacity. Then by a result of [12, 3], we have
\[ \tilde{\sigma}_j = \tilde{\sigma}_0|\{\tilde{u} < j\} + \tilde{\alpha}_j, \]
where $\tilde{\alpha}_j \in \mathcal{M}^+(\mathbb{R}^n)$ is concentrated in the set $\{\tilde{u} = j\}$. On the other hand, since $\tilde{\sigma}_s(\{\tilde{u} < \infty\}) = 0$ (see [19 Lemma 2.9]), we can rewrite (3.6) as
\[ \tilde{\sigma}_j = \tilde{\sigma}|\{\tilde{u} < j\} + \tilde{\alpha}_j. \]

Now using the estimate $\tilde{u} \leq K W_{1,p} \tilde{\sigma}$ and (3.7), we have
\[ \tilde{\sigma}_j \geq \tilde{\sigma}|\{\tilde{u} < j\} \geq \sigma|\{K W_{1,p} \tilde{\sigma} < j\} \geq \sigma|\{W_{1,p} \tilde{\sigma} < k\} \geq \sigma_k', \]
provided $j/K > k$.

Since $u_k' \in W_{0}^{1,p}(B_k(0))$ and $\min\{\tilde{u}, j\} \in W^{1,p}(B_k(0))$, by the comparison principle of [9 Lemma 5.1] we find
\[ u_k' \leq \min\{\tilde{u}, j\} \leq \tilde{u}, \]
provided we choose a $j$ such that $j \geq Kk$. Letting $k \to \infty$, we obtain $u \leq \tilde{u}$ as desired. \qed

Theorem 3.6 justifies the existence (and hence uniqueness) of the minimal $\mathcal{A}$-superharmonic solution to (3.1) provided condition (3.3) holds and $\sigma \ll \text{cap}_p$. It is not known if condition (3.3) alone is enough for the existence of the minimal solution. It is also not known if under condition (3.3) and $\sigma \ll \text{cap}_p$ all $\mathcal{A}$-superharmonic solutions to (3.1) coincide with the minimal solution. For a partial result in this direction, see Theorem 3.12 below.

We now introduce a new notion of a solution so that uniqueness is guaranteed for all nonnegative locally finite measures $\sigma$ such that $\sigma \ll \text{cap}_p$. Our definition is an adaptation of the notion of the reachable solution of [10 Definition 2.3].
Definition 3.8. Let $\sigma \in M^+(\mathbb{R}^n)$. We say that a function $u : \mathbb{R}^n \to [0, +\infty]$ is an $A$-superharmonic reachable solution to equation (3.1) if $u$ is an $A$-superharmonic solution of (3.1), and there exist two sequences $\{u_i\}$ and $\{\sigma_i\}$, $i = 1, 2, \ldots$, such that
(i) Each $\sigma_i \in M^+(\mathbb{R}^n)$ is compactly supported in $\mathbb{R}^n$, and $\sigma_i \leq \sigma$;
(ii) Each $u_i$ is an $A$-superharmonic solution of (3.1) with datum $\sigma_i$ in place of $\sigma$;
(iii) $u_i \to u$ a.e. in $\mathbb{R}^n$.

Remark 3.9. The notion of reachable solution was introduced in [10] for equations over bounded domains with finite measure data. It is also related to the notion of SOLA (Solution Obtained as Limit of Approximations) of [11] for $L^1$ data over bounded domains. By (iii) and the weak continuity result of [36], we see that $\sigma_i \to \sigma$ weakly as measures in $\mathbb{R}^n$. The extra requirement $\sigma_i \leq \sigma$ in our definition plays an important role in the proof of uniqueness in the case when the datum $\sigma$ is absolutely continuous with respect to $\text{cap}_p$.

Theorem 3.10. Suppose $\sigma \in M^+(\mathbb{R}^n)$, and suppose (3.3) holds. Then there exists an $A$-superharmonic reachable solution to (3.1). Moreover, if additionally $\sigma << \text{cap}_p$, then any $A$-superharmonic reachable solution is unique and coincides with the minimal solution.

Proof. Existence: Suppose that (3.3) holds. Then $W_{1,p}\sigma < +\infty$ quasi-everywhere and hence almost everywhere. For each $i = 1, 2, \ldots$, let $u_i^j$ be an $A$-superharmonic renormalized solution (see [12]) to
\[
\begin{cases}
-\text{div} \, A(x, \nabla u_i^j) = \sigma|_{B_i(0)} & \text{in } B_j(0), \\
\quad u_i^j = 0 & \text{on } \partial B_j(0).
\end{cases}
\]

Note that $\sigma|_{B_i(0)} \leq \sigma$ and $\sigma|_{B_i(0)} \to \sigma$ weakly as measures in $\mathbb{R}^n$. Also, by (3.3), we have
\[
u_i^j \leq K W_{1,p}(\sigma|_{B_i(0)}).
\]

Hence, by [20, Theorem 1.17], there exist an $A$-superharmonic function $u_i$ in $\mathbb{R}^n$ with
\[
u_i \leq K W_{1,p}(\sigma|_{B_i(0)}) \leq K W_{1,p}\sigma < +\infty \text{ a.e.,}
\]
and a subsequence $\{u_i^{jk}\}$ such that $u_i^{jk} \to u$ and $Du_i^{jk} \to Du_i$ a.e. as $k \to \infty$. These estimates yield that the Riesz measure of $u_i$ is $\sigma|_{B_i(0)}$ and
\[
\lim_{|x| \to \infty} \inf u_i = 0.
\]

Using again [20, Theorem 1.17] and (3.3), we find a subsequence of $\{u_i\}$ that converges a.e. to an $A$-superharmonic reachable solution $u$ of (3.1).

Uniqueness: We now assume further that $\sigma << \text{cap}_p$. Let $u$ be an $A$-superharmonic reachable solution in the sense of Definition 3.8 with approximating sequences $\{u_i\}$ and $\{\sigma_i\}$. Let us fix an $i \in \{1, 2, \ldots\}$. Then
there exists a positive integer $N = N(i)$ such that $\text{supp}(\sigma_i) \subset B_N(0)$. Let $v$ be the minimal $A$-superharmonic solution to (3.1). Also, let $v_N$ be the minimal $A$-superharmonic solution to (3.1) with datum $\sigma|_{B_N(0)}$ in place of $\sigma$. We have, by Theorem 3.7,

$$u \geq v \geq v_N.$$ 

Thus, as $u_j \to u$ a.e., it is enough to show that

$$v_N \geq u_i.$$ 

Note that since $\sigma_i \leq \sigma$ and $\text{supp}(\sigma_i) \subset B_N(0)$ we have that

$$\sigma_i \leq \sigma|_{B_N(0)}.$$ 

For $R > 0$, let $0 \leq \Theta = \Theta_R \leq 1$ be a cutoff function such that

\[
\Theta \in C_0^\infty(B_R(0)), \quad \Theta \equiv 1 \text{ on } B_{R/2}(0), \quad \text{and } |\nabla \Theta| \leq C/R.
\]

For any $k > 0$, we set

$$T_k^+(t) = \begin{cases} 
t & \text{if } 0 \leq t \leq k, \\
k & \text{if } t > k, \\
0 & \text{if } t < 0.
\end{cases}$$

Also, for any $m > 0$, we define the following Lipschitz function with compact support on $\mathbb{R}$:

$$h_m(t) = \begin{cases} 
1 & \text{if } 0 \leq |t| \leq m, \\
0 & \text{if } |t| \geq 2m, \\
-\frac{t}{m} + 2 & \text{if } m < t < 2m, \\
\frac{t}{m} + 2 & \text{if } -2m < t < -m.
\end{cases}$$

As $u_i$ and $v_N$ are both local renormalized solutions (see [3, 19]), we may use

$$h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta, \quad m, k > 0,$$

as test functions and thus obtaining

$$\int_{\mathbb{R}^n} A(x, Du_i) \cdot \nabla [h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta] \, dx$$

$$= \int_{\mathbb{R}^n} h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta d\sigma_i,$$

and

$$\int_{\mathbb{R}^n} A(x, Dv_N) \cdot \nabla [h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta] \, dx$$

$$= \int_{\mathbb{R}^n} h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta d\sigma|_{B_N(0)}.$$ 

Let

$$I = \int_{\mathbb{R}^n} A(x, Du_i) \cdot \nabla [h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta] \, dx,$$
and
\[
II = \int_{\mathbb{R}^n} A(x, Du_i) \cdot \nabla [h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta] \, dx.
\]

Then by (3.10) we have
\[
(3.11) \quad I - II \leq 0.
\]

On the other hand, we can write
\[
I - II
\]
\[
= \int_{\mathbb{R}^n} [A(x, Du_i) - A(x, Dv_N)] \cdot \nabla T_k^+(u_i - v_N) h_m(u_i) h_m(v_N) \Theta \, dx
\]
\[
+ \int_{\mathbb{R}^n} [A(x, Du_i) - A(x, Av_N)] \cdot Du_i h_m'(u_i) T_k^+(u_i - v_N) h_m(v_N) \Theta \, dx
\]
\[
+ \int_{\mathbb{R}^n} [A(x, Du_i) - A(x, Dv_N)] \cdot Dv_N h_m' (v_N) T_k^+(u_i - v_N) h_m(u_i) \Theta \, dx
\]
\[
+ \int_{\mathbb{R}^n} [A(x, Du_i) - A(x, Dv_N)] \cdot \nabla \Theta T_k^+(u_i - v_N) h_m(u_i) h_m(v_N) \, dx.
\]

Thus, in view of (3.11), it follows that
\[
\int_{\{0 < u_i - v_N < k\}} [A(x, Du_i) - A(x, Dv_N)] \cdot (Du_i - Dv_N) h_m(u_i) h_m(v_N) \Theta \, dx
\]
\[
\leq - \int_{\mathbb{R}^n} [A(x, Du_i) - A(x, Dv_N)] \cdot Du_i h_m'(u_i) T_k^+(u_i - v_N) h_m(v_N) \Theta \, dx
\]
\[
- \int_{\mathbb{R}^n} [A(x, Du_i) - A(x, Dv_N)] \cdot Dv_N h_m' (v_N) T_k^+(u_i - v_N) h_m(u_i) \Theta \, dx
\]
\[
- \int_{\mathbb{R}^n} [A(x, Du_i) - A(x, Dv_N)] \cdot \nabla \Theta T_k^+(u_i - v_N) h_m(u_i) h_m(v_N) \, dx
\]
\[
=: A_m + B_m + C_m.
\]

To estimate \( |A_m| \), we observe that \( |h_m(t)| \leq 1 \) and \( |h_m'(t)| \leq 1/m \). Hence,
\[
|A_m| \leq \beta \frac{k}{m} \int_{\{m < u_i < 2m, 0 < v_N < 2m\}} [|Du_i|^{p-1} + |Dv_N|^{p-1}] |Du_i| \Theta \, dx
\]
\[
\leq C \frac{k}{m} \int_{\{0 < u_i < 2m, 0 < v_N < 2m\}} [|Du_i|^p + |Dv_N|^p] \Theta \, dx.
\]

On the other hand, using \( T_{2m}^+(u_i) \Theta \) as a test function for the equation of \( u_i \) and invoking condition (2.1), we estimate
\[
\alpha \int_{0 < u_i < 2m} |Du_i|^p \Theta \, dx \leq \int_{\mathbb{R}^n} T_{2m}^+(u_i) \Theta \, d\sigma_i
\]
\[
+ \beta \int_{\mathbb{R}^n} |Du_i|^{p-1} T_{2m}^+(u_i) |\nabla \Theta| \, dx.
\]
Since $T_{2m}^+(u_i)/m \leq 2$, and $T_{2m}^+(u_i)/m$ converges to zero quasi-everywhere, we deduce
\[
\lim_{m \to \infty} \frac{1}{m} \int_{0<u_i<2m} |Du_i|^p \Theta \, dx = 0.
\]
Similarly,
\[
\lim_{m \to \infty} \frac{1}{m} \int_{0<v_N<2m} |Dv_N|^p \Theta \, dx = 0.
\]
Hence,
\[
(3.12) \quad \lim_{m \to \infty} |A_m| = 0.
\]
A similar argument gives
\[
\lim_{m \to \infty} |B_m| = 0.
\]
To estimate $|C_m|$, we first use the pointwise bound (3.4) to obtain
\[
|C_m| \leq c \int_{A_R} \left[ |Du_i|^{p-1} + |Dv_N|^{p-1} \right] \min \{ W_{1,p}(\sigma|_{B_N(0)}), k \} \, dx,
\]
where $A_R$ is the annulus
\[
A_R = \{ R/2 < |x| < R \}.
\]
Note that for $R > 4N$ we have
\[
W_{1,p}(\sigma|_{B_N(0)})(x) = \int_{R/4}^{\infty} \left[ \frac{\sigma(B_t(x) \cap B_N(0))}{t^{n-p}} \right] \frac{R^n}{t^{n-p}} \, dt \approx R^{p-n},
\]
for all $x \in A_R$. Thus,
\[
|C_m| \leq c R^{p-n} R^{-1} \int_{A_R} \left[ |Du_i|^{p-1} + |Dv_N|^{p-1} \right] \, dx
\]
\[
\leq c R^{p-n} R^{-1} \left[ (\inf_{A_R} u_i)^{p-1} + (\inf_{A_R} v_N)^{p-1} \right]
\]
\[
\leq c R^{p-n} R^{-1} R^{n-p+1} R^{p-n} = c R^{p-n},
\]
where we used the Caccioppoli inequality and the weak Harnack inequality in the second bound. This gives
\[
(3.13) \quad \lim_{R \to \infty} \limsup_{m \to \infty} |C_m| = 0.
\]
Since $h_m(u_i) h_m(v_N) \to 1$ a.e. as $m \to \infty$, and $\Theta(x) \to 1$ everywhere as $R \to \infty$, it follows from (3.12)–(3.13) and Fatou’s lemma that
\[
\int_{\{0<u_i<v_N<k\}} [A(x, Du_i) - A(x, Dv_N)] \cdot (Du_i - Dv_N) \, dx \leq 0.
\]
Letting $k \to \infty$, we deduce
\[
\int_{\{u_i<v_N>0\}} [A(x, Du_i) - A(x, Dv_N)] \cdot (Du_i - Dv_N) \, dx \leq 0.
\]
Since the integrand is strictly positive whenever $Du_i \neq Du_N$, we infer that $Du_i = Du_N$ a.e. on the set $\{u_i - v_N > 0\}$.

We next claim that the function $T^+_k(u_i - v_N)$ belongs to $W^{1,p}_{loc}(\mathbb{R}^n)$ for any $k > 0$. To see this, for any $k > \kappa$ that satisfies $R$, which violates the condition at infinity, $\lim \inf \frac{|W_k(u_i - v_N)|}{\Theta} = 0$, which yields (3.9), as desired.

On the other hand, using $H_{m,k}(v_N)\Theta$ as a test function for the equation of $v_N$, where

$$H_{m,k}(t) = \begin{cases} 1 & \text{if } 0 \leq |t| \leq m - k, \\ 0 & \text{if } |t| \geq m, \\ \frac{-t + m}{k} & \text{if } m - k < t < m, \\ \frac{t}{k} + \frac{m}{k} & \text{if } -m < t < -(m - k), \end{cases}$$

we have

$$\frac{\alpha}{k} \int_{\{m-k<v_N<m\}} |Dv_N|^p \Theta \, dx \leq \int_{\mathbb{R}^n} \Theta d\sigma|_{B_N(0)} + \beta \int_{\mathbb{R}^n} |Dv_N|^{p-1} |\nabla \Theta| \, dx.$$

Thus, for each fixed $k > 0$, the sequence $\{T^+_k(T^+_m(u_i) - T^+_m(v_N))\}_m$ is uniformly bounded in $W^{1,p}_{loc}(\mathbb{R}^n)$. Since $T^+_k(T^+_m(u_i) - T^+_m(v_N)) \to T^+_k(u_i - v_N)$ a.e. as $m \to \infty$, we see that $T^+_k(u_i - v_N) \in W^{1,p}_{loc}(\mathbb{R}^n)$.

We are now ready to complete the proof of the theorem. Since $T^+_k(u_i - v_N) = T^+_k(T^+_m(u_i) - T^+_m(v_N))$ a.e. on the set $\{u_i < m, v_N < m\}$ and the two functions belong to $W^{1,p}_{loc}(\mathbb{R}^n)$, by (3.14) we have

$$\nabla T^+_k(u_i - v_N) = \nabla T^+_k(T^+_m(u_i) - T^+_m(v_N)) = 0$$

a.e. on the set $\{u_i < m, v_N < m\}$ for any $m > 0$. Thus, $\nabla T^+_k(u_i - v_N) = 0$ a.e. in $\mathbb{R}^n$, which implies the existence of a constant $\kappa \geq 0$ such that

$$\max\{u_i - v_N, 0\} = \kappa$$

a.e. in the entire space $\mathbb{R}^n$. Note that if $\kappa \neq 0$, then $u_i = v_N + \kappa$ in $\mathbb{R}^n$, which violates the condition at infinity, $\lim \inf_{|x| \to \infty} u_i(x) = 0$. It follows that $\kappa = 0$, which yields (3.9), as desired. \hfill \square

The following version of the comparison principle in $\mathbb{R}^n$ is an immediate consequence of Theorems 3.7 and 3.10.
Corollary 3.11. Let \( \sigma, \tilde{\sigma} \in M^+(\mathbb{R}^n) \), where \( \sigma \leq \tilde{\sigma} \) and \( \sigma \ll \text{cap}_p \), 1 < \( p < n \). Let \( u \) be an \( A \)-superharmonic reachable solution of \((3.1)\), and \( \tilde{u} \) any \( A \)-superharmonic solution of \((3.1)\) with datum \( \tilde{\sigma} \) in place of \( \sigma \). Then \( u \leq \tilde{u} \) in \( \mathbb{R}^n \).

For \( \sigma \in M^+(\mathbb{R}^n) \) such that that \( \sigma \ll \text{cap}_p \), sometimes it is desirable to know when an \( A \)-superharmonic solution to \((3.1)\) is also the \( A \)-superharmonic reachable solution to \((3.1)\), and hence also the minimal \( A \)-superharmonic solution to \((3.1)\). The following theorem provides some sufficient conditions in terms of the weak integrability of the gradient of the solution, or in terms of the finiteness of the datum \( \sigma \).

Theorem 3.12. Let \( \sigma \in M^+(\mathbb{R}^n) \), where \( \sigma \ll \text{cap}_p \). Suppose that any one of the following conditions holds:

(i) \( |Du| \in L^{\gamma,\infty}(\mathbb{R}^n) \) for some \( (p-1)n/(n-1) \leq \gamma < p \), where \( L^{\gamma,\infty}(\mathbb{R}^n) \) is the weak \( L^\gamma \) space in \( \mathbb{R}^n \);

(ii) \( |Du| \in L^p(\mathbb{R}^n) \);

(iii) \( \sigma \in M^+_b(\mathbb{R}^n) \).

Then any \( A \)-superharmonic solution \( u \) to the equation \((3.1)\) coincides with the minimal \( A \)-superharmonic solution.

Proof. Let \( v \) be the minimal \( A \)-superharmonic solution of \((3.1)\). Our goal is to show that \( u \leq v \) a.e. Let \( \Theta(x) = \Theta_R(x), R > 0, T_k^+(t), k > 0, \) and \( h_m(t), m > 0 \) be as in the proof of Theorem 3.10. Then arguing as in the proof of Theorem 3.10, with \( u \) in place of \( u_i \) and \( v \) in place of \( v_N \), we have

\[
\int_{\{0 < u_i - v_N < k\}} [A(x, Du) - A(x, Dv)] \cdot (Du - Dv) h_m(u) h_m(v) \Theta \, dx \leq A_m + B_m + C_m,
\]

where

\[
A_m = -\int_{\mathbb{R}^n} [A(x, Du) - A(x, Dv)] \cdot Du h'_m(u) T_k^+(u - v) h_m(v) \Theta \, dx,
\]

\[
B_m = -\int_{\mathbb{R}^n} [A(x, Du) - A(x, Dv)] \cdot Dv h'_m(v) T_k^+(u - v) h_m(u) \Theta \, dx,
\]

and

\[
C_m = -\int_{\mathbb{R}^n} [A(x, Du) - A(x, Dv)] \cdot \nabla \Theta T_k^+(u - v) h_m(u) h_m(v) \, dx.
\]

As in the proof of Theorem 3.10, we have

\[
(3.15) \quad \lim_{m \to \infty} (|A_m| + |B_m|) = 0.
\]

As for \( C_m \), we have

\[
|C_m| \leq \frac{c}{R} \int_{A_R} [|Du|^{p-1} + |Dv|^{p-1}] \min\{W_{1,p}\sigma, k\} \, dx,
\]
where, as above, \( A_R \) is the annulus
\[
A_R = \{ R/2 < |x| < R \}.
\]
Suppose now that condition (i) holds. Then \(|Du|^{p-1} \in L^{q, \infty}(\mathbb{R}^n)\) for some \( q \in (p, n] \). Set
\[
m = \frac{n(p-1)q}{n(q-1) - (p-1)q} > 0,
\]
and note that \( W_{1,p} \sigma \in L^{m, \infty}(\mathbb{R}^n) \). A proof of this fact in the ‘sublinear’ case \((p-1)q/(q-1) \leq 1\) can be found in [32].

We have that either \( m \leq q \) or \( m > q \). In the case \( m \leq q \), for any \( \epsilon > 0 \) we find
\[
\frac{1}{|A_R|} \int_{A_R} |Du|^{p-1} \min\{ W_{1,p} \sigma, k \} dx \leq \frac{1}{R} \left| \begin{array}{c}
|Du|^{p-1} \|_{L^{q, \infty}(A_R)} \| \min\{ W_{1,p} \sigma, k \} \|_{L^{q, \infty}(A_R)} \\
\min\{ W_{1,p} \sigma, k \} \|_{L^{q, \infty}(A_R)} |A_R|^{-1/q}
\end{array} \right|
\]
\[
\frac{4k^{q-1}}{q+1} \left| \begin{array}{c}
|Du|^{p-1} \|_{L^{q, \infty}(\mathbb{R}^n)} \| W_{1,p} \sigma \|_{L^{m, \infty}(\mathbb{R}^n)} \| A_R \|_{|A_R|^{-1/q}}
\end{array} \right|
\]
\[
\leq Ck^{q-1} \left| \begin{array}{c}
|Du|^{p-1} \|_{L^{q, \infty}(\mathbb{R}^n)} \| W_{1,p} \sigma \|_{L^{m, \infty}(\mathbb{R}^n)} R^{\frac{m-q}{qm}}
\end{array} \right| .
\]
Here we shall choose \( \epsilon > 0 \) such that
\[
\frac{n\epsilon}{q(q+\epsilon)} - 1 < 0.
\]
In the case \( m > q \), we have
\[
\frac{1}{|A_R|} \int_{A_R} |Du|^{p-1} \min\{ W_{1,p} \sigma, k \} dx 
\]
\[
\leq \frac{1}{R} \left| \begin{array}{c}
|Du|^{p-1} \|_{L^{q, \infty}(A_R)} \| \min\{ W_{1,p} \sigma, k \} \|_{L^{q, \infty}(A_R)} \\
\min\{ W_{1,p} \sigma \} \|_{L^{m, \infty}(A_R)} |A_R|^{-1/q}
\end{array} \right|
\]
\[
\leq \frac{4k^{q-1}}{q+1} \left| \begin{array}{c}
|Du|^{p-1} \|_{L^{q, \infty}(\mathbb{R}^n)} \| W_{1,p} \sigma \|_{L^{m, \infty}(\mathbb{R}^n)} \| A_R \|_{|A_R|^{-1/q}}
\end{array} \right|
\]
\[
\leq C \left| \begin{array}{c}
|Du|^{p-1} \|_{L^{q, \infty}(\mathbb{R}^n)} \| W_{1,p} \sigma \|_{L^{m, \infty}(\mathbb{R}^n)} R^{\frac{m-q}{qm}}
\end{array} \right| .
\]
Note that, since \( q < p \),
\[
\frac{(m-q)n}{mq} - 1 = \frac{n(p-1) - n(q-1) + (p-1)q}{(p-1)q} - 1 < 0.
\]
Hence, in both cases we have, for any fixed \( k > 0 \),
\[
\lim_{R \to \infty} \frac{1}{R} \int_{A_R} |Du|^{p-1} \min\{ W_{1,p} \sigma, k \} dx = 0,
\]
and likewise,
\[
\lim_{R \to \infty} \frac{1}{R} \int_{A_R} |Dv|^{p-1} \min\{ W_{1,p} \sigma, k \} dx = 0.
\]
On the other hand, suppose now that condition (ii) holds, i.e., \( |Du| \in L^p(\mathbb{R}^n) \). Then \( W_{1,p} \sigma \in L^{\frac{np}{n-p}}(\mathbb{R}^n) \), and as in (3.16) we have
\[
\frac{1}{R} \int_{A_R} |Du|^{p-1} \min\{W_{1,p} \sigma, k\} \, dx
\leq C \left\| \frac{|Du|^{p-1}}{L^{\frac{p}{n-p}}(A_R)} \right\| \left\| W_{1,p} \sigma \right\|_{L^{\frac{np}{n-p}}(A_R)},
\]
and likewise for \( v \). Thus (3.17) and (3.18) also hold under condition (ii).

Finally, suppose that (iii) holds. For any \( 1 < r < \frac{n}{n-1} \) and \( \epsilon \in (0, 1) \) such that \( \epsilon \frac{r}{r-1} < \frac{n(p-1)}{n-1} \), we have
\[
\frac{1}{R} \int_{A_R} |Du|^{p-1} \min\{W_{1,p} \sigma, k\} \, dx
\leq Ck^{1-\epsilon} R^{n-p} \left( \inf_{B_R(0)} u \right)^{p-1} R^{n-1} \left( \inf_{B_R(0)} u \right)^{\epsilon},
\]
where we used the Caccioppoli inequality and the weak Harnack inequality in the last bound (see [18, Theorem 7.46]).

Hence, using [9, Lemma 3.1] we get
\[
(3.19) \quad \frac{1}{R} \int_{A_R} |Du|^{p-1} \min\{W_{1,p} \sigma, k\} \, dx
\leq Ck^{1-\epsilon} R^{n-p} \left( \int_R^{\infty} \left( \frac{\sigma(B_t(0))}{t^{n-p}} \right)^{\frac{1}{p-1}} \, dt \right)^{p-1} \left( \inf_{B_R(0)} u \right)^{\epsilon}
\leq Ck^{1-\epsilon} \sigma(\mathbb{R}^n) \left( \inf_{B_R(0)} u \right)^{\epsilon}.
\]

A similar inequality holds for \( v \) in place of \( u \). Thus, we see that (3.17) and (3.18) hold under condition (iii) as well.

Now (3.17) and (3.18) yield that, for any \( k > 0 \), we have
\[
(3.20) \quad \lim_{R \to \infty} \limsup_{m \to \infty} |C_m| = 0.
\]

Using (3.15) and (3.20), we deduce
\[
\int_{\{0 < u - v < k\}} \left[ A(x, Du) - A(x, Dv) \right] \cdot (Du - Dv) \, dx \leq 0
\]
for any \( k > 0 \). This implies \( Du = Dv \) a.e. on the set \( \{u - v > 0\} \) and, as in the proof of Theorem 3.10, in view of the condition at infinity, we deduce \( u \leq v \) a.e. as desired. \( \square \)

We now provide a criterion for reachability by requiring only the finiteness of the approximating measures \( \sigma_i \).
Corollary 3.13. Let $u$ be an $A$-superharmonic solution of (3.1), where $\sigma \in M^+(\mathbb{R}^n)$, and $\sigma \ll \text{cap}_p$. Suppose that there exist two sequences $\{u_i\}$ and $\{\sigma_i\}$, $i = 1, 2, \ldots$, such that the following conditions hold:
(i) each $\sigma_i \in M^+_b(\mathbb{R}^n)$, and $\sigma_i \leq \sigma$;
(ii) each $u_i$ is an $A$-superharmonic solution of (3.1) with datum $\sigma_i$ in place of $\sigma$;
(iii) $u_i \to u$ a.e. in $\mathbb{R}^n$.
Then $u$ is an $A$-superharmonic reachable solution of (3.1), and thus coincides with the minimal solution.

Proof. By Theorem 3.12, each $u_i$ is a reachable solution. Thus by a diagonal process argument, we see that $u$ is also a reachable solution. Alternatively, this can also be proved by modifying the proof of the uniqueness part in Theorem 3.10, taking into account estimates of the form (3.19). □

Theorem 3.12 formally holds under the condition $|Du| \in L^{\gamma,\infty}(\mathbb{R}^n)$ for $0 < \gamma < (p - 1)n/(n - 1)$ as in this case $\sigma = 0$. The proof of this fact, especially in the case $0 < \gamma \leq p - 1$, requires some results obtained recently in [31].

Theorem 3.14. If $u$ is an $A$-superharmonic function in $\mathbb{R}^n$ such that $|Du| \in L^{\gamma,\infty}(\mathbb{R}^n)$ for some $0 < \gamma < (p - 1)n/(n - 1)$, then $\sigma = 0$ where $\sigma$ is the Riesz measure of $u$.

Proof. Let $Q_r(x)$, $r > 0$, denote the open cube $Q_r(x) := x + (-r, r)^n$ with center $x \in \mathbb{R}^n$ and side-length $2r$. Using $\Phi \in C_0^\infty(Q_r(x))$, $\Phi \geq 0$, $\Phi = 1$ on $Q_{r/2}(x)$, and $|\nabla \Phi| \leq C/r$, as a test function we have
\[
\sigma(Q_{r/2}(x)) \leq C \frac{r}{r} \int_{Q_r(x)} |Du|^{p-1} \, dy.
\]
Thus if $\gamma \in (p - 1, (p - 1)n/(n - 1))$, for any $R > 0$ we use Hölder’s inequality to get
\[
\sigma(Q_{R/2}(0)) \leq C \frac{R}{R} \|Du\|_{L^{\gamma,\infty}(\mathbb{R}^n)}^{p-1} R^{n-\frac{p+1}{\gamma}}.
\]
Note that $n^{\frac{\gamma(p-1)}{p}} < 1$ and thus letting $R \to \infty$ we get $\sigma = 0$.

We now consider the case $0 < \gamma \leq p - 1$. Let $\gamma_1$ be a fixed number in $(p - 1, (p - 1)n/(n - 1))$. By [31, Lemma 2.3], for any cube $Q_\rho(x) \subset \mathbb{R}^n$, we have
\[
\left( \int_{Q_\rho(x)} |Du|^{\gamma_1} \, dy \right)^{\frac{1}{\gamma_1}} \leq \left[ \frac{\sigma(Q_3\rho/2(x))}{\rho^{\gamma_1 - 1}} \right]^{\frac{1}{p-1}} + \frac{1}{\rho} \inf_{q \in \mathbb{R}} \left( \int_{Q_3\rho/2(x)} |u - q|^{p-1} \, dy \right)^{\frac{1}{p-1}}.
\]
On the other hand, by [31, Corollary 1.3] we find
\[
\frac{1}{\rho} \inf_{q \in \mathbb{R}} \left( \int_{Q_{3\rho/2}(x)} |u - q|^{p-1} dy \right)^{\frac{1}{p-1}} \lesssim \left[ \frac{\sigma(Q_{2\rho}(x))}{\rho^{n-1}} \right]^{\frac{1}{p-1}} + \left( \int_{Q_{2\rho}(x)} |Du|^{p-1} dy \right)^{\frac{1}{p-1}}.
\]

Note that [31, Corollary 1.3] is stated for \(1 < p < 3/2\) but the argument there also works for all \(1 < p \leq n\) after taking into account the comparison estimates of [27, 14, 28].

Hence, it follows that
\[
\left( \int_{Q_{\rho}(x)} |Du|^{\gamma_1} dy \right)^{\frac{1}{\gamma_1}} \lesssim \left[ \frac{\sigma(Q_{2\rho}(x))}{\rho^{n-1}} \right]^{\frac{1}{p-1}} + \left( \int_{Q_{2\rho}(x)} |Du|^{p-1} dy \right)^{\frac{1}{p-1}},
\]

where we used (3.21) with \(r = 4\rho\) in the last inequality. This allows us to employ a covering/iteration argument as in [16, Remark 6.12] to obtain that
\[
\left( \int_{Q_{\rho}(x)} |Du|^{\gamma_1} dy \right)^{\frac{1}{\gamma_1}} \lesssim \left( \int_{Q_{4\rho}(x)} |Du|^{p-1} dy \right)^{\frac{1}{p}}
\]
for any \(\epsilon > 0\).

Thus, if \(0 < \gamma < p - 1\), in view of (3.21), (3.22), and Hölder’s inequality, we get
\[
\sigma(Q_{R/2}(0)) \leq C \frac{R^{n-1-\gamma}}{L^{\gamma}(\mathbb{R}^n)} R^{n-\frac{n-1}{\gamma}} \to 0,
\]
as \(R \to \infty\). Hence, \(\sigma = 0\). The case \(\gamma = p - 1\) is treated similarly, starting with the inequality
\[
\sigma(Q_{R/2}(0)) \leq C \frac{R^{n-1-\gamma}}{L^{\gamma}(\mathbb{R}^n)} \left( \int_{Q_{R}(0)} |Du|^{(p-1)(1+\epsilon)} dx \right)^{\frac{1}{p+\epsilon}} R^{n-\epsilon}
\]
for a sufficiently small \(\epsilon > 0\).

Due to the results of [14, 24, 13, 30] (see also [15, 29]), under some additional regularity conditions on the nonlinearity \(A(x, \xi)\), one has
\[
|Du(x)| \leq C \left[ I_1 \sigma(x) \right]^{\frac{1}{p-1}} \quad \text{a.e. } x \in \mathbb{R}^n,
\]
provided \(u\) is an \(A\)-superharmonic solution to the equation (3.1). This gradient estimate holds in particular for \(A(x, \xi) = |\xi|^{p-2} \xi\), i.e., the \(p\)-Laplacian \(\Delta_p\), which yields the following corollary.
Corollary 3.15. Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Suppose that one of the following conditions holds:

(i) $\sigma << \text{cap}_p$ and $I_1 \sigma \in L^{s,\infty}(\mathbb{R}^n)$ for some $n/(n-1) < s < p/(p-1)$. This holds in particular if $\sigma = f \in L^t(\mathbb{R}^n)$ for some $1 < t < np/(np - n + p)$;

(ii) $I_1 \sigma \in L^{p/(p-1)}(\mathbb{R}^n)$, i.e., $\sigma$ is of finite energy.

Then any $p$-superharmonic solution $u$ to the equation

\[
\begin{cases}
-\Delta_p u = \sigma, & u \geq 0 \quad \text{in } \mathbb{R}^n, \\
\lim \inf_{|x| \to \infty} u = 0, & \text{in } \mathbb{R}^n
\end{cases}
\]

coincides with the minimal $p$-superharmonic solution.

Finally, we show that if the condition at infinity, $\lim \inf_{|x| \to \infty} u = 0$ in (3.1), is replaced with the stronger one $\lim_{|x| \to \infty} u = 0$, then all $\mathcal{A}$-superharmonic solutions are indeed reachable.

Theorem 3.16. Suppose that $u$ is an $\mathcal{A}$-superharmonic solution of the equation

\[
(3.23) \begin{cases}
-\text{div}\mathcal{A}(x, \nabla u) = \sigma, & u \geq 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} u = 0, & \text{in } \mathbb{R}^n
\end{cases}
\]

where $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, and $\sigma << \text{cap}_p$. Then $u$ is the unique $\mathcal{A}$-superharmonic solution of (3.23), which coincides with the minimal $\mathcal{A}$-superharmonic reachable solution of (3.1).

Proof. First notice that the condition $\lim_{|x| \to \infty} u = 0$ in (3.1) yields, in view of (3.1),

\[
\lim_{|x| \to \infty} W_{1, p} \sigma(x) = 0.
\]

For any $\epsilon > 0$, let

\[
\Omega_\epsilon := \{ x \in \mathbb{R}^n : u(x) > \epsilon \},
\]

and

\[
u_\epsilon := \max\{ u, \epsilon \} - \epsilon.
\]

Clearly, $\Omega_\epsilon$ is a bounded open set, $\nu_\epsilon = u - \epsilon$ on $\Omega_\epsilon$, and $\nu_\epsilon = 0$ in $\mathbb{R}^n \setminus \Omega_\epsilon$.

Let $v$ be the minimal solution of (3.1), which is also the minimal solution of (3.23), since $v \leq K W_{1, p} \sigma$, and hence $\lim_{|x| \to \infty} v = 0$. It is enough to show that

\[
(3.24) \quad \nu_\epsilon \leq v
\]

in $\Omega_\epsilon$, as this will yield that $u \leq v$ in $\mathbb{R}^n$ after letting $\epsilon \to 0^+$.

Now by Lemma 4.1 below, to verify (3.24), it suffices to show that $\nu_\epsilon$ is a renormalized solution of

\[
\begin{cases}
-\text{div}\mathcal{A}(x, \nabla \nu_\epsilon) = \sigma & \text{in } \Omega_\epsilon, \\
\nu_\epsilon = 0 & \text{on } \partial \Omega_\epsilon.
\end{cases}
\]
Note that, for any $k > 0$, $T_k(u_\epsilon) = T_{k+\epsilon}(u) - \epsilon$. We have $T_k(u_\epsilon) \in W^{1,p}_\text{loc}(\mathbb{R}^n)$, $T_k(u_\epsilon)$ is quasi-continuous in $\mathbb{R}^n$, and $T_k(u_\epsilon) = 0$ everywhere in $\mathbb{R}^n \setminus \Omega_\epsilon$. Thus $T_k(u_\epsilon) \in W^{1,p}_0(\Omega_\epsilon)$ (see [18, Theorem 4.5]).

As $u$ is a local renormalized solution in $\mathbb{R}^n$, for every $k > 0$ there exists a nonnegative measure $\lambda_{k+\epsilon} << \text{cap}_p$, concentrated on the sets $\{u = k + \epsilon\}$, such that $\lambda_{k+\epsilon} \to 0$ weakly as measures in $\mathbb{R}^n$ as $k \to \infty$. Since $\Omega_\epsilon$ is bounded, this implies that $\lambda_{k+\epsilon} \to 0$ in the narrow topology of measures in $\Omega_\epsilon$.

Moreover, for $k > 0$,

$$\int_{\{u < k + \epsilon\}} A(x, Du) \cdot \nabla \varphi dx = \int_{\{u < k + \epsilon\}} \varphi d\sigma_0 + \int_{\mathbb{R}^n} \varphi d\lambda_{k+\epsilon},$$

for every $\varphi \in W^{1,p}_0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$. In particular, we have

$$\int_{\{u_\epsilon < k\} \cap \Omega_\epsilon} A(x, Du) \cdot \nabla \varphi dx = \int_{\{u_\epsilon < k\} \cap \Omega_\epsilon} \varphi d\sigma_0 + \int_{\{u_\epsilon = k\} \cap \Omega_\epsilon} \varphi d\lambda_{k+\epsilon},$$

for every $\varphi \in W^{1,p}_0(\Omega_\epsilon) \cap L^\infty(\Omega_\epsilon)$.

Thus, we conclude that $u_\epsilon$ is a renormalized solution in $\Omega_\epsilon$, as desired. □

4. Quasilinear equations with a sub-natural growth term in $\mathbb{R}^n$

In this section, we study solutions to the equation

\[
\begin{cases}
-\text{div} A(x, \nabla u) = \sigma u^q + \mu, & u \geq 0 \text{ in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u = 0,
\end{cases}
\]

(4.1)

in the sub-natural growth case $0 < q < p - 1$, with $\mu, \sigma \in M^+(\mathbb{R}^n)$.

We consider nontrivial $A$-superharmonic solutions to (4.1) such that $0 < u < \infty$ $d\sigma$-a.e., which implies $u \in L^q(\mathbb{R}^n, \sigma)$, so that $\sigma u^q + \mu \in M^+(\mathbb{R}^n)$ (see [37]).

As was noted in the Introduction, $\sigma << \text{cap}_p$ whenever there exists a nontrivial solution $u$ to (4.1), for any $\mu$ (in particular, $\mu = 0$).

The existence and uniqueness of nontrivial reachable $A$-superharmonic solutions to (4.1), under the additional assumption $\mu << \text{cap}_p$, are proved below. Without this restriction on $\mu$, the existence of nontrivial solutions, not necessarily reachable, was obtained recently in [37], along with bilateral pointwise estimates of solutions in terms of nonlinear potentials.

We use this opportunity to make a correction in the proof of the existence property for (4.1) in the case $\mu = 0$ given in [9, Theorem 1.1], which used a version of the comparison principle ([9], Lemma 5.2). It was invoked in the proof of [37, Theorem 1.1] as well. Some inaccuracies in the statement of this comparison principle and its proof are fixed in the following lemma. The rest of the proofs of [9, Theorem 1.1] and [37, Theorem 1.1] remains valid with this correction. (See the proof of Theorem 4.2 below.)
**Lemma 4.1.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Suppose that $\mu, \nu \in \mathcal{M}_b^+(\Omega)$, where $\mu \leq \nu$ and $\mu \ll \text{Cap}_p$. If $u \geq 0$ is a renormalized solution of

$$
\begin{cases}
-\text{div} \mathcal{A}(x, \nabla u) = \mu & \text{in } \Omega,
\end{cases}
$$

and if $v \geq 0$ is an $\mathcal{A}$-superharmonic function in $\Omega$ with Riesz measure $\nu$ such that $\min\{v, k\} \in W^{1,p}(\Omega)$ for any $k > 0$, then $u \leq v$ a.e.

**Proof.** Let $\nu_j, j > 0$, be the Riesz measure of $\min\{v, j\}$. Since $\min\{v, j\} \in W^{1,p}(\Omega)$ we see that $\nu_j$ belongs to the dual of $W_0^{1,p}(\Omega)$ (see [18, Theorem 21.6]). As in (3.7), we have

$$
\nu_j = \nu|_{\{v < j\}} + \alpha_j
$$

for a measure $\alpha_j \in \mathcal{M}_b^+(\Omega)$ concentrated in the set $\{v = j\}$. Thus the measure $\mu_j := \mu|_{\{v < j\}} \leq \nu|_{\{v < j\}} \leq \nu_j$ for any $j > 0$. This implies that $\mu_j$ also belongs to the dual of $W_0^{1,p}(\Omega)$, and hence there exists a unique solution $u_j$ to the equation

$$
-\text{div} \mathcal{A}(x, \nabla u_j) = \mu_j, \quad u_j \in W_0^{1,p}(\Omega).
$$

Then by the comparison principle (see [9, Lemma 5.1]) we find

$$
0 \leq u_1 \leq u_2 \leq \cdots \leq u_j \leq \min\{v, j\}
$$

for any integer $j > 0$. Thus there is a function $\tilde{u}$ on $\Omega$ such that $0 \leq \tilde{u} \leq v$ a.e. and $u_j \to \tilde{u}$ as $j \to \infty$. We now claim that $\tilde{u}$ is also a renormalized solution to equation (4.2). If this is verified then, as $\mu \ll \text{Cap}_p$, we must have that $\tilde{u} = u$ a.e. (see [25, 12]) and thus $u \leq v$ a.e. as desired.

To show that $\tilde{u}$ is the renormalized solution of (4.2), we first use $T_k^+(u_j)$, $k > 0$, as a test function for (4.3) to obtain

$$
\alpha \int_{\Omega} |\nabla T_k^+(u_j)|^p dx \leq k\mu_j(\Omega) \leq k\mu(\Omega).
$$

Since $T_k^+(u_j) \to T_k^+(\tilde{u})$ a.e. as $j \to \infty$, we see that $T_k^+(\tilde{u}) \in W_0^{1,p}(\Omega)$ and

$$
\alpha \int_{\Omega} |\nabla T_k^+(\tilde{u})|^p dx \leq k\mu(\Omega)
$$

for any $k > 0$. By [2, Lemmas 4.1 and 4.2], this yields

$$
\tilde{u} \in L^{\frac{n(p-1)}{n-p}, \infty}(\Omega) \quad \text{and} \quad D\tilde{u} \in L^{\frac{n(p-1)}{n-1}, \infty}(\Omega).
$$

Moreover, arguing as in Step 4 of the proof of Theorem 3.4 in [12], we see that $\{\nabla u_j\}$ is a Cauchy sequence in measure which converges to $D\tilde{u}$ a.e. in $\Omega$. There is no need to take a subsequence here as the limit is independent of any subsequence.
Moreover, for any Lipschitz function \( h : \mathbb{R} \to \mathbb{R} \) such that \( h' \) has compact support and for any function \( \varphi \in W^{1,r}(\Omega) \cap L^\infty(\Omega), r > n, \) such that \( h(\tilde{u}) \varphi \in W^{1,p}_0(\Omega), \) we have
\[
\int_\Omega A(x, \nabla u_j) \cdot D\tilde{h}(\tilde{u}) \varphi \, dx + \int_\Omega A(x, \nabla u_j) \cdot \nabla \varphi \, h(\tilde{u}) \, dx = \int_\Omega h(\tilde{u}) \varphi \, d\mu_j.
\]
Thus if the support of \( h' \) is in \([-M, M], M > 0, \) then, using \( 0 \leq u_j \leq \tilde{u}, \) we can rewrite the above equality as
\[
\int_\Omega A(x, \nabla T^+_M(u_j)) \cdot \nabla T^+_M(\tilde{u}) \varphi \, dx + \int_\Omega A(x, \nabla u_j) \cdot \nabla \varphi \, h(\tilde{u}) \, dx
= \int_\Omega h(\tilde{u}) \varphi \, d\mu_j = \int_{\{0 \leq u < j\}} h(\tilde{u}) \varphi \, d\mu.
\]

Note that by (4.4) and [2, Lemma 4.2], we have that \( \nabla u_j \) is uniformly bounded in \( L^{n/(p-1)}(\Omega) \) and \( \nabla T^+_M(u_j) \) is uniformly bounded in \( L^p(\Omega) \). Thus by the Vitali Convergence Theorem, the left-hand side of the above equality converges to
\[
\int_\Omega A(x, \nabla T^+_M(\tilde{u})) \cdot \nabla T^+_M(\tilde{u}) \varphi \, dx + \int_\Omega A(x, D\tilde{u}) \cdot \nabla \varphi \, h(\tilde{u}) \, dx
= \int_\Omega A(x, D\tilde{u}) \cdot D\tilde{h}(\tilde{u}) \varphi \, dx + \int_\Omega A(x, D\tilde{u}) \cdot \nabla \varphi \, h(\tilde{u}) \, dx.
\]
On the other hand, by the Lebesgue Dominated Convergence Theorem we have
\[
\lim_{j \to \infty} \int_{\{0 \leq u < j\}} h(\tilde{u}) \varphi \, d\mu = \int_\Omega h(\tilde{u}) \varphi \, d\mu.
\]
Thus, we get
\[
\int_\Omega A(x, D\tilde{u}) \cdot D\tilde{h}(\tilde{u}) \varphi \, dx + \int_\Omega A(x, D\tilde{u}) \cdot \nabla \varphi \, h(\tilde{u}) \, dx = \int_\Omega h(\tilde{u}) \varphi \, d\mu,
\]
which yields that \( \tilde{u} \) is the renormalized solution of (4.2) (see Definition 3.2). \( \square \)

We recall that by \( \kappa = \kappa(\mathbb{R}^n) \) we denote the least constant in the weighted norm inequality (see [9], [37])
\[
(4.5) \quad \left( \int_{\mathbb{R}^n} |\varphi|^q \, d\sigma \right)^{\frac{1}{q}} \leq \kappa \| -\text{div} \, A(x, \nabla \varphi) \|_{M^+(\mathbb{R}^n)}^{-1},
\]
for all \( A \)-superharmonic functions \( \varphi \geq 0 \) in \( \mathbb{R}^n \) such that \( \liminf_{|x| \to \infty} \varphi(x) = 0. \) Notice that by estimates (3.4), \( K^{-1} \varphi \leq W_{1,p}\mu \leq K \varphi, \) where \( \mu = -\text{div} \, A(x, \nabla \varphi) \in M^+(\mathbb{R}^n). \) Here we may assume without loss of generality that \( \mu \in M^+(\mathbb{R}^n), \) so that \( W_{1,p}\mu \not= \infty. \) Consequently, (1.5) is equivalent to the inequality
\[
\left( \int_{\mathbb{R}^n} (W_{1,p}\mu)^q \, d\sigma \right)^{\frac{1}{q}} \leq \kappa \| \mu \|_{M^+(\mathbb{R}^n)}^{-1} \quad \text{for all } \mu \in M^+_b(\mathbb{R}^n),
\]
where $K^{-1} \leq \kappa \leq K$. In particular, one can replace $\text{div} \, A(x, \nabla \varphi)$ in \eqref{4.5} by $\Delta_p$, up to a constant which depends only on $K$.

By $\kappa(B)$, where $B$ is a ball in $\mathbb{R}^n$, we denote the least constant in a similar localized weighted norm inequality with the measure $\sigma_B$ in place of $\sigma$, where $\sigma_B = \chi_B \sigma$ is the restriction of $\sigma$ to $B$.

The so-called intrinsic nonlinear potential $K_{p,q}\sigma$, introduced in \cite{9}, is defined by

$$
K_{p,q}\sigma(x) = \int_0^\infty \left( \frac{\kappa(B(x,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.
$$

Here $B = B(x,t)$ is a ball in $\mathbb{R}^n$ of radius $t > 0$ centered at $x$. As was noticed in \cite{9}, $K_{p,q}\sigma \not\equiv +\infty$ if and only if

$$
\int_1^\infty \left( \frac{\kappa(B(0,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty. \tag{4.6}
$$

By \cite{9} Theorem 1.1], there exists a nontrivial $A$-superharmonic solution to the homogeneous equation \eqref{4.4} in the case $\mu = 0$ if and only if $W_{1,p}\sigma \not\equiv +\infty$ and $K_{p,q}\sigma \not\equiv +\infty$, i.e., conditions (3.3) and (4.6) hold. The next theorem shows that this solution is actually reachable.

**Theorem 4.2.** Let $0 < q < p - 1$, and let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then the nontrivial minimal $A$-superharmonic solution $u$ of

$$
\begin{cases}
-\text{div} \, A(x, \nabla u) = \sigma u^q, & u \geq 0 \quad \text{in} \ \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u = 0,
\end{cases}
\tag{4.7}
$$

constructed in the proof of \cite{9} Theorem 1.1] under the conditions (3.3) and (4.6), is an $A$-superharmonic reachable solution.

**Proof.** We start with the same construction as in the proof of \cite{9} Theorem 1.1] for the minimal $A$-superharmonic solution $u$, but with datum $\sigma|_{B_m(0)}$ in place of $\sigma$ ($m = 1, 2, \ldots$).

For a fixed $m$, let $v_m$ be the minimal $A$-superharmonic solution to the equation

$$
\begin{cases}
-\text{div} \, A(x, \nabla v_m) = \sigma|_{B_m(0)} v_m^q, & v_m \geq 0 \quad \text{in} \ \mathbb{R}^n, \\
\liminf_{|x| \to \infty} v_m = 0,
\end{cases}
$$

We recall from the construction in \cite{9} that

$$
v_m = \lim_{j \to \infty} \left( \lim_{k \to \infty} v_{j,m}^k \right),
$$

where $v_{1,m}^k$ ($k = 0, 1, 2, \ldots$) is the $A$-superharmonic renormalized solution of

$$
\begin{cases}
-\text{div} \, A(x, \nabla v_{1,m}^k) = \sigma|_{B_m(0) \cap B_{2^k}(0)} w_{0,m}^q, & \text{in} \ B_{2^k}(0), \\
v_{1,m}^k = 0 \quad \text{on} \ \partial B_{2^k}(0),
\end{cases}
$$
with \( w_{0,m} = c_0 \left( W_{1,p}(\sigma|_{B_m(0)}) \right)^{\frac{p-1}{p-1-q}} \), and \( v^k_{j,m} \) \((k = 0, 1, 2, \ldots, j = 2, 3, \ldots)\) is the \( \mathcal{A} \)-superharmonic renormalized solution of

\[
\begin{align*}
& -\text{div} \mathcal{A}(x, \nabla v^k_{j,m}) = \sigma|_{B_m(0) \cap B_{2^k}(0)} \left( \lim_{i \to \infty} v^i_{j-1,m} \right)^q \quad \text{in } B_{2^k}(0), \\
& v^k_{j,m} = 0 \quad \text{on } \partial B_{2^k}(0).
\end{align*}
\]

Here \( c_0 \) is a fixed constant such that

\[
0 < c_0 \leq \min \left\{ \left( \epsilon^{\frac{q}{p-1-q}} K^{-1} \right)^{\frac{p-1}{p-1-q}}, CK^{\frac{1-p}{p-1-q}} \right\},
\]

where \( C \) is the constant in (3.9) of [9], and \( \epsilon \) is the constant in (3.10) of [9] with \( \alpha = 1 \).

We also recall from [9] that

\[
u = \lim_{j \to \infty} \left( \lim_{k \to \infty} u^k_j \right),
\]

where \( u^k_j \) are the \( \mathcal{A} \)-superharmonic renormalized solutions of the corresponding problems in \( B_{2^k}(0) \) with \( \sigma \) in place of \( \sigma|_{B_m(0)} \). In particular, \( \min(u^k_j, l) \in W^{1,p}_0(B_{2^k}(0)) \) and \( \min(v^k_{j,m}, l) \in W^{1,p}_0(B_{2^k}(0)) \) for all \( l > 0 \).

Thus, by the above version of the comparison principle (Lemma 4.1) we see that

\[
v^k_{j,m_1} \leq v^k_{j,m_2} \leq u^k_j \quad \text{in } B_{2^k}(0),
\]

whenever \( m_1 \leq m_2 \).

This yields

\[
0 \leq v_1 \leq v_2 \leq \cdots \leq v_m \leq u \quad \text{in } \mathbb{R}^n.
\]

Letting now \( m \to \infty \), we obtain an \( \mathcal{A} \)-superharmonic reachable solution

\[
v := \lim_{m \to \infty} v_m
\]

to (4.7) such that \( v \leq u \) in \( \mathbb{R}^n \). As \( u \) is the minimal \( \mathcal{A} \)-superharmonic solution of (4.7), we see on the other hand that \( u \leq v \), and thus \( u = v \), which completes the proof. \( \square \)

**Remark.** In the proof of [9, Theorem 1.1], there is a misprint in the exponent in inequality (4.8) above for the constant \( c_0 \). This choice of \( c_0 \) ensures the minimality of the solution \( u \) of (4.7) constructed in [9].

We recall that, by [37, Theorem 1.1] and [37, Remark 4.3], a nontrivial \( \mathcal{A} \)-superharmonic solution of (4.11) exists if and only if \( W_{1,p}(\sigma) \neq \infty \), \( K_{p,q}(\sigma) \neq \infty \),
and \( W_{1,p} \mu \neq \infty \), i.e., the following three conditions hold:

\[
\int_1^\infty \left( \frac{\sigma(B(0, \rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty, \\
\int_1^\infty \left( \frac{\varepsilon(B(0, \rho))}{\rho^{n-p}} \right)^{\frac{q(p-1)}{p-1}} \frac{d\rho}{\rho} < \infty, \\
\int_1^\infty \left( \frac{\mu(B(0, \rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty.
\]

(4.9) \hspace{1cm} (4.10) \hspace{1cm} (4.11)

**Theorem 4.3.** Let \( 0 < q < p-1 \), and let \( \mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n) \), where \( \mu \ll \text{cap}_p \). Then, under the conditions (4.9), (4.10), and (4.11), there exists a nontrivial minimal reachable \( \mathcal{A} \)-superharmonic solution of (4.1).

**Proof.** Since the case \( \mu = 0 \) was treated in Theorem 4.2 above, without loss of generality we may assume that \( \mu \neq 0 \). We recall that in the proof of [37, Theorem 1.1], a nontrivial \( \mathcal{A} \)-superharmonic solution \( u \) of (4.1), was constructed using the following iteration process. We set \( u_0 = 0 \), and for \( j = 0, 1, 2, \ldots \) construct the iterations

\[
\begin{cases}
-\text{div} \mathcal{A}(x, \nabla u_{j+1}) = \sigma u_j^q + \mu & \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u_{j+1} = 0,
\end{cases}
\]

(4.12)

where \( u_j \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \). We observe that, for each \( j \), the solution \( u_{j+1} \) was chosen in [37] so that \( u_j \leq u_{j+1} \) \( (j = 0, 1, 2, \ldots ) \) by a version of the comparison principle (see [34, Lemma 3.7 and Lemma 3.9]). Then \( u := \lim_{j \to \infty} u_j \) is a nontrivial \( \mathcal{A} \)-superharmonic solution of (4.1).

We now modify this argument as follows to obtain a minimal nontrivial \( \mathcal{A} \)-superharmonic solution of (4.1). Notice that \( \mu \ll \text{cap}_p \) by assumption, and, as mentioned above, \( \sigma \ll \text{cap}_p \), since a solution exists. Hence, clearly the measure \( \sigma u_j^q + \mu \ll \text{cap}_p \) as well. By Theorem 3.6 \( u_{j+1} \) can be chosen as the minimal \( \mathcal{A} \)-superharmonic solution to (4.12).

It follows by induction that \( u_j \leq u_{j+1} \) \( (j = 0, 1, 2, \ldots ) \). Indeed, this is trivial when \( j = 0 \), and then by the inductive step,

\[
\sigma u_{j-1}^q + \mu \leq \sigma u_j^q + \mu, \quad j = 1, 2, \ldots,
\]

which is obvious when \( j = 1 \). From this, using Theorem 3.7 we deduce \( u_j \leq u_{j+1} \) for all \( j = 1, 2, \ldots \).

Similarly, if \( \tilde{u} \) is any \( \mathcal{A} \)-superharmonic solution of (4.1), then again arguing by induction and using Theorem 3.7 we deduce that \( u_{j+1} \leq \tilde{u} \) \( (j = 0, 1, 2, \ldots ) \), since

\[
\sigma u_{j-1}^q + \mu \leq \sigma \tilde{u}^q + \mu, \quad j = 1, 2, \ldots.
\]

Consequently, \( u \leq \tilde{u} \), i.e., \( u \) is the minimal \( \mathcal{A} \)-superharmonic solution of (4.1).
We next show that $u$ is a reachable solution. Using a similar iteration process with $\sigma|B_m(0)|$ in place of $\sigma$ and $\mu|B_m(0)|$ in place of $\mu$ ($m = 1, 2, \ldots$), we set $v_{0,m} = 0$ and define $v_{j,m}$ to be the minimal $A$-superharmonic solution to the equation

\[
\begin{cases}
-\text{div } A(x, \nabla v_{j+1,m}) = \sigma|B_m(0)|v_{j,m}^q + \mu|B_m(0)|, & v_{j,m} \geq 0, \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} v_{j,m} = 0,
\end{cases}
\]

where $v_{j,m} \leq v_{j+1,m}$ for each $m = 1, 2, \ldots$.

As above, arguing by induction and using Theorem 3.7, we deduce $v_{j,m_1} \leq v_{j,m_2} \leq u$, $j = 1, 2, \ldots$, whenever $m_1 \leq m_2$. It follows that $v_m := \lim_{j \to \infty} v_{j,m} \leq u$ ($m = 1, 2, \ldots$) is an $A$-superharmonic solution of the equation

\[
\begin{cases}
-\text{div } A(x, \nabla v_m) = \sigma|B_m(0)|v_m^q + \mu|B_m(0)|, & v_m \geq 0 \quad \text{in } \mathbb{R}^n, \\
\lim_{|x| \to \infty} v_m = 0,
\end{cases}
\]

where $v_{m_1} \leq v_{m_2} \leq u$ if $m_1 \leq m_2$.

Thus, letting $m \to \infty$, we obtain an $A$-superharmonic reachable solution $v := \lim_{m \to \infty} v_m$ to (4.7) such that $v \leq u$. Since $u$ is the minimal $A$-superharmonic solution of (4.7), we see that $u = v$, which completes the proof.

\[\square\]

We now prove the uniqueness property for reachable solutions of (4.1).

**Theorem 4.4.** Let $0 < q < p - 1$, and let $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$, where $\mu << \text{cap}_p$. Suppose $A$ satisfies conditions (2.1) and (2.2). Then nontrivial $A$-superharmonic reachable solutions of (4.1) are unique.

**Proof.** Let $u, v$ be two nontrivial $A$-superharmonic solutions of (4.1) in $\mathbb{R}^n$. Then by [37 Theorem 1.1] and [37 Remark 4.3], there exists a constant $C \geq 1$, depending only on $p, q$ and $n$, such that

\[C^{-1} u \leq v \leq C u \quad \text{in } \mathbb{R}^n.
\]

Hence, clearly,

\[-\text{div } A(x, \nabla v) = \sigma v^q + \mu \leq C^q (\sigma u^q + \mu) = -\text{div } A(x, \nabla (C^{\frac{1}{p-1}} u)).
\]

Notice that here by definition, $u, v \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma)$. Suppose that $v$ is a reachable solution of (4.1) in $\mathbb{R}^n$. Then by Corollary 3.11 with $\sigma v^q + \mu$ in place of $\sigma$, and $\bar{\sigma} = C^q (\sigma u^q + \mu)$, it follows that $v \leq C^{\frac{q}{p-1}} u$.

By iterating this argument, we deduce

\[v \leq C^{\left(\frac{q}{p-1}\right)^j} u \quad \text{in } \mathbb{R}^n, \quad j = 1, 2, \ldots.
\]

Since $0 < q < p - 1$, letting $j \to \infty$ in the preceding inequality, we obtain $v \leq u$ in $\mathbb{R}^n$. Interchanging the roles of $u$ and $v$, we see that actually $u = v$ in $\mathbb{R}^n$. \[\square\]
Corollary 4.5. Nontrivial \(A\)-superharmonic solutions \(u\) of (4.1) are unique under the assumptions of Theorem 4.4 provided any one of the following conditions holds:
(i) \(u \in L^q(\mathbb{R}^n, d\sigma)\) and \(\mu \in \mathcal{M}^+_b(\mathbb{R}^n)\), or equivalently \(\kappa(\mathbb{R}^n) < \infty\) and \(\mu \in \mathcal{M}^+_b(\mathbb{R}^n)\);
(ii) \(\lim_{|x| \to \infty} u(x) = 0\);
(iii) \(|Du| \in L^p(\mathbb{R}^n)\), or \(|Du| \in L^{q,\infty}(\mathbb{R}^n)\) for some \((p-1)n/(n-1) \leq \gamma < p\).

Proof. Suppose first that (i) holds. By [9, Theorem 4.4], \(\kappa(\mathbb{R}^n) < \infty\) if and only if there exists a nontrivial \(A\)-superharmonic solution \(u \in L^q(\mathbb{R}^n, d\sigma)\) of (4.7). In particular, since by [37, Theorem 4.1],
\[u \geq C \left[ W^{1,p}_\sigma + (K_{p,q}\sigma)^{-1/q} \right],\]
it follows that \(W^{1,p}_\sigma \in L^{q(p-1)/(p-1-q)}(\mathbb{R}^n, d\sigma)\) and \(K_{p,q}\sigma \in L^q(\mathbb{R}^n, d\sigma)\).

Next, we denote by \(\varphi\) an \(A\)-superharmonic solution to the equation
\[-\text{div} A(x, \nabla \varphi) = \mu, \quad \varphi \geq 0 \text{ in } \mathbb{R}^n,\]
\[\lim_{|x| \to \infty} \varphi = 0,\]
where \(\mu\) is the Riesz measure of \(\varphi\). Notice that \(\varphi \geq K^{-1} W^{1,p}_\mu\) by the lower bound in inequality (3.4). Since \(\mu \in \mathcal{M}_b(\mathbb{R}^n)\) and \(\kappa(\mathbb{R}^n) \leq \infty\), using \(\varphi\) as a test function in inequality (4.5) yields \(W^{1,p}_\mu \in L^q(\mathbb{R}^n, d\sigma)\).

Hence, by [37] Theorem 1.1 and [37] Remark 4.3, we deduce that there exists a nontrivial \(A\)-superharmonic solution of (4.1) \(u \in L^q(\mathbb{R}^n, d\sigma)\), and, for any such a solution, \(\sigma u^q + \mu \in \mathcal{M}_b(\mathbb{R}^n)\). It follows that \(u\) is a reachable \(A\)-superharmonic solution of (4.1) by Theorem 3.10 and Theorem 3.12 (iii).

In case (ii), by Theorem 3.16 \(u\) is a reachable solution of (4.1).

In case (iii), \(u\) is a reachable \(A\)-superharmonic solution of (4.1) by Theorem 3.10 and Theorem 3.12 (i), (ii).

In all these cases, reachable \(A\)-superharmonic solutions are unique by Theorem 4.4. \(\square\)

Remark 4.6. Uniqueness of finite energy solutions \(u\) to (4.1) such that \(|Du| \in L^p(\mathbb{R}^n)\) in Corollary 4.5(iii) was established in [35] Theorem 6.1 in the special case of the \(p\)-Laplace operator using a different method. (See also an earlier result [8, Theorem 5.1] in the case \(\mu = 0\).) Solutions of finite energy to (4.1) exist if and only if \(W^{1,p}_\sigma \in L^{1/(1+(p-1)/(p-1-q))}(\mathbb{R}^n, d\sigma)\) and \(W^{1,\mu}_1 \in L^1(\mathbb{R}^n, d\mu)\) (35, Theorem 1.1).

Remark 4.7. Under the assumptions of Theorem 4.4, but without the restriction \(\mu << \text{cap}_p\), it is still possible to prove the existence of an \(A\)-superharmonic reachable solution (not necessarily minimal) of (4.1). The construction of such a solution makes use of an extension of [33] Lemma 6.9 proved below.
Proof. To prove this claim, we shall construct first a nondecreasing sequence \( \{u_m\}_{m \geq 1} \) of \( \mathcal{A} \)-superharmonic solutions of
\[
\begin{aligned}
-\text{div} \mathcal{A}(x, \nabla u_m) &= \sigma|_{B_m(0)} u_m^q + \mu|_{B_m(0)}, \quad u_m \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u_m &= 0.
\end{aligned}
\]
Then by [37, Theorem 1.1 and Remark 4.3],
\[
\begin{aligned}
u_m(x) \leq C \left( W_{1,p} \mu(x) + K_{p,q} \sigma(x) + [W_{1,p} \sigma(x)]^{\frac{p-1}{p-1-q}} \right), \quad x \in \mathbb{R}^n.
\end{aligned}
\]
It follows from [20, Theorem 1.17] that \( u_m \to u \) pointwise everywhere, where \( u \) is an \( \mathcal{A} \)-superharmonic reachable solution of (4.1).

The construction of \( \{u_m\}_{m \geq 1} \) can be done as follows. It suffices to demonstrate only how to construct \( u_1 \) and \( u_2 \) such that \( u_2 \geq u_1 \), since the construction of \( u_m \) for \( m \geq 3 \) is completely analogous. Let \( v_1 \) be an \( \mathcal{A} \)-superharmonic solution of
\[
\begin{aligned}
-\text{div} \mathcal{A}(x, \nabla v_1) &= \mu|_{B_1(0)}, \quad v_1 \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} v_1 &= 0.
\end{aligned}
\]
Here as above \( v_1 \) is an a.e. pointwise limit of a subsequence of \( \{v_1^{(k)}\}_{k \geq 1} \), where each \( v_1^{(k)} \) is a nonnegative \( \mathcal{A} \)-superharmonic renormalized solution of
\[
\begin{aligned}
-\text{div} \mathcal{A}(x, \nabla v_1^{(k)}) &= \mu|_{B_1(0)} \quad \text{in } B_k(0), \\
v_1^{(k)} &= 0 \quad \text{on } \partial B_k(0).
\end{aligned}
\]
Next, for any \( j \geq 1 \), let \( v_{j+1} \) be an \( \mathcal{A} \)-superharmonic solution of
\[
\begin{aligned}
-\text{div} \mathcal{A}(x, \nabla v_{j+1}) &= \sigma|_{B_1(0)} v_q^j + \mu|_{B_1(0)}, \quad v_{j+1} \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} v_{j+1} &= 0.
\end{aligned}
\]
Notice that \( v_{j+1} \) is an a.e. pointwise limit of a subsequence of \( \{v_{j+1}^{(k)}\}_{k \geq 1} \), where each \( v_{j+1}^{(k)} \) is a nonnegative \( \mathcal{A} \)-superharmonic renormalized solution of
\[
\begin{aligned}
-\text{div} \mathcal{A}(x, \nabla v_{j+1}^{(k)}) &= \sigma|_{B_1(0)} v_q^j + \mu|_{B_1(0)} \quad \text{in } B_k(0), \\
v_{j+1}^{(k)} &= 0 \quad \text{on } \partial B_k(0).
\end{aligned}
\]
By [33, Lemma 6.9] we may assume that \( v_2^{(k)} \geq v_1^{(k)} \) for all \( k \geq 1 \), and hence \( v_2 \geq v_1 \). In the same way, by induction we deduce that \( v_{j+1}^{(k)} \geq v_{j}^{(k)} \) for all \( j, k \geq 1 \). It follows that \( v_{j+1} \geq v_j \), and
\[
v_{j+1} \leq C W_{1,p} \left( \sigma v_{j+1}^q + C W_{1,p} \mu \right).
\]
Then by [37, Theorem 4.1], for any \( j \geq 1 \), we obtain the bound
\[
(4.13) \quad v_{j+1}(x) \leq C \left( W_{1,p} \mu(x) + K_{p,q} \sigma(x) + [W_{1,p} \sigma(x)]^{\frac{p-1}{p-1-q}} \right), \quad x \in \mathbb{R}^n.
\]
Thus, the nondecreasing sequence \( \{v_j\}_{j \geq 1} \) converges to an \( \mathcal{A} \)-superharmonic solution \( u_1 \) of

\[
\begin{aligned}
- \text{div} \mathcal{A}(x, \nabla u_1) &= \sigma|B_1(0)| u_1^q + \mu|B_1(0)|, \quad u_1 \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u_1 &= 0.
\end{aligned}
\]

To construct \( u_2 \) such that \( u_2 \geq u_1 \), let \( w_1 \) be an \( \mathcal{A} \)-superharmonic solution of

\[
\begin{aligned}
- \text{div} \mathcal{A}(x, \nabla w_1) &= \mu|B_2(0)|, \quad w_1 \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} w_1 &= 0.
\end{aligned}
\]

Notice that \( w_1 \) is an a.e. pointwise limit of a subsequence of \( \{w_1^{(k)}\}_{k \geq 1} \), where each \( w_1^{(k)} \) is a nonnegative \( \mathcal{A} \)-superharmonic renormalized solution of

\[
\begin{aligned}
- \text{div} \mathcal{A}(x, \nabla w_1^{(k)}) &= \mu|B_2(0)| \quad \text{in } B_k(0), \\
\sigma|B_2(0)| w_1^{(k)} = 0 \quad \text{on } \partial B_k(0).
\end{aligned}
\]

Again, by [33, Lemma 6.9] we may assume that \( w_1^{(k)} \geq v_1^{(k)} \) for all \( k \geq 1 \), and hence \( w_1 \geq v_1 \).

Next, for any \( j \geq 1 \), let \( w_{j+1} \) be an \( \mathcal{A} \)-superharmonic solution of

\[
\begin{aligned}
- \text{div} \mathcal{A}(x, \nabla w_{j+1}) &= \sigma|B_2(0)| w_{j+1}^q + \mu|B_2(0)|, \quad w_{j+1} \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} w_{j+1} &= 0.
\end{aligned}
\]

Notice that \( w_{j+1} \) is an a.e. pointwise limit of a subsequence of \( \{w_{j+1}^{(k)}\}_{k \geq 1} \), where each \( w_{j+1}^{(k)} \) is a nonnegative \( \mathcal{A} \)-superharmonic renormalized solution of

\[
\begin{aligned}
- \text{div} \mathcal{A}(x, \nabla w_{j+1}^{(k)}) &= \sigma|B_2(0)| w_{j+1}^{(k)} + \mu|B_2(0)| \quad \text{in } B_k(0), \\
w_{j+1}^{(k)} = 0 \quad \text{on } \partial B_k(0).
\end{aligned}
\]

We can ensure here that \( w_{j+1}^{(k)} \geq \max\{v_{j+1}^{(k)}, w_j^{(k)}\} \) for all \( j, k \geq 1 \). Indeed, since \( w_1 \geq v_1 \) and \( w_1 \geq 0 \), by Lemma 4.8 below we may assume that \( w_{j+1}^{(k)} \geq \max\{v_{j+1}^{(k)}, w_j^{(k)}\} \) for all \( k \geq 1 \), and hence \( w_2 \geq \max\{v_2, v_1\} \). Repeating this argument by induction we obtain \( w_{j+1}^{(k)} \geq \max\{v_{j+1}^{(k)}, w_j^{(k)}\} \) for all \( j, k \geq 1 \).

It follows that \( w_{j+1} \geq \max\{v_{j+1}, w_j\} \) for all \( j \geq 1 \). As in (4.13) we have

\[
w_{j+1}(x) \leq C \left( W_{1,p} \mu(x) + K_{p,q} \sigma(x) + \left[ W_{1,p} \sigma(x) \right]^{\frac{p-1}{p-1-q}} \right), \quad x \in \mathbb{R}^n,
\]

and hence \( \{w_j\} \) is a nondecreasing sequence which converges to an \( \mathcal{A} \)-superharmonic solution \( u_2 \) of

\[
\begin{aligned}
- \text{div} \mathcal{A}(x, \nabla u_2) &= \sigma|B_2(0)| u_2^q + \mu|B_2(0)|, \quad u_2 \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x| \to \infty} u_2 &= 0.
\end{aligned}
\]

such that \( u_2 \geq u_1 \), as desired. \( \Box \)
The following lemma invoked in the argument presented above is an extension of \cite{33} Lemma 6.9.

**Lemma 4.8.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and let $\mu_1, \mu_2 \in M_b^*(\Omega)$. Suppose that $u_i \, (i = 1, 2)$ is a renormalized solution of
\begin{equation*}
\begin{cases}
-\text{div} \, A(x, \nabla u_i) = \mu_i & \text{in } \Omega, \\
u_i = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation*}
Then for any measure $\nu \in M_b^*(\Omega)$ such that $\nu \geq \mu_1$ and $\nu \geq \mu_2$, there is a renormalized solution $v$ of
\begin{equation*}
\begin{cases}
-\text{div} \, A(x, \nabla v) = \nu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation*}
such that $v \geq u_1$ and $v \geq u_2 \, a.e.$

**Proof.** For $i = 1, 2$, let $u_{i,k} = \min\{u_i, k\} \, (k = 1, 2, \ldots)$. Then $u_{i,k}$ is the bounded renormalized solution of
\begin{equation*}
\begin{cases}
-\text{div} \, A(x, \nabla u_{i,k}) = \mu_{i0}|\{u_i < k}\} + \lambda_{i,k} & \text{in } \Omega, \\
u_{i,k} = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation*}
Here $\mu_i = \mu_{i0} + \mu_{is} \, (i = 1, 2)$ is the decomposition of $\mu_i$ used in Sec. \S above, where $\mu_{i0}, \mu_{is} \in M_b^*(\Omega), \mu_{i0} << \text{cap}_p$, and $\mu_{is}$ is concentrated on a set of zero $p$-capacity. Moreover, $\lambda_{i,k} \in M_b^*(\Omega)$ and $\lambda_{i,k} \to \mu_{is}$ in the narrow topology of measures as $k \to \infty$ (see Definition \S).

Now let $v_k \, (k = 1, 2, \ldots)$ be a renormalized solution of
\begin{equation*}
\begin{cases}
-\text{div} \, A(x, \nabla v_k) = \sum_{i=1}^2 (\mu_{i0} + \lambda_{i,k}) + (\nu - \mu_1) + (\nu - \mu_2) & \text{in } \Omega, \\
u_k = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation*}
Then by \cite{33} Lemma 6.8 we deduce $v_k \geq \max\{u_{1,k}, u_{2,k}\}$ for all $k \geq 1$. Finally, we use the stability results of \cite{12} to find a subsequence of $\{v_k\}$ that converges a.e. to a desired function $v$. \qed

**References**

[1] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der math. Wissenschaften 314, Berlin-Heidelberg-New York: Springer, 1996.

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez. An $L^1$ theory of existence and uniqueness of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4)* 22 (1995), 241–273.

[3] M.-F. Bidaut-Véron. Removable singularities and existence for a quasilinear equation with absorption or source term and measure data, *Adv. Nonlin. Stud.* 3 (2003), 25–63.

[4] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.* 87 (1989), 149–169.

[5] L. Boccardo and Gallouët, Nonlinear elliptic equations with right hand side measures, *Comm. PDE* 17 (1992), 641–655.

[6] L. Boccardo, T. Gallouët, and L. Orsina. Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 13 (1996), 539–551.
[7] H. Brezis and S. Kamin. *Sublinear elliptic equations in $\mathbb{R}^n$*, Manuscr. Math. **74** (1992), 87–106.

[8] D. T. Cao and I. E. Verbitsky. Finite energy solutions of quasilinear elliptic equations with sub-natural growth terms, *Calc. Var. PDE* **52** (2015), 529–546.

[9] D. T. Cao and I. E. Verbitsky. Nonlinear elliptic equations and intrinsic potentials of Wolff type, *J. Funct. Anal.* **272** (2017), 112–165.

[10] G. Dal Maso and A. Malusa. Some properties of reachable solutions of nonlinear elliptic equations with measure data, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4)* **25** (1997), 375–396.

[11] A. Dall’Aglio. Approximated solutions of equations with $L^1$ data. Application to the H-convergence of quasi-linear parabolic equations, *Ann. Mat. Pura Appl.* **170** (1996), 207–240.

[12] G. Dal Maso, F. Murat, A. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4)* **28** (1999), 741–808.

[13] H. Dong and H. Zhu. Gradient estimates for singular $p$-Laplace type equations with measure data, [arXiv:2102.08584](https://arxiv.org/abs/2102.08584).

[14] F. Duzaar and G. Mingione. Gradient estimates via linear and nonlinear potentials, *J. Funct. Anal.* **250** (2010), 2961–2998.

[15] F. Duzaar and G. Mingione. Gradient estimates via non-linear potentials, *Amer. J. Math.* **133** (2011), 1093–1149.

[16] E. Giusti, *Direct Methods in the Calculus of Variations*, River Edge, NJ: World Scientific, 2003.

[17] L. I. Hedberg and T. H. Wolff. Thin sets in nonlinear potential theory, *Ann. Inst. Fourier (Grenoble)* **33** (1983), 161–187.

[18] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford: Oxford Univ. Press, 1993.

[19] T. Kilpeläinen, T. Kuusi, and A. Tuhola-Kujanpää. Superharmonic functions are locally renormalized solutions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** (2011), 775–795.

[20] T. Kilpeläinen and J. Malý. Degenerate elliptic equations with measure data and nonlinear potentials, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4)* **19** (1992), 591–613.

[21] T. Kilpeläinen and J. Malý. The Wiener test and potential estimates for quasilinear elliptic equations, *Acta Math.* **172** (1994), 137–161.

[22] T. Kilpeläinen and X. Xu. On the uniqueness problem for quasilinear elliptic equations involving measures, *Rev. Mat. Iberoamer.* **12** (1996), 461–475.

[23] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Groningen: P. Noordhoff, 1964.

[24] T. Kuusi and G. Mingione. Linear potentials in nonlinear potential theory, *Arch. Ration. Mech. Anal.* **207** (2013), 215–246.

[25] A. Malusa and M. M. Porzio. Renormalized solutions to elliptic equations with measure data in unbounded domains, *Nonlin. Anal.* **67** (2007) 2370–2389.

[26] V. G. Maz’ya, *Sobolev Spaces, with Applications to Elliptic Partial Differential Equations*, Second, Revised and Augm. Ed., Grundlehren der math. Wissenschaften **342**, Heidelberg: Springer, 2011.

[27] G. Mingione. The Calderón-Zygmund theory for elliptic problems with measure data, *Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (5)* **6** (2007), 195–261.

[28] Q.-H. Nguyen and N. C. Phuc. Good-$\lambda$ and Muckenhoupt-Wheeden type bounds in quasilinear measure datum problems, with applications, *Math. Ann.* **374** (2019), 67–98.
[29] Q.-H. Nguyen and N. C. Phuc. Pointwise gradient estimates for a class of singular quasilinear equation with measure data, *J. Funct. Anal.* **278** (2020), no. 5, 108391, 35 pp.

[30] Q.-H. Nguyen and N. C. Phuc. Existence and regularity estimates for quasilinear equations with measure data: the case $1 < p \leq \frac{4n-2}{3n-2}$, *Analysis & PDE* (in press), arXiv:2003.03725.

[31] Q.-H. Nguyen and N. C. Phuc. A comparison estimate for singular p-Laplace equations and its consequences, arXiv:2202.11318.

[32] N. C. Phuc. A sublinear Sobolev inequality for $p$-superharmonic functions, *Proc. Amer. Math. Soc.* **145** (2017), 327–334.

[33] N. C. Phuc and I. E. Verbitsky. Quasilinear and Hessian equations of Lane-Emden type, *Ann. Math.* **168** (2008), 859–914.

[34] N. C. Phuc and I. E. Verbitsky. Singular quasilinear and Hessian equations and inequalities, *J. Funct. Anal.* **256** (2009), 1875–1906.

[35] A. Seesanea and I. E. Verbitsky. Finite energy solutions to inhomogeneous non-linear elliptic equations with sub-natural growth terms, *Adv. Calc. Var.* **13** (2020), 53–74.

[36] N. S. Trudinger and X. J. Wang. On the weak continuity of elliptic operators and applications to potential theory, *Amer. J. Math.* **124** (2002), 369–410.

[37] I. E. Verbitsky. Bilateral estimates of solutions to quasilinear elliptic equations with sub-natural growth terms, *Adv. Calc. Var.* (in press), http://dx.doi.org/10.1515/acv-2021-0004.

[38] I. E. Verbitsky. Global pointwise estimates of positive solutions to sublinear equations, *St. Petersburg Math. J.* (in press), reprint of *Algebra i Analiz* **34**, no. 3 (2022), 297–330, arXiv:2203.0253.