Diffusion on spatial network

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Abstract. In this work, we study the problem of diffusing a product (idea, opinion, disease etc.) among agents on spatial network. The network is constructed by random addition of nodes on the planar. The probability for a previous node to be connected to the new one is inversely proportional to their spatial distance to the power of $\alpha$. The diffusion rate between two connected nodes is inversely proportional to their spatial distance to the power of $\beta$ as well. Inspired from the Fick’s first law, we introduce the diffusion coefficient to measure the diffusion ability of the spatial network. Using both theoretical analysis and Monte Carlo simulation, we get the fact that the diffusion coefficient always decreases with the increasing of parameter $\alpha$ and $\beta$, and the diffusion sub-coefficient follows the power-law of the spatial distance with exponent equals to $-\alpha-\beta+2$. Since both short-range diffusion and long-range diffusion exist, we use anomalous diffusion method in diffusion process. We get the fact that the slope index $\delta$ in anomalous diffusion is always smaller than 1. The diffusion process in our model is sub-diffusion.

1. Introduction

The lastest decade has witnessed an explosive growth of interest in the description of network structure and dynamical properties [1, 2, 3, 4, 5, 6, 7, 8]. In social networks, there are many rumors, opinions, and diseases etc., which often spread gradually through social networks. People always exchange their ideas through face to face chatting, telephone calls, and emails. All these ways are influenced by spatial structure. Previous works have uncovered that people always make calls and sent emails to someone near them [9, 10]. Infectious diseases are spread because of people interaction and traveling, which are constrained by spatial distance as well [11, 12]. Thus, spatial network is a suitable carrier for us to study such problems. Inspired from the Fick’s law, whether we can get the diffusion coefficient to measure the spreading ability of spatial network is the aim of this paper.

In recent years, spatial networks have attracted more and more attention. Recent findings reveal that the spatial distance distribution of links follows a power law or exponential [9, 13, 14, 15, 16]. These distributions are quite natural since, for instance, people tend to have their friends and relatives in their neighborhood, transportation networks often favor shorter distance trips, and many communication networks are mainly dominated by short radio ranges.
To model these systems, scientists have proposed spatially constrained networks embedded in one-dimensional or two-dimensional space with a set of links constructed according to spatial condition [17, 18, 19, 20, 21, 22]. What’s more, various processes take place on spatial networks, and whose guiding idea focuses on the effects of space. V. Colizza et al. applied the spatio-temporal evolution model to the historical case of the Black Death, which occurred in the 14th century. During these periods, only few traveling paths were available and typical trips were limited to relatively short distances in one day time scale [23]. The main difficulty for the work is estimating the diffusion coefficient $D$. In that paper, they assumed that virus spread at a velocity of around 160 kms per year, and obtained $D \approx 10^4 \text{kms}^2/\text{year}$. But $D$ was still an uncertain parameter. Another striking example can be seen in the SARS outbreak in 2003. From the epidemic, scientists thought that pure local spatial diffusion was not a good model anymore and that the global aspect of transportation network needed to be included in the model. They used metapopulation models to discuss the disease spreading in local and with long-range jumps as well and found out that reducing travel is not an efficient strategies for scale-free networks [23]. Hu et al. studied a malware propagation among WIFI routers and Wang et al. discussed the virus spread using Bluetooth and MMS on the spatially constrained networks [24, 25].

In this paper, we analyze this problem in a stylized model of spatial network. The network is fixed on a two-dimensional space and characterized by a set of agents (nodes) and relationships (links) across these agents through which social influence operates. We therefore consider each agent classified as either “active” or “inactive”. If an agent has the product (idea, opinion, disease etc.), we mark it as an active node and set state value equals to 1, otherwise it is an inactive node and set state value equals to 0. Nodes are connected satisfying the social network property, where spatial distance distribution of links follows power law. Both local diffusion and long-range diffusion are discussed in this paper. The diffusion probability of the product is inversely proportional to the spatial distance between active agent and its inactive neighbor.

In the previous model, the spatial distance is ignored [4, 5, 7, 8]. This model is an universal model, considering the effect of spatial network structure itself only. Applying the Fick’s first law, we define the diffusion coefficient. It is an important parameter indicative of the diffusion mobility. Following this process, we want to reveal the relationship between network structure and its diffusion coefficient. We hope this findings will help us to understand diffusion behavior on social network, such as epidemic spreading, opinion dynamics etc.

We use both theoretical analysis and Monte Carlo simulation method. The numerical results of spatial network are the average of 20 simulations for different realization of networks under the same parameters with the network size of 10000 nodes if not mentioned. The simulation results of diffusion coefficient are the average of at least 1000 different starting configurations, performed on the above 20 simulations. The details of the model are described below.

2. Spatial Driven Network

To work on the spatial network, we first map the network in a two-dimensional space with periodic boundaries. The model are constructed in the following way:

(a) Initial condition : The model starts with an initial state ($t = m_0$) of $m_0 + 1$ all-to-all connected nodes on the $1 \times 1$ planar and each of them has a randomly two-dimensional coordinate $(x, y), (-0.5 \leq x < 0.5, -0.5 \leq y < 0.5)$.

(b) Growth : At every time step, a new node $n$ is randomly located in $(x_n, y_n)$ on the planar.

(c) Addition of edges : Following ideas proposed by previous work [19], the new node $n$ connects with $m$ ($m \leq m_0 + 1$) previous nodes. The previous node $i$ is selected by the probability $\pi_i$. 
$$\pi_i = \frac{l_i^{-\alpha}}{\sum_j l_{nj}^{-\alpha}} (0 \leq \alpha), \quad (1)$$

$l_{ni}$ is the spatial distance between new node $n$ and previous node $i$, $\sum_j$ is the sum for all the previous nodes. The growing processes repeat step (b) and (c) until the network reaches the desired size. Accordingly, at each step, the number of nodes increases by one, while the number of edges increases by $m$ ($m = m_0 = 4$ in what follows if not mentioned). There are two limiting cases for the present model: when $\alpha = 0$, the network reduces to the random growing process; when $\alpha \to \infty$, the network is close to the model presented by Ozik et al. (OHO model) [18]. As $\alpha$ increases, the new node prefers to connect with nearer nodes. For different $m$ and $\alpha$, the degree distribution always grows exponentially and is determined by $m$ as follows [22]:

$$p(k) = \frac{1}{m+1} \left( \frac{m}{m+1} \right)^{k-m}, \quad (k \geq m). \quad (2)$$

3. Diffusion on spatial network

3.1. Spatial Diffusion Mechanism

The spatial network is a suitable frame to discuss the human spreading dynamics. Based on some real social communication networks, we find that the communication rate is inversely proportional to the spatial distance [9, 10]. Arouse from the previous works [12, 26, 27], we define the spatial diffusion mechanism: the diffusion rate $\lambda_{sj}$ from the active agent $s$ to its inactive neighbor $j$ is inversely proportional to their spatial distance as follows

$$\lambda_{sj} = \left( \frac{\min_{k \in \Omega_s} l_{sk}}{l_{sj}} \right)^\beta, \quad (3)$$

where $\Omega_s$ is the set of all inactive neighbors of node $s$. The distance between active agent and its closest neighbor is used to rescale $\lambda_{sj}$, changing from 0 to 1. Therefore, for the closest inactive neighbor, the diffusion rate is 1; for other inactive neighbors, the diffusion rate is inversely proportional to the spatial distance from $s$ to them with power $\beta$. We assume $\beta$ is the parameter of diffusion product itself. For smaller $\beta$, the product is easy to diffuse; for larger $\beta$ it is difficult.

3.2. Diffusion Coefficient

Fick’s first law relates the diffusive flux to the concentration under the assumption of steady state [28]. It postulates that the flux goes from regions of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient (spatial derivative), which can be written as:

$$J = -D \nabla c, \quad (4)$$

where $D$ is the diffusion coefficient, $J$ is the diffusion flux and $c$ is the concentration. In spreading networks, the product spreads from active agent to inactive one. When $N$ agents are randomly distributed on $1 \times 1$ two-dimensional space, the density of nodes per unit area is $N$. The state density of the agent is equal to its density multiply its state. For an active agent, the state density is $c = N \times 1 = N$. For inactive agent, the state density is $c = N \times 0 = 0$. The product spreads from high state density agent to low state density agent. So we try to use Fick’s first law to solve this problem. In order to get the diffusion coefficient, we make two hypothesis: the diffusion coefficient of the network is represented by its average values on different agents; it is unrelated to the number of active agents. Thus, in the following discussion, we only set one agent to active state and calculate the diffusion coefficient in one time step diffusion.
At first, we select an agent $s$ randomly as an active seed, set $s$ to the central of the planar and rescale all the nodes. As Figure 1(a) shows, the red node $s$ in the central represents the active agent, and the circles represent inactive agents. Using spatial diffusion mechanism of Eq. (3) and the Fick’s first law of Eq. (4), the diffusion coefficient for node $s$ is given by

$$D_s = \sum_{j \in \Omega_s} \lambda_{sj} \frac{l_{sj}}{N} = \sum_{j \in \Omega_s} \frac{l_{sj}^{-\beta+1}}{(\min_{k \in \Omega_s} l_{ks})^{-\beta} N}. \quad (5)$$

The diffusion flux in this model is the number of successful diffusion from seed $s$. Figure 1(b) is the result of one step spatial diffusion on Figure 1(a).

Then we set agent $s$ as the center and divide the planer to $M$ rings, labeled as $n = 1, 2, \ldots, M$, to calculate $D$ in simulation method (in Fig. 1(c)). The distance between the adjacent two rings is $\frac{1}{2M}$. The distance from the seed to the $n$th ring is $l_n^{M} = \frac{n-\frac{1}{2}}{M}$, which is the average distance from the seed to the two adjacent circles. We assume that the distance between the seed and the nodes in the $n$th ring has the same value and equals to $l_n^{M}$. Since all the agents are randomly located on the planar, the number of agents in the $n$th ring is $N^{M}(n) = \frac{\pi (n^2-(n-1)^2)}{4M^2}N$. The approximation diffusion coefficient of the spatial network is the sum of sub-coefficient for all the parts

$$D_s^{M} = \sum_{n=1}^{M} d_s^{M}(n) + d_{out}^{M} = d_1^{M}(1) + d_2^{M}(2) + \ldots + d_M^{M}(M) + d_{out}^{M}. \quad (6)$$

$d_{out}^{M}$ is the diffusion sub-coefficient of the nodes in the planar but out of the $n$th ring and $l_{out} = \frac{1}{2}$. $d_s^{M}(n)$ is the diffusion sub-coefficient contributed from the nodes in the $n$th ring, satisfying

$$d_s^{M}(n) = -\frac{J_M(n)}{Vc(n)}. \quad (7)$$

In Figure 1(c), $M = 2$, $J^2(1) = 3$, $J^2(2) = 1$, $N = 10$, $l_1^2 = \frac{1}{8}$ and $l_2^2 = \frac{3}{8}$. According to Eq. (7), $d^2(1) = \frac{3}{30}$, $d^2(2) = \frac{1}{30}$, $D_2^2 = \frac{4}{30}$. When $M \to \infty$, $D_s^{M} \to D_s$. 

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**Figure 1.** (a) Example of a network and its original state. (b) One step diffusion on the network (a). (c) Divide the network into some rings to calculate the diffusion coefficient in simulation method.
According to the spatial network evolution mechanism [22], the number of agents, which connect to the seed, in the nth ring is

\[ N^M_C(n) \sim \frac{N^M(n)(l^M_n)^{-\alpha}}{2\pi N(n-\frac{1}{2})^{-\alpha+1}} \sim \frac{1}{(2M)^{\alpha+2}}. \] (8)

Thus, the diffusion flux from the seed to the nth ring satisfies

\[ J^M(n) \sim \frac{N^M_C(n)(l^M_n)^{-\beta}}{2\pi N(n-\frac{1}{2})^{-\alpha-\beta+1}} \sim \frac{1}{(2M)^{\alpha-\beta+2}}. \] (9)

Applying Eq. (4), the diffusion sub-coefficient \( d \) of the nth ring is affected by \( n \):

\[ d^M(n) \sim (n-\frac{1}{2})^{-\alpha-\beta+2}. \] (10)

For large \( M \), each ring has less than two nodes. We use Eq. (4) to calculate \( d \) for each of the nodes, and \( D_s \) follows

\[ D_s = -\sum_{p \in \Theta_s} \frac{1}{l_{sp}^w}, \] (11)

where \( \Theta_s \) is the set of active neighbors, which diffuses from seed \( s \). This process goes for \( N \) times, in each of the process, seed \( s \) is selected randomly. According to the Ergodic hypothesis, the diffusion coefficient \( D \) of the network is the average of \( D_s \).

### 3.3. Numerical Results

For the traditional Fick’s first law, the diffusion coefficient depends on the temperature, viscosity of the fluid and the size of the particles [28]. Now we are looking for which property affects the diffusion coefficient in spatial network. Figure 2 shows the relationship between \( D \) and parameter \( \alpha \). We compare three kinds of results for \( \beta = 0.5 \) and 1: the first one is the simulation result from Eq. (11) on spatial driven network (marked as ‘spatial network’); the second one is the simulation result from Eq. (11) on weighted scale-free network, the network grows following scale-free network rule [1] with the same average degree as the first one, and links are weighted by spatial distance from the first one (marked as ‘scale-free network’); the third one is the analytical result from Eq. (5) with \( M = 100 \) on the same spatial driven network (marked as ‘analytical result’). The simulation result matches the analytical result. In Figure 2, we also find that the scale-free network and spatial driven network have the same diffusion coefficient. These two networks have the same spatial distance distribution of links but other various properties, such as degree distribution, clustering coefficient, shortest path length. Thus, the diffusion coefficient is determined by the spatial distance distribution of links. \( D \) decreases with the increasing of \( \alpha \). We have done it as well in one-dimensional line (\( w = 1 \)), which has the similar relationship. In Figure 3, the left and the lower coordinates correspond to one-dimensional case (\( w = 1 \)), and the right and the upper coordinates correspond to the two-dimensional case (\( w = 2 \)), the dash line satisfies \( \alpha = w \). When \( \alpha \) is less than the network dimension \( w \), \( D \) decreases quickly; when \( \alpha \) is greater than the network dimension, it decreases slowly.

\( \beta \) is the parameter of spatial diffusion mechanism corresponding to the property of diffusion particles in Fick’s first law. In Figure 4, we discuss how \( D \) is affected by parameter \( \beta \) on different spatial driven networks with \( \alpha = 0.5 \) and 1. For different networks, \( D \) always decreases with the
Figure 2. Relation between the diffusion coefficient $D$ and parameter $\alpha$ for $\beta = 0.5, 1$. $D$ is calculated in three different ways.

Figure 3. Diffusion coefficient $D$ as a function of parameter $\alpha$ for $w = 1$ and 2 with $\beta = 1$. The left and the lower coordinates correspond to the one-dimension ($w = 1$) line network space, and the right and the upper coordinates correspond to the two-dimension ($w = 2$) plane network space. The dash line follows $\alpha = w$. 
Figure 4. Diffusion coefficient $D$ as a function of parameter $\beta$ for $\alpha = 0.5$ and $1$. The dashed line has a slope $-1$.

Increasing of $\beta$, which we have got from Eq. (5) as well. When $\beta$ is larger than 1, $D$ decreases following the power of $\beta$ with exponent $-1$, as Figure 4 shows. In Figure 5, the points suggest that, for $\alpha = 1, 2$ and $3$, diffusion sub-coefficient $d$ scales as $d \sim (n - \frac{1}{2})^\gamma$. The variation of the exponent $\gamma$ with $n - \frac{1}{2}$ is shown in Figure 6. The error bars are determined by the fitting with $d \sim (n - \frac{1}{2})^\gamma$. The results suggest that approximately

$$\gamma = -\alpha - \beta + 2$$  \hspace{1cm} (12)

Figures 5 and 6 show consistently between the results of numerical simulation and analytical calculation from Eq. (10) for different values of $\alpha$ and $\beta$. When $\alpha < -\beta + 2$, long-range connection neighbors are in the main role of diffusion coefficient; when $\alpha > -\beta + 2$, short-range connection neighbors are in the main role.

4. Anomalous diffusion on spatial network

Normal diffusion has as basic characteristic the linear scaling of the mean square displacement of the particles with time, $\langle l^2 \rangle \sim Dt$ [29]. Many different experiments though, including the one shown in the previous section, reveal deviations from normal diffusion, in that diffusion is either faster or slower, and which is termed anomalous diffusion [29, 30]. A useful characterization of the diffusion process is again through the scaling of the mean square displacement with time, where though now we are looking for a more general scaling of the form

$$\langle l^2(t) \rangle \sim t^\delta.$$  \hspace{1cm} (13)

Diffusion is then classified through the scaling index $\delta$. The case $\delta = 1$ is normal diffusion, all other cases are termed anomalous. The case $\delta > 1$ forms the family of super-diffusive processes, and the case $\delta < 1$ is sub-diffusion process. Since the long-range connections exist in the spatial network, we try to use the anomalous diffusion model to discuss the diffusion process on the
Figure 5. Diffusion sub-coefficient $d$ as a function of $n - \frac{1}{2}$ for $\alpha = 1, 2, 3$ and $\beta = 1$.

Figure 6. $\alpha$ dependence of the slope $\gamma$ for $\beta = 0.5, 1$. Points are best fit estimates to the simulation data. The straight lines are plots of Eq. (12).
Figure 7. The variation of mean squared displacement for $3 \times 3$ expand spatial driven network with different $\alpha$. inset: The variation of mean squared displacement for $1 \times 1$ spatial driven network with different $\alpha$. The full line satisfies $\langle l^2 \rangle = 0.1667$; the dash line has a slope 1. The nodes represent the ensemble average results for 1000 simulations on a network sample.

spatial network. At first, we select an agent $s$ randomly as an active seed, set $s$ to the central of the planar and rescale all the nodes. The diffusion probability $\lambda_{sj}$ from the active agent $s$ to its inactive neighbor $j$ is inversely proportional to their spatial distance and follows Eq. (3) as well. At each time step, only one inactive neighbor changes its state. For the active agent, it changes its state at the same time. Thus at every time step, there is only one active agent in the network. At time $t$, we mark the active agent as $s_t$. $l(t)$ is the Euclidean distance between active seed $s$ and active agent $s_t$. The variation of mean squared displacement for different spatial driven network with $\alpha = 1, 2, 3, 4$ and 5 is shown in Figure 7 inset. The active agent diffuses from the active seed constraining by diffusion probability. Since the limited space effect, when diffusion time $t$ is large enough, the product with diffuse to any random place on the planar. The limited value of mean squared displacement calculates as

$$\int_{-0.5}^{0.5} \int_{-0.5}^{0.5} (x^2 + y^2) dx dy \approx 0.1667.$$ (14)

The diffusion process only works before the network approaches the limited space effect. The active time increases with the increasing of $\alpha$. When $\alpha = 1$, the active time is less than 20 time steps. In order to reduce the limited space effect, we expand the network to $3 \times 3$ range with periodic boundary condition. The variation of mean squared displacement for different expand spatial driven network is shown in Figure 7. The mean squared displacement of the active agent is a power law function of time as Eq. (13), whose slope $\delta$ is determined by the network parameter $\alpha$. In normal diffusion process, $\delta = 1$.

The relationship between slope $\delta$ and network parameter $\alpha$ is shown in Figure 8. The error bars are determined by the fitting with Eq. (13) in diffusion process. For all the $\alpha$, slope $\delta \leq 1$. The diffusion process on spatial network with $\alpha < 3$ is sub-diffusion. $\delta$ increases with the increasing of $\alpha$ and reaches to 1 (normal diffusion).

In order to reveal the spatial network effect on spatial diffusion, we do realizations of the anomalous diffusion on different spatial networks with $\alpha = 1$ and 4. Figure 9 shows the...
Figure 8. The slope $\delta$ dependence of the parameter $\alpha$. The straight line satisfies $\delta = 1$ for normal diffusion.

Figure 9. Two trajectories of anomalous diffusion on spatial network (a) $\alpha = 1$ (b) $\alpha = 4$ with $N = 10000$. Both trajectories are simulation results for 100 time steps.

comparison of 100 steps simulation trajectories of anomalous diffusion on spatial network with $\alpha = 1$ and spatial network with $\alpha = 4$. When $\alpha = 1$, the network structure is beneficial for diffusion, existing a lot of long-range diffusion. In 100 time steps, the product has diffused on the whole planar. On the contrary, when $\alpha = 4$, it is normal diffusion. The network structure is disadvantage for diffusion, only short-range diffusions are existed. The product diffuses very slowly.
5. Summary
To summarize, we use the Fick’s first law and anomalous diffusion to solve the diffusion dynamics on spatial networks. In the method of Fick’s first law, we introduce a composite diffusion coefficient $D$ in order to measure the diffusion ability of the spatial driven networks. Since our diffusion is not normal in general due to the long-range spreading, this coefficient must take into account the transfers of diffusive substances over all distances. We get diffusion coefficient $D$ using both of theoretical method and simulation method and find that $D$ is determined by spatial distribution of links. In simulation method, we divide the planer to $M$ rings, and apply the Fick’s first law on each ring to get sub-coefficient $d$. For larger $M$, according to the simulation method, $D$ is the sum of diffusion sub-coefficient for all the rings. When $\alpha < w$, where $w$ is the topological dimension of the space, the diffusion coefficient decreases quickly with the increasing of $\alpha$. On the contrary, it decreases slowly. For different diffusion mechanism, $D$ always decreases with the increasing of $\beta$, where $\beta$ is the parameter of diffusion mechanism. When $\beta > 1$, $D$ decreases following the power of $\beta$ with exponent $-1$. Finally, the diffusion sub-coefficient $d(n)$ is determined by the network structure and position of the ring, which follows $d(n) \sim (n - 1)^{-\beta+2}$, where $n$ is the label of the ring. In the method of anomalous diffusion, the diffusion probability is inversely with the Euclidean distance between two connected agents as well. From the variation of mean squared displacement, we get that the slope index $\delta$ in anomalous diffusion increases with the increasing of $\alpha$ and then reaches to 1. The diffusion process in our model is always sub-diffusion.

In anomalous diffusion, the diffusion process is characterized through the scaling of the mean square displacement with time. This is a mature method to discuss the diffusion process. But it works only on the traditional anomalous diffusion process. Once the scaling property of the mean square displacement with time does not exist, the anomalous diffusion method does not work. In this case, the method of Fick’s law is a good choice to discuss the diffusion process. In section 3.1, the diffusion dynamics works on more than one pair of agents. In this condition, the mean square displacement does not easy to get. Thus, using the Fick’s law on all pairs of agents is a more suitable method. At this point we must emphasize that our approach represents a attempt to use the Fick’s first law and anomalous diffusion on complex network. For the future, we will try to apply the diffusion coefficient on different network dynamic processes, such as opinion dynamic (Voter model, Sznajd model), spreading dynamic (SIS model, SIR model) et al.

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References
[1] Barabási A L and Albert R 1999 Science 286 509
[2] Watts D J and Strogatz S H 1998 Nature 393 440
[3] Donetti L, Hurtado P I and Muñoz M A 2005 Phys. Rev. Lett. 95 188701
[4] Sznajd-Weron K 2005 Acta Phys. Pol. B 36 2537
[5] Sznajd-Weron K and Sznajd J 2000 Int. J. Mod. Phys. C 11 1157
[6] Santos F C, Rodrigues J F and Pacheco J M 2005 Phys. Rev. E 72 056128
[7] Pastor-Satorras R and Vespignani A 2001 Phys. Rev. E 63 066117
[8] Pastor-Satorras R and Vespignani A 2002 Phys. Rev. E 65 036104
[9] Lambiotte R, Blondel V D, Kerchove de C, Huens E, Prieur C, Smoreda Z and Dooren Van P 2008 Physica A 387 5317
[10] Goldenberg J and Levy M 2009 arXiv: 0906.3202
[11] Barrat A, Barthélémy M, Pastor-Satorras R and Vespignani A 2004 Proc. Natl. Acad. Sci. 101 3747
[12] Hui Z, Cai X, Greneche J M, Wang Q A 2012 Int. J. Mod. Phys. C 23 1250082
[13] Barrat A, Barthelemy M, and Vespignani A 2005 J. Stat. Mech. P05003
[14] Liben-Nowell D, Novak J, Kumar R, Raghavan P and Tomkins A 2005 Proc. Natl. Acad. Sci. 102 11623-8
[15] Yook S-H, Jeong H and Barabási A-L 2002 Proc. Natl. Acad. Sci. 99 13382
[16] Hayashi Y 2006 IPSJ Trans. Special Issue on Network Ecology 47 (3) 776-85
[17] Sen P and Manna S S 2003 Phys. Rev. E 68 026104
[18] Ozik J, Hunt B R and Ott E 2004 Phys. Rev. E 69 026108
[19] Kosmidis K, Havlin S and Bunde A 2008 Europhys. Lett. 82 48005
[20] Manna S S and Sen P 2002 Phys. Rev. E 66 066114
[21] Xulvi-Brunet R and Sokolov I M 2002 Phys. Rev. E 66 026118
[22] Hui Z, Li W, Cai X, Greneche J-M and Wang Q A 2013 Physica A 392 1909
[23] Colizza V, Barthelemy M, Barrat A and Vespignani A 2007 C. R. Biologies 330 364
[24] Hu H and Myers S, Colizza V and Vespignani A 2009 Proc. Natl. Acad. Sci. 106 1318
[25] Wang P, Gonzalez M, Hidalgo C A, and Barabasi A-L 2009 Science 324 1071
[26] Sun P, Cao X B, Du W B and Chen C L 2010 Phys. Pro. 3 1811
[27] Guo W P, Li X and Wang X F 2007 Physica A 380 684
[28] Crank J 1975 The mathematics of diffusion 2ed, Oxford
[29] Vlahos L, Isliker H, Kominis Y and Hizanidis K 2008 arXiv 0805.0419
[30] Metzler R and Klafter J 2000 Physics Reports. 339 1