On the Casimir of the group $ISL(n, R)$ and its algebraic decomposition

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Abstract

In this paper, an explicit expression for the Casimir operator (or the Casimir invariant) of the inhomogeneous group $ISL(n, R)$ in its enveloping algebra is proposed, which using contractions of the tensorial indices of the generating operators $P^\rho$ and $E^\nu_{\mu}$ may be presented in the following (slightly more comprehensible as equation (1)) form. The Casimir is obtained by symmetrizing this expression. This tensor form is useful in the classification of particles in affine gravitational gauge theories; such as that based on $ISL(4, R)$. It is also proven that the Casimir of $ISL(n, R)$ can be decomposed in terms of the Casimirs of its little groups, a key point in the posterior construction of its irreducible representations.

1. Introduction

The special affine group $ISL(n, R)$ is the semidirect product of the Abelian

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group of translations in $n$ dimensions and the special linear, $SL(n, R)$. The $ISL(4, R)$ group has been chosen as the gauge group in gauge theories of gravity [1]; therefore, the knowledge of its Casimirs will be necessary not only to investigate the irreducible representations of this group, but also to provide these theories with a wave equation.[2] It is has been conjectured that a gauge theory for quantum gravity can be developed by enlarging the Poincare group to the group $IGL(4, R)$. However, the lack of invariants of this group[3], prevents to classify the elementary particles of a theory based on that gauge group. Therefore, $ISL(4, R)$ is the best selection[1][9][8]. The group $ISL(n, R)$ has a subgroup, the Poincare group, from which stems its importance in physics. The eigenvalues of the Casimir of the group $ISL(n, R)$ provide quantum numbers to classify the particles of these theories in the same way that the eigenvalues of the Casimirs of the Poincare groups allow us to classify the particles according to their mass and spin. The eigenvalues of the Casimir of $ISL(n, R)$ label the irreducible representations of the group. The invariants are also useful ingredients in the decomposition of reducible representations into irreducible ones. In the case of gauge theories of gravity based on $ISL(n, R)$, it is important to decompose the unitary irreducible representations of the group $ISL(n, R)$ into the unitary irreducible representations of the Poincare subgroup. This would bring a physical insight into the behavior of the elementary particles of these theories. In section 2, we construct the formula for the Casimir of $ISL(n, R)$. In section 3, we discuss
the induction proof used to guarantee the general validation of the formula for the Casimir of ISL\((n, R)\). Finally in section 4, the algebraic decomposition of ISL\((n, R)\) is achieved.

2. Construction of the Formula for the Casimir of ISL\((n, R)\)

In Ref. 3 it is proved that the group ISL\((n, R)\) has one invariant. And in Ref. 4 it is proved that the order of this invariant is \(\frac{1}{2}n(n + 1)\). Based on this proof, the standard procedure for constructing invariants by contracting tensorial indices with the Levi Civita antisymmetric pseudo tensor and the generators of the Lie group \([5]\), we found a formula for the invariant of ISL\((n, R)\). This expression is given by

\[
\text{Casimir}_{\text{ISL} (n, R)} = \left\{ \xi_{0}^{\rho_{1}}[\theta_{11}] \rho_{2} \ldots [\theta_{n-2,1}, \ldots, \theta_{n-2,n-2}] \rho_{n-1} \right\} P^{\xi_{0}} P^{\rho_{1}}
\]

\[
\ldots P^{\rho_{n-1}} E^{\beta_{1}}_{[\theta_{11}] \rho_{2}} E^{[\beta_{2}(\gamma_{11})]}_{\left\{ \theta_{11} \right\} \rho_{2}} \ldots E^{[\beta_{n-1}(\gamma_{n-2,1}, \ldots, \gamma_{n-2,n-2})]}_{\left\{ \theta_{n-2,1}, \ldots, \theta_{n-2,n-2} \right\} \rho_{n-1}} \}
\]

\[\text{symmetrized} \quad (1)\]

\[\xi_{0}, \theta_{ij}, \gamma_{ij}, \alpha_{i}, \rho_{k}, \beta_{m} = 0, 1, \ldots, n - 1 \quad i, j = 1, 2, \ldots, n - 2 \quad i, k, m = 1, 2, \ldots, n - 1\]

where

\[E^{[\beta_{2}(\gamma_{11})]}_{\left\{ \theta_{11} \right\} \rho_{2}} = E^{\beta_{2}}_{\theta_{11}} E^{\gamma_{11}}_{\rho_{2}}\]

and

\[E^{[\beta_{n-1}(\gamma_{n-2,1}, \ldots, \gamma_{n-2,n-2})]}_{\left\{ \theta_{n-2,1}, \ldots, \theta_{n-2,n-2} \right\} \rho_{n-1}} = E^{\beta_{n-1}}_{\theta_{n-2,1}} E^{\gamma_{n-2,1}}_{\theta_{n-2,2}} \ldots E^{\gamma_{n-2,n-3}}_{\theta_{n-2,n-2}} E^{\gamma_{n-2,n-2}}_{\rho_{n-1}},\]
where \( \zeta \cdots \) is given by the expression

\[
\zeta \cdots = \epsilon_{\xi_0 \beta_1 \cdots \beta_{n-1}} (\delta_{\rho_1 \alpha_1} \cdots \delta_{\rho_{n-1} \alpha_{n-1}})((\delta_{\theta_{11} \gamma_{11}})(\delta_{\theta_{21} \gamma_{21}} \delta_{\theta_{22} \gamma_{22}}) \cdots
\]

\[
(\delta_{\theta_{n-2,1} \gamma_{n-2,1}} \cdots \delta_{\theta_{n-2,n-2} \gamma_{n-2,n-2}})).
\]

\( (2) \)

The \( E_\alpha^\beta \) are the generators of the general linear group \( GL(n, R) \) in the Weyl basis[2] and the \( P_\alpha \) are the generators of the Abelian subgroup of \( ISL(n, R) \).

In formula 1, the following substitution must be carried out

\[
E_\alpha^\alpha = E_\alpha^\alpha - E_{\alpha+1}^\alpha.
\]

\( (3) \)

The commutation relations of \( ISL(n, R) \), are given then

\[
[P_\rho, P_\nu] = 0 \quad [P_\rho, E_\mu^\nu] = \delta_{\rho \mu} P_\nu \quad [E_\nu^\mu, E_\lambda^\tau] = \delta_{\nu \lambda} E_\mu^\tau - \delta_{\mu \tau} E_\nu^\lambda
\]

\( (4) \)

The \( E_\alpha^\alpha \), the generators of the general linear group \( GL(n, R) \), are substituted by the traceless generators \( E_\alpha^\alpha \) of the special linear group \( SL(n, R) \). In equation (2), we define \( \epsilon_{\xi_0} = 1 \).

3. The Induction Proof

In Refs. 6, it is proved that the invariant of \( ISL(n, R) \) can be obtained by solving a system of linear first order partial differential equations (LFPDE). The system of (LFPDE) can trivially be solved for \( n = 1 \). Thus, the first
part of the induction is proven. In order to prove the second part of the induction method, we assume the formula is valid for \( n = k \) and then prove that it is valid for \( n = k + 1 \).

We can construct a scalar of the order required by Lemma 1, in Ref. 4, to be the invariant of \( ISL(k + 1, R) \). According to this Lemma, the order of the invariant in the generators of this group should be \( \frac{1}{2}(k + 1)(k + 2) \). In this same reference, it is proven that the invariant for this group would be of \( k + 1 \) order in the generators of the translations. Therefore the invariant for \( ISL(k + 1, R) \) must be of \( k + 1 \) order in the translations and of \( \frac{1}{2}k(k + 1) \) order in the non-translations generators of the group \( ISL(k + 1, R) \). That is, the invariant of \( ISL(k + 1, R) \) must be different from the invariant for \( ISL(k, R) \) by a factor given by

\[
\mathcal{P}^{\alpha_k} E_{\left[ \beta_k(\gamma_{k-1,1}, \ldots, \gamma_{k-1,k-1}) \right]}_{\left[ \left[ \theta_{k-1,1}, \ldots, \theta_{k-1,k-1} \right] \rho_k \right]}. \tag{5}
\]

If we take into account the form of the invariants of \( ISL(n, R) \) for \( n=1,2,3,4 \), the formula we are assuming valid for \( n = k \) and the factor given above, we can construct a scalar given by

\[
\text{Casimir}_{ISL(k + 1, R)} = \left\{ \mathcal{P}_{\xi_0} \mathcal{P}_{\rho_1} \ldots \mathcal{P}_{\rho_k} \left[ \beta_k(\gamma_{k-1,1}, \ldots, \gamma_{k-1,k-1}) \right] \left[ \left[ \theta_{k-1,1}, \ldots, \theta_{k-1,k-1} \right] \rho_k \right] \right\} \mathcal{P}_{\theta_1} \ldots \mathcal{P}_{\theta_k} \mathcal{P}_{\rho_k}.
\]
This formula coincides with the eqn. (1) for \( n = k + 1 \). However, the proof is not yet complete, since the scalar that we have constructed to be the invariant of \( ISL(k + 1, R) \) could be zero. Therefore, we must prove that the scalar given by eqn. (5) is not zero.

4. The Algebraic Decomposition of \( ISL(N, R) \)

The proof that the scalar given by eqn. (5) does not vanish is based on an algebraic decomposition of the Casimir of \( ISL(n, R) \) in terms of the Casimirs of its little groups. This decomposition allows an immediate classification of the existent particles in a theory based on \( ISL(n_1, R) \), with \( n_1 \) any number.\(^8\)

Then making all the translations equal zero except \( P^0 \) in eqn. (5). That is,

\[
\xi_0 = \alpha_1 = \ldots = \alpha_k = 0,
\]

therefore

\[
\rho_1 = \rho_2 = \ldots = \rho_k = 0.
\]
Hence, the Casimir with all the translations zero except $P^0$ is given by

\[ \text{Casimir}_{\text{ISL}}(k + 1, R) = \]

\[ \{(P^0)^{k+1} \zeta_{00, \ldots, 00(\theta_{11})0, \ldots, (\theta_{k-1,1}, \ldots, \theta_{k-1,k-1})0} \}
\]

\[ E^\beta_{\theta_{11}} E^\gamma_{\theta_{22}} \cdots E^\gamma_{\theta_{k-1,k-1}} \}
\]

\[ \text{symmetrized} \] \hspace{1cm} (7)

where

\[ \zeta_{00, \ldots, 00(\theta_{11})0, \ldots, (\theta_{k-1,1}, \ldots, \theta_{k-1,k-1})0} = \epsilon_{\beta_1 \cdots \beta_k} (\delta_{\theta_{11} 11} \delta_{\theta_{22} 22} \cdots \delta_{\theta_{k-1,k-1} k-1,k-1}) \]

\[ (\delta_{\theta_{21} 21} (\delta_{\theta_{31} 31} \delta_{\theta_{32} 32}) \cdots (\delta_{\theta_{k-1,1} 1,1} \cdots \delta_{\theta_{k-1,k-1} k-1,k-1})). \] \hspace{1cm} (8)

The terms of eqn. (6) with the $\theta'$s = 0 generated by the contraction of the Levi-Civita pseudo tensor cancel out by antisymmetry. Therefore, the indices $\beta, \theta, \gamma$ can be shifted; instead of running from 0, 1, \ldots, $k$ they will run from 0, 1, \ldots, $k - 1$. This defines an isomorphism between the subset of the generators, belonging to the factor which multiply $P^{k+1}$ in eqn. (6), of the Lie algebra of $\text{ISL}(k + 1, R)$ and the Lie algebra of $\text{ISL}(k, R)$.
Hence eqn. (7) can be given by

\[
\zeta_{11}[\theta_{21}\theta_{22}],...,[\theta_{k-1,1},...\theta_{k-1,k-2})\theta_{k-1,k-1}]
\zeta_{21,721,722,...,7k-1,k-1,2}[\theta_{3}(721),...][\theta_{k}(7k-1,1,...,7k-1,k-2)]
= \epsilon_{\beta_{1}...\beta_{k}}(\delta_{\theta_{11},\gamma_{11},\gamma_{22}},...,\gamma_{k-1,k-1})
\]

\[
((\delta_{\theta_{21},721})(\delta_{\theta_{31},731},\delta_{\theta_{32},732})...((\delta_{\theta_{k-1,1},7k-1,1},...,\delta_{\theta_{k-1,k-2},7k-1,k-2}))
\]

(9)

The basis elements of the Lie algebra of the group \text{ISL}(n, R) can be represented by the \(n+1\) by \(n+1\) matrices given below:

\[
\begin{pmatrix}
SL(n, R) & P \\
0 & 0
\end{pmatrix}
\]

where \(P\) are the generators of the group of translations, and \(SL(n, R)\) are the generators of the special linear group in \(n\) dimensions. Therefore, the \(E_{\alpha}^{0}\) generators of the eqn. (6) can be considered as the translations \(P_{\alpha}\) generators of the little group \(ISL(k, R)\) of \((p_{0}^{0}, 0, 0, \ldots, 0_{k})\).[7]

Using eqn. (8), eqn. (6) can be written in the following form:

\[
\text{Casimir}_{\text{ISL}}(k+1, R) =
\]

\[
\{(P_{0})^{k+1}\zeta_{11}[\theta_{21}\theta_{22}],...,[\theta_{k-1,1},...\theta_{k-1,k-2})\theta_{k-1,k-1}]
\zeta_{21,721,722,...,7k-1,k-1,2}[\theta_{3}(721),...][\theta_{k}(7k-1,1,...,7k-1,k-2)]
\epsilon_{\beta_{1}...\beta_{k}}(\delta_{\theta_{11},\gamma_{11},\gamma_{22}},...,\gamma_{k-1,k-1})
\]

\[
E_{\gamma_{11}}^{0}E_{\gamma_{k-1,k-1}}^{0}E_{\gamma_{11}}^{0}...E_{\gamma_{11}}^{0}
\]

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To take advantage of the isomorphism given above, we make the following substitution:

\[ \beta_1 \rightarrow \xi_0, \beta_2 \rightarrow \beta_1, \beta_3 \rightarrow \beta_2, \ldots, \beta_k \rightarrow \beta_{k-1} \]

\[ \theta_{11} \rightarrow \rho_1, \theta_{22} \rightarrow \rho_2, \ldots, \theta_{k-1,k-1} \rightarrow \rho_{k-1} \]

\[ \gamma_{11} \rightarrow \alpha_1, \gamma_{22} \rightarrow \alpha_2, \ldots, \gamma_{k-1,k-1} \rightarrow \alpha_{k-1} \]

\[ \theta_{21} \rightarrow \theta_{11}, (\theta_{31} \rightarrow \theta_{21}, \theta_{32} \rightarrow \theta_{22}), \ldots, (\theta_{k-1,1} \rightarrow \theta_{k-2,1}, \theta_{k-1,2} \rightarrow \theta_{k-2,2}, \ldots, \theta_{k-1,k-2} \rightarrow \theta_{k-2,k-2}) \]

\[ \gamma_{21} \rightarrow \gamma_{11}, (\gamma_{31} \rightarrow \gamma_{21}, \gamma_{32} \rightarrow \gamma_{22}), \ldots, (\gamma_{k-1,1} \rightarrow \gamma_{k-2,1}, \gamma_{k-1,2} \rightarrow \gamma_{k-2,2}, \ldots, \gamma_{k-1,k-2} \rightarrow \gamma_{k-2,k-2}) \]

and by substituting into eqn. (9), we obtain
\[ \text{Casimir}_{ISL}(k+1, R) = \{ (P^0)^{k+1}(\text{Casimir}_{ISL}(k, R)) \}_{\text{symmetrized}} \]  \quad (11)

We have obtained the Casimir of the little group \( ISL(k, R) \) from the Casimir of the group \( ISL(k+1, R) \). We arrive at the same result if we take any of the other translations.

From the above discussion, it is clear that the Casimir of \( ISL(k+1, R) \) given by equation (5) does not vanish, as claimed. This completes the induction proof. We conclude that the formula given by equation (1) is valid for any integer \( n \).

**Conclusion**

Although the formula for the Casimir of \( ISL(n, R) \) has been written in the Weyl basis, this does not limit its application range. The advantage of our formula for \( ISL(n, R) \), over other possible formulation, is its immediate physical and mathematical application as shown above in the little group Casimir decomposition of \( ISL(n, R) \).

In gauge theories of gravity based on the group \( ISL(4, R) \), it should be verified the correct usage of the Casimir operator. The reason is that in these theories, the group \( ISO(1, 3) \) must be a subgroup of the gauge group. This group has a different Lie algebra than that of the group \( ISO(4) \) which is a subgroup of \( ISL(4, R) \). The applications of a deunitarizing inner automorphism,\[9\]
which changes some of the generators of the group $ISL(n, R)$ by a factor $\sqrt{-1}$, is necessary to extend the range of application of our formula. To avoid confusion we suggest using the notation $ISL(1, n-1, R)$ for the group that has as a subgroup $ISO(1, n-1)$.

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