Propagating stress-pulses and wiggling transition revealed in string dynamics

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Abstract – Understanding string dynamics yields insights into the intricate dynamic behaviors of various filamentary thin structures in nature and industry covering multiple length scales. In this work, we investigate the planar dynamics of a flexible string where one end is free and the other end is subject to transverse and longitudinal motions. Under transverse harmonic motion, we reveal the propagating pulse structure in the stress profile over the string, and analyze its role in bringing the system into a chaotic state. For a string where one end is under longitudinal uniform acceleration, we identify the wiggling transition, derive the analytical wiggling solution from the string equations, and present the phase diagram.

Introduction. – An inextensible flexible string is the backbone of many complicated quasi-one-dimensional thin objects, and represents one of the simplest organization of matter [1–3]. Interest in inextensible flexible strings can be traced back to the beginnings of the calculus [4]. When in motion, a flexible string can exhibit a number of counterintuitive dynamic behaviors, ranging from the acceleration of a string when striking a table [5–7], the formation of the chain fountain structure [8–11], to the spontaneous rise-up and lift-off of a pulled string in the plane [12,13] and on a pulley [14,15]. Understanding the intricate dynamics of the filamentary string structure is important as they are ubiquitous in nature and industry covering length scales of several orders of magnitude [3,16–18]. Much has been learnt about the string dynamics by analyzing the equations of motion of the string [4,15,19–22]. However, the analytical solution to the coupled, nonlinear string equations is only limited to some special cases [13,23,24]. Particle simulation based on the spring-bead model has proven to be a powerful tool to study the dynamic states of the string [3,12,25,26].

The goal of this work is to explore the planar dynamics of the string where one end is free and the other end is in transverse and longitudinal motion, respectively. This model system provides the opportunity to clarify a host of questions with broader implications, such as: How will the motion at one end of the string propagate to the other end? Will any dynamic instability occur in the string? What kinds of featured dynamic states will emerge? To address these questions, we resort to the combination of particle simulations and theoretical analysis of the string equations. The main results of this work are presented below. When shaking the string at one end in harmonic motion, we find that the propagation of stress is realized by the oscillating stress-pulse structure across the string. The back-and-forth movement of the stress-pulse induces more pulses and ultimately leads the whole string to a chaotic state. For a traveling string in uniform acceleration, we find a new dynamic state of the string in which it starts to wiggle and deviate from the straight shape after finite duration. We derive an analytical wiggling solution from the string equations which can substantiate the numerical observation. We further characterize the wiggling transition, and present the phase diagram.

Model and method. – An inextensible flexible string can be modeled by the geometric curve $\vec{X}(s,t)$, where $s$ is the natural parameter of the curve and $t$ is the time. The inextensibility condition is $\partial_s \vec{X}(s,t) \cdot \partial_s \vec{X}(s,t) = 1$. The dynamics of the flexible, inextensible string with uniform mass density $\mu$ is governed by the following equation of motion [4,19]:

$$\mu \partial_t^2 \ddot{\vec{X}}(s,t) = \partial_s [\sigma(s,t) \partial_s \vec{X}(s,t)],$$  

(1)
where the stress $\sigma$ arises as a Lagrangian parameter to keep neighbouring parts of the string at a fixed distance [4]. By projecting eq. (1) along the tangent and normal vectors, we obtain the following string equations [24]:

$$\sigma \kappa^2 - \partial_t^2 \sigma = \mu \partial_t \hat{i} \cdot \partial_t \hat{i},$$

$$2\kappa \partial_s \sigma + \sigma \partial_s \kappa = \mu \partial_t^2 \hat{i} \cdot \hat{n},$$

where $(\hat{i}, \hat{n})$ is the dyad of unit tangent and normal vectors on a planar curve, and $\kappa$ is the curvature. It is a challenge to analytically solve the coupled nonlinear differential equations [4]. Furthermore, due to its flexibility, the string may exhibit shapes that are beyond the functional space of $X(s, t) \in C^2(Q_T)$ and $\sigma(s, t) \in C^1(Q_T)$, where $Q_T = J_s \times J_t$, $s \in J_s = [0, L]$, and $t \in J_t = [0, T]$ [4]. The above string equations lay the foundation for the theoretical analysis of relevant simulation results.

In our simulations, the string is modeled by $N + 1$ massive beads connected by high stiffness linear springs lying on the plane. The balance length of each spring is $\ell_0 \equiv 1$. The mass of each bead is $m_0 \equiv 1$. In the initial state, the string is free of stress, and lies along the $x$-axis. The position of each bead is subject to a small quantity of noise $\delta \vec{x}$ whose $x$- and $y$-components conform to the uniform distribution in the interval $[-A_{\text{noise}}, A_{\text{noise}}]$. The introduction of this noise is to trigger the possible instability of a string in longitudinal motion, and also reflects the small fluctuation of the string under various noise sources in reality. We implement the Verlet integration to construct the trajectory of each bead in the discretized string (see the Supplemental Material Supplementarymaterial.pdf (SM)) [26]. We work in the regime of highly inextensible string with large $k_0$. Specifically, $\hat{T} \equiv T/\tau_0 = T\sqrt{k_0}/\sqrt{m_0} \gg 1$ in the transverse harmonic oscillation of period $T$, and $\bar{a} \equiv a/a_0 = am_0/(\ell_0 k_0) \ll 1$ in the case of longitudinal uniform acceleration. $\tau_0 = \sqrt{m_0/k_0} = 1$, $a_0 = \ell_0/\tau_0^2 = 1$.

**Transverse harmonic oscillation.** In this section, we present the main results about the planar dynamics of the string when one end is under transverse harmonic oscillation. The motion of the shaking end (labeled as $i = 0$) is $\{x_0(t) = 0, y_0(t) = A \sin (2\pi t/T)\}$.

We tune the amplitude of the noise in the position of each bead to be a very small fraction of the balance length $\ell_0$ of the spring, and work in the regime of large $T$ (i.e., highly inextensible string). Simulations with varying shaking amplitude $A$ from $\ell_0$ to $5\ell_0$ show that in general the harmonic motion at the head of the string can propagate in the form of a cosine-like wave by only a few wavelengths. In fig. 1(e)–(g), we present the typical case of $A = 2.5\ell_0$ and $T = 100\tau_0$. The entire string consists of the straight and the wavy segments. The horizontal orientation of the tangent vector at the connection of the straight and the wavy parts of the string seems crucial for maintaining the straight segment of the string. Continuously shaking the string finally leads to the chaotic state as shown in fig. 1(h), which is characterized by the large deformation of the waves in the head part and the growing transverse fluctuation in the remaining part of the string.

The formation of the wave structure near the shaking end reduces the longitudinal length of the string due to the rigidity of the spring. The realization of the geometric shrinking of the string relies on the propagation of stress. The question of how the stress propagates across the string naturally arises. In the following, we analyze the evolution of the stress profile over the string in this process. The results are summarized in fig. 1.
Figures 1(a)–(d) show the variation of the stress distribution over the string as it evolves towards the chaotic state. The shaking bead is at $i = 0$. The labeled numbers at the peaks indicate the temporal sequence, some of which correspond to the labeled shapes in fig. 1(e)–(g). Simulations reveal the peak structure in the stress profile. It indicates that the stress propagates in the manner of pulses. The peak structures in the stress profile are named as stress-pulses. The stress-pulse region, where $l - l_0 > 0$, is stretched much more than the remaining part of the string. From fig. 1(a), we see that the region from the location of the stress-pulse to the free end is free of stress. The stress-pulse sharply separates the stretched and the stress-free regions. Here, we emphasize that the word “pulse” specifically refers to the peak structure in the stress profile in fig. 1(a)–(d), but not the wave structure in the string shape as shown in fig. 1(e)–(g). Simulations show the steady propagation of the stress-pulse across the string in a rate that is much faster than the propagation of the wave in the string shape, as shown in fig. 1(a). From fig. 1(a), we obtain the value of the pulse speed to be the characteristic speed of the string $\ell_0/\tau_0$, which is proportional to $\sqrt{\ell_0}$. Further simulations for the case of $T/\tau_0 = 1000$, which is ten times the value for $T/\tau_0$ in fig. 1, confirm that the pulse speed is the same as that in fig. 1 (see SM). Therefore, the pulse propagates infinitely fast in the inextensible limit.

Figure 1(b) shows that when the stress-pulse labeled 4 in fig. 1(a) reaches the free end of the string, it is bounced back, becoming the pulse labeled 5. Remarkably, the stress-pulse is inverted in this process. In other words, the pulse region that is originally stretched becomes compressed. Consequently, the stress distribution over the string is divided into a number of compressed and stretched regions; the shaking end is always stretched.

In contrast, for a pulse whose dynamics is governed by the wave equation, no inversion occurs when reflecting off a free end of the medium [27]. Here, the behavior of the stress-pulse in the highly inextensible string system is governed by the coupled string equations in eqs. (2) and (3) rather than the law of the wave equation. Figure 1(b) shows that the negative stress-pulse continues propagating towards the shaking end. It is finally reflected back, and becomes inverted again (see the pulse labeled 9 in fig. 1(c)). It is of interest to note that propagation and reflection of small waves along a hanging chain subject to an initial velocity have been studied [21] and an interesting pattern of kicks at the free end has been revealed [28].

With the back-and-forth movement of the stress-pulse, we numerically observe the continuous retreat of the free end towards the shaking end along the axis of the string. More information about the axial retreat of the free end is provided in the SM. Furthermore, the oscillation of the stress-pulse across the string induces more pulses as shown in fig. 1(c). Repeating this process ultimately destroys the wavy shape near the shaking end, and the shape of the string becomes chaotic as shown in fig. 1(h). The transition to the chaotic state is also reflected in the stress profile. From fig. 1(d), we see that the stress is highly concentrated in the chaotic segment of the string, and the stress level at the straight segment is significantly reduced. The screening of the stress by the highly curved segment in the chaotic string can be rationalized by the first term in eq. (2). Equation (2) is recognized as the screened Poisson’s equation $(\frac{d^2}{dx^2} - \kappa^2)\psi = f(x)$ for constant $\kappa$; the source term is the temporally varying tangent vector [27]. The corresponding Green’s function is $G(x_1, x_2) = \frac{1}{\kappa} e^{-\kappa|x_1-x_2|}$ under the boundary condition that the Green’s function vanishes for $x \to \pm \infty$. Therefore, the effect of curvature in the string is to screen the stress.

**Longitudinal uniform acceleration.** – We proceed to discuss the planar dynamics of the string under longitudinal uniform acceleration based on simulations and theoretical analysis. The head bead of the string is pulled and maintained in uniform acceleration along the $x$-axis; $x_0(t) = \frac{1}{2}at^2$. For a straight string in longitudinal uniform acceleration, the string equations show that the stress is linear with $s$, increasing from zero to $\mu aL$ from the tail ($s = 0$) to the head ($s = L$) of the string. However, simulations reveal that the traveling string will suddenly deviate from the straight shape after finite duration, and the shape fluctuation persists thereafter. We name such a dynamic transition as the wiggling transition.

In fig. 2, we present typical snapshots of wiggling strings. $y_i$ is the transverse displacement of each bead. The head bead is labeled as $i = 0$. The magnitude of acceleration increases from the green to the red lines. From fig. 2, we see that the tail of the string is generally subject to a stronger shape fluctuation than the head part. Increasing the acceleration enhances the strength of string wiggling. Wiggling transition still occurs by reducing the noise level to as low as $A_{\text{noise}} = 10^{-5} \ell_0$.

Considering that the string in simulations is not strictly inextensible, it is possible that the wiggling of the string is caused by the extensibility of the string? To clarify this.
question, we perform theoretical analysis based on eqs. (2) and (3) for inextensible strings. Furthermore, theoretical analysis based on the string equations allows us to explore the inextensible regime which is beyond the applicability of our numerical simulations. Here, we emphasize that our numerical simulations are based on the spring-bead model with large spring constant (but not strictly inextensible), and the string equations are for inextensible strings.

We focus on the behavior of the string at the onset of wiggling transition when the shape fluctuation is small and varies slowly over the string. This justifies a continuum description of the string based on the equations of motion. The main results are presented below. The detailed information is provided in the SM. The stress distribution can be written as \(\sigma(s, t) = f(t)s + \alpha(t)\). The requirement of a stress-free end at \(s = 0\) sets \(\alpha\) to be zero. The shape of the string is represented by the orientation of the tangent vector which makes angle \(\theta\) with respect to the \(x\)-axis. The shape of the string is represented by \(\theta\). \(\theta(s, t) = \theta_1(s)\theta_2(t)\), where \(\theta_1(s) = \theta_{10}\exp(-qs)\), and \(\theta_2(t)\) satisfies

\[
\dot{\theta}_2(t) + gg(t)\theta_2(t) = 0, \tag{4}
\]

where \(g(t) = 2f(t)/\mu\) and \(q\) is a constant. It is of interest to note that eq. (4) has the same mathematical form as the Schrödinger equation; \(g(t)\), the time-dependent part of the stress \(\sigma(s, t)\), corresponds to the physical potential in the Schrödinger equation. Equation (4) suggests the rich dynamics of the string even in the perturbation regime.

Now, consider the case of interest: \(g(t) = g_0\). \(g_0\) is a constant, and \(g_0 > 0\) without loss of generality. Such a distribution of stress is identical to that over a straight string in uniform acceleration \(a = g_0/2\). By inserting \(\theta_2(t) = \theta_{20}\exp(iwt)\) into eq. (4), we obtain the dispersion relation: \((iw)^2 = -gg_0\). For real positive \(q\), \(w = \sqrt{g_0q}\). Such a solution is plotted in fig. 2(b). The tail of the string wiggles, and the spatial extension \(q^{-1}\) of the wiggling segment is linear with the magnitude of acceleration at fixed frequency \(w\). Therefore, in addition to the trivial straight-string solution, the tail of a uniformly accelerating, inextensible string can wiggle. This analytical result and the preceding simulation results suggest that the extensibility of the string is not a necessary condition for the occurrence of the string wiggling, but it may contribute to the propagation of the wiggling deformation to the entire string. Here, it is of interest to point out that the wiggling transition is an intrinsic property of the string without dependence on any external transverse force.

In the following, we further characterize the wiggling transition by the evolution of its longitudinal length \(X\) and the averaged transverse displacement \(\delta y\). \(\delta y(t) = \sqrt{\sum_{i=0}^{N} y_i^2(t)/N}\). Figure 3 shows that the transition from the straight to the wiggling state is well signified by the entire decline of the oscillations in \(X\), and the simultaneously occurring take-off of the \(\delta y(t)\) curve from the zero line. Long-time observation up to \(t = 10\) millions simulation steps shows the convergence of the string wiggling; the strength of wiggling remains in the interval of \(\delta y \in [0.12\ldots 0.20]\), and \(X \in [50.2\ldots 51.0]\).

In fig. 4, we present the phase diagram of the dynamic state of the string under longitudinal uniform acceleration. The lower (blue) and upper (red) curves are for the cases of \(N = 50\) and \(N = 100\), respectively. The state of the string is characterized by the averaged transverse displacement \(\delta y\). The string is regarded to be in the wiggling state when \(\delta y\) exceeds ten times the initially introduced noise \(A_{\text{noise}}\). Figure 4 shows that a longer string can stay in the straight state for a longer time. The straight-to-wiggling transition becomes insensitive to the magnitude of acceleration when it exceeds about \(10^{-5}a_0\).

**Conclusions.** – To summarize, in this work we have investigated the planar dynamics of a flexible string that is subject to transverse and longitudinal motions at one end. We revealed the pulse structure in the propagation of stress when one end of the string is under transverse harmonic motion, and identified the wiggling transition in a traveling string in uniform acceleration. These results may find applications in the remote control of various filamentary thin structures by manipulating the end.

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