Provably Efficient Generative Adversarial Imitation Learning for Online and Offline Setting with Linear Function Approximation

Zhihan Liu∗ Yufeng Zhang† Zuyue Fu‡ Zhuoran Yang§ Zhaoran Wang¶
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Abstract

In generative adversarial imitation learning (GAIL), the agent aims to learn a policy from an expert demonstration so that its performance cannot be discriminated from the expert policy on a certain predefined reward set. In this paper, we study GAIL in both online and offline settings with linear function approximation, where both the transition and reward function are linear in the feature maps. Besides the expert demonstration, in the online setting the agent can interact with the environment, while in the offline setting the agent only accesses an additional dataset collected by a prior. For online GAIL, we propose an optimistic generative adversarial policy optimization algorithm (OGAP) and prove that OGAP achieves $\tilde{O}(H^2d^{3/2}K^{-1/2} + KH^{3/2}dN_1^{-1/2})$ regret. Here $N_1$ represents the number of trajectories of the expert demonstration, $d$ is the feature dimension, and $K$ is the number of episodes. For offline GAIL, we propose a pessimistic generative adversarial policy optimization algorithm (PGAP). For an arbitrary additional dataset, we obtain the optimality gap of PGAP, achieving the minimax lower bound in the utilization of the additional dataset. Assuming sufficient coverage on the additional dataset, we show that PGAP achieves $\tilde{O}(H^2dK^{-1/2} + H^2d^{3/2}N_2^{-1/2} + H^3/2dN_1^{-1/2})$ optimality gap. Here $N_2$ represents the number of trajectories of the additional dataset with sufficient coverage.

1 Introduction

In imitation learning (IL, Hussein et al. (2017)) (a.k.a apprenticeship learning), the agent remains unknown of the reward, but can learn from an expert demonstration so that the agent learns a policy as good as the expert one. To solve IL problem, there exist mostly three types of methods: behavior cloning (BC, Argall et al. (2009)), inverse reinforcement learning (IRL, Abbeel and Ng (2004)), and online generative adversarial imitation learning (online GAIL). BC regards IL as a supervised learning problem of predicting actions based on states. While appealingly simple, BC suffers from compounding error caused by covariate shift (Ross and Bagnell, 2010; Ross et al., 2011). IRL explicitly solves the true reward function and then accordingly fully solves an RL subproblem at every iteration (Abbeel and Ng, 2004; Ng and Russell, 2000).

∗University of Science and Technology of China; liuzhihan0627@mail.ustc.edu.cn
†Northwestern University; yufengzhang2023@u.northwestern.edu
‡Northwestern University; zuyue.fu@u.northwestern.edu
§Princeton University; zhuoranyang.work@gmail.com
¶Northwestern University; zhaoranwang@gmail.com
Though it has succeeded in tasks involving continuous spaces (Levine and Koltun, 2012; Finn et al., 2016), IRL lacks computational efficiency and the desired true reward function may not be unique. To address these issues, online GAIL (Ho and Ermon, 2016) solves IL through minimax optimization with alternating updates to learn a policy whose performance cannot be discriminated from the expert policy on a certain predefined reward set. The alternating updates in online GAIL mirror the training of generative adversarial networks (Goodfellow et al., 2014; Arjovsky et al., 2017). Specifically, during every iteration, online GAIL first minimizes the discrepancy in expected cumulative reward between the expert policy and the learned policy and then maximizes such a discrepancy over a given reward function class in adversary. Online GAIL achieves tremendous empirical success in a variety of fields, such as autonomous driving (Kuefler et al., 2017), human behavior modeling (Merel et al., 2017), natural language processing (Chen et al., 2017), and robotics control (Tsurumine et al., 2019).

Despite the state-of-art empirical performance of online GAIL, the agent requires a huge amount of interactions with the environment during the training. For some practical problems, it is inconvenient, costly, or risky to get expert data or labeled data, especially when collecting clinical data or developing autonomous driving. Meanwhile, it is available to get other sources of offline data, which may be originated from historical experiments, non-labeled data, and published datasets, etc. Naturally, we desire to utilize these offline data to alleviate the shortage of expert demonstration and aid the agent to mimic the expert policy. To this end, besides online GAIL, we consider the offline generative adversarial imitation learning (offline GAIL) setting. In offline GAIL, we assume that the agent is accessible to an additional dataset besides the expert demonstration, without further interaction with the environment. Some related works (Zolna et al., 2020; Zhang and Wu, 2021) study this setting and provide methods from the aspect of the experiment.

Furthermore, previous theoretical analyses on GAIL either focus on the tabular case (Shani et al., 2021), where the state and action spaces are discrete, or relies on strong assumptions, including access to a well-explored dataset (Zhang et al., 2020), linear-quadratic regulators (Cai et al., 2019), or kernelized nonlinear regulators (Chang et al., 2021). Theoretical analysis for GAIL with linear function approximation either in online or offline settings still remains an open problem, which is crucial for the application of GAIL in the continuous or high dimensional state and action spaces. The cruxes of such an analysis involve: (i) Different from RL, both online GAIL and offline GAIL are minimax optimization problems with respect to the policy and reward function, especially with linear reward set. (ii) For offline GAIL, without assuming the well-exploredness of the additional dataset, the agent may be misled by distribution shift in the additional dataset and shares the suffering with offline RL (Jin et al., 2021; Wang et al., 2020a); (iii) For offline GAIL, we are incapable to update the reward function based on the trajectory of present policy.

Hence in this paper, we aim at tackling these issues and answering the following question:

Can we design provably efficient algorithms for online and offline GAIL with linear function approximation in a unified view?

To answer the above question, we present a unified framework and specialize it as optimistic generative adversarial policy optimization (OGAP) for online GAIL and pessimistic generative adversarial policy optimization (PGAP) for offline GAIL with linear function approximation. This framework is motivated by the alternating update process of GANs and involves two main stages: (i) policy update stage and (ii) reward update stage. (i) In the policy update stage, we apply mirror descent (Beck and Teboulle, 2003; Hazan, 2019) to update the policy and evaluate policy online optimistically for OGAP and offline pessimistically for
PGAP. (ii) In the reward update stage, we first estimate the gradient of GAIL objective function with respect to the reward parameter through the collected trajectory induced by the present policy for OGAP. While for PGAP, we build the estimate through estimated action-value functions during the stage of policy update. Then we use projected gradient ascent to update reward parameters via such an estimate of gradient.

**Contribution** Particularly, we conclude our contributions in the following three aspects.

- For online GAIL with linear function approximation, we propose a new algorithm OGAP and prove that OGAP achieves \( \tilde{O}(H^2d^{3/2}K^{1/2} + KH^{3/2}dN_1^{1/2}) \) regret when applying linear function approximation, demonstrating that OGAP is provably efficient. Here \( N_1 \) represents the number of trajectories of the expert demonstration, \( d \) is the feature dimension, and \( K \) is the episode.

- For offline GAIL with linear function approximation, we design a new algorithm PGAP and obtain the optimality gap of the output policy under the minimal assumption on the additional dataset. Then we decompose the optimality gap into three sources: optimization error, Monte Carlo (MC) estimation error, and intrinsic error. We prove that optimization error and MC estimation error respectively scale to \( \tilde{O}(K^{-1/2}) \) and \( \tilde{O}(N_1^{-1/2}) \), while intrinsic error depends on how well the additional dataset \( \mathcal{D}^A \) covers the expert policy and attains the minimax optimality in the utilization of the additional dataset.

- In addition, we demonstrate that if we further assume that the additional dataset \( \mathcal{D}^A \) has sufficient coverage on the expert policy, we prove PGAP achieves \( \tilde{O}(H^2dK^{-1/2} + H^2d^{3/2}N_2^{-1/2} + H^{3/2}dN_1^{-1/2}) \) optimality gap, thus PGAP has global convergence. Here \( N_2 \) represents the number of trajectories of the additional dataset with sufficient coverage. Furthermore, we discuss the effect of the additional offline dataset \( \mathcal{D}^A \). In particular, facilitated with an additional dataset \( \mathcal{D}^A \) with sufficient coverage, we decrease the dependency for horizon \( H \) and feature dimension \( d \) in the optimality gap.

**Related Works** Our work adds to the body of analysis on GAIL (Cai et al., 2019; Chen et al., 2020; Zhang et al., 2020; Xu et al., 2020; Shani et al., 2021; Chang et al., 2021). Shani et al. (2021) study online GAIL and obtain \( \tilde{O}(H^2|\mathcal{S}||\mathcal{A}|^{1/2}K^{1/2} + H^{3/2}|\mathcal{S}|^{1/2}|\mathcal{A}|^{1/2}KN_1^{-1/2}) \) regret in the tabular case with bounded reward functions but we apply linear function approximation on the transition kernels without assuming the state space or the action space is discrete and we adopt linear reward set. 

Chen et al. (2020) only study the convergence of offline GAIL to a stationary point instead of global convergence (optimality gap) as in this paper. Xu et al. (2020); Zhang et al. (2020) analyze the global convergence of GAIL with neural networks respectively in the tabular case and the continuous case but assume that a well-explored dataset is available (concentrability coefficients are uniformly upper bounded), while our analysis need not such a strict and impractical assumption. Cai et al. (2019) study the global convergence of offline GAIL in the setting of linear-quadratic regulators, which is unnecessary for this paper. As an independent work, Chang et al. (2021) study offline GAIL with bounded reward functions in the continuous kernelized nonlinear regulator (KNR, Kakade et al. (2020)) and Gaussian process (GP, Fisac et al. (2018)) setting. We point out that the KNR (resp. GP) setting is different from linear kernel MDP as analyzed in this paper and each one does not imply the other, which leads to the difference in model estimation and later analysis. In addition, we study the linear reward set instead of bounded reward set and the former case is difficult to handle with (Shani et al., 2021).

\(^1\)Here \( \tilde{O}(\cdot) \) hides the log terms and constant terms.
Our work is related to IRL. Abbeel and Ng (2004); Neu and Szepesvári (2007); Syed et al. (2008); Syed and Schapire (2008) study the convergence of IRL in the tabular case, while they require to solve an RL subproblem every iteration, inefficiently. Our work is also related to BC (Ross and Bagnell, 2010; Rajaraman et al., 2020, 2021; Rashidinejad et al., 2021). BC does not solve a minimax problem as GAIL, but directly mimics the expert policy extracted from the expert demonstration. Rajaraman et al. (2021) propose a BC method which achieves \( \tilde{O}(|S|H^2/N_1) \) suboptimality, attaining \( \Omega(|S|H^2/N_1) \) the lower bound of BC (Rajaraman et al., 2020), when the transition model is unknown. To the best of our knowledge, present analysis of BC only focus on the tabular case and would fail in the continuous state and action space with horizon \( H \geq 2 \), since BC is considered as a classification problem and always faces unseen states in the continuous state space.

Besides, our work is related to the vast body of existing literature on online RL cooperated with optimism (Auer et al., 2002, 2009; Azar et al., 2017; Jin et al., 2018, 2019; Yang and Wang, 2020), offline RL (Fujimoto et al., 2019b; Kumar et al., 2020; Fujimoto et al., 2019a; Duan et al., 2020; Levine et al., 2020; Jin et al., 2021), policy optimization (Beck and Teboulle, 2003; Hazan, 2019; Cai et al., 2020; Nemirovskij and Yudin, 1983), adversarial MDP (Shani et al., 2020b; Rosenberg and Mansour, 2019; Jin et al., 2020), and linear function approximation (Duan et al., 2020; Bradtke and Barto, 1996; Yang and Wang, 2019a; Jin et al., 2019; Ayoub et al., 2020; Yang and Wang, 2019b; Zhou et al., 2021) while they study minimization or maximization problem with known reward through value-based or policy-based method, instead of minimax problem with respect to policy and reward function as GAIL.

Our work is related to a line of study on pessimism. Specifically, the uncertainty quantification for estimated model in PGAP is motivated by the pessimism in offline RL (Chen and Jiang, 2019; Xie et al., 2021a,b; Kumar et al., 2020; Jin et al., 2021; Liu et al., 2020; Yu et al., 2020, 2021; Buckman et al., 2020; Rashidinejad et al., 2021; Uehara and Sun, 2021). Liu et al. (2020) propose a pessimistic variant of fitted Q-learning algorithm (Antos et al., 2007; Munos and Szepesvári, 2008) achieving the optimal policy within a restricted class of policies without assuming the dataset to be well-explored. Jin et al. (2021) propose a provably efficient algorithm with the spirit of pessimism to solve offline RL with linear function approximation, under no coverage assumption on the dataset. Xie et al. (2021a) propose a refined pessimistic estimate and obtain a tighter suboptimality in \( d \) compared with Jin et al. (2021). Rashidinejad et al. (2021) study the offline RL in the tabular case through lower confidence bound (LCB), relying on the partial coverage assumption on the dataset. Uehara and Sun (2021) analyze the constrained pessimistic policy optimization with general function approximation and with the partial coverage assumption of the dataset, then they specialize the case in the KNR setting and give a refined upper bound. The importance of pessimism in offline RL is characterized by Buckman et al. (2020); Zanette (2021) through discussing the lower bound of offline RL when the dataset has no restriction.

**Notations** We denote by \([N] = \{1, \ldots, N\}\). We also denote by \(\| \cdot \|_2\) the \(\ell_2\)-norm of a vector and denote by \(\| \cdot \|_A\) the spectral norm of a matrix \(A\). We denote by \(\Delta(\mathcal{X})\) the set of probability distributions on a set \(\mathcal{X}\) and correspondingly define \(\Delta(\mathcal{X} | S, H) = \{\{\pi_h(\cdot | \cdot)\}_{h \in [H]} : \pi_h(\cdot | s) \in \Delta(\mathcal{X})\text{ for any } (s, h) \in S \times [H]\}\) for all set \(S\) and \(H \in \mathbb{N}_+\). For \(p_1, p_2 \in \Delta(\mathcal{X})\), we denote by \(D_{KL}(p_1 \| p_2)\) the KL-divergence, that is,

\[
D_{KL}(p_1 \| p_2) = \int_{\mathcal{X}} p_1(a) \log \frac{p_1(a)}{p_2(a)} da.
\]

And \(\langle \cdot, \cdot \rangle_{\mathcal{A}}\) is the inner product taken over the action space \(\mathcal{A}\). We also denote by \(\delta_x\) Dirac function centered at \(x\). We denote by \(\text{Vol}(\mathcal{X})\) by the measure of set \(\mathcal{X}\).
2 Preliminary

In this section, we first introduce the notion of the episodic Markov decision process (MDP). Then we introduce generative adversarial imitation learning in the online and offline settings, respectively. Finally we introduce the definition of linear function approximation.

2.1 Episodic Markov Decision Process

We consider an episodic MDP \((S, A, H, \mathcal{P}, r)\), where \(S\) and \(A\) are the state and action spaces, respectively, \(H\) is the length of each episode, \(\mathcal{P}_h\) is the Markov transition kernel of the \(h\)-th step of each episode for any \(h \in [H]\), and \(r_h : S \times A \to [0, 1]\) is the reward function at the \(h\)-th step of each episode for any \(h \in [H]\). We assume without loss of generality that the reward function \(r_h\) is deterministic.

In the episodic MDP, the agent interacts with the environment as follows. At the beginning of each episode, the agent determines a policy \(\pi = \{\pi_h\}_{h \in [H]} \in \Delta(A | S, H)\). Then the agent takes the action \(a_h \sim \pi_h(\cdot | s_h)\) at the \(h\)-th step of the \(k\)-th episode, observes the reward \(r_h(s_h, a_h)\), and transits to the next state \(s_{h+1} \sim \mathcal{P}_h(\cdot | s_h, a_h)\). The episode terminates when the agent reaches the state \(s_{H+1}\). Without loss of generality, we assume that the initial state \(s_1 = x\) is fixed across different episodes. We remark that our algorithms and corresponding analyses readily generalize to the setting where the initial state \(s_1\) is sampled from a fixed distribution.

We now define the value functions in the episodic MDP. For any policy \(\pi = \{\pi_h\}_{h \in [H]}\) and reward function \(r = \{r_h\}_{h \in [H]}\), the state- and action-value functions are defined for any \((s, a, h) \in S \times A \times [H]\) as follows,

\[
V_{h, \pi}^r(s) = \mathbb{E}_\pi \left[ \sum_{i=h}^H r_i(s_i, a_i) \mid s_h = s \right], \quad Q_{h, \pi}^r(s, a) = \mathbb{E}_\pi \left[ \sum_{i=h}^H r_i(s_i, a_i) \mid s_h = s, a_h = a \right],
\]

where the expectation \(\mathbb{E}_\pi[\cdot]\) is taken with respect to the action \(a_i \sim \pi_i(\cdot | s_i)\) and the state \(s_{i+1} \sim \mathcal{P}_i(\cdot | s_i, a_i)\) for any \(i \in \{h, h+1, \ldots, H\}\). With slight abuse of notations, we also denote by \(\mathcal{P}_h\) the operator form of the transition kernel such that \((\mathcal{P}_h f)(s, a) = \mathbb{E}_{s' \sim \mathcal{P}_h(\cdot | s, a)}[f(s')]\) for any \(f : S \to \mathbb{R}\). By the definitions of the value functions in (2.1), for any \((s, a, h) \in S \times A \times [H]\), any policy \(\pi\), and any reward function \(r\), we have

\[
V_{h, \pi}^r(s) = \langle Q_{h, \pi}^r(s, \cdot), \pi_h(\cdot, s) \rangle, \quad Q_{h, \pi}^r(s, a) = r_h(s, a) + \mathcal{P}_h V_{h+1, \pi}^r(s, a), \quad V_{H+1, \pi}^r(s) = 0,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the inner product over the action space \(A\). We further define the expected cumulative reward as follows,

\[
J(\pi, r) = V_{1, \pi}^r(x).
\]

In this paper, we characterize the performance of the agent via the expected cumulative reward \(J(\pi, r)\) defined in (2.3).

2.2 Generative Adversarial Imitation Learning

Given an expert demonstration \(D^E = \{(s_{h, \tau}^E, a_{h, \tau}^E)\}_{h \in [H], \tau \in [N_1]}\) with \(N_1\) trajectories of state-action pairs generated following the underlying MDP and the expert policy \(\pi^E\), the goal of GAIL is to find a policy whose performance is close to that of the expert policy \(\pi^E\) for any reward function in a given set \(\mathcal{R}\) (Ho and Ermon,
We consider the linear setting where the transition kernel is linear in a feature map, which is formalized in

\begin{equation}
\min_{\pi \in \Delta(S,A,H)} \max_{r \in R} J(\pi^E, r) - J(\pi, r),
\end{equation}

where \( J(\pi, r) \) is defined in (2.3).

**Online GAIL.** In online GAIL, the agent interacts with the environment to collect state-action pairs following the underlying MDP and the current policy. For online GAIL, we are interested in the performance of the algorithm during learning. To this end, we compare the expected cumulative reward corresponding to the expert policy during learning with the expected cumulative reward corresponding to the expert policy under the worst-case scenario, which is defined as follows (Shani et al., 2021),

\begin{equation}
\text{Regret}(K) = \max_{r \in R} \sum_{k=1}^{K} [J(\pi^E, r) - J(\pi^k, r)],
\end{equation}

where \( \pi^k \) is the policy of the agent at the \( k \)-th episode.

**Offline GAIL.** To simultaneously utilize non-expert data without further interaction with the environment, we consider offline GAIL, which involves an additional dataset to benefit the policy learning. Specifically, except for the expert demonstration \( D^E = \{(s^E_h, r_h)\}_{h \in [H], r \in [N]} \) collected by the expert policy \( \pi^E \) in the underlying MDP, the agent has access to an additional dataset \( D^A = \{(s^A_h, a^A_h)\}_{h \in [H], r \in [N]} \), which is collected a priori by an experimenter in the underlying MDP. In particular, at each step \( h \in [H] \) of each trajectory \( \tau \in [N] \), the experimenter takes the action \( a^A_h \) at the state \( s^A_h \) and observes the next state \( s^A_{h+1} \sim P_h(\cdot | s^A_h, a^A_h) \). Here \( a^A_h \) is arbitrarily chosen by the experimenter given the filtration

\[ F_{h,\tau} = \sigma(\{(s^n_i, a^n_i): (n-1)H + i \leq (\tau-1)H + h\}) , \]

In other words, in the \( \tau \)-th trajectory, the action the expert takes is only determined by the historical information with randomness. For offline GAIL, we measure the performance of a policy \( \pi \) by the \( R \)-distance (Chen et al., 2020) between the expert policy \( \pi^E \) and \( \pi \), which is defined as follows,

\begin{equation}
D_R(\pi^E, \pi) = \max_{r \in R} [J(\pi^E, r) - J(\pi, r)].
\end{equation}

Here \( R \) is the reward set, which is specified later in §2.3. Optimality gap defined in (2.6) can be considered as one episode regret defined in (2.5). When optimality gap of policy \( \pi \) approaches zero, it implies that the performance difference between the policy \( \pi \) and the expert policy \( \pi^E \) tends to be undistinguishable by the reward set \( R \), which implies that the performance of \( \pi \) is measured by both the optimality gap \( D_R(\pi^E, \pi) \) and the richness of the reward set \( R \).

### 2.3 Linear Function Approximation

We consider the linear setting where the transition kernel is linear in a feature map, which is formalized in the following assumption.

**Assumption 2.1 (Linear Kernel Episodic MDP).** Given measurable sets \( S \) and \( A \) with finite measure, the episodic MDP \((S,A,H,P,r^\mu)\) is a linear MDP with a feature map \( \phi: S \times A \times S \to \mathbb{R}^d \), that is, for any \( h \in [H] \),
there exists \( \theta_h \in \mathbb{R}^d \) with \( \| \theta_h \|_2 \leq \sqrt{d} \) such that \( \mathcal{P}_h(s', | s, a) = \phi(s, a, s')^\top \theta_h \) for any \( (s, a, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S} \). Also, there exists an absolute constant \( R > 0 \) such that
\[
R^{-2} \cdot \sup_{s' \in \mathcal{S}} |\phi(s, a, s')^\top y|^2 \leq \int_{\mathcal{S}} |\phi(s, a, s')^\top y|^2 \, ds' \leq d,
\]
for any \((s, a) \in \mathcal{S} \times \mathcal{A} \) and \( y \in \mathbb{R}^d \) with \( \| y \|_2 \leq 1 \).

Under Assumption 2.1, we further assume that there exists a feature map \( \psi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d \) such that the reward set \( \mathcal{R} \) in (2.4) takes the following form,
\[
\mathcal{R} = \{ r^\mu : r^\mu_h (\cdot, \cdot) = \psi(\cdot, \cdot)^\top \mu_h \text{ for any } (h, \mu) \in [H] \times \mathcal{S} \},
\]
where \( r^\mu = \{ r^\mu_h (\cdot, \cdot) \}_{h \in [H]} \) is the reward function and \( \mu = \{ \mu_h \}_{h \in [H]} \) is the reward parameter. Here \( \mathcal{S} \) is the reward parameter domain, which is defined as follows,
\[
\mathcal{S} = \{ \mu : \mu_h \in B \text{ for any } h \in [H] \}, \quad \text{where } B = \{ u \in \mathbb{R}^d : \| u \|_2 \leq \sqrt{d} \}.
\]
We assume that \( \| \psi(s, a) \|_2 \leq 1 \) for any \((s, a) \in \mathcal{S} \times \mathcal{A} \), which ensures that \( r^\mu_h (s, a) \in [0, \sqrt{d}] \) for any \((s, a, h, \mu) \in \mathcal{S} \times \mathcal{A} \times [H] \times \mathcal{S} \). For notational convenience, for any reward function \( r^\mu \), we denote by the GAIL objective function \( L(\pi, \mu) \) as follow,
\[
L(\pi, \mu) = J(\pi^E, r^\mu) - J(\pi, r^\mu),
\]
where \( J(\pi, r^\mu) \) is defined in (2.3).

Assumption 2.1 corresponds to the linear kernel MDP model in RL. See Ayoub et al. (2020); Zhou et al. (2021); Cai et al. (2020) for various examples of linear kernel MDPs. We remark that the existence of \( R \) in Assumption 2.1 can be guaranteed if for any \((s, a) \in \mathcal{S} \times \mathcal{A} \), the feature map \( \phi(s, a, \cdot) \) is upper bounded and Lipschitz continuous. In particular, a tabular MDP where the state space \( \mathcal{S} \) and the action space \( \mathcal{A} \) are both finite, is a special case of the linear kernel MDP in Assumption 2.1 with \( d = |\mathcal{S}|^2 |\mathcal{A}| \) and the feature map \( \phi(s, a, s') \) being the canonical basis \( e_{(s, a, s')} \) of \( \mathbb{R}^{|\mathcal{S}|^2 |\mathcal{A}|} \). It implies that our analysis for GAIL with linear function approximation also covers the tabular case. We also remark that the range of the reward function is \([0, \sqrt{d}]\) instead of \([0, 1]\). It means that, with the increasing \( d \), we can enrich the reward set \( \mathcal{R} \) and then capture the performance of the policy more meticulously using the optimality gap defined in (2.6). Analysis in GAIL with linear reward set is more challenging than the case with bounded reward set, which is studied in the previous literature (Shani et al., 2021; Chang et al., 2021).

3 Algorithm

We first propose a unified framework in Algorithm 1 to solve GAIL in both online and offline settings. Then we specify the framework for online and offline settings in Optimistic Generative Adversarial Policy optimization (OGAP in Algorithm 2 of §3.1) and Pessimistic Generative Adversarial Policy optimization (PGAP in Algorithm 3 of §3.2), respectively.

This framework in Algorithm 1 involves two stages: (i) policy update stage and (ii) reward update stage. (i) In the policy update stage, we use mirror descent to update the policy based on the estimated action-value function constructed in the previous iteration. For OGAP, we sample a trajectory following the updated policy. Then we construct estimated action-value functions with optimism based on the finite historical data
for OGAP or pessimism based on the additional dataset for PGAP. (ii) In the reward update stage, we first construct an estimate of the gradient based on the collected trajectory induced by the present policy and the finite historical data for OGAP or estimated action-value functions for PGAP, and then we use projected gradient ascent to update reward parameters via such an estimate of gradient. We further detail OGAP and PGAP in §3.1 and §3.2, respectively.

### 3.1 Optimistic Generative Adversarial Policy Optimization

To specialize Algorithm 1 to solve online GAIL, we propose OGAP in Algorithm 2. We detail the policy update stage and reward update stage as follows.

#### 3.1.1 Policy Update Stage

The policy update stage (Lines 4–12 of Algorithm 2) consists of two steps: (i) policy improvement (Lines 4–6) and (ii) policy evaluation (Lines 8–12). In policy improvement, we apply mirror descent in its proximal form to update the current policy via estimated action-value functions, which is specified in policy evaluation stage. In policy evaluation, we employ the optimism principle to construct the estimated action-value functions, which further utilize estimated transition kernels and bonus functions.

**Policy Improvement.** To generate a policy whose performance is close to the expert policy $\pi^E$, we update the policy $\pi^k$ to minimize the GAIL objective function $L(\pi, \mu^{k-1}) = J(\pi^E, r^{k-1}) - J(\pi, r^{k-1})$ in (2.9) under the current reward function $r^{k-1} = r^{d^{k-1}}$. Note that $\pi^E$ is fixed, then we only need to maximize $J(\pi, r^{k-1})$. Applying online mirror descent (Beck and Teboulle, 2003; Hazan, 2019), a standard algorithm to solve online learning problem, we update $\pi$ as follows,

$$\pi^k = \arg\max_{\pi \in \Delta(A | S, H)} \{ \mathcal{L}_{k-1}(\pi) - \alpha^{-1} D(\pi, \pi^{k-1}) \},$$  

(3.1)

where $\alpha$ is the step size, Bregman divergence regularizer $D(\pi, \pi^{k-1})$ is chosen as the expected KL divergence...
Algorithm 2 Optimistic Generative Adversarial Policy Optimization (OGAP)

1: **Input:** Expert demonstration $\mathbb{D}^E$, scaling factor $\kappa$, and step size $\eta$ and $\alpha$.
2: Initialize $\{\hat{Q}_h^0\}_{h \in [H]}$ as zero functions over $S \times A$, $\{\pi_h^0\}_{h \in [H]}$ as uniform distributions over $A$, $\mu^1 = \{\mu_h^1\}_{h \in [H]}$ as zero vectors, and $\{\hat{V}_{h+1}^k\}_{k \in \{0, 1, \ldots, K\}}$ as zero functions over $S$.
3: for $k = 1, \ldots, K$
4:  for $h = 1, \ldots, H$
5:     $\pi_h^k(\cdot | s) \propto \pi_h^{k-1}(\cdot | s) \cdot \exp\{\alpha \cdot \hat{Q}_h^{k-1}(s, \cdot)\}$.
6:  end for
7:  Rollout a trajectory $\{(s_h^k, a_h^k)\}_{h \in [H]}$ following $\pi^k$.
8: for $h = H, \ldots, 1$
9:  Set $\{\hat{P}_h^k\}_{h=1}^H$ and $\{\Gamma_h^k\}_{h=1}^H$ via (3.4) and (3.8), respectively.
10: $\hat{Q}_h^k(\cdot, \cdot) \leftarrow \min\{(r_h^k + \hat{P}_h^k \hat{V}_{h+1}^k + \Gamma_h^k)(\cdot, \cdot), (H - h + 1)\sqrt{d}\} + \hat{Q}_h^k(\cdot, \cdot)$.
11: $\hat{V}^k_h(\cdot) \leftarrow \langle \hat{Q}^k_h(\cdot, \cdot), \pi^k_h(\cdot | \cdot) \rangle_A$.
12: end for
13: Set $\{\nabla_{\mu_h} \tilde{J}(\pi^E, r^u)\}_{h \in [H]}$ via (3.12).
14: for $h = 1, \ldots, H$
15:  $\tilde{\nabla}_{\mu_h} L(\pi^k, \mu^k) \leftarrow \tilde{\nabla}_{\mu_h} \tilde{J}(\pi^E, r^u) \mid \mu = \mu^k - \psi(s_h^k, a_h^k)$.
16: $\mu_h^{k+1} \leftarrow \text{Proj}_B(\mu_h^k + \eta \tilde{\nabla}_{\mu_h} L(\pi^k, \mu^k))$.
17: end for
18: end for

$D(\pi, \pi^{k-1}) = \mathbb{E}_{\pi^{k-1}}[\sum_{h=1}^H D_{KL}(\pi_h(\cdot \mid s_h) || \pi_h^{k-1}(\cdot \mid s_h)) \mid s_1 = x]$, and $L_{k-1}(\pi)$ takes the form as

$$L_{k-1}(\pi) = J(\pi^{k-1}, r^{k-1}) + \mathbb{E}_{\pi^{k-1}}\left[\sum_{h=1}^H \langle \hat{Q}_h^{k-1}(s_h, \cdot), \pi_h(\cdot | s_h) - \pi_h^{k-1}(\cdot | s_h) \rangle_A \mid s_1 = x\right].$$

(3.2)

Here expectation $\mathbb{E}_{\pi}[\cdot]$ is taken with respect to the trajectory induced by $\pi$ and $\hat{Q}_h^{k-1}$ is an estimator of $Q_h^{k-1}$, which is specified later in (3.7). Such policy update formulation defined in (3.1) also corresponds to the policy optimization in online RL (Kakade, 2001; Schulman et al., 2015, 2017; Geist et al., 2019; Shani et al., 2020a; Cai et al., 2020).

By solving (3.1), we obtain the following closed-form update as,

$$\pi_h^k(\cdot | s) \propto \pi_h^{k-1}(\cdot | s) \cdot \exp\{\alpha \cdot \hat{Q}_h^{k-1}(s, \cdot)\} \text{ for any } (s, h) \in S \times [H],$$

(3.3)

which gives Line 5 of Algorithm 2.

**Policy Evaluation.** To evaluate the policy $\pi^k$ under the reward function $r^k$, we first construct estimated transition kernels $\hat{\mathcal{P}}^k = \{\hat{P}_h^k\}_{h \in [H]}$, through value-target regression (Ayoub et al., 2020) on finite historical data in Line 9, and then construct an estimator of the action-value functions by the Bellman equation in (2.2) with an extra bonus term to incorporate exploration in Line 10.

Specifically, in the $k$-th episode, we construct our estimated transition kernels $\hat{\mathcal{P}}^k = \{\hat{P}_h^k\}_{h \in [H]}$ as

$$\hat{P}_h^k(s^\prime \mid s, a) = \phi(s, a, s^\prime) \hat{a}_h^k \text{ for any } (h, s, a, s^\prime) \in [H] \times S \times A \times S,$$

(3.4)
where $\hat{\theta}_h^k$ is the minimizer of the regularized empirical mean-squared Bellman error as follows,

$$
\hat{\theta}_h^k = \arg\min_{\theta \in \mathbb{R}^d} \sum_{\tau=1}^{k-1} \left| \varphi_h^\top (s_h^\tau, a_h^\tau) \theta - \hat{V}_h^\top (s_h^\tau) \right|^2 + \lambda \| \theta \|^2, \quad \text{where } \varphi_h^\top (\cdot, \cdot) = \int_{\mathcal{S}} \phi (\cdot, \cdot, s') \hat{V}_{h+1}^\top (s') \, ds'.
$$

(3.5)

Here $\hat{V}_h^\top$ is constructed in Line 11 of Algorithm 2 and $\lambda > 0$ is the regularization parameter, which is specified later in Theorem 4.1. By solving (3.5), we obtain the closed-form update of $\hat{\theta}_h^k$ as follows,

$$
\hat{\theta}_h^k = (\Lambda_h^k)^{-1} \sum_{\tau=1}^{k-1} \varphi_h^\top (s_h^\tau, a_h^\tau) \hat{V}_{h+1}^\top (s_h^\tau), \quad \text{where } \Lambda_h^k = \sum_{\tau=1}^{k-1} \varphi_h^\top (s_h^\tau, a_h^\tau) \varphi_h^\top (s_h^\tau, a_h^\tau) \, d\cdot + \lambda I.
$$

(3.6)

We use $(r_h^k + \hat{\mathcal{P}}_h^k \hat{V}_{h+1}^k)(s, a)$ as an estimator of $Q_h^\pi^k(s, a)$. To further handle the uncertainty incurred by finite historical data and balance between exploration and exploitation, we employ optimism to incentivize exploration as many no-regret online RL algorithms do (Auer et al., 2002, 2009; Azar et al., 2017; Jin et al., 2018, 2019; Yang and Wang, 2020). Specifically, we incorporate a bonus term into the estimator $(r_h^k + \hat{\mathcal{P}}_h^k \hat{V}_{h+1}^k)(s, a)$ of $Q_h^\pi^k(s, a)$, i.e.,

$$
\hat{Q}_h^k(\cdot, \cdot) = \min \left\{ (r_h^k + \hat{\mathcal{P}}_h^k \hat{V}_{h+1}^k + \Gamma_h^k)(\cdot, \cdot), (H - h + 1) \sqrt{d} \right\}^+, \quad \text{where } \Gamma_h^k(s, a) = H \sqrt{d} \cdot \min \left\{ \kappa \cdot \| \varphi_h^k (s, a) \|_{\Lambda_h^k}^{-1}, 1 \right\}.
$$

(3.7)

(3.8)

Here $\hat{Q}_h^k$ is truncated into $[0, (H - h + 1) \sqrt{d}]$, and $\kappa > 0$ is a scaling parameter.

We highlight that the policy update stage of OGAP (Lines 4–12 of Algorithm 2) corresponds to the no-regret policy optimization in adversarial MDP with full information feedback (Shani et al., 2020b; Rosenberg and Mansour, 2019; Cai et al., 2020; Jin et al., 2020). Such tolerance of arbitrarily chosen reward function every episode paves the way for the alternate update between the policy and reward function.

### 3.1.2 Reward Update Stage

To discriminate the discrepancy between the expert policy $\pi^E$ and the current policy $\pi^k$, we update the reward parameter $\mu^{k+1}$ by maximizing GAIL objective function $L(\pi^k, \mu)$ defined in (2.9). By projected gradient ascent, we obtain the update of the reward parameter as follows,

$$
\mu_h^{k+1} = \text{Proj}_B \{ \mu_h^k + \eta \hat{\nabla}_\mu L(\pi^k, \mu^k) \},
$$

(3.9)

where $\eta$ is the stepsize, $\hat{\nabla}_\mu L(\pi^k, \mu^k)$ is an estimator of $\nabla_\mu L(\pi^k, \mu^k)$, and $\text{Proj}: \mathbb{R}^d \rightarrow B$ is the projection operator to restrict the updated reward parameter $\mu_h^{k+1}$ within the ball $B$ for any $h \in [H]$. Here $B$ is defined in (2.8). Without accessing to the true transition kernels of the underlying MDP and the expert policy $\pi^E$, we need to obtain an estimator $\hat{\nabla}_\mu L(\pi^k, \mu^k)$ in (3.9).

Specifically, to construct an estimator of $\nabla_\mu L(\pi^k, \mu^k)$, we first construct a Monte Carlo (MC) estimator $\hat{L}(\pi^k, \mu^k)$ of $L(\pi^k, \mu^k)$ as follows,

$$
\hat{L}(\pi^k, \mu^k) = \bar{J}(\pi^E, r^k) - \bar{J}(\pi^k, r^k).
$$

(3.10)

Here $\bar{J}(\pi^E, r^k)$ and $\bar{J}(\pi^k, r^k)$ are MC estimators of $J(\pi^E, r^k)$ and $J(\pi^k, r^k)$, respectively, which are defined as follows,

$$
\bar{J}(\pi^E, r^k) = \frac{1}{N_1} \sum_{\tau=1}^{N_1} \sum_{h=1}^{H} \psi(s_h^E, a_h^E) \mu_h, \quad \bar{J}(\pi^k, r^k) = \sum_{h=1}^{H} \psi(s_h^k, a_h^k) \mu_h.
$$

(3.11)
where we use \( N_1 \) trajectories in \( \bar{J}(\pi^E, r^k) \) and one trajectory in \( \bar{J}(\pi^k, r^k) \). Combining (3.10) and (3.11), we obtain that
\[
\hat{\nabla}_{\mu_k} L(\pi^k, \mu^k) = \frac{1}{N_1} \sum_{\tau=1}^{N_1} \psi(s_{h,\tau}^E, a_{h,\tau}^E) - \psi(s_{h,\tau}^k, a_{h,\tau}^k). \tag{3.12}
\]

We use \( \hat{\nabla}_{\mu_k} L(\pi^k, \mu^k) \) as an estimator of \( \nabla_{\mu_k} L(\pi^k, \mu^k) \), which gives Lines 13–17 of Algorithm 2.

### 3.2 Pessimistic Generative Adversarial Policy Optimization

To specialize Algorithm 1 to solve offline GAIL, we propose PGAP in Algorithm 3. Besides the policy update stage and reward update stage, PGAP further contains an initial construction stage, which constructs estimated transition kernels and bonus functions at the beginning of the algorithm. We detail the initial construction, the policy update, and the reward update stage as follows.

**Algorithm 3 Pessimistic Generative Adversarial Policy Optimization (PGAP)**

```
1: **Input:** Expert demonstration \( \mathbb{D}^E \), the additional dataset \( \mathbb{D}^A \), step size \( \eta, \alpha \)
2: Initialize \( \{\hat{Q}_h^k\}_{h \in [H]} \) as zero functions, \( \{\pi_h^k\}_{h \in [H]} \) as uniform distribution, \( \mu^1 = \{\mu_h^1\}_{h \in [H]} \) as zero vectors, and \( \{\hat{V}_{h+1}^k\}_{k \in \{0,1,\ldots,K\}} \) as zero functions over \( S \).
3: Construct \( \{\hat{P}_h\}_{h \in [H]} \) and \( \{\Gamma_h\}_{h \in [H]} \) from \( \mathbb{D}^A \) via (3.19) and (3.16), respectively. \( \triangleright \) Initial Construction
4: for \( k = 1, \ldots, K \) do
5:   for \( h = 1, \ldots, H \) do \( \triangleright \) Policy Improvement
6:     \( \pi_h^k(\cdot | \cdot) \propto \pi_h^{k-1}(\cdot | \cdot) \exp(\alpha \cdot \hat{Q}_h^{k-1}(\cdot, \cdot)) \).
7:   end for
8: for \( h = H, \ldots, 1 \) do \( \triangleright \) Policy Evaluation
9:   \( \hat{Q}_h^k(\cdot, \cdot) \leftarrow \max\{r_h^k + \hat{P}_h \hat{V}_{h+1}^k - \Gamma_h(\cdot, \cdot), 0\} \).
10:  \( \hat{V}_h^k(\cdot) \leftarrow \langle \hat{Q}_h^k(\cdot, \cdot), \pi_h^k(\cdot | \cdot) \rangle_A \).
11: end for
12: Construct \( \{\nabla_{\mu_h} \bar{J}(\pi^E, r^\mu)\}_{h \in [H]} \) via (3.12). \( \triangleright \) Reward Update
13: Construct \( \{\nabla_{\mu_h} \bar{J}(\pi^k, r^\mu)\}_{h \in [H]} \) via Proposition 3.3.
14: for \( h = 1, \ldots, H \) do
15:   \( \nabla_{\mu_h} \hat{L}(\pi^k, \mu^k) \leftarrow \nabla_{\mu_h} \bar{J}(\pi^E, r^\mu) |_{\mu = \mu^k} - \nabla_{\mu_h} \bar{J}(\pi^k, r^\mu) |_{\mu = \mu^k} \).
16:   \( \mu_{h+1}^k \leftarrow \text{Proj}_B[\mu_h^k + \eta \nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)] \).
17: end for
18: end for
19: **Output:** \( \hat{\pi} = \text{Unif}(\{\pi_h^k\}_{h \in [K]}). \)
```

### 3.2.1 Initial Construction Stage

In Line 3 of PGAP, we construct estimated transition kernels \( \{\hat{P}_h\}_{h \in [H]} \) and uncertainty quantifiers \( \{\Gamma_h\}_{h \in [H]} \) via the additional dataset \( \mathbb{D}^A \). Before we detail such construct, we first introduce the following definition of uncertainty quantifiers (Jin et al., 2021) with the confidence parameter \( \xi \in (0, 1) \), which quantifies the uncertainty.
**Definition 3.1 (ξ-Uncertainty Quantifier).** We say \( \{\Gamma_h\}_{h \in [H]} \) with \( \Gamma_h : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \) are ξ-uncertainty quantifiers for estimated kernels \( \tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}_h\}_{h \in [H]} \) with respect to \( \mathbb{P}_D \) if the event

\[
\mathcal{E} = \{ |\tilde{\mathcal{P}}_h \hat{V}(s,a) - \mathcal{P}_h \hat{V}(s,a)| \leq \Gamma_h(s,a) \text{ for any } (s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H] \text{ and any } \hat{V} : \mathcal{S} \to [0,H \sqrt{d}] \}
\]

satisfies \( \mathbb{P}_D(\mathcal{E}) \geq 1 - \xi/2 \). Here \( \mathbb{P}_D \) is with respect to the joint distribution of \( \mathbb{D}^A \cup \mathbb{D}^F \).

We remark that the ξ-uncertainty quantifiers in Definition 3.1 is a counterpart of the bonus functions in OGAP. Recalling that \( |r_h(s,a)| \leq \sqrt{d} \) for any \( (s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H] \), Definition 3.1 implies that with probability at least \( 1 - \xi/2 \), the deviation between the true Bellman equation in (2.2) with \( \mathcal{P} \) and the estimated Bellman equation in (2.2) with \( \tilde{\mathcal{P}} \) is upper bounded by the ξ-uncertainty quantifier \( \{\Gamma_h\}_{h \in [H]} \).

Now, we construct the estimated transition kernels and uncertainty quantifiers as follows. Specifically, we first construct the initial estimated transition kernels \( \tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}_h\}_{h \in [H]} \) as

\[
\tilde{\mathcal{P}}_h(s'|s,a) = \phi(s,a,s') \tilde{\theta}_h \quad \text{for any } (s,a,s',h) \in \mathcal{S} \times \mathcal{A} \times \mathcal{S} \times [H],
\]

where \( \tilde{\theta}_h \) is defined as follows,

\[
\tilde{\theta}_h = \arg\min_{\theta \in \mathbb{R}^d} \sum_{t=1}^{N_2} \int_{\mathcal{S}} \left| \phi(s_h^t, a_h^t, s_t) \theta - \delta_{s_{t+1}}(s') \right|^2 ds' + \lambda \|\theta\|_2^2.
\]

Here \( \lambda > 0 \) is the regularization parameter and \( \delta_x(y) \) is Dirac function. By solving (3.14), we obtain the closed-form solution of \( \tilde{\theta}_h \) as follows,

\[
\tilde{\theta}_h = \Lambda_h^{-1} \sum_{t=1}^{N_2} \phi(s_h^t, a_h^t, s_t'), \quad \text{where } \Lambda_h = \sum_{t=1}^{N_2} \int_{\mathcal{S}} \phi(s_h^t, a_h^t, s') \phi(s_h^t, a_h^t, s')^\top ds' + \lambda I.
\]

Given (3.13) and (3.15), we further construct ξ-uncertainty quantifiers \( \{\Gamma_h\}_{h \in [H]} \) as follows,

\[
\Gamma_h(s,a) = H \sqrt{d} \int_{\mathcal{S}} \Gamma_h^P(s,a,s') ds', \quad \text{where } \Gamma_h^P(s,a,s') = \min \left\{ \kappa \cdot \|\phi(s,a,s')\|_{\Lambda_h^{-1}}^1, 1 \right\}.
\]

Here \( \kappa > 0 \) is a scaling parameter. In the following lemma, we show that \( \{\Gamma_h\}_{h \in [H]} \) in (3.16) are ξ-uncertainty quantifiers for \( \tilde{\mathcal{P}} \) in (3.13) if \( \kappa \) is properly chosen.

**Lemma 3.2.** In (3.16), we set \( \lambda = 1 \) and \( \kappa = c \cdot R \sqrt{d \log(dHN_2/\xi)} \), where \( c > 0 \) is an absolute constant and \( \xi \in (0,1) \) is the confidence parameter. Under Assumption 2.1, \( \{\Gamma_h\}_{h \in [H]} \) in (3.16) are ξ-uncertainty quantifiers for \( \tilde{\mathcal{P}} \), defined in Definition 3.1.

**Proof.** See Appendix B.1 for a detailed proof.

However, given \( (s,a) \in \mathcal{S} \times \mathcal{A} \), the initial estimated transition kernels \( \tilde{\mathcal{P}}_h(\cdot | s,a) \) in (3.13) is not guaranteed to lie within \( \Delta(\mathcal{S}) \). Different from OGAP, we are incapable to update reward functions based on the newly sampled trajectory in offline GAIL. This difference is crucial to the analysis of PGAP (Theorem 4.2), which relies on the fact that estimated GAIL objective function \( \tilde{\mathcal{L}}(\pi,\mu) \) defined in (3.21) is concave for \( \mu \) (we prove it in Lemma 6.3). To address this issue, we define a feasible estimation parameter domain \( \Theta \) and choose the estimated transition kernel parameter \( \tilde{\theta} \) from the feasible domain \( \Theta \) (Zhou et al., 2021). Formally, we take \( \Theta = \Theta_1 \cap \Theta_2 \) with \( \Theta_1 \) and \( \Theta_2 \) defined as follows,

\[
\Theta_1 = \left\{ \tilde{\theta} : \text{if } \xi \text{ holds, then it satisfies that } |\tilde{\mathcal{P}}_h \hat{V}(s,a) - \tilde{\mathcal{P}}_h \hat{V}(s,a)| \leq \Gamma_h(s,a)ight. \\
\left. \text{for any } (s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H] \text{ and any } \hat{V} : \mathcal{S} \to [0,H \sqrt{d}] \right\},
\]

\[
\Theta_2 = \left\{ \tilde{\theta} : \tilde{\mathcal{P}}_h(\cdot | s,a) \in \Delta(\mathcal{S}) \text{ for any } (s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H] \right\},
\]

12
where $\mathcal{E}$ and $\{\mathcal{P}_h\}_{h \in [H]}$ are defined in Definition 3.1 and (3.13), respectively. We remark that under Assumption 2.1, the true transition kernel parameter $\theta = \{\theta_h\}_{h \in [H]}$ lies within the feasible estimation parameter domain $\Theta$, which implies that $\Theta$ is not empty. Thus, similar to (3.14) but enforcing the estimated transition kernel parameter to lie within $\Theta$, we define $\hat{\theta} = \{\hat{\theta}_h\}_{h \in [H]}$ as follows,

$$
\hat{\theta}_h = \arg \min_{\theta \in \Theta} \frac{N}{t=1} \sum \int_S \left[ \phi(s_h, a_h, s')^T \theta - \delta_{s_h+1}(s') \right]^2 ds' + \lambda \|\theta\|_2^2, \tag{3.18}
$$

where the minimization is taken over $\Theta$. Similarly, we construct the estimated transition kernel $\hat{\mathcal{P}} = \{\hat{\mathcal{P}}_h\}_{h \in [H]}$ as follows,

$$
\hat{\mathcal{P}}_h(s', a) = \phi(s, a, s')^T \hat{\theta}_h \quad \text{for any } (s, a, s', h) \in S \times A \times S \times [H], \tag{3.19}
$$

where $\hat{\theta}_h$ is defined in (3.18).

### 3.2.2 Policy Update Stage

As a pessimistic variant of OGAP in Algorithm 2, the policy update stage of PGAP (Lines 5–11 of Algorithm 3) consists of two steps: (i) policy improvement (Lines 5–7) and (ii) policy evaluation (Lines 8–11). In the stage of policy improvement, we adopt the same idea as in OGAP, which employs mirror descent to update the current policy. In the stage of policy evaluation, instead of the optimism principle, we employ the pessimism principle to construct the estimated action-value functions via the additional dataset $\mathcal{D}^A$, which is not assumed to be well-explored as specified later in §4.2. The principle pessimism-in-face-of-uncertainty guides the agent to be conservative to visit the states and actions that are less covered by the additional dataset $\mathcal{D}^A$ (Kumar et al., 2020; Jin et al., 2021; Liu et al., 2020; Yu et al., 2020, 2021; Buckman et al., 2020). Specifically, we construct the estimated action-value functions as follow,

$$
\hat{Q}_h^k(\cdot, \cdot) = \max \{ (r_h^k + \hat{\mathcal{P}}_h \tilde{\Gamma}_h^k)(\cdot, \cdot), 0 \},
$$

where $\{\hat{\mathcal{P}}_h\}_{h \in [H]}$ are the estimated transition kernels and $\{\Gamma_h\}_{h \in [H]}$ are the uncertainty quantifiers defined in Line 3 of PGAP.

### 3.2.3 Reward Update Stage

Similar to the reward update stage of OGAP, in the reward update stage of PGAP, we update the reward parameter as follows,

$$
\mu_{h+1}^k = \text{Proj}_B \{\mu_h^k + \eta \nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)\}. \tag{3.20}
$$

Here $\eta$ is the stepsize, $\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)$ is an estimator of $\nabla_{\mu_h} L(\pi^k, \mu^k)$, and $\text{Proj}: \mathbb{R}^d \rightarrow B$ is the projection operator to restrict the updated reward parameter $\mu_{h+1}^k$ within the ball $B$ for any $h \in [H]$. Here $B$ is defined in (2.8). To achieve (3.20), we also need to obtain the estimated gradient $\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)$ in (3.20). However, since the agent in offline GAIL cannot interact with the environment to collect the state-action pairs following current policy $\pi^k$, the estimator in (3.12) for OGAP is not applicable to PGAP. Instead, we construct an estimator $\hat{L}(\pi^k, \mu^k)$ for $L(\pi^k, \mu^k)$ and use its gradient $\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)$ to estimate $\nabla_{\mu_h} L(\pi^k, \mu^k)$. Specifically, we define the estimator $\hat{L}(\pi^k, \mu^k)$ as

$$
\hat{L}(\pi^k, \mu^k) = \hat{J}(\pi^k, r^k) - \hat{J}(\pi^k, r^k), \tag{3.21}
$$
where \( \tilde{J}(\pi^E, r^k) \) is a MC estimator of \( J(\pi^E, r^k) \). Here we estimate \( J(\pi^k, r^k) \) with \( \tilde{J}(\pi^k, r^k) = \tilde{V}^k_1(x) \), which is constructed in Line 9 of PGAP. Based on (3.21), we construct an estimator \( \hat{\nabla}_{\mu_h} L(\pi^k, \mu^k) \) of \( \nabla_{\mu_h} J(\pi^k, r^k) \) as follows,

\[
\hat{\nabla}_{\mu_h} L(\pi^k, \mu^k) = \nabla_{\mu_h} \tilde{J}(\pi^k, \mu^k) = \nabla_{\mu_h} \tilde{J}(\pi^E, r^k) - \nabla_{\mu_h} \tilde{J}(\pi^k, r^k),
\]

where \( \nabla_{\mu_h} \tilde{J}(\pi^k, r^k) \) is characterized in the following proposition.

**Proposition 3.3.** If we define \( \{\hat{Q}^{k, r^m}_h\}_{h \in [H]} \) and \( \{\hat{V}^{k, r^m}_{h+1}\}_{h \in [H]} \) as

\[
\hat{v}^{k, r^m}_h(\cdot) = 0,
\]

\[
\hat{Q}^{k, r^m}_h(\cdot, \cdot) = \max \{ (r^m_h + \hat{P}_h \hat{V}^{k, r^m}_{h+1} - \Gamma_h)(\cdot, \cdot), 0 \},
\]

\[
\hat{V}^{k, r^m}_h(\cdot, \cdot) = \langle \hat{Q}^{k, r^m}_h(\cdot, \cdot), \hat{\pi}^k(\cdot) \rangle_A,
\]

for all \( h \in [H] \) and \( \mu \in S \). It suggests that \( \hat{Q}^k_h = \tilde{Q}^{k, r^m}_h \) and \( \hat{V}^k_h = \tilde{V}^{k, r^m}_h \), where \( \tilde{Q}^k_h \) and \( \tilde{V}^k_h \) are constructed in the policy evaluation stage in PGAP (Algorithm 3 Lines 8–11). We can solve \( \nabla_{\mu_h} \hat{V}^{k, r^m}_t(x) \) recursively as follows,

\[
\nabla_{\mu_h} \hat{v}^{k, r^m}_{t}(s_t) = \left\{ \begin{array}{ll}
(\hat{\pi}^k(\cdot | s_t)g^k_t(s_t, \cdot), [\hat{P}_t(\nabla_{\mu_h} \hat{V}^{k, r^m}_{t+1})](s_t, \cdot))_A & \text{if } 1 \leq t \leq h - 1, \\
(\hat{\pi}^k(\cdot | s_t)g^k_t(s_t, \cdot), \nabla_{\mu_h} r^m_h(s_t, \cdot))_A & \text{if } t = h,
\end{array} \right.
\]

where \( s_1 = x, [\hat{P}_t f](s,a) \) is a shorthand of \( \int_S f(s') \hat{P}_t(s' | s, a) ds' \) and \( g^k_t : S \times A \to \mathbb{R} \) is defined as

\[
g^k_t(s,a) = 1 \{ \hat{Q}^k_h(s,a) > 0 \},
\]

for all \( (s,a) \in S \times A \). Here \( 1 \{ \cdot \} \) is the indicator function.

**Proof.** Take gradient toward \( \mu_h \) and apply chain rule on (3.23), then we obtain Proposition 3.3. \( \square \)

## 4 Main Results

In this section, we present the theoretical analysis for OGAP and PGAP in §4.1 and §4.2, respectively. Specifically, in §4.1 we upper bound the regret of OGAP. In §4.2, we upper bound the optimality gap of PGAP under no coverage assumption and propose a lower bound to show that PGAP achieves minimax optimality in the utilization of the additional dataset \( \mathbb{D}^A \). Moreover, under the assumption that the additional dataset \( \mathbb{D}^A \) has sufficient coverage, we establish the global convergence guarantee for PGAP and discuss how the additional dataset \( \mathbb{D}^A \) facilitates our policy learning.

### 4.1 Analysis of OGAP

We derive an upper bound of the regret of OGAP in the following theorem.

**Theorem 4.1** (Regret of OGAP). In Algorithm 2, we set

\[
\alpha = \sqrt{2 \log |A| / (H^2 \sqrt{dK})}, \quad \lambda = 1, \quad \kappa = C \sqrt{d \log (HdK/\xi)}, \quad \eta = 1 / \sqrt{HK},
\]

where \( C > 0 \) is a constant. Under Assumption 2.1, it holds with probability at least \( 1 - \xi \) that

\[
\text{Regret}(K) \leq \mathcal{O}(H^2 d^{3/2} K^{1/2} \log (HdK/\xi)) + K \delta_{N_1},
\]

where \( \delta_{N_1} = \mathcal{O}(H^{3/2} d N_1^{-1/2} \log (N_1/\xi)) \).
The first term on the right-hand side of (4.1) scales with $K^{1/2}$, which attains the optimal dependency on $K$ for online RL. The second term on the right-hand side of (4.1) is linear in $K$ and depends on the MC estimation error $\delta_{N_1}$. As the statistical error from the MC estimation, the error term $\delta_{N_1}$ is inevitable and independent of GAIL algorithm, since we cannot access the expert policy but expert demonstration with $N_1$ trajectories. When the number of trajectories $N_1$ in the expert demonstration is sufficiently large such that $N_1 = \Omega(K)$, the first term on the right-hand side of (4.1) dominates the regret upper bound so that the regret of OGAP scales with $K^{1/2}$. The dependency of $H, K$, and $N_1$ correspond to $\tilde{O}(H^2|S||A|^{1/2}K^{1/2} + H^{3/2}|S|^{1/2}|A|^{1/2}K^{-1/2})$ regret in the tabular case, established by Shani et al. (2021). If we consider the case $K = \Omega(N_1^{3/2})$, then the average regret decays at a rate of $N_1^{-1/2}$ and the dependency for $H$ turns from $H^2$ into $H^{3/2}$. As $K$ and $N_1$ both tend to infinity, the average regret also shrinks to zero, meaning that the output policy has the same performance on average with the expert policy with respect to the linear reward set $\mathcal{R}$.

**Relationship with Online RL** According to Assumption 2.1, if we constrain the reward set $\mathcal{R}$ to a fixed reward function $r = \{r_h\}_{h \in [H]}$, then GAIL (2.4) is reduced to RL, with respect to an episodic MDP $(\mathcal{S}, \mathcal{A}, H, \mathcal{P}, r)$, where $\mathcal{S}, \mathcal{A}, H, \mathcal{P}$ are the same as the ones in Assumption 2.1. Hence OGAP can also be considered as RL algorithm for episodic MDP with linear function approximation. From the aspect for information-theory, the lower bound of the regret of any online RL algorithm is $\tilde{O}(K)$ even in tabular case (Jin et al., 2018). Since the reward set is singleton, OGAP needs not MC estimation and has regret $\tilde{O}(K)$, achieving such lower bound and revealing minimax optimality.

### 4.2 Analysis of PGAP

We upper bound the optimality gap of PGAP in the following theorem.

**Theorem 4.2.** (Optimality Gap of PGAP). In Algorithm 3, we set

$$\lambda = 1, \quad \kappa = cR\sqrt{d\log(HdK)/\xi}, \quad \alpha = \sqrt{2\log(\text{vol}(\mathcal{A}))/\left(H^2\sqrt{dK}\right)}, \quad \eta = 1/\sqrt{HK},$$

where $c > 0$ is a constant. Under Assumption 2.1, $\{\Gamma_h\}_{h=1}^H$ constructed in §3.2.1 are $\xi$-uncertainty qualifiers defined in Definition 3.1. It holds with probability at least $1 - \xi$ that

$$D_{\mathcal{R}}(\pi^E, \hat{\pi}) \leq \mathcal{O}(H^2dK^{-1/2}) + \delta_{N_1} + \text{IntUncert}_{\mathcal{D}^A}^{\pi^E}, \quad (4.2)$$

where $\hat{\pi}$ is the output policy of Algorithm 3, $\text{IntUncert}_{\mathcal{D}^A}^{\pi^E} = 2\sum_{h=1}^H E_{\pi^E}[\Gamma_h(s_h, a_h) \mid s_1 = x]$, and $\delta_{N_1} = \mathcal{O}(H^{3/2}dN_1^{-1/2}\log(N_1/\xi))$.

**Proof.** See §6.1 for a proof sketch. 

In Theorem 4.2, the first term on the right-hand side of (4.2) is an optimization error term, which is independent of both expert demonstration $\mathcal{D}^E$ and the additional dataset $\mathcal{D}^A$. The optimization error term decays at a rate of $K^{-1/2}$. The second term on the right-hand side of (4.2) is related to the MC estimation error and also occurs in the upper bound of the regret of OGAP as in Theorem 4.1. The third term on the right-hand side of (4.2) is an intrinsic error $\text{IntUncert}_{\mathcal{D}^A}^{\pi^E}$, which arises from the uncertainty of estimating Bellman equation (2.2) based on the additional dataset $\mathcal{D}^A$. In the structure of the intrinsic error
IntUncert^π_{E,d}, we note that the expectation is taken with respect to the trajectory induced by the expert policy π_E, which measures the quality of the additional dataset D_A and is irrelevant to the training process. We clarify that the occurrence of IntUncert^π_{E,d} implies that the optimality gap only depends on how well the additional dataset D_A covers the trajectories of the expert policy π_E and it is not necessary to assume that the additional dataset D_A is well-explored. Hence, Theorem 4.2 relies on no assumption on the coverage of the additional dataset D_A, such as uniformly lower bound of densities of visitation measures, the behavior policy to be upper bounded uniformly over the state-action space, the concentrability coefficients are uniformly upper bounded, or even the partial coverage assumption (Antos et al., 2007; Munos and Szepesvári, 2008; Yang et al., 2020b,a; Levine et al., 2020; Uehara et al., 2020; Siegel et al., 2020; Wang et al., 2020b; Zhang et al., 2020; Xu et al., 2020).

We also highlight that Theorem 4.2 can be generalized to the case when the transition kernel is non-linear, only if we explicitly find proper uncertainty quantifiers \{\Gamma_h\}_{h=1}^H satisfying Definition 3.1 for the estimated transition kernel.

Pessimism Guarantees Minimax Utilization. With a well-explored and large enough dataset, the full information about the transition kernel can be extracted by the agent and supports the agent to make correct decision. But when we assert no restriction on the dataset, it is challenging to do the same because of the distribution shift and extrapolation error on the states and actions that are less covered by the dataset. This problem has been studied widely in offline RL (Fujimoto et al., 2019a; Kumar et al., 2020; Fujimoto et al., 2019b; Levine et al., 2020; Jin et al., 2021) and Wang et al. (2020a) even propose that the lower bound of offline RL can grow exponentially with the horizon under linear approximation and no assumption on the dataset. Hence how to cooperate the additional dataset D_A to aid the agent to improve the performance in offline GAIL is also difficult since the additional dataset D_A is not assumed to be well-explored. Inspired by the spirit of being conservative in offline RL (Fujimoto et al., 2019b; Kumar et al., 2020; Jin et al., 2021), we propose a pessimistic variant of policy optimization in the policy update stage of PGAP (Lines 5–11 of Algorithm 3), which ensures that PGAP utilize the information of the additional dataset D_A in the sense of minimax optimality. To illustrate it, we present the following proposition, which is adapted from Theorem 4.7 in Jin et al. (2021).

**Proposition 4.3** (Minimax Optimality in Utilizing Additional Dataset). For the output policy \text{Algo}(\overline{D}) of any offline algorithm only based on the dataset \overline{D}, there exists a linear kernel MDP M (S, A, H, P, r) with an initial state x ∈ S, a dataset \overline{D} compliant with M, and a reward set R, such that

\[
\max_{\pi^E \in \Delta(A|S,H)} \mathbb{E}_{\pi^E} \left[ \frac{\mathbb{D}_R(\pi^E, \text{Algo}(\overline{D}))}{\text{Information}_{\overline{D}}^E} \right] \geq c,
\]

where c > 0 is an absolute constant, \mathbb{E}_{\pi^E}[\cdot] is taken expectation with respect to randomness of the dataset \overline{D}, and Information_{\overline{D}}^E is defined as

\[
\text{Information}_{\overline{D}}^E = (\text{Vol}(S))^{-1} \cdot \mathbb{E}_{\pi^E} \left[ \sum_{h=1}^H \int_S \|\phi(s_h, a_h, s')\|_{\Lambda_h^{-1}} \, ds' \bigg| s_1 = x \right],
\]

where \Lambda_h is only determined by the dataset \overline{D} and takes the same form as in (3.15).

**Proof.** See §6.2 for a detailed proof.
According to Proposition 4.3, if we consider the additional dataset $\mathbb{D}^A$ as $\mathbb{B}$, it reveals that $\text{IntUncert}_{\mathbb{D}^A}^{(E)}$ in the upper bound of optimality gap of PGAP (Theorem 4.2) matches the lower bound up to $H, \sqrt{d}, \text{Vol}(S)$, and the scaling parameter $\kappa$ defined in (3.16). Though we do not assume any restriction on the additional dataset $\mathbb{D}^A$, owing to the pessimism principle, PGAP ensures the good utilization of the additional dataset $\mathbb{D}^A$ even in the worst case.

In the sequel, we show that PGAP is provably efficient and attains global convergence under the assumption that the additional dataset $\mathbb{D}^A$ has sufficient coverage. We first impose such an assumption on the additional dataset $\mathbb{D}^A$ as follows.

**Assumption 4.4 (Sufficient Coverage).** The additional dataset $\mathbb{D}^A$ has sufficient coverage with the expert policy $\pi^E$, that is, there exists an absolute constant $c^1 > 0$ such that the event

$$
\mathcal{E}^1 = \left\{ \frac{1}{N_2} \sum_{\tau=1}^{N_2} \int_{S} \phi(s_h^\tau, a_h^\tau, s') \phi(s_h^\tau, a_h^\tau, s')^\top ds' \geq c^1 \cdot E_{\pi^E} \left[ \int_{S} \phi(s_h, a_h, s') \phi(s_h, a_h, s')^\top ds' \right] \right\}
$$

satisfies $\mathbb{P}(\mathcal{E}^1) \geq 1 - \xi/2$. Here the expectation $E_{\pi^E}[\cdot]$ is taken with respect to the trajectory induced by $\pi^E$ and $\xi \in (0, 1)$ is the confidence level.

Assumption 4.4 implies that the additional dataset $\mathbb{D}^A$ with sufficient coverage covers the trajectories of the expert policy $\pi^E$ averagely in the sense of the feature map outer product $\int_{S} \phi(\cdot, \cdot, s') \phi(\cdot, \cdot, s')^\top ds'$. We highlight that sufficient coverage does not assume that the additional dataset $\mathbb{D}^A$ to be well-explored dataset (Antos et al., 2007; Munos and Szepesvári, 2008; Zhang et al., 2020; Xu et al., 2020; Yang et al., 2020b,a; Levine et al., 2020), e.g. restricting that the densities of visitation measures of the behavior policy generating the dataset are uniformly lower bounded, i.e.

$$\inf_{(s,a,h) \in S \times A \times [H]} \left[ \rho^A_h(s,a) \right] = c > 0,$$

where $\rho^A_h$ is the density of visitation measure on $S \times A$ induced by policy $\pi$ at the $h$-step and $\pi^A$ is the policy of the experimenter who collected the additional dataset $\mathbb{D}^A$. We also remark that sufficient coverage is a weaker restriction than the partial coverage assumption in offline RL (Uehara et al., 2020; Siegel et al., 2020; Wang et al., 2020b), which assumes that

$$\sup_{(s,a,h) \in S \times A \times [H]} \left[ \rho^E_h(s,a) / \rho^A_h(s,a) \right] = C^{\pi^E} < \infty.$$

Under Assumption 4.4, we present the following corollary.

**Corollary 4.5.** Under Assumption 4.4 and the same assumptions as in Theorem 4.2, it holds with probability at least $1 - \xi$ that

$$D_R(\pi^E, \hat{\pi}) \leq \bar{\mathcal{O}} \left( H^2 dK^{-1/2} + H^2 d^{3/2} N^{-1/2}_2 + H^{3/2} d N^{-1/2}_1 \right),$$

where $\hat{\pi}$ is the output of Algorithm 3.

**Proof.** See §6.3 for a detailed proof.
Corollary 4.5 proves that under Assumption 4.4 the intrinsic error \( \text{IntUncert}_{\text{opt}}^E \) in Theorem 4.2 decays at a rate of \( N_2^{-1/2} \), showing that PGAP attains global convergence. We remark that this result does not require the additional dataset \( \mathcal{D}^A \) to be well-explored and only relies on a much weaker assumption as sufficient coverage in Assumption 4.4. This improved result also responds to the information-theoretical lower bound \( \Omega(H^2N_2^{-1/2}) \) for offline policy evaluation (Duan et al., 2020). When \( K, N_1, \) and \( N_2 \) all tend to infinity, the optimality gap shrinks to zero as a negative square-root rate, meaning that the output policy has the same performance with the expert policy with respect to the reward set \( \mathcal{R} \). Our subsequent discussion illustrates how the additional dataset \( \mathcal{D}^A \) facilitates our policy learning in PGAP.

**The Additional dataset Contributes.** We illustrate the contribution of the additional dataset \( \mathcal{D}^A \) by considering the following two cases.

1. Without accessing an additional dataset, PGAP is also applicable by simply treating the given expert demonstration \( \mathcal{D}^E \) as the additional dataset \( \mathcal{D}^A \) in PGAP (Algorithm 3), that is, \( \mathcal{D}^A = \mathcal{D}^E \), which satisfies Assumption 4.4. If we set \( K = \Omega(N_1) \), then by Corollary 4.5, we have
   \[
   D_R(\pi^E, \hat{\pi}) = \tilde{O}(H^2d^{3/2}N_1^{-1/2}). \tag{4.3}
   \]
   By comparing (4.3) and (4.4), we observe that the additional dataset \( \mathcal{D}^A \) helps decrease the dependency for \( H \) and \( d \) in the optimality gap by \( H^{1/2} \) and \( d^{1/2} \). It implies that we can use a much smaller expert demonstration \( \mathcal{D}^E \) to learn a policy as good as the expert policy \( \pi^E \), especially when horizon \( H \) and feature space dimension \( d \) are sufficiently large. This improvement is meaningful in the imitation learning tasks, such as autonomous driving and robotics (Demiris* and Johnson, 2003; Hussein et al., 2017; Kebria et al., 2019; Jalali et al., 2019).

2. If we have access to a large enough additional dataset with sufficient coverage, taking \( N_2 = \Omega(d^2HN_1) \) for instance, then by setting \( K = \Omega(N_2) \), we upper bound the optimality gap of PGAP as follows,
   \[
   D_R(\pi^E, \hat{\pi}) = \tilde{O}(H^{3/2}dN_1^{-1/2}). \tag{4.4}
   \]

By comparing (4.3) and (4.4), we observe that the additional dataset \( \mathcal{D}^A \) helps decrease the dependency for \( H \) and \( d \) in the optimality gap by \( H^{1/2} \) and \( d^{1/2} \). It implies that we can use a much smaller expert demonstration \( \mathcal{D}^E \) to learn a policy as good as the expert policy \( \pi^E \), especially when horizon \( H \) and feature space dimension \( d \) are sufficiently large. This improvement is meaningful in the imitation learning tasks, such as autonomous driving and robotics (Demiris* and Johnson, 2003; Hussein et al., 2017; Kebria et al., 2019; Jalali et al., 2019).

5 **Proof Sketch: Analysis of PGAP**

5.1 **Proof of Theorem 4.1**

*Proof.* Recall the definition of regret in (2.5) and GAIL objective function \( L(\pi, \mu) \) in (2.4), we decompose the regret as follows,

\[
\text{Regret}(K) = \sup_{\mu \in \mathcal{S}} \sum_{k=1}^{K} L(\pi^k, \mu)
\leq \sum_{k=1}^{K} [J(\pi^E, r^k) - J(\pi^k, r^k)] + \sup_{\mu \in \mathcal{S}} \sum_{k=1}^{K} [L(\pi^k, r^k) - L(\pi^k, r^k)]. \tag{5.1}
\]

The intuition of decomposition in (5.1) is to respectively deal with regret occurring in the stage of policy update and reward update, which are denoted by term (A) and term (B).
**Upper bound of term (A) in (5.1).** In what follows, we upper bound term (A) in (5.1). For the simplicity of later discussion, we define the model prediction error for estimating Bellman equation (2.2) in the h-th step of k-th episode in Algorithm 3 with reward function \( r^\mu \) as follows,

\[
\epsilon^k_h(s, a) := r^k_h(s, a) + [P_h \hat{V}^k_{h+1}] (s, a) - \hat{Q}^k_h(s, a),
\]

for any \((s, a) \in S \times A, h \in [H], \mu \in S.\)

First, we introduce a regret decomposition lemma to decompose term (A) in (5.1).

**Lemma 5.1** (Regret Decomposition for Policy Update). It holds for any initial state \( x \in S \) that

\[
\sum_{k=1}^{K} (V_{1, \pi^k}(x) - V_{1, \pi_k}(x)) = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ (\hat{Q}^k_h(s_h, \cdot) - \pi^k_h(\cdot | s_h)) \mid s_1 = x \right] + \mathcal{M}_{K, H, 2} \tag{5.3}
\]

Here \( \epsilon^k_h \) is the model prediction error defined in (5.2), and \( \{ \mathcal{M}_{K, H, m} \}_{(k, h, m) \in [K] \times [H] \times [2]} \) is a martingale adapted to the filtration \( \{ \mathcal{F}_{k, h, m} \}_{(k, h, m) \in [K] \times [H] \times [2]} \), with respect to the timestep index \( t(k, h, m) = (k - 1) \cdot 2H + (h - 1) \cdot 2 + m. \)

**Proof.** See Appendix A.1 for a detailed proof.

Lemma 5.1 shows that term (A) in (5.1) can be decomposed into three terms as follows,

\[
\begin{align*}
(A) &= \sum_{k=1}^{K} \left( V_{1, \pi^k}(x) - V_{1, \pi_k}(x) \right) \\
&= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ (\hat{Q}^k_h(s_h, \cdot) - \pi^k_h(\cdot | s_h)) \mid s_1 = x \right] + \mathcal{M}_{K, H, 2} \tag{A1} \\
&+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ \epsilon^k_h(s_h, a_h) \mid s_1 = x \right] - \epsilon^k_h(s_h^*, a_h^*) \tag{A2} \\
&+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ \epsilon^k_h(s_h, a_h) \mid s_1 = x \right] - \epsilon^k_h(s_h^*, a_h^*) \tag{A3} \\
\end{align*}
\]

To upper bound term (A1) and term (A2) in (5.4), we introduce the following two lemmas, respectively.

**Lemma 5.2** (Performance Improvement). If we set \( \alpha = \sqrt{2 \log(\text{vol}(A)) / (H^2 K \sqrt{d})} \) in the policy update stage of OGAP (Line 3 of Algorithm 2), then under Assumption 2.1, for any initial state \( x \in S \), it holds that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ (\hat{Q}^{k-1}_h, \pi^k_h - \pi^{k-1}_h) \mid s_1 = x \right] \leq \sqrt{2H^4 \sqrt{d} K \log(\text{vol}(A))}.
\]

**Proof.** See Appendix A.2 for a detailed proof.

**Lemma 5.3.** It holds that

\[
|\mathcal{M}_{K, H, 2}| \leq 4 \sqrt{H^3 d K \log(8/\xi)}.
\]

with probability at least \( 1 - \xi/4 \), where \( \mathcal{M}_{K, H, 2} \) is the martingale defined in (5.3).
Proof. See Appendix A.3 for a detailed proof.

To upper bound the term (A3) in (5.4), we introduce the following two lemmas.

**Lemma 5.4** (Optimism). Under Assumption 2.1, it holds with probability at least $1 - \xi/4$ that

$$-2\Gamma^k_h(s, a) \leq \iota^k_h(s, a) \leq 0$$

for any $(h, k, s, a) \in [H] \times [K] \times S \times A$,

where $\iota^k_h$ is the model prediction error defined in (5.3).

Proof. See Appendix A.4 for a detailed proof.

**Lemma 5.5.** Under Assumption 2.1, it holds that

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma^k_h(s^k_h, a^k_h) \leq C' \sqrt{H^3dK} \cdot \log(HdK/\xi),$$

where $C' > 0$ is an absolute constant.

Proof. See Appendix A.5 for a detailed proof.

Lemma 5.4 implies that $\mathbb{E}_{\pi^k}(s_h, a_h \mid s_1 = x) \geq 0$ with high probability. Combining Lemmas 5.4 and 5.5, it holds with probability at least $1 - \xi/4$ that

$$(A3) \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \iota^k_h(s^k_h, a^k_h) \leq 2 \sum_{k=1}^{K} \sum_{h=1}^{H} \Gamma^k_h(s^k_h, a^k_h) \leq 2C' \sqrt{H^3dK} \cdot \log(HdK/\xi),$$

with probability at least $1 - \xi/4$.

Now, by plugging Lemma 5.2, Lemma 5.3, and (5.5) into the formulation of term (A) in (5.1), we obtain that

$$(A) \leq \sqrt{2H^4LdK \log(\text{vol}(A))} + 4\sqrt{H^3dK \log(8/\xi)} + 2C' \sqrt{H^3dK} \cdot \log(HdK/\xi)$$

$$\leq C_1 \sqrt{H^4dK} \log(HdK/\xi),$$

with probability at least $1 - \xi/2$, where $C_1$ is an absolute constant.

**Upper bound of term (B) in (5.1).** We decompose term (B) in (5.1) by the following lemma, which characterizes the regret occurring in the reward update.

**Lemma 5.6.** For any $\mu = \{\mu_h\}_{h=1}^{H} \in S$, it holds that

$$\sum_{k=1}^{K} [L(\pi^k, \mu) - L(\pi^k, \mu^k)] \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{2\eta} \left( \|\mu^k_h - \mu_h\|^2 - \|\mu^k_{h+1} - \mu_h\|^2 - \|\mu^k_{h+1} - \mu^k_h\|^2 \right)$$

$$+ \sum_{h=1}^{K} \sum_{h=1}^{H} \left( \mu^k_{h+1} - \mu_h \right)^\top \nabla_{\mu_h} L(\pi^k, \mu^k) + K \left[ \bar{J}(\pi^E, \mu^E) - J(\pi^E, \mu^E) \right]$$

$$+ \sum_{h=1}^{K} \sum_{h=1}^{H} \left( \mu^k_h - \mu_h \right)^\top \left( \psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k) \right),$$

for any $(k, h) \in [K] \times [H]$. Here $\nabla_{\mu_h} L(\pi, \mu^k)$ and $\bar{J}(\pi^E, \mu^E)$ are defined in (3.12) and (3.11), respectively.

Proof. See Appendix A.6 for a detailed proof.
Applying Lemma 5.6, we decompose term (B) in (5.1) into four terms as follows,

\[
(B) \leq \sup_{\mu \in S} \frac{1}{2\eta} \left( \|\mu^k_h - \mu_h\|_2^2 - \|\mu^{k+1}_h - \mu_h\|_2^2 \right)
\]

\[+ \sum_{k=1}^K \sum_{h=1}^H \left[ (\mu^{k+1}_h - \mu^k_h)^\top \hat{\nabla}_{\mu_h} L(\pi^k, \mu^k) \right] \]

\[+ K \cdot \sup_{\mu \in S} \left[ \tilde{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu) \right] \]

\[+ \sup_{\mu \in S} \sum_{k=1}^K \sum_{h=1}^H \left[ (\mu^k_h - \mu_h)^\top (\psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k)) \right]. \]

We upper bound terms (B1), (B2), (B3), and (B4) as follows.

By telescoping the summands in term (B1) of (5.7) with respect to \( k \in [K] \), we have

\[
(B1) = \sup_{\mu \in S} \frac{1}{2\eta} \left( \sum_{h=1}^H \left( \|\mu^1_h - \mu_h\|_2^2 - \|\mu^{K+1}_h - \mu_h\|_2^2 \right) - \sum_{k=1}^K \|\mu^{K+1}_h - \mu^k_h\|_2^2 \right)
\]

\[\leq \sup_{\mu \in S} \frac{1}{2\eta} \sum_{h=1}^H \|\mu^1_h - \mu_h\|_2^2 \leq \frac{2}{\eta} H d,
\]

where the last inequality relies on the fact that \( \|\mu_h\|_2 \leq \sqrt{d} \) for all \( \mu = \{\mu_h\}_{h=1}^H \in S \). This upper bounds term (B1).

By the update process \( \mu^{k+1}_h = \text{Proj}_B \{\mu^k_h + \eta \hat{\nabla}_{\mu_h} L(\pi^k, \mu^k)\} \) in OGAP (Line 16 of Algorithm 2), we have

\[
\|\mu^{k+1}_h - \mu^k_h\|_2 \leq \|\eta \hat{\nabla}_{\mu_h} L(\pi^k, \mu^k)\|_2.
\]

Then we upper bound (B2) in (5.7) as follows,

\[
(B2) = \sum_{k=1}^K \sum_{h=1}^H \left[ (\mu^{k+1}_h - \mu^k_h)^\top \hat{\nabla}_{\mu_h} L(\pi^k, \mu^k) \right]
\]

\[\leq \sum_{k=1}^K \sum_{h=1}^H \|\mu^{k+1}_h - \mu^k_h\|_2 \cdot \|\hat{\nabla}_{\mu_h} L(\pi^k, \mu^k)\|_2
\]

\[\leq \sum_{k=1}^K \sum_{h=1}^H \eta \cdot \|\hat{\nabla}_{\mu_h} L(\pi^k, \mu^k)\|_2 \leq 4\eta H K,
\]

where the first inequality follows from Cauchy-Schwarz inequality, the second inequality follows from (5.9) and the last inequality follows from the fact that \( \|\hat{\nabla}_{\mu_h} L(\pi^k, \mu^k)\|_2 \leq 2 \|\psi(\cdot, \cdot)\|_2 \leq 2 \). This upper bounds term (B2).

We upper bound term (B3) via the following lemma.
Lemma 5.7 (Monte Carlo Estimation). Since reward function class $\mathcal{R}$ is linear as defined in (2.7) and the estimator $\tilde{J}(\pi^E, r^\mu)$ constructed in (3.12), it holds that
\[
\sup_{\mu \in \mathcal{S}} |\tilde{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu)| \leq 4 \sqrt{H^3d^2/N_1} \log(6N_1/\xi),
\]
with probability at least $1 - \xi$.

Proof. See Appendix A.7 for detailed proof.

By Lemma 5.7, it holds with probability at least $1 - \xi/4$ that
\[
(B3) \leq 4K \sqrt{H^3d^2/N_1} \log(24N_1/\xi),
\]
which upper bounds term (B3).

To upper bound term (B4) in (5.7), we introduce the following lemma.

Lemma 5.8 (Unbiased Estimation). It holds that
\[
\sup_{\mu \in \mathcal{S}} \sum_{k=1}^{K} \sum_{h=1}^{H} [(\mu^k_h - \mu_h)^\top (\psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k))] \leq 32 \sqrt{H^3d^2K} \log(36/\xi),
\]
with probability at least $1 - \xi$.

Proof. See Appendix A.8 for a detailed proof.

By Lemma 5.8, it holds with probability at least $1 - \xi/4$ that
\[
(B4) \leq \sup_{\mu \in \mathcal{S}} \sum_{k=1}^{K} \sum_{h=1}^{H} [(\mu^k_h - \mu_h)^\top (\psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k))] \leq 32 \sqrt{H^3d^2K} \log(36/\xi),
\]
which upper bounds term (B4).

Plugging (5.11), (5.10), (5.12), and (5.11) into (5.7), it holds with probability at least $1 - \xi/2$ that
\[
(B) \leq 2\sqrt{H^3d^2K} + 4K \sqrt{H^3d^2K} \log(36/\xi) + 4 \sqrt{H^3d^2/N_1} \log(24N_1/\xi)
\]
\[
\leq 32 \sqrt{H^3d^2K} \log(36/\xi) + 4K \sqrt{H^3d^2/N_1} \log(24N_1/\xi),
\]
where we recall that $\eta = 1/\sqrt{HK}$. Combining (5.1), (5.6), and (5.13), we obtain that
\[
\text{Regret}(K) \leq C_1 \sqrt{H^3d^2K} \log(HdK/\xi) + 32 \sqrt{H^3d^2K} \log(24/\xi) + 4 \sqrt{H^3d^2/N_1} \log(36N_1/\xi)
\]
\[
\leq C_2 \left( H^2d^{3/2}K^{1/2} \log(HdK/\xi) + KH^{3/2}dN_1^{-1/2} \log(N_1/\xi) \right),
\]
with probability at least $1 - \xi$, where $C_2$ is an absolute constant. This concludes the proof of Theorem 4.1.

6 Proof Sketch: Analysis of PGAP

6.1 Proof of Theorem 4.2

Proof. By the property of mixed policy, we can rewrite the optimality gap as
\[
D_{\mathcal{R}}(\pi^E, \hat{\pi}) = \sup_{\mu \in \mathcal{S}} [J(\pi^E, r^\mu) - J(\hat{\pi}, r^\mu)] = \frac{1}{K} \sup_{\mu \in \mathcal{S}} \sum_{k=1}^{K} L(\pi^k, \mu),
\]
with probability at least $1 - \xi/4$. By Lemma 5.7, it holds with probability at least $1 - \xi/4$ that
\[
(B3) \leq 4K \sqrt{H^3d^2/N_1} \log(24N_1/\xi),
\]
which upper bounds term (B3).

To upper bound term (B4) in (5.7), we introduce the following lemma.

Lemma 5.8 (Unbiased Estimation). It holds that
\[
\sup_{\mu \in \mathcal{S}} \sum_{k=1}^{K} \sum_{h=1}^{H} [(\mu^k_h - \mu_h)^\top (\psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k))] \leq 32 \sqrt{H^3d^2K} \log(36/\xi),
\]
with probability at least $1 - \xi$.

Proof. See Appendix A.8 for a detailed proof.

By Lemma 5.8, it holds with probability at least $1 - \xi/4$ that
\[
(B4) \leq \sup_{\mu \in \mathcal{S}} \sum_{k=1}^{K} \sum_{h=1}^{H} [(\mu^k_h - \mu_h)^\top (\psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k))] \leq 32 \sqrt{H^3d^2K} \log(36/\xi),
\]
which upper bounds term (B4).

Plugging (5.11), (5.10), (5.12), and (5.11) into (5.7), it holds with probability at least $1 - \xi/2$ that
\[
(B) \leq 2\sqrt{H^3d^2K} + 4K \sqrt{H^3d^2K} \log(36/\xi) + 4 \sqrt{H^3d^2/N_1} \log(24N_1/\xi)
\]
\[
\leq 32 \sqrt{H^3d^2K} \log(36/\xi) + 4K \sqrt{H^3d^2/N_1} \log(24N_1/\xi),
\]
where we recall that $\eta = 1/\sqrt{HK}$. Combining (5.1), (5.6), and (5.13), we obtain that
\[
\text{Regret}(K) \leq C_1 \sqrt{H^3d^2K} \log(HdK/\xi) + 32 \sqrt{H^3d^2K} \log(24/\xi) + 4 \sqrt{H^3d^2/N_1} \log(36N_1/\xi)
\]
\[
\leq C_2 \left( H^2d^{3/2}K^{1/2} \log(HdK/\xi) + KH^{3/2}dN_1^{-1/2} \log(N_1/\xi) \right),
\]
with probability at least $1 - \xi$, where $C_2$ is an absolute constant. This concludes the proof of Theorem 4.1.
where \( L(\pi^k, \mu) = J(\pi^E, r^\mu) - J(\pi^k, r^\mu) \). Recall that \( J(\pi^k, r^\mu) = V_{\pi^k}(x) \) and \( \tilde{J}(\pi^k, r^k) = \tilde{V}^k(x) \), where \( x \) is the initial state, we upper bound \( D_R(\pi^E, \tilde{\pi}) \) as follows,

\[
D_R(\pi^E, \tilde{\pi}) \leq \frac{1}{R} \left\{ \sum_{k=1}^{K} \left[ J(\pi^E, r^k) - \tilde{J}(\pi^k, r^k) \right] \right\} \tag{6.2} (A) + \sup_{\mu \in S} \sum_{k=1}^{K} \left[ J(\pi^E, r^\mu) - J(\pi^k, r^\mu) - J(\pi^E, r^k) + \tilde{J}(\pi^k, r^k) \right] \right\}. \tag{B}
\]

We upper bound terms (A) and (B) in (6.2) as follows, respectively.

**Upper Bound of Term (A) in (6.2).** To upper bound term (A) in (6.2), we introduce the following lemma.

**Lemma 6.1** (Extended Value Difference (Cai et al., 2020)). Let \( \pi = \{\pi_h\}_{h=1}^{H} \) and \( \pi' = \{\pi'_h\}_{h=1}^{H} \) be any two policies and let \( \{\hat{Q}_h\}_{h=1}^{H} \) be any estimated Q-functions. For any \( h \in [H] \), we define the estimated V-function \( \hat{V}_h : S \rightarrow \mathbb{R} \) by setting \( \hat{V}_h(x) = \langle \hat{Q}_h(x, \cdot), \pi_h(\cdot | x) \rangle_A \) for any \( x \in S \). For any initial state \( x \in S \), we have

\[
\hat{V}_1(x) - V^{x'}_1(x) = \sum_{h=1}^{H} \mathbb{E}_{\pi'} \left[ \langle \hat{Q}_h(s_h, \cdot), \pi_h(\cdot | s_h) - \pi'_h(\cdot | s_h) \rangle_A | s_1 = x \right] + \sum_{h=1}^{H} \mathbb{E}_{\pi'} \left[ \hat{Q}_h(s_h, a_h) - r(s_h, a_h) - \mathbb{P}_h \hat{V}_{h+1}(s_h, a_h) | s_1 = x \right],
\]

where \( \mathbb{E}_{\pi'} \) is taken with respect to the trajectory generated by \( \pi' \) and \( r \) is the reward function.

**Proof.** See Appendix B.1 in Cai et al. (2020) for a detailed proof. \( \Box \)

For the simplicity of later discussion, at the \( h \)-th step of \( k \)-th episode, we define the error for estimating Bellman equation in (2.2) in the policy evaluation stage of PGAP (Lines 8-11 of Algorithm 3) with any reward function \( r^\mu \) as follows,

\[
\epsilon_{h}^{k, r^\mu}(s, a) := r_h^\mu(s, a) + \mathbb{P}_h \hat{V}_{h+1}^{k, r^\mu}(s, a) - \hat{Q}_h^{k, r^\mu}(s, a),
\]

for any \( (s, a, h, \mu) \in S \times A \times [H] \times S \), where \( \hat{Q}_h^{k, r^\mu} \) and \( \hat{V}_{h+1}^{k, r^\mu} \) are defined as

\[
\hat{V}^{k, r^\mu}_{h+1}(\cdot) = 0, \quad \hat{Q}_h^{k, r^\mu}(\cdot, \cdot) = \max \left\{ (r_h^\mu + \mathbb{P}_h \hat{V}_{h+1}^{k, r^\mu} - \Gamma_h)(\cdot, \cdot), 0 \right\}, \quad \hat{V}_{h+1}^{k, r^\mu}(\cdot, \cdot) = \langle \hat{Q}_h^{k, r^\mu}(\cdot, \cdot), \pi_h^k(\cdot | \cdot) \rangle_A, \quad \text{for } h \in [H].
\]

It implies that \( \hat{Q}_h^k = \hat{Q}_h^{k, r^\mu} \) and \( \hat{V}_h^k = \hat{V}_h^{k, r^\mu} \), where \( \hat{Q}_h^k \) and \( \hat{V}_h^k \) are constructed in the policy evaluation stage in PGAP (Lines 8-11 of Algorithm 3).
By applying Lemma 6.1, we decompose term (A) in (6.2) as follows,

\[
(A) = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi E} \left[ (\hat{Q}^k_h(s_h, \cdot), \pi^E_h(\cdot | s_h) - \pi^k_h(\cdot | s_h)) | A \right] s_1 = x \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi E} \left[ \hat{r}^k_h(s_h, a_h) | A \right] s_1 = x.
\]

(A1)

(A2)

We upper bound terms (A1) and (A2) in (6.4) as follows.

By Lemma 5.2, we have

\[
(A1) \leq \sqrt{2H^4d\sqrt{dK}\log(\text{vol}(A))}.
\]

(6.5)

We upper bound term (A2) in (6.4) using the following lemma.

**Lemma 6.2 (Pessimism).** If \( \{\Gamma_h\}_{h \in [H]} \) are \( \xi \)-uncertainty qualifiers defined in Definition 3.1, when conditioned on \( \mathcal{E} \) defined in Definition 3.1, it holds for any \((s, a, h) \in S \times A \times [H]\) and any reward function \( r^\mu \) with \( \mu \in S \) that

\[
0 \leq \epsilon^k_{h,r^\mu}(s, a) \leq 2\Gamma_h(s, a).
\]

*Proof.* See Appendix B.2 for a detailed proof.

By Lemma 6.2, conditioned on \( \mathcal{E} \), we upper bound term (A2) in (6.4) as follows,

\[
(A2) \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi E} \left[ 2\Gamma_h(s_h, a_h) | A \right] s_1 = x = K \cdot \text{IntUncert}^E_D,
\]

(6.6)

where we denote by \( \text{IntUncert}^E_D = \sum_{h=1}^{H} \mathbb{E}_{\pi E} \left[ 2\Gamma_h(s_h, a_h) | A \right] s_1 = x \) for notational convenience. Combining (6.5) and (6.6), we derive an upper bound on term (A) in (6.2) conditioned on \( \mathcal{E} \) as

\[
(A) \leq \sqrt{2H^4d\sqrt{dK}\log(\text{vol}(A))} + K \cdot \text{IntUncert}^E_D.
\]

(6.7)

**Upper Bound of Term (B) in (6.2).** We define \( \hat{L}(\pi, \mu) \) as follows,

\[
\hat{L}(\pi, \mu) = J(\pi^E, r^\mu) \leq \tilde{J}(\pi^k, r^\mu).
\]

Here \( \tilde{J}(\pi, r^\mu) \) is the MC estimation defined in (3.11) and \( \tilde{J}(\pi^k, r^\mu) \) is the estimated cumulative reward defined as \( \tilde{J}(\pi^k, r^\mu) = \tilde{V}^k_1, r^\mu(x) \), where \( \tilde{V}^k_1, r^\mu \) is defined in (6.3) and \( x \) is the initial state. By this, we upper bound term (B) in (6.2) as

\[
(B) = \sup_{\mu \in S} \sum_{k=1}^{K} \left[ J(\pi^E, r^\mu) - J(\pi^k, r^\mu) - J(\pi^E, r^k) + \tilde{J}(\pi^k, r^k) \right]
\]

\[
\leq \sup_{\mu \in S} \sum_{k=1}^{K} \left[ J(\pi^E, r^\mu) - \tilde{J}(\pi^E, r^\mu) \right] + \sup_{\mu \in S} \sum_{k=1}^{K} \left[ - J(\pi^E, r^k) + \tilde{J}(\pi^E, r^k) \right]
\]

(B1)

(B2)

\[
+ \sup_{\mu \in S} \sum_{k=1}^{K} \left[ \tilde{J}(\pi^k, r^\mu) - J(\pi^k, r^\mu) \right] + \sup_{\mu \in S} \sum_{k=1}^{K} \left[ \hat{L}(\pi^k, \mu) - \tilde{L}(\pi^k, \mu^k) \right].
\]

(B3)

(B4)
We upper bounds terms (B1)–(B4) in (6.8) as follows.

By applying Lemma 5.7 on term (B1) and term (B2) in (6.8), it holds with probability at least $1 - \xi/2$ that

$$\text{(B1)} + \text{(B2)} \leq 8K \sqrt{H^3d^2/N_1} \log(24N_1/\xi). \quad (6.9)$$

To upper bound term (B3) in (6.8), we invoke Lemmas 6.1 and 6.2, which imply

$$\text{(B3)} = \sup_{\mu \in S} \left[ \sum_{k=1}^K \sum_{h=1}^H E_{\pi^k} \left[ -t_h r^\mu(s_h^k, a_h^k) \mid s_1 = x \right] \right] \leq 0. \quad (6.10)$$

To upper bound term (B4) in (6.8), we first introduce the following lemma.

**Lemma 6.3.** The function $\hat{L}(\pi^k, \mu)$ defined in (3.21) is concave in $\mu_h$ for any $h \in [H]$, where $\mu = \{\mu_h\}_{h=1}^H \in S$ and $\hat{L}(\pi^k, \mu)$.

**Proof.** See Appendix B.3 for a detailed proof. \hfill \square

By Lemma 6.3, we establish the following lemma to upper bound term (B4) in (6.8), which corresponds to the reward update stage in PGAP (Lines 12–17 of Algorithm 3).

**Lemma 6.4.** For any $\mu \in S$, it holds that

$$\sum_{k=1}^K \left[ \hat{L}(\pi^k, \mu) - \hat{L}(\pi^k, \mu^k) \right] \leq \sum_{k=1}^K \sum_{h=1}^H \frac{1}{2\eta} \|\mu_{h}^k - \mu_h\|_2^2 + \frac{1}{2\eta} \|\mu_{h}^{k+1} - \mu_h\|_2^2 - \frac{1}{2\eta} \|\mu_h^k - \mu_h^{k+1}\|_2^2 + \sum_{k=1}^K \eta \|\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)\|^2_2 \right],$$

which implies that

$$\text{(B4)} \leq \sup_{\mu \in S} \left[ \sum_{h=1}^H \left( \frac{1}{2\eta} \|\mu_{h}^1 - \mu_h\|_2^2 \right) - \frac{1}{2\eta} \|\mu_{h}^{K+1} - \mu_h\|_2^2 - \frac{1}{2\eta} \sum_{k=1}^K \|\mu_h^k - \mu_h^{k+1}\|_2^2 + \sum_{k=1}^K \eta \|\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)\|^2_2 \right]. \quad (6.11)$$

Based on (6.11), we upper bound $\|\nabla_{\mu_h} \hat{L}(\pi^k, r^\mu)\|^2_2$ for any $\mu \in S$ and $h \in [H]$ as follows,

$$\|\nabla_{\mu_h} \hat{L}(\pi^k, r^\mu)\|^2_2 \leq \|\nabla_{\mu_h} \tilde{J}(\pi^E, r^\mu) - \nabla_{\mu_h} \tilde{J}(\pi^k, r^\mu)\|^2_2 \leq 2\|\nabla_{\mu_h} \tilde{J}(\pi^E, r^\mu)\|^2_2 + 2\|\nabla_{\mu_h} \tilde{J}(\pi^k, r^\mu)\|^2_2. \quad (6.12)$$

Recall that $\nabla_{\mu_h} \tilde{J}(\pi^E, r^\mu) = N_1^{-1} \sum_{\tau=1}^{N_1} \psi(s_{h, \tau}^E, a_{h, \tau}^E)$ and $\nabla_{\mu_h} \tilde{J}(\pi^k, r^\mu)$ is characterized in Proposition 3.3, then we have that

$$\|\nabla_{\mu_h} \tilde{J}(\pi^E, r^\mu)\|^2_2 + \|\nabla_{\mu_h} \tilde{J}(\pi^k, r^\mu)\|^2_2 \leq 2\|\psi(\cdot, \cdot)\|^2_2. \quad (6.13)$$
Since it holds that $\|\mu_h\|_2 \leq \sqrt{d}$ for any $\mu \in S$ and $\|\psi(\cdot, \cdot)\|_2 \leq 1$, it also yields that

$$\|\mu'_h - \mu_h\|_2^2 \leq (\|\mu'_h\|_2 + \|\mu_h\|_2)^2 \leq 4d$$  \hspace{1cm} (6.14)

for any $\mu, \mu' \in S$. By setting $\eta = 1/\sqrt{KH}$ and combining (6.11), (6.12), (6.13), and (6.14), we attain that

$$\begin{align*}
(B4) & \leq H \cdot \frac{\sqrt{HK}}{2} \cdot 4d + HK \cdot \frac{1}{\sqrt{HK}} \cdot 2 \cdot 4d \\
& \leq 2H^{3/2}dK^{1/2} + 8H^{1/2}dK^{1/2} \leq 8H^{3/2}dK^{1/2}.
\end{align*}$$  \hspace{1cm} (6.15)

Plugging (6.9), (6.10), and (6.15) into (6.8), conditioned on $E$, it holds with probability at least $1 - \xi/2$ that

$$\begin{align*}
(B) & \leq 8K \sqrt{H^3d^2/N_1 \log(24N_1/\xi)} + 8dH^{3/2}K^{1/2}.
\end{align*}$$  \hspace{1cm} (6.16)

Recall that Definition 3.1 implies $P_D(E) > 1 - \xi/2$. Combining (6.1), (6.7), and (6.16), with probability at least $1 - \xi$, we have

$$\begin{align*}
D_R(\pi^E, \pi) & \leq \sqrt{2H^4\sqrt{d} \log(\text{vol}(A))/K} + \text{IntUncert}^{E}_D \\
& + 8\sqrt{H^3d^2/N_1 \log(24N_1/\xi)} + 8H^{3/2}dK^{-1/2} \\
& \leq 8H^2dK^{-1/2} + \text{IntUncert}^{E}_D + 8H^{3/2}dN_1^{-1/2} \log(24N_1/\xi).
\end{align*}$$

Thus, we conclude the proof of Theorem 4.2. \hfill \Box

### 6.2 Proof of Proposition 4.3

**Proof.** Our proof is based on the following theorem that presents the information-theoretic lower bound.

**Theorem 6.5** (Information-Theoretic Lower Bound, Theorem 4.6 in Jin et al. (2021)). For the output $\text{Algo}(D)$ of any offline RL algorithm based on the dataset $D$, there exists a tabular MDP $M (S, \mathcal{A}, H, P, r)$ with initial state $x \in S$ and a dataset $D$ which is compliant with $M$, such that

$$\mathbb{E}_{\pi} \left[ \frac{J(\pi^*) - J(\text{Algo}(D))}{\sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ 1/\sqrt{1 + n_h(s_h, a_h)} \right] s_1 = x} \right] \geq c$$

where $c$ is an absolute constant, $\pi^*$ is the optimal policy satisfying that $\pi^* := \arg\max_{\pi \in \Delta(A|S, H)} J(\pi)$, and $n_h(s_h, a_h) = \sum_{\tau=1}^{N} 1 \{s_\tau^h = s_h, a_\tau^h = a_h\}$ for $(s_h, a_h) \in S \times A$.

First we show the linear kernel MDP defined in Assumption 2.1 can be reduced to tabular MDP. If we set $d = |S|^2|A|$ and take the feature map as the canonical basis

$$\phi(s, a, s') = e_{(s, a, s')}, \quad \psi(s, a) = e_{(s, a)}, \quad \text{for all } (s, a, s') \in S \times A \times S,$$

then Assumption 2.1 is satisfied with $R = 1$.

Applying Theorem 6.5, we derive a hard instance in tabular MDP with known reward $r$. When we choose the reward set $R$ as the singleton reward $r$, it yields that

$$\begin{align*}
\max_{\pi^E \in \Delta(A|S, H)} \mathbb{E}_{D} \left[ \frac{D_R(\pi^E, \text{Algo}(D))}{\text{Information}^{E}_{D}} \right] & \geq \mathbb{E}_{D} \left[ \frac{D_R(\pi^*, \text{Algo}(D))}{\text{Information}^{E}_{D}} \right] = \mathbb{E}_{D} \left[ \frac{J(\pi^*) - J(\text{Algo}(D))}{\text{Information}^{E}_{D}} \right],
\end{align*}$$  \hspace{1cm} (6.17)
where the last equality is originated from the definition of optimality gap in \((2.6)\).

Next we handle \(\text{IntUncert}_P^\pi\), which takes the form as

\[
\text{Information}_P^\pi = (\text{Vol}(S))^{-1} \cdot \mathbb{E}_\pi \left[ \sum_{h=1}^{H} \int_S \| \phi(s_h, a_h, s') \|_{\Lambda_h^{-1}} ds' \mid s_1 = x \right]
\]

(6.18)

where \(\Lambda_h = \lambda I + \sum_{\tau=1}^{N} \int_S \phi(s_h^\tau, a_h^\tau, s')\phi(s_h^\tau, a_h^\tau, s')^T ds'\),

where \(N\) is the number of the trajectories in the dataset \(\mathcal{D}\). In the tabular setting, we obtain that

\[
\sum_{\tau=1}^{N} \phi(s_h^\tau, a_h^\tau, s')\phi(s_h^\tau, a_h^\tau, s')^T = \sum_{(s, a) \in S \times A} n_h(s, a)W_{(s, a, s')},
\]

where \(W_{(s, a, s')}\) is a symmetric matrix whose non-zero entry is at \((s, a, s')\) and equals to 1. Summing \(s'\) over \(S\), in the tabular case we have that

\[
\Lambda_h = \lambda I + \sum_{\tau=1}^{N} \sum_{s' \in S} (s_h^\tau, a_h^\tau, s')\phi(s_h^\tau, a_h^\tau, s')^T ds' = \lambda I + \sum_{(s, a, s') \in S \times A \times S} n_h(s, a)W_{(s, a, s')},
\]

Choosing \(\lambda = 1\), we obtain that

\[
\phi(s, a, s')^T \Lambda_h^{-1} \phi(s, a, s') = \frac{1}{1 + n_h(s, a)},
\]

for all \((s, a, s') \in S \times A \times S\). Hence we have that

\[
\sum_{h=1}^{H} \sum_{s' \in S} \| \phi(s, a, s') \|_{\Lambda_h^{-1}} = \sum_{h=1}^{H} \frac{|S|}{\sqrt{1 + n_h(s, a)}},
\]

(6.19)

for all \((s, a) \in S \times A\). Taking expectation on (6.19) with respect to the optimal policy \(\pi^*\) and according to (6.18), it holds that

\[
\text{Information}_P^\pi = |S|^{-1} \cdot \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ 1/ \sqrt{1 + n_h(s, a)} \mid s_1 = x \right].
\]

(6.20)

Plugging (6.20) into (6.17), under the hard instance in Theorem 6.5, we obtain that

\[
\max_{\pi^E \in \Delta(A(S, H))} \mathbb{E}_{\pi^E} \left[ \frac{\mathcal{D}_{\mathcal{R}}(\pi^E, Algo(\mathcal{D}))}{\text{Information}_P^\pi} \right] \geq c,
\]

where \(c > 0\) is a positive constant. Then we conclude the proof of Proposition 4.3

\[
\square
\]

6.3 Proof of Corollary 4.5

**Proof.** By the property of trace, we have that

\[
\mathbb{E}_{\pi^E} \left[ \int_S \sqrt{\langle \phi(s_h, a_h, s')^T \Lambda_h^{-1} \phi(s_h, a_h, s') \rangle} ds' \mid s_1 = x \right]
\]

\[
= \mathbb{E}_{\pi^E} \left[ \int_S \sqrt{\text{Tr}(\phi(s_h, a_h, s')^T \Lambda_h^{-1} \phi(s_h, a_h, s'))} ds' \mid s_1 = x \right]
\]

(6.21)

\[
= \mathbb{E}_{\pi^E} \left[ \int_S \sqrt{\text{Tr}(\phi(s_h, a_h, s')\phi(s_h, a_h, s')^T \Lambda_h^{-1})} ds' \mid s_1 = x \right].
\]

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Applying Cauchy-Schwarz inequality on (6.21), we derive that
\[
\mathbb{E}_{x} \left[ \int_{S} \sqrt{\text{Tr}(\phi(s_h, a_h, s')^\top \phi(s_h, a_h, s') \Lambda_h^{-1})} \, ds' \mid s_1 = x \right] 
\leq (\text{Vol}(S))^{1/2} \cdot \left( \int_{S} \text{Tr}(\phi(s_h, a_h, s')^\top \phi(s_h, a_h, s') \Lambda_h^{-1}) \, ds' \mid s_1 = x \right)^{1/2}
\]  
(6.22)
for any \(x, s' \in S\) and all \(h \in [H]\).

To utilize Theorem 4.2, we define the event \(\mathcal{E}^t\) as follows,
\[
\mathcal{E}^t = \left\{ \text{D}_{R}(\pi^E, \widehat{x}) \leq O(H^2 d K^{-1/2}) + \delta_{N_{\lambda}} + \text{IntUncert}_{D}^E \right\},
\]
where \(\text{IntUncert}_{D}^E = 2 \sum_{h=1}^{H} \mathbb{E}_{x} \left[ \text{Vol}(\Gamma_h(s_h, a_h)) \mid s_1 = x \right] \) and \(\Gamma_h\) is defined in (3.16). Conditioned on the event \(\mathcal{E}^t \cap \mathcal{E}^t\), where \(\mathcal{E}^t\) is defined in Assumption 4.4, we obtain that
\[
\text{IntUncert}_{D}^E \leq 2\kappa H \sqrt{d} \sqrt{\text{Vol}(S)} \cdot \sum_{h=1}^{H} \sqrt{\text{Tr} \left( \mathbb{E}_{x} \left[ \int_{S} \phi(s_h, a_h, s')^\top \phi(s_h, a_h, s') \Lambda_h^{-1} \mid s_1 = x \right] \right)},
\]
(6.23)
By plugging (6.21) and (6.22) into (6.23), we have
\[
\text{IntUncert}_{D}^E \leq 2\kappa H \sqrt{d} \sqrt{\text{Vol}(S)} \cdot \sum_{h=1}^{H} \sqrt{\text{Tr} \left( \mathbb{E}_{x} \left[ \int_{S} \phi(s_h, a_h, s')^\top \phi(s_h, a_h, s') \Lambda_h^{-1} \mid s_1 = x \right] \right)},
\]
(6.24)
where \(\text{Vol}(S)\) is the finite measure of the state space \(S\). For notational simplicity, we define
\[
\Sigma_h(x) = \mathbb{E}_{x} \left[ \int_{S} \phi(s_h, a_h, s')^\top \phi(s_h, a_h, s') \Lambda_h^{-1} \mid s_1 = x \right],
\]
(6.25)
for all \(x \in S\) and all \(h \in [H]\).

By Assumption 4.4 and the definition of \(\Sigma_h(x)\) in (6.25), we know that the matrix \((I + c^l N_2 \Sigma_h(x))^{-1} - \Lambda_h^{-1}\) is positive definite conditioned on \(\mathcal{E}^t\). Combining (6.24) and (6.25), we obtain that
\[
\text{IntUncert}_{D}^E \leq 2(\text{Vol}(S))^{1/2} \cdot \kappa H \sqrt{d} \sum_{h=1}^{H} \sqrt{\text{Tr} \left( \Sigma_h(x) \cdot (I + c^l N_2 \cdot \Sigma_h(x))^{-1} \right)}
\]
\[
= 2(\text{Vol}(S))^{1/2} \cdot \kappa H \sqrt{d} \sum_{h=1}^{H} \left( \sum_{j=1}^{d} \frac{\lambda_{h,j}(x)}{1 + c^l \cdot N_2 \cdot \lambda_{h,j}(x)} \right).
\]
(6.26)
Here \(\{\lambda_{h,j}(x)\}_{j=1}^{d}\) are the eigenvalues of \(\Sigma_h(x)\) for any \(x \in S\) and \(h \in [H]\). Meanwhile, under Assumption 2.1, we have \(\|\phi(\cdot, \cdot, \cdot)\|_2 \leq dR\), which is shown in (A.28). By applying Cauchy-Schwarz inequality to (6.25), it holds that
\[
\|\Sigma_h(x)\|_{op} \leq \mathbb{E}_{x} \left[ \left\| \int_{S} \phi(s_h, a_h, s')^\top \phi(s_h, a_h, s') \mid s_1 = x \right\|_{op} \right] \leq d^{1/2} R^2 \cdot (\text{Vol}(S))^{1/2},
\]
for any \(x \in S\) and \(h \in [H]\), where \(\| \cdot \|_{op}\) is the operator norm. As \(\Sigma_h(x)\) is positive semidefinite, we have \(\lambda_{h,j}(x) \in [0, \|\Sigma_h(x)\|_{op}]\) for any \(x \in S\) and \((h, j) \in [H] \times [d]\). Hence, conditioned on \(\mathcal{E}^t \cap \mathcal{E}^t\), combining
(6.26), it holds for any $x \in S$ that

$$\text{IntUncert}_{E \Delta A}^R \leq 2(\text{Vol}(S))^{1/2} \cdot \kappa H \sqrt{d} \sum_{h=1}^{H} \sum_{j=1}^{d} \frac{\lambda_{h,j}(x)}{1 + c^\dagger \cdot N_2 \cdot \lambda_{h,j}(x)} \leq 2(\text{Vol}(S))^{1/2} \cdot \kappa H \sqrt{d} \sum_{h=1}^{H} \sum_{j=1}^{d} \frac{1}{1 + c^\dagger \cdot N_2} \leq c' (\text{Vol}(S))^{1/2} d^{3/2} H^2 N_2^{-1/2} \log(\alpha d N_2/\xi),$$

(6.27)

where the second inequality follows from the fact that $\lambda_{h,j}(x) \in [0, \|\Sigma_h(x)\|_{\text{op}}]$ for any $(x, h, j) \in S \times [H] \times [d]$, while the third inequality follows from the choice of the scaling parameter $\kappa > 0$ stated in Theorem 4.2. Here $c'$ is an absolute constant dependent on $c$ and $c^\dagger$. By the condition in Corollary 4.5, we have $\mathbb{P}_D(\mathcal{E}^\dagger) \geq 1 - \xi/2$. Also, by Theorem 4.2, we have $\mathbb{P}_D(\mathcal{E}^\sharp) \geq 1 - \xi/2$. Hence, by the union bound, we derive that $\mathbb{P}_D(\mathcal{E}^\dagger \cap \mathcal{E}^\sharp) \geq 1 - \xi$. Combining (6.27), we finish the proof of Corollary 4.5.

7 Conclusion

In this paper, we study provably efficient algorithms for GAIL in the online and offline setting with linear function approximation, where both the transition kernel and reward functions are linear. We present a unified framework and specialize it as optimistic generative adversarial policy optimization (OGAP) for online GAIL and pessimistic generative adversarial policy optimization for offline GAIL. With linear function approximation, we derive the upper bound of the regret of OGAP as $\tilde{O}(H^2 d^{3/2} K^{1/2} + KH^{3/2} d N_1^{-1/2})$ and the decomposition of optimality gap of PGAP, without any assumption on the additional dataset. When facilitated with additional dataset with sufficient coverage, we demonstrate that PGAP also has global convergence, achieving $\tilde{O}(H^2 d K^{-1/2} + H^2 d^{5/2} N_2^{-1/2} + H^3/2 d N_1^{-1/2})$ optimality gap. However, provably efficient GAIL with general function approximation (both the transition kernel and reward function) still remains an open problem, which is a challenging but important future direction.

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References

Abbasi-Yadkori, Y., Pál, D. and Szepesvári, C. (2011). Improved algorithms for linear stochastic bandits. In International Conference on Machine Learning.

Abbeel, P. and Ng, A. (2004). Apprenticeship learning via inverse reinforcement learning. International Conference on Machine Learning.

Antos, A., Munos, R. and Szepesvári, C. (2007). Fitted q-iteration in continuous action-space mdps.

Argall, B. D., Chernova, S., Veloso, M. and Browning, B. (2009). A survey of robot learning from demonstration. Robotics and autonomous systems, 57 469–483.
Arjovsky, M., Chintala, S. and Bottou, L. (2017). Wasserstein generative adversarial networks. In *International conference on machine learning*. PMLR.

Auer, P., Cesa-Bianchi, N. and Fischer, P. (2002). Finite-time analysis of the multiarmed bandit problem. *Machine learning*, **47**, 235–256.

Auer, P., Jaksch, T. and Ortner, R. (2009). Near-optimal regret bounds for reinforcement learning. In *Advances in Neural Information Processing Systems*.

Ayoub, A., Jia, Z., Szepesvari, C., Wang, M. and Yang, L. (2020). Model-based reinforcement learning with value-targeted regression. In *International Conference on Machine Learning*.

Azar, M. G., Osband, I. and Munos, R. (2017). Minimax regret bounds for reinforcement learning. In *International Conference on Machine Learning*.

Beck, A. and Teboulle, M. (2003). Mirror descent and nonlinear projected subgradient methods for convex optimization. *Operations Research Letters*, **31**, 167–175.

Bradtke, S. J. and Barto, A. G. (1996). Linear least-squares algorithms for temporal difference learning. *Machine learning*, **22**, 33–57.

Buckman, J., Gelada, C. and Bellemare, M. G. (2020). The importance of pessimism in fixed-dataset policy optimization. *arXiv preprint arXiv:2009.06799*.

Cai, Q., Hong, M., Chen, Y. and Wang, Z. (2019). On the global convergence of imitation learning: A case for linear quadratic regulator. *arXiv preprint arXiv:1901.03674*.

Cai, Q., Yang, Z., Jin, C. and Wang, Z. (2020). Provably efficient exploration in policy optimization. In *International Conference on Machine Learning*.

Chang, J. D., Uehara, M., Sreenivas, D., Kidambi, R. and Sun, W. (2021). Mitigating covariate shift in imitation learning via offline data without great coverage. *arXiv preprint arXiv:2106.03207*.

Chen, J. and Jiang, N. (2019). Information-theoretic considerations in batch reinforcement learning. In *International Conference on Machine Learning*.

Chen, M., Wang, Y., Liu, T., Yang, Z., Li, X., Wang, Z. and Zhao, T. (2020). On computation and generalization of generative adversarial imitation learning. *ArXiv 2001.02792*.

Chen, Z., Zhang, X., Boedihardjo, A. P., Dai, J. and Lu, C.-T. (2017). Multimodal storytelling via generative adversarial imitation learning. In *International Joint Conference on Artificial Intelligence*.

Demiris*, Y. and Johnson, M. (2003). Distributed, predictive perception of actions: A biologically inspired robotics architecture for imitation and learning. *Connection Science*, **15**, 231–243.

Duan, Y., Jia, Z. and Wang, M. (2020). Minimax-optimal off-policy evaluation with linear function approximation. In *International Conference on Machine Learning*.

Finn, C., Levine, S. and Abbeel, P. (2016). Guided cost learning: Deep inverse optimal control via policy optimization. In *International Conference on Machine Learning*.
Fisac, J. F., Akametalu, A. K., Zeilinger, M. N., Kaynama, S., Gillula, J. and Tomlin, C. J. (2018). A general safety framework for learning-based control in uncertain robotic systems. *IEEE Transactions on Automatic Control, 64* 2737–2752.

Fujimoto, S., Conti, E., Ghavamzadeh, M. and Pineau, J. (2019a). Benchmarking batch deep reinforcement learning algorithms. *arXiv preprint arXiv:1910.01708.*

Fujimoto, S., Meger, D. and Precup, D. (2019b). Off-policy deep reinforcement learning without exploration. In *International Conference on Machine Learning.*

Geist, M., Scherrer, B. and Pietquin, O. (2019). A theory of regularized markov decision processes. In *International Conference on Machine Learning.*

Goodfellow, I., Pouget-Abadie, J., Mirza, M., Xu, B., Warde-Farley, D., Ozair, S., Courville, A. and Bengio, Y. (2014). Generative adversarial nets. In *Advances in Neural Information Processing Systems.*

Hazan, E. (2019). Introduction to online convex optimization. *arXiv preprint arXiv:1909.05207.*

Ho, J. and Ermon, S. (2016). Generative adversarial imitation learning. *arXiv preprint arXiv:1606.03476.*

Hussein, A., Gaba, M. M., Elyan, E. and Jayne, C. (2017). Imitation learning: A survey of learning methods. *ACM Computing Surveys (CSUR), 50* 1–35.

Jalali, S. M. J., Kebria, P. M., Khorasaki, A., Saleh, K., Nahavandi, D. and Nahavandi, S. (2019). Optimal autonomous driving through deep imitation learning and neuroevolution. In *2019 IEEE International Conference on Systems, Man and Cybernetics (SMC).* IEEE.

Jin, C., Allen-Zhu, Z., Bubeck, S. and Jordan, M. I. (2018). Is Q-learning provably efficient? In *Advances in Neural Information Processing Systems.*

Jin, C., Jin, T., Luo, H., Sra, S. and Yu, T. (2020). Learning adversarial markov decision processes with bandit feedback and unknown transition. In *International Conference on Machine Learning.*

Jin, C., Yang, Z., Wang, Z. and Jordan, M. I. (2019). Provably efficient reinforcement learning with linear function approximation. In *Annual Conference on Learning Theory.*

Jin, Y., Yang, Z. and Wang, Z. (2021). Is pessimism provably efficient for offline RL? In *International Conference on Machine Learning.*

Kakade, S., Krishnamurthy, A., Lowrey, K., Ohnishi, M. and Sun, W. (2020). Information theoretic regret bounds for online nonlinear control. In *Advances in Neural Information Processing Systems.*

Kakade, S. M. (2001). A natural policy gradient. *Advances in Neural Information Processing Systems.*

Kebria, P. M., Khorasaki, A., Salaken, S. M. and Nahavandi, S. (2019). Deep imitation learning for autonomous vehicles based on convolutional neural networks. *IEEE/CAA Journal of Automatica Sinica, 7* 82–95.

Kuefler, A., Morton, J., Wheeler, T. and Kochenderfer, M. (2017). Imitating driver behavior with generative adversarial networks. In *IEEE Intelligent Vehicles Symposium (IV).* IEEE.
Kumar, A., Zhou, A., Tucker, G. and Levine, S. (2020). Conservative Q-learning for offline reinforcement learning. arXiv preprint arXiv:2006.04779.

Levine, S. and Koltun, V. (2012). Continuous inverse optimal control with locally optimal examples. In International Conference on Machine Learning.

Levine, S., Kumar, A., Tucker, G. and Fu, J. (2020). Offline reinforcement learning: Tutorial, review, and perspectives on open problems. arXiv preprint arXiv:2005.01643.

Liu, Y., Swaminathan, A., Agarwal, A. and Brunskill, E. (2020). Provably good batch reinforcement learning without great exploration. arXiv preprint arXiv:2007.08202.

Merel, J., Tassa, Y., TB, D., Srinivasan, S., Lemmon, J., Wang, Z., Wayne, G. and Heess, N. (2017). Learning human behaviors from motion capture by adversarial imitation. arXiv preprint arXiv:1707.02201.

Munos, R. and Szepesvári, C. (2008). Finite-time bounds for fitted value iteration. Journal of Machine Learning Research, 9.

Nemirovskij, A. S. and Yudin, D. B. (1983). Problem complexity and method efficiency in optimization.

Neu, G. and Szepesvári, C. (2007). Apprenticeship learning using inverse reinforcement learning and gradient methods. In Conference on Uncertainty in Artificial Intelligence. AUAI Press.

Ng, A. Y. and Russell, S. J. (2000). Algorithms for inverse reinforcement learning. In International Conference on Machine Learning.

Rajaraman, N., Han, Y., Yang, L. F., Ramchandran, K. and Jiao, J. (2021). Provably breaking the quadratic error compounding barrier in imitation learning, optimally. arXiv preprint arXiv:2102.12948.

Rajaraman, N., Yang, L., Jiao, J. and Ramchandran, K. (2020). Toward the fundamental limits of imitation learning. In Advances in Neural Information Processing Systems, vol. 33.

Rashidinejad, P., Zhu, B., Ma, C., Jiao, J. and Russell, S. (2021). Bridging offline reinforcement learning and imitation learning: A tale of pessimism. arXiv preprint arXiv:2103.12021.

Rosenberg, A. and Mansour, Y. (2019). Online convex optimization in adversarial markov decision processes. In International Conference on Machine Learning.

Ross, S. and Bagnell, D. (2010). Efficient reductions for imitation learning. In International Conference on Artificial Intelligence and Statistics.

Ross, S., Gordon, G. and Bagnell, D. (2011). A reduction of imitation learning and structured prediction to no-regret online learning. In International Conference on Artificial Intelligence and Statistics.

Schulman, J., Levine, S., Abbeel, P., Jordan, M. and Moritz, P. (2015). Trust region policy optimization. In International Conference on Machine Learning.

Schulman, J., Wolski, F., Dhariwal, P., Radford, A. and Klimov, O. (2017). Proximal policy optimization algorithms. arXiv preprint arXiv:1707.06347.
Shani, L., Efroni, Y. and Mannor, S. (2020a). Adaptive trust region policy optimization: Global convergence and faster rates for regularized mdps. In Proceedings of the AAAI Conference on Artificial Intelligence, vol. 34.

Shani, L., Efroni, Y., Rosenberg, A. and Mannor, S. (2020b). Optimistic policy optimization with bandit feedback. In International Conference on Machine Learning.

Shani, L., Zahavy, T. and Mannor, S. (2021). Online apprenticeship learning. arXiv preprint arXiv:2102.06924.

Siegel, N. Y., Springenberg, J. T., Berkenkamp, F., Abdolmaleki, A., Neunert, M., Lampe, T., Hafner, R., Heess, N. and Riedmiller, M. (2020). Keep doing what worked: Behavioral modelling priors for offline reinforcement learning. arXiv preprint arXiv:2002.08396.

Syed, U., Bowling, M. and Schapire, R. (2008). Apprenticeship learning using linear programming. In International Conference on Machine Learning.

Syed, U. and Schapire, R. E. (2008). A game-theoretic approach to apprenticeship learning. In Advances in Neural Information Processing Systems.

Tsurumine, Y., Cui, Y., Yamazaki, K. and Matsubara, T. (2019). Generative adversarial imitation learning with deep P-network for robotic cloth manipulation. In IEEE-RAS International Conference on Humanoid Robots (Humanoids). IEEE.

Uehara, M., Huang, J. and Jiang, N. (2020). Minimax weight and Q-function learning for off-policy evaluation. In International Conference on Machine Learning.

Uehara, M. and Sun, W. (2021). Pessimistic model-based offline RL: Pac bounds and posterior sampling under partial coverage. arXiv preprint arXiv:2107.06226.

Vershynin, R. (2010). Introduction to the non-asymptotic analysis of random matrices. arXiv preprint arXiv:1011.3027.

Wang, R., Foster, D. P. and Kakade, S. M. (2020a). What are the statistical limits of offline rl with linear function approximation? arXiv preprint arXiv:2010.11895.

Wang, Z., Novikov, A., Zolna, K., Springenberg, J. T., Reed, S., Shahriari, B., Siegel, N., Merel, J., Gulcehre, C., Heess, N. et al. (2020b). Critic regularized regression. arXiv preprint arXiv:2006.15134.

Xie, T., Cheng, C.-A., Jiang, N., Mineiro, P. and Agarwal, A. (2021a). Bellman-consistent pessimism for offline reinforcement learning. arXiv preprint arXiv:2106.06926.

Xie, T., Jiang, N., Wang, H., Xiong, C. and Bai, Y. (2021b). Policy finetuning: Bridging sample-efficient offline and online reinforcement learning. arXiv preprint arXiv:2106.04895.

Xu, T., Li, Z. and Yu, Y. (2020). Error bounds of imitating policies and environments. In Advances in Neural Information Processing Systems.

Yang, L. and Wang, M. (2019a). Sample-optimal parametric Q-learning using linearly additive features. In International Conference on Machine Learning.
Yang, L. and Wang, M. (2019b). Sample-optimal parametric Q-learning using linearly additive features. In *International Conference on Machine Learning*.

Yang, L. and Wang, M. (2020). Reinforcement learning in feature space: Matrix bandit, kernels, and regret bound. In *International Conference on Machine Learning*.

Yang, M., Nachum, O., Dai, B., Li, L. and Schuurmans, D. (2020a). Off-policy evaluation via the regularized lagrangian. *arXiv preprint arXiv:2007.03438*.

Yang, Z., Jin, C., Wang, Z., Wang, M. and Jordan, M. I. (2020b). Bridging exploration and general function approximation in reinforcement learning: Provably efficient kernel and neural value iterations. *arXiv preprint arXiv:2011.04622*.

Yu, T., Kumar, A., Rafailov, R., Rajeswaran, A., Levine, S. and Finn, C. (2021). Combo: Conservative offline model-based policy optimization. *arXiv preprint arXiv:2102.08363*.

Yu, T., Thomas, G., Yu, L., Ermon, S., Zou, J., Levine, S., Finn, C. and Ma, T. (2020). Mopo: Model-based offline policy optimization. *arXiv preprint arXiv:2005.13239*.

Zanette, A. (2021). Exponential lower bounds for batch reinforcement learning: Batch rl can be exponentially harder than online rl. In *International Conference on Machine Learning*.

Zhang, J. and Wu, F. (2021). A method of offline reinforcement learning virtual reality satellite attitude control based on generative adversarial network. *Wireless Communications and Mobile Computing, 2021*.

Zhang, Y., Cai, Q., Yang, Z. and Wang, Z. (2020). GAIL with neural network parameterization: Global optimality and convergence rate. In *International Conference on Machine Learning*.

Zhou, D., He, J. and Gu, Q. (2021). Provably efficient reinforcement learning for discounted mdps with feature mapping. In *International Conference on Machine Learning*.

Zolna, K., Novikov, A., Konyushkova, K., Gulcehre, C., Wang, Z., Aytar, Y., Denil, M., de Freitas, N. and Reed, S. (2020). Offline learning from demonstrations and unlabeled experience. *arXiv preprint arXiv:2011.19885*.
A Proofs of Supporting Lemmas: Analysis of OGAP

A.1 Proof of Lemma 5.1

Proof. For notational simplicity, we define operators $\mathbb{J}_h$ and $\mathbb{J}^k_h$ as
\begin{equation}
(\mathbb{J}_h f)(s) = \langle f(s, \cdot), \pi_h^E(\cdot | s) \rangle, \quad (\mathbb{J}^k_h f)(s) = \langle f(s, \cdot), \pi_h^E(\cdot | s) \rangle,
\end{equation}
for any $s \in \mathcal{S}$, $(k, h) \in [K] \times [H]$, and any function $f : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$. We define $\mathcal{F}_{k, h, 1}, \mathcal{F}_{k, h, 2}$ as follows,
\begin{align}
\mathcal{F}_{k, h, 1} &= \sigma(\{(s_i^{k}, a_i^{k})\}_{(r, i) \in [k-1] \times [H]} \cup \{(s_i^{h}, a_i^{h})\}_{i \in [h]}) \\
\mathcal{F}_{k, h, 2} &= \sigma(\{(s_i^{k}, a_i^{k})\}_{(r, i) \in [k-1] \times [H]} \cup \{(s_i^{h}, a_i^{h})\}_{i \in [h]} \cup \{s_{h+1}^{k}\}),
\end{align}
where $s_{h+1}^{k}$ is defined as a null state for any $k \in [K]$. We define the time index as follows,
\begin{equation}
t(k, h, m) = (k - 1) \cdot 2H + (h - 1) \cdot 2 + m,
\end{equation}
which implies that $\{\mathcal{F}_{k, h, m}\}_{(k, h, m) \in [K] \times [H] \times [2]}$ is a filtration with respect to $t(k, h, m)$.

Now we are ready to prove Lemma 5.1. First we note that for any initial state $x \in \mathcal{S}$, it holds that
\begin{equation}
V_{1,x}^k(x) - V_{1,x}^r(x) = (V_{1,x}^r(x) - V_1(x)) + (\hat{V}_1(x) - V_{1,x}^r(x)),
\end{equation}
where $V_1^k$ is the estimated state value function in the stage of policy evaluation of OGAP (Lines 8–12 of Algorithm 2). We calculate terms (i) and (ii) separately.

**Term (i).** By (2.2), we have
\begin{equation}
V_{h,x}^r(s) = \langle Q_{h,x}^r(s, \cdot), \pi_h^E(\cdot | s) \rangle = \mathbb{J}_h Q_{h,x}^r(s), \quad \hat{V}_h^k(s) = \langle \hat{Q}_h^k(s, \cdot), \pi_h^E(\cdot | s) \rangle = \mathbb{J}^k_h \hat{Q}_h^k(s),
\end{equation}
for any $(k, h) \in [K] \times [H]$. We then have
\begin{equation}
V_{h,x}^r \!-\! \hat{V}_h^k = \mathbb{J}_h Q_{h,x}^r \!-\! \mathbb{J}^k_h \hat{Q}_h^k
= \mathbb{J}_h (Q_{h,x}^r \!-\! \hat{Q}_h^k) + (\mathbb{J}_h \!-\! \mathbb{J}^k_h) \hat{Q}_h^k,
\end{equation}
where $\mathbb{J}_h$ and $\mathbb{J}^k_h$ are defined in (A.1). By the property of state-value function and the definition of $\iota_h^k$, we have
\begin{equation}
Q_{h,x}^r = r_h + \mathcal{P}_h V_{h+1,x}^r, \quad \hat{Q}_h^k = r_h + \mathcal{P}_h \hat{V}_{h+1}^k - \iota_h^k,
\end{equation}
Define $\zeta_h^k = (\mathbb{J}_h - \mathbb{J}^k_h) \hat{Q}_h^k$ and plug (A.7) into (A.6), we have
\begin{equation}
V_{h,x}^r \!-\! \hat{V}_h^k = \mathbb{J}_h \mathcal{P}_h (V_{h+1,x}^r \!-\! \hat{V}_{h+1}^k) + \mathbb{J}_h \iota_h^k + \zeta_h^k,
\end{equation}
for any $(k, h) \in [K] \times [H]$. Here $\iota_h^k$ is the prediction error defined in (5.2). For any $k \in [H]$, note that $V_{h+1,x}^r = \hat{V}_{h+1}^k = 0$, we expand (A.8) across $h \in [H]$ to obtain that
\begin{equation}
V_{1,x}^r \!-\! \hat{V}_1^k = \sum_{h=1}^H \prod_{i=1}^{h-1} \mathbb{J}_i \mathcal{P}_i \mathbb{J}_h \iota_h^k + \sum_{h=1}^H \prod_{i=1}^{h-1} \mathbb{J}_i \mathcal{P}_i \zeta_h^k.
\end{equation}
The effect of composite operator \( \mathcal{P}_h \mathcal{J}_h \) on function \( f \) is to calculate one-step expectation of \( f \) following policy \( \pi_h^E \). Hence we rewrite (A.9) as

\[
V_{1,\pi^E}^k(x) - \tilde{V}_1^k(x) = \sum_{h=1}^{H} \left( \mathbb{E}_{\pi^E} \left[ i_h^k(s_h^k, a_h^k) | s_1 = x \right] \right) \\
+ \sum_{h=1}^{H} \mathbb{E}_{\pi^E} \left[ (\hat{Q}_h^k(s_h, \cdot), \pi_h^E(\cdot | s_h) - \pi_h^E(\cdot | x_h)) | s_1 = x \right].
\]  

(A.10)

This characterize term (i).

**Term (ii).** By (A.4), we have

\[
i_h^k = r_h^k + \mathcal{P}_h \tilde{V}_{h+1}^k - \hat{Q}_h^k \]

\[
= r_h^k + \mathcal{P}_h \tilde{V}_{h+1}^k - Q_{h,\pi^E}^k + (Q_{h,\pi^E}^k - \hat{Q}_h^k) \]

\[
= \mathcal{P}_h (\tilde{V}_{h+1}^k - V_{h+1,\pi^E}^k) + (Q_{h,\pi^E}^k - \hat{Q}_h^k).
\]  

(A.11)

By (A.11), we obtain that

\[
\tilde{V}_h^k - V_{h,\pi^E}^k = J_h^k(\hat{Q}_h^k - Q_{h,\pi^E}^k) + i_h^k - \iota_h^k \]

\[
= (J_h^k(\hat{Q}_h^k - Q_{h,\pi^E}^k) - (\hat{Q}_h^k - Q_{h,\pi^E}^k))(s_h^k, a_h^k) + \mathcal{P}_h (\tilde{V}_{h+1}^k - V_{h+1,\pi^E}^k) - \iota_h^k.
\]  

(A.12)

We define \( D_{k,h,1} \) and \( D_{k,h,2} \) as follows,

\[
D_{k,h,1} = (J_h^k(\hat{Q}_h^k - Q_{h,\pi^E}^k))(s_h^k, a_h^k) - (\hat{Q}_h^k - Q_{h,\pi^E}^k)(s_h^k, a_h^k)
\]

\[
D_{k,h,2} = (\mathcal{P}_h (\tilde{V}_{h+1}^k - V_{h+1,\pi^E}^k))(s_h^k, a_h^k) - (\tilde{V}_{h+1}^k - V_{h+1,\pi^E}^k)(s_h^k). \]

(A.13)

By plugging (A.13) into (A.12), we obtain that

\[
\tilde{V}_h^k(s_h^k) - V_{h,\pi^E}^k(s_h^k) = D_{k,h,1} + D_{k,h,2} + (\tilde{V}_{h+1}^k - V_{h+1,\pi^E}^k)(s_h^k) - \iota_h^k(s_h^k, a_h^k).
\]  

(A.14)

By telescoping (A.14) with respect to \( h \in [H] \), we have

\[
\tilde{V}_1^k(x) - V_{1,\pi^E}^k(x) = \sum_{h=1}^{H} (D_{k,h,1} + D_{k,h,2}) - \sum_{h=1}^{H} \iota_h^k(s_h^k, a_h^k).
\]  

(A.15)

By the definition of \( \mathcal{F}_{k,h,1} \) and \( \mathcal{F}_{k,h,2} \) in (A.2), we have

\[
D_{k,h,1} \in \mathcal{F}_{k,h,1}, \ D_{k,h,2} \in \mathcal{F}_{k,h,1}, \ \mathbb{E}[D_{k,h,1} | \mathcal{F}_{k,h-1,1}] = 0, \ \mathbb{E}[D_{k,h,2} | \mathcal{F}_{k,h,1}] = 0.
\]  

(A.16)

Following from (A.16), we define the martingale

\[
\mathcal{M}_{k,h,m} = \sum_{(r,i) \in [K] \times [H] \times [2], t(r,i) \leq t(k,h,m)} D_{r,i,l},
\]  

(A.17)

with respect to the time index \( t(k,h,m) \) defined in (A.3). It is obvious that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} (D_{k,h,1} + D_{k,h,2}) = \mathcal{M}_{K,H,2}
\]  

(A.18)
Combining (A.4), (A.13), and (A.15) we obtain that
\[
\sum_{k=1}^{K} (V_{x,\pi}^k(x) - V_{1,\pi}^k(x)) = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ (\hat{Q}_h^k(s_h, \cdot), \pi_h^E(\cdot | s_h) - \pi_h^k(\cdot | x_h)) | s_1 = x \right] + \mathcal{M}_{K,H,2} + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ \hat{t}_h^k(s_h, a_h^k) | s_1 = x \right] - \hat{t}_h^k(s_h, a_h^k).
\] (A.19)

By this, we conclude the proof of Lemma 5.1. □

### A.2 Proof of Lemma 5.2

**Proof.** By the update rule of OGAP in (3.3) (Lines 4–6 of Algorithm 2) and the property of the mirror descent, we have
\[
\mathcal{L}_{k-1}(\pi^k) - \alpha^{-1} \cdot D(\pi^E, \pi^{k-1}) \geq \mathcal{L}_{k-1}(\pi^E) - \alpha^{-1} \cdot D(\pi^E, \pi^{k-1}) + \alpha^{-1} \cdot D(\pi^E, \pi^k).
\]

Recalling the definition of \( \mathcal{L}_{k-1}(\pi) \) in (3.2) and rearranging the above inequality, we derive that
\[
\sum_{h=1}^{H} \langle \hat{Q}_h^{k-1}, \pi_h^E - \pi_h^{k-1} \rangle_{\mathcal{A}} \leq \alpha^{-1} \cdot D(\pi^E, \pi^{k-1}) - \alpha^{-1} \cdot D(\pi^E, \pi^k)
\]
\[+ \sum_{h=1}^{H} \langle \hat{Q}_h^{k-1}, \pi_h^k - \pi_h^{k-1} \rangle_{\mathcal{A}} - \alpha^{-1} \cdot D(\pi^k, \pi^{k-1}),
\] (A.20)

where \( D(\pi^E, \pi^{k-1}) = \sum_{h=1}^{H} D_{KL}(\pi_h^k || \pi_h^{k-1}) \). For the last two terms on the right-hand side of (A.20), we have
\[
\sum_{h=1}^{H} \langle \hat{Q}_h^{k-1}, \pi_h^k - \pi_h^{k-1} \rangle_{\mathcal{A}} - \alpha^{-1} \cdot D(\pi^k, \pi^{k-1})
\leq \sum_{h=1}^{H} (\| \hat{Q}_h^{k-1} \|_{\mathcal{A}, \infty} \cdot \| \pi_h^k - \pi_h^{k-1} \|_{\mathcal{A}, 1} - (2\alpha)^{-1} \cdot \| \pi_h^k - \pi_h^{k-1} \|^2_{\mathcal{A}, 1})
\leq \frac{\alpha}{2} \cdot \sum_{h=1}^{H} \| \hat{Q}_h^{k-1} \|^2_{\mathcal{A}, \infty} \leq \alpha H^3 \sqrt{d} / 2,
\]

where the first inequality follows from Holder’s inequality and Pinsker’s inequality, and the last inequality derives from the fact that \( |r_h^k(\cdot, \cdot)| \leq \sqrt{d} \) for any \( h \in [H] \) and \( \mu \in S \). Since \( \pi^0 \) is a uniform distribution on \( \mathcal{A} \), it holds that \( D(\pi^E, \pi^0) \leq H \log(\text{vol}(\mathcal{A})) \). Telescoping (A.2) with respect to \( k \in [K] \), we have
\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \left[ \langle \hat{Q}_h^{k-1}, \pi_h^E - \pi_h^{k-1} \rangle_{\mathcal{A}} \right] \leq \alpha KH^3 \sqrt{d} / 2 + \alpha^{-1} HD(\pi^E, \pi^0)
\leq \alpha KH^3 \sqrt{d} / 2 + \alpha^{-1} H \log(|\mathcal{A}|).
\] (A.21)

Recalling that \( \alpha = \sqrt{2 \log(|\mathcal{A}|) / (H^2 K \sqrt{d})} \) and taking expectation on both side of (A.21), we have
\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^k} \left[ \langle \hat{Q}_h^{k-1}, \pi_h^E - \pi_h^{k-1} \rangle_{\mathcal{A}} | s_1 = x \right] \leq \alpha KH^3 \sqrt{d} / 2 + \alpha^{-1} H \log(|\mathcal{A}|)
\leq \sqrt{2 H^4 \sqrt{d} K \log(\text{vol}(\mathcal{A}))}.
\]

Then we conclude the proof of Lemma 5.2. □
A.3 Proof of Lemma 5.3

Proof. Recalling that we define $D_{k,h,1}$ and $D_{k,h,2}$ in (A.13) and the fact that $|r^k_h(\cdot, \cdot)| \leq \sqrt{d}$ for any $\mu \in S$, we derive that $|D_{k,h,1}| \leq 2H\sqrt{d}$ and $|D_{k,h,2}| \leq 2H\sqrt{d}$ for any $(k, h) \in [K] \times [H]$. Now by Azuma-Hoeffding inequality, we have
\[
P(|M_{K,H,2}| > t) \leq 2 \exp \left( \frac{-t^2}{16H^2Kd} \right),
\]
for any $t > 0$. Setting $t = \sqrt{16H^2dK \cdot \log(8/\xi)}$ with $\xi \in (0, 1)$ in (A.22), we have
\[
M_{K,H,2} \leq \sqrt{16H^2dK \cdot \log(8/\xi)},
\]
with probability at least $1 - 1/4$.

A.4 Proof of Lemma 5.4

Proof. For notational simplicity, we write $\tilde{Q}^k_h(s, a) = r^k_h(s, a) + \bar{\rho}_h \tilde{V}^k_{h+1}(s, a) + \Gamma^k_h(s, a)$. Then, from the policy evaluation stage in Lines 8–12 of Algorithm 3, we have
\[
\tilde{Q}^k_h(s, a) = \min \left\{ \max \left\{ \tilde{Q}^k_h(s, a), 0 \right\}, (H - h + 1)\sqrt{d} \right\},
\]
(\text{A.23})

We introduce the following lemma.

Lemma A.1. Let $\lambda = 1$ in the construction of estimated kernels (3.6) and $\kappa = C \sqrt{d \log(HdK/\xi)}$ in the construction of bonus (3.8). Then it holds with probability at least $1 - \xi/4$ that
\[
|\bar{\rho}_h \tilde{V}^k_{h+1}(s, a) - \bar{\rho}_h \tilde{V}^k_{h+1}(s, a)| \leq \Gamma^k_h(s, a)
\]
for any $(s, a) \in \mathcal{S} \times \mathcal{A}$.

Proof. See Appendix A.9 for a detailed proof.

By Lemma A.1, we obtain that $r^k_h + \bar{\rho}_h \tilde{V}^k_{h+1} \leq \tilde{Q}^k_h$. Moreover, by the fact that $|r^k_h(s, a)| \leq \sqrt{d}$ and $\tilde{V}^k_{h+1}(s) = \langle \tilde{Q}^k_h(s, \cdot), \pi^k_h(\cdot | s) \rangle_{\mathcal{A}} \in [0, (H - h)\sqrt{d}]$ for any $(s, a) \in \mathcal{S} \times \mathcal{A}$, we have $r^k_h + \bar{\rho}_h \tilde{V}^k_{h+1, \pi^k_h} \in [0, (H - h + 1)\sqrt{d}]$. Thus, we have
\[
\tilde{Q}^k_h(s, a) = \min \left\{ \max \left\{ \tilde{Q}^k_h(s, a), 0 \right\}, (H - h + 1)\sqrt{d} \right\}
\]
\[
\geq \min \left\{ \max \left\{ r^k_h(s, a) + \bar{\rho}_h \tilde{V}^k_{h+1}(s, a), 0 \right\}, (H - h + 1)\sqrt{d} \right\}
\]
\[
= r^k_h(s, a) + \bar{\rho}_h \tilde{V}^k_{h+1}(s, a),
\]

which implies that $\iota^k_h \leq 0$.

It remains to establish the lower bound of $r^k_h(s, a)$. By Lemma A.1, we have
\[
\tilde{Q}^k_h(s, a) = r^k_h(s, a) + \bar{\rho}_h \tilde{V}^k_{h+1}(s, a) + \Gamma^k_h(s, a)
\]
\[
\geq r^k_h(s, a) + \bar{\rho}_h \tilde{V}^k_{h+1}(s, a) \geq 0,
\]
(\text{A.24})

where the last inequality follows from the fact that $\tilde{V}^k_{h+1}(s, a) \geq 0$ and $r^k_h(s, a) \geq 0$. By (A.23) and (A.24), we obtain that $\tilde{Q}^k_h(s, a) \leq \tilde{Q}^k_h(s, a)$, which implies that
\[
\iota^k_h(s, a) = (r^k_h + \bar{\rho}_h \tilde{V}^k_h)(s, a) - \tilde{Q}^k_h(s, a)
\]
\[
\geq (\bar{\rho}_h - \bar{\rho}_h) \tilde{V}^k_h(s, a) - \Gamma^k_h(s, a)
\]
\[
\geq -2\Gamma^k_h(s, a).
\]

Here the last inequality follows from Lemma A.1. Thus, we conclude the proof of Lemma 6.2.
A.5 Proof of Lemma 5.5

Proof. By the construction of bonus $\Gamma^k_h$ in (3.8), we have

$$
\sum_{h=1}^H \sum_{k=1}^K \Gamma^k_h(s_h^k, a_h^k) = H \sqrt{d} \cdot \sum_{h=1}^H \sum_{k=1}^K \min \left\{ 1, \kappa \cdot \varphi^k_h(s_h^k, a_h^k) \right\} \left( (\Lambda^k_h)^{-1} \varphi^k_h(s_h^k, a_h^k) \right)^{1/2},
$$

(A.25)

where the last inequality comes from Cauchy-Schwarz inequality. To upper bound the right-hand side of (A.25), we introduce the following lemma.

Lemma A.2 (Elliptical Potential (Abbasi-Yadkori et al., 2011)). Let $\{\phi_t\}_{t=1}^\infty$ be an $\mathbb{R}^d$-valued sequence. Meanwhile, let $\Lambda_0 \in \mathbb{R}^{d \times d}$ be a positive-definite matrix and $\Lambda_t = \Lambda_0 + \sum_{j=1}^{t-1} \phi_j \phi_j^\top$. It holds for any $t \in \mathbb{Z}_+$ that

$$
\sum_{j=1}^t \min \left\{ 1, \phi_j^\top \Lambda_j^{-1} \phi_j \right\} \leq 2 \log \left( \frac{\det (\Lambda_{t+1})}{\det (\Lambda_1)} \right).
$$

Moreover, assuming that $\|\phi_j\|_2 \leq 1$ for any $j \in \mathbb{Z}_+$ and $\lambda_{\min} (\Lambda_0) \geq 1$, it holds for any $t \in \mathbb{Z}_+$ that

$$
\log \left( \frac{\det (\Lambda_{t+1})}{\det (\Lambda_1)} \right) \leq \sum_{j=1}^t \phi_j^\top \Lambda_j^{-1} \phi_j \leq 2 \log \left( \frac{\det (\Lambda_{t+1})}{\det (\Lambda_1)} \right).
$$

Proof. See proof of Lemma 11 in Abbasi-Yadkori et al. (2011) for a detailed proof.

For any fixed $h \in [H]$, by Lemma A.2, we have

$$
\sum_{k=1}^K \varphi^k_h(s_h^k, a_h^k) \left( (\Lambda^k_h)^{-1} \varphi^k_h(s_h^k, a_h^k) \right)^{1/2} \leq 2 \log \left( \frac{\det (\Lambda^K_h)}{\det (\Lambda^1_h)} \right),
$$

(A.26)

where $\Lambda^1_h = \lambda \cdot I$ and $\Lambda^{K+1}_h \in \mathcal{K}_{K,H,2}$, which is defined in (A.2). By Assumption 2.1, we obtain that

$$
\|\varphi^k_h(s, a)\|_2 = \left\| \int_S \phi(s, a, s') \hat{V}^k_h(s') \, ds' \right\|_2 \\
\leq H \sqrt{d} \cdot \left\| \int_S \phi(s, a, s') \, ds' \right\|_2 \leq H d^{3/2} R \cdot \text{Vol}(S).
$$

(A.27)

Here the first inequality comes from the fact that $\hat{V}^k_h \in [0, H \sqrt{d}]$ for any $(k, h) \in [K] \times [H]$, and the last inequality comes from the fact that $\|\phi(\cdot, \cdot, \cdot)\|_2 \leq dR$, which can be verified as follows,

$$
\sup_{(s, a, s') \in S \times \mathcal{A} \times S} \|\phi(s, a, s')\|_2 = \sup_{(s, a, s') \in S \times \mathcal{A} \times S} \left( \sum_{i=1}^d \|\phi(s, a, s')^\top e_i\|_2^2 \right)^{1/2} \leq dR.
$$

(A.28)

Here $\{e_i\}_{i=1}^d$ is a group of orthonormal basis of $\mathbb{R}^d$ and the last inequality follows from Assumption 2.1. By the definition of $\Lambda^k_h$ in (3.6), we can upper bound $\det (\Lambda^{K+1}_h)$ by (A.27) as follows,

$$
\det (\Lambda^{K+1}_h) = \det \left( \sum_{k=1}^K \varphi^k_h(s_h^k, a_h^k) \varphi^k_h(s_h^k, a_h^k)^\top + I \right) \\
\leq \left( \det \left( (H d^{3/2} R \cdot \text{Vol}(S) + 1) \cdot I \right) \right)^d,
$$

(A.29)
which implies that
\[
\log \left( \frac{\det (\Lambda_h^{K+1})}{\det (\Lambda_h^1)} \right) \leq 2d \cdot \log (H^2d^3R^2K \cdot \text{Vol}(S)^2). \tag{A.30}
\]
Recalling that \( \kappa = C \sqrt{d \log (HdK/\xi)} \), combining (A.25), (A.26), and (A.30), we have
\[
\sum_{h=1}^{H} \sum_{k=1}^{K} \Gamma_h^{k}(s_h, a_h) \leq 4H \sqrt{d\kappa} \cdot H \sqrt{dK} \cdot \log (H^2d^3R^2K \cdot \text{Vol}(S)^2) \\
\leq C' \sqrt{H^4d^3K} \cdot \log (HdK/\xi),
\]
where \( C' \) is an absolute constant determined by \( C, R, \) and \( \log (\text{Vol}(S)) \). By this, we conclude the proof of Lemma 5.5. \( \Box \)

A.6 Proof of Lemma 5.6

Proof. By the definition of cumulative reward in (2.3), we observe that
\[
J(\pi, \mu) = \mathbb{E}_\pi \sum_{h=1}^{H} \left[ r_h^\mu(s_h, a_h) \right] \\
= \sum_{h=1}^{H} \mathbb{E}_\pi \left[ r_h^\mu(s_h, a_h) \right] \\
= \sum_{h=1}^{H} \int_{S \times A} \rho_h^\pi(s, a) \cdot r_h^\mu(s, a) dsa,
\]
where \( \rho_h^\pi(s, a) = \mathbb{P}(s_h = s, a_h = a) \) is the density of state-action visitation measure on \( S \times A \). Recall that under Assumption 2.1, we have \( r_h^\mu(s, a) = \psi(s, a)^\top \mu_h \), hence we have
\[
\nabla_{\mu_h} J(\pi^k, \mu^k) = \int_{S \times A} \rho_h^\pi(s, a) \cdot \psi(s, a) dsa.
\]
By (2.4), we obtain that
\[
L(\pi^k, \mu) - L(\pi^k, \mu^k) = \sum_{h=1}^{H} \left( \mu_h - \mu_h^k \right)^\top \nabla_{\mu_h} L(\pi^k, \mu^k),
\]
where \( \nabla_{\mu_h} L(\pi^k, \mu^k) = \nabla_{\mu_h} J(\pi^k, \mu^k) - \nabla_{\mu_h} J(\pi^k, \mu^k). \)

Combining (A.32) and (A.33), we know that \( L(\pi, \mu) \) is a linear function in \( \mu \) for any \( \pi \). Recall that \( \mu_h^{k+1} = \text{Proj}_B\{\mu_h^k + \eta \nabla_{\mu_h} L(\pi^k, \mu^k)\} \) in OGAP (Lines 13–17 of Algorithm 2), by the definition of the projection operator \( \text{Proj}_B(\cdot) \), it holds that
\[
[\mu_h^{k+1} - \mu_h^k - \eta \nabla_{\mu_h} L(\pi^k, \mu^k)]^\top \left( \mu_h - \mu_h^{k+1} \right) \geq 0. \tag{A.34}
\]
Rearranging terms in (A.34), we obtain that
\[
\eta (\mu_h - \mu_h^{k+1})^\top \nabla_{\mu_h} L(\pi^k, \mu^k) \leq (\mu_h^{k+1} - \mu_h^k)^\top (\mu_h - \mu_h^{k+1}) \\
= \frac{1}{2} \left( \|\mu_h^k - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h^k\|_2^2 \right), \tag{A.35}
\]
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which also implies that
\[
\frac{1}{2} \left( \|\mu_h^k - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h^k\|_2^2 \right) - \eta (\mu_h - \mu_h^k)^\top \nabla_{\mu_h} L(\pi^k, \mu^k) \geq 0. \tag{A.36}
\]
By adding a term \(\eta \nabla_{\mu_h} L(\pi^k, \mu^k)^\top (\mu_h - \mu_h^k)\) on both sides of (A.36) and combining (A.33), we obtain that
\[
L(\pi^k, \mu) - L(\pi^k, \mu^k) \leq \sum_{h=1}^H \frac{1}{2\eta} (\|\mu_h^k - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h^k\|_2^2)
+ \sum_{h=1}^H [(\mu_h^{k+1} - \mu_h^k)^\top \nabla_{\mu_h} L(\pi^k, \mu^k)]
+ \sum_{h=1}^H [(\mu_h^k - \mu_h)^\top (\nabla_{\mu_h} L(\pi^k, \mu^k) - \nabla_{\mu_h} L(\pi^k, \mu^k))],
\tag{A.37}
\]
where we take the summation on \(h\) from 1 to \(H\). By the fact that \(\nabla_{\mu_h} L(\pi^k, \mu^k) = \nabla_{\mu_h} \tilde{J}(\pi^E, r^\mu) - \psi(s_h^k, a_h^k)\) and the definition of the GAIL objective function \(L(\pi, \mu)\) in (2.4), we rewrite the third term on the right-hand side of (A.37) as
\[
\sum_{h=1}^H [(\mu_h^k - \mu_h)^\top (\nabla_{\mu_h} L(\pi^k, \mu^k) - \nabla_{\mu_h} L(\pi^k, \mu^k))] = \sum_{h=1}^H [\mu_h^\top (\nabla_{\mu_h} \tilde{J}(\pi^k, \mu^k) - \nabla_{\mu_h} J(\pi^E, r^\mu))]
+ \sum_{h=1}^H [(\mu_h^k - \mu_h)^\top (\psi(s_h^k, a_h^k) - \nabla_{\mu_h} J(\pi^k, \mu^k))].
\tag{A.38}
\]
By (A.32) and the definition of \(\tilde{J}(\pi^E, r^\mu)\) in (3.11), we derive from (A.37) and (A.38) that
\[
L(\pi^k, \mu) - L(\pi^k, \mu^k) \leq \sum_{h=1}^H \frac{1}{2\eta} (\|\mu_h^k - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h^k\|_2^2)
+ \sum_{h=1}^H [(\mu_h^{k+1} - \mu_h^k)^\top \nabla_{\mu_h} L(\pi^k, \mu^k)] + [\tilde{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu)]
+ \sum_{h=1}^H [(\mu_h^k - \mu_h)^\top (\psi(s_h^k, a_h^k) - \nabla_{\mu_h} J(\pi^k, \mu^k))].
\]
Upon telescoping sum on the below inequality for the index \(k \in [K]\), we complete the proof of Lemma 5.6. \(\square\)

A.7 Proof of Lemma 5.7

Proof. For any fixed reward parameter \(\mu \in S\), we define \(J^\tau(\pi^E, r^\mu) = \sum_{h=1}^H \psi(s_{h,\tau}^E, a_{h,\tau}^E)^\top \mu_h\) for any \(\tau \in [N_1]\). Since the expert demonstration \(D = \{s_{h,k}^E, a_{h,k}^E\}_{(k,h)\in[N_1]\times[H]}\) involves \(N_1\) independent trajectories induced by the expert policy \(\pi^E\), we apply Monte Carlo method to estimate \(J(\pi^E, r^\mu)\) by \(N_1\) i.i.d. samples \(\{J^\tau(\pi^E, r^\mu)\}_{\tau=1}^{N_1}\).

Let \(Z_n = \sum_{\tau=1}^n (J^\tau(\pi^E, r^\mu) - J(\pi^E, r^\mu))\) and we have \(|Z_n - Z_{n-1}| \leq 2H\sqrt{d}\), since \(|r_h(\cdot, \cdot)| \leq \sqrt{d}\) for all \(h \in [H]\). Note that \(\{Z_n\}\) is a martingale with zero mean with respect to the filtration \(\mathcal{F}_n = \sigma\{s_h^k, a_h^k\}_{(h,i)\in[H]\times[n]}\), by Azuma-Hoeffding inequality, we have
\[
P_D(|Z_n| > t) \leq 2 \exp \left( \frac{-2t^2}{4H^2dn} \right),
\]
where \(d\) is the dimension of the feature space.

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which implies that
\[ \Pr \left( \left| \frac{Z_{N_1}}{N_1} \right| > m \right) \leq 2 \exp \left( \frac{-2m^2N_1}{4H^2d} \right). \]

Let \( \delta = 2 \exp \{ -m^2N_1/(2H^2d) \} \), it holds with probability at least \( 1 - \delta \) that
\[ |\overline{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu)| = \left| \frac{Z_{N_1}}{N_1} \right| \leq H \sqrt{2d \log(2/\delta)/N_1}. \]  

(A.39)

We union bound \( |\overline{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu)| \) for any \( \mu \in S \) as follows. Since the reward parameter domain \( S \) defined in (2.8) is not a finite set, we apply discretization on \( S \) to derive a union bound on \( |\overline{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu)| \). If we define a normed space \((\mathbb{R}^d, \| \cdot \| \)\), where \( \| \cdot \| \) is defined as
\[ \| \mu \| = \sup_{h \in [H]} \| \mu_h \|_2, \]  

then parameter domain \( S \) belongs to this normed space. Before we continue, we first introduce the definitions of \( \epsilon \)-covering and covering number as follows.

**Definition A.3** (\( \epsilon \)-covering). Let \((V, \| \cdot \| )\) be a normed space, and \( \Theta \subset V \). We say that \( \{V_1, \ldots, V_N\} \) is an \( \epsilon \)-covering of \( \Theta \) if \( \Theta \subset \cup_{i=1}^N B(V_i, \epsilon) \), or equivalently, \( \forall \theta \in \Theta, \exists i \) such that \( \| \theta - V_i \| \leq \epsilon \). Here \( B(V_i, \epsilon) \) denotes a ball centering \( V_i \) with radius \( \epsilon \).

We define the covering number as follows,
\[ \mathcal{N}(\Theta, \| \cdot \|, \epsilon) := \min \left\{ n : \exists \epsilon \text{-covering over } \Theta \text{ of size } n, \Theta \in (V, \| \cdot \| ) \right\}. \]

With Definition A.3, we introduce the following lemma to upper bound the covering number.

**Lemma A.4.** If \((V, \| \cdot \| )\) is a normed space, and (i) \( \Theta \subset V = \mathbb{R}^d \), (ii) \( \Theta \) is convex, (iii) \( \epsilon B_{\text{unit}} \subset \Theta \), where \( \epsilon > 0 \) and \( B_{\text{unit}} \) is the unit ball in \( \mathbb{R}^d \), then it holds that
\[ \mathcal{N}(\Theta, \| \cdot \|, \epsilon) \leq \frac{3^d \text{vol}(\Theta)}{\epsilon^{d \text{vol}(B_{\text{unit}})}}. \]

**Proof.** See Lemma 5.2 of Vershynin (2010) for proof. \( \Box \)

Note that \( S \) is convex as a subset of \( \mathbb{R}^{dH} \), we apply Lemma A.4 with \( V = \mathbb{R}^{dH} \), \( \Theta = S \), \( \| \cdot \| = \| \cdot \|_* \), and an appropriate \( \epsilon > 0 \) satisfying condition (iii) in Lemma A.4, which implies that
\[ \mathcal{N}(S, \| \cdot \|_*, \epsilon) \leq \left( \frac{3}{\epsilon} \right)^{Hd} d^{Hd/2}. \]

By the definition of covering number, there exists an \( \epsilon \)-covering \( V_\epsilon = \{ \mu_1, \ldots, \mu^{\mathcal{N}(S, \| \cdot \|_*, \epsilon)} \} \subset S \). For each \( \mu \in V_\epsilon \), by (A.39), it holds that
\[ |\overline{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu)| \leq H \sqrt{2d \log(2\mathcal{N}(S, \| \cdot \|_*, \epsilon)/\xi)/N_1}, \]

with probability at least \( 1 - \xi/\mathcal{N}(S, \| \cdot \|_*, \epsilon) \). By the union bound, it yields that
\[ \sup_{\mu \in V_\epsilon} |\overline{J}(\pi^E, r^\mu) - J(\pi^E, r^\mu)| \leq H \sqrt{2d \log(2\mathcal{N}(S, \| \cdot \|_*, \epsilon)/\xi)/N_1} \]
\[ \leq H \sqrt{\left( H^2d \log(d) + 2H^2d \log(3/\epsilon) + 2d \log(2/\xi) \right)/N_1}, \]

(A.41)
with probability at least $1 - \xi$. Note that for any $\mu', \mu'' \in S$ satisfying $||\mu' - \mu''||_* \leq \epsilon$, it holds that
\[
\left| \left[ \bar{J}(\pi^E, r^\mu) - J(\pi^E, \mu') \right] - \left[ \bar{J}(\pi^E, r^{\mu''}) - J(\pi^E, \mu'') \right] \right| \leq 4H\epsilon. \tag{A.42}
\]
Combining (A.41) and (A.42) and applying triangle inequality, we derive that
\[
\sup_{\mu \in S} \left| \bar{J}(\pi^E, r^\mu) - J(\pi^E, r^\nu) \right| \leq H \sqrt{(Hd^2 \log(d) + 2Hd^2 \log(3/\epsilon) + 2d \log(2/\xi)) / N_1 + 4H\epsilon}, \tag{A.43}
\]
with probability at least $1 - \xi$. By taking $\epsilon = \sqrt{6d / N_1}$ in (A.43), which satisfies the conditions in Lemma A.4, it holds with probability at least $1 - \xi$ that
\[
\sup_{\mu \in S} \left| \bar{J}(\pi^E, r^\mu) - J(\pi^E, r^\nu) \right| \leq 4\sqrt{Hd^2 / N_1 \log(6N_1 / \xi)}.
\]
We conclude the proof of Lemma 5.7. \qed

A.8 Proof of Lemma 5.8

Proof. First, we fix $\mu \in S$. We define for $(k, h) \in [K] \times [H]$ that
\[
X_h^k = (\mu_h^k - \mu_h)^\top (-\psi(s_h^k, a_h^k) + \nabla_{\mu_h} J(\pi^k, \mu_h)),
\]
\[
Y^k = \sum_{i=1}^k \sum_{h=1}^H X_h^i,
\]
\[
S_h^k = \sigma((s_{h,1}^1, a_{h,1}^1), (s_{h,2}^2, a_{h,2}^2), \ldots, (s_{h,k}^k, a_{h,k}^k)),
\]
\[
E_h = \sigma((s_{h,1}^E, a_{h,1}^E), (s_{h,2}^E, a_{h,2}^E), \ldots, (s_{h,N_1}^E, a_{h,N_1}^E)),
\]
\[
G_h^k = \sigma(S_h^k, E_h),
\]
\[
G^k = \sigma(G_1^k, G_2^k, \ldots, G_H^k),
\]
where $\sigma(\cdot)$ denotes the generated $\sigma$-algebra. It holds that $\{G^k\}_{k \in [K]}$ is a filtration with respect to the time index $k$, since $G^{k_1} \subseteq G^{k_2}$ for $k_1 \leq k_2$.

We first show that $X_h^k \in G^k$ holds for any $(k, h) \in [K] \times [H]$. By the definition of $X_h^k$ in (A.44), it only suffices to prove that $\mu_h^k \in G^k$ for any $(k, h) \in [K] \times [H]$. Here we show this by induction with index $k$. Since $\mu_1^k = 0$ for any $h \in [H]$, the base case where $k = 1$ is trivial. We assume that $\mu_h^k \in G^k$ where $k \geq 1$ is a given integer, then we consider the case $k + 1$. Recall that the update process of reward parameter in OGAP (Lines 13–17 of Algorithm 2) takes the following form,
\[
\mu_h^{k+1} = \text{Proj}_B \{ \mu_h^k + \eta \tilde{\nabla}_{\mu_h} L(\pi^k, \mu_h^k) \}
\]
\[
= \text{Proj}_B \left\{ \mu_h^k + \eta \left[ \frac{1}{N_1} \sum_{\tau=1}^{N_1} \psi(s_{h,\tau}^E, a_{h,\tau}^E) - \psi(s_h^k, a_h^k) \right] \right\}.
\]
First, according to the induction hypothesis, we have $\mu_h^k \in G^k \subseteq G^{k+2}$, which implies that $(s_h^k, a_h^k) \in G^{k+1}$. Then it holds that
\[
\mu_h^k + \frac{1}{N_1} \sum_{\tau=1}^{N_1} \psi(s_{h,\tau}^E, a_{h,\tau}^E) - \psi(s_h^k, a_h^k) \in G^{k+1},
\]
for any \( h \in [H] \). As \( \text{Proj}_B \) is a continuous operator, we obtain that \( \mu_h^{k+1} \in G^{k+1} \). Thus we complete the induction.

Now we construct a martingale to upper bound \( Y^K \). Note that conditioning on the filtration \( G^{k-1} \), the term \( \mu^k_h - \mu_h \) is a constant. Recall that in (A.32) we show that

\[
\nabla_{\mu_h} J(\pi^k, \mu^k) = \int_{S \times A} \psi(s, a) : \rho_h^k(s, a) \text{d}sa,
\]

which implies that

\[
\mathbb{E}_k(X^k_h | G^{k-1}) = (\mu^k_h - \mu_h) \mathbb{E}_k(\psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k) | G^{k-1}) = 0
\]

(A.45)

for any \((k, h) \in [K] \times [H]\). Here the expectation \( \mathbb{E}_k \) is taken with respect to \( a^k_i \sim \pi^k_i(\cdot | s^k_i) \) and \( s^k_{i+1} \sim \mathcal{P}_i(\cdot | s^k_i, a^k_i) \), corresponding to the expectation taken with respect to the state-action visitation measure \( \rho_h^k \) defined in (A.31). Also, we have \( Y^K \in G^k \), since \( X^k_h \in G^k \) for any \((k, h) \times [K] \times [H]\). Moreover, we obtain for any \( k \in [K] \) that

\[
\mathbb{E}_k(Y^K | G^{k-1}) = \mathbb{E}_k(\sum_{i=1}^{H} \sum_{h=1}^{K} X^k_h | G^{k-1})
\]

\[
= \mathbb{E}_k(\sum_{h=1}^{H} X^k_h + \sum_{i=1}^{H} X^k_h | G^{k-1})
\]

\[
= \mathbb{E}_k(\sum_{h=1}^{H} X^k_h | G^{k-1}) + Y^{k-1} = Y^{k-1},
\]

where the last equality follows from (A.45). Thus \( \{Y^K\}_{k=1}^{K} \) is a martingale. Furthermore, by the definition of \( X^k_h \) in (A.44), it holds that

\[
|Y^K - Y^{k-1}| = |\sum_{h=1}^{H} X^k_h| \leq \sum_{h=1}^{H} \|\mu^k_h - \mu_h\|_2 \|\psi(s^k_h, a^k_h) - \nabla_{\mu_h} J(\pi^k, \mu^k)\|_2 \leq 8\sqrt{dH},
\]

where the last inequality follows from the facts that \( \|\psi(\cdot, \cdot)\|_2 \leq 1 \) and \( \|\mu^k_h\|_2 \leq \sqrt{d} \). Therefore, by Azuma-Hoeffding inequality, we obtain that

\[
P(|Y^K| \geq t) \leq \exp\left(\frac{-t^2}{2 \sum_{k=1}^{K} (8\sqrt{dH})^2}\right) = \exp\left(\frac{-t^2}{128KdH^2}\right)
\]

for any \( t > 0 \). Setting \( t = \sqrt{128H^2dK \log(2/\xi)} \) with \( \xi \in (0, 1) \) and by the definition of \( Y^K \) in (A.44), it holds with probability at least \( 1 - \xi \) that

\[
\left| \sum_{k=1}^{K} \sum_{h=1}^{H} (\mu^k_h - \mu_h)^T (\psi(s^k_h, a^k_h) + \nabla_{\mu_h} J(\pi^k, \mu^k)) \right| \leq 8\sqrt{2H^2dK \log(2/\xi)}.
\]

(A.46)

Now we union bound (A.46). This is similar to the proof of Lemma 5.7 in Appendix A.7. By applying Lemma A.4 in the same normed space \((\mathbb{R}^{Hd}, \| \cdot \|_*)\), it holds with probability at least \( 1 - \xi \) that

\[
\sup_{\mu \in \mathcal{V}_i} |M(\mu)| = \left| \sum_{k=1}^{K} \sum_{h=1}^{H} (\mu^k_h - \mu_h)^T (\psi(s^k_h, a^k_h) + \nabla_{\mu_h} J(\pi^k, \mu^k)) \right| \leq 8\sqrt{(H^3d^2 \log(d) + 2Hd^2 \log(3/\epsilon) + 2d \log(2/\xi))K},
\]

(A.47)
where $\mathcal{V}_r$ is the $\epsilon$-covering for $S$ in Definition A.3 and $\| \cdot \|_*$ is defined in (A.40). Here for notational convenience, we denote by $M(\mu) = \sum_{k=1}^{K} \sum_{h=1}^{H} (\mu_k^{k} - \mu_h) \top (-\psi(s_h^{k}, a_h^{k}) + \nabla_{\mu_h} J(\pi, \mu_k^{k}))$. For any $\mu', \mu'' \in S$ satisfying $\|\mu' - \mu''\|_* \leq \epsilon$, it holds that

$$|M(\mu') - M(\mu'')| \leq 4HK\epsilon. \quad (A.48)$$

Combining (A.47) and (A.48) and applying triangle inequality, we have that

$$\sup_{\mu \in S} |M(\mu)| \leq 8\sqrt{(H^3d^2 \log(d) + 2Hd^2 \log(3/\epsilon) + 2d \log(2/\epsilon))K} + 4HK\epsilon, \quad (A.49)$$

with probability at least $1 - \xi$. By taking $\epsilon = \sqrt{d/K}$ in (A.49), which satisfies that $\epsilon B_{\text{unit}} \subset S$, we derive that

$$\sup_{\mu \in S} |M(\mu)| \leq 32\sqrt{H^3d^2K} \log(9K/\xi),$$

with probability at least $1 - \xi$. Hence we conclude the proof of Lemma 5.8. \qed

### A.9 Proof of Lemma A.1

**Proof.** Under Assumption 2.1 and by the definition of $\Lambda_h^k$ in (3.6), we have

$$(\mathcal{P}_h \mathcal{V}_{h+1}^k)(x, a) = \varphi_h^{k}(s, a) \top (\Lambda_h^k)^{-1} \left( \sum_{\tau=1}^{k-1} \varphi_h^{\tau}(s_h^{\tau}, a_h^{\tau}) \varphi_h^{\tau}(s_h^{\tau}, a_h^{\tau}) \top \theta_h + \lambda \cdot \theta_h \right)$$

$$= \varphi_h^{k}(s, a) \top (\Lambda_h^k)^{-1} \left( \sum_{\tau=1}^{k-1} \varphi_h^{\tau}(s_h^{\tau}, a_h^{\tau}) \cdot (\mathcal{P}_h \mathcal{V}_{h+1}^{\tau})(s_h^{\tau}, a_h^{\tau}) + \lambda \cdot \theta_h \right). \quad (A.50)$$

Note that $\mathcal{P}_h \mathcal{V}_{h+1}^k(s, a) = \varphi(s, a) \top \theta_h^k$ by the closed form of $\theta_h^k$ in (B.1), we obtain that

$$\varphi_h^{k}(s, a) \top \theta_h^k - (\mathcal{P}_h \mathcal{V}_{h+1}^k)(s, a)$$

$$= \varphi_h^{k}(s, a) \top (\Lambda_h^k)^{-1} \left( \sum_{\tau=1}^{k-1} \varphi_h^{\tau}(s_h^{\tau}, a_h^{\tau}) \cdot (\mathcal{P}_h \mathcal{V}_{h+1}^{\tau})(s_h^{\tau}, a_h^{\tau}) \right)$$

$$- \lambda \cdot \varphi_h^{k}(s, a) \top (\Lambda_h^k)^{-1} \theta_h, \quad (i)$$

$$\quad (ii)$$

for any $(s, a) \in \mathcal{S} \times \mathcal{A}$. To upper bound the norm of term (i) in (A.51), we introduce the following lemma.

**Lemma A.5.** Let $\lambda = 1$ in the construction of estimated kernels (3.6). It holds with probability at least $1 - \delta/4$ that

$$\left\| \sum_{\tau=1}^{k-1} \varphi_h^{\tau}(s_h^{\tau}, a_h^{\tau}) \cdot \left( \mathcal{P}_h \mathcal{V}_{h+1}^{\tau}(s_h^{\tau}, a_h^{\tau}) - (\mathcal{P}_h \mathcal{V}_{h+1}^k)(s_h^{\tau}, a_h^{\tau}) \right) \right\|_{(\Lambda_h^k)^{-1}} \leq C \sqrt{H^2d^2 \log(HdK/\delta)}$$

for any $(k, h) \in [K] \times [H]$, where $C > 0$ is an absolute constant and $\delta \in (0, 1)$.

**Proof.** See Appendix A.10 for a detailed proof. \qed
By applying Cauchy-Schwarz inequality and Lemma A.5 on term (i) in (A.51), we have

\[
|\langle i \rangle| \leq \|\varphi(s, a)\|_{(A_k)^{-1}} \cdot \left\| \sum_{k=1}^{t-1} \varphi_k^\tau(s_k^\tau, a_k^\tau) \cdot (\hat{V}_{k+1}^\tau(s_{k+1}^\tau) - \left( P_{h_{k+1}} \hat{V}_{k+1}^\tau \right) (s_{k+1}^\tau, a_{k+1}^\tau)) \right\|_{(A_k)^{-1}}
\]

(A.52)

\[
\leq H \sqrt{d} \cdot C \sqrt{d \log(HdK/\xi)} \cdot \|\varphi(s, a)\|_{(A_k)^{-1}}
\]

with probability at least \(1 - \xi/4\).

For term (ii) in (A.51), by setting \(\lambda = 1\), we obtain that

\[
|\langle ii \rangle| \leq \|\varphi(s, a)\|_{(A_k)^{-1}} \cdot \|\theta_h\|_{(A_k)^{-1}}
\]

(A.53)

where the last inequality follows from the fact that \(\|\theta_h\|^2_{(A_k)^{-1}} \leq 1\) and \(\|\theta_h\| \leq \sqrt{d}\) for any \(h \in [H]\).

Combining (A.50), (A.51), (A.52), and (A.53), it holds with probability at least \(1 - \xi/4\) that

\[
|P_h \hat{V}_{h+1}^k(s, a) - \hat{P}_h \hat{V}_{h+1}^k(s, a)| \leq C \sqrt{d \log(HdK/\xi)} \cdot \|\varphi(s, a)\|_{(A_k)^{-1}}
\]

(A.54)

\[
\leq H \sqrt{d} \cdot \|\varphi(s, a)\|_{(A_k)^{-1}} \leq \Gamma_h^k(s, a)
\]

for any \(h \in [H]\) and \((s, a) \in S \times A\). Here \(\kappa = C \sqrt{d \log(HdK/\xi)}\) is the scaling parameter in (3.8) with an absolute constant \(C > 0\). Then we conclude the proof of Lemma A.1. \(\square\)

**A.10 Proof of Lemma A.5**

**Proof.** The proof of Lemma A.5 is adapted from that of Lemma D.1 in Cai et al. (2020).

**Lemma A.6** (Concentration of Self-Normalized Process). Let \(\{\tilde{F}_t\}_{t=0}^\infty\) be a filtration and \(\{\eta_t\}_{t=1}^\infty\) be an \(\mathbb{R}\) -valued stochastic process such that \(\eta_t\) is \(\tilde{F}_t\)-measurable for any \(t \geq 0\). Moreover, we assume that, for any \(t \geq 0\), conditioning on \(\tilde{F}_t\), \(\eta_t\) is a zero-mean and \(\sigma\)-sub-Gaussian random variable with the variance proxy \(\sigma^2 > 0\), that is,

\[
\mathbb{E}\left[ \exp(\lambda \eta_t) \mid \tilde{F}_t \right] \leq e^{\lambda^2 \sigma^2/2}
\]

for any \(\lambda \in \mathbb{R}\). Let \(\{X_t\}_{t=1}^\infty\) be an \(\mathbb{R}^d\) -valued stochastic process such that \(X_t\) is \(\tilde{F}_t\)-measurable for any \(t \geq 0\). Also, let \(Y \in \mathbb{R}^{d \times d}\) be a deterministic and positive-definite matrix. For any \(t \geq 0\), we define

\[
\bar{Y}_t = Y + \sum_{s=1}^t X_s X_s^\top, \quad S_t = \sum_{s=1}^t \eta_s \cdot X_s.
\]

For any \(\delta > 0\), it holds with probability at least \(1 - \delta\) that

\[
\|S_t\|_{(Y_t)^{-1}}^2 \leq 2\sigma^2 \cdot \log \left( \frac{\det(\bar{Y}_t)^{1/2} \det(Y)^{-1/2}}{\delta} \right)
\]

for any \(t \geq 0\).

**Proof.** See Theorem 1 of Abbasi-Yadkori et al. (2011) for a detailed proof. \(\square\)

By the definition of filtration \(\{\mathcal{F}_{k,h,m}\}_{(k,h,m) \in \mathbb{K} \times [H] \times \mathbb{Z}}\) in (A.2) and Markov property, we have

\[
\mathbb{E}[\hat{V}_{h+1}^\tau(s_{h+1}^\tau) \mid \mathcal{F}_{\tau,h,1}] = (P_h \hat{V}_{h+1}^\tau)(s_h^\tau, a_h^\tau).
\]

(A.54)
Conditioning on $\mathcal{F}_{\tau,h,1}$, the only randomness comes from $s_{k+1}^\tau$, while $\hat{V}_{k+1}^\tau$ is a deterministic function determined by $\hat{Q}_{k+1}^\tau$ and $\pi_{k+1}^\tau$, which are further determined by the historical data in $\mathcal{F}_{\tau,h,1}$. For simplicity of notations, we define $\eta_{\tau,h} = \hat{V}_{h+1}^\tau (s_{h+1}^\tau) - (\mathcal{P}_h \hat{V}_{h+1}^\tau)(s_{h}^\tau, a_{h}^\tau)$. By (A.54), conditioning on $\mathcal{F}_{\tau,h,1}$, $\eta_{\tau,h}$ is a zero-mean random variable. Moreover, as $\hat{V}_{h+1}^\tau \in [0, H \sqrt{d}]$, conditioning on $\mathcal{F}_{\tau,h,1}$, $\eta_{\tau,h}$ is an $(H \sqrt{d}/2)$-sub-Gaussian random variable defined in Lemma A.6. Meanwhile, $\eta_{\tau,h}$ is $\mathcal{F}_{k,h,2}$-measurable, since $\mathcal{F}_{\tau,h,1} \subseteq \mathcal{F}_{k,h,2}$ for any $\tau \in [k-1]$. Hence, for any fixed $h \in [H]$, by Lemma A.6, it holds with probability at least $1 - \delta/(4H)$ that

$$\left\| \sum_{\tau=1}^{k-1} \varphi_{\tau}^h (s_{\tau}^h, a_{\tau}^h) \cdot \left( \hat{V}_{h+1}^\tau (s_{h+1}^\tau) - (\mathcal{P}_h \hat{V}_{h+1}^\tau)(s_{h}^\tau, a_{h}^\tau) \right) \right\|^2_{(A_h^k)^{-1}} \leq \frac{H^2d}{2} \left( \frac{1}{2} \log(\det(A_h^k)) - \frac{1}{2} \log(\det(I)) + \log(4H/\delta) \right).$$

(A.55)

Recall that in (A.29) we derive that

$$\det(A_h^{K+1}) \leq \left( \det \left( (Hd^{3/2}R \cdot \text{Vol}(S) + 1) \cdot I \right) \right)^d.$$

(A.56)

By plugging (A.56) into (A.55) and a union bound argument, we obtain with probability at least $1 - \delta/2$ that

$$\left\| \sum_{\tau=1}^{k-1} \varphi_{\tau}^h (s_{\tau}^h, a_{\tau}^h) \cdot \left( \hat{V}_{h+1}^\tau (s_{h+1}^\tau) - (\mathcal{P}_h \hat{V}_{h+1}^\tau)(s_{h}^\tau, a_{h}^\tau) \right) \right\|^2_{(A_h^k)^{-1}} \leq \frac{H^2d}{2} \left( d \cdot \log((Hd^{3/2}R \cdot \text{Vol}(S) + 1) + \log(4H/\delta) \right),$$

which implies that

$$\left\| \sum_{\tau=1}^{k-1} \varphi_{\tau}^h (s_{\tau}^h, a_{\tau}^h) \cdot \left( \hat{V}_{h+1}^\tau (s_{h+1}^\tau) - (\mathcal{P}_h \hat{V}_{h+1}^\tau)(s_{h}^\tau, a_{h}^\tau) \right) \right\|^2_{(A_h^k)^{-1}} \leq C'' \sqrt{H^2d^2 \cdot \log(HdK/\delta)},$$

for any $(k,h) \in [K] \times [H]$ with probability at least $1 - \delta/4$. Here $C'' > 0$ is an absolute constant. By this, we conclude the proof of Lemma A.5. \hfill \Box

### B Proofs of Supporting Lemmas: PGAP

#### B.1 Proof of Lemma 3.2

**Proof.** We show that $\{ \Gamma_h \}_{h=1}^H$ constructed in Lemma 3.2 are the $\xi$-uncertainty qualifiers for the initially estimated transition kernels $\{ \mathcal{P}_h \}_{h=1}^H$ constructed in (3.15). By the definition of $\Lambda_h$ in (3.15), we have

$$\mathcal{P}_h(s' \mid s, a) = \phi(s, a, s')^\top \theta_h = \phi(s, a, s')^\top \Lambda_h^{-1} \left( \sum_{\tau=1}^{N_2} \int_S \phi(s_{\tau}^h, a_{\tau}^h, s') \mathcal{P}_h(s' \mid s_{\tau}^h, a_{\tau}^h) ds' + \lambda \cdot \theta_h \right).$$

(B.1)
By (B.1), we have
\[
\mathcal{P}_h(s' | s, a) - \tilde{\mathcal{P}}_h(s' | s, a) = \mathcal{P}_h(s' | s, a) - \phi(s, a, s')^\top \theta_h
\]
\[
= \phi(s, a, s')^\top \Lambda_h^{-1} \left( \sum_{\tau=1}^{N_h} \left( \int_{S} \phi(s_h^\tau, a_h^\tau, s') \mathcal{P}_h(s' | s_h^\tau, a_h^\tau) ds' - \phi(s_h^\tau, a_h^\tau, s_h^{\tau+1}) \right) \right) \tag{B.2}
\]
\[
+ \lambda \cdot \phi(s, a, s')^\top \Lambda_h^{-1} \theta_h.
\]

We introduce the following lemma to upper bound term (i) on the RHS of (B.2).

**Lemma B.1.** Let \( \lambda = 1 \) in the construction of \( \tilde{\mathcal{P}}_h \) and \( \Gamma_h \) in (3.15) and (3.16). By Assumption 2.1, the event that
\[
\left\| \sum_{\tau=1}^{N_h} \left( \int_{S} \phi(s_h^\tau, a_h^\tau, s') \mathcal{P}_h(s' | s_h^\tau, a_h^\tau) ds' - \phi(s_h^\tau, a_h^\tau, s_h^{\tau+1}) \right) \right\| \leq \sqrt{c_1 R^2 \cdot (d \log(HdN_2/\delta))}
\]
holds for all \((s, a, h) \in S \times A \times [H]\) with probability at least \(1 - \delta\). Here \(c_1\) is an absolute constant.

**Proof.** See proof in Appendix B.5.

For term (i) on the RHS of (B.2), by Cauchy-Schwarz inequality, it holds with probability at least \(1 - \xi/2\) that
\[
|\text{(i)}| \leq \|\phi(s, a, s')\|_{\Lambda_h^{-1}} \cdot \left\| \sum_{\tau=1}^{N_h} \left( \int_{S} \phi(s_h^\tau, a_h^\tau, s') \mathcal{P}_h(s' | s_h^\tau, a_h^\tau) ds' - \phi(s_h^\tau, a_h^\tau, s_h^{\tau+1}) \right) \right\|_{\Lambda_h^{-1}} \quad \tag{B.3}
\]
\[
\leq c_1 R \cdot \sqrt{d \log(dHN_2/\xi)} \cdot \|\phi(s, a, s')\|_{\Lambda_h^{-1}},
\]
where the last inequality follows from Lemma B.1.

For term (ii) in (B.2), setting \( \lambda = 1 \), we have
\[
|\text{(ii)}| \leq \|\phi(s, a, s')\|_{\Lambda_h^{-1}} \cdot \|\theta_h\|_{\Lambda_h^{-1}} \leq \sqrt{d} \cdot \|\phi(s, a, s')\|_{\Lambda_h^{-1}}, \tag{B.4}
\]
where the last inequality follows from the facts that \(\|\Lambda^{-1}\|_2 \leq 1\) and \(\|\theta_h\|_2 \leq \sqrt{d}\) for all \(h \in [H]\). Plugging (B.3) and (B.4) into (B.2), it holds with probability at least \(1 - \xi/2\) that
\[
|\mathcal{P}_h(s' | s, a) - \tilde{\mathcal{P}}_h(s' | s, a)| \leq cR \sqrt{d \log(HdN_2/\xi)} \cdot \|\phi(s, a, s')\|_{\Lambda_h^{-1}}
\]
\[
\leq \kappa \cdot \|\phi(s, a, s')\|_{\Lambda_h^{-1}}
\]
for any \(h \in [H]\) and \((s, a, s') \in S \times A \times S\). Here \(\kappa = cR \sqrt{d \log(dN_2)}\) is the scaling parameter with an absolute constant \(c > 0\). We conclude the proof of Lemma 3.2.

**B.2 Proof of Lemma 6.2**

**Proof.** We establish the lower and upper bounds for \(\iota_h^{k,\mu}\) as follows, respectively.

**Lower Bound.** First we prove by backward induction that \(\hat{V}_h^{k,\mu} \in [0, (H-h+1)\sqrt{d}]\) for any \(h \in [H]\). The base case \(h = H\) holds, since \(\hat{V}_{H+1}^{k,\mu} = 0\) and \(\tau_h^{\mu} \in [0, \sqrt{d}]\). We assume that \(\hat{V}_{h+1}^{k,\mu} \in [0, H-h]\). For the case for
which implies that \( \eta \in E \hat{\pi} \hat{\mu} \), where the last inequality follows from the facts that \( \hat{\pi} \hat{\mu} \). Then it holds that \( \hat{V}_h^{k,r^{m}} \in [0, H-h] \), since \( \hat{V}_h^{k,r^{m}}(s) = (\hat{Q}_h^{k,r^{m}}(s, \cdot), \pi_h^{k,r^{m}}(\cdot | s))_A \) for any \( s \in S \). By induction, it holds that \( \hat{V}_h^{k,r^{m}} \in [0, (H-h+1)\sqrt{d}] \) for any \( h \in [H] \).

For notational simplicity, we write \( \hat{Q}_h^{k,r^{m}}(s, a) = r_h^{m}(s, a) + \hat{P}_h \hat{V}_h^{k,r^{m}}(s, a) - \Gamma_h(s, a) \). From the policy evaluation stage in Algorithm 3, we have

\[
\hat{Q}_h^{k,r^{m}}(s, a) = \max \{ \hat{Q}_h^{k,r^{m}}(s, a), 0 \}.
\]

(B.5)

Meanwhile, by the definition of the \( \epsilon \)-uncertainty qualifiers in Definition 3.1, we have \( r_h^{m} + \hat{P}_h \hat{V}_h^{k,r^{m}} \geq \hat{Q}_h^{k,r^{m}} \). Moreover, by the fact that \( r_h^{m} \in [0, \sqrt{d}] \) and \( \hat{V}_h^{k,r^{m}} \in [0, (H-h)\sqrt{d}] \), we have \( r_h^{m} + \hat{P}_h \hat{V}_h^{k,r^{m}} \in [0, (H-h+1)\sqrt{d}] \). Thus, we derive that

\[
\hat{Q}_h^{k}(s, a) = \max \{ \hat{Q}_h^{k,r^{m}}(s, a), 0 \} \\
\leq \max \{ r_h^{m}(s, a) + \hat{P}_h \hat{V}_h^{k,r^{m}}(s, a), 0 \} \\
= r_h^{m}(s, a) + \hat{P}_h \hat{V}_h^{k,r^{m}}(s, a),
\]

which implies that \( r_h^{k,r^{m}} \geq 0 \).

**Upper Bound.** Since we condition on the event \( E \) defined in Definition 3.1, we have

\[
\hat{Q}_h^{k,r^{m}}(s, a) = r_h^{m}(s, a) + \hat{P}_h \hat{V}_h^{k,r^{m}}(s, a) - \Gamma_h(s, a) \\
\leq r_h^{m}(s, a) + \hat{P}_h \hat{V}_h^{k,r^{m}}(s, a) - H - h + 1,
\]

where the last inequality follows from the facts that \( \hat{V}_h^{k,r^{m}}(s, a) \leq (H-h)\sqrt{d} \) and \( r_h^{m}(s, a) \leq \sqrt{d} \). By (B.5) we have that \( \hat{Q}_h^{k,r^{m}}(s, a) \geq \hat{Q}_h^{k,r^{m}}(s, a) \). Thus, we obtain that

\[
r_h^{k,r^{m}}(s, a) = (r_h^{m} + \hat{P}_h \hat{V}_h^{k})(s, a) - \hat{Q}_h^{k,r^{m}}(s, a) \\
\leq r_h^{m}(s, a) + (\hat{P}_h - \hat{P}_h) \hat{V}_h^{k,r^{m}}(s, a) + \Gamma_h(s, a) \\
\leq 2 \Gamma_h(s, a),
\]

where the last inequality follows from the definition of \( E \). Then we complete the proof of Lemma 6.2. \( \square \)

**B.3 Proof of Lemma 6.3**

**Proof.** Recall that \( \hat{L}(\pi, \mu) = \hat{J}(\pi^E, \mu) - \hat{J}(\pi^k, \mu) \). By Assumption 2.1, we know that the function \( \hat{J}(\pi^E, \mu) = \frac{1}{N_1} \sum_{t=1}^{N_1} \sum_{h=1}^{H} \psi(s_h^E, \pi_h^E, \mu^E) \) is a linear combination of \( \{\mu_h\}_{h=1}^{H} \) and concave. Therefore, to prove that \( \hat{L}(\pi, \mu) \) is concave, it suffices to prove that \( \hat{J}(\pi^k, \mu) \) is convex for any \( \mu \) with \( \mu = \{\mu_h\}_{h=1}^{H} \in S \).

Recall that \( \hat{J}(\pi^k, \mu) = \hat{V}_1^{k,r^{m}}(x) \), where \( x \) is the fixed initial state and \( \hat{V}_1^{k,r^{m}} \) defined in (6.3) is solved by

\[
\hat{V}_h^{k,r^{m}}(\cdot) = 0 \\
\hat{Q}_h^{k,r^{m}}(\cdot, \cdot) = \max \{ \langle (r_h^{m} + \hat{P}_h \hat{V}_h^{k,r^{m}} - \Gamma_h)(\cdot, \cdot), 0 \rangle \} \\
\hat{V}_h^{k,r^{m}}(\cdot, \cdot) = \langle \hat{Q}_h^{k,r^{m}}(\cdot, \cdot), \tau_h^{k,r^{m}}(\cdot | \cdot) \rangle_A, \text{ for } h \in [H].
\]

Our proof relies on the following three basic properties of convex functions:

(i) If \( f(u) \) and \( g(u) \) are both convex function for \( u \), then \( f(u) + g(u) \) is also convex.
(ii) If \( f(u) \) and \( g(u) \) are both convex function for \( u \), then \( \max(f(u), g(u)) = (|f(u) + g(u)| + |f(u) - g(u)|)/2 \) is also convex.

(iii) If \( f(u, s) \) is a convex function for \( u \), then \( \mathbb{E}_{s \sim \rho} f(u, s) \) is also convex function for \( u \), where \( \rho \) is a distribution.

Now we are ready to prove that \( \hat{J}(\pi^k, \mu) \) is convex for \( \{\mu_h\}_{h=1}^H \). For the base case where \( h = 1 \), observing that \( \hat{J}(\pi^k, \mu) = \langle \hat{Q}_1^{\pi^k}(s_1, \cdot), \pi_1^k(\cdot | s_1) \rangle \), by property (ii) and (iii) and (B.6), it suffices to prove that \( r_{11}^\mu + \hat{P}_1 V_{2, r}^{\mu} - \Gamma_1 \) is convex for \( \mu_1 \). Note that \( \{\mu_h\}_{h=1}^H \) are separate reward parameters and \( r_{11}^\mu(\cdot, \cdot) \) is only determined by \( \mu_h \), it shows that \( \hat{P}_1 V_{2, r}^{\mu} - \Gamma_1 \) is a constant regardless of \( \mu_h \), which implies that \( \hat{P}_1 V_{2, r}^{\mu} - \Gamma_1 \) is convex for \( \mu_1 \). Meanwhile, since \( \mathcal{R} \) is linear to \( \psi \) as shown in (2.7), we know that \( r_{11}^\mu = \psi^\top \mu_h \) is also convex for \( \mu_h \). By property (i), we know that \( \hat{J}(\pi^k, \mu) = \hat{V}_{1, r}^{\mu}(s_1) \) is convex for \( \mu_1 \).

For the case when \( h = H' \), where \( 2 \leq H' \leq H \), similar to the analysis in the case when \( h = 1 \), we can prove that \( \hat{V}_{H'}^{\mu} (\cdot) \) is convex for \( \mu_{H'} \). Note that \( \{\Gamma_h\}_{h=1}^H \) defined in (3.1) is independent of \( \{\mu_h\}_{h=1}^H \), we know that \( r_{H'-1}^{\mu} + \hat{P}_{H'-1} V_{H'-1}^{\mu} - \Gamma_{H'-1} \) is convex for \( \mu_{H'} \). By property (ii) and (iii) and (B.6), we know that \( \hat{V}_{H'-1}^{\mu} \) is also convex for \( \mu_{H'} \). By repeating the analysis, we know that \( \hat{J}(\pi^k, \mu) \) is convex for \( \mu_{H'} \). Therefore, we conclude the proof of Lemma 6.3.

\section*{B.4 Proof of Lemma 6.4}

\textbf{Proof.} Since Lemma 6.3 shows that \( \hat{L}(\pi^k, \mu) \) is concave for \( \mu_h \) for any \( h \in [H] \), by the property of concave function, we have

\begin{equation}
\hat{L}(\pi^k, \mu) - \hat{L}(\pi^k, \mu^k) \leq \sum_{h=1}^H \left[ \nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)^\top (\mu_h - \mu_h^k) \right]. \tag{B.7}
\end{equation}

Recall that we apply projected gradient ascent method to update \( \{\mu_h^k\}_{h=1}^H \) in PGAP (Line 16 of Algorithm 3) as

\begin{equation}
\mu_h^{k+1} = \text{Proj}_S \left[ \mu_h^k + \eta \nabla_{\mu_h} \hat{L}(\pi^k, \mu^k) \right], \tag{B.8}
\end{equation}

we obtain that

\begin{equation}
[\mu_h^{k+1} - \mu_h^k - \eta \nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)^\top (\mu_h - \mu_h^{k+1})] \geq 0. \tag{B.9}
\end{equation}

Rearranging terms in (B.9), we have

\begin{equation}
\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)^\top (\mu_h - \mu_h^{k+1}) \leq - \frac{1}{2\eta} (\langle \mu_h^{k+1} - \mu_h^k \rangle^\top (\mu_h - \mu_h^{k+1})) = \frac{1}{2\eta} (\|\mu_h^k - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h^{k+1}\|_2^2). \tag{B.10}
\end{equation}

By adding the term \( \nabla_{\mu_h} \hat{L}(\pi^{k+1}, \mu^k)^\top (\mu_h^{k+1} - \mu_h^k) \) on both sides of (B.10), we obtain that

\begin{equation}
\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)^\top (\mu_h - \mu_h^k) = \frac{1}{2\eta} (\|\mu_h^k - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h\|_2^2 - \|\mu_h^{k+1} - \mu_h^{k+1}\|_2^2) + \nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)^\top (\mu_h^{k+1} - \mu_h^k). \tag{B.11}
\end{equation}

Note that \( \eta \) is positive and by applying Cauchy-Schwarz inequality on the second term of the right-hand side of (B.11), we derive that

\begin{equation}
\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)^\top (\mu_h^{k+1} - \mu_h^k) \leq \|\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)\|_2 \|\mu_h^{k+1} - \mu_h^k\|_2. \tag{B.12}
\end{equation}
From the reward update process in (B.8), we observe that
\[ ||\mu_h^{k+1} - \mu_h^k||_2 \leq ||\nabla_{\mu_h} \hat{L}(\pi^k, \mu^k)||_2. \]  \hspace{1cm} (B.13)

By plugging (B.11), (B.12), and (B.13) into (B.7), we have
\[ \sum_{k=1}^{K} \left[ \hat{L}(\pi^k, \mu) - \tilde{L}(\pi^k, \mu^k) \right] \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \left[ \frac{1}{2\eta} ||\mu_h^{k+1} - \mu_h^k||_2^2 + \frac{1}{2\eta} ||\mu_h^{k+1} - \mu_h^k||_2^2 \right. \\
- \left. \frac{1}{2\eta} ||\mu_h^{k+1} - \mu_h^k||_2^2 + \eta ||\nabla_{\mu_h} \tilde{L}(\pi^k, \mu^k)||_2^2 \right], \]

which concludes the proof of Lemma 6.4. \hfill \square

**B.5 Proof of Lemma B.1**

Proof. Before we prove Lemma B.1, we introduce the following lemma to generalize the concentration of self-normalized vector-valued process in Abbasi-Yadkori et al. (2011) to function-valued process.

**Lemma B.2.** Let \( \Omega \) be a probability space and \( \{\eta_t\}_{t=1}^{\infty} \) be a function-valued stochastic process with a filtration \( \{\mathcal{M}_t\}_{t=0}^{\infty} \), i.e. \( \eta_t : \mathcal{S} \times \Omega \to \mathbb{R} \). We assume that \( \eta_t | \mathcal{G}_{t-1} \) is zero-mean and \( \sigma \)-sub-Gaussian, that is,
\[ \mathbb{E}[\eta_t(s) | \mathcal{G}_{t-1}] = 0, \quad \log \left( \mathbb{E}[\exp(\int_S g(s)\eta_t(s)ds) | \mathcal{G}_{t-1}] \right) \leq \frac{1}{2} ||g||_2^2 \cdot \sigma^2, \]
for any \( s \in \mathcal{S} \) and function \( g : \mathcal{S} \to \mathbb{R} \). Let \( \{X_t\}_{t=0}^{\infty} \) be an vector-function-valued stochastic process where \( X_t : \mathcal{S} \times \Omega \to \mathbb{R}^d, X_t \in \mathcal{M}_{t-1} \). We also assume that \( ||\lambda^\top X_t||_{2,\mathcal{S}} \leq R \cdot ||\lambda^\top X_t||_{2,\mathcal{S}} \) a.s. for all \( \lambda \in \mathbb{R}^d \). Let \( V \in \mathbb{R}^{d \times d} \) be a positive definite matrix and \( \bar{V}_t = \sum_{\tau=1}^{t} \int_{\mathcal{S}} X_\tau(s) X_\tau(s)^\top ds \). We also define
\[ S_t = \sum_{\tau=1}^{t} \int_{\mathcal{S}} X_\tau(s) \eta_\tau(s)ds. \]

Then for any \( \delta > 0 \) and \( t > 0 \), it holds with probability at least \( 1 - \delta \) that
\[ ||S_t||_{\bar{V}_t^{-1}}^2 \leq 2(\sigma R)^2 \cdot \log \left( \frac{\det(\bar{V}_t)^{1/2}}{\delta \det(V)^{1/2}} \right). \]
Proof. See Appendix B.5.1 for a detailed proof. \hfill \square

We consider the filtration \( \{\mathcal{F}_{h,\tau}\}_{h \in [H], \tau \in [N_2]} \) defined in §2.2. For any function \( f : \mathcal{S} \to \mathbb{R} \), by Holder inequality, it holds that
\[ \left| \int_{\mathcal{S}} f(s') \left( \mathcal{P}_h(s' | s_h^\tau, a_h^\tau) - \delta_{s_h^{\tau+1}}(s') \right) ds' \right| \leq 2 ||f||_\infty. \]  \hspace{1cm} (B.14)

By the property of Dirac function, we have
\[ \mathbb{E}[\delta_{s_h^{\tau+1}}(s') | \mathcal{F}_{h,\tau}] = \mathcal{P}_h(s | s_h^\tau, a_h^\tau). \]  \hspace{1cm} (B.15)

Combining (B.14) and (B.15), we verify the condition of Lemma B.2 as follows,
\[ \log \left( \mathbb{E}[\exp(\int_{\mathcal{S}} f(s') \left( \mathcal{P}_h(s' | s_h^\tau, a_h^\tau) - \delta_{s_h^{\tau+1}}(s') \right) ds') | \mathcal{F}_{h,\tau}] \right) \leq 2 ||f||_\infty^2. \]
Note that \( \phi(s^*_h, a^*_h, s') \) is \( \mathcal{F}_{h, r} \)-measurable and \( \delta_{s+1} \) is \( \mathcal{F}_{h+1, r} \)-measurable, we apply Lemma B.2 with \( X_r = \phi(s^*_h, a^*_h, \cdot) \) and \( \eta_r = \mathcal{P}_h(\cdot | s^*_h, a^*_h) - \delta_{s+1} \), which implies that

\[
\left\| \sum_{r=1}^{N_2} \left[ \int_S \phi(s^*_h, a^*_h, s') \mathcal{P}_h(s' | s^*_h, a^*_h) ds' - \phi(s^*_h, a^*_h, s^*_h) \right] \right\|^2_{\Lambda_h^{-1}} \\
\leq 8R^2 \cdot \log \left( \frac{H/\delta}{\lambda \det(\Lambda_h)^{1/2} \det(\lambda I)^{-1/2}} \right),
\]

(B.16)

with probability at least \( 1 - \delta/H \).

We now upper bound the term \( \det(\Lambda_h) \). By the definition of \( \Lambda_h \) in (3.15), it holds for any \( y \in \mathbb{R}^d \) that

\[
y^T \Lambda_h y = \lambda \|y\|^2 + \sum_{r=1}^{N_2} \int_S |y^T \phi(s^*_h, a^*_h, s')|^2 ds' \leq (\lambda + dN_2) \|y\|^2,
\]

where the last inequality follows from Assumption 2.1. Hence we derive that \( \|\Lambda_h\|_2 \leq \lambda + dN_2 \), which implies that

\[
\det(\Lambda_h) \leq \|\Lambda_h\|_2^d \leq (\lambda + dN_2)^d.
\]

(B.17)

Setting \( \lambda = 1 \), combining (B.16) and (B.17), it holds with probability at least \( 1 - p/H \) that

\[
\left\| \sum_{r=1}^{N_2} \left( \int_S \phi(s^*_h, a^*_h, s') \mathcal{P}_h(s' | s^*_h, a^*_h) ds' - \phi(s^*_h, a^*_h, s^*_h) \right) \right\|^2_{\Lambda_h^{-1}} \\
\leq 8R^2 \cdot \left( \frac{1}{2} \cdot d \log(1 + dN_2) + \log(H/p) \right) \leq c_1 R^2 \cdot \left( d \log(HdN_2/p) \right).
\]

Here \( c_1 > 0 \) is an absolute constant. By the union bound for \( h \in [H] \), we know that (B.18) holds for all \( h \in [H] \) with probability at least \( 1 - p \). Thus, we complete the proof of Lemma B.1. \( \square \)

### B.5.1 Proof of Lemma B.2

**Proof.** The proof is a generalization of that in Abbasi-Yadkori et al. (2011). For notational simplicity, we denote by \( \langle f, g \rangle = \int_S f(s)g(s)ds \) the inner product of any functions \( f \) and \( g \). We use the same definitions and notations as Lemma B.1. First, we introduce the following lemmas.

**Lemma B.3.** Let \( \beta \in \mathbb{R}^d \) be a vector and

\[
M^\beta_t = \exp \left\{ \sum_{r=1}^t \left( \frac{\langle \beta^T X_r, \eta_r \rangle}{\sigma R} - \frac{\langle \beta^T X_r, \beta^T X_r \rangle}{2} \right) \right\}.
\]

Let \( T \) be a stopping time with respect to the filtration \( \{\mathcal{M}_t\}_{t=1}^\infty \). Then \( M^\beta_T \) is almost surely well-defined and \( \mathbb{E}[M^\beta_T] \leq 1 \).

**Proof.** We first show that \( \{M^\beta_t\}_{t=0}^\infty \) is a supermartingale. Let

\[
G^\beta_r = \exp \left( \frac{\langle \beta^T X_r, \eta_r \rangle}{\sigma R} - \frac{\|\beta^T X_r\|_2^2}{2} \right).
\]

By the conditional sub-Gaussian property of \( \eta_r \) and the fact that \( \|\beta^T X_t\|_\infty \leq R \cdot \|\beta^T X_t\|_2 \), we have

\[
\mathbb{E}[G^\beta_r | \mathcal{M}_{t-1}] \leq \exp \left( \frac{\|\beta^T X_r\|_\infty^2}{2} - \frac{\|\beta^T X_r\|_2^2}{2} \right) = 1.
\]

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Thus, we have $E[M_t^\beta | \mathcal{M}_{t-1}] = M_{t-1}^\beta \cdot E[G_t^\beta | \mathcal{M}_{t-1}] \leq M_{t-1}^\beta$, which implies that $\{M_t^\beta\}^\infty_{t=0}$ is a supermartingale and $E[M_t^\beta] \leq 1$. We then show that $M_t^\beta$ is well-defined, where $T$ is a stopping time. By the convergence theorem of nonnegative supermartingales, it holds that $M_T^\beta = \lim_{t \to \infty} M_t^\beta$. Thus, $M_T^\beta$ is well-defined whether $T < \infty$ or not. Finally, to show that $E[M_T^\beta] \leq 1$, we apply Fatou’s lemma and obtain that

$$E[M_T^\beta] = E[\lim_{t \to \infty} M_t^\beta] \leq \liminf_{t \to \infty} E[M_t^\beta] \leq 1.$$  

Thus, we conclude the proof of Lemma B.3.  

**Lemma B.4.** Let $T$ be a stopping time with respect to $\{\mathcal{M}_t\}^\infty_{t=0}$, then it holds with probability at least $1 - \delta$ that

$$\|S_T\|^2_{V_{t-1}} > 2(\sigma R)^2 \cdot \log\left(\frac{\det(V_T)^{1/2}}{\sqrt{\det(V)^{1/2}}}\right).$$

**Proof.** Without loss of generality, we assume that $\sigma \cdot R = 1$. We define

$$V_t = \sum_{r=1}^{t} \int X_r(s)X_r(s)^T.$$

Then, we have

$$M_t^\beta = \exp(\beta^T S_t - \|\beta\|^2_{V_t}/2).$$

By Lemma B.3, we have that $E[M_t^\beta] \leq 1$. Let $\Lambda$ be an $\mathbb{R}^d$-valued Gaussian random variable with covariance matrix $V^{-1}$. Moreover, we assume that $\Lambda$ is independent of $\mathcal{M}_t^\infty_{t=0}$. Let $M_t = E[M_t^\beta | \mathcal{M}_\infty]$, where $\mathcal{M}_\infty = \sigma(\cup_{r=0}^\infty \mathcal{M}_r)$. Notice that $E[M_T] = E[E[M_T^\beta | \Lambda]] \leq 1$. We denote by $p$ the density of $\Lambda$ and by $v(A) = \int \exp(-x^T Ax)dx = (2\pi)^d/det(A)$ for positive definite matrix $A \in \mathbb{R}^{d\times d}$. Then we obtain that

$$M_t = \int \exp(\beta^T S_t - \|\beta\|^2_{V_t}/2)p(\beta)\,d\beta$$

$$= \int \exp(-\|\beta - V_t^{-1}S_t\|^2_{V_t}/2 + \|S_t\|^2_{V_{t-1}})p(\beta)\,d\beta$$

$$= v(V)^{-1} \cdot \exp(\|S_t\|^2_{V_{t-1}}/2) \cdot \int \exp(-\|\beta - V_t^{-1}S_t\|^2_{V_t}/2 - \|\beta\|^2_{V_t}/2)\,d\beta. \tag{B.19}$$

Note that

$$\|\beta - V_t^{-1}S_t\|^2_{V_t} + \|\beta\|^2_{V_t}/2 = \|\beta - V_t^{-1}S_t\|^2_{V_t} + \|V_t^{-1}S_t\|^2_{V_t} - \|S_t\|^2_{V_{t-1}}$$

$$= \|\beta - V_t^{-1}S_t\|^2_{V_t} - \|S_t\|^2_{V_{t-1}} - \|S_t\|^2_{V_{t-1}}. \tag{B.20}$$

Plugging (B.20) into (B.19), we have that

$$M_t = v(V)^{-1} \cdot \exp(\|S_t\|^2_{V_{t-1}}/2) \cdot \int \exp(-\|\beta - V_t^{-1}S_t\|^2_{V_t}/2)\,d\beta$$

$$= \frac{v(V_t)}{v(V)} \cdot \frac{\exp(\|S_t\|^2_{V_{t-1}}/2)}{\sqrt{\det(V)/\det(V_t)} \cdot \exp(\|S_t\|^2_{V_{t-1}}/2)}.$$  

Thus, we have

$$\mathbb{P}\left\{\|S_T\|^2_{V_{t-1}} > 2 \log\left(\frac{\det(V_T)^{1/2}}{\sqrt{\det(V)^{1/2}}}\right)\right\} = \mathbb{P}(\delta M_T > 1) \leq E[\delta M_T] \leq \delta,$$

which completes the proof of Lemma B.4.  

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We now prove Lemma B.2 as follows. Define

\[ T = \inf \left\{ t \geq 0 : 2 \log \left( \frac{\det(\bar{V}_t)^{1/2}}{\delta \det(V)^{1/2}} \right) < \|S_t\|_{\bar{V}_t^{-1}}^2 \right\} \]

for a fixed \( \delta > 0 \). Then it holds that

\[
P \left\{ \exists t \geq 0, \|S_t\|_{\bar{V}_t^{-1}}^2 > 2 \log \left( \frac{\det(\bar{V}_t)^{1/2}}{\delta \det(V)^{1/2}} \right) \right\} = P(T < \infty)
\]

\[ = P \left\{ \|S_T\|_{\bar{V}_T^{-1}}^2 > 2 \log \left( \frac{\det(\bar{V}_T)^{1/2}}{\delta \det(V)^{1/2}} \right), T < \infty \right\} \]

\[ \leq P \left\{ \|S_T\|_{\bar{V}_T^{-1}}^2 > 2 \log \left( \frac{\det(\bar{V}_T)^{1/2}}{\delta \det(V)^{1/2}} \right) \right\} \leq \delta, \]

which completes the proof of Lemma B.2. \qed