Application of Statistical Physics to Politics

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Abstract

The concept and technics of real space renormalization group are applied to study majority rule voting in hierarchical structures. It is found that democratic voting can lead to totalitarianism by keeping in power a small minority. Conditions of this paradox are analyzed and singled out. Indeed majority rule produces critical thresholds to absolute power. Values of these thresholds can vary from 50% up to at least 77%. The associated underlying mechanism could provide an explanation for both former apparent eternity of communist leaderships as well as their sudden collapse.
1 Setting the limits

Modern theory of critical phenomena is based on the fundamental concepts of universality and irrelevant variables [1]. These two concepts mean that different physical systems, like for instance a magnet and a liquid, behave the same way when passing from one macroscopic state to another macroscopic state. Well known examples are the magnet becoming a para-magnet, the liquid, a gas, and even may be the creation of the universe from nothing with the big bang. Most of the microscopic properties of the physical compounds involved turn out to be irrelevant for describing the macroscopic change which in turn appears to be universal. While the number of physical systems undergoing phase transitions is infinite, all associated phase transitions can be described in terms of only a small finite number of universality classes. Only a few parameters, like space dimensionality, determine which universality class the system belongs to. The abstract and general nature of the statistical physics framework makes it tempting to extend such notions to non-physical systems, and in particular to social systems, for which, in many cases, there exists an interplay between microscopic properties and macroscopic realities.

Nevertheless the two fields of physical sciences and social sciences are rather different. However, the process of going in parallel from one atom and one human to respectively several atoms in bulk and a social group has much in common. More precisely, it is the hypothesis behind the present approach. It is worth stressing that we are not claiming our models will explain all aspects of human behavior. Like any modeling effort, it is appropriate only to some classes of phenomena in social science and not to others.

However such an approach should be carefully controlled. To just map a physical theory built for a physical reality, onto a social reality, could be at best a nice metaphor, but without predictability, and at worst a misleading and wrong social theory. Physics has been successful in describing macroscopic behavior using properties of the constituent microscopic elements. The task here, is to borrow from physics those techniques and concepts used to tackle the complexity of aggregations. The challenge is then to build a collective theory of social behavior along similar lines, but within the specific constraints of the psycho-social reality. The constant danger is for the theorist to stay in physics, using a social terminology and a physical formalism. The contribution from physics should thus be restricted to qualitative
guidelines for the mathematical modeling of complex social realities. Such a limitation does not make the program less ambitious.

2 Real space: from physics to politics

In this paper we present an application of statistical physics to political sciences [2]. We apply the concept and technics of real space renormalization group [1] to study the majority rule voting process within hierarchical structures. In particular we focus on the conditions for a given political party to get for sure, full power at the hierarchy top level.

We find that majority rule voting produces critical threshold to total power. Having an initial support above the critical threshold guarantees full power at the top. The value of the critical threshold to power is a function of the voting structure, namely the size of voting groups and the number of voting hierarchical levels. Using these results a new explanation is given to the sudden and abrupt fall of eastern former communist parties.

Here we apply the real space renormalization group scheme of collective phenomena in physics to a radically different reality, namely a social one. We emphasize on other technical aspects than the ones usually used in physics. While there it is an abstract and formal tool to study phase transitions, here we associate a political reality to each step of the renormalization group transformation. On this basis it is worth to stress we apply statistical physics to political sciences not as a qualitative metaphor but indeed as a guide to build a quantitative model to study the effect of majority rule voting on the democratic representation of groups within a hierarchical organization.

Like in physics we start from local cells constituted by a small number of degrees of freedom, here individuals. Similarly to Ising spins, to keep the analysis simple, people can choose only between two political tendencies A and B. Associated proportions of A and B support in the system, a political group, a firm, a network, a society, are supposed to be known. We denote them by $p_0$ for the overall A-support and $1 - p_0$ for the B-support. We are assuming each member does have an opinion.

Once formed, each cell elects a representative, either an A or a B using a majority rule. These elected people (the equivalent of the super spin rescaled to an Ising one in real space renormalization group) constitute the first hierarchical level of the organization called level-1. New cells are then formed at
level-1 from these elected people. They in turn elect new representatives to build level-2. This process is repeated again and again.

In physics the rescaled degree of freedom is fictitious while here it is a real person. We are not using a theoretical scheme to embody some complex features but instead we are building up a real organization where each voting step is real. Moreover at odd with physics the number of iterations in the renormalization process are finite and the focus is on the stable fixed points.

The following of the paper is as follows. In next section we present and study the case of 3-person cells. A critical threshold to power is singled out. It equals 50%. A minority is found to self-eliminate within a few voting levels. A making sense bias in the voting rule is then introduced in Section 3. A one vote bonus is added to the tendency already in power in cases of A-B equality. For 4-person cells, this bias shift the critical threshold to power from 50% to 77%. Size effects are analyzed in Section 4. Analytic formulas are then derived. In particular, given an initial support $p_0$ for the A, the number of voting levels necessary to their self-elimination is obtained. Section 5 puts the results in a more practical perspective. Last section contains a discussion about both former apparent eternity of communist leaderships as well as their sudden collapse. Some perspectives are outlined.

3 The simplest fair case

We start from a population distributed among two tendencies A and B with respectively $p_0$ and $1 - p_0$ proportions within the the system. It could be either a political group, a firm, or a whole society. At this stage each member does have an opinion. From now on we will use the political language.

Cells are constituted by randomly aggregating group of 3 persons. It could be by home localization or working place. Each cell then elects a representative using majority rule. To have an A elected requires either 3 A or 2 A in the cell. Otherwise it is a b who is elected. these elected persons constitute the first level of the hierarchy denoted level-1. The same process of cell forming can be repeated within level-1 from the elected persons. Making the cell to vote produces an additional level, namely level-2. We can go on the same way from a level-n to a level-(n+1). The probability to have an A elected at level (n+1) is then,

$$p_{n+1} \equiv P_3(p_n) = p_n^3 + 3p_n^2(1 - p_n) ,$$

(1)
where \( p_n \) is the proportion of elected A persons at level-n.

We call \( P_3(p_n) \) the voting function. It has 3 fixed points \( p_d = 0, p_{c,3} = \frac{1}{2} \)
and \( p_t = 1 \). First one corresponds to the disappearance of the A. Last one \( p_t \) represents the totalitarian situation where only A are present. Both are stable. At contrast \( p_c \) is unstable. It determines the threshold to full power. Starting from \( p_0 < \frac{1}{2} \) repeating voting leads towards 0 while the flow is in direction of 1 for \( p_0 > \frac{1}{2} \).

Therefore majority rule voting produces the self-elimination of any proportion of the A-tendency as long as \( p_0 < \frac{1}{2} \). However the democratic self-elimination to be completed requires a sufficient number of voting levels.

At this stage the instrumental question is to determine the number of levels required to ensure full leadership to the initial larger tendency. The analysis will turn relevant to reality only if this level numbers is small. Most organizations has only a few level, and always less than 10.

For instance starting from \( p_0 = 0.43 \) we get successively \( p_1 = 0.40, p_2 = 0.35, p_3 = 0.28, p_4 = 0.20, p_5 = 0.10, p_6 = 0.03 \) down to \( p_7 = 0.00 \). Therefore 7 levels are sufficient to self-eliminate 43% of the population.

Tough the aggregating voting process eliminate a tendency it stays democratic since it is the leading tendency (more than 50%) which after all gets the total leadership of the organization. It is worth to notice the symmetry with respect to A nd B tendencies.

### 4 The simplest killing case

In real world things are not as fair as above and often it turns out very hard, if not impossible, to change an organization leadership. We will now illustrate this situation.

Considering yet the simplest case we constitute groups of 4 persons instead of 3. The salient new feature is the 2A-2B configuration for which there exists no clear majority. In most social situations it is well admitted that to change a policy required a clear cut majority. In case of no decision, things will stay as they are. It is a bias in favor of the already existing. Often this bias is achieved giving for instance, one additional vote to the committee president.

Along this line, the voting function becomes non symmetrical. Assuming
the B were in power, for an A to be elected at level \( n + 1 \) we have,

\[
p_{n+1} = P_4(p_n) = p_n^4 + 4p_n^3(1 - p_n),
\]

where \( p_n \) is the proportion of elected A persons at level-\( n \). At contrast for a B it is,

\[
1 - P_4(p_n) = p_n^4 + 4p_n^3(1 - p_n) + 2p_n^2(1 - p_n)^2,
\]

where last term embodies the bias in favor of B. From Eqs (2) and (3) the stable fixed points are still 0 and 1. However the unstable one is drastically shifted to,

\[
p_{c,4} = \frac{1 + \sqrt{13}}{6},
\]

which makes the threshold to power to the A about 77%. Moreover the process of self-elimination is accelerated. For instance from \( p_0 = 0.69 \) we have the series \( p_1 = 0.63, p_2 = 0.53, p_3 = 0.36, p_4 = 0.14, p_5 = 0.01, \) and \( p_6 = 0.00 \). It shows that using an a priori reasonable bias in favor of the B turns a majority rule democratic voting to a totalitarian outcome. Indeed to get to power the A must pass over 77% of support which almost out of a possibility. Above series shows how 63% of a population disappears from the leadership within only 5 voting levels.

5 Larger voting groups

Most real organizations work with voting cells larger than 3 or 4. To account for this size variable we generalize above scheme to cells with \( r \) voting persons. We then have to determine the voting function \( p_{n+1} = P_r(p_n) \). Using a majority rule it becomes,

\[
P_r(p_n) = \sum_{l=r}^{m} \frac{r!}{l!(r-l)!} p_n^l (1 + p_n)^{r-l},
\]

where \( m = \frac{r+1}{2} \) for odd \( r \) and \( m = \frac{r+1}{2} \) for even \( r \) which thus accounts for the B-bias.

The two stable fixed points \( p_d = 0 \) and \( p_t = 1 \) are preserved with enlarging the group size. However while the unstable one stays \( p_{c,r} = \frac{1}{2} \) for odd sizes, it varies with \( r \) for even sizes. It starts at \( p_{c,4} = \frac{1 + \sqrt{13}}{6} \) with the limit \( p_{c,r} \rightarrow \frac{1}{2} \) when \( r \rightarrow \infty \).
We can then calculate analytically the critical number of levels \( n_c \) at which \( p_{nc} = \epsilon \) with \( \epsilon \) being a very small number. It determines the level of confidence of the prediction to have no A elected. One way to evaluate \( n_c \) is to expand the voting function \( p_n = P_r(n-1) \) around the unstable fixed point \( p_{c,r} \),

\[
p_n \approx p_{c,r} + (p_{n-1} - p_{c,r})\lambda_r ,
\]

where \( \lambda_r \equiv \frac{dP_r(p_n)}{dp_n} \bigg|_{p_{c,r}} \) with \( P_r(p_c) = p_{c,r} \). Rewriting last equation as,

\[
p_n - p_{c,r} \approx (p_{n-1} - p_{c,r})\lambda_r ,
\]

we can then iterate the process to get,

\[
p_n - p_{c,r} \approx (p_0 - p_{c,r})\lambda_r^n ,
\]

from which we get the critical number of levels \( n_c \) at which \( p_n = \epsilon \). Taking the \( \ln \) on both side of Eq. (8) gives,

\[
n \approx -\frac{\ln(p_c - p_0)}{\ln \lambda_r} + n_0 ,
\]

where \( n_0 \equiv \frac{\ln(p_{c,r} - \epsilon)}{\ln \lambda_r} \) is only valid not too far from \( p_{c,r} \). However it turns out to be a rather good estimate even down to the sable fixed point 0 by making it equal to 1 while taking the integer part of Eq. (9). For a more accurate calculation of \( n_0 \) see [3].

6 The magic formula

Most organizations don’t change their structure at every election or decision event. They are set once and then don’t change any longer. The number of hierarchical levels is thus fixed and constant. Therefore to make our analysis useful we have to invert the question on “how many levels are needed to eliminate a tendency” onto “given \( n \) levels what is the necessary overall support to get full power”.

It is worth to keep in mind that situations for respectively A and B tendencies are not always symetric. Here we stress the dynamics of voting with respect to the A. To implement this operative question, we invert Eq.(7) to,

\[
p_0 = p_{c,r} + (p_n - p_{c,r})\lambda_r^{-n} .
\]
indeed two critical thresholds now appears. First one, the disappearance threshold $p_{\text{d},r}^n$ which gives the value of support under which the A disappears for sure at the top level leadership. It is given by Eq. (10) putting $p_n = 0$, 

$$
p_{\text{d},r}^n = p_{\text{c},r}(1 - \lambda r^{-n}).
$$

In parallel putting $p_n = 1$ give the second threshold $p_{\text{f},r}^n$ above which the A get full and total power. Using Eq.(11), we get,

$$
p_{\text{f},r}^n = p_{\text{d},r}^n + \lambda r^{-n}.
$$

which shows the appearance of a new region for $p_{\text{d},r}^n < p_0 < p_{\text{f},r}^n$. In that region the A neither disappear totally nor get full power ($p_n$ is neither 0 nor 1). It is therefore a coexistence region where some democracy is prevailing since results of the election process are only probabilistic. No tendency is sure of winning making alternating leadership a reality. However as seen from Eq.(12), this democratic region shrinks as a power law $\lambda r^{-n}$ of the number $n$ of hierarchical levels. Having a small number of levels thus puts higher the threshold to a total reversal of power but simultaneously lowers the threshold for non existence.

Again, above formulas are approximates since we have neglected corrections in the vicinity of the stable fixed points. However they give the right qualitative behavior. Actually $p_{\text{d},r}^n$ fits to $n + 1$ and $p_{\text{f},r}^n$ to $n + 2$. For more accurate formulas see [3].

To get a practical feeling of what Eqs. (11) and (12) means, let us illustrate them for the case $r = 4$ where we have $\lambda = 1.64$ and $p_{\text{c},4} = \frac{1 + \sqrt{13}}{6}$. Considering 3, 4, 5, 6 and 7 level organizations, $p_{\text{d},r}^n$ is equals to respectively 0.59, 0.66, 0.70, 0.73 and 0.74. In parallel $p_{\text{f},r}^n$ equals 0.82, 0.80, 0.79, 0.78 and 0.78. These series emphasizes drastically the totalitarian character of the voting process.

## 7 Some prospective

Up to now we have treated very simple cases to single out main trends produced by democratic voting aggregating over several levels. In particular we have singled out the existence of critical thresholds to full power. Moreover these thresholds are not necessarily symmetric for both tendencies in competition. In the biased 4-cell case it is around 0.77% for the A.
Such asymmetries are indeed always present in most realistic situations in which more than two groups are competing. Let us consider for instance the case of three competing groups A, B and C. Assuming a 3-cell case, now the ABC configuration is unsolved using majority rule as it was for the precedent two tendencies AABB 4-cell configuration. For the AB case we made the bias in favor of the group already in power, like giving an additional vote to the comitee president.

For multi-group competitions typically the bias results from parties agreement. Usually the two largest parties, say A and B are hostile while the smallest one C would compromise with either one of them. Then the ABC configuration gives a C elected. In such a case, we need 2A or 2B to elect respectively an A or a B. Otherwise a C is elected. Therefore the elective function for A and B are the same as for the AB 3-cell model. It means that the critical threshold to full power to A and B is 50%. In otherwords for initial A and B supports less than 50% the C get full power. The required number of levels is obtained from above formulas.

It is possible to generalize to as many groups as wanted. The analysis becomes more heavy but the mean features of voting flows towards fixed point are preserved.

8 Conclusion

To conclude we comment on some possible new explanation to the recent generalized auto-collapse of eastern communist parties. Up to this historical and drastic event, communist parties seemed eternal with the same leadership ever. Once they collapsed most explanations were related to both an opportunistic change within the various organizations together with the end of the soviet army threat.

May be the explanation is different and related to our hierachical model. Indeed communist organizations are based on the structutral concept of democratic centralism which nothing else than a tree-like hierachy with a rather high critical threshold to power. Suppose it was of order of 80% like in our 4-cell case. We could than consider that the internal opposition to the orthodox leadership did grew a lot and massively over several decades to eventually reach and pass the critical threshold with the associated sudden outrise of the internal opposition. Therefore the at once collapse of eatern
communist parties would have been the result of a very long and solid phenomena. Such an explanation does not oppose to additional constraints but emphasize on the internal mechanism within these organizations.

At this stage it is of importance to stress that modeling social and political phenomena is not stating some absolute truth but instead to single out some basic trend within very complexe situations.

References

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