The naturalness problem might be studied on the complex two dimensional plane with the technique of dimensional regularization(DREG). The Renormalization group equation(RGE) of the Higgs mass on the plane suggests the Higgs mass approaches zero at ultraviolet (UV) scale, the scale can be Planck scale when the top quark pole mass $M_t = 168$ GeV. The real issue of the naturalness problem in the sense of Wilsonian renormalization group method is not about quadratic divergences but the rescaling effect. The Higgs mass can be considered to be one composed mass. All terms in the lagrangian in this scenario are marginal terms and no relevant terms are left, thus no rescaling effect to cause the naturalness problem anymore. RGE of the vacuum expectation value (VEV) in the Landau gauge up to two-loop order is studied. Scale-dependent behavior of the composed Higgs mass shows that we can have one tiny Higgs mass at high energy scale, even around the Planck scale, when $M_t \leq 170.7$ GeV.

PACS numbers: 11.10.Hi,12.15.Lk,14.80.Bn

I. INTRODUCTION

In the quantum field theory, the naturalness problem assumes new physics around TeV scale. While, no new physics sign has been observed at LHC and no indications have been found which indicates the SM could be a low energy effective theory below the Planck scale sofar. Meanwhile, recent studies on the stability and (meta)stability, with the current Higgs-like mass observed at LHC, suggest that the SM may apply to the Planck scale [1]. Confront this situation, the naturalness problem need to be revisited. To investigate the naturalness problem, two important issues are the way to understand the Higgs mass term and quadratic divergences.

Firstly, we consider the case that the Higgs mass is one gauge invariant quantity. To reveal quadratic divergences, one straightforward way is to use cut-off method, when we calculate quantum corrections to the mass term. The cut-off introduced can be the cut-off in a UV complete theory or the energy scale at which new physics enter into the low energy physics. Quadratic divergences can also been manifested with the DREG method. At one-loop order, a suitable criterion to consider quadratic divergences properly within the framework of DREG, is the occurrence of poles on the complex two dimensional plane [2]. We investigate quadratic divergences calculation with cut-off method, and the way to reveal quadratic divergences with DREG on the complex d dimensional plane at any loop orders. The connections of these two methods are also explored. The quadratic divergences of one-loop level dominates quadratic divergences of the quantum corrections to the Higgs mass term. And with quadratic divergences of one-loop order, the naturalness problem was studied in ’t Hooft-Feynman gauge completely three decades before [2]. In this work, we explore if the problem is gauge dependent through proceeding calculations in $R_\xi$ gauge. The naturalness problem prevents us from calculating the high energy behaviors of the SM up to UV scale. We find that with physics relating to quadratic divergences being attributed to the complex two dimensional plane, one can safely study renormalization group (RG) behaviors of physical parameters with DREG and MS(or $\overline{\text{MS}}$) scheme. After we develop the method to express physical parameters on the complex two dimensional plane through parameters of the $d = 4$ case, we derive the RGE of the Higgs mass on the plane, wherein the Veltman condition can be expressed by running parameters consistently in our renormalization procedure. With energy scale increasing, the Veltman condition is found to be fulfilled at UV scale, and the scale at which the Higgs mass on the complex two dimensional plane approaches zero is slightly higher. Ward identities do hold for all the complex dimensions within DREG [4, 5], thus our study in this part is manifestly gauge invariant.

In the Wilsonian renormalization group method, quadratic divergences is not the real issue of the naturalness problem and can be absorbed to fix the position of critical surface [6]. The role played by the surface is very resemble with that of complex d dimensional plane. For the possibility that the Higgs mass is considered not to be an gauge invariant but one composed quantity, the rescaling effect which causes the naturalness problem in the sense

*Electronic address: lgb@mail.nankai.edu.cn
of Wilsonian renormalization group method is absent, and then we need not to worry about the naturalness problem. The RGEs of VEV and the Higgs term in this situation will be studied.

This paper is organized as follows. In section II, we study the relation between complex $d$ dimensional plane and quadratic divergences, then we give one method to express the physics on the complex two dimensional plane with parameters of $d = 4$ case in DREG. The naturalness problem in the context of the SM is derived in $R_{\xi}$ gauge. We proceed with $DREG$ to handle divergences and the RGE of the Higgs mass on the complex two dimensional plane at one-loop level in the sense of $[2]$ is derived, and been expressed by parameters of $d = 4$ case in section III. In order to make our results general, all calculations in this work are proceeded in $R_{\xi}$ gauge. With the help of the Higgs mechanism, the theory with only marginal terms left is considered, in which the mass term is replaced by the composed mass. And after we generalize this mind to the SM, the RGE of VEV is derived. Then, we take one numerical analysis of the RGE of the composed Higgs mass together with beta functions up to two-loop order. These construct section V. Our conclusions are given in section VI.

II. COMPLEX $d$ DIMENSIONAL PLANE AND QUADRATIC DIVERGENCES

The concept of analytic continuation in the number of dimensions is the basis of the DREG method $[5]$, where the dimension 4 is generalized to be the complex $d$, and the ultraviolet infinities in the original four dimensional momentum integrals manifest themselves as poles on the complex $d$ dimensional plane. As for the mathematic meanings of analytic continuation and complex $d$ dimensional plane, one can refer to $[13]$. To find physical meanings of the complex $d$ dimensional plane, especially the complex two dimensional plane, we investigate the quantum corrections to the mass term with cut-off and DREG method together.

A. Quantum corrections to mass term: quadratic divergences and complex $d$ dimensional plane

A direct relation between power counting and the location of the poles on the complex $d$ plane can be found with the analytic continuation method. Quadratic divergences and logarithmic divergences correspond to poles at $d = 4 - 2/L$ and $d = 4$, with $L$ denotes the number of loops $[1]$. In following paragraphs, we illustrate this relation by investigating one- and two-loop order situations.

At first, we study the one-loop situation which is useful to derive the formula to express naturalness problem in section III C. In the ordinary perturbation calculations of $\lambda \phi^4$ theory, quantum corrections to mass term can be figured out from momentum integral calculations with cut-off method. When the mass term is not negligible but much less than the fundamental scale $\Lambda$ and the lower energy scale $\mu$, i.e., $\Lambda \gg \mu \gg m$, the momentum integral is divergent as

$$\lambda \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \rightarrow \begin{cases} \frac{-i\lambda}{16\pi^2} \log \frac{\Lambda^2}{\mu^2}, & d = 2, \\ \frac{-i\lambda}{16\pi^2} \left(\Lambda^2 - \mu^2 - m^2 \log \frac{\Lambda^2}{\mu^2}\right), & d = 4. \end{cases}$$

With which and the renormalization prescription as will be explored in section III, the relationship between bare and renormalized mass in the four dimensional $\lambda \phi^4$ theory can be written as

$$m^2 - \frac{\Lambda^2}{16\pi^2} (\Lambda^2 - \mu^2 - m^2 \log \frac{\Lambda^2}{\mu^2}) = Z_{\phi} m_0^2.$$  \hspace{1cm} (1)

The quantum contribution $\Lambda^2 - \mu^2$ constructs the source of the naturalness problem in our four dimensional field theory, as will also be shown in Eq. (23).

Poles of momentum integrals, which are involved in self-energy computations and calculated with DREG in $d$ dimensional spacetime, are shown below and listed in Appendix A

$$\lambda \mu^\varepsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \rightarrow \begin{cases} \frac{i\mu^2}{4\pi^2(d-2)}, & d \rightarrow 2, \\ \frac{-i\mu^2}{16\pi^2(d-4)}, & d \rightarrow 4. \end{cases}$$

1 This $\mu$ should be different from the scale parameter $\mu$ introduced in MS scheme when we derive RGEs of physical parameters.
Where the arbitrary scale parameter $\mu$ introduced is to give the running scale of RGEs in MS (or \(\overline{\text{MS}}\)) scheme. $\mu^\varepsilon$ with $\varepsilon = 4 - d$ compensates the dimension of $\lambda$ in $d$ dimensional Lagrangian. And from the above equation, one can achieve poles at $d = 2$ or $d = 4$ when the dimension $d$ continues to 4 or compacts to 2. And these two kinds of poles could not emerge simultaneously.

Compare Eq. (2) with the Eq. (1), the correspondences
\[
\frac{1}{1-d/2} \rightarrow \frac{\Lambda^2}{4\pi}, \quad \frac{1}{2-d/2} \rightarrow \frac{\ln \Lambda^2}{\mu^2}
\]
can be obtained, the quadratic and logarithmic divergences correspond to poles at $d = 2$ and $d = 4$ on the complex $d$ dimensional plane. When the dimension $d$ continues to 4, logarithmic physics can be given. And quadratic divergence might be considered as the physics on the complex two dimensional plane, which will also be explored in the next subsection.

With DREG adopted as the regularization method, the quadratic divergences manifest as the pole at $d = 2$, i.e., the pole on the so-called complex two dimensional plane. The quadratic divergences part of quantum corrections to the Higgs mass at one-loop order can be calculated with this method in the context of the SM. With which, the formula to express the naturalness problem in 't Hooft-Feynman gauge is given in the paper [2]. And we derive the formula in $R_\xi$ gauge in this work.

Now we move onto the two-loop level situation. Momentum integrals, which give quadratic divergences part of quantum corrections of the mass term in $\lambda \phi^4$ theory and the SM, take the forms of
\[
\int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{p^2 q^2 (p^2 q^2 + \Lambda^2)^4}
\]
and
\[
\int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{p^2 q^2 (p + q)^2}
\]
in $d$ dimension. The Eq. (4) calculating with cut-off method in the case of $d = 4$ is divergent as $\Lambda^2 \log(\Lambda^2/\mu^2)$. When we use DREG, the mass terms need to be considered to be associated with momentums in the denominator of the formula to sidestep the Infrared (IR) divergences, and the result is proportional to $\Gamma(1 - d/2)\Gamma(2 - d/2)$. While these kind of momentum integral can be safely neglected when we consider quadratic divergences effects, since these contributions are canceled on the basis of the one-loop renormalization [11]. The Eq. (5) is what we should cares about, which diverges as $\Lambda^2$ (when the higher energy cut-off($\Lambda$) is much larger than the low energy cut-off scale($\mu$)) when we use cut-off method in the case of $d = 4$, and is proportional to $\Gamma(3 - d)\Gamma(2 - d/2)$ when using DREG method and including mass terms in the denominator of the formula. Thus one have poles at $d = 3, 4$ on the complex $d$ plane. The pole at $d = 3$ corresponds to quadratic divergences with the pole at $d = 4$ corresponding to logarithmic divergences.

After we get the relation between quadratic divergences and poles on the complex $d$ dimensional plane, we need to relate quadratic divergences with logarithmic divergences to reveal the physical meaning of the complex $d$ dimensional plane. The DREG may keeps the physics of logarithmic physics as $d \rightarrow 4$ which gives our traditional four dimensional low energy physics. With quadratic divergences effects($\Lambda^2$ relevant physics) being expressed on the complex $4 - 2/L$ dimensional plane, parameters applied on which might be expressed by the parameters of the $d = 4$ case.

With loop order increasing by one, the quadratic divergences contribution of quantum correction to mass term has one more factor $1/16\pi^2$ [12] to multiply$^2$, thus the one-loop order’s contribution dominates the quadratic divergences. Thus in the following paragraphs, we want to explore the relation between the parameters on the complex two dimensional plane and that of $d = 4$ case.

### B. Relations between parameters on the complex two dimensional plane and that of $d = 4$ case

At first, we consider how quadratic divergence shows up with power counting method. For any loop orders, the momentum integral in $d = 4 - 2/L$ dimension Euclidean spacetimes which gives quadratic divergences takes the form
\[
\int d^{4-2/L}k_E \frac{1}{(k_E^2 + m^2)^{2-1/L}}
\]

$^2$ The two-loop level case is $\frac{1}{16\pi^2} \ln \frac{\mu^2}{\Lambda^2}$ times of that of one-loop level case in the method adopt in [10].
except other terms being multiplied to which to give constant terms at UV limit. With the transformation

\[ k_E \to k_E e^{k_E^2 f(M)} \quad m \to m e^{k_E^2 f(M)} , \]

where \( f(M) \) take the responsibility to compensate the dimension of interaction couplings\(^3\) and \( M \) has mass dimension one, we get the momentum integral

\[ \int dk_E \frac{1}{k_E^2} \to \int dk_E \frac{1}{(k_E^2 + m^2)^{1/2}} , \]

for \( k_E \gg m \), which manifests quadratic divergences.

The reasonability of the above method to manifest quadratic divergences can be traced back to the analytic continuation property of the \( \Gamma(x) \) function, with complex \( x \). The function reveals the pole after the definition region of \( \Gamma(x) \) has been analytic generalized to the right of the pole on the complex \( x \) plane to make the function well defined.

For the \( d = 2 \) case, quantum corrections to the mass term at one-loop level calculated with cut-off method is

\[ \lambda \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 - m^2} = -i \frac{\lambda}{4\pi} \log \frac{\Lambda^2}{\mu^2} , \]

where the coupling \( \lambda \) has mass dimensions 2 based on dimensional analysis for the missing of \( \mu^2 \) to compensate the dimension of \( \lambda \) compared with Eq. (2). Firstly, let momentum and mass of the Eq. (9) in Euclidean spacetime take transformation as Eq. (10) with \( f(M) = 1/2M^2 \), then we get \( dk_E^2 \to e^{k_E^2 f(M)} (dk_E^2 + k_E^2 dk_E^2/M^2) \). Secondly, let the scalar quartic coupling transforms as \( \lambda \to \lambda M^2/(4\pi) \). After which, the quantum correction to the mass term in Euclidean spacetime of \( d = 2 \) case

\[ \lambda \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 - m^2} \to \frac{\lambda}{4\pi} \log \frac{\Lambda^2}{\mu^2} , \]

thus we arrive at a nontrivial result: the divergences of \( d = 4 \) case emerges from the \( d = 2 \) case. Changing back to Minkovski spacetimes, no matter \( \Lambda \gg \mu \gg m \) or \( \Lambda \gg \mu \gg m \) with \( M \) comparable to the low energy scale \( \mu \), we have

\[ \lambda \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 - m^2} \to -i \frac{\lambda}{4\pi} \left( \log \frac{\Lambda^2}{\mu^2} \right) . \]

Where the quadratic divergences part of the \( d = 4 \) case emerges from the \( d = 2 \) case, which supports to correspond the quadratic divergences to the pole on the complex two dimensional plane.

Physical implications and discussions: Considering the cut-off as the energy scale in the viewpoint of effective field theory, we achieved that physics in the high-energy region of \( d = 4 \) case can be described by that of the \( d = 2 \) case. When the energy scale \( \Lambda \gg \mu \gg m \), the logarithmic divergences of \( d = 2 \) case gives rise to the quadratic divergences of the \( d = 4 \) case. Thus the quadratic divergences relevant physics up to one-loop level does can live on the complex two dimensional plane as proposed by Veltman. The logarithmic divergences can be considered as low energy physics compared with quadratic divergences from the viewpoint of effective field theory. The \( d = 4 \) case in DREG only preserves logarithmic divergences, thus the \( d = 4 \) case gives rise to low energy physics. Since the logarithmic divergences is multiplicative renormalization in the cut-off method and DREG, the \( \lambda \) on the right hand side of \( \lambda \to \lambda M^2/(4\pi) \) is the \( \lambda \) which is associated with the renormalization of logarithmic divergences in the case of \( d = 4 \) in DREG. Thus the parameters on the complex two dimensional plane (physical parameters at high energy region) can be connected with that of \( d = 4 \) case(low energy region), through which we can study high energy region physics with parameters of the low energy region.

As one important application of the above arguments, we derive the naturalness problem in \( R_\xi \) gauge in the next section. The RGE of the Higgs mass up to one-loop level on the complex two dimensional plane will also be studied. To derive the RGE of the Higgs mass, the factor \( M^2/4\pi \) associated with the \( \lambda \) transformation needs to be replaced by a function of the scale parameter \( \mu \) in the MS(\(\overline{\text{MS}}\)) scheme.

---

\(^3\) The \( f(M) \) can be specialized to be \( 1/2M^2 \) as will be shown in the one-loop level case, and the inverse square of \( M \) can be used to compensate the mass dimension of interaction couplings of the \( d = 4 - 2/L \) and \( d = (4 - 2/L) + 2 \) cases. In addition, the \( f(M) \) needs to be proportional to \( M^{-2} \) to make the exponent in the transformation Eq. (10) dimensionless.
III. RENORMALIZATION OF THE HIGGS MASS INVOLVING QUADRATIC DIVERGENCES

In order to extract some useful physical consequences, it is suitable to explore RGEs of parameters of the renormalizable theory in MS (or \(\text{MS}\)) scheme. The scheme has the remarkable property that beta functions (\(\beta\)) and anomalous dimension of mass (\(\gamma_m\)) derived in this scheme are all gauge-independent \([14, 15]\). In other renormalization schemes, the renormalization coupling constant \((Z_{g,m})\) is gauge dependent in general for including finite terms in which. For no explicit scale parameter (\(\mu\)) dependent in \(Z_{g,m}\), so RGEs of the couplings and \(\gamma_m\) in MS (or \(\text{MS}\)) scheme (which are functions of \(Z_{g,m}\)) carry no explicit \(\mu\)-dependent. Thus MS (or \(\text{MS}\)) is always referred as the mass-independent renormalization scheme. This property of MS (or \(\text{MS}\)) scheme makes it easy to solve RGEs. Based on the above argument, the MS (or \(\text{MS}\)) scheme will be chosen as the renormalization scheme in this paper.

A. Renormalization procedure and the VEV

Considering the Higgs mass of the SM as one physical quantity, the two-point connected Green function of the Higgs field needs to be gauge invariant. When one study the perturbative correction of the Higgs mass of the SM, one should take into account not only standard loop corrections (the two point 1PI self-energies), but also corrections to the definition of VEV via minima of the Higgs potential. The loop corrections to the definition of VEV, entering through the so-called tadpole 1PI (one point 1PI truncated Green function), causes VEV shift. The VEV shift induced by tadpole 1PI values much in the mass renormalization, which induced the 1PR two-point self-energy of the Higgs field \([16]\), with which we can get gauge invariant mass correction \([17, 18]\). This kind of renormalization method has also been adopted in fermion mass renormalization procedure \([39]\). For the renormalization of other mass terms in the SM involving VEV, the same procedure needs to be adopted in order that we get gauge invariant masses.

Furthermore, the tadpole 1PI is related with the Higgs potential \([20]\), with the tadpole 1PI we can arrive at the Higgs potential directly (see Eq. (3.20) and (3.21) of the paper \([17]\)). The paper \([21]\) derived the relation between the tadpole 1PI and the effective potential in Landau gauge, which makes the derivation of the Higgs potential technically easier \([22]\).

B. The Lagrangian and the counter-term method

Relations between renormalized masses and parameters used in this work are

\[
m_H = \sqrt{2} \lambda v , \quad m_W = \frac{g_2 v}{2} , \quad m_Z = \frac{g_1 v}{2 \cos \theta_W} , \quad m_t = \frac{g_t v}{\sqrt{2}} , \quad \cos \theta_W = \frac{g_2}{\sqrt{g_2^2 + g_1^2}} .
\]

(12)

with \(\lambda, g_t, g_2\) and \(g_1\) being scalar quartic coupling, top quark Yukawa coupling, \(SU(2)_L\) and \(U(1)\) gauge couplings, respectively.

After spontaneous symmetry breaking, the bare Lagrangian of the Higgs part of the SM in four dimensional spacetime is

\[
\mathcal{L}_H^0 = \frac{1}{2} (\partial_\mu H^0)^2 - \frac{1}{2} (m_H^0)^2 (H^0)^2 - \lambda^0 v^0 (H^0)^3 - \frac{1}{4} \lambda^0 (H^0)^4 + \text{const. } .
\]

(13)

where superscripts 0 on mass, couplings and the Higgs field stand for bare parameters. Parameters which do not have superscripts represent renormalized parameters. And \(m_H^0 = \sqrt{2 \lambda^0 v^0}\) has been set. Let us first introduce four renormalization constants to relate bare parameters with renormalized ones,

\[
\chi^0 = Z_H^{-2} Z_1 \lambda , \quad (m_H^0)^2 = Z_H^{-1} Z_0 m_H^2 , \quad H^0 = Z_H^{1/2} H.
\]

(14)

Then, the relation between bare and renormalized VEV is given by \(v^0 = Z_1^{-1/2} Z_H^{1/2} Z_0^{-1/2} v\). After renormalization constants being introduced, the bare Lagrangian of the SM, which can be used to extract physics of \(d = 4\) case, is recast as

\[
\mathcal{L}_H^0 = \frac{1}{2} Z_H (\partial_\mu H)^2 - \frac{1}{2} Z_0 m_H^2 H^2 - Z_1^{1/2} Z_0^{1/2} \lambda^0 v^0 H^3 - \frac{1}{4} Z_1 \lambda^0 H^4 + \text{const. } .
\]

(15)
MS) scheme, divergent terms in self-energy of the Higgs field can be subtracted through renormalization constants. Two-point self-energy of the Higgs field comes from 1PI self-energy and tadpole contributions. Where the arbitrary mass parameter \( \mu \) is introduced through \( \lambda^0 = Z_\lambda \lambda \mu^2 \) and \( (g^0)^2_{1,2,t} = Z_{g_1 \cdot g_2 \cdot g_3} g^2_{1,2,t} \mu^2 \) with \( \varepsilon = 4 - d \). With \( Z_\lambda \) and \( Z_{g_1 \cdot g_2 \cdot g_3} \), containing poles at \( d = 4 \), beta functions of all couplings of the SM of \( d = 4 \) case can be given easily 4.

However, to derive RGE of the Higgs mass on the complex two dimensional plane, the corresponding beta functions are needed. These beta functions may be derived from the connections between the parameters on the plane and the parameters of the \( d = 4 \) case, as will be explored below. The bare Lagrangian of the SM, which can be used to extract the physics on the complex two dimensional plane, takes the form of

\[
\mathcal{L}^0_H = \frac{1}{2} Z^0_H (\partial_\mu H)^2 - \frac{1}{2} Z^0_0 m^2_H H^2 - Z^0_1 \lambda^0 \lambda^0 v H^3 - \frac{1}{4} Z^0_1 \lambda^0 \lambda^0 H^4 + \text{const.} . 
\]

(16)

Where the arbitrary mass parameter \( \mu \) is introduced through \( \lambda^0 = Z_\lambda \lambda \mu^2 \) and \( (g^0)^2_{1,2,t} = Z'_{g_1 \cdot g_2 \cdot g_3} g^2_{1,2,t} \mu^2 \) with \( \varepsilon = 2 - d \). The superscript \( t \) is to identify the parameters on the complex two dimensional plane, these parameters are related to parameters of \( d = 4 \) case through \( \lambda' = \lambda \mu^2 \) and \( g^2_{1,2,t} = g^2_{1,2,t} \mu^2 \). The bare Lagrangian of the Higgs part \( \mathcal{L}^0_H \) can be separated to be renormalized \( \mathcal{L}'_H \) and the counter-term \( \mathcal{L}^{ct}_H \) part, with \( \mathcal{L}'_H \) precisely equal to \( \mathcal{L}^0_H \) when bare parameters in \( \mathcal{L}^0_H \) are replaced by renormalized ones. And the counter-term part is

\[
\mathcal{L}^{ct}_H = \frac{1}{2} (Z^0_H - 1)(\partial_\mu H)^2 - \frac{1}{2} (Z^0_0 - 1) m^2_H H^2 - (Z^0_1 \lambda^0 Z^0_1 - 1) \lambda' \lambda' v H^3 - \frac{1}{4} (Z^0_1 - 1) \lambda' \lambda' H^4 + \text{const.} . 
\]

(17)

with renormalization constants \( Z^t_m = Z^t_{Z_H} Z^0_0 \) and \( Z^t_0 = Z^t_{Z_H} Z^0_0 \). We find that the beta functions on the complex two dimensional plane can be expressed by those of the \( d = 4 \) case

\[
\mu \frac{d \lambda'}{d \mu} = \mu^2 \beta(\lambda(\mu)) , \quad \mu \frac{dg^2_{1,2,t}}{d \mu} = \mu^3 \beta(g(\mu)) .
\]

(18)

In MS (or \( \overline{\text{MS}} \)) scheme, divergent terms in self-energy of the Higgs field can be subtracted through renormalization constants. Two-point self-energy of the Higgs field comes from 1PI self-energy and tadpole contributions

\[
\Sigma^H_H (p^2) = \Sigma^{1PI}_H (\mu^2) + \Sigma^T_H (\mu^2) .
\]

(19)

Feynman diagrams which contribute to 1PI self-energy are shown in Fig. 1. The tadpole diagrams contribute to the self-energy of the Higgs boson is

\[
\Sigma^T_H = - i \frac{3 m^2_H}{v} \frac{i}{-m^2_H} T ,
\]

(20)

4 It is customarily considered that the \( d = 4 \) case in DREG corresponds to four dimensional spacetime physics. While, according to our discussions in section 11 the \( d = 2 \) case need to be included to describe high energy region physics of four dimensional spacetime.
where $i/(-m_H^2)$ is the propagator of the Higgs boson carrying zero momentum, and the Higgs three-point vertex is $-i3m_H^2/v$, with amplitude $T$ being depicted in Fig. 2. Up to one-loop level, the counter-term method requires $\Sigma_H(p^2) + i(Z_H' - 1)p^2 - i(Z_H - 1)m_H^2 = 0$. Combining with the relation $(m_H')^2 = Z_m' m_H^2$, we derive the renormalization constant of the Higgs mass $(Z_m')$ on the complex two dimensional plane in next subsection.

C. Renormalization constant and anomalous dimension of the Higgs mass

Proceeding calculations on the complex two dimensional plane with DREG, the renormalization constant for the Higgs field is obtained to be $Z_H' = 1$ for the absence of poles at $d = 2$. The renormalization constant $Z_H$ includes contributions of poles at $d = 2$ and equals to

$$Z_H' = 1 + \frac{2}{(4\pi)m_H^2} \left[ 6\lambda' - \frac{3}{2} Tr[I]g_t^2 + (g_{\mu}^2 - 1)\frac{3g_{\mu}^2}{4} + \frac{g_{\mu}^2}{4} \right].$$

Therefore, the renormalization constant of the Higgs mass is

$$Z_m' = 1 + \frac{2}{(4\pi)m_H^2} \left[ 6\lambda' - \frac{3}{2} Tr[I]g_t^2 + (g_{\mu}^2 - 1)\frac{3g_{\mu}^2}{4} + \frac{g_{\mu}^2}{4} \right].$$

From Eq. (22), it is easy to find that, after one take $Tr[I]=g_t^2=4$ and the replacement $1/(1 - d/2) \rightarrow \Lambda^2/(4\pi)$, the naturalness problem can be expressed by

$$(m_H')^2 = m_H^2 + \frac{2\Lambda^2}{(4\pi)^2v^2} [3m_H^2 - 12m_t^2 + 6m_W^2 + 3m_Z^2]$$

in $R_\xi$ gauge. Thus the naturalness problem is gauge independent. Here one may argue that the naturalness problem is the appearance of physics on the complex two dimensional plane. And from Eq. (23), one may easily find that the naturalness problem disappears when the Veltman condition \cite{2,6} is satisfied, which is the key of SUSY \cite{5}.

From the above arguments, any attempt to analysis naturalness problem with RGEs of the SM which applied to $d = 4$ case may not be appropriate, RGEs that can be applied to the complex two dimensional plane are needed. With renormalization constant of the Higgs mass term on the plane, the RGE of the Higgs mass can be derived. The behavior of the Higgs mass on the plane with respect to the energy scale $\mu$ can be treated as the effect caused by the naturalness problem, i.e., the MS (or $\overline{\text{MS}}$) substraction procedure keeps the structure of the pole terms which is the source of the naturalness problem.

For the bare Higgs mass is independent of $\mu$, RGE of the Higgs mass on the complex two dimensional plane can be derived

$$\mu \frac{dm_H^2}{d\mu} = -m_H^2 \lim_{\epsilon \rightarrow 0} \gamma_m' (m_H(\mu), \epsilon),$$

where

$$\gamma_m' (m_H(\mu), \epsilon) = \frac{\mu}{Z_m'} \left( \frac{\partial Z_m'}{\partial \lambda'}(m_H(\mu), \epsilon) + \frac{\partial Z_m'}{\partial g_t^2}(m_H(\mu), \epsilon) + \frac{\partial Z_m'}{\partial g_t^2}(m_H(\mu), \epsilon) + \frac{\partial Z_m'}{\partial g_t^2}(m_H(\mu), \epsilon) \right)$$

\footnote{For no sign of SUSY being observed at LHC so far, if SUSY exists, it might be realized at high enough scale at which the Veltman condition is satisfied. And the matching between the supersymmetric and non-supersymmetric theories should be done at that scale \cite{19}.}
represents anomalous dimension of the Higgs mass term. Beta functions on the plane can be expressed through beta functions of \( d = 4 \) case as shown by Eq. \((18)\). Since the anomalous dimension of the Higgs mass is the function of \( Z_m' \), it must be gauge independent. While in other renormalization schemes the renormalization constant of the Higgs mass term is gauge dependent in general. This is caused by appearance of the finite terms in addition to the terms given in \( Z_m' \) on the right hand side of Eq. \((22)\) in other schemes, which is the same with the situation discussed at the beginning of this section. After one careful calculation, the anomalous dimension of the Higgs mass term on the plane is derived

\[
\gamma_{m_H'}(m_H(\mu)) = -\frac{1}{(4\pi)m_H^2}(24\lambda' - 12g_t^2 + g_1^2 + 3g_2^2). \tag{26}
\]

Obviously, this equation is gauge independent, and we need to point out that the occurrence of \( m_H^2 \) in the above equation is caused by the form of the pole at \( d = 2 \). While this equation is still dimensionless since couplings in the parentheses on the right hand side of this equation take mass dimension 2. In the derivation of above equation, \( Tr[I] = 2 \) and \( g_\mu^2 = 2 \) have been adopt based on DREG argument \([11]\). With the relations between parameters of \( d = 2 \) and \( d = 4 \) cases, being given under Eq. \((16)\), Eq. \((26)\) can be represented by the parameters of \( d = 4 \) case,

\[
\gamma_{m_H'} = -\frac{\mu^2}{(4\pi)m_H^2}(24\lambda - 12g_t^2 + g_1^2 + 3g_2^2). \tag{27}
\]

While based on Veltman’s argument on the freedom of vector bosons, the \( g_\mu^2 \) and \( Tr[I] \) all need to equals to 4. Thus one can connect the RGE of the Higgs mass on the complex two dimensional plane with the Veltman condition

\[
\mu \frac{dm_H^2}{d\mu} = \frac{2\mu^2}{4\pi} VC(\mu), \tag{28}
\]

with

\[
VC(\mu) = 12\lambda - 12g_t^2 + \frac{3g_1^2 + 9g_2^2}{2}, \tag{29}
\]

and \( VC(\mu) \) equaling to zero is the Veltman condition. Now, we can use Eq. \((28)\) together with beta functions of couplings of the SM of \( d = 4 \) case to study the physics on the complex two dimensional plane.

### D. Scale-dependent property of the Higgs mass on the complex two dimensional plane

To explore the \( \mu \)-dependent property of the Higgs mass on the complex two dimensional plane, all RGEs of couplings of the SM are needed. These beta functions up to two-loop order is listed in Appendix \([3]\). Boundary conditions of \( g_{1,2,t} \) can be obtained as in \([22]\), and the boundary conditions of \( m_H \) is set to be \( m_H(126 \text{ GeV}) = 126 \text{ GeV} \). Considering the theoretical error in derivation of the top quark pole mass from the running one, we choose the top quark pole mass \( M_t = 173.3 \pm 2.8 \text{ GeV} \), which strongly affects the behaviors of RGEs. And other input parameters are chosen to be the central values \([24]\).

From the behavior of RGE of the \( m_H^2 \) as depicted on the left panel of Fig. \(3\) one expect \( m_H^2 \) first decreasing and latter increasing for the change of the sign of \( dm_H^2/d\mu \) with the energy scale increasing\(^6\). The \( |m_H^2| \) running to be very large before one renormalized away the Electroweak scale and the Planck scale\(^7\), because in that energy region \( (dm_H^2/d\mu)/\mu \) (connect with \( VC(\mu) \) through Eq. \((28)\)) is always negative. In fact, this property of the \( m_H^2 \) on the plane is caused by the naturalness problem, i.e., the quadratic divergences which cause the naturalness problem are also multiplicatively

\(^6\) That is caused by contributions which are proportional of the couplings \( (\lambda, g_{1,2,t}) \) in \( \gamma_{m_H} \) changes with the energy scale increasing, i.e., contributions of \( \lambda, g_{1,2,t} \) become bigger than that of \( g_t \), as shown in \([19, 22]\).

\(^7\) Different from the paper \([19]\), we work in the broken phase, where the gauge invariant property of \( m_H \) can be protected as discussed in previous subsections. After we renormalized away the quadratic divergences induced by tadpole diagrams, we have negative corrections to the \( m_H^2 \), that’s why we use \( |m_H^2| \) but \( m_H^2 \) here. And in symmetric phase, one impose \( \langle H \rangle = 0 \), the corrections to the \( m_H^2 \) will be positive after one renormalized away quadratic divergences. Take one-loop corrections as an example, only \( \Sigma_{\mu H}^{(1)}(p^2) \) contribute to \( m_H^2 \) in symmetric phase and the \( \Sigma_{\mu}^{(1)}(p^2) \) needed to be considered in Eq. \((19)\), thus Eq. \((28)\) changes to be \( \mu dm_H^2/d\mu = -\mu^2/(4\pi)VC(\mu) \). And the structure of quadratic divergences in symmetric phase and broken phase are the same, one can expect the UV scales where \( m_H = 0 \) are the same in the two cases.
renormalized as explored by us, thus the effect caused by naturalness problem can be studied through the RGE of the $m_H^2$ on the plane.

The $(dm_H^2/d\mu)/\mu$ (which is proportional to $VC(\mu)$) approaches zero at UV scale, as is plotted on the left panel of Fig. 3. After $VC(\mu)((dm_H^2/d\mu)/\mu)$ changes sign, the $m_H^2$ approaches one nontrivial zero quickly, the energy scale which satisfies $m_H = 0$ is slightly higher than the scale at which $VC = 0$, as is depicted on the right panel of Fig. 3. However, after $m_H^2 = 0$ being achieved, with the increase of energy scale, $m_H^2$ can still get large value due to the sign-flip of $(dm_H^2/d\mu)/\mu$, which is also noted in [24]. This behavior can be explained by the fact that the roles played by the higher energy scale $\Lambda$ where the effective theory can be applied, and the lower energy scale $\mu$ shown in Eq. (1) are interchanged. Above argument is also suitable for our renormalization procedure.

The dependence of energy scale at which $m_H^2(VC) = 0$ on the top quark pole mass, is plotted on the right panel of Fig. 3. Assuming the SM can be valid to arbitrarily large energy scale, one can always expect the value of $m_H$ to arrive at zero. And $m_H^2 = 0$ will be achieved at about the Planck scale for $M_t = 168$ GeV, which is about $2\sigma$ smaller than the central value. The scale increases with the value of $M_t$ increasing. For $M_t = 173.3$ GeV, the scale is around $10^{24}$ GeV. And for the scale exceeding the Planck scale, it was extensively believed that the gravitational contributions need to be considered [10].

For the multiplicative renormalization property of the Higgs mass-square on the complex two dimensional plane, one can have $m_H^2 = 0$ when $m_H = 0$. And the vanishing of the bare Higgs mass is the main result of the paper [10], where it was argued to hint the restoration of SUSY. While the paper [19] argued that the SUSY can match with the SM at the scale $VC = 0$. Based on our analysis, we find that one can not expect $m_H^2 = 0$ and $VC = 0$ to be satisfied at the same scale.

Based on Bardeen’s argument on the naturalness problem, quadratic divergences could be safely removed by imposing boundary condition at the UV (Planck) scale $M_{pl}$ [26, 28], and the behavior of $m_H^2$ plotted in Fig. 3 suggests the natural choice could be $m_H(M_{pl}) = 0$ [8]. Imposing this condition on the RGE of $m_H$ for $d = 4$ case within DREG, which is gauge invariant and multiplicatively renormalized result of logarithmic divergences, we always have one zero mass, with implications: Attributing the naturalness problem to the physics on the complex two dimensional plane, RGEs of $d = 4$ case within DREG method can be described by logarithmic terms safely. And paper [27, 28] supposes the boundary condition imposed on the SM need to be justified in the UV complete theory. The solution of the naturalness problem calls for knowledge of UV complete theory [19]. On the complex two dimensional plane, we can study quadratic divergences effect in a gauge invariant way, and the quadratic divergences correspond to UV physics in the viewpoint of effective field theory. With the naturalness problem lives on the complex two dimensional plane, to solve the problem, one need to investigate the property of the complex two dimensional plane more.

---

8 Based on argument of the paper [27], the natural boundary condition of the mass term at the $M_{pl}$ is chosen to be $m^2(M_{pl}) = 0$, with $m^2$ is the mass term in the tree level Higgs potential with the ‘wrong’ sign, which is equivalent to our result for $m_H^2 = -2m^2$ after considering the SSB. Here, one need to note that the condition $m_H(M_{pl}) = 0$ as was explored by us can be independent of the matter content [24], which is different from the approaching zero of the Veltman condition which depends on the matter content.
If we study the RGE of the $m_H^2$ described by Eq. (27) not Eq. (28) on the complex two dimensional plane, then the UV scale where $m_H$ vanishes will be improved slightly, with other related discussions will not be changed.

### IV. THE VIEWPOINT OF WILSONIAN RENORMALIZATION GROUP ON QUADRATIC DIVERGENCES

In this section, We study the role played by quadratic divergences in the Wilsonian renormalization group method.

From the viewpoint of Wilsonian renormalization group, when the energy scale $\Lambda \gg m$, the correction to the mass term of $\lambda \phi^4$ theory, came from quantum contributions of the energy region $b \Lambda \leq |k_E| \leq \Lambda$.

$$
\frac{1}{2} \lambda \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2} = \frac{1}{(4\pi)^{d/2}} \frac{(1-b^{d-2})}{d-2} \Lambda^{d-2},
$$

where the subscript $E$ denotes the parameters living in Euclidean spacetime. In the $d$ dimensional $\lambda \phi^4$ theory, the relation between masses at energy scale $b^n \Lambda$ (m) and $\Lambda$ ($m'$), with $b < 1$ but very close to 1, is given by

$$
m'^2 = m^2 b^{-2n} + \frac{\lambda n (b^{-2} - 1) \Lambda^2}{32\pi^2},
$$

with the $n$ in the above two equations being constraint by the condition $\Lambda \sim m'$. After iterating $n$ times the Lagrangian from scale $b^n \Lambda$ to $\Lambda$, as shown in [8], the difference between $m'^2$ and $m^2$ comes to be enormous, which calls for delicate choosing of $m'^2$ when one wants to know the $m^2$ at one lower momentum scale $b^k \Lambda$, with $b^n \Lambda \leq b^k \Lambda < \Lambda$.

We would like to point out that the quadratic divergences can be subtracted through one new appropriate choice of coordinates of the theory space, i.e., with which being absorbed into $m_{new}'$. Therefore, the difference between $m^2$ and $m_{new}'^2$ is fully from rescaling of distance and the field $\phi$ [8], i.e., $m^2 = m_{new}'^2 b^{2k}$. The real issue of the naturalness problem is not the quadratic divergences but the rescaling effects. In addition, the operation of absorption of the $\Lambda^2$ terms in the above argument is indeed the attribution of quadratic divergences to the physics on the complex two dimensional plane following our discussions in section II. Furthermore, quadratic divergences up to all loop orders could be absorbed to give the critical surface in the sense of Wilsonian group method [6], this kind of surface plays the role similar with the complex 4 - 2/L dimensional plane. After the quadratic divergences being subtracted, one new coordinate of space of parameters are given, and the RG flows around the critical surface is determined by logarithmic divergences. With DREG, all the four dimensional physics, i.e., RGEs of physical parameters may be achieved at $d \rightarrow 4$, which corresponds to logarithmic divergences. With quadratic divergences living on the complex 4 - 2/L plane, which would not change our section II, the difference between $m^2$ and $m_{new}'^2$ gives rise to $-\lambda \log b/(4\pi)$. Thus $m'^2 = m^2 + \lambda \log b/(4\pi)$. The variance of $m$ is not so rapid now, and the RGE of mass can be derived

$$
\frac{d(m'^2 - m^2)}{d \log b} = \frac{\lambda}{4\pi}.
$$

Generalizing to the SM case, the RGE of the Higgs mass can be computed directly, and the same result with that derived in section III C can be obtained considering the correspondence between $b$ and the $\mu$.

For the case of scalar quartic coupling $\lambda$ in the $\lambda \phi^4$ theory, the relation between quantities at the scale $\Lambda$ and $b^n \Lambda$ is given by

$$
\lambda' = \left( \lambda b^{(d-4)n} - \frac{3\lambda^2}{16\pi^2} \log \frac{1}{b} \sum_{n=1}^n b^{(d-4)n} \right).
$$

Accordingly, when dimension $d$ is continued to 4, this formula will be greatly simplified,

$$
\lambda' = \left( \lambda - \frac{3\lambda^2}{16\pi^2} \log \frac{1}{b} \right).
$$
Where no $n$ shows up as in Eq. (32). Indeed, this is caused by the fact that the mass term $m^2 \phi^2$ and the scalar quartic interaction $\lambda \phi^4/4$ are relevant and marginal terms respectively in four dimension. The $\lambda$ does not have the initial value choosing problem as the mass term when we get its value at one energy scale lower than $\Lambda$. When one study the quantum corrections to $\lambda$ in the $\lambda \phi^4$ theory in four dimensional spacetime, only the correspondence between the pole at $d = 4$ and logarithmic divergences (the second formula of Eq. (3)) can be found to any loop orders. Thus, the quantum corrections of $\lambda$ is proportional to $\log(\Lambda^2/\mu^2)$, which corresponds to the case of Eq. (33) when the scale runs from $\mu$ to $b^n \Lambda$. In the ordinary perturbation calculation as in section III the quantum corrections to $\lambda$ is wholly the same with Eq. (35) when we take $\mu = b^n \Lambda$, which is due to the rescaling distance effect disappearing in four dimensional spacetime, i.e., $b^{(d-4)n} \to b^1 = 1$ as $d \to 4$. We should mention that couplings ($\lambda$, $g_{1,2,3,4}$) of the SM also share the same property as been discussed above.

V. VEV AND THE COMPOSED HIGGS MASS

The rescaling property of the Higgs mass is different from that of the Higgs quartic coupling since it still takes the rescaling factor $b$ as $d \to 4$. The rescaling distance effect in Eq. (32) calls for delicate choosing of the mass of the theory at the scale $\Lambda$, the core of the naturalness problem is the rescaling effect in the Wilsonian sense. In this section, we consider the opposite scenario in which the Higgs mass term has the same rescaling property as that of the Higgs field in the SM, and show that the naturalness problem will not come to us in this case. Then we study RG behaviors of physical parameters up to UV scale.

A. RGE of VEV

The SM has achieved great success, almost fits all experimental results in last decades. In the SM, the Higgs mechanism provides masses to fermions and gauge bosons, and one proper aim of LHC is to check this mechanism. One SM-like Higgs signal with mass about 126 GeV has already been found at LHC [40]. In this paper, we suppose that the signal is just the Higgs of the SM. How to understand the mass term properly is the key to understand the Higgs boson and the naturalness problem. In the following paragraphs, we will show that the naturalness problem does not shows up if the Higgs mass is considered not to be one gauge invariant but one composed quantity.

The key of the Higgs mechanism is the existence of the mass term which has the “wrong” sign, and can be given by $\mu^2 = \lambda \phi^2 |_{\phi = \phi^0}$, with the $\phi^0$ being the value of $\phi$ where the minimum of the potential $V(\phi)$ occurs. Thus the mass term (after SSB) $m^2$ can be given by $2\lambda \phi^2 |_{\phi = \phi^0}$, which is just two times of the product of $\lambda$ and $\phi^2$ when $\phi = \phi^0$, we refer the $m^2$ as the composed mass. Hereafter, we analyze property of the $m^2$ in Wilsonian sense and do not view the $m^2$ as the value with mass dimension two directly. Firstly, imposing the parameters in the Lagrangian at the energy scale $\Lambda$ takes the superscript $\tau$ and the the Lagrangian at the energy scale $b^n \Lambda$ does not. The mass term of the Lagrangian at the energy scale $\Lambda$ is $2\lambda (\phi^0)^2 \phi^2$ with the $\phi^0$ is the value of $\phi'$ at the minima of $V(\phi')$. And, the mass term at the energy scale $b^n \Lambda$ is $2\lambda (\phi^0)^2 b^2 \phi^2$ with $\phi^0$ having the same scaling property as $\phi$. Thus the relationship between $\phi^0$ and $\phi^0$ is the same as that of $\phi$ and $\phi'$, and the relation $\phi^0 = [b^{2-d}(1 + \delta Z)]^{1/2} \phi^0$ holds when we iterating the Lagrangian from energy scale $b \Lambda$ to $\Lambda$. Since we do not consider the $2\lambda \phi^2 |_{\phi = \phi^0}$ as the field of mass in the Lagrangian directly, we will not encounter the enormous fine-tuning. The composed mass term at the energy scale $\Lambda$ can be given by $2(\lambda + \delta \lambda)(\phi^0)^2 \phi^2$ when $d \to 4$. Take RG transformation $n$ times, i.e., successive iterate the transformation procedure on the Lagrangian like from $b \Lambda$ to $\Lambda$ for $n$ times, the relation between the composed mass terms at the energy scale $\Lambda$ and $b^n \Lambda$ can be given by

$$2\lambda (\phi^0)^2 \phi^2 = 2(\lambda + \delta \lambda) b^{(d-4)n} (\phi^0)^2 \phi^2$$

$$\sim 2(\lambda + \delta \lambda) (\phi^0)^2 \phi^2, \text{ when } d \to 4,$$

with the quantities following the symbol “∼” living in energy scale $b^n \Lambda$, and where the $n$ is absent for the same reason as argued in the last section, and the effects caused by $\delta Z$ have been dropped for small contributions.

In the SM, the $\phi^0$ is the VEV, i.e., $v$, which shares the same scaling property as the Higgs field. Translate the Eq. (36) to the usual renormalization method, wherein RGEs are calculated with DREG based on MS (or $\overline{\text{MS}}$) scheme.

\footnote{The $\mu^2$ is used to make the statement in this part clear, which should be different from the scale parameter introduced in the MS($\overline{\text{MS}}$) scheme.}
we have
\[ \lambda v^2 \mu^2 Z_1 Z_H^{-1} = \lambda_0 v_0^2, \]
where the \( v^0 \) and \( \lambda_0 \) are the correspondences of \( \phi^0 \) and \( \lambda' \) respectively. Now the Higgs mass in the SM should be considered as one composed mass. Since \( \lambda_0 v_0^2 \) does not depend on energy scale \( \mu \) introduced in DREG and MS (or \( \overline{\text{MS}} \)) scheme, we can derive the RGE of \( \lambda v^2 \) in our four dimensional spacetime,
\[ \mu \frac{d}{d\mu} (\lambda v^2) = -2 \lambda v^2 \gamma_H + v^2 \beta_\lambda, \]
with one-loop anomalous dimension of the Higgs field is
\[ \gamma_H^{(1)} = \frac{1}{64 \pi^2} (12 g_1^2 - 9 g_2^2 - 3 g_1^2), \]
and the two-loop level contributions to \( \gamma_H \) is
\[ \gamma_H^{(2)} = \frac{1}{(16 \pi^2)^2} \left( 6 \lambda^2 - \frac{27}{4} g_1^4 + 20 g_2^2 g_1^2 + \frac{45}{8} g_1^2 g_2^2 + \frac{85}{24} g_1^2 g_2^2 - \frac{271}{32} g_2^4 + \frac{9}{16} g_1^2 g_2^2 + \frac{431}{96} g_1^2 \right) \]
in Landau gauge, this gauge is appropriate in the sense of \([32]\) based on effective potential argument. And \( \beta_\lambda \) is given in \([23]\). We can now isolate the RGE of VEV from Eq. (38),
\[ \mu \frac{d}{d\mu} v^2 = -2 v^2 \gamma_H, \]
with \( \gamma_H \) are given by Eq. (39,40) at one-loop and two-loop order, and coincide with the anomalous dimension of VEV derived in \([33,34]\) at one- and two-loop order. We want to note that this approach does sidestep the naturalness problem, while the gauge invariant property of the RGE of the composed Higgs mass (still denoted as \( m_H^2 = 2 \lambda v^2 \)) but with different meaning with the \( m_H^2 \) considered in section \( \text{III D} \) disappears, since
\[ \mu \frac{d m_H^2}{d\mu} = 2 \mu \frac{d}{d\mu} (\lambda v^2) \]
and the RGE of the VEV, which is shown in Eq. (41), depends on the gauge parameter.

**B. Scaling property of the composed Higgs mass**

With the RGE of \( v^2 \) (Eq. (41)), we can study the behavior of the VEV with respect to the energy scale \( \mu \). From the Fig. 3 we find that the VEV varies very slowly with the energy scale growing, where \( \beta \) functions of couplings of the SM are considered up to two-loop order. When we solve beta functions \([35,36]\), the boundary (matching) conditions \([25,38,39] \) (matching of \( \overline{\text{MS}} \) coupling constants and pole masses to give boundary conditions of couplings) are used as in section \( \text{III D} \) (matching of \( \overline{\text{MS}} \) coupling constants and pole masses to give boundary conditions of couplings). And the boundary condition of Eq. (11) is chosen to be \( v(M_W) = 246.22 \text{ GeV} \) \([34]\) with \( M_W \) being the pole mass of the W boson. The behavior of \( \lambda \) with respect to the energy scale \( \mu \) is shown in Fig. 3.

We present the composed Higgs mass value as a function of energy scale in Fig. 4 at one-loop and two-loop order, and coincide with the anomalous dimension of VEV derived in \([33,34]\) at one- and two-loop order. We want to note that this approach does sidestep the naturalness problem, while the gauge invariant property of the RGE of the composed Higgs mass (still denoted as \( m_H^2 = 2 \lambda v^2 \)) but with different meaning with the \( m_H^2 \) considered in section \( \text{III D} \) disappears, since
\[ \mu \frac{d m_H^2}{d\mu} = 2 \mu \frac{d}{d\mu} (\lambda v^2) \]
and the RGE of the VEV, which is shown in Eq. (41), depends on the gauge parameter.

\[ \text{10} \] The scalar quartic coupling damps gradually as in Fig. 5 and the VEV has tiny variations with the energy scale growing, as depicted in Fig. 4.
VI. CONCLUSIONS

From our analysis, the naturalness problem induced by quadratic divergences can live on the complex two dimensional plane, and the corresponding physics might does not affect the scaling property of the RGE of $m_H$ derived at $d = 4$ with DREG based on MS(\overline{MS}) scheme. Suppose one UV complete fundamental theory does exist, then quadratic divergences in the SM can be compensated by the same kind of divergences arising from integrations above the cut-off one used. As found by us, we can study the quadratic divergences on the complex two dimensional plane in a gauge invariant scheme, thus opens a new window to explore UV complete theory. With more knowledge of UV complete theory, more detailed quadratic divergences structure of the theory can be studied on the complex two dimensional plane. Therefore, we can expect to achieve one deeper understanding of the naturalness problem.

The meaning of the Higgs mass term of the SM is revisited based on the viewpoint of Wilsonian renormalization group. We derived the RGE of the VEV of the SM in Landau gauge. Numerical analysis up to two-loop order shows that the composed Higgs mass damps gradually with the energy scale growing and eventually to zero within the valid energy region of the SM constrained by vacuum stability condition. And the composed Higgs mass keeps positive even
FIG. 6: The value of the composed Higgs mass-square with energy scale increasing.

FIG. 7: Left: Behavior of Higgs quartic coupling with respect to energy scale; Right: Behavior of the composed Higgs mass-square with respect to energy scale.

up to the Planck scale for $M_t \leq 170.7$ GeV. And in this case, the gauge invariant property of quantum corrections of the composed Higgs mass is absent.

Appendix A: Integration formulas in divergences calculations

Scalar and tensor integrals involved in one-loop calculations on the complex two dimensional plane,

$$
\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2)^2} \to \frac{g^{\mu\nu} i}{2} \frac{1}{4\pi 1 - d/2}.
$$

$$
\int \frac{d^d k}{(2\pi)^d} \frac{k^4}{(k^2 - m^2)^3} \to -\frac{i}{4\pi} \frac{1}{1 - d/2}.
$$

(A1)
Appendix B: Beta functions of couplings of the SM

The beta function for a generic coupling $x$ is given as:

$$ \frac{d x}{d \mu} = \beta_x, $$

(B1)

The list of beta functions up to two-loop order are given below:

\begin{align*}
\beta_\lambda &= \frac{1}{16\pi^2} \left( \lambda (-9g_2^2 - 3g_1^2 + 12g_t^2) + 24\lambda^2 + \frac{3}{4}g_2^4 + \frac{3}{8}(g_1^2 + g_2^2)^2 - 6g_t^4 \right) \\
&\quad + \frac{1}{(16\pi^2)^2} \left( -312\lambda^3 - 144\lambda^2 g_2^2 + 36\lambda (3g_2^2 + g_1^2) - 3\lambda g_t^4 + \frac{80}{2}g_2^2 + \frac{85}{6}g_1^2 \right) \\
&\quad - \frac{73}{8}g_2^6 + \frac{39}{4}g_2^2 g_1^2 + \frac{629}{24}g_1^4 + 30g_6^6 - 32g_1^6g_3^2 - \frac{8}{3}g_1^6g_1^2 - \frac{9}{4}g_1^4g_2^2 \\
&\quad + \frac{21}{2}g_1^2 g_2 g_1^4 - \frac{19}{4}g_1^2 g_2^4 + \frac{305}{16}g_6^6 - \frac{289}{48}g_2 g_1^4 - \frac{559}{48}g_2^2 g_1^4 - \frac{379}{48}g_1^6 
\end{align*}

(B2)

\begin{align*}
\beta_{g_1} &= \frac{1}{16\pi^2} \left( \frac{9}{2}g_1^2 + g_t \left( \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \right) \right) \\
&\quad + \frac{1}{(16\pi^2)^2}g_1^4 \left( \frac{3}{16}g_1^4 + \frac{9}{4}g_2^2 g_1^2 - \frac{23}{4}g_2^4 + 9g_2^2g_3^2 - 108g_4^2 + 6g_1^2 \right) \\
\beta_{g_2} &= \frac{1}{16\pi^2} \left( \frac{41}{6}g_1^2 \right) + \frac{1}{(16\pi^2)^2}g_1^4 \left( \frac{199}{18}g_1^4 + \frac{9}{2}g_2^2 + \frac{44}{3}g_3^2 - \frac{17}{6}g_1^2 \right) \\
\beta_{g_3} &= \frac{1}{16\pi^2} \left( -7g_3^2 \right) + \frac{1}{(16\pi^2)^2}g_1^4 \left( \frac{11}{2}g_1^2 + \frac{9}{2}g_2^2 - 26g_4^2 - 2g_1^2 \right).
\end{align*}

(B3) (B4) (B5) (B6)

[1] S. Alekhin, A. Djouadi and S. Moch, Phys. Lett. B 716, 214 (2012) [arXiv:1207.0980 [hep-ph]].
[2] I. Masina, Phys. Rev. D 87, 053001 (2013) [arXiv:1209.0393 [hep-ph]].
[3] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori and A. Strumia, JHEP 1208, 098 (2012) [arXiv:1205.6497 [hep-ph]].
[4] As for the new physics and metastability, one can refer to V. Branchina and E. Messina, arXiv:1307.5193 [hep-ph].
[5] M. J. G. Veltman, Acta Phys. Polon. B 12, 437 (1981).
[6] R. Decker and J. Pestreau, hep-ph/0512126.
[7] Z. Y. Fang, G. Lopez Castro, J. L. Lucio and J. Pestreau, Mod. Phys. Lett. A 12, 1531 (1997) [hep-ph/9612430].
[8] G. ’t Hooft, Nucl. Phys. B 33, 173 (1971).
[9] G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. B 44, 189 (1972).
[10] H. Aoki and S. Iso, Phys. Rev. D 86, 013001 (2012) [arXiv:1201.0857 [hep-ph]].
[11] K. G. Wilson and J. B. Kogut, Phys. Rept. 12, 75 (1974).
[12] M. E. Peskin, D. V. Schroeder and, “An Introduction to quantum field theory,” Reading, USA: Addison-Wesley (1995) 842 p. p394. (Sec. 1.1)
[13] G. ’t Hooft and M. J. G. Veltman, NATO Adv. Study Inst. Ser. B Phys. 4, 177 (1974).
[14] Y. Hamada, H. Kawai and K. -y. Oda, Phys. Rev. D 87, 053009 (2013) [arXiv:1210.2538 [hep-ph]].
[15] M. Capdequi Peyranere, J. C. Montero and G. Moultaka, Phys. Lett. B 260 (1991) 138.
[16] M. S. Al-sarhi, I. Jack and D. R. T. Jones, Z. Phys. C 55, 283 (1992).
[17] G. Leibbrandt, Rev. Mod. Phys. 47, 849 (1975).
[18] W. E. Caswell and F. Wilczek, Phys. Lett. B 49, 291 (1974).
[19] D. J. Gross, Methods in Field Theory. (eds. R.Balian and J. Zinnjustin), North-Holland, 1976.p.141.
[20] E. Ma, Phys. Rev. D 47, 2143 (1993) [hep-ph/9209221].
[21] S. Weinberg, Phys. Rev. D 7, 2887 (1973).
[22] J. Fleischer and F. Jegerlehner, Phys. Rev. D 23, 2001 (1981).
[23] I. Masina and M. Quiros, arXiv:1305.1242 [hep-ph].
[20] S. R. Coleman and E. J. Weinberg, Phys. Rev. D 7, 1888 (1973).
[21] S. Y. Lee and A. M. Sciacca-luga, Nucl. Phys. B 96, 435 (1975).
[22] M. Sher, Phys. Rept. 179, 273 (1989).
[23] M. Holthausen, K. S. Lim and M. Lindner, JHEP 1202, 037 (2012) [arXiv:1112.2415 [hep-ph]], beta functions for couplings and matching conditions are listed in Appendix A.
[24] J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012) and 2013 partial update for the 2014 edition.
[25] F. Jegerlehner, arXiv:1304.7813 [hep-ph].
[26] W. A. Bardeen, FERMILAB-CONF-95-391-T.
[27] S. Iso, [arXiv:1304.0293 [hep-ph]].
[28] S. Iso and Y. Orikasa, PTEP 2013, 023B08 (2013) [arXiv:1210.2848 [hep-ph]].
[29] K. Fujikawa, Phys. Rev. D 83, 105012 (2011) [arXiv:1104.3396 [hep-th]].
[30] C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B 12, 20 (1972).
[31] M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 236, 221 (1984). M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 222, 83 (1983). M. E. Machacek and M. T. Vaughn, Nucl. Phys. B 249, 70 (1985).
[32] M. Lindner, M. Sher and H. W. Zaglauer, Phys. Lett. B 288, 139 (1989). M. Sher, Phys. Lett. B 317, 159 (1993) [Addendum-ibid. B 331, 448 (1994)] [hep-ph/9307342].
[33] M. F. Zoller, arXiv:1209.5609 [hep-ph]. K. G. Chetyrkin and M. F. Zoller, JHEP 1206 (2012) 033 [arXiv:1205.2892 [hep-ph]]. D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio and A. Strumia, [arXiv:1307.3536 [hep-ph]].