On the Physical Hilbert Space of Loop Quantum Cosmology

Karim Noui*

Laboratoire de Mathematiques et Physique
Theorique Parc de Grandmont 37200 Tours France

Alejandro Perez† and Kevin Vandersloot‡

Institute for Gravitational Physics and Geometry,
The Pennsylvania State University,
104 Davey Lab, University Park, PA 16802, USA

Abstract

In this paper we present a model of Riemannian loop quantum cosmology with a self-adjoint quantum scalar constraint. The physical Hilbert space is constructed using refined algebraic quantization. When matter is included in the form of a cosmological constant, the model is exactly solvable and we show explicitly that the physical Hilbert space is separable consisting of a single physical state. We extend the model to the Lorentzian sector and discuss important implications for standard loop quantum cosmology.

*Electronic address: karim@lmpt.univ-tours.fr
†Electronic address: perez@gravity.psu.edu
‡Electronic address: kfvander@gravity.psu.edu
I. INTRODUCTION

Symmetry reduced models of General Relativity have been extensively studied and applied in the framework of cosmology. In cosmological models the infinite degrees of freedom of the full theory are reduced to a finite number describing the large scale structure of the universe. Current observations seem to be best described by (spatially flat) homogeneous and isotropic Friedman-Robertson-Walker space-times. For that reason, the assumption of homogeneity and isotropy is by now an accepted basic ingredient in the so-called standard model of cosmology.

Under quite general conditions, general relativity predicts the development of space-time singularities. One prime example of such singular behavior is the presence of the initial Big-Bang singularity in cosmological models. Energy densities as well as space time curvature diverge approaching the initial Big-Bang singularity, and any description based on classical space time notions becomes inapplicable. General Relativity is no longer valid and has to be replaced by a quantum theory of gravity.

Although no complete quantum theory of gravity is yet available, loop quantum gravity appears as an promising candidate\[1, 2, 3, 4\]. Loop quantum gravity is a background independent non perturbative approach to the quantization of general relativity. The main prediction of the theory is the discreteness of geometry at the fundamental level: operators corresponding to geometric quantities such as the area and volume have discrete spectra\[5, 6\]. Geometry is quantized and at small scales the smooth notion of space and time simply can no longer be applied. One of the main clear-cut applications of the theory, where the discovery of discreteness plays an important role, is the description of the fundamental degrees of freedom of black hole (isolated) horizons\[7, 8, 9, 10, 11, 12, 13\]. These fundamental excitations give rise to the correct value of black hole entropy that is expected from semiclassical considerations.

The application of loop quantum gravity to cosmological models is known as loop quantum cosmology (LQC)\[14\] (See also\[15\] for an alternative approach). The simplification arising from the symmetry reduction makes it possible to address physical questions that still remain open in the full theory. For instance LQC provides a novel paradigm for the understanding of the fate of the Big-Bang singularity in quantum gravity. The important physical question of whether classical singularities of general relativity are not present in
the quantum theory is answered in the affirmative in LQC\textsuperscript{[16]}. The basic mechanisms leading to this striking result can be traced back to the fundamental discreteness of geometric operators which drastically changes the quantum evolution equation in the deep Planckian regime. In addition, consistency conditions, known as the dynamical initial condition, seem to severely restrict the freedom in the choice of ‘initial’ values for the evolution and it is argued that these conditions may single out a unique wave function of the universe\textsuperscript{[17]}. It has also been shown that loop quantum cosmology leads to modifications of the classical Friedman equation. Recent work has shown that these effects can provide the proper initial conditions for chaotic inflation and may lead to measurable effects on the CMB\textsuperscript{[18]}. 

In spite of these successes, a complete characterization of the physical Hilbert space of loop quantum cosmology remains an open issue\textsuperscript{1}. The definition of the physical inner product is the missing piece of the puzzle in LQC. Without the physical inner product, there is no notion of a probability measure which is an essential ingredient for physical prediction in the quantum theory. Any interpretation of the wave function as a probability amplitude is futile without the probability measure provided by the physical inner product.

In addition the physical inner product provides a means for eliminating pathological solutions to the constraints. For instance it can be shown that the space of solutions of LQC is generally non separable, i.e., an uncountable infinity of solutions to the quantum constraint equation exist. Such a huge physical Hilbert space is certainly not expected for a system with finite number of degrees of freedom. Some of these solutions exhibit peculiar properties such as wild oscillatory or diverging behavior in the regime where one would expect agreement with standard Wheeler-DeWitt quantum cosmology. In the absence of the physical inner product, so far spurious solutions are eliminated using the dynamical initial condition coupled with heuristic arguments based on semi-classicality requirements. Although these requirements are physically motivated, unphysical solutions can be easily identified if a notion of physical inner product is provided. More precisely, on the basis of the notion of physical probability physical states are defined by equivalence classes of solutions up to zero norm states. It is hoped that the ill-behaved solutions mentioned above will be factored out by this means once a suitable notion of physical inner product is provided.

\textsuperscript{1} This remains an open issue in the full theory of LQG. The master constraint approach\textsuperscript{[19]} and the spin foam formulation\textsuperscript{[20]} are current proposals to solve this problem.
There exists a simple method known as group averaging to provide a definition of the physical inner product in generally covariant systems when the constraints generate a group action \cite{21}. In this paper we review and use the technique called \textit{refined algebraic quantization} \cite{22} as a recipe to construct the physical Hilbert space. The physical inner product is constructed in a simple manner if one can define a unitary operator generating ‘translations’ along the gauge orbits generated by the quantum constraints in a suitable kinematical Hilbert space. This technique cannot be applied to the standard formulations of loop quantum cosmology because the quantum Hamiltonian constraint is not a self-adjoint operator (in the kinematical Hilbert space) and thus it does not lead to any well defined unitary ‘evolution’ when exponentiated.

In this paper we present a model of loop quantum gravity that arises from a symmetry reduction of the self dual Plebanski action. In this formulation the symmetry reduction leads to a very simple Hamiltonian constraint that can be quantized explicitly in the framework of loop quantum cosmology. The model is defined in the Riemannian sector which constitutes a limitation when considering physical predictions. However, we will present a method for transforming to the Lorentzian sector and then argue that the main results of the model are manifested in the Lorentzian sector. The model shares the same kinematical Hilbert space with standard loop quantum cosmology, but differs in that the resulting Hamiltonian constraint is self-adjoint and thus the group averaging technique can be used to construct the physical inner product. When matter is incorporated in the form of a cosmological constant, we can solve the model exactly both for the physical wave functions and the physical inner product. The physical inner product reduces the infinite set of solutions of the quantum Hamiltonian constraint equation to one physically normalizable solution. This constitutes an interesting example where the very large (non separable) kinematical Hilbert space is reduced to a (trivially) separable physical Hilbert space.

We end with a discussion of the relevance of the model to standard LQC. We show that despite the simpler constraint the model shares the same qualitative features of standard LQC with one important difference. We will then discuss the implications that can be derived about LQC in general.
II. CLASSICAL SYMMETRY REDUCTION

Our starting point is the classical self-dual Riemannian gravitational action which, as noted by Plebanski\[23\], can be written as a constrained SU(2) BF theory

\[ S[A, \Sigma, \psi] = \int \Sigma_i \wedge F^i(A) - \psi_{ij} \left[ \Sigma^i \wedge \Sigma^j - \frac{1}{3} \delta_{ij} \Sigma^k \wedge \Sigma_k \right] + \Lambda \Sigma_i \wedge \Sigma_i \]  

(1)

where \( \Sigma = \Sigma^i_{\mu\nu} dx^\mu dx^\nu \) is an SU(2) Lie algebra valued two form, \( A = A^i_{\mu} dx^\mu \) is an SU(2) Lie algebra valued connection, \( \tau_i \) are the generators of the Lie algebra of SU(2), and \( \Lambda \) is a cosmological constant. In this paper we set the Planck length \( l_p = \sqrt{8\pi G \hbar} \) equal to one and restrict the model to flat \( k=0 \) cosmologies. The tensor \( \psi_{ij} \) is symmetric and acts as a Lagrange multiplier enforcing the constraint \( \Sigma^i \wedge \Sigma^j - \frac{1}{3} \delta_{ij} \Sigma^k \wedge \Sigma_k = 0 \). Once this constraint is solved, the action becomes equivalent to the self-dual action of general relativity.

In order to reduce the action (1) to spatial homogeneity and isotropy, we write the action explicitly in terms of coordinates separating space and time as

\[ S_{GR}[A, \bar{E}, B] = \int dt \int d^3x \; \dot{A}^i_a \bar{E}^a_i + A^i_0 D_a \bar{E}^a_i + \epsilon^{abc} B^i_{a} F^i_{bc} \]

\[ - \psi_{ij} \left[ \frac{1}{2} B^i_{a} \bar{E}^a_{ij} + \frac{1}{2} B^j_{a} \bar{E}^a_{ai} - \frac{4}{3} \delta^{ij} B^k_{a} \bar{E}^a_k \right] \]

\[ + \Lambda B^i_{a} \bar{E}^a_i \]  

(2)

where we have introduced the definitions \( \bar{E}^a_{ij} \equiv 2 \epsilon^{abc} \Sigma^i_{bc}, \; B^i_{a} \equiv 2 \Sigma^i_{0a}, \) and \( \epsilon^{abc} = \epsilon^{0abc} \).

Spatially homogeneous and isotropic connections can be characterized as

\[ A^i_{a} = A \Lambda^i_{j} \omega^I_{a} \]  

(3)

where the parameter \( A \) encodes the gauge invariant part of the connection, \( \Lambda^i_{j} \) is an SO(3) matrix, and \( \omega^I_{a} = \omega^I_{a} dx^a \) (\( I = 1, 2, 3 \)) are left invariant one forms with respect to the translational symmetry associated with homogeneity. Since we are restricting ourselves to spatially flat cosmologies, the translation symmetry group is isomorphic to \( \mathbb{R}^3 \). Due to isotropy the time component of the connection \( A^0_{a} \) vanishes identically \(^2\). The matrix \( \Lambda^i_{j} \) is pure gauge

\(^2\) According to the general analysis of symmetric connections for arbitrary symmetry groups\[24\], the time component of the connection is a connection on a symmetry reduced principal fiber bundle. This fiber bundle has as its gauge group the centralizer \( Z_{\lambda} := Z_{G}(\lambda(F)) \) where \( G \) is the gauge group of the theory, \( F \) is the isotropy group, and \( \lambda \) is a homomorphism from \( F \) to \( G \). For our model the gauge group as well as the isotropy group is SU(2). The homomorphism \( \lambda \) is thus the identity map and the centralizer \( Z_{\lambda} \) only contains the identity. The time component of the connection thus vanishes.
and satisfies

\[ \Lambda^i_I \Lambda^j_J = \delta^i_j \]
\[ \epsilon_{ijk} \Lambda^i_I \Lambda^j_J \Lambda^k_K = \epsilon_{IJK}. \]  

(4)

Using left invariant vector fields \( \omega_I = \omega^a_I \partial_a \) dual to \( \omega^I \) we can write the momentum canonically conjugate to the connection as

\[ \tilde{E}^a_i = \sqrt{q_0} E \Lambda^I_i \omega^a_I \]

(5)

where the density weight of \( \tilde{E}^a_i \) is provided by the left invariant metric \( q_{0ab} = \omega^a_I \omega^b_J \delta_{IJ} \). Once again, the gauge invariant part of the canonical momentum is given by a single parameter \( E \). Similarly \( B^i_a \) is given by

\[ B^i_a = B \Lambda^i_I \omega^a_I. \]  

(6)

Written in terms of the reduced variables the homogeneous and isotropic gravitational action becomes

\[ S_{GR}[A, E, B] = \int dt \int d^3 x \sqrt{q_0} E A + 6 \sqrt{q_0} B A^2 + 3 \sqrt{q_0} \Lambda B E \]

\[ = V_0 \int dt \ 3 E A + 6 B A^2 + 3 \Lambda B E \]  

(7)

where \( V_0 \equiv \int d^3 x \sqrt{q_0} \). The constraint term \( \psi_{ij} \left[ B^i_a \tilde{E}_{ai}^j + B^j_a \tilde{E}_{ai}^i - \frac{1}{3} \delta^{ij} B^k_a \tilde{E}_{ak}^a \right] \) in the full action (2) vanishes identically for the isotropic model which can be checked by a direct calculation. Technically \( V_0 \) diverges, since the left invariant metric is constant and the integral is over a non compact manifold. To overcome this, we restrict the integral to a finite cell with volume \( V_0 \) and absorb this factor into the variables as \( E \rightarrow V_0^{2/3} E, A \rightarrow V_0^{1/3} A, \) and \( B \rightarrow V_0^{1/3} B, \) whence the action becomes

\[ S_{GR}(A, E, B) = \int dt \ 3 E \dot{A} + B (6 A^2 + 3 \Lambda E). \]  

(8)

It is clear from the action (8) that \( E \) is canonically conjugate to \( A \) with Poisson bracket \( \{A, E\} = \frac{1}{3}, \) and \( B \) acts as a Lagrange multiplier enforcing the Hamiltonian constraint \( H = 0 \) which here for the gravitational part is just \( H = 6 A^2 + 3 \Lambda E. \) Counting degrees of freedom we have two dynamical variables \( A \) and \( E \) with one first class constraint \( H \) thus giving zero degrees of freedom as expected for isotropic general relativity with no matter.
We now show that the action (8) is equivalent to the standard isotropic action written in terms of a single scale factor $a$ and lapse $N$ with metric $ds^2 = N^2 dt^2 + a^2 dx^2$. In the full BF theory, the densitized metric is given by the Urbantke formula

$$\sqrt{g} g_{\mu \nu} = \frac{2}{3} \epsilon_{ijk} \epsilon^{\alpha \beta \gamma \delta} \Sigma_{\mu \alpha}^{i} \Sigma_{\beta \gamma}^{j} \Sigma_{\delta \nu}^{k}.$$  (9)

Reducing this to isotropy gives

$$g_{00} = \frac{2B^2}{E}$$
$$g_{aa} = \frac{E}{2}.$$  (10)

We thus have the relationships between the two sets of variables

$$E = 2a^2$$
$$B = Na.$$  (11)

In the full theory, the connection is a sum of the spin connection plus the extrinsic curvature. Since we are only considering spatially flat models the spin connection vanishes and the connection is equal to the extrinsic curvature which implies for the isotropic model

$$A = \frac{\dot{a}}{2N}.$$  (12)

With the conventions chosen the action becomes

$$S_{GR} = \int dt \frac{3}{N} \left[ a^2 \ddot{a} + \frac{1}{2} a \dot{a}^2 \right] + 6\Lambda Na^3$$  (13)

which coincides with the standard isotropic action of general relativity up to a total derivative.

III. QUANTUM THEORY

We wish to quantize the symmetry reduced theory in the same manner as loop quantum cosmology by using techniques from the full theory adapted to the symmetry. Thus, the Hamiltonian constraint is represented as an operator on a kinematical Hilbert space using holonomies and fluxes as the basic variables. Physical wave functions are those annihilated by the constraint operator. In addition refined algebraic quantization provides a means for
constructing the physical Hilbert space. In this section we start with a formal introduction to refined algebraic quantization (or the group averaging technique) used to calculate the physical inner product. Next we carry out the program for the cosmological model considered here.

A. Refined Algebraic Quantization

The goal of refined algebraic quantization is to find a method for describing the physical Hilbert space, the space of wave functions annihilated by the constraint and normalizable with a physical inner product. Refined algebraic quantization provides a means for determining both. We start with the formal definitions and notation. The discussion will not serve as a general review of the method, but will instead be tailored to the single constraint system at hand.

The idea of refined algebraic quantization is to build the physical Hilbert space $H_{phys}$ starting from a kinematical Hilbert space $H_{kin}$ acquired by quantizing the theory first ignoring the constraint. The constraint is represented as a self-adjoint operator on this kinematical Hilbert space. As in standard Dirac quantization, the physical states are annihilated by the constraint operator. If the eigenstates of the constraint operator are not normalizable then the physical states lie outside of the kinematical Hilbert space. In this scenario refined algebraic quantization provides a recipe for constructing the physical inner product and characterizing the physical Hilbert space.

For the case where the eigenstates of the constraint operator are not normalizable the physical inner product is constructed as follows. A dense subspace $\Phi \subset H_{kin}$ of the kinematical Hilbert space is chosen and solutions to the constraint equation then lie in the topological dual $\Phi^*$ of $\Phi$. We denote the action of $\Phi^*$ on $\Phi$ using Dirac notation; namely, given an element $\langle \psi | \in \Phi^*$ and an element $|\phi \rangle \in \Phi$ the action is denoted $\langle \psi | \phi \rangle$. For the construction of the physical inner product, an anti-linear map $P : \Phi \rightarrow \Phi^*$ is required such that given a state $|\psi_0 \rangle \in \Phi$, $\langle P(\psi_0) | \in \Phi^*$ is a solution to the constraint equation in the sense that $\langle P(\psi_0) | \hat{H} | \phi_0 \rangle = 0$ for any $|\phi_0 \rangle \in \Phi$. Thus $P$ maps onto the kernel of the constraint operator which consists of elements of $\Phi^*$. Technically $P$ is not a projector since $P^2$ is ill defined. The spaces involved satisfy the triple relation $\Phi \subset H_{kin} \subset \Phi^*$.

With the map $P$ we can now define the physical inner product and characterize the
physical Hilbert space. The physical inner product between two states $\langle P(\phi_0) | P(\psi_0) \rangle \in \Phi^*$ is given by
\[
\langle P(\phi_0) | P(\psi_0) \rangle_{\text{phys}} := \langle P(\psi_0) | \phi_0 \rangle.
\]
where the l.h.s is just notation, and due to the anti-linearity of $P$ the order has been reversed on the r.h.s.. A unique physical state is labeled by an equivalence class of states in $\Phi \subset H_{\text{kin}}$. Two states $|\psi_0\rangle$ and $|\psi'_0\rangle$ in $\Phi$ are equivalent if
\[
\langle \psi_0 | = \langle \psi'_0 | + \langle x_0 |
\]
for some $|x_0\rangle \in \Phi$ satisfying
\[
\langle P(x_0) | x_0 \rangle = 0,
\]
ie. when $\langle x_0 \rangle$ has zero physical norm. Under these conditions the physical inner product is independent of the state $|\psi_0\rangle \in \Phi$ used to represent the physical state in $\langle P(\psi_0) | \in \Phi^*$. The elements of the physical Hilbert space can therefore be labeled by an equivalence class of states in $\Phi$ as defined above.

So far, to construct the physical Hilbert space we have demanded the existence of the map $P$ without providing a method of calculating it. We now describe the group averaging technique which provides such a method. The map is given by
\[
\langle P(\phi_0) | = \int dT \langle \phi_0 | e^{-iT\hat{H}}.
\]
Clearly $P$ maps onto the kernel of the Hamiltonian constraint, that is
\[
\langle P(\psi_0) | \hat{H} | \phi_0 \rangle = 0
\]
for any state $|\phi_0\rangle \in \Phi$. This can be shown by inserting a complete set of eigenstates of the constraint operator $\hat{H}|E_n\rangle = E_n|E_n\rangle$ to get
\[
\langle P(\psi_0) | \hat{H} | \phi_0 \rangle = \int dT \langle \psi_0 | e^{-iT\hat{H}} \hat{H} | \phi_0 \rangle
\]
\[
= \sum_n \int dT \langle \psi_0 | E_n \rangle \langle E_n | e^{-iT\hat{H}} \hat{H} | \phi_0 \rangle
\]
\[
= \sum_n \langle \psi_0 | E_n \rangle \int dT e^{-iT\hat{H}} |E_n \rangle \langle E_n | \phi_0 \rangle
\]
\[
= \sum_n \langle \psi_0 | E_n \rangle E_n \delta(E_n) \langle E_n | \phi_0 \rangle = 0
\]
where in the case where the eigenstates are not normalizable the sum over \( n \) is replaced by an integral.

If the eigenstates are normalizable then the construction of the physical Hilbert space is straightforward. The map \( P \) now is a bona fide operator on the kinematical Hilbert space and behaves as a projector satisfying \( P^2 = P \) projecting onto the kernel of the constraint. The physical inner product becomes

\[
\langle P[\phi_0] | P[\psi_0] \rangle_{\text{phys}} = \langle \psi_0 | P | \phi_0 \rangle \tag{20}
\]

\[
= \langle \psi_0 | P^2 | \phi_0 \rangle \tag{21}
\]

thus the physical inner product is equivalent to the kinematical one in the restriction of \( \mathcal{H}_{\text{kin}} \) to the kernel of \( \hat{H} \). The definition of the equivalence relation remains the same with a simplification arising since \( P^2 = P \).

B. Kinematical Hilbert Space of Loop Quantum Cosmology

We now turn to the cosmological model at hand. The first step required in refined algebraic quantization is to build the kinematical Hilbert space by ignoring any constraints. The kinematical Hilbert space for isotropic loop quantum cosmology has been constructed in [26]. We review the results here.

Using ideas from the full theory, configuration variables are constructed from holonomies. For homogeneity and isotropy the holonomy algebra consists of the set of almost periodic functions; namely, those that can be written

\[
f(A) = \sum_j f_j e^{i \nu_j A / 2} \tag{22}
\]

where the sum contains a finite number of terms \( (j = 1, 2, \cdots N \text{ and } N < \infty) \), \( f_j \in \mathbb{C} \), and \( \nu_j \in \mathbb{R} \). The momentum variables consist of fluxes of the triad operator \( E_i^a \) on a 2-surface which after symmetry reduction are trivially proportional to the parameter \( E \). The kinematical Hilbert space is defined by representing this algebra using the Bohr compactification of the real line \( \mathbb{R}_{\text{Bohr}} \). \( \mathbb{R}_{\text{Bohr}} \) is a compact Abelian group and is equipped with a Haar measure denoted by \( d\mu \) with elements being labeled by \( A \in \mathbb{R} \). The Haar measure can be explicitly written as

\[
\int d\mu = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dA. \tag{23}
\]
The kinematical Hilbert space consists of functions which are square integrable with the Haar measure $d\mu$ which corresponds to the completion of the set of almost periodic functions in the norm of this measure. In the notation of section IIIA, $\Phi$ is the space of almost periodic functions. With this measure an orthonormal basis consists of states $|\nu\rangle$ given by

$$\langle A | \nu \rangle = e^{iA\nu/2}$$

where the parameter $\nu$ spans the entire real line. The basis states are normalizable in contrast to a standard quantum mechanical representation and satisfy the orthonormality condition

$$\langle \nu' | \nu \rangle = \delta_{\nu'\nu}$$

which can be shown using the measure (23). The basis states $|\nu\rangle$ are eigenstates of the triad operator $\hat{E} = -i\frac{1}{3}\frac{\partial}{\partial A}$:

$$\hat{E} |\nu\rangle = \frac{\nu}{6} |\nu\rangle.$$  \hspace{1cm} (26)

Geometrical operators can be built from the triad operator. In particular the volume operator is given by $\hat{V} = |\hat{E}|^{3/2}$ thus $|\nu\rangle$ represents a quantum state with definite volume given by $(|\nu|^{6})^{3/2}$. This provides the physical interpretation of the label $\nu$. Finally we can write the decomposition of the identity for both connection and triad representation as

$$1 = \int d\mu |A\rangle \langle A|$$

$$1 = \sum_{\nu=-\infty}^{\infty} |\nu\rangle \langle \nu|$$

where the sum over $\nu$ is a continuous sum over all values of $\nu$ on the real line. It is important to note that the kinematical Hilbert space here is non-separable being spanned by a non-countable set of basis states $|\nu\rangle$.

\textbf{C. Quantization of the Hamiltonian Constraint}

The homogeneous and isotropic model consists of the Hamiltonian constraint $H_{\text{class}} = 6A^2 + 3\Lambda E + \frac{1}{\sqrt{E}}H_{\text{matter}}^3$. In representing the constraint as a self-adjoint operator on the

\footnote{The factor of $1/\sqrt{E}$ is added such that $H_{\text{matter}}$ agrees with the standard form of the matter Hamiltonian used in other formulations. Note that the constraint used here does not have units of energy hence the need for the extra factor.}
kinematical Hilbert space no operator corresponding to the connection $\hat{A}$ exists; instead connection operators need to be represented in terms of the basic variables which are holonomies along edges. In particular the curvature term in the full action (2), which contributes the $A^2$ term after symmetry reduction, can be represented with holonomies around closed loops. Because of isotropy we can consider holonomies around squares on the manifold with edge lengths chosen to be $\nu_0 V_0^{1/3}$ for some positive parameter $\nu_0$. The holonomy along an edge generated by the left invariant vector field $\omega_I$ is given by

$$h_I = \exp(\nu_0 A \Lambda_I) = \cos(\nu_0 A/2) + 2\Lambda_I \sin(\nu_0 A/2).$$

(28)

The $6BA^2$ term in the action is then represented as

$$6BA^2 \to -\frac{2}{\nu_0^2} \epsilon^{IJK} \text{tr} \left[ B \Lambda_I h_J h_K (h_J)^{-1} (h_K)^{-1} \right] = \frac{24B}{\nu_0^2} \sin^2(\nu_0 A/2) \cos^2(\nu_0 A/2).$$

(29)

In the limit where the closed loops are shrunk to a point by taking $\nu_0$ to zero, the classical expression $6BA^2$ is recovered. In the quantum theory since there is no operator corresponding to the connection $\hat{A}$ we cannot remove $\nu_0$ and thus the parameter remains as a quantum ambiguity. In this paper we will not fix it’s value, but it has been argued that it’s value can be determined from the full theory of loop quantum gravity to be equal to $\sqrt{3}/4$. 

The gravitational part of the Hamiltonian constraint thus is given by

$$\hat{H}_{GR} = \frac{24}{\nu_0^2} \sin^2(\nu_0 \hat{A}/2) \cos^2(\nu_0 \hat{A}/2) + 3\Lambda \hat{E}.$$  

(30)

To determine it’s action on the basis states $|\nu\rangle$ we first consider the action of $\sin(\nu_0 \hat{A}/2)$ and $\cos(\nu_0 \hat{A}/2)$. Since we have $\langle A | \nu \rangle = \exp(iA\nu/2)$ we have

$$\sin(\nu_0 \hat{A}/2) |\nu\rangle = \frac{1}{2i} \left( |\nu+\nu_0\rangle - |\nu-\nu_0\rangle \right)$$

$$\cos(\nu_0 \hat{A}/2) |\nu\rangle = \frac{1}{2} \left( |\nu+\nu_0\rangle + |\nu-\nu_0\rangle \right)$$

(31)

and thus the action of $\hat{H}_{GR}$ is

$$\hat{H}_{GR} |\nu\rangle = -\frac{3}{2\nu_0^2} \left( |\nu+4\nu_0\rangle - 2|\nu\rangle + |\nu-4\nu_0\rangle \right) + \frac{\Lambda \nu}{2} |\nu\rangle.$$  

(32)

The action of the curvature part of the constraint operator is thus seen to be a discrete approximation to an operator corresponding to the second derivative with respect to $\nu$ and in the limit of small $\nu_0$ approaches what would be the standard quantum mechanical operator $6\hat{A}^2 = -24\frac{\partial^2}{\partial \nu^2}$. 

12
D. Path Integral Representation of $P$

Now we explicitly compute the group averaging formula (17) and give a definition of the map $P$. The matrix elements of the map $P(\nu'', \nu') = \langle \nu'' | \hat{P} | \nu' \rangle$ can be written in terms of a path integral amplitude. The integrand looks like an ordinary quantum mechanical propagator and thus can discretized using the standard derivation. Decompositions of the identity (27) are inserted at $n$ discrete time slices $t_k = k\epsilon$ with time step $\epsilon = \frac{T}{n+1}$ to get the propagator

$$\langle \nu'' | e^{-i\hat{H}T} | \nu' \rangle = \lim_{\epsilon \to 0} \frac{T}{\epsilon} \prod_{k=1}^{T/\epsilon} \left\{ \sum_{\nu_k, \nu_0} \int d\mu_k \ e^{i \sum_{k=1}^{T/\epsilon} \nu_k (A_{k+1}-A_k)/2 - \epsilon H(A_k, \nu_k)} \right\}$$

with $\nu_1 = \nu'$ and $\nu_{n+1} = \nu''$ and $S$ being the discretized classical action $S = \sum_{k=1}^{T/\epsilon} \nu_k ((A_{k+1}-A_k)/2 - \epsilon H(A_k, \nu_k))$. Notice that the functional integration over the triad $E$ is discretized as continuous sums over it’s eigenvalues at the time slices $\nu_k/6$ and that $H(A_k, \nu_k) = \frac{24}{\nu_0^2} \sin^2(\nu_0 A_k/2) \cos^2(\nu_0 A_k/2) + \Lambda \nu_k/2 + \frac{1}{\sqrt{\nu_k/6}} H_{\text{matter}}$. We note that this path integral amplitude is equivalent to the path integral amplitude arrived at after gauge fixing due to reparameterization invariance of the classical action.

Putting these ideas together, the matrix elements of the map $P$ are calculated through the path integral as

$$P(\nu'', \nu') = \int dT \lim_{\epsilon \to 0} \frac{T}{\epsilon} \prod_{k=1}^{T/\epsilon} \left\{ \sum_{\nu_k, \nu_0} \int d\mu_k \ e^{i S} \right\}.$$  (34)

We then use this to map any state $|\psi_0\rangle = \sum_\nu \psi_{0,\nu} |\nu\rangle \in \Phi$ onto a solution of the constraint equation as in

$$\langle P(\psi_0) | = \int dT \langle \psi_0 | e^{-i\hat{H}T}$$

---

4 Halliwell [27] has considered the path integral for actions that are reparameterization invariant such as the action presented here. Starting with the path integral $P = \int DB \exp\left[i \int_0^{T'} dt 3E \dot{A} + BH \right]$ the one form component $B$ is gauge fixed to be constant in time and after including ghost terms to make the path integral independent of the gauge choice the path integral becomes $P = \int dB(t'' - t') \int DB \exp\left[i \int_0^{T''} dt 3E \dot{A} + BH \right]$ which with the redefinition $T = B(t'' - t)$ and $\tilde{B} = B(t - t')$ takes on the form equivalent to the group averaging one $P = \int dT \int DB \exp\left[i \int_0^{T} dt 3E A + H \right] = \int dT \langle \nu'' | e^{-i\hat{H}T} | \nu' \rangle$.
\[
= \sum_{\nu'', \nu'} \langle \psi_0 | \nu'' \rangle P(\nu'', \nu') \langle \nu' | \\
= \sum_{\nu'', \nu'} \overline{\psi_{0\nu''}} P(\nu'', \nu') \langle \nu' | .
\]

The physical inner product between two physical states \( \langle P(\phi_0) | P(\psi_0) \rangle \) is given by

\[
\langle P(\phi_0) | P(\psi_0) \rangle_{phys} = \langle \psi_0 | P | \phi_0 \rangle = \sum_{\nu''} \langle \psi_0 | \nu'' \rangle \langle \nu'' | P | \nu' \rangle \langle \nu' | \phi_0 \rangle = \sum_{\nu''} \overline{\psi_{0\nu''}} P(\nu'', \nu') \phi_{0\nu'} .
\]

With the equivalence relations of equations (15) and (16), we have a concrete recipe for characterizing the physical Hilbert space of the model.

IV. EXAMPLE - COSMOLOGICAL CONSTANT

A. Physical Hilbert Space

As an example we consider the simplest non-trivial model with a positive cosmological constant and no other forms of matter. In this section we will show that this model can be solved exactly in the connection representation and we will explicitly construct the physical Hilbert space. In the following section we transform to the triad representation in order to discuss the relationship to loop quantum cosmology.

We have already derived the form of the constraint which is given by

\[
\hat{H}_{GR} = \frac{24}{\nu_0^2} \sin^2(\nu_0 \hat{A}/2) \cos^2(\nu_0 \hat{A}/2) + 3\Lambda \hat{E} \\
= \frac{24}{\nu_0^2} \sin^2(\nu_0 \hat{A}/2) \cos^2(\nu_0 \hat{A}/2) - i\Lambda \frac{\partial}{\partial A} .
\]

Because of the simplicity of the constraint operator we can solve exactly for the eigenstates which are given by solutions \( |E_n\rangle \) satisfying \( \hat{H}|E_n\rangle = E_n|E_n\rangle \). This differential equation can be solved to give

\[
E_n(A) = \exp \left[ i \frac{A}{\Lambda} \left( E_n - \frac{3}{\nu_0^2} \right) \right] \exp \left[ i \frac{3 \sin(2\nu_0 A)}{2\Lambda \nu_0^3} \right]
\]

where the eigenvalues \( E_n \) are valued on the real line. The eigenstates are pure phase, but they are normalizable in the kinematical inner product. This is despite the fact that any
$E_n \in \mathbb{R}$ lies in the spectrum of $\hat{H}$. Since the eigenstates are normalizable we expect $P$ to be a bona fide projection operator. The kinematical norm is

$$\langle E_n | E_n \rangle = \int d\mu \overline{E_n(A)} E_n(A)$$

$$= \int d\mu = 1$$

and thus all of the eigenstates have kinematical norm equal to one. Following the general discussion of refined algebraic quantization, the physical Hilbert space consists of the space of zero ‘energy’ eigenstates and the physical inner product is equal to the kinematical one restricted to this space. We see that this space is one dimensional spanned by the solution $|E_0\rangle$ given by

$$E_0(A) = \exp \left( -i \frac{3A}{\nu_0^2 \Lambda} + i \frac{3\sin(2\nu_0 A)}{2\nu_0^3 \Lambda} \right).$$

We can also write out the matrix elements of the map $P(A'', A') = \langle A'' | P | A' \rangle$. Since the map projects onto the zero ‘energy’ eigenstate and satisfies $P^2 = P$ the matrix elements must be of the form

$$P(A'', A') = E_0(A'') \overline{E_0(A')}$$

We thus see that despite the fact that the kinematical Hilbert space is very large, the physical Hilbert space is one-dimensional which is what we expect from a theory with zero degrees of freedom. In the next subsection we will see that in the triad representation there exist a non-countable set of solutions to the constraint equation—this is in fact generally the case in standard loop quantum cosmology—but we will show that the extra solutions all have zero physical norm and thus the physical Hilbert space is one-dimensional.

**B. Triad Representation**

In standard loop quantum cosmology a simple formula for the Hamiltonian constraint in the connection representation does not exist hence the constraint and solutions are more readily given in the triad representation. The space of solutions in standard loop quantum cosmology is non-separable whereas the physical Hilbert space of the model presented here is separable consisting of a single solution. Our aim in this section is to reconstruct the physical Hilbert space in the triad representation to facilitate a comparison with standard loop quantum cosmology. In particular the map $P$ is calculated explicitly using the path integral and picks out a single state among all the solutions to the constraint equation.
In the triad representation the action of the Hamiltonian constraint operator was given in equation (32)

\[ \hat{H} |\nu\rangle = -\frac{3}{2\nu_0^2} \left( |\nu+4\nu_0\rangle - 2|\nu\rangle + |\nu-4\nu_0\rangle \right) + \frac{\Lambda}{2} \nu |\nu\rangle. \]  

We decompose a state lying in the dual of the kinematical Hilbert space \( \langle \psi \mid \) as

\[ \langle \psi \mid = \sum_\nu \psi_\nu \langle \nu \mid. \]  

The constraint equation \( \langle \psi \mid H^\dagger = 0 \) leads to a difference equation for the coefficients \( \psi_\nu \)

\[ \psi_{\nu+4\nu_0} - 2\psi_\nu + \psi_{\nu-4\nu_0} = \frac{1}{3} \Lambda \nu_0^2 \nu \psi_\nu. \]  

It is immediately seen that the difference equation splits into an infinite number of isolated sectors with values of \( \nu \) separated by an integer times \( 4\nu_0 \). We label each sector by the parameter \( \delta \in [0, 4\nu_0) \). Thus the sector labeled by \( \delta \) corresponds to values of \( \nu \) equal to \( \ldots, \delta, 4\nu_0+\delta, 8\nu_0+\delta, \ldots \). The difference equation for each sector is then a second order equation which can be solved exactly and the general solutions are given by

\[ \psi_\nu^\delta = C_1^\delta J_{\nu/(4\nu_0)+3/(2\nu_0^3)} \left( \frac{3}{2\nu_0^3} \right) + C_2^\delta Y_{\nu/(4\nu_0)+3/(2\nu_0^3)} \left( \frac{3}{2\nu_0^3} \right) \]  

where \( J \) and \( Y \) are the Bessel functions of the first and second kind respectively and \( C_{1,2}^\delta \) are constants. There are thus an infinite number of solutions determined by the infinite number of parameters \( C_{1,2}^\delta \) for \( \delta \in [0, 4\nu_0) \).

The matrix elements of the map \( P \) can be solved exactly for this model using the path integral (34). The discretized action is

\[ S = \sum_{k=0}^{T/\epsilon} \frac{\nu_k}{2} (A_{k+1} - A_k) + \epsilon \left[ 6 \sin^2(\nu_0 A_k) / \nu_0^2 + \Lambda \nu k / 2 \right] \]

\[ = \sum_{k=0}^{T/\epsilon} \frac{\nu_k}{2} [A_{k+1} - A_k + \epsilon \Lambda] - 6 \epsilon \sin^2(\nu_0 A_k) / \nu_0^2 \]  

thus the sums over \( \nu_k \) can be performed since they are all of the form \( \exp \left[ i\nu_k (A_{k+1} - A_k + \epsilon \Lambda) / 2 \right] \) which give delta functions. The result is

\[ \langle \nu'' | e^{-i\hat{H}T} | \nu' \rangle = \lim_{\epsilon \to 0} \prod_{k=1}^{T/\epsilon} \left\{ \int_{-\infty}^{\infty} dA_k \delta(A_{k+1} - A_k + \epsilon \Lambda) e^{i\nu'' A_n / 2} e^{-i\nu' A_0 / 2} \right\}. \]  

16
Let us then integrate over the \( A_k \) using the delta functions except for \( A_0 \) and \( A_n \). This leaves us with one remaining delta function

\[
\langle \nu'' | e^{-i\hat{H}T} | \nu' \rangle = \int_{-\infty}^{\infty} dA_0 \, dA_n \, \delta(A_n - A_0 + T\Lambda) \, e^{i\nu''A_n/2} \, e^{-i\nu'A_0/2} \exp\left[ \frac{3i}{\nu_0} \left( T - \frac{\sin(T\nu_0\Lambda)\cos(2\nu_0A' - T\nu_0\Lambda)}{\nu_0\Lambda} \right) \right]. \tag{48}
\]

The path integral is then given by

\[
P(\nu'', \nu') = \int_{-\infty}^{\infty} dA_0 \, dA_n \, e^{i\nu''A_n/2} \, e^{-i\nu'A_0/2} \int dT \, \delta(A'' - A' + T\Lambda) \, e^{i\frac{3\nu}{\nu_0} \left( T - \frac{\sin(T\nu_0\Lambda)\cos(2\nu_0A' - T\nu_0\Lambda)}{\nu_0\Lambda} \right)}
\]

\[
= \int_{-\infty}^{\infty} dA_0 \, dA_n \, e^{i\nu''A_n/2} \, e^{-i\nu'A_0/2} \, e^{\left( \frac{3i\sin(2\nu_0A'' \nu_0)}{2\nu_0\Lambda} - \frac{3i\Lambda'}{\nu_0\Lambda} \right)} \, e^{\left( -\frac{3i\sin(2\nu_0A')}{2\nu_0\Lambda} + \frac{3\Lambda'}{\nu_0\Lambda} \right)}
\]

\[
= \int_{-\infty}^{\infty} dA_0 \, dA_n \, e^{i\nu''A_n/2} \, e^{-i\nu'A_0/2} \, E_0(A'') \, \overline{E_0(A')}, \tag{49}
\]

which is just the Fourier transform of the map from the previous section. \( P \) thus maps onto the single normalizable zero 'energy' eigenstate \(|E_0\rangle\).

We now formulate the physical Hilbert space by using the map \(49\). \( P \) maps onto a single state in the triad representation which is given by the Fourier transform of the zero 'energy' eigenstate \(|E_0\rangle\). Denoting \(|E_0\rangle = \sum_{\nu} p_{\nu} |\nu\rangle\) a calculation gives for the Fourier transform

\[
p_{\nu} = \begin{cases} 
J_{\nu/(4\nu_0)} + 3/(2\Lambda\nu_0^2) & \frac{\nu}{4\nu_0} + \frac{3}{2\Lambda\nu_0^2} \in \mathbb{Z} \\
0 & \frac{\nu}{4\nu_0} + \frac{3}{2\Lambda\nu_0^2} \notin \mathbb{Z}.
\end{cases} \tag{50}
\]

Remarkably, the path integral solution is non zero in precisely one sector \( \delta_c \) corresponding to values of \( \nu \) where the Bessel function order \( \frac{\nu}{4\nu_0} + \frac{3}{2\Lambda\nu_0^2} \) is an integer. The value of the special sector is thus given by \( \delta_c = 4\nu_0 \left[ 1 - \text{mod} \left( \frac{3}{2\Lambda\nu_0^2}, 1 \right) \right] \) where \( \text{mod}(a, b) \) is the remainder of \( a/b \). That the Fourier transform picks out certain modes can be seen from the fact that the state \( E_0(A) \) consists of a phase factor multiplied by a periodic function. The matrix elements of \( P \) in the triad representation are thus given by \( P_{\nu'\nu} = \langle \nu | P | \nu' \rangle = p_{\nu} \overline{p}_{\nu'} \). Using this form of the map, any kinematical state will get projected onto the zero 'energy' eigenstate given in equation \(50\). The uncountable extra solutions \(45\) to the constraint equation in the triad representation are to be mod out of the physical Hilbert space. In this way the projector has reduced what was initially a non-separable space of solutions to the constraint equation to a one dimensional Hilbert space. This is because the generic solutions of the constraint correspond to zero physical norm states and therefore are equivalent to the zero state in the physical Hilbert space.
The previous statement requires a more precise explanation as the generic solutions (45) are not normalizable in terms of the measure (23). These solutions have zero physical norm in the following sense: The generic solutions (45) can be thought of as elements of $\Phi^*$ defined in Section 3.1A. Given a generic solution $\psi^\delta \in \Phi^*$ (as in (45)) with values only in the sector $\delta$, we define the shadow state $\psi^\delta_N$ with $N \in \mathbb{N}$ as

$$\psi^\delta_N = \begin{cases} 
C_1^\delta J_{\frac{\nu}{4\nu_0} + \frac{\delta}{2\nu_0}} \left( \frac{3}{2\Lambda \nu_0} \right) + C_2^\delta Y_{\frac{\nu}{4\nu_0} + \frac{\delta}{2\nu_0}} \left( \frac{3}{2\Lambda \nu_0} \right) & \text{for } \nu = 4n\nu_0 + \delta \text{ with } n \in [-N, N] \\
0 & \text{elsewhere}
\end{cases}$$

for $\nu = 4n\nu_0 + \delta$ with $n \in [-N, N]$. Clearly the shadow state $\psi^\delta_N$ is an almost periodic function and therefore $\psi^\delta_N \in \Phi$. Notice also that as $N \to \infty$, $\psi^\delta_N$ approaches $\psi^\delta \in \Phi^*$. The statement that the generic solutions have zero norm corresponds to the statement that

$$\langle P(\psi^\delta_N) \mid P(\psi^\delta_N) \rangle_{\text{phys}} = 0$$

for $\delta \neq \delta_c$ where again $\delta_c = 4\nu_0 \left[ 1 - \text{mod} \left( \frac{3}{2\Lambda \nu_0}, 1 \right) \right]$ according to (50).

From the form of equation (49) it is clear that $P^2 = P$ and hence the kinematical inner product can be used as the physical inner product. We can see explicitly that the solution (50) is normalizable by calculating it’s norm

$$\langle E_0 \mid E_0 \rangle = \sum_\nu \frac{1}{p_{\nu} p_{\nu}} \sum_{m=-\infty}^{\infty} \left[ J_m \left( \frac{\nu}{2\nu_0} \right) \right]^2 = 1$$

using a known formula for the sum over Bessel functions of integer order.

The physical solution (50) is plotted in figure (1). We see that despite the discrete nature of the solution, it approximates a continuous one which for large values of $\nu$ would approximate the Wheeler-DeWitt equation. For positive values of $\nu$ the solutions decays rapidly which would be expected for the Riemannian model which has no classically allowed region for an effective positive cosmological constant. For negative $\nu$ the solution oscillates with behavior indicating an effective negative cosmological constant. In the discussion that follows we will indicate why this occurs and the interpretation for the Lorentzian regime.

\footnote{Notice that the effect of the cosmological constant term depends on the orientation of the triad which is dynamical for our action.}
FIG. 1: Physical solution. The dots indicate the values of the solution for where $\nu + \frac{\pi}{2}\nu_0 \in \mathbb{Z}$. The actual solution would be zero elsewhere, however the solid lines are interpolated values designed to bring out the behavior of the solution more clearly.

Furthermore, we can see that the classical singularity ($\nu = 0$) does not present a barrier to the evolution as the wave function can be evolved through it.

V. DISCUSSION

The simplicity of the Hamiltonian constraint used here has allowed for explicit calculations to be performed. A rigorous definition of the physical scalar product was given and used to factor out spurious solutions to the constraint equations with zero physical norm. We wish to compare the results of the model with those of standard loop quantum cosmology. To do so, we observe that there is a simple relationship between the Riemannian and Lorentzian theories for homogeneous and isotropic models. Using this simple relation we can compare the two Hamiltonian constraints and the physical implications of the differences. In addition, we will discuss the cosmological predictions of such a model.

The Lorentzian model and relation to standard LQC

A Lorentzian model can be constructed if one uses general properties of the isotropic reduction of general relativity. With homogeneity and isotropy the curvature part of the Hamiltonian constraint changes sign while the cosmological constant remains the same when
going from the Riemannian sector to the Lorentzian one. Using this property the Hamiltonian constraint for the Lorentzian constraint becomes $H_L = -6A^2 + 3\Lambda E$ as opposed to the Riemannian constraint $H_E = 6A^2 + 3\Lambda E$. Equivalently the same effect is obtained by changing the sign of the triad $E$ (which amounts to sending $\nu$ to $-\nu$ in the solutions presented here).

If we extend our model in such a fashion the difference equation becomes

$$
\psi_{\nu+4\nu_0} - 2\psi_\nu + \psi_{\nu-4\nu_0} = \frac{1}{9}\gamma^3 \nu_0^2 l_p^2 \Lambda \nu \psi_\nu
$$

where we have included the various constants needed to compare to standard loop quantum cosmology. Here $\gamma$ is the Barbero-Immirzi parameter, $l_p = \sqrt{\kappa \hbar}$ is the Planck length, and $\kappa = 8\pi G$. The difference equation for standard loop quantum cosmology is given by

$$
(V_{\nu+5\nu_0} - V_{\nu+3\nu_0}) \psi_{\nu+4\nu_0} - 2 (V_{\nu-\nu_0} - V_{\nu+\nu_0}) \psi_\nu + (V_{\nu-3\nu_0} - V_{\nu-5\nu_0}) \psi_{\nu-4\nu_0} = -\frac{1}{3}\gamma^3 \nu_0^2 l_p^2 \Lambda sgn(\nu) V_\nu \psi_\nu
$$

where the volume eigenvalues are $V_\nu = (|\nu| l_p / 6)^{3/2}$, and $sgn$ stands for the sign function. That the two constraints are asymptotically the same can be shown by using the large volume expansion for the volume difference coefficients $V_{\nu+\nu_0} - V_{\nu-\nu_0} \approx (\gamma l_p / 6)^{3/2} 3\sqrt{\nu_0}$. Plugging this approximation into (55) then gives the difference equation (54). The two models would thus share the same large volume semi-classical limit.

The differences between the two difference equations can be traced back to the classical Hamiltonian constraints used. The constraint of standard loop quantum cosmology is

$$
H_{LQC} = -\frac{6}{\kappa \gamma^2} A^2 sgn(E) \sqrt{|E|} + \frac{\Lambda}{\kappa} sgn(E)(|E|)^{3/2}
$$

whereas the constraint used here would be

$$
H = -\frac{6}{\kappa \gamma^2} A^2 + \frac{\Lambda}{\kappa} E.
$$

It is readily seen that for positive values of the triad the two constraints are proportional to each other and thus are classically equivalent on the constraint surface where $H = 0$. The main qualitative difference between the two constraints is that they are not equivalent for negative orientations of the triad. This is indicated in the difference equation by the presence of the $sgn(\nu)$ term on the right hand side of equation (55). The factor $sgn(\nu)$ is put in by hand in standard LQC such that the constraint is classically symmetric for both
orientations of the triad. However, there is a freedom in extending the classical phase space to include negative orientations of the triad. We have shown that the action (11) does not lead to the $\text{sgn}(\nu)$ term. In the next subsection we discuss in more detail the implications of such a change.

The other crucial difference between the constraints is that our constraint is self-adjoint. This allows us to carry out the group averaging technique to construct the physical inner product. The constraint of standard loop quantum cosmology is not, although self-adjoint constraints have been proposed [28, 29].

The triad orientation ambiguity

We have stated the main qualitative difference between the simpler model lies in the classical extension to negative orientations of the triad. This ambiguity, which is manifested by the presence of the $\text{sgn}(\nu)$ term in the difference equation (55), has important consequences in the quantum theory. We now show that absence of the $\text{sgn}(\nu)$ term — forced on us by the action (11) — leads to many attractive features in the quantum theory.

The physical wave function solutions of the two difference equations are shown in figure 2. For large positive values of the triad (positive $\nu$) the solutions behave the same. However, for negative $\nu$ the solutions are entirely different owing to the $\text{sgn}(\nu)$ term. Because of that term, the solution in figure 2(b) is symmetric and thus both orientations of the triad have the same semi-classical limit. On the other hand, the solution of the simplified constraint clearly does not have the right semi-classical limit for negative orientations of the triad. At first one might think that this is undesirable in the quantum theory since a large portion of the quantum configuration space does not have the right semi-classical limit. However, we see that the solution for negative triad is rapidly suppressed indicating a classically forbidden region. Thus, the universe would have very low probability for being in the region and would instead more likely be found in the semi-classical regime of positive $\nu$. Notice that because $P^2 = P$, the standard notion of probability amplitude can be associated to the physical wave function for the single partial observable $E$. Physically the cosmological constant term in (57) changes its sign when $E$ goes to $-E$. The regions $\nu > 0$ and $\nu < 0$ correspond to an effective positive and negative cosmological constant respectively.

It is precisely because of the absence of the $\text{sgn}(\nu)$ term that the model can be solved
FIG. 2: Difference equation solutions for both the simplified constraint (a) and standard loop quantum cosmology (b). The solution of (a) is the physical solution. The solution in (b) is the one picked out by the dynamical initial condition. The solid line are the interpolated values. The actual physical solution of (a) would be zero except for the one sector indicated by the dots. For (b) the dynamical initial condition specifies that the wave function in the other sectors is the one that smoothly interpolates between the dotted solution.

exactly. If we were to use a constraint with the $sgn(\nu)$ term the path integral would not be solvable explicitly and it can be shown that there do not exist any kinematically normalizable solutions. The fact that the physical solution is kinematically normalizable simplified the construction of $H_{phys}$. Without kinematically normalizable solutions the path integral needs to be solved in order to construct the physical inner product. The normalizability of the solutions can be better understand by looking at the large volume behavior of the solutions. As in the Riemannian model the general solution of the simplified model is

$$
\psi^\delta_\nu = C^\delta_1 J_{-\nu/(4\nu_0)+9/(2\gamma^3 l_0^2\Lambda\nu_0^3)} \left( \frac{9}{2\gamma^3 l_0^2\Lambda\nu_0^3} \right) + C^\delta_2 Y_{-\nu/(4\nu_0)+9/(2\gamma^3 l_0^2\Lambda\nu_0^3)} \left( \frac{9}{2\gamma^3 l_0^2\Lambda\nu_0^3} \right).
$$

The Bessel Y solutions all diverge for negative $\nu$ and the Bessel J solutions all diverge for positive $\nu$ except for the one sector where the Bessel function order is an integer as it was in the Riemannian solution (50). The Bessel J solutions are plotted for large $\nu$ in figure 3. It is clear that only in the special sector $\delta = \delta_c$ does the solution not diverge. If we were to include the $sgn(\nu)$ term then the oscillatory solutions would exist for positive and
negative $\nu$, however we could not match a convergent solution for positive $\nu$ with one for negative $\nu$. We thus see that without the sgn($\nu$) term, the model has very attractive features and the construction of the physical Hilbert space is greatly simplified since there exists a normalizable solution.

![Graph](image)

(a) $-\nu/(4\nu_0) + 9/(2\gamma^3 l_p^2 \Lambda \nu_0^3) \in \mathbb{Z}$

(b) $-\nu/(4\nu_0) + 9/(2\gamma^3 l_p^2 \Lambda \nu_0^3) \notin \mathbb{Z}$

FIG. 3: Bessel J solution of (58) for two different sectors. The solution of (a) is for the sector where the Bessel function order is an integer. This solution converges both for large negative and positive $\nu$. The solution of (b) is for a different sector and diverges for large $\nu$. The solid lines are interpolated values.

A question remains as to the relevance of the simplified constraint to that of standard LQC: are these special results specific to the simplified model? If we remove the sgn($\nu$) term from the difference equation of standard LQC, it can be shown that the model with a cosmological constant does indeed exhibit similar behavior. To show this, the difference equation must be solved numerically since it is more complicated. Consider that we start at large negative $\nu$ with some initial conditions of the wave function and we evolve forward toward the classical singularity. For large negative $\nu$ the two independent solutions of the difference equation behave in a characteristic manner. The difference equation for large $|\nu|$ is approximated by the difference equation of our simplified model. Therefore, for large and negative $\nu$ the two independent solutions behave as $f_+(\nu) \approx \nu^{-(\nu+1/2)}$ and $f_-(\nu) \approx \nu^{\nu-1/2}$ which can be shown from the asymptotic expressions of the solutions in equation (58). For generic initial conditions the solution will be a linear combination of $f_+$ and $f_-$. As we evolve toward the singularity $f_+$ rapidly dominates over $f_-$. In this manner numerically the
contribution of the decaying solution $f_-$ becomes negligible in comparison with the contribution of $f_+$ thus effectively selecting the latter. We can perform the same trick starting at large positive $\nu$—where the independent solutions behave as $g_+(\nu) \approx (-1)^{\frac{\nu}{4\nu_0}} \nu^{(\nu-1)/2}$ and $g_-(\nu) \approx (-1)^{\frac{\nu}{4\nu_0}} \nu^{-(\nu+1)/2}$—and evolving backward to select the $g_+$ component. The two solutions can then be tested to see if they match somewhere in the region where the behavior is oscillatory. If the two can be matched then we have found a solution that decays both for positive and negative $\nu$.

When this analysis is performed for the difference equation of LQC with a cosmological constant (without the $\text{sgn}(\nu)$ term) the result is a match only in one sector which corresponds approximately to the one picked out by our simplified model and given by $-\nu/(4\nu_0) + 9/(2\gamma^2 \Lambda \nu_0^3) \in \mathbb{Z}$. In the other sectors the matching cannot be achieved which implies for instance that a solution that decays toward $-\infty$ will evolve into one that diverges toward $+\infty$. Since the falloff behavior at large positive and negative $\nu$ is sufficiently fast we can say that the solution that does not diverge is kinematically normalizable as in our simplified model. The solution is plotted in figure (4). It has the same qualitative behavior as the normalizable solution of the simplified constraint (compare to figures 2(a) and 3(a)).

We are currently investigating whether a normalizable solution exists for a massless scalar field in the closed model. A recent paper has pointed out that in this model all the solutions diverge either for large positive or negative $\nu$ [30]. The question is raised that this implies that the wave function then predicts a large ‘probability’ for the universe being in those regions which are classically forbidden for this model. Notice however that any question about probabilities can only be addressed if a notion of physical inner product is provided. The physical inner product can solve this apparent problem by providing a non trivial probability measure that makes the physical states normalizable. We have shown for our model that most solutions have divergent behavior yet the physical inner product singles out a solution— all the other ill-behaved solutions have zero norm—which is finite everywhere and even kinematically normalizable. Therefore it is interesting to explore whether this might occur with the model of [30]. If a normalizable solution can be found which decays for large volume then the problems raised would be shown to be nonexistent. Preliminary results indicate that one might be able to find normalizable solutions for this model again provided the $\text{sgn}(\nu)$ term is removed.
FIG. 4: Normalizable solution to the difference equation of LQC (55) without the $\text{sgn}(\nu)$ on the r.h.s. Here matter is only in the form of a cosmological constant. The solution is valued only in a single sector which approximately is given by the same sector of the simplified constraint where $-\nu/(4\nu_0) + 9/(2\gamma^3 l_p^2 \Lambda \nu_0^3) \in \mathbb{Z}$.

**Cosmological implications**

We now discuss the cosmological implications of the Lorentzian model. A main question is whether or not the quantum theory cures the classical singularity. A classical singularity would exhibit itself in the quantum theory in two ways: a place where the difference equation breaks down or a place where the operator corresponding to the inverse scale factor is singular. For both these criteria the model presented here does not exhibit singular behavior. We see from the figure 2(a) that the quantum evolution proceeds smoothly through the classical singularity ($\nu = 0$). From the standpoint of the difference equation the singularity point is not special. In addition the operator corresponding to the inverse scale factor is a kinematical operator which has been shown to be bounded in standard loop quantum cosmology [31]. Since the model here shares the same kinematical Hilbert space, the results will be the same. We furthermore see that the physical solution is valued only on discrete values of $\nu$ corresponding to the special sector $\delta_c$. The special sector depends on the value of the cosmological constant and only specific values will select the sector $\delta = 0$ which passes through $\nu = 0$. Thus, it is most likely that the physical solution completely avoids the singularity. The singularity is thus cured in the model.

If the model does not exhibit singular behavior then another question is what happens
when the universe approaches the singularity. The plot in figure 2(a) shows that the solution decays rapidly for negative values of $\nu$ while it oscillates for positive values. Thus it is natural to interpret the region of negative $\nu$ as a classically forbidden region and the singularity as a barrier off which the universe bounces. This is in contrast to standard loop quantum cosmology where semi-classical universes exist for both negative and positive $\nu$ thus opening the possibility for the universe to tunnel through the singularity from one region to the other. Again, the main difference arises from the $\text{sgn}(\nu)$ term in the difference equation. From this we can make the interpretation that a classical collapsing universe approaching the singularity completely avoids it and bounces leading to an expanding universe.

Finally, there is the issue of boundary proposals. In Wheeler-DeWitt quantization there exist two independent solutions for de Sitter space and ad-hoc boundary proposals are added to pick out a solution. In standard loop quantum cosmology the dynamical initial condition is used to pick out a solution. However, the dynamical initial condition only singles out a solution for the sector that passes through the singularity. Semi-classical arguments are then used to fix the wave function in the other sectors (see figure 2(b)). In contrast we see that for the simplified model here, it is the physical inner product that selects a unique wave function. There is no need to supplement the quantum theory with an ad-hoc boundary proposal. This is not surprising for this model as the constraint equation is first order in the connection representation. The question remains as to whether this is relevant in more general situations.

Further implications for LQC

At this stage and in the context of the model presented here one question seems natural: what have we gained by setting up the theory on the non separable kinematical Hilbert space given by the Bohr compactification of the real line? The answer is clear in that considering quasi-periodic functions of $A$ is the analog of considering cylindrical functions of the connection defined on arbitrary graphs in the full theory. Had we defined the quantum theory by using functions of a fixed periodicity (allowing $\nu = \nu_0 n$ for $n \in \mathbb{Z}$) we would have missed the physical solution $E_0(A)$ and the physical Hilbert space would have been 0-dimensional. Only in the special case when $(3/2)\Lambda^{-1}\nu_0^{-3}$ is an integer could we have found the physical state by starting with a formulation on a fixed ‘lattice’, i.e., where
\( \nu = \nu_0 n \) (see Equation (50)). In any other case using a fixed lattice would have resulted in a zero dimensional physical Hilbert space. In our model, the physical inner product selects a given set of ‘graphs’ by selecting a periodicity of the relevant modes. Thus the Bohr compactification is necessary to capture the correct physics.

Another related issue is that the physical Hilbert space separates into orthogonal subspaces. In the standard quantizations of the Hamiltonian constraint in the literature the curvature term is written in terms of holonomies along paths whose length is determined by the parameter \( \nu_0 \). Due to this fact one introduces an intrinsic periodicity to the constraint. It is easy to see that the generalized projection operator associated to such a quantization satisfies the following property

\[
\langle \nu'' | P | \nu' \rangle = \int dT \langle \nu'' | e^{-i\hat{H}T} | \nu' \rangle = 0
\]

for \( \nu'' - \nu' \notin 4\nu_0 n \) for \( n \in \mathbb{Z} \). In this manner the physical Hilbert space is separated into isolated sectors with no quantum interference between them.

![Wave function of standard LQC (dots) compared with the corresponding Wheeler-DeWitt solution (solid line) for positive cosmological constant. The deviation occurs when the extrinsic curvature is of the order of \( \pi/\nu_0 \) which in this case occurs for large volumes.](image)

A direct consequence of this is the fact that in a given sector the physical solutions have a built in periodicity in that \( \Delta \nu = 4\nu_0 \) which implies that physically \( A \leq A_{\text{max}} = \pi/\nu_0 \). This is a puzzling feature as \( A \) is directly related to the extrinsic curvature which classically does not have such a bound. In the model with a cosmological constant, classically the extrinsic curvature grows as the volume of the universe increases until at some volume it
reaches a critical value corresponding to \( A = A_{\text{max}} = \pi/\nu_0 \). At values of \( \nu \) on the order of this critical value the behavior of the wave function changes dramatically and deviates from that expected from standard Wheeler-DeWitt quantization. This is a generic property that applies to all models of LQC (see figure 5). However, it is in this region that both prescriptions should coincide. By decreasing the value of \( \nu_0 \) one can extend the region where LQC and Wheeler-DeWitt quantization agree, yet this parameter is argued to be fixed by the full theory by considering the smallest area eigenvalue [26]. It is surprising that a parameter arising from the fundamental discreetness at Planck scale should have such an important effect at large scales where the universe is expected to behave classically.

**Acknowledgments**

We would like to thank Martin Bojowald for proposing this problem and Abhay Ashtekar and Martin Bojowald for useful discussions. This work has been supported by NSF grants PHY-0354932 and INT-0307569 and the Eberly Research Funds of Penn State University.

[1] C. Rovelli. *Quantum Gravity*. Cambridge, UK: Univ. Pr. (2004) 480 p.
[2] Thomas Thiemann. Introduction to modern canonical quantum general relativity. gr-qc/0110034, 2001.
[3] Abhay Ashtekar and Jerzy Lewandowski. Background independent quantum gravity: A status report. *Class. Quant. Grav.*, 21:R53, 2004.
[4] Alejandro Perez. Introduction to loop quantum gravity and spin foams. 2004.
[5] L. Smolin. *Recent Developments in non Perturbative Quantum Gravity; in Quantum Gravity and Cosmology, Proceedings of the 1991 GIFT International Seminar on Theoretical Physics*. Available at hep-th/9202022 World Scientific, Singapore, 1992.
[6] C. Rovelli and L. Smolin. Discreetness of the area and volume in quantum gravity. *Nucl Phys B*, 442 (1995), Erratum: 456:593,734, 1995.
[7] C. Rovelli. Black hole entropy from loop quantum gravity. *Phys Rev Lett*, 14:3288, 1996.
[8] Kirill V. Krasnov. On statistical mechanics of gravitational systems. *Gen. Rel. Grav.*, 30:53–68, 1998.
[9] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov. Quantum geometry and black hole entropy. *Phys. Rev. Lett.*, 80:904–907, 1998.

[10] A. Ashtekar, John C. Baez, and Kirill Krasnov. Quantum geometry of isolated horizons and black hole entropy. *Adv. Theor. Math. Phys.*, 4:1–94, 2000.

[11] Abhay Ashtekar, Alejandro Corichi, and Kirill Krasnov. Isolated horizons: The classical phase space. *Adv. Theor. Math. Phys.*, 3:419–478, 2000.

[12] Krzysztof A. Meissner. Black hole entropy in loop quantum gravity. *gr-qc/0407052*, 2004.

[13] Marcin Domagala and Jerzy Lewandowski. Black hole entropy from quantum geometry. *gr-qc/0407051*, 2004.

[14] Martin Bojowald. Loop quantum cosmology: Recent progress. *gr-qc/0402053*, 2004.

[15] R. Gambini and J. Pullin, Discrete quantum gravity: A mechanism for selecting the value of fundamental constants Int. J. Mod. Phys. D **12**, 1775 (2003) *gr-qc/0306095*, 2003

[16] Martin Bojowald. Absence of singularity in loop quantum cosmology. *Phys. Rev. Lett.*, 86:5227–5230, 2001.

[17] Martin Bojowald. Dynamical initial conditions in quantum cosmology. *Phys. Rev. Lett.*, 87:121301, 2001.

[18] Shinji Tsujikawa, Parampreet Singh, and Roy Maartens. Loop quantum gravity effects on inflation and the CMB *astro-ph/0311015*, 2003

[19] T. Thiemann, The Phoenix project: Master constraint programme for loop quantum gravity *gr-qc/0305080*, 2003

[20] A. Perez, Spin foam models for quantum gravity *Class. Quant. Grav.* **20**, R43 (2003)

[21] Donald Marolf. Group averaging and refined algebraic quantization: Where are we now? *gr-qc/0011112*, 2000.

[22] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T Thiemann. *J. Math. Phys.*, 36:6456, 1995.

[23] J.F. Plebanski. On the separation of Einsteinian substructures. *J. Math. Phys.*, 18:2511, 1977.

[24] M. Bojowald and H. A. Kastrup. Quantum symmetry reduction for diffeomorphism invariant theories of connections. *Class. Quant. Grav.*, 17:3009–3043, 2000.

[25] H. Urbantke. On integrability properties of su(2) Yang-Yills fields. I. infinitesimal part. UWTHPH-1984-02.

[26] Abhay Ashtekar, Martin Bojowald, and Jerzy Lewandowski. Mathematical structure of loop
quantum cosmology. *Adv. Theor. Math. Phys.*, 7:233–268, 2003.

[27] Jonathan J. Halliwell. Derivation of the Wheeler-De Witt equation from a path integral for minisuperspace models. *Phys. Rev.*, D38:2468, 1988.

[28] Martin Bojowald, Parampreet Singh, and Aureliano Skirzewski. Time dependence in quantum gravity. gr-qc/0408094, 2004.

[29] Joshua Willis. On the low energy ramifications and mathematical extension of loop quantum gravity. *PhD thesis*, 2004.

[30] Daniel Green and William G. Unruh. Difficulties with closed isotropic loop quantum cosmology. gr-qc/0408074, 2004.

[31] Martin Bojowald. The inverse scale factor in isotropic quantum geometry. *Phys. Rev.*, D64:084018, 2001.