ON THE NATURE OF CHARGE CONJUGATION IN QUANTUM THEORY

G.A. Kotel’nikov
Russian Research Center ”Kurchatov Institute”, Moscow 123182, Russia
e-mail: kga@kga.kiae.su

Abstract

On the basis of the invariance of Dirac equation \((\gamma p - mc)\Psi(x, c) = 0\) with respect to the inversion of the speed of light \(Q: x \rightarrow x, c \rightarrow -c\), it is shown that the relationship \([C, PTQ]\Psi(x, c) = 0\) between the transformations of the charge conjugation, the space inversion \(P\), the time reversal \(T\) and the inversion of the speed of light \(Q\) is true. The charge conjugation in quantum theory may be interpreted as the consequence of the discrete symmetries reflecting the fundamental properties of space, time and speed of light.

1 Introduction

The charge conjugation \(C\), the space inversion \(P\), the time reversal \(T\) and the \(CPT\)-theorem connected with them play the important role in quantum theory [1], [2]. Their significance, for example, in the theory of beta-decay, was numerously discussed. As this takes place, the \(P(x \rightarrow -x)\) and the \(T(t \rightarrow -t)\) symmetries are interpreted as the evidence of the fundamental properties of space and time, namely the equivalence of right and left directions in space and the possibility of sign reversal of the time. The nature of the charge conjugation is not established [3]. In the present work we try to connect the charge conjugation \(C\) with the inversion of the speed of light \(Q : c \rightarrow -c\) [4].

The source of the \(Q\)-symmetry seems to be the invariance of the light cone equation relative to the replacement \(c \rightarrow -c, t \rightarrow t, x \rightarrow x:\)

\[
Q : \, c^2t^2 - x^2 = 0 \rightarrow (-c)^2t^2 - x^2 = 0
\]  

(1)

It is shown in the work [4] that the \(Q\)-symmetry is also inherent in the D’Alembert equation, the Maxwell equations, the equation of movement of a charge particle in electromagnetic field, that is, the equations of the classical electrodynamics.

For finding the interrelation between the inversion \(Q\) and the charge conjugation \(C\) in quantum theory we use the Dirac equation.

2 \(P, T, Q\) -symmetry of Dirac equation

Let us introduce the Dirac equation in the form [1,2]

\[
(\gamma^0 p_a - mc)\Psi(x^0, x, c) = (i\hbar\gamma^0 \partial_0 + i\hbar\gamma \cdot \nabla - mc)\Psi(x^0, x, c) = 0
\]  

(2)

Here \(x^a = (ct, x, y, z); g_{ab} = \text{diag}(+1, -1, -1, -1); \gamma^a = (\gamma^0, \gamma); \gamma = (\gamma^1, \gamma^2, \gamma^3)\) are the Dirac matrices; \(p_a = i\hbar \partial_a\); \(a = 0, 1, 2, 3\); the summation is carried out over the twice
repeating index; \( \Psi = \text{column}(\phi, \chi); \phi = \text{column}(\phi_1, \phi_2); \chi = \text{column}(\chi_1, \chi_2) \). The gamma matrices satisfy the relations

\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 2g^{ab}; \quad \gamma^a \gamma^5 + \gamma^5 \gamma^a = 0;
\]

\[
(\gamma^0)^+ = \gamma^0; \quad (\gamma^{1,2,3})^+ = -(\gamma^{1,2,3}); \quad (\gamma^0)^2 = 1; \quad (\gamma^{1,2,3})^2 = -1;
\]

\[
(\gamma^{0,1,3})^* = \gamma^{0,1,3}; \quad (\gamma^2)^* = -\gamma^2; \quad (\gamma^{0,2})^T = \gamma^{0,2}; \quad (\gamma^{1,3})^T = -\gamma^{1,3};
\]

\[
\gamma^5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3; \quad (\gamma^5)^+ = \gamma^5; \quad (\gamma^5)^* = \gamma^5; \quad (\gamma^5)^2 = 1
\]

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^{1,2,3} = \begin{pmatrix} 0 & \sigma_{x,y,z} \\ -\sigma_{x,y,z} & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}
\]

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Here \( I \) is the unit two dimensional matrix. In accordance with Ref. [1,2], we define the action of operators \( P, T, C, Q \) on the solution \( \Psi \) in the following forms:

\[
P\Psi(x^0, x, c) = U_P \Psi(x^0, -x, c);
\]

\[
T\Psi(x^0, x, c) = U_T \Psi^T(-x^0, x, c) = U_T \gamma^0 \Psi^*(-x^0, x, c);
\]

\[
C\Psi(x^0, x, c) = U_C \Psi^T(x^0, x, c) = U_C \gamma^0 \Psi^*(x^0, x, c);
\]

\[
Q\Psi(x^0, x, c) = U_Q \Psi(x^0, x, -c)
\]

Here \( U_P, U_T, U_Q, U_C \) are the corresponding matrices of the transformations; \( \Psi = \Psi^+ \gamma^0; T \) is the transposition; \( * \) is the complex conjugation.

We take Eq. (2) and perform the \( Q \)-inversion in it:

\[
Q : (\gamma^a p_a - mc)\Psi(x^0, x, c) = 0 \rightarrow (U_Q \gamma^a U_Q^{-1} p_a + mc)U_Q \Psi(x^0, x, -c) = 0
\]

Let Eq. (10) coincide with the initial Eq. (2) for the transformed function \( \Psi_Q = U_Q \Psi(x^0, x, -c) \). It can be obtained if the matrix \( U_Q \) has the following property:

\[
U_Q \gamma^a U_Q^{-1} = -\gamma^a
\]

As far as \( U_Q \gamma^a + \gamma^a U_Q = 0 \), it is follows from it and the properties of the \( \gamma^a \)-matrices that

\[
U_Q = \lambda \gamma^5,
\]

where \( \lambda = (\pm 1, \pm i) \). As a result for attaining the invariance of Dirac equation relative to the \( Q \)-inversion it is necessary to transform the function \( \Psi \) following the rule

\[
Q\Psi(x^0, x, c) = U_Q \Psi(x^0, x, -c) = \lambda \gamma^5 \Psi(x^0, x, -c)
\]

We take into account that [3]:

\[
P = i\gamma^0, \quad T = -i\gamma^0 \gamma^1 \gamma^3, \quad C = -\gamma^0 \gamma^2,
\]
use the permutational properties (3) of the $\gamma^a$ - matrices, hold the parameter $\lambda = i$ and make
a table of the transformations of the function $\Psi(x^0, x, c)$ relative to the $P$, $T$, $Q$-inversions
and the charge conjugation $C$.

Berestetski, Lifshits, Pitaevski

The present work

$C, P, T$ symmetries, $c \to +c$ [2]: $P, T, Q$ symmetries, $c \to \pm c$:

$$
P\Psi = i\gamma^0\Psi(x^0, -x, c);
T\Psi = -i\gamma^1\gamma^3\Psi^*(-x^0, x, c);
PT\Psi = \gamma^0\gamma^1\gamma^3\Psi^*(-x^0, -x, c);
CPT\Psi = i\gamma^5\Psi(-x^0, -x, c);
CT\Psi = i\gamma^1\gamma^2\gamma^3\Psi(-x^0, -x, c);
CP\Psi = i\gamma^0\gamma^2\Psi^*(x^0, -x, c);
C\Psi = \gamma^2\Psi^*(x^0, x, c);
Q\Psi = i\gamma^5\Psi(x^0, x, -c);
PQ\Psi = i\gamma^1\gamma^2\gamma^3\Psi(x^0, -x, -c);
TQ\Psi = i\gamma^0\gamma^2\Psi^*(-x^0, x, -c);
PTQ\Psi = -\gamma^2\Psi^*(-x^0, -x, -c);
$$

(15)

One can see that the charge conjugation $C$ with accuracy up to the phase factor $\exp(i\pi)$
corresponds to the $PTQ$-transformation so that

$$
[C, PTQ]\Psi(x^0, x, c) = 0
$$

(16)

Similarly, the operations $CPT$, $CT$ and $CP$ correspond to $Q$, and the $PQ$, $TQ$ compositions.

Let us consider in more details the mechanism of the $C \leftrightarrow PTQ$ correspondence. Following
to [2] we rewrite the function $\Psi$ in explicit form for our case when $c \neq 1$, $h \neq 1$:

$$
\Psi_{\rho\sigma} = \frac{1}{\sqrt{2p^0}}u_{\rho\sigma}e^{-\bar{p}^x x}; \quad \Psi_{-\rho-\sigma} = \frac{1}{\sqrt{2p^0}}u_{-\rho-\sigma}e^{\bar{p}^x x};
$$

(17)

$$
u_{\rho\sigma} = \left( \frac{\sqrt{p^0 + mc w}}{\sqrt{p^0 - mc (n\sigma) w}} \right); \quad u_{-\rho-\sigma} = \left( \frac{\sqrt{p^0 + mc w}}{\sqrt{p^0 + mc (n\sigma) w}} \right)
$$

(18)

Here $p^0 = E/c > 0$, $E = c\sqrt{p^2 + m^2 c^2} > 0$, $p.x = p^0 x^0 - px$, $n = p/p$, $w^+ w = 1$, $w = (n\sigma)w'$,
$p_u = 2mc$, $p_{-u} = -2mc$, $c > 0$. In application to functions $\Psi_{-\rho-\sigma}$ and $\Psi_{\rho\sigma}$ of operators
$C$ and $PTQ$ we have:

$$
C\Psi_{-\rho-\sigma-E}(x^0, x, c) = \gamma^2\Psi_{-\rho-\sigma-E}^*(x^0, x, c) = \Psi_{\rho\sigma E}(x^0, x, c);
P\Psi_{\rho\sigma E}(x^0, x, c) = -\gamma^2\Psi_{\rho\sigma E}^*(-x^0, -x, -c) = -\Psi_{\rho\sigma -E}(x^0, x, -c)
$$

(19)

We take into account in the first expression that $\sigma_y(n\sigma^*) = -(n\sigma)\sigma_y$, $-\sigma_y w^* = u'$, $u'^+ u' = (\sigma_y w^*)^+(\sigma_y w^*) = w^T \sigma_y w^* = (w^+ w) = 1$ and in addition to this in the second expression
that $p^0 = (-E)/(-c) > 0$, $p = (-E)(-v)/c^2 > 0$. In addition to Eq. (16) we have in result

$$
C\Psi_{-\rho-\sigma-E}(x^0, x, c) = -PTQ\Psi_{\rho\sigma E}(x^0, x, c)
$$

(20)

because of function $\Psi_{\rho\sigma E}(x^0, x, c)$ coincides with the function $\Psi_{\rho\sigma -E}(x^0, x, -c)$ due to relations
$E/c = (-E)/(-c)$, $mc = (-m)(-c)$. 

"
3 \textit{PTQ} -symmetry and Dirac equation for a charged particle

Now we consider the Dirac equation for a charge particle with spin 1/2 in electromagnetic field:

\begin{equation}
(\gamma^a p_a - mc)\Psi(x, c) = (e/c)\gamma^a A_a \Psi(x, c) \tag{21}
\end{equation}

where \(x = (x^0, \mathbf{x})\), \(e\) is the charge of a particle, \(A^a = (A^0, \mathbf{A})\) is the 4-potential, \(\gamma^a A_a = \gamma^0 A^0 - \gamma^0 \mathbf{A}\), \(\gamma = (\gamma^1, \gamma^2, \gamma^3)\). Let us subject the equation (21) to the \textit{PTQ} transformation taking into account the formulas (3), (7), (9) on the assumption that an electrical charge is a scalar; the vector potential is a polar vector relative to the replacements both \(x \to -x\), and \(t \to -t\), and \(c \to -c\) \(\square\).

Let us carry out the transformation \(Q\), then \(T\), then \(P\):

\begin{align*}
Q(x^0, \mathbf{x}, c, e, A^0, \mathbf{A}) &= (x^0, \mathbf{x}, -c, e, A^0, \mathbf{A}); \\
Q \Psi_Q(x, -c) &= U_Q \Psi(x^0, \mathbf{x}, -c); \\
Q : (\gamma^a p_a - mc)\Psi &= (e/c)\gamma^a A_a \Psi \\
&= (ihU_Q\gamma^0 U_Q^{-1} \partial_0 + ihU_Q\gamma U_Q^{-1} \mathbf{\nabla} + mc)U_Q \Psi(x^0, \mathbf{x}, -c) = \\
&= (e/c)(U_T\gamma^0 U_T^{-1} A^0 - U_T\gamma U_T^{-1} \mathbf{A})U_T \Psi(x^0, \mathbf{x}, -c) \\
&= (\gamma^a p_a - mc)\Psi_Q = (e/c)\gamma^a A_a \Psi_Q \\
T(x^0, \mathbf{x}, c, e, A^0, \mathbf{A}) &= (x^0, \mathbf{x}, c, e, A^0, -\mathbf{A}); \\
T \Psi_Q(x, -c) &= \Psi_{TQ}(-x^0, \mathbf{x}, -c) = U_T \Psi_Q(-x^0, \mathbf{x}, -c); \\
T : (\gamma^a p_a - mc)\Psi_Q &= (e/c)\gamma^a A_a \Psi_Q \\
&= (ihU_T\gamma^0 U_T^{-1} \partial_0 - ihU_T\gamma U_T^{-1} \mathbf{\nabla} - mc)U_T \Psi_Q(-x^0, \mathbf{x}, -c) = \\
&= (e/c)(U_T\gamma^0 U_T^{-1} A^0 + U_T\gamma U_T^{-1} \mathbf{A})U_T \Psi_Q(-x^0, \mathbf{x}, -c) \\
&= (\gamma^a p_a - mc)\Psi_{TQ} = (e/c)\gamma^a A_a \Psi_{TQ} \\
P(x^0, \mathbf{x}, c, e, A^0, \mathbf{A}) &= (x^0, -\mathbf{x}, c, e, A^0, -\mathbf{A}); \\
P \Psi_{TQ}(-x^0, \mathbf{x}, -c) &= \Psi_{PTQ}(-x^0, -\mathbf{x}, -c) = U_P \Psi_{PTQ}(-x^0, -\mathbf{x}, -c); \\
P : (\gamma^a p_a - mc)\Psi_{TQ} &= (e/c)\gamma^a A_a \Psi_{TQ} \\
&= (ihU_P\gamma^0 U_P^{-1} \partial_0 - ihU_P\gamma U_P^{-1} \mathbf{\nabla} - mc)U_P \Psi_{TQ}(-x^0, -\mathbf{x}, -c) = \\
&= (e/c)(U_P\gamma^0 U_P^{-1} A^0 + U_P\gamma U_P^{-1} \mathbf{A})U_P \Psi_{TQ}(-x^0, -\mathbf{x}, -c) \\
&= (\gamma^a p_a - mc)\Psi_{PTQ} = (e/c)\gamma^a A_a \Psi_{PTQ} \\
\end{align*}

Here matrices \(U\) satisfy the conditions

\begin{align*}
U_Q\gamma^0 U_Q^{-1} &= -\gamma^0, \quad U_Q\gamma U_Q^{-1} = -\gamma; \\
U_T(\gamma^0) U_T^{-1} &= \gamma^0, \quad U_T\gamma U_T^{-1} = -\gamma; \\
U_P\gamma^0 U_P^{-1} &= \gamma^0, \quad U_P\gamma U_P^{-1} = -\gamma; \tag{25}
\end{align*}

which define their explicit forms \(\square\) and \(\square\). On the basis of the formulae \(\square\) and taking into account \(\Psi_{PTQ} = -\gamma^2 \Psi^*(-x^0, -\mathbf{x}, -c)\), we can write

\begin{equation}
(\gamma^a p_a - mc)\gamma^2 \Psi^*(-x, -c) = (e/c)\gamma^a A_a \gamma^2 \Psi^*(-x, -c) \tag{26}
\end{equation}

where \(-x = (-x^0, -\mathbf{x})\). Similarly to the charge conjugation, the equation received coincides with initial Eq. (21) for the electric charge \(-e\) and the transformed function \(-\gamma^2 \Psi^*(-x, -c)\).
In accordance with formula (19) it is possible to admit that Eq. (26) describes a particle with the charge \(-e\), momentum \(p = (p^0 > 0, \mathbf{p} > 0)\) and negative energy \(E < 0\) \((p^0 = (-E)/(-c) > 0, \mathbf{p} = (-E)(-\mathbf{v})/c^2 > 0)\). The energy gap of \(2E\) is present between \(c > 0\) and \(c < 0\)-states.

Thus, the charge conjugation \(C\) puts into correspondence the antiparticle with characteristics \((-e, m, -p, -E, c)\) to the particle with characteristics \((e, m, p, E, c)\). The \(PTQ\)-composition puts into correspondence the particle with characteristics \((-e, m, p, -E, c)\) to the particle with characteristics \((e, m, p, E, c)\). From here it is seen that the particle with characteristics \((-e, m, p, -E, -c)\) may be the redefined antiparticle with respect to the initial particle from Eq. (21).

As in the case of the \(C\)-conjugation [2], the \(PTQ\)-composition is the symmetry transformation of Dirac equation for a charged particle (21) if the 4-potential of electromagnetic field \(A\) is transformed following the rule \(QPT(A) = (-A^0, -\mathbf{A})\).

## 4 Conclusion

The inversion of the speed of light \(Q: x^0 \rightarrow x^0, x \rightarrow x, c \rightarrow -c\) was considered in Dirac equation. As a result the charge conjugation \(C\) in quantum theory may be interpreted as the consequence of the Dirac equation symmetry with respect to the discrete transformations \(P, T, Q\) reflecting the fundamental properties of space, time and speed of light: 

\[ C\Psi_{p=\sigma=-E}(x^0, x, c) = -PTQ\Psi_{p=\sigma=E}(x^0, x, c) \]

We may note the following consequences of the \(PTQ\)-symmetry.

- It is not improbable that the \(V^5(x^0, x, c)\)-space exists in which there are two hyperplanes with \(c = +3.10^{10}\) m/s and \(c = -3.10^{10}\) m/s [4]. In this space the \(c < 0\)-hyperplane with the reversal ”time” \(x^0 \rightarrow -x^0\) and with inverted space \(x \rightarrow -x\) may be interpreted as some antiworld separated from the \(c > 0\)-world by the energy gap of \(2E\). That may be the redefined electron-photon Dirac vacuum. In such vacuum a particle with negative energy is characterized by its placing on another hyperplane of common 5-dimensional physical space of events.

- It is impossible to distinguish between the realization of the 4-dimensional physical event space on the ”+c-hyperplane” and on the ”−c-hyperplane”. From the point of view of an observer placed on the ”+c-hyperplane”, the ”−c-vacuum” is invisible; and the ”+c-vacuum” is invisible for an observer on the ”−c-hyperplane”.

- The additional quantum number, which is connected with the \(Q\)-parity being kept, may exist in quantum interactions.

- The alternative interpretation of the \(K^+\) meson decay scheme \(K^+\pi_3 \rightarrow \pi^+ + \pi^+ + \pi^-; K^+\pi_2 \rightarrow \pi^+ + \pi^0\). The scheme may be explained if the \(K^+\) and \(\pi\) mesons have the negative \(Q\)-parity.
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