Tri-hamiltonian Toda lattice and a canonical bracket for closed discrete curves

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Abstract

Flows on (or variations of) discrete curves in $\mathbb{R}^2$ give rise to flows on a subalgebra of functions on that curve. For a special choice of flows and a certain subalgebra this is described by the Toda lattice hierarchy [HK02]. In the paper it is shown that the canonical symplectic structure on $\mathbb{R}^{2N}$, which can be interpreted as the phase space of closed discrete curves in $\mathbb{R}^2$ with length $N$, induces Poisson commutation relations on the above mentioned subalgebra which yield the tri-hamiltonian poisson structure of the Toda lattice hierarchy.

1 Introduction

The Toda lattice hierarchy is a set of equations, including the Toda equation:

$$\ddot{q}_k = e^{q_{k+1} - q_k} - e^{q_{k-1} - q_k}.$$  

The Toda equation is sometimes also called first flow equation of the Toda lattice hierarchy, it was discovered by Toda in 1967 [To89]. A good overview about the literature about the Toda lattice can be found in [FT86, Sur03]. It is a wellknown fact that the Toda lattice hierachy has a socalled trihamiltonian structure (for an overview about trihamiltonian structures please see [Sur03] and the therein cited references). In particular this means that the Toda lattice as a dynamical system admits three different poisson structures. In the following article it will be shown explicitly how these three different structures can be globally obtained from the the canonical poisson structure

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on $\mathbb{R}^{2N}$. We will confine ourselves to the study of the periodic Toda lattice hierarchy. To do so we define a map from canonical coordinates on $\mathbb{R}^{2N}$ to the $2N$ (periodic) Flaschka-Manakov [Man74, Fla74a, Fla74b] variables that depends on a spectral parameter. The canonical poisson structure on $\mathbb{R}^{2N}$ induces then a Poisson structure for the periodic Flaschka-Manakov variables, which is the tri-hamiltonian Poisson structure of the Toda lattice.

The paper is organized as follows: We will briefly recall the connection between discrete curves in $\mathbb{R}^2$ and the Toda lattice hierarchy, for a more thorough investigation of that connection please see [HK02]. In particular this brief part should serve as a motivation for how the spectral parameter dependent map is derived. The phase space of all closed (real) discrete curves of length $N$ is $\mathbb{R}^{2N}$, hence the canonical poisson structure on $\mathbb{R}^{2N}$ is a Poisson structure on the phase space of all closed discrete curves. In the second part we will then state that this canonical Poisson structure is in fact a Poisson structure which gives the three brackets of the (periodic) Toda lattice.

## 2 Discrete curves and the Toda lattice

**Definition 2.1** A discrete curve in $\mathbb{R}^2$ is a map

$$\gamma : \mathbb{Z} \rightarrow \mathbb{R}^2$$

$$k \mapsto \gamma_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$$ (1)

Define:

$$g_k = \det(\gamma_k, \gamma_{k+1}) = x_k y_{k+1} - y_k x_{k+1}$$ (2)

$$u_k = \det(\gamma_{k-1}, \gamma_{k+1}) = x_{k-1} y_{k+1} - y_{k-1} x_{k+1}$$ (3)

We will from now on consider the generic case $g_k \neq 0$ for all $k$. The following lemma can be straightforwardly obtained by using the above definitions [3]:

**Lemma 2.2**

$$\gamma_{k+1} = \frac{1}{g_{k-1}} (u_k \gamma_k - g_k \gamma_{k-1}).$$ (4)

If the variables $u_k$ and $g_k$ and initial points $\gamma_0$ and $\gamma_1$ are given, then lemma 2.2 is a recursive definition of a discrete curve.
Note that \( \det(\gamma_k, \gamma_{k+1} - \gamma_{k-1}) = 1 \). This means that \( \gamma_k \) and \( \frac{\gamma_{k+1} - \gamma_{k-1}}{g_k + g_{k-1}} \) are linearly independent. Hence an arbitrary flow on (or variation of) \( \gamma \) can be written in the following way:

\[
\frac{d}{dt} \gamma_k = \dot{\gamma}_k = \alpha_k \gamma_k + \frac{\beta_k}{u_k} (\gamma_{k+1} - \gamma_{k-1}) \quad \alpha_k, \beta_k \in \mathbb{R}.
\]

(5)

The variables \( \alpha_k, \beta_k \in \mathbb{R} \) are arbitrary. Equations (4) and (5) can also be reformulated as a zero-curvature condition. Define:

\[
F_k = \begin{pmatrix} \gamma_k^T \\ \gamma_{k-1}^T \end{pmatrix} = \begin{pmatrix} x_k & y_k \\ x_{k-1} & y_{k-1} \end{pmatrix}.
\]

(6)

**Lemma 2.3** Let \( \alpha_k, \beta_k \in \mathbb{R} \) be arbitrary and \( g_k, u_k \) be as defined in (3). Then

\[
F_{k+1} = L_k F_k \quad \dot{F}_k = V_k F_k
\]

with

\[
L_k = \begin{pmatrix} \frac{1}{g_{k-1}} u_k & -\frac{g_k}{g_{k-1}} \\ 1 & 0 \end{pmatrix} \quad V_k = \begin{pmatrix} \alpha_k + \frac{1}{g_{k-1}} \beta_k & -(1 + \frac{g_k}{g_{k-1}}) \frac{\beta_k}{u_k} \\ (1 + \frac{g_{k-2}}{g_{k-1}}) \beta_{k-1} - \frac{1}{g_{k-1}} \beta_{k-1} & \alpha_{k-1} - \frac{1}{g_{k-1}} \beta_{k-1} \end{pmatrix}.
\]

(7)

The compatibility equation

\[
\dot{L}_k = V_{k+1} L_k - L_k V_k
\]

(8)

is satisfied for all \( \alpha_k, \beta_k \in \mathbb{R} \).

The compatibility equation (8) is also called zero curvature equation or condition. We call \( F_k \) a discrete frame.

**Lemma 2.4** A flow on the discrete curve \( \gamma \) given by (5) generates the following flow on the variables \( g_k \):

\[
\dot{g}_k = g_k (\alpha_{k+1} + \alpha_k) + \beta_{k+1} - \beta_k.
\]

(9)

**Lemma 2.5** A flow on the discrete curve \( \gamma \) given by (5) generates the following flow on the variables \( u_k \):

\[
\dot{u}_k = u_k(\alpha_{k-1} + \alpha_{k+1}) + \beta_{k-1} \frac{g_k}{g_{k-1}} (g_{k-2} + g_{k-1}) - \beta_{k+1} \frac{g_{k+1}}{u_{k+1} g_k} (g_k + g_{k+1}) + u_k \left( \frac{1}{g_k} \beta_{k+1} - \frac{1}{g_{k-1}} \beta_{k-1} \right)
\]

(10)
Define
\[
\begin{align*}
a_k &:= g_k^{-2} \\
b_k &:= \frac{u_k}{g_{k-1}g_k} - \lambda,
\end{align*}
\]
(11)
where \(\lambda\) is an arbitrary (but in particular time independent) parameter. Clearly the flows on the variables \(g_k\) and \(u_k\) given in (9) and (10) define flows on the variables \(a_k, b_k\) via definitions (11).

**Theorem 2.6** Denote
\[
V_k := \begin{pmatrix} v_{11}^{11} & v_{12}^{11} \\ v_{21}^{12} & v_{22}^{12} \end{pmatrix}.
\]
Define\[
\begin{align*}
\alpha_k &:= v_{11}^{11} + \frac{v_{12}^{12}u_k}{g_{k-1} + g_k} \\
\beta_k &:= -\frac{v_{12}^{12}g_{k-1}u_k}{g_{k-1} + g_k}.
\end{align*}
\]
(12)
By (5) and with definition (12), \(\alpha_k\) and \(\beta_k\) define a flow on discrete curves in \(\mathbb{R}^2\) depending on the choice of the \(V_k\). This induces via the definition (11) a flow on the variables \(a_k, b_k\).

If the \(\{V_k\}_{k \in [1, \ldots, N]}\) \((V_k\text{ periodic})\) are the two dimensional Lax matrices defining the \(n\)th Toda flow (see e.g. [FT86, Sur03]) with \(\lambda\) being the corresponding spectral parameter, then \(\{a_k, b_k\}_{k \in [1, \ldots, N]}\) \((a_k, b_k\text{ periodic})\) are the Flaschka-Manakov variables obeying the \(n\)th Toda flow.

For a proof of theorem (2.6) and lemmas (10) and (9) please see [HK02].

Note that the Toda flows (as flows on the Flaschka-Manakov variables \(a_k, b_k\)) do not depend on the choice of \(\lambda\), whereas the corresponding flows for the determinants \(g_k\) and \(u_k\) of course depend on the choice of \(\lambda\).

### 3 How to derive Toda Poisson brackets from the canonical Poisson bracket

The following theorem is easy to state and can be verified straightforwardly. Nevertheless the assertion itself is not really suggestive apriori. And indeed it was not found by good guessing and then verifying, but by starting with the brackets of the Toda model (while assuming that they can be derived from the canonical coordinates via the below map) and the help of a computer algebra system.
Theorem 3.1  Let $N > 3$. Let $\{x_i, y_i | i \in \{0...N-1\}\}$ be canonical coordinates on $\mathbb{R}^{2N}$. Define the following symplectic structure on $\mathbb{R}^{2N}$:

$$\Omega := \sum_{i=0}^{N} dx_i \wedge dy_i$$

which leads to the ultralocal Poisson relations:

$$\{x_i, y_i\} = \frac{1}{2}, \quad \text{and zero else.}$$

Let $x, y$ describe a periodic phase space, i.e:

$$x_{k+N} := x_k \quad (13)$$
$$y_{k+N} := y_k \quad (14)$$

Define:

$$g_k := x_k y_{k+1} - y_k x_{k+1}$$
$$u_k := x_{k-1} y_{k+1} - y_{k-1} x_{k+1}$$
$$a_k := \frac{g_k^{(-2)}}{g_k^{-1} g_k}$$
$$b_k := \frac{u_k}{g_k^{-1} g_k} - \lambda,$$

where $\lambda \in \mathbb{R}$ arbitrary.

Then the Poisson relations for the variables $a_i$ and $b_i$, which are given via the ultralocal relations for $x_i$ and $y_i$ read as:

$$\{a_k, a_{k+1}\} = -2a_k a_{k+1} b_{k+1} - 2a_k a_{k+1} \lambda \quad (19)$$
$$\{b_k, b_{k+1}\} = -a_k (b_k + b_{k+1}) - 2a_k \lambda \quad (20)$$
$$\{b_k, a_{k-2}\} = a_{k-2} a_{k-1} \quad (21)$$
$$\{b_k, a_{k-1}\} = a_{k-1} (b_k^2 + a_{k-1}) + 2b_k a_{k-1} \lambda + a_{k-1} \lambda^2 \quad (22)$$
$$\{b_k, a_k\} = -a_k (b_k^2 + a_k) - 2b_k a_k \lambda - a_k \lambda^2 \quad (23)$$
$$\{b_k, a_{k+1}\} = -a_k a_{k+1} \quad (24)$$

and zero for all the remaining commutators. For each power of $\lambda$ this gives one of the three Poisson brackets of the Toda lattice hierarchy (as e.g. given in [Sur03]).

Note that the brackets (19)-(24) make only sense if $N > 3$. Nevertheless the canonical Poisson structure defines also a Poisson structure on the Flaschka-Manakov variables for $N=1,2,3$. 

5
4 Conclusions

The extremely nice Poisson bracket for the coordinates of closed discrete curves, which lead to the brackets of the periodic Toda lattice suggests that the study of discrete curves in the context of the Toda lattice is not an exotic geometrical excursion but rather a very natural choice.

The map from the canonical coordinates $x_k, y_k$ to the Flaschka-Manakov variables $a_k, b_k$ is not one-to-one. This becomes also clear by studying the phase space of quasiperiodic discrete curves, i.e. curves for which $\gamma_{k+N} = M \gamma_k$, where $M \in GL(2, \mathbb{R})$. If $M \in SL(2, \mathbb{R})$ then the corresponding Flaschka-Manakov variables are also periodic, but are not necessarily restricted to the leaves given by the Casimir functions. Nevertheless we expect that Poisson brackets for that general case will be rather nontrivially. We are investigating that phase space right now.

The canonical coordinates are matrix entries of the discrete frame $F_k$ given in (6). In that sense the canonical bracket can be seen as a bracket for the frame. The frame is the (partially discrete) integral of the zero curvature condition in (8) subject to a periodicity condition (i.e. being a closed curve). It is an interesting question whether other integrable systems are admitting such a simple bracket for their corresponding frames.

So it may be also important to find out, whether there is a connection to the work of Gelfand and Zakharovich [GZ], where it’s proven that there exists a local isomorphism between the bihamiltonian bracket of the periodic Toda lattice and a product of two canonical brackets (at generic points).

It would also be interesting to study whether the above could be extended to generalized Toda systems [Kos79].

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