Convergence Analysis of Parallel Multi-block ADMM via Discrete-time Recurrent Neural Networks

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Abstract
We consider two basic problems in parallel multi-block Alternating Direction Method of Multipliers (ADMM): constructing convergence conditions, and controlling the variables among a family of solving the subproblems. To address these problems, we propose a new NeuralADMM framework, which consists of two fundamental steps. First, ADMM is converted into a discrete-time recurrent neural network (DT-RNN) since ADMM recursively updates the variables or solve the subproblems. Second, we study the stability of DT-RNN for the convergence of ADMM by employing quadratic Lyapunov functions. We also propose some convergence conditions and design variable feedbacks for ensuring the convergence of ADMM. Semidefinite programming is easily employed to check these conditions. Numerical experiments further establish the effectiveness of our proposed framework.

Introduction
We consider the following convex minimization problem with \( n \geq 3 \) blocks of variables \( \{x_i\}_{i=1}^n \):

\[
\min_{x_i \in X_i, 1 \leq i \leq n} \sum_{i=1}^n f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^n A_i x_i = c, \tag{1}
\]

where \( A_i \in \mathbb{R}^{m_i \times n}, c \in \mathbb{R}^m, X_i \subset \mathbb{R}^{n_i} \) are closed convex sets, and \( f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R} = \bigcup_{j=1}^{\infty} \mathbb{R} \) are closed proper convex functions. The problem (1) has received increasing attention recently in wide applications, such as in machine learning, statistics, computer vision and signal processing (Boyd et al. 2011, Chang et al. 2016, Hong et al. 2016, Wang, Banerjee, and Luo 2014). To solve the problem (1), we consider the augmented Lagrangian function:

\[
\mathcal{L}_\beta (x_1, \cdots, x_n, \lambda) = \sum_{i=1}^n f_i(x_i) + \lambda^T \left( \sum_{i=1}^n A_i x_i - c \right) + \frac{\beta}{2} \sum_{i=1}^n \|A_i x_i - c\|^2, \tag{2}
\]

with the Lagrange multiplier \( \lambda \in \mathbb{R}^m \), and a penalty parameter \( \beta > 0 \). Since both of the objective function of the problem (2) are the sum of terms on individual \( x_i \), the problem can be decomposed into \( n \) smaller \( x_i \)-subproblems (Deng et al. 2017). Thus, the problem (2) is typically solved by using the Alternating Direction Method of Multipliers (ADMM), which was originally proposed in the early 1970s (Glowinski and Marrocco 1975, Gabay and Mercier 1976) and has since been studied extensively (Eckstein and Bertsekas 1992, Eckstein 1994, Deng et al. 2017, Lu et al. 2017). Roughly, most of ADMMs can be categorized into Gauss-Seidel ADMMs and Jacobian ADMMs. The Gauss-Seidel ADMM updates a variable \( x_i \) by fixing others as their latest versions in a sequential manner (Boyd et al. 2011). However, this manner is not amenable for parallelization as it takes time to wait for the latest versions. To avoid the waiting time, the Jacobian ADMMs update all the \( n \) variables \( x_i \) (1 \leq i \leq n) in a parallel manner (Deng et al. 2017). Moreover, to make the subproblems strictly or strongly convex, a proximal Jacobian ADMM (Deng and Yin 2016, Deng et al. 2017) is considered by adding a proximal term \( \frac{\gamma}{2} \|x_i - x_i^k\|^2 \) (Parikh and Boyd 2014). Thus, we consider the parallel multi-block ADMM as the following scheme:

\[
\begin{align*}
& x_i^{k+1} = \arg \min_{x_i} f_i(x_i) + \frac{\beta}{2} \sum_{j=1}^n A_{ij} x_j^k + A_i x_i + \beta \lambda_k - c_i^T x_i + \frac{\gamma}{2} \|x_i - x_i^k\|^2, \\
& \lambda^{k+1} = \lambda^k - \beta \sum_{j=1}^n A_{ij} x_j^k - c, \tag{3}
\end{align*}
\]

for \( 1 \leq i \leq n \), where \( \|x_i\|^2 = x_i^T D_i x_i, \quad D_i = \alpha_i I - \beta A_i^T A_i \) and \( \gamma > 0 \) is a damping parameter.

Unfortunately, there are the following two limitations in the multi-block ADMM. First, its convergence analysis has been missing for a long time although the convergence of two-block ADMM has been proved (Eckstein and Bertsekas 1992). This would be of great interests to further study convergence conditions for multi-block ADMM (Chen et al. 2016). Recently, some sufficient conditions have been presented to guarantee the convergence of multi-block ADMM under (strongly) convex functions \( f_i \) (Agarwal and Duchi 2011, Lin et al. 2015, Lin et al. 2016, Deng et al. 2017). However, the standard proof methods to analyze multi-block ADMM depend on deep insights by optimization experts, and are designed as an algorithm-by-algorithm basis (Lessard, Recht, and Packard 2016). Thus, it needs a unified framework to more diverse scenarios for the convergence analysis of multi-block ADMM.

Second, some multi-block ADMMs with unfixed matrices \( A_i \) are not convergent. For example, a counter example is constructed to show the failure of three-block ADMM (Chen et al. 2016). The convergence (at least acceptable accuracy) of multi-block ADMM is a basic guarantee for applications although the direct extension of multi-block ADMM is not
necessarily convergent (Chen et al. 2016). It is greatly necessary to make multi-block ADMM convergent when it is not convergent. Corresponding to the above limitations, therefore, two basic problems shown in Fig. 1(a) can be formulated as:

- **Problem 1:** Find sufficient conditions to guarantee that multi-block ADMM is convergent.
- **Problem 2:** Design a controller that drives non-convergent multi-block ADMM to be convergent.

To address the above two basic problems, we convert multi-block ADMM to a discrete-time recurrent neural network (DT-RNN) since it recursively updates the subproblems. All subproblems in (3) can be transformed into DT-RNN. In particular, the parameters (i.e., \( \alpha_i, \beta, \gamma \) or the matrices \( A_i \)) are correspondingly changed into the weight matrix of DT-RNN, and the gradients of functions \( f_i \) can be regarded as the activation functions in DT-RNN. Thus, this translation drives us to analyze the convergence and the controller of multi-block ADMM by studying the stability of DT-RNN as they has been widely studied in neural networks (Hopfield 1982; Hu and Wang 2002). In this paper, the stability of DT-RNN is studied by employing the quadratic Lyapunov function (QLFs) (Hu and Wang 2002) (More discussions are provided in related works). Therefore, we propose an efficient DT-RNN strategy to study multi-block ADMM for solving the problem (1). Our main contributions are as follows:

- We develop a novel NeuralADMM framework shown in Fig. 1(b) to solve the two basic problems of multi-block ADMM. Our framework is to transform it into a DT-RNN system, and study the stability of this system for its convergence and controller.
- We propose efficient conditions to ensure the convergence of multi-block ADMM by using quadratic Lyapunov functions. Simultaneously, we design variable controllers to drive non-convergent ADMM to be convergent. It is worth noting that the counter example (Chen et al. 2016) can be convergent by designing a feedback controller.

The rest of this paper is organized as follows. The related works section reviews existing methods on ADMM and DT-RNN. In the preliminaries section, we provide some preliminaries and notations. Our framework is developed in the NeuralADMM section. Moreover, the numerical experiments are provided in the experiments Section. We conclude the whole paper in the final section.

**Related Work**

In this section, we review multi-block ADMM for solving Eq. (1), the stability of the DT-RNN systems, and the control theory for the convergence analysis of optimization methods, especially ADMM.

It is well-known that the convergence of two-block ADMM has been widely proved (Eckstein and Bertsekas 1992). Unfortunately, no convergence proof is known for \( n \)-block ADMM with \( n \geq 3 \) (Chen et al. 2016) although it appears to work well in many practical applications (Li, Sun, and Toh 2016; Chen, Sun, and Toh 2017). Recently, global (linear) convergence (Han and Yuan 2012) and sublinear convergence (Lin, Ma, and Zhang 2015a) of ADMM have been proved under the condition that functions are (strongly) convex and finally restricted to certain region. Sun et. al. presented a convergent semi-proximal ADMM with 3-block (Sun, Toh, and Yang 2015) and an improved semi-proximal ADMM was proposed for an objective function, which is the sum of two proper closed convex functions plus an arbitrary number of convex quadratic or linear functions (Li, Sun, and Toh 2016). Moreover, ADMM was extended for certain type of nonconvex problems including the consensus and sharing problems without making any assumptions (Hong, Luo, and Razaviyayn 2016; More importantly, ADMM was also extended into parallel or distributed manners (Agarwal and Duchi 2011; Wang, Banerjee, and Luo 2014; Deng et al. 2017). However, the convergence conditions of most ADMM algorithms under a persistent sequence was provided by directly proving that a state sequence \( \| x^{k+1} - x^* \|^2 \) (\( x^* \) is an equilibrium point) is decreasing (Lin, Ma, and Zhang 2015a; Lin, Ma, and Zhang 2016). In this paper, we consider two basic convergence problems of multi-block ADMM shown in Fig. 1(a). Compared to the traditional methods, we transform ADMM into a DT-RNN system, and provide the convergence conditions by proving that a Lyapunov function is decreasing.
Since John Hopfield introduced a very influential associative memory model, known as the discrete-time Hopfield network in 1982, recurrent neural networks (RNN) have been widely studied and applied into the control theory (Hopfield 1982), and natural language processing (Bengio et al. 2003). In fact, there are two main ways to study (deep) RNN. One way is to learn the weights for making RNN convergent by using the data (Hochreiter and Schmidhuber 1997), and now it is a very hot topic in deep learning, more reviews in (Schmidhuber 2015). Another way is to construct Lyapunov functions for (discrete-time) RNN (Hu and Wang 2002). We employ the discrete-time RNN to study the convergence of ADMM since it can be transformed into the discrete-time RNN.

The convergence of two-block ADMM (Nishihara et al. 2015) and some gradient methods (i.e. the Heavy-ball method (Lasserd, Recht, and Packard 2016) [Hu and Lessard 2017]) were proved by using the integral quadratic constraint (IQC) (Megretski and Rantzer 1997). However, they are not applied into multi-block ADMM. In addition, IQC was also used to analyze the stochastic optimization methods (Hu, Seiler, and Rantzer 1997). Recently, an optimized network (OptNet) architecture was presented to integrate differentiable optimization problems (specifically, in the form of quadratic programs) (Amos and Kolter 2017). Compared to OptNet, we propose a NeuralADMM framework to study the convergence problem of multi-block ADMM for subgradient optimization problems.

**Preliminaries**

We give some notations as follows. A P-norm of $y$, $\|y\|_P$, is denoted as $\sqrt{y^TPy}$. The condition number of $A$ is denoted as $\kappa(A) = \sigma_1(A)/\sigma_p(A)$, where $\sigma_1(A) \leq \sigma_p(A)$ denote the largest and smallest singular values of the matrix $A$. The Hadamard product of matrices $A$ and $B$ is denoted as $A \circ B$. The Kronecker product of matrices $A$ and $B$ is denoted as $A \otimes B$. We denote a $m \times m$ identity matrix and zero matrix by I_m and 0_m, respectively. We denote the Kronecker product of matrices $A$ and $B$ by $A \otimes B$.

We suppose that $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is convex and differentiable, and let $\nabla f$ denote the gradient of $f$. We say that $f$ is $\nu$-strongly convex if for all $x, y \in \mathbb{R}^m$, it holds that

$$f(x) \geq f(y) + \nabla f(y)^T(x - y) + \frac{\nu}{2} \|x - y\|^2, \tag{4}$$

where $\lambda > 0$, and $\nabla f$ is $\mu$-Lipschitz continuous if

$$f(x) \leq f(y) + \nabla f(y)^T(x - y) + \frac{\mu}{2} \|x - y\|^2. \tag{5}$$

For $0 < \nu \leq \mu < \infty$, let $F_m(\nu; \mu)$ denote the set of differentiable convex functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ that are $\nu$-strongly convex and whose gradients are $\mu$-Lipschitz continuous. We let $F_m(0; \infty)$ denote the set of convex functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$. In general, we let $\partial f$ denote the subdifferential of $f$. Following the settings (Nishihara et al. 2015), [Lasserd, Recht, and Packard 2016], we give the Lemma.

**Lemma 1.** Suppose that $f_i \in F_m(0; \infty)$ or $F_m(\nu_i; \mu_i)$, where $0 < \nu_i \leq \mu_i < \infty$ and $1 \leq i \leq n$. We denote $g_i \rightarrow \{g_i = \nabla f_i, \text{ for } f_i \text{ is strong convex}, \text{ or } g_i \in \partial f_i, \text{ for } f_i \text{ is convex.} \}$

[Then for nonnegative constants $C = diag(c_1, \ldots, c_n)$ with $c_i > 0$ we have $g(y) \geq C F(y) g(y) \geq 0$.]

In addition, we group our assumptions together in Assumption 1 (Nishihara et al. 2015). 

**Assumption 1.** We assume that $f_i (1 \leq i \leq n)$ are convex, closed, and proper. We also assume that for some $0 < \lambda_i \leq \mu_i < \infty (1 \leq i \leq n), f_i \in F_m(0; \infty)$ or $F_m(\lambda_i; \mu_i)$.

**NeuralADMM Framework**

In this section, we develop a novel NeuralADMM framework for the convergence analysis of parallel multi-block ADMM among a family of solving the subproblems. It consists of two key steps. We first cast ADMM as a discrete-time recurrent neural network (DT-RNN). Second, we study the stability of the DT-RNN for the convergence of ADMM by employing QLFs. Finally, we design variable controllers to make ADMM convergent.

**Parallel Multi-block ADMM as DT-RNN**

Before studying parallel multi-block ADMM, we cast it as a discrete-time recurrent neural network with state sequences $\{\xi^k\}$ and activation function $g(\cdot)$, and satisfying the recursions. First we define the sequences $\{\xi^k\}$ by

$$\xi^k = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_k \\ \lambda^k \end{bmatrix} = \begin{bmatrix} A_1 \lambda_1^k \\ \vdots \\ A_n \lambda_n^k \\ \lambda^k \end{bmatrix} \in \mathbb{R}^{m(n+1)}, \tag{8}$$

and the activation functions $g : \mathbb{R}^{m(n+1)} \rightarrow \mathbb{R}$ by

$$g(\cdot) = \begin{bmatrix} g_1(\cdot) \\ \vdots \\ g_n(\cdot) \\ g_\lambda(\cdot) \end{bmatrix}, \tag{9}$$

where $g_i(\cdot) \rightarrow \{ \begin{cases} \nabla g_i(\cdot), & \hat{f}_i \in F_m(\beta^{-1}; \kappa_i^{-1} \beta^{-1} \kappa_i^{\frac{1}{2}}), \\ \partial f_i(\cdot), & \hat{f}_i \in F_m(0; \infty) \end{cases}, \}$

$$\hat{f}_i = \left\{ \begin{array}{ll} (\beta^{-1} f_i) \circ A_i^{-1}, & f_i \text{ is strong convex}, \\ (\beta^{-1} f_i) + S_{X_i}, & \text{ otherwise.} \end{array} \right.$$
Based on above denoted variables and Assumption 1, ADMM is written as a discrete-time nonlinear dynamical system in following Proposition 1.

**Proposition 1.** Denote system states \( \xi^k \) in (3) with activation function \( g(\cdot) \) in (2). Suppose that Assumption 1 holds. The parallel multi-block ADMM is transformed into a discrete-time recurrent neural network:

\[
\xi^{k+1} = (W \otimes I_n)\xi^k + (W \otimes I_n)g(\xi^k) + h \otimes c,
\]

\[
\xi^0 = \phi(0),
\]

where \( \phi(0) \) is an initialized system state, \( \hat{\alpha}_i = \frac{\alpha_i}{\lambda_i} \), and

\[
W = \mathbf{I}_{n+1} - \frac{\beta}{2} \begin{bmatrix}
\beta & \beta & \cdots & \beta \\
\beta & \beta & \cdots & \beta \\
\cdots & \cdots & \cdots & \cdots \\
\beta & \beta & \cdots & \beta
\end{bmatrix},
\]

\[
h = -\begin{bmatrix}
\frac{1}{\hat{\alpha}_1}, \cdots, \frac{1}{\hat{\alpha}_i}, \cdots, \frac{1}{\hat{\alpha}_n}, 0
\end{bmatrix}^T,
\]

\[
\hat{W} = -\text{diag}\left[\frac{1}{\hat{\alpha}_1}, \cdots, \frac{1}{\hat{\alpha}_i}, \cdots, \frac{1}{\hat{\alpha}_n}, 0\right].
\]

**Proof of Proposition 1:** It is proved by calculating the gradient or the subgradient of \( x_i \), subproblem in proximal Jacobian ADMM (Deng et al. 2017) and set it to zero. Using the fact that \( A_i \) has full rank and denoting that \( \xi_i^k = A_i x_i^k \), the update rule for \( x_i \) from the problem (3) can be rewritten as follows:

\[
x_i^{k+1} = A_i^{-1} \arg \min_{\xi_i} f_i(A_i^{-1}\xi_i) + \frac{\beta}{2} \sum_{j=1, j \neq i}^n (\xi_j^k + \xi_i + \beta^{-1}x_i^k - e)^2
\]

\[
- \frac{e}{\hat{\alpha}_i} + \frac{\alpha_i^{k+1}}{\lambda_i} - \frac{\beta}{2} \|e_i - \xi_i^k - \xi_i^k\|^2. \tag{14}
\]

Since \( \|\xi_i - \xi_i^k\|^2 \leq A_i^2 \|x_i - x_i^k\|^2 \), multiplying through by \( A_i \) the problem (14) is transformed into

\[
\xi_i^{k+1} = \arg \min_{\xi_i} \frac{\beta}{2} \sum_{j=1, j \neq i}^n (\xi_j^k + \xi_i + \beta^{-1}x_i^k - e)^2
\]

\[
+ \frac{\alpha_i}{\lambda_i^2} \|\xi_i - \xi_i^k\|^2 - \frac{\beta}{2} \|\xi_i - \xi_i^k\|^2. \tag{15}
\]

Denote \( \hat{\alpha}_i = \frac{\alpha_i}{\lambda_i} \). The problem (15) implies that

\[
\xi_i^{k+1} = \xi_i^k - \frac{\alpha_i}{\hat{\alpha}_i} \sum_{j=1}^n (\xi_j^k + \frac{\beta}{\alpha_i}x_i^k)
\]

\[
- \frac{1}{\hat{\alpha}_i} g_i(\xi_i^{k+1}) + \frac{\beta}{\hat{\alpha}_i} c,
\]

and since \( g_i(x_i^{k+1}) \) is difficult to be directly calculated, it is approximated by \( g_i(x_i^k) \), and we have,

\[
\xi_i^{k+1} = \xi_i^k - \frac{\alpha_i}{\hat{\alpha}_i} \sum_{j=1}^n (\xi_j^k + \frac{\beta}{\alpha_i}x_i^k)
\]

\[
- \frac{1}{\hat{\alpha}_i} g_i(\xi_i^k) + \frac{\beta}{\hat{\alpha}_i} c. \tag{17}
\]

where \( g_i(\cdot) \) is defined in Eq. (9). Similarly, we rewrite the update rule for the Lagrange multiplier \( \lambda \) in proximal Jacobian ADMM (Deng et al. 2017) as

\[
\lambda^{k+1} = \lambda^k - \gamma \beta \sum_{j=1}^n (\xi_j^k - c). \tag{18}
\]

Together, (17) and (18) confirm the relationship matrices in Proposition 1. The proof is complete. \( \square \)

**Remark 1:** Proposition 1 converts multi-block ADMM into a recurrent neural network system. Thus, the convergence of ADMM is analyzed by studying the stability problems of the recurrent system. Fortunately, these problems have been widely studied in the neural networks (Hopfield 1982, Hu and Wang 2002) and control theory (Liberzon, Hespanha, and Morse 1999, Lin and Antsaklis 2009). This provides an effective way to analyze the convergence of ADMM by using the control methods, especially, quadratic Lyapunov functions (Mason and Shorten 2004, Daafouz, Riedinger, and Jung 2002).

**Remark 2:** When \( D_i = \alpha_i I \) in the problem (3), the parallel multi-block ADMM is also transformed into a discrete-time recurrent neural network system (10) with the following weight matrices

\[
W = \text{diag}\left[\frac{\hat{\alpha}_1}{\alpha_1 + \beta}, \cdots, \frac{\hat{\alpha}_i}{\alpha_i + \beta}, \cdots, \frac{\hat{\alpha}_n}{\alpha_n + \beta}, 0\right],
\]

\[
h = -\begin{bmatrix}
\beta & \beta & \cdots & \beta \\
\cdots & \cdots & \cdots & \cdots \\
\beta & \beta & \cdots & \beta
\end{bmatrix},
\]

\[
\hat{W} = -\text{diag}\left[\frac{1}{\alpha_1 + \beta}, \cdots, \frac{1}{\alpha_i + \beta}, \cdots, \frac{1}{\alpha_n + \beta}, 0\right].
\]

**Remark 3:** Similar to the works (Nishihara et al. 2015, Lessard, Recht, and Packard 2016), the dimensions of the weight matrices in (10) do not depend on any problem parameters although \( \hat{\alpha}_i = \frac{\alpha_i}{\lambda_i} \) in the weight matrices depend on the norm of the parameter \( A_i \). When the number of the parameters in ADMM is small, we can consider the dependent-parameters DT-RNN. Denote system states \( x^k = (x^k_1, \cdots, x^k_n, (\lambda^k)^T) \), with gradients \( g(\cdot) \) in (9).

Suppose that Assumption 1 holds. The parallel multi-block ADMM (3) can be transformed into the following system:

\[
x^{k+1} = Rx^k + \bar{R}g(x^k) + b, \quad x^0 = \phi(0), \tag{22}
\]

where \( \phi(0) \) is an initialized system state, \( N = \sum_{i=1}^n n_i + m \), and

\[
R = I_N - \begin{bmatrix}
\beta I_n A^T A & I_n & I_n \\
\gamma & 0 & 0
\end{bmatrix}, \quad \bar{R} = -\text{diag}\left[I_{\alpha}, 0 I_m\right],
\]

\[
A = [A_1, A_2, \cdots, A_N], I_n = \text{diag}[\alpha_1^{-1} I_{n_1}, \cdots, \alpha_n^{-1} I_{n_n}],
\]

\[
b = \begin{bmatrix}
\beta I_n A^T c \\
\gamma \beta c
\end{bmatrix}. \tag{23}
\]
To further simplify the discrete-time recurrent neural network system (10), the equilibrium point \( \xi^* = (\xi_1^*, \ldots, \xi_n^*)^T \) of (10) is shifted to the origin by the transformation \( \tilde{\xi}_i = \xi_k^* - \xi_i^* \) and \( \lambda^k = \lambda_k - \lambda^* \), which converts the system to the following form:

\[
\tilde{\xi}_{k+1} = (W \otimes I_m)\tilde{\xi}_k + (W \otimes I_m)g(\tilde{\xi}_k),
\]

where \( \tilde{\xi}_k^i = ([\tilde{\xi}_1^i]^T, \ldots, [\tilde{\xi}_n^i]^T, [\lambda^k]^T)^T \) is the state vector of the transformed system, \( g(\tilde{\xi}_k) = [g_1^T(\tilde{\xi}_k), \ldots, g_n^T(\xi_k), \lambda^T(\lambda_k)]^T \), and \( g_i(\xi_i^k - \xi_i^*) - g_i(\xi_i^*) \) and \( \lambda^T(\lambda_k - \lambda^*) - g_\lambda(\lambda^*) \). Due to the transformation, \( g(\cdot) \) also satisfies \( g_i(\cdot) \) in (2).

In addition, the definition of global exponential stability for the dynamical system is now given.

**Definition 1.** (Wu, He, and She 2010) The system (10) is said to be exponential stability, if there exist constants \( \chi, \rho > 0 \) and nonnegative constants \( C = \text{diag}(c_1, \ldots, c_n, c_{n+1}) > 0 \) such that

\[
\|\xi^k - \xi^*\|_2 \leq \chi^k\|\xi^0 - \xi^*\|_2.
\]

Definition 1 shows that ADMM will be convergent as \( \xi^* \) when the system (10) is exponentially stable.

**Finding Convergence Conditions for Problem 1**

In this subsection, we propose convergence conditions for ADMM with arbitrary switching sequences by employing quadratic Lyapunov functions (QLFs) (Mason and Shorten 2004) to ensure the stability of the DT-RNN system (10) as follows.

**Theorem 1.** Suppose that Assumption 1 holds. Fixing \( 0 < \tau < 1 \), if there exist a \( (n + 1) \times (n + 1) \) positive definite matrix \( P = P^T > 0 \) and nonnegative constants \( C = \text{diag}(c_1, \ldots, c_n, c_{n+1}) > 0 \) such that the linear matrix inequalities

\[
\mathcal{Y} = \begin{bmatrix}
W^T PW - \tau_2 P + CF_1 & W^T PW + CF_2 \\
W^T PW + CF_2 & W^T PW + CF_3
\end{bmatrix} \leq 0,
\]

where \( F_1, F_2, \text{ and } F_3 \) are defined in (6), then the system (10) is exponentially stable, that is,

\[
\|\xi^k - \xi^*\|_2 \leq \sqrt{\rho}P^{k}\|\xi^0 - \xi^*\|_2.
\]

**Proof of Theorem 1:** Construct a quadratic Lyapunov function (Mason and Shorten 2004):

\[
V(k) = (\tilde{\xi}_k^T)P\tilde{\xi}_k,
\]

where \( P > 0 \) is to be determined. Defining

\[
\Delta V(k) = V(k+1) - V(k) = (\tilde{\xi}_{k+1}^T)P\tilde{\xi}_{k+1} - (\tilde{\xi}_k^T)P\tilde{\xi}_k,
\]

and adding a convergence rate \( \tau \) into \( \Delta V(k) \) yield

\[
\Delta V(k) \leq (\tilde{\xi}_{k+1}^T)P\tilde{\xi}_{k+1} - \tau^2(\tilde{\xi}_k^T)P\tilde{\xi}_k
\]

From the Lemma 1, adding the terms on the left of (7) and \( \xi^{k+1} \) in (24) to (30) yields

\[
\Delta V(k) \leq (\tilde{\xi}_{k+1}^T)P\tilde{\xi}_{k+1} - \tau^2(\tilde{\xi}_k^T)P\tilde{\xi}_k
\]

Thus, it holds (27). The proof is complete. \( \square \)

**Remark 4:** Our framework is different from the integral quadratic constraint (IQC) method (Lessard, Recht, and Packard 2016; Nishihara et al. 2015) although they come from the control theory. IQC converts the ADMM into a linear dynamical system with nonlinear feedbacks, which are the gradients of the functions \( f_i(\cdot) \), while our framework transforms the multi-block ADMM into a discrete-time recurrent neural network as the gradients of the functions \( f_i(\cdot) \) can be regarded as the activation functions in neural networks. More importantly, our works give a new direction to study multi-block ADMM by using the recurrent neural networks (Hopfield 1982; Wu, He, and She 2010).
Figure 2: (a) plots the convergence rates $\tau$ with different $\beta$ and $\alpha$ when $\gamma = 1$ and $n = 10$. (b) plots the recover error with different $\alpha$ when $\beta = 0.7$ and $\gamma = 1$. (c) also plots the recover error with different $\beta$ when $\alpha = 50$ and $\gamma = 1$.

Remark 5: The sufficient condition in Theorems 1 can ensure the convergence of multi-block ADMM under arbitrary sequences on the variables $x_i$ ($1 \leq i \leq n$). Based on the Proposition 1, different sequences can build the weight matrices with the different rows in the DT-RNN system since switching the variable $x_i$ into $x_j$ correspond to switching the $i$-th row into the $j$-th one. In fact, given a switching sequence, these conditions can also judge that whether multi-block ADMM is convergent or not.

Next, we consider a linear transformation of the DT-RNN system (10). When the gradient functions $g(\xi^k)$ can be (approximately) linearized as $(W \otimes I_m)\xi^k$ by Taylor formula, the DT-RNN system (10) is rewritten as a following linear system (proximately) linearized as $\dot{x} = (W + \hat{W})x - \hat{P} + h \otimes c$.

$\xi^{k+1} = ((W + \hat{W}) \otimes I_m)\xi^k + h \otimes c, \xi^0 = \phi(0), \quad \text{(35)}$

where $W$ and $\hat{W}$ are defined in (11) and (12). Based on the Theorem 1, we have the following corollary.

Corollary 1. Suppose that Assumption 1 holds. Fixing $0 < \tau < 1$, the system (35) is exponentially stable (that is, (27)), if there exist a $(n+1) \times (n+1)$ positive definite matrix $P = P^T > 0$ such that the linear matrix inequalities

$\hat{P} = (W + \hat{W})^T P (W + \hat{W}) - \tau^2 P \preceq 0. \tag{36}$

Designing Feedback Controllers for Problem 2

In order to control ADMM algorithms, we design state feedbacks into the DT-RNN system (10) as follows.

$\xi^{k+1} = (W \otimes I_m)\xi^k + (W \otimes I_m)g(\xi^k) + u^k + h \otimes c, \quad \xi^0 = \phi(0), \tag{37}$

where $W$ and $\hat{W}$ are respectively defined in (11) and (12), and $u^k$ is the control input. In this paper, $u^k$ is set as $(K \otimes I_m)\xi^k$. After the linear transformation and substitution, (37) is rewritten as:

$\dot{\xi} = ((W + \hat{W} + K) \otimes I_m)\tilde{\xi}, \quad \tilde{\xi}^0 = \phi(0), \tag{38}$

where $\tilde{\xi}$ and $\hat{W}$ are defined in (24) and (35), respectively.

Since the switched linear system (35) is transformed by the ADMM algorithm in (3), the state feedbacks $u^k = (K \otimes I_m)\xi^k$ easily control the ADMM algorithm, that means, we add a new controlling term $\frac{1}{2}||\xi^k||^2_{K \otimes I_m}$ into the problem in (3) for obtaining better convergence rates and making ADMM (more) stable. According to corollary 1, designing the feedbacks $K$ reduces to find $P$ and $K$ such that

$(W + \hat{W} + K)^T P (W + \hat{W} + K) - \tau^2 P \preceq 0. \tag{39}$

Generally, the problem of solving numerically (39) for $(P, K)$ is very difficult since it is nonconvex. In order to make this problem numerically well tractable, a sufficient condition is given in the following corollary by employing the Schur complement (Boyd et al. 1994).

Corollary 2. If there exist matrices $N$ and positive definite matrices $Q = Q^T > 0$ such that:

$\begin{bmatrix} -\tau^2 Q & Q \hat{W}^T + N \\ \hat{W} Q + N^T & -Q \end{bmatrix} \preceq 0, \tag{40}$

where $\hat{W} = W + \hat{W}$, then the state feedbacks $K = NQ^{-1}$ stabilizes (37).

Numerical Experiments

In this section, three numerical experiments are used to demonstrate our theoretical results in the Lyapunov framework for the two basic problems in the introduction. In the first two experiments, we will do experiments to confirm the convergence of the parallel ADMM. More importantly, the third experiment shows that we can design state feedbacks to make a non-convergent ADMM problem convergent.

We consider the square of $\ell_2$ or $\ell_1$-minimization problems for finding regression or sparse solutions of a linear system:

$\min_x \frac{1}{2}||x||^2, \quad \text{s.t.} \quad Ax = c, \tag{41}$

$\min_x ||x||_1, \quad \text{s.t.} \quad Ax = c, \tag{42}$

where $x \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times N}$ and $c \in \mathbb{R}^m$. They have been widely used in signal and image processing, statistics, and machine learning. Suppose that the data are partitioned into $n$ blocks: $x = [x_1, x_2, \ldots, x_n]$ and $A = [A_1, A_2, \ldots, A_n]$, $N = \sum_{i=1}^n n_i$, and $n_i$ is the dimensions of the variable $x_i$. 
In the first two experiments, a solution $x^*$ is randomly generated with $p (p < n)$ nonzeros drawn from the standard Gaussian distribution. $A$ is also randomly generated from the standard Gaussian distribution, and its columns are normalized. $x$ and $A$ are partitioned evenly into $n$ blocks. The vector $c$ is then computed by $c = Ax^* + \delta$, where $\delta \sim N(0, \sigma^2I)$. Gaussian noise with standard deviation $\sigma$. We will find the minimal rate $\tau$ by performing a binary search over $\tau$ such that the linear matrix inequalities $\tilde{Y} \preceq 0$ or $\tilde{Y} \succeq 0$ is satisfied. (their feasibilities are a semidefinite program with variables $P$, and they are easily solved by the LMI toolbox in the Matlab.) In addition, we also measure the relative error $\frac{||x^k - x^*||_2}{||x^*||_2}$ in the parallel ADMM.

**Experiment 1.** Consider $N = 1000, m = 2000, p = 1000$ and the standard deviation of noise $\sigma$ is set to be $10^{-6}$, respectively. We set the number of blocks $n = 10$. Since the activation function is $g(x) = x = \nabla \frac{1}{2}||x||_2^2$ in the problem (41), it can be linearly transformed into the DT-RNN system (35). Using the Corollary 1, we search the convergence rates $\tau$ by solving the LMI condition $\tilde{Y} \preceq 0$ in (36) when $\gamma = 1$, $\beta$ changes from 0.1 to 1, and five values of $\alpha = \alpha_i$ $(1 \leq i \leq 10)$ are spaced between 10 and 200. The minimal convergence rates $\tau$ are shown in Fig. 2(a). For example, when $\alpha = 10$ and $\beta = 0.7$, we obtain $\tau = 0.9294$ and $\tau^{180} = 1.8901 \times 10^{-6}$, that is, ADMM will converge, and Fig. 2(b) also verifies the blue curve decreases to $10^{-6}$ after the $k = 180$ iterations. Moreover, the convergence curves in Fig. 2(a) and (b) show ADMM is convergent.

**Experiment 2.** Consider $N = 2000, m = 1000, p = 100$ and the standard deviation of noise $\sigma$ is set to be $10^{-4}$, respectively. We set the number of blocks $n = 20$. The shrinkage function $g(x) = \max(|x| - r, 0) \text{sign}(x)$ can be regarded as an activation function since it is usually used to solve the $\ell_1$-problem (42). Moreover, $g(x)$ satisfies $g(x) = \rho x$, where $0 \leq \rho < 1$ and $\rho = \{ 0, \frac{|x|}{|x|}, \frac{|x|}{|x|} \}$. Therefore, the problem (42) can be linearly transformed into the DT-RNN system (35). Similar to Experiment 1, we obtain the minimal convergence rates $\tau$ shown in Fig. 3(a) by using the Corollary 1. Moreover, Fig. 3(a) and (b) plot the convergence curves to show that ADMM is convergent. We observe that the iterations will decrease with $\beta$ from 0.1 to 1 in Fig. 3(b). However, the convergence rate $\tau$ does not decrease. The plausible reason is the linearization of the shrinkage function. Fortunately, $\tau < 1$ can ensure the convergence of ADMM.

**Experiment 3.** (Chen et al. 2016) Consider the following strongly convex minimization problem with three variables:

$$\min \limits_{x_1, x_2, x_3} 0.05x_1^2 + 0.05x_2^2 + 0.05x_3^2,$$

subject to $A_1 x_1 + A_2 x_2 + A_3 x_3 = 0$, where $A_1 = [1, 1, 1]^T$, $A_2 = [1, 1, 2]^T$ and $A_3 = [1, 2, 2]^T$. This problem (43) with $\beta = 1$ is divergent (Chen et al. 2016). Using the Remark 3 and introducing Lagrange multipliers $\lambda = [\lambda_1, \lambda_2, \lambda_3]^T$, it is transformed into a DT-RNN system (22) with the state $x = [x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3]^T$ as the number of parameters is small in the problem (43). By employing the condition (40) in Corollary 2 with $\gamma = 1$, $\alpha = 1$ and $\tau = 0.9$, we design a variable feedback $K$ to control the non-convergence ADMM to be convergent, where

$$K = \begin{bmatrix} -0.6500 & 0.0400 & 0.0100 & 0.1000 & -0.2000 & -0.0000 \ 0.0400 & -0.7200 & 0.0200 & -0.0000 & -0.2000 & 0.2000 \ 0.0100 & 0.0200 & -0.8800 & 0.1000 & -0.0000 & 0.1000 \ 0.1000 & -0.0000 & 0.1000 & -1.0000 & 0.0000 & 0.0000 \ -0.2000 & 0.2000 & 0.0000 & 0.0000 & -1.0000 & 0.0000 \ -0.0000 & 0.2000 & 0.1000 & -0.0000 & -0.0000 & -1.0000 \ \end{bmatrix}.$$  

We add the norms $||x||_K$ into the DT-RNN system (22) and the problem (43) to make it convergent.

**Conclusion**

In this paper, we developed a novel NeuralADMM framework to study the two basic problems of the multi-block ADMM: finding the convergence conditions, and controlling its convergence. First, we transformed the multi-block ADMM into a discrete-time recurrent neural network. Second, we employed quadratic Lyapunov functions to provide sufficient convergence conditions to guarantee that multi-block ADMM is convergent by studying the stability of the network. Second, we designed feedbacks to make the multi-block ADMM convergent when the convergence sufficient conditions cannot be provided. Three experiments were further conducted to demonstrate the two basic problems were easily solved by our theoretical results. More importantly, this paper provided a new direction to analyze the multi-block ADMM by employing the recurrent neural networks.
