The circular noise is important in connection to Mach’s principle, and also as a possible probe of the Unruh effect. In this letter the power spectrum of the detector following the Trocheries-Takeno motion in the Minkowski vacuum is analytically obtained in the form of an infinite series. A mean distribution function and corresponding energy density are obtained for this particular detected noise. The analogous of a non constant temperature distribution is obtained. And in the end, a brief discussion about the equilibrium configuration is given.

KEY WORDS: circular Unruh effect
yields, in first order perturbation theory the following expression for the transition rate

\[ G^+(x_1, x_2) = \frac{1}{4\pi^2} \frac{1}{-(t_1 - t_2) + (x_1^2 - x_2^2)^2} \]

\[ G^+(x(s_1), x(s_2)) = \langle 0_M | \phi^\dagger(x^\mu(s_1)) \phi(x^\mu(s_2)) | 0_M \rangle \]

\[ P(E) = C(E) \lim_{\tau_0 \to \infty} \frac{1}{2\tau_0} \int_{-\tau_0}^{+\tau_0} \int_{-\tau_0}^{+\tau_0} e^{-iE(s_1 - s_2)} G^+(x(s_1), x(s_2)) ds_1 ds_2, \tag{2} \]

where \( E > 0 \) is the energy difference between the initial and final states of the detector and \( G^+(x, x') \) is the positive frequency Wightman function.

\( C(E) \) depends on the internal constituency of the detector and a detailed discussion of it is given in [3] and [8]. While the second term in (2) corresponds to the noise this detector is submitted to, that’s to say, on the way the field fluctuates as seen by the observer in his trajectory \( x^\mu(r) \).

It is an easy exercise to show that a detector at rest in the Trocheries-Takeno coordinates \((dr' = 0, d\theta' = 0, dz' = 0)\)

\[ t = t', \cos \Omega r' - r' \theta' \sin \Omega r' \]

\[ r = r' \]

\[ \theta = \theta' \cos \Omega r' - \frac{t'}{r'} \sin \Omega r' \]

\[ z = z', \tag{3} \]

according to (2), has a transition rate given by

\[ P(E) = \frac{1}{16\pi^2 \coth^2 \Omega r} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-iEs} (s/2)^2 - |r/cos(\Omega r)|^2 \sin((s/2) \sin(\Omega r) r^{-1})^2 ds, \tag{4} \]

where \( s = t'_0 - t'_1 \) is the proper time between two points in the trajectory of the detector. After a change in the integration variable the above transition rate, is written as

\[ P(E) = \frac{\omega}{8\pi^2 \gamma} \int_{-\infty}^{\infty} \exp[-i(2E/\gamma\omega)x] dx (x^2 - v^2 \sin(x)^2), \tag{5} \]

where

\[ \omega = \tanh(\Omega r), \omega \propto \Omega, \ r \to 0 \]

\[ v = \Omega r, v < 1 \]

\[ \gamma = \cosh(\Omega r) = \frac{1}{\sqrt{1 - v^2}} \]

The fact that \( v < 1 \), comes from the type of trajectory that the detector is following. In this sense, this trajectory is more physical, because the detector is not allowed to travel faster than the speed of light. In (2), the \( 1/(2\tau_0) \) factor is associated to the adiabatic switching of the detector and the usual \(-i\epsilon\) prescription is related to its size. As \( v < 1 \), and the integral is over the real axis, the integrand in (5) can be written as

\[ P(E) = \frac{\omega}{8\pi^2 \gamma} \int_{-\infty}^{+\infty} dx \exp[-i(2E/\gamma\omega)x] \sum_{n=0}^{\infty} \left( \frac{\epsilon \sin(x - i\epsilon)}{x - i\epsilon} \right)^{2n} \sum_{k=1}^{\infty} \left( \frac{\epsilon}{2n + 2k - 1} \right)^{2n+2k-1} \Theta(-E + k\omega\gamma) \tag{6} \]

After a binomial expansion, this last integral is evaluated using ordinary residue calculus

\[ P(E) = \frac{\omega}{2\pi\gamma v} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[ \frac{v(-E/\gamma\omega + k)}{-n - 1}(2n + 2k - 1)(n - 1)! \right]^{2n+2k-1} \Theta(E - k\omega\gamma) \tag{7} \]

where \( \Theta(x) \) is the Heaviside step function, which indicates that the rotational motion is the thermal reservoir. The time scales of the detector \( E \), and the (proper) period of rotation of the detector \( \gamma T' = 2\pi/\omega \) are singled out in (7), a property also shown in [9] and [11].
As a spectrum, \( T \), can be divided by the energy \( E \) and an effective distribution function for the scalar particles can be obtained

\[
n(E) = \frac{\omega}{2\pi\gamma vE} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{[v (-E/(\gamma \omega) + k)]^{2n+2k-1}}{(-1)^{n-1}(2n + 2k - 1)(n-1)!(n + 2k - 1)!} \Theta(-E + k\omega\gamma).
\]  

(8)

Effective here is in a certain sense, that the distribution function is dependent on the measuring apparatus. Other consequences connected to the motion, should be perceived by other types of detectors. Only the detectable properties are taken into account. This distribution function is understood as the mean number of particles per volume, with energy \( E \) perceived by the detector. In the following, density will be assumed to be per unit volume.

In this spirit, the total energy density of the scalar particles is given by the following integral, which can be written in terms of the hypergeometric function

\[
E_t = \int_0^\infty E n(E) dE = \frac{\omega}{2\pi\gamma v} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_0^k dE \frac{[v (-E/(\gamma \omega) + k)]^{2n+2k-1}}{(-1)^{n-1}(2n + 2k - 1)(n-1)!(n + 2k - 1)!}.
\]

(9)

\[
\begin{align*}
E_t &= \frac{\omega^2}{2\pi v} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_0^k dx \frac{[v (-x + k)]^{2n+2k-1}}{(-1)^{n-1}(2n + 2k - 1)(n-1)!(n + 2k - 1)!} \\
E_t &= \frac{1}{2\pi r^2} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\Theta(k(k+1/2)(k+1)!)}{(-1)^{n-1}(2n + 2k - 1)(2n + 2k - 1)(n+1)!(n + 2k - 1)!} + \Theta(k(k+1/2)(k+1)!(k+1)!).
\end{align*}
\]

(10)

where \( \Omega \) plays a role similar to the temperature, and there is an explicit dependence on the coordinate \( r \). This corresponds to a non constant temperature distribution. For instance, in the above expression (10), at the origin when \( r = 0 \), the energy density is \( E_t = 0 \), which makes sense, because the motion of detector when \( r = 0 \) is inertial.

Next, the convergence of the infinite summation (9), is discussed. The second derivative with respect to \( v \) of (9) is of the type

\[
\sum_{k=0}^{\infty} (vk)^{2k+2} \sum_{n=0}^{\infty} \frac{(-v^2k^2)^n}{n!} = \sum_{k=0}^{\infty} (vk)^{2k} \frac{J_{2k}(2kv)}{v^2(kv)^{2k}} = \sum_{k=1}^{\infty} k^2 J_{2k}(2kv)
\]

\[
\sum_{k} k^{3/2} \sqrt{4\pi v} \sqrt{1-v^2} \left( \frac{ve\sqrt{1-v^2}}{1 + \sqrt{1-v^2}} \right)^{2k},
\]

(11)

the last expression is valid asymptotically when \( k \to \infty \) and \( 0 < v < 1 \). In this asymptotic spirit, the summation (11) can be replaced by the integral over \( k \), from 0 to \( \infty \), which results in

\[
g(v) = \frac{3\sqrt{\pi}}{4} \left| \ln \left( \frac{ve\sqrt{1-v^2}}{1 + \sqrt{1-v^2}} \right) \right|^{-5/2}.
\]

The original summation (9) is obtained by integrating the last equation two times in \( v \)

\[
\int_0^v dv' \int_0^{v'} dv'' g(v''),
\]

which is plotted numerically in the following
The above graphic indicates, in a indirect way, that the summation given in (9) and (10) converges, except when \( v \to 1 \). This divergence of the total energy density of the scalar particles perceived by the detector, is expected when the detector is moving (almost) with the velocity of light, \( v \to 1 \).

By summing the first few terms in (10), the following total energy density of scalar particles is obtained at a given \( r \), for \( \Omega = 0.2 \) and \( \Omega = 0.16 \), showing that for a same distance \( r \) from the origin, there is an increase in the energy for larger values of \( \Omega \).

It should be stressed that in spite that (10) describes a non constant temperature distribution, it is an equilibrium configuration. This interesting equilibrium concept is well known and is expected if the space being considered is not homogeneous and isotropic as in Trocheries-Takeno case, see for example [14]. The external condition that originates this motion is responsible for this apparent non equilibrium. The above construction avoids canonical quantization, and it could be usefulness to a further understanding of external conditions.

Acknowledgments

The author acknowledges the brazilian agencies Funpe and Finatec for partial support, and an anonymous referee for improvements.

Note added in proof: It is a pleasure to thank Dr. Marcelo Schiffer which was my Msc. advisor.

[1] Unruh W G 1976 Phys. Rev. D 14 870; Davies P C W 1975 J. Phys. A 8 609; Fulling S A 1973 Phys. Rev. D 10 2850
[2] DeWitt B S 1979 in General Relativity, ed. S. W. Hawking and W. Israel (Cambridge University Press) p. 680
[3] Audretsch J and Müller R 1994 *Phys. Rev. A* **50** 1755
[4] Dalibard J, Dupont-Roc J. and Cohen-Tannoudji C. 1982 *J. Phys.* (Paris) **43** 1617
[5] Takagi S 1988 *Prog. Theor. Phys.* **88** 1
[6] Unruh W G and Wald R M 1984 *Phys. Rev. D* **29** 1047
[7] Bell J S and Leinaas J M 1983 *Nucl. Phys.* **B212** 131; Leinaas J M 2001 Preprint [hep-th/0101054](http://arxiv.org/abs/hep-th/0101054)
[8] Audretsch J, Müller J R and Holzmann M 1995 *Class. Quant. Grav.* **12** 2927
[9] De Lorenci V A and Svaiter N F 1999 *Found. Phys.* **29** 1233; De Lorenci V A, De Paola R D M and Svaiter N F 2000 *Class. Quant. Grav.* **17** 4241
[10] Trocheries M G 1949 *Phyl. Mag.* **40** 1143; Takeno H 1952 *Prog. Theor. Phys.* **7** 367
[11] Davies P C W, Dray T and Manogue C A 1996 *Phys. Rev. D* **53** 4382
[12] Sciama D W, Candelas P and Deutsch D 1981 *Adv. Phys.* **30** 327
[13] Whittaker E T and Watson G N 1963 *A Course of Modern Analysis*, Cambridge at the University Press, p. 369.
[14] Stefani H 1993 *General Relativity: An introduction to the theory of the gravitational field*, Cambridge University Press, P. 80-83.