AN EXACT INTERIOR EXTENSION OF A STATIC SOLUTION FOR AN ELECTRIC CHARGED BALL

A.M. Baranov$^1$ and Z.V. Vlasov$^2$

Dep. of Theoretical Physics, Krasnoyarsk State University, 79 Svobodny Prosp., Krasnoyarsk, 660041, Russia

Received 11 October 2005

We use the model approach to the description of spherical gravitating static fluid ball with an electric charge in general relativity. The metric is written in Bondi’s coordinates. The total energy-momentum tensor (EMT) is chosen as a sum of the EMT of a Pascal perfect fluid and that of the electromagnetic field. An exact solution of the Einstein-Maxwell equations is found, extending a similar solution with parabolic mass density distribution.

1. Introduction

Models of electric charged stars represent special interest despite the exotic nature of the problem. The problem of finding exact solutions of the Einstein-Maxwell equations does not lose its urgency. The electric charge is one of a few quantities which do disappear in stellar collapse. A difficulty of finding exact solutions is connected with the nonlinearity of the gravitational equations.

The Reissner-Nordstrem solution (see [1]) is an exact exterior solution of the Einstein-Maxwell equations. It describes the exterior field of a static charged star. But the problem of finding internal solutions of Einstein-Maxwell equations is harder. One exact solution of this problem is presented in [2], [3]. It describes a charged perfect Pascal fluid with a parabolic distribution of mass density, but without introducing a specific equation of state.

In the present article, we present an extension of the model of [2], [3], connected with an extension of the mass and charge density distributions. A new exact static solution for a charged fluid ball is obtained.

2. The metric and the Einstein-Maxwell equations

We consider an interior static star model as a ball filled with an electrically charged fluid, using the Einstein-Maxwell equations. The metric is

$$ds^2 = F(r) dt^2 + 2L(r) dt dr - r^2 (d\theta^2 + \sin \theta^2 d\phi^2)$$

with radial ($r$) and angular ($\theta$, $\phi$) variables. The vacuum velocity of light is chosen to be equal to unity.

The interior energy-momentum tensor (EMT) is a sum of the perfect fluid and electromagnetic EMTs:

$$T_{\alpha\beta} = T_{\alpha\beta}^{\text{fluid}} + T_{\alpha\beta}^{\text{em}},$$

where

$$T_{\alpha\beta}^{\text{fluid}} = (\mu + p) u_\alpha u_\beta - p g_{\alpha\beta};$$

$$T_{\alpha\beta}^{\text{em}} = \frac{1}{4\pi} \left(-F_{\alpha\sigma} F_{\beta}^{\sigma} + \frac{1}{4} F_{\sigma\tau} F^{\sigma\tau}\right)$$

and $F_{\alpha\beta}$ is the electromagnetic field tensor.

The Einstein-Maxwell equations are

$$(F/xL^2)(\ln L)' = \chi(p + \mu);$$

$$(F/xL^2)(\ln L)' - (1/2L^2)(F'' + 2F'/x - F'(\ln L)') = -\chi(p + W_{el});$$

$$1/x^2(-1 + (F/L^2) + (xF'/L^2 - xF'(\ln L)'/L^2) = -\chi((\mu - p)/2 + W_{el});$$

$$(x^2F/L)' = 4\pi R_0 x^2 L/\sqrt{F},$$

where $t = d/dx$, $x = r/R$, $0 \leq x \leq 1$, $R$ is the stellar radius, $\mu(x)$ is the mass density, $p(x)$ is the pressure, $W_{el} = E^2/(8\pi L^2)$ is the observable electric field energy density, $E$ is the electric field strength, $\rho(x)$ is the electric charge density, $\chi = \varepsilon R^2 = 8\pi R^2$. After the replacement $\varepsilon = F/L^2$ and with the new variable

$$d\zeta = \frac{xdx}{x\sqrt{1 - \Phi(x)}}$$

the set of the Einstein-Maxwell equations is reduced to the nonlinear spatial oscillator equation

$$G'' + \Omega^2(\zeta(x))G = 0,$$

written in terms of the new variable $\zeta$ and $G = \sqrt{F}$ = $\sqrt{\Phi}$.

Here

$$\Omega^2 = -\frac{d}{dy}\left(\frac{\Phi}{y}\right) - \frac{2\chi}{y} W_{el},$$
where the function $\Phi(x)$ plays the role of Newton’s gravitational potential inside the star,
\[
\Phi = 1 - \varepsilon = \frac{\lambda}{x} \int (\mu(x) + W_{el}(x)) x^2 dx. \tag{11}
\]
The pressure can be written as
\[
\chi p = \chi W_{el} - \Phi^2 + \frac{1}{x}(1 - \Phi)(\ln F)'. \tag{12}
\]
In [2], [3], the mass density distribution is taken as parabolic function
\[
\mu(x) = \mu_0(1 - b_0 x^2), \tag{13}
\]
where $b_0 = (\mu_0 - \mu_R)/\mu_0 \leq 1$; $\mu_0 = \mu(x = 0)$; $\mu_R = \mu(x = 1)$.

The energy density of the electric field is taken in [2], [3] by analogy to classical electrostatics:
\[
\varepsilon W_{el} = \lambda_0^2 x^2 \tag{14}
\]
with $\lambda_0 = const$, and electric charge density was found in form
\[
\rho(x) = \rho_0 \sqrt{1 - \Phi(x)}, \tag{15}
\]
where $\rho_0 = \rho(x = 0)$.

If we take the mass density as the function
\[
\tilde{\mu}(x) = \mu_0(1 - b x^2)^3 \tag{16}
\]
and the electric field energy density as
\[
\varepsilon W_{el} = \lambda(x^2) x^2 \tag{17}
\]
with
\[
\lambda(x) = \frac{4\pi R \rho_0}{3}(1 - 3 a x^2/5), \tag{18}
\]
\[
b = 1 - (\tilde{\mu}(x = 1)/\mu_0)^{1/3} = const; \quad a = 80b/63 \quad (as \ an \ extension \ of \ the \ result \ of \ [2], [3]), \]
we obtain an exact static solution of the Einstein-Maxwell equations [4]
\[
G(x) = G_0 \cos(\Omega_0 \cdot \zeta(x) + \alpha) \tag{19}
\]
as the solution of the harmonic spatial oscillator equation
\[
G''_{\zeta \zeta} + \Omega_0^2 G = 0 \tag{20}
\]
with the new electric charge density
\[
\tilde{\rho}(x) = \rho_0(1 - a x^2) \sqrt{1 - \tilde{\Phi}(x)}, \tag{21}
\]
where
\[
a = 1 - \frac{\rho(x = 1)}{\rho_0 \sqrt{1 - \tilde{\Phi}(x = 1)}} = const, \tag{22}
\]
and here
\[
\Omega_0^2 = \frac{\chi}{60^2(1701 \mu_0 a - 1760 \pi \rho_0^2 R^2)}. \tag{23}
\]

\[
\Phi(x = 1) = \eta_0 - \frac{Q^2}{R^2} \equiv \eta^*, \tag{24}
\]
where $Q$ is the integral electric charge of the star; $\eta_0 = 2m/R$ is the compactness of the stellar model without an integral electric charge, and $m$ is the integral mass of charged fluid ball.

The general expression for the metric function $g_{00} = F = G^2$ is
\[
F(x) = G_0^2 \cos^2(\Omega_0 \cdot \zeta(x) + \alpha_0) \tag{25}
\]
and the function $g_{01} = L$ can be found easily from $L = (F/(1 - \tilde{\Phi}))^{1/2}$.

Here the Reissner-Nordstrem solution is the exterior solution. From the boundary conditions we have
\[
G_0 = \left(1 - \eta^* + \frac{2\eta^* - \eta}{2\Omega_0} \right)^{1/2} \tag{26}
\]
and
\[
\tan(\Omega_0 \zeta(x = 1) + \alpha_0) = -\frac{2\eta^* - \eta}{2\Omega_0 \sqrt{1 - \eta^*}}. \tag{27}
\]

3. Summary

This article demonstrates the Einstein-Maxwell’s equations reduction in Bondi’s coordinates to the equation of a nonlinear spatial oscillator. The source of gravitational field is a perfect electrically charged Pascal fluid. The assumption on the behavior of the mass-energy density and that of the of electric field was used here. This solution is an extension of the solution found earlier in [2], [3].

References

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