Can Seiberg-Witten Map Bypass
Noncommutative Gauge Theory No-Go Theorem?

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Abstract

There are strong restrictions on the possible representations and in general on the
matter content of gauge theories formulated on noncommutative Moyal spaces, termed
as noncommutative gauge theory no-go theorem. According to the no-go theorem [1],
matter fields in the noncommutative $U(1)$ gauge theory can only have $\pm 1$ or zero
charges and for a generic noncommutative $\prod_{i=1}^{a} U(N_{i})$ gauge theory matter fields can
be charged under at most two of the $U(N_{i})$ gauge group factors. On the other hand, it
has been argued in the literature that, since a noncommutative $U(N)$ gauge theory can
be mapped to an ordinary $U(N)$ gauge theory via the Seiberg-Witten map, seemingly
it can bypass the no-go theorem. In this note we show that the Seiberg-Witten map
[2] can only be consistently defined and used for the gauge theories which respect the
no-go theorem. We discuss the implications of these arguments for the particle physics
model building on noncommutative space.
1 Introduction

Motivated by string theory considerations (e.g. see [2]), the possibility of having a noncommutative (NC) Moyal spacetime, with the coordinate operators satisfying the commutation relations:

\[ [\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu}, \]

with \( \theta^{\mu\nu} \) being a given constant tensor, as physical spacetime, has been under intense study. Using the Weyl-Moyal correspondence, there is a simple recipe for constructing quantum field theories on NC Moyal spaces: replace the usual product between the fields in the action by the Moyal star product,

\[
(f \star g)(x) = e^{\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial y^{\nu}} f(x)g(y)}|_{y=x} \\
= f(x)g(x) + \frac{i}{2} \theta^{\mu\nu} \partial_{\mu} f(x) \partial_{\nu} g(x) - \frac{1}{8} \theta^{\mu\nu} \theta^{\rho\sigma} \partial_{\mu} \partial_{\rho} f(x) \partial_{\nu} \partial_{\sigma} g(x) + O(\theta^3),
\]

where \( \theta^{\mu\nu} = -\theta^{\nu\mu} \).

For obvious particle physics reasons, as well as string theoretical motivations, gauge field theories on NC Moyal space (NC gauge theories) have also been studied. Due to the fact that the Moyal star product (1.1) is noncommutative, there are severe restrictions on the possible gauge groups and matter contents in NC gauge theories. For example, using the above-mentioned recipe one can easily see that the only possible gauge group is the noncommutative \( U(\mathbb{C}) \), denoted by \( U_\star(\mathbb{C}) \), where the gauge fields are \( \mathbb{C} \times \mathbb{C} \)-valued Hermitian matrices. As will be briefly reviewed in the next section, for similar reasons the matter fields in NC gauge theories can only be in three representations of \( U(\mathbb{C}) \): fundamental, antifundamental or adjoint. For the particular case of the \( U_\star(1) \) gauge theory, the NC QED, due to the noncommutative nature of the products of fields, the allowed charges are limited to 0, \( \pm 1 \) [3]. Moreover, when we have a general gauge group which is composed of a product of several \( U_\star(N_i) \) factors, the matter fields can carry charges under at most two of the \( U_\star(N_i) \) factors. These restrictions on NC gauge theories were crystalized in the NC gauge theory no-go theorem [1] (see also [4, 5]) and used constructively in building a noncommutative version of the Standard Model [6].

On the other hand, motivated by string theory analysis that a NC Moyal space arises from the worldvolume of a D-brane in a constant background Neveu-Schwartz two-form field \( B \), in [2] it was proposed that a NC gauge field theory (which can be thought of as the low energy effective theory of open strings attached to the D-brane) should also have a description in terms of a gauge theory on a commutative spacetime. This ordinary gauge theory will, however, have a complicated action. Based on the intuition coming from string theory, Seiberg and Witten proposed a map, the Seiberg-Witten map [2], between ordinary and noncommutative gauge theories and their fields and gauge transformations. This map can in principle be constructed explicitly, using the defining
equation, as a systematic expansion in powers of $\theta^{\mu\nu}$. A short review on the Seiberg-Witten map will be presented in Appendix A.

It has been argued that the Seiberg-Witten map, which relates a noncommutative gauge theory to an ordinary one, paves the way for constructing the noncommutative version of gauge theories based on generic Lie algebras with matter fields in generic representations [7, 8, 9], thus circumventing the restrictions discussed in [1]. In particular, it has been argued that one can have NC$(su(N))$ gauge theories, as well as having NC $U(1)$ theory with arbitrary charges [8, 9, 10]. The NC$(su(N))$ case is based on a construction relying on the notion of enveloping algebra, which will be reviewed in Appendix B. These ideas have then been employed for the construction of a noncommutative Standard Model [10]. Like the Seiberg-Witten map itself, this model has been defined as a series expansion in powers of $\theta^{\mu\nu}$.

In this letter we study the question:

*Can the Seiberg-Witten map bypass the noncommutative gauge theory no-go theorem?*

We shall argue that indeed the use of the Seiberg-Witten map is limited only to the cases which respect the NC gauge theory no-go theorem. Although the no-go theorem [1], as it stands, does not apply to the NC$(su(N))$ constructed via the use of the enveloping-algebra-valued gauge fields and gauge transformations, we show that the very definition of the Seiberg-Witten map still forbids matter fields being in fundamental representation of more than one NC$(su(N))$ factor (or, equivalently, forbids matter fields from being charged under more than two noncommutative gauge groups/algebras).

This letter is organized as follows. In Section 2 we show that the Seiberg-Witten map can only be used for the cases which respect the noncommutative gauge theory no-go theorem. We also discuss that our results are supported by the string theory intuition and expectations. In Section 3 we discuss the implications of our results for the noncommutative particle physics model building. To be self-contained, in two Appendices we review the Seiberg-Witten map, its definition and consistency conditions, as well as the construction of NC$(su(N))$ gauge theories based on the notion of enveloping algebra and the Seiberg-Witten map.

## 2 Noncommutative gauge theory no-go theorem and the Seiberg-Witten map

The no-go theorem for noncommutative gauge theories, as formulated in [1], states that: 1) the local NC $u(N)$ algebra, denoted as $u_*(N)$, only admits the irreducible $N \times N$ matrix representation. Hence the gauge fields are in $N \times N$ matrix form, while the matter fields can only be in fundamental/antifundamental, adjoint or singlet states; 2) for any gauge group consisting of several simple-group factors, the matter fields can transform nontrivially under at most two NC
gauge group factors, in bi-fundamental representaion. In other words, the matter fields can not carry more than two NC gauge group charges.

The Seiberg-Witten map, on the other hand, relates the NC gauge field theories to ordinary gauge field theories, which do not have the restrictions on the representations of matter fields implied by the no-go theorem. In this section we re-examine, in view of the Seiberg-Witten map, the two parts of the no-go theorem, in particular the charge quantization problem and the property of not-more-than-two charges.

2.1 Ordinary and noncommutative gauge theories

Let us recall some basic facts about ordinary and noncommutative gauge theories and introduce the notations. In what follows, the hatted functions are multiplied with the $\star$-product (1.1), while the unhatted quantities – with ordinary product of functions. The matrix part of the $u_*(N)$ algebra is generated by the $N \times N$ Hermitian matrices $T^a$, $a = 1, 2, \ldots, N^2 - 1$, with the Lie-algebra structure $[T^a, T^b] = i f^{abc} T^c$, normalized as $Tr(T^a T^b) = \frac{1}{2} \delta^{ab}$, to which we add $T^0 = \frac{1}{\sqrt{2N}} 1_{N \times N}$.

In the ordinary $U(N)$ Yang-Mills theory, the gauge fields

$$A_\mu(x) = \sum_{a=0}^{N^2-1} A_\mu^a(x) T^a$$ (2.1)

transform under the infinitesimal gauge transformation $\delta_\Lambda$ as:

$$A_\mu \to A'_\mu = A_\mu + \delta_\Lambda A_\mu = A_\mu + \partial_\mu \Lambda + ig\{\Lambda, A_\mu\} , \quad \Lambda = \sum_{a=0}^{N^2-1} \Lambda^a(x) T^a \in u(N) ,$$ (2.2)

and the corresponding field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\{A_\mu, A_\nu\} ,$$ (2.3)

transforms covariantly

$$F'_{\mu\nu} = F_{\mu\nu} + ig\{\Lambda, F_{\mu\nu}\} .$$ (2.4)

In the above, $g$ is the gauge coupling constant.

Passing to the noncommutative Yang-Mills theory with $U_*(N)$ gauge symmetry, we introduce the $u_*(N)$ algebra, whose elements can be expanded as

$$f = \sum_{a=0}^{N^2-1} f^a(x) T^a ,$$ (2.5)

and the $u_*(N)$ Lie-algebra is defined with the star-matrix bracket:

$$[f, h]_* = f \star h - h \star f , \quad f, h \in u_*(N) .$$ (2.6)
The $U_\star(N)$ gauge theory is described by the $u_\star(N)$-valued gauge fields

$$\hat{A}_\mu(x) = \sum_{a=0}^{N^2-1} \hat{A}^a_\mu(x) T^a,$$  \hspace{1cm} (2.7)

with the noncommutative gauge transformations

$$\hat{A}_\mu \rightarrow \hat{A}'_\mu = \hat{A}_\mu + \hat{\delta}_A \hat{A}_\mu = \hat{A}_\mu + \partial_\mu \hat{\Lambda} + ig[\hat{\Lambda}, \hat{A}_\mu], \quad \hat{\Lambda} = \sum_{a=0}^{N^2-1} \hat{\Lambda}^a(x) T^a \in u_\star(N).$$  \hspace{1cm} (2.8)

The field strength is correspondingly defined with $\ast$-product,

$$\hat{F}_{\mu\nu} = \partial_{[\mu} \hat{A}_{\nu]} + ig[\hat{A}_{\mu}, \hat{A}_{\nu}]_\ast,$$  \hspace{1cm} (2.9)

and it transforms covariantly under the infinitesimal $u_\star(N)$ gauge transformations:

$$\hat{F}_{\mu\nu} \rightarrow \hat{F}'_{\mu\nu} = \hat{F}_{\mu\nu} + ig[\hat{\Lambda}, \hat{F}_{\mu\nu}]_\ast.$$  \hspace{1cm} (2.10)

One of the most striking peculiarities that the $\ast$-product introduces in the structure of the gauge groups appears for the rank 1, where the group $U_\star(1)$ shows a “non-Abelian” behaviour, unlike the ordinary $U(1)$. This feature leads to the charge quantization property [3] in NC QED and in any other model involving $U_\star(1)$ gauge fields coupled to matter, such as the noncommutative versions of the Standard Model [6, 10].

### 2.2 Charge quantization problem

The charge quantization problem arises from the non-Abelian character of the gauge group $U_\star(1)$ and the restriction on the representations of $U_\star(1)$ to the (anti-)fundamental, adjoint and singlet ones, corresponding respectively to $\pm 1$ and zero charges. (For the adjoint representation, although the charge vanishes, the fields have (electric) dipole moment, see e.g. [6, 11].) The Seiberg-Witten map connects the non-Abelian $U_\star(1)$ gauge symmetry to the Abelian $U(1)$ symmetry, therefore the question arises whether the charge quantization problem stands also in this context.

It is well known that for a non-Abelian gauge group the charge is fixed by the representation and two fields in the the same representation can not couple to the same gauge boson with two different coupling constants. Due to its non-Abelian nature, the $U_\star(1)$ gauge field can couple only with a given charge, and therefore in a theory with more than one value for the (electric) charge\(^1\) we have to add in the model as many $U_\star(1)$ gauge fields as the number of matter fields with different charges. For instance, in the Standard Model we have to introduce as many noncommutative

\(^1\)The latter can be thought as $U_\star(1)$ theories with different couplings, the ratio of which is equal to the ratio of the charges in the problem.
hyperphotons as the number of hypercharges, which is six for the Standard Model. This leads to an increase in the number of degrees of freedom in the gauge sector, which has to be reduced consistently, by spontaneous symmetry breaking, to one single hyper photon. The question arises whether, after performing this spontaneous symmetry breaking, the various matter fields will couple with the appropriate hypercharges to the residual $U(1)$ hyper photon.

We can take as a showcase the situation with only two different noncommutative matter fields in (fundamental representation of) $U_\ast(1)$, with two different hypercharges: the field $\hat{\Psi}$, with the hypercharge $q_1$, couples to the hyper photon $\hat{A}^1_\mu$, while the field $\hat{\Phi}$, with the hypercharge $q_2$, couples to the hyper photon $\hat{A}^2_\mu$:

$$D_\mu \hat{\Psi} = \partial_\mu \hat{\Psi} - iq_1 \hat{A}^1_\mu \hat{\Psi},$$
$$D_\mu \hat{\Phi} = \partial_\mu \hat{\Phi} - iq_2 \hat{A}^2_\mu \hat{\Phi}. \quad (2.11)$$

Using the Seiberg-Witten map (see Appendix A) the noncommutative fields $\hat{A}^1_\mu$, $\hat{A}^2_\mu$, as well as $\hat{\Psi}$, $\hat{\Phi}$, are expanded correspondingly in terms of ordinary $U(1)$ gauge fields, $A^1_\mu$ and $A^2_\nu$, and the ordinary matter fields $\Psi$ and $\Phi$, namely,

$$\hat{A}^1_\mu(A) = A^1_\mu - \frac{q_1}{4} \theta^{\sigma\tau} \{ A^1_\sigma, \partial_\tau A^1_\mu \} + O(\theta^2),$$
$$\hat{A}^2_\mu(A) = A^2_\mu - \frac{q_2}{4} \theta^{\sigma\tau} \{ A^2_\sigma, \partial_\tau A^2_\mu \} + O(\theta^2) \quad (2.12)$$

and

$$\hat{\Psi}(\Psi, A_\mu) = \Psi - \frac{q_1}{2} \theta^{\mu\nu} A^1_\mu \partial_\nu \Psi + O(\theta^2),$$
$$\hat{\Phi}(\Psi, A_\mu) = \Phi - \frac{q_2}{2} \theta^{\mu\nu} A^2_\mu \partial_\nu \Phi + O(\theta^2). \quad (2.13)$$

The covariant derivatives of the ordinary fields are:

$$D_\mu \Psi = \partial_\mu \Psi - iq_1 A^1_\mu \Psi,$$
$$D_\mu \Phi = \partial_\mu \Phi - iq_2 A^2_\mu \Phi. \quad (2.14)$$

The ordinary gauge fields $A^1_\mu$ and $A^2_\mu$ can not be imposed to coincide by hand. It should be emphasized that the properties of the noncommutative gauge fields to which one maps the ordinary gauge fields fix the form of the interaction in the ordinary action, i.e. the types of ordinary gauge-invariant terms, as well as the couplings. Since the noncommutative gauge field $\hat{A}^1_\mu$ couples to a matter field with the charge $q_1$, it means that in the ordinary action the field $A^1_\mu$ will couple only with the charge $q_1$, although, as Abelian gauge field, it has the possibility to couple with any other charge. If we wish to couple the ordinary gauge field with another coupling constant than the one specified by the Seiberg-Witten map, the respective coupling will obviously have no correspondent in the noncommutative action, and as such it will have to be discarded.
We can now perform a spontaneous symmetry breaking of the ordinary $U(1) \times U(1)$ symmetry, introducing a complex scalar field $\chi$, with an appropriate Higgs potential leading to a vacuum expectation value $\langle \chi \rangle = v$. As already mentioned, this ordinary field can couple to $A^1_\mu$ only by $\pm q_1$ and to $A^2_\mu$ only by $\pm q_2$. Moreover, it has to satisfy another constraint of the no-go theorem, namely that it can couple simultaneously to the two gauge fields only in the fundamental representation of one and antifundamental representation of the other, i.e.

$$D_\mu \chi = (\partial_\mu - iq_1 A^1_\mu + iq_2 A^2_\mu) \chi.$$  

Using the Higgs mechanism, we obtain a mass term for the combination

$$A^1_\mu = \left( \frac{q_1}{\sqrt{q_1^2 + q_2^2}} A^1_\mu - \frac{q_2}{\sqrt{q_1^2 + q_2^2}} A^2_\mu \right), \quad m^2_A = \frac{v^2}{2}(q_1^2 + q_2^2),$$  

while the orthogonal combination

$$A^2_\mu = \left( \frac{q_2}{\sqrt{q_1^2 + q_2^2}} A^1_\mu + \frac{q_1}{\sqrt{q_1^2 + q_2^2}} A^2_\mu \right)$$

remains massless – the gauge field of the residual $U(1)$ symmetry, i.e. the “true” hyperphoton.

Inverting (2.16) and (2.17), and introducing the result into (2.14), we find the coupling of the matter fields $\Psi$ and $\Phi$ to the residual gauge boson $A^2_\mu$:

$$D_\mu \Psi = \partial_\mu \Psi - i \frac{q_1 q_2}{\sqrt{q_1^2 + q_2^2}} A^2_\mu \Psi + \ldots ,$$

$$D_\mu \Phi = \partial_\mu \Phi - i \frac{q_1 q_2}{\sqrt{q_1^2 + q_2^2}} A^2_\mu \Phi + \ldots .$$

Thus, the two matter fields originally coupled with different charges to the $U(1)$ gauge bosons, will eventually couple with the same charge to the residual gauge boson $A^2_\mu$, upon the reduction of the degrees of freedom by spontaneous symmetry breaking. It is straightforward to see that this result carries over to the noncommutative action via the Seiberg-Witten maps (2.12) and (2.13).

It should be mentioned that the desired couplings with charges $q_1$ and $q_2$, respectively, of the matter fields $\Psi$ and $\Phi$ to the residual gauge bosons could have been obtained only if the combination $A^1_\mu - A^2_\mu$ could be made massive. Arguing backwards, this could have happened only if the Higgs field $\chi$ could couple with the same charge to both $A^1_\mu$ and $A^2_\mu$. However, as emphasized above, such a coupling could not have been taken by Seiberg-Witten map to the noncommutative action, consequently it is not allowed either in the ordinary action.

The result is not surprising, since the residual $U(1)$ symmetry in the language of ordinary fields has to be expressible as a $U_*(1)$ symmetry in the language of noncommutative fields. A $U_*(1)$ symmetry has the charge quantization problem, even if it is a residual one, and the mechanism of spontaneous symmetry breaking takes care of making equal the couplings of the matter fields to the residual gauge boson.
2.3 Maximal number of charges for a matter field

For a gauge group of the form $\prod_{i=1}^{n} U_*(N_i)$, as stated by the no-go theorem, a matter field can be charged under at most two of the group factors, $U_*(N_i)$ and $U_*(N_j)$, the matter field being necessarily in the fundamental representation of one while in the antifundamental of the other.\(^\text{2}\)

The question we would like to address in this section, namely checking if this restriction still persists under the Seiberg-Witten map, in effect reduces to studying the possibility of having a matter field in the fundamental representations of two noncommutative gauge groups.

Let us assume that the field $\hat{\Psi}$ is in the fundamental representation of $U_*(N_1)$ and $U_*(N_2)$:

\[
\begin{align*}
\hat{\delta}_\Lambda \hat{\Psi} &= ig_1 \hat{\Lambda}(\Lambda, A_\mu) \star \hat{\Psi}, & A_\mu, \Lambda &\in u_*(N_1), \quad \hat{\Lambda}(\Lambda, A_\mu) \in u_*(N_1), \\
\hat{\delta}_\Sigma \hat{\Psi} &= ig_2 \hat{\Sigma}(\Sigma, B_\mu) \star \hat{\Psi}, & B_\mu, \Sigma &\in u_*(N_2), \quad \hat{\Sigma}(\Sigma, B_\mu) \in u_*(N_2),
\end{align*}
\]

where $A_\mu$ and $B_\mu$ are the ordinary gauge fields and $\Lambda$ and $\Sigma$ are the gauge parameters. The coupling constants corresponding to the two gauge symmetries are $g_1$ and $g_2$, respectively. Next, consider two successive transformations of $\hat{\Psi}$, corresponding to the two symmetries and their reverse, i.e. $(\hat{\delta}_\Lambda \hat{\delta}_\Sigma - \hat{\delta}_\Sigma \hat{\delta}_\Lambda) \hat{\Psi}(x)$. In the spirit of the Seiberg-Witten map, we have to require the compatibility between the ordinary and the noncommutative gauge transformations (A.9). As two ordinary gauge transformations under two different gauge algebras are independent, we shall require the independence of the corresponding noncommutative gauge transformations, i.e.

\[
(\hat{\delta}_\Lambda \hat{\delta}_\Sigma - \hat{\delta}_\Sigma \hat{\delta}_\Lambda) \hat{\Psi}(x) = 0.
\]

The requirement (2.22) is actually a particular case of (A.9), since

\[
[\Lambda, \Sigma] = 0, \quad \text{when} \quad \Lambda \in u(N_1), \quad \Sigma \in u(N_2).
\]

Thus, the full set of Seiberg-Witten map consistency conditions for a noncommutative matter field in the fundamental representation of two noncommutative $u_*(N)$ algebras is:

\[
\begin{align*}
(\hat{\delta}_\Lambda \hat{\delta}_{\Lambda'} - \hat{\delta}_{\Lambda'} \hat{\delta}_\Lambda) \hat{\Psi}(x) &= \hat{\delta}_{ig_1[\Lambda,\Lambda']} \hat{\Psi}(x), & \Lambda, \Lambda' &\in u(N_1), \\
(\hat{\delta}_\Sigma \hat{\delta}_{\Sigma'} - \hat{\delta}_{\Sigma'} \hat{\delta}_\Sigma) \hat{\Psi}(x) &= \hat{\delta}_{ig_2[\Sigma,\Sigma']} \hat{\Psi}(x), & \Sigma, \Sigma' &\in u(N_2), \\
(\hat{\delta}_\Lambda \hat{\delta}_\Sigma - \hat{\delta}_\Sigma \hat{\delta}_\Lambda) \hat{\Psi}(x) &= 0.
\end{align*}
\]

The conditions (2.24) and (2.25) lead to the expressions for $\hat{\Lambda}(\Lambda, A_\mu)$ and $\hat{\Sigma}(\Sigma, B_\mu)$:

\[
\hat{\Lambda}(\Lambda, A_\mu) = \Lambda + \frac{g_1}{4} \theta^{\mu\nu} \{\partial_\mu \Lambda, A_\nu\} + \ldots,
\]

\(^2\)The only known way of circumventing the no-go theorem in this respect is by the use of noncommutative half-infinite Wilson lines attached to the matter fields [12], in which case the matter fields can be in tensor representations of any number of $U_*(N_i)$ factors [13, 14, 15].
\[ \hat{\Sigma}(\Sigma, B_\mu) = \Sigma + \frac{g_2}{4} \theta^{\mu\nu} \{ \partial_\mu \Sigma, B_\nu \} + \ldots , \]

which have to be inserted into (2.26), rewritten as

\[ \hat{\Lambda}(\Lambda, A_\mu) \star \hat{\Sigma}(\Sigma, B_\mu) - \hat{\Sigma}(\Sigma, B_\mu) \star \hat{\Lambda}(\Lambda, A_\mu) = 0. \]

Up to the first order in \( \theta \), the condition (2.28) becomes

\[ [\Lambda, \Sigma] + \frac{i}{2} \theta^{\mu\nu} [\partial_\mu \Lambda, \partial_\nu \Sigma] + \frac{g_2}{4} \theta^{\mu\nu} [\Lambda, \{ \partial_\mu \Sigma, B_\nu \}] + \frac{g_1}{4} \theta^{\mu\nu} [\{ \partial_\mu \Lambda, A_\nu \}, \Sigma] + O(\theta^2) = 0. \]

The last line in (2.29) contains the commutators \([a, b]\) with \( a \in u(N_1) \) and \( b \in u(N_2) \). Besides the Moyal bracket of \( \Lambda \) and \( \Sigma \), only such ordinary commutators will appear in all orders in \( \theta^{\mu\nu} \), and all of them will vanish. Consequently,

\[ (\hat{\delta}_\Lambda \hat{\delta}_\Sigma - \hat{\delta}_\Sigma \hat{\delta}_\Lambda) \hat{\Psi}(x) = -g_1 g_2 \left( [\Lambda, \Sigma] + \frac{i}{2} \theta^{\mu\nu} [\partial_\mu \Lambda, \partial_\nu \Sigma] + O(\theta^2) \right) \star \hat{\Psi}(x) = -g_1 g_2 [\Lambda, \Sigma] \star \hat{\Psi}(x), \]

which is valid in all orders in \( \theta \). The r.h.s. of the expression (2.30) can not vanish, due to the \( \star \)-product in the Moyal bracket of the gauge parameters belonging to the two independent gauge algebras. This shows that the equations (2.24)-(2.26) can not be simultaneously fulfilled, consequently a noncommutative matter field can not be in the fundamental representation of more than one gauge symmetry. We point out that, for a field transforming in bi-fundamental representation of \( U_\star(N_1) \times U_\star(N_2) \) (i.e. fundamental of \( U_\star(N_1) \) and antifundamental of \( U_\star(N_2) \)), the condition (2.26) is naturally satisfied.

### 2.4 String theory considerations

It is instructive to support the above algebraic arguments by string theory reasoning. Indeed, in the string theory setting it is clear that the Seiberg-Witten map can only be worked out when the no-go theorem holds. To see this, it is enough to revisit the no-go theorem, the Seiberg-Witten map and their connection from the string theory viewpoint. The NC Moyal plane can be obtained as the low energy effective theory of D-branes in the presence of a Neveu-Schwartz two-form \( B \)-field [16]. This is a particular \( \alpha' \to 0 \) limit, while a certain combination of the background \( B \)-field and \( \alpha' \), which appears as the noncommutativity parameter \( \theta^{\mu\nu} \), is held fixed [2]. The Moyal plane is what is seen from such a D-brane when probed by open strings whose end-points are attached to the brane. The first part of the no-go theorem in this picture is related to the simple fact that an open string has two end-points. To see this, we note that when we have \( N \) D-branes on top of each other, one should then associate a \( U(N) \) Chan-Paton factor to each end, i.e. if one end is in the fundamental representation of \( U(N) \), the other end should necessarily be in the antifundamental
representation [17]. (Recall that open strings ending on D-branes are orientable.) This property persists when we have a constant background $B$-field, i.e. the end-points of open strings are in fundamental and antifundamental representations of $U_\ast(N)$ and the whole string is then in the adjoint representation of $U_\ast(N)$.

To argue for the second part of the no-go theorem, namely the not-more-than-two charges, we need to consider a D-brane setup which realizes the $\prod_{i=1}^{n} U_\ast(N_i)$ gauge group. This is done if we consider $n$ stacks of parallel D-branes, each consisting of $N_i$, $i = 1, \ldots, n$ branes, while the stacks are separated by distances much larger than the string scale. One can then recognize two classes of open strings: those which have both ends on the same stack and those which are stretched between different stacks. The former give rise to the $U_\ast(N_i)$ gauge fields, while the latter lead to matter fields in “bi-fundamental” representation of $U_\ast(N_i) \times U_\ast(N_j)$, $i \neq j$. Again, since the open string has two end-points and open strings stretched between D-branes are orientable, there is no room for Chan-Paton factors for the product of more than two $U_\ast(N_i)$.

The Seiberg-Witten map, on the other hand, as explained in [2], comes from the equivalence of two pictures for the dynamics of $D$-branes in the background $B$-field. Explicitly, one can have two different descriptions, the open string description and the closed string description. The open string description is the noncommutative one and the closed string description is the ordinary commutative one. The Seiberg-Witten map in this language expresses the equivalence of the Born-Infeld actions in these two descriptions. If we take this as an equivalent definition for the Seiberg-Witten map, then it clearly states that the Seiberg-Witten map only works for the cases where the noncommutative description in terms of open strings exists, namely the cases which also respect the no-go theorem.

3 Concluding remarks and discussion

We have shown that the Seiberg-Witten map can only be used for the cases which respect the noncommutative gauge theory no-go theorem. In particular, we have been dealing with $U_\ast(N)$ gauge groups or a product of them. On the other hand, using the Seiberg-Witten map as the defining relation, it has been argued that one can construct a noncommutative version of Lie groups/algebras other than $U(N)$ [7, 8, 9]. This, as reviewed in the Appendix B, is achieved by allowing the gauge field to take values in the enveloping algebra of the gauge algebra $\mathcal{A}$, $\mathcal{U}(\mathcal{A})_\ast$, with a combination fixed by the Seiberg-Witten map. The particular case of obvious interest is when $\mathcal{A} = su(N)$.

One may then wonder whether the arguments of Subsection 2.3 would still hold for products of $\mathcal{U}(\mathcal{A}_i)_\ast$. Recalling the discussions reviewed in the Appendices, the key equations to be considered are again (2.24)-(2.26), but now $\Lambda, \Lambda'$ and $\Sigma, \Sigma'$ take values in the enveloping algebras $\mathcal{U}(\mathcal{A}_1)_\ast$ and
\( \mathfrak{u}(A_2)_\ast \), respectively. Eq. (2.27) is still a solution to (2.24) and (2.25), while (2.26) reduces to (2.28). Again we face the same problem as in the \( u_\ast(N) \) case and therefore we conclude that one can not have matter fields which are in the fundamental representation of more than one \( \mathfrak{u}(A)_\ast \) enveloping algebra.

Having shown the persistence of the NC gauge theory no-go theorem within the Seiberg-Witten map expansion and its extension to the cases with the NC gauge theories based on the enveloping algebras, we would like to comment on the physical implications of this result for particle physics model building.

There have been two different proposals for building a NC version of the SM. One which has been proposed by the authors in [6] is based on the \( U_\ast(3) \times U_\ast(2) \times U_\ast(1) \) gauge theory. The matter content is chosen in such a way that it respects the NC gauge theory no-go theorem. Indeed, with this gauge group the most general possible matter content is exactly the one which has been realized in Nature, within the SM. In this sense we have the remarkable property that the matter content is completely specified: fixing the NC gauge group fixes the matter content. With this gauge group, the theory has two extra gauge fields compared to the SM. To reduce this theory to the SM, at least at scales below TeV, we employed the “Higgsac symmetry reduction mechanism” to give masses to these two extra gauge fields. This symmetry reduction mechanism is not spontaneous, but a solution to this problem is under consideration [18]. Besides this point, this model, as it stands and was presented in [6], suffers from the chiral anomaly [5] (a suggestion to cure this problem was made in [19]).

The other proposed formulation for the NC version of the SM is the one based on \( \mathfrak{u}(su(3))_\ast \times \mathfrak{u}(su(2))_\ast \times \mathfrak{u}(u(1))_\ast \) [10], where the applicability of the Seiberg-Witten map has been assumed and used. Although this model has the same particle and gauge field content as the SM, it suffers from other problems: in order for that proposal to work, the NC gauge theory no-go theorem should be bypassed. This is easy to see if we recall that i) in the SM there are six different hypercharges, and this contradicts the \( U_\ast(1) \) charge quantization and ii) the left-handed doublet of quarks is charged under all three gauge symmetries. However, as we have argued here, the no-go theorem still applies despite the usage of the Seiberg-Witten map, upon which that proposal is built.

Therefore, it appears that both of the NC SM proposed have model building problems and constructing a NC version of the SM still remains a challenge. Finding other representations which have not been accounted for in the no-go theorem, in particular the existence of the \( N \)-dimensional totally antisymmetric representation of \( \mathfrak{u}(u(N))_\ast \), i.e. the representations under the \( u_\ast(1) \) part of \( u_\ast(N) \), may prove useful in this respect.

This challenge may be an indication for another possibility that the NC effects could only become important at energies much above the TeV scale and that one should think of a NC GUT
rather than the NC SM.

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A  Noncommutative gauge theories and Seiberg-Witten map

The Seiberg-Witten map [2], as originally proposed, is a map between the NC $U_q(N)$ gauge fields and gauge transformations, respectively denoted by $\hat{A}$ and $\hat{\Lambda}$, and the corresponding ordinary $u(N)$-matrix valued functions $A$ and $\Lambda$.

The Seiberg-Witten map is a field redefinition of the form

$$\hat{A} = \hat{A}(A, \partial A, \partial^2 A, \ldots ; \theta),$$

accompanied by a reparametrization

$$\hat{\Lambda} = \hat{\Lambda}(\Lambda, \partial \Lambda, \partial A, \partial^2 \Lambda, \partial^2 A, \ldots ; \theta).$$

The map of the ordinary gauge field $A$ to the noncommutative gauge field $\hat{A}$ is required to preserve the gauge equivalence relation, though the groups of the noncommutative and the ordinary gauge theories are different. This can be acquired by allowing $\hat{\Lambda}$ to depend on both $\Lambda$ and $A$. The requirement of identification of the gauge equivalence is concisely written as:

$$\hat{A}(A) + \delta_{\hat{\Lambda}} \hat{A}(A) = \hat{A}(A + \delta A),$$

with infinitesimal $\Lambda$ and $\hat{\Lambda}$. The solutions of (A.3) for $\hat{A}$ and $\hat{\Lambda}$ are assumed to be smooth and local to all orders in $\theta$. Eq. (A.3) can be solved as a series expansion in powers of $\theta$. To first order, the map reads [2]:

$$\hat{A}_\mu(A) = A_\mu - \frac{g}{4} \theta^{\sigma\tau} \{ A_\sigma, \partial_\tau A_\mu + F_{\tau\mu} \} + \mathcal{O}(\theta^2),$$

$$\hat{\Lambda}(\Lambda, A) = \Lambda + \frac{g}{4} \theta^{\mu\nu} \{ \partial_\mu \Lambda, A_\nu \} + \mathcal{O}(\theta^2),$$

where $\{ , \}$ is the anticommutator of $u(N)$-valued matrices.

The Seiberg-Witten map can be extended beyond the pure gauge theory. For the NC matter field $\hat{\Psi}$ in the fundamental (or antifundamental) representation of $u_q(N)$, with the gauge transformation rule

$$\delta_{\hat{\Lambda}} \hat{\Psi} = ig \, \hat{\Lambda} \star \hat{\Psi}.$$
and the corresponding ordinary counterpart \( \delta \Psi = ig \Lambda \Psi \), the Seiberg-Witten map is defined by

\[
\hat{\delta}_\Lambda \hat{\Psi} = \delta_\Lambda \hat{\Psi},
\]

where in the r.h.s. of the above equation \( \hat{\Psi} \) should be considered as

\[
\hat{\Psi} = \hat{\Psi}(\Psi, A; \theta)
\]

and \( \delta_\Lambda \) means the ordinary gauge transformation of the ordinary fields on which \( \hat{\Psi} \) depends. (Note that (A.5) implies that (A.7) should be linear in \( \Psi \) and its derivatives.) The above equation, too, can be solved together with (A.3) as a systematic power series expansion in \( \theta \). In the first order the map reads

\[
\hat{\Psi}(\Psi, A) = \Psi - \frac{g}{2} \theta^{\mu\nu} A_\mu \partial_\nu \Psi + \mathcal{O}(\theta^2).
\]

In order that (A.6) has a solution, one should require the closure of the Seiberg-Witten map under the commutator of successive gauge transformations,

\[
(\hat{\delta}_\Lambda \hat{\Sigma} - \delta_\Sigma \hat{\delta}_\Lambda) \hat{\Psi}(x) = \hat{\delta}_{ig[\Lambda, \Sigma]} \hat{\Psi}(x),
\]

which leads to the following consistency condition

\[
i \delta_\Lambda \hat{\Sigma}(\Sigma, A) - i \delta_\Sigma \hat{\Lambda}(\Lambda, A) - g \hat{\Lambda}(\Lambda, A) \ast \hat{\Sigma}(\Sigma, A) + g \hat{\Sigma}(\Sigma, A) \ast \hat{\Lambda}(\Lambda, A) = i \hat{\Omega}(\Omega, A),
\]

where \( \hat{\Omega}(\Omega, A) \) is the gauge parameter of the NC gauge transformation for \( \Omega = ig[\Lambda, \Sigma] \) and the variations of the type \( \delta_\Lambda \hat{\Sigma}(\Sigma, A) \) refer to the ordinary variation \( \delta_\Lambda \) of the gauge field \( A \) on which \( \hat{\Sigma} \) depends.

**B Enveloping algebra valued noncommutative gauge fields and the Seiberg-Witten map**

The Seiberg-Witten map can be used to construct gauge field theories other than \( U_*(N) \) theory [8, 9]. Here we sketch this construction. To start with, consider the definition of the Seiberg-Witten map, (A.3) and (A.6), and let the ordinary gauge fields take values in the Lie algebra \( \mathcal{A} \). Assume that these fields are in some \( N \times N \) matrix form of \( \mathcal{A} \). At zeroth order in \( \theta \), \( \hat{A}_\mu \), \( \hat{\Lambda} \) and \( \hat{\Psi} \) are equal to their ordinary counterparts, hence transforming as representations of the algebra \( \mathcal{A} \). It is readily seen that the solutions to (A.3) and (A.6) at first order in \( \theta \) for a generic algebra \( \mathcal{A} \) are again of the form (A.4) and (A.8). These expressions are not taking values in the algebra \( \mathcal{A} \) (except for \( \mathcal{A} = u(N) \)), but they are in general in the (universal) enveloping algebra of \( \mathcal{A} \), i.e. \( \mathcal{U}(\mathcal{A}) \), with the specific expression which is fixed by the Seiberg-Witten map and will be denoted
by $\mathcal{U}(\mathcal{A})_\star$. If we allow the fields to fall into the representations of $\mathcal{U}(\mathcal{A})_\star$ (rather than $\mathcal{A}$) this will show a way for constructing NC gauge theories based on the algebra $\mathcal{A}$. In this construction the zeroth order NC fields are taken to be in $\mathcal{A}$ and although the defining equations (A.3) and (A.6) have a closed form, the explicit form of the action for these theories can only be given as a power series expansion in $\theta$.

The above can be made more explicit if we choose a specific basis for $\mathcal{A}$. To illustrate the idea, let us focus on the phenomenologically interesting case of $\mathcal{A} = su(N)$ for which, in the notations of the Subsection 2.1, the covariant derivative of the field $\Psi$ is

$$D_\mu \Psi(x) = (\partial_\mu - igA_\mu(x))\Psi(x), \quad (B.1)$$

with the gauge field $A_\mu(x)$ transforming as in (2.2), but this time with $\Lambda = \sum_a \Lambda^a(x) T^a \in su(N)$, $a = 1, 2, \ldots, N^2 - 1$. The noncommutative gauge theory to which one maps the above ordinary gauge theory will be invariant under the $\star$-transformations valued in the enveloping algebra $\mathcal{U}(su(N))_\star$, i.e. a matter field will transform as:

$$\hat{\delta}_\Lambda \hat{\Psi}(x) = ig \hat{\Lambda}(x) \star \hat{\Psi}(x), \quad (B.2)$$

with

$$\hat{\Lambda}(x) = \hat{\Lambda}_a(x) T^a + \hat{\Lambda}_{ab}^1(x) : T^a T^b : + \ldots + \hat{\Lambda}_{a_1 \ldots a_n}^{n-1}(x) : T^{a_1} \ldots T^{a_n} : + \ldots, \quad (B.3)$$

where the dots indicate summation over a basis of $\mathcal{U}(su(N))_\star$, for example the basis of completely symmetrized products of the Lie algebra:

$$: T^a : = T^a, \quad : T^a T^b : = \frac{1}{2} \{T^a, T^b\} = \frac{1}{2} (T^a T^b + T^b T^a),$$

$$: T^{a_1} \ldots T^{a_n} : = \frac{1}{n!} \sum_{\pi \in S_n} T^{a_{\pi(1)}} \ldots T^{a_{\pi(n)}}, \quad (B.4)$$

and

$$\hat{\Lambda}(\Lambda, A) = \Lambda + \frac{g}{4} \theta^{\mu\nu} \{ \partial_\mu \Lambda, A_\nu \} + \mathcal{O}(\theta^2) = \Lambda + \frac{g}{2} \theta^{\mu\nu} \partial_\mu \Lambda^a A^b_\nu : T^a T^b : + \mathcal{O}(\theta^2), \quad (B.5)$$

and similarly for $\hat{\Lambda}$ and $\hat{\Psi}$. The higher orders of the expansions are obtained analogously. In [9] the action of a noncommutative gauge theory with fermionic matter has been derived to the second order in the noncommutativity parameter $\theta$, and the result being written only in terms of ordinary gauge covariant derivatives and field strengths, exhibits beautifully the ordinary gauge invariance of the expansion.

\[3\text{Note that the "gauge invariant" physical observables are also constructed from quantities in } \mathcal{U}(\mathcal{A})_\star \text{ and they involve the parts of the Seiberg-Witten map which are not } \mathcal{A}\text{-valued.}\]
However, one may wonder about the consistent use of the above definition for the $U(A)_*$ theory, in the case of model building, when several $U(A_i)_*$ factors are present. As in the $u_*(N)$ case, one can work out the “integrability condition” for the equations (A.3) and (A.6) by considering two successive gauge transformations. This again leads to (A.10) but now the hatted objects take values in $U(A)_*$. When we have several gauge group factors, like $\prod_{i=1}^n U(A_i)_*$, one should require the consistency condition for each of the $U(A_i)_*$ factors. Moreover, as discussed in Subsection 2.3, we need to examine the consistency relations of the map for matter fields charged under more than two gauge algebra factors – and the result is similar to the one obtained in Subsection 2.3, i.e. matter fields transforming under more than two noncommutative enveloping algebras can not be constructed in this way.

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