POISSON EQUATIONS WITH LOCALLY-LIPSCHITZ COEFFICIENTS AND UNIFORM IN TIME AVERAGING FOR STOCHASTIC DIFFERENTIAL EQUATIONS VIA STRONG EXPONENTIAL STABILITY

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Abstract. We study averaging for Stochastic Differential Equations (SDEs) and Poisson equations. We succeed in obtaining a uniform in time (UiT) averaging result (with a rate) for fully coupled SDE models with super-linearly growing coefficients. This is, to the best of our knowledge, the first UiT multiscale result for slow-fast systems of SDEs (and indeed we are not aware of any other UiT multiscale theorems in the PDE or ODE literature either) and it is the main result of this paper. Key to obtaining both our UiT averaging result and to enable dealing with the super-linear growth of the coefficients (of the slow-fast system and of the associated Poisson equation) is conquering exponential decay in time of the space-derivatives of appropriate Markov semigroups. We refer to semigroups which enjoy this property as being Strongly Exponentially Stable.

There are various approaches in the literature to proving averaging results. The analytic approach we take here requires studying a family of Poisson problems associated with the generator of the (fast component of the) SDE dynamics. The study of Poisson equations in non-compact state space is notoriously difficult. Results exists, but mostly for the case when the coefficients of the PDE are either bounded or satisfy linear growth assumptions (and even the latter case has only recently been achieved). In this paper we treat Poisson equations on non-compact state spaces for coefficients that can grow super-linearly. In particular, we demonstrate how Strong Exponential Stability can be employed not only to prove the UiT result for the slow-fast system but also to overcome some of the technical hurdles in the analysis of Poisson problems. Poisson equations are essential tools in both probability theory and partial differential equations (PDEs). Their vast range of applications includes the study of the asymptotic behaviour of solutions of parabolic PDEs, the treatment of multi-scale and homogenization problems as well as the theoretical analysis of approximations of solutions of Stochastic Differential Equations (SDEs). So our result on Poisson equations is of independent interest as well.

Keywords. Averaging methods for Stochastic differential equations, Poisson equations, Uniform in time approximations, Derivative estimates for Markov Semigroups.

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1. Introduction

This paper is concerned with the problem of obtaining \emph{uniform in time} averaging results for fully coupled systems of Stochastic Differential Equations (SDEs). An important preliminary step to obtain such results is the study of so called ‘Poisson equations with a parameter’. The study of Poisson equations is also of independent interest as such equations play a pivotal role both in PDE theory and in probability theory – to obtain Functional Central Limit theorems [30, 47], in the study of large deviations [16, 41, 50], or in approximation theory, for example as a cornerstone in Stein’s method [3] – so we comment on these two topics, averaging and Poisson equations, separately, starting from the former.

Averaging. Consider the following slow-fast system

\begin{align}
&dX^{\epsilon,x,y}_t = b(X^{\epsilon,x,y}_t, Y^{\epsilon,x,y}_t)dt + \sqrt{2} \sigma(X^{\epsilon,x,y}_t, Y^{\epsilon,x,y}_t) dW_t \quad (1) \\
&dY^{\epsilon,x,y}_t = \frac{1}{\epsilon} g(X^{\epsilon,x,y}_t, Y^{\epsilon,x,y}_t)dt + \sqrt{\frac{2}{\epsilon}} a(X^{\epsilon,x,y}_t, Y^{\epsilon,x,y}_t) dB_t \quad (2)
\end{align}

with initial datum $(X_0, Y_0) = (x, y) \in \mathbb{R}^n \times \mathbb{R}^d$. Here $0 < \epsilon \ll 1$ is a small parameter, $(X^{\epsilon,x,y}_t, Y^{\epsilon,x,y}_t)$ takes values in $\mathbb{R}^n \times \mathbb{R}^d$, $b : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n \times n}$, $a : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$ and, finally, $W_t$ and $B_t$, respectively, are $n$–dimensional and $d$–dimensional standard Brownian motions, respectively, assumed to be independent of each other.

The intuitive description of the classical averaging paradigm proceeds as follows. First, we consider the dynamics in $(2)$ with $X^{\epsilon,x,y}_t = x$ fixed, i.e. we consider the SDE

\[ dY^{\epsilon,x,y}_t = \frac{1}{\epsilon} g(x,Y^{\epsilon,x,y}_t)dt + \frac{\sqrt{2}}{\sqrt{\epsilon}} a(x,Y^{\epsilon,x,y}_t) dB_t, \quad Y^{\epsilon,x,y}_0 = y. \]
Assuming that for every \( x \) the above evolution is ergodic, with invariant measure \( \mu^x(dy) \), under appropriate assumptions on the coefficients, when \( \epsilon \to 0 \) one has

\[
\mathbb{E} f(Y_\epsilon^{x,y}) \to \int_{\mathbb{R}^d} f(y) d\mu^x(y),
\]

for every \( f \in C_b(\mathbb{R}^d) \) (throughout \( C_b(\mathbb{R}^d) \) is the set of continuous and bounded real valued functions on \( \mathbb{R}^d \)), details in Section 4. We emphasize that the invariant measure \( \mu^x \) does depend on the parameter \( x \). Because \( Y_\epsilon^{x,y} \) in (2) equilibrates much faster than \( X_\epsilon^{x,y} \), one expects that, as \( \epsilon \to 0 \), the dynamics (1) should be approximated, at least over finite time horizons \([0, T]\), by the so-called averaged dynamics, i.e. by the following \( \mathbb{R}^n \)-valued SDE:

\[
d\bar{X}_t^x = \bar{b}(\bar{X}_t^x)dt + \sqrt{2}\hat{\sigma}(\bar{X}_t^x)dW_t, \quad \bar{X}_0 = x, \tag{4}
\]

where

\[
\bar{b}(x) := \int_{\mathbb{R}^d} b(x, y) d\mu^x(y), \tag{5}
\]

and \( \hat{\sigma} \) is a square root of the matrix \( \Sigma(x) := \int_{\mathbb{R}^d} \Sigma(x, y) d\mu^x(y) := \int_{\mathbb{R}^d} \sigma(x, y)\sigma^T(x, y) d\mu^x(y) \), where \( \sigma^T \) denotes the transpose of the matrix \( \sigma \).

Averaging (and, more generally, homogenization) results can be obtained in a number of ways but, to the best of our knowledge, the existing techniques can be traced back to a variation of either one of two main approaches, namely either a functional approach, which is the one we adopt in this paper, or a more probabilistic one, as introduced by Khas’minksi in his seminal paper, see also [19,34]. We refer to the former as being a functional approach because it hinges on obtaining preliminary results on so-called Poisson equations with parameter, which we will come to later. Irrespective of the approach taken, the convergence of the slow-fast system to the averaged dynamics (whether weak or strong), has ever only been proved to take place over finite time horizons. That is, broadly speaking, one typically establishes results of the type

\[
\left| \mathbb{E} f(X_\epsilon^{\cdot, x,y}) - \mathbb{E} f(\bar{X}_t^x) \right| \leq \epsilon C(t) \tag{6}
\]

for every \( f \) in a suitable class of functions, where \( C = C(t) \) is a constant dependent on time (and on \( f \) as well as \( x \) and \( y \)). Because of the use of Gronwall’s inequality (or similar arguments), this constant is an increasing function of time \( t \).

Averaging methods are extremely effective and routinely used in applications to engineering, biology, statistics, molecular dynamics to mention just a few application fields, see e.g. [7,11,18,23,37], with no claim to completeness of references; yet, a long-standing criticism of such techniques is the following: while one can typically only prove that the averaged dynamics is a good approximation of the original slow-fast system for finite-time windows (with estimates that deteriorate in time), the averaged dynamics is often used in practice as an approximation of the long-time behaviour of the slow-fast system. The fact that the slow-fast system should, under appropriate assumptions on the coefficients,

\[1\] Clearly \( \hat{\sigma} \) is not univocally determined. This is compatible with the fact that the process associated to a generator is not unique and with the fact that we will study weak-type convergence, see [47, Remark 11.2] on this point.

\[2\] In [6] we are referring to weak convergence, but clearly the literature has plenty of pathwise or other types of results. The mode of convergence is not what we are trying to emphasize here, we take weak convergence only to fix ideas.
converge to the averaged dynamics uniformly in time (i.e. that one should be able to prove that the constant $C$ in (6) is independent of time), has so far only been conjectured on the basis of numerical evidence, see e.g. Section 8 and [47], where the need in applications for multiscale results which hold uniformly in time (and on non-compact state-space) has been explicitly advocated.

In this paper we make it possible to fill this theoretical gap and identify rather general assumptions on the coefficients of the SDEs, under which convergence (as $\epsilon \to 0$) of the slow-fast system to the limiting dynamics is actually uniform in time, thereby providing the many works which implicitly use this fact with a firm theoretical basis. We will show in future work (which is at the moment in progress) how leveraging on the theoretical results that we produce in this paper can be powerful from an applications' perspective, for example to provide guarantees on statistical methodologies.

This UiT result is novel to this paper and to its inspiring work [4]. The difference between the present paper and [4] is in the nature of the system at hand, as in [4] the fast system is a (relatively straightforward) finite state space jump process and the slow system is an SDE with a specific structure, while here we treat general systems of SDEs (i.e. both the fast and slow processes are SDEs). More precisely, besides the fact that our result is uniform in time (UiT), this paper deals with slow-fast systems of SDEs which are fully coupled (i.e. all the coefficients can depend on both $X^{(\epsilon,x,y)}_t$ and $Y^{(\epsilon,x,y)}_t$), evolve in non-compact state space, and whose coefficients are allowed super-linear growth; in particular, the drift coefficients of both the slow and the fast system are locally Lipschitz. Furthermore, our convergence result comes with an explicit rate in $\epsilon$.

Results on multiscale methods typically deal with the case in which the slow-fast system evolves in compact state space, see [19, 22, 31, 47] and references therein. The seminal papers [43, 44, 45] (and, later, [48]) then paved the way for the non-compact setting, and produced results for slow-fast evolutions of the type (1)-(2) in $\mathbb{R}^n \times \mathbb{R}^d$, using the functional analytic approach, as in this paper. The results of [43, 44, 45, 48] refer to the case in which the coefficients of (1)-(2) are bounded. The difficulty in going from compact to non-compact state space and then to unbounded coefficients comes from the fact that the associated Poisson equation inherits such features, i.e. one needs to study a Poisson problem which is posed on non-compact state space and associated with a differential operator with unbounded coefficients. This is a notoriously difficult problem, see e.g. [47, Section 10.2 and 18.4]. We will come back to this when we comment on our results on Poisson equations. For the time being we mention that one way of bypassing this issue is to adopt the diffusion approximation approach instead of the functional analytic one. This has been done recently in [48]. Adopting such an approach has allowed the authors of [48] to produce averaging results for the system (1)-(2) when the coefficients are locally Lipschitz (and indeed our assumptions on the coefficients are analogous to those in [48]), but the scheme of proof of [48] does not give an explicit convergence rate in $\epsilon$ and it does not allow to consider the fully coupled regime (in [48] the diffusion coefficient of the slow variable, which we here denote by $\sigma$, does not depend on the fast variable); on the other hand in [48] all the coefficients are allowed to be time-dependent, a case which we don’t treat here.

The literature on multiscale methods is extremely vast so in the above we have mentioned only the works which are most relevant to our discussion. Other related works
Poisson equations and uniform in time averaging (without any claim to completeness of references) are [25, 32, 51, 52, 53], some of which do cover the case of linearly growing coefficients. We also flag up the very recent [5], which treats the case of SDEs which are non-linear in the sense of McKean.

From a technical standpoint, the idea that allows us to tackle both the problem of obtaining a UiT averaging result and to study Poisson equations with non globally Lipschitz coefficients hinges on using exponentially fast decay (in time) of the derivatives of appropriate Markov semigroups associated with the system (1)-(2) and with (4), i.e. what we will refer to as strong exponential stability. We will explain this further in the last part of this introduction.

**Poisson equations with parameter.** The Poisson equations to be studied in connection with the multiscale system (13)-(14) are of the form

\[
(L^xu)(x, y) = \phi(x, y) - \bar{\phi}(x), \quad x, y \in \mathbb{R}^d, \tag{7}
\]

where \( \phi \) is a regular enough, polynomially growing given function (precise assumptions in Section 3), \( \bar{\phi} := \int_{\mathbb{R}^d} \phi(x, y) \mu^x(dy) \) (where \( \mu^x \) is the measure defined in (10) below) and \( L^x \) is a differential operator in the y variable only, namely

\[
(L^x u)(x, y) = (g(x, y), \nabla_y u(x, y)) + a(x, y) a(x, y)^T : \text{Hess}_y u(x, y). \tag{8}
\]

Observe that \( x \in \mathbb{R}^n \) appears in (7) and in (8) purely as a parameter, so that (7) can be regarded as a family of Poisson equations dependent on the parameter \( x \in \mathbb{R}^n \). In the above and throughout, we use \((\cdot, \cdot)\) to denote the Euclidean scalar product, \( \nabla_x \) and \( \text{Hess}_x \) are used to denote the gradient and Hessian operators respectively. In Section 6.1 we will explain how this Poisson problem emerges in the context of averaging. For the time being it suffices to say that the operator \( L^x \) is the generator of the process \( Y^{x,y}_t := Y^{1,x,y}_t \) which solves the SDE

\[
dY^{x,y}_t = g(x, Y^{x,y}_t) dt + \sqrt{2a(x, Y^{x,y}_t)} dB_t, \tag{9}
\]

and is obtained from (3) by setting \( \epsilon = 1 \); clearly, for each \( x \) fixed, the asymptotic behaviour as \( t \to \infty \) of (9) is the same as the asymptotic behaviour as \( \epsilon \to 0 \) of (3) and the invariant measure \( \mu^x(dy) \) (of either processes) is precisely the only probability measure such that

\[
\int_{\mathbb{R}^d} (L^x f)(x, y) \mu^x(dy) = 0, \tag{10}
\]

for every \( f \) in the domain of \( L^x \). Under our assumptions (see Section 2) such an invariant measure exists and, since we will take \( L^x \) to be elliptic, it is unique and it has a smooth density (for every \( x \in \mathbb{R}^n \)); with abuse of notation, we still denote the density by \( \mu^x(y) \).

As we have already mentioned, the study of Poisson problems in non-compact spaces is far from trivial and was first tackled in the seminal papers [43, 44, 45]. To be more precise, when using Poisson equations to obtain averaging results, one needs to establish, in turn: well-posedness of the Poisson problem (7) and a representation formula for the solution; smoothness of the solution \( u(x, y) \) in both the \( x \) and \( y \) variable; and, finally, one needs to quantify how the solution \( u(x, y) \) changes as the parameter \( x \in \mathbb{R}^n \) varies. Related to this problem is the issue of studying how the invariant measure \( \mu^x(dy) \) of the frozen process varies as the parameter \( x \) varies. Let us comment on these points in turn, starting

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^3For example, in [26] \( x \) is the inverse temperature of the system so tracing the dependence in \( x \) of \( \mu^x \) boils down to tracing the dependence on the inverse temperature.
from well-posedness of (7). The solution to (7) is in general not unique, since constants belong to the kernel of $L^x$. However, once we restrict to the set of mean-zero functions, i.e. to the set of functions $f$ such that $\int f(x,y)\mu^x(dy) = 0$, then the solution, if it exists, is unique (see Section 3 and Section 5). Moreover, one can obtain a useful probabilistic representation formula for such a solution in terms of the process $Y_t^{x,y}$. Indeed, when the equation is posed on a bounded domain $D \subset \mathbb{R}^d$ the solution can be represented as

$$u(x,y) = -\int_0^\tau \left( \mathbb{E}[\phi(x,Y_t^{x,y})] - \bar{\phi}(x) \right) dt, \quad \tau := \inf\{ t > 0 : Y_t^{x,y} \notin D \}.$$  

It is then reasonable to expect that when the equation is posed on $\mathbb{R}^d$ one should have

$$u(x,y) = -\int_0^{\infty} \left( \mathbb{E}[\phi(x,Y_t^{x,y})] - \bar{\phi}(x) \right) dt.$$  

(11)

The above representation formula has been proven in $[43]$, under the assumption that the coefficient $g$ is bounded and $L^x$ is uniformly elliptic. Recently, the authors of $[9]$ have both streamlined the proof of this representation formula and dropped the boundedness assumption on $g$ (as well as the ellipticity assumption on $L^x$), however $[9]$ does not address the issue of the smoothness of the solution $u(x,y)$. Regarding smoothness of the solution, in view of the ellipticity in $y$ of $L^x$ the smoothness of $u(x,y)$ in the variable $y$ is straightforward. The difficulty is in obtaining smoothness in the $x$ variable as a priori there is no obvious reason why the solution should change continuously in $x$. Nonetheless it has been shown in $[44, \text{Section 2}]$ and then more systematically in $[48]$ that, under some assumptions on the coefficients, smoothness in $x$ is inherited from smoothness in $y$; this is done by using so called ‘transfer formulas’, which allow one to express $x$-derivatives in terms of $y$-derivatives, see $[48]$ and Section 3. Moreover $[43, 44, 45, 48]$ obtain a formula for the derivatives in $x$ of $u(x,y)$, which quantifies the way in which the solution changes as $x$ changes. All such results, i.e. smoothness in $x$ and formulas for the $x$-derivatives of $u$, have been obtained so far under the assumptions that the coefficients of $L^x$ are bounded e.g $[43, 44, 45, 48]$. In $[44]$ the author observed that it should be possible to relax such an assumption and the second contribution of this paper consists precisely in removing such an assumption and considering Poisson equations of type (7) with unbounded, locally-Lipschitz coefficients.

As we have already pointed out, the technical tool which allows us to obtain $U_iT$ averaging results and to study Poisson equations with locally Lipschitz coefficients is the validity of certain Derivative Estimates (DE) for Markov Semigroups. In particular, denote by $P_t^x$ the semigroup associated to the process (9) (acting, for each $x$ fixed, on $p \in C(\mathbb{R}^d)$) and by $\mathcal{P}_t$ the semigroup associated with the process (4) (acting on $f \in C(\mathbb{R}^n)$) - precise definitions in the next section. Central to our analysis will be estimates on the time-behaviour of the space derivatives of such semigroups, which can be informally written as follows:

$$|\partial_x(P_tf)(x)| \leq M(x)e^{-Ct}, \quad |\partial_y(P_t^xp)(y)| \leq M(x,y)e^{-Ct}.$$  

(12)

In the above, for simplicity, we only wrote first order derivatives (and we used a rather informal one-dimensional notation) but in reality we will need higher order derivatives.

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$^4$Precise statements in Section 3.
We will come back to this later. If a Markov semigroup enjoys the above property (i.e. the space derivatives of some order decay exponentially fast in time) then we say that **Strong Exponential Stability (SES)** holds for that semigroup. The name is justified by the fact that, assuming the semigroup at hand admits an invariant measure, the above estimates imply exponential convergence to such a measure (see Lemma 5.4 and, for more thorough considerations, [8][13]). The above estimates, which are related to those studied in [13], are different from the ones usually appearing in the literature in two respects: first, similarly to [8][13], these are not smoothing-type estimates, see Note 4.6; second, and further to [8][13][15], the functions \( f \) and \( p \) for which they are obtained in this paper need not be bounded and indeed they will be taken to grow polynomially. As a consequence of the latter fact, the constants \( M(x) \) and \( M(x,y) \) grow polynomially as well, and we trace carefully such a growth.

We specify that, in contexts different from averaging, there are important works in the literature which deal with UiT results, e.g. [12][17][20][21][27][39][40] to mention just a few; we also flag up the related [24], where derivative estimates are used in a vein similar to the present paper. Moreover, the first, second and fourth authors of this paper have been pushing a programme to show how SES is key in proving uniform in time results, in a variety of settings, including convergence of particle systems and numerical methods for SDEs, see e.g. [1][4][8]. An intuitive explanation of why SES is the key concept is given in Note 3.4.

The paper is structured as follows. In Section 2 we introduce the necessary notation and state our assumptions. In Section 3 we state our main results on Poisson equations and then on averaging. We state all our main results in the non-fully-coupled regime first, i.e. in the case in which \( a = a(x), \sigma = \sigma(x) \) (the drift coefficients \( b \) and \( g \) are still allowed to depend on both \( x \) and \( y \)) and then in the fully coupled regime (i.e. in the case in which all the coefficients of the slow-fast system (1)-(2) depend on both \( x \) and \( y \)). In Section 4 we gather results on ergodicity and SES of the semigroup \( P_t^x \). Such results will be needed to study the Poisson problem (7). In particular in Section 4 we prove exponentially fast decay to \( \mu^x \) of the semigroup \( P_t^x \), with rate of exponential decay which is independent of \( x \) and we produce DE of the form (12) for \( y \)-derivatives up to order four of \( P_t^x \). The latter fact will be the key instrument in tackling the issue of the smoothness in \( x \) of the solution of the Poisson equation, when the coefficients of \( \mathcal{L}^x \) grow super-linearly. Section 5 contains the proof of our main result on Poisson equations, Theorem 3.1. For readability, and because some of these results are of independent interest, the proof is split in various statements. In particular, Lemma 5.1 is a well-posedness result for the Poisson problem, and Proposition 5.2 is the key place where we use SES of \( P_t^x \) to prove smoothness in the parameter \( x \) of the solution of the Poisson equation. Several comments on Proposition 5.2 are contained in Note 5.3 (which also explains why we need four derivatives of the semigroup \( P_t^x \)). Section 6 and Section 7 are devoted to the proof of our UiT averaging result. More specifically, in Section 6 we explain our strategy of proof, providing first some heuristics (Section 6.1), then a strategy of proof for the non-fully coupled case (Section 6.2) and then explaining how such a strategy can be extended to the fully coupled case.

\[ \text{In principle we should be more precise and say e.g. that SES of order } k \text{ holds if the derivatives up to order } k \text{ decay exponentially fast; we refrain from doing so, since precise requirements will be spelled out in theorems statements.} \]
The strategy explained in Section 6 clarifies that the two main ingredients in our approach are SES of the averaged semigroup $\bar{P}_t$ and the study of Poisson equations, see Note 6.1. Accordingly, in Section 7 we first obtain DEs for $\bar{P}_t$ and then apply the results on Poisson equations stated in previous sections to the Poisson problem at hand. It is important to notice that the DEs for $\bar{P}_t$ cannot be obtained in the same way as those for $P^x_t$. Indeed, while $\bar{P}_t$ and $P^x_t$ are both Markov Semigroups, the coefficients of the SDE associated with $P^x_t$ are known explicitly, those for the SDE (4) associated to $\bar{P}_t$ are not, in the sense that if $\mu^x$ is not known explicitly then the coefficient $\bar{b}$ (similarly for $\bar{\sigma}$) is only defined via (5), which is a highly non-linear expression in $x$, see Note 2.2. Section 8 contains some numerical experiments. We complement our results with a number of examples, both throughout and in Section 8. Note that when $b$ is in gradient form, our results can be applied to Langevin-type dynamics, which are ubiquitous in applications, see e.g. [47] and references therein, and Section 3.

Finally, with our scheme of proof passing from the non-fully coupled to the fully coupled regime is actually simple and the real difficulty is to understand how to deal with the non-fully coupled case first. The only difference is that in the non-fully coupled case one needs to control four $y$ derivatives of the ‘frozen semigroup’ $P^x_t$ and two $x$-derivatives of the averaged semigroup $\bar{P}_t$, in the fully coupled regime one needs the same number of $y$ derivatives of the frozen semigroup $P^x_t$ and but two additional $x$-derivatives of the ‘averaged semigroup’ $\bar{P}_t$, i.e. the fully coupled case is just more ‘computationally intensive’, but conceptually it does not require any new ideas. Hence in Section 3 we first state all our results in the non-fully coupled case and then in the fully coupled one and the whole paper refers to the non-fully coupled case, with the exception of Section 3.2 and Section 6.3 where the fully coupled regime is treated. Because in many cases of interest in applications the diffusion coefficients are constant, see [7, 11, 26] and references therein (hence falling in the non-fully coupled case) in the non-fully coupled regime we express all our theorems in terms of directly verifiable conditions on the coefficients of (1)–(2) (not just in terms of SES requirements). Similarly, the techniques of this paper can be used, without requiring any new ideas, to study the homogenization regime and the case in which $W_t$ and $B_t$ are not independent of each other. We don’t do it here to contain the length of the paper. The extension that would require more care is the one to the hypoelliptic case. All such questions will be covered in future work.

2. Notation and Assumptions

From now on, unless otherwise stated, we only refer to the non-fully coupled regime, i.e. we consider the following slow-fast system

$$dX_{t,x,y}^e = b(X_{t,x,y}^e, Y_{t,x,y}^e)dt + \sqrt{2}\sigma(X_{t,x,y}^e) dW_t,$$

$$dY_{t,x,y}^e = \frac{1}{\epsilon} g(X_{t,x,y}^e, Y_{t,x,y}^e)dt + \sqrt{2\epsilon a(X_{t,x,y}^e)} dB_t.$$

The corresponding ‘frozen process’ of interest is then

$$dY_{t}^x = g(x, Y_{t}^x)dt + \sqrt{2}a(x) dB_t, \quad Y_0^x = y,$$

and the limiting, averaged process is

$$d\bar{X}_{t} = \bar{b}(\bar{X}_t)dt + \sqrt{2}\sigma(\bar{X}_t)dW_t, \quad \bar{X}_0 = x,$$

(Section 6.3).
where $\bar{b}$ is defined as in \cite{5}. The fully coupled regime is treated separately in Section 3.2 and Section 6.3.

2.1. Notation and Assumptions. Let $\mathcal{P}_t$ be the semigroup associated with the process \cite{10}, i.e.

$$
(\mathcal{P}_t f)(x) := \mathbb{E} \left[ f(X_t) \right], \quad f \in C_b(\mathbb{R}^n),
$$

where $\mathbb{E}$ denotes expectation. It is well-known that under Assumptions A1-A4, which we will state below, such a semigroup is a classical solution to the PDE (see [10, Theorem 1.6.2])

$$
\begin{aligned}
\partial_t (\mathcal{P}_t f)(x) & = \bar{\mathcal{L}} \mathcal{P}_t f(x) \\
(\mathcal{P}_0 f)(x) & = f(x),
\end{aligned}
$$

where $\bar{\mathcal{L}}$ is the second order differential operator formally acting on smooth functions as

$$
(\bar{\mathcal{L}} f)(x) := (\bar{b}(x), \nabla_x f(x)) + \sigma \sigma^T : \text{Hess}_x f(x).
$$

In the above and throughout $M : N := \text{Tr} \{M^T N\} = \sum_{i,j} M_{ij} N_{ij}$ denotes the Frobenius inner product between two matrices $M = (M_{ij})$ and $N = (N_{ij})$ and Tr denotes the trace. Moreover, coherently with $\cite{5}$, for any function $\psi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ we let

$$
\tilde{\psi}(x) := \int_{\mathbb{R}^d} \psi(x, y) d\mu^x(y).
$$

We denote by $\mathcal{P}^\epsilon_t$ the semigroup associated with the slow-fast dynamics \cite{13}-\cite{14}, acting on functions $\psi \in C_b(\mathbb{R}^n \times \mathbb{R}^d)$, i.e.

$$
(\mathcal{P}^\epsilon_t \psi)(x, y) := \mathbb{E} \left[ \psi(X^\epsilon_t, Y^\epsilon_t) \right].
$$

The generator of this semigroup is the second order differential operator formally defined to act on smooth functions as

$$
(\mathcal{L}_\epsilon \psi)(x, y) = (\mathcal{L}_S \psi)(x, y) + \frac{1}{\epsilon} (\mathcal{L}_x \psi)(x, y)
$$

where

$$
(\mathcal{L}_S \psi)(x, y) = (b(x, y), \nabla_x \psi(x, y)) + \sigma(x)\sigma(x)^T : \text{Hess}_x \psi(x, y)
$$

$$
(\mathcal{L}_x \psi)(x, y) = (g(x, y), \nabla_y \psi(x, y)) + a(x)a(x)^T : \text{Hess}_y \psi(x, y).
$$

We emphasise that $\mathcal{L}_S$ and $\mathcal{L}_x$ are differential operators in the $x$ and $y$ variables only, respectively, and they correspond to the slow and fast part of the dynamics, respectively. We let

$$
\Sigma(x) := \sigma \sigma^T(x), \quad A(x) := aa^T(x).
$$

We want to compare the dynamics $\tilde{X}_t$ with the dynamics $X^\epsilon_t, Y^\epsilon_t$, but $X^\epsilon_t, Y^\epsilon_t$ alone does not generate a semigroup. Hence we consider the semigroup $\mathcal{P}^\epsilon_t$, which corresponds to the system $(X^\epsilon_t, Y^\epsilon_t)$, and restrict our attention to the case in which such a semigroup acts on functions $f : \mathbb{R}^n \to \mathbb{R}$ which depend on the variable $x$ only. Note that while $f$ depends only on the variable $x$, the function $(\mathcal{P}^\epsilon_t f)(x, y)$ depends on both variables. In
other words, we restrict to considering initial value problems where the initial profile is a function independent of $y$:

$$
\begin{align*}
\partial_t(P_t^x f)(x,y) &= \Sigma_c(P_t^x f)(x,y) \\
(P_0^x f)(x,y) &= f(x).
\end{align*}
$$

(26)

We will use the following notation:

- We denote by $P_t^x$ the ‘frozen semigroup’, i.e. the semigroup associated with the process $Y_t^{x,y}$ (defined in (15)):

$$
(P_t^x f)(y) := \mathbb{E}_x[f(Y_t^{x,y})], \quad f \in C_b(\mathbb{R}^d).
$$

(27)

- For every fixed $x \in \mathbb{R}^n$, the invariant measure of $Y_t^{x,y}$ is denoted $\mu^x$ (by ergodicity of $Y_t^{x,y}$, see Section 4 such a measure does not depend on $y$). As customary, for any function $\psi : \mathbb{R}^d \to \mathbb{R}$ and any measure $\nu$ on $\mathbb{R}^d$, we write

$$
\nu(\psi) := \int_{\mathbb{R}^d} \psi(\bar{y}) \nu(d\bar{y}).
$$

When we integrate against $\mu^x$, by (20) we have

$$
\mu^x(\psi) = \bar{\psi}(x).
$$

We define the space $L^2(\mu^x)$ as the space of functions $h : \mathbb{R}^d \to \mathbb{R}$ such that

$$
\int_{\mathbb{R}^d} |h(x)|^2 \mu^x(dx) < \infty.
$$

- The partial derivative with respect to the $i$-th coordinate of $x \in \mathbb{R}^n$ is denoted $\partial_{x_i}$ and for higher derivatives we write $\partial_{x_1 x_2}$. We will use the multi-index notation $\partial_x^\gamma = \partial_{x_1}^{\gamma_1} \ldots \partial_{x_n}^{\gamma_n}$ where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is an index of length $n$ and $\gamma_i \in \mathbb{N} \cup \{0\}$. Similarly, we write $\partial_y^\gamma$ where $\gamma = (\gamma_1, \ldots, \gamma_d)$, for partial derivatives in $y$ (if we differentiate with respect to $y$ then it is understood that $\gamma$ is of length $d$). We also let $|\gamma|_s := \sum_{i=1}^d \gamma_i$ (whether the sum is up to $n$ or $d$ will be clear from context).

- For a multivariable function $\phi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ and $i \in \{1, \ldots, n\}$ we refer to the $i$-th coordinate as $\phi_i(x,y)$, and similarly for $\phi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n \times \mathbb{R}^d$ we refer to the $(i,j)$-th element as $\phi_{ij}(x,y)$.

- $C^k(\mathbb{R}^n)$ ($C^k_b(\mathbb{R}^n)$, respectively), will denote the set of continuous (continuous and bounded, respectively) real valued functions on $\mathbb{R}^n$ with continuous (continuous and bounded, respectively) derivatives up to and including order $k$. For $f \in C^k_b(\mathbb{R}^n)$ we define the norm $\|f\|_{C^k_b}$ as

$$
\|f\|_{C^k_b} = \sum_{0 \leq |\gamma|_s \leq k} \|\partial_x^\gamma f\|_{\infty},
$$

where $\|\cdot\|_{\infty}$ denotes the sup-norm. Analogously, the space $C^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^d)$ is the space of functions $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ such that the derivatives $\partial_x^\gamma \partial_y^\ell g$ exist and are continuous for any $\gamma$ such that $|\gamma|_s \leq k$ and $\ell$ such that $|\ell|_s \leq \ell$. 


For any $\phi \in C^{0,k}(\mathbb{R}^n \times \mathbb{R}^d)$ we define the following seminorm

$$|\phi|_{k,m,m'} = \sup_{1 \leq |\gamma|, |\gamma'| \leq k} \sup_{x \in \mathbb{R}^n, y \in \mathbb{R}^d} \frac{\partial_y^\gamma \partial_y'^\gamma \phi(x, y)}{1 + \sum_{m' > 0} |x|^m + \sum_{m > 0} |y|^{m'}};$$

where $\gamma$ varies over all multi-indices of length $d$ and $I$ is the indicator function. Note that $|\phi|_{k,m,m'} < \infty$ if and only if all $y$-derivatives of $\phi$ up to order $k$ are polynomially bounded with exponent at most $m$ in $x$ and $m'$ in $y$, however the function itself need not be bounded. The next norm we define does include derivatives in $x$ and the function itself: for $k \in \mathbb{N} \cup \{0\}$ even, $m, m' \in \mathbb{N} \cup \{0\}$ and $\phi \in C^{k/2,k}(\mathbb{R}^n \times \mathbb{R}^d)$ let

$$\|\phi\|_{k,m,m'} = \sup_{0 \leq 2|\gamma|, +|\gamma'| \leq k} \sup_{x \in \mathbb{R}^n, y \in \mathbb{R}^d} \frac{\partial_y^\gamma \partial_y'^\gamma \phi(x, y)}{1 + \sum_{m' > 0} |x|^m + \sum_{m > 0} |y|^{m'}};$$

where $\gamma$ and $\gamma'$ are indices of length $n$ and $d$ respectively. We say that $\phi \in \text{Poly}_{m,m'}$ if $\phi \in C^{2,4}(\mathbb{R}^n \times \mathbb{R}^d)$ and $\|\phi\|_{k,m,m'} < \infty$. We will often use the sets $\text{Poly}_{0,0}$ and $\text{Poly}_{0,m}$ so let us spell out what they contain. A function $\phi$ is in $\text{Poly}_{0,0}$ if and only if $\phi \in C^{2,4}(\mathbb{R}^n \times \mathbb{R}^d)$ and all the derivatives $\partial_x^\gamma \phi_y^\gamma$ such that $2|\gamma|, +|\gamma'| \leq 4$ are bounded; $\phi \in \text{Poly}_{0,m}$ if and only if $\phi \in C^{2,4}(\mathbb{R}^n \times \mathbb{R}^d)$ and all the derivatives $\partial_x^\gamma \phi_y^\gamma$ such that $2|\gamma|, +|\gamma'| \leq 4$ are bounded in $x$ and polynomially bounded in $y$ by $c(1 + |y|^{m'})$. Motivation for the choice of the norm $\|\cdot\|_{k,m,m'}$ can be found in Note 5.3. For a multivariable function $\phi: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n_0}$, $n_0 \geq 0$, we define $\|\phi\|_{k,m,m'} := \max_{1 \leq i \leq n_0} \|\phi_i\|_{k,m,m'}$, for every $k, m, m' \geq 0$.

For any $0 < \nu < 1$, $C^{4+\nu}(\mathbb{R}^d)$ is the set of four times differentiable functions whose $4^{th}$ order derivatives are locally $\nu$-Hölder continuous.

We now list our main assumptions and then comment on them in Note 2.1.

**Assumption A1** (Growth of coefficients). Recall $\Sigma$ and $A$ defined by (25).

[C1] There exist $m^b_x, m^b_y > 0$ such that $b_i \in \text{Poly}_{m^b_x, m^b_y}$ for all $1 \leq i \leq n$.

[C2] $\Sigma_{ij} \in \text{Poly}_{0,0}$ for all $1 \leq i, j \leq n$.

[C3] There exists $0 < \nu < 1$ such that for each $x \in \mathbb{R}^n$, $g_i(x, \cdot) \in C^{4+\nu}(\mathbb{R}^d)$, and there exists $m^g_y > 0$ such that $g_i \in \text{Poly}_{0,m^g_y}$ for all $1 \leq i \leq d$.

[C4] $A_{ij} \in \text{Poly}_{0,0}$ for all $1 \leq i, j \leq d$.

**Assumption A2** (Uniform ellipticity). There exist constants $\lambda_-$, and $\lambda_+$ such that

$$0 < \lambda_- \leq \left( \frac{A(x)\xi}{|\xi|}, \frac{\xi}{|\xi|} \right) \leq \lambda_+, \quad \text{for every } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^d \setminus \{0\};$$

and

$$0 < \lambda_- \leq \left( \frac{\Sigma(x)\xi}{|\xi|}, \frac{\xi}{|\xi|} \right) \leq \lambda_+, \quad \text{for every } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n \setminus \{0\}.$$
Assumption A3 (Lyapunov condition for frozen process). For every integer \( k \geq 0 \), there exist constants \( r_k, C_k > 0 \) (independent of \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^d \)) such that

\[
(g(x, y), y) + (k - 1)a(x) \leq -r_k|y|^2 + C_k
\]

for every \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^d \).

Assumption A4 (Lyapunov condition for slow process). There exist constants \( \bar{r}, \bar{C} > 0 \) (independent of \( x \in \mathbb{R}^n, y \in \mathbb{R}^d \)), such that

\[
(b(x, y), x) + (4m^b - 1)\sigma(x) : \sigma(x) \leq -\bar{r}|x|^2 + \bar{C}
\]

for every \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^d \).

Assumption A5 (Drift condition for the frozen process). There exist \( \kappa, \zeta_1, \zeta_2, \zeta_3 > 0 \) independent of \( x, y, \xi \) such that for any \( x \in \mathbb{R}^n, y \in \mathbb{R}^d, \xi \in \mathbb{R}^d \) we have

\[
2 \sum_{i,j=1}^{d} \partial_{y,i} g_j(x, y) \xi_j + \sum_{i,j=1}^{d} \zeta_1(\partial_{y,i} g_j(x, y), \xi_j) + \sum_{i,j,k=1}^{d} \zeta_2(\partial_{y,i} g_j(x, y), \xi_j)^2
\]

\[
\quad + \sum_{i,j,k=1}^{d} \zeta_3(\partial_{y,i} \partial_{y,j} g_k(x, y), \xi_j)^2 \leq -\kappa|\xi|^2. \tag{32}
\]

If Assumption [A1][C1][C2] holds, then there exist some constants \( m_1, \ldots, m_8 \geq 0 \) such that the following holds

\[
|\partial^\gamma_y b_i(x, y)| \leq C(1 + |x|^{m_1} + |y|^{m_2}) \quad \text{for all } |\gamma|_* = 4,
\]

\[
|\partial^\gamma_y \partial^\gamma_x b_i(x, y)| \leq C(1 + |x|^{m_3} + |y|^{m_4}) \quad \text{for all } |\gamma|_* = 2, |\tilde{\gamma}|_* = 1,
\]

\[
|\partial^\gamma_x b_i(x, y)| \leq C(1 + |x|^{m_5} + |y|^{m_6}) \quad \text{for all } |\tilde{\gamma}|_* = 2,
\]

\[
|\partial^\gamma_x g_i(x, y)| \leq b_{2,m_7,m_8}(1 + |x|^{m_7} + |y|^{m_8}) \quad \text{for all } 1 \leq |\gamma|_* \leq 2,
\]

\[
|\partial^\gamma_x \Sigma_{ij}(x)| \leq K_\Sigma \quad \text{for all } 1 \leq |\tilde{\gamma}|_* \leq 2,
\]

for some \( C, K_\Sigma \geq 0 \) and for all \( i, j = 1, \ldots, d, x \in \mathbb{R}^n, y \in \mathbb{R}^d \), where in the above \( \gamma, \tilde{\gamma} \) are indices of length \( d \) and \( n \) respectively. Furthermore, by Assumption [A1][C3][C4] there exist constants \( m_9, K_A \geq 0 \) such that for all \( i, j = 1 \ldots d, \)

\[
|\partial^\gamma_x b_i(x, y)| \leq K_g (1 + |y|^{m_9}) \quad \text{for all } |\tilde{\gamma}|_* = 1,
\]

\[
|\partial^\gamma_x A_{ij}(x)| \leq K_A \quad \text{for all } 1 \leq |\gamma|_* \leq 2,
\]

where we set \( K_g := \max_{1 \leq |\gamma|_* \leq 2} \|\partial^\gamma_x g\|_{0,0,m_9}. \)

Assumption A6 (Drift condition for slow process). Let Assumptions A1 to A5 hold. Assume that there exists \( \zeta > 0 \) independent of \( x, y, \xi \) such that for any \( x, \xi \in \mathbb{R}^n, y \in \mathbb{R}^d \) we have

\[
\sum_{i,j=1}^{n} \partial_{x,i} b_j(x, y) \xi_j \leq - \left( \frac{D_0}{\kappa} b_{2,m_7,m_8} D_b(x) + \zeta (1 + |x|^{m_1} + |x|^{m_2} + |x|^{m_3}) + \frac{K^2_\Sigma n^3}{4\lambda_-} \right) |\xi|^2 \tag{34}
\]
where \( \kappa \) is given by (32), \( D_0 := \max\{ (\zeta_1 \lambda_-)^{-1/2}, 1 \} \left( d + \sqrt{\zeta_1 \lambda_- d^2} \right) \) and

\[
D_b(x) := \left( (K_g + K_A) \left( 1 + \frac{2C'_m r_{m8}}{r_{m8}} \right) + K_g \left( \frac{C'_{m8} + C'_{mg}}{r_{m8} + r'_{m8}} + \frac{C'_{mg} + C'_{m8} C'_{mg}}{r'_{m8} r_{m9}} \right) \right) \\
+ \mathbb{I}_{m > 0} \left( K_g + K_A + K_g \frac{C'_{mg}}{r_{m9}} \right) |x|^{m7}.
\]

In the above, for \( m > 0 \), \( C'_m \) and \( r'_m \) are defined in Lemma 4.1; \( C'_0 = 0 \) and \( r'_0 = 1 \).

Note 2.1. Let us comment on each of the above assumptions in turn.

1. Assumption [A1] requires that all the coefficients are \( C^{2,A} \), the drift coefficients \( b \) and \( g \) have at most polynomial growth and the diffusion coefficients \( \Sigma \) and \( A \) are bounded.

2. Assumption [A2] is a uniform ellipticity assumption and is used to ensure differentiability of the semigroups \( P^\epsilon_t \), \( P^x_t \) and \( \bar{P}_t \). More precisely, (29) gives differentiability in \( y \) of \( P^\epsilon_t f \) and \( P^x_t f \), while (30) gives differentiability in \( x \) of \( \bar{P}_t f \) and \( P^\epsilon_t f \).

3. Assumption [A3] is relatively standard in the literature, see for example [48], and (since it is enforced for every \( k \)) it ensures that all the moments of (15) are uniformly bounded in \( t \). In principle we require for Assumption [A3] to hold for every \( k \). However, since \( a \) is bounded (as \( A \) is bounded, by Assumption [A1]), if Assumption [A3] holds with \( k = 1 \) then it holds for every integer \( k \geq 1 \) (if it is true for \( k = 1 \) then we can take \( r_k = r_1 \), \( C_k = C_1 + (k - 1) \lambda_+ \)). When we also impose Assumption [A4] we have that the process (13)-(14) has sufficient number of moments uniformly bounded in \( t \) (and \( \epsilon \)), see Lemma 6.2.

4. Assumption [A3] and Assumption [A2] combined, give existence and uniqueness of the invariant measure \( \mu^x \) of the semigroup \( P^x_t \) (for each \( x \) fixed) and exponential convergence to \( \mu^x \) as well. Such ergodic properties of \( P^x_t \) are known, when \( x \) is fixed. What we will additionally need is to control the way in which the rate of exponential convergence depends on \( x \), see comments after Proposition 4.3 and Section 4.

5. Assumption [A1], Assumption [A3] and Assumption [A4] are sufficient to have pathwise well-posedness of the SDEs (13)-(14), (16) and (15), by [33, Theorem 3.1.1].

6. Assumption [A5] is used to ensure SES (i.e. decay of the derivative) of the ‘frozen semigroup’ \( P^\epsilon_t \), that is to show that (12)2 (or, more precisely, (55)) holds. Therefore we can replace Assumption [A5] by assuming that the frozen semigroup is SES, see Section 3 for more details. Similarly, Assumption [A6] is used to ensure the
decay of the derivative of the averaged semigroup $\bar{P}_t$ and may be replaced by assuming that $\bar{P}_t$ is SES. We emphasize that Assumption A6 is an assumption on the drift $b$ which will be used to derive properties of the averaged coefficient $\bar{b}$.

(7) If the second, third and fourth order derivatives of $g$ are bounded then Assumption A5 reduces to finding $\kappa > 0$ such that

$$2(\xi, \nabla_y g(x, y) \xi) \leq -\kappa |\xi|^2.$$  

(35) Indeed (35) implies that Assumption A5 holds for some $\kappa > 0$ (by taking $\zeta_1, \zeta_2$ and $\zeta_3$ sufficiently small).

(8) Let us give two examples in which Assumption A6 takes a simpler form. Firstly, if $b$ is of the form $b(x, y) = b_1(x) + b_2(y)$, then $m_7$, as defined before Assumption A6, vanishes, implying that $D_b(x)$ is constant. If, further, $b_2$ has bounded (in $y$) first derivatives and hessian, then $D_b$ simplifies to

$$D_b(x) = 3(K_g + K_A).$$

An example satisfying all of the above assumptions is given by the following system:

$$dX_{\epsilon,x,y}^t = (- (X_{\epsilon,x,y}^t)^3 - X_{\epsilon,x,y}^t + b_0(Y_{\epsilon,x,y}^t))dt + \sqrt{2} dW_t,$$

$$dY_{\epsilon,x,y}^t = \frac{1}{\epsilon}(- (Y_{\epsilon,x,y}^t)^3 - Y_{\epsilon,x,y}^t + g_0(X_{\epsilon,x,y}^t))dt + \frac{1}{\sqrt{\epsilon}} \sqrt{2} dB_t$$

where $(X_{\epsilon,x,y}^t, Y_t^t) \in \mathbb{R} \times \mathbb{R}$ and $b_0, g_0 \in C^\infty_b(\mathbb{R})$. Both the slow and fast component are Langevin-type dynamics, we will make more remarks on the above system in Example 3.1. Further examples are given in the next section and in Section 8.

**Note 2.2.** One difficulty in obtaining derivative estimates for the semigroup $\bar{P}_t f$ given by (17), is that the coefficients of the SDE (16) depend on the invariant measure of equation (15), which is itself a function of $x$. Indeed, recall that the coefficient $\bar{b}$ is defined by (5) and depends on $x$ both through the function $b$ itself and through the measure $\mu^x$ so in general has a complicated dependence on $x$. With an explicit expression for $\mu^x$ (which might or might not be available, depending on the specific application at hand, see e.g. [26, 46]), and hence for $\bar{b}$, one could verify an assumption of the type (32) (or, in the case of non-constant drift, use Theorem 7.1 directly) to obtain the desired DEs. In the absence of such an expression, we use a different approach, see Section 7.1.

3. Main Results

In this section we gather our main results. In particular, in Subsection 3.1 we gather our results regarding the non fully coupled regime, in Subsection 3.2 the results in the fully coupled case.

3.1. The non-fully coupled regime. We start by stating results on Poisson equations and then our UiT averaging result for SDEs.

**Poisson equations.** We consider equations of the form (7) where $u : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ is the unknown, while $\phi$ is a given function, assumed to be in the space $\text{Poly}_{m,m'}$ for some $m, m' \geq 0$, and we recall that the operator $\mathcal{L}^\varepsilon$ is a second order differential operator of the form (24). With this in mind, because $x$ in the above appears only as a parameter,
it is sometimes useful to refer to a function \( \phi(x, y) \) as \( \phi^x(y) \) and from now on we will use these notations interchangeably; moreover, when we need to emphasise the dependence of the solution \( u \) on \( \phi \), we denote the solution to the Poisson equation (7) as \( u(x, y) = u^x_\phi(y) \).

As we have already said, such a solution is in general not unique, since constants belong to the kernel of \( \mathcal{L}^x \). However, we restrict to the set of mean-zero functions, i.e. to the set of functions \( f \) such that \( \mu^x(f) = 0 \), to ensure the solution, if it exists, is unique. When we refer to the solution to (7), we understand this to mean the mean-zero solution. Moreover, in order for the Poisson problem (7) to have a solution, it is necessary for the RHS of (7) to be a mean-zero function, as can be seen by integrating the LHS of (7) with respect to \( \mu^x \); this is satisfied in (7) by the definition of \( \bar{\phi} \).

**Theorem 3.1.** Let \( g \) and \( A = aa^T \) satisfy Assumption A1, A2, A3, and Assumption A4. Let \( \bar{\phi} \in \text{Poly}_{m_x, m_y} \), for some \( m_x, m_y \geq 0 \). Then the function \( \bar{\phi} \) is well-defined, the solution \( u_\phi \) of the Poisson equation (7) exists, it is unique (in the class of mean-zero functions) and it is given by

\[
\phi^x_\bar{\phi}(y) := -\int_0^\infty (P^x_\phi \phi^x(y) - \bar{\phi}(x)) \, ds.
\]  

Moreover,

- There exists some \( C > 0 \) (which may depend on \( m_x, m_y \) but is independent of the choice of \( \bar{\phi} \in \text{Poly}_{m_x, m_y} \)) such that the solution \( u_\phi^x \) satisfies the following bound

\[
|u^x_\bar{\phi}(y)| \leq C \|\bar{\phi}\|_{0, m_x, m_y}(1 + |x|^{m_x} + |y|^{m_y}) \text{ for all } x \in \mathbb{R}^n, y \in \mathbb{R}^d. 
\]

- Both the solution \( u_\phi \) and the function \( \bar{\phi} \) are twice differentiable in \( x \) and there exists \( C > 0 \) such that

\[
|\partial_{x_i} \bar{\phi}| \leq C \|\bar{\phi}\|_{2, m_x, m_y}(1 + |x|^{2m_x}), 
\]

\[
|\partial_{x_i, x_j} \bar{\phi}| \leq C \|\bar{\phi}\|_{4, m_x, m_y}(1 + |x|^{4m_x}),
\]

\[
|\partial_{x_i} u^x_\bar{\phi}(y)| \leq C \|\bar{\phi}\|_{2, m_x, m_y}(1 + |y|^{M_y^g} + |x|^{2m_x}),
\]

\[
|\partial_{x_i, x_j} \phi^x_\bar{\phi}(y)| \leq C \|\bar{\phi}\|_{4, m_x, m_y}(1 + |y|^{M_y^{g+m_y}} + |x|^{4m_x}),
\]

for all \( x \in \mathbb{R}^n, y \in \mathbb{R}^d, i, j \in \{1, \ldots, n\}, \text{ where } M_y^g := \max\{2m_y, m_y + m_y\}.\)

**Proof.** Note 4.2 explains that \( \phi \) is integrable with respect to \( \mu^x \) and hence the function \( \bar{\phi} \) is well-defined. The proof of this theorem can be found in Section 5. In particular, well-posedness of the Poisson problem (7) is given in Lemma 5.1. The estimate (39) follows from (38), Lemma 5.1, and Proposition 4.3. For the proof of (40) and (42) see Proposition 5.2, and for the proof of (41) and (43) see Proposition 5.5.

The above result is of independent interest, but also instrumental to solving the averaging problem, see Note 6.1. Explanations on how SES of the semigroup \( P_t^x \) helps to tackle the issue posed by the unboundedness of the drift coefficient of the operator \( \mathcal{L}^x \) in the study of the smoothness in \( x \) of the solution Poisson problem (7) can be found in Note 5.3.

**Averaging for SDEs.** Our main result on averaging for SDEs in the non fully coupled regime (13)-(14) is the following.
Theorem 3.2. Consider the slow-fast system \((13)-(14)\) and the semigroups \(\{P_t^i\}_{t \geq 0}\) and \(\{\bar{P}_t\}_{t \geq 0}\) defined in \((21)\) and \((17)\), respectively. Let Assumption A1 to Assumption A6 hold. Then, for every \(f \in C^2_b(\mathbb{R}^n)\), there exists a constant \(C > 0\), independent of time, such that
\[
|\langle \delta_{P_t} f \rangle(x, y) - \langle \delta_{\bar{P}_t} f \rangle(x)\rangle| \leq C\|f\|_{C^2_b}(1 + |y|^M + |x|^M), \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^d, \tag{44}
\]
where \(M := \max\{2m^g_m, m^g_m + M\} \).

We prove Theorem 3.2 in Section 6. In Example 3.1 we show that system \((36)-(37)\) satisfies all the assumptions of Theorem 3.2. The proof of Theorem 3.2 can be found in Section 6.

Theorem 3.2 contains sufficient conditions, phrased in terms of the coefficients of system \((13)-(14)\), in order for the UiT averaging result to hold. This makes it ready to use - as one needs only check conditions on the coefficients - but it does not help to highlight the role of DEs and SES. We therefore rephrase it below in terms of SES properties of the semigroups \(P_t\) and \(P_t^x\) (we will do a similar thing also for Theorem 3.3 in the fully-coupled case, see Theorem 3.5).

Theorem 3.3. Consider the semigroup \(\{P_t^i\}\) associated to \((1)-(2)\) and the semigroup \(\{\bar{P}_t\}_{t \geq 0}\) associated to \((4)\). Let Assumptions A1 to A4 hold. Suppose, furthermore, that there exists a constant \(K > 0\) and \(m_x, m_y \geq 0\) such that for any \(\psi \in C^0_{\gamma,k}(\mathbb{R}^n \times \mathbb{R}^d)\) with \(|\psi|_{k,m_x,m_y} < \infty\) and for all \(x \in \mathbb{R}^n\) we have
\[
|P_t^x \psi|_{k,m_x,m_y} \leq K|\psi|_{k,m_x,m_y} e^{-\kappa t}, \quad k \in \{2, 4\}, \tag{45}
\]
and that there exist constants \(\bar{K}, C > 0\) such that for any \(\psi \in C^2_b(\mathbb{R}^d)\) we have
\[
\sup_{1 \leq |\gamma| \leq 2} \|\partial_\gamma P_t^x \psi\|_{\infty} \leq \bar{K}e^{-Ct}. \tag{46}
\]
Then \((44)\) holds.

Theorem 3.2 can be seen as a consequence of Theorem 3.3. Indeed, the structure of the proof of Theorem 3.2 clearly shows that \((44)\) is implied by Assumptions A1 to A4 plus \((45)\) and \((46)\). The proof of Theorem 3.2 then goes further and shows that Assumptions A1 to A5 imply \((45)\) (see Proposition 4.5) and that Assumptions A1 to A6 imply \((46)\) (see Proposition 7.2). So, the line of reasoning in the proof of Theorem 3.2 implies Theorem 3.3 and we don’t prove the latter separately. The DEs \((45)\) and \((46)\) are central to the proof for the following reason: using relatively standard tricks from semigroup theory, one can express the difference between \(P_t^i\) and \(P_t\) in terms of two main objects, namely the second derivative of the semigroup \(P_t\) and the second \(x\)-derivative of the solution of a Poisson equation of the form \((7)\) - see \((97)\). Note \((6.1)\) and the calculations in Proposition 7.3. Understanding both these objects requires the analysis of the second \(x\) derivatives of \(P_t^x\). Indeed, because the coefficients of the generator of the averaged semigroup \(\bar{P}_t\) contain the measure \(\mu^x\), one needs to study the second derivative of \(\mu^x\) with respect to \(x\) - see Proposition 5.5. In turn, \(\mu^x\) is the limit as \(t \to \infty\) of the semigroup \(P_t^x\), hence the appearance of the second \(x\) derivative of \(P_t^x\). Moreover, because the solution of Poisson equations of the form \((7)\) can be expressed through the semigroup \(P_t^x\), see \((38)\), it is clear that also the study of the second \(x\) derivatives of the
Poisson equation leads one to consider second $x$ derivatives of $P^x_t$. As discussed in the introduction, the smoothness in $x$ of the semigroup $P^x_t$ is non-trivial and is gained through some “transfer formulas” (see for example (62) and Note 5.3) which allow one to obtain smoothness in $x$ from the smoothness in $y$ (which is instead straightforward). When using such transfer formulas one $x$ derivative comes at the price of two $y$ derivatives, and this is the reason why we will need four $y$-derivatives of $P^x_t$. We note that, as opposed to previous literature, we write these formulas in a way to highlight how they contain appropriate semigroup derivatives and hence the role of such derivatives in our analysis.

Finally let us give an intuitive explanation on how SES of the semigroup $\bar{T}_t$, i.e. (46), is instrumental to obtain an averaging result which is uniform in time in Note 3.4 below.

**Note 3.4 (SES for UiT averaging).** To explain in a simplified setting why SES is key to proving uniform in time convergence, let us consider two Markov semigroups, say $\mathcal{T}_t$ and $\bar{\mathcal{T}}_t$. With standard manipulations, the difference between such semigroups can be expressed in terms of the difference between their respective generators, say $\mathcal{G}$ and $\bar{\mathcal{G}}$, as follows

$$
(\bar{T}_t\varphi)(z) - (\mathcal{T}_t\varphi)(z) = \int_0^t ds \frac{d}{ds} \mathcal{T}_{t-s} \bar{\mathcal{G}} \varphi(z) = \int_0^t ds \mathcal{T}_{t-s}(\mathcal{G} - \bar{\mathcal{G}}) \bar{\mathcal{T}}_s \varphi(z)
$$

$$
\leq \int_0^t ds \| \mathcal{T}_{t-s}(\mathcal{G} - \bar{\mathcal{G}}) \bar{\mathcal{T}}_s \varphi \|_\infty \leq \int_0^t ds \| (\mathcal{G} - \bar{\mathcal{G}}) \bar{\mathcal{T}}_s \varphi \|_\infty
$$

If $\mathcal{G}$ and $\bar{\mathcal{G}}$ are differential operators then the latter difference involves derivatives of the semigroup $\bar{\mathcal{T}}_t$. If such derivatives decay exponentially fast in time, then the difference between such semigroups can be estimated by a constant (independent of time) rather than with exponential growth, which is what would happen by using Gronwall-type arguments. This line of reasoning, when applied to $\mathcal{P}_t$ and $\mathcal{P}^x_t$ rather than $\bar{\mathcal{T}}_t$ and $\mathcal{T}_t$, inspires our approach - though the precise proof does not exactly follow the above calculation and some further manipulations are required (to obtain the correct power of $\epsilon$ on the RHS). Details of our strategy of proof can be found in Section 6. □

**Example 3.1.** Let us come back in more detail to the system (36)-(37). First of all, both the slow and the fast components of (36)-(37) are Langevin-type dynamics for the potential $V(x) = x^4/4 + x^2/2$, with bounded perturbation $b_0$ (or $g_0$). This setup is rather important in applications, see e.g. [47]. However we point out that because of our assumptions on the drifts $b$ and $g$ (Assumptions A5 and A6), the case of Langevin dynamics in double well potentials is not covered in this paper. In the numerics section, Section 8, we provide evidence supporting the idea that a UiT result should be true even in that case, and we reserve this for upcoming work.

Let us now come to show that Assumptions A1-A6 are verified for system (36)-(37) and hence prove that the UiT result of Theorem 3.2 holds for such a system. It is immediate to see that Assumption A1 holds with $m^b_2 = 3, m^b_3 = 0, m^g_2 = 3$ and that Assumption A2 holds with $\lambda_- = \lambda_+ = 1$. As observed in Note 2.1 in order to show that Assumption A3 holds it is sufficient to consider $k = 1$. For $k = 1$, (31) holds with $C_1 = \|g_0\|^2_\infty/2, r_1 = 1/2$. Similarly, Assumption A4 holds with $\bar{C}_1 = \|b_0\|^2_\infty/2, \bar{r}_1 = 1/2$. Assumption A5 holds with $\zeta_1 = \frac{1}{6}$ and $\kappa = 2$. It remains to verify Assumption A6. Observe that $m_1 = m_2 = m_3 = m_4 = m_6 = m_7 = m_8 = m_9 = 0$, and $m_5 = 1$. We also
have \( D_0 = \sqrt{6} + 1 \), \( D_b = \|\partial_x g_0\|_\infty \) and \( K_A = K_\Sigma = 0 \). Therefore, Assumption A6 holds provided
\[
\max\{\|\partial_y b_0\|_\infty, \|\partial_y g_0\|_\infty\} < \frac{1}{(\sqrt{6} + 1)\|\partial_x g_0\|_\infty}.
\]

In conclusion, if (47) holds, then we may apply Theorem 3.2 to (36)-(37) so there exists \( C > 0 \), such that
\[
|\mathcal{P}^t_i f(x, y) - \tilde{\mathcal{P}}^t_i f(x)| \leq \epsilon C\|f\|_{C^2_b}(1 + |y|^6 + |x|^{12}), \text{ for every } f \in C^2_b(\mathbb{R}^n).
\]

\[
\square
\]

3.2. The fully coupled regime. In this subsection and in Subsection 6.3 only we consider the fully coupled system (1)-(2) as opposed to (13)-(14). The only difference between these two systems is that in the former \( a = a(x, y), \sigma = \sigma(x, y) \), i.e. the diffusion coefficients \( a, \sigma \) are allowed to depend on both variables, in the latter \( a = a(x), \sigma = \sigma(x) \).

Therefore, here and in Subsection 6.3 by \( \mathcal{P}^t_i \), \( \tilde{\mathcal{P}}^t_i \) and \( \mathcal{P}^t_i \) we mean the semigroups associated with the processes (9), (4) and (1)-(2), respectively. In particular, the generator \( \mathcal{L}^x \) of \( \mathcal{P}^t_i \) is intended to be given by (38), as opposed to (24) and when we say that Assumptions A1 to Assumption A4 hold, we mean that they hold for \( a = a(x, y), \sigma = \sigma(x, y) \) and Assumption A4 [C4] should be modified to: there exists \( 0 < \nu < 1 \) such that for each \( x \in \mathbb{R}^n \), \( A_{ij}(x, \cdot) \in C^{1+\nu}(\mathbb{R}^d) \), and \( A_{ij} \in \text{Poly}_{0,0} \) for all \( 1 \leq i, j \leq d \).

Theorem 3.5. Let \( q \) and \( A = aa^T \) satisfy Assumption A1, Assumption A2, Assumption A3 and suppose (45) holds. Let \( \phi \in \text{Poly}_{m_x,m_y} \) for some \( m_x, m_y \geq 0 \). Then the solution \( u_\phi \) of the Poisson equation (7) exists and is unique (in the class of mean-zero functions) and the representation formula (38) holds as well. Moreover,

- There exists some \( C > 0 \) (which may depend on \( m_x, m_y \) but is independent of the choice of \( \phi \)) such that the solution \( u_\phi \) satisfies the following bound
  \[
  |u_\phi(y)| \leq C\|\phi\|_{m_x,m_y}(1 + |x|^{m_x} + |y|^{m_y}) \quad \text{for all } x \in \mathbb{R}^n, y \in \mathbb{R}^d.
  \]

- The function \( \tilde{\phi} \) is well-defined, both the solution \( u_\phi \) and the function \( \tilde{\phi} \) are twice differentiable in \( x \) and there exist \( C, m'_x, m'_y > 0 \) such that
  \[
  \sup_{1 \leq |\gamma| \leq 2} |\partial^\gamma_x \phi| \leq C\|\phi\|_{4,m_x,m_y}(1 + |x|^{m'_x}),
  \]
  \[
  \sup_{1 \leq |\gamma| \leq 2} |\partial^\gamma_x u_\phi| \leq C\|\phi\|_{4,m_x,m_y}(1 + |x|^{m'_x} + |y|^{m'_y}),
  \]
  for all \( x \in \mathbb{R}^n, y \in \mathbb{R}^d \).

Before stating the averaging result we require SES of the averaged SDE (4).

Assumption A7. Let \( \{\tilde{\mathcal{P}}^t_i\}_{t \geq 0} \) denote the semigroup associated to (4), and assume that Assumptions A1, A4 hold. Assume there exist constants \( K, C > 0 \) such that for any \( \psi \in C^4_b(\mathbb{R}^n) \) we have
\[
\sup_{1 \leq |\gamma| \leq 4} \|\partial^\gamma_t \tilde{\mathcal{P}}^t_i \psi\|_\infty \leq K e^{-Ct}\|\psi\|_{C^4_b(\mathbb{R}^n)}.
\]
Theorem 3.6. Consider the semigroup \( \{P_t^e\} \) associated to (1), (2) and the semigroup \( \{\tilde{P}_t\}_{t \geq 0} \) associated to (3). Let Assumption A7 to Assumption A4 hold, together with (45) and Assumption A7. Then for every \( f \in C^1_b(\mathbb{R}^n) \) there exist constants \( C, M_x, M_y > 0 \) independent of \( f, t, x, y \) such that

\[
| (P_t^e f)(x, y) - (\tilde{P}_t f)(x) | \leq C \|f\|_{C^1_b} (1 + |y|^{M_y} + |x|^{M_x}), \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^d.
\]

A sketch of the proof of Theorem 3.6 is contained in Subsection 6.3. The main difference between the assumptions of the above theorem and Theorem 3.3 is that in the latter we require \( x \)-derivative estimates of \( \tilde{P}_t \) up to order 2 whereas for Theorem 3.6 we require \( x \)-derivatives of \( P_t^e \) up to order 4. As explained after Theorem 3.3 the two main objects in the proof of Theorem 3.3 are the second \( x \)-derivative of the semigroup \( \tilde{P}_t \) and the second \( x \)-derivative of the solution of a Poisson equation. In contrast, in the setting of Theorem 3.6 we need fourth order \( x \) derivatives of the semigroup \( \tilde{P}_t \) but still only two \( x \)-derivatives of the solution of a Poisson equation. In general to obtain derivative estimates of \( \tilde{P}_t \) up to order 4 of the form in Assumption A7 requires the coefficients of the generator \( \mathcal{L} \) to be \( x \)-differentiable up to order 4. Since these coefficients involve the measure \( \mu^x \), one needs \( \mu^x \) to be 4 times \( x \)-differentiable. As \( \mu^x \) is the limit of \( P_t^e \), one can show the \( x \)-regularity of \( \mu^x \) by first showing the \( x \)-regularity of \( P_t^e \). The \( x \)-derivatives of \( P_t^e \) can be bounded using the \( y \)-derivatives of \( P_t^e \) by using the “transfer formulae” (see for example (62)) however this would then require (55) to hold instead for derivatives up to order 8. We avoid needing \( y \)-derivatives up to order 8 by assuming that Assumption A7 hold. (However in order to obtain bounds on the second \( x \)-derivative of the solution to a Poisson equation we still require bounds on \( y \)-derivatives of \( P_t^e \) up to order 4.) With specific examples at hand, checking that Assumption A7 holds is not difficult, but it can be rather lengthy. An example of a system that satisfies all the assumptions of Theorem 3.6 is the following:

Example 3.2. Let \( d = n = 1 \) and consider the following system:

\[
\begin{align*}
\mathrm{d}X_t^{e,x,y} &= (-X_t^{e,x,y} - Y_t^{e,x,y}) \mathrm{d}t + \sqrt{2} \sqrt{1 + \frac{1}{4} \sin(Y_t^{e,x,y}) - \frac{X_t^{e,x,y}}{4\sqrt{1 + (X_t^{e,x,y})^2}}} \, \mathrm{d}W_t \\
\mathrm{d}Y_t^{e,x,y} &= \left[ \arctan(X_t^{e,x,y}) - Y_t^{e,x,y} \right] \mathrm{d}t + \sqrt{2} \left( 1 + \frac{1}{12} \frac{\sin(b_0(X_t^{e,x,y}) + Y_t^{e,x,y})}{\cos(b_0(X_t^{e,x,y}) + Y_t^{e,x,y})} \right) \mathrm{d}B_t \\
&\quad + \left[ \frac{\cos(b_0(X_t^{e,x,y}) + Y_t^{e,x,y})}{6} \right] \left( 1 + \frac{1}{12} \frac{\sin(b_0(X_t^{e,x,y}) + Y_t^{e,x,y})}{\cos(b_0(X_t^{e,x,y}) + Y_t^{e,x,y})} \right) \mathrm{d}t.
\end{align*}
\]

Here \( b_0 : \mathbb{R} \to \mathbb{R} \) are smooth bounded functions with bounded derivatives of all orders. Observe that square root term in \( X_t^{e,x,y} \) is always real valued since expression in the square root is bounded below by \( 1 - \frac{1}{4} - \frac{1}{4\sqrt{2}} \) which is positive, this follows since the functions \( \sin \) and \( x \mapsto \frac{x}{\sqrt{1 + x^2}} \) are bounded in absolute value by 1. With calculations analogous to those that we will do to obtain the derivative estimates of the next section (which we don’t repeat here for this specific example) one can show that the corresponding frozen semigroup is SES. Moreover we have that \( \mu^x \) is normally distributed with mean \( \arctan(x) \) and variance 1 and hence the averaged equation is given by

\[
\mathrm{d}\bar{X}_t^e = (-\bar{X}_t^e - \arctan(\bar{X}_t^e)) \mathrm{d}t + \sqrt{2} \, \mathrm{d}W_t.
\]
Since the drift is strictly monotone it is immediate to check that it satisfies conditions of the form (35) and hence the averaged semigroup is SES.

4. Preliminary Results on the frozen semigroup $P_t^x$

Looking at the representation formula (38), it is clear that the ergodic properties of the semigroup $P_t^x$ are central to the study of the Poisson problem (1) — trivially, if $P_t^x$ does not decay fast enough to $\bar{\phi}^x$ then (38) makes no sense. With this motivation, Lemma 4.1 and Proposition 4.3 address the ergodic properties of the semigroup $P_t^x$. In Proposition 4.5 we show SES for the same semigroup. Throughout the section we assume that the operator $L_x$ is uniformly elliptic, i.e. $A$ satisfies Assumption A2 and we don’t repeat this fact in every statement.

**Lemma 4.1.** Let $g$ and $A$ satisfy Assumption A1 ($[C3],[C4]$), and Assumption A3. Let $Y_t^{x,y}$ be the solution of the SDE (15) (which does exist under our assumptions, see Note 2.1). Then for any integer $k \geq 1$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^d$ we have

$$E \left[ |Y_t^{x,y}|^k \right] \leq e^{-r_t^k} |y|^k + C_k' r_t^k$$

(51)

$$\int_{\mathbb{R}^d} |y|^k \mu_x(dy) \leq C_k' r_t^k,$$

(52)

where $C_1 = r_1' \sqrt{C_2'/r_2'}$, $r_1' = r_2/2$, $C_2' = C_2/r_2$, $r_2' = 2r_2$, and for $k > 2$

$$C_k' = \frac{C_k}{r_t^k} \left( \frac{2C_k(k-2)}{k r_t^k} \right)^{k-2} r_t^k = \frac{kr_t^k}{2}.$$

We give a sketch of the proof of Lemma 4.1 in Appendix A.1 and there expand on how Assumption A3 implies moment bounds by showing that Assumption A3 implies that the function $V(y) = |y|^k$ is a Lyapunov function for the SDE.

**Note 4.2.** Lemma 4.1 implies that $\bar{\psi} = \bar{\psi}(x)$ (introduced in (20)) is well defined for any $\psi \in \text{Poly}_{m,m'}$ and any $m,m' > 0$, and moreover that $\bar{\psi} \in \text{Poly}_{m,0}$.

**Proposition 4.3.** Let $P_t^x$ be given by (27). Let $g$ and $A$ satisfy Assumption A1 ($[C3],[C4]$), A2 and A3. Let $\phi \in C(\mathbb{R}^n \times \mathbb{R}^d)$ with $\|\phi\|_{0,m_x,m_y} < \infty$ for some $m_x,m_y > 0$; then there exist constants $c,C > 0$ (independent of $x \in \mathbb{R}^n$, $y \in \mathbb{R}^d$) such that the following holds:

$$|(P_s^x \phi^x)(y) - \mu_x(\phi^x)| \leq C \|\phi\|_{0,m_x,m_y} e^{-cs}(1 + |x|^{m_x} + |y|^{m_y})$$

for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^d$, $s > 0$.

(53)

As a consequence, there exists some $C' > 0$ such that

$$\left| \int_0^\infty [(P_s^x \phi^x)(y) - \mu_x(\phi^x)] ds \right| \leq C'\|\phi\|_{0,m_x,m_y} (1 + |x|^{m_x} + |y|^{m_y}).$$

**Proof.** The proof can be found in Appendix A.1.

**Note 4.4.** Lemma 4.1 and Proposition 4.3 still hold if the diffusion coefficient $a$ is also a function of $y$ and indeed we do the proofs in this case. This is relevant in the proofs for the fully coupled case.

□
The above proposition would be standard if we were not tracing the dependence on the parameter $x$ in the main result. That is, using techniques from [38] (see, e.g., [36] Theorem 2.5) one could obtain
\[ |(P_x^t \phi^x)(y) - \mu^x(\phi^x)| \leq C(x)e^{-c(x)s}. \]
This is not sufficient for our purposes, since we need a control on the constants $C, c$ which is uniform in the parameter $x$. Hence the need to adapt the strategy and track the dependence on $x$ in the proofs of [36] Theorem 2.5. The bulk of the work lies in showing that a minorisation condition holds uniformly in $x$, which substantially follows from Assumption A2 (but is still not immediate, as Assumption A2 gives us strict positivity of the transition probabilities for each $x \in \mathbb{R}^n$, but not directly that this strict positivity is uniform in $x$, see Appendix A.1).

We also note that the convergence result of the above proposition can also be obtained using semigroup DE such as those in the conclusion of Proposition 4.5, see Note 5.3 and Lemma 5.4 on this point.

**Proposition 4.5.** Let $\{P_t^x\}_{t \geq 0}$ denote the semigroup defined in (27). Suppose that Assumptions A1, A2, A3 and A5 hold. Let $\psi \in C^{0,2}(\mathbb{R}^n \times \mathbb{R}^d)$ such that $|\psi|_{2,m_x,m_y} < \infty$ for some $m_x,m_y \geq 0$. Then there exists a constant $K > 0$ independent of $x \in \mathbb{R}^n$ such that
\[ |P_t^x \psi^x|_{2,m_x,m_y} \leq K|\psi|_{2,m_x,m_y}e^{-\kappa t}, \quad \text{for every } x \in \mathbb{R}^n, t \geq 0. \] Moreover, for any $\psi \in C^{0,4}(\mathbb{R}^n \times \mathbb{R}^d)$ such that $|\psi|_{4,m_x,m_y} < \infty$ for some $m_x,m_y \geq 0$, we have
\[ |P_t^x \psi^x|_{4,m_x,m_y} \leq K|\psi|_{4,m_x,m_y}e^{-\kappa t}, \quad \text{for every } x \in \mathbb{R}^n, t > 0. \]

We recall that $\kappa$ is the constant appearing in Assumption A3 and that the seminorm $|\cdot|_{4,m_x,m_y}$ (defined in [28]) contains all the $y$ derivatives up to order 4 but no $x$ derivatives and does not contain the function itself.

**Proof.** The proof of this Proposition can be found in Appendix A.1. $\square$

In Section 5 we will require control of both the second and fourth order derivatives of the semigroup $P_t^x$. The bound (55) implies a bound on the second order derivatives of the semigroup; that is, it implies
\[ |P_t^x \psi^x|_{2,m_x,m_y} \leq K|\psi|_{4,m_x,m_y}e^{-\kappa t}. \]
However the above cannot be applied to functions $\psi$ that are only twice differentiable; this fact motivates obtaining (54) as well as (55).

Finally, note that if Assumption A5 holds with $\zeta_2, \zeta_3 > 0$ then it holds with $\zeta_2 = \zeta_3 = 0$ as well and moreover note that the derivative estimate (54) can be proven under Assumption A5 with $\zeta_2 = \zeta_3 = 0$. That is, if there exist $\kappa, \zeta > 0$ independent of $x, y, \xi$ such that for any $x \in \mathbb{R}^n, y, \xi \in \mathbb{R}^d$, we have
\[ 2 \sum_{i,j=1}^n \partial_i g_j(x,y)\xi_i \xi_j + \sum_{i,j=1}^d \zeta_i(\partial_i \partial_j g(x,y), \xi)^2 \leq -\kappa|\xi|^2, \]
then (54) holds. This is shown in the proof of Proposition 4.5.
Note 4.6. The DE of Proposition 4.5 are not smoothing-type estimates; indeed, smoothing estimates for a given Markov Semigroup $P_t$ are, generally, of the form

$$|\mathcal{D}P_t f(x)| \leq \frac{1}{t^\gamma} \|f\|_\infty,$$

where $\mathcal{D}$ is some appropriate differential operator and $\gamma > 0$ depends on $\mathcal{D}$ (for example, in [12] $\mathcal{D}$ is the usual gradient; if the process is hypoelliptic $\gamma$ will depend on the number of commutators needed to obtain the direction $\mathcal{D}$), see [2, 13] and references therein for a comprehensive review. Note that in the above $f \in C_b$ while in this paper the function $f$ will not be bounded but will be $C^2$ at least; smoothing estimates can be seen as quantifying the “explosion” of heat-type semigroups as $t \to 0$. Here we want a specific (i.e. exponential) quantitative estimate for $t$ large. Note that by the semigroup property the short and long-time estimates could be “glued” together, and this is routinely done in the literature; we do not do it here as the smoothing effect is not the main concern of the paper, what we are interested in is the long-time regime.□

5. Poisson equation with parameter

In this section we study the Poisson problem (7). In particular, in Lemma 5.1 we show that the representation formula (38) provides a classical solution to (7) (in the sense that the function in (38) is smooth in $y$) in $L^2(\mu^x)$. In Proposition 5.2 (and in Proposition 5.5) we prove regularity properties in the parameter $x$ of the solution and of the measure $\mu^x$.

Formula (38) is by now standard in the case in which the operator $L^x$ has bounded coefficients. To the best of our knowledge, when the coefficients of $L^x$ are unbounded, such a representation formula has only been studied in [9]. The validity of formula (38) below can heuristically be seen as follows: for $\phi^x \in L^2(\mu^x)$ and for $T > 0$ fixed, let

$$u^{x,T}(y) = -\int_0^T \left((P_s^x \phi^x)(y) - \bar{\phi}(x)\right) ds.$$  (57)

By the fundamental theorem of calculus, and using the fact that $L^x$ is the generator of $P_s^x$ we have

$$L^x u^{x,T} = \lim_{h \to 0} \frac{P_h^x u^{x,T} - u^{x,T}}{h} = -\left[\partial_h \int_{h}^{T+h} (P_s^x \phi^x - \bar{\phi}(x)) ds\right]_{h=0} = \phi^x - P_T^x \phi^x.$$  (58)

Letting $T \to \infty$ and using the convergence $P_T^x \phi^x \to \bar{\phi}(x)$ as $T \to \infty$, yields $L^x u^x_{\phi^x} = \phi^x - \bar{\phi}(x)$, provided $u^{x,T} \to u^x_{\phi^x}$ in $L^2(\mu^x)$ for some function $u^x_{\phi^x} \in L^2(\mu^x)$, since $L^x$ is closed. Lemma 5.1 is largely devoted to proving the latter convergence.

Finally, before stating Lemma 5.1 below, we clarify that the lemma and its proof are a simple adaptation of [9, Corollary 3.10], the main difference being that in Lemma 5.1 we explicitly impose conditions on the coefficients of the operator $L^x$ which give the desired result while [9, Corollary 3.10] is phrased in terms of requirements on the time-behaviour of the semigroup $P_s^x$. 

---

8 This functional choice is quite natural as the semigroup $P_t^x$ can be extended to a $C_0$-semigroup in $L^2(\mu^x)$ (for each $x$, because $\mu^x$ is an invariant measure), and it had already been made in [9].
Lemma 5.1. Consider the Poisson equation \((7)\), and let \(P_t^x\) be given by \((27)\). Assume that
\(g\) and \(A\) satisfy Assumption \([A1]\) \([C3]\) \([C4]\), \([A2]\) and \([A3]\). Then, for any \(m_x, m_y \geq 0\) and
\[\|\phi^x\|_{0,m_x,m_y} < \infty,\]
the function \(u_\phi^x\) defined by \((38)\) is well defined, i.e. for each \(x \in \mathbb{R}^n\) the function \(u_\phi^x\) exists and \(u_\phi^x = u_\phi^x(y) < \infty\) for almost every \(y \in \mathbb{R}^d\), it is a classical solution to \((7)\), and \(u_\phi^x \in L^2(\mu^x)\) for each \(x\). Furthermore this solution is infinitely differentiable in the \(y\) variable.

Proof. The proof of Lemma 5.1 can be found in Appendix A.2.

We now study the properties of the solution \(u_\phi^x\). We use the expression \(\frac{\partial L^x}{\partial x_i}\) to denote the second order differential operator obtained by taking the partial derivative of the coefficients of \(L^x\) by \(x_i\), i.e
\[
\left(\frac{\partial L^x}{\partial x_i}\right) u^x(y) := \partial_{x_i} g(x,y) \cdot \nabla_y u^x(y) + \partial_{x_i} \left( a(x) a(x)^T \right) : \text{Hess}_y u^x(y), \tag{59}
\]
where \(\partial_{x_i} \left(a(x) a(x)^T\right)\) is meant element-wise.

Proposition 5.2. Let \(P_t^x\) be given by \((27)\). Let \(g\) and \(A\) satisfy Assumption \([A1]\) \([C3]\) \([C4]\), \([A2]\), \([A3]\) and \([A5]\). Furthermore let \(\phi \in \text{Poly}_{m_x,m_y}\) for some \(m_x, m_y > 0\). Then:

i) the functions \(x \mapsto (P_t^x \phi^x(y), x \mapsto \partial_{y_i} (P_t^x \phi^x)(y), x \mapsto \partial_{y_iy_j} (P_t^x \phi^x)(y)\) are continuous locally uniformly with respect to \(x\) and \(y\). Indeed, there exist \(C > 0\) such that for every \(x \in \mathbb{R}^n, y \in \mathbb{R}^d\) and \(0 < h < 1\), the following bounds hold for all \(t \geq 0\):
\[
\left| (P_{t+}^x \phi^x-h)(y) - (P_t^x \phi^x) (y) \right| \leq hC\|\phi\|_{2,m_x,m_y}(1 + |y|^{M_y} + |x|^{2m_x}), \tag{60}
\]
and
\[
\left| (\partial_{y_i} P_{t+}^x \phi^x-h)(y) - (\partial_{y_i} P_t^x \phi^x)(y) \right| + \left| (\partial_{y_iy_j} P_{t+}^x \phi^x-h)(y) - (\partial_{y_iy_j} P_t^x \phi^x)(y) \right| \leq hC\|\phi\|_{4,m_x,m_y}(1 + |y|^{M_y} + |x|^{2m_x}), \tag{61}
\]
where \(M_y := \max\{2m_y, m_y + m_x\}\).

ii) the functions \(x \mapsto (P_t^x \phi^x)(y)\) and \(x \mapsto \mu^x(\phi^x)\) are continuously differentiable in \(x\) and we have the following representation formulas for their derivatives:
\[
\partial_{x_i} (P_t^x \phi^x)(y) = (P_t^x \partial_{x_i} \phi^x)(y) + \int_0^t \left[ P_{t-s}^x \frac{\partial L^x}{\partial x_i} P_s^x \phi^x \right](y) ds, \tag{62}
\]
and
\[
\partial_{x_i} \mu^x(\phi^x) = \mu^x(\partial_{x_i} \phi^x) + \int_0^\infty \mu^x \left( \frac{\partial L^x}{\partial x_i} (P_s^x \phi^x) \right) ds, \tag{63}
\]
for all \(i \in \{1, \ldots, n\}\). Furthermore, for all \(i \in \{1, \ldots, n\}\), there exist \(C, C' > 0\) (which are independent of \(x\) and \(y\)) such that \((40)\) and \((42)\) hold.

\(\square\)

\(^9\)The proof of the representation formula \((38)\) that we do in appendix follows the approach of \([9]\). An alternative approach to obtain \((38)\) is to follow steps (a)-(d) in the proof of \([43, \text{Theorem 1}]\) and try and relax the assumption on the boundedness of the coefficients there. This step where removing the boundedness of the coefficients causes the most difficulty is step (c) which concerns proving continuity of the solution. This can be done in our setting, since we assume greater regularity on \(\phi^x\) and therefore can construct a sequence \(u^{x,T}_{T}\), as defined in \((57)\), and one can show that this sequence converges locally uniformly as \(T \to \infty\) to \(u_\phi^x\) and is continuous. We don’t take this approach here as we will need anyway our approach of using DE to remove boundedness of the coefficients when proving the smoothness in \(x\) of higher derivatives of the solution (i.e. not just the representation formula).
Note 5.3. We make a number of comments on Proposition 5.2.

(1) Equation (62) (and similarly (86) below) is called a “transfer formula”, as it allows to write \( x \)-derivatives of the semigroup \( P_t^x \phi^x \) in terms of \( y \)-derivatives of the same semigroup (with the advantage that the latter derivatives exist by ellipticity in \( y \) of the generator \( \mathcal{L}^x \)). Indeed, the left hand side of (62) contains an \( x \)-derivative of \( P_t^x \phi^x \), while the right hand side contains (besides \( x \)-derivatives of \( \phi^x \) and of the coefficients \( g, a \)) two \( y \)-derivatives of the semigroup \( P_t^x \phi^x \) (and no \( x \)-derivatives of \( P_t^x \phi^x \)), as the operator \( \frac{\partial \mathcal{L}^x}{\partial x_i} \) is a second order differential operator in the variable \( y \), see (59). In other words, “one \( x \)-derivative comes at the cost of two \( y \)-derivatives”. In Section 6 we will need bounds on the second \( x \)-derivative of \( P_t^x \phi^x \), hence the need to control \( y \)-derivatives of the semigroup up to order four (see also (86)). This motivates the definition of the seminorm \( \| \cdot \|_{4,m_x,m_y} \), which involves \( y \)-derivatives up to order four, but only \( x \) derivatives up to order 2.

(2) The \( y \)-derivatives of \( P_t^x \phi^x \) not only exist, but they can also be estimated, and indeed in Proposition 4.5 we obtained exponentially fast decay for such derivatives. Since the estimates of Proposition 4.5 can be integrated in time over \( (0, \infty) \), combined with the transfer formula, they can be employed to obtain formula (63) for \( \mu^x \).

(3) The right hand side of (62) can be written in terms of the density of the transition probabilities of the process (15) rather than in terms of the semigroup \( P_t^x \). This is what is done in [44]. If our drift \( g \) was bounded then we could use estimates on the derivatives of the transition density, as is the approach of [44]. In the case of locally Lipschitz coefficients, these estimates are hard to obtain. Hence, we use Proposition 4.5 in order to bound the derivatives of the semigroup directly.

(4) An important ingredient in the proof of Proposition 5.2 (and Proposition 5.5 below) is the convergence of the semigroup \( P_t^x \phi^x \) as \( t \to \infty \). This was obtained in Proposition 4.3 by using Lyapunov techniques, however an alternative strategy of proof is to use the derivative estimates of Proposition 4.5. The latter strategy is used in Lemma 5.4. The main difference between Proposition 4.3 and Lemma 5.4 is the class of functions for which they hold, as Proposition 4.3 holds for \( \phi \) with \( \| \phi \|_{0,m_x,m_y} < \infty \) whereas Lemma 5.4 requires \( |\phi|_{2,m_x,m_y} < \infty \); that is, the latter requires control on higher derivatives of the function to which it is applied (though not of the function itself). This would cause complications if we were to use Lemma 5.4 within the proof of Proposition 5.2 (and Proposition 5.5), as the functions to which the result needs to be applied in that context are quite involved; this is the main reason why Proposition 4.3 is needed in this paper.

□

Proof of Proposition 5.2. Because we need to prove continuity and differentiability, throughout the proof we will consider increments \( he_i \) where \( e_i \) is the \( i \)th element of the standard basis of \( \mathbb{R}^n \) and \( h \in \mathbb{R} \). To simplify notation we write the proof as if in one dimension. The higher dimensional proof follows by replacing \( h \) with \( he_i \).
Before outlining the steps of this proof, we first show some preliminary bounds which will be used repeatedly. First, notice the following:

\[
\left| \frac{\mathcal{L}_{x+h}^x - \mathcal{L}_x^x}{h} \left( P_{s}^{x+h} \phi^{x+h} \right) (y) \right| \leq \sum_{j=1}^{d} \left| \frac{g_j(x+h,y) - g_j(x,y)}{h} \partial_{y_j} P_{s}^{x+h} \phi^{x+h}(y) \right|
\]

\[
+ \sum_{j,k=1}^{d} \left| \frac{A_{jk}(x+h) - A_{jk}(x)}{h} \partial_{y_jy_k} P_{s}^{x+h} \phi^{x+h}(y) \right|
\]

\[
\leq C|\phi|_{2,m_x,m_y} e^{-cs} (1 + |y|^M_y^2 + |x|^{2m_x}),
\]

by (54) and Assumption A1 (C3 [C4]), where we use the fact that the coefficients \( g \) and \( A \) are differentiable with a bound on the derivative. Here \( c \) is a constant which may change line by line. Applying the semigroup to both sides of (64), we get

\[
\left| \left( P_{t-s}^{x} \frac{\mathcal{L}_{x+h}^x - \mathcal{L}_x^x}{h} P_{s}^{x+h} \phi^{x+h} \right) (y) \right| \leq C|\phi|_{2,m_x,m_y} e^{-cs} (1 + \sum_{i=1}^{d} |y|_{t-s}^Y_M^2 + |x|^{2m_x})
\]

\[
\leq C|\phi|_{2,m_x,m_y} e^{-cs} (1 + |y|^M_y^2 + |x|^{2m_x}).
\]

The LHS expression of (65) will appear in the representation formula for the difference quotient that we consider (see Step 1 below). Similarly to what we have done to obtain (64), we get the bounds

\[
\left| \partial_{y_i} \left( \frac{\mathcal{L}_{x+h}^x - \mathcal{L}_x^x}{h} \left( P_{s}^{x+h} \phi^{x+h} \right) \right) (y) \right| \leq C|\phi|_{4,m_x,m_y} e^{-cs} (1 + |y|^M_y^2 + |x|^{2m_x})
\]

and

\[
\left| \partial_{y_iy_j} \left( \frac{\mathcal{L}_{x+h}^x - \mathcal{L}_x^x}{h} \left( P_{s}^{x+h} \phi^{x+h} \right) \right) (y) \right| \leq C|\phi|_{4,m_x,m_y} e^{-cs} (1 + |y|^M_y^2 + |x|^{2m_x})
\]

by (55) and Assumption A1 (C3 [C4]). The final bound we need before outlining the structure of this proof is

\[
\left| \frac{\partial \mathcal{L}_x^x}{\partial x_i} P_{s}^{x} \phi^{x}(y) \right| = \sum_{j=1}^{d} \left| \partial_{x_i} g_j(x,y) \partial_{y_j} P_{s}^{x} \phi^{x}(y) \right| + \sum_{j,k=1}^{d} \left| \partial_{x_i} A(x)_{jk} \partial_{y_jy_k} P_{s}^{x} \phi^{x}(y) \right|
\]

\[
\leq C|\phi|_{2,m_x,m_y} e^{-cs} (1 + |y|^M_y^2 + |x|^{2m_x}),
\]

which follows from (54) and Assumption A1 (C3 [C4]).

Now we outline the strategy of proof, before addressing each step consecutively.

- Step 1: Define the difference quotient

\[
q_{t}^{h,\phi^{x}}(y) := \frac{1}{h} \left[ \left( P_{t}^{x+h} \phi^{x+h} \right) (y) - \left( P_{t}^{x} \phi^{x} \right) (y) \right]
\]

(again, \( q_{t}^{h,\phi^{x}} \) should be \( q_{t}^{h,e_{i},\phi^{x}} \) and obtain the representation formula

\[
q_{t}^{h,\phi^{x}}(y) = \left( P_{t}^{x} q_{0}^{h,\phi^{x}} \right) (y) + \int_{0}^{t} \left[ \frac{\mathcal{L}_{x+h}^x - \mathcal{L}_x^x}{h} \right] \left( P_{s}^{x+h} \phi^{x+h} \right) (y) ds.
\]

The above expression will be instrumental for proving (i) as well as both (62) and (63), which are obtained in Step 2 and Step 3 respectively.
Step 1: We differentiate (69) in time to get

\[
\partial_t q_t^{h, \phi^x}(y) = \frac{1}{h} \left[ (L^{x+h} P_t^{x+h} \phi^{x+h})(y) - (L^x P_t^x \phi^x)(y) \right]
\]

\[
= \left[ \left( L^{x+h} - L^x \right) \frac{P_t^{x+h} \phi^{x+h}}{h} \right] (y) + \left[ L^x \left( \frac{P_t^{x+h} \phi^{x+h} - P_t^x \phi^x}{h} \right) \right] (y)
\]

Step 2: Let \( t \rightarrow \infty \) in (62) to obtain (63). Again, formally it is easy to see that letting \( t \rightarrow \infty \) one obtains (63), so this step primarily consists of justifying the limit. The bound (40) is then a consequence of (63), as we will show.

Step 3: Let \( \xi \in \mathbb{R}^n \) again we obtain

Step 4: Integrate \( \partial_y (P_t^x \phi^x)(y) - \partial_x \mu^x(\phi^x) \) with respect to \( t \) using the representation formulas (62)-(63) to obtain (42).
and
\[
\left| \partial_y q_0^{h, \phi^x} (y) \right|, \left| \partial_y y_0 q_0^{h, \phi^x} (y) \right| \leq C \| \phi \|_{4, m_x, m_y} (1 + |y|^{m_y} + |x|^{m_x}). \tag{75}
\]

From [65], [71] and [74], we obtain (60). Now, using (54) (which we can use because of [66], [67] and [75]), (72) and (73) we have, for $|h| < 1$,
\[
\begin{aligned}
\left| \left( \partial_y P^{x+h}_t \phi^{x+h} \right) (y) - \left( \partial_y P^t_{H_T^h} \phi^x \right) (y) \right| &+ \left| \left( \partial_y y_0 P^{x+h}_t \phi^{x+h} \right) (y) - \left( \partial_y y_0 P^t_{H_T^h} \phi^x \right) (y) \right| \\
&\leq hC \| \phi \|_{4, m_x, m_y} (1 + |y|^{M_T^y} + |x|^{2M_x}),
\end{aligned}
\]
which implies [61].

Step 2: We now justify letting $h \to 0$ in (70). We start from the first addend in (70); by definition
\[
\left( P^t_{H_T^h} q_0^{h, \phi^x} \right) (y) = E \left[ \frac{\phi^{x+h} (Y^{x,y}_t) - \phi^x (Y^{x,y}_t)}{h} \right].
\]
Since $\phi$ is differentiable, the integrand on the RHS of the above can be trivially bounded by
\[
\left| \frac{\phi^{x+h} (Y^{x,y}_t) - \phi^x (Y^{x,y}_t)}{h} \right| \leq |\partial_x \phi (Y^{x,y}_t)|,
\]
where $\xi \in (x, x+h)$ so that now using polynomial growth of $\partial_x \phi$, the bound (51) allows to apply the DCT to the first addend in (70), and conclude that such a term tends to $P^t_{H_T^h} \partial_x \phi^x (y)$ as $h \to 0$.

We now move on to the second addend in (70). For each $x \in \mathbb{R}^n, y \in \mathbb{R}^d$ and $s > 0$ we have
\[
\lim_{h \to 0} \left[ \frac{\mathcal{L}^{x+h} - \mathcal{L}^x}{h} \right] (P^{x+h}_s \phi^{x+h}) (y) = \frac{\partial \mathcal{L}^x}{\partial x_i} P^x_s \phi^x (y),
\]
by (i), the smoothness of $\phi$ and the coefficients of $\mathcal{L}^x$. Hence, if the DCT holds for the integral(s) in (70), then we are done. Notice, indeed, that since $P^{x-s}_t$ is an integral itself, we need to justify using the DCT to pass the limit $h \to 0$ both under the time integral and under the integral implied by the definition of $P^{x-s}_t$. The bound (65) can be used for such tasks. The RHS of (65) is indeed integrable both in time and in space, the latter because of Lemma 4.1.

Step 3: Now we let $t \to \infty$. Letting $t \to \infty$ in the first term on the RHS of (62) is straightforward from (53) of Proposition 4.3. As for the second addend, we write the integral as
\[
\int_0^t \left[ P^x_{t-s} \frac{\partial \mathcal{L}^x}{\partial x_i} P^x_s \phi \right] (y) ds = \int_0^\infty 1_{s < t} \left[ P^x_{t-s} \frac{\partial \mathcal{L}^x}{\partial x_i} P^x_s \phi \right] (y) ds.
\]
Then, by (53) of Proposition 4.3, which we are able to use because of (68), we have
\[
\lim_{t \to \infty} 1_{s < t} \left[ P^x_{t-s} \frac{\partial \mathcal{L}^x}{\partial x_i} P^x_s \phi \right] (y) = \int_{\mathbb{R}^d} \frac{\partial \mathcal{L}^x}{\partial x_i} P^x_s \phi^x (y) \mu^x (dy),
\]
for every $s$ fixed; hence, assuming we can use the DCT, we have
\[
\lim_{t \to \infty} \int_0^\infty 1_{s < t} \left[ P^x_{t-s} \frac{\partial \mathcal{L}^x}{\partial x_i} P^x_s \phi \right] (y) ds = \int_{\mathbb{R}^d} \frac{\partial \mathcal{L}^x}{\partial x_i} P^x_s \phi^x (y) \mu^x (dy) ds. \tag{76}
\]
To justify the use of the DCT in the above, we apply the semigroup to (68) and use (51) to get

$$\left|1_{s < t} \left[ P_{t-s}^{x} \frac{\partial \mathcal{L}^{x}}{\partial x_i} P_{s}^{x} \phi^{x} \right](y) \right| \leq C \|\phi\|_{2, m_x, m_y} e^{-cs}(1 + |y|^{M_y} + |x|^{2m_x}),$$

where the expression on the RHS is integrable in \(s\). This justifies the limit (76). So far we have proved that the RHS of (62) tends to the RHS of (63) as \(t \to \infty\). We now want to do the same for the LHS and to this end we must show that we are allowed to exchange the limit \(t \to \infty\) with the derivative \(\partial_{x_i}\), which we will do by proving that the convergence shown thus far is locally uniform. If we prove this, then (63) follows again from Proposition 4.3. From (62) and (76) we have

$$\partial_{x_i} (P_{t}^{x} \phi^{x})(y) - \lim_{t \to \infty} \partial_{x_i} (P_{t}^{x} \phi^{x})(y) = \begin{aligned}
P_{t}^{x} \partial_{x_i} \phi^{x}(y) - \int_{\mathbb{R}^d} \partial_{x_i} \phi^{x}(y) \mu^{x}(dy) \\
+ \int_{0}^{t} P_{t-s}^{x} \left[ \frac{\partial \mathcal{L}^{x}}{\partial x_i} P_{s}^{x} \phi^{x} \right](y) ds - \int_{0}^{\infty} \int_{\mathbb{R}^d} \frac{\partial \mathcal{L}^{x}}{\partial x_i} P_{s}^{x} \phi^{x}(y) \mu^{x}(dy) ds \\
= \left( P_{t}^{x} \partial_{x_i} \phi^{x} \right)(y) - \mu^{x}(\partial_{x_i} \phi^{x}) \\
+ \int_{0}^{t} \left( P_{t-s}^{x} - \mu^{x} \right) \left( \frac{\partial \mathcal{L}^{x}}{\partial x_i} P_{s}^{x} \phi^{x} \right) ds \\
- \int_{t}^{\infty} \left( \int_{\mathbb{R}^d} \frac{\partial \mathcal{L}^{x}}{\partial x_i} P_{s}^{x} \phi^{x}(\tilde{y}) \mu^{x}(d\tilde{y}) \right) ds,
\end{aligned}$$

We now want to show that each of the addends in the above converge to zero locally uniformly in \(x\) and \(y\). The claim is trivial for the first term, since by Proposition 4.3

$$\left| (P_{t}^{x} \partial_{x_i} \phi^{x})(y) - \mu^{x}(\partial_{x_i} \phi^{x}) \right| \leq C \|\phi\|_{2, m_x, m_y} e^{-cs}(1 + |y|^{M_y} + |x|^{2m_x})$$

(77)

If we show the following two bounds, the proof is concluded:

$$|I| \leq C \|\phi\|_{2, m_x, m_y} e^{-cs}(1 + |y|^{M_y} + |x|^{2m_x})$$

(78)

$$|II| \leq C \|\phi\|_{2, m_x, m_y} e^{-cs}(1 + |x|^{2m_x}).$$

(79)

The bound (78) follows from (68) and Proposition 4.3. We note to the reader that the constants in the exponential seen in (68) and Proposition 4.3 are not necessarily equal. As for (79), using again (68), we have

$$|II| \leq \int_{t}^{\infty} \left( \int_{\mathbb{R}^d} \left| \frac{\partial \mathcal{L}^{x}}{\partial x_i} P_{s}^{x} \phi^{x}(\tilde{y}) \right| \mu^{x}(d\tilde{y}) \right) ds$$

$$\leq C \|\phi\|_{2, m_x, m_y} \int_{t}^{\infty} e^{-cs} \left( \int_{\mathbb{R}^d} (1 + |\tilde{y}|^{M_y} + |x|^{2m_x}) \mu^{x}(d\tilde{y}) \right) ds,$$

so that (79) follows from (52) in Lemma 4.1. From (77), (78) and (79), we have

$$\left| \partial_{x_i} (P_{t}^{x} \phi^{x})(y) - \lim_{t \to \infty} \partial_{x_i} (P_{t}^{x} \phi^{x})(y) \right| \leq C \|\phi\|_{2, m_x, m_y} e^{-cs}(1 + |y|^{M_y} + |x|^{2m_x})$$

(80)
meaning that the convergence of the derivative is locally uniform, and so we can exchange
the limit and derivative. This concludes the proof of (63).

Now we show (40). Using (52), (63) and (68) we have
\[
\left| \partial_{x_i} \int_{\mathbb{R}^d} \phi^x(y) \mu^x(dy) \right| \leq \left| \int_{\mathbb{R}^d} \partial_{x_i} \phi^x(y) \mu^x(dy) \right| + \int_0^\infty \int_{\mathbb{R}^d} \partial_{x_i} \frac{\partial L^x}{\partial x_i} P_s^x \phi^x(y) \mu^x(dy) ds \leq C \|\phi\|_{2,m_x,m_y} (1 + |x|^{m_x}) \quad (81)
\]
\[
+ \int_0^\infty \int_{\mathbb{R}^d} C |\phi|_{2,m_x,m_y} e^{-cs} (1 + |y|^{M_y^2} + |x|^{2m_x}) \mu^x(dy) ds.
\]

We use (52) again to obtain
\[
\int_0^\infty \int_{\mathbb{R}^d} e^{-cs} (1 + |y|^{M_y^2} + |x|^{2m_x}) \mu^x(dy) ds \leq \int_0^\infty e^{-cs} (1 + |x|^{2m_x}) ds \leq 1 + |x|^{2m_x}. \quad (82)
\]
Using (81) and (82) together concludes the proof of (40).

Step 4: Lastly, we show (42). First we write
\[
\left| \partial_{x_i} \int_0^\infty P_t^x (\phi^x - \mu^x(\phi^x))(y) dt \right| = \left| \int_0^\infty \partial_{x_i} \left( P_t^x \phi^x \right) (y) - \partial_{x_i} \left( \mu^x(\phi^x) \right) dt \right|.
\]

This means that we have, using (80)
\[
\left| \partial_{x_i} \int_0^\infty P_t^x (\phi^x - \mu^x(\phi^x))(y) dt \right| \leq \int_0^\infty \left| \partial_{x_i} \left( P_t^x \phi^x \right) (y) - \partial_{x_i} \left( \mu^x(\phi^x) \right) \right| dt \leq C \|\phi\|_{2,m_x,m_y} (1 + |y|^{M_y^2} + |x|^{2m_x}),
\]
and so the proof is done. \(\square\)

**Lemma 5.4.** Let the assumptions of Proposition 5.2 hold. Then for every \(\phi \in C^{0,2}(\mathbb{R}^n \times \mathbb{R}^d)\) such that \(\|\phi\|_{2,m_x,m_y} < \infty\) for some \(m_x, m_y > 0\) we have
\[
((P_t^x \phi^x)(y) - \mu^x(\phi^x)) \leq C \|\phi\|_{2,m_x,m_y} e^{-cs} (1 + |y|^{M_y^2} + |x|^{2m_x}), \quad (83)
\]
where \(M_y^2 := \max\{2m_y^2, m_y^2 + m_y\}.\) Combining (38) and (83) gives
\[
|u^x_\phi(y)| \leq \int_0^\infty \left| \left( P_t^x \phi^x \right) (y) - \mu^x(\phi^x) \right| ds' \leq C \|\phi\|_{2,m_x,m_y} (1 + |y|^{M_y^2} + |x|^{2m_x}). \quad (84)
\]

**Proof.** The proof can be found in Appendix A.2 \(\square\)

We now state the analogous result of Proposition 5.2 for higher derivatives. The proof is analogous to the proof of Proposition 5.2 hence it is deferred to Appendix A.2

**Proposition 5.5.** Let the assumptions of Proposition 5.2 hold and furthermore let \(\phi \in \text{Poly}_{m_x,m_y}\) for some \(m_x, m_y > 0\). Then:
i) for all \(i, j, k \in \{1, \ldots, n\}\)
\[
\left| \partial_{y_j} \partial_{x_k} \left( P_t^x \phi^x \right)(y) \right| + \left| \partial_{y_j y_k} \partial_{x_i} \left( P_t^x \phi^x \right)(y) \right| \leq C \|\phi\|_{4,m_x,m_y} e^{-ct} (1 + |y|^{M_y^2} + |x|^{2m_x}), \quad (85)
\]
where \(M_y^2\) is as defined in Proposition 5.2. The functions \(x \mapsto \partial_{y_j} \partial_{x_k} \left( P_t^x \phi^x \right)(y)\) and \(x \mapsto \partial_{y_j y_k} \partial_{x_i} \left( P_t^x \phi^x \right)(y)\) are continuous in \(x\).
\( \text{ii) the functions } x \mapsto (P^x_t \phi^x)(y) \text{ and } x \mapsto \mu^x(\phi^x) \text{ are twice continuously differentiable} \)

and we have the following representation formulas for their second derivatives:

\[
\begin{align*}
\partial_{x,x_j} \left( P^x_t \phi^x \right)(y) &= \left( P^x_s \partial_{x,x_j} \phi^x \right)(y) + \int_0^t P^x_{t-s} \left[ \frac{\partial^2 \mathcal{L}^x}{\partial x_i \partial x_j} P^x_s \phi^x(y) + \frac{\partial \mathcal{L}^x}{\partial x_j} \partial_{x_i} (P^x_s \phi^x)(y) + \frac{\partial \mathcal{L}^x}{\partial x_i} \partial_{x_j} (P^x_s \phi^x)(y) \right] \, ds. \quad (86)
\end{align*}
\]

and

\[
\begin{align*}
\partial_{x,x_j} \mu^x(\phi^x) &= \int_{\mathbb{R}^d} \partial_{x,x_j} \phi^x(y) \mu^x(dy) \\
&\quad + \int_0^\infty \int_{\mathbb{R}^d} \left[ \frac{\partial^2 \mathcal{L}^x}{\partial x_i \partial x_j} P^x_s \phi^x(y) + \frac{\partial \mathcal{L}^x}{\partial x_j} \partial_{x_i} P^x_s \phi^x(y) + \frac{\partial \mathcal{L}^x}{\partial x_i} \partial_{x_j} P^x_s \phi^x(y) \right] \mu^x(dy) \, ds. \quad (87)
\end{align*}
\]

Furthermore, for all \( i \in \{1, \ldots, n\} \), there exists \( C > 0 \) (which is independent of \( x \) and \( y \)) such that the estimates \((41)\) and \((43)\) hold.

This reasoning and line of proof can be extended to any number of derivatives by an induction step. For an illustration of this in the bounded coefficients context see \([44, \text{Section 4.5}]\).

6. Averaging: Proof of Theorem 3.2 and of Theorem 3.6

This section contains the Proof of Theorem 3.2. In particular, in Section 6.1 we give a heuristic argument which explains the approach we take; Section 6.2 contains the proof itself.

6.1. Heuristics. The structure of the proof of Theorem 3.2 is analogous to the one introduced in \([4, \text{Section 4}]\). Let us start by recalling the heuristic argument which motivates the approach taken in the proof. In particular, the heuristic argument below shows how Poisson equations and their derivatives enter the picture.

To find an expression for the difference \( P^t_\epsilon f - \bar{P}^t f \) in which we are interested, we start by formally expanding the semigroup \( P^t_\epsilon \) in powers of \( \epsilon \):

\[
\mathcal{P}^t_\epsilon f = f^0_t + \epsilon f^1_t + \ldots
\]

From \((26)\), we have

\[
\partial_t \mathcal{P}^t_\epsilon f - \mathcal{L}_\epsilon (\mathcal{P}^t_\epsilon f) = 0.
\]

Plugging the asymptotic expansion \((88)\) into the above, using the decomposition of the generator \((22)\) and collecting terms with the same power of \( \epsilon \) gives us the following set of equations

\[
\mathcal{O} \left( \epsilon^{-1} \right) : \mathcal{L}^x f^0_t = 0, \quad (89)
\]

\[
\mathcal{O} \left( \epsilon^0 \right) : \partial_t f^0_t - \mathcal{L}_S f^0_t = \mathcal{L}^x f^1_t. \quad (90)
\]

From the ergodicity of the process \( Y^{x,y}_t \), defined by \((15)\), (and associated to the generator \( \mathcal{L}^x \) which, we recall, is a differential operator in the \( y \) variable only), equation \((89)\) implies
that \( f_t^0(x, y) \) is constant in \( y \), i.e. \( f_t^0(x, y) = f_t^0(x) \). Now we fix \( x \) and integrate (90) with respect to the invariant measure \( \mu^x(dy) \) to get

\[
\int \partial_t f_t^0(x) \mu^x(dy) - \int (\mathcal{L}_S f_t^0)(x, y) \mu^x(dy) = \int (\mathcal{L}^x f_t^1)(x, y) \mu^x(dy).
\]

The right hand side of the above vanishes because \( \mu^x \) is an invariant measure of the process \( Y_t^{x,y} \), and hence \( \mu^x(\mathcal{L}^x f) = 0 \) for all functions \( f \) in the domain of the generator \( \mathcal{L}^x \). Using the expressions for \( \mathcal{L}_S \) and \( \tilde{\mathcal{L}} \) (namely, equations (5), (19) and (23)-(24)), from the above we obtain

\[
\partial_t f_t^0(x) - (\mathcal{L}^x f_t^0)(x) = 0.
\]

The above equation has a unique solution (see e.g. [35, Proposition 4.1.1]) which, by (18), needs to coincide with the semigroup \( \mathcal{P}_t \), i.e. \( f_t^0 = \mathcal{P}_t f \). Substituting this expression into the expansion (88) and into (90) gives

\[
\mathcal{P}_t^x f - \mathcal{P}_t f = \epsilon f_t^1 + \ldots \tag{91}
\]

\[
\mathcal{L}^x f_t^1 = \left[(\tilde{\mathcal{L}} - \mathcal{L}_S) \mathcal{P}_t f\right](x). \tag{92}
\]

The difference on the LHS of (91) is precisely the quantity which we want to study and (91) indicates that in order to understand the behaviour of such a quantity we need to study \( f_t^1 \). In turn, (92) can be seen as a Poisson equation in the unknown \( f_t^1 \) of the type studied in Section 5. Indeed, let

\[
v_t(x, y) := (\tilde{\mathcal{L}} - \mathcal{L}_S) \mathcal{P}_t f(x)
\]

\[
= \sum_{i=1}^n \left( b_i(x) - b_i(x, y) \right) \partial_{x_i} \mathcal{P}_t f(x), \tag{93}
\]

where in the above the terms involving \( \Sigma \) vanish since \( \mathcal{L}_S \) have the same diffusion coefficient (because of (25)). Then (92) becomes

\[
\mathcal{L}^x f_t^1(x, y) = v_t(x, y), \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}^d. \tag{94}
\]

Recall that \( \mathcal{L}^x \) is a differential operator in the \( y \) variable, with coefficients depending on the parameter \( x \). Since \( \mathcal{P}_t f(x) \) does not depend on \( y \) and \( \mu^x(b) = \hat{b} \) by definition, the right hand side of (94) is of the same form as the right hand side of (7) and, by Assumption A1 ([C1]) and Proposition 7.2, it is of the growth required to apply Lemma 5.1 (indeed for some \( m_x, m_y > 0 \), \( \|b_i \partial_{x_i} \mathcal{P}_t f(x)\|_{m_x, m_y} < \infty \) for every \( i \)). Hence, from Lemma 5.1 there exists a solution to (94).

Finally, we emphasize that \( f_t^1 \) is not the unique solution to the Poisson equation (94) and the proof of Theorem 3.2 does not rely on having a unique solution to such an equation. However, due to such non-uniqueness, \( f_t^1 \) may not represent the entire order \( \epsilon \) term in the expansion (88).

\[\text{10\textsuperscript{For} every fixed} \ x, \text{the semigroup} \ P_t^x \text{has a unique invariant measure; hence, by [14, Proposition 12.27 \\ & Theorem 12.31], the set of stationary points of} \ P_t^x \text{(i.e. the set of functions} \ f \text{such that} \ P_t^x f(y) = f(y) \text{for almost every} \ y \text{comprises only of constant functions. However if} \ L^x f = 0 \text{then} \ \partial_t P_t^x f(y) = P_t^x L^x f(y) = 0, \text{that is,} \ f \text{is a stationary point of} \ P_t^x \text{and hence it must be constant as a function of} \ y.}\]
6.2. Proof of Theorem 3.2.

Proof of Theorem 3.2. Let \( f^1_t \) be the function defined in (94). From Lemma 5.1 this function is well defined and is a classical solution of the Poisson equation (94). Motivated by the heuristics presented in Section 6.1 we define

\[
    r^1_t(x, y) := (\mathcal{P}^1_t f)(x, y) - (\bar{\mathcal{P}}_t f)(x) - \epsilon f^1_t(x, y). \tag{95}
\]

Here we recall \( f \) is any function in \( C^2_0(\mathbb{R}^n) \). With this definition, we have

\[
    |(\mathcal{P}^1_t f)(x, y) - \bar{\mathcal{P}}_t f(x)| = |\epsilon f^1_t(x, y) + r^1_t(x, y)|, \tag{96}
\]

so that we need to study the terms \( f^1_t \) and \( r^1_t \). We first turn our attention to the latter. By differentiating (95) with respect to time we have the following:

\[
    \partial_t r^1_t(x, y) = \partial_t(\mathcal{P}^1_t f)(x, y) - \partial_t(\bar{\mathcal{P}}_t f)(x) - \epsilon \partial_t f^1_t(x, y)
\]

\[
    = \mathcal{L}_x \mathcal{P}^1_t f(x, y) - \partial_t(\bar{\mathcal{P}}_t f)(x) - \epsilon \partial_t f^1_t(x, y).
\]

Rearranging (95), we have \( (\mathcal{P}^1_t f)(x, y) = r^1_t(x, y) + (\mathcal{P}_t f)(x) + \epsilon f^1_t(x, y) \); using this fact, the decomposition \( \mathcal{L}_x = \frac{1}{2} \mathcal{L}^x + \mathcal{L}_S \), and remembering that \( (\mathcal{P}_t f)(x) \) does not depend on the variable \( y \) (so that \( \mathcal{L}_x \mathcal{P}_t = \mathcal{L}_S \mathcal{P}_t \)), we obtain

\[
    \partial_t r^1_t(x, y) = \mathcal{L}_x r^1_t(x, y) + \mathcal{L}_x \mathcal{P}_t f(x) + \epsilon \mathcal{L}_x f^1_t(x, y) - \partial_t \bar{\mathcal{P}}_t f(x) - \epsilon \partial_t f^1_t(x, y)
\]

\[
    = \mathcal{L}_x r^1_t(x, y) + \mathcal{L}_S \bar{\mathcal{P}}_t f(x) + \epsilon \mathcal{L}_x f^1_t(x, y) - \mathcal{L}^x f^1_t(x, y),
\]

where for the penultimate addend we have used (18). We can now use the fact that \( f^1_t \) satisfies the Poisson equation (94), that is, \( (\mathcal{L}_S - \mathcal{L}) \bar{\mathcal{P}}_t f = -\mathcal{L}^x f^1_t \), and again (22) to conclude

\[
    \partial_t r^1_t(x, y) = \mathcal{L}_x r^1_t(x, y) + \epsilon (\mathcal{L}_S f^1_t(x, y) - \partial_t f^1_t(x, y)).
\]

The variation of constants formula then gives

\[
    r^1_t(x, y) = \mathcal{P}^1_t r^1_0(x, y) + \epsilon \int_0^t \mathcal{P}^1_{t-s} (\mathcal{L}_S f^1_s - \partial_s f^1_s)(x, y) ds. \tag{97}
\]

Substituting the above expression for the reminder \( r^1_t(x, y) \) into (96), we obtain

\[
    |\mathcal{P}^1_t f(x, y) - \bar{\mathcal{P}}_t f(x)| = |\epsilon f^1_t(x, y) + \mathcal{P}^1_t r^1_0(x, y) + \epsilon \int_0^t \mathcal{P}^1_{t-s} (\mathcal{L}_S f^1_s - \partial_s f^1_s)(x, y) ds|.
\]

Hence, to prove the statement it suffices to prove the following three estimates:

\[
    |\mathcal{P}^1_{t-s}(\mathcal{L}_S - \partial_s) f^1_s| \leq C \|f\| c_x^2 e^{-\omega t}(1 + |y|^{m_y b + m_y b} + |x|^{4m_x b}), \tag{99}
\]

\[
    |f^1_t(x, y)| \leq C \|f\| c_x^2 e^{-\omega t}(1 + |y|^{m_y b} + |x|^{m_x b}), \tag{100}
\]

\[
    |\mathcal{P}^1_t r^1_0(x, y)| \leq C \|f\| c_x^2 (1 + |y|^{m_y b} + |x|^{m_x b}), \tag{101}
\]

where in the above and throughout we use \( C \) to denote a positive constant that is dependent on \( n \) and \( f \), but is independent of \( x \) and \( y \), and may change line by line, and \( \omega > 0 \) is a positive constant. We prove the above three bounds in turn. To prove (99) it suffices to show the following:

\[
    |(\mathcal{L}_S - \partial_s) f^1_s(x, y)| \leq C \|f\| c_x^2 e^{-\omega s}(1 + |y|^{M_y b + m_y b} + |x|^{4m_x b}). \tag{102}
\]
Indeed, once the above has been shown, we can apply the semigroup \( \{ \mathcal{P}_t^\epsilon \}_{t \geq 0} \) to both sides of the above and bound the right hand side using Lemma 6.2, to obtain
\[
\mathcal{P}_{t-s}^\epsilon \left( C e^{-\omega_s} \| f \|_{c_0^2} (1 + |y|^{M^b_g + m^b_g} + |x|^{4m^b_g}) \right) \leq C e^{-\omega_s} \| f \|_{c_0^2} (1 + |y|^{M^b_g + m^b_g} + |x|^{4m^b_g}),
\]
which implies the desired result. The proof of (102) heavily relies on our results on Poisson equations, i.e. on the results of Section 5 and it is deferred to Proposition 7.3, see (121). Similarly, the proof of (100) is deferred to Proposition 7.3, see (118) below. To obtain (101) note that from (95) we have
\[
r_0^\epsilon(x, y) = f(x) - f(x) - \epsilon f_0 = -\epsilon f_0.
\]
Now we use (100) at \( t = 0 \) to conclude (101), and therefore the proof.

**Note 6.1.** Using linearity of the Poisson equation we can write
\[
f_t^\epsilon(x, y) = -\sum_{i=1}^n u_{b_i}(x, y) \partial_{x_i} \mathcal{P}_t f(x),
\]
where \( u_{b_i}(x, y) \) is the solution to (94) with the right hand side equal to \( b_i \) (compatibly with the notation set at the beginning of Section 3). With the above formula in mind, we can explain how the results of Section 5 on Poisson equations will be used in the averaging proof. The expression for \( f_t^\epsilon \) in the above involves the product of solutions to the Poisson Equation (94) and derivatives of the semigroup \( \mathcal{P}_t \) with respect to \( x \). Moreover, in order to obtain (99) we will need to apply \( \mathcal{L}_S \) (defined in (23)) to \( f_t^\epsilon \); this in turn requires bounds on the \( x \)-derivative of \( u_{b_i}(x, y) \) and \( x \)-derivatives of the averaged semigroup which decay fast enough in time (fast enough to be integrable over \((0, \infty))\). The derivative estimates for the averaged semigroup are obtained in Section 7.1. From what we have said so far, it might seem that, when applying \( \mathcal{L}_S \) (which is a second order differential operator) to \( f_t^\epsilon \) one needs to deal with third order derivatives of \( \mathcal{P}_t \). However this is not the case as in (99) one needs to consider the difference \( \mathcal{L}_S - \partial_t \) applied to \( f_t^\epsilon \). When taking this difference the third order derivatives cancel, see proof of Proposition 7.3.

With regards to the \( x \)-derivatives of the Poisson equation (94), note that \( x \) appears only as a parameter. Derivatives with respect to a parameter of a Poisson equation were studied in Section 5. In Section 7.2 we use Proposition 5.2 and Proposition 5.5 to get the required bounds.

**Lemma 6.2.** Let Assumption [A1] hold and let \( V(x, y) = |x|^{4m^b_g} + |y|^k \); then, for all \( k \geq 0 \) and \( \epsilon \leq 1 \), we have
\[
(\mathcal{P}_t^\epsilon V)(x, y) \leq e^{-r'_t} |x|^{4m^b_g} + e^{-r'_t} |y|^k + \frac{C'_k}{r'_k} + \tilde{C}' \frac{(2m^b_g - 1)}{m^b_g \tilde{r}'},
\]
where the constants \( C'_k, r'_k \) are defined in Lemma 4.1 and
\[
\tilde{C}' = \frac{\tilde{C}}{4m^b_g \tilde{r}}, \quad \tilde{r}' = 2m^b_g \tilde{r},
\]
with \( \tilde{C}, \tilde{r} \) being from Assumption [A4].
Proof. The proof can be found in Appendix A.3. □

6.3. Proof of Theorem 3.6

Proof of Theorem 3.6. This proof follows that of Theorem 3.2, so we point out the places in which it differs, and sketch the rest. The Poisson equation that \( f_t \) solves in this case is slightly different. In particular, the right hand side of (94) will contain two derivatives of the averaged semigroup, instead of just one. This explains why Assumption A7 contains four derivatives, as opposed to the two required in the proof of Theorem 3.2. Indeed, instead of (93)-(94), \( f_t \) is the solution of \( \mathcal{L}^F f_t = v_t \) with \( \mu^x(f_t) = 0 \) where

\[
v_t(x, y) = (\mathcal{L} - \mathcal{L}_S) \bar{P}_t f(x)
= \sum_{i=1}^{n} (\bar{b}_i(x) - b_i(x, y)) \partial_{x_i} \bar{P}_t f(x) + \sum_{i,j=1}^{n} (\bar{\Sigma}_{ij}(x) - \Sigma_{ij}(x, y)) \partial_{x_i} \partial_{x_j} \bar{P}_t f(x).
\]

Again, from Lemma 5.1 there exists a solution to (94) (we observe we can apply Lemma 5.1 to the fully coupled case as such a lemma only relies on Proposition 4.3, which holds also in this setting as explained in Note 4.4). Using the linearity of the Poisson equation we can write

\[
f_t(x, y) = -\sum_{i=1}^{n} u_{b_i}(x, y) \partial_{x_i} \bar{P}_t f(x) - \sum_{i,j=1}^{n} u_{\Sigma_{ij}}(x, y) \partial_{x_i} \partial_{x_j} \bar{P}_t f(x),
\]

where we recall the notation \( u_{b_i}(x, y) \) as the solution to (94) with the right hand side equal to \( b_i \), and analogously for \( u_{\Sigma_{ij}}(x, y) \). Following the proof of Theorem 3.2 it is sufficient to prove that \( f_t \) satisfies the estimates (99)-(101). Since (99) is the most challenging to prove we will give more details for this estimate. Observe that (100) follows directly from (50) with \( \phi = b_i \) or \( \phi = \Sigma_{ij} \) and Assumption A7. The bound (101) follows from (100) setting \( t = 0 \). Let us show that (99) holds, which follows once we have that (102) holds. Differentiating (105) with respect to \( t \) we get

\[
\partial_t f_t(x, y) = -\sum_{i=1}^{n} u_{b_i}(x, y) \partial_{x_i} \mathcal{L} \bar{P}_t f(x) - \sum_{i,j=1}^{n} u_{\Sigma_{ij}}(x, y) \partial_{x_i} \partial_{x_j} \mathcal{L} \bar{P}_t f(x).
\]
Further, using (19) and (23) we can write
\[
(\mathcal{L}_S - \partial_s) f^1_s(x, y) = \\
- \sum_{i,j=1}^{n} b_i(x, y) \partial_x u_{bij}(x, y) \partial_x \bar{P}_s f(x) - \sum_{i,j=1}^{n} \left( b_i(x, y) - \bar{b}_i(x) \right) u_{bij}(x, y) \partial_{x, x_j} \bar{P}_s f(x) \\
- \sum_{i,j,k=1}^{n} \Sigma_{ij}(x) \partial_{x, x_j} u_{bij}(x, y) \partial_{x_k} \bar{P}_s f(x) - 2 \sum_{i,j,k=1}^{n} \Sigma_{ij}(x) \partial_x u_{bij}(x, y) \partial_{x_j x_k} \bar{P}_s f(x) \\
+ \sum_{i,j=1}^{n} \partial_x \bar{b}_j(x) u_{bij}(x, y) \partial_x \bar{P}_s f(x) + \sum_{i,j=1}^{n} \partial_x \Sigma_{jk}(x) u_{bij}(x, y) \partial_{x_j x_k} \bar{P}_s f(x) \\
- \sum_{i,j,k=1}^{n} b_i(x, y) \partial_x u_{\Sigma_{jk}}(x, y) \partial_{x, x_k} \bar{P}_s f(x) - \sum_{i,j,k=1}^{n} b_i(x, y) u_{\Sigma_{jk}}(x, y) \partial_{x_j x_k} \bar{P}_s f(x) \\
- \sum_{i,j,k,l=1}^{n} \Sigma_{ij}(x) \partial_{x, x_j} u_{\Sigma_{kl}}(x, y) \partial_{x_k} \bar{P}_s f(x) - 2 \sum_{i,j,k,l=1}^{n} \Sigma_{ij}(x) \partial_x u_{\Sigma_{kl}}(x, y) \partial_{x_j x_k} \bar{P}_s f(x) \\
- \sum_{i,j,k,l=1}^{n} \left( \Sigma_{ij}(x, y) - \bar{\Sigma}_{ij}(x) \right) u_{\Sigma_{kl}}(x, y) \partial_{x, x_k x_l} \bar{P}_s f(x) \\
+ \sum_{i,j,k=1}^{n} \partial_{x, x_j} \bar{b}_k(x) u_{\Sigma_{ij}}(x, y) \partial_{x_k} \bar{P}_s f(x) + 2 \sum_{i,j,k=1}^{n} \partial_x \bar{b}_k(x) u_{\Sigma_{ij}}(x, y) \partial_{x_j x_k} \bar{P}_s f(x) \\
+ \sum_{i,j,k,l=1}^{n} \partial_{x, x_j} \Sigma_{kl}(x) u_{\Sigma_{ij}}(x, y) \partial_{x_k} \bar{P}_s f(x) + 2 \sum_{i,j,k,l=1}^{n} \partial_x \Sigma_{kl}(x) u_{\Sigma_{ij}}(x, y) \partial_{x_j x_k} \bar{P}_s f(x) \\
+ \sum_{i,j,k=1}^{n} \bar{b}_k(x) u_{\Sigma_{ij}}(x, y) \partial_{x, x_j x_k} \bar{P}_s f(x) - \sum_{i,j,k=1}^{n} \left( \Sigma_{ij}(x, y) - \bar{\Sigma}_{ij}(x) \right) u_{bij}(x, y) \partial_{x, x_j x_k} \bar{P}_s f(x).
\]

We use Assumption A7, along with (48)-(50) (with \( \phi = b_i \) and \( \phi = \Sigma_{ij} \)) and Assumption A1 to get
\[
| (\mathcal{L}_S - \partial_s) f^1_s(x, y) | \leq C e^{-\omega s} \| f \|_{C^3_s} (1 + |y|^{M_y} + |x|^{M_x}).
\]

Hence, as in the proof of Theorem 3.2 we obtain (99)-(101), and conclude the proof. \( \square \)

7. Strong ergodicity of the averaged semigroup and application of Section 5 to Averaging

In this section we prove the remaining results needed to prove Theorem 3.2. In Section 7.1 we address the decay in time of the space derivative of the averaged semigroup \( \bar{P}_t \). In Section 7.2 we will use the results of Section 5 and Section 7.1 to prove bounds on \( f^1_t \) (solution to the Poisson problem (94)) and its \( x \) derivatives. This is the section where we most heavily make use of our results on Poisson equations within the proof of the averaging result, Theorem 3.2. The role of these results within the proof of Theorem 3.2 has been explained in Note 6.1.
7.1. Derivative estimates for the Averaged Semigroup and Examples. We now prove that the Assumption A6 made on the coefficients implies the required derivative estimates. In order to prove the derivative estimates we make use of [35, Theorem 7.1.5], with an added mollification argument. Indeed, [35, Theorem 7.1.5] requires in particular that the averaged drift \( \bar{b} \) (see (5)) is in \( C^{2+\nu}(\mathbb{R}^n) \), meaning that the second derivative is Hölder continuous. From our results, see Proposition 5.5, we only have that \( \bar{b} \) is twice continuously differentiable. Hence, to prove Theorem 7.1 we follow a mollification argument (similar to that in [35, Theorem 6.1.9]) which relaxes [35, Theorem 7.1.5] (in our setting) to twice continuously differentiable coefficients.

**Theorem 7.1.** Let Assumptions A1 and A2 hold. Assume that there exists a polynomial \( R: \mathbb{R}^n \to \mathbb{R} \) and a constant \( K_\Sigma > 0 \) such that for all \( x \in \mathbb{R}^n \)

\[
|\partial_\gamma^2 b(x)| \leq R(x) \quad \text{for all multi-indices } \gamma \text{ of length } n \text{ such that } |\gamma|_* = 2,
\]

and \( |\partial_\gamma^2 \Sigma(x)| \leq K_\Sigma \) for any \( \gamma \) such that \( |\gamma|_* = 1, 2 \). Moreover there exists a polynomial \( \tilde{R}: \mathbb{R}^n \to \mathbb{R} \) and a constants \( L > 0 \) such that

\[
\sum_{i,j=1}^n \partial_\xi_i \partial_\xi_j (\delta b)(x) \leq -\tilde{R}(x)|\xi|^2,
\]

\[
-C_1 := \sup_{x \in \mathbb{R}^n} \left( -\tilde{R}(x) + LR(x) + \frac{K_\Sigma^2 n^3}{4\lambda_-} \right) < 0.
\]

Then, there exist positive constants \( M, \omega > 0 \) such that for all \( f \in C^2_b(\mathbb{R}^n) \)

\[
\sum_{1 \leq |\gamma|_* \leq 2} \|\partial_\gamma^2 \mathcal{P}_t f(x)\|_\infty^2 \leq M e^{-\omega t} \sum_{1 \leq |\gamma|_* \leq 2} \|\partial_\gamma^2 f(x)\|_\infty.
\]

**Proof of Theorem 7.1.** In order to use [35, Theorem 7.1.5], we need \( \bar{b} \in C^{2+\nu} \) for some \( \nu > 0 \). Since this is not necessarily true under our assumptions, we apply a classical procedure and first smooth the coefficient \( \bar{b} \), apply [35, Theorem 7.1.5] to such regularised coefficients and then obtain estimates independent of the regularization parameter. We follow the smoothing procedure outlined in [35, Theorem 6.1.9]. To this end, for any \( \delta > 0 \) let \( \varphi(x) = \delta^{-n}\varphi(x/\delta) \), where \( \varphi \in C^\infty(\mathbb{R}^n) \) is any non-negative even function compactly supported in the ball centred at 0 with radius 1 and such that \( \int_{\mathbb{R}^n} \varphi(x)dx = 1 \). Denote by \( \bar{b}^\delta \) the convolution between \( \bar{b} \) and \( \varphi^\delta \), i.e.

\[
\bar{b}^\delta := (\bar{b} * \varphi^\delta)(x) := \int_{\mathbb{R}^n} \bar{b}(\bar{x}) \varphi^\delta(x - \bar{x}) d\bar{x}.
\]

We let \( \mathcal{L}^\delta \) be defined as \( \mathcal{L} \) (see (19)) with \( b \) replaced by \( \bar{b}^\delta \) for all \( i = 1, \ldots, n \). We will first prove the estimate (109) for the smoothed semigroup \( \mathcal{P}_t^\delta \), corresponding to the operator \( \mathcal{L}^\delta \), and then argue that we can take \( \delta \to 0 \). To prove the required derivative estimate, we can use [35, Theorem 7.1.5] (where \( p = 2, k = 2 \) in their notation). It is important in order to be able to take \( \delta \to 0 \) that the constants in the hypotheses of [35, Theorem 7.1.5] should be uniform in \( \delta \). Let us first observe that the hypotheses of [35, Theorem 7.1.5] are given by [35, Hypothesis 6.1.1 (i)-(iii), Hypothesis 7.1.3 (ii-2)]. For the readers convenience, we gather the assumptions that involve \( \bar{b}^\delta \) explicitly in Assumption A8 modified to match our notation. Then we will explain how the remaining assumptions are satisfied (the remaining ones are straightforward), before verifying Assumption A8 in detail.
Assumption A8. Here we collect [35, Hypothesis 6.1.1 (iii), Equation (6.1.2)] and [35, Hypothesis 7.1.3 (ii-2)], noting that we can take the function $\kappa = \kappa(x)$ in [35] to be $\kappa(x) = \lambda_-$.

(1) There exists a constant $C > 0$ such that
\[
\sum_{i=1}^{n} \bar{b}_i(x)x_i \leq C(1 + |x|^2), \quad \text{for any } x \in \mathbb{R}^n. \tag{110}
\]

(2) $\Sigma_{ij}, \bar{b}_j \in C^{2+\nu}$ for some $\nu \in (0, 1)$ and any $i, j = 1, \ldots, n$ and there exist $m^\Sigma$, a constant $K^\Sigma > 0$ and a positive function $r: \mathbb{R}^n \to \mathbb{R}$ such that
\[
\sup_{|\gamma| = 2} |\partial^\gamma \bar{b}_j(x)| \leq r(x), \tag{111}
\]
and $\sup_{1 \leq |\gamma| \leq 2} |\partial^\gamma \Sigma_{ij}(x)| \leq K^\Sigma$ for any $x \in \mathbb{R}^n$ and $i, j = 1, \ldots, n$. Moreover, there exist a function $\ell: \mathbb{R}^n \to \mathbb{R}$ such that
\[
\sum_{i,j=1}^{n} \partial_{xi} \bar{b}_j(x)\xi_i \xi_j \leq \ell(x)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \tag{112}
\]
and a constant $\tilde{L} > 0$ such that
\[
0 > -C_2 := \sup_{x \in \mathbb{R}^n} \left( \ell(x) + \tilde{L}r(x) + \frac{K^2_2n^3}{4\lambda_-} \right). \tag{113}
\]

Finally, there exists a constant $K_1 \in \mathbb{R}$ such that
\[
\sum_{i,j,h,k} \partial_{x_i,x_h} \Sigma_{ij}(x)M_{ij}M_{hk} \leq K_1 \sum_{h,k=1}^{n} M_{hk}^2 \tag{114}
\]
for any symmetric matrix $M = (M_{hk})_{h,k=1}^n$ and any $x \in \mathbb{R}^n$.

Let us now explain why Assumptions A1, A2 and A8 implies that we can apply [35, Theorem 7.1.5]. Indeed, [35, Hypothesis 6.1.1 (i)] follows from Assumption A2, [35, Hypothesis 6.1.1 (ii)] follows from (110), with $\varphi = |x|^2$. [35, Hypothesis 6.1.1 (iii)] follows since $\Sigma$ is bounded (Assumption A1) and (110). It is now left to show that Assumption A8 holds under the assumptions of Theorem 7.1.

We start with Assumption A8 (2). Notice that for any polynomial $R(x) := c(1 + |x|^m)$ with constants $c, m > 0$, we have that $(R \ast \varphi^\delta)(x) \leq R(x) + c\delta^m$. Indeed, using the fact that $\varphi^\delta$ is compactly supported on the ball centred at 0 with radius $\delta$ and that $\varphi^\delta$ integrates to 1, we can write $$(R \ast \varphi^\delta)(x) = \int_{\mathbb{R}^n} c(1 + |x - \bar{x}|^m)\varphi^\delta(\bar{x})d\bar{x} \leq c(1 + |x|^m + \delta^m) \int_{\mathbb{R}^n} \varphi^\delta(\bar{x})d\bar{x} \leq c(1 + |x|^m) + c\delta^m.$$ By linearity of the convolution we can generalise this to polynomials with non-negative coefficients to find $(R \ast \varphi^\delta)(x) \leq R(x) + R(\delta \cdot e_1)$, where $e_1$ is the first element of the standard basis of $\mathbb{R}^n$. Convolving both sides of (106) and bounding the convolution of the polynomial on the right hand side using the above, we obtain (111) with $r(x) = c_1(1 + \ldots$
\[ |x|^{m_1} + c_1 \delta_0^{m_1}; \text{ similarly (107) implies that (112) holds with } \ell(x) = -c_2(1 + |x|^{m_2}) + c_2 \delta_0^{m_2}, \text{ for all } \delta \leq \delta_0, \text{ with } \delta_0 \text{ to be chosen later.} \]

Also,

\[
\sup_{1 \leq |y| \leq 2} |\partial_x^\gamma \Sigma_{ij}(x)| \leq K_\Sigma,
\]

immediately by Assumption A1. Using (108) and (112) (as well as our definitions of \(r(x), \ell(x)\) above), we have that for all \(\bar{L} < L,\)

\[ -C_2 \leq -C_1 + \bar{L} c_1 \delta_0^{m_1} + c_2 \delta_0^{m_2}. \]

We pick \(\delta_0 = \min\{1, \frac{C_1}{2 c_1}\}\) and \(\bar{L} = \min\{L, \frac{c_2}{2 c_1}\}\) to obtain

\[ -C_2 \leq -C_1 + \frac{C_1}{4} + \frac{C_1}{2} = -\frac{C_1}{4} < 0, \]

so that (111)-(113) hold, independent of \(\delta\), for \(\bar{\delta} < 0\). By Assumption A1 (114) holds uniformly in \(\delta\) because we are not smoothing the diffusion coefficient \(\Sigma\), and hence Assumption A8 (2) holds with constants that are independent of \(\delta\). Now we show that in our setting, Assumption A8 (2) implies Assumption A8 (1). First, (113) implies that \(\ell(x) < -C_2 < 0\); using this and (112) with \((x, \xi) = (z x, x)\) for \(z \in [0, 1]\), we have

\[
\frac{d}{dz} \sum_{j=1}^{n} \bar{b}_j(z x_j) x_j \leq -C_2 |x|^2. \tag{116}
\]

Integrating (116) over \(z \in [0, 1]\), we have

\[
\sum_{j=1}^{n} \bar{b}_j(x) x_j \leq \sum_{j=1}^{n} \bar{b}_j(0) x_j - C_2 |x|^2 \\
\leq |\bar{b}_j(0)||x| - C_2 |x|^2 \\
\leq C |x| - C_2 |x|^2,
\]

for some \(C > 0\) and all \(0 < \delta < 1\), where we use Assumption A1 with the bound on a convolved polynomial, as in (115). This gives Assumption A8 (1).

Now, since the conditions do hold for each \(\delta\), we obtain (109) for the semigroup corresponding to the mollified coefficients. That is, there exist positive constants \(M, \omega, \delta_0 > 0\) (independent of \(\delta\)) such that for all \(f \in C^2_0(\mathbb{R}^n)\) and \(\delta < \delta_0\)

\[
\sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma \bar{P}_t^\delta f(x)\|_\infty^2 \leq M e^{-\omega t} \sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma f(x)\|_\infty. \tag{117}
\]

We observe that by Arzela-Ascoli, we can find a sequence \(\delta_k \to 0\) such that \((\bar{P}_t^{\delta_k} f)(x)\) and its first two derivatives converge locally uniformly (in \(t\) and \(x\)) to some \(Q_t f(x)\) (and its first two derivatives, respectively) satisfying

\[
\sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma Q_t f(x)\|_\infty^2 \leq M e^{-\omega t} \sum_{1 \leq |\gamma| \leq 2} \|\partial_x^\gamma f(x)\|_\infty,
\]

with the same positive constants \(M, \omega > 0\) as (117). Indeed, we can apply Arzela-Ascoli to the set \(\{\bar{P}_t^{\delta} f(x), \partial_x \bar{P}_t^{\delta} f(x), \partial_x x_j \bar{P}_t^{\delta} f(x)\}_\delta\) for all \(i, j = 1, \ldots, n\), noting that this set is uniformly bounded (by (117)) and equicontinuous, (by Schauder estimates; see [35 Theorem C.1.4]). It remains to show that \(Q_t f = \bar{P}_t f\). We leave the details.
to [35, Theorem 6.1.9], which uses that \( \bar{b}^t \) converges locally uniformly to \( \bar{b} \) to show \( Q_t f \) solves the PDE (18), and hence \( Q_t f = \mathcal{P}_t f \).

Under Assumptions [A1 A6] the conditions of Theorem 7.1 are satisfied. Hence we get the following result.

**Proposition 7.2.** Let \( \{ \mathcal{P}_t \}_{t \geq 0} \) denote the semigroup described by (17), and assume that Assumptions [A1 A6] hold. Then there exists constants \( \bar{K}, C > 0 \) such that for any \( \psi \in C^2_b(\mathbb{R}^d) \) we have

\[
\sup_{1 \leq |\gamma| \leq 2} \| \partial_x^\gamma \mathcal{P}_t \psi(x) \|_\infty \leq \bar{K} e^{-2Ct}.
\]

**Proof.** The proof of this result can be found in Appendix A.4

Below, we gather another example for which our main result, Theorem 3.2, holds, and in particular for which the semigroup derivative estimates concluded in Proposition 7.2 hold. We will use this example to illustrate the discussion in Note 2.2.

**Example 7.1.** Let us consider following the system

\[
dX^t_{x,y} = (-X^t + b_0(Y^t_{x,y})) dt + \sqrt{2} dW_t
\]

\[
dY^t_{x,y} = \frac{1}{\epsilon}(-Y^t + g_0(X^t_{x,y})) dt + \frac{1}{\sqrt{\epsilon}} \sqrt{2} dB_t
\]

for some \( b_0, g_0 \in C^\infty_b(\mathbb{R}) \). Though we wish to emphasise the verification of Assumption A6 in this example, we first show that Assumptions A1-A5 hold. It is immediate to see that Assumption A1 holds with \( m = 1 \), and that Assumption A2 holds with \( \lambda_- = \lambda_+ = 1 \). As commented in Note 2.1 it is sufficient to consider \( k = 1 \). For \( k = 1 \), (31) holds with \( C_1 = \| g_0 \|_2^2 / 2, \tilde{r}_1 = 1/2 \). Similarly, Assumption A4 holds with \( \tilde{C}_1 = \| b_0 \|_2^2 / 2, \tilde{r}_1 = 1/2 \). Assumption A5 holds with \( \zeta_1 = \delta \) and \( \kappa = 2 \) for any \( \delta > 0 \). It remains to verify Assumption A6. Observe that \( m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8 = 0, D_0 = 2 \) (taking \( \delta = \lambda_- \)), and \( D_b = \| \partial_x g_0 \|_\infty \). Therefore, Assumption A6 holds provided

\[
\max \{ \| \partial_y b_0 \|_\infty, \| \partial_{yy} b_0 \|_\infty \} < \frac{1}{\| \partial_x g_0 \|_\infty}.
\]

It may not be immediately obvious that there should be such a relation between the \( y \)-derivatives of the drift of the slow equation and the \( x \)-derivative of the drift of the fast equation, so let us motivate it with a specific choice of \( b_0 \) and \( g_0 \). As noted in Note 2.2 it is in some cases possible to obtain an explicit expression for \( \bar{b} \), allowing one to verify a condition of the type Assumption A5. If we set \( g_0(x) = \sin(x) \) and \( b_0(y) = -r \cos(y) \), for some \( r \in \mathbb{R} \) then the averaged equation for this setting becomes

\[
dX^t_x = -X^t_x - r \sqrt{\frac{\pi}{\epsilon}} \cos(\sin(X^t_x)) dt + \sqrt{2} dW_t.
\]

From the above, this system would satisfy Assumption A6 when \( |r| < 1 \). For \( |r| < 2 \) the averaged coefficient \( \bar{b}(x) \) is monotonic. In particular it would satisfy a drift condition of the form A5 with \( g \) replaced by \( \bar{b} \) and taking \( x \) derivatives instead of \( y \) derivatives. Therefore we do have derivative estimates of the form given by the conclusion of Proposition 7.2. Hence we see, in both cases, that \( r \) is limited to a bounded set of values, but that our
Assumption A6 is more restrictive. For $1 < |r| < 2$, our Assumption 46 does not hold, but we can still obtain derivative estimates of the form given by the conclusion of Proposition 7.2 since we have a usable expression for $b$. This partly motivates the discussion in Note 7.2.2.

7.2. Application of Section 5 to Averaging. Here we use the results of Section 5 to obtain bounds on $f_t^1$, defined by (103), which are required for the proof of Theorem 3.2. We recall the notation of Section 3 and Section 5 where we refer to the solution of (7) as $u_\phi$ for $\phi \in \text{Poly}_{m_x,m_y}$.

**Proposition 7.3.** Let Assumptions [A1-A6] hold. Then, for all $t > 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^d$, we have

\[
|f_t^1(x, y)| \leq C\|f\|_{c^2}(\|b\|_{4,4,4}, m_y) e^{-\omega t}(1 + |y|^m_y + |x| m_x) \tag{118}
\]

\[
|\sum_{i=1}^n \partial_{x_i} f_t^1(x, y)| \leq C\|f\|_{c^2}(\|b\|_{4,4,4}, m_y) e^{-\omega t}(1 + |y|^{M_y^b} + |x| m_x) \tag{119}
\]

\[
|\sum_{i,j=1}^n \partial_{x_{ij}} f_t^1(x, y)| \leq C\|f\|_{c^2}(\|b\|_{4,4,4}, m_y) e^{-\omega t}(1 + |y|^{M_y^b} + |x| m_x) \tag{120}
\]

where $M_y^b$ is defined in Theorem 3.2. Furthermore,

\[
|\langle \mathcal{L}_S - \partial_t \rangle f_t^1(x, y)| \leq C e^{-\omega t}\|f\|_{c^2}(\|b\|_{4,4,4}, m_y) + \|\Sigma\|_{4,0,0}) (1 + |y|^{M_y^b} + |x| m_x) \tag{121}
\]

**Proof of Proposition 7.3.** By (39), (42) and (43) with $\phi = b_i$ we have for each $i \in \{1, \ldots, n\}$

\[
|u_{b_i}| \leq C\|b_i\|_{a,m_x,m_y} (1 + |y| m_y + |x| m_x) \tag{122}
\]

\[
|\partial_{x_i} u_{b_i}| \leq C\|b_i\|_{a,m_x,m_y} (1 + |y|^{M_y^b} + |x| m_x) \tag{123}
\]

\[
|\partial_{x_{ij}} u_{b_i}| \leq C\|b_i\|_{a,m_x,m_y} (1 + |y|^{M_y^b} + |x| m_x) \tag{124}
\]

We use Proposition 7.2 along with (122)-(124) to conclude (118)-(120). Let us now consider (121). We differentiate (103) with respect to time to get

\[
\partial_t f_t^1(x, y) = - \sum_{i=1}^n u_{b_i}(x, y) \partial_{t_i} \tilde{\mathcal{P}}_t f(x).
\]

Hence, by (118)

\[
\partial_t f_t^1(x, y) = - \sum_{i=1}^n u_{b_i}(x, y) \partial_{x_i} \tilde{\mathcal{L}} \tilde{\mathcal{P}}_t f(x). \tag{125}
\]

The term $u_{b_i}$ is bounded by (39). So we turn our attention to $\partial_{x_i} \tilde{\mathcal{L}} \tilde{\mathcal{P}}_t f(x)$:

\[
\partial_{x_k} (\tilde{\mathcal{L}} \tilde{\mathcal{P}}_t f(x)) = \sum_{i=1}^n (\partial_{x_k} \tilde{b}_i(x)) \partial_{x_i} \tilde{\mathcal{P}}_t f(x) + \sum_{i,j=1}^n (\partial_{x_k} \tilde{\Sigma}_{ij}(x)) \partial_{x_{ij}} \tilde{\mathcal{P}}_t f(x) + \tilde{\mathcal{L}} \partial_{x_k} \tilde{\mathcal{P}}_t f(x). \tag{126}
\]
Further, using \( \{19\}, \{23\}, \) and \( \{125\}-\{126\} \) we can write
\[
(L_S - \partial_s) f_s^1(x, y) =
\]
\[
- \sum_{i,j=1}^{n} b_i(x, y) \partial x_i u_{b_j}(x, y) \partial x_j \tilde{P}_s f(x) - \sum_{i=1}^{n} \left( b_i(x, y) - \bar{b}_i(x) \right) u_{b_j}(x, y) \partial x_i \tilde{P}_s f(x)
\]
\[
- \sum_{i,j,k=1}^{n} \Sigma_{ij}(x) \partial x_i x_j u_{b_k}(x, y) \partial x_k \tilde{P}_s f(x) - 2 \sum_{i,j,k=1}^{n} \Sigma_{ij}(x) \partial x_i u_{b_k}(x, y) \partial x_j x_k \tilde{P}_s f(x)
\]
\[
+ \sum_{i,j=1}^{n} \partial x_i \bar{b}_j(x) u_{b_i}(x, y) \partial x_j \tilde{P}_s f(x) + \sum_{i,j,k=1}^{n} \partial x_i \Sigma_{jk}(x) u_{b_i}(x, y) \partial x_j x_k \tilde{P}_s f(x),
\]
and using Proposition \(7.2\), \(40\), \(122\), \(123\), \(124\) and Assumption \(A1\) \(\{C1\}, \{C2\}\)
\[
|(L_S - \partial_s) f_s^1(x, y)| \leq Ce^{-\omega s} ||f||_{C^2_b} ||b||_{4,m_b^1,m_b^2} (1 + ||\Sigma||_{4,0,0})(1 + |y|^{M_b^1+m_b^2} + |x|^{4m_b^2})
\]

\[\square\]

8. Numerics and Examples

In this section we provide numerical evidence for the validity of Theorem \(3.2\) and illustrate the applicability of the result. The three systems that we numerically solve in this section have different drift terms for the fast equation. The first two differ in how they are coupled, but both have monotonic drifts (corresponding to a convex potential), and the third does not. The aim of this is to produce a set of numerical results that fit the conditions of Theorem \(3.2\) as well as a set of numerical results that can motivate or support future work, being outside the conditions of Theorem \(3.2\). The second example will illustrate that we believe the semigroup derivative estimates to be an important part in obtaining a UiT bound. We begin with an example where \(g\) in \(14\) is monotonic.

Example 8.1 (Convex). Consider the following coupled slow-fast system
\[
dX_t^{e,x,y} = - [Y_t^{e,x,y} + X_t^{e,x,y}] dt + dW_t,
\]
\[
dY_t^{e,x,y} = - \frac{1}{\epsilon} [Y_t^{e,x,y}] dt + \sqrt{\frac{2}{\epsilon}} dB_t,
\]
and the corresponding averaged equation
\[
d\tilde{X}_t^{x} = - \tilde{X}_t^{x} dt + d\tilde{B}_t,
\]
where \(W_t, B_t, \tilde{B}_t\) are all Brownian motions independent to each other.

We numerically solve both the coupled system \(127,128\) and \(129\) using the Euler-Maruyama scheme (see, e.g, \(28\)), with a time step \(\Delta t \ll \epsilon\), to ensure negligible discretisation error in the time interval (the exact \(\Delta t\) will be given later). Halving the time step resulted in no observable change of results. There are two approximations being made here– one being the method of averaging, and one being discretising the equations to produce numerical results. Since we are testing a result that is UiT, it is important that the discretisation of the system that we use to solve it is itself UiT. For this, we refer the reader to \(12\). Using the results of \(12\), one can see that both the coupled system and the averaged equation are approximated uniformly in time by the Euler-Maruyama scheme.
In Figure 1, we show that the method of averaging produces a \( \text{UiT} \) approximation to the system (127)-(128). The left hand plot indicates that the difference between the second moment of \( \bar{X}_t^\epsilon \) and the second moment of \( X_t^{\epsilon,x,y} \) is bounded uniformly in time. The left hand plot is for one value of \( \epsilon \), and is indeed not surprising on its own for a system in which the second moment is bounded uniformly in time itself. The middle plot indicates the preservation of the convergence in \( \epsilon \). Here, for each value of \( \epsilon \), maximum of the differences in second moment over the time. We have, then, that one statistic of the approximation induced by the method of averaging stays uniformly close in time to the coupled system seen in Example 8.1 and that this closeness increases as \( \epsilon \) decreases. In theory, the \( f \) from Theorem 3.2 needed to produce the second moment (i.e the quadratic \( f(x) = x^2 \)) is unbounded, meaning that the results in this paper, strictly speaking, do not apply. In practice, however, this does not particularly pose a problem, since we discretise. Indeed, because we discretise, the domain sampled is finite meaning the function is bounded. In the right hand side plot one can see that this does generalise to the distribution, giving indication that for any bounded statistic \( f \), the bound in Theorem 3.2 would hold. We conclude that these results are in agreement with the theory developed in this paper.

Now we introduce an example without corresponding numerics, to illustrate why we believe that the semigroup derivatives in Assumption A7 are important in obtaining \( \text{UiT} \) bounds for averaging.

\( ^{11} \)We choose the second moment as the comparison statistic because of the Gaussian nature of the considered processes, and because the first moment is trivial.
**Example 8.2.**

\[
\begin{align*}
\frac{dX_t^{\epsilon,x,y}}{dt} &= -\left[ Y_t^{\epsilon,x,y} + X_t^{\epsilon,x,y} \right] dt + dW_t \quad (130) \\
\frac{dY_t^{\epsilon,x,y}}{dt} &= -\frac{1}{\epsilon} \left[ Y_t^{\epsilon,x,y} + X_t^{\epsilon,x,y} \right] dt + \frac{\sqrt{2}}{\sqrt{\epsilon}} dB_t \quad (131)
\end{align*}
\]

which yields

\[
\frac{d\bar{X}_t^{\epsilon}}{dt} = d\tilde{B}_t, \quad (132)
\]

where \(W_t, B_t, \tilde{B}_t\) are all Brownian motions independent of each other.

It is easy to see that Assumptions A1-A5 hold. However, Assumption A6 does not hold, and moreover the averaged semigroup is not SES in the sense of Assumption A7. In fact our method of proof would still work if the bound of the averaged semigroup derivatives was merely integrable in time over the positive real line, but again this would not hold, since the averaged equation (132) is a brownian motion. Hence, this is an example in which both the ‘fast’ and ‘slow’ equation are well-behaved but the corresponding averaged equation is not so, and this rules out the possibility of a UiT bound under our mechanism of proof. Indeed, the authors of this paper believe that conditions on the averaged semigroup derivatives (or conditions that imply conditions on the averaged semigroup derivatives) are necessary to obtain UiT convergence since they appear in \(f^1\), see (103). This example would satisfy the conditions of prior, non uniform-in-time results (see e.g [48, Theorem 2.3]). That is, Assumption A6 (or Assumption A7) is the ‘further’ condition that we need in order to obtain a UiT bound.

\\[\square\]

Now we introduce an example of a system that does not satisfy the conditions under which we have proven the UiT bound, with numerical experiments similar to those in Example 8.1.

**Example 8.3 (Double Well).** Consider the following coupled slow-fast system

\[
\begin{align*}
\frac{dX_t^{\epsilon,x,y}}{dt} &= -\left[ Y_t^{\epsilon,x,y} + X_t^{\epsilon,x,y} \right] dt + dW_t \quad (133) \\
\frac{dY_t^{\epsilon,x,y}}{dt} &= -\frac{1}{\epsilon} \left[ Y_t^{\epsilon,x,y} (Y_t^{\epsilon,x,y} + 2)(Y_t^{\epsilon,x,y} - 2) \right] dt + \frac{\sqrt{2}}{\sqrt{\epsilon}} dB_t \quad (134)
\end{align*}
\]

and the associated averaged equation

\[
\frac{d\bar{X}_t^{\epsilon}}{dt} = -\bar{X}_t^{\epsilon} dt + d\tilde{B}_t, \quad (135)
\]

where \(W_t, B_t, \tilde{B}_t\) are all Brownian motions independent to each other. Note that, while the above system satisfies A1-A4, it does not satisfy Assumption A5.

We use Euler-Maruyama, as in the previous example, to solve this system numerically. Again, we can verify that \(Y_t^{\epsilon,x,y}\) satisfies the conditions in [12, Hypothesis 3.1]. This is not immediate, but one can utilise Proposition 4.5 to obtain the semigroup derivative decay. The resulting plots can be seen in Figure 2, and they were produced in the same way as Figure 1. From the plots in Figure 2a, one might conjecture that there can indeed be a similar UiT bound that holds when \(g\) in (11) is non-convex. That is, Assumption A5 may be sufficient rather than necessary (though, as discussed in the previous example, the authors expect that decay of the derivatives of the semigroup in some form is necessary). We note to the reader that we numerically solved the system for a shorter and shorter time frame (see the caption of Figure 2 for details) as \(\epsilon\) decreased to allow for the finer
discretisation needed, but that all distributions were checked to be in or very close to equilibrium at the point of stopping.

Though not directly relevant to the existence of a UiT bound, we include Figure 2b as an example of a transition in the qualitative behaviour of the method of averaging as \( \epsilon \) varies. It is simply a restriction to the middle \( \epsilon \)-regime of Figure 2a. From the right hand side plot, we see that for large \( \epsilon \) the settled distribution of the averaged equation and the fully coupled system are qualitatively distinct; that is, they are single-modal and bi-modal respectively. On the left hand side plot, it is clear that this is also a quantitative difference, though it does also portray a smooth transition as the convergence in \( \epsilon \) is shallower in this large \( \epsilon \) regime than in the small regime. While the order of convergence is 1 as \( \epsilon \to 0 \) (as we have proved in Theorem 3.2), the order of convergence may be far slower if \( \epsilon \) is too large. Hence, it is important when using the method of averaging to ensure that \( \epsilon \) is small enough that there is not such a qualitative difference between the behaviour of the averaged equation and the coupled system.

\[ \square \]

**Appendix A. Proofs**

**A.1. Proofs of Section 4.** Let us recall (see Note 4.4) that we need Lemma 4.1 and Proposition 4.3 to hold if the diffusion coefficient \( a \) is also a function of \( y \), as is relevant in the proofs for the fully coupled case. So the proofs of Lemma 4.1 and Proposition 4.3 below are done in this more general setting.

**Proof of Lemma 4.1.** The proof of this result can be found in [34, Lemma 3.6]. We observe that Assumption A3 implies [34, Assumption A] for every \( k \geq 2 \) which is the main assumption of [34, Lemma 3.6]. We include the calculation here to give an explicit form of the constants \( C'_k \) and \( r'_k \). The result holds for \( k = 1 \) by Jensen’s inequality and the case \( k = 2 \). Indeed,

\[
\mathbb{E} [ |Y_{x,y}^t|^k ] \leq \left( \mathbb{E} [ |Y_{x,y}^t|^2 ] \right)^{1/2} \leq e^{-r'_2 t/2} |y| + \sqrt{C'_2 r'_2}. \tag{136}
\]

To obtain the last inequality we have used Young’s inequality.

Now let us consider the case \( k > 2 \). Assumption A3 implies that the function \( V(y) = |y|^k \) is a Lyapunov function for the process \( Y_{x,y}^t \) (see [34, Lemma 3.6] for details) in the sense that for all \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^d \)

\[
\mathcal{L}^x V(y) \leq -k r_k |y|^k + C_k k |y|^{k-2}
\]

\[
\leq -\frac{k r_k}{2} V(y) + 2 C_k \left( \frac{2 C(k-2)}{k r_k} \right)^{k-2}. \tag{137}
\]

To obtain the last inequality we have used Young’s inequality.

Then, using the positivity of Markov semigroups, we have

\[
\frac{d}{dt} (e^{k r_k t/2} P_t^x V(y)) = \left( \frac{1}{2} k r_k P_t^x V(y) + P_t^x \mathcal{L}^x V(y) \right) e^{k r_k t/2} \leq 2 C_k \left( \frac{2 C_k (k-2)}{k r_k} \right)^{k-2} e^{k r_k t/2}.
\]

We can then integrate and obtain

\[
\mathbb{E} [ |Y_{t,x,y}^t|^k ] = P_t^x V(y) \leq e^{-k r_k t/2} V(y) + \frac{4 C_k}{k r_k} \left( \frac{2 C_k (k-2)}{k r_k} \right)^{k-2}.
\]
(A) Left. Plot of $\left| \mathbb{E} \left[ X_t^{\epsilon,x,y} \right] - \mathbb{E} \left[ X_t^x \right] \right|^2$, plotted until time $t = 10^3$, with $\epsilon = 0.1$. Middle. The maximum difference in second moment, plotted against $\epsilon$. Plotted alongside a line of slope 1 for comparison. Right. The corresponding distributions of $X_t^{\epsilon,x,y}$ for different $\epsilon$ values. Plotted alongside the distribution of $X_t^x$, which is independent of $\epsilon$. These plots were produced using the behaviour of $10^5$ particles, with $\epsilon = 10^{-m}$ and $\Delta t = 10^{-m-3}$ for values $m \in \{1, 1.28, 1.56, 1.83, 2.11, 2.39, 2.67, 2.94, 3.22, 3.5 \}$. All realisations were run for $N = 10^6$ steps (hence run until $T = 10^{-m-3}$) and were initialised at $X_0^{\epsilon,x,y} = X_0^x = Y_0^{\epsilon,x,y} = 0$. The authors checked that the distributions had reached equilibrium for all values of $\epsilon$.

(b) Left. The maximum difference in second moment, plotted against $\epsilon$. Plotted alongside a line of slope 1 for comparison. The red-dashed lines indicate points at which the coupled system is clearly bimodal (right) and clearly single-modal (left). Right. The distributions of $X_T^{\epsilon,x,y}$ for different $\epsilon$ values. Plotted alongside the distribution of $X_T^x$ (in orange), which is independent of $\epsilon$. These plots were produced with the same parameters as above, using $m \in \{1.5, 1.56, 1.61, 1.67, 1.72, 1.78, 1.83, 1.89, 1.94, 2 \}$. Figure 2
By an analogous argument for $k = 2$ we have
\[ \mathbb{E} \left[ |Y_t^{y,x}|^2 \right] = P_t^x V(y) \leq e^{-2r_2 t} V(y) + \frac{C_2}{r_2}. \]

\[ \square \]

Proof of Proposition 4.3. Using [36, Theorem 2.5], which holds under our Assumptions A2 and A3, we obtain (53) for some $C = C(x), c = c(x)$ \[12\] What we need within our scheme of proof, is that $C(x)$ and $c(x)$ are uniformly bounded above and below respectively. To prove this fact, we will show that the conditions of [36, Theorem 2.5] hold uniformly in $x$. In particular, in [36, Theorem 2.5], the authors require two conditions to hold: a Minorisation condition, and a Lyapunov condition. For the readers convenience we gather these two conditions and then write a modified Minorisation condition, which we will use for our proof.

Assumption A9 (Minorisation condition). Let $Y_t^{x,y}$ be the process (15) with semigroup \( \{P_t^x\}_{t \geq 0} \) and transition probability denoted by $P_t^x(y, A) = (P_t^x \mathbb{1}_A)(y)$, where $A \in \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sets of $\mathbb{R}^d$. For all fixed compact sets $\bar{U} \in \mathcal{B}(\mathbb{R}^d)$, there exist $\eta, T > 0$ independent of $x \in \mathbb{R}^n$ such that
\[ P_T^x(y, A) \geq \eta \lambda(A \cap \bar{U}) \quad \forall A \in \mathcal{B}(\mathbb{R}^d), y \in \bar{U}, \]
where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Assumption A10 (Lyapunov Condition). There is a function $V : \mathbb{R}^d \to [1, \infty)$, with \( \lim_{y \to \infty} V(y) = \infty \), and real numbers $a, d \in (0, \infty)$, independent of $x$, such that
\[ \mathcal{L}^x V(y) \leq -aV(y) + d, \]
where $\mathcal{L}^x$ is the generator for the process outlined in (24).

Assumption A10 is exactly [36, Assumption 2.4]. Assumption A9 implies the conclusion of [36, Lemma 2.3], with $\nu(A) = \lambda(A \cap \bar{U})$. Now we write a modified Minorisation condition that implies Assumption A9 in our setting.

Assumption A11 (Minorisation condition I). Let $Y_t^{x,y}$ be the process (15) with semigroup \( \{P_t^x\}_{t \geq 0} \) and density $p_t^x(y,y')$ such that $P_t^x(y, A) = \int_A p_t^x(y,y')dy'$. For all fixed compact sets $\bar{U} \in \mathcal{B}(\mathbb{R}^d), y \in \bar{U}$ and $T > 0$, there exists $\eta(y) > 0$ independent of $x \in \mathbb{R}^n$ such that
\[ \inf_{x \in \mathbb{R}^n} \inf_{y' \in \bar{U}} p_T^x(y,y') \geq \eta(y). \]

Indeed, if Assumption A11 holds, then we have, for each $A \in \mathcal{B}(\mathbb{R}^d), y \in \bar{U}$ and $x \in \mathbb{R}^n$
\[ P_T^x(y, A) = \int_A p_T^x(y,y')dy' \geq \int_{A \cap \bar{U}} p_T^x(y,y')dy' \geq \eta(y) \lambda(A \cap \bar{U}). \]
Since $\bar{U}$ is a compact subset of $\mathbb{R}^d$, $\eta := \inf_{y \in \bar{U}} \eta(y) > 0$ and so we have that Assumption A11 implies Assumption A9. Hence, if we show that both Assumption A10 and

\[ \text{Strictly speaking [36, Theorem 2.5]} \text{ refers to a hypoelliptic generator with constant diffusion matrix.} \]

The process to which we apply this theorem is elliptic, with non-constant diffusion matrix. Under our assumptions the conclusion of [36, Theorem 2.5] still holds, as one can easily see by following the proof of that theorem. Nonetheless, because we need to control the dependence of the various constants on $x$, in what follows we anyway retrace a good part of that proof.
Assumption [A11] hold uniformly in $x \in \mathbb{R}^n$, then we are done. The Lyapunov condition stated in Assumption [A10] is uniform in $x$ and follows directly from Assumption [A3] with $V(y) = |y|^k$ for any $k > 0$, see the proof of Lemma 4.1. The remaining problem is in verifying Assumption [A11].

We prove that Assumption [A11] holds uniformly in $x$ by contradiction. Suppose that there exists a compact set $\bar{U} \in \mathcal{B}(\mathbb{R}^d)$, a point $y \in \bar{U}$, and a time $T > 0$ (which from this point onwards we fix) such that $\inf_{x \in \mathbb{R}^n} \inf_{y' \in \mathcal{U}} p^x_T(y, y') = 0$. That is, there exists a sequence $(x_n)_{n=1}^\infty$ such that

$$\inf_{y' \in \mathcal{U}} p^x_T(y, y') \to 0, \quad \text{as } n \to \infty. \quad (138)$$

We will arrive at a contradiction by first considering a process whose coefficients are the limit of a subsequence of the coefficients corresponding to the sequence of processes $(Y^x_{t,y})_{n=1}^\infty$; in particular we will show that the limiting process still satisfies uniform ellipticity and so has strictly positive density. This strict positivity will contrast the limit of the densities necessarily being not strictly positive, and we will argue that these two densities must agree; hence a contradiction.

We first consider the sequence of drift coefficients $(g^x(z))_{k=1}^\infty$, as functions of $z \in \mathbb{R}^d$. This sequence is pointwise bounded due to Assumption [A1], [C3] and equicontinuous when restricted to compact sets by local Lipschitzianity. Hence, by Arzela-Ascoli (see e.g. [49]) we have a subsequence of drifts $(g^{x_{nk}}(z))_{k=1}^\infty$ which converges locally uniformly to a function $g^*(z)$. Similarly, we take a further subsequence to obtain local uniform convergence of the sequence of diffusion coefficients $(a^{x_{nk}}(z))_{k=1}^\infty$ to some $a^*(z)$. Note that $a^*(z)$ still satisfies Assumption [A2]. Now, we consider the $\mathbb{R}^d$-valued process defined by

$$dY^x_{t,y} = g^*(Y^x_{t,y}) dt + \sqrt{2}a^*(Y^x_{t,y}) d\tilde{W}_t$$

with corresponding semigroup $\{P^*_t\}_{t \geq 0}$ and generator

$$(\mathcal{L}^*h)(y) = g^*(y) \cdot \nabla_y h(y) + a^*(y)a^*(y)^T : \text{Hess}_y h(y).$$

By the ellipticity of $a^*(z)$, the density $p^*_T(y, z)$ associated with the process $Y^x_{t,y}$ at time $t = T$ is strictly positive everywhere; that is

$$p^*_T(y, z) > 0, \quad \text{for all } z \in \mathbb{R}^d. \quad (139)$$

Now, we consider the sequence of the continuous densities $(p^{x_{nk}}_T(y, z))_{k=1}^\infty$ (densities of the process $Y^x_{t,y}$) as functions of $z \in \mathbb{R}^d$. This sequence is uniformly bounded and equicontinuous when restricted to compact sets (uniform boundedness follows from [6], Theorem 7.3.3] equicontinuity follows from the fact that the derivative is uniformly locally bounded using [6], Theorem 6.4.5], where we use that the density is uniformly bounded, which itself is from [6, Theorem 7.3.3]). Hence, we take a subsequence that converges locally uniformly to some density $\tilde{p}_T(y, z)$. By (138), and since $\bar{U}$ is compact, we have $\inf_{y' \in \mathcal{U}} \tilde{p}_T(y, y') = 0$; this infimum is attained at some $y_0 \in \bar{U}$, i.e. $\tilde{p}_T(y, y_0) = 0$. By this and (139), we have a contradiction (and thus the proof is complete) if we can show $p^*_T(y, \cdot) = p_T(y, \cdot).$ To do this, we consider the semigroup associated with the density $\tilde{p}_T(y, \hat{y})$. i.e $\tilde{P}_T f(y) = \int_{\mathbb{R}^d} f(\hat{y}) \tilde{p}_T(y, \hat{y}) d\hat{y}$. For $f \in C^2_b(\mathbb{R}^d)$ we have
\[ |\mathcal{P}_T^*f(y) - P_{T}^n f(y)| = \left| \int_0^T (\mathcal{P}_{T-s}^*(\mathcal{L}^x - \mathcal{L}^*) P_{s}^{x_n} f)(y) \, ds \right| \]

\[
\leq \sum_{i=1}^d \int_0^T \left| (\mathcal{P}_{T-s}^*(g_i^{x_n} - g_i^*) \partial_{y_i} P_{s}^{x_n} f)(y) \right| \, ds \\
+ \sum_{i,j=1}^d \int_0^T \left| (\mathcal{P}_{T-s}^*(A_{ij}^{x_n} - A_{ij}^*) \partial_{y_{ij}} P_{s}^{x_n} f)(y) \right| \, ds, \tag{140}
\]

where \( A^*(y) = a^*(y)a^*(y)^T \). From [35, Theorem 6.1.7] (or also from our own results), we have that \( \|\partial_{y_i} P_{s}^{x_n} f\|_\infty, \|\partial_{y_{ij}} P_{s}^{x_n} f\|_\infty < \infty \). Indeed, since the hypotheses of [35, Theorem 6.1.7] hold for us uniformly in \( x \in \mathbb{R}^n \) (for details of this verification, see the proof of Theorem 7.1 above), we have

\[ \|\partial_{y_i} P_{s}^{x_n} f\|_\infty + \|\partial_{y_{ij}} P_{s}^{x_n} f\|_\infty < C, \]

for some \( C > 0 \) independent of \( n \). Then, by (140)

\[ |\mathcal{P}_T^*f(y) - P_{T}^n f(y)| \leq C \sum_{i=1}^d \int_0^T \left| (\mathcal{P}_{T-s}^*|g_i^{x_n} - g_i^*\right)(y) \, ds \\
+ C \sum_{i,j=1}^d \int_0^T \left| (\mathcal{P}_{T-s}^*|A_{ij}^{x_n} - A_{ij}^*\right)(y) \, ds. \]

By Assumption A11[C3][C4] and Lemma 4.1, we can use the DCT to take the limit inside both the time integral and the integral implied by the semigroup. This yields

\[ |\mathcal{P}_T^*f(y) - P_{T}^n f(y)| \to 0 \]

as \( n \to \infty \). Then \( \mathcal{P}_T^*f(y) = \tilde{\mathcal{P}}_T f(y) \) (for every \( f \in C_c^2 \) and, by density, for every \( f \in C_b \)) and, so in particular \( 0 < \tilde{p}_t^*(y, y_0) = \tilde{p}_t(y, y_0) = 0 \). This is a contradiction. Thus, Assumption A11 holds uniformly in \( x \).

Hence, both Assumption A10 and Assumption A11 hold independent of \( x \), and this entails that all constants in the proof of [36, Theorem 2.5] are independent of \( x \), and we have the result as required. \( \square \)

**Proof of Proposition 4.5** We first prove (55), before we explain how to adapt the proof to obtain (54). Fix \( \psi \in \text{Poly}_{m_s, m_y} \) and set \( f_t^x(y) = P_t^x \psi(x) \). Define the function

\[ \Gamma(f_t^x) = \sum_{i=1}^d |\partial_{y_i} f_t^x|^2 + \gamma_1 \sum_{i,j=1}^d |\partial_{y_i} \partial_{y_j} f_t^x|^2 + \gamma_2 \sum_{i,j,k=1}^d |\partial_{y_i y_j y_k} f_t^x|^2 + \gamma_3 \sum_{i,j,k,l=1}^d |\partial_{y_i y_j y_k y_l} f_t^x|^2. \]

Here \( \gamma_1, \gamma_2, \gamma_3 \) are positive constants which will be chosen later in the proof. If we show that

\[ \partial_s P_{t-s}^x \Gamma(f_s^x) \leq -\kappa P_{t-s}^x \Gamma(f_s^x) \tag{141} \]

then by Gronwall’s inequality we have

\[ P_{t-s}^x \Gamma(f_s^x) \leq e^{-\kappa s} P_t^x \Gamma(f_0^x). \tag{142} \]
By Lemma 4.1, we have
\[ P_t^x (1 + |x|^{m_x} + |y|^{m_y}) \leq 1 + |x|^{m_x} + |y|^{m_y} + \frac{C'_m}{m_y}. \]

Since \( \Gamma(f_0^x) \leq (d + \gamma_1 d^2 + \gamma_2 d^3 + \gamma_3 d^4) |\psi|_{4,m_x,m_y} (1 + |x|^{m_x} + |y|^{m_y}) \) and using the positivity of Markov semigroups we have
\[ P_t^x \Gamma(f_0^x) \leq |\psi|_{4,m_x,m_y} (d + \gamma_1 d^2 + \gamma_2 d^3 + \gamma_3 d^4) \left( 1 + |x|^{m_x} + |y|^{m_y} + \frac{C'_m}{m_y} \right). \]

Therefore taking \( s = t \) in (142) we have
\[ \Gamma(f_t^x) \leq |\psi|_{4,m_x,m_y} (d + \gamma_1 d^2 + \gamma_2 d^3 + \gamma_3 d^4) e^{-\kappa t} \left( 1 + |x|^{m_x} + |y|^{m_y} + \frac{C'_m}{m_y} \right). \]

Thus (55) holds for \( K > 0 \). It remains to prove (141), note that (141) follows (see [13, Proposition 3.4] for details) provided there exists \( \kappa > 0 \) with
\[ (\partial_t - \mathcal{L}^x) \Gamma(f_t^x) \leq -\kappa \Gamma(f_t^x). \]

We prove that (143) holds by expanding each term in \( \Gamma \) and using Young's inequality, ellipticity and Assumption A5. Since the left hand side of (143) involves the generator \( \mathcal{L}^x \) applied to \( \Gamma(f_t^x) \) we expect derivatives up to order 6 of \( f_t^x \) to appear. However the 6-th order derivatives cancel and we see that the 5-th order derivatives appear with a minus sign and hence can be bounded above by zero.

Let us first consider the first order derivative terms
\[ (\partial_t - \mathcal{L}^x) (|\partial_y f_t^x|^2) = 2(\partial_y (\mathcal{L}^x f_t^x))(\partial_y f_t^x) - 2(\mathcal{L}^x \partial_y f_t^x)(\partial_y f_t^x) - \sum_{j=1}^d (a^j(x), \nabla_y \partial_y f_t^x)^2. \]

(144)

Here \( a^j(x) \) denotes the \( j \)-th column of \( a(x) \). Observe that since the diffusion coefficient \( a \) is independent of \( y \) we can write
\[ \partial_y \mathcal{L}^x f_t^x - \mathcal{L}^x \partial_y f_t^x = (\partial_y g(x), \nabla_y f_t^x). \]

(145)

Substituting (145) into (144) we have
\[ (\partial_t - \mathcal{L}^x) (|\partial_y f_t^x|^2) = 2(\partial_y g(x), \nabla_y f_t^x)(\partial_y f_t^x) - \sum_{j=1}^d (a^j(x), \nabla_y \partial_y f_t^x)^2. \]

Summing over \( i \) we have
\[ \sum_{i=1}^d (\partial_t - \mathcal{L}^x) (|\partial_y f_t^x|^2) = 2 \sum_{i=1}^d (\partial_y g(x), \nabla_y f_t^x)(\partial_y f_t^x) - \sum_{i=1}^d \sum_{j=1}^d (a^j(x), \nabla_y \partial_y f_t^x)^2. \]
Next consider the second order derivatives

\[
(\partial_t - \mathcal{L}^x)(|\partial_y \partial_y f_t^x|^2) = 2(\partial_y \partial_y \mathcal{L}^x f_t^x)(\partial_y \partial_y f_t^x) - 2\mathcal{L}^x \partial_y \partial_y f_t^x \partial_y \partial_y f_t^x - \sum_{k=1}^d (a^k, \nabla_y \partial_y \partial_y f_t^x)^2.
\]

Using that \(\mathcal{L}^x\) is given by (24) we have

\[
\partial_y \partial_y \mathcal{L}^x f_t^x - \mathcal{L}^x \partial_y \partial_y f_t^x = (\partial_y \partial_y g(x), \nabla_y f_t^x) + (\partial_y g(x), \nabla_y \partial_y f_t^x) + (\partial_y g(x), \partial_y \nabla_y f_t^x).
\]

Here \(\partial_y \nabla_y\) is meant component wise, i.e. \(\partial_y \nabla_y f_t^x\) is a vector whose \(j\)-th entry is \(\partial_y \partial_y f_t^x\). Note that

\[
\sum_{j=1}^d (\partial_y g(x), \partial_y \nabla_y f_t^x)(\partial_y \partial_y f_t^x) = (\nabla_y \partial_y f_t^x)^T(\nabla_y g(x))(\nabla_y \partial_y f_t^x).
\]

Then using Young’s inequality and the ellipticity assumption, Assumption A2 we have for any \(\nu_0 > 0\) which will be chosen later

\[
\sum_{i,j=1}^d (\partial_t - \mathcal{L}^x)(|\partial_y \partial_y f_t^x|^2) \leq \sum_{i,j=1}^d \nu_0(\partial_y \partial_y g(x), \nabla_y f_t^x)^2 + \nu_0^{-1}(\partial_y \partial_y f_t^x)^2 + 4 \sum_{i=1}^d (\nabla_y \partial_y f_t^x)^T(\nabla_y g(x))(\nabla_y \partial_y f_t^x) - \lambda \sum_{i,j=1}^d (\nabla_y \partial_y \partial_y f_t^x)^2.
\]

Similarly for the third order derivatives,

\[
\sum_{i,j,k=1}^d (\partial_t - \mathcal{L}^x)(|\partial_y \partial_y \partial_y f_t^x|^2)
\]
Using Young’s inequality and ellipticity we have for any \( \nu_1, \nu_2 > 0 \) to be chosen later

\[
\sum_{i,j,k=1}^{d} (\partial_t - L^x)(|\partial_{y_i,y_j,y_k} f_t^x|)^2
\]

\[
\leq \sum_{i,j,k=1}^{d} \nu_1 (\partial_{y_i,y_j,y_k} g(x), \nabla_y f_t^x)^2 + (\nu_1^{-1} + 3\nu_2^{-1})(\partial_{y_i,y_j,y_k} f_t^x)^2 + 3\nu_2 (\partial_{y_i,y_j} g(x), \nabla_y \partial_{y_k} f_t^x)^2
\]

\[
+ \sum_{j,k=1}^{d} 6(\nabla_y \partial_{y_j,y_k} f_t^x)^T (\nabla_y g(x)) (\nabla_y \partial_{y_j,y_k} f_t^x) - \lambda \sum_{i,j,k=1}^{d} (\nabla_y \partial_{y_i,y_j,y_k} f_t^x)^2.
\]

Similarly,

\[
\sum_{i,j,k,\ell=1}^{d} (\partial_t - L^x)(|\partial_{y_i,y_j,y_k,y_\ell} f_t^x|)^2 = \sum_{i,j,k,\ell=1}^{d} 2(\partial_{y_i,y_j,y_k,y_\ell} f_t^x) (\partial_{y_i,y_j,y_k,y_\ell} f_t^x)
\]

\[
- \sum_{i,j,k,\ell,p=1}^{d} (a^p, \nabla_y \partial_{y_i,y_j,y_k,y_\ell} f_t^x)^2
\]

\[
= 2 \sum_{i,j,k,\ell=1}^{d} ((\partial_{y_i,y_j,y_k,y_\ell} g(x), \nabla_y f_t^x) + 4(\partial_{y_i,y_j,y_k} g(x), \nabla_y \partial_{y_\ell} f_t^x) + 6(\partial_{y_i,y_j} g(x), \nabla_y \partial_{y_k,y_\ell} f_t^x)
\]

\[
+ 4\partial_{y_i} g(x), \nabla_y \partial_{y_j,y_k,y_\ell} f_t^x) (\partial_{y_i,y_j,y_k,y_\ell} f_t^x) - \sum_{i,j,k,\ell,p=1}^{d} (a^p, \nabla_y \partial_{y_i,y_j,y_k,y_\ell} f_t^x)^2
\]

Using Young’s inequality and ellipticity once more we have

\[
\sum_{i,j,k,\ell=1}^{d} (\partial_t - L^x)(|\partial_{y_i,y_j,y_k,y_\ell} f_t^x|)^2
\]

\[
\leq \sum_{i,j,k,\ell=1}^{d} \nu_2 (\partial_{y_i,y_j,y_k,y_\ell} g(x) \cdot \nabla_y f_t^x)^2 + (\nu_3^{-1} + 4\nu_4^{-1} + 6\nu_5^{-1})(\partial_{y_i,y_j,y_k,y_\ell} f_t^x)^2
\]

\[
+ 4\nu_4 (\partial_{y_i,y_j,y_k} g(x) \cdot \nabla_y \partial_{y_\ell} f_t^x)^2 + 6\nu_5 (\partial_{y_i,y_j} g(x) \cdot \nabla_y \partial_{y_k,y_\ell} f_t^x)^2
\]

\[
+ 8 \sum_{i,j,k,\ell=1}^{d} (\partial_{y_i} g(x) \cdot \nabla_y \partial_{y_j,y_k,y_\ell} f_t^x) (\partial_{y_i,y_j,y_k,y_\ell} f_t^x) - \lambda \sum_{i,j,k,\ell=1}^{d} (\nabla_y \partial_{y_i,y_j,y_k,y_\ell} f_t^x)^2.
\]
Combining these terms we have

\[(\partial_t - \mathcal{L}^x)\Gamma(f^x) \leq 2(\nabla_y f^x)^T \nabla_y g(x)(\nabla_y f^x) - \lambda \sum_{i=1}^{d} (\nabla_y \partial_y \partial_y f^x_i)^2 \]

\[+ \sum_{i,j=1}^{d} \gamma_1 \nu_0 (\partial_y \partial_y g(x), \nabla_y f^x_i)^2 + \gamma_1 \nu_0^{-1} (\partial_y \partial_y f^x_i)^2 \]

\[+ 4 \sum_{i=1}^{d} \gamma_1 (\nabla_y \partial_y f^x_i)^T (\nabla_y g(x))(\nabla_y \partial_y f^x_i) - \gamma_1 \lambda \sum_{i,j=1}^{d} \gamma_1 (\nabla_y \partial_y \partial_y f^x_i)^2 \]

\[+ \sum_{i,j,k=1}^{d} \gamma_2 \nu_1 (\partial_y g(x), \nabla_y f^x_i)^2 + \gamma_2 (\nu_4^{-1} + 3\nu_2^{-1})(\partial_y \partial_y, \partial_y f^x_i)^2 + 3\gamma_2 \nu_2 (\partial_y g(x), \nabla_y \partial_y f^x_i)^2 \]

\[+ \sum_{j,k=1}^{d} \gamma_2 \nu_3 (\partial_y g(x), \nabla_y f^x_i)^2 + \gamma_2 (\nu_3^{-1} + 3\nu_4^{-1} + 6\nu_5^{-1})(\partial_y \partial_y, \partial_y f^x_i)^2 \]

\[+ 4 \nu_4 \gamma_3 (\partial_y g(x), \nabla_y \partial_y f^x_i)^2 + 6\gamma_3 \nu_5 (\partial_y g(x), \nabla_y \partial_y f^x_i)^2 \]

\[+ 8 \sum_{i,j,k,\ell=1}^{d} \gamma_3 (\partial_y g(x), \nabla_y \partial_y \partial_y f^x_i)(\partial_y \partial_y \partial_y f^x_i) - \gamma_3 \lambda \sum_{i,j,k,\ell=1}^{d} (\nabla_y \partial_y \partial_y f^x_i)^2. \]
Rearranging and bounding the fifth order derivative terms by zero we have

\[
(\partial_t - \mathcal{L}^x)\Gamma(f^x_t) \leq 2(\nabla_y f^x_t)^T \nabla_y g(x)(\nabla_y f^x_t) + \sum_{i,j=1}^d \gamma_1 \nu_0 (\partial_{y_i} \partial_{y_j} g(x), \nabla_y f^x_t)^2 \\
+ \sum_{i,j,k=1}^d \gamma_2 \nu_1 (\partial_{y_i,y_j,y_k} g(x), \nabla_y f^x_t)^2 + \sum_{i,j,k,\ell=1}^d \gamma_3 \nu_2 (\partial_{y_i,y_j,y_k,y_\ell} g(x), \nabla_y f^x_t)^2 \\
- \lambda_\gamma - \sum_{i=1}^d (\nabla_y \partial_{y_i} f^x_t)^2 + \sum_{i=1}^d \gamma_1 \nu_0^{-1} (\partial_{y_i} \partial_{y_j} f^x_t)^2 + 4 \sum_{i=1}^d \gamma_1 (\nabla_y \partial_{y_i} f^x_t)(\nabla_y \partial_{y_i} f^x_t) \\
+ 3 \sum_{i,j,k=1}^d \gamma_2 \nu_2 (\partial_{y_i,y_j} g(x), \nabla_y \partial_{y_k} f^x_t)^2 + \sum_{i,j,k,\ell=1}^d 4 \nu_3 \gamma_1 \partial_{y_i,y_j,y_k,y_\ell} g(x), \nabla_y \partial_{y_i} f^x_t)^2 \\
- \gamma_1 \lambda_\gamma - \sum_{i=1}^d \gamma_1 (\nabla_y \partial_{y_i} \partial_{y_j} f^x_t)^2 + \sum_{i=1}^d \gamma_2 (\nu_1^{-1} + 3 \nu_2^{-1})(\partial_{y_i,y_j} f^x_t)^2 \\
+ \sum_{j,k=1}^d \gamma_2 \nu_2 (\nabla_y \partial_{y_j,y_k} f^x_t)^T (\nabla_y g(x)) \nabla_y \partial_{y_j,y_k} f^x_t + \sum_{i,j,k,\ell=1}^d 6 \nu_3 \gamma_1 \partial_{y_i,y_j,y_k,y_\ell} g(x), \nabla_y \partial_{y_k,y_\ell} f^x_t)^2 \\
- 2 \gamma_2 \lambda_\gamma - \sum_{i,j,k=1}^d (\nabla_y \partial_{y_i,y_j,y_k} f^x_t)^2 + \gamma_3 (\nu_3^{-1} + 4 \nu_4^{-1} + 6 \nu_5^{-1})(\partial_{y_i,y_j,y_k,y_\ell} f^x_t)^2 \\
+ 8 \sum_{i,j,k,\ell=1}^d \gamma_3 (\nabla_y \partial_{y_j,y_k,y_\ell} f^x_t)^T (\nabla_y g(x))(\nabla_y \partial_{y_j,y_k,y_\ell} f^x_t) 
\]

By making the choice

\[
\nu_0 = \zeta_1/\gamma_1, \nu_1 = \zeta_2/\gamma_2, \nu_3 = \zeta_3/\gamma_3 \\
\nu_2 = \frac{\gamma_1 \zeta_1}{3 \gamma_2}, \nu_4 = \frac{\gamma_1 \zeta_2}{2 \gamma_3}, \nu_5 = \frac{\gamma_2}{2 \gamma_3}, \\
\gamma_1 = \sqrt{\zeta_1 \lambda_-}, \quad \gamma_2 = \sqrt{\frac{\gamma_1 \lambda_-}{\zeta_2^{-1} + 9 \gamma_1^{-1} \zeta_1^{-1}}}, \quad \gamma_3 = \sqrt{\frac{\gamma_2 \lambda_-}{\zeta_3^{-1} + 8 \gamma_1 \zeta_1^{-1} + 12 \gamma_2 \zeta_1^{-1}}}.
\]
we obtain

\[
(\partial_t - \mathcal{L}^x) \Gamma(f^x_t) \leq 2(\nabla_y f^x_t)^T \nabla_y g(x)(\nabla_y f^x_t) + \sum_{i=1}^{d} \zeta_1(\partial_{y_i} g(x), \nabla_y f^x_t)^2
\]
\[
+ \sum_{i,j,k=1}^{d} \zeta_2(\partial_{y_i,y_j,y_k} g(x), \nabla_y f^x_t)^2 + \sum_{i,j,k=1}^{d} \zeta_3(\partial_{y_i,y_j,y_k} g(x), \nabla_y f^x_t)^2
\]
\[
+ 4 \sum_{i=1}^{d} \gamma_1(\nabla_y \partial_{y_i} f^x_t)^T (\nabla_y g(x))(\nabla_y \partial_{y_i} f^x_t)
\]
\[
+ \sum_{i,j,k=1}^{d} 2\gamma_1(\partial_{y_i,y_j,y_k} g(x), \nabla_y \partial_{y_i} f^x_t)^2 + \sum_{i,j,k=1}^{d} 2\gamma_2(\partial_{y_i,y_j,y_k} g(x), \nabla_y \partial_{y_i} f^x_t)^2
\]
\[
+ \sum_{j,k=1}^{d} 2\gamma_6(\nabla_y \partial_{y_j,y_k} f^x_t)^T (\nabla_y g(x))(\nabla_y \partial_{y_j,y_k} f^x_t) + \sum_{i,j,k=1}^{d} 3\gamma_2(\partial_{y_i,y_j} g(x), \nabla_y \partial_{y_i} f^x_t)^2
\]
\[
+ 8 \sum_{i,j,k,l=1}^{d} \gamma_3(\nabla_y \partial_{y_i,y_j,y_k,y_l} f^x_t)^T (\nabla_y g(x))(\nabla_y \partial_{y_i,y_j,y_k,y_l} f^x_t).
\]

Now by applying (32) we have

\[
(\partial_t - \mathcal{L}^x) \Gamma(f^x_t) \leq -\kappa |\nabla_y f^x_t|^2 - 2\kappa \sum_{i=1}^{d} \gamma_1 |\nabla_y \partial_{y_i} f^x_t|^2 - 3\kappa \sum_{j,k=1}^{d} \gamma_2 |\nabla_y \partial_{y_j,y_k} f^x_t|^2
\]
\[
- 4\kappa \sum_{i,j,k,l=1}^{d} \gamma_3 |\nabla_y \partial_{y_i,y_j,y_k,y_l} f^x_t|^2
\]
\[
\leq -\kappa \Gamma(f^x_t).
\]

Therefore (141) holds.

Let us now explain how to prove (54). The proof follows the same argument as above with the difference that we set \( \gamma_2 = \gamma_3 = 0 \). in the above argument. That is we replace \( \Gamma(f^x_t) \) by

\[
\Gamma(f^x_t) = \sum_{i=1}^{d} |\partial_{y_i} f^x_t|^2 + \gamma_1 \sum_{i,j=1}^{d} |\partial_{y_i} \partial_{y_j} f^x_t|^2.
\]
Following the previous argument with $\zeta_2$, and $\zeta_3$ set to zero leads to \((47)\) becoming
\[
(\partial_t - \mathcal{L}^x)\Gamma(f_t^x) \leq 2(\nabla_y f_t^x)^T \nabla_y g(x) (\nabla_y f_t^x) + \sum_{i,j=1}^d \zeta_1 (\partial_y_i \partial_y_j g(x), \nabla_y f_t^x)^2
+ 4 \sum_{i=1}^d \gamma_1 (\nabla_y \partial_y_i f_t^x)^T (\nabla_y g(x)) (\nabla_y \partial_y_i f_t^x)
+ \sum_{i,j,k=1}^d 2\gamma_1 \zeta_1 (\partial_y_i \gamma_j g(x), \nabla_y \partial_y_k f_t^x)^2.
\]

Therefore, applying \((56)\), we have that \((\partial_t - \mathcal{L}^x)\Gamma(f_t^x) \leq -\kappa \Gamma(f_t^x)\) and we obtain the $\Gamma(f_t^x)$ decays exponentially by setting $s = t$ in \((142)\). We can express this in terms of the derivatives of $f_t^x$ by using that for any $m_x, m_y \geq 0$ and $\psi \in C^{0,2}(\mathbb{R}^n \times \mathbb{R}^d)$ with $|\psi|_{2,m_x,m_y} < \infty$ and for all $x \in \mathbb{R}^n, y \in \mathbb{R}^d$ we have
\[
\sum_{i=1}^d |\partial_y_i P_t \psi^x(y)|^2 + \sum_{i,j=1}^d |\partial_y_i \partial_y_j P_t \psi^x(y)|^2
\leq \max\{(\zeta_1 \lambda_-)^{-1/2}, 1\} \left(d + \sqrt{\zeta_1 \lambda_- d^2}\right) |\psi|_{2,m_x,m_y} e^{-\kappa t} \left(1 + |x|^{m_x} + |y|^{m_y} + \frac{C'}{r_{m_y}}\right),
\]
and so \((54)\) holds. Here the term $\max\{(\zeta_1 \lambda_-)^{-\frac{1}{2}}, 1\}$ arises inverse of the smallest coefficient in $\Gamma$, i.e. $(\min\{1, \gamma_1\})^{-1}$ and recalling that $\gamma_1 = \sqrt{\kappa \lambda_-}$.

\section*{A.2. Proofs of Section 5} Before we prove Lemma 5.1 we give two preliminary lemmas. The lemma below is a specific case of \cite{42} Note 3.1.

\begin{lemma}
Let $([T, \infty), \lambda)$ be the truncated real line (for some $T \in \mathbb{R}$) equipped with the Lebesgue measure, $(\mathbb{R}^d, \mu)$ be Euclidean space equipped with some measure $\mu$ and $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ a positive function. Suppose
\[
F(y) := \int_T^\infty f(s, y) ds < \infty \quad \mu\text{-a.e}
\]
and
\[
\int f(s, y) ds \leq m \mu(y) < \infty \text{ for some } m > 1.
\]

Then
\[
\int_0^\infty \left(\int_T^\infty f(s, y) ds\right)^m d\mu(y) \leq \left(\int f(s, y) d\mu(y)\right)^{\frac{1}{m}} \left(\int f(s, y) ds\right)^{\frac{m}{m}} d\mu(y)
\]
\end{lemma}

\begin{proof}
We refer the reader to \cite{42} Note 3.1.
\end{proof}

In Lemma A.2 we will prove that under certain conditions the solution to the Poisson equation, \((7)\), is given by \((38)\). The following is a slight generalisation of \cite{9} Corollary 3.10, but less general than Proposition 3.21 from the same paper.
Lemma A.2. Let \( \phi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R} \) be such that for each \( x \in \mathbb{R}^n \) \( y \mapsto \phi^x(y) \in L^2(\mu) \). Let Assumption A.2 hold. Then the function \( u \) defined in \( (38) \) is a classical solution to the Poisson equation \( (7) \).

Proof. Fix \( x \in \mathbb{R}^n \). As argued in [9, Section 3], it is sufficient to show that for \( u^{x,T} \) defined by \( (57) \) we have \( u^{x,T} \to u^x \) as \( T \to \infty \) in \( L^2(\mu^x) \), as this implies that \( u^x \) is a solution of \( \mathcal{L}^x u^x(y) = \phi^x - \bar{\phi}(x) \). Using Lemma A.1,

\[
\|u^{x,T} - u^x\|_{L^2(\mu^x)}^2 = \left[ \int_0^\infty \left( \int_T^\infty (P_t^x \phi^x)(y) - \bar{\phi}(x) \right) ds \right] \frac{1}{2}
\leq \int_T^\infty \left( \int |P_t^x \phi^x(y) - \bar{\phi}(x)|^2 d\mu^x(y) \right) \frac{1}{2} ds
= \int_T^\infty \|P_t^x \phi^x - \bar{\phi}(x)\|_{L^2(\mu^x)} ds.
\]

Taking \( T \to \infty \) concludes the proof. Smoothness follows by ellipticity. Indeed, by considering the PDE \( \mathcal{L} u^x = \phi^x - \bar{\phi}(x) \) restricted to a ball, one can use standard results to obtain smoothness. This is done in [43, Theorem 1], where we make the observation that since we are restricting to a ball, our coefficients satisfy their conditions. Assume, conversely, that \( \mathcal{L}^x u^x = \phi^x - \bar{\phi}(x) \) with \( \int u^x d\mu^x = 0 \). Then

\[
P_t^x u^x - u^x = \int_0^t \partial_s P_s^x u^x ds = \int_0^t \mathcal{L}^x P_s^x u^x ds = \int_0^t P_s^x \mathcal{L} u^x ds = \int_0^t P_s^x \phi^x - \bar{\phi}(x) ds.
\]

From Proposition 4.3 we have that \( \lim_{t \to \infty} P_t^x u^x = \int u^x d\mu^x = 0 \) then we have

\[
u^x = -\int_0^\infty (P_t^x \phi^x - \bar{\phi}(x)) dt,
\]

so that the solution is unique in the class \( L^2(\mu^x) \) when restricted to mean-zero functions.

An alternative proof to Lemma A.2 can be obtained by using the methods in [9], in particular by combining [9, Section 5.5], [9, Remark 5.7] and [9, Proposition 3.21].

Proof of Lemma 5.1. It is sufficient to show that the conditions of Lemma A.2 hold. To see this, we use (52) from Lemma 4.1 to write

\[
\int_0^\infty \|P_s^x \phi^x - \bar{\phi}(x)\|_{L^2(\mu^x)} ds \leq \int_0^\infty \left[ \int |(P_s^x \phi^x)(y) - \bar{\phi}(x)|^2 \mu^x(dy) \right] \frac{1}{2} ds
\leq \int_0^\infty \left[ \int C e^{-2cs} (1 + |x|^{2m_x} + |y|^{2m_y}) \mu^x(dy) \right] \frac{1}{2} ds.
\]

Then we use (52) from Lemma 4.1

\[
\int_0^\infty \|P_s^x \phi^x - \bar{\phi}(x)\|_{L^2(\mu^x)} ds \leq C(1 + |x|^{m_x}) \int_0^\infty e^{-cs} ds < \infty.
\]
Hence Lemma A.2 holds and we conclude. 

\textbf{Proof of Lemma 5.4.} This is a method of proof seen in the literature elsewhere (see e.g. \cite{15}) so we only sketch it. We write

\[ |(P_t^x \phi^x)(y) - (P_s^x \phi^x)(y)| = \int_s^t \partial_u (P_t^x \phi^x)(y) du = \int_s^t (L^x P_t^x \phi^x)(y) du. \]

Now we use Assumption A1 (\cite{C3}, \cite{C4}) and Proposition 4.5 to have

\[ |(P_t^x \phi^x)(y) - (P_s^x \phi^x)(y)| \leq C|\phi|_{2,m_x,m_y} \int_s^t (1 + |y|^{m_y})(1 + |x|^{m_z} + |y|^{m_y}) e^{-c u} du \]

\[ + C|\phi|_{2,m_x,m_y} \int_s^t (1 + |x|^{m_z} + |y|^{m_y}) e^{-c u} du \]

\[ \leq C|\phi|_{2,m_x,m_y} (1 + |y|^{m_y} + |y|^{2m_y} + |x|^{2m_x})(e^{-cs} - e^{-ct}). \]

Taking \( t \to \infty \) on both sides concludes (83). Here we use Assumption A3 to apply Lemma 4.3 so that \((P_t^x \phi^x)(y) \to \mu^x(\phi^x)\). Integrating over \( s \) yields (84). \qed

\textbf{Proof of Proposition 5.5.} This proof will follow a similar outline to the proof of Proposition 5.2 and in the same way we will consider increments \( h e_j \) where \( e_j \) is the \( j \)-th element of the standard basis of \( \mathbb{R}^n \). We begin by showing (85) and some preliminary bounds, before outlining the steps to proving the continuity and the second \( x \)-derivative representations (86) and (87). We use (62) to write

\[ \partial_{y_j}(\partial_{x_i} P_t^x \phi^x(y)) = \partial_{y_j} P_t^x \partial_{x_i} \phi^x(y) + \int_0^t \partial_{y_j} \left[ P_t^x \frac{\partial L^x}{\partial x_i} P_s^x \phi^x \right](y) ds. \quad (149) \]

Similarly,

\[ \partial_{y_k} \partial_{y_j}(\partial_{x_i} P_t^x \phi^x(y)) = \partial_{y_k} \partial_{y_j} P_t^x \partial_{x_i} \phi^x(y) + \int_0^t \partial_{y_k} \partial_{y_j} \left[ P_t^x \frac{\partial L^x}{\partial x_i} P_s^x \phi^x \right](y) ds. \quad (150) \]

Addressing the first terms of (149) and (150) we have that, from Proposition 4.5 since \( |\partial_{x_i} \phi|_{2,m_x,m_y} \leq \|\phi\|_{4,m_x,m_y} \),

\[ |\partial_{y_j} P_t^x \partial_{x_i} \phi^x(y)| + |\partial_{y_k} \partial_{y_j} P_t^x \partial_{x_i} \phi^x(y)| \leq C\|\phi\|_{4,m_x,m_y} e^{-ct}(1 + |y|^{m_y} + |x|^{m_z}). \quad (151) \]

Here \( C \) and \( c \) are constants which may change sign by line. Addressing the integral terms in (149) and (150) we have, using Proposition 4.5 (we bound \( \partial_{y_j} \frac{\partial L^x}{\partial x_i} P_s^x \phi^x \) and \( \partial_{y_k} \partial_{y_j} \frac{\partial L^x}{\partial x_i} P_s^x \phi^x \) by taking the limit of (66) and (67) as \( h \to 0 \)) again that,

\[ \left| \int_0^t \partial_{y_j} \left[ P_t^x \frac{\partial L^x}{\partial x_i} P_s^x \phi^x \right](y) ds \right| \]

\[ \leq \int_0^t C|\phi|_{4,m_x,m_y} e^{-c(t-s)} e^{-cs}(1 + |y|^{\max\{2m_y,m_y+m_y\}} + |x|^{2m_x}) ds \]

\[ \leq C|\phi|_{4,m_x,m_y} t e^{-ct}(1 + |y|^{M_y} + |x|^{2m_x}). \]

as well as

\[ \left| \int_0^t \partial_{y_k} \partial_{y_j} \left[ P_t^x \frac{\partial L^x}{\partial x_i} P_s^x \phi^x \right](y) ds \right| \leq C|\phi|_{4,m_x,m_y} t e^{-ct}(1 + |y|^{M_y} + |x|^{2m_x}). \]

\[ \quad (152) \]

\[ \quad (153) \]
Finally, with (154) and (155) together with (149) and (150) to write
\[ |\partial y_j (\partial x_i P^x_t \phi^x(y)) | + |\partial y_j y_k (\partial x_i P^x_{ik} \phi^x(y)) | \leq C(1 + t)\|\phi\|_{4,m_z,m_y} e^{-ct}(1 + |y|^{M^g_y} + |x|^{2m_z}), \]
which implies the required inequality (85). Now we show that \( \partial y_j (\partial x_i P^x_t \phi^x(y)) \) and \( \partial y_j y_k (\partial x_i P^x_{ik} \phi^x(y)) \) are continuous in \( x \). In order to see this, notice that we can extend the result (i) from Proposition 5.2 [to all \( \psi \in C^2(\mathbb{R}^n \times \mathbb{R}^d) \)] with \( |\psi|_{2,m_z,m_y} < \infty \) by taking a sequence \( \phi_k \in \text{Poly}_{m_z,m_y} \) that converges pointwise to \( \psi \) and such that \( |\psi - \phi|_{2,m_z,m_y} \to 0 \) as \( k \to \infty \). Indeed, using the DCT (justified by the uniform bound obtained from Proposition 4.5) concludes that we can extend (i) from Proposition 5.2. Next, use this along with the DCT (justified by the uniform bounds (152) and (153) to obtain that (149) and (150) are continuous in \( x \) as required. This concludes the proof of (i). We move on to proving some preliminary bounds that will be useful for proving (ii). First, similarly to (68),
\[ \left| \frac{\partial^2 L^x}{\partial x_i x_j} P^x_t \phi^x(y) \right| = \sum_{k=1}^{d} \left| \partial x_i, g_k(x, y) \partial y_k P^x_t \phi^x(y) \right| + \sum_{k,l=1}^{d} \left| \partial x_i, A(x)_{kl} \partial y_k y_l P^x_t \phi^x(y) \right| \leq C\|\phi\|_{4,m_z,m_y} e^{-cs}(1 + |y|^{M^g_y} + |x|^{2m_z}). \]
We also have, by direct calculation and (85)
\[ \left| \frac{\partial^2 L^x}{\partial x_i x_j} P^x_t \phi^x(y) \right| = \sum_{j=1}^{d} \left| \partial x_i, g_j(x, y) \partial y_j \partial x_i P^x_t \phi^x(y) \right| + \sum_{j,k=1}^{d} \left| \partial x_i, A(x)_{jk} \partial y_k \partial x_i P^x_t \phi^x(y) \right| \leq C\|\phi\|_{4,m_z,m_y} e^{-cs}(1 + |y|^{M^g_y+m^g_y} + |x|^{4m_z}). \]
Finally, with (154) and (155) we have
\[ \left| \frac{\partial^2 L^x}{\partial x_i x_j} P^x_t \phi^x(y) + \frac{\partial^2 L^x}{\partial x_j x_i} P^x_t \phi^x(y) + \frac{\partial^2 L^x}{\partial x_i x_j} P^x_t \phi^x(y) \right| \leq \left| \frac{\partial^2 L^x}{\partial x_i x_j} P^x_t \phi^x(y) \right| + \left| \frac{\partial^2 L^x}{\partial x_j x_i} P^x_t \phi^x(y) \right| + \left| \frac{\partial^2 L^x}{\partial x_i x_j} P^x_t \phi^x(y) \right| \leq C\|\phi\|_{4,m_z,m_y} e^{-cs}(1 + |y|^{M^g_y+m^g_y} + |x|^{4m_z}). \]
Now we outline the strategy of proof for (ii), making the note similarly to the start of the proof of Proposition 5.2 that here and throughout \( h \) is in fact \( h \varepsilon_j \).

- Step 1: Define the difference quotient
\[ q^h_{t,\phi^x}(y) := \frac{\partial x_i (P^{x+h}_t \phi^{x+h})(y) - \partial x_i (P^x_t \phi^x)(y)}{h} \]
and obtain the representation formula
\[ \tilde{q}^h_{t,\phi^x}(y) = P^x_t q^h_0 \phi^x(y) + \int_0^t P^x_{t-s} \left[ \frac{\partial^2 L^x}{\partial x_i x_j} q^h_{t-s,\phi^x}(y) + \frac{1}{h} \left( \frac{\partial L^x}{\partial x_i} - \frac{\partial L^x}{\partial x_i} \right) P^x_{t-s} \phi^x(y) \right] ds. \]
• Step 2: Let $h \to 0$ in (158) to obtain (86), and prove the continuity of the LHS of (86).

• Step 3: Let $t \to \infty$ in (86) to obtain (87). The bound (41) is then a consequence of (87).

• Step 4: Integrate $\partial_{x_i} (P^x_t \phi^x) (y) - \partial_{x_i} \mu^x (\phi^x)$ with respect to $t$ using the representation formulas (86)-(87) to obtain (43).

Step 1: We differentiate (157) in time using the formulas from Proposition 5.2 to get

$$\partial_t q_t^h \phi^x (y) = \frac{1}{h} \left( \frac{\partial L^{x+h}}{\partial x_i} P^{x+h}_t \phi^{x+h} (y) + L^{x+h} \partial_{x_i} (P^{x+h}_t \phi^{x+h}) (y) \right) - \frac{1}{h} \left( \frac{\partial L^x}{\partial x_i} P^x_t \phi^x (y) + L^x \partial_{x_i} (P^x_t \phi^x) (y) \right)$$

$$= \frac{\partial L^{x+h}}{\partial x_i} \left( \frac{P^{x+h}_t \phi^{x+h} - P^x_t \phi^x}{h} \right) (y) + \frac{1}{h} \left( \frac{\partial L^{x+h}}{\partial x_i} - \frac{\partial L^x}{\partial x_i} \right) P^x_t \phi^x (y)$$

$$+ \left( \frac{L^{x+h} - L^x}{h} \right) \partial_{x_i} \left( P^{x+h}_t \phi^{x+h} \right) (y)$$

$$+ \frac{L^x}{h} \left( \partial_{x_i} \left( P^{x+h}_t \phi^{x+h} \right) (y) - \partial_{x_i} \left( P^x_t \phi^x \right) (y) \right),$$

which, using (157) and recalling the definition of $q_t^h \phi^x (y)$ from (69), gives us

$$\partial_t q_t^h \phi^x (y) = \frac{\partial L^{x+h}}{\partial x_i} q_t^h \phi^x (y) + \frac{1}{h} \left( \frac{\partial L^{x+h}}{\partial x_i} - \frac{\partial L^x}{\partial x_i} \right) P^x_t \phi^x (y)$$

$$+ \left( \frac{L^{x+h} - L^x}{h} \right) \partial_{x_i} \left( P^{x+h}_t \phi^{x+h} \right) (y) + L^x q_t^h \phi^x (y).$$

Using variation of constants we get (158).

Step 2: We let $h \to 0$ in a very similar way to the proof of Proposition 5.2

$$\lim_{h \to 0} q_t^h \phi^x (y) = (P^x_t \partial_{x_i} \phi^x) (y)$$

$$+ \int_0^t P^{s}_s \left[ \frac{\partial^2 L^x}{\partial x_i \partial x_j} P^x_s \phi^x (y) + \frac{\partial L^x}{\partial x_j} \partial_{x_j} (P^x_s \phi^x) (y) + \frac{\partial L^x}{\partial x_i} \partial_{x_i} (P^x_s \phi^x) (y) \right] ds$$

to conclude (86). Indeed, to take $h \to 0$ we get pointwise convergence under the integral by the smoothness of our coefficients, and the continuity in $x$ of $\partial_{y_i} P^x_s \phi^x (y), \partial_{y_j} P^x_s \phi^x (y), \partial_{y_i} \partial_{x_i} P^x_s \phi^x (y)$ and $\partial_{y_i} \partial_{x_j} P^x_s \phi^x (y)$, and then use the DCT.

We use (156) with (86) to prove continuity in $x$ of $\partial_{x_i} (P^x_t \phi^x) (y)$. Indeed, this is the uniform bound needed to apply the DCT along with the smoothness of our coefficients as well as the continuity in $x$ of $\partial_{y_i} P^x_s \phi^x (y), \partial_{y_j} P^x_s \phi^x (y), \partial_{y_i} \partial_{x_i} P^x_s \phi^x (y)$ and $\partial_{y_i} \partial_{x_j} P^x_s \phi^x (y)$. Hence, we have continuity.

Step 3: To let $t \to \infty$ in (86), we must justify the use of the DCT as we had to in the proof of Proposition 5.2. Letting $t \to \infty$ in the first term of (86) is straightforward from Proposition 4.3. With respect to the second addend, it is very similar to the same limit for the second addend of (62) in the proof of Proposition 5.2. This time, we use (156) to justify the use of the DCT to pass the limit inside the time integral. Therefore,
we have proved that the RHS of (62) tends to the RHS of (63) as \( t \to \infty \). As in Step 3 in the proof of Proposition 5.2, we now wish to do the same for the LHS which involves exchanging the limit \( t \to \infty \) with the derivative \( \partial_{x_i} \). We have

\[
\lim_{t \to \infty} \partial_{x_i} \left( P_t \phi^x \right) (y) = \partial_{x_i} \left( P_t \phi^x \right) (y)
\]

\[
= \int_{\mathbb{R}^d} \partial_{x_i} \phi^x (y) \mu^x (dy) - (P_t \phi^x) (y)
\]

\[
+ \int_0^{\infty} \left[ \frac{\partial^2 L^x}{\partial x_i \partial x_j} P_s \phi^x (y) + \frac{\partial L^x}{\partial x_j} P_s \phi^x (y) + \frac{\partial L^x}{\partial x_i} \partial_{x_i} P_s \phi^x (y) \right] \mu^x (dy) ds
\]

\[
- \int_0^t \left[ \frac{\partial^2 L^x}{\partial x_i \partial x_j} P_s \phi^x (y) + \frac{\partial L^x}{\partial x_j} P_s \phi^x (y) + \frac{\partial L^x}{\partial x_i} \partial_{x_i} P_s \phi^x (y) \right] ds
\]

\[
= \mathbb{E} \left[ \mu^x (\partial_{x_i} \phi^x) - \partial_{x_i} \phi^x (y) \right]
\]

\[
+ \int_t^{\infty} \mu^x \left( \frac{\partial^2 L^x}{\partial x_i \partial x_j} P_s \phi^x + \frac{\partial L^x}{\partial x_j} \partial_{x_i} P_s \phi^x + \frac{\partial L^x}{\partial x_i} \partial_{x_j} P_s \phi^x \right) ds
\]

\[
= I + II
\]

We now show that each of the addends in the above converge to zero locally uniformly in \( x \) and \( y \). Similarly to what we have done in the analogous stage of the proof of Proposition 5.2, the claim is trivial for the first term, since by Proposition 4.3

\[
\left( P_s \partial_{x_i} \phi^x \right) (y) - \mu^x (\partial_{x_i} \phi^x) \leq C \| \phi \|_{4, m_x, m_y} e^{-cs} (1 + |y|^{m_y} + |x|^{m_x})
\]

(159)

If we show the following bounds, the proof is concluded by integrating in \( t \):

\[
|I| \leq C \| \phi \|_{4, m_x, m_y} t e^{-ct} (1 + |y|^{M^y + m_y} + |x|^{4m_x}),
\]

(160)

\[
|II| \leq C \| \phi \|_{4, m_x, m_y} e^{-ct} (1 + |x|^{4m_x}),
\]

(161)

Now we proceed with proving (160). With (156) and Proposition 4.3 we write (with \( \tilde{c} \) denoting the constant in (4.3))

\[
|I| \leq e^{-ct} \int_0^t C e^{(c-\tilde{c})s} \| \phi \|_{4, m_x, m_y} (1 + |y|^{M^y + m_y} + |x|^{4m_x}) ds
\]

\[
\leq C e^{-ct} \| \phi \|_{4, m_x, m_y} (1 + |y|^{M^y + m_y} + |x|^{4m_x}),
\]

and so (160) is shown. Now we show (161). By (156) and subsequently (52) from Lemma 4.1, we have

\[
|II| \leq \| \phi \|_{4, m_x, m_y} \int_t^{\infty} \int_{\mathbb{R}^d} \left[ C e^{-cs} (1 + |y|^{M^y + m_y} + |x|^{4m_x}) \right] \mu^x (dy) ds
\]

\[
\leq \| \phi \|_{4, m_x, m_y} \int_t^{\infty} C e^{-cs} (1 + |x|^{4m_x}) ds
\]

\[
\leq C \| \phi \|_{4, m_x, m_y} e^{-ct} (1 + |x|^{4m_x}),
\]
so that (161) is shown. Hence, with (159) (160) and (161), and subsequently Proposition 4.3 we have

\[
\left| \partial_{x_i} \left( P^x \phi^x \right)(y) - \lim_{t \to \infty} \partial_{x_i} \left( P^x_t \phi^x \right)(y) \right| \\
\leq C \| \phi \|_{4,m_y} e^{-ct} (1 + |y|^{m_y} + |x|^{m_x}) \\
+ C \| \phi \|_{4,m_y} e^{-ct} (1 + |y|^{M_y^0 + m_y^0} + |x|^{4m_x}) \\
\leq C \| \phi \|_{4,m_y} e^{-ct} (1 + |y|^{M_y^0 + m_y^0} + |x|^{4m_x})
\]

(162)

for some constants \( C, c \). Hence, the convergence is locally uniform and we have that

\[
\lim_{t \to \infty} \partial_{x_i, j} \left( P^x \phi^x \right)(y) = \partial_{x_i, j} \mu^x(\phi^x).
\]

Hence (87) holds. Now we show (41). Using (52) of Lemma 4.1, (87), (156), and the assumption that \( \phi \in \text{Poly}_{m_1, m_2} \) we have

\[
\left| \partial_{x_i, j} \mu^x(\phi^x) \right| \\
\leq \int_{\mathbb{R}^d} \partial_{x_i, j} \phi^x(\mu^x(dy)) \\
+ \int_0^\infty \int_{\mathbb{R}^d} \left| \frac{\partial^2 \mathcal{L}^x}{\partial x_i, j} P^x \phi^x(y) \right| + \frac{\partial \mathcal{L}^x}{\partial x_i} \partial_{x_i} P^x \phi^x(y) + \frac{\partial \mathcal{L}^x}{\partial x_j} \partial_{x_j} P^x \phi^x(y) \mid \mu^x(dy)ds \\
\leq C \| \phi \|_{4,m_y} (1 + |x|^{m_x}) \\
+ C \| \phi \|_{4,m_y} \int_0^\infty \int_{\mathbb{R}^d} e^{-cs} (1 + |y|^{M_y^0 + m_y^0} + |x|^{4m_x}) \mu^x(dy)ds
\]

Now we use (52) from Lemma 4.1 again to obtain

\[
\int_0^\infty \int_{\mathbb{R}^d} Ce^{-cs} (1 + |y|^{M_y^0 + m_y^0} + |x|^{4m_x}) \mu^x(dy)ds \leq \int_0^\infty Ce^{-cs} (1 + |x|^{4m_x})ds \\
\leq C(1 + |x|^{4m_x})
\]

so that we have shown

\[
\left| \partial_{x_i, j} \int_{\mathbb{R}^d} \phi^x(y) \mu^x(dy) \right| \leq C \| \phi \|_{4,m_y} (1 + |x|^{4m_x}),
\]

which proves (41). Since the convergence as \( t \to \infty \) is locally uniform, we also have continuity of \( \partial_{x_i, j} \mu^x(\phi^x) \).

Step 4: We proceed in a similar vein to the proof of Proposition 5.2. We note that the expression we wish to control is

\[
\left| \partial_{x_i, j} \int_0^\infty \mathbb{E} \left[ \mu^x(\phi^x) - \phi^x(Y_t^{x,y}) \right] dt \right| = \left| \int_0^\infty \partial_{x_i, j} \mu^x(\phi^x) - \partial_{x_i, j} \left( P^x_t \phi^x \right)(y) dt \right|.
\]

This means that we have, using (162),

\[
\sum^n_{i,j=0} \left| \partial_{x_i, j} \int_0^\infty \mathbb{E} \left[ \mu^x(\phi^x_j) - \phi_j(x, Y_t^{x,y}) \right] dt \right| \\
\leq \sum^n_{i,j=0} \left| \partial_{x_i, j} \mu^x(\phi^x_j) - \partial_{x_i, j} \left( P^x_t \phi^x \right)(y) dt \right| \\
\leq C \| \phi \|_{4,m_y} (1 + |y|^{M_y^0 + m_y^0} + |x|^{4m_x}),
\]

and so the proof is complete. \( \square \)
A.3. Proofs of Section 6

Proof of Lemma 6.2. Let \( V_1(x) = |x|^{4n_k^p} \) and \( V_2(y) = |y|^k \) for some \( k \geq 0 \). Using Assumption A3, A4 and (22), there exist \( r', \tilde{C}', \tilde{r}_k, \tilde{C}_k > 0 \) such that

\[
(\mathcal{G}_r V_1)(x, y) \leq -\tilde{r}' V_1(x) + \tilde{C}', \quad (\mathcal{G}_r V_2)(y) \leq -\frac{r_k}{\epsilon} V_2(y) + \frac{C_k'}{\epsilon}.
\]

By the same argument as in the proof of Lemma 4.1 this gives

\[
\mathcal{P}_r V_1(x, y) \leq e^{-\tilde{r}' t} V_1(x) + \frac{\tilde{C}'}{\tilde{r}'}, \quad \mathcal{P}_r V_2(y) \leq e^{-\frac{r_k}{\epsilon} t} V_2(y) + \frac{C_k'}{r_k'},
\]

(163)

Hence, taking \( V(x, y) = V_1(x) + V_2(y) \), using (163) with \( \epsilon \leq 1 \) we can conclude (104). \( \square \)

A.4. Proofs of Section 7

Proof of Proposition 7.2. Observe that Proposition 7.2 follows from Theorem 7.1 provided the conditions of Theorem 7.1 hold. Therefore, it is sufficient to verify (106)–(108).

First, we will find a polynomial \( R(x) \) that satisfies (107). Then we will find the polynomial \( R(x) \) that satisfies (106). Finally, we will verify that these polynomials satisfy (108).

We write the LHS of (107), using (63) with \( \phi = b_j \), as

\[
\sum_{i,j=1}^{n} \partial_i b_j(x) \xi_i \xi_j = \mu^x \left( \sum_{i,j=1}^{n} \partial_i b_j^x(y) \xi_i \xi_j \right) + \sum_{i,j=1}^{n} \left( \int_{0}^{\infty} \mu^x \left( \frac{\partial L^x}{\partial x_i} (P_s^x b_j^x) \right) ds \right) \xi_i \xi_j.
\]

(164)

Now we bound \( I \) from (164). We have that

\[
|I| \leq \sum_{i,j=1}^{n} \left| \int_{0}^{\infty} \mu^x \left( \frac{\partial L^x}{\partial x_i} (P_s^x b_j^x) \right) ds \right| |\xi_i| |\xi_j|
\]

(165)

and

\[
\left| \int_{0}^{\infty} \mu^x \left( \frac{\partial L^x}{\partial x_i} (P_s^x b_j^x) \right) ds \right| \leq \int_{0}^{\infty} \mu^x \left( \sum_{k=1}^{d} |\partial_{x_k} g_k(x, y) \partial_{y_k} P_s^x b_j^x(y)| + \sum_{k,l} |\partial_{x_k} A(x)_{kl} \partial_{y_k} P_s^x b_j^x(y)| \right) ds
\]

Using Assumption A1 [C3] and (148) from Proposition 4.5 (with \( \psi = b_j \), since \( b_j \in C^{0,2}(\mathbb{R}^n \times \mathbb{R}^d) \) with \( |b_j|_{2,m_k^p} < \infty \) by Assumption A1 [C1]) we write

\[
\sum_{k=1}^{d} |\partial_{x_k} g_k(x, y) \partial_{y_k} P_s^x b_j^x(y)|
\]

\begin{align}
&\leq D_0 e^{-\kappa s} b_{2,m_7^p} K_g \left( 1 + |y|^{m_9^p} \right) \left( 1 + |x|^{m_7^p} + |y|^{m_8^p} + \frac{C_{ms}'}{r_{ms}'} \right) \\
&\leq D_0 e^{-\kappa s} b_{2,m_7^p} K_g \left[ 1 + |x|^{m_7^p} + |y|^{m_8^p} + \frac{C_{ms}'}{r_{ms}'} \right]
\end{align}

(166)
and similarly

\[ \sum_{k,l=1}^{d} |\partial_{x_k} A_{kl}(x) \partial_{y_l} y | P_s b_k(x) | \leq D_0 e^{-\kappa s} K_A \left( 1 + |x|^{m_7} + |y|^{m_8} + \frac{C_m}{r_m} \right). \]  

(167)

Using (166), (167) and Proposition 4.1

\[ \int_0^\infty \mu^x \left( \frac{\partial \mathcal{L}^x}{\partial x_i} \left( P_s b_i^y \right) \right) ds \leq \frac{D_0}{\kappa} |b|_{2,m_7,m_8} \left[ (K_g + K_A) \left( 1 + \frac{2C_m}{r_m} \right) 
+ K_g \left( \frac{C_m}{r_m} + \frac{C_m}{r_m} \right) + \frac{C_m}{r_m} \right] \]

so that, using (165),

\[ |I| \leq n|\xi^2| \frac{D_0}{\kappa} |b|_{2,m_7,m_8} \left[ (K_g + K_A) \left( 1 + \frac{2C_m}{r_m} \right) 
+ K_g \left( \frac{C_m}{r_m} + \frac{C_m}{r_m} \right) + \frac{C_m}{r_m} \right] \]  

(168)

By (34), (164) and (168), (107) holds with

\[ \tilde{R}(x) = - \left( \zeta \left( 1 + |x|^{m_1} + |x|^{m_2} + |x|^{m_5} \right) + \frac{K^2 n^3}{4\lambda_1} \right). \]  

(169)

The cases where at least one of $m_7, m_8$, and $m_9$ are equal to 0 follows with the same $\tilde{R}(x)$ from the subsequent simplified version of (168), where we recall that we define $\frac{C_m}{r_0} = 0$.

Now we move onto showing (106). From (87), we have that

\[ \left| \partial_{x_i x_j} b_k(x) \right| \]

\[ \leq \left| \mu^x \left( \partial_{x_i} \partial_{x_j} b_k^x \right) \right| + \int_0^\infty \mu^x \left( \left| \frac{\partial^2 \mathcal{L}^x}{\partial x_i x_j} P_s b_k^x \right| + \left| \frac{\partial \mathcal{L}^x}{\partial x_i} \partial_{x_j} b_k^x \right| + \left| \frac{\partial \mathcal{L}^x}{\partial x_j} \partial_{x_i} b_k^x \right| \right) ds \]

\[ \leq \left( 1 + |x|^{m_5} + \frac{C_m}{r_m} \right) \leq \tilde{C} (1 + |x|^{m_5}). \]  

(170)

Here $\tilde{C}$ is a generic constant which may change line by line and is independent of $x \in \mathbb{R}^n, y \in \mathbb{R}^d$ but may depend on norms of the coefficients. Considering the first term of $\Pi$, ...
similarly to the term I,
\[
\int_0^\infty \mu^x \left( \frac{\partial^2 \mathcal{L}^x}{\partial x_i \partial x_j} P_s^x b_k^x \right) ds \leq \hat{C} \left( 1 + |x|^{m_7} \right)
\]  
(171)

Now, we consider the second term of II. Making the observation that \( \frac{\partial \mathcal{L}^x}{\partial x_i} \) is a second order differential operator in the \( y \) variable, we need a bound on the terms \( \partial_y \partial_{x_i} P_t^x b_k^x \) and \( \partial_{yy} \partial_{x_i} P_t^x b_k^x \). For these terms, we recall the formulae (149)–(150), setting \( \phi^x(y) = b_k(x, y) \). Notice first that by Assumption A1 [C1] \( |\partial_{x_i} b_k|_{2,m_3,m_4} < \infty \) for all \( i \in \{1, \ldots, d\} \), and hence applying the semigroup derivative estimates to this term is immediate. For the next, we wish to apply (54) to \( \frac{\partial^2 \mathcal{L}^x}{\partial x_i^2} P_s^x b_k^x \). To this end, we write the following bound
\[
\max \left\{ \left| \frac{\partial_y \partial_{x_i} P_s^x b_k^x}{\partial x_i} \right|, \left| \frac{\partial_y \partial_{x_i} P_t^x b_k^x}{\partial x_i} \right| \right\} 
\leq \hat{C} e^{-\kappa s} (1 + |y|^{m_2+m_9}|x|^{m_1} + |y|^{m_7+m_9} + |x|^{m_1})
\]
\[
\leq \hat{C} e^{-\kappa s} (1 + |x|^{m_1}) (1 + |y|^{m_7+m_9} + |y|^{m_2+m_9})
\]

meaning we can use Proposition 4.5 with \( \psi = \frac{\partial \mathcal{L}^x}{\partial x_i} P_s^x b_k^x \) to obtain
\[
\max \left\{ \left| \frac{\partial_y \partial_{x_i} P_s^x b_k^x}{\partial x_i} \right|, \left| \frac{\partial_y \partial_{x_i} P_t^x b_k^x}{\partial x_i} \right| \right\} 
\leq \hat{C} e^{-\kappa t} (1 + |x|^{m_1}) (1 + |y|^{m_7+m_9} + |y|^{m_2+m_9})
\]  
(172)

By differentiating (62) twice, using Proposition 4.5 with \( \psi = \partial_{x_i} b_k^x \) for the first addend and integrating (172) for the second, we have
\[
\max \left\{ \left| \partial_y \partial_{x_i} P_s^x b_k^x \right|, \left| \partial_y \partial_{x_j} P_s^x b_k^x \right| \right\} \leq K e^{-\kappa s} (1 + |y|^{m_4} + |x|^{m_3})
\]
\[
+ \hat{C} e^{-\kappa s} (1 + |x|^{m_1}) (1 + |y|^{m_7+m_9} + |y|^{m_2+m_9})
\]

so, using Lemma 4.1
\[
\int_0^\infty \mu^x \left( \frac{\partial \mathcal{L}^x}{\partial x_i} \partial_{x_j} P_s^x b_k^x \right) ds \leq \hat{C}(1 + |x|^{m_1} + |x|^{m_3}).
\]  
(173)

Now we can use (170), (171) and (173) to write
\[
|\partial_{x_i} b_k(x)| \leq \hat{C} (1 + |x|^{m_1} + |x|^{m_3} + |x|^{m_5})
\]
so that (106) holds, with
\[
R(x) = \hat{C} (1 + |x|^{m_1} + |x|^{m_3} + |x|^{m_5}).
\]  
(174)

We conclude (108) with \( L = \zeta/\hat{C} \) by using (169), (174) and Assumption A6  

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