MAJORIZATION AND SPHERICAL FUNCTIONS

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Abstract. Majorization is a partial order on real vectors which plays an important role in a variety of subjects, ranging from algebra and combinatorics to probability and statistics. A basic goal in the study of majorization is to construct functions which characterize this order. In this paper, we introduce a generalized notion of majorization associated to an arbitrary root system $\Phi$, and show that it admits a natural characterization in terms of the values of spherical functions on any Riemannian symmetric space with restricted root system $\Phi$.

1. Introduction

Given vectors $\lambda, \mu \in W^N$, where

$$W^N = \{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N : \lambda_1 \geq \cdots \geq \lambda_N\}$$

is the fundamental type $A$ Weyl chamber, we declare $\lambda \succeq \mu$ if and only if

$$\sum_{i=1}^j \lambda_i \geq \sum_{i=1}^j \mu_j, \quad 1 \leq j \leq N,$$

with equality holding for $j = N$. This defines a partial order on $W^N$ known as majorization. This partial order was introduced by Hardy, Littlewood, and Pólya in 1934, and has since come to play an important role in a variety of subjects, ranging from algebra and combinatorics to probability and statistics; see [2] for a comprehensive treatment.

Majorization admits an intuitive geometric characterization: we have $\lambda \succeq \mu$ if and only if $P_\lambda \supseteq P_\mu$, where $P_\lambda, P_\mu \subset \mathbb{R}^N$ are the convex sets whose extreme points are obtained by permuting the coordinates of $\lambda$ and $\mu$. This paper is about a numerical characterization of majorization which carries over to a much more general context. To state this, we begin by considering a second geometric object associated to each $\lambda \in W^N$, namely the compact symplectic manifold $O_\lambda$ of Hermitian matrices $X$ whose eigenvalues are the coordinates of $\lambda$. According to the classical Schur–Horn theorem [2], the function that sends a Hermitian matrix to its diagonal entries is a surjection $O_\lambda \to P_\lambda$. Since $O_\lambda$ is a homogeneous space for the conjugation action of the unitary group $U(N)$, it carries a unique conjugation-invariant probability measure often referred to as the orbital measure on $O_\lambda$. The pushforward of this measure under the Schur–Horn map gives a probability measure $m_\lambda$ on $P_\lambda$ known as the Duistermaat–Heckman measure [8]. Let $L_\lambda$ denote the Laplace...
transform of the Duistermaat–Heckman measure, i.e. the function \( L_\lambda : \mathbb{C}^N \rightarrow \mathbb{C} \) defined by
\[
L_\lambda(a_1, \ldots, a_N) = \int_{P_\lambda} e^{\sum_{i=1}^N a_i x_i} \, dm_\lambda(x).
\]
Thus, the restriction of \( L_\lambda \) to real arguments is the moment generating function of \( m_\lambda \), while restricting to imaginary arguments gives the characteristic function.

**Theorem 1.1.** For any \( \lambda, \mu \in W^N \), we have \( \lambda \succeq \mu \) if and only if \( L_\lambda \geq L_\mu \) pointwise on \( \mathbb{R}^N \).

Theorem 1.1 generalizes an ex-conjecture of Cuttler, Greene, and Skandera [6] which characterizes the majorization order on Young diagrams via nonnegative specializations of Schur polynomials (the original conjecture was proved by Sra [33], and refined by Khare and Tao [20]). The purpose of this paper is to prove a general principle that yields a large family of majorization-characterizing inequalities in a very general context; in particular, Theorem 1.1 emerges as a special case of this construction. Briefly put, we consider an extended notion of majorization associated to the Weyl group of a given root system \( \Phi \), and show that it is characterized by a pointwise inequality for spherical functions on any Riemannian symmetric space with restricted root system \( \Phi \).

The paper is organized as follows.

In Section 2, we introduce the notion of Weyl group majorization, and prove our main result in the special case of spherical functions on the Lie algebra \( \mathfrak{g} \) of a compact group \( G \) (Theorem 2.3). We treat this case separately because spherical functions reduce to Harish-Chandra orbital integrals in this setting, leading to a very natural and direct generalization of Theorem 1.1. Moreover, the proofs in the Lie algebra case do not require a discussion of symmetric spaces, and are thus somewhat more elementary.

In Section 3, we treat the general case of spherical functions on a Riemannian symmetric space of non-compact type, and prove our main result (Theorem 3.1). We then use this theorem to deduce majorization-characterizing inequalities for spherical functions on symmetric spaces of Euclidean type (Proposition 3.1) and compact type (Proposition 3.2). Furthermore, we discuss how the result for the compact case implies inequalities for various families of orthogonal polynomials, such as Schur polynomials.

In Section 4, we develop an even more general framework based on Heckman–Opdam hypergeometric functions [15]. We state a conjectural characterization of majorization in this context, and prove one direction of this conjecture. Moreover, we show that the full conjecture holds in rank one, where it reduces to an inequality for the classical Gauss hypergeometric function.

## 2. Lie Algebras

Let \( G \) be a connected, compact Lie group with Lie algebra \( \mathfrak{g} \). Let \( T \subset G \) be a maximal torus, \( \mathfrak{t} = \text{Lie}(T) \subset \mathfrak{g} \) the corresponding Cartan subalgebra, and \( W \) the Weyl group generated by reflections in the root hyperplanes in \( \mathfrak{t} \).

**Definition 2.1.** Let \( \lambda, \mu \) be two vectors in \( \mathfrak{t} \), or in any other vector space on which \( W \) acts by reflections. We say that \( \lambda \ W-\text{majorizes} \ \mu \), written \( \lambda \succeq W \mu \), if \( \mu \) lies in the convex hull of the Weyl orbit of \( \lambda \).

\( \triangle \)
The relation $\succeq$ defines a preorder on $t$ and a partial order on each Weyl chamber. When $G = U(N)$, so that $W \cong S_N$ and $t \cong \mathbb{R}^N$, $W$-majorization coincides with the usual notion of majorization for vectors, as described in the Introduction. Generalized majorization orders associated to group actions have been studied since the 1960’s; see [2, 3, 10, 25].

In this section, we prove that $W$-majorization may be characterized by comparing the pointwise behavior of Laplace transforms of invariant probability measures on coadjoint orbits of $G$. Concretely, choose an invariant inner product $\langle \cdot, \cdot \rangle$ identifying $\mathfrak{g} \cong \mathfrak{g}^*$, and for $\lambda \in \mathfrak{g}$ let $O_\lambda = \{ \text{Ad}_g \lambda \mid g \in G \}$ denote its (co)adjoint orbit. Define

$$L_\lambda(x) = \int_{O_\lambda} e^{\langle y, x \rangle} dy = \int_G e^{(\text{Ad}_g \lambda, x)} dg, \quad \lambda, x \in \mathfrak{t}_C,$$

where $dy$ is the unique invariant probability measure on $O_\lambda$, $dg$ is the Haar probability measure on $G$, and $\mathfrak{t}_C \cong t \oplus i t$ is a Cartan subalgebra of the complexified Lie algebra $\mathfrak{g}_C$.

The functions $L_\lambda(x)$ are ubiquitous objects that arise in many different areas of mathematics and physics. They were originally studied by Harish-Chandra in the context of harmonic analysis on Lie algebras [12], and they play an important role in the orbit method in representation theory [21]. When $G = U(N)$, the transform $L_\lambda(x)$ is known as the Harish-Chandra–Itzykson–Zuber (HCIZ) integral; it has been widely studied in theoretical physics and random matrix theory since the 1980’s [10]. More recently, the HCIZ integral has become an important object in combinatorics [11, 28] and probability [3, 27]. For further background on these functions and their diverse applications, we refer the reader to [23, 24].

The main result of this section is the following characterization theorem.

**Theorem 2.2.** For any $\lambda, \mu \in \mathfrak{t}$, the following are equivalent:

1. $\lambda \succeq \mu$,
2. $L_\lambda(x) \geq L_\mu(x)$ for all $x \in \mathfrak{t}$.

**Proof.** We first show (ii) implies (i), by proving the contrapositive. The discriminant of $\mathfrak{g}$ is the polynomial $\Delta_\mathfrak{g}(x) = \prod_{\alpha \in \Phi^+} (\alpha, x)$, where $\Phi^+$ is the set of positive roots. Let $x \in \mathfrak{t}$ with $\Delta_\mathfrak{g}(x) \neq 0$. We assume for now that $\Delta_\mathfrak{g}(\lambda), \Delta_\mathfrak{g}(\mu) \neq 0$ as well; later we will remove this assumption. The Laplace transform (1) admits an exact expression, due to Harish-Chandra [12]:

$$L_\lambda(x) = \frac{\Delta_\mathfrak{g}(\rho)}{\Delta_\mathfrak{g}(\lambda)} \sum_{w \in W} \epsilon(w) e^{\langle w(\lambda), x \rangle},$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and $\epsilon(w)$ is the sign of $w \in W$. Taking $t > 0$ and using (2), we can write

$$L_\mu(tx) - L_\lambda(tx) = \frac{\Delta_\mathfrak{g}(\rho)}{\Delta_\mathfrak{g}(tx)} \sum_{w \in W} \epsilon(w) \left( e^{\langle t \epsilon(w(\mu), x) \rangle} - e^{\langle t \epsilon(w(\lambda), x) \rangle} \right).$$

1. The $W$-majorization preorder is not the same as the height partial order on $t$. If $\lambda, \mu \in \mathfrak{t}$, we say that $\lambda$ is higher than $\mu$ if $\lambda - \mu$ can be written as a linear combination of positive roots with nonnegative coefficients. The resulting order coincides with $W$-majorization when $\lambda$ and $\mu$ are both dominant, but the height partial order is not $W$-invariant.

2. Here we take the roots to be real-valued linear functionals in $t^*$, which we identify with $t$ via the inner product. As a result, our roots differ by a factor of $i$ from those typically used in the setting of complex semisimple Lie algebras.
This expression is manifestly $W$-invariant in $\lambda$, $\mu$ and $x$, so we may assume without loss of generality that all three are dominant. Then as $t \to \infty$ we have:

\[
L_\mu(tx) - L_\lambda(tx) = \frac{\Delta_\rho(x)}{\Delta_\rho(tx)} \left( \frac{e^{t(\mu, x)}}{\Delta_\rho(\mu)} - \frac{e^{t(\lambda, x)}}{\Delta_\rho(\lambda)} \right) + \text{(lower-order terms)}.
\]

Now suppose $\lambda \not\preceq \mu$. Then $\mu$ lies outside the convex hull of the $W$-orbit of $\lambda$, so by the hyperplane separation theorem there is some $x_0 \in t$ and $C > 0$ such that $\langle \mu, x_0 \rangle > C$ while $\langle w(\lambda), x_0 \rangle < C$ for all $w \in W$. By making a small perturbation to $x_0$ if necessary, we can ensure that $\Delta_\rho(x_0) \neq 0$. Take $x$ in (4) to be the dominant representative of the Weyl orbit of $x_0$. Then we still have $\langle \mu, x \rangle > C$, $\langle \lambda, x \rangle < C$, and from (4) we find:

\[
\lim_{t \to \infty} \left[ L_\mu(tx) - L_\lambda(tx) \right] \geq \lim_{t \to \infty} \frac{\Delta_\rho(x)}{\Delta_\rho(tx)} \left( \frac{e^{t(\mu, x)}}{\Delta_\rho(\mu)} - \frac{e^{tC}}{\Delta_\rho(\lambda)} \right) = \infty,
\]

which implies $L_\mu(tx) > L_\lambda(tx)$ for some $t > 0$, so that (ii) cannot hold.

Now we remove the assumption that $\Delta_\rho(\lambda), \Delta_\rho(\mu) \neq 0$. In this case the expression (4) may be singular, so we instead take a limit:

\[
L_\mu(tx) - L_\lambda(tx) = \lim_{\eta \to 0} \frac{\Delta_\rho(x)}{\Delta_\rho(tx)} \sum_{w \in W} \epsilon(w) \left( \frac{e^{t(w(\mu + \eta\rho), x)}}{\Delta_\rho(\mu + \eta\rho)} - \frac{e^{t(w(\lambda + \eta\rho), x)}}{\Delta_\rho(\lambda + \eta\rho)} \right).
\]

To evaluate this limit, we apply l’Hôpital’s rule as many times as needed, treating the $\lambda$ and $\mu$ terms separately. After $j$ applications to the $\lambda$ terms and $k$ applications to the $\mu$ terms for some $j, k \geq 0$, in place of (4) we find:

\[
L_\mu(tx) - L_\lambda(tx) = \frac{\Delta_\rho(x)}{\Delta_\rho(tx)} \left( \frac{t^k(\mu, x)k e^{t(\mu, x)}}{\partial^k_\rho \Delta_\rho(\mu)} - \frac{t^j(\lambda, x)j e^{t(\lambda, x)}}{\partial^j_\rho \Delta_\rho(\lambda)} \right) + \text{(lower-order terms)},
\]

where $\partial^k_\rho \Delta_\rho(\mu) = \frac{d^k}{d\eta^k} \Delta_\rho(\mu + \eta\rho)|_{\eta=0}$. The remainder of the argument then goes through as before, and we conclude that (ii) implies (i).

The other direction of the proof, (i) implies (ii), amounts to showing that for all $x \in t$, the function $\lambda \mapsto L_\lambda(x)$ is $W$-convex. This function is clearly $W$-invariant, and by [10 Theorem 1], a $W$-invariant, convex function is $W$-convex. It therefore remains only to show that $L_\lambda(x)$ is convex in $\lambda$, and for this it is sufficient to show midpoint convexity. For $u, v \in t$ we have

\[
L_{(u+v)/2}(x) = \int_G e^{(\mathrm{Ad}_u(u+v)/2, x)} d\rho = \int_G \sqrt{e^{(\mathrm{Ad}_u u, x)} e^{(\mathrm{Ad}_v v, x)}} d\rho \\
\leq \int_G \frac{1}{2} \left( e^{(\mathrm{Ad}_u u, x)} + e^{(\mathrm{Ad}_v v, x)} \right) d\rho = \frac{1}{2} L_u(x) + \frac{1}{2} L_v(x),
\]

where in the final line we have applied the inequality of arithmetic and geometric means. This proves the theorem.

\[\square\]

**Remark 2.3.** It is easily verified from the definition (1) that $L_\lambda(x) = L_x(\lambda)$, so condition (ii) in Theorem 2.2.2 could equivalently be written:

\[
L_x(\lambda) \geq L_x(\mu) \text{ for all } x \in t.
\]

\[\triangle\]
3. Symmetric spaces

This section contains our main results, which are majorization inequalities for spherical functions on Riemannian symmetric spaces. After introducing some background on symmetric spaces and spherical functions, we state and prove separate inequalities for each of the three types of irreducible symmetric space. In each case the theorem takes the form of an inequality between the pointwise values of any two spherical functions, which reflects the $W$-majorization order on the space of vectors that index the spherical functions. As we explain below, these results imply Theorem 2.2 and discretizations thereof, such as the Schur function inequality studied in [6, 33].

3.1. Background on symmetric spaces and spherical functions. We introduce only the minimum background on symmetric spaces and spherical functions that is required to state and prove the theorems. We refer the reader to [29, Appendices B and C] for a concise introduction to these topics, and to [17, 18] for detailed references. The definitions below mostly follow [18, ch. 4]

Definition 3.1. Let $G$ be a connected Lie group, and $K \subset G$ a compact subgroup. We say that $(G, K)$ is a symmetric pair if $K$ is the fixed-point set of an involutive automorphism $\sigma : G \to G$. For our purposes, a Riemannian symmetric space is a quotient $X = G/K$, where $(G, K)$ is a symmetric pair. When $G$ is non-compact and semisimple with finite center, and $K$ is a maximal compact subgroup, we say that $X$ is of non-compact type. △

Below, the term “symmetric space” always means a Riemannian symmetric space as defined above.

Definition 3.2. Let $X = G/K$ be a Riemannian symmetric space, and write $[g] \in X$ for the image of $g \in G$ under the quotient map $G \to G/K$. Let $D(X)$ be the algebra of differential operators on $X$ that are invariant under all translations $[x] \mapsto [gx], g \in G$. A complex-valued function $\phi \in C^\infty(X)$ is called a spherical function if all of the following hold:

(i) $\phi([\text{id}]) = 1$,
(ii) $\phi([kx]) = \phi([x])$ for all $k \in K$,
(iii) $D\phi = \gamma_D\phi$ for each $D \in D(X)$, where $\gamma_D$ is some complex eigenvalue. △

Spherical functions play a central role in the theory of harmonic analysis on symmetric spaces, and many important families of special functions can be realized as spherical functions on some symmetric space.

Example 3.3. Let $G$ be a compact connected Lie group, and $K \subset G \times G$ the diagonal subgroup. Then $(G \times G, K)$ is a symmetric pair, and we can identify $(G \times G)/K \cong G$ via $(g_1, g_2)K \mapsto g_1g_2^{-1}$. The spherical functions on $G$ are precisely the functions of the form

$$\phi_\lambda(g) = \frac{\chi_\lambda(g)}{\dim V_\lambda},$$

where $V_\lambda$ is the irreducible representation of $G$ with highest weight $\lambda$, and $\chi_\lambda$ is its character. △
Example 3.4. Let $G$ again be a compact connected Lie group. If we regard its Lie algebra $\mathfrak{g}$ as an abelian Lie group, we can form the semidirect product $G \ltimes \mathfrak{g}$ with multiplication $(g_1, x_1) \cdot (g_2, x_2) = (g_1 g_2, \text{Ad}_{g_1} x_2 + x_1)$. Then $(G \ltimes \mathfrak{g}, G)$ is a symmetric pair, and we can identify $(G \ltimes \mathfrak{g})/G \cong \mathfrak{g}$ via $(g, x) \mapsto \text{Ad}_g x$. Thus $\mathfrak{g}$ is a symmetric space, and the spherical functions on $\mathfrak{g}$ reduce to the Laplace transforms studied in Section 2.

$$L_\lambda(x) = \int_G e^{\langle \text{Ad}_x \lambda, x \rangle} \, dg, \quad x \in \mathfrak{g}, \quad \lambda \in \mathfrak{t}_C,$$

where $\mathfrak{t}_C$ is the complexification of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. \hfill \triangle

If $X$ is a symmetric space then its universal cover $\tilde{X}$ is also symmetric, and the spherical functions on $X$ may be identified with spherical functions on $\tilde{X}$ that are constant on the fibers of the covering map. In this sense the spherical functions on $\tilde{X}$ subsume those on $X$, so that in what follows we may assume without loss of generality that $X$ is simply connected. We say that $X$ is irreducible if it cannot be written as a nontrivial product of symmetric spaces. A simply connected, irreducible symmetric space is always:

1. of non-compact type; or,
2. a Euclidean space; or,
3. compact.

These three types correspond respectively to the cases in which $X$ is negatively curved, flat, or positively curved. There is a a well-known correspondence between the three types, which we now describe. If $X^- = G/K$ is a symmetric space of non-compact type, $\sigma : G \to G$ is the associated involution fixing $K$, and $\mathfrak{g} = \text{Lie}(G)$, then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, where $\mathfrak{t} = \text{Lie}(K)$ is the fixed-point set of $d\sigma$. From these data we can construct both a Euclidean symmetric space and a compact symmetric space. First define $\mathfrak{g}^+ = \mathfrak{t} + i\mathfrak{p} \subset \mathfrak{g}_C$, which is the Lie algebra of the compact real form $G^+$ of $G$. The symmetric space $X^+ = G^+/K$ is obviously compact. Next define the algebra $\mathfrak{g}^0$, which is equal to $\mathfrak{g}$ as a vector space but is endowed with a different Lie bracket $[\cdot, \cdot]_0$ defined by

$$[x, y]_0 = \begin{cases} 0, & x, y \in \mathfrak{p}, \\ [x, y], & \text{otherwise}. \end{cases}$$

Then the group $G^0 = \exp(\mathfrak{g}^0) \cong K \ltimes \mathfrak{p}$ acts on $\mathfrak{p}$ by affine transformations, $(k, p) \cdot x = \text{Ad}_k x + p$, and $X^0 = G^0/K \cong \mathfrak{p}$ is a Euclidean symmetric space.

Thus we have constructed a triple of symmetric spaces $(X^-, X^0, X^+)$ that belong respectively to the three types listed above. Moreover, every simply connected, irreducible symmetric space occurs in such a triple. In the following subsections, we study spherical functions on the spaces $X^-, X^0$, and $X^+$.

3.2. Symmetric spaces of non-compact type. When $X^- = G/K$ is a symmetric space of non-compact type, the spherical functions admit a convenient integral representation, due to Harish-Chandra [13, 14]. The version that we use here is proved in [13] ch. 4, Theorem 4.3. Let $G = NAK$, $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{t}$ be the Iwasawa decompositions of $G$ and $\mathfrak{g}$. For $g \in G$, let $a(g)$ be the unique element of $\mathfrak{a}$ such that $g \in Ne^{a(g)} K$. The Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ restricts to an inner product on $\mathfrak{a}$. For $\alpha \in \mathfrak{a}$, define

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x \text{ for all } h \in \mathfrak{a} \}.$$
The restricted root system $\Phi$ of $X^{-}$ consists of all nonzero $\alpha \in \mathfrak{a}$ for which $\mathfrak{g}_\alpha$ is nontrivial. Fix a choice $\Phi^+$ of positive roots, and let $W$ be the Weyl group generated by reflections in the root hyperplanes. For $\alpha \in \Phi$, define $m_\alpha = \dim \mathfrak{g}_\alpha$, and set $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} m_\alpha \alpha$. Write $dk$ for the normalized Haar measure on $K$.

**Theorem 3.5** (Harish-Chandra). The spherical functions on $X^{-}$ are exactly the functions of the form

$$
(7) \quad \phi_{i\lambda}([g]) = \int_K e^{i\lambda + \rho, a(kg)} dk, \quad g \in G,
$$

as $\lambda$ ranges over $\mathfrak{a}_C$. Moreover, two such functions $\phi_{\lambda}$ and $\phi_{\mu}$ are identical if and only if $\mu = w(\lambda)$ for some $w \in W$.

The following theorem is the main result of this paper.

**Theorem 3.6.** Let $X^{-} = G/K$ be a Riemannian symmetric space of non-compact type. For any $\lambda, \mu \in \mathfrak{a}$, the following are equivalent:

(i) $\lambda \succeq \mu$,

(ii) $\phi_{i\lambda}(x) \succeq \phi_{i\mu}(x)$ for all $x \in X$.

**Proof.** The argument generalizes the proof of Theorem 3.22. We first show that (ii) implies (i) by proving the contrapositive, and then that (i) implies (ii) using the integral representation (7) for the spherical functions.

Suppose $\lambda \not\succeq \mu$. Since the map $\lambda \mapsto -\lambda$ is an isometry of $\mathfrak{a}$, we have $\lambda \succeq \mu$ if and only if $-\lambda \succeq -\mu$. Accordingly, to prove that (ii) implies (i), it suffices to show that $\phi_{-i\mu}(x) > \phi_{-i\lambda}(x)$ for some $x \in X$. By hyperplane separation, we can obtain $y \in \mathfrak{a}$ and $C_1 > 0$ such that $\langle \mu, y \rangle > C_1$ and $\langle \lambda, y \rangle < C_1$ for all $w \in W$. Clearly both of these inequalities still hold if we replace $y$ with the dominant representative of its Weyl orbit, and by Theorem 3.5, we have $\phi_{-i\mu} = \phi_{-i\lambda}$ for $w \in W$. Therefore without loss of generality we may take all three of $\lambda$, $\mu$ and $y$ to be dominant.

With these assumptions, we will study the asymptotic behavior of the spherical functions $\phi_{-i\lambda}$ and $\phi_{-i\mu}$ at infinity. This topic is well understood; see e.g. [7]. In particular we have the following sharp estimate as $t \to +\infty$, which is also a special case of (22) below:

$$
(8) \quad \phi_{-i\lambda}([e^{ty}]) \asymp e^{t(\lambda - 2\rho, y)} \prod_{\alpha \in \Phi^+ \atop \langle \alpha, \lambda \rangle = 0} \left(1 + 4t\langle \alpha, y \rangle\right).
$$

This estimate implies that

$$
\phi_{-i\mu}([e^{ty}]) - \phi_{-i\lambda}([e^{ty}]) > e^{-2t\langle \rho, y \rangle} \left(C_2 e^{t\langle \mu, y \rangle} - C_3 e^{tC_1} \prod_{\alpha \in \Phi^+ \atop \langle \alpha, \lambda \rangle = 0} \left(1 + 4t\langle \alpha, y \rangle\right)\right)
$$

for some constants $C_2, C_3 > 0$. For $t$ sufficiently large, the quantity on the right-hand side above is positive, proving that $\phi_{-i\mu}(x) > \phi_{-i\lambda}(x)$ for some $x \in X$, as desired.

We next prove that (i) implies (ii). It suffices to show that the function $f_x(\lambda) = \phi_{i\lambda}(x)$ is $W$-convex. As in the proof of Theorem 2.2 we use the result of [10] Theorem 1], which states that a $W$-invariant, convex function is $W$-convex. By Theorem 3.5, $f_x$ is $W$-invariant, so we need only prove that $f_x$ is convex, for which it suffices to check midpoint convexity. Write $x = [g]$ for some $g \in G$. Using the
integral representation \[ (7) \] and the inequality of arithmetic and geometric means, we find:

\[
\begin{align*}
  f_x \left( \frac{1}{2}(\lambda + \mu) \right) &= \int_K e^{\langle \rho - (\lambda + \mu)/2, a(k) \rangle} dk \\
  &= \int_K e^{\langle \rho, a(k) \rangle} e^{-\langle \lambda, a(k) \rangle} e^{-\langle \mu, a(k) \rangle} dk \\
  &\geq \frac{1}{2} \int_K e^{\langle \rho, a(k) \rangle} \left( e^{-\langle \lambda, a(k) \rangle} + e^{-\langle \mu, a(k) \rangle} \right) dk \\
  &= \frac{1}{2} \left( f_x(\lambda) + f_x(\mu) \right),
\end{align*}
\]

which shows that \( f_x \) is convex, completing the proof. \( \square \)

3.3. Euclidean symmetric spaces. The spherical functions on the Euclidean symmetric space \( X^0 \cong \mathfrak{p} \) are precisely the functions

\[
\phi^0_\lambda(x) = \lim_{\epsilon \to 0} \phi^{-\lambda/\epsilon}_{\lambda/\epsilon}\left(e^{\epsilon x}\right) = \int_K e^{\langle \lambda, \text{Ad}_k x \rangle} dk, \quad x \in \mathfrak{p},
\]

as \( \lambda \) ranges over \( \mathfrak{a}_\mathbb{C} \); see [18, ch. 4, Proposition 4.8]. Taking the limit \[ (9) \] in the proof of Theorem 3.6, we obtain the following.

**Proposition 3.7.** Let \( X^0 \) be a Euclidean symmetric space. For any \( \lambda, \mu \in \mathfrak{a} \), the following are equivalent:

(i) \( \lambda \succeq \mu \),

(ii) \( \phi^0_\lambda(x) \geq \phi^0_\mu(x) \) for all \( x \in X \).

In particular, Theorem 2.2 is a special case of Proposition 3.7 corresponding to the Euclidean symmetric space described in Example 3.4.

3.4. Compact symmetric spaces. We now consider the compact symmetric space \( X^+ = G^+/K \). Let \( V_\lambda \) be the irreducible \( G^+ \)-representation with highest weight \( \lambda \) and \( \chi_\lambda \) its character. If \( V_\lambda \) contains a nontrivial \( K \)-fixed vector, we say that \( V_\lambda \) is a spherical representation and \( \lambda \) is a spherical highest weight. By [18, ch. 4, Theorem 4.2], the spherical functions on \( X^+ \) are precisely the functions

\[
\phi^+_\lambda([g]) = \int_K \chi_\lambda(g^{-1} k) dk, \quad g \in G^+,
\]

where \( \chi_\lambda \) is the character of an irreducible spherical representation of \( G^+ \).

Here we depart in two ways from the conventions used above in Section 2: First, we now use the notation \( \langle \cdot, \cdot \rangle \) to indicate the Killing form, which restricts to a negative-definite form on \( \mathfrak{g}^+ \) rather than an inner product. Second, we now regard the roots and weights of \( G^+ \) as imaginary-valued linear functionals on a Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{g}^+ \) with \( i\mathfrak{a} \subset \mathfrak{t} \). We then use the Killing form to identify the weights and roots with elements of \( i\mathfrak{t} \).

With these conventions, the spherical highest weights of \( G^+ \) correspond to certain lattice points in \( \mathfrak{a} \subset i\mathfrak{t} \); see [18 ch. 5 §4]. By [18 ch. 5, Theorem 4.1 and Corollary 4.2], if \( G^+ \) is simply connected and semisimple then the spherical highest weights are exactly those \( \lambda \in \mathfrak{a} \) satisfying

\[
\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}_{\geq 0} \quad \text{for all } \alpha \in \Phi^+,
\]
where $\Phi^+$ are the positive restricted roots of $X^-$. Given $\lambda, \mu \in a$, we write $\lambda \succeq \mu$ to indicate that $\lambda W$-majorizes $\mu$, where $W$ is the Weyl group generated by reflections in the restricted roots.

The function $\phi^+_{\lambda}$ can be analytically continued to the complexification $G_C$, so that we may evaluate $\phi^+_{\lambda}([e^{ix}])$ for any $x \in t_C$. We then have the following majorization inequality.

**Proposition 3.8.** Let $\lambda, \mu \in a$ be two spherical highest weights of $G^+$. The following are equivalent:

(i) $\lambda \succeq \mu$,

(ii) $\phi^+_{\lambda}([e^{ix}]) \geq \phi^+_{\mu}([e^{ix}])$ for all $x \in t$.

**Proof.** Consider the spherical function $\phi^{-i(\lambda - \rho)}$ on $X^-$, regarded as a function on the non-compact group $G$. When $\lambda$ is a spherical highest weight, this function also admits an analytic continuation to $G_C$, which coincides with $\phi^+_{\lambda}$; see [35, §4]. Since $t = t \cap t + ia$ and $[e^{tx}] = [e^x] \in X^+$ for $y \in t \cap t$, we can take $x \in ia$, so that $e^{ix} \in G$. The desired result is then immediate from Theorem 3.6. □

### 3.5. Application to Schur polynomials.

Let us explain how Proposition 3.8 implies the results of [6, 33] relating majorization and Schur polynomials. Let $\mathbb{Y}^N \subset \mathbb{W}^N$ be the set of points in the type $A$ Weyl chamber whose coordinates are nonnegative integers. Associated to each $\lambda \in \mathbb{Y}^N$ is the corresponding Schur polynomial

\[
(12) \quad s_{\lambda}(x_1, \ldots, x_N) = \det[x_{j-i}^N + \lambda_i]^N_{i,j=1} / \det[x_{j-i}^N]^N_{i,j=1}.
\]

Each $\lambda \in \mathbb{Y}^N$ corresponds to the highest weight of an irreducible polynomial representation of $U(N)$, and we can identify $\mathbb{R}^N$ with a Cartan subalgebra in $u(N)$ such that

\[
(13) \quad s_{\lambda}(e^{iy_1}, \ldots, e^{iy_N}) = \chi_{\lambda}(e^y), \quad y = (y_1, \ldots, y_N) \in \mathbb{R}^N,
\]

and $s_{\lambda}(1, \ldots, 1) = \dim V_{\lambda}$. If we then regard the group $U(N)$ as a compact symmetric space as in Example 3.3, we find

\[
\phi^+_{\lambda}([e^y]) = s_{\lambda}(e^{y_1}, \ldots, e^{y_N}) / s_{\lambda}(1, \ldots, 1).
\]

Writing $x_i = e^{yi}$, Proposition 3.8 then yields Conjecture 7.4 in [6] under the stricter assumption that all $x_1, \ldots, x_N > 0$. Since Schur polynomials are continuous, we can relax this to $x_1, \ldots, x_N \geq 0$, which proves Conjecture 7.4 in [6] in full. We note that a characterization of weak majorization in terms of Schur polynomials was recently obtained in [20].

Many families of orthogonal polynomials can be realized as spherical functions on a compact symmetric space [35]. In all such cases, Proposition 3.8 gives an inequality for the orthogonal polynomials that is analogous to the Schur function case.
4. HYPERGEOMETRIC FUNCTIONS

The Heckman–Opdam hypergeometric functions are a family of special functions associated to root systems, which generalize the classical Gauss hypergeometric function to higher dimensions. They are eigenfunctions of the hyperbolic quantum Calogero–Sutherland Hamiltonian and were introduced in the paper [15] in order to prove the complete integrability of quantum Calogero–Sutherland models. Many special functions of interest can be expressed via limits or specializations of Heckman–Opdam hypergeometric functions, including the spherical functions on all Riemannian symmetric spaces of non-compact type. Also in [15], Heckman and Opdam defined the multivariable Jacobi polynomials, now known as Heckman–Opdam polynomials. These are closely related to hypergeometric functions and generalize numerous widely studied families of orthogonal polynomials, such as Schur and Jack polynomials.

In this final section, we conjecture that Heckman–Opdam hypergeometric functions satisfy a fundamental monotonicity property with respect to $W$-majorization. If true, this conjecture would unify and generalize all of the majorization results discussed in this paper. We prove one of the two directions of implication that comprise the conjecture, and we show that the full conjecture holds in rank one.

Just as Heckman–Opdam hypergeometric functions generalize the spherical functions $\phi^{-\lambda}$ on different symmetric spaces of non-compact type, the Heckman–Opdam polynomials generalize of the functions $\phi^{+\lambda}$, up to some differences in normalization. Similarly, another related class of functions, the generalized Bessel functions, interpolate between the functions $\phi^{0\lambda}$ on different Euclidean symmetric spaces. Accordingly, if the conjecture is true, then we should expect that analogous results hold for both generalized Bessel functions and Heckman–Opdam polynomials. This intuition appears to be correct: we show below that the conjecture would immediately imply an analogue for Heckman–Opdam polynomials of the Schur polynomial inequality proved in [6, 33].

To define the Heckman–Opdam hypergeometric functions, we first must fix some preliminary data. Here we largely follow the conventions of Anker [1] and Heckman and Schlichtkrul [16]. Let $V \cong \mathbb{R}^r$ be a Euclidean space, $\Phi \subset V$ a crystallographic root system spanning $V$, and $W$ the Weyl group acting on $V$ by reflections in the root hyperplanes. The Heckman–Opdam hypergeometric function $F_{k,\lambda}$ depends on a point $\lambda$ in the complexification $V \subset \mathbb{C}$, as well as on a multiplicity parameter, which is a function $k : \Phi \rightarrow \mathbb{C}$ such that $k_{w \cdot \alpha} = k_{\alpha}$ for $w \in W$. Unless stated otherwise, we assume in what follows that $\lambda \in V$ and that all $k_{\alpha}$ are nonnegative real numbers.

We now define $F_{k,\lambda}$ in terms of solutions to certain differential-difference equations. Fix a choice of positive roots $\Phi^+$. For $\alpha \in \Phi^+$ let $s_{\alpha}$ be the reflection through the hyperplane $\{x \in V \mid \langle \alpha, x \rangle = 0\}$, and define $\rho(k) = \frac{1}{2} \sum_{\alpha \in \Phi^+} k_{\alpha} \alpha$.

**Definition 4.1.** For $y \in V$, the Cherednik operator $D_{k,y}$ is the differential-difference operator

$$D_{k,y} = \partial_y + \sum_{\alpha \in \Phi^+} \langle y, \alpha \rangle k_{\alpha} \frac{1}{1 - e^{-\alpha}} (1 - s_{\alpha} - \langle y, \rho(k) \rangle).$$

The Cherednik operators were originally defined and studied in [15]. For details of their properties, we refer the reader to these papers as well as to [11 §4] and [30].
we need only the following fact: when all $k_\alpha$ are nonnegative, for any $\lambda \in V_\mathbb{C}$ there is a unique smooth function $G_{k,\lambda}$ on $V$ satisfying the system of differential-difference equations

\begin{equation}
D_{k,y}G_{k,\lambda} = \langle y, \lambda \rangle G_{k,\lambda} \quad \text{for all } y \in V
\end{equation}

and normalized so that $G_\lambda(0) = 1$.

**Definition 4.2.** For $k \geq 0$ and any $\lambda \in V_\mathbb{C}$, the Heckman–Opdam hypergeometric function $F_{k,\lambda}$ is defined as

\begin{equation}
F_{k,\lambda} = \frac{1}{|W|} \sum_{w \in W} G_{k,\lambda}(w(x)).
\end{equation}

The functions $F_{k,\lambda}$ unify and interpolate between many widely studied special functions, as illustrated in the following examples.

**Example 4.3.** In the 1-dimensional case where $V \cong \mathbb{R}$, the root system $\Phi$ can be either $A_1$ or $BC_1$. For $BC_1$ there are two Weyl orbits $\{\pm 1\}, \{\pm 2\} \subset \mathbb{R}$, and the Heckman–Opdam hypergeometric function reduces to the Gauss hypergeometric function:

\begin{equation}
F_{k,\lambda}(x) = 2F_1\left(\frac{k_1}{2} + k_2 + \lambda, \frac{k_1}{2} + k_2 - \lambda; k_1 + k_2 + \frac{1}{2}; -\sinh^2 \frac{x}{2}\right).
\end{equation}

The Heckman–Opdam hypergeometric function for $A_1$ corresponds to the special case $k_1 = 0$.

**Example 4.4.** Suppose $\Psi$ is the restricted root system of a symmetric space $X = G/K$ of noncompact type, and $m_\alpha = \dim g_\alpha$ for each $\alpha \in \Psi$. Take $V = a$, $\Phi = \{2\alpha \mid \alpha \in \Psi\}$ and $k_{2\alpha} = \frac{1}{2}m_\alpha$. Then

\begin{equation}
\phi_\lambda ([e^x]) = F_{k,\lambda}(2x), \quad x \in a, \quad \lambda \in a_\mathbb{C}.
\end{equation}

See [1, §4].

**Example 4.5.** The generalized Bessel function $J_{k,\lambda}$ on $V$ can be obtained as the rational limit of $F_{k,\lambda}$:

\begin{equation}
J_{k,\lambda}(x) = \lim_{\varepsilon \to 0} F_{k,\lambda/\varepsilon}(\varepsilon x).
\end{equation}

From the previous example and the relation (9), it is clear that $J_{k,\lambda}$ generalizes the spherical functions on Euclidean symmetric spaces in the same way that $F_{k,\lambda}$ generalizes the spherical functions on symmetric spaces of non-compact type. See [1, §3 and §4.4] for details on generalized Bessel functions and the rational limit.

For any multiplicity parameter $k$, we define the function

\begin{equation}
\delta_k(x) = \prod_{\alpha \in \Phi^+} (e^{(\alpha,x)/2} - e^{-(\alpha,x)/2})^{k_\alpha}.
\end{equation}

**Example 4.6.** When $\Phi$ is reduced, we can identify $V$ with a Cartan subalgebra $t$ of a compact semisimple Lie algebra $g$ with root system $\Phi$. We write $k = \overline{\Gamma}$ for the multiplicity parameter with $k_\alpha = 1$ for all $\alpha \in \Phi$. Then

\begin{equation}
F_{\overline{\Gamma},\lambda}(x) = \frac{\Delta_g(x)}{\delta_{\overline{\Gamma}}(x)} L_{\lambda}(x), \quad x \in t.
\end{equation}
We conjecture the following monotonicity property for Heckman–Opdam hypergeometric functions.

**Conjecture 4.7.** Let $\lambda, \mu \in V$. The following are equivalent:

(i) $\lambda \succeq \mu$,

(ii) $F_{k,\lambda}(x) \geq F_{k,\mu}(x)$ for all $x \in V$.

Here we show one half of the conjecture, namely that (ii) implies (i), using Schapira’s sharp asymptotics for $F_{k,\lambda}$ [32]. We then give an elementary proof of the conjecture in rank one, where it amounts to an inequality for the Gauss hypergeometric function.

**Proposition 4.8.** Let $\lambda, \mu \in V$. If $F_{k,\lambda}(x) \geq F_{k,\mu}(x)$ for all $x \in V$, then $\lambda \succeq \mu$.

**Proof.** As in the proof of Theorem 3.6, we show the contrapositive. Suppose $\lambda \not\succeq \mu$, and again use hyperplane separation to obtain a $y \in V$ and $C_1 > 0$ such that $\langle \mu, y \rangle > C_1$ and $\langle w(\lambda), y \rangle < C_1$ for all $w \in W$. Without loss of generality, we take $\lambda, \mu$ and $y$ to be dominant. We have the following sharp asymptotic estimate, due to Schapira [32, Theorem 3.1 and Remark 3.1]:

\[
F_{k,\lambda}(ty) \approx e^{t(\lambda - \rho(k), y)} \prod_{\alpha \in \Phi^+ \setminus \{\langle \alpha, \lambda \rangle = 0\}} (1 + t\langle \alpha, y \rangle)
\]

as $t \to +\infty$. We thus find that

\[
F_{k,\mu}(ty) - F_{k,\lambda}(ty) > e^{-t(\rho(k), y)} \left( C_2 e^{t(\mu, y)} - C_3 e^{tC_1} \prod_{\alpha \in \Phi^+ \setminus \{\langle \alpha, \mu \rangle = 0\}} (1 + t\langle \alpha, y \rangle) \right)
\]

for some constants $C_2, C_3 > 0$. For $t$ sufficiently large, the quantity on the right-hand side above is positive, which implies that $F_{k,\lambda}(x) < F_{k,\mu}(x)$ for some $x \in V$, completing the proof. □

**Remark 4.9.** As in the proof of Theorem 3.6 to complete the proof of Conjecture 4.7 it suffices to check that the function $\lambda \mapsto F_{k,\lambda}(x)$ is midpoint-convex. However, integral representations analogous to (7) for general Heckman–Opdam hypergeometric functions are not known, so we cannot directly apply the same technique. Although there are known integral expressions in certain cases where the multiplicity parameter does not correspond to a symmetric space (see e.g. [31, 34]), these are more complicated than (7) and have so far resisted a similar analysis. A more promising approach to a general proof of Conjecture 4.7 might be to use hypergeometric differential equations, as illustrated in the following proposition.

**Proposition 4.10.** When $\dim V = 1$, Conjecture 4.7 is true.

**Proof.** In light of Proposition 4.8, we need only show that (i) implies (ii) in Conjecture 4.7. Following the discussion in Example 4.3, it is sufficient to consider the case $\Phi = BC_1$. It is a classical result that the Gauss hypergeometric function $F(z) = {}_2F_1(a, b; c; z)$ satisfies Euler’s hypergeometric equation:

\[
z(1-z)\frac{d^2F}{dz^2} + [c - (a + b + 1)z] \frac{dF}{dz} - abF = 0.
\]

Comparing to (17), we find:

\[
F_{k,\lambda}''(x) + \left( k_1 \coth \frac{x}{2} + 2k_2 \coth x \right) F_{k,\lambda}'(x) + \left[ \left( \frac{k_1}{2} + k_2 \right)^2 - \lambda^2 \right] F_{k,\lambda}(x) = 0,
\]

as required.
for \( \lambda, x \in \mathbb{R} \). The function \( F_{k, \lambda} \) is determined by (23) and by the initial conditions

\[
F'_{k, \lambda}(0) = F'_{k, \mu}(0) = 0, \quad F''_{k, \lambda}(0) = F''_{k, \mu}(0) = 0,
\]

which follow respectively from the fact that \( F_{k, \lambda} \) is \( W \)-invariant (i.e. even) and from the normalization of the function \( G_{k, \lambda} \) in the definition (10).

Suppose \( \lambda \geq \mu \), which in dimension one just means that \( |\lambda| \geq |\mu| \). Since the equation (23) depends only on \( \lambda^2 \) and not on the sign of \( \lambda \), we find \( F_{k,-\lambda} = F_{k,\lambda} \), so we can in fact take \( |\lambda| > |\mu| \). We will show that \( F_{k,\lambda}(x) \geq F_{k,\mu}(x) \) for all \( x \in \mathbb{R} \), with equality only at \( x = 0 \).

Since \( F_{k,\lambda} \) is even, it suffices to consider \( x \geq 0 \). From (23) and the initial conditions (24), we have:

\[
\begin{align*}
F_{k,\lambda}(0) &= F_{k,\mu}(0) = 1, \\
F'_{k,\lambda}(0) &= F'_{k,\mu}(0) = 0, \\
F''_{k,\lambda}(0) &= \lambda^2 - \left( \frac{k_1}{2} + k_2 \right)^2 > \mu^2 - \left( \frac{k_1}{2} + k_2 \right)^2 = F''_{k,\mu}(0).
\end{align*}
\]

Therefore there is some \( \varepsilon > 0 \) such that for all \( x \in (0, \varepsilon) \),

\[
F_{k,\lambda}(x) > F_{k,\mu}(x), \quad F'_{k,\lambda}(x) > F'_{k,\mu}(x), \quad F''_{k,\lambda}(x) > F''_{k,\mu}(x).
\]

Consider the set \( E = \{ x > 0 \mid F_{k,\lambda}(x) = F_{k,\mu}(x) \} \). If \( E \) is empty, then \( F_{k,\lambda}(x) > F_{k,\mu}(x) \) for all \( x > 0 \), and there is nothing to prove. Assume for the sake of contradiction that \( E \) is not empty, and let \( x_0 = \inf E \). Clearly \( x_0 > \varepsilon \). Since \( F_{k,\lambda} \) and \( F_{k,\mu} \) are continuous, we have \( F_{k,\lambda}(x_0) = F_{k,\mu}(x_0) \), and

\[
(25) \quad F_{k,\lambda}(x) > F_{k,\mu}(x) \quad \text{for all} \quad x \in (0, x_0).
\]

However, since

\[
F_{k,\lambda}(x_0) = 1 + \int_0^{x_0} F'_{k,\lambda}(\tau) \, d\tau, \quad F_{k,\mu}(x_0) = 1 + \int_0^{x_0} F'_{k,\mu}(\tau) \, d\tau,
\]

and \( F'_{k,\lambda} > F'_{k,\mu} \) on \((0, \varepsilon)\), in order to have \( F_{k,\lambda}(x_0) = F_{k,\mu}(x_0) \) there must be some \( x_1 \in (\varepsilon, x_0) \) such that \( F'_{k,\lambda}(x_1) < F'_{k,\mu}(x_1) \). By the intermediate value theorem and by (24), there must then be some \( x_2 \in (\varepsilon, x_1) \) such that the following hold:

\[
F_{k,\lambda}'(x_2) = F_{k,\mu}'(x_2), \quad F_{k,\lambda}''(x_2) < F_{k,\mu}''(x_2), \quad \text{and} \quad F_{k,\lambda}(x_2) > F_{k,\mu}(x_2).
\]

But by direct inspection of the equation (23) for \( F_{k,\lambda} \) and the corresponding equation for \( F_{k,\mu} \), it is impossible for all three of these conditions to hold at a single point, yielding a contradiction and completing the proof. \( \square \)

We next turn our attention from hypergeometric functions to the closely related family of Heckman–Opdam polynomials, which we now define. For \( \alpha \in \Phi \), write

\[
\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.
\]

The fundamental weights \( \omega_1, \ldots, \omega_r \) are defined by \( \langle \omega_i, \alpha^\vee \rangle = \delta_{ij} \), where \( \alpha_1, \ldots, \alpha_r \) are the simple roots. They span the weight lattice \( P \subset V \). The dominant integral weights are the lattice points \( P^+ \subset P \) that lie in the dominant Weyl chamber.

The Heckman–Opdam polynomials \( P_{k,\lambda} \) depend on a nonnegative multiplicity parameter \( k \) and a dominant integral weight \( \lambda \in P^+ \). They are elements of \( \mathbb{R}[P] \), the group algebra of the weight lattice, and are therefore polynomials in an abstract-algebraic sense. However, it is typical to identify \( \mathbb{R}[P] \) with the algebra spanned by the functions \( e^{(\lambda, x)} \), \( \lambda \in P \), so that as functions on \( V \) the Heckman–Opdam polynomials are actually exponential polynomials.
We write an element \( f \in \mathbb{R}[P] \) as \( f = \sum_{\lambda \in P} f_{\lambda} e^{\lambda} \), where only finitely many \( f_{\lambda} \) are nonzero, and set
\[
\bar{f} = \sum_{\lambda \in P} f_{-\lambda} e^{\lambda}.
\]
Define a bilinear form \((\cdot, \cdot)_k\) on \( \mathbb{R}[P] \) by
\[
(f, g)_k = (f \bar{g} \delta_k \bar{\delta}_k)_0,
\]
which extracts the constant term (i.e. the coefficient of \( e^0 = 1 \)) in \( f \bar{g} \delta_k \bar{\delta}_k \), where \( \delta_k \) is the function defined in (20). This bilinear form is symmetric and positive definite, and therefore defines an inner product on \( \mathbb{R}[P] \).

For \( \lambda \in P_+ \), let
\[
M_{\lambda} = \frac{|W \cdot \lambda|}{|W|} \sum_{w \in W} e^{w(\lambda)}
\]
be the monomial \( W \)-invariant (exponential) polynomial, and define \( \text{low}(\lambda) \) as the set of \( \mu \in P_+ \) such that \( \lambda - \mu \) can be written as a linear combination of positive roots with non-negative integer coefficients.

**Definition 4.11.** For \( \lambda \in P_+ \), the *Heckman–Opdam polynomial* \( P_{k,\lambda} \) is defined by
\[
P_{k,\lambda} = \sum_{\mu \in \text{low}(\lambda)} c_{\lambda\mu} M_{\mu}, \quad c_{\lambda\lambda} = 1,
\]
and by the orthogonality relations
\[
(P_{k,\lambda}, M_{\mu})_k = 0, \quad \mu \in \text{low}(\lambda), \; \mu \neq \lambda.
\]
\( \triangle \)

As \( \lambda \) ranges over \( P^+ \) with \( k \) fixed, the \( P_{k,\lambda} \) form an orthogonal \( \mathbb{R} \)-basis of the \( W \)-invariant elements \( \mathbb{R}[P]^W \).

**Example 4.12.** When all \( k_\alpha \) are 0, \( P_{0,\lambda} = M_{\lambda} \).

Let \( g \) be a compact semisimple Lie algebra with root system \( \Phi \), and identify \( V \) with a Cartan subalgebra \( t \subset g \). Let \( G \) be the connected, simply connected Lie group with \( \text{Lie}(G) = g \). When \( k = \tilde{1} \), \( P_{1,\lambda}(ix) = \chi_{\lambda}(e^x) \) is the character of the irreducible \( G \)-representation with highest weight \( \lambda \).

When \( \Phi = A_{N-1} \), if we identify \( M_{\lambda} \) with the monomial symmetric polynomial
\[
m_{\lambda} = \frac{|S_N \cdot \lambda|}{N!} \sum_{\sigma \in S_N} \prod_{i=1}^N x_i^{\lambda_{\sigma(i)}},
\]
then Heckman–Opdam polynomials are Jack polynomials. In particular, for \( k = \tilde{1} \), they are Schur polynomials. In light of Proposition 4.13 below, Conjecture 4.7 therefore subsumes both the Schur polynomial inequality of [6, 33] and the classical Muirhead inequality for monomial symmetric polynomials [26]. \( \triangle \)

Up to a normalizing factor, the Heckman–Opdam polynomials turn out to be specializations of the Heckman–Opdam hypergeometric function. In particular, for \( \lambda \in V \), define
\[
\hat{c}(\lambda, k) = \prod_{\alpha \in \Phi^+} \frac{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2} k_\alpha)}{\Gamma((\lambda, \alpha^\vee) + \frac{1}{2} k_\alpha + k_\alpha)^{1/2}},
\]
where \( k_{\pm \alpha} = 0 \) if \( \frac{1}{2} \alpha \not\in \Phi \). Observe that if \( \Phi \) is the root system of a compact Lie algebra \( \mathfrak{g} \), we have \( \tilde{c}(\lambda, \overline{1}) = \Delta_{\mathfrak{g}}(\lambda)^{-1} \). Set

\[
c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}.
\]

We then have the following relation between Heckman–Opdam polynomials and hypergeometric functions [16, eq. 4.4.10]:

\[
F_{k, \lambda + \rho(k)}(x) = c(\lambda + \rho(k), k) P_{k, \lambda}(x), \quad x \in V.
\]

This relation generalizes the relation between the spherical functions \( \phi_{-i(\lambda - \rho)} \) and \( \phi_{+} \) discussed in Section 3.3. It immediately yields the following proposition.

**Proposition 4.13.** Let \( \lambda, \mu \in \mathbb{P}^+ \). If Conjecture 4.7 holds, then the following are equivalent:

(i) \( \lambda \preceq \mu \),

(ii) \( \tilde{c}(\lambda + \rho(k), k) P_{k, \lambda}(x) \geq \tilde{c}(\mu + \rho(k), k) P_{k, \mu}(x) \) for all \( x \in V \).

In other words, Conjecture 4.7 would imply a generalization of Proposition 3.8 to the case of Heckman–Opdam polynomials.

It is well known that Heckman–Opdam polynomials can be realized as a limit of Macdonald polynomials [22]. It is interesting to speculate about whether Conjecture 4.7, if it holds, is itself a manifestation of an even more general monotonicity property of Macdonald polynomials with respect to \( W \)-majorization.

**Acknowledgements**

Colin McSwiggen would like to thank Patrick McSwiggen for helpful discussions during the preparation of this manuscript. The work of Colin McSwiggen is partially supported by the National Science Foundation under Grant No. DMS 1714187.

The work of Jonathan Novak is partially supported by a Lattimer Fellowship, as well as by the Natural Science Foundation under Grant No. DMS 1812288.

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