Liouville Theory: Quantum Geometry of Riemann Surfaces

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Abstract

Inspired by Polyakov’s original formulation \cite{1, 2} of quantum Liouville theory through functional integral, we analyze perturbation expansion around a classical solution. We show the validity of conformal Ward identities for puncture operators and prove that their conformal dimension is given by the classical expression. We also prove that total quantum correction to the central charge of Liouville theory is given by one-loop contribution, which is equal to 1. Applied to the bosonic string, this result ensures the vanishing of total conformal anomaly along the lines different from those presented by KPZ \cite{3} and Distler-Kawai \cite{4}.

According to Polyakov \cite{1, 2}, basic properties of quantum Liouville theory can be read from the correlation function of puncture operators, which is depicted by the following functional integral

\[
< X > = \int_{\mathcal{C}(X)} D\phi \ e^{-(1/2\pi h)S(\phi)}.
\]

Here \( X \) is an \( n \)-punctured sphere, i.e. a Riemann sphere \( \hat{\mathbb{C}} \) with \( n \) removed distinct points, called punctures; \( \mathcal{C}(X) \) is “domain of integration”, consisting of all smooth conformal metrics \( ds^2 = e^{\phi(w, \bar{w})}|dw|^2 \) on \( X \) satisfying asymptotics

\[
e^{\phi} \approx \frac{1}{r_i^2 \log^2 r_i}, \quad i = 1, \ldots, n.
\]
\[ r_i = |w - w_i|, \ i = 1, \ldots, n - 1, \text{ and } r = |w| \text{ for } i = n, \text{ near the punctures } w_1, \ldots, w_{n-1}, w_n = \infty, \] where \( w \) is the global complex coordinate on \( X \); functional \( S(\phi) \) is the Liouville action, and positive \( h \) is a coupling constant. Because of asymptotics (2), naive form of the Liouville action is ill-defined. Its proper definition is given by the following regularization of the naive action [5]

\[ S(\phi) = \lim_{\epsilon \to 0} \left\{ \int_{X_\epsilon} (|\phi_w|^2 + e^\phi) d^2 w + 2\pi n \log \epsilon + 4\pi(n - 2) \log |\log \epsilon| \right\}, \]

where \( X_\epsilon = X \setminus \bigcup_{i=1}^{n-1} \{|w - w_i| < \epsilon\} \cup \{|w| > 1/\epsilon\} \). Classical equations of motion \( \delta S = 0 \) yield Liouville equation, the equation for complete conformal metric on \( X \) of constant negative curvature \(-1\). It has a unique solution, called Poincaré metric, and is denoted by \( \phi_{cl} \).

We define functional integral \( \langle X \rangle \) by its perturbation expansion around the classical solution \( \phi = \phi_{cl} \) (cf. [6], where \( \phi_{cl} \) corresponds to the standard metric of positive constant curvature on Riemann sphere). It can be obtained from (1) using the “integration measure” \( \mathcal{D}\phi \) (which is not translation-invariant!), defined by the norm

\[ ||\delta\phi||^2 = \int_X |\delta\phi|^2 e^\phi d^2 w. \]

(cf. [4]). Corresponding propagator \( G(w, w') \) is given by the Green’s function of the operator \( 2\Delta + 1 \), where

\[ \Delta = e^{-\phi_{cl}} \partial^2_{w\bar{w}} \]

is the hyperbolic Laplacian on \( X \). Its logarithmic divergence at coincident points is renormalized in a reparametrization invariant way using the geodesic distance in the Poincaré metric

\[ G(w, w) = \lim_{w' \to w} \left\{ G(w, w') + \frac{1}{2\pi} (\log |w - w'|^2 + \phi_{cl}(w)) \right\}. \]

In case of three punctures \( w_1, w_2, w_3 = \infty \), it is easy to see that

\[ \langle X \rangle = \frac{c}{|w_1 - w_2|^{1/h}}, \] (3)

where \( c \) stands for the value of normalized thrice-punctured sphere with punctures at 0, 1, \( \infty \). Formula (3) supports the interpretation of \( \langle X \rangle \) as a correlation function of puncture operators \( e^{\phi/2h} \) with conformal dimensions \( \Delta = \bar{\Delta} = 1/2h \). Note that the puncture at \( \infty \) plays the role of global charge in accordance with Gauss-Bonnet theorem. From now on we will assume that Riemann surface \( X \) is normalized, i.e. \( X = \mathbb{C} \setminus \{w_1, \ldots, w_{n-3}, 0, 1\} \).
Conformal invariance of Liouville theory implies that its stress-energy tensor is traceless. Namely, its (2, 0)-component $T(\phi)(w)$ is given by

$$T(\phi) = \frac{1}{h} (\phi_{ww} - \frac{1}{2} \phi^2 w),$$  \hspace{1cm} (4)$$

and is conserved on classical equations of motion. It has the following transformation law under holomorphic change of coordinates $w = f(\tilde{w})$

$$\tilde{T}(\tilde{w}) = T(f(\tilde{w}))(f'(\tilde{w}))^2 + \frac{1}{h} S(f(\tilde{w})),
$$

where $S$ stands for the Schwartzian derivative of function $f$

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$

Geometrically, $T$ has the meaning of a projective connection (times $1/h$).

We emphasize that the modification of Liouville’s stress-energy tensor, i.e. addition of a total derivative to the “free-field” term in (4), is the crucial feature of the theory. This form of $T(\phi)$ can be obtained through variation of the generalized Liouville action with respect to the fiducial metric (see, e.g., [7]).

The expectation value of the stress-energy tensor is defined as follows

$$< T(w)X > = \int_{\mathcal{C}(X)} D\phi \, T(\phi)(w) \, e^{(-1/2\pi h)S(\phi)}.$$

Introducing its normalized value,

$$<< T(w)X >> = < T(w)X > / < X >,$$

one should expect the validity of conformal Ward identities, which we write in the form

$$<< T(w)X >> = \sum_{i=1}^{n-1} \frac{\Delta(h)}{(w-w_i)^2} + \sum_{i=1}^{n-3} \left( \frac{1}{w-w_i} + \frac{w_i-1}{w} - \frac{w_i}{w-1} \right) \frac{\partial}{\partial w_i} \log < X >,$$$$

(5)$$

where overall $SL(2, \mathbb{C})$ symmetry is already fixed.

According to Belavin-Polyakov-Zamolodchikov [8], conformal Ward identities (5) constitute the main axiom of the conformal field theory in two dimensions and, in principle, should lead to its solution. Since in our formulation correlation functions $< X >$ and $<< T(w)X >>$ were already defined by functional integrals, one needs to prove the validity of (5) and to calculate conformal dimension $\Delta(h)$.  

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It turns out that conformal Ward identities are non-trivial even at the tree level. Indeed, one has

\[ \langle\langle T(w)X\rangle\rangle_{cl} = T(\phi_{cl})(w) = T_{cl}(w) = \frac{1}{\hbar} \mathcal{S}(J^{-1})(w) \]

\[ = \sum_{i=0}^{n-1} \frac{1}{2\hbar(w-w_i)^2} + \frac{1}{\hbar} \sum_{i=0}^{n-3} \frac{1}{w-w_i} + \frac{w_i-1}{w} - \frac{w_i}{w-1} c_i \]

and

\[ \langle X \rangle_{cl} = e^{-1/(2\pi\hbar)} \mathcal{S}_{cl}. \]

Here \( J^{-1} \) is the inverse of the uniformization map \( J \), which maps the hyperbolic plane \( H \) onto the Riemann surface \( X \) so that \( X \simeq H/\Gamma \), where \( \Gamma \) is the Fuchsian group uniformizing \( X \), \( c_i \) are the so-called accessory parameters of the Fuchsian uniformization (introduced by Poincaré), and \( S_{cl} = S(\phi_{cl}) \) is the classical Liouville action. Therefore, conformal Ward identity at the tree level should imply that

\[ \Delta_{cl} = \frac{1}{2\hbar} \quad \text{and} \quad c_i = -\frac{1}{2\pi} \frac{\partial S}{\partial w_i}. \]

The latter formula, conjectured by Polyakov [2], is quite a non-trivial statement on accessory parameters (unknown to Poincaré!), which was proved in [3] (see also [4] for additional discussion) using Teichmüller theory. Note that arguments given above confirm our definition of the Liouville action and the arrangement of \( \pi \)'s and \( h \)'s elsewhere in the formulas.

Moreover, using the principle that “central charge of the theory equals 12 times the number of Schwarzians in transformation law of the stress-energy tensor \( \langle\langle T(w)\rangle\rangle \)” [8], we see that

\[ c_{cl} = \frac{12}{\hbar} = 24 \Delta_{cl}. \]

Also note that, according to [10],

\[ S_{cl} = 2\pi \log |w_i - w_j| + O(1), \]

as \( w_i \to w_j, \ j = 1, \ldots, n - 1 \), and

\[ S_{cl} = 2\pi \log |w| + O(1), \]

as \( w \to \infty \), so that \( \langle X \rangle_{cl} \) indeed has singularities of a correlation function of primary fields of conformal dimension \( \Delta = \bar{\Delta} = \Delta_{cl} = 1/2\hbar \).
It is also possible to perform one-loop calculations exactly. In doing so one should use reparametrization invariant regularization of propagator \(G(w, w')\) at coincident points, a similar regularization scheme for its partial derivatives (using short-distance behavior of Green's functions established by Hadamard [11]), and the definition of determinant of operator \((2\Delta + 1)\) through Selberg zeta function of a Riemann surface \(X\). As a result we obtain that [12]

\[
\langle \langle T(w) \rangle \rangle = \frac{1}{\hbar} S(J^{-1})(w) + T_1(w) + O(\hbar),
\]

where

\[
T_1(w) = -\pi \delta_{ww'} G(w, w') \bigg|_{w' = w} - \pi \int_X (G_{ww}(w, w') - \phi_w(w) G_w(w, w')) G(w', w') e^{\phi(w')} d^2 w'.
\]

Using regularization scheme described above one can prove the following properties.

(1) One-loop contribution \(T_1(w)\) is holomorphic on \(X\). It follows from the definition of Green's function and regularization scheme; the second term in expression for \(T_1\), which is the artifact of the modification of the stress-energy tensor (the first term in (4)), is crucial for this property.

(2) Quantity \(T_1(w)\) has a transformation law of \(1/12\) times projective connection. This follows from the analysis of the first term for \(T_1\), based on the formula

\[
\lim_{w' \to w} \left( \frac{f'(w) f'(w')}{(f(w) - f(w'))^2} - \frac{1}{(w - w')^2} \right) = \frac{1}{6} S(f)(w).
\]

The same property is inherent in the free theory and is related with the second term in (4).

(3) The difference

\[
Q_1(w) = T_1(w) - \frac{\hbar}{12} T_{cl}(w),
\]

when lifted to hyperbolic plane \(H\) via map \(J^{-1}\), defines the holomorphic \(\Gamma\)-automorphic form of weight 4 with constant terms \(\pi^2/12\) at every cusp.

(4) Conformal Ward identities are valid in the one loop approximation with

\[
\Delta_{\text{loop}} = 0.
\]

This follows from the property (3), formula

\[
\log < X > = -\frac{1}{2\pi \hbar} S_{cl} - \frac{1}{2} \log \det(2\Delta + 1) + O(\hbar),
\]
and explicit expression for the first derivative of Selberg zeta function with respect to moduli parameters \([13]\).

Using property (2) and the “exchange rate” central charge/12 times Schwartzians, we obtain

\[
c_{\text{loop}} = 12 \times \frac{1}{12} = 1,
\]

and, therefore,

\[
c_{\text{Liouv}} = c_{\text{cl}} + c_{\text{loop}} = \frac{12}{h} + 1, \quad \Delta = \Delta_{\text{cl}} = \frac{1}{2h}.
\]

**4** It is remarkable that simple expression

\[
c_{\text{Liouv}} = 1 + \frac{12}{h}
\]

for the central charge of the theory is exact. Indeed, one can prove this theorem analyzing every term in perturbation expansion of \(<< T(w)X >>\). Although it looks like almost an impossible task, we observe that Schwartzians in higher loops can only appear through contribution of the second term in definition (4) of the stress-energy tensor. This is the term which corresponds to the free theory and it is easy to keep its track in all orders of perturbation theory, which completes the proof.

It should be noted that this formula for the central charge was conjectured in \([14]\) on quantum group symmetry of the Liouville model.

Our results look different from traditional ones (cf. \([3, 4, 16]\)). We note that, contrary to the approach in \([4]\), we do not make any a priori assumptions about the change of the “integration measure” in functional integral. Instead, we systematically use properties of the classical solution—Poincaré metric—and unambiguously define our main quantities—correlation functions \(<X>\) and \(<< T(w)X >>\)—through perturbation expansion. However, we do not know yet how to treat these basic objects of “quantum geometry of Riemann surfaces” non-perturbatively. We also observe that whereas in approaches of KPZ and Distler-Kawai, one always has \(c < 1\), in our approach, it is always \(c > 1\). This might suggest that we are actually dealing with two different phases of \(2-D\) quantum gravity, separated by the “\(c = 1\) barrier” (cf. discussion in \([15]\)).

Details of calculations, generalization to the higher genus case, and precise formulation of auxiliary mathematical results we have used, will be presented elsewhere.
Here, as a final speculation, we point out that if
\[
\frac{1}{2\pi h} = \frac{25 - D}{24\pi},
\]
as one should expect from the bosonic string in \(D\) dimensions, then
\[
c_{\text{Liou}} = 25 - D + 1 = 26 - D.
\]
This is the expression one really needs: it cancels the contribution from string modes and ghosts and ensures the vanishing of total conformal anomaly for any \(D\).

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