RANDOM DATA THEORY FOR THE CUBIC FOURTH-ORDER NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the cubic nonlinear fourth-order Schrödinger equation
\[ i\partial_t u - \Delta^2 u + \mu \Delta u = \pm |u|^2 u, \quad \mu \geq 0 \]
on \(\mathbb{R}^N, N \geq 5\) with random initial data. We prove almost sure local well-posedness below the scaling critical regularity. We also prove probabilistic small data global well-posedness and scattering. Finally, we prove the global well-posedness and scattering with a large probability for initial data randomized on dilated cubes.

1. Introduction.

1.1. Introduction. We consider the Cauchy problem for the cubic fourth-order nonlinear Schrödinger equation
\[
\begin{cases}
    i\partial_t u - \Delta^2 u + \mu \Delta u = \pm |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
    u|_{t=0} = u_0,
\end{cases}
\]
(1.1)
where \(u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, u_0 : \mathbb{R}^N \to \mathbb{C}\), and \(\mu \geq 0\). The plus (resp. minus) sign in front of the nonlinearity corresponds to the defocusing (resp. focusing) case. The fourth-order Schrödinger equation has been introduced by Karpman [21] and Karpman-Shagalov [22] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity.

It is well-known that the equation (1.1) with \(\mu = 0\) enjoys the scaling invariance
\[
u_\lambda(t, x) := \lambda^2 u(\lambda^4 t, \lambda x), \quad \lambda > 0.
\]
(1.2)
A direct computation shows
\[
\|u_\lambda(0)\|_{\dot{H}^{\gamma}} = \lambda^{\gamma + 2 - \frac{N}{2}} \|u_0\|_{\dot{H}^{\gamma}}.
\]
We thus define the critical exponent
\[
\gamma_c := \frac{N - 4}{2}.
\]
(1.3)

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For initial data $u_0 \in H^\gamma(\mathbb{R}^N)$, we say that the Cauchy problem (1.1) is subcritical, critical or supercritical if $\gamma > \gamma_c$, $\gamma = \gamma_c$ or $\gamma < \gamma_c$ respectively.

1.2. Known results. In the last decade, the fourth-order Schrödinger equation has attracted a lot of interest in mathematics, numerics and physics. Let us recall some known results related to (1.1) in both deterministic and probabilistic settings.

(1) Deterministic setting. Artzi-Koch-Saut [1] established sharp dispersive estimates for the fourth-order Schrödinger operator. Thanks to these dispersive estimates, Pausader [31] showed the local well-posedness for fourth-order nonlinear Schrödinger equations in the (sub)critical cases. Pausader [31, 32, 33] and Miao-Xu-Zhao [27, 28] investigated the asymptotic behavior of global $H^2$-solutions in the energy-critical case. In the mass and energy intercritical case, the energy scattering for the defocusing problem was shown by Pausader [31] in dimensions $N \geq 5$ and Pausader-Xia [35] in dimensions $1 \leq N \leq 4$. The energy scattering for the focusing problem was studied by Guo [16] and author [14]. In the mass-critical case, the asymptotic behavior of global $L^2$-solutions was proved by Pausader-Shao [34] in dimensions $N \geq 5$. The asymptotic behavior of global solutions below the energy space was studied by Miao-Wu-Zhang [29] and author [12]. In [5], Boulenger-Lenzmann established the existence of finite time blow-up $H^2$-solutions for the focusing problem. Dynamical properties such as mass-concentration and limiting profile of blow-up $H^2$-solutions were studied by Zhu-Yang-Zhang [37] and author [13].

(2) Probabilistic setting. In the supercritical case, it was shown in [31, 11] that (1.1) is ill-posed in the sense that the solution map fails to be continuous at 0. Recently, the probabilistic techniques have been exploited to show almost sure local well-posedness for nonlinear dispersive equations below the critical regularity threshold. This approach was initiated by Bourgain [6]. More precisely, he considered random initial data of the form

$$f^\omega(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1 + |n|^2}} e^{inx},$$

where $(g_n)_{n \in \mathbb{Z}^2}$ is a sequence of independent standard complex-valued Gaussian random variables. By combining deterministic PDE techniques and probabilistic arguments, he showed that the (Wick ordered) cubic NLS on $\mathbb{T}^2$ with initial data $f^\omega$ is almost sure well-posed. Later, Burq-Tzvetkov [7] considered a more general class of random initial data on compact Riemannian manifolds $M$ of the form

$$f^\omega(x) = \sum_{n=1}^{\infty} g_n(\omega) c_n(x), \quad c_n = \langle u, e_n \rangle_{L^2(M)} = \int_M u(x) e_n(x) d\text{vol}(x),$$

where $(c_n)_{n \geq 1}$ is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of the Laplace-Beltrami operator, and $(g_n)_{n \geq 1}$ is a sequence of independent mean-zero random variables with uniform bound on the fourth moments. They proved the almost sure local well-posedness for the cubic nonlinear wave equation on the three-dimensional compact manifolds. After, Burq-Thomann-Tzvetkov [8] and Deng [10] showed the almost sure well-posedness for nonlinear Schrödinger equation with harmonic potential in dimensions 1 and 2 respectively. Recently, Bényi-Oh-Pocovnicu [3] and Lührmann-Mendelson [26] independently introduced randomizations on $\mathbb{R}^N$. As consequences, the almost sure well-posedness for cubic nonlinear Schrödinger equation was shown in [3], and the almost sure well-posedness for energy subcritical nonlinear wave equations was established in [26]. There are several works on random Cauchy theory followed these results (see e.g. [19, 2, 30, 4, 24, 15]).
Concerning the random data Cauchy problem for fourth-order nonlinear Schrödinger equations, we mention recent works of Hirayama-Okamoto [20], Chen-Zhang [9], and Zhang-Xu [36]. Motivated by aforementioned results, in this paper, we study the random data Cauchy problem for (1.1). Before stating our results, let us recall the definition of Wiener randomization on \(\mathbb{R}^N\) due to [3]. Let \(\psi \in C_0^\infty(\mathbb{R}^N)\) be such that \(0 \leq \psi \leq 1\), \(\text{supp}(\psi) \subset [-1, 1]^N\) and

\[
\sum_{n \in \mathbb{Z}^N} \psi(\xi - n) = 1, \quad \forall \xi \in \mathbb{R}^N.
\]

(1.4)

Given a function \(f\) on \(\mathbb{R}^N\), we have

\[
f(x) = \sum_{n \in \mathbb{Z}^N} \psi(D-n)f(x),
\]

where

\[
\psi(D-n)f(x) = (2\pi)^{-N} \int e^{ix \cdot \xi} \psi(\xi - n) \hat{f}(\xi) d\xi.
\]

The Wiener randomization of \(f\) on \(\mathbb{R}^N\) is defined by

\[
f_\omega(x) = \sum_{n \in \mathbb{Z}^N} g_n(\omega) \psi(D-n)f(x),
\]

(1.5)

where \((g_n)_{n \in \mathbb{Z}^N}\) is a sequence of independent mean-zero complex-valued random variables on a probability space \((\Omega, \mathcal{F}, P)\), where the real and imaginary parts of \(g_n\) are independent and endowed with probability distributions \(\mu_n^{(1)}\) and \(\mu_n^{(2)}\).

In the sequel, we make the following assumption: there exists \(c > 0\) such that

\[
\left| \int e^{\delta x} d\mu_n^{(j)}(x) \right| \leq e^{c\delta^2}
\]

for all \(\delta \in \mathbb{R}, n \in \mathbb{Z}^N\) and \(j = 1, 2\). Note that (1.6) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

1.3. Main results. Denote \(U_\mu(t) = e^{-it(\Delta^2 - \mu \Delta)}\) the Schrödinger operator associated to (1.1) and define

\[
\gamma_N := \max \left\{ \frac{(N-1)(N-4)}{2(N+5)}, \frac{N-4}{4} \right\}.
\]

(1.7)

Our first result is the following almost sure local well-posedness.

**Theorem 1.1** (Almost sure local well-posedness). Let \(N \geq 5, \mu \geq 0\) and \(\gamma \in (\gamma_N, \gamma_c)\). Let \(f \in H^\gamma(\mathbb{R}^N)\), and \(f_\omega\) be the Wiener randomization defined in (1.5) satisfying (1.6). Then the equation (1.1) is almost surely locally well-posed with respect to the randomization data \(f_\omega\). More precisely, there exist \(C, c, \theta > 0\) such that for each \(0 < T \ll 1\), there exists a set \(\Omega_T \subset \Omega\) with the following properties:

- \(\mathcal{P}(\Omega \backslash \Omega_T) \leq C \exp \left( -cT^{-\theta} \|f\|_{H^\gamma(\mathbb{R}^N)}^{-2} \right)\).
- For each \(\omega \in \Omega_T\), there exists a unique solution \(u\) to (1.1) with the initial data \(f_\omega\) in the class \(U_\mu(t)f_\omega + C([0, T], H^\gamma(\mathbb{R}^N)) \subset C([0, T], H^\gamma(\mathbb{R}^N))\).
We will prove Theorem 1.1 by considering the equation satisfied by the nonlinear part of $u$. More precisely, let

$$z(t) = z^\omega(t) := U_\mu(t)f^\omega, \quad v(t) := u(t) - U_\mu(t)f^\omega. \quad (1.8)$$

Then the equation (1.1) with initial data $f$ is reduced to

$$\begin{cases}
  i\partial_t v - \Delta^2 v + \mu \Delta v = \pm |v + z|^2(v + z), \\
v|_{t=0} = 0.
\end{cases} \quad (1.9)$$

We will prove the Cauchy problem (1.9) is almost sure locally well-posed by viewing $z$ as a random forcing term. This is done by using variants of the Bourgain $X^{\gamma,b}$-spaces adapted to the $U^p$- and $V^p$-spaces introduced by Tataru, Koch and their collaborators [17, 18, 25]. Since we are considering algebraic nonlinearity, we use the Littlewood-Paley decomposition to decompose the nonlinearity into dyadic pieces, and then carefully perform the case-by-case analysis. We refer the reader to Sections 3 and 4 for more details.

The next result is the probabilistic small data global well-posedness and scattering.

**Theorem 1.2** (Probabilistic small data global well-posedness and scattering). Let $N \geq 5$, $\mu \geq 0$, and $\gamma \in (\gamma_N, \gamma_c)$. Let $f \in H^\gamma(\mathbb{R}^N)$ and $f^\omega$ be the Wiener randomization defined in (1.5) satisfying (1.6). Then there exist $C, c > 0$ such that for each $0 < \varepsilon \ll 1$, there exists a set $\Omega_\varepsilon \subset \Omega$ with the following properties:

- $\mathcal{P}(\Omega \setminus \Omega_\varepsilon) \leq C \exp(-c\varepsilon^{-2}\|f\|_{H^\gamma(\mathbb{R}^N)}^{-2}) \to 0$ as $\varepsilon \to 0$.
- For each $\omega \in \Omega_\varepsilon$, there exists a unique global in time solution $u$ to (1.1) with initial data $\varepsilon f^\omega$ in the class
  \[ \varepsilon U_\mu(t)f^\omega + C\left(\mathbb{R}, H^{2\gamma}(\mathbb{R}^N)\right) \subset C\left(\mathbb{R}, H^\gamma(\mathbb{R}^N)\right). \]

- For each $\omega \in \Omega_\varepsilon$, there exists $f^\omega_+ \in H^{\gamma_c}(\mathbb{R}^N)$ such that
  \[ \left\| u(t) - \varepsilon U_\mu(t)f^\omega - U_\mu(t)f^\omega_+ \right\|_{H^{\gamma_c}(\mathbb{R}^N)} \to 0 \]
  as $t \to \infty$. A similar statement holds for $t \to -\infty$.

**Remark 1.** In [9], Chen-Zhang considered the Cauchy problem of the fourth-order nonlinear Schrödinger equation of the form

$$i\partial_t u + \mu \Delta u + \Delta^2 u = P_m\left(\left(\partial_x^\alpha u\right)_{|\alpha| \leq 2}, \left(\partial_x^\alpha \overline{u}\right)_{|\alpha| \leq 2}\right),$$

where $\mu \in \mathbb{R}$, $P_m$ is a homogeneous polynomial of degree $m \geq 3$ containing the second order derivative. More precisely, $P_m$ is of the form

$$c_1 \prod_{k=1}^m u_k + c_2 \sum_{i=1}^N \partial_i u_1 \prod_{k=2}^m u_k + c_3 \sum_{i,j=1}^N \partial_i u_1 \partial_j u_2 \prod_{k=3}^m u_k + c_4 \sum_{i,j=1}^N \partial_i^2 u_1 ^m \prod_{k=2}^m u_k,$$

where $u_k = u$ or $\overline{u}$ for $k = 1, \cdots, m$. They established almost sure local well-posedness and probabilistic small data global existence and scattering for random
initial data in $H^\gamma$ with $\gamma \in (\beta_{N,m}, \beta_{c,m}]$, where

$$\beta_{c,m} := \frac{N}{2} - \frac{2}{m-1}, \beta_{N,m} := \begin{cases} \beta_{c,m} - \frac{1}{2} + \frac{m-2}{3m-4} & \text{if } N = 2, m \geq 4, \\ \beta_{c,m} - \frac{1}{2} + \frac{5-m}{2(N-1)(m-1)} & \text{if } N \geq 3, 3 \leq m < 5, \\ \beta_{c,m} & \text{if } N \geq 3, m \geq 5. \end{cases}$$

In particular, when $m = 3$, we have

$$\beta_{c} := \beta_{N,3} = \frac{N}{2} - \frac{2}{3}, \beta_{N} := \beta_{c,3} = \frac{(N-2)^2}{2(N-1)}.$$ 

It is easy to see that $\beta_{N} > \gamma_{c}$. Thus, the result in [9] does not apply to show almost sure well-posedness for (1.1) below the critical regularity threshold.

Finally, we have the almost sure global well-posedness and scattering with a large probability. This is done by considering the randomization based on a partition of the frequency space by dilated cubes. More precisely, given $\lambda > 0$, we define

$$\psi_{\lambda}(\xi) := \psi(\lambda \xi),$$

where $\psi$ is as in (1.4). We can write a function $f$ on $\mathbb{R}^N$ as

$$f(x) = \sum_{n \in \mathbb{Z}^N} \psi_{\lambda} \left( D - \lambda^{-1} n \right) f(x).$$

We now introduce the randomization $f^{\omega,\lambda}$ of $f$ on dilated cubes of scale $\lambda$ by

$$f^{\omega,\lambda}(x) := \sum_{n \in \mathbb{Z}^N} g_n(\omega) \psi_{\lambda} \left( D - \lambda^{-1} n \right) f(x), \quad (1.10)$$

where $(g_n)_{n \in \mathbb{Z}^N}$ is a sequence of independent mean-zero complex-valued random variables satisfying (1.6). We have the following global well-posedness and scattering with a large probability.

**Theorem 1.3** (Large probability global well-posedness and scattering). Let $N \geq 5$, $\mu = 0$, and $\gamma \in (\gamma_N, \gamma_c)$. Let $f \in H^\gamma(\mathbb{R}^N)$ and $f^{\omega,\lambda}$ be the Wiener randomization on dilated cubes of scale $\lambda \gg 1$ defined in (1.10). Then the equation (1.1) is globally well-posed with a large probability. More precisely, for each $0 < \varepsilon < 1$, there exists a large dilation scale $\lambda_0 = \lambda_0(\varepsilon, \|f\|_{H^\gamma}) > 0$ such that for each $\lambda > \lambda_0$, there exists a set $\Omega_\lambda \subset \Omega$ with the following properties:

- $P(\Omega \setminus \Omega_\lambda) < \varepsilon$.
- For each $\omega \in \Omega_\lambda$, there exists a unique global-in-time solution to (1.1) with initial data $f^{\omega,\lambda}$ in the class

$$U_0(t) f^{\omega,\lambda} + C \left( \mathbb{R}, H^{\gamma_c}(\mathbb{R}^N) \right) \subset C \left( \mathbb{R}, H^{\gamma}(\mathbb{R}^N) \right).$$

- For each $\omega \in \Omega_\lambda$, there exists $f^+_\omega \in H^{\gamma_c}(\mathbb{R}^N)$ such that

$$\left\| u(t) - U_0(t) f^{\omega,\lambda} - U_0(t) f^+_\omega \right\|_{H^{\gamma}(\mathbb{R}^N)} \to 0$$

as $t \to \infty$. A similar statement holds for $t \to -\infty$. 

This paper is organized as follows. In Section 2, we give some preliminaries needed in the sequel including some basic properties of the Wiener randomization, probabilistic Strichartz estimates and function spaces. In Section 3, we prove probabilistic nonlinear estimates which are key ingredients of the proof. In Section 4, we prove the almost sure well-posedness given in Theorems 1.1, 1.2 and 1.3.

2. Preliminaries.

2.1. Notations. Let $1 \leq r \leq \infty$ and $\gamma \in \mathbb{R}$. We denote the Lebesgue space and Sobolev space by $L^r(\mathbb{R}^N)$ and $H^\gamma(\mathbb{R}^N)$ respectively. The notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq C B$. Similarly, $A \gtrsim B$ means $A \geq c B$ for some constant $c > 0$. We also use $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$. Let $I \subset \mathbb{R}$ and $1 \leq q, r \leq \infty$. We define the mixed norm

$$
\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^N)} := \left( \int_I \left( \int_{\mathbb{R}^N} |u(t,x)|^r \, dx \right) \frac{1}{t^{\frac{q-r}{q}}} \, dt \right)^{\frac{1}{q}}
$$

with a usual modification when either $q$ or $r$ are infinity. When $q = r$, we use the notation $L^q_t L^q_x(I \times \mathbb{R}^N)$ instead of $L^q_t L^q_x(I \times \mathbb{R}^N)$. Let $\varphi$ be a smooth real-valued radial function on $\mathbb{R}^N$ satisfying

$$
\varphi(\xi) = \begin{cases} 
1 & \text{if } |\xi| \leq 1, \\
0 & \text{if } |\xi| \geq 2.
\end{cases}
$$

We define the Littlewood-Paley operators

$$
P_M f(\xi) = \varphi(\xi) \hat{f}(\xi)
$$

and for dyadic numbers $M = 2^m, m \geq 1$,

$$
P_M f(\xi) = (\varphi(2^{-1} \xi) - \varphi(2M^{-1} \xi)) \hat{f}(\xi).
$$

We also define

$$
P_{\leq M_1} = \sum_{1 \leq M \leq M_1} P_M, \quad P_{\geq M_1} = \sum_{M \geq M_1} P_M.
$$

2.2. Wiener randomization and Probabilistic Strichartz estimates. In this subsection, we first recall some basic properties of the Wiener randomization and probabilistic Strichartz estimates related to the fourth-order Schrödinger equation.

The first property of the Wiener randomization is that it preserves the differentiability in the sense: if $f \in H^\gamma(\mathbb{R}^N)$, then $f^\omega \in H^\gamma(\mathbb{R}^N)$ almost surely.

**Lemma 2.1** ([2, Lemma 3]). Let $f \in H^\gamma(\mathbb{R}^N)$ and $f^\omega$ be the Wiener randomization defined in (1.5) satisfying (1.6). Then it holds that

$$
\mathcal{P} \left( \|f^\omega\|_{H^\gamma(\mathbb{R}^N)} > \lambda \right) \leq C \exp \left( -c \lambda^2 \|f\|_{H^\gamma(\mathbb{R}^N)}^{-2} \right)
$$

for all $\lambda > 0$. In particular, $f^\omega \in H^\gamma(\mathbb{R}^N)$ almost surely.

**Remark 2.** It was shown in [7, Appendix] that the Wiener randomization does not gain differentiability in the sense: if $f \in H^\gamma(\mathbb{R}^N) \setminus H^{\gamma+\varepsilon}(\mathbb{R}^N)$, then $f^\omega \in H^\gamma(\mathbb{R}^N) \setminus H^{\gamma+\varepsilon}(\mathbb{R}^N)$ almost surely.

The second property of the Wiener randomization is that it gains the integrability in the sense: if $f \in L^2(\mathbb{R}^N)$, then $f^\omega \in L^p(\mathbb{R}^N)$ for all $2 \leq p < \infty$ almost surely.
Lemma 2.2 ([2, Lemma 4]). Let \( f \in L^2(\mathbb{R}^N) \) and \( f^\omega \) be the Wiener randomization defined in (1.5) satisfying (1.6). Then it holds that
\[
P \left( \| f^\omega \|_{L^p(\mathbb{R}^N)} > \lambda \right) \leq C \exp \left( -c \lambda^2 \| f \|_{L^2(\mathbb{R}^N)}^{-2} \right)
\]
for all \( p \in [2, \infty) \) and all \( \lambda > 0 \). In particular, \( f^\omega \in L^p(\mathbb{R}^N), p \in [2, \infty) \) almost surely.

Remark 3. Comparing to the Sobolev embedding \( H^N \hookrightarrow L^p(\mathbb{R}^N) \) for all \( p \in [2, \infty) \), the Wiener randomization makes a gain of \( N^2 \) derivatives.

The Wiener randomization also allows us establish some improvements of Strichartz estimates which are essential tools to the almost sure well-posedness of (1.1). Let us recall Strichartz estimates for (1.1) on \( \mathbb{R}^N \). A pair \((q,r)\) is Biharmonic admissible, or \((q,r) \in B\) for short, if
\[
\frac{4}{q} + \frac{N}{r} = \frac{N}{2}, \quad \begin{cases} r \in \left[2, \frac{2N}{N-4}\right] & \text{if } N \geq 5, \\ r \in [2, \infty) & \text{if } N = 4, \\ r \in [2, \infty) & \text{if } N \leq 3. \end{cases}
\]

Let \( \mu \in \mathbb{R} \). We denote \( U_\mu(t) := e^{-it(\Delta^2 - \mu \Delta)} \) the propagator for the free fourth-order Schrödinger equation
\[
i \partial_t - \Delta^2 u + \mu \Delta u = 0.
\]
We have the following dispersive estimates due to Ben Artzi-Koch-Saut [1].

Lemma 2.3 ([1, Theorem 1]). Let \( N \geq 1 \) and \( \mu \in \mathbb{R} \). It holds that
\[
\| U_\mu(t)f \|_{L^\infty(\mathbb{R}^N)} \lesssim |t|^{-\frac{N}{4}} \| f \|_{L^1(\mathbb{R}^N)}
\]
for all \( t \neq 0 \), and if \( \mu < 0 \), it requires \( |t| \leq 1 \).

Using this dispersive estimate and the abstract theory of Keel-Tao [23], we have the following Strichartz estimates for (1.1).

Lemma 2.4 (Strichartz estimates [31, Proposition 3.1]). Let \( N \geq 1, \mu \in \mathbb{R} \) and \( I \subset \mathbb{R} \) be an interval. It holds that
\[
\| U_\mu(t)f \|_{L^q_t L^r_x(I \times \mathbb{R}^N)} \lesssim \| f \|_{L^2(\mathbb{R}^N)}
\]
for all Biharmonic admissible pairs \((q,r)\), and if \( \mu < 0 \), it requires \(|I| < \infty\).

Remark 4. We have from Sobolev embedding and Strichartz estimates that
\[
\| U_\mu(t)f \|_{L^q_t L^r_x(I \times \mathbb{R}^N)} \lesssim \| \nabla^{\frac{N}{2} - \frac{N+4}{r}} f \|_{L^2(\mathbb{R}^N)}
\]
for \( p \geq \frac{2(N+4)}{N} \). Note that the derivative loss in (2.1) depends only on the size of the frequency support and not its location. Namely, if \( \hat{f} \) is supported on a cube \( Q \) of side length \( M \), then
\[
\| U_\mu(t)f \|_{L^q_t L^r_x(I \times \mathbb{R}^N)} \lesssim M^{\frac{N}{2} - \frac{N+4}{r}} \| f \|_{L^2(\mathbb{R}^N)}
\]
which follows from Bernstein’s inequalities.

We have the following improvements of Strichartz estimates under the Wiener randomization.
Lemma 2.5 (Local-in-time probabilistic Strichartz estimates). Let \( N \geq 1 \) and \( \mu \in \mathbb{R} \). Let \( f \in L^2(\mathbb{R}^N) \) and \( f^\omega \) be the Wiener randomization defined in (1.5) satisfying (1.6). Then for \( 2 \leq q, r < \infty \), there exist \( C, c > 0 \) such that
\[
\mathcal{P} \left( \| U_\mu (t) f^\omega \|_{L_t^q L_x^r([0,T] \times \mathbb{R}^N)} > \lambda \right) \leq C \exp \left( -c \lambda^2 T^{-\frac{2}{q}} \| f \|_{L_x^2(\mathbb{R}^N)}^{-2} \right)
\]
for all \( T > 0 \) and all \( \lambda > 0 \).

Remark 5. Taking \( \lambda = T^\theta R \), we have
\[
\| U_\mu (t) f^\omega \|_{L_t^q L_x^r([0,T] \times \mathbb{R}^N)} \leq T^\theta R
\]
outside a set of probability at most
\[
C \exp \left( -c R^2 T^{-2} (\theta^{-1} - \theta) \| f \|_{L_x^2(\mathbb{R}^N)}^{-2} \right)
\]
for all \( T, \theta, R > 0 \). Note that for \( R > 0 \) fixed, this probability can be made arbitrarily small by letting \( T \to 0 \) as long as \( \theta < \frac{1}{7} \).

Lemma 2.6 (Global-in-time probabilistic Strichartz estimates). Let \( N \geq 1 \) and \( \mu \geq 0 \). Let \( f \in L^2(\mathbb{R}^N) \) and \( f^\omega \) be the Wiener randomization defined in (1.5) satisfying (1.6). Let \( (q,r) \) be a Biharmonic admissible pair with \( q, r < \infty \). Let \( \tilde{r} \geq r \). Then there exist \( C, c > 0 \) such that
\[
\mathcal{P} \left( \| U_\mu (t) f^\omega \|_{L_t^q L_x^r(\mathbb{R}^N)} > \lambda \right) \leq C \exp \left( -c \lambda^2 \| f \|_{L_x^2(\mathbb{R}^N)}^{-2} \right)
\]
for all \( \lambda > 0 \).

The proofs of Lemmas 2.5 and 2.6 follow the same argument as in [2] using Strichartz estimates given in Lemma 2.4. We thus omit the details.

2.3. Function spaces and their properties. In this subsection, we recall the definitions and basic properties of the \( U^p \)- and \( V^p \)-spaces developed by Tataru, Koch and their collaborators [17, 18, 25]. These spaces have been very effective in establishing well-posedness of various dispersive PDEs in critical regularities.

Let \( Z \) be the the collection of finite partitions \( (t_k)_{k=0}^K \) of \( \mathbb{R} \), i.e. \(-\infty < t_0 < \cdots < t_K \leq \infty \). If \( t_K = \infty \), we use the convention \( u(t_K) := 0 \) for all functions \( u : \mathbb{R} \to H^\gamma(\mathbb{R}^N) \).

Definition 2.7. Let \( 1 \leq p < \infty \) and \( \gamma \in \mathbb{R} \).

- A \( U^p \)-atom is defined by a step function \( a : \mathbb{R} \to H^\gamma(\mathbb{R}^N) \) of the form
  \[
a(t) = \sum_{k=1}^K \phi_k(t_{k-1}, t_k)(t), \quad \sum_{k=0}^{K-1} \| \phi_k \|_{H^\gamma(\mathbb{R}^N)} = 1,
\]
  where \( (t_k)_{k=0}^K \in Z \), \( (\phi_k)_{k=0}^{K-1} \subset H^\gamma(\mathbb{R}^N) \) and \( \chi_I \) is the characteristic function of \( I \).

- We define the atomic space \( U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \) to be the collection of functions \( u : \mathbb{R} \to H^\gamma(\mathbb{R}^N) \) of the form
  \[
u(t) = \sum_{j=1}^\infty \lambda_j a_j
\]
with the norm
\[
\| u \|_{U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : (2.3) \text{ holds} \right\},
\]
Lemma 2.9

Suppose that $T$ is a bounded operator. Suppose that we have

\[ \|u\|_{V^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))} := \sup_{t \in \mathbb{R}} \left( \sum_{k=1}^{K} \|u(t_k) - u(t_{k-1})\|_{H^\gamma(\mathbb{R}^N)}^p \right)^{\frac{1}{p}}. \]

We also define $V^p_\Delta H^\gamma(\mathbb{R}^N)$ to be the closed subspace of all right-continuous functions in $V^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))$ such that $\lim_{t \to -\infty} u(t) = 0$.

We define $U^p_\Delta H^\gamma(\mathbb{R}^N)$ (resp. $V^p_\Delta H^\gamma(\mathbb{R}^N)$) to be the space of all functions $u : \mathbb{R} \to H^\gamma(\mathbb{R}^N)$ such that the following norm is finite:

\[ \|u\|_{U^p_\Delta H^\gamma(\mathbb{R}^N)} := \|u(-t)\|_{V^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))}. \]

We use $V^p_\Delta H^\gamma(\mathbb{R}^N)$ to denote the subspace of right-continuous functions in $V^p_\Delta H^\gamma(\mathbb{R}^N)$.

Remark 6. It was shown in [17] that the spaces $U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))$, $V^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))$ and $V^p_\Delta H^\gamma(\mathbb{R}^N)$ are Banach spaces. The closed subspace of continuous functions in $U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))$ is also a Banach space. Moreover, we have the following embeddings:

\[ U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \hookrightarrow V^p_\Delta(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \hookrightarrow U^q(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \hookrightarrow L^\infty(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \]

for $1 \leq p < q < \infty$. Similar embeddings hold for $U^p_\Delta H^\gamma(\mathbb{R}^N)$ and $V^p_\Delta H^\gamma(\mathbb{R}^N)$.

We have the following tranference principle.

Lemma 2.8 (Tranference principle [17]). Let $N \geq 1$ and $\mu \geq 0$. Let $T$ be a $k$-linear operator. Suppose that we have

\[ \|T(U_\mu(t)f_1, \ldots, U_\mu(t)f_k)\|_{L^p_\mu L^q_\mu(\mathbb{R} \times \mathbb{R}^N)} \lesssim \prod_{j=1}^{k} \|f_j\|_{L^2(\mathbb{R}^N)} \]

for some $1 \leq p, q \leq \infty$. Then, we have

\[ \|T(u_1, \ldots, u_k)\|_{L^p_\mu L^q_\mu(\mathbb{R} \times \mathbb{R}^N)} \lesssim \prod_{j=1}^{k} \|u_j\|_{U^p_\mu L^2(\mathbb{R}^N)}. \]

We also have the following interpolation inequality.

Lemma 2.9 (Interpolation lemma [17]). Let $\gamma \in \mathbb{R}$ and $E$ be a Banach space. Suppose that $T : U^{p_1}(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \times \cdots \times U^{p_k}(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \to E$ is a bounded $k$-linear operator such that

\[ \|T(u_1, \ldots, u_k)\|_{E} \leq C_1 \prod_{j=1}^{k} \|u_j\|_{U^{p_j}(\mathbb{R}, H^\gamma(\mathbb{R}^N))} \]

for some $p_1, \ldots, p_k > 2$. Moreover, assume that there exists $C_2 \in (0, C_1]$ such that

\[ \|T(u_1, \ldots, u_k)\|_{E} \leq C_2 \prod_{j=1}^{k} \|u_j\|_{L^2(\mathbb{R}, H^\gamma(\mathbb{R}^N))}. \]

\[ \text{Remark: Since } U^2(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \hookrightarrow U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N)), \text{ we have } C_2 \leq C_1. \]
Then we have
\[ \|T(u_1, \cdots, u_k)\|_E \leq C_2 \left( \ln \frac{C_1}{C_2} + 1 \right)^k \prod_{j=1}^{k} \|u_j\|_{V^2(\mathbb{R}, H^\gamma(\mathbb{R}^N))} \]
for \( u_j \in V^2_{rec}(\mathbb{R}, H^\gamma(\mathbb{R}^N)), j = 1, \cdots, k. \)

We refer the reader to [17] (see also [25]) for the proof of above results.

**Definition 2.10.** Let \( \gamma \in \mathbb{R}. \)

- We define \( X^\gamma(\mathbb{R}) \) to be the space of all tempered distributions \( u : \mathbb{R} \to H^\gamma(\mathbb{R}^N) \) such that the norm
  \[ \|u\|_{X^\gamma(\mathbb{R})} := \left( \sum_{M \geq 1 \text{ dyadic}} M^{2\gamma} \|P_M u\|_{L^2(\mathbb{R}^N)}^2 \right)^{1/2} \]
is finite.

- We define \( Y^\gamma(\mathbb{R}) \) to be the space of all tempered distributions \( u : \mathbb{R} \to H^\gamma(\mathbb{R}^N) \) such that for every dyadic number \( M = 2^m, m \geq 0, \) the map \( t \mapsto P_M u(t) \) is in \( V^2_{rec, \Delta} H^\gamma(\mathbb{R}^N) \) and the norm
  \[ \|u\|_{Y^\gamma(\mathbb{R})} := \left( \sum_{M \geq 1 \text{ dyadic}} M^{2\gamma} \|P_M u\|_{V^2_{rec, \Delta} L^2(\mathbb{R}^N)}^2 \right)^{1/2} \]
is finite.

By definition, we have
\[ \|U_\mu(t)f\|_{X^\gamma(\mathbb{R})} \sim \|f\|_{H^\gamma(\mathbb{R}^N)}. \] (2.4)

Moreover, we have the following embeddings:
\[ U^p_{\Delta} H^\gamma(\mathbb{R}^N) \hookrightarrow X^\gamma(\mathbb{R}) \hookrightarrow Y^\gamma(\mathbb{R}) \hookrightarrow V^2_{\Delta} H^\gamma(\mathbb{R}^N) \hookrightarrow U^p_{\Delta} H^\gamma(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \] (2.5)
for \( p > 2. \)

Given an interval \( I \subset \mathbb{R}, \) we define the local-in-time versions \( X^\gamma(I) \) and \( Y^\gamma(I) \) of these spaces as restriction norms. For example, we define the \( X^\gamma(I)\)-norm by
\[ \|u\|_{X^\gamma(I)} := \inf \left\{ \|v\|_{X^\gamma(\mathbb{R})} : v|_I = u \right\}. \]

We have the following lemmas due to Bényi-Oh-Pocovnicu [3, Appendix].

**Lemma 2.11 ([3, Appendix]).** Let \( \gamma \in \mathbb{R} \) and \( 1 \leq p < \infty. \) Let
\[ u = \sum_{j=1}^{\infty} \lambda_j a_j, \]
where \( (\lambda_j)_{j \geq 1} \in \ell^1(\mathbb{C}) \) and \( a_j \) are \( U^p \)-atoms. Given an interval \( I \subset \mathbb{R}. \) Then we can write
\[ u \cdot \chi_I = \sum_{j=1}^{\infty} \lambda_j \tilde{a}_j \]
for some $(\tilde{\lambda}_j)_{j \geq 1} \in \ell^1(\mathbb{C})$ and $\tilde{a}_j$ are $U^p$-atoms satisfying
\[
\sum_{j=1}^{\infty} |\tilde{\lambda}_j| \leq \sum_{j=1}^{\infty} |\lambda_j|.
\]
As a consequence, we have
\[
\|u \cdot \chi_I\|_{U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))} \leq \|u\|_{U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))}
\]
for any $u \in U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))$ and any $I \subset \mathbb{R}$.

Given an interval $I \subset \mathbb{R}$. We define the local-in-time $U^p$-norm in the usual manner as a restriction norm
\[
\|u\|_{U^p(I, H^\gamma(\mathbb{R}^N))} = \inf \{ \|v\|_{U^p(\mathbb{R}, H^\gamma(\mathbb{R}^N))} : v|_I = u \}.
\]
Note that this infimum achieve by $v = u \cdot \chi_I$ in view of Lemma 2.11.

We have the following Strichartz estimates adapted to the $X^\gamma$- and $Y^\gamma$-spaces.

**Lemma 2.12.** Let $N \geq 1$ and $\mu \geq 0$. Let $(q, r)$ be a Biharmonic admissible pair with $q > 2$ and $p \geq \frac{2(N+4)}{N}$. Then for any $0 < T \leq \infty$, we have
\[
\begin{align*}
\|u\|_{L_t^q L_x^r([0,T) \times \mathbb{R}^N)} &\lesssim \|u\|_{Y^0([0,T])}, \quad (2.6) \\
\|u\|_{L_t^p L_x^{\infty}([0,T) \times \mathbb{R}^N)} &\lesssim \|\nabla \|^{\frac{N}{2} - \frac{N+4}{p}} u\|_{Y^0([0,T])}. \quad (2.7)
\end{align*}
\]

**Proof.** By Strichartz estimates, we have
\[
\|U_{\mu}(t)f\|_{L_t^q L_x^r([0,T) \times \mathbb{R}^N)} \lesssim \|f\|_{L^2(\mathbb{R}^N)}.
\]
It follows from the tranference principle that
\[
\|u\|_{L_t^q L_x^r([0,T) \times \mathbb{R}^N)} \lesssim \|u\|_{U_\mu^q([0,T), L^2(\mathbb{R}^N))} \leq \|u\|_{Y^0([0,T])},
\]
where we have used the embedding (2.5).

Similarly, by (2.1) and the tranference principle, we have
\[
\|u\|_{L_t^p L_x^{\infty}([0,T) \times \mathbb{R}^N)} \lesssim \|\nabla \|^{\frac{N}{2} - \frac{N+4}{p}} u\|_{L_\mu^p([0,T), L^2(\mathbb{R}^N))} \leq \|\nabla \|^{\frac{N}{2} - \frac{N+4}{p}} u\|_{Y^0([0,T])}.
\]
\[\Box\]

**Remark 7.**
- The derivative loss in (2.7) depends only on the size of the spatial frequency support and not its location. Namely, if the spatial frequency support of $\hat{u}(t, \xi)$ is contained in a cube of side length $M$ for all $t \in \mathbb{R}$, then
\[
\|u\|_{L_t^p L_x^{\infty}([0,T) \times \mathbb{R}^N)} \lesssim M^{\frac{N}{2} - \frac{N+4}{p}} \|u\|_{Y^0([0,T])}
\]
by Bernstein’s inequalities.
- By (2.5), we can replace the norm $Y^0([0,T))$ in (2.6) and (2.7) by $X^0([0,T))$.

We also have the following bilinear estimate related to the fourth-order Schrödinger equation.

**Lemma 2.13** (Bilinear estimate). Let $N \geq 5$ and $\mu \geq 0$. Let $M_1, M_2 \in 2^\mathbb{N}$ be such that $M_1 \leq M_2$. Then it holds that
\[
\| [U_{\mu}(t)P_{M_1}f][U_{\mu}(t)P_{M_2}g] \|_{L_t^2(\mathbb{R} \times \mathbb{R}^N)} \lesssim M_1^{\frac{N-4}{2}} \left( \frac{M_1}{M_2} \right)^{\frac{2}{N}} \|f\|_{L^2(\mathbb{R}^N)} \|g\|_{L^2(\mathbb{R}^N)}. \quad (2.8)
\]
Proof. For simplifying the notation, we denote \( L_t^p L_x^q, L_t^p \) and \( L_t^q \) instead of \( L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^N), L_t^p(\mathbb{R} \times \mathbb{R}^N) \) and \( L_t^q(\mathbb{R}^N) \) respectively.

We first consider the case \( M_1 \sim M_2 \). By Hölder’s inequality, Sobolev embedding and Strichartz estimates, we have

\[
\| [U_\mu(t) P_M f][U_\mu(t) P_M g] \|_{L_t^2 L_x^q} \lesssim \| [U_\mu(t) P_M f][U_\mu(t) P_M g] \|_{L_t^\infty L_x^2} \lesssim \left| \int_{\mathbb{R}^N} F(t, x) \dot{G}(t, x) dx \right| \int_{\mathbb{R}} |F(t, x)| L^2 dt.
\]

By Parseval’s identity, we have

\[
\text{LHS}(2.8) = \sup_{\| G \|_{L_t^2 L_x^q} = 1} \left| \left\langle \left( [U_\mu(t) P_M f][U_\mu(t) P_M g] \right), \dot{G}(t) \right\rangle_{L_t^2 L_x^q} \right|.
\]

where

\[
F(t, x) = \int_{\mathbb{R}} \int_{\mathbb{R}^N} e^{-it(\xi - \eta)^4 + |\eta|^4 - \mu((\xi - \eta)^2 + |\eta|^2)} \hat{P}_M f(\xi - \eta) \hat{P}_M g(\eta) d\eta.
\]

It follows that

\[
\int_{\mathbb{R}} \left\langle \left( [U_\mu(t) P_M f][U_\mu(t) P_M g] \right), \dot{G}(t) \right\rangle_{L_t^2 L_x^q} dt
\]

\[
= \int_{\mathbb{R}} \left\langle \left( \hat{P}_M f(\xi - \eta) \hat{P}_M g(\eta) \right), \hat{G}(\xi - \eta) \hat{P}_M f(\xi - \eta) \hat{P}_M g(\eta) \right\rangle_{L_t^2 L_x^q} d\eta
\]

\[
= \int_{\mathbb{R}^N} \hat{P}_M f(\xi - \eta) \hat{P}_M g(\eta) \hat{G}(\xi - \eta) \hat{P}_M f(\xi - \eta) \hat{P}_M g(\eta) d\eta.
\]

where \( \hat{G} \) is the space-time Fourier transform of \( G \). Thus, the estimate (2.8) is reduced to show

\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \hat{G}(\xi - \eta) \hat{P}_M f(\xi - \eta) \hat{P}_M g(\eta) d\xi d\eta \right|
\]

\[
\lesssim \left( \frac{M_1}{M_2} \right)^\frac{3}{2} \| \hat{G} \|_{L_t^2 L_x^q} \| \hat{f} \|_{L_t^2 L_x^q} \| \hat{g} \|_{L_t^2 L_x^q}.
\]
By renaming the components, we can assume that $|\xi| \sim |\xi_1| \sim M_1$ and $|\eta| \sim |\eta_1| \sim M_2$, where $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2)$ with $\xi, \eta \in \mathbb{R}^{N-1}$. By the change of variables
\[
\begin{align*}
\tau &= |\xi|^4 + |\eta|^4 - \mu(|\xi|^2 + |\eta|^2), \\
\zeta &= \xi + \eta,
\end{align*}
\]
a direct computation shows
\[d\tau d\zeta = J d\xi_1 d\eta,
\]
where
\[J = 4(|\xi|^2|\xi_1| - |\eta|^2|\eta_1| - 2\mu(\xi_1 - \eta_1)) \sim |\eta|^3 \sim M_2^3.
\]
By the Cauchy-Schwarz inequality with the fact $|\xi| \lesssim M_1$, we get
\[
\text{LHS}(2.9) = \left| \int \int \int_{\mathbb{R}^N \times \mathbb{R}^N} G(\tau, \zeta) \hat{P}_M \hat{f}(\xi) \hat{P}_N g(\eta) J^{-1} d\tau d\xi d\zeta \right|
\]
\[
\leq \| \hat{G} \|_{L^2_{C,\xi}} \left( \int \int \int_{\mathbb{R}^N \times \mathbb{R}^N} |\hat{P}_M \hat{f}(\xi)|^2 |\hat{P}_N g(\eta)|^2 J^{-2} d\tau d\xi d\zeta \right)^{1/2} d\tau d\xi d\zeta
\]
\[
\lesssim M_1^{N-4} |\hat{G}|_{L^2_{C,\xi}} \left( \int \int \int_{\mathbb{R}^N \times \mathbb{R}^N} |\hat{P}_M \hat{f}(\xi)|^2 |\hat{P}_N g(\eta)|^2 J^{-2} d\tau d\xi d\zeta \right)^{1/2}
\]
\[
\lesssim M_1^{N-4} |\hat{G}|_{L^2_{C,\xi}} \left( \int \int \int_{\mathbb{R}^N \times \mathbb{R}^N} |\hat{P}_M \hat{f}(\xi)|^2 |\hat{P}_N g(\eta)|^2 J^{-2} d\tau d\xi d\zeta \right)^{1/2}
\]
\[
\lesssim M_1^{N-4} M_2^{-\frac{2}{3}} \| \hat{G} \|_{L^2_{C,\xi}} \| \hat{P}_M \hat{f} \|_{L^2_{\xi}} \| \hat{P}_N g \|_{L^2_{\zeta}}
\]
which proves (2.9), and the proof is complete.

We have the following bilinear estimate adapted to the $X^\gamma$- and $Y^\gamma$-spaces.

**Lemma 2.14.** Let $N \geq 5$ and $\mu \geq 0$. Then for any $0 < T \leq \infty$ and $M_1 \leq M_2$, we have
\[
\| P_{M_1} u_1 P_{M_2} u_2 \|_{L^2_s((0,T) \times \mathbb{R}^N)} \lesssim M_1^{N-4} M_2^{-\frac{2}{3}} \| P_{M_1} u_1 \|_{Y^0((0,T))} \| P_{M_2} u_2 \|_{Y^0((0,T))}.
\]

(2.10)

**Remark 8.** By (2.5), we can replace the $Y^0((0,T))$-norm of the above estimate by the $X^0((0,T))$-norm.

**Proof of Lemma 2.14.** We follow the argument of [3]. By (2.8) and the tranference principle, we have
\[
\| P_{M_1} u_1 P_{M_2} u_2 \|_{L^2_s((0,T) \times \mathbb{R}^N)} \lesssim M_1^{N-4} M_2^{-\frac{2}{3}} \| P_{M_1} u_1 \|_{U^{\frac{4}{3}}((0,T),L^2(\mathbb{R}^N))} \| P_{M_2} u_2 \|_{U^{\frac{4}{3}}((0,T),L^2(\mathbb{R}^N))}.
\]

(2.11)

Since $\left( 4, \frac{2N}{N-2} \right)$ is Biharmonic admissible, we have
\[
\| e^{i\Delta} P_{M_1} f \|_{L^2_s((0,T) \times \mathbb{R}^N)} \lesssim \| \nabla \|_{\frac{N-4}{2}} e^{i\Delta} P_{M_1} f \|_{L^2_s L^2_{t,x}((0,T) \times \mathbb{R}^N)} \lesssim M_1^{N-4} \| f \|_{L^2_s(\mathbb{R}^N)}.
\]

The tranference principle gives
\[
\| P_{M_1} u \|_{L^2_s((0,T) \times \mathbb{R}^N)} \lesssim M_1^{N-4} \| u \|_{U^{\frac{4}{3}}((0,T),L^2(\mathbb{R}^N))}.
\]
By Hölder’s inequality, we get
\[ \|P_{M_1} u_1 P_{M_2} u_2\|_{L^2_{x,t}([0,T] \times \mathbb{R}^N)} \]
\[ \lesssim M_{1}^{\frac{N-4}{2}} M_{2}^{\frac{N-4}{2}} \|u_1\|_{L^2_{x,t}([0,T], L^2(\mathbb{R}^N))} \|u_2\|_{L^2_{x,t}([0,T], L^2(\mathbb{R}^N))}. \]  
(2.12)

By the interpolation lemma, we have from (2.11) and (2.12) that
\[ \|P_{M_1} u_1 P_{M_2} u_2\|_{L^2_{x,t}([0,T] \times \mathbb{R}^N)} \]
\[ \lesssim M_{1}^{\frac{N-4}{2}} \left( \frac{M_1}{M_2} \right)^{\frac{N+2}{2}} \left( \ln \left( \frac{M_2}{M_1} \right)^{\frac{N+2}{2}} + 1 \right)^2 \|u_1\|_{L^2_{x,t}([0,T], L^2(\mathbb{R}^N))} \|u_2\|_{L^2_{x,t}([0,T], L^2(\mathbb{R}^N))} \]
\[ \lesssim M_{1}^{\frac{N-4}{2}} \left( \frac{M_1}{M_2} \right)^{\frac{N-4}{2}} \|u_1\|_{L^2_{x,t}([0,T], L^2(\mathbb{R}^N))} \|u_2\|_{L^2_{x,t}([0,T], L^2(\mathbb{R}^N))}. \]

This shows (2.10) and the proof is complete. ☐

To finish this section, we recall the following linear estimates which are needed in the sequel.

\textbf{Lemma 2.15} (Linear estimates \cite[Propositions 2.10 and 2.11]{18}). \textit{Let} \( N \geq 1 \) \textit{and} \( \mu \geq 0. \) \textit{Let} \( \gamma \geq 0 \) \textit{and} \( 0 < T \leq \infty. \) \textit{Then it holds that}
\[ \|U_\mu(t) f\|_{X^{\gamma}([0,T])} \leq \|f\|_{H^\gamma(\mathbb{R}^N)} \]  
(2.13)
\[ \text{and} \]
\[ \left\| \int_0^t U_\mu(t-s) F(s) ds \right\|_{X^{\gamma}([0,T])} \leq \sup_{\|v\|_{X^{\gamma}(\mathbb{R}^N)} = 1} \left| \int_0^T \int_{\mathbb{R}^N} F(t,x) \tilde{v}(t,x) dx dt \right| \]
(2.14)
\[ \text{for all} \ f \in H^\gamma(\mathbb{R}^N) \text{ and} \ F \in L^1([0,T], H^\gamma(\mathbb{R}^N)). \]

\textbf{3. Probabilistic nonlinear estimates.} \textit{In this section, we will prove probabilistic nonlinear estimates needed to show the almost sure well-posedness. Denote}
\[ \Phi(v(t)) := \mp \int_0^t U_\mu(t-s) \tilde{N}(v+z)(s) ds \]
(3.1)
\[ \text{and} \]
\[ \tilde{\Phi}(v(t)) := \mp \int_0^t U_\mu(t-s) \tilde{N}(v+\varepsilon z)(s) ds, \]
(3.2)
\[ \text{where} \ z, v \text{ are as in} \ (1.8) \text{ and} \ \tilde{N}(f) = f \mathcal{T} f. \] \textit{We have the following probabilistic nonlinear estimates.}

\textbf{Proposition 1.} \textit{Let} \( N \geq 5, \mu \geq 0 \) \textit{and} \( \gamma \in (\gamma_N, \gamma_c) \), \textit{where} \( \gamma_c \) \textit{and} \( \gamma_N \) \textit{are as in} \ (1.3) \textit{and} \ (1.7) \textit{respectively. Let} \( f \in H^\gamma(\mathbb{R}^N) \) \textit{and} \( f^w \) \textit{be the Wiener randomization defined in} \ (1.5) \textit{satisfying} \ (1.6).}

\textbullet \ \textit{Let} \( 0 < T \leq 1. \) \textit{Then there exists} \( 0 < \vartheta \ll 1 \) \textit{such that}
\[ \|\Phi(v)\|_{X^{\frac{N-4}{2}}([0,T])} \leq C_1 \left( \|v\|_{X^{\frac{N-4}{2}}([0,T])}^3 + T^\vartheta R^3 \right), \]
(3.3)
\[ \|\Phi(v_1) - \Phi(v_2)\|_{X^{\frac{N-4}{2}}([0,T])} \leq C_2 \left( \sum_{j=1}^2 \|v_j\|_{X^{\frac{N-4}{2}}([0,T])}^2 + T^\vartheta R^2 \right) \|v_1 - v_2\|_{X^{\frac{N-4}{2}}([0,T])}, \]
(3.4)
for all \( v, v_1, v_2 \in X^{\frac{N-4}{2}}([0,T]) \) and all \( R > 0 \), outside a set of probability at most 
\[
C \exp \left( -cR^2 \| f \|^2_{H^\gamma(\mathbb{R}^N)} \right).
\]

- For \( 0 < \varepsilon \ll 1 \), we have 
\[
\| \tilde{\Phi}(v) \|_{X^{\frac{N-4}{2}}(\mathbb{R})} \leq C_3 \left( \|v\|_{X^{\frac{N-4}{2}}(\mathbb{R})}^3 + R^3 \right),
\]
\[
\| \tilde{\Phi}(v_1) - \tilde{\Phi}(v_2) \|_{X^{\frac{N-4}{2}}(\mathbb{R})} \leq C_4 \left( \sum_{j=1}^2 \|v_j\|_{X^{\frac{N-4}{2}}(\mathbb{R})}^2 + R^2 \right) \|v_1 - v_2\|_{X^{\frac{N-4}{2}}(\mathbb{R})},
\]
for all \( v, v_1, v_2 \in X^{\frac{N-4}{2}}(\mathbb{R}) \) and all \( R > 0 \), outside a set of probability at most 
\[
C \exp \left( -cR^2 \varepsilon^{-2} \| f \|^2_{H^\gamma(\mathbb{R}^N)} \right).
\]

**Proof.** We mainly follow the argument of Bényi-Oh-Pocovnicu [3].

- Let \( 0 < T \leq 1 \). We only prove (3.3), and the one for (3.4) is treated similarly.

Given \( M \geq 1 \), we define 
\[
\Phi_M(v(t)) := \mp i \int_0^t U_\mu(t-s)P_{\leq M} \mathcal{N}(v+z)(s)ds.
\]

By Bernstein’s and Hölder inequalities, we have 
\[
\|P_{\leq M} \mathcal{N}(v+z)\|_{L^1_tH_x^{\frac{N-4}{2}}([0,T] \times \mathbb{R}^N)} 
\leq M \|v\|_{L^3_tL^6_x([0,T] \times \mathbb{R}^N)} \|\mathcal{N}(v+z)\|_{L^1_tL^2_x([0,T] \times \mathbb{R}^N)} 
\leq M \|v\|^3_{L^1_tL^6_x([0,T] \times \mathbb{R}^N)} + M \|z\|^2_{L^3_tL^6_x([0,T] \times \mathbb{R}^N)},
\]
for all \( v \in X^{\frac{N-4}{2}}(\mathbb{R}) \) and all \( R > 0 \), outside a set of probability at most 
\[
C \exp \left( -cR^2 \varepsilon^{-2} \| f \|^2_{H^\gamma(\mathbb{R}^N)} \right).
\]

Thanks to the local in time probabilistic Strichartz estimates, the second term in (3.7) is finite almost surely. On the other hand, by the Sobolev embedding and (2.6), we have 
\[
\|v\|_{L^1_tL^6_x([0,T] \times \mathbb{R}^N)} \leq \|\nabla\|_{L^1_tL^6_x([0,T] \times \mathbb{R}^N)} \|v\|_{X^{\frac{N-4}{2}}([0,T])} \lesssim \|v\|_{X^{\frac{N-4}{2}}([0,T])} < \infty.
\]

This shows that for each \( M \geq 1 \), \( P_{\leq M} \mathcal{N}(v+z) \in L^1_tH_x^{\frac{N-4}{2}}([0,T] \times \mathbb{R}^N) \) almost surely. Thus, by Lemma 2.15,
\[
\|\Phi_M(v)\|_{X^{\frac{N-4}{2}}([0,T])} \lesssim \sup_{v_4 \in \mathcal{Y}_M([0,T])} \left| \int_0^T \int_{\mathbb{R}^N} \langle \nabla \|_{X^{\frac{N-4}{2}}([0,T])} \|v_4(t)\|_{X^{\frac{N-4}{2}}([0,T])} \right|, \]
where \( v_4 = P_{\leq M}v_4 \). In the following, we estimate the right hand side of (3.8) independently of the cutoff size \( M \geq 1 \), by performing a case-by-case analysis of expressions of the form 
\[
\left| \int_0^T \int_{\mathbb{R}^N} \langle \nabla \|_{X^{\frac{N-4}{2}}([0,T])} (w_1w_2w_3)v_4dxdt \right|,
\]
where \( \|v_4\|_{\mathcal{Y}_M([0,T])} \leq 1 \) and \( w_j = v \) or \( z \), \( j = 1, 2, 3 \). As a result, by letting \( M \to \infty \), the same estimate holds for \( \Phi(v) \) without any cutoff, thus yielding (3.3).

Before proceeding further, let us simplify some of the notation. In the following, we drop the complex conjugate sign since it plays no role. We also denote \( X^\gamma([0,T]) \)
Lastly, in most of the cases, we dyadically decompose and similarly for \( v \) and \( H \), we have
\[
\| v_1 \|_{L_t^\infty L_x^{\infty+4}} \lesssim \| v_1 \|_{X_{L_t^4 L_x^{N+4}}} \lesssim \| v_4 \|_{L_t^2 L_x^{2N(N+4)}}.
\]
By Sobolev embedding and (2.6), we have
\[
\| v_2 \|_{L_t^{N+4} L_x^{N+4}} \lesssim \| v_2 \|_{X_{L_t^4 L_x^{N+4}}} \lesssim \| v_2 \|_{L_t^{N+4} L_x^{N+4}}.
\]
Similarly for \( v_3 \). We thus get
\[
\int_0^T \int_{\mathbb{R}^N} \langle \nabla \rangle^{\frac{N+4}{2}} v_1 v_2 v_3 v_4 dx dt \lesssim \prod_{j=1}^3 \| v_j \|_{X_{L_t^4 L_x^{N+4}}}.
\]
**Case 2.** \( M_2 \sim M_3 \). By Hölder’s inequality, we have
\[
\left| \int_0^T \int_{\mathbb{R}^N} z_1 z_2 \langle \nabla \rangle^{\frac{N+4}{2}} z_3 v_4 dx dt \right|
\leq \| z_1 \|_{L_t^4 L_x^{N+4}} \| z_2 \|_{L_t^4 L_x^{N+4}} \| \langle \nabla \rangle^{\frac{N+4}{2}} z_3 \|_{L_t^4 L_x^{N+4}} \| v_4 \|_{L_t^2 L_x^{2N(N+4)}}
\sim \| z_1 \|_{L_t^4 L_x^{N+4}} \| z_2 \|_{L_t^4 L_x^{N+4}} \| \langle \nabla \rangle^{\frac{N+4}{2}} z_3 \|_{L_t^4 L_x^{N+4}} \| v_4 \|_{L_t^2 L_x^{2N(N+4)}}.
\]
By the Littlewood-Paley decomposition and the local-in-time probabilistic Strichartz estimates, we observe that for \( r \geq 2 \),
\[
\sum_{M_1 \geq 1} \| P_{M_1} z_1 \|_{L_t^r L_x^r} = \sum_{M_1 \geq 1} M_1^{0+} \| P_{M_1} z_1 \|_{L_t^{r_1} L_x^{r_1}} \leq \left( \sum_{M_1 \geq 1} M_1^{0+} \right)^{\frac{2}{3}} \left( \sum_{M_1 \geq 1} \| \langle \nabla \rangle^{0+} P_{M_1} z_1 \|_{L_t^{r_1} L_x^{r_1}} \right)^{\frac{1}{3}}.
\]
\footnote{We have \( \int_{\mathbb{R}^N} f_1(x) f_2(x) f_3(x) f_4(x) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_1(\xi_1) f_2(\xi_2) f_3(\xi_3) f_4(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 \), where \( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \) denotes the integration with respect to the hyperplane’s measure \( \delta_0(\xi_1 + \xi_2 + \xi_3 + \xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 \).}
Similarly, by (2.10), we have
\[ \left( \sum_{M_1 \geq 1} M_1^{0-} \right) \frac{1}{2} \left\| \langle \nabla \rangle^{0+} P_{M_1} z_1 \right\|_{L_{t,x}^2} \leq \left( \sum_{M_1 \geq 1} M_1^{0-} \right) \frac{1}{2} \left\| \langle \nabla \rangle^{0+} P_{M_1} z_1 \right\|_{L_{t,x}^2} \]
outside a set of probability at most
\[ C \exp \left( -cR^2T^{-\frac{3}{2}} \| f \|_{H^{0+}}^{-2} \right). \]
Applying the above observation to \( r = \frac{N+4}{4} \) and \( 4 \), we obtain
\[ \left| \int_0^T \int_{\mathbb{R}^N} z_1 z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 v_4 dx dt \right| \lesssim T^{0+} R^3 \]
outside a set of probability at most
\[ C \exp \left( -cR^2T^{-\frac{3}{2}} \| f \|_{H^{0+}}^{-2} \right) + C \exp \left( -cR^2T^{-\frac{3}{2}} \| f \|_{H^{0+}}^{-2} \right). \]
Here we have extracted a negative power of \( M_3 \) and used \( M_4 \lesssim M_3 \) to estimate
\[ \sum_{M_4 \geq 1} M_4^{0-} \| P_{M_4} v_4 \|_{L_{t,x}^{2(n+4)}} \lesssim \| v_4 \|_{L_{t,x}^{2(n+4)}} \lesssim \| v_4 \|_{H^{0+}} \leq 1. \]
Note that since \( T \leq 1 \) and \( \gamma > \frac{N-4}{4} \), the above probability can be bounded by
\[ C \exp \left( -cR^2 \| f \|_{H^{0+}}^{-2} \right). \]

**Subcase 2b.** \( M_3 \gg M_2 \gg M_1 \). Note that we must have \( M_4 \sim M_3 \).

**Subcase 2b.i.** \( \frac{N-4}{4} \gg M_2 \gg M_1 \). By Hölder’s inequality, we have
\[ \left| \int_0^T \int_{\mathbb{R}^N} z_1 z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 v_4 dx dt \right| \leq \| z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 \|_{L_{t,x}^2} \| z_1 v_4 \|_{L_{t,x}^2}. \]
Let \( a > 0 \) be a small constant to be chosen shortly, we estimate
\[ \| z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 \|_{L_{t,x}^2} \lesssim M_3^{\frac{(N-4)a}{4}} \| z_2 \|_{L_{t,x}^2}^{\frac{a}{4}} \| z_3 \|_{L_{t,x}^2}^{\frac{a}{4}} \lesssim M_3^{\frac{(N-4)a}{4}} \| z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 \|_{L_{t,x}^2}^{1-a} \]
\[ \lesssim M_2^{-\gamma} M_3^{\frac{(N-4)a}{4}} \| \langle \nabla \rangle \|_{L_{t,x}^2}^{\gamma} \| z_2 \|_{L_{t,x}^2}^{\gamma} \| z_3 \|_{L_{t,x}^2}^{\gamma} \lesssim M_2^{-\gamma} M_3^{\frac{(N-4)a}{4}} \| \langle \nabla \rangle \|_{L_{t,x}^2}^{\gamma} \| z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 \|_{L_{t,x}^2}^{1-a}. \]
We next use (2.10) and (2.4) to have
\[ \| z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 \|_{L_{t,x}^2} \lesssim M_2^{-\gamma} M_3^{\frac{(N-4)a}{4}} \| z_2 \|_{X^0} \| \langle \nabla \rangle^{\frac{N-4}{4}} z_3 \|_{X^0} \lesssim M_2^{-\gamma} M_3^{\frac{(N-4)a}{4}} \| P_{M_2} f_w \|_{L^2} \| P_{M_3} f_w \|_{H^\gamma} \]
\[ \lesssim M_2^{-\gamma} M_3^{\frac{(N-4)a}{4}} \| P_{M_2} f_w \|_{L^2} \| P_{M_3} f_w \|_{H^\gamma}. \]
It follows that
\[ \| z_2 \langle \nabla \rangle^{\frac{N-4}{4}} z_3 \|_{L_{t,x}^2} \lesssim M_2^{-\gamma} M_3^{\frac{(N-4)a}{4}} \| P_{M_2} f_w \|_{L^2} \| P_{M_3} f_w \|_{H^\gamma} \]
\[ \lesssim M_2^{-\gamma} M_3^{\frac{(N-4)a}{4}} \| P_{M_2} f_w \|_{L^2} \| P_{M_3} f_w \|_{H^\gamma}. \]
Similarly, by (2.10), we have
\[ \| z_1 v_4 \|_{L_{t,x}^2} \lesssim M_1^{\frac{N-4}{4}} \| z_1 \|_{X^0} \| v_4 \|_{Y^0} \lesssim M_1^{\frac{N-4}{4}} \| P_{M_1} f_w \|_{L^2}. \]
$$\leq M_1^{\frac{N-1}{2} - \gamma} M_4^{-\frac{3}{2} +} \| P M_1 f^\omega \|_{H^\gamma}.$$ 

Since $M_3^{\frac{3}{N+4}} \gg M_2 \geq M_1$ and $M_3 \sim M_4$, we obtain

$$\left| \int_0^T \int_{\mathbb{R}^N} z_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 dxdt \right| \leq M_3^{-\frac{4}{N+4}} \| P M_1 f^\omega \|_{H^\gamma} \prod_{j=2}^3 \| \langle \nabla \rangle^\gamma z_j \|_{L_{t,x}^4} \| P M_1 f^\omega \|_{L_{t,x}^4} \| v_4 \|_{Y^0},$$

provided

$$N - 1 = \gamma - N - 1 a > 0 \quad \text{or} \quad a < 1 - \frac{2 \gamma}{N - 1}.$$ 

We want the largest frequency $M_3$ to have a negative power so that we can sum over dyadic blocks. This requires

$$\frac{N - 4}{2} - \frac{N + 5}{N - 1} \gamma < 0$$

which is satisfied as

$$\gamma > \gamma_N \geq \frac{(N - 1)(N - 4)}{2(N + 5)}.$$ 

Under this condition, we can sum over dyadic blocks as in Case 2a. We thus get

$$\left| \int_0^T \int_{\mathbb{R}^N} z_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 dxdt \right| \lesssim T^{0 +} R^3$$

outside a set of probability at most

$$C \exp \left( -c R^2 T^{-\frac{1}{2} +} \| f \|_{H^\gamma}^{-2} \right) + C \exp \left( -c R^2 \| f \|_{H^\gamma}^{-2} \right).$$

**Subcase 2b.ii.** $M_2 \gtrsim M_3^{\frac{3}{N+4}} \gg M_1$. By Hölder’s inequality and (2.10), we have

$$\left| \int_0^T \int_{\mathbb{R}^N} z_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 dxdt \right| \leq \| z_2 \|_{L_{t,x}^4} \| \langle \nabla \rangle^{\frac{N-4}{2}} z_3 \|_{L_{t,x}^4} \| z_1 v_4 \|_{L_{t,x}^4}$$

$$\lesssim M_1^{-\frac{N-4}{2} - \gamma} M_2^{-\gamma} M_3^{\frac{3}{N+4} - \gamma} M_4^{-\frac{3}{2} +} \| P M_1 f^\omega \|_{H^\gamma} \| \langle \nabla \rangle^\gamma z_2 \|_{L_{t,x}^4} \| \langle \nabla \rangle^\gamma z_3 \|_{L_{t,x}^4} \| v_4 \|_{Y^0}$$

$$\lesssim M_3^{-\frac{N-4}{N+4} - \gamma} \| P M_1 f^\omega \|_{H^\gamma} \| \langle \nabla \rangle^\gamma z_2 \|_{L_{t,x}^4} \| \langle \nabla \rangle^\gamma z_3 \|_{L_{t,x}^4},$$

outside a set of probability at most

$$C \exp \left( -c R^2 T^{-\frac{1}{2} +} \| f \|_{H^\gamma}^{-2} \right) + C \exp \left( -c R^2 \| f \|_{H^\gamma}^{-2} \right)$$

as long as (3.11) holds.

**Subcase 2b.iii.** $M_2 \geq M_1 \gtrsim M_3^{\frac{3}{N+4}}$. By Hölder’s inequality, we have

$$\left| \int_0^T \int_{\mathbb{R}^N} z_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 dxdt \right| \lesssim \| z_1 \|_{L_{t,x}^{6(N+4)}} \| z_2 \|_{L_{t,x}^{6(N+4)}} \| \langle \nabla \rangle^{\frac{N-4}{2}} z_3 \|_{L_{t,x}^{6(N+4)}} \| v_4 \|_{L_{t,x}^{6(N+4)}}$$

$$\lesssim M_1^{-\gamma} M_2^{-\gamma} M_3^{-\frac{3}{2} - \gamma} \prod_{j=1}^3 \| \langle \nabla \rangle^\gamma z_j \|_{L_{t,x}^{6(N+4)}} \| v_4 \|_{Y^0}.$$
\[
\lesssim M_3^{\frac{N-4}{N+3}} \cdot \frac{1}{N+3} \sum_{j=1}^{3} \| (\nabla)^{\gamma} z_j \|_{L^{2(N+4)}_{t,x}} \lesssim T^{0+} R^3
\]
outside a set of probability at most
\[
C \exp \left( -cR^2 T^{-\frac{N+3}{2(N+4)}} \| f \|_{H^\gamma}^2 \right)
\]
as long as (3.11) holds.

**Case 3.** \(vuv\) case. Without loss of generality, we assume \(M_1 \geq M_2\).

**Subcase 3a.** \(M_1 \gtrsim M_3\). In this case, we only perform the dyadic decomposition on \(v_1, v_2\) and \(z_3\), Note that \(M_1 \gtrsim \max\{M_2, M_3, |\xi_4|\}\), where \(\xi_4\) is the spatial frequency of \(v_4\). By Hölder’s inequality and (2.10), we have
\[
\left| \int_0^T \int_{\mathbb{R}^N} (\nabla)^{\frac{N-4}{N+3}} v_1 v_2 z_3 v_4 dx dt \right| \lesssim \sum_{M_1 \geq M_2} \| (\nabla)^{\frac{N-4}{N+3}} P_{M_1} v_1 P_{M_2} v_2 \|_{L^2_{t,x}} \sum_{M_3} \| P_{M_3} z_3 \|_{L^{2(N+4)}_{t,x}} \| v_4 \|_{L^{2(N+4)}_{t,x}}.
\]
\[
\lesssim \sum_{M_1 \geq M_2} M_2^{\frac{N-4}{M_1}} \left( \frac{M_1}{M_2} \right)^{\frac{1}{2}} \| (\nabla)^{\frac{N-4}{N+3}} P_{M_1} v_1 \|_{X^{0}} \| P_{M_2} v_2 \|_{X^{0}} \sum_{M_3} \| P_{M_3} z_3 \|_{L^{2(N+4)}_{t,x}} \| v_4 \|_{X^{0}}^\gamma.
\]
If \(M_1 \sim M_2\), then we simply remove \(\left( \frac{M_1}{M_2} \right)^{\frac{1}{2}}\) and use Cauchy-Schwarz inequality to bound
\[
\sum_{M_1 \sim M_2} \| P_{M_1} v_1 \|_{X^{\frac{N-4}{N+4}}} \| P_{M_2} v_2 \|_{X^{\frac{N-4}{N+4}}} \lesssim \left( \sum_{M_1} \| P_{M_1} v_1 \|_{X^{\frac{N-4}{N+4}}}^2 \right)^{\frac{1}{2}} \left( \sum_{M_2} \| P_{M_2} v_2 \|_{X^{\frac{N-4}{N+4}}}^2 \right)^{\frac{1}{2}} \lesssim \| v_1 \|_{X^{\frac{N-4}{N+4}}} \| v_2 \|_{X^{\frac{N-4}{N+4}}}.
\]
If \(M_1 \gg M_2\), then we can extract from \(\left( \frac{M_1}{M_2} \right)^{\frac{1}{2}}\) a negative power of \(M_1\) which allows to sum over \(M_1\). By extracting a negative power of \(M_1\), we can sum over \(M_2\). We thus get
\[
\left| \int_0^T \int_{\mathbb{R}^N} (\nabla)^{\frac{N-4}{N+3}} v_1 v_2 z_3 v_4 dx dt \right| \lesssim T^{0+} R \| v_1 \|_{X^{\frac{N-4}{N+4}}} \| v_2 \|_{X^{\frac{N-4}{N+4}}}
\]
outside a set of probability at most
\[
C \exp \left( -cR^2 T^{-\frac{N+3}{2(N+4)}} \| f \|_{H^\gamma}^2 \right).
\]

**Subcase 3b.** \(M_3 \gg M_1 \geq M_2\). Note that we must have \(M_3 \sim M_4\).

**Subcase 3b.i.** \(M_1 \gtrsim M_3^{\frac{1}{N+3}}\). By Hölder’s inequality, Lemma 2.12 and (2.10), we have
\[
\left| \int_0^T \int_{\mathbb{R}^N} v_1 v_2 (\nabla)^{\frac{N-4}{N+3}} z_3 v_4 dx dt \right| \lesssim \| v_1 \|_{L^{2(N+4)}_{t,x}} \| (\nabla)^{\frac{N-4}{N+3}} z_3 \|_{L^{\frac{N+4}{N+3}}_{t,x}} \| v_2 v_4 \|_{L^2_{t,x}}.
\]
We want the power of \( \frac{N-4}{2(N-1)} \) which is less restrictive than (3.11). Outside a set of probability at most

\[
C \exp \left( -cR^2 T \frac{\frac{N-4}{2(N-1)}}{\frac{N-4}{2(N-1)}} \right)
\]

provided

\[
\frac{(N-4)(N-7)}{2(N-1)} - \gamma < 0 \quad \text{or} \quad \gamma > \frac{(N-4)(N-7)}{2(N-1)}
\]

which is less restrictive than (3.11).

**Subcase 3b.ii.** \( M_3^{\frac{N-4}{2(N-1)}} \gg M_1 \). By Hölder’s inequality, we have

\[
\left| \int_0^T \int_{\mathbb{R}^N} v_1 v_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 dxdt \right| \leq \| v_1 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 \|_{L^1_{t,x}} \| v_2 \|_{L^2_{t,x}} \| v_4 \|_{L^2_{t,x}}.
\]

Let \( a > 0 \) be a small constant to be chosen later, we use Lemma 2.12 and (2.10) to estimate

\[
\| v_1 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 \|_{L^1_{t,x}} \leq M_3 \langle \frac{N-4}{2(N-1)} \rangle M_4 \langle \frac{2(N-4)}{2(N-1)} \rangle \| v_1 \|_{L^2_{t,x}} \| v_2 \|_{L^\infty_{t,x}} \| v_3 \|_{L^1_{t,x}} \| v_4 \|_{L^2_{t,x}}.
\]

Similarly, we have

\[
\| v_2 v_4 \|_{L^2_{t,x}} \leq M_2 \langle \frac{N-4}{2(N-1)} \rangle M_4 \langle \frac{3}{2} \rangle + \| v_2 \|_{L^\infty_{t,x}} \| v_4 \|_{L^2_{t,x}} \leq M_2 \langle \frac{N-4}{2(N-1)} \rangle M_4 \langle \frac{3}{2} \rangle + \| v_2 \|_{L^\infty_{t,x}} \| v_4 \|_{L^2_{t,x}}.
\]

It follows that

\[
\left| \int_0^T \int_{\mathbb{R}^N} v_1 v_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 dxdt \right| \leq M_3 \langle \frac{(N-4)a}{2(N-1)} \rangle M_4 \langle \frac{3}{2} \rangle + M_3 \langle \frac{(N-4)a}{2(N-1)} \rangle \| v_1 \|_{L^2_{t,x}} \| v_2 \|_{L^\infty_{t,x}} \| v_3 \|_{L^1_{t,x}} \| v_4 \|_{L^2_{t,x}}.
\]

provided

\[
\frac{3}{2} - \frac{(N-1)a}{2} > 0 \quad \text{or} \quad a < \frac{3}{N-1}.
\]

We want the power of \( M_3 \) is strictly negative in order to sum over dyadic blocks. This requires

\[
\gamma > \frac{(N-4)(N-7)}{2(N-1)}
\]
which is less restrictive than (3.11). It follows that
\[ \left| \int_0^T \int_{\mathbb{R}^N} v_1 v_2 \left( \nabla \right)^{\frac{N-4}{2}} z_3 v_4 \, dx \, dt \right| \lesssim T^0 R \| v_1 \| X^{\frac{N-4}{2}} \| v_2 \| X^{\frac{N-4}{2}} \]
outside a set of probability at most
\[ C \exp \left( -c R^2 T^{-\frac{1}{12}} \| f \|_{H^{\gamma}}^{-2} \right) + C \exp \left( -c R^2 \| f \|_{H^{\gamma}}^{-2} \right). \]

**Case 4.** vzz case. Without loss of generality, we assume that \( M_3 \geq M_2. \)

**Subcase 4a.** \( M_1 \geq M_3. \) By Hölder’s inequality and Lemma 2.12, we have
\[ \left| \int_0^T \int_{\mathbb{R}^N} v_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 \, dx \, dt \right| \lesssim \| v_1 \|_{X^{\frac{N-4}{2}}} \| z_2 \|_{X^{\frac{N-4}{2}}} \| z_3 \|_{X^{\frac{N-4}{2}}} \| v_4 \|_{X^{\frac{N-4}{2}}} \lesssim T^0 R \| v_1 \| X^{\frac{N-4}{2}} \]
outside a set of probability at most
\[ C \exp \left( -c R^2 T^{-\frac{1}{12}} \| f \|_{H^{\gamma}}^{-2} \right). \]

Here we have used that if \( M_3 \geq \max\{M_1, M_4\}, \) then we can extract a negative power of \( M_3 \) to sum over \( M_1 \) and \( M_4. \) Otherwise, we have \( M_1 \sim M_4 \gg M_3. \) In this case, we can use Cauchy-Schwarz inequality to sum over \( M_1 \) and \( M_4. \)

**Subcase 4b.** \( M_3 \gg M_1. \)

**Subcase 4b.1.** \( M_3 \sim M_2 \gg M_1. \) We must have \( M_1 \sim M_4. \) By Hölder's inequality and Lemma 2.12, we have
\[ \left| \int_0^T \int_{\mathbb{R}^N} v_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 \, dx \, dt \right| \lesssim \| v_1 \|_{X^{\frac{N-4}{2}}} \| z_2 \|_{X^{\frac{N-4}{2}}} \| z_3 \|_{X^{\frac{N-4}{2}}} \| v_4 \|_{X^{\frac{N-4}{2}}} \lesssim T^0 R \| v_1 \| X^{\frac{N-4}{2}} \]
outside a set of probability at most
\[ C \exp \left( -c R^2 T^{-\frac{1}{12}} \| f \|_{H^{\gamma}}^{-2} \right) \]
as long as \( \gamma > \frac{N-4}{4} \) which is less restrictive than (3.11).

**Subcase 4b.2.** \( M_3 \gg M_2, M_1. \) In this case, we must have \( M_3 \sim M_4. \)

**Subcase 4b.2.i.** \( M_1, M_2 \ll M_3^{-\frac{1}{2}}. \) By Hölder’s inequality, (3.10) and Lemma 2.12, we have
\[ \left| \int_0^T \int_{\mathbb{R}^N} v_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 \, dx \, dt \right| \lesssim \| z_2 \|_{X^{\frac{N-4}{2}}} \| v_1 \|_{L_t^6 L_x^6} \| v_4 \|_{L_t^6 L_x^6} \lesssim \left( M_1 \right)^{\frac{N-4}{4}} \left( M_2 \right)^{\frac{N-4}{4}} - \gamma \left( \frac{N-4}{4} \right) \left( \frac{N-4}{4} \right) + M_3^{\frac{N-4}{4}} - \gamma + \frac{3}{2} + M_4^{-\frac{1}{2}} \]
\[ \times \| v_1 \|_{X^0} \prod_{j=2}^3 \left( \| \langle \nabla \rangle^j z_j \|_{L_t^6 L_x^6} \| P_M f^a \|_{H^{\gamma}}^{1-a} \right) \| v_4 \|_{Y^0} \]
provided
\[ a < 1 - \frac{2\gamma}{N-1}. \]
We want the largest frequency to have a negative power in order to sum over dyadic blocks. This requires
\[ \gamma > \frac{(N - 4)^2}{2(N + 2)} \]
which is again less restrictive than (3.11). We thus get
\[ C \exp \left(-cR^2T^{-\frac{1}{2}}\|f\|_{H^s}^{-2}\right) + C \exp \left(-cR^2\|f\|_{H^s}^{-2}\right). \]

**Subcase 4b.2.ii.** \( M_1 \ll M_3^{3-\gamma} \ll M_2. \) By Hölder’s inequality, Lemma 2.12 and (2.10), we have
\[ \left| \int_0^T \int_{\mathbb{R}^N} v_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 \, dx \, dt \right| \]
\[ \leq m \| v_2 \|_{L^1_{t,x}} \| \nabla \|^{\frac{N-4}{2}} z_3 \| v_1 v_4 \|_{L^1_{t,x}} \]
\[ \leq M_1^{\frac{N-4}{2}} - M_2^\gamma M_3^{\frac{N-4}{2} - \gamma} M_4^{\frac{1}{2} +} \| v_1 \|_{X_0} \prod_{j=2}^3 \| \nabla \|^{\gamma} z_j \| v_4 \|_{Y^0} \]
\[ \leq M_1^{\frac{N-4}{2}} - M_2^\gamma M_3^{\frac{N-4}{2} - \gamma} m \| v_1 \|_{X_0} \prod_{j=2}^3 \| \nabla \|^{\gamma} z_j \| L^1_{t,x} \]
\[ \leq M_3^{\frac{(N-4)^2}{N-1} \gamma} \prod_{j=2}^3 \| \nabla \|^{\gamma} z_j \| L^1_{t,x} \leq T^{0+} R^2 \| v_1 \|_{X_0^{\frac{N-4}{2}}} \]
outside a set of probability at most
\[ C \exp \left(-cR^2T^{-\frac{1}{2}}\|f\|_{H^s}^{-2}\right) \]
as long as
\[ \gamma > \frac{(N - 4)^2}{2(N + 2)} \]
which is satisfied if (3.11) holds.

**Subcase 4b.2.iii.** \( M_2 \ll M_3^{3-\gamma} \ll M_1. \) By Hölder’s inequality, Lemma 2.12 and (2.10), we have
\[ \left| \int_0^T \int_{\mathbb{R}^N} v_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 \, dx \, dt \right| \]
By Bernstein’s inequality and Hölder’s inequality, we have

\[ \| P_\leq M \mathcal{N}(v + \varepsilon z) \|_{L^1_t L^2_x \mathbb{R}^N} \lesssim M^{\frac{N-4}{2}} \| \mathcal{N}(v + \varepsilon z) \|_{L^1_t L^2_x (\mathbb{R} \times \mathbb{R}^N)} \]

outside a set of probability at most

\[ C \exp \left( -c R^2 \| f \|_{H^s}^{-2} \right) + C \exp \left( -c R^2 T^{-\frac{4}{N-4}} \| f \|_{H^s}^{-2} \right) \]

as long as

\[ \gamma > \frac{(N - 4)^2}{2(N + 2)} \]

which is satisfied if (3.11) holds.

**Subcase 4b.2.iv.** \( M_1, M_2 \gtrsim M_3^{\frac{4}{N-4}} \). By Hölder’s inequality and Lemma 2.12, we have

\[
\left| \int_0^T \int_{\mathbb{R}^N} v_1 z_2 \langle \nabla \rangle^{\frac{N-4}{2}} z_3 v_4 dx dt \right| \\
\leq \| v_1 \|_{L^2_t L^{\frac{N-4}{2}}_x} \| z_2 \|_{L^\infty_t L^{\frac{N-4}{2}}_x} \| \langle \nabla \rangle^{\frac{N-4}{2}} z_3 \|_{L^1_t L^{\frac{N-4}{2}}_x} \| v_4 \|_{L^2_t L^{\frac{N-4}{2}}_x} \\
\lesssim M_1^{\frac{N-4}{2}} M_2^{\frac{N-4}{2}} M_3^{\frac{N-4}{2}} \| v_1 \|_{L^\infty_t L^{\frac{N-4}{2}}_x} \prod_{j=2}^3 \| \langle \nabla \rangle^\gamma z_j \|_{L^\infty_t L^{\frac{N-4}{2}}_x} \| v_4 \|_{L^2_t L^{\frac{N-4}{2}}_x} \\
\lesssim M_3^{\frac{(N-4)^2}{N-1}} M_2^{\frac{(N-4)^2}{N-1}} \| v_1 \|_{L^\infty_t L^{\frac{N-4}{2}}_x} \prod_{j=2}^3 \| \langle \nabla \rangle^\gamma z_j \|_{L^\infty_t L^{\frac{N-4}{2}}_x} \lesssim T^0 + R^2 \| v_1 \|_{L^\infty_t L^{\frac{N-4}{2}}_x} \\
\]

outside a set of probability at most

\[ C \exp \left( -c R^2 T^{-\frac{4}{N-4}} \| f \|_{H^s}^{-2} \right) \]

as long as

\[ \gamma > \frac{(N - 4)^2}{2(N + 2)} \]

which is again satisfied if (3.11) holds.

Collecting the above cases, we prove (3.3).

- We next estimate (3.5), the estimate (3.6) is treated in a similar manner. Given \( M \geq 1 \), we define

\[ \Phi_M(v(t)) := \mp i \int_0^t U_\mu(t-s) P_\leq M \mathcal{N}(v + \varepsilon z)(s) ds. \]

By Bernstein’s inequality and Hölder’s inequality, we have

\[
\| P_\leq M \mathcal{N}(v + \varepsilon z) \|_{L^1_t L^2_x (\mathbb{R} \times \mathbb{R}^N)} \lesssim M^{\frac{N-4}{2}} \| \mathcal{N}(v + \varepsilon z) \|_{L^1_t L^2_x (\mathbb{R} \times \mathbb{R}^N)} \\
\lesssim M^{\frac{N-4}{2}} \| v \|_{L^1_t L^2_x (\mathbb{R} \times \mathbb{R}^N)} + M^{\frac{N-4}{2}} \| \varepsilon z \|_{L^1_t L^2_x (\mathbb{R} \times \mathbb{R}^N)}. 
\]
Thanks to the global in time probabilistic Strichartz estimates and the fact \(6N \geq \frac{6N}{3N-8}\), we see that the second term in the right hand side is finite almost surely. On the other hand, by Sobolev embedding and Lemma 2.12, we have
\[
\|v\|_{L_t^4L_x^6(\mathbb{R}^N)} \lesssim \|\nabla^\frac{N-4}{2}v\|_{L_t^\frac{6N}{N-8}(\mathbb{R}^N)} \lesssim \|v\|_{X_t^\frac{N-4}{2}(\mathbb{R}^N)} < \infty.
\]
This shows that for each \(M \geq 1\), \(P_{\leq M}N(v + \varepsilon z) \in L_t^1H_x^{\frac{2N}{N-2}}(\mathbb{R} \times \mathbb{R}^N)\) almost surely. Thus, by Lemma 2.15,
\[
\|\hat{\Phi}_M(v)\|_{X_t^\frac{N-4}{2}(\mathbb{R})} \lesssim \sup_{v_4 \in \mathcal{V}^{y_4}(\mathbb{R})} \left| \int_\mathbb{R} \int_\mathbb{R}^N \langle \nabla \rangle^\frac{N-4}{2} \left[ N(v + \varepsilon z)(t,x) \right] \psi_4(t,x) dx dt \right| \quad (3.12)
\]
almost surely, where \(v_4 = P_{\leq M}v_4\). As above, we will estimate the right hand side of (3.12) independent of the cutoff size \(M \geq 1\) by performing a case-by-case analysis of expressions of the form
\[
\left| \int_\mathbb{R} \int_\mathbb{R}^N \langle \nabla \rangle^\frac{N-4}{2} (w_1w_2w_3) v_4 \right| dx dt, \quad (3.13)
\]
where \(\|v_4\|_{Y^{y_4}(\mathbb{R})} \leq 1\) and \(w_j = v\) or \(z, j = 1, 2, 3\). Then, letting \(M \to \infty\), the same estimate holds for \(\hat{\Phi}(v)\) without any cutoff.

The rest of the proof follows in a similar manner as the proof of the first part by changing the time interval from \([0,T]\) to \(\mathbb{R}\) and replacing \(z\) by \(\varepsilon z\). Note that \(\left(\frac{N+4}{2}, \frac{2N(N+4)}{N^2+4N-16}\right), (4, \frac{2N}{N-2})\) and \(\left(\frac{6N+4}{N+8}, \frac{6N(N+4)}{3N^2+8N-32}\right)\) are Biharmonic admissible and \(\frac{N+4}{2} \geq \frac{2N(N+4)}{N^2+4N-16}, 4 \geq \frac{2N}{N-2}\) and \(\frac{6N+4}{N+8} \geq \frac{6N(N+4)}{3N^2+8N-32}\), we can use the global in time probabilistic Strichartz estimates for \((q, \tilde{r}) = (\frac{N+4}{2}, \frac{N+4}{N+8})\), \((4, 4)\) and \((\frac{6N+4}{N+8}, \frac{6N+4}{N+8})\), for instance
\[
\|\langle \nabla \rangle^{\gamma z}\|_{L_t^{\frac{N+4}{2}}} \leq \frac{R}{\varepsilon}
\]
outside a set of probability at most
\[
C \exp \left( -cR^2 \varepsilon^{-2} \|f\|_{H^{7/2}}^2 \right).
\]
We see that the contribution to (3.13) is given by

Case 2: \(R^3\), Case 3: \(R^2 \prod_{j=1}^2 \|v_j\|_{X_t^\frac{N-4}{2}(\mathbb{R})}\), Case 4: \(R^2 \|v_1\|_{X_t^\frac{N-4}{2}(\mathbb{R})}\)

outside a set of probability at most
\[
C \exp \left( -cR^2 \varepsilon^{-2} \|f\|_{H^{7/2}}^2 \right)
\]
in all cases as long as \(\gamma > \gamma_N\). The proof is complete.

4. Probabilistic well-posedness. We are now able to prove the almost sure local well-posedness for (1.1) given in Theorem 1.1.

Proof of Theorem 1.1. We will show that (1.9) is almost sure locally well-posed. To this end, we consider
\[
\mathcal{X} := \left\{ v \in X_t^\frac{N-4}{2}([0,T]) \cap C([0,T], H_x^{\frac{N-4}{2}}(\mathbb{R}^N)) : \|v\|_{X_t^\frac{N-4}{2}([0,T])} \leq \eta \right\}
\]
equipped with the distance
\[ d(v_1, v_2) := \|v_1 - v_2\|_{X^{N/4}([0,T])} \]
for some \( T, \eta > 0 \) to be chosen later. It is enough to show that the functional
\[ \Phi(v(t)) = \mp i \int_0^t U_\mu(t-s) N(v + z)(s) \, ds \]
is a contraction on \((\mathcal{X}, d)\). By Proposition 1, we have for \( 0 < T \leq 1 \), there exists \( 0 < \vartheta \ll 1 \) such that
\[ \|\Phi(v)\|_{X^{N/4}([0,T])} \leq C_1 \left( \|v\|_{X^{N/4}([0,T])}^3 + T^\vartheta R^3 \right), \]
\[ \|\Phi(v_1) - \Phi(v_2)\|_{X^{N/4}([0,T])} \leq C_2 \left( \sum_{j=1}^{2} \|v_j\|_{X^{N/4}([0,T])}^2 + T^\vartheta R^2 \right) \|v_1 - v_2\|_{X^{N/4}([0,T])} \]
for all \( v, v_1, v_2 \in X^{N/4}([0,T]) \) and \( R > 0 \), outside a set of probability at most
\[ C \exp \left( -cR^2 \|f\|_{H^\gamma(\mathbb{R}^N)}^2 \right). \]
It follows that for \( v, v_1, v_2 \in \mathcal{X} \),
\[ \|\Phi(v)\|_{X^{N/4}([0,T])} \leq C_1 \left( \eta^3 + T^\vartheta R^3 \right), \]
\[ d(\Phi(v_1), \Phi(v_2)) \leq C_2 \left( 2\eta^2 + T^\vartheta R^2 \right) d(v_1, v_2) \]
outside a set of probability at most
\[ C \exp \left( -cR^2 \|f\|_{H^\gamma(\mathbb{R}^N)}^2 \right). \]
We choose \( \eta > 0 \) small so that
\[ C_1 \eta^2 \leq \frac{1}{2}, \quad 2C_2 \eta^2 \leq \frac{1}{4}. \]
For given \( R \gg 1 \), we choose \( T = T(R) \) so that
\[ C_1 T^\vartheta R^3 \leq \frac{\eta}{2}, \quad C_2 T^\vartheta R^2 \leq \frac{1}{4}, \]
hence
\[ T^\vartheta = \min \left\{ \frac{\eta}{2C_1 R^3}, \frac{1}{4C_2 R^2} \right\}. \]
With such choices, we see that \( \Phi \) is a contraction on \((\mathcal{X}, d)\) outside a set of probability at most
\[ C \exp \left( -cT^{-\theta} \|f\|_{H^\gamma(\mathbb{R}^N)}^2 \right) \sim C \exp \left( -cT^{-\theta} \|f\|_{H^\gamma(\mathbb{R}^N)}^2 \right) \]
for some \( \theta > 0 \). The proof is complete. \( \square \)

We next prove the probabilistic small data global well-posedness and scattering for (1.1) given in Theorem 1.2.

**Proof of Theorem 1.2.** We consider
\[ \mathcal{Y} := \left\{ v \in X^{N/4}(\mathbb{R}) \cap C(\mathbb{R}, H^{N/4}(\mathbb{R}^N)) : \|v\|_{X^{N/4}(\mathbb{R})} \leq \delta \right\} \]
equipped with the distance
\[ d(v_1, v_2) := \|v_1 - v_2\|_{X^{N/4}(\mathbb{R})} \]
for some $\delta > 0$ to be chosen later. For $0 < \varepsilon \ll 1$, we will show that the functional

$$\hat{I}(v(t)) := \mp i \int_0^t U_\mu(t - s)N(v + \varepsilon z)(s) ds$$

is a contraction on $(\mathcal{Y}, d)$. By Proposition 1 with $R = \delta$, we have for any $v, v_1, v_2 \in \mathcal{Y}$,

$$\|\hat{I}(v)\|_{X^{\frac{N-4}{2}}(\mathbb{R})} \leq 2C_3\delta^3,$$

$$d(\hat{I}(v_1), \hat{I}(v_2)) \leq 3C_4\delta^2d(v_1, v_2)$$

outside a set of probability at most

$$C \exp \left(-c\delta^2\varepsilon^{-2}\|f\|_{H^\gamma(\mathbb{R}^N)}^{-2}\right).$$

By choosing $\delta > 0$ small so that

$$2C_3\delta^2 \leq 1, \quad 3C_4\delta^2 \leq \frac{1}{2},$$

we see that $\hat{I}$ is a contraction on $(\mathcal{Y}, d)$ outside a set of probability at most

$$C \exp \left(-c\delta^2\varepsilon^{-2}\|f\|_{H^\gamma(\mathbb{R}^N)}^{-2}\right).$$

Noting that $\delta$ is an absolute constant, we conclude that for each $0 < \varepsilon \ll 1$, there exists a set $\Omega_\varepsilon \subset \Omega$ such that

- $\mathcal{P}(\Omega \setminus \Omega_\varepsilon) \leq C \exp(-c\varepsilon^{-2}\|f\|_{H^\gamma(\mathbb{R}^N)}^{-2})$;
- For each $\omega \in \Omega_\varepsilon$, there exists a unique global in time solution to (1.1) with initial data $\varepsilon f_\omega$ in the class

$$\mathcal{E}U_\mu(t)f_\omega + C(\mathbb{R}, H^\gamma(\mathbb{R}^N)) \subset C(\mathbb{R}, H^\gamma(\mathbb{R}^N)).$$

It remains to show the scattering. Fix $\omega \in \Omega_\varepsilon$ and let $v = v(\varepsilon, \omega)$ be the global in time solution to (1.9) constructed above. We will show that there exists $f_\omega^* \in H^{\frac{N-4}{2}}(\mathbb{R}^N)$ such that

$$U_\mu(-t)v(t) = \mp i \int_0^t U_\mu(s)N(v + \varepsilon z)(s) ds \to f_\omega^*$$

in $H^{\frac{N-4}{2}}(\mathbb{R}^N)$ as $t \to \infty$. Set $w(t) = U_\mu(-t)v(t)$. For $0 < t_1 \leq t_2 < \infty$, we have

$$U_\mu(t_2)(w(t_2) - w(t_1)) = \mp i \int_{t_1}^{t_2} U_\mu(t_2 - s)N(v + \varepsilon z)(s) ds$$

$$= \mp i \int_0^{t_2} U_\mu(t_2 - s)\chi_{[t_1, \infty)}(s)N(v + \varepsilon z)(s) ds =: I(t_1, t_2).$$

In the following, we view $I(t_1, t_2)$ as a function on $t_2$ and estimate its $X^{\frac{N-4}{2}}([0, \infty))$-norm. We now revisit the computation in the proof of Proposition 1.

In Case 1, we proceed slightly differently. By Lemma 2.15 and Hölder’s inequality, we have

$$\|I(t_1, t_2)\|_{X^{\frac{N-4}{2}}([0, \infty))} \leq \sup_{v \in \mathcal{E}U_\mu(\mathbb{R}^N)} \left|\int_0^\infty \int_{\mathbb{R}^N} \chi_{[0, \infty)}(t) \langle \nabla \rangle^{\frac{N-4}{2}} \varepsilon \nabla v \nabla v dx dt\right|$$

$$\leq \|\langle \nabla \rangle^{\frac{N-4}{2}} v\|_{L^1([0, \infty) \times \mathbb{R}^N)} \|v\|_{L^{\frac{N}{N-4}}([0, \infty) \times \mathbb{R}^N)}.$$  \hspace{1cm} (4.3)
By Lemma (2.12), we have
\[
\| \langle \nabla \rangle^{\frac{N-4}{2}} v \|_{L_t^{2(N+4)}(\mathbb{R}^N)} + \| v \|_{L_t^{\frac{N-4}{2},\infty}(\mathbb{R}^N)} \lesssim \| v \|_{X^{\frac{N-4}{2}}(\mathbb{R})} \leq \delta.
\]
Then the monotone convergence theorem, (4.3) tends to 0 as \( t_1 \to \infty \).

In Cases 2, 3 and 4, we had at least one factor \( z \). We multiply the cutoff function \( \chi \) only on the \( \varepsilon z \)-factors but not on the \( v \)-factors. Note that \( \| v \|_{X^{\frac{N-4}{2}}(\mathbb{R})} \leq \delta \).

As in the proof of Proposition 1, we estimate at least a small portion of these \( z \)-factors in \( \| \langle \nabla \rangle^q \varepsilon z^\omega \|_{L_t^r((t_1,\infty)\times\mathbb{R}^N)} \) with \( q = \frac{N+4}{2} \) or 4 or \( \frac{6(N+4)}{N+8} \). Recall that \( \| \langle \nabla \rangle^q \varepsilon z^\omega \|_{L_t^r((t_1,\infty)\times\mathbb{R}^N)} \leq \delta \) for \( \omega \in \Omega_c \). Hence, by the monotone convergence theorem, we have \( \| \langle \nabla \rangle^q \varepsilon z^\omega \|_{L_t^r((t_1,\infty)\times\mathbb{R}^N)} \to 0 \) as \( t_1 \to \infty \). Thus, the contributions from Cases 2, 3 and 4 tend to 0 as \( t_1 \to \infty \). Therefore,
\[
\lim_{t_1 \to \infty} \| (t_1, t_2) \|_{X^{\frac{N-4}{2}}((0,\infty))} = 0.
\]

In conclusion, we obtain
\[
\lim_{t_1 \to \infty} \sup_{t_2 > t_1} \| w(t_2) - w(t_1) \|_{H^{\frac{N-4}{2}}(\mathbb{R}^N)} = \lim_{t_1 \to \infty} \sup_{t_2 > t_1} \| U_\mu(t)(w(t_2) - w(t_1)) \|_{H^{\frac{N-4}{2}}(\mathbb{R}^N)} = \lim_{t_1 \to \infty} \| (t_1, t_2) \|_{L_t^2H^{\frac{N-4}{2}}((0,\infty)\times\mathbb{R}^N)} \lesssim \lim_{t_1 \to \infty} \| (t_1, t_2) \|_{X^{\frac{N-4}{2}}((0,\infty))} = 0.
\]
This proves (4.2) and the scattering of \( w^\omega(t) = \varepsilon U_\mu(t)f^\omega + v^\omega(t) \), which completes the proof of Theorem 1.2.

Finally, we prove the global well-posedness and scattering with a large probability given in Theorem 1.3. We follow the idea of [3], that is to exploit the dilation symmetry (1.2) of the cubic 4NLS (1.1). Denote
\[
f_\lambda(x) := \lambda^2 f(\lambda x), \quad \lambda > 0.
\]
We have
\[
\| f_\lambda \|_{H^7(\mathbb{R}^N)} = \lambda^7 \frac{N-4}{2} \| f \|_{H^7(\mathbb{R}^N)}.
\]
If \( \gamma < \gamma_c = \frac{N-4}{2} \), then we can make the \( H^7 \)-norm of the scaled function \( f_\lambda \) small by taking \( \lambda \gg 1 \). The issue is that the Strichartz estimates we employ in proving probabilistic well-posedness are (sub)critical and do not become small even if we take \( \lambda \gg 1 \). It is for this reason that we consider the randomization \( f_\lambda \) on dilated cubes.

**Proof of Theorem 1.3.** Fix \( f \in H^7(\mathbb{R}^N) \) with \( \gamma \in (\gamma_N, \gamma_c) \), where \( \gamma_N \) is as in (1.7). Let \( f^{\omega,\lambda} \) be its randomization on dilated cubes of scale \( \lambda \) as in (1.10). Instead of considering (1.1) with \( u_0 = f^{\omega,\lambda} \), we consider the scaled Cauchy problem
\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t u_\lambda - \Delta^2 u_\lambda = \pm |u_\lambda|^2 u_\lambda, \\
u_\lambda|_{t=0} = u_{0,\lambda} = (f^{\omega,\lambda})_\lambda,
\end{array} \right.
\end{align*}
\]
where \( u_\lambda \) is as in (1.2) and \( (f^{\omega,\lambda})_\lambda(x) = \lambda^2 f^{\omega,\lambda}(\lambda x) \) is the scaled randomization. For simplicity, we denote \( (f^{\omega,\lambda})_\lambda \) by \( f^{\omega,\lambda}_\lambda \) in the following. We denote the linear and nonlinear part of \( u_\lambda \) by \( u_\lambda(t) = u_\lambda^N(t) := U_0(t)f^{\omega,\lambda}_\lambda \) and \( u_\lambda(t) := u_\lambda(t) - U_0(t)f^{\omega,\lambda}_\lambda \).
We reduce (4.5) to
\[
\begin{aligned}
&i\partial_t v_\lambda - \Delta^2 v_\lambda = \pm |v_\lambda + z_\lambda|^2 (v_\lambda + z_\lambda), \\
&v_\lambda|_{t=0} = 0.
\end{aligned} \tag{4.6}
\]

Note that if u satisfies (1.1) with initial data \(u(0) = f^{\omega,\lambda}\), then \(u_\lambda, z_\lambda\) and \(v_\lambda\) are the scalings of \(u, z := U_0(t) f^{\omega,\lambda}\) and \(v := u - z\) respectively. In fact, it is clear for \(u_\lambda\) by using (1.2). For \(z_\lambda\) and \(v_\lambda\), this follows from the following observation:
\[
\mathcal{F}_x [U_0(t) f^{\omega,\lambda}_\lambda]\big| (\xi) = \lambda^2 \mathcal{F}_x \big| (\xi) = e^{-it\xi^4} f^{\omega,\lambda}_\lambda(\lambda^{-1} \xi) = e^{-it\xi^4} f^{\omega,\lambda}_\lambda(\xi) = \tilde{z}_\lambda(t, \xi). \tag{4.7}
\]

Define
\[
\hat{\Phi}_\lambda(v_\lambda(t)) := \mp i \int_0^t U_0(t - s) \mathcal{N}(v_\lambda + z_\lambda)(s) ds.
\]

We will show that there exists \(\lambda_0 = \lambda_0(\varepsilon, \|f\|_{\mathcal{H}^\gamma(\mathbb{R}^N)}) > 0\) such that, for \(\lambda > \lambda_0\), the estimates (3.5) and (3.6) in Proposition 1 (with \(\tilde{\Phi}\) replaced by \(\hat{\Phi}_\lambda\)) hold with \(R = \delta\) outside a set of probability strictly smaller than \(\varepsilon\), where \(\delta\) is as in (4.1). In fact, we first observe that
\[
\psi(D-n)f_\lambda = (\psi(D-\lambda^{-1}n)f)_\lambda.
\]

Hence, we have
\[
f^{\omega,\lambda}_\lambda = (f^{\omega,\lambda})_\lambda = \sum_{n \in \mathbb{Z}^N} g_n(\omega) \psi(D-n)f_\lambda. \tag{4.8}
\]

Given \(\delta\) as in (4.1) and \(\lambda > 0\), we define
\[
\Omega_{1,\lambda} := \left\{ \omega \in \Omega : \|U_0(t) f^{\omega,\lambda}_\lambda\|_{L^t_t L^2_x W^{2,\gamma} (\mathbb{R} \times \mathbb{R}^N)} \leq \delta, \quad r = \frac{N + 4}{2}, 4, \frac{6(N + 4)}{N + 8} \right\}.
\]

We also define
\[
\Omega_{2,\lambda} := \left\{ \omega \in \Omega : \|f^{\omega,\lambda}_\lambda\|_{\mathcal{H}^\gamma(\mathbb{R}^N)} \leq \delta \right\}.
\]

Now, set \(\Omega_\lambda := \Omega_{1,\lambda} \cap \Omega_{2,\lambda}\). It follows from (4.8), Lemma 2.1 and Lemma 2.6 that
\[
\mathcal{P}(\Omega \setminus \Omega_\lambda) \leq C \exp \left( -c\delta^2 \|f\|_{\mathcal{H}^\gamma(\mathbb{R}^N)}^2 \right) \leq C \exp \left( -c\delta^2 \lambda^{-2\gamma + N - 4} \|f\|_{\mathcal{H}^\gamma(\mathbb{R}^N)}^2 \right)
\]
for \(\lambda \geq 1\). Note that
\[
\|f\|_{\mathcal{H}^\gamma}^2 = \|f\|_{L^2}^2 + \|f\|_{H^\gamma}^2 = \lambda^{-N+4} \|f\|_{L^2}^2 + \lambda^{2\gamma - N+4} \|f\|_{H^\gamma}^2
\]
\[
= \lambda^{2\gamma - N+4} (\lambda^{-2\gamma} \|f\|_{L^2}^2 + \|f\|_{H^\gamma}^2) \leq \lambda^{2\gamma - N+4} \|f\|_{H^\gamma}^2
\]
since \(\lambda \geq 1\). By setting
\[
\lambda_0 \sim \left( \frac{\log \left( \frac{1}{\varepsilon^2} \|f\|_{\mathcal{H}^\gamma(\mathbb{R}^N)} \right)}{\delta^2} \right)^{\frac{\gamma - N + 4}{N + 8}},
\]
we have
\[
\mathcal{P}(\Omega \setminus \Omega_\lambda) < \varepsilon
\]
for all \(\lambda > \lambda_0\). Note that \(\lambda_0 \to \infty\) as \(\varepsilon \to 0\).

Recall that the pairs \((\frac{N+4}{N+8}, \frac{N+4}{N+8}), (4, 4)\) and \((\frac{6(N+4)}{N+8}, \frac{6(N+4)}{N+8})\) are the only relevant values of the space-time Lebesgue indices controlling the random forcing term in the proof of Proposition 1. Hence, the estimates (3.5) and (3.6) in Proposition 1 (with \(\tilde{\Phi}\) replaced by \(\hat{\Phi}_\lambda\)) hold with \(R = \delta\) for each \(\omega \in \Omega_\lambda\). Then repeating the proof of Theorem 1.2, we see that for each \(\omega \in \Omega_\lambda\), there exists a unique global solution \(u_\lambda\) to (4.5) which scatters both forward and backward in time. By undoing
the scaling, we obtain a unique global solution $u$ to (1.1) with initial data $f^{\omega, \lambda}$ for each $\omega \in \Omega_{\lambda}$. Moreover, scattering for $u_{\lambda}$ implies the scattering for $u$. Indeed, as in Theorem 1.2, there exists $f^{\omega, \lambda}_+ \in H^{\frac{N-2}{4}}(\mathbb{R}^N)$ such that
\[
\lim_{t \to \infty} \| u_{\lambda}(t) - U_0(t) f^{\omega, \lambda}_+ - U_0(t) f^{\omega, \lambda}_- \|_{H^{\frac{N-2}{4}}(\mathbb{R}^N)} = 0.
\]
A computation similar to (4.7) gives
\[
U_0(t) f^{\omega, \lambda}_+ + U_0(t) f^{\omega, \lambda}_- = (U_0(t) f^{\omega, \lambda} + U_0(t) f^{\omega}_+)_{\lambda},
\]
where $f^{\omega}_+ = (f^{\omega, \lambda})_{\lambda-1} \in H^{\frac{N-2}{4}}(\mathbb{R}^N)$. Then, by (4.4), we obtain
\[
\lim_{t \to \infty} \| u(t) - U_0(t) f^{\omega, \lambda} - U_0(t) f^{\omega}_+ \|_{H^{\frac{N-2}{4}}(\mathbb{R}^N)} = 0.
\]
This proves that $u$ scatters forward in time. The proof is complete.

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\textbf{REFERENCES}

[1] M. Ben-Artzi, H. Koch and J. C. Saut, Dispersion estimates for fourth-order Schrödinger equations, \textit{C. R. Acad. Sci.}, \textbf{330} (2000), 87–92.

[2] A. Bényi, T. Oh and O. Pocovnicu, Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS, in \textit{Excursions in Harmonic Analysis}, Birkhäuser, Cham, 2015.

[3] A. Bényi, T. Oh and O. Pocovnicu, On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$, \textit{T. Am. Math. Soc. Ser. B}, \textbf{2} (2015), 1–50.

[4] A. Bényi, T. Oh and O. Pocovnicu, Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^1$, \textit{T. Am. Math. Soc. B}, \textbf{6} (2019), 144–160.

[5] T. Boulenger and E. Lenzmann, Blowup for biharmonic NLS, \textit{Ann. Sci. Éc. Norm. Supér.}, \textbf{50} (2017), 503–544.

[6] J. Bourgain, Invariant measures for the 2D-defocusing nonlinear Schrödinger equation, \textit{Commun. Math. Phys.}, \textbf{176} (1996), 421–445.

[7] N. Burq and N. Tzvetkov, Random data Cauchy theory for supercritical wave equations. I. Local theory, \textit{Invent. Math.}, \textbf{173} (2008), 449–475.

[8] N. Burq, L. Thomann and N. Tzvetkov, Long time dynamics for the one dimensional nonlinear Schrödinger equation, \textit{Ann. Inst. Fourier}, \textbf{63} (2013), 2137–2198.

[9] M. Chen and S. Zhang, Random data Cauchy problem for the fourth order Schrödinger equation with the second order derivative nonlinearities, \textit{Nonlinear Anal.}, \textbf{190} (2020), 111608.

[10] Y. Deng, Two-dimensional nonlinear Schrödinger equation with random radial data, \textit{Anal. PDE}, \textbf{5} (2012), 913–960.

[11] V. D. Dinh, On well-posedness, regularity and ill-posedness for the nonlinear fourth-order Schrödinger equation, \textit{Bull. Belg. Math. Soc. Simon Stevin}, \textbf{25} (2018), 415–437.

[12] V. D. Dinh, Global existence and scattering for a class of nonlinear fourth-order Schrödinger equation below the energy space, \textit{Nonlinear Anal.}, \textbf{172} (2018), 115–140.

[13] V. D. Dinh, On blowup solutions to the focusing intercritical nonlinear fourth-order Schrödinger equation, \textit{J. Dynam. Differ. Equ.}, \textbf{31} (2019), 1793–1823.

[14] V. D. Dinh, Dynamics of radial solutions for the focusing fourth-order nonlinear Schrödinger equations, \texttt{arXiv:2001.03022}.

[15] B. Dodson, J. Lührmann and D. Mendelson, Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation, \textit{Adv. Math.}, \textbf{347} (2019), 619–676.

[16] Q. Guo, Scattering for the focusing $L^2$-supercritical and $H^2$-subcritical biharmonic NLS equations, \textit{Commun. Partial Differ. Equ.}, \textbf{41} (2016), 185–207.

[17] M. Hadac, S. Herr and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, \textit{Ann. Inst. H. Poincaré Anal. Non Linéaire}, \textbf{26} (2009), 917–941.
[18] S. Herr, D. Tataru and N. Tzvetkov, Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in $H^1(T^3)$, *Duke Math. J.*, **159** (2011), 329–349.

[19] H. Hirayama and M. Okamoto, Random data Cauchy problem for the nonlinear Schrödinger equation with derivative nonlinearity, *Discrete Contin. Dyn. Syst.*, **36** (2016), 6943–6974.

[20] H. Hirayama and M. Okamoto, Random data Cauchy theory for the fourth order nonlinear Schrödinger equation with cubic nonlinearity, arXiv:1505.06497.

[21] V. I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrödinger-type equations, *Phys. Rev. E*, **53** (1996), 1336–1339.

[22] V. I. Karpman and A. G. Shagalov, Stability of soliton described by nonlinear Schrödinger-type equations with higher-order dispersion, *Phys. D*, **144** (2000), 194–210.

[23] M. Keel and T. Tao, Endpoint Strichartz estimates, *Am. J. Math.*, **120** (1998), 955–980.

[24] R. Killip, J. Murphy and M. Visan, Almost sure scattering for the energy-critical NLS with radial data below $H^1(\mathbb{R}^4)$, *Commun. Partial Differ. Equ.*, **44** (2019), 51–71.

[25] H. Koch, D. Tataru and M. Visan, *Dispersive equations and nonlinear waves*, Birkhäuser 45, Springer Basel, 2014.

[26] J. Lührmann and D. Mendelson, Random data Cauchy theory for the nonlinear wave equations of power-type on $\mathbb{R}^3$, *Commun. Partial Differ. Equ.*, **39** (2014), 2262–2283.

[27] C. Miao, G. Xu and L. Zhao, Global well-posedness and scattering for the defocusing energy critical nonlinear Schrödinger equation of fourth order in the radial case, *J. Differ. Equ.*, **246** (2009), 3715–3749.

[28] C. Miao, G. Xu and L. Zhao, Global well-posedness and scattering for the focusing energy critical nonlinear Schrödinger equations of fourth order in dimensions $d \geq 9$, *J. Differ. Equ.*, **251** (2011), 3381–3402.

[29] C. Miao, H. Wu and J. Zhang, Scattering theory below energy for the cubic fourth-order Schrödinger equation, *Math. Nachr.*, **288** (2015), 798–823.

[30] T. Oh, M. Okamoto and O. Pocovnicu, On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities, *Discrete Contin. Dyn. Syst.*, **39** (2019), 3479–3520.

[31] B. Pausader, Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case, *Dyn. Partial Differ. Equ.*, **4** (2007), 197–225.

[32] B. Pausader, The focusing energy-critical fourth-order Schrödinger equation with radial data, *Discrete Contin. Dyn. Syst.*, **24** (2009), 1275–1292.

[33] B. Pausader, The cubic fourth-order Schrödinger equation, *J. Funct. Anal.*, **256** (2009), 2473–2517.

[34] B. Pausader and S. Shao, The mass-critical fourth-order Schrödinger equation in high dimensions, *J. Hyper. Differ. Equ.*, **7** (2010), 651–705.

[35] B. Pausader and S. Xia, Scattering theory for the fourth-order Schrödinger equation in low dimensions, *Nonlinearity*, **26** (2013), 2175–2191.

[36] S. Zhang and S. Xu, The probabilistic Cauchy problem for the fourth order Schrödinger equation with special derivative nonlinearities, *Commun. Pure Appl. Anal.*, **19** (2020), 3367–3385.

[37] S. Zhu, H. Yang and J. Zhang, Limiting profile of the blow-up solutions for the fourth-order nonlinear Schrödinger equation, *Dyn. Partial Differ. Equ.*, **7** (2010), 187–205.

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