On lacunary Toeplitz determinants.

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Abstract

By using Riemann–Hilbert problem based techniques, we obtain the asymptotic expansion of lacunary Toeplitz determinants \( \det_N \left[ c_{\ell_a - m_b} \left[ f \right] \right] \) generated by holomorphic symbols, where \( \ell_a = a \) (resp. \( m_b = b \)) except for a finite subset of indices \( a = h_1, \ldots, h_n \) (resp. \( b = t_1, \ldots, t_r \)). In addition to the usual Szegő asymptotics, our answer involves a determinant of size \( n + r \).

Introduction

A lacunary Toeplitz determinant generated by a symbol \( f \) refers to the below determinant

\[
\det_N \left[ c_{\ell_a - m_b} \left[ f \right] \right] \quad \text{where} \quad c_n \left[ f \right] = \oint_{\partial \mathcal{D}_1} \frac{f(z)}{z^{n+1}} \cdot \frac{dz}{2i\pi}
\]  

(0.1)

and \( \partial \mathcal{D}_\eta \) is the counter clockwise oriented boundary of the disc of radius \( \eta \) centred at 0. The sequences \( \ell_a, m_b \) appearing in (0.1) are such that

\[
\ell_a = a \quad \text{for} \quad a \in \{1, \ldots, N\} \setminus \{h_1, \ldots, h_n\} \quad \text{and} \quad \ell_{h_a} = p_a \quad a = 1, \ldots, n \tag{0.2}
\]

\[
m_a = a \quad \text{for} \quad a \in \{1, \ldots, N\} \setminus \{t_1, \ldots, t_r\} \quad \text{and} \quad m_{t_a} = k_a \quad a = 1, \ldots, r \tag{0.3}
\]

The integers \( h_a \in \{1 ; N\} \) and \( p_a \in \mathbb{Z} \setminus \{1 ; N\} \), \( a = 1, \ldots, n \) (resp. \( t_a \in \{1 ; N\} \) and \( k_a \in \mathbb{Z} \setminus \{1 ; N\} \), \( a = 1, \ldots, r \)) are assumed to be pairwise distinct. The large-\( N \) asymptotic behaviour of such determinants has been first considered by Tracy and Widom [5] and Bump and Diaconis [1]. More or less at the same time, these authors have obtained two formulae of a very different kind for these large-\( N \) asymptotics. In fact, both collaborations expressed the large-\( N \) behaviour of the lacunary Toeplitz determinant in terms of the unperturbed determinant \( \det_N \left[ c_{a-b} \left[ f \right] \right] \) times an extra term whose representations took a very different form. The expression found Bump and Diaconis was based on characters of the symmetric group associated with the partitions \( \lambda \) and \( \mu \).

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that can be naturally associated with the sequences $\ell_a$ and $m_a$. The answer involved the sum over the symmetric groups of $|\lambda|$ and $\mu l$ elements. In their turn, Tracy and Widom obtained a determinant representation of the type
\[
\det_{N}[c_{\ell_a-b}[f]] = \det_{N-m}[c_{\ell_b-\ell_a}[f]] \cdot \det q[W_{jk}] \cdot \left(1 + o(1)\right) \quad q = \max\{t_1, \ldots, t_r, h_1, \ldots, h_n\},
\]
where $W_{jk}$ was an explicit $q \times q$ sized matrix depending on the symbol $f$ and the numbers $h_1, \ldots, h_n, p_1, \ldots, p_n, t_1, \ldots, t_r$ and $k_1, \ldots, k_r$. In $[2]$, Dehaye proved, by a direct method, the equivalence between the two aforementioned formulae. One should also mention that the large-$N$ asymptotic behaviour of some generalizations of lacunary Toeplitz determinants have been obtained by Lions in $[4]$. The drawbacks of the aforementioned asymptotic expansions was that the answer depended on the magnitude of the lacunary parameters $p_a, k_b, h_a, t_b$. As soon as these parameters were also growing with $N$, the form of the answer did not allow for an easy access to the large-$N$ asymptotic behaviour of the lacunary determinant. Indeed, in Bump-Diaconis’ case, the number of summed up terms was growing as $\sum(p_a - h_a) + \sum(k_a - t_a)$ whereas in Tracy-Widom’s case, the non-trivial determinant part involved a matrix of size $\max|h_a, t_b|$. In the present note we obtain an asymptotic expansion solely in terms of a $(n+r)\times(n+r)$ matrix and show that the latter is enough so as to treat certain cases of lacunary parameters $p_a, k_b, h_a, t_b$ going to infinity. The structure of the asymptotics when $r \neq 0$ (ie $m_a \neq a$) is slightly more complex, so that we postpone the statement of the corresponding results to the core of the paper and present the asymptotic expansion we obtain on the example of line-lacunary Toeplitz determinants.

**Theorem 0.1** Let $f$ be a non-vanishing function on $\partial D_1$ such that $f$ and $\ln f$ are holomorphic on some open neighbourhood of $\partial D_1$. Let $\ell_a$ be defined as (0.2) and $\alpha$ be the piecewise analytic function
\[
\alpha(z) = \exp\left\{-\sum_{n \geq 0} c_n [\ln f] z^n\right\} \quad \text{for} \quad z \in D_1 \quad \text{and} \quad \alpha(z) = \exp\left\{\sum_{n \geq 1} c_n [\ln f] z^{-n}\right\} \quad \text{for} \quad z \in \mathbb{C}\setminus D_1. \quad (0.5)
\]
Then, provided that the matrix $M$ given below is non-singular, the line-lacunary Toeplitz determinant $\det_{N}[c_{\ell_a-b}[f]]$ admits the representation
\[
\det_{N}[c_{\ell_a-b}[f]] = \det_{N}[c_{a-b}[f]] \cdot \det_{n}[M_{ab}] \cdot \left(1 + O(N^{-\infty})\right),
\]
where the $n \times n$ matrix $M$ reads
\[
M_{ab} = -\mathbf{1}_{\mathcal{C}_a}(p_a) \cdot \oint_{\partial D_{\ell}} \frac{dz}{2i\pi} \cdot \oint_{\partial D_{t}} \frac{ds}{2i\pi} \cdot \frac{\alpha(z)}{\alpha(s)} \cdot \frac{s^{N-p_a} \cdot z^{h_a-N-1}}{z - s} + \mathbf{1}_{\mathcal{C}_b}(p_a) \cdot \oint_{\partial D_{\ell}} \frac{dz}{2i\pi} \cdot \oint_{\partial D_{t}} \frac{ds}{2i\pi} \cdot \frac{\alpha(s)}{\alpha(z)} \cdot \frac{s^{-p_a} \cdot z^{h_a-1}}{z - s},
\]
and $1 > \eta_{z} > \eta_{s} > 0$.

The theorem above allows one to obtain the large $N$-asymptotic expansion of the line-lacunary Toeplitz determinant independently on the magnitude (in respect to $N$) of the lacunary parameters $\{h_a\}$ and $\{p_a\}$. Indeed, since the size of the matrix $M$ does not depend on the integers $\{h_a\}$ or $\{p_a\}$, the problem boils down to a classical asymptotic analysis of one-dimensional integrals. Still, in order to provide one with an explicit answer, some more data on these parameters is needed. For instance, one has the...
Corollary 0.1 Let
\[
\begin{align*}
p_a &= 1 - p_a^- & a = 1, \ldots, n_- & \quad \text{and} \quad p_{a+n_-} &= p_a^+ + N & a = 1, \ldots, n_+ \quad (0.8) \\
h_a &= h_a^- & a = 1, \ldots, n_- & \quad \text{and} \quad h_{a+n_-} &= N + 1 - h_a^+ & a = 1, \ldots, n_+ \quad (0.9)
\end{align*}
\]
where \(p_a^+\) and \(h_a^-\) are assumed to be independent of \(N\) and \(n = n_- + n_+\). Provided that the matrices \(M^{(\pm)}\) given below are not singular, one has
\[
\det_n [M_{ab}] = \det_{n_-} [M_{ab}^{(+)}] \cdot \det_{n_+} [M_{ab}^{(-)}] \cdot \left(1 + O(N^{-\infty})\right), \quad (0.10)
\]
where
\[
M_{ab}^{(+)} = - \oint_{\partial \mathcal{D}_a} \frac{dz}{2\pi i} \oint_{\partial \mathcal{D}_b} \frac{ds}{2\pi i} \frac{s^{-p_a^+} \cdot z^{-h_a^-}}{z - s} \frac{\alpha(z)}{\alpha(s)} \quad \text{and} \quad M_{ab}^{(-)} = \oint_{\partial \mathcal{D}_{a+1}} \frac{dz}{2\pi i} \oint_{\partial \mathcal{D}_{b+1}} \frac{ds}{2\pi i} \frac{s^{p_a^+} \cdot z^{h_a^-}}{z - s} \frac{\alpha(s)}{\alpha(z)}. \quad (0.11)
\]
We obtain the asymptotic expansion (0.6) by interpreting the lacunary Toeplitz determinant as the determinant of a finite rank perturbation of an integrable integral operator acting on the unit circle. The inverse of the integrable integral operator, in the large-\(N\) regime, can be constructed by means of an asymptotic resolution of a Riemann–Hilbert problem. We have restricted the study of the present paper to holomorphic symbols. However, in principle, one could apply the method to less regular symbols, e.g., those containing Fischer-Hartwig singularities. Of course, the price of such generalisation would be to deal with certain technicalities related with the more complex structure of the large-\(N\) approximant to the associated resolvent operator.

The paper is organized as follows. We prove Theorem 0.1 in Section 1. In Section 2 we establish the large-\(N\) asymptotic expansion of general line and row lacunary Toeplitz determinants subordinate to the sequences (0.2)–(0.3). Technical details related to the large-\(N\) inversion of integrable integral operators arising in the analysis of Toeplitz determinant generated by holomorphic non-vanishing on \(\partial \mathcal{D}_1\) symbols are recalled in appendix A.

1 The line lacunary Toeplitz determinants

In this section, we first prove a preliminary factorisation result that allows one to express the lacunary Toeplitz determinant \(\det_{n} [c_{a-b}[f]]\) in terms of the non-perturbed Toeplitz determinant \(\det_{n} [c_{a-b}[f]]\) and of the determinant of a \(n \times n\) matrix. We subsequently analyse the large-\(N\) behaviour of this finite-size \(n\) determinant.

1.1 The factorisation

Lemma 1.1 Let \(f\) be non-vanishing on \(\partial \mathcal{D}_1\) and such that \(f\) and \(\ln f\) are holomorphic in some open neighbourhood of \(\mathcal{D}\). Let \(V_0\) be the integral kernel
\[
V_0(z, s) = (f(z) - 1) \cdot \frac{z^{\frac{p}{2}} \cdot s^{\frac{h}{2}} - z^{-\frac{p}{2}} \cdot s^{-\frac{h}{2}}}{2\pi i(z - s)} \quad (1.1)
\]
of the integrable integral operator \(I + V_0\) acting on \(L^2(\partial \mathcal{D}_1)\). Then, provided that \(N\) is large enough, \(I + V_0\) is invertible with inverse \(I - R_0\) and the below factorization holds
\[
\det_{n} [c_{a-b}[f]] = \det_{n} [c_{a-b}[f]] \cdot \det_{n} [M_{ab}] \quad (1.2)
\]
corrections, one can trade the kernel

As it has been recalled in the appendix, the resolvent kernel

Proof —

Let $I + V$ be the integral operator on $L^2(\partial D_1)$ with a kernel given by

$$V(z, s) = \sum_{a=1}^N \kappa_a(z) \cdot \tau_a(s) \quad \text{where} \quad \tau_a(z) = \frac{1}{2i\pi} \cdot z^{\alpha_a - \frac{N}{2}} \quad (1.4)$$

and

$$\kappa_a(z) = \begin{cases} (f(z) - 1) \cdot \frac{N}{2} - a, & a \in \{1, \ldots, N\} \setminus \{h_1, \ldots, h_n\} \\ f(z) \cdot \frac{N}{2} - \frac{N}{2} - h_a, & a = 1, \ldots, n \end{cases} \quad (1.5)$$

Since $V$ is a finite rank $N$ operator, the Fredholm determinant of $I + V$ reduces to one of an $N \times N$ matrix

$$\text{det}_{\partial D_1}(I + V) = \text{det}_N \left[ \delta_{ab} + \int_{\partial D_1} \kappa_a(z) \cdot \tau_b(z) \cdot dz \right] = \text{det}_N \left[ c_{a-b}[f] \right]. \quad (1.6)$$

One can decompose the kernel $V$ as $V = V_0 + V_1$ where $V_0$ has been introduced in (1.1) whereas $V_1$ is the finite rank $n$ perturbation of $V_0$ given by

$$V_1(z, s) = -\frac{f(z)}{2i\pi} \sum_{a=1}^N (\frac{N}{2} - h_a - \frac{N}{2}) \cdot s^{\alpha_a - \frac{N}{2}} \quad (1.7)$$

It follows from the strong Szegő limit theorem and from the identity $\text{det}_{\partial D_1}(I + V_0) = \text{det}_N \left[ c_{a-b}[f] \right]$ that, provided $N$ is taken large enough, the operator $I + V_0$ is invertible. Hence, all-in-all, we get that

$$\text{det}_{\partial D_1}(I + V) = \text{det}_{\partial D_1}(I + V_0) \cdot \text{det}_{\partial D_1}(I + (I - R_0) \cdot V_1) = \text{det}_N \left[ c_{a-b}[f] \right] \cdot \text{det}_n[M_{kl}] \quad (1.8)$$

where the matrix $M_{kl}$ is as defined in (1.3).

1.2 Asymptotic analysis of $\text{det}_n[M]$—Proof of theorem 0.1

As it has been recalled in the appendix, the resolvent kernel $R_0$ of the operator $I + V_0$ can be recast as

$$R_0 = R_0^{(0)} + R_0^{(\infty)} \quad (1.9)$$

where

$$R_0^{(0)}(z, s) = \frac{f(z) - 1}{2i\pi} \cdot \left( \frac{N}{2} \cdot s^{\frac{N}{2}} \cdot \alpha_+(s) \cdot \alpha_-(z) \cdot \frac{\alpha_-(z)^{-1}(s)}{z - s} \right) \quad (1.10)$$

and

$$\|R_0^{(\infty)}\|_{L^\infty(\partial D_1 \times \partial D_1)} \leq C \cdot N \cdot e^{-\kappa N} \quad (1.11)$$

Above, $\alpha_\pm$ are the $+ (ie from within)$ and $- (ie from the outside)$ non-tangential limits on $\partial D_1$ of the piecewise analytic function $\alpha$ defined in (1.5). The decomposition (1.9) ensures that, for the price of exponentially small corrections, one can trade the kernel $R_0$ for $R_0^{(0)}$ in (1.3). Using that $\alpha_+$ (resp. $\alpha_-$) admit an analytic continuation to some open neighbourhood of $\partial D_1$ in $C \setminus \overline{D}_1$ (resp. interior $D_1$) we deform the contours in the double integral associated with $R_0^{(0)}$ to
• $\partial D_{\eta_1} \times \partial D_{\eta_1}$ in what concerns the part of the integrand containing $\alpha_+ (s)/\alpha_- (z)$;

• $\partial D_{\eta_2} \times \partial D_{\eta_2}$ in what concerns the part of the integrand containing $\alpha_+ (z)/\alpha_- (s)$.

The resulting residue cancels out the pre-factors in (1.12) leading to

$$M_{kl} = \oint_{\partial D_{\eta_1}} \oint_{\partial D_{\eta_1}} \alpha_- (s) \cdot (\alpha_+^{-1} (z) - \alpha_-^{-1} (z)) \cdot (s^{-h_k} - s^{-p_k}) \cdot \frac{z^{h_l-1}}{z-s} + \oint_{\partial D_{\eta_2}} \oint_{\partial D_{\eta_2}} \alpha_+ (s) \cdot (\alpha_+ (z) - \alpha_- (z)) \cdot (s^{N-h_k} - s^{N-p_k}) \cdot \frac{z^{h_l-1-N}}{z-s} + O(N^{-\infty}).$$

The term $s^{-h_k}$ (resp. $s^{N-h_k}$) do not contribute to the integral as can be seen by deforming the contour of integration to $\eta_k$ (resp. $\eta_1^{-1} = 0$). Further, the first line of (1.12) only gives non-vanishing contributions if $p_k \leq 0$ (resp. the last line of (1.12) only gives non-vanishing contributions if $p_k \geq N + 1$). This yields (0.7).

2 The asymptotic expansion of line and row lacunary Toeplitz determinants

2.1 The factorisation in the general case

The factorized representation in the general case depends, in particular, on whether there are some overlaps between the integers parametrising the lacunary line and columns. We thus need a definition so as to be able to distinguish between the different cases.

**Definition 2.1** The sets $[h_a]_{t_a}^1$ and $[t_b]_{b}^1$ with $h_a, t_b \in [[1; N]]$ are said to be well-ordered with overlap $c \in [[0; \min (r, n)]]$ if

$$h_a = t_a \quad \text{for} \quad a = 1, \ldots, c \quad \text{whereas} \quad [h_{c+1}, \ldots, h_a] \cap [t_{c+1}, \ldots, t_r] = \emptyset. \quad (2.1)$$

It is clear that given two not well ordered sequences $\ell_a$ and $m_a$, one can always relabel the indices of the lacunary integers $[p_a, h_a, k_a, t_b]$ so that (2.1) holds. There is thus no restriction in assuming that the sequences $\ell_a$ and $m_a$ are well ordered, so that we are going to do so in the following.

**Proposition 2.1** Let $\ell_a$ and $m_a$ be sequences as defined in (0.2)-(0.3) and $f$ a non-vanishing holomorphic function on some open neighbourhood of $\partial D_1$ such that $\ln f$ is also holomorphic on this neighborhood. Then, the lacunary line and row Toeplitz determinant admits the representation

$$\det N [c_{a-m_b}, f] = \det [c_{a-b}; f] \cdot \det [N]. \quad (2.2)$$

The matrix $N$ appearing above admits the blocks structure

$$N = \left( \begin{array}{cc} N_{I,I} & N_{I,II} \\ N_{II,I} & N_{II,II} \end{array} \right) \quad (2.3)$$

with blocks being given by

$$(N_{A,I})_{ab} = \delta_{A,I} \delta_{ab} \delta_{b>c} + \oint_{\partial D_h} U_{A,a}(z) \cdot \eta_{I,b}(z) \cdot dz \quad \text{and} \quad (N_{A,II})_{ab} = \delta_{A,II} \delta_{ab} \delta_{b\leq c} + \oint_{\partial D_1} U_{A,a}(z) \cdot \eta_{II,b}(z) \cdot dz \quad (2.4)$$
in which $A \in \{I, II\}$. The functions $U_{A,a}$ are built in terms of the resolvent $R_0$ to the integral operator $I + V_0$ defined in (1.1) and of the functions $u_{A,a}$

\[ u_{I,a}(z) = \frac{f(z)}{2i\pi} \cdot z^{-\frac{N}{2} - a} - \frac{z^{-\frac{N}{2} - a}}{2i\pi} \left( \delta_{a\leq c} + \delta_{a>c} f(z) \right) \quad \text{and} \quad u_{II,a}(z) = \frac{f(z) - 1}{2i\pi} \cdot z^{-\frac{N}{2} - \tau} \]

(2.5)

as $U_{A,a} = (I - R_0)[u_{A,a}]$. Finally, the function $v_{A,b}$ appearing in (2.4) read

\[ v_{I,b}(z) = \delta_{b\leq c} \cdot z^{b_0 - \frac{N}{2} - 1} + \delta_{b>c} \cdot z^{b_0 - 1 - \frac{N}{2}} \quad \text{and} \quad v_{II,b}(z) = -\delta_{b\leq c} \cdot z^{b_0 - \frac{N}{2} - 1} + \delta_{b>c} \cdot z^{b_0 - 1 - \frac{N}{2}}. \]

(2.6)

**Proof** —

Let $I + \widetilde{V}$ be the integral operator acting on $L^2(\partial \mathcal{D}_1)$ with the kernel

\[
\widetilde{V}(z,s) = \sum_{a=1}^{N} \tilde{k}_a(z) \cdot \tilde{\tau}_a(s) \quad \text{where} \quad \begin{cases} 
\tilde{\tau}_a(z) = z^{a-1-\frac{N}{2}}/(2i\pi) & a \in \{1, \ldots, N\} \setminus \{t_1, \ldots, t_r\} \\
\tilde{\tau}_a(z) = z^{a-1-\frac{N}{2}}/(2i\pi) & a = 1, \ldots, r
\end{cases}
\]

and

\[
\tilde{k}_a(z) = (f(z) - 1) \cdot z^{a-\frac{N}{2}} - \sum_{s=1}^{r} \delta_{a,s} \cdot z^{-\frac{N}{2} - k_s} \quad a \in \{1, \ldots, N\} \setminus \{t_1, \ldots, t_r\}
\]

\[
\tilde{k}_a(z) = f(z) \cdot z^{a-\frac{N}{2} - a} - \sum_{s=1}^{r} \delta_{a,s} \cdot z^{-\frac{N}{2} - h_s} \quad a = 1, \ldots, n.
\]

(2.8)

It is readily seen that

\[ \det_{\partial \mathcal{D}_1}[I + \widetilde{V}] = \det_N \left[ \delta_{ab} + \int_{\partial \mathcal{D}_1} \tilde{k}_a(z) \cdot \tilde{\tau}_b(z) \cdot dz \right] = \det_N \left[ c_{\ell_a - m_b} f \right]. \]

(2.9)

The kernel $\widetilde{V}$ can be recast as $\widetilde{V} = V_0 + \widetilde{V}_1$ where $V_0$ has been introduced in (1.1) and $\widetilde{V}_1$ is the finite rank $n + 2r$ perturbation of $V_0$ given by

\[ \widetilde{V}_1(z,s) = \sum_{a=1}^{n} u_{I,a}(z) \cdot v_{I,a}(s) + \sum_{a=1}^{r} \left[ u_{II,a}(z) \cdot \widetilde{v}_{II,a}(s) + u_{III,a}(z) \cdot \widetilde{u}_{III,a}(s) \right]. \]

(2.10)

The functions $u_{I,a}, u_{II,a}$ and $v_{I,a}$ are as defined in (2.5)-(2.6) whereas

\[
\widetilde{v}_{II,b}(z) = \delta_{b\leq c} \cdot z^{b_0 - \frac{N}{2} - 1} - z^{b_0 - \frac{N}{2} - 1}, \quad \widetilde{v}_{III,b}(z) = z^{b_0 - \frac{N}{2} - 1} \quad \text{and} \quad \widetilde{u}_{III,a}(z) = \frac{z^{b_0 - \frac{N}{2} - 1}}{2i\pi}.
\]

(2.11)

Proceeding as in the proof of lemma (1.1) we get that

\[ \det_{\partial \mathcal{D}_1}[I + V] = \det_{\partial \mathcal{D}_1}[I + V_0] \cdot \det_{\partial \mathcal{D}_1}[I + (I - R_0) \cdot \widetilde{V}_1] = \det_N \left[ c_{j-k} [b] \right] \cdot \det_{n+2r}[M] \]

(2.12)

where $M$ is the $(n + 2r) \times (n + 2r)$ block matrix

\[
M = \begin{pmatrix}
M_{I,I} & M_{I,II} & M_{I,III} \\
M_{II,I} & M_{II,II} & M_{II,III} \\
M_{III,I} & M_{III,II} & M_{III,III}
\end{pmatrix}
\]

with $(M_{A,B})_{ab} = \delta_{A,B} \delta_{ab} + \int_{\partial \mathcal{D}_1} U_{A,a}(z) \cdot v_{B,b}(z) \cdot dz$.

(2.13)
The upper-case entries \( A, B \) belong to \( \{ I, II, III \} \) whereas the lower-case entries \( a, b \) subordinate to the upper-case entry \( I \) run from 1 to \( n \) and those subordinate to the upper-case entries \( II \) or \( III \) run from 1 to \( r \). Since \( \widetilde{u}_{III,a} \in \ker(V_0) \), it follows that \( \widetilde{u}_{III,a} \in \ker(R_0) \). Then, a straightforward calculation shows that, in fact, the block structure of the lines of type \( III \) simplifies leading to

\[
M = \begin{pmatrix}
M_{I,I} & M_{I,II} & M_{I,III} \\
M_{II,I} & M_{II,II} & M_{II,III} \\
-I_c & 0 & 0 \\
0 & 0 & -I_{r-c}
\end{pmatrix}.
\]

(2.14)

There, \( I_k \) refers to the identity matrix in \( k \) dimensions. In order to reduce the size of the determinant of \( M \) it is enough to exchange the first \( c \) columns of the block \( I \) with the first \( c \) ones of the block \( III \), and then exchange the \( r-c \) last columns of block \( II \) with the \( r-c \) last columns of the block \( III \). This produces, all in all, an overall \((-1)^r\) sign. The latter cancels out with the one issuing from \(\det_r[-I_r] \), thus leading to \(\det_{n+2r}[M] = \det_{n+r}[N] \). ■

2.2 A special case of a large \( N \) asymptotics

Replacing the exact resolvent \( R_0 \) by its approximate resolvent \((1.10)\) in the definition of the matrix entries of \( N\) (2.2) leads to exponentially small in \( N \) corrections. Upon such a replacement, Proposition [2.1] basically yields the most general expression for the large-\( N \) asymptotics of the ratio

\[
\det_N \left[ c_{a-mb}[^f J] \right] / \left( \det_N \left[ c_{a-b}[^f J] \right] \right).
\]

(2.15)

Although there should, quite probably, exist a direct transformation that would allow to connect Tracy-Widom’s answer to ours (and hence Bump and Diaconis one due to \([2]\)) we have not succeeded in finding it. The formula can, of course, be simplified as soon as one provides some more informations on the lacunary integers \( p_a, h_a, k_b, t_b \). Below, we treat a specific example of such a simplification, much in the spirit of the one outlined in Corollary \([1]\). In order to state the theorem, we first need to introduce a matrix \( N_{A,b}^{(e)}(J;c) \) depending on the sets of integers

\[
J = \{p_a; \{h_a; \{k_a; \{t_a\}\}\}\}.
\]

(2.16)

The set \( J \) parametrizes its entries according to

\[
\left( N_{I,I}^{(e)}(J;c) \right)_{ab} = -\varepsilon \int_{\partial D_1} \frac{1}{z(1 - \varepsilon 0^+ - s)} \left( b_{\varepsilon; s_{\varepsilon} a; s} \frac{a_{\varepsilon}(z)}{a_{\varepsilon}(s)} \right)^{\varepsilon} s^{\varepsilon h_a} z^{\varepsilon h_b} + s^{\varepsilon p_a} z^{\varepsilon t_b} \cdot s \cdot d\zeta
\]

(2.17)

and

\[
\left( N_{I,I}^{(e)}(J;c) \right)_{ab} = -\varepsilon \int_{\partial D_1} \frac{1}{z(1 - \varepsilon 0^+ - s)} \left( b_{\varepsilon; s_{\varepsilon} a; s} \frac{a_{\varepsilon}(z)}{a_{\varepsilon}(s)} \right)^{\varepsilon} s^{\varepsilon h_a} z^{\varepsilon h_b} + s^{\varepsilon p_a} z^{\varepsilon t_b} \cdot s \cdot d\zeta
\]

(2.18)
and finally

\[ (\mathcal{N}^{(e)}_{II,I}(\mathcal{J}; c))_{ab} = -e \int_{\partial D_1} \frac{s^{\epsilon_{k_0} - 1}}{z(1 - e0^+)} - s \left( \delta_{b \leq c} \left( \frac{\alpha_{\epsilon}(z)}{\alpha_{\epsilon}(s)} \right)^\epsilon z^{\epsilon_{k_0} - 1} - \delta_{b > c} \left( \frac{\alpha_{\epsilon}(z)}{\alpha_{\epsilon}(s)} \right)^\epsilon z^{-\epsilon_{k_0}} \right) \cdot \frac{ds \cdot dz}{(2\pi)^2} \]  

(2.19)

\[ (\mathcal{N}^{(e)}_{II,II}(\mathcal{J}; c))_{ab} = -e \int_{\partial D_1} \frac{s^{\epsilon_{k_0} - 1}}{z(1 - e0^+)} - s \left( \delta_{b \leq c} \left( \frac{\alpha_{\epsilon}(z)}{\alpha_{\epsilon}(s)} \right)^\epsilon z^{-\epsilon_{k_0}} + \delta_{b > c} \left( \frac{\alpha_{\epsilon}(z)}{\alpha_{\epsilon}(s)} \right)^\epsilon z^{\epsilon_{k_0} - 1} \right) \cdot \frac{ds \cdot dz}{(2\pi)^2} \]  

(2.20)

The (1 - eO^+) prescription means that the integral should be understood as the limit when z approaches a point on \( \partial D_1 \) from the inside (\( \epsilon = +1 \)) or outside (\( \epsilon = -1 \)) of the unit disk.

**Theorem 2.1** Assume that the lacunary integers \( p_a, h_a \) are given as in (2.21) and (2.22) and, likewise, that the lacunary integers \( k_b, t_b \) are given as:

\[ \begin{align*}
  k_a &= 1 - k_a^- \quad \text{for } a = 1, \ldots, r_- \quad \text{and} \quad k_{a+r_-} = k_a^+ + N \quad \text{for } a = 1, \ldots, r_+ \\
  t_a &= t_a^- \quad \text{for } a = 1, \ldots, r_- \quad \text{and} \quad t_{a+r_-} = N + 1 - t_a^+ \quad \text{for } a = 1, \ldots, r_+.
\end{align*} \]

(2.21) \hspace{1cm} (2.22)

Further, let the sets \( \{h_a^+\}^{r_+}_{a=1} \) and \( \{k_a^+\}^{r_+}_{a=1} \) be well ordered with overlap \( c_+ \) (resp. \( c_- \)). Then, provided that the matrices \( \mathcal{N}^{(e)}(\mathcal{J}^{(e)}; c) \) have maximal rank, one has the asymptotic expansion

\[ \det_N [c_{a-b}[f]] \cdot \left( \det_N [c_{a-b}[f]] \right)^{-1} = \det_{n+r^+} \left[ \mathcal{N}^{(e)}(\mathcal{J}^{(e)}; c_+) \right] \cdot \det_{n+r^-} \left[ \mathcal{N}^{(e)}(\mathcal{J}^{(e)}; c_-) \right] \cdot \left( 1 + O(N^{-\infty}) \right). \]

(2.23)

The sets \( \mathcal{J}^{(e)} \) appearing above are defined as

\[ \mathcal{J}^{(e)} = \left\{ [p_a^+]^{r_+}_{a=1} ; [h_a^+]^{r_+}_{a=1} ; [k_a^+]^{r_+}_{a=1} ; [t_a^+]^{r_+}_{a=1} \right\} \quad \text{and} \quad \mathcal{J}^{(e)} = \left\{ [1-p_a^-]^{r_-}_{a=1} ; [1-h_a^-]^{r_-}_{a=1} ; [1-k_a^-]^{r_-}_{a=1} ; [1-t_a^-]^{r_-}_{a=1} \right\}. \]

**Proof** —

The representation obtained in proposition 2.1 is invariant under a permutation of the integers \( h_a, t_a \) along with a simultaneous permutation of the associated integers \( p_a, k_a \). Hence, reorganising the entries of the matrix \( \mathcal{N} \) in each block so that the natural order imposed by the \( \pm \) splitting of the lacunary integers is respected and a repeated application of the manipulations outlined in the proof of Theorem 0.1 leads to \( \det_{n+r} [\mathcal{N}] = \det_{n+r} [\tilde{\mathcal{N}}] \) where

\[ \tilde{\mathcal{N}} = \begin{pmatrix}
  \mathcal{N}^{(e)}_{II,I}(\mathcal{J}^{(e)}; c_-) & 0 & \mathcal{N}^{(e)}_{II,I}(\mathcal{J}^{(e)}; c_+)& 0 \\
  0 & \mathcal{N}^{(e)}_{II,I}(\mathcal{J}^{(e)}; c_-) & 0 & \mathcal{N}^{(e)}_{II,I}(\mathcal{J}^{(e)}; c_+)
\end{pmatrix} + O(N^{-\infty}). \]

(2.24)

Finally, \( O(N^{-\infty}) \) appearing above refers to an \( (n + r) \times (n + r) \) matrix whose all entries are a \( O(N^{-\infty}) \). It is then enough to exchange the appropriate lines and columns in \( \det_{n+r} [\tilde{\mathcal{N}}] \) and invoke the maximality of the rank of the matrices \( \mathcal{N}^{(e)} \).
Conclusion

In this paper we have proposed a Riemann–Hilbert problem based approach to the analysis of the large-size asymptotic behaviour of lacunary Topelitz determinants having a finite number of modified lines an rows. Our approach allows one to obtain an alternative to the ones obtained in [1, 5] representation for its large-$N$ asymtotics. Our answer involves a determinant that solely depends on the number of modified rows and lines and not on the index of the largest modified line or column. In particular, this allows one to investigate the asymptotics in the case when the locii of some of the modified lines and columns go to infinity. We have treated certain instances of such a situation in the present paper. It is clear from the very setting of our analysis that our method allows one to treat also generalisations of lacunary Toeplitz determinants such as those considered in [4]. To do so, one should simply replace the functions $\hat{\tau}_t$ and $\hat{\kappa}_h$ arising in our analysis by more general ones.

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A Asymptotic inversion of $I + V_0$–The Riemann–Hilbert approach

A.1 The Riemann–Hilbert problem associated with $I + V_0$

Consider the Riemann–Hilbert problem for a piecewise analytic $2 \times 2$ matrix $\chi$ having a jump on the unit circle $\partial D_1$:

- $\chi$ is analytic on $\mathbb{C} \setminus \partial D_1$;
- $\chi(z) = I_2 + \frac{1}{z} \cdot O\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)$ when $z \to \infty$;
- $\chi$ admits continuous $\pm$-boundary values on $\partial D_1$;
- $\chi^+(z) \cdot \left(\begin{array}{cc} 2 - f(z) & (f(z) - 1) \cdot z^{-N} \\ (1 - f(z)) \cdot z^{-N} & f(z) \end{array}\right) = \chi^-(z) ; \quad z \in \partial D_1$.

In the formulation of the Riemann–Hilbert problem, we have adopted the following notations. Given an oriented Jordan curve $\Gamma \subset \mathbb{C}$ and a function $f$ on $\mathbb{C} \setminus \Gamma$, $f_\pm$ refer to the $\pm$-boundary values of the function $f$ on $\Gamma$ where $+$ (resp. $-$) refers to approaching a point on $\Gamma$ non-tangentially from the left (resp. right) side of the curve. Finally, a matrix domination of the sort $A = O(B)$ is to be understood entry-wise viz $A_{jk} = O(B_{jk})$.

We also remind that the unit circle $\partial D_1$ is oriented canonically (ie the $+$ side of the contour corresponds to the interior of the circle). It is a standard fact that the above Riemann–Hilbert problem admits a unique solution.

A.2 Transformation to a perturbatively solvable Riemann–Hilbert problem for $\Gamma$

We now define a new matrix $\Upsilon$ according to Fig. [A] ie

- $\Upsilon = \chi \alpha^{\epsilon z} \cdot \chi^-$, for $\epsilon$ being in the exterior of $\Gamma_{\text{ext}}$ and the interior of $\Gamma_{\text{int}}$. 

• $\Upsilon = \chi \alpha^{\sigma_3} M^{-1}_{\text{ext}}$, for $z$ between $\Gamma_{\text{ext}}$ and $\partial \mathcal{D}_1$.

• $\Upsilon = \chi \alpha^{\sigma_3} M_{\text{int}}$, for $z$ between $\Gamma_{\text{int}}$ and $\partial \mathcal{D}_1$.

Here $\alpha$ is as defined in (0.5). It is readily seen that it solves the scalar RHP

\[ \alpha \text{ analytic on } \mathbb{C} \setminus \partial \mathcal{D}_1, \quad \alpha = f \alpha_+ \quad \text{on } \partial \mathcal{D}_1, \quad \alpha(z) \to 1 \text{ when } z \to \infty. \quad (A.1) \]

The matrices $M_{\text{int}}/\text{ext}$ appearing in the definition of $\Upsilon$ read

\[ M_{\text{int}}(z) = \begin{pmatrix} 1 & 0 \\ (1 - f^{-1}(z))\alpha^{-2}(z) \cdot z^N \end{pmatrix}, \quad M_{\text{ext}}(z) = \begin{pmatrix} 1 & 0 \\ (f^{-1}(z) - 1)\alpha^2(z) \cdot z^{-N} & 1 \end{pmatrix}. \quad (A.2) \]

The curves $\Gamma_{\text{ext}}$ and $\Gamma_{\text{int}}$ are chosen in such a way that they are located inside of the open neighbourhood of $\partial \mathcal{D}_1$ on which $f$ is holomorphic. One readily sees that $\Upsilon$ satisfies the RHP

\[ \Upsilon = \chi \alpha^{\sigma_3} \Upsilon = \chi \alpha^{\sigma_3} \cdot M_{\text{ext}}^{-1} \Upsilon = \chi \alpha^{\sigma_3} \cdot M_{\text{int}} \Gamma_{\text{ext}} \Gamma_{\text{int}} \partial \mathcal{D}_1 \Upsilon = \chi \alpha^{\sigma_3} \cdot M_{\text{int}} \Gamma_{\text{int}} \Gamma_{\text{ext}} \partial \mathcal{D}_1 \]

Figure 1: Contour for the RHP $\Upsilon$ and the associated contour $\Gamma_{\Upsilon} = \Gamma_{\text{int}} \cup \Gamma_{\text{ext}}$.

• $\Upsilon$ is analytic in $\mathbb{C} \setminus \Gamma_{\Upsilon}$;

• $\Upsilon(z) = I_2 + \frac{1}{z} \cdot O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ when $z \to \infty$;

• $\Upsilon$ admits continuous ±-boundary values on $\Gamma_{\Upsilon}$;

• $\Upsilon_+(z) \cdot G_{\Upsilon}(z) = \Upsilon_-(z), \quad z \in \Gamma_{\Upsilon}$, where $G_{\Upsilon}(z) = M_{\text{ext}}(z) \cdot 1_{\Gamma_{\text{ext}}}(z) + M_{\text{int}}(z) \cdot 1_{\Gamma_{\text{int}}}(z)$.

and $1_A$ stands for the indicator function of the set $A$. Since

\[ ||G_{\Upsilon} - I_2||_{L^\infty(\Gamma_{\Upsilon})} + ||G_{\Upsilon} - I_2||_{L^1(\Gamma_{\Upsilon})} + ||G_{\Upsilon} - I_2||_{L^2(\Gamma_{\Upsilon})} \leq C_1 e^{-\kappa N} \quad (A.3) \]

for some constants $C_1 > 0$ and $\kappa > 0$, it follows from the equivalence of the Riemann–Hilbert problem for $\Upsilon$ with the singular integral equation satisfied by $\Upsilon_+$ that, for any compact $K \supset \Gamma_{\Upsilon}$,

\[ ||\Upsilon - I_2||_{L^\infty(\mathbb{C} \setminus K)} \leq C'_1 e^{-\kappa N} \quad (A.4) \]

for some new constant $C'_1 > 0$. 
A.3 The resolvent operator and factorisation of the determinant

It is well known [3] that the resolvent kernel $R_0$ associated with the integrable integral operator $I + V_0$ takes the form

$$R_0(z, s) = \frac{(F_L(z), F_R(z))}{z - s}$$ (A.5)

where given vector $x, y, (x, y) = x_1 y_1 + x_2 y_2$ and

$$F_L^T(z) = E_L^T(z) \cdot \chi^{-1}(z) \quad F_R(z) = \chi(z) \cdot E_R(z)$$ (A.6)

where the two-dimensional vectors $E_R(z), E_L(z)$ take the form

$$E_L^T(z) = (f(z) - 1) \cdot \left(-z^{-N}, z^N \right) \quad \text{and} \quad E_R^T(z) = \frac{1}{2i\pi} \cdot \left(z^N, z^{-N} \right).$$ (A.7)

It follows from the factorisation of $\chi$ that $F_R$ can be recast as

$$F_R(z) = F_R^{(0)}(z) + F_R^{(\infty)}(z) \quad \text{with} \quad \begin{cases} F_R^{(0)}(z) = \quad M^{-1}_{int}(z) \cdot \alpha^{(0)}(z) \cdot E_R(z) \\ F_R^{(\infty)}(z) = (\Upsilon - I_2) \cdot M^{-1}_{int}(z) \cdot \alpha^{(\infty)}(z) \cdot E_R(z) \end{cases}$$ (A.8)

The uniform bounds on $\Upsilon - I_2$ ensure that

$$\|F_R^{(\infty)}\|_{L^\alpha(\mathcal{C})} = C \cdot e^{-KN}$$ (A.9)

for some $C > 0$. Thus, for $z \in \mathcal{C}$,

$$F_R^{(0)}(z) = \frac{1}{2i\pi} \cdot \left(z^N \alpha^{-1}(z), z^{-N} \alpha^+(z) \right) \quad \text{and} \quad E_L^{(0)}(z) = (b(z) - 1) \cdot \left(z^{-N} \alpha^+(z), z^N \alpha^{(0)}(z) \right)$$ (A.10)

As a consequence, the resolvent kernel $R_0$ decomposes exactly as given in (1.9).

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