Entropy Points and Applications for Free Semigroup Actions

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Received: 16 June 2021 / Accepted: 24 November 2021 / Published online: 6 December 2021
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Abstract
The aim of this manuscript is to study some local properties of the topological entropy of a free semigroup action. In order to do that we focus on the set of entropy points of a free semigroup action, show that this set carries the full entropy of the system (which, with respect to the chaoticity of the system, gives a fundamental relevance to such set). We prove that the topological entropy of a free semigroup action focuses on a countable set of points. Our results are inspired by the ones presented in [1].

Keywords Free semigroup action · Topological entropy · Entropy points · Skew-products

Mathematics Subject Classification Primary: 37B05 · 37B40 Secondary: 37D20 · 37D35 · 37C85

1 Introduction
The concept of entropy, introduced into the realm of dynamical systems more than fifty years ago in [2], has become an important ingredient in the characterization of the complexity of dynamical systems. Topological entropy represents the exponential growth rate of the number of orbit segments distinguishable with arbitrarily fine but finite precision, and describes in a crude but suggestive way the total exponential complexity of the orbit structure with a single number, so naturally has a connection with complexity and chaos, which have been areas of intense research over the last half-century.

There have been myriad investigations involving comparisons with other forms of entropy as well, such as Kolmogorov-Sinai (metric) entropy, fundamental group entropy, Shannon entropy, and algebraic entropy. In addition, very active research has continued on connections,
largely by way of estimates, with other aspects of the spaces such as measurable dynamical system structure as in the variational principle, smooth manifold structures, Riemannian metrics (where Newhouse has played a major role [3–5]), homotopy and homology as in the Shub conjecture [6] (which was confirmed for smooth maps by Yomdin [7]). For instance, due to the variational principle, topological entropy is a fundamental tool in obtaining statistical and ergodic information for a wide class of dynamical systems. Also, it represents an important topological invariant in the study of thermodynamic properties of certain systems that have physical motivation (see [8]) (for instance, in the works of James Maxwell and Ludwig Boltzmann (see [9] and references given therein), related to statistical mechanics, the concept of metric entropy plays a fundamental role).

The increasing activity in applications involving topological entropy is most likely due to and catalyzed by its intimate association with complexity, communication efficiency, unpredictability, and dynamical chaos. For example, applications to fluid mechanics and granular dynamics related to mixing are now rather well established and still being intensely investigated. Other application areas have also seen intensifying activity in recent years. These include biological applications where, for instance, a generalized form of topological entropy has been used in the analysis of DNA sequences (see [10,11] and references therein), industrial engineering (see [12] and references therein), where manufacturing networks have been studied using topological entropy (see [13] and references therein), communication engineering, control theory and engineering (see [14] and references therein), condensed matter and quantum physics (see [15] and references therein), group theory, information theory, and even in the social sciences (see [16] and references therein). With the success of topological entropy as a tool in these efforts, it is likely that such research is bound to grow apace.

Inasmuch as topological entropy has proven so important in dynamical systems theory and numerous related application areas and is in general rather intricate, there have been numerous research efforts dedicated to finding efficient, effective schemes for its approximate or exact computation. Considerable progress has been made in these endeavors, especially in the development of algorithmic schemes for computing lower bounds in two and three dimensions, but much work remains to be done.

In this manuscript we will consider a notion of topological entropy for a free semigroup action introduced by Bufetov [17] in the end of the last century. In such work the author considers a free semigroup $G$ generated by a finite set of continuous maps $G_1 = \{ \text{id}, f_1, \ldots, f_p \}$ acting on a compact metric space $X$ and the fixed random walk $\eta_p = (\frac{1}{p}, \ldots, \frac{1}{p})^N$ and obtain a relation between the topological entropy of the free semigroup action induced by $G$, the topological entropy of the shift map $\sigma : \Sigma^+ \to \Sigma^+$ and the induced continuous skew product $\mathcal{F}_G : \Sigma^+_p \times X \to \Sigma^+_p \times X$. In addition to representing a natural extension of the dynamic system concept, due to its relationship with skew product-type applications, understanding the dynamic behavior of free semigroup actions has attracted several researchers, see for example [18–28]. Recently, in [29–31], a more general definition of topological entropy inspired by the one introduced in [17] was exploited. In that context the random walk considered in $G$ is any Borel probability measure $\mathbb{P}$ in $\Sigma^+_p$ (see Sect. 2 for a precise definition), the authors introduce a definition of metric entropy of a free semigroup action with respect to a Borel probability measure on the phase space and a variational principle is obtained. One of the main ingredients to get such variational principle is the relation between the free semigroup action and the skew product.

Starting with the study of the topological analogue of Kolmogorov systems, in recent years much attention has been paid to the so called local properties of entropy and its many interesting results (see [1,32–36]). In [1] it was explored, using Bowen’s definition of topological
entropy, the concept of entropy point (point for which the topological entropy is positive for any neighborhood that contains it) for a single map. Through the results presented there, it was possible to see the importance of such a notion for a better understanding of the local behavior of the dynamics, its topological entropy and also its metric entropy, since they showed that the support of ergodic measures is contained in the set of entropy points and that the metric entropy of an ergodic measure represents a lower bound for the topological entropy of the Borelian sets, among other important results, as the ones about the entropy function (see Definition 2.4) which contribute to a better understanding of the local behavior of topological entropy. A somewhat surprising consequence of the study of uniform entropy points (see Sect. 2.2 for the definition) is that for any topological dynamical system there is a countable closed subset whose entropy is equal to the entropy of the original system. In fact, the countable subset $F$ can be chosen such that the set of the limit points of $F$ has at most one limit point.

The main objective of this work is to obtain important properties that help to describe the local behavior of the free semigroup action and its topological entropy. To do this we focus on the set of entropy points for a free semigroup action (the existence of such points was guaranteed for free semigroup action with positive topological entropy in [37]). Our results are motivated by those presented in [1] and we show that the set of entropy points of the free semigroup action contains all the topological entropy of the system (see Theorem A); given a probability measure in the phase space, under certain conditions, the metric entropy of the free semigroup action with respect to that measure represents a lower bound for the topological entropy of any Borelian set (see Theorem B); if the free semigroup action satisfies the strong orbital specification property then its entropy function is constant (see Theorem E); as in the classical setting, there is always a closed enumerable set with total entropy, such set has at most one accumulation point and, in the case of the existence of a limit point, the entropy function evaluated at that point coincides with the entropy of the action (see Theorem F).

The paper is organized as follows. In Sect. 2 we present the main definition and our main results. In Sects. 3, 4, 5, 6, 7, and 8 we prove the main results. In Sect. 9 we present some examples which are applications of our result.

1.1 Setting

Given a finite set of continuous maps $g_i : X \to X$, $i \in \mathcal{P} = \{1, 2, \ldots, p\}$, $p \geq 1$, and the finitely generated semigroup $(G, \circ)$ with the finite set of generators $G_1 = \{id, g_1, g_2, \ldots, g_p\}$, we write $G = \bigcup_{n \in \mathbb{N}_0} G_n$ where $G_0 = \{id\}$ and $g \in G_n$ if and only if $g = g_{i_n} \cdots g_{i_2} g_{i_1}$, with $g_{i_j} \in G_1$ (for notational simplicity’s sake we will use $g_j g_i$ instead of the composition $g_j \circ g_i$). A semigroup can have multiple generating sets. We will assume that the generator set $G_1$ is minimal, meaning that no function $g_j$, for $j = 1, \ldots, p$, can be expressed as a composition of the remaining generators. We shall consider different concatenations instead of the elements in $G$ they create. One way to interpret this statement is to consider the itinerary map $\iota : F_p \to G$ given by $\hat{i} = i_n \cdots i_1 \mapsto g_{\hat{i}} := g_{i_n} \cdots g_{i_1}$, where $F_p$ is the free semigroup with $p$ generators, and to regard concatenations on $G$ as images by $\iota$ of paths on $F_p$. Set $G_1^* = G_1 \setminus \{id\}$ and, for every $n \geq 1$, let $G_n^*$ denote the space of concatenations of $n$ elements in $G_1^*$. To summon each element $g$ of $G_n^*$, we will write $|g| = n$ instead of $g \in G_n^*$. In $G$, one consider the semigroup operation of concatenation defined as usual: if $g = g_{i_n} \cdots g_{i_2} g_{i_1}$ and $h = h_{i_m} \cdots h_{i_1} h_{i_0}$, where $n = |g|$ and $m = |h|$, then $g h = g_{i_n} \cdots g_{i_2} h_{i_m} \cdots h_{i_1} h_{i_0} \in G_{m+n}$. The finitely generated semigroup $G$ induces an action in $X$, say.
We say that $S$ is a semigroup action if, for any $g, h \in G$ and every $x \in X$, we have $S(g h, x) = S(g, S(h, x))$. The action $S$ is continuous if the map $g : X \to X$ is continuous for any $g \in G$.

## 2 Preliminaries and Main Results

Consider a finitely generated free semigroup $(G, G_1)$ acting on a compact metric space $X$. Let $K \subset X$ be a compact set. Given $g = g_{i_n} \ldots g_{i_1} \in G_n$, we say a set $E \subset K$ is $(g, \varepsilon)$-separated if $d_p(x_1, x_2) > \varepsilon$ for any distinct $x_1, x_2 \in E$. When no confusion is possible with the notation for the concatenation of semigroup elements, the maximum cardinality of a $(g, \varepsilon)$-separated sets of $K$ will be denoted by $s(K, g, \varepsilon)$. We say that $F \subset K$ is a $(g, \varepsilon)$-spanning set if given $x \in K$ there exists $y \in F$ so that $d_p(x, y) < \varepsilon$. When no confusion is possible with the notation for the concatenation of semigroup elements, the minimum cardinality of a $(g, \varepsilon)$-spanning sets of $K$ will be denoted by $b(K, g, \varepsilon)$ (whenever we need to emphasize the subset $K$ we will wright $(K, g, \varepsilon)$-separated set or $(K, g, \varepsilon)$-spanning set). We now recall the notion of topological entropy introduced in [29], which is a generalization of the notion of topological entropy of a free semigroup action introduced by Bufetov in [17].

Let $\mathbb{P}$ be a Borel probability measure in $\Sigma^+_p$.

**Definition 2.1** Given a compact set $K \subset X$, we define

$$h_{top}(K, S, \mathbb{P}) = \lim_{\varepsilon \to 0} S(K, S, \mathbb{P}, \varepsilon)$$

where

$$S(K, S, \mathbb{P}, \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log S_n(K, S, \mathbb{P}, \varepsilon)$$

and

$$S_n(K, S, \mathbb{P}, \varepsilon) = \int_{\Sigma^+_p} s(K, g_{\omega_n} \ldots g_{\omega_1}, \varepsilon) d\mathbb{P}(\omega).$$

The topological entropy $h_{top}(K, S, \mathbb{P})$ is defined for $K = X$. In the case we have $\mathbb{P} = \eta_p := (\frac{1}{p}, \ldots, \frac{1}{p})^\mathbb{N}$ we recover the definition given in [17]. From now on we fix $\eta_p$ as the probability measure on $\Sigma^+_p$ and we write $h_{top}(K, S, \eta_p)$ as $h_{top}(K, S)$, for any $K \subset X$.

With Definition 2.1 we can can talk about entropy points:

1. We say that $x_0 \in X$ is an entropy point if for any closed neighbourhood $K$ of $x_0$ we have $h_{top}(K, S) > 0$.

2. We say that $x_0 \in X$ is a full entropy point if for any closed neighbourhood $K$ of $x_0$ we have $h_{top}(K, S) = h_{top}(X, S)$.

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Entropy points are those for which local neighborhoods reflect the complexity of the entire dynamical system. If one measures the chaos of the system by its topological entropy, the presence of an entropy point in closed neighbourhood means that, when restricted to any closed neighbourhood of such, the dynamics is chaotic.

In [37], as in the case of a single map, the authors proved that the set of full entropy points is not empty. More precisely, it holds the following:

**Theorem 2.2** Let $S: G \times X \to X$ be a finitely generated free semigroup action with positive topological entropy. Then $E^f_f(X, S) \neq \emptyset$.

Although not entirely related, the dynamics of free semigroup actions has a strong connection with skew products. Indeed, if $X$ is a compact metric space and one considers a finite set of continuous maps $g_i: X \to X$, $i \in \{1, 2, \ldots, p\}$, $p \geq 1$ consider the step skew product

$$\mathcal{F}_G: \Sigma^+_p \times X \to \Sigma^+_p \times X \quad (\omega, x) \mapsto (\sigma(\omega), g_{\omega_1}(x))$$

where $\omega = (\omega_1, \omega_2, \ldots, )$ is an element of the full unilateral space of sequences $\Sigma^+_p = \{1, 2, \ldots, p\}^\mathbb{N}$ and $\sigma$ denotes the shift map on $\Sigma^+_p$. With this notation we will write $\mathcal{F}_G(\omega, x) = (\sigma^n(\omega), f_{\omega}^n(x))$ for every $n \geq 1$.

### 2.1 Metric Entropy and Entropy Points

In what follows we denote by $\mathcal{M}(X)$ the set of measures on $X$, by $\mathcal{M}_1(X)$ the set of Borel probability measures on $X$ and for a given continuous map $f: X \to X$ we denote by $\mathcal{M}_f(X)$ the subset of $f$-invariant probability measures in $X$. In [31] the authors considered $\mathbb{P} = \eta_{\underline{a}} := (a_1, \ldots, a_p)^\mathbb{N}$, with $(a_1, \ldots, a_p)$ a probability vector, and defined the metric entropy of a free semigroup action $S$ with respect to $\nu \in \mathcal{M}_1(X)$ as

$$h_\nu(S, \eta_{\underline{a}}) = \sup_{\mu \in \mathcal{M}_f(X) : \Pi(\nu, \sigma)} \left\{ h_\mu(\mathcal{F}_G) + \int_{\Sigma^+_p} \log a_{\omega_1} \, d\eta_{\underline{a}}(\omega) \right\},$$

where $\Pi(\nu, \sigma)$ is the subset of $\mathcal{M}_1(\Sigma^+_p \times X)$ consisting of probability measures which are $\mathcal{F}_G$-invariant, $(\pi_{\Sigma^+_p})^* \mu$ is $\sigma$-invariant and $(\pi_X^*)^* \nu = \nu$. They proved that it satisfies a variational principle given by

$$h_{\text{top}}(X, S, \eta_{\underline{a}}) = \sup_{\nu \in \mathcal{M}_1(X) : \Pi(\nu, \sigma) \neq \emptyset} h_\nu(S, \eta_{\underline{a}}).$$

In the case where $\eta_{\underline{a}} = \eta_{\underline{p}} := \left(\frac{1}{p}, \ldots, \frac{1}{p}\right)^\mathbb{N}$ we have the following

$$h_\nu(S) := h_\nu(S, \eta_{\underline{p}}) = \sup_{\mu \in \Pi(\nu, \sigma)} h_\mu(\mathcal{F}_G) - \log p.$$

Denote by $E_p(S, X)$ the set of entropy points of $S$, by $E^f_p(S, X)$ the set of all full entropy points of $S$. We also denote by $E_p(S^+_p \times X, \mathcal{F}_G)$ the set of entropy points of $\mathcal{F}_G$ and by $E^f_p(S^+_p \times X, \mathcal{F}_G)$ the set of full entropy points of $\mathcal{F}_G$.

Our first result shows that the metric entropy and the topological entropy of a free semigroup action are concentrated on the the set of entropy points.

**Theorem A** Let $S: G \times X \to X$ be a finitely generated free semigroup action. Then
i. $E_p(X, S)$ and $E^f_p(X, S)$ are both $S$-invariant;

ii. If $v \in M_1(X)$ is so that $\Pi(\sigma, v)_{erg} \neq \emptyset$ and $h_v(S) > 0$, then $\text{supp}(v) \subset E_p(X, S)$;

iii. $htop(E_p(X, S), S) = htop(X, S)$.

As proved in [1], it is possible to get an upper bound for the metric of $v \in M_1(X)$ for which $\Pi(\sigma, v)_{erg} := \{ \mu \in \Pi(\sigma, v) : \mu \text{ is } F_\mu \text{-ergodic} \} \neq \emptyset$, in terms of the local information of the topological entropy.

**Theorem B** Let $S : G \times X \to X$ be a finitely generated free semigroup action and $v \in M_1(X)$ so that $\Pi(\sigma, v)_{erg} \neq \emptyset$. Then

$$\liminf_{\varepsilon \to 0} \{ S(K, S, \varepsilon) : K \in B_X \text{ with } v(K) > 0 \} \geq h_v(S).$$

In particular, for any $K \in B_X$, $htop(K, S) \geq h_v(S)$.

As an immediate application of Theorem B we have

**Corollary 2.3** Let $S : G \times X \to X$ be a finitely generated free semigroup action with positive topological entropy. Then

$$\bigcup \{ \text{supp}(v) : \Pi(\sigma, v)_{erg} \neq \emptyset \text{ and } h_v(S) = htop(X, S) \} \subset E^f_p(X, S).$$

In particular, if there exists $v \in M_1(X)$ of maximal entropy and satisfying $\Pi(\sigma, v)_{erg} \neq \emptyset$, then $htop(E^f_p(X, S), S) = htop(X, S)$.

### 2.2 Entropy Function

For each $\varepsilon > 0$ and $x \in X$, define

$$h_d(x, \varepsilon) = \inf \{ B(K, S, \varepsilon) : K \text{ is compact neighbourhood of } x \},$$

where the subscript $d$ is to emphasize the metric $d$,

$$B(K, S, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log B_n(K, S, \varepsilon),$$

$$B_n(K, S, \varepsilon) = \frac{1}{p^n} \sum_{g \in G_n^*} b(K, g, \varepsilon)$$

and $b(K, g, \varepsilon)$ denotes the minimum cardinality of a $(g, \varepsilon)$-spanning set in the metric $d$.

As $h_d(x, \varepsilon)$ increases as $\varepsilon$ decreases to zero, it is well defined the following

$$h_d(x) = \lim_{\varepsilon \to 0^+} h_d(x, \varepsilon)$$

(4)

and it is less or equal to $htop(X, S)$. Adapting the proof of Proposition 2.5.3 of [38] we can conclude that $h_d(x)$ depends only on the topology of $X$ and we can denote $h_d(x)$ by $htop(x)$.

**Definition 2.4** Let $S : G \times X \to X$ be a continuous finitely generated free semigroup action. The function $htop : X \to [0, htop(X, S)]$, $x \mapsto htop(x)$ is called the entropy function of $S$.

Since, by [37], $B(K, S, \varepsilon) \leq S(K, S, \varepsilon) \leq B(K, S, \varepsilon/2)$, we have

$$htop(x) = \liminf_{\varepsilon \to 0} \{ S(K, S, \varepsilon) : K \text{ is compact neighbourhood of } x \}.$$
The next theorem gives a lower bound, depending on the metric entropy, for the entropy function in each point. Moreover, it is proved that the supremum of the entropy function is given by the topological entropy.

**Theorem C** Let \( S : G \times X \to X \) be a continuous finitely generated free semigroup action. Then

i. \( h_{top}(x) \geq \sup \{ h_v(\mathcal{S}) : v \in \mathcal{M}_1(X), \, \Pi(\sigma, v)_{erg} \neq \emptyset \text{ and } x \in supp(v) \} \);

ii. For each \( \varepsilon > 0 \), \( h_d(\cdot, \varepsilon) \) is upper semi continuous, consequently, \( h \) is Borel measurable;

iii. The entropy function \( h_{top} : X \to [0, \infty) \) is \( S \)-invariant;

iv. If \( K \subset X \) is closed, then \( \sup_{x \in K} h_{top}(x) \geq h_{top}(K, S) \). In particular,

\[
\sup_{x \in X} h_{top}(x) = h_{top}(X, S);
\]

v. \( h_{top} \) is \( S \)-invariant and for \( v \in \mathcal{M}_1(X) \) satisfying \( \Pi(\sigma, v)_{erg} \neq \emptyset \), we have

\[
\int_X h_{top}(x) \, dv \geq h_v(S).
\]

We say \( x \in X \) is a uniform entropy point if \( h_{top}(x) > 0 \). Denote by \( E_{up}(X, S) \) the set of all uniform entropy points. We say \( x \in X \) is a uniform full entropy point if \( h_{top}(x) = h_{top}(X, S) \).

Denote by \( E_{up}^f(X, S) \) the set of all uniform entropy points. Our next result guarantees that \( E_{up}(X, S) \) and \( E_{up}^f(X, S) \) are \( S \)-invariant and that for \( v \in \mathcal{M}_1(X) \) satisfying \( \Pi(\sigma, v)_{erg} \neq \emptyset \) and \( h_v(S) > 0 \) we have \( supp(v) \subset h_{top}^{-1}([0, \infty]) = E_{up}(X, S) \). In the case where \( h_v(S) = h_{top}(X, S) \), \( supp(v) \subset h_{top}^{-1}(h_{top}(X, S)) = E_{up}^f(X, S) \).

**Theorem D** Let \( S : G \times X \to X \) be a continuous finitely generated free semigroup action.

i. \( E_{up}^f(X, S) \subset E_{up}(X, S) \cap E_{up}^f(X, S), E_{up}(X, S) \subset E_p(X, S) \);

ii. If \( K \) is a closed subset with \( h_{top}(K, S) > 0 \), then \( K \cap E_{up}(X, S) \neq \emptyset \). In particular \( E_{up}(X, S) \neq \emptyset \) when \( h_{top}(X, S) > 0 \);

iii. \( E_{up}(X, S) \) and \( E_{up}^f(X, S) \) are both \( S \)-invariant;

iv. \( E_{up}(X, S) \subset X \) is a \( F_\sigma \) subset, i.e., it is a countable union of closed subsets of \( X \).

The subset \( E_{up}^f(X, S) \) is a \( F_{\sigma \delta} \) subset, i.e., it is a countable intersection of \( F_\sigma \) subsets of \( X \);

v. If \( v \in \mathcal{M}_1(X) \) is so that \( \Pi(\sigma, v)_{erg} \neq \emptyset \) and \( h_v(S) > 0 \), then \( supp(v) \subset E_{up}(X, S) \);

vi. If \( v \in \mathcal{M}_1(X) \) is so that \( \Pi(\sigma, v)_{erg} \neq \emptyset \) and \( h_v(S) = h_{top}(X, S) \), then \( supp(v) \subset E_{up}^f(X, S) \);

vii. \( h_{top}(E_{up}(X, S), S) = h_{top}(X, S) \).

**Definition 2.5** We say that a continuous free semigroup action \( S : G \times X \to X \) associated to the finitely generated semigroup \( G \) satisfies the strong orbital specification property (concept introduced in [37]) if for any \( \varepsilon > 0 \) there exists \( p(\varepsilon) > 0 \) such that for any \( h_{p_j} \in G_{p_j}^* \) (with \( p_j \geq p(\delta) \) for \( 1 \leq j \leq k \)) any points \( x_1, \ldots, x_k \in X \) and any natural numbers \( n_1, \ldots, n_k \), any semigroup elements \( g_{n_j,j} = g_{i_{n_j,j}} \cdots g_{i_{2,j}} g_{i_{1,j}} \in G_{n_j} \) \((j = 1 \ldots k)\) there exists \( x \in X \) so that \( d(g_{\ell,1}(x), g_{\ell,1}(x_1)) < \delta \) for every \( \ell = 1 \ldots n_1 \) and

\[
d( g_{\ell,j} h_{p_{j-1}} \cdots g_{p_2} h_{p_1} g_{p_1,1}(x), g_{\ell,j}(x_j) ) < \varepsilon
\]

for every \( j = 2 \ldots k \) and \( \ell = 1 \ldots n_j \) (here \( g_{\ell,j} := g_{i_{\ell,j}} \cdots g_{i_{1,j}} \)).

In the next theorem we show that the strong orbital specification property implies that every point is a uniform full entropy point.
Theorem E Let $S: G \times X \to X$ be a continuous finitely generated free semigroup action so that every element $g \in G_1$ is a local homeomorphism. If $S$ satisfies the strong orbital specification property then $h_{top}(x) = h_{top}(X, S)$ for every $x \in X$.

Our last result shows that the topological entropy of a free semigroup action may be computed in terms of countable sets.

Theorem F Let $S: G \times X \to X$ be a continuous finitely generated free semigroup action. Then there exists a countable closed subset $K \subset X$ such that $h_{top}(K, S) = h_{top}(X, S)$. Moreover,

i. $K$ can be chosen such that the set of limit points of $K$ has at most one limit point;

ii. $K$ has a unique limit point if and only if there is $x \in X$ with $h_{top}(x) = h_{top}(X, S)$.

3 Proof of Theorem A

In order to prove Theorem A we need some auxiliary results. In what follows, for $\ell \in \mathbb{N}$ and $\omega_1, \ldots, \omega_\ell \in \{1, \ldots, p\}$, $[\omega_1 \ldots \omega_\ell] \subset \Sigma_+^\ell$ denotes the cylinder set

$$[\omega_1 \ldots \omega_\ell] := \{ \theta \in \Sigma_+^\ell : \theta_{[1,n]} = \omega_1 \ldots \omega_n \}.$$ 

Proposition 3.1 Let $\omega = \omega_1 \omega_2 \cdots \in \Sigma_+^\ell$. Then

$$h_{top}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G) = h_{top}(K, S) + \log p, \text{ for all } \ell \in \mathbb{N}.$$ 

Proof Let $n$ be a positive integer and consider $N = p^n$. There are $N$ distinct words of length $n$ in $\mathbb{F}_p$. Denote these words by $\theta^1, \ldots, \theta^N$. Let $(\theta(i))_{i=1}^N \subset \Sigma_+^\ell$ be a sequence such that $\theta(i)_{[1,n]} = \theta^i$. We notice that, for $0 < \varepsilon < \frac{1}{2}$, the sequence $(\omega_1 \ldots \omega_\ell \theta(i))_{i=1}^N \subset [\omega_1 \ldots \omega_\ell]$ is $(n + \ell, \varepsilon, \sigma)$-separated subset of $[\omega_1 \ldots \omega_\ell]$.

Denoting $\theta^i = \theta^1 \cdots \theta^N$ and $g^{(i)} = g_{\theta^1} \cdots g_{\theta^N} \cdots g_{\omega_1}$. Set $s_i = s(K, g^{(i)}, \varepsilon)$, and let the points $x^i_1, \ldots, x^i_{b_i}$ form a $(g^{(i)}, \varepsilon)$-separated subset of $K$. Then the points

$$(\omega_1 \ldots \omega_\ell \theta(i), x^i_j) \in [\omega_1 \ldots \omega_\ell] \times K,$$

$i = 1, \ldots, N$ $j = 1, \ldots, Z_i$,

form a $(n + \ell, \varepsilon, \mathcal{F}_G)$-separated subset of $[\omega_1 \ldots \omega_\ell] \times K$. It implies that

$$S_{n+\ell}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G, \varepsilon) \geq \sum_{i=1}^N s(K, g^{(i)}, \varepsilon),$$

and then $h_{top}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G) \geq h_{top}(K, S) + \log p$.

For the converse inequality, consider $\varepsilon > 0$ and take $C(\varepsilon)$ an arbitrary positive integer such that $p^{-C(\varepsilon)} < \frac{\varepsilon}{100}$. We notice that there exist $N = p^{n+2C(\varepsilon)}$ distinct words of length $n + 2C(\varepsilon)$. Denote these words by $\omega^1, \ldots, \omega^N$.

Let $(\omega(i))_{i=1}^N \subset \Sigma_+^\ell$ be an arbitrary sequence satisfying

$$\omega(i)_{[1,n+C(\varepsilon)+\ell]} = \omega_1 \omega_2 \ldots \omega_\ell \omega^i.$$ 

This sequence forms a $(n, \varepsilon, \sigma)$-spanning set of $[\omega_1 \ldots \omega_\ell]$. Denote $\theta^i = \omega(i)_{[1,n+\ell]}$, by $g^{(i)}$ the concatenation associated to $\theta^i$ and $b_i = b(K, g^{(i)}, \varepsilon)$ and assume that the points $x^i_1, \ldots, x^i_{b_i}$ form a $(g^{(i)}, \varepsilon)$-spanning subset (of minimal cardinality, since the cardinality of the set of
such points is equal to \( b_i \) of \( K \). It follows that the points \((\omega(i), x^i_j)\) with \( i = 1, \ldots, N \) and \( j = 1, \ldots, b_i \), form a \((n + \ell, \varepsilon)\)-spanning subset of \([\omega_1, \ldots, \omega_\ell] \times K\). In that case, there exists a positive constant \( T = T(\varepsilon) \) depending only on \( \varepsilon \) so that

\[
B_{n+\ell}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G, \varepsilon) \leq \sum_{i=1}^{n+2C(\varepsilon)+\ell} b(K, g^{(i)}, \varepsilon) \leq T(\varepsilon) \sum_{|g|=n+\ell} b(K, g, \varepsilon).
\]

From the last inequalities we obtain

\[
h_{top}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G) \leq h_{top}(K, \mathcal{S}) + \log p.
\]

It concludes the proof. \( \square \)

**Remark 3.2** If instead of considering the symmetric random walk \( \eta_p \) we take \( \eta_{\mathcal{S}} \) any Bernoulli probability measure on \( \Sigma_p^+ \) we obtain, for \( \ell \in \mathbb{N} \) and \( \omega_1 \ldots \omega_\ell \in \Sigma_p^+ \),

\[
P_{top}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G, \phi_\mathcal{S}) = h_{top}(K, \mathcal{S}, \phi_\mathcal{S}),
\]

where \( \phi_\mathcal{S} : (\omega, x) \in \Sigma_p^+ \times X \mapsto \log a_{\omega_1} \in \mathbb{R} \) and \( P_{top}(\cdot, \mathcal{F}_G, \cdot) \) denotes the topological pressure (see [31]).

As an immediate consequence of Proposition 3.1 we have the following.

**Corollary 3.3** Under the above notations we have that \( E_p(\Sigma_p^+ \times X, \mathcal{F}_G) = \Sigma_p^+ \times X \).

**Proof** Given \((\omega, x) \in \Sigma_p^+ \times X\) and \( F \subset X \) closed neighbourhood containing \((\omega, x)\), by the definition of the product topology of \( \Sigma_p^+ \times X \), there exist closed subsets \([\omega_1 \ldots \omega_\ell] \subset \Sigma_p^+ \) and \( K \subset X \), so that \([\omega_1 \ldots \omega_\ell] \times K \subset F\). As \( h_{top}(F, \mathcal{F}_G) \geq h_{top}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G) \) and, by Proposition 3.1, \( h_{top}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G) \geq \log p > 0 \). \( \square \)

Let \( \pi_X \) be the canonical projection of \( \Sigma_p^+ \times X \) on the first coordinate.

**Corollary 3.4** Under the above assumptions, \( x \in X \) is a full entropy point for \( \mathcal{S} \) if and only \((\omega, x)\) is a full entropy point for \( \mathcal{F}_G \), for every \( \omega \in \Sigma_p^+ \).

**Proof** Let \((\omega, x) \in \Sigma_p^+ \times X\) be a full entropy point for \( \mathcal{F}_G \). Let \( K \) be a closed neighbourhood of \( x \) and notice that \( \Sigma_p^+ \times K \) is a closed neighbourhood of \((\omega, x)\). By Proposition 3.1 we have that

\[
h_{top}(X, \mathcal{S}) + \log p = h_{top}(\Sigma_p^+ \times X, \mathcal{F}_G) = h_{top}(\Sigma_p^+ \times K, \mathcal{F}_G) = h_{top}(K, \mathcal{S}) + \log p,
\]

which implies that \( h_{top}(X, \mathcal{S}) = h_{top}(K, \mathcal{S}) \) and then \( x \in E_p^f(X, \mathcal{S}) \).

By the other hand, given \( x \in E_p^f(X, \mathcal{S}) \) and \( \omega \in \Sigma_p^+ \), any closed neighbourhood \( V \) of \((\omega, x)\) contains a closed neighbourhood of the form \([\omega_1 \ldots \omega_\ell] \times K\), for some \( \ell \in \mathbb{N} \) and some \( K \) closed neighbourhood of \( x \). Then, by Proposition 3.1 we have that

\[
h_{top}(V, \mathcal{F}_G) \geq h_{top}([\omega_1 \ldots \omega_\ell] \times K, \mathcal{F}_G)
\]

\[
= h_{top}(K, \mathcal{S}) + \log p
\]

\[
= h_{top}(X, \mathcal{S}) + \log p
\]

\[
= h_{top}(\Sigma_p^+ \times X, \mathcal{F}_G),
\]

and it implies that \( h_{top}(V, \mathcal{F}_G) = h_{top}(\Sigma_p^+ \times X, \mathcal{F}_G) \). Hence \((\omega, x) \in E_p^f(\Sigma_p^+ \times X, \mathcal{F}_G) \) and it ends the proof. \( \square \)
Another important consequence of Proposition 3.1 and the existence of full entropy points is the following.

**Corollary 3.5** Let $\mathbb{S} : G \times X \to X$ be a finitely generated free semigroup action. Then $E_p^\mathbb{S}(X, \mathbb{S}) \neq \emptyset$.

The next result guarantees that the sets $E_p(X, \mathbb{S})$ and $E_p^f(X, \mathbb{S})$ are invariant under the free semigroup action $\mathbb{S}$, i.e., $g(E_p(X, \mathbb{S})) \subset E_p(X, \mathbb{S})$ and $g(E_p^f(X, \mathbb{S})) \subset E_p^f(X, \mathbb{S})$ for every $g \in G$.

**Theorem 3.6** Let $\mathbb{S} : G \times X \to X$ be a finitely generated free semigroup action. Then $g(E_p(X, \mathbb{S})) \subset E_p(X, \mathbb{S})$ and $g(E_p^f(X, \mathbb{S})) \subset E_p^f(X, \mathbb{S})$ for every $g \in G$.

To prove Theorem 3.6 we need the following lemma.

**Lemma 3.7** Let $(X, d)$ be a compact metric space and $f : X \to X$ be a continuous map. Given $K \subset X$ and $\alpha > 0$, if $h_{top}(K, f) \geq \alpha$ then $h_{top}(f(K), f) \geq \alpha$.

**Proof** Let $K \subset X$ and fix $\alpha > 0$. If $h_{top}(f(K), f) < \alpha$, there exists $\gamma > 0$ such that $h_{top}(f(K), f) \leq \alpha - \gamma$.

Given $\eta > 0$ there exists $\delta > 0$ so that
\[
\limsup_{n \to \infty} \frac{1}{n} \log s_n(f(K), f, \varepsilon) \leq (\alpha - \gamma) + \eta, \text{ for every } 0 < \varepsilon < \delta.
\]

It implies that for $0 < \varepsilon < \delta$ there exists $N_0 = N_0(\varepsilon)$ so that
\[
s_n(f(K), f, \varepsilon) \leq e^{n(\eta + (\alpha - \gamma))}, \text{ for every } n \geq N_0.
\]

Since $K \subset f^{-1}(f(K))$ we have that
\[
s_n(K, f, \varepsilon) \leq s_n(f(K), f, \varepsilon) + s_1(K, f, \varepsilon) \leq e^{n(\eta + (\alpha - \gamma))} + 1, \text{ for every } n \geq N_0
\]
and so,
\[
\limsup_{n \to \infty} \frac{1}{n} \log s_n(K, f, \varepsilon) \leq \eta + (\alpha - \gamma),
\]
which gives, $h_{top}(K, f) \leq \eta + (\alpha - \gamma)$. As $\eta > 0$ may be taken arbitrarily small we conclude that $h_{top}(K, f) \leq \alpha - \gamma$, which contradicts the hypothesis and ends the proof. \qed

Let us proceed to the proof of Theorem 3.6. We will only prove that $g(E_p(X, \mathbb{S})) \subset E_p(X, \mathbb{S})$ holds for every $g \in G$, since the inclusion $f(E_p^f(X, \mathbb{S})) \subset E_p^f(X, \mathbb{S})$, for every $g \in G$, may be obtained by a very similar argument to the first one. We observe that it is enough to prove that $f(E_p(X, \mathbb{S})) \subset E_p(X, \mathbb{S})$ and $f(E_p^f(X, \mathbb{S})) \subset E_p^f(X, \mathbb{S})$ for every $f \in G_1$. Fix $x_0 \in E_p(X, \mathbb{S})$. We $f \in G_1^*$. Let $\omega = \omega_1\omega_2 \cdots \in \Sigma_p^+$ so that $g_{\omega_1}^1 = g_{\omega_1} = f$ and consider the cylinder set $[\omega_1]$, which contains $\omega$, and take $K \subset X$ any closed neighbourhood of $f(x_0)$. As $f$ is continuous and $x_0 \in E_p(X, \mathbb{S})$, we have that $f^{-1}(K)$ is a closed neighbourhood of $x_0$ and $h_{top}(f^{-1}(K), \mathbb{S}) > 0$. By Proposition 3.1 we have that
\[
h_{top}([\omega_1] \times f^{-1}(K), \mathcal{F}_G) > \log p
\]
and so, as $\mathcal{F}_G([\omega_1] \times f^{-1}(K)) \subset \Sigma_p^+ \times K$, by Lemma 3.7 and Proposition 3.1 we have
\[
\log p < h_{top}(\mathcal{F}_G([\omega_1] \times f^{-1}(K)), \mathcal{F}_G) \leq h_{top}(\Sigma_p^+ \times K, \mathcal{F}_G) = h_{top}(K, \mathbb{S}) + \log p.
\]
Hence, $h_{\text{top}}(K, S) > 0$ which implies that $f(x_0) \in E_p(X, S)$ and finishes the proof.

The next lemma is an important tool and will be useful for the rest of the work.

**Lemma 3.8** Let $S : G \times X \to X$ be a finitely generated free semigroup action and $K_1, K_2, \ldots, K_m \subset X$ ($m \in \mathbb{N}$). Then $B \left( \bigcup_{i=1}^{m} K_i, S, \varepsilon \right) = \max_i B(K_i, S, \varepsilon)$, for every $\varepsilon > 0$ and so $h_{\text{top}} \left( \bigcup_{i=1}^{m} K_i, S \right) = \max_i h_{\text{top}}(K_i, S)$.

**Proof** Fix $\varepsilon > 0$. By definition, given $n \in \mathbb{N}$ and $g \in G^*_n$, there exists $i(g, \varepsilon) \in \{1, \ldots, m\}$ so that

$$s(K, g, \varepsilon) \leq \sum_{i=1}^{m} s(K_i, g, \varepsilon) \leq m \cdot s(K_{i(g, \varepsilon)}, g, \varepsilon),$$

where $K = \bigcup_{i=1}^{m} K_i$ and $s(K_{i(g, \varepsilon)}, g, \varepsilon) = \max_{1 \leq i \leq m} s(K_i, g, \varepsilon)$. It implies that

$$S_n(K, S, \varepsilon) \leq m \cdot \frac{1}{p^n} \sum_{g \in G^*_n} s(K_{i(g, \varepsilon)}, g, \varepsilon).$$

Choose an increasing sequence $\{n_j\}_{j \in \mathbb{N}}$ such that $\left\{ \frac{1}{n_j} \log S_{n_j}(K, S, \varepsilon) \right\}_{j \in \mathbb{N}}$ tends to $\limsup_{n \to \infty} \frac{1}{n} \log S_n(K, S, \varepsilon)$ with $j \to \infty$. At least one element of the set $\{K_1, \ldots, K_m\}$ appears infinitely many times in the sequence $\left\{K_{i(g, \varepsilon)}\right\}_{g \in G^*_n, j \in \mathbb{N}}$, say $K_{i(\varepsilon)}$ such element. Again choosing a subsequence of the sequence $\{n_j\}_{j \in \mathbb{N}}$, for simplicity denoting it by $\{n_j\}_{j \in \mathbb{N}}$, we may assume $K_{i(g, \varepsilon)} = K_{i(\varepsilon)}$ for all $g \in G^*_n$ and $j \in \mathbb{N}$. It yields

$$\lim_{j \to \infty} \frac{1}{n_j} \log S_{n_j}(K, S, \varepsilon) \leq \lim_{j \to \infty} \frac{1}{n_j} \log \left( m \cdot \frac{1}{p^n} \sum_{g \in G^*_n} s(K_{i(g, \varepsilon)}, g, \varepsilon) \right) = \lim_{j \to \infty} \frac{1}{n_j} \log \left( \frac{1}{p^n} \sum_{g \in G^*_n} s(K_{i(\varepsilon)}, g, \varepsilon) \right). \quad (5)$$

Now take a sequence of positive real numbers $\{\varepsilon_t\}_{t \in \mathbb{N}}$, convergent to zero. At least one element of the set $\{K_1, \ldots, K_m\}$, say $K^*$, appears infinitely many times in the infinite sequence $\left\{K_{i(\varepsilon_t)}\right\}_{t \in \mathbb{N}}$. So taking a subsequence $\{\varepsilon_{t_i}\}_{t_i \in \mathbb{N}}$ of $\{\varepsilon_t\}_{t \in \mathbb{N}}$ we get the equality $K_{i(\varepsilon_{t_i})} = K^*$, which holds for any $t \in \mathbb{N}$. By (5) we conclude that

$$h_{\text{top}}(K, S) = \lim_{t \to \infty} \lim_{j \to \infty} \frac{1}{n_j} \log S_{n_j}(K, S, \varepsilon_{t_i}) \leq \lim_{t \to \infty} \lim_{j \to \infty} \frac{1}{n_j} \log \left( \frac{1}{p^n} \sum_{g \in G^*_n} s(K^*, g, \varepsilon_{t_i}) \right) \leq h_{\text{top}}(K^*, S).$$
In particular we obtain \( h_{\text{top}}\left( \bigcup_{i=1}^{m} K_i, \mathcal{S} \right) \leq \max_{i} h_{\text{top}}(K_i, \mathcal{S}) \). The inequality
\[
\max_{i} h_{\text{top}}(K_i, \mathcal{S}) \leq h_{\text{top}}\left( \bigcup_{i=1}^{m} K_i, \mathcal{S} \right)
\]
is obvious. \( \square \)

The fact of the semigroup action having positive entropy when restricted to a subset of \( X \) is related to the presence of entropy points in such subset is proved in the next proposition.

**Proposition 3.9** Let \( \mathcal{S} : G \times X \to X \) be a finitely generated free semigroup action and \( K \) be a closed subset of \( X \).

i. If \( h_{\text{top}}(K, \mathcal{S}) > 0 \), then \( K \cap E_{\rho}(X, \mathcal{S}) \neq \emptyset \);

ii. If \( h_{\text{top}}(K, \mathcal{S}) = h_{\text{top}}(X, \mathcal{S}) \), then \( K \cap E^{\ell}_{\rho}(X, \mathcal{S}) \neq \emptyset \).

**Proof** We start proving (i). Cover \( K \) by finitely many closed balls \( B^1_1, \ldots, B^1_{\ell_1} \) with diameter at most 1. We may assume that \( K = \bigcup_{j=1}^{\ell_{1}} B^1_{j} \), otherwise we take \( A^1_{j} = B^1_{j} \cap K \) for \( j = 1, \ldots, \ell_1 \), which are closed subsets of \( X \) and \( K = \bigcup_{j=1}^{\ell_{1}} A^1_{j} \). By Lemma 3.8 we have that
\[
0 < h_{\text{top}}(K, \mathcal{S}) = h_{\text{top}}\left( \bigcup_{j=1}^{\ell_{1}} B^1_{j}, \mathcal{S} \right) = \max_{j} h_{\text{top}}(B^1_{j}, \mathcal{S})
\]
and it implies the existence of \( j_1 \) so that \( h_{\text{top}}(K, \mathcal{S}) = h_{\text{top}}(B^1_{j_1}, \mathcal{S}) \). Cover \( B^1_{j_1} \) by finitely many closed balls \( B^2_1, \ldots, B^2_{\ell_2} \) with diameter at most \( \frac{1}{2} \). By the same reasoning, there exists \( j_2 \) for which \( h_{\text{top}}(K, \mathcal{S}) = h_{\text{top}}(B^1_{j_1}, \mathcal{S}) = h_{\text{top}}(B^2_{j_2}, \mathcal{S}) \). By induction, for each \( n \geq 2 \), there is a closed ball \( B^n_{j_n} \subset B^n_{j_{n-1}} \) with diameter at most \( \frac{1}{n} \) such that \( h_{\text{top}}(K, \mathcal{S}) = h_{\text{top}}(B^n_{j_n}, \mathcal{S}) \). Set \( x \) the unique point in the intersection of the closed balls \( B^n_{j_n} \). By definition, \( x \) is an entropy point for \( \mathcal{S} \).

To prove (ii) we notice that, since \( h_{\text{top}}(K, \mathcal{S}) = h_{\text{top}}(X, \mathcal{S}) \), then \( h_{\text{top}}(\Sigma^+_p \times K, \mathcal{F}_G) = h_{\text{top}}(\Sigma^+_p \times X, \mathcal{F}_G) \). By [1, Theorem 3.5] there exists \( (\omega, x) \in \Sigma^+_p \times K \cap E^\ell_p(\Sigma^+_p \times X, \mathcal{F}_G) \). By Corollary 3.4 we have that \( \pi_X(\omega, x) \in K \cap E^\ell_p(X, \mathcal{S}) \), and it proves (ii). \( \square \)

### 3.1 Proof of Theorem A

Before we start the proof we recall the definition of Katok’s entropy of a probability measure in \( X \), which was extended for the free semigroup action setting in [31].

#### 3.1.1 Katok’s Entropy

**Definition 3.10** Given a Borel probability measure \( \nu \) on \( X \), \( \delta \in (0, 1) \) and \( \varepsilon > 0 \), define
\[
h^K_{\nu}(\mathcal{S}, \varepsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log S^\nu(X, \mathcal{S}, n, \varepsilon, \delta),
\]
where
\[
S^\nu(X, \mathcal{S}, n, \varepsilon, \delta) = \frac{1}{p^n} \sum_{g \in G^*_\delta} s^\nu(g, \varepsilon, \delta),
\]
\( \mathcal{S} \) Springer
and for \( g \in G_n^* \)

\[
\nu(g, \varepsilon, \delta) = \inf_{E \subseteq X : \nu(E) > 1 - \delta} s(E, g, \varepsilon)
\]

and \( s(E, g, \varepsilon) \) denotes the maximal cardinality of the \((g, \varepsilon)\)-separated subsets of \( E \).

The \( K \)-metric entropy of the free semigroup action \( S \) with respect to \( \nu \) is defined by

\[
h^K_\nu (S) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} h^K_\nu (S, \varepsilon, \delta)
\]

Observe that the previous limit is well defined due to the monotonicity of the function \((\varepsilon, \delta) \mapsto \frac{1}{n} \log S_\nu(X, S, n, \varepsilon, \delta)\) on the variables \( \varepsilon \) and \( \delta \). Moreover, if the set of generators is \( G_1 = \{\text{Id}, f\} \), we recover the notion proposed by Katok for a single dynamics \( f \).

**Remark 3.11** We observe that Definition 3.10 could be made in terms of spanning sets. More precisely, given \( \varepsilon > 0 \), a positive integer \( n \) and \( g \in G_n^* \), we set

\[
b_\nu(g, \varepsilon, \delta) = \inf_{E \subseteq X : \nu(E) > 1 - \delta} b(E, g, \varepsilon).
\]

It is not difficult to prove that

\[
h^K_\nu (S) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log B_\nu(X, S, n, \varepsilon, \delta),
\]

where

\[
B_\nu(X, S, n, \varepsilon, \delta) = \frac{1}{p^n} \sum_{g \in G_n^*} b_\nu(g, \varepsilon, \delta).
\]

By [31, Theorem C] we have that

\[
h_{top}(X, S) = \sup_{\nu \in \mathcal{M}_1(X)} h^K_\nu (S).
\]

Let us proceed to the proof of Theorem A. To prove part (i) we notice that it is a consequence of Theorem 3.6.

We recall that for \( \nu \in \mathcal{M}_1(X) \), we denote the set of ergodic measures in \( \Pi(\sigma, \nu) \) by \( \Pi(\sigma, \nu)_{\text{erg}} \).

Since \( h_\nu(\mathcal{S}) > 0 \) and \( \Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset \), by the definition of the metric entropy, there exists \( \mu \in \Pi(\sigma, \nu)_{\text{erg}} \) for which \( h_\mu(\mathcal{F}_G) > \log p \). If \( x \in \text{supp}(\nu) \), for any closed neighbourhood \( N_x \) of \( x \), \( \nu(N_x) = \mu(\Sigma^+_p \times N_x) > 0 \). Since \( \mu \) is \( \mathcal{F}_G \)-ergodic and \( \mu(\Sigma^+_p \times N_x) > 0 \), by [1, Theorem 3.7] we have that

\[
h_{top}(N_x, \mathcal{S}) + \log p = h_{top}(\Sigma^+_p \times N_x, \mathcal{F}_G) \geq h_\mu(\mathcal{F}_G) + \log p,
\]

which implies that \( h_{top}(N_x, \mathcal{S}) > 0 \) and then, \( x \in E_p(X, \mathcal{S}) \), concluding the proof item (ii).

For (iii) we notice that

\[
h_{top}(X, \mathcal{S}) = \sup_{\nu \in \mathcal{M}_1(X) : \Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset} h_\nu(\mathcal{S}).
\]
By (ii) we have that the supremum is taken on the probability measures on $X$ with support contained in $E_p(X, S)$ (which is $S$-invariant by Theorem 3.6). By the proof of item (iii) of Theorem C of [31, p. 485] we have that

$$h_\nu(S) \leq h^K_\nu(S),$$

and it implies that $h_{top}(X, S) = h_{top}(E_p(X, S), S)$, since supp($\nu$) $\subset E_p(X, S)$.

4 Proof of Theorem B

Let $0 < \delta < 1$ fixed. Let $\varepsilon > 0$ and $K \in B_X$ satisfying $\nu(K) > 0$. As $\Pi(\sigma, \nu)_{\text{erg}} \neq \emptyset$, for $\mu \in \Pi(\sigma, \nu)_{\text{erg}}$, by the ergodicity of $\mu$, there exists $m = m(K) \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{i=0}^m \mathcal{F}_G^i\left(\Sigma_p^+ \times K\right)\right) \geq 1 - \delta.$$

**Claim.** If $K^{(m)} := \pi_X\left(\bigcup_{i=0}^m \mathcal{F}_G^i\left(\Sigma_p^+ \times K\right)\right)$, then $S(K, \varepsilon) = S(K^{(m)}, S, \varepsilon)$.

**Proof of the Claim.** First of all we notice that $\nu(K^{(m)}) \geq 1 - \delta$. Fix $n \in \mathbb{N}$ and take $g \in G_n^*$. Let $E_g \subset K^{(m)}$ be a $(g, \varepsilon)$-separated set. In particular there exists $0 \leq i_0(n) \leq m$ so that

$$\left|E_g \cap \pi_X\left(\mathcal{F}_G^i\left(\Sigma_p^+ \times K\right)\right)\right| \geq \frac{|E_g|}{m+1}. \quad (9)$$

Otherwise, since $E_g \subset \pi_X\left(\Sigma_p^+ \times K\right) \cup \pi_X\left(\mathcal{F}_G(K)\right) \cup \cdots \cup \pi_X\left(\mathcal{F}_G^m(K)\right)$, we have $E_g = \bigcup_{i=0}^m E_g \cap \pi_X\left(\mathcal{F}_G^i(K)\right)$, which gives

$$|E_g| \leq \sum_{i=0}^m \left|E_g \cap \pi_X\left(\mathcal{F}_G^i\left(\Sigma_p^+ \times K\right)\right)\right| < (m+1) \frac{|E_g|}{m+1} = |E_g|,$$

which is a contradiction. Then, if $\tilde{g} = g_{j_0} \cdots g_{j_l}$, we have that $g^{-1}_{\Sigma_{i_0}(n)}(E) \cap K$ is a $(\tilde{g}, \varepsilon)$-separated set in $K$. Taking $\tilde{g} = g_{\Sigma_{i_0}(n)}$, we have that $g^{-1}_{\Sigma_{i_0}(n)}(E) \cap (\tilde{g}, \varepsilon)$-separated, with $\tilde{g} \in G_{n+m}$. By (9)

$$s(K, \tilde{g}, \varepsilon) \geq \frac{1}{m+1} s(K^{(m)}, \tilde{g}, \varepsilon) \geq \frac{1}{m+1} s(K, g, \varepsilon).$$

Denote by $\tilde{g}^{m+n}_{m+n}$ the subset of $\tilde{g}$ obtained in the previous construction. It shows that

$$S_{m+n}(K, S, \varepsilon) = \frac{1}{p^{m+n}} \sum_{\tilde{g} \in \tilde{g}^{m+n}_{m+n}} s(K, \tilde{g}, \varepsilon)$$

$$\geq \frac{1}{p^{m+n}} \frac{1}{m+1} \sum_{\tilde{g} \in \tilde{g}^{m+n}_{m+n}} s(K, \tilde{g}, \varepsilon)$$

$$\geq \frac{1}{p^{m+n}} \frac{1}{m+1} \sum_{g \in G^*_n} s(K^{(m)}, g, \varepsilon)$$

$$\geq \frac{1}{p^{m+n}} \frac{1}{m+1} \sum_{g \in G^*_n} s(K, g, \varepsilon).$$
and it implies
\[ S_{m+n}(K, S, \varepsilon) \geq \frac{1}{p^m} \frac{1}{m+1} S_n(K^{(m)}, S, \varepsilon) \geq \frac{1}{p^m} \frac{1}{m+1} S_n(K, S, \varepsilon). \]

So,
\[
S(K, S, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n+m} \log S_{m+n}(K, S, \varepsilon) \\
\geq \limsup_{n \to \infty} \frac{1}{n+m} \log \left( \frac{1}{p^m} \frac{1}{m+1} S_n(K^{(m)}, S, \varepsilon) \right) \\
= \limsup_{n \to \infty} \frac{1}{n+m} \log S_n(K^{(m)}, S, \varepsilon) \\
\geq \limsup_{n \to \infty} \frac{1}{n+m} \log \left( \frac{1}{p^m} \frac{1}{m+1} S_n(K, S, \varepsilon) \right) \\
= S(K, S, \varepsilon)
\]

which guarantees \( S(K, S, \varepsilon) = S(K^{(m)}, S, \varepsilon), \) since
\[
\limsup_{n \to \infty} \frac{1}{n+m} \log S_n(K^{(m)}, S, \varepsilon) = S(K^{(m)}, S, \varepsilon),
\]

and this proves the claim. Let us proceed to finalize the proof of the theorem. As, for any \( \varepsilon > 0, n \in \mathbb{N} \) and \( R \subset X \) it holds \( B_n(R, S, \varepsilon) \leq S_n(R, S, \varepsilon), \) we have for \( 0 < \delta < 1 \)
\[
\inf \{ S(K, S, \varepsilon) : K \in B_X \text{ with } \nu(K) > 0 \} \\
= \inf \{ S(K^{(m)}, S, \varepsilon) : K \in B_X \text{ with } \nu(K) > 0 \} \\
\geq \inf \left\{ \limsup_{n \to \infty} \frac{1}{n} \log B_n(K^{(m)}, S, \varepsilon) : K \in B_X \text{ with } \nu(K) > 0 \right\} \\
\geq \limsup_{n \to \infty} \frac{1}{n} \inf \left\{ \log B_n(K^{(m)}, S, \varepsilon) : K \in B_X \text{ with } \nu(K) > 0 \right\} \\
\geq \limsup_{n \to \infty} \frac{1}{n} \log B_{\nu}(S, n, \varepsilon, \delta).
\]

Letting \( \varepsilon \to 0 \) and \( \delta \to 0 \) and noticing that
\[
h^K_{\nu}(S) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{p^n} B_{\nu}(S, n, \varepsilon, \delta) \geq h_{\nu}(S),
\]
we conclude the proof.

### 5 Proof of Theorem C

We notice that (i) is an immediate consequence of Theorem B.

For (ii), let \( \varepsilon > 0. \) For \( r \in \mathbb{R}, \) if \( h_{\text{top}}(x_0, \varepsilon) < r, \) then \( B(K, S, \varepsilon) < r, \) for some closed neighbourhood \( K \) of \( x_0. \) In particular, \( h_{\text{top}}(x, \varepsilon) < r \) for each \( x \in K. \) So, \( h_{\text{top}}(\cdot, \varepsilon) \) is upper semi continuous. As we can obtain \( h \) as the limit of \( \{h_{\text{top}}(x, \frac{1}{n})\}_{n \in \mathbb{N}}, \) sequence of upper semi continuous maps, it is a Borel map.
(iv). Cover \( K \) by finitely many closed balls \( B_1, \ldots, B_k \) with diameter at most 1. We know that \( h_{top}(K, S) = \max_j h_{top}(B_j, S) \) and it implies the existence of \( j \) so that \( B(K, S, \varepsilon) = B(B_j \cap K, S, \varepsilon) \). Cover \( B_1 \) by finitely many closed balls \( B_2, \ldots, B_k \) with diameter at most \( 1/2 \). By the same reasoning, there exists \( j \) for which \( B(K, S, \varepsilon) = B(B_j \cap K, S, \varepsilon) \). By induction, for each \( n \geq 2 \), there is a closed ball \( B_j \cap K \subset B_{jn}^{n-1} \) with diameter at most \( 1/n \) such that \( B(K, S, \varepsilon) = B(B_j \cap K, S, \varepsilon) \). Set \( x_0 \) the unique point in the intersection of the closed balls \( B_{jn}^{n} \), which belongs to \( K \), since \( K \) is closed. If \( K' \) is any closed neighbourhood of \( x_0 \), for \( n \in \mathbb{N} \) sufficiently large, \( B_{jn}^{n} \cap K \subset K' \), which implies
\[
B(K', S, \varepsilon) \geq B(B_{jn}^{n} \cap K, S, \varepsilon) = B(K, S, \varepsilon)
\]
and so
\[
h_{top}(x_0) \geq h_d(x_0, \varepsilon) \geq B(B_{jn}^{n} \cap K, S, \varepsilon).
\]
Hence, letting \( \varepsilon \to 0^+ \),
\[
\sup_{x \in K} h_{top}(x) \geq h_{top}(K, S).
\]
For the last part of Theorem C we need of the following proposition, which is a consequence of the definition of the entropy function of \( S \) and \( \mathcal{F}_G \). In order to emphasize the dynamics under consideration, the free semigroup action or the skew product map, we will denote the entropy function associated to \( S \) by \( h_{top}(\cdot, S) \) and the entropy function associated to \( \mathcal{F}_G \) by \( h_{top}(\cdot, \mathcal{F}_G) \). We recall that for any compact topological space \( Y \), if \( d' \) is metric on \( Y \) which generates the topology on \( Y \), then \( h_{top}(\cdot) = h_{d'}(\cdot) \).

**Proposition 5.1** Let \( S : G \times X \rightarrow X \) be a finitely generated free semigroup action. For \( x \in X \) and \( \omega \in S_p^+ \),
\[
h_{top}(\omega, x, \mathcal{F}_G) = h_{top}(x, S) + \log p.
\]

**Proof** Fix \( \varepsilon > 0 \). Let \( x \in X \) and \( \delta > 0 \). There exists \( Z = Z(\varepsilon) \subset X \) closed neighbourhood of \( x \) so that \( h_d(x, \varepsilon) \leq B(Z, S, \varepsilon) \leq h_d(x, \varepsilon) + \delta \). Given \( \omega \in S_p^+ \), we notice that \( S_p^+ \times Z \) is a closed neighbourhood of \( (\omega, x) \). It follows that
\[
h_{D \times d}(\omega, x, \varepsilon) = B(S_p^+ \times Z, \mathcal{F}_G, \varepsilon)
\leq B(Z, S, \varepsilon) + \log p
\leq h_d(x, \varepsilon) + \log p + \delta.
\]

As \( \delta \) was taken arbitrary, \( h_{D \times d}(\omega, x, \varepsilon) \leq h_d(x, \varepsilon) + \log p \), which implies
\[
h_{top}(\omega, x, \mathcal{F}_G) \leq h_{top}(x, S) + \log p.
\]
By the other hand, given \( X \times K \) closed neighbourhood of \( (\omega, x) \), where \( X \subset S_p^+ \) is a cylinder and \( K \subset X \) is a closed neighbourhood of \( x \) (notice that any closed neighbourhood of \( (\omega, x) \) contains a neighbourhood of the form \( S \times K \)), by the proof of Proposition 3.1, we have that \( S(K, S, \varepsilon) + \log p \leq S(X \times K, \mathcal{F}_G, \varepsilon) \), for every \( \varepsilon > 0 \). So,
\[
h_{D \times d}(\omega, x, \varepsilon) = \inf\{S(V, \mathcal{F}_G, \varepsilon) : V \text{ is a closed of } (\omega, x)\}
\geq \inf\{S(K, S, \varepsilon) : K \text{ is a closed of } x\} + \log p
= h_d(x, \varepsilon) + \log p,
\]
which implies \( h_{top}(\omega, x, \mathcal{F}_G) \geq h_{top}(x, S) + \log p \) and finishes the proof. \( \square \)
The next lemma guarantees that the entropy function associated to $F_G$ is $F_G$-invariant.

**Lemma 5.2** Let $F_G : \Sigma_p^+ \times X \rightarrow \Sigma_p^+ \times X$ be the skew-product map given in (3). Then the entropy function associated to $F_G$ is $F_G$-invariant.

**Proof** Let $(\omega, x) \in \Sigma_p^+ \times X$ and fix $\varepsilon > 0$. Let $K \subset \Sigma_p^+ \times X$ be a closed neighbourhood of $F_G(\omega, x)$. We notice that, by the continuity of $F_G$, there exists a closed neighbourhood $M \subset F_G^{-1}(K)$ of $(\omega, x)$ of diameter less than $\varepsilon$. So, given $n \in \mathbb{N}$, if $E \subset M$ is a $(n, \varepsilon)$-separated set then $F_G(E) \subset K$ is a $(n, \varepsilon)$-separated set in $K$. It implies that $s_n(M, F_G, \varepsilon) \leq s_{n-1}(K, F_G, \varepsilon)$, which gives $S(M, F_G, \varepsilon) \leq S(K, F_G, \varepsilon)$. Then, by the definition of $h_{top}$ we have that

$$h_{top}((\omega, x), F_G) = \lim_{\varepsilon \to 0} \inf \{ S(U, F_G, \varepsilon) : U \text{ is a closed neighbourhood of } (\omega, x) \}$$

$$\leq \lim_{\varepsilon \to 0} \inf \{ S(V, F_G, \varepsilon) : V \text{ is a closed neighbourhood of } F_G(\omega, x) \}$$

$$= h_{top}(F_G(\omega, x), F_G).$$

(10)

For the converse inequality take $K \subset \Sigma_p^+ \times X$ closed neighbourhood of $F_G(\omega, x)$ and consider $F_G^{-1}(K)$, which is a closed neighbourhood of $(\omega, x)$ and satisfies $F_G(F_G^{-1}(K)) = K$. Moreover, if $F \subset F_G^{-1}(K)$ is a $(n, \varepsilon)$-spanning set then $F_G(F) \subset K$ is a $(n-1, \varepsilon)$-spanning. So, $B(F, F_G, \varepsilon) \leq B(F_G^{-1}(K), F_G, \varepsilon)$ and it implies that

$$h_{top}(F_G(\omega, x), F_G) = \lim_{\varepsilon \to 0} \inf \{ B(K, F_G, \varepsilon) : K \text{ is a closed neighbourhood of } F_G(\omega, x) \}$$

$$\leq \lim_{\varepsilon \to 0} \inf \{ B(K', F_G, \varepsilon) : K' \text{ is a closed neighbourhood of } (\omega, x) \}$$

$$= h_{top}((\omega, x), F_G).$$

(11)

By Equations (10) and (11) we conclude that $h_{top}(F_G(\omega, x), F_G) = h_{top}((\omega, x), F_G)$ and it ends the proof.

The next corollary proves item (iii) of Theorem C.

**Corollary 5.3** Let $S : G \times X \rightarrow X$ be a finitely generated free semigroup action and $h_{top}$ its entropy function. Then $h_{top}$ is $S$-invariant.

**Proof** Let $f \in G_1^+$ and consider $\omega = \omega_1 \omega_2 \cdots \in \Sigma_p^+$ so that $\sigma_1 \omega = f$. Then, by Proposition 5.1 and Lemma 5.2 we have that

$$h_{top}(x, S) + \log p = h_{top}((\omega, x), F_G) = h_{top}(F_G(\omega, x), F_G)$$

$$= h_{top}((\sigma(\omega), f(x)), F_G)$$

$$= h_{top}(f(x), S) + \log p,$$

which implies that $h_{top}(x, S) = h_{top}(f(x), S)$, for any $f \in G_1^+$, and we conclude the proof.

Let us proceed to prove item (v) of Theorem C. By part (ii) we have that $\sup_{(\omega, x) \in \Sigma_p^+ \times X} h_{top}((\omega, x), F_G) = h_{top}(\Sigma_p^+ \times X, F_G)$. Let $\alpha \in \mathcal{M}_{\text{erg}}(\Sigma_p^+ \times X)$ then, as, by Lemma 5.2, $h_{top}((\cdot, \cdot), F_G)$ is $F_G$-invariant, there exists $c \geq 0$ with $h_{top}(\Sigma_p^+ \times X, F_G) \geq c$ such that $h_{top}((\omega, x), F_G) = c$ for $\alpha$-a.e. $(\omega, x) \in \Sigma_p^+ \times X$. By (i) applied to $F_G$ we have that $c \geq h_\alpha(F_G)$. So,

$$\int_{\Sigma_p^+ \times X} h_{top}((\omega, x), F_G) d\alpha \geq h_\alpha(F_G).$$
For \( \nu = (\pi_X)_a \alpha \) we obtain, by Proposition 5.1,
\[
h_\nu(S) = \sup_{\mu \in \Pi(\sigma, \nu)_{\text{erg}}} h_\mu - \log p
\leq \sup_{\mu \in \Pi(\sigma, \nu)_{\text{erg}}} \int_{\Sigma^+_p \times X} h_{\text{top}}((\omega, x), F_G) d\mu - \log p
\leq \sup_{\mu \in \Pi(\sigma, \nu)_{\text{erg}}} \int_{\Sigma^+_p \times X} h_{\text{top}}(x, S) d\mu - \log p
= \int_{\Sigma^+_p \times X} h_{\text{top}}(x, S) d\nu,
\]
and it ends the proof.

6 Proof of Theorem D

The next proposition shows that the notions of uniform entropy points and uniform full entropy points are related to the notions of entropy points and full entropy points.

Proposition 6.1 Let \( S : G \times X \to X \) be a finitely generated free semigroup action.

i. \( E_{\text{up}}(X, S) \subset E_{\text{up}}(X, S) \cap E_{\text{fp}}(X, S) \), \( E_{\text{up}}(X, S) \subset E_p(X, S) \);

ii. If \( K \) is a closed subset set with \( h_{\text{top}}(K, S) > 0 \), then \( K \cap E_{\text{up}}(X, S) \neq \emptyset \). In particular \( E_{\text{up}}(X, S) \neq \emptyset \) when \( h_{\text{top}}(X, S) > 0 \);

iii. \( E_{\text{up}}(X, S) \) and \( E_{\text{fp}}(X, S) \) are both \( S \)-invariant;

iv. \( E_{\text{up}}(X, S) \subset X \) is a \( F_\sigma \) subset, i.e., it is a countable union of closed subsets of \( X \). The subset \( E_{\text{up}}(X, S) \) is a \( F_{\sigma\delta} \) subset, i.e., it is a countable intersection of \( F_\sigma \) subsets of \( X \).

Proof Part (i) follows directly from the definitions. Part (ii) is a consequence of the fourth item of Theorem C. Part (iii) follows from the third item of Theorem C. Finally to prove part (iv) we can assume, without loss of generality, that \( h_{\text{top}}(X, S) > 0 \). Then,
\[
E_{\text{up}}(X, S) = \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \left\{ x \in X : h_d \left( x, \frac{1}{n} \right) \geq \frac{1}{m} \right\}.
\]
As, for each \( \varepsilon > 0, h_d(\cdot, \varepsilon) \) is upper semicontinuous on \( X \) we have that \( \{ x \in X : h_d \left( x, \frac{1}{n} \right) \geq \frac{1}{m} \} \) is a closed subset of \( X \), for each \( m, n \in \mathbb{N} \) and then \( E_{\text{up}}(X, S) \) is a \( F_\sigma \) subset of \( X \). Now, for \( E_{\text{fp}}(X, S) \) we have that:

If \( h_{\text{top}}(X, S) < \infty \), \( E_{\text{fp}}(X, S) = \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{ x \in X : h_d \left( x, \frac{1}{n} \right) \geq h_{\text{top}}(X, S) - \frac{1}{m} \right\} \).

If \( h_{\text{top}}(X, S) = \infty \), \( E_{\text{fp}}(X, S) = \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{ x \in X : h_d \left( x, \frac{1}{n} \right) \geq m \right\} \).

In both cases, as , for each \( \varepsilon > 0, h_d(\cdot, \varepsilon) \) is u.s.c on \( X \), we conclude that \( E_{\text{fp}}(X, S) \) is a \( F_{\sigma\delta} \) subset of \( X \).

Combining Proposition 6.1 and the next result we obtain the proof of Theorem D.

Proposition 6.2 Let \( S : G \times X \to X \) be a finitely generated free semigroup action.
i. If \( v \in \mathcal{M}_1(X) \) is so that \( \Pi(\sigma, v)_{\text{erg}} \neq \emptyset \) and \( h_\nu(S) > 0 \), then \( \text{supp}(v) \subset E_{up}(X, S) \);  
ii. If \( v \in \mathcal{M}_1(X) \) is so that \( \Pi(\sigma, v)_{\text{erg}} \neq \emptyset \) and \( h_\nu(S) = h_{top}(X, S) \), then \( \text{supp}(v) \subset E_{up}(X, S) \);  
iii. \( h_{top}(E_{up}(X, S), S) = h_{top}(X, S) \).

**Proof** Part (i) and part (ii) follow from Theorem C. For part (iii) we mimic the proof of item (iii) of Theorem A.

### 7 Proof of Theorem E

Since \( S \) has the strong orbital specification property by [37, Theorem 11] we have \( h_{top}(X, S) > 0 \) and every point is full entropy point. Fix \( \zeta > 0 \). There exists \( \varepsilon_0 > 0 \) so that

\[
S(X, S, \varepsilon) \geq h_{top}(X, S) - \zeta, \text{ for any } \varepsilon \in (0, \varepsilon_0).
\]

By the definition of the entropy function, there exists \( \eta \in (0, \varepsilon_0) \) so that for every \( 0 < \varepsilon < \eta \) there exists a closed neighbourhood of \( x \), say \( K = K(\frac{x}{4}) \), for which we have

\[
\quad h_{top}(x) - \zeta < S(K, S, \frac{\varepsilon}{4}) < h_{top}(x) + \zeta.
\]

Fix \( \varepsilon \in (0, \eta) \) and let \( p(\varepsilon) \in \mathbb{N} \) be given by the strong orbital specification property. Since there are finitely many elements in \( G_{p(\varepsilon)} \), finitely many of its concatenations and the local inverse branches of elements \( g : X \to X \) are uniformly continuous there exists a uniform constant \( C_\varepsilon \) (that tends to zero as \( \varepsilon \to 0 \)) so that \( \text{diam}(h^{-1}_\varepsilon(B(y, \varepsilon))) \leq C_\varepsilon \) for \( h \in G_{p(\varepsilon)} \) and \( y \in X \).

Fix \( h = h_{i(p(\varepsilon))} \ldots h_1 \in G^{*}_{p(\varepsilon)} \). Take \( n \geq 1 \) and \( g = g_{i_n} \ldots g_{i_1} \in G_n \) arbitrary, let \( E = \{x_1, \ldots, x_{\ell}\} \subset X \) be a maximal \((g, \varepsilon)\)-separated set and consider the open set \( W \subset K \) defined by the set of points \( y \in K \) so that \( d(y, \partial K) > C_\eta \). Assume that \( 0 < \varepsilon \leq \eta \) satisfies \( \varepsilon + C_\varepsilon < C_\eta \). Given a maximal \( \varepsilon \)-separated set \( F = \{z_1, \ldots, z_m\} \subset W \), by the specification property, for any \( x_i \in E \) and \( z_j \in F \) there exists

\[
y_j^i \in B(z_j, \frac{\varepsilon}{4}) \cap h^{-1}_\varepsilon(B(x_i, g, \frac{\varepsilon}{4})),
\]

where, if \( g = g_{i_n} \ldots g_{i_1} \),

\[
B(x_i, g, \frac{\varepsilon}{4}) = \left\{ y \in X : d(g_{i_m} \ldots g_{i_1}(x_i), g_{i_m} \ldots g_{i_1}(y)) < \frac{\varepsilon}{4}, \text{ for every } 0 \leq m \leq n \right\}.
\]

Since \( \text{diam}(h^{-1}_\varepsilon(B(y, \frac{\varepsilon}{4}))) \leq C_\varepsilon \frac{1}{4} \), we obtain

\[
d(h^{-1}_\varepsilon(B(x_i, g, \frac{\varepsilon}{4})), \partial K) \geq C_\eta - \frac{\varepsilon}{4} - C_\varepsilon \frac{1}{4} > 0,
\]

provided that \( \varepsilon < \eta \). It implies that \( h^{-1}_\varepsilon(B(x_i, g, \frac{\varepsilon}{4})) \subset K \) for every \( i \). By construction, the dynamical balls \( \{B(x_i, g, \frac{\varepsilon}{4})\}_{i=1} \) are pairwise disjoint and the points \( y_j^i \in K \) are \((gh, \frac{\varepsilon}{4})\)-separated. This proves that

\[
s(K, gh, \frac{\varepsilon}{4}) \geq s(X, g, \varepsilon)s(K, id, \varepsilon) \geq s(X, g, \varepsilon).
\]
Since the elements \( g \) and \( h \) were chosen arbitrary then, summing over all possible concatenations and letting \( n \rightarrow \infty \), we deduce by (12) that

\[
S(K, S, \varepsilon) \geq S(X, S, \varepsilon) \geq h_{\text{top}}(X, S) - \zeta.
\]

It implies that

\[
h_{\text{top}}(x) + \zeta \geq S(K, S, \varepsilon) \geq h_{\text{top}}(X, S) - \zeta,
\]

which gives \( h_{\text{top}}(x) \geq h_{\text{top}}(X, S) - 2\zeta \). As \( \zeta \) can be chosen arbitrarily small we conclude that \( h_{\text{top}}(x) = h_{\text{top}}(X, S) \), ending the proof.

### 8 Proof of Theorem F

We will get the proof of the theorem as a consequence of the following proposition.

**Proposition 8.1** Let \( S : G \times X \to X \) be a finitely generated free semigroup action and \( h_{\text{top}} \) its entropy function.

i. If \( K \) is a countable closed subset with a unique limit point \( x_0 \), then \( h_{\text{top}}(x_0) \geq h_{\text{top}}(K, S) \).

ii. Let \( x_0 \in X \). Then there exists a countable closed subset \( K \subset X \) such that \( x_0 \in K \) is its unique limit point in \( X \) and \( h_{\text{top}}(x_0) = h_{\text{top}}(K, S) \).

**Proof** Let \( d \) be a metric on \( X \). We start with (i). Fix \( \varepsilon > 0 \). By hypotheses, for any closed neighbourhood \( Z \) of \( x_0 \), \( K \setminus Z \) is a finite set and it implies \( B(Z, S, \varepsilon) \geq B(K, S, \varepsilon) \). By the definition of \( h \) we obtain \( h_{\text{top}}(x_0) \geq h_d(x_0, \varepsilon) \geq B(K, S, \varepsilon) \). Letting \( \varepsilon \to 0 \) we get the conclusion.

For (ii) we assume \( h_{\text{top}}(x_0) < \infty \). For each \( m \in \mathbb{N} \) we set

\[
K_n = \left\{ x \in X : d(x, x_0) \leq \frac{1}{n} \right\}.
\]

For each \( m \in \mathbb{N} \) choose \( \varepsilon_m \) such that

\[
h_{\text{top}}(x_0) - \frac{1}{m} < \inf_{n \in \mathbb{N}} S(K_n, S, \varepsilon_m) = \inf_{n \in \mathbb{N}} \limsup_{k \to \infty} \frac{\log S_k(K_n, S, \varepsilon_m)}{k}.
\]

So, there exists an increasing sequence \( \{k_{n,m}\}_{n \in \mathbb{N}} \subset \mathbb{N} \) such that for all \( n \in \mathbb{N} \)

\[
\frac{1}{p^{k_{n,m}}} \sum_{g \in G_{k_{n,m}}^*} S(K_n, g, \varepsilon_m) = S_{k_{n,m}}(K_n, S, \varepsilon_m) \geq e^{k_{n,m}\left(h_{\text{top}}(x_0) - \frac{1}{m}\right)}.
\]

Now take \( g \in G_{k_{n,m}}^* \) and denote by \( E_{g,(m,n)} \) a \( (g, \varepsilon_m) \)-separated set in \( K_n \) of maximum cardinality. Define

\[
E_{m,n} = \bigcup_{g \in G_{k_{n,m}}^*} E_{g,(m,n)} \cup \{x_0\} \text{ and } K = \bigcup_{m \geq 1} \bigcup_{n \geq m} E_{n,m}.
\]

If \( V \) is a neighbourhood of \( x_0 \), by the definition of the sets \( K_n \), we have that \( K_{n_0} \subset V \) for \( n_0 \) large enough. So,

\[
K \setminus V \subset K \setminus K_{n_0} \subset \bigcup_{m=1}^{n_0-1} \bigcup_{n=m}^{n_0-1} E_{n,m}.
\]
As each $E_{n,m}$ is a finite set, $K \setminus V$ is finite and it guarantees that $x_0$ is the unique possible limit point of $K$ in $X$.

Let us prove that $h_{top}(x_0) = h_{top}(K, S)$. To do that fix $m \in \mathbb{N}$. For $n \geq m$, take $g \in G_{k_{n,m}}^*$, and notice that $E_{g_{n,m}}$ is a $(g_{n,m})$-separated set in $K$. It gives

$$
\frac{1}{p^{k_{n,m}}} \sum_{g \in G_{k_{n,m}}} s(K; g, \varepsilon_m) \geq \frac{1}{p^{k_{n,m}}} \sum_{g \in G_{k_{n,m}}} |E_{g_{n,m}}| \geq \epsilon_{k_{n,m}}(h_{top}(x_0) - \frac{1}{m}).
$$

In such case we obtain

$$
h_{top}(K, S) \geq S(K, S, \varepsilon_m) = \limsup_{k \to \infty} \frac{\log S_k(K, S, \varepsilon_m)}{k} \geq \limsup_{n \to \infty} \frac{\log S_{k_{n,m}}(K, S, \varepsilon_m)}{k_{n,m}} \geq \limsup_{n \to \infty} \frac{\log \epsilon_{k_{n,m}}(h_{top}(x_0) - \frac{1}{m})}{k_{n,m}} = h_{top}(x_0) - \frac{1}{m}.
$$

As the equality holds for all $m \in \mathbb{N}$ we obtain $h_{top}(K, S) \geq h_{top}(x_0)$. By (i) we get the desired equality in (ii).

\[ \square \]

Let us proceed to the proof of Theorem F. For $\varepsilon > 0$ we consider $B_{\varepsilon}(x)$ the open ball of radius $\varepsilon > 0$ and center $x$. As $X$ is compact, by Theorem C, there exists $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that

$$
\lim_{n \to \infty} x_n = x_0 \text{ and } \lim_{n \to \infty} h_{top}(x_n) = h_{top}(X, S).
$$

Let $\{r_n\}_{n \in \mathbb{N}}$ be any given sequence of positive real numbers which converges to $0$. Applying Proposition 8.1, given $n \in \mathbb{N}$ it is possible to take a countable closed subset $K_n$ such that $h_{top}(K_n, S) = h_{top}(x_n)$ and $x_n$ is its unique limit point in $X$. Moreover, $K_n \setminus B_{r_n}(x_n)$ is a finite subset and, under such observation, without loss of generality, we assume $K_n \subset B_{r_n}(x_n)$.

Define $K = \{x_0\} \bigcup_{n \in \mathbb{N}} K_n$. As each $K_n$ is a closed countable set, $K$ is a countable closed subset of $X$ and the set of limit points of $K$ in $X$ is just $\{x_0\} \cup \{x_n : n \in \mathbb{N}\}$, as $x_n \to x_0$ and $r_n \to 0$ when $n \to \infty$. Finally

$$
h_{top}(K, S) \geq h_{top}(K_n, S) = h_{top}(x_n) \Rightarrow h_{top}(K, S) \geq \lim_{n \to \infty} h_{top}(x_n) = h_{top}(X, S),
$$

which ends the proof by taking $K$ as the desired set, since $h_{top}(K, S) \leq h_{top}(X, S)$.

9 Examples

Our first example consider a free semigroup action where the generating set is given by expanding maps.

**Example 9.1** We say that a $C^1$-local diffeomorphism $f : M \to M$ on a compact Riemannian manifold is an expanding map if there are constants $C > 0$ and $0 < \lambda < 1$ such that $\|(Df^n(x))^{-1}\| \leq C \lambda^n$ for every $n \geq 1$ and $x \in X$. In [37, Theorems 13 and 16] the authors
proved that if \( G_1^* = \{g_1, g_2, \ldots, g_k\} \) is a finite set of expanding maps acting on \( M \) and \( G \) is the free semigroup generated by \( G_1 \) then \( S \) has the strong orbital specification property and then every point \( x \in M \) is a full entropy point for the free semigroup action \( S \). Moreover, by Theorem \( E \), \( h_{\text{top}}(x) = h_{\text{top}}(M, S) \) for all \( x \in M \).

The next example shows that it is possible to get a free semigroup action where the fixed generating set is not given by expanding maps, but the set of full entropy points is still the whole phase space.

**Example 9.2** For any \( \beta > 0 \), consider the interval map \( f_\beta : [0, 1] \to [0, 1] \) given by
\[
f_\beta(x) = \begin{cases} 
    x(1 + (2x)^\beta), & \text{if } x \in [0, \frac{1}{2}] \\
    2x - 1, & \text{if } x \in (\frac{1}{2}, 1]
\end{cases}
\]
also known as Maneville-Pomeau map. Although \( f_\beta \) is not continuous it induces a continuous and topologically mixing circle map \( \tilde{f}_\beta \) taking \( S^1 = [0, 1]/\sim \) with the identification \( 0 \sim 1 \).

Let \( G \) be the semigroup generated by \( G_1 = \{id, f_\beta, R_\alpha\} \) where \( R_\alpha \) is the rotation of angle \( \alpha \). Clearly no element of \( G_1 \) is an expanding map. Again, by [37] we have that every \( x \in S^1 \) is a full entropy point.

In the two previous examples we have that the set of entropy points is the whole phase space. In the next we present a free semigroup action where the set of entropy points is different of the ambient space.

**Example 9.3** Let \( X_1 \) and \( X_2 \) be compact metric spaces. For \( i \in \{1, 2\} \) take \( f_i : X_i \to X_i \) and \( g_i : X_i \to X_i, i \in \{1, 2\} \), continuous maps. Then define \( X = X_1 \cup X_2 \) and
\[
f(x) = \begin{cases} 
    f_1(x), & \text{if } x \in X_1, \\
    f_2(x), & \text{if } x \in X_2
\end{cases}
\] and
\[
g(x) = \begin{cases} 
    g_1(x), & \text{if } x \in X_1, \\
    g_2(x), & \text{if } x \in X_2
\end{cases}
\]
If \( f_2 = g_2 = id_{X_2} \) we have that the topological entropy of the semigroup action generated by \( G_1 = \{id_{X_1}, f_1, g_1\} \) coincides with the topological entropy of the semigroup generated by \( H_1 = \{id_{X_1}, f_1, g_1\} \). In particular, \( E_p(X, S) \subset X_1 \neq X \) in the case where the free semigroup action of the free semigroup generated by \( H_1 \) has positive topological entropy.

In Proposition \( 3.9 \) we see that if the topological entropy of a closed subset is positive then this set contains an entropy point. In what follows we are going to show that the converse, in general, is not true.

**Example 9.4** Consider
\[
C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Define
\[
A = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix},
\]
matrices in \( M_{5 \times 5}(\mathbb{Z}) \). We have \( AB = BA \) and, since for any \( x = (x_1, x_2, x_3, x_4, x_5) \in T^5 \) and \( m, n \in \mathbb{N} \cup \{0\} \)
\[
A^n B^m(x) = (C^n(x_1, x_2), C^m(x_3, x_4), 0),
\]
these matrices induce non transitive linear endomorphisms on the torus \( \mathbb{T}^5 = \mathbb{R}^5 / \mathbb{Z}^5 \).

The action given by the semigroup generated by \( G_1 = \{ id_{\mathbb{T}^5}, A, B \} \) does not admit a point with dense orbit. Another important consequence of the last inequality is that \( h_{top}(\pi((0, 0, 0, 0, x) : x \in \mathbb{R})) = 0 \), where \( \pi : \mathbb{R}^5 \rightarrow \mathbb{R}^5 / \mathbb{Z}^5 \) is the canonical projection.

By the other hand, given \( z \in \pi((0, 0, 0, 0, x) : x \in \mathbb{R}) \), we have that \( h_{top}(z) > 0 \), i.e., \( z \in E_{top}(\mathbb{T}^5, \mathcal{S}) \). In fact, given \( z \in \pi((0, 0, 0, 0, x) : x \in \mathbb{R}) \) and a closed neighbourhood \( K \subset \mathbb{T}^5 \), there exist \( y \in \mathbb{R} \) so that \( \pi(0, 0, 0, 0, y) = z, U \subset \mathbb{R} \) closed neighbourhood of 0 and \( V \subset \mathbb{R} \) closed neighbourhood of \( y \) so that \( \pi(U \times U \times U \times U \times V) \subset K \) is a closed neighbourhood of \( z \). We notice that by \([39]\), given \( \alpha \in (0, 1) \), for every \( \varepsilon > 0 \) small enough we have that

\[
B(\pi(U \times U \times U \times U \times V), \mathcal{S}, \varepsilon) \geq \log \frac{4\varepsilon}{2},
\]

which implies \( B(K, \mathcal{S}, \varepsilon) \geq B(\pi(U \times U \times U \times U \times V), \mathcal{S}, \varepsilon) \geq \log 2 \). Hence, as for any closed neighbourhood \( K \) of \( z \) we have that \( B(K, \mathcal{S}, \varepsilon) \geq \log 2 \) and so \( h_{top}(z) \geq \log 2 > 0 \).

In the next example we present a free semigroup action which is topologically transitive, has positive topological entropy and the set of uniform full entropy points does not coincide with the hole space.

**Example 9.5** Let \( X = [0, 2] \) and consider \( f : X \rightarrow X \) defined as

\[
f_1(x) = \begin{cases} 
1 - |1 - 3x|, & \text{if } x \in [0, 2/3], \\
3x - 2, & \text{if } x \in [2/3, 1], \\
x, & \text{if } x \in [1, 2].
\end{cases}
\]

In this case \( f_1([0, 1]) = [0, 1], f_1|_{[0,1]} \) is topologically transitive and \( h_{top}([0, 2], f_1) = \log 3 \).

Let \( f_2 : X \rightarrow X \) be the continuous map defined as

\[
f_2(x) = \begin{cases} 
x, & \text{if } x \in [0, 1], \\
2 - |3 - 2x|, & \text{if } x \in [1, 2].
\end{cases}
\]

We notice that \( f_2|_{[1,2]} \) is topologically transitive and \( h_{top}([0, 2], f_2) = \log 2 \). Let \( G \) be the free semigroup generated by \( G_1 = \{ id, f_1, f_2 \} \) and let \( \mathcal{S} : G \times X \rightarrow X \) be the induced free semigroup action by \( G \) on \( X \). Since for any two opens sets \( U, V \subset X \) there exists \( n \in \mathbb{N} \) for which \( f_1^n(U) \cap V \neq \emptyset \) or \( f_2^n(U) \cap V \neq \emptyset \), we have that \( \mathcal{S} \) is topologically transitive. As

\[
f_1|_{[0,1]} \circ f_2|_{[0,1]} = f_2|_{[0,1]} \circ f_1|_{[0,1]} = f_1|_{[0,1]},
\]

and \( f_1|_{[0,1]} \) satisfies the specification property (which coincides with the strong orbital specification property when we consider the free semigroup generated by \( f_1|_{[0,1]} \)), by Theorem E, we have that \( h_{top}(x) = \log 3 \) for any \( x \in [0, 1] \). By the other hand, as \( f_1|_{[1,2]} \circ f_1|_{[1,2]} = f_2|_{[1,2]} \circ f_1|_{[1,2]} = f_2|_{[1,2]} \), by Theorem E (again using the fact that \( f_2|_{[1,2]} \) has the specification property), we have that \( h_{top}(x) = \log 2 \) for any \( x \in [1, 2] \). Hence, \( E_{up}([0, 2], \mathcal{S}) = [0, 1] \neq [0, 2] \).

In the next example was given in \([1]\) and presents have a continuous map with positive topological entropy with a unique full entropy point.

**Example 9.6** Let \( \alpha \in [0, \infty) \), there exists a compact metric space \( X \) and a continuous map \( f : X \rightarrow X \) so that \( h_{top}(X, f) = \alpha \) and \( E_{\alpha}^f(X, f) \). In fact, given \( \alpha \in [0, \infty) \) let \( (\alpha_n)_{n \in \mathbb{N}} \) be a sequence so that \( 0 < \alpha_n < \alpha \) for every \( n \in \mathbb{N} \) and \( \lim_{n \rightarrow \infty} \alpha_n = \alpha \). Let \( f_n : X_n \rightarrow X_n \)
be a continuous map with \( X_n \subset \mathbb{R}^2 \) a compact subset and \( h_{top}(X_n, f_n) = \alpha_n \) (see [40, pp. 178–179]). Let \( B_n \) be a sequence of disjoint closed balls in \( \mathbb{R}^2 \) with \( \text{diam}(B_n) \to 0 \), \((0,0) \not\in B_n \) and \( B_n \to (0,0) \). Embed \( X_n \) into \( B_n \) for each \( n \) and put \( X = \cup_{n \in \mathbb{N}} X_n \cup \{(0,0)\} \).

Define \( f : X \to X \) such that \( f|_{X_n} = f_n \) and \((0,0)\) is a fixed point. Then \( f : X \to X \) is a continuous map and \( h_{top}(X, f) = \alpha \).

It is clear that there is no ergodic measure with maximal entropy for \( f : X \to X \). Moreover, by definition \( E^f_p(X, f) \subset X \cup \cup_{n \in \mathbb{N}} X_n \). Then \((0,0)\) is the unique full entropy point.

**Acknowledgements** We thank Paulo Varandas and Lucas Backes for the valuable discussions that helped to improve the quality of this work.

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