Deformations of $G_2$-structures with torsion

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Abstract

We consider non-infinitesimal deformations of $G_2$-structures on 7-dimensional manifolds and derive an exact expression for the torsion of the deformed $G_2$-structure. We then specialize to a case when the deformation is defined by a vector $v$ and we explicitly derive the expressions for the different torsion components of the new $G_2$-structure in terms of the old torsion components and derivatives of $v$. In particular this gives a set of differential equations for the vector $v$ which have to be satisfied for a transition between $G_2$-structures with particular torsions. For some specific torsion classes we find that these equations have no solutions.

1 Introduction

Seven-dimensional manifolds with $G_2$-structure have been studied for more than 40 years. Already in 1969, Alfred Gray studied vector cross products on manifolds [7], which on 7-manifolds do actually correspond to $G_2$-structures. Later on, Fernández and Gray classified the possible torsion classes of $G_2$-structures [5]. The concept of $G_2$-structures provides a classification for a large class of 7-manifolds. In fact, it is well-known that a 7-manifolds admits a $G_2$-structure if and only if the first two Stiefel-Whitney classes $w_1$ and $w_2$ vanish [5, 6]. Alternatively, a 7-manifold admits a $G_2$-structure if and only if it is orientable and admits a spin structure. A very important special case of a $G_2$-structure is when the torsion vanishes. This implies that the holonomy group lies in $G_2$. In Section 2 we give a more precise definition and an overview of the properties of $G_2$-structures.

Suppose we are given a 7-manifold that admits a $G_2$-structure, we can ask the question - $G_2$-structures of which torsion classes exist on it? This is of course a very difficult question, and it is still not clear how to approach this. However, we could start with some given $G_2$-structure, deform it and then require that the new $G_2$-structure lies in some particular torsion class. This is precisely what we attempt in this paper. In Section 5 we first derive an expression for the $G_2$-structure torsion for a general (non-infinitesimal) deformation, and then in Section 6 we specialize to a particular type of deformation - deformations that are defined by a vector (that is, the $G_2$ invariant 3-form is deformed by a 3-form lying in the 7-dimensional component of $\Lambda^3$). In this case, we obtain an explicit expression for the torsion of the deformed $G_2$-structure in terms of the old $G_2$-structure, its torsion and the vector which defines the deformation.
We then proceed to show that a deformation of this type takes a torsion-free $G_2$-structure to another torsion-free $G_2$-structure if and only if the vector that defines the deformation is parallel. Moreover we also show that on closed, compact manifolds there are no such deformations from strict torsion classes $W_1, W_7, W_1 \oplus W_7$ to the vanishing torsion class $W_0$, and vice versa.

Such deformations of $G_2$-structures have been first considered by Karigiannis in [12], where he wrote down the deformed metric and Hodge star operation, and indeed asked the question whether it is possible to deform a $G_2$-structure to a strictly smaller torsion class. This paper aims to give a partial answer to this question. Here we are mainly concerned with non-infinitesimal deformations, but infinitesimal deformations and flows of $G_2$-structures, and in particular properties of the moduli space of manifolds with $G_2$ holonomy have also been studied by Karigiannis [12, 13], Karigiannis and Leung [14] and by Grigorian and Yau [8, 9].

2 $G_2$-structures

The 14-dimensional group $G_2$ is the smallest of the five exceptional Lie groups and is closely related to the octonions. In particular, $G_2$ can be defined as the automorphism group of the octonion algebra. The restriction of octonion multiplication to just the imaginary octonions defines a vector cross product on $V = \mathbb{R}^7$ and vice versa. Moreover, the Euclidean inner product on $V$ can also be defined in terms of octonion multiplication. The group that preserves the vector cross product is precisely $G_2$ and since it preserves the inner product as well, we can see that it is a subgroup of $SO(7)$. For more on the relationship between octonions and $G_2$, see [1, 8]. The structure constants of the vector cross product define a particular 3-form on $\mathbb{R}^7$, hence $G_2$ can alternatively be defined in the following way.

**Definition 1** Let $(e^1, e^2, ..., e^7)$ be a basis for $V^*$, and denote $e^i \wedge e^j \wedge e^k$ by $e^{ijk}$. Then define $\varphi_0$ to be the 3-form on $\mathbb{R}^7$ given by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \quad (2.1)$$

Then $G_2$ is defined as the subgroup of $GL(7, \mathbb{R})$ which preserves $\varphi_0$.

Suppose for some 3-form $\varphi$ on $V$ we define a bilinear form by

$$B_\varphi(u, v) = \frac{1}{6} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \quad (2.2)$$

Here the symbol $\lrcorner$ denotes contraction of a vector with the differential form:

$$(u \lrcorner \varphi)_{mn} = u^a \varphi_{a mn}.$$  

Note that we will also use this symbol for contractions of differential forms using the metric. So for a $p$-form $\alpha$ and a $(p + q)$-form $\beta$, for $q \geq 0$,

$$(\alpha \lrcorner \beta)_{b_1...b_q} = \alpha_{a_1...a_p} \beta_{a_1...a_p b_1...b_q} \quad (2.3)$$

where the indices on $\alpha$ are raised using the metric.

Following Hitchin ([10]), $B_\varphi$ is a symmetric bilinear form on $V$ with values in the one-dimensional space $\Lambda^7 V^*$. Hence it defines a linear map $K_\varphi : V \to V^* \times \Lambda^7 V^*$. Then taking
the determinant we get $\det K \varphi \in (\Lambda^7 V^*)^9$, so if this does not vanish, we choose a positive root
- $(\det K \varphi)^{\frac{1}{9}} \in \Lambda^7 V^*$. Then we obtain an inner product on $V$

$$g_\varphi (u, v) = B_\varphi (u, v) (\det K \varphi)^{-\frac{1}{9}}$$ (2.4)

and the volume form of this inner product is then $(\det K \varphi)^{\frac{1}{9}}$. In components we can rewrite this as

$$(g_\varphi)_{ab} = (\det s)^{-\frac{1}{9}} s_{ab}. \tag{2.5}$$

with

$$s_{ab} = \frac{1}{144} \varepsilon_{amn} \varepsilon_{bpq} \varepsilon_{rst} \varepsilon^{mnpqrst} \tag{2.6}$$

where $\varepsilon^{mnpqrst}$ is the alternating symbol with $\varepsilon^{12...7} = +1$.

Applying (2.2) to $\varphi_0$ as defined by (2.1), we recover the standard Euclidean metric on $V$:

$$g_0 = (e^1)^2 + ... + (e^7)^2. \tag{2.7}$$

As we know, the stabilizer of $\varphi_0$ in $GL (7, \mathbb{R})$ is $G_2$, which is 14-dimensional. Since $GL (7, \mathbb{R})$ is 49-dimensional, we find that the orbit of $\varphi_0$ in $\Lambda^3 V^*$ has dimension $49 - 14 = 35 = \dim \Lambda^3 V^*$. Hence the orbit of $\varphi_0$ is an open subset $\Lambda_+^3 \subset \Lambda^3 V^*$.

**Definition 2** Let $V$ be a 7-dimensional real vector space. Then a 3-form $\varphi$ is said to be positive if lies in the $GL (7, \mathbb{R})$ orbit of $\varphi_0$.

In fact, in $\Lambda^3 V^*$ there are two open orbits of $GL (7, \mathbb{R})$ [2]. The second open orbit consists of 3-forms for which the metric defined by (2.4) has indefinite signature (4, 3), and the corresponding stabilizer is the so-called split $G_2$. The 3-form that it stabilizes can be obtained by changing the minus signs to plus signs in the expression (2.1) for $\varphi_0$. The existence of these open orbits gives a notion of a non-degenerate 3-form on $V$ - that is, a 3-form which lies in one of the open orbits [10]. Moreover, it turns out that non-degeneracy of a 3-form is equivalent to non-degeneracy of the corresponding metric. Thus if the determinant of the metric (2.5), or equivalently $\det (s_{ab})$ for $s_{ab}$ in (2.6), is non-zero, then the 3-form is in one of the open orbits. If moreover, the metric is positive-definite, then the 3-form is positive.

Now, given a $n$-dimensional manifold $M$, a $G$-structure on $M$ for some Lie subgroup $G$ of $GL (n, \mathbb{R})$ is a reduction of the frame bundle $F$ over $M$ to a principal subbundle $P$ with fibre $G$. The concept of a $G$-structure gives a convenient way of encoding different geometric structures. For example, an $O (n)$-structure is a reduction of the frame bundle to a subbundle with fibre $O (n)$. This defines an orthonormal frame at each point at $M$ and thus we can define a Riemannian metric on $M$. Hence there is a 1-1 correspondence between $O (n)$-structures and Riemannian metrics. Similarly, an almost complex structure on a $2m$-dimensional manifold $M$ is equivalent to a $GL (m, \mathbb{C})$-structure.

A $G_2$-structure is then a reduction of the frame bundle on a 7-dimensional manifold $M$ to a $G_2$ principal subbundle. It turns out that there is a 1-1 correspondence between $G_2$-structures and positive 3-forms on the manifold. Define the bundle of positive 3-forms on $M$ as the subset of 3-forms $\varphi$ in $\Lambda^3 T^* M$ such that for every point $p$ in $M$, $\varphi|_p \in \Lambda^3 T_p^* M$ is a positive 3-form in the sense of Definition [2]. Using the $G_2$ principal bundle we can then define a positive 3-form $\varphi$ on the whole manifold. Conversely, suppose we are given a positive 3-form. Then at each
point $p$ the subset of $GL(7, \mathbb{R})$ that identifies $\varphi|_p$ with $\varphi_0$ is isomorphic to $G_2$. Overall the whole manifold this will be a subset of the frame bundle $F$, and it is easy to show that this does give a principal subbundle with fibre $G_2$, and hence a $G_2$-structure.

Once we have a $G_2$-structure, since $G_2$ is a subgroup of $SO(7)$, we can also define a Riemannian metric. More concretely, if $\varphi$ is the 3-form which defines the $G_2$-structure, then using (2.4) we can define a corresponding metric $g$. Following Joyce ([11]), we will adopt the following abuse of notation

**Definition 3** Let $M$ be an oriented 7-manifold. The pair $(\varphi, g)$ for a positive 3-form $\varphi$ and corresponding metric $g$ will be referred to as a $G_2$-structure.

Since a $G_2$-structure defines a metric, it also defines a Hodge star. Thus we can construct another $G_2$-invariant object - the 4-form $^*\varphi$. Since the Hodge star is defined by the metric, which in turn is defined by $\varphi$, the 4-form $^*\varphi$ depends non-linearly on $\varphi$. For convenience we will usually denote $^*\varphi$ by $\psi$. On $\mathbb{R}^7$, when $\varphi$ is given by its canonical form $\varphi_0$ (2.1), $\psi$ takes the following canonical form

$$\psi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \quad (2.8)$$

By considering the canonical forms $\varphi_0$ and $\psi_0$, we can write down various contraction identities for a $G_2$-structure $(\varphi, g)$ and its corresponding 4-form $\psi$ [3, 9, 13].

**Proposition 4** The 3-form $\varphi$ and the corresponding 4-form $\psi$ satisfy the following identities:

$$\varphi_{abc} \varphi_{mn} = g_{am}g_{bn} - g_{an}g_{bm} + \psi_{abmn} \quad (2.9a)$$

$$\varphi_{abc} \psi_{mnp} = 3 \left( g_{[a}m \varphi_{np]b} - g_{b[m} \varphi_{np]}a \right) \quad (2.9b)$$

$$\psi_{abcd} \psi_{mnpq} = 24 \delta_{a}^{[m} \delta_{b}^{n} \delta_{c}^{p} \delta_{d}^{q} + 72 \psi_{[ab} \delta_{c}^{[m} \delta_{d}^{n]} \delta_{p]} - 16 \varphi_{[abc} \varphi_{d]mn} \delta_{a}^{[m} \delta_{b}^{n]} \delta_{c}^{p} \delta_{d}^{q} \quad (2.9c)$$

where $[m n p]$ denotes antisymmetrization of indices and $\delta_{a}^{a} = 1$ if $a = b$ and 0 otherwise.

The above identities can be of course further contracted - the details can be found in [9, 13]. These identities and their contractions are crucial whenever any calculations involving $\varphi$ and $\psi$ have to be done.

For a general $G$-structure, the spaces of $p$-forms decompose according to irreducible representations of $G$. Given a $G_2$-structure, we have the following decomposition of $p$-forms:

$$\Lambda^1 = \Lambda^1_7 \quad (2.10a)$$

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \quad (2.10b)$$

$$\Lambda^3 = \Lambda^3_7 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} \quad (2.10c)$$

$$\Lambda^4 = \Lambda^4_7 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \quad (2.10d)$$

$$\Lambda^5 = \Lambda^5_7 \oplus \Lambda^5_{14} \quad (2.10e)$$

$$\Lambda^6 = \Lambda^6_7 \quad (2.10f)$$

The subscripts denote the dimension of representation and components which correspond to the same representation are isomorphic to each other. We have the following characterization of the various components [2, 3]:
**Proposition 5** Let $M$ be a 7-manifold with a $G_2$-structure $(\varphi, g)$. Then the components of spaces of 2-, 3-, 4-, and 5-forms are given by

- $\Lambda_7^2 = \{\alpha \lrcorner \varphi : \alpha \in \Lambda_7^1\}$
- $\Lambda_{14}^2 = \{\omega \in \Lambda^2 : (\omega_{ab}) \in \mathfrak{g}_2\} = \{\omega \in \Lambda^2 : \omega \lrcorner \varphi = 0\}$
- $\Lambda_7^3 = \{f \varphi : f \in C^\infty(M)\}$
- $\Lambda_7^3 = \{\alpha \lrcorner \psi : \alpha \in \Lambda_7^1\}$
- $\Lambda_{27}^3 = \{\chi \in \Lambda^3 : \chi_{abc} = h_{[a}^d \varphi_{bc]}d \text{ for } h_{ab} \text{ traceless, symmetric}\}$
- $\Lambda_1^4 = \{f \psi : f \in C^\infty(M)\}$
- $\Lambda_7^4 = \{\alpha \wedge \varphi : \alpha \in \Lambda_7^1\}$
- $\Lambda_{14}^4 = \{\omega \wedge \psi : \omega \in \Lambda_{14}^2\}$
- $\Lambda_{27}^4 = \{\chi \in \Lambda^4 : \chi_{abcd} = h_{[a}^e \psi_{bcd]e} \text{ for } h_{ab} \text{ traceless, symmetric}\}$
- $\Lambda_7^5 = \{\alpha \wedge \psi : \alpha \in \Lambda_7^1\}$
- $\Lambda_{14}^5 = \{\omega \wedge \varphi : \omega \in \Lambda_{14}^2\}$
- $\Lambda_{27}^5 = \{\omega \lrcorner \varphi : \alpha \in \Lambda_{27}^1\}$

In particular, we see that the 7-dimensional component of $\Lambda^3$ is defined by 1-forms (or equivalently, vectors), and the 27-dimensional component is given by traceless symmetric tensors. For convenience, and following [3], we will adopt the following notation for the map from symmetric tensors into $\Lambda^3$:

$$i_\varphi : \text{Sym}^2(V^*) \longrightarrow \Lambda^3 \text{ given by } i_\varphi(h)_{abc} = h_{[a}^d \varphi_{bc]d}$$  \hspace{1cm} (2.11)

and similarly, for the map from symmetric tensors into $\Lambda^4$:

$$i_\psi : \text{Sym}^2(V^*) \longrightarrow \Lambda^4 \text{ given by } i_\psi(h)_{abcd} = h_{[a}^e \psi_{bcd]e}.$$  \hspace{1cm} (2.12)

It is sometimes useful to be able to find projections of given $p$-form onto the different components. Here we collect some of these results [13, 9]:

**Proposition 6** Suppose $\omega$ is a 2-form. Then the projections $\pi_7(\omega)$ and $\pi_{14}(\omega)$ onto $\Lambda_7^2$ and $\Lambda_{14}^2$, respectively, are given by

$$\pi_7(\omega) = \alpha \lrcorner \varphi \text{ where } \alpha = \frac{1}{6} \omega \lrcorner \varphi$$  \hspace{1cm} (2.13a)

$$\pi_{14}(\omega) = \frac{2}{3} \omega - \frac{1}{6} \omega \lrcorner \psi$$  \hspace{1cm} (2.13b)

**Proposition 7** Suppose $\chi$ is a 3-form. Then the projections $\pi_1(\chi)$, $\pi_7(\chi)$ and $\pi_{27}(\chi)$ onto $\Lambda_1^3$, $\Lambda_7^3$ and $\Lambda_{27}^3$, respectively, are given by

$$\pi_1(\chi) = a \varphi \text{ where } a = \frac{1}{42} \chi \lrcorner \varphi$$  \hspace{1cm} (2.14a)

$$\pi_7(\chi) = \omega \lrcorner \psi \text{ where } \omega = -\frac{1}{24} \chi \lrcorner \psi$$  \hspace{1cm} (2.14b)

$$\pi_{27}(\chi) = i_\varphi(h) \text{ where } h_{ab} = \frac{3}{4} \chi_{mn(a} \varphi_{b)}^{mn} - \frac{3}{28} (\chi \lrcorner \varphi) g_{ab}$$  \hspace{1cm} (2.14c)
Similarly, if $\chi$ is a 4-form the corresponding 4-form, the projections are

$$\pi_1(\chi) = a\psi$$ where $a = \frac{1}{168}\chi \wedge \psi \quad (2.15a)$$
$$\pi_7(\chi) = \omega \wedge \varphi$$ where $\omega = \frac{1}{24}\varphi \wedge \chi \quad (2.15b)$$
$$\pi_{27}(\chi) = i_\psi(h)$$ where $h_{ab} = \frac{1}{3}\chi_{mnp}(a\psi_b)_{mnp} + \frac{1}{21}(\chi \wedge \psi)_{gab} \quad (2.15c)$$

Proposition 8 Suppose $\eta$ is a 5-form. Then the projections $\pi_7(\eta)$ and $\pi_{14}(\eta)$ onto $\Lambda^5_7$ and $\Lambda^5_{14}$, respectively, are given by

$$\pi_7(\eta) = \alpha \wedge \psi$$ where $\alpha = \frac{1}{72}\psi \wedge \eta \quad (2.16a)$$
$$\pi_{14}(\eta) = \omega \wedge \varphi$$ where $\omega = \frac{1}{9}\varphi \wedge \eta - \frac{1}{36}(\varphi \wedge \eta) \wedge \psi \quad (2.16b)$$

Proof. Consider $\eta = \alpha \wedge \psi + \omega \wedge \varphi \quad (2.17)$

where $\alpha$ is a 1-form and $\omega \in \Lambda^2_{14}$. Then,

$$\psi \eta)_m = 5\psi^{abcd}\alpha_{a} \psi_{bcd} + 10\psi^{abcd} \omega_{[ab} \varphi_{cd]} \quad (2.18)$$

Using the contractions between $\varphi$ and $\psi$ from Proposition 4, we find that

$$\psi \eta)_m = 72\alpha_m + 24\varphi_{m} \omega_{ab}$$
$$= 72\alpha_m$$

since $\omega \wedge \varphi = 0$ for $\omega \in \Lambda^2_{14}$. Hence we get the $\pi_7$ projection.

Now for $\eta$ as in (2.17), from definition of the Hodge star, we have,

$$\varphi \eta \eta)_{ab} = \frac{1}{2}\varepsilon_{abcd} \eta_{mn} (\ast \eta)_{mn} \varphi_{abc}$$
$$= 3(\ast \eta \varphi)_{ab}$$
$$= 12(\pi_7 \ast \eta)_{ab} - 6(\pi_{14} \ast \eta)_{ab} \quad (2.19)$$

In particular, from Proposition 6,

$$\pi_{14}(\varphi \eta) = \frac{2}{3}(\varphi \eta) - \frac{1}{6}(\varphi \eta) \wedge \psi$$

and so,

$$\pi_{14}(\ast \eta) = -\frac{1}{9}(\varphi \eta) + \frac{1}{36}(\varphi \eta) \wedge \psi \quad (2.20)$$

However,

$$\ast (\pi_{14} \ast \eta \wedge \varphi)_{mn} = \frac{1}{12}\varepsilon_{mn} \pi_{14} \ast \eta_{ab} \varphi_{cde}$$
$$= \frac{1}{2}(\pi_{14} \ast \eta \wedge \psi)_{mn}$$
$$= -\pi_{14} \ast \eta_{mn}$$
Hence,
\[ \pi_{14}(\eta) = -\pi_{14}(\ast\eta) \wedge \varphi. \]  

3 \ G_2\text{-structure torsion}

As before, suppose \( M \) is a 7-dimensional manifold with a \( G_2 \)-structure \((\varphi, g)\). The metric \( g \) defines a reduction of the frame bundle to a principal \( SO(7) \)-subbundle \( Q \), that is, a subbundle of oriented orthonormal frames. The metric also defines a Levi-Civita connection \( \nabla \) on the tangent bundle \( TM \), and hence on \( F \). However, the \( G_2 \)-invariant 3-form \( \varphi \) reduces the orthonormal bundle further to a principal \( G_2 \)-subbundle \( Q \). We can then pull back the Levi-Civita connection to \( Q \). On \( Q \) we can uniquely decompose \( \nabla \) as
\[ \nabla = \bar{\nabla} + \mathcal{T} \]  
where \( \bar{\nabla} \) is a \( G_2 \)-compatible canonical connection \( \bar{\nabla} \) on \( P \), taking values in \( g_2 \subset so(7) \), while \( \mathcal{T} \) is a 1-form taking values in \( g_2^\perp \subset so(7) \). This 1-form \( \mathcal{T} \) is known as the intrinsic torsion of the \( G_2 \)-structure. The intrinsic torsion is precisely the obstruction to the Levi-Civita connection being \( G_2 \)-compatible. Note that \( so(7) \) splits according to \( G_2 \) representations as
\[ so(7) \cong \Lambda^2V \cong \Lambda^2_7 \oplus \Lambda^2_{14} \]
but \( \Lambda^2_{14} \cong g_2 \), so the complement \( g_2^\perp \cong \Lambda^2_7 \cong V \). Hence \( \mathcal{T} \) can be represented by a tensor \( T_{ab} \) which lies in \( W \cong V \otimes V \). Now, since \( \varphi \) is \( G_2 \)-invariant, it is \( \bar{\nabla} \)-parallel, so the torsion is determined by \( \nabla \varphi \).

Following [13], consider the 3-form \( \nabla_X \varphi \) for some vector field \( X \). It is easy to see
\[ \nabla_X \varphi \in \Lambda^3_7 \]
and thus overall,
\[ \nabla \varphi \in \Lambda^1_7 \otimes \Lambda^3_7 \cong W. \]  
Thus \( \nabla \varphi \) lies in the same space as \( T_{ab} \) and thus completely determines it. Given (3.3), we can write
\[ \nabla a \varphi_{bcd} = T_a{}^e \psi_{ebcd} \]  
where \( T_{ab} \) is the full torsion tensor. From this we can also write
\[ T_a{}^m = \frac{1}{24} (\nabla a \varphi_{bcd}) \psi^{mbcd}. \]  
This 2-tensor fully defines \( \nabla \varphi \) since pointwise, it has 49 components and the space \( W \) is also 49-dimensional (pointwise). In general we can split \( T_{ab} \) into torsion components as
\[ T = \tau_1 g + \tau_7 \varphi + \tau_{14} + \tau_{27} \]  
where \( \tau_1 \) is a function, and gives the 1 component of \( T \). We also have \( \tau_7 \), which is a 1-form and hence gives the 7 component, and, \( \tau_{14} \in \Lambda^2_{14} \) gives the 14 component and \( \tau_{27} \) is traceless.
symmetric, giving the 27 component. Note that the normalization of these components is different from [13]. Hence we can split $W$ as

$$W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}. \quad (3.7)$$

Originally the torsion of $G_2$-structures was studied by Fernández and Gray [5], and their analysis revealed that there are in fact a total of 16 torsion classes of $G_2$-structures. Later on, Karigiannis reproduced their results using simple computational arguments [13]. The 16 torsion classes arise as the subsets of $W$ which $\nabla \phi$ belongs to.

Note that our notation differs from Fernández and Gray. Our $\tau_1$ corresponds to their $\tau_0$, $\tau_7$ corresponds to $\tau_4$, $\tau_{14}$ corresponds to $\tau_2$ and $\tau_{27}$ corresponds to $\tau_3$.

Moreover, as shown in [13], the torsion components $\tau_i$ relate directly to the expression for $d\phi$ and $d\psi$. In fact, in our notation,

$$d\phi = 4\tau_1 \psi - 3\tau_7 \wedge \varphi - *\tau_{27} \quad (3.8a)$$

$$d\psi = -4\tau_7 \wedge \psi - 2* \tau_{14}. \quad (3.8b)$$

Similarly to (3.4), we can express the covariant derivative of $\psi$ in terms of $T$.

**Lemma 9** Given a $G_2$-structure defined by 3-form $\varphi$, with torsion $T_a^m$ given by (3.5), the covariant derivative of the corresponding 4-form $\psi$ is given by

$$\nabla_a \psi_{bcde} = -4T_{a[b} \varphi_{cde]} \quad (3.9)$$

**Proof.** Consider the identity (2.9a):

$$\varphi_{abc} \varphi^c_{\ mn} = g_{am} g_{bn} - g_{an} g_{bm} + \psi_{abmn}$$

Applying the covariant derivative to both sides, we get

$$\nabla_e \psi_{abmn} = (\nabla_e \varphi_{abc}) \varphi^c_{\ mn} + \varphi_{abc} (\nabla_e \varphi^c_{\ mn})$$

Now using (3.4) and using contraction identities between $\varphi$ and $\psi$, we get (3.9). ■

Suppose $d\varphi = d\psi = 0$. Then this means that all four torsion components vanish and hence $T = 0$, and as a consequence $\nabla \varphi = 0$. The converse is trivially true. This result is originally due to Fernández and Gray [5]. Moreover, a $G_2$-structure is torsion-free if and only if the holonomy of the corresponding metric is contained in $G_2$ [11].

The torsion tensor $T_{ab}$ and hence the individual components $\tau_1, \tau_7, \tau_{14}$ and $\tau_{27}$ must also satisfy certain differential conditions. For the exterior derivative $d$, $d^2 = 0$, so from (3.8), must have

$$d (4\tau_1 \psi - 3\tau_7 \wedge \varphi - *\tau_{27}) = 0$$

$$d (4\tau_7 \wedge \psi + 2* \tau_{14}) = 0$$

Alternatively, note that we have

$$(d^2 \varphi)_{abcde} = 20 \nabla_{[a} \nabla_{b} \varphi_{cde]}$$

$$= 20 \nabla_{[a} T_{b} \psi_{f]cde]}$$
and

\[(d^2 \psi)_{\text{abcef}} = 30 \nabla_{[a} \nabla_{b} \psi_{\text{cdef}}] = 30 \nabla_{[a} T_{bc} \varphi_{\text{def}]}.\]

So in particular, we get conditions

\[
\begin{align*}
\nabla_{[a} T_{b]} f \psi_{[f|cde]} & = 0 \quad (3.10a) \\
\nabla_{[a} T_{bc} \varphi_{\text{def}]} & = 0 \quad (3.10b)
\end{align*}
\]

From these, we get the following conditions.

**Proposition 10** The torsion tensor $T_{ab}$ of a $G_2$-structure $\varphi$ satisfies the following consistency conditions

1. \[
\varphi^{abc} T_{bc} T_{am} - T^{bd} T^c c \varphi_{mdc} - \psi_m^{abc} \nabla_a T_{bc} - (\text{Tr } T) \varphi_m^{ab} T_{ab} = 0 \quad (3.11)
\]

2. \[
\nabla_m (\text{Tr } T) - \nabla_a T_{m} a - T_{mc} \varphi^{abc} T_{ab} = 0 \quad (3.12)
\]

3. \[
0 = -\varphi_{mn} c \nabla_c (\text{Tr } T) + 6 T_{a[m} \psi_{n]}^{abc} T_{bc} + 2 \left( \nabla_{a} T_{[m|b]} \right) \varphi_{n]}^{ab} + 2 \varphi_{mn} T_{[ac} T_{bc} + 2 (\text{Tr } T) \psi_{mn}^{ab} T_{ab} + 2 \varphi_{mn} \varphi_{bcd} T^{cd} T^{ba} - 2 \varphi_{mn} \varphi_{bcd} T^{cd} T^{ab} + 2 T_{[m}^{a} T_{[a|n]} - 4 \varphi_{mn}^{ab} \nabla_{[a} T_{b|n]} - 2 (\text{Tr } T) T_{[mn]} + \varphi_{mn}^{a} \nabla_{b} T_{a}^{b} \quad (3.13)
\]

**Proof.** Let us first look at \((d^2 \varphi)_{\text{abcede}} = 20 \nabla_{[a} T_{b]} f \psi_{[f|cde]}.\) We have

\[
\begin{align*}
\nabla_{a} T_{b} f \psi_{f|cde} & = (\nabla_{a} T_{b} f) \psi_{f|cde} + T_{b} f \left( \nabla_{a} \psi_{f|cde} \right) \\
& = (\nabla_{a} T_{b} f) \psi_{f|cde} - 4 T_{b} f T_{f|a} \psi_{cde} \\
& = (\nabla_{a} T_{b} f) \psi_{f|cde} - \frac{4}{5} T_{b} f T_{[a|f]} \psi_{cde} - \frac{4}{5} T_{b} f T_{f[a] \varphi_{cde]} + \frac{12}{5} T_{b} f T_{[ac] \varphi_{de]} f}.
\end{align*}
\]

Anti-symmetrizing we obtain

\[
\begin{align*}
\nabla_{[a} T_{b] f} \psi_{[f|cde]} = (\nabla_{[a} T_{b]} f) \psi_{[f|cde]} + \frac{4}{5} T_{[a} f T_{b]} \psi_{[f|cde]} - \frac{4}{5} T_{[a} f T_{f}[b] \psi_{cde]} - \frac{12}{5} T_{[a} f T_{bc} \varphi_{de] f} \quad (3.14)
\end{align*}
\]

Using Proposition 8, we find the projections of (3.14) onto $\Lambda^5_{14}$ and $\Lambda^5_{7}$. Considering the corresponding 2-form in $\Lambda^2_{14}$ we obtain (3.13). From the $\Lambda^5_{7}$ component, we get

\[
0 = 2 \nabla_m (\text{Tr } T) - 2 \nabla_a T_{m} a - 2 T_{mc} \varphi^{abc} T_{ab} + \varphi^{abc} T_{bc} T_{am} - T^{bd} T^c c \varphi_{mdc} - \psi_m^{abc} \nabla_a T_{bc} - (\text{Tr } T) \varphi_m^{ab} T_{ab} \quad (3.15)
\]
However, let us now look at \((d^2\psi)_{abcdef} = 30\nabla_{[a}T_{bc\varphi_{de}]f}\). This is now a 6-form, so taking the Hodge star we get 1-form and hence automatically another 7 component. From this we immediately obtain

\[
\varphi^{abc}T_{bc}T_{am} - T^{bd}T^{c}_{b}\varphi_{mdc} - \psi_{m}^{abc}\nabla_{a}T_{bc} - (\text{Tr} T)\varphi_{m}^{ab}T_{ab} = 0
\]

that is, (3.11). Subtracting this condition from (3.15), we obtain (3.12).

The Ricci curvature of a \(G_2\)-structure manifold is determined by the torsion tensor and its derivatives. General expressions for the Ricci curvature have previously been given by Bryant in [3] and Karigiannis in [13]. Our expression differs from the expression in ([13]) due different sign convention for \(\psi\). This also leads to a different sign for \(T_{ab}\).

\[
R_{ab} = (\nabla_{a}T_{nm} - \nabla_{n}T_{am})\varphi^{nm}_{b} - T_{an}T_{n}^{b} + \text{Tr} (T) T_{ab} + T_{ac}T_{nm}\varphi^{nm}_{c} b
\]

(3.16)

This expression is a priori not symmetric in indices \(a\) and \(b\), as it should be. However we can consider projections of the antisymmetric part of (3.16) into the 7 and 14 representations. It turns out that the 7 component is a combination of (3.11) and (3.12), and hence vanishes. Similarly, the 14 component is proportional to (3.13), and so also vanishes. Thus indeed, given the conditions (3.11)-(3.13), the expression (3.16) for the Ricci curvature is indeed symmetric.

Since we are interested in particular torsion classes, which are given by torsion components \(\tau_{1}, \tau_{7}, \tau_{14}\) and \(\tau_{27}\).

**Proposition 11** Given the decomposition (3.6) of the full torsion tensor \(T_{ab}\) into components \(\tau_{1}, \tau_{7}, \tau_{14}\) and \(\tau_{27}\), these components satisfy the following consistency conditions:

1. \[
\nabla_{a}(\tau_{14})^{a}_{m} + 2\varphi_{m}^{ab}\nabla_{a}(\tau_{7})^{b}_{a} + 4(\tau_{7})^{a}_{m}(\tau_{14})^{a}_{m} = 0
\]

2. \[
\nabla_{m}\tau_{1} - \frac{1}{2}\varphi_{m}^{ab}\nabla_{a}(\tau_{7})^{b}_{a} - \frac{1}{6}\nabla_{a}(\tau_{27})^{a}_{m} - (\tau_{7})^{a}_{m}(\tau_{27})^{a}_{m} - (\tau_{7})^{m}_{m}\tau_{1} = 0
\]

3. \[
0 = \varphi_{mnla}\nabla_{b}(\tau_{27})^{ab} + 6\nabla_{a}(\tau_{27})_{b[m}\varphi_{n]}^{ab} - 24\tau_{1}(\tau_{14})_{mn}
\]

\[
-18\left(\frac{2}{3}\nabla_{[m}(\tau_{7})_{n]} - \frac{1}{6}\psi_{mn}^{ab}\nabla_{a}(\tau_{7})_{b}\right)
\]

\[
-18\left(\frac{2}{3}(\tau_{14})_{a[m}(\tau_{27})_{n]}^{a} - \frac{1}{6}\psi_{mn}^{ab}(\tau_{14})_{c}^{a}(\tau_{27})_{bc}\right)
\]

10
For some of the torsion classes, these conditions simplify, which we summarize below.

| Torsion class | Condition | In coordinates |
|---------------|-----------|----------------|
| $W_1$         | $d\tau_1 = 0$ | $\nabla_m \tau_1 = 0$ |
| $W_7$         | $d\tau_7 = 0$ | $\nabla_{[m} (\tau_7)_{n]} = 0$ |
| $W_{14}$      | $d^* \tau_{14} = 0$ | $\nabla_a (\tau_{14})^m_{\ m} = 0$ |
| $W_{27}$      | $d^* i_3(\tau_{27}) = 0$ | $\nabla_a (\tau_{27})^m_{\ m} = 0$ |
| $W_1 \oplus W_7$ | $\tau_7 = d \log \tau_1$ if $\tau_1$ nowhere zero | $\nabla_{a} (\tau_{27})^a_{\ b} [\tau_{27}]_{ab} = 0$ |
| $W_1 \oplus W_{14}$ | $\tau_1 = 0$ or $\tau_{14} = 0$ | $\nabla_{a} (\tau_{27})^a_{\ b} [\tau_{27}]_{ab} = 0$ |
| $W_1 \oplus W_{27}$ | $\pi_7 (d^* i_3(\tau_{27})) = \frac{1}{6} (d\tau_1) \wedge \varphi$ | $\nabla_{a} (\tau_{27})^a_{\ m} = 6 \nabla_{m} \tau_{1}$ |
| $W_7 \oplus W_{14}$ | $d\tau_7 = 0$ | $\nabla_{a} (\tau_{27})^a_{\ m} + 4 \tau_7 (\tau_{14})^a_{\ m} = 0$ |
| $W_7 \oplus W_{14}$ | $d^* \tau_{14} = 4 \tau_7 \wedge \tau_{14}$ | $\nabla_{a} (\tau_{27})^a_{\ b} [\tau_{27}]_{ab} = 0$ |

To obtain these conditions, we simply use the expressions (3.17) to (3.19), and set the relevant torsion components to zero. For the class $W_1 \oplus W_7$, the characterization that either $\tau_7$ is the gradient of $\log \tau_1$ or $\tau_1$ is zero everywhere was given by Cleyton and Ivanov in (4).

4 Deformations of $G_2$-structures

Suppose we have a $G_2$-structure on $M$ defined by the 3-form $\varphi$. In [9, 8] we considered small deformations of $G_2$-structures and then expanded related quantities such as the metric $g$, the volume form $\sqrt{\det g}$ and the 4-form $\psi$ up to a certain order in the small parameter. We will now deduce some results about more general deformations. Suppose we have a deformation for some 3-form $\chi$

$$\varphi \rightarrow \tilde{\varphi} = \varphi + \chi$$

(4.1)

In [9, 8] it was pointed out that generically it is difficult to obtain a closed form expression for $\tilde{g}$ and $\tilde{\psi}$. One of the challenges was to obtain a closed form expression for $\det \tilde{g}$. However it turns out that there is an easy way to do this, even if obtaining the full explicit expression is still computationally challenging. Note that we will use upper indices with tilde to denote indices raised with the deformed metric $\tilde{g}$.

Lemma 12 Given a deformation of $\varphi$ as in (4.1), the related quantities $\tilde{g}$, $\tilde{\psi}$ and $\det g$ are given by:

$$\tilde{g}_{ab} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} s_{ab}$$

(4.2a)

$$\tilde{\psi} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} \tilde{\eta}$$

(4.2b)

where

$$s_{ab} = g_{ab} + \frac{1}{2} \chi_{mna} (\varphi_b)^{mn} + \frac{1}{8} \chi_{amn} x_{bpq} \psi^mnpq + \frac{1}{24} \chi_{amn} x_{bpq} (\ast \chi)^{mnpq}$$

(4.3)
Moreover,
\[ \tilde{\psi}^a_{bcd} = \tilde{g}^{am} \tilde{g}^{bn} \tilde{g}^{cp} \tilde{g}^{dq} \tilde{\psi}_{mnpq} = \left( \frac{\det g}{\det \tilde{g}} \right)^\frac{1}{2} (\psi^{mnpq} + * \chi^{mnpq}) \]  
\hspace{2cm} (4.4)

**Proof.** From [9] we know the expression (4.2a) with \( \tilde{s}_{ab} \) as in (4.3). Now,
\[ \tilde{\psi}^a_{abcd} = \tilde{s} (\varphi + \chi)_{abcd} = \frac{1}{3!} \frac{1}{\sqrt{\det g}} \epsilon^{mnprst} (\varphi_{rst} + \chi_{rst}) \tilde{g}_{ma} \tilde{g}_{nb} \tilde{g}_{pc} \tilde{g}_{qd} \]
Here \( \epsilon^{abcdrst} \) refers to the alternating symbol which takes values 0 and \pm 1. Hence, using (4.2a), we get
\[ \tilde{\psi}^a_{abcd} = \left( \frac{\det g}{\det \tilde{g}} \right)^\frac{1}{2} (\psi^{mnpq} + * \chi^{mnpq}) \tilde{g}_{ma} \tilde{g}_{nb} \tilde{g}_{pc} \tilde{g}_{qd} \]  
\hspace{2cm} (4.5)
which is precisely (4.2b). Incidentally, by raising indices in (4.5) using \( \tilde{g} \), we obtain (4.4). 

Another possible simplification is to use contraction formulae for \( \tilde{\varphi} \) and \( \tilde{\psi} \). Since \( \tilde{\varphi} \) defines a \( G_2 \)-structure, \( \tilde{\varphi} \) and \( \tilde{\psi} \) satisfy the same identities as \( \varphi \) and \( \psi \) in Proposition 4. So, in particular, we have
\[ \tilde{\varphi}^a_{\ b} = \frac{1}{4} \tilde{\varphi}_{amn} \tilde{\psi}^{mnbc} = \frac{1}{4} \left( \frac{\det g}{\det \tilde{g}} \right)^\frac{1}{2} (\varphi_{amn} + \chi_{amn}) (\psi^{mnbc} + * \chi^{mnbc}) \]
\[ = \frac{1}{4} \left( \frac{\det g}{\det \tilde{g}} \right)^\frac{1}{2} (4 \varphi^a_{bc} + \varphi_{amn} * \chi^{mnbc} + \chi_{amn} \psi^{mnbc} + \chi_{amn} * \chi^{mnbc}) \]  
\hspace{2cm} (4.6)
This expression is very simple from computational point of view, since it does not involve the quantity \( s_{ab} \) (apart from the determinant factor). In our example for \( \chi \in \Lambda_2^3 \), this becomes
\[ \tilde{\varphi}^a_{\ bc} = (1 + M)^{-\frac{1}{2}} \left( \varphi^a_{bc} - \chi^a_{bc} + b^b v_m \varphi^a_{cm} - v^c v_m \varphi^a_{bm} \right) \]  
\hspace{2cm} (4.7)
We can use (4.6) together with other contraction identities to get closed expressions for the inverse metric \( \tilde{g}^a_{\ b} \) and the determinant \( \det \tilde{g} \).

**Proposition 13** Given a deformation of \( \varphi \) as in (4.1), and the corresponding deformation of the metric (4.2a), the deformed inverse metric is given by
\[ \tilde{g}^{am} = \left( \frac{\det g}{\det \tilde{g}} \right) \gamma^{am} \]  
\hspace{2cm} (4.8)
where
\[ \gamma^{am} = g^{am} - \frac{1}{96} * \chi^a_{bcd} * \chi^m_{pqr} \varphi_{bcp} \varphi_{dqr} - \frac{1}{48} * \chi^a_{bcd} * \chi^m_{pqr} \varphi_{bcp} \varphi_{dqr} \]
\[ - \frac{1}{48} * \chi^a_{bcd} \varphi^m_{pqr} \varphi_{bcp} \varphi_{dqr} - \frac{1}{96} * \chi^a_{bcd} * \chi^m_{pqr} \varphi_{bcp} \varphi_{dqr} - \frac{1}{96} * \chi^a_{bcd} * \chi^m_{pqr} \varphi_{bcp} \varphi_{dqr} \]
\[ - \frac{1}{12} * \chi^a_{bcd} \varphi^m_{bcp} \varphi_{dqr} + \frac{1}{6} * \chi^a_{bcd} \varphi^m_{bcp} \varphi_{dqr} - \frac{1}{12} * \chi^a_{bcd} \varphi^m_{bcp} \varphi_{dqr} + \frac{1}{12} * \chi^a_{bcd} \varphi^m_{bcp} \varphi_{dqr} \]  
\hspace{2cm} (4.9)
and
\[
\left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{3}{2}} = \frac{1}{7} \gamma^{am} s_{am}
\] (4.10)

for $s_{am}$ as in (4.3).

**Proof.** We have the following $G_2$-structure contraction identity for $\tilde{\varphi}$:

\[
\tilde{\varphi}_{abc} \tilde{\varphi}_{m} \tilde{b} \tilde{c} = 6 \tilde{g}_{am}
\]

Hence,

\[
\tilde{\varphi}_{c} \tilde{a} \tilde{b} \tilde{c} \tilde{m} \tilde{c} = -6 \tilde{g}^{\tilde{a} \tilde{m}}
\]

From (4.6), we thus have

\[
\tilde{g}^{\tilde{a} \tilde{m}} g_{am} = \left( \frac{\det g}{\det \tilde{g}} \right) \left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{1}{2}} \gamma^{am} s_{am} = 7
\]

Expanding this, and simplifying using $G_2$ contraction identities, we get (4.8). Now note that

\[
\tilde{g}^{\tilde{a} \tilde{m}} g_{am} = \left( \frac{\det g}{\det \tilde{g}} \right) \left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{3}{2}} \gamma^{am} s_{am} = 7
\]

with $s_{am}$ given by (4.3). Hence,

\[
\left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{3}{2}} = \frac{1}{7} \gamma^{am} s_{am}.
\]

The simplest deformation would be one where $\chi$ lies in $\Lambda_1^3$, and is hence proportional to $\varphi$. So suppose

\[
\chi = (f^3 - 1) \varphi.
\]

This way,

\[
\tilde{\varphi} = f^3 \varphi.
\]

From (4.3), we get

\[
s_{ab} = f^9 g_{ab}.
\]

Therefore, from (4.12a),

\[
\tilde{g}_{ab} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} f^9 g_{ab}.
\]

Taking the determinant on both sides, we find that

\[
\det \tilde{g} = f^{14} \det g.
\]

13
\( \tilde{g}_{ab} = f^2 g_{ab} \) \hspace{1cm} (4.14) \\
\( \tilde{g}^{ab} = f^{-2} g^{ab} \) \hspace{1cm} (4.15)

and so this defines a conformation transformation. We can also then show that

\[ \tilde{\psi} = f^4 \psi. \] \hspace{1cm} (4.16)

The next simplest case is when \( \chi = v \psi \in \Lambda_{27}^{3} \). Then \( *\chi = -v^\flat \wedge \varphi \). From this we can obtain \( s_{ab} \) and as it was shown by Karigiannis in [12],

\[ s_{ab} = (1 + M) g_{ab} - v_a v_b \] \hspace{1cm} (4.17)

where \( M = |v|^2 \) with the norm taken using \( g_{ab} \). Also, using the expressions for \( \chi \) and \( *\chi \) we can now get \( \gamma^{ab} \) from (4.9):

\[ \gamma^{ab} = (1 + M) \left( g^{ab} + v^a v^b \right) \] \hspace{1cm} (4.18)

Substituting this into (4.10), we get:

\[ \left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{3}{2}} = (1 + M)^2 \] \hspace{1cm} (4.19)

and therefore,

\[ \tilde{g}_{ab} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} s_{ab} = (1 + M)^{\frac{2}{3}} ( (1 + M) g_{ab} - v_a v_b ) \] \hspace{1cm} (4.20a)

\[ \tilde{g}^{ab} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\gamma^{am}} = (1 + M)^{-\frac{1}{3}} ( g^{mu} + v^m v^u ) \] \hspace{1cm} (4.20b)

As expected, these are precisely the results obtained in [12]. However the method used here does not depend on the particular form of \( s_{ab} \) and hence theoretically is applicable in the case when \( \chi \in \Lambda_{27}^{3} \) as well.

In fact, when \( \chi \in \Lambda_{27}^{3} \), we can write it as

\[ \chi_{abc} = h_{[a}^{\ d} \varphi_{bc]d} \] \hspace{1cm} (4.21)

for some traceless, symmetric \( h_{ab} \). Then it can be shown that

\[ *\chi_{abcd} = -\frac{4}{3} h^e_{[a} \psi_{[c|e][b|d]}. \] \hspace{1cm} (4.22)

Now we can substitute both \( \chi \) and \( *\chi \) into the expression (4.3) for \( s_{ab} \), and after some manipulations get

\[ s_{ab} = g_{ab} + \frac{2}{3} h_{ab} + \frac{2}{9} h_{a}^{\ e} h_{eb} - \frac{1}{18} \text{Tr} \left( h^2 \right) g_{ab} - \frac{1}{18} \varphi_{amn} \varphi_{bpq} h^{mn} h^{pq} \] \hspace{1cm} (4.23)

This expression for \( s_{ab} \) is already rather complicated, and so even getting an explicit closed expression for \( \left( \frac{\det \tilde{g}}{\det g} \right) \) becomes a very tough task, which is beyond the scope of this paper.

Now let us consider what happens to the Levi-Civita connection of the deformed metric.
Lemma 14 Given a deformation of $\varphi$ as in (4.1), and the corresponding deformation of the metric (4.2a), the components of the Levi-Civita connection $\tilde{\Gamma}_{a b c}$ corresponding to the new metric $\tilde{g}$ are given by

$$\tilde{\Gamma}_{a b c} = \Gamma_{a b c} + \delta \Gamma_{a b c}$$

where

$$\delta \Gamma_{a b c} = \frac{1}{2} \left( \frac{\text{det } g}{\text{det } \tilde{g}} \right)^{\frac{1}{2}} \left[ g^{bd} \left( \nabla_c s_{ad} + \nabla_a s_{cd} - \nabla_d s_{ac} \right) - \frac{1}{9} \left( \delta^b_a \delta^c_d + \delta^b_c \delta^d_a - \tilde{g}_{ac} \tilde{g}_{bd} \right) \tilde{g}^{\tilde{m}\tilde{n}} \nabla_{\tilde{c}} s_{\tilde{m}\tilde{n}} \right]$$

(4.24)

for $s_{ab}$ given by (4.3).

Proof. As it is well known, the components of the Levi-Civita connection are given by

$$\Gamma_{a b c} = \frac{1}{2} g^{bd} \left( g_{da,c} + g_{dc,a} - g_{ac,d} \right)$$

(4.25)

and hence for the modified metric, we have

$$\tilde{\Gamma}_{a b c} = \frac{1}{2} \tilde{g}^{bd} \left( \tilde{g}_{da,c} + \tilde{g}_{dc,a} - \tilde{g}_{ac,d} \right)$$

(4.26)

The difference between the two connections $\delta \Gamma_{a b c}$ is then given by

$$\delta \Gamma_{a b c} = \tilde{\Gamma}_{a b c} - \Gamma_{a b c} = \frac{1}{2} \tilde{g}^{bd} \left( \nabla_c \tilde{g}_{ad} + \nabla_a \tilde{g}_{cd} - \nabla_d \tilde{g}_{ac} \right)$$

(4.27)

So consider $\nabla_c \tilde{g}_{ad}$. Let

$$\gamma = \left( \frac{\text{det } \tilde{g}}{\text{det } g} \right)^{\frac{3}{2}}$$

then, from (4.2a), we have

$$\nabla_c \tilde{g}_{ad} = \nabla_c \left( \gamma^{-\frac{1}{2}} s_{ad} \right) = -\frac{1}{3} \gamma^{-\frac{1}{2}} \left( \nabla_c \gamma \right) s_{ad} + \gamma^{-\frac{1}{2}} \nabla_c s_{ad}$$

$$= -\frac{1}{3} \gamma^{-1} \left( \nabla_c \gamma \right) \tilde{g}_{ad} + \gamma^{-\frac{1}{2}} \nabla_c s_{ad}$$

Now let us look at $\nabla_c \gamma$. Note that by definition of the determinant, we get

$$\gamma^3 = \left( \frac{\text{det } \tilde{g}}{\text{det } g} \right)^{\frac{3}{2}} = \frac{1}{7!} \frac{1}{\text{det } g} \tilde{z}^{mpqrst} \tilde{z}^{abcdefg} s_{am} s_{bn} s_{cp} s_{dq} s_{er} s_{fs} s_{gt}.$$ 

and similarly,

$$\tilde{g}^{mu} = \frac{1}{6!} \left( \frac{\text{det } g}{\text{det } \tilde{g}} \right)^{\frac{4}{2}} \frac{1}{\text{det } g} \tilde{z}^{mpqrst} \tilde{z}^{abcdefg} s_{bn} s_{cp} s_{dq} s_{er} s_{fs} s_{gt}$$

So, in particular,

$$\nabla_c (\gamma^3) = \nabla_c \left( \frac{1}{7!} \frac{1}{\text{det } g} \tilde{z}^{s s s s s s s s} \right)$$

$$= \frac{1}{6!} \frac{1}{\text{det } g} \tilde{z}^{s s s} \left( \nabla_c s \right) s s s s s s$$

$$= \gamma^\frac{8}{3} \tilde{g}^{\tilde{m}\tilde{n}} \nabla_{\tilde{c}} s_{\tilde{m}\tilde{n}}$$

15
Therefore,
\[ \nabla_c \gamma = \frac{1}{3} \gamma \tilde{g}^{\tilde{m}\tilde{n}} \nabla_c s_{mn} \] (4.28)
where for brevity we have omitted the contracted indices. Hence,
\[ \nabla_c \tilde{g}_{ad} = -\frac{1}{9} \gamma \frac{1}{3} \tilde{g}_{ad} \tilde{g}^{\tilde{m}\tilde{n}} \nabla_c s_{mn} + \gamma \frac{1}{3} \nabla_c s_{ad}. \] (4.29)
Substituting (4.29) into (4.27), we find that
\[ \delta \Gamma^b_{a c} = \frac{1}{2} \gamma \frac{1}{3} \left( \tilde{g}^{\tilde{m}\tilde{n}} (\nabla_c s_{ad} + \nabla_a s_{cd} - \nabla_d s_{ac}) - \frac{1}{9} \left( \delta_a^b \delta_c^e + \delta_e^b \delta_a^c - \tilde{g}_{ac} \tilde{g}_{de} \right) \tilde{g}^{\tilde{m}\tilde{n}} \nabla_e s_{mn} \right) \]
and then after substituting \( \gamma^{-\frac{1}{3}} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} \) we get the result.

For the conformal deformation with \( \chi \) given by (4.11), we find that
\[ \delta \Gamma^b_{a c} = \frac{1}{9} f^{-\frac{9}{2}} \partial_e f \left( \delta_e^b \delta_a^c + \delta_a^b \delta_e^c - \tilde{g}_{ac} \tilde{g}_{de} \right) \tilde{g}_{de} \tilde{g}^{\tilde{m}\tilde{n}} \nabla_e s_{mn} \] (4.30)

5 Torsion deformations

Suppose we have a deformation of \( \phi \) given by (4.1). Using the results from Sect. 4, we can calculate the deformed torsion.

**Lemma 15** Given a deformation of \( \phi \) as in (4.1), the full torsion \( \tilde{T} \) of the new \( G_2 \)-structure \( \tilde{\phi} \) is given by
\[ \tilde{T}^a_{\tilde{m}} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} \left( T^a_m + \frac{1}{24} T^e_{\psi_{ebcd}} \chi^{mbcd} + \frac{1}{24} \psi_{mbcd} \nabla_a \chi_{ebcd} \right) \] (5.31)
with \( \delta \Gamma^e_{a b} \) given by (4.24).

**Proof.** Starting from (3.5) for \( \tilde{\phi} \) and \( \tilde{\psi} \), we get
\[ \tilde{T}^a_{\tilde{m}} = \frac{1}{24} \left( \nabla_a \tilde{\phi}_{bcd} \right) \tilde{\psi}^{\tilde{m}\tilde{n}\tilde{b}\tilde{d}} = \frac{1}{24} \left( \nabla_a \tilde{\phi}_{bcd} - 3 \delta \Gamma^e_{a b cde} \tilde{\psi}^{\tilde{m}\tilde{n}\tilde{b}\tilde{d}} \right) \]
\[ = \frac{1}{24} \left( \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} \nabla_a \tilde{\phi}_{bcd} * \tilde{\phi}^{mbcd} - 3 \delta \Gamma^e_{a b cde} \tilde{\psi} * \tilde{\phi}_{cd e} \tilde{\psi}^{mbcd} \right) \]
We can write
\[ \tilde{\phi}_{cd e} \tilde{\psi}^{mb} = 4 \tilde{\phi}^{\tilde{m} \tilde{n}} \]
and we can expand
\[ \nabla_a \tilde{\phi}_{bcd} * \tilde{\phi}^{mbcd} = \left( \nabla_a \tilde{\phi}_{bcd} + \nabla_a \chi_{bcd} \right) \left( \psi_{mbcd} + \chi_{mbcd} \right) \]
\[ = 24 T^a_m + T^e_{\psi_{ebcd}} \chi^{mbcd} \]
\[ + \psi_{mbcd} \nabla_a \chi_{bcd} + \nabla_a \chi_{ebcd} \]
Hence the result. ■

The torsion classes $W_i$ were originally defined by the $G_2$-structure $\varphi$, so once we have deformed $\varphi$ to $\tilde{\varphi}$ we will also get new torsion classes. Denote the new space by $\tilde{W}$ which splits as

$$\tilde{W} = \tilde{W}_1 \oplus \tilde{W}_7 \oplus \tilde{W}_{14} \oplus \tilde{W}_{27}. \quad (5.32)$$

The new torsion $\tilde{T}$ should now split as

$$\tilde{T}_{ab} = \tilde{\tau}_1 \tilde{g}_{ab} + \left( \tilde{\tau}_7^\# \cdot \tilde{\varphi} \right)_{ab} + (\tilde{\tau}_{14})_{ab} + (\tilde{\tau}_{27})_{ab} \quad (5.33)$$

Note that $\tilde{\tau}_7^\#$ refers to the vector obtained from the 1-form $\tilde{\tau}_7$ by raising indices using the deformed inverse metric $\tilde{g}^{-1}$. In general, determining these new torsion components $\tilde{\tau}_i$ is quite complicated. First we would have to lower one of the indices in (5.31) using $\tilde{g}$ and extract the different components. It is however easy to extract the the $\tilde{W}_1$-component directly from (5.31) by just contracting the indices.

**Lemma 16** The $\tilde{W}_1$-component $\tilde{\tau}_1$ of the deformed torsion $\tilde{T}$ is given by

$$\tilde{\tau}_1 = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} \left( \tau_1 + \frac{1}{168} T_a e \psi_{ebcd} * \chi^{abcd} + \frac{1}{168} \psi^{abcd} \nabla_a \chi_{bcd} + \frac{1}{168} \nabla_a \chi_{bcd} * \chi^{abcd} \right)$$

**Proof.** Contracting the indices in (5.31) we get

$$\tilde{T}_a \tilde{\tau} = \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} \left( T_a a + \frac{1}{24} T_a e \psi_{ebcd} * \chi^{abcd} + \frac{1}{24} \psi^{abcd} \nabla_a \chi_{bcd} + \frac{1}{24} \nabla_a \chi_{bcd} * \chi^{abcd} \right) - \frac{1}{2} \delta \Gamma_a e \ b \tilde{\varphi} c \ a \ b$$

Note that since the Christoffel symbols are symmetric in the bottom two indices, $\delta \Gamma_a e \ b \tilde{\varphi} c \ a \ b = 0$

Since

$$\tilde{T}_a \tilde{\tau} = 7 \tilde{\tau}_1,$$

we get the result. ■

Consider now what happens to $\tilde{T}_{an}$ with lowered indices.

$$\tilde{T}_{an} = \tilde{\tau}_1 \tilde{g}_{mn} = \left( \frac{\det g}{\det \tilde{g}} \right) \left( T_a m + \frac{1}{24} T_a e \psi_{ebcd} * \chi^{mbcd} + \frac{1}{24} \psi^{mbcd} \nabla_a \chi_{bcd} \right) + \frac{1}{24} \nabla_a \chi_{bcd} * \chi^{mbcd} \right) s_{mn} - \frac{1}{2} \delta \Gamma_a e \ b \tilde{\varphi} c \ a \ b$$

Now using the expression for $\delta \Gamma_a e \ b \tilde{\varphi} c \ a \ b$ (4.24), we get

$$\delta \Gamma_a e \ b \tilde{\varphi} c \ a \ b = \frac{1}{2} \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{2}} \left( \tilde{g}^e d \tilde{\varphi} c \ a \ b \ (\nabla_b s_{ad} + \nabla_a s_{bd} - \nabla_d s_{ab}) \right) - \frac{1}{9} \left( \delta^e d \ a \ b + \delta^e d \ a \ b \ - \tilde{g}_{ab} \tilde{g}^e f \tilde{\varphi} \ a \ b \tilde{g}^e f \left( \nabla s_{pq} \right) \right) - 17$$
Simplifying further, we eventually get

\[
\delta \Gamma^e_{\ b\ \tilde{e}n} = \left( \frac{\det g}{\det \tilde{g}} \right)^{1/2} \left( \tilde{\varphi}_n \tilde{\delta}^d \nabla_b s_{ad} - \frac{1}{9} \tilde{\varphi}_{an} \tilde{f} \tilde{g}^{\tilde{p}q} \nabla_f s_{pq} \right)
\]

\[
= \left( \frac{\det g}{\det \tilde{g}} \right)^{1/2} \tilde{\varphi}_c \tilde{\delta}^d \left( \delta^c_n \nabla_b s_{ad} - \frac{1}{9} \delta^c_{a\tilde{b}n} \tilde{g}^{\tilde{p}q} \nabla_d s_{pq} \right)
\]

\[
= \frac{1}{4} \left( \frac{\det g}{\det \tilde{g}} \right) \left( 4\varphi_c \tilde{b}d + \varphi_{cmn} * \chi^{mnbcd} + \chi_{cmn} \psi^{mnbcd} + \chi_{cmn} * \chi^{mnbcd} \right) \times
\]

\[
\times \left( \delta^c_n \nabla_b s_{ad} - \frac{1}{9} \delta^c_{a\tilde{b}n} \tilde{g}^{\tilde{p}q} \nabla_d s_{pq} \right)
\]

Thus, overall we have

\[
\tilde{T}_{an} = \frac{1}{24} \left( \frac{\det g}{\det \tilde{g}} \right) \left( 24T_a^m + T_a^e \psi_{eabcd} * \chi^{mnbcd} + \psi^{mnbcd} \nabla_a \chi_{bcd} + \nabla_a \chi_{bcd} * \chi^{mnbcd} \right) \times (5.35)
\]

\[
\times s_{mn} - \frac{1}{8} \left( \frac{\det g}{\det \tilde{g}} \right) \left( 4\varphi_c \tilde{b}d + \varphi_{cpq} * \chi^{pqbcd} + \chi_{cpq} \psi^{pqbcd} + \chi_{cpq} * \chi^{pqbcd} \right) \times
\]

\[
\times \left( \delta^c_n \nabla_b s_{ad} - \frac{1}{9} \delta^c_{a\tilde{b}n} \tilde{g}^{\tilde{p}q} \nabla_d s_{pq} \right).
\]

In the particular case of conformal deformations we can simply plug in \( \chi, s \) and \( \tilde{g} \) as in (4.11), (4.12) and (4.14) into (5.34) and obtain the deformed torsion.

**Proposition 17** Let \((\varphi, g)\) be a \(G_2\)-structure with torsion \(T\). Then define \((\tilde{\varphi}, \tilde{g})\) to be a new \(G_2\)-structure given by a conformal transformation of \((\varphi, g)\):

\[
\tilde{\varphi} = f^3 \varphi
\]

\[
\tilde{g} = f^2 g.
\]

Then the full torsion tensor \(\tilde{T}\) is given by

\[
\tilde{T} = fT - df \wedge \varphi
\]

Thus from Proposition 17 we see that a conformal transformation only affects the \(W_7\) torsion class, while the torsion classes in \(W_1, W_{14}\) and \(W_{27}\) are simply scaled. Therefore, the only conformally invariant torsion classes are the ones that contain a \(W_7\) component. This was previously shown in [5] and [12] but here we have an explicit expression for the torsion from which this conclusion follows trivially.

The expression (5.36) also shows that if the \(W_7\) component of the original torsion is an exact form, then it is possible to remove this component by applying a particular conformal transformation. Note that this implies that the class \(W_1 \oplus W_7\) is conformal to the class \(W_1\). As we know from (!!!), if \(\tau_1 \neq 0\), then

\[
\tau_7 = d \left( \log \tau_1 \right)
\]

Hence in order to remove this torsion component, need

\[
d \left( \log \tau_1 \right) = \frac{1}{f} df
\]

18
hence,
\[ f = \tau_0 \tau_1 \]
is a solution for a constant \( \tau_0 \). The original torsion is
\[ T = \tau_1 g + \frac{1}{f} df \wedge \varphi \]
so, under the change (5.36), the new torsion will become
\[ \tilde{T} = \tau_0 \tau_1^2 g \]
However, under the transformation
\[ \varphi \longrightarrow \tau_1^3 \varphi, \quad (5.37) \]
the metric changes as
\[ g \longrightarrow \tau_1^2 g \]
Hence in terms of the new metric, the new torsion is
\[ \tilde{T} = \tau_0 \tilde{g} \]
and so the constant \( \tau_0 \) is the new \( W_1 \) torsion component. Thus the conformal transformation (5.37) reduces the class \( W_1 \oplus W_7 \) to \( W_1 \). Conversely, a conformal transformation of the \( W_1 \) class will result in \( W_1 \oplus W_7 \). Since \( G_2 \)-structures in the \( W_1 \) class are sometimes called nearly \( G_2 \) or nearly parallel, the \( G_2 \)-structures in the strict \( W_1 \oplus W_7 \) class are referred to as conformally nearly parallel. If \( W_1 = 0 \), then we just have the \( W_7 \) class. In this case, we just know that \( \tau_7 \) is closed. So by Poincaré Lemma, we can at least locally find a function \( h \) such that \( dh = \tau_7 \). By taking a conformal transformation with \( f = e^h \), we can thus locally fully remove the torsion. Hence the \( W_7 \) class is sometimes called locally conformally parallel.

For the class \( W_1 \oplus W_7 \) we can also explicitly write out the Ricci curvature.

**Corollary 18** Suppose the 3-form \( \varphi \) defines a \( G_2 \)-structure with torsion contained in the class \( W_1 \oplus W_7 \). The the Ricci curvature of the corresponding metric is given by
\[ R_{ab} = (\nabla^c (\tau_7)_c + 5 (\tau_7)^c (\tau_7)_c + 6 \tau_1^2) g_{ab} - 5 (\tau_7)_a (\tau_7)_b + 5 \nabla_a (\tau_7)_b \quad (5.38) \]

**Proof.** In the general expression for the Ricci curvature, (3.16), substitute
\[ T_{ab} = \tau_1 g_{ab} + (\tau_7)^c \varphi_{cab}. \]
We then get
\[ R_{ab} = (\nabla^c (\tau_7)_c + 5 (\tau_7)^c (\tau_7)_c + 6 \tau_1^2) g_{ab} - 5 (\tau_7)_a (\tau_7)_b + 5 \nabla_b (\tau_7)_a - \psi_{cd} \nabla_c (\tau_7)_d + \varphi_{ab} \nabla_c \tau_1 - \tau_1 \varphi_{ab} \nabla^c (\tau_7)_c \]
However using the fact that \( d\tau_7 = 0 \), and hence that \( \nabla_a (\tau_7)_b \) is symmetric, and moreover that \( \nabla_c \tau_1 = \tau_1 (\tau_7)_c \), we obtain (5.38). \( \blacksquare \)
6 Torsion for $\Lambda_7$ deformations

Now consider in detail the case when we have a deformation in $\Lambda_7$. Here we have

$$h_{ab} = v^e \varphi_{cab} \quad (6.39)$$

Then,

$$\chi_{bcd} = h_{[b} \varphi_{cde]} = v^e \psi_{bcde}$$

$$*\chi_{mnpq} = 4v_{[m} \varphi_{npq]}$$

So we take a $G_2$-structure $\varphi$ and deform it to

$$\tilde{\varphi} = \varphi + v^e \psi_{bcde} \quad (6.40)$$

For convenience, let

$$M = |v|^2$$

then, as we know,

$$s_{ab} = g_{ab} (1 + M) - v_a v_b$$

We also know that

$$\left( \frac{\text{det } \tilde{g}}{\text{det } g} \right)^{\frac{1}{2}} = (1 + M)^{\frac{1}{2}}$$

$$\tilde{g}_{ab} = (1 + M)^{-\frac{2}{3}} (g_{ab} (1 + M) - v_a v_b)$$

$$\tilde{g}^{\tilde{a} \tilde{b}} = (1 + M)^{-\frac{1}{3}} (g^{ab} + v^a v^b)$$

$$\tilde{\varphi}^{\tilde{a} \tilde{b} \tilde{c} \tilde{d} \tilde{e}}_{\tilde{a} \tilde{b}} = (1 + M)^{-\frac{2}{3}} (\varphi^{bc}_{a} - \chi_{a}^{bc} + v^{b} v_{m} \varphi^{cm}_{a} - v^{c} v_{m} \varphi^{bm}_{a})$$

Note that the deformed metric defined above is always positive definite. To see this, suppose $\xi^a$ is some vector, then

$$\tilde{g}_{ab} \xi^a \xi^b = \left( 1 + |v|^2 \right)^{-\frac{2}{3}} \left( |\xi|^2 + |v|^2 |\xi|^2 - (v_a \xi^a)^2 \right) \geq 0 \quad (6.41)$$

since $(v_a \xi^a)^2 \leq |v|^2 |\xi|^2$. Therefore, under such a deformation, the 3-form $\tilde{\varphi}$ is always a positive 3-form, and thus indeed defines a $G_2$-structure. The deformation is also invertible. Suppose we want to use a vector $\tilde{v}$ to get from $\tilde{\varphi}$ back to $\varphi$. So we have

$$\varphi = \tilde{\varphi} + \tilde{v}^e \tilde{\psi}_{bcde} \quad (6.42)$$

Then, from (6.40), we obtain

$${\tilde{v}^e \tilde{\psi}_{bcde} = -v^e \psi_{bcde}}$$

Now we multiply both sides by $\tilde{\psi}^{\tilde{b} \tilde{c} \tilde{d} \tilde{a}}$. For the left hand side we obtain, using standard contraction identities

$$\tilde{v}^e \psi_{bcde} \tilde{\psi}^{\tilde{b} \tilde{c} \tilde{d} \tilde{a}} = 24 \tilde{v}^a$$

20
For the right hand side, we use the expression for \( \tilde{\psi} \):

\[
- v^e \psi_{bcde} \tilde{\psi}^{bced} = - (1 + M)^{-\frac{2}{3}} v^e \psi_{bcde} \left( \psi^{bced} + \ast \chi^{bced} \right)
\]

\[
= - (1 + M)^{-\frac{2}{3}} v^e \psi_{bcde} \left( \psi^{bced} + 4v^b \varphi^{cde} \right)
\]

\[
= - 24 (1 + M)^{-\frac{2}{3}} v^a
\]

Thus we have the following lemma. Here we decompose \( \nabla v \) in terms of representations of \( G_2 \) as

\[
\nabla_a v_b = v_1 g_{ab} + v_2 e \varphi_{cab} + (v_{14})_{ab} + (v_{27})_{ab}
\]

where \( v_{14} \in \Lambda^2_{14} \) and \( v_{27} \) is traceless symmetric.

**Lemma 19** Suppose we have a deformation of a \( G_2 \)-structure \( \varphi \) given by

\[
\varphi \rightarrow \tilde{\varphi} = \tilde{\varphi} + v^e \psi_{bcde}
\]

Then conversely, the deformation of the new \( G_2 \)-structure \( \tilde{\varphi} \) given by

\[
\tilde{\varphi} \rightarrow \tilde{\varphi} + \tilde{v}^a \tilde{\psi}_{bcde}
\]

results in the original \( G_2 \)-structure \( \varphi \) if and only if

\[
\tilde{v}^a = - (1 + M)^{-\frac{2}{3}} v^a
\]

Moreover, \( \tilde{v} \) has the following properties:

1. Denote the norm squared of \( \tilde{v} \) with respect to the deformed metric \( \tilde{g} \) by \( \tilde{M} \). Then, \( \tilde{M} \) is given by

\[
\tilde{M} = |\tilde{v}|^2_{\tilde{g}} = \tilde{v}^a \tilde{v}^b \tilde{g}_{ab} = M (1 + M)^{-2}
\]

2. The covariant derivative \( \tilde{\nabla} \) of \( \tilde{v} \) with respect to the deformed metric \( \tilde{g} \) is given by

\[
\tilde{\nabla}_a \tilde{v}_c = \frac{2}{3} (1 + M)^{-\frac{2}{3}} v_a v_c \left( (5 + 2M) v_1 - (v_{27})_{mn} v^m v^n \right) - (1 + M)^{-\frac{1}{3}} (v_{27})_{ac}(6.45)
\]

\[
- \frac{1}{3} (1 + M)^{-\frac{2}{3}} g_{ac} \left( (3 + 4M) v_1 + (v_{27})_{mn} v^m v^n \right) - (1 + M)^{-\frac{4}{3}} (v_{14})_{ac}
\]

\[
+ \frac{1}{3} (1 + M)^{-\frac{2}{3}} \left( 3v^b (v_{27})_{ba} v_c + (1 + 3M) v^b (v_{27})_{bc} v_a \right)
\]

\[
+ \frac{2}{3} (1 + M)^{-\frac{2}{3}} \left( 3\varphi_{abd} v^b (v_7)^d v_c - \varphi_{cde} v^b (v_7)^d v_a \right)
\]

\[
+ \frac{2}{3} (1 + M)^{-\frac{2}{3}} \left( v_a v^b (v_{14})_{bc} - 3v_c v^b (v_{14})_{ba} \right) - (1 + M)^{-\frac{4}{3}} (v_7)^b \varphi_{bac}
\]

3. Moreover, \( \tilde{\nabla} \tilde{v} = 0 \) if and only if \( \nabla v = 0 \).

**Proof.** The calculation of \( \tilde{M} \) is immediate. For the second part, we apply Lemma 14 to calculate \( \tilde{\nabla}_a \tilde{v}_c \). Finally for the third part, it is trivial that \( \tilde{\nabla} \tilde{v} = 0 \) if \( \nabla v = 0 \). For the “only if” statement, we can invert (6.45) to get \( \nabla v \) in terms of \( \tilde{\nabla} \tilde{v} \), in which case it becomes clear that \( \nabla v = 0 \) if \( \tilde{\nabla} \tilde{v} = 0 \).
Now let us use the expression for deformed torsion \( \text{(5.35)} \) to write it down in terms of \( v \).

First, we have

\[
\nabla_d s_{pq} = 2 g_{pq} v_m (\nabla_d v^m) - (\nabla_d v^p) v_q - (\nabla_d v^q) v_p
\]

and thus,

\[
\delta \nabla_b s_{ad} - \frac{1}{9} \delta \tilde{\gamma}_a g_{bn} \tilde{\gamma}_b \nabla_d s_{pq} = \nabla e s_{pq} \left( \delta \delta \tilde{\gamma}_b \delta \tilde{\gamma}_a \nabla_d s_{pq} - \frac{1}{9} \delta \delta \tilde{\gamma}_a g_{bn} \tilde{\gamma}_n \right)
\]

\[
= (1 + M)^{-1} \nabla e s_{pq} \left( (1 + M) \delta \delta \tilde{\gamma}_b \delta \tilde{\gamma}_a \nabla_d s_{pq} - \frac{1}{9} \delta \delta \tilde{\gamma}_a (g_{bn} (1 + M) - v_b v_n) (g^{pq} + v^p v^q) \right)
\]

So, overall, we have

\[
\tilde{T}_{an} = \frac{1}{24} (1 + M)^{-4} \left( (24 T_a^m + T_a^e \psi_{ebcd} \chi^{mabcd} + \psi^{mabcd} \nabla a \chi_{ebcd}) \right.
\]

\[
+ \nabla a \chi_{ebcd} \chi^{mabcd} s_{mn} - 3 (1 + M)^{1-1} \left( \phi \phi_{bd} - \chi_{bd} + v^b v_m \phi_{bm} - v^d v_m \phi_{bd} \right)
\]

\[
\times \nabla a \chi_{ebcd} \left( \delta \delta \tilde{\gamma}_b \delta \tilde{\gamma}_a \nabla_d s_{pq} - \frac{1}{9} \delta \delta \tilde{\gamma}_a (g_{bn} (1 + M) - v_b v_n) (g^{pq} + v^p v^q) \right)
\]

It makes sense to expand \( \nabla v \) also in terms of \( G_2 \)-representations:

\[
\nabla a v_b = v_1 g_{ab} + v_7 \phi_{cab} + (v_{14})_{ab} + (v_{27})_{ab}
\]

where \( v_{14} \in A_{14}^2 \) and \( v_{27} \) is traceless symmetric. Together with the similar expansion of \( T_{an} \) \( \text{(3.6)} \), after some manipulations, we obtain:

**Theorem 20** Given a \( G_2 \)-structure \( \phi \) with full torsion tensor \( T_{ab} \), a deformation of \( \phi \) which lies \( A_{14}^2 \) given by \( \phi \rightarrow \phi + v_e \psi_{ebcd} \) results in a new \( G_2 \)-structure \( \tilde{\phi} \) with torsion tensor \( \tilde{T}_{an} \) given by

\[
\tilde{T}_{an} = (1 + M)^{-4} \left( v_1 (v_a v_n - (1 + M) g_{an}) - \frac{4}{3} (1 + M) v_1 \phi_{anm} v^m 
\right.
\]

\[
- \left[ \frac{4}{3} (1 + M) \phi_{anm} v^m + \frac{5}{3} v_a \phi_{nmp} v^m v^p + \frac{4}{3} v_n \phi_{amp} v^m v^p 
\right.
\]

\[
+ \frac{1}{3} v^m v^p v^q v^r v_{an} v_{np} + \frac{3}{3} v_n (v_7)_{an} + \frac{8}{3} v_a (v_7)_{n} - (1 + M) (v_{14})_{an} 
\]

\[
-2 v_m (v_{14})_{an} v_n + \frac{1}{3} \phi_{anm} v_{14} v_p + \frac{1}{3} \phi_{anm} v q v^m v^p_{14} - (1 + M) (v_{27})_{an} 
\]

\[
+ v_m (v_{27})_{an} v_n - (1 + M) \phi_{anm} v_{14} v_m - \frac{1}{3} \phi_{anm} v_{27} v_p 
\]

\[
+ \frac{1}{3} v_{anmp} v^m v_{27} v_q + v_a \phi_{nmp} v^m v_{27} v_q - \frac{1}{3} \phi_{anm} v_m v_{27} v_p v_q 
\]

\[
+ (1 + M)^{-4} \left( \tau_1 g_{an} + (v_7)_{an} v_m + \phi_{anm} (v_7)_{n} + v_n (v_7)_{n} - g_{an} (v_7)_{m} v_m 
\right.
\]

\[
+ \left( v_{anmp} (v_7)_{n} + (v_{14})_{an} - \phi_{anm} (v_{14})_{n} + (v_{27})_{an} + \phi_{anm} (v_{27})_{n} + (v_{27})_{n} \right)
\]

22
From this we can also extract the individual components of $\tilde{T}_{an}$ in the representations of $G_2$. So first we have the component of $\tilde{T}_{an}$ in $\tilde{W}_1$:

$$\tilde{\tau}_1 = \frac{1}{7} \tilde{T}_{ab} \tilde{g}^{\tilde{a} \tilde{b}} = (1 + M)^{-\frac{1}{3}} \left( \tilde{T}_{ab} \tilde{g}^{\tilde{a} \tilde{b}} + v^a v^b \tilde{T}_{ab} \right)$$

$$= (1 + M)^{-\frac{2}{3}} \left( \left( 1 + \frac{1}{7} M \right) \tau_1 - v_1 - \frac{6}{7} (\tau_7)^a v_a + \frac{3}{7} (v_7)^a v_a + \frac{1}{7} (\tau_{27})_{ab} v^a v^b \right) \quad (6.49)$$

The 7-dimensional component is given by

$$(\tilde{\tau}_7)_c = \frac{1}{6} \tilde{T}_{ab} \tilde{\varphi}^{\tilde{a} \tilde{b}} c = \frac{1}{6} (1 + M)^{-\frac{2}{3}} \tilde{T}_{ab} \left( \varphi^{ab}_c - v^m \psi^{ab}_{cm} + v^a v_m \varphi^{bm}_{c} - v^b v_m \varphi^{am}_{c} \right) \quad (6.50)$$

where we have used $\tilde{\varphi}^{\tilde{a} \tilde{b}}$. Now using the expression for $\tilde{T}_{ab}$, after some manipulations, we obtain

$$(\tilde{\tau}_7)_c = (\tau_7)_c - \frac{1}{6} \varphi^{ab}_c (\tau_7)_a v_b - \frac{1}{6} v^a (\tau_{27})_{ac} - \frac{1}{6} v^a (\tau_{14})_{ac}$$

$$+ \frac{v_c}{6 (1 + M)} \left( (\tau_{27})_{ab} v^a v^b + 6 \tau_1 - 6 (\tau_7)_a v^a - 8 v_1 + 3 (v_7)_a v^a \right)$$

$$- \frac{1}{6} \frac{v_c}{(1 + M)} \left( 3 (M + 2) (v_7)_c + v^a (v_{27})_{ac} + \varphi_{ca} b^a (v_{27})_{bc} v^d + 3 \varphi_{cab} v^a (v_7)_b \right) \quad (6.51)$$

Let us now find the $\tilde{W}_{14}$ component. We have $\tilde{T}_{[an]} \in \Lambda^2$, so

$$\pi_{14} \left( \tilde{T}_{[an]} \right) = \frac{2}{3} \tilde{T}_{[an]} - \frac{1}{6} \tilde{T}_{mp} \tilde{\psi}^{\tilde{m} \tilde{p}} \quad (6.52)$$

The skew-symmetric part of $[6.48]$ is given by:

$$\tilde{T}_{[an]} = (1 + M)^{-\frac{4}{3}} \left( -\frac{4}{3} (1 + M) v_1 \varphi_{anm} v^m - \left( \frac{1 + 4}{3} M \right) \varphi_{anm} v^m \right) \quad (6.53)$$

$$- \frac{1}{3} \psi_{anmp} v^m v^p_{\tilde{q}} + \psi_{n \varphi \tilde{a} \tilde{m} p} v^m v^p_{\tilde{q}} + \frac{1}{3} v^m v_m \varphi_{an} v^p_{\tilde{q}} + \frac{7}{3} v_{[a} (v_7)_{n]}$$

$$- (1 + M) (v_{14})_{an} - 2 v_m (v_{14})_{an} v^m_{[a} + \frac{1}{3} \varphi_{anmp} v^m v^p_{\tilde{q}} + \frac{1}{3} \psi_{anmp} v^m v^p_{\tilde{q}}$$

$$+ v_m (v_{27})_{an} v^m - (1 + M) \varphi_{anmp} (v_{27})_{an} v^m - \frac{1}{3} \varphi_{anmp} v^m v^p_{\tilde{q}}$$

$$+ \frac{1}{3} \psi_{anmp} v^m v^p_{\tilde{q}} v^q_{[a} \varphi_{n \tilde{m} \tilde{p} \tilde{q}]} v^m v^p_{\tilde{q}} - \frac{1}{3} \varphi_{anmp} v^m v^p_{\tilde{q}} v^q_{[a} \varphi_{n \tilde{m} \tilde{p} \tilde{q}]} v^m v^p_{\tilde{q}}$$

$$+ (1 + M)^{-\frac{2}{3}} \left( \tau_1 \varphi_{anm} v_m + \varphi_{anm} \tau_7 v^m + \psi_{anmp} v^m \tau_7 v^p + (\tau_{14})_{an} + \varphi_{anmp} (\tau_{14})_{an} \right)$$

Now note that

$$\tilde{\psi}_{an} = \tilde{\psi}_{an} \tilde{g}_{qr} g_{an} = (1 + M)^{-2} \left( \psi^{mnpq} + 4 v^m v^p v^q \varphi_{mnpq} \right) \quad (6.54)$$

$$\left( g_{aq} (1 + M) - v_a v_q \right) \left( g_{nr} (1 + M) - v_n v_r \right) \quad (6.55)$$
Hence the 14-dimensional component is

\[
(\tilde{\tau}_{14})_{an} = (1 + M)^{-\frac{4}{3}} \left( \frac{10}{3} (v_7)_n [a, v_n] + \frac{4}{3} v_{[a} \varphi_{mn]} v_m (v_7)_p - \left( \frac{5}{6} + \frac{1}{2} M \right) \psi_{mn} v_m (v_7)_p \right) - \frac{1}{3} (v_7)_m v^m v_p \varphi_{an} - \frac{1}{3} M (v_7)_m \varphi^m_{an} - (1 + M) (v_{14})_{an} - \frac{2}{3} v_m (v_{14})^m_{[a, v_n]} \\
+ \frac{1}{3} \varphi_{an} v_p (v_{14})^p_m + \frac{1}{3} \psi_{an} v_m (v_{14})_{pq} v^q - \frac{1}{3} \varphi_{an} v_m (v_{27})_{pq} v^p v^q \\
+ (M + 1) \varphi_{[a \varphi_{(v_{27})]p}} v_m + \frac{1}{6} (M - 1) \varphi^m_{an} (v_{27})^p m v_p \\
+ \frac{2}{3} v_m (v_{27})^m_{[a, v_n]} - \frac{4}{3} \varphi_{an} v_m (v_{27})_{pq} v^q + \frac{1}{3} \psi_{an} v_m (v_{27})_{pq} v^q \\
+ (1 + M)^{-\frac{2}{3}} \left( -\frac{1}{6} M \varphi^m_{an} \varphi^m_{an} (v_7)_m + \frac{1}{6} \psi_{an} \varphi^m_{an} (v_7)_m v_p - \frac{1}{3} \varphi^m_{an} v_m (v_{14})^m_{[a, v_n]} (v_{14})^m_m v_p + \frac{2}{3} v_7 [a (v_7)_n] \\
+ \frac{1}{6} \varphi^m_{an} v_m v_p (v_7)^p + (v_{14})^m v_p + \frac{1}{6} \psi_{an} (v_{14})^m m v_p v^q - \frac{1}{3} \varphi^m_{an} v_p (v_{14})^p m \\
- \varphi^m_{an} (v_{27})^p m v_p + \frac{1}{6} \varphi^m_{an} (v_{27})^p m v_p v^q + \frac{1}{6} \psi_{an} (v_{27})^p m v_p v^q \right) \\
+ (1 + M)^{-\frac{2}{3}} \left( -\frac{1}{6} M \varphi^m_{an} \varphi^m_{an} (v_7)_m + \frac{1}{6} \psi_{an} \varphi^m_{an} (v_7)_m v_p - \frac{1}{3} \varphi^m_{an} v_m (v_{14})^m_{[a, v_n]} (v_{14})^m_m v_p + \frac{2}{3} v_7 [a (v_7)_n] \\
+ \frac{1}{6} \varphi^m_{an} v_m v_p (v_7)^p + (v_{14})^m v_p + \frac{1}{6} \psi_{an} (v_{14})^m m v_p v^q - \frac{1}{3} \varphi^m_{an} v_p (v_{14})^p m \\
- \varphi^m_{an} (v_{27})^p m v_p + \frac{1}{6} \varphi^m_{an} (v_{27})^p m v_p v^q + \frac{1}{6} \psi_{an} (v_{27})^p m v_p v^q \right)
\]

Finally, the component in $\tilde{W}_{27}$ is now given by

\[
(\tilde{\tau}_{27})_{an} = \tilde{T}_{(an)} - \tau_1 \tilde{g}_{an}
\]

where $\tilde{T}_{(an)}$ is the symmetric part of ($6.48$):

\[
\tilde{T}_{(an)} = (1 + M)^{-\frac{4}{3}} \left( v_7 (v_7)_{an} - (1 + M) g_{an} + 3 v_{[a} \varphi_{mn]} v_m v_7 \right) + 3 v_{(a} (v_7)_n) - (1 + M) (v_27)_{an} + v_m (v_27)^m_{(a, v_n)} \\
- (1 + M) \varphi_{[a \varphi_{(v_27)}]p} v_m + v_{(a \varphi_{[v_m]})} v_m (v_27)^p_{pq} v^q \\
+ (1 + M)^{-\frac{1}{3}} \left( \tau_1 g_{an} + v_{(a} (v_7)_n) - g_{an} \varphi_{(v_7)}^m v_m \\
- \varphi_{[a \varphi_{(v_27)}]p} v_m + \tau_{(v_7)_n} + \varphi_{(v_27)}^m (v_27)^p_{an} \right)
\]

and

\[
\tilde{T}_{1} = (1 + M)^{-\frac{4}{3}} \left( ((1 + M) g_{an} - v_n v_a) \left( \left( 1 + \frac{1}{6} M \right) \tau_1 - v_1 - \frac{6}{7} \tau_{(v_7)}^m v_m + \frac{3}{7} \tau_{(v_7)}^m v_m - \frac{1}{7} \tau_{(v_27)}^m v_m v_p \right) \right)
\]

Thus overall, we have

\[
(\tilde{\tau}_{27})_{an} = (1 + M)^{-\frac{4}{3}} \left( -\frac{3}{7} \left( (1 + M) g_{an} - v_n v_a \right) v_7^m v_m + 3 \psi_{(a \varphi_{[v_m]})} \psi_{[v_7]} v_7 \right) + 3 v_{(a} (v_7)_n) - (1 + M) (v_27)_{an} + v_m (v_27)^m_{(a, v_n)} - (1 + M) \varphi_{[a \varphi_{(v_27)}]p} v_m \\
+ v_{(a \varphi_{[v_m]})} v_m (v_27)^p_{pq} v^q + \left( 1 + \frac{1}{6} M \right) \tau_1 v_a v_n - \frac{6}{7} \tau_{(v_7)}^m v_m v_a v_n + \frac{1}{7} \tau_{(v_27)}^m v_m v_p v_a v_n \\
+ (1 + M)^{-\frac{1}{3}} \left( -\frac{1}{7} M \tau_1 g_{an} + v_{(a} (v_7)_n) - \frac{1}{7} v_{(a} (v_7)_n) \varphi_{(v_7)}^m v_m \\
- \varphi_{[a \varphi_{(v_27)}]p} v_m + \tau_{(v_7)_n} + \varphi_{[a \varphi_{(v_27)}]p} v_m \right)
\]
The expressions (6.49), (6.51), (6.54) and (6.56) give us the components of the new torsion \( \tilde{T} \) in \( \tilde{W}_1, \tilde{W}_7, \tilde{W}_{14} \) and \( \tilde{W}_{27} \), respectively. As we can see these expressions are quite complicated, so for a generic deformation vector \( v \), in general we would obtain

**Theorem 21** Given a \( G_2 \)-structure \( \varphi \) with full torsion tensor \( T_{ab} \), a deformation of \( \varphi \) which lies \( \Lambda^3 \) given by \( \varphi \to \varphi + v_c \psi_{bcd} \) results in a new \( G_2 \)-structure \( \tilde{\varphi} \) with a torsion tensor \( \tilde{T}_{an} \) if only if the components \( v_1, v_7, v_{14} \) and \( v_{27} \) of \( \nabla v \) satisfy the following equations

\[
v_1 = \tau_1 - \frac{3}{7} (\tau_7)_a v^a - \frac{1}{7} (\tau_7)_a v^a - \frac{7}{7} (\tau_7)_a v^a + \frac{14}{14} (1 + M)^{\frac{1}{2}} (\tau_{27})_{ab} v^a v^b \tag{6.57a}
\]

\[
(v_7)^c = (\tau_7)^c - \frac{1}{3} \tau_1 v^c + \frac{1}{3} \varphi^c_{ab} (\tau_7)^a v^b - \frac{1}{6} (\tau_1)_{ab} v^a v^b \tag{6.57b}
\]

\[
(v_{14})_{ab} = \frac{1}{(M + 9)} \left( \frac{4}{3} (M - 27) (\tau_7)|v_{ab}| - \frac{1}{3} (M - 27) \psi^{mn}_{ab} (\tau_7)_m v_n - 4 M \varphi^m_{ab} (\tau_7)_m \right) v_{mn} \tag{6.57c}
\]

\[
+ 4 (\tau_7)^m (\varphi^m_{(a b)} v_n v_n - 24 \varphi^m_{ab} |v_{ab}| (\tau_7)_m v_n + \frac{1}{2} (M + 2) (M + 9) (\tau_4)_{ab})
\]

\[
+ \frac{1}{2} (M + 9) \varphi^m_{ab} \varphi^b_{pq} v_m v_p (\tau_1)_{mn} + (M - 7) v_m (\tau_1)_m |v_{ab}|)
\]

\[
+ 8 \varphi^m_{(a b)} v_m v_p (\tau_1)_m v_p (\tau_4)_{m} \varphi^m_{ab} + 4 \psi^{mn}_{ab} v_{n p} (\tau_1)_m v_p v_{m n} \tag{6.57c}
\]

\[
+ 16 v_m (\tau_27)_{m} |v_{ab}| - \frac{4}{3} v_m \varphi^{ab} (\tau_27)_m v_{n p} v^p v^p + \frac{4}{3} M v_m (\tau_27)_{m} \varphi^m_{ab} - 4 \psi^{mn}_{ab} v_{n p} (\tau_27)_m v_p v_{m n} \tag{6.57c}
\]

\[
+ 8 \varphi^m_{ab} v_m v_p (\tau_27)_{n} v_{mn} = \frac{1}{6} (M + 17) (1 + M)^{\frac{1}{2}} (\tau_7)_{m} v_m - 4 \varphi^{ab} (\tau_7)_m v_n \tag{6.57c}
\]

\[
+ 24 \varphi^m_{ab} |v_{ab}| (\tau_7)_m v_n - 2 (M - 15) (\tau_7)_m |v_{ab}| + \frac{1}{2} (M - 15) \psi^{mn}_{ab} (\tau_7)_m v_n \tag{6.57c}
\]

\[
+ 4 M \varphi^m_{ab} (\tau_7)_m - (1 + M)^{\frac{1}{2}} (M + 9) \left( \psi^{mn}_{ab} (\tau_7)_m v_n \right) \tag{6.57c}
\]

\[
+ 16 (1 + M)^{\frac{1}{2}} \psi^{m}_{(a b)} v_m (\tau_7)_m |v_{ab}| + 8 (1 + M)^{\frac{1}{2}} \psi^{mn}_{ab} |v_{ab}| v_m v_p (\tau_4)_{m} \tag{6.57c}
\]

\[
+ (M - 3) (1 + M)^{\frac{1}{2}} \psi^m_{(a b)} v_m (\tau_7)_m \varphi^m_{ab} - 4 (1 + M)^{-\frac{1}{2}} \psi^{mn}_{ab} v_{n p} (\tau_4)_{m} \tag{6.57c}
\]

\[
- 8 (1 + M)^{-\frac{1}{2}} \varphi^{mn}_{ab} v_m v_p (\tau_7)_{m} v_p v_{m n} \tag{6.57c}
\]

\[
+ \frac{1}{6} (M - 9) (1 + M)^{\frac{1}{2}} \psi^m_{(a b)} v_m (\tau_7)_m \varphi^m_{ab} - 4 (1 + M)^{\frac{1}{2}} \psi^{mn}_{ab} v_{n p} (\tau_27)_m \tag{6.57c}
\]

\[
\]

\[
(v_{27})_{ab} = \left( (\tau_7)_{ab} + 4 (\tau_7)_{(a b)} \right) v_{ab} + \left( 4 (1 + M)^{-\frac{1}{2}} \tau_1 - \frac{1}{2} (1 + M)^{-\frac{1}{2}} (\tau_7)_{mn} v^m v^n \right) v_a v_b \tag{6.57d}
\]

\[
- \frac{1}{7} \left( (\tau_7)_{m} v^m - 3 (\tau_7)_{m} v^m - 4 M (1 + M)^{-\frac{1}{2}} \tau_1 - \frac{1}{2} (1 + M)^{-\frac{1}{2}} (\tau_7)_{mn} v^m v^n \right) g_{ab} \tag{6.57d}
\]

\[
- 3 (\tau_7)_{ab} v^m - \varphi^{mn}_{(a} (\tau_7)_{b)mn} v^m - \frac{1}{2} (1 + M)^{-\frac{1}{2}} \varphi^m_{ab} \varphi^b_{pq} (\tau_7)_{mp} v_n v_q \tag{6.57d}
\]

\[
- (1 + M)^{-\frac{1}{2}} \left( \frac{1}{2} (2 + M) (\tau_7)_{ab} + v_m (\tau_7)_m (a b) + v^m \varphi_{mn} (\tau_7)_n (a \right) \tag{6.57d}
\]

\[
+ \varphi^{mn}_{(a b)} v_m v_p (\tau_27)_{p} \tag{6.57d}
\]
and moreover, the following necessary condition is satisfied:

\[ d((v_7)_\mu \varphi + v_{14}) = 0. \quad (6.58) \]

**Proof.** In order to obtain the equations which the components \( v_1, v_7, v_{14} \) and \( v_{27} \) of \( \nabla v \) must satisfy, we have to invert the expressions for \( \tilde{\tau}_1, \tilde{\tau}_7, \tilde{\tau}_{14} \) and \( \tilde{\tau}_{27} \) that are given by (6.49), (6.50), (6.54) and (6.56), respectively. So we solve for \( v_1, (v_7)^a, (v_{14})_a \) and \( (v_{27})_{ab} \) in terms of the original torsion components \( \tau_1, \tau_7, \tau_{14} \) and \( \tau_{27} \) and the new torsion components \( \tilde{\tau}_1, \tilde{\tau}_7, \tilde{\tau}_{14} \) and \( \tilde{\tau}_{27} \). Therefore pointwise we have the same number of variables as equations, so generically we should be able to solve it.

For convenience, let us denote the left hand sides of equations (6.49), (6.50), (6.54) and (6.56), by \( \tilde{\tau}_1, \tilde{\tau}_7, \tilde{\tau}_{14} \) and \( \tilde{\tau}_{27} \), respectively. Hence these equations can be rewritten as

\[
\begin{align*}
\tilde{\tau}_1 &= \tilde{\tau}_1 \\
\tilde{\tau}_7 &= \tilde{\tau}_7 \\
\tilde{\tau}_{14} &= \tilde{\tau}_{14} \\
\tilde{\tau}_{27} &= \tilde{\tau}_{27}
\end{align*}
\]

Let us first look at the \( \tilde{\tau}_1 \) equation. Note that the expression for \( \tilde{\tau}_1 \) contains the scalars \( v_1 \) and \( (v_7)^a v_a \). So in order to find \( v_1 \), we would also need to find \( (v_7)^a v_a \). We can get another equation that has \( (v_7)^a v_a \) by constructing the scalar \( (\tilde{\tau}_7)^c v_c \). However that now also has the scalar \( (v_{27})^{ab} v_a v_b \). So we would need another equation - this time from by \( (\tilde{\tau}_{27})_{ab} v^a v^b \). Now we can solve the system

\[
\begin{align*}
\tilde{\tau}_1 &= (1 + M)^{-\frac{2}{3}} \left( -v_1 + \frac{3}{l} (v_7)^a v_a + \left( 1 + \frac{1}{l} M \right) \tau_1 - \frac{6}{l} (\tau_7)^a v_a \right) \\
&\quad + \frac{1}{l} (\tau_{27})^{ab} v_a v_b \\
(\tilde{\tau}_7)^c v_c &= (\tilde{\tau}_7)^c v_c = (1 + M)^{-1} \left( -\frac{4}{3} M v_1 - (v_7)^a v_a - \frac{1}{6} (v_{27})^{ab} v_a v_b + M \tau_1 \right) \\
&\quad + (\tau_7)^a v_a - \frac{1}{6} (\tau_{27})^{ab} v_a v_b \\
(\tilde{\tau}_{27})_{ab} v^a v^b &= (\tilde{\tau}_{27})_{ab} v^a v^b = (1 + M)^{-\frac{2}{3}} \left( \frac{18}{l} M (v_7)^a v_a - (v_{27})^{ab} v_a v_b + \frac{6}{l} M^2 \tau_1 \right) \\
&\quad + \frac{6}{l} M (\tau_7)^a v_a + \left( 1 + \frac{6}{l} M \right) (\tau_{27})^{ab} v_a v_b 
\end{align*}
\]

We have three equations, and the three variables, \( v_1, (v_7)^a v_a \) and \( (v_{27})^{ab} v_a v_b \), which we treat as being independent. The determinant of this system is proportional to \( (M + 1) \), but \( M = |v|^2 > 0 \), so we always have a solution. Solving, we get the solution (6.57a) for \( v_1 \) and also solutions for \( (v_7)^a v_a \) and \( (v_{27})^{ab} v_a v_b \).

Note that we could have also considered \( (\tilde{\tau}_{27})_{ab} g^{ab} \). However, since \( \tilde{\tau}_{27} \) is traceless with respect to \( g^{ab} = (1 + M)^{-\frac{2}{3}} (g^{ab} + v^a v^b) \),

\[
(\tilde{\tau}_{27})_{ab} g^{ab} = - (\tilde{\tau}_{27})_{ab} v^a v^b
\]
so we would get no new independent equation.

Next, we look at the $\tilde{\tau}_7$ equation. We now have expressions for $(v_7)^a v_a$ and $(v_{27})^{ab} v_a v_b$, so we can replace any instances of these scalars by the solutions of the above scalar equations. Our remaining variables are now $(v_7)^c, (v_{14})^{a c} v^a, (v_{27})^{a c} v^a, \varphi^{a c} (v_7)^a v_b, \varphi^{a b} (v_{14})^{e b} v^a v_e$ and $\varphi^{a b} (v_{27})^{b e} v^a v_e$. To solve for these variables, we construct six equations

\[
(\tilde{\tau}_7)_a = (\tilde{\tau})_a \quad (6.60a)
\]
\[
(\tilde{\tau}_{14})^{ab} v^a = (\tilde{\tau}_{14})^{ab} v^a \quad (6.60b)
\]
\[
(\tilde{\tau}_{27})^{ab} v^a = (\tilde{\tau}_{27})^{ab} v^a \quad (6.60c)
\]
\[
\varphi^{a b} (\tilde{\tau}_7)^{b c} v^c = \varphi^{a b} (\tilde{\tau}_7)^{b c} v^c \quad (6.60d)
\]
\[
\varphi^{a b} (\tilde{\tau}_{14})^{b d} v^d v^c = \varphi^{a b} (\tilde{\tau}_{14})^{b d} v^d v^c \quad (6.60e)
\]
\[
\varphi^{a b} (\tilde{\tau}_{27})^{b d} v^d v^c = \varphi^{a b} (\tilde{\tau}_{27})^{b d} v^d v^c \quad (6.60f)
\]

The left hand side of each of these equations is now some function of $v, \tau_1, \tau_7, \tau_{14}$ and $\tau_{27}$ constructed from the expressions for $\tilde{\tau}_1, \tilde{\tau}_7, \tilde{\tau}_{14}$ and $\tilde{\tau}_{27}$ and with any instances of $v_1, (v_7)^a v_a$ and $(v_{27})^{ab} v_a v_b$ replaced by the solutions of equations (6.59). It turns out that we do not get any new variables, and so we get six equations for six variables. The determinant of this system is positive, so we can solve this, and in particular, get the solution for $(v_7)^c$ (6.57b). We also get solutions for the other vectors constructed above.

Now we can look at the last two equation - $(\tilde{\tau}_{14})^{ab} = 0$ and $(\tilde{\tau}_{27})^{ab} = 0$. We now have solutions for scalars and vectors, so we can substitute them into these equations. Then, the variables in the first equation are skew-symmetric quantities, and in the second equation we have symmetric quantities.

In the $\tilde{\tau}_{14}$ equation the quantities are $(v_{14})^{ab}$ and $\varphi^{cd} [a (v_{27})^{bd} v_c$, while in the $\tilde{\tau}_{27}$ equation we have $(v_{27})^{ab}$ and $\varphi^{cd} [a (v_{27})^{bd} v_c$. Hence we can construct quantities $\varphi^{cd} [a (\tilde{\tau}_{27})^{bd} v_c$ and $\varphi^{cd} (a \tilde{\tau}_{27})^{bd} v_c$ which give us one extra equation for both skew-symmetric and symmetric quantities. For the skew-symmetric equations we get no new variables, thus our equations are

\[
(\tilde{\tau}_{14})^{ab} = (\tilde{\tau}_{14})^{ab} \\
\varphi^{cd} [a (\tilde{\tau}_{27})^{bd} v_c = \varphi^{cd} [a (\tilde{\tau}_{27})^{bd} v_c
\]

(6.61a)
\]
\]

(6.61b)

Here we solve for $(v_{14})^{ab}$ and $\varphi^{cd} [a (v_{27})^{bd} v_c$, and immediately get the solution (6.57c). It can be checked that this expression does indeed give a 2-form lying in $\Lambda^2_{14}$.

Going back to the symmetric equations, from $\varphi^{cd} (a \tilde{\tau}_{27})^{bd} v_c$ we get a new symmetric variable $\varphi_a^{cd} \varphi_b^{ef} v_c v_e (v_{27})^{df}$. We then construct the quantity $\varphi_a^{cd} \varphi_b^{ef} v_c v_e (\tilde{\tau}_{27})^{df}$ and get no new variables. Therefore, the symmetric equations are

\[
(\tilde{\tau}_{27})^{ab} = (\tilde{\tau}_{27})^{ab} \\
\varphi^{cd} (a \tilde{\tau}_{27})^{bd} v_c = \varphi^{cd} (a \tilde{\tau}_{27})^{bd} v_c \\
\varphi_a^{cd} \varphi_b^{ef} v_c v_e (\tilde{\tau}_{27})^{df} = \varphi_a^{cd} \varphi_b^{ef} v_c v_e (\tilde{\tau}_{27})^{df}
\]

(6.62a)
\]
\]

(6.62b)
\]
\]

(6.62c)

where we solve for $(v_{27})^{ab}, \varphi^{cd} (a \tilde{\tau}_{27})^{bd} v_c$ and $\varphi_a^{cd} \varphi_b^{ef} v_c v_e (\tilde{\tau}_{27})^{df}$. We have three equations with three variable, and the determinant is again positive, so we solve it and get the solution (6.57d) for $(v_{27})^{ab}$. Note that it is always traceless, hence indeed always corresponds to the component in the 27-dimensional representation.
To get the necessary condition (6.58), first note that
\[ dv^b = 2 (v_7)_a \varphi + 2v_{14}. \] (6.63)
Therefore, we must have
\[ d^2 v^b = 0, \]
which gives us (6.58). So far, we have only considered the algebraic constraints on the components of \( \nabla v \), so the differential condition (6.58) is not automatically satisfied in general, and must be imposed separately.

Note that the condition (6.58) in Theorem 21 involves second derivatives of \( v \) - in particular, derivatives of the 7 and the 14 components of \( \nabla v \), that is, \( v_7 \) and \( v_{14} \). However, from the equations (6.57b) and (6.57c), \( v_7 \) and \( v_{14} \) are expressed in terms of \( v \) and the torsion components. However, derivatives of \( v \) can be reduced again to expressions just involving \( v \) and the torsion components, using all of the equations (6.57a) to (6.57d). So overall, (6.58) gives a relationship between \( v \), the torsion components and the derivatives of the torsion components. Moreover, we can also apply the conditions on the derivatives of torsion components from Proposition 11 in order to relate some of the torsion derivatives to the torsion components themselves. In the general case, the resulting expressions are extremely long, and not very helpful, so we will consider individual torsion classes in order to gain more insight.

The simplest case is when the original torsion vanishes.

**Corollary 22** Suppose the 3-form \( \varphi \) defines a torsion-free \( G_2 \)-structure, then a deformation of \( \varphi \) which lies in \( \Lambda^3 \) and is given by \( \varphi \rightarrow \varphi + v^e \psi_{bcde} \) results in a new torsion-free \( G_2 \)-structure \( \tilde{\varphi} \) if and only if
\[ \nabla v = 0 \]

**Proof.** We get this immediately by setting \( \tau_1 = \tau_7 = \tau_{14} = \tau_{27} = 0 \) in (6.57a) to (6.57d) in Theorem 21. The condition (6.58) is then automatically satisfied.

This is equivalent to saying that \( d\chi = 0 \) and \( d^* \chi = 0 \) for \( \chi = v^e \psi_{bcde} \). This is however exactly the same condition as the one for an infinitesimal deformation.

**Theorem 23** Suppose \((\varphi, g)\) is a \( G_2 \)-structure on a closed, compact manifold \( M \). Consider a deformation of the \( G_2 \)-structure \( \varphi \) given by
\[ \varphi \rightarrow \varphi + v^e \psi_{bcde} \] (6.64)
If the torsion \( T \) lies in the class \( W_1 \oplus W_7 \), then this deformation results in a torsion-free \( G_2 \)-structure if and only if \( T = 0 \) and \( \nabla v = 0 \).

**Proof.** If \( T = 0 \), from Corollary 22 we know that the deformation (6.64) results in a torsion-free \( G_2 \)-structure if and only if \( \nabla v = 0 \). So assume now \( T \neq 0 \).

Now let us assume that \( T \in W_1 \subset W_1 \oplus W_7 \) and suppose the deformation (6.64) results in \( \tilde{T} = 0 \). Thus here we have \( \tau_7 = \tau_{14} = \tau_{27} = 0 \) and \( \tilde{\tau}_1 = \tilde{\tau}_7 = \tilde{\tau}_{14} = \tilde{\tau}_{27} = 0 \). Then from Theorem 21 we have
\[ \nabla_a v_b = \tau_{1gab} - \frac{1}{3} \tau_1 v^c \varphi_{cab} \] (6.65)
and in particular,
\[ dv^b = -\frac{2}{3} \tau_1 v_a \varphi. \]
Now, the consistency condition \( d^2 v^b = 0 \) is equivalent to either \( \tau_1 = 0 \) or 

\[
d(v \wedge \varphi) = 0.
\]

Using (6.65) and the fact that 

\[
\nabla \varphi = \tau_1 \psi,
\]

we find that 

\[
\pi_1 (d (v \wedge \varphi)) = 3 \tau_1 \varphi = 0
\]

So we must have \( \tau_1 = 0 \), which gives a contradiction. Hence there are no deformations from torsion class \( W_1 \) to \( W_0 \).

Next we assume that \( T \in W_7 \subset W_1 \oplus W_7 \), so that only \( \tau_7 \) is non-vanishing. In this case, 

\[
\nabla_a v_b = (M + 9)^{-1} (-g_{ab} (\tau_7)_c v^c + 3 (1 + M) v_b (\tau_7)_a + (33 + M) v_a (\tau_7)_b
\]

\[
-3 \varphi_{ab} (\tau_7)_c (M - 3) + 4 (\tau_7)_c v^c \varphi_{abc} v^d - 24 \varphi_{d[a} (\tau_7)_c v^d v_b)
\]

\[
+12 \psi d_{ab} (\tau_7)_c v^d)
\]

and correspondingly we can also get \( dv^b \) from this. As before, we consider \( d (dv^b) \) and the projections of it on to \( \Lambda^3_1 \), \( \Lambda^3_2 \) and \( \Lambda^3_3 \). Let \( \xi_1 \) be the scalar corresponding to the \( \Lambda^3_1 \) projection, \( \xi_7 \) - the projection corresponding to the \( \Lambda^3_7 \) projection and \( \xi_{27} \) - the symmetric 2-tensor corresponding to the \( \Lambda^3_{27} \) component. As before, we can obtain scalars \( (\xi_7)_a v^a \) and \( (\xi_{27})_{ab} v^a v^b \). Hence we get three scalar equations

\[
0 = 16 (M - 15) ((\tau_7)_a v^a)^2 - 6 (3M^2 - 34M + 27) (\tau_7)_a (\tau_7)_a
\]

\[
- (M + 9)^2 \nabla^a (\tau_7)_a
\]

\[
0 = 4 (M - 3) ((\tau_7)_a v^a)^2 - 9M (M + 3) (\tau_7)_a (\tau_7)_a + (\nabla_a (\tau_7)_b) v^a v^b
\]

\[
-M \nabla^a (\tau_7)_a
\]

\[
0 = 6 (1 + M) (M - 39) ((\tau_7)_a v^a)^2 - 4M (5M - 3) (\tau_7)_a (\tau_7)_a
\]

\[
+2 (M - 3) (\nabla_a (\tau_7)_b) v^a v^b - 3M (1 + M) \nabla^a (\tau_7)_a
\]

We can solve these equations to get \( \nabla^a (\tau_7)_a \), \( (\nabla_a (\tau_7)_b) v^a v^b \) and \( (\tau_7)_a (\tau_7)_a = |\tau_7|^2 \) in terms of \( ((\tau_7)_a v^a)^2 = \langle \tau_7, v \rangle^2 \). So in particular, we get

\[
|\tau_7|^2 = \frac{3 \langle \tau_7, v \rangle^2 (3M^2 - 10M + 51)}{(7M^2 - 66M - 9) M}
\]

(6.70)

Further, from \( \xi_7^d = 0 \), \( \varphi_{abc} v^b \xi_7^c = 0 \), \( (\xi_{27})_{mn} v^n = 0 \) and \( \varphi_{abc} (\xi_{27})^{b} v^n v^c = 0 \), we actually find that

\[
v = \frac{M}{\langle \tau_7, v \rangle} \tau_7
\]

(6.71)

and after contracting with \( \tau_7 \), we get

\[
|\tau_7|^2 = \frac{\langle \tau_7, v \rangle^2}{M}
\]

(6.72)
Comparing (6.70) and (6.72), we get

\[ \langle \tau_7, v \rangle^2 (M + 9)^2 = 0 \]

Hence \( \langle \tau_7, v \rangle = 0 \) and so must have \( \tau_7 = 0 \). Therefore, there are no deformations from \( W_7 \) to \( W_0 \).

Finally, suppose \( T_{ab} \) lies in the strict class \( W_1 \oplus W_7 \), so that the \( W_1 \) component of the torsion is \( \tau_1 \) and the \( W_7 \) component is \( \tau_7 \). In this case, from Theorem 21, we have

\[
\nabla_a v_b = (\tau_1 - (\tau_7)_c v^c) g_{ab} + \frac{1}{(M + 9)} (-3 (M - 3) (\tau_7)_c \varphi^c_{ab})
- (M + 33) v_a (\tau_7)_b + 3 (1 + M) (\tau_7)_a v_b
- \frac{1}{3} v^c \varphi^b_{ac} \left( 9 \tau_1 + \tau_1 M - 12 (\tau_7)_d v^d \right)
+ 12 v_a \varphi^{cd}_{\ b} (\tau_7)_e v^e - 12 v_b \varphi^{cd}_{\ a} (\tau_7)_e v^e + 12 (\tau_7)_c v^c \psi_{cd}^{\ ab} \tag{6.73}
\]

Following the general procedure outlined above, we used Maple to expand the necessary condition (6.58). Again, as before, we consider the projections of \( d ((\tau_7)_c v^c + v_{14}) \). As outlined above we first consider the \( \pi_1, \pi_7 \) and \( \pi_{27} \) projections of \( d ((\tau_7)_c v^c + v_{14}) \).

Denote by \( \xi_1 \) the scalar corresponding to the \( \Lambda^1_1 \) component, let \( \xi_7 \) and \( \xi_{27} \) be the vector and symmetric tensor components. Then by considering the equations \( \xi_1 = 0, (\xi_7)_a v_a = 0 \) and \( (\xi_{27})_{mn} v^m v^n = 0 \), we can express \( (\nabla_a (\tau_7)_b) v^a v^b, (\nabla^a (\tau_7)_b)_a \) and \( |\tau_7|^2 \) in terms of \( M, \tau_1 \) and \( \langle \tau_7, v \rangle \). In particular, we find that

\[
|\tau_7|^2 = \frac{3 \langle \tau_7, v \rangle^2 (3 M^2 - 10 M + 51)}{(7 M^2 - 66 M - 9) M} - \frac{4 \tau_1 \langle \tau_7, v \rangle (M + 9)^2}{3 (7 M^2 - 66 M - 9)} + \frac{2 \tau_7^2 M (M + 9)^2}{97 M^2 - 66 M - 9} \tag{6.74}
\]

Further, we can consider the vector equations \( \xi_7^d = 0, \varphi_{abc} v^b \xi_7^c = 0, (\xi_{27})_{mn} v^m v^n = 0 \) and \( \varphi_{abc} (\xi_{27})^b \ v^m v^n = 0 \). From these, in particular, we find

\[
\tau_7 = \frac{\langle \tau_7, v \rangle}{M} v \tag{6.75}
\]

So as before, we get

\[
|\tau_7|^2 = \frac{\langle \tau_7, v \rangle^2}{M} \tag{6.76}
\]

Now if we equate (6.74) and (6.76), and then solve for \( \langle \tau_7, v \rangle \), we obtain an expression for \( \langle \tau_7, v \rangle \) in terms of \( \tau_1, \tau_7 \) and \( v \).

\[
\langle \tau_7, v \rangle = \frac{M \tau_1}{3} \tag{6.77}
\]

Hence,

\[
v = \frac{3}{\tau_1} \tau_7. \tag{6.78}
\]
and,

\[ M = \frac{9}{\tau_1^1} |\tau_7|^2 \]  
\[ \langle \tau_7, v \rangle = \frac{3}{\tau_1^1} |\tau_7|^2 \]  

Next, from equations \((\xi_{27})_{ab} = 0, \varphi^{cd}_{(a}(\xi_{27})_{b)d} v_c = 0\) and \(\varphi_a^{cd} \varphi_b^{ef} v_c v_e (\xi_{27})_{df} = 0\), we finally obtain an expression for \(\nabla_a (\tau_7)_b\). Using (6.78) and (6.79) to completely eliminate \(v\) from the resulting expression, we overall get:

\[ \nabla \tau_7 = \left( \frac{1}{3} \tau_1^2 - |\tau_7|^2 \right) g + 5\tau_7 \otimes \tau_7 \]  

By first considering the trace of this, we find that we get the condition

\[ \nabla_a \tau_7^a + 2 (\tau_7)_a (\tau_7)^a - \frac{7}{3} \tau_1^2 = 0 \]  

Recall however, that a \(G_2\)-structure in the strict torsion class \(W_1 \oplus W_7\) has

\[ \tau_7 = \nabla (\log \tau_1) \]

So we can rewrite (6.81) as

\[ \nabla^2 (\log \tau_1) + 2 |\nabla (\log \tau_1)|^2 - \frac{7}{3} \tau_1^2 = 0 \]  

Now note that if we let \(F = \tau_1^2\), then

\[ \nabla^2 F = \frac{14}{3} F^2 \]

Multiplying by \(F\), integrating over the whole manifold \(M\), and applying Stokes’s Theorem (since \(M\) is closed), we get

\[ \int_M F (\nabla^2 F) \, \text{vol} = - \int_M |\nabla F|^2 \, \text{vol} = \frac{14}{3} \int_M F^3 \sqrt{\det g} \, \text{vol} \]

However, \(F = \tau_1^2\) is a positive function, so the right-hand side is non-negative, while the left-hand side is non-positive, and we can only have equality when both sides vanish. This happens only if \(F = 0\) and this implies that both \(\tau_1\) and \(\tau_7\) vanish. Therefore we cannot have a deformation from \(W_1 \oplus W_7\) into \(W_0\).

**Corollary 24** *Given a deformation \(\varphi \rightarrow \varphi + v^e \psi_{bced}\) of a torsion-free \(G_2\)-structure \(\varphi\), the torsion of the new \(G_2\)-structure \(\tilde{\varphi}\) will necessarily have a non-trivial component in \(W_{14}\) or \(W_{27}\) unless \(\nabla v = 0\).*

**Proof.** Suppose the vector \(v\) defines a deformation a torsion-free \(G_2\)-structure results in a new \(G_2\)-structure with torsion \(\tilde{T}\) lying in \(W_1 \oplus W_7\), that is, there is no \(W_{14}\) or \(W_{27}\) component. Then by Lemma 19 there exists a corresponding deformation, defined by vector \(\tilde{v}\), in the opposite direction from \(W_1 \oplus W_7\) to a torsion-free \(G_2\)-structure. However by Theorem 23 such a deformation exists if and only if \(\nabla \tilde{v} = 0\) (and equivalently, by Lemma 19 \(\nabla v = 0\)) and the torsion \(\tilde{T}\) vanishes. ■
Theorem 25 There is no deformation of the form \((6.64)\) within the strict torsion class \(W_1\).

Proof. We consider a \(G_2\)-structure \((\varphi, g)\) where the only non-zero component of torsion \(T\) is \(\tau_1\). Suppose \((6.64)\) gives a deformation to a \(G_2\)-structure \((\tilde{\varphi}, \tilde{g})\) with torsion \(\tilde{T}\) with the only non-zero component being \(\tilde{\tau}_1\). Then from Theorem \([21]\)

\[
\nabla_a v_b = \left(\tau_1 - (1 + M)\frac{5}{2} \tilde{\tau}_1\right) g_{ab} + 4 (1 + M)^{-\frac{1}{3}} \tilde{\tau}_1 v_a v_b - \frac{1}{3} v^c \varphi_{cab} \left(\tau_1 - 4 (1 + M)^{-\frac{1}{3}} \tilde{\tau}_1\right)
\]

(6.84)

and in particular,

\[
dv^b = -\frac{2}{3} v^c \varphi \left(\tau_1 - 4 (1 + M)^{-\frac{1}{3}} \tilde{\tau}_1\right)
\]

(6.85)

Then we take \(d (dv^b)\), and decompose it into \(\Lambda_1^3\), \(\Lambda_2^3\) and \(\Lambda_3^3\) components. Since \(d (dv^b)\) must vanish, so must each of these components. We hence get the following equations:

\[
0 = \tau_1^2 - \frac{1}{21} \frac{9 M^2 + 106 M + 105}{(1 + M)^{\frac{4}{3}}} \tau_1 \tilde{\tau}_1 + 4 \frac{(15 M^2 + 21 + 28 M) \tilde{\tau}_1^2}{(1 + M)^{\frac{4}{3}}}
\]

(6.86a)

\[
0 = \left(\tau_1^2 - 5 \frac{\tau_1 \tilde{\tau}_1}{(1 + M)^{\frac{4}{3}}} + \frac{4 \tilde{\tau}_1^2}{(1 + M)^{\frac{4}{3}}}\right) v^a
\]

(6.86b)

\[
0 = \left(\tau_1^2 - \frac{1}{27} \frac{(15 M^2 + 14 M + 135) \tau_1 \tilde{\tau}_1}{(1 + M)^{\frac{4}{3}}} + 4 \frac{(21 M^2 + 10 M + 27) \tilde{\tau}_1^2}{(1 + M)^{\frac{4}{3}}}
\]

\[\]

+ \left(\frac{8}{27} \frac{(3 M + 5) \tau_1 \tilde{\tau}_1}{(1 + M)^{\frac{4}{3}}} - \frac{16}{27} \frac{(3 M + 7) \tilde{\tau}_1^2}{(1 + M)^{\frac{4}{3}}}\right) v_a v_b
\]

(6.86c)

Now if we contract \((6.86b)\) with \(v_a\) and \((6.86c)\) with \(v^a v^b\), we get three scalar equations:

\[
0 = \tau_1^2 - \frac{1}{21} \frac{(9 M^2 + 106 M + 105) \tau_1 \tilde{\tau}_1}{(1 + M)^{\frac{4}{3}}} + 4 \frac{(15 M^2 + 21 + 28 M) \tilde{\tau}_1^2}{(1 + M)^{\frac{4}{3}}}
\]

(6.87)

\[
0 = \tau_1^2 - 5 \frac{\tau_1 \tilde{\tau}_1}{(1 + M)^{\frac{4}{3}}} + \frac{4 \tilde{\tau}_1^2}{(1 + M)^{\frac{4}{3}}}
\]

(6.88)

\[
0 = \tau_1^2 + \frac{1}{9} \frac{(3 M^2 - 34 M - 45) \tau_1 \tilde{\tau}_1}{(1 + M)^{\frac{4}{3}}} + 4 \frac{(3 M^2 + 4 M + 9) \tilde{\tau}_1^2}{(1 + M)^{\frac{4}{3}}}
\]

(6.89)

Here our unknowns are \(\tau_1^2\), \(\tau_1 \tilde{\tau}_1\) and \(\tilde{\tau}_1^2\). The determinant of the system is \(\frac{32 M^2}{21 (1 + M)} > 0\), so the only solution is \(\tau_1 = \tilde{\tau}_1 = 0\). \(\blacksquare\)

7 Concluding remarks

In this paper we have studied the deformations of \(G_2\) structures on 7-manifolds. Given a general deformation \(\chi\) of a \(G_2\)-structure \((\varphi, g)\), we obtained a new \(G_2\)-structure \((\tilde{\varphi}, \tilde{g})\), and for this new \(G_2\)-structure we calculated its torsion tensor \(\tilde{T}\) in terms of the old \(G_2\)-structure \((\varphi, g)\), its torsion \(T\) and the deformation \(\chi\). We then specialized to \(\chi\) lying in \(\Lambda_2^3\), given by \(\chi = -v \varphi\). For such a deformation, in Theorem \([20]\) we computed \(\tilde{T}\) in terms of \(v\) and its derivatives. So then, given a \(G_2\)-structure \((\varphi, g)\) with particular torsion components \(\tau_i\), we found the equations that \(v\) must
satisfy so that the the torsion of \((\tilde{\varphi}, \tilde{g})\) has particular components \(\tilde{\tau}_i\). For particular cases of \(\tau_i\) and \(\tilde{\tau}_i\), we analysed these equations. In particular, we found that a deformation within the zero torsion class is possible if and only if \(\nabla v = 0\). In other cases, it was found that there are no deformations of this type that take the torsion from the classes \(W_1, W_7\) and \(W_1 \oplus W_7\) to zero. Also, it was found that there are no deformations which take the \(W_1\) torsion class to itself. Although these are all mostly negative results, since deformations in \(\Lambda^3_7\) are invertible, we can conclude that any such deformation with non-zero \(\nabla v\) from a torsion-free \(G_2\)-structure will yield a \(G_2\)-structure whose torsion contains at least a \(W_{14}\) or \(W_{27}\) component.

So far we have developed a technique for computing the deformed torsion, and of course there is a significant amount of work to be done to fully understand deformations of other torsion classes. Deformations that lie in \(\Lambda^3_7\) are of course the simplest possible deformations, apart from conformal deformations, since they are defined by just a vector. Also, as we note in Section 6 such a deformation of a positive 3-form still yield a positive 3-form, which does define a \(G_2\)-structure. The ultimate aim would be to make sense of non-infinitesimal deformations that lie in \(\Lambda^3_{27}\). These are then defined by traceless symmetric tensors, and moreover, not all such deformations yield positive 3-forms, so extra conditions need to be imposed. On the other hand, these deformations have many more degrees of freedom than the \(\Lambda^3_7\) deformations, so we could expect to get more interesting results and unlock many of the mysteries of \(G_2\) manifolds. In particular, one of the aims would be to show if there are any obstructions to deformations of \(G_2\) holonomy manifolds. A even more ambitious program would be to try and understand which \(G_2\)-structures exist on a given manifold and what is the smallest torsion class.

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