ON EIGENFUNCTION EXPANSIONS OF DIFFERENTIAL EQUATIONS WITH DEGENERATING WEIGHT

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Abstract. Let $A$ be a symmetric operator. By using the method of boundary triplets we parameterize in terms of a Nevanlinna parameter $\tau$ all exit spaces $\overline{A} = \overline{A}^*$ of $A$ with the discrete spectrum $\sigma(A)$ and characterize the Shtraus family of $\overline{A}$ in terms of abstract boundary conditions. Next we apply these results to the eigenvalue problem for the $2r$-th order differential equation $l[y] = \lambda \Delta(x)y$ on an interval $[a, b]$, $-\infty < a < b \leq \infty$, subject to $\lambda$-depending separated boundary conditions with entire operator-functions $C_0(\lambda)$ and $C_1(\lambda)$, which form a Nevanlinna pair $(C_0, C_1)$. The weight $\Delta(x)$ is nonnegative and may vanish on some intervals $(\alpha, \beta) \subset I$. We show that in the case when the minimal operator of the equation has the discrete spectrum (in particular, in the case of the quasiregular equation) the set of eigenvalues of the eigenvalue problem is an infinite subset of $\mathbb{R}$ without finite limit points and each function $y \in L^2_\Delta(I)$ admits the eigenfunction expansion $y(x) = \sum_{k=1}^{\infty} y_k(x)$ converging in $L^2_\Delta(I)$. Moreover, we give an explicit method for calculation of eigenfunctions $y_k$ in this expansion and specify boundary conditions on $y$ implying the uniform convergence of the eigenfunction expansion of $y$. These results develop the known ones obtained for the case of the positive weight $\Delta$ and the more restrictive class of $\lambda$-depending boundary conditions.

1. Introduction

We study the eigenvalue problem for the differential equation of an even order $2r$

\begin{equation}
 l[y] = \sum_{k=0}^{r} (-1)^k \left( p_{r-k}(x) y^{(k)} \right)^{(k)} = \lambda \Delta(x)y, \quad x \in I = [a, b], \quad -\infty < a < b \leq \infty
\end{equation}

subject to separated boundary conditions

\begin{equation}
 (\cos B) y^{(1)}(a) + (\sin B) y^{(2)}(a) = 0, \quad C_0(\lambda) \Gamma_{0b} y + C_1(\lambda) \Gamma_{1b} y = 0
\end{equation}

depending on the parameter $\lambda \in \mathbb{C}$. It is assumed that the coefficients $p_j$ and the weight $\Delta$ in (1.1) are real-valued functions on an interval $I = [a, b]$ such that $\Delta(x) \geq 0$ a.e. on $I$ and $p_0^{-1}$, $p_1, \ldots, p_r, \Delta$ are integrable on each compact interval $[a, b'] \subset I$ (the latter means that the endpoint $a$ is regular). Below we denote by $B(\mathbb{C}^m)$ the set of all linear operators in $\mathbb{C}^m$ or equivalently the set of all $m \times m$-matrices. In (1.2) $B = B^* \in B(\mathbb{C}^r)$, $y^{(1)}$ and $y^{(2)}$ are vectors of quasi-derivatives of a function $y$, $\Gamma_{ab} y \in \mathbb{C}^m$ are singular boundary values of $y^{(j)}$ at the endpoint $b$ and $C_0, C_1$ are entire $B(\mathbb{C}^m)$-valued functions which form a Nevanlinna pair $(C_0, C_1)$. The particular case of (1.2) are the following selfadjoint separated boundary conditions with $B_1 = B_1^* \in B(\mathbb{C}^{d-r})$:

\begin{equation}
 (\cos B) y^{(1)}(a) + (\sin B) y^{(2)}(a) = 0, \quad (\cos B_1) \Gamma_{0b} y + (\sin B_1) \Gamma_{1b} y = 0
\end{equation}

Note that eigenfunction expansions generated by problems of the type (1.1), (1.2) have been studied by many authors (see, e.g., [3, 7, 10, 12, 23] and their references).

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Denote by $L^2_\Delta(I)$ the Hilbert space of all functions $f$ on $I$ such that $\int_I \Delta(x)|f(x)|^2\,dx < \infty$ and by $(\cdot, \cdot)_\Delta$ the inner product in $L^2_\Delta(I)$. Recall that equation (1.1) is called quasiregular if each solution $y$ of (1.1) belongs to $L^2_\Delta(I)$ and regular if $I = [a, b]$ is a compact interval (clearly, each regular equation is quasiregular). For regular equation one can put in (1.2) and (1.3) $\Gamma_{ab} = y^{(1)}(b)$ and $\Gamma_{1b} = y^{(2)}(b)$.

**Definition 1.1.** The weight $\Delta$ in (1.1) is called positive, if $\Delta(x) > 0$ a.e. on $I$ and nontrivial, if the set $\{x \in I : \Delta(x) > 0\}$ has the positive Borel measure.

Clearly, non triviality is the weakest restriction on $\Delta$, which saves the interest to studying of (1.1).

According to [20] boundary problem (1.1), (1.3) for equation (1.1) with the nontrivial weight $\Delta$ generates an operator $T = T^*$ in $L^2_\Delta(I)$ (for the case of the positive Weight see e.g. [24]). Equation (1.1) is said to have the discrete spectrum if for some (and hence any) boundary problem (1.1), (1.3) the operator $T$ has the discrete spectrum.

An eigenvalue of the problem (1.1), (1.2) is defined as $\lambda \in \mathbb{C}$ for which this problem has a solution $y \in L^2_\Delta(I)$, $y \neq 0$; this solution is called an eigenfunction. The following eigenfunction expansion theorem is the classical result for the selfadjoint eigenvalue problem (1.1), (1.3) (see e.g. [4, 9]).

**Theorem 1.2.** Assume that equation (1.1) with the positive weight $\Delta$ has the discrete spectrum. Then:

(i) The set of all eigenvalues of the problem (1.1), (1.3) is an infinite countable subset $\{t_k\}$ of $\mathbb{R}$ without finite limit points. Moreover, eigenfunctions for different $t_k$ are mutually orthogonal in $L^2_\Delta(I)$.

(ii) Each absolutely continuous function $y \in L^2_\Delta(I)$ with absolutely continuous quasiderivatives $y^{[k]}$ satisfying $\Delta^{-1}[y] \in L^2_\Delta(I)$ and the boundary conditions (1.3) admits the eigenfunction expansion

$$y(x) = \sum_{k=1}^{\infty} y_k(x),$$

which converges absolutely and uniformly on each compact interval $[a, b^'] \subset I$. In (1.4) $y_k(x) = (y, v_k)_{\Delta} v_k(x)$, where $\{v_k\}$ is an orthonormal system of eigenfunctions.

F. Atkinson in [1, Theorem 8.9.1] extended Theorem 1.2 to regular Sturm-Liouville equations

$$l[y] = -(p(x)y')' + q(x)y = \lambda \Delta(x)y, \quad x \in I = [a, b]$$

such that $0 \leq p(x) \leq \infty$ and the following conditions are satisfied: (i) $\Delta(x) \geq 0$ a.e. on $I$; (ii) there is no interval $(a, b') \subset I$ such that $\Delta(x) = 0$ a.e. on $(a, b')$ and there is no interval $(a', b) \subset I$ such that $\Delta(x) = 0$ a.e. on $(a', b)$; (iii) if $\Delta(x) = 0$ a.e. on an interval $(a', b') \subset I$, then $q(x) = 0$ a.e. on $(a', b')$.

Eigenfunction expansions for the regular Sturm-Liouville equations (1.5) with the positive weight $\Delta$ subject to the $\lambda$-depending boundary conditions of the special form

$$\cos B \cdot y(a) + \sin B \cdot (py')(a) = 0, \quad (M_0 - \lambda N_0)y(b) + (M_1 - \lambda N_1) \cdot (py')(b) = 0$$

were studied in the papers [23, 10, 12]. It is assumed there that $p > 0$ and the coefficients $M_j$ and $N_j$ in (1.6) are such that $(M_0 - \lambda N_0)(M_1 - \lambda N_1)^{-1}$ is a Nevanlinna function. In these papers problem (1.5), (1.6) is associated with a certain self-adjoint operator $\tilde{A}$ in a Hilbert space $H \supset L^2_\Delta(I)$. It follows from the results of [10, 12] that eigenvalues of the problem (1.5), (1.6) form a strictly increasing unbounded sequence $\{\lambda_k\}$, $\lambda_k \in \mathbb{R}$, and each function $y \in AC(I)$ such that $py' \in AC(I)$ and $\Delta^{-1}l[y] \in L^2_\Delta(I)$ admits the eigenfunction expansion (1.4) converging absolutely and uniformly on $I$. The results
of [10, 12] were extended to boundary problems for regular equations (1.1) of an order 2r with the positive weight $\Delta$ subject to linear boundary conditions of the type (1.6) in [3] and boundary conditions (1.2) with polynomial matrices $C_j(\lambda)$ in [7]. Note that eigenfunctions of the problem (1.5), (1.6) are not mutually orthogonal and the method of [3, 7, 10, 12] does not give explicit formulas for calculation of $y_k$ in (1.4).

In the present paper the above results are developed in the following two directions:

(i) Instead of positivity we assume that the weight $\Delta \geq 0$ in (1.1) is nontrivial, so that the equality $\Delta = 0$ may hold on some intervals $(a, \beta) \subset I$.

(ii) It is assumed that $C_j(\lambda)$ in the boundary conditions (1.2) are entire operator functions which form a Nevanlinna pair $(C_0, C_1)$. This assumption generalizes the known results to a wider class of $\lambda$-depending boundary conditions.

In the case of the Sturm-Liouville equation (1.5) our main results can be formulated in the form of the following two theorems.

**Theorem 1.3.** Assume that equation (1.5) with the nontrivial weight $\Delta$ is quasiregular.

Let $D_{\text{max}}$ be the set of all functions $y \in AC(I) \cap L^2_\Delta(I)$ such that $y^{[1]} := py' \in AC(I)$ and $-(y^{[1]})' + qu = \Delta(x)f_y$ (a.e. on $I$) with some $f_y \in L^2_\Delta(I)$. For a given $B \in \mathbb{R}$ denote by $\varphi_B(\cdot, \lambda)$ and $\psi_B(\cdot, \lambda)$ the solutions of (1.5) with $\varphi_B(a, \lambda) = \sin B$, $\varphi_B^{[1]}(a, \lambda) = -\cos B$ and $\psi_B(a, \lambda) = \cos B$, $\psi_B^{[1]}(a, \lambda) = \sin B$. Denote also by $\Gamma_{0b}$ and $\Gamma_{1b}$ singular boundary values of a function $y \in D_{\text{max}}$ at the endpoint $b$ given by

$$
\Gamma_{0b}y = \lim_{x \to b} (\psi_B^{[1]}(x, 0)y(x) - \psi_B(x, 0)y^{[1]}(x)), \quad \Gamma_{1b}y = \lim_{x \to b} (-\psi_B^{[1]}(x, 0)y(x) + \varphi_B(x, 0)y^{[1]}(x)).
$$

Let $C = (C_0, C_1)$ be an entire Nevanlinna pair, i.e., a pair of entire functions $C_0$ and $C_1$ without common zeros and such that

$$
\text{Im} \lambda \cdot \text{Im} (C_1(\lambda)\overline{C_0(\lambda)}) \geq 0, \quad C_1(\lambda)\overline{C_0(\lambda)} - \overline{C_1(\lambda)}C_0(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
$$

Consider the eigenvalue problem for equation (1.5) subject to the boundary conditions

$$
\cos B \cdot y(a) + \sin B \cdot y^{[1]}(a) = 0, \quad C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0.
$$

and let $EV$ be the set of all eigenfunctions of this problem. Then:

(i) $EV = \{t_k\}$ is an infinite countable subset of $\mathbb{R}$ without finite limit points and the linear-fractional transform

$$
m(\lambda) = \frac{w_2(\lambda)C_0(\lambda) + w_4(\lambda)C_1(\lambda)}{w_1(\lambda)C_0(\lambda) + w_3(\lambda)C_1(\lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
$$

with the coefficients

$$
w_1(\lambda) = 1 + \lambda \int_I \varphi_B(x, 0)\Delta(x)\varphi_B(x, \lambda) \, dx, \quad w_2(\lambda) = \lambda \int_I \psi_B(x, 0)\Delta(x)\psi_B(x, \lambda) \, dx
$$

$$
w_3(\lambda) = -\lambda \int_I \varphi_B(x, 0)\Delta(x)\varphi_B(x, \lambda) \, dx, \quad w_4(\lambda) = 1 - \lambda \int_I \varphi_B(x, 0)\Delta(x)\psi_B(x, \lambda) \, dx.
$$

defines a meromorphic Nevanlinna function $m$ such that $EV$ coincides with the set of all poles of $m$.

(ii) Assume that $y \in L^2_\Delta(I)$, $\hat{y}_k = \int_I \varphi_B(x, t_k)\Delta(x)y(x) \, dx$ is the Fourier coefficient of $y$ and $\hat{\xi}_k$ is the residue of $m$ at the pole $t_k$. Then the equality

$$
y_k(x) = -\hat{\xi}_k \hat{y}_k \varphi_B(x, t_k)
$$

defines eigenfunctions $y_k$ of the problem (1.5), (1.7) such that eigenfunction expansion (1.4) of $y$ converging in $L^2_\Delta(I)$ is valid.

If $C = (C_0, C_1)$ is an entire Nevanlinna pair and $C_1(\lambda) \neq 0, \lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\tau := -C_0C_1^{-1}$ is a Nevanlinna function. Therefore there exist the limits $B_\infty =$
\[ \lim_{y \to \infty} \tau(iy) \in \mathbb{R}, \quad \hat{D}_\infty = \lim_{y \to \infty} y \text{Im} \tau(iy) \leq \infty \] and one of the following alternative cases holds:

Case 1. \( \mathcal{B}_\infty \neq 0 \); Case 2. \( \mathcal{B}_\infty = 0 \) and \( \hat{D}_\infty < \infty \); Case 3. \( \mathcal{B}_\infty = 0 \) and \( \hat{D}_\infty = \infty \). Moreover, in Case 2 there exists the limit \( D_\infty = \lim_{y \to \infty} \tau(iy) \).

**Theorem 1.4.** Let under the assumptions of Theorem 1.3 \( C_1(\lambda) \neq 0 \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), and let a function \( y \in \mathcal{D}_{\text{max}} \) satisfies one of the following boundary conditions (1.10) – (1.12) depending on Cases 1 – 3:

\begin{align}
(1.10) & \quad \text{in Case 1:} \quad \cos B \cdot y(a) + \sin B \cdot y^{[1]}(a) = 0, \quad \Gamma_{0b} y = 0; \\
(1.11) & \quad \text{in Case 2:} \quad \cos B \cdot y(a) + \sin B \cdot y^{[1]}(a) = 0, \quad \Gamma_{1b} y = D_\infty \Gamma_{0b} y; \\
(1.12) & \quad \text{in Case 2:} \quad \cos B \cdot y(a) + \sin B \cdot y^{[1]}(a) = 0, \quad \Gamma_{0b} y = \Gamma_{1b} y = 0.
\end{align}

Then the eigenfunction expansion (1.4) of \( y \) converges absolutely and uniformly on each compact interval \([a, b]\) \( \subseteq \mathcal{I} \).

**Remark 1.5.** If in addition to the assumptions of Theorems 1.3 and 1.4 the equation (1.5) is regular, then statements of these theorems are valid with \( \Gamma_{0b} y = y(b), \quad \Gamma_{1b} y = y^{[1]}(b) \) and the equality (1.8) should be replaced with

\[ (1.13) \quad m(\lambda) = \frac{\psi_B(b, \lambda) C_0(\lambda) + \psi_B^{[1]}(b, \lambda) C_1(\lambda)}{\varphi_B(b, \lambda) C_0(\lambda) + \varphi_B^{[1]}(b, \lambda) C_1(\lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

Similar theorems are proved in the paper for equation (1.1) of an even order \( 2r \) with the nontrivial weight \( \Delta \) and the discrete spectrum. We study eigenvalue problem for this equation subject to the boundary conditions (1.2) with an entire Nevanlinna pair \( C = (C_0, C_1) \), i.e., a pair of entire operator functions \( C_0 \) and \( C_1 \) such that \( \text{ran} C(\lambda) = \mathbb{C}^m \), \( i \text{Im} \lambda \cdot C(\lambda) J C^*(\lambda) \leq 0 \) and \( C(\lambda) J C^*(\overline{\lambda}) = 0 \), \( \lambda \in \mathbb{C} \), with \( J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \). The operator functions \( C_0 \) and \( C_1 \) take on values in \( \mathcal{B}(\mathbb{C}^m) \), where \( m = d - r \) and \( d \) is the deficiency index of the equation (1.1). We show that the set of eigenvalues of the problem (1.1), (1.2) is an infinite countable subset of \( \mathbb{R} \) without finite limit points and each function \( y \in L^2_\Delta(\mathcal{I}) \) admits the eigenfunction expansion (1.4) converging in \( L^2_\Delta(\mathcal{I}) \) (the eigenfunctions \( y_k \) in (1.4) are not generally speaking mutually orthogonal). Moreover, similarly to Theorems 1.3 and 1.4 we give an explicit method for calculation of \( y_k \) in terms of the original entire Nevanlinna pair \( C = (C_0, C_1) \) and specify boundary conditions on \( y \) implying the uniform convergence of (1.4) (see Theorems 4.9, 4.11 and 4.12). In the case of the selfadjoint eigenvalue problem (1.1), (1.3) these results extend Theorem 1.2 to equations (1.1) with the nontrivial weight \( \Delta \).

Our approach is based on the method of boundary triplets in the extension theory of a symmetric operator \( A \) in the Hilbert space \( \mathfrak{H} \) (see [2, 6, 11] and references therein). By using this method we parameterize in terms of a Nevanlinna parameter \( \tau \) all exit space extensions \( \tilde{A} \) \( = \tilde{A}^* \) of \( A \) with the discrete spectrum \( \sigma(\tilde{A}) \). Moreover, we characterize the Shtraus family of the extension \( \tilde{A} \) in terms of abstract boundary conditions and describe extensions \( \tilde{A} \) with certain special properties of their Shtraus family. These results enables us to define an abstract eigenvalue problem for \( A^* \), which generates the abstract eigenvector expansion in \( \mathfrak{H} \). In the case when \( A \) is the minimal operator of the equation (1.1) the abstract eigenvalue problem for \( A^* \) takes the form (1.1), (1.2) and the abstract eigenvector expansion turns into (1.4).

In conclusion note that this paper is a continuation of [20], where ”continuous” eigenfunction expansions in the form of the generalized Fourier transform were considered.
2. Preliminaries

2.1. Some notations and definitions. The following notations will be used throughout the paper: \( \mathfrak{H}, \mathcal{H} \) denote separable Hilbert spaces; \( B(\mathcal{H}_1, \mathcal{H}_2) \) is the set of all bounded linear operators defined on \( \mathcal{H}_1 \) with values in \( \mathcal{H}_2 \); \( B(\mathcal{H}) = B(\mathfrak{H}, \mathcal{H}) \); \( A \upharpoonright \mathcal{L} \) is a restriction of the operator \( A \in B(\mathcal{H}_1, \mathcal{H}_2) \) to the linear manifold \( \mathcal{L} \subset \mathcal{H}_1 \); \( P_{\mathcal{L}} \) is the orthoprojection in \( \mathfrak{H} \) onto the subspace \( \mathcal{L} \subset \mathfrak{H} \); \( \mathbb{C}_+ (\mathbb{C}_-) \) is the open upper (lower) half-plane of the complex plane.

A non-decreasing strongly left continuous operator function \( \xi : \mathbb{R} \to B(\mathcal{H}) \) with \( \xi(0) = 0 \) is called a distribution function or shortly a distribution.

We denote by \( \mathcal{F} \) the class of all sets \( F \subset \mathbb{R} \) satisfying at least one (and hence all) of the following equivalent conditions: (i) the set \( F \cap [a, b] \) is finite or empty for any compact interval \([a, b] \subset \mathbb{R} \); (ii) \( F \) is closed and \( F \) has no finite limit points. Clearly, each set \( F \in \mathcal{F} \) admits the representation in the form of a finite or infinite increasing sequence \( F = \{ t_k \}_k \), where \( \nu_{-} \in \mathbb{Z} \cup \{-\infty\} \), \( \nu_{+} \in \mathbb{Z} \cup \{\infty\} \), \( -\infty \leq \nu_{-} \leq \nu_{+} \leq \infty \) and \( k \) is an integer between \( \nu_{-} \) and \( \nu_{+} \). We assume that \( \emptyset \in \mathcal{F} \).

Recall that a linear manifold \( T \) in the Hilbert space \( \mathcal{H}_0 \oplus \mathcal{H}_1 \) (resp. \( \mathfrak{H} \)) is called a linear relation from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) (resp. \( \mathfrak{H} \)). The set of all closed linear relations from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) (in \( \mathfrak{H} \)) will be denoted by \( \tilde{C}(\mathcal{H}_0, \mathcal{H}_1) \) (resp. \( \tilde{C}(\mathfrak{H}) \)). Clearly for each linear operator \( T : \text{dom} \, T \to \mathcal{H}_1 \), \( \text{dom} \, T \subset \mathcal{H}_0 \), its graph \( \text{gr} \, T = \{ \{ f, T f \} : f \in \text{dom} \, T \} \) is a linear relation from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \). This fact enables one to consider an operator \( T \) as a linear relation. In the following we denote by \( \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) \) the set of all closed linear operators \( T : \text{dom} \, T \to \mathcal{H}_1 \) with \( \text{dom} \, T \subset \mathcal{H}_0 \). Moreover, we let \( \mathcal{C}(\mathfrak{H}) = \mathcal{C}(\mathcal{H}, \mathcal{H}) \).

For a linear relation \( T \) from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) we denote by \( \text{dom} \, T \), \( \ker \, T \), \( \text{ran} \, T \) and \( \text{mul} \, T := \{ h_1 \in \mathcal{H}_1 : \{ 0, h_1 \} \subset \mathcal{H}_1 \} \) the domain, kernel, range and multivalued part of \( T \) respectively; moreover, we let \( \text{mul} \, T = \{ 0 \} \oplus \text{mul} \, T \). Clearly \( T \) is an operator if and only if \( \text{mul} \, T = \{ 0 \} \).

Denote also by \( T^{-1} \) and \( T^* \) the inverse and adjoint linear relations of \( T \) respectively.

A linear relation \( T \) in \( \mathfrak{H} \) is called symmetric (self-adjoint) if \( T \subset T^* \) (resp. \( T = T^* \)).

As is known, for a symmetric relation \( T \in \tilde{C}(\mathfrak{H}) \) the decompositions

\[
\mathfrak{H} = \mathfrak{H}_0 \oplus \text{mul} \, T, \quad T = \text{gr} \, T_{\text{op}} \oplus \widehat{\text{mul} \, T}
\]

hold with \( \mathfrak{H}_0 = \mathfrak{H} \oplus \text{mul} \, T \) and a symmetric operator \( T_{\text{op}} \in \mathcal{C}(\mathfrak{H}_0) \) (the operator part of \( T \)). Clearly, \( T^* = T \) if and only if \( T_{\text{op}} = T_{\text{op}} \).

For a linear relation \( T \in \tilde{C}(\mathfrak{H}) \) and \( \lambda \in \mathbb{C} \) we let

\[
\mathcal{N}_\lambda(T) = \ker \left( T - \lambda \right) = \{ f \in \mathfrak{H} : \{ f, \lambda f \} \subset \mathfrak{H} \}, \quad \widehat{\mathcal{N}_\lambda(T)} = \{ \{ f, \lambda f \} : f \in \mathcal{N}_\lambda(T) \}.
\]

Moreover, we use the following notations: \( \rho(T) = \{ \lambda \in \mathbb{C} : (T - \lambda)^{-1} \in B(\mathfrak{H}) \} \) is the resolvent set, \( \sigma(T) = \mathbb{C} \setminus \rho(T) \) is the spectrum, \( \sigma_{p}(T) = \{ \lambda \in \mathbb{C} : \mathcal{N}_\lambda(T) \neq 0 \} \) is the point spectrum and \( \widehat{\sigma}(T) = \{ \lambda \in \mathbb{C} : \mathcal{N}_\lambda(T) = \{ 0 \} \} \) and \( \text{ran} \left( T - \lambda \right) \) is closed is the set of regular type points of \( T \).

A subspace \( \mathfrak{H}_1 \subset \mathfrak{H} \) reduces a relation \( T \in \tilde{C}(\mathfrak{H}) \) if \( T = T_1 \oplus T_2 \), where \( T_1 \in \tilde{C}(\mathfrak{H}_1) \) and \( T_2 \in \tilde{C}(\mathfrak{H} \ominus \mathfrak{H}_1) \).

For an operator \( T = T^* \in B(\mathfrak{H}) \) we write \( T \geq 0 \) if \( \langle T f, f \rangle \geq 0 \), \( f \in \mathfrak{H} \), and \( T > 0 \) if \( T - \alpha I \geq 0 \) with some \( \alpha > 0 \).

2.2. Nevanlinna functions. Recall that a holomorphic operator function \( \tau : \mathbb{C} \setminus \mathbb{R} \to B(\mathfrak{H}) \) is called a Nevanlinna function if \( \text{Im} \, \lambda \cdot \text{Im} \, \tau(\lambda) \geq 0 \) and \( \tau^*(\lambda) = \overline{\tau(\lambda)} \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). The class of all Nevanlinna \( B(\mathfrak{H}) \)-valued functions will be denoted by \( \mathcal{R}[\mathfrak{H}] \).
As is known (see e.g. [2, 6]), an operator function $\tau : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H})$ belongs to the class $R[\mathcal{H}]$ if and only if it admits the integral representation

$$
\tau(\lambda) = A + \lambda B + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\xi(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
$$

with self-adjoint operators $A$, $B \in \mathcal{B}(\mathcal{H})$, $B \geq 0$, and a distribution $\xi : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H})$ such that $\int_{\mathbb{R}} \left( t^2 + 1 \right)^{-1} (d\xi(t)h, h) < \infty$, $h \in \mathcal{H}$. The distribution $\xi$ in (2.2) is called a spectral function of $\tau$ (it defines uniquely by $\tau$).

The operator-function $\tau \in R[\mathcal{H}]$ belongs to the class: (i) $R_c[\mathcal{H}]$, if $\text{ran Im } \tau(\lambda)$ is closed for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$; (ii) $R_w[\mathcal{H}]$ if $\text{Im } \lambda \cdot \text{Im } \tau(\lambda) > 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Clearly, in the case $\dim \mathcal{H} < \infty$ one has $R_c[\mathcal{H}] = R[\mathcal{H}]$.

The following proposition is well known (see e.g. [17]).

**Proposition 2.1.** Let $\tau \in R[\mathcal{H}]$. Then:

(i) The equalities

$$
B_{\tau,\infty} = s\lim_{y \to 0} \frac{1}{iy} \tau(iy),
$$

$$
dom D_{\tau,\infty} = \{ h \in \mathcal{H} : \lim_{y \to \infty} y \text{Im } (\tau(iy)h, h) < \infty \}, \quad D_{\tau,\infty}h = \lim_{y \to \infty} \tau(iy)h, \quad h \in \text{dom } D_{\tau,\infty}
$$

correctly define the operator $B_{\tau,\infty} \in \mathcal{B}(\mathcal{H})$, $B_{\tau,\infty} \geq 0$, the (not necessarily closed) linear manifold $\text{dom } D_{\tau,\infty} \subset \mathcal{H}$ and the linear operator $D_{\tau,\infty} : \text{dom } D_{\tau,\infty} \to \mathcal{H}$. Moreover, $\text{dom } D_{\tau,\infty} \subset \text{ker } B_{\tau,\infty}$.

(ii) For each $t \in \mathbb{R}$ the equalities

$$
B_\tau(t) = s\lim_{y \to 0} (-iy) \tau(t + iy)), \quad \text{dom } D_\tau(t) = \{ h \in \mathcal{H} : \lim_{y \to 0} \frac{1}{y} \text{Im } (\tau(t + iy)h, h) < \infty \}
$$

$$
D_\tau(t)h = \lim_{y \to 0} \tau(t + iy)h, \quad h \in \text{dom } D_\tau(t)
$$

correctly define the operator $B_\tau(t) \in \mathcal{B}(\mathcal{H})$, $B_\tau(t) \geq 0$, the (not necessarily closed) linear manifold $\text{dom } D_\tau(t) \subset \mathcal{H}$ and the linear operator $D_\tau(t) : \text{dom } D_\tau(t) \to \mathcal{H}$.

**Definition 2.2.** [6, 2] A function $\tau : \mathbb{C} \setminus \mathbb{R} \rightarrow \tilde{\mathcal{C}}(\mathcal{H})$ is referred to the class $\tilde{R}(\mathcal{H})$ of Nevanlinna relation valued functions if: (i) $\text{Im } (f^t, f') \geq 0$, $\{ f, f' \} \in \tau(\lambda)$, $\lambda \in \mathbb{C}_+$; (ii) $(\tau(\lambda) + i)^{-1} \in \mathcal{B}(\mathcal{H})$ for any $\lambda \in \mathbb{C}_+$ and $(\tau(\lambda) + i)^{-1}$ is a holomorphic operator-function in $\mathcal{C}_+$; (iii) $\tau^*(\lambda) = \tau(\overline{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

It is clear that $R[\mathcal{H}] \subset \tilde{R}(\mathcal{H})$.

According to [14] for each function $\tau$ in $\tilde{R}(\mathcal{H})$ the multivalued part $\mathcal{K} := \text{mul } \tau(\lambda)$ of $\tau(\lambda)$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and for each such $\lambda$ the decompositions

$$
\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{K}, \quad \tau(\lambda) = \tau_0(\lambda) \oplus \tilde{\mathcal{K}} = \{ h_0, \tau_0(\lambda)h_0 \oplus k \} : h_0 \in \mathcal{H}_0, k \in \mathcal{K}
$$

hold with subspaces $\mathcal{H}_0 = \mathcal{H} \oplus \mathcal{K}$, $\tilde{\mathcal{K}} = \{ 0 \} \oplus \mathcal{K}$ and a function $\tau_0 \in \tilde{R}(\mathcal{H}_0)$, whose values are densely defined operators. The operator function $\tau_0$ is called the operator part of $\tau$.

A function $\tau$ in $\tilde{R}(\mathcal{H})$ will be referred to the class $\tilde{R}_c(\mathcal{H})$ if $\tau_0 \in R_c[\mathcal{H}_0]$. In the case $\dim \mathcal{H} < \infty$ one has $\tilde{R}_c(\mathcal{H}) = \tilde{R}(\mathcal{H})$.

2.3. **Meromorphic Nevanlinna functions and entire Nevanlinna pairs.** Below within this subsection we assume that $\mathcal{H}$ is a Hilbert space with $\dim \mathcal{H} < \infty$.

**Definition 2.3.** A distribution $\xi : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H})$ will be called discrete if there exist sequences $F = \{ t_k \}_{k=0}^\infty \in \mathcal{F}$ and $\Xi = \{ \xi_k \}_{k=0}^\infty$ with $0 \leq \xi_k \in \mathcal{B}(\mathcal{H})$, $\xi_k \neq 0$, such that $\xi(t)$ is constant on each interval $(t_{k-1}, t_k)$ and $\xi(t_k + 0) - \xi(t_k) = \xi_k$. For a discrete distribution $\xi$ defined by sequences $F$ and $\Xi$ we write $\xi = \{ F, \Xi \}$. 
Definition 2.4. An operator-valued function \( \tau_0 \in R[H_0] \) will be referred to the class \( R_{\text{mer}}[H_0] \) if it admits a continuation to an entire or meromorphic function in \( \mathbb{C} \) (this function is denoted by \( \tau_0 \) as well). A relation-valued function \( \tau \in \tilde{R}(\mathcal{H}) \) will be referred to the class \( \tilde{R}_{\text{mer}}(\mathcal{H}) \) if its operator part \( \tau_0 \) (see (2.3)) belongs to \( R_{\text{mer}}[H_0] \).

For a function \( \tau_0 \in R_{\text{mer}}[H_0] \) we denote by \( F_{\tau_0} \) the set of all poles of \( \tau_0 \). Clearly, \( F_{\tau_0} \) admits the representation as a sequence \( F_{\tau_0} = \{ t_k \}^{\nu}_{\nu=1} \in \mathcal{F} \).

As is known (see e.g. [2, Proposition A.4.5]) a function \( \tau_0 \in R[H_0] \) admits a continuation to an interval \( (\alpha, \beta) \subset \mathbb{R} \) if and only if the spectral function \( \xi \) of \( \tau \) is constant on \( (\alpha, \beta) \). This fact and (2.2) yield the following assertion.

Assertion 2.5. (i) Let \( \tau_0 \in R[H_0] \). Then the following assertions are equivalent: (a) \( \tau_0 \in R_{\text{mer}}[H_0] \); (b) there is a set \( F \in \mathcal{F} \) such that \( \tau_0 \) admits a holomorphic continuation to \( \mathbb{R} \setminus F \); (c) the spectral function \( \xi \) of \( \tau \) is discrete.

(i) Let \( \tau_0 \in R_{\text{mer}}[H_0] \) and let \( \xi = \{ F, \Xi \} \) be a (discrete) spectral function of \( \tau_0 \) with \( \Xi = \{ \xi_k \}^{\nu}_{\nu=1} \) and \( F = \{ t_k \}^{\nu}_{\nu=1} \). Then \( F = F_{\tau_0} \) and for any \( t_k \in F_{\tau_0} \) there exists a function \( \tau_k \in \tilde{R}[H_0] \) holomorphic at \( t_k \) and such that

\[
\tau_0(\lambda) = 1_{t_k=\lambda} \xi_k + \tau_k(\lambda), \quad \lambda \in \mathbb{C} \setminus F_{\tau_0}
\]

Equality (2.4) means that each \( t_k \in F_{\tau_0} \) is a pole of \( \tau_0 \) of the first order and

\[
\xi_k = -\lim_{\lambda \to t_k} (\lambda - t_k) \tau_0(\lambda) = -\text{res}_{t_k} \tau_0,
\]

where \( \text{res}_{t_k} \tau_0 \) denotes the residue of \( \tau_0 \) at \( t_k \).

In the following we deal with pairs \((C_0, C_1)\) of holomorphic operator functions \( C_j : \mathcal{D} \to B(\mathcal{H}), \mathcal{D} \subset \mathbb{C}, j \in \{0, 1\} \). Clearly, the equality

\[
C(\lambda) = (C_0(\lambda), C_1(\lambda)) : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}, \quad \lambda \in \mathcal{D}
\]

enables one to identify such a pair with a holomorphic operator function \( C : \mathcal{D} \to B(\mathcal{H}^2, \mathcal{H}) \).

Recall (see e.g. [5, 6]) that a pair \( C = (C_0, C_1) \) of holomorphic operator functions \( C_j : \mathbb{C} \setminus \mathbb{R} \to B(\mathcal{H}), j \in \{0, 1\} \), is called a Nevanlinna pair in \( \mathcal{H} \) if for any \( \lambda \in \mathbb{C} \setminus \mathbb{R} \)

\[
\text{Im} \lambda \cdot \text{Im} (C_1(\lambda)C_0^*(\lambda)) \geq 0, \quad C_1(\lambda)C_0^*(\lambda) - C_0(\lambda)C_1^*(\lambda) = 0, \quad \text{ran} C(\lambda) = \mathcal{H}.
\]

Let \( J_\mathcal{H} \in B(\mathcal{H}^2) \) be the operator given by

\[
J_\mathcal{H} = \begin{pmatrix} 0 & -I_\mathcal{H} \\ I_\mathcal{H} & 0 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}.
\]

Then the conditions (2.6) can be written as

\[
i \text{Im} \lambda \cdot C(\lambda)J_\mathcal{H}C^*(\lambda) \leq 0, \quad C(\lambda)J_\mathcal{H}C^*(\lambda) = 0, \quad \text{ran} C(\lambda) = \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Definition 2.6. A pair \( C = (C_0, C_1) \) of entire operator functions \( C_j : \mathbb{C} \to B(\mathcal{H}) \) is said to be an entire Nevanlinna pair in \( \mathcal{H} \) if its restriction onto \( \mathbb{C} \setminus \mathbb{R} \) is a Nevanlinna pair and \( \text{ran} C(t) = \mathcal{H}, t \in \mathbb{R} \). The set of all entire Nevanlinna pairs in \( \mathcal{H} \) is denoted by \( \text{ENP}(\mathcal{H}) \).

Clearly, for a pair \( C = (C_0, C_1) \in \text{ENP}(\mathcal{H}) \) the relations (2.6) (or equivalently (2.8)) hold for any \( \lambda \in \mathbb{C} \).

As is known (see e.g. [5, 6]) for each Nevanlinna pair \( C = (C_0, C_1) \) in \( \mathcal{H} \) the equality

\[
\tau_C(\lambda) := \ker (C(\lambda)) = \{ (h, h') \in \mathcal{H}^2 : C_0(\lambda)h + C_1(\lambda)h' = 0 \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]

defines a relation-valued function \( \tau_C \in \tilde{R}(\mathcal{H}) \).

Recall that an operator \( X \in B(\mathcal{H}^2) \) is called \( J_\mathcal{H} \)-unitary if \( X^*J_\mathcal{H}X = 0 \). If \( X \in B(\mathcal{H}) \) is \( J_\mathcal{H} \)-unitary, then \( 0 \in \rho(X) \) and \( X^{-1} \) is \( J_\mathcal{H} \)-unitary as well.
Lemma 2.7. Let $C = (C_0, C_1)$ be a Nevanlinna pair in $\mathcal{H}$ and let $X \in B(\mathcal{H}^2)$ be a $J_\mathcal{H}$-
unitary operator. Then the equality $\tilde{C}(\lambda) = C(\lambda)X^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, defines a Nevanlinna
pair $\tilde{C} = (\tilde{C}_0, \tilde{C}_1)$ in $\mathcal{H}$ such that $\tau_{\tilde{C}}(\lambda) = X\tau_C(\lambda)$. If in addition $C \in \text{ENP}(\mathcal{H})$, then the
equality $\tilde{C}(\lambda) = C(\lambda)X^{-1}$, $\lambda \in \mathbb{C}$, defines a pair $\tilde{C} = (\tilde{C}_0, \tilde{C}_1) \in \text{ENP}(\mathcal{H})$.

Proof. The required statements directly follows from conditions (2.8) which define a Nevanlinna pair. \hfill $\square$

Lemma 2.8. Let $C = (C_0, C_1)$ be a Nevanlinna pair in $\mathcal{H}$, let $\tau = \tau_C \in \tilde{R}(\mathcal{H})$ be given by (2.9) and let $\tau_0 \in R[\mathcal{H}_0]$ and $\mathcal{K}$ be the operator and multivalued parts of $\tau$ respectively (see (2.3)). Then:

(i) $\mathcal{K} = \ker C_1(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(ii) if $C_0(\lambda) = (C_{00}(\lambda), C_{01}(\lambda)): \mathcal{H}_0 \oplus \mathcal{K} \to \mathcal{H}$

\begin{equation}
C_1(\lambda) = (C_{10}(\lambda), 0): \mathcal{H}_0 \oplus \mathcal{K} \to \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\end{equation}

are block representations of $C_j(\lambda)$ and $\tilde{C}_1(\lambda) \in B(\mathcal{H})$ is given by

\begin{equation}
\tilde{C}_1(\lambda) = (C_{10}(\lambda), -C_{01}(\lambda)): \mathcal{H}_0 \oplus \mathcal{K} \to \mathcal{H},
\end{equation}

then $\ker \tilde{C}_1(\lambda) = \{0\}$ and

\begin{equation}
\text{ran} (\tilde{C}_1^{-1}(\lambda)C_{00}(\lambda)) \subset \mathcal{H}_0, \quad \tau_0(\lambda) = -\tilde{C}_1^{-1}(\lambda)C_{00}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

Proof. (i) It follows from (2.9) that $\{0, h'\} \in \tau_C(\lambda) \iff h' \in \ker C_1(\lambda)$. This yields statement (i).

(ii) One can easily verify that the equalities

\begin{equation}
X = \begin{pmatrix}
P_{\mathcal{H}_0} & P_{\mathcal{K}} \\
-P_{\mathcal{K}} & P_{\mathcal{H}_0}
\end{pmatrix}, \quad X^{-1} = \begin{pmatrix}
P_{\mathcal{H}_0} & -P_{\mathcal{K}} \\
P_{\mathcal{K}} & P_{\mathcal{H}_0}
\end{pmatrix}
\end{equation}

defines a $J_\mathcal{H}$-unitary operator $X \in B(\mathcal{H} \oplus \mathcal{H})$ and its inverse $X^{-1}$. Therefore by Lemma 2.7 the equality $\tilde{C}(\lambda) = C(\lambda)X^{-1}$ defines a Nevanlinna pair $\tilde{C} = (\tilde{C}_0, \tilde{C}_1)$ in $\mathcal{H}$ such that $\tau_{\tilde{C}}(\lambda) = X\tau_C(\lambda)$. It follows from (2.10) and the second equality in (2.13) that

\[\tilde{C}_0(\lambda) = C_0(\lambda)P_{\mathcal{H}_0} + C_1(\lambda)P_{\mathcal{K}} = C_{00}(\lambda)P_{\mathcal{H}_0},\]

\[\tilde{C}_1(\lambda) = -C_0(\lambda)P_{\mathcal{K}} + C_1(\lambda)P_{\mathcal{H}_0} = -C_{01}(\lambda)P_{\mathcal{K}} + C_{10}(\lambda)P_{\mathcal{H}_0} = \tilde{C}_1(\lambda),\]

and, consequently,

\begin{equation}
\tau_{\tilde{C}}(\lambda) = \{\{h, h'\} \in \mathcal{H}^2 : C_{00}(\lambda)P_{\mathcal{H}_0}h + \tilde{C}_1(\lambda)h' = 0\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

On the other hand, (2.3) and the first equality in (2.13) yield $\tau_{\tilde{C}}(\lambda) = \{\{h_0 \oplus k, \tau_0(\lambda)h_0\} : h_0 \in \mathcal{H}_0, k \in \mathcal{K}\}$. Hence

\begin{equation}
\tau_{\tilde{C}}(\lambda) = \tau_0(\lambda)P_{\mathcal{H}_0} \in B(\mathcal{H}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

It follows from (2.14) that $\ker \tilde{C}_1(\lambda) = \text{mul} \tau_{\tilde{C}}(\lambda) = \{0\}$ and consequently $\tau_{\tilde{C}}(\lambda) = -\tilde{C}_1^{-1}(\lambda)C_{00}(\lambda)P_{\mathcal{H}_0}$. Combining this equality with (2.15) one gets (2.12). \hfill $\square$

Proposition 2.9. Let $C = (C_0, C_1) \in \text{ENP}(\mathcal{H})$. Then the equality (2.9) defines the relation-valued function $\tau_C \in \tilde{R}_{\text{mer}}(\mathcal{H})$.

Proof. Since the restriction of $C$ onto $\mathbb{C} \setminus \mathbb{R}$ is a Nevanlinna pair, it follows that $\tau_C \in \tilde{R}(\mathcal{H})$. Let $\tau_0$ be the operator part of $\tau_C$. Then by Lemma 2.8 $\tau_0$ is defined by the equality in (2.12) with entire operator functions $\tilde{C}_1$ and $C_{00}$. Let $\det(\lambda)$, $\lambda \in \mathbb{C}$, be the determinant of the matrix of the operator $\tilde{C}_1(\lambda)$ in some orthonormal basis of $\mathcal{H}$ and let $Z = \{\lambda \in \mathbb{C} : \det(\lambda) = 0\}$. Since $\det$ is an entire function and $\det(\lambda) \neq 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, it
follows that $Z \in \mathcal{F}$ and ker $\tilde{C}_1(\lambda) = \{0\}$, $\lambda \in \mathbb{C} \setminus Z$. Therefore $\tilde{C}_1^{-1}$ is holomorphic on the set $\mathbb{C} \setminus Z$ and by inclusion in (2.12) ran $(\tilde{C}_1^{-1}(\lambda)C_{00}(\lambda)) \subset \mathcal{H}_0$, $\lambda \in \mathbb{C} \setminus Z$. Hence the equality in (2.12) with $\lambda \in \mathbb{C} \setminus Z$ defines a holomorphic continuation of $\tau_0$ onto $\mathbb{R} \setminus Z$ and by Assertion 2.5, (i) $\tau_0 \in \overline{R_{\text{mer}}}[\mathcal{H}_0]$. □

2.4. Boundary triplets and self-adjoint extensions. In the following we denote by $A$ a closed symmetric linear relation (in particular closed not necessarily densely defined symmetric operator) in a Hilbert space $\mathcal{H}$. Denote also by $n_{\pm}(A) := \dim \mathcal{R}(A^*)$, $\lambda \in \mathbb{C}_{\pm}$, the deficiency indices of $A$, by $\text{ext}(A)$ the set of all proper extensions of $A$ (i.e., the set of all linear relations $\tilde{A}$ in $\mathcal{H}$ such that $A \subset \tilde{A} \subset A^*$) and by $\overline{\text{ext}}(A)$ the set of closed extensions $\tilde{A} \in \text{ext}(A)$.

Below in this subsection we recall some definitions and results concerning boundary triplets for symmetric relations (see e.g. [2, 6, 11]).

**Definition 2.10.** A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_j : A^* \to \mathcal{H}$, $j \in \{0, 1\}$, is called a boundary triplet for $A^*$, if the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ from $A^*$ into $\mathcal{H} \oplus \mathcal{H}$ is surjective and the following Green’s identity holds:

$$(f' \cdot g) - (f' \cdot g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}), \quad \hat{f} = \{f, f'\}, \quad \hat{g} = \{g, g'\} \in A^*.$$ 

**Remark 2.11.** As is known mul $A^* = \{0\}$ if and only if $\overline{\text{dom}} A = \mathcal{H}$. In this case $A$ and $A^*$ are densely defined operators and operators $\Gamma_j : A^* \to \mathcal{H}$ in Definition 2.10 can be replaced with operators $\Gamma_j : \text{dom} A^* \to \mathcal{H}$, $j \in \{0, 1\}$.

**Proposition 2.12.** If $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $A^*$, then $n_{+}(A) = n_{-}(A) = \dim \mathcal{H}$. Conversely, for each symmetric relation $A \in \tilde{C}(\mathcal{H})$ with $n_{+}(A) = n_{-}(A)$ there exists a boundary triplet for $A^*$.

With a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $A^*$ one associates an extension $A_0 = A_0^* \in \overline{\text{ext}}(A)$ given by

$$(2.16) \quad A_0 = \ker \Gamma_0 = \{\hat{f} \in A^* : \Gamma_0 \hat{f} = 0\}.$$ 

**Proposition 2.13.** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $A^*$ an let $A_0 = A_0^* \in \overline{\text{ext}}(A)$ be given by (2.16). Then the equality $\Gamma_1 \upharpoonright \tilde{\mathcal{H}}_\lambda(A^*) = M(\lambda) \Gamma_0 \upharpoonright \tilde{\mathcal{H}}_\lambda(A^*)$, $\lambda \in \rho(A_0)$, correctly defines the operator function $M : \rho(A_0) \to \mathcal{B}(\mathcal{H})$ belonging to the class $R_{\text{u}}[\mathcal{H}]$.

The operator-functions $M$ defined in Proposition 2.13 is called the Weyl function of the triplet $\Pi$.

As is known a linear relation $\tilde{A} = A^*$ in a Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$ is called an exit space extension of $A$ if $A \subset \tilde{A}$ and there is no nontrivial subspace in $\tilde{\mathcal{H}} \ominus \mathcal{H}$ reducing $\tilde{A}$ (in the case $\overline{\text{dom}} A = \mathcal{H}$ each exit space extension $\tilde{A}$ is an operator). Let $\tilde{A}_j = \tilde{A}_j^* \in \tilde{C}(\tilde{\mathcal{H}})$ be exit space extensions of $A$ and let $\tilde{\mathcal{H}}_{rj} = \tilde{\mathcal{H}}_j \ominus \tilde{\mathcal{H}}_{rj}$, so that $\tilde{\mathcal{H}}_j = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}_{rj}$, $j \in \{1, 2\}$. The extensions $\tilde{A}_1$ and $\tilde{A}_2$ are said to be equivalent if there is a unitary operator $V \in \mathcal{B}(\tilde{\mathcal{H}}_{1}, \tilde{\mathcal{H}}_{2})$ such that $\tilde{A}_2 = U \tilde{A}_1$ with the unitary operator $U = (I_0 \oplus V) \oplus (I_0 \oplus V) \in \mathcal{B}(\tilde{\mathcal{H}}_{1}, \tilde{\mathcal{H}}_{2})$. In the following we do not distinguish equivalent extensions.

An exit space extension $\tilde{A} = \tilde{A}^* \in \tilde{C}(\tilde{\mathcal{H}})$ of $A$ is called canonical if $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}$ (i.e., if $\tilde{A} \in \overline{\text{ext}}(A)$).

A description of all exit space (in particular canonical) self-adjoint extensions $\tilde{A}$ of $A$ in terms of the boundary triplet is given by the following theorem.

**Theorem 2.14.** Assume that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $A^*$. Then:
Remark of the Krein formula for generalized resolvents

\[(2.20)\]

establishes a bijective correspondence \(\tilde{A} = A_\theta\) between all linear relations \(\theta\) in \(\mathcal{H}\) and all extensions \(\tilde{A} \in \text{ext}(A)\). Moreover, the following holds: (a) \(A_\theta \in \text{ext}(A)\) if and only if \(\theta \in \tilde{C}(\mathcal{H})\); (b) \(A_\theta\) is symmetric (self-adjoint) if and only if \(\theta\) is symmetric (resp. self-adjoint). In this case \(n_+(A) = n_+(\theta)\).

(ii) The equality

\[(2.18)\]

establishes a bijective correspondence \(\tilde{A} = A_\tau\) between all relation valued functions \(\tau = \tau(\lambda) \in \tilde{R}(\mathcal{H})\) and all exit space self-adjoint extensions \(\tilde{A}\) of \(A\). Moreover, an extension \(\tilde{A}_\tau\) is canonical if and only if \(\tau(\lambda) \equiv \theta(\theta^*)\), \(\lambda \in \mathbb{C} \setminus \mathbb{R}\), in which case \(\tilde{A}_\tau = A_{\tau^*}\).

**Proposition 2.15.** Let \(\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}\) be a boundary triplet for \(A^*\) and let \(X \in \mathcal{B}(\mathcal{H}^2)\) be a \(J_\mathcal{H}\)-unitary operator with \(J_\mathcal{H}\) of the form (2.7). Then:

(i) \(\tilde{\Pi} := \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}\) with \((\tilde{\Gamma}_0, \tilde{\Gamma}_1)^\top = X(\Gamma_0, \Gamma_1)^\top\) is a boundary triplet for \(A^*\);

(ii) if \(\theta \in \tilde{C}(\mathcal{H})\) and \(\tilde{A} = A_\theta \in \text{ext}(A)\) (in the triplet \(\Pi\)), then \(\tilde{A} = \tilde{A}_\theta\) (in the triplet \(\tilde{\Pi}\)) with \(\tilde{\theta} = X\theta\);

(iii) if \(\tau \in \tilde{R}(\mathcal{H})\) and \(\tilde{A} = \tilde{A}_\tau\) is an exit space extension of \(A\) (in the triplet \(\Pi\)), then \(\tilde{A} = \tilde{A}_{\tau^*}\) (in the triplet \(\tilde{\Pi}\)) with \(\tilde{\tau} \equiv \tilde{R}(\mathcal{H})\) given by \(-\tilde{\tau}(\lambda) = X(-\tau(\lambda)), \lambda \in \mathbb{C} \setminus \mathbb{R}\).

**Remark 2.16.** (i) Let \(\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}\) be a boundary triplet for \(A^*\), \(\theta \in \tilde{C}(\mathcal{H})\) and \(\tilde{A} = A_{\tau^*} \in \text{ext}(A)\). Consider the abstract boundary value problem

\[(2.19)\]

where the second relation is the abstract boundary condition depending on the boundary parameter \(\lambda \in \mathbb{C}\). It is clear that for a given \(\lambda\) the set of all solutions of (2.19) (i.e., the set of all \(f \in \text{dom} A^*\) satisfying (2.19)) coincides with \(\mathcal{N}_\lambda(\tilde{A})\). Hence \(\sigma_B(\tilde{A})\) coincides with the set of all \(\lambda \in \mathbb{C}\) for which the set of all solutions of (2.19) is nontrivial.

(ii) The same parametrization \(\tilde{A} = \tilde{A}_\tau\) as in Theorem 2.14, (ii) can be given by means of the Krein formula for generalized resolvents

\[(2.20)\]

where \(A_0 = A_0^* \in \text{ext}(A)\) is a (basic) extension (2.16), \(M(\lambda)\) is the Weyl function of the triplet \(\Pi\) and \(\gamma(\lambda)\) is a so-called \(\gamma\)-field (for more details see [15, 17]).

3. Shtraus family and abstract eigenvector expansion

3.1. Shtraus family of linear relations. Assume that \(\tilde{\mathfrak{H}} \supset \mathfrak{H}\) is a Hilbert space, \(\mathfrak{H}_r := \tilde{\mathfrak{H}} \ominus \mathfrak{H}_r^\top P_0 \tilde{\mathfrak{H}}\) is the orthoprojection in \(\tilde{\mathfrak{H}}\) onto \(\mathfrak{H}\) and \(\tilde{A} = \tilde{A}^* \in \tilde{C}(\tilde{\mathfrak{H}})\) is an exit space extension of \(A\). Recall that a linear relation \(C_{\tilde{A}}\) in \(\tilde{\mathfrak{H}}\) given by

\[(3.1)\]

is called the compression of \(\tilde{A}\). It is easy to see that \(C_{\tilde{A}}\) is a (not necessarily closed) symmetric extension of \(A\). Note also that the equality

\[(3.2)\]

defines a linear relation \(T_{\tilde{A}}\) in \(\tilde{\mathfrak{H}}\) and \(A \subset C_{\tilde{A}} \subset T_{\tilde{A}} \subset A^*\).
Definition 3.1. [2, 8, 22] A family of linear relations $S_A(\lambda)$ in $\mathfrak{H}$ defined by

$$S_A(\lambda) = \{ (f, f') \in \mathfrak{H}^2 : (f + f_r) + \lambda f_r \in \tilde{A} \text{ with some } f_r \in \mathfrak{H}_r \}, \quad \lambda \in \mathbb{C}$$

is called the Shtraus family of $\tilde{A}$.

It is easy to see that $S_A(\lambda) = (P_0(\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H})^{-1} + \lambda$, which is equivalent to

$$S_A(\lambda) - \lambda^{-1} = P_0(\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C}.$$

Moreover, $S_A(t)$ is a symmetric extension of $A$ and $A \subset S_A(t) \subset T_{\tilde{A}} \subset A^*$ for any $t \in \mathbb{R}$.

Lemma 3.2. Let $\Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$ be a boundary triplet for $A^*$. Then for any $t \in \mathbb{R}$ the following statements hold:

(i) $A^t := (A - t)^{-1}$ is a closed symmetric relation in $\mathfrak{H}$, $(A^t)^* = (A^* - t)^{-1}$ and the collection $\Pi_t = \{ \mathcal{H}, \Gamma_{0t}, \Gamma_{1t} \}$ with

$$\Gamma_{0t}(f, f') = \Gamma_0(f^t, f + tf')^t, \quad \Gamma_{1t}(f, f') = -\Gamma_1(f^t, f + tf'), \quad \{ f, f' \} \in (A^t)^*$$

is a boundary triplet for $(A^t)^*$. 

(ii) If $\tilde{A} \in \text{ext}(A)$, then $(A - t)^{-1} \in \text{ext}(A^t)$ and $\Gamma_t(A - t)^{-1} = -\Gamma_{\tilde{A}}$ (here $\Gamma = (\Gamma_0, \Gamma_1)^T$ and $\Gamma_{1t} = (\Gamma_{0t}, \Gamma_{1t})^T$).

(iii) If $t \in \mathcal{R}(\mathcal{H})$ and $\tilde{A} = \tilde{A}_r$ is the exit space extension of $A$, then $\tilde{A}^t := (\tilde{A} - t)^{-1}$ is an exit space extension of $A^t$ and $\tilde{A}^t = \tilde{A}_{r,t}$ (in the triplet $\Pi_t$) with $\tau^t \in \mathcal{R}(\mathcal{H})$ given by

$$\tau^t = \tau^t(\lambda) = -\tau(t + \frac{1}{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. Statement (i) and (ii) are obvious.

(iii) Let $t \in \mathcal{R}(\mathcal{H})$ and $\tilde{A} = \tilde{A}_r$. We show that

$$S_{\tilde{A}}(\lambda) = (S_{\tilde{A}}(t + \frac{1}{\lambda}) - t)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then by (3.3) $\{ f, f' \} \in S_{\tilde{A}}(\lambda)$ if and only if $\{ f + f_r, f' + \lambda f_r \} \in \tilde{A}^t$ with some $f_r \in \mathfrak{H}_r$. Now the equivalences

$$\{ f + f_r, f' + \lambda f_r \} \in \tilde{A}^t \iff \{ f' + \lambda f_r, (f + tf')^t \oplus (1 + \lambda t)f_r \} \in \tilde{A}^t \iff$$

$$\{ f', f + tf' \} \in S_{\tilde{A}}(t + \frac{1}{\lambda}) \iff \{ f, f' \} \in (S_{\tilde{A}}(t + \frac{1}{\lambda}) - t)^{-1},$$

proves (3.7). Next, by (2.18) and (3.4) the equality $\tilde{A} = \tilde{A}_r$ is equivalent to $S_{\tilde{A}}(\lambda) = A_{-\tau(\lambda)}$, that is $\Gamma S_{\tilde{A}}(\lambda) = -\tau(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence by (3.7) and statement (ii)

$$\Gamma_{1t} S_{\tilde{A}}(\lambda) = -\Gamma_{1t} S_{\tilde{A}}(t + \frac{1}{\lambda}) = \tau(t + \frac{1}{\lambda}) = -\tau^t(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

which implies that $\tilde{A}^t = \tilde{A}_{r,t}$. 

In the following theorem we characterize in terms of the parameter $\tau$ the Shtraus family $S_{\tilde{A}}(t)$, $t \in \mathbb{R}$, corresponding to the exit space extension $\tilde{A} = \tilde{A}_r$ of $A$.

Theorem 3.3. Assume that $\Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$ is a boundary triplet for $A^*$, $\tau \in \mathcal{R}_c(\mathcal{H})$, $\tilde{A} = \tilde{A}_r$ is the corresponding exit space self-adjoint extension of $A$ and $S_{\tilde{A}}(\lambda)$ is the Shtraus family of $\tilde{A}$. Moreover, let $\tau_0 \in R_c[\mathcal{H}_0]$ and $\mathcal{K}$ be the operator and multivalued parts of $\tau$ respectively (see (2.3)), let $t \in \mathbb{R}$ and let $\mathcal{B}_{\tau_0}(t) \in \mathcal{B}(\mathcal{H}_0)$ and $D_{\tau_0}(t) : \text{dom} D_{\tau_0}(t) \rightarrow \mathcal{H}_0$ (dom $D_{\tau_0}(t) \subset \mathcal{H}_0$) be operators corresponding to $\tau_0$ in accordance with Proposition 2.1, (ii). Assume also that ran $\mathcal{B}_{\tau_0}(t)$ is closed. Then $S_{\tilde{A}}(t) = A_{\eta(t)} = \{ \tilde{f} \in A^* : \{ \Gamma_0 \tilde{f}, \Gamma_1 \tilde{f} \} \in \eta(t) \}$, where $\eta(t) = \eta_r(t)$ is given by

$$\eta_r(t) = \{ (h, -D_{\tau_0}(t) h + \mathcal{B}_{\tau_0}(t) h_0 + \kappa) : h \in \text{dom} D_{\tau_0}(t), h_0 \in \mathcal{H}_0, \kappa \in \mathcal{K} \}. $$
Proof. Let \( \tau^t \in \tilde{R}(\mathcal{H}) \) be given by (3.6). Then according to Lemma 3.2 \( A^t = (A - t)^{-1} \) is a symmetric relation in \( \mathcal{H} \). \( \mathcal{A}^t = (\mathcal{A} - t)^{-1} \) is an exit space self-adjoint extension of \( A^t \) and \( \mathcal{A}^t = \mathcal{A}_{\tau^t} \) in the triplet \( \Pi_t = \{ \mathcal{H}, \Gamma_{\mathcal{O}t}, \Gamma_{\mathcal{U}t} \} \) for \( (A^t)^* \) given by (3.5). It is clear that the multivalued part of \( \tau^t \) coincides with \( \mathcal{K} \), while the operator part \( \tau_0^t \) of \( \tau^t \) is

\[
\tau_0^t(\lambda) = -\tau_0 \left( t + \frac{1}{\lambda} \right), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Hence \( \tau_0^t \in R_c[\mathcal{H}] \) and, consequently, \( \tau^t \in \tilde{R}_c(\mathcal{H}) \). Moreover, by (3.9)

\[
dom D_{\tau_0^t}\infty = dom D_{\tau_0}(t), \quad D_{\tau_0^t}\infty = -D_{\tau_0}(t), \quad B_{\tau_0^t}\infty = B_{\tau_0}(t),
\]

where \( B_{\tau_0^t}\infty \in B(\mathcal{H}_0) \) and \( D_{\tau_0^t}\infty : \dom D_{\tau_0^t}\infty \to \mathcal{H}_0 \) (dom \( D_{\tau_0^t}\infty \subset \mathcal{H}_0 \)) are the operators corresponding to \( \tau_0^t \) in accordance with Proposition 2.1, (i). Hence ran \( B_{\tau_0^t}\infty \) is closed.

Let \( \Gamma_t = (\Gamma_{\mathcal{O}t}, \Gamma_{\mathcal{U}t})^\top \) and let \( C_{\tilde{A}^t} \) be the compression of \( \tilde{A}^t \). Then by [19, Theorem 3.8] \( \Gamma_t C_{\tilde{A}^t} = -\eta(t) \), where

\[
\eta(t) = \{ \{ h, D_{\tau_0^t}\infty h + B_{\tau_0^t}\infty h_0 + k : h \in \dom D_{\tau_0^t}\infty, h_0 \in \mathcal{H}_0, k \in \mathcal{K} \} \}
\]

It follows from (3.4) that \( (S_{\tilde{A}}^t)(t)^{-1} = C_{\tilde{A}^t} \) and by Lemma 3.2, (ii) \( \Gamma S_{\tilde{A}}^t(t) = -\Gamma_t C_{\tilde{A}^t} = \eta(t) \). Hence \( S_{\tilde{A}}^t(t) = A_0(t) \) with \( \eta(t) = \eta_0(t) \) given by (3.11) and (3.10) yields (3.8). \( \square \)

Recall that two extensions \( \tilde{A}_1, \tilde{A}_2 \in \overline{\text{ext}(A)} \) are called transversal if \( \tilde{A}_1 \cap \tilde{A}_2 = A \) and \( \tilde{A}_1 + \tilde{A}_2 := \{ \tilde{f} + \tilde{g} : \tilde{f} \in A_1, \tilde{g} \in A_2 \} = A^* \).

Let as before \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \) and let \( A_0(= \ker \Gamma_0) \) be a basic self-adjoint extension of \( A \) in the Krein formula (2.20). In the following two Theorems 3.4 and 3.5 we characterise in terms of the Nevanlinna parameter \( \tau \) exit space extensions \( \tilde{A} = \tilde{A}_r \) of \( A \) such that the Schraub family \( S_{\tilde{A}}(t) \) satisfies one of the following extremal conditions: (i) \( S_{\tilde{A}}(t) \subset A_0 \) (in particular, \( S_{\tilde{A}}(t) = A_0 \) ); (ii) \( S_{\tilde{A}}(t) \) is self-adjoint and transversal with \( A_0 \). For the compression \( C(A_r) \) of \( A_r \) similar results were obtained in our paper [19, Theorems 3.9 and 3.16]. Similarly to Theorem 3.3 the proof of Theorems 3.4 and 3.5 can be easily obtained by application of results from [19] to the symmetric relation \( A^t = (A - t)^{-1} \) and the boundary triplet \( \Pi_t = \{ \mathcal{H}, \Gamma_{\mathcal{O}t}, \Gamma_{\mathcal{U}t} \} \) for \( (A^t)^* \) (see (3.5)). Therefore we omit the proof.

**Theorem 3.4.** Assume that \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) is a boundary triplet for \( A^* \), \( A_0 = \ker \Gamma_0 \), \( \tau \in \tilde{R}(\mathcal{H}) \), \( \tilde{A} = \tilde{A}_r \) is the exit space extension of \( A \) and \( t \) is a real point. Then:

(i) The following statements (a1) and (a2) are equivalent:

(a1) \( T_{\tilde{A}} \) is closed and \( S_{\tilde{A}}(t) \subset A_0 \) (for \( T_{\tilde{A}} \) see (3.2));

(a2) \( \tau \in \tilde{R}_c(\mathcal{H}) \) and the operator part \( \tau_0 \in R_c[\mathcal{H}_0] \) of \( \tau \) satisfies

\[
\lim_{y \to 0^+} \frac{1}{y} \text{Im} (\tau_0(t + iy)h, h) = \infty, \quad h \in \mathcal{H}_0, \ h \neq 0.
\]

Moreover, if statement (a1) (equivalently (a2) ) is valid and ran \( B_{\tau_0}(t) \) is closed, then \( S_{\tilde{A}}(t) \) is closed and \( S_{\tilde{A}}(t) = \{ \tilde{f} \in A^* : \Gamma_0 \tilde{f} = 0, \Gamma_1 \tilde{f} \in \text{ran} B_{\tau_0}(t) \oplus \mathcal{K} \} \) (for \( \mathcal{K} \) see (2.3)).

(ii) The following statements (a1') and (a2') are equivalent:

(a1') \( T_{\tilde{A}} \) is closed and \( S_{\tilde{A}}(t) = A_0 \); (a2') \( \tau \in \tilde{R}_c(\mathcal{H}) \) and \( \ker B_{\tau_0}(t) = \{ 0 \} \).

Moreover, if statement (a1') (equivalently (a2') ) is valid and ran \( B_{\tau_0}(t) \) is closed, then \( S_{\tilde{A}}(t) = A_0 \).

(iii) \( T_{\tilde{A}} \) is closed and \( S_{\tilde{A}}(t) = A \) if and only if \( \tau \in R_c[\mathcal{H}] \), \( B_\tau(t) = 0 \) and (3.12) holds with \( \tau \) in place of \( \tau_0 \).

**Theorem 3.5.** Let the assumptions be the same as in Theorem 3.4. Then:
(i) If $\tau \in \tilde{R}_e(\mathcal{H})$, $\ker \mathcal{B}_{\tau_0}(t)$ is closed and

$$\lim_{y \to 0} \frac{1}{y} \text{Im} (\tau_0(t + iy)h, h) < \infty, \quad h \in \ker \mathcal{B}_{\tau_0}(t),$$

then $S_{\tilde{A}}(t)$ is self-adjoint (in (3.13) $\tau_0 \in R_e[\mathcal{H}_0]$ is the operator part of $\tau$).

(ii) The following statements (b1) and (b2) are equivalent:

(b1) $T_{\tilde{A}}$ is closed, $S_{\tilde{A}}(t)$ is self-adjoint and transversal with $A_0$;

(b2) $\tau \in R_e[\mathcal{H}]$ and

$$\lim_{y \to 0} \frac{1}{y} \text{Im} (\tau(t + iy)h, h) < \infty, \quad h \in \mathcal{H}.$$

Moreover, if (b1) (equivalently (b2)) is satisfied, then

$$S_{\tilde{A}}(t) = A_{-\mathcal{K}}^* = \{ \hat{f} \in A^* : \Gamma_0 \hat{f} = -K \Gamma_0 \hat{f} \},$$

with the operator $K = K^* \in B(\mathcal{H})$ given by

$$\hat{K} = s - \lim_{y \to 0} \tau(t + iy).$$

### 3.2. The case of finite deficiency indices.

**Theorem 3.6.** Assume that:

(i) $A \in \tilde{C}(\mathcal{S})$ is a symmetric relation with equal finite deficiency indices $n_+ (A) = n_- (A) < \infty$, $\Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \}$ is a boundary triplet for $A^*$, $\tau \in \tilde{R}(\mathcal{H})$ and $\tilde{A} = \tilde{A}_\tau$ is the corresponding exit space self-adjoint extension of $A$;

(ii) $\tau_0 \in \tilde{R}[\mathcal{H}_0]$ and $\mathcal{K}$ are the operator and multivalued parts of $\tau$ respectively (see (2.3)).

Then the Shtraus family $S_{\tilde{A}}(t)$ of $\tilde{A}$ is given in the triplet $\Pi$ by

$$S_{\tilde{A}}(t) = A_{\eta(t)}^* = \{ \hat{f} \in A^* : \Gamma_0 \hat{f}, \Gamma_1 \hat{f} \} \in \eta(t), \quad t \in \mathbb{R},$$

where $\eta = \eta_\tau : \mathbb{R} \to \tilde{C}(\mathcal{H})$ is the relation-valued function (3.8).

**Proof.** Since by Proposition 2.12 dim $\mathcal{H} < \infty$, it follows that $\tilde{R}(\mathcal{H}) = \tilde{R}_e(\mathcal{H})$ and $\ker \mathcal{B}_{\tau_0}(t)$ is closed. Now the required statement is implied by Theorem 3.3. \qed

**Remark 3.7.** Theorems 3.4, 3.5 and 3.6 readily yield the results obtained in [8] for the Shtraus family of the extension $\tilde{A} \supset A$ in the case $n_+ (A) < \infty$.

**Corollary 3.8.** Let under the assumption (i) of Theorem 3.6 $\tau \in \tilde{R}[\mathcal{H}]$ and $\tau$ admits a holomorphic continuation at the point $t_0 \in \mathbb{R}$. Then $S_{\tilde{A}}(t_0) = A_{-\tau(t_0)}$ (in the triplet $\Pi$)

**Proof.** The equality $\text{Im} \tau(t_0) = 0$ yields

$$\lim_{y \to 0} \frac{1}{y} \text{Im} (\tau(t_0 + iy)h, h) = \text{Re} (\tau(t_0)h, h) < \infty, \quad h \in \mathcal{H}.$$

Hence $\ker D_{\tau}(t_0) = \mathcal{H}$ and $D_{\tau}(t_0) = \tau(t_0)$. Moreover, $\mathcal{B}_{\tau}(t_0) = 0$ and by (3.8)

$$\eta(t_0) = \eta_\tau(t_0) = -\tau(t_0).$$

This and Theorem 3.6 yield the result. \qed

In the following proposition we characterize in terms of abstract boundary conditions the Shtraus family of the exit space extension $\tilde{A}$ with the parameter $\tau \in \tilde{R}(\mathcal{H})$ generated by an entire Nevanlinna pair.

**Proposition 3.9.** Let under the assumption (i) of Theorem 3.6 $\tau = \tau_C \in \tilde{R}_\text{mea}(\mathcal{H})$ be a relation-valued function defined by (2.9) with an entire Nevanlinna pair $C = (C_0, C_1) \in \text{ENP}(\mathcal{H})$. Then the equality (the abstract boundary condition)

$$S_{\tilde{A}_C}(t) = \{ \hat{f} \in A^* : C_0(t) \Gamma_0 \hat{f} - C_1(t) \Gamma_1 \hat{f} = 0 \}, \quad t \in \mathbb{R}.$$
defines the Shtraus family $S_{A_r}(t)$ of $\tilde{A}_r$.

Proof. Assume that $t \in \mathbb{R}$. Since $\text{Im} \, C_1(t)C_0^*(t) = 0$ and $\text{ran}(C_0(t), C_1(t)) = \mathcal{H}$, it follows from [6, Proposition 6.46] that the equality
\[
\theta = \ker C(t) = \{\{h, h'\} \in \mathcal{H}^2 : C_0(t)h + C_1(t)h' = 0\}
\]
defines a relation $\theta = \theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$. Hence by (2.1)
\[
(3.20) \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{K}, \quad \theta = \{\{h_0, \theta_{op}h_0 + k\} : h_0 \in \mathcal{H}_0, k \in \mathcal{K}\},
\]
where $\mathcal{K} = \text{mul} \, \theta$ and $\theta_{op} = \theta_{op}^* \in \mathcal{B}(\mathcal{H}_0)$ is the operator part of $\theta$. It was shown in the proof of Lemma 2.8 that the equality (2.13) defines a $J_\mathcal{H}$-unitary operator $X \in \mathcal{B}(\mathcal{H}^2)$. Therefore by Lemma 2.7 the equality $\tilde{\mathcal{C}}(\lambda) = C(\lambda)X^{-1}$, $\lambda \in \mathbb{C}$, defines a pair $\tilde{\mathcal{C}} = (\tilde{C}_0, \tilde{C}_1) \in \text{ENP}(\mathcal{H})$ such that $\tau_{\tilde{\mathcal{C}}} = X\tau_{\mathcal{C}}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Next we put $\tilde{\theta} := X\theta$. Then
\[
(3.21) \quad \tilde{\theta} = \ker \tilde{C}(t) = \{\{h, h'\} \in \mathcal{H}^2 : \tilde{C}_0(t)h + \tilde{C}_1(t)h' = 0\}
\]
and, consequently, $\text{mul} \, \tilde{\theta} = \ker \tilde{C}_1(t)$. On the other hand, by (3.20) $\tilde{\theta} = \theta_{op}P_{\mathcal{H}_0} \in \mathcal{B}(\mathcal{H})$. Thus ker $\tilde{C}_1(t) = \{0\}$ and, consequently, there is a neighbourhood $U(t)$ of $t$ in $\mathbb{C}$ such that ker $\tilde{C}_1(\lambda) = \{0\}$, $\lambda \in U(t)$. This and formula (2.9) (for $\tilde{\mathcal{C}}$) imply that $\text{mul} \, \tau_{\tilde{\mathcal{C}}}(\lambda) = \{0\}$, $\lambda \in U(t) \setminus \mathbb{R}$, and, consequently, the multivalued part $\tilde{K}$ of $\tau_{\tilde{\mathcal{C}}} \in \tilde{\mathcal{R}}(\mathcal{H})$ is $\tilde{K} = \text{mul} \, \tau_{\tilde{\mathcal{C}}}(\lambda) = \{0\}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (this means that $\tau_{\tilde{\mathcal{C}}} \in \tilde{\mathcal{R}}(\mathcal{H})$). Therefore ker $\tilde{C}_1(\lambda) = \{0\}$ and $\tau_{\tilde{\mathcal{C}}}(\lambda) = -\tilde{C}_1^{-1}(\lambda)\tilde{C}_0(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, $\tau_{\tilde{\mathcal{C}}}$ admits a holomorphic continuation at the point $t$ and $\tau_{\tilde{\mathcal{C}}}(t) = -\tilde{C}_1^{-1}(t)\tilde{C}_0(t) = \tilde{\theta} \in \mathcal{B}(\mathcal{H})$.

Let $\tilde{J} = \text{diag} (I_{\mathcal{H}}, -I_{\mathcal{H}}) \in \mathcal{B}(\mathcal{H}^2)$. Then $\tilde{X} := \tilde{J}X\tilde{J} \in \mathcal{B}(\mathcal{H}^2)$ is a $J$-unitary operator and by Proposition 2.15, (i) the equality $(\tilde{\Gamma}_0, \tilde{\Gamma}_1)^\top = \tilde{X}(\Gamma_0, \Gamma_1)^\top$ defines a boundary triplet $\tilde{\Pi} = \{\mathcal{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ for $A^\ast$. Moreover, $\tilde{X}(-\tau_{\mathcal{C}}(\lambda)) = -\tau_{\tilde{\mathcal{C}}}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and by Proposition 2.15, (iii) $\tilde{A} = \tilde{A}_{\tilde{\Pi}}$ (in the triplet $\tilde{\Pi}$). Therefore by Corollary 3.8 $S_{\tilde{A}}(t) = A_{-\tau_{\mathcal{C}}(t)} = A_{-\tilde{\theta}}$ (in the triplet $\tilde{\Pi}$). Finally, the equality $-\tilde{\theta} = \tilde{X}(-\theta)$ and Proposition 2.15, (ii) yield $S_{\tilde{A}}(t) = A_{-\theta}$ (in the triplet $\Pi$), which is equivalent to (3.19). □

3.3. Abstract eigenvector expansion. Assume that $A \in \tilde{\mathcal{C}}(\mathcal{S})$ is a symmetric relation with finite deficiency indices $n_+(A) = n_-(A) =: d$ and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $A^\ast$ (hence by Proposition 2.12 dim $\tilde{\mathcal{H}} = d < \infty$). Moreover, let $\tau \in \tilde{\mathcal{R}}(\mathcal{H})$ and let $\eta(t) = \eta_{\tau}(t)$ be a $\tilde{\mathcal{C}}(\mathcal{H})$-valued function given for any $t \in \mathbb{R}$ by (3.8). We consider the abstract eigenvalue problem
\[
(3.22) \quad \{f, tf\} \in A^\ast
\]
\[
(3.23) \quad \{\Gamma_0\{f, tf\}, \Gamma_1\{f, tf\}\} \in \eta_{\tau}(t)
\]
with the abstract boundary condition (3.23) depending on the parameter $t \in \mathbb{R}$. The set of all solutions $f \in \text{dom} \, A^\ast$ of the problem (3.22), (3.23) will be denoted by $\tilde{\mathcal{N}}_t$. Clearly, $\tilde{\mathcal{N}}_t$ is a linear manifold in $\mathcal{N}_t(A^\ast)$.

Definition 3.10. A point $t \in \mathbb{R}$ is called an eigenvalue of the problem (3.22), (3.23) if $\tilde{\mathcal{N}}_t \neq \emptyset$. The set of all such eigenvalues is denoted by $\tilde{EV}$. An element $f \in \tilde{\mathcal{N}}_t$ for $t \in \tilde{EV}$ is called an eigenvector of the problem (3.22), (3.23) corresponding to $t$.

Remark 3.11. In the case of a densely defined operator $A$ the operators $\Gamma_0$ and $\Gamma_1$ of the triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ are defined on dom $A^\ast$ (see Remark 2.11) and the eigenvalue
problem (3.22), (3.23) takes the form
\begin{equation}
A^* f = tf
\end{equation}
\begin{equation}
\{\Gamma_0 f, \Gamma_1 f\} \in \eta(t)
\end{equation}
Assume now that \( C = (C_0, C_1) \) is an entire Nevanlinna pair in \( \mathcal{H} \) and \( \tau = \tau_C \in \tilde{\mathcal{R}}(\mathcal{H}) \) is given by (2.9). Then by (3.17) and (3.19) \( \eta(t) = \{\{h, h'\} \in \mathcal{H}^2 : C_0(t)h - C_1(t)h' = 0\} \) and the boundary condition (3.25) can be written as
\begin{equation}
C_0(t)\Gamma_0 f - C_1(t)\Gamma_1 f = 0.
\end{equation}

**Proposition 3.12.** Let under the assumptions of Theorem 3.6 \( \tilde{A} \) be a linear relation in a Hilbert space \( \tilde{\mathfrak{H}} \supset \mathfrak{H} \). Moreover, let \( \eta(r) \) be given by (3.8), let \( \tilde{\mathfrak{N}}_t \) (\( t \in \mathbb{R} \)) be the set of all solutions of the problem (3.22), (3.23) and let \( \mathcal{E}^V \) be the set of all eigenfunctions of the same problem. Then
\begin{equation}
P_\delta \mathfrak{N}_t(\tilde{A}) = \tilde{\mathfrak{N}}_t, \quad t \in \mathbb{R}, \quad \text{and} \quad \sigma_p(\tilde{A}) = \mathcal{E}^V.
\end{equation}

**Proof.** Let \( \tilde{\mathfrak{S}}_r := \tilde{\mathfrak{S}} \oplus \mathfrak{S} \) and let \( t \in \mathbb{R} \). Then \( f \in P_\delta \mathfrak{N}_t(\tilde{A}) \) if and only if \( f \oplus f_r \in \mathfrak{N}_t(\tilde{A}) \) or, equivalently, \( \{f \oplus f_r, tf \oplus tf_r\} \in \tilde{A} \) with some \( f_r \in \mathfrak{S}_r \). Therefore by (3.3) \( P_\delta \mathfrak{N}_t(\tilde{A}) = \{f \in \tilde{\mathfrak{S}} : \{f, tf\} \in S_{\tilde{A}}(t)\} = \mathfrak{N}_t(S_{\tilde{A}}(t)) \). Moreover, by Theorem 3.6 \( S_{\tilde{A}}(t) = \tilde{A}_r(t) \) and Remark 2.16, (i) implies that \( \mathfrak{N}_t(S_{\tilde{A}}(t)) = \tilde{\mathfrak{N}}_t \). This yields the first equality in (3.27).

Next, in view of (2.1) one has \( \tilde{\mathfrak{S}} = \tilde{\mathfrak{S}}_0 \oplus \text{mul} \tilde{A} \) and \( \tilde{A} = \text{gr} \tilde{A}_0 \oplus \text{mul} \tilde{A} \), where \( \tilde{A}_0 = \tilde{A}_0 = A_0 \) is an operator in \( \tilde{\mathfrak{S}}_0 \). Let \( t \in \mathbb{R} \). Since \( \mathfrak{N}_t(\tilde{A}) = \mathfrak{N}_t(\tilde{A}_0) \), it follows that \( \mathfrak{N}_t(\tilde{A}) \cap \mathfrak{N}_r \subset \mathfrak{N}_t(\tilde{A}_0) \). Therefore the subspace \( \mathfrak{N}_t(\tilde{A}) \cap \mathfrak{N}_r \) reduces \( \tilde{A}_0 \) and, consequently, \( \tilde{A} \). Hence \( \mathfrak{N}_t(\tilde{A}) \cap \mathfrak{S}_r = \{0\} \) and therefore \( \mathfrak{N}_t = \{0\} \equiv \mathfrak{N}_t(\tilde{A}) = \{0\} \), which yields the second equality in (3.27). \( \square \)

**Remark 3.13.** Clearly in the case \( \tau(\lambda) \equiv \theta(= \theta^*) \) (i.e., in the case of the canonical extension \( \tilde{A} = A_{-\theta} \)) Proposition 3.12 turns into the statements of Remark 2.16, (i).

**Lemma 3.14.** Assume that \( T, \tilde{T} \in \tilde{\mathcal{C}}(\mathfrak{S}) \) and \( T \subset \tilde{T} \). Then:
\begin{itemize}
    \item[(i)] If \( \text{dim ker} \tilde{T} < \infty \) and \( \text{ran} \tilde{T} \) is closed, then the same statements hold for \( T \);
    \item[(ii)] If \( \text{dim ker} T < \infty \), \( \text{ran} T \) is closed and \( \text{dim} \tilde{T}/T < \infty \), then \( \text{dim ker} \tilde{T} < \infty \) and \( \text{ran} \tilde{T} \) is closed.
\end{itemize}

**Proof.** (i) Assume that \( \text{dim ker} \tilde{T} < \infty \) and \( \text{ran} \tilde{T} \) is closed. Since \( \text{ker} T \subset \text{ker} \tilde{T} \), it follows that \( \text{dim ker} T < \infty \). Next, \( T \) and \( \tilde{T} \) admit the representations
\begin{equation}
T = T_1 \oplus \tilde{\mathfrak{N}}_0(T), \quad \tilde{T} = \tilde{T}_1 \oplus \tilde{\mathfrak{N}}_0(T),
\end{equation}
where \( T_1, \tilde{T}_1 \in \tilde{\mathcal{C}}(\mathfrak{S}) \), \( T_1 \subset \tilde{T}_1 \) and \( \text{ker} T_1 = \{0\} \). Since \( \text{dim} \tilde{\mathfrak{N}}_0(\tilde{T}_1) < \infty \), it follows that \( T_1 + \tilde{\mathfrak{N}}_0(\tilde{T}_1) \) is a closed subspace in \( \tilde{T}_1 \) and hence \( \tilde{T}_1 \) admits the representation
\begin{equation}
\tilde{T}_1 = T_0 \oplus (T_1 + \tilde{\mathfrak{N}}_0(\tilde{T}_1))
\end{equation}
with some \( T_0 \in \tilde{\mathcal{C}}(\mathfrak{S}) \). Let \( \tilde{T}_2 := T_0 \oplus T_1 \). Then \( \tilde{T}_2 \in \tilde{\mathcal{C}}(\mathfrak{S}) \), \( T_1 \subset \tilde{T}_2 \subset \tilde{T}_1 \) and by (3.29) \( \tilde{T}_2 \cap \tilde{\mathfrak{N}}_0(\tilde{T}_1) = \{0\} \), which implies that \( \text{ker} \tilde{T}_2 = \{0\} \). Moreover, by (3.29) \( \text{ran} \tilde{T}_2 = \text{ran} \tilde{T}_1 \) and the second equality in (3.28) yields \( \text{ran} \tilde{T}_1 = \text{ran} \tilde{T} \). Hence \( \text{ran} \tilde{T}_2 \) is closed and consequently \( \tilde{T}_2^{-1} \) is a bounded operator from \( \text{ran} \tilde{T}_2 \) into \( \mathfrak{S} \). Since \( \tilde{T}_1^{-1} \subset \tilde{T}_2^{-1} \) and \( \tilde{T}_1^{-1} \in \tilde{\mathcal{C}}(\mathfrak{S}) \), it follows that \( \text{ran} \tilde{T}_1 (= \text{dom} \tilde{T}_2^{-1}) \) is closed. Moreover, by the first equality in (3.28) \( \text{ran} T = \text{ran} T_1 \), which implies that \( \text{ran} T \) is closed.
Assume that \( \dim \ker T < \infty \), \( \ker T \) is closed and \( \dim \tilde{T}/T < \infty \). It is clear that \( \dim (\ker \tilde{T}/\ker T) \leq \dim (\tilde{T}/T) < \infty \) and, consequently, \( \ker \tilde{T} \) is closed. Moreover, \( \tilde{\mathcal{N}}_0(T) = \tilde{\mathcal{N}}_0(\tilde{T}) \cap T \) and hence \( \dim (\tilde{\mathcal{N}}_0(\tilde{T})/\tilde{\mathcal{N}}_0(T)) \leq \dim (\tilde{T}/T) < \infty \). Since \( \dim \tilde{\mathcal{N}}_0(T) = \dim \ker T < \infty \) and \( \dim \tilde{\mathcal{N}}_0(\tilde{T}) = \dim \ker \tilde{T} \), it follows that \( \dim \ker \tilde{T} < \infty \). \( \square \)

**Definition 3.15.** A symmetric relation \( A \in \tilde{C}(\tilde{\mathcal{S}}) \) has a discrete spectrum if \( \dim \mathcal{N}_t(A) < \infty \) and \( \ker (A - t) \) is closed for any \( t \in \mathbb{R} \).

Clearly, \( A \) has a discrete spectrum if and only if so is the operator part \( A_{\text{op}} \) of \( A \).

In the following we denote by \( \text{Sym}_d(\tilde{\mathcal{S}}) \) (\( \text{Self}_d(\tilde{\mathcal{S}}) \)) the set of all symmetric (resp. self-adjoint) linear relations \( A \in \tilde{C}(\tilde{\mathcal{S}}) \) with the discrete spectrum.

If \( A \in \text{Sym}_d(\tilde{\mathcal{S}}) \), then \( n_+(A) = n_-(A) \) and
\[
\mathbb{R} \setminus \tilde{\rho}(A) = \sigma_p(A) \subset F, \quad \dim \mathcal{N}_t(A) < \infty, \quad t \in \sigma_p(A)
\]
In the case \( A = A^* \in \tilde{C}(\tilde{\mathcal{S}}) \) the inclusion \( A \in \text{Self}_d(\tilde{\mathcal{S}}) \) is equivalent to conditions
\[
\sigma(A) = \sigma_p(A) \subset F, \quad \dim \mathcal{N}_t(A) < \infty, \quad t \in \sigma_p(A).
\]

**Proposition 3.16.** Let \( A \in \tilde{C}(\tilde{\mathcal{S}}) \) be a symmetric relation. Then:

(i) if there exists an exit space (in particular, canonical) extension \( \tilde{\tilde{A}} \in \text{Self}_d(\tilde{\mathcal{S}}) \) of \( A \), then \( A \in \text{Sym}_d(\tilde{\mathcal{S}}) \);

(ii) if \( A \in \text{Sym}_d(\tilde{\mathcal{S}}) \) and \( n_+(A) < \infty \), then each symmetric extension \( \tilde{\tilde{A}} \in \text{ext}(A) \) belongs to \( \text{Sym}_d(\tilde{\mathcal{S}}) \).

**Proof.** If \( n_+(A) = n_-(A) < \infty \), then for each \( \tilde{\tilde{A}} \in \text{ext}(A) \) and \( t \in \mathbb{R} \) one has \( \dim (\tilde{\tilde{A}} - t)/(A - t) = \dim \tilde{\tilde{A}}/A \leq 2n_+(A) < \infty \). Now application of Lemma 3.14 to relations \( A - t \) and \( \tilde{\tilde{A}} - t \) yields the result. \( \square \)

**Remark 3.17.** For densely defined operators \( A \) with finite deficiency indices and canonical extensions \( \tilde{\tilde{A}} = \tilde{\tilde{A}}^* \) if \( A \) statements of Proposition 3.16 are well known (see e.g. [21, §14.9]).

In the following theorem we describe in terms of the parameter \( \tau \in \tilde{R}(\mathcal{H}) \) exit space extensions \( \tilde{A}_r = \tilde{A}_r^* \) of \( A \) with the discrete spectrum.

**Theorem 3.18.** Let under the assumptions of Theorem 3.6 \( A \in \text{Sym}_d(\tilde{\mathcal{S}}) \) and let \( \tilde{\tilde{A}} = \tilde{A}_r \) be a linear relation in a Hilbert space \( \tilde{\mathcal{S}} \supset \mathcal{S} \). Then \( \tilde{\tilde{A}} \in \text{Self}_d(\tilde{\mathcal{S}}) \) if and only if \( \tau \in \tilde{R}_{\text{mer}}(\mathcal{H}) \).

**Proof.** Let \( \tau_0 \in \mathcal{R}[\mathcal{H}_0] \) be the operator part of \( \tau \) (see (2.3)). Since by Proposition 2.12 \( \dim \mathcal{H}_0 < \infty \), it follows from [19, Proposition 2.3] that \( \tau_0 \) admits the representation
\[
(3.30) \quad \tau_0(\lambda) = \left( \begin{array}{cc} \tau_1(\lambda) & -B_1 \\ -B_1^* & -B_2 \end{array} \right) : \mathcal{H} \oplus \mathcal{H}'' \to \mathcal{H}' \oplus \mathcal{H}'', \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]
with \( \tau_1 \in \mathcal{R}_u(\mathcal{H}') \), \( B_1 \in B(\mathcal{H}', \mathcal{H}''') \) and \( B_2 = B_2^* \in B(\mathcal{H}'') \). Next, assume that \( \tilde{\mathcal{S}}_r := \tilde{\mathcal{S}} \oplus \mathcal{S} \) and let \( S \in \text{ext}(A) \) be a symmetric relation in \( \tilde{\mathcal{S}} \) given by \( S = \tilde{\tilde{A}} \cap \tilde{\mathcal{S}}^2 \). It was shown in the proof of Theorem 3.8 in [19] that there exists a boundary triplet \( \Gamma' = \{ \mathcal{H}', \Gamma_0', \Gamma_1' \} \) for \( S^* \) such that \( \tilde{\tilde{A}} = \tilde{S}_{\tau_1} \) (in the triplet \( \Pi' \)) and the results of [5] imply that there exist a simple symmetric operator \( A_r \) in \( \tilde{\mathcal{S}}_r \) and a boundary triplet \( \Pi_r = \{ \mathcal{H}', \Gamma_0^r, \Gamma_1^r \} \) for \( A_r^* \) such that \( \tau_1 \) is the Weyl function of \( \Pi_r \) and \( \tilde{\tilde{A}} \in \text{ext}(S \oplus A_r) \). Hence \( \tilde{\tilde{A}} \in \text{ext}(A \oplus A_r) \), \( n_{\pm}(A \oplus A_r) < \infty \) and by Proposition 3.16 the following equivalences hold:
\[
\tilde{\tilde{A}} \in \text{Self}_d(\tilde{\mathcal{S}}) \iff A \oplus A_r \in \text{Sym}_d(\tilde{\mathcal{S}}) \iff A_r \in \text{Sym}_d(\tilde{\mathcal{S}}_r) \iff A_{0r} \in \text{Self}_d(\tilde{\mathcal{S}}_r),
\]
where \(A_{0r} = \ker \Gamma_0^r\). Moreover, according to \[2, \text{Corollary 3.6.2}\] \(A_{0r} \in \text{Self}_d(\mathcal{H}_r)\) if and only if \(\tau_1 \in R_{\text{mer}}[\mathcal{H}']\). Finally, by (3.30) \(\tau_1 \in R_{\text{mer}}[\mathcal{H}']\) if and only if \(\tau \in R_{\text{mer}}(\mathcal{H})\). These equivalences yield the statement of the theorem. \(\square\)

In the following theorem we show that in the case of a symmetric operator \(A\) in \(\mathcal{H}\) with the discrete spectrum each element \(f \in \mathcal{H}\) admits an eigenvector expansion due to the eigenvalue problem (3.24), (3.26).

**Theorem 3.19.** Assume that \(\dim \mathcal{H} = \infty\), \(A \in \text{Sym}_d(\mathcal{H})\) is a densely defined operator, \(n_+(A) = n_-(A) < \infty\) and \(\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}\) is a boundary triplet for \(A^*\). Moreover, let \(C = (C_0, C_1)\) be an entire Nevanlinna pair in \(\mathcal{H}\), let \(\tau = \tau_C \in R_{\text{mer}}(\mathcal{H})\) be given by (2.9), let \(\mathcal{N}_t\) (\(t \in \mathbb{R}\)) be the set of all solutions of the problem (3.24), (3.26) and let \(\overline{EV}\) be the set of all eigenvalues of the same problem. Then:

(i) \(\overline{EV}\) is an infinite set without finite limit points, so that it can be written as an increasing infinite sequence \(\overline{EV} = \{\nu_k\}_0^\infty\). Moreover, \(\dim \mathcal{N}_t < \infty\) for any \(t \in \overline{EV}\).

(ii) for any \(f \in \mathcal{H}\) there exists a sequence \(\{f_k\}_0^\infty\) of eigenvectors \(f_k \in \mathcal{N}_t\) (\(t \in \overline{EV}\)) of the problem (3.24), (3.26) such that the following eigenvector expansion of \(f\) is valid:

\[
f = \sum_{k=\nu_-}^{\nu_+} f_k.
\]

Moreover, \(\tilde{A}_r \in \text{Self}_d(\tilde{\mathcal{H}})\) with \(\tilde{\mathcal{H}} \supset \mathcal{H}\) (for \(\tilde{A}_r\) see Theorem 2.14, (ii)), \(\overline{EV} = \sigma(\tilde{A}_r)(= \sigma_p(\tilde{A}_r))\) and the eigenvector \(f_k\) in (3.31) can be calculated via

\[
f_k = P_\mathcal{H}E(\{t_k\})f,
\]

where \(E(\cdot)\) is the orthogonal spectral measure of \(\tilde{A}_r\) and \(P_\mathcal{H}\) is the orthoprojection in \(\tilde{\mathcal{H}}\) onto \(\mathcal{H}\).

**Proof.** (i) It follows from Theorem 3.18 that \(\tilde{A}_r \in \text{Self}_d(\tilde{\mathcal{H}})\). Moreover, by (3.27) \(\overline{EV} = \sigma(\tilde{A}_r)(= \sigma_p(\tilde{A}_r))\). This yields statement (i).

(ii) Let \(f \in \mathcal{H}\). Then for any \(t_k \in \overline{EV}\) one has \(E(\{t_k\})f \in \mathcal{N}_{t_k}(\tilde{A})\) and by (3.27) \(f_k := P_\mathcal{H}E(\{t_k\})f \in \mathcal{N}_{t_k}\). Since \(\tilde{A}_r \in \text{Self}_d(\tilde{\mathcal{H}})\), it follows that \(f = \sum_{k=\nu_-}^{\nu_+} E(\{t_k\})f\). Therefore

\[
f = P_\mathcal{H}f = \sum_{k=\nu_-}^{\nu_+} P_\mathcal{H}E(\{t_k\})f = \sum_{k=\nu_-}^{\nu_+} f_k,
\]

which yields (3.31). \(\square\)

4. **Eigenfunction expansions for differential equations**

4.1. **Notations.** Let \(I = [a, b) (-\infty < a < b \leq \infty)\) be an interval of the real line (the endpoint \(b < \infty\) might be either included to \(I\) or not). Denote by \(AC(I)\) the set of functions \(f : I \rightarrow \mathbb{C}\) which are absolutely continuous on each compact interval \([a, b'] \subset I\).

Assume that \(\Delta : I \rightarrow \mathbb{R}\) is a nonnegative function integrable on each compact interval \([a, b'] \subset I\). Denote by \(L^2_{\Delta}(I)\) the semi-Hilbert space of Borel measurable functions \(f : I \rightarrow \mathbb{C}\) satisfying \(\|f\|_{L^2_{\Delta}} := \int_I \Delta(x)|f(x)|^2 dx < \infty\). The semi-definite inner product \((\cdot, \cdot)_{\Delta}\) in \(L^2_{\Delta}(I)\) is defined by \((f, g)_{\Delta} = \int_I \Delta(x)f(x)g(x) dx, \quad f, g \in L^2_{\Delta}(I)\). Moreover, let \(L^2(I)\) be the Hilbert space of the equivalence classes in \(L^2_{\Delta}(I)\) with respect to the semi-norm \(\|\cdot\|_{\Delta}\). Denote also by \(\pi_{\Delta}\) the quotient map from \(L^2_{\Delta}(I)\) onto \(L^2(I)\). Clearly, \(\ker \pi_{\Delta}\) coincides with the set of all Borel measurable functions \(f : I \rightarrow \mathbb{C}\) such that \(\Delta(x)f(x) = 0\) (a.e. on \(I\)).
As is known [9, Ch13.5] each distribution $\xi : \mathbb{R} \to B(\mathbb{C}^r)$ gives rise to the semi-Hilbert space $L^2(\xi; \mathbb{C}^r)$ of all Borel-measurable functions $g : \mathbb{R} \to \mathbb{C}^r$ such that $||g||^2_{L^2(\xi; \mathbb{C}^r)} = \int_\mathbb{R} (d\xi(t)g(t), g(t)) < \infty$. In the following we denote by $L^2(\xi; \mathbb{C}^r)$ the Hilbert space of all equivalence classes in $L^2(\xi; \mathbb{C}^r)$ with respect to the seminorm $|| \cdot ||_{L^2(\xi; \mathbb{C}^r)}$. Moreover, we denote by $\pi_\xi$ the quotient map from $L^2(\xi; \mathbb{C}^r)$ onto $L^2(\xi; \mathbb{C}^r)$.

With a $B(\mathbb{C}^r)$-valued distribution $\xi$ one associates the multiplication operator $\Lambda_\xi(= \Lambda_\xi^*)$ in $L^2(\xi; \mathbb{C}^r)$. The orthogonal spectral measure $E_\xi(\cdot)$ of $\Lambda_\xi$ is given on Borel sets $\delta \subset \mathbb{R}$ by

$$E_\xi(\delta)\tilde{g} = \pi_\xi(\chi_\delta g), \quad \tilde{g} \in L^2(\xi; \mathbb{C}^r), \quad g \in \tilde{g},$$

where $\chi_\delta$ is the indicator of $\delta$.

4.2. **Differential equations with the nontrivial weight.** Assume that $\mathcal{I} = [a, b)$ ($-\infty < a < b \leq \infty$) is an interval in $\mathbb{R}$ and let

$$l[y] = \sum_{k=1}^{r} (-1)^k (p_{r-k}(x)y^{(k)})^{(k)} + p_r(x)y$$

be a symmetric differential expression of an even order $n = 2r$ on $\mathcal{I}$ with real-valued coefficients $p_j(\cdot) : \mathcal{I} \to \mathbb{R}$. We assume that functions $p_0^{-1}$ and $p_j$, $j \in \{1, \ldots, r\}$ are integrable on each compact interval $[a, b] \subset \mathcal{I}$ (this means that the endpoint $a$ is regular for $l[y]$).

Following to [21, 24] we denote by $y[j]$, $j \in \{0, 1, \ldots, 2r\}$, the quasi-derivatives of a function $y : \mathcal{I} \to \mathbb{C}$ (here $y[0] = y$). Denote also by $\text{dom} l$ the set of all functions $y : \mathcal{I} \to \mathbb{C}$ such that $y[j] \in AC(\mathcal{I})$ for $j \leq 2r - 1$ and let $l[y] = y^{[2r]}$, $y \in \text{dom} l$. With a function $y \in \text{dom} l$ one associates the vector-functions $y^{(j)} : \mathcal{I} \to \mathbb{C}^r$, $j \in \{1, 2\}$, given by

$$y^{(1)} = y \oplus y^{[1]} \oplus \cdots \oplus y^{[r-1]}, \quad y^{(2)} = y^{[2r-1]} \oplus y^{[2r-2]} \oplus \cdots \oplus y^{[r]}.$$

We consider the differential equation

$$l[y] = \lambda \Delta(x)y, \quad x \in \mathcal{I}, \quad \lambda \in \mathbb{C}$$

with the weight $\Delta : \mathcal{I} \to \mathbb{R}$ integrable on each compact interval $[a, b] \subset \mathcal{I}$ and satisfying $\Delta(x) \geq 0$ a.e.on $\mathcal{I}$. In the following we assume that the weight $\Delta$ is nontrivial and not necessarily positive (see Definition 1.1). A function $y \in \text{dom} l$ is a solution of (4.4), if it satisfies (4.4) a.e. on $\mathcal{I}$. An $m$-component operator function

$$Y(x, \lambda) = (Y_1(x, \lambda), Y_2(x, \lambda), \ldots, Y_m(x, \lambda)) : \mathbb{C} \oplus \mathbb{C} \cdots \oplus \mathbb{C} \to \mathbb{C}, \quad x \in \mathcal{I}$$

with values in $B(\mathbb{C}^m, \mathbb{C})$ is called an operator solution of (4.4), if each component $Y_j(x, \lambda)$ is a (scalar) solution of (4.4). With each such a solution $Y(x, \lambda)$ we associate the operator functions $Y^{(j)} : \mathcal{I} \to B(\mathbb{C}^m, \mathbb{C}^r)$, $j \in \{1, 2\}$, given by $Y^{(1)}(x, \lambda) = (Y^{[j-1]}_k(x, \lambda))_{r \times m}$ and $Y^{(2)}(x, \lambda) = (Y^{[2r-j]}_k(x, \lambda))_{r \times m}$, $x \in \mathcal{I}$.

Denote by $\mathcal{D}_{\text{max}}$ the linear manifold in $L^2_\Delta(\mathcal{I})$ given by

$$\mathcal{D}_{\text{max}} = \{y \in \text{dom} l \cap L^2_\Delta(\mathcal{I}) : l[y] = \Delta(x)f_y(x) \text{ (a.e. on } \mathcal{I}) \}$$

with some $f_y \in L^2_\Delta(\mathcal{I})$.

Clearly if $y \in \mathcal{D}_{\text{max}}$ and $f_{1y}$ and $f_{2y}$ are two functions from (4.6), then $\pi_\Delta f_{1y} = \pi_\Delta f_{2y}$. Therefore for a given $y \in \mathcal{D}_{\text{max}}$ the function $f_y$ in (4.6) is defined uniquely up to $\Delta$-equivalence.

As is known, for any $y, z \in \mathcal{D}_{\text{max}}$ there exists the limit

$$[y, z]_b := \lim_{x \to b} ((y^{(1)}(x), z^{(2)}(x))_{\mathbb{C}^r} - (y^{(2)}(x), z^{(1)}(x))_{\mathbb{C}^r}).$$
This fact enables one to define the linear manifold $\mathcal{D}_{\min}$ in $L^2_\Delta(\mathcal{I})$ by setting

$$\mathcal{D}_{\min} = \{ y \in \mathcal{D}_{\max} : y^{(1)}(a) = y^{(2)}(a) = 0 \text{ and } [y, z]_b = 0 \text{ for every } z \in \mathcal{D}_{\max} \}$$

For $\lambda \in \mathbb{C}$ denote by $\mathcal{N}_\lambda$ the linear space of all solutions $y$ of (4.4) belonging to $L^2_\Delta(\mathcal{I})$ (clearly, $\mathcal{N}_\lambda \in \mathcal{D}_{\max}$). It turns out that the number $N_+ = \dim \mathcal{N}_\lambda$, $\lambda \in \mathbb{C}_+$ ($N_- = \dim \mathcal{N}_\lambda$, $\lambda \in \mathbb{C}_-$) does not depend on $\lambda \in \mathbb{C}_+$ (resp. $\lambda \in \mathbb{C}_-$). The numbers $N_\pm$ are called the formal deficiency indices of the equation (4.4). It turns out that $r \leq N_+ = N_- \leq 2r$. In the following we put $d := N_\pm$.

Below within this subsection we recall some results from our paper [20] concerning equation (4.4) with the nontrivial Weight $\Delta$.

**Theorem 4.1.** For the differential equation (4.4) the equalities

$$\tilde{y} = \pi_\Delta y, \quad S_{\min} \tilde{y} = \pi_\Delta f_y, \quad y \in \mathcal{D}_{\min};$$

$$\tilde{y} = \pi_\Delta y, \quad S_{\min} \tilde{y} = \pi_\Delta f_y, \quad y \in \mathcal{D}_{\min}$$

correctly define the linear operators $S_{\max} : \text{dom } S_{\max} \to L^2_\Delta(\mathcal{I})$ (the maximal operator) and $S_{\min} : \text{dom } S_{\min} \to L^2_\Delta(\mathcal{I})$ (the minimal operator) with the domains $\text{dom } S_{\max} = \pi_\Delta \mathcal{D}_{\max} \subset L^2_\Delta(\mathcal{I})$ and $\text{dom } S_{\min} = \pi_\Delta \mathcal{D}_{\min} \subset L^2_\Delta(\mathcal{I})$ respectively. Moreover, $S_{\min}$ is a closed densely defined symmetric operator with equal deficiency indices $n_{\pm}(S_{\min}) = d < \infty$ and $S_{\max} = S_{\min}^*$.

**Proposition 4.2.** Assume that $B = B^* \in \mathcal{B}(\mathbb{C}^r)$ and let $\mathcal{D}$ and $\mathcal{D}_*$ be linear manifolds in $L^2_\Delta(\mathcal{I})$ given by

$$\mathcal{D} = \{ y \in \mathcal{D}_{\max} : \cos B \cdot y^{(1)}(a) + \sin B \cdot y^{(2)}(a) = 0 \text{ and } [y, z]_b = 0, z \in \mathcal{D}_{\max} \}$$

$$\mathcal{D}_* = \{ y \in \mathcal{D}_{\max} : \cos B \cdot y^{(1)}(a) + \sin B \cdot y^{(2)}(a) = 0 \}$$

Then the equalities

$$\tilde{y} = \pi_\Delta y, \quad S \tilde{y} = \pi_\Delta f_y, \quad y \in \mathcal{D}; \quad \tilde{y} = \pi_\Delta y, \quad S^* \tilde{y} = \pi_\Delta f_y, \quad y \in \mathcal{D}_*$$

correctly define the linear operators $S : \text{dom } S \to L^2_\Delta(\mathcal{I})$ and $S^* : \text{dom } S^* \to L^2_\Delta(\mathcal{I})$ with the domains $\text{dom } S = \pi_\Delta \mathcal{D} \subset L^2_\Delta(\mathcal{I})$ and $\text{dom } S^* = \pi_\Delta \mathcal{D}_* \subset L^2_\Delta(\mathcal{I})$ respectively. Moreover, $S$ is a closed symmetric extension of $S_{\min}$ with the deficiency indices $n_{\pm}(S) = d - r$ and $S^*$ is the adjoint of $S$.

**Remark 4.3.** In the case of the equation (4.4) with the positive weight $\Delta$ statements of Theorem 4.1 and Proposition 4.2 are the well-known classical results (see e.g. [21, 24]).

**Proposition 4.4.** Let $B = B^* \in \mathcal{B}(\mathbb{C}^r)$, let $S$ and $S^*$ be operators defined in Proposition 4.2 and let $\Gamma_b = (\Gamma_{0b}, \Gamma_{1b})^T : \mathcal{D}_{\max} \to (\mathbb{C}^{d-r})^2$ be a surjective linear operator satisfying

$$[y, z]_b = (\Gamma_{0b} y, \Gamma_{1b} z) - (\Gamma_{1b} y, \Gamma_{0b} z), \quad y, z \in \mathcal{D}_{\max}$$

(according to [20] such an operator exists). Then:

(i) for each $\tilde{y} \in \text{dom } S^*$ there exists a unique $y \in \mathcal{D}_*$ such that $\pi_\Delta y = \tilde{y}$ and $\pi_\Delta f_y = S^* \tilde{y}$ (here $f_y$ is taken from (4.6));

(ii) the collection $\Pi = \{ \mathbb{C}^{d-r}, \Gamma_0, \Gamma_1 \}$ with operators $\Gamma_j : \text{dom } S^* \to \mathbb{C}^{d-r}$ given by

$$\Gamma_0 \tilde{y} = \Gamma_{0b} y, \quad \Gamma_1 \tilde{y} = -\Gamma_{1b} y, \quad \tilde{y} \in \text{dom } S^*$$

is a boundary triplet for $S^*$ (in (4.11) $y \in \tilde{y}$ is a function from statement (i)).

In the following with an operator $B = B^* \in \mathcal{B}(\mathbb{C}^r)$ we associate the $r$-component operator solutions $\varphi_y(x, \lambda) = (\varphi_1(x, \lambda), \varphi_2(x, \lambda), \ldots, \varphi_r(x, \lambda)) \in \mathcal{B}(\mathbb{C}^r, \mathbb{C}^r)$ and $\psi_B(x, \lambda) = (\psi_1(x, \lambda), \psi_2(x, \lambda), \ldots, \psi_r(x, \lambda)) \in \mathcal{B}(\mathbb{C}^r, \mathbb{C})$ of (4.4) defined by the initial values

$$\varphi_B^{(1)}(a, \lambda) = \sin B, \quad \varphi_B^{(2)}(a, \lambda) = -\cos B; \quad \psi_B^{(1)}(a, \lambda) = \cos B, \quad \psi_B^{(2)}(a, \lambda) = \sin B.$$
It easy to see that for each functions $f \in \mathcal{L}_2^2(\mathcal{I})$ with compact support the equality

$$
\hat{f}(t) = \int_{\mathcal{I}} \varphi_B^*(x,t) \Delta(x)f(x) \, dx
$$

defines a continuous functions $\hat{f} : \mathbb{R} \to \mathbb{C}^r$.

**Definition 4.5.** A distribution $\xi : \mathbb{R} \to \mathcal{B}(\mathbb{C}^r)$ is called a spectral function of the equation (4.4) if for each function $f \in \mathcal{L}_2^2(\mathcal{I})$ with compact support the Parseval equality $\|\hat{f}\|_{L^2(\xi;\mathbb{C}^r)} = \|f\|_{\Delta}$ holds.

If $\xi$ is a spectral function, then for each $f \in \mathcal{L}_2^2(\mathcal{I})$ the integral in (4.13) converges in $\mathcal{L}^2(\xi;\mathbb{C}^r)$ to a function $\hat{f} \in \mathcal{L}^2(\xi;\mathbb{C}^r)$, which is called the generalized Fourier transform of $f$. Moreover, the equality

$$
V_\xi \hat{f} = \pi_\xi \hat{f}, \quad \hat{f} \in \mathcal{L}_2^2(\mathcal{I}),
$$

where $\hat{f}$ is the Fourier transform of a function $f \in \tilde{\mathcal{I}}$, defines an isometry $V_\xi \in \mathcal{B}(\mathcal{L}_2^2(\mathcal{I}), \mathcal{L}^2(\xi;\mathbb{C}^r))$.

The $m$-component operator solution $Y(x, \lambda)$ of (4.4) given by (4.5) will be referred to the class $\mathcal{L}_2^2(\mathcal{I};\mathbb{C}^m)$ if $Y_j(\cdot, \lambda) \in \mathcal{L}_2^2(\mathcal{I})$ for any $j \in \{1, \ldots, m\}$. With each such a solution one associates the operator $Y(\lambda) : \mathbb{C}^m \to \mathcal{N}_\lambda$, given by $(Y(\lambda)h)(x) = Y(x, \lambda)h$, $h \in \mathbb{C}^m$.

A description of all spectral functions of the equation (4.4) is given by the following theorem.

**Theorem 4.6.** [20] Let $(\Gamma_0, \Gamma_1)^\top$ be the operator from Proposition 4.4. Then:

(i) For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique pair of operator solutions $v(\cdot, \lambda) \in \mathcal{L}_2^2(\mathcal{I};\mathbb{C}^r)$ and $u(\cdot, \lambda) \in \mathcal{L}_2^2(\mathcal{I};\mathbb{C}^{d-r})$ of (4.4) satisfying the boundary conditions:

$$
\cos B \cdot v^{(1)}(a, \lambda) + \sin B \cdot v^{(2)}(a, \lambda) = -I_r, \quad \Gamma_0 v(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
$$

$$
\cos B \cdot u^{(1)}(a, \lambda) + \sin B \cdot u^{(2)}(a, \lambda) = 0, \quad \Gamma_0 u(\lambda) = I_{d-r}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
$$

Moreover, the equalities

$$
M(\lambda) = \begin{pmatrix}
m_0(\lambda) & M_2(\lambda) \\
M_3(\lambda) & M_4(\lambda)
\end{pmatrix} : \mathbb{C}^r \oplus \mathbb{C}^{d-r} \to \mathbb{C}^r \oplus \mathbb{C}^{d-r}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
$$

$$
m_0(\lambda) = \sin B \cdot v^{(1)}(a, \lambda) - \cos B \cdot v^{(2)}(a, \lambda)
$$

$$
M_2(\lambda) = \sin B \cdot u^{(1)}(a, \lambda) - \cos B \cdot u^{(2)}(a, \lambda), \quad M_3(\lambda) = -\Gamma_1 v(\lambda), \quad M_4(\lambda) = -\Gamma_1 u(\lambda)
$$

define a function $M \in R[\mathbb{C}^r \oplus \mathbb{C}^{d-r}]$ with $M_4 \in R_u[\mathbb{C}^{d-r}]$.

(ii) For any $\tau \in \hat{R}(\mathbb{C}^{d-r})$ the equality

$$
m(\lambda) = m_0(\lambda) - M_2(\lambda)(\tau(\lambda) + M_4(\lambda))^{-1} M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
$$

defines a function $m \in R[\mathbb{C}^r]$ such that the spectral function $\xi = \xi_\tau$ of $m$ is a spectral function of the equation (4.4) and, conversely, for any spectral function $\xi$ of (4.4) there is a unique $\tau \in \hat{R}(\mathbb{C}^{d-r})$ such that $\xi$ is a spectral function of $m$ given by (4.16).

**Theorem 4.7.** [20] Let under the assumptions of Proposition 4.4 $\Pi = \{\mathbb{C}^{d-r}, \Gamma_0, \Gamma_1\}$ be the boundary triplet (4.11) for $S^*$. Assume also that $\tau \in \hat{R}(\mathbb{C}^{d-r})$, $\widehat{S}_\tau = \widehat{S}_\tau^* \subseteq \mathcal{C}(\tilde{\mathcal{S}})$ ($\tilde{\mathcal{S}} \subseteq \mathcal{L}_2^2(\mathcal{I})$) is the corresponding exit space extension of $S$, $\tau = \xi_\tau$ is the spectral function of the equation (4.4) (see Theorem 4.6, (ii)) and $\Lambda_\xi$ is the multiplication operator in $\mathcal{L}^2(\xi;\mathbb{C}^r)$.

Then there exists a unitary operator $\widehat{V} \in \mathcal{B}(\mathcal{L}_2^2(\mathcal{I};\mathbb{C}^r))$ such that $\widehat{V} \mathcal{L}_2^2(\mathcal{I}) = V_\xi$ and the operators $\widehat{S}_\tau$ and $\Lambda_\xi$ are unitarily equivalent by means of $\widehat{V}$. 
4.3. Eigenfunction expansions: calculation of eigenfunctions and uniform convergence. Below we suppose that for equation (4.4) the following assumptions hold:

(A1) $D_{\text{max}} \subset L^2_{\Delta}(I)$ is the linear manifold (4.6) and $(\Gamma_0, \Gamma_1)^{\top} : D_{\text{max}} \to (C^{d-r})^2$ is a surjective operator satisfying (4.10).

(A2) $B = B^* \in B(C)$ and $C = (C_0, C_1)$ is an entire Nevanlinna pair in $C^{d-r}$.

Consider the eigenvalue problem

\begin{align}
(4.17) & \\ l[y] = t\Delta y & \quad (4.18) & (\cos B)y^{(1)}(a) + (\sin B)y^{(2)}(a) = 0, \quad C_0(t)\Gamma_0 y + C_1(t)\Gamma_1 y = 0
\end{align}

with separated boundary conditions (4.18) depending on the parameter $t \in \mathbb{R}$. The set of all solutions of the problem (4.17), (4.18) for a given $t \in \mathbb{R}$ (i.e., the set of all $y \in \mathcal{N}_t$ satisfying (4.18)) will be denoted by $\mathcal{N}_t$. Clearly, $\mathcal{N}_t$ is a finite dimensional subspace in $L^2_{\Delta}(I)$.

**Definition 4.8.** A point $t \in \mathbb{R}$ such that $\mathcal{N}_t \neq \{0\}$ is called an eigenvalue of the problem (4.17), (4.18). The subspace $\mathcal{N}_t(\subset L^2_{\Delta}(I))$ for an eigenvalue $t$ is called an eigenspace and a function $y \in \mathcal{N}_t$ is called an eigenfunction of the problem (4.17), (4.18).

**Theorem 4.9.** Assume that for equation (4.4) with the nontrivial weight $\Delta$ the operator $S_{\text{min}}$ has the discrete spectrum. Moreover let the assumptions (A1) and (A2) at the beginning of the subsection hold, let $\mathcal{N}_t (t \in \mathbb{R})$ be the set of all solutions of the eigenvalue problem (4.17), (4.18) and let $EV$ be the set of all eigenvalues of the same problem. Then:

(i) $EV$ is an infinite countable subset of $\mathbb{R}$ without finite limit points, so that it can be written as an increasing infinite sequence $EV = \{t_k\}_{\nu^+}$, $t_k \in \mathbb{R}$;

(ii) for any $y \in L^2_{\Delta}(I)$ there exists a sequence $\{y_k\}_{\nu^+}$ of eigenfunctions $y_k \in \mathcal{N}_{t_k}$ ($t_k \in EV$) of the problem (4.17), (4.18) such that the following eigenfunction expansion holds:

\begin{equation}
(4.19) \quad y(x) = \sum_{k=\nu^{-}}^{\nu^{+}} y_k(x).
\end{equation}

The series in (4.19) converges in $L^2_{\Delta}(I)$, that is

\begin{equation}
(4.20) \quad \lim_{\nu^{-} \to \nu^{+}} \left\| y - \sum_{k=\nu^{-}}^{\nu^{+}} y_k \right\|_{\Delta} = 0.
\end{equation}

(iii) Let $\tau = \tau_C \in \tilde{R}_{\text{mer}}(C^{d-r})$ be given by (2.9). Then the function $y_k$ in (4.19) can be defined as a unique function in $\mathcal{N}_{t_k}$ such that

\begin{equation}
(4.21) \quad \pi_\Delta y_k = PE(\{t_k\})\pi_\Delta y.
\end{equation}

Here $E(\cdot)$ is the orthogonal spectral measure of the exit space extension $\tilde{S}_\tau = \tilde{S}_\tau^* \in \mathcal{C}(\tilde{S})$ of $S$ $(\tilde{S} \supset L^2_{\Delta}(I))$ corresponding to $\tau$ in the boundary triplet $\Pi = \{C^{d-r}, \Gamma_0, \Gamma_1\}$ for $S^*$ (see Propositions 4.2 and 4.4) and $P$ is the orthoprojection in $\tilde{S}$ onto $L^2_{\Delta}(I)$. Moreover, $\tilde{S}_\tau \in \text{Self}_{d}(\tilde{S})$ and $EV = \sigma(\tilde{S}_\tau)(= \sigma_p(\tilde{S}_\tau))$.

**Proof.** Together with (4.17), (4.18) consider the abstract eigenvalue problem

\begin{equation}
(4.22) \quad S^*\tilde{y} = t\tilde{y}, \quad C_0(t)\Gamma_0 \tilde{y} - C_1(t)\Gamma_1 \tilde{y} = 0.
\end{equation}

Denote by $\mathcal{N}_t (t \in \mathbb{R})$ the set of all solutions of the problem (4.22) and by $\overline{EV}$ the set of all eigenvalues of the same problem. In view of Theorem 3.19 the following assertions are valid:

(a1) $\tilde{S}_\tau \in \text{Self}_{d}(\tilde{S})$ and $\overline{EV} = \sigma(\tilde{S}_\tau)(= \sigma_p(\tilde{S}_\tau))$;
By using (4.9) and (4.11) one can easily verify that the series converges in \( L^2(\mathcal{I}) \) according to [20, Proposition 5.11] the equation (4.4) is definite, that is the equalities
\[
\xi = \sum_{k=\nu_-}^{\nu_+} \tilde{y}_k
\]
(the series converges in \( L^2(\mathcal{I}) \)). Moreover, \( \tilde{y}_k \) in (4.23) can be defined via
\[
\tilde{y}_k = PE(\{t_k\})\tilde{y}.
\]
By using (4.9) and (4.11) one can easily verify that \( \pi_\Delta \mathfrak{N}_t = \widetilde{\mathfrak{N}}_t, \ t \in \mathbb{R} \). Moreover, according to [20, Proposition 5.11] the equation (4.4) is definite, that is the equalities \( l[y] = t\Delta(x)y(x) \) and \( \Delta(x)y(x) = 0 \) (a.e. on \( \mathcal{I} \)) yields \( y = 0 \). Hence \( \ker (\pi_\Delta \upharpoonright \mathfrak{N}_t) = \{0\} \), so that \( \pi_\Delta \upharpoonright \mathfrak{N}_t (t \in \mathbb{R}) \) is an isomorphism of \( \mathfrak{N}_t \) onto \( \widetilde{\mathfrak{N}}_t \). Therefore \( EV = \widetilde{EV} \), which in view of (a1) and (a2) yields statement (i) and the equality \( EV = \sigma(\widetilde{S}_r) \).

Next assume that \( y \in L^2(\mathcal{I}) \). Then by assertion (a2) \( \tilde{y} := \pi_\Delta y \) admits the representation (4.23) with \( \tilde{y}_k \in \mathfrak{N}_k \) given by (4.24). As was shown, there exists a unique \( y_k \in \mathfrak{N}_k \) such that \( \pi_\Delta y_k = \tilde{y}_k \), that is (4.21) holds. Moreover, (4.23) yields (4.20). This proves statements (ii) and (iii).

Our next goal is to provide an explicit method for calculation of eigenfunctions \( y_k \) in the expansion (4.19). To this end we need the following definition.

**Definition 4.10.** A discrete distribution \( \xi : \mathbb{R} \to B(\mathbb{C}^r) \) which is a spectral function of the equation (4.4) in the sense of Definition 4.5 is called a discrete spectral function of (4.4).

If \( \xi = \{F, \Xi\} \) is a discrete spectral function of (4.4) with \( F = \{t_k\}_{\nu_-}^{\nu_+} \in F \) and \( \Xi = \{\xi_k\}_{\nu_-}^{\nu_+} \), \( 0 \leq \xi_k \in B(\mathbb{C}^r) \), \( \xi_k \neq 0 \), then a function \( g \in L^2(\xi; \mathbb{C}^r) \) can be identified with a sequence \( g = \{g_k\}_{\nu_-}^{\nu_+} \) \( (g_k = g(t_k) \in \mathbb{C}^r) \) such that \( \sum_{k=\nu_-}^{\nu_+} (\xi_k g_k, g_k) \leq \infty \).

Assume that \( \xi = \{F, \Xi\} \) is a discrete spectral function of (4.4) with \( F = \{t_k\}_{\nu_-}^{\nu_+} \) and \( \Xi = \{\xi_k\}_{\nu_-}^{\nu_+} \). Then the generalized Fourier transform \( \hat{f} \) of a function \( f \in L^2(\mathcal{I}) \) with compact support (see (4.13)) can be identified with a sequence \( \hat{f} = \{\hat{f}_k\}_{\nu_-}^{\nu_+} \), \( \hat{f}_k = \hat{f}(t_k) \in \mathbb{C}^r \), given by
\[
\hat{f}_k = \int_{\mathcal{I}} \varphi^*_B(x, t_k) \Delta(x) f(x) \, dx, \quad t_k \in F.
\]
(Here the integral exists as the Lebesgue integral). The sequence \( \hat{f} = \{\hat{f}_k\}_{\nu_-}^{\nu_+} \) satisfies the Parseval equality
\[
(||f||_\Delta^2 =) \int_{\mathcal{I}} \Delta(x)|f(x)|^2 \, dx = \sum_{k=\nu_-}^{\nu_+} (\xi_k \hat{f}_k, \hat{f}_k) \quad = ||\hat{f}||_{L^2(\xi; \mathbb{C}^r)}^2.
\]
For an arbitrary function \( f \in L^2(\mathcal{I}) \) its Fourier transform is a sequence \( \hat{f} = \{\hat{f}_k\}_{\nu_-}^{\nu_+} \in L^2(\xi; \mathbb{C}^r) \) with \( \hat{f}_k = \hat{f}(t_k) \in \mathbb{C}^r \), such that
\[
\lim_{\nu_- \to a} \sum_{k=\nu_-}^{\nu_+} (\xi_k(\hat{f}_k - \int_{[a,b]} \varphi^*_B(x, t_k) \Delta(x) f(x) \, dx), \hat{f}_k - \int_{[a,b]} \varphi^*_B(x, t_k) \Delta(x) f(x) \, dx) = 0
\]
The last equality will be written in the form of the integral (4.25) as well (one says that this integral converges in \( L^2(\xi; \mathbb{C}^r) \)). The vectors \( \hat{f}_k \in \mathbb{C}^r \) (see (4.25)) will be called the Fourier coefficients of a function \( f \in L^2(\mathcal{I}) \) (with respect to the spectral function \( \xi \)).
For a discrete spectral function $\xi = \{F, \Xi\}$ with $F = \{t_k\}_{\nu_+}^\nu$ and $\Xi = \{\xi_k\}_{\nu_-}^\nu$, the isometry $V_\xi \in B(L^2_\Delta(\mathcal{I}), L^2(\xi; \mathbb{C}^r))$ is defined by the same formula (4.14), but now $\hat{f} = \{\hat{f}_k\}_{\nu_+}^\nu$ is a sequence (4.25). Moreover, [20, Proposition 3.8] implies that

$$
V_\xi \hat{g} = \pi_\Delta \left( \sum_{k=\nu_-}^{\nu_+} \varphi_B(\cdot, t_k)\xi_k g_k \right), \quad \hat{g} \in L^2(\xi; \mathbb{C}^r), \quad g = \{g_k\}_{\nu_-}^{\nu_+} \in \hat{g},
$$

where the series converges in $L^2_\Delta(\mathcal{I})$.

In the following theorem we specify explicit formulas for calculations of eigenfunctions $y_k$ in the expansion (4.19) of $y$.

**Theorem 4.11.** Let the assumptions be the same as in Theorem 4.9. Moreover, let $M \in R[\mathbb{C}^r \oplus \mathbb{C}^{d^*}]$ be the operator function (4.15). Then:

(i) The equality

$$
m(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
$$

defines a function $m \in R_{\text{mer}}[\mathbb{C}^r]$ and according to Assertion 2.5 the (discrete) spectral function $\xi$ of $m$ is $\xi = \{F_m, \Xi\}$, where $F_m = \{t_k\}_{\nu_+}^\nu$ is the set of all poles of $m$ and $\Xi = \{\xi_k\}_{\nu_-}^\nu$ with $\xi_k \in B(\mathbb{C}^r)$ given by $\xi_k = -\text{res} m_k$. Moreover, $\xi$ is a discrete spectral function of the equation (4.4).

(ii) $EV = F_m = \{t_k\}_{\nu_+}^\nu$ and an eigenfunction $y_k$ in the expansion (4.19) of $y \in L^2_\Delta(\mathcal{I})$ admits the representation

$$
y_k(x) = \varphi_B(x, t_k)\xi_k \hat{g}_k,
$$

where $\hat{g}_k$ is the Fourier coefficient (4.25) of $y$ (with respect to the spectral function $\xi = \xi$).

**Proof.** Let $\tau = \tau_C \in R_{\text{mer}}(\mathbb{C}^{d^*})$ be given by (2.9). One can easily verify that

$$-(\tau(\lambda) + M_4(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
$$

and therefore (4.16) can be written in the form (4.27). According to Theorem 4.6 $m \in R[\mathbb{C}^r]$ and the spectral function $\xi$ of $m$ is a spectral function of the equation (4.4).

Assume that $\Pi = \{\mathbb{C}^{d^*}, \Gamma_0, \Gamma_1\}$ is the boundary triplet (4.11) for $S^*$ (see Propositions 4.2 and 4.4) and let $\tilde{S}_r = \tilde{S}_r^* \in \mathcal{C}(\tilde{S})$ ($\tilde{S} \supset L^2_\Delta(\mathcal{I})$) be an exit space extension of $S$ corresponding to $\tau$. Then according to Theorem 4.9 $\tilde{S}_r \in \text{Self}_d(\tilde{S})$ and $EV = \sigma(\tilde{S}_r)$. Moreover, by Theorem 4.7 $\tilde{S}_r$ is unitarily equivalent to the multiplication operator $\Lambda_\xi$ in $L^2(\xi; \mathbb{C}^r)$ and hence $\Lambda_\xi$ has the discrete spectrum. Therefore $\xi = \{F, \Xi\}$ is a discrete spectral function with $F = \sigma(\Lambda_\xi) \in F$ and Assertion 2.5 implies that $m \in R_{\text{mer}}[\mathbb{C}^r]$ and $F = F_m$. On the other hand, $\sigma(\Lambda_\xi) = \sigma(\tilde{S}_r) = EV$ and, consequently, $EV = F_m$.

Now it remains to prove (4.28). Let $E(\cdot)$ be the orthogonal spectral measure of $\tilde{S}_r$ and let $P$ be the orthoprojection in $\tilde{S}$ onto $L^2_\Delta(\mathcal{I})$. Then by Theorem 4.7 for any $t_k \in EV$ one has

$$
PE(\{t_k\}) \upharpoonright L^2_\Delta(\mathcal{I}) = V_\xi^* E(\{t_k\})V_\xi,
$$

where $V_\xi \in B(L^2_\Delta(\mathcal{I}), L^2(\xi; \mathbb{C}^r))$ is the isometry (4.14) and $E(\cdot)$ is the orthogonal spectral measure of $\Lambda_\xi$ given by (4.1). According to Theorem 4.9 $y_k$ is defined by (4.21). Combining (4.29) with (4.21) one gets

$$
\pi_\Delta y_k = V_\xi^* E(\{t_k\})V_\xi \pi_\Delta y, \quad y \in L^2_\Delta(\mathcal{I}).
$$

It follows from (4.14) that $V_\xi \pi_\Delta y = \pi_\xi \hat{y}$, where $\hat{y} = \{\hat{y}_k\}_{\nu_-}^{\nu_+}$ ($\hat{y}_k = \hat{y}(t_k))$ is the sequence of the Fourier coefficients of $y$. Substituting this equality into (4.30) and taking (4.1) and (4.26) into account one obtains $\pi_\Delta y_k = \pi_\Delta(\varphi_B(\cdot, t_k)\xi_k \hat{g}_k)$. This yields (4.28).
Let \( \dim \mathcal{H} < \infty \) and let \( C = (C_0, C_1) \) be a Nevanlinna pair in \( \mathcal{H} \). Then according to Lemma 2.8 the subspace \( \mathcal{K} := \ker C_1(\lambda) \subset \mathcal{H} \) does not depend on \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Let \( \mathcal{H}_0 := \mathcal{H} \ominus \mathcal{K} \), let \( C_{0j}(\lambda), j \in \{0, 1\} \), and \( C_{10}(\lambda) \) be entries of the block representations (2.10) of \( C_0(\lambda) \) and \( C_1(\lambda) \) and let \( \tilde{C}_1(\lambda) \in \mathcal{B}(\mathcal{H}) \) be given by (2.11). Then according to Lemma 2.8 \( \ker \tilde{C}_1(\lambda) = \{0\} \) and the relations (2.12) define a Nevanlinna function \( \tau_0 \in R[\mathcal{H}_0] \). Let \( B_{\tau_0,\infty} \in \mathcal{B}(\mathcal{H}_0) \) and \( D_{\tau_0,\infty} : \text{dom} D_{\tau_0,\infty} \to \mathcal{H}_0 \) (\( \text{dom} D_{\tau_0,\infty} \subset \mathcal{H}_0 \)) be operators corresponding to \( \tau_0 \) in accordance with Proposition 2.1, (i). In the following with a pair \( C \) we associate a linear relation \( \eta_C \) in \( \mathcal{H} \) given by

\[
\eta_C = \{\{h_0, -D_{\tau_0,\infty}h + B_{\tau_0,\infty}h_0 + k\} : h \in \text{dom} D_{\tau_0,\infty}, h_0 \in \mathcal{H}_0, k \in \mathcal{K}\}.
\]

If \( \ker C_1(\lambda) = \{0\} \) for some (and hence all) \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), then definition of \( \eta_C \) can be rather simplified. Namely, in this case \( \tilde{C}_1(\lambda) = C_1(\lambda) \), \( C_{00}(\lambda) = C_0(\lambda) \) and hence \( \tau_0(\lambda) = \tau_C(\lambda) = -C_1^{-1}(\lambda)C_0(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), and

\[
\eta_C = \{\{h_0, -D_{\tau_0,\infty}h + B_{\tau_0,\infty}h_0\} : h \in \text{dom} D_{\tau_0,\infty}, h_0 \in \mathcal{H}_0\}.
\]

In the following theorem we provide sufficient conditions for the uniform convergence of the eigenfunction expansion (4.19).

**Theorem 4.12.** Let under the assumptions of Theorem 4.9 \( \eta_C \) be the linear relation in \( \mathbb{C}^{d-r} \) defined by (4.31) and let \( y \in \text{dom} I \cap L^2_\Delta(\mathcal{I}) \) be a function such that: (i) the equality \( l[y] = \Delta(x)f_y(x) \) (a.e. on \( \mathcal{I} \)) holds with some \( f_y \in L^2_\Delta(\mathcal{I}) \); (ii) the boundary conditions

\[
(\cos B)y^{(1)}(a) + (\sin B)y^{(2)}(a) = 0, \quad \{\Gamma_{0y}, -\Gamma_{1y}\} \in \eta_C
\]

are satisfied. Then

\[
y^{(j)}(x) = \sum_{k=0}^{\nu_+} y^{(j)}_k(x), \quad j \in \{0, 1, \ldots, 2r - 1\},
\]

where \( y_k \) are eigenfunctions from the expansion (4.19) of \( y \). The series in (4.34) converges absolutely for any \( x \in \mathcal{I} \) and uniformly on each compact interval \([a, b'] \subset \mathcal{I}\).

**Proof.** Let \( \tau = \tau_C \in R_{nec}(\mathbb{C}^{d-r}) \) be given by (2.9) and let \( \tau_0 \) and \( \mathcal{K} \) be the operator and multivalued parts of \( \tau \) respectively. Then by Lemma 2.8 \( \mathcal{K} = \ker C_1(\lambda) \) and \( \tau_0(\lambda) = -\tilde{C}_1^{-1}(\lambda)C_0(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \). This implies that \( \eta_C = \eta_\tau \), where \( \eta_\tau \in \tilde{C}(\mathbb{C}^{d-r}) \) is the linear relation defined in [20, Theorem 2.4].

Assume that \( \xi \) is a discrete spectral function of (4.4) defined in Theorem 4.11. As was shown in the proof of this theorem equality (4.27) is equivalent to (4.16) and according to Theorem 4.6 \( \xi \) is a spectral function corresponding to \( \tau \) in the sense of [20] (see Definition 4.5). Moreover, in view of (4.28) the expansion (4.19) can be written as the inverse Fourier transform

\[
y(x) = \int_{\mathbb{R}} \varphi_B(x, t) d\xi(t)\hat{y}(t),
\]

where \( \hat{y} \) is the Fourier transform (4.13) of \( y \). Now the required statement follows from [20, Theorem 5.14].}

In the case of a selfadjoint eigenvalue problem (1.1), (1.3) the above results take a rather simpler form. Namely, the following corollary holds.

**Corollary 4.13.** Assume that for equation (4.4) with the nontrivial weight \( \Delta \) the operator \( S_{\min} \) has the discrete spectrum and let the assumption (A1) at the beginning of Section 4.3 be satisfied. Moreover, let \( \Re t \ (t \in \mathbb{R}) \) be the set of all solutions of the selfadjoint eigenvalue problem (1.1), (1.3) with \( \lambda = t \in \mathbb{R} \), \( B = B^* \in \mathcal{B}(\mathbb{C}^r), B_1 = B_1^* \in \mathcal{B}(\mathbb{C}^{d-r}) \) and let \( EV \) be the set of all eigenvalues of this problem. Then:
(i) Statements of Theorem 4.9 are valid and, moreover, eigenspaces $\mathfrak{N}_{tk}$ are mutually orthogonal in $L^2_\lambda (I)$.

(ii) Statements of Theorem 4.11 hold with $C_0(\lambda) = \cos B_1$ and $C_1(\lambda) = \sin B_1$ in (4.27).

(iii) Assume that $y \in \text{dom} l \cap L^2_\lambda (I)$ is a function such that the equality $[y] = \Delta(x)f_y(x)$ holds a.e. on $I$ with some $f_y \in L^2_\lambda (I)$ and the boundary conditions (1.3) are satisfied. Then statements of Theorem 4.12 are valid.

Proof. Clearly, the equalities $C_0(\lambda) = \cos B_1$, $C_1(\lambda) = \sin B_1$, $\lambda \in \mathbb{C}$, defines a pair $C = (C_0, C_1) \in \text{ENP}(C^{d-r})$, for which the boundary conditions (4.18) take the form (1.3). Therefore statements (i) and (ii) are implied by Theorems 4.9 and 4.11 respectively.

Next, $\tau_C(\lambda) = \theta$, $\lambda \in \mathbb{C}$, with $\theta = \theta^* = \{(h, h') \in H^2 : (\cos B_1)h + (\sin B_1)h' = 0\}$ and hence $K = \ker (\sin B_1) = \text{mul} \theta$, $\tau_0(\lambda) = \theta_\text{op}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Clearly, $B_{\tau_0 \infty} = 0$ and the equality $\text{Im} \tau_0(\lambda) = 0$ yields $\text{dom} D_{\tau_0 \infty} = H_0$ and $D_{\tau_0 \infty} = \theta_\text{op}$. Therefore by (4.31) $\eta_C = -\theta$ and the second boundary condition in (4.33) takes the form $\{\Gamma_0y, \Gamma_1y\} \in \theta$, which is equivalent to the second equality in (1.3). Now statement (iii) follows from Theorem 4.12. \qed

4.4. The case of a quasiregular equation. Recall that differential equation (4.4) is called quasiregular if it has the maximal formal deficiency indices $d = 2r$ and regular if it is given on a compact interval $I = [a, b]$ (the latter implies that $p^{-1}_0$, $p_j$, $j \in \{1, 2, \ldots, r\}$ and $\Delta$ are integrable on $I$). Clearly, each regular equation (4.4) is quasiregular.

Proposition 4.14. If equation (4.4) with the nontrivial weight $\Delta$ is quasiregular, then the operator $S_{\text{min}}$ has the discrete spectrum.

Proof. Assume that (4.4) is quasiregular. Then according to [20, Proposition 5.2] there exists a quasiregular definite Hamiltonian system on $I$ such that the minimal operator $T_{\text{min}}$ generated by this system is unitarily equivalent to $S_{\text{min}}$. Since $T_{\text{min}}$ has the discrete spectrum (see e.g. [2, 16]), so is $S_{\text{min}}$. \qed

Remark 4.15. For the quasiregular equation (4.4) $\varphi_C(\cdot, t_k) \in L^2_\lambda (I; C^s)$ and hence for any $f \in L^2_\lambda (I)$ the integral in (4.25) exists as the Lebesgue integral. Therefore in this case the Fourier coefficients $\hat{f}_k$ of $f$ are defined by (4.25) independently on the spectral function $\xi$.

According to [13] for quasiregular equation (4.4) the equalities

\begin{equation}
\Gamma_{0b}y := \lim_{x \to b}((\psi^{(2)}_C(x, 0))^*y^{(1)}(x) - (\psi^{(1)}_C(x, 0))^*y^{(2)}(x))
\end{equation}

\begin{equation}
\Gamma_{1b}y := \lim_{x \to b}[-(\varphi^{(2)}_C(x, 0))^*y^{(1)}(x) + (\varphi^{(1)}_C(x, 0))^*y^{(2)}(x)], \quad y \in D_{\text{max}}.
\end{equation}

define a surjective operator $\Gamma_{0b, \Gamma_{1b}}^\top : D_{\text{max}} \to (C^s)^2$ satisfying (4.10).

In the following two propositions we provide some peculiarities of the previous results for quasiregular equations.

Proposition 4.16. Assume that equation (4.4) with the nontrivial weight $\Delta$ is quasiregular and let assumptions (A1) and (A2) at the beginning of Section 4.3 be satisfied with the operator $\Gamma_{0b, \Gamma_{1b}}^\top$ given by (4.35) and (4.36). Moreover, let $\mathfrak{N}_t (t \in \mathbb{R})$ be the set of all solutions of the eigenvalue problem (4.17), (4.18) and let $\text{EV}$ be the set of all eigenvalues of the same problem. Then statements of Theorems 4.9, 4.11 and 4.12 are valid. Moreover, in this case the operator-function $m \in R_{\text{mer}}[\mathbb{C}^s]$ in Theorem 4.11 can be calculated via the linear-fractional transform

\begin{equation}
m(\lambda) = (C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda))^{-1}(C_0(\lambda)w_2(\lambda) + C_1(\lambda)w_4(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\end{equation}
with the operator coefficients $w_j(\lambda) \in B(C^r)$ given by

$$
w_1(\lambda) = I_r + \lambda \int \varphi_C^*(x, 0) \Delta(x) \varphi_C(x, \lambda) \, dx, \quad w_2(\lambda) = \lambda \int \varphi_C^*(x, 0) \Delta(x) \psi_C(x, \lambda) \, dx
$$

$$
w_3(\lambda) = -\lambda \int \varphi_C^*(x, 0) \Delta(x) \varphi_C(x, \lambda) \, dx, \quad w_4(\lambda) = I_r - \lambda \int \varphi_C^*(x, 0) \Delta(x) \psi_C(x, \lambda) \, dx.
$$

**Proof.** Application of [18, Theorem 6.16 and Proposition 6.11] to the same Hamiltonian system as in the proof of Proposition 4.14 gives (4.37). This and Proposition 4.14 yield the result. \[\square\]

For regular equation (4.4) one can put in the assumption (A1) $\Gamma_{0b} y = y^{(1)}(b)$ and $\Gamma_{1b} y = y^{(2)}(b), \ y \in D_{\max}$. In this case the boundary conditions (4.18) take the form

$$
(\cos B)y^{(1)}(a) + (\sin B)y^{(2)}(a) = 0, \quad C_0(t)y^{(1)}(b) + C_1(t)y^{(2)}(b) = 0.
$$

**Proposition 4.17.** Let equation (4.4) with the nontrivial weight $\Delta$ be regular and let the assumption (A2) be satisfied. Then statements of Theorems 4.9, 4.11 and 4.12 are valid for the eigenvalue problem (4.17), (4.38). Moreover, in this case the boundary conditions (4.33) take the form

$$
(\cos B)y^{(1)}(a) + (\sin B)y^{(2)}(a) = 0, \quad \{y^{(1)}(b), -y^{(2)}(b)\} \in \eta_C
$$

and the operator function $m \in R_{\text{mer}}[C^r]$ in Theorem 4.11 can be calculated via

$$m(\lambda) = (C_0(\lambda)\varphi_B^{(1)}(b, \lambda) + C_1(\lambda)\varphi_B^{(2)}(b, \lambda))^{-1}(C_0(\lambda)\psi_B^{(1)}(b, \lambda) + C_1(\lambda)\psi_B^{(2)}(b, \lambda)), \ \lambda \in \mathbb{C} \setminus \mathbb{R}.
$$

**Proof.** The required statements are implied by Proposition 4.14 and [20, Theorem 5.15]. \[\square\]

In the case $r = 1$ (4.4) takes the form of the Sturm-Liouville equation (1.5). Below we prove Theorems 1.3, 1.4 and Remark 1.5 concerning quasiregular equation (1.5).

**Proof.** In view of (4.32) the linear relation $\eta_C$ in $\mathbb{C}$ is defined as follows: (i) in Case 1 $\eta_C = \{0\} \oplus \mathbb{C}$; (ii) in Case 2 $\eta_C = \{\{h, -D_x h\} : h \in \mathbb{C}\}$; (iii) in Case 3 $\eta_C = \{0\}$. Now application of Proposition 4.16 to the eigenvalue problem (1.5), (1.7) gives Theorems 1.3 and 1.4. Finally, Remark 1.5 is implied by Proposition 4.17. \[\square\]

### 4.5. Example

Consider the eigenvalue problem

$$
y'' = \lambda y, \quad x \in I = [0, 1], \ \lambda \in \mathbb{C}
$$

$$
y'(0) = 0, \quad \lambda y(1) - y'(1) = 0,
$$

i.e., the eigenvalue problem (1.5), (1.7) with $p(x) \equiv 1$, $q(x) \equiv 0$, $\Delta(x) \equiv 1$, $x \in I = [0, 1]$, in (1.5) and $B = \frac{\pi}{2}$, $C_0(\lambda) = \lambda$, $C_1(\lambda) = -1$ in (1.7). Since the equation (4.39) is regular and $C = (C_0, C_1) \in \text{ENP}(\mathbb{C})$, we may apply Theorems 1.3, 1.4 and Remark 1.5 to the problem (4.39),(4.40).

The immediate checking shows that the solutions $\varphi_B(\cdot, \lambda)$ and $\psi_B(\cdot, \lambda)$ of (4.39) are $\varphi_B(x, \lambda) = \cos(\sqrt{\lambda} x)$ and $\psi_B(x, \lambda) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} x)$. Hence

$$
\varphi_B(1, \lambda) = \cos \sqrt{\lambda}, \quad \varphi_B'(1, \lambda) = -\sqrt{\lambda} \sin \sqrt{\lambda}
$$

$$
\psi_B(1, \lambda) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}, \quad \psi_B'(1, \lambda) = \cos \sqrt{\lambda}
$$

and according to Theorem 1.3 and Remark 1.5 the equality $m(\lambda) = \frac{\Phi(\lambda)}{\Psi(\lambda)}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, with entire functions

$$
\Phi(\lambda) = \psi_B(1, \lambda) C_0(\lambda) + \psi_B'(1, \lambda) C_1(\lambda) = \sqrt{\lambda} \sin \sqrt{\lambda} - \cos \sqrt{\lambda}
$$

$$
\Psi(\lambda) = \varphi_B(1, \lambda) C_0(\lambda) + \varphi_B'(1, \lambda) C_1(\lambda) = \lambda \cos \sqrt{\lambda} + \sqrt{\lambda} \sin \sqrt{\lambda}, \ \lambda \in \mathbb{C}
$$
defines a meromorphic Nevanlinna function \( m \) such that the set \( \text{EV} \) of all eigenvalues of the problem \((4.39), (4.40)\) coincides with the set of all poles of \( m \). Let \( \Phi \) and \( \Psi \) be the sets of all real zeros of \( \Phi \) and \( \Psi \) respectively. Assume also that

\[
S = \{s_k\}_0^\infty, \quad 0 = s_0 < s_1 < \cdots < s_k < \ldots
\]

is the set of all nonnegative solutions of the equation \( s = -\tan s \). It is easy to see that \( \Phi \) defines a meromorphic Nevanlinna function \( B \). Let \( \Phi(0) = 0 \) and \( \Phi'(0) = 1 \) and consequently Case 1 in Theorem 1.4 holds.

Next, the derivative \( \Psi' \) is

\[
\Psi'(t) = \frac{1}{2}(3\cos \sqrt{t} + \frac{1}{\sqrt{t}}\sin \sqrt{t} - \sqrt{t}\sin \sqrt{t}), \quad t \in (0, \infty).
\]

Hence \( \Psi'(0) = \lim_{t \to +0} \Psi'(t) = 2 \) and the equality \( \frac{\sin s_k}{s_k} = -\cos s_k \) yields

\[
\Psi'(s_k^2) = \frac{1}{2}(2\cos s_k - s_k \sin s_k), \quad k \in \mathbb{N}.
\]

Let \( \hat{\xi}_k = \text{res}_{s_k^2} m \). Then \( \hat{\xi}_k = \frac{\Phi(s_k^2)}{\Psi'(s_k^2)} \) and, consequently, \( \hat{\xi}_0 = -\frac{1}{2} \).

\[
\hat{\xi}_k = \frac{2(s_k \sin s_k - \cos s_k)}{2\cos s_k - s_k \sin s_k} = \frac{2(s_k\tan s_k - 1)}{2 - s_k\tan s_k} = -\frac{2s_k^2 + 1}{s_k^2 + 2}, \quad k \in \mathbb{N}.
\]

Hence by (1.9) the eigenfunctions \( y_k \) in the expansion (1.4) are

\[
y_0(x) = \frac{1}{2}\hat{y}_0, \quad y_k(x) = \frac{2s_k^2 + 1}{s_k^2 + 2}\hat{y}_k \cos(s_kx), \quad k \in \mathbb{N},
\]

where \( \hat{y}_k \) are the Fourier coefficients \( y \):

\[
\hat{y}_0 = \int_{[0,1]} y(x) \, dx, \quad \hat{y}_k = \int_{[0,1]} y(x) \cos(s_kx) \, dx.
\]

Note also that the Nevanlinna function \( \tau \) defined before Theorem 1.4 is \( \tau(\lambda) = \lambda \). Hence \( B_\infty = 1 \) and consequently Case 1 in Theorem 1.4 holds.

Now applying Theorems 1.3, 1.4 and Remark 1.5 we arrive at the following assertion.

**Assertion 4.18.** Each function \( y \in L^2([0,1]) \) admits the representation

\[
y(x) = b_0 + \sum_{k=1}^\infty b_k \cos(s_kx),
\]

where \( \{s_k\}_1^\infty \) \((s_k < s_{k+1})\) is the set of all positive solutions of the equation \( s = -\tan s \) and

\[
b_0 = \frac{1}{2} \int_{[0,1]} y(x) \, dx, \quad b_k = \frac{2s_k^2 + 1}{s_k^2 + 2} \int_{[0,1]} y(x) \cos(s_kx) \, dx.
\]

(the series in (4.41) converges in \( L^2([0,1]) \)). If in addition \( y, y' \in AC([0,1]), y'' \in L_2([0,1]) \) and \( y'(0) = 0, y(b) = 0 \), then the series in (4.41) converges uniformly on \([0,1]\).

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