Asymptotic optimality of twist-untwist protocols for Heisenberg scaling in atomic interferometry

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Twist-untwist protocols for quantum metrology consist of a serial application of: 1. unitary nonlinear dynamics (e.g., spin squeezing or Kerr nonlinearity), 2. parameterized dynamics $U(\phi)$ (e.g., a collective rotation or phase space displacement), 3. time reversed application of step 1. Such protocols are known to produce states that allow Heisenberg scaling for experimentally accessible estimators of $\phi$ even when the nonlinearities are applied for times much shorter than required to produce Schrödinger cat states. In this work, we prove that twist-untwist protocols provide the lowest estimation error among quantum metrology protocols that utilize two calls to a weak evolution and a readout involving only measurement of a spin operator $\vec{n} \cdot \hat{J}$, asymptotically in the number of particles. We consider the following physical settings: all-to-all interactions generated by one-axis twisting $J_z^2$ (e.g., interacting Bose gases), constant finite range spin-spin interactions of distinguishable or bosonic atoms (e.g., trapped ions or Rydberg atoms, or lattice bosons). In these settings, we further show that the optimal twist-untwist protocols asymptotically achieve 85% and 92% of the respective quantum Cramér-Rao bounds. We show that the error of a twist-untwist protocol can be decreased by a factor of $L$ without an increase in the noise of the spin measurement if the twist-untwist protocol can be noiselessly iterated as an $L$ layer quantum alternating operator ansatz.

I. INTRODUCTION

Advances in experimental implementations of squeezing-enhanced quantum metrology protocols [1–5] emphasize the fact that non-classicality and many-body entanglement are resources for near-term quantum technologies. In particular, quantum circuits consisting of alternating squeezing and unsqueezing operations have allowed to amplify signals of microwave photons [6], cooled mechanical oscillators [7], and atomic ensembles (twist-untwist protocols) [8, 9]. The possibility of integrating these circuits as modules of variational quantum sensing algorithms further suggests that squeezing-enhanced sensing could be utilized in near-term quantum computers [10].

Controllable interatomic interactions such as magnetic or optical Feshbach resonances [11] are central to entanglement generation in hybrid atomic-optical systems [12–14]. In particular, control of two-body elastic scattering allows to generate entanglement by the phenomenon of spin squeezing [15–17]. Although spin squeezing can be analyzed by employing analogies with continuous-variable quadrature squeezing, it is a many-body quantum effect, in the sense that there are spin squeezing quantifiers that imply particle entanglement of a given many-body state [18]. In contrast, continuous-variable squeezing is not sufficient for entanglement despite it being the cause of several non-classical optical phenomena.

The fact that squeezing-unsqueezing protocols can enhance measurement sensitivity for quantum estimation protocols beyond the standard quantum limit was first elucidated in the context of photonic Mach-Zehnder and four-wave-mixing interferometers [19]. By replacing the non-passive elements of these photonic interferometers by entanglement-generating atomic interactions such as one-axis or two-axis twisting, one is led to protocols that achieve analogous scaling of the sensitivity with respect to the number of atoms. Specifically, in the context of phase sensing with atomic ensembles, a single layer one-axis twist-untwist protocol is defined by the parameterized $N$ particle quantum state $|\psi_\phi\rangle$ where

$$|\psi_\phi\rangle = e^{i\chi t J_z^2} e^{-i\phi J_y} e^{-i\chi t J_z^2} |\uparrow\rangle^\otimes N.$$  (1)

In (1), the spin operators satisfy the $su(2)$ relation $[J_x, J_y] = i\epsilon_{ijk} J_k$, and $|\uparrow\rangle$ is the maximal $J_z$ eigenvector in the spin $\frac{N}{2}$ representation of $SU(2)$. A major advantage of the the protocol (1) is that large values of the effective interaction time $\chi t$ are not required in order to achieve Heisenberg scaling for the estimation precision of $\phi$ [8]. In particular, generation of the state that minimizes the quantum Cramér-Rao bound for estimation of a $y$-rotation, viz., an equal amplitude Schrödinger cat state of the minimal and maximal $J_y$ eigenvectors, is not required. However, the question of the conditions under which protocol (1) is an optimal strategy for achieving Heisenberg scaling for estimation of $\phi$ remains open. In this work, we show that the protocol (1) achieves the minimal error possible among all protocols that apply a weak one-axis twisting before and after the rotation parameter in the limit $N \to \infty$ (Section II). In Section III and Section IV, respectively, we obtain analogous results for the multilayer improvements of the protocol (1), and its implementation in systems with uniform, finite range atom-atom interactions. The principal physical constraint in these results is our requirement of an asymptotically van-
ishing interaction time $\chi t \to 0$ as $N \to \infty$. This constraint is consistent with the fidelity losses encountered when generating coherence and entanglement of a many-atom system over long times.

The setting of the quantum metrology problem at hand consists of: 1. preparation of a spin-$\frac{1}{2}$ coherent state of $N$ two-level atoms, 2. application of layers of the protocol (1) or its finite-range generalizations, and 3. measurement of $J_y$. An operator-valued estimator of the phase $\phi$ is given by $\hat{\phi} = \frac{\int \langle \psi_\phi | J_y | \psi_\phi \rangle}{\int \langle \psi_\phi | J_y | \psi_\phi \rangle^2}$, and we define the empirical error as $(\Delta \phi)^2 := \text{Var}[\hat{\psi}_\phi] \hat{\phi}$. It follows that

$$
(\Delta \phi)^2 := \frac{\text{Var}[\hat{\psi}_\phi] J_y}{(\partial_\phi \langle \psi_\phi | J_y | \psi_\phi \rangle)^2}.
$$

(2)

This practical approach can achieve empirical errors that scale similarly to a $N^{-2}$ quantum Cramér-Rao bound for estimation of $\phi$ on a quantum state manifold that is related to (11) (see Proposition 1 for a rigorous statement). However, such Heisenberg scaling for (2) does not require saturation of the quantum Fisher information by implementation of an optimal measurement. Since the error (2) is invariant under $\chi \to -\chi$, the protocol can be defined with $\chi > 0$ without loss of generality. A physical explanation of the fact that Heisenberg scaling is obtained for small interaction time $\chi t$ for the protocol (1) is that the initial one-axis twisting drives the coherent state $|+\rangle^\otimes N$ toward a Schrödinger cat state of $J_x$ according to the Yurke-Stoler dynamics driven by the one-axis twisting, but symmetrically about the $x$-axis in spin phase space. For small $\chi t$, this process actually creates a pseudo-cat state with respect to the $J_y$ generator, which is sensitive to the rotation $\phi$. The untwisting acts to amplify the phase of the rotation. Note that the protocol does not return the initial spin coherent state to the manifold of spin coherent states.

The fact that the nonlinearity of an interaction can compensate for weak interaction strength to achieve Heisenberg scaling in quantum sensing can also be observed for continuous variable displacement sensing. The continuous variable analogue of the twist-untwist protocol is given by applications of the Kerr nonlinearity with opposite signs: $|\psi_\phi\rangle = e^{i\chi t(\alpha^*a)^2} D(\phi)e^{-i\chi t(\alpha a)^2} |\phi\rangle$ where $D(\phi) = e^{-i\phi p}$ is a unitary displacement operator and $|\alpha\rangle$ is a Heisenberg-Weyl coherent state with $\text{Im} \alpha = 0$. One finds that for a homodyne readout of the $p$-quadrature

$$
(\Delta \phi)^2|_{\phi=0} := \left[ \frac{\text{Var}[\hat{\psi}_\phi] P}{(\partial_\phi \langle \psi_\phi | J_y | \psi_\phi \rangle)^2} \right]_{\phi=0}
$$

$$
= \frac{1}{2\alpha^4 e^{-2\alpha^2(1-cos 2\chi t)} \sin^2(\alpha^2 \sin(2\chi t)) + 1}
$$

(3)

which, for large $\alpha$ has an approximate minimum at $\chi t = \frac{\pi - 2}{\alpha^2 + 1}$, an interaction time at which (3) scales as the inverse square of the intensity $\alpha^2$.

### II. Asymptotic Optimality of Twist-Untwist Protocols

Our main result in Theorem 1 shows that when using the fixed measurement scheme defined by the estimator $\hat{\phi}$, and for two calls to a low interaction time one-axis twisting evolution separated by the rotation to be sensed, the twist-untwist protocol (1) gives an optimal probe state in the limit of large $N$. However, it is useful to first understand how, given the twist-untwist protocol (1), the measurement scheme defined by $\hat{\phi}$ performs when compared to the ultimate precision obtainable by an optimal unbiased estimator in one-shot quantum estimation theory.

For this, we recall that the Heisenberg limit for the scaling of the mean squared error of an unbiased estimator of $\phi$ is defined by the quantum Fisher information appearing in the the one-shot quantum Cramér-Rao bound as $\mathcal{O}(N^2)$. However, measurement of the operator-valued estimator $\hat{\phi}$ in Section I does not give values in $[-\pi, \pi]$ and so there is not a unique way to relate (2) to a quantum Cramér-Rao bound [20]. Therefore we first provide a basic, but rigorous, statement that relates the optimal scaling of (2) for twist-untwist protocols to Heisenberg scaling of the quantum Fisher information of the twist-untwist protocol (1).

**Proposition 1.** The minimum of (2) with respect to twist-untwist protocol (1) occurs at $\chi t = \tan^{-1} \frac{1}{\sqrt{N-2}}$ with minimum value asymptotically given by $\frac{1}{\sqrt{N-2}}$ as $N \to \infty$. With $\text{QFI}(\psi_\phi)$ defined as the quantum Fisher information for the protocol (1), the function

$$
f(\chi t) := \frac{\text{QFI}(\psi_\phi)}{(\Delta \phi)^2|_{\phi=0}}
$$

satisfies $f(\tan^{-1} \frac{1}{\sqrt{N-2}}) \sim e^2 \chi t$. Further, $f(\chi t) \leq 1$ and the maximum value 1 is asymptotically attained when $\chi t$ is a function of $N$ that goes to zero as $N \to \infty$.

**Proof.** The value $\tan^{-1} \frac{1}{\sqrt{N-2}}$ for the critical interaction time is proven in Ref. [8]. Note that the numerator of $f$ is the lower bound appearing in the quantum Cramér-Rao inequality (QCRI), so $f = 1$ implies that the measurement saturating the QCRI for the protocol $|\psi_\phi\rangle$ has the same error as the $J_y$ measurement defined in (2). The fact that $f \leq 1$ follows from the fact that $((\Delta \phi)^2|_{\phi=0})^{-1}$ is at most the classical Fisher information with respect to the $J_y$ measurement at $|\psi_\phi=0\rangle$ [21], and the existence of a measurement for which the classical Fisher information saturates the quantum Fisher information [22]. From the symmetric logarithmic derivative for the state manifold $|\psi_\phi\rangle$ (see (14)), it follows that

$$
\text{QFI}(\psi_\phi) = 4\text{Var}_{e^{i\chi x u \hat{J}_x}|+\rangle^\otimes N} J_y = \frac{1}{8} \left[ N^2 + N - N(N-1) \cos(N-2) + 2\chi t \right].
$$

(5)
Note that QFI(\(\psi_\phi\)) is independent of \(\phi\). The value of (2) with respect to twist-untwist protocol (1) is calculated from a more general formula (8) below, so the final result is
\[
f(\chi t) = \frac{2N(N - 1)^2 \sin^2 \chi t \cos^{2N-4} \chi t}{N^2 (1 - \cos^{N-2} 2\chi t) + N (1 + \cos^{N-2} 2\chi t)}.
\]
Using \(\tan^{-1} \frac{1}{\sqrt{N-2}} \sim \frac{1}{\sqrt{N}} \cos^N \frac{2}{\sqrt{N}} \sim e^{-2}, \cos^{2N-4} \frac{1}{\sqrt{N-2}} \sim e^{-1}, \) and \(\sin^2 \frac{1}{\sqrt{N}} \sim 1, \) gives the asymptotic result \(f(\chi t) \sim 1\), which implies that at the interaction time that minimizes (2), the lowest possible achievable error is only about 15% lower than (2) in the limit \(N \to \infty\). Taking the derivative of (6) with respect to \(\chi t\) and using the assumption \(\chi t \to 0\) as \(N \to \infty\) to replace \(\sin \chi t \approx \chi t\) and \(\cos \chi t \approx 1\), one gets the critical interaction time \(\chi t_\ast \sim \frac{1}{\sqrt{N(N-2)}} \sim \frac{1}{N}\).

Then \(\lim_{N \to \infty} f((\chi t)_\ast) = 1\).

The function \(f\) is plotted in Fig. for \(N = 10^3\). For \(\chi t > \tan^{-1} \frac{1}{\sqrt{N-2}}\), the one-axis twisting probe state \(e^{i\chi t J_z^2} \begin{bmatrix} 1 \end{bmatrix} \otimes N\) begins to enter the Schrödinger cat domain, which is not accessible by the twist-untwist protocol (1). However, for \(\chi t \leq \tan^{-1} \frac{1}{\sqrt{N-2}}\), the twist-untwist protocol with error (2) scales similarly to the optimal error achievable with the one-axis twisting probe, with both quantities scaling as \(O(N^{-2})\) when \(\chi t \approx \tan^{-1} \frac{1}{\sqrt{N-2}}\).

Although Proposition 1 suggests how to interpret the optimal \(N^{-2}\) scaling of (2) for the twist-untwist protocol (1), it remains unclear whether similar protocols involving one-axis twisting before and after the rotation would be able to achieve the same scaling. Therefore, we now consider more general protocols of the form
\[
|\psi_\phi\rangle = e^{ia_2 J_z^2} e^{-ia_1 J_z^2} \begin{bmatrix} 1 \end{bmatrix} \otimes N
\]
with \(a_j \in \mathbb{R}\). Numerical optimization of the signal-to-noise ratio (2) and effects of dephasing noise for such protocols were considered in [23]. A closed formula for (2) is obtained for this protocol:
\[
\frac{2(N+1) - 2(N-1) \cos^{N-2} (2a_1 + 2a_2)}{N(N-1)^2 \sin^2 a_2 [\cos^{N-2} 2a_2 + \cos^{N-2} (2a_1 + 2a_2)]}.
\]

The formula (8) demands that we refine the parameter space of (7) so as to have a well-defined sensing protocol. In particular, we restrict to \(a_2 \in (-\pi/2, 0) \cup (0, \pi/2)\) and \(a_1 \in (-\pi/2, 0)\) without loss of generality. In experimental implementations of (7), the range of available interaction times \(a_j\) will depend on \(N\), due to decoherence.

We now aim to show that when the interaction times \(a_1\) and \(a_2\) go to zero as \(N \to \infty\), (8) is asymptotically minimized for \(a_2 = -a_1\), i.e., at a point at which (7) defines a twist-untwist protocol. The key observation is that for fixed \(a_1\), (8) has an asymptotic extremum at \(a_2 = -a_1\). This is shown in Theorem 1 below.

**Theorem 1.** Let \((\Delta \phi)^2|_{\phi=0}\) be defined with respect to \(|\psi_\phi\rangle\) as in (7) and let \(a_1\) and \(a_2\) be functions of \(N\) that go to zero as \(N \to \infty\). Then, as \(N \to \infty\) the unique critical point of \(N^2(\Delta \phi)^2|_{\phi=0}\) is given by
\[
a_1 \sim \frac{N}{(N+1)(N-2)} \sim \frac{1}{\sqrt{N}}, \\
a_2 \sim -\frac{N+1}{N(N-2)} \sim -\frac{1}{\sqrt{N}}.
\]

**Proof.** Call \(f(a_1, a_2) := N^2(\Delta \phi)^2|_{\phi=0}\) and note that we restrict to \(a_1 < 0\). The factor of \(N^2\) in the definition of \(f\) is so that the \(N \to \infty\) limit of \(f\) is not zero pointwise; without this factor, \(f\) is asymptotically constant. For example, with \(c := \tan^{-1} \frac{1}{\sqrt{N}}\), it follows that \(\lim_{N \to \infty} f(-c,c) = \epsilon\). The components of \(\nabla f(a_1, a_2)\) are rational functions of \(N\) with polynomial coefficients consisting of powers of \(\sin(g(a_1, a_2)), \cos(g(a_1, a_2))\) where \(g(a_1, a_2) \in \{2a_1 + 2a_2, 2a_1 + a_2, 2a_2\}\) are linear functions of \(a_1, a_2\). Using the assumption that \(a_1\) and \(a_2\) go to zero as \(N \to \infty\), we linearize the coefficients by setting \(\sin(g(a_1, a_2)) \sim g(a_1, a_2), \cos(g(a_1, a_2)) \sim 1\). The extremum condition is then asymptotically given by
\[
(N+1)a_1 + Na_2 = 0 \\
N(N-2)(a_1 + a_2) - a_2^{-1} = 0
\]
which has the required solution pair. Uniqueness is due to the assumption \(a_1 < 0\).

The fact that the asymptotic critical point (9) defines an asymptotic minimum of \(f\) can be checked numerically or by taking second derivatives. Note that with the values in (9), \(\lim_{N \to \infty} a_2^{-1} = 1\) so that, under the assumptions of Theorem 1, a twist-untwist protocol is an optimal strategy for estimation of \(\phi\) in (7). Further, note that the asymptotically optimal parameters exhibit \(N^{-1/2}\) decay. This decay is modified when the interaction has finite range, as discussed in Section IV.

**III. LAYERED TWIST-UNTWIST PROTOCOLS**

An \(L\) layer twist-untwist protocol can be defined by the parameterized state
\[
|\psi_{\phi}^{(L)}\rangle = e^{i\phi J_y} \left( e^{-i\phi J_y} e^{i\chi t J_z^2} e^{-i\phi J_y} e^{-i\chi t J_z^2} \right)^L |\zeta = 1\rangle.
\]

The state (11) is motivated by the consideration of a twist-untwist layer \(e^{ix\phi} e^{-ix\phi} e^{-i\chi t C} \otimes N\) as a module for interferometer. Similarly structured circuits appear in asymptotically optimal variational quantum algorithms for quantum unstructured search [24]. It is clear that for
$L$ layers, the denominator of (2) is given by

$$
\left( \partial_\phi \langle J_y \rangle_{\psi^{(L)}} \right)^2_{\phi=0} = L^2 \left( \partial_\phi \langle J_y \rangle_{\psi^{(1)}} \right)^2.
$$ (12)

The numerator of (2) at $\phi = 0$ is invariant with respect to the number of layers $L$. Because a probe state consisting of $L$ independent copies of (1) would have variance of the total $y$-spin component increased by a factor of $L$, we conclude that an $L$ layer protocol (11) allows the value of $(\Delta \phi)^2$ to be decreased by a factor of $L^{-1}$ compared to the protocol consisting of $L$ independent copies of (1). As an alternative to the layered protocol (11) one may consider the parameterized state $|\psi_\phi\rangle = e^{ixC} e^{-iL\phi J_y} e^{-i\chi t} |+\rangle \otimes |\psi_0\rangle$, which allows to obtain an $L^2$ increase in the derivative of the signal, similarly as in (12). The difficulty with this proposal is that the map $e^{-i\phi J_y} |\psi\rangle \mapsto e^{-iL\phi J_y} |\psi\rangle$ cannot be carried out unitarily on all $|\psi\rangle$. A proof of this fact can be provided which is similar to proofs of “no-go” theorems for noiseless parametric amplification in the continuous-variable setting (i.e., that the map $|\alpha\rangle \mapsto |L\alpha\rangle$ cannot be achieved by a unitary operation).

In fact, the multiplicative improvement obtained from the $L$ layer twist-untwist protocol extends to the quantum Fisher estimation.

**Proposition 2.** The quantum Fisher information $\text{QFI}(\psi^{(L)}_\phi)$ at $\phi = 0$ for $L$ layer twisting-untwisting protocol (11) with $C = J_z$ is given by

$$
L^2 \text{QFI}(\psi^{(1)}_\phi) + 2(L - 1)N \cos^{N-1} \chi t + (L - 1)^2 N.
$$ (13)

**Proof.** The symmetric logarithmic derivative at $\phi = 0$ for the one layer protocol is given by

$$
\mathcal{L}^{(1)}_{\phi=0} = -2ie^{iC} J_y e^{-iC} |+\rangle \langle +| \otimes |\psi_0\rangle + 2i |+\rangle \langle +| \otimes e^{iC} J_y e^{-iC} |\psi_0\rangle.
$$ (14)

from which it follows that

$$
\mathcal{L}^{(L)}_{\phi=0} = L \mathcal{L}^{(1)}_{\phi=0} + (\pm 1)e^{iC} J_y e^{-iC} |+\rangle \langle +| \otimes |\psi_0\rangle + h.c.
$$ (15)

One then calculates

$$
\text{QFI}(\psi^{(L)}_\phi) := \langle + | (\mathcal{L}^{(L)}_{\phi=0})^2 | + \rangle \otimes |\psi_0\rangle = L^2 \text{QFI}(\chi, t) + N(L - 1)^2 + 4L(L - 1) \left( \langle + | e^{iC} J_y e^{-iC} J_y | + \rangle \otimes |\psi_0\rangle + h.c. \right)
$$ (16)

and the last term can be evaluated explicitly by using

$$
e^{i\alpha J_z^2} J_y e^{-i\alpha J_z^2} = \frac{1}{2} \frac{j}{m} e^{2i\alpha^2 J_z^2} + h.c.
$$

A generalization of (11) that allows one-axis twisting to alternate with rotations is given by

$$
|\psi^{(L)}_\phi(\vec{a})\rangle = e^{i\phi J_y} \prod_{j=1}^L e^{-i\alpha_j J_y} e^{i\alpha_j J_y} e^{i\alpha_j J_y} e^{i\alpha_j J_y} |\zeta = 1\rangle
$$ (17)

where $\vec{a} := (a_{L,2}, a_{L,1}, a_{L-1,2}, a_{L-1,1}, \ldots)$ is the row vector of parameters. Defining the partial sums $\varphi_\ell = \sum_{j=1}^\ell \alpha_j$ allows one to evaluate

$$
N^2(\Delta \phi)^2|_{\phi=0} = \frac{A(\varphi_{2\ell})}{B(\varphi_j^2_{\ell=1})}
$$

$$
A(\varphi_{2\ell}) = 2(N + 1) - 2(N - 1) \cos^{N-2}(2\varphi_{2\ell})
$$

$$
B(\varphi_j^2_{\ell=1}) = N(N - 1)^2 \left( \sum_{j=1}^{2\ell - 1} \sin \varphi_j \cos^{N-2} \varphi_j + \cos^{N-2}(2\varphi_{2\ell} - \varphi_j) \right)^2
$$ (18)

Solving for extrema of (18) without restriction of the protocol (17) are complicated, even when the linearization
method in Theorem 1 is used. However, constraining each layer in (17) to satisfy the twist-untwist condition \( a_{\ell,1} = -a_{\ell,2} \) for all layers \( \ell \) results in a simplification of the partial sums, viz., \( \varphi_{2\ell} = 0 \) and \( \varphi_{2\ell+1} = a_{\ell-\ell,2} \). One then finds that (18) is minimized for equal strength twist-untwist in each layer, i.e., \( \varphi_{2\ell+1} = \tan^{-1} \frac{1}{\sqrt{N-2}} \).

IV. FINITE RANGE TWIST-UNTWIST PROTOCOLS

Rydberg-Rydberg atom interactions between trapped neutral atoms provide a platform for scalable many-body entanglement generation via spin-squeezing [25]. For \( N \) atoms on a one-dimensional lattice with periodic boundary condition, a finite range, two-local Hamiltonian generalizing the one-axis twisting generator \( J^z_N \) is given by

\[
H_K = \frac{1}{4} \sum_{j=0}^{N-1} \sum_{i=j-K \mod N}^{j+K \mod N} V_{i,j} Z_i \otimes Z_j \tag{19}
\]

where \( K \) is an integer in \([1, \frac{N}{2}]\) representing the interaction range, and where \( Z_j \) denotes the Pauli \( Z \) matrix acting on atom \( j \) and the identity operator on all other atoms.

A twist-untwist protocol with respect to \( H_K \) is defined as in (11) with the substitution \( J^z \rightarrow H_K \). For such a protocol, the expression for the denominator of (2) can be calculated analytically by using the identity

\[
e^{i\chi t} H_K (\sigma^+)_r e^{-i\chi t} H_K = (\sigma^+)_r \exp \left[ i\chi t \sum_{i=0}^{N-1} V_{r,i} Z_i \right]. \tag{20}
\]

where \( (\sigma^+)_r \) denotes the \( \sigma^+ \) matrix acting on atom \( r \) and the identity operator on all other atoms. The result is

\[
\partial_\phi \langle \psi_\phi | J_y | \psi_\phi \rangle \big|_{\phi=0} = \frac{1}{2} \sum_{r=0}^{N-1} \sum_{\ell=0}^{K} \left[ \sin (\chi V_{r,r-\ell}) \prod_{i=0}^{r-K} \cos (\chi V_{r,r-i}) \right] \tag{21}
\]

where in the above equation, \( r - \ell \) and \( r - i \) are modulo \( N \). We now consider the case of a constant, finite range interaction

\[
V_{i,j} = \begin{cases} 1 & 0 < |i - j| \leq K \\ 0 & |i - j| > K \text{ or } i = j \end{cases} \tag{22}
\]

where the inequalities are interpreted modulo \( N \). In this case, \( V_{i,j} \) is the adjacency matrix of the \( K \)-nearest neighbors graph. For this choice of \( V_{i,j} \), the expression (21) becomes \( NK \sin \chi t \cos^{2K-1} \chi t \), which obtains a maximum at \( \chi t = \tan^{-1} \frac{1}{2K-1} \). At this maximum, one obtains the minimal value of \( (\Delta \phi)^2 \big|_{\phi=0} = \frac{1}{2NK(1-\frac{2}{2K-1})^{K-1}} \)

Furthermore, we obtain an analytical formula for (2) for the general protocol

\[
|\psi_\phi\rangle = e^{i\alpha_2 H_K} e^{-i\phi J_y} e^{i\alpha_1 H_K} |+\rangle^\otimes N, \tag{23}
\]

restricting to the regime of short-range interactions \( K \leq \frac{N}{4} \). The formula is given in Appendix A.

In analogy with Proposition 1, one can use the aforementioned formulas (A1) to show that the ratio \( \text{QFI}^{-1}/(\Delta \phi)^2 \big|_{\phi=0} \) evaluated at the critical interaction time \( \chi t = \tan^{-1} \frac{1}{\sqrt{2K-1}} \) asymptotes to \((e + e^{-1} - 2)^{-1} \approx 0.92\) as \( K \rightarrow \infty \). Therefore, the one-shot quantum Cramér-Rao bound is asymptotically only about 8% lower that the empirical error achieved using \( J_y \) measurement and propagation of error. Further, the formu-
las (A1) allow to prove asymptotic optimality for finite-range one-axis twist-untwist protocols among the protocols (23).

**Theorem 2.** Let $(\Delta \phi)^2|_{\phi=0}$ be defined with respect to $|\psi_0\rangle$ as in (23). Further, let $K = r(N - 2)$ for a fixed locality parameter $0 < r \leq \frac{1}{4}$ and let $a_1$ and $a_2$ be functions of $N$ that go to zero as $N \to \infty$. Then, as $N \to \infty$ the unique critical point of $N^2(\Delta \phi)^2|_{\phi=0}$ is given by

$$a_2 a_1 \sim -\frac{8K^2 - 2K - 1}{8K^2 - 3K - \frac{1}{2}}$$

$$a_1 a_2 \sim \frac{-1}{2K - 1} \quad (24)$$

**Proof.** The condition $K = r(N - 2)$ for a fixed $0 < r \leq \frac{1}{4}$ allows to apply (A1) to calculate $N^2(\Delta \phi)^2|_{\phi=0}$. The assumption of asymptotically vanishing $a_1, a_2$ allows to substitute the trigonometric functions appearing in the equation $\nabla_a \left(N^2(\Delta \phi)^2|_{\phi=0}\right) = 0$ by their first order Maclaurin expansions as in Theorem 1. Applying this to $\partial a_1 \left(N^2(\Delta \phi)^2|_{\phi=0}\right) = 0$ gives the first asymptotic criticality condition. The second asymptotic criticality condition is obtained by combining $\partial a_1 a_2 \left(N^2(\Delta \phi)^2|_{\phi=0}\right) = 0$.

Note that in Theorem 2 we have taken $K$ to be a function of $N$. Unlike the $N^{-1/2}$ decay of $a_{1(2)}$ in the case of asymptotically optimal twist-untwist protocols for full range interactions in Theorem 1, a $K^{-1/2}$ decay is observed in the case of range $K$ interactions. As a consequence, the rate of convergence of the optimal protocol of the form (23) to a finite-range one-axis twist-untwist protocol is controlled by the locality parameter $r$. It is also possible to obtain formulas analogous to (A1) and (24) in dimension $D$ when the size of the interaction region scales as $O(rD^N)$. However, we have not observed any change in the rate of optimality with respect to the locality parameter $r$. For instance, in the case of an $N \times N$ square lattice with periodic boundary condition and with constant interaction on squares of $K \times K$ sites (with $K$ odd and $K \leq \frac{N+2}{4}$), we find that the optimal one-axis twisting parameters satisfy an asymptotic relation of the form

$$a_2 a_1 \sim -\frac{cK^4 + f(K)}{cK^4 + g(K)} \quad (25)$$

with $f(K), g(K) \in O(K^3)$.

**A. Translation-invariant finite range twist-untwist protocols**

We now consider physical systems that exhibit Bose statistics as in Section II, but have spatial interactions inherited from the system of distinguishable spins in Section IV. Specifically, we consider the physical scenario of two-level bosonic atoms in a ring-shaped optical lattice of $N$ sites. The internal states of the bosons are assumed to interact pairwise (e.g., magnetically) over a distance of $K$ sites. The two body interaction can be written as a Heisenberg model by using the boson operators $a$ and $b$ for the internal states:

$$\left\{ \begin{array}{l}
\frac{1}{4} \sum_{i=0}^{N-1} r+jK mod N \sum_{i\neq j} V_{j,i} \left(a_i^\dagger a_j - b_i^\dagger b_j\right) \left(a_i^\dagger a_i - b_i^\dagger b_i\right) .
\end{array} \right. \quad (26)$$

We assume that $a_i^\dagger a_j + b_i^\dagger b_j = 1$ for all $j$, so that the sites have unit occupancy. When restricted to this subspace, it is clear that (26) is equal to $H_K$ in (19). However, it is possible to further restrict the interaction Hamiltonian (26) to describe dynamics in the translation invariant subspace of spin waves, i.e., the states spanned by the basis

$$\{\{0, 1, \ldots, 0, 1\}\} \cup \{\sum_{i,\ell=1}^{M} a_{i,\ell}^\dagger b_{i,\ell} |0, 1, \ldots, 0, 1\rangle \}_{M=1}^N$$

where $M$ is the excitation number of the spin wave and $|n_{a,1}, n_{a,2}, \ldots, n_{a,N-1}, n_{b,N-1}\rangle$ is an insulating state with $a_i^\dagger a_j = n_{a,i} b_i^\dagger b_j = n_{b,j}$. In terms of matrix elements, the Hamiltonian (26) after the projection to the spin wave subspace is equal to the Hamiltonian $\tilde{H}_K := P_B H_K P_B$, where $P_B$ is the projection of $(C^2)^\otimes N$ to the symmetric subspace. For simplicity, we again restrict to the constant interaction potential $V_{i,j} = 1$. In the special case of $N \equiv 1 mod 2$, $K = \frac{N - 1}{2}$, and $V_{i,j} = 1$, then (19) is equal to $J^2 - \frac{N}{4}$, so no projection is needed. For the general case, one finds that the matrix elements of $H_K$ in the orthonormal basis of Dicke states are given by

$$\langle \psi_n | H_K | \psi_n \rangle = \frac{\delta_{n,n'} \langle n | H | n \rangle}{\langle n | x : \text{Ham}(x) = n \rangle}$$

$$| \psi_n \rangle := \frac{1}{\sqrt{\langle n | x : \text{Ham}(x) = n \rangle}} |x\rangle , \ n = 0, \ldots, N$$

(28)

where $|x\rangle$ is a state in the computational basis and $\text{Ham}(x)$ is its associated Hamming weight. The twist-untwist protocol in this subspace is defined by the parameterized state

$$|\psi_{\phi} \rangle := e^{i\chi t H_K} e^{-i\phi} |\psi_{\phi} \rangle$$

where the state $|\rangle \otimes N$ corresponds to a superposition of the spin wave basis states in (27).

Because the exact computation of the matrix elements of $H_K$ takes exponential time in $N$, it is useful to define
a model bosonic Hamiltonian for \( \tilde{H}_K \) that helps to analyze the metrological gain obtained in the protocol (29). Specifically, consider the Hamiltonian

\[
\tilde{H}_K^{(2)} = \frac{1}{4} \sum_{j=0}^{N-1} \sum_{i,j-K \mod N} V_{ij} P_BP_j P_BP_i. \tag{30}
\]

The properties of \( \tilde{H}_K^{(2)} \) that make it useful as a model interaction for \( \tilde{H}_K \) are established in the following proposition.

**Proposition 3.** If \( V_{i,j} = 1 \), then \( \tilde{H}_K^{(2)} = \frac{2KJ^2}{N} \). If \( V_{i,j} > 0 \) for all \( i, j \), then \( \| \tilde{H}_K^{(2)} \| = \| \tilde{H}_K \| \).

**Proof.** First, note that \( P_BP_j P_BP_i = \frac{1}{N} \sum_{i=1}^{N} Z_i \), which is proven by considering computational basis states \( |x\rangle, |x'\rangle \): if \( \text{Ham}(x) \neq \text{Ham}(x') \) then \( \langle x'|P_BP_j P_BP|x\rangle = 0 \); if \( \text{Ham}(x) = \text{Ham}(x') = n \) then

\[
\langle x'|P_BP_j P_BP|x\rangle = \frac{1}{\binom{n}{x}} \sum_{x: \text{Ham}(x) = n} (-1)^{x_j} = 1 - \frac{2K}{N} = \langle N - n, n | \frac{2J}{N} | N - n, n \rangle, \tag{31}
\]

and \( \frac{2J}{N} = \frac{1}{N} \sum_{i=0}^{N-1} Z_i \), which proves the first statement. Note that the numerator of the second line of (31) is the difference between the number of ways for \( n \) ones \((n - 1 \) ones) to appear in \( x \) given that \( x_j = 0 \) \((x_j = 1) \). To prove the second statement, note that for \( V_{i,j} > 0 \),

\[
\| \tilde{H}_K^{(2)} \| = \left( |0\rangle \otimes_N \tilde{H}_K^{(2)} |0\rangle \otimes_N \right) = \| \tilde{H} \|. \tag{32}
\]

\( \square \)

Numerical computation of \( (\Delta \phi)^2 |\phi = 0 \) for the protocol (29) using both \( \tilde{H}_K \) and \( \tilde{H}_K^{(2)} \) for \( N = 16 \) is shown in Fig. 2. The \( K^{-1/2} \) decay of the optimal twist-untwist parameters which appeared in the analysis of the protocol (23) for distinguishable two-level atoms is not observed. Instead, we conclude that the translation invariant twist-untwist protocol (29) effectively converts a finite interaction range to a multiplicative factor in the interaction strength. This results in longer interaction times required to reach maximal sensitivity for short-range bosonic twist-untwist protocols, but independence of the maximal sensitivity on the interaction range.

\section{V. Discussion}

Our theorems show that the twist-untwist protocol (1), and its generalization to constant, finite range one-axis twisting generators, is asymptotically optimal among protocols that apply two calls to asymptotically weak one-axis twisting evolutions separated by a call to the rotation parameter of interest. We expect that proof methods also allow to obtain analogous results for more general spin squeezing interactions, e.g., two-axis twisting, or twist-and-turn generators [26]. In practice, the assumption of perfect generation of the Bose-Einstein condensed state \( |\phi \rangle \otimes_N \) is an experimentally demanding one. Further, even for large \( N \), a high fidelity one axis twisted state for times on the order \( O(N^{-1/2}) \) is another practical challenge. The present work provides a foundation for future analyses of imperfect or noisy twist-untwist protocols for atomic interferometry.

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Appendix A: Formula for (2) for finite range one axis twist-untwist protocol

The formulas for the numerator and denominator of (2) that allow to prove Theorem 2 are given by

\[
\text{Var}_{\psi_0} J_y |_{\phi = 0} = \frac{N}{4} \left[ 1 + \sum_{j=0}^{K-1} \left( 1 - \cos^{K+j-1} (2a_1 + 2a_2) \cos^{2K-2j} (a_1 + a_2) \right) \right.
\]

\[
+ \left. \left( 1 - \cos^{j+1} (2a_1 + 2a_2) \cos^{4K-2j-2} (a_1 + a_2) \right) \right]
\]

\[
\partial_{\phi} \langle \psi_0 | J_y | \psi_0 \rangle |_{\phi = 0} = \frac{N}{2} \sin a_2 \sum_{j=0}^{K-1} (\cos^{K+j-1} a_2 + \cos^{K+j-1} (2a_1 + a_2)) (\cos a_1 \cos (a_1 + a_2))^{K-j}.
\]