Global Existence and the Decay of Solutions to the Prandtl System with Small Analytic Data

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Abstract

In this paper, we prove the global existence and the large time decay estimate of solutions to Prandtl system with small initial data, which is analytical in the tangential variable. The key ingredient used in the proof is to derive a sufficiently fast decay-in-time estimate of some weighted analytic energy estimate to a quantity, which consists of a linear combination of the tangential velocity with its primitive one, and which basically controls the evolution of the analytical radius to the solutions. Our result can be viewed as a global-in-time Cauchy–Kowalevsakya result for the Prandtl system with small analytical data, which in particular improves the previous result in Ignatova and Vicol (Arch Ration Mech Anal 220:809–848, 2016) concerning the almost global well-posedness of a two-dimensional Prandtl system.

1. Introduction

Describing the behavior of boundary layers is one of the most challenging and important problems in mathematical fluid mechanics. The governing equations of the boundary layer obtained by the vanishing viscosity of the Navier–Stokes system with a Dirichlet boundary condition was proposed by Prandtl [32] in order to explain the disparity between the boundary conditions verified by an ideal fluid and a viscous fluid with small viscosity. Heuristically, these boundary layers are of amplitude $O(1)$ and of thickness $O(\sqrt{\nu})$, where there is a transition from the interior flow governed by Euler equation to the Navier–Stokes flow with a vanishing viscosity $\nu > 0$. One may check [11, 26] and references therein for more on boundary layer theory. We refer in particular to [19] for a comprehensive recent survey.

One of the key steps to rigorously justify this inviscid limit of the Navier–Stokes system with a Dirichelt boundary condition is to deal with the well-posedness of the Prandtl system
\[
\begin{align*}
\frac{\partial_t U + U \partial_x U + V \partial_y U - \partial_y^2 U + \partial_x p}{\partial_x U + \partial_y V} &= 0, 
& (t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\
U|_{y=0} &= V|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} U(t, x, y) = w(t, x), \\
U|_{t=0} &= U_0,
\end{align*}
\] (1.1)

where \( U \) and \( V \) represent the tangential and normal velocities of the boundary layer flow, \((w(t, x), p(t, x))\) are the traces of the tangential velocity and pressure of the outflow on the boundary which satisfy Bernoulli’s law:

\[
\frac{\partial_t w + w \partial_x w + \partial_x p}{0} = 0.
\] (1.2)

Since there is no horizontal diffusion in the \( U \) equation of (1.1), the nonlinear term \( V \partial_y U \) (which almost behaves like \( -y \partial_x U \partial_y U \)) loses one horizontal derivative in the process of energy estimate, and therefore the question of whether or not the Prandtl system with general data is well-posed in Sobolev spaces is still open. In fact, E and ENQUIST [12] constructed a class of initial data which generate solutions with finite time singularities in case the solutions exist locally in time. GÉRARD-VARET and DORMY [13] proved the ill-posedness in Sobloev spaces for the linearized Prandtl system around non-monotonic shear flows. The nonlinear ill-posedness was also established in [15,19] in the sense of non-Lipschitz continuity of the flow. Nevertheless, we have the following positive results for two classes of special data:

- Under a monotonic assumption on the tangential velocity of the outflow, OLEINIK [26] first introduced the Crocco transformation and then proved the local existence and uniqueness of classical solutions to (1.1). With the additional “favorable” condition on the pressure, XIN and ZHANG [35] obtained the global existence of weak solutions to this system. Recently, by ingenious use of the cancelation property of the bad terms containing the tangential derivative, the authors of [1] and [25] succeeded in proving the existence of local smooth solution to (1.1) in Sobolev space via performing energy estimates in weighted Sobolev spaces.

- For the data which is analytic in both \( x \) and \( y \) variables, SAMMARTINO and CAFLISCH [33] established the local well-posedness result of (1.1). The analyticity in \( y \) variable was removed by LOMBARDO ET AL. [24]. The main argument used in [24,33] is to apply the abstract Cauchy–Kowalewskaya (CK) theorem. More recently, GÉRARD-VARET and MASMOUDI [14] proved the well-posedness of (1.1) for a class of data with Gevrey regularity. This result was improved to be optimal in the sense of [13] in [9] by Dietert and Gérvard-Varet.

On the other hand, in order to study the vanishing viscosity limit of Navier–Stokes equations in a bounded domain, it is crucial to understand the solution of the system (1.1) on a longer time interval than the one which causes the instability used to prove ill-posedness (see for instance [10,16–18]). The authors of [8] observed that to control Navier–Stokes equations in a rectangle domain, one needs the long time existence of solutions to a system, which is similar to the system (1.1), with small analytical initial data (see also [22]). Indeed the question of the long time existence for Prandtl system with small analytic data was first addressed in [37] and an almost global existence result was provided in [21].
In this paper, we investigate the global existence and the large time decay estimates of the solutions to the Prandtl system with small data which is analytic in the tangential variable. For simplicity, here we take \( w(t, x) \) in (1.1) to be \( \varepsilon f(t) \) with \( f(0) = 0 \), which, along with (1.2), implies \( \partial_x p = -\varepsilon f'(t) \). Let us take a cut-off function \( \chi \in C^\infty[0, \infty) \) with \( \chi(y) = \begin{cases} 1 & \text{if } y \geq 2 \\ 0 & \text{if } y \leq 1. \end{cases} \) we denote \( W \equiv U - \varepsilon f(t) \chi(y) \). Then \( W \) solves

\[
\begin{align*}
&\partial_t W + (W + \varepsilon f(t) \chi(y)) \partial_x W + V \partial_y (W + \varepsilon f(t) \chi(y)) - \partial_y^2 W = \varepsilon m, \\
&\partial_x W + \partial_y V = 0, \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^2, \\
&W|_{y=0} = V|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} W(t, x, y) = 0, \\
&W|_{t=0} = U_0,
\end{align*}
\]

where \( \mathbb{R}_+^2 \overset{\text{def}}{=} \mathbb{R} \times \mathbb{R}_+ \) and \( m(t, y) \overset{\text{def}}{=} (1 - \chi(y)) f'(t) + f(t) \chi''(y) \).

In order to get rid of the source term in the \( W \) equation of (1.3), we introduce \( u^s \) via

\[
\begin{align*}
&\partial_t u^s - \partial_y^2 u^s = \varepsilon m(t, y), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}_+^2, \\
&u^s|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} u^s(t, y) = 0, \\
&u^s|_{t=0} = u_0 \equiv U_0.
\end{align*}
\]

With \( u^s \) being determined by (1.4), we set \( u \overset{\text{def}}{=} W - u^s \) and \( v \overset{\text{def}}{=} V \). Then \((u, v)\) verifies

\[
\begin{align*}
&\partial_t u + (u + u^s + \varepsilon f(t) \chi(y)) \partial_x u + v \partial_y (u + u^s + \varepsilon f(t) \chi(y)) - \partial_y^2 u = 0, \\
&\partial_x u + \partial_y v = 0, \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^2, \\
&u|_{y=0} = v|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} u(t, x, y) = 0, \\
&u|_{t=0} = u_0 \equiv U_0.
\end{align*}
\]

On the other hand, due to \( \partial_x u + \partial_y v = 0 \), there exists a potential function \( \varphi \) so that \( u = \partial_y \varphi \) and \( v = -\partial_x \varphi \). Then by integrating the \( u \) equation of (1.5) with respect to \( y \) variable over \([y, \infty)\), we obtain

\[
\partial_t \varphi + (u + u^s + \varepsilon f(t) \chi(y)) \partial_x \varphi \\
+ 2 \int_y^\infty (\partial_y (u + u^s + \varepsilon f(t) \chi(y'))) \partial_x \varphi \, dy' - \partial_y^2 \varphi = Q(t, x)
\]

for some function \( Q(t, x) \). However, since we assume that \( \varphi \) decays to zero sufficiently fast as \( y \) approaches \(+\infty\), we find that \( Q(t, x) = 0 \). Therefore, by virtue of (1.5), \( \varphi \) satisfies

\[
\begin{align*}
&\partial_t \varphi + (u + u^s + \varepsilon f(t) \chi(y)) \partial_x \varphi \\
&+ 2 \int_y^\infty (\partial_y (u + u^s + \varepsilon f(t) \chi(y'))) \partial_x \varphi \, dy' - \partial_y^2 \varphi = 0, \\
&\partial_y \varphi|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} \varphi(t, x, y) = 0, \\
&\varphi|_{t=0} = \varphi_0.
\end{align*}
\]
In order to globally control the evolution of the analytic band to the solutions of (1.5), we introduce the following key quantity:

\[
G \overset{\text{def}}{=} u + \frac{y}{2(t)} \varphi \quad \text{and} \quad g \overset{\text{def}}{=} \partial_y G = \partial_y u + \frac{y}{2(t)} u + \frac{\varphi}{2(t)}. \tag{1.7}
\]

We emphasize that the introduction of those “good unknowns” \( G \) and \( g \) in (1.7) is in fact inspired by the function \( g \overset{\text{def}}{=} \partial_y u + \frac{y}{2(t)} u \), which was introduced by Ignatova and Vicol in [21], and which has deep roots in [25] by Masmoudi and Wong, where the authors of [21] basically proved that the weighted analytical norm of \( g(t) \) decays like \( t^{-\frac{1}{2}} \), which decays faster than the weighted analytical norm of \( u \) itself. We observe that \( g - g = \frac{y}{2(t)} \). One novelty of this paper is to prove that the analytical norm of \( g \) is almost decays like \( t^{-\frac{1}{2}} \).

At the beginning of Section 7, we shall show that \( G \) verifies

\[
\begin{align*}
\partial_t G - \partial_y^2 G + (t)^{-1}G + (u + u^\varepsilon + \varepsilon f(t) \chi(y)) \partial_x G + v \partial_y G \\
+ v \partial_y (u^\varepsilon + \varepsilon f(t) \chi(y)) - \frac{1}{2}(t)^{-1}v \partial_y (y \varphi) \\
+ \frac{y}{(t)} \int_0^\infty \int_y \langle \partial_y (u + u^\varepsilon + \varepsilon f(t) \chi(y')) \partial_x \varphi \rangle \, dy' = 0, \tag{1.8}
\end{align*}
\]

\( G|_{y=0} = 0 \) and \( \lim_{y \to +\infty} G(t, x, y) = 0 \),

\( G|_{t=0} = G_0 \overset{\text{def}}{=} u_0 + \frac{y}{2} \varphi_0 \).

The main result of this paper is as follows:

**Theorem 1.1.** Let \( \delta > 0 \) and \( f \in H^1(\mathbb{R}_+) \), satisfying that

\[
C_f \overset{\text{def}}{=} \int_0^\infty \langle t \rangle^{\frac{5}{2}} \left( |f(t)| + |f'(t)| \right) \, dt \\
+ \left( \int_0^\infty \langle t \rangle^{\frac{7}{2}} \left( f^2(t) + (f'(t))^2 \right) \, dt \right)^{\frac{1}{2}} < \infty. \tag{1.9}
\]

Let \( u_0 = \partial_y \varphi_0 \) satisfy \( u_0(x, 0) = 0 \), \( \int_0^\infty u_0 \, dy = 0 \) and \( \|e^{\frac{y^2}{2t}} e^{\delta |D_x|}(\varphi_0, u_0)\|_{B^{\frac{1}{2}, 0}} < \infty \). We assume moreover that \( G_0 = u_0 + \frac{y}{2} \varphi_0 \) satisfies

\[
\|e^{\frac{y^2}{2t}} e^{\delta |D_x|} G_0\|_{B^{\frac{1}{2}, 0}} \leq c_0 \tag{1.10}
\]

for some \( c_0 \) sufficiently small. Then (1.4) has a solution \( u^\varepsilon \) and there exists \( \varepsilon_0 > 0 \) so that for \( \varepsilon \leq \varepsilon_0 \), the system (1.5) has a unique global solution \( u \) which satisfies

\[
\|e^{\frac{y^2}{2t}} e^{\delta |D_x|} u\|_{L^\infty(\mathbb{R}_+; B^{\frac{1}{2}, 0})} + \|e^{\frac{y^2}{2t}} e^{\delta |D_x|} \partial_y u\|_{L^2(\mathbb{R}_+; B^{\frac{1}{2}, 0})} \\
\leq C \|e^{\frac{y^2}{2t}} e^{\delta |D_x|} u_0\|_{B^{\frac{1}{2}, 0}}. \tag{1.11}
\]
Furthermore, for any \( t > 0 \), there hold
\[
\| \langle t \rangle \frac{1}{2} \left\| e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} u(t) \right\|_{B^1_2,0}^{\frac{1}{2} + \gamma} + \left\| \langle t \rangle \frac{1}{2} e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} \partial_y u \right\|_{L^2(t/2; t; B^1_2,0)}^{\frac{1}{2} + \gamma} \leq C \| e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} (\varphi_0, u_0) \|_{B^1_2,0}^{\frac{1}{2} + \gamma},
\]
(1.12)
\[
\| \langle t \rangle \frac{1}{2} \left\| e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} G(t) \right\|_{B^1_2,0}^{\frac{1}{2} + \gamma} + \left\| \langle t \rangle \frac{1}{2} e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} \partial_y G \right\|_{L^2(t/2; t; B^1_2,0)}^{\frac{1}{2} + \gamma} \leq C \| e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} G_0 \|_{B^1_2,0}^{\frac{1}{2} + \gamma},
\]
and
\[
\| \langle t \rangle \frac{1}{2} \left\| e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} \partial_t u \right\|_{B^1_2,0}^{\frac{1}{2} + \gamma} + \left\| \langle t \rangle \frac{1}{2} e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} \partial_y u \right\|_{L^2(t/2; t; B^1_2,0)}^{\frac{1}{2} + \gamma} \leq C \| e^{\frac{t^2}{8}} e^{\frac{3}{2} |dx|} G_0 \|_{B^1_2,0}^{\frac{1}{2} + \gamma},
\]
(1.13)
for any \( \gamma \in (0, 1) \).

The anisotropic Besov spaces \( B^1_2,0 \) will be recalled in Section 2. Here and all
in that follows, we always denote \( \langle t \rangle \overset{\text{def}}{=} 1 + t \).

**Remark 1.1.** (1) In the previous results concerning the long time well-posedness
of the Prandtl system in [21,37], only a lower bound of the lifespan to the
solution was obtained. We also mention that similar type of result as in
[21,37] for the lifespan of MHD boundary layer equation was obtained in
[34].

(2) Our global well-posedness result does not contradict the blow-up result in
[12]. In fact, Theorem 1.1 of [12] claims that if \( u_0(0, y) = 0 \) and \( a_0(y) = -\partial_x u_0(0, y) \) is nonnegative and of compact support such that
\[
E(a_0) < 0 \quad \text{with} \quad E(a) \overset{\text{def}}{=} \int_0^\infty \left( \frac{1}{2} (\partial_y a(y))^2 - \frac{1}{4} a^3(y) \right) \, dy < 0. \tag{1.14}
\]

Then any smooth solution of (1.1) does not exist globally in time.

For small initial data \( u_0(x, y) = \eta \phi(x, y) \), we have \( a_0(y) = -\eta \partial_x \phi(0, y) \) and
\[
E(a_0) = \frac{\eta^2}{2} \int_0^\infty (\partial_x \phi(0, y))^2 \, dy - \frac{\eta^3}{4} \int_0^\infty (\partial_x \phi(0, y))^3 \, dy,
\]
which cannot satisfy \( E(a_0) < 0 \) for \( \eta \) sufficiently small except that \( \partial_x \phi(0, y) = 0 \). However, in the latter case, due to the fact that the solution decays to zero
as \( y \) approaching to \( +\infty \), we have \( a_0(y) = \partial_x \phi(0, y) = 0 \), which implies
\( E(a_0) = 0 \) so that (1.14) can not be satisfied in both cases.

(3) We also remark that the exponential weight that appears in the norm of (1.10)
excludes the possibility of taking initial data of (1.5), which is slowly varying in
the normal variable. Indeed we consider an initial data of the form \( u_0^\varepsilon(x, y) = \eta \phi(x, \varepsilon y) \) with \( \eta, \varepsilon \) being sufficiently small such that
\[
E(a_0) = \frac{\varepsilon \eta^2}{2} \int_0^\infty (\partial_x \phi(0, y))^2 \, dy - \frac{\eta^3}{4\varepsilon} \int_0^\infty (\partial_x \phi(0, y))^3 \, dy < 0.
\]
Then it is easy to check that $u_0^\varepsilon$ defined above cannot verify our smallness condition (1.10).

**Remark 1.2.** (1) The idea of closing the analytic energy estimate (1.11) for solutions of (1.5) goes back to [4], where Chemin introduced a tool to make analytical type estimates and controlling the size of the analytic radius simultaneously. It was used in the context of anisotropic Navier–Stokes system [5] (see also [30,31]), which implies the global well-posedness of three dimensional Navier–Stokes system with a class of “ill prepared data”, which is slowly varying in the vertical variable, namely of the form $\varepsilon x_3$, and the $B_{-1,-1}^{-1}(\mathbb{R}^3)$ norm of which blow up as the small parameter goes to zero.

(2) We mention that in our previous paper with Zhang in [29], we used the weighted analytic norm of $\partial_y u$ to control the analytic band of the solutions, which seems more obvious than the weighted analytic norm of $g$, which is defined by (1.7). Since in [29], we worked on a Prandtl type system in a strip with homogenous boundary condition so that we can use the classical Poincaré inequality to derive the exponential decay estimates for the solutions. Therefore we have a global control for the analytic band. Here in the upper space, by using another type of Poincaré inequality, (3.1), yields decay of a sort of weighted analytic norm to $\partial_y u$ like $(t)^{-\frac{1}{4}}$ as the time $t$ going to $\infty$. However this estimate cannot guarantee that the quantity $\int_0^\infty (t)^{\frac{1}{4}} \| e^{y} \partial_y u(t) \|_{B_{2,0}^{-1}} \, dt$ is finite, which will be crucial to globally control the analytic band of the solutions to (1.5).

**Remark 1.3.** We would like to mention two recent papers concerning the global well-posedness of the boundary layer equations after the submission of this paper.

(1) Liu and the second author [23] extended a similar result as that of Theorem 1.1 for the MHD boundary layer equations where we could also allow the far field velocity and magnetic field to depend on the tangential variable. There of course, the analytic norms of far fields are assumed to be small and to decay to zero with some rates as the time approaching to $\infty$.

(2) By combining the method in [9], which is based on both a tricky change of unknown and appropriate choice of test function, and the time weighted energy estimate method used in the proof of Theorem 1.1, Wang, Wang and the second author [36] proved the global well-posedness of the system (1.1) with initial data in the optimal Gevrey class 2. However due to the instabilities described in [9], the solution will lose initial regularities, and there will be loss of decay rates for the solutions in [36] compared with that in (1.12). Furthermore, the smallness condition is posed on the initial data $(\varphi_0, u_0)$ instead of the “good-unknown” $G_0$ in (1.10).

Let us end this introduction by mentioning the notations that we shall use in this article.

For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq C b$. $(a \mid b)_{L^2_v} \overset{\text{def}}{=} \int_{\mathbb{R}^2_v} a(x, y) b(x, y) \, dx \, dy$ (resp. $(a \mid b)_{L^2_v} \overset{\text{def}}{=} \int_{\mathbb{R}^2_v} a(y) b(y) \, dy$) stands for the $L^2$ inner product of $a, b$ on $\mathbb{R}_+^2$ (resp.
2. Littlewood-Paley Theory and Functional Framework

Throughout the rest of this paper, we shall frequently use Littlewood-Paley decomposition in the horizontal variable, $x$. For the convenience of the reader we shall collect some basic facts on the anisotropic Littlewood-Paley theory in this section. Let us first recall from [2] that

\[
\Delta_k^h a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a}), \quad S_k^h a = \mathcal{F}^{-1}(\chi(2^{-k}|\xi|)\hat{a}),
\]

(2.1)

where, and in all that follows, $\mathcal{F}a$ and $\hat{a}$ always denote the partial Fourier transform of the distribution $a$ with respect to $x$ variable, that is, $\hat{a}(\xi, y) = \mathcal{F}x\to\hat{a}(a)(\xi, y)$ and $\chi(\tau)$, $\varphi(\tau)$ are smooth functions such that

\[
\text{Supp } \varphi \subset \left\{ \tau \in \mathbb{R} / \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{k \in \mathbb{Z}} \varphi(2^{-k}\tau) = 1,
\]

\[
\text{Supp } \chi \subset \left\{ \tau \in \mathbb{R} / |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) + \sum_{k \geq 0} \varphi(2^{-k}\tau) = 1.
\]

**Definition 2.1.** Let $s$ be in $\mathbb{R}$. For $u$ in $S'_h(\mathbb{R}^2_+)$, which means that $u$ is in $S'(\mathbb{R}^2_+)$ and satisfies $\lim_{k \to -\infty} \|S_k^h u\|_{L^\infty} = 0$, we set

\[
\|u\|_{B^{s,0}} \defeq \left\| \left(2^{ks}\|\Delta_k^h u\|_{L^2_+}\right)_{k \in \mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})}.
\]

- For $s \leq \frac{1}{2}$, we define $B^{s,0}(\mathbb{R}^2_+) \defeq \{ u \in S'_h(\mathbb{R}^2_+) \mid \|u\|_{B^{s,0}} < \infty \}$.
- If $\ell$ is a positive integer and if $\ell - \frac{1}{2} < s \leq \ell + \frac{1}{2}$, then we define $B^{s,0}(\mathbb{R}^2_+)$ as the subset of distributions $u$ in $S'_h(\mathbb{R}^2_+)$ such that $\partial_\ell^s u$ belongs to $B^{s-\ell,0}(\mathbb{R}^2_+)$.

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner type spaces $\tilde{L}^p(\mathbb{R}^2_+)$.

**Definition 2.2.** Let $p \in [1, +\infty]$ and $T_0, T \in [0, +\infty]$. We define $\tilde{L}^p(T_0, T; B^{s,0}(\mathbb{R}^2_+))$ as the completion of $C((T_0, T); S(\mathbb{R}^2_+))$ by the norm

\[
\|a\|_{\tilde{L}^p(T_0, T; B^{s,0})} \defeq \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_{T_0}^T \|\Delta_k^h a(t)\|_{L^2_+}^p \, dt \right)^{\frac{1}{p}},
\]

with the usual change if $p = \infty$. In particular, when $T_0 = 0$, we shall denote $\|a\|_{\tilde{L}^p_p(B^{s,0})} \defeq \|a\|_{\tilde{L}^p(0, T; B^{s,0})}$ for simplicity.
In order to overcome the difficulty that one cannot use a Gronwall type argument in the framework of Chemin-Lerner space $\tilde{L}^2_T(B^s,0)$, we also need to use the time-weighted Chemin-Lerner type norm, which was introduced by the authors in [28].

**Definition 2.3.** Let $f(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$ be a nonnegative function and $t_0, t \in [0, \infty]$. We define

$$\|a\|_{\tilde{L}^p_{t_0,t}([B^s,0])} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_{t_0}^{t} f(t') \|\Delta_h^k a(t')\|_{L^p_{\mathbb{R}_+}}^p \frac{dt'}{\Delta^1_h} \right)^{\frac{1}{p}}.$$  

(2.2)

When $t_0 = 0$, we simplify the notation $\|a\|_{\tilde{L}^p_{0,t}([B^s,0])}$ as $\|a\|_{\tilde{L}^p_{t}([B^s,0])}$.

We also recall the following anisotropic Bernstein lemma from [6,27]:

**Lemma 2.1.** Let $B^k$ be a ball of $\mathbb{R}_h$, and $C_h$ a ring of $\mathbb{R}_h$; let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q \leq \infty$. Then it holds that

If the support of $\hat{a}$ is included in $2^k B^k$, then

$$\|\partial^\ell_x a\|_{L^{p_1}_{\mathbb{R}_h}(L^q)} \lesssim 2^k \left( \ell + \frac{1}{p_2} - \frac{1}{p_1} \right) \|a\|_{L^{p_1}_{\mathbb{R}_h}(L^q)}.$$  

If the support of $\hat{a}$ is included in $2^k C_h$, then

$$\|a\|_{L^{p_1}_{\mathbb{R}_h}(L^q)} \lesssim 2^{-k\ell} \|\partial^\ell_x a\|_{L^{p_1}_{\mathbb{R}_h}(L^q)}.$$  

Finally, to deal with the estimate concerning the product of two distributions, we shall frequently use Bony’s decomposition (see [3]) in the horizontal variable

$$fg = T^h_{f,g} + T^h_g f + R^h(f, g),$$  

(2.3)

where

$$T^h_{f,g} \overset{\text{def}}{=} \sum_k S^h_{k-1} f \Delta^h_k g, \quad R^h(f, g) \overset{\text{def}}{=} \sum_k \widetilde{\Delta}^h_k f \Delta^h_k g \quad \text{with} \quad \widetilde{\Delta}^h_k f \overset{\text{def}}{=} \sum_{|k-k'|\leq 1} \Delta^h_k f.$$  

3. Sketch of the Proof to Theorem 1.1

We point out that a key ingredient used in the proof of Theorem 1.1 is the following Poincaré type inequality, which is a special case of Treves inequality that can be found in [20] (see also Lemma 3.3 of [21]):

**Lemma 3.1.** Let $\Psi(t,y) \overset{\text{def}}{=} \frac{y^2}{8(t)}$ and $d$ be a nonnegative integer. Let $u$ be a smooth enough function on $\mathbb{R}_d \times \mathbb{R}_+$ which decays to zero sufficiently fast as $y$ approaching to $+\infty$. Then one has

$$\int_{\mathbb{R}_d \times \mathbb{R}_+} |\partial_y u(X,y)|^2 e^{2\Psi} \, dX \, dy \geq \frac{1}{2(t)} \int_{\mathbb{R}_d \times \mathbb{R}_+} |u(X,y)|^2 e^{2\Psi} \, dX \, dy.$$  

(3.1)
Proof. We remark that compared with Lemma 3.3 of [21], here we do not need any boundary condition for $u$ on the boundary $y = 0$. For completeness, we outline its proof here. As a matter of fact, for any fixed $X \in \mathbb{R}^d$, we first get, by using integration by parts, that

$$
\int_{R_+} u^2(X, y) e^{\frac{y^2}{4t}} \, dy = \int_{R_+} (\partial_y u) u^2(X, y) e^{\frac{y^2}{4t}} \, dy
$$

$$
= -2 \int_{R_+} yu(X, y) \partial_y u(X, y) e^{\frac{y^2}{4t}} \, dy
$$

$$
- \frac{1}{2(t)} \int_{R_+} y^2 u^2(X, y) e^{\frac{y^2}{4t}} \, dy.
$$

By integrating the above inequality over $\mathbb{R}^d$ with respect to the $X$ variables, we find

$$
\int_{\mathbb{R}^d \times R_+} u^2 e^{\frac{y^2}{4t}} \, dX \, dy + \frac{1}{2(t)} \int_{\mathbb{R}^d \times R_+} y^2 u^2 e^{\frac{y^2}{4t}} \, dX \, dy
$$

$$
= -2 \int_{\mathbb{R}^d \times R_+} yu \partial_y u e^{\frac{y^2}{4t}} \, dX \, dy
$$

$$
\leq 2 \left( \frac{1}{2(t)} \int_{\mathbb{R}^d \times R_+} y^2 u^2 e^{\frac{y^2}{4t}} \, dX \, dy \right)^{1/2} \left( 2(t) \int_{\mathbb{R}^d \times R_+} (\partial_y u)^2 e^{\frac{y^2}{4t}} \, dX \, dy \right)^{1/2}
$$

$$
\leq \frac{1}{2(t)} \int_{\mathbb{R}^d \times R_+} y^2 u^2 e^{\frac{y^2}{4t}} \, dX \, dy + 2(t) \int_{\mathbb{R}^d \times R_+} (\partial_y u)^2 e^{\frac{y^2}{4t}} \, dX \, dy.
$$

This leads to (3.1). \qed

By virtue of Lemma 3.1, we get, by using a standard argument of energy estimate to the system (1.4), that

$$
\| e^{\frac{y^2}{4t}} \partial_y u^\varepsilon(t) \|_{L^2_y} \leq C(t)^{-\frac{3}{4}}.
$$

We remark that, intuitively, the quantity $\| e^{\Psi} \partial_y u^\varepsilon(t) \|_{L^2_y}$ is a natural part to control the time evolution of the analytical radius to the analytic solutions of (1.1), yet it is obvious that (3.2) is not enough to guarantee that the quantity $\int_0^\infty \langle t \rangle^{\frac{1}{4}} \| e^{\frac{y^2}{8t}} \partial_y u^\varepsilon(t) \|_{L^2_y} \, dt$ is finite, which will be required to go through our process below.

To overcome the above difficulty, we are going to construct a special solution of (1.4) via its primitive function, that is, $u^\varepsilon(t, y) = \partial_y \psi^\varepsilon(t, y)$. We define $\psi^\varepsilon$ through

$$
\begin{aligned}
\frac{\partial_t \psi^\varepsilon}{c_M} - \partial^2_y \psi^\varepsilon = & \varepsilon M(t, y), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\
\psi^\varepsilon|_{y=0} = & 0 \quad \text{and} \quad \lim_{y \to +\infty} \psi^\varepsilon = 0,
\end{aligned}
$$

(3.3)

where

$$
M(t, y) \overset{\text{def}}{=} - \int_y^\infty (1 - \chi(y')) \, dy' f'(t) + f(t) \chi'(y),
$$

(3.4)
so that $m$ in (1.3) equals to $\partial_y M$. We observe that $\int_{y}^{\infty} (1 - \chi(y')) \, dy' = 0$ for $y \geq 2$, that is, $M(t, y)$ is supported on the interval $[0, 2]$ with respect to $y$ variable. It is crucial to observe that the quantity

$$G^s \overset{\text{def}}{=} u^s + \frac{y}{2(t)} \psi^s$$

(3.5)

decays faster than $u^s$, which is inspired the definition of the function $g$ in [21]. Indeed we first observe from (3.3) that

$$\partial_t \left( \frac{y}{2(t)} \psi^s \right) - \partial_y^2 \left( \frac{y}{2(t)} \psi^s \right) + \frac{1}{(t)} G^s = \varepsilon \frac{y}{2(t)} M,$$

from which, along with (1.4) and (3.5), we find

$$\left\{ \begin{array}{l}
\partial_t G^s - \partial_y^2 G^s + \langle t \rangle^{-1} G^s = \varepsilon H \quad \text{with} \quad H \overset{\text{def}}{=} m + \frac{y}{2(t)} M, \\
G^s|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} G^s(t, y) = 0, \\
G^s|_{t=0} = 0.
\end{array} \right.$$ 

(3.6)

With $G^s$ being determined by (3.6), by virtue of (3.5) and $\psi^s|_{y=0} = 0$, we obtain

$$\psi^s(t, y) = e^{-\frac{y^2}{4\langle t \rangle}} \int_0^y e^{\frac{y'}{4\langle t \rangle}} G^s(t, y') \, dy' \quad \text{and} \quad u^s(t, y) \overset{\text{def}}{=} \partial_y \psi^s(t, y).$$

(3.7)

We observe that $u^s$ defined above satisfies the boundary condition $u^s(t, 0) = 0$ although the boundary condition in (3.3) does not match with that in (1.4).

As we already mentioned in (3.2), a similar decay estimate for the weighted analytical norm to the solutions of (1.5) can also be derived, which will not be enough to go through our process below. Inspired by the function $g \overset{\text{def}}{=} \partial_y u + \frac{y}{2(t)} u$, which was introduced by Ignatova and Vicol in [21], here we introduce the function $G$ and $g$ in (1.7). It is a crucial observation here that the weighted analytic norm of $g$ can control the evolution of the analytic norm to the solutions of (1.5).

Next, as in [4, 5, 8, 29–31, 37], for any locally bounded function $\Phi$ on $\mathbb{R}^+ \times \mathbb{R}$, we define

$$u_{\Phi}(t, x, y) = \mathcal{F}_{\xi \to -x}^{-1} \left( e^{\Phi(t, \xi)} \widehat{u}(t, \xi, y) \right).$$

(3.8)

Let $G$ and $G^s$ be determined by (1.7) and (3.5), respectively, we introduce a key quantity $\theta(t)$ to describe the evolution of the analytic band to the solutions of (1.5):

$$\left\{ \begin{array}{l}
\dot{\theta}(t) = \langle t \rangle^{\frac{1}{2}} \left( \| e^{\Phi} \partial_y G^s(t) \|_{L^1_x} + \varepsilon f(t) \| e^{\Phi} \chi' \|_{L^2_x} + \| e^{\Phi} \partial_y G(t) \|_{B_{2,1}^{1,0}} \right), \\
\theta|_{t=0} = 0.
\end{array} \right.$$ 

(3.9)

Here $(t) \overset{\text{def}}{=} 1 + t$, and the phase function $\Phi$ is defined by

$$\Phi(t, \xi) \overset{\text{def}}{=} (\delta - \lambda \theta(t)) |\xi|,$$

(3.10)
and the weighted function $\Psi(t, y)$ is determined by

$$
\Psi(t, y) = \frac{y^2}{8\langle t \rangle},
$$

which satisfies

$$
\partial_t \Psi(t, y) + 2(\partial_y \Psi(t, y))^2 = 0.
$$

We present now a more precise statement of our result in this paper.

**Theorem 3.1.** Let $\Phi$ and $\Psi$ be defined respectively by (3.10) and (3.11). Then under the assumptions of Theorem 1.1, there exist positive constants $c_0$, $\varepsilon_0$ and $\lambda$, so that for $u^s$ determined by (3.7) and $\varepsilon \leq \varepsilon_0$, the system (1.5) has a unique global solution $u$ which satisfies $\sup_{t \in [0, \infty)} \theta(t) \leq \frac{\delta}{2\lambda}$, and

$$
\|e^{\Psi}u\|_{\tilde{L}^\infty(R^+; B^{1,0}_{2,0})} + \|e^{\Psi} \partial_y u\|_{\tilde{L}^2(R^+; B^{1,0}_{2,0})} \leq C\|e^{\frac{\varepsilon}{c_0}} e^{\delta |D_x|} u_0\|_{B^{1,0}_{2,0}}.
$$

Moreover, for $G$ given by (1.7), there exists a positive constant $C$ so that for any $t > 0$ and $\gamma \in (0, 1)$, it holds that

$$
\|e^{\Psi} G\|_{\tilde{L}^\infty(R^+; B^{1,0}_{2,0})} + \|e^{\Psi} \partial_y G\|_{\tilde{L}^2(R^+; B^{1,0}_{2,0})} \leq C\|e^{\frac{\varepsilon}{c_0}} e^{\delta |D_x|} G_0\|_{B^{1,0}_{2,0}}.
$$

We remark that one of the crucial step to prove Theorem 3.1 is to control the time evolution of $\theta(t)$, which basically determines the analytical radius of the solutions to (1.5).

Let us now sketch the structure of this paper below.

In Section 4, we shall prove the following proposition concerning the large time decay estimate of $\|e^{\Psi} G^s(t)\|_{L^2}$, which in particular guarantees

**Proposition 3.1.** Let $f(t) \in H^1(R_+)$ and satisfy (1.9). Then for $G^s$ being determined by (3.6), one has

$$
\int_0^\infty \langle t \rangle^{\frac{1}{2}} \|e^{\Psi} \partial_y G^s(t)\|_{L^2}^2 \, dt \leq C C_f \varepsilon
$$

for the constant $C_f$ given by (1.9).

In what follows, we shall always assume that $t < T^*$ with $T^*$ being determined by

$$
T^* \defeq \sup \{ t > 0, \theta(t) < \delta/\lambda \}.
$$
so that by virtue of (3.10), for any $t < T^*$, there holds the following convex inequality:

$$
\Phi(t, \xi) \leq \Phi(t, \xi - \eta) + \Phi(t, \eta) \quad \text{for} \quad \forall \xi, \eta \in \mathbb{R}.
$$

(3.17)

In Section 5, we shall deal with the \textit{a priori} decay estimates for the analytic solutions of (1.6).

\textbf{Proposition 3.2.} Let $\varphi$ be a smooth enough solution of (1.6). Then there exists a large enough constant $\lambda$, so that for any nonnegative and non-decreasing function $n \in C^1(\mathbb{R}_+)$ and any $t_0 \in [0, t]$ with $t < T^*$, one has

$$
\| (t')^{\frac{1}{2}} e^{\Psi} \varphi \|_{L^\infty L^\infty(B^{1,0})} \leq C \| e^{\frac{2}{\lambda}} e^{\delta |D_1|} |\varphi_0| \|_{B^{1,0}},
$$

and

$$
\| h^\frac{1}{2} e^{\Psi} \varphi \|_{L^\infty (0, t; B^{1,0})} + \| h^\frac{1}{2} e^{\Psi} \partial_y \varphi \|_{L^2 (0, t; B^{1,0})} \leq \| h^\frac{1}{2} e^{\Psi} \varphi (t_0) \|_{B^{1,0}} + \| \sqrt{h^\frac{1}{2}} e^{\Psi} \varphi \|_{L^2 (0, t; B^{1,0})}.
$$

(3.19)

Section 6 is devoted to the \textit{a priori} decay estimates for the analytic solutions of (1.5):

\textbf{Proposition 3.3.} Let $u$ be a smooth enough solution of (1.5). Then there exists a large enough constant $\lambda$, so that for any $t < T^*$, we have

$$
\| (t')^{\frac{1}{2}} e^{\Psi} u \|_{L^\infty L^\infty(B^{1,0})} + \| (t')^{\frac{1}{2}} e^{\Psi} \partial_y u \|_{L^2 (0, t; B^{1,0})} \leq C \| e^{\frac{2}{\lambda}} e^{\delta |D_1|} (\varphi_0, u_0) \|_{B^{1,0}}.
$$

(3.20)

\textbf{Remark 3.1.} Just as the decay of solutions to the classical and inhomogeneous incompressible Navier–Stokes system (see [7] and the references therein), if the solutions decay to zero with the rate $\langle t \rangle^{-s}$ for $s > 0$ as time going to $\infty$, the space derivatives of the solutions should decay to zero with rate $\langle t \rangle^{-\left(\frac{s}{2} + \frac{1}{2}\right)}$. Here the scenario is the same. The derivation of the decay-in-time estimate (3.20) will be based on the decay-in-time estimate for $\varphi$ obtained in (3.19). Otherwise, if we perform directly the analytic energy estimate for $u$, the decay rate will be $(t)^{-\frac{1}{2}}$ instead of $(t)^{-\frac{3}{4}}$ as claimed in Proposition 3.3.

In Section 7, we shall deal with the \textit{a priori} decay estimates of $G$, which will be the most crucial ingredient used in the proof of Theorem 3.1.

\textbf{Proposition 3.4.} Let $G$ be determined by (1.7). Then there exists a large enough constant $\lambda$, so that for any $t < T^*$, we have

$$
\| (t')^{\frac{5}{2}} e^{\Psi} G \|_{L^\infty L^\infty(B^{1,0})} + \| (t')^{\frac{5}{2}} e^{\Psi} \partial_y G \|_{L^2 (0, t; B^{1,0})} + \int_0^t \langle t' \rangle^{\frac{1}{2}} e^{\Psi} \partial_y G (t') \|_{B^{1,0}} \, dt' \leq C \| e^{\frac{2}{\lambda}} e^{\delta |D_1|} G_0 \|_{B^{1,0}}.
$$

(3.21)
We remark that the most natural replacement of \( \|e^\Psi \partial_y G_\Phi(t)\|_{B^\frac{1}{2},0}^1 \) in (3.9) would be \( \|e^\Psi \partial_y u_\Phi(t)\|_{B^\frac{1}{2},0}^1 \) as was the case is in [29]. However we observe from Proposition 3.3 that the decay-in-time estimate (3.20) cannot guarantee the convergence of the integral \( \int_0^\infty (t)^{\frac{3}{2}} \|e^\Psi \partial_y G_\Phi(t)\|_{B^\frac{1}{2},0}^1 \, dt \). The idea here is control \( u \) via \( G \), which in particular induces a faster decay rate for \( u \) as time going to \( \infty \). The price to pay is that we lose decay in the \( y \) variable for \( u \) as \( y \) is approaching \( +\infty \). This is the content of the following lemma, which will be frequently used in the subsequent sections:

**Lemma 3.2.** Let \( G \) and \( \Psi \) be defined respectively by (1.7) and (3.11). Let \( \varphi \) and \( u \) be smooth enough solution of (1.6) and (1.5) respectively on \([0, T]\). Then, for any \( \gamma \in (0, 1) \) and \( t \leq T \), one has

\[
\begin{align*}
\|e^\Psi \Delta_k^h u_\Phi(t)\|_{L^2_+} &\lesssim \|e^\Psi \Delta_k^h G_\Phi(t)\|_{L^2_+}; \quad (3.22) \\
\|e^\Psi \Delta_k^h \partial_y u_\Phi(t)\|_{L^2_+} &\lesssim \|e^\Psi \Delta_k^h \partial_y G_\Phi(t)\|_{L^2_+}; \quad (3.23) \\
\langle t \rangle^{-1} \|e^\Psi \Delta_k^h \partial_y (y\varphi) \Phi(t)\|_{L^2_+} + \langle t \rangle^{-\frac{3}{4}} \|e^\Psi \Delta_k^h \partial_y (y\varphi) \Phi(t)\|_{L^\infty(L^2_+)} &\lesssim \|e^\Psi \Delta_k^h \partial_y G_\Phi\|_{L^2_+}. \quad (3.24)
\end{align*}
\]

**Remark 3.2.** Although we can extract some information of \( (\varphi, u) \) from \( G \) according to Lemma 3.2, we lose the decay of \( (\varphi, u) \) at \( y = \infty \). In order to propagate the analytic regularity as well as the decay properties of \( (\varphi, u) \), we need to perform the analytic energy estimate for \( \varphi \) and \( u \) in Propositions 3.2 and 3.3. We remark that most of the technical estimates used in the proof of Propositions 3.2 and 3.3 will also be employed in the proof of Proposition 3.4.

Let us postpone the proof of this lemma till the end of this section.

We are now in a position to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** The general strategy to prove the existence result for a non-linear partial differential equation is first to construct appropriate approximate solutions, then perform uniform estimates for such approximate solution sequence, and finally pass to the limit in the approximate problem. For simplicity, here we only present the *a priori* estimates for smooth enough solutions of (1.5) in the analytical framework.

Indeed let \( u \) and \( \varphi \) be smooth enough solutions of (1.5) and (1.6) on \([0, T^*]\), where \( T^* \) is the maximal time of existence of the solutions. Let \( G \) be defined by (1.7). For any \( t < T^* \) (of course here \( T^* \leq T^* \)) with \( T^* \) being defined by (3.16), we deduce from (3.9) that

\[
\theta(t) \leq \int_0^t \langle t' \rangle^\frac{3}{2} \left( \|e^\Psi \partial_y G^s(t')\|_{L^2_+} + \|e^\Psi \partial_y G_\Phi(t')\|_{B^\frac{1}{2},0}^1 \right) \, dt' + \varepsilon \int_0^t \|e^\Psi(t') \chi'\|_{L^2_+}^\frac{1}{2} f(t') \, dt'.
\]
Notice that $\text{Supp } \chi' \subset [1, 2]$, one has

$$\| e^{\Psi(t')} \chi' \|_{L^2_v} \leq e^{\|\chi'\|_{L^2_v}} \leq e^{\frac{1}{2} \|\chi'\|_{L^2_v}},$$

from which, with Propositions 3.1 and 3.4, we infer

$$\theta(t) \leq C \left( \| e^{\frac{y^2}{4\sigma^2}} G_{0 \sigma} \|_{L^2_v} + \epsilon C_f \right) \text{ for } t < T^*, \quad (3.25)$$

where the constant $C_f$ is determined by (1.9).

In particular, if we take $c_0$ in (1.10) and $\varepsilon_0$ so small that

$$C \left( c_0 + \varepsilon_0 C_f \right) \leq \frac{\delta}{2\lambda}, \quad (3.26)$$

then we deduce from (3.25) that

$$\sup_{t \in [0, T^*)} \theta(t) \leq \frac{\delta}{2\lambda} \text{ for } \varepsilon \leq \varepsilon_0.$$

Thus, in view of (3.16), we get by a continuous argument that $T^* = \infty$. Propositions 3.3 and 3.4 ensure the first two inequalities of (3.14). Moreover, (6.11) holds for $t = \infty$, which implies (3.13). Finally Proposition 3.4 and Lemma 3.2 ensure the last inequality of (3.14). This completes the existence part of Theorem 3.1. The uniqueness part follows from Theorem 1.1 of [37]. This concludes the proof of Theorem 3.1. $\Box$

Let us end this section with the proof of Lemma 3.2.

Proof of Lemma 3.2. As a matter of fact, due to $\partial_x u + \partial_y v = 0$ and $v(t, x, 0) = 0$, we find that

$$\partial_x \int_0^\infty u(t, x, y) \, dy = - \int_0^\infty \partial_y v(t, x, y) \, dy = v(t, x, 0) = 0,$$

which implies $\int_0^\infty u(t, x, y) \, dy = C(t)$. However since $u$ decays to zero as $|x|$ tends to $\infty$, we have $C(t) = 0$, that is

$$\int_0^\infty u(t, x, y) \, dy = 0. \quad (3.27)$$

Due to $u = \partial_y \varphi$, we deduce that

$$\varphi(t, x, 0) = - \int_0^\infty u(t, x, y) \, dy = 0. \quad (3.28)$$

Thanks to (3.28), we deduce from (1.7) and $u = \partial_y \varphi$ that

$$\varphi(t, x, y) = e^{-\frac{y^2}{4\sigma^2}} \int_0^y e^{\frac{(y')^2}{4\sigma^2}} G(t, x, y') \, dy'. \quad (3.29)$$
which implies
\[ u = \partial_y \varphi = -\frac{y}{2(t)} e^{-\frac{y^2}{4t}} \int_0^y e^{\frac{(y')^2}{4t}} G(t, x, y') \, dy' + G, \] (3.30)
and
\[ \partial_y u = \partial_y^2 \varphi = -\frac{y}{2(t)} G + \partial_y G(t, y) + \left(-\frac{1}{2(t)} + \frac{y^2}{4(t)^2}\right) e^{-\frac{y^2}{4t}} \int_0^y e^{\frac{(y')^2}{4t}} G(t, x, y') \, dy'. \] (3.31)

In view of (3.11), (3.30) and
\[ \sup_{y \in [0, \infty)} (e^{-y^2} \int_0^y e^{z^2} \, dz) < \infty, \] (3.32)
we infer that
\[ \| e^{\gamma} \Delta_{\bar{h}} \rho \Phi(t) \|_{L^2_+} \]
\[ \lesssim \| e^{\gamma} \Delta_{\bar{h}} G(t) \|_{L^2_+} \]
\[ + \langle t \rangle^{-\frac{1}{2}} \left\| y e^{(\gamma - 2)\Psi} \left( \int_0^y e^{2\gamma \Psi} \, dy' \right)^{\frac{1}{2}} \left( \int_0^\infty | e^{\gamma} \Delta_{\bar{h}} G |^2 \, dy \right)^{\frac{1}{2}} \right\|_{L^2_+} \]
\[ \lesssim \| e^{\gamma} \Delta_{\bar{h}} G(t) \|_{L^2_+} + \langle t \rangle^{-\frac{3}{2}} \left\| y e^{-(1 - \gamma)\Psi} \left( \int_0^\infty | e^{\gamma} \Delta_{\bar{h}} G |^2 \, dy \right)^{\frac{1}{2}} \right\|_{L^2_+}, \]
from which, with \( \gamma \in (0, 1) \), we deduce (3.22).

Due to \( \lim_{y \to \infty} G(t, x, y) = 0 \), we write \( G = -\int_y^\infty \partial_y G \, dy' \), and hence, for \( \gamma \in (0, 1) \), we infer
\[ \| \Delta_{\bar{h}} G(t, \cdot, y) \|_{L^2_+} \leq \left( \int_y^\infty e^{-2\gamma \Psi} \, dy' \right)^{\frac{1}{2}} \left( \int_0^\infty | e^{\gamma} \Delta_{\bar{h}} \partial_y G |^2 \, dy \right)^{\frac{1}{2}} \]
\[ \lesssim \langle t \rangle^{\frac{1}{2}} e^{-\gamma \Psi} \left( \int_0^\infty | e^{\gamma} \Delta_{\bar{h}} \partial_y G |^2 \, dy \right)^{\frac{1}{2}}, \]
from which, with (3.31), we infer
\[ \| e^{\gamma} \Delta_{\bar{h}} \partial_y \Phi(t) \|_{L^2_+} \]
\[ \lesssim \langle t \rangle^{-\frac{1}{2}} \left\| e^{(\gamma - 2)\Psi} \int_0^y e^{\frac{3\gamma \Psi}{2}} \, dy' \left( \int_0^\infty | e^{\gamma} \Delta_{\bar{h}} \partial_y G \Phi |^2 \, dy \right)^{\frac{1}{2}} \right\|_{L^2_+} \]
\[ + \langle t \rangle^{-\frac{3}{2}} \left\| ye^{(\gamma - 2)\Psi} \int_0^y e^{\frac{3\gamma \Psi}{2}} \, dy' \left( \int_0^\infty | e^{\gamma} \Delta_{\bar{h}} \partial_y G \Phi |^2 \, dy \right)^{\frac{1}{2}} \right\|_{L^2_+}, \] (3.33)
\[ + \langle t \rangle^{-\frac{3}{2}} \left\| y e^{\gamma \Psi} \int_0^\infty | e^{\gamma} \Delta_{\bar{h}} \partial_y G \Phi |^2 \, dy \right\|_{L^2} \]
\[ + \| e^{\gamma} \Delta_{\bar{h}} \partial_y G(t) \|_{L^2_+}. \]
(3.33) together with the fact that
\[
\sup_{y \in \mathbb{R}^+} \left( e^{-\frac{3\gamma}{2} \Psi} \int_0^y e^{\frac{3\gamma}{2} \Psi} \, dy' \right) \leq C \langle t \rangle^{\frac{1}{2}}
\] (3.34)
implies (3.23).

Finally, let us turn to the proof of (3.24). We first observe from (3.29) that
\[
\partial_y (y \varphi) = \left( 1 - \frac{y^2}{2\langle t \rangle} \right) \varphi + y G.
\] (3.35)

Then along the same lines as the proof of (3.33), we deduce that
\[
\| e^{\gamma \Psi} \left( 1 - \frac{1}{2} \langle t \rangle^{-1} y^2 \right) \Delta_k^h \varphi \Phi (t) \|_{L^2_+} \\
= \| e^{(\gamma - 2) \Psi} \left( 1 - \frac{1}{2} \langle t \rangle^{-1} y^2 \right) \int_0^y e^{2\Psi} \int_{y'}^\infty \Delta_k^h \partial_y G \, dz \, dy' \|_{L^2_+} \\
\leq \| e^{(\gamma - 2) \Psi} \left( 1 - \frac{1}{2} \langle t \rangle^{-1} y^2 \right) \int_0^y e^{2\Psi} \left( \int_{y'}^\infty e^{-2\Psi} \, dz \right)^{\frac{1}{2}} \\
\times \left( \int_0^\infty |e^{\Psi} \Delta_k^h \partial_y G \Phi|^2 \, dy \right)^{\frac{1}{2}} \, dy' \|_{L^2_+} \\
\lesssim \langle t \rangle \cdot \left( e^{\gamma \Psi} \left( 1 - \frac{1}{2} \langle t \rangle^{-1} y^2 \right) \int_0^y e^{\frac{3\gamma}{2} \Psi} \left( \int_0^\infty |e^{\Psi} \Delta_k^h \partial_y G \Phi|^2 \, dy \right)^{\frac{1}{2}} \, dy' \right)^{\frac{1}{2}} \|_{L^2_+} \\
\lesssim \langle t \rangle \| e^{\Psi} \Delta_k^h \partial_y G \Phi \|_{L^2_+}.
\]

A direct computation ensures that
\[
\| e^{\gamma \Psi} y \Delta_k^h G \Phi (t) \|_{L^2_+} \lesssim \| e^{\gamma \Psi} y \left( \int_0^y e^{2\Psi} \, dy' \right)^{\frac{1}{2}} \left( \int_0^\infty |e^{\Psi} \Delta_k^h \partial_y G \Phi|^2 \, dy \right)^{\frac{1}{2}} \|_{L^2_+} \\
\lesssim \langle t \rangle \cdot \| e^{\frac{3\gamma}{2} \Psi} \left( \int_0^\infty |e^{\Psi} \Delta_k^h \partial_y G \Phi|^2 \, dy \right)^{\frac{1}{2}} \|_{L^2_+} \\
\lesssim \langle t \rangle \| e^{\Psi} \Delta_k^h \partial_y G \Phi \|_{L^2_+}.
\]

This, along with (3.35), ensures that
\[
\| e^{\gamma \Psi} \Delta_k^h \partial_y (y \varphi) \Phi (t) \|_{L^2_+} \lesssim \langle t \rangle \| e^{\Psi} \Delta_k^h \partial_y G \Phi \|_{L^2_+}.
\] (3.36)

By exactly the same procedure as in the proof of (3.36), we find that
\[
\| e^{\gamma \Psi} \Delta_k^h \partial_y \left( y \varphi \right) \Phi (t) \|_{L^2_+} \lesssim \langle t \rangle \cdot \| e^{\Psi} \Delta_k^h \partial_y G \Phi \|_{L^2_+}.
\]

This, together with (3.36), ensures (3.24). We thus conclude the proof of Lemma 3.2.
4. The Decay-in-Time Energy Estimate of $G^s$

The goal of this section is to present the proof of Proposition 3.1. The particular, we are going to prove in the classical weighted energy space that $G^s$ determined by (3.6) decays faster than $u^s$, with the decay rate being given by (3.2). We start the proof of Proposition 3.1 by the following lemma:

**Lemma 4.1.** Let $G^s(t, y)$ and $\Psi(t, y)$ be defined by (3.5) and (3.11), respectively.

Then for any $t > 0$, one has

$$\| \langle t \rangle^{\frac{5}{2}} e^\Psi G^s \|_{L^\infty(\mathbb{R}^+; L^2_\nu)} \leq C \varepsilon \| \langle t \rangle^{\frac{5}{2}} H \|_{L^1(\mathbb{R}^+; L^2_\nu)},$$

(4.1)

and

$$\int_0^t \| \langle t' \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\nu}^2 \, dt' \leq e^2 \left( \| \langle t \rangle^{\frac{5}{2}} H \|_{L^1(\mathbb{R}^+; L^2_\nu)}^2 + \| \langle t \rangle^{\frac{5}{2}} H \|_{L^2(\mathbb{R}^+; L^2_\nu)}^2 \right),$$

(4.2)

for $H$ given by (3.6).

**Proof.** By taking $L^2_\nu$ as the inner product of the $G^s$ equation of (3.6) with $e^{2\Psi} G^s$, we obtain

$$\left( \partial_t G^s | e^{2\Psi} G^s \right)_{L^2_\nu} - \left( \partial_y^2 G^s | e^{2\Psi} G^s \right)_{L^2_\nu} + \langle t \rangle^{-1} \| e^\Psi G^s(t) \|_{L^2_\nu}^2 = \varepsilon \left( H | e^{2\Psi} G^s \right)_{L^2_\nu}.$$

It is easy to observe that

$$\left( \partial_t G^s | e^{2\Psi} G^s \right)_{L^2_\nu} = \frac{1}{2} \frac{d}{dt} \| e^\Psi G^s(t) \|_{L^2_\nu}^2 - \int_{\mathbb{R}^+} e^{2\Psi} \partial_t \Psi | G^s |^2 \, dy.$$

Due to $G^s|_{y=0} = 0$, we get, by using integration by parts and Young’s inequality, that

$$- \left( \partial_y^2 G^s | e^{2\Psi} G^s \right)_{L^2_\nu} = \| e^\Psi \partial_y G^s \|_{L^2_\nu}^2 + \frac{1}{2} \int_{\mathbb{R}^+} e^{2\Psi} \partial_y \Psi \partial_y G^s G^s \, dy$$

$$\geq \frac{1}{2} \| e^\Psi \partial_y G^s \|_{L^2_\nu}^2 - 2 \int_{\mathbb{R}^+} e^{2\Psi} (\partial_y \Psi)^2 | G^s |^2 \, dy.$$

As a result, thanks to (3.12), we obtain

$$\frac{1}{2} \frac{d}{dt} \| e^\Psi G^s(t) \|_{L^2_\nu}^2 + \frac{1}{2} \| e^\Psi \partial_y G^s(t) \|_{L^2_\nu}^2 + \langle t \rangle^{-1} \| e^\Psi G^s(t) \|_{L^2_\nu}^2$$

$$\leq \varepsilon \| e^\Psi G^s(t) \|_{L^2_\nu} \| e^\Psi H(t) \|_{L^2_\nu},$$

(4.3)

Applying Lemma 3.1 for $d = 0$ yields

$$\| e^\Psi \partial_y G^s(t) \|_{L^2_\nu}^2 \geq \frac{1}{2} \| e^\Psi G^s(t) \|_{L^2_\nu}^2,$$

so that we deduce from (4.3) that

$$\frac{1}{2} \frac{d}{dt} \| e^\Psi G^s(t) \|_{L^2_\nu}^2 + \frac{5}{4\langle t \rangle} \| e^\Psi G^s(t) \|_{L^2_\nu}^2 \leq \varepsilon \| e^\Psi G^s(t) \|_{L^2_\nu} \| e^\Psi H(t) \|_{L^2_\nu},$$

$$\frac{1}{2} \frac{d}{dt} \| e^\Psi G^s(t) \|_{L^2_\nu}^2 + \frac{5}{4\langle t \rangle} \| e^\Psi G^s(t) \|_{L^2_\nu}^2 \leq \varepsilon \| e^\Psi G^s(t) \|_{L^2_\nu} \| e^\Psi H(t) \|_{L^2_\nu},$$
which implies
\[
\frac{d}{dt} \| e^\Psi G^s(t) \|_{L^2} + \frac{5}{4(t)} \| e^\Psi G^s(t) \|_{L^2} \leq \varepsilon \| e^\Psi H(t) \|_{L^2},
\]
and
\[
\frac{d}{dt} \left( (t) \frac{5}{2} \| e^\Psi G^s(t) \|_{L^2} \right) \leq \varepsilon (t) \frac{5}{2} \| e^\Psi H(t) \|_{L^2}.
\]
Integrating the above inequality over \([0, t]\) gives rise to (4.1).

On the other hand, we deduce from (4.3) and Young’s inequality that
\[
\begin{align*}
d \| e^\Psi G^s(t) \|_{L^2}^2 + \| e^\Psi \partial_y G^s(t) \|_{L^2}^2 + 2(t)^{-\frac{1}{2}} \| e^\Psi G^s(t) \|_{L^2}^2 & \leq 2\varepsilon(t) \frac{5}{2} \| e^\Psi H(t) \|_{L^2} (t)^{-\frac{1}{2}} \| e^\Psi G^s(t) \|_{L^2} \\
& \leq \varepsilon^2(t) \| e^\Psi H(t) \|_{L^2}^2 + (t)^{-1} \| e^\Psi G^s(t) \|_{L^2}^2. 
\end{align*}
\]
Multiplying the above inequality by \((t) \frac{5}{2}\) and then integrating the resulting inequality over \([t/2, t]\), we obtain
\[
\begin{align*}
\int_{t/2}^{t} \langle t' \rangle \frac{5}{2} e^\Psi \partial_y G^s(t') \|_{L^2}^2 \, dt' & \leq \int_{t/2}^{t} \langle t' \rangle \frac{5}{2} e^\Psi G^s(t') \|_{L^2}^2 \, dt' + \varepsilon^2 \int_{t/2}^{t} \langle t' \rangle \frac{5}{2} e^\Psi H(t') \|_{L^2}^2 \, dt' \\
& \leq \max_{t' \in [0, t]} \int_{t/2}^{t} \langle t' \rangle \frac{5}{2} e^\Psi G^s(t') \|_{L^2}^2 1 + \frac{5 \ln 2}{2} + \varepsilon^2 \| e^\Psi H \|_{L^2(L^2)}^2.
\end{align*}
\]
Inserting (4.1) into the above inequality leads to (4.2). This finishes the proof of Lemma 4.1. □

**Remark 4.1.** By integrating (4.4) over \([0, t]\), we obtain
\[
\| e^\Psi \partial_y G^s \|_{L^2(L^2)}^2 \leq \varepsilon^2 \int_{0}^{\infty} \langle t \rangle \| e^\Psi H(t) \|_{L^2}^2 \, dt.
\]

Let us now present the proof of Proposition 3.1.

**Proof of Proposition 3.1.** In view of (1.3) and (3.4), both \(m\) and \(M\) are supported in \([0, 2]\) for any \(t \geq 0\), so that we observe from (3.6) that
\[
\| (t) \frac{5}{4} H \|_{L^1(\mathbb{R}^+; L^2)} \leq \| (t) \frac{5}{4} m \|_{L^1(\mathbb{R}^+; L^2)} + \| (t) \frac{5}{4} m \|_{L^1(\mathbb{R}^+; L^2)} \leq C \int_{0}^{\infty} \langle t \rangle \frac{5}{4} (|f(t)| + |f'(t)|) \, dt \leq CC_f,
\]
and
\[ \| \langle t \rangle^{\frac{3}{2}} H \|_{L^2(\mathbb{R}_+;L^2_\xi)} \lesssim \| \langle t \rangle^{\frac{5}{2}} y M \|_{L^1(\mathbb{R}_+;L^2_\xi)} + \| \langle t \rangle^{\frac{3}{2}} m \|_{L^1(\mathbb{R}_+;L^2_\xi)} \]
\[ \leq C \left( \int_0^\infty \langle t \rangle^{\frac{3}{2}} \left( f^2(t) + (f'(t))^2 \right) \, dt \right)^{\frac{1}{2}} \leq C C_f, \]
for \( C_f \) given by (1.9). Hence, for any \( t > 0 \), we deduce from (4.2) and (4.5) that
\[ \| e^\Psi \partial_y G^s \|_{L^2_\xi}^2 + \int_{\frac{t}{2}}^t \langle t' \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' \leq C C_f e^2. \] (4.6)

Meanwhile, for any \( t > 1 \), we fix an integer \( N_t \) so that \( 2^{N_t-1} \leq t < 2^{N_t} \), which implies \( t/2 < 2^{N_t-1} \). Then we deduce from (4.6) that
\[ \int_{2^{N_t-1}}^t \langle t' \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' \leq \left( \int_{2^{N_t-1}}^t \langle t' \rangle^{-\frac{1}{2}} \, dt' \right)^{\frac{1}{2}} \left( \int_{t/2}^t \langle t' \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' \right)^{\frac{1}{2}} \]
\[ \leq C 2^{-\frac{N_t}{2}} C_f e. \]

Along the same lines, for any \( j \in [0, N_t - 2] \), we have
\[ \int_{2^j}^{2^{j+1}} \langle t' \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' \leq C 2^{-\frac{j}{2}} C_f e. \]

As a result, it turns out that
\[ \int_0^t \langle t \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' \leq 2 \int_0^1 \| e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' \]
\[ + \int_{2^{N_t-1}}^t \langle t' \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' + \sum_{j=0}^{N_t-2} \int_{2^j}^{2^{j+1}} \langle t' \rangle^{\frac{5}{2}} e^\Psi \partial_y G^s(t') \|_{L^2_\xi}^2 \, dt' \]
\[ \leq CC_f e \left( 1 + \sum_{j=0}^{\infty} 2^{-\frac{j}{2}} \right) \leq CC_f e. \]

This completes the proof of Proposition 3.1. \( \square \)

Motivated by the proof of Lemma 3.2, we have the following corollary of Proposition 3.1:

**Corollary 4.1.** Let \( u^s \) be determined by (3.7). Then for any \( \gamma \in (0, 1) \), we have
\[ \int_0^\infty \langle t \rangle^{\frac{1}{2}} \| e^{\gamma \Psi} \partial_y u^s(t) \|_{L^2_\xi} \, dt \leq C C_f e. \] (4.7)

**Proof.** In view of (3.7), we have
\[ u^s(t, y) = \partial_y \psi^s(t, y) = -\frac{\gamma}{2\langle t \rangle} e^{-\frac{\gamma^2}{4\langle t \rangle}} \int_0^\gamma y e^{\frac{(\gamma')^2}{4\langle t \rangle}} G^s(t, y') \, dy' + G^s(t, y) \]
and
\[
\partial_y u^s(t, y) = \frac{1}{2(t)} + \frac{y^2}{4(t^2)} e^{-\frac{y^2}{4t}} \int_0^y e^{\frac{(y')^2}{4t}} G^s(t, y') dy' - \frac{y}{2(t)} G^s(t, y) + \partial_y G^s(t, y).
\]

(4.8)

Since \( \lim_{y \to \infty} G^s(t, y) = 0 \), we write
\[
G^s(t, y) = -\int_{\infty}^y \partial_y G^s(t, y') dy'.
\]

Then due to \( \gamma \in (0, 1) \) we find that
\[
|G^s(t, y)| \leq \left( \int_{\infty}^y e^{-2\Psi} dy' \right)^{\frac{1}{2}} \|e^\Psi \partial_y G^s(t)\|_{L^2_v}
\]
\[
\lesssim (t)^{\frac{1}{4}} e^{-\frac{1+\gamma}{2}\Psi} \|e^\Psi \partial_y G^s(t)\|_{L^2_v}
\]

and
\[
\left| \int_0^y e^{\frac{(y')^2}{4t}} G^s(t, y') dy' \right| \lesssim (t)^{\frac{1}{4}} \int_0^y e^{-\frac{3-\gamma}{2}\Psi} dy' \|e^\Psi \partial_y G^s(t)\|_{L^2_v}.
\]

Hence, by virtue of (4.8), we infer that
\[
\|e^{\gamma \Psi} \partial_y u^s(t)\|_{L^2_v} \lesssim \left( (t)^{-\frac{3}{4}} \left\| e^{(\gamma-2)\Psi} \int_0^y e^{\frac{3-\gamma}{2}\Psi} dy' \right\|_{L^2_v} + (t)^{-\frac{3}{4}} \left\| ye^{-\frac{1-\gamma}{2}\Psi} \right\|_{L^2_v} + 1 \right) \|e^\Psi \partial_y G^s(t)\|_{L^2_v},
\]

which, together with (3.34), ensures that
\[
\|e^{\gamma \Psi} \partial_y u^s(t)\|_{L^2_v} \lesssim \left( (t)^{-\frac{3}{4}} \left\| e^{-\frac{1-\gamma}{2}\Psi} \right\|_{L^2_v} + (t)^{-\frac{3}{4}} \left\| ye^{-\frac{1-\gamma}{2}\Psi} \right\|_{L^2_v} + 1 \right) \|e^\Psi \partial_y G^s(t)\|_{L^2_v}
\]

(4.9)

from which and (3.15), we conclude the proof of (4.7).

\[\Box\]

5. Analytic Energy Estimate to the Primitive Function of \( u \)

The goal of this section is to present the a priori weighted analytic energy estimate to the primitive function \( \phi \) to the solution of (1.5), namely, the proof of Proposition 3.2. The key ingredient lies in the following proposition:

**Proposition 5.1.** Let \( \phi \) be a smooth enough solution of (1.6). Let \( \Phi(t, \xi) \) and \( \Psi(t, y) \) be given by (3.10) and (3.11) respectively. Then for any nonnegative and
non-decreasing function \( h \in C^1(\mathbb{R}_+) \), there exists a large enough constant \( \lambda \) so that

\[
\| h^2 e^{\Psi} \Delta^h_{k} \varphi \|_{L^2_{\infty}(t_0, t; L^2_+)}^2 + 2c\lambda 2^k \int_{t_0}^{t} \dot{t} (t') \| h^2 e^{\Psi} \Delta^h_{k} \varphi (t') \|_{L^2_+}^2 \, dt' + \| h^2 e^{\Psi} \Delta^h_{k} \partial_y \varphi \|_{L^2_{\infty}(t_0, t; L^2_+)}^2 \leq \| h^2 e^{\Psi} \Delta^h_{k} \varphi (t_0) \|_{L^2_+}^2 \\
+ \int_{t_0}^{t} h' (t') \| e^{\Psi} \Delta^h_{k} \varphi (t') \|_{L^2_+}^2 \, dt' + C d^2_k 2^{-k} \| h^2 e^{\Psi} \varphi \|_{L^2_{t_0, t; \dot{t}(t)}(B^1, 0)}^2
\]  

(5.1)

for any \( t_0 \in [0, t] \) with \( t < T^* \), which is defined by (3.16).

**Proof.** In view of (1.6) and (3.8), we write

\[
\partial_t \varphi - \partial_{yy} \varphi + \lambda \dot{t} (t) \partial_{h} \varphi + [ (u + u^x + \varepsilon f (t) \chi (y)) \partial_x \varphi ] \partial_t \varphi + 2 \int_y \partial_y (u + u^x + \varepsilon f (t) \chi (y')) \partial_x \varphi \, dy' = 0.
\]  

(5.2)

Here and in all that follows, we shall always denote \( |D_h| \) to be the Fourier multiplier in the \( x \) variable with symbol \( |\xi| \).

By applying the dyadic operator \( \Delta^h_{k} \) to (5.2) and then taking the \( L^2_+ \) inner product of the resulting equation with \( h(t) e^{\Psi} \Delta^h_{k} \varphi \), we find that

\[
\begin{align*}
\dot{h}(t) (e^{\Psi} \Delta^h_{k} (\partial_t \varphi - \partial_{yy} \varphi) \ | e^{\Psi} \Delta^h_{k} \varphi)_{L^2_+} + \lambda \dot{t} (t) h(t) (e^{\Psi} |D_h| \Delta^h_{k} \varphi \ | e^{\Psi} \Delta^h_{k} \varphi)_{L^2_+} \\
+ \dot{h}(t) (e^{\Psi} \Delta^h_{k} [(u + u^x + \varepsilon f (t) \chi (y)) \partial_x \varphi \ | e^{\Psi} \Delta^h_{k} \varphi)_{L^2_+} \\
+ 2 \dot{h}(t) (e^{\Psi} \int_y \Delta^h_{k} [\partial_y (u + u^x + \varepsilon f (t) \chi (y')) \partial_x \varphi \ | e^{\Psi} \Delta^h_{k} \varphi)_{L^2_+} = 0.
\end{align*}
\]  

(5.3)

In the rest of this section, we shall always assume that \( t < T^* \) with \( T^* \) being determined by (3.16) so that by virtue of (3.10), for any \( t < T^* \), there holds the convex inequality (3.17).

The proof of Proposition 5.1 relies on the following lemmas:

**Lemma 5.1.** Under the assumptions of Proposition 5.1, for any \( t_0 \in [0, t] \) with \( t < T^* \), we have

\[
\int_{t_0}^{t} \dot{h}(t') (e^{\Psi} \Delta^h_{k} (\partial_t \varphi - \partial_{yy} \varphi) \ | e^{\Psi} \Delta^h_{k} \varphi)_{L^2_+} \, dt' \\
\geq \frac{1}{2} \left( \| h^2 e^{\Psi} \Delta^h_{k} \varphi (t) \|_{L^2_+}^2 - \| h^2 e^{\Psi} \Delta^h_{k} \varphi (t_0) \|_{L^2_+}^2 \\
- \int_{t_0}^{t} h' (t') \| e^{\Psi} \Delta^h_{k} \varphi (t') \|_{L^2_+}^2 \, dt' + \| e^{\Psi} \Delta^h_{k} \partial_y \varphi \|_{L^2_{t_0, t; \dot{t}(t)}(B^1, 0)}^2 \right).
\]  

(5.4)

**Lemma 5.2.** Under the assumptions of Proposition 5.1, for any \( t_0 \in [0, t] \) with \( t < T^* \), we have

\[
\int_{t_0}^{t} \dot{h}(t') \| (e^{\Psi} \Delta^h_{k} [(u + u^x + \varepsilon f (t') \chi (y)) \partial_x \varphi) \ | e^{\Psi} \Delta^h_{k} \varphi)_{L^2_+} \, dt' \\
\lesssim d^2_k 2^{-k} \| h^2 e^{\Psi} \varphi \|_{L^2_{t_0, t; \dot{t}(t)}(B^1, 0)}^2.
\]  

(5.5)
Lemma 5.3. Under the assumptions of Proposition 5.1, for any $t_0 \in [0, t]$ with $t < T^*$, we have
\[
\int_{t_0}^{t} h(t') \left| \left( e^{\Psi} \int_{y}^{\infty} \Delta_{k}^{\Phi} \mathcal{D}_{y} \varphi \right) dy' \right| e^{\Psi} \Delta_{k}^{\Phi} L_{+}^{2} \, dt' \lesssim d_{k}^{2} 2^{-k} \| h \| \| e^{\Psi} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^2 \quad (5.6)\]

Let us admit the above lemmas for the time being and continue our proof of Proposition 5.1. Indeed it follows from Lemma 2.1 that
\[
\lambda \vartheta(t) \left( e^{\Psi} |D_{h} \Delta_{k}^{\Phi} | e^{\Psi} \Delta_{k}^{\Phi} \right) L_{+}^{2} \lesssim c \lambda \vartheta(t) 2^{k} \| e^{\Psi} \Delta_{k}^{\Phi} (t) \|_{L_{+}^{2}}^{2} \quad (5.7)
\]
is easy to observe that
\[
\int_{t_0}^{t} h(t') \left| \left( e^{\Psi} \int_{y}^{\infty} \Delta_{k}^{\Phi} \mathcal{D}_{y} \varphi \right) dy' \right| e^{\Psi} \Delta_{k}^{\Phi} L_{+}^{2} \, dt' \lesssim \int_{t_0}^{t} h(t') \left( e^{-\frac{1}{2} \Psi} \int_{y}^{\infty} e^{\frac{1}{2} \Psi} \mathcal{D}_{y} \varphi \right) dy' \left( e^{\Psi} \Delta_{k}^{\Phi} \right) L_{+}^{2} \, dt'
\]
from which, with (3.9), (4.9) and Definition 2.3, we infer
\[
\int_{t_0}^{t} h(t') \left| \left( e^{\Psi} \int_{y}^{\infty} \Delta_{k}^{\Phi} \mathcal{D}_{y} \varphi \right) dy' \right| e^{\Psi} \Delta_{k}^{\Phi} L_{+}^{2} \, dt' \lesssim 2^{k} \int_{t_0}^{t} \vartheta(t') \left( e^{\frac{1}{2} \Psi} \Delta_{k}^{\Phi} \right) L_{+}^{2} \, dt' \lesssim d_{k}^{2} 2^{-k} \| h \| \| e^{\Psi} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} \quad (5.8)
\]

By integrating (5.3) over $[t_0, t]$ and then inserting the estimates, (5.4–5.8) into the resulting inequality, we obtain (5.1). This completes the proof of Proposition 5.1.

With Proposition 5.1, we now present the proof of Proposition 3.2

Proof of Proposition 3.2. we first observe from (3.1) that
\[
\frac{1}{2} \int_{0}^{t} \langle t \rangle^{-\frac{1}{2}} \| e^{\Psi} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} \, dt' \leq \int_{0}^{t} \| \langle t \rangle^{-\frac{1}{2}} e^{\Psi} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} \, dt',
\]
so that by taking $t_0 = 0$ and $h(t) = \langle t \rangle^{\frac{1}{2}}$ in (5.1), we obtain
\[
\| \langle t \rangle^{\frac{1}{2}} e^{\Psi} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} \lesssim 2 c d_{k}^{2} \| \langle t \rangle^{\frac{1}{2}} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} \quad (5.9)
\]
\[
\| \langle t \rangle^{\frac{1}{2}} e^{\Psi} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} \lesssim 2 c d_{k}^{2} \| \langle t \rangle^{\frac{1}{2}} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} + C d_{k}^{2} 2^{-k} \| \langle t \rangle^{\frac{1}{2}} e^{\Psi} \Delta_{k}^{\Phi} \|_{L_{+}^{2}}^{2} \quad (5.10)
\]
By taking square root of the above inequality and then multiplying the resulting one by $2^k$ and finally summing over $k \in \mathbb{Z}$, we find, for any $t < T^*$,

$$\| \langle t' \rangle^{1/2} e^\Psi \phi \|_{L^\infty_t (B^n_t)} + \sqrt{2c \lambda} \| \langle t' \rangle^{1/2} e^\Psi \phi \|_{L^2_t (B^n, B^{1,0}_t)} \leq \left\| e^{-\delta_1} e^{\delta_1 D} |\phi|_{B^n_t} \right\|_{B^{1,0}_t} + \sqrt{C} \| \langle t' \rangle^{1/2} e^\Psi \phi \|_{L^2_t (B^n_t, B^{1,0}_t)}.$$  \hspace{1cm} (5.9)

By taking $\lambda$ in (5.9) to be so large that $c \lambda \geq C$, we achieve (3.18).

On the other hand, in view of (5.1), we get, by using a similar derivation of (5.9), that

$$\| \int_t^\infty e^\Psi \phi \|_{L^\infty_t (B^n_{t'; t})} + \sqrt{2c \lambda} \| \int_t^\infty e^\Psi \phi \|_{L^2_t (B^n_{t'; t}; B^{1,0}_t)} \leq \left\| \int_t^\infty e^\Psi \phi \right\|_{B^{1,0}_t} + \sqrt{C} \| \int_t^\infty e^\Psi \phi \|_{L^2_t (B^n_{t'; t}; B^{1,0}_t)}.$$  \hspace{1cm} (5.10)

Taking $c \lambda \geq C$ in the above inequality gives rise to (3.19). This concludes the proof of Proposition 3.2. \hfill \Box

Let us end this section with the proofs of Lemmas 5.1–5.3.

Proof of Lemma 5.1. We first get, by using integration by parts, that

$$(e^\Psi \partial_t \Delta^h_k \phi \mid e^\Psi \Delta^h_k \phi)_{L^2_t} = (\partial_t (e^\Psi \Delta^h_k \phi) \mid e^\Psi \Delta^h_k \phi)_{L^2_t} - (\partial_t e^\Psi \Delta^h_k \phi \mid e^\Psi \Delta^h_k \phi)_{L^2_t}.$$  \hspace{1cm} (5.11)

By multiplying the above equality by $\tilde{h}(t)$ and then integrating the resulting one over $[t_0, t]$, we find

$$\int_{t_0}^t \tilde{h}(t') (e^\Psi \partial_t \Delta^h_k \phi \mid e^\Psi \Delta^h_k \phi)_{L^2_t} \, dt' = \frac{1}{2} \left( \| \tilde{h} e^\Psi \Delta^h_k \phi(t) \|_{L^2_t}^2 - \| \tilde{h} e^\Psi \Delta^h_k \phi(t_0) \|_{L^2_t}^2 \right) - \frac{1}{2} \int_{t_0}^t \tilde{h}'(t') \| e^\Psi \Delta^h_k \phi(t') \|_{L^2_t}^2 \, dt' - \int_{t_0}^t \int_{\mathbb{R}_+^n} \tilde{h} \partial_t |\Psi| e^\Psi \Delta^h_k \phi \, dx \, dy \, dt'.$$  \hspace{1cm} (5.12)

Whereas due to $\partial_y \phi|_{y=0} = 0$, by using integration by parts and Young’s inequality, we achieve

$$-\int_{t_0}^t (e^\Psi \partial_{yy} \Delta^h_k \phi \mid e^\Psi \Delta^h_k \phi)_{L^2_t} \, dt' = \| e^\Psi \Delta^h_k \partial_{yy} \phi \|_{L^2 (t_0; t, L^2^n)}^2 + 2 \int_{t_0}^t \int_{\mathbb{R}_+^n} \partial_y \Psi e^{2\Psi} \Delta^h_k \phi \partial_{yy} \phi \, dx \, dy \, dt' \geq \left( \frac{1}{2} \| e^\Psi \Delta^h_k \partial_{yy} \phi \|_{L^2 (t_0; t, L^2^n)}^2 - 2 \int_{t_0}^t \int_{\mathbb{R}_+^n} (\partial_y \Psi)^2 |e^\Psi \Delta^h_k \phi|^2 \, dx \, dy \, dt', \right.$$

which, together with (3.12) and (5.10), ensures (5.4). This finishes the proof of Lemma 5.1. \hfill \Box
Proof of Lemma 5.2. By applying Bony’s decomposition (2.3) in the horizontal variable to \( u \partial_x \varphi \), we write
\[
u \partial_x \varphi = T_u^h \partial_x \varphi + T_{\partial_x \varphi}^h u + R^h(u, \partial_x \varphi).
\]

Considering (3.17) and the support properties to the Fourier transform of the terms in \( T_u^h \partial_x \varphi \), we write
\[
\int_{t_0}^t h(t') \left| \left( e^{\Psi} \Delta_h^k [T_u^h \partial_x \varphi]_\Phi \mid e^{\Psi} \Delta_h^k \varphi \Phi \right)_{L^2_+} \right| \, dt' \leq \sum_{|k'-k| \leq 4} \int_{t_0}^t \| S_{k-1}^h u \Phi(t') \|_{L^\infty_+} \| h^{1/2} e^{\Psi} \Delta_h^k \partial_x \varphi \Phi(t') \|_{L^2_+} \| h^{1/2} e^{\Psi} \Delta_h^k \varphi \Phi(t') \|_{L^2_+} \, dt' \leq d_k(t') 2^{-k} \langle t' \rangle^{1/2} \| e^{\Psi} \partial_y G \Phi(t') \|_{B^{1/2,0}}.
\]

where \( \{ d_k(t') \}_{k \in \mathbb{Z}} \) designates a non-negative generic element in the unit sphere of \( \ell^1(\mathbb{Z}) \) for any \( t' > 0 \). Then we get, by applying Lemma 2.1, that
\[
\| S_{k-1}^h u \Phi(t') \|_{L^\infty_+} \leq \sum_{|k'-k| \leq 2} 2^{k'} \| \Delta_h^k u \Phi(t') \|_{L^\infty_+(L^2_h)} \leq \langle t' \rangle^{1/2} \| e^{\Psi} \partial_y G \Phi(t') \|_{B^{1/2,0}},
\]

which, together with (3.9), ensures that
\[
\| S_{k-1}^h u \Phi(t') \|_{L^\infty_+} \leq \hat{\theta}(t').
\]

Then in view of Definition 2.3, by applying Lemma 2.1 and Hölder’s inequality, we obtain
\[
\int_{t_0}^t h(t') \left( e^{\Psi} \Delta_h^k [T_u^h \partial_x \varphi]_\Phi \mid e^{\Psi} \Delta_h^k \varphi \Phi \right)_{L^2_+} \, dt' \leq \sum_{|k'-k| \leq 4} 2^{k'} \left( \int_{t_0}^t \hat{\theta}(t') \| h^{1/2} e^{\Psi} \Delta_h^k \varphi \Phi(t') \|_{L^2_+}^2 \, dt' \right)^{1/2} \times \left( \int_{t_0}^t \hat{\theta}(t') \| h^{1/2} e^{\Psi} \Delta_h^k \varphi \Phi(t') \|_{L^2_+}^2 \, dt' \right)^{1/2} \leq d_k^2 2^{-k} \| h^{1/2} e^{\Psi} \varphi \Phi \|_{L^2_+}^2 \| B^{1,0}(\hat{\theta}(t)) \).
\]

Similarly, we have
\[
\int_{t_0}^{t} h(t') \left| \left( e^{\Psi} \Delta^h_{k} \left[ T^h_{\partial_x \psi} u \right] \Phi + e^{\Psi} \Delta^h_{k} \phi \right) \right| L^2_+ \left| dt' \right.
\]
\[
\lesssim \sum_{|k'| - k | \leq 4} \int_{t_0}^{t} \left[ \| h^{\frac{1}{2}} e^{\Psi} S^h_{k'-1} \partial_x \phi(t')\| L^2_h(L^\infty_x) \| \Delta^h_{k} u \phi(t')\| L^\infty_h(L^\infty_x) \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{k} \phi(t')\| L^2_+ \right] dt'.
\]
from which, with (5.11), we infer
\[
\int_{t_0}^{t} h(t') \left| \left( e^{\Psi} \Delta^h_{k} \left[ T^h_{\partial_x \psi} u \right] \Phi + e^{\Psi} \Delta^h_{k} \phi \right) \right| L^2_+ \left| dt' \right.
\]
\[
\lesssim \sum_{|k'| - k | \leq 4} 2^{-\frac{\nu'}{2}} \int_{t_0}^{t} |\dot{\theta}(t')| h^{\frac{1}{2}} e^{\Psi} S^h_{k'-1} \partial_x \phi(t')\| L^2_h(L^\infty_x) \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{k} \phi(t')\| L^2_+ \left| dt' \right.
\]
\[
\lesssim \sum_{|k'| - k | \leq 4} 2^{-\frac{\nu'}{2}} \left( \int_{t_0}^{t} |\dot{\theta}(t')| h^{\frac{1}{2}} e^{\Psi} S^h_{k'-1} \partial_x \phi(t')\| L^2_h(L^\infty_x) \| dt' \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_{t_0}^{t} |\dot{\theta}(t')| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{k} \phi(t')\| L^2_+ \| dt' \right)^{\frac{1}{2}}.
\]
However it follows from Lemma 2.1 and Definition 2.3 that
\[
\left( \int_{t_0}^{t} |\dot{\theta}(t')| h^{\frac{1}{2}} e^{\Psi} S^h_{k'-1} \partial_x \phi(t')\| L^2_h(L^\infty_x) \| dt' \right)^{\frac{1}{2}}
\]
\[
\lesssim \sum_{j \leq k'-2} 2^{3j} \left( \int_{t_0}^{t} |\dot{\theta}(t')| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{j} \phi(t')\| L^2_+ \| dt' \right)^{\frac{1}{2}}
\]
\[
\lesssim d_{k'} 2^{\frac{\nu'}{2}} \| h^{\frac{1}{2}} e^{\Psi} \phi \| L^2_{t_0, t; \dot{\theta}(t)} (B^{1, 0}).
\]
As a result, it turns out that
\[
\int_{t_0}^{t} h(t') \left| \left( e^{\Psi} \Delta^h_{k} \left[ T^h_{\partial_x \psi} u \right] \Phi + e^{\Psi} \Delta^h_{k} \phi \right) \right| L^2_+ \left| dt' \right.
\]
\[
\lesssim d_{k'} 2^{-\frac{\nu'}{2}} \| h^{\frac{1}{2}} e^{\Psi} \phi \| L^2_{t_0, t; \dot{\theta}(t)} (B^{1, 0}).
\]
Finally again due to (5.11) and the support properties to the Fourier transform of the terms in \(R^h(u, \partial_x \psi)\), we get, by applying Lemma 2.1, that
\[
\int_{t_0}^{t} h(t') \left| \left( e^{\Psi} \Delta^h_{k} \left[ T^h(u, \partial_x \psi) \right] \Phi + e^{\Psi} \Delta^h_{k} \phi \right) \right| L^2_+ \left| dt' \right.
\]
\[
\lesssim 2^k \sum_{k' \geq k-3} \int_{t_0}^{t} |\tilde{\Delta^h_{k}} u \phi(t')\| L^\infty_h(L^2_x) \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{k} \partial_x \phi(t')\| L^2_+ \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{k} \phi(t')\| L^2_+ \left| dt' \right.
\]
\[
\lesssim 2^k \sum_{k' \geq k-3} 2^\frac{k'}{2} \int_{t_0}^{t} |\dot{\theta}(t')| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{k} \phi(t')\| L^2_+ \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_{k} \phi(t')\| L^2_+ \left| dt' \right.
\]
\[ \lesssim 2^k \sum_{k' \geq k-3} 2^{k'} \left( \int_{t_0}^t \dot{h}(t') \left\| \mathcal{L}_{k'} \left( m(x, \dot{\varphi}) - 1 \right) \right\|_{L^2_t} dt' \right)^{\frac{1}{2}} \]
\[ \times \left( \int_{t_0}^t \dot{h}(t') \left\| \mathcal{L}_{k'} \left( m(x, \dot{\varphi}) - 1 \right) \right\|_{L^2_t} dt' \right)^{\frac{1}{2}}, \]

which, together with Definition 2.3, ensures that

\[ \int_{t_0}^t \dot{h}(t') \left( e^{\Psi} \mathcal{L}_{k'} \left[ (u(x) \cdot \mathbf{v}(x)) \partial_x \varphi \right] \right) \geq \left\| e^{\Psi} \mathcal{L}_{k'} \varphi \right\|_{L^2_t} dt' \lesssim d_k 2^{-\frac{k}{2}} \left( \sum_{k' \geq k-3} d_{k'} 2^{-k'} \right) \left\| e^{\Psi} \mathcal{L}_{k'} \varphi \right\|_{L^2_t} \left[ (B^1, 0) \right] \]  
(5.15)

By summing up (5.12), (5.14) and (5.15), we achieve

\[ \int_{t_0}^t \dot{h}(t') \left( e^{\Psi} \mathcal{L}_{k'} \left[ (u(x) \cdot \mathbf{v}(x)) \partial_x \varphi \right] \right) \geq \left\| e^{\Psi} \mathcal{L}_{k'} \varphi \right\|_{L^2_t} \left[ (B^1, 0) \right], \]
(5.16)

It is easy to observe that

\[ \int_{t_0}^t \left( e^{\Psi} \mathcal{L}_{k'} \left[ (u(x) + e^{f(t)} \chi(y)) \partial_x \varphi \right] \right) \geq \left\| e^{\Psi} \mathcal{L}_{k'} \varphi \right\|_{L^2_t} \left[ (B^1, 0) \right] \]
(5.17)

Combining (5.16) with (5.17) leads to (5.5). This finishes the proof of Lemma 5.2.

\[ \square \]

**Proof of Lemma 5.3.** By applying Bony’s decomposition in the horizontal variable (2.3) to \( \partial_x u \partial_x \varphi \), we write

\[ \partial_x u \partial_x \varphi = T^h_{\partial_x u} \partial_x \varphi + T^h_{\partial_x \varphi} \partial_x u + R^h (\partial_x u, \partial_x \varphi). \]

Considering (3.17) and the support properties to the Fourier transform of the terms in \( T^h_{\partial_x u} \partial_x \varphi \), we write

\[ \int_{t_0}^t \dot{h}(t') \left( e^{\Psi} \mathcal{L}_{k'} \left[ T^h_{\partial_x \varphi} \partial_x \varphi \right] \right) \geq \left\| e^{\Psi} \mathcal{L}_{k'} \varphi \right\|_{L^2_t} \left[ (B^1, 0) \right] \]
(5.18)

Combining (5.16), (5.17) and (5.18) leads to (5.5). This finishes the proof of Lemma 5.2.

\[ \square \]
where we used Lemma 2.1 and (3.23) in the last step so that
\[
\|e^{\frac{3}{2}}\Psi S_{k'-1}^h \partial_y u_\Phi(t')\|_{L^2_y(L^\infty_h)} \lesssim \|e^{\Psi} \partial_x G_\Phi(t')\|_{BL.^{1.0}}.
\]
Then we get, by applying Hölder’s inequality and (3.9), that
\[
\int_{t_0}^t h(t') \left( (e^{\Psi} \int_{t_0}^\infty \Delta^h_k \left[ T_{\partial_y u}^h \partial_x \Phi \right] \Phi \, dy' \mid e^{\Psi} \Delta^h_k \partial_x \Phi \right)_{L^2_y} \right| \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \left( \int_{t_0}^t \dot{\theta}(t') \| \dot{h}^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \Phi(t') \|_{L^2_y}^2 \right)^\frac{1}{2} \\
\times \left( \int_{t_0}^t \dot{\theta}(t') \| \dot{h}^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \Phi(t') \|_{L^2_y}^2 \right)^\frac{1}{2} \\
\lesssim d_k^2 2^{-k} \| \dot{h}^\frac{1}{2} e^{\Psi} \partial_x \Phi \|_{L^2_y}^2 \left( B_{t_0, \dot{\theta}(t')}^{1.0} \right).
\]
Along the same lines, by virtue of (3.23), we infer
\[
\int_{t_0}^t h(t') \left( (e^{\Psi} \int_{t_0}^\infty \Delta^h_k \left[ T_{\partial_y u}^h \partial_x \Phi \right] \Phi \, dy' \mid e^{\Psi} \Delta^h_k \partial_x \Phi \right)_{L^2_y} \right| \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} \int_{t_0}^t \langle t' \rangle^\frac{3}{2} |h|^\frac{1}{2} e^{\Psi} S_{k'-1}^h \partial_x \Phi(t') \|_{L^2_y(L^\infty_h)} \| e^{\frac{3}{2}} \Psi \Delta^h_k \partial_y u \Phi(t') \|_{L^2_y} \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \Phi(t') \|_{L^2_y} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \int_{t_0}^t \langle t' \rangle^\frac{3}{2} \| e^{\Psi} \partial_y G_\Phi(t') \|_{BL.^{1.0}} \| h^\frac{1}{2} e^{\Psi} S_{k'-1}^h \partial_x \Phi(t') \|_{L^2_y(L^\infty_h)} \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \Phi(t') \|_{L^2_y} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \left( \int_{t_0}^t \dot{\theta}(t') \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \Phi(t') \|_{L^2_y(L^\infty_h)}^2 \right)^\frac{1}{2} \\
\times \left( \int_{t_0}^t \dot{\theta}(t') \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \Phi(t') \|_{L^2_y(L^\infty_h)}^2 \right)^\frac{1}{2}.
\]
from which, with (5.13), we deduce that
\[
\int_{t_0}^t h(t') \left( (e^{\Psi} \int_{t_0}^\infty \Delta^h_k \left[ T_{\partial_y u}^h \partial_x \Phi \right] \Phi \, dy' \mid e^{\Psi} \Delta^h_k \partial_x \Phi \right)_{L^2_y} \right| \, dt' \\
\lesssim d_k^2 2^{-k} \| h^\frac{1}{2} e^{\Psi} \partial_x \Phi \|_{L^2_y}^2 \left( B_{t_0, \dot{\theta}(t')}^{1.0} \right).
\]
Finally due to the support properties to the Fourier transform of the terms in \( R^h(u, \partial_x \phi) \), we get, by applying Lemma 2.1 and (3.23), that
\[
\int_{t_0}^t h(t') \left( (e^{\Psi} \int_{t_0}^\infty \Delta^h_k \left[ R^h(\partial_y u, \partial_x \phi) \right] \Phi \, dy' \mid e^{\Psi} \Delta^h_k \partial_x \phi \right)_{L^2_y} \right| \, dt' \\
\lesssim 2^k \sum_{k' \geq k-3} \int_{t_0}^t \langle t' \rangle^\frac{3}{2} \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_y u \Phi(t') \|_{L^2_y} \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \phi(t') \|_{L^2_y} \, dt' \\
\lesssim 2^k \sum_{k' \geq k-3} 2^{-\frac{k'}{2}} \int_{t_0}^t \dot{\theta}(t') \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \phi(t') \|_{L^2_y(L^\infty_h)} \| h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_x \phi(t') \|_{L^2_y} \, dt'.
\]
Then a similar derivation of (5.15) leads to
\[
\int_{t_0}^t h'(t') \left( \left< e^\Psi \Delta_k^h \left[ R^h(\partial_y u, \partial_x \varphi) \right], \right> \Phi \right|_{L^2_+} \right| \ dt' \\
\leq 2^{-k} \left< h^\frac{1}{2} e^\Psi \Delta_k^h u \Phi \right|_{L^2_{(t_0, t) \cap \mathcal{B}^{1,0}_+}}^2 + \left< h^\frac{1}{2} e^\Psi \partial_y u \Phi \right|_{L^2_{(t_0, t) \cap \mathcal{B}^{1,0}_+}}^2.
\]

As a consequence, we arrive at (5.6). This finishes the proof of Lemma 5.3.

6. Analytic Energy Estimate of \( u \)

In this section, we are going to present the weighted analytic energy estimate of \( u \) and to obtain its decay-in-time estimate, namely, we shall present the proof of Proposition 3.3. The key ingredient will be the following proposition:

**Proposition 6.1.** Let \( \Phi(t, \xi) \) and \( \Psi(t, y) \) be given by (3.10) and (3.11) respectively. Let \( u \) be a smooth enough solution of (1.5). Then for any nonnegative and non-decreasing function \( h \in C^1(\mathbb{R}_+) \), there exits a large enough constant \( \lambda \) so that

\[
\| h^\frac{1}{2} e^\Psi u \Phi \|_{L^\infty(0, t; \mathcal{B}^{\frac{1}{2},0})} + \| h^\frac{1}{2} e^\Psi \partial_y u \Phi \|_{L^2(0, t; \mathcal{B}^{\frac{1}{2},0})} \\
\leq \| h^\frac{1}{2} e^\Psi u \Phi(t_0) \|_{\mathcal{B}^{\frac{1}{2},0}} + \| \sqrt{h} e^\Psi u \Phi \|_{L^2(0, t; \mathcal{B}^{\frac{1}{2},0})}
\]

for any \( t_0 \in [0, t] \) with \( t < T^* \), which is defined by (3.16).

**Proof.** In view of (1.5), we get, by a similar derivation of (5.3), that

\[
\begin{aligned}
h(t) \left( e^\Psi \Delta_k^h \left( \partial_t u \Phi - \partial_{yy} u \Phi \right) \left| e^\Psi \Delta_k^h u \Phi \right|_{L^2_+}
\right.

&+ \lambda \dot{h}(t) \left( e^\Psi \left| D_h \Delta_k^h u \Phi \right| e^\Psi \Delta_k^h u \Phi \right)_{L^2_+}

&+ h(t) \left( e^\Psi \Delta_k^h \left[ \left( u + u^s + \epsilon f(t) \chi(y) \right) \partial_x u \right] \Phi \left| e^\Psi \Delta_k^h u \Phi \right|_{L^2_+}

&+ h(t) \left( e^\Psi \Delta_k^h \left[ v \partial_y \left( u + u^s + \epsilon f(t) \chi(y) \right) \right] \Phi \left| e^\Psi \Delta_k^h u \Phi \right|_{L^2_+} = 0.
\end{aligned}
\]

In what follows, we shall always assume that \( t < T^* \) with \( T^* \) being determined by (3.16) so that by virtue of (3.10), for any \( t < T^* \), there holds the convex inequality (3.17).

Let us now handle term by term in (6.2).

Firstly due to \( u \big|_{y=0} = 0 \), we get, by a similar proof of Lemma 5.1, that

\[
\int_{t_0}^t h'(t') \left( e^\Psi \Delta_k^h \left( \partial_t u \Phi - \partial_{yy} u \Phi \right) \left| e^\Psi \Delta_k^h u \Phi \right|_{L^2_+} \right) \ dt'

\geq \frac{1}{2} \left( \| h^\frac{1}{2} e^\Psi \Delta_k^h u \Phi(t_0) \|_{L^2_+}^2 - \| h^\frac{1}{2} e^\Psi \Delta_k^h u \Phi(t) \|_{L^2_+}^2 \right)

- \int_{t_0}^t h'(t') \left| e^\Psi \Delta_k^h u \Phi(t') \right|_{L^2_+}^2 \ dt' + \| h^\frac{1}{2} e^\Psi \Delta_k^h \partial_y u \Phi \|_{L^2_+}^2.
\]

The proof is completed.
It follows from Lemma 5.2 that
\[
\int_{t_0}^{t} h(t) \left( e^{\psi} \Delta_k^h [ (u + u^s + \epsilon f(t) \chi(y) ) \partial_y u \Phi ] \right) \left| e^{\psi} \Delta_k^h u \Phi \right|_{L^2_+} \ \mathrm{d}t' \leq d_k^2 2^{-k} \left\| h^\frac{1}{2} \epsilon \psi u \Phi \right\|_{L_{t_0:t; \dot{\theta}(t)}^2(B^{1,0})}^2. \tag{6.4}
\]

To deal with the estimate of \( \int_{t_0}^{t} h(t') \left( e^{\psi} \Delta_k^h [ v \partial_y u \Phi ] \right) \left| e^{\psi} \Delta_k^h u \Phi \right|_{L^2_+} \ \mathrm{d}t' \), we need the following lemma, the proof of which will be postponed at the end of this section:

**Lemma 6.1.** Under the assumptions of Proposition 6.1, for any \( t_0 \in [0, t] \) with \( t < T^* \), we have
\[
\int_{t_0}^{t} h(t') \left( e^{\psi} \Delta_k^h [ v \partial_y u \Phi ] \right) \left| e^{\psi} \Delta_k^h u \Phi \right|_{L^2_+} \ \mathrm{d}t' \leq d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\psi} u \Phi \right\|_{L_{t_0:t; \dot{\theta}(t)}^2(B^{1,0})}^2. \tag{6.5}
\]

On the other hand, due to \( \partial_x u + \partial_y v = 0 \), we have \( v = -\int_y^\infty \partial_y v \ dy' = \int_y^\infty \partial_x u \ dy' \), so that it follows from Lemma 2.1 that, for any \( \gamma \in (0, 1) \),
\[
\left\| e^{\psi} \Delta_k^h v \Phi (t') \right\|_{L^\infty(L^2_+)} \leq 2^k \left\| e^{\psi} \left( \int_y^\infty e^{-2\psi} \ dy' \right)^\frac{1}{2} \left( \int_0^\infty \left| e^{\psi} \Delta_k^h u \Phi (t') \right|^2 \ dy \right)^\frac{1}{2} \right\|_{L^\infty(L^2_+)} \tag{6.6}
\]
\[
\leq 2^k' (t')^\frac{1}{2} \left\| e^{-\frac{1}{2}\psi} \right\|_{L^\infty} \left\| e^{\psi} \Delta_k^h u \Phi (t') \right\|_{L^2_+} \leq 2^k' (t')^\frac{1}{2} \left\| e^{\psi} \Delta_k^h u \Phi (t') \right\|_{L^2_+}.
\]

Then we deduce from (4.9) and (6.6) that
\[
\int_{t_0}^{t} h(t') \left( e^{\psi} \Delta_k^h v \Phi \partial_y (u^s + \epsilon f(t) \chi(y)) \right) \left| e^{\psi} \Delta_k^h u \Phi \right|_{L^2_+} \ \mathrm{d}t' \leq 2^k \int_{t_0}^{t} h(t') \left( \left\| e^{\psi} \partial_y u^s \right\|_{L^2_+} + \epsilon f(t') \left\| e^{\psi} \chi \right\|_{L^2_+} \right) \left\| e^{\psi} \Delta_k^h v \Phi \right\|_{L^\infty(L^2_+)} \left\| e^{\psi} \Delta_k^h u \Phi \right\|_{L^2_+} \ \mathrm{d}t' \leq 2^k \int_{t_0}^{t} h(t') \left( \right\| e^{\psi} \partial_y u^s \right\|_{L^2_+} + \epsilon f(t') \left\| e^{\psi} \chi \right\|_{L^2_+} \right) \left\| e^{\psi} \Delta_k^h u \Phi \right\|_{L^2_+}^2 \ \mathrm{d}t'.
\]

As a result, thanks to (3.9) and Definition 2.3, we achieve
\[
\int_{t_0}^{t} h(t') \left( e^{\psi} \Delta_k^h v \Phi \partial_y (u^s + \epsilon f(t) \chi(y)) \right) \left| e^{\psi} \Delta_k^h u \Phi \right|_{L^2_+} \ \mathrm{d}t' \leq 2^k \int_{t_0}^{t} \dot{\theta}(t') \left\| h^\frac{1}{2} e^{\psi} \Delta_k^h u \Phi \right\|_{L^2_+}^2 \ \mathrm{d}t' \leq d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\psi} u \Phi \right\|_{L_{t_0:t; \dot{\theta}(t)}^2(B^{1,0})}^2. \tag{6.7}
\]

It follows from Lemma 2.1 that
\[
\lambda \dot{\theta}(t) \left( e^{\psi} \right) \left( D_h | \Delta_k^h u \Phi \right) \left| e^{\psi} \Delta_k^h u \Phi \right|_{L^2_+} \geq c \lambda \dot{\theta}(t) 2^k \left\| e^{\psi} \Delta_k^h u \Phi (t) \right\|_{L^2_+}^2. \tag{6.8}
\]
By integrating (6.2) over $[t_0, t]$ and then inserting the estimates, (6.3), (6.4), (6.5), (6.7) and (6.8), into the resulting inequality, we conclude that
\[
\| h^\frac{1}{2} e^\Psi \Delta_k h \|_{L^\infty_t((t_0, t); L^2_x)}^2 + 2c\lambda^2 \int_{t_0}^t \| e^\Psi h^\frac{1}{2} \partial_y u \|_{L^2_t((t_0, t); L^2_x)}^2 \, dt' \\
+ \| h^\frac{1}{2} e^\Psi \Delta_k^h \phi \|_{L^2_t((t_0, t); L^2_x)}^2 \\
\leq \| h^\frac{1}{2} e^\Psi \Delta_k h \|_{L^2((t_0, t_0); L^2)}^2 + \int_{t_0}^t h'(t') \| e^\Psi \Delta_k^h \phi \|_{L^2_t((t_0, t); L^2_x)}^2 \, dt' \\
+ C d^2 \| h^\frac{1}{2} e^\Psi \|_{L^2_{t_0, t_0}((t_0, t); B^1_0)}^2.
\]  
(6.9)

By taking square root of (6.9) and then multiplying the resulting inequality by $2^{\frac{1}{2}}$ and finally summing over $k \in \mathbb{Z}$, we find, for any $t < T^*$, that
\[
\| h^\frac{1}{2} e^\Psi \|_{L^\infty_t((t_0, t); B^1_0)} + \sqrt{2c\lambda} \| h^\frac{1}{2} e^\Psi \|_{L^2_t((t_0, t); B^1_0)} \\
+ \| h^\frac{1}{2} e^\Psi \Delta_k h \|_{L^2_t((t_0, t); B^1_0)} \\
\leq \| h^\frac{1}{2} e^\Psi \|_{L^2((t_0, t_0); B^1_0)} + \| \sqrt{h'} e^\Psi \|_{L^2_t((t_0, t); B^1_0)} \\
+ C \| h^\frac{1}{2} e^\Psi \|_{L^2_{t_0, t_0}((t_0, t); B^1_0)}.
\]  
(6.10)

By taking $\lambda$ in (6.10) to be a large enough positive constant so that $c\lambda \geq C$, we deduce (6.1). This completes the proof of Proposition 6.1. \qed

Now we are in a position to complete the proof of Proposition 3.3.

**Proof of Proposition 3.3.** Taking $h(t) = 1$ and $t_0 = 0$ in (6.1) gives rise to
\[
\| e^\Psi \phi \|_{L^\infty_t(B^1_0)} + \| e^\Psi \partial_y \phi \|_{L^2_t(B^1_0)} \leq C \| e^\frac{x^2}{8} e^{\delta |D|} \|_{B^1_0},
\]  
(6.11)

while by taking $h(t) = (t - t_0)$ and $t_0 = \frac{t}{2}$ in (6.1), we find
\[
\| (t - t_0) \|_{B^1_0} \approx \| (t' - t/2) \|_{B^1_0} \leq \| e^\Psi \phi \|_{L^2((t/2, t); B^1_0)} \leq \| e^\Psi \phi \|_{L^2((t/2, t); B^1_0)}. \]

Note that $u = \partial_y \phi$, by virtue of (3.18) and (3.19), so we achieve
\[
\| (t - t_0) \|_{B^1_0} \approx \| e^\Psi \partial_y \phi \|_{B^1_0} \leq C \| e^\frac{x^2}{8} e^{\delta |D|} \|_{B^1_0}. \]  
(6.12)

Finally, thanks to (6.12), we get, by taking $h(t) = t$ and then $t_0 = \frac{t}{2}$ in (6.1),
\[
\| (t') \|_{B^1_0} \approx \| e^\Psi \phi \|_{L^2((t/2, t); B^1_0)} \leq C \| e^\frac{x^2}{8} e^{\delta |D|} \|_{B^1_0}. \]  
(6.12)

which, together with (6.11) and (6.12), ensures (3.20). This ends the proof of Proposition 3.3. \qed
Proposition 6.1 has been proved provided that we present the proof of Lemma 6.1.

**Proof of Lemma 6.1.** Once again we first get, by applying Bony’s decomposition in the horizontal variable (2.3) to \( v \partial_y u \), that

\[
v \partial_y u = T_v^h \partial_y u + T_{\partial_y u}^h v + R^h(v, \partial_y u).
\]  

(6.13)

Considering (3.17) and the support properties to the Fourier transform of the terms in \( T_v^h \partial_y u \), and thanks to (3.23), we get

\[
\int_{t_0}^t h(t') \left| \left( e^x \Delta_k^h \left[ T_v^h \partial_y u \right]_{\Phi} \right) e^y \Delta_k^h u \Phi \right|_{L_+^2} \, dt'
\]

\[
\lesssim \sum_{|k' - k| \leq 4} \int_{t_0}^t \left\| \frac{1}{2} e^x \mathcal{S}^h_{k' - 1} v \Phi(t') \right\|_{L_+^\infty} \left\| e^y \Delta_k^h \partial_y u \Phi(t') \right\|_{L_+^2} \left\| h^\frac{1}{2} e^y \Delta_k^h u \Phi(t') \right\|_{L_+^2} \, dt'
\]

\[
\lesssim \sum_{|k' - k| \leq 4} 2^{-k'} \left( \int_{t_0}^t \left\| \frac{1}{2} \partial_y(t') \right\|_{L_+^\infty} \left\| h^\frac{1}{2} e^y \mathcal{S}^h_{k' - 1} v \Phi(t') \right\|_{L_+^2} \, dt' \right)^{\frac{1}{2}} \times \left( \int_{t_0}^t \left\| e^y \Delta_k^h u \Phi(t') \right\|_{L_+^2} \, dt' \right)^{\frac{1}{2}}.
\]

In view of Definition 2.3 and (6.6), we get, by applying Lemma 2.1, that

\[
\left( \int_{t_0}^t \left\langle t' \right\rangle^{-\frac{1}{2}} \partial_y(t') \right\|_{L_+^\infty} \frac{1}{2} e^y \mathcal{S}^h_{k' - 1} v \right\|_{L_+^2} \, dt' \right)^{\frac{1}{2}}
\]

\[
\lesssim \sum_{\ell \leq k' - 2} 2^{-\frac{k'}{2}} \left( \int_{t_0}^t \partial_y(t') \left\| e^y \Delta_k^h u \Phi(t') \right\|_{L_+^2} \, dt' \right)^{\frac{1}{2}}
\]

\[
\lesssim d_k 2^{-\frac{k'}{2}} \left\| e^y \Delta_k^h u \Phi \right\|_{L_{t, \partial_y(t')}^{2} \left( B^1_0 \right)}.
\]

whence we obtain

\[
\int_{t_0}^t h(t') \left| \left( e^x \Delta_k^h \left[ T_v^h \partial_y u \right]_{\Phi} \right) e^y \Delta_k^h u \Phi \right|_{L_+^2} \, dt' \lesssim d_k 2^{-k} \left\| h^\frac{1}{2} e^y \Delta_k^h u \Phi \right\|_{L_{t, \partial_y(t')}^{2} \left( B^1_0 \right)}.
\]

In the same manner, in view of (3.23) and (6.6), we infer that
\[
\int_{t_0}^{t} h(t') \left| (e^{\Psi} \Delta^h_k [T^h_{\partial_3} u] \Phi | e^{\Psi} \Delta^h_k u \Phi) \right|_{L^2_{+}} \, dt' \\
\lesssim \sum_{|k'| - |k| \leq 4} \int_{t_0}^{t} \| e^{\Psi} \frac{1}{\ell} \Delta^h_k \Phi \|_{L^\infty_{+}} \| e^{\Psi} \Delta^h_k u \Phi(t') \|_{L^2_{+}} \, dt' \\
\times \| e^{\Psi} \frac{1}{\ell} \Delta^h_k \Phi \|_{L^\infty_{+}} \| e^{\Psi} \Delta^h_k u \Phi(t') \|_{L^2_{+}} \, dt' \\
\lesssim \sum_{|k'| - |k| \leq 4} 2^k \int_{t_0}^{t} \| e^{\Psi} \partial_3 G \Phi(t') \|_{B_{1/2}} \| e^{\Psi} \Delta^h_k \Phi(t') \|_{L^2_{+}} \| e^{\Psi} \Delta^h_k u \Phi(t') \|_{L^2_{+}} \, dt',
\]

from which, we get, by a similar derivation as to that of (5.12), that

\[
\int_{t_0}^{t} h(t') \left| (e^{\Psi} \Delta^h_k [T^h_{\partial_3} u] \Phi | e^{\Psi} \Delta^h_k u \Phi) \right|_{L^2_{+}} \, dt' \lesssim d^2 2^{-k} \| e^{\Psi} \Delta^h_k u \Phi \|_{L^2_{+}}^{2}.
\]

Finally, considering the support properties to the Fourier transform of the terms in \( R^h(v, \partial_3 u) \), we deduce from Lemma 2.1 that

\[
\int_{t_0}^{t} h(t') \left| (e^{\Psi} \Delta^h_k [R^h(v, \partial_3 u)] \Phi | e^{\Psi} \Delta^h_k u \Phi) \right|_{L^2_{+}} \, dt' \\
\lesssim 2^k \sum_{k' \geq k - 3} \int_{t_0}^{t} \| e^{\Psi} \Delta^h_k \Phi \|_{L^\infty_{+}} \| e^{\Psi} \Delta^h_k \Phi(t') \|_{L^2_{+}} \| e^{\Psi} \Delta^h_k u \Phi(t') \|_{L^2_{+}} \, dt',
\]

from which, with (3.23) and (6.6), we get, by a similar derivation as to that of (5.15), that

\[
\int_{t_0}^{t} h(t') \left| (e^{\Psi} \Delta^h_k [R^h(v, \partial_3 u)] \Phi | e^{\Psi} \Delta^h_k u \Phi) \right|_{L^2_{+}} \, dt' \\
\lesssim 2^k \sum_{k' \geq k - 3} 2^k \int_{t_0}^{t} \| e^{\Psi} \partial_3 G \Phi(t') \|_{B_{1/2}} \| e^{\Psi} \Delta^h_k \Phi(t') \|_{L^2_{+}} \| e^{\Psi} \Delta^h_k u \Phi(t') \|_{L^2_{+}} \, dt' \\
\lesssim 2^k \sum_{k' \geq k - 3} 2^k \left( \int_{t_0}^{t} \| \hat{\partial}(t') \|_{L^2_{+}} \| e^{\Psi} \Delta^h_k u \Phi(t') \|_{L^2_{+}} \, dt' \right)^{1/2} \\
\times \left( \int_{t_0}^{t} \| \hat{\partial}(t') \|_{L^2_{+}} \| e^{\Psi} \Delta^h_k u \Phi(t') \|_{L^2_{+}} \, dt' \right)^{1/2} \\
\lesssim d^2 2^{-k} \| e^{\Psi} \Delta^h_k u \Phi \|_{L^2_{+}}^{2}.
\]

As a consequence, we achieve (6.5). This finishes the proof of Lemma 6.1. \( \square \)

7. The Analytic Energy Estimate of the Good Quantity \( G \)

One key observation of this paper is that the weighted analytical norm of the function \( g = \partial_3 G \) introduced in (1.7) can control the evolution of the analytic radius to the solutions of (1.5). In order to have a globally in time estimate of the
loss to the analytic radius of $u$, we need the weighted analytical norm of $\partial_y G$ to decay fast enough as time goes to $\infty$. The goal of this section is to derive such a decay estimate of $G$, namely, (3.21).

Before proceeding, we first derive the equation satisfied by $G$, which is defined by (1.7). Indeed we observe from (1.6) that

$$
\partial_t \left[ \frac{y}{2(t)} \varphi \right] - \partial_y^2 \left[ \frac{y}{2(t)} \varphi \right] + \langle t \rangle^{-1} \left[ u + \frac{y}{2(t)} \varphi \right] + \left( u + u^s + \varepsilon f(t) \chi(y) \right) \partial_x \left[ \frac{y}{2(t)} \varphi \right] + \frac{y}{(t)} \int_y^\infty \left( \partial_y \left( u + u^s + \varepsilon f(t) \chi(y') \right) \partial_x \varphi \right) \, dy' = 0.
$$

(7.1)

Then by summing up the $u$ equation of (1.5) with (7.1), we obtain the $G$ equation of (1.8). Moreover, due to $u|_{y=0} = 0$, we find $G|_{y=0} = 0$. As a consequence, $G$ verifies (1.8).

The key ingredient used in the proof Proposition 3.4 lies in the following proposition:

**Proposition 7.1.** Let $\Phi(t, \xi)$ and $\Psi(t, y)$ be given by (3.10) and (3.11), respectively. Let the function $G$ be defined by (1.7). Then for any nonnegative and non-decreasing function $h \in C^1(\mathbb{R}_+)$, there exists a large enough constant $\lambda$ so that

$$
\frac{1}{2} \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_k G_{\Phi} \|_{L^2_t(L^3)(\mathbb{R}^3)}^2 + \| \langle t \rangle^{-\frac{1}{2}} h^{\frac{1}{2}} e^{\Psi} \Delta^h_k G_{\Phi} \|_{L^2_t(L^3)(\mathbb{R}^3)}^2 + \frac{1}{2} \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_k \partial_y G_{\Phi} \|_{L^2_t(L^3)(\mathbb{R}^3)}^2 + c \lambda 2^k \int_0^t \dot{h}(t') \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_k G_{\Phi}(t') \|_{L^2(L^3)}^2 \, dt'
$$

$$
\leq \frac{1}{2} \| h^{\frac{1}{2}} e^{\Psi} \Delta^h_k G_{\Phi}(0) \|_{L^2(L^3)}^2 + \frac{1}{2} \| \sqrt{h} e^{\Psi} \Delta^h_k G_{\Phi} \|_{L^2_t(L^3)(\mathbb{R}^3)}^2 + C d^2 2^{-k} \| h^{\frac{1}{2}} e^{\Psi} G_{\Phi} \|_{L^2_t(L^3)(B^{1,0})}^2,
$$

(7.2)

for any $t < T^*$, which is defined by (3.16).

**Proof.** In view of (1.8), we get, by a similar derivation as to that of (5.3), that

$$
h(t)(e^{\Psi} \Delta^h_k \left( \partial_y G_{\Phi} - \partial_{yy} G_{\Phi} + (t)^{-1} G_{\Phi} \right) | e^{\Psi} \Delta^h_k G_{\Phi} )_{L^2_t(L^3)} + \lambda \dot{h}(t) h(t) (e^{\Psi} | D_h | \Delta^h_k G_{\Phi} | e^{\Psi} \Delta^h_k G_{\Phi} )_{L^2_t(L^3)} + h(t) (e^{\Psi} | D_h | \Delta^h_k [v \partial_y G_{\Phi}] | e^{\Psi} \Delta^h_k G_{\Phi} )_{L^2_t(L^3)} + h(t) (e^{\Psi} \Delta^h_k [(u + u^s + \varepsilon f(t) \chi(y)) \partial_x G_{\Phi}] | e^{\Psi} \Delta^h_k G_{\Phi} )_{L^2_t(L^3)} + h(t) (e^{\Psi} \Delta^h_k [\partial_y (u^s + \varepsilon f(t) \chi(y)) \partial_x G_{\Phi}] | e^{\Psi} \Delta^h_k G_{\Phi} )_{L^2_t(L^3)} + (t)^{-1} h(t) (e^{\Psi} y \int_y^\infty \Delta^h_k [\partial_y (u + u^s + \varepsilon f(t) \chi(y')) \partial_x G_{\Phi}] \, dy' \, e^{\Psi} \Delta^h_k G_{\Phi} )_{L^2_t(L^3)} = 0.
$$

(7.3)
In what follows, we shall always assume that \( t < T^* \), with \( T^* \) being determined by (3.16) so that, by virtue of (3.10), for any \( t < T^* \), there holds the convex inequality (3.17).

Next let us handle things term by term in (7.3).

Due to \( G|_{y=0} = 0 \), it follows from a similar proof of Lemma 5.1 that

\[
\int_0^t h(t')(e^\Psi \Delta_k^h (\partial_t G\Phi - \partial_{yy} G\Phi) | e^\Psi \Delta_k^h G\Phi)_{L^2_+} \, dt' \\
\geq \frac{1}{2} \left( \| h^\perp e^\Psi \Delta_k^h G\Phi(t) \|_{L^2_+}^2 - \| h^\perp e^\Psi \Delta_k^h G\Phi(0) \|_{L^2_+}^2 \right) - \int_0^t h(t') \| e^\Psi \Delta_k^h G\Phi(t') \|_{L^2_+}^2 \, dt' + \| e^\Psi \Delta_k^h \partial_y G\Phi \|_{L^2_+(L^2_+)}^2,
\]

which together with (3.9), (4.9) and Definition 2.3 ensures that

\[
\int_0^t h(t') (e^\Psi \Delta_k^h (u^x + e f(t) \chi(y)) \partial_t G\Phi | e^\Psi \Delta_k^h G\Phi)_{L^2_+} \, dt' \\
\leq d_k^{-2} 2^{-k} \| h^\perp e^\Psi G\Phi \|_{L^2_{t,\partial(\mathcal{B}^1,0)}}^2.
\]

On the other hand, we observe from the proof of (6.6) that

\[
\| e^\Psi \Delta_k^h v\Phi(t') \|_{L^\infty_+(L^2_+)} \lesssim 2^{k'} (t')^{1/2} \| e^\Psi \Delta_k^h u\Phi(t') \|_{L^2_+},
\]

which, together with (3.22), implies that

\[
\| e^\Psi \Delta_k^h v\Phi(t') \|_{L^\infty_+(L^2_+)} \lesssim (t')^{1/2} 2^k \| e^\Psi \Delta_k^h G\Phi(t) \|_{L^2_+}.
\]

As a result, we have that

\[
\int_0^t h(t')(e^\Psi \partial_y (u^x + e f(t) \chi(y)) \Delta_k^h v\Phi | e^\Psi \Delta_k^h G\Phi)_{L^2_+} \, dt' \\
\lesssim \int_0^t h(t') e^\Psi \partial_y (u^x + e f(t) \chi(y)) \|_{L^2_v} \| e^\Psi \Delta_k^h v\Phi(t') \|_{L^\infty_+(L^2_+)} \| e^\Psi \Delta_k^h G\Phi(t') \|_{L^2_+} \, dt' \\
\lesssim 2^k \int_0^t h(t')(t') \| e^\Psi \partial_y (u^x + e f(t) \chi(y)) \|_{L^2_v} \| e^\Psi \Delta_k^h G\Phi(t') \|_{L^2_+}^2 \, dt'.
\]

This together with (3.9), (4.9) and Definition 2.3 ensures that

\[
\int_0^t \dot{\theta}(t') \| \sqrt{h} e^\Psi \Delta_k^h G\Phi(t') \|_{L^2_+}^2 \, dt' \\
\leq 2^k \int_0^t \dot{\theta}(t') \| \sqrt{h} e^\Psi \Delta_k^h G\Phi(t') \|_{L^2_+}^2 \, dt' \\
\leq d_k^{-2} 2^{-k} \| h^\perp e^\Psi G\Phi \|_{L^2_{t,\partial(\mathcal{B}^1,0)}}^2.
\]

The estimate of the remaining terms in (7.3) relies on the following lemmas:
Lemma 7.1. For any \( t < T^* \), it holds that
\[
\int_0^t h(t') \left| (e^{\psi} \Delta_k^h [v \partial_y G] \Phi \mid e^{\psi} \Delta_k^h G \Phi)_{L^2_+} \right| \, dt' \lesssim d_k^2 2^{-k} \| h \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{1}{2} \| G \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{2}{3} \tag{7.8}
\]

Lemma 7.2. For any \( t < T^* \), it holds that
\[
\int_0^t h(t')(t')^{-\frac{1}{2}} \left| (e^{\psi} \Delta_k^h [v \partial_y (y \varphi)] \Phi \mid e^{\psi} \Delta_k^h G \Phi)_{L^2_+} \right| \, dt' \lesssim d_k^2 2^{-k} \| h \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{1}{2} \| G \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{2}{3} \tag{7.9}
\]

Lemma 7.3. For any \( t < T^* \), it holds that
\[
\int_0^t (t')^{-\frac{1}{2}} h(t') \left| (e^{\psi} \Delta_k^h [\int_y^\infty \Delta_k^h [\partial_y (u + u^c + \varepsilon f(t') \chi(y')) v \Phi \partial_y v'] | e^{\psi} \Delta_k^h G \Phi)_{L^2_+} \right| \, dt' \lesssim d_k^2 2^{-k} \| h \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{1}{2} \| G \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{2}{3} \tag{7.10}
\]

The proof of the above lemmas involves tedious calculations, which we shall postpone until the Appendix A. For now we admit Lemmas 7.1–7.3 for the time being and continue the proof of Proposition 7.1. As a matter of fact, by integrating (7.3) over \([0, t]\) and then inserting the estimates (7.4–7.10) into the resulting inequality, we achieve (7.2). This completes the proof of Proposition 7.1. \( \square \)

Now we present the proof of Proposition 3.4.

Proof of Proposition 3.4. It follows from Lemma 3.1 that
\[
\frac{1}{2} \| (t')^{-\frac{1}{2}} \| h \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{1}{2} \| G \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{2}{3} \leq \| h \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{1}{2} \| G \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{2}{3}.
\]

Inserting the above inequality into (7.2) and taking \( h(t) = \langle t \rangle^\frac{3}{2} \) in the resulting inequality, we find, for any \( t < T^* \),
\[
2^k \| \langle t \rangle^\frac{3}{2} e^{\psi} \Delta_k^h G \Phi \|_{L^2_{t, \dot{q}(\Phi)}}^2 + c \lambda 2^{2k} \int_0^t \| \hat{\Theta}(t') \|_{L^2_{t, \dot{q}(\Phi)}}^\frac{3}{2} e^{\psi} \Delta_k^h G \Phi(t') \|_{L^2_{t, \dot{q}(\Phi)}}^2 \, dt' \\
\leq 2^k \| e^{\psi} \Delta_k^h G \Phi(0) \|_{L^2_{t, \dot{q}(\Phi)}}^2 + C d_k^2 \| \langle t \rangle^\frac{3}{2} e^{\psi} \Delta_k^h G \Phi \|_{L^2_{t, \dot{q}(\Phi)}}^2.
\]

Taking the square root of the above inequalities and then summing up the resulting ones gives rise to
\[
\| \langle t \rangle^\frac{3}{2} e^{\psi} G \Phi \|_{L^2_{t, \dot{q}(\Phi)}}^2 + \sqrt{c} \lambda \| \langle t \rangle^\frac{3}{2} e^{\psi} G \Phi \|_{L^2_{t, \dot{q}(\Phi)}}^2 \\
\leq \| e^{\psi} e^{\delta |D_h| G_0} \|_{L^2_{t, \dot{q}(\Phi)}}^2 + C \| \langle t \rangle^\frac{3}{2} e^{\psi} G \Phi \|_{L^2_{t, \dot{q}(\Phi)}}^2. \tag{7.11}
\]
In particular, taking \( \lambda \) in (7.11) so large that \( c \lambda \geq C \), we achieve
\[
\| \langle t \rangle^{\frac{5}{4}} e^{\psi} G \Phi \|_{L^{\infty}(B_T^{1,0})} \leq \| e^{\frac{x^2}{2}} e^{\delta |D_h|} G_0 \|_{B_T^{1,0}} \text{ for any } t < T^*, \tag{7.12}
\]
while by taking \( h(t) = 1 \) in (7.2), we get, by a similar derivation as to that of (7.11), that
\[
\sqrt{c \lambda} \| e^{\psi} G \Phi \|_{L^2_t(t,\partial_t)(B^{1,0})} + \| e^{\psi} \partial_y G \Phi \|_{L^2_t(B^{1,0})}
\leq \| e^{\frac{x^2}{2}} e^{\delta |D_h|} G_0 \|_{B_T^{1,0}} + C \| e^{\psi} G \Phi \|_{L^2_t(t,\partial_t)(B^{1,0})}.
\]
By taking \( \lambda \) so large that \( c \lambda \geq C \) in the above inequality, we obtain
\[
\| e^{\psi} \partial_y G \Phi \|_{L^2_t(t,\partial_t)(B^{1,0})} \leq \| e^{\frac{x^2}{2}} e^{\delta |D_h|} G_0 \|_{B_T^{1,0}} \text{ for any } t < T^*. \tag{7.13}
\]
On the other hand, along exactly the same lines as to the proof of (7.2), for any \( t \in (0, T^*) \), we can show that
\[
\frac{1}{2} \| \langle t \rangle^{\frac{5}{8}} e^{\psi} \Delta_h^k G \Phi(t) \|_{L^2}^2 + \frac{1}{2} \int_t^t \| \langle t \rangle^{\frac{5}{8}} e^{\psi} \Delta_h^k \partial_y G \Phi \|_{L^2}^2 \, dt'
+ c \lambda 2^k \int_t^t \dot{\theta}(t) \| \langle t \rangle^{\frac{5}{8}} e^{\psi} \Delta_h^k G \Phi(t') \|_{L^2}^2 \, dt'
\leq \frac{1}{2} \| \langle t/2 \rangle^{\frac{5}{8}} e^{\psi} \Delta_h^k G \Phi(t/2) \|_{L^2}^2 + \frac{1}{4} \int_t^t \langle t' \rangle^{\frac{3}{2}} \| e^{\psi} \Delta_h^k G \Phi(t') \|_{L^2}^2 \, dt'
+ C d_k 2^{-k} \| \langle t \rangle^{\frac{5}{8}} e^{\psi} G \Phi \|_{L^2_{t/2,t}(B^{1,0})}^2,
\]
from which, with (7.12), we get, by a similar derivation as to that of (7.11), that
\[
\| \langle t \rangle^{\frac{5}{8}} e^{\psi} G \Phi(t) \|_{B_T^{1,0}} + \sqrt{2c \lambda} \| \langle t \rangle^{\frac{5}{8}} e^{\psi} G \Phi \|_{L^2_{t/2,t}(B^{1,0})}
+ \| \langle t \rangle^{\frac{5}{8}} e^{\psi} \partial_y G \Phi \|_{L^2_{t/2,t,B_T^{1,0}}} \leq \| e^{\frac{x^2}{2}} e^{\delta |D_h|} G_0 \|_{B_T^{1,0}} + \| \langle t \rangle^{\frac{3}{2}} e^{\psi} G \Phi \|_{L^2_{t/2,t,B_T^{1,0}}}
+ \sqrt{C} \| \langle t \rangle^{\frac{5}{8}} e^{\psi} G \Phi \|_{L^2_{t/2,t}(B^{1,0})}.
\]
However, it follows from (7.12) that for any \( t \in (0, T^*) \),
\[
\| \langle t \rangle^{\frac{3}{2}} e^{\psi} G \Phi \|_{L^2_{t/2,t,B_T^{1,0}}} \lesssim \| \langle t \rangle^{\frac{3}{2}} e^{\psi} G \Phi \|_{L^2_{t/2,t}(B_T^{1,0})} \lesssim \| e^{\frac{x^2}{2}} e^{\delta |D_h|} G_0 \|_{B_T^{1,0}}.
\]
As a consequence, as long as \( c \lambda \geq C \), we arrive at
\[
\| \langle t \rangle^{\frac{5}{8}} \partial_y G \Phi \|_{L^2_{t/2,t,B_T^{1,0}}} \lesssim \| e^{\frac{x^2}{2}} e^{\delta |D_h|} G_0 \|_{B_T^{1,0}} \text{ for any } t < T^*. \tag{7.14}
\]
With (7.12) and (7.14), to finish the proof of (3.21), it remains to show that, for any \( t < T^* \),

\[
\int_0^t \langle t' \rangle^{\frac{1}{2}} \| e^{\Psi} \partial_y G_\Phi(t') \|_{B^{\frac{1}{2},0}_2} \, dt' \leq C \| e^{\frac{\lambda^2}{2} \cdot D_{\mathcal{H}} | G_0 \|_{B^{\frac{1}{2},0}_2}. \tag{7.15}
\]

Indeed for any \( t < T^* \) and \( t > 1 \), there exists a unique integer \( N_t \) so that \( 2^{N_t - 1} < t \leq 2^{N_t} \). Then we have \( \frac{t}{2} \leq 2^{N_t - 1} \), so that it holds that

\[
\int_{2^{N_t - 1}}^t \langle t' \rangle^{\frac{1}{2}} \| e^{\Psi} \partial_y G_\Phi(t') \|_{B^{\frac{1}{2},0}_2} \, dt' \leq \left( \int_{2^{N_t - 1}}^t \langle t' \rangle^{-2} \, dt' \right)^{\frac{1}{2}} \times \left( \int_{t/2}^t \langle t' \rangle^{\frac{5}{2}} \| e^{\Psi} \partial_y G_\Phi(t') \|_{B^{\frac{1}{2},0}_2}^2 \, dt' \right)^{\frac{1}{2}} \leq C 2^{-\frac{N_t}{2}} \| \langle t' \rangle^{\frac{5}{2}} \partial_y G_\Phi \|_{L^2(t/2,t;B^{\frac{1}{2},0}_2)}.
\]

Along the same lines, for any \( j \in [0, N_t - 2] \), we find that

\[
\int_{2^j}^{2^{j+1}} \langle t' \rangle^{\frac{1}{2}} \| e^{\Psi} \partial_y u_\Phi(t') \|_{B^{\frac{1}{2},0}_2} \, dt' \leq \left( \int_{2^j}^{2^{j+1}} \langle t' \rangle^{-2} \, dt' \right)^{\frac{1}{2}} \times \left( \int_{2^j}^{2^{j+1}} \langle t' \rangle^{\frac{5}{2}} \| e^{\Psi} \partial_y u_\Phi(t') \|_{B^{\frac{1}{2},0}_2}^2 \, dt' \right)^{\frac{1}{2}} \leq C 2^{-j} \| \langle t' \rangle^{\frac{5}{2}} \partial_y G_\Phi \|_{L^2(2^j,2^{j+1};B^{\frac{1}{2},0}_2)}.
\]

As a consequence, we deduce from (7.13), (7.14) and the above inequalities that

\[
\int_0^t \langle t' \rangle^{\frac{1}{2}} \| e^{\Psi} \partial_y G_\Phi(t') \|_{B^{\frac{1}{2},0}_2} \, dt' \leq \| e^{\Psi} \partial_y G_\Phi \|_{L^2(0,1;B^{\frac{1}{2},0}_2)} + \sum_{j=0}^{N_t - 2} \int_{2^j}^{2^{j+1}} \langle t' \rangle^{\frac{1}{2}} \| e^{\Psi} \partial_y G_\Phi(t') \|_{B^{\frac{1}{2},0}_2} \, dt' + \int_{2^{N_t - 1}}^t \langle t' \rangle^{\frac{1}{2}} \| e^{\Psi} \partial_y G_\Phi(t') \|_{B^{\frac{1}{2},0}_2} \, dt' \leq C \| e^{\frac{\lambda^2}{2} \cdot D_{\mathcal{H}} | G_0 \|_{B^{\frac{1}{2},0}_2} \left( 1 + \sum_{j=0}^{\infty} 2^{-\frac{j}{2}} \right) \leq C \| e^{\frac{\lambda^2}{2} \cdot D_{\mathcal{H}} | G_0 \|_{B^{\frac{1}{2},0}_2}.
\]

This leads to (7.15). We thus complete the proof of Proposition 3.4. \( \square \)

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Appendix A: The proof of Lemmas 7.1–7.3

In this appendix, we shall present the proof of Lemmas 7.1–7.3.

**Proof of Lemma 7.1.** We first get, by applying Bony’s decomposition (2.3) in the horizontal variable to \(v \partial_y G\), that

\[
v \partial_y G = T^h_v \partial_y G + T^h_{\partial_y} G v + R^h(v, \partial_y G).
\]

Considering (3.17) and the support properties to the Fourier transform of the terms in \(T^h_v \partial_y G\), we write

\[
\int_0^t h(t') \left| \left( e^{\Psi} \Delta^h_k [T^h_v \partial_y G] \Phi \right) \left| \left( e^{\Psi} \Delta^h_k G \Phi \right) \right| \right|_{L^2_*} \, dt'
\]

\[
\lesssim \sum_{|k'-k| \leq 4} \int_0^t \left| h \frac{1}{2} S^h_{k'-1} v \Phi(t') \right| \left| \left| e^{\Psi} \Delta^h_k \partial_y G \Phi(t') \right| \right|_{L^2_*} \left| \left| e^{\Psi} \Delta^h_k G \Phi(t') \right| \right|_{L^2_*} \, dt'
\]

\[
\lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \left| h \frac{1}{2} S^h_{k'-1} v \Phi(t') \right| \left| \left| e^{\Psi} \partial_y G \Phi(t') \right| \right|_{B^2_{2,0}} \left| \left| e^{\Psi} \Delta^h_k G \Phi(t') \right| \right|_{L^2_*} \, dt'.
\]

Then, in view of (3.9), by applying Hölder’s inequality, we find that

\[
\int_0^t h(t') \left| \left( e^{\Psi} \Delta^h_k [T^h_v \partial_y G] \Phi \right) \left| \left( e^{\Psi} \Delta^h_k G \Phi \right) \right| \right|_{L^2_*} \, dt'
\]

\[
\lesssim \sum_{|k'-k| \leq 4} 2^{-k'} \left( \int_0^t \left| \dot{\theta}(t') \left( t' \right) \right| - \frac{1}{2} \left| \left| h \frac{1}{2} S^h_{k'-1} v \Phi(t') \right| \right|_{L^2_*} \, dt' \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_0^t \left| \dot{\theta}(t') \right| \left| h \frac{1}{2} e^{\Psi} \Delta^h_k G \Phi(t') \right| \left| L^2_* \, dt' \right| \right)^{\frac{1}{2}}.
\]

In view of (7.6), we get, by a similar derivation as to that of (5.13), that

\[
\left( \int_0^t \left| \dot{\theta}(t') \right| \left| h \frac{1}{2} S^h_{k'-1} v \Phi(t') \right| \left| L^2_* \, dt' \right| \right)^{\frac{1}{2}} \lesssim d_k 2^{-k} \left| h \frac{1}{2} e^{\Psi} G \Phi \left| L^2_{t, \theta(t)} \right( B^{1,0} \right), \quad \left( A.1 \right)
\]

As a result, we deduce from Definition 2.3 that

\[
\int_0^t h(t') \left| \left( e^{\Psi} \Delta^h_k [T^h_v \partial_y G] \Phi \right) \left| \left( e^{\Psi} \Delta^h_k G \Phi \right) \right| \right|_{L^2_*} \, dt' \lesssim d_k 2^{-k} \left| h \frac{1}{2} e^{\Psi} G \Phi \left| L^2_{t, \theta(t)} \right( B^{1,0} \right).
\]

Along the same lines, we de deduce from (3.9) and (7.6) that

\[
\int_0^t h(t') \left| \left( e^{\Psi} \Delta^h_k [T^h_{\partial_y} G] \Phi \right) \left| \left( e^{\Psi} \Delta^h_k G \Phi \right) \right| \right|_{L^2_*} \, dt'
\]

\[
\lesssim \sum_{|k'-k| \leq 4} \int_0^t \left| e^{\Psi} S^h_{k'-1} \partial_y G \Phi(t') \right| \left| L^2_* \left( L^\infty \right) \right| \left| h \frac{1}{2} \Delta^h_k v \Phi(t') \right| \left| L^\infty \left( L^2_* \right) \right| \left| h \frac{1}{2} e^{\Psi} \Delta^h_k G \Phi(t') \right| \left| L^2_* \, dt' \right|
\]

\[
\lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \left| \dot{\theta}(t') \right| \left| h \frac{1}{2} e^{\Psi} \Delta^h_k G \Phi(t') \right| \left| h \frac{1}{2} e^{\Psi} \Delta^h_k G \Phi(t') \right| \left| L^2_* \, dt' \right|.
\]
Finally again due to (7.6) and the support properties to the Fourier transform of the terms in $R^h (v, \partial_y G)$, we get, by applying Lemma 2.1, that

$$\int_0^t h(t') \left| \left( e^{\Psi} \Delta_k^h [T^h_0 \partial_y G] \right)_{\Phi} - e^{\Psi} \Delta_k^h G \Phi \right|_{L^2_+} \, dt' \lesssim d_k^2 2^{-k} \| h^2 e^{\Psi} G \Phi \|^2_{L^2_{t,0} (B^{1,0})}.$$  

Then a similar derivation as to that of (5.12) yields that

$$\int_0^t h(t') \left| \left( e^{\Psi} \Delta_k^h [R^h (v, \partial_y G)] \right)_{\Phi} - e^{\Psi} \Delta_k^h G \Phi \right|_{L^2_+} \, dt' \lesssim d_k^2 2^{-k} \| h^2 e^{\Psi} G \Phi \|^2_{L^2_{t,0} (B^{1,0})}.$$  

from which we get, by a similar derivation as to that of (5.15), that

$$\int_0^t h(t') \left| \left( e^{\Psi} \Delta_k^h [R^h (v, \partial_y G)] \right)_{\Phi} - e^{\Psi} \Delta_k^h G \Phi \right|_{L^2_+} \, dt' \lesssim d_k^2 2^{-k} \| h^2 e^{\Psi} G \Phi \|^2_{L^2_{t,0} (B^{1,0})}.$$  

Summing up the above estimates gives rise to (7.8). This finishes the proof of Lemma 7.1. □

**Proof of Lemma 7.2.** Applying Bony’s decomposition (2.3) in the horizontal to $v \partial_y (y \varphi)$ yields

$$v \partial_y (y \varphi) = T^h_v \partial_y (y \varphi) + T^h_{\partial_y (y \varphi)} v + R^h (v, \partial_y (y \varphi)).$$

In view of (3.17) and (3.24), we infer that

$$\int_0^t h(t') (t')^{-1} \left| \left( e^{\Psi} \Delta_k^h \left[ T^h_v \partial_y (y \varphi) \right] \right)_{\Phi} - e^{\Psi} \Delta_k^h G \Phi \right|_{L^2_+} \, dt' \lesssim \sum_{|k' - k| \leq 4} \int_0^t h(t') (t')^{-1} \left| e^{\Psi} \Delta_{k'}^h v \varphi (t') \right|_{L^2_+} \left| e^{\Psi} \Delta_k^h \partial_y (y \varphi) (t') \right|_{L^2_+} \left| e^{\Psi} \Delta_k^h G \Phi (t') \right|_{L^2_+} \, dt'$$  

$$\lesssim \sum_{|k' - k| \leq 4} \int_0^t h(t') \left( e^{\Psi} \Delta_{k'}^h v \varphi (t') \right)_{L^2_+} \left( e^{\Psi} \Delta_k^h \partial_y (y \varphi) (t') \right)_{L^2_+} \left( e^{\Psi} \Delta_k^h G \Phi (t') \right)_{L^2_+} \, dt'$$  

$$\lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \left( \int_0^t (t') \left( e^{\Psi} \Delta_k^h G \Phi (t') \right)_{L^2_+} \, dt' \right)^{\frac{1}{2}}$$  

$$\times \left( \int_0^t \left( h^2 e^{\Psi} \Delta_k^h \partial_y (y \varphi) (t') \right)_{L^2_+} \, dt' \right)^{\frac{1}{2}},$$

which, together with Definition 2.3 and (A.1), ensures that

$$\int_0^t h(t') (t')^{-1} \left| \left( e^{\Psi} \Delta_k^h \left[ T^h_0 \partial_y (y \varphi) \right] \right)_{\Phi} - e^{\Psi} \Delta_k^h G \Phi \right|_{L^2_+} \, dt' \lesssim d_k^2 2^{-k} \| h^2 e^{\Psi} G \Phi \|^2_{L^2_{t,0} (B^{1,0})}.$$
Similarly, by virtue of (3.24) and (7.6), we have

\[
\int_0^t \hat{h}(t') \langle t' \rangle^{-1} \left( e^\Psi \Delta^h_k [T^h \partial_y (y \varphi)] \right)_\Phi | e^\Psi \Delta^h_k G \Phi \rangle_{L^2_+} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} \int_0^t \hat{h}(t') \langle t' \rangle^{-1} \left| e^\Psi S^h_{k' - 1} \partial_y (y \varphi \phi)(t') \right|_{L^2_+(L^\infty_+)} \\
\times \left\| e^\Psi \Delta^h_k v_\Phi (t') \right\|_{L^\infty_+(L^2_+)} \left\| e^\Psi \Delta^h_k G \Phi (t') \right\|_{L^2_+} \, dt' \\
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} \int_0^t \hat{\theta}(t') \| h \frac{1}{2} e^\Psi \Delta^h_k G \Phi (t') \|_{L^2_+} \| h \frac{1}{2} e^\Psi \Delta^h_k G \Phi (t') \|_{L^2_+} \, dt'.
\]

As a result, we deduce, by a similar derivation as to that of (5.12), that

\[
\int_0^t \hat{h}(t') \langle t' \rangle^{-1} \left( e^\Psi \Delta^h_k [T^h \partial_y (y \varphi)] \right)_\Phi | e^\Psi \Delta^h_k G \Phi \rangle_{L^2_+} \, dt' \lesssim d_k^2 2^{-k'} \| h \frac{1}{2} e^\Psi G \Phi \|_{L^2_{1, \delta(\Phi)}(B^1_0)}.
\]

Finally again thanks to (3.24) and (7.6), we get, by applying Lemma 2.1, that

\[
\int_0^t \hat{h}(t') \langle t' \rangle^{-1} \left( e^\Psi \Delta^h_k [R^h (v, \partial_y (y \varphi))] \right)_\Phi | e^\Psi \Delta^h_k G \Phi \rangle_{L^2_+} \, dt' \\
\lesssim 2^{k'} \sum_{k' \geq k - 3} \int_0^t \hat{h}(t') \langle t' \rangle^{-1} \left\| e^\Psi \Delta^h_k v_\Phi (t') \right\|_{L^\infty_+(L^2_+)} \\
\times \left\| e^\Psi \Delta^h_k \partial_y (y \varphi \phi)(t') \right\|_{L^2_+} \left\| e^\Psi \Delta^h_k G \Phi (t') \right\|_{L^2_+} \, dt' \\
\lesssim 2^{k'} \sum_{k' \geq k - 3} 2^{k'} \int_0^t \hat{\theta}(t') \| e^\Psi \Delta^h_k G \Phi (t') \|_{L^2_+} \| e^\Psi \Delta^h_k G \Phi (t') \|_{L^2_+} \, dt'.
\]

Then it follows from a similar derivation as to that of (5.15) that

\[
\int_0^t \hat{h}(t') \langle t' \rangle^{-1} \left( e^\Psi \Delta^h_k [R^h (v, \partial_y (y \varphi))] \right)_\Phi | e^\Psi \Delta^h_k G \Phi \rangle_{L^2_+} \, dt' \\
\lesssim d_k^2 2^{-k'} \| h \frac{1}{2} e^\Psi G \Phi \|_{L^2_{1, \delta(\Phi)}(B^1_0)}.
\]

By summarizing the above estimates, we conclude the proof of (7.9). This ends the proof of Lemma 7.2. \( \square \)

**Proof of Lemma 7.3.** We first observe from \( \partial_x u + \partial_y v = 0 \) that \( v = \int_0^\infty \partial_x u \, dy' \), so that one has

\[
|\Delta^h_k v_\Phi (t)| \lesssim e^{-\frac{5}{8} \Psi} \int_0^\infty e^{-\frac{1}{8} \Psi} \times e^{\frac{3}{4} \Psi} |\Delta^h_k \partial_x u_\Phi (t)| \, dy',
\]
from which, with (3.22) and Lemma 2.1, we infer that
\[ \|e^\frac{\psi}{2} \Delta_k^h u \phi(t)\|_{L^2} \lesssim \|e^{-\frac{\psi}{2}} \|_{L^2_t} \|e^{\frac{3}{4} \psi} \Delta_k^h \partial_x u \phi(t)\|_{L^2_+} \]
\[ \lesssim 2^k \langle t \rangle^\frac{1}{2} \|e^{\frac{3}{4} \psi} \Delta_k^h u \phi(t)\|_{L^2_+} \]
\[ \lesssim 2^k \langle t \rangle^\frac{1}{2} \|e^\psi \Delta_k^h G \phi(t)\|_{L^2_+}. \] (A.2)

In view of (3.9), (4.9) and (A.2), we infer that
\[ \int_0^t \langle t' \rangle^{-1} h(t') \bigg| \bigg( e^\psi y \int_y^\infty (\partial_y u^x + \epsilon f(t') \chi(y)) \Delta_k^h u \phi \, dy' \bigg| e^\psi \Delta_k^h G \phi \bigg)_{L^2_+} \, dt' \]
\[ \lesssim \sum_{|k' - k| \leq 4} \int_0^t \langle t' \rangle^{-1} \|e^{-\frac{\psi}{2}} y\|_{L^2_t} \|e^{\frac{3}{4} \psi} \Delta_k^h u \phi(t')\|_{L^2_+} \]
\[ \times \|e^{\frac{3}{4} \psi} \Delta_k^h G \phi(t')\|_{L^2_+} \, dt' \]
\[ \lesssim \sum_{|k' - k| \leq 4} 2^k \int_0^t \langle t' \rangle \|e^{\frac{3}{4} \psi} \Delta_k^h G \phi(t')\|_{L^2_+} \, dt'. \] (A.3)

where in the last step, we used Definition 2.3.

On the other hand, due to (3.17), (3.23), (A.2) and the support properties to the Fourier transform of the terms in \( T_{\partial_y u}^h \), we find that
\[ \int_0^t \langle t' \rangle^{-1} \bigg| \bigg( e^\psi y \int_y^\infty \Delta_k^h \left[ T_{\partial_y u}^h \right] \phi \, dy' \bigg| e^\psi \Delta_k^h G \phi \bigg)_{L^2_+} \, dt' \]
\[ \lesssim \sum_{|k' - k| \leq 4} \int_0^t \langle t' \rangle^{-1} \|e^{-\frac{\psi}{2}} y\|_{L^2_t} \|e^{\frac{3}{4} \psi} \Delta_k^{h} S_{k' - 1}^h \partial_y u \phi(t')\|_{L^2_+} \]
\[ \times \|e^{\frac{3}{4} \psi} \Delta_k^h G \phi(t')\|_{L^2_+} \, dt' \]
\[ \lesssim \sum_{|k' - k| \leq 4} 2^k \int_0^t \langle t' \rangle \|e^{\frac{3}{4} \psi} \Delta_k^h G \phi(t')\|_{L^2_+} \, dt'. \]

Then a similar derivation as to that of (5.12) yields
\[ \int_0^t \langle t' \rangle^{-1} \bigg| \bigg( e^\psi y \int_y^\infty \Delta_k^h \left[ T_{\partial_y u}^h \right] \phi \, dy' \bigg| e^\psi \Delta_k^h G \phi \bigg)_{L^2_+} \, dt' \]
\[ \lesssim d \|e^{\frac{3}{4} \psi} \Delta_k^h G \phi\|_{L^2_+} \, dt'. \] (A.3)

Again thanks to (3.23), we get, by a similar procedure, that
\[ \int_0^t \langle t' \rangle^{-1} \bigg| \bigg( e^\psi y \int_y^\infty \Delta_k^h \left[ T_{\partial_y u}^h \right] \phi \, dy' \bigg| e^\psi \Delta_k^h G \phi \bigg)_{L^2_+} \, dt' \]
In view of (A.2), we get, by a similar derivation as to that of (5.13), that

\[ \sum_{|k' - k| \leq 4} \int_0^t \langle t' \rangle^{-\frac{1}{2}} \| e^{-\frac{\Psi}{4} y} \|_{L_h^2} \| \frac{\partial}{\partial t} e^{\frac{\Psi}{4} z} \Delta_k^h v(t') \|_{L_h^2} \langle L_n^\infty \rangle \langle t \rangle d t' \]

\times \| e^{\frac{3}{4} \Psi} \Delta_k^h \partial_y u \Phi(t') \|_{L_n^2} \| \frac{1}{2} e^{\Psi} \Delta_k^h G \Phi(t') \|_{L_n^2} \langle t \rangle d t' \]

\[ \lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \int_0^t \langle t' \rangle^{-\frac{1}{2}} \| e^{-\frac{\Psi}{4} y} \|_{L_h^2} \| e^{\frac{3}{4} \Psi} \Delta_k^h \partial_y u \Phi(t') \|_{L_h^2} \langle L_n^\infty \rangle \langle t \rangle d t' \]

\times \| e^{\frac{1}{2} \Psi} \Delta_k^h v(t') \|_{L_n^2} \| e^{\frac{3}{4} \Psi} \Delta_k^h G \Phi(t') \|_{L_n^2} \langle t \rangle d t' \]

\[ \lesssim \sum_{|k' - k| \leq 4} 2^{-\frac{k'}{2}} \int_0^t \langle t' \rangle^{-\frac{1}{2}} \| e^{-\frac{\Psi}{4} y} \|_{L_h^2} \| e^{\frac{3}{4} \Psi} \Delta_k^h \partial_y u \Phi(t') \|_{L_h^2} \langle L_n^\infty \rangle \langle t \rangle d t' \]

Finally, again due to (3.23), (A.2) and the support properties to the Fourier transform of the terms in $R^h(\partial_y u, \partial_x \varphi)$, we get, by applying Lemma 2.1, that

\[ \int_0^t h(t') \langle t' \rangle^{-\frac{1}{2}} \| e^{\Psi} y \int_y^\infty \Delta_k^h \big[ R_h(\partial_y u, \partial_x G) \big]_\Phi \| e^{\Psi} \Delta_k^h G \Phi \|_{L_h^2} \langle t \rangle d t' \]

\[ \lesssim 2^k \sum_{k' \geq k-3} \int_0^t \langle t' \rangle^{-\frac{1}{2}} \| e^{\Psi} y \|_{L_h^2} \| e^{\frac{3}{4} \Psi} \Delta_k^h \partial_y u \Phi(t') \|_{L_h^2} \langle L_n^\infty \rangle \langle t \rangle d t' \]

\times \| e^{\frac{1}{2} \Psi} \Delta_k^h v(t') \|_{L_n^2} \| e^{\frac{3}{4} \Psi} \Delta_k^h G \Phi(t') \|_{L_n^2} \langle t \rangle d t' \]

\[ \lesssim 2^k \sum_{k' \geq k-3} 2^{-\frac{k'}{2}} \int_0^t \langle t' \rangle \| e^{\frac{1}{2} \Psi} \Delta_k^h G \Phi(t') \|_{L_n^2} \| e^{\frac{1}{2} \Psi} \Delta_k^h G \Phi(t') \|_{L_n^2} \langle t \rangle d t' \]

from which, along with a similar derivation as to that of (5.15), we obtain that

\[ \int_0^t \langle t' \rangle^{-\frac{1}{2}} \| e^{\Psi} y \int_y^\infty \Delta_k^h \big[ R_h(\partial_y u, \partial_x G) \big]_\Phi \| e^{\Psi} \Delta_k^h G \Phi \|_{L_h^2} \langle t \rangle d t' \]

\[ \lesssim d_k^2 2^{-k} \| e^{\frac{1}{2} \Psi} G \Phi \|_{L_{t,r}^2(B^{1.0})} \]

Therefore, by virtue of (6.13), we conclude that

\[ \int_0^t \langle t' \rangle^{-\frac{1}{2}} \| e^{\Psi} y \int_y^\infty \Delta_k^h \big[ R_h(\partial_y u, \partial_x G) \big]_\Phi \| e^{\Psi} \Delta_k^h G \Phi \|_{L_h^2} \langle t \rangle d t' \]

\[ \lesssim d_k^2 2^{-k} \| e^{\frac{1}{2} \Psi} G \Phi \|_{L_{t,r}^2(B^{1.0})} \]

This, together with (A.3), ensures (7.10). We thus finish the proof of Lemma 7.3. \qed
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