Chiral Rings and Integrable Systems for Models of Topological Gravity

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We review the superconformal properties of matter coupled to 2d gravity, and W-extensions thereof. We show in particular how the \( N=2 \) structure provides a direct link between certain matter-gravity systems and matrix models. We also show that much, probably all, of this can be generalized to \( W \)-gravity, and this leads to an infinite class of new exactly solvable systems. These systems are governed by certain integrable hierarchies, which are generalizations of the usual KdV hierarchy and whose algebraic structure is given in terms of quantum cohomology rings of grassmannians.
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1. Introduction

There has been some recent progress in understanding theories describing conformal matter coupled to 2d gravity. Matter-plus-gravity systems are interesting to study because they are, for certain choices of matter theories, supposed to be exactly solvable. More precisely, they are supposed to be equivalent to matrix models\(^1\), which are exactly solvable by themselves as a consequence of an underlying structure of KdV-type integrable hierarchies. To deduce this equivalence directly from Liouville theory is quite difficult, largely due to technical complications. We will review how the $N = 2$ superconformal structure helps to provide a manifest and direct relationship of (certain of) such models to matrix models, by making use of a connection to topological Landau-Ginzburg theory,\(^2\) and to integrable hierarchies.

Since theories of $W$-gravity coupled to matter appear by now\(^3\)-\(^4\) to be on a footing similar to ordinary gravity, one might suspect, by analogy, that there should exist a corresponding infinite sequence of new types of matrix models that describe these theories, governed by certain integrable hierarchies. In a recent paper\(^5\), we made some progress in understanding these new integrable systems in terms of chiral rings, and we will briefly review the main ingredients of this construction as well.

2. $N = 2$ superconformal symmetry of the matter-gravity system

We like to briefly recapitulate ordinary gravity coupled to conformal 2d matter. For simplicity, we will consider mainly minimal matter models, but this is not really important. These matter models, denoted by $M_{p,q}$, where $p, q = 1, 2, \ldots$ are coprime
integers, have central charges \( c_M = 13 - 6(t + \frac{1}{t}) \), where \( t \equiv q/p \). We thus consider tensor products

\[
M_{p,q}^{\text{matter}} \otimes M_{p,-q}^{\text{Liouville}} \otimes \{b,c\},
\]

where \( M_{p,-q}^{\text{Liouville}} \) denotes a Liouville theory with appropriate central charge, and \( \{b,c\} \) denotes the fermionic ghost system with spins \( \{2, -1\} \).

In \textit{BRST} quantization the physical states of the combined matter-gravity system are given by the non-trivial cohomology classes of a \textit{BRST} operator,

\[
Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} J_{\text{BRST}}, \quad J_{\text{BRST}} = c [T_M + T_L + \frac{1}{2}T_{gh}],
\]

which is nilpotent for \( c_L + c_M = 26 \). The most prominent physical states correspond to the tachyon operators\(^6\):

\[
T_{r,s} = c V_{r,s}^L V_{r,s}^M,
\]

where \( V_{r,s} \) denotes exponential vertex operators in the usual notation. By convention, the tachyons have \( bc \)-ghost number equal to one. In addition, there exist\(^7\) extra physical states whose number and precise structure depends on the specific value of \( t \). For unitary minimal models, where \( t = (p + 1)/p \), there exist infinitely many of such extra states for each matter primary, whereas for generic \( t \), there exists basically only one extra sort of states besides the tachyons: these are the operators with vanishing ghost charge. They form what is called\(^8\) the ground ring, which we will denote by \( R_{\text{gr}} \). It is precisely because these operators have zero ghost charge (and zero dimension like all physical operators), that the set of ground ring operators closes into itself under operator products. In fact, even though this ring is in general infinite, it is finitely generated, i.e., it has two generators by whose action all other ring elements can be generated\(^8\):

\[
x = \left[ bc - \frac{t}{\sqrt{2t}} (\partial \phi_L - i \partial \phi_M) \right] V_{1,2}^L V_{1,2}^M,
\]

\[
\gamma^0 = \left[ bc - \frac{1}{\sqrt{2t}} (\partial \phi_L + i \partial \phi_M) \right] V_{2,1}^L V_{2,1}^M.
\]

(Here, \( \phi_{L,M} \) denotes the Liouville field and the matter free field, respectively). The properties of the ground ring elements remind very much to the typical features of chiral fields in \( N = 2 \) superconformal theories. The whole point is, of course, that the matter-gravity-ghost system is essentially nothing but a (twisted) \( N = 2 \) superconformal theory. More precisely, it is known\(^9,10\) that one can improve the \textit{BRST} current (2.2) by a total derivative piece,

\[
G^+ = J_{\text{BRST}} - \partial \left( \sqrt{\frac{t}{2}}(c \partial \phi_L) + \frac{1}{2}(1 - \frac{2}{t}) \partial c \right),
\]

such that \( G^+ \) together with
\[ G^- = b, \quad T = T_L + T_M + T_{gh}, \quad J = cb + \sqrt{\frac{2}{\ell}} \partial \phi_L, \quad (2.6) \]

indeed generates the (topologically twisted\textsuperscript{11,12}) \( \mathcal{N}=2 \) superconformal algebra,

\[
\begin{align*}
T(z) \cdot T(w) & \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \\
T(z) \cdot G^\pm(w) & \sim \frac{1}{2}(3 \mp 1)G^\pm(w) + \frac{\partial G^\pm(w)}{(z-w)}, \\
T(z) \cdot J(w) & \sim \frac{1}{3}c^{\mathcal{N}=2} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)}, \\
J(z) \cdot J(w) & \sim \frac{1}{3}c^{\mathcal{N}=2} + \frac{J(z) \cdot G^\pm(w) \sim \pm \frac{G^\pm(w)}{(z-w)}}{\frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)}} , \\
G^+(z) \cdot G^-(w) & \sim \frac{1}{3}c^{\mathcal{N}=2} + \frac{J(w)}{(z-w)^2} + \frac{T(w) + \partial J(w)}{(z-w)}, \\
G^\pm(z) \cdot G^\pm(w) & \sim 0,
\end{align*}
\]

with anomaly
\[
c^{\mathcal{N}=2} = 3(1 - \frac{2}{\ell}) . \quad (2.8)
\]

Upon untwisting, \( T \rightarrow T - \frac{1}{2} \partial J \), \( c^{\mathcal{N}=2} \) becomes the central charge of an ordinary \( \mathcal{N}=2 \) algebra. Note that the free-field realization (2.5), (2.6) of the \( \mathcal{N}=2 \) algebra is different from the usual one\textsuperscript{13}. This is however irrelevant, and one can show\textsuperscript{10} that the above realization can be obtained by hamiltonian reduction\textsuperscript{14} from a \( SL(2|1) \) WZW model in a way that is analogous and equivalent to the way of deriving the usual free-field realization of the \( \mathcal{N}=2 \) algebra. Alternatively, one can show that the two free field realizations of the \( \mathcal{N}=2 \) algebra can be obtained as two different gauge choices in a topological gauged WZW model\textsuperscript{15}.

Actually, the construction of the twisted \( \mathcal{N}=2 \) algebra is a priori not unique\textsuperscript{16,17}. Indeed, one may replace the Liouville field \( \phi_L \) in (2.5), (2.6) by some appropriate combination of \( \phi_L \) with the matter free field, \( \phi_M \). However, for describing minimal models coupled to gravity, the above choices for \( G^+ \) and \( J \) are the unique, correct ones\textsuperscript{*}. Namely, if \( G^+ \) and \( J \) depended on \( \phi_M \), then the various different vertex operator representatives \( V_{r,s}^M \) that describe the same given physical state of the minimal model would have different properties under the \( \mathcal{N}=2 \) algebra, which clearly would not make any sense. However, for non-minimal models, where these vertex operators describe distinct physical states, there are other possible choices. For example, in order to describe black holes in \( \mathcal{N}=2 \) language, one chooses\textsuperscript{16} the \( \mathcal{N}=2 \) currents to depend only on the matter field, \( \phi_M \).

\textsuperscript{*} Our choice for \( J \) implies that it is not holomorphically conserved if the theory is perturbed by the cosmological constant\textsuperscript{16}, and one might get the impression that this is not desirable. For our application to minimal models, there is however nothing wrong with that.
An immediate question is about the significance of the twisted $N=2$ superconformal symmetry. For generic $t$, the mere presence of an $N=2$ algebra doesn’t really seem to provide any important new insights, since the representation theory for arbitrary $c_{N=2}$ is not very restrictive. On the other hand, for integer $t \equiv k + 2, k \geq 0$, a lot can be learned: namely then the anomaly (2.8) becomes equal to the anomaly of the twisted $N=2$ minimal models, $A_{k+1}^{\text{top}}$, $c_{N=2} = \frac{3k}{k+2}$. This is a powerful statement, since minimal models tend to be easily solved entirely by representation theory. (For $t = -1$, which describes $c=1$ matter coupled to gravity and which can be related to the 2d black hole$^{16}$, the theory is solvable as well, but we will focus on $t \geq 2$ in the following.)

However, this does not yet imply that the minimal models $M_{1,2+k}$ coupled to gravity are the same as the topological minimal models $A_{k+1}^{\text{top}}$. What we have shown is simply that these theories have the same free field realization with the same central charges. A priori, they don’t have even the same spectra. The spectrum of a topological $N=2$ model is well known$^{12}$: it is given by the chiral ring$^{18}$, which is the finite set of primary chiral fields. For $A_{k+1}^{\text{top}}$, this is a nilpotent, polynomial ring generated by one element $x$:

$$R^{A_{k+1}^{\text{top}}} = \frac{P(x)}{[x^{k+1}] \equiv 0} = \{1, x, x^2, \ldots, x^k\}. \quad (2.9)$$

One can check that powers of the ground ring generator $x$ in (2.4) are indeed primary and chiral with respect to the $N=2$ currents (2.5) and (2.6), and that $R^{A_{k+1}^{\text{top}}}$ is identical to the subring of the ground ring $R^{\text{gr}}$ that is generated by $x$. (For $t = k + 2$, it turns out that the corresponding tachyons (2.3) have the same $N=2$ quantum numbers as the ground ring elements, so that they can be viewed as different representatives of the same set of physical fields.) On the other hand, the full ground ring $R^{\text{gr}}$ of the matter-gravity system contains infinitely many more operators$^{19}$:

$$R^{\text{gr}} = R^{A_{k+1}^{\text{top}}} \otimes \{ (\gamma^0)^n, \quad n = 0, 1, 2, \ldots \}. \quad (2.10)$$

These extra operators simply do not exist in the topological minimal models. The difference between the spectra (2.9) and (2.10) can be accounted for as follows: it turns out that the extra operators are exact with respect to an additional BRST like operator, $\tilde{Q}$:

$$\gamma^0 = -\{ \tilde{Q}, (\frac{t+1}{t} \partial c + \frac{1}{\sqrt{2t}} c \partial \phi_L) \} \text{, \quad where \quad } \tilde{Q} = \oint \frac{dz}{2\pi i} be^{-\frac{t}{\sqrt{2t}}(\phi_L - i\phi_M)} . \quad (2.11)$$

One can show$^{10}$ that $\tilde{Q}$ is one of the Felder-like screening operators that arise in our particular free field realization of the minimal models. That is, by definition the full BRST operator of the topologically twisted $N=2$ minimal models is the sum
of $\mathcal{Q}_{BRST}$, $\tilde{Q}$ and the other screening operators, such that it maximally truncates the infinite free field spectrum precisely to the finite set of physical operators (2.9).

We thus see that the full $BRST$ operator of the topological minimal models is not the correct one if we wish to describe the minimal models $M_{1,2+k}$ coupled to gravity. The correct operator obtains if we drop $\tilde{Q}$ as an extra piece of the full $BRST$ operator, and it can be shown that then indeed the “missing” operators $\gamma^0$ become physical. This can be actually be better formulated in terms of equivariant cohomology. Roughly speaking, imposing equivariant cohomology means that one restricts the Hilbert space to states $|X\rangle$ that satisfy $b_0|X\rangle = 0$. The operator $(t^{1+1} \partial c + \frac{1}{\sqrt{2l}} c \partial \phi_L)$ in (2.11) does not obey this condition, and this means that the ground ring generator $\gamma^0$ is not the $BRST$ variation of a physical operator – hence, it is physical. It is actually well-known\textsuperscript{20,21,22} that in pure topological gravity one has to require equivariant cohomology, in order to obtain a non-empty theory.

The situation can be summarized in Fig.1. What we have discussed so far corresponds to step A.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Models describing gravity coupled to conformal minimal matter of type $(1, k+2)$.}
\end{figure}
It can be shown$^{23}$ that the modified minimal topological models, which contain the operators $(\gamma^0)^n$ and which are equivalent to the minimal models $M_{1,2+k}$ coupled to gravity, are in fact also equivalent to the un-modified models $A_{k+1}^{\text{top}}$ coupled to topological gravity$^{20}$. A priori, the building blocks of $A_{k+1}^{\text{top}}$ and of the same models coupled$^{24}$ to topological gravity appear to be quite different. There is, however, the remarkable fact that the total BRST operator of the topological matter plus topological gravity system obeys$^{23}$

$$Q^{\text{tot}} \equiv Q^{N=2} + Q_{BRST} = U^{-1} Q^{N=2} U ,$$

where $U = e^{\oint c [G_M + G_L + \frac{i}{2} G_{gh}]}$ is a homotopy operator. The upshot is that the cohomologies of $A_{k+1}^{\text{top}}$ coupled to topological gravity and of the modified minimal topological models are isomorphic, so that at least at the level of Fock spaces the theories are equivalent. This refers to step B in Fig.1. Step C is an expression of the fact$^{24}$ that the recursion relations of $[A_{k+1}^{\text{top}} \otimes \text{topological gravity}]$ are the same as those of the corresponding matrix models$^{25}$.

3. Relation to dispersionless KdV hierarchy

The relationship between the matter-gravity system and matrix models can be exhibited also via a more direct route. One can make use of the fact that the Landau-Ginzburg realization$^{2}$ (step E in Fig.1) of the topological matter models can be directly related$^{26}$ to the KdV integrable structure of the matrix models (step F). More precisely, one considers the dependence of correlators on perturbation parameters $t_j$, defined by$^\star \langle \ldots e^{\int d^2z d^2\theta \sum_{i=0}^{k} t_{i+2} x^{k-i}} \rangle$. It was shown$^{26}$ that the effect of such perturbations can be described in terms of a Landau-Ginzburg superpotential of the form

$$W(x,g(t)) = \frac{1}{k+2} x^{k+2} - \sum_{i=0}^{k} g_{i+2}(t) x^{k-i} .$$

The coupling constants $g_{i+2}(t_j)$ are certain, in general non-trivial functions of the perturbation parameters. Since the correlation functions can easily be computed$^{26}$ once one knows $W(x,g(t))$, solving the theory just amounts to determining these functions. This can be done by making use of the fact that $t$ are very particular, namely flat$^{27}$ coordinates on the LG deformation space. Requiring that the appropriate Gauß-Manin connection vanishes, leads to the following differential equations for $g(t)$:

$$- \partial_{t_{i+2}} W(x,g(t)) = \partial_x \Omega_{k+1-i}(x,g(t)) ,$$

$^\star$ Note that such perturbations lead away from the conformal point. We restrict here to the “small phase space”, i.e., to perturbations generated by the primary fields.
\((i = 0, \ldots, k)\), which involve the hamiltonians

\[
\Omega_i(x, g(t)) = \left( (k + 2)W \right)_{k+2}^+(x, g(t)) .
\]

(3.3)

(Here, the subscript "\(^+\)" denotes, as usual, the truncation to non-negative powers of \(x\).) The crucial observation\(^{26,28}\) is that under the substitutions \(x \to D\) and \(W(x, g) \to L(D, g)\), these equations are nothing but the dispersionless limit of the KdV flow equations

\[
\partial_{t_{i+2}} L(D, g(t)) = \left[ (L_{k+1}^{k+2})_+, L \right](D, g(t)) .
\]

(3.4)

if one imposes as boundary condition the string equation: \(D g_{k+2} = 1\). These equations describe\(^1,29\)(step G in the figure) the dynamics of the matrix models of type \((1, k + 2)\). This immediately proves the equality of correlation functions (as functions of the small phase space variables \(t\)) of the primary fields with the corresponding correlators of the matrix models (step F). These arguments, which involve only \(N=2\) Landau-Ginzburg theory, can also be extended to the gravitational descendants and to some of the recursion relations they obey\(^{30,23}\)(in the small phase space). In fact, (2.12) implies that all states of the matter-gravity system have BRST representatives in the matter sector alone. That is, the gravitational descendants can be expressed in terms of the LG field \(x\) (in equivariant cohomology) as well:\(^{23}\)

\[
\sigma_n(\phi_i)(x, t) = \partial_x \Omega_{i+(k+2)n+1}(x, g(t))
\]

(3.5)

(where \(\sigma_n(\phi_i)(x, 0) \equiv (\gamma^0)^n x^i\) in previous notation). This is precisely in the spirit of what was said above: the ingredients of the coupling of \(A_{k+1}^{\text{top}}\) to topological gravity are already built in the structure of the models \(A_{k+1}^{\text{top}}\) themselves\(^{30}\). All what is necessary to describe the coupling of these models to topological gravity is to modify their cohomological definition. The fact that topological gravity coupled to \(A_{k+1}^{\text{top}}\) can be described purely in terms of LG theory corresponds to step D in Fig.1.

Of particular interest is the perturbation of these models by the "cosmological constant" term. In our language\(^{10,16}\), it is the perturbation by the top element of \(\mathcal{R}^{A_{k+1}^{\text{top}}},\)

\[
S_{\text{cosm}} = \mu \int d^2 z e^{\sqrt{\frac{2}{t}} \phi_L} \equiv t_2 \int d^2 z d^2 \theta \, x^k .
\]

(3.6)

It is known\(^{31}\) that this perturbation is integrable and leads to the massive quantum \(N=2\) sine-Gordon model; although not invariant under the full (twisted) \(N=2\)
superconformal symmetry, it is supersymmetric, and the corrected supercharge \( \oint \mathcal{G} \) still serves as a BRST operator. Under the perturbation, the \( N = 2 \ U(1) \) current \( J \) ceases to be holomorphically conserved, which is a typical feature of perturbations leading away from the conformal point. The effective superpotential is given by a Chebyshev polynomial:

\[
W(x, t_2 = \mu) = \frac{2}{k+2} \mu \frac{x^{k+2}}{k+2} T_k(x) = \frac{1}{k+2} x^{k+2} - \mu x^k + O(\mu^2). \tag{3.7}
\]

At \( \mu = 1 \), the deformed chiral ring becomes identical\(^3\) to the fusion ring of the \( SU(2)_k \) WZW model, which it is also the same as the operator product algebra of the \( SU(2)_k/SU(2)_k \) topological field theory. This observation then allows to finally make contact to the formulation of matter-plus-gravity models in terms of topological \( G/G \) theories\(^3\): it is known\(^3,16\) that at the level of Fock space cohomology, the (suitably defined) \( SU(2)/SU(2) \) model is indeed equivalent to the matter-gravity system. We thus have the relation:

\[
\left[ M_{1,2+k} \otimes \text{Liouville gravity} \right] \bigg|_{\mu=1} \cong \left[ \frac{SU(2)_k}{SU(2)_k} \right]_{\text{modified cohomology}} \tag{3.8}
\]

4. Extension to \( W \)-gravity

One obvious motivation for investigating generalizations is the wish to step beyond the \( c_M = 1 \) barrier of ordinary gravity. This can be achieved by considering matter theories with extended symmetries, coupled to the corresponding extended geometry. The prime candidates for such models are those related to \( W \)-algebras. (One may also consider supersymmetric versions: it turns out\(^10\) that \( N = 1 \) matter coupled to \( N = 1 \) supergravity yields \( N = 3 \) superconformal models, etc.).

For a given theory of \( W_n \)-gravity coupled to matter, there is a barrier at \( c_M = n-1 \), below of which there is a finite number of (dressed) primary fields and below of which the theory should be solvable. In analogy to ordinary gravity, one would expect that such theories should be solvable also at the accumulation points \( c_M = n-1 \) (where there exists an extra \( SU(n) \) current algebra symmetry). At these points, such models are presumably related to black hole type of objects in spacetimes with signature \((n-1,n-1)\) and are characterized by topological field theories based on non-compact versions of \( \text{CP}^{n-1}_{n-1,k} \).

The physical models in question are tensor products

\[
W_n^{\text{matter}} \otimes W_n^{\text{Liouville}} \otimes_{j=1}^{n-1} \{b_j, c_j\}, \tag{4.1}
\]

which might be called “non-critical \( W \)-strings”\(^3\). Above, \( W_n^{\text{matter}} \) denotes conformal field theories that have a \( W \)-algebra as their chiral algebra, which can be for example \( W_n \) minimal models \( M_{p,q}^{(n)} \) with central charges \( c_M = (n-1)(1-n(n+}
1) \((t-1)^2\), \(t=q/p\). Furthermore, \(W_n^{\text{Liouville}}\) denotes a \((n-1)\)-component generalization of Liouville theory (Toda theory), and \(\{b_j, c_j\}\) denotes the Hilbert space of a ghost system with spins \(j+1\) and \(-j\), respectively. As it turns out, the structure of these theories for arbitrary \(n\) is very similar to \(n=2\), which corresponds to ordinary gravity. However, only for \(n=3\) the generalization of the \(\text{BRST}\) current is explicitly known:

\[
J_{\text{BRST}} = c_2 \left[ \frac{1}{b_L} W_L + \frac{1}{b_M} W_M \right] + c_1 \left[ T_L + T_M + \frac{1}{2} T_{gh}^1 + T_{gh}^2 \right] + \left[ T_L - T_M \right] b_1 c_2 (\partial c_2) + \mu (\partial b_1) c_2 (\partial^2 c_2) + \nu b_1 c_2 (\partial^3 c_2) ,
\]

(4.2)

where \(b_{L,M}^2 \equiv \frac{16}{5c_{L,M} + 22}\) and \(\mu = \frac{3}{5} \nu = \frac{1}{10b_L^2} (1 - 17b_L^2)\). In this equation, \(T_{L,M}\) and \(W_{L,M}\) denote the usual stress tensors and \(W\)-generators of the Liouville and matter sectors, and \(T_{gh}^i\) are the stress tensors of the ghosts.

Using this \(\text{BRST}\) current, one can study the spectrum of physical operators of \(W_3\) matter coupled to \(W_3\) gravity, and one finds that the analogs of ground ring elements and tachyons are states with ghost numbers equal to 0, 1, 2, 3 (the first number corresponds to ground ring elements, and the last one to tachyons). The explicit expressions are however too complicated to be written down here.

The interesting point is that there appears an \(N=2\) superconformal symmetry for all \(n\). For example, for \(W_3\) gravity one finds that

\[
G^+ = J_{\text{BRST}} + \partial \left[ -c_1 J + 2i \sqrt{\frac{1}{3}} b_1 c_1 c_2 J + i \frac{(1+t)}{2} \sqrt{\frac{3}{t}} b_1 c_1 (\partial c_2) \right.
- i \frac{(3+2t)}{\sqrt{3t}} b_1 (\partial c_1) c_2 - \frac{(7t^2-10t-15)}{4t} b_1 (\partial^2 c_2) c_2 + i \frac{(t-9)}{\sqrt{3t}} b_2 (\partial c_2) c_2
- i \frac{(3+4t)}{\sqrt{3t}} (\partial b_1) c_1 c_2 - \frac{3(4t^2-2t-3)}{2t} (\partial b_1) c_2 (\partial c_2) c_2 + \frac{(t-3)}{t} (\partial c_1)
+ i \frac{1}{2\sqrt{3t}} c_2 (2t J^2 - 3(t-5) T_L - 3(t-1) T_M - 6(1+t) \partial J)
+ i \frac{(1+t)}{2} \sqrt{\frac{3}{t}}(\partial c_2) J - i \frac{(t^2-4t-1)}{2t} \sqrt{\frac{3}{t}} (\partial^2 c_2) + tb_1 (\partial c_2) c_2 J \left. \right]
\]

(4.3)

together with

\[
G^- = b_1 , \quad T = T_L + T_M + T_{gh} , \quad J = c_1 b_1 + c_2 b_2 + \frac{3}{\sqrt{t}} (\lambda_1 \cdot \partial \phi_L) + \frac{i}{2} \sqrt{\frac{3}{t}} (t-1) \partial [b_1 c_2] \]

(4.4)
gives a non-standard free field realization of the topological algebra (2.7) with

\[
c^{N=2} = 6(1 - \frac{3}{t}) .
\]

(4.5)

Since we are dealing here with theories with an extended symmetry, coupled to an extended “\(W\)-geometry”, it is perhaps not too surprising to find that these

\* The existence of \(\text{BRST}\) currents for arbitrary \(n\) can be inferred from indirect arguments\(^{10,3}\).
topological algebras actually extend to topologically twisted $N=2$ $W$-algebras. For $t = n + k$, which corresponds to $W_n$-minimal matter models $M_{1,n+k}^{(n)}$, the anomaly indeed becomes equal to the central charges of the minimal $N=2$ $W_n$ models at level $k$: $c^{N=2} = 3\frac{(n-1)k}{n+k}$. These models are just the well-known Kazama-Suzuki models based on cosets $\frac{SU(n)}{U(n-1)}$, which are known to have an $N = 2$ $W_n$ chiral algebra. The models that arise here are of course the topologically twisted versions, which we will denote by $\text{CP}^{\text{top}}_{n-1,k}$; $n = 2$ corresponds to ordinary gravity coupled to matter: $\text{CP}^{\text{top}}_{1,k} \equiv A^{k+1}_{k+1}$.

The chiral rings of these topological minimal $W_n$ matter models are well understood, and are described further below. They are generated by primary chiral fields $x_i, i = 1, \ldots, (n-1)$ (with $U(1)$ charges equal to $i/(n+k)$), and have elements

$$R^{\text{CP}^{\text{top}}_{n-1,k}} = \left\{ \prod_{i=1}^{n-1} (x_i)^{n_i}, \sum n_i \leq k \right\}.$$  

The full ground rings of the minimal models $M_{1,n+k}^{(n)}$ coupled to $W_n$-gravity contain in addition generators $\gamma_i^0, i = 1, \ldots, (n-1)$ (with $U(1)$ charges equal to $i$) and are the “$W$-gravitationally extended” chiral rings of the Kazama-Suzuki models:

$$R^\text{gr} = R^{\text{CP}^{\text{top}}_{n-1,k}} \otimes \left\{ \prod_{i=1}^{n-1} (\gamma_i^0)^{n_i}, n_i = 0, 1, 2, \ldots \right\}.$$  

These rings have an obvious interpretation in terms of topological minimal $W_n$-matter $\text{CP}^{\text{top}}_{n-1,k}$ coupled to topological $W_n$-gravity. Like for ordinary gravity, the ground ring generators $x_i$ can be interpreted as the fields of topological LG models, with superpotentials given in refs. and in eq. (5.20) below. It would be very interesting to investigate as to what extent also the $W$-gravitational descendants ($\gamma_i^0$) can be expressed in terms of LG fields. Ideally, the whole topological $W_n$-matter-gravity system can be described in terms of Landau-Ginzburg theory.

Although this has not yet been thoroughly investigated for general $n$, our considerations seem so far to indicate that the structure for general $n$ is indeed very much parallel to the one of $n = 2$. Accordingly, one would have for $W_n$-matter models of type $(1,k+n)$ coupled to $W_n$-gravity a scheme that is analogous to Fig.1. It would be exciting to verify the remaining links in the figure for $W$-gravity. In particular, by analogy to step F in Fig.1 one would expect the existence of an infinite sequence of new integrable systems, whose Lax operators are given in terms of the Kazama-Suzuki superpotentials, and in analogy to step G one would expect the existence of an infinite class of new matrix models. While this latter assertion is more difficult to prove, we were so far indeed successful to get an idea about the structure of the new integrable systems, and this is what we like to briefly outline next.
5. Quantum rings and integrable systems for $W$-gravity

The issue is to find a multi-variable generalization\(^5\) of the dispersionless KdV hierarchy, which describes the models $M_{1,n+k}^{(n)}$ coupled to $W_n$-gravity, as well as the models $CP_{n-1,k}^{\top}$ coupled to topological $W_n$-gravity. Our strategy is inspired by a general relationship between topological LG theory, chiral rings and Drinfeld-Sokolov types of integrable systems. At the heart of our construction is the generalization of the above-mentioned relationship between LG superpotential and dispersionless Lax operator to many variables $x_i$.

We like first to reformulate the ordinary, dispersionless\(^2\) KdV hierarchy\(^*\) (pertaining to the LG models $CP_{1,k}^{\top} \equiv A_{k+1}^{\top}$) in matrix language, because it is this form of the hierarchy that is most suitable for our generalization. One starts with the linear Drinfeld-Sokolov system\(^4\)

\[
\left[ D \mathbb{1} - L_1 \right] \cdot \Psi = 0 , \tag{5.1}
\]

where the “Lax operator” $L_1$ is given by the $(k+2) \times (k+2)$ dimensional matrix

\[
L_1(g) = \Lambda_1^{(z)} + Q_1(g) , \tag{5.2}
\]

where $\Lambda_1^{(z)}$ has the familiar form

\[
\Lambda_1^{(z)} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
z & 0 & 0 & \ldots & 0
\end{pmatrix} , \tag{5.3}
\]

with $z$ representing the spectral parameter. In (5.2), $Q_1$ is usually taken to be a lower triangular matrix that is determined only up to gauge transformations belonging to the nilpotent subgroup $N^-$. Upon recursively solving for the components of $\Psi$ in favor to the first component $\Psi_0$, the system (5.1) is equivalent to the gauge invariant, scalar spectral equation

\[
L(D,g) \Psi_0 = \frac{1}{k+2} z \Psi_0 . \tag{5.4}
\]

In the dispersionless limit, where $D \to x$ and $L(D,g) \to W(x,g)$ (cf., (3.1)), this is precisely the characteristic equation of the Lax operator $L_1$, which therefore must satisfy

\[
W(L_1(g),g) = \frac{1}{k+2} z \mathbb{1} . \tag{5.5}
\]

\* With “KdV hierarchy” we will always mean the $(k+1)$th generalized KdV hierarchy.
This “superpotential spectral equation” can be taken as the definition of $\mathcal{L}_1$ in terms of the Landau-Ginzburg superpotential $W$, and (non-uniquely) determines $Q_1(g)$. The gauge freedom can be fixed by going to any particular gauge. The choice that is most appropriate for us is however not given by taking $Q_1$ to be a lower triangular matrix, but by taking $Q_1$ to belong to the Heisenberg subalgebra generated by $\Lambda_1^{(z)}$. That is, we have an infinite expansion

$$
\mathcal{L}_1(g) = \Lambda_1^{(z)} + \sum_{l=1}^{\infty} q_l(g) (\Lambda_1^{(z)})^{-l},
$$

whose coefficients can be computed from (5.5) in a recursive way.

The KdV flow equations (3.2) that determine the LG couplings $g(t)$ take the form

$$
\partial_{t_{k+2}} \Omega_{k+1-j} (g(t)) = \partial_{t_{j+2}} \Omega_{k+1-i} (g(t)),
$$

and involve the following, matrix-valued hamiltonians:

$$
\Omega_i (\mathcal{L}_1(g), g) = \frac{1}{i} (\Lambda_1^{(z)} (\mathcal{L}_1(g), \mathcal{L}^{-1}_1(g), g))^i_+, \\
\text{with } [\Omega_i, \Omega_j] \equiv 0, \quad \Omega_1 \equiv L_1,
$$

where the subscript “+” denotes the truncation to positive powers of $\mathcal{L}_1$ in the expansion of the constant matrix $(\Lambda_1^{(z)})^i$. It is clear that the constant flows associated with $\Omega_{n(k+2)} = \frac{1}{n(k+2)} z^n \mathbb{1}$ are trivial and correspond to perturbations by the null operators $\sigma_n (\phi_{k+1})$; the hamiltonians $\Omega_i$ with $i > k+1$ correspond to the gravitational descendants (cf., (3.5)).

It is well-known that the basic underlying structure of the KdV integrable system is the algebra generated by $\Lambda_1^{(z)}$, which is the principal Heisenberg subalgebra of $\hat{\mathfrak{s}}\ell(k+2)$. Its positive part,

$$
\mathcal{H}^+ \equiv \{ (\Lambda_1^{(z)})^m, m \in \mathbb{Z}_+ \},
$$

is precisely what determines the hamiltonians, $\Omega = (\mathcal{H}^+)_+$. In view of our later generalization, it is very helpful to note that $\Lambda_1^{(z)}$ is identical to the chiral ring structure constant $C_1(z)$ that pertains to the following LG potential “at one level higher”:

$$
W^{A_{k+2}}_{A_{k+2}} (x, t_{k+2} = z, t_l = 0) = \frac{1}{k+3} x^{k+3} - z x.
$$

This means that the underlying algebraic structure of the $A_{k+1}$ type matter-gravity system is that of a specifically deformed chiral ring pertaining to the LG theory $A_{k+2}$:

$$
\mathcal{H}^+ \cong \mathcal{R}^{A_{k+2}} (t_{k+2} = z, t_l = 0).
$$
For $g = 0$, the superpotential spectral equation (5.5) represents a specific relation in this ring, and can be viewed as the equation of motion associated with the LG potential (5.10):

\[
W^{A_{k+1}}(x,0) - \frac{1}{k+2} z = \frac{1}{k+2} \partial_x W^{A_{k+2}}(x,z) = 0 .
\] (5.12)

This important fact, namely that $\Lambda^{(z)} = C_1(z)$ so that the matrix-valued spectral equation $W(\Lambda^{(z)}) \equiv \frac{1}{k+2}(\Lambda^{(z)})^{k+2} = \frac{1}{k+2} z I$ can be interpreted as some chiral ring vanishing relation (associated to a different LG theory), is our starting point of the generalization to many variables. More precisely, our plan is to use appropriate ring structure constants $C_i(z)$ to construct Hamiltonians and Lax operators for the models $\text{CP}^{n-1}_{k+1}$ coupled to gravity. This is motivated by the fact that their chiral rings have a common underlying structure for all $n$: it is the structure of principal embeddings\(^{44}\) of $s(2)$. Such kind of embeddings is also precisely what underlies the construction of the $W_n$-algebras.\(^{45}\)

Specifically, it is well-known that the Heisenberg algebra generator $\Lambda^{(z)}_1$ that figures in the Drinfeld-Sokolov matrix system\(^{42,43}\) is nothing but an $s(2)$ step generator $I_+$ (principally embedded in $s(2)$),

\[
\Lambda^{(z)}_1 = \Lambda_1 + z \Lambda_-,(k+1) , \quad \text{where} \quad \Lambda_1 = I_+ \equiv \sum_{\text{simple roots } \alpha} E_\alpha , \quad \Lambda_-,(k+1) = E_- ,
\] (5.13)

perturbed by the spectral parameter $z$ ($\psi$ denotes the highest root). The point is, as mentioned above, that this is also the structure of the perturbed chiral ring $\mathcal{R}_{\text{CP}^{1}_{k+1}}(z)$: it is known\(^{18,39}\) that the unperturbed ring $\mathcal{R}_{\text{CP}^{1}_{k+1}}(z)$ is isomorphic to the cohomology ring $H^*(\text{CP}^{k+1})$, and there is a theorem by Kostant\(^{46}\) that says that $H^*(\text{CP}^{k+1})$ is generated by an $s(2)$ step generator $I_+$. The deformation by the spectral parameter is then precisely what deforms the cohomology ring $H^*$ into the quantum cohomology ring $QH^*$, whence

\[
\mathcal{H}^+ \cong \mathcal{R}_{\text{CP}^{1}_{k+1}}(z) \cong QH^*_\partial(\text{CP}^{k+1}, \mathbb{R}) .
\] (5.14)

(The word “quantum” indicates that the deformation of the classical cohomology ring by the spectral parameter $z$ is precisely the effect of the instanton corrections in a supersymmetric $\text{CP}_{k+1} \sigma$-model\(^{47}\)).

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For the more general models $\text{CP}^{top}_{n-1,k}$ that are related to $W_n$-gravity, the story is very similar: it is known \textsuperscript{18,39} that the chiral rings are isomorphic to the Dolbeault cohomology rings of certain grassmannians:

$$R_{\text{CP}^{top}_{n-1,k}} \cong H^*_\partial \left( SU/(n+k-1) \over SU/(n-1) \times SU(k) \times U(1) \right).$$  \hfill (5.15)

These chiral rings are generated by ring structure constants $C_i$, $i = 1, \ldots, (n-1)$, which represent the LG fields $x_i$. The important point is that these ring structure constants are determined by principal embeddings of $\mathfrak{sl}(2)$ as well!

More precisely, consider the following matrices:

$$\Lambda_p = \sum_{\{\alpha; p\alpha = p\}} a^{(p)}_{\alpha} E_{\alpha}, \quad \text{for each } p \in \{1, 2, \ldots, (n+k-2)\},$$  \hfill (5.16)

where the coefficients $a^{(p)}_{\alpha}$ are determined through $[\Lambda_p, \Lambda_p'] = 0$, with $\Lambda_1 \equiv I_+$ as in (5.13). Kostant’s theorem \textsuperscript{46} now tells that when taken in the $(n-1)$th fundamental representation of $\mathfrak{sl}(n+k-1)$, the matrices $\Lambda_i$, $i = 1, \ldots, (n-1)$, generate $H^*_\partial \left( SU/(n+k-1) \over SU/(n-1) \times SU(k) \times U(1) \right)$, and this means that they are precisely the ring structure constants of the Kazama-Suzuki models:

$$C_i = \Lambda_i.$$  \hfill (5.17)

Our idea is to employ the $\Lambda_i$ to construct Lax operators and hamiltonians for generalized Drinfeld-Sokolov systems. The relevant objects are of course matrices $\Lambda^{(z)}_i$ that are perturbed by a spectral parameter; they are uniquely defined by requiring $[\Lambda^{(z)}_i, \Lambda^{(z)}_{i'}] = 0$ where $\Lambda^{(z)}_1$ is as in (5.13) (but now in the $(n-1)$th fundamental representation of $\mathfrak{sl}(n+k-1)$). They generate precisely the quantum deformation \textsuperscript{47} of the grassmannian cohomology rings, which are the same as specifically perturbed chiral rings of the models $\text{CP}^{top}_{n-1,k}$:

$$R_{\text{CP}^{top}_{n-1,k}}(t_{k+n-1} = z) \cong QH^*_\partial \left( SU/(n+k-1) \over SU/(n-1) \times SU(k) \times U(1) \right).$$  \hfill (5.18)

These perturbed chiral rings are associated with the LG superpotentials

$$W_{\text{CP}^{top}_{n-1,k}}(x_i, z) = W_{\text{CP}^{top}_{n-1,k}}(x_i, 0) - z x_1,$$  \hfill (5.19)

which were investigated previously \textsuperscript{48} in the context of integrable perturbations of the models $\text{CP}^{top}_{n-1,k}$. Such superpotentials were first explicitly written down in refs.\textsuperscript{18,49}, and have the form:

$$W_{\text{CP}^{top}_{n-1,k}}(x_i, 0) = \sum_{l=1}^{k} (\xi_l)^{n+k}(x_i),$$  \hfill (5.20)

where $x_i = \sum_{1 \leq l_1 \leq \ldots \leq l_i \leq k} \xi_{l_1} \xi_{l_2} \ldots \xi_{l_i}$.
are the elementary symmetric polynomials. This formula was obtained by making use of the fact that, in the Borel-Weil picture, the cohomology of the grassmannian $G/H$ is generated by Chern classes $c_i$ of certain $H$-valued vector bundles, which satisfy relations of the form
\[
\text{Ch}_{(n-1,1)}(c_i, t) \cdot \text{Ch}_{(1,k)}(c_i, t) = 1 ,
\]
(5.21)
where
\[
\text{Ch}_v(c_i, t) = \dim v \sum_{j=0}^{\dim v} c_j(\xi) t^j
\]
(5.22)
is the total graded Chern form associated with the $H$-representation $v$. The relations among the $c_i \cong x_i$ generated by (5.21) lead precisely to the vanishing relations associated with the potentials (5.20). The formula (5.20) for the superpotentials was subsequently used in in ref.\(^41\), where the following generating function was found:
\[
- \log \left[ \sum_{i=1}^{n-1} (-t)^i x_i \right] = \sum_{k=-n+1}^{\infty} t^{n+k} W^{\text{CP}^\text{top}}_{n-1,k}(x_i, 0) .
\]
(5.23)

From this it is easy to prove that
\[
W^{\text{CP}^\text{top}}_{n-1,k}(x_i, 0) = \frac{1}{n+k} \left( \sum_{i=1}^{n-1} (n - i) x_{i-1} \partial x_i \right) W^{\text{CP}^\text{top}}_{n-1,k+1}(x_i, 0) ,
\]
(5.24)
which means that a given superpotential can be written as a vanishing relation of the superpotential “at one level higher”. This is the key point which makes the whole construction fly. Namely, (5.19) and (5.24) imply that
\[
W^{\text{CP}^\text{top}}_{n-1,k}(x_i, 0) - \frac{1}{n+k} z = \frac{1}{n+k} \left( \sum_{i=1}^{n-1} (n - i) x_{i-1} \partial x_i \right) W^{\text{CP}^\text{top}}_{n-1,k+1}(x_i, z) ,
\]
(5.25)
and this means that the ring structure constants $\Lambda_i^{(z)} = C_i$ of the models $\text{CP}^\text{top}_{n-1,k}$ satisfy $W^{\text{CP}^\text{top}}_{n-1,k}(\Lambda_i^{(z)}, 0) = \frac{1}{n+k} z \mathbb{1}$. This is precisely what we have been looking for: namely we can take for the Lax operators of the integrable hierarchies just the perturbed versions of these $\Lambda_i^{(z)}$,
\[
\mathcal{L}_i(g) = \Lambda_i^{(z)} + Q_i(g) = \Lambda_i^{(z)} + \sum_{l_j} q_i^{l_1, \ldots, l_{n-1}}(g) (\Lambda_1^{(z)})^{-l_1} \ldots (\Lambda_{n-1}^{(z)})^{-l_{n-1}} ,
\]
(5.26)
whose coefficients $q_i(g)$ are such that
\[
W^{\text{CP}^\text{top}}_{n-1,k}(\mathcal{L}_1(g), \ldots, \mathcal{L}_{n-1}(g), g) = \frac{1}{n+k} z \mathbb{1} .
\]
(5.27)
This is the desired generalization of the matrix-valued superpotential spectral equation (5.5).

To obtain a hierarchy of differential equations, we need to construct appropriate hamiltonians. By analogy to (5.8), we simply take the commuting matrices

\[ \Omega_{l_1, \ldots, l_{n-1}}(\mathcal{L}_i, g) = \left( (\Lambda_{l_1}^{(z)})^{l_1} \cdots (\Lambda_{l_{n-1}}^{(z)})^{l_{n-1}} \right)_+, \quad l_i \geq 0 , \]  

(5.28)

where “+” denotes projection to positive grade. That is, we take as relevant Heisenberg algebra

\[ \mathcal{H}^+ \cong \mathcal{H}^+_{\mathcal{G}}\left( \frac{SU(n+k)}{SU(n-1) \times SU(k+1) \times U(1)}, \mathbb{R} \right) , \]  

(5.29)

which just means, like previously for \( n = 2 \), that the underlying algebraic structure of the \( \text{CP}_{n-1,k}^{\text{top}} \) matter-gravity integrable system is given by the quantum ring associated with the matter model “at one level higher”, \( \text{CP}_{n-1,k+1}^{\text{top}} \). The perturbation by the spectral parameter \( z \) deforms the finite, nilpotent ring \( \mathcal{R}_{\text{CP}_{n-1,k+1}^{\text{top}}} \) into an infinite dimensional, affine algebra, which reflects the extension of the matter ring (4.6) to the gravitationally extended ground ring (4.7) of the matter-gravity system. It would be very interesting to study in more detail the structure \( \mathcal{H}^+ \) in relation with the \( W \)-gravity descendants of (4.7). How this precisely works is not so clear because the number of hamiltonians (5.28) per grade does not grow indefinitely with increasing grade, since there are relations between polynomials of the \( \Lambda_{l_i}^{(z)} \) (for example the superpotential spectral equation). These relations are just the multigenerator analogs of the well-known condition that reduces the KP to the KdV hierarchy.

Strictly speaking, \( \mathcal{H}^+ \) in (5.29) is the enveloping algebra of the principal Heisenberg algebra. That is, since the generators \( \Lambda_{l_i}^{(z)} \) are in general in a higher fundamental representation of \( \mathfrak{sl}(n+k) \), powers of the \( \Lambda_{l_i}^{(z)} \) will in general not belong to the principal Heisenberg subalgebra of \( \hat{\mathfrak{sl}}(n+k) \), but to its enveloping algebra. This is precisely how this construction makes it possible to have more commuting hamiltonians at a given grade as compared to the usual KdV type of systems\(^{42,43} \), where one considers only the flows associated with the algebra, which are representation-independent.

The flow equations that determine the small phase space couplings \( g(t) \) have then supposedly the generic form

\[ \left[ D_{l_1, \ldots, l_{n-1}}, D'_{l'_1, \ldots, l'_{n-1}} \right] = 0 , \]

\[ D_{l_1, \ldots, l_{n-1}} \equiv \frac{\partial}{\partial t_{l_1, \ldots, l_{n-1}}} - \sum_{k_j} Z^{k_1, \ldots, k_{n-1}}_{l_1, \ldots, l_{n-1}} \Omega_{k_1, \ldots, k_{n-1}}(\mathcal{L}_i(g(t)), g(t)) \]  

(5.30)

(where \( Z \) are normalization constants), but whether these equations really determine the correct LG couplings \( g(t) \) in terms of the flat coordinates \( t \), is a problem.
that we don’t know how to answer yet in general. All what we have done so far was to check these equations for a couple of examples, where they indeed produced the correct results\(^5\).

But these results as well as the general structure strongly suggest that that the kind of integrable systems we proposed makes sense and correctly describes the quasi-classical dynamics of the models \(M_{n+k}^{(n)}\) coupled to \(W_n\)-gravity, which are supposedly equivalent to the models \(\text{CP}^{n+k}_{n-1}\) coupled to topological \(W_n\)-gravity. This would correspond to the completion of step F in Fig.1 for \(W\)-gravity, and make step G in the figure appear feasible.

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