We show that the strong coupling limit of d-dimensional quantum electrodynamics with $2^d/2^{d/2}$ flavors of fermions can be mapped onto the s=1/2 quantum Heisenberg antiferromagnet in d-1 space dimensions. The staggered Néel order parameter is the expectation value of a mass operator in QED and the spin-waves are pions. We speculate that the chiral symmetry breaking phase transition corresponds to a transition between the flux phase and the conventional Néel ordered phase of an antiferromagnetic t-J model.

The possibility that quantum electrodynamics (QED) can have a nonperturbative ultraviolet fixed point has been investigated by many authors [1-8]. Such a fixed point would give QED sensible ultraviolet behavior and save it from triviality by avoiding the Landau ghost (or Moscow zero) [9-10]. The most popular candidate for an ultraviolet fixed point arises from a second order phase transition which occurs when the electromagnetic

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coupling is increased to a critical value where the theory breaks chiral symmetry.

The physical mechanism for such a phase transition advocated by Miranksy [3] is that of ‘collapse’ of the electron-positron wave-function when the charge reaches a super-critical value, $e^2 \sim 4\pi$. There are two ways to screen super-critical charges. In the case of a super-critical nucleus the high electric field produces electron-positron pairs, ejects the positron and absorbs the electron to screen its charge. Alternatively, the pair production is suppressed by fermion mass, so the system can stabilize itself by increasing the electron mass, thus the tendency to break chiral symmetry. These ideas are supported by studies of the Schwinger-Dyson equations for QED in the quenched ladder approximation [3-5] which find a line of fixed points in the $e^2$-plane between 0 and $e^2_c = 4\pi^2 / 3$ where the theory breaks chiral symmetry dynamically and has a nontrivial continuum limit. This critical point has other interesting behavior such as large negative anomalous dimensions for fermion composite operators so that some four-fermion operators are relevant [4]. There is also some numerical evidence for this phase transition [7].

In this Letter we shall present a lattice model whose weak coupling continuum limit is 4-flavor QED with light fermions and which, in its strong coupling limit, exhibits spontaneously broken chiral symmetry. Conventional analysis of Euclidean lattice gauge theory formulates it as a classical statistical mechanics problem and seeks second order phase transitions so that a nontrivial continuum limit exists [11]. Here we formulate a Hamiltonian version of lattice QED as quantum statistical mechanics [12]. In the latter a relativistic continuum limit exists if there are gapless degrees of freedom and if those degrees of freedom have a relativistic dispersion relation, $\omega(k) \sim |k|$. This is, of course, true for the weak
coupling continuum limit of the lattice model which produces QED with 4 flavors of electron (reminiscent of the standard model with four generations). We show that the strong coupling limit is equivalent to the spin 1/2 quantum Heisenberg antiferromagnet and that the Néel order coincides with chiral symmetry breaking. In three or greater dimensions, there is a rigorous proof that the ground state of the s=1/2 antiferromagnet has Néel order [13]. Also, there is evidence from numerical simulations and large spin expansions for Néel order in two dimensions [14]. Furthermore, these systems exist in nature and can be studied by experiment. In dimension 3 and higher the order persists for some range of temperature.

This result indicates that in strong coupling 4-flavor QED has a nontrivial continuum limit. The light excitations are the ‘pions’ which coincide with the spin waves of the antiferromagnet. They have a relativistic dispersion relation and interactions which are commonly represented by a nonlinear sigma model. Corrections to the strong coupling limit take into account fermion hopping terms similar to those in a gauge invariant t-J model. We conjecture that if $1/e^2$ is increased to some critical value the chiral symmetry of QED is restored. The resulting phase of the t-J model is known as the flux phase which is a plaquette-centered antiferromagnetic gapless semiconductor rather than site-centered antiferromagnetic insulator.

We shall use staggered fermions on a (d-1)-dimensional lattice and continuum time which are obtained by spin-diagonalization [15] of the naively latticed Dirac Hamiltonian

$$H_f = \frac{i}{2} \sum_{x,j} \left( \psi^\dagger(x) \alpha^j \nabla_j \psi(x) - \nabla_j \psi^\dagger(x) \alpha^j \psi(x) \right) = -\frac{i}{2} \sum_{x,j} \left( \psi^\dagger(x + \hat{j}) \alpha^j \psi(x) - \psi^\dagger(x) \alpha^j \psi(x + \hat{j}) \right)$$

(1)
where $\nabla^j$ is the forward lattice difference operator, $\nabla^j f(x) = f(x + \hat{j}) - f(x)$, $\alpha^j$ are the $2^{[d/2] \times 2^{[d/2]}$ Hermitean Dirac matrices, $x, y, \ldots$ refer to sites on a hypercubic lattice, and $\hat{i}, \hat{j}, \ldots$ refer to unit vectors. (Here $[d/2]$ is the largest integer $\leq d/2$.) The second form of the Hamiltonian in (1) describes a fermion hopping problem in a U$(2^{[d/2]}$) background gauge field given by the unitary matrices $\alpha^j$. The crucial observation which allows spin diagonalization is that this background field has only U(1) curvature, i.e. if we consider the product around any plaquette $\alpha^j \alpha^k (\alpha^j)\dagger (\alpha^k)\dagger = -1$. This allows diagonalization using the gauge transformation $\psi(x) \to (\alpha^1)^{x_1} (\alpha^2)^{x_2} \ldots (\alpha^{(d-1)})^{x(d-1)} \psi(x)$ resulting in the Hamiltonian

$$H_f = -\frac{i}{2} \sum_{x,j} (-1)\sum_{1}^{j-1} x_p \left( \psi\dagger(x + \hat{j})\psi(x) - \psi\dagger(x)\psi(x + \hat{j}) \right)$$

which describes $2^{[d/2]}$ species of lattice fermions with background U(1) magnetic flux $\pi$ through every plaquette of the lattice. Each species of fermion must have the same spectrum as the original one given by the Dirac Hamiltonian (1). This allows reduction of the fermion multiplicity by a factor of $2^{[d/2]}$. The result resembles a condensed matter hopping problem with a single species of fermion where there is a background magnetic field $\pi$ per plaquette.

Chiral symmetries are obtained by lattice translations by one site. This translation interchanges the even ($\sum_1^{d-1} x_p$=even) and odd ($\sum_1^{d-1} x_p$=odd) sublattices. The substitutions

$$\psi(x) \to (-1)^{\sum_{j+1}^{d-1} x_p} \psi(x + \hat{j})$$

leaves the Hamiltonian in (2) invariant. A candidate for Dirac mass operator, which
changes sign under the transformations in (3), is the staggered charge density operator

\[ \mu = \sum_x (-1)^{x_p} \psi^\dagger(x) \psi(x) \quad (4) \]

If we introduce \( N \) species of lattice fermions, the continuum limit of (2) describes \( 2^{d-1} N/2^{[d/2]} \) species of massless Dirac fermions.

To obtain the continuum limit and the number of fermion species, we first divide the lattice into \( 2^{d-1} \) sublattices according to whether the components of their coordinates are even or odd. For example, when \( (d-1)=3 \), we label 8 fermion species as \( \psi(\text{even, even, even}) \equiv \psi_1, \psi(\text{even, odd, odd}) \equiv \psi_2, \psi(\text{odd, even, odd}) \equiv \psi_3, \psi(\text{odd, odd, even}) \equiv \psi_4, \psi(\text{even, even, odd}) \equiv \psi_5, \psi(\text{even, odd, even}) \equiv \psi_6, \psi(\text{odd, even, even}) \equiv \psi_7, \psi(\text{odd, odd, odd}) \equiv \psi_8. \) Then, if we add the mass operator in (4), in momentum space the Hamiltonian has the form

\[ H_f = \int_{\Omega_B} d^3k \, \psi^\dagger(k) \left( A^i \sin k_i + B m \right) \psi(k) \quad (5) \]

where the \( 8 \times 8 \) Dirac matrices are \( \psi(k) = (\psi_1, \ldots, \psi_8), \quad A^i = \begin{pmatrix} 0 & \alpha^i \\ \alpha^i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \quad (6) \]

\( \sigma^i \) are Pauli matrices, we have used the Fourier transform \( \psi(x) = \int_{\Omega_B} \frac{d^3k}{(2\pi)^3} \frac{e^{-ik\cdot x}}{2} \psi(k) \) and \( \Omega_B \) is the Brillouin zone of the (even,even,even) sublattice, \(-\pi/2 < k_i \leq \pi/2\). The fermion spectrum is \( \omega(k) = \sqrt{\sum_i \sin^2 k_i + m^2} \) and only the region \( k_i \sim 0 \) is relevant to the continuum limit. We have normalized \( \psi(k) \) so that

\[ \{ \psi(x), \psi^\dagger(y) \} = \delta(x-y), \quad \{ \psi(k), \psi^\dagger(l) \} = \delta(k-l) \quad (7) \]

If we define \( \beta = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \) and the unitary matrix \( M = \frac{1}{2} \begin{pmatrix} 1-\beta & 1+\beta \\ 1+\beta & 1-\beta \end{pmatrix} \) and \( \psi = M \psi' \) with \( \psi' = (\psi_a, \psi_b) \) the Hamiltonian is

\[ H_f = \int_{\Omega_B} d^3k \begin{pmatrix} \psi_a^\dagger & \psi_b^\dagger \end{pmatrix} \begin{pmatrix} \alpha^i \sin k_i - \beta m & 0 \\ 0 & \alpha^i \sin k_i + \beta m \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \quad (8) \]
In the low momentum limit, \( \sin k_i \sim k_i \), with fermion density 1/2 per site, we obtain 2 continuum Dirac fermions. Furthermore, the staggered charge operator gives a Dirac mass of differing sign for the two species. In \( d \) dimensions the procedure is similar to this. In general, we shall consider \( N \) lattice species in \( d \) dimensions which yields \( 2^{(d-1)}N/2^{[d/2]} \) continuum species of Dirac fermions where the lattice fermion density is \( N/2 \) per site.

In lattice electrodynamics the gauge field \( A_i(x) \) and electric field \( E_i(x) \) associated with the link between \( x \) and \( x + \hat{i} \) are conjugate variables, \([A_i(x), E_i(y)] = ie^2 \delta_{ij} \delta(x - y)\). We shall represent this commutator by taking the quantum states as functions of \( A_i(x) \) and \( E_i(x) = -ie^2 \partial / \partial A_i(x) \). The anticommutator algebra (7) is represented by a \( 2N \)-level fermion system at each site with cyclic vector defined by \( \psi(x)|0 >= 0 \) \( \forall x \). The Hamiltonian is

\[
H = \sum_{x,j} -\frac{e^2}{2} \frac{\partial^2}{\partial A_j(x)^2} + \sum_{x,i,j} \frac{1}{2e^2} F_{ij}^2 + \sum_{x,i,j} \left( t_{x,j} \psi^\dagger(x + \hat{j}) e^{iA_j(x)} \psi(x) + h.c. \right) \tag{9}
\]

where \( F_{ij} = \nabla_i A_j - \nabla_j A_i \) for noncompact QED, \( F_{ij} = \sin(\nabla_i A_j - \nabla_j A_i) \) for compact QED, and \( t_{x,i} \) contains a background field of \( \pi \) (mod 2\( \pi \)) through every plaquette of the lattice. The Hamiltonian is invariant under the gauge transformation \( A_i(x) \rightarrow A_i(x) + \nabla_i \chi(x) \), \( \psi(x) \rightarrow e^{i\chi(x)} \psi(x) \) which is generated by

\[
G(x) = \sum_j \nabla_j \frac{1}{i} \frac{\partial}{\partial A_j(x - \hat{j})} + \psi^\dagger(x) \psi(x) - N/2 \tag{10}
\]

Gauge invariance is imposed as a physical state condition, \( G(x) \Psi_{\text{phys}}(A) = 0 \).

We shall first consider the case \( N = 2 \). Then there are 4 flavors of 4 component fermions in 4 dimensions and 2 flavors of 4 component fermions in 2+1 dimensions. For strong coupling perturbation theory we write the Hamiltonian as, \( H = H_0 + H_1 + H_2 \) where \( H_0 = -\frac{e^2}{2} \sum \frac{\partial^2}{\partial x^2} \), \( H_1 = \sum t\psi^\dagger \psi + h.c. \) and \( H_2 = \frac{1}{4e^2} \sum F_{ij}^2 \) are each gauge invariant.
operators. The leading order ground state is the ground state of $H_0$ which is $A$-independent and is gauge invariant when each state has fermion occupation number one, $|\Psi_0[\tau_x]\rangle = \prod_x \psi_{\tau_x}(x)|0\rangle$. This is a normalizable state in compact QED where $A_i(x)$ is integrated from 0 to $2\pi$ and is a non-normalizable component of a continuum spectrum in non-compact QED. It is highly degenerate: each of the $2^V$ (where $V$ is volume) components labelled by $[\tau_x]$ has the same eigenvalue of $H_0$. The degeneracy is resolved by diagonalizing the matrix elements of perturbations in $1/e^2$. First order perturbations to the vacuum energy vanish. Thus, the leading term in the vacuum energy is of order $1/e^2$, and is given by the lowest eigenvalue of the matrix

$$\delta_2 E_0 = -\langle \Psi_0[\tau_x]|H_1|\frac{1}{H_0 - E_0}H_1|\Psi_0[\tau'_x]\rangle + \langle \Psi_0[\tau_x]|H_2|\Psi_0[\tau'_x]\rangle$$

(11)

The second term is diagonal and therefore is irrelevant to resolving the degeneracy. Diagonalizing the matrix in the first term is equivalent to solving the eigenvalue problem for the four-fermion Hamiltonian

$$H_{\text{eff}} = -\frac{4}{e^2} \sum_{x,i} |t_{x,i}|^2 \psi^\dagger(x + \hat{i})\psi(x)\psi^\dagger(x)\psi(x + \hat{i})$$

(12)

restricted to the subspace of the Hilbert space where each site is singly occupied. With the identity $-2\psi^\dagger(x)\psi(y)\psi^\dagger(y)\psi(x) = \psi^\dagger(x)\bar{\sigma}\psi(x) \cdot \psi^\dagger(y)\bar{\sigma}\psi(y) + \psi^\dagger(x)\psi(x)\psi^\dagger(y)\psi(y)$ we obtain the Hamiltonian of the $s=1/2$ quantum antiferromagnet,

$$H_{\text{eff}} = \frac{2}{e^2} \sum_{x,i} |t_{x,i}|^2 \psi^\dagger(x + \hat{i})\bar{\sigma}\psi(x + \hat{i}) \cdot \psi^\dagger(x)\bar{\sigma}\psi(x) + \text{const.}$$

(13)

Fermion mass operators are staggered charge density operators. For $N=2$, consider a mass operator with differing signs for the two lattice flavors, i.e. $m \sum_x (-1)^x \psi^\dagger \sigma^3 \psi$
which in the naive continuum limit corresponds to the mass matrix
\[
\begin{pmatrix}
  m & 0 & 0 & 0 \\
  0 & -m & 0 & 0 \\
  0 & 0 & -m & 0 \\
  0 & 0 & 0 & m
\end{pmatrix}
\].

(Note that there is no chiral anomaly in this channel.) This corresponds to a staggered magnetization operator and its expectation value is the order parameter of the antiferromagnet. There is a proof that for \( s=1/2 \) and \( (d-1) \geq 3 \), (and it is widely believed to also hold in \( d-1=2 \) although there is no rigorous proof) that the Heisenberg antiferromagnet has a Néel ordered ground state, i.e.

\[
\lim_{m \to 0} \lim_{V \to \infty} < \sum_x (-1)^x \psi_x^\dagger \sigma^3 \psi > \neq 0
\]  \hspace{1cm} (14)

In quantum electrodynamics, this is the chiral limit and implies that there is a nontrivial expectation value of the mass operator and thus spontaneous chiral symmetry breaking. In \( (d-1) \geq 3 \) this persists for some finite range of temperature whereas in \( (d-1)=2 \) it can only be true at zero temperature.

We conclude that, in the infinite coupling limit, \( 2^d/2^{[d/2]} \) flavor (4 flavor in \( d=4 \) QED breaks chiral symmetry. The only light particles in the spectrum are spin-waves which are the Goldsone bosons for breaking of the \( \text{SU}(2) \) symmetry of the Heisenberg model. In QED they are the ‘pions’ corresponding to the breaking of the chiral symmetry and electrons and all other charged excitations are confined. There are not enough spin waves to account for the breaking of \( \text{SU}_L(4) \times \text{SU}_R(4) \) chiral symmetry. The reason for this mismatch of the number of Goldstone bosons and broken symmetries arises from the presence of relevant operators which reduce some of the apparent continuous chiral symmetries of the naive continuum theory to discrete ones of the lattice. These discrete symmetries are broken but do not require Goldstone bosons. The only true flavor symmetry of the lattice theory
is SU(2). The apparent $SU_L(4) \times SU_R(4)$ symmetry of the weak coupling limit stems from the fact that all of the operators which could break $SU_L(4) \times SU_R(4)$ to SU(2) plus discrete chiral symmetries are irrelevant in that limit. However, they can become relevant at strong coupling and the full $SU_L(4) \times SU_R(4)$ symmetry is absent. Also, recall that Goldstone’s theorem applies to the continuous symmetries of the bare Hamiltonian. The effective Hamiltonian (13) is invariant under the discrete chiral transformations in (3). However, the ordered ground state is not - therefore the discrete chiral symmetries which involve translations by one lattice site are also broken as $1/e^2 \rightarrow 0$.

We can think of the higher order in $1/e^2$ corrections as perturbations of the Heisenberg model. When we increase $1/e^2$ we eventually should arrive at a critical coupling where the chiral symmetry is restored. A future technical problem will be to estimate the critical coupling. Also, spin-wave analysis which is exceptionally good for the quantum antiferromagnet can also be used to look at the dynamics of strong-coupling QED.

We expect that chiral symmetry breaking should persist for $N$ in the vicinity of $N = 2$. However, when $N=1$ (and whenever $N$ is odd), since operator $\psi^\dagger \psi$ has integer eigenvalues, it is not possible to find states which are annihilated by the gauge generator in (10) without nontrivial electric fields. The strong coupling ground state energy is necessarily of order $e^2$ and the ground state is the lowest eigenstate of the effective coulomb Hamiltonian,

$$H_c = \sum_x \frac{e^2}{2} (\psi^\dagger \psi - 1/2) \frac{1}{-\nabla^2} (\psi^\dagger \psi - 1/2)$$ (15)

We expect that the ground state of $H_c$ is a chiral symmetry breaking Wigner lattice where either the even or odd sublattice is completely occupied and the other sublattice is completely empty. Thus, we expect chiral symmetry breaking in this case too, with
critical behavior in the universality class of the Ising model. However, we point out that the vacuum energy is not a smooth function of $N$, being of order $e^2$ when $N$ is odd and of order $1/e^2$ when $N$ is even.

In 2+1-dimensions, our results suggest chiral symmetry breaking, at least for $N=1$ and 2, in agreement with the analysis of quenched Schwinger-Dyson equations [16]. It is also known that at large $N$ other phases, such as the flux phase itself [17] compete with the Néel ordered phase of the antiferromagnet. It would be interesting to examine this further, in particular to obtain an estimate of upper critical $N$, if one exists, in the context of the present model.

Salmhofer and Seiler [8] have given a proof that QED defined on a spacetime lattice with $\geq 4$ flavors and in $\geq 4$ dimensions breaks chiral symmetry. Their model differs from ours in the order of symmetry of the strong coupling versus the weak coupling limit. Their 4-flavor QED has only $Z_2$ symmetry at strong coupling and no Goldstone bosons. Furthermore, the order parameters differ in the two cases, theirs being in the chiral U(1) channel. It is also a puzzle to us that the vacuum energy of their model in the large $e^2$ limit is of order one, whereas we find that it is always either of order $e^2$ or $1/e^2$ depending on whether $N$ is even or odd. Resolution of these differences is an important problem.

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