On a distribution property of the residual order of $a \pmod{p}$

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Abstract

Let $a$ be a positive integer with $a \neq 1$ and $Q_a(x;k,l)$ be the set of primes $p \leq x$ such that the residual order of $a$ in $\mathbb{Z}/p\mathbb{Z}^\times$ is congruent to $l \pmod{k}$. It seems that no one has ever considered the density of $Q_a(x;k,l)$ for $l \neq 0$ when $k \geq 3$. In this paper, the natural densities of $Q_a(x;4,l)$ ($l = 0,1,2,3$) are considered. We assume $a$ is square free and $a \equiv 1 \pmod{4}$. Then, for $l = 0,2$, we can prove unconditionally that their natural densities are equal to $1/3$. On the contrary, for $l = 1,3$, we assume Generalized Riemann Hypothesis, then we can prove their densities are equal to $1/6$.

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1 Introduction

Let $\mathcal{P}$ be the set of all prime numbers.

For a fixed natural number $a \geq 2$, we can define two functions, $I_a$ and $D_a$, from $\mathcal{P}$ to $\mathbb{N}$:

\begin{align*}
I_a: & \quad p \mapsto I_a(p) = |(\mathbb{Z}/p\mathbb{Z})^\times : (a \pmod{p})| \\
& \quad \text{(the residual index mod $p$ of $a$),}

D_a: & \quad p \mapsto D_a(p) = \sharp\langle a \pmod{p} \rangle \\
& \quad \text{(the order of the class $a \pmod{p}$ in $(\mathbb{Z}/p\mathbb{Z})^\times$),}
\end{align*}

where $(\mathbb{Z}/p\mathbb{Z})^\times$ denotes the set of all invertible residue classes mod $p$, and $\sharp$ : $\cdot$ the index of the subset.

We have a simple relation

\begin{equation}
I_a(p)D_a(p) = p - 1, \quad \text{(1.2)}
\end{equation}

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but both of these functions fluctuate quite irregularly. C. F. Gauss already noticed that \( I_{10}(p) = 1 \) happens rather frequently. And the famous Artin’s conjecture for primitive roots asks whether the cardinality of the set

\[
N_a(x) := \{p \leq x \mid I_a(p) = 1\}
\]  

(1.3)
tends to \( \infty \) or not as \( x \to \infty \). On the assumption of the Generalized Riemann Hypothesis for a certain type of Dedekind zeta functions, C. Hooley \[3\] succeeded in calculating the natural density of \( N_a(x) \). There are various variations of Artin’s Conjecture, among which two papers Lenstra \[5\] and Murata \[7\] considered the surjectivity of the map \( I_a \). For any natural number \( n \), we define

\[
N_a(x; n) := \{p \leq x \mid I_a(p) = n\}
\]  

(1.4)

Then their results show that, for a square free \( a \) with \( a \not\equiv 1 \pmod{4} \), we have, under GRH, an asymptotic formula

\[
\#N_a(x; n) \sim C_a^{(n)} \text{li} x
\]  

(1.5)

and \( C_a^{(n)} > 0 \), where \( \text{li} x := \int_2^x (\log t)^{-1} dt \) and the constant \( C_a^{(n)} \) depends on \( a \) and \( n \). Therefore, for such an \( a \), the map \( I_a \) is surjective from \( \mathbb{P} \) onto \( \mathbb{N} \).

And the surjectivity of the map \( D_a \) is also proved by many authors. They proved that, except for at most finitely many \( n \)'s, the map \( D_a \) is surjective from \( \mathbb{P} \) onto \( \mathbb{N} \).

Thus these two maps are surjective for those \( a \)'s, but between their surjective-properties we notice a big difference. Under GRH, for any \( n \in \mathbb{N} \), \((1.3)\) means that

\[
I_a^{-1}(n) = \{p \in \mathbb{P} \mid I_a(p) = n\}
\]  

(1.6)

contains infinite elements, but on the contrary, the set

\[
D_a^{-1}(n) = \{p \in \mathbb{P} \mid D_a(p) = n\}
\]  

(1.7)

contains only a finite number of elements. In fact, if \( D_a(p) = n \), then

\[ n + 1 \leq p \leq a^n. \]

And recent study on cryptography shows that characterizing \( D_a \) is very difficult.

For the purpose of considering the distribution property of the map \( D_a \), here we take an arbitrary natural number \( k \geq 2 \) and an arbitrary residual class \( l \pmod{k} \) and consider the asymptotic behavior of the cardinality of the following set:

\[
Q_a(x; k, l) := \{p \leq x \mid D_a(p) \equiv l \pmod{k}\}.
\]  

(1.8)

It is more than 50 years ago, W. Sierpinski first considered this problem and H. Hasse proved, by our notations, that, for an odd prime \( q \),

\[
\text{the Dirichlet density of } Q_a(x; q, 0) = \frac{q}{q^2 - 1}
\]  

(1 and 2). Odoni \[8\] proved the existence of the natural density of \( Q_a(x; q, 0) \), and he obtained a similar results on \( Q_a(x; k, 0) \) for a composite square free moduli \( k \).

In this paper we take \( k = 4 \) and consider the distribution property of \( Q_a(x; 4, l) \) for all residue classes \( l = 0, 1, 2, 3 \). Our results consist of two theorems:
Theorem 1.1 We assume $a$ is a square free positive integer with $a \geq 3$. When $l = 0, 2$, we have
\[ \# Q_a(x; 4, l) = \frac{1}{3} \text{Li} x + O \left( \frac{x}{\log x \log \log x} \right). \]

Theorem 1.2 Let $a$ be as above. We assume GRH and further assume $a \equiv 1 \pmod{4}$. Then, for $l = 1, 3$, we have
\[ \# Q_a(x; 4, l) = \frac{1}{6} \text{Li} x + O \left( \frac{x}{\log x \log \log x} \right). \]

Here GRH means:

Hypothesis 1.3 (Generalized Riemann Hypothesis) For any positive integers $k$ and $m$ with $k \mid m$, we assume that the Riemann Hypothesis holds for the Dedekind zeta function $\zeta_K(s)$ for the field $K = \mathbb{Q}(\zeta_m, a^{1/k})$ where $\zeta_m = \exp(2\pi i/m)$.

As we mentioned above, Hasse and Odoni investigated $\# Q_a(x; q, l)$ with $l = 0$, and for $l \neq 0$, the distribution property of $\# Q_a(x; q, l)$ remains unknown so far.

When $l = 0$, the condition "$D_a(p) \equiv 0 \pmod{q}$" can be reformulated in the notation of algebraic number theory without much difficulty. In fact, for a prime $p$ with $q \mid |p - 1|$, using the relation (1.2),
\[ D_a(p) \equiv 0 \pmod{q} \iff q \nmid I_a(p), \]
and we can count the number of such primes $p$’s by
\[ \# \{p \leq x ; q \mid |p - 1| \} - \# \{p \leq x ; q \mid |p - 1|, q \nmid I_a(p)\}. \]

The last condition $q \mid I_a(p)$ means that $a$ is a $q$-th power residue modulo $p$, so we can utilize the prime ideal theorem etc.

On the contrary, when $l \neq 0$, the reformulation of "$D_a(p) \equiv l \pmod{q}$" needs a rather complicated procedure (see Lemma 3.1 and compare (i) and (iii) ). Moreover, e.g. the calculation of each $\# N_a(x; 2f + l \cdot 2f^2 + 1 + 2f \pmod{2f^2+2})$ which appears in (3.4) requires a consideration on the generalized Artin’s conjecture $N_a(x; 2f + l \cdot 2f^2 + 2)$ in the special residue class $p \equiv 1 + 2f \pmod{2f^2+2}$. That is why we need GRH in Theorem 1.2. It seems an interesting phenomenon that, after all, we arrived at such a simple result as Theorem 1.2.

And our result as well as Odoni’s result shows that the value distribution of the map $D_a$ is rather irregular.

We prepare some preliminary lemmas in Section 2, prove Theorem 1.1 in Section 3, and prove the conditional result Theorem 1.2 in Section 4. In Section 5, we mention some numerical examples. For our results, see also [9].

2 Preliminaries

Throughout this paper we fix a square free integer $a \geq 3$, and $p$ denotes an odd prime which does not divide $a$. For $k \in \mathbb{Z}$, let $\zeta_k = \exp(2\pi i/k)$. We denote Euler’s totient and the Möbius function by $\varphi(k)$ and $\mu(k)$, respectively. For a prime power $q^e$, $q^e \mid m$ means that $q^e \mid m$ and $q^{e+1} \nmid m$. 
Let $K$ be an algebraic number field. Then we define
\[
\pi(x, K) = \sharp \{ p : \text{a prime ideal in } K, \ Np \leq x \} \tag{2.1}
\]
and
\[
\pi^{(1)}(x, K) = \sharp \{ p : \text{a prime ideal of degree 1 in } K, \ Np \leq x \} \tag{2.2}
\]
where $Np$ is the (absolute) norm of $p$. Moreover let $L/K$ be a finite Galois extension. Then for a prime ideal $p$ in $K$, we define the Frobenius symbol by
\[
(p, L/K) = \left\{ \sigma \in \text{Gal}(L/K) ; \ q^\sigma = q \text{ for some prime } q \text{ in } L \text{ above } p, \ \alpha^\sigma \equiv \alpha^{Np} \pmod{q} \text{ for all } \alpha \in L \right\}. \tag{2.3}
\]
This notation is due to Lenstra \[5\].

Next we introduce some preliminary results. In the course of our proof, we need the exact value of the extension degree of a certain type of Kummer fields. In the following lemma only, we include the case $a$ is not a square free integer.

**Lemma 2.1** Let $k, r \in \mathbb{N}$ with $k \mid r$. We assume $a$ is not a perfect $h$-th power with $h \geq 2$. And $a_1$ being the square free part of $a$ (i.e. $a = a_1 a_2^2$ with $a_1$: square free), we put
\[
h_1 = \left\{ \begin{array}{ll}
2a_1, & \text{if } a_1 \equiv 1 \pmod{4}, \\
4a_1, & \text{otherwise}.
\end{array} \right.
\]
Then we have
\[
[\mathbb{Q}(\zeta_r, a^{1/k}) : \mathbb{Q}] = \left\{ \begin{array}{ll}
k \varphi(r), & \text{if } k \text{ is odd}, \\
k \varphi(r), & \text{if } k \text{ is even and } h_1 \nmid r, \\
k \varphi(r)/2, & \text{if } k \text{ is even and } h_1 \mid r.
\end{array} \right. \tag{2.4}
\]
**Proof.** See Moree \[6, \text{Lemma 2}\] or Murata \[7, \text{Section 3}\].

**Theorem 2.2** For a prime $q$ and $i, j \in \mathbb{N} \cup \{0\}$, we define an extension field
\[
K_{i,j}^{(q)} = \mathbb{Q}(\zeta_{q^i}, \zeta_{q^j}, a^{1/q^i}),
\]
and we put
\[
n = [K_{i,j}^{(q)} : \mathbb{Q}],
\]
\[
D = \text{the discriminant of } K_{i,j}^{(q)}.
\]
Then, under the condition
\[
x \geq \exp(10n \log^2 |D|),
\]
we have
\[
\pi^{(1)}(x, K_{i,j}^{(q)}) = \text{li } x + O(nx e^{-c \sqrt{\log x/n^2}}),
\]
where the constant implied by $O$-symbol and the positive constant $c$ depend only on $a$ and $q$. 

\[\]
Proof. For the field $K_{i,j}$, we have an estimate
\[ |D| \leq (n^2|a|)^n. \]
Then Theorems 1.3 and 1.4 of Lagarias-Odlyzko [4] give the desired formula.

And we need the Chebotarev density theorem with GRH:

**Theorem 2.3 (Chebotarev density theorem, GRH)** Let $K$ be an algebraic number field, $L/K$ be a finite Galois extension and $C$ be a conjugacy class in $G = \text{Gal}(L/K)$. We define $\pi(x; L/K, C)$ by
\[ \pi(x; L/K, C) = \# \{ p : \text{a prime ideal in } K, \text{unramified in } L, (p, L/K) = C, Np \leq x \}. \quad (2.5) \]
Then, under GRH for the field $L$, we have
\[ \pi(x; L/K, C) = \frac{n}{\#G} \# C \cdot \log n \quad (2.6) \]
where $d_L$ is the discriminant of $L$ and $n_L = [L : \mathbb{Q}]$.

**Proof.** Lagarias-Odlyzko [4, Theorem 1.1].

Here we recall the set of primes $N_a(x; n)$ which we defined in (1.4).

**Lemma 2.4** We assume GRH. Let $a$ be a square free integer $\geq 2$, $\psi(x)$ be a monotone increasing positive function which satisfies
\[ \lim_{x \to \infty} \psi(x) = +\infty \quad \text{and} \quad \psi(x) \ll (\log x)^{\frac{3}{4}}. \]
Then we have
\[ \# \{ p \leq x ; I_a(p) \geq \psi(x) \} \ll \frac{\pi(x)}{\psi(x)}, \]
where the constant implied by $\ll$-symbol is absolute.

**Proof.** Let $y$ be the largest integer not exceeding $\psi(x)$. We have
\[ \{ p \leq x ; I_a(p) \geq \psi(x) \} = \{ p \leq x ; p \nmid a \} - \bigcup_{n=1}^{y-1} N_a(x; n), \quad (2.7) \]
where $\bigcup_{n=1}^{y-1}$ is a disjoint union. Then Theorems 1 and 2 of Murata [7] prove, with $\varepsilon = 1$,
\[ \# N_a(x; n) = C_a^{(n)} \text{li} x + O \left( \left\{ n \log \log x + \log a \right\} \frac{x}{\log^2 x} \right) \quad (2.8) \]
and
\[ \sum_{n \leq y} C_a^{(n)} = 1 + O \left( \frac{1}{y} \right). \quad (2.9) \]
Thus from (2.7), we have
\[ \# \{ p \leq x ; I_a(p) \geq \psi(x) \} = \pi(x) - \left( 1 + O \left( \frac{1}{y} \right) \right) \text{li} x + O \left( \frac{x \log \log x}{\log^2 x} \cdot \sum_{n \leq y} n \right) \]
\[ = O \left( \frac{1}{y} \pi(x) \right) + O \left( \frac{x \log \log x}{(\log x)^{3/2}} \right) = O \left( \frac{\pi(x)}{\psi(x)} \right). \]
3 Proof of Theorem 1.1

Generally speaking, the condition “$D_a(p) \equiv j \pmod{4}$” is rather difficult to handle. So, using the relation (1.2), we transform the condition on $D_a(p)$ into some conditions on $I_a(p)$. Here we introduce the set

$$N_a(x; n; s \pmod{t}) := \{ p \leq x ; \ p \in N_a(x; n), \ p \equiv s \pmod{t} \}, \quad (3.1)$$

which is a generalization of the set $N_a(x; n)$ which appeared in Artin’s conjecture for primitive roots.

We can prove the following lemma, which is the starting point of our proof:

**Lemma 3.1** For any $x > 0$ we have

(i)

$$\#Q_a(x; 4, 0) = \#\{ p \leq x ; \ p \equiv 1 \pmod{4} \}$$

$$- \sum_{j \geq 1} \#\{ p \leq x ; \ p \equiv 1 \pmod{2^{j+1}}, \ 2^j I_a(p) \}$$

$$+ \sum_{j \geq 1} \#\{ p \leq x ; \ p \equiv 1 \pmod{2^{j+2}}, \ 2^j I_a(p) \}, \quad (3.2)$$

(ii)

$$\#Q_a(x; 4, 2) = \sum_{j \geq 1} \#\{ p \leq x ; \ p \equiv 1 \pmod{2^j}, \ 2^{j-1} I_a(p) \}$$

$$- \sum_{j \geq 1} \#\{ p \leq x ; \ p \equiv 1 \pmod{2^{j+1}}, \ 2^{j-1} I_a(p) \}$$

$$- \sum_{j \geq 1} \#\{ p \leq x ; \ p \equiv 1 \pmod{2^j}, \ 2^j I_a(p) \}$$

$$+ \sum_{j \geq 1} \#\{ p \leq x ; \ p \equiv 1 \pmod{2^{j+1}}, \ 2^j I_a(p) \}, \quad (3.3)$$

(iii)

$$\#Q_a(x; 4, 1) = \sum_{f \geq 1} \sum_{l \geq 0} \#N_a(x; 2^f + l \cdot 2^{f+2}; 1 + 2^f \pmod{2^{f+2}})$$

$$+ \sum_{f \geq 1} \sum_{l \geq 0} \#N_a(x; 3 \cdot 2^f + l \cdot 2^{f+2}; 1 + 3 \cdot 2^f \pmod{2^{f+2}}), \quad (3.4)$$

(iv)

$$\#Q_a(x; 4, 3) = \sum_{f \geq 1} \sum_{l \geq 0} \#N_a(x; 3 \cdot 2^f + l \cdot 2^{f+2}; 1 + 2^f \pmod{2^{f+2}})$$

$$+ \sum_{f \geq 1} \sum_{l \geq 0} \#N_a(x; 2^f + l \cdot 2^{f+2}; 1 + 3 \cdot 2^f \pmod{2^{f+2}}). \quad (3.5)$$
**Proof.** We can prove these formulas in a similar manner. Here we show only the proof of (i).

From the condition

\[ D_a(p) \equiv 0 \pmod{4}, \]

it is necessary that \( p \equiv 1 \pmod{4} \). So we consider a prime such that \( 2^j || p - 1, \ j \geq 2 \). Then, with the relation (1.2), we have

\[ D_a(p) \equiv 0 \pmod{4} \iff 2^{j-1} \nmid I_a(p). \]

Hence we have

\[ Q_a(x; 4, 0) = \bigcup_{j \geq 2} \{ p \leq x ; 2^j || p - 1, \ 2^{j-1} \nmid I_a(p) \} \]

\[ = \bigcup_{j \geq 2} \left( \{ p \leq x ; p \equiv 1 \pmod{2^j}, \ 2^{j-1} \nmid I_a(p) \} \right. \]

\[ - \left. \{ p \leq x ; p \equiv 1 \pmod{2^{j+1}}, \ 2^{j-1} \nmid I_a(p) \} \right) \]

\[ = \{ p \leq x ; p \equiv 1 \pmod{4} \} \]

\[ - \bigcup_{j \geq 2} \{ p \leq x ; p \equiv 1 \pmod{2^j}, \ 2^{j-1} | I_a(p) \} \]

\[ + \bigcup_{j \geq 2} \{ p \leq x ; p \equiv 1 \pmod{2^{j+1}}, \ 2^{j-1} | I_a(p) \}, \]

which gives (3.2). \( \blacksquare \)

It seems that our result on \( Q_a(x; 4, 0) \) cannot be derived from Odoni's result, because 4 is not square free. So we describe our proof briefly here.

**Proof of Theorem 1.1.**

The first term of the right hand side of (3.2) is calculated by the Siegel-Walfisz theorem. As to the other terms, we take

\[ \eta_1 = \log \log x \quad \text{and} \quad \eta_2 = \sqrt{x \log x}. \]

Then

\[ \sum_{j \geq 2} \#\{ p \leq x ; p \equiv 1 \pmod{2^j}, \ 2^{j-1} | I_a(p) \} \]

\[ = \left( \sum_{2^j \leq \eta_1} + \sum_{\eta_1 < 2^j \leq \eta_2} + \sum_{\eta_2 < 2^j \leq x} \right) \#\{ p \leq x ; p \equiv 1 \pmod{2^j}, \ 2^{j-1} | I_a(p) \} \]

\[ = I_1 + I_2 + I_3, \text{ say.} \]

By the Siegel-Walfisz Theorem again, we have

\[ I_2 \ll \frac{x}{\log x \log \log x}, \]
and in a similar way to Hooley [3], we have

\[ I_3 \ll \frac{x}{\log^3 x}. \]

For a prime \( p \equiv 1 \pmod{2^i} \) and \( 2^j | I_a(p) \) if and only if \( p \) splits completely in the field \( K_{i,j}^{(2)} \). Thus

\[ I_1 = \sum_{2^j \leq \eta_1} \frac{1}{[K_{j,j-1}^{(2)} : \mathbb{Q}]} \pi^{(1)}(x; K_{j,j-1}^{(2)}), \]

and Theorem 2.2 gives

\[ I_1 = \sum_{j=2}^{\infty} \frac{1}{n_j} \text{li} x - \sum_{2^j > \eta_1} \frac{1}{n_j} \text{li} x + O \left( \sum_{2^j \leq \eta_1} x e^{-c \sqrt{\log x/n_j}} \right), \]

where \( n_j = [K_{j,j-1}^{(2)} : \mathbb{Q}] \). When \( 2^j \leq \eta_1 \),

\[ n_j^2 \ll 2^{4j} \ll (\log \log x)^4, \]

and we have

\[ \sum_{2^j > \eta_1} \frac{1}{n_j} \text{li} x \ll \frac{x}{\log x (\log \log x)^2}, \]

\[ \sum_{2^j \leq \eta_1} x e^{-c \sqrt{\log x/n_j}} \ll \frac{x}{\log^2 x}. \]

Consequently, (3.2) turn into

\[ \#Q_a(x; 4, 0) = \left\{ \frac{1}{\varphi(4)} - \sum_{j \geq 1} \left( \frac{1}{[K_{j+1,j}^{(2)} : \mathbb{Q}]} - \frac{1}{[K_{j+2,j}^{(2)} : \mathbb{Q}]} \right) \right\} \text{li} x \]

\[ + O \left( \frac{x}{\log x \log \log x} \right). \hspace{1cm} (3.6) \]

Now, using Lemma 2.1, we can easily verify that the coefficient of \( \text{li} x \) is equal to 1/3. When \( l = 2 \), we notice that

\[ \#Q_a(x; 4, 2) = \#Q_a(x; 2, 0) - \#Q_a(x; 4, 0). \]

We already have the asymptotic formula for \( \#Q_a(x; 4, 0) \), and from Odoni’s result, we have

\[ \#Q_a(x; 2, 0) = \frac{2}{3} \text{li} x + O \left( \frac{x}{\log x \log \log x} \right). \]

This completes the proof of Theorem 1.1. (Of course we can derive the same result from (3.3) directly.)

Remark. It is clear from the proof described above that the assumptions \( a \) is square free and \( a \neq 2 \) are not essential. In fact, if we spare no effort to calculate the degrees in (3.3) (and in the corresponding formula for \( \#Q_a(x; 2, 0) \)), we can find the densities \( \#Q_a(x; 4, 0) \) and \( \#Q_a(x; 4, 2) \) for other types of \( a \)’s.
4 Proof of Theorem 1.2

We shall prove here Theorem 1.2. Our proof consists of two parts:

**Part I.** In Proposition 4.5, under the assumptions of Theorem 1.2, we prove the existence of the natural densities

\[ \delta_j = \lim_{x \to \infty} \frac{\#Q_a(x; 4, j)}{\pi(x)} \]

for \( j = 1, 3 \).

**Part II.** We prove in Proposition 4.7 that

\[ \delta_1 = \delta_3. \] (4.1)

Then our unconditional result Theorem 1.1 shows that

\[ \delta_1 + \delta_3 = 1 - (\delta_0 + \delta_2) = \frac{1}{3}, \]

and (4.1) proves Theorem 1.2.

**Part I.**

We start our proof from formulas (3.4) and (3.5).

1° A reduction of the double-infinite-sum in (3.4).

In order to simplify the double-infinite-sum in (3.4), we apply Lemma 2.4. We take \( \psi(x) = \log \log x \), then

\[ \sum_{2^f + l \cdot 2^f \leq \log \log x} \#N_a(x; 2^f + l \cdot 2^{f+2}; 1 + 2^f \pmod{2^{f+2}}) \leq \{ p \leq x ; I_a(p) \geq \log \log x \} \]

\[ \ll \pi(x) (\log \log x)^{-1}. \]

So we have

\[ \#Q_a(x; 4, 1) = \sum_{f \geq 1, l \geq 0 \atop 2^f + l \cdot 2^{f+2} \leq \log \log x} \#N_a(x; 2^f + l \cdot 2^{f+2}; 1 + 2^f \pmod{2^{f+2}}) \]

\[ + \sum_{f \geq 1, l \geq 0 \atop 2^f + l \cdot 2^{f+2} \leq \log \log x} \#N_a(x; 3 \cdot 2^f + l \cdot 2^{f+2}; 1 + 3 \cdot 2^f \pmod{2^{f+2}}) \]

\[ + O \left( \pi(x) \frac{1}{\log \log x} \right). \] (4.2)

This formula shows that, in our proof, the calculation of \( \#N_a(x; 2^f + l \cdot 2^{f+2}; 1 + 2^f \pmod{2^{f+2}}) \) is very important. In the literature, \( \#N_a(x; n) \) is already calculated by Lenstra [5] and Murata [7] for arbitrary \( n \in \mathbb{N} \). And \( \{ p \leq x ; I_a(p) = 1, p \equiv s \pmod{t} \} \) is already considered in Lenstra [5] (see also Moree [6]). So, in what follows, we combine Lenstra’s idea about the control of residue classes and Murata’s method of obtaining an asymptotic formula of \( \#N_a(x; n) \).

2° A decomposition of \( \#N_a(x; 2^f + l \cdot 2^{f+2}; 1 + 2^f \pmod{2^{f+2}}) \).
In the formulas (3.4) and (3.5), we find four terms of the same type: 
\[ \#N_a(x; j \cdot 2^f + l \cdot 2^{f+2}; 1 + j' \cdot 2^f \pmod{2^{f+2}}) \]
for \((j, j') = (1, 1), (1, 3), (3, 1)\) and \((3, 3)\). We can calculate these terms in the same way, and in \(2^0 - 5^0\) we think about only the case \((j, j') = (1, 1)\).

Here we need some new notations. The letter \(q\) always means a prime number, and we put
\[ k = 2^f + l \cdot 2^{f+2}, \]
\[ k_0 = \prod_{q | k} q \] (i.e. the core of \(k\)).

We define the algebraic number field
\[ K_k = \mathbb{Q}(\zeta_{k_0}, a^{1/k}) \]
and define two sets of prime ideals of \(K_k\):
\[ B(x; K_k; a^{1/k}; N) = \left\{ \begin{array}{l} \text{a prime ideal in } K_k, \ Np = p^1 \leq x, p \equiv 1 \pmod{N} \\ \text{a}^{1/k} \text{ is a primitive root mod } p \end{array} \right\} \]
and
\[ B(x; K_k; a^{1/k}; N; s \pmod{t}) = \{ p \in B(x; K_k; a^{1/k}; N) \ ; \ p \equiv s \pmod{t} \}. \]

First we decompose \( \#N_a(x; k; 1 + 2^f \pmod{2^{f+2}}) \) into the sum of some \( \#B(x; K_k; a^{1/k}; N; s \pmod{t}) \)'s and here we use Murata’s method \([7]\).

As a consequence of \([7, Lemma 3]\), we have, if \( p \in B(x; K_k; a^{1/k}; N; s \pmod{t}) \), then \( Np = p \in N_a(x; k; s \pmod{t}) \). And the same argument with \([7, Lemma 4]\), we get easily that, if \( p \in N_a(x; k; s \pmod{t}) \), then \( p \) gives rise to \([K_k : \mathbb{Q}] \varphi(\tilde{p})/\tilde{p} \) elements of \( B(x; K_k; a^{1/k}; k; s \pmod{t}) \), where \( \tilde{p} \) is defined in \([7, p. 559] \). In fact, we can prove these results only by limiting the proofs of \([7]\) into the residue class \( s \pmod{t} \).

From these relations, making use of the Möbius inversion formula, we can deduce the following decomposition:

**Proposition 4.1**

\[ \#N_a(x; k; 1 + 2^f \pmod{2^{f+2}}) = \frac{1}{[K_k : \mathbb{Q}] \varphi(k_0)} \sum_{d | k_0} \frac{\mu(d)}{d} \#B(x; K_k; a^{1/k}; kd; 1 + 2^f \pmod{2^{f+2}}). \]

**3° Calculation of** \( \#B(x; K_k; a^{1/k}; kd; 1 + 2^f \pmod{2^{f+2}}) \). 

*We can simplify* \([7, Proposition 1]\) *into*

\[ |N_a^{(n)}(x)| = \frac{1}{[K_n : \mathbb{Q}] \varphi(n_0)} \sum_{d | n_0} \frac{\mu(d)}{d} |B(\sqrt{a}; K_n; x; nd)|. \]

This simplification is due to Dr. R. Takeuchi.
The cardinality \( \mathcal{B}(x; K_k; a^{1/k}; m) \) is already calculated in [7, Proposition 2], and this calculation is carried out along Hooley’s work [3]. In our present case, we have to take into account the condition “\( p \equiv 1 + 2f (\text{mod } 2^{f+2}) \)”, but the calculation itself needs only some slight modifications. We omit the detail and present our result in Proposition 4.2.

We define

\[
P(x; K_k; a^{1/k}; kd; n) = \left\{ \mathfrak{p} : \text{a prime ideal in } K_k, N\mathfrak{p} = p^l \leq x, p \equiv 1 (\text{mod } kd), \right. \]
\[
\text{the equation } x^q \equiv a^{1/k} (\text{mod } p) \text{ is solvable in } O_{K_k} \text{ for any } q|n \left. \right\},
\]

where \( O_{K_k} \) is the ring of integers of the field \( K_k \), and

\[
P(x; K_k; a^{1/k}; kd; s (\text{mod } t); n) = \{ \mathfrak{p} \in P(x; K_k; a^{1/k}; kd; n) : p \equiv s (\text{mod } t) \}.
\]

Then we have

**Proposition 4.2**

\[
\mathcal{B}(x; K_k; a^{1/k}; kd; 1 + 2f (\text{mod } 2^{f+2}))
\]
\[
= \sum_n' \mu(n) \mathcal{B}(x; K_k; a^{1/k}; kd; 1 + 2f (\text{mod } 2^{f+2}); n) + O \left( \frac{x(\log \log x)^3}{\log^2 x} \right), \quad (4.3)
\]

where the \( \sum_n' \) means the sum over such an \( n \leq x \) which is either 1 or a positive square free integer composed entirely of prime factors not exceeding \( (1/8) \log x \), and the constant implied by the \( O \)-symbol is absolute (In Hooley’s paper [3], he made use of \( (1/6) \log x \) instead of \( (1/8) \log x \)).

4° Calculation of \( \mathcal{B}(x; K_k; a^{1/k}; kd; 1 + 2f (\text{mod } 2^{f+2}); n) \).

Here we need GRH.

We define algebraic extension fields

\[
G_{k,n,d} = K_k(\zeta_n, a^{1/kn}, \zeta_{kd}),
\]
\[
\tilde{G}_{k,n,d} = G_{k,n,d}(\zeta_{2^{f+2}}),
\]

and we take an automorphism \( \sigma \in \text{Aut}(\mathbb{Q}(\zeta_{2^{f+2}})/\mathbb{Q}) \) which is defined by

\[
\sigma : \zeta_{2^{f+2}} \mapsto (\zeta_{2^{f+2}})^{1+2f}.
\]
Let \( \sigma^* \in \text{Aut}(\tilde{G}_{k,n,d}/K_k) \) be the automorphism defined by

\[
\begin{align*}
\sigma^*|_{G_{k,n,d}} &= \text{id}_{G_{k,n,d}}, \\
\sigma^*|_{Q(\zeta_{2f+2})} &= \sigma.
\end{align*}
\] (4.4)

Here we remark that such a \( \sigma^* \) does not always exist. When we can construct this \( \sigma^* \) from \( \sigma \), we can prove the following Lemma 4.3 and Proposition 4.4:

**Lemma 4.3** When \( \sigma^* \) exists, \( \{\sigma^*\} \) is a conjugacy class of \( \text{Aut}(\tilde{G}_{k,n,d}/K_k) \) by itself.

**Proposition 4.4** We assume GRH, and \( \sigma^* \in \text{Aut}(\tilde{G}_{k,n,d}/K_k) \) exists. Then

\[
\%P(x; K_k; a^{1/k}; kd; 1 + 2^f \pmod{2^f+2}; n) = \pi(x; \tilde{G}_{k,n,d}/K_k, \{\sigma^*\}) + O(k^2 \sqrt{x} (\log \log x)^4), \] (4.5)

where \( \pi(x; \tilde{G}_{k,n,d}/K_k, \{\sigma^*\}) \) is defined in (2.3).

When the automorphism \( \sigma^* \in \text{Aut}(\tilde{G}_{k,n,d}/K_k) \) does not exist, we regard \( \pi(x; \tilde{G}_{k,n,d}/K_k, \{\sigma^*\}) = 0 \).

**Proof of Lemma 4.3.** We take an arbitrary \( \tau \in \text{Aut}(\tilde{G}_{k,n,d}/K_k) \). Since \( \tau(a^{1/nk}) = a^{1/nk} \zeta_n^i \) for some \( i \in \mathbb{N} \), we have

\[
\sigma^* \circ \tau(a^{1/nk}) = a^{1/nk} \zeta_n^i = \tau \circ \sigma^*(a^{1/nk}).
\]

Similarly we can prove

\[
\sigma^* \circ \tau(\zeta_n) = \tau \circ \sigma^*(\zeta_n) \quad \text{and} \quad \sigma^* \circ \tau(\zeta_{kd}) = \tau \circ \sigma^*(\zeta_{kd}).
\]

Moreover, since \( \tau(\zeta_{2f+2}) = (\zeta_{2f+2})^{i'} \) for some \( i' \in \mathbb{N} \), we have

\[
\sigma^* \circ \tau(\zeta_{2f+2}) = (\zeta_{2f+2})^{i'(1+2^f)} = \tau \circ \sigma^*(\zeta_{2f+2}).
\]
These prove our assertion.

**Proof of Proposition 4.4.** We notice that the next two conditions are equivalent:

(a) \( p \in P(x; K_k; a^{1/k}; kd; n), \)
(b) \( p \) splits completely in the extension \( G_{k,n,d}/K_k \) and \( Np \leq x, \)

and also the following two are equivalent:

(c) \( p \equiv 1 + 2f \pmod{2^{f+2}}, \)
(d) the Frobenius map \( (p, Q(\zeta_{2f+2})/Q) = \sigma. \)

Now let \( p \in P(x; K_k; a^{1/k}; kd; 1 + 2f \pmod{2^{f+2}}; n), \) \( G_{k,n,d}/K_k \) is a normal extension, and we can define \([p, \tilde{G}_{k,n,d}/K_k],\) the conjugacy class of Frobenius automorphisms corresponding to prime ideals \( \mathfrak{p} \subset G_{k,n,d} \) over \( p. \) We take \( \tau \in [p, \tilde{G}_{k,n,d}/K_k]. \) Then the equivalent relation \( (a) \iff (b) \) implies that the ideal \( p \) splits completely in the extension \( \tilde{G}_{k,n,d}/K_k, \) thus \( \tau|_{G_{k,n,d}} = \text{id}_{G_{k,n,d}}. \) Also the equivalent relation \( (c) \iff (d) \) implies \( \tau|_{Q(\zeta_{2f+2})} = \sigma. \) Consequently \( \tau = \sigma^* \) and, with Lemma 4.3, \([p, \tilde{G}_{k,n,d}/K_k] = \{\sigma^*\}. \) This proves

\[
P(x; K_k; a^{1/k}; kd; 1 + 2f \pmod{2^{f+2}}; n)
\subset \{p: \text{a prime ideal in } K_k, [p, \tilde{G}_{k,n,d}/K_k] = \{\sigma^*\}, Np \leq x\}.
\]

Now we consider the prime ideal \( p \) of \( K_k \) which satisfies two conditions:

\[
[p, \tilde{G}_{k,n,d}/K_k] = \{\sigma^*\} \quad \text{and} \quad Np \leq x.
\]

It is easy to see that

\[
\# \left\{ p: \text{a prime ideal of } K_k, p \text{ satisfies the conditions (4.13),} \right. \\
\left. \quad Np = p^s \text{ with } s \geq 2 \right\} 
\ll \left[ K_k : Q \right] \cdot O \left( \sum_{i=2}^{\infty} x^{1/i} \right) 
\ll k^2 \sqrt{x} (\log \log x)^4.
\]

This means that, except for at most \( O(k^2 \sqrt{x} (\log \log x)^4) \) of primes, we can assume \( Np = p \leq x. \) Then the property \( \sigma^*|_{G_{k,n,d}} = \text{id}_{G_{k,n,d}} \) implies that \( p \) satisfies the condition (b), and \( p \in P(x; K_k; a^{1/k}; kd; n). \) Furthermore, the property \( \sigma^*|_{Q(\zeta_{2f+2})} = \sigma \) implies (d). These show that \( p \in P(x; K_k; a^{1/k}; kd; 1 + 2f \pmod{2^{f+2}}; n), \) and we proved our assertion.

**5° Existence of the densities \( \delta_1 \) and \( \delta_3. \)**

We can now prove the main result of Part I.

**Proposition 4.5** We assume GRH. Then for \( j = 1, 3, \) \( \#Q_a(x; 4,j) \) has the natural density \( \delta_j. \)

Before we prove this proposition, we prepare some estimates:
Lemma 4.6 Under the above notations, let $d_{\tilde{G}_{k,n,d}}$ be the discriminant of the field $\tilde{G}_{k,n,d}$. Then

(i) 

$$[\tilde{G}_{k,n,d} : K_k] = \delta \frac{d}{k_0 \varphi((n,k_0))} \cdot kn \varphi(n),$$

where $\delta$ is one of the five numbers $\{8, 4, 2, 1, 1/2\}$.

(ii) 

$$\log |d_{\tilde{G}_{k,n,d}}| \ll (nkd)^3 \log(nkd).$$

Proof. It is already proved in Murata [7] that $[K_k : Q] = \eta_1 k \varphi(k_0)$, where $\eta_1 = 1$ or $1/2$, and that (cf. formulas (11) and (12) of [7])

$$[G_{k,n,d} : Q] = \eta_2 \frac{\varphi(k_0)k^2d}{k_0} \frac{n \varphi(n)}{\varphi((n,k_0))},$$

where $\eta_2 = 1$ or $1/2$. Moreover, since $\tilde{G}_{k,n,d} = G_{k,n,d}(\zeta^{2f+2})$ and $\zeta^{2f} \in G_{k,n,d}$,

$$[\tilde{G}_{k,n,d} : G_{k,n,d}] \big| 4.$$ 

Combining these formulas, we get (i) easily.

We now prove (ii). Let $L_1/Q$ and $L_2/Q$ be two extension fields, $L$ be the composite field $L_1 \cdot L_2$, and $d_{L_1}$, $d_{L_2}$, $d_L$ be the discriminants of $L_1$, $L_2$, $L$, respectively. Then we have the following relation:

$$|d_L| \big| |d_{L_1}|^{[L_1:L]} |d_{L_2}|^{[L_2:L]}.$$ 

From this, we have an estimate

$$|d_L| \leq |d_{L_1}|^{[L_2:Q]} |d_{L_2}|^{[L_1:Q]}.$$ 

(4.7)

Here we take

$$L_1 = Q(a^{1/nk}),$$

$$L_2 = Q(\zeta_n, \zeta_{kd}, \zeta^{2f+2}).$$

It is known that the discriminant of the cyclotomic field $Q(\zeta_{p^r})$ is given by

$$|d_{Q(\zeta_{p^r})}| = p^{r-1}(p^{r-1}p^{r-1}).$$

From this, it is easy to prove that, for any $m \in \mathbb{N}$,

$$\log |d_{Q(\zeta_m)}| \leq m^2 \log m.$$

Thus we have firstly,

$$\log |d_{L_2}| \leq \left(4nk^d\right)^2 \log(4nk^d).$$ 

(4.8)
We also have
\[ |d_{L_1}| \leq |\text{the discriminant of the polynomial } X^{nk} - a| \]
\[ = \prod_{0<i<j<nk} a^{2/nk}(\zeta_{nk}^i - \zeta_{nk}^j)^2 \]
\[ \leq n^k |d_{Q(\zeta_{nk})}|, \]
and then
\[ \log |d_{L_1}| \leq nk \log a + (nk)^2 \log(nk). \] (4.9)

Since \([L_1 : Q] \leq nk\) and \([L_2 : Q] \leq 4nkd\), we now prove (ii) from (4.7), (4.8) and (4.9).

**Corollary 4.7** We assume GRH. The numbers \(k, n, d\) are as above, and let \(k \leq \log \log x\). Then, for any \(\varepsilon > 0\), we have
\[ \pi(x; \tilde{G}_{k,n,d}/K_k, \{\sigma^*\}) = \frac{1}{[\tilde{G}_{k,n,d} : K_k]} \text{li} x + O\left(x^{3/4 + 2\varepsilon}(\log x)^2\right), \] (4.10)
where the constant involved by the \(O\)-symbol depends only on \(\varepsilon\).

**Proof.** Lemma 4.3 says that, when \(\sigma^*\) exists, we can take the conjugacy class \(C = \{\sigma^*\}\) and \(\#C = 1\). Let us apply Theorem 2.3 for \(L = \tilde{G}_{k,n,d}, K = K_k, C = \{\sigma^*\}\), then, in order to prove Corollary 4.7 it is now sufficient to estimate the next two terms:
\[ \sqrt{x} \left[ \frac{[\tilde{G}_{k,n,d} : K_k]}{[\tilde{G}_{k,n,d} : K_k]} \log d_{\tilde{G}_{k,n,d}} \right] \text{ and } \log d_{\tilde{G}_{k,n,d}}. \]

Here we recall that \(n\) is a square free integer composed entirely of prime factors not exceeding \((1/8) \log x\). Then we can estimate
\[ \log n \leq \sum_{p \leq x^{1/8}} \log p \ll \left(\frac{1}{8} + \varepsilon\right) \log x, \]
and we have
\[ n \ll x^{3/8 + \varepsilon} \]
for any \(\varepsilon > 0\). Then by Lemma 4.7 (i) and (ii),
\[ \sqrt{x} \left[ \frac{[\tilde{G}_{k,n,d} : K_k]}{[\tilde{G}_{k,n,d} : K_k]} \log d_{\tilde{G}_{k,n,d}} \right] \ll \sqrt{x} k_0 (nk)^2 \log(nkd) \]
\[ \ll x^{3/4 + 2\varepsilon} \log x (\log \log x)^5, \]
\[ \log d_{\tilde{G}_{k,n,d}} \ll x^{3/8 + 3\varepsilon} \log x (\log \log x)^6, \]
and these prove (4.10).
Proof of Proposition 4.5. We define here the number $c(n)$ by

$$c(n) = \begin{cases} 1, & \text{if the Frobenius map } \sigma^* \text{ defined by (4.4) exists}, \\ 0, & \text{if not}. \end{cases}$$

Combining (4.3), (4.5) and Corollary 4.7, we have, provided $k \leq \log \log x$,

$$\sharp B(x; K_k; a^{1/k}; kd; 1 + 2^f \pmod{2^{f+2}})$$

$$= \sum_{n} \mu(n)c(n) \left( \pi(x; \tilde{G}_{k,n,d}/K_k, \{\sigma^*\}) + O(\sqrt{x}(\log \log x)^6) \right) + O \left( \frac{x(\log \log x)^3}{\log^2 x} \right)$$

$$= \sum_{n} \frac{\mu(n)c(n)}{[G_{k,n,d}:K_k]} \text{li} x + O \left( \frac{x^{\frac{7}{2}} + 3\epsilon(\log x)^2}{\log^2 x} \right) + O \left( \frac{x(\log \log x)^3}{\log^2 x} \right).$$

Now, from Lemma 4.6 (i), it is seen that the leading coefficient is an absolutely convergent series. Making use of an estimate

$$\sum_{n} \mu(n)c(n) \left[ \tilde{G}_{k,n,d}:K_k \right] \text{li} x + O \left( \frac{1}{\log x} \right),$$

we obtain a formula

$$\sharp B(x; K_k; a^{1/k}; kd; 1 + 2^f \pmod{2^{f+2}}) = \tilde{\delta}_{k,d} \text{li} x + O \left( \frac{x(\log \log x)^3}{\log^2 x} \right),$$

where

$$\tilde{\delta}_{k,d} = \sum_{n=1}^{\infty} \frac{\mu(n)c(n)}{[G_{k,n,d}:K_k]}.$$

Then Proposition 4 gives

$$\sharp N_a(x; k; 1 + 2^f \pmod{2^{f+2}}) = \left( \frac{1}{[K_k:Q]} \varphi(k_0) \sum_{d|k_0} \frac{\mu(d)}{d} \tilde{\delta}_{k,d} \right) \text{li} x$$

$$+ O \left( \frac{k_0}{k_0 \varphi(k_0)^2} \sum_{d|k_0} \frac{1}{d} \frac{x(\log \log x)^3}{\log^2 x} \right)$$

$$= \tilde{\delta}_k \text{li} x + O \left( \frac{x(\log \log x)^3}{\log^2 x} \right),$$

with

$$\tilde{\delta}_k = \frac{1}{[K_k:Q]} \varphi(k_0) \sum_{d|k_0} \frac{\mu(d)}{d} \tilde{\delta}_{k,d}.$$

Similarly for $m = 3 \cdot 2^f + l \cdot 2^{f+2}$, we have

$$\sharp N_a(x; m; 1 + 3 \cdot 2^f \pmod{2^{f+2}}) = \tilde{\delta}_m \text{li} x + O \left( \frac{x(\log \log x)^3}{\log^2 x} \right).$$
Then, our formula (4.2) yields

\[ \#Q_a(x; 4, 1) = \left( \sum_{k \leq \log \log x} \tilde{\delta}_k \right) \text{li} x \]
\[ + \left( \sum_{m \leq \log \log x} \tilde{\delta}_m \right) \text{li} x + O \left( \frac{x}{\log x \log \log x} \right). \]  
\hspace{1cm} (4.11)

From the definition, \( \tilde{\delta}_k \) and \( \tilde{\delta}_m \) are non-negative numbers, and a priori, \( \sum_{k=1}^{\infty} \tilde{\delta}_k \leq 1 \) and \( \sum_{m=1}^{\infty} \tilde{\delta}_m \leq 1 \). Thus the two leading coefficients which appeared in (4.11) converge, namely,

\[ \sum_{k \leq \log \log x} \tilde{\delta}_k + \sum_{m \leq \log \log x} \tilde{\delta}_m = \delta_1 + o(1). \]

This proves the existence of the density \( \delta_1 \), and similarly, we can show the existence of \( \delta_3 \).

**Part II.**

In this paragraph, we present our proof for \( \delta_1 = \delta_3 \).

Our proof is based on the following expressions (4.13) and (4.14) which we obtained in Part I, but prior to the details, we need a few new notations:

\[ k = k(l, f) = 2^f + l \cdot 2^{f+2}, \]
\[ m = m(l, f) = 3 \cdot 2^f + l \cdot 2^{f+2}, \]
\[ k_0 = \prod_{q | k} q, \quad m_0 = \prod_{q | m} q \quad (\text{the cores of } k \text{ and } m), \]

and accordingly

\[ \tilde{G}_{k,n,d} = K_k(\zeta_n, a^{1/nk}; \zeta_{nd}, \zeta_{2^{f+2}}), \quad (\text{the same as in Part I}) \]
\[ \tilde{G}_{m,n,d} = K_m(\zeta_n, a^{1/nm}; \zeta_{nd}, \zeta_{2^{f+2}}). \]

Furthermore,

\[ c_1(k, n, d) = \begin{cases} 1, & \text{if we can construct } \sigma_1^* \in \text{Aut}(\tilde{G}_{k,n,d}/K_k) \\
0, & \text{if not,} \end{cases} \]
\[ c_3(k, n, d) = \begin{cases} 1, & \text{if we can construct } \sigma_3^* \in \text{Aut}(\tilde{G}_{k,n,d}/K_k) \\
0, & \text{if not,} \end{cases} \]

where

\[ \begin{align*}
\sigma_3^*|_{G_{k,n,d}} &= \text{id}_{G_{k,n,d}}, \\
\sigma_3^*|_{Q(\zeta_{2^{f+2}})} &= \sigma' : \zeta_{2^{f+2}} \mapsto (\zeta_{2^{f+2}})^{1+3 \cdot 2^f}. \end{align*} \]  
\hspace{1cm} (4.12)
\[ \begin{align*}
\delta_1 &= \sum_{f \geq 1} \sum_{l \geq 0} \frac{1}{[K_k : \mathbb{Q}]} \frac{k_0}{\varphi(k_0)} \sum_{d | k_0} \mu(d) \sum_{n} \mu(n)c_1(k, n, d) \left( [G_{k,n,d} : K_k] \right) \\
&\quad + \sum_{f \geq 1} \sum_{l \geq 0} \frac{1}{[K_m : \mathbb{Q}]} \frac{m_0}{\varphi(m_0)} \sum_{d | m_0} \mu(d) \sum_{n} \mu(n)c_3(m, n, d) \left( [G_{m,n,d} : K_m] \right), \\
\delta_3 &= \sum_{f \geq 1} \sum_{l \geq 0} \frac{1}{[K_k : \mathbb{Q}]} \frac{k_0}{\varphi(k_0)} \sum_{d | k_0} \mu(d) \sum_{n} \mu(n)c_3(k, n, d) \left( \bar{G}_{k,n,d} : K_k \right) \\
&\quad + \sum_{f \geq 1} \sum_{l \geq 0} \frac{1}{[K_m : \mathbb{Q}]} \frac{m_0}{\varphi(m_0)} \sum_{d | m_0} \mu(d) \sum_{n} \mu(n)c_1(m, n, d) \left( \bar{G}_{m,n,d} : K_m \right).
\end{align*} \]

(4.13)

We remark here that, when
\[ \begin{align*}
c_1(k, n, d) &= c_3(k, n, d), \\
c_1(m, n, d) &= c_3(m, n, d),
\end{align*} \]

(4.15)

then the first term of the right hand side of (4.13) is equal to the first term of (4.14), and the second terms of (4.13) and (4.14) coincide with each other. This means \( \delta_1 = \delta_3 \).

Now we prove (4.15).

**Proposition 4.8** We assume \( a_1 \equiv 1 \pmod{4} \). Then the relations (4.13) hold.

**Proof.** Here we give the proof only for the first relation.

**Case 1.** If \( f \geq 2 \), then it is easy to see that, when \( c_1(k, n, d) = 1 \), i.e. \( \sigma_1^* \) exists, then \( (\sigma_1^*)^3 \) satisfies (4.12), i.e. \( c_3(k, n, d) = 1 \), and vice versa. Thus \( c_1(k, n, d) = c_3(k, n, d) \).

**Case 2.** If \( d \) is even, then we can prove
\[ c_1(k, n, d) = c_3(k, n, d) = 0. \]

In fact, in this case, \( 2^{f+1} | kd \), and two conditions
\[ \begin{align*}
\sigma_1^*(\zeta_{kd}) &= \zeta_{kd}, \\
\sigma_1^*(\zeta_{2^{f+2}}) &= (\zeta_{2^{f+2}})^{1+2f}
\end{align*} \]

contradict each other. Thus \( c_1(k, n, d) = 0 \), and similarly \( c_3(k, n, d) = 0 \).

**Case 3.** Here we assume \( f = 1 \), \( d \) is odd and furthermore \( a_1 \equiv 1 \pmod{4} \). We calculate the extension degree \([G_{k,n,d} \cap \mathbb{Q}(\zeta_8) : \mathbb{Q}]\).

We have \( G_{k,n,d} = K_k(\zeta_n, \zeta_{kd}, a^{1/kn}) \)
Let \( \langle a, b \rangle \) mean the least common multiple of \( a \) and \( b \). We have

\[
[G_{k,n,d} : \mathbb{Q}] = \begin{cases} 
  nk\varphi(\langle n, kd \rangle), & \text{if } 2a_1 \nmid \langle n, kd \rangle, \\
  \frac{1}{2}nk\varphi(\langle n, kd \rangle), & \text{if } 2a_1 | \langle n, kd \rangle,
\end{cases}
\]

and \([Q(\zeta_8) : \mathbb{Q}] = 4\). Since the two conditions “\( 2a_1 \nmid \langle n, kd \rangle \)” and “\( 2a_1 \nmid 4\langle n, kd \rangle \)” are equivalent, we have

\[
[G_{k,n,d} \cap Q(\zeta_8) : \mathbb{Q}] = \frac{[G_{k,n,d} : \mathbb{Q}] [Q(\zeta_8) : \mathbb{Q}]}{[G_{k,n,d} : \mathbb{Q}]} = 1.
\]

This means we can construct \( \sigma^*_1 \) with the properties (4.4), i.e. \( c_1(k, n, d) = 1 \). Similarly \( c_3(k, n, d) = 1 \). This completes the proof of (4.15).

As we described in the above, now we can conclude that \( \delta_1 = \delta_3 = \frac{1}{6} \).

### 5 Numerical Examples

In this section we look at some numerical calculations of the densities of \( Q_a(x; 4, l) \), including those for \( a \)'s which are not dealt with in the previous sections. We did computer calculations of \( \sharp Q_a(x; 4, l) / \pi(x) \) up to \( x = 10^7 \), where \( \pi(x) \) denotes the number of primes not exceeding \( x \).

The tables below exhibit what the densities of \( Q_a(x; 4, l) \) are like for various square free \( a \)'s. The exact densities of \( Q_a(x; 4, 0) \) and \( Q_a(x; 4, 2) \) can be found unconditionally by Theorem 1.1. When \( a \equiv 1 \pmod{4} \) (the cases \( a = 5 \) and 21 in the tables), the exact densities of \( Q_a(x; 4, 1) \) and \( Q_a(x; 4, 3) \) can be proved to be 1/6 under GRH (Theorem 1.2). On the other hand, when \( a \not\equiv 1 \pmod{4} \), the exact densities of \( Q_a(x; 4, 1) \) and \( Q_a(x; 4, 3) \) are unknown even if we assume GRH. Among such \( a \)'s, in the case \( a = 3 \), both the densities of \( Q_3(x; 4, 1) \) and \( Q_3(x; 4, 3) \) seem very close to 1/6. But the calculation for \( a = 6 \) shows that there really exists a case when the densities of \( Q_a(x; 4, 1) \) and \( Q_a(x; 4, 3) \) seem different values from 1/6. This observation shows that the condition \( a \equiv 1 \pmod{4} \) is not just for technical reasons, but plays an essential role in Theorem 1.2 for determining of the natural densities of \( \sharp Q_a(x; 4, l) \), \( l = 1, 3 \).

| \( x \)   | \( l = 0 \)      | \( l = 1 \)      | \( l = 2 \)      | \( l = 3 \)      |
|----------|-----------------|-----------------|-----------------|-----------------|
| \( 10^3 \) | 0.319277        | 0.156627        | 0.349398        | 0.174699        |
| \( 10^4 \) | 0.327628        | 0.167074        | 0.340668        | 0.164629        |
| \( 10^5 \) | 0.334619        | 0.167049        | 0.333055        | 0.165276        |
| \( 10^6 \) | 0.333227        | 0.167155        | 0.332934        | 0.16684        |
| \( 10^7 \) | 0.333320        | 0.166771        | 0.333099        | 0.166810        |
Table 5.2. The densities of $Q_{21}(x; 4, l)$

| $x$ | $l = 0$ | $l = 1$ | $l = 2$ | $l = 3$ |
|-----|---------|---------|---------|---------|
| $10^3$ | 0.339394 | 0.133333 | 0.339394 | 0.187879 |
| $10^4$ | 0.329527 | 0.160685 | 0.334421 | 0.175367 |
| $10^5$ | 0.333507 | 0.166494 | 0.333194 | 0.166649 |
| $10^6$ | 0.332582 | 0.165972 | 0.334110 | 0.167335 |
| $10^7$ | 0.332836 | 0.166527 | 0.333917 | 0.166720 |

Table 5.3. The densities of $Q_3(x; 4, l)$

| $x$ | $l = 0$ | $l = 1$ | $l = 2$ | $l = 3$ |
|-----|---------|---------|---------|---------|
| $10^3$ | 0.331325 | 0.150602 | 0.331325 | 0.186747 |
| $10^4$ | 0.331703 | 0.163814 | 0.339038 | 0.165444 |
| $10^5$ | 0.334411 | 0.167362 | 0.332325 | 0.165902 |
| $10^6$ | 0.332488 | 0.166607 | 0.333762 | 0.167142 |
| $10^7$ | 0.333298 | 0.166757 | 0.333397 | 0.166548 |

Table 5.4. The densities of $Q_6(x; 4, l)$

| $x$ | $l = 0$ | $l = 1$ | $l = 2$ | $l = 3$ |
|-----|---------|---------|---------|---------|
| $10^3$ | 0.331325 | 0.126506 | 0.325301 | 0.216867 |
| $10^4$ | 0.334963 | 0.133659 | 0.333333 | 0.198044 |
| $10^5$ | 0.333785 | 0.133577 | 0.332847 | 0.199791 |
| $10^6$ | 0.333151 | 0.132249 | 0.333507 | 0.201093 |
| $10^7$ | 0.333331 | 0.132179 | 0.333019 | 0.201471 |

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