ON THE HOMOTOPY CLASSIFICATION OF MAPS

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Abstract. We establish certain conditions which imply that a map \( f : X \to Y \) of topological spaces is null homotopic when the induced integral cohomology homomorphism is trivial; one of them is: \( H^*(X) \) and \( \pi_*(Y) \) have no torsion and \( H^*(Y) \) is polynomial.

1. Introduction

We give certain classification theorems for maps via induced cohomology homomorphism. Such a classification is based on new aspects of obstruction theory to the section problem in a fibration beginning in [4], [5] and developed in some directions in [24], [25]. Given a fibration \( F \to E \xrightarrow{\xi} X \), the obstructions to the section problem of \( \xi \) naturally lay in the groups \( H^{i+1}(X; \pi_i(F)) \), \( i \geq 0 \). A basic method here is to use the Hurewicz homomorphism \( u_i : \pi_i(F) \to H_i(F) \) for passing the above obstructions into the groups \( H^{i+1}(X; H_i(F)) \), \( i \geq 0 \). In particular, this suggests the following condition on a fibration: The induced homomorphism

\[
(1.1)_m \quad u^* : H^{i+1}(X; \pi_i(F)) \to H^{i+1}(X; H_i(F)), \quad 1 \leq i < m
\]

is an inclusion (assuming \( u_1 : \pi_1(F) \to H_1(F) \) is an isomorphism). Note also that the idea of using the Hurewicz map in the obstruction theory goes back to the paper [23]. (Though its main result was erroneous, it became one crucial point for applications of characteristic classes (see [7]).)

For the homotopy classification of maps \( X \to Y \), the space \( F \) in (1.1)_m is replaced by \( \Omega Y \) and we establish the following statements. Below all topological spaces are assumed to be path connected (hence, \( Y \) is also simply connected) and the ground coefficient ring is the integers \( \mathbb{Z} \).

Given a commutative graded algebra (cga) \( H^* \) and an integer \( m \geq 1 \), we say that \( H^* \) is \( m \)-relation free if \( H^i \) is torsion free for \( i \leq m \) and also there is no multiplicative relation in \( H^i \) for \( i \leq m+1 \); in particular, \( H^{2i-1} = 0 \) for \( 1 \leq i \leq \lfloor \frac{m+2}{2} \rfloor \). We also allow \( m = \infty \) for \( H \) to be polynomial on even degree generators.

Theorem 1. Let \( f : X \to Y \) be a map such that the pair \((X, \Omega Y)\) satisfies \((1.1)_m\), \( X \) is an \( m \)-dimensional polyhedron and \( H^*(Y) \) is \( m \)-relation free. Then \( f \) is null homotopic if and only if

\[
0 = H^*(f) : H^*(Y) \to H^*(X).
\]
Theorem 2. Let $X$ and $Y$ be spaces such that the Hurewicz map $u_i : \pi_i(\Omega Y) \to H_i(\Omega Y)$ is an inclusion for $1 \leq i < m$, and $\operatorname{Tor}(H^{i+1}(X), H_i(\Omega Y)/\pi_i(\Omega Y)) = 0$ when $\pi_i(\Omega Y) \neq 0$. Then $X$ is an $m$-dimensional polyhedron and $H^*(Y)$ is $m$-relation free. Then a map $f : X \to Y$ is null homotopic if and only if
\[
0 = H^*(f) : H^*(Y) \to H^*(X).
\]

Theorem 3. Let $X$ be an $m$-dimensional polyhedron and $G$ a topological group such that $\pi_i(G)$ is torsion free for $1 \leq i < m$, and $\operatorname{Tor}(H^{i+1}(X), \operatorname{Coker} u_i) = 0$, $u_i : \pi_i(G) \to H_i(G)$ when $\pi_i(G) \neq 0$. Suppose that the cohomology algebra $H^*(BG)$ of the classifying space $BG$ is $m$-relation free. Then a map $f : X \to BG$ is null homotopic if and only if
\[
0 = H^*(f) : H^*(BG) \to H^*(X).
\]

In fact the two last Theorems follow from the first one, since their hypotheses imply $(1.1)_m$, too. A main example of $G$ in Theorem 3 is the unitary group $U(n)$ with $m = 2n$, since $u_{2i}$ is a trivial inclusion and $u_{2i-1}$ is an inclusion given by multiplication by the integer $(i - 1)!$ for $1 \leq i \leq n$. A $U(n)$-principal fibre bundle over $X$ is classified by a map $X \to BU(n)$. Suppose that all its Chern classes are trivial, then $H^*(f) = 0$ and by Theorem 3, $f$ is null homotopic. Therefore the $U(n)$-principal fibre bundle is trivial. Thus, we have in fact deduced the following statement, the main result of [22] (compare also [29]).

Corollary 1. Let $\xi$ be a $U(n)$-principal fibre bundle over $X$ with $\dim X \leq 2n$ and the only torsion in $H^{2i}(X)$ is relatively prime to $(i - 1)!$. Then $\xi$ is trivial if and only if the Chern classes $c_k(\xi) = 0$ for $1 \leq k \leq n$.

While the proof of this statement in [22] does not admit an immediate generalization for an infinite dimensional $X$, Theorem 3 does by taking $m = \infty$. Furthermore, for $G = U$ and $X = BU$ recall that $[BU, BU]$ is an abelian group, so we get that two maps $f, g : BU \to BU$ are homotopic if and only if $H^*(f) = H^*(g) : H^*(BU; \mathbb{Q}) \to H^*(BU; \mathbb{Q})$ (compare [14], [21]). Note also that when $m = \infty$ in Theorem 3, $H^*(Y)$ must have infinitely many polynomial generators (e.g. $Y = BU, BSp$) as it follows from the solution of the Steenrod problem for finitely generated polynomial rings [1] (the underlying spaces do not have torsion free homotopy groups in all degrees).

Finally, note that beside obstruction theory we apply a main ingredient of the proof of Theorem 1 is an explicit form of minimal multiplicative (non-commutative) resolution of an $m$-relation free cga (of a polynomial algebra when $m = \infty$ in total degrees $\leq m$ (compare [24], [26]). Namely, the generator set of the resolution in the above range only consists of monomials formed by $\smile_1$ products. Remark that the idea of using $\smile_1$ product when dealing with polynomial cohomology, especially in the context of homogeneous spaces, has been realized by several authors [17], [9], [20], [12] (see also [18] for further references).

In sections 2 and 3 we recall certain basic definitions and constructions, including the functor $D(X; H_*)$ [2], [3], for the aforementioned obstruction theory, and in section 4 prove Theorems 1-3.

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2. Functor $D(X; H)$

Given a bigraded differential algebra $A = \{A^{i,j}\}$ with $d : A^{i,j} \to A^{i+1,j}$ and total degree $n = i + j$, let $D(A)$ be the set \(^3\) defined by $D(A) = M(A) / G(A)$ where

$$M(A) = \{ a \in A^1 | da = -aa, a = a^{2,-1} + a^{3,-2} + \cdots \},$$

$$G(A) = \{ p \in A^0 | p = 1 + p^{1,-1} + p^{2,-2} + \cdots \},$$

and the action $M(A) \times G(A) \to M(A)$ is given by the formula

$$a * p = p^{-1}ap + p^{-1}dp.$$

In other words, two elements $a,b \in M(A)$ are on the same orbit if there is $p \in G(A)$, $p = 1 + p'$, with

$$b - a = ap' - p'b + dp'.$$

Note that an element $a = \{a^{*,*}\}$ from $M(A)$ is of total degree 1 and referred to as \textit{twisting}; we usually suppress the second degree below. There is a distinguished element in the set $D(A)$, the class of 0 \in $A$, and denoted by the same symbol.

There is simple but useful (cf. \cite{24})

\textbf{Proposition 1.} Let $f, g : A^{*,*} \to B^{*,*}$ be two dga maps that preserve the bigrading. If they are $(f,g)$-derivation homotopic via $s : A^{*,*} \to B^{*,*}$, i.e., $f - g = sd + ds$ and $s(ab) = (-1)^{|a|}fasb + sgb$, then $D(f) = D(g) : D(A) \to D(B)$.

\textbf{Proof.} Given $a \in M(A)$, apply the $(f,g)$-derivation homotopy $s$ to get $fa - ga = dsa + sda = dsa + s(-aa) = dsa + fasa - saga$. From this we deduce that $fa$ and $ga$ are equivalent by \textbf{(2.2)} for $p' = -sa$. $\Box$

Another useful property of $D$ is fixed by the following comparison theorem \cite{2, 3}:

\textbf{Theorem 4.} If $f : A \to B$ is a cohomology isomorphism, then $D(f) : D(A) \to D(B)$ is a bijection.

For our purposes the main example of $D(A)$ is the following (cf. \cite{2, 3})

\textbf{Example 1.} Fix a graded (abelian) group $H_\ast$. Let

$$\rho : (R_{\geq 0}H_\ast, \partial^R) \to H_\ast, \partial^R : R_iH_q \to R_{i-1}H_q,$$

be its free group resolution. Form the bigraded Hom complex

$$(R^{*,*}, d^R) = (\text{Hom}(RH_\ast, RH_\ast), d^R), \quad d^R : R^{s,t} \to R^{s+1,t};$$

an element $f \in R^{*,*}$ has bidegree $(s,t)$ if $f : R_iH_q \to R_{j-i}H_{q-j}$. Note also that $R^{*,*}$ becomes a dga with respect to the composition product.

Given a topological space $X$, consider the dga

$$(\mathcal{H}, \nabla) = (C^\ast(X; \mathbb{R}), \nabla = d^C + d^R)$$

which is bigraded via $\mathcal{H}^{r,t} = \prod_{i+j=r,t} C^{i}(X; \mathbb{R}^{j,t})$. Thus we get

$$\mathcal{H} = \{\mathcal{H}^{n}\}, \quad \mathcal{H}^{n} = \prod_{n=r+t} \mathcal{H}^{r,t}, \quad \nabla : \mathcal{H}^{r,t} \to \mathcal{H}^{r+1,t}.$$

We refer to $r$ as the perturbation degree which is mainly exploited by inductive arguments below. For example, for a twisting cochain $h \in M(\mathcal{H})$, we have

$$h = h^{2} + \cdots + h^{r} + \cdots, \quad h^{r} \in \mathcal{H}^{r,1-r},$$

$\cdots$. \hfill \blacksquare$
is a free module. In \cite{[2]} the Hirsch model was extended for arbitrary Brown’s twisting tensor product model (\text{C}) \text{RH} \text{by replacing the chains RH}

\text{d} \text{in which (3.1) }

\n
\begin{equation}
\n\n\alpha \gamma \text{H}_\text{h} \sim \text{H}_\text{h} \text{(3.1)}
\end{equation}

\n
\text{Define}

\n
\text{D}(X; H_\text{s}) = D(\mathcal{H}, \nabla).

Then \text{D}(X; H_\text{s}) becomes a functor on the category of topological spaces and continuous maps to the category of pointed sets.

\n
\text{Example 2. Given two dga’s } B^* \text{ and } C^\bullet, \text{ with } d^B : B^i \rightarrow B^{i+1} \text{ and } d^C_j : C^{j,t} \rightarrow C^{j+1,t}, \text{ where } A = B \otimes C. \text{ View } (A, d) \text{ as bigraded via } A = \{A^r, d\}, A^{r,t} = \prod_{r+i+j} B^i \otimes C^{j,t}, \text{ d = } d^B \otimes 1 + 1 \otimes d^C_j. \text{ Note also that the dga } (\mathcal{H}, \nabla) \text{ in the previous example can also be viewed as a special case of the above tensor product algebra by setting } B^* = C^* (X) \text{ and } C^\bullet = RH^\bullet \text{.}

3. Predifferential \text{d(\xi)} \text{ of a fibration}

Let \text{F} \rightarrow \text{E} \xrightarrow{\xi} \text{X} \text{ be a fibration}. In \cite{[2]} a unique element of \text{D}(X; H_\text{s}(F)) \text{ is naturally assigned to } \xi; \text{ this element is denoted by } \text{d(\xi)} \text{ and referred to as the predifferential of } \xi. \text{ The naturalness of } \text{d(\xi)} \text{ means that for a map } f : \text{Y} \rightarrow \text{X},

\n
\begin{equation}
\n\text{d(f(\xi)) = D(f)(d(\xi))},
\end{equation}

\n
\text{where } f(\xi) \text{ denotes the induced fibration on } \text{Y}.

\n
\text{Originally } d(\xi) \text{ appeared in homological perturbation theory for measuring the non-freeness of the Brown-Hirsch model: First, in \cite{[11]} G. Hirsch modified E. Brown’s twisting tensor product model } (C_\text{s}(X) \otimes C_\text{s}(F), d_\text{phi}) \rightarrow (C_\text{s}(E), d_\text{E}) \text{ \cite{[3]}, \text{ by replacing the chains } C_\text{s}(F) \text{ by its homology } H_\text{s}(F) \text{ provided the homology is a free module. In \cite{[2]} the Hirsch model was extended for arbitrary } H_\text{s}(F) \text{ by replacing it by a free module resolution } RH_\text{s}(F) \text{ to obtain } (C_\text{s}(X) \otimes RH_\text{s}(F), d_\text{h}) \text{ in which } d_\text{h} = d_\text{X} \otimes 1 + 1 \otimes d_\text{F} + - \cap h \text{ and } h \text{ is just an element of } M(\mathcal{H}) \text{ in Example } \text{[1]} \text{ with } H_\text{s} = H_\text{s}(F). \text{ Furthermore, to an isomorphism } p : (C_\text{s}(X) \otimes RH_\text{s}(F), d_\text{h}) \rightarrow (C_\text{s}(X) \otimes RH_\text{s}(F), d_\text{phi}) \text{ between two such models answers an equivalence relation } h \sim_p h' \text{ in } M(\mathcal{H}), \text{ and the class of } h \text{ in } D(X; H_\text{s}(F)) \text{ is identified as } d(\xi). \text{ More precisely, we recall some basic constructions for the definition of } d(\xi) \text{ we need for the obstruction theory in question.}

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\text{For convenience, assume that } X \text{ is a polyhedron and that } \pi_1(X) \text{ acts trivially on } H_\text{s}(F). \text{ Then } \xi \text{ defines the following colocal system of chain complexes over } \text{X: To each simplex } \sigma \in \text{X} \text{ is assigned the singular chain complex } (C_\text{s}(F_\sigma), \gamma_\sigma) \text{ of the space } F_\sigma = \xi^{-1}(\sigma) : \text{X} \ni \sigma \rightarrow (C_\text{s}(F_\sigma), \gamma_\sigma) \subset (C_\text{s}(E), d_\text{E}), \text{ and to a pair } \tau \subset \sigma \text{ of simplices an induced chain map } C_\text{s}(F_\tau) \rightarrow C_\text{s}(F_\sigma). \text{ Set } C_\sigma = \{C_\sigma^{s,t}\}, \text{ C}_\sigma^{s,t} = \text{Hom}_\text{E}(R_\sigma \text{H}_\text{s}(F), C_\text{s}(F_\sigma)) \text{ where } C_\text{s} \text{ is regarded as bigraded via } C_\text{s}_{i,s} = C_{i,s}, C_\text{s}_{i,s} = 0, i \neq 0, \text{ and } f : R_\sigma \text{H}_\text{s}(F) \rightarrow C_{j-s-q-t}(F_\sigma) \text{ is of bidegree } (s,t). \text{ Then we obtain a colocal system of cochain complexes } \mathcal{C} = \{C_\sigma^{s,t}\} \text{ on } X. \text{ Define } \mathcal{F} \text{ as the simplicial cochain complex } C^*(X; \mathcal{C}) \text{ of } X \text{ with coefficients in the colocal system } \mathcal{C}. \text{ Then } \mathcal{F} = \{\mathcal{F}_{i,j-t}\}, \text{ } \mathcal{F}_{i,j-t} = C^i(X; C^{j,t}) \text{.}
Furthermore, obtain the bicomplex $\mathcal{F} = \{\mathcal{F}^{r,t}\}$ via
\[
\mathcal{F}^{r,t} = \prod_{r_i + t_j = r} \mathcal{F}^{r_i,t_j}, \quad \delta : \mathcal{F}^{r,t} \to \mathcal{F}^{r+1,t}, \quad \gamma : \mathcal{F}^{r,t} \to \mathcal{F}^{r,t+1}, \quad \delta = d^C + \partial^R, \quad \gamma = \{\gamma_0\},
\]
and finally set
\[
\mathcal{F} = \{\mathcal{F}^m\}, \quad \mathcal{F}^m = \prod_{m=r+t} \mathcal{F}^{r,t}.
\]
We have a natural dg pairing
\[
(\mathcal{F}, \delta + \gamma) \otimes (\mathcal{H}, \nabla) \to (\mathcal{F}, \delta + \gamma)
\]
defined by $\gamma$ product on $C^\ast(X;\gamma)$ and the obvious pairing $C_\sigma \otimes R \to C_\sigma$ in coefficients; in particular we have $\gamma(fh) = \gamma(f)h$ for $f \otimes h \in F \otimes H$. Denote $\mathcal{R}_\sigma = Hom(RH_\ast(F), H_\ast(F))$ and define
\[
(\mathcal{F}_\sigma, \delta_\sigma) := (H(\mathcal{F}, \gamma), \delta_\sigma) = (C^\ast(X; \mathcal{R}_\sigma), \delta_\sigma).
\]
Clearly, the above pairing induces the following dg pairing
\[
(\mathcal{F}_\sigma, \delta_\sigma) \otimes (\mathcal{H}, \nabla) \to (\mathcal{F}_\sigma, \delta_\sigma).
\]
In other words, this pairing is also defined by $\gamma$ product on $C^\ast(X;\gamma)$ and the pairing $\mathcal{R}_\sigma \otimes R \to \mathcal{R}_\sigma$ in coefficients. Note that $\rho$ induces an epimorphism of chain complexes
\[
\rho^\ast : (\mathcal{H}, \nabla) \to (\mathcal{F}_\sigma, \delta_\sigma).
\]
In turn, $\rho^\ast$ induces an isomorphism in cohomology.

Consider the following equation
\[
(\delta + \gamma)(f) = fh
\]
with respect to a pair $(h, f) \in H^1 \times F^0$,
\[
h = h^2 + \cdots + h^r + \cdots, \quad h^r \in H^{r-1-r},
\]
\[
f = f^0 + \cdots + f^r + \cdots, \quad f^r \in F^{r-r},
\]
satisfying the initial conditions:
\[
\nabla(h) = -hh
\]
\[
\gamma(f^0) = 0, \quad [f^0]_\gamma = \rho^\ast(1) \in F^{0,0}, \quad 1 \in \mathcal{H}.
\]
Let $(h, f)$ be a solution of the above equation. Then $d(\xi) \in D(X; H_\ast(F))$ is defined as the class of $h$. Moreover, the transformation of $h$ by (2.1) is extended to pairs $(h, f)$ by the map
\[
(M(H) \times F^0) \times (G(H) \times F^{-1}) \to M(H) \times F^0
\]
given for $((h, f), (p, s)) \in (M(H) \times F^0) \times (G(H) \times F^{-1})$ by the formula
\[
(h, f) \ast (p, s) = (h \ast p, fp + p(h \ast p) + (\delta + \gamma)(s)),
\]
We have that a solution $(h, f)$ of the equation exists and is unique up to the above action. Therefore, $d(\xi)$ is well defined.

Note that action (3.3) in particular has a property that if $(\tilde{h}, \tilde{f}) = (h, f) \ast (p, s)$ and $h^r = 0$ for $2 \leq r \leq n$, then in view of (2.2) one gets the equalities
\[
\tilde{h}^{n+1} = h \ast (1 + p^n) = h^{n+1} + \nabla(p^n).
\]
3.1. Fibrations with $d(\xi) = 0$. The main fact of this subsection is the following theorem from [4]:

**Theorem 5.** Let $F \to E \xrightarrow{\xi} X$ be a fibration such that $(X, F)$ satisfies (1.1)$_m$. If the restriction of $d(\xi) \in D(X; H_*(F))$ to $d(\xi)|_{X^n} \in D(X^n; H_*(F))$ is zero, then $\xi$ has a section on the $m$-skeleton of $X$. The case of $m = \infty$, i.e., $d(\xi) = 0$, implies the existence of a section on $X$.

**Proof.** Given a pair $(h, f) \in \mathcal{H} \times \mathcal{F}$, let $(h_{tr}, f_{tr})$ denote its component that lies in $C^*(X; \text{Hom}(H_0(F), RH_*(F))) \times C^*(X; \text{Hom}(H_0(F), C_*(F)))$.

Below $(h_{tr}, f_{tr})$ is referred to as the transgressive component of $(h, f)$. Observe that since $RH_0(F) = H_0(F) = \mathbb{Z}$, we can view $(h_{tr}^{r+1}, f_{tr}^r)$ as a pair of cochains laying in $C^{r+1}(X; RH_*(F)) \times C^r(X; C_*(F))$. Such an interpretation allows us to identify a section $\gamma: X \to E$ on the $r$-skeleton $X^r \subset X$ with a cochain, denoted by $c^r_\gamma$, in $C^r(X; C_*(F))$ via $c^r_\gamma(\sigma) = \gamma|_\sigma: \Delta^r \to F_\sigma \subset E$, $\sigma \subset X^r$ is an $r$-simplex, $r \geq 0$.

The proof of the theorem just consists of choosing a solution $(h, f)$ of (3.2) so that the transgressive component $f_{tr} = \{f_{tr}^r\}_{r \geq 0}$ is specified by $f_{tr}^r = c^r_\chi$ with $\chi$ a section of $\xi$. Indeed, since $F$ is path connected, there is a section $\chi^1$ on $X^1$; consequently, we get the pairs $(0, f_{tr}^0) := (0, c^0_\chi)$ and $(0, f_{tr}^1) := (0, c^1_\chi)$ with $\gamma(f_{tr}^1) = \delta(f_{tr}^0)$. Then $\delta(f_{tr}^1) \in C^2(X; C_1(F))$ is a $\gamma$-cocycle and $[\delta(f_{tr}^1)]_\gamma \in C^2(X; H_1(F))$ becomes the obstruction cocycle $c(\chi^1) \in C^2(X; \pi_1(F))$ for extending of $\chi^1$ on $X^2$; moreover, one can choose $h^2_{tr}$ to be satisfying $\rho^*(h^2_{tr}) = [\delta(f_{tr}^1)]_\gamma$ (since $\rho^*$ is an epimorphism and a weak equivalence).

Suppose by induction that we have constructed a solution $(h, f)$ of (3.2) and a section $\chi^n$ on $X^n$ such that $h^n = 0$ for $2 \leq r \leq n$, $f_{tr}^n = c^n_\chi$ and

$$\rho^*(h_{tr}^{n+1}) = [\delta(f_{tr}^n)]_\gamma \in C^{n+1}(X; H_n(F)).$$

In view of (2.3) we have $\nabla(h^{n+1}) = 0$ and from the above equality immediately follows that

$$u^\#(c_\chi^n) = \rho^*(h_{tr}^{n+1})$$

in which $c_\chi^n \in C^{n+1}(X; \pi_n(F))$ is the obstruction cocycle for extending of $\chi^n$ on $X^{n+1}$ and $u^\#: C^{n+1}(X; \pi_n(F)) \to C^{n+1}(X; H_n(F))$.

Since $d(\xi)|_{X^n} = 0$, there is $p \in G(\mathcal{H})$ such that $(h + p)|_{X^n} = 0$; in particular, $(h + p)^{n+1} = 0 \in H^{n+1, -n}$ and in view of (3.3) we establish the equality $h^{n+1} = -\nabla(p^n)$, i.e., $[h^{n+1}] = 0 \in H^{n+1}(\mathcal{H}, \nabla)$; in particular, $[h_{tr}^{n+1}] = 0 \in H^{n+1}(X; H_n(F))$. Consequently, $[u^\#(c_\chi^n)] = 0 \in H^{n+1}(X; H_n(F))$. Since (1.1)$_n$ is an inclusion induced by $u^\#$, $[c_\chi^n] = 0 \in H^{n+1}(X; \pi_n(F))$. Therefore, we can extend $\chi^n$ on $X^{n+1}$ without changing it on $X^n$ in a standard way. Finally, put $f_{tr}^{n+1} = c^{n+1}_\chi$ and choose a $\nabla$-cocycle $h_{tr}^{n+2}$ satisfying $\rho^*(h_{tr}^{n+2}) = [\delta(f_{tr}^{n+1})]_\gamma$. The induction step is completed.

4. Proof of Theorems 1, 2 and 3

First we recall the following application of Theorem 4 (4)

**Theorem 6.** Let $f : X \to Y$ be a map such that $X$ is an $m$-polyhedron and the pair $(X, \Omega Y)$ satisfies (1.1)$_m$. If $0 = D(f) : D(Y; H_*(\Omega Y)) \to D(X; H_*(\Omega Y))$, then $f$ is null homotopic.
Proof. Let $\Omega \to PY \xrightarrow{\pi} Y$ be the path fibration and $f(\pi)$ the induced fibration. It suffices to show that $f(\pi)$ has a section. Indeed, (3.1) together with $D(f) = 0$ implies $d(f(\pi)) = 0$, so Theorem 5 guarantees the existence of the section.

Now we are ready to prove the theorems stated in the introduction. Note that just below we shall heavily use multiplicative, non-commutative resolutions of cga’s that are enriched with $\sim_1$ products. Namely, given a space $Z$, recall its filtered model $f_Z : (RH(Z), d_h) \to C^*(Z)$ \cite{24, 26} in which the underlying differential (bi)graded algebra $(RH(Z), d)$ is a non-commutative version of Tate-Jozefiak resolution of the cohomology algebra $H^*(Z)$ \cite{25, 15}, while $h$ denotes a perturbation of $d$ similar to \cite{10}. Moreover, given a map $X \to Y$, there is a dga map $RH(f) : (RH(Y), d_h) \to (RH(X), d_h)$ (not uniquely defined!) such that the following diagram

\[
\begin{array}{cccc}
(RH(Y), d_h) & \xrightarrow{RH(f)} & (RH(X), d_h) \\
\rho_Y & \downarrow & \downarrow f_X \\
C^*(Y) & \xrightarrow{C(f)} & C^*(X)
\end{array}
\]

commutes up to $(\alpha, \beta)$-derivation homotopy with $\alpha = C(f) \circ f_Y$ and $\beta = f_X \circ RH(f)$ (see, \cite{12, 24}).

Proof of Theorem 7 The non-trivial part of the proof is to show that $H(f) = 0$ implies $f$ is null homotopic. In view of Theorem 6 it suffices to show that $D(f) = 0$. By (4.1) and Proposition 4 we get the commutative diagram of pointed sets

\[
\begin{array}{cccc}
D(\mathcal{H}_Y) & \xrightarrow{D(f_Y)} & D(\mathcal{H}_X) \\
\downarrow D(f_Y) & & \downarrow D(f_X) \\
D(Y; H_*(\Omega Y)) & \xrightarrow{D(f)} & D(X; H_*(\Omega Y))
\end{array}
\]

in which

\[
\mathcal{H}_X = RH^*(X) \otimes \text{Hom}(RH_*(\Omega Y), RH_*(\Omega Y)),
\]
\[
\mathcal{H}_Y = RH^*(Y) \otimes \text{Hom}(RH_*(\Omega Y), RH_*(\Omega Y))
\]

(see Example 2) and the vertical maps are induced by $f_X \otimes 1$ and $f_Y \otimes 1$; these maps are bijections by Theorem 4. Below we need an explicit form of $RH(f)$ to see that $H(f) = 0$ necessarily implies $RH(f)|_{V'(m)} = 0$ with $V'(m) = \bigoplus_{1 \leq i+j \leq m} V^i,j$; hence, the restriction of the map $\mathcal{H}(f) := RH(f) \otimes 1$ to $RH'(m) \otimes 1$, $RH'(m) = \bigoplus_{1 \leq i+j \leq m} R^iH^j(Y)$, is zero, and, consequently,

\[
D(f_X) \circ D(\mathcal{H}(f)) = 0.
\]

First observe that any multiplicative resolution $(RH, d) = (T(V^*, *), d), V = \langle V \rangle$, of a cga $H$ admits a sequence of multiplicative generators, denoted by

\[
a_1 \sim \cdots \sim a_{n+1} \in V^{-n,*}, \quad a_i \in V^0, \quad n \geq 1,
\]

where $a_i \sim a_j = (-1)^{|a_i||a_j|+1}a_j \sim a_i$ and $a_i \neq a_j$ for $i \neq j$. Furthermore, the expression $ab \sim uv$ also has a sense by means of formally (successively) applying the Hirsch formula

\[
c \sim (ab) = (c \sim a)b + (-1)^{|a||c|+1}a(c \sim b).
\]

The resolution differential $d$ acts on (4.3) by iterative application of the formula

\[
d(a \sim b) = da \sim b - (-1)^{|a|}a \sim db + (-1)^{|a|}ab - (-1)^{|a||b|}ba.
\]
Consequently, we get
\[ d(a_1 \sim \cdots \sim a_n) = \sum_{(i,j)} (-1)^j (a_{i_1} \sim \cdots \sim a_{i_k}) \cdot (a_{j_1} \sim \cdots \sim a_{j_l}) \]
where the summation is over unshuffles \((i,j) = (i_1 < \cdots < i_k \mid j_1 < \cdots < j_l)\) of \(\underline{n}\).

In the case of \(H\) to be \(m\)-relation free with a basis \(U^1 \subset H^1, i \leq m\), we have that the minimal multiplicative resolution \(RH\) of \(H\) can be built by taking \(V\) with \(V^0, i \leq m\), and \(V^{-n}, n > 0\), to be the set consisting of monomials \([19]\) for \(1 \leq i - n \leq m\) (compare [20]). The verification of the acyclicity in the negative resolution degrees of \(RH\) restricted to the range \(RH^{(m)}\) is straightforward (see also Remark [1]). Regarding the map \(RH(f)\), we can choose it on \(RH^{(m)}\) as follows. Let \(R_0H(f) : R_0H(Y) \to R_0H(X)\) be determined by \(H(f)\) in an obvious way and then define \(RH(f)\) for \(a \in V^{(m)}\) by
\[
RH(f)(a) = \begin{cases} 
R_0H(f)(a), & a \in V^{0,*}, \\
R_0H(f)(a_1) \sim \cdots \sim R_0H(f)(a_n), & a = a_1 \sim \cdots \sim a_n, \\
R_0H(f)(a_1) \sim \cdots \sim R_0H(f)(a_n), & a \in V^{-n,*}, a_i \in V^{0,*}, n \geq 1,
\end{cases}
\]
and extend to \(RH^{(m)}\) multiplicatively. Furthermore, \(f_X\) and \(f_Y\) are assumed to be preserving the generators of the form \([19]\) with respect to the right most association of \(\sim \) products in question. Since \(h\) annihilates monomials \([19]\) and the existence of formula \([19]\) in a simplicial cochain complex, \(f_X\) and \(f_Y\) are automatically compatible with the differentials involved. Then the maps \(\alpha\) and \(\beta\) in \([19]\) also preserve \(\sim \) products, and become homotopic by an \((\alpha, \beta)\)-derivation homotopy \(s : RH(Y) \to C^*(X)\) defined as follows: choose \(s\) on \(V^{0,*}\) by \(ds = \alpha - \beta\) and extend on \(V^{-n,*}\) inductively by
\[
s(a_0 \sim z_n) = -\alpha(a_0) \sim s(z_n) + \beta(z_n) + \beta(z_n)s(a_0), \quad n \geq 1,
\]
in which \(z_1 = a_1\) and \(z_k = a_1 \sim \cdots \sim a_k\) for \(k \geq 2, a_i \in V^{0,*}\). Clearly, \(H(f) = 0\) implies \(RH(f)|_{V^{(m)}} = 0\). Since \([12]\), \(D(f) = 0\) and so \(f\) is null homotopic by Theorem [3]. Theorem is proved.

**Remark 1.** Let \(V^{(m)}_n\) be a subset of \(V^{(m)}\) consisting of all monomials formed by the \(\cdot\) and \(\sim \) products evaluated on a string of variables \(a_1, \ldots, a_n\). Then there is a bijection of \(V^{(m)}_n\) with the set of all faces of the permuatahedron \(P_n\) \([19, 27]\) such that the resolution differential \(d\) is compatible with the cellular differential of \(P_n\) (compare \([16]\)). In particular, the monomial \(a_1 \sim \cdots \sim a_n\) is assigned to the top cell of \(P_n\), while the monomials \(a_{\sigma(1)} \cdots a_{\sigma(n)}, \sigma \in S_n, \) to the vertices of \(P_n\) (see Fig. 1 for \(n = 3\)). Thus, the acyclicity of \(P_n\) immediately implies the acyclicity of \(RH^{(m)}\) in the negative resolution degrees as desired.
Remark 2. An example provided by the Hopf map $f : S^3 \to S^2$ shows that the implication $H(f) = 0 \Rightarrow RH(f) \mid_{\nu(i)} = 0$, $k < m$ for $RH(f)$ making \((1.7)\) commutative up to $(\alpha, \beta)$-derivation homotopy is not true in general. More precisely, let $x \in R^3H^2(S^2)$ and $y \in R^3H^3(S^3)$ with $px \in H^2(S^2)$ and $py \in H^3(S^3)$ to be the generators, and let $x_1 \in R^{-1}H^4(S^2)$ with $dx_1 = x^2$. Then $s(x_2) = \alpha(x)s(x)$ is a cocycle in $C^3(S^3)$ with $dsy(x) = \alpha(x)$ (since $\beta = 0$) and $[\alpha(x)s(x)] = py$. Consequently, while $H(f) = 0 = R^0H(f)$, a map $RH(f) : RH(S^2) \to RH(S^3)$ required in \((4.7)\) has a non-trivial component increasing the resolution degree: Namely, $R^{-1}H^4(S^2) \to R^0H^3(S^3)$, $x_1 \to y$.

Proof of Theorem 2. The conditions that $u_i : \pi_i(\Omega Y) \to H_i(X)$ is an inclusion and $\text{Tor}(H^{i+1}(X), H_i(\Omega Y)/\pi_i(\Omega Y)) = 0$ for $1 \leq i < m$, immediately implies \((1.1)_m\). So the theorem follows from Theorem 1.

Proof of Theorem 3. Since the homotopy equivalence $\Omega BG \simeq G$, the conditions of Theorem 2 are satisfied: Indeed, there is the following commutative diagram

$$
\begin{array}{ccc}
\pi_k(G) & \xrightarrow{u_k} & H_k(G) \\
\downarrow i_\pi & & \downarrow i_H \\
\pi_k(G) \otimes \mathbb{Q} & \xrightarrow{u_k \otimes 1} & H_k(G) \otimes \mathbb{Q}
\end{array}
$$

where $i_\pi, i_H$ and $u_k \otimes 1$ are the standard inclusions (the last one is a consequence of a theorem of Milnor-Moore). Consequently, $u_k : \pi_k(\Omega BG) \to H_k(\Omega BG), k < m$, is an inclusion, too. Theorem is proved.

\[\square\]

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