Hall current effect in bioconvection Oldroyd-B nanofluid flow through a porous medium with Cattaneo-Christov heat and mass flux theory

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Bioconvection due to microorganisms is important area of research, considerably importance for environment and sustainable fuel cell technologies. Buongiorno nanofluid model for Cattaneo-Christov heat and mass flux theory taken into account the Oldroyd-B nanofluid and gyrotactic microorganisms in a rotating system with the effects of Hall current, and Darcy porous medium is scrutinized. The constitutive equations of the problem are transformed into nondimensional equations with the help of similarity transformations. Homotopy analysis method is used to obtain the solution. Graphs and table support the comprehensive representation of the achieved results. Radial velocity is reduced with the increasing values of relaxation time, retardation time and magnetic field parameters while heat transfer is augmented with thermal relaxation time parameter. The nanoparticles concentration is reduced with the increasing values of Schmidt number and the gyrotactic microorganisms concentration is enhanced with the increasing values of Peclet number. A nice agreement is obtained while comparing the present results numerically with the published results. The proposed mathematical model is used in biochemical engineering, meteorology, power and transportation production, optoelectronic and sensing microfabrication.

Heat transfer performance is enhanced on account of adding the micro size particles in the base liquids. The improved heating conduction of nanofluids has shown better results. Nanofluids are used in cooling of metallic plates, paper productions, drawing of plastic sheets, aerodynamics, etc. When the nanofluids are regulated by magnetic field, then these have potential applications such as anticancer drugs with the magnetic nanoparticles. Considering the applied magnetic field, Mandal and Pal1 analyzed the carbon nanotubes nanofluid with homogeneous-heterogeneous chemical reactions, variant heating transportation and mass diffusion, binary chemical reaction and activation energy using the convective boundary conditions. Yaseen et al.2 worked on the magnetohydrodynamic hybrid and mono nanoliquids within two shrinking and revolving discs through a Darcy-Forchheimer permeable medium considering Cattaneo-Christov (C-C) heat flux theory, viscous dissipation, Ohmic heating, solar radiation and heating generating/absorbing phenomena. Mandal3 used the fifth order Runge-Kutta-Fehlberg method in the presence of shooting procedure to solve the problem of convection boundary layer flow and heat transfer of micropolar nanofluid containing four types of nanoparticles on a nonlinear expanding source. Garia et al.4 scrutinized the magnetohydrodynamic motion of water based hybrid suspension on two surfaces by implementing the Das and Tiwari concept by considering the C-C heating transportation theory. They computed the coefficients of correlation of heating transportation and skin friction processed by the evaluation of probable error and statistical declaration. Pal and Mandal5 obtained the dual solutions of mixed convection-radiation stagnation point motion of three nanofluids through a porous medium over an expanding/contracting space by using the Runge-Kutta-Fehlberg procedure in connection to shooting algorithm which shows that the size of boundary layer is high for second solution compared to that of first solution. Rawat and Kumar6 worked on the water based copper nanofluid stagnation point flow with C-C concept and heating generating/absorbing phenomena, thermal radiation, activation energy, suction and slip condition which proved

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that thermal relaxation and radiation parameters enhance the heat transfer while slip and suction parameters reduce the mass transfer. Some nanofluids literature can be read from the references27–30.

Recently, non-Newtonian liquids have gained a wide range of importance in manufacturing, commerce, and mechanics which are seen in food processing, material handling, oil storage, warehousing, etc. In general, non-Newtonian suspensions possess three leading classifications as differential, rate and integral types in which the rate type shows the stress relaxation. Oldroyd-B class holds the rate-type fluid which possess the generalization of the upper convected viscoelastic Maxwell fluid class in the possession of retardation time and presents the motion of viscoelastic fluids proposed by Oldroyd. Hafeez et al.8 looked over the Oldroyd-B fluid in the existence of C-C heating transportation theory, homogeneous-heterogeneous reactions by using the BVP Midrich technique in which the flow speed becomes slow due to the effect of relaxation time factor. Shahzad et al.19 worked on the Das and Tiwari nanoliquid concept to investigate the thermal characteristics of Oldroyd-B fluid with engine oil as a conventional base suspension. They proved that the heating conduction of molybdenum disulfide engine oil nanofluid is greater than the copper engine oil nanofluid. Irfan et al.20 performed the analysis of double stratification in nonlinear radiative flow of Oldroyd-B nanofluid in stagnation region with the effects of magnetohydrodynamic and heat source/sink, Brownian diffusion and thermophoresis. Anwar et al.21 explained the transient free convection motion of an Oldroyd-B liquid through a vertical porous channel with nonlinear solar radiation in which the one side flow direction has unsteady velocity, while other side performs no movement on the basis of momentum conservation law and Fourier’s principle of heat transfer. Irfan et al.22 investigated the Oldroyd-B fluid with isothermal/exothermic reaction incorporating the features of Ohmic dissipation and Joule heating effects. The different characteristics of Oldroyd-B nanofluid are investigated by Irfan et al.23–25. Hafeez et al.26 investigated the Oldroyd-B liquid taken into account the C-C heat and mass flux theory using Darcy-Forchheimer effect to probe the bioconvection heating and nanoparticles concentration retaining gyrotactic microbes through a permeable source. Hafeez et al.27 worked on the modeling of heating transportation theory, homogeneous-heterogeneous reactions by using the BVP Midrich technique in which the flow speed becomes slow due to the effect of relaxation time factor. Shahzad et al.19 considered the steady heating and mass transferring motion of Oldroyd-B fluid with isothermal/exothermic reaction incorporating the features of Ohmic dissipation and Joule heating. The non-Newtonian features and others properties of various suspensions are exist in the studies23–28.

Bioconvection takes place in the suspension on account of up-swimming microorganisms which leads to the unstable density stratification, shows considerably importance for environment and sustainable fuel cell technologies having uses in environmental systems like ocean algae, fuel cells and biological polymer synthesis. The addition of nanofluid and bioconvection has obtained very important achievements in micro-fluidic devices including micro-reactors and micro-channel. Waqas et al.30–33 addressed the bioconvection effect due to microorganisms in Jeffery nanofluid past an expanding surface by considering the magnetic dipole effect. Raja et al.40 applied the soft computing based backpropagation neural network with Wu's slip effects to a bioconvection second-grade nanofluid model with C-C heat and mass flux theory. Hall current effect and porous medium. Waqas et al.41 discussed the numerical study of Oldroyd-B nanofluid flow with heat and mass transfer, gyrotactic microorganisms past a rotating disk through bvp4c built-in function of MATLAB software. Khan et al.42 analyzed the bioconvection nano-suspension motion and entropy generation in rotating system with the application of Homotopy Analysis Method. Rashad and Nabwey43 described the bioconvection of a nanofluid pertaining to motile microorganisms over a horizontal circular cylinder under the convective boundary conditions using Buongiorno's nanofluid model with the Oberbeck-Bousinesq approximation. Alwabaty et al.44 analyzed the bioconvection in magnetic nanoparticles using Wu's slip effects near the surface. Lv et al.45 introduced a new form of non-Newtonian liquid called Reiner-Rivlin nanofluid past a rotating disk with C-C heat flux through a porous media in the presence of gyrotactic microorganisms. Shehzad et al.46 studied the bioconvection flow of Maxwell fluid past the isolated disk by considering the Buongiorno nanofluid model with Cattaneo-Christov energy and mass species flux models. Related studies are mentioned in the references47–50.

The aforementioned literature contains different aspects associated with different fluids and are interesting studies. Strong applied magnetic field with permeable media associated to gyrotactic microorganisms along a revolving disk in the presence of Oldroyd-B nano-suspension needs attention to be investigated. So the authors studied the Oldroyd-B nano-suspension motion over an expandable revolving disc in the light of C-C model solved through51.

**Procedure**

**Problem formulation.** The swirl of Oldroyd-B nanofluid retaining gyrotactic microorganisms in porous space for three dimensions is modelled. Heat and mass transfer modeling is obtained through C-C concept. For Hall current effect, strong magnetic field is in implementation to perpendicular side (consider the Fig. 1).

Considered problem has the dynamic mechanisms

\[
\frac{\partial u}{\partial t} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0,
\]

(1)
Figure 1. A plot presents the modeling.

\[ \rho_f \left( \frac{\partial u}{\partial t} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} \right) = \rho_f \frac{\partial^2 u}{\partial z^2} + \lambda_1 \left( \frac{u^2}{r^2} + \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + 2uv \frac{\partial u}{\partial z} - \frac{2}{r} uv \frac{\partial u}{\partial r} - \frac{v^2}{r} uv \right) \]

\[ - \lambda_2 \mu f \left[ \frac{1}{r} \left( \frac{\partial u}{\partial r} \right)^2 - \frac{1}{r} \left( \frac{\partial u}{\partial r} \right) \left( \frac{\partial w}{\partial r} \right) + \frac{\partial u}{\partial z} \left( \frac{\partial w}{\partial z} \right) - \frac{\partial u}{\partial z} \left( \frac{\partial w}{\partial r} \right) - \frac{\partial w}{\partial r} \left( \frac{\partial w}{\partial z} \right) + \frac{\partial u}{\partial z} \left( \frac{\partial w}{\partial r} \right) \right] \]

\[ - \frac{\mu f^2 B_0^2}{r} \left( u - mv + w \lambda_1 \frac{\partial \theta}{\partial z} \right) \]

\[ - \frac{\mu f^2}{k} u - k^2 u^2, \]

\[ \rho_f \left( u \frac{\partial v}{\partial t} + w \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \]

\[ = - \lambda_1 \left( \frac{u^2}{r^2} + \frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial z^2} + 2uv \frac{\partial v}{\partial z} - \frac{2}{r} uv \frac{\partial v}{\partial r} + \frac{1}{r} \left( \frac{\partial u}{\partial r} \right) \left( \frac{\partial w}{\partial r} \right) - \frac{\partial u}{\partial z} \left( \frac{\partial w}{\partial z} \right) \right) \]

\[ + \mu f \frac{\partial^2 v}{\partial z^2} + \lambda_2 \mu f \left[ - \frac{u}{r} \frac{\partial v}{\partial r} + \frac{\partial v}{\partial z} \left( \frac{\partial w}{\partial z} \right) - \frac{\partial v}{\partial z} \left( \frac{\partial w}{\partial r} \right) - \frac{\partial w}{\partial r} \left( \frac{\partial w}{\partial z} \right) + \frac{\partial u}{\partial z} \left( \frac{\partial w}{\partial r} \right) \right] \]

\[ - \frac{\sigma f^2 B_0^2 (v + mu + w \lambda_1 \frac{\partial \theta}{\partial z})}{1 + m^2} \]

\[ - \frac{\sigma f^2}{k} v = k^2 w^2, \]

\[ \left( u \frac{\partial T}{\partial t} + w \frac{\partial T}{\partial z} \right) \]

\[ = \frac{\partial^2 T}{\partial z^2} + 2 \gamma_1 \tau D_T \left[ \frac{\partial T}{\partial z} \frac{\partial T}{\partial r} + \frac{\partial T}{\partial z} \right] + \tau \left[ D_B \left( \frac{\partial^2 C}{\partial z^2} \right) + \frac{D_C}{T_\infty} \left( \frac{\partial T}{\partial z} \right)^2 \right] \]

\[ + \gamma_1 \tau D_B \left[ \frac{\partial^2 C}{\partial z^2} + \frac{\partial C}{\partial z} \left( \frac{\partial T}{\partial r} \right) + \frac{\partial C}{\partial z} \left( \frac{\partial T}{\partial z} \right) \right] \]

\[ + \gamma_1 \left[ \frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial z} \right] + \frac{D_T}{T_\infty} \left( \frac{\partial^2 T}{\partial z^2} \right) \]

\[ + \frac{\partial T}{\partial r}, \]

\[ \frac{\partial C}{\partial r} + \frac{\partial C}{\partial z} \]

\[ = D_B \frac{\partial^2 C}{\partial z^2} + \gamma_2 \tau D_T \left[ \frac{\partial^2 C}{\partial z^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial z^2} \]

\[ - \gamma_2 \left[ \frac{\partial^2 C}{\partial z^2} + \frac{\partial C}{\partial z} \right], \]
where $t' = t/T_w$, $z' = z/T_{\infty}$, $u_0, v_0, w_0$ are the initial conditions, and $u$, $v$, $w$ are the velocity components in the $x$, $y$, and $z$ directions, respectively.

The system of partial differential equations is solved using a finite difference method. The boundary conditions are applied at $t = 0$ and $z = 0$.

Substituting the values from Eq. (9) in Eqs. (1–8), the below eight equations (10–17) are obtained.

$$2f^2 + h' = 0,$$

$$f^2 - g^2 + f'h - f'' + \beta_1 (h^2 f'' + 2ff'h + 2gg'h) + \beta_2 (2f'^2 + 2f'h'' - f''''h)$$

$$- M(f' - mg - 2\beta_1 h'f')$$

$$= \lambda_f h',$$

$$- \lambda_f h' = 0,$$

$$2fg + g'h - g'' + \beta_1 (h^2 g'' + 2 (fg' + gf') h) - \beta_2 (hg'' - 2f'g' - 2h'g')$$

$$- M(mf' + g - 2\beta_1 hg')$$

$$= \lambda g - \lambda g^2 = 0,$$

$$\frac{1}{Pr} \theta'' - h\theta' + Nb \phi'' \gamma_1 (h\theta'\phi'' + h\theta''\phi') + Ni \left(\theta''\right)^2 + 2\gamma_3 h\theta'\theta''$$

$$\gamma_3 \left(h^2 \theta'' + hh'\phi''\gamma_1 \right) = 0,$$

$$\phi'' - Sc h\phi' - \gamma_4 \left(h^2 \phi'' + hh'\phi'\gamma_4 \right) + \frac{Nt}{Nb} \left(\theta'' + 2\gamma_4 h\theta''\phi''\gamma_4 \right) = 0,$$

$$\chi'' - Lb h\chi' - Pe \left(\chi'\phi' + \phi''(\gamma_5 + \chi)\right) = 0,$$

$$f = \Omega_1, g = 1, h = 0, \theta = 1, \phi = 1, \chi = 1 \text{ at } \zeta = 0,$$

$$f \rightarrow 0, g \rightarrow 0, h \rightarrow 0, \theta \rightarrow 0, \phi \rightarrow 0, \chi \rightarrow 0 \text{ as } \zeta \rightarrow \infty,$$

where $\gamma$ is the reaction rate constant, $D$ is the diffusion coefficient, $k$ is the partition coefficient, and $Sc$, $St$, $Pe$, and $Bi$ are the Schmidt number, Steady number, Peclet number, and Biot number, respectively.
\[ C_F = \frac{\tau_{|z=0}}{\rho_f (r\Omega)^2}, \]  

(18)

where

\[ \tau = \sqrt{(\tau_r)^2 + (\tau_\theta)^2}, \]  

(19)

exhibits the shear stress.

\[ Nu_r = \frac{-rq_1}{\alpha (T_w - T_\infty)}, \quad Sh_r = \frac{-rq_2}{D_h (C_w - C_\infty)}, \quad Nn_r = \frac{-rq_3}{D_m (N_w - N_\infty)}, \]  

(20)

where \( q_1, q_2 \) and \( q_3 \) exhibit fluxes due to heat, mass and motile microorganisms having the formulations

\[ q_1 = -\alpha T_z|z=0, \quad q_2 = -D_h C_z|z=0, \quad q_3 = D_m N_z|z=0. \]  

(21)

Upon employing the Eqs. (9), (18) goes to simplification as

\[ C_F = Re_r^{-2} \left[ (f'(0))^2 + (g'(0))^2 \right], \]  

(22)

where \( Re_r = \frac{r\Omega}{v} \) exhibits the Reynolds number.

Upon employing the Eq. (9) in Eq. (20), calculations are

\[ Nu_r = -Re_r^{1.5} \phi'(0), \quad Sh_r = -Re_r^{0.5} \phi'(0), \quad Nn_r = -Re_r^{0.5} \chi'(0). \]  

(23)

**Computational work**

Under the scenario of HAM\(^2\), the initial approximations and auxiliary linear operators are

\[ h_0(\zeta) = 0, f_0(\zeta) = \Omega \exp(-\zeta), g_0(\zeta) = \exp(-\zeta), \theta_0(\zeta) = \exp(-\zeta), \phi_0(\zeta) = \exp(-\zeta), \chi_0(\zeta) = \exp(\zeta), \]  

(24)

\[ L_h = h', L_f = f'' - f, L_g = g'' - g', L_\theta = \theta'' - \theta, L_\phi = \phi'' - \phi, L_\chi = \chi'' - \chi. \]  

(25)

The linear operators are associated with

\[ L_h \left[ E_1 \right] = 0, \quad L_f \left[ E_2 \exp(\zeta) + E_3 \exp(-\zeta) \right] = 0, \quad L_g \left[ E_4 \exp(\zeta) + E_5 \exp(-\zeta) \right] = 0, \quad L_\theta \left[ E_6 \exp(\zeta) + E_7 \exp(-\zeta) \right] = 0, \quad L_\phi \left[ E_8 \exp(\zeta) + E_9 \exp(-\zeta) \right] = 0, \quad L_\chi \left[ E_{10} \exp(\zeta) + E_{11} \exp(-\zeta) \right] = 0, \]  

(26)

where \( E_i (i = 1-11) \) present the constants.

**Equations of deformation for zeroth order.** Deformation equations of zeroth order are

\[ (1 - q)L_h[h(\zeta, q) - h_0(\zeta)] = q h_0 N_h[f(\zeta, q), h(\zeta, q)], \]  

(27)

\[ (1 - q)L_f[f(\zeta, q) - f_0(\zeta)] = q h_0 N_f[f(\zeta, q), g(\zeta, q), h(\zeta, q)], \]  

(28)

\[ (1 - q)L_g[g(\zeta, q) - g_0(\zeta)] = q h_0 N_g[f(\zeta, q), g(\zeta, q), h(\zeta, q)], \]  

(29)

\[ (1 - q)L_\theta[\theta(\zeta, q) - \theta_0(\zeta)] = q h_0 N_\theta[h(\zeta, q), \theta(\zeta, q), \phi(\zeta, q)], \]  

(30)

\[ (1 - q)L_\phi[\phi(\zeta, q) - \phi_0(\zeta)] = q h_0 N_\phi[h(\zeta, q), \theta(\zeta, q), \phi(\zeta, q)], \]  

(31)

\[ (1 - q)L_\chi[\chi(\zeta, q) - \chi_0(\zeta)] = q h_0 N_\chi[h(\zeta, q), \chi(\zeta, q), \phi(\zeta, q)]. \]  

(32)

where \( q \) is presented as an embedding parameter and \( h_f, h_g, h_\theta, h_\phi \) and \( h_\chi \) are the non-zero auxiliary parameters. The nonlinear operators \( N_f, N_g, N_h, N_\theta, N_\phi, \) and \( N_\chi \) are defined as

\[ N_h[f(\zeta, q), h(\zeta, q)] = 2f(\zeta, q) + \frac{\partial h(\zeta, q)}{\partial \zeta}. \]  

(33)
\[ \mathcal{N}_f[f(\zeta, q), g(\zeta, q), h(\zeta, q), \theta(\zeta, q)] = -\frac{\partial^2 f(\zeta, q)}{\partial \zeta^2} + (f(\zeta, q))^2 - (g(\zeta, q))^2 - \frac{\partial f(\zeta, q)}{\partial \zeta} h(\zeta, q) \\
+ \beta_1 \left[ (h(\zeta, q))^2 \frac{\partial^2 f(\zeta, q)}{\partial \zeta^2} + 2f(\zeta, q) \frac{\partial f(\zeta, q)}{\partial \zeta} h(\zeta, q) - 2g(\zeta, q) \frac{\partial g(\zeta, q)}{\partial \zeta} h(\zeta, q) \right] \\
+ \beta_2 \left[ 2 \left( \frac{\partial f(\zeta, q)}{\partial \zeta} \right)^2 + 2 \frac{\partial f(\zeta, q)}{\partial \zeta} \frac{\partial^2 h(\zeta, q)}{\partial \zeta^2} - \frac{\partial f(\zeta, q)}{\partial \zeta} \frac{\partial h(\zeta, q)}{\partial \zeta} \right] \\
- \frac{M}{1 + m^2} \left[ \frac{\partial f(\zeta, q)}{\partial \zeta} - mg(\zeta, q) - 2\beta_1 h(\zeta, q) \frac{\partial g(\zeta, q)}{\partial \zeta} \right] \\
- \lambda_3 f(\zeta, q) - \lambda_4 (f(\zeta, q))^2, \quad (34) \]

\[ \mathcal{N}_g[f(\zeta, q), g(\zeta, q), h(\zeta, q)] = 2f(\zeta, q)g(\zeta, q) + \frac{\partial g(\zeta, q)}{\partial \zeta} h(\zeta, q) - \frac{\partial^2 g(\zeta, q)}{\partial \zeta^2} \\
+ \beta_1 \left[ (h(\zeta, q))^2 \frac{\partial^2 g(\zeta, q)}{\partial \zeta^2} + 2f(\zeta, q) \frac{\partial g(\zeta, q)}{\partial \zeta} h(\zeta, q) + 2g(\zeta, q) \frac{\partial f(\zeta, q)}{\partial \zeta} h(\zeta, q) \right] \\
- \beta_2 \left[ h(\zeta, q) \frac{\partial^2 g(\zeta, q)}{\partial \zeta^2} - 2 \frac{\partial f(\zeta, q)}{\partial \zeta} \frac{\partial g(\zeta, q)}{\partial \zeta} \right] \\
- \frac{M}{1 + m^2} \left[ \frac{\partial f(\zeta, q)}{\partial \zeta} + g(\zeta, q) - 2\beta_1 h(\zeta, q) \frac{\partial g(\zeta, q)}{\partial \zeta} \right] \\
- \lambda_3 g(\zeta, q) - \lambda_4 (g(\zeta, q))^2, \quad (35) \]

\[ \mathcal{N}_\theta[f(\zeta, q), h(\zeta, q), \theta(\zeta, q), \phi(\zeta, q)] = \frac{1}{Re} \frac{\partial^2 \theta(\zeta, q)}{\partial \zeta^2} - h(\zeta, q) - \frac{\partial \theta(\zeta, q)}{\partial \zeta} \\
+ \mathcal{N}_t \left[ \frac{\partial h(\zeta, q)}{\partial \zeta} \frac{\partial \phi(\zeta, q)}{\partial \zeta} + \gamma_3 \left( h(\zeta, q) \frac{\partial h(\zeta, q)}{\partial \zeta} \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} + h(\zeta, q) \frac{\partial \phi(\zeta, q)}{\partial \zeta} \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} \right) \right] \\
+ \mathcal{N}_t \left[ \frac{\partial h(\zeta, q)}{\partial \zeta} \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} + 2\gamma_3 h(\zeta, q) \frac{\partial h(\zeta, q)}{\partial \zeta} \frac{\partial \phi(\zeta, q)}{\partial \zeta} \right] \\
- \gamma_3 \left( h(\zeta, q) \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} + h(\zeta, q) \frac{\partial h(\zeta, q)}{\partial \zeta} \frac{\partial \phi(\zeta, q)}{\partial \zeta} \right), \quad (36) \]

\[ \mathcal{N}_\phi[h(\zeta, q), \theta(\zeta, q), \phi(\zeta, q)] = \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} - Sch(\zeta, q) \frac{\partial \phi(\zeta, q)}{\partial \zeta} \\
- \gamma_4 \left[ (h(\zeta, q))^2 \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} + h(\zeta, q) \frac{\partial h(\zeta, q)}{\partial \zeta} \frac{\partial \phi(\zeta, q)}{\partial \zeta} \right] \quad (37) \]

\[ + \frac{N_t}{N_b} \left[ \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} + \gamma_4 h(\zeta, q) \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} \right], \]

\[ \mathcal{N}_\chi[h(\zeta, q), \phi(\zeta, q), \chi(\zeta, q)] = \frac{\partial^2 \chi(\zeta, q)}{\partial \zeta^2} - Lbh(\zeta, q) \frac{\partial \chi(\zeta, q)}{\partial \zeta} \\
- Pe \left[ \frac{\partial \phi(\zeta, q)}{\partial \zeta} \frac{\partial \chi(\zeta, q)}{\partial \zeta} + \frac{\partial^2 \phi(\zeta, q)}{\partial \zeta^2} \right] \left( \gamma_5 + \chi(\zeta, q) \right), \quad (38) \]

Eq. (27) retains the B.C.

\[ h(0, q) = 0. \quad (39) \]

Eq. (28) retains B.C.

\[ f(0, q) = \Omega_1, \quad f(\infty, q) = 0. \quad (40) \]

Eq. (29) retains B.C.

\[ g(0, q) = 1, \quad g(\infty, q) = 0. \quad (41) \]

Eq. (30) retains B.C.

\[ \theta(0, q) = 1, \quad \theta(\infty, q) = 0. \quad (42) \]

Eq. (31) retains B.C.

\[ \phi(0, q) = 1, \quad \phi(\infty, q) = 0. \quad (43) \]

Eq. (32) retains B.C.

\[ \chi(0, q) = 1, \quad \chi(\infty, q) = 0. \quad (44) \]

Upon \( q = 0 \) and \( q = 1 \), Eqs. (27–32) become
\[ q = 0 \Rightarrow h(\zeta, 0) = h_0(\zeta) \text{ and } q = 1 \Rightarrow h(\zeta, 1) = h(\zeta), \quad (51) \]

\[ f(\zeta, 0) = f_0(\zeta) \text{ and } f(\zeta, 1) = f(\zeta), \quad (52) \]

\[ g(\zeta, 0) = g_0(\zeta) \text{ and } g(\zeta, 1) = g(\zeta), \quad (53) \]

\[ \theta(\zeta, 0) = \theta_0(\zeta) \text{ and } \theta(\zeta, 1) = \theta(\zeta), \quad (54) \]

\[ \phi(\zeta, 0) = \phi_0(\zeta) \text{ and } \phi(\zeta, 1) = \phi(\zeta), \quad (55) \]

\[ \chi(\zeta, 0) = \chi_0(\zeta) \text{ and } \chi(\zeta, 1) = \chi(\zeta). \quad (56) \]

Expanding \( h(\zeta, q), f(\zeta, q), g(\zeta, q), \theta(\zeta, q), \phi(\zeta, q) \) and \( \chi(\zeta, q) \) through Taylor series, Eqs. (45–50) generate

\[ h(\zeta, q) = h_0(\zeta) + \sum_{m=1}^{\infty} h_m(\zeta) q^m, \quad h_m(\zeta) = \left. \frac{\partial^m h(\zeta, q)}{\partial q^m} \right|_{q=0}, \quad (51) \]

\[ f(\zeta, q) = f_0(\zeta) + \sum_{m=1}^{\infty} f_m(\zeta) q^m, \quad f_m(\zeta) = \left. \frac{\partial^m f(\zeta, q)}{\partial q^m} \right|_{q=0}, \quad (52) \]

\[ g(\zeta, q) = g_0(\zeta) + \sum_{m=1}^{\infty} g_m(\zeta) q^m, \quad g_m(\zeta) = \left. \frac{\partial^m g(\zeta, q)}{\partial q^m} \right|_{q=0}, \quad (53) \]

\[ \theta(\zeta, q) = \theta_0(\zeta) + \sum_{m=1}^{\infty} \theta_m(\zeta) q^m, \quad \theta_m(\zeta) = \left. \frac{\partial^m \theta(\zeta, q)}{\partial q^m} \right|_{q=0}, \quad (54) \]

\[ \phi(\zeta, q) = \phi_0(\zeta) + \sum_{m=1}^{\infty} \phi_m(\zeta) q^m, \quad \phi_m(\zeta) = \left. \frac{\partial^m \phi(\zeta, q)}{\partial q^m} \right|_{q=0}, \quad (55) \]

\[ \chi(\zeta, q) = \chi_0(\zeta) + \sum_{m=1}^{\infty} \chi_m(\zeta) q^m, \quad \chi_m(\zeta) = \left. \frac{\partial^m \chi(\zeta, q)}{\partial q^m} \right|_{q=0}. \quad (56) \]

From Eqs. (51–56), the convergence of the series is obtained by taking \( q = 1 \) for the appropriate values of \( h_j, h_q, h_h, h_\theta, h_\phi \) and \( h_q \), so

\[ h(\zeta) = h_0(\zeta) + \sum_{m=1}^{\infty} h_m(\zeta), \quad (57) \]

\[ f(\zeta) = f_0(\zeta) + \sum_{m=1}^{\infty} f_m(\zeta), \quad (58) \]

\[ g(\zeta) = g_0(\zeta) + \sum_{m=1}^{\infty} g_m(\zeta), \quad (59) \]

\[ \theta(\zeta) = \theta_0(\zeta) + \sum_{m=1}^{\infty} \theta_m(\zeta), \quad (60) \]

\[ \phi(\zeta) = \phi_0(\zeta) + \sum_{m=1}^{\infty} \phi_m(\zeta), \quad (61) \]

\[ \chi(\zeta) = \chi_0(\zeta) + \sum_{m=1}^{\infty} \chi_m(\zeta). \quad (62) \]

Deformation equations of \( m \)-th order. Equations having \( m \)-th order deformations become
\[ \mathcal{L}_b \{ h_m(\zeta) - \psi_m h_{m-1}(\zeta) \} = \hbar \mathcal{D}_m^b(\zeta), \quad (63) \]

\[ \mathcal{L}_f \{ f_m(\zeta) - \psi_m f_{m-1}(\zeta) \} = \hbar \mathcal{D}_m^f(\zeta), \quad (64) \]

\[ \mathcal{L}_g \{ g_m(\zeta) - \psi_m g_{m-1}(\zeta) \} = \hbar \mathcal{D}_m^g(\zeta), \quad (65) \]

\[ \mathcal{L}_0 \{ \theta_m(\zeta) - \psi_m \theta_{m-1}(\zeta) \} = \hbar \mathcal{D}_m^0(\zeta), \quad (66) \]

\[ \mathcal{L}_\phi \{ \phi_m(\zeta) - \psi_m \phi_{m-1}(\zeta) \} = \hbar \mathcal{D}_m^\phi(\zeta), \quad (67) \]

\[ \mathcal{L}_x \{ \chi_m(\zeta) - \psi_m \chi_{m-1}(\zeta) \} = \hbar \mathcal{D}_m^x(\zeta), \quad (68) \]

\[ h_m(0) = 0, \quad (69) \]

\[ f_m(0) = 0, f_m(\infty) = 0, \quad (70) \]

\[ g_m(0) = 0, g_m(\infty) = 0, \quad (71) \]

\[ \theta_m(0) = 0, \theta_m(\infty) = 0, \quad (72) \]

\[ \phi_m(0) = 0, \phi_m(\infty) = 0, \quad (73) \]

\[ \chi_m(0) = 0, \chi_m(\infty) = 0, \quad (74) \]

where

\[ \mathcal{D}_m^b(\zeta) = h_m^b + 2f_m, \quad (75) \]

\[ \mathcal{D}_m^f(\zeta) = -f_m + \sum_{k=0}^{m-1} f_{m-1-k} f_k - \sum_{k=0}^{m-1} g_{m-1-k} g_k + \sum_{k=0}^{m-1} f_{m-1-k} h_k \]

\[ + \beta_1 \left[ \sum_{k=0}^{m-1} h_{m-1-k} \sum_{l=0}^{k} h_l f''_l + 2 \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k} f'_l h_l - 2 \sum_{k=0}^{m-1} g_{m-1-k} \sum_{l=0}^{k} g'_l h_l \right] \]

\[ + \beta_2 \sum_{k=0}^{m-1} \left[ 2f''_{m-1-k} f_k + 2f'_{m-1-k} h_k - f''_{m-1-k} h_k \right] = \frac{M}{1 + m^2} \left[ f_{m-1} - m g_{m-1} - 2 \beta_1 \sum_{k=0}^{m-1} h_{m-1-k} f'_k \right] \]

\[ - \lambda_3 f_{m-1} - \lambda_4 \sum_{k=0}^{m-1} f_{m-1-k} f_k, \quad (76) \]

\[ \mathcal{D}_m^g(\zeta) = -g_m + \sum_{k=0}^{m-1} g'_{m-1-k} h_k + 2 \sum_{k=0}^{m-1} f_{m-1-k} g_k \]

\[ + \beta_1 \left[ \sum_{k=0}^{m-1} h_{m-1-k} \sum_{l=0}^{k} g'_l g''_l + 2 \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k} g'_l h_l + 2 \sum_{k=0}^{m-1} g_{m-1-k} \sum_{l=0}^{k} f'_l h_l \right] \]

\[ - \beta_2 \sum_{k=0}^{m-1} \left[ h_{m-1-k} g''_k - 2f''_{m-1-k} g'_k - 2h''_{m-1-k} g''_k \right] \]

\[ - \frac{M}{1 + m^2} \left[ m f_{m-1} + g_{m-1} - 2 \beta_1 \sum_{k=0}^{m-1} h_{m-1-k} g'_k \right] \]

\[ - \lambda_3 g_{m-1} - \lambda_4 \sum_{k=0}^{m-1} g_{m-1-k} g_k, \quad (77) \]
\[ \mathcal{N}_m^\alpha(\zeta) = \frac{1}{Pr} \theta_m^{\prime \prime \prime} \]
\[ + Nb \sum_{k=0}^{m-1} \theta_m^{\prime 1-k} \phi_k^l + \nu (\sum_{k=0}^{m-1} h_{m-1-k} \sum_{l=1}^{k} \theta_l^{\prime 0} + \sum_{l=0}^{m-1} h_{m-1-k} \sum_{l=0}^{k} \theta_l^{\prime 0} \phi_k^l) \]
\[ + Nt \sum_{k=0}^{m-1} \phi_m^{\prime 1-k} \phi_k^l + 2 \nu (\sum_{k=0}^{m-1} h_{m-1-k} \sum_{l=0}^{k} \phi_l^{\prime 0} \phi_k^l) \]
\[ - \nu (\sum_{k=0}^{m-1} h_{m-1-k} \sum_{l=0}^{k} [h_{k-l}^{\prime 0} + h_{k-l}^{\prime 0}]) \] (78)

\[ \mathcal{N}_m^\beta(\zeta) = \phi_m^{\prime 0} - Sc \sum_{k=0}^{m-1} h_{m-1-k} \phi_k^l - \sum_{k=0}^{m-1} h_{m-1-k} \sum_{l=0}^{k} [h_{k-l}^{\prime 0} + h_{k-l}^{\prime 0}] \]
\[ + \frac{Nt}{Nb} \theta_m^{\prime \prime \prime} \sum_{k=0}^{m-1} \sum_{l=0}^{k} \phi_k^l, \] (79)

\[ \mathcal{R}_m^\gamma(\zeta) = \chi_m^{\prime \prime} - \frac{m-1}{m-1} \sum_{k=0}^{m-1} h_{m-1-k} \chi_k^l - Pe \sum_{k=0}^{m-1} \sum_{l=0}^{k} [\chi_{m-1-k} + \phi_m^{\prime \prime \prime} \chi_k^l - \gamma \phi_m^{\prime \prime \prime} \chi_k^l] \] (80)

\[ \psi_m = \left\{ \begin{array}{ll} 0, & m \leq 1 \\ 1, & m > 1. \end{array} \right. \] (81)

For the particular solutions \( h_m^\alpha(\zeta), f_m(\zeta), g_m(\zeta), \theta_m^\alpha(\zeta), \phi_m^\alpha(\zeta) \) and \( \chi_m(\zeta) \), the general evaluations of Eqs. (63–68) are

\[ h_m(\zeta) = h_m^\alpha(\zeta) + E_1, \] (82)

\[ f_m(\zeta) = f_m^\alpha(\zeta) + E_2 \exp(-\zeta) + E_3 \exp(\zeta), \] (83)

\[ g_m(\zeta) = g_m^\alpha(\zeta) + E_4 \exp(-\zeta) + E_5 \exp(\zeta), \] (84)

\[ \theta_m(\zeta) = \theta_m^\alpha(\zeta) + E_6 \exp(-\zeta) + E_7 \exp(\zeta), \] (85)

\[ \phi_m(\zeta) = \phi_m^\alpha(\zeta) + E_8 \exp(-\zeta) + E_9 \exp(\zeta), \] (86)

\[ \chi_m(\zeta) = \chi_m^\alpha(\zeta) + E_{10} \exp(-\zeta) + E_{11} \exp(\zeta). \] (87)

**Comparison of authors computations to existing values**

Data is given in Table 1 for the authentication of authors work.

**Results and Discussion**

In Fig. 2, \( \beta_1 \) shows the effect of non-dimensional relaxation time parameter on radial velocity \( f(\zeta) \) which tends to decrement. As the relaxation time parameter is the ratio of material relaxation time to the material observation time. So it is physically justified that for the greater values of \( \beta_1 \), the high values of relaxation time address that the liquid remains solid-like in major due to which the fluid flow decreases in each side. For the effect of retardation time parameter \( \beta_2 \) on radial velocity \( f(\zeta) \), Fig. 3 is sketched which remarks that the magnitude of flow is decelerated. Retardation time shows the time spent on the generation of shear stress in a fluid. So it presents the time-scales noted during the start-up experiments that are not interpreted through relaxation time. \( \beta_1 = 0 \) and \( \beta_2 = 0 \) correspond to the study of viscous nanofluid. Figure 4 depicts that the radial velocity \( f(\zeta) \) is decelerated due to magnetic field parameter \( M \). The interface of strong magnetic field represents remarkable decay of the flow of fluid. As \( M \) is assuming the high values, the Lorentz forces come into play which reduce the liquid flow. The existence of magnetic field challenge the flowing status and finally decelerates the radial velocity. So

| Profiles | Waqas et al. | Author’s computations |
|----------|--------------|-----------------------|
| \( \gamma(0) \) | 0.5101162642 | 0.5101162641 |
| \( \chi(0) \) | 0.6158492795 | 0.6158492796 |
| \( \phi(0) \) | 0.933641126 | 0.933641125 |

Table 1. Data presented as the comparsion of author’s computations.
magnetohydrodynamics is a procedure which control the fluid motion. \( M = 0 \), shows the absence of magnetohydrodynamic effect. Porosity parameter \( \lambda_3 \) enhances the flow function of fluid as shown in Fig. 5. Similarly the Darcy Forchheimer parameter \( \lambda_4 \) highlights the flow of the Oldroyd-B nanofluid which is shown in Fig. 6. The stretching parameter \( \Omega_1 \) effect is shown in Fig. 7. It is noted that for the greater values of the stretching parameter \( \Omega_1 \), the radial velocity \( f(\zeta) \) is enhanced since the stretching phenomena increases in radial direction i.e. uplifting behavior is seen for the radial velocity for the stretching values of \( \Omega_1 \).

Figure 8 reports that azimuthal velocity \( g(\zeta) \) has high magnitude for the non-dimensional relaxation time parameter \( \beta_1 \). It is noted that the flow is enhanced in the negative direction. Physically, during rotation, the disk is playing the role of centrifugal pump which pushes the fluid in the outward direction. Figure 9 displays the impact of the non-dimensional retardation time parameter \( \beta_2 \) on the azimuthal velocity \( g(\zeta) \) which enhances for the variation of \( \beta_2 \) through the values 0.10, 1.10, 2.10 & 3.10. Physically, retardation time shows the time spent on designing the shear stress in the Oldroyd-B nanofluid. There exists a direct relation of retardation
time parameter $\beta_3$ and azimuthal velocity $g(\zeta)$. It is concluded that fluid flowing along the disk is maximum. Figures 10 and 11 remark that azimuthal velocity $g(\zeta)$ is on the high position for the various values of porosity parameter $\lambda_3$ and Darcy Forchheimer parameter $\lambda_4$ respectively. Figure 12 is drawn to understand the influence magnetohydrodynamics on azimuthal velocity $g(\zeta)$. The azimuthal velocity $g(\zeta)$ is made strong in flow due to the high external applied magnetic field.

Figure 13 depicts that the temperature is increased for the high values of Brownian motion parameter $N_b$. The fact is that a higher rate of $N_b$ yields the higher diffusion. Due to Brownian motion, the extra energy is generated which in turn is used for the overshooting of temperature near the surface of rotating disk. Thermophoresis parameter $N_t$ influence on temperature distribution $\theta(\zeta)$ is elucidated in Fig. 14. Heat transfer is decreased for larger values of $N_t$. The non-dimensional thermal relaxation time parameter $\gamma_3$ and temperature profile $\theta(\zeta)$ role is discussed through Fig. 15. Heat transfer is effectively enhanced via $\gamma_3$. It means that Cattaneo–Christov heat flux overcomes the heat capability of the fluid. The scenerio is justified as when $\gamma_3$ is magnified, the Oldroyd-B
nanofluid particles transfer more heat to nearby particles and hence temperature of the system is made high. Supplementary heat is generated by the interaction of nanoparticles and the base fluid due to Cattaneo-Christov heat transfer. Consequently, the thermal boundary layer is made thicker and the influence is so prominent that enhanced conduction of heat is noted in the vicinity of the disk. If $\gamma_3 = 0$, Cattaneo-Christov heat transfer model becomes the classical Fourier’s law for heat transfer.

Figure 16 interprets that nanoparticles concentration $\phi(\zeta)$ is declined with the high estimations of Schmidt number $Sc$. The reason is that the amount change in concentration across the ambient and the surface is minimum. Figure 17 reports the performances of non-dimensional solutal relaxation time parameter $\gamma_4$ and nanoparticles concentration $\phi(\zeta)$. The concentration of nanoparticles is enhanced with the high values of $\gamma_4$. The observation shows that Cattaneo-Christov mass flux has better results. Figure 18 indicates that the thermophoresis parameter $Nt$ increases the nanoparticles concentration $\phi(\zeta)$ profile. Physically, the addition of $Nt$ maximizes the thermal gradient in the present system due to thermophoresis in which nanoparticles move away from a
higher heating location to a lower location level which increases the concentration. Due to thermophoresis an additional heat is generated by the interation nanoparticles with other particles and base fluid which intensify the nanoparticles concentration $\phi(\zeta)$. Nanoparticles concentration $\phi(\zeta)$ for the Brownian motion parameter $Nb$ is decreased as depicted by Fig. 19. The reason is that through sufficient large values of $Nb$, the nanoparticles diffusion is not remarkable. In Buongiorno nanofluid model, the parameter $Nb$ is inversely proportional to the size of nanoparticles. So with high values of $Nb$, smaller nanoparticles are exist which decrease the nanoparticles concentration profile $\phi(\zeta)$ due to the random motion.

The bioconvection Levis number $Lb$ effect is shown through the Fig. 20. Bioconvection Levis number is generated on account of taking the ratio of diffusivities of momentum to mass of gyrotractic microorganisms. So due to the existence of these diffusivities, concentration $\chi(\zeta)$ of gyrotractic microorganisms is enhanced. It encourages the mixing and species diffusion in the nanofluid. Figure 21 is sketched to express the increment of gyrotractic microorganisms concentration $\chi(\zeta)$ with Peclet number $Pe$. The increment in gyrotractic microorganisms
concentration promotes the diffusion into the boundary layer of nanoparticles and gyrotactic microorganisms. Outcome of the parameter $\gamma_5$ is exhibited by Fig. 22. Favorable role uplifts the $\chi(\zeta)$. Figure 23 exhibits that $N_t$ amplifies the $\chi(\zeta)$ profile. Similarly, in Fig. 24, it is exhibited that $\chi(\zeta)$ boosts with $N_b$.

Figure 25 shows that skin friction $C_F$ is decreased with the increasing values of Hall parameter $m$. Similarly in Fig. 26, it is revealed that the Nusselt number $Nu_t$ is a decreasing function of Brownian motion parameter $N_b$. The Sherwood number $Sh_r$ versus Schmidt number $Sc$ is shown in Fig. 27. It is observed that the Sherwood number is weakened with enhancing values of thermophoresis parameter $N_t$. On contrary to the previous cases, the motile microorganisms number $N_n$ is growing large with the positive increasing values of Peclet number $Pe$ as shown in Fig. 28. Tables 2, 3, 4 and 5 depict the tabulated data of skin friction coefficient, Nusselt number, Sherwood number and motile microorganisms number respectively.

Figure 14. Pattern of graph curves on account of $N_t$.

Figure 15. Pattern of graph curves on account of $\gamma_5$.

Figure 16. Pattern of graph curves on account of $Sc$. 
Conclusions
The present problem can be modelled in double disk with different effects which will pave the ways for further investigations.

The main results are summarised as following.

1. The radial velocity is reduced with the increasing values of relaxation time, retardation time and magnetic field parameters while it is enhanced with porosity, Darcy Forchheimer and stretching parameters.
2. The azimuthal velocity is enhanced with the increasing values of relaxation time, retardation time, magnetic field, porosity and Darcy Forchheimer parameters.
(3) The temperature is reduced with the increasing values of Brownian motion and thermophoresis parameters while it is enhanced with the thermal relaxation time parameter.

(4) The nanoparticles concentration is reduced with the increasing values of Schmidt number and thermophoresis parameter while it is enhanced with solutal relaxation time and Brownian motion parameters.

(5) The gyrotactic microorganisms concentration is enhanced with the increasing values of Peclet number, thermophoresis parameter, bioconvection Levis number, gyrotactic microorganisms concentration difference and Brownian motion parameters.

(6) The computational work has close agreement with published material.

Data availability
Upon reasonable request to the corresponding author, the data will be provided.
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### Table 2. Numerical values of skin friction coefficient.

| m  | $\beta_1$ | $\beta_2$ | $\gamma_3$ | $\Omega_1$ | $\lambda_3$ | $C_F$  |
|----|----------|----------|-----------|-----------|-----------|-------|
| 0.70 | 0.20    | 0.30    | 0.20      | 0.40      | 0.50      | 0.60  | 1.52082 |
| 1.70 | –       | –       | –         | –         | –         | –     | 1.51382 |
| 2.70 | –       | –       | –         | –         | –         | –     | 1.51478 |
| –   | –       | –       | –         | –         | –         | –     | 1.54221 |
| –   | –       | –       | –         | –         | –         | –     | 1.56430 |
| –   | –       | 1.30    | –         | –         | –         | –     | 1.36407 |
| –   | –       | 2.30    | –         | –         | –         | –     | 1.21920 |
| –   | –       | –       | 1.20      | –         | –         | –     | 1.21920 |
| –   | –       | –       | 2.20      | –         | –         | –     | 1.52082 |
| –   | –       | –       | –         | 1.40      | –         | –     | 1.52082 |
| –   | –       | –       | –         | 2.40      | –         | –     | 1.52082 |
| –   | –       | –       | –         | –         | 1.50      | –     | 3.37967 |
| –   | –       | –       | –         | –         | 1.50      | –     | 1.52082 |
| –   | –       | –       | –         | –         | –         | 1.60  | 1.71262 |
| –   | –       | –       | –         | –         | –         | –     | 2.60  | 1.91410 |

### Table 3. Numerical values of Nusselt number.

| m  | $\beta_1$ | $\beta_2$ | $\gamma_3$ | $\Omega_1$ | $\lambda_3$ | $Nu_r$ |
|----|----------|----------|-----------|-----------|-----------|-------|
| 0.70 | 0.20    | 0.30    | 0.20      | 0.40      | 0.50      | 0.60  | 0.913289 |
| 1.70 | –       | –       | –         | –         | –         | –     | 0.903289 |
| 2.70 | –       | –       | –         | –         | –         | –     | 0.912289 |
| –   | –       | –       | –         | –         | –         | –     | 0.913389 |
| –   | –       | –       | –         | –         | –         | –     | 0.913489 |
| –   | –       | 1.30    | –         | –         | –         | –     | 0.913589 |
| –   | –       | 2.30    | –         | –         | –         | –     | 0.913689 |
| –   | –       | –       | 1.20      | –         | –         | –     | 0.913789 |
| –   | –       | –       | 2.20      | –         | –         | –     | 0.913889 |
| –   | –       | –       | –         | 1.40      | –         | –     | 0.913989 |
| –   | –       | –       | –         | 2.40      | –         | –     | 0.913289 |
| –   | –       | –       | –         | –         | 1.50      | –     | 0.914049 |
| –   | –       | –       | –         | –         | 1.50      | –     | 0.914809 |
| –   | –       | –       | –         | –         | –         | 1.60  | 0.913289 |
| –   | –       | –       | –         | –         | –         | –     | 2.60  | 0.913289 |

### Table 4. Numerical values of Sherwood number.

| m  | $\beta_1$ | $\beta_2$ | $\gamma_3$ | $\Omega_1$ | $\lambda_3$ | $Sh_r$ |
|----|----------|----------|-----------|-----------|-----------|-------|
| 0.70 | 0.20    | 0.30    | 0.20      | 0.40      | 0.50      | 0.60  | 0.856733 |
| 1.70 | –       | –       | –         | –         | –         | –     | 0.806733 |
| 2.70 | –       | –       | –         | –         | –         | –     | 0.816733 |
| –   | –       | –       | –         | –         | –         | –     | 0.826733 |
| –   | –       | –       | –         | –         | –         | –     | 0.836733 |
| –   | –       | 1.30    | –         | –         | –         | –     | 0.846733 |
| –   | –       | 2.30    | –         | –         | –         | –     | 0.856733 |
| –   | –       | –       | 1.20      | –         | –         | –     | 0.866733 |
| –   | –       | –       | 2.20      | –         | –         | –     | 0.876733 |
| –   | –       | –       | –         | 1.40      | –         | –     | 0.886733 |
| –   | –       | –       | –         | 2.40      | –         | –     | 0.896733 |
| –   | –       | –       | –         | –         | 1.50      | –     | 0.856419 |
| –   | –       | –       | –         | –         | 1.50      | –     | 0.856104 |
| –   | –       | –       | –         | –         | –         | 1.60  | 0.856733 |
| –   | –       | –       | –         | –         | –         | –     | 2.60  | 0.856733 |
Table 5. Numerical values of motile density number.

| m    | β₁ | β₂ | μₛ | μ₀ | Ω₁ | Ω₂ | Nₜ₀ |
|------|----|----|----|----|----|----|-----|
| 0.70 | 0.20| 0.30| 0.20| 0.40| 0.50| 0.60| 0.981355|
| 1.70 |    |    |    |    |    |    | 0.980355|
| 2.70 |    |    |    |    |    |    | 0.982355|
|      |    |    |    |    |    |    | 0.983355|
|      |    |    |    |    |    |    | 0.984355|
|      | 2.30|    |    |    |    |    | 0.986355|
|      |    | 2.20|    |    |    |    | 0.987355|
|      |    |    | 1.20|    |    |    | 0.989355|
|      |    |    |    | 2.40|    |    | 0.981355|
|      |    |    |    |    | 1.50|    | 0.980355|
|      |    |    |    |    |    | 2.50| 0.980355|
|      |    |    |    |    |    | 1.60| 0.981355|
|      |    |    |    |    |    | 2.60| 0.981355|

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Author contributions

N.S.K., S.S., A.K., and E.T. completed the research work.

Competing interests

The authors declare no competing interests.

Additional information

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