ON THE STRUCTURE OF THE MODULI OF JETS OF
G-STRUCTURES WITH LINEAR CONNECTION.

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ABSTRACT. The moduli space of jets of certain G-structures (basically those which admit a canonical linear connection) is shown to be isomorphic to the quotient of a natural G-module by G.

1. INTRODUCTION

Given a closed subgroup $G$ of the general linear group $GL(n, \mathbb{R})$, a $G$-structure is a reduced bundle $P(M, G)$ with structure group $G$ of the bundle of linear frames $FM \to M$.

The main types of geometries arise from different choices of $G$. For example, there is a one-to-one correspondence between the set of Riemannian metrics on $M$ and the set of $O(n)$-structures on $M$. Analogously, almost Hermitian geometries correspond to $U(n/2)$-structures, almost symplectic geometries to $Sp(n/2)$-structures, and so on.

It is a well-known fact that $G$-structures on $M$ are in one-to-one correspondence with smooth sections $s \in \Gamma(FM/G)$ of the quotient bundle $FM/G \to M$. Denoting by $q : FM \to FM/G$ the natural projection, the fiber over each $x \in M$ of the $G$-structure $P_s \to M$ associated to $s \in \Gamma(FM/G)$ is $(P_s)_x = \{u = (x; X_1, \ldots, X_n) \in F_xM \mid q(u) = s(x)\}$.

The group $\text{Diff}(M)$ of diffeomorphisms of $M$ acts in a natural way on the set of $G$-structures as follows. For each diffeomorphism $f : M \to M'$, there is an associated diffeomorphism $\tilde{f} : FM \to FM'$ given by $\tilde{f}(x; X_1, \ldots, X_n) = (f(x); f_*(X_1), \ldots, f_*(X_n))$, which defines a diffeomorphism $\tilde{f} : FM/G \to FM/G$ by $\tilde{f}[u] = [\tilde{f}(u)]$. The action of $\text{Diff}(M)$ on $\Gamma(FM/G)$ is defined by:

$$f \cdot s = \tilde{f} \circ s \circ f^{-1}, \quad f \in \text{Diff}(M), \ s \in \Gamma(FM/G).$$

Two $G$-structures $s$ and $s'$ are said to be equivalent if they are related by a diffeomorphism $f \in \text{Diff}(M)$, which amounts to the fact that $\tilde{f}(P_s) = P_{s'}$, and they are said to be locally equivalent at points $p \in M$ and $p' \in M'$ if they are equivalent in some open neighborhoods of $p$ and $p'$ by a diffeomorphism which maps $p$ to $p'$. We call the quotient $\mathfrak{M}_G(M) = \Gamma(FM/G)/\text{Diff}(M)$ the moduli space of $G$-structures on $M$. The description of this space is a basic problem in differential geometry.

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It is possible to state an analogous problem for analytic $G$-structures. To study this category, it is natural to introduce the spaces $J^r(FM/G)$ of $r$-jets of $G$-structures. The action of $\text{Diff}(M)$ on $\Gamma(FM/G)$ induces a natural action of the groupoid $J^r_{\text{inv}}(M,M)$ of $(r+1)$-jets of diffeomorphisms of $M$ on the space $J^r(M,M)$ as follows:

$$(j^r_{x+1} f) \cdot (j^r_s) = j^r_{f(x)} (f \cdot s), \quad j^r_{x+1} f \in J^r_{\text{inv}}(M,M), \quad j^r_s \in J^r(M,M).$$

We will call the quotient $\mathcal{M}_G^r(M) = J^r(FM/G)/J^r_{\text{inv}}(M,M)$ the moduli space of $r$-jets of $G$-structures on $M$. There are natural projections $\mathcal{M}_G^{r+k}(M) \to \mathcal{M}_G^r(M)$, $k \geq 0$, so that we can define the moduli space of jets of $G$-structures as the projective limit $\mathcal{M}_G^\infty(M) = \lim_{\leftarrow} \mathcal{M}_G^r(M)$. The local equivalence problem for analytic $G$-structures can be reduced to the study of this moduli space.

Another reason for analyzing $\mathcal{M}_G^\infty(M)$ is that it is the space where the geometric objects associated to $G$-structures are defined.

In general, the description of $\mathcal{M}_G^\infty(M)$ is an extremely complicated problem. The aim of this paper is to describe, to some extent, the structure of this space in the particular case of $G \subset GL(n,\mathbb{R})$ being a closed subgroup such that the first prolongation $\mathfrak{g}^{(1)}$ vanishes and there exists a supplementary $G$-submodule $W$ of $\delta ((\mathbb{R}^n)^* \otimes \mathfrak{g})$ in $\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$. As we will explain later, this is a technical condition which assures that $G$-structures have a canonical linear connection attached.

More precisely, we will prove the following theorem.

**Theorem 1.1.** Let $M$ be a $n$-dimensional smooth manifold, and let $G \subset GL(n,\mathbb{R})$ be a closed subgroup such that the first prolongation $\mathfrak{g}^{(1)}$ of its Lie algebra $\mathfrak{g}$ vanishes and there exists a supplementary $G$-submodule $W$ of $\delta ((\mathbb{R}^n)^* \otimes \mathfrak{g})$ in $\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$. Then, there exists a family of $G$-modules $S'$ and homomorphisms $S'^{r+k} \to S'$, $k \geq 0$, such that each space $\mathcal{M}_G^r(M)$ of $r$-jets of $G$-structures is canonically isomorphic to the quotient $S'/G$.

The moduli space $\mathcal{M}_G^\infty(M)$ is then canonically isomorphic to the quotient $S'^\infty/G$, where $S'^\infty = \lim_{\leftarrow} S'$.
The natural action of $G$ on $V$ defines linear $G$-actions on the spaces $V^* \otimes g$ and $\wedge^2 V^* \otimes V$, with respect to which the operator $\delta$ is an homomorphism of $G$-modules. The image of this operator is a $G$-submodule and we assume that it admits a supplementary $G$-submodule $W$:

$$\wedge^2 V^* \otimes V = \delta (V^* \otimes g) \oplus W.$$ (1)

Given a $G$-structure $s \in \Gamma(FM/G)$, the bundle $\wedge^2 T^* M \otimes TM$ is identified with the associated bundle $P_s \times_G \left( \wedge^2 V^* \otimes V \right)$. The decomposition (1) allows to express this bundle as the sum:

$$\wedge^2 T^* M \otimes TM = (P_s \times_G \delta (V^* \otimes g)) \oplus (P_s \times_G W).$$

Let $\theta : T(FM) \to V$ be the canonical 1-form of $FM$, which maps each tangent vector $X_u$ at a frame $u \in FM$ to $\theta (X_u) = u^{-1} (\pi_\star (X_u)) \in V$, where $\pi : FM \to M$ is the canonical projection and the frame $u$ is understood as a linear isomorphism $u : V \to T_{\pi(u)}M$. We also denote by $\theta$ the restriction of the canonical form to the subbundle $P_s$. The restriction of the projection $\pi_\star$ to each horizontal complement $H_u \subset T_u P_s$ to the vertical tangent space at $u \in P_s$ is an isomorphism $H_u \cong T_{\pi(u)}M$, so that for any frame $u$ and any vector $v \in V$ there is a unique $B_u (v) \in H_u$ such that $\theta (B_u (v)) = v$. The horizontal complement $H_u$ defines then an element $t(H_u) \in \wedge^2 V^* \otimes V$ by:

$$t(H_u)(v, w) = d\theta (B_u(v), B_u(w)), \quad v, w \in V.$$ 

If $H_u$ are the horizontal subspaces of a linear connection on $M$ adapted to $P_s$, then there is a well-defined and $G$-equivariant torsion function: $P_s \to \wedge^2 V^* \otimes V$, given by $u \mapsto t(H_u)$, which is related to the torsion tensor $T \in \Gamma \left( \wedge^2 T^* M \otimes TM \right)$ by: $t(H_u)(v, w) = u^{-1} (T_x (u(v), u(w)))$ for each $u \in P_s$ and $v, w \in V$.

It can be shown (see, e.g., [2] and [3]) that the condition $t(H_u) \in W$ characterizes the horizontal subspaces defining a linear connection adapted to $P_s$. In other words:

**Theorem 2.1** ([4]). Let $G \subset GL(n, \mathbb{R})$ be a closed Lie subgroup such that the first prolongation $g^{(1)}$ vanishes and there exists a supplementary $G$-submodule $W$ of $\delta (V^* \otimes g)$ in $\wedge^2 V^* \otimes V$. Then, for each $G$-structure $s \in \Gamma(FM/G)$ and each $u \in P_s := s(M)$ there exists a unique horizontal space $H_u \subset T_u P$ such that $t(H_u) \in W$. These horizontal spaces define a linear connection $\nabla(s)$ adapted to $P_s$ which is characterized by the condition that its torsion is a section of the vector bundle $P_s \times_G W$.

Next two lemmas provide some properties of this canonical connection which will be used to describe the structure of the moduli spaces in later sections.

**Lemma 2.2.** The assignment $s \mapsto \nabla(s)$ defines an operator $\nabla : \Gamma(FM/G) \to \Gamma(C(M))$, with $C(M) \to M$ being the affine bundle of linear connections on $M$, which satisfies:

1. $\nabla$ is natural, i.e., for each diffeomorphism $f : M \to M'$, the direct image of $\nabla(s)$ onto $\tilde{f}(P_s)$ is $\nabla(f \cdot s)$.
2. $\nabla(s)$ is an operator of order 1, i.e., if $j^1 s = j^1 s'$, then $\nabla(s)(x) = \nabla(s')(x)$. Thus, $\nabla$ defines a map, which we will also denote $\nabla : J^1 (FM/G) \to C(M)$, by $\nabla(j^1 s) = \nabla(s)(x)$. 


Proof. In order to prove the naturality of the map \( \nabla : \Gamma(FM/G) \to \Gamma(C(M)) \), it suffices to show that \( t(H_u) \in W \) implies \( t(\tilde{f}_*(H_u)) \in W \) for each diffeomorphism \( f : M \to M' \) because, then, the direct image \( \tilde{f}(H_u) \subset \tilde{f}_*(T_uP) = T_{\tilde{f}(u)}P_{f*,} \) of each horizontal space of the connection \( \nabla(s) \) by \( f \) is, by Theorem 2.1, a horizontal space of the connection \( \nabla(f \cdot s) \). But this follows easily from the functoriality of the canonical 1-form of the bundle of linear frames. Explicitly, the canonical 1-form \( \theta' : TFM' \to V \) of \( FM' \) is related to \( \theta \) by \( \theta = \tilde{f}^*\theta' \) as one can easily check. From this it follows that \( \theta' (\tilde{f}_*(B_u)(v)) = \theta ((B_u)(v)) = v \), for all \( v \in V \), and therefore we have that \( B'_{\tilde{f}(u)} = f_*(B_u) \) and so

\[
\sum t(\tilde{f}_*(H_u))(v, w) = d\theta' \left( B'_{\tilde{f}(u)}(v), B'_{\tilde{f}(u)}(w) \right) \\
= d\theta' (\tilde{f}_*(B_u)(v), \tilde{f}_*(B_u)(w)) \\
= d\left( \tilde{f}^*\theta' \right) ((B_u)(v), (B_u)(w)) \\
= d\theta ((B_u)(v), (B_u)(w)) = t(H_u)(v, w).
\]

To prove that \( \nabla \) is an operator of order 1, first notice that the condition \( j^2_i s = j^2_i \) is equivalent to the fact that the mappings \( s_* : T_xM \to T_{s(x)}(FM/G) \) and \( s'_* : T_xM \to T_{s'(x)}(FM/G) \) coincide. But, then, the equality \( T_uP_s = T_uP_{s'} \) holds for any \( u \in s(x) = s'(x) \). To see this, notice that if \( q : FM \to FM/G \) is, as before, the natural projection, then:

\[
T_uP_s = q^{-1}(T_{q(u)}(s(M))) \\
= q^{-1}(s_*(T_xM)) \\
= q^{-1}(s'_*(T_xM)) \\
= T_uP_{s'}.
\]

Since \( t(H_u) \in W \) is an algebraic condition depending only on the vector spaces \( T_uP_s \) and \( W \), it follows that the equality \( T_uP_s = T_uP_{s'} \) implies the equality of the horizontal subspaces defining the connections: \( H_u = H'_u \) for \( u \in s(x) = s'(x) \).}

Now, let \( \sigma : \mathcal{U} \subset M \to P_s \) be a section defined in a coordinate neighborhood \( (\mathcal{U}, (x^1, \ldots, x^n)) \) of \( x \in M \). We will denote \( \sigma = (X_1, \ldots, X_n) \), with \( X_i = \sum_j \sigma_i^j \frac{\partial}{\partial y^j} \) for each \( i = 1, \ldots, n \). Let \( \omega \in \Gamma(T^*P_s \otimes \mathfrak{g}) \) be the connection form of \( \nabla := \nabla(s) \).

If we consider, as before, each \( \sigma(x) \in P_s \) as a linear isomorphism \( \sigma(x) : V \to T_xM \), we can define a function \( \eta \in C^\infty(\mathcal{U}, V^* \otimes \mathfrak{g}) \) by: \( \eta(x) = (\sigma^*\omega)_x \circ \sigma(x) \). The components of \( \eta \) in the standard basis \( \{v_1, \ldots, v_n\} \) of \( V \) are related to those of the local connection 1-form \( \sigma^*\omega \in \Gamma(T^*M \otimes \mathfrak{g}) \) by: \( \eta^i_j = (\sigma^*\omega)^i_j(X_i) \).

Let \( P_{\text{im}s} : \bigwedge^2 V^* \otimes V \to \delta (V^* \otimes \mathfrak{g}) \) be the projection onto \( \delta (V^* \otimes \mathfrak{g}) \) according to the decomposition [4]. Since \( \ker \delta = \{0\} \), it makes sense to consider the homomorphism:

\[
\delta^{-1} \circ P_{\text{im}s} : \bigwedge^2 V^* \otimes V \to V^* \otimes \mathfrak{g}.
\]

Lemma 2.3. The local connection 1-form \( \sigma^*\omega \in \Gamma(T^*M \otimes \mathfrak{g}) \) is determined by the equation: \( \eta(x) = -\left( \delta^{-1} \circ P_{\text{im}s} \right) \left( \tilde{t}(\sigma(x)) \right) \), where \( \tilde{t} \) is the torsion function of the linear flat connection making parallel \( \sigma \).
Proof. Using the identity $\nabla X X_j = \sum_k X_k (\sigma^* \omega)_{ij}^k (X)$, $X \in \Gamma(TM)$, $1 \leq j \leq n$, we see that the components of the torsion tensor of $\nabla$ in the moving frame $\sigma$, which are given by $T(X_i, X_j) = \nabla X_i X_j - \nabla X_j X_i - [X_i, X_j] = \sum_k T^k_{ij} X_k$, $1 \leq i, j \leq n$, can be written as $T^k_{ij} (x) = \eta^k_{ij} (x) - \eta^k_{ji} (x) + \tilde{t}^k_{ij} (\sigma(x))$. Using the relation $t(H_{\sigma(x)}) (v_i, v_j) = \sum_k T^k_{ij} (x) v_k$, last equation reads:

$$t(H_{\sigma(x)}) = \delta \eta (x) + \tilde{t} (\sigma(x)).$$

The condition $t(H_{\sigma(x)}) \in W$ can be written as $\big( \delta^{-1} \circ P_{\text{im} \delta} \big) \big( t(H_{\sigma(x)}) \big) = 0$ or, equivalently, $\eta (x) = - \big( \delta^{-1} \circ P_{\text{im} \delta} \big) \big( \tilde{t} (\sigma(x)) \big)$. \[\blacksquare\]

Remark 2.1. Notice that $\tilde{t}^k_{ij} (\sigma(x)) = - \sum_{h,l} \sigma^{kl} \left( \sigma_{hi} \frac{\partial \sigma_{lj}}{\partial x^h} - \sigma_{hj} \frac{\partial \sigma_{li}}{\partial x^h} \right)$, where $(\sigma^{ij})$ stands for the inverse matrix of $(\sigma_{ij})$. Thus, the components $(\sigma^* \omega)_{ij}^k \in \Gamma(T^* M)$ of the local connection 1-form $\sigma^* \omega \in \Gamma(T^* M \otimes \mathfrak{g})$ are given by:

$$\tag{2} (\sigma^* \omega)_{ij}^k (X_n) = \sum_{i,j,k,h,l} A_{\alpha \beta \gamma}^{ij} \sigma^{kl} \left( \sigma_{hi} \frac{\partial \sigma_{lj}}{\partial x^h} - \sigma_{hj} \frac{\partial \sigma_{li}}{\partial x^h} \right), 1 \leq \alpha, \beta, \gamma \leq n,$$

where the real coefficients $A^{ij}_{\alpha \beta \gamma}$ are determined by the homomorphism $\delta^{-1} \circ P_{\text{im} \delta}$. These coefficients are then universal, in the sense that they only depend on the group $G \subset GL(n, \mathbb{R})$ and the supplementary $W$ chosen to define the connections, but not on the particular $G$-structure considered.

Remark 2.2. Equation (2) shows that the map $\tilde{\nabla} : J^1 (FM) \to C(M)$ defined as $\tilde{\nabla} (j^1_{\sigma}) = \nabla (q \circ \sigma) (x)$ is smooth. Since the diagram

$$\begin{array}{ccc}
J^1 (FM) & \xrightarrow{q^1} & J^1 (FM/G) \\
\n \nabla \downarrow & & \nabla \downarrow \\
\n\n & & C(M)
\end{array}$$

is commutative, and the projection $q^1 : J^1 (FM) \to J^1 (FM/G)$ is a surjective submersion, we conclude that the map $\nabla : J^1 (FM/G) \to C(M)$ is also smooth.

To end this section, we illustrate the construction of canonical connections with some examples. In all cases we will take $W$ as the $G$-submodule of tensors $T \in \bigwedge^2 V^* \otimes V$ such that trace $(A \circ i_v T) = 0$ for every $A \in \mathfrak{g}$ and every $v \in V$. This is a supplementary $G$-submodule of the image of $\delta$ whenever $\mathfrak{g}^{(1)} = 0$ and $\mathfrak{g}$ is closed under transposition (see [10]).

Example 2.1. It is well-known that the canonical connection (in the sense of Theorem 2.1) associated to each $O(n)$-structure or, equivalently, to each Riemannian metric is the Levi-Civita connection. That is because for $G = O(n)$ the alternator operator $\delta$ is surjective and, therefore, the supplementary submodule in equation (3) is $W = \{ 0 \}$. The condition $\eta (x) = - \delta^{-1} \big( \tilde{t} (\sigma(x)) \big)$ is equivalent to Koszul formula, as we next explain.

The operator $\delta^{-1} \circ P_{\text{im} \delta} = \delta^{-1} : \bigwedge^2 V^* \otimes V \to V^* \otimes \mathfrak{g}$ is easily obtained as follows. Let us denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $V$, and take any $T = \delta \tau \in \bigwedge^2 V^* \otimes V$, $\tau \in V^* \otimes \mathfrak{g}$. Then, for any $u, v, w \in V$, the following relations
hold:
\[
\begin{align*}
\langle T(u, v), w \rangle &= \langle \tau(u)v, w \rangle - \langle \tau(v)u, w \rangle \\
\langle T(w, u), v \rangle &= \langle \tau(w)u, v \rangle - \langle \tau(u)w, v \rangle \\
\langle T(w, v), u \rangle &= \langle \tau(w)v, u \rangle - \langle \tau(v)w, u \rangle 
\end{align*}
\]
Adding the three equations, taking into account that \( g = \{ A \in V^* \otimes V \mid \langle Au, v \rangle + \langle Av, u \rangle = 0 \ \forall u, v \in V \} \), we obtain the expression of \( \tau = \delta^{-1}T \):
\[
\langle \tau(u)v, w \rangle = \frac{1}{2} (\langle T(u, v), w \rangle + \langle T(w, u), v \rangle + \langle T(w, v), u \rangle)
\]
for all \( u, v, w \in V \). Hence, the condition \( \eta(x) = -\delta^{-1} (\tilde{\iota}(\sigma(x))) \) can be written, in components, as:
\[
\eta^k_i(x) = -\frac{1}{2} \left( \tilde{\iota}^k_j(\sigma(x)) + \tilde{\iota}^j_k(\sigma(x)) + \tilde{\iota}^j_i(\sigma(x)) \right).
\]

Now, let \((M, g)\) be a Riemannian manifold and \( P_{\delta} \to M \) the corresponding \( O(n) \)-structure, that is, the subbundle of orthonormal frames with respect to \( g \). For any section \( \sigma = (X_1, \ldots, X_n) : \mathcal{U} \to P_{\delta} \), the components of \( \eta \in C^\infty(\mathcal{U}, V^* \otimes g) \) are given by: \( \eta^k_i = \langle \sigma^\ast \omega \rangle^k_j (X_i) = g(\nabla X_j, X_k, X_i) \), whereas those of \( \tilde{\iota}(\sigma(x)) \) are:
\[
\tilde{\iota}^k_i = -g([X_i, X_j], X_k).
\]
Substitution of these expressions in (4) yields:
\[
2g(\nabla X_j, X_k, X_k) = g([X_i, X_j], X_k) + g([X_k, X_j], X_i) + g([X_k, X_j], X_i),
\]
which is nothing but Koszul formula in the orthonormal frame \( \{X_1, \ldots, X_n\} \).

**Example 2.2.** Consider now the case of \((O(p) \times O(q))\)-structures, i.e., those with structure group \( G = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \mid A_1 \in O(p), A_2 \in O(q) \right\} \). These structures correspond to Riemannian almost product structures.

Let \( \{v_1, \ldots, v_n\} \) be the canonical basis of \( V \), and let us denote \( I_1 = \{1, \ldots, p\} \), \( I_2 = \{p + 1, \ldots, p + q\} \), \( V_\alpha = \text{Span}\{v_i\}_{i \in I_\alpha} \). It can be easily seen that \( W \) is given as the \( G \)-submodule of tensors \( T \in \Lambda^2 V^* \otimes \mathcal{V} \) such that:
\[
\langle T(u, v), w \rangle = \langle T(u, w), v \rangle \text{ if } v, w \in V_\alpha, \alpha = 1, 2,
\]
for all \( u \in V \).

In order to determine an expression of the operator \( \delta^{-1} \circ P_{\text{mor}} \), let us decompose any \( T \in \Lambda^2 V^* \otimes \mathcal{V} \) as \( T = \tilde{T} + \delta \tau \), with \( \tilde{T} \in W \) and \( \tau \in V^* \otimes g \).

If \( v \in V_\alpha, w \in V_\beta \), with \( \alpha \neq \beta \), then \( \langle \tau(u)v, w \rangle = 0 \) for each \( u \in V \), because \( \tau(u) \) belongs to \( g = o(p) \oplus o(q) \).

If \( u, v, w \in V_\alpha, \alpha = 1, 2 \), then \( \langle \tilde{T}(u, v), w \rangle = 0 \), as follows easily from (3). In this case, one can compute, as in the previous example:
\[
\langle \tau(u)v, w \rangle = \frac{1}{2} (\langle T(u, v), w \rangle + \langle T(w, u), v \rangle + \langle T(w, v), u \rangle).
\]

Finally, if \( u \in V_\alpha \) and \( v, w \in V_\beta, \alpha \neq \beta \), we have:
\[
\begin{align*}
\langle T(u, v), w \rangle &= \langle \tau(u)v, w \rangle + \langle \tilde{T}(u, v), w \rangle \\
\langle T(u, w), v \rangle &= \langle \tau(u)w, v \rangle + \langle \tilde{T}(u, w), v \rangle.
\end{align*}
\]
Substracting the second equation from the first one, we obtain:
\[ \langle \tau(u)v, w \rangle = \frac{1}{2} \left( \langle T(u,v), w \rangle - \langle T(u,w), u \rangle \right). \]

Thus, in components, we have:
\[
((\delta^{-1} \circ P_{\text{im}^3})(T))_{ij}^k = \begin{cases} 
\frac{1}{2} \left( T_{ij}^k + T_{ki}^j + T_{kij} \right) & \text{if } i,j,k \in I_\alpha, \alpha = 1,2, \\
\frac{1}{2} \left( T_{ij}^k - T_{kj}^i \right) & \text{if } i \in I_\alpha, j,k \in I_\beta, \alpha \neq \beta, \\
0 & \text{if } j \in I_\alpha, k \in I_\beta, \alpha \neq \beta.
\end{cases}
\]

Let us consider now a Riemannian almost product manifold, i.e., a Riemannian manifold \((M, g)\) and a tensor field \(\phi \in \mathfrak{T}^1_1(M)\) such that \(\phi^2 = \text{id}\) and \(g(\phi X, \phi Y) = g(X,Y)\) for any vector fields \(X, Y\) on \(M\). The tensor \(\phi\) gives rise to two mutually orthogonal distributions \(V_1\) and \(V_2\), corresponding to its eigenvalues 1 and -1, called the vertical and horizontal distributions, respectively. Denoting \(p = \dim V_1, q = \dim V_2\), the corresponding \((O(p) \times O(q))\)-structure is the subbundle of orthonormal frames \((X_1, \ldots, X_n)\) such that \(X_1, \ldots, X_p\) are vertical vectors and \(X_{p+1}, \ldots, X_n\) are horizontal vectors.

From the expression of \(\delta^{-1} \circ P_{\text{im}^3}\), we obtain, as in the preceding example:
\[ g(\nabla_X X_j, X_k) = \frac{1}{2} (g([X_i, X_j], X_k) + g([X_k, X_i], X_j) + g([X_j, X_k], X_i)) \]
if \(i, j, k \in I_\alpha, \alpha = 1,2, \)
\[ g(\nabla_X X_j, X_k) = \frac{1}{2} (g([X_i, X_j], X_k) - g([X_k, X_i], X_j)) \]
if \(i \in I_\alpha, j,k \in I_\beta, \alpha \neq \beta, \) and
\[ g(\nabla_X X_j, X_k) = 0 \]
if \(j \in I_\alpha, k \in I_\beta, \alpha \neq \beta. \)

The third equation implies that \(\nabla_X\) leaves the vertical and horizontal distributions invariant for all \(X\). The first one gives:
\[ g(\nabla_X Y, Z) = \frac{1}{2} (X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X)), \]
if \(X, Y, Z \in V_\alpha, \alpha = 1,2.\) Denoting by \(\nabla^\text{LC}\) the Levi-Civita connection, this means that, if \(X, Y \in V_\alpha\), then \(\nabla_X Y\) is the orthogonal projection of \(\nabla^\text{LC} X Y\) onto \(V_\alpha\) according to the decomposition \(TM = V_1 \oplus V_2\). Finally, the second equation yields:
\[ g(\nabla_X Y, Z) = \frac{1}{2} (X(g(Y, Z)) + g([X, Y], Z) - g([X, Z], Y)), \]
if \(X \in V_\alpha, Y, Z \in V_\beta, \alpha \neq \beta. \)

**Example 2.3.** Next, we consider the case \(G = \mathbb{R}^* \cdot I_n\) with \(n \geq 2\). If \(n = 2\) we obtain classical webs over surfaces (see \([3]\)).

The Lie algebra \(\mathfrak{g}\) of \(G\) is generated by the identity matrix \(I_n\), hence we can take:
\[ W = \{ T \in \wedge^2 V^* \otimes V \mid \text{trace}(i_v T) = 0 \ \forall v \in V \}. \]
A short computation shows that the operator \(\delta^{-1} \circ P_{\text{im}^3}\) is defined by:
\[ ((\delta^{-1} \circ P_{\text{im}^3})(T))(v) = \left( \frac{1}{n-1} \text{trace}(i_v T) \right) I_n, \]
for all \( T \in \Lambda^2 V^* \otimes V \), \( v \in V \). The condition \( \eta(x) = -\delta^{-1} (\tilde t(\sigma(x))) \) is written, in components:

\[
\eta^k_{ij}(x) = -\frac{\delta^k_j}{n-1} \sum_i \tilde I^i_{ij}(\sigma(x)).
\]

From this expression, it follows that the canonical connection is given, in any local section \( \sigma = (X_1, \ldots, X_n) \), by:

\[
\nabla_{X_i}X_j = \left( \frac{1}{n-1} \sum_k C^k_{ij} \right) X_j,
\]

with the coefficients \( C^k_{ij} \) being defined by \([X_i, X_j] = \sum_k C^k_{ij}X_k\).

### 3. Canonical representation of the moduli spaces

In this section we are going to describe the underlying manifolds of the \( G \)-modules \( S^r \) to which we refer in Theorem 3.1 and the action of the group \( G \) on them. However, the definition of linear structures on these \( G \)-manifolds will be postponed until section 5. We also describe the isomorphisms \( \mathcal{M}_G(M) \cong S^r / G \).

Let us denote by \( \mathfrak{G}^{r+1}_0 \subset J^{r+1}_{\text{inv}}(V, V) \) the Lie group of \((r+1)\)-jets of diffeomorphisms of \( V \) which leave the origin \( 0 \in V \) fixed. The restriction to \( \mathfrak{G}^{r+1}_0 \) of the action of the Lie groupoid \( J^{r+1}_{\text{inv}}(V, V) \) on \( J^r(FV/G) \) defines an action of \( \mathfrak{G}^{r+1}_0 \) on \( J^r_0(FV/G) \).

**Proposition 3.1.** There exists a canonical bijection \( \mathcal{M}_G(M) \cong J^r_0(FV/G) / \mathfrak{G}^{r+1}_0 \).

**Proof.** Let us recall that \( \mathcal{M}_G(M) \) has been defined as the quotient

\[
J^r(FM/G) / J^{r+1}_{\text{inv}}(M, M).
\]

For each element \([j^r_x s] \in \mathcal{M}_G(M)\), we choose a chart \( \varphi : \mathcal{U} \subset M \rightarrow V \), centered at \( x \in M \), and define the element \([j^{r+1}_{x, \varphi}(s)] \in J^r_0(FV/G)\). If we take another representative \([j^{r+1}_{x, \varphi'}(s)] \in J^r_0(FV/G)\), and a chart \((\mathcal{U}', \varphi')\) centered at \( f(x) \in M \), we have:

\[
(j^{r+1}_{x, \varphi'}(s)) = (j^{r+1}_{x, \varphi}(s)).
\]

Thus, the element of \( J^r_0(FV/G) \) assigned to \([j^r_x s]\) is determined up to an \((r+1)\)-jet of the form \( j^{r+1}_0(\varphi' \circ f \circ \varphi^{-1}) \), which is an element of \( \mathfrak{G}^{r+1}_0 \) and, hence, there is a well defined mapping \( \mathcal{M}_G(M) \rightarrow J^r_0(FV/G) / \mathfrak{G}^{r+1}_0 \). It is easy to find the inverse of this mapping. Given a class \([j^r_x t] \in J^r_0(FV/G) / \mathfrak{G}^{r+1}_0 \) and a chart \( \varphi : \mathcal{U} \subset M \rightarrow V \) centered at \( x \in M \) the element \([j^{r+1}_0(\varphi^{-1}) \cdot (j^r_x t)] \in J^r_0(FM/G) / J^{r+1}_{\text{inv}}(M, M)\) is well-defined. The mapping \([j^r_x t] \mapsto [j^{r+1}_0(\varphi^{-1}) \cdot (j^r_x t)]\) is the required inverse \( J^r_0(FV/G) / \mathfrak{G}^{r+1}_0 \rightarrow \mathcal{M}_G(M)\).

From now on, we will denote the spaces \( \mathcal{M}_G(M) \) just by \( \mathcal{M}_G \), because we have seen that they actually do not depend on the base manifold \( M \).

Now, we define the spaces \( \mathcal{E}^r(V) \) of framed \( r \)-jets of \( G \)-structures as:

\[
\mathcal{E}^r(V) := J^r_0(FV/G) \times F_{0V/G} F_0V = \{(j^r_0 s, u) \in J^r_0(FV/G) \times F_0V : [u] = s(0)\}.
\]

We consider the left action of \( \mathfrak{G}^{r+1}_0 \) on \( \mathcal{E}^r(V) \) given by:

\[
(j^{r+1}_0 f) \cdot (j^r_0 s, u) = ((j^{r+1}_0 f) \cdot (j^r_0 s), \tilde f(u)), \quad j^{r+1}_0 f \in \mathfrak{G}^{r+1}_0, \ (j^r_0 s, u) \in \mathcal{E}^r(V).
\]
Besides, we consider the right action of $G$ on $E'(V)$ induced by the action of $G$ on $F_0 V$, i.e.,

$$(j_0^s, u) \cdot g = (j_0^s, u \cdot g), \quad g \in G, \ (j_0^s, u) \in E'(V).$$

The quotient $E'(V)/G$ can obviously be identified with $J^r_0(FV/G)$.

It is easy to check that the group actions just defined commute. Therefore, we have the bijections:

$$\mathcal{M}_G^{r} \cong \frac{J_0^r(FV/G)}{G_{r+1}} \cong \frac{E'(V)/G}{G_{r+1}} \cong \frac{E'(V)/\mathcal{G}_{r+1}^+}{G_{r+1}}.$$

Thus, if there is a manifold $S'$ such that $S'/G \cong \mathcal{M}_G^{r}$, the natural candidate is $S' := E'(V)/\mathcal{G}_{r+1}^+$, if it is indeed a smooth manifold without singularities. In general, this would not be true. However, the existence of linear connections functorially attached to $G$-structures makes each $E'(V)$ a trivial principal bundle with structure group $\mathcal{G}_{r+1}^+$, as we will see below, and so $E'(V)/\mathcal{G}_{r+1}^+$ is a manifold in our particular cases.

Given $(j_0^s, u) \in E'(V)$, let $s : U \subset V \to FV/G$ be a representative of $j_0^s$, defined on an open neighborhood $U$ of $0 \in V$. The connection $\nabla(s)$ provides an exponential mapping $\exp_s : W_0 \subset T_0 V \to V$ defined on some neighborhood $W_0$ of $0 \in T_0 V$. The composition of this mapping with the isomorphism $u : V \to T_0 V$ yields a diffeomorphism defined on a neighborhood of $0 \in V$, which is nothing but the set of normal coordinates associated to the connection $\nabla(s)$ and the frame $u$.

**Lemma 3.2.** The assignment $\text{Exp}' : E'(V) \to \mathcal{G}_{r+1}^+$ given by $\text{Exp}' (j_0^s, u) = j_0^r (\exp_s u)$ is a well-defined smooth map. Moreover, it is $\mathcal{G}_{r+1}^+$-equivariant with respect to the action of the group on $E'(V)$ defined above and the natural left action of $\mathcal{G}_{r+1}^+$ on itself.

**Proof.** Let $(V, (x^1, \ldots, x^n))$ be the standard chart of $V$. Let us denote $f = \exp_s u$, and $u = (X_1, \ldots, X_n)$, with $X_i = \sum_{j=1}^n a_{ij}^s (\frac{\partial}{\partial x_j})_0$, $i = 1, \ldots, n$.

The map $t \mapsto f(tx)$ is the geodesic from $0 \in V$ with initial speed $u(x)$, i.e., the solution of the system of second order differential equations:

$$\frac{d^2 f^k(tx)}{dt^2} = - \sum_{j_1, j_2} \Gamma^k_{j_1 j_2} (f(tx)) \frac{df^{j_1}(tx)}{dt} \frac{df^{j_2}(tx)}{dt}, \quad 1 \leq k \leq n,$$

with initial conditions $f(0) = 0$, $\frac{df(tx)}{dt}(0) = u(x)$. We can write this system in an equivalent way as follows:

$$f(0) = 0$$

$$\sum_i x^i \frac{\partial f^k}{\partial x^i}(0) = \sum_i x^i a_{ik}^s, \quad 1 \leq k \leq n,$$

$$\sum_{i_1, i_2} x^{i_1} x^{i_2} \frac{\partial^2 f^k}{\partial x^{i_1} \partial x^{i_2}}(tx) = - \sum_{i_1, i_2, j_1, j_2} x^{i_1} x^{i_2} \Gamma_{j_1 j_2}^k (f(tx)) \frac{\partial f^{j_1}}{\partial x^{i_1}}(tx) \frac{\partial f^{j_2}}{\partial x^{i_2}}(tx), \quad 1 \leq k \leq n.$$
evaluating in $t = 0$ leads to the identity:
\[
\sum_{i_1, \ldots, i_{r+1}} x^{i_1} \cdots x^{i_{r+1}} \frac{\partial^{r+1} f^k}{\partial x^{i_1} \cdots \partial x^{i_{r+1}}}(0) = \sum_{i_1, \ldots, i_{r+1}} x^{i_1} \cdots x^{i_{r+1}} R_{i_1, \ldots, i_{r+1}}, \quad 1 \leq k \leq n,
\]
where each $R_{i_1, \ldots, i_{r+1}}$ is a polynomial in the derivatives of the Christoffel symbols, at $x = 0$, up to order $r - 1$, and the derivatives at $x = 0$ of the components $f^k$ up to order $r$.

Last equation implies:
\[
\frac{\partial^{r+1} f^k}{\partial x^{i_1} \cdots \partial x^{i_{r+1}}}(0) = \frac{1}{(r+1)!} \sum_{\tau \in S_{r+1}} R_{\tau(i_1) \cdots \tau(i_{r+1})}, \quad 1 \leq k, i_1, \ldots, i_{r+1} \leq n.
\]
Since the map $j^r s \mapsto \nabla(s)(x)$ is smooth (see Remark 2.2), the derivatives up to order $r - 1$ of the Christoffel symbols $\Gamma^k_i(x)$ are smooth functions of $j^r s$. Thus, the $(r+1)\text{th}$ order derivatives at $x = 0$ of $f$ are smooth functions of $j^r s$ and $j^0 f$.

Finally, using induction, it follows that $\text{Exp}^r : (j^r s, u) \mapsto j^{r+1} f$ is a well-defined smooth map.

In order to prove the $\mathcal{G}^{r+1}_0$-equivariance, let us point that, due to the naturality of the assignment $s \mapsto \nabla(s)$, the equality $f \circ \exp_s \circ u = \exp_{f,s} \circ \bar{f}(u)$ holds for each diffeomorphism $f : M \rightarrow M'$ and each $G$-structure $s \in \Gamma(FM/G)$. Therefore:
\[
\text{Exp}^r \left( (j^{r+1} f) \cdot (j^r s, u) \right) = \text{Exp}^r \left( j^r_0 (f \cdot s), \bar{f}(u) \right)
= j^{r+1}_0 \left( \exp_{f,s} \circ \bar{f}(u) \right)
= j^{r+1}_0 (f \circ \exp_s \circ u)
= (j^{r+1} f) \cdot \text{Exp}^r(j^r s, u),
\]
for each $j^{r+1} f \in \mathcal{G}^{r+1}_0$ and each $(j^r s, u) \in \mathcal{E}^r(V)$.

Next, we define $\mathcal{E}^r(V)_1$ as the fiber over $1_{\mathcal{G}^{r+1}_0} \in \mathcal{G}^{r+1}_0$ of the map $\text{Exp}^r : \mathcal{E}^r(V) \rightarrow \mathcal{G}^{r+1}_0$, i.e., $\mathcal{E}^r(V)_1 := (\text{Exp}^r)^{-1}(1_{\mathcal{G}^{r+1}_0})$.

**Lemma 3.3.** $\mathcal{E}^r(V)_1$ is a smooth submanifold of $\mathcal{E}^r(V)$.

**Proof.** We prove first that, due to equivariance, the map $\text{Exp}^r$ is a surjective submersion, which implies that $\mathcal{E}^r(V)_1$ is in fact a submanifold of $\mathcal{E}^r(V)$. It is surjective because the image of each orbit in $\mathcal{E}^r(V)$ is an orbit of left translations in the group, namely, $\mathcal{G}^{r+1}_0$. To see that it is a submersion, note that if $(j^r s, u) \in \mathcal{E}^r(V)$ were a critical point of $\text{Exp}^r$, then every element of its orbit would also be a critical point, and, being surjective restricted to each orbit, the map $\text{Exp}^r$ would not have any regular value, contradicting Sard’s Theorem. Thus, each fiber of $\text{Exp}^r$, in particular $\mathcal{E}^r(V)_1$, is a smooth submanifold of $\mathcal{E}^r(V)$.

On the other hand, the equivariance of $\text{Exp}^r$ allows to define a map $p^r : \mathcal{E}^r(V) \rightarrow \mathcal{E}^r(V)_1$ as
\[
p^r(j^r s, u) = (\text{Exp}^r(j^r s, u))^{-1} \cdot (j^r s, u).
\]
A straightforward calculation shows that this map is invariant under the right action associated to the left action of $\mathcal{G}^{r+1}_0$ on $\mathcal{E}^r(V)$, which is given by:
\[
(j^r s, u) \cdot (j^{r+1} f) = (j^{r+1} f)^{-1} \cdot (j^r s, u), \quad j^{r+1} f \in \mathcal{G}^{r+1}_0, \quad (j^r s, u) \in \mathcal{E}^r(V).
\]
Indeed, the following lemma holds:

**Lemma 3.4.** $p^r : \mathcal{E}^r(V) \to \mathcal{E}^r(V)_1$ is a trivial principal bundle with structure group $\mathfrak{G}_0^{r+1}$.

**Proof.** The map $\Lambda^r : \mathcal{E}^r(V) \to \mathcal{E}^r(V)_1 \times \mathfrak{G}_0^{r+1}$ defined by: $\Lambda^r(j_0^r s, u) = \left(p^r(j_0^r s, u), (\text{Exp}^r(j_0^r s, u))^{-1}\right)$ is a $\mathfrak{G}_0^{r+1}$-equivariant diffeomorphism. The inverse $(\Lambda^r)^{-1} : \mathcal{E}^r(V)_1 \times \mathfrak{G}_0^{r+1} \to \mathcal{E}^r(V)$ is given by $(\Lambda^r)^{-1}(j_0^r s, u, j_0^r s, u f) = (j_0^r s, u)$.

This diffeomorphism endows $\mathcal{E}^r(V)$ with the structure of a trivial principal $\mathfrak{G}_0^{r+1}$-bundle over $\mathcal{E}^r(V)_1$.

**Corollary 3.5.** The space $S^r := \mathcal{E}^r(V)/\mathfrak{G}_0^{r+1}$ is a smooth manifold canonically diffeomorphic to $\mathcal{E}^r(V)_1$.

**Remark 3.1.** Explicitly, the diffeomorphism $\overline{p}^r : S^r \to \mathcal{E}^r(V)_1$ is given by

$$\overline{p}^r([(j_0^r s, u)]) = p^r(j_0^r s, u).$$

Its inverse is the composed map: $\mathcal{E}^r(V)_1 \to \mathcal{E}^r(V) \to \mathcal{E}^r(V)/\mathfrak{G}_0^{r+1}$ of the inclusion (which is the map trivializing the bundle) with the natural projection.

**Remark 3.2.** Let us denote by $u^0 : V \to T_0V$ the canonical frame. The condition $j_0^{r+1}(\exp_s o u) = 1_{\mathfrak{G}_0^{r+1}}$ defining $\mathcal{E}^r(V)_1$ implies $D(\exp_s o u)(0) = id_V$, where $D$ stands for the standard Fréchet derivative in $V$. Using that

$$D(\exp_s o u)(0) = D(\exp_s o u^0)(0) \circ D((u^0)^{-1} \circ u)(0) = D((u^0)^{-1} \circ u)(0) = (u^0)^{-1} \circ u,$$

we conclude that $u = u^0$, so that $\mathcal{E}^r(V)_1 \subset J_0^r(FV/G) \times \{u^0\}$. Therefore from now on, we will consider $\mathcal{E}^r(V)_1$ as a subspace of $J_0^r(FV/G)$.

**Remark 3.3.** Notice that an $r$-jet $j_0^r s \in J_0^r(FV/G)$ belongs to $\mathcal{E}^r(V)_1$ if and only if $s(0) = [u^0]$ and

$$\frac{d^k}{dt^k} \sum_{i_1, i_2} x^{i_1, i_2} \gamma_{i_1, i_2} (tx)(t)(0) = 0,$$

for any $x$ in a neighborhood of $0 \in V$, as follows from (6).

The action of $G$ on $S^r$ can be translated now to $\mathcal{E}^r(V)_1$.

**Lemma 3.6.** The action of $G$ on $S^r$ induces, by means of the diffeomorphism $S^r \cong \mathcal{E}^r(V)_1$, the following action of $G$ on $\mathcal{E}^r(V)_1$:

$$(j_0^r s) \cdot g = (j_0^{r+1} g^{-1}) \cdot j_0^r s, \quad j_0^r s \in \mathcal{E}^r(V)_1, \quad g \in G \subset \mathfrak{G}_0^{r+1}.$$

**Proof.** For each $j_0^r s \in \mathcal{E}^r(V)_1$ and each $g \in G$, we have, taking into account Remark 3.1 we have that

$$\begin{align*}
(j_0^r s) \cdot g & \equiv \overline{p}^r([j_0^r s, u^0]) \cdot g \\
& = \overline{p}^r([j_0^r s, u^0 \circ g]) \\
& = p^r(j_0^r s, u^0 \circ g) \\
& = (\text{Exp}^r(j_0^r s, u^0 \circ g))^{-1} \cdot (j_0^r s, u^0 \circ g).
\end{align*}$$
From the fact that:

\[
\text{Exp}^r(j_0^r s, u \circ g) = j_0^{r+1} (\text{exp}_s \circ u^0 \circ g) = j_0^{r+1} (\text{exp}_s u^0) \cdot j_0^{r+1} g = j_0^{r+1} g,
\]

it follows the desired expression of \((j_0^r s) \cdot g \):

\[
(j_0^r s) \cdot g = (j_0^{r+1} g^{-1}) \cdot (j_0^r s, u^0 \circ g) = ((j_0^{r+1} g^{-1}) \cdot (j_0^r s), u^0)
\]

which is identified with \((j_0^{r+1} g^{-1}) \cdot (j_0^r s)\).

It is immediate that the projections \(J_0^{r+k} (FV/G) \to J_0^r (FV/G) \) induce projections \(S^{r+k} \to S^r, k \geq 0\), which allow to define \(S^\infty = \lim_{\leftarrow} S^r\) as the projective limit of the family \(\{S^r\}\). The action of \(G\) induces an action on the limit \(S^\infty\) in a natural way.

Thus, we have proved the following theorem of reduction of the group:

**Theorem 3.7.** The spaces \(S^r = \mathcal{E}^r(V)/\mathfrak{S}_0^{r+1}\) have natural smooth \(G\)-module structures, so that each moduli space \(\mathcal{M}_{\mathcal{G}}\) is canonically isomorphic to the quotient \(S^r/G\). Taking projective limits yields a canonical isomorphism \(\mathcal{M}_{\mathcal{G}}^\infty \cong S^\infty/G\).

4. Embedding of the moduli of framed \(r\)-jets \(S^r\) in the space of jets of moving frames \(J_0^r(FV)\)

Let us denote by \(J_0^r (FV/G)_{u^0}\) the manifold of \(r\)-jets \(j_0^r s \in J_0^r (FV/G)\) such that \(s(0) = [u^0]\). We will define an embedding of this manifold in \(J_0^r (FV)\). Then, from the diffeomorphism \(S^r \cong \mathcal{E}^r(V)\) and the inclusion \(\mathcal{E}^r(V) \to J_0^r (FV/G)_{u^0}\), we will obtain an embedding of \(S^r\) in \(J_0^r (FV)\). This construction is a technical tool to define the \(G\)-module structure on \(S^r\) in the next section.

We define \(\Upsilon^r : J_0^r (FV/G)_{u^0} \to J_0^r (FV)\) as follows. Given \(j_0^r s \in J_0^r (FV/G)_{u^0}\), let \(s : \mathcal{U} \subset V \to FV/G\) be a local representative of \(j_0^r s\). We define a moving frame \(\sigma = (X_1, \ldots, X_n)\) in a neighborhood of \(0 \in V\) as the \(\nabla(s)\)-parallel transport of the frame \(u^0\) along each ray \(x(t) = (t \cdot \lambda^1, \ldots, t \cdot \lambda^n) \in V\), and we set \(\Upsilon^r (j_0^r s) = j_0^r \sigma\).

**Lemma 4.1.** The assignment \(\Upsilon^r : J_0^r (FV/G)_{u^0} \to J_0^r (FV)\) is a well-defined embedding.

**Proof.** Let \(s\) be a representative of \(j_0^r s\) defined in a neighborhood of \(0 \in V\), and denote by \(\Gamma^k_{ij}\) the Christoffel symbols of the connection \(\nabla(s)\). Let \(\sigma = (X_1, \ldots, X_n)\) be the moving frame defined above using the connection \(\nabla(s)\), and let \(X_i = \sum_k \sigma_{ki} (x) \partial/\partial x^k\) be the expression of the vector field \(X_i\) with respect to the standard chart of \(V\).

Since, by definition, each \(X_i\) is parallel along the rays \(x(t) = t \lambda, \lambda = (\lambda^1, \ldots, \lambda^n)\), the functions \(\sigma_{ki}\) satisfy the differential equations:

\[
\sum_\beta \left( \frac{\partial \sigma_{\gamma i}}{\partial x^\beta} (t \lambda) + \sum_\alpha \Gamma^\gamma_{\alpha i} (t \lambda) \sigma_{\alpha i} (t \lambda) \right) \lambda^\beta = 0
\]

\[
\sigma_{ij}(0) = \delta_{ij}
\]
for any $\lambda \in V$, and any indices $1 \leq \gamma, i, j \leq n$. Multiplying these equations by $t$, we obtain that $\sigma$ is characterized by the initial condition $\sigma(0) = u^0$ and the equations:

$$\sum_{\beta} \left( \frac{\partial \sigma_{\gamma i}}{\partial x^\beta} + \sum_{\alpha} \Gamma^\gamma_{\beta \alpha} \sigma_{\alpha i} \right) x^\beta = 0, \ 1 \leq i, \gamma \leq n.$$

Iterated derivation of (8) and evaluation in $x = 0$ leads to:

$$\frac{\partial^k \sigma_{\gamma i}}{\partial x^{i_1} \ldots \partial x^{i_k}} (0) = -\frac{1}{k} \sum_{(i_1, \ldots, i_k)} \mathfrak{S} \left( \frac{\partial^{k-1} \sigma}{\partial x^{i_2} \ldots \partial x^{i_k}} \sum_{\alpha} \Gamma^\gamma_{\alpha \gamma} \sigma_{\alpha i} \right) (0),$$

for any $k \geq 1$ and any $1 \leq \gamma, i, i_1, \ldots, i_r \leq n$, were $\mathfrak{S}$ stands for the cyclic sum with respect to the corresponding indices.

For $r = 1$ we have $\frac{\partial \sigma_{\gamma i}}{\partial x} (0) = -\Gamma^\gamma_{\gamma i} (0)$, so that $j_0^1 \sigma = \Upsilon^1 (j_0^1 s)$ depends smoothly on $j_0^1 s$. By induction, using (8), it follows that $j_0^r \sigma = \Upsilon^r (j_0^r s)$ depends smoothly on $j_0^r s$.

Finally, it is obvious, by its very definition, that $\Upsilon^r$ takes its values in the closed submanifold: $J^r_0 (FV)_{u^0} = \{ j_0^r \sigma \in J^r_0 (FV) \mid \sigma(0) = u^0 \}$, and that it is a section of the projection $J^r_0 (FV)_{u^0} \to J^r_0 (FV/G)_{u^0}$ induced by $q : FV \to FV/G$. Therefore, $\Upsilon^r$ is indeed an embedding.■

Next we define a right action of $G$ on $J^r_0 (FV)_{u^0}$ by:

$$(j_0^r \sigma) \cdot g = j_0^r \left( R_g \circ (g^{-1} \cdot \sigma) \right), \quad j_0^r \sigma \in J^r_0 (FV)_{u^0}, \ g \in G,$$

where $R_g$ stands for right translations in $FV$. Besides, we have the right action of $G$ on $J^r_0 (FV/G)_{u^0}$:

$$(j_0^r s) \cdot g = (j_0^{r+1} g^{-1}) \cdot j_0^r s, \quad j_0^r s \in J^r_0 (FV/G)_{u^0}, \ g \in G.$$

Lemma 4.2. The map $\Upsilon^r : J^r_0 (FV/G)_{u^0} \to J^r_0 (FV)_{u^0}$ is $G$-equivariant with respect to the right actions of $G$ previously defined.

Proof. Due to the naturality of the assignment $s \mapsto \nabla (s)$, if $\sigma$ is the parallel moving frame associated to the $G$-structure $s$, then $R_g \circ (g^{-1} \cdot \sigma)$ is the parallel moving frame associated to $s \cdot g = \tilde{g}^{-1} \circ s \circ g$. From this fact, the result follows easily.

Denoting $S^r = \Upsilon^r (E^r (V)_1) \subset J^r_0 (FV)$ we have now the identifications between $G$-manifolds $S^r \cong E^r (V)_1 \cong S^r$.

We derive now a characterization of the submanifolds $S^r \subset J^r_0 (FV)$, which will be used in next section to define inductively a $G$-module structure on each of them. First of all, we introduce some definitions.

Let us denote $V^{r,p}_q := S^r (V^*) \otimes V^* \otimes V^\otimes q$ with its natural $G$-module structure. Consider the homomorphisms of $G$-modules $\text{sym}^{r,p}_q : V^{r,p}_q \to V^{r+1,p-1}_q$, $p \geq 1$, given by symmetrization on the first $r + 1$ covariant indices. We will denote $\bar{V}^{r,p}_q := \ker (\text{sym}^{r,p}_q)$.

Let $\sigma = (X_1, \ldots, X_n)$, with $X_i = \sum \sigma_{ji} \partial / \partial x^j$, $1 \leq i \leq n$, be a representative of the $r$-jet $j_0^r \sigma \in J^r_0 (FV)$. Then, for each $g \in G$, the derivatives of the components of the vectors in the representative $R_g \circ (g^{-1} \cdot \sigma)$ of $(j_0^r \sigma) \cdot g$ are given by

$$\frac{\partial^k \sigma_{rs}}{\partial x^{p_1} \ldots \partial x^{p_k}} (0) = g^{ra} \frac{\partial^k \sigma_{\alpha \gamma}}{\partial x^{l_1} \ldots \partial x^{l_k}} (0) g_{l_1 p_1} \cdots g_{l_k p_k} g_{rs},$$

for any $\alpha, \beta, l_1, \ldots, l_k \in \{ 1, \ldots, n \}$, $1 \leq r, s \leq n$, and any indices $1 \leq \gamma, a, \alpha, \beta \leq n$. Multiplying these equations by $t$, we obtain that $\sigma$ is characterized by the initial condition $\sigma(0) = u^0$ and the equations:

$$\sum_{\beta} \left( \frac{\partial \sigma_{\gamma i}}{\partial x^\beta} + \sum_{\alpha} \Gamma^\gamma_{\beta \alpha} \sigma_{\alpha i} \right) x^\beta = 0, \ 1 \leq i, \gamma \leq n.$$
Lemma 4.3. The mapping given in Remark 2.1, it follows easily that these components have the form:

\[ F_{ij} \stackrel{\sigma}{\rightarrow} \]

with \( \{v_1, \ldots, v_n\} \) being the canonical basis of \( V \) and \( \{v^*1, \ldots, v^*n\} \) its dual basis.

Next lemma is an immediate consequence of the above remarks:

**Lemma 4.4.** The mapping \( (\pi_r^{r+1}, \Sigma^{(r+1)}) : J_0^{r+1}(F^V) \to J_0^{r+1}(F^V) \times V_1^{r+1} \) is a \( G \)-equivariant diffeomorphism.

Now, let \( \nabla \) and \( \bar{\nabla} \) be the canonical connections adapted to the \( \{e\} \)-structure \( \sigma \) and the \( G \)-structure \( s = g \circ \sigma \), respectively. The connection \( \nabla \) is characterized by the equations \( \nabla_X X_j = 0 \), \( 1 \leq i, j \leq n \), so that the corresponding Christoffel symbols are given by \( \bar{\Gamma}_{ij}^k = -\sum \frac{\partial \sigma_{ij}}{\partial x^k} \sigma_{k\beta} \). We define the tensor field \( F \) as the difference of both connections, i.e., by \( F(X, Y) = \nabla_X Y - \nabla_Y X, X, Y \in \mathcal{X}(V) \), and denote by \( F_{ij}^k = \Gamma_{ij}^k - \bar{\Gamma}_{ij}^k \), its components with respect to the canonical coordinates in \( V \). From the expression in local coordinates of the connection \( \bar{\nabla} \) given in Remark 2.1, it follows easily that these components have the form:

\[ F_{ij}^k = \sum_{\lambda, \mu, \rho} \sigma^{\alpha \lambda} \sigma^{\beta \mu} \sigma^{\gamma \rho} (\sigma^r \omega)^{\rho}_\mu (X_\lambda) = \]

\[ = \sum_{\lambda, \mu, \rho, i, j, k, h, \xi} A_{ijkl}^\rho \sigma^{\alpha \lambda} \sigma^{\beta \mu} \sigma^{\gamma \rho} \sigma^{k \xi} \left( \sigma_{hi} \frac{\partial \sigma_{ij}}{\partial x^k} - \sigma_{hj} \frac{\partial \sigma_{i\lambda}}{\partial x^k} \right) \]

Due to the naturality of \( \nabla \) and \( \bar{\nabla} \), for each \( g \in G \), the moving frame \( R_g \circ (g^{-1} \circ \sigma) \) has an associated tensor field \( F \) whose components are related to those of \( F \) by:

\[ F_{ij}^k(x) = g^{k\gamma} F_{ij}^\gamma(g \cdot x) g_{\alpha i} g_{\beta j}. \]

Iterated derivation of these equations and evaluation at \( x = 0 \) yields:

\[ \frac{\partial^k F_{ij}^k}{\partial x^{p_1} \cdots \partial x^{p_k}}(0) = g^{k\gamma} \frac{\partial^k F_{ij}^\gamma}{\partial x^{p_1} \cdots \partial x^{p_k}}(0) g_{i \alpha j} \cdots g_{k \alpha i} g_{\gamma j}. \]

So, we can also define an equivariant mapping \( F^{(r)} : J_0^{r+1}(F^V) \to V_1^{r+1} \) as follows:

\[ F^{(r)}(j_0^{r+1}) = \sum_{i_1, \ldots, i_r, \alpha, \beta, \gamma} \sum_{\beta} \frac{\partial^r F_{ij}^\gamma}{\partial x^{i_1} \cdots \partial x^{i_r} \partial x^j}(0) v^{i_1} \cdots v^{i_r} \sigma v^\gamma \sigma v^{\beta} \sigma v_{\gamma}. \]

**Lemma 4.4.** There exist a linear map \( L^{(r)} : V_1^{r+1} \to V_1^{r+2} \) and a polynomial function \( Q^{(r)} : J_0^{r+1}(F^V) \to V_1^{r+2} \) in the standard coordinates of \( J_0^{r+1}(F^V) \) such that \( F^{(r)} = L^{(r)} \circ \Sigma^{(r+1)} + Q^{(r)} \circ \pi^{r+1}_r \). Moreover, \( L^{(r)} \) and \( Q^{(r)} \) are \( G \)-equivariant.

**Proof.** Derivation of \( (10) \) and evaluation at \( x = 0 \) show that the derivatives of \( F^{(r)}_{\alpha \beta} \) at \( x = 0 \) are of the form:

\[ \frac{\partial^{r+1} F_{ij}^\gamma}{\partial x^{i_1} \cdots \partial x^{i_r} \partial x^{j}}(0) = \sum_{i_1, \ldots, i_r, j, k} A^{ij}_{\alpha \beta} \left( \frac{\partial^{r+1} \sigma_{kj}}{\partial x^{i_1} \cdots \partial x^{i_r} \partial x^{j}}(0) - \frac{\partial^{r+1} \sigma_{ki}}{\partial x^{i_1} \cdots \partial x^{i_r} \partial x^{j}}(0) \right) O_r \]

where \( O_r \) stands for a polynomial in the derivatives up to order \( r \) of \( \sigma \) at \( x = 0 \).

Accordingly, we define \( L^{(r)} \) as the linear map sending each element of \( V_1^{r+1} \):

\[ \sum \sum_{i_1, \ldots, i_r, j, k} A^{ij}_{\alpha \beta} \left( \frac{\partial^{r+1} \sigma_{kj}}{\partial x^{i_1} \cdots \partial x^{i_r} \partial x^{j}}(0) - \frac{\partial^{r+1} \sigma_{ki}}{\partial x^{i_1} \cdots \partial x^{i_r} \partial x^{j}}(0) \right) v^{i_1} \cdots v^{i_r} v^j \sigma v_{\gamma} \]
Theorem 4.5. The submanifold $S^{r+1}$ of $(j^r \sigma_\gamma)_{ij,k}$ as the map $\delta(r) \in E$ to the element of $V_1^{r-2}$:

$$\sum_{i,j,k} A_{ij,k} \left(t^k_{i-j,i-j} - t^k_{i-j,i-j}\right) v^\alpha \otimes v^\beta \otimes v^\gamma.$$

Thus, $L(r)$ is the composition:

$$V_1^{r+1, \delta(r+1)} \xrightarrow{\delta^{r+1}} S^r(V^r) \otimes \bigwedge V^r \otimes V_1 \xrightarrow{\delta \circ \pi^{r+1}} S^r(V^r) \otimes \bigwedge V^r \otimes V \subset V_1^{r,2},$$

where the first arrow stands for the Spencer's operator, which is defined in general as the map $\delta^{r,1} : S^{r+1} \otimes \bigwedge V^r \otimes V \to S^r \otimes \bigwedge V^r \otimes V$ given by:

$$(\delta^{r+1,1}(u)) \left(u_1, \ldots, u_r, u'_1, \ldots, u'_1\right) = \sum_{h=1}^l (-1)^{h+1} t(u_1, \ldots, u_r, u'_1, \ldots, u'_1),$$

for each $t \in S^{r+1}(V^r) \otimes \bigwedge V^r \otimes V$ and any vectors $u_1, \ldots, u_r, u'_1, \ldots, u'_1 \in V$.

The difference $F(r) - L(r) \circ \pi^{r+1} : J_0^r(FV) \to V_1^{r,2}$ is a function depending polynomially in the derivatives up to order $r$ of $\sigma$ at 0, and hence there is a well-defined polynomial function $Q(r) : J_0^r(FV) \to V_1^{r,2}$ such that $F(r) - L(r) \circ \pi^{r+1} = Q(r) \circ \pi^{r+1}$.

Finally, it is clear that $L(r)$ is $G$-equivariant, since it has been defined as a composition of homomorphisms of $G$-modules. The equivariance of the maps $F(r)$, $L(r)$, $\pi^{r+1}$ and $\pi^{r+1}$ implies that $Q(r)$ is also $G$-equivariant.

The desired characterization of the submanifold $S^{r+1}$ is the following.

**Theorem 4.5.** The submanifold $S^{r+1} \subset J_0^r(FV)$ can be characterized as the set of $(r+1)$-jets $j_0^{r+1} \sigma \in (\pi^{r+1}, \pi^{r+1})^{-1} \left(S^{r} \times \tilde{V}_1^{r+1,1}\right)$ such that

$$\text{sym}^{r+1}_1 \left((L(r) \circ \pi^{r+1}) (j_0^{r+1} \sigma)\right) = -\text{sym}^{r+1}_1 \left((Q(r) \circ \pi^{r+1}) (j_0^{r+1} \sigma)\right).$$

**Proof.** By definition, an $r$-jet $j_0^0 \sigma \in J_0^r(FV)$ belongs to $S^r$ if and only if $j_0^0 \sigma = \Upsilon^r (j_0^0 s)$ with $j_0^0 s \in E(V)_1$. Since $\Upsilon^r$ is a section of the projection $q' : J_0^r(FV)_{\gamma} \to J_0^r(FV/G)_{\gamma}$, we have that $j_0^0 s = q' (j_0^0 \sigma)$. So that $j_0^0 \sigma \in S^r$ if and only if $q'(j_0^0 \sigma) \in E(V)_1$ and $(\Upsilon^r \circ q') (j_0^0 \sigma) = j_0^0 \sigma$.

Let us first study the second condition, which means that, up to order $r$, the moving frame $\sigma$ is obtained by the parallel displacement of the canonical frame $u^0$ at $0 \in V$ using the canonical connection of the $G$-structure determined by $\sigma$. For each $j_0^0 \sigma \in J_0^r(FV)$, let $\sigma$ and $s$ be local representatives of $j_0^0 \sigma$ and $q'(j_0^0 \sigma)$, respectively, and let $\Gamma_{ij}^k$ be the Christoffel symbols of the canonical connection $\nabla(s)$. Then, $(\Upsilon^r \circ q') (j_0^0 \sigma) = j_0^0 \sigma$ if and only if $j_0^0 \sigma$ satisfies equation (3) for any 1 $\leq k \leq r$, 1 $\leq \gamma, i, i_1, \ldots, i_r \leq n$.

Writing $\Gamma_{ij}^k = \Gamma_{ij}^k + F_{ij}^k = -\sum_{\alpha} \delta_{ij} \frac{\partial \sigma_{\alpha}}{\partial x^k} + F_{ij}^k$, equation (3) reads:

$$\left(\theta_{i_2 \ldots i_k} - \sum_{\alpha} \frac{\partial^{k-1}}{\partial x_{i_2} \ldots \partial x_{i_k}} \sum_{\alpha} F_{i_2 \ldots i_k}^\gamma \sigma_{\alpha i} \right) (0) = 0,$$

But this is equivalent to

$$\sum_{i_1, \ldots, i_k} x^{i_1} \ldots x^{i_k} \left(\frac{\partial^{k-1}}{\partial x_{i_2} \ldots \partial x_{i_k}} \sum_{\alpha} F_{i_2 \ldots i_k}^\gamma \sigma_{\alpha i} \right) (0) = 0,$$
and hence to the equations

\[
\left( \frac{d^{k-1}}{dt^{k-1}} \sum_{\alpha}^{\beta} x^\beta F^\gamma_{\alpha\beta}(tx) \right)(0) = 0,
\]

for any \(1 \leq k \leq r\), \(1 \leq \gamma, i, \leq n\) and any \(x\) in a neighborhood of \(0 \in V\). Using induction, these equations are seen to be equivalent to the following:

\[
\left( \frac{d^{k-1}}{dt^{k-1}} \sum_{\beta} x^\beta F^\gamma_{\beta i}(tx) \right)(0) = 0,
\]

for any \(1 \leq k \leq r\), \(1 \leq \gamma, i, \leq n\) and any \(x\), which can be written in terms of the partial derivatives at \(0 \in V\) of \(F\) (and hence of \(\sigma\)) as:

\[
S_{(i_1 \cdots i_k+1)} \frac{\partial^{k-1} F^\gamma_{i_1 \cdots i_k}}{\partial x^{i_1} \cdots \partial x^{i_k}}(0) = 0,
\]

for each \(1 \leq k \leq r\), and for any indices \(1 \leq \alpha, \gamma, i_1, \ldots, i_k \leq n\).

Let us now find the equations provided by the condition \(q^r(j_0^r \sigma) \in E^r(V)\), which means that, up to order \(r+1\), the normal coordinates of the connection attached to the \(G\)-structure \(G \cdot \sigma\) are just the canonical coordinates of \(V\). Setting \(\Gamma_{i_1i_2} = F_{i_1i_2} = -\sum_j \sigma^{ij_2} \frac{\partial \sigma_{ij_2}}{\partial x^{i_1}} + \sum_j \sigma_{ij_2} \frac{\partial \sigma_{ij_2}}{\partial x^{i_1}}\) in (11) and using (11), we obtain that \(q^r(j_0^r \sigma) \in E^r(V)\) if and only if:

\[
\left( \frac{d^{k-1}}{dt^{k-1}} \sum_{j}^{\beta} \sigma_{ij_2}(tx) \frac{d}{dt} \left( \sum_{i_2} x^{i_2} \sigma^{ij_2}(tx) \right) \right)(0) = 0,
\]

for any \(1 \leq k \leq r\), \(1 \leq \gamma \leq n\), and any \(x\) in a neighborhood of \(0 \in V\). By induction, it follows again that these equations are equivalent to:

\[
\left( \frac{d^{k}}{dt^{k}} \sum_{\beta} x^\beta \sigma^{\gamma\beta}(tx) \right)(0) = 0,
\]

for any \(1 \leq k \leq r\), \(1 \leq \gamma \leq n\), and any \(x\), which can be written as:

\[
S_{(i_1 \cdots i_{k+1})} \frac{\partial^k \sigma^{\gamma\gamma_1}}{\partial x^{i_2} \cdots \partial x^{i_{k+1}}}(0) = 0
\]

or, equivalently, as:

\[
(12)
S_{(i_1 \cdots i_{k+1})} \frac{\partial^k \sigma^{\gamma\gamma_1}}{\partial x^{i_2} \cdots \partial x^{i_{k+1}}}(0) = 0
\]

for each \(1 \leq k \leq r\), and for any indices \(1 \leq \gamma, i_1, \ldots, i_{k+1} \leq n\). Finally, an \(r\)-jet \(j_0^r \sigma \in J_0^r(FV)\) belongs to \(S^r\) if and only if it satisfies \(\sigma_{\alpha\beta}(0) = \delta_{\alpha\beta}\), together with equations (12) and (11).

For the inductive definition of the submanifolds \(S^r\) appearing in the statement of the theorem, notice that an \((r+1)\)-jet \(j_0^{r+1} \sigma \in J_0^{r+1}(FV)\) lies in \(S^{r+1}\) if and
only if $\pi_{r+1}^{-1} \left(j_0^{r+1}\sigma\right) \in S^r$ and equations (12) and (14) are satisfied at the top level, i.e.:

$$
\mathcal{G} \left. \frac{\partial (F_{r+1})_{\alpha_1 \ldots \alpha_{i+2}}}{\partial x^{i_2} \ldots \partial x^{i_{r+2}}} \right|_{(i_1, \ldots, i_{r+2})} (0) = 0,
$$

$$
\mathcal{G} \left. \frac{\partial F_{r+1}}{(i_1, \ldots, i_{r+1})} \right|_{\partial x^{i_2} \ldots \partial x^{i_{r+1}}} (0) = 0,
$$

for any indices $1 \leq \alpha, \gamma, i_1, \ldots, i_{r+2} \leq n$.

Using the diffomorphism given in Lemma 4.3 and the definition of the operators $F^{(r)}$, and taking into account that

$$
\text{sym}^r q \left( v^{*i_1} \circ \cdots \circ v^{*i_r} \circ v^{*i_{r+1}} \circ \cdots \circ v^{*i_{r+p}} \circ v_{j_1} \circ \cdots \circ v_{j_n} \right)
= \frac{1}{r+1} \mathcal{G} \left( v^{*i_1} \circ \cdots \circ v^{*i_r} \circ v^{*i_{r+1}} \circ \cdots \circ v^{*i_{r+p}} \circ v_{j_1} \circ \cdots \circ v_{j_n} \right)
$$

we see that equations (13) and (14) are equivalent to $\text{sym}^{r+1}(\Sigma^{(r+1)}(j_0^{r+1}\sigma)) = 0$ and $\text{sym}^{r+2}(F^{(r)}(j_0^{r+1}\sigma)) = 0$, so each submanifold $S^{r+1} \subset J_0^{r+1}(FV)$ is characterized by these conditions together with $\pi_{r+1}^{-1} \left(j_0^{r+1}\sigma\right) \in S^r$. That is:

$$
S^{r+1} = \left\{ j_0^{r+1}\sigma \in \left(\pi_{r+1}^{-1}, \Sigma^{(r+1)}\right)^{-1} \left(S^r \times V_1^{r+1, 1}\right) \mid \text{sym}^{r+2}(F^{(r)}(j_0^{r+1}\sigma)) = 0 \right\}
$$

Finally, using the decomposition of $F^{(r)} = L^{(r)} \circ \Sigma^{(r+1)} + Q^{(r)} \circ \pi_{r+1}^{-1}$ given in Lemma 4.4, we obtain the desired result. ■

5. Definition of the G-module structure

In this section, we will define a $G$-module structure on each $S^r$, inductively on $r$, such that the projections are homomorphisms of $G$-modules. The corresponding $G$-manifolds $S^r = E^r(V)/\mathcal{G}_{r+1}^{-1}$ will inherit $G$-module structures, and the natural projections between them will also be homomorphisms of $G$-modules. Therefore we will obtain a $G$-module structure on the projective limit $S^\infty = \lim_{\rightarrow r} S^r$.

A main tool to define a $G$-module structure on each $S^r$ (and therefore on each $S^r$) is next lemma.

Lemma 5.1. The projections $\pi_{r+1} : S^{r+1} \rightarrow S^r$ admit smooth $G$-equivariant sections $\rho_{r+1} : S^r \rightarrow S^{r+1}$.

Proof. Let us denote by $\tilde{L}^{(r)} : \tilde{V}_1^{r+1, 1} \rightarrow V_1^{r+1, 1}$ the restriction of $L^{(r)}$ to $\tilde{V}_1^{r+1, 1}$, and $W^{r+1} := \ker \left( \text{sym}^{r+2} \circ \tilde{L}^{(r)} \right)$. Let us assume for the moment that we can choose a $G$-submodule $Z^{r+1} \subset \tilde{V}_1^{r+1, 1}$ such that $\tilde{V}_1^{r+1, 1} = W^{r+1} \oplus Z^{r+1}$. Then, the restriction $\left( \text{sym}^{r+2} \circ \tilde{L}^{(r)} \right)|_{Z^{r+1}} : Z^{r+1} \rightarrow \text{im} \left( \text{sym}^{r+2} \circ \tilde{L}^{(r)} \right)$ is isomorphism of $G$-modules. Using that the projections $\pi_{r+1} : S^{r+1} \rightarrow S^r$ are onto, together with the fact that, by Theorem 4.3 we have that $\text{sym}^{r+2} \circ L^{(r)} \circ \Sigma^{(r+1)}(\Sigma^{(r+1)}(j_0^{r+1}\sigma)) = -\text{sym}^{r+2} \circ Q^{(r)} \circ \pi_{r+1}^{-1}$ on $S^{r+1}$ and $\Sigma^{(r+1)}(\Sigma^{(r+1)}(j_0^{r+1}\sigma)) \subset V_1^{r+1, 1}$, we obtain that

$$
\left( -\text{sym}^{r+2} \circ Q^{(r)} \right)(S^r) = \left( -\text{sym}^{r+2} \circ Q^{(r)} \circ \pi_{r+1}^{-1} \right)(S^{r+1})
$$

$$
= \left( \text{sym}^{r+2} \circ L^{(r)} \circ \Sigma^{(r+1)} \right)(S^{r+1}) \subset \text{im} \left( \text{sym}^{r+2} \circ \tilde{L}^{(r)} \right).
$$

THE MODULI OF G-STRUCTURES WITH LINEAR CONNECTION. 17
Thus, for each $j_0^r \sigma \in S^r$ there exists a unique $v^{r+1} \in Z^{r+1}$ such that
\[(\text{sym}^r_1 \circ \bar{L}(r)) (v^{r+1}) = -\text{sym}^r_1 \left( Q^r (j_0^r \sigma) \right) .\]

Now, using the isomorphism of Lemma (3), we set
\[\mu^{r+1}_r (j_0^r \sigma) = \left( \pi^{r+1}_r \times \Sigma^{(r+1)} \right)^{-1} (j_0^r \sigma, v^{r+1}) .\]

It is clear that this defines a section $\mu^{r+1}_r : S^r \to S^{r+1}$ of the projection $\pi^{r+1}_r$. Moreover, from the $G$-equivariance of $\pi^{r+1}_r$, $\Sigma^{(r+1)}$, $\text{sym}^r_1$, $Q^r$, $L(r)$ and the fact that $Z^{r+1}$ is a $G$-submodule, it follows that $\mu^{r+1}_r$ is $G$-equivariant too.

It only remains to prove that it is indeed possible to choose a $G$-submodule $Z^{r+1} \subset V^{r+1}_1$ such that $V^{r+1}_1 = W^{r+1} \oplus Z^{r+1}$. Note first that
\[V^{r+1,1}_1 = \ker (\text{sym}^{r+1,1}_1) \oplus (S^{r+2} (V^* ) \otimes V) = \tilde{V}^{r+1,1}_1 \oplus \ker \delta^{r+1,1} .\]

Therefore, $\delta|_{\tilde{V}^{r+1,1}_1} : \tilde{V}^{r+1,1}_1 \to \text{im}\delta^{r+1,1}$ is an isomorphism of $G$-modules. It is easy to check that its inverse is given by \[(r+1)\text{sym}^{r+2}_1 : \text{im}\delta^{r+1,1} \to \tilde{V}^{r+1,1}_1.\]

Hence, it will enough to find a supplementary $G$-submodule $\tilde{Z}^{r+1}_1$ of
\[\ker (\text{sym}^{r+2}_1 \circ (1 \otimes \delta^{-1} \circ P_{\text{im}\delta}))|_{\text{im}\delta^{r+1,1}}\]
in $\text{im}\delta^{r+1,1}$. In order to do this, let us first prove that, if we consider $S^{r+1} (V^*) \otimes g$ as a $G$-submodule of $V^{r+1}_1$, then
\[\text{sym}^{r+2}_1 \circ L(r)|_{S^{r+1} (V^*) \otimes g} = 1_{S^{r+1} (V^*) \otimes g} .\]

So let $\tau \in S^{r+1} (V^*) \otimes g$. Given vectors $u_1, \ldots, u_{r+1}, u_{r+2} \in V$, we can consider $\tau (u_1, \ldots, u_{r+1}) \in g$, and so $\tau (u_1, \ldots, u_{r+1}) u_{r+2} \in V$. We have that
\[(\delta^{r+1,1} \tau) (u_1, \ldots, u_{r+1}, u_{r+2}) = \tau (u_1, \ldots, u_{r+1}) u_{r+2} - \tau (u_1, \ldots, u_{r+2}) u_{r+1} = \delta (\tau (u_1, \ldots, u_{r+2})(u_{r+1}, u_{r+2}) - \tau (u_1, \ldots, u_{r+1}) u_{r+2} .\]

Therefore, $\delta^{r+1,1} \tau = (1_{S^{r} (V^*)} \otimes \delta) \tau$, whence
\[(16) \text{sym}^{r+2}_1 \circ L(r) \tau = \text{sym}^{r+2}_1 \circ (1_{S^{r} (V^*)} \otimes \delta^{-1} \circ P_{\text{im}\delta}) \circ \delta^{r+1,1} \tau = \text{sym}^{r+2}_1 \tau = \tau .\]

Thus,
\[\text{im} (\text{sym}^{r+2}_1 \circ L(r)) = S^{r+1} (V^*) \otimes g\]
and, according to (15), we also have that
\[\text{im} (\text{sym}^{r+2}_1 \circ L(r)) = \text{im} (\text{sym}^{r+2}_1 \circ L(r)) = S^{r+1} (V^*) \otimes g.\]

Taking into account that, by (6), $\delta^{r+1,1}|_{S^{r+1} (V^*) \otimes g}$ is injective, we conclude that
\[
\dim \ker (\text{sym}^{r+2}_1 \circ (1 \otimes \delta^{-1} \circ P_{\text{im}\delta}) |_{\text{im}\delta^{r+1,1}}) = \dim (\text{im}\delta^{r+1,1}) - \dim (S^{r+1} (V^*) \otimes g) = \dim (\text{im}\delta^{r+1,1}) - \dim (\delta^{r+1,1} (S^{r+1} (V^*) \otimes g)) .
\]
Let us define $\hat{Z}^{r+1} = \delta^{r+1,1} (S^{r+1} (V^* \otimes \mathfrak{g})$. Obviously it is a $G$-submodule of $\text{im} \delta^{r+1,1}$ of the required dimension. Moreover, (16) implies that
\[
\hat{Z}^{r+1} \cap \ker \left( \text{sym}^{r,2}_1 \circ (1 \otimes \delta^{-1} \circ P_{\text{im} \delta}) \right|_{\text{im} \delta^{r+1,1}} = \{0\}.
\]
Hence, $\hat{Z}^{r+1}$ is the desired supplementary.

**Proof of Theorem 1.1.** The proof will be by induction on $r$. The case $r = 0$ is trivial: since $S^0$ has a unique element, determined by the conditions $\sigma_{ij} (0) = \delta_{ij}$, it admits the trivial $G$-module structure.

Now, assume that a $G$-module structure has been already defined on $S^r$ and let us define one on $S^{r+1}$.

Let us consider the $G$-equivariant map $f^r = \Sigma^{(r+1)} \circ \mu^r_{r+1} : S^r \to \mathcal{V}_1^{r+1,1}$, and define a $G$-equivariant bijection $\psi^r : S^r \times \mathcal{V}_1^{r+1,1} \to S^r \times \mathcal{V}_1^{r+1,1}$ by $\psi^r (j^r_0 \sigma, v^{r+1}) = (j^r_0 \sigma, v^{r+1} - f^r (j^r_0 \sigma))$, where the action of $G$ on both sides is just the diagonal one.

After Lemma 5.1, we see that the image $(\pi^{r+1}_1 \times \Sigma^{(r+1)}) (S^{r+1})$ can be characterized as the set of pairs $(j^r_0 \sigma, v^{r+1}) \in S^r \times \mathcal{V}_1^{r+1,1}$ satisfying:
\[
\left( \text{sym}^{r,2}_1 \circ L^{(r)} \right) (v^{r+1}) = \left( \text{sym}^{r,2}_1 \circ L^{(r)} \right) (f^r (j^r_0 \sigma))
\]
(both members being equal to $-\text{sym}^{r,2}_1 (Q^{(r)} (j^r_0 \sigma))$). But this set is just the preimage by $\psi^r$ of the $G$-submodule $S^r \oplus W^{r+1} \subset S^r \oplus \mathcal{V}_1^{r+1,1}$.

Now, we define the $G$-module structure on $S^{r+1}$ as the one that makes linear the bijection:
\[
\delta^r := \psi^r \circ (\pi^{r+1}_1 \times \Sigma^{(r+1)}) : S^{r+1} \to S^r \oplus W^{r+1},
\]
where we are considering the direct sum $G$-module structure on the right hand side.

The following diagram
\[
\begin{array}{ccc}
S^{r+1} & \xrightarrow{\delta^r} & S^r \oplus W^{r+1} \\
\pi^{r+1}_1 \downarrow & & \downarrow \text{pr}_1 \\
S^r
\end{array}
\]
where $\text{pr}_1$ denotes the projection onto the first summand, is commutative, so that the projections $\pi^{r+1}_1$ are linear. Using induction, as well as the fact that $\delta^r$ is $G$-equivariant, we conclude that the action of $G$ on each $S^r$ is also linear. Moreover, the smoothness of the maps $f^r$ implies the compatibility of the $G$-module structure with the manifold structure on each $S^r$.

Finally, the resulting $G$-module structure on each $G$-manifold $S^r$ does not depend either on the supplementary subspace $W$ used to define the canonical connections or on the sections $\mu^r_{r+1}$ between the corresponding spaces $S^r$. In fact, changing any of them would define isomorphic $G$-module structures, as follows from next general lemma:

**Lemma 5.2.** Let $M$ and $N$ be two smooth $n$-dimensional manifolds endowed with smooth $G$-module structures. If $M$ and $N$ are diffeomorphic as $G$-manifolds, then they also are isomorphic as $G$-modules.

**Proof.** Let $f : M \to N$ be a $G$-equivariant diffeomorphism. By composing, if necessary, with a translation in $N$ we can assume that $f(0) = 0$. Due to the smoothness of the module structures, any linear isomorphisms $\varphi_1 : M \to \mathbb{R}^n$,
\[ \phi : N \to \mathbb{R}^n \text{ (onto } \mathbb{R}^n \text{ with its standard linear structure) define global charts of } M \text{ and } N. \] The map \( \phi \circ f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism leaving 0 fixed, and it is equivariant with respect to the linear actions of \( G \) on \( \mathbb{R}^n \) induced by \( \phi_1 \) and \( \phi_2 \). Then, the Fréchet derivative \( D \left( \phi_2 \circ f \circ \phi_1^{-1} \right) (0) : \mathbb{R}^n \to \mathbb{R}^n \) is a linear \( G \)-equivariant isomorphism, and so is the composed map: \( \phi_2^{-1} \circ D \left( \phi_2 \circ f \circ \phi_1^{-1} \right) (0) \circ \phi_1 : M \to N. \]

By definition, the \( G \)-module structure on \( S^{r+1} \) is isomorphic to \( S^r \oplus W^{r+1} \). This fact (together with the obvious identity \( S_1^0 = \{0\} \)) yields an isomorphism of \( G \)-modules:

\[
S^r \cong W^1 \oplus W^2 \oplus \cdots \oplus W^r,
\]

and the \( G \)-module structure of \( S^r \) is determined by that of the spaces \( W^k, k = 1, \ldots, r \).

An element \( t \in W^{r+1} \) lies in \( \tilde{V}_1^{r+1,1} := \ker \left( \text{sym}_{1}^{r+1,1} \right) \), whence its components must satisfy the equations

\[
(18) \quad \bigwedge_{(i_1, \ldots, i_{r+1+i_{r+2}})} t_{i_1 \cdots i_{r+1+i_{r+2}}} = 0, \quad 1 \leq k, i_1, \ldots, i_{r+2} \leq n.
\]

Denoting by \( (L^r t)_{i_1 \cdots i_{r+1+i_{r+2}}} \) the components of \( L^r t \), it follows that an element \( t \in \tilde{V}_1^{r+1,1} \) lies in \( W^{r+1} \) if and only if the following equations are also satisfied:

\[
(19) \quad \bigwedge_{(i_1 \cdots i_{r+1})} (L^r t)_{i_1 \cdots i_{r+1+i_{r+2}}} = 0, \quad 1 \leq k, i_1, \ldots, i_{r+2} \leq n.
\]

Therefore, \( W^{r+1} \) is the \( G \)-submodule of \( V_1^{r+1,1} = S^{r+1}(V^*) \otimes V^* \otimes V \) characterized by equations (18) and (19).

Notice that, in particular, \( W^1 = \ker \left( \text{sym}_1^{0,2} \circ L^0 \right) \). By definition, \( \text{sym}_1^{0,2} \) is the identity in \( V^* \otimes V^* \otimes V \). The mapping \( \tilde{L}^0 : \tilde{V}_1^{1,1} \to V_1^{0,2} \) is defined on \( \tilde{V}_1^{1,1} = \bigwedge^2 V^* \otimes V \) as

\[
\bigwedge^2 V^* \otimes V \stackrel{\text{sym}_1^{0,2}}{\longrightarrow} \delta (V^* \otimes \mathfrak{g}) \stackrel{\delta^{-1}}{\longrightarrow} V^* \otimes \mathfrak{g}.
\]

Thus \( W^1 = \ker P_{\text{im}\delta} = W \).

Next, we will describe the \( G \)-modules \( W^r \) for some different choices of \( G \).

**Example 5.1** \((e)\)-structures. For complete parallelisms, \( W^{r+1} \) is characterized by equation (18). Equation (19) is empty because the tensor \( F \), and hence the operator \( L^r \), are zero. Therefore, \( W^{r+1} = \tilde{V}_1^{r+1,1} \).

**Example 5.2** \((O(n))\)-structures. We have seen (Example 2.1) that, for \( G = O(n) \), the map \( \delta \) is an isomorphism, with inverse \( \delta^{-1} : \bigwedge^2 V^* \otimes V \to V^* \otimes \mathfrak{g} \) given in components, by:

\[
(\delta^{-1}(T))_{ij}^k = \frac{1}{2} \left( T_{ij}^k + T_{ki}^j + T_{kj}^i \right)
\]

The operator \( L^r \) is then given by:

\[
(L^r t)_{i_1 \cdots i_{r+1+i_{r+2}}} = \frac{1}{2} \left( t_{i_1 \cdots i_{r+1+i_{r+2}}} - t_{i_1 \cdots i_{r+1+i_{r+2}}} + t_{i_1 \cdots i_k i_{r+1} - t_{i_1 \cdots i_{r+1} k} + t_{i_1 \cdots i_k i_{r+2} - t_{i_1 \cdots i_{r+2} k}} \right).
\]
Taking the cyclic sum of this expression, and using \((18)\), we obtain:

\[
\mathcal{G}_{(i_1 \ldots i_{r+1})} (L^{(r)}) t^{k}_{i_1 \ldots i_{r+1} i_{r+2}} = \frac{1}{2} \left( (r+1)t^{k}_{i_1 \ldots i_{r+1} i_{r+2}} + t^{k}_{i_1 \ldots i_{r+1} i_{r+2}} - t^{r+2}_{i_1 \ldots i_{r+1} k} - (r+1)t^{r+2}_{i_2 \ldots i_{r+1} k} + \mathcal{G}_{(i_1 \ldots i_{r+1})} \left( t^{r+1}_{i_1 \ldots i_{r+1} k} - t^{r+1}_{i_2 \ldots i_{r+2}} \right) \right)
\]

Therefore, equations \((18)\) can be written:

\[
(20) \quad t^{k}_{i_1 \ldots i_{r+1} i_{r+2}} - t^{r+2}_{i_1 \ldots i_{r+1} k} = \frac{1}{r+2} \left( \mathcal{G}_{(i_1 \ldots i_{r+2})} t^{r+2}_{i_1 \ldots i_{r+1} k} + \mathcal{G}_{(i_1 \ldots i_{r+2})} t^{r+2}_{i_1 \ldots i_{r+1} k} \right) = \mathcal{G}_{(i_1 \ldots i_{r+2})} t^{r+2}_{i_1 \ldots i_{r+1} k},
\]

from which it follows that:

\[
\mathcal{G}_{(i_1 \ldots i_{r+2})} t^{r+2}_{i_1 \ldots i_{r+1} k} = 0.
\]

Last equation can be introduced in \((20)\) giving:

\[
\frac{1}{r+2} \left( -t^{r+2}_{i_1 \ldots i_{r} i_{r+1} k} + t^{r+2}_{i_1 \ldots i_{r+1} i_{r+2}} \right),
\]

from which one can conclude that the equations characterizing the subspace \(W^{r+1} \subset V^{r+1}_{1} \) are \((18)\) and

\[
(21) \quad t^{k}_{i_1 \ldots i_{r+1} i_{r+2}} - t^{r+2}_{i_1 \ldots i_{r} i_{r+2}} = 0,
\]

for any indices \(1 \leq k, i_1, \ldots, i_{r+2} \leq n\).

It should be pointed that, in \(4\), Epstein describes the space of \(\infty\)-jets of Riemannian metrics \(g\) at a point, as follows. He writes the Taylor series for \(g_{ij} (x)\) in normal coordinates: \(\delta_{ij} + \sum_{r \geq 1} g_{ij i_1 \ldots i_r} x^{i_1} \ldots x^{i_r}\). The coefficients \(g_{ij i_1 \ldots i_r}\) satisfy the following conditions:

1. They are symmetric in the first two indices.
2. They are symmetric in the last \(r\)-indices.
3. \(\mathcal{G}_{(i_1 \ldots i_{r+1})} g_{ii_1 \ldots i_{r+1}} = 0\) for \(1 \leq i, i_1, \ldots, i_{r+1} \leq n\).

Then, he defines \(f_r \in (V^*)^{\otimes (r+2)}\) by \(f_r (v_i, v_j, v_{i_1}, \ldots, v_{i_r}) = g_{ij i_1 \ldots i_r}\), and proves that the set of \(f_r \in (V^*)^{\otimes (r+2)}\) satisfying the symmetry conditions corresponding to \((18)\) is an irreducible \(GL(n, \mathbb{R})\)-module \(Y_r\) with Young diagram having \(r\) squares in the first row and 2 squares in the second one, except that if \(r = 1\) then \(Y_r = \{0\}\). Moreover, for any sequence \(f_r \in Y_r, 2 \leq r < \infty\), there is a Riemannian metric whose Taylor series gives the elements \(f_r\), so that one can regard \(\Pi_{r \geq 2} Y_r\) as the space of \(\infty\)-jets of Riemannian metrics at a point.

The identification between Riemannian metrics \(g\) and sections \(s\) of the bundle \(FV/O(n) \to V\) together with Lemma 5.2 yield an isomorphism of \(O(n)\)-modules:
\[ S^r \cong \Pi_{2 \leq k \leq r} Y_k. \] Moreover, it is straightforward to check that the mapping \( g_{jk i_1 \ldots i_{r+1}} \mapsto t^k_{i_1 \ldots i_{r+1}} \) also defines an isomorphism of \( O(n) \)-modules \( W^{r+1} \cong Y_{r+1} \).

**Example 5.3** \((O(p) \times O(q))\)-structures. From the expression of \( \delta^{-1} \circ P_{im} \delta \) given in Example 2.3, it is immediate that the components of \( L^{(r)} t \) are:

\[ (L^{(r)} t)^k_{i_1 \ldots i_{r+1} i_{r+2}} = \frac{1}{2} \left( t^k_{i_1 \ldots i_{r+1} i_{r+2}} - t^k_{i_1 \ldots i_{r+1} i_{r+2}} \right) \]

\[ + t^k_{i_1 \ldots i_{r+1} i_{r+1} i_{r+2}} - t^k_{i_1 \ldots i_{r+1} i_{r+1} i_{r+2}} \]

\[ - t^k_{i_1 \ldots i_{r+1} i_{r+2}} \]

if \( i_{r+1}, i_{r+2}, k \in I_\alpha, \alpha = 1, 2 \),

\[ (L^{(r)} t)^k_{i_1 \ldots i_{r+1} i_{r+2}} = \frac{1}{2} \left( t^k_{i_1 \ldots i_{r+1} i_{r+2}} - t^k_{i_1 \ldots i_{r+1} i_{r+2}} \right) \]

\[ + t^k_{i_1 \ldots i_{r+1} i_{r+2}} - t^k_{i_1 \ldots i_{r+1} i_{r+2}} \]

\[ - t^k_{i_1 \ldots i_{r+1} i_{r+2}} \]

if \( i_{r+1} \in I_\alpha, i_{r+2}, k \in I_\beta, \alpha \neq \beta, \) and

\[ (L^{(r)} t)^k_{i_1 \ldots i_{r+1} i_{r+2}} = 0 \]

if \( i_{r+2} \in I_\alpha, k \in I_\beta, \alpha \neq \beta. \)

As in the previous example, relation (22) leads to the equations (21) if \( i_{r+1}, i_{r+2}, k \in I_\alpha, \alpha = 1, 2. \) On the other hand, taking the cyclic sum with respect to \( i_1, \ldots, i_{r+1} \) in (22), and using (18), we obtain that equations (21) are also satisfied if \( i_{r+1} \in I_\alpha, i_{r+2}, k \in I_\beta, \alpha \neq \beta. \) In this way, the equations of \( W^{r+1} \subset V^{r+1}_{1} \) turn out to be:

\[ t^k_{i_1 \ldots i_{r+1} i_{r+2}} - t^k_{i_1 \ldots i_{r+1} i_{r+2}} = 0, \]

for any indices \( i_1, \ldots, i_{r+1} \in I_1 \cup I_2, i_{r+2}, k \in I_\alpha, \alpha = 1, 2. \)

**Example 5.4** \((\mathbb{R}^*)\)-structures. The expression of \( \delta^{-1} \circ P_{im} \delta \) given in Example 2.3 leads to the following expression of \( L^{(r)} t \):

\[ (L^{(r)} t)^k_{i_1 \ldots i_{r+1} i_{r+2}} = \frac{1}{n-1} g_{i_{r+2}}^{i_1} \sum_j \left( t^j_{i_1 \ldots i_{r+1} i_{r+2}} - t^j_{i_1 \ldots i_{r+1} i_{r+2}} \right). \]

Taking the cyclic sum with respect to \( i_1, \ldots, i_{r+1} \) (for \( k = i_{r+2} \)), and using (18), we obtain:

\[ \mathcal{G}_{(i_1 \ldots i_{r+1})} (L^{(r)} t)^k_{i_1 \ldots i_{r+1} i_{r+2}} = \]

\[ = \frac{1}{n-1} \left( (r+1) \sum_j t^j_{i_1 \ldots i_{r+1} i_{r+2}} + \sum_j t^j_{i_1 \ldots i_{r+1} i_{r+2}} \right) = \]

\[ = \frac{r+2}{n-1} \sum_j t^j_{i_1 \ldots i_{r+1} i_{r+2}} \]

The resulting equations of \( W^{r+1} \) are then (18) and:

\[ \sum_k t^k_{i_1 \ldots i_{r+1} i_{r+2}} = 0, \]

\( 1 \leq k, i_1, \ldots, i_{r+1} \leq n. \)
THE MODULI OF G-STRUCTURES WITH LINEAR CONNECTION.

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