Persistence of chirality in the Su-Schrieffer-Heeger model in the presence of on-site disorder

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We consider the effects of on-site and hopping disorder on zero modes in the Su-Schrieffer-Heeger model. In the absence of disorder a domain wall gives rise to two chiral fractionalized bound states, one at the edge and one bound to the domain wall. On-site disorder breaks the chiral symmetry, in contrast to hopping disorder. By using the polarization we find that on-site disorder has little effect on the chiral nature of the bound states for weak to moderate disorder. We explore the behaviour of these bound states for strong disorder, contrasting on-site and hopping disorder and connect our results to the localization properties of the bound states and to recent experiments.

I. INTRODUCTION

The Su-Schrieffer-Heeger (SSH) model [1] was introduced in the context of polyacetylene but has attracted much interest as a model of non-interacting fermions in one dimension that displays charge fractionalization [2]. The SSH model also gives a simple example of a model with topologically distinct states which arise for opposite hopping dimerization patterns. Fractionalization arises when domain walls are introduced that separate the two dimerization patterns and give rise to zero energy modes with specific chiralities bound at the domain walls.

Recently there have been several experimental realizations of the SSH model: in cold atom systems [3–6] and graphene nanoribbons [7–9]. Condensed matter implementations of the SSH model generically break the chiral (sublattice) symmetry that gives rise to zero modes through e.g. next-nearest-neighbour hopping [10] (as occurs in polyacetylene) or disorder. In this work we explore the effect of broken chiral symmetry due to disorder on the states that are zero modes in the absence of disorder. Specifically, this is important for the interpretation of experimental results on signatures of chirality in real systems where disorder is inevitable and chiral symmetry is broken [7–9].

Much of the previous work on disorder in the SSH model has focused on the case in which there is disorder in hopping amplitudes [11–13]. For this special class of disorder, the chiral symmetry of the model is preserved, and hence zero modes in the clean model remain zero modes in the disordered model for weak disorder, only disappearing at a critical disorder value [? ]. On-site disorder explicitly breaks chiral symmetry so that the zero modes in the clean limit are no longer topologically protected and have a non-zero energy for infinitesimal disorder. In the limit of infinitesimal disorder we expect there to be states that closely resemble the zero energy modes in the clean limit. Previous work [10, 14] has investigated how on-site disorder affects the localization properties of edge states, but not their chiral properties. In particular, for a SSH model with a domain wall, if the system is large enough, we might expect localized states at the wall and the edge to retain some of their chiral properties.

In this work we study the disordered SSH model with a domain wall and explore the extent to which the localized states retain their chiral nature even though chiral symmetry has been broken by on-site disorder. Our main tool to do this is the polarization – it has been shown that in a system with chiral symmetry there is a relationship between the winding number and the polarization [15–21]. Our main result is that we find that for even quite sizeable disorder strengths, the bound states can be viewed as being chiral from a practical point of view, even if not perfectly so. We relate the changes in polarization as a function of on-site disorder strength to changes in the localization properties of the electronic states.

The structure of this paper is as follows: in Sec. II we introduce the disordered SSH model and show numerical calculations of its spectrum and of localized states. In Sec. III we discuss our results and conclude.

II. THE DISORDERED SSH MODEL

The Hamiltonian for the SSH model on a $N$ site chain may be written as

$$H_{SSH} = -t_0 \sum_{n=1}^{N-1} [1 + (-1)^n u] \left\{ c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1} \right\},$$

(1)

where $c_n$ and $c_n^\dagger$ are annihilation and creation operators for fermions on site $n$ respectively, $t_0$ is the hopping strength and $u$ is the dimensionless strength of the stagger in the hopping. Fractionalized states arise if a domain wall is introduced into the parameter $u$ [1, 2]. Here we consider domain walls of the form:

$$u = u_0 \tanh \left[ \frac{(n - n_0)}{\xi/a} \right],$$

(2)

where $u_0$ specifies the amplitude of the domain wall, $n_0$ is the centre of the domain wall and $\xi$ is the width, with $a$ the lattice spacing. In the presence of such a domain wall,
the SSH model develops fractionalized zero modes which have support on a single sublattice. In a finite chain with open boundary conditions, and a domain wall of the form Eq. (2), one of these zero modes will be localized at an edge, and the other will be localized at the domain wall, as illustrated in Fig. 1.

We introduce disorder in the form of a random on-site potential with Hamiltonian

$$H_{\text{dis}} = \sum_{n=1}^{N} \epsilon_n c_n^\dagger c_n,$$

where $\epsilon_n$ is a random variable drawn from a uniform distribution on $[-W, W]$. Such a potential breaks chiral symmetry and hence the fractionalization seen at $W = 0$ will no longer be present. However, it is still of interest to study how the chirality of the bound states that form the $W = 0$ zero modes evolve with increasing disorder. In particular the question we want to investigate is how they lose their chirality with increasing disorder.

We diagonalized the Hamiltonian $H = H_{\text{SSH}} + H_{\text{dis}}$ for a $N = 500$ site chain and found the ordered list of energy eigenvalues. We averaged over 50000 disorder configurations to obtain Fig. 2. Once chiral symmetry is broken by a disorder potential ($W \neq 0$), the zero modes seen at $W = 0$ move away from being exactly at zero energy, but they are clearly identifiable in the gap out to disorder strengths of $W/t_0 \sim 1.3$. Several other bound states are visible in the gap out to $W/t_0 \sim 0.8$ for the particular choice of parameters in Fig. 2.

We also calculated the disorder averaged density of states, shown for $W/t_0 = 0.1, 0.7,$ and $1.5$ in Fig. 3. Peaks corresponding to the bound states are clearly visible up to moderate disorder ($W/t_0 \lesssim 1$) but for stronger disorder the bands broaden sufficiently to obscure them. While the $W = 0$ zero modes do not continue to have zero energy for $W \neq 0$, we can ask whether they can be treated as chiral for practical purposes as the disorder is increased.

In the case of disorder that preserves chiral symmetry (e.g. hopping disorder) [11] one can consider a real-space calculation of a topological invariant which is closely related to the polarization [21]. In the case of on-site disorder that we consider here, there is no strict topological protection, so we instead focus on the polarization of bound states, which can change continuously as disorder increases. Specifically, we introduce projection operators $\hat{P}_A$ and $\hat{P}_B$, which project a bound state $|\psi\rangle$ on to either the $A$ or $B$ sublattices respectively. We can use these projection operators to calculate the polarization, i.e. the density imbalance between $A$ and $B$ sublattices

$$P = \langle \psi | \hat{P}_A - \hat{P}_B | \psi \rangle,$$

for bound states $|\psi\rangle$ localized at the domain wall and the edge. When $W \neq 0$ we select the bound states by projecting disordered bound states $|\psi(W \neq 0)\rangle$ onto the $W = 0$ bound states $|\phi(W = 0)\rangle$. The results we show are for the states that have the maximum overlap with the clean bound states i.e. those that maximize $|\langle \psi(W \neq 0) | \phi(W = 0) \rangle|$. We calculate the polarization $P$ for these bound states in the presence of both chiral symmetry preserving and chiral symmetry breaking disorder. We introduce chiral symmetry preserving disorder via the Hamiltonian

$$H_{\text{chiral dis}} = \sum_{n=1}^{N-1} \tau_n \left\{ c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1} \right\},$$

where $\tau_n$ is a random variable drawn from a uniform distribution on $[-W, W]$. Similarly to Ref. [11], we find that

![FIG. 1. Wavefunction for the zero mode localized on the domain wall in the centre of a $N = 500$ site chain in the clean SSH model for $\xi/a = 10$ and $u_0 = 0.2$.](image)

![FIG. 2. Ordered energy eigenvalues of a $N = 500$ site chain in the SSH model as they evolve with increasing $W/t_0$, for $\xi/a = 10$ and $u_0 = 0.2$.](image)
$P$ goes to zero for large $W$ for the $W \neq 0$ bound states at the domain wall and the edge, consistent with the transition in winding number with $W$ identified in Ref. [11], as illustrated in Fig. 4. Even though the states lose their polarization, they remain localized for all disorder strengths [22].

We performed similar calculations for the SSH model with on-site disorder and display the results in Fig. 5. We found that for $N \gtrsim 300$ that our results appear to be independent of $N$. We also see that $|P|$ decays more quickly with $W/t_0$ than for hopping disorder, but that $1 - |P|$ $\ll$ 1 for values of $W/t_0$ that are an appreciable fraction of 1, demonstrating that small amounts of on-site disorder do not greatly alter the chiral nature of the states.

Unlike the situation in which there is hopping disorder, $|P|$ does not approach zero with increasing $W/t_0$ and in fact increases towards 1 with increasing $W/t_0$. The reason for this behaviour can be illuminated with the inverse participation ratio, defined by

$$\text{IPR} = \frac{\sum_i |\psi(r_i)|^4}{\left(\sum_i |\psi(r_i)|^2\right)^2}, \quad (6)$$

which gives a measure of localization. The value of the IPR differs significantly between localized and extended states. For localized states, the IPR takes a constant value, whereas for extended states, the IPR scales like $1/L^d$ where $d$ is the spatial dimension. The IPR is illustrated in Fig. 6 and illustrates that the localization length increases up to a disorder strength of $W/t_0 \sim 1 - 2$, consistent with results obtained for edge states in smaller systems [10, 14]. The localization length decreases at larger values of disorder, consistent with Anderson localization becoming more important. The states are always localized, as expected for a one dimensional disordered fermion system [23], but the degree of localization varies with disorder strength.

The behaviour seen in $P$ can be understood from a picture in which increasing on-site disorder breaks chiral symmetry so that the zero disorder zero mode states start to have some support on both sublattices, but unlike the hopping disorder case, $P$ does not go to zero, because with increasing on-site disorder strength, the states become sufficiently localized that most of their support is on a single site. Figure 6 illustrates that there is a crossover from a localized state that retains much of its $W = 0$ chiral character to a strongly Anderson localized state as a function of $W/t_0$. 

FIG. 3. Disorder averaged density of states (DOS) for a SSH chain with $N = 500$ sites for a domain wall centred at site 250 with width $\zeta/a = 100$, strength $u_0 = 0.2$ and $N = 500$ for disorder strengths a) $W/t_0 = 0.1$, b) $W/t_0 = 0.7$, and c) $W/t_0 = 1.5$.

FIG. 4. Polarization $P$ for the states bound at the domain wall (Wall state) and the edge (Edge state) in a chain with $N = 500$ sites with $\zeta/a = 10$ and $u_0/t_0 = 0.2$ for the SSH model with hopping disorder.
III. DISCUSSION

We studied the SSH model with on-site disorder and compare our results to those obtained for hopping disorder. Our results demonstrate that even though chiral symmetry is broken by the introduction of on-site disorder, the zero energy states at zero disorder evolve so that they continue to be strongly polarized for $W/t_0 \lesssim 1$ and can be treated as chiral for practical purposes for moderate on-site disorder. The IPR illustrates that this is a crossover from topology-induced localization to Anderson localization with increasing disorder.

FIG. 6. Inverse Participation Ratio (IPR) for the states bound at the domain wall (Wall state) and the edge (Edge state) in a chain with 500 sites and a domain wall width of $\xi/a = 10$ and $u_0 = 0.2$ as a function of disorder strength.

We note that our calculations here have direct relevance to recent experiments. In particular, two groups used graphene nanoribbons [7–9] to engineer the SSH model and studied edge states in these systems. Our results here show that the edge states that are topologically protected in the clean limit persist to large values of disorder. Hence, given the inevitability of some level of on-site disorder in experiment, the edge states observed in experiment are still meaningful approximations to the clean case. From a theoretical perspective, the fractionization [24] seen in the SSH model in one dimension has been generalized to two dimensions [25–28] and it would be very interesting to see how disorder affects the zero energy modes in those models.

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