Chaotic mechanism description by an elementary mixer for the template of an attractor

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Abstract

Templates can be used to describe the topological properties of chaotic attractors. For attractors bounded by genus one torus, these templates are described by a linking matrix. For a given attractor, it has been shown that the template depends on the Poincaré section chosen to perform the analysis. The purpose of this article is to present an algorithm that gives the elementary mixer of a template in order to have a unique way to describe a chaotic mechanism. This chaotic mechanism is described with a linking matrix and we also provide a method to generate and classify all the possible chaotic mechanisms made of two to five strips.

1 Introduction

The topological characterization method has been introduced in 1980’s. Its purpose is to use the properties of the periodic orbits to describe the topological structure of chaotic attractors. Birman and Williams [1] are the first to apply this technique on the Lorenz system [2] where the orbits are considered as the skeleton of the attractor to describe its template. This method and its applications are described by Gilmore and co-workers [3, 4, 5] where the linking matrix describe how attractor’s couple of orbits are entwined. This linking number between two orbits is numerically computed. The linking matrix also permits to obtain this topological invariant between each pair of orbits with the method described by Le Sceller et al. [6].

Some algebraic relations are introduced to perform operations between the linking matrix. Gilmore et al. propose the concatenation of linking matrix [3] and we recently propose algorithms to perform this concatenation [7]. Tufillaro and co-workers [8, 9] propose another algebraic representation of a linking matrix: only a matrix satisfying the orientation convention to know how strips are ordered. Using this representation convention for templates, we present an algebraical relation between two templates symmetric by inversion [10]. Other representations have been introduced to describe the chaotic mechanism. For instance, Martín & Used [11] proposed some links between templates focusing on how branches are organised after stretching and folding and before squeezing. Towards the same goal, Cross & Gilmore [12] proposed to use return maps to compare chaotic mechanisms.

In this article, we use conventions to describe the template of a chaotic attractor with only one linking matrix. This representation depends on the Poincaré section chosen to perform the analysis [10]. The topological characterization of one attractor of the Malasoma system [13] leads to obtain four templates (and four linking matrix) to describe one attractor from four non equal but equivalent Poincaré sections [14]. Here we propose to find a unique linking matrix, named elementary mixer, to describe the chaotic mechanism. This elementary mixer represents the chaotic mechanism of the template without constraint introduced by the Poincaré section choice to perform the analysis. We first introduce the conventions to obtain a template of an attractor from a given Poincaré section. The second part is dedicated to the algebraical relations between linking matrix. We then propose an algorithm to extract the chaotic mechanism of a linking matrix; this chaotic mechanism is described with an elementary mixer. Finally, we provide a classification containing all the existing chaotic mechanisms for templates composed of less than six strips.
These results permits to prove that the Burke-Shaw attractor have the same chaotic mechanism as the Malasoma attractor.

2 Template of chaotic attractors

Definition 1 [15] A template is a compact branched two-manifold with boundary and smooth expansive semiflow built locally from two types of charts: joining and splitting. Each chart, as illustrated in Figure 1, carries a semiflow, endowing the template with an expanding semiflow, and gluing maps between charts must respect the semiflow and act linearly on the edges.

![Figure 1](image)

(a) (b)

Figure 1: A template is a branched two-manifold with two types of charts: (a) joining chart; (b) splitting chart.

A linker is a synthesis of the relative organization of $n$ strips: torsions and permutations in a planar projection (Fig. 2). A mixer is a linker ended by a joining chart that stretches and squeezes strips to a branch line.

![Figure 2](image)

Figure 2: Convention representation of oriented crossings. The permutation between two branches is positive if the crossing generated is equal to +1, otherwise it is a negative permutation. We use the same convention for torsions.

The purpose of the topological characterization method is to describe the topological properties of a chaotic attractor using topological invariants: the linking number between each couple of orbits. The orbits are the skeleton of the attractor and structure the flow. The linking number is an integer indicating how many times orbits are entwined one around the other. In the literature, this method permits to successfully describe a wide range of attractors. Here we give some references where the authors give the templates of attractors bounded by genus one torus: Rössler attractors by Letellier et al. [16], Duffing oscillator attractor by Gilmore & McCallum [17], Burke-Shaw attractor by Letellier et al. [18] and recently we propose the template of Malasoma attractors [10, 14]. This method can also be applied to attractors bounded by torus with an higher genus mainly by Letellier & co-workers, for instance, it is the case for templates of the Lorenz attractors [19], Chen attractor [20], Chua attractors [21] and multiscroll attractor [22].

In this article, we focus our attention on attractors bounded by genus one torus. This is a first step before considering the templates of attractors bounded by higher genus torus. In order to compare all templates previously cited, we already introduce a method that ensures us to obtain a template described by a unique linking matrix [10]. This is performed using conventions:

1. Consider clockwise evolution of the flow;
2. Poincaré section is oriented from the inside to the outside;

3. Template is described with a unique linking matrix that satisfy the Tufillaro convention [8] when the strips stretch and squeeze: their order from the left to the right corresponds to the bottom to top order.

This definition of the clockwise flow refers to the definition (3.1) of [23] where only one point is considered around which the flow evolves clockwise. However, this definition have to be extended to consider as many points with clockwise evolution of the flow as holes without singularities of the toroidal boundary of the template. We already applied the topological characterization of one attractor solution to the Malasoma system [13] with these conventions [14]. To explore its symmetry properties, we use several equivalent Poincaré sections to propose the template of this applied the topological characterization of one attractor solution to the Malasoma system [13] with these conventions [14]. To explore its symmetry properties, we use several equivalent Poincaré sections to propose the template of this.

\[ L_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad L_b = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad L_c = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad L_d = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \] (1)

In the next sections of this article, we will provide tools and algorithm to obtain only one way to describe the chaotic mechanism of attractor’s template.

3 Algebraic relation between linking matrix

A mixer \( \mathcal{M} \) is defined by a linking matrix \( M \) where the right side is “]” to represent the merging at the branch line with respect to the Tufillaro convention [8] (see [7] to distinguish the two linking matrix representation conventions). For the following section, we named linkers and mixers of template as actions and also consider torsions and permutations are linkers.

**Definition 2** In a template \( T \), Given two actions \( D_1 \) and \( D_2 \), when \( D_1 \) precedes \( D_2 \), without any other action between, the resulting action is the concatenation of \( D_1 \) before \( D_2 \), noted \( D_1 \oplus D_2 \).

3.1 Concatenation of torsion and mixer

**Lemma 1** [10] Given a template \( T \), containing a torsion \( t \) defined by \( T = |\tau| \) and a mixer \( \mathcal{M} \) defined by \( M \) with \( n \) strips. If \( \mathcal{M} \) and \( t \) are concatenated, then

- \( \mathcal{M} \oplus t \equiv \mathcal{M}' \)
- \( t \oplus \mathcal{M} \equiv \mathcal{M}' \), if \( \tau \) is even
- \( t \oplus \mathcal{M} \equiv \mathcal{M}'' \), if \( \tau \) is odd

\( \mathcal{M}' \) is defined by \( M' \) and \( \mathcal{M}'' \) is defined by \( M'' \)

\[ M' = \begin{bmatrix} M_{1,1} + \tau & \cdots & M_{1,n} + \tau \\ \vdots & \ddots & \vdots \\ M_{n,1} + \tau & \cdots & M_{n,n} + \tau \end{bmatrix}, \quad M'' = \begin{bmatrix} M_{n,1} + \tau & \cdots & M_{1,n} + \tau \\ \vdots & \ddots & \vdots \\ M_{1,1} + \tau & \cdots & M_{1,1} + \tau \end{bmatrix}. \] (2)
3.2 Concatenation of mixers

**Theorem 1** [7] Given a template \( T \) containing a mixer \( A \) of \( n_a \) strips defined by the linking matrix \( A \) and a mixer \( B \) of \( n_b \) strips defined by the linking matrix \( B \). If \( A \) is concatenated before \( B \), then \( A \oplus B \equiv C \), where \( C \) is a mixer of \( n_a \times n_b \) strips defined by the linking matrix \( C \) that is the sum of the expanded linking matrices \( A \) and \( B \) with additional permutations due to the insertion of \( A \)

\[
C = A_{\text{expand}} + A_{\text{insertion}} + B_{\text{expand}}
\]  

with the respect of the strips order at the beginning of \( A \).

**Example 1** In the article [18], from an attractor solution to the Burke-Shaw system, the template of a symmetrical attractor is obtained and characterised by the linking matrix

\[
\begin{bmatrix}
3 & 2 & 2 & 3 \\
2 & 2 & 2 & 3 \\
2 & 2 & 3 & 4 \\
3 & 3 & 4 & 4
\end{bmatrix}.
\]  

(4)

Also, the image attractor mixer defined by the linking matrix

\[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]  

(5)

is obtained by the classical method [16]. As shown in [10], the return map of the image attractor have to be built with a variable oriented from the inside to the outside to satisfy the conventions. This is not the case for the return map of the image system because \( |x| \) is the chosen variable to build it (figure 10 of [18]). Thus, the order of the strips is reversed to get the exact mixer \( A \) of the image attractor defined by

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.
\]  

(6)

Then, the concatenation of \( A \oplus A = B \), with \( B \) a mixer defined by

\[
B = \begin{bmatrix} s_1 & 1 & 1 \\ s_2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} u_2 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s_1} & 0 & 1 \\ \frac{1}{s_2} & 0 & 0 \end{bmatrix} + \begin{bmatrix} u_2 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 3 \\ 2 & 2 & 3 & 4 \end{bmatrix}
\]  

(7)

Figure 3 shows the different steps of the Theorem 1 applied to this example. This concatenation permits to obtain the mixer of the symmetrical attractor that can be confirmed by the linking numbers between pairs of orbits [6].

3.3 Symmetry

Finally we also present the algebraical relation between two symmetric templates [10]. First, when two templates \( T \) and \( T' \) are symmetric by reflection, the linking matrix \( T \) of \( T \) is the transposed linking matrix \( T' \) of \( T' \); thus, it is noted \( T' = T^p \).

**Definition 3** [10] Given two templates \( T \) and \( T' \) with respectively linking matrix \( T \) and \( T' \), if

\[
T' = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix} - T
\]  

(8)

then \( T' \) is symmetric to \( T \) by inversion. We choose to note it \( T' = T^p \).
As a consequence, using the previous relation, we can obtain the following relations between the linking matrix of the templates of the Malasoma attractor \( \mathbf{H} \)

\[
L_a = L_b = L_c = L_d.
\]  

(9)

4 Elementary mixer

The template of an attractor bounded by a genus one torus is an ordered series of torsions and mixers. The reduced template \( [14] \) of an attractor is the concatenation of the torsions and the mixers of a template; thus it is a mixer described by a linking matrix. Because it exists at the most as many reduced template as the number \( n \) of its components (torsions or mixers), for a given template, there are up to \( n \) possible reduced templates.

Proposition 1 Two templates bounded by genus one torus have the same chaotic mechanism if it exists an algebraical relation between one of their reduced templates.

Given \( \mathcal{M} \), a set of mixers with the same chaotic mechanism. \( \mathcal{M}_e \in \mathcal{M} \) is an elementary mixer and its linking matrix \( \mathcal{M}_e \) satisfies the following conditions:

- \( \| \mathcal{M}_e \|_{\infty} = \min_{N \in \mathcal{M}} \| N \|_{\infty} \);
- \( \text{Tr}(\mathcal{M}_e) \geq 0 \);
- \( M_{n,n} - M_{1,1} \geq 0 \);

For specific cases, two matrix can satisfy these criteria: this is the case when two mixers are symmetric by reflection. Complementary to this third criterion, we thus perform a line by line (from 1 to \( n \)) comparison; the elementary mixer is the mixer whose the first to have a line where the sum of the absolute values is lower than the corresponding line of the other mixer. For instance, between
\[ M_1 = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} \tag{10} \]

the third criteria with complementary computation indicates that the elementary mixer is defined by the linking matrix \( M_1 \).

**Proposition 2** A chaotic mechanism is described by a unique elementary mixer.

**Proof** The use of the infinite norm (first criterion) permits to obtain one mixer with the values closest to zero as elements of the linking matrix. Thus it removes all the matrix symmetric by inversion \( \mathcal{S} \) and all the matrix with extra torsions. The second criterion using the trace eliminates the equivalent chaotic mechanism with the same infinite norm but with only another orientation (transposed matrix). The purpose of the third criterion is to remove two transposed linking matrix (same infinite norm and equivalent orientation). The simplest way is to compute the difference between the first and last diagonal elements that sometimes require the extension using the comparison line by line in addition, see \( \mathcal{T} \). Thus, only one mixer remains to describe a chaotic mechanism. ■

### 5 Tables of chaotic mechanisms

#### 5.1 Algorithm

In order to generate a large amount of chaotic mechanisms we propose an iterative process using concatenation. We remind that concatenating two mixers made of \( n \) and \( m \) strips gives a chaotic mechanism made of \( m \times n \) strips. The algorithm we used starts with a list of \( l \) elementary mixers with \( n \) strips. We concatenate all possible couples of this list to obtain a set of mixers made of \( n^2 \) strips. In this list we extract all potential mixers made of \( n + 1 \) strips and compute their elementary mixer. It gives \( l' \), a set of elementary mixers made of \( n + 1 \) strips. Contrary to the work of \([11]\), we do not try to add potential branches but generate more complex chaotic mechanism and take all small parts inside.

#### 5.2 Chaotic mechanisms

The non trivial part of the topological characterization method is to provide a template (and its linking matrix) in order to compare if the linking numbers it generates corresponds to the linking numbers numerically obtained (see \([24]\) for this methodology applied on eight attractors). The construction of a template is generally made strip by strip, starting to find the relative organisation of two adjacent strips and adding a new strip to the left or to the right. As a consequence, we organise the chaotic mechanism made of \( n \) strips depending on the chaotic mechanisms made of \( n - 1 \) strips. This permits to easily link and build templates in regards of the two chaotic mechanisms made of \( n - 1 \) strips they contains. For genus one bounded attractor, there is only one chaotic mechanism made of two strips, it is defined by the elementary mixer

\[ a = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{11} \]

As it exists only one chaotic mechanism made of two branches, its “sons” (mechanisms generated from this mixer) contains this mechanism on the left and on the right. If two mechanisms have the same “parents” at the same place, they are ordered with respect to the ordered criteria (proposition 1). In the following tables \([2]\) and \([3]\) the chaotic mechanism associated to the left strips is indicated in the line and the mechanism associated to the right is in the column. This gives the left and right parents of son with \( n + 1 \) strips.
Table 1: Table B3 containing the chaotic mechanisms made of three strips.

| B3 | a= | 0 0 |
|----|----|-----|
| a= | 0 0 |
| 0 1 |

| c= | 0 0 0 |
|----|-----|
| 0 1 0 |
| 0 0 0 |

| d= | -1 -1 0 |
|----|-----|
| -1 0 0 |
| 0 0 1 |

Table 2: Table E4 containing the chaotic mechanisms made of four strips.

| E4 | c= | 0 0 0 |
|----|----|-----|
| 0 1 0 |
| 0 0 0 |

| f= | 0 0 0 |
|----|-----|
| 0 1 0 |
| 0 0 0 |

| g= | 0 -1 -1 0 |
|----|-----|
| -1 -1 -1 0 |
| -1 -1 0 0 |
| 0 0 0 1 |

| h= | -1 -1 0 0 |
|----|-----|
| -1 0 0 0 |
| 0 0 1 0 |
| 0 0 0 0 |

| i= | -1 -1 0 0 |
|----|-----|
| -1 0 0 0 |
| 0 0 1 1 |
| 0 0 1 2 |

| j= | -1 -1 -1 -1 |
|----|-----|
| -1 0 0 0 |
| -1 0 1 1 |
| -1 0 1 2 |

Table 3: Table K5 containing the chaotic mechanisms made of five strips.

| K5 | f= | 0 0 0 0 |
|----|----|-----|
| 0 1 0 0 |
| 0 0 0 0 |
| 0 0 0 0 |

| t= | 0 0 0 0 |
|----|-----|
| 0 1 0 0 |
| 0 0 1 0 |
| 0 0 0 0 |

| m= | -1 -1 -1 0 |
|----|-----|
| -1 0 0 0 |
| -1 0 1 0 |
| -1 0 0 0 |

| r= | 0 0 0 0 |
|----|-----|
| 0 0 0 0 |
| 0 0 1 0 |
| 0 0 0 0 |

| s= | -1 -1 0 0 |
|----|-----|
| -1 0 0 0 |
| -1 0 1 0 |
| -1 0 0 0 |

| v= | -1 -1 -1 -1 |
|----|-----|
| -1 0 0 0 |
| -1 0 1 1 |
| -1 0 1 2 |

| n= | 0 0 0 0 |
|----|-----|
| 0 0 0 0 |
| 0 1 1 0 |
| 0 0 1 1 |

| w= | -1 -1 -1 -1 |
|----|-----|
| -1 0 0 0 |
| -1 0 1 1 |
| -1 0 1 2 |

| z= | -1 -1 -1 -1 |
|----|-----|
| -1 0 0 0 |
| -1 0 1 1 |
| -1 0 1 2 |

| y= | -1 -1 -1 -1 |
|----|-----|
| -1 0 0 0 |
| -1 0 1 1 |
| -1 0 1 2 |
With these tables, if one is trying to build a template, he only needs to inverse or add torsions to these chaotic mechanisms in order to obtain the chaotic mechanism that fits its linking numbers. Compared to the work [11], the mechanism $2A$ and $2C$ have the same chaotic mechanism $d$ and $2B$ and $2D$ have the same chaotic mechanism $c$.

### 5.3 Comparison

In the example illustrating the concatenation, we found that the template of the Burke-Shaw attractor is described with the linking matrix $B$ (7). The elementary mixer $B_e$ associated to this link matrix is $g$ because

$$B_e = B \oplus | -3| = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = g .$$

We also know that the $g = L_b$ [1]. As a consequence, the elementary mixer of the Burke-Shaw is the same as the Malasoma one. The Burke-Shaw attractor and the Malasoma attractor have the same chaotic mechanism.

### 6 Conclusion

As the obtention of a template for an attractor is depending on the Poincaré section choice [14], we provide a new framework to compare attractor’s templates. In this article, we defined the elementary mixer describing the chaotic mechanism of a template bounded by a genus one torus. The algebraic relation between linking matrix are used to make links between linking matrix describing the same chaotic mechanism. We also provide all possible chaotic mechanisms with their elementary mixer for templates made of two to five strips. These results permits first to compare templates. We thus demonstrate that the Burke-Shaw attractor and the Malasoma attractor have the same chaotic mechanism. The main asset of this classification is to reduce the number of attractors to use if we want to compare the impact of their dynamical properties when they are included in other systems. For instance in optimization systems [25], the chaotic maps can be replaced by first return maps of chaotic attractors with specific chaotic mechanism (see [26] for an application to an ant colony algorithm).

The tables 1, 2 and 3 containing all possible chaotic mechanisms can also be used to reduce the number of articles in literature claiming that a new chaotic attractor has been found. Using this classification, an attractor is a new one only if there is no previous attractor with the same chaotic mechanism. In future works, we plan to use this classification to list the chaotic mechanisms that can be produced by a differential equations system further to our work on the bifurcation diagram of the Rössler system [24]. Also, this classification is a first step before adressing more complex attractors made of toroidal chaos, eg. Li attractor [27].

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