Bound on Hardy’s non-locality from the principle of Information Causality

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Recently, the principle of non-violation of Information Causality [Nature 461, 1101 (2009)], has been proposed as one of the foundational properties of nature. We explore the Hardy’s non-locality theorem for two qubit systems, in the context of generalized probability theory, restricted by the principle of non-violation of Information Causality. Applying, a sufficient condition for Information causality violation, we derive an upper bound on the maximum success probability of Hardy’s nonlocality argument. We find that the bound achieved here is higher than that allowed by quantum mechanics, but still much less than what the no-signalling condition permits. We also study the Cabello type non-locality argument (a generalization of Hardy’s argument) in this context.

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Introduction

Understanding quantum non-locality is a fundamental problem facing science [1]. Quantum non-locality is profoundly manifested via the correlations between the measured values of physical quantities pertaining to different (spatially separated) parts of the quantum system, which cannot be generated locally. That is, quantum correlations violate Bell’s inequality [2], which any local hidden variable theory has to satisfy. Despite this, quantum non-locality cannot be used for signalling between distant quantum systems [3].

In order to quantify and explain the effects of quantum non-locality, we have to relate it to some fundamental principle(s). These principle(s) must ensue from violation of Bell type inequalities (a basic manifestation of quantum non-locality), and the no-signalling principle. One such approach, initiated by Popescu and Rohrlich [4], is to find the class of models following exclusively from non-locality and no-signalling requirements, and then ask which of them coincides with quantum mechanics. They showed that it is possible to write down sets of “superquantum” correlations that are more nonlocal than that allowed by quantum mechanics, yet satisfying nonsignalling. They invented a hypothetical device now known as PR box, which generates a set of correlations that return a value of $\sqrt{2}$ for the Clauser-Horne-Shimony-Holt (CHSH) expression [4], the maximum algebraic value possible, yet are nonsignalling. The maximum quantum value is given by Tsirelson’s theorem [5] as $2\sqrt{2}$. Thus only the non-signalling requirement does not give a non-local theory which uniquely determines quantum mechanics. Several important works [2,3] thereafter, show that numerous properties which were originally thought to be typically quantum, are also shared by general nonsignalling models. Although, W. van Dam [14] showed that the strongest nonsignalling correlations (viz. PR boxes) make communication complexity trivial, which seems highly implausible in Nature. An intriguing question to ask is, what are characterizing features of quantum mechanics (Nature), which makes it unique.

Very recently, Pawłowski et al [15] give a bold new proposal for understanding the quantum mechanical bound on non-local correlations. They showed that if non-violation of Information Causality (IC) is accepted as a foundational principle of nature, then it follows that, a class of non-local correlations which are postquantum violate this principle, while Quantum and Classical correlations respect it. The principle states that communication of $m$ classical bits causes information gain of at most $m$ bits, this is a generalization of the no-signalling principle, the case $m = 0$ corresponds to no-signalling. On applying IC principle to non-local correlations, Tsirelson’s bound [5] naturally emerges, all correlations exceeding Tsirelson’s bound violate the principle of information causality. However, there are also nonquantum correlations which are bounded by $2\sqrt{2}$ (Tsirelson’s bound), so it remains interesting to see whether various quantum correlations follows from IC condition or not. Alcock et al [16] showed that a part of the boundary between quantum and nonquantum correlations can indeed be recovered from the principle of Information Causality. Studying quantum nonlocality from different perspectives (like non-locality without inequalities), in the context of IC may lead to a deeper understanding of the issues involved.

Hardy [17,18] gave a non-locality theorem (in contrast to Bell’s inequality) which provides a manifestation of quantum non-locality, without using statistical inequalities involving expectation values. Later, Cabello [19] has introduced a logical structure to prove Bell’s theorem without inequality for three particles GHZ and W states and the argument is also applicable for two qubits [20,21]. Hardy’s logical structure is a special case of Cabello’s structure. The maximum probability of success of Hardy’s non-locality argument for two qubit
systems is 0.09 whereas in case of Cabello’s argument it is 0.1078 \cite{20}. Maximum success probability of Hardy’s non-locality argument for the class of generalized no signalling theories in two qubit systems is 0.5 \cite{22,23}.

Barrett et al \cite{24} with a aim to shed light on the range of quantum possibilities by placing them in a wider context (generalized probability theory), investigated the set of correlations that are constrained only by the no-signaling principle. They found that these correlations form a polytope, which contains the quantum correlations as a (proper) subset. Their work sets an elegant mathematical framework for understanding the general structure of these nonlocal correlations.

In the present paper we study Hardy’s non-locality arguments for two qubit systems in the context of non-violation of information causality. We adopt the characterization of no signalling boxes and notations used in Barrett et al \cite{24}. We have found that unlike CHSH inequality, Hardy’s bound on the success of non-local probabilities cannot be recovered from the condition \cite{15} which follows from the principle of non-violation of information causality.

**Hardy’s/Cabello’s argument for two qubits**

Consider two spin-1/2 particles 1 and 2 with spin observable \(A\), \(A’\) on particle 1 and \(B\), \(B’\) on particle 2. These observable gives the eigenvalues \(\pm 1\). Now consider the following joint probabilities:

\[
\begin{align*}
P(A = +1, B = +1) &= q_1, \\
P(A' = -1, B = -1) &= 0, \\
P(A = -1, B' = -1) &= 0, \\
P(A' = -1, B' = -1) &= q_4.
\end{align*}
\]

Here equation (1) tells that, if \(A\) is measured on particle 1 and \(B\) is measured on particle 2, then the probability that both get value \(+1\) is \(q_1\), remaining equations can also be interpreted in a similar fashion. These equations form the basis of Cabello’s nonlocality argument. It can easily be seen that these equations contradict local-realism if \(q_1 < q_4\). To show this, let us consider those hidden variable states \(\lambda\) for which \(A' = -1\) and \(B' = -1\). For these states, equations (2) and (3) tell that the values of \(A\) and \(B\) must be equal to \(+1\). Thus according to local realism \(P(A = +1, B = +1)\) should be at least equal to \(q_4\). This contradicts equation (1) as \(q_1 < q_4\). It should be noted here that \(q_1 = 0\) reduces this argument to that of Hardy’s. So by Cabello’s argument, we specifically mean that the above argument runs, even with nonzero \(q_1\).

**Hardy’s correlations from no-signalling polytope**

Given a set of observables \(X,Y \in \{0,1\}\) and outcomes \(a,b \in \{0,1\}\) joint probabilities \(p_{ab|XY}\) thus form an entire correlation table with \(2^4\) entries, which can be regarded as a point of \(2^5\)-dimensional vector space. The positivity, normalization and non-signalling constraints lead the entire correlation table to a convex subset in the form of a polytope which is known as no-signalling polytope \(P\), which is eight dimensional \cite{24}. There are 24 vertices of the polytope \(P\), 16 of which represent local correlations (called “local vertices”) and 8 represent nonlocal correlations. The local vertices can be expressed as

\[
P_{ab|XY} = \begin{cases} 1, & \text{if } a = \alpha X \oplus \beta, \\ 0, & \text{otherwise} \end{cases} \quad (5)
\]

where \(\alpha, \beta, \gamma, \delta \in \{0,1\}\) and \(\oplus\) denotes addition modulo 2. The eight nonlocal vertices have the form:

\[
P_{ab|XY} = \begin{cases} \frac{1}{2^n} & \text{if } a + b = \gamma Y \oplus \beta X \oplus \gamma, \\ 0, & \text{otherwise} \end{cases} \quad (6)
\]

where \(\alpha, \beta, \gamma \in \{0,1\}\).

Now we use the correspondence \((X = 0) \leftrightarrow A, (X = 1) \leftrightarrow A', (Y = 0) \leftrightarrow B, (Y = 1) \leftrightarrow B'\) and \(a,b = 0(1) \leftrightarrow +1(-1)\). Then it is straightforward to check that five of the 16 local vertices and one of the 8 nonlocal vertices satisfy Hardy’s equations (1)-(4) (when \(q_1 = 0\), namely those given by \(p_{ab|XY} = p_{0001}^H, p_{0011}^H, p_{1000}^H, p_{1010}^H, p_{1100}^H, p_{1111}^H\) and \(p_{0011}^H\)). The other vertices can be covered by another set of Hardy’s equations. Then the joint probabilities satisfying Hardy’s conditions can be written as a convex combination of the above 6 vertices (five local vertices and one nonlocal vertex). Then

\[
P_{ab|XY} = c_1 p_{ab|XY} + c_2 p_{ab|XY} + c_3 p_{ab|XY} + c_4 p_{ab|XY} + c_5 p_{ab|XY} + c_6 p_{ab|XY} \quad (7)
\]

where \(\sum_{j=1}^{6} c_i = 1\).

Now if we consider \(q_1 \neq 0\) (but \(q_1 < q_4\)), then the equations (1) - (4) is known as Cabello’s nonlocality conditions, which can be written as a convex combination of the above 6 vertices which satisfies Hardy’s conditions along with another four local vertices \(p_{0000}^C, p_{0010}^C, p_{1000}^C, p_{1010}^C\) and one nonlocal vertex \(p_{1111}^C\). So we get,

\[
P_{ab|XY} = c_1 p_{ab|XY} + c_7 p_{ab|XY} + c_8 p_{ab|XY} + c_9 p_{ab|XY} + c_{10} p_{ab|XY} + c_{11} p_{ab|XY} \quad (8)
\]

where the expression \(p_{ab|XY}^C\) is given in equation (7) and coefficients \(c_i\)’s satisfy the condition \(\sum_{j=1}^{11} c_i = 1\).

One can check from equation (7) that the success probability for Hardy’s argument is given by \(p_{1111}^H = \frac{1}{2} c_6\). From
here, one can obviously see that under the no-signaling constraint, the maximum success probability of Hardy’s argument \( p_{11|11}^{H} \) is achieved for \( c_6 = 1 \) and \( c_1 = c_2 = c_3 = c_4 = c_5 = 0 \). Similarly the success probability for Cabello’s argument follows from equation (8) and can be written as, \( p_{11|11}^{C} - p_{00|00}^{C} = \frac{1}{2}c_6 + c_{10} - C \), where \( C = c_7 + c_8 + c_9 + c_{10} + \frac{1}{2}c_{11} \), and, here too we obtain that \( p_{11|11}^{C} - p_{00|00}^{C} \) for \( c_6 = 1 \) and rest of the \( c_i \)'s = 0. This maximum success probability of Hardy’s/Cabello’s argument, restricted by the no-signalling condition, has also been derived in [22, 23]. One should note that the probability set for which this maximum is achieved coincides with PR correlation for both the cases. In the following sections we will derive an upper bound on the maximum value of these success probabilities from the principle of non-violation of information causality.

**Information Causality**

Let us now briefly review the principle of information causality (IC). Alice and Bob, who are separated in space, have access to non-signalling resources such as shared randomness, entanglement or (in principle) PR boxes. Alice receives a random variable \( C = c_1, a_2, ..., a_N \) while Bob receives a random variable \( b \in \{1, 2, ..., N\} \). Alice then sends \( m \) classical bits to Bob, who must output a single bit \( \beta \) with the aim of guessing the value of Alice’s \( b \)-th bit \( a_b \). Their degree of success at this task is measured by

\[
I = \sum_{K=1}^{N} I(a_K : \beta | b = K),
\]

where \( I(a_K : \beta | b = K) \) is Shannon mutual information between \( a_K \) and \( \beta \). The principle of information causality states that physically allowed theories must have

\[
I \leq m. \tag{9}
\]

It was proved in [15] that both classical and quantum correlations satisfy this condition. A condition under which IC is violated was derived in [15], based on a construction by Van Dam and Wolf and Wullschleger- for a specific realization of the Alice-Bob channel. It goes as follows. Define \( P_1 \) and \( P_2 \):

\[
P_1 = \frac{1}{2} \left[ p(a=b|00) + p(a=b|10) \right]
= \frac{1}{2} \left[ p_{00|00} + p_{11|00} + p_{00|10} + p_{11|10} \right]

P_2 = \frac{1}{2} \left[ p(a=b|01) + p(a\neq b|11) \right]
= \frac{1}{2} \left[ p_{00|01} + p_{11|01} + p_{01|11} + p_{10|11} \right] \tag{10}
\]

Then, the IC condition (9) is violated for all boxes for which

\[
E_1^2 + E_2^2 > 1, \tag{11}
\]

where \( E_j = 2P_j - 1 \) \((j = 1, 2)\). Here it is important to note that the condition (11) is only a sufficient condition (based on the protocol given in [15]) for violating the IC principle.

**Hardy’s nonlocality and Information Causality**

In this section we derive an upper bound on the maximum probability of success of Hardy’s non-locality argument for a two qubit system in the context of non-violation of information causality. Let Alice and Bob share non-signalling nonlocal correlation satisfying Hardy’s condition i.e. the joint probability \( P_{ab|XY}^H \) given in equation (7). Then for this nonlocal correlation we have

\[
P_1 = \frac{1}{2}(c_5 + c_4),
P_2 = \frac{1}{2}(c_1 + c_2 + c_3) \tag{12}
\]

To satisfy the IC condition equation (12) has to satisfy the condition

\[
E_1^2 + E_2^2 \leq 1
\]

i.e

\[
(c_5 + c_4 - 1)^2 + (c_1 + c_2 + c_3 - 1)^2 \leq 1,
\]

which implies

\[
c_5^2 + 2c_4c_5 + 2c_4c_5c_6 + 2c_4c_5c_6c_5 - 1 \leq 0 \tag{13}
\]

The above equation gives the maximum value of \( c_6 = \sqrt{2} - 1 \). Then an upper bound on the maximum probability of success of Hardy’s non-locality is given by \( P_{11|11}^{H} = \frac{1}{2}c_6 \leq \frac{1}{4}(\sqrt{2} - 1) = 0.20717 \).

**Cabello’s nonlocality and Information Causality**

Now we try to find an upper bound on the maximum probability of success in Cabello’s case in the context of non-violation of information causality. Let Alice and Bob share non-signalling nonlocal correlation satisfying Cabello’s condition i.e joint probability \( P_{ab|XY}^C \) given in equation (8). Then for this nonlocal correlation we have:

\[
P_1 = \frac{1}{2}[C + c_5 + \frac{c_{11}}{2} + c_4 + c_7 + c_8] \tag{14}
\]

\[
P_2 = \frac{1}{2}[1 + c_9 - (c_4 + c_5 + c_6 + c_{10})]
\]

where \( C = c_7 + c_8 + c_9 + c_{10} + \frac{1}{2}c_{11} \). Then

\[
E_1 = c_7 + c_8 - c_1 - c_2 - c_3 - c_6
E_2 = c_9 - c_4 - c_5 - c_6 - c_{10} \tag{15}
\]

To satisfy the IC condition equation (15) has to satisfy the condition

\[
E_1^2 + E_2^2 \leq 1.
\]
One can easily check that
\[ E_1 + E_2 = -(1 + 2x) \]
where \( x = (c_10 + \frac{1}{2}c_6) - C \). It follows that
\[ E_1^2 + E_2^2 = 4x^2 + 4(1 + E_2)x + 2(1 + E_2)E_2 + 1, \]
so in order to satisfy the IC condition,
\[ x^2 + (1 + E_2)x + \frac{1}{2}E_2(1 + E_2) \leq 0 \]
writing \( E_2 \) in terms of \( P_2 \) we obtain,
\[ x^2 + 2P_2x + P_2(2P_2 - 1) \leq 0. \]
Since we are to find \( x_{max} \) it is sufficient to consider only the equality. Then,
\[ x = -P_2 + \sqrt{P_2(1 - P_2)}; \]
\[ 0 \leq P_2 \leq \frac{1}{2}. \]
The maximum value of \( x \) we obtained from here is,
\[ x_{max} = \frac{1}{2}(\sqrt{2} - 1) = 0.20717. \]
This value is same as in the Hardy’s case. We conclude that on applying the IC condition, maximum probability of success of the Cabello’s argument is same as that of the Hardy’s argument, both achieving the same numerical value 0.20717.

**Conclusion**

The maximum probability of success of the Hardy’s and Cabello’s non-locality (for the two qubits system) in Quantum mechanics is 0.09 and 0.1078 respectively [20]. Interestingly, for generalized nonlocal no-signalling theories we find that this bound is 0.5 in both the cases and the probability set for which this is achieved coincides with the PR correlation. We showed that on applying the principle of Information causality this bound decreases from 0.5 to 0.20717 in both the cases, but could not reach their respective Quantum mechanical bounds. Interestingly, in quantum mechanics the maximum probability of success for the Cabello’s case is not same as the Hardy’s case [20]. Since the condition given by equation (11) of the present paper is a sufficient condition derived in [17] for the violating IC, the probability derived here are therefore, strictly an upper bound on the maximum probabilities allowed in an IC-respecting no-signaling theory. Restricting the no-signalling probability set by the full power of IC principle may reduce the probability to the quantum limit. However, it is curious that the same sufficient condition for violating the IC, gives the quantum bound for the CHSH expression [15].

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