Mixed Convection Flow of Brinkman Type Hybrid Nanofluid Based on Atangana-Baleanu Fractional Model

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Abstract. The industrial and engineering consumption of nanofluids is increased day by day due to successful implementation. The improved thermophysical properties play a vital role in the efficiency of nanofluids in convection processes. But this technology is not stopped here and reached to the next level by introducing hybrid nanofluids. Hence, this article is dedicated to focus on the mixed convection flow hybrid nanofluid. The hybridized nanoparticles of copper and alumina are dissolved in water as a base fluid to form a suspension. The Atangana-Baleanu fractional model is considered for flow demonstration over a vertical plate. The fractional PDE’s of the model is subjected to physical initial and boundary conditions. It is assumed that the electrically conducting laminar incompressible flow is under the influence of a magnetic field of variable direction. The Laplace transform technique is implemented to develop exact solutions for the problem under consideration. To explore the behavior of flow parameters, the obtained solutions are numerically computed and displayed in various figures with a physical explanation.

It is found that the velocity and temperature profiles behave alike for fractional parameter $\alpha$. Both the profiles decrease with increasing values of $\alpha$. However, the trend of these profiles is opposite for volume concentration $\phi_{hf}$ of hybrid nanofluid. The velocity profile decreases with increasing values of $\phi_{hf}$ whereas, the temperature profile increases with increasing values of $\phi_{hf}$.

1. Introduction
Mixed convection flows appear in various industrial and engineering processes which included paper production, extraction of polymer and rubber, melt spinning and continuous casting and wire drawing and glass-fiber [1]. Mixed convection flows correspondingly plays a vital rule in a thermal manufacturing application such as in heated or cooled enclosures and electric power supply, ventilating and air conditioning systems in atmospheric and oceanic circulations, electronic machinery and thermal distribution in buildings [2, 3]. The idea of nanofluids is extensively investigated in the literature and succeeded in many developments. Nanofluids have categorized in various classes among which a very recent one is hybrid nanofluid [4]. The physical combination of two dissimilar nanoparticles dissolved in a base fluid form hybrid nanofluid. The main motive of preparing hybrid nanofluid is to further advance the thermal physical properties. Izadi et al. [5] studied the convective flow together with a variable magnetic field of electrically conducting hybrid nanofluid. Shalasavar et al. [6] investigated the entropy generation optimization and heat transfer flow of viscoelastic hybrid nanofluid. Farooq et al. [7] demonstrated the entropy generation of hybrid nanofluid flow over a radially non-Linear stretching sheet.
In the view of the above-discussed literature, the solutions for nanofluid models are obtained either using approximate or any numerical scheme. There are rare cases where the exact analytical solutions were obtained using integral transforms. Furthermore, the studies on fractional Brinkman type models with hybrid nanoparticles over an oscillating vertical plate and magnetic field with variable direction are not reported yet. To fill this gap, the magnetohydrodynamics (MHD) mixed flow of hybrid nanofluid over an oscillating vertical plate based on fractional Brinkman type fluid model is considered. Brinkman type fluid model was invented by Brinkman in his pioneer work while studying the fluid flow due to the exertion of viscous force, on a dense swarm particles surface. Using the Stokes’ formula, it was noticed that the viscous force acting on dense swarm particles is significantly greater than that of on a remote particle [8, 9]. The classical model of Brinkman type fluid is fractional here using the Atangana-Baleanu fractional operator in Caputo sense [10]. The Atangana-Baleanu fractional operator is based on non-singular and non-local kernel. The result obtained from fractional models based on Atangana-Baleanu fractional operators is additional realistic, reliable and more general. Jan et al. [11] investigated oil based MoS$_2$ nanofluid for Stock’s second problem using Atangana-Baleanu fractional operator and obtained exact solutions via the Laplace transform technique. Sheikh et al. [12, 13] carried out a comparative analysis using Atangana-Baleanu fractional operator taking into consideration the flow of Casson fluid.

Keeping in mind the importance of Atangana-Baleanu fractional operator and hybrid nanofluid, this article deals with MHD mixed convection flow of hybrid nanofluid. The fractional Brinkman type fluid model together with energy equation is utilized to demonstrate the flow phenomena over an oscillating infinite vertical plate. The uniform magnetic field with variable direction is assumed. The governing equations of the problem are transformed to dimensionless form using non-similarity variables. The exact analytical solutions are obtained using the Laplace transform technique.

2. Description of the Problem

Considered the unsteady MHD mixed convection flow of hybrid nanofluid based on fractionalized Brinkman type fluid model, past over an infinite oscillating vertical plate. The fluid and plate are assumed stationary at $t \leq 0$ with ambient temperature $T_\infty$. After a small instant of time ($t = 0^+$) the temperature of the plate rises from $T_\infty$ to $W_T$ and the plates start oscillation. At this moment of time, the mixed convection takes place due to temperature gradient which strengthens buoyancy forces and oscillation of the plate. Hence, the fluid starts motion in $x -$ direction. It is considered that the hybrid nanofluid is electrically conducting and a uniform magnetic field of the magnitude $B_0$ of variable direction is applied to the plate. The flow of electrically conducting fluid under the effect of applied magnetic field experience electromagnetic force and the induced magnetic field is neglected because of small Reynold number. Bearing in mind the above assumptions, the governing equations in the absence of external pressure gradient were derived by using Maxwell’s relations, Ohm’s law [14] and Boussinesq’s approximation as [11].

\[
\rho_{nf}\left(\frac{\partial u(y,t)}{\partial t} + \beta_s u(y,t)\right) = \mu_{nf}\frac{\partial^2 u(y,t)}{\partial y^2} - \sigma_{nf} B_0^2 \sin \gamma u(y,t) + g(\rho \beta_T)_{nf} (T(y,t) - T_\infty),
\]

\[
\left(\rho C_p\right)_{nf} \frac{\partial T(y,t)}{\partial t} = k_{nf} \frac{\partial^2 T(y,t)}{\partial y^2},
\]

subject to the following appropriate physical initial and boundary conditions

\[
\left\{\begin{array}{ll}
\rho_{nf}\left(\frac{\partial u(y,0)}{\partial t} + \beta_s u(y,0)\right) = \mu_{nf}\frac{\partial^2 u(y,0)}{\partial y^2}, & y \geq 0, \\
\rho_{nf}\left(\frac{\partial u(0,t)}{\partial t} + \beta_s u(0,t)\right) = \mu_{nf}\frac{\partial^2 u(0,t)}{\partial y^2}, & t > 0, \\
\rho_{nf}\left(\frac{\partial u(y,t)}{\partial t} + \beta_s u(y,t)\right) = \mu_{nf}\frac{\partial^2 u(y,t)}{\partial y^2}, & y \to \infty, t > 0, \\
\end{array}\right.
\]

subject to the following appropriate physical initial and boundary conditions

\[
\left\{\begin{array}{ll}
\rho_{nf}\left(\frac{\partial u(y,0)}{\partial t} + \beta_s u(y,0)\right) = \mu_{nf}\frac{\partial^2 u(y,0)}{\partial y^2}, & y \geq 0, \\
\rho_{nf}\left(\frac{\partial u(0,t)}{\partial t} + \beta_s u(0,t)\right) = \mu_{nf}\frac{\partial^2 u(0,t)}{\partial y^2}, & t > 0, \\
\rho_{nf}\left(\frac{\partial u(y,t)}{\partial t} + \beta_s u(y,t)\right) = \mu_{nf}\frac{\partial^2 u(y,t)}{\partial y^2}, & y \to \infty, t > 0, \\
\end{array}\right.
\]

where $\rho_{nf}$ is the density, $\mu_{nf}$ is the dynamic viscosity, $\sigma_{nf}$ is the electrical conductivity, $\left(C_p\right)_{nf}$ is the heat capacitance, $T(y,t)$ is the temperature and $k_{nf}$ is the thermal
conductivity of hybrid nanofluid. The mathematical expressions for improved thermophysical properties are presented in Table 1 [7, 15].

| Properties                                      | Mathematical Expression                                                                 |
|------------------------------------------------|----------------------------------------------------------------------------------------|
| Density                                         | \( \rho_{hf} = (1 - \phi_{hf}) \rho_f + \phi_{Cu} \rho_{Cu} + \phi_{Al_2O_3} \rho_{Al_2O_3} \),   |
| Dynamic viscosity                               | \( \mu_{hf} = \frac{\mu_f}{(1 - (\phi_{Cu} + \phi_{Al_2O_3}))^{2.5}} \)                |
| Thermal expansion                               | \( (\rho \beta_f)_{hf} = (1 - \phi_{hf})(\rho \beta_f)_f + \phi_{Cu} (\rho \beta_{Cu}) + \phi_{Al_2O_3} (\rho \beta_{Al_2O_3}) \) |
| Heat capacitance                                | \( (\rho C_p)_{hf} = (1 - \phi_{hf})(\rho C_p)_f + \phi_{Cu} (\rho C_{Cu}) + \phi_{Al_2O_3} (\rho C_{Al_2O_3}) \) |
| Electrical conductivity                         | \( \sigma_{hf} = 1 + \left\{ \frac{\phi_{Cu} \sigma_{Cu} + \phi_{Al_2O_3} \sigma_{Al_2O_3}}{\phi_{hf}} + 2 - \left( \frac{\phi_{Cu} \sigma_{Cu} + \phi_{Al_2O_3} \sigma_{Al_2O_3}}{\sigma_f} - \phi_{hf} \right) \right\} \) |
| Thermal conductivity                            | \( k_{hf} = \frac{\phi_{Cu} k_{Cu} + \phi_{Al_2O_3} k_{Al_2O_3} + 2 k_f + 2 \left( \phi_{Cu} k_{Cu} + \phi_{Al_2O_3} k_{Al_2O_3} \right)}{\phi_{hf}} - k_f \phi_{hf} \) |

It is worth mentioning here that the subscripts \( hf \), \( f \), \( Cu \) and \( Al_2O_3 \) are respectively used for hybrid nanofluid, base fluid, copper nanoparticles, and alumina nanoparticles. \( \phi_{hf} \) is the volume concentration of nanoparticles such that \( \phi_{hf} = \phi_{Cu} + \phi_{Al_2O_3} \). In this research, the hybridized nanoparticles \( Cu \) and \( Al_2O_3 \) are dissolved in water (\( H_2O \)) as a base fluid to form hybrid nanofluid. The numerical values of the base fluid and nanoparticles are given in Table 2 [7, 15].

| Material                                      | Base fluid   | Nanoparticles |
|-----------------------------------------------|--------------|---------------|
|                                               | H_2O         | Al_2O_3       | Cu            |
| \( \rho (kg/m^3) \)                           | 997.1        | 3970          | 8933          |
| \( C_p (J/kg K) \)                            | 4179         | 765           | 385           |
| \( K (W/mK) \)                                | 0.613        | 40            | 400           |
| \( \beta_f \times 10^{-5} (K^{-1}) \)        | 21           | 0.85          | 1.67          |
| \( \sigma \)                                  | 0.05         | \( 1 \times 10^{-10} \) | \( 1 \times 10^{-7} \) |

To reduce the number of variables and get rid of units, the following non-similarity variables

\[
\nu = \frac{\mu}{\rho_f}, \quad \xi = \frac{U_0}{V_f}, \quad \tau = \frac{U_0^2}{V_f}, \quad \theta = \frac{T - T_w}{T_w - T_c}
\]

are introduce into equations (1)-(4) which yield to the following dimensionless system

\[
a_{41} \left( \frac{\partial v(\xi, \tau)}{\partial \tau} + \beta_1 v(\xi, \tau) \right) = a_{41} \frac{\partial^2 v(\xi, \tau)}{\partial \xi^2} - a_{22} M \sin \gamma v(\xi, \tau) + a_{43} Gr \theta(\xi, \tau)
\]

(5)

\[
a_{44} Pr \frac{\partial \theta(\xi, \tau)}{\partial \tau} = \lambda_{hf} \frac{\partial^2 \theta(\xi, \tau)}{\partial \xi^2},
\]

(6)

\[
v(\xi, 0) = 0, \theta(\xi, 0) = 0, \forall \xi \geq 0,
\]

(7)
\[ v(0, r) = H(\tau) \cos \omega t, \quad \theta(0, r) = 1, \text{for } r > 0 \]  
\[ v(\xi, r) \rightarrow 0, \theta(\xi, r) \rightarrow 0; \quad \xi \rightarrow \infty, \text{for } r > 0 \]  
\[ (8) \]

where

\[ \beta_b = \frac{v_f \beta_e^*}{U_0 - \frac{2}{k}}, \quad M = \frac{v_f \sigma B_0^2}{\mu U_0 - \frac{2}{k}}, \quad Gr = \frac{U_0^3 g (\beta_e - T_e)}{T_w - T_e}, \quad Pr = \left( \frac{\mu C_p}{k} \right) , \]

\[ \lambda_{\text{inf}} = \frac{k_{\text{inf}}}{k_f}, \quad a_{10} = (1 - \phi_{\text{inf}}) + \frac{\phi_{\text{inf}} \rho_0 c_p}{\rho_f}, \quad a_{11} = \frac{1}{\left( 1 - (\phi_{\text{inf}} + \phi_c) \right)^2}, \quad \sigma_f = \frac{\sigma_{\text{inf}}}{\sigma_f} , \]

\[ a_{31} = (1 - \phi_{\text{inf}}) + \frac{\phi_{\text{inf}} \rho_0 c_p}{\rho_f}, \quad a_{32} = (1 - \phi_{\text{inf}}) + \frac{\phi_{\text{inf}} \rho_0 c_p}{\rho_f} , \]

is the Brinkman type fluid parameter, magnetic parameter, thermal Grashof number, and Prandtl number respectively. \( \lambda_{\text{inf}}, a_{10}, a_{11}, a_{31}, a_{32}, a_{33} \) are constant terms. The dimensionless governing Equations (5) and (6) are artificially transform into time fractional form in terms of Atangana-Baleanu fractional derivatives in Caputo sense as [11].

\[ \eta_l \left( ^{\alpha} D^{\alpha} v(\xi, r) + \beta_b v(\xi, r) \right) = a_{10} - \frac{\partial^2 v(\xi, r)}{\partial \xi^2} - a_{22} M \sin v(\xi, r) + a_{31} Gr \theta(\xi, r), \]

\[ (9) \]

\[ a_{31} \left( ^{\alpha} D^{\alpha} \theta(\xi, r) \right) = \lambda_{\text{inf}} \frac{\partial^2 \theta(\xi, r)}{\partial \xi^2} , \]

\[ (10) \]

where \( ^{\alpha} D^{\alpha} f(\eta) \) is Atangana-Baleanu fractional operator in Caputo sense defined by [10]

\[ D_{\alpha}^\mu f(\eta, r) = \frac{N(\alpha)}{1 - \alpha} \left[ E_{\alpha} \left[ -\frac{\alpha (\tau - t)^\alpha}{1 - \alpha} \right] f(\eta, r) dt \right], \]

\[ (11) \]

where \( N(\alpha) \) is the normalization function such that \( N(1) = N(0) = 1 \) and \( E_{\alpha}(\cdot) \) is the Mittag-Leffler function which is given by

\[ E_{\alpha} \left( -t^\alpha \right) = \sum_{k=0}^{\infty} \left( -t \right)^{\alpha k} / \Gamma(\alpha K + 1). \]

The Laplace transform of equation (11) is obtained as

\[ L \left\{ f(\eta, r); q \right\} = \frac{q^\alpha Lf(\xi, r) - f(\eta, 0)}{(1 - \alpha) q^\alpha + \alpha} \]

\[ (13) \]

where \( q \) is the time variable in the Laplace transform domain, \( \alpha \) is the fractional order and \( f(\eta, 0) \) is the initial value of the function. The normalization function \( N(1) = 1 \) is chosen.

3. Solution of the Problem

In this section, the Laplace transform technique is used as a tool of the solution to the proposed problem of hybrid nanofluid. The exact analytical solutions are obtained here for velocity and temperature profiles.

3.1. Solutions of the energy equation

Applying the Laplace transform to equation (10) and using the corresponding initial and boundary conditions from equations (7) and (8) yield to

\[ \frac{a_{31} Pr b_0 q^\alpha}{\lambda_{\text{inf}}} \mathcal{L}^\alpha \left( \frac{\partial^2 v(\xi, q)}{\partial \xi^2} \right) = \frac{d^2 \mathcal{L}^\alpha (\xi, q)}{d \xi^2}. \]

\[ (14) \]
\[ \bar{\sigma}(\xi, q) = \frac{1}{q} \exp \left\{ -\xi \left( \frac{h_q q''}{q'' + h_{\xi}} \right)^\frac{1}{2} \right\}, \]  

(15)

where

\[ h_0 = \frac{a_{h_s} \text{Pr} h_{\xi}}{\lambda_{mf}}. \]

To obtain the inverse Laplace transform, equation (15) can be written in a more simplified and convenient form as

\[ \bar{\sigma}(\xi, q) = \bar{\Theta}(q) \bar{\Phi}(\xi, q; h_\alpha, h_\beta), \]

(16)

where

\[ \bar{\Phi}(\xi, q; h_\alpha, h_\beta) = \frac{1}{q} \exp \left\{ -\xi \left( \frac{h_q q''}{q'' + h_{\xi}} \right)^\frac{1}{2} \right\}, \]

(17)

and

\[ \bar{\Theta}(q) = \frac{1}{q^{1 - \alpha}}. \]

(18)

Taking the inverse Laplace transform equation (16) yield to

\[ \theta(\xi, \tau) = \Theta(\tau) * \Phi(\xi, \tau; h_\alpha, h_\beta), \]

(19)

where

\[ \Phi(\xi, \tau; h_\alpha, h_\beta) = L^{-1} \left\{ \bar{\Phi}(\xi, q; h_\alpha, h_\beta) \right\} = \frac{1}{\pi} \int_0^\infty \Phi_1(\xi, \tau; h_\alpha, h_\beta)(u\tau\sin \alpha \pi) \exp(-\tau r - u\tau \cos \alpha \pi) dr du \]

(20)

\[ \Phi_1(\xi, \tau; h_\alpha, h_\beta) = L^{-1} \left\{ \frac{1}{q} \exp \left\{ -\xi \left( \frac{h_q q''}{q'' + h_{\xi}} \right)^\frac{1}{2} \right\} \right\} = 1 - \frac{2h_\alpha}{\pi} \int_0^\infty \frac{\sin(\xi s)}{s(h_\alpha + s^2)} \exp\left( -\frac{h_{\xi} \tau s^2}{h_\alpha + s^2} \right) ds, \]

(21)

and

\[ \Theta(\tau) = \frac{1}{\tau^\alpha \Gamma(1 - \alpha)}, \]

(22)

It is worth mentioning here that the solutions obtained here satisfy all the imposed conditions of the problem and * represents the convolution product.

3.2. Solution of momentum equation

Applying the Laplace transform to equation (9) and using the corresponding initial and boundary conditions from equations (7) and (8) yield to

\[ \frac{d^2 \bar{\nabla}(\xi, q)}{d\xi^2} = \left( \frac{h_q q'' + h_\alpha}{q'' + h_{\xi}} \right) \bar{\nabla}(\xi, q) = -h_{\slip} \bar{\sigma}(\xi, q) \]

(23)

where

\[ h_\alpha = \frac{a_{h_s} \text{Gr}}{a_{\slip}}, \quad h_\gamma = \frac{a_{h_s} \tan \gamma}{a_{\slip}} + h_\beta, \]

and

\[ h_\alpha = \frac{a_{h_s} \text{Gr}}{a_{\slip}} + h_\beta, \]

By solving equation (16) analytically leads to
\[ \bar{v}(\xi, q) = \left[ \frac{q}{q^2 + \omega^2} + h_1 \left( \frac{q'' + h_1}{h_1 q'' + h_1} \right) \right] \frac{1}{q} \exp \left( -\xi \left( \frac{h_1 q'' + h_1}{q'' + h_1} \right)^{\frac{1}{2}} \right) \]

\[ -h_1 \left( \frac{q'' + h_1}{h_1 q'' + h_1} \right) \frac{1}{q} \exp \left( -\xi \left( \frac{h_1 q''}{q'' + h_1} \right)^{\frac{1}{2}} \right) \]

(24)

where

\[ h_1 = h_0 + h_2. \]

After further simplification equation (24) yield to

\[ \bar{v}(\xi, q) = \frac{q}{q^2 + \omega^2} + h_1 \left( \frac{q''}{h_1 q'' + h_1} + \frac{h_1}{h_1 q'' + h_1} \right) \tilde{\Theta}(q) \tilde{\Phi}(\xi, q, h_1, h_1) \]

\[ -h_1 \left( \frac{q''}{h_1 q'' + h_1} + \frac{h_1}{h_1 q'' + h_1} \right) \tilde{\Theta}(q) \tilde{\Phi}(\xi, q, h_1, h_1) \]

(25)

Upon taking the inverting the Laplace transform, equation (25) takes the following form

\[ \bar{v}(\xi, q) = \cos \omega t \frac{h_0}{h_1} \left( R_{\alpha, \nu} (-h_0, t) + h_1 F_{\alpha} (-h_0, t) \right) * \tilde{\Phi}(\xi, \tau, h_1, h_1) \]

\[ -\frac{h_1}{h_1} \left( R_{\alpha, \nu} (-h_0, t) + h_1 F_{\alpha} (-h_0, t) \right) * \tilde{\Phi}(\xi, \tau, h_1, h_1) \]

(26)

where

\[ h_1 = h_0 / h_1, \]

and the function \( \Phi(\xi, \tau, h_0, h_1) \) is already defined in equations (20) and (21), \( R_{\alpha, \nu}(\ldots) \) is the Lorenzo and Hartley's function and \( F_{\alpha}(\ldots) \) Robotnov and Hartley's function which are given by [14, 16, 17]

\[ R_{\alpha, \nu}(\ldots) = \sum_{\nu=0}^{n} \left( \frac{(-m)^n}{\Gamma((n+1)\alpha-\nu)} \right) \]

(27)

\[ F_{\alpha}(\ldots) = \sum_{\nu=0}^{n} \left( \frac{(-m)^n}{\Gamma((n+1)\alpha)} \right) \]

(28)

The newly introduced function \( \Psi'(\xi, \tau, h_0, h_1, h_1) \) is given by

\[ \Phi'(\xi, \tau, h_0, h_1, h_1) = L^\nu \left[ \tilde{\Phi}(\xi, q, h_1, h_1) \right] = \]

\[ \frac{1}{\pi} \int_{0}^{\pi} \Psi_{1}(\xi, \tau, h_0, h_1, h_1) \right| \cos(\alpha \pi) \exp(-\nu \cos \alpha \pi) \right| drdu \]

(29)

where

\[ \Psi_{1}(\xi, \tau, h_0, h_1, h_1) = L^\nu \left[ \frac{1}{q} \exp \left( -\xi \left( \frac{h_1 q + h_1}{q + h_1} \right)^{\frac{1}{2}} \right) \right] = \exp \left( -\xi \left( h_1 \right)^{\frac{1}{2}} \right) \]

\[ -\int_{0}^{\infty} \frac{\xi^{(h_1 - h_0) \frac{1}{2}}}{2 (\xi s)^{\frac{1}{2}}} \exp(h_1 s) \times \frac{1}{u} \exp \left( -\frac{\xi u^2 - h_1 u}{4u} \right) I_1 \left( 2 (u (h_1 - h_0) \frac{1}{2}) \right) \]

and \( I_1(\ldots) \) is Bessel function of the first kind. It is important to highlight here that the solutions obtained in equations (19) and (26) for temperature and velocity profiles satisfy all the imposed physical conditions. Furthermore, these equations can be reduced to the solutions obtained by Jan et al. [11]
making $\gamma = \pi / 2$ and taking engine oil based molybdenum disulphide nanofluid instead of hybrid nanofluid.

4. Results and discussion

In this article the unsteady mixed convection MHD flow of hybrid nanofluid over an infinite oscillating plate is studied. The hybridized nanofluid is characterized by dissolving Cu and Al$_2$O$_3$ in with equal concentration in pure water as a base fluid. The problem is modeled in terms of fractional PDE’s using Atangana-Baleanu fractional operator. The exact analytical solutions are obtained for velocity and temperature profile by using the Laplace transform technique. The influence of pertinent flow parameters which are $\alpha, \phi_{nf}, \beta, M$ and $\gamma$ is studied on the corresponding velocity and temperature profiles physically.

Figures 1 and 2 are plotted to show the effect of $\alpha$ on velocity and temperature profiles. It is found that both the velocity and temperature profiles decreased with increasing values of $\alpha$. This can be physically justified as when $\alpha$ is increased, the momentum and thermal boundary layer decreased and became thinnest at $\alpha = 1$ as a result the velocity and temperature profiles decreased. The similar trend of $\alpha$ is reported in [18, 19] which validate the present solutions.

Figures 3 and 4 depict the influence of $\phi_{nf}$ on velocity and temperature profiles. It is noticed that the velocity and temperature profiles behave oppositely with variation in $\phi_{nf}$. The velocity profile decreases with increasing values of $\phi_{nf}$. By increasing values of $\phi_{nf}$ the hybrid nanofluid became more concentrated and denser. In the case of the velocity profile, the effective density of hybrid nanofluid dominates the thermal conductivity which leads to a decrease in the velocity profile. However, in the
energy equation, the effective thermal conductivity plays a significant rule. When $\phi_{nf}$ is increased as a result the thermal conductivity of the fluid is increased which enhance the capacity of hybrid nanofluid to conduct more heat. Consequently, the temperature profile shows an increasing trend. The same effect of volume concentration of nanoparticles in case of nanofluid and hybrid nanofluid can be observed in [7, 15, 20] and the references therein.

Figure 5 displays the comparison of temperature profile for $Cu - Al_2O_3 - H_2O$, $Cu - H_2O$, $Al_2O_3 - H_2O$ and pure water. In this figure, small variation is observed. To study the comparison more clearly, the temperature profile is computed for different values of $\xi$ and presented in Table 3. From this table it can be clearly seen that the temperature profile of $Cu - H_2O$ higher followed by $Cu - Al_2O_3 - H_2O$, $Al_2O_3 - H_2O$ and pure water. As expected the thermal heat transfer ability of $Al_2O_3 - H_2O$ is improved and the temperature profile of $Cu - Al_2O_3 - H_2O$ higher than $Al_2O_3 - H_2O$ and pure water.

![Figure 4. Comparison of temperature profile against $\xi$ for different nanofluid and pure water](image1)

![Figure 5. Velocity profile against $\xi$ for different values of $b\beta$](image2)

**Table 3.** Comparison of temperature profile against $\xi$ tabular form for different fluids

| $\xi$ | $v(\xi, \tau)$ profile of $Cu - Al_2O_3 - H_2O$ | $v(\xi, \tau)$ profile of $Cu - H_2O$ | $v(\xi, \tau)$ profile of $Al_2O_3 - H_2O$ | $v(\xi, \tau)$ profile of Pure water |
|-------|---------------------------------------------|-------------------------------------|------------------------------------------|------------------------------------|
| 0.1   | 1                                           | 1                                   | 1                                        | 1                                  |
| 0.2   | 0.7872                                      | 0.7887                              | 0.7853                                   | 0.7527                             |
| 0.3   | 0.6194                                      | 0.6217                              | 0.6163                                   | 0.5661                             |
| 0.4   | 0.4871                                      | 0.4898                              | 0.4835                                   | 0.4255                             |
| 0.5   | 0.3829                                      | 0.3858                              | 0.3791                                   | 0.3196                             |
| 0.6   | 0.3008                                      | 0.3037                              | 0.2971                                   | 0.2399                             |
| 0.7   | 0.2363                                      | 0.2389                              | 0.2328                                   | 0.18                               |
| 0.8   | 0.1855                                      | 0.1879                              | 0.1823                                   | 0.1349                             |
| 0.9   | 0.1455                                      | 0.1477                              | 0.1427                                   | 0.1011                             |
| 1     | 0.1141                                      | 0.1161                              | 0.1116                                   | 0.0757                             |

Figure 6 depicts the impact of $b\beta$ on velocity profile. Increasing the values of $b\beta$ decreasing the velocity profile. $b\beta$ represents the porosity of highly porous media. The effect of $b\beta$ is similar to drag forces [11]. When $b\beta$ increased consequently the drag force increased subsequently the velocity profile decreases.
The effect of $M$ and $\gamma$ are presented in Figures 7 and 8. In Figure 7, it is found that increasing the values of $M$ corresponds to increase Lorentz forces (Lorentz forces are like drag forces) which retard the velocity profile. $\gamma$ is the angle of inclination of the magnetic field. It is obvious that normal magnetic forces are stronger than the inclined magnetic force. Hence, it is achieved in Figure 8 that the velocity profiles is lowest for $\gamma = \pi / 2$ due to stronger magnetic field.

5. Conclusion
In this article, the MHD mixed convection flow of hybrid nanofluid over an oscillating infinite vertical plate is investigated. The exact analytical solutions developed using the Laplace transform technique. To explore the physics of flow parameters the obtained solutions are computed and plotted in various figures. The major key points extracted from the study are as follow

- The velocity and temperature profiles show the same trend for $\alpha$. Both the velocity and temperature profiles decrease with increasing values of $\alpha$.
- The velocity profile retards for greater values of $\phi_{nf}$ in contrast the temperature profile enhances with enchantment in $\phi_{nf}$.
- The temperature profile for $Cu - H_2O$ nanofluid higher followed by $Cu - Al_2O_3 - H_2O$, $Al_2O_3 - H_2O$ and pure water.
- The velocity profile increases with increasing values of $\beta$, $M$ and $\gamma$.

Acknowledgment
The authors would like to acknowledge Ministry of Education (MOE) and Research Management Centre-UTM, Universiti Teknologi Malaysia UTM for the financial support through vote numbers 5F004, 07G70, 07G72, 07G76 and 07G77 for this research.

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