Counting and enumerating optimum cut sets for hypergraph $k$-partitioning problems for fixed $k^*$

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Abstract

We consider the problem of enumerating optimal solutions for two hypergraph $k$-partitioning problems—namely, Hypergraph-$k$-Cut and Minmax-Hypergraph-$k$-Partition. The input in hypergraph $k$-partitioning problems is a hypergraph $G = (V, E)$ with positive hyperedge costs along with a fixed positive integer $k$. The goal is to find a partition of $V$ into $k$ non-empty parts $(V_1, V_2, \ldots, V_k)$—known as a $k$-partition—so as to minimize an objective of interest.

1. If the objective of interest is the maximum cut value of the parts, then the problem is known as Minmax-Hypergraph-$k$-Partition. A subset of hyperedges is a minmax-$k$-cut-set if it is the subset of hyperedges crossing an optimum $k$-partition for Minmax-Hypergraph-$k$-Partition.

2. If the objective of interest is the total cost of hyperedges crossing the $k$-partition, then the problem is known as Hypergraph-$k$-Cut. A subset of hyperedges is a min-$k$-cut-set if it is the subset of hyperedges crossing an optimum $k$-partition for Hypergraph-$k$-Cut.

We give the first polynomial bound on the number of minmax-$k$-cut-sets and a polynomial-time algorithm to enumerate all of them in hypergraphs for every fixed $k$. Our technique is strong enough to also enable an $n^{O(k^2)}p$-time deterministic algorithm to enumerate all min-$k$-cut-sets in hypergraphs, thus improving on the previously known $n^{O(k^2)}p$-time deterministic algorithm, where $n$ is the number of vertices and $p$ is the size of the hypergraph. The correctness analysis of our enumeration approach relies on a structural result that is a strong and unifying generalization of known structural results for Hypergraph-$k$-Cut and Minmax-Hypergraph-$k$-Partition.

We believe that our structural result is likely to be of independent interest in the theory of hypergraphs (and graphs).

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1 Introduction

In hypergraph \( k \)-partitioning problems, the input consists of a hypergraph \( G = (V, E) \) with positive hyperedge-costs \( c : E \rightarrow \mathbb{R}_+ \) and a fixed positive integer \( k \) (e.g., \( k = 2, 3, 4, \ldots \)). The goal is to find a partition of the vertex set into \( k \) non-empty parts \( V_1, V_2, \ldots, V_k \) so as to minimize an objective of interest. There are several natural objectives of interest in hypergraph \( k \)-partitioning problems. In this work, we focus on two particular objectives: **Minmax-Hypergraph-\( k \)-Partition** and **Hypergraph-\( k \)-Cut**:

1. In **Minmax-Hypergraph-\( k \)-Partition**, the objective is to minimize the maximum cut value of the parts of the \( k \)-partition—i.e., minimize \( \max_{i=1}^{k} c(\delta(V_i)) \); here \( \delta(V_i) \) is the set of hyperedges intersecting both \( V_i \) and \( V \setminus V_i \) and \( c(\delta(V_i)) = \sum_{e \in \delta(V_i)} c(e) \) is the total cost of hyperedges in \( \delta(V_i) \).

2. In **Hypergraph-\( k \)-Cut**, the objective is to minimize the cost of hyperedges crossing the \( k \)-partition—i.e., minimize \( c(\delta(V_1, \ldots, V_k)) \); here \( \delta(V_1, \ldots, V_k) \) is the set of hyperedges that intersect at least two sets in \( \{V_1, \ldots, V_k\} \) and \( c(\delta(V_1, \ldots, V_k)) = \sum_{e \in \delta(V_1, \ldots, V_k)} c(e) \) is the total cost of hyperedges in \( \delta(V_1, \ldots, V_k) \).

If the input \( G \) is a graph, then we will refer to these problems as **Minmax-Graph-\( k \)-Partition** and **Graph-\( k \)-Cut** respectively. We note that the case of \( k = 2 \) corresponds to global minimum cut in both objectives. In this work, we focus on the problem of enumerating all optimum solutions to **Minmax-Hypergraph-\( k \)-Partition** and **Hypergraph-\( k \)-Cut**.

**Motivations and Related Problems.** We consider the problem of counting and enumerating optimum solutions for partitioning problems over hypergraphs for three reasons. Firstly, hyperedges provide more powerful modeling capabilities than edges and consequently, several problems in hypergraphs become non-trivial in comparison to graphs. Although hypergraphs and partitioning problems over hypergraphs (including **Minmax-Hypergraph-\( k \)-Partition**) were discussed as early as 1973 by Lawler [43], most of these problems still remain open. The powerful modeling capability of hyperedges has been useful in a variety of modern applications, which in turn, has led to a resurgence in the study of hypergraphs with recent works focusing on min-cuts, cut-sparsifiers, spectral-sparsifiers, etc. [6,7,11,18,19,23,24,35,42,56]. Our work adds to this rich and emerging theory of hypergraphs.

Secondly, hypergraph \( k \)-partitioning problems are special cases of submodular \( k \)-partitioning problems. In submodular \( k \)-partitioning problems, the input is a finite ground set \( V \), a submodular function\(^1\) \( f : 2^V \rightarrow \mathbb{R} \) provided by an evaluation oracle\(^2\) and a positive integer \( k \) (e.g., \( k = 2, 3, 4, \ldots \)). The goal is to partition the ground set \( V \) into \( k \) non-empty parts \( V_1, V_2, \ldots, V_k \) so as to minimize an objective of interest. Two natural objectives are of interest: (1) In **Minmax-Submod-\( k \)-Partition**, the objective is to minimize \( \max_{i=1}^{k} f(V_i) \) and (2) In **Minsum-Submod-\( k \)-Partition**, the objective is to minimize \( \sum_{i=1}^{k} f(V_i) \). If the given submodular function is symmetric\(^3\), then we denote the resulting problems as **Minmax-SymSubmod-\( k \)-Partition** and **Minsum-SymSubmod-\( k \)-Partition** respectively. Since the hypergraph cut function is symmetric submodular, it follows that **Minmax-Hypergraph-\( k \)-Partition** is a special case of **Minmax-SymSubmod-\( k \)-Partition**. Moreover, **Hypergraph-\( k \)-Cut** is a special case of **Minsum-Submod-\( k \)-Partition** (this reduction is slightly non-trivial with the submodular function in the reduction being asymmetric—e.g., see [50] for the

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\(^1\)We emphasize that the objective of Hypergraph-\( k \)-Cut is not equivalent to minimizing \( \sum_{i=1}^{k} c(\delta(V_i)) \).

\(^2\)A real-valued set function \( f : 2^V \rightarrow \mathbb{R} \) is submodular if \( f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \forall A, B \subseteq V \).

\(^3\)An evaluation oracle for a set function \( f \) over a ground set \( V \) returns the value of \( f(S) \) given \( S \subseteq V \).
Queyranne claimed, in 1999, a polynomial-time algorithm for Minsum-SymSubmod-$k$-Partition for every fixed $k$ \cite{52}, however the claim was retracted subsequently (see \cite{27}). The complexity status of submodular $k$-partitioning problems (for fixed $k \geq 4$) are open, so recent works have focused on hypergraph $k$-partitioning as a stepping stone towards submodular $k$-partitioning \cite{7,11,12,27,50,60,61}. Our work contributes to this stepping stone by advancing the state of the art in hypergraph $k$-partitioning problems. We emphasize that the complexity status of two other variants of hypergraph $k$-partitioning problems which are also special cases of Minsum-Submod-$k$-Partition are still open (see \cite{50,60,61} for these variants).

Thirdly, counting and enumeration of optimum solutions for graph $k$-partitioning problems are fundamental to graph theory and extremal combinatorics. They have found further reaching applications than initially envisioned. We discuss some of the results and applications for $k = 2$ and $k > 2$ now. For $k = 2$ in connected graphs, it is well-known that the number of min-cuts and the number of $\alpha$-approximate min-cuts are at most $\binom{n}{2}$ and $O(n^{2\alpha})$ respectively, and they can all be enumerated in polynomial time for constant $\alpha$. These combinatorial results have been the crucial ingredients of several algorithmic and representation results in graphs. On the algorithmic front, these results enable fast randomized construction of graph skeletons which, in turn, plays a crucial role in fast algorithms to solve graph min-cut \cite{37}. On the representation front, counting results form the backbone of cut sparsifiers which in turn have found applications in sketching and streaming \cite{2,4,12}. A polygon representation of the family of $6/5$-approximate min-cuts in graphs was given by Benczur and Goemans in 1997 (see \cite{8–10})—this representation was used in the recent groundbreaking $(3/2 - \epsilon)$-approximation for metric TSP \cite{39}. On the approximation front, in addition to the $(3/2 - \epsilon)$-approximation for metric TSP \cite{39}, counting results also led to the recent 1.5-approximation for path TSP \cite{59}. For $k > 2$, we note that fast algorithms for Graph-$k$-Cut have been of interest since they help in generating cutting planes while solving TSP \cite{5,20}. A recent series of works aimed towards improving the bounds on the number of optimum solutions for Graph-$k$-Cut culminated in a drastic improvement in the run-time to solve Graph-$k$-Cut \cite{28,31,32}. Given the status of counting and enumeration results for $k$-partitioning in graphs and their algorithmic and representation implications that were discovered subsequently, we believe that a similar understanding in hypergraphs could serve as an important ingredient in the algorithmic and representation theory of hypergraphs.

**The Enumeration Problem.** There is a fundamental structural distinction between hypergraphs and graphs that becomes apparent while enumerating optimum solutions to $k$-partitioning problems. In connected graphs, the number of optimum $k$-partitions for Graph-$k$-Cut and for Minmax-Graph-$k$-Partition are $n^{O(k)}$ and $n^{O(k^2)}$ respectively and they can all be enumerated in polynomial time, where $n$ is the number of vertices in the input graph \cite{13,17,28,32,38,57}. In contrast, a connected hypergraph could have exponentially many optimum $k$-partitions for both Minmax-Hypergraph-$k$-Partition and Hypergraph-$k$-Cut even for $k = 2$—e.g., consider the hypergraph with a single hyperedge containing all vertices; we will denote this as the spanning-hyperedge-example. Hence, enumerating all optimum $k$-partitions for hypergraph $k$-partitioning problems in polynomial time is impossible. Instead, our goal in the enumeration problems is to enumerate $k$-cut-sets corresponding to optimum $k$-partitions. We will call a subset $F \subseteq E$ of hyperedges to be a $k$-cut-set if there exists a $k$-partition $(V_1, \ldots, V_k)$ such that $F = \delta(V_1, \ldots, V_k)$; we will call a 2-cut-set as a cut-set. In the enumeration problems that we will consider, the input consists of a hypergraph $G = (V, E)$ with positive hyperedge-costs $c : E \rightarrow \mathbb{R}_+$ and a fixed positive integer $k$ (e.g., $k = 2, 3, 4, \ldots$).

1. For an optimum $k$-partition $(V_1, \ldots, V_k)$ for Minmax-Hypergraph-$k$-Partition in $(G, c)$, we will denote $\delta(V_1, \ldots, V_k)$ as a minmax-$k$-cut-set. In Enum-MinMax-Hypergraph-$k$-
We observe that in the spanning-hyperedge-example, although the number of optimum $G$ hypergraphs is exponential, in contrast to graphs, whose representation size is the number of edges, the representation size of a hypergraph is\footnote{For an optimum $k$-partition $(V_1,\ldots,V_k)$ for Hypergraph-$k$-Cut in $(G,c)$, we will denote $\delta(V_1,\ldots,V_k)$ as a min-$k$-cut-set. In Enum-Hypergraph-$k$-Cut, the goal is to enumerate all min-$k$-cut-sets.} for which the number of optimum $k$-partitions for Minmax-Hypergraph-$k$-Partition (as well as Hypergraph-$k$-Cut) is exponential, the number of minmax-$k$-cut-sets (as well as min-$k$-cut-sets) is only one.

1.1 Results\footnote{We emphasize that our result shows the first polynomial bound on the number of minmax-$k$-cut-sets.}

In contrast to graphs, whose representation size is the number of edges, the representation size of a hypergraph $G = (V,E)$ is $p := \sum_{e \in E} |e|$. Throughout, our algorithmic discussion will focus on the case of fixed $k$ (e.g., $k = 2,3,4,\ldots$). There are no prior results regarding Enum-MinMax-Hypergraph-$k$-Partition in the literature. We recall the status of Minmax-Hypergraph-$k$-Partition. As mentioned earlier, Minmax-Hypergraph-$k$-Partition was discussed as early as 1973 by Lawler\footnote{There exists a deterministic algorithm to solve Enum-MinMax-Hypergraph-$k$-Partition and as a consequence, obtained the first polynomial-time algorithm to solve Minmax-Hypergraph-$k$-Partition. Their algorithm does not show any bound on the number of minmax-$k$-cut-sets since it solves the more general problem of Minmax-SymSubmod-$k$-Partition for which the number of optimum $k$-partitions can indeed be exponential (recall the spanning-hyperedge-example). Focusing on hypergraphs raises the question of whether all $k$-cut-sets corresponding to optimum solutions can be enumerated efficiently for every fixed $k$. We answer this question affirmatively by giving the first polynomial-time algorithm for Enum-MinMax-Hypergraph-$k$-Partition.}}

\begin{theorem}\
There exists a deterministic algorithm to solve Enum-MinMax-Hypergraph-$k$-Partition that runs in time $O(kn^{4k^2-2k+1}p)$, where $n$ is the number of vertices and $p$ is the size of the input hypergraph. Moreover, the number of minmax-$k$-cut-sets in a $n$-vertex hypergraph is $O(n^{4k^2-2k})$.
\end{theorem}

We emphasize that our result shows the first polynomial bound on the number of minmax-$k$-cut-sets in hypergraphs for every fixed $k$ (in addition to a polynomial-time algorithm to enumerate all of them for every fixed $k$). Our upper bound of $n^{O(k^2)}$ on the number of minmax-$k$-cut-sets is tight—there exist $n$-vertex connected graphs for which the number of minmax-$k$-cut-sets is $n^{\Theta(k^2)}$ (see Section\footnote{Next, we briefly recall the status of Hypergraph-$k$-Cut and Enum-Hypergraph-$k$-Cut. Hypergraph-$k$-Cut was shown to be solvable in randomized polynomial time only recently\footnote{Chandrasekaran and Chekuri’s techniques were extended to design the first deterministic polynomial-time algorithm to solve Enum-Hypergraph-$k$-Cut in\footnote{The algorithm for Enum-Hypergraph-$k$-Cut given in\footnote{This run-time has a quadratic dependence on}}}. The randomized algorithms also showed that the number of min-$k$-cut-sets is $O(n^{2k-2})$ and they can all be enumerated in randomized polynomial time. A subsequent deterministic algorithm was designed to solve Hypergraph-$k$-Cut in time $n^{O(k)}p$ by Chandrasekaran and Chekuri\footnote{There exists a deterministic algorithm to solve Enum-MinMax-Hypergraph-$k$-Partition, the goal is to enumerate all min-$k$-cut-sets.}}.\footnote{We emphasize that our result shows the first polynomial bound on the number of minmax-$k$-cut-sets.}
k in the exponent of n although the number of \(\text{min-}k\text{-cut-sets}\) has only linear dependence on k in the exponent of n (since it is \(O(n^{2k-2})\)). So, an open question that remained from [7] is whether one can obtain an \(n^{O(k)}\)-time deterministic algorithm for \(\text{Enum-Hypergraph-}k\text{-cut}\). We resolve this question affirmatively.

**Theorem 1.2.** There exists a deterministic algorithm to solve \(\text{Enum-Hypergraph-}k\text{-cut}\) that runs in time \(O(n^{16k-25}p)\), where \(n\) is the number of vertices and \(p\) is the size of the input hypergraph.

Our algorithms for both \(\text{Enum-MinMax-Hypergraph-}k\text{-partition}\) and \(\text{Enum-Hypergraph-}k\text{-cut}\) are based on a structural theorem that allows for efficient recovery of optimum \(k\)-cut-sets via minimum \((s,t)\)-terminal cuts (see Theorem 1.4). Our structural theorem builds on structural theorems that have appeared in previous works on \(\text{Minmax-Hypergraph-}k\text{-partition}\) and \(\text{Hypergraph-}k\text{-cut}\) [7,11,12]. Our structural theorem may appear to be natural/incremental in comparison to ones that appeared in previous works, but formalizing the theorem and proving it is a significant part of our contribution. Moreover, our single structural theorem is strong enough to enable efficient algorithms for both \(\text{Enum-Hypergraph-}k\text{-cut}\) as well as \(\text{Enum-MinMax-Hypergraph-}k\text{-partition}\) in contrast to previously known structural theorems. In this sense, our structural theorem can be viewed as a strong and unifying generalization of structural theorems that have appeared in previous works. We believe that our structural theorem will be of independent interest in the theory of cuts and partitioning in hypergraphs (as well as graphs).

### 1.2 Technical overview and main structural result

We focus on the unit-cost variant of \(\text{Enum-Hypergraph-}k\text{-Cut}\) and \(\text{Enum-MinMax-Hypergraph-}k\text{-Partition}\) in the rest of this work for the sake of notational simplicity—i.e., the cost of every hyperedge is 1. Throughout, we will allow multigraphs and hence, this is without loss of generality. Our algorithms extend in a straightforward manner to arbitrary hyperedge costs. They rely only on minimum \((s,t)\)-terminal cut computations and hence, they are strongly polynomial-time algorithms.

**Notation and background.** Let \(G = (V,E)\) be a hypergraph. Throughout this work, \(n\) will denote the number of vertices in \(G\), \(m\) will denote the number of hyperedges in \(G\), and \(p := \sum_{e \in E} |e|\) will denote the representation size of \(G\). We will denote a partition of the vertex set into \(h\) non-empty parts by an ordered tuple \((V_1,...,V_h)\) and call such an ordered tuple as an \(h\)-partition. For a partition \(P = (V_1, V_2,...,V_h)\), we will say that a hyperedge \(e\) crosses the partition \(P\) if it intersects at least two parts of the partition. We will refer to a 2-partition as a cut. For a non-empty proper subset \(U\) of vertices, we will use \(\overline{U}\) to denote \(V \setminus U\), \(\delta(U)\) to denote the set of hyperedges crossing the cut \((U,\overline{U})\), and \(d(U) := |\delta(U)|\) to denote the cut value of \(U\). We observe that \(\delta(U) = \delta(\overline{U})\), so we will use \(d(U)\) to denote the value of the cut \((U,\overline{U})\). More generally, given a partition \(P = (V_1, V_2,...,V_h)\), we denote the set of hyperedges crossing the partition by \(\delta(V_1, V_2,...,V_h)\) (also by \(\delta(P)\) for brevity) and the number of hyperedges crossing the partition by \(|\delta(V_1, V_2,...,V_h)|\). We will denote the optimum value of \(\text{Minmax-Hypergraph-}k\text{-partition}\) and \(\text{Hypergraph-}k\text{-cut}\) respectively by

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\begin{align*}
\text{OPT}_{\text{minmax-}k\text{-partition}} & := \min \left\{ \max_{i \in [k]} |\delta(V_i)| : (V_1,...,V_k) \text{ is a } k\text{-partition of } V \right\} \text{ and } \\
\text{OPT}_{k\text{-cut}} & := \min \left\{ |\delta(V_1,...,V_k)| : (V_1,...,V_k) \text{ is a } k\text{-partition of } V \right\}.
\end{align*}
\]

A key algorithmic tool will be the use of fixed-terminal cuts. Let \(S, T\) be disjoint non-empty subsets of vertices. A 2-partition \((U,\overline{U})\) is an \((S,T)\)-terminal cut if \(S \subseteq U \subseteq V \setminus T\). Here, the set \(U\) is known as the source set and the set \(\overline{U}\) is known as the sink set. A minimum-valued \((S,T)\)-terminal cut
cut is known as a minimum \((S, T)\)-terminal cut. Since there could be multiple minimum \((S, T)\)-terminal cuts, we will be interested in source minimal minimum \((S, T)\)-terminal cuts. For every pair of disjoint non-empty subsets \(S\) and \(T\) of vertices, there exists a unique source minimal minimum \((S, T)\)-terminal cut and it can be found in deterministic polynomial time via standard maxflow algorithms. In particular, the source minimal minimum \((S, T)\)-terminal cut can be found in time \(O(np)\) \cite{18}.

Our technique to enumerate all minmax-\(k\)-cut-sets and all min-\(k\)-cut-sets will build on the approaches of Chandrasekaran and Chekuri for Hypergraph-\(k\)-Cut and Minmax-SymSubmod-
\(k\)-Partition \cite{7,11,12}. We need the following structural theorem that was shown in \cite{7}.

**Theorem 1.3.** \cite{7} Let \(G = (V, E)\) be a hypergraph and let \(OPT_{k\text{-cut}}\) be the optimum value of Hypergraph-\(k\)-Cut in \(G\) for some integer \(k \geq 2\). Suppose \((U, \overline{U})\) is a 2-partition of \(V\) with \(d(U) < OPT_{k\text{-cut}}\). Then, for every pair of vertices \(s \in U\) and \(t \in \overline{U}\), there exist subsets \(S \subseteq U \setminus \{s\}\) and \(T \subseteq \overline{U} \setminus \{t\}\) with \(|S| \leq 2k - 3\) and \(|T| \leq 2k - 3\) such that \((U, \overline{U})\) is the unique minimum \((S \cup \{s\}, T \cup \{t\})\)-terminal cut in \(G\).

**Enum-Hypergraph-\(k\)-Cut.** We first focus on Enum-Hypergraph-\(k\)-Cut. We note that Theorem 1.3 will allow us to recover those parts \(V_i\) of an optimum \(k\)-partition \((V_1, \ldots, V_k)\) for which \(d(V_i) < OPT_{k\text{-cut}}\). However, recall that our goal is not to recover all optimum \(k\)-partitions for Hypergraph-\(k\)-Cut, but rather to recover all min-\(k\)-cut-sets (i.e., not to recover the parts of every optimum \(k\)-partition, but rather only to recover the \(k\)-cut-set of every optimum \(k\)-partition).

The previous work \cite{7} that designed an \(n^{O(k^2)}p\)-time deterministic enumeration algorithm achieved this by proving the following structural result: suppose \((V_1, \ldots, V_k)\) is an optimum \(k\)-partition for Hypergraph-\(k\)-Cut for which \(d(V_i) = OPT_{k\text{-cut}}\). Then, they showed that for every subset \(T \subseteq V_1\) satisfying \(T \cap V_j \neq \emptyset\) for all \(j \in \{2, \ldots, k\}\), there exists a subset \(S \subseteq V_1\) with \(|S| \leq 2k\) such that the source minimal minimum \((S, T)\)-terminal cut \((A, \overline{A})\) satisfies \(\delta(A) = \delta(V_i)\). This structural theorem in conjunction with Theorem 1.3 allows one to enumerate a candidate family \(F\) of \(n^{O(k^2)}\) subsets of hyperedges such that every min-\(k\)-cut-set is present in the family. The drawback of their structural theorem is that it is driven towards recovering the cut-set \(\delta(V_i)\) of every part \(V_i\) of every optimum \(k\)-partition \((V_1, \ldots, V_k)\). Hence, their algorithmic approach ends up with a run-time of \(n^{O(k^2)}p\). In order to improve the run-time, we prove a stronger result: we show that for an arbitrary cut \((U, \overline{U})\) with cut value \(OPT_{k\text{-cut}}\) (as opposed to only those sets \(V_i\) of an optimum \(k\)-partition \((V_1, \ldots, V_k)\)), its cut-set \(\delta(U)\) can be recovered as the cut-set of any minimum \((S, T)\)-terminal cut for some \(S\) and \(T\) of small size. The following is the main structural theorem of this work.

**Theorem 1.4.** Let \(G = (V, E)\) be a hypergraph and let \(OPT_{k\text{-cut}}\) be the optimum value of Hypergraph-\(k\)-Cut in \(G\) for some integer \(k \geq 2\). Suppose \((U, \overline{U})\) is a 2-partition of \(V\) with \(d(U) = OPT_{k\text{-cut}}\). Then, there exist sets \(S \subseteq U\), \(T \subseteq \overline{U}\) with \(|S| \leq 2k - 1\) and \(|T| \leq 2k - 1\) such that every minimum \((S, T)\)-terminal cut \((A, \overline{A})\) satisfies \(\delta(A) = \delta(U)\).

We encourage the reader to compare and contrast Theorems 1.3 and 1.4. The former helps to recover cuts whose cut value is strictly smaller than \(OPT_{k\text{-cut}}\) while the latter helps to recover cut-sets whose size is equal to \(OPT_{k\text{-cut}}\). So, the latter theorem is weaker since it only recovers cut-sets, but we emphasize that this is the best possible that one can hope to do (as seen from the spanning-hyperedge-example). However, proving the latter theorem requires us to work with cut-sets (as opposed to cuts) which is a technical barrier to overcome. Indeed, our proof of Theorem 1.4 deviates significantly from the proof of Theorem 1.3 since we have to work with cut-sets. Our proof also deviates from the structural result in \cite{7} that was mentioned in the paragraph above Theorem 1.4 since our result is stronger than their result—our result helps to recover the cut-set \(\delta(U)\) of an
arbitrary cut \((U, \overline{U})\) whose cut value is \(d(U) = OPT_{k\text{-cut}}\) while their result helps only to recover the cut-set \(\delta(V_i)\) of a part \(V_i\) of an optimum \(k\)-partition \((V_1, \ldots, V_k)\) for \(\text{HYPERGRAPH-}k\text{-CUT}\) whose cut value is \(d(V_i) = OPT_{k\text{-cut}}\); moreover, their proof technique crucially relies on a containment property with respect to the part \(V_i\), whereas under the hypothesis of our structural theorem, the containment property fails with respect to the set \(U\) and consequently, our proof technique differs from theirs.

Theorems 1.3 and 1.4 lead to a deterministic \(n^{O(k)}\)-time algorithm to enumerate all \(\text{MIN-}k\text{-CUT-SETS}\) via a divide-and-conquer approach. We describe this algorithm now: For each pair \((S, T)\) of disjoint subsets of vertices \(S\) and \(T\) with \(|S|, |T| \leq 2k - 1\), compute the source minimal minimum \((S, T)\)-terminal cut \((A, \overline{A})\); (i) if \(G - \delta(A)\) has at least \(k\) connected components, then add \(\delta(A)\) to the candidate family \(F\); (ii) otherwise, add the set \(A\) to a collection \(C\). We note that the sizes of the family \(F\) and the collection \(C\) are \(O(n^{4k-2})\). Next, for each subset \(A\) in the collection \(C\), recursively enumerate all \(\text{MIN-}k/2\text{-CUT-SETS}\) in the subhypergraphs induced by \(A\) and \(\overline{A}\) respectively denoted \(G[A]\) and \(G[\overline{A}]\) respectively—and add \(\delta(A) \cup F_1 \cup F_2\) to the family \(F\) for each \(F_1\) and \(F_2\) being \(\text{MIN-}k/2\text{-CUT-SET}\) in \(G[A]\) and \(G[\overline{A}]\) respectively. Finally, return the subfamily of \(k\)-cut-sets from the family \(F\) that are of smallest size.

We sketch the correctness analysis of the above approach: let \(F = \delta(V_1, \ldots, V_k)\) be a \(\text{MIN-}k\text{-CUT-SET}\) with \((V_1, \ldots, V_k)\) being an optimum \(k\)-partition for \(\text{HYPERGRAPH-}k\text{-CUT}\). We will show that the family \(F\) contains \(F\). Let \(U := \bigcup_{i=1}^{k/2} V_i\). We note that \(\delta(U) \subseteq F\). We have two possibilities: (1) Say \(d(U) = |F|\). Then, \(d(U) = OPT_{k\text{-cut}}\). Consequently, by Theorem 1.3, the \(\text{MIN-}k\text{-CUT-SET}\) \(F\) will be added to the family \(F\) by step (i). (2) Say \(d(U) < |F|\). Then, by Theorem 1.3, the set \(U = \bigcup_{i=1}^{k/2} V_i\) will be added to the collection \(C\) by step (ii); moreover, \(F_1 := F \cap E(G[U])\) and \(F_2 := F \cap E(G[\overline{U}])\) are \(\text{MIN-}k/2\text{-CUT-SETS}\) in \(G[U]\) and \(G[\overline{U}]\) respectively and they would have been enumerated by recursion, and hence, the set \(\delta(U) \cup F_1 \cup F_2 = F\) will be added to the family \(F\). The size of the family \(F\) can be shown to be \(n^{O(k \log k)}\) and the run-time is \(n^{O(k \log k)} p\) (see Theorem 4.1). Using the known fact that the number of \(\text{MIN-}k\text{-CUT-SETS}\) in a \(n\)-vertex hypergraph is \(O(n^{2k-2})\), we can improve the run-time analysis of this approach to \(n^{O(k)} p\) (see Lemma 4.1).

**Enum-MinMax-Hypergraph-\(k\)-Partition.** Next, we focus on **Enum-MinMax-Hypergraph-\(k\)-Partition**. There is a fundamental technical issue in enumerating \(\text{MINMAX-}k\text{-CUT-SETS}\) as opposed to \(\text{MIN-}k\text{-CUT-SETS}\). We highlight this technical issue now. Suppose we find an optimum \(k\)-partition \((V_1, \ldots, V_k)\) for **MINMAX-HYPERGRAPH-\(k\)-PARTITION** (say via Chandrasekaran and Chekuri’s algorithm \[12\]) and store only the \(\text{MINMAX-}\) \(k\text{-CUT-SET}\) \(F = \delta(V_1, \ldots, V_k)\) but forget to store the partition \((V_1, \ldots, V_k)\); now, by knowing a \(\text{MINMAX-}k\text{-CUT-SET}\) \(F\), can we recover some optimum \(k\)-partition for **MINMAX-HYPERGRAPH-\(k\)-PARTITION** (not necessarily \((V_1, \ldots, V_k)\))? Or by knowing a \(\text{MINMAX-}k\text{-CUT-SET}\) \(F\), is it even possible to find the value \(OPT_{\text{MINMAX-}k\text{-partition}}\) without solving the problem again—i.e., is there an advantage to knowing a \(k\)-partition in order to solve **MINMAX-HYPERGRAPH-\(k\)-PARTITION**? We are not aware of such an advantage. This is in stark contrast to **HYPERGRAPH-\(k\)-CUT** where knowing a \(\text{MIN-}k\text{-CUT-SET}\) enables a linear-time solution to **HYPERGRAPH-\(k\)-CUT**.

Why is this issue significant while solving **Enum-MinMax-Hypergraph-\(k\)-Partition**? We recall that in our approach for **Enum-Hypergraph-\(k\)-Cut**, the algorithm computed a polynomial-sized family \(F\) containing all \(\text{MIN-}k\text{-CUT-SETS}\) and returned the ones with smallest size—the smallest size ones will exactly be \(\text{MIN-}k\text{-CUT-SETS}\). It is unclear if a similar approach could work for enum-
merating MINMAX-k-CUT-SETS: suppose we do have an algorithm to enumerate a polynomial-sized family $\mathcal{F}$ containing all MINMAX-k-CUT-SETS; now, in order to return all MINMAX-k-CUT-SETS (which is a subfamily of $\mathcal{F}$), note that we need to identify them among the ones in the family $\mathcal{F}$—i.e., we need to verify if a given subset $F \in \mathcal{F}$ of hyperedges is a MINMAX-k-CUT-SET; this verification problem is closely related to the question mentioned in the previous paragraph. We do not know how to address this verification problem directly. So, our algorithmic approach for ENUM-MINMAX-HYPERGRAPH-k-PARTITION has to overcome this technical issue.

Our ingredient to overcome this technical issue is to enumerate representatives for MINMAX-k-CUT-SETS. For a $k$-partition $(V_1, \ldots, V_k)$ and disjoint subsets $U_1, \ldots, U_k \subseteq V$, we will call the $k$-tuple $(U_1, \ldots, U_k)$ to be a $k$-cut-set representative of $(V_1, \ldots, V_k)$ if $U_i \subseteq V_i$ and $\delta(U_i) = \delta(V_i)$ for all $i \in [k]$. We note that a fixed $k$-partition $(V_1, \ldots, V_k)$ could have several $k$-cut-set representatives and a fixed $k$-tuple $(U_1, \ldots, U_k)$ could be the $k$-cut-set representative of several $k$-partitions. Yet, it is possible to efficiently verify if a given $k$-tuple $(U_1, \ldots, U_k)$ is a $k$-cut-set representative (Theorem 5.1). Moreover, knowing a $k$-cut-set representative $(U_1, \ldots, U_k)$ of a $k$-partition $(V_1, V_2, \ldots, V_k)$ allows one to recover the $k$-cut-set $F := \delta(V_1, \ldots, V_k)$ since $F = \bigcup_{i=1}^k \delta(U_i)$. Thus, in order to enumerate all MINMAX-k-CUT-SETS, it suffices to enumerate $k$-cut-set representatives of all optimum $k$-partitions for MINMAX-HYPERGRAPH-k-PARTITION. At this point, the astute reader may wonder if there exists a polynomial-sized family of $k$-cut-set representatives of all optimum $k$-partitions for MINMAX-HYPERGRAPH-k-PARTITION given that the number of optimum $k$-partitions for MINMAX-HYPERGRAPH-k-PARTITION could be exponential. For example, is there a polynomial-sized family of $k$-cut-set representatives of all optimum $k$-partitions for MINMAX-HYPERGRAPH-k-PARTITION in the spanning-hyperedge-example? Indeed, in the spanning-hyperedge-example, even though the number of optimum $k$-partitions for MINMAX-HYPERGRAPH-k-PARTITION is exponential, there exists a $(k! \binom{n}{k})$-sized family of $k$-cut-set representatives of all optimum $k$-partitions: consider the family $\{(\{v_1\}, \ldots, \{v_k\}) : v_1, \ldots, v_k \in V, v_i \neq v_j \forall \text{ distinct } i, j \in [k]\}$.

It turns out that Theorems 1.3 and 1.4 are strong enough to enable efficient enumeration of $k$-cut-set representatives of all optimum $k$-partitions for MINMAX-HYPERGRAPH-k-PARTITION. We describe the algorithm to achieve this: For each pair $(S, T)$ of disjoint subsets of vertices with $|S|, |T| \leq 2k - 1$, compute the source minimal minimum $(S, T)$-terminal cut $(U, \overline{U})$ and add $U$ to a candidate collection $\mathcal{C}$. We note that the size of the collection $\mathcal{C}$ is $O(n^{k-2})$. Next, for each $k$-tuple $(U_1, \ldots, U_k) \in \mathcal{C}^k$, verify if $(U_1, \ldots, U_k)$ is a $k$-cut-set representative (using Theorem 5.1) and if so, then add the $k$-tuple to the candidate family $\mathcal{D}$. Finally, return $\arg\min\{\max_{i=1}^k d(U_i) : (U_1, \ldots, U_k) \in \mathcal{D}\}$, i.e., prune and return the subfamily of $k$-cut-set representatives $(U_1, \ldots, U_k)$ from the family $\mathcal{D}$ that have minimum max$_{i=1}^k d(U_i)$.

We note that the size of the family $\mathcal{D}$ is $n^{O(k^2)}$ and consequently, the run-time is $n^{O(k^2)}p$. We sketch the correctness analysis of the above approach: let $(V_1, \ldots, V_k)$ be an optimum $k$-partition for MINMAX-HYPERGRAPH-k-PARTITION. We will show that the family $\mathcal{D}$ contains a $k$-cut-set representative of $(V_1, \ldots, V_k)$. By noting that $OPT_{\text{minmax-k-cut}} \leq OPT_{\text{k-cut}}$ and by Theorems 1.3 and 1.4 for every $i \in [k]$, we have a set $U_i$ in the collection $\mathcal{C}$ with $U_i \subseteq V_i$ and $\delta(U_i) = \delta(V_i)$. Hence, the $k$-tuple $(U_1, \ldots, U_k) \in \mathcal{C}^k$ is a $k$-cut-set representative and it will be added to the family $\mathcal{D}$. The final pruning step will not remove $(U_1, \ldots, U_k)$ from the family $\mathcal{D}$ and hence, it will be in the subfamily returned by the algorithm.

**Significance of our technique.** As mentioned earlier, our techniques build on the structural theorems that appeared in previous works [7,11,12]. The main technical novelty of our contribution lies in Theorem 1.4 which can be viewed as the culmination of structural theorems developed in those previous works. We also emphasize that using minimum $(s,t)$-terminal cuts to solve global partitioning problems is not a new technique per se (e.g., minimum $(s,t)$-terminal cut is the first
and most natural approach to solve global minimum cut). This technique of using minimum (s, t)-
terminal cuts to solve global partitioning problems has a rich variety of applications in combinatorial
optimization: e.g., (1) it was used to design the first efficient algorithm for GRAPH-k-CUT for fixed
k [26], (2) it was used to design efficient algorithms for certain constrained submodular minimization
problems [25,49], and (3) more recently, it was used to design fast algorithms for global minimum cut
in graphs as well as to obtain fast Gomory-Hu trees in unweighted graphs [1,44]. The applicability
of this technique relies on identifying and proving appropriate structural results. Our Theorem 1.4
is such a structural result. The merit of the structural result lies in its ability to solve two different
enumeration problems in hypergraph k-partitioning which was not possible via structural theorems
that were developed before. Moreover, it leads to the first polynomial bound on the number of
MINMAX-k-CUT-SETS in hypergraphs for every fixed k.

Organization. We discuss related work in Section 1.3. In Section 1.4, we recall properties of the
hypergraph cut function. In Section 2 we prove a special case of Theorem 1.4. In Section 3, we use
this special case to prove Theorem 1.4. In Section 4, we design an efficient algorithm for ENUM-
HYPERGRAPH-k-CUT and prove Theorem 1.2. In Section 5, we design an efficient algorithm for
ENUM-MINMAX-HYPERGRAPH-k-PARTITION and prove Theorem 1.1. We present a lower bound
example in Section 6. We conclude with an open question in Section 7.

1.3 Related work

We briefly discuss the status of the enumeration problems for k = 2 followed by the status for k ≥ 2
in graphs and hypergraphs.

Enumeration problems for k = 2. For k = 2, both ENUM-MINMAX-HYPERGRAPH-k-PARTITION
and ENUM-HYPERGRAPH-k-CUT are equivalent to enumerating optimum solutions for global minimum
cut in hypergraphs. For graphs that are connected, it is well-known that the number of minimum cuts and the number of α-approximate minimum cuts are at most \( \binom{n}{2} \) and \( O(n^{2\alpha}) \) respectively and they can all be enumerated in polynomial time for constant \( \alpha \) [17,21,25,33,36,48]. For connected hypergraphs, the number of minimum cuts can be exponential as seen from the spanning-hyperedge-example. However, recent results have shown that the number of minimum cut-sets in a hypergraph is at most \( \binom{n}{2} \) via several different techniques and they can all be enumerated in polynomial time [7,14,18,23,24]. On the other hand, there exist hypergraphs with exponential number of 2-approximate minimum cut-sets [4].

GRAPH-k-CUT and ENUM-GRAPH-k-CUT. When k is part of input, GRAPH-k-CUT is NP-hard [26] and admits a 2(1 – 1/k)-approximation [30,53,54,55,61]. Manurangsi [17] showed that there is no polynomial-time \( (2 - \epsilon) \)-approximation for any constant \( \epsilon > 0 \) assuming the Small Set Expansion Hypothesis [53]. We note that GRAPH-k-CUT is \( \text{W}[1] \)-hard when parameterized by k [22] and admits a fixed-parameter approximation scheme when parameterized by k [29,30,41,45], and is fixed-parameter tractable when parameterized by k and the solution size [40].

GRAPH-k-CUT for fixed k was shown to be polynomial-time solvable by Goldschmidt and Hochbaum [26]. Subsequently, Karger and Stein [38] gave a randomized polynomial-time algorithm whose analysis also showed that the number of optimum k-partitions in a connected graph is \( O(n^{2k-2}) \) and they can all be enumerated in polynomial time for every fixed k (also see [17,34,57,58]). The number of optimum k-partitions has recently been improved to \( O(n^k) \) for fixed k thereby leading to a faster algorithm for GRAPH-k-CUT for fixed k [28,31,32].

Consider the n-vertex hypergraph \( G = (V, E) \) obtained from the complete bipartite graph \( K_{n/2,n/2} = (L \cup R, E') \) by adding \( n^2/4 \) copies of two hyperedges \( e_1 \) and \( e_2 \), where \( e_1 := L \) and \( e_2 := R \).
Hypergraph-$k$-Cut and Enum-Hypergraph-$k$-Cut. When $k$ is part of input, Hypergraph-$k$-Cut is at least as hard as the densest $k$-subgraph problem [16]. Combined with results in [46], this implies that Hypergraph-$k$-Cut is unlikely to have a sub-polynomial factor approximation ratio. Moreover, Hypergraph-$k$-Cut is $W[1]$-hard even when parameterized by $k$ and the solution size (see [11]). These two hardness results illustrate that Hypergraph-$k$-Cut differs significantly from Graph-$k$-Cut in complexity.

Hypergraph-$k$-Cut for fixed $k$ was recently shown to be polynomial-time solvable via a randomized algorithm [14, 23]. The analysis of the randomized algorithm also showed that the number of $\min-k$-cut-sets is $O(n^{2k-2})$ and they can all be enumerated in randomized polynomial time. A deterministic polynomial-time algorithm was given by Beideman, Chandrasekaran, and Wang [7]. Subsequently, a deterministic polynomial-time algorithm for Enum-Hypergraph-$k$-Cut for fixed $k$ was given by Beideman, Chandrasekaran, and Wang [7].

Minmax-Graph-$k$-Partition and Enum-MinMax-Graph-$k$-Partition. When $k$ is part of input, Minmax-Graph-$k$-Partition is NP-hard [13] while its approximability is open—we do not yet know if it admits a constant factor approximation. When parameterized by $k$, it is $W[1]$-hard and admits a fixed-parameter approximation scheme [13].

Minmax-Graph-$k$-Partition for fixed $k$ is polynomial-time solvable via the following observation (see [12, 13]): in connected graphs, an optimum $k$-partition for Minmax-Graph-$k$-Partition is a $k$-approximate solution to Graph-$k$-Cut. The randomized algorithm of Karger and Stein implies that the number of $k$-approximate solutions to Graph-$k$-Cut is $n^{O(k^2)}$ and they can all be enumerated in polynomial time [17, 28, 32, 38]. These two facts together imply that Minmax-Graph-$k$-Partition can be solved in time $n^{O(k^2)}$ and moreover, the number of optimum $k$-partitions for Minmax-Graph-$k$-Partition in a connected graph is $n^{O(k^2)}$ and they can all be enumerated in polynomial time for constant $k$.

Minmax-Hypergraph-$k$-Partition and Enum-MinMax-Hypergraph-$k$-Partition. Minmax-Hypergraph-$k$-Partition was discussed as early as 1973 by Lawler [43]. When $k$ is part of input, Minmax-Hypergraph-$k$-Partition is at least as hard as the densest $k$-subgraph problem (this follows from the reduction in [16] and was observed by [15]). For fixed $k$, the approach for Minmax-Graph-$k$-Partition described above does not extend to Minmax-Hypergraph-$k$-Partition. This is because, the number of $k$-approximate solutions to Hypergraph-$k$-Cut can be exponential and hence, they cannot be enumerated efficiently (e.g., we have already seen that the number of 2-approximate minimum cut-sets in a hypergraph can be exponential). Chandrasekaran and Chekuri [12] gave a deterministic polynomial-time algorithm for the more general problem of Minmax-SymSubmod-$k$-Partition for fixed $k$ which in turn, implies that Minmax-Hypergraph-$k$-Partition is also solvable efficiently for fixed $k$. Their algorithm finds an optimum $k$-partition for Minmax-Hypergraph-$k$-Partition and is not conducive to enumerate all Minmax-$k$-cut-sets. We emphasize that no polynomial bound on the number of Minmax-$k$-cut-sets for fixed $k$ was known prior to our work.

For a detailed discussion on other hypergraph $k$-partitioning problems that are special cases of Minsum-Submod-$k$-Partition, we refer the reader to [50, 60, 61].

1.4 Preliminaries

Let $G = (V, E)$ be a hypergraph. Throughout, we will follow the notation mentioned in the second paragraph of Section 1.2. For disjoint $A, B \subseteq V$, we define $E(A, B) := \{e \in E : e \subseteq A \cup B, e \cap A \neq \emptyset, e \cap B \neq \emptyset\}$, and $E[A] := \{e \in E : e \subseteq A\}$. We will repeatedly rely on the fact that the hypergraph cut function $d : 2^V \to \mathbb{R}_+$ is symmetric and submodular. We recall that a
set function $f : 2^V \to \mathbb{R}$ is symmetric if $f(U) = f(\overline{U})$ for all subsets $U \subseteq V$ and is submodular if $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for all subsets $A, B \subseteq V$.

We will need the following partition uncrossing theorem that was proved in previous works on Hypergraph-$k$-Cut and Enum-Hypergraph-$k$-Cut (see Figure 1 for an illustration of the sets that appear in the statement of Theorem 1.5):

**Theorem 1.5.** Let $G = (V, E)$ be a hypergraph, $k \geq 2$ be an integer and $\emptyset \neq R \subseteq U \subseteq V$. Let $S = \{u_1, \ldots, u_p\} \subseteq U \setminus R$ for $p \geq 2k - 2$. Let $(A_i, A_i)$ be a minimum $((S \cup R) \setminus \{u_i\}, \overline{U})$-terminal cut. Suppose that $u_i \in A_i \setminus (\cup_{j \in [p] \setminus \{i\}} A_j)$ for every $i \in [p]$. Then, the following two hold:

1. There exists a $k$-partition $(P_1, \ldots, P_k)$ of $V$ with $U \subseteq P_k$ such that $|\delta(P_1, \ldots, P_k)| \leq \frac{1}{2} \min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j\}$.

2. Moreover, if there exists a hyperedge $e \in E$ such that $e$ intersects $W := \cup_{1 \leq i < j \leq p} (A_i \cap A_j)$, $e$ intersects $Z := \cap_{i \in [p]} A_i$, and $e$ is contained in $W \cup Z$, then the inequality in the previous conclusion is strict.

![Figure 1: Illustration of the sets that appear in the statement of Theorem 1.5](image)

**2 A special case of Theorem 1.4**

The following is the main theorem of this section. Theorem 2.1 implies Theorem 1.4 in the special case where the 2-partition $(U, \overline{U})$ of interest to Theorem 1.4 is such that $|\overline{U}| \leq 2k - 1$.

**Theorem 2.1.** Let $G = (V, E)$ be a hypergraph and let $OPT_{k\text{-cut}}$ be the optimum value of Hypergraph-$k$-Cut in $G$ for some integer $k \geq 2$. Suppose $(U, \overline{U})$ is a 2-partition of $V$ with $d(U) = OPT_{k\text{-cut}}$. Then, there exists a set $S \subseteq U$ with $|S| \leq 2k - 1$ such that every minimum $(S, \overline{U})$-terminal cut $(A, \overline{A})$ satisfies $\delta(A) = \delta(U)$.

**Proof.** Consider the collection

$$\mathcal{C} := \{Q \subseteq V : \overline{U} \subseteq Q, d(Q) \leq d(U), \text{ and } \delta(Q) \neq \delta(U)\}.$$ 

Let $S$ be an inclusion-wise minimal subset of $U$ such that $S \cap Q \neq \emptyset$ for all $Q \in \mathcal{C}$, i.e., the set $S$ is completely contained in $U$ and is a minimal transversal of the collection $\mathcal{C}$. Proposition 2.1 and Lemma 2.1 complete the proof of Theorem 2.1 for this choice of $S$. \qed
Proposition 2.1. Every minimum $(S, \overline{U})$-terminal cut $(A, \overline{A})$ has $\delta(A) = \delta(U)$.

Proof. Let $(A, \overline{A})$ be a minimum $(S, \overline{U})$-terminal cut. If $A = U$, then we are done, so we may assume that $A \neq U$. This implies that $S \subseteq A$ and $\overline{U} \subseteq \overline{A}$. Since $(U, \overline{U})$ is a $(S, \overline{U})$-terminal cut, we have that $d(\overline{A}) = d(A) \leq d(U)$. Since $S$ intersects every set in the collection $C$, we have that $\overline{A} \not\subseteq C$. Hence, $\delta(\overline{A}) = \delta(U)$, and by symmetry of cut-sets, $\delta(A) = \delta(U)$.

\[ \square \]

Lemma 2.1. The size of the subset $S$ is at most $2k - 1$.

Proof. For the sake of contradiction, suppose $|S| \geq 2k$. Our proof strategy is to show the existence of a $k$-partition with fewer crossing hyperedges than $OPT_{k\text{-cut}}$, thus contradicting the definition of $OPT_{k\text{-cut}}$. Let $S := \{u_1, u_2, \ldots, u_p\}$ for some $p \geq 2k$. For notational convenience, we will use $S - u_i$ to denote $S \setminus \{u_i\}$ and $S - u_i - u_j$ to denote $S \setminus \{u_i, u_j\}$. For a subset $X \subseteq U$, we denote the source minimal $(X, \overline{U})$-terminal cut by $(H_X, \overline{H_X})$.

Our strategy to arrive at a $k$-partition with fewer crossing hyperedges than $OPT_{k\text{-cut}}$ is to apply the second conclusion of Theorem 1.5. The next few claims will set us up to obtain sets that satisfy the hypothesis of Theorem 1.5.

Claim 2.1. For every $i \in [p]$, we have $\overline{H_{S - u_i}} \in C$.

Proof. Let $i \in [p]$. Since $S$ is a minimal transversal of the collection $C$, there exists a set $B_i \in C$ such that $B_i \cap S = \{u_i\}$. Hence, $(\overline{B_i}, B_i)$ is a $(S - u_i, \overline{U})$-terminal cut. Therefore,

$$d(\overline{H_{S - u_i}}) \leq d(B_i) \leq d(U).$$

Since $(H_{S - u_i}, \overline{H_{S - u_i}})$ is a $(S - u_i, \overline{U})$-terminal cut, we have that $\overline{U} \subseteq \overline{H_{S - u_i}}$. If $d(\overline{H_{S - u_i}}) < d(U)$, then $\delta(\overline{H_{S - u_i}}) \neq \delta(U)$ and $\overline{U} \not\subseteq \overline{H_{S - u_i}}$, and consequently, $\overline{H_{S - u_i}} \not\subseteq C$. So, we will assume henceforth that $d(\overline{H_{S - u_i}}) = d(U)$.

Since $(H_{S - u_i} \cap B_i, \overline{H_{S - u_i}} \cap \overline{B_i})$ is a $(S - u_i, \overline{U})$-terminal cut, we have that

$$d(H_{S - u_i} \cap \overline{B_i}) \geq d(H_{S - u_i}).$$

Since $(H_{S - u_i} \cup B_i, \overline{H_{S - u_i}} \cup \overline{B_i})$ is a $(S - u_i, \overline{U})$-terminal cut, we have that

$$d(H_{S - u_i} \cup \overline{B_i}) \geq d(H_{S - u_i}).$$

Therefore, by submodularity of the hypergraph cut function, we have that

$$2d(U) \geq d(H_{S - u_i}) + d(B_i) \geq d(H_{S - u_i} \cap \overline{B_i}) + d(H_{S - u_i} \cup \overline{B_i}) \geq 2d(H_{S - u_i}) = 2d(U).$$

Therefore, all inequalities above should be equations. In particular, we have that $d(H_{S - u_i} \cap \overline{B_i}) = d(U) = d(B_i) = d(H_{S - u_i})$ and hence, $(H_{S - u_i} \cap \overline{B_i}, H_{S - u_i} \cap \overline{B_i})$ is a minimum $(S - u_i, \overline{U})$-terminal cut. Since $(H_{S - u_i}, \overline{H_{S - u_i}})$ is a source minimal $(S - u_i, \overline{U})$-terminal cut, we must have $H_{S - u_i} \cap \overline{B_i} = H_{S - u_i}$, and thus $H_{S - u_i} \subseteq \overline{B_i}$. Therefore, $B_i \subseteq \overline{H_{S - u_i}}$. Since $B_i \in C$, we have $\delta(B_i) = \delta(U)$. However, $d(B_i) = d(U)$. Therefore $\delta(U) \setminus \delta(B_i) \neq \emptyset$. Let $e \in \delta(U) \setminus \delta(B_i)$. Since $e \in \delta(U)$, but $e \not\in \delta(B_i)$, and $U \subseteq B_i$, we have that $e \subseteq B_i$, and thus $e \subseteq \overline{H_{S - u_i}}$. Thus, we conclude that $\delta(U) \setminus \delta(H_{S - u_i}) \neq \emptyset$, and so $\delta(U) \setminus \delta(H_{S - u_i}) \neq \emptyset$. This also implies that $U \not\subseteq \overline{H_{S - u_i}}$. Thus, $\overline{H_{S - u_i}} \not\subseteq C$. 

Claim 2.1 implies the following Corollary.
Corollary 2.1. For every \( i \in [p] \), we have \( u_i \in H_{S-u_i} \).

Proof. By definition, \( S - u_i \subseteq H_{S-u_i} \), so \( S \cap H_{S-u_i} \subseteq \{u_i\} \). By Claim 2.1 we have that \( H_{S-u_i} \in C \). Since \( S \) is a transversal of the collection \( C \), we have that \( S \cap H_{S-u_i} \neq \emptyset \). So, the vertex \( u_i \) must be in \( H_{S-u_i} \).

Having obtained Corollary 2.1 the next few claims (Claims 2.2, 2.3, 2.4, and 2.5) are similar to the claims appearing in the proof of a structural theorem that appeared in [7]. Since the hypothesis of the structural theorem that we are proving here is different from theirs, we present the complete proofs of these claims here. The way in which we use the claims will also be different from [7].

The following claim will help in showing that \( u_i, u_j \not\in H_{S-u_i-u_j} \), which in turn, will be used to show that the hypothesis of Theorem 1.5 is satisfied by suitably chosen sets.

Claim 2.2. For every \( i, j \in [p] \), we have \( H_{S-u_i-u_j} \subseteq H_{S-u_i} \).

Proof. We may assume that \( i \neq j \). We note that \( (H_{S-u_i-u_j} \cap H_{S-u_i}, \overline{H_{S-u_i-u_j} \cap H_{S-u_i}}) \) is a \((S - u_i - u_j, U)\)-terminal cut. Therefore,

\[ d(H_{S-u_i-u_j} \cap H_{S-u_i}) \geq d(H_{S-u_i-u_j}). \] (2)

Also, \( (H_{S-u_i-u_j} \cup H_{S-u_i}, \overline{H_{S-u_i-u_j} \cup H_{S-u_i}}) \) is a \((S - u_i, U)\)-terminal cut. Therefore,

\[ d(H_{S-u_i-u_j} \cup H_{S-u_i}) \geq d(H_{S-u_i}). \] (3)

By submodularity of the hypergraph cut function and inequalities (2) and (3), we have that

\[
d(H_{S-u_i-u_j}) + d(H_{S-u_i}) \geq d(H_{S-u_i-u_j} \cap H_{S-u_i}) + d(H_{S-u_i-u_j} \cup H_{S-u_i}) \\
\geq d(H_{S-u_i-u_j}) + d(H_{S-u_i}).
\]

Therefore, inequality (2) is an equation, and consequently, \( (H_{S-u_i-u_j} \cap H_{S-u_i}, \overline{H_{S-u_i-u_j} \cap H_{S-u_i}}) \) is a minimum \((S - u_i - u_j, U)\)-terminal cut. If \( H_{S-u_i-u_j} \setminus H_{S-u_i} \neq \emptyset \), then \( (H_{S-u_i-u_j} \cap H_{S-u_i}, \overline{H_{S-u_i-u_j} \cap H_{S-u_i}}) \) contradicts source minimality of the minimum \((S - u_i - u_j, U)\)-terminal cut \( (H_{S-u_i-u_j}, \overline{H_{S-u_i-u_j}}) \). Hence, \( H_{S-u_i-u_j} \setminus H_{S-u_i} = \emptyset \) and consequently, \( H_{S-u_i-u_j} \subseteq H_{S-u_i} \).

Claim 2.2 implies the following Corollary.

Corollary 2.2. For every \( i, j \in [p] \), we have \( u_i, u_j \not\in H_{S-u_i-u_j} \).

Proof. By Corollary 2.1 we have that \( u_i \not\in H_{S-u_i} \). Therefore, \( u_i, u_j \not\in H_{S-u_i} \cap H_{S-u_j} \). By Claim 2.2, \( H_{S-u_i-u_j} \subseteq H_{S-u_i} \) and \( H_{S-u_i-u_j} \subseteq H_{S-u_j} \). Therefore, \( H_{S-u_i-u_j} \subseteq H_{S-u_i} \cap H_{S-u_j} \), and thus, \( u_i, u_j \not\in H_{S-u_i-u_j} \).

The next claim will help in controlling the cost of the \( k \)-partition that we will obtain by applying Theorem 1.5.

Claim 2.3. For every \( i, j \in [p] \), we have \( d(H_{S-u_i}) = d(U) = d(H_{S-u_i-u_j}) \).
Thus, we have that $d(H_{S-u_a}) = d(U) = d(H_{S-u_{a-b}})$. Since $(U, \overline{U})$ is a $(S-u_a, \overline{U})$-terminal cut, we have that $d(H_{S-u_a}) \leq d(U)$. Since $(H_{S-u_a}, \overline{H_{S-u_a}})$ is a $(S-u_a, \overline{U})$-terminal cut, we have that $d(H_{S-u_a-u_b}) \leq d(H_{S-u_a}) \leq d(U)$. Thus, in order to prove the claim, it suffices to show that $d(H_{S-u_a-u_b}) \geq d(U)$.

Suppose for contradiction that $d(H_{S-u_a-u_b}) < d(U)$. Let $\ell \in [p] \setminus \{a, b\}$ be an arbitrary element (which exists since we have assumed that $p \geq 2k$ and $k \geq 2$). Let $R := \{ u_\ell \}$, $S' := S - u_a - u_\ell$, and $A_i := \overline{H_{S-u_a-u_i}}$ for every $i \in [p] \setminus \{a, \ell\}$. We note that $|S'| = p - 2 \geq 2k - 2$. By definition, $(A_i, A_j)$ is a minimum $(S - u_a - u_i, \overline{U})$-terminal cut for every $i \in [p] \setminus \{a, \ell\}$. Moreover, by Corollary 2.2, we have that $u_i \in A_i \setminus (\cup_{j \in [p] \setminus \{a, \ell\}} A_j)$ for every $i \in [p] \setminus \{a, \ell\}$. Hence, the sets $U$, $R$, and $S'$, and the cuts $(A_i, A_j)$ for $i \in [p] \setminus \{a, \ell\}$ satisfy the conditions of Theorem 1.5. Therefore, by the first conclusion of Theorem 1.5, there exists a $k$-partition $\mathcal{P}'$ with

$$|\delta(\mathcal{P}')| \leq \frac{1}{2} \min\{d(H_{S-u_a-u_i}) + d(H_{S-u_{a-b}}) : i, j \in [p] \setminus \{a, \ell\}\}.$$ 

By assumption, $d(H_{S-u_a-u_b}) < d(U)$ and $b \in [p] \setminus \{a, \ell\}$, so $\min\{d(H_{S-u_a-u_i}) : i \in [p] \setminus \{a, \ell\}\} < d(U)$. Since $(U, \overline{U})$ is a $(S - u_a - u_i, \overline{U})$-terminal cut, we have that $d(H_{S-u_a-u_i}) \leq d(U)$ for every $i \in [p] \setminus \{a, \ell\}$. Therefore,

$$\frac{1}{2} \min\{d(H_{S-u_a-u_i}) + d(H_{S-u_{a-b}}) : i, j \in [p] \setminus \{a, \ell\}\} < d(U) = OPT_{k\text{-cut}}.$$ 

Thus, we have that $|\delta(\mathcal{P}')| < OPT_{k\text{-cut}}$, which is a contradiction.

The next two claims will help in arguing the existence of a hyperedge satisfying the conditions of the second conclusion of Theorem 1.5. In particular, we will need Claim 2.5. The following claim will help in proving Claim 2.5.

**Claim 2.4.** For every $i, j \in [p]$, we have

$$d(H_{S-u_i} \cap H_{S-u_j}) = d(U) = d(H_{S-u_i} \cup H_{S-u_j}).$$

**Proof.** Since $(H_{S-u_i} \cap H_{S-u_j}, \overline{H_{S-u_i} \cap H_{S-u_j}})$ is a $(S - u_i - u_j, \overline{U})$-terminal cut, we have that $d(H_{S-u_i} \cap H_{S-u_j}) \geq d(H_{S-u_i-u_j})$. By Claim 2.3, we have that $d(H_{S-u_i-u_j}) = d(U) = d(H_{S-u_i})$. Therefore,

$$d(H_{S-u_i} \cap H_{S-u_j}) \geq d(H_{S-u_i}). \tag{4}$$

Since $(H_{S-u_i} \cup H_{S-u_j}, \overline{H_{S-u_i} \cup H_{S-u_j}})$ is a $(S - u_j, \overline{U})$-terminal cut, we have that

$$d(H_{S-u_i} \cup H_{S-u_j}) \geq d(H_{S-u_j}). \tag{5}$$

By submodularity of the hypergraph cut function and inequalities (4) and (5), we have that

$$d(H_{S-u_i}) + d(H_{S-u_j}) \geq d(H_{S-u_i} \cap H_{S-u_j}) + d(H_{S-u_i} \cup H_{S-u_j}) \geq d(H_{S-u_i}) + d(H_{S-u_j}).$$

Therefore, inequalities (4) and (5) are equations. Thus, by Claim 2.3, we have that

$$d(H_{S-u_i} \cap H_{S-u_j}) = d(H_{S-u_i}) = d(U),$$

and

$$d(H_{S-u_i} \cup H_{S-u_j}) = d(H_{S-u_j}) = d(U).$$

\[\square\]
Claim 2.5. For every $i, j, \ell \in [p]$ with $i \neq j$, we have $H_{S-u_\ell} \subseteq H_{S-u_i} \cup H_{S-u_j}$.

Proof. If $\ell = i$ or $\ell = j$ the claim is immediate. Thus, we assume that $\ell \notin \{i, j\}$. Let $Q := H_{S-u_\ell} \setminus (H_{S-u_i} \cup H_{S-u_j})$. We need to show that $Q = \emptyset$. We will show that $(H_{S-u_\ell} \setminus Q, \overline{H_{S-u_\ell}} \setminus Q)$ is a minimum $(S - u_\ell, \overline{U})$-terminal cut. Consequently, $Q$ must be empty (otherwise, $H_{S-u_\ell} \setminus Q \subseteq H_{S-u_\ell}$ and hence, $(H_{S-u_\ell} \setminus Q, \overline{H_{S-u_\ell}} \setminus Q)$ contradicts source minimality of the minimum $(S - u_\ell, \overline{U})$-terminal cut $(H_{S-u_\ell}, \overline{H_{S-u_\ell}})$.

We now show that $(H_{S-u_\ell} \setminus Q, \overline{H_{S-u_\ell}} \setminus Q)$ is a minimum $(S - u_\ell, \overline{U})$-terminal cut. Since $H_{S-u_\ell} \setminus Q = H_{S-u_\ell} \cap (H_{S-u_i} \cup H_{S-u_j})$, we have that $S - u_i - u_j - u_\ell \subseteq H_{S-u_\ell} \setminus Q$. We also know that $u_i$ and $u_j$ are contained in both $H_{S-u_\ell}$ and $H_{S-u_i} \cup H_{S-u_j}$. Therefore, $S - u_\ell \subseteq H_{S-u_\ell} \setminus Q$. Thus, $(H_{S-u_\ell} \setminus Q, \overline{H_{S-u_\ell}} \setminus Q)$ is a $(S - u_\ell, \overline{U})$-terminal cut. Therefore,

$$d(H_{S-u_\ell} \cap (H_{S-u_i} \cup H_{S-u_j})) = d(H_{S-u_\ell} \setminus Q) \geq d(H_{S-u_\ell}).$$

(6)

We also have that $(H_{S-u_\ell} \cup (H_{S-u_i} \cup H_{S-u_j}), \overline{H_{S-u_\ell}} \cup (H_{S-u_i} \cup H_{S-u_j}))$ is a $(S - u_\ell, \overline{U})$-terminal cut. Therefore, $d(H_{S-u_\ell} \cup (H_{S-u_i} \cup H_{S-u_j})) \geq d(H_{S-u_\ell})$. By Claims 2.3 and 2.4 we have that $d(H_{S-u_\ell}) = d(V_1) = d(H_{S-u_i} \cup H_{S-u_j})$. Therefore,

$$d(H_{S-u_\ell} \cup (H_{S-u_i} \cup H_{S-u_j})) \geq d(H_{S-u_i} \cup H_{S-u_j}).$$

(7)

By submodularity of the hypergraph cut function and inequalities (6) and (7), we have that

$$d(H_{S-u_\ell}) + d(H_{S-u_i} \cup H_{S-u_j}) \geq d(H_{S-u_\ell} \cap (H_{S-u_i} \cup H_{S-u_j})) + d(H_{S-u_\ell} \cup (H_{S-u_i} \cup H_{S-u_j})) \geq d(H_{S-u_\ell}) + d(H_{S-u_i} \cup H_{S-u_j}).$$

Therefore, inequalities (6) and (7) are equations, so $(H_{S-u_\ell} \setminus Q, \overline{H_{S-u_\ell}} \setminus Q)$ is a minimum $(S - u_\ell, \overline{U})$-terminal cut. 

Let $R := \{u_p\}$, $S' := S - u_p$, and $(\overline{A_i}, A_i) := (H_{S-u_i}, \overline{H_{S-u_i}})$ for every $i \in [p - 1]$. By definition, $(\overline{A_i}, A_i)$ is a minimum $(S - u_i, \overline{U})$-terminal cut for every $i \in [p - 1]$. Moreover, by Corollary 2.1 we have that $u_i \in A_i \setminus (\cup_{j \in [p-1]} \{j\})A_j).$ Hence, the sets $U$, $R$, and $S'$, and the cuts $(\overline{A_i}, A_i)$ for $i \in [p-1]$ satisfy the conditions of Theorem 1.5. We will use the second conclusion of Theorem 1.5. We now show that there exists a hyperedge satisfying the conditions mentioned in the second conclusion of Theorem 1.5. We will use Claim 2.6 below to prove this. Let $W := \cup_{1 \leq i < j \leq p-1} (A_i \cap A_j)$ and $Z := \cap_{i \in [p-1]} A_i$ as in the statement of Theorem 1.5.

Claim 2.6. There exists a hyperedge $e \in E$ such that $e \cap W \neq \emptyset$, $e \cap Z \neq \emptyset$, and $e \subseteq U \cup Z$.

Proof. We note that $S \subseteq (S - u_i) \cup (S - u_j) \subseteq H_{S-u_i} \cup H_{S-u_j}$ for every distinct $i, j \in [p - 1]$. Therefore, $S \cap (A_i \cap A_j) = \emptyset$ for every distinct $i, j \in [p - 1]$, and thus $S \cap W = \emptyset$. Since $S$ is a transversal of the collection $\mathcal{C}$, it follows that the set $W$ is not in the collection $\mathcal{C}$.

By definition, $\overline{U} \subseteq A_i$ for every $i \in [p-1]$, and thus $\overline{U} \subseteq W$. Since $W \notin \mathcal{C}$, either $d(W) > d(U)$ or $\delta(W) = \delta(U)$. By Claim 2.1, we have that $\overline{H_{S-u_p}} \subseteq \mathcal{C}$, and thus, $d(\overline{H_{S-u_p}}) \leq d(U)$ and $\delta(\overline{H_{S-u_p}}) \neq \delta(U)$. Consequently, $d(W) \geq d(\overline{H_{S-u_p}})$, and $\delta(W) \neq \delta(\overline{H_{S-u_p}})$, and thus, $d(W) \setminus \delta(\overline{H_{S-u_p}}) \neq \emptyset$. Let $e \in d(W) \setminus \delta(\overline{H_{S-u_p}})$. We will show that this choice of $e$ achieves the desired properties.

For each $i \in [p]$, let $Y_i := \overline{H_{S-u_i}} \setminus W$. By Claim 2.5, for every $i, j, \ell \in [p]$ with $i \neq j$ we have that $H_{S-u_\ell} \subseteq H_{S-u_i} \cup H_{S-u_j}$. Therefore $\overline{H_{S-u_\ell}} \cap \overline{H_{S-u_i}} \subseteq \overline{H_{S-u_\ell}}$ for every such $i, j, \ell \in [p]$, and hence $W \subseteq \overline{H_{S-u_\ell}}$ for every $\ell \in [p]$. Thus, $W \subseteq \overline{H_{S-u_\ell}}$. Since $e \in d(W) \setminus \delta(\overline{H_{S-u_p}})$, we have that $e \subseteq W \cup Y_p$ and $e \cap W \neq \emptyset$ and $e \cap Y_p \neq \emptyset$. Therefore, in order to show that $e$ has the three desired properties as in the claim, it suffices to show that $Y_p \subseteq Z$. We prove this next.
Then, there exist sets 

By Theorem 2.1, there exists a subset

Proof. By Theorem 2.1 there exists a subset \( S \subseteq U \) with \( |S| \leq 2k - 1 \) such that every minimum \((S, \overline{U})\)-terminal cut \((A, \overline{A})\) satisfies \( \delta(A) = \delta(U) \).

Claim 3.1. Let \((Y, Y')\) be the source minimal \((S, T)\)-terminal cut. Then \( \delta(Y) = \delta(U) \).

Proof. Since \((U, \overline{U})\) is a \((S, T)\)-terminal cut, and \((Y, Y')\) is a minimum \((S, T)\)-terminal cut, we have that

\[
\delta(U) \geq \delta(Y).
\]

Since \((U \cap Y, \overline{U} \cap Y)\) is a \((S, \overline{U})\)-terminal cut, we have that

\[
d(U \cap Y) \geq d(U).
\]
Since \((U \cup Y, \overline{U} \cup Y)\) is a \((U, T)\)-terminal cut, we have that
\[
d(U \cup Y) \geq d(U).
\]
Thus, by the submodularity of the hypergraph cut function we have that
\[
2d(U) \geq d(U) + d(Y) \geq d(U \cap Y) + d(U \cup Y) \geq 2d(U).
\]
Therefore, we have that \(d(U \cap Y) = d(U)\), so \((U \cap Y, \overline{U} \cap Y)\) is a minimum \((S, T)\)-terminal cut. Since \((Y, \overline{Y})\) is the source minimal \((S, T)\)-terminal cut, we have that \(U \cap Y = Y\), and hence \(Y \subseteq U\). Therefore, \((Y, \overline{Y})\) is a minimum \((S, \overline{U})\)-terminal cut. By the choice of \(S\), we have that \(\delta(Y) = \delta(U)\).

Applying Claim 3.1 to both sides of the partition \((U, \overline{U})\), we have that the source minimal minimum \((S, T)\)-terminal cut \((Y, \overline{Y})\) has \(\delta(Y) = \delta(U)\), and the source minimal minimum \((T, S)\)-terminal cut \((Z, \overline{Z})\) has \(\delta(Z) = \delta(U)\). Therefore, for every \(e \in \delta(U)\), we have that \(e \cap Y \neq \emptyset\) and \(e \cap Z \neq \emptyset\).

Let \((A, \overline{A})\) be a minimum \((S, T)\)-terminal cut. Since \((Y, \overline{Y})\) is the source minimal minimum \((S, T)\)-terminal cut, we have that \(Y \subseteq A\). Since \((Z, \overline{Z})\) is the source minimal minimum \((T, S)\)-terminal cut, we have that \(Z \subseteq \overline{A}\). Since every \(e \in \delta(U)\) intersects both \(Y\) and \(Z\), it follows that every \(e \in \delta(U)\) intersects both \(A\) and \(\overline{A}\), and hence, \(\delta(U) \subseteq \delta(A)\). Since \((A, \overline{A})\) is a minimum \((S, T)\)-terminal cut, \(d(A) \leq d(U)\), and thus we have that \(\delta(A) = \delta(U)\).

4 Algorithm for Enum-Hypergraph-\(k\)-Cut

In this section, we design a deterministic algorithm for Enum-Hypergraph-\(k\)-Cut that is based on divide and conquer and has a run-time of \(n^{O(k)}\) source minimal minimum \((s, t)\)-terminal cut computations, where \(n\) is the number of vertices in the input hypergraph. The high-level idea is to use minimum \((S, T)\)-terminal cuts to enumerate a collection of candidate cuts such that for every optimum \(k\)-partition for Hypergraph-\(k\)-Cut, either the union of some \(k/2\) parts of the optimum \(k\)-partition is contained in the candidate collection or we find the set of hyperedges crossing this optimum \(k\)-partition. This helps in cutting the recursion depth to \(\log k\) which saves on overall run-time. We describe the algorithm in Figure 2 and its guarantees in Theorem 4.1. We recall that for a hypergraph \(G = (V, E)\) and a subset \(A \subseteq V\), the subgraph \(G[A]\) induced by \(A\) is given by \(G[A] = (A, E')\), where \(E' := \{e \in E : e \subseteq A\}\).

Theorem 4.1 is a self-contained proof that the number of MIN-\(k\)-CUT-SETS in a \(n\)-vertex hypergraph is \(O(n^{8k \log k})\) and the run-time of the algorithm in Figure 2 is \(O(n^{8k \log k})\) source minimal minimum \((s, t)\)-terminal cut computations. In Lemma 4.1 we improve the run-time analysis of the same algorithm to \(O(n^{16k})\) source minimal minimum \((s, t)\)-terminal cut computations. For this, we exploit the known fact that the number of MIN-\(k\)-CUT-SETS in a \(n\)-vertex hypergraph is \(O(n^{2k-2})\) (via the randomized algorithm in [14]).

Theorem 4.1 and Lemma 4.1 together imply Theorem 4.2 since the source minimal minimum \((s, t)\)-terminal cut in a \(n\)-vertex hypergraph of size \(p\) can be computed in time \(O(np)\) [18].

**Theorem 4.1.** Let \(G = (V, E)\) be a \(n\)-vertex hypergraph of size \(p\) and let \(k\) be a positive integer. Then, Algorithm Enum-Cut-Sets\((G, k)\) in Figure 3 returns the family of all MIN-\(k\)-CUT-SETS in \(G\) and it can be implemented to run in time \(O(n^{(8k-6) \log k})T(n, p)\), where \(T(n, p)\) denotes the time complexity for computing the source minimal minimum \((s, t)\)-terminal cut in a \(n\)-vertex hypergraph of size \(p\). Moreover, the cardinality of the family returned by the algorithm is \(O(n^{(8k-6) \log k})\).
U := F that minimize the returned family. We now show the induction step. Assume that subfamily is indeed a min-cut-set. We begin by showing correctness. The last step of the algorithm considers only min-cut-sets. We show this by induction. For the base case of $k = 1$, the only min-cut-set is the empty set which is contained in the returned family. We now show the induction step. Assume that $k \geq 2$. Let $F \subseteq E$ be a min-cut-set in $G$ and let $(V_1, \ldots, V_k)$ be an optimum $k$-partition for Hypergraph-$k$-Cut such that $F = \delta(V_1, \ldots, V_k)$. We will show that $F$ is in the family returned by the algorithm. Let $U := \bigcup_{i=1}^{\lfloor k/2 \rfloor} V_i$. We distinguish between the following two cases:

1. Suppose $d(U) < OPT_{k\text{-cut}}$.

   By Theorem 1.3 there exist disjoint subsets $S, T \subseteq V$ with $|S|, |T| \leq 2k - 2$ such that $(U, \overline{U})$ is the unique minimum $(S, T)$-terminal cut. Hence, the set $U$ is in the collection $C$. Moreover, $U$ contains $\lfloor k/2 \rfloor$ non-empty sets $V_1, V_2, \ldots, V_{\lfloor k/2 \rfloor}$, so we have $|U| \geq \lfloor k/2 \rfloor$. Similarly, we have $|\overline{U}| \geq k - \lfloor k/2 \rfloor$. Since $(V_1, \ldots, V_k)$ is an optimum $k$-partition for Hypergraph-$k$-Cut, the set $\{e \in F \setminus \delta(U) : e \subseteq U\}$ is a min-$\lfloor k/2 \rfloor$-cut-set in $G[U]$. Similarly, the set $\{e \in F \setminus \delta(U) : e \subseteq \overline{U}\}$ is a min-$(k - \lfloor k/2 \rfloor)$-cut-set in $G[\overline{U}]$. Since $d(U) < OPT_{k\text{-cut}}$, we know that $G - \delta(U)$ has less than $k$ connected components. Therefore, the set $U$ is in the collection $C$. By induction hypothesis, we know that the set $\{e \in F \setminus \delta(U) : e \subseteq U\}$ is contained in the family $\mathcal{F}_U$ and the set $\{e \in F \setminus \delta(U) : e \subseteq \overline{U}\}$ is contained in the family $\mathcal{F}_U$. Therefore, the set $F$ is added to the family $\mathcal{F}$ in the second for-loop.

2. Suppose $d(U) = OPT_{k\text{-cut}}$.

   By Theorem 1.4 there exist sets $S \subseteq U$ and $T \subseteq \overline{U}$ with $|S|, |T| \leq 2k - 1$ such that the source minimal minimum $(S, T)$-terminal cut $(A, \overline{A})$ satisfies $\delta(A) = \delta(U) = F$. Therefore, the set $A$ is in the collection $C$. Since $F = \delta(A)$, the hypergraph $G - \delta(A)$ contains at least $k$ connected components. Therefore, the set $F = \delta(A)$ is added to the family $\mathcal{F}$ in the first for-loop.

Thus, in both cases, we have shown that the set $F$ is contained in the family $\mathcal{F}$. Since the algorithm
returns the subfamily of hyperedge sets in \( \mathcal{F} \) that are \textsc{min-}\(k\)-cut-\textsc{sets}, the set \( F \) is in the family returned by the algorithm.

Next, we bound the run time of the algorithm. Let \( N(k,n) \) denote the run time of the algorithm for a \( n \)-vertex hypergraph. Then, we have \( N(1,n) = O(1) \). For \( k \geq 2 \), there are \( O(n^{4k-2}) \) pairs of subsets \( S,T \subseteq V \) with \( |S|, |T| \leq 2k-1 \) and \( S \cap T = \emptyset \). Hence, the first for-loop performs \( O(n^{4k-2}) \) source minimal minimum \((S,T)\)-terminal cut computations. The collection \( \mathcal{C} \) and the family \( \mathcal{F} \) at the end of the first for-loop each have \( O(n^{4k-2}) \) sets. This implies that the first for-loop can be implemented to run in \( O(n^{4k-2})T(n,p) \) time. For each \( A \in \mathcal{C} \), the computation of Enum-Cut-Sets(\( G[A], [k/2] \)) in the second for-loop runs in \( N(|k/2|,n) \) time. The computation of Enum-Cut-Sets(\( G[\overline{A}], k-[k/2] \)) in the second for-loop runs in \( N(k-[k/2],n) \) time. Hence, the second for-loop can be implemented to run in \( O(n^{4k-2})N(|k/2|,n)N(k-[k/2],n) \) time. The last step to prune the family \( \mathcal{F} \) can be implemented to run in time that is linear in the time to implement the first and second for-loops: this is because for each member of \( \mathcal{F} \), we can decide whether it is a minimum \(k\)-cut-set in time linear in the time to write this member in \( \mathcal{F} \). Therefore, we have

\[
N(k,n) = O\left(n^{4k-2}\right)T(n,p) + O\left(n^{4k-2}\right)N\left(\left\lfloor \frac{k}{2} \right\rfloor , n\right)N\left(k - \left\lfloor \frac{k}{2} \right\rfloor , n\right).
\]

Since \( N(1,n) = O(1) \), we have that \( N(k,n) = O\left(n^{(8k-6)\log k}\right)T(n,p) \).

Finally, we bound the cardinality of the family returned by the algorithm. Let \( f(k,n) \) be the cardinality of the family returned by the algorithm for a \( n \)-vertex hypergraph. We note that \( f(k,n) \) is at most the cardinality of the family \( \mathcal{F} \) computed by the algorithm. There are \( O(n^{4k-2}) \) pairs of subsets \( S,T \subseteq V \) with \( |S|, |T| \leq 2k-1 \) and \( S \cap T = \emptyset \). Hence, the total cardinality of the collection \( \mathcal{C} \) and the family \( \mathcal{F} \) at the end of the first for-loop is \( O(n^{4k-2}) \). Consequently, for \( k \geq 2 \), by the recursion, we have that

\[
f(k,n) = O\left(n^{4k-2}\right)f\left(\left\lfloor \frac{k}{2} \right\rfloor , n\right)f\left(k - \left\lfloor \frac{k}{2} \right\rfloor , n\right)
\]

and \( f(1,n) = 1 \). So, \( f(k,n) = O\left(n^{(8k-6)\log k}\right) \).

We recall that the number of \textsc{min-}\(k\)-cut-\textsc{sets} in a \( n \)-vertex hypergraph is \( O\left(n^{2k-2}\right) \) \[14\]. Assuming this bound improves the run-time of Algorithm Enum-Cut-Sets(\( G,k \)) in Figure \[2\].

**Lemma 4.1.** Algorithm Enum-Cut-Sets(\( G,k \)) in Figure \[3\] can be implemented to run in time \( O(n^{16k-26})T(n,p) \), where \( n \) is the number of vertices, \( p \) is the size of the input hypergraph \( G \), and \( T(n,p) \) denotes the time complexity for computing the source minimal minimum \((s,t)\)-terminal cut in a \( n \)-vertex hypergraph of size \( p \).

**Proof.** Let \( N(k,n) \) denote the run time of the algorithm for a \( n \)-vertex hypergraph. Then, we have \( N(1,n) = O(1) \). For \( k \geq 2 \), there are \( O(n^{4k-2}) \) pairs of subsets \( S,T \subseteq V \) with \( |S|, |T| \leq 2k-1 \) and \( S \cap T = \emptyset \). Hence, the first for-loop performs \( O(n^{4k-2}) \) source minimal minimum \((S,T)\)-terminal cut computations. The collection \( \mathcal{C} \) and the family \( \mathcal{F} \) at the end of the first for-loop each have \( O(n^{4k-2}) \) sets. This implies that the first for-loop can be implemented to run in \( O(n^{4k-2})T(n,p) \) time. For each \( A \in \mathcal{C} \), the computation of Enum-Cut-Sets(\( G[A], [k/2] \)) in the second for-loop runs in \( N(|k/2|,n) \) time. The computation of Enum-Cut-Sets(\( G[\overline{A}], k-[k/2] \)) in the second for-loop runs in \( N(k-[k/2],n) \) time. We recall that \( \mathcal{F}_A \) consists of all minimum \([k/2]\)-cut-sets in a \( n \)-vertex...
graph and hence, has size $O(n^{2(k/2)^2})$. Similarly, $\mathcal{F}_A'$ has size $O(n^{2(k-\lceil k/2 \rceil)^2})$. Hence, the second for-loop can be implemented to run in time

$$O \left( n^{4k-2} \right) \left( N \left( \left\lfloor \frac{k}{2} \right\rfloor, n \right) + N \left( k - \left\lfloor \frac{k}{2} \right\rfloor, n \right) + O(n^{2(k/2)^2})O(n^{2(k-\lceil k/2 \rceil)^2}) \right)$$

$$= O \left( n^{4k-2} \right) \left( N \left( \left\lfloor \frac{k}{2} \right\rfloor, n \right) + N \left( k - \left\lfloor \frac{k}{2} \right\rfloor, n \right) + O(n^{2k-4}) \right).$$

Moreover, the size of the family $\mathcal{F}$ at the end of the second for-loop is $O(n^{4k-2}) + O(n^{4k-2})|\mathcal{F}_A'|$. $|\mathcal{F}_A'| = O(n^{-2})|\mathcal{F}^{2k-4}| = O(n^{6k-6})$. Hence, the last step to prune the family $\mathcal{F}$ can be implemented to run in time $O(n^{6k-6})$. Hence, the second for-loop and the last step can together be implemented to run in time

$$O \left( n^{4k-2} \right) \left( N \left( \left\lfloor \frac{k}{2} \right\rfloor, n \right) + N \left( k - \left\lfloor \frac{k}{2} \right\rfloor, n \right) + O(n^{2k-4}) \right).$$

Therefore, we have

$$N(k, n) = O \left( n^{4k-2} \right) \left( T(n, p) + N \left( \left\lfloor \frac{k}{2} \right\rfloor, n \right) + N \left( k - \left\lfloor \frac{k}{2} \right\rfloor, n \right) + O(n^{2k-4}) \right).$$

Solving the recursive relation gives $N(k, n) = O(n^{16k-26})T(n, p)$.

\[\Box\]

5 Algorithm for Enum-MinMax-Hypergraph-$k$-Partition

In this section, we design a deterministic algorithm for Enum-MinMax-Hypergraph-$k$-Partition that runs in time $n^{O(k^2)}p$, where $n$ is the number of vertices and $p$ is the size of the input hypergraph. For this, we rely on the notion of $k$-cut-set representatives.

We recall that for a $k$-partition $(V_1, \ldots, V_k)$ and disjoint subsets $U_1, \ldots, U_k \subseteq V$, the $k$-tuple $(U_1, \ldots, U_k)$ is defined to be a $k$-cut-set representative of $(V_1, \ldots, V_k)$ if $U_i \subseteq V_i$ and $\delta(U_i) = \delta(V_i)$ for all $i \in [k]$. We first show that there exists a polynomial-time algorithm to verify whether a given $k$-tuple $(U_1, \ldots, U_k)$ is a $k$-cut-set representative.

**Theorem 5.1.** Let $G = (V, E)$ be a $n$-vertex hypergraph of size $p$ and let $k$ be a positive integer. Then, there exists an algorithm that takes as input the hypergraph $G$ and disjoint subsets $U_1, \ldots, U_k \subseteq V$ and runs in time $O(knp)$ to decide if $(U_1, \ldots, U_k)$ is a $k$-cut-set representative of some $k$-partition $(V_1, \ldots, V_k)$ and if so, then return such a $k$-partition.

**Proof.** We will use Algorithm Recover-Partition$(G, U_1, \ldots, U_k)$ in Figure 3

We begin by showing correctness. Since Algorithm 3 maintains $U_i \subseteq P_i$ and $\delta(U_i) = \delta(P_i)$ for all $i \in [k]$, if it returns a $k$-partition, then the $k$-partition necessarily satisfies the required conditions. Next, we show that if $(U_1, \ldots, U_k)$ is a $k$-cut-set representative of a $k$-partition $(V_1, \ldots, V_k)$, then the algorithm will indeed return a $k$-partition $(P_1, \ldots, P_k)$ with $U_i \subseteq P_i$ and $\delta(U_i) = \delta(P_i)$ for all $i \in [k]$ (however, $(P_1, \ldots, P_k)$ may not necessarily be the same as $(V_1, \ldots, V_k)$).

Let $(V_1, \ldots, V_k)$ be a $k$-partition such that $U_i \subseteq V_i$ and $\delta(U_i) = \delta(V_i)$ for all $i \in [k]$. Let $(P_1, \ldots, P_k)$ be the sequence of subsets at the end of the for-loop. Moreover, for each $j \in [t]$, let $(P^j_1, \ldots, P^j_k)$ be the sequence of subsets at the end of the $j$th iteration of the for-loop. For notational convenience, for $i \in [k]$, we will define $P^0_i := U_i$. We note that $(P_1, \ldots, P_k) = (P^1_1, \ldots, P^1_k)$ and that $P^0_i \subseteq P^1_i \subseteq \cdots \subseteq P^t_i$ for every $i \in [k]$. We observe that $U_i \subseteq P^j_i$ and $\delta(U_i) = \delta(P^j_i)$ for all $j \in \{0,1,2,\ldots,t\}$. Moreover, the subsets $P^j_1, \ldots, P^j_k$ are pairwise disjoint for each $j \in \{0,1,2,\ldots,t\}$. Therefore, it suffices to show that $\bigcup_{i=1}^k P_i = V$. 

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a sub-problem—namely Enum-MinMax-Hypergraph (1) every family $G \cup j G$ of $(\ldots)$ holds by definition. We now prove the induction step. By induction hypothesis, we have that $(\ldots)$ Moreover, each hyperedge in $(\ldots)$ $C$ be the components of $G - \cup_{i=1}^{k} \delta(U_i)$ that are disjoint from $\cup_{i=1}^{k} U_i$ For $j = 1, \ldots, t$
If $\exists i \in [k]$ such that $\delta(P_i \cup C_j) = \delta(P_i)$
$P_i \leftarrow P_i \cup C_j$
If $(P_1, \ldots, P_k)$ is a $k$-partition of $V$
Return $(P_1, \ldots, P_k)$
Else
Return NO

Figure 3: Algorithm in Theorem 5.1

We claim that $C_1 \cup \ldots \cup C_j \subseteq \cup_{i=1}^{k} P_i$ for each $j \in \{0, 1, \ldots, t\}$. Applying this claim for $j = t$ gives that $\cup_{i=1}^{k} P_i = V$ as desired. We now show the claim by induction on $j$. The base case of $j = 0$ holds by definition. We now prove the induction step. By induction hypothesis, we have that $C_1 \cup \ldots \cup C_{j-1} \subseteq \cup_{i=1}^{k} P_i^{j-1}$. We will show that there exists $i \in [k]$ such that $\delta(P_i^{j-1} \cup C_j) = \delta(P_i^{j-1})$. We know that $C_j$ is contained in one of the sets in $\{V_1, \ldots, V_k\}$, say $C_j \subseteq V_\ell$ for some $\ell \in [k]$. We will prove that $\delta(P_\ell^{j-1} \cup C_j) = \delta(P_\ell)$ to complete the proof of the claim. Since $C_j$ is a component of $G - \cup_{i=1}^{k} \delta(U_i) = G - \cup_{i=1}^{k} \delta(V_i)$, we know that each hyperedge in $\delta(C_j)$ crosses the $k$-partition $(V_1, \ldots, V_k)$. Moreover, each hyperedge in $\delta(C_j)$ intersects $C_j \subseteq V_\ell$, and hence, $\delta(C_j) \subseteq \delta(V_\ell)$. Therefore,

$$\delta(P_\ell^{j-1} \cup C_j) - \delta(P_\ell^{j-1}) \subseteq \delta(C_j) \subseteq \delta(V_\ell) = \delta(U_\ell) = \delta(P_\ell^{j-1}).$$

This is possible only if $\delta(P_\ell^{j-1} \cup C_j) - \delta(P_\ell^{j-1}) = \emptyset$, i.e., $\delta(P_\ell^{j-1} \cup C_j) \subseteq \delta(P_\ell^{j-1})$. We now show the reverse inclusion. We have that

$$\delta(P_\ell^{j-1}) - \delta(P_\ell^{j-1} \cup C_j) = \delta(U_\ell) - \delta(P_\ell^{j-1} \cup C_j)$$
$$= \delta(V_\ell) - \delta(P_\ell^{j-1} \cup C_j)$$
$$= E(V_\ell - P_\ell^{j-1} - C_j, V - V_\ell - P_\ell^{j-1}) \cup E(V_\ell \cap P_\ell^{j-1}, P_\ell^{j-1} - V_\ell)$$
$$\subseteq E[V - P_\ell^{j-1}] \cup E[P_\ell^{j-1}].$$

We note that the LHS is a subset of $\delta(P_\ell^{j-1})$ while the RHS is disjoint from $\delta(P_\ell^{j-1})$ since $E[V - P_\ell^{j-1}] \cap \delta(P_\ell^{j-1}) = \emptyset$ and $E[P_\ell^{j-1}] \cap \delta(P_\ell^{j-1}) = \emptyset$. Hence, the above containment is possible only if $\delta(P_\ell^{j-1}) - \delta(P_\ell^{j-1} \cup C_j) = \emptyset$ and hence, $\delta(P_\ell^{j-1}) \subseteq \delta(P_\ell^{j-1} \cup C_j)$. Consequently, $\delta(P_\ell^{j-1} \cup C_j) = \delta(P_\ell^{j-1})$.

We now bound the run-time. We can verify if there exists $i \in [k]$ such that $\delta(P_i \cup C_j) = \delta(P_i)$ in time $O(kp)$. The number of iterations of the for-loop is $t \leq n$. Hence, the total run-time is $O(knp)$.

Next, we address the problem of enumerating all $\text{MINMAX-k-CUT-SETS}$. For this, we define a sub-problem—namely $\text{ENUM-MINMAX-HYPERGRAPH-k-CUT-SET-REPS}$. The input here is a hypergraph $G = (V, E)$ and a fixed positive integer $k$ (e.g., $k = 2, 3, 4, \ldots$). The goal is to enumerate a family $\mathcal{F}$ of $k$-cut-set representatives satisfying the following two properties:

1. every $k$-tuple $(U_1, \ldots, U_k)$ in the family $\mathcal{F}$ is a $k$-cut-set representative of some optimum $k$-partition $(V_1, \ldots, V_k)$ for $\text{MINMAX-HYPERGRAPH-k-PARTITION}$ and
(2) for every optimum $k$-partition $(V_1,\ldots, V_k)$ for MINMAX-HYPERGRAPH-$k$-PARTITION, the family $\mathcal{F}$ contains a $k$-cut-set representative $(U_1,\ldots, U_k)$ of $(V_1,\ldots, V_k)$.

We note that if a family $\mathcal{F}$ is a solution to ENUM-MINMAX-HYPERGRAPH-$k$-CUT-SET-REPS, then returning $\left\{ \bigcup_{i=1}^{k} \delta(U_i) : (U_1,\ldots, U_k) \in \mathcal{F} \right\}$ solves ENUM-MINMAX-HYPERGRAPH-$k$-PARTITION. Hence, it suffices to solve ENUM-MINMAX-HYPERGRAPH-$k$-CUT-SET-REPS in order to solve ENUM-MINMAX-HYPERGRAPH-$k$-PARTITION. We describe our algorithm for ENUM-MINMAX-HYPERGRAPH-$k$-CUT-SET-REPS in Figure 4 and its guarantees in Theorem 5.2. Theorem 1.1 follows from Theorem 5.2.

**Theorem 5.2.** Let $G = (V, E)$ be a $n$-vertex hypergraph of size $p$ and let $k$ be a positive integer. Then, Algorithm Enum-MinMax-Reps($G, k$) in Figure 4 solves ENUM-MINMAX-HYPERGRAPH-$k$-CUT-SET-REPS and it can be implemented to run in time $O(kn^{4k^2-2k+1}p)$. Moreover, the cardinality of the family returned by the algorithm is $O(n^{4k^2-2k})$.

```latex
\begin{figure}[h]
\begin{center}
\begin{algorithm}
\begin{algorithmic}
   \State \textbf{Input:} Hypergraph $G = (V, E)$ and an integer $k \geq 2$
   \State \textbf{Output:} Family $\mathcal{F}$ of $k$-cut-set representatives of all optimum $k$-partitions for MINMAX-HYPERGRAPH-$k$-PARTITION
   \State Initialize $C \leftarrow \emptyset$, $D \leftarrow \emptyset$, and $\mathcal{F} \leftarrow \emptyset$
   \State For each pair $(S, T)$ such that $S, T \subseteq V$ with $S \cap T = \emptyset$ and $|S|, |T| \leq 2k - 1$
      \State Compute the source minimal minimum $(S,T)$-terminal cut $(U, \overline{U})$
      \State $C \leftarrow C \cup \{U\}$
   \State For all $(U_1,\ldots, U_k) \in C^k$ such that $U_1,\ldots, U_k$ are pairwise disjoint
      \State If Recover-Partition($G, U_1,\ldots, U_k$) returns a $k$-partition
         \State $D \leftarrow D \cup \{(U_1,\ldots, U_k)\}$
      \State $\lambda \leftarrow \min\{\max_{i\in[k]} d(U_i) : (U_1,\ldots, U_k) \in D\}$
   \State For all $(U_1,\ldots, U_k) \in D$ such that $\max_{i\in[k]} d(U_i) = \lambda$
      \State $\mathcal{F} \leftarrow \mathcal{F} \cup \{(U_1,\ldots, U_k)\}$
   \State Return $\mathcal{F}$
\end{algorithmic}
\end{algorithm}
\end{center}
\caption{Algorithm in Theorem 5.2}
\end{figure}
```

**Proof.** We begin by showing correctness—i.e., the family $\mathcal{F}$ returned by the algorithm satisfies properties (1) and (2) mentioned in the definition of ENUM-MINMAX-HYPERGRAPH-$k$-CUT-SET-REPS. By the second for-loop, each $k$-tuple added to the collection $D$ is a $k$-cut-set representative of some $k$-partition (it need not necessarily be a $k$-cut-set representative of an optimum $k$-partition for MINMAX-HYPERGRAPH-$k$-PARTITION). The algorithm returns a subfamily of $D$ and hence, it returns a subfamily of $k$-cut-set representatives. We only have to show that a $k$-cut-set representative of an arbitrary optimum $k$-partition for MINMAX-HYPERGRAPH-$k$-PARTITION is present in the family $D$; this will guarantee that the value $\lambda$ computed by the algorithm will exactly be $OPT_{\text{minmax-}k\text{-partition}}$ and owing to the way in which the algorithm constructs the family $\mathcal{F}$ from the family $D$, it follows that the family $\mathcal{F}$ satisfies properties (1) and (2).

Let $OPT_{\text{minmax-}k\text{-partition}}$ denote the optimum value of a minmax $k$-partition in $G$ and let $OPT_{k\text{-cut}}$ denote the optimum value of a minimum $k$-cut in $G$. We note that $OPT_{\text{minmax-}k\text{-partition}} \leq OPT_{k\text{-cut}}$. This is because, if $(P_1,\ldots, P_k)$ is a $k$-partition with minimum $|\delta(P_1,\ldots, P_k)|$ (i.e., an optimum $k$-partition for HYPERGRAPH-$k$-CUT), then

$$OPT_{\text{minmax-}k\text{-partition}} \leq \max_{i \in [k]} |\delta(P_i)| \leq |\delta(P_1,\ldots, P_k)| = OPT_{k\text{-cut}}.$$
Let \((V_1, \ldots, V_k)\) be an arbitrary optimum \(k\)-partition for MINMAX-HYPERGRAPH-\(k\)-PARTITION. We will show that the family \(\mathcal{F}\) returned by the algorithm contains a \(k\)-cut-set representative of \((V_1, \ldots, V_k)\). We have that \(d(V_i) \leq OPT_{\text{minmax-}k\text{-partition}} \leq OPT_{k\text{-cut}}\) for all \(i \in [k]\). Hence, by Theorems 1.3 and 1.7 there exist subsets \(S_i \subseteq V_i, T_i \subseteq V - V_i\) with \(|S_i|, |T_i| \leq 2k - 1\) such that the source minimal \((S_i, T_i)\)-terminal cut \((U_i, \overline{U}_i)\) satisfies \(\delta(U_i) = \delta(V_i)\) for all \(i \in [k]\). Source minimality of the cut \((U_i, \overline{U}_i)\) also guarantees that \(U_i \subseteq V_i\) for all \(i \in [k]\). Hence, the \(k\)-tuple \((U_1, \ldots, U_k)\) is a \(k\)-cut-set representative of \((V_1, \ldots, V_k)\). It remains to show that this \(k\)-tuple is indeed present in the families \(\mathcal{D}\) and \(\mathcal{F}\). We note that the sets \(U_1, \ldots, U_k\) are added to the collection \(\mathcal{C}\) in the first for-loop. Since the \(k\)-tuple \((U_1, \ldots, U_k)\) is a \(k\)-cut-set representative of the \(k\)-partition \((V_1, \ldots, V_k)\), the \(k\)-tuple \((U_1, \ldots, U_k)\) will be added to the family \(\mathcal{D}\) in the second for-loop. Since the family \(\mathcal{D}\) contains only \(k\)-cut-set representatives of \(k\)-partitions, it follows that \(\lambda = OPT_{\text{minmax-}k\text{-partition}}\) and \((U_1, \ldots, U_k)\) will be added to the family \(\mathcal{F}\) in the third for-loop. Hence, the \(k\)-cut-set representative \((U_1, \ldots, U_k)\) of the optimum \(k\)-partition \((V_1, \ldots, V_k)\) for MINMAX-HYPERGRAPH-\(k\)-PARTITION is present in the family \(\mathcal{F}\) returned by the algorithm.

The bound on the size of the family \(\mathcal{F}\) returned by the algorithm is

\[
|\mathcal{F}| \leq |\mathcal{D}| \leq |\mathcal{C}|^k = O(n^k(4k^2-2)).
\]

Next, we bound the run time of the algorithm. The first for-loop can be implemented to run in time \(O(n^{4k-2})T(n, p)\). The second for-loop executes the algorithm from Theorem 5.1 \(O(n^{4k^2-2k})\) times and hence, the second for-loop can be implemented to run in time \(O(kn^{4k^2-2k+1}p)\). The computation of \(\lambda\) and the third for-loop can be implemented to run in time \(O(|\mathcal{D}|) = O(n^{4k^2-2k})\). Hence, the total run-time is \(O(n^{4k-2})T(n, p) + O(kn^{4k^2-2k+1}p)\). We recall that \(T(n, p) = O(np)\) and hence, the total run-time is \(O(kn^{4k^2-2k+1}p)\).

6 A lower bound on the number of MINMAX-\(k\)-CUT-SETS

In this section, we show that there exist \(n\)-vertex connected graphs for which the number of MINMAX-\(k\)-CUT-SETS is \(n^{\Omega(k^2)}\). In particular, we show the following result.

**Lemma 6.1.** For every positive integer \(k \geq 2\), there exists a positive integer \(n\) such that the number of optimum \(k\)-partitions for MINMAX-GRAPH-\(k\)-PARTITION in the \(n\)-vertex complete graph is \(n^{\Omega(k^2)}\).

**Proof.** Let \(k \geq 2\) be fixed, and let \(G = (V, E)\) be the complete graph on \(n = k(k-1)\) vertices (with all edge weights being uniformly 1). We will show that \(OPT_{\text{minmax-}k\text{-partition}} = (k-1)^3\) and every partition of \(V\) into \(k\) parts of equal size is an optimum \(k\)-partition for MINMAX-GRAPH-\(k\)-PARTITION. Since the number of partitions of \(V\) into \(k\) parts of equal size is \(\Omega(k^n) = \Omega(n^{k^2/2})\), the lemma follows.

First, we show that \(OPT_{\text{minmax-}k\text{-partition}} \geq (k-1)^3\). For every partition of \(V\) into \(k\) non-empty parts, the largest part has at least \(k - 1\) vertices by pigeonhole principle, and at most \(k(k-1) - (k-1) = (k-1)^2\) vertices since each of the remaining \(k - 1\) parts contain at least one vertex. Therefore, the cut value of the largest part is at least

\[
\min_{x \in (k-1, \ldots, (k-1)^2)} x(n - x) = (k-1)^3.
\]

The equality follows since \(n = k(k-1)\) and the function \(f(x) = x(n - x)\) is convex and is minimized at the boundaries. This implies that \(OPT_{\text{minmax-}k\text{-partition}} \geq (k-1)^3\).
Next, we show that $OPT_{\text{minmax-}k\text{-partition}} \leq (k - 1)^3$ and every partition of $V$ into $k$ parts of equal size is an optimum $k$-partition for $\text{MINMAX-GRAPH-}k\text{-PARTITION}$. Let $(V_1, \ldots, V_k)$ be an arbitrary $k$-partition of $V$ such that $|V_i| = k - 1$ for all $i \in [k]$. The $\text{MINMAX-GRAPH-}k\text{-PARTITION}$ objective value of this $k$-partition is $(k - 1)(n - (k - 1)) = (k - 1)^3$. Thus, $(V_1, \ldots, V_k)$ is an optimum $k$-partition for $\text{MINMAX-GRAPH-}k\text{-PARTITION}$ in $G$.

We note that our example exhibiting $n^{\Omega(k^2)}$ optimum $k$-partitions for $\text{MINMAX-GRAPH-}k\text{-PARTITION}$ has the number of vertices $n$ upper bounded by a function of $k$. We are not aware of examples that exhibit $n^{\Omega(k^2)}$ optimum $k$-partitions for $\text{MINMAX-GRAPH-}k\text{-PARTITION}$ for fixed $k$ but arbitrary $n$ (e.g., $k = 2, 3, 4, \ldots$ but $n$ is arbitrary).

7 Conclusion

We showed the first polynomial bound on the number of $\text{MINMAX-}k\text{-CUT-SETS}$ in hypergraphs for every fixed $k$ and gave a polynomial-time algorithm to enumerate all $\text{MINMAX-}k\text{-CUT-SETS}$ as well as all $\text{MIN-}\text{-}k\text{-CUT-SETS}$ in hypergraphs for every fixed $k$. Our main contribution is a structural theorem that is the backbone of the correctness analysis of our enumeration algorithms. In order to enumerate $\text{MINMAX-}k\text{-CUT-SETS}$ in hypergraphs, we introduced the notion of $k$-cut-set representatives and enumerated $k$-cut-set representatives of all optimum $k$-partitions for $\text{MINMAX-HYPERGRAPH-}k\text{-PARTITION}$. Our technique builds on known structural results for $\text{HYPERGRAPH-}k\text{-CUT}$ and $\text{MINMAX-HYPERGRAPH-}k\text{-PARTITION}$.

The technique underlying our enumeration algorithms is not necessarily novel—we simply rely on minimum $(s,t)$-terminal cuts. Using fixed-terminal cuts to address global partitioning problems is not a novel technique by itself—it is common knowledge that minimum $(s,t)$-terminal cuts can be used to solve global minimum cut. However, there are several problems where naive use of this technique fails to lead to efficient algorithms; e.g., multiway cut does not help in solving $\text{GRAPH-}k\text{-CUT}$ since multiway cut is NP-hard. Adapting this technique for specific partitioning problems requires careful identification of structural properties. In fact, beautiful structural properties have been shown for a rich variety of partitioning problems in combinatorial optimization in order to exploit this technique: for example, it was used (1) to design the first efficient algorithm for $\text{GRAPH-}k\text{-CUT}$ [26], (2) to solve certain constrained submodular minimization problems [25,49], and (3) more recently, to design fast algorithms for global minimum cut in graphs and for $\text{GOMORY-HU TREE}$ in unweighted graphs [1,44]. Our use of this technique also relies on identifying and proving a suitable structural property, namely Theorem[4]. The advantage of our structural property is that it simultaneously enables enumeration of $\text{MIN-}\text{-}k\text{-CUT-SETS}$ as well as $\text{MINMAX-}k\text{-CUT-SETS}$ in hypergraphs which was not possible via structural theorems that were developed before. Furthermore, it helps in showing the first polynomial bound on the number of $\text{MINMAX-}k\text{-CUT-SETS}$ in hypergraphs for every fixed $k$.

We also emphasize a limitation of our technique. Although it helps in solving $\text{ENUM-HYPERGRAPH-}k\text{-CUT}$ and $\text{ENUM-MINMAX-HYPERGRAPH-}k\text{-PARTITION}$, it does not help in solving a seemingly related hypergraph $k$-partitioning problem—namely, given a hypergraph $G = (V,E)$ and a fixed integer $k$, find a $k$-partition $(V_1, \ldots, V_k)$ of the vertex set that minimizes $\sum_{i=1}^{k} |\delta(V_i)|$. Natural variants of our structural theorem fail to hold for this objective. Resolving the complexity of this variant of the hypergraph $k$-partitioning problem for $k \geq 5$ remains open.

We mention an open question concerning $\text{HYPERGRAPH-}k\text{-CUT}$ and the enumeration of $\text{MIN-}\text{-}k\text{-CUT-SETS}$ in hypergraphs for fixed $k$. We recall the status in graphs: the number of minimum $k$-partitions in a connected graph was known to be $O(n^{2k-2})$ via Karger-Stein’s algorithm [38] and
$\Omega(n^k)$ via the cycle example, where $n$ is the number of vertices; recent works have improved on the upper bound to match the lower bound for fixed $k$—this improvement in upper bound also led to the best possible $O(n^k)$-time algorithm for Graph-$k$-Cut for fixed $k$ \[28\][31][32]. For hypergraphs, the number of min-$k$-cut-sets is known to be $O(n^{2k-2})$ and $\Omega(n^k)$. Can we improve the upper/lower bound? Is it possible to design an algorithm for Hypergraph-$k$-Cut that runs in time $O(n^k p)$?

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