Asymptotic approximations for Bloch waves and topological mode steering in a planar array of Neumann scatterers

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Abstract

We study the canonical problem of wave scattering by periodic arrays, either of infinite or finite extent, of Neumann scatterers in the plane; the characteristic lengthscale of the scatterers is considered small relative to the lattice period. We utilise the method of matched asymptotic expansions, together with Fourier series representations, to create an efficient and accurate numerical approach for finding the dispersion curves associated with Floquet-Bloch waves through an infinite array of scatterers. The approach lends itself to direct scattering problems for finite arrays and we illustrate the flexibility of these asymptotic representations on topical examples from topological wave physics.

1 Introduction

A fundamental understanding of wave propagation through periodic media underpins several areas of modern wave physics particularly photonic, and phononic, crystal devices [51, 15] and topological photonics [37] such as in valleytronics [2] - the latter relying upon the detailed orientations of multiple inclusions within the cell that is repeated. Although the precise setting varies between electromagnetism or acoustics, many of these periodic problems reduce to the study of the wave equation, and in the frequency domain this becomes the Helmholtz equation with periodic arrangements of inclusions. The essential computation becomes that of dispersion curves which help characterise essential details of the wave spectrum such as band-gaps of forbidden frequencies, flat-bands for slow-light or slow-sound, or Dirac points exhibiting locally dispersionless waves. The overwhelming approach in engineering and physics is to compute these curves with finite elements such as the commercial package Comsol [5], although there are numerous numerical alternatives such as the plane wave expansion method [16] which can be highly effective. However, such numerical methods can become a distraction particularly when dealing with topological effects, where it is the geometrical arrangement of the scatterers that matters, whereby faster or more flexible simulation methods are valuable for optimisation. For flexural waves in elastic plates (that can be modelled using the Kirchhoff-Love equations [12], fourth-order partial differential equations and, unlike for the Helmholtz equation here, have non-singular Green’s function [7]) very rapid numerical methods for dispersion curve evaluation [47] can be created that are well suited to studies
of topological media \cite{43, 29, 28, 30, 27, 42, 38}. Our aim here is to extend this rapid solution methodology to the Helmholtz system, with its singular Green’s function, by using matched asymptotic expansions to build in the presence of the small Neumann inclusions so we can again arrive at a rapidly convergent eigenvalue problem. This setting also enables rapid scattering simulations for finite crystals as an extension of the classical Foldy method \cite{31}.

Matched asymptotic expansions are the natural mathematical language in which to couch wave scattering problems involving a small parameter; the technique in the context of waves is neatly summarised in \cite{6}, more extensively in \cite{45}, and in a more modern context in \cite{31}. The aim is to take advantage of the small parameter, the ratio of, say, defect size to wavelength or other natural lengthscales, and then create an inner problem valid in the neighbourhood of the scatterer that is matched to an outer problem; these inner and outer problems being, hopefully, relatively straightforward to determine, such that rapid, accurate, and insightful solutions can follow. There are two limiting situations to consider, Dirichlet or Neumann (sound-soft or sound-hard) inclusions and, for periodic media, these were considered by McIver and co-workers \cite{19, 33} respectively with the inner following from Laplacian or Poisson equations and complex variable methods; the outer constructed using a doubly-periodic Green’s function based around multipole methods and Bessel functions. Although effective in generating limited dispersion relations these lack the flexibility to easily treat multiple inclusions within a cell or to be extended to scattering by finite arrays. As outlined for the Dirichlet case in \cite{40} modifying the outer solution to one based around a conditionally convergent Fourier series representation of the Green’s function, and subsequent manipulation, yields a convergent eigenvalue problem; this is the natural way to proceed, exits Bessel function approximations and special functions from the problem, and the matching between the inner and outer problems tie together very neatly. The Dirichlet problem only requires a monopole source, at leading order, whereas the weak scattering by Neumann inclusions requires further analysis including additional dipole source terms, see \cite{31, 32}, and this is the case treated here.

Importantly, for practical purposes a plane wave expansion approach, modelled around that used in flexural waves \cite{47, 29, 43}, leads to a highly effective semi-analytical numerical method posed in reciprocal, i.e. Fourier, space for extracting dispersion curves; herein we require asymptotic matching to be explicitly built into that formulation. Such matching removes any singularities observed within the wavefield from the numerics, subsequently we are not constrained by any convergence issues created by said singularities. Another practical benefit is that this leads naturally into a Foldy-like approach for scattering. Foldy’s approach \cite{9} was initially derived for isotropic scatterers as implemented via matching in \cite{40}. For Neumann scatterers, additional information about the gradients of the scattered field are required to close the system; this extension to account for anisotropy was introduced by Martin \cite{31} \cite{32} and is the generalized or extended Foldy approach.

These fast Foldy-like schemes complement the finite element schemes typically used in the physics and engineering literature for scattering calculations dependent upon some incident field, additionally it can also be used to generate eigensolutions by setting the incident field to zero and analysing the homogeneous Foldy problem whose solutions then cleanly identify the modes that form the scattered field.
Whilst the primary thesis of this article focuses on the method of matched asymptotics to provide a general and systematic approach to handle scattering by small defects, there are other semi-analytical alternatives. In the present context, for the special case of circular inclusions, typically a single inclusion resides within a cell that repeats; multipole expansions [36] provide a route to dispersion relation calculations, requiring knowledge of the convergence of the various lattice sums that appear in the generalised Rayleigh identity, and have been approximated in the dilute limit for doubly-periodic media [48, 35]. However, this approach becomes cumbersome for multiple inclusions within a cell and the settings in, say, topological photonics that have delicate dependence upon the inclusions, and their relative orientation within each periodic cell. Multipole methods are also feasible and popular for scattering problems involving circular cylinders, c.f. [21] yielding systems of linear algebraic equations as an extension of Foldy’s approach; it is nonetheless instructive to arrive at the system for small scatterers from matching and allow for non-circular scatterers.

The outline of the article is such that we first mathematically model the problem in section 2. Once the inclusions have being approximated by a series of monopoles and dipoles in the outer region, we are in a position to perform a traditional matching procedure about an inclusion in which the singular Green’s function is matched to the solution in the inner region - the solution of which satisfies the Neumann condition exactly. Section 3 utilises a divergent Fourier series to represent the solution - numerically we truncate the divergent sum where the matched asymptotic analysis allows us to determine the finite error in doing so; this allows a convergent, generalized eigenvalue problem to be written down determining the dispersion relation detailing the dispersive properties of the constructed media.

Once these properties are known, we wish to demonstrate these effects in physical space, this is examined in section 4 where scattering coefficients and matrices are determined for the generalized Foldy approach, using the inner solution in a similar vein to [40]. Customarily one provides an incident field to determine how the structure scatters said field; we go one step further and outline a scheme which visualises the excited modes throughout the structure at a certain frequency completely independent of the source, that is the solution of the homogeneous Foldy problem. Our scheme determining the dispersion relation is tested against the full finite element computations in section 5. Finally, in section 6 we demonstrate the utility and efficiency of our succinct formulae by applying them to a few topical examples in topological physics.

2 Formulation

Assuming harmonic waves with \( \exp(-i\Omega t) \) dependence being understood (and suppressed henceforth) and with \( \Omega \) being the frequency, we consider the dimensionless Helmholtz equation

\[
(\nabla^2 + \Omega^2) \phi(x) = 0
\]

for a two dimensional wavefield \( \phi(x) \). Consider some \( IJ \)th inclusion introduced within the field, the boundary of which satisfies the Neumann condition. Denoting \( L \) as some characteristic length scale (defined in (5)), our attention is
restricted to the case of small circular inclusions, whose radius \( \epsilon_{IJ} \ll L \). Subsequently, for some \( IJ \)th scatterer centered on \( x = X_{IJ} \)

\[
\frac{\partial \phi}{\partial r_{IJ}} \bigg|_{r_{IJ}=\epsilon_{IJ}} = 0, \quad \text{where } r_{IJ} = |r_{IJ}| = |x - X_{IJ}|. \tag{2}
\]

The works of Crighton et al. [6], Martin [31] and McIver [33] indicate that the \( IJ \)th Neumann inclusion, being weak, should act to perturb the wavefield at order \( \epsilon_{IJ}^2 \). The former two sources assume further that the inclusions act as some combination of monopoles and dipoles; therefore, (1) subject to (2) may be approximated by

\[
(\nabla^2 + \Omega^2) \phi = 4i \infty \sum_{I=1}^{P} \sum_{J=1}^{Q} \epsilon_{IJ}^2 \{ a_{IJ} \delta(x - X_{IJ}) - b_{IJ} \cdot \nabla \delta(x - X_{IJ}) \}. \tag{3}
\]

Here \( \delta(x) \) denotes the Dirac delta function. When we consider an infinite, doubly periodic, arrangement of Neumann inclusions we consider primitive cells in physical space that are spanned by the lattice vectors \( \alpha_{i=1,2} \), as in figure 1. In such an arrangement the double sum within (3) is enumerated by considering the \( I \)th primitive cell, containing \( P \) Neumann scatterers. The centroids of each cell form a two dimensional Bravis lattice exemplified with physical position vector

\[
R = n\alpha_1 + m\alpha_2 \quad \text{for some } n, m \in \mathbb{Z}. \tag{4}
\]

The aforementioned characteristic length \( L \) is defined to be that of the lattice period

\[
L = \min(|\alpha_1|, |\alpha_2|). \tag{5}
\]

The periodic nature of the material allows us to utilise Bloch’s theorem [18], where one may assume

\[
\phi(x) = \Phi(x) \exp(i\kappa \cdot x), \quad \text{where } \Phi(x + R) = \Phi(x). \tag{6}
\]

Here \( \kappa \) denotes the Bloch-wave vector and \( \Phi \) inherits the periodicity of the lattice. Subsequently, we express \( \Phi(x) \) by means of a Fourier series, therefore

\[
\phi(x) = \sum_{G} \Phi_G \exp(iK_G \cdot x) \quad \text{where } K_G = \kappa + G, \tag{7}
\]

\( \Phi_G \) denotes the Fourier coefficients which represent the amplitude of the \( G \)th excited mode characterising incoming and outgoing Bloch waves, propagating throughout the primitive cells. The reciprocal space is spanned by the lattice vectors \( \beta_{i=1,2} \), which satisfy the following orthogonality conditions

\[
\alpha_i \cdot \beta_j = 2\pi \tilde{\delta}_{ij} \tag{8}
\]

where \( \tilde{\delta}_{ij} \) denotes the Kronecker delta function. The reciprocal lattice vector is explicitly given by

\[
G = n\beta_1 + m\beta_2, \quad \text{for some } n, m \in \mathbb{Z}. \tag{9}
\]

Due to the underlying periodicity we consider the solution throughout the reduced \( \kappa \) space known as the first Brillouin zone, symmetry allows us to reduce this further to the irreducible Brillouin zone.
Throughout this article, scattered waves are considered purely outgoing; therefore, any Hankel functions present are of the first kind and subsequently we drop their superscripts and denote

$$H_\nu^{(1)}(x) = H_\nu(x),$$

for the Hankel function of the first kind and $\nu$th order.

3 A convergent scheme in Fourier space

The solution in reciprocal space utilises Fourier series expansions for the outer solution. However the solution is singular about the location of the point source terms, therefore the series expansions must be divergent within its inner limits. Unfortunately the coefficients of the monopole and dipole terms are determined a posteriori and require inspection within the inner limit in which we expect the series expansions to diverge. The singularities are removed by considering the residual error from artificially truncating the diverging sum; the singularities present in the diverging piece of the sum coincide with the singularities observed from the outer limit of the inner solution. Remarkably the residual error allows a convergent generalized eigenvalue problem to be written down with such flexibility as to determine non-trivial wave propagation throughout the primitive cells containing an arbitrary arrangement of Neumann inclusions.

For brevity we shall assume $P = 1$, Bloch expanding equation (3), by making use of (7) one finds throughout the fundamental cell (dropping the now redundant subscripts)

$$\Phi_G = \frac{4}{i\omega} \frac{c^2 \{a - i b \cdot K_G\} \exp(-iK_G \cdot X)}{(K_G \cdot K_G - \Omega^2)},$$

Fig. 1: A doubly periodic crystalline structure of Neumann inclusions, with primitive cells represented by black squares with centroids (●). (i) The crystal in physical space, spanned by $\alpha_{i=1,2}$. (ii) An arrangement of three Neumann inclusions (white circles) in physical space. (iii) The first Brillouin zone (a primitive cell in reciprocal space) and irreducible Brillouin zone ($\Gamma X M \Gamma$), in reciprocal space. (iv) The crystal in reciprocal space, spanned by $\beta_{i=1,2}$. 
where $\mathcal{A}$ denotes the area of the primitive cell. The series representation of $\phi$ is subsequently

$$
\phi = \frac{4}{i \mathcal{A}} \sum_G e^{2 \{a - i b \cdot K_G\} \exp[i K_G \cdot r]} (K_G \cdot K_G - \Omega^2).
$$

(12)

Similarly to Schnitzer & Craster [40], we see that this series is conditionally convergent when $r \neq 0$ and diverges as $r \to 0$. Moreover the isotropic piece of this series is, other than its relative order, identical to that considered in [40] and subsequently diverges to some logarithmically weak singularity. Unfortunately the anisotropy present within this system behaves much worse. The outer limit of the inner solution is derived in $C$, where we find

$$
\lim_{r \to 0} \phi = \frac{4i}{\pi} \frac{a}{\Omega^2} \left[ 1 - \frac{\Omega^2 r^2}{4} \right] + \frac{4 b \cdot \hat{r}}{i \pi} \left( r \Omega - \frac{\Omega^3 r^3}{16} \right) + \epsilon^2 \left\{ \frac{2ia}{\pi} \left[ \log \frac{r}{\epsilon} + \frac{3}{4} \right] + \frac{b \cdot \hat{r}}{i \pi} \left( \frac{2}{r} - \Omega^2 r \left[ \log \frac{r}{\epsilon} - \frac{7}{4} \right] \right) \right\}.
$$

(13)

Here $\hat{r} = r / |r|$. The above equation implies the series given in (12) will diverge like $1/r$ as $r \to 0$. Lastly, we see in comparison to the Dirichlet inclusions [40], we have $2P$ more unknowns due to the requirement of the dipole source term. To resolve this anisotropic behaviour we require not only the solution to $\phi$ but also $\nabla \phi$ to close the system; moreover if one considers the gradient of (13) then it is apparent that $\nabla (12)$ will diverge like $1/r^2$ as $r \to 0$.

This presents an intriguing numerical question - how can one perform a series expansion to evaluate all unknowns of a system when such a series does not converge?

### 3.1 Leveraging the residual error

The solution in the outer region is given by the series expansion (12), numerically we require some ‘large’ finite radius $R'$ in Fourier space within which we truncate the Bloch expansion (7)

$$
\phi = \left\{ \sum_{|G| < R'} \phi_t + \sum_{|G| > R'} \phi_r \right\} \Phi_G \exp(i K_G \cdot x).
$$

(14)

Here, we split the infinite sum up into two pieces:

- $\phi_t$ represents the truncated portion of the series expansion, where one considers $\Phi_G$ from (11)
- $\phi_r$ denotes the residual portion of the sum, the left over piece beyond truncation - we approximate $\phi_r$ from (12) provided $|G| > R' \gg 1$

The issue here is $a$ and $b$ are, as yet, undetermined constants. Their evaluation requires matching the inner solution in the outer limit (left) to the outer region in the inner limit (right) as follows

$$
\lim_{r \to 0} \phi = \lim_{r \to 0} \{ \phi_t + \phi_r \} \sim \frac{1}{r}.
$$

(15)
How can $\phi_t$ hope to capture this, or any other weaker singularities, within its expansion? The answer is it cannot, since a finite sum of finite values will be unsurprisingly finite. But both sides of the equality in (15) must be identical - it is therefore logical to assume any singularities must be present in $\phi_r$, the infinite portion of the series. Therefore we shall equate

$$\phi_t = \phi - \phi_r \quad \text{as } r \to 0,$$

and expect any singularities present in $\phi$ and $\phi_r$ in the above limit to cancel, leaving behind some finite residual error arising from artificially truncating the divergent sum. Remarkably we can utilise the residual error in (16) to write down a generalized eigenvalue problem - the solution of which determines all unknowns in the system.

### 3.2 The convergent Eigenvalue problem

$\phi_r$ is derived in B - equation (46) can be expressed as

$$\phi_r \sim e^{\epsilon \exp(i\kappa \cdot r)} \left\{ \frac{2b \cdot \hat{r}}{r} J_0(Rr') + \frac{\epsilon r}{i} J_0(Rr') \left[ i(\epsilon b\Omega + 2a\kappa) \cdot \hat{r} - 2a \right] - \frac{J_1(Rr')}{Rr'} \left[ 2i \left( b \cdot 2\kappa \cos(2\phi - 2\theta_\kappa) - (b \times \kappa) \cdot e_z \sin(2\phi - 2\theta_\kappa) \right) + \epsilon r (ib\Omega^2 + 2a\kappa) \cdot \hat{r} \right] - 2\epsilon \left[ b \cdot \kappa \cos(3\theta_\kappa - 3\phi) + (b \times \kappa) \cdot e_z \sin(3\theta_\kappa - 3\phi) \right] \frac{J_2(Rr')}{R^2 r'} \right\}.$$  

Throughout $b$ and $\kappa$ are considered to be some two dimensional quantities with components in $e_z$ and $e_y$ only. Here $J_{10}$ refers to Dr Van der Pol’s Bessel-integral function of zero order*, the properties of which are examined in Humbert [14]. Therefore inserting (13) and (17) into (16) we find

$$\lim_{r \to 0} \frac{4ia}{\pi \Omega^2} + \epsilon^2 \left\{ \frac{2a\epsilon}{\pi} \left[ \log \frac{2 \epsilon R}{\Omega} + \frac{3}{4} - \gamma_E \right] - \frac{\kappa \cdot b}{\pi} \right\} + O(r, r \log r).$$

The Euler-Mascheroni constant is denoted by $\gamma_E$ throughout. The above equation holds for any $P$ inclusions placed into the fundamental cell, in the limit as $r_{1,J} \to 0$ for $J = 1, \ldots, P$ - so long as the inner solutions do not interfere with one another. That is to say that the inclusions are not placed close enough to one another such that the analysis in C breaks down.

Thus far, equations $(KG \cdot KG - \Omega^2) \cdot (11)$ and $\Omega^2 \cdot (19)$ will form a generalized eigenvalue problem whose matrices have dimensions $(N + P) \times (N + 3P)$, where $N$ represents all of the combinations of $G$ such that $|G| < R^2$. The eigenvalues found will represent frequency $\Omega$. Rather interestingly we expect $\Omega$ to be real, and subsequently the matrices forming the eigenvalue problem must be

\*For the evaluation of $J_{10}(x)$ for small parameters refer to equation (47). When constructing the eigensolution in the fundamental cell in physical space a useful identity is

$$J_{10}(z) = -\frac{x^2}{8} F_0(1,1;2,2; -\frac{x^2}{2}) + \log \frac{x}{2} + \gamma_E,$$

where $F_0(v_1, \ldots, v_l; w_1, \ldots, w_m; z)$ is the generalized hypergeometric function.
Hermitian; thus motivating the necessity to consider $\nabla \phi$ to close the problem - precisely what was proposed by Martin [31] [32] to consider anisotropy in the generalized Foldy system. Consider,

$$\nabla \phi_t = \nabla \phi - \nabla \phi_r \quad \text{as } r \to 0$$

(20)

For multiple objects it is easiest to consider the gradients, in the above, in polar coordinates about the center of each inclusion. Then transforming to a global Cartesian system we find the following components

$$\begin{align*}
e_x : \lim_{r \to 0} \sum_{G < R'} i K_1 \Phi_G \exp(i K_G \cdot x) &= \\
&= \left[ \frac{2}{i \pi} + i \frac{\varepsilon^2}{\pi} \left( \log \frac{2}{\varepsilon R^2} - \frac{5}{4} - \gamma_E \right) + \frac{\varepsilon^2}{2 \pi R^2} \right] b_1 + \\
&\quad + \frac{2}{i \pi} \frac{\varepsilon_1}{\pi} - \frac{\varepsilon_2}{4\pi} \left( \kappa_1 b \cdot \kappa - \kappa_2 (b \times \kappa) \cdot e_z \right),
\end{align*}$$

(21)

$$\begin{align*}
e_y : \lim_{r \to 0} \sum_{G < R'} i K_2 \Phi_G \exp(i K_G \cdot x) &= \\
&= \left[ \frac{2}{i \pi} + i \frac{\varepsilon^2}{\pi} \left( \log \frac{2}{\varepsilon R^2} - \frac{5}{4} - \gamma_E \right) + \frac{\varepsilon^2}{2 \pi R^2} \right] b_2 + \\
&\quad + \frac{2}{i \pi} \frac{\varepsilon_2}{\pi} - \frac{\varepsilon_1}{4\pi} \left( \kappa_2 b \cdot \kappa + \kappa_1 (b \times \kappa) \cdot e_z \right)
\end{align*}$$

(22)

The complete generalized eigenvalue problem may now be formed from equations $(K_G \cdot K_G - \Omega^2) \cdot (11), \Omega^2 \cdot (19), (21)$ and $(22)$; explicitly expressed as

$$(A - \Omega^2 B) \Phi = 0$$

(23)

The scheme works by looping through each of the required values of $\kappa$, solving the generalized eigenvalue problem whose eigenvalues yield $\Omega^2$ and the entries of the eigenvector $\Phi$ contain all previously unknown constants for a wave of fixed phase - the constants being $\Phi_G$ (for $|G| < R'$), $a_{1J}$ and $b_{1J}$ where $J = 1, \ldots, P$.

Refer to $E$ for the explicit matrix factorization of the proposed eigenvalue problem. Pictorially, refer to figure 2. Typically in convergent schemes, one defines some arbitrary $M$ and considers $\Phi_G$, contained within $N$ positions enumerated by $G$’s in (9), where each of the coordinates $(n, m)$ ranges from $-M \to M$. Contrastingly, we opt to define our truncation position as shown in figure 2.

The eigensolution can be exploited to construct the solution in physical space, utilising (12) for $\phi_t$ and (17) for $\phi_r$ one can construct $\phi$ and its associated gradients or flux. This is precisely what is plotted in figure 4 (iv) and similar figures. Note the wavefield is constructed over the entire space, but due to the singularities occurring inside the inclusions we white them out as not to afflict the colour scale.

It should be said that $\phi_r$ has been constructed for one object only - in fact the analysis for the eigensolution works for multiple objects since we only require $\phi_r$ in the limit as $r_{1J} \to 0$ for each $J = 1, \ldots, P$ where the $J$th inclusion will dominate the matching. The question of how exactly (17) can be used within $\lim_{r_{1J} \to 0} \phi_r$ for multiple scatterers per cell is a valid question. We note, however, that the series (12) is convergent so long as $r_{1J} \neq 0$; subsequently we observe $\phi_r$ only having an appreciable influence on the eigensolution (in physical space) in the vicinity of the inclusions.
4 Generalized Foldy solution

Here we seek to extend the scattering computations performed by Schnitzer & Craster [40], in which Foldy’s method [9] can be applied directly due to the isotropic nature of the sound-soft scatterers considered.

Foldy’s method [9] considers the effect of some incident field interacting with multiple isotropic scatterers by naturally resolving the singularities at the $n$th scatterer by means of calculating the external field. Since Foldy’s method does not rely upon a periodic arrangement of scatterers, we replace subscript $IJ$ with $n$ and consider the $n$th scatterer in some arbitrary arrangement. The external field, at the $n$th sound hard inclusion, is defined as

$$\phi_n(x_n) = \lim_{x \to x_n} \left\{ \phi - c_n^2 (a_n H_0(\Omega |x - x_n|) + b_n \cdot \hat{r} \Omega H_1(\Omega |x - x_n|)) \right\}. \quad (24)$$

Foldy [9] hypothesised that the scattered field arising from some isotropic source (monopole) will have strength, explicitly from the $n$th scatterer, $a_n$ and will be proportional to the external field incident upon the $n$th scatterer. The proportionality constant, the monopole scattering coefficient denoted $\tau_n$, is given by

$$a_n = \tau_n \phi_n(x_n). \quad (25)$$

Linton & Martin [22] state that anisotropic scatterers can be considered by modifying Foldy’s original method. The so-called generalized Foldy approach has been outlined in Martin [32] [31], where one may utilize the gradient of the external field to determine the dipole scattering coefficient following identical logic to that of the monopole scattering coefficient. That is

$$b_n = T_n \nabla \phi_n(x_n). \quad (26)$$

Here $T_n$ is the dipole scattering coefficient matrix, a $2 \times 2$ matrix fully encapsulating any anisotropy introduced by the dipole contribution of the sound hard scatterer.
The \( \lim_{r_n \to 0} \phi \) is given by equation (62), therefore by (25) and (26) we find that

\[
\begin{align*}
\tau_n & \sim \frac{4i}{\pi \Omega_0^2} c_n^2 \left[ 1 - \frac{2i}{\pi} \left( \log \frac{2}{\epsilon_n \Omega} + \frac{3}{4} - \gamma_E \right) \right] + O(r_n, r_n \log r_n) \\
T_n & \sim \frac{1}{2 + \frac{2 \epsilon^2}{\Omega^2} \left( \log \frac{2}{\epsilon_n \Omega} - \frac{5}{4} - \gamma_E \right)} + O(r_n, r_n \log r_n)
\end{align*}
\] as \( r_n \to 0 \), (27)

Here \( I \) is the 2-by-2 identity matrix. We consider some incident point source placed at \( x = X_{\text{inc}} \), with a strength comparable to that of the scatterers; subsequently,

\[
(\nabla^2 + \Omega^2) \phi = \epsilon_{\min}^2 \phi_{\text{inc}} + 4i \sum_j c_j^2 \left\{ a_j \delta(x - X_j) - b_j \cdot \nabla \delta(x - X_j) \right\}
\] (28)

Here \( \epsilon_{\min} \) denotes the smallest inclusion present within the field and \( \phi_{\text{inc}} \) denotes the incident field source term. It is rather natural for us to consider two sources \( \epsilon_{\min}^2 \phi_{\text{inc}} \) by \( \phi_{\text{inc}} \) then

\[
\phi = \epsilon_{\min}^2 \phi_{\text{inc}}(x) + \sum_{j=1}^{m} c_j^2 \left\{ a_j H_0(\Omega|x - X_j|) + b_j \cdot r \Omega H_1(\Omega|x - X_j|) \right\},
\] (30)

Substituting (30) into (25) and \( \nabla(30) \) into (26), one finds

\[
\frac{4i}{\pi \Omega_0^2} c_n^2 \left[ 1 - \frac{2i}{\pi} \left( \log \frac{2}{\epsilon_n \Omega} + \frac{3}{4} - \gamma_E \right) \right] - \sum_{j=1}^{m} c_j^2 \left\{ a_j H_0(\Omega r_{nj}) + b_j \cdot r \Omega H_1(\Omega r_{nj}) \right\} = \epsilon_{\min}^2 \phi_{\text{inc}}
\] (31)

\[
\frac{2}{\pi \Omega_0^2} c_n^2 \Omega^2 \left[ \log \frac{2}{\epsilon_n \Omega} - \frac{5}{4} - \gamma_E \right] - \sum_{j=1}^{m} c_j^2 \left\{ - a_j \Omega H_1(\Omega r_{nj}) + \right. \\
+ \frac{b_j \cdot e_{r_{nj}}}{2} \Omega^2 \left[ H_0(\Omega r_{nj}) - H_2(\Omega r_{nj}) \right] e_{r_{nj}} + \\
\left. + \left( \frac{b_j \cdot e_{\phi_{nj}}}{r_{nj}} \Omega H_1(\Omega r_{nj}) \right) e_{\phi_{nj}} \right\} = \epsilon_{\min}^2 \nabla \phi_{\text{inc}}.
\] (32)

Where \( r_{nj} = |X_n - X_j| \). If one compares equation (27) with that used by Martin [32], we see that our \( \Omega \) is equivalent to Martin’s \( k \). Moreover the ‘good choices’ for the monopole and dipole scattering coefficients given by Martin [32] are nothing more than \( \tau_n \) and \( T_n \) to leading order. However, since the left
hand sides of (25) and (26) are order unity, one needs to retain the inverse scattering coefficients correct to order $\epsilon^2$ such that they balance correctly with the scattered field.

To avoid confusion between local radial coordinate systems around each $j$th scatterer, we define a global Cartesian basis in which

$$e_{rj} = \cos \varphi_{ij} e_x + \sin \varphi_{ij} e_y, \quad e_\varphi j = -\sin \varphi_{ij} e_x + \cos \varphi_{ij} e_y$$

Here $\varphi_{ij}$ is the polar angle of $X_i - X_j$ - the angle from the $j$th to the $i$th scatterer, centred on the $j$th scatterer. Subsequently considering equation (31), (32) $e_x$ and (32) $e_y$ we form the following matrix system. If we consider a total $m$ number of scatterers placed in any arbitrary arrangement

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} a_s \\ b_{1s} \\ b_{2s} \end{pmatrix} = \begin{pmatrix} \phi_{inc,s} \\ \nabla \phi_{inc1,s} \\ \nabla \phi_{inc2,s} \end{pmatrix}$$

(34)

Here

$$a^\dagger_s = (a_1, a_2, \ldots, a_m), \quad b^\dagger_{i=1,2,s} = (b_{i1}, b_{i2}, \ldots, b_{im}).$$

Superscript $\dagger$ denotes the transpose operation. Refer to the appendix section F for the contents of $\otimes$ and the incident vectors and associated gradients $\phi_{inc,s}$ corresponding to $\phi_{inc}$ evaluated at various inclusions. Left multiplying (34) by $\otimes^{-1}$, one finds the $3m$ unknowns, $a_j$, $b_{1j}$ and $b_{2j}$ for each $j = 1, \ldots, m$ scatterer. Substituting these into (30) or $\nabla(30)$ the field displacement and gradient can be constructed over the entire space analytically.

### 4.1 Solutions to the homogeneous Foldy problem

When considering the dispersive nature of materials, the exploration of non-trivial dynamics occurring throughout the structure are of utmost importance in predicting the propagation of waves and the transport of energy occurring throughout the structure when some incident source is applied upon it. We have considered such nature for doubly and singly periodic infinite media, the dispersion relations determined in figures 4(iii) and 6 for example.

These relations are assumed to hold for the Foldy solutions provided that one builds a large enough finite region of doubly or singly periodic (or even graded if the grading is considered gently enough) region of scatterers to replicate the conditions through which the dispersion relations were found. The Foldy simulations are dependent upon some source incident upon the structure; non-trivial behaviour of the medium at fixed frequency can not be determined. One can consider comparable “eigen-like”-generalized Foldy solutions if one can find a solution to the homogenous generalized Foldy problem - that is setting $\phi_{inc,s}$, $\nabla \phi_{inc1,s}$ and $\nabla \phi_{inc2,s}$ equal to zero in (34) and seeking the solutions to all unknown scattering coefficients.

The issue is, if one considers the entries of $\otimes$ in (34) (appendix section F) the rows and columns of which are nothing more than Hankel functions evaluated at some frequency and differing positions of inclusions throughout the lattice; the rows or columns are far from being linearly dependent on one another. Therefore
is a matrix of full rank, and by the rank-nullity theorem our non-trivial null-space is subsequently empty and we doomed to never find a perfect non-trivial solution to our homogeneous problem.

However, if one allows for an approximate solution then we may utilise the singular value decomposition of . If one picks the right-singular vector as the unknown monopole and dipole coefficients in (34), the right-singular vector chosen should correspond to the smallest singular value of such a decomposition. Provided the singular value is small and the total number of scatterers considered (hence the dimension of the matrix) is large, one will find that each row of multiplied by the column vector full of the unknown coefficients is negligible. The error involved in considering the right-singular vector as a valid non-trivial solution would introduce errors, in many cases, smaller than the truncation error in considering the asymptotic solution to the extended Foldy accurate to order . We demonstrate the reliability of this choice for an approximate non-trivial homogeneous solution by the demonstration that it is capable of finding the edge modes existing between the two media refer to figure 8.

Fig. 3: Comparison between eigenvalues from (23), and finite element generated dispersion relation for a single inclusion in either square or hexagonal unit cells. The full dispersion diagram is shown in (iii) for with asymptotic (squares) and FE (solid lines). For a fixed wavenumber, , we show the frequency variation with on the first branch (iv) and second branch (v). The regular and dashed Roman enumerated quantities denote a square and hexagonal primitive cell configuration respectively.
5 Dispersion curves

Having developed the asymptotic technique and implemented this methodology numerically, we now compare and contrast with dispersion curves obtained from full numerical simulations using the open source finite element (FE) package FreeFEM++ [13]. We begin as shown in figure 3, by considering a single inclusion within a square, or hexagonal, cell with dispersion curves shown for both FE (solid), and from the asymptotics (squares). We choose $\epsilon = 0.1$ (i.e. relatively large for such an asymptotic scheme) and note that there is still a pleasing agreement even for the higher branches in the dispersion diagram; the discrepancy as $\epsilon$ increases is illustrated in figure 3 for a typical wavevector. As expected, from the matching procedure, both (57) and (62) lose their validity as $\epsilon \Omega$ approaches order unity - the asymptotic solution ultimately breaks down.

Although such agreement is pleasing, our primary aim is to employ the asymptotic scheme for clusters of inclusions within a primitive cell, and in particular use the scheme as a rapid route for prototyping or optimising effects. Figure 4 shows an arrangement of inclusions, chosen to have specific symmetries such that a symmetry-induced Dirac point occurs at the $K$ point in the dispersion diagram; this underlies so-called valley-Hall edge states and we use this geometrical arrangement to illustrate that the asymptotic scheme is capable of generating these, along with the underlying numerics required to interpret them. [29] used group theoretic arguments to demonstrate that by having point group symmetries of $C_{3v}$, at both $\Gamma$ and $K$, it would guarantee the presence of a Dirac cone; the geometry chosen here is case (ii) of [29]. The topological effects occur due to the breaking of the mirror symmetry by rotating the system of inclusions, lowering the point group symmetry to $C_{3}$, and gapping the Dirac point to open a band-gap. For the purposes of the asymptotic scheme it is interesting to note that in figure 4, the lowest dispersion curves permits the symmetry induced Dirac point, which is at a low enough frequency to be well captured by the scheme - certainly well enough that one can explore the topological valley-Hall effect.

6 Topological mode steering in a planar array of Neumann scatterers

6.1 Time-averaged energy flux for a symmetry-induced topological system

The earlier sections focused on deriving asymptotically accurate formulae dealing with the singular behaviour present within a divergent sum. Here we give a topical example on how the use of these formulae allows one to expedite computations of the energy flux; this is a physically useful quantity that allows us to determine whether a state is topologically protected. The energy flux is defined as follows,

$$
\langle F \rangle = -\frac{1}{2} \Re \left\{ \left[ -i \Omega \phi \right] \left[ \nabla \phi \right] \right\} .
$$

(36)

By utilising (14) and $\nabla(14)$ or (30) and $\nabla(30)$ we are able to rapidly compute (36); as demonstrated in figure 4(iv). Upon symmetry reduction of the cellular structure we obtain the dispersion curves shown in figure 5(iii). Notably, the
...energy fluxes of the modes that demarcate the band gap have opposite chirality (figures 5(iv, v)) and it is precisely this property that imbues the ensuing edge modes with their protective property. The two distinct interfaces, that are constructable using the cells in figures 5(iv, v), are shown in figure 6.

The resulting pair of concave and convex dispersion curves (figure 6) yield modes that are of either even or odd-parity, and are hence physically distinguishable. The coupling between the even and odd-parity modes around different angled bends has been explored in [29, 42] and also in the context of more complicated topological domains in [28]. The time-averaged energy flux of the two distinct edge modes is shown in figure 7 where the accuracy of our numerical scheme is exemplified by the clarity of the orbital motion in these figures. The right/left propagating modes shown are often said to have right/left chiral pseudospins. The near orthogonality of the forward and backwards propagating pseudospin modes (figure 7) is inherited from the bulk solutions (figure 5) via the bulk-boundary correspondence [39]. Protection against backscattering depends upon the orthogonality of these opposite pseudospin states and it has been shown to be approximately valid for small band gaps [8].

6.2 Modal coupling between topologically distinct domains

To demonstrate the utility of our extended Foldy (section 4) and homogeneous Foldy (section 4.1) computations, we show how the distinct modes in figure 6 couple around different angled bends. Transport of energy around corners in structured media is of inherent interest across wave physics [34, 4, 26]. The modal symmetries are indispensable for determining whether or not energy will couple around a bend or along parallel interfaces [28, 42]. The majority of the valley-Hall literature, to name but a few [50, 49, 41, 25, 23, 17, 10, 3], uses a Z-
Fig. 5: Gapping the Dirac point at $K$ \((iii)\) by perturbing the structure present in figure 4 via a $-\frac{\pi}{6}$ rotation as shown in \((i)\). Panels \((iv)\) and \((v)\) show the eigensolution and associated flux at the blue and red cross respectively.

shaped interface to demonstrate robustness of the modes. However this design, that solely contains $\pi/3$ bends, does not result in modal conversion between the even and odd-parity edge modes (figure 6). To clearly demonstrate both the modal conversion and modal preservation cases by the use of one all encompassing figure, we use a double Z configuration (figure 8). The displacement pattern, shown here, uses the homogeneous-Foldy method developed in section 4.1. Interestingly it is solely along the gentle $2\pi/3$ bend (♣) in which the edge state undergoes modal conversion; along the left-sided interface (♠) there is modal preservation as the energy traverses a $\pi/3$ bend. The modal differences between the edge states along the two vertical interfaces (♣ and ♠) is further exemplified by the fluxes shown.

For completeness, we also perform the usual Foldy scattering calculation (section 4) which generates figure 9. Expectedly, this scattering solution mirrors the homogeneous-Foldy solution shown in figure 8. The examples in this subsection demonstrate how our semianalytic convergent expressions allow us to obtain highly resolved and precise edge states. The clarity of the solutions obtained is of paramount importance as they allow us to interpret the relative interface orientations with ease.
Fig. 6: The dispersion relation along the singly periodic infinite ribbon constructed of two medium derived from the primitive cells in figure 5. Medium 1, built of blue cells consisting of a $-\frac{\pi}{6}$ rotation of the original structure. Medium 2, similarly for the red, consisting of a $\frac{\pi}{6}$ rotation of the original structure. The grey section represents the band gap in figure 5 in which we observe the purple and green edge states coexisting for a certain range of frequencies. The wave field $\phi$ has been plotted for the edge states within the extremities of the figure next to the arrangement (medium 1 over 2 or 2 over 1) over which the edge state resides. The leftward and rightward wavefields are that of the green and purple interfacial modes respectively - specifically at the crosses where $\Omega = 3.73$.

Fig. 7: Fluxes corresponding to the ribbons shown in figure 6. Rightward ($+\kappa$) and leftward ($-\kappa$) propagating modes for both of the geometrically distinct interfaces are shown.

6.3 Chiral beaming in the propagating regime

The modal conversion and preservation effect outlined in the preceding section occurs within the bulk band-gap frequency range (figure 6). Contrastingly, in this subsection, we operate within the propagating regime of the bulk. We use our succinct asymptotic formulae to show how opposite pseudospin modes are
Fig. 8: Double-Z schematic containing 3760 cells half blue half red forming a structure with a total of 15040 scatterers (top left). The homogeneous-Foldy solution (top right) with enlarged sections ♠ and ♣ showing the associated flux across these sections clearly showing different interfacial modes (same frequency different $\kappa$) as in figure 7. Here $\Omega = 3.73$.

Fig. 9: Extended Foldy solution for the example shown in figure 8. Isotropic source placed within the white circle ignites a leftward and rightward propagating even-parity edge mode. The white arrow indicates the sole interface that hosts an odd-parity state. Again $\Omega = 3.73$. 
ousted from a carefully placed isotropic source (figure 10). This effect is more commonly referred to as chiral beaming and, ordinarily, the source is placed along the boundary between a topologically nontrivial domain and a homogeneous region [24]. Here, we opt to place our source along the interface between two topologically distinct domains, as shown by the schematic in figure 10. We clearly see two highly localized pulses beam into the upper (blue) and lower (red) domains. The angular difference between the pulses, within either the blue or red domain, is attributed to the \( \pi/3 \) rotational difference between the \( K \) and \( K' \) isofrequency contours; the contours for the upper domain, superimposed onto the Brillouin zone, are shown in the rightmost panel of figure 10. Interestingly, the contours for the lower domain are \( \pi/3 \) rotations of the upper domain; hence, in the upper medium a right (left) pseudospin mode, associated with \( K(\bar{K}') \) beams off to the right (left) whilst in the lower region the pseudospin directions are switched. The modes with common chirality or pseudospin are indicated by identically coloured arrows. This chiral beaming phenomenon occurs near the standing wave frequency that demarcates the band gap and hence residual energy associated with the edge state is also shown to propagate along the interface.

Fig. 10: Schematic of domain (left) containing 3780 cells, half blue and half red, forming a structure with a total of 15120 scatterers. An isotropic source positioned along the interface and excited at the frequency \( \Omega = 3.06 \) results in the central displacement pattern. This was calculated using the extended Foldy’s method (section 4). The anisotropic behaviour is explained by examining the isofrequency contours of the upper medium (right). The lower medium’s contours are a \( \pi/3 \) rotation of the upper medium’s.

7 Concluding remarks

Herein we have used matched asymptotic expansions to derive a rapidly convergent numerical scheme, which allows us to derive eigensolutions for a Helmholtz system consisting of an array of Neumann scatterers. Subsequently, we extended the scattering computations in [40] and outlined how the scattering coefficients and matrices were deducible through an extended Foldy scheme. A simplified
A Suitable Green’s functions

We consider the method taken by Graff [12] pp. 284-285, which is easily extended to consider dipole source terms. The approach is based upon considering the Fourier transform of the system which is pertinent to us since, if one considers the Fourier series representation (refer to B), then one may apply this technique to evaluate the integrals required to find \( \phi_r \). Considering the Green’s function only for the monopole source term

\[
(\nabla^2 + \Omega^2)\phi_{\text{mono}} = a\delta(x - X).
\]

(37)

We define the two dimensional Fourier transform and inverse pair as follows

\[
\hat{f}(\xi) = \mathcal{F}[f(x); \xi] = \int_{-\infty}^{\infty} f(x) e^{-i \xi \cdot x} dx,
\]

\[
f(x) = \mathcal{F}^{-1}[\hat{f}(\xi); x] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i \xi \cdot x} d\xi.
\]

(38)

Following Graff [12] and making the above changes, one finds the axisymmetric component of the Green’s function

\[
\phi_{\text{mono}} = \frac{a}{4i} H_0(\Omega r).
\]

(39)

This technique can be extended for dipole sources, the Green’s function being given in Martin [31]; however we derive it so there can be no doubt in consistency with the form used. Consider

\[
(\nabla^2 + \Omega^2)\phi_{\text{di}} = b \cdot \nabla \delta(x - X).
\]

(40)

We take the Fourier transform of the above equation, using the definition (38) referring to figure 11 (i) it may be shown

\[
\phi_{\text{di}} = \frac{b \cos(\varphi - \sigma)}{4i\pi^2} \int_0^\infty \int_{-\infty}^{2\pi - \varphi} \xi^2 \cos(\Theta) \exp[i\xi r \cos(\Theta)] d\Theta d\xi,
\]

(41)

where \( \Theta = \theta - \varphi \). Integrating over one period, referring to Abramovitz & Stegun [1], we utilise the integral representations of the Bessel functions of integer order. Various infinite integrals of combinations between powers and Bessel functions are known, and referring to Gradshteyn and Ryzhik [11] one requires

\[
\int_0^\infty J_\nu(bx)x^{\mu+1} dx = \frac{a^{\nu-\mu}b^\mu}{2\nu(\mu!)^{\nu+1}} K_{\nu-\mu}(ab),
\]

when \(-1 < \Re\{\nu\} < \Re\{2\mu + 1\}\), \(a > 0, b > 0\).
where $K_n$ refers to the modified Bessel function of the second kind and $n$th order. The relationship between $K_n$ and $H_n$ is given in Watson [46], and utilising the above it can be shown that

$$\phi_{di} = -\frac{b \cdot \hat{r}}{4i} \Omega H_1(\Omega r).$$

(43)

![Diagram](image)

Fig. 11: The required vector quantities in Fourier space to derive:

(i) – The Green’s functions (39) and (43).

(ii) – The residual portion of the Fourier series, that is $\phi_r$, in (46).

B Determining $\phi_r$

The residual part of the sum will be calculated by means of extending the method in A as follows - referring to figure (11) (ii) - assuming $R' \gg 1$, we may utilize a Laurent series expansion to the frequency dependent denominator within the definition of $\phi_r$ (14), denote $|G| = G$, $|b| = b$ and $|\kappa| = \kappa$

$$\phi_r = \frac{4}{iG^2} \exp(i\kappa \cdot r) \sum_{G > R'} \epsilon^2 (a - i b \cdot K_G) \cdot \left[ 1 - \frac{2\kappa \cos(\theta_G - \theta_\kappa)}{G^3} + \frac{4\kappa^2 \cos^2(\theta_G - \theta_\kappa) + \Omega^2 - \kappa^2}{G^4} + O\left(\frac{1}{G^5}\right) \right] \exp[iGr \cos(\theta_G - \phi)]$$

(44)

The sum (44) may be expressed as an integral over $G$ considered in radial polars - in a similar fashion to the “Double-sum asymptotics” within [40] and A - the integral over $\theta_G$ is evaluated by means of the definition of integral representations of the Bessel functions of integer order [1]. Subsequent direct
integration over \( G \), one finds\(^1\)

\[
\phi_r \sim \exp(i k \cdot r) \left\{ \frac{2 b \cdot \hat{r}}{i \pi} J_0(R' r) + J_{i0}(R' r) \left[ ir(\text{i} b \Omega^2 + 2 a \kappa) \cdot \hat{r} - 2 a \right] - J_i(R' r) \left[ 2 i b \kappa \cos(2 \varphi - \theta_R - \sigma) + ir(\text{i} b \Omega^2 + 2 a \kappa) \cdot \hat{r} \right] - 2 b c^2 \cos(\sigma + 2 \theta_R - 3 \varphi) J_j(R' r) \right\}.
\]

(46)

Note \( R' \) is some large but finite value. To build the eigenvalue problem, one requires (16), that is the above in the inner limit as \( r \to 0 \) - here \( r \) should dominate the limit, that is to say \( r R' \ll 1 \). The behaviour of Bessel functions in such a limit is well known. Moreover from Humbert [14]

\[
J_{i0}(x) = Ci(x) - \log 2 = \log x/2 + \gamma_E - \frac{x^2}{4} + O(x^4) \quad \text{as} \quad x \to 0,
\]

(47)

Here \( Ci(x) \) is the cosine integral function.

\section{Matched asymptotics}

The matching procedure is defined within section 6.3 “Formal approach to matching” in Crighton et al. [6]. The outer solution to the wavefield is defined \( \phi = \phi(r, \varphi; \epsilon) \) and is a function of outer variable \( r \). The inner solution to the wavefield is defined \( \Phi = \Phi(R, \varphi; \epsilon) \) and is a function of the inner variable \( R \). Both \( \phi \) and \( \Phi \) are to be expressed as asymptotic expansions dependent upon the small parameter \( \epsilon \), where

\[
r = \epsilon R.
\]

(48)

Denote \( \phi^{(n)}(r, \varphi; \epsilon) \) to represent \( \phi \) correct up to and including \( O(\epsilon^n) \); moreover, the inner limit of \( \phi^{(n)}(r, \varphi; \epsilon) \), correct up to and including \( O(\epsilon^n) \) is denoted \( \phi^{(n,m)} \) - explicitly

\[
\phi^{(n)}(r = \epsilon R, \varphi; \epsilon) = \phi^{(n,m)}(R, \varphi; \epsilon) + O(\epsilon^m).
\]

(49)

Similarly for the outer solution, \( \Phi \) correct up to \( O(\epsilon^m) \) is denoted \( \Phi^{(m)}(R, \varphi; \epsilon) \), where it is appropriate to examine the outer limit of the inner solution correct up to and including \( O(\epsilon^n) \)

\[
\Phi^{(m)}(R = \epsilon R, \varphi; \epsilon) = \Phi^{(m,n)}(r, \varphi; \epsilon) + O(\epsilon^n).
\]

(50)

The matching procedure describes the overlap of \( \phi^{(n,m)} \) and \( \Phi^{(m,n)} \) as follows

\[
\phi^{(n,m)} \equiv \Phi^{(m,n)}
\]

(51)

The above procedure is naturally consistent with Van Dyke’s matching rule [44], in which \( \log \epsilon \) terms are regarded as order unity [40, 6].

\(^1\)Here we utilise the aforementioned Dr Van der Pol’s Bessel-integral function of zero order [14], \( J_{i0} \). Interestingly \( J_{i0} \) can be used to evaluate integrals containing combinations of Bessel functions and powers, of the form (45), integrating by parts we find a recursive relation which can be exploited to find

\[
\int_{x}^{\infty} J_n(\xi) \xi^{n+1} d\xi = \left\{ \sum_{m=1}^{n} \left( \frac{(m-1)!}{n!} \right) \frac{(m)!J_m(x)}{m^{2(n-m+1)}} \right\} - \frac{1}{2^m n!} J_{i0}(x), \quad \text{where} \quad n \in \mathbb{N} : n \neq 0.
\]

(45)
C.1 Outer region

The matching procedure is eased with a traditional outer and inner expansion around one inclusion. Subsequently (3) is modified

$$(\nabla^2 + \Omega^2)\phi = 4i\epsilon^2 \{a\delta(x - X) - b \cdot \nabla [\delta(x - X)]\}. \quad (52)$$

Motivated by [40] we consider the outer solution into two parts

$$\phi = \chi + \psi = \chi_0 + \epsilon^2 (\chi_1 + \psi_1) + \ldots. \quad (53)$$

where $\chi$ and $\psi$ denote the complimentary and particular solution of (52). The form of the complementary solution is obtained via separation of variables,

$$\chi_{i=0,1} = \sum_n (B_{1n} \cos n\varphi + B_{2n} \sin n\varphi)J_n(\Omega r) + (C_{1n} \cos n\varphi + C_{2n} \sin n\varphi)Y_n(\Omega r), \quad (54)$$

$\chi$ should not be singular hence the coefficients of $Y_n(\Omega r)$ are set to zero. $\phi_1$ is given by the Green’s functions (39) and (43)

$$\psi_1 = aH_0(\Omega r) + (b \cdot \hat{r})\Omega H_1(\Omega r). \quad (55)$$

One is granted certain free reign over the solution, so long as (52) is satisfied. To best emulate the sound hard inclusions it is best that $\frac{\partial \phi}{\partial r} \bigg|_{r = \epsilon} = 0$ is satisfied. Subsequently the traditional outer solution is given by

$$\phi^{(2)}(r, \varphi) = \frac{4i}{\pi} \left\{ \frac{a}{\Omega^2} J_0(\Omega r) - \frac{b \cdot \hat{r}}{\Omega} J_1(\Omega r) \right\} + \epsilon^2 \left\{ -a \left[ 1 + \frac{2i}{\pi} \left( \log \frac{\epsilon \Omega}{2} + \gamma_E - \frac{3}{4} \right) \right] J_0(\Omega r) - \right.$$

$$- \left. \Omega b \cdot \hat{r} \left[ \frac{2i}{\pi} \left( \log \frac{\epsilon \Omega}{2} + \gamma_E + \frac{5}{4} \right) + 1 \right] J_1(\Omega r) + aH_0(\Omega r) + (b \cdot \hat{r})\Omega H_1(\Omega r) \right\}. \quad (56)$$

The inner limit of the outer solution is examined below

$$\phi^{(2,3)}(R, \varphi) = \frac{4i}{\pi} \left\{ \frac{a}{\Omega^2} - \frac{b \cdot \hat{r}}{\Omega} \frac{\Omega}{2} \left[ \frac{R + 1}{R} \right] \right\} + \epsilon^2 a \left[ \frac{2i}{\pi} \left( \log R + \frac{3}{4} - \frac{R^2}{2} \right) \right] +$$

$$+ \epsilon^3 b \cdot \hat{r} \left[ \frac{i \Omega^2 R}{\pi} \left( \log R - \frac{7}{4} + \frac{R^2}{4} \right) \right]. \quad (57)$$

C.2 Inner region

We infer, from equation (57), that the inner solution should be asymptotically expanded as

$$\Phi(R, \varphi; \epsilon) = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \epsilon^3 \Phi_3 + \ldots, \quad (58)$$

and should satisfy the following problem

$$\left( \nabla^2 + \epsilon^2 \Omega^2 \right) \Phi = 0, \quad (59)$$

$$\frac{\partial \Phi}{\partial R} = 0, \quad \text{on } R = 1. \quad (60)$$
Formally, we work within a low-frequency regime - the asymptotics developed will only hold if the $\epsilon^2 \Omega^2$ term within (59) is of $o(1)$. The solution in the inner limit is found by considering (59) to the orders indicated in (58). Solving each $\Phi_i$ in polar coordinates is not a difficult task, since the dependence on $\varphi$ is known whether each term matches with a monopole or dipole like term. Matching to (57) at various orders determines all unknown constants in the inner solution. Subsequently

$$\Phi^{(3)} = \frac{4i}{\pi} \frac{\alpha}{\Omega^2} + \epsilon \left[ \frac{2}{i \pi} \left( \frac{1}{R} + R \right) \left[ b \cdot \hat{r} \right] + \epsilon^2 \frac{2a_i}{\pi} \left\{ \log R + \frac{3}{4} - \frac{1}{2} R^2 \right\} + \epsilon^3 \frac{i b \cdot \hat{r}}{\pi} \Omega^2 \left\{ R \log R - \frac{7}{4} R + \frac{1}{4} R^3 \right\} \right].$$

Therefore $\Phi^{(3,2)}$ may be determined, which yields the inner solution in the outer limit - and is tentatively expressed as

$$\lim_{r \to 0} \phi = \Phi^{(3,2)} = \frac{4i}{\pi} \frac{\alpha}{\Omega^2} \left[ \log \frac{7}{4} - \frac{1}{2} \right] + \frac{4}{i \pi} \frac{b \cdot \hat{r}}{\Omega} \left\{ \log R - \frac{7}{4} R + \frac{1}{4} R^3 \right\}. \quad (62)$$

### D Computing the time-averaged energy flux

Lighthill [20] remarks that in linear systems equations of motion are dominated by small quantities whose square terms are deemed negligible; however statements expressing energy are dominated by second order terms. Additionally any expressions determined should be consistent with the conservation of acoustic energy equation

$$\frac{\partial E}{\partial t} + \nabla \cdot F = 0. \quad (63)$$

Here $E$ is the acoustic energy with corresponding flux $F$. Consider the two dimensional wave equation - the analogue of equation 1 - multiplying by $\frac{\partial \phi'}{\partial t}$ allows for a factorization concurrent with (63). Assuming the wavefield is time-harmonic the period averaged energy flux may be expressed as

$$\langle \Phi \rangle = \frac{1}{2} \Re \left\{ \left[ -i \Omega \phi \right] [\nabla \phi] \right\}, \quad (64)$$

where the overbar denotes complex conjugation. The expression in (64) can be utilized both in the the primitive cell in physical space from the eigensolution and in the Foldy solution.
The Explicit components within equation (23)

The components of $\mathbf{A}$, $\mathbf{B}$ and $\Phi$ are

$$\mathbf{A}_{[(N+3P)\times(N+3P)]} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}, \quad (65)$$

$$\mathbf{B}_{[(N+3P)\times(N+3P)]} = \begin{pmatrix} 1' & 2' & 3' & 4' \\ 5' & 6' & 7' & 8' \\ 9' & 10' & 11' & 12' \\ 13' & 14' & 15' & 16' \end{pmatrix}. \quad (66)$$

Here the subscript $[P \times Q]$ denotes the dimensions of a matrix with $P$ rows and $Q$ columns. We have factorised the eigenvalue problem with the eigenvector $\Phi_{[(N+3P)\times1]}$, containing all of the unknowns, as follows

$$\Phi^\dagger = \begin{pmatrix} \Phi_{G_1} & \ldots & \Phi_{G_N} & a_1 & \ldots & a_P & b_1 & \ldots & b_{1P} & b_2 & \ldots & b_{2P} \end{pmatrix}. \quad (67)$$

The block matrices forming 65 and 66 are

$$\begin{align*}
1_{rc} &= \mathbf{K}_{G_r} \cdot \mathbf{K}_{G_c} \cdot \delta_{rc} \quad r, c = 1, \ldots, N. \\
2_{rc} &= \frac{4ic_1^2}{\sigma} \exp(-i\mathbf{K}_{G_r} \cdot \mathbf{X}_{1c}) \quad r = 1, \ldots, N \quad c = 1, \ldots, P. \\
3_{rc} &= \frac{4c_2^2}{\sigma} \mathbf{K}_{G_r} \cdot \mathbf{K}_{G_c} \exp(-i\mathbf{K}_{G_r} \cdot \mathbf{X}_{1c}) \quad r = 1, \ldots, N \quad c = 1, \ldots, P. \\
4_{rc} &= \frac{4c_2^2}{\sigma} \mathbf{K}_{G_r} \cdot \mathbf{K}_{G_c} \cdot \exp(-i\mathbf{K}_{G_r} \cdot \mathbf{X}_{1c}) \quad r = 1, \ldots, N \quad c = 1, \ldots, P. \\
6_{rc} &= \frac{4}{i\pi} \delta_{rc} \quad r, c = 1, \ldots, P. \\
9_{rc} &= i\mathbf{K}_{G_r} \cdot \mathbf{K}_{G_c} \cdot \exp(i\mathbf{K}_{G_r} \cdot \mathbf{X}_{1r}) \quad r = 1, \ldots, P \quad c = 1, \ldots, N. \\
10_{rc} &= -\frac{\sigma c_1^2}{\pi} \delta_{rc} \quad r, c = 1, \ldots, P. \\
11_{rc} &= \frac{i}{\pi} \left[ 2 + \frac{c_1^2 R^2}{2} + \frac{\sigma^2}{4} (\kappa_1^2 - \kappa_2^2) \right] \delta_{rc} \quad r, c = 1, \ldots, P. \\
12_{rc} &= \frac{c_1^2}{2\pi} \kappa_1 \kappa_2 \delta_{rc} \quad r, c = 1, \ldots, P. \\
13_{rc} &= i\mathbf{K}_{G_r} \cdot \mathbf{K}_{G_c} \cdot \exp(i\mathbf{K}_{G_r} \cdot \mathbf{X}_{1r}) \quad r = 1, \ldots, P \quad c = 1, \ldots, N. \\
14_{rc} &= -\frac{\sigma c_1^2}{\pi} \delta_{rc} \quad r, c = 1, \ldots, P.
\end{align*} \quad (68)-(78)
the matrices, and all zero. Here \( r \) of any block matrices, forming \( F \) within figure 2.

\[
\rho_{rc} = \frac{\epsilon_c^2}{2m} \kappa_1 \kappa_2 \delta_{rc} \quad r, c = 1, \ldots, P. \quad (79)
\]

\[
\sigma_{rc} = \frac{i}{\pi} \left[ 2 + \frac{\epsilon_c^2 R^2}{2} - \frac{\epsilon_c^2}{4} (\kappa_1^2 - \kappa_2^2) \right] \delta_{rc} \quad r, c = 1, \ldots, P. \quad (80)
\]

\[
\tau_{rc} = \delta_{rc} \quad r, c = 1, \ldots, N. \quad (81)
\]

\[
\rho_r = -\exp(iK_G \cdot \mathbf{X}_r) \quad r = 1, \ldots, P \quad c = 1, \ldots, N. \quad (82)
\]

\[
\sigma_{rc} = \frac{\epsilon_c^2 2i}{\pi} \left[ \log \frac{2}{\epsilon_c R} + \frac{3}{4} - \gamma_E \right] \delta_{rc} \quad r, c = 1, \ldots, P. \quad (83)
\]

\[
\tau_{rc} = -\frac{\epsilon_c^2}{\pi} \kappa_1 \delta_{rc} \quad r, c = 1, \ldots, P. \quad (84)
\]

\[
\omega_{rc} = -\frac{\epsilon_c^2}{\pi} \kappa_2 \delta_{rc} \quad r, c = 1, \ldots, P. \quad (85)
\]

\[
\rho_r = \frac{\epsilon_c^2}{\pi} \left[ \log \frac{2}{\epsilon_c R} - \frac{5}{4} - \gamma_E \right] \delta_{rc} \quad r, c = 1, \ldots, P. \quad (86)
\]

In (67) the superscript \( \dagger \) denotes the transpose operation. The components of any block matrices, forming \( A \) or \( B \), which are “missing” from this list are all zero. Here \( r \) and \( c \) denotes the row and column number respectively of the matrices, and \( \delta_{rc} \) denotes the Kronecker delta function. \( G_{i} \) denotes the \( i \)th arbitrary reciprocal position vector for the \( i = 1, \ldots, N \) Bloch modes considered within figure 2.

**F The Explicit components within the extended Foldy system**

The constituent matrices forming \( \otimes \) from equation (34) are

\[
\rho_{rc} = \left( \frac{4i}{\pi \Omega} - \epsilon_c^2 \left[ 1 - \frac{2i}{\pi} \left( \log \frac{2}{\epsilon_c \Omega} + \frac{3}{4} - \gamma_E \right) \right] \right) \delta_{rc} - \epsilon_c^2 H_0(\Omega r_c) \left\{ 1 - \delta_{rc} \right\} \quad (87)
\]

\[
\sigma_{rc} = -\epsilon_c^2 \Omega \cos \varphi_{rc} H_1(\Omega r_c) \left\{ 1 - \delta_{rc} \right\} \quad (88)
\]

\[
\tau_{rc} = -\epsilon_c^2 \Omega \sin \varphi_{rc} H_1(\Omega r_c) \left\{ 1 - \delta_{rc} \right\} \quad (89)
\]

\[
\omega_{rc} = -\frac{\epsilon_c^2}{\pi} \kappa_1 \delta_{rc} \quad (90)
\]

\[
\rho_{rc} = \left[ \frac{2}{i \pi} + \frac{\epsilon_c^2 \Omega^2}{\pi} \left( \log \frac{2}{\epsilon_c \Omega} - \frac{5}{4} - \gamma_E \right) - \frac{\epsilon_c^2 \Omega^2}{2} \right] \delta_{rc} - \epsilon_c^2 \left\{ \frac{\Omega^2 \cos^2 \varphi_{rc}}{2} \left[ H_0(\Omega r_c) - H_2(\Omega r_c) \right] + \frac{\Omega}{r_c} \sin^2 \varphi_{rc} H_1(\Omega r_c) \right\} \left\{ 1 - \delta_{rc} \right\} \quad (91)
\]

\[
\sigma_{rc} = -\epsilon_c^2 \cos \varphi_{rc} \sin \varphi_{rc} \left\{ \frac{\Omega^2}{2} \left[ H_0(\Omega r_c) - H_2(\Omega r_c) \right] + \frac{\Omega}{r_c} H_1(\Omega r_c) \right\} \cdot \left\{ 1 - \delta_{rc} \right\} \quad (92)
\]
\( \mathbf{J}_{fc} = - \mathbf{3}_{fc} \)  
(93) 
\( \mathbf{S}_{fc} = \mathbf{6}_{fc} \)  
(94) 
\[
\mathcal{G}_{fc} = \left[ \frac{2}{\pi} + \frac{e_\phi^2 \Omega^2}{\pi} \left( \log \frac{2}{\epsilon_\phi \Omega} - \frac{5}{4} - \gamma_E \right) - \frac{e_\phi^2 \Omega^2}{2} \right] \delta_{fc} - e_\phi^2 \left\{ \frac{\Omega^2 \sin^2 \varphi_{inc}}{2} \left[ H_0(\Omega r_{fc}) - H_2(\Omega r_{fc}) \right] + \frac{\Omega}{r_{fc}} \cos^2 \varphi_{inc} \cos \varphi_{inc} H_1(\Omega r_{fc}) \right\} \left\{ 1 - \delta_{fc} \right\}
\]  
(95) 

In the above \( \tilde{r}, c = 1, \ldots, m \), denotes the row and column of the block matrices assembling the scheme (34). The incident field is inserted into the scheme as follows

F.0.1 Monopole incident source

\[
\phi_{incs} \tilde{r} = \frac{e_{\min} a_{inc}}{4i} H_0(\Omega|\mathbf{X}_\tilde{r} - \mathbf{X}_{inc}|), \quad \tilde{r} = 1, \ldots, m.
\]  
(96) 

\[
\nabla \phi_{inc1s} \tilde{r} = \frac{e_{\min} a_{inc}}{4 \cos \varphi_{inc} \Omega} H_1(\Omega|\mathbf{X}_\tilde{r} - \mathbf{X}_{inc}|), \quad \tilde{r} = 1, \ldots, m.
\]  
(97) 

\[
\nabla \phi_{inc2s} \tilde{r} = \frac{e_{\min} a_{inc}}{4 \sin \varphi_{inc} \Omega} H_1(\Omega|\mathbf{X}_\tilde{r} - \mathbf{X}_{inc}|), \quad \tilde{r} = 1, \ldots, m.
\]  
(98) 

F.0.2 Dipole incident source

\[
\phi_{incs} \tilde{r} = \frac{e_{\min} a_{inc}}{4 \Omega} \left[ b_{inc} \cos \varphi_{inc} H_1(\Omega|\mathbf{X}_\tilde{r} - \mathbf{X}_{inc}|) \right]
\]  
(99) 

Similarly to (96)-(98) the required gradient terms \( \nabla \phi_{inc1s} \) and \( \nabla \phi_{inc2s} \) can be determined for the Dipole source term.

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