CIRCULAR LAW, EXTREME SINGULAR VALUES AND
POTENTIAL THEORY

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ABSTRACT. Consider the empirical spectral distribution of complex random \( n \times n \) matrix whose entries are independent and identically distributed random variables with mean zero and variance \( 1/n \). In this paper, via applying potential theory in the complex plane and analyzing extreme singular values, we prove that this distribution converges, with probability one, to the uniform distribution over the unit disk in the complex plane, i.e. the well known circular law, under the finite fourth moment assumption on matrix elements.

1. INTRODUCTION

Let \( \{X_{kj}\}, k, j = \cdots \), be a double array of independent and identically distributed (i.i.d.) complex random variables (r.v.'s) with \( E X_{11} = 0 \) and \( E|X_{11}|^2 = 1 \). The complex eigenvalues of the matrix \( n^{-1/2}X = n^{-1/2}(X_{kj}) \) are denoted by \( \lambda_1, \cdots, \lambda_n \). The two-dimensional empirical spectral distribution \( \mu_n(x, y) \) is defined as

\[
\mu_n(x, y) = \frac{1}{n} \sum_{k=1}^{n} I(\text{Re}(\lambda_k) \leq x, \text{Im}(\lambda_k) \leq y).
\]

The study of \( \mu_n(x, y) \) is related to understanding the random behavior of slow neutron resonances in nuclear physics. See [16]. Since 1950’s it has been conjectured that, under the unit variance condition, \( \mu_n(x, y) \) converges to the so-called circular law, i.e. the uniform distribution over the unit disk in the complex plane. Up to now, this conjecture is only proved in some partial cases.

The first answer for complex normal matrices was given in [16] based on the joint density function of the eigenvalues of \( n^{-1/2}X \). Huang in [11] reported that this result was obtained in an unpublished paper of Silverstein (1984). After

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more than one decade, Edelman [7] also showed that the expected empirical spectral distribution converges to the circular law for real normal matrices. It is Girko who investigated the circular law for general matrix with independent entries for the first time in [8]. But Girko imposed, not only moment conditions, but also strong smooth conditions on matrix entries. Later on, he further published a series of papers (for example, [9]) about this problem. However, as pointed out in [1] and [10], Girko’s argument includes serious mathematical gaps. The rigorous argument of the conjecture was given by Bai in his 1997 celebrated paper [1] for general random matrices. In addition to the finite \( (4 + \varepsilon)th \) moment condition Bai still assumed that the joint density of the real and imaginary part of the entries is bounded. Again, the result was further improved by Bai and Silverstein under the assumption \( E|X_{11}|^{2+\eta} < \infty \) in their comprehensive book [2], but the finiteness condition of the density of matrix entries is still there. Recently, Götze and Tikhomirov [10] gave a proof of the convergence of \( E\mu_n(x, y) \) to the circular law under the strong moment assumption that the entries have sub-Gaussian tails or are sparsely non-zero instead of the condition about the density of the entries in [1].

Generally speaking, there are five approaches to studying the spectral distribution of random matrices. The difficulty of the circular conjecture is that the methodologies used in Hermitian matrices do not work well in non-Hermitian ones. There was no powerful tool to attack this conjecture.

1. **Moment method.** Moments are very important characteristics of r.v.’s. They have many applications in probability and statistics. For example, we have moment estimators in statistics. As far as we know, it is Wigner [23] [24] who introduced moment method into random matrices. Since then, the moment method has been very successful in establishing the convergence of the empirical spectral distribution of Hermitian matrices. Bai did a lot of important work. One can refer to [2]. But moment method fails to work in non-Hermitian ones, because for any complex r.v. \( Z \) uniformly distributed over any disk centered at 0, one can verify that for any \( m \geq 1 \)

\[
EZ^m = 0.
\]

2. **Stieltjes transform.** Another powerful tool in random matrices theory is the Stieltjes transform, which is defined by

\[
m_G(z) := \frac{1}{\lambda - z}dG(\lambda), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C}, \ Im(z) > 0\},
\]

for any distribution function \( G(x) \). The basic property of Stieltjes transform is that it is a representing class of probability measures. This property offers one a strong analytic machine. Still see [2] and the references therein. However, the Stieltjes transform of \( n^{-1/2}X \) is unbounded if \( z \) coincides with one eigenvalue.
So this leads to serious difficulties when dealing with the Stieltjes transform of \( n^{-1/2}X \).

3. **Orthogonal polynomials.** The study of orthogonal polynomials goes back as far as Hermite. For the deep connections between orthogonal polynomials and random matrices, one can refer to [3]. Orthogonal polynomials are usually limited to Gaussian random matrices. Moreover, orthogonal polynomials are only suitable to deriving the spacing between consecutive eigenvalues for large classes of random matrices (see [4]).

4. **Characteristic functions.** There is a long history of characteristic functions. In 1810, Laplace used Fourier transform, i.e. characteristic functions to prove central limit theorem for bounded r.v.’s. Then in 1934 P. Lévy reproved Linderberg central limit theorem by characteristic functions. From that time on, characteristic functions are well known to almost every mathematician. Surprisingly, one can not see any application of characteristic functions in random matrices until 1984. Girko combined together the characteristic function of \( \mu_n(x, y) \) and the Stieltjes transform, trying to prove the conjecture in [8]. Developing ideas proposed by Girko [8], Bai reduced the conjecture to estimating the smallest singular value of \( n^{-1/2}X - zI \) in [1]. However, one should note that some uniform estimate of the smallest singular values of \( n^{-1/2}X - zI \) with respect to \( z \) will be required if the method in [1] is employed.

5. **Potential theory.** Potential theory is the terminology given to the wide area of analysis encompassing such topics as harmonic and subharmonic functions, the boundary problem, harmonic measure, Green’s function, potentials and capacity. Since Doob’s famous book [5] appeared, it is widely accepted that potential theory and probability theory are closely related. For example, superharmonic functions correspond to supermartingales.

The logarithmic potential of a measure \( \mu \) (see [19]) is defined by

\[
U^\mu(z) := \int \log \frac{1}{|z - t|} d\mu(t),
\]

where \( \mu(t) \) is any positive finite Borel measure with support in a compact subset of the complex plane. There is also an inversion formula, i.e. \( \mu \) can be defined through \( U^\mu \) as \( d\mu = -(2\pi)^{-1} \Delta U^\mu \), where \( \Delta \) is the two dimensional Laplacian operator. This relation makes Khoruzhenko in [12] suggest to use potential theory to derive the circular law. Then Götze and Tikhomirov in [10] used the logarithmic potential of \( E\mu_n \) convoluted by a smooth distribution to provide a proof for the convergence of \( E\mu_n \) to the circular law with entries being sub-Gaussian or sparsely non-zero.

In this paper, the conjecture, the convergence of \( \mu_n(x, y) \) to the circular law with probability one, is established under the assumption that the underlying
r.v.’s have finite fourth moment. Compared with [10], we work on the logarithmic potential of \( \mu_n(x, y) \) directly, while [10] depends on the logarithmic potential of a convolution of \( E\mu_n(x, y) \) and the uniform distribution on the disk of radius \( r \).

The main result of this paper is formulated as follows.

**Theorem 1.** Suppose that \( \{X_{jk}\} \) are i.i.d. complex r.v.’s with \( EX_{11} = 0 \), \( E|X_{11}|^2 = 1 \) and \( E|X_{11}|^4 < \infty \). Then, with probability one, the empirical spectral distribution function \( \mu_n(x, y) \) converges to the uniform distribution over the unit disk in two dimensional space.

**Remark 1.** The bounded density condition in [1] and the sub-Gaussian assumption in [10] are not needed any more.

Theorem 1 will be handled by potential theory in conjunction with estimates for the smallest singular value of \( n^{-1/2}X - 2I \).

The research of the smallest singular values originates from von Neumann and his colleagues. They guessed that

\[(1.4) \quad s_n(X) \sim n^{-1/2} \quad \text{with high probability},\]

with \( s_n(X) \) being the smallest singular value of \( X \). Edelman in [6] proved it for random Gaussian matrices, i.e., for each \( \varepsilon \geq 0 \)

\[(1.5) \quad P(s_n(X) \leq \varepsilon n^{-1/2}) \sim \varepsilon.\]

Rudelson and Vershynin in [18] solved it for real random matrices, i.e., for every \( \delta > 0 \) there exist \( \varepsilon > 0 \) and \( n_0 \) depending only on \( \delta \) and the fourth moment of \( X_{jk} \) so that

\[(1.6) \quad P(s_n(X) \leq \varepsilon n^{-1/2}) \leq \delta \quad \text{for all} \quad n \geq n_0.\]

Moreover, since (1.5) fails to hold for the random sign matrices (\( X_{jk} \) being symmetric \( \pm 1 \) r.v.’s), Spielman and Teng [20] speculated that for random sign matrices for any \( \varepsilon \geq 0 \)

\[(1.7) \quad P(s_n(X) \leq \varepsilon n^{-1/2}) \leq \varepsilon + c^n \quad 0 < c < 1.\]

Again, (1.7) has been proved for real random matrices with i.i.d. subgaussian entries in [18].

We will adapt Rudelson and Vershynin’s method to obtain the order of the smallest singular value for complex matrices perturbed by a constant matrix.

Formally, let \( W = X + A_n \), where \( A_n \) is a fixed complex matrix and \( X = (X_{jk}) \), a random matrix. Denote the singular values of \( W \) by \( s_1, \ldots, s_n \).
arranged in the non-increasing order. Particularly, the smallest singular value is
\[ s_n(W) = \inf_{x \in \mathbb{C}^n: \|x\|_2 = 1} \|Wx\|_2, \]
where \( \| \cdot \|_2 \) means Euclidean norm, and we denote the spectral norm of a matrix by \( \| \cdot \| \).

**Theorem 2.** Let \( \{X_{jk}\} \) be i.i.d. complex r.v.’s with \( EX_{11} = 0, E|X_{11}|^2 = 1 \) and \( E|X_{11}|^3 < B \). Let \( K \geq 1 \). Then for every \( \epsilon \geq 0 \),
\[(1.8) \quad P(s_n(W) \leq \epsilon n^{-1/2}) \leq C\epsilon + c^n + P(\|W\| > Kn^{1/2}),\]
where \( C > 0 \) and \( c \in (0, 1) \) depend only on \( K, B, E(Re(X_{11}))^2, E(Im(X_{11}))^2, \) and \( ERe(X_{11})Im(X_{11}) \).

**Remark 2.** In Theorem 2, \( \epsilon \) is arbitrary. It can depend on \( n \). \( K \) is a constant not smaller than 1. In Section 3 when we apply (1.8) in the proof of Theorem 1, we will select \( \epsilon = n^{-1-\delta}, K > 4 \).

**Remark 3.** Theorem 2 includes Theorem 5.1 in [18] as a special case, where \( A_n = 0 \), the r.v.’s are real and have finite fourth moment. Therefore, (1.6) is true with \( X \) replaced by \( W \) when \( X_{11} \) has finite fourth moment and \( \|A_n\| \leq C\sqrt{n} \) (\( C \geq 0 \)), i.e.,
\[ P(s_n(W) \leq \epsilon n^{-1/2}) \leq \delta \quad \text{for all} \quad n \geq n_0. \]
Moreover, if \( X \) is a subgaussian matrix and \( \|A_n\| \leq C\sqrt{n} \), by Lemma 2.4 of [18] or Fact 2.4 of [14], (1.7) holds with \( X \) replaced by \( W \), i.e.,
\[ P(s_n(W) \leq \epsilon n^{-1/2}) \leq \epsilon + c^n \quad 0 < c < 1. \]
This exponential rate is better than the polynomial rate in Tao and Vu [22].

Furthermore, for general random matrices, similar to steps (3.3)-(3.4) in Section 3 one can conclude that

**Corollary 1.** In addition to the assumptions of Theorem 2, suppose that \( |X_{ij}| \leq \sqrt{n}\epsilon_n \) and \( \|A_n\| \leq C\sqrt{n} \) with \( 0 \leq C < \infty \), then for any \( \epsilon \geq 0 \)
\[(1.9) \quad P(s_n(W) \leq \epsilon n^{-1/2}) \leq C\epsilon + n^{-l}, \]
where \( l \) is any positive number and \( \epsilon_n \to 0 \) with the convergence rate slower than any preassigned one as \( n \to \infty \).

**Remark 4.** Taking \( \epsilon = 0 \), Corollary 1 then leads to a polynomial bound for the singularity probability:
\[ P(W_n \text{ is singular}) \leq n^{-l}, \]
with \( l \) being any positive number.
Remark 5. For random sign matrices Tao and Vu [21] showed that for every $A > 0$ there exists $B > 0$ so that
\[
P(s_n(X) \leq n^{-B}) \leq n^{-A}.
\]
Recently, Tao and Vu [22] reported a result concerning the smallest singular value of a perturbed matrix too. Under some mild conditions, they proved that
\[
P(s_n(W) \leq n^{-B}) \leq n^{-A}.
\]
Compared with their results, (1.9) gives an explicit dependence between the bound on $s_n(W)$ and probability, while the relationship between $A$ and $B$ in [21] and [22] is implicit. In addition, (1.9) holds for general random matrices, while Tao and Vu’s theorem basically applies to discrete random matrices.

Remark 6. In this paper, we will use the letters $B, K_1, K_2$ to denote some finite absolute constants.

The argument of Theorem 2 is presented in the next section and the proof of the circular law is given in the last section.

2. Smallest singular value

In this section the smallest singular value of the matrix $X$ perturbed by a constant matrix will be characterized. We begin first with the estimation of the so-called small ball probability.

2.1. Small ball probability. The small ball probability is defined as
\[
P_\varepsilon(b) = \sup_{v \in \mathbb{C}} P(|S_n - v| \leq \varepsilon),
\]
where
\[
S_n = \sum_{k=1}^{n} b_k \eta_k
\]
with $\eta_1, \cdots, \eta_n$ being i.i.d. r.v.’s and $b = (b_1, \cdots, b_n) \in \mathbb{C}^n$ (see [13]). If each $\eta_k$ is perturbed by a constant $a_k \in \mathbb{C}$, then $P_\varepsilon(b)$ does not change, i.e.
\[
P_\varepsilon(b) = \sup_{v \in \mathbb{C}} P(| \sum_{k=1}^{n} b_k (\eta_k - a_k) - v | \leq \varepsilon)
\]

We first establish a small ball probability for big $\varepsilon$ via central limit theorem for complex r.v.’s $\eta_1, \cdots, \eta_n$. Before we state the next result, let us introduce some more notation and terminology. $Re(z)$ and $Im(z)$ will denote the real and imaginary part of a complex number $z$. Write $\eta_{1k} = Re(\eta_k), \eta_{2k} = Im(\eta_k), \sigma_1^2 = \sigma_{1k}^2 = E(\eta_{1k} - E\eta_{1k})^2, \sigma_2^2 = \sigma_{2k}^2 = E(\eta_{2k} - E\eta_{2k})^2, \sigma_{12} = \sigma_{12k} =$
Let \( \xi \) and \( \eta \), if \((E(\xi - E\xi)(\eta - E\eta))^2 = E(\xi - E\xi)^2E(\eta - E\eta)^2 > 0\), then we will say that \( \xi \) and \( \eta \) are linearly correlated.

**Theorem 3.** Let \( \eta_1, \cdots, \eta_n \) be i.i.d. complex r.v.'s with variances at least 1, \( E|\eta|^3 < B \) and let \( b_1, \cdots, b_n \) be complex numbers such that \( 0 < K_1 \leq |b_k| \leq K_2 \) for all \( k \). Then for every \( \varepsilon > 0 \),

\[
P_\varepsilon(b) \leq \frac{C}{\sqrt{n}} \left( \frac{\varepsilon}{K_1} + \left( \frac{K_2}{K_1} \right)^3 \right),
\]

where \( C \) is a finite constant depending only on \( B, \sigma_1, \sigma_2 \) and \( \sigma_{12} \).

**Proof.** Suppose first that \( Re(\eta_k) \) and \( Im(\eta_k) \) are linearly correlated, \( k = 1, \cdots, n \). Then \( \eta_k - E\eta_k = \xi_k(1 + ib_0)/(1 + b_0^2)^{1/2} \) a.s., where \( \xi_k = (1 + b_0^2)^{1/2}Re(\eta_k - E\eta_k) \) and \( b_0 \) is an absolute real constant. Write \( \tilde{b}_k = b_k(1 + ib_0)/(1 + b_0^2)^{1/2} \) which satisfies \( K_1 \leq |\tilde{b}_k| \leq K_2 \). Let \( \tilde{b}_{1k} = Re(\tilde{b}_k) \) and \( \tilde{b}_{2k} = Im(\tilde{b}_k) \). Noting that

\[
\sup_{v \in \mathbb{C}} P(|S_n - v| \leq \varepsilon) \leq \sup_{v \in \mathbb{C}} P(\left| \sum_{k=1}^n \tilde{b}_{1k} \xi_k - Re(v) \right| \leq \varepsilon, \left| \sum_{k=1}^n \tilde{b}_{2k} \xi_k - Im(v) \right| \leq \varepsilon)
\]

and either \( \sum_{k=1}^n \tilde{b}_{1k}^2 \geq nK_1^2/2 \) or \( \sum_{k=1}^n \tilde{b}_{2k}^2 \geq nK_2^2/2 \), we can complete the proof for the linearly correlated case by Berry-Esseen inequality.

The case where \( Re(\eta_k) = 0 \) or \( Im(\eta_k) = 0 \) a.s. follows from Berry-Esseen inequality directly.

Now suppose \( Re(\eta_k) \) and \( Im(\eta_k) \) are not linearly correlated, and \( P(Re(\eta_k) = 0) < 1, P(Im(\eta_k) = 0) < 1 \). Let \( b_k = b_{1k} + ib_{2k} \) and \( v = v_1 + iv_2 \). Define \( \hat{\eta}_{1k} = b_{1k} \eta_{1k} - b_{2k} \eta_{2k} \) and \( \hat{\eta}_{2k} = b_{1k} \eta_{2k} + b_{2k} \eta_{1k} \). Obviously, \( \sum_{k=1}^n E|\hat{\eta}_{jk} - E\hat{\eta}_{jk}|^3 \leq \sum_{k=1}^n E|b_k(\eta_k - E\eta_k)|^3 \leq 8B\|b\|^3, j = 1, 2 \), where \( \|b\|^3 = \sum_{k=1}^n |b_k|^3 \). In order to apply Berry-Esseen inequality, we need to get a lower bound for \( E|\hat{\eta}_{jk} - E\hat{\eta}_{jk}|^2 \). For \( j = 1 \), we have

\[
E|\hat{\eta}_{1k} - E\hat{\eta}_{1k}|^2 = b_{1k}^2 \sigma_{1k}^2 + b_{2k}^2 \sigma_{2k}^2 - 2b_{1k}b_{2k}\sigma_{12k} = |b_k|^2 \left( |b_{1k}||\sigma_{1k}|/|b_k| - |b_{2k}|\sigma_{2k}/|b_k| \right)^2 + 2|b_{1k}b_{2k}| |b_k|^{-2}(\sigma_{1k}\sigma_{2k} - \text{sign}(b_{1k}b_{2k})\sigma_{12k}) \right).
\]

For \( t \in [0, 1] \), let \( f(t) = (t\sigma_1 - \sqrt{1 - t^2}\sigma_2)^2 + 2t\sqrt{1 - t^2}(\sigma_1\sigma_2 \pm \sigma_{12}) \). So the smallest value \( a = \min_{t \in [0, 1]} f(t) \) of \( f(t) \) in \( [0, 1] \) is attained at 0 or 1 or some
$t_0 \in (0, 1)$. Therefore, $a$ is a positive constant depending only on $\sigma_1$, $\sigma_2$ and $\sigma_{12}$. Hence $E|\hat{\eta}_{1k} - E\hat{\eta}_{1k}|^2 \geq a|b_k|^2$. Similarly, $E|\hat{\eta}_{2k} - E\hat{\eta}_{2k}|^2 \geq a|b_k|^2$. By Berry-Esseen inequality, one can then conclude that

\[
(2.5) \quad \sup_{v_1 \in \mathbb{R}} P \left( |\sum_{k=1}^{n} (\hat{\eta}_{1k} - E\hat{\eta}_{1k}) - v_1| \leq \varepsilon \sqrt{2} \right) \leq C \varepsilon \frac{1}{\|b\|_2} + C \left( \frac{\|b\|_3}{\|b\|_2} \right)^3
\]

and

\[
(2.6) \quad \sup_{v_2 \in \mathbb{R}} P \left( |\sum_{k=1}^{n} (\hat{\eta}_{2k} - E\hat{\eta}_{2k}) - v_2| \leq \varepsilon \sqrt{2} \right) \leq C \varepsilon \frac{1}{\|b\|_2} + C \left( \frac{\|b\|_3}{\|b\|_2} \right)^3,
\]

where $C$ is a constant depending only on $B$, $\sigma_1$, $\sigma_2$ and $\sigma_{12}$.

Thus (2.4) follows from (2.5), (2.6) and the following inequality

\[
\sup_{v \in \mathbb{C}} P(|S_n - v| \leq \varepsilon) \leq \sup_{v_1 \in \mathbb{R}} P \left( |\sum_{k=1}^{n} (\hat{\eta}_{1k} - E\hat{\eta}_{1k}) - v_1| \leq \varepsilon \sqrt{2} \right) + \sup_{v_2 \in \mathbb{R}} P \left( |\sum_{k=1}^{n} (\hat{\eta}_{2k} - E\hat{\eta}_{2k}) - v_2| \leq \varepsilon \sqrt{2} \right).
\]

□

Theorem 3 only yields a polynomial rate $n^{-1/2}$. Next, an improved small ball probability is needed for our future use. To this end, some concepts will be presented which are parallel to those of [18].

Denote the unit sphere in $\mathbb{C}^n$ by $S^{n-1}$.

**Definition 1.** Let $\alpha \in (0, 1)$ and $\tau \geq 0$. The essential least common denominator of a vector $b \in \mathbb{C}^n$, denoted by $D(b) = D_{\alpha, \tau}(b)$, is defined to be the infimum of $t > 0$ so that all coordinates of the vector $t b$ are of distance at most $\alpha$ from nonzero integers except $\tau$ coordinates.

**Definition 2.** Suppose that $\gamma, \rho \in (0, 1)$. A vector $b \in \mathbb{C}^n$ is sparse if $|\text{supp}(b)| \leq \gamma n$. A vector $b \in S^{n-1}$ is compressible if $b$ is within Euclidean distance $\rho$ from the set of all sparse vectors. All vectors $b \in S^{n-1}$ except compressible vectors are called incompressible. Let $\text{Sparse} = \text{Sparse}(\gamma)$, $\text{Comp} = \text{Comp}(\gamma, \rho)$ and $\text{Incomp} = \text{Incomp}(\gamma, \rho)$ denote, respectively, the sets of sparse, compressible and incompressible vectors.

**Definition 3.** For some $K_1, K_2 > 0$, the spread part of a vector $b \in \mathbb{C}^n$ is defined as

\[
\hat{b} = (\sqrt{n} b_k)_{k \in \sigma(b)},
\]
where the subset $\sigma(b) \subseteq \{1, \ldots, n\}$ is given by \( k : K_1 \leq \sqrt{n}|b_k| \leq K_2 \). Similarly, for \( j = 1, 2 \), define
\[
\mathbf{b}_j = (\sqrt{n}|b_{jk}|)_{k \in \sigma(b)}, \quad |\mathbf{b}_j| = (\sqrt{n}|b_{jk}|)_{k \in \sigma(b)}, \quad |\mathbf{b}| = (\sqrt{n}|b_k|)_{k \in \sigma(b)},
\]
where \( b_{1k} \) and \( b_{2k} \) denote, respectively, the real part and imaginary part of \( b_k \).

Similar to the real case, the complex incompressible vector are also evenly spread, i.e. many coordinates are of the order \( n^{-1/2} \).

**Lemma 1.** Let \( b \in \text{Incomp}(\gamma, \rho) \). Then there is a set \( \sigma_1(b) \subseteq \{1, \ldots, n\} \) of cardinality \(|\sigma_1(b)| \geq cn \) with \( c \geq \rho^2 \gamma / 4 \) so that for \( j = 1 \) or 2,
\[
(2.7) \quad \frac{\rho}{2\sqrt{2n}} \leq |b_{jk}| \leq \frac{1}{\sqrt{\gamma n}} \quad \text{for all } k \in \sigma_1(b).
\]

**Proof.** By Lemma 3.4 in [18], for \( b \in \text{Incomp}(\gamma, \rho) \), there is a set \( \sigma(b) \) of cardinality \(|\sigma(b)| \geq \frac{1}{2} \rho^2 \gamma n \) so that
\[
\frac{\rho}{\sqrt{2n}} \leq |b_k| \leq \frac{1}{\sqrt{\gamma n}} \quad \text{for all } k \in \sigma(b).
\]
Hence \( |b_{1k}| \leq 1 / \sqrt{\gamma n} \) and \( |b_{2k}| \leq 1 / \sqrt{\gamma n} \) if \( k \in \sigma(b) \). On the other hand, either \( b_{1k} \) or \( b_{2k} \) must be bigger than \( \rho(2\sqrt{2n})^{-1} \). The assertion follows. \( \square \)

The following result refines Theorem 3.

**Theorem 4.** Let \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \) whose spread part \( \hat{b} \) is well defined (for some fixed truncation levels \( K_1, K_2 > 0 \)). Suppose \( 0 < \alpha < K_1 / 6K_2 \) and \( 0 < \beta < 1 / 2 \).

1. Suppose that \( \eta_1, \ldots, \eta_n \) are i.i.d. real r.v.’s, or imaginary r.v.’s, or complex ones with linearly correlated \( \text{Re}(\eta_k) \) and \( \text{Im}(\eta_k) \), \( k = 1, 2, \ldots, n \). If \( E|\eta_k - E\eta_k|^2 = 1 \) and \( E|\eta_k|^3 < B \), for any \( \varepsilon \geq 0 \), then
\[
(2.8) \quad P_\varepsilon(b) \leq \frac{C}{\sqrt{\beta}} \left( \varepsilon + \frac{1}{\sqrt{n} \max\{D_{\alpha,\beta n}(b_1), D_{\alpha,\beta n}(b_2)\}} \right) + C \exp(-c\alpha^2 \beta n),
\]
where \( C, c > 0 \) depend only on \( B, K_1, K_2 \).

2. Let \( \eta_1, \ldots, \eta_n \) be i.i.d. complex r.v.’s with \( E|\eta_k - E\eta_k|^2 = 1 \) and \( E|\eta_k|^3 < B \), then (2.8) holds or
\[
(2.9) \quad P_\varepsilon(b) \leq \frac{C}{\sqrt{\beta}} \left( \varepsilon + \frac{1}{\sqrt{n} D_{\alpha,\beta n}(|b|)} \right) + C \exp(-c\alpha^2 \beta n),
\]
where \( C, c > 0 \) depend only on \( B, K_1, K_2, \sigma_1, \sigma_2 \) and \( \sigma_{12} \).
Proof. Since \( P_{\varepsilon}(b) = \sup_{v \in \mathbb{C}} P(|S_n - ES_n - v| \leq \varepsilon) \), we can assume that \( E\eta_k = 0 \).

(1). We only consider the case where the r.v.'s \( \{\eta_k\} \) are real. The other two cases follow from the real case. Let \( b_k = b_{1k} + ib_{2k} \) and \( v = v_1 + iv_2 \). Noting that

\[
\sup_{v \in \mathbb{C}} P(|S_n - v| \leq \varepsilon) \leq \min \left( \sup_{v_1 \in \mathbb{R}} P\left( \left| \sum_{k=1}^{n} b_{1k} \eta_k - v_1 \right| \leq \varepsilon \right), \sup_{v_2 \in \mathbb{R}} P\left( \left| \sum_{k=1}^{n} b_{2k} \eta_k - v_2 \right| \leq \varepsilon \right) \right)
\]

Then Corollary 4.9 in [18] leads to (2.8).

(2). For the moment we assume that

\[ 1 \leq |b_k| \leq K \quad \text{for all } k \]

Let \( b_k = b_{1k} + ib_{2k}, \eta_k = \eta_{1k} + i\eta_{2k} \) and \( v = v_1 + iv_2 \). It is observed that Theorem 3 implies Theorem 4 for big values of \( \varepsilon \) (constant order or even larger). Therefore we can suppose in what follows that

\[ \varepsilon \leq \varepsilon_l \]

where \( \varepsilon_l \) is a constant which will be specified later.

If the real part of \( \eta_1 \) is linearly correlated to the imaginary part of \( \eta_1 \), then we have (2.8). Therefore we assume in the sequel that \( \eta_{11} \) is not linearly correlated to \( \eta_{21} \).

Set \( \zeta_k = \frac{1}{|b_k|} |\xi_k - \xi'_k| \) where \( \xi_k = b_{1k} \eta_{1k} - b_{2k} \eta_{2k} \) and \( \xi'_k \) is an independent copy of \( \xi_k \). Then

\[
\frac{1}{2} E\zeta_k^2 = \frac{1}{|b_k|^2} E|b_{1k} \eta_{1k} - b_{2k} \eta_{2k}|^2.
\]

As in the proof of Theorem 3

\[ E\zeta_k^2 \geq 2a > 0, \]

where \( a \) is some positive constant depending only on \( \sigma_1, \sigma_2 \) and \( \sigma_{12} \).

On the other hand, \( E\zeta_k^3 \leq 64B \). The Paley-Zygmund inequality ([14]) gives that

\[ P(\zeta_k > \sqrt{a}) \geq \frac{(E\zeta_k^2 - a)^3}{(E\zeta_k^2)^2} \geq \frac{a^3}{64a^2 B^2} =: \beta, \]

which is a positive constant depending only on \( B, \sigma_1, \sigma_2 \) and \( \sigma_{12} \). Following [18] we introduce a new r.v. \( \hat{\zeta}_k \) conditioned on \( \zeta_k > \sqrt{a} \), that is, for any measurable function \( g \)

\[ Eg(\hat{\zeta}_k) = \frac{Eg(\zeta_k)I(\zeta_k > \sqrt{a})}{P(\zeta_k > \sqrt{a})}, \]

where \( I(\cdot) \) is the indicator function.
which entails
\begin{equation}
E_g(\zeta_k) \geq \beta E_g(\hat{\zeta}_k).
\end{equation}

From Esseen inequality, one has
\begin{equation}
P_{\varepsilon}(b) \leq \sup_{v_1 \in \mathbb{R}} P\left( \left| \sum_{k=1}^{n} \xi_k - v_1 \right| \leq \varepsilon \right)
\leq C \int_{-\pi/2}^{\pi/2} |\phi(t/\varepsilon)| dt,
\end{equation}
where
\[ \phi(t) := E \exp(i \sum_{k=1}^{n} \xi_k t). \]
With the notation \( \phi_k(t) = E \exp(i \xi_k t) \), it is observed that
\[ |\phi_k(t)|^2 = E \cos(|b_k| \zeta_k t), \]
and we then have
\[ |\phi(t)| \leq \prod_{k=1}^{n} \exp\left( - \frac{1}{2} (1 - |\phi_k(t)|^2) \right) \]
\[ = \exp\left( - E \sum_{k=1}^{n} \frac{1}{2} (1 - \cos(|b_k| \zeta_k t)) \right) = \exp\left( - E g(\hat{\zeta}_k t) \right), \]
where
\[ g(t) := \sum_{k=1}^{n} \sin^2 \left( \frac{1}{2} |b_k| t \right). \]
This, together with (2.11), gives
\[ |\phi(t)| \leq \exp\left( - \beta E g(\hat{\zeta}_k t) \right). \]
Consequently, (2.12) becomes
\begin{equation}
P_{\varepsilon}(b) \leq C \int_{-\pi/2}^{\pi/2} \exp\left( - \beta E g(\hat{\zeta}_k t/\varepsilon) \right) dt
\leq C E \int_{-\pi/2}^{\pi/2} \exp\left( - \beta g(\hat{\zeta}_k t/\varepsilon) \right) dt
\leq C \sup_{z \geq \sqrt{\pi}} \int_{-\pi/2}^{\pi/2} \exp\left( - \beta g(zt/\varepsilon) \right) dt.
\end{equation}
Let
\[ M := \max_{|t| \leq \pi/2} g(zt/\varepsilon) = \max_{|t| \leq \pi/2} \sum_{k=1}^{n} \sin^2(|b_k| zt/2 \varepsilon). \]
and the level sets of $g$ be

$$T(m, r) := \{t : |t| \leq r, \ g(zt/\varepsilon) \leq m\}.$$  

As in [18], one can prove that

$$\frac{n}{4} \leq M \leq n,$$

by taking $\varepsilon < (\pi\sqrt{a})/4 = l_1$. All the remaining arguments including the analysis for the level sets $T(m, r)$ are similar to those of [18] and so we here omit the details. Thus, one can conclude that for every $\varepsilon \geq 0$

$$|P_{\varepsilon}(b)| \leq C \sqrt{\tau} \left(\varepsilon + \frac{1}{D_{\alpha,\tau}(|b|)}\right) + C \exp\left(-\frac{c\alpha^2\tau}{A^2}\right),$$

where $0 < \tau < n$, $|b| = (|b_1|, \cdots, |b_n|)$ and $C, c > 0$ are positive constants depending only on $B, \sigma_1, \sigma_2$ and $\sigma_{12}$.

Finally, combining (2.14) and Lemma 2.1 in [18] one can obtain the small ball probability for complex case (when applying (2.14) to the spread part of the vector $b$ one can suppose that $K_1 = 1$ by re-scaling $b_k$ and $\alpha$). Thus we complete the proof. $\blacksquare$

To treat the compressible vector, the following lemma is needed.

**Lemma 2.** Suppose that $\eta_1, \cdots, \eta_n$ are i.i.d. centered complex r.v.'s with $E|\eta_k|^2 = 1$ and $E|\eta_k|^3 \leq B$. Let $\{a_{jk}, j, k = 1, \cdots, n\}$ be complex numbers. Then for $0 < \lambda < 1$ and any vector $b = (b_1, \cdots, b_n) \in S^{n-1}$ there is $\mu \in (0,1)$ such that the sum $S_{nj} = \sum_{k=1}^{n} b_k(\eta_k - a_{jk})$ satisfy

$$P(|S_{nj}| > \lambda) \geq \mu$$

where $\mu$ depends only on $\lambda$ and $B$.

**Proof.** Simple calculation indicates that

$$E|S_{nj}|^2 = |\sum_{k=1}^{n} b_k a_{jk}|^2 + 1.$$
On the other hand by Burkholder inequality we have

\[ E|S_{n1}|^3 \leq 4 \left( \sum_{k=1}^{n} |b_k a_{jk}|^3 + E \sum_{k=1}^{n} |b_k \eta_k|^3 \right) \]

\[ \leq C \left( \sum_{k=1}^{n} |b_k a_{jk}|^3 + \left( \sum_{k=1}^{n} |b_k|^2 E |\eta_k|^2 \right)^{3/2} + \sum_{k=1}^{n} |b_k|^3 E |\eta_k|^3 \right) \]

\[ \leq C \left( \sum_{k=1}^{n} |b_k a_{jk}|^3 + 1 + B \right). \]

Hence Paley-Zygmund inequality gives that

\[ P(|S_{nj}| > \lambda) \geq \frac{(E|S_{nj}|^2 - \lambda^2)^3}{(ES_{nj}^3)^2} \geq \frac{(c_{nj}^2 + 1 - \lambda^2)^3}{C(c_{nj}^3 + 1 + B)^2}, \]

where

\[ c_{nj} = \left| \sum_{k=1}^{n} b_k a_{jk} \right|. \]

Take

\[ f(t) = \frac{(t^2 + 1 - \lambda^2)^3}{(t^3 + 1 + B)^2}, \quad t \in (0, \infty). \]

Then one can conclude that

\[ \mu := \min_{t \in (0, \infty)} f(t) > 0 \]

and then

\[ P(|S_{nj}| > \lambda) \geq \mu > 0 \]

where \( \mu \) depends only on \( \lambda \) and \( B \). \( \square \)

2.2. Proof of Theorem 2. The whole argument is similar to that of [18] and we only sketch the proof. For more details one can refer to [18].

Since \( S^{n-1} \) can be decomposed as the union of \( \text{Comp} \) and \( \text{Incomp} \), we then consider the smallest singular value on each set separately.

By Lemma 2 there are \( c_1 > 0 \) and \( v \in (0, 1) \) depending on \( \mu \) only so that

\[ P(\|Wb\|_2 < c_1 \sqrt{n}) \leq v^n, \quad b \in S^{n-1}. \]

Actually, the proof is similar to that of Proposition 3.4 in [14]. The only difference is that we should use our Lemma 2 instead of Lemma 3.6 in [14]. Therefore similar to Lemma 3.3 in [18], we have, there exist \( \gamma, \rho, c_2, c_3 > 0 \) so that

\[ P\left( \inf_{b \in \text{Comp}(\gamma, \rho)} \|Wb\|_2 \leq c_2 n^{1/2} \right) \leq e^{-c_3 n} + P(\|W\| > K n^{1/2}), \]

(2.15)
where \( K \geq 1 \).

Let \( X_1, \cdots, X_n \) denote the column vectors of \( W \) and \( H_k \) the span of all columns except the \( k \)-th column. One can check that Lemma 3.5 in [18] is still true in complex case and hence

\[
P\left( \inf_{b \in \text{Incomp}(\gamma, \rho)} \|Wb\|_2 \leq \varepsilon \rho n^{-1/2} \right) \leq \frac{1}{\gamma n} \sum_{k=1}^{n} P(\text{dist}(X_k, H_k) < \varepsilon)
\]

\[
\leq \frac{1}{\gamma n} \sum_{k=1}^{n} P(|\langle Y_k, X_k \rangle| < \varepsilon),
\]

(2.16)

where \( Y_k \) is any unit vector orthogonal to \( H_k \) and can be chosen to be independent of \( X_k \). Here \( \langle \cdot, \cdot \rangle \) is the canonical inner product in \( \mathbb{C}^n \).

When all \( \{X_{jk}\} \) are real r.v.’s, or when \( \text{Re}(X_{jk}) \) and \( \text{Im}(X_{jk}) \) are linearly correlated or when \( \text{Re}(X_{jk}) = 0 \) we have

\[
P(|\langle Y_k, X_k \rangle| < \varepsilon \text{ and } U_K) \leq P(Y_k \in \text{Comp} \text{ and } U_K)
\]

\[
+ P(|\langle Y_k, X_k \rangle| < \varepsilon, Y_k \in \text{Incomp} \text{ and } U_K),
\]

(2.17)

where \( U_K \) denotes the event that \( \|W\| \leq Kn^{1/2} \). One can check that Lemma 3.6 in [18] applies to complex case and hence

\[
P(Y_k \in \text{Comp} \text{ and } U_K) \leq e^{-c_4n},
\]

where \( c_4 \) is a constant depending only on \( B, K, \sigma_1, \sigma_2 \) and \( \sigma_{12} \). Further,

\[
P(|\langle Y_k, X_k \rangle| < \varepsilon, Y_k \in \text{Incomp} \text{ and } U_K)
\]

\[
\leq \sum_{j=1}^{2} P\left(V_{jk}, U_K, D_{\alpha,\beta n}(\hat{Y}_{jk}) < \varepsilon \right. \text{ and } Y_k \in \text{Incomp}\right)
\]

\[
+ \sum_{j=1}^{2} E \left[ I(D_{\alpha,\beta n}(\hat{Y}_{jk}) \geq \varepsilon \text{ and } Y_k \in \text{Incomp}) P(|\langle Y_k, X_k \rangle| < \varepsilon|Y_k) \right]
\]

where \( V_{1k} \) and \( V_{2k} \) denote, respectively, the events that the real part and imaginary part of the vector \( Y_k \in \text{Incomp} \) satisfy (2.7) in Lemma \( \mathbf{1} \). \( \hat{Y}_{1k} \) and \( \hat{Y}_{2k} \) denote, respectively, the spread part of the real part and imaginary part of the vector \( Y_k \). By (2.8) in Theorem \( \mathbf{4} \) and (2.3) we have

\[
I(D_{\alpha,\beta n}(\hat{Y}_{jk}) \geq \varepsilon \varepsilon) P(|\langle Y_k, X_k \rangle| < \varepsilon|Y_k) \leq c_5 \varepsilon + c_6 e^{-c_7 n},
\]

where \( c_5, c_6, c_7 \) are positive constants depending only on \( B, \sigma_1, \sigma_2 \) and \( \sigma_{12} \).
On the other hand,

\[
P\left( V_{1k}, U_K, D_{\alpha,\beta n}(\hat{Y}_{1k}) < e^{cn} \text{ and } Y_k \in \text{Incomp} \right) \leq \sum_{D \in \mathcal{D}} P(Y_k \in S_D, U_K \text{ and } V_{1k}).
\]

Here the level set \( S_D \subseteq S^{n-1} \) is defined as

\[
S_D := \{ Y_k \in \text{Incomp} : D \leq D_{\alpha,\beta n}/2(\hat{Y}_{1k}) < 2D \}.
\]

and

\[
\mathcal{D} = \{ D : D_0 \leq D < e^{cn}, D = 2^k, k \in \mathbb{Z} \},
\]

where \( \alpha \) and \( D_0 \) are some constants. For more details about \( \alpha \) and \( D_0 \), see [18]. Further, one can similarly prove that Lemma 5.8 in [18] holds in our case and therefore we obtain

\[
P(Y_k \in S_D \text{ and } U_K) \leq e^{-n},
\]

which, combined with the fact that the cardinal number \( |\mathcal{D}| \) is of order \( n \), then implies that

\[
P\left( V_{1k}, U_K, D_{\alpha,\beta n}(\hat{Y}_{1k}) < e^{cn} \text{ and } Y_k \in \text{Incomp} \right) \leq e^{-c_8 n},
\]

where \( c_8 > 0 \). Similarly, one may also show that

\[
P\left( V_{2k}, U_K, D_{\alpha,\beta n}(\hat{Y}_{2k}) < e^{cn} \text{ and } Y_k \in \text{Incomp} \right) \leq e^{-c_8 n}.
\]

Picking up the above argument one can conclude that

\[
P(\langle Y_k, X_k \rangle < \varepsilon \text{ and } \|W\| \leq Kn^{1/2}) \leq C\varepsilon + e^{-c' n},
\]

which further gives that

\[
(2.18) \quad P\left( \inf_{x \in \text{Incomp}(\gamma, \rho)} \|Wx\|_2 \leq \varepsilon \rho n^{-1/2} \right) \leq \frac{C}{\delta}(\varepsilon + c^n) + P(\|W\| > Kn^{1/2}),
\]

where \( C > 0 \) and \( c \in (0, 1) \) depend only on \( K, B, \sigma_1, \sigma_2 \) and \( \sigma_{12} \).

For all the remaining case, i.e. \( Re(X_{jk})Im(X_{jk}) \neq 0 \), and \( Re(X_{jk}), Im(X_{jk}) \) are not linearly correlated, one has

\[
P(\langle Y_k, X_k \rangle < \varepsilon \text{ and } U_k) \leq P(D_{\alpha,\beta n}(|Y_k|) < e^{cn} \text{ and } U_K)
\]

\[
+ E \left[ I(D_{\alpha,\beta n}(|Y_k|) \geq e^{cn})P\left( |\langle Y_k, X_k \rangle| < \varepsilon \left| Y_k \right| \right) \right],
\]

(2.19)

and one can similarly obtain (2.13) for complex case. Theorem 2 follows from (2.15)–(2.19) immediately.
3. THE CONVERGENCE OF LOGARITHMIC POTENTIAL AND CIRCULAR LAW

In this part the logarithmic potential will be used to show that the circular law is true. According to Lower Envelop Theorem and Unicity Theorem (see Theorem 6.9, p.73, and Corollary 2.2, p.98, in [19]), it suffices to show that the corresponding potential converges to the potential of the circular law.

To make use of Theorem 2 one needs to bound the maximum singular value of $W$. To this end, we would like to present an important fact which was proved in [25], that is, if (1) $EX_{jk} = 0$, (2) $|X_{jk}| \leq \sqrt{n}\varepsilon_n$, (3) $E|X_{jk}|^2 \leq 1$ and $1 \geq E|X_{jk}|^2 \to 1$ and (4) $E|X_{jk}|^l \leq c(\sqrt{n}\varepsilon_n)^{l-3}$ for $l \geq 3$, where $\varepsilon_n \to 0$ with the convergence rate slower than any preassigned one as $n \to \infty$. Then for any $K > 4$

\begin{equation}
(3.1) \quad P(\|XX^*\| > Kn) = o(n^{-l}),
\end{equation}

where $l$ is any positive number (proved for real case in [25], for complex case see Chapter 5 of [2]).

Let the random matrix $\hat{X} = (\hat{X}_{jk})$ with $\hat{X}_{jk} = X_{jk}I(|X_{jk}| \leq \sqrt{n}\varepsilon_n)$. Then one can show that

\begin{equation}
(3.2) \quad P(\hat{X} \neq X, i.o.) = 0,
\end{equation}

see Lemma 2.2 of [25] (the argument of the complex case is similar to that of the real one). Here the notation $i.o.$ means infinitely often. Thus it is sufficient to consider the random matrix $\hat{X}$ in order to prove the conjecture.

Taking $A_n = E\hat{X} - z\sqrt{n}I$ in Theorem 2 one can obtain that

\begin{equation}
(3.3) \quad P(s_n(\hat{X} - z\sqrt{n}I) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + c^n + P(\|\hat{X} - z\sqrt{n}I\| > Kn^{1/2}),
\end{equation}

where $EX = (EX_{kj})$. Here one should note that from (3.3) re-scaling the underlying r.v.’s is trivial. Moreover

$$\|EX - z\sqrt{n}I\| \leq |z|\sqrt{n} + \frac{E|X_{11}|^4}{n\varepsilon^3_n}.$$ 

Therefore, applying (3.1) and choosing an appropriate $K$ in (3.3), we have

\begin{equation}
(3.4) \quad P(s_n(\hat{X} - z\sqrt{n}I) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + c^n + n^{-l}
\end{equation}

where both $C > 0$ and $c \in (0,1)$ depend only on $K$, $E|X_{11}|^3$, $E(Re(X_{11}))^2$, $E(Im(X_{11}))^2$, and $ERe(X_{11})Im(X_{11})$.

In the sequel, to simplify the notation, we still use the notation $X$ instead of $\hat{X}$ and $\mu_n(x, y)$ instead of the empirical spectral distribution corresponding to $\hat{X}$. But one should keep in mind that $\{X_{kj}\}$ are non-centered and $|X_{kj}| \leq \sqrt{n}\varepsilon_n$. 
Let 
\[ H_n = (n^{-1/2}X - zI)(n^{-1/2}X - zI)^* \]
for each \( z = s + it \in \mathbb{C} \). Here \((\cdot)^*\) denotes the transpose and complex conjugate of a matrix. Let \( v_n(x, z) \) be the empirical spectral distribution of Hermitian matrix \( H_n \).

Before we prove the convergence of the logarithmic potential of \( \mu_n(x, y) \), we will characterize the relation between the potential of the circular law \( \mu(x, y) \) and the integral of logarithmic function with respect to \( v(x, z) \), the limiting distribution of \( v_n(x, z) \) as below.

**Lemma 3.**
\[
\int \int \log \frac{1}{|x + iy - z|} d\mu(x, y) = -\frac{1}{2} \int_0^\infty \log xv(dx, z).
\]

**Proof.** Let \( x + iy = re^{i\theta}, r > 0 \). One can then verify that
\[
\int_{-\pi}^{\pi} \log |z - re^{i\theta}| d\theta = \begin{cases} 
2\pi \log r & \text{if } |z| \leq r, \\
2\pi \log |z| & \text{if } |z| > r.
\end{cases}
\]

It follows that
\[
\int \int \log \frac{1}{|x + iy - z|} d\mu(x, y) = \begin{cases} 
2^{-1}(1 - |z|^2) & \text{if } |z| \leq 1, \\
-\log |z| & \text{if } |z| > 1.
\end{cases}
\]

On the other hand by Lemma 4.4 in [1] one has
\[
\frac{d}{ds} \int_0^\infty \log xv(dx, z) = g(s, t),
\]
where
\[
g(s, t) = \begin{cases} 
\frac{2s}{s^2 + t^2} & \text{if } s^2 + t^2 > 1, \\
2s & \text{otherwise}.
\end{cases}
\]

Therefore for any \( z = s + it, z_1 = s_1 + it \) with \( |z_1| > 1 \), we have
\[
\int_0^\infty \log xv(dx, z) - \int_0^\infty \log xv(dx, z_1) + \log |z_1|^2 = \int_{s_1}^s g(u, t)du + \log |z_1|^2.
\]

Let \( s_1 \to \infty \) and then \( |z_1| \to \infty \). Therefore, from Lemma 4.2 of [1] the left and right end point, \( x_1 \) and \( x_2 \), of the support of \( v(\cdot, z_1) \) satisfy
\[
\frac{x_j}{|z_1|^2} = 1 + o(1), \quad j = 1, 2,
\]
which implies that
\[ \int_0^\infty \log xv(dx, z_1) - \log |z_1|^2 = \int_{x_1}^{x_2} \log \frac{x}{|z_1|^2} v(dx, z_1) \to 0, \]
as \( s_1 \to \infty \). In addition,
\[ \int_{s_1}^s g(u, t)du + \log |z_1|^2 = \begin{cases} |z|^2 - 1 & \text{if } |z| \leq 1 \\ \log |z|^2 & \text{if } |z| > 1. \end{cases} \]
Thus Lemma 3 is complete. \( \square \)

We now proceed to prove the convergence of the potential of \( \mu_n(x, y) \). The potential of \( \mu_n(x, y) \) is
\[ U_{\mu_n(x,y)} = -\frac{1}{n} \log \left| \det \left( n^{-1/2}X - z I \right) \right| \]
\[ = -\frac{1}{2n} \log \left| \det(H_n) \right| \]
\[ = -\frac{1}{2} \int_0^\infty \log xv_n(dx, z), \]
where \( I \) is the identity matrix. We will prove
\[ \int_0^\infty \log xv_n(dx, z) \overset{a.s.}{\longrightarrow} \int_0^\infty \log xv(dx, z) \]
as \( n \to \infty \). Observe that by the fourth moment condition
\[ \lambda_{\max}(H_n) \leq 2(\lambda_{\max}(n^{-1}XX^*) + |z|^2) \overset{a.s.}{\longrightarrow} 8 + 2|z|^2, \]
where \( \lambda_{\max}(H_n) \) denotes the maximum eigenvalue of \( H_n \). It follows that for any \( \delta > 0 \) and sufficiently large \( n \)
\[ \left| \int_{n^{-4-2\delta}}^{n^{-4}} \log x \left( v_n(dx, z) - v(dx, z) \right) \right| \]
\[ = \left| \int_{n^{-4-2\delta}}^{n^{-4}+2|z|^2+\delta} \log x \left( v_n(dx, z) - v(dx, z) \right) \right| \]
\[ \leq \left( |\log(n^{-4-2\delta})| + \log(8 + 2|z|^2 + \delta) \right) \| v_n(x, z) - v(x, z) \| \overset{a.s.}{\longrightarrow} 0. \]
Here we do not present the proof of the convergence of \( v_n(x, z) \) to \( v(x, z) \) with the desired convergence rate for each \( z \). Indeed, the rank inequality (see Theorem 11.43 in [2]) can be used to re-centralize \( X_{jk} \) and then Lemma 10.15 in [2] provides the convergence rate under the assumption \( E|X_{11}|^2+\delta < \infty \).
On the other hand, by (3.4) and Borel-Cantelli lemma,
\[
\frac{1}{2n} \log \left| \det (H_n) \right| I(s_n (X - z \sqrt{n} I) < n^{-3/2 - \delta}) \overset{a.s.}{\rightarrow} 0.
\]
Here we take \( \varepsilon = n^{-1 - \delta}, \delta > 0 \) in (3.4). One should observe that \( \varepsilon \) in Theorem 5.1 in [18] can be dependent on \( n \), so does \( \varepsilon \) in Theorem 2. Moreover, from Lemma 4.2 in [2] one can conclude that
\[
\int_0^{n^{-4-2\delta}} \log xv(dx, z) \rightarrow 0.
\]
Therefore
\[
(3.10) \quad U^{\mu_n (x,y)} \overset{a.s.}{\rightarrow} \frac{1}{2} \int_0^\infty \log xv(dx, z).
\]
Again by the fourth moment condition
\[
|\lambda_1 (X)| \leq \left( \lambda_{\max} (n^{-1} XX^*) \right)^{1/2} \overset{a.s.}{\rightarrow} 2.
\]
So for all large \( n \), almost surely \( \mu_n \) is compactly supported on the disk \( \{ z : |z| \leq 2 + \delta \} \). Here we have used the fact that all the eigenvalues of an \( n \times n \) matrix are dominated by the largest singular value of the same matrix. Consequently Theorem 1 follows from Lemma 3 combined with Lower Envelope Theorem and Unicity Theorem for logarithmic potential of measures (see Theorem 6.9, p.73, and Corollary 2.2, p.98, in [19]).

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