On representation categories of $A_\infty$-algebras and $A_\infty$-coalgebras

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Abstract

In this paper, we use the language of monads, comonads and Eilenberg-Moore categories to describe a categorical framework for $A_\infty$-algebras and $A_\infty$-coalgebras, as well as $A_\infty$-modules and $A_\infty$-comodules over them respectively. The resulting formalism leads us to investigate relations between representation categories of $A_\infty$-algebras and $A_\infty$-coalgebras. In particular, we relate $A_\infty$-comodules and $A_\infty$-modules by considering a rational pairing between an $A_\infty$-coalgebra $C$ and an $A_\infty$-algebra $A$. The categorical framework also motivates us to introduce $A_\infty$-contramodules over an $A_\infty$-coalgebra $C$.

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1 Introduction

A monad on a category \( \mathcal{C} \) consists of an endofunctor \( U : \mathcal{C} \to \mathcal{C} \) along with natural transformations \( \eta : 1 \to U \) and \( \epsilon : U \circ U \to U \) satisfying conditions similar to an ordinary ring. As with a ring, one can study a monad by looking at its representation category, i.e., its category of modules. The role of modules over the monad \( U \) is played by the classical Eilenberg-Moore category \( EM_U \) (see [20], [21]). For instance, Borceux, Caenepeel and Janelidze [11] have shown that Eilenberg-Moore categories can be used to study a categorical form of Galois descent, extending from the usual notion of Galois extension of commutative rings. The dual notion of comonad resembles that of a coalgebra, and we have Eilenberg-Moore categories of comodules over them. For instance, Eilenberg-Moore categories of comodules can be used to study notions such as relative Hopf modules, or Doi-Hopf modules (see [14]).

In this paper, we consider \( A_\infty \)-versions of monads, comonads and their representation categories. We develop Eilenberg-Moore categories for \( A_\infty \)-monads and \( A_\infty \)-comonads. We study a bar construction for \( A_\infty \)-monads. We also study versions of distributive laws that enable us to lift \( A_\infty \)-monads to the Eilenberg-Moore category, or extend \( A_\infty \)-monads to the Kleisli category, of an ordinary monad. This gives us a categorical framework for \( A_\infty \)-algebras and \( A_\infty \)-coalgebras, as well as \( A_\infty \)-modules and \( A_\infty \)-comodules over them respectively, using the language of monads, comonads and Eilenberg-Moore categories. We are then motivated by this formalism to consider rational pairings between \( A_\infty \)-coalgebras and \( A_\infty \)-algebras, and also to introduce \( A_\infty \)-contramodules.

We recall that an \( A_\infty \)-algebra \( A \) consists of a graded vector space \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) and a sequence \( \{ m_k : A^{\otimes k} \to A \}_{k \geq 1} \) of homogeneous linear maps satisfying higher order associativity conditions. The \( A_\infty \)-algebras were introduced by Stasheff [42], [43], and applied to the study of homotopy associative structures on \( H \)-spaces. In later years, \( A_\infty \)-structures have found widespread applications in algebra, topology, geometry as well as in physics (see, for instance, [1], [9], [22], [23], [26], [27], [29], [30], [35], [41], [44]).

We now describe the paper in more detail. We begin in Section 2 with an abelian category \( \mathcal{C} \) that is \( K \)-linear (for a field \( K \)) and satisfies (AB5). An \( A_\infty \)-monad \((U, \Theta)\) consists of a sequence \( U = \{ U_n \}_{n \in \mathbb{Z}} \) of endofunctors on \( \mathcal{C} \) along with a collection \( \Theta \) of natural transformations

\[
\Theta = \{ \theta_k(\bar{n}) : U_{\bar{n}} := U_{n_1} \circ \ldots \circ U_{n_k} \to U_{n_1 + \ldots + n_k + 2k} = U_{\bar{n} + (2k)} \mid k \geq 1, \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \} \quad (1.1)
\]

satisfying certain relations. We say that \((U, \Theta)\) is even if \( \theta_k(\bar{n}) = 0 \) whenever \( k \) is odd. Dually, an \( A_\infty \)-comonad \((V, \Delta)\) consists of a sequence \( V = \{ V_n \}_{n \in \mathbb{Z}} \) of endofunctors and a collection \( \Delta \) of natural transformations

\[
\Delta = \{ \delta_k(\bar{n}) : V_{\bar{n} + (2k)} := V_{n_1 + \ldots + n_k + (2k)} \to V_{n_1} \circ \ldots \circ V_{n_k} = V_{\bar{n}} \mid k \geq 1, \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \} \quad (1.2)
\]
satisfying dual relations. Our first result is that if \( \{ (U_n, V_n) \}_{n \in \mathbb{Z}} \) is a sequence of adjoint functors, then \( U = \{ U_n \}_{n \in \mathbb{Z}} \) can be equipped with the structure of an even \( A_{\infty} \)-monad (resp. an even \( A_{\infty} \)-comonad) if and only if \( V = \{ V_n \}_{n \in \mathbb{Z}} \) can be equipped with the structure of an even \( A_{\infty} \)-comonad (resp. an even \( A_{\infty} \)-monad). Additionally, if \( U = \{ U_n \}_{n \in \mathbb{Z}} \) is an even \( A_{\infty} \)-monad, we describe a process that gives a collection \( V^\oplus = \{ V^\oplus_n \}_{n \in \mathbb{Z}} \) of subfunctors of \( V = \{ V_n \}_{n \in \mathbb{Z}} \) carrying the structure of an \( A_{\infty} \)-comonad \((\mathcal{V}^\oplus, \Delta^\oplus)\) that is locally finite. In other words, for each \( k \geq 1, T \in \mathbb{Z} \), the induced morphism

\[
\mathcal{V}^\oplus_{T+(2-k)} = \mathcal{V}^\oplus_{n_1 + \ldots + n_k + (2-k)} \rightarrow \prod_{n \in \mathbb{Z}, \Sigma k = T} \mathcal{V}^\oplus_{n_1} \circ \ldots \circ \mathcal{V}^\oplus_{n_k}
\]

factors through the direct sum \( \bigoplus_{n \in \mathbb{Z}, \Sigma k = T} \mathcal{V}^\oplus_{n_1} \circ \ldots \circ \mathcal{V}^\oplus_{n_k} \).

If \((U, \Theta)\) is an \( A_{\infty} \)-monad, an \( A_{\infty} \)-module \((M, \pi)\) consists of a collection \( M = \{ M_n \}_{n \in \mathbb{Z}} \) of objects of \( C \) along with morphisms

\[
\pi = \{ \pi_l(n, l) : \mathcal{U}_n M_l \rightarrow \mathcal{M}_{\mathcal{U}l+(2-k)} | k \geq 1, n = (n_1, \ldots, n_k) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \}
\]

satisfying certain conditions described in Definition \([3, 4]\). This gives the Eilenberg-Moore category \( EM(U, \Theta) \) of \( A_{\infty} \)-modules over \((U, \Theta)\). We also describe a canonical functor \( \mathcal{C} \rightarrow EM(U, \Theta) \). Further, if the functors \( \{ (U_n, V_n) \}_{n \in \mathbb{Z}} \) preserve direct sums, this extends to a functor \( \mathcal{C}^\oplus \rightarrow EM(U, \Theta) \) from the category \( \mathcal{C}^\oplus \) of \( \mathbb{Z} \)-graded objects over \( C \). The \( A_{\infty} \)-comodules over an \( A_{\infty} \)-comonad \((\mathcal{V}, \Delta)\) are defined in a dual manner. If \((U_n, V_n)\) is a sequence of adjoint functors, we show (see Proposition \([3, 5]\)) that the category of even \( A_{\infty} \)-comonads over \((U, \Theta)\) is isomorphic to the category of even \( A_{\infty} \)-comodules over \((\mathcal{V}, \Delta)\). We define \( A_{\infty} \)-morphisms \( \alpha = \{ a_l(n, l) : \mathcal{U}_n M_l \rightarrow \mathcal{M}_{\mathcal{U}l+(2-k)} \} \) between \( A_{\infty} \)-modules \((M^1, \pi^1), (M^2, \pi^2)\) (see Definition \([3, 6]\)). Then, we show in Theorem \([3, 8]\) that even \( A_{\infty} \)-modules over \((U, \Theta)\) with odd \( A_{\infty} \)-morphisms between them correspond to even \( A_{\infty} \)-comodules over \((\mathcal{V}, \Delta)\) with odd \( A_{\infty} \)-morphisms between them.

We study locally finite \( A_{\infty} \)-comodules over \( A_{\infty} \)-comonads in Section 4. Let \((\mathcal{V}, \Delta)\) be an \( A_{\infty} \)-comonad such that each \( V_n \) is exact. If \( \mathcal{P} = \{ P_n \}_{n \in \mathbb{Z}} \) is an \( A_{\infty} \)-comodule, we describe a process of restricting the structure maps \( \rho_l(n, l) : \mathcal{P}_{\Sigma l+(2-k)} \rightarrow \mathcal{V}_l P_l \) of \( P \) to a subobject \( P^\oplus \) such that for each \( k \geq 1, l \in \mathbb{Z}, T \in \mathbb{Z} \), the induced morphism

\[
\bigoplus_{n \in \mathbb{Z}, \Sigma k = T} \mathcal{V}_l P^\oplus_{n_1 + \ldots + n_k + (2-k)} \rightarrow \bigoplus_{n \in \mathbb{Z}, \Sigma k = T} \mathcal{V}_l P^\oplus_{n_1} \circ \ldots \circ \mathcal{V}_l P^\oplus_{n_k}
\]

factors through the direct sum \( \bigoplus_{n \in \mathbb{Z}, \Sigma k = T} \mathcal{V}_l P^\oplus_{n_1} \circ \ldots \circ \mathcal{V}_l P^\oplus_{n_k} \). Further, we show (see Theorem \([4, 6]\)) that this construction gives a right adjoint to the inclusion of locally finite \( A_{\infty} \)-comodules into the Eilenberg-Moore category \( EM(\mathcal{V}, \Delta) \) of \( A_{\infty} \)-comodules over \((\mathcal{V}, \Delta)\).

In Section 5, we develop the bar construction for an \( A_{\infty} \)-monad \((U, \Theta)\). This gives a comonad \( Bar(U, \Theta) \) which is graded and equipped with a coderivation. If \((\mathcal{W}, \delta_1, \delta_2)\) is a conilpotent dg-comonad (see Definition \([5, 2]\)), we show that morphisms from \((\mathcal{W}, \delta_1, \delta_2)\) to \( Bar(U, \Theta) \) correspond to families of natural transformations satisfying conditions similar to the classical case of twisting morphisms. In Section 6, we study distributive laws between an \( A_{\infty} \)-monad \((U, \Theta)\) and an ordinary monad \( S \). We show how these distributive laws can be used to lift \((U, \Theta)\) to an \( A_{\infty} \)-monad \((\bar{U}, \bar{\Theta})\) on the Eilenberg-Moore category \( EM(S) \) of \( S \). Similarly, we show how distributive laws can be used to extend \((U, \Theta)\) to an \( A_{\infty} \)-monad \((\bar{U}, \bar{\Theta})\) on the Kleisli category \( Kl(S) \) of \( S \).

In Section 7, we relate this formalism of monads and comonads to \( A_{\infty} \)-algebras and \( A_{\infty} \)-coalgebras. Motivated by a classical construction of Popescu \([46]\) (see also Artin and Zhang \([2]\) on the noncommutative base change of an algebra, we construct examples of \( A_{\infty} \)-monads and \( A_{\infty} \)-comonads starting with a \( K \)-linear Grothendieck category \( C \). If \( A \) is an \( A_{\infty} \)-algebra and \( F \) is a monad on \( C \) that preserves direct sums, the collection of functors \( \mathcal{U}^A_{\mathcal{C}} = \{ \mathcal{U}^A_{\mathcal{C}, n} : A \otimes \mathcal{C} \rightarrow \mathcal{C} \}_{n \in \mathbb{Z}} \) is an \( A_{\infty} \)-monad on \( C \). Additionally, if \( A \) is even (i.e., the operations \( m_k : A^k \rightarrow A \) vanish whenever \( k \) is odd) and \( F \) has a right adjoint \( G \), the collection of functors \( \mathcal{V}^A_{\mathcal{C}} = \{ \mathcal{V}^A_{\mathcal{C}, n} : Hom(A_n, G(\_)) : \mathcal{C} \rightarrow \mathcal{C}_{n \in \mathbb{Z}} \) is an \( A_{\infty} \)-comonad on \( C \) (see Proposition \([7, 2]\)). We also mention classes of \( A_{\infty} \)-algebras that satisfy this evenness condition and have been studied extensively in the literature (see \([19, 24, 28, 34]\)).

Henceforth, we fix \( \mathcal{C} = Vect \), the category of \( K \)-vector spaces. If \( A \) is a \( \mathbb{Z} \)-graded vector space, we show that the collection of functors \( \mathcal{U}^A = \{ \mathcal{U}^A_n : A_n \otimes \mathcal{C} \rightarrow Vect \} \) can be equipped with the structure of an \( A_{\infty} \)-monad if and only if \( A \) can
be equipped with the structure of an $A_\infty$-algebra. In that case, the Eilenberg-Moore category $EM_{U_A}$ of $A_\infty$-modules over $U_A$ corresponds to $A_\infty$-modules over the $A_\infty$-algebra $A$. On the other hand, if $C$ is an $A_\infty$-coalgebra that is even, it follows from our earlier result in Theorem 2.5 that the right adjoints $(\text{Hom}(C_{-n}, -)) : \text{Vect} \rightarrow \text{Vect}_{\in\mathbb{Z}}$ can be equipped with the structure of an $A_\infty$-monad. This is the key observation that motivates our definition of $A_\infty$-contramodules in Section 9.

By a pairing of an $A_\infty$-algebra $A$ and an $A_\infty$-coalgebra $C$, we will mean an $\infty$-morphism of $A_\infty$-algebras $f = \{f_k : A^{\otimes k} \rightarrow C^*\}_{k \geq 1}$ from $A$ to the graded dual $C^*$ of $C$. We show that such a pairing induces a functor $\iota(C, A) : M^C \rightarrow \mathcal{A} \mathcal{M}$ from right $C$-comodules to left $A$-modules. Additionally, when the pairing is rational in the sense of Definition 8.3, the functor $\iota(C, A)$ embeds $M^C$ as a full subcategory of $\mathcal{A} \mathcal{M}$. Our main result in Section 8 is that a rational pairing $f = \{f_k : A^{\otimes k} \rightarrow C^*\}_{k \geq 1}$ of an $A_\infty$-algebra with an $A_\infty$-coalgebra gives rise to an adjunction (see Theorem 8.6)

$$\mathcal{A} \mathcal{M}(\iota(C, A)(M'), M) \cong M^C(M', R_f(M)) \quad (1.6)$$

for any $M' \in M^C$ and $M \in \mathcal{A} \mathcal{M}$.

We introduce $A_\infty$-contramodules in Section 9. If $A'$ is an ordinary algebra, we note that the structure map $A' \otimes M' \rightarrow M'$ of a module $M'$ may be expressed equivalently as a morphism $M' \rightarrow [A', M'] := \text{Vect}(A', M')$. Accordingly, a contramodule over an ordinary coalgebra $C'$, as defined by Eilenberg and Moore in [21] IV.5, consists of a space $M''$ along with a structure map $\text{Vect}(C', M'') = [C', M''] \rightarrow M''$ satisfying certain coassociativity conditions. In particular, if $V$ is any vector space, then $[C, V]$ is a $C'$-contramodule. In a categorical sense, it may be said therefore that both comodules and contramodules dualize the notion of module over an algebra. Even though the study of contramodules in the literature is not as developed as that of comodules, the topic has seen a lot of renewed interest in recent years (see, for instance, [5], [6], [7], [10], [37], [38], [39], [40], [46]).

Accordingly, our notion of an $A_\infty$-contramodule over an $A_\infty$-coalgebra $C$ consists of a graded vector space $M \in \text{Vect}^\infty$ along with a collection of structure maps

$$t^M_k : \text{Vect}^\infty(C^{\otimes k-1}, M) = [C^{\otimes k-1}, M] \rightarrow M \quad k \geq 1 \quad (1.7)$$

satisfying certain conditions set out in Definition 9.1. We show that there is a canonical functor $[C, \_] : \text{Vect}^\infty \rightarrow M_{[C, \_]}$ from graded vector spaces to the category $M_{[C, \_]}$ of $A_\infty$-contramodules over $C$. Further, we describe a canonical functor that relates $A_\infty$-contramodules to the Eilenberg-Moore category of $A_\infty$-modules over the $A_\infty$-monad $(\text{Hom}(C_{-n}, \_)) : \text{Vect} \rightarrow \text{Vect}_{\in\mathbb{Z}}$ described in Section 7. There is also a faithful functor $k^C$ that takes $M_{[C, \_]}$ to $A_\infty$-modules over $C^*$.

The final result of Section 9 relates $A_\infty$-contramodules to $A_\infty$-comodules, using contratensor products. For contramodules over coassociative coalgebras, the contratensor product was defined in [37], along with its right adjoint. We let $C$, $D$ be $A_\infty$-coalgebras. Let $N$ be a space equipped with an $A_\infty$-left $C$-coaction as well as an $A_\infty$-right $D$-coaction (a $(C, D)$-bicomodule for instance) satisfying certain conditions. Using $N$, we define a contratensor product (see 9.22)

$$\_ \otimes_C N : M_{[C, \_]} \rightarrow M^D \quad (1.8)$$

We conclude with Theorem 9.8 which gives an adjunction of functors

$$M^D(M \otimes_C N, Q) \cong M_{[C, \_]}(M, [N, Q]^D) \quad (1.9)$$

for $M \in M_{[C, \_]}$ and $Q \in M^D$. We also mention that for the convenience of the reader and in order to fix notation, we have collected the basic definitions of $A_\infty$-algebras, $A_\infty$-coalgebras, $A_\infty$-modules and $A_\infty$-comodules in an appendix in the final section of this paper (see, for instance, [27], [31]).

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2 $A_\infty$-monads, $A_\infty$-comonads and adjoints

Let $K$ be a field. Throughout this section and the rest of this paper, we let $\mathcal{C}$ be a $K$-linear abelian category. We will suppose that $\mathcal{C}$ satisfies the (AB5) axiom. This means in particular that for any family $\{M_i\}_{i \in I}$ of objects in $\mathcal{C}$, we have an inclusion
For any tuple \( \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \), we set \( |n| := k \) and \( \Sigma \bar{n} := n_1 + \ldots + n_k \). We also write \( \bar{n}^{op} \) for the opposite tuple \( (n_k, \ldots, n_1) \in \mathbb{Z}^k \). On the other hand, for \( T \in \mathbb{Z} \) and \( k \geq 1 \), we denote by \( \mathbb{Z}(T, k) \) the set of all \( \bar{n} \in \mathbb{Z}^k \) satisfying \( \Sigma \bar{n} = T \).

For any collection \( \mathcal{U} = \{ \mathcal{U}_n : \mathcal{C} \to \mathcal{C} \}_{n \in \mathbb{Z}} \) of endofunctors on \( \mathcal{C} \) indexed by \( \mathbb{Z} \) and \( \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \), we will denote by \( \mathcal{U}_{\bar{n}} \) the composition \( \mathcal{U}_n \circ \cdots \circ \mathcal{U}_{n_1} \). We start by considering an \( \mathbb{A}_\infty \)-monad on \( \mathcal{C} \).

**Definition 2.1.** An \( \mathbb{A}_\infty \)-monad \((\mathcal{U}, \Theta)\) on \( \mathcal{C} \) consists of the following data:

(a) A collection \( \mathcal{U} = \{ \mathcal{U}_n : \mathcal{C} \to \mathcal{C} \}_{n \in \mathbb{Z}} \) of endofunctors on \( \mathcal{C} \).

(b) A collection \( \Theta \) of natural transformations

\[
\Theta = \{ \theta_\bar{n}(\bar{n}) : \mathcal{U}_{\bar{n}} = \mathcal{U}_{n_1} \circ \cdots \circ \mathcal{U}_{n_k} = \mathcal{U}_{n_1 + \ldots + n_k} | k \geq 1, \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \} \tag{2.1}
\]

satisfying, for each \( \bar{z} \in \mathbb{Z}^N, N \geq 1 \):

\[
0 = \sum (-1)^{p+\bar{q}+\bar{r}+r} \mathcal{U}_{\bar{n}(\bar{n})} \circ \cdots \circ \mathcal{U}_{\bar{n}(\bar{n})} : \mathcal{U}_{\bar{n}(\bar{n})} \circ \cdots \circ \mathcal{U}_{\bar{n}(\bar{n})} \to \mathcal{U}_{\bar{n}(\bar{n})} \circ \cdots \circ \mathcal{U}_{\bar{n}(\bar{n})} \tag{2.2}
\]

where the sum runs over partitions \( \bar{z} = (\bar{n}, \bar{n}', \bar{n}'') \) with \( |\bar{n}| = p, |\bar{n}'| = q, |\bar{n}''| = r \). We will say that \((\mathcal{U}, \Theta)\) is even if each \( \theta_\bar{n}(\bar{n}) = 0 \) whenever \( k \) is odd.

**Definition 2.2.** An \( \mathbb{A}_\infty \)-comonad \((\mathcal{V}, \Delta)\) on \( \mathcal{C} \) consists of the following data:

(a) A collection \( \mathcal{V} = \{ \mathcal{V}_n : \mathcal{C} \to \mathcal{C} \}_{n \in \mathbb{Z}} \) of endofunctors on \( \mathcal{C} \).

(b) A collection \( \Delta \) of natural transformations

\[
\Delta = \{ \delta_\bar{n}(\bar{n}) : \mathcal{V}_{\bar{n}} = \mathcal{V}_{n_1} \circ \cdots \circ \mathcal{V}_{n_k} = \mathcal{V}_{n_1 + \ldots + n_k} | k \geq 1, \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \} \tag{2.3}
\]

such that, for each \( \bar{z} \in \mathbb{Z}^N, N \geq 1 \), we have

\[
0 = \sum (-1)^{p+\bar{q}+\bar{r}+r} \mathcal{V}_{\bar{n}(\bar{n})} \circ \cdots \circ \mathcal{V}_{\bar{n}(\bar{n})} : \mathcal{V}_{\bar{n}(\bar{n})} \circ \cdots \circ \mathcal{V}_{\bar{n}(\bar{n})} \to \mathcal{V}_{\bar{n}(\bar{n})} \circ \cdots \circ \mathcal{V}_{\bar{n}(\bar{n})} \tag{2.4}
\]

where the sum runs over partitions \( \bar{z} = (\bar{n}, \bar{n}', \bar{n}'') \) with \( |\bar{n}| = p, |\bar{n}'| = q, |\bar{n}''| = r \). We will say that an \( \mathbb{A}_\infty \)-comonad \((\mathcal{V}, \Delta)\) is locally finite if for each \( k \geq 1, T \in \mathbb{Z} \), the induced morphism

\[
\bigotimes_{\bar{n} \in \mathbb{Z}(T, k)} \delta_\bar{n}(\bar{n}) : \mathcal{V}_{T+2-k} = \mathcal{V}_{n_1 + \ldots + n_k} \to \bigotimes_{\bar{n} \in \mathbb{Z}(T, k)} \mathcal{V}_{n_1} \circ \cdots \circ \mathcal{V}_{n_k} \tag{2.5}
\]

factors through the direct sum \( \bigoplus_{\bar{n} \in \mathbb{Z}(T, k)} \mathcal{V}_{\bar{n}} \).

We will say that \((\mathcal{V}, \Delta)\) is even if each \( \delta_\bar{n}(\bar{n}) = 0 \) whenever \( k \) is odd.

In any category \( \mathcal{A} \) containing products and direct sums, we will say that a collection \( \{ \eta_i : X \to X_i \}_{i \in I} \) of morphisms is locally finite if the induced morphism \( \prod \eta_i : X \to \prod X_i \) factors through the direct sum \( \bigoplus X_i \). We now make two conventions that we will use for notation throughout the paper.

(1) Suppose \((U, V)\) is a pair of adjoint functors between categories \( \mathcal{A} \) and \( \mathcal{B} \), so that we have natural isomorphisms

\[
\mathcal{B}(UX, Y) \cong \mathcal{A}(X, VY) \tag{2.6}
\]

for objects \( X \in \mathcal{A}, Y \in \mathcal{B} \). For any morphism \( f \in \mathcal{B}(UX, Y) \), we denote by \( f^R \) the corresponding morphism in \( \mathcal{A}(X, VY) \). Conversely, for any \( g \in \mathcal{A}(X, VY) \), we denote by \( g^L \) the corresponding morphism in \( \mathcal{B}(UX, Y) \).

(2) Suppose \((U_1, V_1)\) and \((U_2, V_2)\) are pairs of adjoint functors between categories \( \mathcal{A} \) and \( \mathcal{B} \). Then, we know that there are isomorphisms

\[
\text{Nat}(U_1, U_2) \cong \text{Nat}(V_2, V_1) \tag{2.7}
\]

For any natural transformation \( \eta \in \text{Nat}(U_1, U_2) \), we denote by \( \eta^R \) the corresponding natural transformation in \( \text{Nat}(V_2, V_1) \). Conversely, for any \( \zeta \in \text{Nat}(V_2, V_1) \), we denote by \( \zeta^L \) the corresponding natural transformation in \( \text{Nat}(U_1, U_2) \).

We will say that \((U = \{U_n : \mathcal{A} \to \mathcal{B}\}_{n \in \mathbb{Z}}, V = \{V_n : \mathcal{B} \to \mathcal{A}\}_{n \in \mathbb{Z}})\) is a \( \mathbb{Z} \)-system of adjoints between categories \( \mathcal{A} \) and \( \mathcal{B} \) if \((U_n, V_n)\) is a pair of adjoint functors for each \( n \in \mathbb{Z} \).
Theorem 2.3. Let \( \mathbb{U} = \{ \mathbb{U}_n : C \rightarrow \mathbb{C}_n \} \), \( \mathbb{V} = \{ \mathbb{V}_n : C \rightarrow \mathbb{C}_n \} \) be a \( \mathbb{Z} \)-system of adjoints on \( C \). Then,

(a) \( \mathbb{U} \) can be equipped with the structure of an even \( A_\infty \)-monad if and only if \( \mathbb{V} \) can be equipped with the structure of an even \( A_\infty \)-comonad.

(b) \( \mathbb{U} \) can be equipped with the structure of an even \( A_\infty \)-comonad if and only if \( \mathbb{V} \) can be equipped with the structure of an even \( A_\infty \)-monad.

Proof. We only prove (a) because (b) is similar. Suppose that \( \mathbb{U}, \Theta = \{ \theta_k(\tilde{n}) \} \) is an even \( A_\infty \)-monad. Accordingly, we have transformations

\[
(\theta_k(\tilde{n}) : \mathbb{U}_n \rightarrow \mathbb{U}_{\Sigma n + (2-k)}) \mapsto (\theta^\prime_k(\tilde{n}) : \mathbb{V}_{\Sigma n + (2-k)} \rightarrow \mathbb{V}_{\tilde{n}^\prime})
\]

Using the notation of (2.2), we see that the \( \theta^\prime_k(\tilde{n}) \) satisfy, for any \( \tilde{z} \in \mathbb{Z}^N, N \geq 1 \),

\[
\sum_{\tilde{z} = (\tilde{n}, \tilde{r}, \tilde{r}^\prime)} (-1)^F(\tilde{n}^{\rightarrow r} \ast \theta^\prime_k(\tilde{n}^{\rightarrow r}) \ast \tilde{n}^\prime) \delta_{p+1, +}(\tilde{n}, \Sigma \tilde{n}^\prime + (2 - q), \tilde{n}^{\prime \prime}) = 0
\]

as a transformation \( \mathbb{V}_{\Sigma n + \Sigma n + \Sigma n^\prime + (3-N)} \rightarrow \mathbb{V}_{\tilde{n}^{\prime \prime}} \circ \mathbb{V}_{\tilde{n}^{\prime \prime}} \circ \mathbb{V}_{\tilde{n}^{\prime \prime}} \) (note that \( q \) must be even). We now set \( \delta_k(\tilde{n}) := \theta^\prime_k(\tilde{n}^{op}) \).

Accordingly, the condition in (2.9) now becomes

\[
\sum_{\tilde{z} = (\tilde{n}, \tilde{r}, \tilde{r}^\prime)} (-1)^F(\tilde{n}^{\rightarrow r} \ast \delta_k(\tilde{n}^{op}) \ast \tilde{n}^{\prime \prime}) \delta_{p+1, +}(\tilde{n}^{op}, \Sigma \tilde{n}^{op} + (2 - q), \tilde{n}^{op}) = 0
\]

Comparing with (2.4), we see that \( (\mathbb{V}, \Delta = \{ \delta_k(\tilde{n}) \}) \) satisfies the conditions for being an even \( A_\infty \)-comonad. Further, these arguments can be reversed, showing that if \( \mathbb{V} \) carries the structure of an even \( A_\infty \)-comonad, then \( \mathbb{U} \) carries structure of an even \( A_\infty \)-monad. This proves the result.

We will now describe the connection between \( A_\infty \)-monads and locally finite \( A_\infty \)-comonads. For this, we will first prove some simple lemmas on the collection \( \text{End}(\mathbb{C}) \) of endofunctors in \( \mathbb{C} \). If \( F \in \text{End}(\mathbb{C}) \), a subfunctor \( F' \hookrightarrow F \) is a natural transformation such that \( F'(M) \rightarrow F(M) \) is a monomorphism in \( \mathbb{C} \) for each \( M \in \mathbb{C} \). In order to avoid set theoretic complications, we will assume wherever necessary that the category \( \text{End}(\mathbb{C}) \) is well-powered, i.e., the collection of subfunctors of any \( F \in \text{End}(\mathbb{C}) \) forms a set.

Lemma 2.4. Let \( F, G \in \text{End}(\mathbb{C}) \). Suppose that \( G \) preserves monomorphisms. Then, if \( F'' \hookrightarrow F' \hookrightarrow F \) and \( G'' \hookrightarrow G' \hookrightarrow G \) are subfunctors, the composition \( G'' \circ F'' \) is a subfunctor of \( G' \circ F' \).

Proof. We take some \( M \in \mathbb{C} \) and consider the following commutative diagram

\[
\begin{array}{ccc}
G \circ F''(M) & \xrightarrow{\text{mono}} & G \circ F'(M) \\
\uparrow & & \uparrow \text{mono} \\
G'' \circ F''(M) & \xrightarrow{\text{mono}} & G'' \circ F'(M)
\end{array}
\]

Since \( G \) preserves monomorphisms, the morphisms in the top row of (2.11) are all monomorphisms. Further, the commutativity of the square in (2.11) shows that \( G' \circ F''(M) \rightarrow G' \circ F'(M) \) is a monomorphism. The composition in the bottom row now gives the desired result.

Lemma 2.5. Let \( \eta_i : F \rightarrow F_i \) \( i \in I \) be a family of morphisms in \( \text{End}(\mathbb{C}) \). Fix an indexing set \( A \). For each \( i \in I \), let \( \{ F^a \}_{a \in A} \) be a family of subfunctors of \( F_i \) and let \( \{ F^a_i \}_{a \in A} \) be a family of subfunctors of \( F \) such that \( \eta_i : F \rightarrow F_i \) restricts to \( \eta^a_i : F^a \rightarrow F^a_i \). For each \( a \in A \), suppose that \( \{ \eta^a_i : F^a \rightarrow F^a_i \}_{i \in I} \) is locally finite in \( \text{End}(\mathbb{C}) \). Then, the family

\[
\left\{ \eta^a = \sum_{\alpha \in A} \eta^a_i : F^a \rightarrow \sum_{\alpha \in A} F^a_i = F^a \right\}_{i \in I}
\]

is locally finite.
Proof. Since \( \eta^\alpha : F^\alpha \to F^\alpha \) is locally finite for each \( \alpha \in A \), we have morphisms \( F^\alpha \to \bigoplus_{i \in I} F^\alpha_i \to \prod_{i \in I} F^\alpha_i \). Summing over all \( \alpha \in A \), we can consider for each \( i_0 \in I \) the following composition

\[
\sum_{\alpha \in A} F^\alpha \to \sum_{\alpha \in A} \bigoplus_{i \in I} F^\alpha_i = \bigoplus_{i \in I} \sum_{\alpha \in A} F^\alpha_i \to \sum_{\alpha \in A} F^\alpha_{i_0}.
\]

The second morphism in (2.13) is given by the canonical projection corresponding to \( i_0 \in I \). The interchange in the middle term in (2.13) is justified by the fact that \( \mathcal{C} \) satisfies (AB5). As we vary over all \( i_0 \in I \), the compositions in (2.13) give us a factorization

\[
\sum_{\alpha \in A} F^\alpha \to \bigoplus_{i \in I} \sum_{\alpha \in A} F^\alpha_i \to \prod_{i \in I} \sum_{\alpha \in A} F^\alpha_i
\]

This proves the result. \( \square \)

Lemma 2.6. Let \( F, G \in \text{End}(\mathcal{C}) \). Suppose that \( G \) preserves monomorphisms. Let \( \{F^\alpha\}_{\alpha \in A} \) (resp. \( \{G^\alpha\}_{\alpha \in A} \)) be a family of subfunctors of \( F \) (resp. \( G \)). Then, \( \sum_{\alpha \in A} G^\alpha F^\alpha \) is a subfunctor of \( (\sum_{\alpha \in A} G^\alpha)(\sum_{\alpha \in A} F^\alpha) \).

Proof. For each fixed \( \alpha_0 \in A \), we have subfunctors \( F^{\alpha_0} \hookrightarrow (\sum_{\alpha \in A} F^\alpha) \larrow F \) and \( G^{\alpha_0} \hookrightarrow (\sum_{\alpha \in A} G^\alpha) \larrow G \). Applying Lemma 2.4 it follows that each \( G^{\alpha_0} F^{\alpha_0} \) is a subfunctor of \( (\sum_{\alpha \in A} G^\alpha)(\sum_{\alpha \in A} F^\alpha) \). The result is now clear by summing over all \( \alpha_0 \in A \). \( \square \)

We now consider a \( \mathbb{Z} \)-system \( (U = \{U_n : \mathcal{C} \to \mathcal{C}\}_{n \in \mathbb{Z}}, V = \{V_n : \mathcal{C} \to \mathcal{C}\}_{n \in \mathbb{Z}}) \) of adjoints on \( \mathcal{C} \). Suppose that \( (U, \Theta) \) is an \( A_{\infty} \)-monad, given by a collection of natural transformations

\[
\Theta = \{\delta_k(\tilde{n}) : U_{\tilde{n}} = U_{n_1} \circ \ldots \circ U_{n_k} \to U_{n_1 + \ldots + n_k} = U_{\sum n + (k-1)} \mid k \geq 1, \tilde{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k\}
\]

satisfying the condition in (2.2). If \( (U, \Theta) \) is even, setting \( \delta_k(\tilde{n}) = \theta^k_{\tilde{n}}(\tilde{n}^{op}) \) as in the proof of Theorem 2.3, we know that \( (V, \Delta) \) is an \( A_{\infty} \)-comonad, where

\[
\Delta = \{\delta_k(\tilde{n}) : V_{\sum n + (k-1)} = V_{n_1 + \ldots + n_k + (k-1)} \to V_{n_1} \circ \ldots \circ V_{n_k} = V_{\tilde{n}} \mid k \geq 1, \tilde{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k\}
\]

We now consider a family \( \mathcal{V}^\alpha = \{V^\alpha_n \hookrightarrow V^\alpha\}_{n \in \mathbb{Z}} \) of subfunctors satisfying the following conditions

(a) For any \( k \geq 1 \) and \( \tilde{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \), each of the transformations \( \delta_k(\tilde{n}) \) in (2.16) restricts to the corresponding subfunctors

\[
\Delta^\alpha = \{\delta^\alpha_k(\tilde{n}) : V^\alpha_{\sum n + (k-1)} = V^\alpha_{n_1 + \ldots + n_k + (k-1)} \to V^\alpha_{n_1} \circ \ldots \circ V^\alpha_{n_k} = V^\alpha_{\tilde{n}} \mid k \geq 1, \tilde{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k\}
\]

We note here that since each \( V^\alpha_n \) is a right adjoint, it preserves monomorphisms. As such, it follows from Lemma 2.4 that each \( V^\alpha_n \circ \ldots \circ V^\alpha_{n_k} \) is a subfunctor of \( V^\alpha_{\tilde{n}} \).

(b) For each \( k \geq 1 \) and \( T \in \mathbb{Z} \), the collection

\[
\{\delta^\alpha_k(\tilde{n}) : V^\alpha_{T + (k-1)} = V^\alpha_{n_1 + \ldots + n_k + (k-1)} \to V^\alpha_{n_1} \circ \ldots \circ V^\alpha_{n_k} = V^\alpha_{\tilde{n}} \}_{\tilde{n} \in \mathbb{Z}^k(T,k)}
\]

is locally finite.

Lemma 2.7. \( (\mathcal{V}^\alpha, \Delta^\alpha) \) is a locally finite \( A_{\infty} \)-comonad.

Proof. By assumption, the natural transformations \( \delta_k(\tilde{n}) \) restrict to transformations \( \delta^\alpha_k(\tilde{n}) \). Since each \( V^\alpha_n \) preserves monomorphisms, we see that all the terms appearing in (2.4) restrict to corresponding subfunctors. As such, these transformations \( \delta^\alpha_k(\tilde{n}) \) satisfy the condition in (2.4). Hence, \( (\mathcal{V}^\alpha, \Delta^\alpha) \) is an \( A_{\infty} \)-comonad. The fact that \( (\mathcal{V}^\alpha, \Delta^\alpha) \) is locally finite follows directly from the assumption in (2.18). \( \square \)

We now let \( \{\mathcal{V}^\alpha, \Delta^\alpha\}_{\alpha \in A} \) denote the collection of such families of subfunctors. We note that this collection is non-empty since it contains the family of zero subfunctors. We now define

\[
\mathcal{V}^\alpha = \sum_{\alpha \in A} \mathcal{V}^\alpha_{\mathcal{V}^\alpha_{\mathcal{V}^\alpha}} \tag{2.19}
\]
Proposition 2.8. The collection \( V^\otimes = \{ V^p \}_{n \in \mathbb{Z}} \) is equipped with the structure of a locally finite \( A_\infty \)-comonad.

Proof. For any \( k \geq 1 \) and \( \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k \), we consider

\[
\delta^p_k(\bar{n}) = \sum_{a \in A} \delta^p_{ka}(\bar{n}) : \sum_{a \in A} V^a_{\Sigma k + 2(k-1)} \to \sum_{a \in A} V^a_{n_1} \circ ... \circ V^a_{n_k} \to \left( \sum_{a \in A} V^a_{n_1} \right) \circ ... \circ \left( \sum_{a \in A} V^a_{n_k} \right)
\]

where the inclusion in (2.20) follows from Lemma 2.6. Again since each \( V^p \) preserves monomorphisms, we see that all the transformations appearing in (2.4) restrict to corresponding subfunctors. It follows that the transformations \( \delta^p_k(\bar{n}) : V^p_{\Sigma k + 2(k-1)} \to V^p_{n_1} \circ ... \circ V^p_{n_k} \) satisfy the condition in (2.4), making \( \left( V^\otimes, A^\otimes = \{ \delta^p_k(\bar{n}) \mid k \geq 1, \bar{n} \in \mathbb{Z}^k \} \right) \) an \( A_\infty \)-comonad.

Additionally, using Lemma 2.5 we see that for each \( k \geq 1 \) and \( T \in \mathbb{Z} \), we have a factorization

\[
\sum_{a \in A} V^a_{T + (2-k)} \to \prod_{\bar{n} \in \mathbb{Z}(T,k)} \sum_{a \in A} V^a_{n_1} \circ ... \circ V^a_{n_k} \to \prod_{\bar{n} \in \mathbb{Z}(T,k)} \left( \sum_{a \in A} V^a_{n_1} \right) \circ ... \circ \left( \sum_{a \in A} V^a_{n_k} \right)
\]

(2.21)

From (2.21), it is clear that \( (V^\otimes, A^\otimes) \) is locally finite.

\[\square\]

Remark 2.9. On the other hand, suppose that \( \{ U_n : \mathcal{C} \to \mathcal{C} \}_{n \in \mathbb{Z}}, \mathcal{V} = \{ V_n : \mathcal{C} \to \mathcal{C} \}_{n \in \mathbb{Z}} \) is a \( \mathbb{Z} \)-system of adjoints such that \( \mathcal{V} \) carries the structure of an even \( \mathcal{A}_\infty \)-comonad. By Theorem 2.3 we know that \( \{ U_n \}_{n \in \mathbb{Z}} \) is canonically equipped with an \( \mathcal{A}_\infty \)-comonad structure. Additionally, if we suppose that each \( U_n \) preserves monomorphisms, we can construct by a process similar to (2.19) a family \( \mathcal{U}^\otimes = \{ U^\otimes_n \rightarrow U_n \}_{n \in \mathbb{Z}} \) of subfunctors equipped with a locally finite \( \mathcal{A}_\infty \)-comonad structure.

3 Eilenberg-Moore categories for \( A_\infty \)-monads and \( A_\infty \)-comonads

We now consider modules over \( A_\infty \)-monads and comonads over \( A_\infty \)-comonads.

Definition 3.1. Let \( (U, \Theta) \) be an \( A_\infty \)-monad over \( \mathcal{C} \). A \((U, \Theta)\)-module \((M, \pi)\) consists of the following data:

(a) A collection \( M = \{ M_n \}_{n \in \mathbb{Z}} \) of objects of \( \mathcal{C} \).

(b) A collection of morphisms

\[
\pi = \{ \pi_\alpha(\tilde{n}, l) : \tilde{U}_n M_l \to M_{\Sigma \tilde{n} + (2-k)} \mid k \geq 1, \tilde{n} = (n_1, ..., n_k) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \}.
\]

(3.1)

satisfying, for each \( \tilde{z} \in \mathbb{Z}^N \), \( N \geq 0 \) and \( l \in \mathbb{Z} \),

\[
\sum_{\tilde{z}: \tilde{n}, \tilde{n}' \in \tilde{z}} (-1)^{\rho_1 + q + S_2 \rho_3} \pi_{\rho_1 + 1}(\tilde{n}, \Sigma \tilde{n}' + (2 - q), \tilde{n}'', l) \circ (\bar{U}_{\tilde{n}} \ast \bar{\theta}_q(\tilde{n}''))(\bar{U}_{\tilde{n}} M_l) = - \sum_{\tilde{z}: \tilde{m}, \tilde{m}' \in \tilde{z}} (-1)^{\rho_1 + S_2 \rho_3} \pi_{\rho_1 + 1}(\tilde{m}, \Sigma \tilde{m}' + l + (2 - b)) \circ \bar{U}_{\tilde{m}}(\pi_{\tilde{m}}(\tilde{m}'', l))
\]

(3.2)

On the left hand side of (3.2), we have \( |n| = p, |n'| = q \) and \( |n''| = r - 1 \), while on the right hand side, we have \( |m| = a \) and \( |m'| = b - 1 \). Accordingly, both sides of (3.2) represent morphisms

\[
\tilde{U}_{\tilde{n}} \tilde{U}_{\tilde{n}'} M_l = \tilde{U}_{\tilde{m}} \tilde{U}_{\tilde{m}'} M_l \to M_{\Sigma \tilde{n} + \Sigma \tilde{n}' + 1 + (2-N)} = M_{\Sigma \tilde{m} + \Sigma \tilde{m}' + 1 + (2-N)}
\]

(3.3)
A morphism \( f : (M, \pi) \rightarrow (M', \pi') \) of \( \mathbb{U} \)-modules consists of a collection \( f = \{ f_i : M_i \rightarrow M'_i \}_{i \in \mathbb{Z}} \) such that \( \pi'_i(\tilde{n}, l) \circ \epsilon_0(\bar{h}_i) = f_{\Sigma^{h_i+1}(2-k)} \circ \pi_i(\tilde{n}, l) \) for every \( k \geq 1, \tilde{n} = (n_1, \ldots, n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \). We denote the category of \( (\mathbb{U}, \Theta) \)-modules by \( E M_{(\mathbb{U}, \Theta)} \).

We will say that a \( \mathbb{U} \)-module \( (M, \pi) \) is even if \( \pi_k(\tilde{n}, l) = 0 \) whenever \( k \) is odd. The full subcategory of even \( \mathbb{U} \)-modules will be denoted by \( E M_{(\mathbb{U}, \Theta)}^{e} \).

**Definition 3.2.** Let \((\mathbb{V}, \Delta)\) be an \( \mathbb{A}_\infty \)-comonad over \( \mathbb{C} \). A \((\mathbb{V}, \Delta)\)-comodule \((P, \rho)\) consists of the following data:

1. A collection \( P = \{ P_n \}_{n \in \mathbb{Z}} \) of objects of \( \mathbb{C} \).
2. A collection of morphisms
   \[
   \rho = \{ \rho_k(\tilde{n}, l) : P_{\Sigma^{h_i+1}(2-k)} \rightarrow \mathbb{U}_n P_l \mid k \geq 1, \tilde{n} = (n_1, \ldots, n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \}\]
   satisfying, for each \( \tilde{z} \in \mathbb{Z}^N, N \geq 0 \) and \( l \in \mathbb{Z} \),
   \[
   \sum_{\tilde{z}=(\tilde{n}, \tilde{r}, \tilde{s})} (-1)^{r_1+q_1+q} \mathbb{V}_n(\tilde{n} \ast \delta_q(\tilde{r}))(\mathbb{V}_l P_l) \circ \rho_{p+1+r}(\tilde{n}, \Sigma \tilde{r} + (2 - q), \tilde{s}, l) \\
   = - \sum_{\tilde{z}=(\tilde{n}, \tilde{r}, \tilde{s})} (-1)^{r_1+q_1+q} \mathbb{V}_n(\tilde{n} \ast \delta_q(\tilde{r}))(\mathbb{V}_l P_l) \circ \rho_{p+1+r}(\tilde{n}, \Sigma \tilde{r} + (2 - q), \tilde{s}, l) \tag{3.5}
   
   On the left hand side of (3.5), we have \( |\tilde{n}| = p, \ |\tilde{r}| = q \) and \( |\tilde{s}| = r - 1 \), while on the right hand side, we have \( |\tilde{n}| = a \) and \( |\tilde{r}| = b - 1 \). Accordingly, both sides of (3.5) represent morphisms
   \[
   P_{\Sigma^{h_i+1}(2-k)} P_{\Sigma^{h_i+1}(2-k)} = P_{\Sigma^{h_i+1}(2-k)} P_{\Sigma^{h_i+1}(2-k)} \rightarrow \mathbb{U}_{\tilde{n}} \mathbb{U}_{\tilde{s}} P_l = \mathbb{U}_{\tilde{n}} \mathbb{U}_{\tilde{s}} P_l \tag{3.6}
   
   A morphism \( g : (P, \rho) \rightarrow (P', \rho') \) of \( \mathbb{V} \)-comodules consists of a collection \( g = \{ g_i : P_i \rightarrow P'_i \}_{i \in \mathbb{Z}} \) such that \( \rho'_k(\tilde{n}, l) \circ (\mathbb{U}_n \ast \theta_q(\tilde{r}))(\mathbb{n}_l P_l) = \mathbb{V}_n(\tilde{n} \ast \delta_q(\tilde{r}))(\mathbb{V}_l P_l) \circ \rho_{p+1+r}(\tilde{n}, \Sigma \tilde{r} + (2 - q), \tilde{s}, l) \) for every \( k \geq 1, \tilde{n} = (n_1, \ldots, n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \). We denote the category of \( (\mathbb{V}, \Delta) \)-comodules by \( E M_{(\mathbb{V}, \Delta)} \).

We will say that a \( \mathbb{V} \)-comodule \((P, \rho)\) is even if \( \rho_k(\tilde{n}, l) = 0 \) whenever \( k \) is odd. The full subcategory of even \( \mathbb{V} \)-comodules will be denoted by \( E M_{(\mathbb{V}, \Delta)}^{e} \).

**Proposition 3.3.**

(a) Let \((\mathbb{U}, \Theta)\) be an \( \mathbb{A}_\infty \)-monad over \( \mathbb{C} \). Then, for any object \( M \in \mathbb{C} \), the collection \( \{ \mathbb{U}_n M \}_{n \in \mathbb{Z}} \) can be equipped with a canonical \((\mathbb{U}, \Theta)\)-module structure.

(b) Let \((\mathbb{V}, \Delta)\) be an \( \mathbb{A}_\infty \)-comonad over \( \mathbb{C} \). Then, for any object \( M \in \mathbb{C} \), the collection \( \{ \mathbb{V}_n M \}_{n \in \mathbb{Z}} \) can be equipped with a canonical \((\mathbb{V}, \Delta)\)-module structure.

**Proof.** We prove only (a) because (b) is similar. For the sake of convenience, we set \( Q_n := \mathbb{U}_n M \) for each \( n \in \mathbb{Z} \). For \( \tilde{n} = (n_1, \ldots, n_{k-1}) \in \mathbb{Z}^{k-1}, k \geq 1, l \in \mathbb{Z} \), we set
   \[
   \pi_k(\tilde{n}, l) : \mathbb{U}_n Q_l = \mathbb{U}_n (\mathbb{U}_l M) \xrightarrow{\theta_{p+1+r}(\tilde{n}, l)} \mathbb{U}_{\Sigma^{h_i+1}(2-k)} M = Q_{\Sigma^{h_i+1}(2-k)} \tag{3.7}
   
   We note that the condition in (2.2) for \((\mathbb{U}, \Theta)\) to be an \( \mathbb{A}_\infty \)-monad implies in particular that for each \( \tilde{z} \in \mathbb{Z}^N, N \geq 0 \) and \( l \in \mathbb{Z} \), we must have
   \[
   \sum_{\tilde{z}=(\tilde{n}, \tilde{r}, \tilde{s})} (-1)^{r_1+q_1+q} \pi_{p+1+r}(\tilde{n}, \Sigma \tilde{r} + (2 - q), \tilde{s}, l) \circ (\mathbb{U}_n \ast \theta_q(\tilde{r}))(\mathbb{U}_l M) \\
   = - \sum_{\tilde{z}=(\tilde{n}, \tilde{r}, \tilde{s})} (-1)^{r_1+q_1+q} \pi_{p+1+r}(\tilde{n}, \Sigma \tilde{r} + (2 - q), \tilde{s}, l) \circ (\mathbb{U}_n \ast \theta_q(\tilde{r}))(\mathbb{U}_l M) \tag{3.8}
   
   On the left hand side of (3.8), we have \( |\tilde{n}| = p, |\tilde{r}| = q \) and \( |\tilde{s}| = r - 1 \), while on the right hand side, we have \( |\tilde{n}| = a \) and \( |\tilde{r}| = b - 1 \). Putting \( \pi_k(\tilde{n}, l) = \theta_k(\tilde{n}, l) \) and \( Q_n = \mathbb{U}_n M \), it follows from (3.8) that
   \[
   \sum_{\tilde{z}=(\tilde{n}, \tilde{r}, \tilde{s})} (-1)^{r_1+q_1+q} \pi_{p+1+r}(\tilde{n}, \Sigma \tilde{r} + (2 - q), \tilde{s}, l) \circ (\mathbb{U}_n \ast \theta_q(\tilde{r}))(\mathbb{U}_l Q_l) \\
   = - \sum_{\tilde{z}=(\tilde{n}, \tilde{r}, \tilde{s})} (-1)^{r_1+q_1+q} \pi_{p+1+r}(\tilde{n}, \Sigma \tilde{r} + (2 - q), \tilde{s}, l) \circ (\mathbb{U}_n \ast \theta_q(\tilde{r}))(\mathbb{U}_l Q_l) \tag{3.9}
   
   This proves the result. \( \square \)
Let \((\U, \Theta)\) be an \(A_\infty\)-monad over \(\mathcal{C}\). We now let \(\mathcal{C}^\mathbb{Z}\) denote the category of \(\mathbb{Z}\)-graded objects over \(\mathcal{C}\). Then, for any \(M = \{M_i\}_{i \in \mathbb{Z}}\) in \(\mathcal{C}^\mathbb{Z}\), we set
\[
\hat{\U} M = \left\{ \left( \hat{\U} M_i \right)_l := \bigoplus_{x+y=l} \U_x M_y \right\}_{l \in \mathbb{Z}} \quad (3.10)
\]
In particular, for \(M \in \mathcal{C}\), we denote by \(\hat{\U} M\) the object \([\U_x M]_{l \in \mathbb{Z}} \in EM_{\U, \Theta}\) determined by Proposition 3.3.

**Proposition 3.4.** Let \((\U, \Theta)\) be an \(A_\infty\)-monad over \(\mathcal{C}\) such that each of the functors \([\U]_{l \in \mathbb{Z}}\) preserves direct sums. Then, \(\hat{\U}\) determines a functor
\[
\hat{\U} : \mathcal{C}^\mathbb{Z} \longrightarrow EM_{\U, \Theta} \quad (3.11)
\]
**Proof.** For the sake of convenience, we set \(Q_l := (\hat{\U} M)_l\) for each \(l \in \mathbb{Z}\). For \(n = (n_1, \ldots, n_k) \in \mathbb{Z}^{k-1}, k \geq 1, l \in \mathbb{Z}\), we define \(\pi_k(n, l) : \hat{\U}_l Q_l \longrightarrow Q_{\Sigma^l + l + (2-k)}\) by setting
\[
\hat{\U}_l Q_l = \hat{\U}_l \left( \bigoplus_{x+y=l} \U_x M_y \right) = \left( \bigoplus_{x+y=l} \U_{x+y} \right) \left( \bigoplus_{x+y=l} \U_x M_y \right) \xrightarrow{\sum l \rightarrow \pi_k(n, l)} \left( \bigoplus_{x+y=l} \U_{x+y} + (2-k) M_y \right) = Q_{\Sigma^l + l + (2-k)} \quad (3.12)
\]
In (3.12), we have used the fact that each \(\U_n\), and hence any \(\hat{\U}_n\), preserves direct sums. As in (3.8), we note that the condition in (2.2) for \((\U, \Theta)\) to be an \(A_\infty\)-monad implies in particular that for each \(z \in \mathcal{C}^\mathbb{Z}\), \(N \geq 0\) and \(x, y \in \mathbb{Z}\), we must have
\[
\sum_{\hat{\U}_l Q_l} (-1)^{p+q+\Sigma^l} \theta_{p+1+r}(\hat{\U}_l Q_l) = \sum_{\hat{\U}_l Q_l} (-1)^{p+q+\Sigma^l} \theta_{p+1+r}(\hat{\U}_l Q_l) = 0 \quad (3.13)
\]
Applying the definition of \(\pi_k(n, l)\) in (3.12) and taking the direct sum of the equalities in (3.13) over all \(x, y \in \mathbb{Z}\) such that \(l = x + y\), we obtain
\[
\sum_{\hat{\U}_l Q_l} (-1)^{p+q+\Sigma^l} \pi_{l+1+r}(\hat{\U}_l Q_l) = \sum_{\hat{\U}_l Q_l} (-1)^{p+q+\Sigma^l} \pi_{l+1+r}(\hat{\U}_l Q_l) = 0 \quad (3.14)
\]
It is now clear that we have a functor \(\hat{\U} : \mathcal{C}^\mathbb{Z} \longrightarrow EM_{\U, \Theta}\).

We mention here the following two simple facts on the formalism of adjoint functors, which we will use in the proof of the next result.

1. Let \((U_1, V_1)\) and \((U_2, V_2)\) be pairs of adjoint endofunctors on a category \(\mathcal{A}\). We consider a transformation \(\theta : U_1 \longrightarrow U_2\) and the corresponding transformation \(\theta^\# : V_2 \longrightarrow V_1\) between right adjoints. Let \(X, Y \in \mathcal{A}\) and let \(f : U_2 X \longrightarrow Y\) be a morphism. Then, we have commutative diagrams
\[
\begin{array}{ccc}
U_2 X & \xrightarrow{\theta^\#} & V_2 Y \\
\xrightarrow{f} \downarrow & \Longrightarrow & \xrightarrow{(f \circ \theta^\#)} \downarrow \\
U_1 X & \xrightarrow{(f \circ \theta^\#)} & V_1 Y
\end{array} \quad (3.15)
\]

2. Let \((U_1, V_1)\) and \((U_2, V_2)\) be pairs of adjoint endofunctors on a category \(\mathcal{A}\). We consider objects \(X, Y, Z \in \mathcal{A}\) as well as morphisms \(f : U_1 X \longrightarrow Y\) and \(g : U_2 Y \longrightarrow Z\). Then, we have commutative diagrams
\[
\begin{array}{ccc}
U_2 f & \xrightarrow{\theta^\#} & V_1 g \\
\xrightarrow{(g \circ U_2 f)} \downarrow & \Longrightarrow & \xrightarrow{(g \circ U_2 f)} \downarrow \\
U_2 U_1 X & \xrightarrow{(g \circ U_2 f)} & V_1 V_2 Z
\end{array} \quad (3.16)
\]
Proposition 3.5. Let \((U = \{U_n : \mathbb{C} \rightarrow \mathbb{C}\})_{n \in \mathbb{N}}\) and \((V = \{V_n : \mathbb{C} \rightarrow \mathbb{C}\})_{n \in \mathbb{N}}\) be a \(\mathbb{Z}\)-system of adjoints on \(\mathbb{C}\). Let \((U, \Theta)\) be an even \(A_{\infty}\)-monad and \((V, \Delta)\) be the corresponding \(A_{\infty}\)-comonad. Then, the categories \(EM^U_{(U, \Theta)}\) and \(EM^V_{(V, \Delta)}\) are isomorphic.

Proof. Let \((M, \pi)\) be an even \((U, \Theta)\)-module as in Definition 3.1. Accordingly, for \(n \in \mathbb{Z}^{k-1}, k \geq 1, l \in \mathbb{Z}\), we have morphisms

\[
\pi_\ell(n, l) : U_n M_l \rightarrow M_{\Sigma \ell^i + l + (2-k)} \Rightarrow (\sigma^R_k(n, l) : M_l \rightarrow V_{\Sigma \ell^i + l + (2-k)} )
\]  

(3.17)

We now define \(P = \{P_l\}_{l \in \mathbb{Z}}\) by setting \(P_l = M_{-l}\) for each \(l \in \mathbb{Z}\) as well as

\[
p_\ell(n, l) : P_{\Sigma \ell^i + l + (2-k)} = M_{\Sigma \ell^i - l + (2-k)} \xrightarrow{\pi^R_{\Sigma \ell^i - l + (2-k)}} V_{\Sigma \ell^i - l + (2-k)}
\]

(3.18)

for \(l \in \mathbb{Z}\) and \(n \in \mathbb{Z}^{k-1}\). From the proof of Theorem 2.3, we know that the collection \(\Delta = \{\delta_\ell(n)\}\) is given by setting \(\delta_\ell(n) = \theta^R_k(n)^p\). In the notation of (3.2), we now see that for each \(\bar{z} \in \mathbb{Z}^N, N \geq 0, l \in \mathbb{Z}\), we have

\[
\sum_{\bar{z} = (\bar{n}, \bar{r}, \bar{r}')} (-1)^{p + p' + 2q} \pi_{p + 1 + r}(\bar{n}, \Sigma \bar{n}' + (2 - q), \bar{r}', l) \circ (U_{\bar{n}} * \theta_{q}(\bar{r}')) (U_{\bar{r}} M_l) = - \sum_{\bar{z} = (\bar{n}, \bar{r})} (-1)^{p + 2q} \pi_{p + 1 + r}(\bar{n}, \Sigma \bar{m}' + l + (2 - b)) \circ (U_{\bar{m}} \pi_{\ell}(\bar{m}' , l))
\]

(3.19)

Taking adjoints on both sides of (3.19), we obtain

\[
\sum_{\bar{z} = (\bar{n}, \bar{r}, \bar{r}')} (-1)^{p + p' + 2q} (\pi_{p + 1 + r}(\bar{n}, \Sigma \bar{n}' + (2 - q), \bar{r}', l) \circ (U_{\bar{n}} * \theta_{q}(\bar{r}')) (U_{\bar{r}} M_l))^R = - \sum_{\bar{z} = (\bar{n}, \bar{r})} (-1)^{p + 2q} (\pi_{p + 1 + r}(\bar{n}, \Sigma \bar{m}' + l + (2 - b)) \circ (U_{\bar{m}} \pi_{\ell}(\bar{m}' , l))^R
\]

(3.20)

We now consider one by one the terms in (3.20). For any term on the left hand side of (3.20), we apply (3.15) with

\[
X = M_l, \quad Y = M_{\Sigma \ell^i + \Sigma \ell^i + l + (2-N)}
\]

\[
f = \pi_{p + 1 + r}(\bar{n}, \Sigma \bar{n}' + (2 - q), \bar{r}', l) = U_2 X = U_2 U_{\bar{n}} U_{\bar{r'}} U_{\bar{r}} = U_2 U_{\bar{n} \Sigma \ell^i + (2-q)} U_{\bar{r'}} = U_2
\]

(3.21)

It follows therefore from (3.15) that the left hand side of (3.20) gives

\[
\sum_{\bar{z} = (\bar{n}, \bar{r}, \bar{r}')} (-1)^{p + p' + 2q} (\pi_{p + 1 + r}(\bar{n}, \Sigma \bar{n}' + (2 - q), \bar{r}', l) \circ (U_{\bar{n}} * \theta_{q}(\bar{r}')) (U_{\bar{r}} M_l))^R
\]

(3.22)

If we shift degrees by setting \(l' = - \Sigma - l + (2 - N)\) and use (3.18), then (3.22) becomes

\[
- \sum_{\bar{z} = (\bar{n}, \bar{r}, \bar{r}')} (-1)^{p + 1} (\pi_{\ell^i} (\Sigma \bar{n}' + (2 - q), \bar{r}', l')) \circ \rho_{p + 1 + r}(\bar{n}, \Sigma \bar{n}' + (2 - q), \bar{r}', l')
\]

(3.23)

where the sign is obtained from the fact that \(q\) and \(p + 1 + r\) are both even. For any term on the right hand side of (3.20), we now apply (3.16) with

\[
X = M_l, \quad Y = M_{\Sigma \ell^i + l + (2-b)} \quad Z = M_{\Sigma \ell^i + \Sigma \ell^i + l + (2-N)}
\]

\[
f = \pi_{\ell}(\bar{m}', l) : U_1 X = U_1 U_{\bar{m}} M_l \rightarrow M_{\Sigma \ell^i + l + (2-b)} = Y
\]

(3.24)

It follows therefore from (3.16) that the right hand side of (3.20) gives

\[
- \sum_{\bar{z} = (\bar{m}, \bar{r}, \bar{r}')} (-1)^{p + 2q} \pi_{p + 1 + r}(\bar{m}, \Sigma \bar{m}' + l + (2 - b)) \circ (U_{\bar{m}} \pi_{\ell}(\bar{m}' , l))^R
\]

(3.25)
Again, if we shift degrees by setting \( p' = -\Sigma z - l - (2 - N) \), then (3.25) becomes

\[
\begin{align*}
&\sum_{z \in \mathbb{Z}^N} (1)^{a+b\Sigma n'} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^N} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^N} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^{k-1}} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^{k-1}} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^{k-1}} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^{k-1}} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^{k-1}} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a)) \\
&= \sum_{z \in \mathbb{Z}^{k-1}} \nu_{\bar{\alpha} + 1} (\bar{\alpha}^{op}, l) \circ \rho_{\bar{\alpha}} (\bar{\alpha}^{op}, \Sigma n + l' + (1 - a))
\end{align*}
\]

(3.26)

where the last equality in (3.25) follows from the fact that \( a + 1 \) and \( b \) must both be even. By inspecting (3.25) and (3.26), we see that the morphisms \( \rho_{\bar{\alpha}} (\bar{\alpha}, l) \) together make \( P = \{ P_n = M_{a,n} \}_{a \in \mathbb{Z}} \) into a \((\mathcal{V}, \Delta)\)-comodule. These arguments can be reversed, showing that even \((\mathcal{U}, \Theta)\)-modules correspond to even \((\mathcal{V}, \Delta)\)-comodules and vice-versa.

\[\square\]

**Definition 3.6.** Let \((\mathcal{U}, \Theta)\) be an \( A_\infty \)-monad over \( \mathcal{C} \) and let \((M^1, \pi^1), (M^2, \pi^2)\) be \((\mathcal{U}, \Theta)\)-modules. An \( \infty \)-morphism \( \alpha : (M^1, \pi^1) \rightarrow (M^2, \pi^2) \) consists of a collection

\[
\alpha = \{ \alpha_k (\bar{n}, l) : \mathcal{U}_k M^1_l \rightarrow M^2_{\Sigma^{k+1}z + (1-k)} | k \geq 1, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \}
\]

(3.27)

satisfying for each \( \bar{z} \in \mathbb{Z}^N, N \geq 0, l \in \mathbb{Z}, \)

\[
\sum_{z \in \mathbb{Z}^{k-1}} (-1)^{a+b \Sigma n'} \alpha_{\bar{\alpha} + 1} (\bar{\alpha}, \Sigma n' + l + (2 - b)) \circ \mathcal{U}_\bar{z} (\pi_1^1 (\bar{\alpha}, l))
\]

\[
+ \sum_{z \in \mathbb{Z}^{k-1}} (-1)^{a+b \Sigma n'} \alpha_{\bar{\alpha} + 1} (\bar{\alpha}, \Sigma n' + l + (2 - b), \bar{\alpha}, l) \circ (\mathcal{U}_\bar{z} + \theta_b (\bar{n}) \circ \mathcal{U}_l) (M^1_l)
\]

(3.28)

where:

1. In the first term on left hand side of (3.28), we have \( |\bar{n}| = a \) and \( |\bar{n}'| = b - 1 \).
2. In the second term on left hand side of (3.28), we have \( |\bar{\alpha}| = a, |\bar{\alpha}'| = b \) and \( |\bar{n}'| = c - 1 \).
3. On the right hand side of (3.28), we have \( |\bar{\alpha}| = a \) and \( |\bar{\alpha}'| = b - 1 \).

We observe that both sides of (3.28) represent morphisms \( \mathcal{U}_z M^1_l \rightarrow M^2_{\Sigma^{k+1}z + (1-k)} \). We will denote by \( EM_{(\mathcal{U}, \Theta)} \) the category of \((\mathcal{U}, \Theta)\)-modules equipped with \( \infty \)-morphisms. For any \((M, \pi)\) in \( EM_{(\mathcal{U}, \Theta)} \), the identity \( \iota : (M, \pi) \rightarrow (M, \pi) \) is given by taking \( \iota_1 (l) = id : M_l \rightarrow M_l \) for each \( l \in \mathbb{Z} \) and \( \iota_k (\bar{z}, l) = 0 \) otherwise. Given morphisms \( \alpha : (M^1, \pi^1) \rightarrow (M^2, \pi^2) \) and \( \beta : (M^2, \pi^2) \rightarrow (M^3, \pi^3) \), we define their composition \( \gamma = \beta \alpha \) as follows

\[
\gamma = \{ \gamma_k (\bar{z}, l) : \mathcal{U}_l M^1_l \rightarrow M^3_{\Sigma^{k+1}z + (1-k)} | k \geq 1, \bar{z} = (z_1, ..., z_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \}
\]

(3.29)

\[
\gamma_k (\bar{z}, l) := \sum_{z \in \mathbb{Z}^{k-1}} \beta_{\bar{\alpha} + 1} (\bar{\alpha}, \Sigma n' + l + (1 - b)) \circ \mathcal{U}_z (\alpha_k (\bar{n}, l))
\]

where \( |\bar{\alpha}| = a \) and \( |\bar{n}'| = b - 1 \) in (3.29). We will say that a morphism \( \alpha : (M^1, \pi^1) \rightarrow (M^2, \pi^2) \) is odd if \( \alpha_k (\bar{n}, l) = 0 \) for any even \( k \). In particular, any identity morphism is odd. From (3.29), we also notice that the composition of odd morphisms in \( EM_{(\mathcal{U}, \Theta)} \) must be odd.

**Definition 3.7.** Let \((\mathcal{V}, \Delta)\) be an \( A_\infty \)-comonad over \( \mathcal{C} \) and let \((P^1, \rho^1), (P^2, \rho^2)\) be \((\mathcal{V}, \Delta)\)-comodules. An \( \infty \)-morphism \( \alpha : (P^1, \rho^1) \rightarrow (P^2, \rho^2) \) consists of a collection

\[
\alpha = \{ \alpha_k (\bar{n}, l) : P^1_{\Sigma^{k+1}z + (1-k)} \rightarrow \mathcal{V}_l P^2_l | k \geq 1, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \}
\]

(3.30)

satisfying for each \( \bar{z} \in \mathbb{Z}^N, N \geq 0, l \in \mathbb{Z}, \)

\[
\sum_{z \in \mathbb{Z}^{k-1}} (-1)^{ab + b \Sigma n'} \mathcal{V}_l (\rho^2_{\bar{n}} (\bar{n}', l)) \circ \alpha_{\bar{\alpha} + 1} (\bar{\alpha}, \Sigma n' + l + (2 - b))
\]

\[
+ \sum_{z \in \mathbb{Z}^{k-1}} (-1)^{c + ab + b \Sigma n'} (\mathcal{V}_z \circ \delta_b (\bar{n}') \circ \mathcal{V}_l) (P^2_l) \circ \alpha_{\bar{\alpha} + 1} (\bar{\alpha}, \Sigma n' + (2 - b), \bar{n}', l)
\]

(3.31)

\[
= \sum_{z \in \mathbb{Z}^{k-1}} \mathcal{V}_l (\rho^1_{\bar{n}} (\bar{n}', l)) \circ \rho^2_{\bar{\alpha} + 1} (\bar{\alpha}, \Sigma n' + l + (1 - b))
\]
where:

1. In the first term on left hand side of (3.31), we have $|\bar{n}| = a$ and $|\bar{n}'| = b - 1$.
2. In the second term on left hand side of (3.31), we have $|\bar{n}| = a$, $|\bar{n}'| = b$ and $|\bar{n}''| = c - 1$.
3. On the right hand side of (3.31), we have $|\bar{p}| = a$ and $|\bar{p}'| = b - 1$.

We observe that both sides of (3.31) represent morphisms $P^1_{\Sigma^{l+1}(1-N)} \to \mathbb{P}_2^2$. We will denote by $EM^{(V, \Delta)}$ the category of $(V, \Delta)$-comodules equipped with $\omega$-morphisms. The composition of morphisms in $EM^{(V, \Delta)}$ is defined in a manner dual to (3.29). We will say that a morphism $\alpha$ in $EM^{(V, \Delta)}$ is odd if $\alpha_k(\bar{n}, l) = 0$ whenever $k$ is even. We may also verify that the collection of odd morphisms in $EM^{(V, \Delta)}$ is closed under composition.

For an $A_\omega$-monad $(\cup, \Theta)$ we denote by $EM^{(\cup, \Theta)}$ the subcategory of $EM^{(\cup, \Theta)}$ whose objects are even $(\cup, \Theta)$-modules with odd $\omega$-morphisms between them. Similarly, for an $A_\omega$-comonad $(V, \Delta)$, we denote by $EM^{(V, \Delta)}$ the subcategory of $EM^{(V, \Delta)}$ whose objects are even $(V, \Delta)$-comodules with odd $\omega$-morphisms between them. We now have the final result of this section.

Theorem 3.8. Let $(\cup = \{\cup_n : C \to C\}_{n \geq 0}, \mathcal{V} = \{\mathcal{V}_n : C \to C\}_{n \geq 0})$ be a $\mathcal{V}$-system of adjoints on $C$. Let $(\cup, \Theta)$ be an even $A_\omega$-monad and $(V, \Delta)$ be the corresponding $A_\omega$-comonad. Then, the categories $EM^{(\cup, \Theta)}$ and $EM^{(V, \Delta)}$ are isomorphic.

Proof. From Proposition 3.5 we already know that there is a one-one correspondence between the objects of $EM^{(\cup, \Theta)}$ and $EM^{(V, \Delta)}$. We consider therefore a morphism $\alpha : (M^1, \pi') \to (M^2, \pi')$ in $EM^{(\cup, \Theta)}$. Accordingly for any $k \geq 1, \bar{n}, l \in \mathbb{Z}$ we have morphisms

\[(a_k(\bar{n}, l): \cup_{\bar{n}}M^1_l \to M^2_{\Sigma^{l+1}(1-k)}) = (a_k^R(\bar{n}, l): M^1_l \to \mathbb{V}_{\bar{n}'}M^2_{\Sigma^{l+1}(1-k)})\]  

(3.32)

From the proof of Proposition 3.5 we know that the object $(P^1, \rho^1)$ (resp. $(P^2, \rho^2)$) in $EM^{(V, \Delta)}$ corresponding to $(M^1, \pi')$ (resp. $(M^2, \pi')$) is given by setting

\[P^1_l := M^1_l \quad \rho^1_{\bar{n}}(\bar{l}, l): P^1_{\Sigma^{l+1}(1-k)} = M^1_{\Sigma^{l+1}(1-k)} \to \mathbb{V}_{\bar{n}'}M^2_{\Sigma^{l+1}(1-k)} = \mathbb{V}_{\bar{n}'}P^2_l\]  

(3.33)

For $k \geq 1, \bar{n}, l \in \mathbb{Z}$, we now define

\[\beta_k(\bar{n}, l): P^1_{\Sigma^{l+1}(1-k)} = M^1_{\Sigma^{l+1}(1-k)} \to \mathbb{V}_{\bar{n}'}P^2_l\]  

(3.34)

We claim that (3.34) determines a morphism $(P^1, \rho^1) \to (P^2, \rho^2)$ in $EM^{(V, \Delta)}$. From the proof of Theorem 2.3, we know that the collection $\Delta = \{\delta_k(\bar{n})\}$ is given by setting $\delta_k(\bar{n}) = \theta_k^R(\bar{n}').$ In the notation of (3.28), we now see that for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 0$, $l \in \mathbb{Z}$, we have

\[\sum_{z = (\bar{n}', \bar{n}')} (-1)^{r + v} \alpha_{\bar{a}+1}(\bar{n}, \Sigma\bar{n}' + l + (2 - b)) \circ \cup_{\bar{n}}(\pi^1_{\bar{n}'}) l \o + \sum_{z = (\bar{n}', \bar{n}')} (-1)^{a + b + v} \alpha_{\bar{a}+1}(\bar{n}, \Sigma\bar{n}' + (2 - b), \bar{n}'', l) \o (\cup_{\bar{n}} \circ \theta_k(\bar{n}''))(M^1_l) = \sum_{z = (\bar{p}', \bar{p}'')} \pi^2_{\bar{a}+1}(\bar{p}, \Sigma\bar{p}' + l + (1 - b)) \o \cup_{\bar{p}}(\alpha_{\bar{a}}(\bar{p}', l))\]  

(3.35)

We now consider one by one the terms in (3.35). For those in the first term on the left hand side of (3.35), we apply (3.16) with

\[X = M^1_l \quad Y = M^1_{\Sigma^{l+1}(1-b-2)} \quad Z = M^2_{\Sigma^{l+1}(1-N)} \quad U_1 = \cup_{\bar{n}'} \quad U_2 = \cup_{\bar{n}} \quad f = \pi^1_{\bar{n}'}(\bar{n}', l) : U_1X = \cup_{\bar{n}}M^1_l \to \mathbb{V}_{\bar{n}'}M^2_{\Sigma^{l+1}(2-b)} = Y \]  

(3.36)

\[g = \alpha_{\bar{a}+1}(\bar{n}, \Sigma\bar{n}' + (2 - b)) : U_2Z = \cup_{\bar{n}}M^1_{\Sigma^{l+1}(1-N)} \to \mathbb{V}_{\bar{n}'}M^2_{\Sigma^{l+1}(1-N)} = Z\]
It follows from (3.16) that the first term on the left hand side of (3.35) gives
\[
\sum_{\bar{z}=(\bar{m},\bar{m}')} (1)^{a+b+2\sum} (\alpha_{a+1} (\bar{m}, \Sigma \bar{m}'+ l+(2-b)) \circ \bigcup_{\bar{m}} (\sigma_{b}^R (\bar{m}, l)))^R
\]
\[
= \sum_{\bar{z}=(\bar{m},\bar{m}')} (1)^{a+b+2\sum} (\bigvee_{\bar{m}''} (\alpha_{a+1} (\bar{m}, \Sigma \bar{m}'+ l+(2-b)) \circ \pi_{b}^R (\bar{m}', l))
\]
\[
= \sum_{\bar{z}=(\bar{m},\bar{m}')} (1)^{a+b+2\sum} (\beta_{a+1} (\bar{m}, \bar{m}'', -\Sigma (l-(1-N)))) \circ \beta_{b}^R (\bar{m}''',\Sigma \bar{m'}) - (2-b))
\]
\[
= \sum_{\bar{z}=(\bar{m},\bar{m}')} (\bigvee_{\bar{m}''} (\beta_{a+1} (\bar{m}, l')) \circ \beta_{b}^R (\bar{m}''',p', l'))
\]
where we have set \(l' = -\Sigma (l-(1-N))\), used the replacements in (3.33), (3.34) as well as the fact that \(a, b\) are both even. For those in the second term on the left hand side of (3.35), we apply (3.13) with
\[
X = M_1^1 \quad Y = M_2^{\Sigma+1-N}
\]
\[
f = a_{a+1+c} (\bar{n}, \Sigma \bar{n}'+(2-b), \bar{n}'', l) : U_{2} \longrightarrow \bigcup_{\bar{m}} U_{\Sigma} \bigcup_{\Sigma} U_{\Sigma} = U_{2}
\]
It follows from (3.15) that the second term on the left hand side of (3.35) gives
\[
\sum_{\bar{z}=(\bar{m},\bar{m}')} (1)^{a+b+2\sum} (\alpha_{a+1} (\bar{n}, \Sigma \bar{n}'+ (2-b), \bar{n}'', l) \circ (\bigcup_{\bar{m}} \theta_{b} (\bar{m}') \circ \bigcup_{\bar{m}'}) (M_1^1))^R
\]
\[
= \sum_{\bar{z}=(\bar{m},\bar{m}')} (1)^{a+b+2\sum} (\bigvee_{\bar{m}''} \theta_{b} (\bar{m}', \Sigma \bar{m}') \circ \alpha_{a+1+c} (\bar{n}, \Sigma \bar{n}'+ (2-b), \bar{n}'', l)) (3.39)
\]
where we have set \(l' = -\Sigma (l-(1-N))\), used the replacements in (3.33), (3.34) as well as the fact that \(b, a+c\) are both even. For those on the right hand side of (3.35), we apply (3.16) with
\[
X = M_1^1 \quad Y = M_2^{\Sigma+1-N}\]
\[
f = a_{a+1+c} (\bar{n}, \Sigma \bar{n}'+ (2-b), \bar{n}'', l) : U_{2} = \bigcup_{\bar{m}} U_{\Sigma} \bigcup_{\Sigma} U_{\Sigma} = U_{2}
\]
It follows from (3.16) that the the right hand side of (3.35) gives
\[
\sum_{\bar{z}=(\bar{p},\bar{p}')} (\pi_{a+1}^2 (\bar{p}, \Sigma \bar{p}'+l+(1-b)) \circ \bigcup_{\bar{p}} (\sigma_{b}^R (\bar{p}', l)))^R
\]
\[
= \sum_{\bar{z}=(\bar{p},\bar{p}')} \bigvee_{\bar{p}''} (\pi_{a+1}^2 (\bar{p}, \Sigma \bar{p}'+l+(1-b)) \circ \beta_{b}^R (\bar{p}', l)) (3.41)
\]
where we have set \(l' = -\Sigma (l-(1-N))\) and used the replacements in (3.33), (3.34). By inspecting (3.37), (3.39) and (3.41) and comparing with (3.31), we conclude that \(\varphi_{b} (\bar{n}, l)\) together determine a morphism \((P_1, \rho_1) \longrightarrow (P_2, \rho_2)\) in \(EM^{a}(\nabla, \Delta)\). Finally, these arguments can be reversed, which shows that there is an isomorphism between categories \(EM_{eo}^{a}(\nabla, \Delta)\) and \(EM_{eo}^{a}(\nabla, \Delta)\).

## 4 Locally finite comodules over \(A_{\infty}\)-comonads

We suppose throughout that \((\nabla, \Delta)\) is an \(A_{\infty}\)-comonad. We now define what it means for a \((\nabla, \Delta)\)-comodule to be locally finite.

**Definition 4.1.** Let \((P, \rho)\) be a \((\nabla, \Delta)\)-comodule. In the notation of Definition 3.2 we will say that \((P, \rho)\) is locally finite if for any \(T \in \mathbb{Z}\) and \(k \geq 1\), the family of morphisms
\[
\rho_k (\bar{n}, l) : P_{T+(2-k)} \longrightarrow \bigvee_{\bar{n}} P_{T+(2-k)}_{\bar{n}, l} \text{ in } \mathcal{C} \text{ is locally finite.}
\]
In the rest of this section, we suppose that each \( V_{\alpha} \) preserves monomorphisms. Given a \((\mathcal{V}, \Delta)\)-comodule \((P, \rho)\), we now consider, in the notation of Definition 3.2, a family of subobjects \( P^p_n \to P^p_n \) satisfying the following conditions:

(a) For any \( \bar{n} \in \mathbb{Z}^{k-1} \), \( l \in \mathbb{Z} \), the morphisms in (4.4) restrict to the corresponding subobjects:

\[
\rho^p = (\rho^p_{k}(\bar{n}, l) : P^p_{\Sigma_{k+1}(2-k)} \to \mathbb{V}_n P^p_l) \quad | \quad k \geq 1, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z} \tag{4.2}
\]

(b) For any \( T \in \mathbb{Z} \) and \( k \geq 1 \), the family of morphisms:

\[
\rho^p = (\rho^p_{k}(\bar{n}, l) : P^p_{T+(2-k)} \to \mathbb{V}_n P^p_l)_{(\bar{n}, l) \in \mathbb{Z}(T,k)} \tag{4.3}
\]

is locally finite in \( C \).

**Lemma 4.2.** \((P^p, \rho^p)\) is a locally finite \((\mathcal{V}, \Delta)\)-comodule.

**Proof.** Because the morphisms \( \rho = (\rho_k(\bar{n}, l) | k \geq 1, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z}) \) in (3.4) restrict to

\[
\rho^p = (\rho^p_{k}(\bar{n}, l) | k \geq 1, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1}, l \in \mathbb{Z}) \tag{4.4}
\]

the morphisms \( \rho^p(\bar{n}, l) \) also satisfy the relations in (3.5). The local finiteness condition is clear from the assumption in (4.3). \( \square \)

We now let \( \{(P^p, \rho^p)\}_{\alpha \in A} \) denote the collection of such families of subobjects. We note that this collection is non-empty since it contains the zero family. We now define

\[
P^p = \left\{ P^p_n := \sum_{\alpha \in A} P^p_{\alpha} \right\}_{n \in \mathbb{Z}} \tag{4.5}
\]

**Proposition 4.3.** The collection \( P^p = \{P^p_n\}_{n \in \mathbb{Z}} \) is a locally finite \( A_{\alpha} \)-comodule over \((\mathcal{V}, \Delta)\).

**Proof.** Since each \( \mathbb{V}_n \) preserves monomorphisms, for \( \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1} \) and \( l \in \mathbb{Z} \), we have induced maps

\[
\rho^p_{k}(\bar{n}, l) : P^p_{\Sigma_{k+1}(2-k)} = \sum_{\alpha \in A} P^p_{\Sigma_{k+1}(2-k)} \to \mathbb{V}_n P^p_l \to \mathbb{V}_n \left( \sum_{\alpha \in A} P^p_{\alpha} \right) = \mathbb{V}_n(P^p) \tag{4.6}
\]

Again since the morphisms in (4.6) are restrictions of \( \rho_k(\bar{n}, l) \) to subobjects, it follows that they also satisfy the relations in (3.5). This makes \((P^p, \rho^p)\) a \((\mathcal{V}, \Delta)\)-comodule. Since each \((P^p, \rho^p)\) is locally finite, we have for each \( \alpha \in A, T \in \mathbb{Z} \) and \( k \geq 1 \), the following factorisation

\[
P^p_{T+(2-k)} \to \left( \prod_{(\bar{n}, l) \in \mathbb{Z}(T,k)} \mathbb{V}_n P^p_l \right) \to \left( \prod_{(\bar{n}, l) \in \mathbb{Z}(T,k)} \mathbb{V}_n P^p_l \right) \tag{4.7}
\]

From (4.7), it follows that each of the compositions \( P^p_{T+(2-k)} \to P^p_{T+(2-k)} \prod_{(\bar{n}, l) \in \mathbb{Z}(T,k)} \mathbb{V}_n P^p_l \) factors through the subobject \( \bigoplus_{(\bar{n}, l) \in \mathbb{Z}(T,k)} \mathbb{V}_n P^p_l \). Since \( C \) is abelian and \( P^p_{T+(2-k)} = \sum_{\alpha \in A} P^p_{\alpha} \), the result follows. \( \square \)
Lemma 4.4. Let $(\mathcal{V}, \Delta)$ be an $A_\omega$-comonad such that each $(\mathcal{V}_n)_{n \in \mathbb{Z}}$ is exact. Suppose that $g : (P, \rho^P) \to (Q, \rho^Q)$ is a morphism in EM$^{(\mathcal{V}, \Delta)}$. Then,

$$R = \{ R_l := \text{Ker}(g_l : P_l \to Q_l) \}_{l \in \mathbb{Z}} 
S = \{ S_l := \text{Cok}(g_l : P_l \to Q_l) \}_{l \in \mathbb{Z}}$$

are canonically equipped with the structure of $(\mathcal{V}, \Delta)$-comodules.

**Proof.** Since the functors $\mathcal{V}_n$ are all exact, it is clear from (4.8) that we have induced morphisms

$$\rho^P_k(\bar{n}, l) : R_{\mathcal{V}_1 + (2-k)} \to \mathcal{V}_n P_l$$

(4.10)

Again since the functors $\mathcal{V}_n$ are all exact, it follows that the morphisms in (4.10) satisfy the conditions in (3.5) for $R$ and $S$ to be $(\mathcal{V}, \Delta)$-comodules.

□

Lemma 4.5. Let $(\mathcal{V}, \Delta)$ be an $A_\omega$-comonad such that each $(\mathcal{V}_n)_{n \in \mathbb{Z}}$ is exact. Suppose that $g : (P, \rho^P) \to (Q, \rho^Q)$ is a morphism in EM$^{(\mathcal{V}, \Delta)}$ such that each $g_l : P_l \to Q_l$ is an epimorphism. Then if $P$ is locally finite, so is $Q$.

**Proof.** For any $T \in \mathbb{Z}$ and $k \geq 1$, we consider the families

$$\rho^P_k(\bar{n}, l) : P_{\mathcal{V}_1 + (2-k)} \to \mathcal{V}_n P_l$$

(4.11)

It is immediate that

$$\left( \prod_{(n, l) \in \mathcal{Z}(T, k)} \rho^Q_k(\bar{n}, l) \right) \circ g_{\mathcal{V}_1 + (2-k)} = \left( \prod_{(n, l) \in \mathcal{Z}(T, k)} \mathcal{V}_n(g_l) \right) \circ \left( \prod_{(n, l) \in \mathcal{Z}(T, k)} \rho^P_k(\bar{n}, l) \right).$$

We now consider the commutative diagram

$$\begin{array}{rcl}
R_{\mathcal{V}_1 + (2-k)} & \xrightarrow{\rho^P_k(\bar{n}, l)} & P_{\mathcal{V}_1 + (2-k)} \\
\Bigoplus_{(n, l) \in \mathcal{Z}(T, k)} \mathcal{V}_n P_l & \xrightarrow{\bigoplus \mathcal{V}_n(g_l)} & \Bigoplus_{(n, l) \in \mathcal{Z}(T, k)} \mathcal{V}_n P_l \\
\Bigoplus_{(n, l) \in \mathcal{Z}(T, k)} \mathcal{V}_n Q_l & \xrightarrow{g_{\mathcal{V}_1 + (2-k)}} & \Bigoplus_{(n, l) \in \mathcal{Z}(T, k)} \mathcal{V}_n Q_l \\
\end{array}$$

(4.12)

where $R_{\mathcal{V}_1 + (2-k)}$ is the kernel of $g_{\mathcal{V}_1 + (2-k)} : P_{\mathcal{V}_1 + (2-k)} \to Q_{\mathcal{V}_1 + (2-k)}$ and the factorization $\iota^P_{T,k} \circ \left( \bigoplus \rho^P_k(\bar{n}, l) \right)$ is due to the fact that $(P, \rho^P)$ is locally finite. From (4.12) we see that

$$\iota^P_{T,k} \circ \left( \bigoplus \rho^P_k(\bar{n}, l) \right) \circ u_{\mathcal{V}_1 + (2-k)} = \left( \prod \mathcal{V}_n(g_l) \right) \circ \iota^P_{T,k} \circ \left( \bigoplus \rho^P_k(\bar{n}, l) \right) \circ u_{\mathcal{V}_1 + (2-k)} = \left( \prod \mathcal{V}_n(g_l) \right) \circ \left( \prod \rho^P_k(\bar{n}, l) \right) \circ u_{\mathcal{V}_1 + (2-k)} = \left( \prod \rho^Q_k(\bar{n}, l) \right) \circ g_{\mathcal{V}_1 + (2-k)} \circ u_{\mathcal{V}_1 + (2-k)} = 0$$

(4.13)

Since $\iota^P_{T,k}$ is a monomorphism, it follows from (4.13) that $\left( \bigoplus \mathcal{V}_n(g_l) \right) \circ \left( \bigoplus \rho^P_k(\bar{n}, l) \right) \circ u_{\mathcal{V}_1 + (2-k)} = 0$. Since $R_{\mathcal{V}_1 + (2-k)}$ is the kernel of the epimorphism $g_{\mathcal{V}_1 + (2-k)} : P_{\mathcal{V}_1 + (2-k)} \to Q_{\mathcal{V}_1 + (2-k)}$, we have an induced map $Q_{\mathcal{V}_1 + (2-k)} \to \bigoplus_{(n, l) \in \mathcal{Z}(T, k)} \mathcal{V}_n Q_l$ which fits into the commutative diagram (4.12). Again since $g_{\mathcal{V}_1 + (2-k)}$ is an epimorphism, it is easily seen that this gives a factorization of $\prod \rho^Q_k(\bar{n}, l)$. 

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Theorem 4.6. Let $(\mathcal{V}, \Delta)$ be an $A_\infty$-comonad such that each $(\mathcal{V}_n)_{n \in \mathbb{Z}}$ is exact. Then, the association $Q \mapsto Q^\oplus$ determines a functor from $EM^{(\mathcal{V}, \Delta)}$ to the full subcategory $EM^{(\mathcal{V}, \Delta)}_{\text{loc}}$ of comodules in $EM^{(\mathcal{V}, \Delta)}$ that are locally finite. Further, this functor is right adjoint to the inclusion $EM^{(\mathcal{V}, \Delta)}_{\text{loc}} \hookrightarrow EM^{(\mathcal{V}, \Delta)}$, i.e., there are natural isomorphisms

$$EM^{(\mathcal{V}, \Delta)}(P, Q) \cong EM^{(\mathcal{V}, \Delta)}_{\text{loc}}(P, Q^\oplus)$$

(4.14)

for $P \in EM^{(\mathcal{V}, \Delta)}_{\text{loc}}$ and $Q \in EM^{(\mathcal{V}, \Delta)}$.

Proof. We consider some $P \in EM^{(\mathcal{V}, \Delta)}_{\text{loc}}$, $Q \in EM^{(\mathcal{V}, \Delta)}$ and a morphism $g : P \to Q$ in $EM^{(\mathcal{V}, \Delta)}$. Applying Lemma 4.4 we can consider the $(\mathcal{V}, \Delta)$-comodule $R$ determined by setting

$$P_n \to R_n := \text{Im}(g_n : P_n \to Q_n) \hookrightarrow Q_n \quad n \in \mathbb{Z}$$

(4.15)

Since each $P_n \to R_n$ is an epimorphism and $P$ is locally finite, it follows from Lemma 4.5 that $R$ is also locally finite. As such, the family $\{R_n \to Q_n\}_{n \in \mathbb{Z}}$ of subobjects determines a locally finite $(\mathcal{V}, \Delta)$-comodule. From the definition in (4.15), it is now clear that each $R_n \subset Q_n^\oplus$.

It remains to show that the association $Q \mapsto Q^\oplus$ is a functor. For this, we consider a morphism $Q' \to Q$ in $EM^{(\mathcal{V}, \Delta)}$. Then $Q'^\oplus \in EM^{(\mathcal{V}, \Delta)}_{\text{loc}}$ and if we apply the above reasoning to the composition $Q'^\oplus \to Q' \to Q$ in $EM^{(\mathcal{V}, \Delta)}$, we see that there is an induced morphism $Q'^\oplus \to Q^\oplus$. This proves the result. \qed

5 Bar construction for $A_\infty$-monads

In this section, we will obtain the bar construction $\text{Bar}(U, \Theta)$ of an $A_\infty$-monad $(U, \Theta)$, which gives a differential graded comonad on $\mathcal{C}$. Further, we show that morphisms to $\text{Bar}(U, \Theta)$ from a conilpotent dg-comonad correspond to families of natural transformations that behave in a manner similar to the classical case of twisting morphisms (see, for instance, [27] § 2, [28] § 4, [32] § 2)). We mention here that the cobar construction of an $A_\infty$-comonad can be obtained in an analogous manner. We begin with the following definition.

Definition 5.1. A graded comonad $(\mathcal{W}, \delta_2)$ on $\mathcal{C}$ is given by the following

(a) A collection $(\mathcal{W}_n)_{n \in \mathbb{Z}}$ of endofunctors on $\mathcal{C}$.

(b) A collection of natural transformations

$$\delta_2 = \{\delta_2(n_1, n_2) : \mathcal{W}_{n_1+n_2} \to \mathcal{W}_{n_1} \circ \mathcal{W}_{n_2} | (n_1, n_2) \in \mathbb{Z}^2\}$$

(5.1)

such that for each $T \in \mathbb{Z}$, the induced morphism

$$\prod_{n_1+n_2=T} \delta_2(n_1, n_2) : \mathcal{W}_T \to \prod_{n_1+n_2=T} \mathcal{W}_{n_1} \circ \mathcal{W}_{n_2}$$

(5.2)

factors through the direct sum \(\bigoplus_{n_1+n_2=T} \mathcal{W}_{n_1} \circ \mathcal{W}_{n_2}\) and for every $n_1, n_2, n_3 \in \mathbb{Z}$, we have

$$\bigl(\mathcal{W}_{n_1} \circ \delta_2(n_2, n_3)\bigr) \delta_2(n_1, n_2+n_3) = \bigl(\delta_2(n_1, n_2) \circ \mathcal{W}_{n_3}\bigr) \delta_2(n_1+n_2, n_3) : \mathcal{W}_{n_1+n_2+n_3} \to \mathcal{W}_{n_1} \circ \mathcal{W}_{n_2} \circ \mathcal{W}_{n_3}$$

(5.3)

A morphism $\beta : (\mathcal{W}, \delta_2) \to (\mathcal{W}', \delta'_2)$ of graded comonads on $\mathcal{C}$ is a collection of natural transformations $\beta = (\beta_n : \mathcal{W}_n \to \mathcal{W}'_n)_{n \in \mathbb{Z}}$ such that $(\beta_n \circ \delta_2(n_2, n_3)) \delta_2(n_1, n_2+n_3) = (\delta_2(n_1, n_2) \circ \mathcal{W}_n) \delta'_2(n_1+n_2, n_3) : \mathcal{W}_{n_1+n_2+n_3} \to \mathcal{W}_{n_1} \circ \mathcal{W}_{n_2} \circ \mathcal{W}_{n_3}$.

A differential graded comonad (or dg-comonad) $(\mathcal{W}, \delta_1, \delta_2)$ on $\mathcal{C}$ consists of a graded comonad $(\mathcal{W}, \delta_2)$ as well as a collection of natural transformations $\delta_1 = \{\delta_1(n) : \mathcal{W}_n \to \mathcal{W}_{n+1} | n \in \mathbb{Z}\}$ satisfying the following conditions (for $n, n_1, n_2 \in \mathbb{Z}$)

$$\delta_1(n+1) \delta_1(n) = 0$$

$$\delta_2(n_1, n_2) \delta_1(n_1+n_2-1) = \bigl(\delta_1(n_1-1) \circ \mathcal{W}_{n_2}\bigr) \delta_2(n_1-1, n_2) + (-1)^n \bigl(\mathcal{W}_n \circ \delta_1(n_2-1)\bigr) \delta_2(n_1, n_2-1) : \mathcal{W}_{n_1+n_2-1} \to \mathcal{W}_{n_1} \circ \mathcal{W}_{n_2}$$

(5.4)
A morphism $\beta : (\mathcal{W}, \delta_1, \delta_2) \to (\mathcal{W}', \delta_1', \delta_2')$ of dg-comonads on $\mathcal{C}$ is a collection of natural transformations $\beta = \{ \beta_n : \mathcal{W}_n \to \mathcal{W}'_n \}_{n \in \mathbb{Z}}$ such that

$$\delta_1'(n) \beta_n = \beta_{n+1} \delta_1(n) \quad (\beta_{n_1} \ast \beta_{n_2}) \delta_2(n_1, n_2) = \delta_2'(n_1, n_2) \beta_{n_1+n_2}$$ \hspace{1cm} (5.5)

for $n, n_1, n_2 \in \mathbb{Z}$. We will denote by $dg\text{-}Comont(\mathcal{C})$ the category of dg-comonads over $\mathcal{C}$.

If $(\mathcal{W}, \delta_2)$ is a graded comonad over $\mathcal{C}$, we set for any $\vec{n} = (n_1, ..., n_k) \in \mathbb{Z}^k, k \geq 1$

$$\delta_2(\vec{n}) := (\mathcal{W}_{n_1} \ast \cdots \ast \mathcal{W}_{n_k}) \delta_2(n_1, ..., n_k) : \mathcal{W}_{\Sigma \vec{n}} \to \mathcal{W}_{n_1} \circ \cdots \circ \mathcal{W}_{n_k} = \mathcal{W}_{\vec{n}}$$ \hspace{1cm} (5.6)

We note that due to the coassociativity condition in (5.3), the natural transformation $\delta_2(\vec{n})$ appearing in (5.6) can be expressed in a number of equivalent ways, one for each way the sum $(n_1 + \cdots + n_k)$ can be “broken up” into the partition $(n_1, ..., n_k)$.

From the conditions in Definition 5.1 we know that for each $T \in \mathbb{Z}$ and $k \geq 1$, the morphism $\prod_{\vec{n} \in \mathbb{Z}^k \setminus \{0\}} \delta_2(\vec{n}) : \mathcal{W}_T \to \prod_{\vec{n} \in \mathbb{Z}^k \setminus \{0\}} \mathcal{W}_{\vec{n}}$ factors through the direct sum $\bigoplus_{\vec{n} \in \mathbb{Z}^k} \mathcal{W}_{\vec{n}}$. We now have the following definition.

**Definition 5.2.** Let $(\mathcal{W}, \delta_2)$ be a graded comonad over $\mathcal{C}$. We will say that $(\mathcal{W}, \delta_2)$ is conilpotent if for each object $M \in \mathcal{C}$ and $L \in \mathbb{Z}$, we have

$$\mathcal{W}_L(M) = \bigoplus_{k \geq 1} \text{Ker} \left( \mathcal{W}_L(M) \to \bigoplus_{\vec{n} \in \mathbb{Z}^k} \mathcal{W}_{\vec{n}}(M) \right)$$ \hspace{1cm} (5.7)

We now let $(\mathcal{U}, \Theta)$ be an $A_{\infty}$-monad, and let $(\mathcal{W}, \delta_1, \delta_2)$ be a conilpotent dg-comonad over $\mathcal{C}$. For each $n \in \mathbb{Z}$, we set

$$A_n := \prod_{k \in \mathbb{Z}} \text{Nat}(\mathcal{W}_k, \mathcal{U}_{k+n}) = (\zeta = (\xi^{(k)})_{k \in \mathbb{Z}} \mid \xi^{(k)} \in \text{Nat}(\mathcal{W}_k, \mathcal{U}_{k+n}))$$ \hspace{1cm} (5.8)

where $\text{Nat}(\mathcal{W}_k, \mathcal{U}_{k+n})$ is the collection of natural transformations from $\mathcal{W}_k$ to $\mathcal{U}_{k+n}$. We set $A := \bigoplus_{n \in \mathbb{Z}} A_n$. We now have maps $\{ m_k : A^{\otimes k} \to A \}_{k \geq 1}$ with $m_k$ of degree $(2 - k)$, given by components

$$m_k(\vec{n}) : A_{n_1} \otimes A_{n_2} \otimes \cdots \otimes A_{n_k} \to A_{\Sigma \vec{n} + (2 - k)} \quad (\zeta_{n_1} \otimes \cdots \otimes \zeta_{n_k}) \mapsto \left( \sum_{\vec{t} \in \mathbb{Z}^k} \theta_k(\vec{t} + \vec{n})(\xi^{(n_1)}_{\Sigma \vec{t}} * \cdots * \xi^{(n_k)}_{\Sigma \vec{t}}) \delta_2(\vec{t}) \right)$$ \hspace{1cm} (5.9)

for $\vec{n} = (n_1, n_2, ..., n_k) \in \mathbb{Z}^k$. In (5.9), we have suppressed the sign of the summand $\theta_k(\vec{t} + \vec{n})(\xi^{(n_1)}_{\Sigma \vec{t}} * \cdots * \xi^{(n_k)}_{\Sigma \vec{t}}) \delta_2(\vec{t})$, which is $(-1)^{(n_1 + \cdots + n_k) + \Sigma(n_1 + \cdots + n_k) + \Sigma(n_1, ..., n_k)}$ as given by Koszul sign rule. For ease of notation, we set $A_{\vec{n}} := A_{n_1} \otimes A_{n_2} \otimes \cdots \otimes A_{n_k}$. Accordingly, an element $(\zeta_{n_1} \otimes \cdots \otimes \zeta_{n_k}) \in A_{n_1} \otimes A_{n_2} \otimes \cdots \otimes A_{n_k} = A_{\vec{n}}$ will often be denoted by $\zeta_{\vec{n}}$. Similarly, we write $\xi^{(k)}_{\vec{n}} = \xi^{(n_1)}_{\Sigma \vec{t}} * \cdots * \xi^{(n_k)}_{\Sigma \vec{t}}$ for $\vec{n}, \vec{t} \in \mathbb{Z}^k$. We now have the following result.

**Proposition 5.3.** The collection $(A, \{ m_k : A^{\otimes k} \to A \}_{k \geq 1})$ is an $A_{\infty}$-algebra.

**Proof.** We choose $\vec{z} \in \mathbb{Z}^N$ for some $N \geq 1$ and consider $\zeta_{\vec{z}} \in A_{\vec{z}}$. In the following, we have suppressed the signs for sake of convenience. With the sum running over partitions $\vec{z} = (n, \vec{n}', \vec{n}'')$ with $|n| = p, |\vec{n}'| = q, |\vec{n}''| = r$, we see that we have for any $l \in \mathbb{Z}$

$$\left( \sum_{\zeta \in \mathcal{U}_{\vec{z}}} m_{p+1+l}(\vec{n}, \vec{n}', \vec{n}'')(n \otimes m_l(\vec{n}')(\Sigma \xi)) \right)^{(l)}$$

$$= \left( \sum_{\zeta \in \mathcal{U}_{\vec{z}}} m_{p+1+l}(\vec{n}, \vec{n}', \vec{n}'')(\zeta \otimes m_l(\vec{n}'')(\Sigma \xi)) \right)^{(l)}$$

$$= \sum_{l \in \mathbb{Z}^k \setminus \{0\}} \sum_{\zeta \in \mathcal{U}_{\vec{z}}} \theta_{p+1+l}(\vec{t} + \vec{n}, \vec{n}', \vec{n}'') \zeta^{(n_1)}_{\Sigma \vec{t}} * \cdots * \zeta^{(n_k)}_{\Sigma \vec{t}} * m_l(\vec{n}'')(\Sigma \xi^{(l_{p+1+l})}) \delta_2(\vec{t})$$

$$= \sum_{l \in \mathbb{Z}^k \setminus \{0\}} \sum_{\zeta \in \mathcal{U}_{\vec{z}}} \theta_{p+1+l}(\vec{t}_1, ..., \vec{t}_p, \vec{t}_{p+q+1}, ..., \vec{t}_{p+q+r}) \delta_2(\vec{t}) \left( \sum_{\zeta \in \mathcal{U}_{\vec{z}}} \theta_{p+1+l}(\vec{t}_1, ..., \vec{t}_p, \vec{t}_{p+q+1}, ..., \vec{t}_{p+q+r}) \delta_2(\vec{t}) = 0 \right)$$

$\square$
As in [27] § 3.6, we recall here that any \( A\)-algebra \( (\{m_k : A^\otimes k \to A\}_{k \geq 1}) \) can be equivalently described using structure maps \( (sA, \{b_j : (sA)^\otimes j \to (sA)^\otimes j\}_{j \geq 1}) \) given by
\[
 b_j(n) := (-1)^{n(j-1)+...+n_1}m_k(n_1 + 1, ..., n_k + 1): (sA)_{n_1} \otimes ... \otimes (sA)_{n_k} \to (sA)^{\otimes_{j+1}}
 (5.10)
\]
for \( n = (n_1, n_2, ..., n_k) \in \mathbb{Z}^k \), where \( sA = \{(sA)_j := A_{j+1}\}_{j \in \mathbb{Z}} \) is the suspension of \( A \).

We will need the following basic example of a conilpotent graded comonad for describing the bar construction. Let \( F = \{F_n : \mathcal{C} \to \mathcal{C}\}_{n \in \mathbb{Z}} \) be a family of endofunctors. As always, for any \( k \geq 1 \) and \( \tilde{n} = (n_1, ..., n_k) \in \mathbb{Z}^k \), we set \( F_{\tilde{n}} := F_{n_1} \circ ... \circ F_{n_k} \). We now consider the graded comonad \( T^c(F) \) given by setting
\[
 T^c(F)_L := \bigoplus_{\tilde{n} \in \mathbb{Z}^k, k \geq 1} F_{\tilde{n}} \quad L \in \mathbb{Z}
 (5.11)
\]
The “comultiplication” \( \delta^c_2 : T^c(F) \to T^c(F) \circ T^c(F) \) is given on components by means of identity maps \( F_{\tilde{n}} \to F_{n_1} \circ F_{n_2} \) for each partition \( \tilde{n} = (n_1, n_2) \) of \( \tilde{n} \). If we consider some \( m \in \mathcal{C} \), we see that \( \delta^c_2(\tilde{n})(M)F_{\tilde{n}}(M) = 0 \) for any \( i \) such that \( |\tilde{n}| > |\tilde{n}| \). From the condition in \( 5.7 \), it is now clear that \( (T^c(F), \delta^c_2) \) is a conilpotent graded comonad.

**Proposition 5.4.** Let \( (\mathcal{W}, \delta_2) \) be a conilpotent graded comonad over \( \mathcal{C} \) and \( F = \{F_n : \mathcal{C} \to \mathcal{C}\}_{n \in \mathbb{Z}} \) be a family of endofunctors. Then, there is a one-one correspondence
\[
 \prod_{L \in \mathbb{Z}} \text{Nat}(\mathcal{W}_L, F_L) \xrightarrow{g} \text{gr-Comon}(\mathcal{C})(\mathcal{W}, \delta_2), (T^c(F), \delta^c_2))
 (5.12)
\]

**Proof.** For \( \zeta = \{\zeta(n)\}_{n \in \mathbb{Z}} \in \prod_{L \in \mathbb{Z}} \text{Nat}(\mathcal{W}_L, F_L) \), we get \( \chi := h(\zeta) \) by setting for each \( M \in \mathcal{C} \)
\[
 \chi_L(M) = \bigoplus_{\tilde{n} \in \mathbb{Z}^k, k \geq 1} \chi_{\tilde{n}}(M) = \bigoplus_{\tilde{n} \in \mathbb{Z}^k, k \geq 1} \zeta(\tilde{n})\delta^c_2(\tilde{n})(M) : \mathcal{W}_L(M) \to T^c(F)_L(M) = \bigoplus_{\tilde{n} \in \mathbb{Z}^k, k \geq 1} F_{\tilde{n}}(M)
 (5.13)
\]
where \( \zeta(\tilde{n}) = \zeta(n_1) \ast ... \ast \zeta(n_k) \) in \( 5.13 \) for \( \tilde{n} = (n_1, ..., n_k) \in \mathbb{Z}^k \). It is clear from the conilpotence condition in \( 5.7 \) that the morphism in \( 5.13 \) is well defined. On the other hand, for \( \chi \in \text{gr-Comon}(\mathcal{C})(\mathcal{W}, \delta_2), (T^c(F), \delta^c_2)) \), we get \( \zeta := g(\chi) \) by setting \( \zeta(n) \) to be the composition of \( \chi_L(M) : \mathcal{W}_L(M) \to T^c(F)_L(M) \) with the projection to \( F_L(M) \) for each \( M \in \mathcal{C} \). It may be verified that these two associations are inverse to each other. \( \square \)

**Lemma 5.5.** Let \( F = \{F_n : \mathcal{C} \to \mathcal{C}\}_{n \in \mathbb{Z}} \) be a family of endofunctors. Then, there is a one-one correspondence between the following
(a) Families of natural transformations
\[
 \Gamma = \{g_{\tilde{n}}(\bar{\tilde{n}}) : F_{\tilde{n}} = F_{n_1} \circ ... \circ F_{n_k} \to F_{\Sigma_{n+1}} \mid k \geq 1 \ \tilde{n} = (n_1, ..., n_k) \in \mathbb{Z}^k \}
 (5.14)
\]
(b) Families of natural transformations \( \delta^c_1 = (\delta^c_1(L) : T^c(F)_L \to T^c(F)_{L+1} \mid L \in \mathbb{Z}) \) which satisfy
\[
 \delta^c_2(L_1, L_2)\delta^c_1(L_1 + L_2 - 1) = (\delta^c_1(L_1 - 1) \ast T^c(F)_{L_2})\delta^c_2(L_1 - 1, L_2) + (-1)^{L_1}(T^c(F)_{L_1} \ast \delta^c_1(L_2 - 1))\delta^c_2(L_1, L_2 - 1)
 (5.15)
\]
for \( L_1, L_2 \in \mathbb{Z} \).

**Proof.** Given a family \( \Gamma \) as in (a), we set for each \( \bar{\tilde{n}} \in \mathbb{Z}^N, N \geq 1 \):
\[
 \delta^c_1(\Sigma\bar{\tilde{n}}) : \bigoplus_{\tilde{z} \in \tilde{n}} (-1)^{\tilde{z}}(F_{\tilde{n}} \ast g_{\tilde{z}}(\bar{\tilde{n}})) \to \bigoplus_{\tilde{z} \in \tilde{n}} F_{\tilde{n}} \circ F_{\Sigma_{n+1}} \circ F_{\bar{\tilde{n}}} \leftarrow T^c(F)_{\Sigma_{n+1}}
 (5.16)
\]
where the sum in \( 5.16 \) is taken over all partitions \( \bar{\tilde{n}} = (\tilde{n}, \tilde{n'}, \tilde{n''}) \). It may be verified that the family \( \delta^c_1 \) satisfies the condition in \( 5.15 \). Conversely, given a family \( \delta^c_1 \) as in (b), we set for each \( \bar{\tilde{n}} \in \mathbb{Z}^N, N \geq 1 \)
\[
 g_{\tilde{n}}(\bar{\tilde{n}}) : F_{\tilde{n}} \to T^c(F)_{\Sigma_{n+1}} \leftarrow F_{\Sigma_{n+1}}
 (5.17)
\]
where the second arrow in \( 5.17 \) is the canonical projection. It may be checked that these two associations are inverse to each other. \( \square \)
Given a collection \( F = \{ F_n : \mathcal{C} \to \mathcal{C} \}_{n \in \mathbb{Z}} \) of endofunctors, we now set \( sF = \{(sF)_n := F_{n+1} \}_{n \in \mathbb{Z}} \). If \((\mathcal{U}, \Theta)\) is an \( A_\infty \)-monad, we can now consider the family \( \Gamma_1^{\mathcal{U}} \) of natural transformations

\[
\gamma_1^{\mathcal{U}}(\bar{n}) : (s\mathcal{U})_{\bar{n}} \circ \cdots \circ (s\mathcal{U})_1 \to (s\mathcal{U})_{\bar{n}+1} = U_{\bar{n}+2} \\
\gamma_1^{\mathcal{U}}(\bar{n}) := (-1)^{n_1(k-1)+\cdots+n_{k-1}d_k}(n_1 + 1, \ldots, n_k + 1) \tag{5.18}
\]

for \( \bar{n} = (n_1, \ldots, n_k) \), \( k \geq 1 \). Applying Lemma 5.5, the family of natural transformations in (5.18) corresponds to \( \delta_1^{\mathcal{U}} = \{ \delta_1^{\mathcal{U}}(L) : T^c(s\mathcal{U})_L \to T^c(s\mathcal{U})_{L+1} \mid L \in \mathbb{Z} \} \). Finally, using the fact that \((\mathcal{U}, \Theta)\) is an \( A_\infty \)-monad, it can be seen from the relations in (2.2) that each \( \delta_1^{\mathcal{U}}(L + 1) \delta_1^{\mathcal{U}}(L) = 0 \). We will now say that the dg-cononad given by

\[
Bar(\mathcal{U}, \Theta) := (T^c(s\mathcal{U}), \delta_1^{\mathcal{U}}, \delta_2^{\mathcal{U}}) \tag{5.19}
\]

is the bar construction on the \( A_\infty \)-monad \((\mathcal{U}, \Theta)\).

**Theorem 5.6.** Let \((\mathcal{W}, \delta_1, \delta_2)\) be a dg-cononad over \( \mathcal{C} \) that is conilpotent. Let \((\mathcal{U}, \Theta)\) be an \( A_\infty \)-monad over \( \mathcal{C} \) and let \((A_1, \{ m_l : A_0^\mathcal{U} \to A_1 \}_{l \in \mathbb{Z}} \) be the \( A_\infty \)-algebra given by setting \( A_1 := \prod_{l \in \mathbb{Z}} \text{Nat}(\mathcal{W}_l, \mathcal{U}_{l+1}) \). Then, there is a one-one correspondence between the following:

(a) Morphisms \( \chi : (\mathcal{W}, \delta_1, \delta_2) \to Bar(\mathcal{U}, \Theta) = (T^c(s\mathcal{U}), \delta_1^{\mathcal{U}}, \delta_2^{\mathcal{U}}) \) of dg-cononads over \( \mathcal{C} \)

(b) Elements \( \zeta = (\xi(l))_{l \in \mathbb{Z}} \in \prod_{l \in \mathbb{Z}} \text{Nat}(\mathcal{W}_l, (s\mathcal{U})_l) = A_1 \) satisfying

\[
\sum_{i=1}^{k} \sum_{j=1}^{n_i} (-1)^{i_1+\cdots+i_{k-1}d_k}(n_1 - 1, \ldots, n_i - 1, 1) \delta_1(M) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (-1)^{i_1+\cdots+i_{k-1}d_k}(n_1 - 1, \ldots, n_i - 1, \bar{n}, n_{i+1} - 1, \ldots, n_k - 1) \delta_1(M) \tag{5.20}
\]

for each fixed \( k \geq 1 \), \( (n_1, \ldots, n_k) \in \mathbb{Z}^k \) and \( M \in \mathcal{C} \).

**Proof.** We consider a morphism \( \chi : (\mathcal{W}, \delta_1, \delta_2) \to (T^c(s\mathcal{U}), \delta_1^{\mathcal{U}}, \delta_2^{\mathcal{U}}) \) of dg-cononads over \( \mathcal{C} \). In particular, we see that \( \chi \in gr - \text{Comon}(\mathcal{C})(\mathcal{W}, \delta_2), (T^c(s\mathcal{U}), \delta_2^{\mathcal{U}}) \). Since \((\mathcal{W}, \delta_2)\) is conilpotent, it follows from Proposition 5.4 that the morphism \( \chi \) corresponds to an element \( \zeta = (\xi(l))_{l \in \mathbb{Z}} \in \prod_{l \in \mathbb{Z}} \text{Nat}(\mathcal{W}_l, (s\mathcal{U})_l) = A_1 \). Additionally, the morphism \( \chi \) of graded cononads is well behaved with respect to the coderivations \( \delta_1 \) and \( \delta_1^{\mathcal{U}} \). We now fix \( k \geq 1 \), \( \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k \) and \( M \in \mathcal{C} \). Composing \( \delta_1^{\mathcal{U}}(M) \) with the morphisms induced by \( \chi \), we have the sum of the following compositions over \( 0 \leq i \leq k - 1 \), \( j \geq 1 \)

\[
\sum_{i=1}^{k-1} \sum_{j=1}^{m} (-1)^{i_1+\cdots+i_{k-1}d_k}(n_1 - 1, \ldots, n_i - 1, 1) \delta_1(M) \tag{5.22}
\]
On the other hand, composing $\delta_1$ with the morphisms induced by $\chi$, we have the composition
\[(\xi^{(m-1)} \ast \ldots \ast \xi^{(n-1)})\delta_2(n_1 - 1, \ldots, n_k - 1)\delta_1(M) \tag{5.23}\]
The result is now clear from (5.22) and (5.23).

6 Distributive laws and lifting of $A_\infty$-monads to Eilenberg-Moore categories

Let $(\mathcal{S}, \theta_S, \epsilon_S)$ be a monad on $\mathcal{C}$, with “multiplication” $\theta_S : \mathcal{S} \circ \mathcal{S} \to \mathcal{S}$ and “unit” $\epsilon_S : id \to \mathcal{S}$. The standard Eilenberg-Moore category $EM_S$ for the monad $\mathcal{S}$ consists of pairs $(M, \pi_M : \mathcal{S}(M) \to M)$ satisfying $\pi_M \circ \mathcal{S}(\pi_M) = \pi_M \circ \theta_S(M)$ and $id_M = \pi_M \circ \epsilon_S(M)$ (see, for instance, [10]). The category $EM_S$ is equipped with a pair $(L_S, R_S)$ of adjoint functors, where $R_S : EM_S \to \mathcal{C}$ is the forgetful functor and its left adjoint $L_S$ is given by
\[ L_S : \mathcal{C} \to EM_S \quad M \mapsto (\mathcal{S}(M), \theta_S(M)) \tag{6.1} \]

If $T$ is another monad on $\mathcal{C}$, it is a classical fact that liftings of $T$ to a monad $\overline{T}$ on the Eilenberg-Moore category of $\mathcal{S}$ over $\mathcal{T}$ (see Beck [8]). In this section, we will prove similar results, showing how an $A_\infty$-monad on $\mathcal{C}$ can be lifted to an $A_\infty$-monad on $EM_S$. We note that dual results may be proved for lifting of $A_\infty$-comonads to Eilenberg-Moore categories of comonads on $\mathcal{C}$.

More explicitly, let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. A lifting of $F$ to the Eilenberg-Moore category $EM_S$ is given by a functor $\overline{F} : EM_S \to EM_S$ that fits into the following commutative diagram
\[
\begin{array}{ccc}
EM_S & \xrightarrow{\overline{F}} & EM_S \\
R_S \downarrow & & \downarrow R_S \\
\mathcal{C} & \xrightarrow{F} & \mathcal{C}
\end{array}
\tag{6.2}
\]

We know (see [8]) that such liftings are in one-one correspondence with distributive laws of the monad $\mathcal{S}$ over the endofunctor $F$. Such a distributive law consists of a natural transformation $\lambda : SF \to FS$ such that the following two diagrams commute
\[
\begin{array}{ccc}
SF & \xrightarrow{\lambda} & FS \\
\downarrow t_SF & & \downarrow t_S \lambda \\
S^2F & \xrightarrow{\lambda \circ \lambda} & FS^2
\end{array}
\quad \begin{array}{ccc}
S^2F & \xrightarrow{\lambda \circ \lambda} & FS^2 \\
\downarrow t_S \lambda & & \downarrow t_S \lambda \\
SFS & \xrightarrow{\lambda \circ \lambda} & FS^2
\end{array}
\tag{6.3}
\]

Given a distributive law as in (6.3), the lift of $F$ to $EM_S$ is defined by setting $\overline{F}(M, \pi_M) := (F(M), \pi(M) \circ \lambda(M))$. Conversely, given a lifting $\overline{F}$ of $F$ as in (6.2), the corresponding distributive law is defined by setting
\[
\lambda : SF \xrightarrow{SF\epsilon_S} SF = SF R_S L_S = R_S L_S R_S L_S \xrightarrow{R_S L_S} \overline{F} R_S L_S = \overline{F} R_S L_S = FS \tag{6.4}
\]

where the morphism $R_S L_S R_S \overline{F} L_S \to R_S \overline{F} L_S$ in (6.4) is induced by the counit $L_S \epsilon_S \to id$ of the adjunction $(L_S, R_S)$.

Now suppose that $\tau : F \to F'$ is a natural transformation between endofunctors on $\mathcal{C}$. Suppose that $F$ (resp. $F'$) lifts to an endofunctor $\overline{F}$ (resp. $\overline{F}'$) on $EM_S$, corresponding to a distributive law $\lambda : SF \to FS$ (resp. $\lambda' : SF' \to FS$). The natural transformation $\tau$ is said to lift to $EM_S$ (see [15, Definition 3.7]) if for each $(M, \pi_M) \in EM_S$, the morphism $\tau(M) : F(M) \to F'(M)$ gives a morphism $\overline{\tau}(M, \pi_M) : \overline{F}(M, \pi_M) \to \overline{F}'(M, \pi_M)$ in $EM_S$. In other words, the following diagram must commute
\[
\begin{array}{ccc}
SF(M) & \xrightarrow{\lambda(M)} & FS(M) \xrightarrow{F(\pi_M)} F(M) \\
\downarrow S(\tau(M)) & & \downarrow \tau(M) \\
SFS(M) & \xrightarrow{\lambda'(M)} & FSF(M) \xrightarrow{F'(\pi_M)} F'(M)
\end{array}
\tag{6.5}
\]
Additionally, we know from [45, Proposition 3.13] that such a lifting of \( \tau \) exists if and only if the following diagram commutes

\[
\begin{array}{ccc}
SF & \rightarrow & FS \\
\downarrow \tau & & \downarrow \tau S \\
SF' & \rightarrow & FS'
\end{array}
\]  

(6.6)

We need one more convention. If \( F_1, \ldots, F_k \) is a family of endofunctors, along with distributive laws \( \{ \lambda_i : SF_i \rightarrow F_i S \}_{i \leq k} \), we write

\[
(\lambda_1, \ldots, \lambda_k) : S(F_1 \ldots F_k) \xrightarrow{\lambda_1 F_2 \ldots F_k} F_1 S F_2 \ldots F_k \xrightarrow{F_1 \lambda_2 F_3 \ldots F_k} \ldots \xrightarrow{F_1 \cdots F_{k+1} \lambda_k} (F_1 \ldots F_k) S
\]

(6.7)

for the composition of the distributive laws \( \lambda_1, \ldots, \lambda_k \).

**Definition 6.1.** Let \((S, \theta_S, t_S)\) be a monad and let \((U, \Theta)\) be an \(A_\infty\)-monad on \(C\). A distributive law of the monad \((S, \theta_S, t_S)\) over the \(A_\infty\)-monad \((U, \Theta)\) is given by a family \( \Lambda = \{ \lambda_n : S U_n \rightarrow U_n S \}_{n \in \mathbb{Z}} \) such that

(a) Each \( \lambda_n : S U_n \rightarrow U_n S \) is a distributive law for the monad \( S \) over the endofunctor \( U_n \).

(b) For each \( \bar{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k, k \geq 1 \), the following diagram commutes

\[
\begin{array}{ccc}
SU_{\bar{n}} = SU_{n_1} \ldots SU_{n_k} \xrightarrow{\lambda_{\bar{n}}(n_1, \ldots, n_k)} U_{n_1} \ldots U_{n_k} S = U_{\bar{n}} S \\
\downarrow \theta_{\bar{n}} & & \downarrow \theta_{\bar{n}} S \\
SU_{\Sigma \bar{n} + (2-\bar{\xi})} \xrightarrow{\lambda_{\Sigma \bar{n} + (2-\bar{\xi})}} U_{\Sigma \bar{n} + (2-\bar{\xi})} S
\end{array}
\]

(6.8)

**Lemma 6.2.** Let \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) be a distributive law of the monad \( S \) over the \( A_\infty\)-monad \((U, \Theta)\). Then, \((U, \Theta)\) lifts to an \( A_\infty\)-monad \((\bar{U}, \bar{\Theta})\) on the category \( EM_S \).

**Proof.** Since each \( \lambda_n : SU_n \rightarrow U_n S \) is a distributive law, it follows from (6.2) and (6.3) that \( U_n \) lifts to \( \bar{U}_n : EM_S \rightarrow EM_S \).

Further, since (6.8) commutes, it follows from the criterion described in (6.6) that the natural transformations \( \theta_n(\bar{n}) \) lift to \( \theta_{\bar{n}} : \bar{U}_n = \bar{U}_{n_1} \ldots \bar{U}_{n_k} \rightarrow \bar{U}_{\Sigma \bar{n} + (2-\bar{\xi})} \).

Accordingly, for any \((M, \pi_M) \in EM_S\) and \( \bar{\xi} \in \mathbb{Z}^N, N \geq 1 \), it follows from (6.2) that the morphism in \( C \) underlying the sum

\[
\sum (-1)^{p+q+r+q^2} \theta_{p+1+q}(\bar{n}, \Sigma \bar{n}', 2 - q) \cdot (\bar{\pi}_M \circ \bar{\theta}_q(\bar{n}) \circ \bar{\pi}_{\bar{\xi}'}): \bar{U}_n \circ \bar{U}_{\bar{\xi}'} \circ \bar{\pi}_{\bar{\xi}'} \rightarrow \bar{U}_{\Sigma \bar{\xi} + \Sigma \bar{\xi}' + (3-N)}(M, \pi_M)
\]

is zero, where the sum runs over partitions \( \bar{\xi} = (\bar{n}, \bar{n}', \bar{\xi}') \) with \(|p| = p, |q| = q, |\bar{\xi}'| = r \).

\[ \square \]

**Theorem 6.3.** Let \((U, \Theta)\) be an \( A_\infty\)-monad and let \((S, \theta_S, t_S)\) be a monad on \( C \). Then, there is a one one correspondence between the following

(a) Liftings of the \( A_\infty\)-monad \((U, \Theta)\) to an \( A_\infty\)-monad \((\bar{U}, \bar{\Theta})\) on the category \( EM_S \).

(b) Distributive laws of the monad \( S \) over the \( A_\infty\)-monad \((U, \Theta)\).

**Proof.** We know from Lemma [6.2] that a distributive law of the monad \( S \) over the \( A_\infty\)-monad \((U, \Theta)\) gives rise to a lifting of the \( A_\infty\)-monad \((U, \Theta)\) to \( EM_S \). On the other hand, let \((\bar{U}, \bar{\Theta})\) be a lifting of the \( A_\infty\)-monad \((U, \Theta)\) to \( EM_S \). In particular, the lifting \( \bar{U}_n \) of the endofunctor \( U_n \) leads to a distributive law \( \lambda_n : SU_n \rightarrow U_n S \). Also, for any \( k \geq 1 \) and \( \bar{n} \in \mathbb{Z}^k \), we know that the natural transformation \( \theta_k(\bar{n}) : \bar{U}_n \rightarrow \bar{U}_{\Sigma \bar{n} + (2-\bar{\xi})} \) lifts to \( EM_{\bar{S}} \).

It now follows from (6.6) that we must have \( (\theta_k(\bar{n}) S) \circ (\lambda_n, \ldots, \lambda_n) = \lambda_{\Sigma \bar{n} + (2-\bar{\xi})} \circ (\theta_k(\bar{n}) S) \).

Hence, the collection \( \Lambda = \{ \lambda_n : SU_n \rightarrow U_n S \}_{n \in \mathbb{Z}} \) gives a distributive law of the monad \((S, \theta_S, t_S)\) over the \( A_\infty\)-monad \((U, \Theta)\) in the sense of Definition 6.1. Since there is a one one correspondence between distributive laws and liftings of an endofunctor to \( EM_S \), it follows that these two associations are inverse to each other. \[ \square \]

We recall (see, for instance, [45, Proposition 3.30]) that a distributive law of \( S \) over a monad \( T \) gives rise to a new monad \( TS \). We will now study something similar for \( A_\infty\)-monads.
Proposition 6.4. Let $\Lambda = \{\lambda_n : \mathbb{S}U_n \to \mathbb{S}U_n\}_{n \in \mathbb{Z}}$ be a distributive law of the monad $\mathbb{S}$ over the $A_\infty$-monad $(U, \Theta)$. Then, the pair $(U^{(\mathbb{S})}, \Theta^{(\mathbb{S})})$ defined by setting $\mathbb{S}U_n^{(\mathbb{S})} := U_n \mathbb{S}$ and $\theta^{(\mathbb{S})}_n(\bar{\alpha}) : \mathbb{S}U_n^{(\mathbb{S})} \to \mathbb{S}U_{n+k-1}^{(\mathbb{S})}$ as follows

$$
\mathbb{S}U_n^{(\mathbb{S})} = (U_n \mathbb{S}) \ldots (U_n \mathbb{S}) \quad \text{where} \quad U_n \mathbb{S}_{n+1} \ldots U_n \mathbb{S}_{n+k-1} \\
\vdots \\
U_n (\lambda_{n+1} \ldots \lambda_{n+k-1}) \Theta^{(\mathbb{S})}_{n+k-1} \\
\mathbb{S}U_n^{(\mathbb{S})} = U_{n+k} \ldots U_n \mathbb{S} \\
\theta^{(\mathbb{S})}_n(\bar{\alpha}) = \Theta^{(\mathbb{S})}_{n+k-1} \\
\mathbb{S}U_{n+k} \mathbb{S} = \mathbb{S}U_{n+k}^{(\mathbb{S})}
$$

(6.10)

gives an $A_\infty$-monad on $\mathcal{C}$, where $\theta^{(\mathbb{S})}_n : \mathbb{S}^k \to \mathbb{S}$ is obtained by iterating the multiplication $\bar{\theta} : \mathbb{S}^2 \to \mathbb{S}$ on the monad $\mathbb{S}$.

Proof. We fix any $\bar{z} \in \mathbb{Z}^N$ and any partition $\bar{z} = (\bar{n}, \bar{n}', \bar{n}'')$ with $|\bar{n}| = p$, $|\bar{n}'| = q$, $|\bar{n}''| = r$. Since the distributive laws are well behaved both with respect to the multiplication $\bar{\theta}$ on the monad $\mathbb{S}$ as in (6.3) and the structure maps $\theta_{\bar{\theta}}(\bar{n})$ of the $A_\infty$-monad $(U, \Theta)$ as in (6.5), we note that the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{S}U_n^{(\mathbb{S})} \circ \mathbb{S}U_{n'}^{(\mathbb{S})} \circ \mathbb{S}U_{n''}^{(\mathbb{S})} & \longrightarrow & \mathbb{S}U_n \circ \mathbb{S}U_{n'} \circ \mathbb{S}U_{n''} \circ \mathbb{S}^N \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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For $k \geq 1$, $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^{k-1}$, $l \in \mathbb{Z}$, we now consider the composition

$$
\pi_k^\infty(\bar{n}, l) : U_{\bar{n}} \rightarrow U_{\bar{n} \delta^{k-1} M_l} \rightarrow U_{\bar{n} \delta^{k-1} M_l} \rightarrow U_{\bar{n} \delta^{k-1} M_l} \rightarrow U_{\bar{n} \delta^{k-1} M_l} \rightarrow M_{\Sigma^{l+1}(2-k)}
$$

where the first morphism in (6.15) is obtained by repeatedly applying the distributive law $\Lambda = \{\lambda_n : U_{\bar{n}} \rightarrow U_{\bar{n}} S \}_{n \in \mathbb{Z}}$ as in top horizontal arrow in (6.11). As in the proof of Proposition 6.2, it may now be verified that the collection $\{M_n\}_{n \in \mathbb{Z}}$ along with the morphisms $\pi_k^\infty(\bar{n}, l)$ satisfies the relations for being a $(U^S, \Theta^S)$-module in the sense of Definition 3.1.

We conclude this section with the following result. By a distributive law of the $A_\infty$-monad $(U, \Theta)$ over the monad $(S, \theta_S, t_S)$, we mean a collection of natural transformations $\Lambda' = \{\lambda_n' : U_{\bar{n}} S \rightarrow S(U_{\bar{n}} S) \}_{n \in \mathbb{Z}}$ such that

(a) The following two diagrams commute for each $n \in \mathbb{Z}$

\[
\begin{array}{ccc}
U_{\bar{n} S} & \xrightarrow{\lambda_n} & SU_n \\
\downarrow{\theta_{\bar{n} S}} & & \downarrow{\theta_S} \\
U_{\bar{n} S} & \xrightarrow{\lambda_{\bar{n}} S} & S(U_{\bar{n} S}) \\
\end{array}
\]

(b) For each $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, $k \geq 1$, the following diagram commutes

\[
\begin{array}{ccc}
U_{\bar{n} S} = U_{n_1} \cdots U_{n_k} S & \xrightarrow{U_{n_k} \cdots U_{n_1}} & SU_{n_1} \cdots SU_{n_k} \\
\downarrow{\theta_{\bar{n} S}} & & \downarrow{\theta_S} \\
U_{\bar{n} S} & \xrightarrow{\lambda_{\bar{n}} S} & S(U_{\bar{n} S}) \\
\end{array}
\]

We now recall (see, for instance, [25 § 5.2]) that the Kleisli category $Kl(S)$ of the monad $(S, \theta_S, t_S)$ is the category whose objects are the same as those of $\mathcal{C}$, but a morphism $M \rightarrow M'$ in $Kl(S)$ corresponds to a morphism $M \rightarrow S(M')$ in $\mathcal{C}$. The composition of $\alpha : M \rightarrow S(M')$ with $\beta : M' \rightarrow S(M'')$ in $Kl(S)$ is given by

$$
M \xrightarrow{\alpha} S(M') \xrightarrow{\beta} S(M'')
$$

There is a canonical functor $J_S : \mathcal{C} \rightarrow Kl(S)$ that is identity on objects and which takes the morphism $\alpha : M \rightarrow M'$ in $\mathcal{C}$ to $M \xrightarrow{\alpha} M' \xrightarrow{\epsilon_S(M')}$ in $Kl(S)$. From [25 Proposition 5.2.5], we know that a natural transformation $\lambda_n' : U_{\bar{n}} \rightarrow SU_{\bar{n}}$ satisfying the conditions in (6.16) corresponds to an extension $\tilde{U}_{\bar{n}}$ of the endofunctor $U_{\bar{n}}$ to the Kleisli category $Kl(S)$, i.e., $\tilde{U}_{\bar{n}} \circ J_S = J_S \circ U_{\bar{n}}$. From [45 Theorem 4.16], we know that the condition in (6.17) ensures that the natural transformations $\theta_{\bar{n}}(\bar{n}) : \tilde{U}_{\bar{n}} \rightarrow \tilde{U}_{\Sigma^{l+1}(2-k)}$ extend to natural transformations $\theta_{\bar{n}}(\bar{n}) : \tilde{U}_{\bar{n}} \rightarrow \tilde{U}_{\Sigma^{l+1}(2-k)}$. Proceeding as in the proof of Theorem 6.3, we can now prove the following result.

**Theorem 6.6.** Let $(U, \Theta)$ be an $A_\infty$-monad and let $(S, \theta_S, t_S)$ be a monad on $\mathcal{C}$. Then, there is a one one correspondence between the following

(a) Extensions of the $A_\infty$-monad $(U, \Theta)$ to an $A_\infty$-monad $(\tilde{U}, \tilde{\Theta})$ on the category $Kl(S)$.

(b) Distributive laws of the $A_\infty$-monad $(U, \Theta)$ over the monad $S$.

### 7 $A_\infty$-(co)algebras and $A_\infty$-(co)monads

In this section, we will show how to give examples of $A_\infty$-monads and $A_\infty$-comonoids on any $K$-linear Grothendieck category. For basic definitions such as $A_\infty$-algebras, $A_\infty$-coalgebras, $A_\infty$-modules over $A_\infty$-algebras and $A_\infty$-comodules over $A_\infty$-coalgebras, we refer the reader to the appendix in Section 10.
Let $\mathcal{C}$ be a $K$-linear Grothendieck category. If $R$ is a $K$-algebra, an $R$-object in $\mathcal{C}$ consists of $M \in \mathcal{C}$ along with a morphism $\xi_M : R \to \mathcal{E}(M, M)$ of $K$-algebras. The category $\mathcal{C}_R$ of $R$-objects in $\mathcal{C}$ is a classical construction of Popescu [30, p 108]. This abstract module theory has also been developed extensively by Artin and Zhang [2], who studied Hilbert Basis Theorem, completions, localizations and Grothendieck’s theory of flat descent in this context. In particular, if $\mathcal{C} = \text{Mod-S}$, the category of right modules over a $K$-algebra $S$, then $\mathcal{C}_R = \text{Mod}(S \otimes K R)$. Accordingly, the abstract module category $\mathcal{C}_p$ is often understood as a noncommutative base change of $R$ by means of the category $\mathcal{C}$ (see Lowen and Van den Bergh [33]). If $V$ is a right $R$-module, then there is a pair of adjoint functors (see [2, § B3, B4])

$$\otimes : \mathcal{C} \to \mathcal{C}_R \quad \text{Hom}_R(V, \_ : \mathcal{C} \to \mathcal{C}$$

(7.1)

Therefore, for $\mathcal{C} = \mathcal{C}_K$, an $R$-module object in $\mathcal{C}$ is determined by $M \in \mathcal{C}$ and a structure map $M \otimes R \to M$ in $\mathcal{C}$ satisfying the usual associativity and unit conditions. We observe that $\otimes \otimes : \mathcal{C} \to \mathcal{C}$ is an ordinary monad on $\mathcal{C}$. Similarly, if $C'$ is a $K$-coalgebra, a $C'$-comodule object in $\mathcal{C}$ is determined by $P \in \mathcal{C}$ and a structure map $P \to P \otimes C'$ satisfying coassociativity and counit conditions (see Brzeziński and Wisbauer [13, § 39.1]). Again, $\otimes \otimes C'$ is an ordinary comonad on $\mathcal{C}$. This suggests how we can create $A_{\infty}$-monads and $A_{\infty}$-comonads on $\mathcal{C}$.

We let $\text{Vect}^T$ be the category of $\mathbb{Z}$-graded vector spaces. We know that $\text{Vect}^T$ is a closed symmetric monoidal category, with internal hom $[W', W]$ of two objects given by

$$[W', W]_p := \prod_{k \in \mathbb{Z}} \text{Vect}(W'_k, W_{k+p}) \quad p \in \mathbb{Z}$$

(7.2)

Additionally, we consider the endofunctor $T : \text{Vect}^T \to \text{Vect}^T$ that inverts the grading, i.e., $(TW)_k := W_{-k}$ for any $k \in \mathbb{Z}$ and $W \in \text{Vect}^T$.

We let $(\mathbb{F}, \theta\mathbb{F})$ be a non-unital monad on $\mathcal{C}$. In other words, we have an endofunctor $\mathbb{F} \in \text{End}(\mathcal{C})$ along with a “multiplication” $\theta\mathbb{F} : \mathbb{F} \circ \mathbb{F} \to \mathbb{F}$ satisfying the usual associativity conditions. We note that any monad on $\mathcal{C}$ is equipped in particular with the structure of a non-unital monad. Similarly, a non-counital comonad $(\mathcal{G}, \Delta\mathcal{G})$ consists of an endofunctor $\mathcal{G} \in \text{End}(\mathcal{C})$ along with a “comultiplication” $\Delta\mathcal{G} : \mathcal{G} \to \mathcal{G} \circ \mathcal{G}$ satisfying the usual coassociativity conditions. The following result can be used to construct several examples of $A_{\infty}$-monads on $\mathcal{C}$.

**Proposition 7.1.** Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be an $A_{\infty}$-algebra and let $(\mathbb{F}, \theta\mathbb{F})$ be a non-unital monad on $\mathcal{C}$. Suppose that $\mathbb{F}$ preserves direct sums. Then, the collection of functors

$$\bigcup_{\mathbb{C}}^\mathbb{F} A = \{ \bigcup_{\mathbb{C}, n} A_n \otimes \mathbb{F}(\_ : \mathcal{C} \to \mathcal{C}) \otimes \mathbb{F}(\_ : \mathcal{C} \to \mathcal{C}) \}$$

(7.3)

is an $A_{\infty}$-monad on $\mathcal{C}$. In particular, the collection of functors $\bigcup_{\mathbb{C}}^\mathbb{F} A = \{ \bigcup_{\mathbb{C}, n} A_n \otimes \_ : \mathcal{C} \to \mathcal{C} \otimes \_ \}$ is an $A_{\infty}$-monad on $\mathcal{C}$.

**Proof.** Let $m_k : A^\otimes k \to A$ of degree $(2 - k)$ for $k \geq 1$ be the structure maps for $A$ as an $A_{\infty}$-algebra as in Definition [10.1]. Since $\mathbb{F}$ preserves direct sums, we note that for $k \geq 1$ and $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, we have

$$\bigcup_{\mathbb{C}, n_1}^\mathbb{F} \circ \cdots \circ \bigcup_{\mathbb{C}, n_k}^\mathbb{F} = A_{n_1} \otimes \cdots \otimes A_{n_k} \otimes \mathbb{F}(\_ : \mathcal{C} \to \mathcal{C})$$

(7.4)

We can now define natural transformations as in (2.1)

$$\Theta_{\mathbb{C}, n}^\mathbb{F} = \{ \Theta_{\mathbb{C}, n}^\mathbb{F}(\_ : \mathcal{C} \to \mathcal{C}) : \bigcup_{\mathbb{C}, n} A_n \otimes \_ \to \bigcup_{\mathbb{C}, n} A_n \otimes \_ \otimes \mathbb{F}(\_ : \mathcal{C} \to \mathcal{C}) \}$$

(7.5)

induced by the structure maps $A_{n_1} \otimes \cdots \otimes A_{n_k} \to A_{n_1 + \cdots + n_k + (2 - k)}$ of the $A_{\infty}$-algebra $A$ and the $k$-fold multiplication $\theta_{\mathbb{F}}^{k-1} : \mathbb{F}^{k} \to \mathbb{F}$ on the monad $\mathbb{F}$. The relations in (2.2) now follow directly from the relations satisfied by the structure maps of $A$ (see [10.1]).

We will say that an $A_{\infty}$-algebra $A$ is even if the structure maps $m_k : A^\otimes k \to A$ vanish whenever $k$ is odd. For instance, this condition is satisfied by $A_{2,p}$-algebras (for $p$ even), which have been extensively studied in the literature (see, for instance, [19], [23], [28], [44]). An $A_{2,p}$-algebra is an $A_{\infty}$-algebra with all the operations $m_k$ vanishing except for $k = 2$ and $k = p$. The
$A_{2,p}$-algebras were introduced by He and Lu in [24] and studied further by Dotsenko and Vallette in [19]. If $A$ is a $p$-Koszul algebra, then the Ext-algebra $\text{Ext}^* (K_A, K_A)$ carries the structure of an $A_{2,p}$-algebra (see [24]). Here, $K_A$ refers to the $A$-module structure on $K$ obtained from the fact that $A_0 = K$. The authors in [24] also describe a general procedure for constructing an $A_{2,p}$-algebra starting from any positively graded associative algebra. Other motivating examples for $A_{2,p}$-algebras, such as the $A_{5,0}$-Koszul dual of $K[t]/(t^p)$ for $p \geq 3$, appear in Lu, Palmieri, Wu, and Zhang [34] and in Keller [28 § 2.6].

**Proposition 7.2.** Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be an $A_{\infty}$-algebra that is even and let $(\mathcal{F}, \theta \mathcal{F})$ be a non-unital monad on $\mathcal{C}$. Suppose that $\mathcal{F}$ has a right adjoint $\mathcal{G}$. Then, the collection of functors

$$\mathcal{V}^A_{\mathcal{G}} = \{ \mathcal{V}^A_{\mathcal{G}, n} := \text{Hom}(A_n, \mathcal{G}(\underline{1})) : \mathcal{C} \to \mathcal{C}_{\otimes \mathbb{Z}} \}_{n \in \mathbb{Z}}$$

(7.6)

is an $A_{\infty}$-comonad on $\mathcal{C}$. In particular, the collection of functors

$$\mathcal{V}^A_{\mathcal{G}} = \{ \mathcal{V}^A_{\mathcal{G}, n} := \text{Hom}(A_n, \underline{1}) : \mathcal{C} \to \mathcal{C}_{\otimes \mathbb{Z}} \}_{n \in \mathbb{Z}}$$

(7.7)

is an $A_{\infty}$-comonad on $\mathcal{C}$. Further, we have isomorphisms of categories $\text{EM}^e_{\mathcal{U} \otimes \mathbb{Z}} \cong \text{EM}^{A_{\infty}}_{\mathcal{V}}$ and $\text{EM}^{eo}_{\mathcal{U} \otimes \mathbb{Z}} \cong \text{EM}^{A_{\infty}}_{\mathcal{V}}$.

**Proof.** Since $\mathcal{F}$ is a left adjoint (and hence it preserves direct sums), it follows by (7.1) that the functors $\text{Hom}(A_n, \mathcal{G}(\underline{1})) : \mathcal{C} \to \mathcal{C}$ are right adjoint to the functors $A_n \otimes \mathcal{F}(\underline{1}) = \mathcal{F}(A_n \otimes \underline{1}) : \mathcal{C} \to \mathcal{C}$. The result now follows from Theorem 2.3, Proposition 3.5 and Theorem 3.8.

The following result can be used to give several examples of $A_{\infty}$-comonads.

**Proposition 7.3.** Let $C = \bigoplus_{n \in \mathbb{Z}} C_n$ be an $A_{\infty}$-coalgebra with structure maps $\{ w_k : C \to C^{\otimes k} \}_{k \geq 1}$ and let $(\mathcal{G}, \Delta \mathcal{G})$ be a non-countinuous comonad on $\mathcal{C}$. Suppose that $\mathcal{G}$ preserves direct sums. Then, the collection of functors

$$\mathcal{U}^T_{\mathcal{G}} = \{ \mathcal{U}^T_{\mathcal{G}, n} := C_{\Delta n} \otimes \mathcal{G}(\underline{1}) : \mathcal{C} \to \mathcal{C}_{\otimes \mathbb{Z}} \}_{n \in \mathbb{Z}}$$

(7.8)

can be equipped with the structure of an $A_{\infty}$-coalgebra on $\mathcal{C}$. In particular, $\mathcal{U}^T_{\mathcal{G}} = \{ \mathcal{U}^T_{\mathcal{G}, n} := C_{\Delta n} \otimes \underline{1} : \mathcal{C} \to \mathcal{C}_{\otimes \mathbb{Z}} \}_{n \in \mathbb{Z}}$ is an $A_{\infty}$-comonad on $\mathcal{C}$.

If $C$ is even, i.e., $w_k = 0$ whenever $k$ is odd and $\mathcal{G}$ has a right adjoint $\mathcal{I}$, the collection of functors

$$\mathcal{V}^T_{\mathcal{G}} = \{ \mathcal{V}^T_{\mathcal{G}, n} := \text{Hom}(C_{\Delta n}, \mathcal{I}(\underline{1})) : \mathcal{C} \to \mathcal{C}_{\otimes \mathbb{Z}} \}_{n \in \mathbb{Z}}$$

(7.9)

is an $A_{\infty}$-coalgebra on $\mathcal{C}$. In particular, if $C$ is even, $\mathcal{V}^T_{\mathcal{G}} = \{ \mathcal{V}^T_{\mathcal{G}, n} := \text{Hom}(C_{\Delta n}, \underline{1}) : \mathcal{C} \to \mathcal{C}_{\otimes \mathbb{Z}} \}_{n \in \mathbb{Z}}$ is an $A_{\infty}$-comonad on $\mathcal{C}$.

**Proof.** In a manner similar to the proof of Proposition 7.1, it is clear from the definition of an $A_{\infty}$-coalgebra in Definition 10.3 and the fact that $\mathcal{G}$ preserves direct sums that $\mathcal{U}^T_{\mathcal{G}}$ is an $A_{\infty}$-comonad on $\mathcal{C}$. If $C$ is even, so is $\mathcal{U}^T_{\mathcal{G}}$. By (7.1), the functors $\text{Hom}(C_{\Delta n}, \mathcal{I}(\underline{1})) = \text{Hom}(\mathcal{I}(TC_n) \otimes \underline{1})$ are right adjoint to the functors $\mathcal{G}((TC_n) \otimes \underline{1}) = C_{\Delta n} \otimes \mathcal{G}(\underline{1})$ and the result now follows directly from Theorem 2.3.

For the rest of this section, we suppose that $\mathcal{C} = \text{Vect}$, the category of vector spaces over the field $K$. In this paper, for an $A_{\infty}$-algebra $A$, the category of left $A_{\infty}$-modules will always be denoted by $\text{L}_A \mathcal{M}$ (see Definition 10.3). For an $A_{\infty}$-coalgebra $C$, the category of left $A_{\infty}$-comodules will be denoted by $\text{C}_C \mathcal{M}$ (see Definition 10.6).

**Proposition 7.4.** Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded vector space. Consider the collection of functors

$$\mathcal{U}^A = \{ \mathcal{U}^A_n := A_n \otimes \underline{1} : \text{Vect} \to \text{Vect}_{\otimes \mathbb{Z}} \}_{n \in \mathbb{Z}}$$

(7.10)

Then, the family $\{ \mathcal{U}^A_n \}$ can be equipped with the structure of an $A_{\infty}$-comonad if and only if $A = \bigoplus_{n \in \mathbb{Z}} A_n$ can be equipped with the structure of an $A_{\infty}$-algebra.
Corollary 7.5. Let \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) be an \( A_\infty \)-algebra. Then, we have

(a) The category \( EM_{U_A} \) of modules over the \( A_\infty \)-monad \( U^A = [U^A_n := A_n \otimes \_ : \text{Vect} \rightarrow \text{Vect}]_{n \in \mathbb{Z}} \) is isomorphic to the category \( \mathcal{M} \) of \( A_\infty \)-modules over \( A \).

(b) Suppose that \( U^A \) is even. Then, the collection of functors

\[
\mathcal{F}^A = \{ \mathcal{F}^A_n := \text{Hom}(A_n, \_ : \text{Vect} \rightarrow \text{Vect}) \}_{n \in \mathbb{Z}}
\]

is an \( A_\infty \)-comonad.

(c) Suppose that \( U^A \) is even. Then, we have isomorphisms of categories \( EM_{U_A} \cong EM_{\mathcal{F}^A} \) and \( EM_{U_A} \cong EM_{\mathcal{F}^A} \).

Proof. By inspecting the conditions in Definition 3.1, we can see directly that \( U_n \)-modules correspond to \( A_\infty \)-modules over \( A \) in the sense of Definition 10.6. This proves (a). Since the functors \( \{\text{Hom}(A_n, \_ : \text{Vect} \rightarrow \text{Vect})\}_{n \in \mathbb{Z}} \) are right adjoint to the functors \( (A \otimes \_ : \text{Vect})_{n \in \mathbb{Z}} \), the result of (b) follows from Theorem 3.7. The result of (c) is also clear from Proposition 3.5 and Theorem 3.8.

Proposition 7.6. Let \( C = \bigoplus_{n \in \mathbb{Z}} C_n \) be an \( A_\infty \)-coalgebra. Then, there is a canonical functor

\[
\mathcal{F}^C : \mathcal{CM} \rightarrow EM^{TC}
\]

that embeds \( A_\infty \)-comodules as a subcategory of the Eilenberg-Moore category of comodules over the \( A_\infty \)-comonad \( UT_C \).

Proof. Let \( P \) be an \( A_\infty \)-comodule over \( C \) in the sense of Definition 3.1. It is equipped with structure maps \( \{w^k_P : P \rightarrow C^\otimes k \otimes P\}_{k \geq 1} \). We set \( Q = TP \), i.e., \( Q_n := P_n \) for \( n \in \mathbb{Z} \). For any \( k \geq 1 \), \( n \in \mathbb{Z}^{k-1} \), \( l \in \mathbb{Z} \), we have

\[
\rho_k(n, l) : Q_{2\bar{n} + l + (2-k)} \rightarrow P_{-\Sigma_{-l}-(2-k)} \rightarrow (C^\otimes k \otimes P)_{-\Sigma_{-l}}
\]

where the second map in (7.13) is the canonical projection. Due to the relations in Definition 10.6, \( \rho_k(n, l) \) satisfies the relations in \( \mathcal{M} \) for \( Q \) to be a \( UT_C \)-comodule. As such, setting \( \mathcal{H}^C(P) = Q = TP \) defines a functor \( \mathcal{H}^C : \mathcal{CM} \rightarrow EM^{TC} \).

It remains to show that \( \mathcal{H}^C \) determines a full embedding. For this, we consider \( P' \in \mathcal{CM} \) along with structure maps \( \{w^k_P : P' \rightarrow C^\otimes k \otimes P'\}_{k \geq 1} \) as well as \( Q' = TP' = \mathcal{H}^C(P') \in EM^{TC} \) with structure maps \( \rho'_k(n, l) : Q'_{2\bar{n} + l + (2-k)} \rightarrow \bigoplus_{\bar{n} \in \mathbb{Z}^{k-1}} (C^\otimes k \otimes P')_{-\Sigma_{-l}} \).

For each fixed \( T \in \mathbb{Z} \) and \( k \geq 1 \), we now observe that the outer square as well as the lower square in the following diagram commute:

\[
\begin{array}{ccc}
Q_{2\bar{n} + l + (2-k)} & \xrightarrow{\rho'_k(n, l)} & \bigoplus_{\bar{n} \in \mathbb{Z}^{k-1}} (C^\otimes k \otimes P')_{-\Sigma_{-l}} \\
\downarrow w^k_{\bar{T}-l,-(2-k)} & & \downarrow \bigoplus_{\bar{n} \in \mathbb{Z}^{k-1}} (C^\otimes k \otimes P')_{-\Sigma_{-l}} \\
\bigoplus_{\bar{n}, \bar{l} \in \mathbb{Z}^{k-1}} (C_{-\bar{n}_{-l}} \otimes \_ \otimes P_{-l}) & \xrightarrow{\bigoplus_{(\bar{n}, \bar{l} \in \mathbb{Z}^{k-1})} (C_{-\bar{n}_{-l}} \otimes \_ \otimes P_{-l})} & \bigoplus_{\bar{n} \in \mathbb{Z}^{k-1}} (C_{-\bar{n}_{-l}} \otimes \_ \otimes P'_{-l}) \\
\downarrow & & \downarrow \\
\prod_{\bar{n}, \bar{l} \in \mathbb{Z}^{k-1}} (C_{-\bar{n}_{-l}} \otimes \_ \otimes P_{-l}) & \xrightarrow{\prod_{(\bar{n}, \bar{l} \in \mathbb{Z}^{k-1})} (C_{-\bar{n}_{-l}} \otimes \_ \otimes P_{-l})} & \prod_{\bar{n} \in \mathbb{Z}^{k-1}} (C_{-\bar{n}_{-l}} \otimes \_ \otimes P'_{-l})
\end{array}
\]
Since the direct sums in (7.14) embed into the direct products, it follows that the upper square in (7.14) also commutes. We now see that \( h = [h_n]_{n \in \mathbb{Z}} \) also determines a morphism in \( C \).

8 Rational pairings between \( A_\infty \)-algebras and \( A_\infty \)-coalgebras

If \( W \in Vect^\mathbb{Z} \) is a graded vector space, we note that by (7.2), its graded dual \( W^* = [W,K] \) is given by \( W_n^* = Vect(W_n,K) \) for \( n \in \mathbb{Z} \). If \( C \) is an \( A_\infty \)-coalgebra, equipped with structure maps \( [w_k : C \rightarrow C^\otimes k]_{k \geq 1} \), we know that its graded dual \( C^* \) is an \( A_\infty \)-algebra with \( m_k : (C^*)^\otimes k \rightarrow C^* \) determined by

\[
m_k(\phi_1 \otimes \cdots \otimes \phi_k) = (-1)^{k|\phi_1| + \cdots + |\phi_k| + 1}(\mu \circ (\phi_1 \otimes \cdots \otimes \phi_k) \circ w_k)
\]

where \( \mu \) denotes the multiplication on \( K \). If \( M \in M^C \) is a right \( C \)-comodule, we know that \( M \) may be treated as a left \( C^* \)-module with structure maps given by

\[
(C^*)^\otimes k \otimes M \xrightarrow{\mu^\otimes k} (C^*)^\otimes k \otimes C \xrightarrow{\phi_1 \otimes \cdots \otimes \phi_k} M
\]

(8.1)

where \( \{w_k^M : M \rightarrow M \otimes C^\otimes k\}_{k \geq 1} \) comes from the right \( C \)-comodule structure of \( M \).

**Definition 8.1.** Let \( C \) be an \( A_\infty \)-coalgebra and let \( A \) be an \( A_\infty \)-algebra. A pairing between \( C \) and \( A \) is an \( A_\infty \)-algebra morphism \( f = \{f_k : A^\otimes k \rightarrow C^*\}_{k \geq 1} \) from \( A \) to the graded dual \( C^* \) of \( C \).

**Theorem 8.2.** Let \( C \) be an \( A_\infty \)-coalgebra and let \( A \) be an \( A_\infty \)-algebra. Then, a pairing \( f = \{f_k : A^\otimes k \rightarrow C^*\}_{k \geq 1} \) between \( C \) and \( A \) induces a functor \( (C,A) : M^C \rightarrow A^M \).

**Proof.** From (8.2), it follows that we have a functor \( M^C \rightarrow C^M \). Considering the morphism \( f' = \{f_k : A^\otimes k \rightarrow C^*\}_{k \geq 1} \) of \( A_\infty \)-algebras, we also have a functor \( C^M \rightarrow A^M \) given by restriction of scalars (see, for instance, (27) § 6.2)). The result is now clear.

We now want to define a rational pairing between an \( A_\infty \)-coalgebra and an \( A_\infty \)-algebra. For this, we will need to refine the result of Theorem 8.2. First, we consider a morphism \( f = \{f_k : A^\otimes k \rightarrow B\}_{k \geq 1} \) of \( A_\infty \)-algebras as in Definition 10.2. Let \( M \) be a left \( B \)-module, equipped with structure maps \( \{m^B_k : B^\otimes k \otimes M \rightarrow M\}_{k \geq 1} \). Then, the restriction of scalars makes \( M \) into an \( A \)-module, determined by setting (see, for instance, (27) § 6.2))

\[
m^A_{k+1} : A^\otimes k \otimes M \rightarrow M \quad m^A_{k+1} = \sum (-1)^s m^B_{i+1}(f_i \otimes \cdots \otimes f_i \otimes M)
\]

(8.3)

for \( k \geq 0 \), where the sum is taken over all decompositions \( k = i_1 + \cdots + i_j \) and \( s = (l-1)(i_1-1) + (l-2)(i_2-1) + \cdots + 2(i_{j-2}-1) + (i_{j-1}-1) \). We also note that \( f = \{f_k : A^\otimes k \rightarrow B\}_{k \geq 1} \) determines a morphism

\[
\tilde{f} : \bigoplus_{l \geq 0} A^\otimes l \rightarrow \bigoplus_{l \geq 0} B^\otimes l
\]

(8.4)

of graded vector spaces, where the action of \( \tilde{f} \) on the component \( A^\otimes k \) is given by \( \sum (-1)^s (f_i \otimes \cdots \otimes f_i) \), where the signs are as in (8.3). Accordingly, for any \( V \in Vec^\mathbb{Z} \), we have an induced morphism

\[
[\tilde{f}, V] : \prod_{l \geq 0} B^\otimes l \rightarrow \prod_{l \geq 0} A^\otimes l \quad \left( [\tilde{f}, V] = \left[ \bigoplus_{l \geq 0} B^\otimes l \right] \otimes V \right) = \left( \bigoplus_{k \geq 0} A^\otimes k \right) \otimes V
\]

(8.5)

In particular, we consider \( f = \{f_k : A^\otimes k \rightarrow C^*\}_{k \geq 1} \) determining the pairing between the \( A_\infty \)-coalgebra \( C \) and the \( A_\infty \)-algebra \( A \). For any \( V \in Vec^\mathbb{Z} \), we note that the closed symmetric structure on \( Vec^\mathbb{Z} \) induces a canonical morphism

\[
\alpha(C,A,V) : \prod_{l \geq 0} V \otimes C^\otimes l \xrightarrow{\prod [\tilde{f}, V]} \prod_{l \geq 0} [A^\otimes l, V]
\]

(8.6)
Definition 8.3. Let \( f = \{ f_k : A^\otimes k \to C^* \}_{k \geq 1} \) be a pairing of an \( A_{\infty} \)-coalgebra \( C \) and an \( A_{\infty} \)-algebra \( A \). We will say that the pairing is rational if the canonical morphism \( \alpha(C,A,V) \) as defined in (8.6) is a monomorphism in \( \text{Vec}^2 \) for any \( V \in \text{Vec}^2 \).

Proposition 8.4. Let \( f = \{ f_k : A^\otimes k \to C^* \}_{k \geq 1} \) be a rational pairing of an \( A_{\infty} \)-coalgebra \( C \) and an \( A_{\infty} \)-algebra \( A \). Then, the functor \( i(C,A) : M^C \to _A M \) embeds \( M^C \) as a full subcategory of \( _A M \).

Proof. We consider \( M, N \in M^C \) with structure maps \( \{ w^M_k : M \to M \otimes C^\otimes k \}_{k \geq 1} \) and \( \{ w^N_k : N \to N \otimes C^\otimes k \}_{k \geq 1} \) respectively. If we treat \( M, N \) as left \( A \)-modules via the functor \( i(C,A) \), we have to show that any \( A \)-module morphism \( h : M \to N \) is also a morphism in \( M^C \). We consider therefore the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{h} & N \\
\prod w^M_k \downarrow & & \downarrow \prod w^N_k \\
\prod M \otimes C^\otimes k & \xrightarrow{\prod \otimes C^\otimes} & \prod N \otimes C^\otimes k \\
\alpha(C,A,M) \downarrow & & \downarrow \alpha(C,A,N) \\
\prod [A^\otimes, M] & \xrightarrow{\prod [A^\otimes, h]} & \prod [A^\otimes, N] \\
\end{array}
\]

(8.7)

Considering the explicit definition of restriction of scalars in (8.3) and the construction of the functor \( i(C,A) \) in Theorem 8.2, we realize that the outer square in (8.7) commutes because \( h \) is an \( A \)-module morphism. The lower square in (8.7) commutes because \( h \) is a morphism of graded vector spaces. Accordingly, we have

\[
\alpha(C,A,N) \circ (\prod w^N_k) \circ h = (\prod w^M_k) \circ \alpha(C,A,M) \circ (\prod \otimes C^\otimes) = (\prod h \circ \otimes C^\otimes) \circ (\prod w^M_k) = \alpha(C,A,N) \circ (\prod h \circ \otimes C^\otimes) \circ (\prod w^M_k)
\]

(8.8)

Since the pairing is rational by assumption, we know that \( \alpha(C,A,N) \) is a monomorphism. Accordingly, it follows from (8.8) that the upper square in (8.7) also commutes. This proves that \( h \) is a morphism of right \( C \)-comodules.

Proposition 8.5. Let \( f = \{ f_k : A^\otimes k \to C^* \}_{k \geq 1} \) be a rational pairing of an \( A_{\infty} \)-coalgebra \( C \) and an \( A_{\infty} \)-algebra \( A \). Then, the category \( M^C \), treated as a full subcategory of \( _A M \), is closed under direct sums, subobjects and quotients.

Proof. It is clear that \( M^C \), treated as a full subcategory of \( _A M \), is closed under direct sums. Suppose we have a short exact sequence

\[
0 \longrightarrow M' \xrightarrow{g} M \xrightarrow{h} M'' \longrightarrow 0
\]

(8.9)
in \( _A M \) such that \( M \in M^C \) with structure maps \( \{ w^M_{k+1} : M \to M \otimes C^\otimes k \}_{k \geq 0} \). Since products of short exact sequences in \( \text{Vec}^2 \) are exact, we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & M' \\
& \xrightarrow{g} & M \\
& \downarrow \prod w^M_{k+1} & \downarrow \prod w^M_k \\
\prod (M' \otimes C^*) & \xrightarrow{\prod (g \otimes C^*)} & \prod (M \otimes C^*) \\
\alpha(C,A,M') \downarrow & & \downarrow \alpha(C,A,M) \\
\prod [A^\otimes, M'] & \xrightarrow{\prod [A^\otimes, g]} & \prod [A^\otimes, M] \\
\end{array}
\]

(8.10)

Since \( g \) is an \( A \)-module morphism, we note that the composition \( \alpha(C,A,M) \circ (\prod w^M_{k+1}) \circ g \) factors through \( \prod [A^\otimes, g] \). Accordingly, we have \( (\prod [A^\otimes, h]) \circ \alpha(C,A,M) \circ (\prod w^M_{k+1}) \circ g = 0 \). It follows that

\[
\alpha(C,A,M'') \circ (\prod (h \otimes C^*)) \circ (\prod w^M_{k+1}) \circ g = (\prod [A^\otimes, h]) \circ \alpha(C,A,M) \circ (\prod w^M_{k+1}) \circ g = 0
\]

(8.11)
Since \( f \) is a rational pairing, we know that \( \alpha(C, A, M') \) is a monomorphism, whence it follows from \( \{ \prod (h \otimes C^*) \} \circ (\prod w_{M_1} \otimes ) \circ g = 0 \). Since the middle row in \((8.10)\) is short exact, there are now induced maps \( M' \rightarrow \prod (M' \otimes C^*) \) and \( M'' \rightarrow \prod (M'' \otimes C^*) \) making the diagram commutative and giving \( M', M'' \) the structure of \( C \)-comodules. This proves the result.

\[ \square \]

Given the rational pairing \( f = \{ f_k : A^{\otimes k} \rightarrow C^* \}_{k \geq 1} \), we now set for any \( M \in \mathcal{A} \)
\[ R_f(M) := \sum_{g \in \mathcal{A}(\mathcal{C}(A)(N), M), N \in M^C} \text{Im}(g) \subseteq M \quad (8.12) \]

We now have the main result on rational pairings of \( A_{\infty} \)-coalgebras and \( A_{\infty} \)-algebras.

**Theorem 8.6.** Let \( f = \{ f_k : A^{\otimes k} \rightarrow C^* \}_{k \geq 1} \) be a rational pairing of an \( A_{\infty} \)-coalgebra \( C \) and an \( A_{\infty} \)-algebra \( A \). Then, we have a functor \( R_f : \mathcal{A} \rightarrow \mathcal{M} \) that is right adjoint to \( \iota(C, A) : \mathcal{M} \rightarrow \mathcal{A} \), i.e., we have natural isomorphisms
\[ \mathcal{A} \mathcal{M}(\iota(C, A)(M'), M) \cong \mathcal{M}(M', R_f(M)) \quad (8.13) \]

for any \( M' \in \mathcal{M} \) and \( M \in \mathcal{A} \).

**Proof.** We consider some \( g \in \mathcal{A} \mathcal{M}(\iota(C, A)(N), M), with N \in \mathcal{M} \). Since \( \mathcal{M} \) is closed under quotients, we see that \( \text{Im}(g) \in \mathcal{M} \). Applying Proposition 8.3, again, since \( \mathcal{M} \) is closed under direct sums and quotients in \( \mathcal{A} \), we see that the sum \( R_f(M) = \sum \text{Im}(g) \) defined as in \( (8.12) \) must in \( \mathcal{M} \).

Further, from the definition in \((8.12)\), it follows that the image of any morphism \( \iota(C, A)(M') \rightarrow M \) in \( \mathcal{A} \) must lie in \( R_f(M) \). From this it is clear that we have an isomorphism \( \mathcal{A} \mathcal{M}(\iota(C, A)(M'), M) \cong \mathcal{M}(M', R_f(M)). \)

It remains to show that \( R_f \) is a functor. For this, we consider a morphism \( h : M_1 \rightarrow M_2 \) in \( \mathcal{A} \). Since \( R_f(M_1) \) lies in \( \mathcal{M} \), it follows from the definition in \((8.12)\) that the image of the composition \( R_f(M_1) \rightarrow M_1 \xrightarrow{h} M_2 \) lies in \( R_f(M_2) \). This proves the result.

\[ \square \]

### 9 \( A_{\infty} \)-contramodules

Let \( C = \bigoplus_{n \in \mathbb{Z}} C_n \) be an \( A_{\infty} \)-coalgebra, equipped with structure maps \( \{ w_k : C \rightarrow C^{\otimes k} \}_{k \geq 1} \), where \( w_k \) has degree \( 2 - k \) (see Definition 10.5). For any \( k \geq 1 \) and \( \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k \), let \( w_k(\bar{n}) : C_{-n_1} \otimes \ldots \otimes C_{-n_k} \rightarrow C^{\otimes k} \) be a component of the structure map \( w_k : C \rightarrow C^{\otimes k} \) of \( C \). We define \( w'_k : C \rightarrow C^{\otimes k} \) whose components are given by \( w'_k(\bar{n}) := (-1)^{k+1} w_k(\bar{n}) : C_{-n_1} \otimes \ldots \otimes C_{-n_k} \rightarrow C^{\otimes k} \). We are now ready to introduce \( A_{\infty} \)-contramodules over \( C \).

**Definition 9.1.** Let \( C \) be an \( A_{\infty} \)-coalgebra. An \( A_{\infty} \)-contramodule is a \( \mathbb{Z} \)-graded vector space \( M \) equipped with structure maps
\[ t^M_k : [C^{\otimes k-1}, M] \rightarrow M, \quad k \geq 1 \quad (9.1) \]

with \( t^M_k \) of degree \((2 - k)\) satisfying, for each \( n \geq 1 \),
\[ \sum (-1)^{p+qr} t^M_{p+1+r} \circ [C^{\otimes p} \otimes w'_q \otimes C^{\otimes r-1}, M] + \sum (-1)^q t^M_{p+1} \circ [C^{\otimes a}, t^M_b] = 0 \quad (9.2) \]

where the first sum in \((9.2)\) is taken over all \( p \geq 0, q, r \geq 1 \) such that \( p + q + r = n \) and the second sum in \((9.2)\) is taken over all \( a \geq 0, b \geq 1 \) with \( a + b = n \). In \((9.2)\), we choose the isomorphism \([C^{\otimes a}, [C^{\otimes b-1}, M]] \cong [C^{\otimes a+b-1}, M]\) making \( M \) a right \( A_{\infty} \)-contramodule.

A morphism \( h : M \rightarrow M' \) of \( A_{\infty} \)-contramodules is a morphism of graded vector spaces that is compatible with the respective structure maps. We will denote by \( \mathcal{M} \) the category of \( A_{\infty} \)-contramodules over \( C \).
We say that $M \in \mathbf{M}_1$ is even if $r^M_k = 0$ whenever $k$ is odd. We denote by $\mathbf{M}'_1$ the full subcategory of $\mathbf{M}_1$ consisting of even objects. By Proposition 7.3, we know that when $C$ is even, the collection of functors

$$V^TC = [\forall^TC := \text{Hom}(C, -)] : \text{Vect} \rightarrow \text{Vect}$$

is an $A_\infty$-monad. The following result is now the $A_\infty$-contramodule counterpart of Proposition 7.6.

**Proposition 9.2.** Let $C$ be an even $A_\infty$-coalgebra. Then, there is a canonical functor

$$\mathcal{A}_1 : \mathbf{M}'_1 \rightarrow EM_{\forall TC}$$

that embeds $\mathbf{M}'_1$ as a subcategory of even modules over the $A_\infty$-monad $\forall TC$. If $C$ is such that $C_n \neq 0$ for only finitely many $n \in \mathbb{Z}$, then the categories $\mathbf{M}'_1$ and $EM_{\forall TC}$ are isomorphic.

**Proof.** Let $M \in \mathbf{M}'_1$. For $k \geq 1$, $\bar{n} \in \mathbb{Z}^{-1}$, $l \in \mathbb{Z}$, we define

$$\pi_k(\bar{n}, l) : \forall^T_\bar{n} M_l = [C_{-n} \otimes \ldots \otimes C_{-n-l}] \rightarrow [C^{\otimes k-1}, M]_{\bar{n}+t} \xrightarrow{\iota^M_{\bar{n}+t}} M_{\Sigma^+ l}(2-k)$$

We denote by $\{\delta_k(\bar{n})\}$ the structure maps of the $A_\infty$-monad $\forall TC$. For $N \geq 0$, $\bar{z} = (\bar{m}, \bar{m}') \in \mathbb{Z}^N$ with $|\bar{m}| = p$, $|\bar{m}'| = q$, $|\bar{n}'| = r-1$, we observe that the following diagram is commutative

$$\begin{array}{ccc}
\forall^T_\bar{n} \otimes \bar{m}' & \xrightarrow{\forall^T_n \otimes \delta_k(\bar{n}')} & \forall^T_\bar{n} \otimes \bar{m}'
\downarrow & & \downarrow \\
[C^{\otimes p}, M]_{\Sigma^+ l+k} & \xrightarrow{[C^{\otimes p} \otimes \delta_k(\bar{m}'), M]_{\Sigma^+ l}} & [C^{\otimes (p+r)}, M]_{\Sigma^+ l+k}
\end{array}$$

Similarly, for $\bar{z} = (\bar{m}, \bar{m}') \in \mathbb{Z}^N$ with $|\bar{m}| = a$, $|\bar{m}'| = b-1$, we observe that the following diagram is commutative

$$\begin{array}{ccc}
\forall^T_\bar{n} \otimes \bar{m}' & \xrightarrow{\forall^T_n \otimes \delta_k(\bar{n}')} & \forall^T_\bar{n} \otimes \bar{m}'
\downarrow & & \downarrow \\
[C^{\otimes a}, [C^{\otimes b-1}, M]]_{\Sigma^+ l+k} & \xrightarrow{[C^{\otimes a} \otimes \delta_k(\bar{m}'), M]_{\Sigma^+ l+k}} & [C^{\otimes (a+b-1)}, M]_{\Sigma^+ l+k}
\end{array}$$

Using the fact that the expressions in (9.2) are zero, it follows from (9.6) and (9.7) that the $\pi_k(\bar{n}, l)$ defined in (9.5) make $M$ into a module over the $A_\infty$-monad $\forall TC$. We therefore have the functor $\mathbf{M}'_1 \rightarrow EM_{\forall TC}$.

Now suppose that $C_n \neq 0$ for only finitely many $n \in \mathbb{Z}$. Then, it follows that for any $T \in \mathbb{Z}$, we have

$$[C^{\otimes k-1}, M]_{\bar{m}} = \bigoplus_{n_1 + \ldots + n_{k-1} = T} [C_{-m_1} \otimes \ldots \otimes C_{-m_{k-1}}, M]_{\bar{m}} \xrightarrow{\forall^T_{n_k}} M_{\Sigma^+ l}(2-k)$$

In other words, the structure maps $\iota^M_k : [C^{\otimes k-1}, M] \rightarrow M$ of $M \in \mathbf{M}'_1$ are completely determined by the induced morphisms $\forall^T_{n_1} \ldots \forall^T_{n_{k-1}}(M) \rightarrow M_{n_1 + \ldots + n_{k-1}+l}(2-k)$ on components. It follows that the categories $\mathbf{M}'_1$ and $EM_{\forall TC}$ are isomorphic. □

**Proposition 9.3.** Let $C$ be an $A_\infty$-coalgebra. Then, there exists a faithful functor $\kappa^C : \mathbf{M}'_1 \rightarrow \mathbf{M}_1$. If the $A_\infty$-coalgebra $C$ is such that the canonical map $W \otimes C \rightarrow [C, W]$ is an epimorphism for each $W \in \text{Vect}^c$, then $\kappa^C$ is an isomorphism.

**Proof.** Let $M \in \mathbf{M}_1$, with structure maps as in (9.1). Then, for any $k \geq 1$, we consider the compositions

$$\kappa^M_k : M \otimes (C^*)^{\otimes k-1} \rightarrow [C^{\otimes k-1}, M] \xrightarrow{\iota^M_k} M$$

(9.9)
where the morphism \( M \otimes (C^* \otimes 1) \rightarrow [C \otimes 1, M] \) in (9.9) is induced by the canonical morphism \( M \otimes (C^* \otimes 1) \otimes C^\otimes 1 \rightarrow M \). This makes \( M \) into a \( C^* \)-module. It is also clear that the functor \( \kappa^C \) is faithful.

In particular, suppose that the \( A^*_\infty \)-coalgebra \( C \) is such that the canonical map \( W \otimes C^* \rightarrow [C, W] \) is an epimorphism for each \( W \in \text{Vec} \). We note that for each \( n \in \mathbb{Z} \), the canonical map

\[
(W \otimes C^*)_n = \bigoplus_{k \in \mathbb{Z}} (W_{k+n} \otimes (C^*)_k) \longrightarrow \bigoplus_{k \in \mathbb{Z}} [C_k, W_{k+n}] \rightarrow \prod_{k \in \mathbb{Z}} [C_k, W_{k+n}] = [C, W]_n
\]  

(9.10)

is always a monomorphism, i.e. \( W \otimes C^* \rightarrow [C, W] \) is a monomorphism for each \( W \in \text{Vec} \). By assumption, the morphisms in (9.10) are also epimorphisms. Then, the maps \( M \otimes (C^* \otimes 1) \rightarrow [C \otimes 1, M] \) which appear in (9.9) are all isomorphisms and it is clear that the functor \( \kappa^C \) is an isomorphism.

\[ \square \]

**Proposition 9.4.** Let \( C \) be an \( A^*_\infty \)-coalgebra. Then, there is a functor \( [C, -] : \text{Vec} \rightarrow \text{M}_C \).

**Proof.** We consider \( W \in \text{Vec} \) and put \( N = [C, W] \). By setting \( t^N_k := [w^k, W] : [C^\otimes 1, N] = [C^\otimes 1, [C, W]] = [C^\otimes , W] \rightarrow [C, W] = N \) where \( w^k \in C \rightarrow C^\otimes \) is as at the beginning of this section, we obtain \( N = [C, W] \in \text{M}_C \).

\[ \square \]

We will now construct functors between categories of \( A^*_\infty \)-comodules and \( A^*_\infty \)-contramodules. Let \( (C, w^L_0), (D, w^R_0) \) be \( A^*_\infty \)-coalgebras. By a \((C, D)\)-space \((N, w^L_k, w^R_k)\), we will mean a graded vector space \( N \) equipped with families of morphisms

\[
w^L_k = [w^L_p : N \rightarrow C^\otimes 1 \otimes N]_{p \geq 1} \quad w^R_k = [w^R_q : N \rightarrow N \otimes D^\otimes 1 \otimes q \geq 1]
\]

(9.11)

such that \((N, w^L_0)\) is a left \( C \)-comodule and \((N, w^R_0)\) is a right \( D \)-comodule. In particular, any \((C, D)\)-bicomodule becomes a \((C, D)\)-space in an obvious manner. Additionally, for \( Q, Q' \in \text{M}^D \), we set \([Q, Q']^D = \text{M}^D(Q[-p], Q')\) for \( p \in \mathbb{Z} \).

**Proposition 9.5.** Let \((C, w^L_0), (D, w^R_0)\) be \( A^*_\infty \)-coalgebras and let \((N, w^L_k, w^R_k)\) be a \((C, D)\)-space. Then, for each \( Q \in \text{M}^D \) and each \( k \geq 1 \), there is an isomorphism

\[
[C^\otimes 1, [N, Q]^D] \cong [C^\otimes 1 \otimes N, Q]^D
\]

(9.12)

**Proof.** We consider \( f \in [C^\otimes 1, [N, Q]^D] \). Then, \( f \) corresponds to

\[
f' : C^\otimes 1 \otimes N \rightarrow Q \quad (\tilde{c} \otimes n) \mapsto f(\tilde{c})(n)
\]

(9.13)

In order to prove that \( f' \in [C^\otimes 1 \otimes N, Q]^D \), we have to show that the following diagram commutes for each \( l \geq 1 \)

\[
\begin{array}{ccc}
C^\otimes 1 \otimes N & \xrightarrow{c^\otimes 1 \otimes w^L_{\otimes D_{\otimes 1}}} & C^\otimes 1 \otimes N \otimes D^\otimes 1 \\
f' \downarrow & & \downarrow (f' \otimes D^\otimes 1) \\
Q & \xrightarrow{w^L_{\otimes D_{\otimes 1}}} & Q \otimes D^\otimes 1
\end{array}
\]

(9.14)

where we have adapted the Sweedler notation in (9.14). But since \( f(\tilde{c}) \) is a \( D \)-comodule morphism, we already have

\[
w^L_{\otimes D_{\otimes 1}}(f(\tilde{c})(n))_0 \otimes w^L_{\otimes D_{\otimes 1}}(f(\tilde{c})(n))_1 = f(\tilde{c})(w^L_{\otimes D_{\otimes 1}}(f(\tilde{c})(n)))
\]

(9.15)

From (9.14) and (9.15), it follows that \( f' \in \text{M}^D(C^\otimes 1 \otimes N, Q) \). Conversely, we consider \( g \in [C^\otimes 1 \otimes N, Q]^D \). Then, \( g \) corresponds to

\[
g' : C^\otimes 1 \rightarrow [N, Q] \quad g'(\tilde{c}) := g(\tilde{c} \otimes n)
\]

(9.16)

In order to show that \( g' \) takes values in \([N, Q]^D\), we have to show that for each \( l \geq 1 \) we must have

\[
w^L_{\otimes D_{\otimes 1}}(g(\tilde{c} \otimes n))_0 \otimes w^L_{\otimes D_{\otimes 1}}(g(\tilde{c} \otimes n))_1 = w^L_{\otimes D_{\otimes 1}}(g'(\tilde{c})(n))_0 \otimes w^L_{\otimes D_{\otimes 1}}(g'(\tilde{c})(n))_1
\]

(9.17)

However since \( g \in [C^\otimes 1 \otimes N, Q]^D \), we already know that

\[
w^L_{\otimes D_{\otimes 1}}(g(\tilde{c} \otimes n))_0 \otimes w^L_{\otimes D_{\otimes 1}}(g(\tilde{c} \otimes n))_1 = g(\tilde{c} \otimes w^L_{\otimes D_{\otimes 1}}(n))_0 \otimes w^L_{\otimes D_{\otimes 1}}(n)_1
\]

(9.18)

From (9.17) and (9.13) it is clear that \( g' \in [C^\otimes 1, [N, Q]^D] \). This proves the result.

\[ \square \]
We will say that \((N, w^C_k, w^D_k)\) is a commuting \((C, D)\)-space if the \(C\) and \(D\)-coactions commute with each other, i.e., for each \(k \geq 1, l \geq 1\), we have a commutative diagram
\[
\begin{array}{ccc}
N & \xrightarrow{w^D_k} & N \otimes D^{\otimes l-1} \\
\downarrow w^C_k & & \downarrow w^C_k \otimes D^{\otimes l-1} \\
C^{\otimes k-1} \otimes N & \xrightarrow{C^{\otimes k-1} \otimes w^D_k} & C^{\otimes k-1} \otimes N \otimes D^{\otimes l-1}
\end{array}
\] (9.19)

We note in particular that this means that each \(w^D_k : N \rightarrow N \otimes D^{\otimes l-1}\) is a morphism of left \(C\)-comodules, while each \(w^C_k : N \rightarrow C^{\otimes k-1} \otimes N\) is a morphism of right \(D\)-comodules. For the rest of this section, we will assume that both the \(C\)-comodule and \(D\)-comodule structures on \(N\) are even.

**Proposition 9.6.** Let \((C, w^C_*), (D, w^D_*)\) be even \(A_{\text{co}}\)-coalgebras. Then, \((N, w^C_*, w^D_*)\) induces a functor \([N, _]^{D} : M^{D} \rightarrow M_{[C, \_]}^{1}\).

**Proof.** We consider \(Q \in M^{P}\). We claim that \([N, Q]^{D} \in M_{[C, \_]}^{1}\). From Proposition 9.5, we already have that \([C^{\otimes k-1}, [N, Q]]^{D} \cong [C^{\otimes k-1} \otimes N, Q]^{D}\) for each \(k \geq 1\). We now have structure maps
\[
t_k : [C^{\otimes k-1}, [N, Q]]^{D} \cong [C^{\otimes k-1} \otimes N, Q]^{D} \xrightarrow{[w^C_k, Q]^{D}} [N, Q]^{D}
\] (9.20)
where we have used the fact that each morphism \(w^C_k : N \rightarrow C^{\otimes k-1} \otimes N\) is a morphism of \(D\)-comodules. It may be verified that the morphisms in (9.20) make \([N, Q]^{D}\) into an \(A_{\text{co}}\)-contramodule over \(C\).

Let \((M, t^M_*)\) be an \(A_{\text{co}}\)-contramodule over \(C\). Then, for each \(k \geq 1\), we have a pair of canonical maps
\[
[C^{\otimes k-1}, M] \otimes N \xrightarrow{e_{\{C^{\otimes k-1}, M\} \otimes [N, Q]^{D}}} M \otimes N
\] (9.21)
where \(e_{\{C^{\otimes k-1}, M\}} : [C^{\otimes k-1}, M] \otimes C^{\otimes k-1} \rightarrow M\) is the canonical evaluation map. We now define the contratensor product \(M \otimes_C N\) to be the coequalizer
\[
M \otimes_C N := \text{Coeq} \left( \bigoplus_{k \geq 1} \left( [C^{\otimes k-1}, M] \otimes N \xrightarrow{[C^{\otimes k-1}, M] \otimes [N, Q]^{D}} M \otimes N \right) \right)
\] (9.22)

We note that (9.22) extends the contratensor product on contramodules over coassociative coalgebras defined in (37). We now have the following result.

**Proposition 9.7.** Let \((C, w^C_*), (D, w^D_*)\) be even \(A_{\text{co}}\)-coalgebras. Then, \((N, w^C_*, w^D_*)\) induces a functor \(_{\otimes_C} : M_{[C, \_]}^{1} \rightarrow M^{D}\).

**Proof.** By (9.22), we know that for any \(M \in M_{[C, \_]},\) the contratensor product \(M \otimes_C N\) is a quotient of \(M \otimes N\). We claim that the maps \(M \otimes w^D_1 : M \otimes N \rightarrow M \otimes N \otimes D^{\otimes l-1}\) descend to the quotient \(M \otimes_C N\). For this, we note that for any \(k, l \geq 1\), we have the following commutative diagram
\[
\begin{array}{ccc}
[C^{\otimes k-1}, M] \otimes N & \xrightarrow{[C^{\otimes k-1}, M] \otimes [N, Q]^{D}} & [C^{\otimes k-1}, M] \otimes N \otimes D^{\otimes l-1} \\
\downarrow & & \downarrow \\
[C^{\otimes k-1}, M] \otimes D^{\otimes l-1} & \xrightarrow{[C^{\otimes k-1}, M] \otimes [N, Q]^{D}} & M \otimes N \otimes D^{\otimes l-1}
\end{array}
\] (9.23)
We note that the left side square in (9.23) commutes because \( N \) is a commuting \((C, D)\)-space. Also, for any \( k, l \geq 1 \), we have the commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
[C^{<k-1}, M] \otimes N & \stackrel{[C^{<k-1}, M] \otimes t^D_0 N}{\longrightarrow} & [C^{<k-1}, M] \otimes N \otimes D^{[l-1]} \\
\rightarrow
\end{array}
\end{array}
\]

(9.24)

\[
\begin{array}{c}
\begin{array}{ccc}
t^D_0 N & \longrightarrow & [t^D_0 N \otimes D^{[l-1]}] \\
\rightarrow & & \\
M \otimes N & \stackrel{M \otimes t^D_0 N}{\longrightarrow} & M \otimes N \otimes D^{[l-1]} \\
\rightarrow
\end{array}
\end{array}
\]

Considering (9.23), (9.24) and the definition in (9.22), we have an induced map on the coequalizers \( M \otimes C N \longrightarrow (M \otimes C N) \otimes D^{[l-1]} \) for each \( l \geq 1 \), making \( M \otimes C N \) into an \( A_\infty \)-comodule over \( D \).

The following simple observation will be used in the proof of the next theorem. If \((Q, w^Q_1)\) and \((Q', w^{Q'}_1)\) are \( D \)-comodules, then \( M^D(Q, Q') \) may be expressed as the equalizer

\[
M^D(Q, Q') = Eq \left\{ [Q, Q'] \stackrel{f - \prod [t^D_0 w^Q_1 \circ f]}{\longrightarrow} \prod ([Q, Q'] \otimes D^{[l-1]}) \right\} 
\]

(9.25)

Similarly, if \((M, t^M_1)\) and \((M', t^{M'}_1)\) are \( C \)-contramodules, then \( M_{|C|\setminus}(M, M') \) may be expressed as the equalizer

\[
M_{|C|\setminus}(M, M') = Eq \left\{ [M, M'] \stackrel{f - \prod [t^{M'}_1 \circ [C^{-1}, f]]}{\longrightarrow} \prod ([C^{<k-1}, M], M') \right\} 
\]

(9.26)

We are now ready to prove the main result of this section.

**Theorem 9.8.** Let \((C, w^C_1), (D, w^D_1)\) be even \( A_\infty \)-coalgebras. Then, any commuting \((C, D)\)-space \((N, w^N_1, w^N_1)\) that is even leads to an adjunction of functors

\[
M^D(N \otimes_C N, Q) \cong M_{|C|\setminus}(M, [N, Q]^D)
\]

(9.27)

for \( M \in M_{|C|\setminus} \) and \( Q \in M^D \).

**Proof.** Using the definition in (9.22) as well as the expressions in (9.25) and (9.26), we can now compute directly

\[
M^D(M \otimes_C N, Q)
\]

\[
= Eq \left\{ [M \otimes_C N, Q] \stackrel{f - \prod [t^D_0 w^Q_1 \circ f]}{\longrightarrow} \prod ([M \otimes_C N, Q \otimes D^{[l-1]}]) \right\}
\]

\[
= Eq \left\{ \bigwedge_{k, l} \Coeq \left( \bigwedge_{k, l} [C^{<k-1}, M] \otimes N \rightarrow M \otimes N, Q \right) \bigwedge_{k, l} \Coeq \left( \bigwedge_{k, l} [C^{<k-1}, M] \otimes N \rightarrow M \otimes N, Q \otimes D^{[l-1]} \right) \right\}
\]

\[
= Eq \left\{ \bigwedge_{k, l} [M \otimes N, Q] \bigwedge_{k, l} \bigwedge_{k, l} [C^{<k-1}, M] \otimes N \rightarrow M \otimes N, Q \otimes D^{[l-1]} \right\}
\]

\[
= Eq \left\{ \bigwedge_{k, l} [M, [N, Q] \otimes D^{[l-1]}] \bigwedge_{k, l} [C^{<k-1}, M] \otimes N \rightarrow M \otimes N, Q \otimes D^{[l-1]} \right\}
\]

\[
= Eq \left\{ \bigwedge_{k, l} [M, [N, Q] \otimes D^{[l-1]}] \bigwedge_{k, l} [C^{<k-1}, M] \otimes N \rightarrow M \otimes N, Q \otimes D^{[l-1]} \right\}
\]

We conclude with the following result.
Corollary 9.9. Let \((C, w^C), (D, w^D)\) be even \(A_\infty\)-coalgebras. Let \(N_1\) be an even left \(A_\infty\)-comodule over \(C\) and \(N_2\) be an even right \(A_\infty\)-comodule over \(D\). Then, we have an induced adjunction of functors, i.e., natural isomorphisms
\[
M^D(M \otimes_C (N_1 \otimes N_2), Q) \cong M_{\mathcal{C}_C}((N_1 \otimes N_2), Q^D) \tag{9.28}
\]
for \(M \in M_{\mathcal{C}_C}\) and \(Q \in M^D\).

Proof. If \(N_1\) is a left \(C\)-comodule and \(N_2\) is a right \(D\)-comodule, then \(N_1 \otimes N_2\) carries left \(C\)-coactions and right \(D\)-coactions in the obvious manner. It is clear that the left \(C\)-coactions commute with the right \(D\)-coactions, making \(N_1 \otimes N_2\) a commuting \((C, D)\)-space. The result now follows from Theorem 9.8. \(\square\)

10 Appendix: Basic Definitions

For the convenience of the reader, we recall in this section the well known definitions of \(A_\infty\)-algebras, \(A_\infty\)-coalgebras as well as modules and comodules over them respectively.

Definition 10.1. An \(A_\infty\)-algebra over \(K\) is a \(\mathbb{Z}\)-graded vector space \(A = \bigoplus_{n \in \mathbb{Z}} A_n\) along with a family of graded \(K\)-linear maps \(\{m_k : A^\otimes k \to A\}_{k \geq 1}\) with \(m_k\) of degree \(2 - k\), satisfying for each \(n \geq 1\):
\[
\sum (-1)^{p+qr} m_{p+1+r} (1^\otimes p \otimes m_q \otimes 1^\otimes r) = 0 \tag{10.1}\]
where 1 denotes the identity map and the sum is over all triples \((p, q, r)\) such that \(n = p + q + r\), \(p, r \geq 0, q \geq 1\).

We should mention that the terms in (10.1) will have additional signs when applied to elements of \(A\), following the usual Koszul sign conventions.

Definition 10.2. Let \(A\) and \(B\) be \(A_\infty\)-algebras. A morphism of \(A_\infty\)-algebras \(f : A \to B\) is a family of graded \(K\)-linear maps \(\{f_k : A^\otimes k \to B\}_{k \geq 1}\) with \(f_k\) of degree \(1 - k\), satisfying for each \(n \geq 1\):
\[
\sum (-1)^{p+qr} f_{p+1+r} (1^\otimes p \otimes m_q \otimes 1^\otimes r) = \sum (-1)^s m_i (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_s}) \tag{10.2}\]
where the first sum is over all \((p, q, r)\) such that \(n = p + q + r\) and the second sum is over all \(1 \leq t \leq n\) and all decompositions \(n = i_1 + i_2 + \cdots + i_t\), and the sign \(s\) is given by \(s = (t-1)(i_1-1) + (t-2)(i_2-1) + \cdots + 2(i_{t-2} - 1) + (i_{t-1} - 1)\).

Definition 10.3. Let \(A\) be an \(A_\infty\)-algebra over \(K\). A (left) \(A_\infty\)-module over \(A\) is a \(\mathbb{Z}\)-graded space \(M\) with maps \(\{m_k^M : A^\otimes (k-1) \otimes M 	o M\}_{k \geq 1}\) with \(m_k^M\) of degree \(2 - k\), satisfying for each \(n \geq 1\):
\[
\sum (-1)^{p+qr} m_{p+1+r}^M (1^\otimes A \otimes m_q \otimes 1^\otimes A) + \sum (-1)^a m_{a+1}^M (1^\otimes A \otimes m_b^M) = 0,
\]
where the first sum is over all \(p, q, r\) such that \(p + q + r = n\), \(p \geq 0, q, r \geq 1\) and the second sum is over all \(a, b\) such that \(a + b = n\), \(a \geq 0\) and \(b \geq 1\).

A morphism of left \(A_\infty\)-modules \(f : M \to N\) is a family of maps \(f = \{f_k : M \to N\}_{k \geq 2}\) that is well behaved with respect to the structure maps of \(M\) and \(N\). The category of left \(A_\infty\)-modules over \(A\) will be denoted by \(A\mathcal{M}\). Similarly, we may define the category \(A\mathcal{A}\) of right \(A\)-modules.

Definition 10.4. Let \(M, N\) be left \(A_\infty\)-modules over the \(A_\infty\)-algebra \(A\), with respective structure maps \(\{m_k^M : A^\otimes (k-1) \otimes M \to M\}_{k \geq 1}\) and \(\{m_k^N : A^\otimes (k-1) \otimes N \to N\}_{k \geq 1}\). An \(A_\infty\)-morphism of (left) \(A_\infty\)-modules \(f : M \to N\) is a family of maps
\[
f = \{f_k : A^\otimes (k-1) \otimes M \to N\}_{k \geq 1}
\]
with \(f_k\) of degree \(1 - k\), satisfying for each \(n \geq 1\):
\[
\sum (-1)^a f_{a+1} (1^\otimes A \otimes m_b^M) + \sum (-1)^{a+bc} f_{a+1+c} (1^\otimes A \otimes m_b \otimes 1^\otimes A) = \sum m_{a+1}^N (1^\otimes A \otimes f_b).
\]
where:
(1) the first sum on the left side is taken over all \(a, b\) such that \(n = a + b, a \geq 0, b \geq 1\)
(2) the second sum on the left side is taken over all \(a, b, c\) such that \(n = a + b + c, a \geq 0, b, c \geq 1\)
(3) the sum on the right side is taken over all \(a, b\) such that \(n = a + b, a \geq 0\) and \(b \geq 1\).

**Definition 10.5.** An \(A_{\infty}\)-coalgebra over \(K\) is a \(\mathbb{Z}\)-graded vector space \(C = \bigoplus C_n\) along with a family of graded \(K\)-linear maps \(\{w_k : C \rightarrow C^{\otimes k}\}_{k \geq 1}\) with \(w_k\) of degree \(2 - k\), satisfying the following two conditions:
(a) For each \(n \geq 1\), we have
\[
\sum (-1)^{pq+r}(1^{\otimes p} \otimes w_q \otimes 1^{\otimes r})w_{p+q+1} = 0, \quad \forall \ k \geq 1,
\]
where \(1\) denotes the identity map, and the sum is over all \(p, q, r\) such that \(n = p + q + r, p, r \geq 0, q \geq 1\).
(b) The induced morphism \(\prod_{k \geq 1} w_k : C \rightarrow \prod_{k \geq 1} C^{\otimes k}\) factors through the direct sum \(\bigoplus_{k \geq 1} C^{\otimes k}\).

**Definition 10.6.** Let \(C\) be an \(A_{\infty}\)-coalgebra over \(K\) with structure maps \(\{w_k : C \rightarrow C^{\otimes k}\}_{k \geq 1}\). A (left) \(A_{\infty}\)-comodule over \(C\) is a \(\mathbb{Z}\)-graded space \(P\) with maps \(\{w_k^P : P \rightarrow C^{\otimes k-1} \otimes P\}_{k \geq 1}\) with \(w_k^P\) of degree \(2 - k\), satisfying, for each \(n \geq 1\), we have
\[
\sum (-1)^{pq+r}(1^{\otimes p} \otimes w_q \otimes 1^{\otimes r})w_{p+q+1} + \sum (-1)^{ab}(1^{\otimes a} \otimes w_b \otimes 1^{\otimes r})w_{a+1} = 0,
\]
where the first sum is over all \(p, q, r\) such that \(n = p + q + r, p \geq 0, q, r \geq 1\) and the second sum is over all \(a, b\) such that \(a + b = n, a \geq 0, b \geq 1\).

A morphism of left \(A_{\infty}\)-comodules \(g : P \rightarrow Q\) is a family of maps \(g = \{g_l : P_l \rightarrow Q_l\}_{l \in \mathbb{Z}}\) that is well behaved with respect to the structure maps of \(P\) and \(Q\). The category of left \(A_{\infty}\)-comodules over \(C\) will be denoted by \(\mathcal{C}M\). Additionally, we will say that \(P \in \mathcal{C}M\) is strongly finite if the induced morphism \(\prod_{k \geq 1} w_k^P : P \rightarrow \prod_{k \geq 1} (C^{\otimes k-1} \otimes P)\) factors through the direct sum \(\bigoplus_{k \geq 1} (C^{\otimes k-1} \otimes P)\).

Similarly, we may define the category \(\mathcal{MC}\) of right \(C\)-comodules.

**Definition 10.7.** Let \(C\) be an \(A_{\infty}\)-coalgebra. Let \(P, Q\) be left \(A_{\infty}\)-comodules over \(C\) with respective structure maps \(\{w_k^P : P \rightarrow C^{\otimes k-1} \otimes P\}_{k \geq 1}\) and \(\{w_k^Q : Q \rightarrow C^{\otimes k-1} \otimes Q\}_{k \geq 1}\). An \(A_{\infty}\)-morphism of comodules \(g : P \rightarrow Q\) is a sequence of graded morphisms \(\{g_k : P \rightarrow C^{\otimes k-1} \otimes Q\}_{k \geq 1}\), with \(g_k\) of degree \(1 - k\), satisfying for each \(n \geq 1\):
\[
\sum (-1)^{ab}(1^{\otimes a} \otimes g_b \otimes 1^{\otimes r})g_{a+1} = \sum (1^{\otimes a} \otimes g_b)w_{a+1},
\]
where
(1) the first sum on the left is over all \(a, b\) such that \(n = a + b, a \geq 0, b \geq 1\).
(2) the second sum on the left is over all \(a, b, c\) such that \(n = a + b + c, a \geq 0, b, c \geq 1\).
(3) the sum on the right side is over all \(a, b\) such that \(n = a + b, a \geq 0, b \geq 1\).

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