\textbf{$\mathfrak{sl}(n, H)$-Current Algebras on $S^3$}

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Abstract

Let $H$ be the quaternion numbers and $\mathfrak{gl}(n, H) = H \otimes \mathfrak{gl}(n, C)$ be the general linear algebra over $H$. We introduce three non-trivial 2-cocycles $c_k$, $k = 0, 1, 2$, on the Lie algebra $S^3 H = \text{Map}(S^3, H)$ with the aid of the corresponding vector fields that form a basis of the vector fields on $S^3$. We extend them to 2-cocycles on the Lie algebra $S^3 \mathfrak{gl}(n, H) = S^3 \mathfrak{h} \otimes \mathfrak{gl}(n, C)$. Then we have the corresponding central extension $S^3 \mathfrak{gl}(n, H)$ of $S^3 \mathfrak{gl}(n, H)$, as well as its central extension by the 2-cocycles $\{ c_k \}$: $\mathfrak{gl}(n, \hat{H})(a) = \mathfrak{gl}(n, \hat{H}) \oplus \mathfrak{sl}(n, \mathbb{C})$. Then we have the second extension of $\mathfrak{gl}(n, \hat{H})(a)$ by adding a derivation $d$ that acts on $\mathfrak{gl}(n, \hat{H})$ by the Euler vector field $d_0$: $\mathfrak{gl} = \mathfrak{gl}(n, \hat{H}) \oplus (\mathbb{C} \oplus \mathbb{C}) \oplus C \mathfrak{d}$. The Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is defined as a Lie subalgebra of $\mathfrak{gl}(n, \hat{H})$ generated by $C[\phi^+] \otimes \mathfrak{sl}(n, \mathbb{C})$. We have the corresponding central extension of $\mathfrak{sl}(n, \mathbb{C})$ by the 2-cocycles $\{ c_k \}$ and the derivation $d_0$, which becomes a Lie subalgebra of $\mathfrak{gl}$. Let $\hat{\mathfrak{sl}}$ be a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ and $\hat{\mathfrak{h}} = \mathfrak{h} \oplus (\mathbb{C} \oplus \mathbb{C}) \oplus C \mathfrak{d}$. $\mathfrak{h}$ is a commutative subalgebra of $\hat{\mathfrak{sl}}$. The root space decomposition of the $\text{ad}(\hat{\mathfrak{h}})$-representation of $\hat{\mathfrak{sl}}$ is obtained. The set of roots is $\hat{\Delta} = \{ \frac{m}{2} \alpha + \alpha; \ \alpha \in \Delta, \ m \in \mathbb{Z} \} \cup \{ \frac{m}{2} \delta; \ m \in \mathbb{Z} \}$. And the root spaces are $\hat{\mathfrak{g}}_{\delta + \alpha} = \mathbb{C}[\phi^\pm; \ m] \otimes \mathfrak{g}_\alpha$ for $\alpha \neq 0$, $\hat{\mathfrak{g}}_{\delta} = \mathbb{C}[\phi^\pm; \ m] \otimes \mathfrak{g}_0$ for $m \neq 0$, and $\hat{\mathfrak{g}}_{\delta} = (\mathbb{C}[\phi^\pm; 0] \otimes \mathfrak{g}_0) \oplus (\mathbb{C} \oplus \mathbb{C}) \mathfrak{d} \supset \hat{\mathfrak{h}}$. Here $\mathbb{C}[\phi^\pm; \ m]$ is the subspace of
those spinors that are of homogeneous degree $m$. Finally the Chevalley generators
of $\hat{s}l$ are given.

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0 Introduction

Current algebras are certain central extensions of Lie algebras of smooth mappings of
a given manifold into a finite dimensional Lie algebra. Loop algebras are the simplest
example where the manifold is 1-dimensional loop. Loop algebras appear in the simplified
model of quantum field theory where the space is one-dimensional and many important
facts in the representation theory of loop algebra were first discovered by physicists. A
loop algebra with the value in the complexification of a simple Lie algebra turned out
to behave like a simple Lie algebra and highly developed theory of finite dimensional Lie
algebra was extended to such loop algebras, [C], [K], [P-S] and [W]. A loop algebra is
also called an affine Lie algebra and the highest weight theory of finite dimensional Lie
algebra was extended to affine algebras. In this paper we shall investigate a generalization
of affine Lie algebras to the Lie algebra of mappings from three-sphere $S^3$ to a Lie algebra.
As an affine Lie algebra is a central extension of the Lie algebra of smooth mappings from
$S^1$ to the complexification of a Lie algebra, so our objective is a central extension of the
Lie algebra of smooth mappings from $S^3$ to the quaternionification of a Lie algebra.

Section 1 is devoted to a rather long introduction to our previous results on analysis
of quaternion valued functions (spinors) on $R^4$ that were developed in [G-M] [Ko1, Ko2]
and [K-I], since these subjects seem not to be familiar. Let $H$ be the quaternion numbers.
$H$ becomes an associative algebra and the Lie algebra structure is induced on it. The
space $\Delta = H \otimes C = H \oplus H$ gives an irreducible representation of the Clifford algebra
Clif$_4(C)$. The Dirac matrices

$$
\gamma_k = \begin{pmatrix}
0 & -i\sigma_k \\
-i\sigma_k & 0
\end{pmatrix}, \quad k = 1, 2, 3, \quad \gamma_4 = \begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix},
$$

(0.1)

where $\sigma$'s are Pauli matrices, gives the generators of Clif$_4(C) \simeq \text{End}(\Delta) = \mathfrak{gl}(2, H)$. The
representation $\Delta$ decomposes into irreducible representations $\Delta^\pm \simeq H$ of Spin(4). Let
$S = C^2 \times \Delta$ be the trivial spinor bundle on $C^2$. The corresponding bundle $S^+ = C^2 \times \Delta^+$.
(resp. $S^c = \mathbb{C}^2 \times \Delta^-$) is called the even (resp. odd) spinor bundle and the sections are called even (resp. odd) spinors. The Dirac operator is given by

$$D = -\frac{\partial}{\partial x_1}\gamma_4 - \frac{\partial}{\partial x_2}\gamma_3 - \frac{\partial}{\partial x_3}\gamma_2 - \frac{\partial}{\partial x_4}\gamma_1 : C^\infty(M, S) \to C^\infty(M, S).$$

(0.2)

Now we fix the following basis of the vector fields on $\{|z| = 1\} \simeq S^3$:

$$\theta = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2},$$

$$e_+ = -z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2}, \quad e_- = -\bar{z}_2 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_2}.$$

The radial vector field is defined by

$$\frac{\partial}{\partial n} = \frac{1}{2|z|}(\nu + \bar{\nu}), \quad \nu = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$  

(0.3)

We shall denote by $\gamma$ the Clifford multiplication of the radial vector $\frac{\partial}{\partial n}$. $\gamma$ changes the chirality: $\gamma = \gamma_+ \oplus \gamma_- : S^+ \oplus S^- \to S^- \oplus S^+$, and $\gamma^2 = 1$. Then the half spinor Dirac operator $D = D|S^+$ has the polar decomposition:

$$D = \gamma_+ \left( \frac{\partial}{\partial n} - \hat{\nabla} \right),$$

(0.4)

with the tangential (nonchiral) Dirac operator $\hat{\nabla}$ given by

$$\hat{\nabla} = -\left[ \sum_{i=1}^3 \left( \frac{1}{|z|} \theta_i \right) \cdot \nabla_{\frac{\partial}{\partial \theta_i}} \right] = \frac{1}{|z|} \left( \begin{array}{c} -\frac{1}{2} \theta \quad e_+ \\ -e_- \quad \frac{1}{2} \theta \end{array} \right).$$

The tangential Dirac operator $\hat{\nabla}$ on the sphere $S^3 = \{|z| = 1\}$ is a self adjoint elliptic differential operator. We proved in [Ko1, Ko2] the following facts. The eigenvalues of $\hat{\nabla}$ are $\left\{ \frac{m}{2}, -\frac{m+3}{2} ; m = 0, 1, \cdots \right\}$ with multiplicity $(m+1)(m+2)$. We have an explicitly written polynomial formula for eigenspinors $\left\{ \phi^{+(m,l,k)}, \phi^{-(m,l,k)} \right\}_{0 \leq l \leq m, 0 \leq k \leq m+1}$ corresponding to the eigenvalue $\frac{m}{2}$ and $-\frac{m+3}{2}$ respectively, and they give rise to a complete orthonormal system in $L^2(S^3, S^+)$. A spinor $\phi$ on a domain $G \subset \mathbb{C}^2$ is called a harmonic spinor on $G$ if $D\phi = 0$. Each $\phi^{+(m,l,k)}$ is extended to a harmonic spinor on $\mathbb{C}^2$, while each $\phi^{-(m,l,k)}$ is extended to a harmonic spinor on $\mathbb{C}^2 \setminus \{0\}$. Every harmonic spinor $\varphi$ on $\mathbb{C}^2 \setminus \{0\}$ has a
Laurent series expansion by the basis $\phi^{\pm(m,l,k)}$:

$$\varphi(z) = \sum_{m,l,k} C_{+,(m,l,k)} \phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)} \phi^{-(m,l,k)}(z). \tag{0.5}$$

If only finitely many coefficients are non-zero it is called a spinor of Laurent polynomial type. The set of spinors of Laurent polynomial type is denoted by $\mathbb{C}[\phi^\pm]$. In [K-I] we proved that the restriction of $\mathbb{C}[\phi^\pm]$ to $S^3$ becomes an associative subalgebra of $\text{Map}(S^3, \mathbb{H})$. We must note that $\mathbb{C}[\phi^\pm]$ itself is not an algebra.

Let $S^3\mathbb{H} = \text{Map}(S^3, \mathbb{H}) = C^\infty(S^3, S^+)$ be the set of smooth even spinors on $S^3$. There is a correspondence $S^3\mathbb{H} \ni u + jv \longleftrightarrow \phi = \begin{pmatrix} u \\ v \end{pmatrix} \in C^\infty(S^3, S^+)$. We define the Lie algebra structure on $S^3\mathbb{H}$ by the Lie bracket

$$[ u_1 + jv_1, u_2 + jv_2 ] = (v_1 \bar{v}_2 - \bar{v}_1 v_2) + j((u_2 - \bar{u}_2)v_1 - (u_1 - \bar{u}_1)v_2). \tag{0.6}$$

Or we write, for even spinors $\phi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$,

$$[ \phi_1, \phi_2 ] = \begin{pmatrix} v_1 \bar{v}_2 - \bar{v}_1 v_2 \\ (u_2 - \bar{u}_2)v_1 - (u_1 - \bar{u}_1)v_2 \end{pmatrix}, \tag{0.7}$$

Let $\theta_0 = \sqrt{-1}\theta$, $\theta_1 = e_+ + e_-$, $\theta_2 = \sqrt{-1}(e_+ - e_-)$. For $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$ we put

$$\Theta_k \phi = \Theta_k \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \theta_k u \\ \theta_k v \end{pmatrix}, \quad k = 0, 1, 2.$$

And we introduce the following non-trivial 2-cocycles $c_k$, $k = 0, 1, 2$ on the Lie algebra $S^3\mathbb{H}$:

$$c_k(\phi_1, \phi_2) = \frac{1}{2\pi^2} \int_{S^3} \text{tr}(\Theta_k \phi_1 \cdot \phi_2) \, d\sigma, \tag{0.8}$$

for $\phi_1$ and $\phi_2 \in S^3\mathbb{H}$.

Let $\mathfrak{gl}(n, \mathbb{H}) = \mathbb{H} \otimes \mathfrak{gl}(n, \mathbb{C})$ be the algebra of $n \times n$-matrices with entries in $\mathbb{H}$. We
have the $R$-algebra isomorphism:

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{m}_j(2n, \mathbb{C}).$$ (0.9)

$\mathfrak{gl}(n, \mathbb{H})$ is a Lie algebra defined by the $R$-linear bracket:

$$[A_1 + JB_1, A_2 + JB_2] = (A_1 A_2 - A_2 A_1) - (\overline{B}_1 B_2 - \overline{B}_2 B_1) + J(\bar{A}_1 B_2 - B_2 A_1 + B_1 A_2 - \overline{A}_2 B_1),$$ (0.10)

for $A_1 + JB_1, A_2 + JB_2 \in \mathfrak{gl}(n, \mathbb{H})$. More conveniently, if we write $E_{ij}$ the $n \times n$-matrix with entry 1 at $(i,j)$-place and 0 otherwise, $\{E_{ij}\}_{i,j}$ is the basis of $\mathfrak{gl}(n, \mathbb{C})$ and we have

$$[z_1 \otimes E_{ij}, z_2 \otimes E_{kl}] = (z_1 z_2) \otimes \delta_{jk} E_{ij} - (z_2 z_1) \otimes \delta_{il} E_{kl},$$ (0.11)

for $z_1, z_2 \in \mathbb{H}$.

Let $S^3 \mathfrak{gl}(n, \mathbb{H}) = S^3 \mathbb{H} \otimes \mathfrak{gl}(n, \mathbb{C})$ be the Lie algebra endowed with the bracket:

$$[\phi \otimes E_{ij}, \psi \otimes E_{kl}]_{S^3 \mathfrak{gl}(n, \mathbb{H})} = (\phi \cdot \psi) \otimes \delta_{jk} E_{ij} - (\psi \cdot \phi) \otimes \delta_{il} E_{kl},$$ (0.12)

where $\phi, \psi \in S^3 \mathbb{H}$ and $\{E_{ij}; 1 \leq i, j \leq n\}$ is the basis of $\mathfrak{gl}(n, \mathbb{C})$. Each 2-cocycle $c_k, k = 0, 1, 2$, is extended to $S^3 \mathfrak{gl}(n, \mathbb{H})$ by the formula

$$c_k(\phi \otimes X, \psi \otimes Y) = \text{Trace}(XY) c_k(\phi, \psi), \quad \phi, \psi \in S^3 \mathbb{H}, \, X, Y \in \mathfrak{gl}(n, \mathbb{C}).$$

Then we have the associated central extension: $S^3 \mathfrak{gl}(n, \mathbb{H}) \oplus (\oplus_{k=0}^2 \mathbb{C} a_k)$.

As a Lie subalgebra of $S^3 \mathfrak{gl}(n, \mathbb{H})$ we have the Lie algebra $\mathbf{C}[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C})$ of $\mathfrak{gl}(n, \mathbb{C})$-valued Laurent polynomial spinors on $S^3$. We denote it by $\hat{\mathfrak{gl}}(n, \mathbb{H})$. $\hat{\mathfrak{gl}}(n, \mathbb{H})$ has the central extension by the 2-cocycles $c_k, k = 0, 1, 2$, as well:

$$\hat{\mathfrak{gl}}(n, \mathbb{H})(a) = \hat{\mathfrak{gl}}(n, \mathbb{H}) \oplus (\oplus_{k=0}^2 \mathbb{C} a_k).$$

Next the radial derivative is by definition $d_0 = \frac{1}{2} (\nu + \overline{\nu})$ and acts on $C^\infty(S^3)$ and then on $\text{Map}(S^3, \mathbb{H})$. We see that $d_0$ preserves the space $\mathbf{C}[\phi^\pm]$ of spinors of Laurent polynomial type on $S^3$. In fact $d_0$ acts on $\mathbf{C}[\phi^\pm]|S^3$ as a derivation. We extend the derivation $d_0$ on
\(C[\phi^\pm]\) to that on \(\widehat{\mathfrak{gl}(n, H)}\). Then we have the central extension of \(\widehat{\mathfrak{gl}(n, H)}(a)\) by \(d_0\):

\[
\widehat{\mathfrak{gl}} = C[\phi^\pm] \otimes \mathfrak{gl}(n, C) \oplus \left( \bigoplus_{k=0}^2 Ca_k \right) \oplus C d. \tag{0.13}
\]

Here we used the relation

\[
c_k( d_0 \phi_1, \phi_2 ) + c_k( \phi_1, d_0 \phi_2 ) = 0. \tag{0.14}
\]

As we have seen hitherto the Lie bracket on \(\widehat{\mathfrak{gl}}\) is given by

\[
\begin{align*}
[\phi \otimes X, \psi \otimes Y]_{\widehat{\mathfrak{gl}}} &= (\phi \cdot \psi) \otimes (XY) - (\psi \cdot \phi) \otimes (YX) + (X|Y) \sum_{k=0}^2 c_k(\phi, \psi) a_k, \\
[a_k, \phi \otimes X]_{\widehat{\mathfrak{gl}}} &= 0, \quad [d, a_k]_{\widehat{\mathfrak{gl}}} = 0, \quad k = 0, 1, 2, \tag{0.15} \\
[d, \phi \otimes X]_{\widehat{\mathfrak{gl}}} &= d_0 \phi \otimes X. \tag{0.16}
\end{align*}
\]

for \(\phi, \psi \in C[\phi^\pm]\) and any base \(X, Y\) of \(\mathfrak{gl}(n, C)\).

In section 2 we construct (rather give a convincing definition of) a quaternionification of \(\text{Map}(S^3, \mathfrak{sl}(n, C))\) and its central extensions. Let \(\mathfrak{sl}(n, H)\) denote the quaternion special linear algebra. We have the \(\mathbb{R}\)-linear isomorphism

\[
\mathfrak{sl}(n, H) \ni A + JB \xrightarrow{\sim} \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{gl}(2n, C) : \ A \in \mathfrak{sl}(n, C) \right\}, \tag{0.17}
\]

where

\[
\mathfrak{sl}(n, C) = \{ A \in \mathfrak{gl}(n, C) ; \ \text{Trace} A \in \sqrt{-1} \mathbb{R} \}. \tag{0.18}
\]

So \(\mathfrak{sl}(n, H) = \mathfrak{sl}(n, C) + J \mathfrak{gl}(n, C)\). Note that \(H \otimes \mathfrak{sl}(n, C) = \mathfrak{sl}(n, C) + J \mathfrak{sl}(n, C)\) is not a Lie algebra. We see that the \(H\)-module \(H \otimes \mathfrak{sl}(n, C)\) generates the Lie algebra \(\mathfrak{sl}(n, H)\). Things being so we shall introduce the current algebra \(\widehat{\mathfrak{sl}(n, H)}\) as the Lie subalgebra of \(\widehat{\mathfrak{gl}(n, H)}\) that is generated by \(C[\phi^\pm] \otimes \mathfrak{sl}(n, C)\). By the 2-cocycles \(c_k, k = 0, 1, 2,\) of (0.8) we have the central extension

\[
\widehat{\mathfrak{sl}(n, H)}(a) = \mathfrak{sl}(n, H) \oplus \left( \bigoplus_{k=0}^2 Ca_k \right). \tag{0.19}
\]
Further we have the central extension of $\mathfrak{sl}(n, \mathbb{H})(a)$ by the derivation $d_0$:

$$\hat{\mathfrak{s}\mathfrak{l}} = \mathfrak{sl}(n, \mathbb{H}) \oplus (\bigoplus_{k=0}^{2} \mathbb{C} a_k) \oplus (\mathbb{C} d).$$

(0.20)

These are Lie subalgebras of $\hat{\mathfrak{g}l} = \mathfrak{gl}(n, \mathbb{H}) \oplus (\bigoplus_{k=0}^{2} \mathbb{C} a_k) \oplus (\mathbb{C} d)$.

Finally we discuss the root space decomposition of $\hat{\mathfrak{s}\mathfrak{l}}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$, and let $\{\alpha_i; i = 1, \cdots, r = n - 1\}$ be the set of simple roots of $\mathfrak{sl}(n, \mathbb{C})$ and $\{\alpha_i^\vee; i = 1, \cdots, r\}$ be the set of simple coroots. Let $(a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle)_{i,j=1,\cdots,r}$ be the Cartan matrix. Let

$$\hat{\mathfrak{h}} = \left( (\mathbb{C} \phi^+(0.0,1)) \otimes \mathfrak{h} \right) \oplus (\bigoplus_{k=0}^{2} \mathbb{C} a_k) \oplus (\mathbb{C} d) = \mathfrak{h} \oplus (\bigoplus_{k=0}^{2} \mathbb{C} a_k) \oplus (\mathbb{C} d).$$

(0.21)

Since $\hat{\mathfrak{h}}$ is a commutative subalgebra of $\hat{\mathfrak{s}\mathfrak{l}}$, $\hat{\mathfrak{s}\mathfrak{l}}$ is decomposed into a direct sum of the simultaneous eigenspaces of $ad(\hat{\mathfrak{h}})$, $\hat{\mathfrak{h}} \in \hat{\mathfrak{h}}$. An element $\lambda$ of the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ is regarded as an element of $\hat{\mathfrak{h}}^*$ by putting $\langle \lambda, a \rangle = \langle \lambda, d \rangle = 0$. So $\Delta \subset \mathfrak{h}^*$ is a subset of $\hat{\mathfrak{h}}^*$.

We define $\delta$ and $\Lambda_k \in \hat{\mathfrak{h}}^*$ for $k = 0, 1, 2$, as follows: $\langle \delta, \alpha_i \rangle = \langle \Lambda_k, \alpha_i \rangle = 0$ for $1 \leq i \leq r$, $\langle \delta, a_k \rangle = 0, \langle \delta, d \rangle = 1, \langle \Lambda_k, a_k \rangle = 1$ and $\langle \Lambda_k, d \rangle = 0$. Then the set $\{\alpha_1, \cdots, \alpha_r, \Lambda_0, \delta\}$ forms a basis of $\hat{\mathfrak{h}}^*$. For $\lambda = \alpha + k_0 \delta \in \hat{\mathfrak{h}}^*$, $\alpha = \sum_{i=1}^{r} k_i \alpha_i \in \Delta$, $k_i \in \mathbb{Z}$, $i = 0, 1, \cdots, r$, we put,

$$\hat{\mathfrak{g}}_\lambda = \left\{ \xi \in \mathfrak{sl}; [\hat{\mathfrak{h}}, \xi] = \langle \lambda, \hat{\mathfrak{h}} \rangle \xi \quad \text{for all} \ \hat{\mathfrak{h}} \in \hat{\mathfrak{h}} \right\}.$$ (0.22)

$\lambda$ is a root of $\hat{\mathfrak{s}\mathfrak{l}}$ if $\hat{\mathfrak{g}}_\lambda \neq 0$. $\hat{\mathfrak{g}}_\lambda$ is called the root space of $\lambda$.

The set of roots of the representation $\left(\hat{\mathfrak{s}\mathfrak{l}}, ad(\hat{\mathfrak{h}})\right)$ is

$$\hat{\Delta} = \left\{ \frac{m}{2} \delta + \alpha; \ \alpha \in \Delta, m \in \mathbb{Z} \right\} \cup \left\{ \frac{m}{2} \delta; \ m \in \mathbb{Z} \right\}.$$ (0.23)

$\hat{\mathfrak{s}\mathfrak{l}}$ has the weight space decomposition:

$$\hat{\mathfrak{s}\mathfrak{l}} = \bigoplus_{m \in \mathbb{Z}} \hat{\mathfrak{g}}_{\frac{m}{2} \delta} \bigoplus \bigoplus_{\alpha \in \Delta, m \in \mathbb{Z}} \hat{\mathfrak{g}}_{\frac{m}{2} \delta + \alpha}. $$ (0.24)

Each weight space is given as follows.

$$ \hat{\mathfrak{g}}_{\frac{m}{2} \delta + \alpha} = \mathbb{C}[\phi^\pm; m] \otimes \mathfrak{g}_\alpha, \quad \text{for } \alpha \neq 0 \text{ and } m \in \mathbb{Z}, $$

$$ \hat{\mathfrak{g}}_{0 \delta} = (\mathbb{C}[\phi^\pm; 0] \otimes \mathfrak{g}_0) \oplus (\bigoplus_{k=0}^{2} \mathbb{C} a_k) \oplus (\mathbb{C} d) \supset \hat{\mathfrak{h}}, $$

$$ \hat{\mathfrak{g}}_{\frac{m}{2} \delta} = \mathbb{C}[\phi^\pm; m] \otimes \mathfrak{g}_0, \quad \text{for } 0 \neq m \in \mathbb{Z}. $$
Where $C[\phi^\pm; m]$ is the subspace of $C[\phi^\pm]$ consisting of those elements that are of homogeneous order $m$: $\phi(z) = |z|^m \phi(\frac{z}{|z|})$.

Finally the generators of $\hat{sl}$ will be given by the set of generators $(h_j, x_i, y_i)$ plus $(h_\theta, x_\theta, y_\theta)$, where $\theta$ is the highest root of $\hat{sl}(n, C)$, that is combined with the coefficients consisting the generators of the algebra $C[\phi^\pm]$.

1 Preliminaries on quaternionic analysis on $S^3$

Here we prepare a sufficiently long preliminary because I think various subjects belonging to quaternion analysis or detailed properties of harmonic spinors of the Dirac operator on $C^2$ are not so familiar to the readers.

1.1 Quaternions $H$

Let $H$ be the quaternion numbers. A general quaternion is of the form $x = x_1 + x_2i + x_3j + x_4k$ with $x_1, x_2, x_3, x_4 \in \mathbb{R}$. By taking $x_3 = x_4 = 0$ the complex numbers $C$ are contained in $H$ if we identify $i$ as the usual complex number. Every quaternion $x$ has a unique expression $x = z_1 + jz_2$ with $z_1, z_2 \in C$. This identifies $H$ with $C^2$ as $C$-vector spaces. The quaternion multiplication will be from the right $x \rightarrow xy$ where $y = w_1 + jw_2$ with $w_1, w_2 \in C$:

$$xy = (z_1 + jz_2)(w_1 + jw_2) = (z_1w_1 - z_2w_2) + j(z_1w_2 + z_2w_1).$$

(1.1)

Correspondingly $C^2$ becomes an associative algebra with the multiplication

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} z_1w_1 - z_2w_2 \\ \bar{z}_1w_2 + z_2w_1 \end{pmatrix}. \quad (1.2)$$

The multiplication of a $g = a + jb \in H$ to $H$ from the left yields an endomorphism in $H$: $\{x \rightarrow gx\} \in End_H(H)$. Under the identification $H \simeq C^2$ mentioned above it is the $C$-linear map

$$C^2 \ni \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in C^2. \quad (1.3)$$
This establishes the $\mathbb{R}$-linear isomorphism

$$\mathbf{H} \ni a + jb \xrightarrow{\sim} \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathfrak{mj}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \right\}. \quad (1.4)$$

It becomes a ring isomorphism over $\mathbb{R}$:

$$\mathbf{H} \simeq \text{End}_{\mathbf{H}}(\mathbf{H}) \simeq \mathfrak{mj}(2, \mathbb{C}). \quad (1.5)$$

The matrices corresponding to $i, j, k \in \mathbf{H}$ are

$$e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \quad (1.6)$$

For our later convenience we note also the relation to the Pauli matrices: $e_1 = -i\sigma_1, e_2 = -i\sigma_2, e_3 = i\sigma_3$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.7)$$

so that $\mathbf{H}$ is the Clifford algebra $\text{Clif}_{4}(\mathbb{C}) = \text{Clif}(\mathbb{R}^2, x_1^2 + x_2^2) \otimes \mathbb{C}$ that is generated by \{\sigma_1, \sigma_2\}.

1.2 $\mathfrak{gl}(2, \mathbf{H})$

Let $\Delta = \mathbf{H} \otimes \mathbb{C} = \mathbf{H} \oplus \mathbf{H}$ be an irreducible representation of the Clifford algebra $\text{Clif}_{4}(\mathbb{C})$:

$$\text{Clif}_{4}(\mathbb{C}) = \text{Clif}(\mathbb{R}^4) \otimes \mathbb{C} \simeq \text{End}(\Delta) = \mathfrak{gl}(2, \mathbf{H}). \quad (1.8)$$

We put

$$\mathfrak{mj}(4, \mathbb{C}) = \left\{ \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} : A, B \in \mathfrak{gl}(2, \mathbb{C}) \right\}. \quad (1.9)$$

As in the previous section there is a ring isomorphism over $\mathbb{R}$:

$$\mathfrak{gl}(2, \mathbf{H}) \simeq \mathfrak{mj}(4, \mathbb{C}), \quad (1.10)$$
given by

\[ A + jB \rightarrow \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : \ A, B \in \mathfrak{gl}(2, \mathbb{C}) \right\}. \tag{1.11} \]

We have the isomorphism of \( \mathbb{C} \)-algebra:

\[ \text{Clif}^c_4(\mathbb{C}) = \text{Mat}_2(\text{Clif}^c_2(\mathbb{C})), \tag{1.12} \]

where \( \text{Mat}_2(E) \) is the algebra of \( 2 \times 2 \) matrices with entries in the algebra \( E \). \( \text{Clif}^c_2(\mathbb{C}) \) being generated by \( \{ \sigma_i ; i = 1, 2 \} \), \( \text{Clif}^c_4(\mathbb{C}) \) is generated by the following Dirac matrices:

\[ \gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad \gamma_4 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}. \tag{1.13} \]

The set

\[ \{ \gamma_p, \gamma_p\gamma_q, \gamma_p\gamma_q\gamma_r, \gamma_p\gamma_q\gamma_r\gamma_s ; \ 1 \leq p, q, r, s \leq 4 \} \tag{1.14} \]

gives a 16-dimensional basis of the representation \( \text{Clif}^c_4(\mathbb{C}) \simeq \text{End}(\Delta) \) with the following relations:

\[ \gamma_p\gamma_q + \gamma_q\gamma_p = 2\delta_{pq}. \tag{1.15} \]

The representation \( \Delta \) decomposes into irreducible representations \( \Delta^\pm = H \) of Spin(4). Let \( S = \mathbb{C}^2 \times \Delta \) be the trivial spinor bundle on \( \mathbb{C}^2 \). The corresponding bundle \( S^+ = \mathbb{C}^2 \times \Delta^+ \) ( resp. \( S^- = \mathbb{C}^2 \times \Delta^- \) ) is called the even ( resp. odd ) spinor bundle and the sections are called even ( resp. odd ) spinors. In the following we look a even spinor on a subset \( M \subset \mathbb{C}^2 \) like as a smooth function on \( M \) valued in \( H : \mathcal{C}\infty(M, H) = \mathcal{C}\infty(M, S^+) \).

We feel free to use the alternative notation as in (1.4) or (1.11) to write a spinor:

\[ \mathcal{C}\infty(M, H) \ni u + jv \longleftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{C}\infty(M, S^+). \tag{1.16} \]

The Dirac operator is defined by

\[ \mathcal{D} = c \circ d : \mathcal{C}\infty(M, S) \rightarrow \mathcal{C}\infty(M, S). \tag{1.17} \]

where \( d : S \rightarrow S \otimes T^*\mathbb{C}^2 \simeq S \otimes T\mathbb{C}^2 \) is the exterior differential and \( c : S \otimes T\mathbb{C}^2 \rightarrow S \) is the bundle homomorphism coming from the Clifford multiplication. By means of the
decomposition $S = S^+ \oplus S^-$ the Dirac operator has the chiral decomposition:

$$
D = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} : C^\infty(\mathbb{C}^2, S^+ \oplus S^-) \to C^\infty(\mathbb{C}^2, S^+ \oplus S^-).
$$

(1.18)

With respect to the Dirac matrices $\gamma_j, j = 1, 2, 3, 4$, the Dirac operator has the formula

$$
D = -\frac{\partial}{\partial x_1} \gamma_4 - \frac{\partial}{\partial x_2} \gamma_3 - \frac{\partial}{\partial x_3} \gamma_2 - \frac{\partial}{\partial x_4} \gamma_1.
$$

(1.19)

If we adopt the notation

$$
\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_3} - i \frac{\partial}{\partial x_4},
$$

$D$ and $D^\dagger$ have the following coordinate expressions;

$$
D = \begin{pmatrix} \frac{\partial}{\partial z_1} & -\frac{\partial}{\partial z_2} \\ \frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1} \end{pmatrix}, \quad D^\dagger = \begin{pmatrix} \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} \\ -\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1} \end{pmatrix}.
$$

(1.20)

By virtue of (1.10) we have also the following quaternion expression:

$$
D = \frac{\partial}{\partial z_1} + j \frac{\partial}{\partial z_2}, \quad D^\dagger = \frac{\partial}{\partial z_1} - j \frac{\partial}{\partial z_2}.
$$

(1.21)

1.3 Harmonic spinors

An even (resp. odd) spinor $\varphi$ is called a harmonic spinor if $D\varphi = 0$ (resp. $D^\dagger \varphi = 0$).

We shall introduce a set of harmonic spinors which, restricted to $S^3$, forms a complete orthonormal system of $L^2(S^3, S^+)$. First we fix the following basis of the vector fields on $\{|z| = 1\} \simeq S^3$.

$$
e_+ = -z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2}, \quad e_- = -\bar{z}_2 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial z_2}
$$

(1.22)

$$
\theta = (z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2})
$$

(1.23)
We have the commutation relations;

\[
[\theta, e_+] = 2e_+, \quad [\theta, e_-] = -2e_-, \quad [e_+, e_-] = -\theta.
\] (1.24)

The dual basis of \((e^*_+, e^*_-, \theta^*)\) are the differential 1-forms

\[
e^*_+ = \frac{1}{|z|^2}(-\overline{z}_2 dz_1 + \overline{z}_1 dz_2), \quad e^*_- = \frac{1}{|z|^2}(-z_2 dz_1 + z_1 dz_2),
\] (1.25)

\[
\theta^* = \frac{1}{2|z|^2}(\overline{z}_1 dz_2 - \overline{z}_2 dz_1 - z_1 dz_2 + z_2 dz_1).
\] (1.26)

We have the integrable condition:

\[
d\theta^* = e^*_+ \wedge e^*_- = 2e^*_+ \wedge \theta^*, \quad de^*_+ - 2\theta^* \wedge e^*_-.
\] (1.27)

It holds that \(\theta^* \wedge e^*_+ \wedge e^*_- = d\sigma_{S^3}\).

In the following we denote a function \(f(z, \bar{z})\) of variables \(z, \bar{z}\) simply by \(f(z)\). For \(m = 0, 1, 2, \cdots, \) and \(l, k = 0, 1, \cdots, m\), we define the polynomials:

\[
v^{k}_{(l,m-l)} = (e_-)^k z_1^{l} z_2^{m-l},
\] (1.28)

\[
w^{k}_{(l,m-l)} = (-1)^k l! \frac{1}{(m-k)!} v^{m-l}_{(k,m-k)}.
\] (1.29)

Then \(v^{k}_{(l,m-l)}\) and \(w^{k}_{(l,m-l)}\) are harmonic polynomials on \(C^2\); \(\Delta v^{k}_{(l,m-l)} = \Delta w^{k}_{(l,m-l)} = 0\),

\[
\text{where } \Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}.
\]

In fact, \(\left\{ \frac{1}{\sqrt{2\pi}} v^{k}_{(l,m-l)} ; m = 0, 1, \cdots, 0 \leq k, l \leq m \right\}\) forms a \(L^2(S^3)\)-complete orthonormal system of the space of harmonic polynomials, as well as \(\left\{ \frac{1}{\sqrt{2\pi}} w^{k}_{(l,m-l)} ; m = 0, 1, \cdots, 0 \leq k, l \leq m \right\}\).

The radial vector field is defined by

\[
\frac{\partial}{\partial n} = \frac{1}{2|z|} (\nu + \bar{\nu}), \quad \nu = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.
\] (1.30)

We shall denote by \(\gamma\) the Clifford multiplication of the radial vector \(\frac{\partial}{\partial n}\). \(\gamma\) changes the chirality: \(\gamma = \gamma_+ \oplus \gamma_- : S^+ \oplus S^- \rightarrow S^- \oplus S^+\), and \(\gamma^2 = 1\).
Proposition 1.1. The Dirac operators $D$ and $D^\dagger$ have the following polar decompositions:

$$ D = \gamma_+ \left( \frac{\partial}{\partial n} - \vartheta \right), $$

$$ D^\dagger = \left( \frac{\partial}{\partial n} + \vartheta + \frac{3}{2|z|} \right) \gamma_-, $$

where the tangential (nonchiral) Dirac operator $\vartheta$ is given by

$$ \vartheta = - \left[ \sum_{i=1}^{3} \left( \frac{1}{|z|^{\theta_i}} \right) \cdot \nabla_{\frac{3}{2}|z|^{\theta_i}} \right] = \frac{1}{|z|} \begin{pmatrix} -\frac{1}{2} \theta & e_+ \\ -e_- & \frac{1}{2} \theta \end{pmatrix}. $$

The tangential Dirac operator on the sphere $S^3 = \{|z| = 1\}$:

$$ \vartheta|S^3 : C^\infty(S^3, S^+), \longrightarrow C^\infty(S^3, S^+) $$

is a self adjoint elliptic differential operator.

Now we introduce a basis of the space of even harmonic spinors by the following formula. For $m = 0, 1, 2, \ldots; l = 0, 1, \ldots, m$ and $k = 0, 1, \ldots, m + 1$, we put

$$ \phi^+(m,l,k)(z) = \sqrt{\frac{(m + 1 - k)!}{k!!(m - l)!}} \begin{pmatrix} k \psi_{(l,m-l)}^k \\ -v_{(l,m-l)}^k \end{pmatrix}, \quad (1.31) $$

$$ \phi^-(m,l,k)(z) = \sqrt{\frac{(m + 1 - k)!}{k!!(m - l)!}} \begin{pmatrix} 1 \\ \frac{1}{|z|^2} \end{pmatrix}^{m+2} \begin{pmatrix} u_{(m+1-l,l)}^k \\ w_{(m-l+1)}^k \end{pmatrix}. \quad (1.32) $$

We have the following

Proposition 1.2. 1. $\phi^+(m,l,k)$ is a harmonic spinor on $C^2$ and $\phi^-(m,l,k)$ is a harmonic spinor on $C^2 \setminus \{0\}$ that is regular at infinity.

2. On $S^3 = \{|z| = 1\}$ we have:

$$ \vartheta \phi^+(m,l,k) = \frac{m}{2} \phi^+(m,l,k), \quad \vartheta \phi^-(m,l,k) = -\frac{m + 3}{2} \phi^-(m,l,k). \quad (1.33) $$
3. The eigenvalues of $\hat{\partial}$ are

\[ \frac{m}{2}, \quad -\frac{m + 3}{2}; \quad m = 0, 1, \ldots, \]

and the multiplicity of each eigenvalue is equal to $(m + 1)(m + 2)$.

4. The set of eigenspinors

\[ \left\{ \frac{1}{\sqrt{2\pi}} \phi^+(m,l,k), \frac{1}{\sqrt{2\pi}} \phi^-(m,l,k) ; \quad m = 0, 1, \ldots, 0 \leq l \leq m, \quad 0 \leq k \leq m + 1 \right\} \]

forms a complete orthonormal system of $L^2(S^3, S^+)$. 

1.4 Spinors of Laurent polynomial type

If $\varphi$ is a harmonic spinor on $\mathbb{C}^2 \setminus \{0\}$ then we have the expansion

\[ \varphi(z) = \sum_{m,l,k} C_{+(m,l,k)} \phi^+(m,l,k)(z) + \sum_{m,l,k} C_{-(m,l,k)} \phi^-(m,l,k)(z), \]

that is uniformly convergent on any compact subset of $\mathbb{C}^2 \setminus \{0\}$. The coefficients $C_{\pm(m,l,k)}$ are given by the formula:

\[ C_{\pm(m,l,k)} = \frac{1}{2\pi^2} \int_{S^3} \langle \varphi, \phi^{\pm(m,l,k)} \rangle \, d\sigma, \]

where $\langle , \rangle$ is the inner product of $S^+$. 

**Definition 1.3.** We call the series (1.36) a spinor of Laurent polynomial type if only finitely many coefficients $C_{\pm(m,l,k)}$ are non-zero. The space of spinors of Laurent polynomial type is denoted by $\mathbb{C}[\varphi^\pm]$. 

The multiplication of two even spinors is defined by

\[ \phi_1 \cdot \phi_2 = \left( \begin{array}{c} u_1 u_2 - \overline{u_1} v_2 \\ v_1 u_2 + \overline{v_1} v_2 \end{array} \right), \]

for $\phi = \left( \begin{array}{c} u_i \\ v_i \end{array} \right), i = 1, 2,$ see (1.2).

**Proposition 1.4.** The restriction of $\mathbb{C}[\varphi^\pm]$ to $S^3$ is an associative subalgebra of
\[ S^3\mathbb{H} \text{ generated by the spinors:} \]

\[
\begin{align*}
\phi^{+(0,0,1)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \phi^{+(0,0,0)} &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & \phi^{+(1,0,1)} &= \begin{pmatrix} z_2 \\ -\bar{z}_1 \end{pmatrix}, & \phi^{-(0,0,0)} &= \begin{pmatrix} z_2 \\ \bar{z}_1 \end{pmatrix}.
\end{align*}
\]

We must note that \( \mathbb{C}[\phi^\pm] \) over \( \mathbb{C}^2 \setminus \{0\} \) is not an algebra.

### 1.5 2-cocycles on \( S^3\mathbb{H} \)

Let \( S^3\mathbb{H} = \text{Map}(S^3, \mathbb{H}) = \mathcal{C}^\infty(S^3, S^+) \) be the set of smooth even spinors on \( S^3 \). We continue to adopt the notation:

\[
\mathcal{C}^\infty(S^3, S^+) \ni \phi = \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow u + jv \in S^3\mathbb{H}.
\]  

(1.39)

We define the Lie algebra structure on \( S^3\mathbb{H} \) by the Lie bracket

\[
\left[ \phi_1, \phi_2 \right] = \begin{pmatrix} v_1\bar{v}_2 - \bar{v}_1v_2 \\ (u_2 - \bar{u}_2)v_1 - (u_1 - \bar{u}_1)v_2 \end{pmatrix},
\]  

(1.40)

for even spinors \( \phi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \) and \( \phi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \). This is equivalent to the following Lie bracket:

\[
\left[ u_1 + jv_1, u_2 + jv_2 \right] = (v_1\bar{v}_2 - \bar{v}_1v_2) + j((u_2 - \bar{u}_2)v_1 - (u_1 - \bar{u}_1)v_2).
\]  

(1.41)

The trace of a spinor \( \phi = \begin{pmatrix} u \\ v \end{pmatrix} \) is by definition

\[
tr \phi = 2\text{Re}u = u + \bar{u}.
\]  

(1.42)

Evidently we have \( tr [\phi, \psi] = 0 \).

In the following we introduce three 2-cocycles on \( S^3\mathbb{H} \) that come from the basis of vector fields on \( S^3 \), \( [1.22] \). We put

\[
\theta_0 = \sqrt{-1}\theta, \quad \theta_1 = e_+ + e_-, \quad \theta_2 = \sqrt{-1}(e_+ - e_-).
\]  

(1.43)
For a $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in S^3\mathbb{H}$, we put
\[
\Theta_k \varphi = \begin{pmatrix} \frac{1}{2} \theta_k & 0 \\ 0 & \frac{1}{2} \theta_k \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \theta_k u \\ \theta_k v \end{pmatrix}, \quad k = 0, 1, 2.
\]

**Lemma 1.5.** For any $k = 0, 1, 2$, and $\phi, \psi \in S^3\mathbb{H}$, we have
\[
\Theta_k (\phi \cdot \psi) = (\Theta_k \phi) \cdot \psi + \phi \cdot (\Theta_k \psi). \tag{1.44}
\]
\[
\int_{S^3} \Theta_k \varphi \, d\sigma = 0. \tag{1.45}
\]

For the first equation we use the fact $\theta_k = \theta_k$. The second assertion follows from
\[
\int_{S^3} \theta_k f \, d\sigma = 0,
\]
for any function $f$ on $S^3$. This is true by virtue of (1.27).

**Definition 1.6.** For $\phi_1$ and $\phi_2 \in S^3\mathbb{H}$, we put
\[
c_k(\phi_1, \phi_2) = \frac{1}{2\pi^2} \int_{S^3} tr (\Theta_k \phi_1 \cdot \phi_2) \, d\sigma. \tag{1.46}
\]

**Proposition 1.7.** For each $k = 0, 1, 2$, $c_k$ defines a non-trivial 2-cocycle on the algebra $S^3\mathbb{H}$. That is, $c_k$ satisfies the equations:
\[
c_k(\phi_1, \phi_2) = -c_k(\phi_2, \phi_1), \tag{1.47}
\]
\[
c_k(\phi_1 \cdot \phi_2, \phi_3) + c_k(\phi_2 \cdot \phi_3, \phi_1) + c_k(\phi_3 \cdot \phi_1, \phi_2) = 0, \tag{1.48}
\]
for any $\phi_1, \phi_2, \phi_3 \in S^3\mathbb{H}$, and there is no 1-cochain $b$ such that $c_k(\phi_1, \phi_2) = b([\phi_1, \phi_2])$.

**Proof.** By (1.45) and the Leibnitz rule (1.44) we have
\[
0 = \int_{S^3} tr (\Theta_k (\phi_1 \cdot \phi_2)) \, d\sigma = \int_{S^3} tr (\Theta_k \phi_1 \cdot \phi_2) \, d\sigma + \int_{S^3} tr (\phi_1 \cdot \Theta_k \phi_2) \, d\sigma
\]
Hence $c_k(\phi_1, \phi_2) + c_k(\phi_2, \phi_1) = 0$. The following calculation proves (1.48).

$$c_k(\phi_1 \cdot \phi_2, \phi_3) = \int_{S^3} \text{tr} \left( \Theta_k(\phi_1 \cdot \phi_2) \cdot \phi_3 \right) d\sigma$$

$$= \int_{S^3} \text{tr} \left( \Theta_k \phi_1 \cdot \phi_2 \cdot \phi_3 \right) d\sigma + \int_{S^3} \text{tr} \left( \Theta_k \phi_2 \cdot \phi_3 \cdot \phi_1 \right) d\sigma$$

$$= c_k(\phi_1, \phi_2 \cdot \phi_3) + c_k(\phi_2, \phi_3 \cdot \phi_1) = -c_k(\phi_2 \cdot \phi_3, \phi_1) - c_k(\phi_3 \cdot \phi_1, \phi_2).$$

Suppose now that $c_0$ is the coboundary of a 1-cochain $b : S^3 \mathbb{H} \rightarrow \mathbb{C}$. Then

$$c_0(\phi_1, \phi_2) = (\delta b)(\phi_1, \phi_2) = b(\phi_1, \phi_2)$$

for any $\phi_1, \phi_2 \in S^3 \mathbb{H}$. Take $\phi_1 = \frac{1}{\sqrt{2}} \phi^{+(1,1,2)} = \left( \begin{array}{c} -z_2 \\ 0 \end{array} \right)$ and $\phi_2 = \frac{1}{2}(\phi^{+(1,0,1)} + \phi^{-(0,0,0)}) = \left( \begin{array}{c} z_2 \\ 0 \end{array} \right)$. Then $[\phi_1, \phi_2] = 0$, so $(\delta b)(\phi_1, \phi_2) = 0$. But $c_0(\phi_1, \phi_2) = \sqrt{-1}$. Therefore $c_0$ can not be a coboundary. For $\phi_1$ and $\phi_3 = \phi^{+(1,0,2)} = \sqrt{2} \left( \begin{array}{c} z_1 \\ 0 \end{array} \right)$, we have $[\phi_1, \phi_3] = 0$ and $c_1(\phi_1, \phi_3) = -\frac{1}{\sqrt{2}}$. So $c_1$ can not be a coboundary by the same reason as above. Similarly for $c_2$. 

Definition 1.8. On $S^3 \mathbb{H}$ there are three 2-cocycles defined by

$$c_k(\phi_1, \phi_2) = \frac{1}{2\pi^2} \int_{S^3} \text{tr} \left( \Theta_k \phi_1 \cdot \phi_2 \right) d\sigma, \quad k = 0, 1, 2. \quad (1.49)$$

1.6 Radial derivative on $S^3 \mathbb{H}$

We define the following operator $d_0$ on $C^\infty(S^3)$:

$$d_0 f(z) = |z| \frac{\partial}{\partial n} f(z) = \frac{1}{2}(\nu + \bar{\nu}) f(z). \quad (1.50)$$

For an even spinor $\varphi = \left( \begin{array}{c} u \\ v \end{array} \right)$ we put

$$d_0 \varphi = \left( \begin{array}{c} d_0 u \\ d_0 v \end{array} \right). \quad (1.51)$$
The radial derivative $d_0$ preserves $\mathbb{C}[^\pm \phi]$, that is, $d_0\varphi \in \mathbb{C}[^\pm \phi]$ for $\varphi \in \mathbb{C}[^\pm \phi]$.

**Proposition 1.9.**

1. \[
    d_0(\phi_1 \cdot \phi_2) = (d_0\phi_1) \cdot \phi_2 + \phi_1 \cdot (d_0\phi_2). \quad (1.52)
\]

2. \[
    d_0\phi^{+(m,l,k)} = \frac{m}{2} \phi^{+(m,l,k)}, \quad d_0\phi^{-(m,l,k)} = -\frac{m+3}{2} \phi^{-(m,l,k)}. \quad (1.53)
\]

3. Let $\varphi$ be a spinor of Laurent polynomial type:

\[
    \varphi(z) = \sum_{m,l,k} C_{+(m,l,k)} \phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)} \phi^{-(m,l,k)}(z). \quad (1.54)
\]

Then $d_0\varphi$ is a spinor of Laurent polynomial type and

\[
    \int_{S^3} \text{tr} \left( d_0 \varphi \right) d\sigma = 0. \quad (1.55)
\]

**Proof.** The formula (1.53) follows from the definition (1.31). The last assertion follows from the fact that the coefficient of $\phi^{+(0,0,1)}$ in the Laurent expansion of $d_0\varphi$ vanishes.

**Proposition 1.10.** Let $c$ denote one of the 2-cocycles $c_k$, $k = 0, 1, 2$. Then we have

\[
    c( d_0\phi_1, \phi_2 ) + c( \phi_1, d_0\phi_2 ) = 0. \quad (1.56)
\]

In fact, since $\theta d_0 = (\nu - \bar{\nu})(\nu + \bar{\nu}) = \nu^2 - \bar{\nu}^2 = d_0 \theta$, we have

\[
    0 = \int_{S^3} \text{tr} \left( d_0(\Theta\phi_1 \cdot \phi_2) \right) d\sigma = \int_{S^3} \text{tr} \left( (d_0\Theta\phi_1) \cdot \phi_2 + \Theta\phi_1 \cdot d_0\phi_2 \right) d\sigma
\]

\[
    = \int_{S^3} \text{tr} \left( (\Theta d_0\phi_1) \cdot \phi_2 \right) d\sigma + \int_{S^3} \text{tr} \left( \Theta\phi_1 \cdot d_0\phi_2 \right) d\sigma.
\]

Let $\mathbb{C}[^\pm \phi; N]$ be the subspace of $\mathbb{C}[^\pm \phi]$ consisting of those elements that are of homogeneous order $N$: $\varphi(z) = |z|^N \varphi(\frac{z}{|z|})$. $\mathbb{C}[^\pm \phi; N]$ is spanned by the spinors $\varphi = \phi_1 \cdots \phi_n$ such that each $\phi_i$ is equal to $\phi_i = \phi^{+(m_i,l_i,k_i)}$ or $\phi_i = \phi^{-(m_i,l_i,k_i)}$, where $m_i \geq 0$ and
0 \leq l_i \leq m_i + 1, 0 \leq k_i \leq m_i + 2, and such that

\[ N = \sum_{i: \phi_i = \phi^+(m_i, l_i, k_i)} m_i - \sum_{i: \phi_i = \phi^-(m_i, l_i, k_i)} (m_i + 3). \]

It holds that \( d_0 \varphi = \frac{N}{2} \varphi \), so the eigenvalues of \( d_0 \) on \( C[\phi^\pm] \) are \( \{ \frac{N}{2}; N \in \mathbb{Z} \} \) and \( C[\phi^\pm; N] \) is the space of eigenspinors for the eigenvalue \( \frac{N}{2} \). For example \( \phi = \phi^+(2,0,0) \cdot \phi^-(0,0,0) \in C[\phi^\pm; -1] \), and \( d_0 \phi = -\frac{1}{2} \phi \). We have the eigenspace decomposition:

\[ C[\phi^\pm] = \bigoplus_{N \in \mathbb{Z}} C[\phi^\pm; N]. \quad (1.57) \]

## 2 Extensions of the Lie algebra \( C[\phi^\pm] \otimes \mathfrak{gl}(n, H) \)

### 2.1 Extension of the Lie algebra \( \text{Map}(S^3, \mathfrak{gl}(n, H)) \)

Let \( \mathfrak{gl}(n, H) = H \otimes \mathfrak{gl}(n, C) \) be the algebra of \( n \times n \)-matrices with entries in \( H \). We have the isomorphisms \( H^n \simeq H \otimes C^n \simeq C^{2n} \) and the quaternionic structure is defined by the conjugate linear map \( J \in \text{End}_C(C^{2n}) : \)

\[ J(x \otimes v) = (jx) \otimes v, \quad x \in H; v \in C^n. \quad (2.1) \]

So that \( H^n \simeq C^n + JC^n \). We put

\[ \text{mj}(2n, C) = \{ Z \in \mathfrak{gl}(2n, C), \quad JZ = \overline{Z}J \} \quad (2.2) \]

\[ = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right\}; \quad A, B \in \mathfrak{gl}(n, C) \quad (2.3) \]

Where the complex conjugation is defined with respect to a fixed \( C \)-basis:

\[ A = \sum_i a_i e_i \rightarrow \overline{A} = \sum_i \overline{a_i} e_i, \]

It is independent on the \( \mathbb{R} \)-base change. Then we have the \( \mathbb{R} \)-algebra isomorphism:

\[ \mathfrak{gl}(n, H) \simeq \text{mj}(2n, C), \quad A + JB \rightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \quad (2.4) \]
The left multiplication of $H$ on $\mathfrak{gl}(n,H) \simeq \mathfrak{mj}(2n,C)$ is given by

$$(a+jb) \cdot (X + JY) = (aX - bY) + J(aY + bX) \xrightarrow{\sim} \begin{pmatrix} aX - bY & -aY - bX \\ aY + bX & aX - bY \end{pmatrix},$$

for $a, b \in C$ and $X, Y \in \mathfrak{gl}(n,C)$. The multiplication in $\mathfrak{gl}(n,H)$ becomes

$$(X_1 + JY_1) \cdot (X_2 + JY_2) = (X_1X_2 - Y_1Y_2) + J(Y_1X_2 + X_1Y_2). \quad (2.5)$$

The corresponding description in $\mathfrak{mj}(2n,C)$ is easy to see.

$\mathfrak{gl}(n,H)$ is a Lie algebra defined by the $R$-linear bracket:

$$[X_1 + JY_1, X_2 + JY_2] = (X_1X_2 - X_2X_1 - Y_1Y_2 + Y_2Y_1) + J(Y_1X_2 - Y_2X_1 + X_1Y_2 - X_2Y_1), \quad (2.6)$$

for $X_1 + JY_1, X_2 + JY_2 \in \mathfrak{gl}(n,H)$. This bracket has a more convenient description that is easy to handle. For any bases $X, Y$ of $\mathfrak{gl}(n,C)$ and $z_1, z_2 \in H$, we put

$$[z_1 \otimes X, z_2 \otimes Y] = (z_1z_2) \otimes XY - (z_2z_1) \otimes YX. \quad (2.7)$$

It is easy to see that thus defined ( $R$-linear ) bracket satisfies the antisymmetry equation and the Jacobi identity, and gives the same one as (2.6). More conveniently, if we write $E_{ij}$ the $n \times n$-matrix with entry 1 at $(i, j)$-place and 0 otherwise, $\{E_{ij}\}_{i,j}$ is the basis of $\mathfrak{gl}(n,C)$ and we have

$$[z_1 \otimes E_{ij}, z_2 \otimes E_{kl}] = (z_1z_2) \otimes \delta_{jk}E_{ij} - (z_2z_1) \otimes \delta_{il}E_{kl}, \quad (2.8)$$

for $z_1, z_2 \in H$.

We proceed to the current algebra $\text{Map}(S^3, \mathfrak{gl}(n,H)) = S^3H \otimes \mathfrak{gl}(n,C)$, which we shall write simply by $S^3\mathfrak{gl}(n,H)$. We may also write $F + JG \in S^3\mathfrak{gl}(n,H)$ with $F, G \in S^3\mathfrak{gl}(n,C)$:

$$S^3\mathfrak{gl}(n,H) = S^3\mathfrak{gl}(n,C) + J(S^3\mathfrak{gl}(n,C)). \quad (2.9)$$

**Proposition 2.1.** $S^3\mathfrak{gl}(n,H)$ endowed with the following bracket $[,]_{S^3\mathfrak{gl}(n,H)}$ becomes a
Lie algebra.

\[[\phi \otimes E_{ij}, \psi \otimes E_{kl}]_{S^3 gl(n,H)} = (\phi \cdot \psi) \otimes \delta_{jk} E_{ij} - (\psi \cdot \phi) \otimes \delta_{il} E_{kl}, \quad (2.10)\]

for \(\phi, \psi \in S^3 H\).

The symmetric bilinear form \((X,Y) \mapsto (X|Y), X,Y \in gl(n,C)\) is given by

\[(X|Y) = tr(XY).\]

It holds \((E_{ij}|E_{kl}) = \delta_{ij}\delta_{jk}\). Then we define \(C\)-valued 2-cocycles on the Lie algebra \(S^3 gl(n,H)\) by

\[c_k(\phi_1 \otimes X, \phi_2 \otimes Y) = (X|Y)c_k(\phi_1, \phi_2), \quad k = 0, 1, 2. \quad (2.11)\]

The 2-cocycle property follows from the fact \((XY|Z) = (YZ|X)\) and Proposition 1.7.

Let \(a_k, k = 0, 1, 2\), be three indefinite numbers. For each \(k = 0, 1, 2\) there is a central extension of the Lie algebra \(S^3 gl(n,H)\) by the 1-dimensional center \(Ca_k\) associated to the cocycle \(c_k\). Summing-up the above we arrive at the following theorem.

**Theorem 2.2.** The \(H\)-vector space

\[S^3 gl(n,H)(a) = (S^3 H \otimes gl(n,C)) \oplus (\oplus_{k=0,1,2} Ca_k), \quad (2.12)\]

defined with the following bracket becomes a Lie algebra.

\[\left[\phi \otimes X, \psi \otimes Y\right] = (\phi \cdot \psi) \otimes XY - (\psi \cdot \phi) \otimes YX + (X|Y) \sum_{k=0}^{2} c_k(\phi, \psi) a_k, \quad (2.13)\]

for \(\phi, \psi \in S^3 H\) and any bases \(X, Y \in gl(n,H)\).

As a Lie subalgebra of \(S^3 gl(n,H)\) we have \(C[\phi^\pm] \otimes gl(n,C)\). The basis of \(C[\phi^\pm] \otimes gl(n,C)\) consists of

\[\phi^\pm(m,l,k) \otimes E_{ij}. \quad (2.14)\]

**Definition 2.3.**
1. We put
\[
\mathfrak{gl}(n, \mathbb{H}) = C[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C}).
\]
(2.15)

2. We denote by \( \mathfrak{gl}(n, \mathbb{H})(a) \) the extension of the Lie algebra \( \mathfrak{gl}(n, \mathbb{H}) \) by the three 1-dimensional centers \( C_{a_k} \) associated to the cocycle \( c_k \):
\[
\mathfrak{gl}(n, \mathbb{H})(a) = C[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C}) \oplus (\oplus_{k=0,1,2} C_{a_k}).
\]
(2.16)

2.2 Extension of \( \mathfrak{gl}(n, \mathbb{H})(a) \) by the derivation

We introduced the radial derivative \( d_0 \) acting on \( S^3 \mathbb{H} \) in (1.51). \( d_0 \) preserves the space of spinors of Laurent polynomial type \( C[\phi^\pm] \), Proposition 1.9. The derivation \( d_0 \) on \( C[\phi^\pm] \) is extended to a derivation of the Lie algebra \( C[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C}) \):
\[
d_0 (\phi \otimes X) = (d_0 \phi) \otimes X, \quad \phi \in C[\phi^\pm], X \in \mathfrak{gl}(n, \mathbb{C}).
\]
(2.17)

In fact we have
\[
\begin{align*}
[d_0 (\phi_1 \otimes X_1), \phi_2 \otimes X_2] &+ [\phi_1 \otimes X_1, d_0 (\phi_2 \otimes X_2)] \\
&= (d_0 \phi_1 \cdot \phi_2) \otimes (X_1 X_2) - (\phi_2 \cdot d_0 \phi_1) \otimes (X_2 X_1) + (\phi_1 \cdot d_0 \phi_2) \otimes (X_1 X_2) \\
&\quad - (d_0 \phi_2 \cdot \phi_1) \otimes (X_2 X_1) + (X_1 X_2) \sum_k (c_k(d_0 \phi_1, \phi_2) + c_k(\phi_1, d_0 \phi_2)) a_k.
\end{align*}
\]

Since \( c_k(d_0 \phi_1, \phi_2) + c_k(\phi_1, d_0 \phi_2) = 0 \) from Proposition 1.10, the right-hand side is equal to
\[
d_0 \left( [\phi_1 \otimes X_1, \phi_2 \otimes X_2] \right)
\]
by virtue of (1.52). Hence
\[
d_0 \left( [\phi_1 \otimes X_1, \phi_2 \otimes X_2] \right) = [d_0 (\phi_1 \otimes X_1), \phi_2 \otimes X_2] + [\phi_1 \otimes X_1, d_0 (\phi_2 \otimes X_2)].
\]

\( d_0 \) is a derivation that acts on the Lie algebra \( C[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C}) \).

We denote by \( \mathfrak{g}l \) the Lie algebra that is obtained by adjoining a derivation \( d \) to \( \mathfrak{gl}(n, \mathbb{H})(a) \) which acts on \( C[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C}) \) as \( d_0 \) and kills \( a \). More explicitly we have the following

**Theorem 2.4.** Let \( a_k, k = 0,1,2 \), and \( d \) be indefinite elements. We consider the \( \mathbb{C} \)
vector space:

\[ \hat{\mathfrak{gl}} = (\mathbb{C}[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C})) \oplus (\bigoplus_{k=0}^2 \mathbb{C} a_k) \oplus (\mathbb{C} d), \]  

and define the following bracket on \( \hat{\mathfrak{gl}} \). For \( X, Y \in \mathfrak{gl}(n, \mathbb{C}) \) and \( \phi, \psi \in \mathbb{C}[\phi^\pm] \),

\[
[ \phi \otimes X , \psi \otimes Y ]_{\hat{\mathfrak{gl}}} = [ \phi \otimes X , \psi \otimes Y ]^\sim = (\phi \cdot \psi) \otimes (XY) - (\psi \cdot \phi) \otimes (YX) + (X[Y) \sum_{k=0}^2 c_k(\phi, \psi) a_k,
\]

\[
[a_k, \phi \otimes X]_{\hat{\mathfrak{gl}}} = 0,
\]

\[
[d, \phi \otimes X]_{\hat{\mathfrak{gl}}} = d_0 \phi \otimes X, \ [d, a_k]_{\hat{\mathfrak{gl}}} = 0, \quad k = 0, 1, 2.
\]

Then \( (\hat{\mathfrak{gl}}, [\cdot, \cdot]_{\hat{\mathfrak{gl}}}) \) becomes a Lie algebra.

**Proof**

It is enough to prove the following Jacobi identity:

\[
[[d, \phi_1 \otimes X_1], \phi_2 \otimes X_2]_{\hat{\mathfrak{g}}} + [[\phi_1 \otimes X_1, \phi_2 \otimes X_2]_{\hat{\mathfrak{g}}}, d]_{\hat{\mathfrak{g}}} = 0.
\]

In the following we shall abbreviate the bracket \( [\cdot, \cdot]_{\hat{\mathfrak{gl}}} \) simply to \( [\cdot, \cdot] \). We have

\[
[[d, \phi_1 \otimes X_1], \phi_2 \otimes X_2] = [d_0 \phi_1 \otimes X_1, \phi_2 \otimes X_2] = (d_0 \phi_1 \cdot \phi_2) \otimes (X_1 X_2) - (\phi_2 \cdot d_0 \phi_1) \otimes (X_2 X_1)
\]

\[
+ (X_1 | X_2) \sum_k c_k(\phi_1, \phi_2) a_k.
\]

Similarly

\[
[[\phi_2 \otimes X_2, d], \phi_1 \otimes X_1] = (\phi_1 \cdot d_0 \phi_2) \otimes (X_1 X_2) - (d_0 \phi_2 \cdot \phi_1) \otimes (X_2 X_1)
\]

\[
+ (X_1 | X_2) \sum_k c_k(\phi_1, d_0 \phi_2) a_k.
\]

\[
[[\phi_1 \otimes X_1, \phi_2 \otimes X_2], d] = - [d, (\phi_1 \cdot \phi_2) \otimes (X_1 X_2) - (\phi_2 \cdot \phi_1) \otimes (X_2 X_1)]
\]

\[
+ (X_1 | X_2) \sum_k c_k(\phi_1, \phi_2) a_k
\]

\[
= - d_0(\phi_1 \cdot \phi_2) \otimes (X_1 X_2) + d_0(\phi_2 \cdot \phi_1) \otimes (X_2 X_1).
\]
Proposition 2.5. The centralizer of $d$ in $\hat{\mathfrak{gl}}$ is given by

$$\left( C[\phi^\pm; 0] \otimes \mathfrak{gl}(n, \mathbb{C}) \right) \oplus \left( \bigoplus_k C a_k \right) \oplus C d. \quad (2.19)$$

We remember that $C[\phi^\pm; 0]$ is the subspace in $C[\phi^\pm]$ generated by $\phi_1 \cdots \phi_n$ with $\phi_i$ being $\phi_i = \phi^{\pm(m_i,l_i,k_i)}$ such that

$$\sum_{i; \phi_i=\phi^+(m_i,l_i,k_i)} m_i - \sum_{i; \phi_i=\phi^-(m_i,l_i,k_i)} m_i = 0.$$

3 $\mathfrak{sl}(n, \mathbb{H})$-current algebras on $S^3$

3.1 Preliminaries on $\mathfrak{sl}(n, \mathbb{H})$

Let $\mathfrak{sl}(n, \mathbb{H})$ denote the quaternion special linear algebra:

$$\mathfrak{sl}(n, \mathbb{H}) \simeq \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in mj(2n, \mathbb{C}) : \text{Trace} A \in \sqrt{-1} \mathbb{R} \right\} \quad (3.1)$$

We put

$$\mathfrak{st}(n, \mathbb{C}) = \{ A \in \mathfrak{gl}(n, \mathbb{C}) ; \text{Trace} A \in \sqrt{-1} \mathbb{R} \}. \quad (3.2)$$

Then the correspondence (3.1) is given by

$$\mathfrak{sl}(n, \mathbb{H}) \ni A + JB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in mj(2n, \mathbb{C}), \quad A \in \mathfrak{st}(n, \mathbb{C}). \quad (3.3)$$

In the following $\mathfrak{sl}(n, \mathbb{C})$, $J\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{st}(n, \mathbb{C})$ are considered as submodules of $\mathfrak{gl}(n, \mathbb{H}) \simeq mj(2n, \mathbb{C})$. So $\mathfrak{sl}(n, \mathbb{H}) = \mathfrak{st}(n, \mathbb{C}) + J\mathfrak{gl}(n, \mathbb{C})$. We have

$$\mathbb{H} \otimes \mathfrak{sl}(n, \mathbb{C}) \subset \mathbb{H} \otimes \mathfrak{st}(n, \mathbb{C}) \subset \mathfrak{st}(n, \mathbb{C}) + J\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{H}) \quad (3.4)$$

$\mathbb{H} \otimes \mathfrak{sl}(n, \mathbb{C})$ is not a Lie algebra, nor $\mathbb{H} \otimes \mathfrak{st}(n, \mathbb{C})$.

Let $\mathfrak{h}$ be the diagonal matrices of $\mathfrak{sl}(n, \mathbb{C})$. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. $ad(H)E_{ij} = (\lambda_i - \lambda_j)E_{ij}$ for $H = \sum \lambda_i E_{ii} \in \mathfrak{h}$ such that $\sum \lambda_j = 0$. So the set of roots of $\mathfrak{sl}(n, \mathbb{C})$ with respect to $\mathfrak{h}$ consist of the $(n - 1)n + 1$ elements: $0$, $\Delta = \{ \alpha_{ij} \}$, where $\alpha_{ij}(H) = \lambda_i - \lambda_j$.
for any $H = \sum_{i=1}^{n} \lambda_i E_{ii} \in \mathfrak{h}$. Let $\mathfrak{sl}(n, C) = \mathfrak{h} \oplus \sum_{i \neq j} \mathfrak{g}_{\alpha_{ij}}$ be the root space decomposition with $\mathfrak{g}_{\alpha_{ij}} = C E_{ij}, i \neq j$. Let $\Pi = \{ \alpha_i = \lambda_i - \lambda_{i+1}; i = 1, \ldots, r = n - 1 \} \subset \mathfrak{h}^*$ be the set of simple roots and let $\{ \alpha_i^\vee; i = 1, \ldots, r \} \subset \mathfrak{h}$ be the set of simple coroots. It holds that

$$\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}, \text{ for } i < j, \quad \alpha_{ij} = - (\alpha_j + \cdots + \alpha_{i-1}), \text{ for } j < i,$$

The Cartan matrix $A = (a_{ij})_{i,j=1,\ldots,r}$ is given by $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. Fix a standard set of generators of $\mathfrak{sl}(n, C)$:

$$h_i = \alpha_i^\vee, x_i \in \mathfrak{g}_{\alpha_i}, y_i \in \mathfrak{g}_{-\alpha_i}, \quad 1 \leq i \leq n - 1,$$

so that

$$[x_i, y_j] = h_j \delta_{ij}, \quad [h_i, x_j] = a_{ji} x_j, \quad [h_i, y_j] = - a_{ji} y_j. \quad (3.5)$$

If $a_{ij} = 0$, then $[x_i, x_j] = 0$ and $[y_i, y_j] = 0$. We see that $\{ E_{ij}; 1 \leq i \neq j \leq n \}$ are generated as follows;

$$E_{ij} = [x_i, [x_{i+1}, \cdots [x_{j-2}, x_{j-1}] \cdots]], \quad \text{for } i < j,$$

$$E_{ij} = [y_j, [y_{j+1}, \cdots [y_{i-2}, y_{i-1}] \cdots]], \quad \text{for } i > j. \quad (3.6)$$

**Proposition 3.1.** The following relations hold in $\mathfrak{sl}(n, H)$:

$$[x_i, Jy_j] = Jh_j \delta_{ij}, \quad [h_i, Jx_j] = a_{ji} Jx_j, \quad [h_i, Jy_j] = - a_{ji} Jy_j.$$

$$[Jx_i, y_j] = - Jh_j \delta_{ij}, \quad [Jh_i, x_j] = a_{ji} Jx_j, \quad [Jh_i, y_j] = - a_{ji} Jy_j.$$

$$[Jx_i, Jy_j] = - h_j \delta_{ij}, \quad [Jh_i, Jx_j] = - a_{ji} x_j, \quad [Jh_i, Jy_j] = - a_{ji} y_j.$$

$$[Jh_i, E_{jk}] = (a_{ij} + \cdots a_{i,k-1}) J E_{jk}, \quad j < k$$

$$[Jh_i, E_{jk}] = -(a_{ij} + \cdots a_{i,k-1}) J E_{jk}, \quad k < j.$$

**Proposition 3.2.** The $H$-module $H \otimes \mathfrak{sl}(n, C) = \mathfrak{sl}(n, C) + J \mathfrak{sl}(n, C)$ generates the Lie algebra $\mathfrak{sl}(n, H)$. The generators are given by

$$h_i, x_i, y_i, Jh_i, Jx_i, Jy_i; \quad (i = 1, \cdots, n - 1). \quad (3.7)$$
Proof Prove precisely! The basis \( \{ h_i, 1 \leq i \leq n - 1; \ E_{jk}, 1 \leq j \neq k \leq n, \} \) of \( \mathfrak{sl}(n, \mathbb{C}) \) augmented by \( [\sqrt{-1} Jh_1, Jh_2] = 2\sqrt{-1} E_{22} \in \mathfrak{sl}(n, \mathbb{C}) \) gives a basis of \( \mathfrak{sl}(n, \mathbb{C}) \). Similarly, for any \( c \in \mathbb{C} \), \( [J(cE_{ni}), E_{in}] = cJ(E_{ii} + E_{nn}) \) augments the basis of \( \mathfrak{sl}(n, \mathbb{C}) + J\mathfrak{sl}(n, \mathbb{C}) = H \otimes \mathfrak{sl}(n, \mathbb{C}) \) to a basis of \( \mathfrak{sl}(n, H) = \mathfrak{sl}(n, \mathbb{C}) + J\mathfrak{gl}(n, \mathbb{C}) \).

### 3.2 Lie algebra \( \widehat{\mathfrak{sl}}(n, H) \) and its central extensions

**Definition 3.3.** \( \widehat{\mathfrak{sl}}(n, H) \) is the Lie subalgebra of \( \widehat{\mathfrak{gl}}(n, H) = \mathbb{C}[\phi^\pm] \otimes \mathfrak{gl}(n, \mathbb{C}) \) generated by \( \mathbb{C}[\phi^\pm] \otimes \mathfrak{sl}(n, \mathbb{C}) \).

Note that \( \mathbb{C}[\phi^\pm] \otimes \mathfrak{sl}(n, \mathbb{C}) \) is not a Lie algebra.

By the 2-cocycle \( c_k \), \( k = 0, 1, 2 \), of (2.11) we have the central extension

\[
\mathfrak{sl}(n, H)(a) = \mathfrak{sl}(n, H) \oplus (\oplus_k \mathbb{C}a_k). \tag{3.8}
\]

Further we have the central extension of \( \mathfrak{sl}(n, H)(a) \) by the derivation \( d_0 \) of (1.51):

\[
\widehat{\mathfrak{sl}} = \mathfrak{sl}(n, H) \oplus (\oplus_k \mathbb{C}a_k) \oplus (Cd). \tag{3.9}
\]

These are Lie subalgebras of \( \widehat{\mathfrak{gl}} = \mathfrak{gl}(n, H) \oplus (\oplus_k \mathbb{C}a_k) \oplus (Cd) \).

As a Lie subalgebras of \( \widehat{\mathfrak{gl}} \), the Lie bracket of \( \widehat{\mathfrak{sl}} \) is given as follows:

\[
[\phi \otimes X, \psi \otimes Y]_{\widehat{\mathfrak{sl}}} = (\phi \psi) \otimes (XY) - (\psi \phi) \otimes (YX) + (X|Y) \sum_k c_k(\phi, \psi) a_k, \\
[a_k, \phi \otimes X]_{\widehat{\mathfrak{sl}}} = 0, \quad [a_k, d]_{\widehat{\mathfrak{sl}}} = 0, \\
[d, \phi \otimes X]_{\widehat{\mathfrak{sl}}} = d_0 \phi \otimes X,
\]

for any bases \( X, Y \in \mathfrak{sl}(n, \mathbb{C}) \) and \( \phi, \psi \in \mathbb{C}[\phi^\pm] \). Since \( \phi^{(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) we identify a \( X \in \mathfrak{sl}(n, \mathbb{C}) \) with \( \phi^{(0,0,1)} \otimes X \in \mathfrak{sl}(n, H) \). Thus we look \( \mathfrak{sl}(n, \mathbb{C}) \) as a Lie subalgebra of \( \mathfrak{sl}(n, H) \):

\[
[\phi^{(0,0,1)} \otimes X, \phi^{(0,0,1)} \otimes Y]_{\mathfrak{sl}(n, H)} = [X, Y]_{\mathfrak{sl}(n, \mathbb{C})}, \quad \tag{3.10}
\]

and we shall write \( \phi^{(0,0,1)} \otimes X \) simply as \( X \).
3.3 Root space decomposition of $\hat{s}l$

Let

$$\hat{h} = ( (C\phi^{(0,0,1)}) \otimes h) \oplus (\oplus_k C a_k) \oplus (C d) = h \oplus (\oplus_k C a_k) \oplus (C d). \quad (3.11)$$

We write $\hat{h} = h + \sum s_k a_k + td \in \hat{h}$ with $h \in h$ and $s_k, t \in C$. Then the adjoint actions of

$$ad(\hat{h})(\xi) = ad(\hat{h}) (\phi \otimes X + \sum \mu_j a_j + \nu d) = \phi \otimes (hX - Xh) + t\nu \phi \otimes X. \quad (3.12)$$

An element $\lambda$ of the dual space $h^*$ of $h$ is regarded as a element of $\hat{h}^*$ by putting

$$\langle \lambda, a_k \rangle = 0, \quad \langle \lambda, d \rangle = 0, \quad k = 0, 1, 2. \quad (3.13)$$

So $\Delta \subset h^*$ is seen to be a subset of $\hat{h}^*$. We define $\delta, \Lambda_k \in \hat{h}^*, k = 0, 1, 2$, by

$$\langle \delta, \alpha \rangle = \langle \Lambda_k, \alpha \rangle = 0,$$

$$\langle \delta, a_k \rangle = 0, \quad \langle \delta, d \rangle = 1,$$

$$\langle \Lambda_k, a_k \rangle = 1, \quad \langle \Lambda_k, d \rangle = 0, \quad 1 \leq i \leq r, \quad k = 0, 1, 2. \quad (3.14)$$

Then the set $\{ \alpha_1, \cdots, \alpha_r, \Lambda_0, \Lambda_1, \Lambda_2, \delta \}$ forms a basis of $\hat{h}^*$.

Since $\hat{h}$ is a commutative subalgebra of $\hat{s}l$, $\hat{s}l$ is decomposed into a direct sum of the simultaneous eigenspaces of $ad(\hat{h})$, $\hat{h} \in \hat{h}^*$. For $\lambda = \alpha + k\delta \in \hat{h}^*$, $\alpha = \sum_{i=1}^r k_i \alpha_i \in \Delta, k_i \in Z$, $i = 0, 1, \cdots, r$, we put,

$$\hat{g}_\lambda = \{ \xi \in \hat{s}l; \quad [\hat{h}, \xi] = \langle \lambda, \hat{h} \rangle \xi \quad \text{for} \forall \hat{h} \in \hat{h} \}.$$

$\lambda$ is a root of $\hat{s}l$ if $\hat{g}_\lambda \neq 0$. $\hat{g}_\lambda$ is called the root space of $\lambda$.

Let $\hat{\Delta}$ denote the set of roots of the representation $\left(\hat{s}l, ad(\hat{h})\right)$.

**Theorem 3.4.** 1.

$$\hat{\Delta} = \left\{ \begin{array}{l} m \delta + \alpha; \quad \alpha \in \Delta, \ m \in Z \end{array} \right\} \cup \left\{ \begin{array}{l} \frac{m}{2} \delta; \quad m \in Z \end{array} \right\}.$$
2. For $\alpha \in \Delta$, $\alpha \neq 0$ and $m \in \mathbb{Z}$, we have
\[ \hat{g}_{\frac{m}{2}, \delta + \alpha} = C[\phi^\pm; m] \otimes g_\alpha. \] (3.18)

3. \[ \hat{g}_0 = \hat{h}, \]
\[ \hat{g}_{\frac{m}{2}, \delta} = C[\phi^\pm; m] \otimes \hat{h}, \quad \text{for } 0 \neq m \in \mathbb{Z}. \]

4. $\hat{sl}$ has the following decomposition:
\[ \hat{sl} = \bigoplus_{m \in \mathbb{Z}} \hat{g}_{m, \delta} \bigoplus_{\alpha \in \Delta, m \in \mathbb{Z}} \hat{g}_{\frac{m}{2}, \delta + \alpha} \] (3.19)

Proof
First we prove the second assertion. Let $X \in g_\alpha$ for a $\alpha \in \Sigma$, $\alpha \neq 0$, and let $\varphi \in C[\phi^\pm; m]$ for a $m \in \mathbb{Z}$. We have, for any $h \in \mathfrak{h}$,
\[
[\phi^{+(0,0,1)} \otimes h, \varphi \otimes X]_\mathfrak{g} = \varphi \otimes (hX - Xh) = (\alpha, h) \varphi \otimes X,
\]
\[
[d, \varphi \otimes X]_\mathfrak{g} = \frac{m}{2} \varphi \otimes X,
\]
that is, for every $\hat{h} \in \hat{\mathfrak{h}}$, we have
\[
[\hat{h}, \varphi \otimes X]_\mathfrak{g} = \left( \frac{m}{2} \delta + \alpha, \hat{h} \right) (\varphi \otimes X). \] (3.20)
Therefore we have $\varphi \otimes X \in \hat{g}_{\frac{m}{2}, \delta + \alpha}$.

Conversely, for a given $m \in \mathbb{Z}$ and a $\xi \in \hat{g}_{\frac{m}{2}, \delta + \alpha}$, we shall show that $\xi$ has the form $\phi \otimes X$ with $\phi \in C[\phi^\pm; m]$ and $X \in g_\alpha$. Let $\xi = \phi \otimes X + \mu a + \nu d$ for $\phi \in C[\phi^\pm]$, $X \in \mathfrak{sl}(n, \mathbb{C})$ and $\mu, \nu \in \mathbb{C}$. $\phi$ is decomposed to the sum
\[
\phi = \sum_{n \in \mathbb{Z}} \phi_n
\]
by the homogeneous degree; \( \phi_n \in C[\phi^\pm; n] \). We have

\[
[\hat{h}, \xi] = [\phi^{+(0,0,1)} \otimes h + \sum s_k a_k + td, \phi \otimes X + \sum \mu_k a_k + \nu d] = \phi \otimes [h, X]
\]

\[+
  t\left(\sum_{n \in \mathbb{Z}} \frac{n}{2} \phi_n \otimes X\right)
\]

for any \( \hat{h} = \phi^{+(0,0,1)} \otimes h + \sum s_k a_k + td \in \hat{h} \). From the assumption we have

\[
[\hat{h}, \xi] = \langle \frac{m}{2} \delta + \alpha, \hat{h} \rangle \xi
\]

\[= <\alpha, h > \phi \otimes X + \left(\frac{m}{2} t+ <\alpha, h >\right)(\sum \mu_k a_k + \nu d)
\]

\[+ \frac{m}{2} t (\sum n \phi_n) \otimes X.
\]

Comparing the above two equations we have \( \mu_k = \nu = 0 \), and \( \phi_n = 0 \) for all \( n \) except for \( n = m \). Therefore \( \phi \in C[\phi^\pm; m] \). We also have \( [\hat{h}, \xi] = \phi \otimes [h, X] = \langle \alpha, h \rangle \phi \otimes X \) for all \( \hat{h} = \phi^{+(0,0,1)} \otimes h + \sum s_k a_k + td \in \hat{h} \). Hence \( X \in g_\alpha \) and \( \xi = \phi_m \otimes X \in \hat{g}_{\frac{m}{2} \delta + \alpha} \). We have proved

\( \hat{g}_{\frac{m}{2} \delta + \alpha} = C[\phi^\pm; m] \otimes g_\alpha \).

The proof of the third assertion is also carried out by the same argument as above if we revise it for the case \( \alpha = 0 \). The above discussion yields the first and the fourth assertions.

**Proposition 3.5.** We have the following relations:

1. \[
[\hat{g}_{\frac{m}{2} \delta + \alpha} , \hat{g}_{\frac{n}{2} \delta + \beta}] \subseteq \hat{g}_{\frac{m+n}{2} \delta + \alpha + \beta} ,
\]
   for \( \alpha, \beta \in \hat{\Delta} \) and for \( m, n \in \mathbb{Z} \).

2. \[
[\hat{g}_{\frac{m}{2} \delta} , \hat{g}_{\frac{n}{2} \delta}] \subseteq \hat{g}_{\frac{m+n}{2} \delta} ,
\]
   for \( m, n \in \mathbb{Z} \).
3.4 Chevalley generators of $\hat{sl}$

By the natural embedding of $sl(n, C)$ in $\hat{sl}$ we have the vectors

$$\hat{h}_i = \phi^{+(0,0,1)} \otimes h_i \in \hat{h},$$

$$\hat{x}_i = \phi^{+(0,0,1)} \otimes x_i \in \hat{g}_{0\delta + \alpha_i}, \quad \hat{y}_i = \phi^{+(0,0,1)} \otimes y_i \in \hat{g}_{0\delta - \alpha_i}, \quad i = 1, \cdots, r = n - 1.$$ Then

$$[\hat{x}_i, \hat{y}_j]_{\hat{sl}} = \delta_{ij} \hat{h}_i,$$

$$[\hat{h}_i, \hat{x}_j]_{\hat{sl}} = a_{ij} \hat{x}_j, \quad [\hat{h}_i, \hat{y}_j]_{\hat{sl}} = -a_{ij} \hat{y}_j, \quad 1 \leq i, j \leq r. \quad (3.23)$$

We have obtained a part of generators of $\hat{sl}$ that come naturally from $sl(n, C)$. We want to augment these generators to the Chevalley generators of $\hat{sl}$. We take the following set of generators of the algebra $C[\phi^+] | S^3$:

$$I = \phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad J = \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad (3.24)$$

$$\kappa = \frac{1}{\sqrt{2}} \phi^{+(1,0,2)} + \phi^{+(1,1,1)} + \frac{1}{\sqrt{2}} J \phi^{+(1,1,0)} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}, \quad (3.25)$$

$$\lambda = -\frac{1}{\sqrt{2}} \phi^{+(1,1,2)} + \phi^{+(1,0,1)} + \frac{1}{\sqrt{2}} J \phi^{+(1,0,0)} = \begin{pmatrix} \bar{z}_2 \\ -\bar{z}_1 \end{pmatrix}. \quad (3.26)$$

We put

$$\kappa_* = -\phi^{-(0,0,1)} = \frac{1}{|z|^4} \begin{pmatrix} z_1 \\ -\bar{z}_2 \end{pmatrix}, \quad \lambda_* = -\phi^{-(0,0,0)} = \frac{1}{|z|^4} \begin{pmatrix} z_2 \\ \bar{z}_1 \end{pmatrix}. \quad (3.27)$$

Lemma 3.6.

1. $\kappa, \lambda \in C[\phi^+; 1]$ and $\kappa_*, \lambda_* \in C[\phi^+; -3]$

2. $\kappa \kappa_* = \kappa_* \kappa = \lambda \lambda_* = \lambda_* \lambda = \frac{1}{|z|^2} I. \quad (3.28)$
Let $\theta$ be the highest root of $\mathfrak{sl}(n, \mathbb{C})$ and suppose that $x_\theta \in \mathfrak{g}_\theta$ and $y_\theta \in \mathfrak{g}_{-\theta}$ satisfy the relations $[x_\theta, y_\theta] = h_\theta$ and $(x_\theta | y_\theta) = 1$. We introduce the following vectors of $\hat{\mathfrak{sl}}$:

$$
\begin{align*}
\hat{y}_J &= J \otimes y_\theta \in \hat{\mathfrak{g}}_{0\delta-\theta}, & \hat{x}_J &= (-J) \otimes x_\theta \in \hat{\mathfrak{g}}_{0\delta+\theta}, \\
\hat{y}_\kappa &= \kappa \otimes y_\theta \in \hat{\mathfrak{g}}_{\frac{1}{2}\delta-\theta}, & \hat{x}_\kappa &= \kappa^* \otimes x_\theta \in \hat{\mathfrak{g}}_{-\frac{1}{2}\delta+\theta}, \\
\hat{y}_\lambda &= \lambda \otimes y_\theta \in \hat{\mathfrak{g}}_{-\frac{3}{2}\delta-\theta}, & \hat{x}_\lambda &= \lambda^* \otimes x_\theta \in \hat{\mathfrak{g}}_{\frac{1}{2}\delta+\theta}.
\end{align*}
$$

Then we have the generators of $\hat{\mathfrak{sl}}(n, H)(a)$ that are given by the following three tuples:

$$
\left( \hat{x}_i, \hat{y}_i, \hat{h}_i \right), \quad i = 1, 2, \ldots, r,
$$

$$
\left( \hat{x}_\lambda, \hat{y}_\lambda, \hat{h}_\theta \right), \quad \left( \hat{x}_\kappa, \hat{y}_\kappa, \hat{h}_\theta \right), \quad \left( \hat{x}_J, \hat{y}_J, \hat{h}_\theta \right).
$$

These three tuples satisfy the following relations.

**Proposition 3.7.**

1. 

$$
[\hat{x}_\pi, \hat{y}_i]_{\hat{\mathfrak{sl}}} = [\hat{y}_\pi, \hat{x}_i]_{\hat{\mathfrak{sl}}} = 0, \quad \text{for } 1 \leq i \leq r, \text{ and } \pi = J, \kappa, \lambda.
$$

2. 

$$
[\hat{x}_J, \hat{y}_J]_{\hat{\mathfrak{sl}}} = \hat{h}_\theta.
$$

3. 

$$
[\hat{x}_\lambda, \hat{y}_\lambda]_{\hat{\mathfrak{sl}}} = \sqrt{-1}\hat{h}_\theta, \quad [\hat{x}_\kappa, \hat{y}_\kappa]_{\hat{\mathfrak{sl}}} = \sqrt{-1}\hat{h}_\theta + \sqrt{-1}a_0.
$$

Adding $d$ to these generators of $\hat{\mathfrak{sl}}(n, H)(a)$ we obtain the Chevalley generators of $\hat{\mathfrak{sl}}$.

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