Differentiation properties of class $L^1([0,1]^2)$ with respect to two different basis of rectangles

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Abstract

It is a well known result by Saks [8] that there exist a function $f \in L^1(\mathbb{R}^2)$ so that for almost every $(x,y) \in \mathbb{R}^2$

$$\lim_{\text{diam} R \to 0} \left| \frac{1}{|R|} \int_R f(x,y) \, dx dy \right| = \infty,$$

where $R = \{(a,b) \times (c,d) : a < b, c < d\}$. In this note we address the following question: assume we have two different collections of rectangles; under which conditions there exists a function $f \in L^1(\mathbb{R}^2)$ so that its integral averages are divergence with respect to one collection and convergence with respect to another? More specifically, let $\mathcal{D}, \mathcal{C} \subset (0,1]$ and consider rectangles with side lengths in $\mathcal{D}$ and respectively in $\mathcal{C}$. We show that if the sets $\mathcal{D}$ and $\mathcal{C}$ are sufficiently “far” from each other, then such a function can be constructed. We also show that in the class of positive functions our condition is also necessary for such a function to exist.

1 Introduction

Let $\mathcal{R} = \{(a,b) \times (c,d)\}$ be the set of all rectangles with their sides parallel to the coordinate axis. Given a collection $\mathcal{C} \subset (0,1]$, let $\mathcal{R}_\mathcal{C} \subset \mathcal{R}$ be the collection of all rectangles $[a,b) \times (c,d]$ so that $b-a \in \mathcal{C}$ and $d-c \in \mathcal{C}$.

Definition 1.1. A family of rectangles $\mathcal{M} \subset \mathcal{R}$ is said to be a basis of differentiation (or simply a basis), if for any point $z \in \mathbb{R}^2$ there exists a sequence of rectangles $R_k \in \mathcal{M}$ such that $z \in R_k$, $k \in \mathbb{N}$, and $\text{diam} R_k \to 0$ as $k \to \infty$.

Let $\mathcal{C}, \mathcal{D} \subset (0,1]$ be two collections. Thus $\mathcal{R}_\mathcal{C}$ and $\mathcal{R}_\mathcal{D}$ will be basis of differentiation if and only if $\liminf \mathcal{C} = 0$ and $\liminf \mathcal{D} = 0$. Let $\mathcal{M} \subset \mathcal{R}$ be a differentiation basis. For any function $f \in L^1(\mathbb{R}^2)$ we define

$$\delta_{\mathcal{M}}(z, f) = \limsup_{\text{diam} R \to 0, \ z \in R \subset \mathcal{M}} \left| \frac{1}{|R|} \int_R f \, dm - f(z) \right|.$$

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Here and below, we denote by \( m \) or \( | \cdot | \) the Lebesgue measure on \( \mathbb{R}^2 \). The function \( f \in L^1(\mathbb{R}^2) \) is said to be differentiable at a point \( z \in \mathbb{R}^2 \) with respect to the basis \( \mathcal{M} \), if \( \delta_{\mathcal{M}}(z, f) = 0 \). Denote 
\[
\mathcal{F}(\mathcal{M}) = \{ f \in L^1(\mathbb{R}^2) : \delta_{\mathcal{M}}(z, f) = 0, m\text{-a.e. } z \in \mathbb{R}^2 \}.
\]

Let \( \Phi: \mathbb{R}^+ \to \mathbb{R}^+ \) be a convex function. Denote by \( \Phi(L)(\mathbb{R}^2) \) the class of measurable functions \( f \) defined on \( \mathbb{R}^2 \) such that \( \Phi(|f|) \in L^1(\mathbb{R}^2) \). If \( \Phi \) satisfies the \( \Delta_2 \)-condition \( \Phi(2t) \leq k \Phi(t) \), then \( \Phi(L)(\mathbb{R}^2) \) turns to be an Orlicz space with the norm 
\[
\|f\|_{\Phi} = \inf\left\{ c > 0 : \int_{\mathbb{R}^2} \Phi\left( \frac{|f|}{c} \right) \, dm \leq 1 \right\}.
\]

The following classical theorems of Jessen, Marcinkiewicz, and Zygmund \([5]\), and Saks \([8]\) determine the optimal Orlicz space, which functions have a.e. differentiable integrals with respect to the entire family of rectangles \( \mathcal{R} \) is the space 
\[
L(1 + \log L)(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)
\]
corresponding to the case \( \Phi(t) = t(1 + \log^+ t) \). See also \([2]\).

**Theorem 1.2** (Jessen-Marcinkiewicz-Zygmund \([5]\)). If 
\[
f \in L(1 + \log L)(\mathbb{R}^2),
\]
then 
\[
\delta_{\mathcal{R}}(z, f) = 0
\]
for \( m \)-almost every \( z \in \mathbb{R}^2 \).

**Theorem 1.3** (Saks \([8]\)). If 
\[
\Phi(t) = o(t \log t) \text{ as } t \to \infty,
\]
then \( \Phi(L)(\mathbb{R}^2) \not\subset \mathcal{F}(\mathcal{R}) \). Moreover, there exists a positive function \( f \in \Phi(L)(\mathbb{R}^2) \) such that 
\[
\delta_{\mathcal{R}}(z, f) = \infty \text{ everywhere.}
\]

In this paper, we are interested in differentiability property of class \( L^1([0, 1]^2) \) with respect to two basis \( \mathcal{R}_C \) and \( \mathcal{R}_D \). We investigate conditions under which there exists a function \( f \in L^1([0, 1]^2) \), so that 
\[
\delta_{\mathcal{R}_C}(z, f) = 0 \text{ for } m\text{-almost every } z \in [0, 1]^2
\]
and 
\[
\delta_{\mathcal{R}_D}(z, f) \neq 0 \text{ for } m\text{-almost every } z \in [0, 1]^2.
\]

To the best of the authors knowledge there is only one result that has some relation to the problem considered in this note. In \([7]\) the authors study equivalence of differentiation basis of dyadic rectangles. More specifically consider the basis 
\[
\mathcal{R}^\text{dyadic} = \left\{ \left[ \frac{i - 1}{2^m}; \frac{i}{2^m} \right] \times \left[ \frac{j - 1}{2^m}; \frac{j}{2^m} \right) : i, j, n, m \in \mathbb{Z} \right\}.
\]

Let \( \Delta = \{ \nu_k : k = 1, 2, \ldots \} \) be an increasing sequence of positive integers. This sequence generates the rare basis \( \mathcal{R}_\Delta^\text{dyadic} \) of dyadic rectangles of the form \([1]\) with \( n, m \in \Delta \). This kind of bases have
also been considered in several papers [3, 4, 6, 9]. In [7] the authors study under which conditions the basis $R_{\Delta}$ will be equivalent to the basis given by $R_{\Delta}$. We remark that although in [7] one only considers dyadic rectangles, this case can be compared with $D = [0, 1] \setminus C$ in our case.

Unlike the results considered above, in this note we consider the problem in full generality, namely we and are interested in all rectangles with their sides in $C$ or $D$ that contain the point of differentiation.

The paper is self contained and uses some methods from analysis and probability theory.

## 2 Main results

Intuitively, in order to have divergence and convergence phenomena simultaneously, the two set $C$ and $D$ have to be far from each other. We now formalize this intuition. For each $x \in D$, we define the following two numbers

$$
\overline{x} = \sup\{ a \in C : a < x \},
$$

$$
\underline{x} = \inf\{ a \in C : a > x \}.
$$

The ratios $\overline{x}/x$ and $x/\underline{x}$ denote that distance of $x$ from the set $C$ from below and above. We prove the following theorem.

**Theorem 2.1.** Let $C, D \subset (0, 1]$ and assume

$$
\liminf_{x \to 0, x \in D} \left( \max \left\{ \frac{\overline{x}}{x}, \frac{\underline{x}}{x} \right\} \right) = 0.
$$

Then there exists a function $f \in L^1([0, 1]^2, m)$ so that for $m$-almost all $z \in [0, 1]^2$

$$
\delta_{R_C}(z, f) = 0,
$$

and

$$
\delta_{R_D}(z, f) = \infty.
$$

The function that will be constructed in Theorem 2.1 is unbounded both from above and below. This leads as to the following question

**Question.** Does there exist a positive function $f$ satisfying the conditions of Theorem 2.1?

We have the following theorem in the opposite direction

**Theorem 2.2.** Let $C, D \subset (0, 1]$ and assume that

$$
\liminf_{x \to 0, x \in D} \left( \max \left\{ \frac{\overline{x}}{x}, \frac{\underline{x}}{x} \right\} \right) > 0.
$$

Let $f \in L^1([0, 1]^2, m)$, with $f \geq 0$ almost surely. If

$$
\delta_{R_C}(z, f) = 0 \text{ for } m\text{-almost every } z \in [0, 1]^2,
$$

then

$$
\delta_{R_D}(z, f) = 0 \text{ for } m\text{-almost every } z \in [0, 1]^2.
$$
The proof of Theorem 2.1 is based on the following theorem.

**Theorem 2.3.** For every \( \varepsilon, \delta > 0 \) and \( n > 0 \), there exist a function \( f \in L^\infty([0,1]^2, m) \), \( \eta = \eta(\varepsilon, \delta, n) \in (0, \delta) \) and a set \( Q \subset [0,1]^2 \), so that

\[
\|f\|_{L^1(m)} \leq 4 + o(1),
\]

and \( |Q| > 1 - \varepsilon \) such that

(i) If \( z \in Q \) and \( z \in A \in \mathcal{R}_C \), then

\[
\left| \frac{1}{|A|} \int_A f \, dm \right| \leq 2.
\]

(ii) If \( z \in Q \), then there exists \( B \in \mathcal{R}_D \), with \( z \in B \) and \( |B| > \eta \), so that

\[
\left| \frac{1}{|B|} \int_B f \, dm \right| > n.
\]

(iii) For every \( R \in \mathcal{R} \) with \( |R| > \delta \), we have

\[
\left| \frac{1}{|R|} \int_R f \, dm \right| < 1.
\]

### 3 Preliminaries

#### 3.1 Auxiliary construction

Let \( n \in \mathbb{N} \). Given a decreasing sequence \( b_1 > b_2 > \cdots > b_n > \cdots > b_{2n} \), let

\[
B_1 = [0, b_1] \times [0, b_{2n}], B_2 = [0, b_2] \times [0, b_{2n-1}], \ldots, B_n = [0, b_n] \times [0, b_{n+1}].
\]

Here the first factor is the height and the second one is the width. Suppose that the sequence \((b_j)\) satisfies

\[
b_1 b_{2n} \leq b_2 b_{2n-1} \leq \cdots \leq b_{n-1} b_{n+2} \leq b_n b_{n+1}.
\]

In other words, we suppose that

\[
|B_1| \leq |B_2| \leq \cdots \leq |B_{n-1}| \leq |B_n|.
\]

For \( j = 1, \ldots, n-1 \), let \( q_j \in \mathbb{N} \) be such that \( q_j = \lceil |B_{j+1}|/|B_j| \rceil \).

Let \( \Theta \) be the set of \((\theta_{n-1}, \theta_{n-2}, \ldots, \theta_2, \theta_1)\) with \( \theta_j \in \{0,1,\ldots,q_j\} \) satisfying the following condition: if \( \theta_k = 0 \) for some \( k \), then \( \theta_l = 0 \) for every \( l \in \{k, k-1, \ldots, 1\} \). For each \( \theta = (\theta_{n-1}, \theta_{n-2}, \ldots, \theta_1) \in \Theta \), let

\[
|\theta| = \begin{cases} 
1, & \text{if all } \theta_j \neq 0 \\
\max\{1 \leq j \leq n-1 : \theta_j = 0\} + 1, & \text{otherwise}
\end{cases}
\]

and define rectangles \( B(\theta) \) as follows. For \( \theta \in \Theta \) with \( |\theta| = n \), that is \( \theta_{n-1} = 0 \), we let

\[
B(\theta) = [0, b_n] \times [0, b_{n+1}] = B_n.
\]
For $\theta \in \Theta$ with $|\theta| \in \{1, \ldots, n - 1\}$, we let

$$B(\theta) = [0, b_{|\theta|}] \times \left[ \sum_{j=|\theta|}^{n-1} (\theta_j - 1)b_{2n+1-j}, \sum_{j=|\theta|}^{n-1} (\theta_j - 1)b_{2n+1-j} + b_{2n+1-|\theta|} \right],$$

where note that $\theta_{n-1} \in \{1, \ldots, q_{n-1}\}$ as $|\theta| \neq n$. Note also that for each $\theta \in \Theta$, one has $|B(\theta)| = b_{|\theta|}b_{2n+1-|\theta|}$.

Define subsets of $[0, 1]^2$ by

$$E = \bigcup_{\theta \in \Theta} B(\theta) \quad \text{and} \quad F = \bigcup_{\theta \in \Theta: |\theta| < n} B(\theta).$$

One has

$$E = \bigcup_{j=1}^{n} \bigcup_{|\theta| = j} B(\theta) = \left( \bigcup_{j=1}^{n-1} \bigcup_{|\theta| = j} B(\theta) \right) \cup B_n,$$

$$F = \bigcup_{j=1}^{n-1} \bigcup_{|\theta| = j} B(\theta), \tag{4}$$

where for every $j \in \{1, \ldots, n - 1\}$ there are $q_{n-1} \cdots q_j$-many rectangles $B(\theta)$ of $|\theta| = j$.

### 3.2 Area estimates

**Lemma 3.1.** Suppose that there is $\lambda \in (0, 1)$ such that $b_{k+1}/b_k < \lambda$ for every $k = 1, \ldots, 2n - 1$. Then one has

$$\frac{1}{n} + \frac{1 - \lambda}{n} \sum_{j=1}^{n-1} \frac{q_{n-1} \cdots q_j}{(q_{n-1} + 1) \cdots (q_j + 1)} \leq \frac{|E|}{n|B_n|} \leq 1, \tag{5}$$

and

$$\frac{1 - \lambda}{n - 1} \sum_{j=1}^{n-1} \frac{q_{n-1} \cdots q_j}{(q_{n-1} + 1) \cdots (q_j + 1)} \leq \frac{|F|}{(n-1)|B_n|} \leq 1. \tag{6}$$

**Proof.** By construction, one has

$$|E| = b_nb_{n+1} + q_{n-1}b_{n+2}(b_{n-1} - b_n) + q_{n-1}q_{n-2}b_{n+3}(b_{n-2} - b_{n-1}) + \cdots + q_{n-1} \cdots q_1 b_{2n}(b_1 - b_2)$$

$$= |B_n| + q_{n-1}|B_{n-1}| + q_{n-1}q_{n-2}|B_{n-2}| + \cdots + q_{n-1} \cdots q_1 |B_1|$$

$$- (q_{n-1}b_{n+2}b_n + q_{n-1}q_{n-2}b_{n+3}b_{n-1} + \cdots + q_{n-1} \cdots q_1 b_{2n}b_2).$$

Since

$$q_{n-1} \cdots q_{j+1}q_j |B_j| \leq q_{n-1} \cdots q_{j+1} |B_{j+1}| \leq \cdots \leq q_{n-1} |B_{n-1}| \leq |B_n|$$

for every $j = 1, \ldots, n - 1$, it follows that $|E| \leq n|B_n|$. On the other hand, one has

$$\frac{|E|}{n|B_n|} = \frac{|B_n| + \sum_{j=1}^{n-1} q_{n-1} \cdots q_j b_{2n+1-j}(b_j - b_{j+1})}{n|B_n|}.$$
\[ = \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} \frac{q_{n-1} \cdots q_j b_{2n+1-j} (b_j - b_{j+1})}{|B_n|}. \]

Since
\[ |B_n| \leq (q_{n-1} + 1)|B_{n-1}| \leq \cdots \leq (q_{n-1} + 1) \cdots (q_j + 1)|B_j| \]

for each \( j = 1, \ldots, n - 1 \), it follows that
\[ \frac{q_{n-1} \cdots q_j b_{2n+1-j} (b_j - b_{j+1})}{|B_n|} \leq \cdots \leq \frac{q_{n-1} \cdots q_j b_{2n+1-j} (b_j - b_{j+1})}{(q_{n-1} + 1) \cdots (q_j + 1)} \]

and hence
\[ \frac{|E|}{n|B_n|} \geq \frac{1}{n} + \frac{1 - \lambda}{n} \sum_{j=1}^{n-1} \frac{q_{n-1} \cdots q_j}{(q_{n-1} + 1) \cdots (q_j + 1)}. \]

We have obtained (5). Since the proof of (6) is same as that of (5), we omit it. \( \square \)

Let
\[ K_E = \left( \frac{1}{n} + \frac{1 - \lambda}{n} \sum_{j=1}^{n-1} \frac{q_{n-1} \cdots q_j}{(q_{n-1} + 1) \cdots (q_j + 1)} \right)^{-1}, \]
\[ K_F = \left( \frac{1 - \lambda}{n-1} \sum_{j=1}^{n-1} \frac{q_{n-1} \cdots q_j}{(q_{n-1} + 1) \cdots (q_j + 1)} \right)^{-1}. \]

Then it follows from (5) and (6) in Lemma 3.1 that
\[ K_E^{-1} \frac{n}{n-1} \leq \frac{|E|}{|F|} \leq K_F \frac{n}{n-1}. \]  \( \tag{7} \)

Notice that \( K_E, K_F \geq 1 \) and they can be arbitrarily close to one by taking \( \lambda \in (0, 1) \) small and \( (b_n) \) relevantly.

4 Main Lemma

Without loss of generality we can assume that \( \mathcal{D} \) is a sequence, i.e. \( \mathcal{D} = \{b_n\}_{n \geq 1} \). For each \( b_n \in \mathcal{D} \), we define
\[ \overline{a}_n = \sup \{ a \in \mathcal{C} : a < b_n \}, \]
\[ \underline{a}_n = \inf \{ a \in \mathcal{C} : a > b_n \}. \]
Recall that we are given $C, D \subseteq (0, 1]$, where $D = \{b_n\}_{n \in \mathbb{N}}$, and assume
\[
\liminf_{n \to \infty} \left( \max \left\{ \frac{\bar{a}_n}{b_n}, \frac{b_n}{a_n} \right\} \right) = 0. \tag{8}
\]
Throughout this section, we assume (8). In this section, we prove the following lemma which will play a fundamental role for the proof of Theorems.

**Lemma 4.1.** Given $\varepsilon, \delta > 0$ and $n \in \mathbb{N}$, there exist sets $F, D \subseteq [0, 1]^2$, a function $h \in L^1([0, 1]^2, m)$, and $\eta = \eta(\varepsilon, \delta, n) \in (0, \delta)$ satisfying $|D| \leq \varepsilon |F|$ and
\[
\frac{\|h\|_{L^1(m)}}{|F|} = 2 + o(1), \tag{9}
\]
such that for every $z \in [0, 1]^2$ we have the following.

(i) If $z \notin D$ and $z \in A \in \mathcal{R}_C$, then
\[
\left| \frac{1}{|A|} \int_A h \, dm \right| \leq 1.
\]

(ii) If $z \in F$, then there exists $B \in \mathcal{R}_D$, with $z \in B$ and $|B| > \eta$, so that
\[
\frac{1}{|B|} \int_B h \, dm > n.
\]

(iii) For every $R \in \mathcal{R}$ with $|R| > \delta$, we have
\[
\frac{1}{|R|} \int_R h \, dm < 1/2.
\]

### 4.1 Proof of Lemma 4.1

We start with construction of a function and sets in the following two sections. Then we complete the proof of Lemma 4.1 in Section 4.1.3.

#### 4.1.1 Construction of the function

Define $\Theta^* = \{\theta \in \Theta : |\theta| = 1\}$. For $\tau \in (0, b_{2n})$ and $\theta \in \Theta^*$, define a subset of $B(\theta)$ by
\[
B_\tau(\theta) = [0, \tau] \times \left[ \sum_{j=1}^{n-1} (\theta_j - 1)b_{2n+1-j}, \sum_{j=1}^{n-1} (\theta_j - 1)b_{2n+1-j} + \tau \right].
\]
Namely $B_\tau(\theta)$ is the square with side lengths $\tau$ at the bottom left corner of $B(\theta)$ for $\theta \in \Theta^*$. Define also $\sigma : \Theta^* \to \{1, -1\}$ by
\[
\sigma(\theta) = \begin{cases} +1, & \text{if } \theta_{n-1} \text{ is odd}, \\ -1, & \text{if } \theta_{n-1} \text{ is even}. \end{cases}
\]

Then define $h : [0, 1]^2 \to \mathbb{R}$ by
\[
h(z) = \begin{cases} \sigma(\theta) \frac{n|B_n|}{\tau^2 q_1 \cdots q_{n-1}}, & \text{if } z \in B_\tau(\theta) \text{ for some } \theta \in \Theta^*, \\ 0, & \text{otherwise}. \end{cases} \tag{10}
\]
Lemma 4.2. For every $\theta \in \Theta$, one has

$$n \leq \frac{1}{|B(\theta)|} \left| \int_{B(\theta)} h \, dm \right| \leq n \prod_{j=\theta}^{n-1} \left( 1 + \frac{1}{q_j} \right).$$

Proof. By construction, one has

$$\left| \int_{B(\theta)} h \, dm \right| = \frac{n|B_n|}{\tau^2 q_1 \cdots q_{n-1}} \times |B(\theta) \cap \text{supp} h|$$

$$= \frac{n|B_n|}{\tau^2 q_1 \cdots q_{n-1}} \times \begin{cases} \tau^2 q_1 \cdots q_{|\theta|-1}, & |\theta| > 1 \\ \tau^2, & |\theta| = 1 \end{cases}$$

$$= \frac{n|B_n|}{q_{|\theta|} \cdots q_{n-1}}.$$

and hence

$$\frac{1}{|B(\theta)|} \left| \int_{B(\theta)} h \, dm \right| = \frac{1}{|B(\theta)|} \frac{n|B_n|}{q_{|\theta|} \cdots q_{n-1}} \geq \frac{n|B_n|}{|B_n|} = n.$$

Since $|B_n| \leq (q_{n-1}+1)|B_{n-1}| \leq \cdots \leq (q_{n-1}+1) \cdots (q_{|\theta|}+1)|B(\theta)|$, one also has

$$\frac{1}{|B(\theta)|} \left| \int_{B(\theta)} h \, dm \right| = \frac{1}{|B(\theta)|} \frac{n|B_n|}{q_{|\theta|} \cdots q_{n-1}} \leq n \prod_{j=\theta}^{n-1} \left( 1 + \frac{1}{q_j} \right).$$

Lemma 4.3. Let $n \in \mathbb{N}$. Then we have

$$\left\| h \right\|_{L^1([0,1]^2,m)} \leq 2 K_E,$$  \hfill (11)

and

$$2 K_E^{-1} \frac{n}{n-1} \leq \frac{\left\| h \right\|_{L^1([0,1]^2,m)}}{|F|} \leq 2 K_E K_F \frac{n}{n-1}. \hfill (12)$$

Proof. Note first that

$$\left\| h \right\|_{L^1([0,1]^2,m)} = 2 \int_F |h| \, dm.$$

It follows from the construction that

$$\int_F |h| \, dm = \frac{n|B_n|}{\tau^2 q_1 \cdots q_{n-1}} \times |F \cap \text{supp} h| = \frac{n|B_n|}{\tau^2 q_1 \cdots q_{n-1}} \times \tau^2 q_1 \cdots q_{n-1} = n|B_n|,$$

and hence Lemma 3.1-(5) implies (11). By using (7), we have (12) from (11).
4.1.2 Construction of the exceptional set $D$

Let $h$ be the function of the form (10) defined in Section 4.1.1. In this section, we will construct a set $D$ while examining the integral averages of $h$ with respect to $R_C$ by making use of the condition (8). Indeed, one uses the following property on the two sets $C$ and $D = \{b_j\}_{j \in \mathbb{N}}$. The proof is a direct consequence of the condition (8), and is omitted.

**Lemma 4.4.** Assume the condition (8). Given $b_1 > \cdots > b_n$, and $\lambda_k \in (0, 1)$ for $k = 1, \ldots, n$, one can choose $b_{n+1} > \cdots > b_{2n}$ such that

$$\frac{\overline{a}_{n+k}}{b_{n+k}} \leq \lambda_k \frac{b_{n-k+1}}{\overline{a}_{n-k}},$$

and

$$\frac{b_{n+k}}{\overline{a}_{n+k}} \leq \lambda_k \frac{\overline{a}_{n-k+1}}{b_{n-k+1}}$$

for $k = 1, \ldots, n$, where $\overline{a}_0 = 1$ as a convention.

Henceforth, we denote rectangles with side lengths $x$ and $y$ by $A_{xy}$, where $x$ is the length of the vertical side and $y$ is that of the horizontal one. In other words, we assume that $x$ is the height and $y$ is the width of our rectangle. Let

$$A_C = \{A_{xy} \subset [0, 1]^2 : A_{xy} \cap \text{supp } h \neq \emptyset , \ x, y \in C\}.$$

We primarily divide our argument into the following cases with respect to the heights of $A_{xy} \in A_C$: 1) $x \leq b_{n+1}$, 2) $x \in (b_n, b_1]$, 3) $x > b_1$, and 4) $x \in (b_{n+1}, b_n]$.

**Case 1) $x \leq b_{n+1}$:** By assumption (8), take $b_{n+1}$ so that

$$\frac{\overline{a}_{n+1}}{b_{n+1}} \ll b_n.$$

(Here and below, we will sometimes write $X \ll Y$ if for a given $\lambda \in (0, 1)$ one can take $X$ so small that $X \leq \lambda Y$.) Then for every $y \in (0, 1]$ one has

$$xy \leq x \leq \overline{a}_{n+1} \ll b_nb_{n+1} = |B_n|.$$

Let

$$D_1 = \bigcup_{A_{xy} \in A_C} \{\text{int } A_{xy}, (x, y) \in (0, b_{n+1}] \times (0, 1]\}.$$

**Case 2) $b_n < x \leq b_1$:** There is $r = r(x) \in \{1, \ldots, n-1\}$ such that

$$b_{r+1} < x \leq b_r.$$

We consider two cases depending on the side $y$.

2-i) **The case $y > b_{2n-r}$:** We begin with the case where $A = A_{xy}$ contains $B(\theta) \in \mathcal{R_D}$ with $|\theta| = r+1$, that is height $b_{r+1}$ and width $b_{2n-r}$. More precisely, there is $p = p(y) \in \mathbb{N}$ such
that $A$ contains $p$-many disjoint copies of $B_{|\theta|} = B_{r+1}$ and does not contain $(p+1)$-many disjoint copies of it. We then have by Lemma 4.2 that

$$
\int_A h \, dm \leq |B(\theta)|(p + 2)n \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{q_j}\right).
$$

Since $A$ contains a rectangle of height $x$ and width $pb_{2n-r}$, it follows that

$$
\int_A h \, dm \leq \frac{(p + 2)b_{r+1}b_{2n-r}}{xp_{b_{2n-r}}} \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{q_j}\right)
$$

$$
= \frac{n}{p} \frac{b_{r+1}}{x} \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{q_j}\right)
$$

$$
\leq \frac{n}{p} \frac{b_{r+1}}{a_{r+1}} \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{q_j}\right). \quad (13)
$$

Next, suppose that $A$ contains no rectangles $B(\theta) \in \mathcal{R}_D$ with $|\theta| = r + 1$, that is the case of $p = 0$. One still has

$$
\int_A h \, dm \leq \frac{2n b_{r+1} b_{2n-r}}{xy} \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{q_j}\right) \leq 2n \frac{b_{r+1}}{a_{r+1}} \prod_{j=r+1}^{n-1} \left(1 + \frac{1}{q_j}\right). \quad (14)
$$

2-ii) The case $y \leq b_{2n-r}$. By definition, one has

$$
\frac{y}{b_{2n-r}} \leq \frac{\overline{a}_{2n-r}}{b_{2n-r}} \quad \text{and} \quad \frac{b_{r+1}}{x} \geq \frac{b_{r+1}}{a_r}.
$$

By applying Lemma 4.4 with $k = n - r$, one obtains

$$
\frac{\overline{a}_{2n-r}}{b_{2n-r}} \leq \lambda \frac{b_{r+1}}{a_r},
$$

and thus

$$
\frac{y}{b_{2n-r}} \leq \frac{\overline{a}_{2n-r}}{b_{2n-r}} \leq \lambda \frac{b_{r+1}}{a_r} \leq \lambda \frac{b_{r+1}}{x},
$$

which means $xy \leq \lambda b_{r+1} b_{2n-r} = |B_{r+1}|$.

Let

$$
D_2 = \bigcup_{r=1}^{n-1} \bigcup_{A_{xy} \in A'_C} \{\text{int}A_{xy}: (x, y) \in (b_{r+1}, b_r) \times (b_{2n-r+1}, b_{2n-r})\}.
$$

Case 3) $x > b_1$: We divide into two cases as follows.

3-i) The case $y > b_{2n}$. By the same argument as Case 2-i), one can show the convergence of integral averages of $h$ over $A_{xy}$.
3-ii) The case $y \leq b_{2n}$. As in Case 2-ii), one obtains
\[ \frac{y}{b_{2n}} \leq \frac{a_{2n}}{b_{2n}} \leq \frac{b_1}{\delta_0} \leq \frac{b_1}{\lambda x} \]
with making use of Lemma 4.4 with $k = n$, and hence $xy \leq \lambda b_1 b_{2n} = \lambda |B_1|$. Let
\[ D_3 = \bigcup_{A_{xy} \in A_C} \{ \text{int} A_{xy} : (x, y) \in (b_1, 1) \times (0, b_{2n}) \} . \]

Case 4) $b_{n+1} < x \leq b_n$: We divide into two cases as follows.

4-i) The case $y > b_{n+1}$. For such an $A = A_{xy} \in A_C$, it follows from the definition of $h$ (10) that either $\int_A h \, dm = 0$ or there are at most two $\theta, \theta' \in \Theta$ with $|\theta| = |\theta'| = n - 1$ such that
\[ \left| \int_A h \, dm \right| \leq \int_A |h| \, dm \leq \int_{B(\theta)} |h| \, dm + \int_{B(\theta')} |h| \, dm . \]
In the latter case, one has
\[ \frac{1}{|A|} \left| \int_A h \, dm \right| \leq \frac{2}{|A|} \int_{B(\theta)} |h| \, dm \leq 2 \frac{|B(\theta)|}{|A|} n \left( 1 + \frac{1}{\lambda n} \right) \]
by Lemma 4.2. Since
\[ |B(\theta)| = b_{\theta}|b_{2n+1} - |\theta| = b_{n-1}b_{n+2} \leq b_{n+2} \]
it follows that
\[ \frac{1}{|A|} \left| \int_A h \, dm \right| \leq 4n \frac{b_{n+2}}{|A|} \leq 4n \frac{b_{n+2}}{b_{n+1}^2} . \]

4-ii) The case $y \leq b_{n+1}$. One has $xy \ll b_{n+1} b_{n+1} = |B_n|$ by (8). Let
\[ D_4 = \bigcup_{A_{xy} \in A_C} \{ \text{int} A_{xy} : (x, y) \in (b_{n+1}, b_n) \times (0, b_{n+1}) \} . \]

Now we set
\[ D = D_1 \cup D_2 \cup D_3 \cup D_4. \]
Observe here that $D_4$ is covered by $D_2$, hence one has $D = D_1 \cup D_2 \cup D_3$.

Lemma 4.5. Given $\varepsilon \in (0, 1)$, one can define $F$ and $D$ such that $|D| \leq \varepsilon |F|$.

Proof. We see that $D_1$ is covered by a rectangle $R$ with height $2(\bar{\pi}_{n+1} + \tau q_{n-1})$ and width 1. Then by the assumption (8),
\[ |D_1| \leq |R| = 2(\bar{\pi}_{n+1} + \tau q_{n-1}) \ll 2(b_n b_{n+1} + \tau q_{n-1}) = 2(|B_n| + \tau q_{n-1}) . \]

Next, we estimate $|D_2|$. As we have seen in Case 2-ii, the area of each rectangle $A_{xy}$ with $(x, y) \in (b_{r+1}, b_r) \times (b_{2n-r+1}, b_{2n-r})$ is estimated as
\[ |A_{xy}| = xy \leq \bar{\pi} \bar{\pi}_{2n-r} \leq \lambda |B_{r+1}| \]
for each \( r \in \{1, \ldots, n-1\} \). Here, it follows from Lemma 4.4 with \( k = n - r \) that

\[
\overline{a}_r \overline{a}_{2n-r} \leq \lambda b_{r+1} b_{2n-r}
\]

and

\[
b_{r+1} b_{2n-r} \leq \lambda \overline{a}_{r+1} \overline{a}_{2n-r}.
\]

Since \( \overline{a}_{2n-r} \leq \overline{a}_{2n-r-1} \) by definition, one has

\[
\overline{a}_r \overline{a}_{2n-r} \leq \lambda^2 \overline{a}_{r+1} \overline{a}_{2n-r-1} \tag{17}
\]

for each \( r \in \{1, \ldots, n-1\} \). Note also that

\[
\frac{\overline{a}_{r+1}}{\overline{a}_r} = \frac{b_{r+1}}{b_r} \frac{\overline{a}_r}{\overline{a}_{r+1}} \leq \frac{b_r}{b_{r+1}} \frac{\overline{a}_{r+1}}{\overline{a}_r} \ll 1 \tag{18}
\]

by (18). By making use of (18) and (17), we can apply the same argument for the proof of Lemma 3.1 to the sequence \( \{\overline{a}_1, \ldots, \overline{a}_{n-1}, \overline{a}_{n+1}, \ldots, \overline{a}_{2n-1}\} \). As a result, we have

\[
|D_2| \leq (n-1) \left| \bigcup_{A_{xy} \in \mathcal{A}_C} \{x, y \in (b_{r+1}, b_r) \times (b_{2n-r+1}, b_{2n-r})\} \right|
\]

\[
\ll (n-1) \left| \bigcup_{|q|=r+1} B(\theta) \right| \leq (n-1)|B_n|.
\]

In the same way as above, we have \( |D_3| \ll |B_n| \).

Consequently, we have

\[
|D| \leq |D_1| + |D_2| + |D_3| \ll |B_n| + (n-1)|B_n| + |B_n| \leq \frac{n+1}{n} K_E |E|
\]

by Lemma 3.1 (5), and thus

\[
|D| \ll K_E K_F \frac{n+1}{n-1} |F|
\]

by (7).

\section*{4.1.3 Proof of Lemma 4.1}

**Lemma 4.6.** Let \( R \) be a rectangle with \( |R| > 4n|B_n| \). Then one has

\[
\frac{1}{|R|} \left| \int_R h \, dm \right| < \frac{1}{2}
\]

**Proof.** We have

\[
\frac{1}{|R|} \left| \int_R h \, dm \right| \leq \frac{1}{|R|} \|h\|_{L^1(m)} = \frac{2}{|R|} n|B_n| < \frac{1}{2}
\]

by (8).

**Proof of Lemma 4.7.** One can suppose, with the aid of (8), that \( b_1 > \cdots > b_n > b_{n+1} > \cdots > b_{2n} \) will satisfy

1. Condition (3),
2. \( \frac{b_n}{a_n} < \cdots < \frac{b_1}{a_1} < \frac{1}{3n2^{n-1}} \),

3. Lemma 4.4

4. \( \delta > 4nb_n b_{n+1} \),

5. \( b_{n+2} < \frac{b_{n+1}^2}{4n} \).

For such a sequence, let \( F \) and \( D \) be the sets and \( h \) be the function defined as (4), (16) and (10), respectively. Then we have Lemma 4.5 and the property (9) follows from (12). Let \( \eta = |B_1| \). Then \( \eta \in (0, \delta) \), and the property (ii) follows from Lemma 4.2.

The property (i) follows from the consequences of Cases 2-i), 3-i), and 4-i). Indeed, we have for (13) that

\[
\frac{1}{|A|} \int_A h \, dm \leq \frac{n}{p} \frac{b_{r+1}}{a_{r+1}} \prod_{j=r+1}^{n-1} \left( 1 + \frac{1}{q_j} \right) \leq 3n \frac{1}{3n2^{n-1}2^{n-r-1}} < 1
\]

for every \( r \in \{1, \ldots, n-1\} \) by the condition 2. Similarly, we have for (14) that

\[
\frac{1}{|A|} \int_A h \, dm \leq 2n \frac{b_{r+1}}{a_{r+1}} \prod_{j=r+1}^{n-1} \left( 1 + \frac{1}{q_j} \right) < 1
\]

for every \( r \in \{1, \ldots, n-1\} \) by the condition 2. For (15), it follows from by the condition 5 that

\[
\frac{1}{|A|} \int_A h \, dm \leq 4n \frac{b_{n+2}}{b_{n+1}^2} < 1.
\]

Lemma 4.6 yields the property (iii). Lemma 4.1 is obtained. \( \square \)

5 Proof of Theorem 2.3

5.1 Random translations

Let

\[ \omega = (\alpha, \beta), \]

where \( \alpha \) and \( \beta \) are uniformly distributed on \([0, 1]^2\). Let \( \omega_1, \ldots, \omega_N \) be a set of independent, uniformly distributed vectors on \([0, 1]^2\). For \( z \in [0, 1]^2 \) and \( \omega = (\omega_1, \ldots, \omega_N) \), consider the product

\[ f_N(z, \omega) = \prod_{k=1}^N (1 - \mathbb{1}_F(z + \omega_k)). \]

Note that \( f_N(z, \omega) = 0 \) if and only if \( 1 - \mathbb{1}_F(z + \omega_k) = 0 \) for some \( k \), i.e., \( z \in F + \omega_k \). Hence \( \text{supp} f_N(\cdot, \omega) \subseteq [0, 1]^2 \) is the set which is not covered by the set

\[ F_0(\omega) = \bigcup_{k=1}^N \{ F + \omega_k \}. \]
Therefore, the support of $1 - f_N(\cdot, \omega)$ will be the set that is covered by the set $F_0(\omega)$. Next, consider also
\[ g_N(z, \omega) = \prod_{k=1}^{N} \left( 1 - 1_D(z + \omega_k) \right), \]
and denote
\[ D_0(\omega) = \bigcup_{k=1}^{N} \{ D + \omega_k \}. \]
Note that the support of the function $((1 - f_N) \cdot g_N)(\cdot, \omega)$ is the set of points that is covered by $F_0(\omega)$ but not by $D_0(\omega)$, namely $F_0(\omega) \setminus D_0(\omega)$. For each $\omega = (\omega_1, \ldots, \omega_N)$, let $|Q_0(\omega)| = |Q_0(\omega)|$. Then we have
\[ |Q_0(\omega)| = \int_{[0,1]^2} ((1 - f_N) \cdot g_N)(z, \omega) \, dm(z). \]
Below, we denote the integration with respect to $\omega = (\omega_1, \ldots, \omega_N)$ by $\mathcal{E}$ for notational simplicity. Namely,
\[ \mathcal{E}(u) = \int_{([0,1]^2)^N} u(\omega) \, d\omega = \int_{[0,1]^2} \cdots \int_{[0,1]^2} u(\omega_1, \ldots, \omega_N) \, d\omega_1 \cdots d\omega_N \]
for a function $u$ on $([0,1]^2)^N$.

**Remark 5.1.** Precisely, the integral (19) above should be written as
\[ \mathcal{E}(u) = \int_Y u(\omega(y)) \, dP(y) = \int_{([0,1]^2)^N} u \, dP^\omega \]
where $(Y, P)$ denotes the underlying probability space on which the $N$-tuple $\omega = (\omega_1, \ldots, \omega_N)$ of independent random variables is defined, and $P^\omega$ is the joint distribution. By independency,
\[ \mathcal{E}(u) = \int_{([0,1]^2)^N} u \, dP^\omega = \int_{[0,1]^2} \cdots \int_{[0,1]^2} u \, dP^\omega_1 \cdots dP^\omega_N, \]
and denote (20) by (19) for an abuse of notation.

Henceforth, we construct a function defined on (a neighborhood of) a set $Q_0(\omega) = F_0(\omega) \setminus D_0(\omega)$ with suitably chosen $\omega$. Here we have the following lemma on such $\omega$.

**Lemma 5.2.** Let $N = \lceil 1/|F| \rceil \in \mathbb{N}$. There is an open set $\Omega \subset ([0,1]^2)^{1/|F|}$ with $|\Omega| > 0$ such that for any $\omega \in \Omega$ we have
\[ |Q_0(\omega)| = |F_0 \setminus D_0|(\omega) \geq \frac{99}{100} \left( \frac{1}{(2e)^2} - \frac{1}{e} \right) > 0 \]
for sufficiently small $e \in (0,1)$.

In fact, we will prove a more general result which will be of independent interest. See also Remark 5.6.
Proposition 5.3. Let \(A, B \subset [0, 1]^2\) be sets such that \(0 < |B| \leq c|A|\) for some \(c > 0\), and let \(N = \lfloor 1/|A| \rfloor \in \mathbb{N}\). Suppose that for any \(\omega \in ([0, 1]^2)^N\) we have \(|A \cap (B + \omega)| \leq \varepsilon|A|\) for some \(\varepsilon \in (0, \min\{1, c_0\})\) where \(c_0 = |B|/|A| \in (0, c]\). Then there exist \(\kappa = \kappa(c_0, |A|) \in (0, 1)\) and an open set \(\Omega \subset ([0, 1]^2)^N\) with \(|\Omega| > 0\) such that for any \(\omega \in \Omega\) we have

\[
\left| \bigcup_{k=1}^{N} \{A + \omega_k\} \setminus \bigcup_{k=1}^{N} \{B + \omega_k\} \right| > \frac{99}{100} \left( \frac{1}{(\kappa^{-1}e)^{c_0}} - \frac{1}{e^{(1+c_0-\varepsilon)}} \right).
\]

Note that \(\kappa = \kappa(c_0, |A|) \in (0, 1)\) in Proposition 5.3 can be taken such that \(\kappa \to 1\) as \(c_0|A| \to 0\). Once Proposition 5.3 is obtained, we have Lemma 5.2.

Proof of Lemma 5.2. Take \(A = F\) and \(B = D\) in Proposition 5.3. One can take \(c = \varepsilon\) by Lemma 4.5. Hence Proposition 5.3 yields the result with \(\kappa = 1/2\).

We postpone the proof of Proposition 5.3 for a short while, and proceed the argument. Henceforth, we let \(N = \lfloor 1/|F| \rfloor\). Fix sufficiently small \(\varepsilon \in (0, 1)\), and set

\[
\chi = \frac{99}{100} \left( \frac{1}{(2e)^{\varepsilon}} - \frac{1}{e} \right) > 0.
\]

As a result, we obtain the following lemma.

Lemma 5.4. Let \(\delta > 0\) and \(n \in \mathbb{N}\). There exist a function \(h_0 \in L^1([0, 1]^2, m)\) with

\[
\|h_0\|_{L^1} \leq 4K_EK_F \frac{n}{n - 1},
\]

a set \(Q_0 \subset [0, 1]^2\) with \(|Q_0| > \chi\), and an \(\eta = \eta(\varepsilon, \delta, n) \in (0, \delta)\) such that we have the following.

(i) If \(z \in Q_0\) and \(z \in A \in \mathcal{R}_C\), then

\[
\frac{1}{|A|} \left| \int_A h_0 \, dm \right| \leq 2.
\]

(ii) If \(z \in Q_0\), then there exists \(B \in \mathcal{R}_D\) with \(B \ni z\) and \(|B| > \eta\) such that

\[
\frac{1}{|B|} \left| \int_B h_0 \, dm \right| > n.
\]

(iii) For every \(R \in \mathcal{R}\) with \(|R| > \delta\), we have

\[
\frac{1}{|R|} \left| \int_R h \, dm \right| < 1.
\]

Proof. Let \(\Omega\) be the set as in Lemma 5.2. Define

\[
\Omega_0 = \{\omega \in \Omega : |\omega_i - \omega_j| \notin \{b_1, \ldots, b_n\}\}.
\]

We see \(|\Omega_0| = |\Omega| > 0\).

Since \(\omega \mapsto Q_0(\omega) = F_0(\omega) \setminus D_0(\omega)\) is continuous with respect to the Hausdorff metric on the space of all closed subsets of \([0, 1]^2\), there exists \(\omega_0 = ((\omega_0)_1, \ldots, (\omega_0)_N) \in \Omega_0\) such that for any

\[
\omega_0 \mapsto Q_0(\omega) = F_0(\omega) \setminus D_0(\omega)
\]
rectangle $A = A_{xy}$ with sides $x, y \in \mathcal{C}$ containing $z \in Q_0(\omega_0)$, there will be cancellations for all but at most two rectangles from $\mathcal{R}_D$, say $B = B(\theta) + (\omega_0)_i$ and $B' = B(\theta') + (\omega_0)_j$ for some $\theta, \theta' \in \Theta$ and $i, j \in \{1, \ldots, N\}$. Namely, let

$$h_0 = h_{F+(\omega_0)_1} + \cdots + h_{F+(\omega_0)_N},$$

then it follows that either $\int_A h_0 \, dm = 0$ or

$$\left| \frac{1}{|A|} \int_A h_0 \, dm \right| \leq \left| \frac{1}{|A|} \int_{A \cap B} h_{F+(\omega_0)_i} \, dm(z) \right| + \left| \frac{1}{|A|} \int_{A \cap B'} h_{F+(\omega_0)_j} \, dm \right| \leq 2$$

by Lemma 4.1(i). Hence the first property follows.

The argument same as above shows the third property by Lemma 4.1(ii).

Since $\omega_0 \in \Omega_0$, Lemma 4.1(iii) yields the property (ii) by taking $\tau > 0$ small if necessary. (Recall that $\tau > 0$ determines the height of the support of $h_0$.)

One has

$$\|h_0\|_{L^1(m)} \leq \|h_{F+(\omega_0)_1}\|_{L^1(m)} + \cdots + \|h_{F+(\omega_0)_N}\|_{L^1(m)}$$

$$\leq 2K_E K_F |F| \frac{n}{n - 1} \left( \frac{1}{|F|} + 1 \right) \leq 4K_E K_F \frac{n}{n - 1},$$

by (12) and (7) with noting $N = \lceil 1/|F| \rceil$. Hence Lemma holds for $Q_0 = Q_0(\omega_0)$ and $h_0$ defined as above. \qed

### 5.1.1 Proof of Proposition 5.3

In this section, we denote $X = [0, 1]^2$ for notational simplicity. Given $\omega = (\omega_1, \ldots, \omega_N) \in X^N$, denote

$$A_0(\omega) = \bigcup_{k=1}^N \{A + \omega_k\} \quad \text{and} \quad B_0(\omega) = \bigcup_{k=1}^N \{B + \omega_k\}.$$

For each $\omega = (\omega_1, \ldots, \omega_N)$, let $|A_0 \setminus B_0|(\omega) = |A_0(\omega) \setminus B_0(\omega)|$.

**Lemma 5.5.** We have

$$\mathcal{E}(|A_0 \setminus B_0|) \geq (1 - c_0 |A|)^N - (1 - (1 + c_0 - \varepsilon)|A|)^N. \quad (21)$$

**Proof.** We see

$$\mathcal{E}(|A_0 \setminus B_0|) = \int_{X^N} |A_0 \setminus B_0|(\omega) \, d\omega$$

$$= \mathcal{E} \left( \int_X b_N(z, \omega) \, dm(z) \right) - \mathcal{E} \left( \int_X (a_N \cdot b_N)(z, \omega) \, dm(z) \right), \quad (22)$$

where

$$a_N(z, \omega) = \prod_{k=1}^N (1 - 1_A(z + \omega_k)) \quad \text{and} \quad b_N(z, \omega) = \prod_{k=1}^N (1 - 1_B(z + \omega_k)).$$
For the first term in the right-hand side of (22), by the Fubini theorem,

\[
\mathcal{E} \left( \int_X b_N(z,\omega) \, dm(z) \right) = \mathcal{E} \left( \int_X \prod_{k=1}^N (1 - \mathbb{1}_B(z + \omega_k)) \, dm(z) \right) = \int_X \mathcal{E} \left( \prod_{k=1}^N (1 - \mathbb{1}_B(z + \omega_k)) \right) \, dm(z).
\]

Since \(|B| = c_0|A|\) by assumption, we obtain

\[
\mathcal{E} \left( \prod_{k=1}^N (1 - \mathbb{1}_B(z + \omega_k)) \right) = (1 - |B|)^N = (1 - c_0|A|)^N
\]

for every \(z \in X\), and thus

\[
\mathcal{E} \left( \int_X b_N(z,\omega) \, dm(z) \right) = (1 - c_0|A|)^N.
\] (23)

Next, for the second term in the right-hand side of (22), by the Fubini theorem again, one has

\[
\mathcal{E} \left( \int_X (a_N \cdot b_N)(z,\omega) \, dm(z) \right) = \int_X \mathcal{E}(a_N \cdot b_N)(z) \, dm(z),
\]

where

\[
\mathcal{E}(a_N \cdot b_N)(z) = \mathcal{E} \left( \prod_{k=1}^N (1 - \mathbb{1}_A(z + \omega_k)) \prod_{k=1}^N (1 - \mathbb{1}_B(z + \omega_k)) \right)
\]

\[
= \mathcal{E} \left( \prod_{k=1}^N (1 - \mathbb{1}_A(z + \omega_k) - \mathbb{1}_B(z + \omega_k) + \mathbb{1}_{A \cap B}(z + \omega_k)) \right)
\]

\[
= (1 - |A| - |B| + |A \cap B|)^N
\]

\[
\leq (1 - |A| - c_0|A| + \varepsilon|A|)^N
\] (24)

for every \(z \in [0,1]^2\). Hence we have

\[
\mathcal{E} \left( \int_X (a_N \cdot b_N)(z,\omega) \, dm(z) \right) \leq (1 - (1 + c_0 - \varepsilon)|A|)^N
\] (25)

Lemma 5.5 follows from (22), (23) and (24).

\[ \square \]

**Proof of Proposition 5.3.** Since \(1/|A| \leq N < (1 + |A|)/|A|\), it follows from Lemma 5.5 that

\[
(1 - c_0|A|)^N \geq (1 - c_0|A|)^{(1+|A|)/|A|} \geq (\kappa \cdot e^{-1})^{c_0}
\]

for some \(\kappa = \kappa(c_0|A|) \in (0,1)\) such that \(\kappa \to 1\) as \(c_0|A| \to 0\), and

\[
(1 - (1 + c_0 - \varepsilon)|A|)^N \leq (1 - (1 + c_0 - \varepsilon)|A|)^{1/|A|} \leq e^{-(1+c_0-\varepsilon)}.
\]
Hence, by Lemma 5.5, we arrive at
\[ E(|A_0 \setminus B_0|) = \int_{X_N} |A_0 \setminus B_0| \, d\omega \geq \frac{1}{(\kappa - 1)e^{c_0}} - \frac{1}{e^{(1+c_0-\epsilon)}}. \] (26)

Since \( \omega \mapsto |A_0 \setminus B_0| (\omega) \) is a non-negative and continuous function, it follows from (26) that there is an open set \( \Omega \subset X_N = X[^1/\vert A \vert] \) with \( \vert \Omega \vert > 0 \) such that for any \( \omega \in \Omega \) we have
\[ |A_0 \setminus B_0| (\omega) \geq \frac{99}{100} \left( \frac{1}{(\kappa - 1)e^{c_0}} - \frac{1}{e^{(1+c_0-\epsilon)}} \right). \]

Proposition is obtained. \( \square \)

**Remark 5.6.** Proposition 5.3 concerns the case where \( c_0 > 0 \) (or \( c \)) is rather large. If \( c > 0 \) is small, this is exactly the case of Lemma 5.2, then we have a better upper bound
\[ E(a_N \cdot b_N)(z) \leq (1 - |A|)^N \]
in stead of (24), and thus
\[ E(|A_0 \setminus B_0|) \geq (1 - c|A|)^N - (1 - |A|)^N \]
in stead of (21).

### 5.2 Proof of Theorem 2.3

**Proof.** Denote \( X_0 = [0, 1]^2 \), and let \( h_0 \in L^1(X_0) \) and \( Q_0 = F_0(\omega_0) \setminus D_0(\omega_0) \subset X_0 \) be as in Lemma 5.4. Hence \( |Q_0| > \chi \) by Lemma 5.4. Let \( h_0^* = |Q_0|h_0 \). Then \( \|h_0^*\|_{L^1(m)} \leq (4 + o(1))|Q_0| \). Since we fix such an \( \omega_0 \in \Omega_0 \), we will omit \( \omega_0 \) and write, say \( D_0 \) instead of \( D_0(\omega_0) \) for notational simplicity. By Lemma 4.3 one has
\[ |D_0| \leq |D| \left( \frac{1}{|F|} + 1 \right) < \varepsilon_0 = \frac{\varepsilon}{2^3}. \]

Here and below, we let \( \varepsilon_k = \varepsilon/2^{k+3} \) for \( k \in \mathbb{N} \cup \{0\} \).

Let
\[ Y_1 = X_0 \setminus (Q_0 \cup D_0) = X_0 \setminus (F_0 \cup D_0). \]

Since the boundary of the set \( Q_0 \cup D_0 \) consists of straight lines, we can partition \( Y_1 \) into squares of small size. Precisely, there exist \( a_1 = a_1(\varepsilon) \in (0, 1) \) and a finite family of squares \( S_{1,a} \) with side lengths \( a \in [a_1, 1) \) such that
\[ \left| Y_1 \setminus \bigcup_{a \geq a_1} S_{1,a} \right| < \varepsilon_0. \]
(Note that the disjoint union includes squares of same side lengths.) Let
\[ X_1 = \bigcup_{a \geq a_1} S_{1,a}. \]
Inside each square $S_{1,a}$, we can find a function $h_{1,a} \in L^1(X_0)$ and a subset $Q_{1,a} = F_{1,a} \setminus D_{1,a} \subset S_{1,a}$ for which all the properties of Lemma 5.4 hold, and hence

$$\bigcup_{a \geq a_1} Q_{1,a} > \chi \bigcup_{a \geq a_1} S_{1,a} = \chi |X_1|.$$ 

Define

$$Q^*_1 = \bigcup_{a \geq a_1} Q_{1,a} \quad \text{and} \quad D^*_1 = \bigcup_{a \geq a_1} D_{1,a}.$$ 

Then $Q^*_1, D^*_1 \subset X_1$, and one has

$$|D^*_1| < \varepsilon_1.$$ 

Notice that each support $\text{supp} h_{1,a}$ consists of finitely many horizontal line segments. Hence we may and do assume that the supports $\text{supp} h_{1,a}$ in different squares $S_{1,a}$ never lay on the same horizontal line. We may do assume further that no two such line segments, one from $\text{supp} h^*_0$ and the other from any of $\text{supp} f_{1,a}$, lay on the same horizontal line. In short, one can assume that any two horizontal line segments in $\text{supp} h^*_0 \cup \{ \text{supp} h_{1,a} : a \geq a_1 \}$ have a “vertical gap”. Let

$$h^*_1 = \sum_{a \geq a_1} |Q_{1,a}| h_{1,a} \in L^1(X_0).$$ 

Then, by construction, we still have all the properties of Lemma 5.4 for $Q_0 \cup Q^*_1$ and $h^*_0 + h^*_1$ by the same argument of the proof for Lemma 5.4. Here each function $h_{1,a}$ is multiplied by the area $|Q_{1,a}|$ just to have $\|h^*_1\|_{L^1(m)} \leq (4+o(1))|Q^*_1|$. Hence it follows that $\|h^*_0 + h^*_1\|_{L^1(m)} \leq (4+o(1))(|Q_0| + |Q^*_1|)$.

Next, define

$$Y_2 = X_1 \setminus (Q^*_1 \cup D^*_1),$$

and repeat the procedure described above. Namely, there exist $a_2 = a_2(\varepsilon) \in (0, a_1)$ and a finite family of squares $S_{2,a}$ with side lengths $a \in [a_2, a_1)$ with

$$\bigg| Y_2 \setminus \bigcup_{a \in [a_2, a_1)} S_{2,a} \bigg| < \varepsilon_1$$

such that for each $S_{2,a}$, there are $h_{2,a} \in L^1(X_0)$ and $Q_{2,a} = F_{2,a} \setminus D_{2,a} \subset S_{2,a}$ for which all the properties of Lemma 5.4 hold such that

$$\bigg| \bigcup_{a \in [a_2, a_1)} Q_{2,a} \bigg| > \chi \bigg| \bigcup_{a \in [a_2, a_1)} S_{2,a} \bigg| \quad \text{and} \quad \bigg| \bigcup_{a \in [a_2, a_1)} D_{2,a} \bigg| < \varepsilon_2.$$ 

Define

$$X_2 = \bigcup_{a \in [a_2, a_1)} S_{2,a}, \quad Q^*_2 = \bigcup_{a \in [a_2, a_1)} Q_{2,a}, \quad D^*_2 = \bigcup_{a \in [a_2, a_1)} D_{2,a}.$$ 

Note that one has $Q^*_2, D^*_2 \subset X_2$, and $|D^*_2| < \varepsilon_2$. We let

$$h^*_2 = \sum_{a \in [a_2, a_1)} |Q_{2,a}| h_{2,a} \in L^1(X_0),$$

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where $\text{supph}_2^a$ are placed such that any two horizontal line segments in $\text{supph}_0^i \sqcup \text{supph}_1^i \sqcup \text{supph}_2^*$ have a vertical gap. Hence, all the properties of Lemma 5.4 still hold for $Q_0 \sqcup Q_1^* \sqcup Q_2^*$ and $h_0^* + h_1^* + h_2^*$. Notice that $\| h_0^* + h_1^* + h_2^* \|_{L^1(m)} \leq (4 + o(1))(|Q_0| + |Q_1^*| + |Q_2^*|)$ holds.

Once $X_k$, and $Q_k^* \sqcup D_k^* \subset X_k$ are defined, by letting

\[ Y_{k+1} = X_k \setminus (Q_k^* \sqcup D_k^*), \]

one can define $X_{k+1} \subset Y_{k+1}$, and $Q_{k+1}^* \sqcup D_{k+1}^* \subset Y_{k+1}$ with

\[ |Y_{k+1} \setminus X_{k+1}| < \varepsilon_k, \quad |Q_{k+1}^*| > \chi |X_{k+1}|, \quad |D_{k+1}^*| > \varepsilon_{k+1} \]

and a function $h_{k+1}^* \in L^1(X_0)$, with $\| h_{k+1}^* \|_{L^1(m)} \leq (4 + o(1))|Q_{k+1}^*|$, having the vertical gap property among $\text{supph}_0^i \sqcup \text{supph}_1^i \sqcup \cdots \sqcup \text{supph}_{k+1}^*$ such that all the properties of Lemma 5.4 hold for $Q_0 \sqcup Q_1^* \sqcup \cdots \sqcup Q_{k+1}^*$ and $h_0^* + h_1^* + \cdots + h_{k+1}^*$.

Now, we show that the area of $\bigcup_{k=0}^{n-1} Q_k^*$, where $Q_0^* = Q_0$, can be arbitrarily close to one.

**Lemma 5.7.** One can take $n \in \mathbb{N}$ so large that

\[ \left| \bigcup_{k=0}^{n-1} Q_k^* \right| > 1 - \varepsilon. \]

**Proof.** One has

\[ |Q_0^*| = |Q_0| = |X_0| - |D_0| - |Y_1| > 1 - \varepsilon_0 - (|X_1| + \varepsilon_0) = 1 - |X_1| - 2\varepsilon_0, \]

and

\[ |Q_0^* \sqcup Q_1^*| = |Q_0^*| + |Q_1^*| > (1 - |X_1| - 2\varepsilon_0) + (|X_1| - |D_1^*| - |Y_2|) \]
\[ > 1 - 2\varepsilon_0 - \varepsilon_1 - |Y_2| \]
\[ > 1 - 2\varepsilon_0 - \varepsilon_1 - (|X_2| + \varepsilon_1) \]
\[ > 1 - |X_2| - 2\varepsilon_0 - 2\varepsilon_1. \]

Hence by induction, one obtain

\[ \left| \bigcup_{k=0}^{n-1} Q_k^* \right| > 1 - |X_n| - 2 \sum_{k=0}^{n-1} \varepsilon_k = 1 - |X_n| - 2 \sum_{k=0}^{n-1} \frac{\varepsilon}{2^{k+3}} > 1 - |X_n| - \frac{\varepsilon}{2}. \]

Next, one sees that $|X_n|$ strictly decreases to 0 as $n$ grows. Indeed,

\[ |X_{n+1}| \leq |Y_{n+1}| = |X_n| - |Q_n^*| - |D_n^*| \]
\[ < |X_n| - |Q_n^*| \]
\[ < |X_n| - \chi |X_n| \]
\[ = (1 - \chi)|X_n| < \cdots < (1 - \chi)^{n+1}|X_0| = (1 - \chi)^{n+1}. \]

Take $n \in \mathbb{N}$ so large that $(1 - \chi)^n < \varepsilon/2$. Then it follows that

\[ \left| \bigcup_{k=0}^{n-1} Q_k^* \right| > 1 - |X_n| - \frac{\varepsilon}{2} > 1 - \varepsilon. \]

Lemma is obtained. \qed

Take an $n \in \mathbb{N}$ so large as in Lemma 5.7. Letting $Q = \bigcup_{k=0}^{n-1} Q_k^*$ and $f = h_0^* + h_1^* + \cdots + h_{n-1}^*$ yields the Theorem 2.3. \qed
6 Proof of Theorem 2.1

6.1 Setup

To prove Theorem 2.1, we use Theorem 2.3 recursively to find sequences of sets \( \{Q_n\}_{n \geq 1} \) and functions \( \{f_n\}_{n \geq 1} \) as follows. Let \( \varepsilon_1 = 1/2, \delta_1 = 1/2, \) and \( a_1 = 1. \) Then there are \( Q_1 \subset [0,1]^2 \) with \( |Q_1| > 1 - \varepsilon_1 \) and \( f_1 \in L^\infty(m) \) with \( \|f_1\|_{L^1(m)} \leq 2 \) and \( \eta_1 \in (0, \delta_1) \) such that

(i) If \( z \in Q_1 \) and \( z \in A \in \mathcal{R}_C, \) then
\[
\left| \frac{1}{|A|} \int_A f_1 \, dm \right| \leq 2.
\]

(ii) If \( z \in Q_1, \) then there exists \( B \in \mathcal{R}_D, \) with \( z \in B \) and \( |B| > \eta_1, \) so that
\[
\left| \frac{1}{|B|} \int_B f_1 \, dm \right| > a_1 (= 1).
\]

(iii) For every \( R \in \mathcal{R} \) with \( |R| > \delta_1, \) we have
\[
\left| \frac{1}{|R|} \int_R f_1 \, dm \right| < 1.
\]

Suppose that \( Q_{n-1} \subset [0,1]^2, f_{n-1} \in L^\infty(m) \) and \( \eta_{n-1} \) are defined such that the corresponding properties (i\(_{n-1}\)), (ii\(_{n-1}\)), (iii\(_{n-1}\)) hold. Let \( \varepsilon_n = 1/(n + 1)^2, \delta_n = \eta_{n-1}, \) and
\[
a_n = \left( 2 \sup_{k \leq n-1} \|f_k\|_{L^\infty(m)} + n \right) n^2.
\]

Then there are \( Q_n \subset [0,1]^2 \) with \( |Q_n| > 1 - \varepsilon_n \) and \( f_n \in L^\infty(m) \) with \( \|f_n\|_{L^1(m)} \leq 2 \) and \( \eta_n \in (0, \delta_n) \) such that

(i\(_n\)) If \( z \in Q_n \) and \( z \in A \in \mathcal{R}_C, \) then
\[
\left| \frac{1}{|A|} \int_A f_n \, dm \right| \leq 2.
\]

(ii\(_n\)) If \( z \in Q_n, \) then there exists \( B \in \mathcal{R}_D, \) with \( z \in B \) and \( |B| > \eta_n, \) so that
\[
\left| \frac{1}{|B|} \int_B f_n \, dm \right| > a_n.
\]

(iii\(_n\)) For every \( R \in \mathcal{R} \) with \( |R| > \delta_n, \) we have
\[
\left| \frac{1}{|R|} \int_R f_n \, dm \right| < 1.
\]

Notice that \( \delta_n > \eta_n = \delta_{n+1} > \eta_{n+1} = \delta_{n+2} \) by construction, and thus both \( \eta_n \) and \( \delta_n \) are decreasing sequences in particular.
6.2 Proof of Theorem 2.1

Proof of Theorem 2.1 Define

\[ f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n. \]

Note that by property (2) of Theorem 2.3, we have

\[ \left\| \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \right\|_{L^1(m)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \left\| f_n \right\|_{L^1(m)} < \infty. \]

Hence \( f \) is well defined and \( f \in L^1(m) \). By the Borel-Cantelli lemma, we also have that

\[ m \left( \liminf_{n \to \infty} Q_n \right) = 1. \]

Since \( f_k \in L^\infty(m) \), then by Theorem 1.2 we have that for all \( k \in \mathbb{N} \), there is \( \Gamma_k \subset [0,1]^2 \) with \( m(\Gamma_k) = 1 \) such that

\[ \lim_{\delta R \to 0, z \in R} \frac{1}{|R|} \int_{R} f_k dm = f_k(z) \quad (27) \]

for every \( z \in \Gamma_k \). It follows from (27) and Theorem 2.3(i) that for every \( z \in Q_k \cap \Gamma_k \) we have

\[ |f_k(z)| \leq 2. \quad (28) \]

Denote \( \Gamma_\infty = \bigcap_{k=1}^{\infty} \Gamma_k \) and

\[ \Lambda = \left( \liminf_{n \to \infty} Q_n \right) \cap \Gamma_\infty. \]

Clearly \( m(\Lambda) = 1 \).

First, we prove convergence, namely \( \delta_{R^C}(z,f) = 0 \) for \( m \)-almost every \( z \in [0,1]^2 \). Let \( z \in \Lambda \).

Hence there exists \( M = M_z \in \mathbb{N} \) so that \( z \in Q_n \) for all \( n \geq M \). Then for every \( A \in \mathcal{R}_C \) with \( A \ni z \) we have

\[ \left| \frac{1}{|A|} \int_{A} f dm - f(z) \right| = \left| \sum_{n=1}^{M} \frac{1}{n^2} \frac{1}{|A|} \int_{A} f_n dm - \sum_{n=1}^{M} \frac{f_n(z)}{n^2} \right| \]

\[ \leq \sum_{n=1}^{M} \frac{1}{n^2} \left| \frac{1}{|A|} \int_{A} f_n dm - \frac{f_n(z)}{n^2} \right| \]

\[ + \sum_{n=M+1}^{\infty} \frac{1}{n^2} \left| \frac{1}{|A|} \int_{A} f_n dm \right| + \sum_{n=M+1}^{\infty} \frac{|f_n(z)|}{n^2}. \quad (29) \]

Since \( z \in Q_n \cap \Gamma_\infty \) for all \( n \geq M \), it follows from (28) that \( |f_n(z)| \leq 2 \) for all \( n \geq M \). Note also that

\[ \left| \frac{1}{|A|} \int_{A} f_n dm \right| \leq 2 \]

by the property (i_n). Thus for the last two terms (29), we have

\[ \sum_{n=M+1}^{\infty} \frac{1}{n^2} \left| \frac{1}{|A|} \int_{A} f_n dm \right| + \sum_{n=M+1}^{\infty} \frac{|f_n(z)|}{n^2} \leq \sum_{n=M}^{\infty} \frac{4}{n^2} < \frac{4}{M-1}. \]

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For the first term \([29]\), since \(x \in \Gamma_\infty\), then we have by \([27]\) that
\[
\lim_{\text{diam} A \to 0, \ z \in A \in \mathcal{R}_c} \left| \sum_{n=1}^{\infty} \frac{1}{n^2} \left| A \right| \int_A f_n \, dm - \sum_{n=1}^{M} \frac{f_n(z)}{n^2} \right| = 0.
\]

Consequently, it follows that
\[
\lim_{\text{diam} A \to 0, \ z \in A \in \mathcal{R}_c} \left| \sum_{n=1}^{\infty} \frac{1}{n^2} \left| A \right| \int_A f_n \, dm - \sum_{n=1}^{\infty} \frac{f_n(z)}{n^2} \right| = 0.
\]

In other words \(\delta_{\mathcal{R}_c}(z, f) = 0\) for \(m\)-almost every \(z \in [0, 1]^2\).

We now prove divergence. Let \(z \in A\). Hence there exists \(M_z \in N\) so that \(z \in Q_n\) for all \(n \geq M_z\). Take \(N > M_z\). Then by the property (ii\(N\)), there exists \(B \in \mathcal{R}_D\), with \(z \in B\) and \(|B| > \eta_N\) such that
\[
\left| \frac{1}{|B|} \int_B f_N \, dm \right| > a_N.
\]

Hence for such a rectangle \(B \in \mathcal{R}_D\), we have
\[
\left| \frac{1}{|B|} \int_B f \, dm \right| = \left| \frac{1}{|B|} \int_B \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \, dm \right| = \left| \frac{1}{|B|} \int_B f_n \, dm \right| - \left| \sum_{n \neq N} \frac{1}{n^2} \frac{1}{|B|} \int_B f_n \, dm \right| > a_N - \frac{\sum_{n \neq N} \frac{1}{n^2} \frac{1}{|B|} \int_B f_n \, dm}{N^2}.
\]

Here we have
\[
\sum_{n=1}^{N-1} \frac{1}{n^2} \frac{1}{|B|} \int_B |f_n| \, dm \leq \left( \sup_{n \leq N-1} \|f_n\|_{L^\infty(m)} \right) \cdot \sum_{n=1}^{N-1} \frac{1}{n^2} \leq 2 \sup_{n \leq N-1} \|f_n\|_{L^\infty(m)}.
\]

Notice that we have \(|B| > \delta_n\) for all \(n > N\) since \(|B| > \eta_N = \delta_{N+1}\) and \(\delta_k\) is a decreasing sequence in \(k\). Thus by the property (iii\(n\)) with \(n > N\), we have
\[
\left| \frac{1}{|B|} \int_B f_n \, dm \right| \leq 1
\]
for all \(n > N\), and hence
\[
\left| \sum_{n > N} \frac{1}{n^2} \frac{1}{|B|} \int_B f_n \, dm \right| \leq \sum_{n = N+1}^{\infty} \frac{1}{n^2} \leq \frac{1}{N}.
\]

It follows that
\[
\left| \frac{1}{|B|} \int_B f \, dm \right| > \frac{a_N}{N^2} - 2 \sup_{n \leq N-1} \|f_n\|_{L^\infty(m)} - \frac{1}{N} = N - \frac{1}{N}.
\]

Consequently, we have
\[
\left| \frac{1}{|B|} \int_B f \, dm - f(z) \right| \geq \left| \frac{1}{|B|} \int_B f \, dm \right| - |f(z)| > N - \frac{1}{N} - |f(z)|.
\]

Letting \(N \to \infty\), the divergence property \(\delta_{\mathcal{R}_D}(z, f) = \infty\) holds for \(m\)-almost every \(z \in [0, 1]^2\). \(\Box\)
7 Proof of Theorem 2.2

Proof. By assumption, one can take \( c \in (0,1) \) so small that

\[ 1 \geq \max \left\{ \frac{\alpha}{a}, \frac{\beta}{a} \right\} > c > 0 \]

holds for every sufficiently small \( a \in \mathcal{D} \). Suppose \( \delta_{\mathcal{R}_c}(z,f) = 0 \) for \( m \)-almost every \( z \in \mathbb{R}^2 \). For a given \( \delta > 0 \) consider the set

\[ E_\delta = \left\{ z \in \mathbb{R}^2 : \sup_{\text{diam} A < \delta, z \in A \in \mathcal{R}_c} \left| f \int_A dm - f(z) \right| < 1 \right\}. \]

We have \( |E_\delta| > 0 \) for \( \delta > 0 \) small enough. Then for the characteristic function \( \mathbb{1}_{E_\delta} \) of \( E_\delta \), we will have by Theorem 1.2 that almost all \( z \in E_\delta \) are Lebesgue, namely for almost every \( z \in E_\delta \) we have

\[ \lim_{\text{diam} R \to 0, z \in R \in \mathbb{R}} \frac{|R \cap E_\delta|}{|R|} = 1. \]

Take \( B \in \mathcal{R}_D \) with \( z \in B \) and \( \text{diam} B \) so small such that

\[ \frac{|B \cap E_\delta|}{|B|} > 1 - c^2. \] (31)

We now wish to cover the rectangle \( B \) with rectangles of sides \( x, y \in \mathcal{C} \). More specifically, we have the following lemma.

**Lemma 7.1.** There is a collection \( \{ A_q \in \mathcal{R}_c \}_{q=1}^n \) for some \( n \leq \left( \frac{1}{c} + 1 \right)^2 \) such that

1. \( B \subset \bigcup_{q=1}^n A_q \),
2. \( |B \cap A_q| \geq c^2 |B| \) for every \( q \in \{1, \ldots, n\} \),
3. \( \frac{|A_q|}{|B|} \leq \frac{1}{c^2} \) for every \( q \in \{1, \ldots, n\} \).

**Proof.** Let \( B = [s, s+a] \times [t, t+b] \) for some small \( a, b \in \mathcal{D} \) and some \( (s,t) \in \mathbb{R}^2 \). Then by assumption, there exist \( x, y \in \mathcal{C} \) such that

\[ (1a) \quad 1 \geq \frac{x}{a} > c \quad \text{or} \quad (2a) \quad 1 \geq \frac{a}{x} > c, \]

and

\[ (1b) \quad 1 \geq \frac{y}{b} > c \quad \text{or} \quad (2b) \quad 1 \geq \frac{b}{y} > c. \]

Hence, consider the following four cases. For the case where \( x \in \mathcal{C} \) satisfies (2a) and \( y \in \mathcal{C} \) does (2b), by taking

\[ A_1 = [s, s+x] \times [t, t+y], \]
we have $|B \cap A_1| = |B|$ and
\[
\frac{|A_1|}{|B|} = \frac{xy}{ab} < \frac{1}{c^2}.
\]

For the case where $x \in C$ satisfies (2a) and $y \in C$ does (1b), we consider
\[
A_\ell = [s, s + x] \times [t + (\ell - 1)y, t + \ell y]
\]
for $\ell \in \{1, \ldots, [b/y]\}$, and also
\[
A_{[b/y]+1} = [s, s + x] \times [t + b - y, t + b]
\]
to cover up. Then we have
\[
\frac{|B \cap A_\ell|}{|B|} = \frac{xy}{ab} > c,
\]
and
\[
\frac{|A_\ell|}{|B|} = \frac{xy}{ab} \leq \frac{1}{c}
\]
for every $\ell \in \{1, \ldots, [b/y] + 1\}$. The case where $x \in C$ satisfies (1a) and $y \in C$ does (2b) is similar, hence omit this case.

For the case where $x \in C$ satisfies (1a) and $y \in C$ does (1b), we define
\[
A_{k, \ell} = [s + (k - 1)x, s + kx] \times [t + (\ell - 1)y, t + \ell y]
\]
for $k \in \{1, \ldots, [a/x]\}$ and $\ell \in \{1, \ldots, [b/y]\}$. Define also
\[
A_{[a/x]+1, \ell} = [s + a - x, s + a] \times [t + (\ell - 1)y, t + \ell y]
\]
for $\ell \in \{1, \ldots, [b/y]\}$;
\[
A_{k, [b/y]+1} = [s + (k - 1)x, s + kx] \times [t + b - y, t + b]
\]
for $k \in \{1, \ldots, [a/x]\}$, and
\[
A_{[a/x]+1, [b/y]+1} = [s + a - x, s + a] \times [t + b - y, t + b]
\]
Denote the whole $\{A_{k, \ell}\}$ by $\{A_q\}_{q=1}^{n}$, where we have
\[
n \leq \left( \left\lceil \frac{a}{x} \right\rceil + 1 \right) \left( \left\lceil \frac{b}{y} \right\rceil + 1 \right) \leq \left( \frac{1}{c} + 1 \right)^2.
\]
We also have
\[
\frac{|B \cap A_q|}{|B|} = \frac{xy}{ab} > c^2,
\]
and
\[
\frac{|A_q|}{|B|} = \frac{xy}{ab} \leq 1
\]
for every $q \in \{1, \ldots, n\}$.

In all cases above, we have a collection $\{A_q \in \mathcal{R}_C\}_{q=1}^{n}$ such that
\[
B \subset \bigcup_{q=1}^{n} A_q.
\]
Lemma is obtained.
\[\square\]
It follows from (31) and Lemma 7.1 (iii) that \( E_\delta \cap A_q \neq \emptyset \) for all \( q \in \{1, \ldots, n\} \). Hence
\[
\left| \frac{1}{|A_q|} \int_{A_q} f \, dm \right| \leq 1 + f(z)
\]
for all \( q \in \{1, \ldots, n\} \). Thus we have by Lemma 7.1 that
\[
\int_B f \, dm \leq \sum_{q=1}^n \int_{A_q} f \, dm \leq (1 + f(z)) \sum_{q=1}^n |A_q| \leq (1 + f(z)) \left( \frac{1}{c} + 1 \right)^2 \frac{1}{c^2} |B|,
\]
and which implies
\[
\frac{1}{|B|} \int_B f \, dm \leq \left( \frac{1}{c} + 1 \right)^2 \frac{1}{c^2} (1 + f(z)).
\]
Consequently, we have
\[
\limsup_{\text{diam} B \to 0, \ z \in \mathbb{R}^2} \frac{1}{|B|} \int_B f \, dm < \infty.
\]
Thus, for almost every Lebesgue points \( z \in E_\delta \) we have that the upper differential is bounded. Then, due to a theorem of Besicovitch [1], it follows that \( \delta_{\mathcal{R}_D}(z, f) = 0 \) for \( m \)-almost every \( z \in \mathbb{R}^2 \).

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