

Graph States as a Resource for Quantum Metrology

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Quantum metrology describes the framework for estimation strategies which surpass the precision limit of classical strategies [1][4]. A classical estimation strategy can be emulated by using non-entangled single qubit classical strategies [1–4]. A classical estimation strategy which surpasses the precision limit of θ can be effective for practical quantum processes including cryptography [14, 15], quantum networks [16, 17], computation [18, 19], and quantum error correction [20, 21]. Furthermore, they can be implemented using different techniques, such as, ion traps [22], superconducting qubits [23], NV centers [24] and discrete [25, 27] and continuous variable [28, 29] optics. Here we show that they can also be effective resources for practical quantum metrology in the presence of noise.

In our study we consider the canonical case of phase estimation, where an unknown phase θ is encoded using non-interacting Hamiltonians

\[ U_\theta = \exp \left(-i \theta \sum_{j=1}^{n} X_j \right). \] (1)

In this scenario the QFI for an arbitrary quantum state \( \rho = \sum_{j} \lambda_j |j\rangle\langle j| \) is [11, 30]

\[ Q(\rho) = \frac{1}{2} \sum_{j,k} \frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} |\langle j| \sum_{i=1}^{n} X_i |k\rangle|^2. \] (2)

Which can be simplified for pure states ψ

\[ Q(\psi) = \sum_{i,j=1}^{n} \text{Tr}(X_i X_j \psi) - \text{Tr}(X_i \psi) \text{Tr}(X_j \psi). \] (3)

An n qubit stabilizer state, \( ψ \), is defined to be the unique +1-eigenvalue of n independent Pauli operators \( g_1, \ldots, g_n \)

\[ ψ = 2^{-n} \prod_{i=1}^{n} (g_i + 1) = 2^{-n} \sum_{S \in S} S, \] (4)

where S is the stabilizer group generated by \( g_1, \ldots, g_n \), containing all Pauli operators S which stabilize \( S \psi = ψ \). Aaronson and Gottesman [31] compute the number of n qubit stabilizer states, \( N_n \), by counting the number of ways \( n \) independent generators can be chosen from the Pauli group. It is clear from Eq. (3) that if the generators are chosen from the Pauli group such that there are no stabilizers of the forms \( \pm X_i \) or \( -X_i X_j \), then the QFI of the stabilizer state is equal to the number of stabilizers of the form \( X_i X_j \). Defining \( k = n^{1-\epsilon/2} \), we show in Appendix B that there are at least

\[ \sum_{j=k}^{n} \binom{n - 1}{j - 1} 2^j N_{n-j} \geq \binom{n - 1}{k - 1} 2^k N_{n-k} \] (5)

stabilizer states with a QFI of at least \( n^{2-\epsilon} \).
One class of stabilizer states which have no stabilizers of the form \( \pm X_i \) or \(-X_iX_j\) are graph states with no isolated vertices. Formally, an \( n \) qubit graph state \( G = (V,E) \) can be defined in correspondence to a simple graph with \( n \) vertices \( V \) and edges \( E \). The corresponding generators are

\[
g_i = X_i \bigotimes_{j \in N(i)} Z_j, \tag{6}
\]

where \( N(i) \) is the neighbourhood of the \( i \)th vertex. A graph has no isolated vertices if \( |N(i)| \geq 1 \ \forall i \). A graph state is stabilized by \( X_iX_j \) if \( N(i) = N(j) \), i.e. if the \( i \)th and \( j \)th vertices have the same neighbourhood. Hence, the QFI of a graph state \( G \) with no isolated vertices is equal to the number of pairs of vertices \( (i,j) \) such that \( N(i) = N(j) \). Alternatively we can write

\[
Q(G) = \sum_{i,j=1}^{n} \delta_{N(i),N(j)}, \tag{7}
\]

where we use Kronecker delta notation to signify that \( \delta_{A,B} = 1 \) if \( A = B \), and \( \delta_{A,B} = 0 \) otherwise.

Eq. (7) makes it immediately evident whether a specific graph state is a good resource for quantum metrology. Consider a cluster state, where the associated graph is in the shape of a lattice. Since no two vertices have identical neighbourhoods we get that \( \delta_{N(i),N(j)} = 1 \iff i = j \), thus

\[
Q(G_{\text{cluster}}) = n. \tag{8}
\]

This result corresponds to what Friis et al. state in [8]: unmodified cluster states do not provide a scaling advantage for quantum metrology. If we instead consider an \( n \) qubit star graph, where one central vertex is connected to the remaining \( n-1 \) exterior vertices, (which is equivalent to the GHZ state up to local unitaries) we have that \( \delta_{N(i),N(j)} = 1 \) when \( i \) and \( j \) are one of the exterior vertices or \( i = j \) is the central vertex, thus

\[
Q(G_{\text{star}}) = (n-1)^2 + 1. \tag{9}
\]

The QFI of graph states can be further improved by performing local Clifford (LC) operations on the graph state first. Two vertices \((i,j)\) of a graph are said to be in a clique if \( N(i) \cup \{i\} = N(j) \cup \{j\} \), the associated graph state is also stabilized by the Pauli operator \( Y_iY_j \). Therefore, performing a LC operation \( L \) on all vertices which are a member of a clique such that

\[
LYLY^\dagger = X, \tag{10}
\]

will increase the number of stabilizers of the form \( X_iX_j \). If this is done, the QFI is now equal to the number of pairs of vertices which have identical neighbourhoods or are a member of the same clique. Alternatively we can write

\[
Q(G^{\text{LC}}) = \sum_{i,j=1}^{n} \delta_{N(i),N(j)} + \delta_{N(i)\cup\{i\},N(j)\cup\{j\}} - \delta_{i,j}. \tag{11}
\]

The only scenario for a pair of vertices \((i,j)\) to have identical neighbourhoods and be a member of the same clique is when \( i = j \); this is the reason for the \(-\delta_{i,j}\) term in Eq. (11). The complete graph, where each vertex is connected to every other vertex, satisfies \( N(i) \cup \{i\} = N(j) \cup \{j\} \ \forall i,j, \) thus

\[
Q(G^{\text{complete}}) = n^2. \tag{12}
\]

Ozmenaic et al. showed that most entangled states are not good for quantum metrology [7]. However, they also showed that most symmetric states are good quantum metrology. Eq. (7) and (11) show a similar result; the higher the QFI, the more internal symmetry present within a graph state, the higher the QFI.

![Bundled graph state constructed from a 3 qubit cyclic graph](image)

**FIG. 1:** Bundled graph state constructed from a 3 qubit cyclic graph, \( Q(G_{\text{bundle}}) = n_1^2 + n_2^2 + n_3^2 \geq n^2 \log n \).
We illustrate the construction of a bundled graph state in Fig. 1 by transforming a 3 qubit graph state into an n qubit bundled graph state.

We chose the term bundled graph states because the qubits are divided into bundles of vertices \(v_{1}^{j}, \ldots, v_{n}^{j}\) which all share a common neighbourhood. Using Eq. (7) to compute that

\[
Q(G_{\text{bundle}}) \geq \sum_{i=1}^{k} n_{i}^{2} = \frac{n^{2}}{k} = n^{2-\log_{n}k}.
\]  

(15)

Note that the shape of graph effects the usefulness of the bundled graph state counterpart. An n qubit bundled cyclic graph divided into k bundles of n/k qubits has a QFI of

\[
Q(G_{\text{bundled cyclic}}) = \frac{n^{2}}{k}.
\]  

(16)

Hence, the QFI decreases with the number of bundles when n is kept constant. However, an analogous bundled star graph has a QFI of

\[
Q(G_{\text{bundled star}}) = \frac{n^{2}}{k^{2}} + (n - \frac{n}{k})^{2}.
\]  

(17)

Here the QFI increases as the number of bundles increases. This is because a star graph has a naturally high QFI, and bundling qubits together has a negative impact. Both of these effects are illustrated in Fig. 2.

Next we explore the robustness of graph states subjected to two different noise models. First we investigate the robustness of graph states subjected to independent and identically distributed (iid) dephasing. This transforms a graph state \(G\) into a mixed state

\[
G \rightarrow G_{\text{dephasing}} = \sum_{k} p^{k} (1-p)^{n-k} Z_{k} G Z_{k}^{\dagger}.
\]  

(18)

where \(p\) is the dephasing probability. A closed form expression for \(Q(G_{\text{dephasing}})\) is derived in Appendix C. If the size of the neighbourhoods, \(N\), are large enough and the dephasing probability is small enough such that

\[
(2p(1-p) + 1/2)^{N} \approx 0
\]  

(19)

then we can approximate that

\[
Q(G_{\text{dephasing}}) \approx (1 - 2p)^{2} Q(G) + 4np(1-p).
\]  

(20)

Substituting the above in Eq. (15)

\[
Q(G_{\text{bundle}}) \geq (1-2p)^{2} \frac{n^{2}}{k} = n^{2-\log_{n}k - \frac{1}{2} np + \mathcal{O}(p^{2})}.
\]  

(21)

In Fig. 2 we see that bundled star graph states and bundled cyclic graph states surpass the SQL for \(p \leq 0.2\). Additionally, the subplot of the bundled star graph subjected to iid dephasing demonstrates the viability of the approximation in Eq. (19). The set of qubits all connected to the central node has a neighbourhood of \(n/k\) and for intermediate values of \(p \in [0.1,0.2]\) the term

\[
(2p(1-p) + 1/2)^{\frac{1}{2}}
\]  

(22)

becomes non-negligible when \(k\) is large; hence a difference in slope among the family of curves.

The second noisy system we explore is the robustness of graph states subjected to a small number of erasures. In Appendix D we derive a closed form expression for the QFI of a general graph state with no isolated vertices subjected to erasures occurring at vertices \(y_{1}, \ldots, y_{e}\). The general form is extremely dependent on the shape of the graph as well as the choice of \(y_{1}, \ldots, y_{e}\). To obtain any sort of meaningful value to quantify the robustness we compute \(\bar{Q}\); the average QFI of the system over all \(\binom{n}{e}\) permutations of losing \(e\) qubits.

For a bundled star graph of \(n\) nodes of \(j = n/k\) qubits subjected to \(1 \leq e < j\) erasures:

\[
\bar{Q} = \binom{n-j}{e} \frac{n}{k} \binom{j}{e} \frac{k - 1}{n}
\]  

(23)

And for a bundled cyclic graph of \(n\) nodes of \(j = n/k\) qubits subjected to \(1 \leq e < 2j\) erasures:

\[
\bar{Q} = 2 \binom{n-j}{e} - \binom{n-3j}{e-2j} + 2 \binom{n-3j}{e-2j}
\]  

(24)

The above expressions are derived by computing Eq. (D.5) for all \(\binom{n}{e}\) permutations of losing \(e\) qubits. Evidently, we see from Eq. (23) that bundled star graphs lose all quantum advantages after a single erasure, similar to that of the GHZ state. However, Eq. (24) has a term which scales quadratically with \(n\), implying that (on average) bundled cyclic graphs retain a quantum advantage in scaling after a small number of erasures, this can be seen in Fig. 2.

We have shown that certain graph states have a QFI which surpasses the classical limit. To realize a precision of \(\Delta \theta^{2} = 1/Q\) a POVM which maximizes the Fisher information must be made (see Appendix A). Generally, this POVM is dependent on the unknown parameter or is highly entangled, making the ideal measurement infeasible. The obvious question is: can a precision of \(\Delta \theta^{2} = 1/Q(G)\) be achieved with single qubit measurements when using a graph state \(G\) as a resource?

If the following conditions are satisfied then we can achieve the desired precision:

1. The phase we are trying to estimate is small.

2. There exists a stabilizer, \(S_{M}\), for the graph which consists entirely of \(Y\) and \(Z\) operators.
**FIG. 2**: Robustness of $n = 100$ qubit bundled cyclic graphs (a, c) and bundled star graphs (b, d) subjected to iid dephasing (a, b) and a small number of erasures (c, d). In every scenario the graph is divided into $k$ bundles of $n/k$ qubits: $\bullet \rightarrow k = 5$, $\bullet \rightarrow k = 10$ and $\bullet \rightarrow k = 20$. The standard quantum limit $\bullet$ and Heisenberg limit $\bullet$ are also displayed for clarity. For small $p$, we observe that $\log n Q$ decreases linearly, which is expected from Eq. (21). We also see that bundled cyclic graph states retain a quantum advantage after a small number of erasures, in contrast the QFI of bundled star graphs fall below the SQL after a single erasure, which is expected from Eq. (23) and (24).

The desired precision can be attained by measuring in the $S_M$ basis

$$
\langle S_M \rangle = \text{Tr} \left( e^{i \frac{\theta}{2} \sum_{i=0}^{n} X_i} S_M e^{-i \frac{\theta}{2} \sum_{i=0}^{n} X_i} G \right)
$$

$$
= \text{Tr} \left( e^{i \theta} \sum_{i=0}^{n} X_i G \right)
$$

$$
= \sum_{m=0}^{\infty} \frac{(i \theta)^m}{m!} \text{Tr} \left( (\sum_{i=0}^{n} X_i)^m G \right)
$$

$$
= 1 - \frac{\theta^2}{2} Q(G) + O(\theta^3).
$$

Using the error propagation formula

$$
\Delta \theta^2 = \frac{\Delta S_M^2}{\partial^2 \langle S_M \rangle} = \frac{\theta^2 Q(G) + O(\theta^3)}{\theta^2 Q(G)^2 + O(\theta^3)} \approx \frac{1}{Q(G)}
$$

Measuring a small phase is naturally the regime where quantum metrology is most interesting. The second condition above is always satisfied for any bundled star graph, and is satisfied for bundled cyclic graphs when the number of bundles is a multiple of four. We thus see that even with fixed, local measurements, a precision scaling of $\Delta \theta^2 = 1/Q(G)$ can be achieved for these states.

In the scenario in which a graph state $G$ does not have a stabilizer consisting entirely of $Y$ and $Z$ operators, we can still achieve a precision of $\Delta \theta^2 = 1/Q(G)$ by using a graph state with one additional qubit, $G^+$. First we find a stabilizer $S$ such that if vertex $i$ and $j$ have identical neighbourhoods then the Pauli operator in the $i$th and $j$th position of $S$ are either both $Z$ or $Y$ or are both $X$ or $I$. We denote the set of vertices where the Pauli operator of $S$ are either both $Z$ or $Y$ or are both $X$ or $I$. We denote the set of vertices where the Pauli operator of $S$ is $X$ or $I$ by $C_S$. Next, we append an additional vertex to $G$ and connect it to all of the vertices in the set.
where $C_S$. This is equivalent to adding a new generator

$$g_{n+1} = X_{n+1} \bigotimes Z_j.$$  \hspace{1cm} (27)

Repeating the same computation in Eq. 25 and 26, we achieve a precision of $\Delta^{2^\alpha} = 1/Q(G)$ by measuring $G^\dagger$ in the basis of $S_M = g_{n+1}S$. This is a result of the fact that if $X_iX_j$ stabilizes $G$ then it also stabilizes $G^\dagger$ and that $S_M$ contains only Pauli $Y$ and $Z$ operators with the exception of the $n + 1$ vertex.

In this study we have presented families of graph states which achieve better than classical scaling in precision, and can be robust against iid dephasing and a small number of erasures. These results compare favorably with other resource states for tolerating noise \cite{7,8}. One may also consider broader noise models, for example collective dephasing \cite{34}, however, this is beyond our current scope, and for realistic scenarios we do not expect big changes. Another approach is to feed forward quantum error correction strategy, however this can greatly complicate implementation \cite{35,36}. We have also found an expression for the QFI of any graph state. A main advantage of graph states is that they are already a fundamental resource across quantum information. As such we inherit all the benefits of the work that has gone into their generation \cite{22,29} and distribution over quantum networks \cite{16,17}, as well as the flexibility and potential for integration into more elaborate tasks where sensing might be a subroutine. In this sense graph states make a natural choice for integrating quantum sensing into future quantum networks, and our work demonstrates their capacity to do so.

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Appendix A: Fundamentals of Estimation Theory

Consider an observable random variable $X$ which is dependent on some unknown parameter $\theta$. The goal is to construct an estimate $\hat{\theta}$ based off of the observed outcomes $x_1, \ldots, x_n$. Since $\hat{\theta}$ is dependent on $X$, it is a random variable as well and different moments of $\hat{\theta}$ can be computed. An estimator is unbiased if

$$\langle \hat{\theta} \rangle = \sum_x p(\vec{x}|\theta) \hat{\theta}(\vec{x}) = \theta.$$  \hfill (A.1)

Here $p(\vec{x}|\theta)$ is the probability of observing outcome $\vec{x} = x_1, \ldots, x_n$, otherwise known as the likelihood function. For any unbiased estimator, the variance can be bounded below via the Cramer-Rao bound [30, 38]

$$\Delta \hat{\theta}^2 \geq \frac{1}{\mathcal{I}(X, \theta)}.$$  \hfill (A.2)

Where $\mathcal{I}(X, \theta)$ is the Fisher information

$$\mathcal{I}(X, \theta) = \langle (\partial_{\theta} \log p(X|\theta))^2 \rangle = \sum_x \frac{(\partial_{\theta} p(\vec{x}|\theta))^2}{p(\vec{x}|\theta)}.$$  \hfill (A.3)

In a scenario where the unknown parameter is encoded into a quantum state $\rho \rightarrow \rho_\theta$, our observable is dependent on the choice of POVM $\{E_k\}$ and the probability of observing the $k$th outcome is $\text{Tr}(E_k \rho_\theta)$. The Fisher information can be written as

$$\mathcal{I}(\{E_k\}, \theta) = \sum_k \frac{(\partial_{\theta} \text{Tr}(E_k \rho_\theta))^2}{\text{Tr}(E_k \rho_\theta)}.$$  \hfill (A.4)

We now define a new quantity, the quantum Fisher Information (QFI), which is the Fisher information maximized over all POVM’s [30]

$$Q = \max_{\{E_k\}} \mathcal{I}(\{E_k\}, \theta).$$  \hfill (A.5)

For a simple encoding $\rho_\theta = e^{-i\theta H} \rho e^{i\theta H}$ with $\rho = \sum_j \lambda_j |j\rangle\langle j|$ the QFI has a closed form expression [11, 30]

$$Q = 2 \sum_{j,k} \frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k} |\langle j|H|k\rangle|^2.$$  \hfill (A.6)

Which can be simplified further for pure states $|\psi\rangle$

$$Q = 4\Delta H^2 \geq 4 \langle \psi|H^2|\psi\rangle - 4 \langle \psi|H|\psi\rangle^2.$$  \hfill (A.7)

Appendix B: Lower bound of $\bar{N}_{n,\epsilon}$

We begin by defining two sets of Pauli matrices

$$A = \{X_1 X_i \mid 1 < i \leq n\}, \quad B_k = \{A_1 \ldots A_k \mid A_i \in \{Y, Z\} \forall i\}.$$  \hfill (B.1)

Next we construct the following stabilizer group

$$S = \langle X_1 X_2, \ldots, X_1 X_k, P, P g_1, \ldots, P g_{n-k} \rangle.$$  \hfill (B.2)
Where $P \in B_k$ and $g_1, \ldots, g_{n-k}$ are the generators for any $n-k$ qubit stabilizer state. Notice that the stabilizer group of $S$ does not contain any stabilizer of the form $\pm X_i$ or $-X_iX_j$, thus the QFI is equal to the number of stabilizers of the form $X_iX_j$, which there are $k^2$ by construction. We define $\tilde{N}_{n,\epsilon}$ to be the number of unique ways in which we can choose a set of generators from $A$, a Pauli operator from $B_k$ and generators $g_1, \ldots, g_{n-k}$ such that the constructed stabilizer state has a QFI of at least $n^{2-\epsilon}$. By choosing $k = n^{1-\epsilon/2}$, we can set the following lower bound

$$\tilde{N}_{n,\epsilon} \geq \sum_{j=k}^{n} \binom{n-1}{j-1} |B_j| N_{n-j}$$

$$= \sum_{j=k}^{n} \binom{n-1}{j-1} 2^j N_{n-j}$$

$$\geq \binom{n-1}{k-1} 2^k N_{n-k}. \quad \text{(B.3)}$$

### Appendix C: QFI of graph states subjected to iid dephasing

We model an $n$ qubit graph state $G$ undergoing iid dephasing via

$$G \rightarrow G^{\text{dephasing}} = \sum_k p^k (1 - p)^{n-k} Z_k G Z_k, \quad \text{(C.1)}$$

where $p$ is the probability that a qubit undergoes a phase flip. This effectively maps the graph state onto the orthonormal basis $\{Z_k | G\}_k$. We compute the QFI using Eq. $\text{(A.7)}$

$$Q(G^{\text{dephasing}}) = \frac{1}{2} \sum_{j,k} \frac{(|\lambda_j - \lambda_k|)^2}{\lambda_j + \lambda_k} |\langle G | Z_j \sum_i X_i Z_k | G \rangle|^2. \quad \text{(C.2)}$$

The only non-zero terms in the sum is if $n \leq j + k = N_\bar{x}$, where $N_\bar{x}$ is the shared neighbourhood of a set of vertices $\bar{x} \subseteq V$. Define $|\bar{x}| = x$ and $|N_\bar{x}| = N_\bar{x}$. We split $\bar{k}$ into three potions, $a$ qubits with a flipped phase from $\bar{x}$, $b$ qubits with a flipped phase from $N_\bar{x}$ and $c$ qubits with a phase flipped from the remaining $n - x - N_\bar{x}$ qubits,

$$Q(G^{\text{dephasing}}) = \frac{1}{2} \sum_{x, a, b, c=0} \frac{(|\lambda_a - \lambda_b + \lambda_c|)^2}{\lambda_a - \lambda_b + \lambda_c} |\langle G | Z_k \sum_i X_i Z_k | G \rangle|^2 \quad \text{(C.3)}$$

Where

$$f(\bar{x}, p) = x^2 (1 - 2p)^2 + 4xp(1 - p), \quad \text{(C.4)}$$

and

$$g(\bar{x}, p) = \sum_{j=0}^{N_\bar{x}} \binom{N_\bar{x}}{j} \left( \frac{|p^{N_\bar{x}-j} (1 - p)^j - p^j (1 - p)^{N_\bar{x}-j}|^2}{p^{N_\bar{x}-j} (1 - p)^j + p^j (1 - p)^{N_\bar{x}-j}} \right)$$

$$= 2 - 4p^{N_\bar{x}} (1 - p)^{N_\bar{x}} \sum_{j=0}^{N_\bar{x}} \binom{N_\bar{x}}{j} \frac{p^j (1 - p)^j}{(1 - p)^{N_\bar{x}} p^j + p^j (1 - p)^{N_\bar{x}}} \quad \text{(C.5)}$$

$$\geq 2 - 2 \left( 2p(1 - p) + \frac{1}{2} \right)^{N_\bar{x}}.$$
FIG. C.3: Plot of Eq. (C.5) using varying values of $N_x$. Regardless of the value of $N_x$, $g = 0$ when $p = 1/2$, this is because $G_{\text{dephasing}}$ is the maximally mixed state at $p = 1/2$, which is useless for quantum metrology.

From equation Eq. (C.5) we see that $g \approx 2$ when $(2p(1-p) + 1/2)^N_x \approx 0$; this is illustrated in Fig. C.3. Using this approximation, the QFI of $G_{\text{dephasing}}$ can be written as

$$Q(G_{\text{dephasing}}) \approx \sum_{\bar{x}} f(\bar{x}, p) = (1 - 2p)^2 Q(G) + 4np(1 - p). \quad (C.6)$$

Appendix D: QFI of graph states subjected finite erasures

We model an $n$ qubit graph state $G$ subjected to finite erasures $\bar{y} = \{y_1, \ldots, y_e\}$ via

$$G \rightarrow G_{\bar{y}} = \text{Tr}_{\bar{y}} G. \quad (D.1)$$

This maps $G$ into an equally weighted mixed state

$$G_{\bar{y}} = 2^{-|L_{\bar{y}}|} \sum_{j \subseteq L_{\bar{y}}} Z_j |G\rangle |G\rangle Z_j. \quad (D.2)$$

Where the set $L_{\bar{y}}$ is the set which contains all of the lost qubits as well as all of their respective neighbourhoods

$$L_{\bar{y}} = \bigcup_{i=1}^{e} \{y_i\} \cup N(y_i). \quad (D.3)$$

Similarly to in Appendix C, we denote a set of vertices with a shared neighbourhood as $\bar{x} \subseteq V$. Using Eq. (A.7) we compute the QFI of the new state

$$Q(G_{\bar{y}}) = \frac{1}{2} \sum_{j,k} \frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} |\langle G| Z_j \sum_i X_i Z_k |G\rangle|^2$$

$$= \frac{1}{2} \sum_{\bar{x}} \sum_k \frac{((\lambda_k + N_x) - \lambda_k)^2}{\lambda_k + N_x + \lambda_k} |\langle G| Z_{\bar{k}+N_x} \sum_i X_i Z_k |G\rangle|^2. \quad (D.4)$$
Note that $\lambda_{\vec{k} + \mathcal{N}_x} - \lambda_{\vec{k}} = 0$ if $\vec{k} \subseteq L_{\vec{y}}$ and $\vec{k} + \mathcal{N}_x \subseteq L_{\vec{y}}$. Regardless of $\vec{k}$, this only occurs if $\mathcal{N}_x \subseteq L_{\vec{y}}$. In the scenario where $\mathcal{N}_x \notin L_{\vec{y}}$ the value of $Q$ depends on whether $\vec{x} \subseteq L_{\vec{y}}$ or $\vec{x} \nsubseteq L_{\vec{y}}$. We can thus simplify the above expression

$$Q(G_{\vec{y}}) = \sum_{\vec{x}} l(\vec{x}, \vec{y}),$$

where

$$l(\vec{x}, \vec{y}) = \begin{cases} 
    x^2 & \text{if } \mathcal{N}_x \notin L_{\vec{y}} \text{ and } \vec{x} \nsubseteq L_{\vec{y}} \\
    x & \text{if } \mathcal{N}_x \nsubseteq L_{\vec{y}} \text{ and } \vec{x} \subseteq L_{\vec{y}} \\
    0 & \text{otherwise}
\end{cases}$$

(D.6)