1. Introduction

Certain representation spaces have been investigated by algebraic geometers as moduli spaces of holomorphic bundles over a Riemann surface. Such moduli spaces exhibit symplectic and Kähler structures as well as gauge theory interpretations. The purpose of this article is to elucidate the local structure of such a space, and the focus will be on the singularities. Among the tools will be the interconnection between the theory of algebraic and symplectic quotients and, furthermore, Poisson structures, a concept which has been known in mathematical physics for long and is currently of much interest in mathematics as well.

On $\mathbb{R}^{2n}$ with its standard coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$, the formula

$$\{f, h\} = \sum \left( \frac{\partial f}{\partial p_j} \frac{\partial h}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial h}{\partial p_j} \right)$$

yields a bracket $\{\cdot, \cdot\}$ on the algebra of smooth functions, nowadays referred to as a Poisson bracket. This bracket was introduced by Poisson around 1809, and he observed that, given three functions $f, g, h$ with $\{f, g\} = 0$ and $\{f, h\} = 0$, one also has $\{f, \{g, h\}\} = 0$. This means that if $g$ and $h$ are integrals of motion for (the hamiltonian vector field of) $f$, so is $\{g, h\}$. See e.g. [1 p. 196], [4 p. 216]. Abstractly, a Poisson algebra is a commutative algebra $A$ together with the additional structure of a Lie bracket $\{\cdot, \cdot\}$ which behaves as a derivation in each variable with respect to the algebra structure. A smooth Poisson structure on an ordinary smooth manifold is symplectic if and only if it is locally of the kind (1.1).
Poisson structures provide some insight into the geometry of certain representation spaces. Thus, let \( G \) be a compact Lie group and \( \pi \) the fundamental group of a closed surface \( \Sigma \), and consider the \textit{representation space} \( \text{Rep}(\pi, G) = \text{Hom}(\pi, G)/G \); here \( G \) acts on \( \text{Hom}(\pi, G) \) via conjugation. Pick an invariant symmetric non-degenerate bilinear form on the Lie algebra \( \mathfrak{g} \) of \( G \). The following result may be found in [24]; see also [26–28] for a leisurely introduction and for more references.

\textbf{Theorem 1.} The decomposition of \( \text{Rep}(\pi, G) \) into orbit types of representations is a stratification, and the data induce a Poisson algebra

\[ (C^\infty(\text{Rep}(\pi, G)), \{\cdot, \cdot\}) \]

of continuous functions on \( \text{Rep}(\pi, G) \) turning the latter into a stratified symplectic space in such a way that a choice of complex structure on \( \Sigma \) induces a Kähler structure on each stratum. Furthermore, the space \( \text{Rep}(\pi, G) \) is locally semi-algebraic. Finally, the Poisson structure detects the stratification.

Here the algebra \( C^\infty(\text{Rep}(\pi, G)) \) encapsulates the \textit{real} geometry of the space \( \text{Rep}(\pi, G) \). With this structure, \( \text{Rep}(\pi, G) \) is \textit{not} a real variety in the usual sense; its singularity behaviour is alluded to by the wording “locally semi-algebraic”. The meaning of “(locally) semi-algebraic” will be explained in Section 2 below where we will also explain how the Poisson structure detects the stratification, and the decomposition into orbit types will be reproduced in Section 4 below.

There is a version of the above result involving twisted representation spaces [24] as well as one involving surfaces with non-empty boundary [14]. In algebraic geometry, after a choice of complex structure on \( \Sigma \) has been made, for \( G = \text{U}(n) \), the space \( \text{Rep}(\pi, G) \) arises as the (real space underlying the) moduli space of semi-stable holomorphic rank \( n \) vector bundles on \( \Sigma \) of degree zero, and the twisted representation spaces correspond to the case of arbitrary degree [39–42, 45, 46]. The Poisson structure then gives rise to some interesting Poisson geometry. In particular, for \( G = \text{SU}(2) \), the space \( \text{Rep}(\pi, G) \) amounts to the moduli space of semi-stable holomorphic vector bundles on \( \Sigma \) of rank 2, degree 0, and trivial determinant. To describe the stratification in this case, let \( Z \subseteq G \) denote the center and \( T \subseteq G \) a maximal torus. The space \( N = \text{Rep}(\pi, G) \) is a union

\[ N = N_G \cup N_T \cup N_Z \]

of three strata where \( N_\mathcal{K} \) denotes the points of orbit type \( (\mathcal{K}) \). The piece \( N_Z \) is called the top stratum.

For genus \( \ell \geq 2 \), \textsc{Narasimhan-Ramanan} proved that the complement \( \mathcal{K} \) of the top stratum is the \textit{Kummer} variety of \( \Sigma \) associated with its Jacobian \( J \) and the canonical involution thereupon [39]. This has the following consequence, established in [23].

\textbf{Theorem 2.} For \( G = \text{SU}(2) \), when \( \Sigma \) has genus \( \ell \geq 2 \), the Poisson algebra \( (C^\infty(\text{Rep}(\pi, G)), \{\cdot, \cdot\}) \) detects the Kummer variety \( \mathcal{K} \) in \( \text{Rep}(\pi, G) \) together with its \( 2^{2\ell} \) double points. More precisely, \( \mathcal{K} \) consists of the points where the rank of the Poisson structure is not maximal, the double points being those where the rank is zero.

For genus \( \geq 3 \), the Kummer variety \( \mathcal{K} \) is precisely the (complex analytic) singular locus of \( \text{Rep}(\pi, G) \), a result due to \textsc{Narasimhan-Ramanan} [39]. This has been
reproved in [23] within our framework. When $\Sigma$ has genus two the space $\text{Rep}(\pi, G)$ equals complex projective 3-space [39] and $K$ is the Kummer surface associated with the Jacobian of $\Sigma$. In the literature, this case has been considered somewhat special: The algebra of ordinary smooth functions on $\text{Rep}(\pi, G)$, realized as complex projective 3-space, is a smooth structure on $\text{Rep}(\pi, G)$ in the above sense as well—call it the standard structure—and the Kummer surface is certainly not the singular locus for this structure. However, from our point of view there is no exception. Our smooth structure $C^\infty(\text{Rep}(\pi, G))$ is not the standard one, and even in the genus two case, as a stratified symplectic space, $\text{Rep}(\pi, G)$ still has singularities: The Poisson algebra $(C^\infty(\text{Rep}(\pi, G)), \{\cdot, \cdot\})$ detects (the real space underlying) a Kummer surface together with its 16 singularities and hence the underlying algebra of functions can plainly not be that of ordinary smooth functions; in particular, the symplectic structure on the top stratum does not extend to the whole space. It is interesting to observe that the stratification mentioned in Theorem 1 is finer than the standard complex analytic one on complex projective 3-space.

Under the circumstances of Theorem 1, the Kähler structures on the strata of $\text{Rep}(\pi, G)$ fit together in a very precise way. Abstracting from this situation, we isolated the notion of stratified Kähler space in [31]. The structure on complex projective 3-space just explained is a particular case of a stratified Kähler structure which is not the standard Kähler structure. In [31] it is shown that, in particular, such “exotic” structures on complex projective spaces abound.

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2. Singularities, real vs complex; an (almost toy) example

What is a singularity? “A singular point is one which behaves differently from the other points close to it.”

For illustration consider $\mathbb{R}^2$, but viewed as a half-cone. What kind of structure can we put on $\mathbb{R}^2$ to distinguish the cone point, $o$, from the other points? A possible answer: Consider the ordinary coordinates $x, y$ in the plane and introduce a third variable $\rho$, subject to the relation

$$x^2 + y^2 = \rho^2.$$  

Let $A$ be the algebra of smooth functions in these variables, subject to the relation (2.1). Notice that $\rho$ is not a smooth function on the plane in the usual sense, and the algebra $A$ is strictly larger than that of ordinary smooth functions on the plane. The situation is entirely parallel to that of an algebra of continuous functions on complex projective 3-space strictly larger than the ordinary one mentioned in Theorem 2 above. Under the present circumstances, the real algebraic variety $Z$ defined by (2.1), that is, the set of points $(x, y, \rho)$ satisfying this equation, is a (double) cone. The half cone we are interested in is characterized by the additional constraint $\rho \geq 0$. Technically one says that the half cone $C$ is a semi-algebraic subset of $Z$, cf. e. g. [7]. Why do we distinguish $C$ from $Z$? The answer is that a space of the kind $\text{Rep}(\pi, G)$ is locally semi-algebraic.

Given a space $X$ which is decomposed into pieces which are smooth manifolds, a smooth structure on $X$ is an algebra $C^\infty(X)$ of (real or complex valued) continuous functions which, on each piece, restrict to ordinary smooth functions. A space $X$ together with a smooth structure is called a smooth space.
The half cone \( C \), together with the algebra \( A = C^\infty(C) \) of functions defined above, is a smooth space, and so is the (double) cone \( Z \), together with \( A = C^\infty(Z) \). Viewed abstractly, \( Z \) is just a realization of the real maximal spectrum of \( A \) and, as algebras, \( C^\infty(C) \) and \( C^\infty(Z) \) coincide. We can view \( A = C^\infty(C) = C^\infty(Z) \) as the algebra of ordinary smooth functions on \( \mathbb{R}^3 \), subject to the relation (2.1), which means that every function belonging to this smooth structure is the restriction of an ordinary smooth function on the ambient space. More generally: Given a closed connected subset \( Y \subseteq \mathbb{R}^n \) (not necessarily a smooth manifold), the continuous functions on \( Y \) which are restrictions of ordinary smooth functions on \( \mathbb{R}^n \) endow \( Y \) with a smooth structure; this construction goes back to Whitney [53], and the resulting functions on \( Y \) are nowadays called Whitney-smooth functions. Thus, in the present situation, the algebra \( A = C^\infty(C) = C^\infty(Z) \) is that of Whitney smooth functions on \( Z \) as well as on \( C \).

Given a smooth space \( (X,C^\infty(X)) \), for each point \( x \) of \( X \), the ideal \( m_x \) of \( x \) in \( C^\infty(X) \) consists of the smooth functions which vanish at \( x \). See e. g. [43,44]. The Zariski tangent space \( T_xX \) of \( X \) at \( x \) is the vector space \((m_x/m_x^2)^* \) [7,54]; see e. g. [15] for the notion of Zariski tangent space in algebraic geometry.

Another well known description of the Zariski tangent space is this: Let \( x \in X \) and view \( R \) as a \( C^\infty(X) \)-module, written \( R_x \), by means of the evaluation mapping from \( C^\infty(X) \) to \( R \) which assigns to a function \( f \) its value \( f(x) \) at \( x \in X \); now a \textit{derivation} at \( x \in X \) is a linear map \( d \) from \( C^\infty(X) \) to \( R \) satisfying the usual \textit{Leibniz} rule

\[
(d(fh))(x) = (df)(x)h(x) + f(x)dh.
\]

We denote the real vector space of all derivations at \( x \) by \( \text{Der}(C^\infty(X),R_x) \). For \( x \in X \), the assignment to \( \phi \in T_xX \) of the derivation \( d_\phi \) at \( x \) given by \( d_\phi(f) = \phi(f-f_x) \) identifies \( T_xX \) with \( \text{Der}(C^\infty(X),R_x) \); here \( f \in C^\infty(X) \) and \( f_x \) denotes the function having constant value \( f \). Thus, when \( X \) is a smooth manifold near a point \( x \) in the usual sense, with standard smooth structure near \( x \), the Zariski tangent space boils down to the ordinary smooth tangent space whence there is no risk of confusion in notation. Given smooth spaces \( (X,C^\infty(X)) \), \( (Y,C^\infty(Y)) \), and a smooth map \( \phi \) from \( X \) to \( Y \), the \textit{derivative} at a point \( x \in X \) is the dual \( d\phi_x : T_xX \to T_{\phi x}Y \) of the linear map from \( m_{\phi(x)}/m_{\phi(x)}^2 \) to \( m_x/m_x^2 \) induced by \( \phi \).

Intuitively, the Zariski tangent space at \( x \) is the linear span of all the \textit{tangents} to \( x \). Lack of space does not allow us to elaborate on this here; see e. g. [36]. In the above example of a (half) cone: At every point different from \( o \), the Zariski tangent space is a plane while, at \( o \), it is a copy of \( \mathbb{R}^3 \). In fact, this copy of \( \mathbb{R}^3 \) is the span of the cone. Thus we may distinguish the singular point \( o \) from the other points by means of the notion of Zariski tangent space defined in terms of the non-standard smooth structure involving the additional function \( \rho \).

There is yet another way to distinguish the cone point from the other points: Introduce a Poisson structure \{ , \} by setting

\[
\{x,y\} = 2\rho, \quad \{x,\rho\} = 2y, \quad \{y,\rho\} = -2x.
\]

This Poisson structure is symplectic everywhere except at the cone point where it degenerates. Thus it distinguishes the cone point from the other points.

Recall that a \textit{symplectic} manifold is a smooth manifold \( M \) together with a closed non-degenerate 2-form \( \omega \). Given \( f \), the identity \( \omega(X_f,\cdot) = df \) then associates a
uniquely determined vector field $X_f$ to $f$, the Hamiltonian vector field of $f$, and the assignment $\{f, h\} = X_f h$ yields a Poisson bracket on the algebra $C^\infty(M)$ of ordinary smooth functions on $M$. Such a Poisson bracket is called symplectic. The Poisson structure (1.1) is a special case thereof.

A decomposition of a space $X$ into pieces which are smooth manifolds such that these pieces fit together in a certain precise way is called a stratification [13]. More precisely: Let $X$ be a Hausdorff paracompact topological space and let $\mathcal{I}$ be a partially ordered set with order relation denoted by $\leq$. An $\mathcal{I}$-decomposition of $X$ is a locally finite collection of disjoint locally closed manifolds $S_i \subseteq X$ called pieces (recall that a collection $\mathcal{A}$ of subsets of $X$ is said to be locally finite provided every $x \in X$ has a neighborhood $U_x$ in $X$ such that $U_x \cap A \neq \emptyset$ for at most finitely many $A \in \mathcal{A}$) such that the following hold:

$$X = \cup S_i$$

$$S_i \cap \overline{S}_j \neq \emptyset \iff S_i \subseteq \overline{S}_j \iff i \leq j.$$  

The space $X$ is then called a decomposed space. A decomposed space $X$ is called a stratified space if the pieces of $X$, called strata, satisfy the following condition: Given a point $x$ in a piece $S$ there is an open neighborhood $U$ of $x$ in $X$, an open ball $B$ around $x$ in $S$, a stratified space $L$, called the link of $x$, and a decomposition preserving homeomorphism from $B \times C^\infty(L)$ onto $U$. Here $C^\infty(L)$ refers to the open cone on $L$ and, as a stratified space, $L$ is less complicated than $C^\infty(L)$ whence the definition is not circular; the idea of complication is here made precise by means of the notion of depth. A stratified symplectic space [47] is a stratified space $X$ together with a Poisson algebra $(C^\infty(X), \{ , \})$ of continuous functions which, on each piece, restricts to an ordinary smooth symplectic Poisson structure; in particular, $C^\infty(X)$ is a smooth structure on $X$.

The half cone, endowed with the above smooth structure and Poisson algebra (2.2) is an example of a stratified symplectic space with two strata, the cone point and its complement, the “top” stratum. The Poisson structure detects the stratification: It has “rank” zero at the cone point and “rank” two on the top stratum whence it is symplectic there. Given a point $x$ of $X$, the rank at $x$ refers here to the rank of the linear map from $\Omega_x(X)$ to $\text{Der}(C^\infty(X), \mathbb{R}_x)$ induced by the Poisson structure; more precisely, the canonical map from $\Omega(x) \otimes_{C^\infty(X)} \mathbb{R}_x$ to $\Omega_x(X) \cong \mathfrak{m}_x / \mathfrak{m}_x^2$ (see what is said above,) is an isomorphism of vector spaces, the Poisson structure induces a $C^\infty(X)$-morphism from the $C^\infty(X)$-module $\Omega(X)$ of differentials with respect to $C^\infty(X)$, endowed with the $(\mathbb{R}, C^\infty(X))$-Lie algebra structure coming from the Poisson structure, cf. [17], to the $C^\infty(X)$-module $\text{Der}(C^\infty(X))$ of derivations of $C^\infty(X)$, and this morphism, in turn, induces the linear map from $\Omega_x(X)$ to $\text{Der}(C^\infty(X), \mathbb{R}_x)$ under discussion. More details about the detection of the stratification by means of the notion of rank may be found in [23].

The ordinary complex structure of the plane, combined with the symplectic structure on the top stratum, turns the latter into a Kähler manifold. But the cone point is not a singularity for the complex structure whence the latter cannot be used to detect the cone point. This reflects the remark made earlier that the symplectic stratification is finer than the standard complex analytic one.

For completeness, we recall at this stage that a Kähler manifold is a smooth complex manifold together with a hermitian metric whose imaginary part is a symplectic structure. Equivalently, a smooth symplectic manifold $(M, \omega)$ together
with a complex structure $J$ which is compatible with $\omega$ (i.e. $\omega(JX,JY) = \omega(X,Y)$) is called a Kähler manifold provided the (real) symmetric bilinear form $g$ given by $g(X,Y) = \omega(X,JY)$ is positive definite. See e.g. [35] for details.

3. Symplectic reduction in a nutshell

Given a compact Lie group $G$, a hamiltonian $G$-space is a smooth symplectic $G$-manifold $(M,\omega)$ together with a smooth $G$-equivariant map $\mu$ from $M$ to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $G$ satisfying the formula

$$\omega(X_M,\cdot) = X \circ d\mu$$

for every $X \in \mathfrak{g}$; here $X_M$ denotes the vector field on $M$ induced by $X \in \mathfrak{g}$ via the $G$-action, and $X$ is viewed as a linear map on $\mathfrak{g}^*$. The map $\mu$ is called a moment mapping (or moment map). We recall that (3.1) says that, given $X \in \mathfrak{g}$, the vector field $X_M$ is the hamiltonian vector field for the smooth function $X \circ \mu$ on $M$. See [1, 4] for details. Given a hamiltonian $G$-space $(M,\omega,\mu)$, the space $M_{\text{red}} = \mu^{-1}(0)/G$ is called the reduced space or symplectic quotient. It carries the smooth structure $C^\infty(M_{\text{red}}) = (C^\infty(M))^G/IG$, where $IG$ refers to the ideal in the algebra $(C^\infty(M))^G$ of smooth $G$-invariant functions on $M$ which vanish on the zero locus $\mu^{-1}(0)$. Arms-Cushman-Gotay [2] established the fact that the symplectic Poisson algebra on $C^\infty(M)$ descends to a Poisson structure $\{\ ,\ \}_\text{red}$ on $C^\infty(M_{\text{red}})$, and Sjamaar-Lerman [47] have shown that the decomposition of $M_{\text{red}}$ according to orbit types is a stratification in such a way that $(M_{\text{red}},C^\infty(M_{\text{red}}),\{\ ,\ \}_\text{red})$ is a stratified symplectic space. When $M_{\text{red}}$ is smooth, i.e. has a single stratum, the structure comes down to that of a smooth symplectic manifold on $M_{\text{red}}$, the ordinary Marsden-Weinstein reduced space [38].

4. Representation spaces

Denote the genus of the closed surface $\Sigma$ under consideration by $\ell$. The standard presentation

$$\langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle$$

of $\pi = \pi_1(\Sigma)$ where $r = [x_1, y_1] \ldots [x_\ell, y_\ell]$ determines a word map $r: G^{2\ell} \to G$, and the choice of generators identifies $\text{Hom}(\pi,G)$ with the subset $r^{-1}(e)$ of $G^{2\ell}$ (where $e \in G$ is the neutral element) whence $\text{Rep}(\pi,G)$ may be realized as the quotient

$$\text{Rep}(\pi,G) = r^{-1}(e)/G.$$ 

The situation is very similar to that of a hamiltonian $G$-space. Indeed a corresponding hamiltonian $G$-space, called extended moduli space, has been constructed in [24, 32]. The extended moduli space then determines the requisite smooth structure, and symplectic reduction yields the stratified symplectic space structure on $\text{Rep}(\pi,G)$ coming into play in Theorem 1.

The decomposition which underlies the stratification of $\text{Rep}(\pi,G)$ is that by orbit types. For the reader’s convenience, we explain this briefly: Write $N = \text{Rep}(\pi,G)$ and, for a closed subgroup $K \subseteq G$ let $N_{(K)} \subseteq N$ denote the subspace of classes $[\phi]$ of representations $\phi$ having stabilizer $Z_\phi$ conjugate to $K$. Thus the representation space $N$ decomposes into a disjoint union of orbit types $N_{(K)}$, $(K) \in \mathcal{I}$, the indexing set $\mathcal{I}$ being that of all possible stabilizer subgroups modulo conjugacy, and each orbit type decomposes further into its connected components. Sometimes the decomposition into connected components is the more appropriate one.
5. The local structure

Consider a homomorphism \( \phi: \pi \to G \). It represents a point \([\phi]\) of the space \( \text{Rep}(\pi, G) \). By means of \( \phi \) we endow \( \mathfrak{g} \) with a \( \pi \)-module structure. The Lie bracket on \( \mathfrak{g} \) induces a graded Lie bracket \([\ ,\ ]_\phi\) on the group cohomology group \( H^*_\phi = H^*(\pi, \mathfrak{g}_\phi) \), and the chosen invariant symmetric bilinear form on \( \mathfrak{g} \), combined with the isomorphism from \( H^2(\pi, \mathbb{R}) \) onto \( \mathbb{R} \) given by a choice of fundamental class of \( \pi \), induces a non-degenerate graded commutative pairing

\[
H^*_\phi \otimes H^{2-*}_\phi \to \mathbb{R}.
\]

In degree 1, (5.1) amounts to a symplectic structure \( \sigma_\phi \) on the vector space \( H^1_\phi \). Moreover, \( H^0_\phi \) is the Lie algebra of the stabilizer \( Z_\phi \subseteq G \) of \( \phi \), and the assignment \( \Theta_\phi(\eta) = \frac{1}{2} [\eta, \eta]_\phi \) \((\eta \in H^1_\phi)\) yields a momentum mapping \( \Theta_\phi: H^1_\phi \to H^2_\phi \) for the action of \( Z_\phi \) on \( H^1_\phi \) which is therefore hamiltonian; here \( H^2_\phi \) is identified with the dual of \( H^0_\phi \) by means of (5.1). We then have the following result, given in Theorem 6.3 of [21]; for the analogous gauge theory situation, it has been spelled out in [19].

**Theorem 3.** The reduced space \((H^1_\phi)_{\text{red}}\), with its stratified symplectic space structure, is a local model for \( \text{Rep}(\pi, G) \) near \([\phi]\) as a stratified symplectic space. More precisely, the choice of \( \phi \) (in its class \([\phi]\)) induces a diffeomorphism of an open neighborhood \( W_\phi \) of \([0]\) in \((H^1_\phi)_{\text{red}}\) onto an open neighborhood \( U_\phi \) of \([\phi]\) in \( \text{Rep}(\pi, G) \), where \( W_\phi \) and \( U_\phi \) are endowed with the induced stratified symplectic structures \((C^\infty(W_\phi), \{\cdot,\cdot\})\) and \((C^\infty(U_\phi), \{\cdot,\cdot\})\), respectively (the notation \( \{\cdot,\cdot\} \) being abused somewhat).

Thus locally, that is, near the point \([\phi]\) of \( \text{Rep}(\pi, G) \), it suffices to study the reduced space \((H^1_\phi)_{\text{red}}\) (written \( H^1_\phi \) in [21]). The latter may be understood by means of geometric invariant theory: Write \( W = H^1_\phi \) and \( K = Z_\phi \). By a theorem of Hilbert and Hurwitz [52] (VIII §14), the (graded) algebra \( \mathbb{R}[W]^K \) of \( K \)-invariant polynomials on \( W \) has a finite set \( p_1, \ldots, p_d \) of generators. Let

\[
p = (p_1, \ldots, p_d): W \to \mathbb{R}^d
\]

be the Hilbert map, let \( I \) be the ideal of relations among the \( p_j \) in \( \mathbb{R}[y_1, \ldots, y_d] \), and let \( Z \) be the corresponding algebraic subset of \( \mathbb{R}^d \). By the Tarski-Seidenberg Theorem, \( X := \text{Im}(p) \) is a semi-algebraic subset of \( Z \), cf. e. g. [7], and the induced map \( \overline{p}: W/K \to X \) is a homeomorphism. With the appropriate structures, it is even a diffeomorphism of smooth spaces [43]. If \( S \) is a real affine \( K \)-subvariety of \( W \), it may be shown that there is an algebraic subset \( Z' \) of \( Z \) such that the inequalities determining \( X \subseteq Z \) determine \( p(S) = S/K : = X' = X \cap Z' \subseteq Z' \). See e. g. [44] for details. Since the zero locus of \( \Theta_\phi \) is a real affine \( Z_\phi \)-subvariety of \( H^1_\phi \), this shows that \( \text{Rep}(\pi, G) \) is locally semi-algebraic.

Near any of its points, by means of the local model, we can now elucidate the stratified symplectic structure of \( \text{Rep}(\pi, G) \) (locally) as follows: After identification of \( H^*_\phi \) with the cohomology of \( \Sigma \) with the appropriate coefficients, and after a choice of complex structure on \( \Sigma \) has been made, the star operator endows \( H^1_\phi \) with a complex structure compatible with the symplectic structure \( \sigma_\phi \), and in this way \( H^1_\phi \) becomes a unitary \( Z_\phi \)-representation such that \( \Theta_\phi \) is its unique momentum mapping having
the value zero at the origin. In (2.4) on p. 52 of Arms-Gotay-Jennings [3], this is called the “the standard example”. The upshot is that, locally, the space \( \text{Rep}(\pi,G) \) may be studied by techniques coming from constrained systems in mechanics.

To simplify the exposition, consider a general finite dimensional unitary representation \( W \) of a compact Lie group \( K \). Associated with it is the unique momentum mapping \( \mu \) from \( W \) to \( k^* \) having the value zero at the origin [3]. In terms of complex coordinates \( z = (z_1, \ldots, z_m) \) on \( W \), when \( K \) is a subgroup of \( U(m) \), this momentum mapping is given by the formula

\[
(\xi \circ \mu)(z) = \frac{i}{2} \sum \xi_{j,k} z_j \overline{z}_k
\]

where \( \xi = (\xi_{j,k}) = (-\xi_{k,j}) \) is a complex \((m \times m)\)-matrix in \( k \); here \( \xi \in k \) is viewed as a coordinate function on \( k^* \).

The \( K \)-action extends to an action of the complexification \( K^C \) of \( K \) on \( W \). The idea is now to relate the reduced space \( W_{\text{red}} \) with the affine categorical quotient \( W // K^C \), cf. e. g. [44]. Recall that, in general, given a reductive group \( \Gamma \) and an affine \( \Gamma \)-variety \( V \), the affine categorical quotient, also referred to as the algebraic quotient in the literature, is a morphism \( \tau: V \to V // \Gamma \) of affine varieties which is constant on \( \Gamma \)-orbits and has the following universal property: Given a morphism \( \psi: V \to V' \) of affine varieties which is constant on \( \Gamma \)-orbits, there is a unique morphism \( \Psi: V // \Gamma \to V' \) of affine varieties such that \( \Psi \circ \tau = \psi \). With a slight abuse of language, the variety \( V // \Gamma \) is then referred to as the categorical quotient as well. Under our circumstances, the categorical quotient is the complex affine variety corresponding to the algebra \( \mathbb{C}[W]^K \) of \( K \)-invariant complex polynomials; actually the algebra of \( K^C \)-invariant complex polynomials coincides with that of \( K \)-invariant complex polynomials. By the theorem of Hilbert and Hurwitz, this algebra has a finite set \( f_1, \ldots, f_t \) of generators. These yield a \( K^C \)-invariant algebraic map \( f \) from \( W \) to \( \mathbb{C}^t \) which, by construction, factors through the space of \( K^C \)-orbits in \( W \), and the image \( Y \) in \( \mathbb{C}^t \) is the variety defined by the relations among the \( f_1, \ldots, f_t \). This variety is a model for \( W // K^C \). See e. g. [44] (§3) for details. An observation of Kempf-Ness [34], cf. §4 of [44], where the zero locus \( \mu^{-1}(0) \) is referred to as a Kempf-Ness set, implies that the canonical map \( W_{\text{red}} \to W // K^C \) from the reduced space \( W_{\text{red}} \) to the (affine) categorical quotient \( W // K^C \) induced by the inclusion of \( \mu^{-1}(0) \) into \( W \) is a homeomorphism. As a space, in fact, as a complex affine variety, the reduced space \( W_{\text{red}} \) thus looks like the affine categorical quotient \( W // K^C \). As a stratified symplectic space it looks somewhat different, though. We mention in passing that the stratified symplectic and the Kähler structures combine to what we called a stratified Kähler structure in [31].

We now illustrate this local model analysis for the special case where \( G = SU(2) \). We remind the reader that the decomposition

\[
N = N_G \cup N_T \cup N_Z
\]

has been described earlier.

**Theorem 4.** Near a point of \( N(K) \), \( N \) and \( (C^\infty(N), \{\cdot, \cdot\}) \) may be described in the following way:
$K = Z$: the space $\mathbb{C}^{3(\ell-1)}$ with its standard symplectic Poisson structure;
$K = T$: a product of $\mathbb{C}^\ell$ with its standard symplectic Poisson structure and of the reduced space and reduced Poisson algebra of a system of $\ell-1$ particles in the plane with total angular momentum zero;
$K = G$: the reduced space and reduced Poisson algebra of a system of $\ell$ particles in 3-space with total angular momentum zero.

The proof of this theorem relies on the local models: For example, when $\ell = 2$, near a point of the “middle” stratum $N(T)$, the space $N$ looks like the product of a copy of $\mathbb{C}^2$ with the reduced system of a single particle in the plane $\mathbb{R}^2$. The latter is $\mathbb{R}^2$, endowed with the smooth structure and Poisson algebra (2.2).

In some more detail: For a point $[\phi]$ in the top stratum, $H^0_\phi$ and $H^2_\phi$ are zero, and hence near $[\phi]$, the moduli space looks like a neighborhood of zero in $H^1_\phi$, with the symplectic structure $\sigma_\phi$. The latter, in turn, amounts to a copy of $\mathbb{C}^{3(\ell-1)}$, with the standard symplectic structure.

For a point $[\phi]$ in the middle stratum $N(T)$, $\mathfrak{g}_{\phi}$ decomposes into a direct sum of $t$ and $t^\perp$ where $t$ is the Lie algebra of $T$ which is a copy of the reals (with trivial $T$-action) and $t^\perp$ amounts to $\mathbb{R}^2$, with circle action through the 2-fold covering map onto $SO(2, \mathbb{R})$. Moreover, $H^1_\phi$ decomposes into a direct sum of $(t \otimes \mathbb{C})^\ell$ and $(t^\perp \otimes \mathbb{C})^{\ell-1}$. The action on $(t \otimes \mathbb{C})^\ell$ is trivial – in fact, this summand corresponds to the points in the stratum $N(T)$ (locally), while the $SO(2, \mathbb{R})$-representation on $(t^\perp \otimes \mathbb{C})^{\ell-1}$ is hamiltonian, with momentum mapping $\Theta_\phi$, restricted to $(t^\perp \otimes \mathbb{C})^{\ell-1}$. The latter boils down to the classical constrained system of $\ell-1$ particles moving in the plane with constant total angular momentum. In particular, reduction at total angular momentum zero yields the reduced space we are looking for (locally).

When $\ell = 2$, $W = t^\perp \otimes \mathbb{C} \cong \mathbb{R}^2 \times \mathbb{R}^2$. The $SO(2, \mathbb{R})$-representation is the obvious one, that is, $SO(2, \mathbb{R})$ acts as rotation group on each copy of $\mathbb{R}^2$. With the usual coordinates $(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, the momentum mapping $\mu$ from $W$ to $\mathbb{R}$ is given by the assignment to $(q, p)$ of the determinant $|qp|$. The algebra of (real) invariants in $\mathbb{R}[W]$ is generated by the three scalar products $qq, qp, pp$, and the determinant $|qp|$. However, on the zero locus $\mu^{-1}(0)$, the determinant vanishes whence the algebra $C^\infty(W_{\text{red}})$ is generated by the three scalar products.

To illustrate how we can understand $W_{\text{red}}$ as a space by means of the corresponding categorical affine quotient we observe that the extension of the $SO(2, \mathbb{R})$-representation to its complexification amounts to the standard $SO(2, \mathbb{C})$-representation on $\mathbb{C}^2$. This representation has a single invariant, the complex scalar product which, with the notation $w = q + ip$, we write $ww$, and the algebra of complex invariants is free. Consequently the affine categorical quotient $\mathbb{C}^2//SO(2, \mathbb{C})$ is a copy of the complex line $\mathbb{C}$. By virtue of the observation of Kempf-Ness, the canonical map from $W_{\text{red}}$ to $\mathbb{C}$ is a homeomorphism. Under this homeomorphism, with the notation $ww = x_1 + ix_2$, so that $x_1$ and $x_2$ are the coordinate functions on $\mathbb{C}$, viewed as the real plane, we have

$$x_1 = qq - pp, \ x_2 = 2qp, \ \rho = qq + pp.$$ 

It is obvious that the algebra $C^\infty(W_{\text{red}})$ is as well generated by the coordinate functions $x_1, x_2$ and the radius function $\rho$. Moreover, the complex picture tells us that the single obvious relation (2.1) between the coordinate functions and the
radius function is defining, that is, the relation $x_1^2 + x_2^2 = \rho^2$ suffices; hence the latter is a defining relation for $C^\infty(W_{\text{red}})$. Finally, a straightforward calculation of the Poisson brackets between $x_1, x_2, \rho$, viewed as functions on the original space $W$, yields the formulas already spelled out as (2.2). See [23] for details. Since the Zariski tangent space for the non-standard structure $C^\infty(W_{\text{red}})$ on $\mathbb{R}^2 = W_{\text{red}}$ is a copy of $\mathbb{R}^3$, the Zariski tangent space at a point $[\phi]$ of the middle stratum $N(T)$ is a copy of $\mathbb{R}^7$.

A similar reasoning yields a model for a neighborhood of a point in the “bottom” stratum $N_G$. This stratum consists of $2^{2\ell}$ isolated points and, locally, the Poisson algebra is that of the reduced classical constrained system of $\ell$ particles in $\mathbb{R}^3$ with total angular momentum zero. Even for $\ell = 2$, the reduced Poisson algebra is already rather complicated: It has ten generators; in fact, the elements of a basis of $\mathfrak{sp}(2, \mathbb{R})$ may be taken as coordinate functions on $\mathfrak{sp}(2, \mathbb{R})^*$, and the reduced space may be described as the closure of a certain nilpotent orbit in $\mathfrak{sp}(2, \mathbb{R})^*$. This relies on the theory of dual pairs [16] and is worked out in [37]; see our paper [23] for details. A more general theory explaining how and why nilpotent orbits come into play here has been developed in [31]. Suffice it to mention at this stage that, once $\mathfrak{sp}(2, \mathbb{R})$ has been identified with its dual $\mathfrak{sp}(2, \mathbb{R})^*$ by means of the Killing form, the $\text{Sp}(2, \mathbb{R})$-momentum mapping

$$(T^*\mathbb{R}^3)^\times 2 \cong \mathbb{R}^3 \otimes \mathbb{R}^4 \to \mathfrak{sp}(2, \mathbb{R})^*$$

for the induced hamiltonian $\text{Sp}(2, \mathbb{R})$-action on $(T^*\mathbb{R}^3)^\times 2$ coming from the standard representation of $\text{Sp}(2, \mathbb{R})$ on $\mathbb{R}^4$ identifies the $O(3, \mathbb{R})$-reduced space, that is, the reduced space of two particles in $\mathbb{R}^3$ with total angular momentum zero, with the closure of the nilpotent orbit generated by

$$\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \in \mathfrak{sp}(2, \mathbb{R}).$$

The Zariski tangent space at any of the 16 double points of the Kummer surface, for the non-standard smooth structure on $\text{Rep}(\pi, \text{SU}(2))$ we are presently discussing, has dimension 10; see what is said in the next section. Topologically, these 16 points are not singularities, though: as a space, $\text{Rep}(\pi, \text{SU}(2))$ is just complex projective 3-space which, with its ordinary smooth structure, at any of these points, is plainly non-singular.

6. Cleaning up the Zariski tangent space

The notions of singularity and Zariski tangent space refer to a particular structure. The representation spaces described above carry real (locally semi-algebraic) structures as well as complex analytic ones; at a point of such a space, the Zariski tangent space for the complex structure will in general not be the complexification of the Zariski tangent space for the real structure, and the lack of distinction between various structures seems to have created some confusion in the literature. We take the opportunity to try to “clean up” the situation.

As before, let $G$ be a compact Lie group and $\pi$ the fundamental group of a closed surface $\Sigma$. In [9] (p. 205), [33] (p. 1091), and elsewhere in the literature, for
a homomorphism $\phi: \pi \to G$ (where $G$ could be any real algebraic group and $\pi$ any finitely presented discrete group), the vector space $H^1(\pi, g_\phi)$ is referred to as the Zariski tangent space of $\text{Rep}(\pi, G)$ at $[\phi]$; in [9], ‘Zariski tangent space’ is actually put in quotation marks. In [8] (proof of Lemma 11 on p. 49), it is asserted that, given any homomorphism $\phi$ from $\pi$ to $G$, the vector space $H^1(\pi, g_\phi)$ is the ‘tangent space’ of $\text{Rep}(\pi, G)$ at $[\phi]$ (whether or not $\text{Rep}(\pi, G)$ is smooth near $[\phi]$). None of the cited references makes precise, with reference to which structure, i.e. algebra of functions, the Zariski tangent space is to be taken. What seems intended is this: Write $\ell$ for the genus of $\Sigma$, let $\phi \in \text{Hom}(\pi, G) \subseteq G^{2\ell}$, and consider the derivative $dr_\phi$ of the word map $r: G^{2\ell} \to G$ at $\phi$; right translation identifies the kernel $\ker(dr_\phi)$ of $dr_\phi$ with the space of $1$-cocycles for $\pi$ in $g_\phi$ [9] and, by Theorem 7.14 of [21], $\ker(dr_\phi)$ amounts to the Zariski tangent space of $\text{Hom}(\pi, G)$ with reference to the smooth structure induced from the embedding into $G^{2\ell}$. Moreover, right translation identifies the tangent space $T_{\phi}G\phi$ at $\phi$ to the $G$-orbit $G\phi$ in $\text{Hom}(\pi, G) \subseteq G^{2\ell}$ with the space of $1$-coboundaries for $\pi$ in $g_\phi$; in the cited references, $H^1(\pi, g_\phi)$ is apparently viewed as the normal space to $T_{\phi}G\phi$ inside $\ker(dr_\phi)$, and the Zariski tangent space of $\text{Rep}(\pi, G)$ at $[\phi]$ seems to have been confused with this normal space. At a point $[\phi]$ of the top stratum, the notion of normal space indeed makes sense and $H^1(\pi, g_\phi)$ comes down to the ordinary tangent space, in particular, may be identified with the normal space to $T_{\phi}G\phi$.

However, the first objection to taking $H^1(\pi, g_\phi)$ as the Zariski tangent space in general is the purely formal observation that $H^1(\pi, g_\phi)$ is not even invariantly defined in terms of the point $[\phi]$ of $\text{Rep}(\pi, G)$ unless $[\phi]$ belongs to the top stratum of $\text{Rep}(\pi, G)$. More precisely, given the two representatives $\phi$ and $\phi' = \text{Ad}(x)\phi$ of $[\phi]$, where $x \in G$, the association $w \mapsto \text{Ad}(x)w$ ($w \in g$) yields an isomorphism $x^*g_\phi \to g_{\phi'}$ of $\pi$-modules but this isomorphism depends on the choice of $x$; in particular, when $\phi' = \phi$, that is, when $x$ lies in the stabilizer $Z_{\phi} \subseteq G$ of $\phi$, there is no need for the automorphism $x^*: g_\phi \to g_\phi$ to be the identity; indeed, the resulting $Z_{\phi}$-representation on $g$ (= $g_\phi$ but the fact that $g$ carries the $\pi$-representation $\phi$ is not relevant at this point) will in general be non-trivial. For example, under the circumstances of Theorem 4, the “bottom” stratum consists of $2^{2\ell}$ isolated points, and the stabilizer $Z_{\phi}$ for any representative $\phi$ of any of these points coincides with the whole group $G = \text{SU}(2)$, and the adjoint representation of $G$ on $g$ is plainly non-trivial. Likewise, under these circumstances, the stabilizer $Z_{\phi}$ for any representative $\phi$ of a point $[\phi]$ of the middle stratum is a maximal torus $T$ in $\text{SU}(2)$, and the resulting $T$-representation is plainly non-trivial.

We now explain briefly the relationship between $H^1(\pi, g_\phi)$ and the Zariski tangent space $T_{[\phi]}\text{Rep}(\pi, G)$. More details and proofs may be found in Section 7 of [21]. Theorem 3 reduces the relationship under discussion to that between $H^1(\pi, g_\phi)$ and the Zariski tangent space $T_{[\phi]}(H^1_\phi)_{\text{red}}$. To elucidate the latter relationship, let $V_\phi \subseteq H^1_\phi$ be the zero locus of the momentum mapping $\Theta_\phi$. The projection from $V_\phi$ to $(H^1_\phi)_{\text{red}}$ induces a linear map $T_0V_\phi \to T_{[\phi]}(H^1_\phi)_{\text{red}}$ between the Zariski tangent spaces. Furthermore, by Lemma 7.6 in [21], $V_\phi$ spans $H^1_\phi$ whence $T_0V_\phi = H^1_\phi$, thus the projection from $V_\phi$ to $(H^1_\phi)_{\text{red}}$ induces a linear map

$$\lambda: H^1_\phi \to T_{[\phi]}(H^1_\phi)_{\text{red}}.$$ 

By Theorem 3, the choice of $\phi$ (in its class $[\phi]$) induces a diffeomorphism of an open
neighborhood $W_{\phi}$ of $[0] \in (H^1_{\phi})_{\text{red}}$ onto an open neighborhood $U_{\phi}$ of $[\phi] \in \text{Rep}(\pi,G)$ and hence an isomorphism from $T_{[0]}(H^1_{\phi})_{\text{red}}$ onto $T_{[\phi]}\text{Rep}(\pi,G)$; the latter combines with $\lambda$ to a linear map

$$\lambda_{\phi}: H^1_{\phi} \to T_{[\phi]}\text{Rep}(\pi,G).$$

The map $\lambda_{\phi}$ has the following properties (see p. 214 of [21]):

1. It is independent of the choice of $\phi$ in the sense that, for every $x \in G$, the composite

$$H^1(\pi, g_{\phi}) \xrightarrow{\text{Ad}_x(x)} H^1(\pi, g_{x\phi}) \xrightarrow{\lambda_{x\phi}} T_{[\phi]}\text{Rep}(\pi,G)$$

defined by $x \mapsto \lambda_{x\phi}$ is an isomorphism if and only if

$$\text{Ad}_x(x) \circ \lambda_{\phi} = \lambda_{x\phi}.$$

2. Its kernel equals the subspace of $H^1(\pi, g_{\phi})$ which is the linear span of the elements $xw - w$ where $w \in H^1(\pi, g_{\phi})$ and $x \in Z_{\phi}$.

3. Its image equals the (smooth) tangent space $T_{[\phi]}(\text{Rep}(\pi,G)_{(K)})$, viewed as a subspace of $T_{[\phi]}\text{Rep}(\pi,G)$, which is both smooth and the (smooth) tangent space $T_{[\phi]} \text{Rep}(\pi,G)_{(K)}$.

4. It is an isomorphism if and only if $[\phi]$ is a non-singular point of $\text{Rep}(\pi,G)$, i.e., belongs to the top stratum.

Under the circumstances of Theorem 4, for the special case where the genus $\ell$ of the surface $\Sigma$ equals 2, these observations entail the following insight into the Zariski tangent spaces of the corresponding space $N = \text{Rep}(\pi,G)$: For a point $[\phi]$ of the stratum $N_{(T)}$, the Zariski tangent space $T_{[\phi]}N$ has dimension $4 + 3 = 7$. On the other hand, the dimension of $H^1_{\phi}$ equals 8, and the linear map $\lambda_{\phi}$ from $H^1_{\phi}$ to $T_{[\phi]}N$ has rank four. Thus the Zariski tangent space $T_{[\phi]}N$ can in no way be identified with the cohomology group $H^1_{\phi}$. Likewise, let $[\phi]$ be a point in $N_G$. Then the Zariski tangent space $T_{[\phi]}N$ has dimension 10 and hence the minimal number of generators of $C^\infty(N)$ near $[\phi]$ is 10; see p. 217 of [21] for details. Moreover, a closer look reveals that the Zariski tangent space $T_{[\phi]}N$ equals that of $T_{[\phi]}N_{(T)}$, with reference to the induced smooth structure $C^\infty(N_{(T)})$. In fact, in the language of constrained systems, in the local model, $N_{(T)}$ corresponds to reduced states where each of the two particles individually has angular momentum zero, cf. what is said in our paper [23]. In particular, the minimal number of generators of the induced smooth structure $C^\infty(N_{(T)})$ near $[\phi]$ is 10. Finally, the linear map $\lambda_{\phi}$ from $H^1_{\phi}$ to $T_{[\phi]}N$ is zero since the derivative of $\lambda$ at the origin is zero. Thus the Zariski tangent space $T_{[\phi]}N$ can in no way be identified with the cohomology group $H^1_{\phi}$, which has dimension 12. As for the complex analytic structure, we recall that, cf. [23], as a complex variety, near a point $[\phi]$ in $N_G$, the stratum $N_{(T)}$ looks like the quadric $Y^2 = XZ$ in complex 3-space. Hence, at a point $[\phi]$ in $N_G$, the complex Zariski tangent space of $N_{(T)}$ has dimension 3, and this Zariski tangent space coincides with the smooth complex tangent space of $N$ at $[\phi]$. But, as noted above, the real Zariski tangent space $T_{[\phi]}N$ at the point $[\phi]$ has dimension 7, and $T_{[\phi]}N$ actually coincides with $T_{[\phi]}N_{(T)}$. Thus, as a smooth space, with smooth structure $C^\infty(\text{Rep}(\pi,G))$ given in Theorem 1, the space $N = \text{Rep}(\pi,G)$ looks rather different from complex projective 3-space with its standard smooth structure.

The observation that the tangent cones for varieties of spaces of homomorphisms $\text{Hom}(\pi,G)$ for suitable discrete groups $\pi$ (e.g. fundamental groups of compact Kähler manifolds) and appropriate Lie groups $G$ are quadratic as well as the rigidity
results for such spaces, due to Goldman and Millson and others, see [11] and the references there, were influential in the development of the subject, and these results are unaffected by our remarks. In particular, the vector space $H^1_{\phi}$ or what corresponds to it under certain more general circumstances is a constituent of a differential graded Lie algebra controlling the corresponding deformation problem under consideration; see e. g. [9, 12, 33].

We conclude with a comment on another abuse of language: In the literature, it is common to refer to a space of the kind $\text{Rep}(\pi, G)$, $G$ being a real algebraic Lie group, as a “representation variety”. As explained above, $\text{Rep}(\pi, G)$ is not a real variety in an obvious way; in particular, $\text{Rep}(\pi, G)$ does not consist of the real points of the corresponding complex representation variety $\text{Hom}(\pi, G_C)//G_C$ (where the notation $//$ refers to the corresponding categorical quotient). The “naive” representation space $\text{Rep}(\pi, G) = \text{Hom}(\pi, G)/G$ lies inside the real categorical quotient $\text{Hom}(\pi, G)//G$, and the latter is indeed a real variety. Locally, the difference between $\text{Rep}(\pi, G)$ and $\text{Hom}(\pi, G)//G$ is of the kind as that between a half cone and a (double) cone explained in Section 2 above.

References

1. R. Abraham and J. E. Marsden, Foundations of Mechanics, Benjamin-Cummings Publishing Company, 1978.
2. J. M. Arms, R. Cushman, and M. J. Gotay, A universal reduction procedure for Hamiltonian group actions, in: The geometry of Hamiltonian systems, T. Ratiu, ed., MSRI Publ. 20 (1991), Springer, Berlin · Heidelberg · New York · Tokyo, 33–51.
3. J. M. Arms, M. J. Gotay, and G. Jennings, Geometric and algebraic reduction for singular momentum mappings, Advances in Mathematics 79 (1990), 43–103.
4. V. I. Arnold, Mathematical Methods of classical mechanics, Graduate Texts in Mathematics, No. 60, Springer, Berlin · Heidelberg · New York · Tokyo, 1978, 1989 (2nd edition).
5. M. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. London A 308 (1982), 523–615.
6. C. Berger and J. Huebschmann, Comparison of the geometric bar and W-constructions, J. of Pure and Applied Algebra 131 (1998), 109–123.
7. J. Bochnak, M. Coste, and M.-F. Roy, Géométrie algébrique réelle, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 12. A series of modern surveys in Mathematics, vol. 12, Springer, Berlin · Heidelberg · New York · Tokyo, 1987.
8. J. L. Brylinski and D. A. McLaughlin, Holomorphic quantization and unitary representations, in: Lie Theory and Geometry, In honor of B. Kostant, J. L. Brylinski, R. Brylinski, V. Guillemin, V. Kac, eds., Progress in Mathematics 1994 (1994), Birkhäuser, Boston · Basel · Berlin, 21–64.
9. W. M. Goldman, The symplectic nature of the fundamental group of surfaces, Advances 54 (1984), 200–225.
10. W. M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Inventiones 85 (1986), 263–302.
11. W. M. Goldman and J. Millson, The deformation theory of representations of fundamental groups of compact Kaehler manifolds, Publ. Math. I. H. E. S. 67
12. W. M. Goldman and J. Millson, *Differential graded Lie algebras and singularities of level set momentum mappings*, Commun. Math. Phys. 131 (1990), 495–515.
13. M. Goresky and R. MacPherson, *Intersection homology theory*, Topology 19 (1980), 135–162.
14. K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles*, Duke Math. J. 89 (1997), 377–412.
15. R. Hartshorne, *Algebraic Geometry*, Graduate texts in Mathematics No. 52, Springer, Berlin-Göttingen-Heidelberg, 1977.
16. R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. 313 (1989), 539–570.
17. J. Huebschmann, *Poisson cohomology and quantization*, J. für die Reine und Angewandte Mathematik 408 (1990), 57–113.
18. J. Huebschmann, *On the quantization of Poisson algebras*, in: Symplectic Geometry and Mathematical Physics, Actes du colloque en l’honneur de Jean-Marie Souriau, P. Donato, C. Duval, J. Elhadad, G.M. Tuynman, eds., Progress in Mathematics 99 (1991), Birkhäuser, Boston · Basel · Berlin, 204–233.
19. J. Huebschmann, *The singularities of Yang-Mills connections for bundles on a surface. I. The local model*, Math. Z. 220 (1995), 595–605.
20. J. Huebschmann, *The singularities of Yang-Mills connections for bundles on a surface. II. The stratification*, Math. Z. 221 (1996), 83–92.
21. J. Huebschmann, *Smooth structures on moduli spaces of central Yang-Mills connections for bundles on a surface*, J. of Pure and Applied Algebra 126 (1998), 183–221.
22. J. Huebschmann, *Poisson structures on certain moduli spaces for bundles on a surface*, Annales de l’Institut Fourier 45 (1995), 65–91.
23. J. Huebschmann, *Poisson geometry of flat connections for SU(2)-bundles on surfaces*, Math. Z. 221 (1996), 243–259.
24. J. Huebschmann, *Symplectic and Poisson structures of certain moduli spaces*, Duke Math. J. 80 (1995), 737–756.
25. J. Huebschmann, *Symplectic and Poisson structures of certain moduli spaces. II. Projective representations of cocompact planar discrete groups*, Duke Math. J. 80 (1995), 757–770.
26. J. Huebschmann, *Poisson geometry of certain moduli spaces for bundles on a surface*, “A translation of algebra”, International Geometrical Colloquium, Moskau, 1993; Vseross. Inst. Nauchn. i. Tekhn. Inform. (VINITI) Moscou 1995, edited by E. I. Kuznetsova, J. Math. Sci. 82 (1996), 3780–3784.
27. J. Huebschmann, *Poisson geometry of certain moduli spaces*, Lectures delivered at the “14th Winter School”, Srni, Czeque Republic, January 1994, Rendiconti del Circolo Matematico di Palermo, Serie II 39 (1996), 15–35.
28. J. Huebschmann, *On the Poisson geometry of certain moduli spaces*, in: Proceedings of an international workshop on “Lie theory and its applications in physics”, Clausthal, 1995 H. D. Doebner, V. K. Dobrev, J. Hilgert, eds. (1996), World Scientific, Singapore · New Jersey · London · Hong Kong, 89–101.
29. J. Huebschmann, *Extended moduli spaces, the Kan construction, and lattice gauge theory*, Topology 38 (1999), 555–596.
30. J. Huebschmann, *On the variation of the Poisson structures of certain moduli spaces*, [dg-ga/9710033](http://arxiv.org/abs/dg-ga/9710033), Math. Ann. (to appear).

31. J. Huebschmann, *Kaehler spaces, nilpotent orbits, and singular reduction*, in preparation.

32. J. Huebschmann and L. Jeffrey, *Group Cohomology Construction of Symplectic Forms on Certain Moduli Spaces*, Int. Math. Research Notices **6** (1994), 245–249.

33. M. Kapovich and J. J. Millson, *On the deformation theory of representations of fundamental groups of compact hyperbolic 3-manifolds*, Topology **35** (1996), 1085–1106.

34. G. Kempf and L. Ness, *The length of vectors in representation spaces*, Algebraic geometry, Copenhagen, 1978, Lecture Notes in Mathematics **732** (1978), Springer, Berlin · Heidelberg · New York, 233–244.

35. S. Kobayashi and K. Nomizu, *Foundations of differential geometry, I* (1963), *II* (1969), Interscience Tracts in Pure and Applied Mathematics, No. 15, Interscience Publ., New York-London-Sydney.

36. E. Kunz, *Einführung in die kommutative Algebra und algebraische Geometrie*, Vieweg Studium Band 46 Aufbautkurs Mathematik, Friedrich Vieweg & Sohn, Braunschweig/Wiesbaden, 1980; *engl. translation: Introduction to commutative algebra and algebraic geometry*, Birkhäuser, Boston, 1985.

37. E. Lerman, R. Montgomery and R. Sjamaar, *Symplectic Geometry*, Warwick, 1990, ed. D. A. Salamon, London Math. Soc. Lecture Note Series, vol. 192, Cambridge University Press, Cambridge, UK, 1993, pp. 127–155.

38. J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetries*, Rep. on Math. Phys. **5** (1974), 121–130.

39. M. S. Narasimhan and S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. **89** (1969), 19–51.

40. M. S. Narasimhan and S. Ramanan, *θ-linear systems on abelian varieties*, Bombay colloquium (1985), 415–427.

41. M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. **82** (1965), 540–567.

42. P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute Lecture Notes, Springer, Berlin–Göttingen–Heidelberg, 1978.

43. G.W. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63–68.

44. G. W. Schwarz, *The topology of algebraic quotients*, In: Topological methods in algebraic transformation groups, Progress in Math. **80** (1989), Birkhäuser, Boston · Basel · Berlin, 135–152.

45. C. S. Seshadri, *Spaces of unitary vector bundles on a compact Riemann surface*, Ann. of Math. **85** (1967), 303–336.

46. C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque Vol. 96, Soc. Math. de France, 1982.

47. R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. **134** (1991), 375–422.

48. A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964), 149–157.
49. A. Weinstein, *The local structure of Poisson manifolds*, J. of Diff. Geom. 18 (1983), 523–557.

50. A. Weinstein, *Poisson structures*, in E. Cartan et les Mathématiciens d’aujourd’hui, Lyon, 25–29 Juin, 1984, Astérisque, hors-serie, (1985), 421–434.

51. A. Weinstein, *The symplectic structure on moduli space*, in: The Andreas Floer Memorial Volume, H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds., Progress in Mathematics 133 (1995), Birkhäuser, Boston · Basel · Berlin, 627–635.

52. H. Weyl, *The classical groups*, Princeton University Press, Princeton NJ, 1946.

53. H. Whitney, *Analytic extensions of differentiable functions defined on closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.

54. H. Whitney, *Complex analytic varieties*, Addison-Wesley Pub. Comp., Reading, Ma, Menlo Park, Ca, London, Don Mills, Ontario, 1972.