Torus actions on free associative algebras, lifting and Bia\l ynicki-Birula type theorems

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Abstract

We examine several instances of algebraic torus action on the free associative algebra. We prove the free algebra analogue of a classical theorem of A. Bia\l ynicki-Birula, which establishes linearity of maximal torus action. We also formulate and prove linearity theorems in a few specific situations, as well as provide a framework for construction of non-linearizable torus actions.

1 Introduction

The group action linearity problem asks, generally speaking, whether any action of a given algebraic group on an affine space is linear in some suitable coordinate system (or, in other words, whether for any such action there exists an automorphism of the affine space such that it conjugates the action to a representation). This subject owes its existence largely to the classical work of A. Bia\l ynicki-Birula [1], who considered

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regular (i.e. by polynomial mappings) actions of the \(n\)-dimensional torus on the affine space \(\mathbb{A}^n\) (over algebraically closed ground field) and proved that any faithful action is conjugate to a representation (or, as we sometimes say, linearizable). The result of Bia/ynicki-Birula had motivated the study of various analogous instances, such as those that deal with actions of tori of dimension smaller than that of the affine space, or, alternatively, linearity conjectures that arise when the torus is replaced by a different sort of algebraic group. In particular, Bia/ynicki-Birula himself \[2\] had proved that any effective action of \((n-1)\)-dimensional torus on \(\mathbb{A}^n\) is linearizable, and for a while it was believed \[7\] that the same was true for arbitrary torus and affine space dimensions. Eventually, however, the negation of this generalized linearity conjecture was established, with counter-examples due to Asanuma \[9\].

Evidently, to define a group action on the affine space is (modulo duality) the same as to define an action on the affine coordinate algebra, which is just \(K[x_1, \ldots, x_n]\). Various instances of the group action linearity problem therefore admit algebraic formulations.

More recently, the linearity of effective torus actions has become a stepping stone in the study of geometry of automorphism groups. In the paper \[24\], the following result was obtained.

**Theorem 1.1.** Let \(K\) be algebraically closed, and let \(n \geq 3\). Then any Ind-variety automorphism \(\Phi\) of the Ind-group \(\text{Aut}(K[x_1, \ldots, x_n])\) is inner.

The notions of Ind-variety (or Ind-group in this context) and Ind-variety morphism were introduced by Shafarevich \[10\]: an Ind-variety is the direct limit of a system whose morphisms are closed embeddings. Automorphism groups of algebras with polynomial identities, such as the (commutative) polynomial algebra and the free associative algebra, are archetypal examples; the corresponding direct systems of varieties consist of sets \(\text{Aut}_{\leq N}\) of automorphisms of total degree less or equal to a fixed number, with the degree induced by the grading. The morphisms are inclusion maps which are obviously closed embeddings.

Theorem \[1.1\] is proved by means of tame approximation (stemming from the main result of \[11\]), with the following Proposition, originally due to E. Rips, constituting one of the key results.

**Proposition 1.2.** Let \(K\) be algebraically closed and \(n \geq 3\) as above, and suppose that \(\Phi\) preserves the standard maximal torus action on the commutative polynomial algebra \[1\]. Then \(\Phi\) preserves all tame automorphisms.

The proof relies on the Bialynicki-Birula theorem on the maximal torus action. In a similar fashion, the paper \[24\] examines the Ind-group \(\text{Aut}(K\langle x_1, \ldots, x_n \rangle)\) of automorphisms of the free associative algebra \(K\langle x_1, \ldots, x_n \rangle\) in \(n\) variables, and establishes results completely analogous to Theorem \[1.1\] and Proposition \[1.2\]. In accordance with that, the free associative analogue of the Bialynicki-Birula theorem was required.

Such an analogue is indeed valid, and we have established it in our previous note \[15\] on the subject. We will provide the proof of this result in the sequel.

Given the existence of a free algebra version of the Bialynicki-Birula theorem, one may inquire whether various other instances of the linearity problem (such as the Bialynicki-Birula theorem on the action of the \((n-1)\)-dimensional torus on the affine space.

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1. That is, the action of the \(n\)-dimensional torus on the polynomial algebra \(K[x_1, \ldots, x_n]\), which is dual to the action on the affine space.
2. The free associative case was amenable to the above approach when \(n > 3\).
\[ \mathbb{K}[x_1, \ldots, x_n] \) can be studied. As it turns out, direct adaptation of proof techniques from the commutative realm is sometimes possible. There are certain limitations, however. For instance, Bialynicki-Birula’s proof [2] of linearity of \((n - 1)\)dimensional torus actions uses commutativity in an essential way. Nevertheless, a neat workaround of that hurdle can be performed when \(n = 2\), as we show in this note. Also, a special class of torus actions (positive-root actions) turns out to be linearizable. Finally, some of the proof techniques developed by Asanuma [9] admit free associative analogues; this will allow us to prove the existence of non-linearizable torus actions in positive characteristic, in complete analogy with Asanuma’s work.

Acknowledgments

The free associative version of the Bialynicki-Birula theorem (1) was conceived in the prior work [24] of A. K.-B., J.-T. Y. and A. E.. Theorem 2.5 is due to A. E. and A. K.-B.; Lemma 3.1 and the review of known results for the linearization problem is due to F. R., J.-T. Y. and W. Z.. The linearity of actions of \(\mathbb{K}^\times\) on the free algebra in two variables is due to A. E..

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2 Actions of algebraic tori

In this section we recall basic definitions of the theory of torus actions, as formulated by Bialynicki-Birula [13] and others.

Let \(\mathbb{K}\) be the ground field. Let \(I\) be a finite or a countable index set and let

\[ Z = \{z_i : i \in I\} \]

be the set of variables, which is sometimes referred to as the alphabet.

The free associative algebra \(F_I(\mathbb{K}) = \mathbb{K} \langle Z \rangle\) is the algebra generated by words in the alphabet \(Z\) (as usually, word concatenation gives the multiplication of monomials and extends linearly to define the multiplication in the algebra).

Any element of \(\mathbb{K} \langle Z \rangle\) can be written uniquely in the form

\[ \sum_{k=0}^{\infty} \sum_{i_1, i_2, \ldots, i_k \in I} a_{i_1, i_2, \ldots, i_k} z_{i_1} z_{i_2} \cdots z_{i_k}, \]

where the coefficients \(a_{i_1, i_2, \ldots, i_k}\) are elements of the field \(\mathbb{K}\) and all but finitely many of these elements are zero.

In our context, the alphabet \(Z\) is the same as the set of algebra generators, therefore the terms “monomial” and “word” will be used interchangeably.

In the sequel, we employ the following short-hand notation for a free algebra monomial. For an element \(z\), its powers are defined intuitively. Any monomial
$z_1 z_2 \ldots z_k$ can then be written in a reduced form with subwords $zz\ldots z$ replaced by powers. We then write

$$z^I = z^{i_1}_{j_1} z^{i_2}_{j_2} \ldots z^{i_k}_{j_k}$$

where by $I$ we mean an assignment of $i_k$ to $j_k$ in the word $z^I$. Sometimes we refer to $I$ as a multi-index, although the term is not entirely accurate. If $I$ is such a multi-index, its absolute value $|I|$ is defined as the sum $i_1 + \cdots + i_k$.

For a field $\mathbb{K}$, let $\mathbb{K}^\times = \mathbb{K}\{0\}$ denote the multiplicative group of its non-zero elements viewed.

**Definition 2.1.** An $n$-dimensional algebraic $\mathbb{K}$-torus is a group

$$\mathbb{T}_n \simeq (\mathbb{K}^\times)^n$$

(with obvious multiplication).

Denote by $\mathbb{A}^n$ the affine space of dimension $n$ over $\mathbb{K}$.

**Definition 2.2.** A (left, geometric) torus action is a morphism

$$\sigma : \mathbb{T}_n \times \mathbb{A}^n \to \mathbb{A}^n.$$

that fulfills the usual axioms (identity and compatibility):

$$\sigma(1, x) = x, \quad \sigma(t_1, \sigma(t_2, x)) = \sigma(t_1 t_2, x).$$

The action $\sigma$ is effective if for every $t \neq 1$ there is an element $x \in \mathbb{A}^n$ such that $\sigma(t, x) \neq x$.

In [1], Białynicki-Birula proved the following two theorems, for $\mathbb{K}$ algebraically closed.

**Theorem 2.3.** Any regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ has a fixed point.

**Theorem 2.4.** Any effective and regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ is a representation in some coordinate system.

The term "regular" is to be understood here as in the algebro-geometric context of regular function (Białynicki-Birula also considered birational actions).

In the following section (dedicated to the proof of the free algebra version of Theorems 2.3 and 2.4), the ground field is algebraically closed.

As was mentioned in the introduction, an algebraic group action on $\mathbb{A}^n$ is the same as the corresponding action by automorphisms on the algebra

$$\mathbb{K}[x_1, \ldots, x_n]$$

of coordinate functions. In other words, it is a group homomorphism

$$\sigma : \mathbb{T}_n \to \text{Aut} \ \mathbb{K}[x_1, \ldots, x_n].$$

An action is effective if and only if $\text{Ker} \ \sigma = \{1\}$.

The polynomial algebra is a quotient of the free associative algebra

$$F_n = \mathbb{K}\langle z_1, \ldots, z_n \rangle$$

by the commutator ideal $I$ (it is the two-sided ideal generated by all elements of the form $fg - gf$). The definition of torus action on the free algebra is thus purely algebraic.

In this short note we establish the free algebra version of the Białynicki-Birula theorem. The latter is formulated as follows.
Theorem 2.5. Suppose given an action $\sigma$ of the algebraic $n$-torus $\mathbb{T}_n$ on the free algebra $F_n$. If $\sigma$ is effective, then it is linearizable.

The linearity (or linearization) problem, as it has become known since Kambayashi, asks whether all (effective, regular) actions of a given type of algebraic groups on the affine space of given dimension are conjugate to representations. According to Theorem 2.5, the linearization problem extends to the noncommutative category. Several known results concerning the (commutative) linearization problem are summarized below.

1. Any effective regular torus action on $\mathbb{A}^2$ is linearizable (Gutwirth [4]).
2. Any effective regular torus action on $\mathbb{A}^n$ has a fixed point (Bialynicki-Birula [1]).
3. Any effective regular action of $\mathbb{T}_{n-1}$ on $\mathbb{A}^n$ is linearizable (Bialynicki-Birula [2]).
4. Any (effective, regular) one-dimensional torus action (i.e., action of $\mathbb{K}^\times$) on $\mathbb{A}^3$ is linearizable (Koras and Russell [8]).
5. If the ground field is not algebraically closed, then a torus action on $\mathbb{A}^n$ need not be linearizable. In [9], Asanuma proved that over any field $\mathbb{K}$, if there exists a non-rectifiable closed embedding from $\mathbb{A}^m$ into $\mathbb{A}^n$, then there exist non-linearizable effective actions of $(\mathbb{K}^\times)^r$ on $\mathbb{A}^{1+n+m}$ for $1 \leq r \leq 1 + m$.
6. When $\mathbb{K}$ is infinite and has positive characteristic, there are examples of non-linearizable torus actions on $\mathbb{A}^n$ (Asanuma [9]).

Remark 2.6. A closed embedding $\iota : \mathbb{A}^m \rightarrow \mathbb{A}^n$ is said to be rectifiable if it is conjugate to a linear embedding by an automorphism of $\mathbb{A}^n$.

As can be inferred from the review above, the context of the linearization problem is rather broad, even in the case of torus actions. The regulating parameters are the dimensions of the torus and the affine space. This situation is due to the fact that the general form of the linearization conjecture (i.e., the conjecture that states that any effective regular torus action on any affine space is linearizable) has a negative answer.

Transition to the noncommutative geometry presents the inquirer with an even broader context: one now may vary the dimensions as well as impose restrictions on the action in the form of preservation of the PI-identities. Caution is well advised. Some of the results are generalized in a straightforward manner – the proof in the next section being the typical example, others require more subtlety and effort. Of some note to us, given our ongoing work in deformation quantization (see, for instance, [16]) is the following instance of the linearization problem, which we formulate as a conjecture.

Conjecture 2.7. For $n \geq 1$, let $P_n$ denote the commutative Poisson algebra, i.e. the polynomial algebra

$$\mathbb{K}[z_1, \ldots, z_{2n}]$$

equipped with the Poisson bracket defined by

$$\{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}.$$

Then any effective regular action of $\mathbb{T}_n$ by automorphisms of $P_n$ is linearizable.
A version of Theorem 1.1 for the commutative Poisson algebra is a conjecture of significant interest. It turns out that the algebra $P_n$ admits a certain augmentation by central variables which distort the Poisson structure, such that the automorphism group of the resulting algebra admits the property of Theorem 1.1. The case is studied in the paper [25].

3 Maximal torus action on the free algebra

In this section, we provide proof to the free algebra version (Theorem 2.5) of the Bialynicki-Birula theorem [1].

The proof proceeds along the lines of the original commutative case proof of Bialynicki-Birula.

If $\sigma$ is the effective action of Theorem 2.5, then for each $t \in T_n$ the automorphism $\sigma(t) : F_n \to F_n$ is given by the $n$-tuple of images of the generators $z_1, \ldots, z_n$ of the free algebra:

$$(f_1(t, z_1, \ldots, z_n), \ldots, f_n(t, z_1, \ldots, z_n)).$$

Each of the $f_1, \ldots, f_n$ is a polynomial in the free variables.

**Lemma 3.1.** There is a translation of the free generators $(z_1, \ldots, z_n) \to (z_1 - c_1, \ldots, z_n - c_n), \ (c_i \in K)$ such that (for all $t \in T_n$) the polynomials $f_i(t, z_1 - c_1, \ldots, z_n - c_n)$ have zero free part.

**Proof.** This is a direct corollary of Theorem 2.3. Indeed, any action $\sigma$ on the free algebra induces, by taking the canonical projection with respect to the commutator ideal $I$, an action $\bar{\sigma}$ on the commutative algebra $K[x_1, \ldots, x_n]$. If $\sigma$ is regular, then so is $\bar{\sigma}$. By Theorem 2.3, $\bar{\sigma}$ (or rather, its geometric counterpart) has a fixed point, therefore the images of commutative generators $x_i$ under $\bar{\sigma}(t)$ (for every $t$) will be polynomials with trivial degree-zero part. Consequently, the same will hold for $\sigma$. \hfill $\square$

We may then suppose, without loss of generality, that the polynomials $f_i$ have the form

$$f_i(t, z_1, \ldots, z_n) = \sum_{j=1}^{n} a_{ij}(t) z_j + \sum_{j,l=1}^{n} a_{ijl}(t) z_j z_l + \sum_{k=3}^{N} \sum_{J, |J| = k} a_{i,J}(t) z^J$$

where by $z^J$ we denote, as in the introduction, a particular monomial

$$z_{i_1}^{k_1} z_{i_2}^{k_2} \ldots$$

(a word in the alphabet $\{z_1, \ldots, z_n\}$ in the reduced notation; $J$ is the multi-index in the sense described above); also, $N$ is the degree of the automorphism (which is finite) and $a_{ij}, a_{ijl}, \ldots$ are polynomials in $t_1, \ldots, t_n$.

As $\sigma_t$ is an automorphism, the matrix $[a_{ij}]$ that determines the linear part is non-singular. Therefore, without loss of generality we may assume it to be diagonal (just as in the commutative case [1]) of the form

$$\text{diag}(t_1^{m_{11}}, \ldots, t_{n1}^{m_{1n}}, \ldots, t_1^{m_{n1}}, \ldots, t_{nn}^{m_{nn}}).$$

Now, just as in [1], we have the following
Lemma 3.2. The power matrix \([m_{ij}]\) is non-singular.

Proof. Consider a linear action \(\tau\) defined by

\[
\tau(t) : (z_1, \ldots, z_n) \mapsto (t_1^{m_{11}} \cdots t_n^{m_{nn}} z_1, \ldots, t_1^{m_{1n}} \cdots t_n^{m_{nn}} z_n), \quad (t_1, \ldots, t_n) \in \mathbb{T}_n.
\]

If \(T_1 \subset T_n\) is any one-dimensional torus, the restriction of \(\tau\) to \(T_1\) is non-trivial. Indeed, were it to happen that for some \(T_1\), \(\tau(t)z = z, \quad t \in T_1, \quad (z = (z_1, \ldots, z_n))\)

then our initial action \(\sigma\), whose linear part is represented by \(\tau\), would be identity modulo terms of degree \(> 1\):

\[
\sigma(t)(z_i) = z_i + \sum_{j,l} a_{ijl}(t) z_j z_l + \cdots.
\]

Now, equality \(\sigma(t^2)(z) = \sigma(t)(\sigma(t)(z))\) implies

\[
\sigma(t)(\sigma(t)(z_i)) = \sigma(t) \left( z_i + \sum_{j,l} a_{ijl}(t) z_j z_l + \cdots \right)
= z_i + \sum_{j,l} a_{ijl}(t) z_j z_l + \sum_{j,l} a_{ijl}(t) \left( z_j + \sum_{k,m} a_{km}(t) z_k z_m + \cdots \right)
= z_i + \sum_{j,l} a_{ijl}(t^2) z_j z_l + \cdots
\]

which means that

\[
2a_{ijl}(t) = a_{ijl}(t^2)
\]

and therefore \(a_{ijl}(t) = 0\). The coefficients of the higher-degree terms are processed by induction (on the total degree of the monomial). Thus

\[
\sigma(t)(z) = z, \quad t \in T_1
\]

which is a contradiction since \(\sigma\) is effective. Finally, if \([m_{ij}]\) were singular, then one would easily find a one-dimensional torus such that the restriction of \(\tau\) were trivial. \(\square\)

Consider the action

\[
\varphi(t) = \tau(t^{-1}) \circ \sigma(t).
\]

The images under \(\varphi(t)\) are

\[
(g_1(z,t), \ldots, g_n(z,t)), \quad (t = (t_1, \ldots, t_n))
\]

with

\[
g_i(z,t) = \sum g_{m_1 \cdots m_n}(z) t_1^{m_1} \cdots t_n^{m_n}, \quad m_1, \ldots, m_n \in \mathbb{Z}.
\]

Define \(G_i(z) = g_{i,0 \ldots 0}(z)\) and consider the map \(\beta : F_n \to F_n\),

\[
\beta : (z_1, \ldots, z_n) \mapsto (G_1(z), \ldots, G_n(z)).
\]
Lemma 3.3. $\beta \in \text{Aut} F_n$ and

$$\beta = \tau(t^{-1}) \circ \beta \circ \sigma(t).$$

Proof. This lemma mirrors the final part in the proof in [11]. The conjugation is straightforward, since for every $s, t \in \mathbb{T}_n$ one has

$$\varphi(st) = \tau(t^{-1}s^{-1}) \circ \sigma(st) = \tau(t^{-1}) \circ \sigma(s) \circ \sigma(t) = \tau(t^{-1}) \circ \varphi(s) \circ \sigma(t).$$

Denote by $\hat{F}_n$ the power series completion of the free algebra $F_n$, and let $\hat{\sigma}, \hat{\tau}$ and $\hat{\beta}$ denote the endomorphisms of the power series algebra induced by corresponding morphisms of $F_n$. The endomorphisms $\hat{\sigma}, \hat{\tau}, \hat{\beta}$ come from (polynomial) automorphisms and therefore are invertible.

Let

$$\hat{\beta}^{-1}(z_i) \equiv B_i(z) = \sum_J b_{i,j} z^J$$

(just as before, $z^J$ is the monomial with multi-index $J$). Then

$$\hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1}(z_i) = B_i(t_1^{m_{11}} \ldots t_n^{m_{1n}} G_1(z), \ldots, t_1^{m_{n1}} \ldots t_n^{m_{nn}} G_n(z)).$$

Now, from the conjugation property we must have

$$\hat{\beta} = \hat{\sigma}(t^{-1}) \circ \hat{\beta} \circ \hat{\tau}(t),$$

therefore $\hat{\sigma}(t) = \hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1}$ and

$$\hat{\sigma}(t)(z_i) = \sum_J b_{i,j}(t_1^{m_{11}} \ldots t_n^{m_{1n}})^{j_1} \ldots (t_1^{m_{n1}} \ldots t_n^{m_{nn}})^{j_n} G(z)^J;$$

here the notation $G(z)^J$ stands for a word in $G_i(z)$ with multi-index $J$, while the exponents $j_1, \ldots, j_n$ count how many times a given index appears in $J$ (or, equivalently, how many times a given generator $z_i$ appears in the word $z^J$).

Therefore, the coefficient of $\hat{\sigma}(t)(z_i)$ at $z^J$ has the form

$$b_{i,j}(t_1^{m_{11}} \ldots t_n^{m_{1n}})^{j_1} \ldots (t_1^{m_{n1}} \ldots t_n^{m_{nn}})^{j_n} + S$$

with $S$ a finite sum of monomials of the form

$$c_{L}(t_1^{m_{11}} \ldots t_n^{m_{1n}})^{l_1} \ldots (t_1^{m_{n1}} \ldots t_n^{m_{nn}})^{l_n}$$

with $(j_1, \ldots, j_n) \neq (l_1, \ldots, l_n)$. Since the power matrix $[m_{ij}]$ is non-singular, if $b_{i,j} \neq 0$, we can find a $t \in \mathbb{T}_n$ such that the coefficient is not zero. Since $\sigma$ is an algebraic action, the degree

$$\sup_t \deg(\sigma)$$

is a finite integer $N$. With the previous statement, this implies that

$$b_{i,j} = 0,$$

whenever $|J| > N$.

Therefore, $B_i(z)$ are polynomials in the free variables. What remains is to notice that

$$z_i = B_i(G_1(z), \ldots, G_n(z)).$$

Thus $\beta$ is an automorphism. \[\square\]

From Lemma 3.3 it follows that

$$\tau(t) = \beta^{-1} \circ \sigma(t) \circ \beta$$

which is the linearization of $\sigma$. Theorem 2.5 is proved.
4 Action of $\mathbb{K}^\times$ on $\mathbb{K}\langle z_1, z_2 \rangle$

The proof of linearity property in the case of maximal torus action on the free algebra is obtained from the original proof of Białynicki-Birula in a straightforward manner. Having done that, one could try to prove the free version of the other result of Białynicki-Birula, on the linearity of the action of $T_{n-1}$. However, taking that path, one quickly runs into trouble. It is nonetheless possible that the free analogue of the main result of [2] exists. We have then the following conjecture.

**Conjecture 4.1.** Any effective regular action of $T_{n-1}$ on the free algebra $F_n(\mathbb{K})$ is linearizable, provided that $\mathbb{K}$ is algebraically closed.

Despite the lack of an attack strategy on Conjecture 4.1, something can be done right away. Namely, we can prove Conjecture 4.1 for the exceptional case $n = 2$ as a corollary to the original Białynicki-Birula theorem [2]. The case $n = 2$ is exceptional thanks to the isomorphism between the automorphism groups of $\mathbb{K}[x_1, x_2]$ and $\mathbb{K}\langle z_1, z_2 \rangle$ – a result due to Makar-Limanov [12] (see also [13]). Precisely, we have the following.

**Theorem 4.2** (Makar-Limanov, [12]). Let $\mathbb{K}$ be a field. Then the homomorphism

$$\Phi : \text{Aut} \mathbb{K}\langle z_1, z_2 \rangle \to \text{Aut} \mathbb{K}[x_1, x_2]$$

induced by abelianization (i.e. for an automorphism $\varphi \in \text{Aut} \mathbb{K}\langle z_1, z_2 \rangle$, the polynomials $\Phi(\varphi)(x_i)$ that define the image under $\Phi$ are images of $\varphi(z_i)$ under the projection map $\mathbb{K}\langle z_1, z_2 \rangle \to \mathbb{K}(z_1, z_2)/I$, $I$ is the commutator ideal) is an isomorphism.

**Remark 4.3.** The inverse $\Theta$ to the isomorphism $\Phi$ is called the lifting map.

As a corollary of Makar-Limanov’s theorem, we immediately get the following lemma.

**Lemma 4.4.** 1. Let $\varphi$ be an automorphism of the free algebra $\mathbb{K}\langle z_1, z_2 \rangle$ defined by

$$(z_1, z_2) \mapsto (z_1 + f(z_1, z_2), z_2).$$

If $f$ is non-zero, then $f$ cannot be an element of the commutator ideal.

2. Let $\varphi$ be an automorphism of the free algebra $\mathbb{K}\langle z_1, z_2 \rangle$ defined by

$$(z_1, z_2) \mapsto (z_1 + f(z_1, z_2), z_2 + g(z_1, z_2)).$$

Suppose that both $f$ and $g$ are not zero. Then it is impossible for both $f$ and $g$ to be (simultaneously) elements of the commutator ideal.

**Proof.** 1. Indeed, if $f \neq 0$ and $f \in I$, then the identity automorphism of $\mathbb{K}[x_1, x_2]$ has two distinct pre-images under the map $\Phi$ – namely the identity automorphism of the free algebra and the automorphism $\varphi$, which contradicts Theorem 4.2.

2. Again, if both $f$ and $g$ were in the commutator ideal, then the identity automorphism of the commutative polynomial algebra would have had two distinct pre-images under the map $\Phi$ – namely $\varphi$ and the identity automorphism of the free algebra, in contradiction with Theorem 4.2.

We are now able to prove the main theorem of this section.

**Theorem 4.5.** Let $\mathbb{K}$ be algebraically closed. Any effective regular action of (the one-dimensional torus) $\mathbb{K}^\times$ on the free algebra $\mathbb{K}\langle z_1, z_2 \rangle$ is linearizable.
Proof. We first show that an effective action

$$\sigma : \mathbb{K}^\times \to \text{Aut} \mathbb{K}\langle z_1, z_2 \rangle$$

maps under $\Phi$ to an effective action. Indeed, suppose that $\Phi \circ \sigma$ is not effective; then there is a non-trivial element $\lambda$ in $\text{Ker} \Phi \circ \sigma$. As $\Phi$ is an isomorphism, we must have $\sigma(\lambda) = \text{Id}$, which is a contradiction to $\sigma$ being effective.

The rest of the proof is straightforward. Given an effective regular action $\sigma$, we map it to an action $\overline{\sigma}$ (whose associated group homomorphism is given by $\Phi \circ \sigma$) which is effective. The action $\overline{\sigma}$ fulfills the conditions of the Białyńcki-Birula’s theorem [2]; therefore, there exists an automorphism $\beta \in \text{Aut} \mathbb{K}[x_1, x_2]$ such that

$$\beta^{-1} \circ \overline{\sigma}(\lambda) \circ \beta$$

is a linear map for every $\lambda \in \mathbb{K}^\times$. The lifted automorphism

$$\hat{\beta} = \Theta(\beta) \ (= \Phi^{-1}(\beta))$$

is the conjugation map for $\sigma$. Indeed, if

$$\hat{\beta}^{-1} \circ \sigma(\lambda) \circ \hat{\beta}$$

is not linear for some $\lambda$, then its higher-degree terms must be polynomials (either both non-zero or one zero and one non-zero) in the commutator ideal (because the abelianization $\Phi$ is one-to-one and maps this automorphism to a linear change of commuting variables), which contradicts one of the two parts of Lemma 4.4.

5 Positive-root torus actions

As was mentioned in the introduction, direct adaptation of proofs in the commutative category to the free associative case (as well as other associative algebras) has its limitations. Nevertheless, sometimes imposition of additional assumptions paves the way for a novel proof.

In this section, we consider positive-root torus actions and prove the linearity property analogous to the Białyńcki-Birula theorem. The assumption on the actions turns out to be strong enough for the linearization to be achieved regardless of the torus’s dimension. We employ the notion of generic matrices in the proof.

Given an action $\phi$ of $r$-dimensional torus $T_r$ on the (commutative) polynomial algebra, denote by $\phi_1$ its linear part, i.e. the mapping constructed from degree one components of the images $\phi(T_1, \ldots, T_r, x_i)$. If the action is regular, then the eigenvalues of $\phi_1$ are

$$\lambda_i = \prod_j T_j^{k_{ij}}$$

with $k_{ij}$ integers.

Definition 5.1. The action $\phi$ has positive roots, if all $k_{ij}$ are positive integers.

The positive-root actions we studied in [6, 7] (under a slightly different name). In particular, the following theorem is a consequence of [2].
Theorem 5.2. Any (effective, regular) positive-root action of $T_r$ on $\mathbb{K}[x_1, \ldots, x_n]$ is linearizable.

In order to prove the free associative version of this theorem, we devise a way to reduce positive-root case to the commutative one. To that end, we introduce the generic matrices and induce the action on the rings of coefficients.

Definition 5.3. If $A$ is an associative $\mathbb{K}$-algebra, then a two-sided ideal $I$ of $A$ is called a $T$-ideal if it is stable under all $\mathbb{K}$-algebra endomorphisms of $A$.

Definition 5.4. A generic $n$ by $n$ matrix $[18, 19]$ is a matrix

$$
\begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nn}
\end{bmatrix}
$$

whose entries are mutually commuting indeterminates $x_{ij}$. The $\mathbb{K}$-algebra of $n \times n$ generic matrices of order $m$ is the algebra generated over $\mathbb{K}$ by $m$ distinct associative generic $n$ by $n$ matrices. It is a subalgebra in the algebra

$$M_n(\mathbb{K}[\{x_{ij}^{(1)}\}, \ldots, \{x_{ij}^{(m)}\}])$$

of matrices with entries polynomial in $x_{ij}^{(k)}$, $1 \leq i, j \leq n$, $1 \leq k \leq m$.

Remark 5.5. The algebra of generic matrices is a basic object in the study of the polynomial identities and invariants of $n \times n$ matrices. Various aspects of the framework can be found in [18, 19] and [29]. The connection with the Jacobian conjecture is due to Yagzhev [30, 31]. In [26, 27], the free associative analog of the Jacobian conjecture was studied, and a criterion was formulated and proved.

Let $h$ be an endomorphism of the free associative algebra $F_n$ (over $n$ generators), and let $I_n$ be the $T$-ideal of the algebra of generic matrices of order $n$. Then we have $h(I_n) \subseteq I_n$ for all $n$ by definition of $I_n$. Hence $h$ induces an endomorphism $h_{I_n}$ of $F_n$ modulo $I_n$. If $h$ is invertible, then $h_{I_n}$ is invertible, but the converse is not true. For proof cf. [18].

Now, consider $h$ to be an element of the group of transformations $\Delta_n \subset \text{End}_\mathbb{K}(F_n)$ defined by the requirement that its elements have invertible Jacobian and

$$h(z_i) = \alpha_i z_i + \varphi_i(z_1, \ldots, z_{i-1})$$

for $\alpha_i \in \mathbb{K}^\times$ with $\varphi_i$ polynomial of order at least 2.

In [26, 27, 29], it has been shown that the invertibility of the Jacobian implies invertibility of the mapping $h$. See also [18] for an exposition.

Our argument is based on the aforementioned techniques and proceeds as follows. We need to demonstrate that the endomorphism of the free algebra sending the generators to the eigenvectors of the positive-root action is an automorphism (and thus the desired linearizing coordinate change). To that end, we, expanding upon the ideas of Yagzhev [30, 31] (again, see also the exposition in [18]), induce the endomorphism of the algebra of $n$ generic $N$ by $N$ matrices (for arbitrary $N$). This in turn induces a (well-defined) endomorphism on the ring of coefficients, which is the commutative polynomial algebra over $nN^2$ variables. The induced mapping corresponds to the positive-root torus action on the commutative algebra, and by Theorem 5.2 is linearizable by the eigenvector map. Therefore, the induced mappings themselves are automorphisms. This implies (a non-trivial fact that utilizes certain techniques from general algebra) that the Jacobian of the initial endomorphism is invertible, which together with the free associative Jacobian conjecture, proved in ([26, 27, 29]), shows invertibility of the initial endomorphism.

More precisely, we have the following.
**Theorem 5.6.** Let $\sigma : \mathbb{T}_r \times F_n \to F_n$ be a regular torus action with positive roots. Then it is linearizable.

To demonstrate that, consider the eigenvalue mapping

$$x_i \mapsto v_i$$

corresponding to the positive-root torus action (it comes from the linear part of the action). It induces an endomorphism

$$\phi : x_i \mapsto v_i(x_i)$$

of the free algebra $F_n$.

By replacing $z_1, \cdots, z_n$ by $N \times N$ generic matrices and passing to the coefficient algebras, we construct a polynomial mapping

$$F_{(N)} : \mathbb{A}_K^{N^2} \to \mathbb{A}_K^{N^2}.$$ 

Here one could use a theorem of Formanek, Halpin and Li (referenced in [19]), which states that for $G$ the ring of $N \times N$ generic matrices as a subring of $R = M_n(\mathbb{K}[x_{ij}])$ generated by the generic matrices, we have

$$G/[G,G] = \mathcal{O}(\mathbb{A}_K^{nN^2})$$

the polynomial ring in $nN^2$ variables.

Therefore, the mapping $\phi$ induces, for each $N$, an endomorphism

$$\phi_N : \mathcal{O}(\mathbb{A}_K^{nN^2}) \to \mathcal{O}(\mathbb{A}_K^{nN^2})$$

of the polynomial algebra.

This mapping corresponds to a positive-root torus action on the commutative polynomial algebra. By Theorem 5.2, it is invertible and corresponds to action linearization.

We shall need following

**Proposition 5.7.** Gröbner basis of finitely generated bimodule $M$ over free associative algebra $A$ is finite and can be obtained from basis of $G$ by finite number of compositions. Same is true for algebra of generic matrices.

**Proof.** Let $m_1 \prec m_2 \prec \cdots \prec m_s$ be generators of $M$, $a_1 \prec \cdots \prec a_s$ be generators of $A$. Let us order monomials of the form $u_i m_j v_k$; $u_i, v_k$ are monomials over $a_1, \ldots, a_s$.

First we compare inner part $m_j$, then $u_i$, afterwards $v_k$ (first by length, then lexicographically). We get linear order. Note that composition of every two elements has highest term of the form $umv$ such that $|u| \leq \max(|u_i|), |v| \leq \max(|v_k|)$. It is clear that compositions produce finite set. According to the Bergman Diamond lemma [23] it follows that Gröbner basis is finite. \qed

**Remark 5.8.** Same is true for left (right) ideals, modules of algebra of generic matrices and tensor products of free algebras.
Remark 5.9. In fact, Gröbner (or Gröbner-Shirshov) basis was invented by A.I. Shirshov when he proved that one-relator Lie Algebras like groups have solvable equality problem. Any regular word $W$ admit in unique way Lie bracket arrangement such that if one open them then get $W$ as a highest term. Free Lie algebra has basis (Hall basis or Hall-Shirshov basis) of regular words with this Lie bracket arrangement. From other hand, regular word can not overlap with itself and composition of relation with itself is impossible. See [28] for details.

In order to go back to the initial map, we need the following.

Proposition 5.10. If the Jacobian of the induced endomorphism at any reduction modulo the $T$-ideal is invertible, then the Jacobian of the initial free associative algebra endomorphism is invertible.

Proof. One needs to prove that the submodule of relations corresponding to the Jacobian has a finite Groebner basis. This result follows from the results of [21]. Alternatively, one can use the diamond lemma (as in Latyshev [22], see also [23]).

Given an endomorphism $\phi$ of the free algebra $F_n$, the images $D_i(x_j)$ of the generators under the mapping to the module of differentials span a $F_n$-$F_n$-bimodule. The Jacobian of $\phi$ corresponds to a submodule, which by diamond lemma admits a finite Groebner basis. The maximal length among the monomials that constitute the basis elements is therefore a constant $m$. By passing to generic matrices of size $N > m$ and noting that the submodules corresponding to the Jacobian in the two cases are the same, we conclude that the Jacobian of the endomorphism $\phi$ is invertible.

As a consequence of Proposition 5.10 and the free associative Jacobian conjecture [26,27,29], the mapping $\phi$ is an automorphism. This yields the desired linearization of the positive-root torus action.

Remark 5.11. The negative-root torus action, defined similarly to the positive-root one, with the requirement $k_{ij} > 0$ replaced by $k_{ij} < 0$, is also linearizable, which can be seen by composing the action with the group inversion, thus reducing to the positive-root case.

6 Non-linearizable torus actions

The examples of non-linearizable torus actions, as well as a way to study those, were developed by Asanuma [9]. It is not difficult to observe that most of Asanuma’s proof technique can be carried to the free associative case without loss. As in Asanuma’s case, the existence of non-linearizable torus actions is tied to the existence of so-called non-rectifiable ideals in the appropriate algebras. This section establishes that fact and discusses consequences. We will be brief, referring the reader to the original work [9] for a more thorough exposition. One rather remarkable feature of Asanuma’s technique is the fact that, modulo minor details and replacements, it may be repeated almost verbatim in the associative category – a situation similar to the one we have observed in the Białynicki-Birula’s theorem on the action of the maximal torus. This seems to be a peculiar circumstance, in light of the negative answer to the automorphism lifting problem provided in [24].

Let $K$ be a field and let $A, B, C, D$ be associative $K$-algebras. Let $I$ be a two-sided ideal of $A$, $J$ be a two-sided ideal of $B$.

Definition 6.1. The ideal $I$ is equivalent $J$ if there exists an algebra homomorphism $\sigma : A \to B$ such that $\sigma(I) = J$. The equivalence relation is denoted by $I \sim J$. 

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Definition 6.2. If $I$ is equivalent to the ideal $(z_{m+1}, \ldots, z_n)$ of the free algebra $K\langle z_1, \ldots, z_n \rangle$ for some $m \leq n$ (when $m = n$, $(z_{m+1}, \ldots, z_n)$ is the zero ideal), then the ideal $I$ is said to be rectifiable.

Definition 6.3. An algebra homomorphism $\alpha : A \to C$ is equivalent to a homomorphism $\beta : B \to D$, if there exist homomorphisms $\gamma, \delta$ that together with $\alpha$ and $\beta$ form a commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\gamma} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
C & \xrightarrow{\delta} & D
\end{array}
$$

In particular, if $\alpha$ is equivalent to the projection $\beta : K\langle z_1, \ldots, z_n \rangle \to K\langle z_1, \ldots, z_m \rangle$ defined by $\beta(z_i) = z_i$ for $i = 1, \ldots, m$ and $\beta(z_i) = 0$ otherwise, then we call $\alpha$ rectifiable.

Remark 6.4. Note that if $\alpha$ is equivalent to $\beta$, then the ideals $\text{Ker} \alpha$ and $\text{Ker} \beta$ are equivalent.

Lemma 6.5. Let $C$ be an associative algebra, and let

$$
\alpha, \gamma : K\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle \to C
$$

be two surjective $K$-algebra homomorphisms. If $\alpha(y_j) = 0$ ($j = 1, \ldots, m$) and $\gamma(x_i) = 0$ ($i = 1, \ldots, n$), then there exist polynomials $f_j \in K\langle x_1, \ldots, x_n \rangle$ ($j = 1, \ldots, m$) and $g_i \in K\langle y_1, \ldots, y_m \rangle$ ($i = 1, \ldots, n$) such that

$$
\alpha(x_i) = \gamma(g_i), \quad \alpha(f_j) = \gamma(y_j)
$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. If we set $\tau = \tau_1 \circ \tau_2$ ($\tau_i \in \text{Aut} K\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$ where

$$
\tau_1 : (x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x_1 + g_1, \ldots, x_n + g_n, y_1, \ldots, y_m)
$$

and

$$
\tau_2 : (x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x_1, \ldots, x_n, y_1 - f_1, \ldots, y_m - f_m)
$$

then

$$
\alpha = \gamma \circ \tau.
$$

In particular, $\alpha$ is equivalent to $\gamma$.

Proof. The images of $\alpha$ and $\gamma$ coincide, therefore the polynomials $g_i$ and $f_j$ exist. As

$$
\gamma \circ \tau_1 : (x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (\alpha(x_1), \ldots, \alpha(x_n), \gamma(y_1), \ldots, \gamma(y_m))
$$

it follows that

$$
\gamma \circ \tau_1 \circ \tau_2(x_i) = \gamma \circ \tau_1(x_i) = \alpha(x_i)
$$

and

$$
\gamma \circ \tau_1 \circ \tau_2(y_j) = \gamma(y_j) - \alpha(f_j) = 0.
$$

Therefore $\gamma \circ \tau = \alpha$. □
Corollary 6.6. Let $C$ be an associative $\mathbb{K}$-algebra generated over $\mathbb{K}$ by $m$ elements and let
$$\alpha, \beta : \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle \to C$$
be surjective with
$$\alpha(y_j) = \beta(y_j) = 0$$
for all $j$. Then $\alpha$ is equivalent to $\beta$. In particular, if $C = \mathbb{K}\langle y_1, \ldots, y_m \rangle$, then $\alpha$ and $\beta$ are both rectifiable.

Proof. The algebra $C$ is of the form $\mathbb{K}\langle z_1, \ldots, z_m \rangle$ with $z_j \in \alpha(\mathbb{K}\langle x_1, \ldots, x_n \rangle)$. Define the homomorphism
$$\gamma : \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle \to C$$
by
$$\gamma(x_i) = 0, \quad \gamma(y_j) = z_j.$$  
Then $\gamma$ is surjective. By Lemma 6.5, both $\alpha$ and $\beta$ are equivalent to $\gamma$, which means that $\alpha$ is equivalent to $\beta$.

Corollary 6.7. Let $I$ and $J$ be (two-sided) ideals of $A = \mathbb{K}\langle z_1, \ldots, z_n \rangle$ such that
$$A/I \cong_{\mathbb{K}} A/J.$$  
If $A/I$ is generated over $\mathbb{K}$ by $m$ elements as a $\mathbb{K}$-algebra, then
$$\langle I, y_1, \ldots, y_m \rangle \sim \langle J, y_1, \ldots, y_m \rangle$$
as ideals of $\mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$. In particular, if $A/I \cong \mathbb{K}\langle z_1, \ldots, z_m \rangle$, then both $\langle I, y_1, \ldots, y_m \rangle$ and $\langle J, y_1, \ldots, y_m \rangle$ are rectifiable.

Proof. Immediately follows from Corollary 6.6.

Definition 6.8. Given an associative $\mathbb{K}$-algebra $A$ and a commutative monoid $M$, the algebra $A$ is called $M$-graded, if it can be represented as a direct sum
$$A = \bigoplus_{m \in M} \Gamma_m$$
of $\mathbb{K}$-modules, such that $\Gamma_{m_1}\Gamma_{m_2} \subseteq \Gamma_{m_1+m_2}$. The map
$$\Gamma : M \to \{ \Gamma_m \mid m \in M \}$$
is called the $M$-grading of $A$.

Definition 6.9. Given an associative $\mathbb{K}$-algebra $A$ and its two-sided ideal $I$, the (extended) Rees algebra is
$$\mathcal{R}_A(I) = A[t, t^{-1}I] = \bigoplus_{n=-\infty}^{+\infty} I^n t^n.$$  
The Rees algebra is a $\mathbb{Z}$-graded (according to powers of $t$) $\mathbb{K}$-algebra and a $\mathbb{K}[t]$-algebra.
Proposition 6.10. If $\Gamma$ and $\Delta$ denote the Rees $\mathbb{Z}$-gradings of $\mathcal{R}_A(I)$ and $\mathcal{R}_B(J)$, then the following are equivalent:

1. $\Gamma \simeq_K \Delta$.
2. $\Gamma \simeq_{K[t]} \Delta$.
3. $I \sim J$.

(Isomorphisms are graded.)

Proof. (1) $\Rightarrow$ (3): Suppose there is a graded $K$-isomorphism

$$\sigma : \mathcal{R}_A(I) \rightarrow \mathcal{R}_B(J).$$

Then, in particular,

$$\sigma(t^n A) = \sigma(\Gamma_n) = \Delta_n = t^n B$$

when $n = 0$ or $n = 1$, and

$$\sigma(t^{-1} I) = \sigma(\Gamma_{-1}) = \Delta_{-1} = t^{-1} J.$$

Therefore

$$\sigma|_A : A \rightarrow B$$

is a $K$-isomorphism, such that

$$\sigma|_A(I) = \sigma(tAt^{-1} I) = tBt^{-1}J = J$$

which realizes the equivalence.

(3) $\Rightarrow$ (2): if $I \sim J$, then there is a $K$-isomorphism

$$\theta : A \rightarrow B$$

such that $\theta(I) = J$. The map $\theta$ extends uniquely to

$$\theta' : A[t, t^{-1}] \rightarrow B[t, t^{-1}]$$

whose restriction to the Rees algebra (which is a subalgebra of the algebra of Laurent polynomials) furnishes the required $\mathbb{Z}$-graded $K$-isomorphism.

(2) $\Rightarrow$ (1) is immediate. \qed

Proposition 6.11. Let $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ be the free algebra, and let $I$ and $J$ be two (two-sided) ideals of $A$, such that $A/I \simeq A/J$.

If $A/I$ is generated by $m$ elements over $\mathbb{K}$, then

$$\mathcal{R}_A(I)\langle y_1, \ldots, y_m \rangle \simeq_{\mathbb{K}[t]} \mathcal{R}_A(J)\langle y_1, \ldots, y_m \rangle.$$ 

If, in particular, $A/I \simeq \mathbb{K}\langle y_1, \ldots, y_m \rangle$, then

$$\mathcal{R}_A(I)\langle y_1, \ldots, y_m \rangle \simeq_{\mathbb{K}[t]} \mathbb{K}[t]\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle.$$
Proof. Let $B = A(y_1, \ldots, y_m)$, $I' = \langle I, y_1, \ldots, y_m \rangle$ and $J' = \langle J, y_1, \ldots, y_m \rangle$. Then, by Corollary 6.7, $I' \sim J'$, and by Proposition 6.10

$$R_B(I') \simeq \mathbb{K}[t] R_B(J').$$

But

$$R_B(I') = R_A(I) \langle t^{-1}y_1, \ldots, t^{-1}y_m \rangle$$

and

$$R_B(J') = R_A(J) \langle t^{-1}y_1, \ldots, t^{-1}y_m \rangle.$$

As $t^{-1}y_1, \ldots, t^{-1}y_m$ are free variables, we must have

$$R_A(I) \langle \ y_1, \ldots, y_m \ \rangle \simeq \mathbb{K}[t] R_A(J) \langle \ y_1, \ldots, y_m \ \rangle.$$

In particular, when $A/I \simeq \mathbb{K}(y_1, \ldots, y_m)$, we may take $J = \langle x_{m+1}, \ldots, x_n \rangle$, which, together with

$$R_A(J) \langle y_1, \ldots, y_m \rangle \simeq \mathbb{K}[t] \mathbb{K}[t] \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$$

yields

$$R_A(I) \langle y_1, \ldots, y_m \rangle \simeq \mathbb{K}[t] \mathbb{K}[t] \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle.$$  

Any regular action of the $r$-torus $T_r = (\mathbb{K}^\times)^r$ on an associative $\mathbb{K}$-algebra $A$ is equivalent to a homomorphism

$$\phi : A \to A \otimes_\mathbb{K} \mathbb{K}[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] = A[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}].$$

Given an element $f \in A$, its image under $\phi$ can be written as

$$\phi(f) = \sum m f_m T_1^{m_1} \ldots T_r^{m_r}$$

with $m = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ and $f_m \in A$. The map $f \mapsto f_m$ induces a $\mathbb{Z}^r$-grading

$$\Gamma : m \mapsto \Gamma_m = \{ f_m, \ | \ f \in A \}$$

of the algebra $A$.

**Definition 6.12.** Two (regular) $T_r$-actions $\phi$ and $\psi$, respectively, on $A$ and $B$ are equivalent, if there exists a $\mathbb{K}$-homomorphism $\sigma : A \to B$, such that the diagram

$$\begin{array}{ccc}
A[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] & \xrightarrow{\sigma \otimes \text{Id}} & B[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] \\
\phi & \uparrow & \psi \\
A & \xrightarrow{\sigma} & B
\end{array}$$

commutes.

**Proposition 6.13.** A regular $T_r$-action $\phi$ on $A$ is linearizable (in the sense of the previous sections), if and only if it is equivalent, in the sense of Definition 6.12, to a linear action on $A$. (An action $\psi : A \to A[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}]$ is linear if the images $\psi(x_i)$ of the generators of $A$ are linear in $x_i$).
Proof. Straightforward. 

**Definition 6.14.** Given a two-sided ideal $I$ and an action $\phi$ on $A$, $I$ is called fixed (by $\phi$), if the image $\phi(I)$ is contained in the ideal $I[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}]$.

If $I$ is fixed by $\phi$, then $\phi$ induces a canonical $\mathbb{K}$-homomorphism

$$\bar{\phi} : \text{gr}_I(A) \to \text{gr}_I(A)[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}]$$

on the associated graded ring, which therefore defines a torus action. $A/I$ is a subring in $\text{gr}_I(A)$, and we have

$$\bar{\phi}(A/I) \subset A/T.$$

If $I$ is maximal, then $A/I$ is a simple ring, therefore

$$\bar{\phi}(A/I) \subset A/T.$$

Therefore, any homogeneous element of $A$ of non-zero degree (with respect to the $\mathbb{Z}^r$-grading induced by the torus action $\phi$) is contained in the fixed maximal ideal $I$, which means that any maximal two-sided ideal $I$ fixed by $\phi$ is of the form

$$(I \cap \Gamma_0) \oplus \left( \bigoplus_{m \neq 0} \Gamma_m \right).$$

**Proposition 6.15.** If $\phi$ is a linearizable torus action on the free algebra $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle$, then there exist a maximal two-sided ideal $M$ fixed by $\phi$, such that the induced torus action $\bar{\phi}$ on $\text{gr}_M(A)$ is equivalent to $\phi$.

Proof. Take $M = \langle y_1, \ldots, y_n \rangle$, where $y_1, \ldots, y_n$ is the set of generators such that $\phi$ is linear with respect to them. Then $M$ is obviously fixed by $\phi$, and $\phi$ is equivalent to the induced action. 

**Proposition 6.16.** Let $A$ be an associative $\mathbb{K}$-algebra and let $\phi$ be an action of $T_r$ on $\mathcal{R}_A(I)\langle y_1, \ldots, y_m \rangle$ defined by the requirement that $f \in A\langle y_1, \ldots, y_m \rangle$ is in the zeroth component with respect to the induced $\mathbb{Z}^r$-grading, while $t$ has non-zero degree and for some $s$ and all $i = s + 1, \ldots, m$ the degrees of $y_i$ are not in the subgroup of $\mathbb{Z}^r$ defined by the degree of $t$. Then:

1. If $B$ is another algebra and $\psi$ is a $T_r$-action on $\mathcal{R}_B(J)\langle y_1, \ldots, y_m \rangle$ defined by the same requirement as above on the grading, then $\phi$ is equivalent to $\psi$ if and only if the ideals $I\langle y_1, \ldots, y_s \rangle$ and $J\langle y_1, \ldots, y_s \rangle$ are equivalent.

2. Let $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ be the free algebra. Then $\phi$ is linearizable if and only if $I\langle y_1, \ldots, y_s \rangle$ is rectifiable.

Proof. We may suppose $s = 1$ in the assumption from the start, for we have

$$\mathcal{R}_A(I)\langle y_1, \ldots, y_m \rangle = \mathcal{R}_C(1C)\langle y_{s+1}, \ldots, y_m \rangle$$

with $C = A\langle y_1, \ldots, y_s \rangle$, and therefore can descend to $C = A$.

Let $R$ and $S$ denote the algebras $\mathcal{R}_A(I)\langle y_1, \ldots, y_m \rangle$ and $\mathcal{R}_B(J)\langle y_1, \ldots, y_m \rangle$, respectively, and let $\Gamma$ and $\Delta$ be the gradings induced on $R$ and $S$ by the corresponding torus actions.

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To prove the first statement, it is enough to demonstrate the \((\Rightarrow)\) direction (the other one is obvious). The equivalence of actions means the existence of a graded \(\mathbb{K}\)-isomorphism \(\bar{\sigma}\). Denote by \(P\) and \(Q\) the ideals in \(R\) and \(S\) generated by \(y_1, \ldots, y_m\). The ideals \(P\) and \(Q\) are fixed, respectively, by \(\phi\) and \(\psi\). Then, the \(\mathbb{Z}^t\) subgroup assumption on the degrees implies, as is fairly easy to verify, that \(\bar{\sigma}(P) = Q\). Therefore, \(\bar{\sigma}\) descends to \(\sigma : \mathcal{R}_A(I) \to \mathcal{R}_B(J)\) and the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\bar{\sigma}} & S \\
\downarrow{\pi_A} & & \downarrow{\pi_B} \\
\mathcal{R}_A(I) & \xrightarrow{\sigma} & \mathcal{R}_B(J)
\end{array}
\]

commutes (\(\pi_A\) and \(\pi_B\) are natural projections \(y_i \mapsto 0\)). The map \(\sigma\) is a \(\mathbb{Z}\)-graded (with respect to the Rees grading) \(\mathbb{K}\)-isomorphism, and part 1 follows from Proposition 6.10.

To prove part 2, it is again sufficient to demonstrate the \((\Rightarrow)\) direction. Let \(\phi\) be linearizable. By Proposition 6.15 there exists a two-sided maximal ideal \(M\) of \(R\) fixed by \(\phi\) such that the induced action \(\bar{\phi}\) on the associated graded ring \(\text{gr}_M(R)\) is equivalent to \(\phi\). The ideal \(M\) is generated by the subset \((A \cap M) \cup t^{-1}I \cup \{t, y_1, \ldots, y_m\}\).

Consider the left module \(M/M^2\) over the simple ring \(R/M\). It is clear from the proof of Proposition 6.15 that \(R/M = \mathbb{K}\). The image of the set \(\{t, y_1, \ldots, y_m\}\) under the projection

\[
M \to M/M^2
\]

is thus a linearly independent system over \(\mathbb{K}\). Therefore, there exist \(f_i \in A \cap M\) \((i = 1, \ldots, u)\) and \(f_{u+j} \in I\) \((j = 1, \ldots, n - u)\) such that the set of images under the quotient map,

\[
\{\bar{t}, \bar{y}_1, \ldots, \bar{y}_m, \bar{f}_1, \ldots, \bar{f}_u, \bar{t^{-1}f_{u+1}}, \ldots, \bar{t^{-1}f_n}\}
\]

is a basis of the \(\mathbb{K}\)-vector space \(M/M^2\).

Let \(F = \mathbb{K}\langle x_1, \ldots, x_n \rangle\) be the free algebra and let \(J = \langle x_{u+1}, \ldots, x_n \rangle\). Define the isomorphism

\[
\theta : \text{gr}_R(M) \to S
\]

by

\[
\theta(t) = t, \ \theta(f_i) = x_i, \ \theta(t^{-1}f_j) = t^{-1}x_j, \ \theta(y_k) = y_k
\]

(where \(i = 1, \ldots, u, \ j = u + 1, \ldots, n, \ k = 1, \ldots, m\)). Then \(\theta\) induces the action \(\psi\) on \(S\) which fulfills the conditions of part 1. Therefore, \(I\) is equivalent to \(J\) and \(I\langle y_1, \ldots, y_s \rangle\) is rectifiable. \(\square\)

This Proposition, together with Proposition 6.11, constitutes the foundation of Asanuma’s counter-examples; in our analogy, this translates to the following statement: if there exists a non-rectifiable ideal \(I\) of the free algebra \(F_n = \mathbb{K}\langle x_1, \ldots, x_n \rangle\) such that \(F_n/I \cong_{\mathbb{K}} F_m\), then there are examples of non-linearizable \(\mathbb{T}_r\)-actions on free associative algebras.

If \(A = \mathcal{R}_F_n(I)\langle y_1, \ldots, y_m \rangle\), then by Proposition 6.11 there exists an isomorphism of \(\mathbb{K}\)-algebras

\[
\beta : A \to F_n[t]\langle y_1, \ldots, y_m \rangle.
\]
If \( \phi \) is a (regular) torus action, define \( \delta \) by the commutative diagram

\[
\begin{array}{ccc}
A[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] & \xrightarrow{\delta} & F_n[t][T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] \langle y_1, \ldots, y_m \rangle \\
\phi & \downarrow & \\
A & \xrightarrow{\beta} & F_n[t] \langle y_1, \ldots, y_m \rangle.
\end{array}
\]

Then \( \delta \) is equivalent to \( \phi \) and is linearizable if and only if \( I \langle y_1, \ldots, y_s \rangle \) (\( s \) is as in Proposition 6.16) is rectifiable. This can be furthered by obtaining a class of actions which are linearizable if and only if \( I \) itself is rectifiable, at which point the construction of \( I \) reduces to lifting to the free algebra of ideals obtained by Asanuma.

In fact, it is clear from our generalization of Asanuma’s work that some of his results on the cancellation problem (Section 8 of [9]) also have free associative analogs. These are formulated as follows.

**Definition 6.17.** Let \( D \) be an integral domain. An (associative) \( D \)-algebra \( A \) is \( D \)-invariant, if for any \( D \)-algebra \( B \), such that for some \( m \) the free products

\( A \ast \mathbb{K}\langle y_1 \rangle \ast \ldots \ast \mathbb{K}\langle y_m \rangle \cong_D B \ast \mathbb{K}\langle y_1 \rangle \ast \ldots \ast \mathbb{K}\langle y_m \rangle \)

are isomorphic as \( D \)-algebras, then \( A \cong_D B \).

The main problem of interest is the free associative analogue of the so-called Cancellation conjecture, as formulated by Drensky and Yu [17]:

**Conjecture 6.18.** Let \( R \) be a \( \mathbb{K} \)-algebra. If

\( R \ast \mathbb{K}\langle y \rangle \cong_K \mathbb{K}\langle x_1, \ldots, x_n \rangle \)

then

\( R \cong_K \mathbb{K}\langle x_1, \ldots, x_{n-1} \rangle \).

Asanuma’s results on the Rees algebras (and their associative analogues given in Proposition 6.11) allow us to establish a version of the Cancellation conjecture for co-products over a (commutative) \( \mathbb{K} \)-algebra \( D \). The following statement holds.

**Theorem 6.19.** Let \( D \) be an integral domain which is a \( \mathbb{K} \)-algebra, and let \( x \) be an indeterminate over \( D \). Suppose given a non-zero element \( t \in D \) and monic polynomials \( f(x) \) and \( g(x) \) in \( \mathbb{K}[x] \) of degree greater than 1. Set \( A = D[x, t^{-1}f(x)] \) and \( B = D[x, t^{-1}g(x)] \). Then, if

\( \mathbb{K}[x]/(f(x)) \cong_K \mathbb{K}[x]/(g(x)) \)

then

\( A \ast_D \mathbb{K}\langle y \rangle \cong_D B \ast_D \mathbb{K}\langle y \rangle \),

where the product \( R \ast_D S \) is the quotient of the free product \( R \ast S \) over \( \mathbb{K} \) by the ideal generated by all elements of the form

\( r \ast (ds) - d(r \ast s) \).
Proof. Let
\[ K[x]/(f(x)) \simeq_K K[x]/(g(x)) \,.
\]
The element \( t \) is transcendent over \( K \), therefore by Proposition 6.11 we have\(^3\)
\[ K[x, t, t^{-1}f(x)] * K\langle y \rangle \simeq_{K[t]} K[x, t, t^{-1}g(x)] * K\langle y \rangle \,.
\]
The algebra \( A \) is the quotient
\[ A \simeq_{D[x]} D\langle x, y \rangle / (ty - f(x)) \]
which implies
\[ A \simeq_{D} K\langle t, x, y \rangle / (ty - f(x)) \otimes_{K[t]} D \simeq_{D} K\langle x, t, t^{-1}f(x) \rangle \otimes_{K[t]} D \]
and a similar isomorphism for \( B \). It follows that
\[ A *_{D} K\langle y \rangle \simeq K\langle x, t, t^{-1}f(x) \rangle \otimes_{K[t]} D \simeq_{D} K\langle x, t, t^{-1}g(x) \rangle \otimes_{K[t]} D \simeq_{D} B \]
as required. \( \square \)

7 Discussion

The noncommutative toric action linearity property has several useful applications. In the work [24], it is used to investigate the properties of the group \( \text{Aut} F_2 \) of automorphisms of the free algebra. As a corollary of Theorem 2.5, one gets

Corollary 7.1. Let \( \theta \) denote the standard action of \( \mathbb{T}_n \) on \( K[x_1, \ldots, x_n] \) – i.e., the action
\[ \theta_t : (x_1, \ldots, x_n) \mapsto (t_1x_1, \ldots, t_nx_n). \]

Let \( \tilde{\theta} \) denote its lifting to an action on the free associative algebra \( F_2 \). Then \( \tilde{\theta} \) is also given by the standard torus action.

This statement plays a part, along with a number of results concerning the induced formal power series topology on \( \text{Aut} F_2 \), in the establishment of the free associative analogue of Theorem 1.1.

The proofs in this paper, for the most part, were based upon the techniques from the commutative category. It is, however, a problem of legitimate interest to try and obtain proofs for various linearity statements using tools specific to the category of associative algebras, bypassing the known commutative results. As one outstanding example of this problem, we expect the free associative analogue of the second Białyńcki-Birula theorem to hold and formulate it here as a conjecture.

Conjecture 7.2. Any effective action of \( \mathbb{T}_{n-1} \) on \( F_n \) is linearizable.

Also of independent interest is the following instance of the linearity problem.

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\(^3\)Or rather, by a consequence of Proposition 6.11 when commutation relations are imposed on both sides.
Conjecture 7.3. For $n \geq 1$, let $P_n$ denote the commutative Poisson algebra, i.e. the polynomial algebra
\[ \mathbb{K}[z_1, \ldots, z_{2n}] \]
equipped with the Poisson bracket defined by
\[ \{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}. \]
Then any effective regular action of $\mathbb{T}_n$ by automorphisms of $P_n$ is linearizable.

This problem is loosely analogous to the Białynicki-Birula theorem, in the sense of maximality of torus with respect to the dimension of the configurations space (spanned by $x_i$). There seems to be no straightforward way of finding the linearizing canonical coordinates on the phase space, however. For the Ind-variety $\text{Aut} P_n$, a version of Theorem 1.1 may be stated. The geometry of $\text{Aut} P_n$ is relevant to problems of deformation quantization.

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