A GEOMETRIC PROOF OF THE FLYPING THEOREM

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Abstract. In 1898, Tait asserted several properties of alternating knot diagrams. These assertions became known as Tait’s conjectures and remained open until the discovery of the Jones polynomial in 1985. The new polynomial invariants soon led to proofs of all of Tait’s conjectures, culminating in 1993 with Menasco–Thistlethwaite’s proof of Tait’s flyping conjecture.

In 2017, Greene (and independently Howie) answered a long-standing question of Fox by characterizing alternating links geometrically. Greene then used his characterization to give the first geometric proof of part of Tait’s conjectures. We use Greene’s characterization, Menasco’s crossing ball structures, and re-plumbing moves to give the first entirely geometric proof of Menasco–Thistlethwaite’s flyping theorem.

1. Introduction

P.G. Tait asserted in 1898 that all reduced alternating diagrams of a given prime nonsplit link in $S^3$ minimize crossings, have equal writhe, and are related by flype moves (see Figure 1) [13]. Tait’s conjectures remained unproven for almost a century, until the 1985 discovery of the Jones polynomial, which quickly led to proofs of Tait’s conjectures about crossing number and writhe. Tait’s flyping conjecture remained open until 1993, when Menasco–Thistlethwaite gave its first proof [9, 10], which they described as follows:

The proof of the Main Theorem stems from an analysis of the [chessboard surfaces] of a link diagram, in which we use geometric techniques [introduced in [8]]... and properties of the Jones and Kauffman polynomials.... Perhaps the most striking use of polynomials is... where we “detect a flype” by using the fact that if just one crossing is switched in a reduced alternating diagram of $n$ crossings, and if the resulting link also admits an alternating diagram, then the crossing number of that link is at most $n - 2$. Thus, although the proof of the Main Theorem has a strong geometric flavor, it is not entirely geometric; the question
remains open as to whether there exist purely geometric proofs of this and other results that have been obtained with the help of new polynomial invariants.

We answer part of Menasco–Thistlethwaite’s question by giving the first entirely geometric proof of the flyping theorem.

**Flyping theorem.** All reduced alternating diagrams of a given prime, nonsplit link are flype-related.

As a corollary, we obtain a new geometric proof of other parts of Tait’s conjectures, which were first proven independently by Kauffman [6], Murasugi [11], and Thistlethwaite [14] using the Jones polynomial, and were first proved geometrically by Greene [3]:

**Theorem 5.24.** All reduced alternating diagrams of a given link have the same crossing number and writhe.

It does not follow a priori that reduced alternating link diagrams minimize crossings. This and other parts of Menasco–Thistlethwaite’s question remain open; see Problems 2.7 2.9.

Like Menasco–Thistlethwaite’s proof, ours stems from an analysis of chessboard surfaces and uses the geometric techniques introduced in [8]. Recent insights of Greene and Howie regarding these chessboard surfaces also play a central role in our analysis [3, 4]. (Those insights answered another longstanding question, this one from Ralph Fox: “What [geometrically] is an alternating knot [or link]?”)

The most striking difference between our proof and the original proof in [10] is that we “detect flypes” via re-plumbing moves. Specifically, suppose $D$ and $D’$ are reduced alternating diagrams of a prime nonsplit link $L$ with respective chessboard surfaces $B,W$ and $B’,W’$, where $B$ and $B’$ are positive-definite. With this setup:

**Theorem 4.11.** $D$ and $D’$ are flype-related if and only if $B$ and $B’$ are related by isotopy and re-plumbing moves, as are $W$ and $W’$.

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$^1$B and W are essential, since D is reduced alternating and L is nonsplit.
Theorem 4.11 is the first of two intermediate results which, together with background from [3], immediately imply the flyping theorem. With the same setup as Theorem 4.11, the second result is:

**Theorem 5.22.** Any essential positive-definite surface spanning \( L \) is related to \( B \) by isotopy and re-plumbing moves.

Our proof of the flyping theorem is entirely geometric, not just in the formal sense that it does not use the Jones polynomial, but also in the more genuine sense that it conveys a geometric way of understanding why the flyping theorem is true. Namely, using Greene’s characterization, we leave the diagrammatic context for a geometric context; in that context, we describe moves that correspond to flypes; finally we prove that these moves are sufficiently robust:

- Reduced alternating diagrams \( D \) correspond to geometric objects: pairs \((B, W)\) of essential definite surfaces of opposite sign (this is Greene’s characterization);
- Flype moves on \( D \) correspond to isotopy and re-plumbing moves on \((B, W)\) (this is Theorem 4.11); and
- The pair \((B, W)\) is related to any other such pair \((B', W')\) by some sequence of these moves (this is Theorem 5.22).

An outline: Section 2 reviews Greene’s characterization and other facts about spanning surfaces, then establishes further properties of definite surfaces, culminating with Lemma 2.15, which describes how a positive-definite surface and a negative-definite surface can intersect. Section 3 addresses generalized plumbing and re-plumbing moves, with a brief interlude about proper isotopy of spanning surfaces through the 4-ball. Section 4 sets up crossing ball structures and uses them, together with facts from §3, to prove Theorem 4.11. In particular, Figure 14 shows the type of re-plumbing move associated with flyping. Section 5 uses crossing balls structures and Lemma 2.15 to prove Theorem 5.22, and thus the flyping theorem and Theorem 5.24. Several of the arguments in §§4-5 are adapted from [1, 5].

Thank you to Colin Adams for posing a question about flypes and chessboard surfaces during SMALL 2005 which eventually led to the insight behind Figure 14 and Theorem 4.11. Thank you to Josh Howie and Alex Zupan for helpful discussions. Thank you to Josh Greene for helpful discussions and especially for encouraging me to think about this problem.

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2Likewise for negative-definite surfaces and \( W \).

3Generalized plumbing is also called Murasugi sum.
2. Definite surfaces

2.1. Background. Let \( L \) be a link in \( S^3 \) with a closed regular neighborhood \( \nu L \) and projection \( \pi_L : \nu L \to L \). One can define spanning surfaces for \( L \) in two ways; in both definitions, \( F \) is compact, but not necessarily orientable, and each component of \( F \) has nonempty boundary.\(^1\) First, a spanning surface for \( L \) is an embedded surface \( F \subset S^3 \) with \( \partial F = L \). Alternatively, a spanning surface is a properly embedded surface \( F \) in the link exterior \( S^3 \setminus \nu L \) such that \( \partial F \) intersects each meridian on \( \partial \nu L \) transversally in one point.\(^2\) We use the latter definition throughout, except in \( \S \) 3.2 and Corollary 5.23.

The rank \( \beta_1(F) \) of the first homology group of a spanning surface \( F \) counts the number of “holes” in \( F \). When \( F \) is connected, \( \beta_1(F) = 1 - \chi(F) \) counts the number of cuts — along disjoint, properly embedded arcs — required to reduce \( F \) to a disk. Thus:

Observation 2.1. If \( \alpha \) is a properly embedded arc in a spanning surface \( F \) and \( F' = F \setminus \nu \alpha \), then \( \beta_1(F) - |F| = \beta_1(F') + 1 - |F'| \). In particular, if \( F' \) connected, then \( \beta_1(F) = \beta_1(F') + 1 \).

A spanning surface \( F \) is (geometrically) incompressible if any simple closed curve in \( F \) that bounds a disk in \( S^3 \setminus (F \cup \nu L) \) also bounds a disk in \( F \).\(^3\) A spanning surface \( F \) is \( \partial \)-incompressible if any properly embedded arc in \( F \) that is \( \partial \)-parallel in \( S^3 \setminus (F \cup \nu L) \) is also

\(^1\)Convention: Given \( X \subset S^3 \), \( \nu X \) denotes a closed regular neighborhood of \( X \) and is taken in \( S^3 \) unless stated otherwise.

\(^2\)When \( L \) is nonsplit and alternating, \( F \) is connected, by Corollary 5.2 of [1].

\(^3\)Definition: A meridian on \( \partial \nu L \) is a circle \( \pi_{L}^{-1}(x) \cap \partial \nu L \) for a point \( x \in L \).

\(^4\)For compact \( X,Y \subset S^3 \), \( X \setminus \setminus Y \) denotes the metric closure of \( X \setminus Y \). If each \( x \in X \cap Y \) has a neighborhood \( \nu(x) \) such that \( Z \cap \nu(x) \) is connected or empty for each component \( Z \) of \( X \setminus Y \), then \( X \setminus \setminus Y \) is the disjoint union of the closures in \( S^3 \) of the components of \( X \setminus Y \). Hence, each component of \( X \setminus \setminus Y \) embeds naturally in \( S^3 \), although \( X \setminus \setminus Y \) may not.

For a general construction for \( X \setminus \setminus Y \), let \( \{(U_\alpha, \phi_\alpha)\} \) be a maximal atlas for \( X \). About each \( x \in X \), choose a sufficiently tiny chart \( (U_x, \phi_x) \) and construct \( U_x \setminus \setminus Y \) as above. Denote the components of \( U_x \cap (U_x \setminus \setminus Y) \) by \( U_\alpha \), \( \alpha \in I_x \), and denote

Figure 2. Constructing chessboard surfaces; chessboards near a vertical arc (yellow) at a crossing
\( \partial \)-parallel in \( F \). If \( F \) is incompressible and \( \partial \)-incompressible, then \( F \) is essential. This geometric notion of essentiality is weaker than the algebraic notion of \( \pi_1 \)-essentiality, which holds \( F \) to be essential if inclusion \( F \hookrightarrow S^3 \setminus \nu L \) induces an injective map on fundamental groups and \( F \) is not a Möbius band spanning the unknot.

**Proposition 2.2.** If an essential surface \( F \) contains a properly embedded arc \( \beta \) which is parallel in \( S^3 \setminus (F \cup \nu L) \) to a properly embedded arc \( \alpha \subset \partial \nu L \setminus \partial F \), then \( \alpha \) is parallel in \( \partial \nu L \) to \( \partial F \).

**Proof.** Since \( F \) is essential and \( \beta \) is parallel in the link exterior to \( \alpha \subset \partial \nu L \), \( \beta \) is also parallel in \( F \) to an arc \( \alpha' \subset \partial F \). Thus, \( \alpha \) and \( \alpha' \) are both parallel in \( S^3 \setminus (F \cup \nu L) \) to \( \beta \), and so \( \alpha \) is parallel through a disk \( X \subset S^3 \setminus (F \cup \nu L) \) to \( \alpha' \subset \partial F \). Since \( L \) is nontrivial and nonsplit, \( X \) is parallel in \( S^3 \setminus (F \cup \nu L) \) to a disk \( X' \subset \partial \nu L \), through which \( \alpha \) is parallel to \( \alpha' \subset \partial F \). \( \square \)

Given any diagram \( D \) of \( L \), one may a color the complementary regions of \( D \) in the projection sphere \( S^2 \) black and white in chessboard fashion. See Figure 2.\(^8\) One may then construct spanning surfaces \( B \) and \( W \) for \( L \) such that \( B \) projects into the black regions, \( W \) projects into the white, and \( B \) and \( W \) intersect in vertical arcs which project to the the crossings of \( D \). Call the surfaces \( B \) and \( W \) the chessboard surfaces from \( D \). A connected alternating diagram is reduced iff both chessboard surfaces are essential.

The *euler number* \( e(F) \) of a spanning surface \( F \) is the algebraic self-intersection number of the properly embedded surface in the 4-ball obtained by perturbing \( F \). Alternatively, \( -e(F) \) can be computed by summing the component-wise boundary slopes of \( F \).\(^8\) For brevity, we call \( -e(F) \) the slope of \( F \) and denote \( -e(F) = s(F) \).

Let \( F \) and \( F' \) be spanning surfaces for a link \( L \subset S^3 \) with \( F \equiv F' \) and \( \partial F \equiv \partial F' \) on \( \partial \nu L \). Orient \( L \) arbitrarily, and orient \( \partial F \) and \( \partial F' \) so that each is homologous in \( \nu L \) to \( L \). Then the algebraic intersection number \( i(\partial F, \partial F')_{\partial \nu L} = s(F) - s(F') \).

Given any spanning surface \( F \) for a link \( L \subset S^3 \), Gordon-Litherland construct a symmetric, bilinear pairing \( \langle \cdot, \cdot \rangle : H_1(F) \times H_1(F) \to \mathbb{Z} \) as follows.\(^2\) See Figure 3. Let \( \nu F \) be a neighborhood of \( F \) in the link exterior \( S^3 \setminus \nu L \) with projection \( p : \nu F \to F \), such that \( p^{-1}(\partial F) = \nu F \cup \partial \nu L \); denote the frontier \( \partial \nu F \setminus \partial \nu L \) by \( \tilde{F} \). Given any oriented simple closed curve \( \gamma \subset F \), denote \( \tilde{\gamma} = \partial(p^{-1}(\gamma)) \), and each natural embedding \( f_\alpha : U_\alpha \to U_x \). Then \( \bigcup_{x \in X} \{(U_\alpha, \phi_x \circ f_\alpha)\}_{\alpha \in I} \) is an atlas for \( X \setminus \nu L \). Gluing all the maps \( f_\alpha \) yields a natural map \( f : X \setminus \nu L \to X \subset S^3 \).

\(^7\)That is, so that regions of the same color meet only at crossing points.

\(^8\)If \( L_1, \ldots, L_m \) are the components of \( \partial F \), then the boundary slope of \( F \) along each \( L_i \) equals the framing of \( L_i \) in \( F \), given by the linking number \( \text{lk}(L_i, \tilde{L}_i) \), where \( \tilde{L}_i \) is a co-oriented pushoff of \( L_i \) in \( F \). Hence, \( s(F) = \sum_{i=1}^m \text{lk}(L_i, \tilde{L}_i) \).
Figure 3. An oriented curve $\gamma$ on $F$, with $\overline{\gamma}$ on $\overline{F}$.

orient $\overline{\gamma}$ following $\gamma$. Let $\tau : H_1(F) \to H_1(\overline{F})$ be the transfer map. The Gordon-Litherland pairing
\[
\langle \cdot, \cdot \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}
\]
is the symmetric, bilinear mapping given by the linking number
\[
\langle a, b \rangle = \text{lk}(a, \tau(b)).
\]

Any projective homology class $g \in H_1(F)/\pm$ has a well-defined self-pairing $\langle g, g \rangle$. When there is a simple closed curve $\gamma \subset F$ representing $g$, (such as when $g$ is primitive and $L$ is a knot) $\frac{1}{2}\langle g, g \rangle$ equals the framing of $\gamma$ in $F$.

Given an ordered basis $B = (a_1, \ldots, a_m)$ on $H_1(F)$, the Goeritz matrix $G = (x_{ij}) \in \mathbb{Z}^{m \times m}$ given by $x_{ij} = \langle a_i, a_j \rangle$ represents $\langle \cdot, \cdot \rangle$ with respect to $B$. The signature of $G$ is called the signature of $F$ and is denoted $\sigma(F)$. Gordon-Litherland show that the quantity $\sigma(F) - \frac{1}{2}s(F)$ is independent of $F$, and in fact equals the Murasugi invariant $\xi(L)$, which is the average signature of $L$ across all orientations.

They also show that $\sigma(F)$ is the signature of the 4-manifold obtained by pushing the interior of $F$ into the interior of the 4-ball $B^4$, while fixing $\partial F$ in $\partial B^4 = S^3$, and taking the double-branched cover of $B^4$ along this surface. In particular, when $L$ is a knot, $\xi(L)$ is the signature of $L$ and of the 4-manifold obtained as a double-branched cover of $B^4$ along any perturbed Seifert surface.

A spanning surface $F$ is positive-definite if $\langle \alpha, \alpha \rangle > 0$ for all nonzero $\alpha \in H_1(F)$ \[12\]. Equivalently, $F$ is positive-definite iff $\sigma(F) = \beta_1(F)$. Negative-definite surfaces are defined analogously.

**Proposition 2.3.** If $F_+$ and $F_-$, respectively, are positive- and negative-definite spanning surfaces for the same link $L \subset S^3$ \[12\] then
\[
s(F_+) - s(F_-) = 2(\beta_1(F_+) + \beta_1(F_-)).
\]

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\[10\]Given any primitive $g \in H_1(F)$, choose an oriented simple closed curve $\gamma \subset \text{int}(F)$ representing $g$, and construct $\overline{\gamma}$ as above; then, $\tau(g) = [\overline{\gamma}]$.

\[11\]That is, any $y = \sum_{i=1}^{m} y_i a_i$ and $z = \sum_{i=1}^{m} z_i a_i$ satisfy
\[
(y, z) = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} G \begin{bmatrix} z_1 & \cdots & z_m \end{bmatrix}^T.
\]

\[12\]Note that this is true even if $L$ is split.
Proof. Definiteness implies that $\beta_1(F_{\pm}) = \pm \sigma(F_{\pm})$, and \cite{2} gives $s(F_{\pm}) = 2(\sigma(F_{\pm}) - \xi(L))$.\cite{13} Thus:

\begin{align*}
s(F_+) - s(F_-) &= 2(\sigma(F_+) - \xi(L)) - 2(\sigma(F_-) - \xi(L)) \\
&= 2(\beta_1(F_+) + \beta_1(F_-)).
\end{align*}

\qed

Greene characterized alternating links in terms of definite surfaces:

**Theorem 1.1 of \cite{3}**. If $B$ and $W$ are positive- and negative-definite spanning surfaces for a link $L$ in a homology $\mathbb{Z}/2\mathbb{Z}$ sphere with irreducible complement, then $L$ is an alternating link in $S^3$, and it has an alternating diagram whose chessboards are isotopic to $B$ and $W$.

Moreover, this diagram is reduced if and only if neither $B$ nor $W$ has a projective homology class with self-pairing $\pm 1$.\cite{14}

**Convention 2.4.** Isotopies of properly embedded surfaces and arcs are always taken rel boundary.\cite{15} Two properly embedded surfaces or arcs are parallel if they have the same boundary and are related by an isotopy which fixes this boundary.

The converse of the first sentence of the theorem is also true:

**Proposition 4.1 of \cite{3}**. A connected link diagram is alternating if and only if its chessboard surfaces are definite and of opposite signs.\cite{16}

**Convention 2.5.** If $D$ is a connected alternating link diagram, then its chessboard surfaces $B$ and $W$ are labeled such that $B$ is positive-definite and $W$ is negative-definite. Likewise for chessboard surfaces $B'$ and $W'$ (resp. $B_i$ and $W_i$) from such a diagram $D'$ (resp. $D_i$).

**Observation 2.6.** Theorem 1.1 of \cite{3} extends to split links in $S^3$ as follows:

If $B$ and $W$ are positive- and negative-definite spanning surfaces for a link $L \subset S^3$, then $L$ has an alternating diagram $D \subset S^2$ such that each checkerboard surface of each connected component of $D$ is isotopic to a connected component of $B$ or $W$.

In particular, $B$ and $W$ have the same number of connected components, and this equals the number of split components of $L$.

Greene used Theorem 1.1 of \cite{3} and lattice flows to give a geometric proof of part of Tait’s conjectures:

\cite{13}This holds for each split component; slopes and signatures are additive under split union.

\cite{14}A diagram $D$ is reduced if each crossing abuts four distinct regions of $S^2 \setminus D$.

\cite{15}For example, an isotopy of a spanning surface $F \subset S^3 \setminus \nu L$ is a homotopy $h : F \times I \to S^3 \setminus \nu L$ for which $h(F \times \{t\})$ is a spanning surface at each $t \in I$.

\cite{16}Murasugi’s proof of Tait’s second conjecture \cite{12} proves and uses the forward direction of this proposition.
Theorem 1.2 of [3]. Any two reduced alternating diagrams of the same link have the same crossing number and writhe.

Alternatively, this theorem follows from the flyping theorem, since flypes preserve crossing number and writhe. Thus, our proof of the flyping theorem will give a new geometric proof of this theorem.

Remark. Theorem 1.2 of [3] does not imply, a priori, that a reduced alternating diagram realizes the underlying link’s crossing number. All existing proofs of this fact [6, 11, 14, 17] use the Jones polynomial.

Problem 2.7. Give an entirely geometric proof that any reduced alternating diagram realizes the underlying link’s crossing number.

More generally, Thistlethwaite proved that any adequate link diagram minimizes crossings. See Corollary 3.4 of [15] (or Corollary 5.14 of [7] for Lickorish’s simpler proof). Thistlethwaite then deduced that any reduced alternating tangle diagram minimizes crossings. See Definition 2.2 and Theorem 3.1 of [16].

Problem 2.8. Prove Corollary 3.4 of [15] geometrically.

Problem 2.9. Give a geometric proof of Theorem 3.1 of [16].

2.2. Operations on definite surfaces; arcs of intersection.

Definite surfaces behave well under boundary-connect sum:

Proposition 2.10. If \( F \) is a boundary-connect sum of positive-definite surfaces \( F_1 \) and \( F_2 \), then \( F \) is positive-definite.

Proof. Let \( G_i \) be a Goeritz matrix for \( F_i \) for \( i = 1, 2 \). Then each \( G_i \) has \( \beta_1(F_i) \) positive eigenvalues (counted with multiplicity). Further,

\[
G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}
\]

is a Goeritz matrix for \( F \) with \( \beta_1(F) \) positive eigenvalues. \( \square \)

Next, consider subsurfaces of definite surfaces. Greene proved:

Lemma 3.3 of [3]. If \( F \) is a definite surface and \( F' \subset F \) is a compact subsurface with connected boundary, then \( F' \) is definite.

Here is a related fact:

Proposition 2.11. If \( F' \) is a compact subsurface of a definite surface \( F \), and every component of \( F \setminus F' \) intersects \( \partial F \), then every component of \( F' \) is definite.

Proof. Let \( F'_0 \) be a component of \( F' \), let \( j : F'_0 \to F \) denote inclusion, and let \( g \in H_1(F'_0) \) be primitive with \( j_*(g) = 0 \in H_1(F) \). Choose an oriented multicurve \( \gamma \subset F'_0 \) representing \( g \). Then \( \gamma = \partial F'' \) for some orientable \( F'' \subset F \). The fact that \( \gamma \subset F'_0 \) implies that \( F'' \subset F' \), or
Figure 4. Adding twists to a spanning surface

else $F''$ would intersect, and therefore contain, some component of $F \setminus F'$, implying contrary to assumption that $\partial F'' \neq \gamma$. Ergo, $j_*$ is injective, so $F'$ is definite.

In particular, Proposition 2.11 immediately implies:

**Lemma 2.12.** If $\alpha$ is a system of disjoint properly embedded arcs in a definite surface $F$, then each component of $F \setminus \nu\alpha$ is definite.

Next, consider the operation of adding (half) twists, shown in Figure 4. It works like this. Let $F$ be a spanning surface for a link $L$, $\alpha \subset F$ a properly embedded arc, and $m$ an integer. Let $A$ be an unknotted annulus or Möbius band whose core circle has framing $\frac{m}{2}$, and let $\alpha'$ be a properly embedded arc in $A$ such that $A \setminus \nu\alpha'$ is a disk. Construct $F \natural A$ in such a way that $\alpha$ and $\alpha'$ are glued together to form an arc $\alpha''$. Depending on the sign of $m$, the surface $F' = (F \natural A) \setminus \nu\alpha''$ is said to be obtained from $F$ by adding $\left| \frac{m}{2} \right|$ (positive or negative) twists along $\alpha$.

**Proposition 2.13.** If $F'$ is obtained by adding positive twists to a positive-definite surface $F$, then $F'$ is positive-definite.\footnote{Likewise for adding negative twists to a negative-definite surface.}

Indeed, if $G$ is a positive-definite symmetric matrix and $G'$ is obtained by increasing a diagonal entry of $G$, then $G'$ is also positive-definite. Alternatively, here is a geometric proof:

**Proof.** Let $A$ be an unknotted annulus or Möbius band with $m$ half-twists for some $m > 0$. Then $A$ is also positive-definite, as are $F \natural A$ and $F'$, by Proposition 2.10 and Lemma 2.12.

In the following way, one can compute the slope difference $s(F) - s(F')$ by counting, with signs, the arcs of $F \cap F'$. Given an arc $\alpha$ of $F \cap F'$, let $\nu\partial\alpha$ be a regular neighborhood of the endpoints $\partial\alpha$ in $\partial\nu L$. Then $\partial F \cap \nu\partial\alpha$ and $\partial F' \cap \nu\partial\alpha$ each consist of two properly embedded arcs, one in each disk of $\nu\partial\alpha$. The two arcs in each disk intersect transversally in a single point, giving

$$i(\partial F, \partial F')_{\nu\partial\alpha} \in \{0, \pm2\}.$$
Figure 5. A positive-definite surface $F$ cannot intersect a negative-definite surface $W$ along an arc $\alpha$ with $\langle \partial F, \partial W \rangle_{\nu \partial \alpha} = -2$.

Note:

\[
s(F) - s(F') = \sum_{\text{arcs } \alpha \text{ of } F \cap F'} i(\partial F, \partial F')_{\nu \partial \alpha}
\]

Observation 2.0 implies:

**Observation 2.14.** Suppose $F$ and $W$ respectively are positive- and negative-definite surfaces spanning a link $L \subset S^3$, and $\alpha$ is an arc of $F \cap W$ with $\langle \partial F, \partial W \rangle_{\nu \partial \alpha} = 0$. Denote $F' = F \setminus \nu \alpha$, $L' = \partial F'$, and $W' = W \setminus \nu \alpha$. Then the following are equivalent:

- $\alpha$ is separating on $F$;
- $\alpha$ is separating on $W$;
- $L$ has (one) more split component than $L$.

**Lemma 2.15.** Suppose $F$ and $W$ respectively are positive- and negative-definite surfaces spanning a link $L \subset S^3$, and $\alpha$ is an arc of $F \cap W$. Then $\langle \partial F, \partial W \rangle_{\nu \partial \alpha} \neq -2$. Moreover, denoting $F' = F \setminus \nu \alpha$, if $|F'| = |F|$ then $i(\partial F, \partial W)_{\nu \partial \alpha} = 2$.

**Proof.** Observation 2.1 gives $\beta_1(F) - |F| = \beta_1(F') + 1 - |F'|$, and Lemma 2.12 implies that $F'$ is positive-definite. Denote $\partial F' = L'$. Suppose first that $i(\partial F, \partial W)_{\nu \partial \alpha} = -2$. Construct a surface $W'$ by adding one negative half-twist to $W$ along $\alpha$. Then $W'$ also spans $L'$, and $\beta_1(W') = \beta_1(W)$. Proposition 2.13 implies that $W'$ is negative-definite. Hence, $L'$ is alternating, by Theorem 1.1 of [3]. Moreover, since $|W'| = |W|$, Observation 2.14 implies that $|F'| = |F|$, hence $\beta_1(F) = \beta_1(F') + 1$. Proposition 2.3 now gives:

\[
s(F') - s(W') = s(F) - s(W) + 2
\]

\[
= 2(\beta_1(F) + \beta_1(W)) + 2
\]

\[
> 2(\beta_1(F') + \beta_1(W')).
\]

\footnote{Note that $\partial W'$ is isotopic to $L'$.}

\footnote{Or equivalently, using Observation 2.14 if $L$ and $L'$ have the same number of split components.}
This contradicts Proposition 2.3.

Now, suppose that \( i(\partial F, \partial W)_{\nu \alpha} = 0 \) and \( |F'| = |F| \). The argument here is identical to the first case, except that we define \( W' = W \setminus \nu \alpha \), giving \( \beta_1(W') = \beta_1(W) - 1 \), hence:

\[
\begin{align*}
    s(F') - s(W') &= s(F) - s(W) \\
                 &= 2(\beta_1(F) + \beta_1(W)) \\
                &> 2(\beta_1(F') + \beta_1(W')).
\end{align*}
\]

Again, this contradicts Proposition 2.3. \( \Box \)

3. Generalized plumbing

3.1. Basic definitions. Let \( F \) be a spanning surface for a nonsplit link \( L \) (not necessarily alternating). A plumbing cap for \( F \) is a properly embedded disk \( V \subset S^3 \setminus (F \cup \nu L) \) with the following properties:

- The natural map \( f : S^3 \setminus (F \cup \nu L) \to S^3 \) restricts to an embedding \( f : V \to S^3 \), and \( f(\partial V) \) bounds a disk \( \hat{U} \subset F \cup \nu L \).
- Denoting the 3-balls of \( S^3 \setminus (\hat{U} \cup V) \) by \( Y_1, Y_2 \), neither subsurface \( F_i = F \cap Y_i \) is a disk.

The disk \( U = \hat{U} \cap F \) is the shadow of \( V \). If the first property holds but the second fails, we call \( V \) a fake plumbing cap for \( F \); we still call \( U \) the shadow of \( V \).

The decomposition \( F = F_1 \cup F_2 \) is a plumbing decomposition or de-plumbing of \( F \) along \( U \) and \( V \), denoted \( F = F_1 * F_2 \). See Figure 6. The reverse operation, in which one glues \( F_1 \) and \( F_2 \) along \( U \) to produce \( F \), is called generalized plumbing or Murasugi sum.

If \( V \) is a plumbing cap for \( F \) with shadow \( U \), then one can construct another spanning surface \( F' = (F \setminus U) \cup V \); we call the operation of changing \( F \) to \( F' \) re-plumbing \( F \) along \( U \) and \( V \). See Figure 7. Call the analogous operation along a fake plumbing cap a fake re-plumbing. This is an isotopy move. Two spanning surfaces are plum-related if there is a sequence of re-plumbing and isotopy moves taking one surface to the other.
Figure 7. Re-plumbing a spanning surface replaces a plumbing shadow with its cap.

**Proposition 3.1.** If \( V \) is a plumbing cap or fake plumbing cap for a spanning surface \( F \), and \( U \) is the shadow of \( V \), then \( F \) is plumb-related to \( (F \setminus U) \cup V \).

**Proof.** If \( V \) is a fake plumbing cap, then \( F \) is isotopic to \( (F \setminus U) \cup V \). Otherwise, a re-plumbing move takes \( F \) to \( (F \setminus U) \cup V \). \( \square \)

3.2. **Re-plumbing in \( S^3 \) and proper isotopy through \( B^4 \).** As an aside, it is interesting to note that plumb-related surfaces, viewed as embedded surfaces with boundary in \( S^3 \), rather than as properly embedded surfaces in link exteriors, are properly isotopic in \( B^4 \).

More precisely, let \( L \) be a link in \( S^3 \), and for \( i = 1, 2 \), let \( F_i \subset S^3 \) be an embedded surface with \( \partial F_i = L \subset S^3 \). With this setup:

**Proposition 3.2.** If \( F_1 \setminus \nu L \) and \( F_2 \setminus \nu L \) are plumb-related, then:

- \( F_1' \) and \( F_2' \) are related by an ambient isotopy of \( B^4 \) which fixes \( S^3 \cup L \) pointwise.
- There is an isomorphism \( \phi : H_1(F_1) \to H_1(F_2) \) satisfying \( \langle \alpha, \beta \rangle_{F_1} = \langle \phi(\alpha), \phi(\beta) \rangle_{F_2} \) for all \( \alpha, \beta \in H_1(F_1) \).
- \( F_1 \) and \( F_2 \) have the same (net) slope: \( s(F_1) = s(F_2) \).
- If \( F_1 \) is definite, then \( F_2 \) is definite of the same sign.
- In particular, if \( F_1 \) is a chessboard surface from a reduced alternating diagram, then so is \( F_2 \).

**Proof.** The first statement follows from the observation that any re-plumbing move can be realized as an isotopy through \( B^4 \) in which one fixes the entire surface except the plumbing shadow and pushes the plumbing shadow through \( B^4 \) to the plumbing cap. The second then follows from Theorem 3 of [2], which states that the pairing on \( F_i \) corresponds to the intersection form on the 2-fold branched cover of \( B^4 \) with branch set \( F_i' \), since \( F_1' \) and \( F_2' \) give the same branched cover. The last three statements follow immediately, using [3]. \( \square \)

\[ ^{20} \text{If } L \text{ has multiple components, then } F_1 \text{ and } F_2 \text{ may have different component-wise slopes.} \]
3.3. Acceptable and irreducible plumbing caps.

**Definition 3.3.** A plumbing cap \( V \) is **acceptable** if both:

- No arc of \( \partial V \cap \partial \nu L \) is parallel in \( \partial \nu L \) to \( \partial F \).
- No arc of \( \partial V \cap F \) is parallel in \( F \setminus U \) to \( \partial F \).

Any, possibly fake, plumbing cap \( V \) for \( F \) can be adjusted near any arcs described in Definition 3.3 so as to remove those arcs one at a time. Doing so for a plumbing cap eventually yields an acceptable plumbing cap \( V' \). With this setup:

**Observation 3.4.** The surfaces obtained by re-plumbing \( F \) along \( V \) and \( V' \) are isotopic.

In the fake case, the same procedure gives an \( F \)-parallel disk, so there is no such thing as an acceptable fake plumbing cap:

**Observation 3.5.** No fake plumbing cap satisfies both properties from Definition 3.3.

The first bulleted property in Definition 3.3 implies that if \( V \) is an acceptable plumbing cap with shadow \( U \) then \( (\partial U \cup \partial V) \cap \partial \nu L \) is a system of circles on \( \partial \nu L \), each isotopic to a meridian. Consider one such circle \( \alpha \cup \beta \), where \( \alpha \) is an arc of \( \partial V \cap \partial \nu L \) and \( \beta \) an arc of \( \partial U \cap \partial \nu L \). Since \( \beta \subset \partial F \) is transverse to each meridian on \( \partial \nu L \), we may assign opposite local orientations to \( \beta \) and \( \pi_L(\beta) \subset L \). Extend these local orientations to global orientations on \( L \).

**Convention 3.6.** With the setup above, \( \text{sign}(\alpha) = \text{lk}(L, \alpha \cup \beta) \); \( \alpha \) is called **positive** if \( \text{sign}(\alpha) = +1 \) and **negative** if \( \text{sign}(\alpha) = -1 \).

**Proposition 3.7.** For any acceptable plumbing cap \( V \), the arcs of \( \partial V \cap \partial \nu L \) alternate in sign around \( \partial V \).

**Proof.** Consider any arc \( c \) of \( \partial V \cap F \). Orient \( c \) so that \( U \cap \nu c \) lies to the left of \( c \) when viewed from \( V \cap \nu c \). Then the initial point of \( c \) has sign +1 and the terminal point has sign −1. See Figure 8. \( \square \)

A plumbing cap \( V \) is **reducible** if there is a properly embedded disk \( X \subset S^3 \setminus (\nu L \cup F \cup V) \) such that \( \partial X = \alpha \cup \beta \) for arcs \( \alpha \subset U \), \( \beta \subset V \), and both disks \( V_1, V_2 \) obtained by surgering \( V \) along \( X \) are (non-fake) plumbing caps for \( F \); \( X \) is a reducing disk. If no such \( X \) exists, then \( V \) is **irreducible**. See Figure 9.

**Proposition 3.8.** If \( V \) is an acceptable, irreducible plumbing cap for \( F \) with shadow \( U \), and if \( \varepsilon \) is a properly embedded arc in \( U \) with neither endpoint on \( \partial \nu L \) which is parallel in \( S^3 \setminus (F \cup \nu L \cup V) \) to an arc \( \varepsilon' \subset V \), then \( \varepsilon \) is parallel in \( U \) to \( \partial V \).

**Proof.** Let \( W \) be a properly embedded disk in \( S^3 \setminus (F \cup \nu L \cup V) \) with \( \partial W = \varepsilon \cup \varepsilon' \). Surger \( V \) along \( W \) to obtain two disks \( V_1, V_2 \); since \( V \)
Given an acceptable plumbing cap $V$ for $F$, each arc of $\partial V \cap F$ joins arcs of $V \cap \partial \nu L$ with opposite signs, here $-1$ left and $+1$ right.

A reducing disk for a reducible plumbing cap; the arc $\rho$ in the proof of Proposition 3.8 and the neighborhood $Y$ in the proof of Lemma 3.10.

is irreducible, we may assume wlog that $V_1$ is a fake plumbing cap. Denote the shadow of each $V_i$ by $U_i$.

If $\partial V_1 \subset F$, then $\varepsilon$ is parallel through $U_1$ to $\partial F$. Assume otherwise. Then, because $V_1$ is fake, Observation 3.5 implies that either some arc of $\partial V_1 \cap \partial \nu L$ is parallel in $\partial \nu L$ to $\partial F$, or some arc $\rho$ of $\partial V_1 \cap F$ is parallel in $F \setminus \setminus U_1$ to $\partial F$. Yet, $V$ is acceptable, and $\partial V$ contains all arcs of $\partial V_1 \cap \partial \nu L$ and all but one arc of $\partial V_1 \cap F$. Therefore, the one arc $\rho$ of $\partial V_1 \cap F$ which $\partial V$ does not contain must cut off a disk from $F \setminus \setminus U$. That disk, however, must contain $U_2$, implying that $V_2$ is also a fake plumbing cap. This implies, contrary to assumption, that $V$ was a fake plumbing cap as well.

Proposition 3.9. If $X$ is a reducing disk for an acceptable plumbing cap $V$, then the disks $V_1, V_2$ obtained by surgering $V$ along $X$ both...
satisfy $|V_i \cap F| < |V \cap F|$. Moreover, the surface obtained by re-plumbing along $V$ is isotopic to the surface obtained by re-plumbing along $V_1$ and then along $V_2$.

**Proof.** Let $\partial X = \alpha \cup \beta$, where $\alpha \subset U$ and $\beta \subset V$. If necessary, perturb $X$ so that neither point of $\partial \alpha = \partial \beta$ lies on $\partial F$. Then $|V_1 \cap F| + |V_2 \cap F| = |V \cap F|$. Moreover, both $|V_i \cap F| > 0$, since $\alpha$ is not parallel in $U$ to $\partial F$. This confirms the first statement.

Next, let $U_i$ denote the shadow of $V_i$ for $i = 1, 2$, and let $Y$ be a bicollared neighborhood of $X$ in $S^3 \setminus (F \cup \nu L \cup V)$, so that $\partial Y \cap F = \nu \alpha$ is a regular neighborhood in $U$ of $\alpha$ and $\partial Y \cap V = \nu \beta$ is a regular neighborhood in $V$ of $\beta$. Note that $\partial Y \setminus \nu (\alpha \cup \beta)$ consists of two disks, $X_1 \subset V_1$ and $X_2 \subset V_2$. See Figure 9.

Re-plumb $F$ along $V_1$ and then $V_2$ by replacing $U_1$ with $V_1$ and then $U_2$ with $V_2$; next, push the disk $X_1 \cup \nu \alpha \cup X_2$ through $Y$ to $\nu \beta$. These two moves together replace $U_1 \cup U_2 \cup \nu \alpha = U$ with

$$(V_1 \cup V_2) \setminus \setminus (X_1 \cup X_2) \cup \nu \beta = V,$$

the same as re-plumbing $F$ along $V$.  

Together, Observation 3.4 and Proposition 3.9 immediately imply:

**Lemma 3.10.** If $F$ and $F'$ are plumb-related, then there is a sequence of re-plumbing and isotopy moves taking $F$ to $F'$ in which each re-plumbing move follows an acceptable irreducible plumbing cap.

4. Crossing balls and plumbing caps

Section 4 uses the crossing ball structures introduced in 8 to study plumbing caps for chessboards from alternating link diagrams.

4.1. Crossing ball setup. Let $D$ be a reduced diagram (not necessarily alternating) of a prime nonsplit link $L$ with crossings $c_1, \ldots, c_n$, and let $\nu S^2$ be a neighborhood of $S^2$ with projection $\pi : \nu S^2 \to S^2$. Insert disjoint closed crossing balls $C_i$ into $\nu S^2$, each centered at the respective crossing point $c_i$. Denote $C = \bigsqcup_{i=1}^n C_i$. Perturb each $C_i$ near its equator $\partial C_i \cap S^2$ so that, for the balls $Y_+, Y_-$ comprising $S^3 \setminus \setminus (S^2 \cup C)$, each $\pi|_{\partial Y_\pm}$ is a diffeomorphism.

Construct a (smooth) embedding of $L$ in $(S^2 \setminus \text{int}(C)) \cup \partial C$ by perturbing the arcs of $D \cap C$ following the over-under information at the crossings, while fixing $D \cap S^2 \setminus \text{int}(C)$. Call the arcs of $L \cap S^2$ edges, and those of $L \cap \partial C \cap \partial Y_\pm$ overpasses and underpasses, respectively. Near each crossing, this looks like Figure 10. center.

Let $\nu L$ be a closed regular neighborhood of $L$ in $\nu S^2$ with projection map $\pi_L : \nu L \to L$. For each edge $e \subset L$, call the cylinder

\footnote{Here and throughout, $\mid - \mid$ counts the connected components of $-$.}

\footnote{The assumption that $L$ is prime implies that $L$ is nontrivial.}
Figure 10. A link near a crossing ball with $S_+$ and $S_-$. 

$E = \pi_L^{-1}(e) \cap \partial \nu L$ an edge of $\partial \nu L$; call the rectangles $E \cap Y_+$ the top and bottom of the edge, respectively. For each over/underpass $e_\pm$ of $L$, call $E_\pm = \pi_L^{-1}(e_\pm) \cap \partial \nu L$ an over/underpass of $\partial \nu L$. Call $E_+ \cap Y_+$ and $E_- \cap Y_-$ the top and bottom of the overpass, respectively, and call $E_- \cap Y_-$ and $E_+ \cap Y_-$ the bottom and top of the underpass, respectively. Assume that the meridia comprising $\pi_L^{-1}(L \cap \partial C \cap S_+ \cap S_-)$ also comprise $\partial \nu L \cap \pi_L^{-1}(\partial C \cap S_+ \cap S_-)$. Then these meridia cut $\partial \nu L$ into its edges, overpasses, and underpasses.

Denote the two balls of $S^3 \setminus \left(S^2 \cup C \cup \nu L\right)$ by $H_\pm$, so that each $H_\pm = Y_\pm \setminus \hat{\nu}L$. Also denote $\partial H_\pm = S_\pm$. See Figure 10.

For each crossing $c_t$, denote the vertical arc $\pi^{-1}(c_t) \cap C_t \setminus \hat{\nu}L$ by $v_t$, and denote $v = \bigcup_t v_t$. For each $t$, $\partial C_t \cap S^2 \setminus \hat{\nu}L$ consists of four arcs on the equator of $\partial C$. Two of these arcs lie in black regions of $S^2 \setminus D$, two in white. For the two arcs $\alpha, \beta$ in black regions, a core circle in $\alpha \cup \beta \cup (\partial \nu L \cap C_t)$ bounds a disk $B_t \subset C_t$ such that $\pi(B_t)$ is disjoint from the white regions of $S^2 \setminus D$ and intersects $D$ only at $c_t$. This disk $B_t$ contains the vertical arc $v_t$ and is called the standard positive crossing band at $C_t$. The arcs in the white regions likewise give rise to a standard negative crossing band $W_t$ in $C_t$; note that $B_t \cap W_t = v_t$. Any properly embedded disk in $C_t \setminus \hat{\nu}L$ which contains $v_t$ and is isotopic in $C_t \setminus \hat{\nu}L$ to $B_t$ (resp. $W_t$) is called a positive (resp. negative) crossing band. See Figure 2.

Denote the union of the black (resp. white) regions of $S^2 \setminus D$ by $\hat{B}$ (resp. $\hat{W}$). Then

(1) \[ B = \left(\hat{B} \setminus \text{int}(C \cup \nu L)\right) \cup \bigcup_t B_t \]

and

(2) \[ W = \left(\hat{W} \setminus \text{int}(C \cup \nu L)\right) \cup \bigcup_t W_t \]
are the black and white chessboard surfaces from $D^2$.

Denote the two balls of $S^3 \setminus (B \cup W \cup \nu L)$ by $\tilde{H}_\pm$, such that each $\tilde{H}_\pm \supset H_\pm$; also denote $\partial \tilde{H}_\pm = \tilde{S}_\pm$.

4.2. **Plumbing cap setup.** Keeping the setup from §4.1, assume that $D$ is alternating and $V$ is an acceptable plumbing cap for $B$.

**Definition 4.1.** $V$ is in **standard position** if:

- $V$ is transversal in $S^3$ to $B, W, \partial C, v$, and $\partial V \cap \partial \nu L$ is transverse in $\partial \nu L$ to each meridian on $\partial \nu L$;
- no arc of $\partial V \cap B \setminus \partial C$ is parallel in $B$ to $\partial C$;
- $\partial V \cap \partial \nu L \cap C = \emptyset$; and
- $W$ intersects $V$ only in arcs, hence cuts $V$ into disks.

Let us check that it is possible to position $V$ in this way. The first three conditions are straightforward. Achieve the fourth condition by removing any simple closed curves (“circles”) of $V \cap W$ one at a time through the following procedure.

Choose a circle $\gamma$ of $V \cap W$ which bounds a disk $V'$ of $V \setminus W$. Since $W$ is incompressible, $\gamma$ also bounds a disk $W' \subset W$. Choose a circle $\gamma' \subset W' \cap V$ which bounds a disk $W''$ of $W' \setminus V$. (If $\text{int}(W') \cap V = \emptyset$, then $\gamma' = \gamma$.) The circle $\gamma'$ also bounds a disk $V'' \subset V$. The sphere $V'' \cup W''$ bounds a ball $Y$ in the link exterior, since $L$ is nonsplit. Push $V''$ through $Y$ past $W''$, removing $\gamma'$ from $V \cap W$. Repeat this process until $W$ intersects $V$ only in arcs, and thus cuts $V$ into disks.

**Observation 4.2.** If $V$ is in standard position, then all components of $V \cap \tilde{H}_\pm$ are disks, and all components of $\tilde{H}_\pm \setminus \setminus V$ are balls.

**Observation 4.3.** If $V$ is in standard position, then each arc of $\partial V \cap \partial \nu L$ which is disjoint from $W$ is a negative arc that traverses exactly one overpass or underpass (along the top or bottom, respectively).

This follows from Convention 3.6 and the fact that $D$ is alternating; see Figure 11. Observation 4.3 implies, in particular:

**Observation 4.4.** Any positive arc of $\partial V \cap \partial \nu L$ lies entirely on a single edge of $\partial \nu L$ and contains an endpoint of an arc of $V \cap W$.

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23Note that $B \cap W = v$. Also:

$S_+ \cap S_- = S^2 \setminus \text{int}(C \cup \nu L) = (B \cup W) \setminus \text{int}(C)$,

$S_+ \cup S_- = (S^2 \setminus \text{int}(C \cup \nu L)) \cup (\partial C \setminus \nu L) \cup (\partial \nu L \setminus \text{int}(C))$.

24That is, $\partial V$ is disjoint from the top of each overpass and bottom of each underpass of $\partial \nu L$. Hence, for any overpass that $\partial V$ intersects, $\partial V$ traverses the entire top of the overpass; likewise for underpasses of $\partial \nu L$. 

Figure 11. Each arc of $\partial V \cap \partial \nu L$ disjoint from $W$ is negative and traverses exactly one over/underpass.

Figure 12. The possible configurations for $V$ near an innermost circle of $V \cap \hat{S}_+$. 

**Proposition 4.5.** $V \cap W \neq \emptyset$, and each arc $\alpha$ of $\partial V \setminus W$ intersects at most two disks of $B \setminus W$.

**Proof.** Observation 4.4 and Proposition 3.7 imply that $V \cap W \neq \emptyset$. Consider an arc $\alpha$ of $\partial V \setminus W$. Proposition 3.7 and Observation 4.4 imply that $\alpha$ intersects at most one negative arc of $\partial V \cap \partial \nu L$. Also, each arc of $\alpha \setminus \partial \nu L$ lies in a single disk of $B \setminus W$. The result now follows from Observations 4.3 and 4.4. □

Next, consider an outermost disk $V_0 \subset V \setminus W$. Its boundary consists of an arc $\alpha \subset \partial V \setminus W$ and an arc $\beta \subset V \cap W$. Let $\nu \alpha$ be a regular neighborhood of $\alpha$ in $\partial V$ with $\nu \alpha \cap \partial B \subset \alpha$.

**Lemma 4.6.** If such $V_0 \subset \hat{H}_+$, then $V_0$ appears as in Figure 12. In particular, the endpoints of $\nu \alpha$ lie in opposite disks of $\hat{S}_+ \setminus \partial V_0$.

**Proof.** Let $\partial V_0 = \alpha \cup \beta$ as above. Since $V$ is in standard position, the endpoints of $\alpha$ cannot lie on the same vertical arc or edge of $\partial \nu L$. This and the fact that $D$ is prime and reduced imply that $\alpha$ must traverse an overpass. Hence, by Proposition 4.5 $\alpha$ intersects exactly two disks of $B \setminus W$, and its configuration depends only on its endpoints, each of which lies either on $v$ or on a positive arc of $\partial V \cap \partial \nu L$. Figure 12 shows the three possibilities. □

With $V_0$ as in Lemma 4.6.
Assume wlog circle $\alpha$ $\beta$ that each interior intersects $W$, boundary consists of two arcs, $\alpha$ $\beta$. Proof. Proposition 4.5 implies that $|\partial V_1 \cap \partial V| < |\partial V \cap \partial L| = 4$. Since both $|\partial V_1 \cap \partial V| \leq |\partial V \cap \partial L| = 4$, they must both equal two. But then $V$, $V_1$ $V_2$ would each decompose $F$ as a boundary connect sum. This is impossible because $F$ is essential and $L$ is prime.

**Proposition 4.8.** If $V$ is apparent, then $V$ is irreducible.

*Proof.* If $V$ were reducible, then it would be possible to surger $V$ to give two plumbing caps $V_1$, $V_2$ with both $|\partial V_1 \cap \partial V| < |\partial V \cap \partial L| = 4$. Since both $|\partial V_1 \cap \partial V| \leq |\partial V \cap \partial L| = 4$, they must both equal two. But then $V_1$, $V_2$ would each decompose $F$ as a boundary connect sum. This is impossible because $F$ is essential and $L$ is prime. □

**Theorem 4.9.** If $V$ is an acceptable, irreducible plumbing cap for $B$ in standard position, then $V$ is apparent in $D$.

*Proof.* Proposition 4.5 implies that $V \cap W \neq \emptyset$. Assume for contradiction that $|V \cap W| > 1$. Then there is a disk $V_1 \subset V$ whose boundary consists of two arcs, $\alpha \subset \partial V$ and $\beta \subset V \cap W$, and whose interior intersects $W$ in a nonempty collection of arcs $\beta_1$, $\beta_2$, $\beta_k$, such that each $\beta_i$ is parallel through a disk $X_i \subset V_1$ to an arc $\alpha_i \subset \alpha$.

Assume wlog that the disks $X_i$ all lie in $H^-$. Denote $\gamma = \alpha \cup \beta = \partial V_1$, and for each $i = 1, \ldots, k$, denote the circle $\alpha_i \cup \beta_i = \partial X_i$ by $\gamma_i$. Then $\gamma_1$, $\gamma_2$, $\gamma_k$ are disjoint simple closed curves in $\hat{S}_+$. Relabel if necessary so that $\gamma_1$ bounds a disk $Y \subset \hat{S}_+$, which is disjoint from $\beta \cup \gamma_2 \cup \cdots \cup \gamma_k$. Let $\delta, \delta'$ be the arcs of $\partial V \setminus W$ that each share an endpoint with $\alpha_1$. The other endpoints of $\delta, \delta'$ must both lie in the disk $\hat{S}_+ \setminus \gamma_1$. Lemma 4.6 implies wlog that the endpoints of $\delta$ lie in opposite disks of $\hat{S}_+ \setminus \partial \gamma_1$. Hence, Observation 4.7 implies that $\delta$ traverses the underpass at the same crossing where $\gamma_1$ traverses the underpass. Moreover, $\partial V$ is disjoint from the vertical arc at this crossing, or else an arc of $\partial V \setminus W$ would have both endpoints on that vertical arc.

Therefore, there is a properly embedded arc $\varepsilon$ in the shadow $U$ of $V$ which is disjoint from $W$ and has one endpoint on $\delta \cap \text{int}(B)$ and the other on $\delta' \cap \text{int}(B)$, as in Figure 13 right. Further, since $\delta$ and $\delta'$ both lie on the boundary of the disk $V_1 \setminus (X_1 \cup \cdots \cup X_k) \subset V \setminus W$, that disk contains an arc $\varepsilon'$ with the same endpoints as $\varepsilon$. The arcs

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25This is possible because at least two circles among $\gamma_1$, $\gamma_2$, $\gamma_k$ bound disks of $\hat{S}_+ \setminus \bigcup_{i=1}^k \gamma_i$, and at most one of these disks intersects $\beta$.

26This is because these endpoints lie on $\beta \cup \gamma_2 \cup \cdots \cup \gamma_k \subset \hat{S}_+ \setminus Y$. 

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Figure 13. Every plumbing cap for $W$ is either isotopically apparent (left) or reducible (right).

Figure 14. A flype move corresponds to an isotopy of one chessboard surface (here, $W$) and a re-plumbing of the other.

$\varepsilon$ and $\varepsilon'$ lie on the boundary of the same ball of $\widehat{H}\setminus V$. Thus, $\varepsilon$ is parallel in $S^3\setminus (F \cup \nu L \cup V)$ to the arc $\varepsilon' \subset V$, and so Proposition 3.8 implies that $\varepsilon$ is parallel in $U$ to one of the two arcs of $\partial V \setminus \partial \varepsilon$.

The arc of $\partial V \setminus \partial \varepsilon$ which contains $\alpha_1$ is not a subset of $\partial U$, and so $\varepsilon$ must be parallel in $U$ to the other arc of $\partial V \setminus \partial \varepsilon$. But then that arc is disjoint from $W$ and so $|V \cap W| = 1$, contrary to assumption. □

Given an apparent plumbing cap $V$ for $B$ (or $W$), there is a corresponding flype move, as shown in Figure 14. Namely, the flype move proceeds along an annular neighborhood of a circle $\gamma \subset S^2$ comprised of the arc $V \cap W$ together with an arc in $U \cup \nu L$. (The resulting link diagram might be equivalent to $D$.) Conversely, if $D \to D'$ is a flype move along an annulus $A \subset S^2$, then there is an apparent plumbing cap $V$ for $B$ (or $W$) with $V \cap W \subset A$ (resp. $V \cap B \subset A$).

Proposition 4.10. Let $V$ be an apparent plumbing cap for $B$, $D \to D'$ the flype move corresponding to $V$, $B'$ and $W'$ the chessboard surfaces from $D'$, and $B''$ the surface obtained by re-plumbing $B$ along $V$. Then $B'$ and $B''$ are isotopic, as are $W'$ and $W''$.

Proof. Figure 14 demonstrates the isotopies. □

27 An analogous statement holds for apparent plumbing caps for $W$. 
Theorem 4.11. Let $D$ and $D'$ be reduced alternating diagrams of a prime, nonsplit link $L$ with respective chessboard surfaces $B,W$ and $B',W'$. Then $D$ and $D'$ are flype-related if and only if $B$ and $B'$ are plumb-related as are $W$ and $W'$.

Greene’s characterization in [3] implies that another reduced alternating diagram $D''$ of $L$ has chessboard surfaces which are isotopic to $B'$ and $W$. We will see in the proof of Theorem 4.11 that $D''$ (which will be denoted $D_m$) is related to $D$ (resp. $D'$) by a sequence of flypes each of which corresponds to an isotopy of $W$ (resp. $B'$).

Proof. If $D$ and $D'$ are flype-related, then $B$ and $B'$ are plumb-related as are $W$ and $W'$, by Proposition 4.10.

Assume conversely that $B$ and $B'$ are plumb-related, as are $W$ and $W'$. Then, by Lemma 3.10, there are two sequences of re-plumbing moves $B = B_0 \to \cdots \to B_m = B'$ and $W = W_0 \to \cdots \to W_n = W'$, such that each re-plumbing move $B_i \to B_{i+1}$ (resp. $W_i \to W_{i+1}$) follows an acceptable, irreducible plumbing cap $V_i$ (resp. $V'_i$).

Let $D_0 = D$. Put the plumbing cap $V_0$ in standard position with respect to $D$. Theorem 4.9 implies that $V_0$ is now apparent. Let $D_0 \to D_1$ be the flype move corresponding to $V_0$. By Proposition 4.10, the negative-definite chessboard surface from $D_1$ is isotopic to $W$, and the positive-definite chessboard surface from $D_1$ is isotopic to $B_1$.

Repeat in this manner. Ultimately, this gives a sequence of flypes from $D = D_0$ to a diagram $D_m$ of $L$ whose positive-definite chessboard surface is isotopic to $B'$ and whose negative-definite chessboard surface is isotopic to $W$. To complete the proof, construct a sequence of flypes $D_m = D'_0 \to \cdots \to D'_n = D'$ in the same manner, but now with the flypes coming from the (apparent) acceptable, irreducible plumbing caps $V'_i$ for $W_i$.

5. Spanning surfaces in the crossing ball setting

Adopt the setup from §4.1 of $D$, $B$, $W$, $C$, $\nu L$, $H_{\pm}$, $S_{\pm}$, etc. Now, also let $F$ be an incompressible spanning surface for the prime nonsplit link $L$. Keep this setup throughout §5.

The opening of each subsection of §5 will declare any additional hypotheses for that subsection. Starting in §5.4, these hypotheses will become increasingly specific regarding $F$ and $L$. Namely:

- Starting in §5.3 $F$ is essential.
- Starting in §5.5 $L$ and $D$ are alternating, hence $B$ is positive-definite, and $W$ is negative-definite.
- Starting in §5.6 $F$ is positive-definite.

5.1. Fair position for $F$.

Definition 5.1. $F$ is in fair position if:
Figure 15. Positive (left) and negative (right) crossing bands in a surface $F$ in fair (or good) position.

- $F$ is transverse in $S^3$ to $B$, $W$, $\partial C$, and $v$;
- $\partial F$ is transverse in $\partial \nu L$ to each meridian on $\partial \nu L$;
- whenever $C_t \cap \partial F \neq \emptyset$, $C_t$ contains a crossing band in $F$;
- each crossing band in $F$ is disjoint from $S_+$; and
- $S_+ \cup S_-$ cuts $F$ into disks.

(Later, we will define a slightly more restrictive good position for $F$.) It is certainly possible to position $F$ so that the first four conditions hold. One way to achieve the fifth condition is by minimizing $|F \setminus (S_+ \cup S_-)|$, subject to the first four conditions.

The fourth condition implies that every crossing band in $F$ appears as in Figure 15. This creates an asymmetry between $F \cap S_-$ versus $F \cap S_+$ which will be strategically useful. The idea is that by pushing $F \cap (S_+ \cup S_-)$ into $S_-$ as much as possible, the remaining circles of $F \cap S_+$ will be more constrained; ultimately, these added constraints will enable the key re-plumbing move in the proof of Theorem 5.22.

**Proposition 5.2.** If $F$ is in fair position, then all components of $C \setminus F$ and $H_\pm \setminus F$ are balls, and all components of $F \cap S_+ \cap S_-$, $F \cap \partial C \cap S_\pm$, and $\partial F \cap S_\pm$ are arcs.

**Proof.** By assumption, $F$ intersects each crossing ball in disks, hence it cuts these crossing balls into balls. Likewise for $H_\pm$.

All components of $\partial C \cap S_\pm$, $\partial C \cap S_\pm$, and $S_+ \cap S_-$ are disks. If $F$ intersected one of these disks in a circle, $\gamma$, then, since all components

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28If some component $X$ of $F \setminus (S_+ \cup S_-)$ is not a disk, choose a component of $\partial X$ and push it into $\text{int}(X)$. The resulting circle $\gamma$ is 0-framed in $F$, bounds no disk in $X$, and does bound a disk in $S^3 \setminus (S_+ \cup S_- \cup \nu L)$. Among all such disks, choose one $Z \cap F$ with minimal $|Z \cap F|$. Since $\gamma$ is 0-framed in $F$, $Z \cap F$ is comprised of circles. Choose an innermost disk $Z_0 \subset Z \setminus F$, and denote $\partial Z_0 = \gamma_0$. The incompressibility of $F$ implies that $\gamma_0$ bounds a disk $X_0$ in $F$. The 2-sphere $X_0 \cup Z_0$ bounds a ball $Y \subset S^3 \setminus \nu L$ whose interior is disjoint from $F$. The minimality of $|Z \cap F|$ implies that $X_0 \cap (S_+ \cup S_-) \neq \emptyset$; also, $X_0$ must be disjoint from all crossing bands and must contain any saddle disks that it intersects. Therefore, it is possible to push $F$ near $X_0$ through $Y$ past $Z_0$, thus decreasing $|F \setminus (S_+ \cup S_-)|$, while ensuring that the resulting positioning satisfies the first four conditions. This contradicts the minimality of $|F \setminus (S_+ \cup S_-)|$. 

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of $F \setminus (S_+ \cup S_- \cup \nu L)$ are disks, $\gamma$ would bound disks of $F$ in both components of $F \setminus (S_+ \cup S_- \cup \nu L)$ whose boundaries contain $\gamma$. This contradicts the assumption that $F$ is connected.

The assumption that $D$ is nontrivial and nonsplit implies that each component of $\partial F \cap S_\pm$ is an arc.

5.2. Bigon Moves.

**Definition 5.3.** Let $F$ be in fair position and $\alpha \subset S_\pm$ an arc that:

- intersects $F$ precisely on its endpoints, which lie on the same circle $\gamma$ of $F \cap S_\pm$, but not on the same arc of $F \cap S_+ \cap S_-;
- is disjoint from over/underpasses; and
- intersects $S_+ \cap S_-$ in exactly one component.

Then $\alpha$ is parallel through a properly embedded (“bigon”) disk $Z \subset H_\pm \setminus F$ to an arc $\beta \subset F \cap H_\pm$, and one can push $F \cap \nu \beta$ through $\nu Z$ past $\alpha$, as in Figures 16 and 17. This is called a **bigon move**.

**Proposition 5.4.** Given an arc $\alpha \subset S_\pm$ as in Definition 5.3, the endpoints of $\alpha$ do not lie on the same circle of $F \cap S_\pm$.

**Proof.** If either endpoint of $\alpha$ is on $\partial \nu L$, instead of in $S_+ \cap S_-$, there is nothing to prove. Assume that $\alpha \subset S_+ \cap S_-$, and assume for contradiction that $\gamma_\pm$, respectively, are circles of $F \cap S_\pm$, both of which contain both endpoints of $\alpha$. Let $X_\pm$, respectively, denote the disks of $F \cap H_\pm$ with $\partial X_\pm = \gamma_\pm$. Choose arcs $\beta_\pm \subset X_\pm$ with the same endpoints as $\alpha$, and let $\delta_\pm = \alpha \cup \beta_\pm$.

The circles $\delta_\pm$ bound properly embedded disks $Z_\pm \subset H_\pm \setminus F$. Gluing $Z_+$ and $Z_-$ along $\alpha$ produces a disk $Z$ with $Z \cap F = \partial Z = \beta_+ \cup \beta_-$. Since $F$ is incompressible, $\beta_+ \cup \beta_-$ must bound a disk $X \subset F$. The boundary of this disk $X$ is the curve $\beta_+ \cup \beta_-$, which intersects
Figure 18. Removing arcs of $\partial F \cap S_\pm$ with both endpoints on the same arc of $S_+ \cap S_- \cap \partial \nu L$.

Figure 19. Removing arcs of $F \cap \partial C \cap S_\pm$ with both endpoints on the same arc of $\partial C \cap S_+ \cap S_-$. 

$S_+ \cap S_-$ only in two points. Therefore, $X$ intersects $S_+ \cap S_-$ in a single arc, and this arc has the same endpoints as $\alpha$. Contradiction. □

**Lemma 5.5.** Performing a bigon move on a surface in fair position produces a surface in fair position.

**Proof.** Consider a bigon move along an arc $\alpha$. Assume wlog that the endpoints of $\alpha$ lie on the same circle $\gamma_+$ of $F \cap S_+$. Then, by Proposition 5.4, the endpoints of $\alpha$ lie on distinct circles $\gamma_-, \gamma_-'$ of $F \cap S_-$. Let $X_+$ be the disk of $F \cap H_+$ bounded by $\gamma_+$, and let $X_-$ and $X_-'$ be the disks of $F \cap H_- \cong \gamma_-$ and $\gamma_-'$, respectively.

Since $X_- \neq X_-'$, the bigon move along $\alpha$ must involve pushing an arc $\beta_+ \subset X_+$ past $\alpha$. This has the effect of splitting $X_+$ into two distinct disks of $F \cap H_+$, while merging $X_-$ and $X_-'$ into a single disk of $F \cap H_-$. The rest of $F$ is unaffected. Thus, this move preserves the fact that $S_+ \cup S_-$ cuts $F$ into disks. Hence, it produces a surface in fair position. □

5.3. **Good position for $F$.**

**Definition 5.6.** $F$ is in **good position** if:
Figure 20. When $F$ is in good position, $F \cap C$ is comprised of crossing bands and saddle disks.

- $F$ is in fair position,
- no arc of $\partial F \cap S_\pm$ has both endpoints in the same arc of $S_+ \cap S_- \cap \partial \nu L$, and
- no arc of $F \cap \partial C \cap S_\pm$ has both endpoints on the same arc of $\partial C \cap S_+ \cap S_-$. 

Given a surface $F$ in fair position, it is possible to use the moves in Figures 18 and 19 to remove any arcs described in Definition 5.6, all while preserving the conditions of fair position. That is, any surface in fair position admits local adjustments that put it in good position.

**Proposition 5.7.** If $F$ is in good position, then each component of $F \cap C$ is either a crossing band or a saddle disk as in Figures 15, 20.

**Proof.** Consider a crossing ball $C_t$ where $F$ does not have a crossing band. Each component $\gamma$ of $F \cap C_t$ is a circle because $F$ does not have a crossing band at $C_t$, and so, by assumption, $\partial F \cap \partial C_t = \emptyset$. Also, by assumption, each such circle $\gamma$ bounds a disk of $F \cap C_t$.

Moreover, Proposition 5.2 implies that $S^2$ cuts each such $\gamma$ into arcs; by assumption, the endpoints of each arc of $\gamma \setminus S^2$ are on distinct arcs of $\partial C_t \cap S_+ \cap S_-$. Since each disk of $\partial C_t \cap S_\pm$ contains only two arcs of of $\partial C_t \cap S_+ \cap S_-$, the result follows.

5.4. **Essential spanning surfaces.** Assume in §5.4 that $F$ is an essential surface in good position. Define the complexity of $F$ to be

$$||F|| = |v \setminus \nu F| = \#\text{saddle disks} + \#\text{crossings} - \#\text{crossing bands}.$$ (3)

**Proposition 5.8.** For each arc $\beta$ of $\partial F \cap S_\pm$ that traverses no over/underpasses, the endpoints of $\beta$ lie on distinct circles of $F \cap S_\pm$.

**Proof.** Assume for contradiction wlog that an arc $\beta$ of $\partial F \cap S_-$ traverses no underpasses and has both its endpoints on the same circle $\gamma$ of $F \cap S_-$. The disk of $F \cap H_-$ bounded by $\gamma$ contains a properly embedded arc $\alpha$ with the same endpoints as $\beta$. The fact
Figure 21. Left: If an arc $\beta$ of $\partial F \cap S_-$ is disjoint from underpasses, then its endpoints lie on distinct circles of $F \cap S_+$. Right: In particular, no arc of $F \cap S_+ \cap S_-$ is parallel in $S_+ \cap S_-$ to $\partial \nu L$.

Figure 22. No arc of $\partial F \cap S_+ \cap S_-$ has both endpoints on the same crossing ball.

that $\beta$ traverses no underpasses implies that $\alpha$ is parallel in $S^3 \setminus \nu L$ to an arc $\alpha'$ on $\partial \nu L \cap S_+$ (drawn yellow in Figure 21 left). Since $F$ is in good position, the shared endpoints of these arcs lie in distinct disks of $S_+ \cap S_-$. Therefore, $\alpha' \cup \beta$ is a meridian on $\partial \nu L$; hence, $\alpha'$ is not parallel in $\partial \nu L$ to $\partial F$. This contradicts Proposition 2.2.

Proposition 5.9. No arc of $F \cap S_+ \cap S_-$ is parallel in $S_+ \cap S_-$ to $\partial \nu L$.

Proof. If such an arc exists, then choose one, $\alpha$ which is outermost in $S_+ \cap S_-$. Up to mirror symmetry, $\alpha$ must appear as in Figure 21 right. This is impossible, by Proposition 5.8.

Proposition 5.10. If $\|F\|$ is minimized, then no arc of $\partial F \cap S_+ \cap S_-$ has both endpoints on the same crossing ball.

Proof. If both endpoints of an arc of $\partial F \cap S_+ \cap S_-$ lie on the same crossing ball, then there is an outermost such arc $\alpha$ in $S_+ \cap S_-$. Both endpoints of $\alpha$ lie on saddle disks, by 5.7, so $\alpha$ abuts two saddle disks. The sequence of isotopy moves shown in Figure 22 decreases $\|F\|$, contrary to assumption.

Proposition 5.11. If $\|F\|$ is minimized and a circle $\gamma$ of $F \cap S_\pm$ traverses the over/underpass at a crossing ball $C_t$, then $\gamma \cap \partial C_t = \emptyset$.

Proof. Assume for contradiction that $\gamma$ traverses the overpass at $C_t$ and $\gamma \cap \partial C_t \neq \emptyset$. Then there is a disk $R$ of $S_+ \cap S_-$ containing a
If $||F||$ is minimized and $\gamma \subset F \cap S_+$ traverses the overpass at $C_t$, then $\gamma \cap C_t = \emptyset$.

In particular, Proposition 5.11 implies the following two facts:

**Proposition 5.12.** If $||F||$ is minimized, no arc $\alpha$ of $F \cap S_+ \cap S_-$ has one endpoint on $C$ and the other on an incident edge of $\partial \nu L$.

**Proof.** If such an arc exists, choose one, $\alpha$, which is outermost in $S_+ \cap S_-$. Let $x$ denote the endpoint of $\alpha$ on a crossing ball $C_t$, and assume WLOG that the edge of $\partial \nu L$ containing the other endpoint $y$ of $\alpha$ abuts the overpass at $C_t$. Consider the arc $\beta$ of $\partial F$ on this edge between $y$ and the overpass at $C_t$.

The arc $\beta$ must intersect $S_+ \cap S_-$ in exactly one point, due to Propositions 5.11 and 5.9 and the fact that $\alpha$ is outermost. Hence, there is a bigon move which causes the circle of $F \cap S_+$ traversing the overpass at $C_t$ to intersect $\partial C_t$, contradicting Proposition 5.11. \[\square\]

**Proposition 5.13.** If $||F||$ is minimized, then for each crossing ball $C$, where $F$ does not have a crossing band, the arcs of $F \cap \partial C_t \cap S_\pm$ all lie on distinct circles of $F \cap S_\pm$ from one another and from the arc of $\partial F \cap S_\pm \cap \partial \nu L$ which traverses the over/underpass at $C_t$.

**Proof.** Otherwise, a bigon move near $C_t$ gives an immediate contradiction to Proposition 5.10 or Proposition 5.12. \[\square\]

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29 The proof of Proposition 5.14 employs a similar argument and provides greater detail on this point.

30 To see the bigon move, let $\rho$ be the arc of $F \cap S_+ \cap S_-$ with an endpoint on $\text{int}(\beta)$; let $\alpha'$ be the arc of $F \cap \partial C_t \cap S_-$ sharing an endpoint with $\alpha$; and let $\sigma$ be the arc of $F \cap S_+ \cap S_-$, other than $\alpha$ that shares an endpoint with $\alpha'$. The bigon move is along an arc in $S_+ \cap S_-$ with one endpoint on $\rho$ and the other on $\sigma$. 

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Figure 23. If $||F||$ is minimized and $\gamma \subset F \cap S_+$ traverses the overpass at $C_t$, then $\gamma \cap C_t = \emptyset$. 

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A GEOMETRIC PROOF OF THE FLYPING THEOREM 27
Figure 24. No circle of $F \cap S_+$ intersects both the top of the overpass at $C_t$ and an edge of $\partial \nu L$ incident to the underpass at $C_t$.

Proposition 5.14. If $||F||$ is minimized and a circle $\gamma$ of $F \cap S_+$ traverses the over/underpass of $\partial \nu L$ at a crossing $C_t$, then $\gamma$ is disjoint from both edges of $\partial \nu L$ incident to the under/overpass at $C_t$.

Proof. Assume for contradiction that a circle $\gamma$ of $F \cap S_+$ traverses the overpass of $\partial \nu L$ at $C_t$ and intersects an edge of $\partial \nu L$ incident to the underpass at $C_t$. Then this edge of $\partial \nu L$ contains an endpoint of an arc of $\gamma \cap S_+ \cap S_-$. Denote the disk of $S_+ \cap S_-$ containing this endpoint by $R$. As shown in Figure 24 there is an arc $\alpha \subset S_+$ disjoint from overpasses with $\alpha \cap S_+ \cap S_- \subset R$ and $\partial \alpha \subset \gamma$. Choose $\alpha$ such that $\alpha \pitchfork F$ and all points of $\text{int}(\alpha) \cap F$ lie on distinct arcs of $F \cap S_+$.

We claim that $\alpha$ is properly embedded in $S_+ \setminus \setminus F$, i.e. $\alpha \cap F = \partial \alpha$. Suppose otherwise. Then there is an arc $\alpha'$ of $\alpha \setminus \setminus F$ sharing neither endpoint with $\alpha$, such that both endpoints of $\alpha'$ lie on the same circle of $F \cap S_+$. Perform a bigon move along $\alpha'$. Consider the resulting arc $\beta$ that lies in the disk of $R \setminus \setminus (\gamma \cup \alpha)$ incident to $C_t$. Each endpoint of $\beta$ lies either on the arc $\partial C_t \cap R$ or on the arc of $\partial \nu L \cap R$ incident to the underpass at $C_t$. This is impossible, by Propositions 5.10 and 5.12. Therefore, $\alpha$ is properly embedded in $S_+ \setminus \setminus F$.

Following Figure 24 perform a bigon move along $\alpha$, and then adjust $F$ to have a crossing band at $C_t$. This decreases $||F||$, contrary to assumption. \[\square\]

5.5. Alternating links. Assume again in §5.5 that $F$ is an essential surface in good position. Assume also that its complexity $||F|| = |v \setminus \setminus F|$ is minimized, and that $D$ is a reduced alternating diagram (of a prime nonsplit link $L$) with chessboard surfaces $B$ and $W$.

Much of the work in §5.5 will address an innermost circle $\gamma$ of $F \cap S_+$, in the following setup. Let $T_+$ be an innermost disk of

\[\text{Recall Convention 2.5.}\]
Figure 25. How $F$ appears near each edge of $\partial \nu L$ in §§5.5-5.7 (up to mirror symmetry below the top crossing ball).

$S_+ \setminus F$, and denote $\partial T_+ = \gamma$. Let $X$ be the disk that $\gamma$ bounds in $F$. Denote

$$T_- = S_- \cap \pi^{-1}(\pi(T_+)) .$$

Before addressing $\gamma$, we note that, because $D$ is alternating and $F$ is in good position:

**Observation 5.15.** Near each edge of $\partial \nu L$, $F$ appears as in Figure 25 (up to mirror symmetry below the top crossing ball).

Note in particular that an edge of $\partial \nu L$ contains an odd number of points of $F \cap S_+ \cap S_-$ if and only if $\partial F$ traverses the top of the overpass incident to that edge. We now turn our attention to the innermost circle $\gamma$ of $F \cap S_+$.

**Proposition 5.16.** Any arc of $\gamma \cap \partial C$ appears as in Figure 26.

**Proof.** Suppose $\gamma$ intersects a crossing ball $C_t$. Then Proposition 5.13 implies that $\gamma$ must intersect $C_t$ in one arc only and must not traverse the overpass at $C_t$. Proposition 5.12 further implies that $\gamma$ is disjoint from the edge $E$ of $\partial \nu L$ which abuts both the underpass at $C_t$ and the disk of $\partial C_t \setminus (S^2 \cup \nu L)$ incident to $\gamma$. Hence, $E \cap F \cap S_+ \cap S_- = \emptyset$, and so Observation 5.15 implies that $F$ must have a crossing band at the opposite end of this edge. 

**Proposition 5.17.** No arc of $\gamma \cap S_+ \cap S_-$ has both endpoints on $C$.

**Proof.** This follows immediately from Proposition 5.16 and the fact that $D$ is alternating. 

Say that $\gamma$ encloses a crossing ball $C_t$ if $\pi(T_+) \supset \pi(C_t)$.

**Lemma 5.18.** $F$ has a crossing band in every crossing ball which $\gamma$ encloses, and $\gamma$ encloses at least one crossing ball.

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32Recall that $\pi$ denotes the projection map $\nu S^2 \to S^2$.

33This gives an alternate proof of the crossing band lemma from [1]:
Figure 26. The vicinity of an arc of $\gamma \cap \partial C$. (The crossing ball to the right may instead have a negative crossing band.) Inward-pointing arrows along the innermost circle $\gamma$ of $F \cap S_+$ identify black and white regions of $S_+ \cap S_-$. 

Proof. If $\gamma$ encloses a crossing ball $C_t$, then $\partial F$ does not traverse the overpass at $C_t$, since $\gamma$ is an innermost circle of $F \cap S_+$. Fair position implies that $F$ has a crossing band at $C_t$.

Assume, contrary to the second claim, that $\gamma$ encloses no crossing balls. Then $\gamma$ cannot intersect a crossing ball, due to Proposition 5.16. Further, $\gamma$ cannot traverse an overpass, because of Proposition 5.14 and Observation 5.15. Therefore, $T_+$ is disjoint from all crossing balls and overpasses. Hence, $\pi(\gamma)$ bounds a disk in $S^2 \setminus C$. This implies, contrary to assumption, that some arc of $\gamma \cap S_+ \cap S_-$ is parallel in $S_+ \cap S_-$ to $\partial \nu L$. $\square$

5.6. Definite surfaces. Assume again in §5.6 that $F$ is an essential surface in good position, $||F|| = |v \setminus F|$ is minimized, and $D$ is a reduced alternating diagram of $L$ with chessboard surfaces $B$ and $W$. Also maintain the setup from §5.5 of the innermost circle $\gamma \subset F \cap S_+$ and the associated disks $T_+ \subset S_+$ and $T_- \subset S_-$. Now assume also that $F$ is positive-definite.

Much of the work in §5.6 will address the possibilities for arcs $\delta$ of $F \cap T_-$. First, note the following more general consequences of the assumption that $F$ is positive-definite:

Lemma 5.19. Every crossing band in $F$ is positive, and any arc $\alpha$ of $F \cap W$ that is disjoint from $C$ satisfies

$$i(\partial F, \partial W)_{\nu \partial \alpha} = 2.$$ 

Crossing band lemma from [1]. Given a reduced alternating diagram $D$ of a prime nonsplit link $L$ and an essential spanning surface $F$ for $L$, it is possible to isotope $F$ so that it has a crossing band at some crossing in $D$. 

Proof. Regarding the second claim, suppose that \( i(∂F, ∂W)_{να} \neq 2 \), and denote \( F' = F \setminus να \) and \( W' = W \setminus να \); then \( F' \) and \( W' \) span the same link \( L' \). Lemma 2.15 implies that \( L' \) is split and \( i(∂F, ∂W)_{να} = 0 \). Yet, \( D \) is connected, prime, and alternating, so banding \( D \) along \( να \cap W \) gives a diagram \( D' \) of \( L' \) which is alternating and connected. This contradicts the fact that \( L' \) is split [8]. The first claim follows from the same argument, since smoothing a crossing in a reduced alternating diagram of a prime nonsplit link gives a connected alternating diagram. □

Now consider \( F \cap T_− \). Each component of \( F \cap \text{int}(T_−) \) is disjoint from \( S_+ \) (since \( γ \) is an innermost circle of \( F \cap S_+ \)) and thus consists alternately of arcs on \( ∂νL \cap S_- \) and arcs on \( ∂C \cap S_- \) which abut crossing bands in \( F \). This and the fact (due to Lemma 5.19) that every crossing band in \( F \) is positive imply that every circle of \( F \cap \text{int}(T_−) \) encloses some disk of \( B \setminus C \) in \( T_− \) and is parallel in \( (∂νL \cup ∂C) \cap S_- \) to the boundary of this disk. Likewise, for each arc \( δ \) of \( F \cap \text{int}(T_−) \), some disk \( B_0 \) of \( B \cap T_− \setminus C \) abuts all components of \( ∂νL \cap S_- \) and \( ∂C \cap S_- \) that intersect \( δ \).

**Convention 5.20.** Orient each arc \( δ \) of \( F \cap \text{int}(T_−) \) so that this nearby disk \( B_0 \) lies to the right of \( δ \), when viewed from \( H_+ \). Denote the initial point of \( δ \) by \( x \) and the terminal point by \( y \).

**Proposition 5.21.** Let \( δ \) be an arc of \( F \cap \text{int}(T_−) \), following Convention 5.20.

1. There are three possibilities for \( δ \)'s initial point \( x \) and two for its terminal point \( y \), namely (also see Figure 27):

| endpoint type \( z \) | initial or terminal |
|----------------------|-------------------|
| \( z \in ∂νL \) \( π(z) = π(w) \) for \( w \in ∂F \) | initial |
| \( z \in ∂νL \) \( π(z) = π(w) \) for \( w \) on saddle | initial |
| \( z \in ∂B \) \( i(∂F, ∂B)_{νz} = 1 \) | initial |
| \( z \in ∂B \) \( i(∂F, ∂B)_{νz} = -1 \) | terminal |
| \( z \in ∂W \) \( i(∂F, ∂W)_{νz} = 1 \) | terminal |
| \( z \in ∂W \) \( i(∂F, ∂W)_{νz} = -1 \) | initial |

2. \( x \in ∂B \) if and only if \( y \in ∂B \). Thus, there are just three possibilities for the pair \( (x,y) \) (see Figure 30).

3. There are just three possibilities for \( δ \) (see Figure 30), each of which is parallel in \( T_- \) to ∂\( T_- \).

*Proof.* Part (i) follows almost entirely from Conventions 2.5 and 5.20. We only need to show that the sixth possibility, an initial point \( z \in W \) with \( i(∂F, ∂W)_{νz} = -1 \) (struck though in the table and shown far right in Figure 27) is impossible. Indeed, by Lemma 5.19, such a point \( z \) must be an endpoint of an arc of \( γ \cap W \), the other endpoint of
FIGURE 27. From left: the five possible types of endpoints of an arc $\delta$ of $F \cap \text{int}(T_-)$, oriented as in Convention 5.20. Far right: the impossible type of such endpoint.

FIGURE 28. The contradiction in the proof of part (i) of Proposition 5.21 which lies on $C$. With Proposition 5.16, this gives the configuration shown in Figure 28 contradiction.

For (ii), we argue by contradiction. Suppose first that $x \in \partial B$ and $y \notin \partial B$. Let $\gamma_x \subset \gamma$ be the arc of $F \cap S_+ \cap S_-$ that has $x$ as an endpoint, and let $\gamma_y \subset \gamma$ be the arc of $F \cap S_+ \cap \partial \nu L$ that has $y$ as an endpoint. Let $x' \in \text{int}(\gamma_x)$ and $y' \in \text{int}(\gamma_y)$. See Figure 29 left. Perform a bigon move along the arc $\alpha$. Then consider the arc of $\partial F \cap S_+$ that contains $y$. Both endpoints of this arc lie on the circle of $F \cap S_-$ that contains $\delta$, contradicting Proposition 5.8.

The two remaining cases, with $x \notin \partial B$ and $y \in \partial B$, are similar. Figure 29 shows in each case that a bigon move yields a contradiction, using either Proposition 5.14 (center) or Proposition 5.11 (right).

For part (iii), it suffices to check that the initial points and terminal points of the arcs of $F \cap T_i$ alternate around $\gamma$. Indeed, by considering an initial point $x$ and proceeding along $\gamma$ counterclockwise around $T_-$ (viewed from $H_+$), this follows in all cases from Convention 5.20 and the fact that $D$ is alternating. □

5.7. Proof of Theorem 5.22 and the flyping theorem. Let $L$ be a prime nonsplit link, and let $D$ be a reduced alternating diagram of

\footnote{Note that $x' \in \partial B_0$, and $y'$ is on the same edge of $\partial \nu L$ that contains $y$. This edge abuts $B_0$, so there is a properly embedded arc $\alpha \subset S_+ \setminus F$ with endpoints $x', y' \in \gamma$ which is disjoint from overpasses and intersects $S_+ \cap S_-$ only in $B_0$. Since $y' \in \partial \nu L$, it is possible to perform a bigon move along $\alpha$.}
These bigon moves show that the initial point $x$ of $\delta$ lies on $\partial B$ iff the terminal point $\partial y$ does.

$L$ with chessboard surfaces $B$ and $W$, which are respectively positive- and negative-definite.

**Theorem 5.22.** Any essential positive-definite surface $F$ spanning $L$ is plumb-related to $B$.

As an aside, note that Proposition 3.2 and Theorem 5.22 imply:

**Corollary 5.23.** Any two essential positive-definite surfaces spanning $L$ are related by proper isotopy through $B^4$.

**Proof of Theorem 5.22.** While fixing $B$ and $W$, transform $F$ by isotopy so that it is in good position and $||F|| = |v \setminus F|$ is minimal. If $F \cap H_+ = \emptyset$, then $F$ traverses the top of no overpass, hence has a crossing band at every crossing. Each crossing band is positive, due to Lemma 5.19, and so $F$ is isotopic to $B$.

Assume instead that $F \cap H_+ \neq \emptyset$, and let $T_+$ be an innermost disk of $S_+ \setminus \setminus F$, set up as at the beginning of §5.5 with $T_+ = S_- \cap \pi^{-1}(\pi(T_+))$ and $\gamma = \partial T_+ = \partial X$ for $X \subset F \cap H_+$. Proposition 5.21 implies that each arc $\delta$ of $F \cap \text{int}(T_-)$ has one one the three types (I, II, or III from left to right) shown in Figure 30. Note:

(4) At least one arc $\delta$ of $F \cap \text{int}(T_-)$ is of type II or III.

Fixing $F \cap (S_+ \cup S_- \cup C)$, adjust $F \cap H_\pm$ so that (i) $\pi$ restricts to an injection on each disk of $F \cap H_\pm$, and (ii) for any point $z \in W \cap T_+$, $F \cap \pi^{-1}(z)$ consists of a single point, which lies in $X$.\footnote{This may involve pushing some disks of $F \cap H_-$ through the origin and some disks of $F \cap H_+$ through the point at infinity.}

\footnote{This is so that Figure 31 will indeed be generic.}
Take an annular neighborhood $A$ of $\pi(\gamma)$ in $S^2$, such that $A$ intersects only the crossing balls that $\pi(\gamma)$ intersects, $\partial A \cap C = \emptyset$, and each arc of $F \cap S_+ \cap S_-$ that intersects $A$ lies on $\gamma$ or has an endpoint on a saddle disk at a crossing where $\gamma$ abuts a saddle disk. Denote $\partial A = \gamma_0 \cup \gamma_1$, such that $\gamma_0 \subset \pi(T_+)$. Denote $S^2 \setminus A = Z_0 \cup Z_1$, where each $\partial Z_i = \gamma_i$. Denote $\pi^{-1}(Z_0) = Y_0$.

Denote the arcs of $\gamma \cap W$ by $\omega_1, \ldots, \omega_m$. Each $\omega_i$ has a dual arc $\alpha_i \subset A \cap W \setminus C$. Denote the rectangles of $A \setminus (\alpha_1 \cup \cdots \cup \alpha_m)$ by $A_1, \ldots, A_m$ such that each $\partial A_i \supset \alpha_i \cup \alpha_{i+1}$, taking indices modulo $m$. Denote each prism $\pi^{-1}(A_i)$ by $P_i$.

Each prism $P_i$ intersects $F$ in one of the four ways indicated in the left column of Figure 31; by (4), not all $P_i$ appear as bottom-left in the figure. For each $i$, let $F_i$ denote the component of $F \cap P_i$ which abuts $\gamma$. Each $F_i$ is a disk. Observe that $F_i$ and $F_j$ intersect in an arc when $i \equiv j \pm 1 \pmod{m}$ and are disjoint when $i \not\equiv j, j \pm 1 \pmod{m}$. Denote $F_A = F_1 \cup \cdots \cup F_m$. The disk $X \cap Y_0$ attaches to $F_A$ along its boundary. Therefore, $F_A$ is an annulus, and the following subsurface of $F$ is a disk:

$$U = (X \cap Y_0) \cup F_A.$$  

There is a properly embedded disk $V \subset \nu S^2 \setminus (F \cup \nu L)$ which intersects $Y_0$ in a disk (in $H_-$) and intersects each prism $P_i$ as indicated in the right column of Figure 31. Note that $\partial V \cap F = \partial U \cap F$ and $(\partial V \cap \partial \nu L) \cup (\partial U \cap \partial \nu L)$ is a system of meridia and inessential circles on $\partial \nu L$. Thus $V$ is a, possibly fake, plumbing cap for $F$, and $U$ is its shadow. Whether or not $V$ is fake, Proposition 3.1 implies that $F$ is plumb-related to $(F \setminus U) \cup V$. Figure 31 shows the move $F \to (F \setminus U) \cup V$ within each prism $P_i$. [80]

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[73] The arc $\alpha_i$ has one endpoint on $\gamma_0$ and one on $\gamma_1$, with $|\omega_i \cap \alpha_i| = 1$.

[78] The green arcs top-left describe a disk in $R \setminus \nu L$, which is parallel through a ball $Y \subset P_i$ to $\pi^{-1}(\gamma_i)$; $F$ intersects $P_i$, as shown, and in an arbitrary number of additional disks in $Y_i$, each containing a saddle disk.

[80] A caveat analogous to the one in Note 38 applies here as well.

[81] Alternatively, one can visualize the re-plumbing move as follows. Replace the disk $U \cap H_+$ with its reflection in the projection sphere. The resulting surface
Whereas the disk $U$ intersects $C$ in some number of saddle disks and no crossing bands, the disk $V$ intersects $C$ in no saddle disks and some number of crossing bands. Moreover, \[4\] implies that either $U \cap C \neq \emptyset$ or $V$ contains a crossing band. Hence, the move $F \rightarrow F_1 = (F \setminus \setminus U) \cup V$ decreases the complexity of $F$: $||F_1|| < ||F||$. \[41\]

Isotope $F_1$ into good position so as to minimize its complexity. If $||F_1|| > 0$, then the same argument gives another re-plumbing move

\[41\]This implies that $V$ was not a fake plumbing cap.
$F_1 \to F_2$ with $||F_2|| < ||F_1||$. Repeat, performing isotopy and re-plumbing moves $F \to F_1 \to \cdots \to F_r$ until $||F_r|| = 0$. Then $F_r$ is disjoint from $H_+$, hence isotopic to $B$. □

**Flyping theorem.** Any two reduced alternating diagrams $D, D'$ of the same prime, nonsplit link $L$ are flype-related.

**Proof.** Let $B, W$ and $B', W'$ be the respective chessboard surfaces of $D$ and $D'$, where $B$ and $B'$ are positive-definite. Then $B$ and $B'$ are plumb-related, by Theorem 5.22, as are $W$ and $W'$. Therefore, $D$ and $D'$ are flype-related, by Theorem 4.11. □

Since crossing number and writhe are invariant under flypes and additive under diagrammatic connect sum and disjoint union, we obtain a new geometric proof of more parts of Tait’s conjectures [3, 6, 11, 14, 17]:

**Theorem 5.24.** All reduced alternating diagrams of a given link have the same crossing number and writhe.

Problems 2.7–2.9 remain open. Another corollary is worth noting:

**Corollary 5.25.** If $D$ and $D'$ are reduced alternating diagrams of the same link, and if $B, W$ and $B', W'$ are their respective chessboard surfaces, then $s(B) = s(B')$ and $s(W) = s(W')$.

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