New Higgs Transitions between Dual N=2 String Models

Per Berglund  
*Theory Division, CERN*  
*CH-1211 Geneva 23, Switzerland*  
Sheldon Katz  
*Department of Mathematics*  
*Oklahoma State University*  
*Stillwater, OK 74078, USA*  
Albrecht Klemm  
*Department of Mathematics, Harvard University*  
*Cambridge MA 02138, USA*  
Peter Mayr  
*Theory Division, CERN*  
*CH-1211 Geneva 23, Switzerland*

We describe a new kind of transition between topologically distinct $N = 2$ type II Calabi–Yau vacua through points with enhanced non-abelian gauge symmetries together with fundamental charged matter hyper multiplets. We connect the appearance of matter to the local geometry of the singularity and discuss the relation between the instanton numbers of the Calabi–Yau manifolds taking part in the transition. In a dual heterotic string theory on $K3 \times T^2$ the process corresponds to Higgsing a semi-classical gauge group or equivalently to a variation of the gauge bundle. In special cases the situation reduces to simple conifold transitions in the Coulomb phase of the non-abelian gauge symmetries.

Email: berglund@nxth04.cern.ch, katz@math.okstate.edu, klemm@abel.math.harvard.edu, mayr@surya11.cern.ch
1. Introduction

During the last few years there has been a great deal of progress in the understanding of non-perturbative phenomena in supersymmetric field theories as well as in various string theories \[1\]. In particular the idea of duality has proven to be crucial. The basic point here is that one underlying theory might have several descriptions in terms of physical variables. In the current setting these ideas originated in the work by Seiberg and Witten in the context of \(N = 2\) supersymmetric Yang-Mills theory \[2\]. For all gauge groups the global prepotential can be derived from the periods of a suitable auxiliary Riemann-surface \[3\]. Combining the properties of \(N = 2\) gauge theories with the conjectured duality between \(N = 4\) string theories in four dimensions \[4\] it was suggested that also \(N = 2\) string theories in four dimensions have a duality structure; the heterotic string theory compactified on \(K3 \times T^2\) and the type II theory compactified on a Calabi-Yau manifold, could be dual to each other after including non-perturbative states \[5\][6].

In \[5\] concrete pairs of dual \(N = 2\) theories were constructed in which non-perturbative properties of the heterotic string can be investigated exactly. The key idea is the absence of neutral couplings between vector multiplets and hyper multiplets in \(N = 2\) theories \[7\]. As the heterotic dilaton sits in a vector multiplet the vector multiplet moduli space can receive space-time perturbative and non-perturbative corrections. Under \(N = 2\) string-string-duality it is identified with the vector multiplet moduli space of the Type IIA string, which corresponds to the Kähler moduli space of the compactified Calabi-Yau manifold. The latter has to have Hodge numbers \(h_{11} = N_v\) and \(h_{21} = N_h - 1\), where \(N_v, N_h\) are the number of vector and hyper multiplets and the \(-1\) corresponds to the the dilaton of the type II string. (Note however, that the rank of the gauge group is \(N_v + 1\) as the graviphoton contributes a \(U(1)\)-factor.) As the Type IIA dilaton is in a hyper multiplet the vector multiplet moduli space is exact at tree level and so is not corrected by space-time instantons. It is, however, corrected by worldsheet instantons. By mirror symmetry we can identify it with the hyper multiplet moduli space of the Type IIB theory on the mirror Calabi-Yau manifold, which receives neither world-sheet nor space-time corrections. The upshot is that the exact non-perturbative vector multiplet moduli space of the heterotic string is modelled by the complex structure moduli space of a specific Calabi-Yau manifold \[\text{1}\]. In this sense the complex moduli space of Calabi-Yau manifold replaces the complex moduli space of the auxiliary Riemann surface, which one had in the \(N = 2\) Yang-Mills field theory case.

Following the initial work \[5\][6] a number of consistency checks have been made further establishing the conjectured type II/heterotic string duality; comparison of the perturbative region of the potentials \[4\] of the heterotic couplings (gauge and gravitational) with

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1 Similarly, the structure of the non-perturbative hyper moduli space of the type IIA theory, which is a quaternionic manifold \[8\] can be investigated via the heterotic string.
that of the dual type IIA vacuum \cite{3}, \cite{10}, as well as non-perturbative consistency checks in the point-particle limit \cite{11}.

Furthermore, and what will be the focus of this paper, aspects of the perturbative enhancements of the gauge symmetry can be studied by considering the Picard lattice of the generic $K_3$-fiber \cite{12}. Using the Higgs mechanism as a way of lowering the rank of the gauge group and thus finding a way in which the various moduli spaces can be connected\cite{3} \cite{13}, chains of such transitions have been studied extensively \cite{3} \cite{14} \cite{15}.

Here we will focus on a particular chain and investigate in detail the transition from the point of view of the local geometry as well as in terms of the the worldsheet instanton sums in the type IIA theory. In particular, we will not only be able to identify the enhanced perturbative gauge symmetry but also the matter representations. The geometrical transition is of a slightly generalized type compared to the conifold transitions on one hand and the strong coupling transitions on the other hand. In addition to having N-1 divisors being contracted to a singular curve $C$, of genus zero, giving rise to an SU(N) theory with no adjoint hyper multiplets \cite{10}, there are singular fibers, which when contracted to $C$ give rise to massless hyper multiplets in the fundamental representation. The general structure of these transitions is discussed in section 2. We then turn to a specific set of models in section 3. This chain of Calabi-Yau manifolds can be viewed as $K_3$ fibrations over $P^1$ as well as elliptic fibrations over the Hirzebruch surface $F_2$. Finally, we end with conclusions and discussions in section 4.

2. Structure of the extremal transitions

2.1. Physical spectrum and flat directions

It will be useful to consider the purely field theoretic problem of which massless particles will appear in the moduli space. At a generic point in the moduli space of vacuum expectation values of the scalar component of the vector multiplet, the gauge symmetry is $U(1)^{N-1}$. By a suitable gauge transformation we can then write the above scalar in the diagonal form $diag(\phi_1, ..., \phi_N)$, where $\sum_{i=1}^{N} \phi_i = 0$ and $\phi_i \neq \phi_j$, for all $i, j$. Clearly, all the matter fields are massive with masses proportional to $\phi_i$. If we now set $\phi_i = 0$, $i = 1, ..., N - 1$, the gauge symmetry is enhanced to SU($N$) and only matter fields charged with respect to the SU($N$) become massless. Let us for simplicity assume that there are $M$ fields which all transform in the fundamental representation. Because of the tracelessness condition there exists a surface for which one can have $SU(N - 1) \times U(1)$,\footnote{This so called Higgs-branch has to be distinguished from the Higgs breaking mechanism into the Coulomb-branch by vector multiplets, in which the gauge bosons become massive as short vector multiplets under spontaneous generation of central charge, as in the Seiberg-Witten theory.}
with \( \phi_i = \phi_j \neq 0, i, j = 1, \ldots, N - 1 \) but without massless charged matter. If we reduce the symmetry enhancement one step further there is however room for massless matter in the fundamental of \( SU(N - 2) \); e.g. choose \( \phi_i = 0, i = 1, \ldots, N - 2 \) and \( \phi_{N - 1} = -\phi_N \neq 0 \). In general one will have several \( SU(k_j) \); however, only the one for which \( \phi_i = 0, i = 1, \ldots, k_j \) will have \( M \) massless matter multiplets. In particular, there exists a codimension one surface in which only e.g. \( \phi_1 = 0 \), such that there is no gauge enhancement. However, because of \( \phi_1 = 0 \) we will have \( M \) massless singlets. It is very gratifying that, as will now be seen, the Calabi-Yau moduli space exactly reproduces this kind of behaviour.

2.2. Local geometry of the Calabi–Yau singularity

During the recent developments it has become clear that the physical singularities associated to massless solitonic BPS states are essentially encoded in the geometry of the singularity of the compactified manifold. The role played by the geometry can be understood from the interpretation of the massless states as solitonic p-branes wrapped around the vanishing cycles of the singularity; the gauge and Lorentz quantum numbers depend then on some characteristic properties of the homology cycles, in particular their dimension, topology and intersection numbers \[16\].

The simplest case (in the type IIB picture) is that of a vanishing three-cycle leading to a massless hyper multiplet, the case considered originally by Strominger \[17\]. On the other hand, if the three-cycle shrinks to a curve, rather than to a point, one obtains enhanced gauge symmetries, as has been argued in \[18\]. More precisely, if the local geometry is that of an ALE space with \( A_N \) singularity over a curve of genus \( g \) one obtains an enhanced \( SU(N + 1) \) gauge symmetry \[19\] together with \( g \) hyper multiplets in the adjoint representation \[16\]. The case \( g = 0 \) is exceptional in that the enhanced gauge symmetry is asymptotically free and broken to its abelian factor due to strong coupling effects in the infrared; this case has been considered in \[11\] \[20\].

Let us now assume that we have a collection of curves \( C_i \) with the transverse space that of an ALE-manifold with \( A_{N_i} \) type singularity and consider further a point of intersection between two of these curves \[21\]. The singularity structure of the transverse space has been analyzed in detail in \[22\] in the context of elliptic fibrations with the following result: if along the curves \( C_1 \) and \( C_2 \) the elliptic fiber is of Kodaira type \( I_{N_1} \) and \( I_{N_2} \) respectively, then above the point where \( C_1 \) intersects \( C_2 \) the elliptic fiber is of type \( I_{N_1 + N_2} \). Indeed the examples we will consider are all elliptically fibered Calabi–Yau manifolds. This allows in particular for a simple interpretation in terms of 5-branes \[18\] \[16\] located at the points where the fibration becomes singular. However this special structure is not necessary; the general configuration is that of a collection of curves \( C_i \) with transverse \( A_{N_i} \) singularities colliding in a set of \( M \) points over which the singularity structure jumps to \( A_{N_i + N_j} \).

A simple D-brane arrangement based on a collection of \( N_i \) coinciding D-branes intersecting a second collection of \( N_j \) coinciding D-branes has been given in \[18\]. In this
picture additional matter in the fundamental representation arises from open string states with one end attached to the first and the other end to the second collection. There is a special configuration in which one of the two collections consists of a single D-brane only, say $N_j = 1$. In this case one expects a single non-abelian factor of $SU(N_i)$ together with matter in the fundamental representation.

This is the physical situation whose realization we will consider in the context of Calabi–Yau compactifications. Specifically the case of $SU(N+1)$ gauge symmetry with $M$ fundamental matter multiplets arises from the following local data: there is a curve $C$, which in our case will be a $P^1$, over which one has a bundle structure where the generic fiber is a Hirzebruch-Jung tree of the resolution of the $A_N$ singularity, that is a collection of $P^1$'s, $E_i$, $i = 1..N$ with intersection matrix proportional to the Cartan matrix of $A_N$. In addition, above the $M$ exceptional points, the singularity structure of the fiber becomes $A_{N+1}$ because one component of the fiber factorizes. More precisely out of the $N$ generic components $N - 1$ are toric divisors $D_i$ of the manifold $X_i$ which are ruled surfaces, while the $N$-th component, $\hat{E}$ is only birationally ruled, having $M$ degenerate fibers. The last component $\hat{E}$ is actually a conic bundle, which means that the fibers are all plane conics. These conics are smooth over a generic points of $C$ while they split into line pairs over the $M$ exceptional points.

Let us describe now how the appearance of this structure will lead to geometrical transitions between two Calabi–Yau manifolds $X_i$ and $X_{i-1}$ (where $i$ denotes the number of $N_V = h_{11}$ of vector multiplets). First we can contract the $N - 1$ divisors together with the $M$ degenerate fibers of the conic bundle. According to our previous discussion, the degenerate fibers contain a collection of $N + 1$ rational curves $E_i$ with intersection matrix of $A_{N+1}$ which are all contracted. $N$ of them are again associated to the Cartan subalgebra of the gauge group while the additional one is related to the matter hyper multiplets; there is a natural action of the $A_N$ Weyl group on this class which generates the components of the fundamental representation. The gauge quantum numbers of the solitonic p-brane states are determined by the reduction of the Cartan matrix of $A_{N+1}$ to that of $A_N$; the components of the matter fields arise naturally from wrappings of the cycles $E_{N+1} - E_i$, $i = 0,\ldots,N$; however there is no independent modulus associated to the volume of $E_{N+1}$ and correspondingly no additional vector multiplet. This is expected from the fact that the additional rational curves are isolated rather than being fibers of a ruled surface. After the $N$ surfaces have been contracted to the base the singularities can be simplified by a deformation of the complex structure in such a way that the resulting singularity is the contraction of $N - 1$ surfaces arising from $X_{i-1}$. This completes the transition from $X_i$ to $X_{i-1}$ at the enhanced symmetry points.

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3 Here $E_0$ is the homology cycle which fulfills the relation $\sum_{i=0}^{N} E_i \sim 0$, reflecting the tracelessness condition of $SU(N + 1)$, as discussed in the previous section.
Clearly, in the generic situation one will only contract subsets $S_i$ of the $N - 1$ divisors and/or the conic bundle with the $M$ degenerate fibers. The result will depend on whether a given subset $S_i$ contains $\hat{E}$; if it does we will get from that factor an enhanced gauge symmetry $SU(k_i + 1)$ together with $M$ fundamental representations; however if it does not, the result is that of a non-abelian gauge symmetry without matter, broken down to the Cartan subalgebra by strong infrared dynamics. In fact if $N > 1$ we can contract only $\hat{E}$ and the result is $M$ representations of $U(1)$, that is we are back to the familiar case of the conifold singularity.

2.3. String moduli space

We turn now to a discussion of the string moduli spaces involved in the transition. The latter is described by a motion in the vector multiplet moduli $M$ space of the Calabi–Yau manifold $X_i$ to a locus where the vev of the scalar superpartner of a vector field vanishes and then turning on vevs in the new flat directions of the Higgs branch corresponding to a motion in the hyper multiplet moduli space of the Calabi–Yau manifold $X_{i-1}$; the new Calabi-Yau manifold $X_{i-1}$ will therefore have fewer vector moduli while the number of hyper moduli has increased; the associated change in the Hodge numbers $h_{11}$ and $h_{12}$ indicates that the two manifolds are of different topological type. There are two types of natural coordinates on $M$, the algebraic coordinates $z_n$ and the special coordinates $t_n$ [2], where the two are related by the mirrormaps $z_m(t_n)$. We are interested in the relations between these two types of coordinates on $M^i$ and $M^{i-1}$.

$N = 2$ supersymmetry puts strong restrictions on these relations; in particular the special geometry of the vector multiplets in the type IIA compactification constrains the map between the two set of coordinates $t_n^i$ and $t_n^{i-1}$ to be linear. Moreover we will find simple relations between the Gromov–Witten invariants $n_{i_1,\ldots,i_{h_{11}}}$ on $M^i$ and $M^{i-1}$, which are defined in terms of the instanton corrected Yukawa couplings $y_{abc}$ as

$$y_{abc} = y_{abc}^0 + \sum_{d_1,\ldots,d_{h_{11}}} \frac{n_{d_1,\ldots,d_{h_{11}}}^r d_a d_b d_c}{1 - \prod_{n=1}^{h_{11}} q_n^{d_n}} \prod_{n=1}^{h_{11}} q_n^{d_n},$$

where the $n_{d_1,\ldots,d_{h_{11}}}^r$ is the virtual fundamental class of the moduli space of rational curves of multidegree $d_1, \ldots, d_{h_{11}}$. Such a relation between the instanton numbers are of course a special property of the type of singularity we consider and will not be present in other types of transitions proceeding e.g. through non-canonical singularities.

In a way similar to that the special geometry of the vector moduli space restricts the relations between the coordinates on $M^i_V$ and $M^{i-1}_V$, the quaternionic structure of the hyper moduli space of the type IIB theory compactified on the same pair of manifold implies simple relations between the coordinates on the hyper multiplet moduli spaces, $\xi_n$. They are related to the algebraic moduli $z_n$ by rational functions which in turn depend on
the special representation of the Calabi–Yau manifolds. In particular there are in general
different reflexive polyhedra describing the same Calabi–Yau space, however in different
algebraic coordinates related by rational transformations. It is convenient to choose a
preferred representation in which the relation between the algebraic moduli on $\mathcal{M}^i$ and
$\mathcal{M}^{i-1}$ becomes particularly simple:

$$z_{n}^{i-1} = \prod_{m} (z_{m}^{i})^{\delta_{m}^{n}}$$

From the definition of the algebraic coordinates this relation translates to linear relations
between the Mori vectors $l_{n}^{i}$, $l_{n}^{i-1}$, generating the dual of the Kähler cone. As a consequence
the smaller dual polyhedron $\Delta_{i-1}^{*}$ is obtained from the larger one $\Delta_{i}^{*}$ by omitting one of
the vertices 4.

A new feature of the corresponding transition in the moduli space is that, rather than
being located at the zero locus of the principal discriminant or a restricted discriminant,
it can take place at one of the boundary divisors $z_{n} = 0$. Such transitions where first
discussed in [25]; indeed this is the situation in the examples described in section 3 5.
Similar examples have been considered independently [27].

In the case when the Calabi–Yau is a K3 fibration [28], the vector moduli space of the
pair of Calabi–Yau manifolds can be mapped to that of the dual heterotic theory, the map
again being linear by the above mentioned argument. Of course this relation will depend
on the special compactification under consideration and will be discussed further in the
examples.

3. Global description of the Calabi-Yau spaces and examples

Given a four-dimensional $\text{N}=2$ heterotic vacuum it is far from trivial to find the dual
type IIA model. Although there has been some progress recently as far as understanding
the perturbative gauge structure on the type IIA side [12][29][26] there is still some guess
work to be done in terms of finding a Calabi-Yau manifold that will fit the bill. In fact,
recent work [30] indicates that there will in general exist more than one type IIA theory
which agrees with the perturbative heterotic vacuum under consideration. From now on,
we will assume that there is a set of models which to lowest order corresponds to their
counterparts in the heterotic chain, i.e. the number of vector multiplets and singlet hyper
multiplets agree, and the models are all K3-fibrations.

4 This kind of transition was originally studied in [25] and more recently in [26].
5 A model with similar properties has been observed in [25].
3.1. A chain of connected Calabi–Yau manifolds

We will here consider an example first discussed by Aldazabal et. al [14]. Let us first briefly review the structure of the chain of heterotic vacua with the least number of vector multiplets, as it is the most suitable for the consideration in this paper; for more details see [14]. Starting from a $Z_6$ orbifold with a specific embedding in $E_8 \times E_8$ it is possible to Higgs away most of the gauge group, leaving just $SU(5) \times U(1)^4$. The spectrum consists of $4 \ 10$, $22 \ 5$ and $118 \ 1$ hyper multiplets. In fact one can further Higgs the $SU(5)$, giving a set of models with the following spectra. (In what follows we will refer to the models by the number of vector multiplets and the number of singlet hyper multiplets.)

\[
\begin{array}{|c|c|c|c|}
\hline
(N_v + 1, N_h) & \text{gauge group} & \text{spectrum} & \text{Calabi – Yau} \\
\hline
(8, 118) & SU(5) \times U(1)^4 & 4 \ 10 + 22 \ 5 + 118 \ 1 & (P^5(1, 1, 2, 5, 9)[14, 11])^{7,17,18}_{220} \\
(7, 139) & SU(4) \times U(1)^4 & 4 \ 6 + 24 \ 4 + 139 \ 1 & (P^5(1, 1, 2, 6, 9, 10)[16, 12])^{6,13,17}_{264} \\
(6, 162) & SU(3) \times U(1)^4 & 30 \ 3 + 162 \ 1 & (P^4(1, 1, 2, 6, 8)[18])^{5,161}_{312} \\
(5, 191) & SU(2) \times U(1)^4 & 28 \ 2 + 191 \ 1 & (P^4(1, 1, 2, 6, 10)[20])^{4,190}_{372} \\
(4, 244) & U(1)^4 & 244 \ 1 & (P^4(1, 1, 2, 8, 12)[24])^{3,243}_{480} \\
\hline
\end{array}
\]

Table 3.1: Chain of heterotic - type IIA duals

We have analyzed the matter structure and transitions for most of these models. Rather than give this analysis for the hypersurfaces and complete intersections in weighted projective spaces here, we will find it more convenient to report on the transition using Batyrev’s construction of Calabi-Yau mirror pairs $(X_n, X_n^*)$ as hypersurfaces in more general toric varieties $(P_{\Delta_n}, P_{\Delta_n}^*)$ [32]. An exhaustive list of polyhedra $(\Delta_n, \Delta_n^*)$ for the models in [14] can be found in [26]. In general there will be several different candidates for $(X_n, X_n^*)$, differing presumably only by birational maps.

3.2. Phases of the Kähler moduli space

Given a singular ambient space $P_{\Delta}$, we have in general many phases in the associated extended Kähler moduli space of the nonsingular space $\tilde{P}_\Delta$. They correspond to the different ways to resolve $P_{\Delta}$ and are defined by the different regular triangulations of

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*An alternate way of constructing this chain of models starts from the asymmetric instanton embedding $(10, 14)$ which breaks $E_8 \times E_8$ to $E_7 \times E_7$, completely Higgsing one of the $E_7$ and the second $E_7$ to $SU(5)$ [31].

*We will indicate the number of vector multiplets of the models by the subscript $n$. 

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the polytope $\Delta^*$. If $(\Delta, \Delta^*)$ are reflexive there is a canonical way\textsuperscript{8} to embed a Calabi-hypersurface $(X, X^*)$ in $(\hat{P}_{\Delta}, \hat{P}_{\Delta^*})$ \textsuperscript{32}. Among the phases of $\hat{P}_{\Delta}$ are the ones that give rise to Calabi-Yau varieties, when the Kähler classes are restricted to $X$. They correspond to triangulations\textsuperscript{4} involving all points of $\Delta^*$ on dimension 0, 1, 2-faces. Even restricting to phases which correspond to manifolds which are K3-fibrations does not in general narrow down the choice to a unique model. Furthermore depending on the particular situation at hand, it may be the case that there exists more than one type IIA vacuum to a given heterotic theory. Finally, there is a technical problem in finding the true Calabi-Yau phases. Frequently the Kähler cones of $\hat{X}$ at hand, it may be the case that there exists more than one type IIA vacuum to a given heterotic theory. Finally, there is a technical problem in finding the true Calabi-Yau phases. Frequently the Kähler cones of $\hat{P}_{\Delta}$ are narrower than the Kähler cone of $X$, because the former are bounded by curves in $\hat{P}_{\Delta}$ which vanish on $X$. We describe in Appendix A how to deal with this situation.

3.3. The vector moduli space of our examples

We will now use mirror symmetry and toric geometry to investigate the vector moduli space for the models in the chain described in the previous section. Candidates of type II models were constructed as chains of nested polyhedra $\Delta^*_1 \subset \ldots \subset \Delta^*_9$ in \textsuperscript{26}. For simplicity, we now turn to studying the extremal transitions connecting the three models with the fewest number of vector multiplets discussed above. By extremal we refer to a general transition obtained by contracting curves corresponding to edges of the Mori cone, and then deforming the resulting singular Calabi-Yau threefold to get a smooth Calabi-Yau manifold. As will be shown this transition is not necessarily of the simple conifold type \textsuperscript{38}.

We first have to calculate one valid Mori cone for each of the models\textsuperscript{8}. Let us therefore start by considering $X_5$; as we will show, $X_4$ and $X_3$ can then be obtained by taking a particular limit in the Kähler moduli space of $X_5$. Inside the polyhedron $\Delta^*_5$ one has the following relevant points (inside dimension 0, 1, 2, 4-faces) \textsuperscript{26}

$$
\begin{align*}
\nu^0 &= (0, 0, 0, 0), \\
\nu^1 &= (-1, 0, 2, 3), \\
\nu^2 &= (0, 0, -1, 0), \\
\nu^3 &= (0, 0, 0, -1), \\
\nu^4 &= (0, 0, 2, 3), \\
\nu^5 &= (0, 1, 2, 3), \\
\nu^6 &= (1, 2, 2, 3), \\
\nu^7 &= (0, -1, 2, 3), \\
\nu^8 &= (0, -1, 1, 2), \\
\nu^9 &= (0, -1, 1, 1).
\end{align*}
$$

The Mori cone can be found by repeated application of the procedure described in Appendix A. This leads to the following set of Mori generators

$$
\begin{align*}
l^{(1)} &= (-1; 0, 0, 0, 1, 0, 0, -2, 1, 1), \\
l^{(2)} &= (0; 1, 0, 0, 0, -2, 1, 0, 0, 0), \\
l^{(3)} &= (0; 0, 0, 0, -2, 1, 0, 1, 0, 0), \\
l^{(4)} &= (-1; 0, 1, 0, 0, 0, 0, 1, -2, 1), \\
l^{(5)} &= (-1; 0, 0, 0, 1, 0, 0, 0, 1, -1)
\end{align*}
$$

\textsuperscript{8} Even if we fix the triangulation of $\Delta^*$ this does not fix $(X_n, X^*_n)$ uniquely. A simple counterexample with a non-toric phase is the $X_9(3, 2, 2, 1, 1)^{2, 85}$ case discussed in \textsuperscript{25} \textsuperscript{26} \textsuperscript{30}.

\textsuperscript{9} We have used PUNTOS \textsuperscript{34} to find the triangulations

\textsuperscript{10} We content ourselves with those phases arising from the triangulations of $\Delta^*$. 

8
The mirror manifold of $X_5$ is given by the Laurent polynomial 

$$P = \sum_{i=0}^{9} \prod_{j=1}^{4} a_i X_i^{\nu_j} = 0$$  \hspace{1cm} (3.3)$$

in $P_{\Delta^*_5}$. A crucial insight [32, 36, 37], that the large complex structure variables of the mirror $X_5^*$ are defined by the Mori cone of $X$, specifically for $X_5$ above:

$$w_i = (-1)^l \prod_{j=1}^{9} a_j^{l(i)}.$$  \hspace{1cm} (3.4)$$

By the construction of [26] the models $X_4, X_3$ are given as hypersurfaces in toric varieties whose dual polytopes, $\Delta^*_{4,3}$ respectively are obtained by deleting the point $\nu_9$ or points $(\nu_8, \nu_9)$ from $\Delta^*_5$; this corresponds to the restriction of the moduli space of $X_5$ to $a_9 = 0$ and $a_9 = a_8 = 0$ respectively.

| Table 3.2: The Calabi-Yau manifolds which correspond to reflexive polyhedra inside $\Delta^*_5$. The polyhedra are specified as convex hulls of the points given in (3.1). Furthermore, we list the number of Kähler $h^{1,1}$ and complex structure deformations $h^{2,1}$ of $X_i$ (the number of non-algebraic deformations is indicated in parentheses) as well as the vanishing coefficients in the Laurent polynomial (3.3) and the canonical large complex structure coordinates $z_k$ of $X_i^*$, which are related to the Mori cones by (3.4). |
|----------------|----------------|----------------|
| $P$            | $X_3$           | $X_4$           | $X_5$           |
| $a_9 = a_9 = 0$| $a_8 = a_9 = 0$ | $a_8 = 0$       |
| $\Delta^*$     | conv($\nu_1, \ldots, \nu_7$) | conv($\nu_1, \ldots, \nu_8$) | conv($\nu_1, \ldots, \nu_9$) |
| $h^{1,1}$       | $3(0)$          | $4(0)$          | $5(0)$          |
| $h^{2,1}$       | $243(1)$        | $190(1)$        | $161(1)$        |
| $z_1$           | $w_1 w_4 w_5^3$ | $w_1 w_5$       | $w_1$           |
| $z_2$           | $w_2$           | $w_2$           | $w_2$           |
| $z_3$           | $w_3$           | $w_3$           | $w_3$           |
| $z_4$           | $w_4 w_5$       | $w_4$           | $w_4$           |
| $z_5$           | $-$             | $-$             | $w_5$           |

Using toric geometry one can calculate the classical intersections corresponding to the Kähler classes, $J_i$, which are dual to the Mori generators (3.2),

$$8J_1^3 + 2J_1^2 J_2 + 4J_1^2 J_3 + J_1 J_2 J_3 + 2J_1 J_3^2 +$$

$$16J_1^2 J_4 + 4J_1 J_2 J_4 + 8J_1 J_3 J_4 + 2J_2 J_3 J_4 + 4J_2^2 J_4 + 24J_1 J_4^2 + 6J_2 J_4^2 + 12J_3 J_4^2 + 36J_4^3 +$$

$$48J_5^2 J_1 + 24J_5 J_1^2 + 36J_5 J_4 J_1 + 6J_5 J_2 J_1 + 12J_5 J_3 J_1 + 6J_5 J_3^2 J_1 + 18J_5 J_4 J_3 +$$

$$3J_5 J_3 J_2 + 9J_5 J_4 J_2 + 12J_5^2 J_2 + 54J_5 J_4^2 + 24J_5^2 J_3 + 72J_5^2 J_4 + 96J_5^3$$  \hspace{1cm} (3.5)$$
as well as the evaluation of the Chern class on the $(1,1)$ forms $J_i$,

$$c_2(J_1) = 92, \quad c_2(J_2) = 24, \quad c_2(J_3) = 48, \quad c_2(J_4) = 132, \quad c_2(J_5) = 168 \quad (3.6)$$

Obviously the intersection numbers of $X_4$ and $X_3$ are simply given from these by the restriction to the first four respectively three Kähler classes. For further studying the transition we also need the Gromov-Witten invariant for the rational curves. These are obtained from the solutions of the Picard-Fuchs equations using the mirror hypothesis and listed in Appendix B.

### 3.4. Local geometry and summation of the instanton corrections

Let us begin with $X_3$, and identify the associated toric variety $P_3$. A partial list on the number of rational curves of low degree, including those of importance for studying the transitions described in this paper, can be found in table A.1.

We will now show that this model is primitive, i.e. it does not admit a geometric transition to a model with fewer Kähler parameters. Let us therefore discuss the edges of the Mori cone one at a time, to determine whether their contraction admits a birational smooth deformation. We start by studying the first edge of the Mori cone. This edge describes curves contained in an elliptic fibration over a surface, so the contraction of these curves is not a birational map (note that $n_{1,0,0} = 480$, the negative of the Euler characteristic, as explained in [25]).

For our purposes, it is best to think of $F_2$ as a complete toric variety with edges $(-1,-2), (1,0), (0,1), (0,-1)$. Its Mori cone is given by $(-2,0,1,0,1), (0,1,-2,1,0)$. As such, it can be thought of as $\mathbb{C}^4 - \{x_1 = x_6 = 0\} \cup \{x_5 = x_7 = 0\}$ (in terms of the $x_i$ of $P_3$), modulo the $(\mathbb{C}^*)^2$ identification

$$(x_1, x_5, x_6, x_7) \sim (tx_1, st^{-2}x_5, tx_6, sx_7). \quad (3.7)$$

The hyperplane class of $F_2$ is the toric divisor $x_7 = 0$, and will be denoted by $H$. We can also think of $F_2$ as the minimal desingularization of $\mathbb{P}(1,1,2)$, so it makes sense to talk of the degree of a curve on $F_2$. Note that a curve in the class $dH$ has degree $2d$. In passing, we note that the exceptional divisor of this blowup, $x_5 = 0$, is the section of self-intersection $-2$.

---

11 This model has been studied earlier [38][12].

12 In this section, we will frequently perform intersection calculations in the Calabi-Yau manifolds $X_k$. These calculations are often inferred from the Mori cone; at times they may also be performed by Schubert [39].
Returning to the first edge of the Mori cone, we note that the curve is contracted by $D_1, D_5, D_6, D_7$. The relations $D_1 \cdot D_6 = 0$ and $D_5 \cdot D_7 = 0$, together with the $G^*$ actions defining the toric variety, show that there is a map $X_3 \to F_2$. The fibers are elliptic curves with typical equation $x_3^2 + x_6^3 + x_2 f_{16} + f_{24} = 0$, where the $f_i$ have degree $i$ in the variables $x_1, x_6, x_7$ of $\mathbb{P}(1,1,2)$, where $x_7$ is the variable of degree 2.

The divisor $D_5$ describes a ruled surface over an elliptic curve. This is the second edge of the Mori cone. In this situation, the Gromov-Witten invariant is $2g - 2 = 0$. There is no extremal transition in this case, although there is an $SU(2)$ gauge symmetry that is broken after a non-polynomial deformation.

Now we turn our attention to the divisor $D_4$. The $K3$ fibration defined by $(x_1, x_6)$ restricts to $D_4$ to describe $D_4$ as a ruled surface over a genus 0 curve; the Gromov-Witten invariant is $n_{0,0,1} = 2g - 2 = -2$, and there is no transition.

In summary, the Mori cone of $X_3$ coincides with that of $P_3$, and this model is primitive, i.e. does not admit a geometric transition to a model with fewer Kähler parameters. This checks against the heterotic side, where the model with $(n_H, n_V) = (244, 4)$ is at the bottom of the chain.

We next turn to the 4 parameter model $X_4$. Some of the instanton numbers for this model appear in table A.2. We will see that the fourth edge of the Mori cone is represented by a conic bundle containing 28 line pairs, and that after contraction, there is a transition to $X_3$. From this transition, the Mori cone of $X_3$ is the quotient of the Mori cone of $X_4$ after modding out by the edge $(0,0,0,1)$. It remains to match up the edges from the above geometry. The edges $(0,1,0,0)$ and $(0,1,0)$ correspond to the ruled surface over the elliptic curve, so are to be identified. The elliptic fibration identifies $(1,0,0,2)$ with $(1,0,0)$; or equivalently identifies $(1,0,0,0)$ with $(1,0,0)$ due to the quotient. Finally, the remaining edges $(0,0,1,0)$ and $(0,0,1)$ are identified as ruled surfaces over rational curves. From this, we infer the relation

$$n_{a,b,c} = \sum_k n_{a,b,c,k}$$

which checks against the instanton numbers that we have provided, for example $n_{1,0,0} = -2 + 56 + 372 + 56 + -2 = 480$.

---

13 Throughout our discussion, we will denote by $D_k$ the restriction to $X_i$ of the toric divisor with equation $x_k = 0$, when the model under discussion is clear from context.

14 In [41][42] it was shown that turning on the non-polynomial deformation connects the moduli space for $X_3$ defined as an elliptic fibration over $F_2$ (our case) with that of an elliptic fibration over $F_0$.

15 Recent relevant geometric results about primitive Calabi-Yau threefolds have been given in [43].
As we saw for $X_3$, the model $X_4$ also admits a map $\pi : X_4 \to F_2$ defined by $(x_1, x_5, x_6, x_7 x_8)$. The fibers have type $(1, 0, 0, 2)$ and are again elliptic. To calculate this type, note that since $x_1 = x_5 = 0$ defines a point of $F_2$, the same equation defines an elliptic fiber of $X_4$. We accordingly calculate the intersection numbers $D_1 \cdot D_5 \cdot D_k$ for $1 \leq k \leq 8$, obtaining the Mori vector $(-6; 0, 2, 3, 1, 0, 0, 0, 0)$, where the $-6$ arises because the coordinates must sum to 0. In our basis for the Mori cone given in (A.5), this is just $(1, 0, 0, 2)$. Throughout this section, other classes have been computed in this manner; the classes will be given without further comment. A curve $C$ of type $(0, 0, 0, 1)$ is also contracted by $\pi$. Since $C \cdot D_8 = -1$, we see that $C$ is necessarily contained in $D_8$. Since $C \cdot D_7 = 1$, there is a unique fiber of $D_7$, a curve of type $(1, 0, 0, 0)$ which meets $C$. This curve is also contracted by $\pi$. The fiber of $\pi$ containing both of these curves contains a third component of type $(0, 0, 0, 1)$; thus there are two curves of type $(0, 0, 0, 1)$ in the same fiber.

Because of $C \subset D_8$, we restrict attention to $D_8$, which admits a map to $\mathbb{P}^1$ by restricting the K3 fibration defined by $(x_1, x_6)$. After restricting to $D_8$ (hence putting $x_8 = 0$), the equation of $X_4$ becomes $x_2^2 f_8 + x_3^2 + x_7^2 f_16 + x_2 x_3 f_4 + x_2 x_7 f_{12} + x_3 x_7 f_8 = 0$ where the $f_i$ have degree $i$ in the variables $x_1, x_6$ of $\mathbb{P}^1$. We interpret the above as a family of conics in the $\mathbb{P}^2$ with coordinates $(x_3, x_4, x_7)$, with $(x_1, x_6) \in \mathbb{P}^1$ as a parameter. The discriminant of this family has degree 24. Thus the general fiber of $D_8$ is a smooth conic of type $(0, 0, 0, 2)$, while there are 28 fibers where the conic splits into line pairs, each of type $(0, 0, 0, 1)$. As a check, note that $n_{0,0,0,2} = -2$ and $n_{0,0,0,1} = 56$.

The transition to $X_3$ is found by writing down monomials in the variables of $P_4$ which have intersection number 0 with $(0, 0, 0, 1)$. Choosing them in the order $(x_1, x_2 x_8, x_3 x_8, x_4, x_5, x_6, x_7 x_8)$, we see that the assignment

$$(x_1, \ldots, x_8) \mapsto (x_1, x_2 x_8, x_3 x_8, x_4, x_5, x_6, x_7 x_8)$$

defines a mapping from $X_4$ to $P_3$ which contracts the conic bundle, and takes $X_4$ to a singular form of $X_3$. The transition is produced simply by deforming the equation of $X_3$.

As a final comment on this model, we observe that the class of the elliptic fiber is $(1, 0, 0, 2)$. We have seen that the fiber of $D_7$ is of type $(1, 0, 0, 0)$, while the fiber of $D_8$ is of type $(0, 0, 0, 2)$. The intersection of $D_7$ and $D_8$ meets either fiber in two points. Thus $D_7 \cup D_8$ is a fibration over $\mathbb{P}^1$ whose general fiber is a union of two $\mathbb{P}^1$s intersecting in two points, while there are 28 special fibers which form triangles of curves. By itself, $D_8$ is contracted to get $X_3$, and this is the case $N = 2, M = 28$ of the geometry described in Section 2.2. We accordingly expect to see $28 \cdot 2$ hyper multiplets becoming massless at the transition, and that is in perfect agreement with Table 3.1.

Finally, we turn to the 5 parameter model $X_5$. A partial list on low degree instanton numbers, including those of importance for studying the transitions described in this paper can be found in Table A.3.
The key to understanding this transition is the contraction of the curves of type $(0, 0, 0, 0, 1)$. From table A.3 we have $n_{0,0,0,0,1} = 30$. This class has Mori vector $(-1, 0, 0, 1, 0, 0, 0, 0, 1, -1)$, so we see that this curve is contained in $D_9 = 0$ and is contracted by the divisors $D_1, D_2, D_4, D_5, D_6, D_7$. We also note from the first entry of the Mori generators that the equation of $X_5$ is in the class $J_1 + J_4 + J_5$, where the $J_k$ are the dual generators of the Kähler cone. We accordingly write the equation of $X_5$ in the form

$$x_3f + x_8g,$$

where $f$ is a polynomial with cohomology class $J_1 + J_4$ and $g$ is a polynomial with cohomology class $3J_4$. A curve is contracted by the divisors listed above if and only if $f = g = x_9 = 0$. We calculate $D_9 \cdot (J_1 + J_4) \cdot 3J_4 = 30$. Thus $n_{0,0,0,0,1} = 30$. The transition to $X_4$ is now visible—$X_4$ is obtained from $X_5$ by the map $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$.

The Mori cone of $X_4$ is thus the quotient of the Mori cone of $X_5$ by the vector $(0, 0, 0, 0, 1)$. We now match up the other edges. The edge $(0, 1, 0, 0, 0)$ is the fiber of a ruled surface over an elliptic curve, so corresponds to $(0, 1, 0, 0)$. There is again an elliptic fibration over $\mathbb{P}(1, 1, 2)$; we calculate that the fiber has class $(1, 0, 0, 2, 3)$, which is equivalent to $(1, 0, 0, 2, 0)$ under the quotient. This must match with the elliptic class $(1, 0, 0, 2)$ of $X_4$. This implies that $(0, 0, 0, 1, 0)$ corresponds to $(0, 0, 0, 1)$, and $(1, 0, 0, 0, 0)$ corresponds to $(1, 0, 0, 0)$. The remaining edges $(0, 0, 1, 0, 0)$ and $(0, 0, 1, 0)$ are therefore related; they are fibers of ruled surfaces over rational curves.

This gives the formula

$$n_{a,b,c,d} = \sum_k n_{a,c,b,d,k}$$

which checks against the instanton numbers that we have provided, for example $n_{1,0,0,0} = -2 + 30 + 30 - 2 = 56$.

As a final comment on this model, we observe that the class of the elliptic fiber is $(1, 0, 0, 2, 3)$. We have seen that the fiber of $D_7$ is of type $(1, 0, 0, 0, 0)$, while the fiber of $D_8$ is of type $(0, 0, 0, 1, 0)$. We can also calculate that the fiber of $D_9$ has class $(0, 0, 0, 1, 3)$. The curves of type $(0, 0, 0, 0, 1)$ are contained in degenerate fibers of $D_9$; unlike the $X_4$ situation, the other component of this fiber is of a different type $(0, 0, 0, 1, 2)$. The $(0, 0, 0, 0, 1)$ curve meets $D_8$ but not $D_7$, while the $(0, 0, 0, 1, 2)$ curve meets $D_7$ but not $D_8$. Thus $D_7 \cup D_8 \cup D_9$ is a degenerate elliptic fibration over $\mathbb{P}^1$ whose general fiber is a triangle. There are 30 special fibers where there is an extra component, and the fiber is a square of curves. Together, $D_8 \cup D_9$ contract to give $X_3$ (note that if we contract $(0, 0, 0, 0, 1)$ together with $D_8$ to get $X_3$, then we are contracting $(0, 0, 0, 1, 0)$, hence the entire fibration $D_9$ with class $(0, 0, 0, 1, 0) + 3(0, 0, 0, 0, 1))$. This is the case $N = 3$, $M = 30$ of the geometry described in Section 2.2. We accordingly expect to see 30 3 hyper multiplets becoming massless at the transition, and that is in perfect agreement with Table 3.1.
We can also explicitly see for the $X_5$ to $X_4$ transition how the sets of three curves change to sets of two curves as we go to $X_4$. After contracting $(0, 0, 0, 0, 1)$ to get to $X_4$, the two curves $(0, 0, 0, 1, 0)$ and $(0, 0, 0, 1, 2)$ pair up to become 30 pairs of $(0, 0, 0, 1)$ curves. In fact, it can be shown that the curve $(0, 0, 0, 1, 2)$ is the “partner” of $(0, 0, 0, 0, 1)$ under the conic bundle. As the conic bundle gets smoothed out by the deformation process, there are fewer lines pairs left.

We expect similar phenomena to arise for $X_6$ and $X_7$. For example, using the $\mathbb{P}^5(1, 1, 2, 6, 8, 10)[16, 12]$ model for $X_6$, we have checked that the transition to the $\mathbb{P}^4(1, 1, 2, 6, 8)[18]$ model for $X_5$ occurs by projection on the first 5 coordinates, and there are $M = 24$ exceptional curves lying on a birationally ruled surface.

3.5. Physical interpretation of the examples

Let us now try to understand the above described transitions in a physical context. We start with the $X_4$ model. At the codimension one surface where the conic bundle is contracted we have a singular $\mathbb{P}^1$ of type $A_1$ with 28 double points of type $A_2$. As we will now argue, this corresponds to an infrared free theory $SU(2)$ gauge theory with 28 2. First, it has been shown in [19] [16] that a $\mathbb{P}^1$ bundle over a curve with singularity type $A_1$ gives an enhanced $SU(2)$ gauge symmetry in the type IIA string theory. If the base curve of the family is rational as in our case, it is also shown that there is no new matter [16]. What we find here is that the contraction of the isolated curves, corresponding to solitonic 2-branes wrapped around the curves becoming massless, gives rise to non-abelian charged matter. The non-abelian charges arise from the fact that these isolated curves originate from the same conic bundle as does the continuous family of rational curves leading to the non-abelian gauge bosons. This is the non-abelian generalization of the conifold singularity. As the $SU(2)$ is Higgsed, the rank of the gauge group is reduced by one, while the number of hyper multiplets increase by 53. This is seen very nicely from the expression of the instanton numbers (3.8). Note how the fact that $n_{0,0,0,2} = -2$ fits with losing two of the hyper multiplets as the $W^\pm$ of the $SU(2)$ becoming massive. There is of course as usual one hyper multiplet which is “eaten” as the $U(1)$ gauge boson of the Cartan subalgebra of $SU(2)$ become massive. Note how this exactly matches the heterotic description. At the transition from $(191, 5)$ to $(244, 4)$, the gauge group is $SU(2) \times U(1)^4$, with 28 2 hyper multiplets under the $SU(2)$. The transition occurs by Higgsing the $SU(2)$.

In fact we can give a quite explicit map of the transition of the type II theory to that on the heterotic side by analyzing the physical quantities such as the mirror maps, discriminants and periods which determine the $N = 2$ effective action. The relation between the Mori generators

$$l_1^{(3)} = l_1^{(4)} + 2 l_4^{(4)}, \quad l_2^{(3)} = l_2^{(4)}, \quad l_3^{(3)} = l_3^{(4)}$$

(3.11)
implies the following relations between the special and algebraic coordinates, in an obvious notation:

\begin{align}
z^{(3)}_1 &= z_1 z_4^2, \quad z^{(3)}_2 = z_2, \quad z^{(3)}_3 = z_3 \\
t^{(3)}_1 &= t_1 + 2 t_4, \quad t^{(3)}_2 = t_2, \quad t^{(3)}_3 = t_3.
\end{align}

(3.12)

From the mirror maps we find

\[ z_4 \sim \frac{1}{(1 - q_4)^2}, \quad z_1 \sim (1 - q_4)^4 \]

that is the transition takes indeed place at the boundary divisor \( z_1 = 1/z_4 = 0 \). In this limit the data of \( X_4 \) such as the periods and discriminants reduce to those of \( X_3 \). It is instructive to see the connection to the heterotic moduli. On the heterotic side, which is realized in terms of a simple orbifold construction of \( K3 \) \[14\] one has in addition to the dilaton \( S \), the moduli of the torus \( T, U \) a Wilson line \( B \). On these moduli acts the perturbative T-duality group which in a similar model has been determined to be \[14\]:

\begin{align}
T &\rightarrow T + 1, \quad U \rightarrow U + 1, \quad B \rightarrow B + 1, \quad B \rightarrow -B, \quad T \leftrightarrow U; \\
B &\rightarrow B - U, \quad T \rightarrow T + U - 2B, \quad U \rightarrow U,
\end{align}

(3.13)

together with the generalization of the inversion element \( T \rightarrow -1/T \) which is however realized in a less obvious way on the physical expressions. We expect a similar modular group realized in the present model, possibly up to some coefficients which depend on the details of the compactification lattice. Indeed, matching (3.13) to the symmetries realized on the physical couplings of the Calabi–Yau compactification, we find the identifications

\[ t_1 = T, \quad t_2 = S - T, \quad t_3 = U - T, \quad t_4 = B + U \]

where \( B \) has in fact period 2 \( U \) rather than \( U \). As a check on our physical picture we note that the Weyl symmetry element of the \( SU(2) \) subgroup of the \( E_8 \) factor is not corrected by non-perturbative string effects contrary to the mirror symmetry of the heterotic torus, as expected.

The situation for \( X_5 \) is similar but as the rank is larger there is now room for more interesting phenomena. As for \( X_4 \) we can obtain an \( SU(2) \) by shrinking down the divisor \( D_8 \). As this is not the conic bundle there are no degenerate fibers, i.e. we have a family of curves parameterized by \( \mathbb{P}^1 \) which is reflected in \( n_{0,0,1,0} = -2 \). Hence, there is no matter, and the unbroken \( SU(2) \) is present only in the perturbative theory. This agrees with the general field theoretic picture, as discussed in section 2.1. When the rank of the gauge group is 2, there is an \( SU(2) \) for \( \phi_1 = \phi_2 \neq 0 \), where the \( \phi_i \) are the scalar vevs of the vector multiplets. If we in addition to contracting the instantons of degree \((0, 0, 0, 1, 0)\) also shrink those of type \((0, 0, 0, 0, 1)\) we get a further enhancement, to \( SU(3) \) as well as 30
massless triplets. This can be seen as we are now forced to shrink down any combination of instantons which have degree \((0, 0, 0, k, l)\). Among the non-zero entries we find three components which all have \(n_{(0,0,0,k,l)} = 30\). Thus, 3 times 30 massless particles, forming 30 3s under \(SU(3)\). As we Higgs the \(SU(3)\), the three components split into a set of two which gives the 28 2s of the remaining \(SU(2)\) and 29 new singlet hyper multiplets. We have then arrived at the \(SU(2)\) point of \(X_4\) discussed above. Once more this agrees perfectly with the heterotic picture. At the transition from \((162, 6)\) to \((191, 5)\), the gauge group is \(SU(3) \times U(1)^4\), with 30 3 hyper multiplets under the \(SU(3)\). The transition occurs by breaking the \(SU(3)\) to \(SU(2)\) just as described above.

Finally, it is possible to avoid the enhanced gauge symmetry, and restrict to a codimension one surface where 30 isolated instantons of degree \((0,0,0,0,1)\) shrink to zero. This is the type IIA analog of the conifold transition in the type IIB string discussed by Greene, Morrison and Strominger [45]. Here, there are 29 flat directions among the 30 hyper multiplets which become massless as the size of the instantons go to zero. Thus, after Higgsing we are left with \(U(1)^5\) and 29 new singlets. This IIA transition description was given in [46], and applies as well to a similar transition occurring in [25].

4. Discussion and Conclusions

In this paper we have given strong evidence for extremal transitions between type II Calabi-Yau vacua, where the dual process in the heterotic string corresponds to Higgsing [16]. In particular, we have found evidence for a very nice correspondence between the appearance of enhanced \(SU(N)\) gauge symmetries and the corresponding matter structure on one hand, and the existence of a particular type of singularities in the Calabi-Yau manifold. The geometrical structure in question is that of \(N-2\) rationally ruled surfaces and a conic bundle with \(M\) degenerate fibers. As the fibers, which are \(P^1\)s, shrink to zero, particles appearing as BPS-saturated states of 2-branes wrapped around the \(P^1\)s become massless. The crucial point is the existence of the degenerate fibers as they are the source of the massless matter transforming in the fundamental of the relevant gauge group. The particular models considered in this paper are elliptically fibered Calabi-Yau manifolds. However, it seems as if the existence of the conic bundles is independent of this fact. We thus believe that this scenario is more general, and we are currently investigating such models [17].

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16 In a recent paper, transitions between type II vacua related to the dual \(SO(32)\) heterotic string have been discussed [46].

17 It has become known to us that similar problems are to be discussed in a forthcoming paper by Bershadsky et. al [47].
Finally, recall that the transitions are taking place in the perturbative region of the heterotic string, i.e. in the limit of large base in the K3-fibration in the type II theory. Thus, it still remains the possibility that there exist other K3-fibrations with the same behaviour as the radius of the $\mathbb{P}^1$ becomes large $[30]$.

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**Appendix A. The Kähler Cone: Toric Variety vs. Calabi-Yau Hypersurface**

We will now explain the relation between phases, although different when thought of as toric varieties, in fact are identical when restricted to the Calabi-Yau hypersurface. Let us assume that we have two toric varieties, $P_I$ and $P_{II}$, which are related to each other by a flop, i.e. a surface/curve, $C$, is blown down on $P_I$, and when passing through the wall of the Kähler cone where $P_I$ and $P_{II}$ meet $C$ reemerges in $P_{II}$ as a surface/curve, $\tilde{C}$. (In terms of the complex structure moduli space of the mirror theory the flop is merely an analytic continuation beyond the radius of convergence, corresponding to the walls of the Kähler cone.) We are still, however, to restrict this process to that of the hypersurfaces $X_I$ and $X_{II}$. Indeed, if the restriction of $C$ and $\tilde{C}$ to $X_I$ and $X_{II}$ respectively is empty there is nothing to be flopped and $X_I$ is isomorphic to $X_{II}$. We then have to consider the new Kähler cone as that of the union of the Kähler cones of the $P_{I,II}$. This process is repeated until we have a distinct set of inequivalent models. (In the example we will consider we always find just one K3-fibration phase after applying the above scheme.)

Let us now apply the above idea to that of toric variety $P_4$ From the dual polytope $\Delta_4^*$ one finds three Calabi-Yau phases which all are K3-fibrations. Their respective Mori generators are given by

\[
(0, 0, -1, 0, 1, 0, 0, -3, 3), (0, 1, 0, 0, 0, -2, 1, 0, 0),
\]
\[
(0, 0, 0, 0, -2, 1, 0, 1, 0), (-2, 0, 1, 1, 0, 0, 0, 1, -1)
\]
I hypersurface in (A.4) and cone is generated by with just one Calabi-Yau phase given as a hypersurface in a toric variety where the Mori cone is the intersection of the two Mori cones (A.2) and (A.3), which we calculate to be

\[
(0, 1, 0, 0, 0, -2, 1, 0, 0), (-2, 0, 0, 1, 1, 0, 0, -2, 2), \quad (A.2)
\]

and

\[
(0, 0, 1, 0, -1, 0, 0, 3, -3), (0, 0, -1, 0, -5, 3, 0, 0, 3) \quad (A.3)
\]

The second toric variety is obtained from the third one by a birational transformation which contracts the surface \(x_2 = x_4 = 0\) on the second toric variety and resolves the resulting singularity to the surface \(x_5 = x_8 = 0\) on the third toric variety. But on the second model for \(X_4\) we have \(D_2 \cdot D_4 = 0\), and on the second we have \(D_5 \cdot D_8 = 0\). This says that the birational transformation does not affect the hypersurface, which are therefore isomorphic.

So there is really just one Calabi-Yau phase coming from the two toric varieties described by (A.2) and (A.3). Therefore the Kähler cone of the hypersurface is the union of the two Kähler cones. Since the Mori cone is dual to the Kähler cone, we conclude that the actual Mori cone is the intersection of the two Mori cones (A.2) and (A.3), which we calculate to be

\[
(0, 1, 0, 0, 0, -2, 1, 0, 0), (-2, 0, 0, 1, 1, 0, 0, -2, 2), \quad (A.4)
\]

However, this new toric variety is related to that of (A.4) by a flop as well; contracting \(x_4 = x_8 = 0\) in the above phase and then resolving the surface \(x_2 = x_7 = 0\) in phase I. However, just as in the previous case \(D_4 \cdot D_8 = 0\) when restricted to the Calabi-Yau hypersurface in (A.4) and \(D_2 \cdot D_7 = 0\) on the hypersurface in phase I. Thus we are left with just one Calabi-Yau phase given as a hypersurface in a toric variety where the Mori cone is generated by

\[
(-2, 0, 0, 1, 1, 0, 0, -2, 2), (0, 1, 0, 0, 0, -2, 1, 0, 0) \quad (A.5)
\]

\[
(0, 0, 0, 0, -2, 1, 0, 1, 0), (-2, 0, 1, 1, 0, 0, 0, 1, -1).
\]

**Appendix B. The Gromov-Witten invariants for \(X_{3,4,5}\)**

| \(a_1\) | \(a_2\) | \(a_3\) | Instanton Numbers |
|--------|--------|--------|------------------|
| 0, 0, 1 | -2    | 0, 1, 1 | -2              |
| 0, 1, 4 | -8    | 0, 1, 5 | -10             |
| 1, 0, 0 | 480   | 1, 0, 1 | 480             |
| 1, 1, 3 | 2400  | 1, 1, 4 | 3360            |
| 2, 0, 2 | 480   | 2, 2, 2 | 480             |
| 4, 0, 0 | 480   | 5, 0, 0 | 480             |

Table B.1: A list of instanton numbers for rational curves of degree \([a_1, a_2, a_3]\) on \(X_3\).
| [a_1, a_2, a_3] | 56 | [a_1, a_2, a_3] | 30 | [a_1, a_2, a_3] | 2 | [a_1, a_2, a_3] | 0 |
|-----------------|----|-----------------|----|-----------------|----|-----------------|----|
| [0, 0, 0, 1]    | 56 | [0, 0, 0, 2]    | -2 | [0, 0, 0, 3]    | 0  | [0, 0, 1, 0]    | -2 |
| [0, 1, 0, 0]    | 0  | [0, 1, 1, 0]    | -2 | [0, 1, 2, 0]    | -4 | [0, 1, 3, 0]    | -6 |
| [0, 1, 4, 0]    | -8 | [0, 1, 5, 0]    | -10| [0, 2, 3, 0]    | -6 | [0, 2, 4, 0]    | -32|
| [0, 2, 5, 0]    | -110| [0, 2, 6, 0]    | -288| [0, 3, 4, 0]    | -8 | [0, 3, 4, 0]    | -8 |
| [0, 3, 5, 0]    | -110| [1, 0, 0, 0]    | -2 | [1, 0, 0, 1]    | 56 | [1, 0, 0, 2]    | 372|
| [1, 0, 0, 3]    | 56 | [1, 0, 0, 4]    | -2 |                   |    |                   |    |

Table B.2: A list of instanton numbers for rational curves of degree \([a_1, a_2, a_3, a_4]\) on \(X_4\).

| [a_1, a_2, a_3, a_4, a_5] | 30 | [a_1, a_2, a_3, a_4, a_5] | 0 | [a_1, a_2, a_3, a_4, a_5] | -2 | [a_1, a_2, a_3, a_4, a_5] | 30 |
|---------------------------|----|---------------------------|----|---------------------------|----|---------------------------|----|
| [0, 0, 0, 1]              | 30 | [0, 0, 0, 2]              | 0  | [0, 0, 0, 1, 0]           | -2 | [0, 0, 0, 1, 1]           | 30 |
| [0, 0, 1, 2]              | 30 | [0, 0, 1, 3]              | -2 | [0, 0, 1, 0, 0]           | -2 | [0, 1, 0, 0, 0]           | 0  |
| [0, 1, 1, 0]              | -2 | [0, 1, 2, 0]              | -4 | [0, 1, 3, 0, 0]           | -6 | [0, 1, 4, 0, 0]           | -8 |
| [0, 2, 3, 0]              | -6 | [0, 2, 4, 0, 0]           | -32| [0, 2, 5, 0, 0]           | -110| [1, 0, 0, 0, 0]           | -2 |
| [1, 0, 0, 1, 0]           | -2 | [1, 0, 0, 1, 1]           | 30 | [1, 0, 0, 1, 2]           | 30 | [1, 0, 0, 1, 3]           | -2 |
| [1, 0, 0, 2, 2]           | 30 | [1, 0, 0, 2, 3]           | 312| [1, 0, 0, 2, 4]           | 30 | [1, 0, 1, 0, 0]           | -2 |
| [1, 0, 1, 1, 0]           | -2 | [1, 0, 1, 1, 1]           | 30 | [1, 0, 1, 1, 2]           | 30 | [1, 0, 1, 1, 3]           | -2 |

Table B.3: A list of instanton numbers for rational curves of degree \([a_1, a_2, a_3, a_4, a_5]\) on \(X_5\).
References

[1] E. Witten, Nucl. Phys. B443 (1995) 85, [hep-th/9503124].
[2] N. Seiberg, E. Witten, Nucl. Phys. B426 (1994) 19, [hep-th/9407087].
[3] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. 344 (1995) 169, P. Argyres and A. Faraggi, Phys. Rev. Lett. 74 (1995) 3931; Ulf H. Danielsson, Bo Sundborg, Phys. Lett. 358B (1995) 273; A. Brandhuber, K. Landsteiner, Phys.Lett.B358 (1995) 73, R.G. Leigh, M.J. Strassler, Phys.Lett.B 356 (1995) 492; E. Martinec, N. Warner, Nucl. Phys. B459 (1996) 97
[4] see e.g. C. Hull and P. Townsend, Nucl. Phys. B438 (1995) 109, [hep-th/9410167].
[5] S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69, [hep-th/9505105].
[6] S. Ferrara, J. Harvey, A. Strominger, and C. Vafa Second-Quantized Mirror Symmetry, [hep-th/9505162].
[7] B. de Wit, P.G. Lauwers, R. Philippe, S.Q. Su, A. van Proeyen, Phys. Lett. 134B (1984) 37; B. de Wit, A. van Proeyen, Nucl. Phys. B245 (1984) 89; J.P. Derendinger, S. Ferrara, A. Masiero, A. van Proeyen, Phys. Lett. 140B (1984) 307; B. de Wit, P.G. Lauwers, A. van Proeyen, Nucl. Phys. B255 (1985) 569; E. Cremmer, C. Kounnas, A. van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit, L. Girardello, Nucl. Phys. B250 (1985) 385.
[8] J. Bagger, E. Witten, Nucl. Phys. B222 (1983) 1; J. Bagger, A. Galperin, E. Ivanov, V. Ogievetsky, Nucl. Phys. B303 (1988) 522.
[9] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B451 (1990) 53; I. Antoniadis, S. Ferrara, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. B447 (1995) 35.
[10] V. Kaplunovsky, J. Louis, and S. Theisen, Phys. Lett. 357 B (1995) 71; A. Klemm, W. Lerche and P. Mayr, Phys. Lett. 357 B (1995) 313; C. Vafa and E. Witten, preprint HUTP-95-A023; [hep-th/9507050]; I. Antoniadis, E. Gava, K. Narain and T. Taylor, Nucl. Phys. B455 (1995) 109, B. Lian and S.T. Yau, Mirror Maps, Modular Relations and Hypergeometric Series I,II, [hep-th/9507151], [hep-th/9507153]; P. Aspinwall and J. Louis, Phys. Lett. 369 B (1996) 233; I. Antoniadis, S. Ferrara and T. Taylor, Nucl. Phys. B460 (1996) 489; G. Curio, Phys. Lett. 366 B (1996) 131, Phys. Lett. 366 B (1996) 78; G. Lopes Cardoso, G. Curio, D. Lüst and T. Mohaupt, Instanton Numbers and Exchange Symmetries in N = 2 Dual String Pairs, [hep-th/9603108].
[11] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B459 (1996) 537.
[12] P. Aspinwall, Phys. Lett. 357 B (1995) 329, [hep-th/9507012]; Enhanced Gauge Symmetries and Calabi-Yau Threefolds, [hep-th/9511171].
[13] A. C. Avram, P. Candelas, D. Jancic, M. Mandelberg, On the connectedness of moduli spaces of Calabi-Yau manifolds [hep-th/9511230]; T. Chiang, B. R. Greene, M. Gross,
Y. Kanter, *Black hole condensation and the web of Calabi-Yau manifolds*. hep-th 9511204

[14] G. Aldazabal, L.E. Ibanez, A. Font, F. Quevedo, *Chains of N=2, D=4 heterotic/type II duals*, hep-th/9510093.

[15] P. Candelas and A. Font, *Duality Between Webs of Heterotic and Type II Vacua*, hep-th/9603170.

[16] S. Katz, D. R. Morrison, M. R. Plesser, *Enhanced Gauge Symmetry in Type II String Theory*, hep-th/9601108.

[17] A. Strominger, *Massless Black Holes and Conifolds in String Theory*, hep-th/9504090.

[18] M. Bershadsky, V. Sadov, C. Vafa, *D-Strings on D-Manifolds*, hep-th/9510225.

[19] A. Klemm and P. Mayr, *Strong Coupling Singularities and Non-abelian Gauge Symmetries in N = 2 String Theory*, hep-th/9601014.

[20] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Selfdual Strings and N = 2 Supersymmetric Field Theory*, hep-th/9601014.

[21] D. R. Morrison and M. R. Plesser, private communication.

[22] R. Miranda, in *The Birational Geometry of Degenerations*, Progress in Math. vol. 29, Birkhäuser, 19983, 85.

[23] J. Harris, *Algebraic Geometry*, Springer (1992) New York

[24] P. Aspinwall, B. Greene and D. Morrison, *Nucl. Phys. B420* (1994) 184

[25] P. Berglund, S. Katz and A. Klemm, *Nucl. Phys. B456* (1995) 153, hep-th/9506091.

[26] P. Candelas and A. Font, *Duality Between Webs of Heterotic and Type II Vacua*, hep-th/9603170.

[27] D.R. Morrison, *Through the looking glass*, to appear.

[28] K. Oguiso, Internat. J. Math. 4 (1993) 439.

[29] B. Hunt and R. Schimmrigk, *Heterotic Gauge Structure of Type II K3 Fibrations*, hep-th/9512138.

[30] P. Berglund, S. Katz, A. Klemm and P. Mayr, *Type II Strings on Calabi-Yau Manifolds and String Duality*, in preparation.

[31] G. Aldazabal, A. Font, L.E. Ibanez, F. Quevedo, *Heterotic/Heterotic Duality in D=6,4*, hep-th/9602097.

[32] V. Batyrev, Duke Math. Journal 69 (1993) 349, V. Batyrev, Journal Alg. Geom. 3 (1994) 493, V. Batyrev, *Quantum Cohomology Rings of Toric Manifolds*, preprint 1992.

[33] S. Hosono, B.H.Lian, S.-T.Yau, *GKZ Hypergeometric Systems and Applications to Mirror Symmetry*, alg-geom/9511001.

[34] J. DeLoera, PUNTOS, ftp.geom.umn.edu/pub/software

[35] P. Aspinwall, *An N=2 Dual Pair and a Phase Transition*, hep-th/9510142.

[36] D. R. Morrison, hep-th/9311049; D. R. Morrison, P. Aspinwall, B.R. Greene, *The monomial divisor mirror map*, hep-th/9407087.
[37] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Commun. Math. Phys. 167 (1995) 301, Nucl. Phys. B433 (1995) 501.

[38] A. Klemm, W. Lerche, P. Mayr, as cited in [10].

[39] S. Katz and S.A. Stromme. Schubert: a Maple package for intersection theory, ftp.math.okstate.edu/pub/schubert

[40] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. B416 (1994) 481, hep-th/9308083.

[41] D. R. Morrison and C. Vafa, Compactifications of F-Theory on Calabi–Yau Threefolds, hep-th/9602114.

[42] P. S. Aspinwall and Mark Gross, Heterotic-Heterotic String Duality and Multiple K3 Fibrations, hep-th/9602118.

[43] M. Gross, Primitive Calabi-Yau Threefolds, alg-geom/9512002.

[44] P. Mayr and S. Stieberger, Phys. Lett. 355 B (1995) 107; G. Lopes Cardoso, D. Lüst and T. Mohaupt, Nucl. Phys. B432 (1994) 68

[45] B.R. Greene, D.R. Morrison, A. Strominger, Black Hole Condensation and the Unification of String Vacua, hep-th/9504145.

[46] P. S. Aspinwall and M. Gross, The SO(32) Heterotic String on a K3 Surface, hep-th/9605131.

[47] Bershadsky et. al, Geometric Singularities and Enhanced Gauge Symmetries, to appear.