WEAK LEFSCHETZ PROPERTY OF PL-SPHERES

FEIFEI FAN

Abstract. A recent result of Papadakis-Petrotou shows that every simplicial sphere has the weak Lefschetz property in characteristic 2. In this paper, we give a simpler proof of this result for PL-spheres by showing that the weak Lefschetz property in characteristic 2 is preserved by bistellar moves. Several applications are given.

1. Introduction

Our motivating problem is the well-known $g$-conjecture for simplicial spheres. It was first proposed by McMullen [11] in 1971 for a complete characterization of the $f$-vectors (i.e. face numbers) of simplicial polytopes. Less than ten years later the $g$-conjecture was proved as a theorem. To describe this theorem, let us review some notions.

For a $(d-1)$-dimensional simplicial complex $\Delta$, the $f$-vector of $\Delta$ is

\[(f_0, f_1, \ldots, f_{d-1}),\]

where $f_i$ is the number of the $i$-dimensional faces of $\Delta$. Sometimes it is convenient to set $f_{-1} = 1$ corresponding to the empty set. The $h$-vector of $\Delta$ is the integer vector $(h_0, h_1, \ldots, h_d)$ defined from the equation

\[h_0 t^d + \cdots + h_{d-1} t + h_d = f_{-1} (t-1)^d + f_0 (t-1)^{d-1} + \cdots + f_{d-1}.\]

The $f$-vector and the $h$-vector contain equivalent combinatorial information about $\Delta$, and determine each other by means of linear relations

\[h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}, \quad f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{d-i} h_j, \quad \text{for } 0 \leq i \leq d.\]

The sequence $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor}) = (h_0, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is usually called the $g$-vector of $\Delta$.

For any two positive integers $a$ and $i$ there is a unique way to write

\[a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}.\]

2010 Mathematics Subject Classification. Primary 13F55; Secondary 05E45, 14M25.

The author is supported by the National Natural Science Foundation of China (Grant Nos. 11801580, 11871284).
with \( a_i > a_{i-1} > \cdots > a_j \geq j \geq 1 \). Define the \( i \)th pseudopower of \( a \) as
\[
a^{(i)} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}.
\]
Here \( 0^{(i)} = 0 \) for all \( i \).

**Theorem 1** (\( g \)-theorem). An integer vector \((h_0, h_1, \ldots, h_d)\) is the \( h \)-vector of a polytopal \((d - 1)\)-sphere (the boundary complex of a simplicial \( d \)-polytope) if and only if

(a) \( h_i = h_{d-i} \) for \( i = 0, 1, \ldots, d \) (the Dehn-Sommerville relations);
(b) \( 1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor} \);
(c) the \( g \)-vector satisfies \( g_{i+1} \leq g_i^{(i)} \) for \( i \geq 1 \).

Condition (a) says that the \( h \)-vector of a simplicial polytope is symmetric. Condition (b) says that the \( h \)-vector is unimodal, growing up to the middle, while (c) gives a restriction on the rate of this growth. Both (b) and (c) can be reformulated by saying that the \( g \)-vector is an \( M \)-vector, explaining the name ‘\( g \)-theorem’. (A sequence of integers \((k_0, k_1, k_2, \ldots)\) satisfies \( k_0 = 1 \) and \( 0 \leq k_{i+1} \leq k_i^{(i)} \) for \( i \geq 1 \) is called an \( M \)-sequence. Finite \( M \)-sequences are \( M \)-vectors.)

Both necessity and sufficiency parts of the \( g \)-theorem were proved almost simultaneously (around 1980). The sufficiency of McMullen’s conditions was proved by Billera and Lee [2], while their necessity was proved by Stanley [19] by using the theory of toric varieties. In 1993, McMullen gave another proof of the necessity part of \( g \)-theorem using complicated and difficult convex geometry [12]. Later, McMullen simplified his approach in [13]. The idea behind both Stanley’s and McMullen’s proofs was to find a ring whose Hilbert function equals the \( h \)-vector of the polytope, and satisfying the Hard Lefschetz theorem (Theorem 2.5).

The \( g \)-theorem gives a complete characterisation of integral vectors arising as the \( f \)-vectors of polytopal spheres. It is therefore natural to ask whether the \( g \)-theorem extends to all sphere triangulations. This question has been regarded as one of the main open problems in the theory of face enumeration:

**Conjecture 2** (\( g \)-conjecture). The \( g \)-theorem holds for all simplicial spheres, or even for all rational homology spheres.

In a very recent preprint [1], Adiprasito announced a proof of this conjecture on its highest level of generality. More recently, Papadakis and Petrotau [15] announced another proof of this conjecture by showing that simplicial spheres have the weak Lefschetz property (see Definition 2.7) in characteristic 2. In this paper, we focus on the PL-sphere case, and give a simpler proof of Papadakis-Petrotau result in this case. The following theorem is the main result of this paper.
Theorem 3. Every PL-sphere has the weak Lefschetz property over any infinite field of characteristic 2 or 0.

We remark that the statement for characteristic 0 is implied by the statement for characteristic 2. This follows from a general result in [8, Lemma 2.7].

The proof is elementary and essentially self-contained. Following McMullen’s proof of the Hard Lefschetz theorem for simplicial polytopes, the main idea of the proof of Theorem 3 is to show that the weak Lefschetz property is preserved by bistellar move operations on PL-spheres. For even dimensional spheres, this is known for all bistellar moves except the middle dimensional bistellar move (see the survey article by Swartz [22]), which is the case we solve in Section 3.

Theorem 3 has several other applications, one of which is to the Grünbaum-Kalai-Sarkaria conjecture. This conjecture is about the embeddability of a simplicial complex $\Delta$ of dimension $d$ into $\mathbb{R}^{2d}$. Note that the dimension $2d$ is the highest nontrivial case in a sense that every $d$-complex embeds into $\mathbb{R}^{2d+1}$.

It is well known that if a simple planar graph has $f_0$ vertices and $f_1$ edges, then $f_1 \leq 3f_0$. Actually Euler’s combinatorial formula shows that if $f_0 \geq 3$ this inequality can be strengthened to $f_1 \leq 3f_0 - 6$ (see e.g. [7, Lemma 1.3.5]). Grünbaum [6] asked if there are generalizations of this inequality for $d$-dimensional simplicial complexes that allow PL embedding into $\mathbb{R}^{2d}$. Kalai and Sarkaria [9, 17] suggested a precise formula:

**Conjecture 4** (Grünbaum-Kalai-Sarkaria). Let $\Delta$ be a simplicial complex of dimension $d$. If there is a PL embedding $\Delta \hookrightarrow \mathbb{R}^{2d}$ then

$$f_d(\Delta) \leq (d + 2)f_{d-1}(\Delta).$$

As observed by Kalai-Sarkaria, the number $d+2$ in the inequality is the best possible constant in every dimension. They also showed that if every even-dimensional PL-sphere has the weak Lefschetz property, then the conjecture is true. So Theorem 3 implies the validity of Conjecture 4 (see Theorem 4.5).

2. Preliminaries

2.1. Notations and conventions. Throughout this paper, we assume that $\Delta$ is a simplicial complex, and we use $|\Delta|$ to denote the geometric realization of $\Delta$. If $\Delta$ has $m$ vertices, we usually identify the vertices of $\Delta$ with $[m] = \{1, \ldots, m\}$. We refer to $i$-dimensional faces as $i$-faces. Let $\mathcal{F}_i(\Delta)$ be the set of $i$-faces of $\Delta$. A simplicial complex is pure if all of its facets (maximal faces) have the same dimension. By $\Delta^{m-1}$ we denote the simplicial complex consisting of all subsets of $[m]$; its boundary $\partial \Delta^{m-1}$ will be the subcomplex of all proper subsets of $[m]$. By abuse of notation, sometimes the symbol $\sigma$ will be used ambiguously to denote a face $\sigma \in \Delta$ and also the simplicial complex consisting of $\sigma$ and all its faces.
The link, star and deletion of a face $\sigma \in \Delta$ are respectively the subcomplexes
\[
\text{lk}_\sigma \Delta = \{ \tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset \};
\]
\[
\text{st}_\sigma \Delta = \{ \tau \in \Delta : \tau \cup \sigma \in \Delta \};
\]
\[
\Delta - \sigma = \{ \tau \in \Delta : \sigma \nsubseteq \tau \}.
\]

The join of two simplicial complexes $\Delta$ and $\Delta'$, where the vertex set $F_0(\Delta)$ is disjoint from $F_0(\Delta')$, is the simplicial complex
\[
\Delta \ast \Delta' = \{ \sigma \cup \sigma' : \sigma \in \Delta, \sigma' \in \Delta' \}.
\]
In particular, we say that $\Delta^0 \ast \Delta$ is the cone over $\Delta$ and denoted cone $\Delta$.

Let $\Delta$ be a pure simplicial complex of dimension $d$ and $\sigma \in \Delta$ a $(d-i)$-face such that $\text{lk}_\sigma \Delta = \partial \Delta^i$ and the subset $\tau = F_0(\Delta^i) \subset F_0(\Delta)$ is not a face of $\Delta$. Then the operation $\chi_{\sigma}$ on $\Delta$ defined by
\[
\chi_{\sigma} \Delta = (\Delta - \sigma) \cup (\partial \sigma \ast \Delta^i)
\]
is called a bistellar $i$-move. Obviously we have $\chi_{\tau} \chi_{\sigma} \Delta = \Delta$. Two pure simplicial complexes are bistellarly equivalent if one is transformed to another by a finite sequence of bistellar moves.

A simplicial complex $\Delta$ is called a triangulated manifold (or simplicial manifold) if $|\Delta|$ is a topological manifold. More generally, a $d$-dimensional simplicial complex $\Delta$ is a $k$-homology manifold ($k$ is a field) if
\[
H_*(|\Delta|; k) = \widetilde{H}_*(S^d; k) \quad \text{for all } x \in |\Delta|,
\]
or equivalently,
\[
H_*(\text{lk}_\sigma \Delta; k) = H_*(S^{d-|\sigma|}; k) \quad \text{for all } \emptyset \neq \sigma \in \Delta.
\]

$\Delta$ is a $k$-homology $d$-sphere if it is a $k$-homology $d$-manifold with the same $k$-homology as $S^d$. (Remark: Usually, the terminology “homology sphere” means a manifold having the homology of a sphere. Here we take it in a more relaxed sense than its usual meaning.)

A piecewise linear (PL for short) $d$-sphere is a simplicial complex which has a common subdivision with $\partial \Delta^{d+1}$. A PL $d$-manifold is a simplicial complex $\Delta$ of dimension $d$ such that $\text{lk}_\sigma \Delta$ is a PL-sphere of dimension $d - |\sigma|$ for every nonempty face $\sigma \in \Delta$. It is obvious that two bistellarly equivalent PL-manifolds are PL homeomorphic, i.e., they have a common subdivision. The following fundamental result shows that the converse is also true.

**Theorem 2.1** (Pachner [14, (5.5)]). Two PL-manifolds are bistellarly equivalent if and only if they are PL homeomorphic.
As we have seen in Section 1, the \( g \)-theorem relates the \( g \)-vectors of polytopal spheres to \( M \)-sequences. The importance of \( M \)-sequence comes from the following fundamental result of Macaulay in commutative algebra.

**Theorem 2.2** (Macaulay [10], [3, Theorem I.4.2.10]). Let \((k_0, k_1, k_2, \ldots)\) be a sequence of nonnegative integers. Then \((k_0, k_1, k_2, \ldots)\) is the Hilbert function of a homogeneous quotient of a polynomial ring if and only if \(k_0 = 1\) and \(0 \leq k_{i+1} \leq k_i^{(i)}\) for all \(i \geq 1\).

2.2. Face rings and l.s.o.p. For a commutative ring \(k\) with unit, let \(k[x_1, \ldots, x_m]\) be the polynomial algebra with one generator for each vertex in \(\Delta\). It is a graded algebra by setting \(\deg x_i = 1\). The Stanley-Reisner ideal of \(\Delta\) is \(I_{\Delta} := (x_{i_1}x_{i_2} \cdots x_{i_k} : \{i_1, i_2, \ldots, i_k\} \not\in \Delta)\). The Stanley-Reisner ring (or face ring) of \(\Delta\) is the quotient \(k[\Delta] := k[x_1, \ldots, x_m]/I_{\Delta}\).

Since \(I_{\Delta}\) is a monomial ideal, the quotient ring \(k[\Delta]\) is graded by degree.

For a face \(\sigma = \{i_1, \ldots, i_k\} \in \mathcal{F}_{k-1}(\Delta)\), denote by \(x_{\sigma} = x_{i_1} \cdots x_{i_k} \in k[\Delta]\) the face monomial corresponding to \(\sigma\).

Assuming \(k\) is a field, a set \(\Theta = \{\theta_1, \ldots, \theta_d\}\) consisting of \(d = \dim \Delta + 1\) linear forms in \(k[\Delta]\) is called an l.s.o.p. (linear system of parameters), if \(k(\Delta; \Theta) := k[\Delta]/(\Theta)\) has Krull dimension zero, i.e., it is a finite-dimensional \(k\)-space. We will use the simplified notation \(k(\Delta)\) for \(k(\Delta; \Theta)\) whenever it creates no confusion, and write the component of degree \(i\) of \(k(\Delta)\) as \(k(\Delta)_i\). For a subcomplex \(\Delta' \subset \Delta\), let \(I\) be the ideal of \(k[\Delta]\) generated by faces in \(\Delta \setminus \Delta'\), and denote \(I/I\Theta\) by \(k(\Delta, \Delta'; \Theta)\) or simply \(k(\Delta, \Delta')\).

A linear sequence \(\theta_1, \ldots, \theta_d\) is an l.s.o.p if and only if the restriction \(\Theta_\sigma = r_\sigma(\Theta)\) to each face \(\sigma \in \Delta\) generates the polynomial algebra \(k[x_i : i \in \sigma]\); here \(r_\sigma : k[\Delta] \rightarrow k[x_i : i \in \sigma]\) is the projection homomorphism (see [3, Theorem II.5.1.16]).

It is a general fact that if \(k\) is an infinite field, then \(k[\Delta]\) admits an l.s.o.p. (Noether normalization lemma). If \(\Theta\) is an l.s.o.p. for \(k[\Delta]\), then \(k(\Delta)\) is spanned by the face monomials (see [20, Lemma III.2.4]).

Suppose \(\Theta = \{\theta_i = \sum_{j=1}^{m} a_{ij}x_j\}_{i=1}^{d}\) is an l.s.o.p. for \(k[\Delta]\). Then there is an associated \(d \times m\) matrix \(M_\Theta = (a_{ij})\). Let \(\lambda_i = (a_{i1}, a_{i2}, \ldots, a_{im})^T\) denote the column vector corresponding to the vertex \(i \in [m]\). For any ordered subset \(S = (i_1, \ldots, i_k) \subset [m]\), the submatrix \(M_\Theta(S)\) of \(M_\Theta\) is defined to be \(M_\Theta(S) = (\lambda_{i_1}, \ldots, \lambda_{i_k})\).
2.3. Cohen-Macaulay and Gorenstein complexes. In this subsection we review some basic combinatorial and algebraic concepts used in the rest of this paper. Throughout this subsection, \( k \) is an infinite field of arbitrary characteristic.

Let \( \Delta \) be a simplicial complex of dimension \( d - 1 \). The face ring \( k[\Delta] \) is a Cohen-Macaulay ring if for any l.s.o.p \( \Theta = \{\theta_1, \ldots, \theta_d\} \), \( k(\Delta) \) is a free \( k[\theta_1, \ldots, \theta_d] \) module. In this case, \( \Delta \) is called a Cohen-Macaulay complex over \( k \).

If \( \Delta \) is Cohen-Macaulay, the following result of Stanley shows that the \( h \)-vector of \( \Delta \) has a pure algebraic description.

\textbf{Theorem 2.3} (Stanley [18]). Let \( \Delta \) be a \((d-1)\)-dimensional Cohen-Macaulay complex and let \( \Theta = \{\theta_1, \ldots, \theta_d\} \) be an l.s.o.p. for \( k[\Delta] \). Then

\[ \dim_k k(\Delta)_i = h_i(\Delta), \quad \text{for all } 0 \leq i \leq d. \]

Let \( M \) be a finitely-generated graded \( k[x_1, \ldots, x_m] \)-module. The socle of \( M \) is the following graded submodule of \( M \)

\[ \text{Soc } M := \{ a \in M : x_i \cdot a = 0 \text{ for all } i \}. \]

The face ring \( k[\Delta] \) is a Gorenstein ring if \( \dim_k \text{Soc } k(\Delta) = 1 \) for any l.s.o.p \( \Theta \), or in other words, \( k(\Delta) \) is a Poincaré duality \( k \)-algebra. In this case, \( \Delta \) is called a Gorenstein complex over \( k \). Further, \( \Delta \) is called Gorenstein* if \( k[\Delta] \) is Gorenstein and \( \Delta \) is not a cone, i.e., \( \Delta \neq \text{cone } \Delta' \) for any \( \Delta' \).

These algebraic properties of face rings have combinatorial-topological characterisations as follows.

\textbf{Theorem 2.4}. Let \( \Delta \) be a simplicial complex. Then

(a) (Reisner [16]) \( \Delta \) is Cohen-Macaulay (over \( k \)) if and only if for all faces \( \sigma \in \Delta \) (including \( \sigma = \emptyset \)) and \( i < \dim lk_\sigma \Delta \), we have \( \bar{H}_i(lk_\sigma \Delta ; k) = 0 \).

(b) (Stanley [20, Theorem II.5.1]) \( \Delta \) is Gorenstein* (over \( k \)) if and only if it is a \( k \)-homology sphere.

2.4. Strong and weak Lefschetz property. Both Stanley’s and McMullen’s proof the necessity of the \( g \)-theorem is by proving the following theorem.

\textbf{Theorem 2.5} ([19],[12]). If \( \Delta \) is the boundary of a simplicial \( d \)-polytope, then for a certain l.s.o.p. \( \Theta \) of \( \mathbb{Q}[\Delta] \), there exists a linear form \( \omega \) in \( \mathbb{Q}[\Delta] \) such that the multiplication map

\[ \cdot \omega^{d-2i} : \mathbb{Q}(\Delta)_i \to \mathbb{Q}(\Delta)_{d-i} \]

is an isomorphism for all \( i \leq d/2 \).

Let us see why Theorem 2.5 implies the \( g \)-theorem. If \( \omega \) satisfies the condition in Theorem 2.5, then the map \( \omega : \mathbb{Q}(\Delta)_i \to \mathbb{Q}(\Delta)_{i+1} \) is obviously injective for all \( i <
Let $\Delta$ be a $k$-homology sphere. We say $\Delta$ has the \textit{strong Lefschetz property} over $k$ if there exists an l.s.o.p. $\Theta$ for $k[\Delta]$ and a linear form $\omega$ such that the multiplication map
\[ \cdot \omega^{d-2i} : k(\Delta)_i \to k(\Delta)_{d-i} \]
is an isomorphism for all $i \leq d/2$.

\textbf{Conjecture 2.6 (Algebraic $g$-conjecture).} \textit{If $k$ is an infinite field, then every $k$-homology sphere has the strong Lefschetz property.}

Recently, Adiprasito posted a highly technical preprint \cite{1} claiming to prove Conjecture 2.6. More recently, Papadakis and Petrotou \cite{15} posted another purely algebraic proof of Conjecture 2.6 for the case that $k$ has characteristic 2. The idea in Papadakis-Petrotou proof is to view the entries $a_{ij}$ of $M_\Theta$ as variables and use a partial differential operator defined on the field of fractions of the polynomial ring
\[ k[a_{ij} : 1 \leq i \leq d, 1 \leq j \leq m]. \]

Our proof of Theorem 3 is inspired by this idea.

In fact, to prove the $g$-conjecture, it is enough to prove the following weaker property holds.

\textbf{Definition 2.7.} Let $\Delta$ be a Cohen-Macaulay complex (over $k$). We say that $\Delta$ has the \textit{weak Lefschetz property} (WLP) if there is an l.s.o.p. $\Theta$ for $k[\Delta]$ and a linear form $\omega$ such that the multiplication map $\cdot \omega : k(\Delta)_i \to k(\Delta)_{i+1}$ is either injective or surjective for all $i$. Such a linear form $\omega$ is called a \textit{weak Lefschetz element} (WLE).

\textbf{Proposition 2.8.} \textit{Let $\Delta$ be a $k$-homology $(d-1)$-sphere, then $\omega$ is a WLE if the multiplication map $k(\Delta)_{[d/2]} \xrightarrow{\omega} k(\Delta)_{[d/2]+1}$ is surjective.}

\textbf{Proof.} Suppose the multiplication map $k(\Delta)_{[d/2]} \xrightarrow{\omega} k(\Delta)_{[d/2]+1}$ is surjective. Then $k(\Delta)/\omega)_{[d/2]+1} = 0$. This implies that $(k(\Delta)/\omega)_i = 0$ for all $i \geq [d/2] + 1$, and therefore $k(\Delta)_i \xrightarrow{\omega} k(\Delta)_{i+1}$ is surjective for all $i \geq [d]$. Since $\Delta$ is Gorenstein, $k(\Delta)$ is a Poincaré duality algebra. Namely, there is an nondegenerate bilinear paring
\[ k(\Delta)_j \times k(\Delta)_{d-j} \to k(\Delta)_d = k, \quad (a, b) \mapsto ab. \]
It follows that $k(\Delta)_i \xrightarrow{\omega} k(\Delta)_{i+1}$ is injective for $i < [d/2]$. \hfill $\square$

We define a set of pairs $\mathcal{W}(\Delta) \subset k^{d_0} \oplus k^{d_0}$ to be
\[ \mathcal{W}(\Delta) = \{ (\omega, \Theta) : \Theta \text{ is an l.s.o.p. for } k[\Delta] \text{ and } \omega \text{ is a WLE } \}. \]
It is well known that \( W(\Delta) \) is a Zariski open set (see e.g. [21, Proposition 3.6]). We will use the term ‘generic choice’ of \( \Theta \) or \( \omega \) to mean that these elements are chosen from a non-empty Zariski open set.

2.5. A criterion of WLP for even spheres. In this subsection we recall some result about the canonical module (see [20, I.12] for the definition) of \( k[\Delta] \) when \( \Delta \) is a homology ball, and see how it relates to the WLP of the boundary sphere of \( \Delta \) for the case that \( \dim \Delta \) is odd.

Let \( \Delta \) be a \( k \)-homology \((d − 1)\)-ball with boundary \( \partial \Delta \). Then there is an exact sequence

\[
0 \to I \to k[\Delta] \to k[\partial \Delta] \to 0,
\]

where \( I \) is the ideal of \( k[\Delta] \) generated by all faces in \( \Delta \setminus \partial \Delta \). By a theorem of Hochster [20, Theorem II.7.3] \( I \) is the canonical module of \( k[\Delta] \). Then from [3, Theorem I.3.3.4 (d) and Theorem I.3.3.5 (a)] we have the following lemma.

**Lemma 2.9.** Let \( \Delta \) be a \( k \)-homology \((d − 1)\)-ball, and \( \Theta \) be an l.s.o.p. for \( k[\Delta] \). Then there is a nondegenerate bilinear pairing

\[
k(\Delta)_i \times k(\Delta, \partial \Delta)_{d−i} \to k(\Delta, \partial \Delta)_d = k.
\]

**Proposition 2.10.** Let \( \Delta \) be a \( k \)-homology \((2n − 1)\)-ball, and \( \Theta \) be a generic l.s.o.p for \( k[\Delta] \). Then the following are equivalent.

(a) \( \partial \Delta \) has the WLP.

(b) The natural map \( \phi : k(\Delta, \partial \Delta)_n \to k(\Delta)_n \) is an isomorphism.

**Proof.** Since \( \Theta = \{\theta_1, \ldots, \theta_{2n}\} \) is generic, we may assume that \( \Theta_0 = \{\theta_1, \ldots, \theta_{2n−1}\} \) is an l.s.o.p. for \( k[\partial \Delta] \). The exact sequence (2.1) induces an exact sequence:

\[
0 \to I/(I \cap \Theta) \to k(\Delta; \Theta) \to k[\partial \Delta]/\Theta \to 0.
\]

Note that \( \phi \) factors through the surjection \( k(\Delta, \partial \Delta) := I/I\Theta \to I/(I \cap \Theta) \), and \( \dim k(\Delta, \partial \Delta)_n = \dim k(\Delta, \partial \Delta)_n \) by Lemma 2.9. Thus \( \phi \) is an isomorphism \( \iff \) \( (k[\partial \Delta]/\Theta)_n = 0 \iff \) the multiplication map \( k(\partial \Delta; \Theta_0)_{n−1} \to k(\partial \Delta; \Theta_0)_n \) is an isomorphism \( \iff \) \( (\theta_{2n}, \Theta_0) \in W(\partial \Delta) \). \( \square \)

**Remark 2.11.** From the proof of Proposition 2.10 we know that the conditions in Proposition 2.10 only depend on the restriction of \( \Theta \) to \( \partial \Delta \). In other word, if \( \Theta' \) is another l.s.o.p. for \( k[\Delta] \) such that \( M_\Theta(V(\partial \Delta)) = M_{\Theta'}(V(\partial \Delta)) \), then the conditions in Proposition 2.10 hold for \( \Theta \) if and only if they hold for \( \Theta' \).

2.6. The canonical function. In this subsection, we recall a useful result in [15].

**Lemma 2.12** ([15, Corollary 4.5]). Let \( \Delta \) be a \((d−1)\)-dimensional \( k \)-homology sphere or ball, \( \Theta \) be an l.s.o.p. for \( k[\Delta] \). Assume \( \sigma_1 \) and \( \sigma_2 \) are two ordered facets of \( \Delta \), which have the same orientation in \( \Delta \). Then

\[
\det(M_\Theta(\sigma_1))x_{\sigma_1} = \det(M_\Theta(\sigma_2))x_{\sigma_2}
\]
in \( k(\Delta)_d \) or \( k(\Delta, \partial \Delta)_d \) respectively.

When \( \Delta \) is a \((d - 1)\)-dimensional \( k \)-homology sphere or ball, \( k(\Delta)_d \) or \( k(\Delta, \partial \Delta)_d \) is \( k \), which is spanned by a facet. So each facet \( \sigma \in \Delta \) defines a map

\[
\Psi_\sigma : k(\Delta)_d \text{ or } k(\Delta, \partial \Delta)_d \to k
\]

such that for all \( f \) in \( k(\Delta)_d \) or \( k(\Delta, \partial \Delta)_d \),

\[
f = \Psi_\sigma(f) \det(M_\Theta(\sigma))x_\sigma.
\]

Lemma 2.12 says that \( \Psi_\sigma = \pm \Psi_\tau \) for any two facets \( \sigma, \tau \in \Delta \). Particularly, if we fix an orientation on \( \Delta \), this map is independent of the choice of the oriented facet, giving a canonical function \( \Psi_\Delta : k(\Delta)_d \text{ or } k(\Delta, \partial \Delta)_d \to k \) (see [15, Remark 4.6]).

### 3. WLP and bistellar moves

According to Pachner’s theorem (Theorem 2.1), if we can show that the WLP is preserved by bistellar moves, then Theorem 3 holds. Thanks to Swartz’s survey article [23], we only need to verify this for bistellar moves in a special dimension.

**Theorem 3.1** (Swartz [23, Theorem 3.1 and 3.2]). Let \( \Delta \) be a \( k \)-homology \((d - 1)\)-sphere and suppose that \( \Delta' \) is obtained from \( \Delta \) via a bistellar \( i \)-move with \( i \neq \lfloor d/2 \rfloor \). Then \( \Delta \) has the WLP over \( k \) if and only if \( \Delta' \) does.

Hence Theorem 3 will follow from:

**Theorem 3.2.** If \( k \) has characteristic 2, then Theorem 3.1 also holds for \( d = 2n - 1 \) and \( i = \lfloor d/2 \rfloor = n - 1 \).

Before proving Theorem 3.2, let us see how it implies Theorem 3. The even-dimensional case is obvious. For the odd-dimensional case, the link of each vertex of \( \Delta \) is an even-dimensional PL-sphere, so we can deduce the WLP of \( \Delta \) by applying the following result of Swartz to the links of all vertices of \( \Delta \).

**Theorem 3.3** (Swartz [22, Theorem 4.26]). Let \( \Delta \) be a \((d - 1)\)-dimensional \( k \)-homology manifold on \([m] \). If for at least \( m - d \) of the vertices \( v \) of \( \Delta \) and generic linear form \( \omega_v \) and l.s.o.p. \( \Theta_v \) of \( k[\text{lk}_v \Delta] \), the multiplication

\[
\cdot_{\omega_v} : k(\text{lk}_v \Delta)_{i-1} \to k(\text{lk}_v \Delta)_i
\]

is surjective, then for generic linear form \( \omega \) and l.s.o.p. \( \Theta \) of \( k[\Delta] \), the multiplication

\[
\cdot_{\omega} : k(\Delta)_i \to k(\Delta)_{i+1}
\]

is surjective.
3.1. Proof of Theorem 3.2. Let $\Delta$ be a $k$-homology $(2n-2)$-sphere on $[m]$, which has the WLP, $\Delta' = \chi_\sigma \Delta$ with $\dim \sigma = n - 1$ and $\lk_\sigma \Delta = \partial \tau$. Without loss of generality, we may assume that $\sigma = \{1, \ldots, n\}$ and $\tau = \{n + 1, \ldots, 2n\}$. We need to show that $\Delta'$ also has the WLP. First we will give two lemmas that hold in arbitrary characteristic.

In the above notation, let

$$D = \cone \Delta = \{v\} \ast \Delta, \quad D_1 = D - \sigma, \quad K = D \cup \st_\sigma \Delta (\sigma \ast \tau).$$

Then $D, D_1$ and $K$ are all $k$-homology $(2n-1)$-balls, and $\partial K = \Delta'$. Let $\Theta = \{\theta_1, \ldots, \theta_{2n}\}$ be a generic l.s.o.p for $k[\sigma \ast \tau]$. After a linear trasformation if necessary, we may assume that

$$M_\Theta = \begin{pmatrix} v & \sigma & \tau & U \\ a_1 & I_n & T & * \\ \vdots \\ a_{2n} & 0 & I_n & * \end{pmatrix}. \tag{3.1}$$

where $U = \{2n + 1, \ldots, m\}$, $I_n$ is the $n \times n$ identity matrix, and $T$ is a diagonal matrix:

$$T = \begin{pmatrix} t_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_n \end{pmatrix}.$$

To make the subsequent calculation easier, we further require that $a_1, \ldots, a_n = 0$ and $a_{n+1}, \ldots, a_{2n} = 1$ in (3.1). This requirement is reasonable by remark 2.11.

Lemma 3.4. Let $\tau' = \{n + 1, \ldots, 2n - 1\}$, and let $\sigma_1 = \{v\} \cup \tau'$. Then $k(K, \partial K)_n$ has a basis of the form: $\{x_\tau, x_{\sigma_1}, \ldots, x_{\sigma_s}\}$, where $\sigma_i \in D_1 \setminus \partial D_1$ for $2 \leq i \leq s$.

Proof. Let $I_0$ be the ideal of $k[\sigma \ast \tau]$ generated by $x_\tau$. Then we have a short exact sequence:

$$0 \to k(D, \partial D)_n \to k(K, \partial K)_n \to (I_0/I_0 \Theta)_n \to 0. \tag{3.2}$$

Similarly, let $D_0 = \st_\sigma D = \{v\} \ast \st_\sigma \Delta$, and let $J_0$ be the ideal of $k[D_0]$ generated by $x_v$. Then there is another short exact sequence

$$0 \to k(D_1, \partial D_1) \to k(D, \partial D) \to J_0/J_0 \Theta \to 0. \tag{3.3}$$

Since $D_0 = \cone(\sigma \ast \partial \tau)$, it is easy to see that $k(D_0)_{n-1} = k$ is generated by $x_{\tau'}$. Note that there is an obvious isomorphism

$$k(D_0)_{n-1} \xrightarrow{x_v} (J_0/J_0 \Theta)_*.$$

Using (3.3) and this isomorphism, we can pick a basis $\{\sigma_1, \ldots, \sigma_s\}$ for $k(D, \partial D)_n$ with $\sigma_i \in D_1 \setminus \partial D_1$ for $2 \leq i \leq s$. (3.2) shows that $\{\sigma, \sigma_1, \ldots, \sigma_s\}$ is a basis for $k(K, \partial K)_n$. \hfill $\square$
Lemma 3.5. If the map \( \phi : k(K, \partial K) \to k(K) \) is not an isomorphism, then \( \ker \phi \) is spanned by an element of the form:

\[
\alpha = x_\sigma + \sum_{i=1}^{s} l_i x_{\sigma_i}, \text{ where } l_i \in k \text{ and } l_1 = (-1)^n \prod_{i=1}^{n} t_i.
\]

Proof. Since \( \partial D = \Delta \) has the WLP, Proposition 2.10 shows that the map \( k(D, \partial D) \to k(D) \) is an isomorphism. Note that this isomorphism factors through

\[
k(D, \partial D) \to k(K, \partial K) \xrightarrow{\phi} k(K).
\]

Hence if \( \phi \) is not an isomorphism, then \( \dim_k \ker \phi = 1 \), and by Lemma 3.4, \( \ker \phi \) is spanned by an element \( \alpha \) of the form in the lemma.

Since \( st_\sigma K = \sigma * \partial(\tau \cup \{v\}) \), the expression of \( M_\Theta \) implies that \( x_\sigma \) restricts to \( (-1)^{n-1} (\prod_{i=1}^{n} t_i) x_{\sigma_1} \) in \( k(st_\sigma K) \). Thus, the equation

\[
\phi_\sigma \circ \phi(\alpha) = x_\sigma + l_1 x_{\sigma_1} = 0, \text{ where } \phi_\sigma : k(K) \to k(st_\sigma K),
\]

gives \( l_1 = (-1)^n \prod_{i=1}^{n} t_i \).

□

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. By Proposition 2.10, we only need to show that the map \( \phi : k(K, \partial K) \to k(K) \) is an isomorphism, since \( \partial K = \Delta' \).

Set \( \beta = \sum_{i=1}^{s} l_i x_{\sigma_i} \). If \( \phi \) is not an isomorphism, then Lemma 3.5 implies that \( x_\sigma = \beta \) in \( k(D) \). Note that char \( k = 2 \). Hence we have

\[
\Psi_D(\beta^2) = \Psi_D(x_\sigma \beta) = l_1 \Psi_D(x_\sigma x_{\sigma_1}) = t_1 \cdots t_n,
\]

where the second equality follows by the fact that \( \sigma_i \notin st_\sigma \Delta \) for \( i \geq 2 \) by Lemma 3.4, and the third equality uses Lemma 3.5 and the fact that \( \det(M_\Theta(\sigma_1 \cup \sigma)) = 1 \).

Viewing \( t_i \) as variables, the canonical function \( \Psi_D \) and the above coefficients \( l_i \) are then functions on \( t_1, \ldots, t_n \). Define a partial differential operator \( P : = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \).

Then by (3.4), we have

\[
P \Psi_D(\beta^2) = P \sum_{i=1}^{s} l_i^2 \Psi_D(x_{\sigma_i}^2) = \sum_{i=1}^{s} l_i^2 P \Psi_D(x_{\sigma_i}^2) = 1,
\]

where the first and second equality come from the assumption that char \( k = 2 \). However, since \( \tau \notin D \), it follows that for any \( \sigma_i \), there is some \( n_i \in n + 1, \ldots, 2n \) with \( n_i \notin F_0(L_i) \), where \( L_i = st_\sigma D \). This implies that \( t_{n_i} \) dose not appear in the function \( \Psi_D(x_{\sigma_i}^2) = \Psi_{L_i}(x_{\sigma_i}^2) \). Therefore \( P \Psi_D(x_{\sigma_i}^2) = 0 \) for all \( 1 \leq i \leq s \), a contradiction. □
4. Applications of Theorem 3

4.1. Enumeration of bistellar moves. $g$-theorem for PL-spheres can be applied to the enumeration problem of bistellar moves on a PL-sphere.

**Theorem 4.1.** Let $\Delta$ be a PL-sphere of dimension $d$. Then the number of $k$-moves in the sequence of bistellar moves taking $\Delta$ to $\partial \Delta^{d+1}$ cannot exceed the number of $(d-k)$-moves for $k \leq \lfloor d/2 \rfloor$.

This is a direct consequence of the non-negativity of the $g$-vector of a PL-sphere and the following elementary result that describe the effect of a bistellar move on the $h$-vector. Note that the $g$-vector of $\partial \Delta^{d+1}$ is $(1, 0, \ldots, 0)$.

**Proposition 4.2** (see i.e. [22, Proposition 5.1]). If a triangulated $d$-manifold $\Delta'$ is obtained from $\Delta$ by a bistellar $k$-move, $0 \leq k \leq \lfloor (d-1)/2 \rfloor$, then

\[
g_{k+1}(\Delta') = g_{k+1}(\Delta) + 1;
\]
\[
g_i(\Delta') = g_i(\Delta) \text{ for } i \neq k + 1.
\]

Furthermore, if $d$ is even and $\Delta'$ is obtained from $\Delta$ by a bistellar $d/2$-move, then

\[
g_i(\Delta') = g_i(\Delta) \text{ for all } i.
\]

4.2. WLP of simplicial toric varieties. Here is another application of Theorem 3 to toric algebraic geometry. Consider $\Sigma$ a complete simplicial fan in $\mathbb{R}^d$. Here we require $\Sigma$ to be rational, i.e., each ray of $\Sigma$ is generated by a primitive vector $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{id}) \in \mathbb{Z}^d$. Let $\Delta_\Sigma$ be the underlying simplicial complex of $\Sigma$. By definition, $\{i_1, \ldots, i_k\} \subset [m]$ is a face of $\Delta_\Sigma$ if and only if $\lambda_{i_1}, \ldots, \lambda_{i_k}$ span a cone of $\Sigma$. It is easy to see that the vectors $\lambda_1, \ldots, \lambda_m$ define an l.s.o.p. for $\mathbb{Q}[\Delta_\Sigma]$:

\[
\Theta_\Sigma = \{\theta_i = \lambda_{i1}x_1 + \cdots + \lambda_{im}x_m\}_{i=1}^d.
\]

The rational cohomology of the associated toric variety $X_\Sigma$ can be calculated as follows:

**Theorem 4.3** ([5, Danilov], see also [4, Theorem 12.4.1]). Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^d$. Then there is a ring isomorphism

\[
H^*(X_\Sigma; \mathbb{Q}) \cong \mathbb{Q}(\Delta_\Sigma; \Theta_\Sigma),
\]
\[
H^{2i}(X_\Sigma; \mathbb{Q}) \cong \mathbb{Q}(\Delta_\Sigma; \Theta_\Sigma)_i, \quad H^{2i+1}(X_\Sigma; \mathbb{Q}) = 0,
\]

Note that when $\Sigma$ is complete and simplicial, $\Delta_\Sigma$ is a PL $(d-1)$-sphere, so from Theorem 3 and Theorem 4.3 we get the following theorem, which is a ‘weak’ generalization of the hard Lefschetz theorem on projective toric varieties to all simplicial toric varieties.
Theorem 4.4. Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^d$, $X_\Sigma$ the associated toric variety, $\mu_i = \dim H^{2i}(X_\Sigma; \mathbb{Q})$. Then the vector
$$\{\mu_0, \mu_1 - \mu_0, \ldots, \mu_{\lfloor d/2 \rfloor} - \mu_{\lfloor d/2 \rfloor - 1}\}$$
is a $M$-vector. Moreover, if the rays of $\Sigma$ are in generic positions, then $H^*(X_\Sigma; \mathbb{Q})$ has the WLP.

4.3. GKS conjecture. While it is well known that the Grünbaum-Kalai-Sarkaria conjecture is a consequence of Theorem 3, we include its proof here by following [1, Corollary 4.8], for the sake of completeness.

Theorem 4.5. Conjecture 4 holds.

Proof. Suppose $\Delta$ can be piecewise linearly embedded into $\mathbb{R}^{2d}$. Then there exists a PL $2d$-sphere $K$ containing $\Delta$ as a subcomplex. By Theorem 3, there exists $(\omega, \Theta) \in \mathcal{W}(K)$ for any infinite field $k$ of characteristic 2. Hence in the following commutative diagram, the top horizontal map is a surjection
$$
\begin{array}{ccc}
k(K)_d & \xrightarrow{\omega} & k(K)_{d+1} \\
\downarrow & & \downarrow \\
k(K)_d & \xrightarrow{\omega} & k(\Delta)_{d+1}
\end{array}
$$
Since the vertical projections are clearly surjective, the bottom horizontal map is also surjective. So we have
$$\dim_k k(K)_{d+1} \leq \dim_k k(K)_d \leq f_d(\Delta). \quad (4.1)$$

The second inequality comes from the fact that $k(\Delta)$ is spanned by face monomials.

Furthermore, for any $(d-1)$-face $\sigma \in \Delta$, choose a facet $\tau \in K$ such that $\sigma \subset \tau$. Without loss of generality, we may assume $\sigma = \{1, \ldots, d\}$, $\tau = \{1, \ldots, 2d+1\}$ and
$$M_\Theta = (I_{2d+1} \mid A).$$
Hence the generators $x_1, \ldots, x_d$ do not appear in the expressions of the $d+1$ linear forms $\theta_{d+1}, \ldots, \theta_{2d+1}$. Multiplying $x_\sigma$ by $\theta_i$ for each $i \geq d+1$ produces exactly $d+1$ linearly independent relations among the face monomials in $k(K)_{d+1}$. So there are at most $(d+1)f_{d-1}(\Delta)$ linearly independent relations between the generators, i.e. $d$-face monomials, of $k(K)_{d+1}$. It follows that
$$\dim_k k(K)_{d+1} \geq f_d(\Delta) - (d+1)f_{d-1}(\Delta). \quad (4.2)$$
Combining (4.1) with (4.2) gives the desired inequality. \qed
REFERENCES

[1] K. Adiprasito, Combinatorial Lefschetz theorems beyond positivity, arXiv:1812.10454, 2018.
[2] L. Billera and C. Lee, A proof of the sufficiency of McMullen’s conditions for f-vectors of simplicial convex polytopes, J. Combin. Theory Ser. A 31 (1981), no. 3, 237–255.
[3] W. Bruns and J. Herzog, Cohen-Macaulay Rings, revised ed., Cambridge Studies in Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, 1998.
[4] D. Cox, J. Little, and H. Schenck, Toric varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.
[5] V. Danilov, The geometry of toric varieties, Uspekhi Mat. Nauk. 33 (1978), no. 2, 85–134 (Russian); Russian Math. Surveys 33 (1978), 97–154 (English translation).
[6] B. Grünbaum, Higher-dimensional analogs of the four-color problem and some inequalities for simplicial complexes, J. Combinatorial Theory 8 (1970), 147–153.
[7] A. Gundert, On the complexity of embeddable simplicial complexes, Diplomarbeit, Freie Universität Berlin, available at arXiv:1812.08447, 2009.
[8] D. Cook II and U. Nagel, Enumerations deciding the weak lefschetz property, arXiv:1105.6062, 2011.
[9] G. Kalai, The diameter of graphs of convex polytopes and f-vector theory, In: Applied geometry and discrete mathematics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 387–411.
[10] F. Macaulay, Some properties of enumeration in the theory of modular systems, Proc. London Math. Soc. 26 (1927), 531–555.
[11] P. McMullen, The number of faces of simplicial polytopes, Israel J. Math. 9 (1971), 559–570.
[12] _______, On simple polytopes, Invent. Math. 113 (1993), no. 2, 419–444.
[13] _______, Weights on polytopes, Disc. Comp. Geom. 15 (1996), no. 4, 363–388.
[14] U. Pachner, P.L. homeomorphic manifolds are equivalent by elementary shellings, European J. Combin. 12 (1991), no. 2, 129–145.
[15] S. A. Papadakis and V. Petrotou, The characteristic 2 anisotropicity of simplicial spheres, arXiv:2012.09815, 2020.
[16] G. Reisner, Cohen-Macaulay quotients of polynomial rings, Adv. Math. 21 (1976), no. 1, 30–49.
[17] K. S. Sarkaria, Shifting and embeddability of simplicial complexes, Technical report, Max-Planck-Institut für Mathematik, Bonn, 1992.
[18] R. Stanley, The upper bound conjecture and Cohen-Macaulay rings, Studies in Appl. Math. 54 (1975), no. 2, 135–142.
[19] _______, The number of faces of a simplicial convex polytope, Adv. Math. 80 (1980), no. 3, 251–258.
[20] _______, Combinatorics and Commutative Algebra, 2nd ed., Progress in Math., vol. 41, Birkhauser, Boston, 1996.
[21] E. Swartz, g-elements, finite buildings and higher Cohen-Macaulay connectivity, J. Combin. Theory Ser. A 113 (2006), no. 7, 1305–1320.
[22] _______, Face enumeration–from spheres to manifolds, J. Eur. Math. Soc. 11 (2009), no. 3, 449–485.
[23] _______, Thirty-five years and counting, arXiv:1411.0987, 2014.

FEIFEI FAN, SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, GUANGZHOU, 510631, CHINA.

Email address: fanfeifei@mail.nankai.edu.cn