EXPONENTIAL STABILITY FOR INFINITE-DIMENSIONAL NON-AUTONOMOUS PORT-HAMILTONIAN SYSTEMS

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Abstract. We study the non-autonomous version of an infinite-dimensional port-Hamiltonian system on an interval \([a, b]\). Employing abstract results on evolution families, we show \(C^1\)-well-posedness of the corresponding Cauchy problem, and thereby existence and uniqueness of classical solutions for sufficiently regular initial data. Further, we demonstrate that a dissipation condition in the style of the dissipation condition sufficient for uniform exponential stability in the autonomous case also leads to a uniform exponential decay of the energy in this non-autonomous setting.

Key words: Infinite-dimensional port-Hamiltonian system, non-autonomous Cauchy problem, evolution family, well-posedness, uniform exponential stability.

MSC: 47D06, 35L50, 35B64

1. Introduction

Consider the following non-autonomous partial differential equation

\[
\frac{\partial x}{\partial t}(t, \zeta) = \left( P_1 \frac{\partial}{\partial \zeta} + P_0 \right) (\mathcal{H}(t, \zeta)x(t, \zeta)) \quad \zeta \in [a, b], \ t \geq 0,
\]

\[
x(0, \zeta) = x_0(\zeta), \quad \zeta \in [a, b],
\]

on the state space \(X = L^2(a, b; \mathbb{K}^n)\) (for \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\)), and with boundary conditions of the type

\[
\tilde{W}_B \left[ \begin{array}{c} (\mathcal{H}x)(t, a) \\ (\mathcal{H}x)(t, b) \end{array} \right] = 0, \quad t \geq 0,
\]

where \(x(t, \zeta)\) takes its values in \(\mathbb{K}^n\), \(\mathcal{H}(t, \zeta)\), \(P_0, P_1\) are \(n \times n\) matrices and \(\tilde{W}_B\) is an \(n \times 2n\) matrix. This class of PDE is called a (infinite-dimensional, linear) non-autonomous port-Hamiltonian system and covers, among others, the wave equation, the transport equation, beam equations as well as certain networks all with possibly time- and spatial dependent parameters. As an example, let us consider the model of vibrating string on the compact interval \([a, b]\). We assume that the string is fixed at the left end point \(\zeta = a\) and at the right end point \(\zeta = b\) a damper is attached. In addition, Young’s modulus \(T\) and the mass density \(\rho\) of the string are assumed to be time- and spatial dependent. Let us denote by \(\omega(t, \zeta)\) the vertical position of the string at position \(\zeta \in [a, b]\) and time \(t \geq 0\). Then the evolution of the vibrating string can be modelled by the non-autonomous wave equation

\[
\frac{\partial}{\partial t} \left( \rho(t, \zeta) \frac{\partial w}{\partial t}(t, \zeta) \right) = \frac{\partial}{\partial \zeta} \left( T(t, \zeta) \frac{\partial w}{\partial \zeta}(t, \zeta) \right), \quad \zeta \in [a, b], \ t \geq 0,
\]

\[
\frac{\partial w}{\partial t}(t, a) = 0, \quad t \geq 0
\]

\[
T(b, t) \frac{\partial w}{\partial \zeta}(t, b) = -k \frac{\partial w}{\partial t}(t, b), \quad t \geq 0.
\]

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Taking the momentum-strain couple \( \left( \rho \frac{\partial v}{\partial t}, \frac{\partial w}{\partial \zeta} \right) \) as the state variable one sees that the non-autonomous vibrating string can be written as a system of the form (1)-(3) with \( P_0 = 0 \),
\[
P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{H}(t, \zeta) = \begin{bmatrix} \frac{1}{\sqrt{1 - T(t, \zeta)}} \\ 0 \end{bmatrix} T(t, \zeta) \quad \text{and} \quad \tilde{W}_B = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Autonomous port-Hamiltonian systems, that is system (1)-(3) with time-independent \( \mathcal{H}(t, \zeta) = \mathcal{H}(\zeta) \), have been investigated recently, e.g. in [18, 15, 6, 28, 32, 29]. Well-posedness and uniform exponential (or, asymptotic) stability for autonomous port-Hamiltonian systems can in most cases be tested via a simple matrix condition [15, 14].

If the Hamiltonian density \( \mathcal{H} \) is coercive as a matrix multiplication operator on \( L^2(a, b; \mathbb{K}^{n \times n}) \), the energy (or, Hamiltonian) of the system
\[
\|x\|_H^2 := (x | x)_H := \int_a^b x^*(\zeta)\mathcal{H}(\zeta)x(\zeta) \, d\zeta
\]
defines an equivalent (to the usual \( L^2 \)-norm) norm on \( L^2(a, b; \mathbb{C}^n) \). In this case, the existence of weak or classical solutions with non-increasing energy can be tested via a simple matrix condition. More precisely, consider the linear operator
\[
A \mathcal{H} = P_1 \frac{\partial}{\partial \zeta} \mathcal{H} + P_0 \mathcal{H}
\]
\[
D(A \mathcal{H}) = \left\{ x \in L^2(a, b; \mathbb{K}^n) | \mathcal{H}x \in H^1(a, b; \mathbb{K}^n), \tilde{W}_B \left( \frac{\mathcal{H}x(t, b)}{\mathcal{H}x(t, a)} \right) = 0 \right\}.
\]

Then \( A \mathcal{H} \) generates a contractive \( C_0 \)-semigroup on the energy state space \( X_\mathcal{H} := L^2_\mathcal{H}(a, b; \mathbb{K}^n) := (L^2(a, b; \mathbb{K}^n), \| \cdot \|_\mathcal{H}) \) if and only if \( W_B \Sigma W_B^* \geq 0 \) and \( P_0 + P_0^* \leq 0 \), where
\[
W_B := \sqrt{2} \tilde{W}_B \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1} \quad \text{and} \quad \Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},
\]
and this is exactly the case, if the operator \( A \) is dissipative on \( L^2(a, b; \mathbb{K}^{n \times n}) \) (equipped with the standard \( L^2 \)-norm). Moreover, the asymptotic behaviour of the solution of (1)-(3) has also been studied, e.g. in [30, 15, 6]. The authors give a result on exponential stability using a Lyapunov method in [30, 15] and using a frequency domain method in [6] based on classical stability theorems by Gearhart, Prüss and Huang, and by Arendt, Batty, Lyubich and Vă. Note that port-Hamiltonian systems of order \( N \geq 2 \) also have been investigated with similar results in [6, 28], E. g., it has been proved in [30, Theorem III.2] that the \( A \mathcal{H} \) generates an exponentially stable \( C_0 \)-semigroup if for some constant \( c > 0 \) one of the following conditions is satisfied for all \( x \in D(A \mathcal{H}) \).

\[
(7) \quad \Re(A \mathcal{H}x | x)_\mathcal{H} \leq -c|\mathcal{H}(b)x(b)|^2.
\]
\[
(8) \quad \Re(A \mathcal{H}x | x)_\mathcal{H} \leq -c|\mathcal{H}(a)x(a)|^2.
\]

This equalities hold, for example, if \( W_B \Sigma W_B^* > 0 \) [15, Lemma 9.4.1], i.e. \( W_B \Sigma W_B^* \) is a symmetric, positive definite matrix.

In contrast to autonomous port-Hamiltonian systems, the non-autonomous situation with \( \mathcal{H} \) and/or \( \tilde{W}_B \) depending on the time variable has not been considered so far. The main purpose of this paper is to generalize the known results on well-posedness and exponential stability for autonomous port-Hamiltonian systems to the non-autonomous setting. In particular, we show that the technique used in [30, 15] for the proof of uniform exponential stability can be applied to non-autonomous port-Hamiltonian systems as well.

To do so, we write system (1)-(3) as an abstract non-autonomous evolution equation of the form
\[
(9) \quad \dot{x}(t) - AB(t)x(t) = 0 \quad \text{a.e. on} \ [0, \infty),
\]
\[
(10) \quad x(0) = x_0, \quad 0 > 0,
\]
where \( A : D(A) : X \rightarrow X \) is the generator of a contractive \( C_0 \)-semigroup and \( B : [0, +\infty) \rightarrow \mathcal{L}(X) \) is a time-dependent multiplicative perturbation. The well-posedness of this abstract class has been studied by Schnaubelt and Weiss [25]. The parabolic case has been investigated in [5].
Recall that a continuous function \( u : [0, \infty) \rightarrow X \) is called a classical solution of (9)-(10) if \( u(t) \in D(AB(t)) \) for all \( t \geq 0 \), \( u \in C^1((0, \infty), X) \) and \( u \) satisfies (9)-(10), so that in particular \( ABu \in C((0, \infty); X) \). As in the autonomous case, well-posedness means that (9)-(10) has a unique classical solution which continuously depends on the initial data \( x_0 \in D(AB(0)) \).

The study of non-autonomous evolution equations has a long history which goes back to Vito Volterra in 1938 [31]. However, it was only in 1950–1970 that a general theory has been developed by T. Kato [17], [16], H. Tanabe [27], P. E. Sobolevsky [26] and others. P. Acquistapace and B. Terreni [3] extended the previous work by Kato, Tanabe, Sobolevsky and obtained some of the most powerful results. Their approach is based on the discretization of the given equation and the use of semigroup theory. Another approach to these equations using semigroup theory and evolution families has been used by J. S. Howland [13]. This approach is presented in the monograph [8] by C. Chicone and Y. Latushkin, and has been further developed by R. Nagel and G. Nickel [22], R. Schnaubelt [24] and many other authors. For the Hilbert space setting a variational approach has been developed essentially by Lions’s school, leading to the existence and uniqueness of weak solutions [20, 21].

The existence and uniqueness for solutions of a non-autonomous Cauchy problem is closely related to the existence of a (strongly continuous) evolution family

\[
U := \{U(t, s) : t \geq s \geq 0\} \subset \mathcal{L}(X)
\]

i.e., a family that has the following properties: \( U(t, t) = I \) and \( U(t, s) = U(t, r)U(r, s) \) for every \( 0 \leq s \leq r \leq t \) and \( U(\cdot, \cdot) : \Delta \rightarrow \mathcal{L}(X) \) is strongly continuous where \( \Delta := \{(t, s) \in \mathbb{R}^2 \mid t \geq s \geq 0\} \). More precisely, if the abstract Cauchy problem is well-posed for all initial data \( x_s \in D(AB(s)) \) and initial times \( s \geq 0 \), i.e.

\[
\dot{x}(t) - AB(t)x(t) = 0, \quad t \geq s \quad x(s) = x_s \in D(AB(s))
\]

has a unique classical solution which depends continuously on the initial data, then the solutions \( x(t, s, x_s) \) define an evolution family \( U \subset \mathcal{L}(X) \) by \( U(t, s)x_s := x(t, s, x_s) \). On the other hand, for an evolution family to be the solution operator (for classical solutions) of an abstract Cauchy problem, \( U \) needs to satisfy further properties then just being an evolution family, e.g. \( U(\cdot, s)x \in C^1((s, \infty); X) \) for all \( x_s \in D_s \) for some dense subsets \( D_s \subset X \) [11, IV.8].

The (exponential) growth bound of an evolution family \( U \) is defined by

\[
w_0(U) := \inf \left\{ w \in \mathbb{R} \mid \text{there is } M_w \geq 1 \text{ with } \|U(t, s)\| \leq M_w e^{w(t-s)} \text{ for } t \geq s \right\}.
\]

The evolution family is called exponentially bounded if \( w_0(U) < +\infty \) and exponentially stable if \( w_0(U) < 0 \). If \( (S(s))_{s \geq 0} \) is a \( C_0 \)-semigroup on \( X \) then \( U(t, s) := T(t-s) \) yields a strongly continuous evolution family. In contrast to \( C_0 \)-semigroups which are always exponentially bounded, i.e. \( \|T(t)\| \leq M e^{\omega t} \) \((T \geq 0)\) for some \( M \geq 1, \omega \in \mathbb{R} \), see e.g. [11, Proposition I.5.5], the same cannot be said about evolution families in general. Moreover, in many cases uniform exponential stability for a \( C_0 \)-semigroup (or, the growth bound) can be determined via the spectrum of its generator, e.g. for analytic semigroups. In contrast, for evolution families this fails to be true even in the finite dimensional case, see e.g. [11, Example VI.9.9]. Nevertheless, the asymptotic of an exponential evolution family can be characterized in terms of the associated evolution semigroup. Indeed, it is well known [8, Section.3.3] that to each exponential bounded evolution family \( U \) one may associate a unique \( C_0 \)-semigroup \( T \) on \( L^p([0, +\infty); X) \) \((p \in [1, \infty))\) defined for each \( f \in L^p([0, +\infty); X) \) by setting

\[
(T(t)f)(s) := \begin{cases} U(s, s-t)f(s-t) & \text{for } s, s-t \in [0, +\infty), \\
0 & \text{for } s \in [0, +\infty), t - s \not\in [0, +\infty), \end{cases}
\]

Denoting by \( G \) the generator of \( T \), the following characterisation is well known: Let \( U \) be an exponentially bounded evolution family on \( X \) and let \( p \in [1, \infty) \). Then the following assertions are equivalent.

(i) \( U \) is exponentially stable.

(ii) The generator \( G \) of the associated evolution semigroup is surjective.
(iii) For all $x \in X$ and $s \geq 0$ there exists a constant $M > 0$ such that
\[ \int_s^\infty \|U(t, s)x\|^p \, dt \leq M\|x\|^p. \]

(iv) For all $f \in L^p([0, +\infty); X)$ one has $U \ast f \in L^p([0, +\infty); X)$.

For the proof and other concepts of stability we refer e.g., to [24, 7, 12, 19] and the references therein.

We abstain from this route towards exponential stability for non-autonomous port-Hamiltonian systems and follow a different approach for the study of exponential stability by mimicking the techniques used in [30, 15] for the autonomous case. These are based on an idea of Cox and Zuazua [9].

This paper is organized as follows. In Section 2 we recall some abstract results on the theory of evolution families and the well-posedness for non-autonomous evolution equations and prove some preliminary results. In Section 3 we provide sufficient conditions for which the non-autonomous port-Hamiltonian system is well-posed and the corresponding evolution family is exponentially stable. The last section is devoted to some examples.

2. Background on evolution families and preliminary results

Throughout this section $X$ is a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We denote by $(\cdot, \cdot)$ the scalar product and by $\| \cdot \|$ the norm on $X$. Let $\{A(t) | t \geq 0\}$ be a family of linear, closed operators with domains $\{D(A(t)) | t \geq 0\}$. Consider the non-autonomous Cauchy problems
\[ \dot{u}(t) - A(t)u(t) = 0 \quad \text{a.e. on } [s, \infty), \quad u(s) = x_s, (s > 0). \]
Recall that a continuous function $u : [s, \infty) \to X$ is called a classical solution of (11) if $u(t) \in D(A(t))$ for all $t \geq s, u \in C^1((s, \infty), X)$ and $u$ satisfies (11).

**Definition 2.1.** (a) The non-autonomous Cauchy problem (11) is called $C^1$-well posed if there is a family $\{Y_t | t \geq 0\}$ of dense subspaces of $X$ such that:

(i) $Y_t \subseteq D(A(t))$ for all $t \geq 0$.

(ii) For each $s \geq 0$ and $x_s \in Y_s$ the Cauchy problem (11) has a unique classical solution $u(\cdot, s, x_s)$ with $u(t, s, x_s) \in Y_t$ for all $t \geq s$.

(iii) The solutions depend continuously on the initial data $s, x_s$.

In this case we also say that (11) is $C^1$-well posed on $Y_t$ if we want to specify the regularity subspaces $Y_t, t \geq 0$.

(b) We say that the family $\{A(t), t \geq 0\}$ generates an evolution family $\mathcal{U}$ if there is a family $\{Y_t | t \geq 0\}$ of dense subspaces of $X$ with $Y_t \subseteq D(A(t))$, $U(t, s)Y_s \subseteq Y_t$ and for every $x_s \in Y_s$ the function $U(\cdot, s)x_s$ is a classical solution of (11).

In the autonomous situation, i.e., if $A(t) = A$ is a time-independent operator, it is well known that the associated Cauchy problem is $C^1$-well posed if and only if $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$. In this case the unique classical solution to (11) is given by $T(\cdot - s)x_s$ for each $x_s \in D(A)$.

As mentioned in the introduction, the evolution law (i) do not guarantees that the evolution family is strongly differentiable in the first component and that $\mathcal{U}$ is generated by a family of linear closed operators. In fact, may happen that the trajectory $U(\cdot, s)x$ is differentiable only for $x = 0$. The standard counterexample is given by $U(t, s) = \frac{p(t)}{p(s)}$ with $X = \mathbb{C}$ and $p$ is a nowhere differentiable function such that $p$ and $1/p \in C_b(\mathbb{R})$. However, the following characterization holds:

**Proposition 2.2.** The Cauchy problem (11) is $C^1$-well posed if and only if $\{A(t), t \geq 0\}$ generates a unique evolution family.

For this statement, we refer to [11, Proposition 9.3] or [23, Proposition 3.10].

Let us consider the special case where the domains $D(A(t)) = D$ are time independent. Then it is well known that the family of closed, linear operators $\{A(t) | t \geq 0\}$ generates a unique evolution family with $Y_t = D$ for all $t \geq 0$ if the following assumptions are satisfied:
(H1) \( \{ A(t) \mid t \geq 0 \} \) is Kato-stable: i.e., for \( T \geq 0 \) there are \( \omega \in \mathbb{R}, M \geq 1 \) such that
\[
\| (\lambda - A(t_n))^{-1}(\lambda - A(t_{n-1}))^{-1} \cdots (\lambda - A(t_0))^{-1} \|_{\mathcal{L}(X)} \leq M (\lambda - \omega)^{-n},
\]
for all \( 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq T, n \in \mathbb{N} \) and all \( \lambda > \omega \).

(H2) For each \( T \geq 0 \) and \( x \in D \) the function \( A(t)x \in C^1([0, T]; X) \).

This result is due to Kato [16], we refer to the survey paper [24] for further reading. The stability condition (H1) is always fulfilled if for each \( t \geq 0 \) the operator \( A(t) \) generates a contractive \( C_0 \)-semigroup. For general semigroups, we recall the following two useful stability tests [27, Propositions 4.3.2 and 4.3.3]:

Proposition 2.3. Assume that there is a family \( \{ \| \cdot \|_t, t \geq 0 \} \) of norms on \( X \) that are equivalent to the original time independent norm \( \| \cdot \| \) such that for each \( T \geq 0 \) there exists a constant \( c \geq 0 \) such that
\[
\| x \|_{t} \leq e^{c(t-s)} \| x \|_s
\]
for all \( x \in X \) and \( t, s \in [0, T] \). If \( A(t) \) generates a contractive semigroup on \( X_t := (X, \| \cdot \|_t) \) for all \( t \geq 0 \) then the family \( \{ A(t), t \geq 0 \} \) is Kato-stable. If moreover \( M : [0, \infty) \rightarrow \mathcal{L}(X) \) is a locally bounded function, then the perturbed family \( \{ A(t) + M(t), t \geq 0 \} \) is again Kato-stable.

2.1. A class of non-autonomous evolution equations. In this section we consider the special case where \( A(t) \) is defined as a bounded non-autonomous multiplicative perturbation of a dissipative operator. More precisely, let \( B : [0, \infty) \rightarrow \mathcal{L}(X) \) be a function of class \( C^2 \). Assume that \( B \) is self-adjoint and uniformly coercive, i.e., \( B(t)^* = B(t) \) and
\[
(B(t)x|x) \geq \beta \| x \|^2
\]
for some constant \( \beta > 0 \) and for all \( t \geq 0 \) and \( x \in H \). Then for each \( t \in [0, \infty) \) the function
\[
\| x \|_t := \sqrt{(B(t)x|x)} = \| B^{1/2}(t)x \|
\]
defines a norm which is equivalent to the time-independent reference norm \( \| \cdot \| \). Moreover, the norms \( \{ \| \cdot \|_t, t \geq 0 \} \) are uniformly equivalent to \( \| \cdot \| \) on each compact interval \( [0, T] \subset [0, \infty) \). Indeed, let \( T > 0 \) and set \( \beta_T := \max_{t \in [0, T]} \| B(t) \|_{\mathcal{L}(X)} \) then we have
\[
\frac{1}{\beta_T} \| x \|_s \leq \| x \|_t \leq \frac{1}{\beta} \| x \|_t \quad \text{for all } t, s \in [0, T] \quad \text{and } x \in X.
\]

Let \( A : D(A) \subset X \rightarrow X \) be the infinitesimal generator of a contractive \( C_0 \)-semigroup on \( H \) and consider the class of non-autonomous problems
\[
x(t) - AB(t)x(t) = 0 \quad \text{a.e. on } [s, \infty), \quad x(s) = x_s, s \geq 0.
\]
Here the operators \( AB(t) \) are defined on their natural domains
\[
D(AB(t)) = \{ x \in X \mid B(t)x \in D(A) \}
\]
which may depend on \( t \). Then for each \( t \geq 0 \) the operator \( AB(t) \), and thus \( B(t)A \), generates a contractive semigroup on \( X_t \) [15, Lemma 7.2.3]. Note that \( AB(t) \) and \( B(t)A \) are similar since \( B(t)AB(t)B^{-1}(t) = B(t)A \) and \( B^{-1}(t) \in \mathcal{L}(X) \) for every \( t \geq 0 \).

In Theorem 2.4 we show that (16) is \( C^1 \)-well posedness and that the associated evolution family satisfies an exponential decay estimate. The latter will be needed in the next section where we study the exponential stability of the evolution family generated by non-autonomous port-Hamiltonian systems. We point out that the \( C^1 \)-well posedness for non-autonomous evolution equations of the form (16) has been studied by Schnaubelt and Weiss [25, Proposition 2.8-(a)]. In the following we include a (slightly modified) proof for the sake of completeness and in order to make reading this paper easier. In fact, the proof of [25, Proposition 2.8-(a)] is based on a perturbation argument due to Curtain and Pritchard, see [25, Proposition 2.7], which is not needed here.

Theorem 2.4. The Cauchy problem (16) is \( C^1 \)-well posed with regularity space \( \{ D(AB(t)), t \geq 0 \} \).

Further, for each compact interval \( [0, T] \subset [0, \infty) \) there exist constants \( c_T, \kappa_T > 0 \) such that
\[
\| U(t, \tau)x \|_s^2 \leq e^{\frac{M_T}{\kappa_T} \tau-s} \| U(s, \tau)x \|_s^2 \quad (0 \leq \tau \leq s \leq t \leq T),
\]
for each $x \in X$ where $M_T = \max_{t \in [0,T]} \| \hat{B}(t) \|_{\mathcal{L}(X)}$.

Proof. (i) We first claim that $\hat{A} := \{ B(t)A + \hat{B}(t)B(t)^{-1} \mid t \geq 0 \}$ generates an evolution family $V$ with constant regularity space $D(A)$ if and only if $A := \{ A(t) \mid t \geq 0 \}$ generates an evolution family $U$ with regularity spaces $Y_t = D(AB(t)), t \geq 0$. Moreover, both evolution families $U, V$ are related as follows

$$U(t, s) = B(t)^{-1}V(t, s)B(s), \quad (t, s) \in \Delta.$$ 

Note that $D(AB(t))$ is dense in $X$ since $D(AB(t)) = B^{-1}(t)D(A)$ and $B(t) \in \mathcal{L}(X)$ for all $t \geq 0$. Assume that $A$ generates an evolution family $U$ and let $V$ be defined by (18). Then for each $x \in D(A)$ we have $V(\cdot, s)x \in C^1((s, \infty);X), V(t, s)x \in D(AB(t))$ and

$$\frac{d}{dt}V(t, s)x = \frac{d}{dt}B(t)U(t, s)B^{-1}(s)x = \frac{d}{dt}B(t)U(t, s)B^{-1}(s)x + B(t)AB(t)U(t, s)B^{-1}(s)x = [B(t)A + \hat{B}(t)B(t)^{-1}]B(t)U(t, s)B^{-1}(s)x = \hat{B}(t)U(t, s)x$$

for all $t \geq s$. Thus $V$ is generated by $\hat{A}$. The converse implication can be similarly proved.

Now, to finish the proof of the first assertion of Theorem 2.4 it remains to check that $\hat{A}$ satisfies Kato’s conditions (H1)-(H2). For each $t \geq 0$ the operator $B(t)A$ generates a contractive $C_0$-semigroup on $H_t = (H, \| \cdot \|_t)$ as remarked above. Further, the family $\{ \| \cdot \|_t \mid t \geq 0 \}$ defined by (14) satisfies the stability test condition (12) with $c = cr = \max_{t \in [0,T]} (\| B(t) \|_t)$. Moreover, $\hat{B}(\cdot)B(\cdot)^{-1} : [0, \infty) \rightarrow \mathcal{L}(X)$ is locally bounded by assumption. We deduce from Proposition 2.3 that the family $\{ B(t)A + \hat{B}(t)B(t)^{-1} \}$ is Kato-stable. Moreover, for every $x \in D = D(A)$ one has that $Ax \in X$ is a constant vector and since $B$ is of class $C^2$ it follows that $\hat{B}(\cdot)Ax \in C^1([0, \infty);X)$, in particular $B(\cdot)Ax \in C^1([0,T];X)$ for each $x \in D$ and $T > 0$. Thus, Kato’s conditions (H1)-(H2) are satisfied,

(ii) Let us now prove (17). For each $t \geq 0$ we have seen that $AB(t)$ generates a contractive $C_0$-semigroup on $X_t$. Thus, in particular, $AB(t)$ is dissipative, i.e.,

$$\Re(\langle AB(t)x \mid x \rangle), \leq 0 \quad \text{for every } x \in D(AB(t)).$$

Let $[0, T]$ be a compact interval. Let $\tau \in [0, T)$ and $x_\tau \in D(AB(\tau))$. Thus using the first part of the proof we have

$$\frac{d}{dt}\|U(t, \tau)x_\tau\|^2_{L_2} = \frac{d}{dt}B(t)U(t, \tau)x_\tau \mid U(t, \tau)x_\tau$$

$$= (\hat{B}(t)U(t, \tau)x_\tau \mid U(t, \tau)x_\tau) + 2 \Re(B(t)U(t, \tau)x_\tau \mid AB(t)U(t, \tau)x_\tau)
\leq (\hat{B}(t)U(t, \tau)x_\tau \mid U(t, \tau)x_\tau)
\leq \frac{M_T}{\beta} \|U(t, \tau)x_\tau\|^2_{L_2}, \quad \tau \leq t \leq T,$$

where $M_T = \max_{t \in [0,T]} \| \hat{B}(t) \|_{\mathcal{L}(X)}$. Integrating on $(s, t)$ the above inequality with $s \in \left[ \tau, t \right)$ and using Grönwall Lemma we obtain that

$$\|U(t, s)x_\tau\|^2_{L_2} \leq e^{\frac{M_T}{\beta}(t-s)} \|U(s, \tau)x_\tau\|^2_{L_2},$$

holds for all $t, s \in [0,T]$ with $t \geq s \geq \tau$. This gives the desired estimates, since $D(A(s))$ is dense in $X$, and completes the proof the theorem.

The inequality (17) will be needed in the next section where we study the exponential stability of the evolution family $U$ generated by non-autonomous port-Hamiltonian systems.

Remark 2.5. (i) Under the assumption of Theorem 2.4 it easy to see that the evolution family $U$ generated by $A := \{ A(t) \mid t \geq 0 \}$ is locally exponentially bounded. In fact, taking $s = \tau$ in (17) and using
we obtain that
\[
\|U(t,s)x\|^2 \leq \frac{1}{\beta} \|U(t,s)x\|^2 \leq \frac{1}{\beta} e^{\frac{\beta}{2}e^{\|x\|_a^2}} \leq \frac{\beta}{2} e^{\frac{\beta}{2}e^{\|x\|_a^2}}
\]
for all \( T > 0 \) and each \( 0 \leq s \leq t \leq T \). Recall that all strongly continuous semigroups are exponentially bounded. This is however not the case for general evolution families, cf. [EN00, Section VI.9].

(ii) If in addition \( \dot{B}(t) \leq 0 \) for all \( t \geq 0 \) then (19) and (20) imply that \( t \mapsto \|B^{1/2}(t)U(t,s)x\| \) is decreasing on \([s, \infty)\) for each \( s \geq 0 \) and \( x \in X \). This can be seen as generalization of [15, Lemma 7.2.3] to the non-autonomous setting.

3. Non-autonomous Port-Hamiltonian systems

In this section we are concerned with the linear non-autonomous Port-Hamiltonian system (1)-(3) introduced in Section 1. Recall that in this case we have \( X = L^2(a,b;\mathbb{K}^n) \). Throughout this section we always assume the following:

Assumption 3.1.

(i) \( \text{Re } P_0 := \frac{P_0 + P_0^*}{2} \leq 0 \).

(ii) \( P_1 \) is invertible and self-adjoint.

(iii) \( W_B \Sigma W_B^* \geq 0 \) and rank \( \bar{W}_B = n \).

(iv) \( H \in C^2([0,\infty);L^\infty(a,b;\mathbb{K}^{n \times n})) \) and there exist \( m, M \geq 0 \) such that

\[
\beta \leq \|H(t,\xi) - H(t,0)\| \leq M, \quad \text{a.e. } \xi \in [a,b], t \geq 0.
\]

Remark 3.2. (i) In principle, it is also possible to consider \( P_0 \in C([0,\infty);L^\infty(a,b;\mathbb{K}^{n \times n})) \) without any restriction on the dissipativity of \( P_0 \). In this case the operator \( A(t) \) may depend on the time variable \( t \), yet its domain stays independent of \( t \) as \( P_0(t,\cdot) \) is just a bounded perturbation. However, \( A(t) \) will not generate a contractive \( C_0 \)-semigroup on \( L^2(a,b;\mathbb{K}^n) \) unless \( \text{Re } P_0(t,\cdot) \leq 0 \) a.e. cf. [6].

(ii) \( P_1 \) being invertible ensures that \( H^1(a,b;\mathbb{K}^n) \) is the maximal domain for the differential operator \( P_1 \frac{\partial}{\partial t} + P_0 \) on \( L^2(a,b;\mathbb{K}^n) \), i.e. \( A \) is a closed operator (otherwise it could never be the generator of a semigroup).

(iii) \( A \) is dissipative if and only if \( \text{Re } P_0 \leq 0 \) and \( W_B \Sigma W_B^* \geq 0 \), and in this case \( A \) defined below already generated a contractive \( C_0 \)-semigroup on \( L^2(a,b;\mathbb{K}^{n}) \).

(iv) The assumption on \( H \) ensures that the family of operators \( B(t) := H(t,\cdot) \) as matrix multiplication operators on \( L^2(a,b;\mathbb{K}^n) \) satisfies the assumptions of Section 2.

Theorem 3.3. The non-autonomous port-Hamiltonian system (1)-(3) is \( C^1 \)-well posed with regularity spaces

\[
Y_1 = \left\{ x \in L^2(a,b;\mathbb{K}^n) | (H(t,\cdot),x) \in H^1(a,b;\mathbb{K}^n) \text{ and } \bar{W}_B \begin{pmatrix} H(t,b)x(t,b) \\ H(t,a)x(t,a) \end{pmatrix} = 0 \right\}
\]

Moreover, for each compact interval \([0,T]\) and classical solution \( x \) of (1)-(3) we have

\[
\|x(t)\|_a^2 \leq e^{c_T(t-s)}\|x(s)\|_a^2 \quad (0 \leq s \leq t \leq T),
\]

for some constant \( c_T \geq 0 \) that depends only on \( m \) and \( \max_{t \in [0,T]} \|\dot{H}(t,\cdot)\| \).

Proof. Consider the linear operator \( A = P_1 \frac{\partial}{\partial t} + P_0 \) with domain

\[
D(A) = \left\{ x \in H^1(a,b;\mathbb{K}^n), \bar{W}_B \begin{pmatrix} x(t,b) \\ x(t,a) \end{pmatrix} = 0 \right\}.
\]

The operator \( A \) with domain \( D(A) \) generates an contraction \( C_0 \)-semigroup on \( X \) by [14, Theorem 1.1]. The claim follows then from Theorem 2.4 since \( L^\infty(a,b;\mathbb{K}^{n \times n}) \) can be seen as a subspace of \( L(X) \) by identifying a function \( F \in L^\infty(a,b;\mathbb{K}^{n \times n}) \) with its multiplication operator : \( x \mapsto Fx \).  \( \square \)
Lemma 3.4. Assume that the conditions of Theorem 3.3 hold. In addition we assume that \( \mathcal{H}(t, \cdot) \) is Lipschitz continuous uniformly in time on \([a, b]\), i.e., \( \frac{\partial}{\partial \zeta} \mathcal{H} \in L^\infty_{loc}([0, \infty) \times [a, b]; \mathbb{K}^{n \times n}) \). Then there exist constants \( \tau > 0 \) and \( C_\tau > 0 \) such that for each classical solution \( x \) of (1)-(3) we have

\[
\| x(\tau) \|^2 \leq C_\tau \int_0^\tau | \mathcal{H}(t, b)x(t, b) |^2 \, dt,
\]

\[
\| x(\tau) \|^2 \leq C_\tau \int_0^\tau | \mathcal{H}(t, a)x(t, a) |^2 \, dt.
\]

**Proof.** For the proof we follow the same strategy as in [30, Lemma III.1], also see [15, Lemma 9.1.2]. Let \( \gamma > 0 \) and \( \tau > 0 \) be chosen such that \( \tau > 2\gamma(b - a) \). Let \( x \) be a classical solution of (1)-(3) and define the function \( F : [a, b] \rightarrow \mathbb{R} \) via

\[
F(\zeta) := \int_{\gamma(b - \zeta)}^{\tau - \gamma(b - \zeta)} x^*(t, \zeta) \mathcal{H}(t, \zeta)x(t, \zeta) \, dt, \quad \zeta \in [a, b].
\]

Note that \( F(b) = \int_0^\tau x^*(t, b) \mathcal{H}(t, b)x(t, b) \, dt \), and hence

\[
\frac{1}{M} \int_0^\tau | \mathcal{H}(t, b)x(t, b) |^2 \, dt \leq F(b) \leq \frac{1}{m} \int_0^\tau | \mathcal{H}(t, b)x(t, b) |^2 \, dt.
\]

For simplicity we sometimes write \( x, \mathcal{H} \) instead of \( x(t, \zeta), \mathcal{H}(t, \zeta) \). Then we have

\[
\frac{d}{d\zeta} F(\zeta) = \int_{\gamma(b - \zeta)}^{\tau - \gamma(b - \zeta)} \left[ x^* \left( \frac{\partial}{\partial \zeta} (\mathcal{H}x) + (\frac{\partial}{\partial \zeta} x)^* \mathcal{H}x \right) + \gamma(x^* \mathcal{H}x)(\tau - \gamma(b - \zeta), \zeta) + \gamma(x^* \mathcal{H}x)(\gamma(b - \zeta), \zeta) \right] \, d\zeta
\]

\[
= \int_{\gamma(b - \zeta)}^{\tau - \gamma(b - \zeta)} \left[ x^* P_{1}^{-1} \left( \frac{\partial x}{\partial t} - P_0 \mathcal{H}x \right) + \left( P_{1}^{-1} \frac{d}{dt} x - \frac{\partial \mathcal{H}}{\partial \zeta} x - P_{1}^{-1} P_0 \mathcal{H}(t, \zeta)x \right)^* \right] \, d\zeta
\]

\[
= \int_{\gamma(b - \zeta)}^{\tau - \gamma(b - \zeta)} \left( x^* P_{1}^{-1} \frac{\partial x}{\partial t} + \frac{\partial x^*}{\partial \zeta} P_{1}^{-1} x \right) \, d\zeta
\]

Here we have used that \( P_1 \) is invertible and self adjoint, \( x \) solves (1) and that \( \mathcal{H} \) is self-adjoint. Next, by the fundamental theorem of calculus we have

\[
\int_{\gamma(b - \zeta)}^{\tau - \gamma(b - \zeta)} \left( x^* P_{1}^{-1} \frac{\partial x}{\partial t} + \frac{\partial x^*}{\partial \zeta} P_{1}^{-1} x \right) \, d\zeta = [x^* P_{1}^{-1} x]_{\gamma(b - \zeta)}^{\tau - \gamma(b - \zeta)}.
\]

Therefore,

\[
\frac{d}{d\zeta} F(\zeta) = - \int_{\gamma(b - \zeta)}^{\tau - \gamma(b - \zeta)} x^* \left( \mathcal{H} \mathcal{H}^* P_{1}^{-1} + P_0 P_{1}^{-1} \mathcal{H} + \frac{\partial \mathcal{H}}{\partial \zeta} \right) x \, d\zeta
\]

\[
+ \left[ x(x \mathcal{H} + P_{1}^{-1})x^* \right] (\tau - \gamma(b - \zeta), \zeta) + \left[ x(x \mathcal{H} - P_{1}^{-1})x^* \right] (\gamma(b - \zeta), \zeta).
\]

Now, thanks to (21) we can choose \( \gamma \) large enough such that

\[
\pm P_{1}^{-1} + \gamma \mathcal{H}(t, \zeta) \geq 0, \quad (a.e. \zeta \in [a, b], t \geq 0).
\]

Moreover, since \( [\gamma(b - \zeta), \tau - \gamma(b - \zeta)] \subset [0, \tau] \) and since \( \frac{\partial}{\partial \zeta} \mathcal{H} \in L^\infty_{loc}([0, \infty) \times [a, b]; \mathbb{K}^{n \times n}) \) there exists \( \kappa_\tau > 0 \) such that

\[
\mathcal{H}(t, \zeta) P_0 P_{1}^{-1} + P_0 P_{1}^{-1} \mathcal{H}(t, \zeta) + \frac{\partial \mathcal{H}}{\partial \zeta} (t, \zeta) \leq \kappa_\tau \mathcal{H}(t, \zeta)
\]

\[
\mathcal{H}(t, \zeta) P_0 P_{1}^{-1} + P_0 P_{1}^{-1} \mathcal{H}(t, \zeta) + \frac{\partial \mathcal{H}}{\partial \zeta} (t, \zeta) 
\]
for a.e. $\zeta \in [a, b]$ and all $t \in [0, \tau]$. For example, we may choose

$$\kappa_\tau := 1 + \|P_0^* P_1^{-1}\| + \frac{1}{m} \| \frac{\partial \mathcal{H}}{\partial \zeta} \|_{L^\infty([0, \tau]; L^\infty(a, b, k^n))} > 0.$$ 

Inserting (27) and (28) into (25)-(26) we obtain that

$$\frac{d}{d\zeta} F(\zeta) \geq -\kappa_\tau F(\zeta)$$

holds for a.e. $\zeta \in [a, b]$ and all $t \in [0, \tau]$. This implies

$$(29) \quad F(\zeta) \leq e^{\kappa_\tau (b-a)} F(b) \quad \text{for all } \zeta \in [a, b], t \in [0, \tau]$$

Using (22), there exists a constant $c_\tau > 0$ (which depends on the interval $[0, \tau]$, more precisely on $\max_{t \in [0, \tau]} \| \mathcal{H}(t, \cdot) \|$ and $\frac{1}{m}$) such that

$$(\tau - 2\gamma(b-a)) \| \mathcal{H}(\tau - \gamma(b-\zeta), \cdot) x(\tau - \gamma(b-\zeta)) \|^2 \leq e^{c_\tau \tau} \int_{\gamma(b-\zeta)}^{\tau - \gamma(b-\zeta)} \| x(t) \|^2 dt$$

$$= e^{c_\tau \tau} \int_{\gamma(b-\zeta)}^{\tau - \gamma(b-\zeta)} \int_a^b \mathcal{H}(t, \zeta) x(t, \zeta) d\zeta dt$$

$$= e^{c_\tau \tau} \int_a^b \int_{\gamma(b-\zeta)}^{\tau - \gamma(b-\zeta)} \mathcal{H}(t, \zeta) x(t, \zeta) d\zeta dt$$

$$= e^{c_\tau \tau} \int_a^b F(\zeta) d\zeta \leq F(b)e^{c_\tau \tau} e^{\kappa_\tau (b-a)} (b-a).$$

Here we have used (29) to obtain the last inequality. Taking $\zeta = b$ and using that $\tau > 2\gamma(b-a)$ we conclude

$$\| x(t) \|^2 \leq \frac{F(b)e^{c_\tau \tau + \kappa_\tau (b-a)}(b-a)}{\tau - 2\gamma(b-a)} \int_0^\tau | \mathcal{H}(t, b)x(t, b) |^2 dt.$$

This completes the proof of the desired inequality (23) for the constant

$$C_\tau := \frac{e^{c_\tau \tau + \kappa_\tau (b-a)}(b-a)}{\tau - 2\gamma(b-a)} > 0.$$

The second inequality (24) can be obtained by the same technique.

**Remark 3.5.** If $\frac{\partial \mathcal{H}}{\partial \zeta}$ is not only locally bounded with values in $L^\infty(a, b; k^n)$, i.e.

$$\mathcal{H} \in C^2([0, \infty); L^\infty(a, b; k^n)) \cap C_0^1([0, \infty); L^\infty(a, b; k^n))$$

then

$$\kappa_\tau := 1 + \|P_0^* P_1^{-1}\| + \frac{1}{m} \| \frac{\partial \mathcal{H}}{\partial \zeta} \|_{L^\infty([0, \infty); L^\infty(a, b, k^n))}$$

and the constant $c_\tau$ in (22) can be chosen independent of the interval $[s, s + \tau]$ for all $s \geq 0$. Therefore, one obtains the finite observability estimates

$$\| x(s + \tau) \|^2 \leq C_\tau \int_s^{s + \tau} | \mathcal{H}(t, b)x(t, b) |^2 dt, \quad s \geq 0$$

and

$$\| x(s + \tau) \|^2 \leq C_\tau \int_s^{s + \tau} | \mathcal{H}(t, a)x(t, a) |^2 dt, \quad s \geq 0$$

respectively, where the constant $C_\tau > 0$ does neither depend on $x$ nor on $s \geq 0$. We will use this property in the next result, the Stability Theorem 3.6.

Under slightly more restrictive regularity conditions we are now able to state the following uniform exponential stability theorem, provided dissipative boundary conditions are imposed.
Theorem 3.6. Assume that the assumptions of Lemma 3.4 are satisfied. In addition we assume that
\[ H(s, \cdot) \leq H(t, \cdot) \text{ for every } t \geq s \]
and \( \frac{\partial H}{\partial s}, \frac{\partial H}{\partial \kappa} \in L^\infty([0, \infty) \times [a, b]) \), i.e.
\[ H \in C^2([0, \infty); C([a, b]; \mathbb{R}^n)) \cap C_1^0([0, \infty); C([a, b]; \mathbb{R}^n)) \cap L^\infty([0, \infty); \text{Lip}([a, b]; \mathbb{R}^n)). \]
Assume that there exists \( \kappa > 0 \) such that for every classical solution \( x \) of (1)-(3) the following two conditions holds
\[ \text{Re}(AH(t)x(t))x(t) \leq -\kappa |(Hx)(t,b)|^2 \quad t \geq 0. \]
\[ \text{Re}(AH(t)x(t))x(t) \leq -\kappa |(Hx)(t,a)|^2 \quad t \geq 0. \]
Then the system (1)-(3) is uniformly exponentially stable, i.e. there are constants \( \omega < 0 \) and \( L \geq 1 \) such that for all classical solutions \( x \) of (1)-(3)
\[ \|x(t + s)\| \leq Le^{\omega t} \|x(s)\| \quad \text{for all } s, t \geq 0. \]

Proof. By (22), the regularity and boundedness assumptions on \( H \) there exists a constant \( c_0 > 0 \) such that
\[ \|x(t + s)\|^2 \leq e^{c_0 t} \|x(s)\|^2, \quad s, t \geq 0. \]
According to Lemma 3.4 and Remark 3.5 there exists \( \tau > 0 \) and a constant \( C_\tau > 0 \) such that
\[ \|x(\tau + s)\|^2 \leq C_\tau \int_0^{\tau + s}|H(t,b)x(t,b)|^2 \, dt, \quad s \geq 0. \]
This inequality together with (30) implies that
\[ \|x(s + \tau)\|^2 - \|x(s)\|^2 \leq \int_s^{s + \tau} \frac{d}{dt} \|x(t)\|^2 \, dt \\
= 2 \text{Re} \int_s^{s + \tau} (AH(t)x(t))x(t) \, dt + \int_s^{s + \tau} |H(t,b)|^2 \, dt \\
\leq -2\kappa \int_s^{s + \tau} |(Hx)(t,b)|^2 \, dt \\
\leq -\frac{2\kappa}{C_\tau} \|x(s + \tau)\|^2. \]
holds for every \( s \geq 0 \). We deduce that
\[ \|x(s + \tau)\|^2 \leq \rho_\tau \|x(s)\|^2 := \frac{1}{1 + \frac{\kappa}{C_\tau}} \|x(s)\|^2, \quad s \geq 0. \]
Using (34) and (35) we obtain iteratively for all \( s \geq 0 \) and \( t = n\tau + r, r \in [0, \tau), n \in \mathbb{N} \), that
\[ \|x(s + t)\|^2 \leq \|x(s + n\tau + r)\|^2 \leq e^{c_0 t} \|x(s + n\tau)\|^2 \leq e^{c_0 \rho_\tau^r} \|x(s)\|^2 \leq e^{c_0 \rho_\tau^r} \|x(s)\|^2. \]
Finally, according to (15) and (21) we obtain the desired estimate (33) with
\[ \omega := \log(\rho_\tau) \quad \text{and} \quad L := e^{c_0 \rho_\tau^r} \frac{M}{m}. \]
This completes the proof of the asserted statement. \( \square \)

Corollary 3.7. Assume that the assumptions of Lemma 3.4 are satisfied. In addition we assume \( W_B \Sigma W_B^* > 0 \). Then the classical solution \( x \) of (1)-(3) is uniformly exponentially stable.

Proof. Since \( W_B \Sigma W_B^* > 0 \), both conditions (31) and (32) hold, e.g. by the proof of [15, Lemma 9.1.4]. Now the claim follows from Theorem 3.6. \( \square \)

Remark 3.8. The previous results have been proved in [30, Theorem III.2] and [15, Theorem 9.1.3, Theorem 7.2.4] in the case where \( H \) is independent of \( t \).
Remark 3.9. It would be also possible to consider port-Hamiltonian systems of higher order $N \geq 2$, i.e.

$$A = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial t^k}$$

on an appropriate domain $D(A) \subseteq H^N(a, b; \mathbb{K}^N)$ (including, say, dissipative boundary conditions), where now the conditions on the matrices $P_k$ read: $P_k \in \mathbb{K}^{n \times n}$ with $P_k = (-1)^{k+1} P_k$ for $k \geq 1$ and $P_N$ invertible. The $C^1$-well posedness result Theorem 3.3 directly transfers to that situation. However, a final observability estimate as in Lemma 29 is not (yet) known for that situation, and proofs for uniform exponential stability in the autonomous situation rather rely on particular semigroup techniques (the ABLV-Theorem and the stability theorem of Gearhart, Prüss and Huang) which are not at hand for non-autonomous problems.

4. Examples

4.1. A non-autonomous vibrating string. Consider a vibrating string described by the time-dependent wave equation (4)-(6) introduced in Section 1. The Young’s modulus function $T$ and the mass density function $\rho$ are assumed to be measurable and satisfy the following conditions:

(i) $T, \rho \in C^2([0, \infty); L^\infty(a, b)) \cap C_0([0, \infty); L^\infty(a, b))$

(ii) There is a constant $\alpha > 0$ such that for $a.e. \zeta \in [a, b]$ and all $t \geq 0$

$$\alpha^{-1} \leq \rho(t, \zeta), T(t, \zeta) \leq \alpha$$

Recall that (4)-(6) can be reformulated in the port-Hamiltonian form by choosing the momentum-strain function $\rho$ invertible. The exponential stability in the autonomous situation rather rely on particular semigroup techniques (the final observability estimate as in Lemma 29 is not (yet) known for that situation, and proofs for uniform exponential stability in the autonomous situation rather rely on particular semigroup techniques (the ABLV-Theorem and the stability theorem of Gearhart, Prüss and Huang) which are not at hand for non-autonomous problems.

Moreover, the boundary conditions (5)-(6) can be reformulated as follows

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad H(t, \zeta) = \begin{bmatrix} \rho(t, \zeta) & 0 \\ 0 & T(t, \zeta) \end{bmatrix}$$

Moreover, the boundary conditions (5)-(6) can be reformulated as follows

$$W_B = \sqrt{2 \tilde{W}_B} \begin{bmatrix} P_1 & -P_1 \end{bmatrix}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ k & k \end{bmatrix}$$

The $2 \times 4$ matrix

$$W_B = \sqrt{2 \tilde{W}_B} \begin{bmatrix} P_1 & -P_1 \end{bmatrix}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ k & k \end{bmatrix}$$

has rank 2 and $W_B \Sigma W_B^T = \begin{bmatrix} 2k & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. From [15, Lemma 7.2.1] we now that

$$\text{Re}(\mathcal{A} \mathcal{H}(\cdot, t)x | x)_t = \frac{1}{2} \left( (\mathcal{H}x)^*(t, b) P_1(\mathcal{H}x)(t, b) - (\mathcal{H}x)^*(t, a) P_1(\mathcal{H}x)(t, a) \right)$$

holds for all $x \in D(\mathcal{A} \mathcal{H}(\cdot, t))$ and all $t \geq 0$. By this equality, together with the boundary conditions (5)-(6), we observe that for $x = \begin{bmatrix} \rho \frac{\partial u}{\partial \zeta} \\ \frac{\partial u}{\partial \zeta} \end{bmatrix}$

$$2 \text{Re}(\mathcal{A} \mathcal{H}(\cdot, t)x | x)_t \leq -\frac{k}{1 + k^2} \|H(b, t)x(b)\|^2 \quad (t \geq 0).$$

Thus the following well-posedness and stability results follows from Theorem 3.6 and Corollary 3.7.

Proposition 4.1. Let $\omega_0 \in H^1(a, b; \mathbb{C})$ and $\omega_1 \in H^1(a, b; \mathbb{C})$ be such that $T(0, \cdot) \frac{\partial \omega}{\partial t} \in H^1(a, b; \mathbb{C})$ and $T(b, 0) \frac{\partial \omega}{\partial t} = 0$. Then (4) with boundary conditions (5)-(6) and initial conditions

$$\omega(0, \cdot) = \omega_0, \quad \frac{\partial \omega}{\partial t}(0, \cdot) = \omega_1$$

has a unique solution $\omega$ such that

$$t \mapsto \begin{bmatrix} \frac{\partial \omega}{\partial t}(t, \cdot) \\ \frac{\partial \omega}{\partial \zeta}(t, \cdot) \end{bmatrix} \in C^1((0, \infty); L^2(a, b; \mathbb{C}^2)) \cap C((0, \infty); L^2(a, b; \mathbb{C}^2))$$
If, in addition, \( T, \rho \in C^2([0, \infty); C([a, b]) \cap C^1([0, \infty); C([a, b])) \cap L^\infty([0, \infty); \text{Lip}([a, b])) \) and \( T, \rho^{-1} \) are decreasing with respect to the time variable then we have
\[
\| \frac{\partial w}{\partial t}(t, \cdot) \|^2_{L^2(a,b)} + \| T(t, \cdot) \frac{\partial w}{\partial \zeta}(t, \cdot) \|^2_{L^2(a,b)} \leq M e^{\omega t} \left( \| \frac{\partial w}{\partial \zeta}(\cdot) \|^2_{L^2(a,b)} + \| \omega_1 \|^2_{L^2(a,b)} \right)
\]
for all \( t \geq 0 \) and some constants \( M \geq 1 \) and \( \omega < 0 \) that are independent of \( t \geq 0 \) and the initial data. More precisely, for all \( s, t \geq 0 \) one has the estimate
\[
\| \frac{\partial w}{\partial t}(t+s, \cdot) \|^2_{L^2(a,b)} + \| T(t+s, \cdot) \frac{\partial w}{\partial \zeta}(t+s, \cdot) \|^2_{L^2(a,b)} \leq M e^{\omega t} \left( \| \frac{\partial w}{\partial \zeta}(s, \cdot) \|^2_{L^2(a,b)} + \| T(t, \cdot) \frac{\partial w}{\partial \zeta}(s, \cdot) \|^2_{L^2(a,b)} \right).
\]

### 4.2. Time dependent Timoshenko Beam

Consider the non-autonomous version of the Timoshenko beam model given by the equations:
\[
\frac{\partial}{\partial t} \left( \frac{\partial w}{\partial t}(t, \zeta) \right) = \frac{\partial}{\partial \zeta} \left( K(t, \zeta) \left( \frac{\partial}{\partial \zeta} w(t, \zeta) + \phi(t, \zeta) \right) \right)
\]
\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial t}(t, \zeta) \right) = \frac{\partial}{\partial \zeta} \left( EI(t, \zeta) \frac{\partial^2 \phi}{\partial \zeta^2}(t, \zeta) + K(t, \zeta) \left( \frac{\partial}{\partial \zeta} w(t, \zeta) - \phi(t, \zeta) \right) \right)
\]
where \( \zeta \in (a, b), t \geq 0, w(t, \zeta) \) is the transverse displacement of the beam and \( \phi(t, \zeta) \) is the rotation angle of the filament of the beam. Dropping the coordinates \( \zeta \) and \( t \) and taking as state variable \( x := \left( \frac{\partial w}{\partial \zeta} - \phi, \frac{\partial^2 \phi}{\partial \zeta^2}, \frac{\partial w}{\partial \zeta} \right) \) one can see that the Timoshenko beam model may be written as a system of the form (1)-(3) with
\[
P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\]
and
\[
\mathcal{H} = \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & \rho^{-1} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & I_\rho^{-1} \end{bmatrix}
\]

Further, we impose the following boundary conditions
\[
\frac{\partial w}{\partial t}(t, a) = 0, \quad t \geq 0
\]
\[
\frac{\partial \phi}{\partial t}(t, a) = 0, \quad t \geq 0
\]
\[
K(t, b) \left[ \frac{\partial w}{\partial \zeta}(t, b) - \phi(t, b) \right] = -\alpha_1 \frac{\partial w}{\partial t}(t, b), \quad t \geq 0
\]
\[
EI(t, b) \frac{\partial \phi}{\partial \zeta}(t, b) = -\alpha_2 \frac{\partial \phi}{\partial t}(t, b), \quad t \geq 0
\]
for some positive constants \( \alpha_1, \alpha_2 \geq 0 \), i.e. we impose conservative boundary conditions at the left end \( \zeta = a \) and dissipative feedback at the right end \( \zeta = b \) of the beam. These boundary conditions can be written as
\[
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} H(t, b)x \\ H(t, a)x \end{bmatrix} := W_B \begin{bmatrix} H(t, b)x \\ H(t, a)x \end{bmatrix}
\]
The corresponding \( 4 \times 8 \) matrix \( W_B \) is given by
\[
W_B = \sqrt{2} \mathcal{W}_B \begin{bmatrix} P_1 & -P_1 \end{bmatrix}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 1 & 0 & 0 & 0 & 0 & \alpha_2 \end{bmatrix}
\]
has full rank and \( W_B \Sigma W_B^* = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} \geq 0. \]
We conclude that the non-autonomous Cauchy problem associated with the Timoshenko beam (39)-(39) is $C^1$-well posed provided the physical parameters that defining the Hamiltonian $H$ satisfy similar conditions to those described for the vibrating string, i.e. $K, \rho, EI, I_p \in C^2([0, \infty); L^\infty((a, b))] \cap C^1([0, \infty); L^\infty((a, b)))$. Moreover, by Corollary 3.7 the associated evolution family is uniformly exponentially stable if in addition $\alpha_1, \alpha_2 > 0$ are both strictly positive and $K, \rho, EI, I_p \in C^2([0, \infty); C([a, b])) \cap C^1([0, \infty); C([a, b])) \cap L^\infty([0, \infty); \mathrm{Lip}([a, b]))$ and $K, \rho^{-1}, EI, I_p^{-1}$ are decreasing with respect to the time variable $t$.

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