Hydrodynamic Stability Analysis of Particle-Laden Solid Rocket Motors

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Abstract. Fluid-wall interactions within solid rocket motors can result in parietal vortex shedding giving rise to hydrodynamic instabilities, or unsteady waves, that translate into pressure oscillations. The oscillations can result in vibrations observed by the rocket, rocket subsystems, or payload, which can lead to changes in flight characteristics, design failure, or other undesirable effects. For many years particles have been embedded in solid rocket propellants with the understanding that their presence increases specific impulse and suppresses fluctuations in the flowfield. This study utilizes a two dimensional framework to understand and quantify the aforementioned two-phase flowfield inside a motor case with a cylindrical grain perforation. This is accomplished through the use of linearized Navier-Stokes equations with the Stokes drag equation and application of the biglobal ansatz. Obtaining the biglobal equations for analysis requires quantification of the mean flowfield within the solid rocket motor. To that end, the extended Taylor-Culick form will be utilized to represent the gaseous phase of the mean flowfield while the self-similar form will be employed for the particle phase. Advancing the mean flowfield by quantifying the particle mass concentration with a semi-analytical solution the finalized mean flowfield is combined with the biglobal equations resulting in a system of eight partial differential equations. This system is solved using an eigensolver within the framework yielding the entire spectrum of eigenvalues, frequency and growth rate components, at once. This work will detail the parametric analysis performed to demonstrate the stabilizing and destabilizing effects of particles within solid rocket combustion.

1. Introduction

It is generally accepted that the work of Varapaev and Yagodikin [1] has led the way in hydrodynamic stability investigations of rocket chambers. Over time several dedicated researchers have focused on hydrodynamic stability models and their ability to predict oscillations in solid rocket flowfields, such as Casalis, Avalon, and Pineau [2], Ugurtas et al. [3], Féraill and Casalis [4], Elliott, Batterson, and Majdalani [5], and Boyer et al. [6, 7]. Instabilities such as parietal, obstacle, and angle vortex shedding have been identified in these models where a link has been made between experimentally reported frequencies and stability eigenmodes. Cold-flow measurements obtained using two experimental
facilities, VECLA (Veine d’Etude de la Couche Limite Acoustique) and VALDO (Veine Axisymétrique pour Limiter le Développement des Oscillations), have been used to validate findings of the aforementioned hydrodynamic stability analyses [8]. The most noteworthy of these solid rocket motor (SRM) instability analyses can be attributed to Chedevergne and Casalis [9-11] with their biglobal framework.

The present work aims to advance previous biglobal stability analyses through the creation of a framework which can analyze particle-laden rocket flows. It has been long understood that the addition of particles increases specific impulse and suppresses high frequency instabilities. Pioneering work in this area can be attributed to Saffman [12], who extended the conventional single-phase stability theory. Following Saffman, Féraille and Casalis [4] have applied two-phase theory to the local stability approach using Stokes drag force to couple the gaseous and particle phases. The present work addresses the particle mass concentration in a different manner, thus resulting in a semi-analytical solution. In what follows, the biglobal equations are developed using the semi-analytical solution for particle mass concentration and mean flow using a self-similar form. The result is a framework to analyze particle-laden rocket internal motion. In short, the addition of particles will be shown to have both stabilizing and destabilizing effects as it will be described in the results and discussion section.

2. Physical geometry

The solid rocket core flow field is the primary geometry of interest for this study as illustrated in Figure 1. The motor is idealized as a cylindrical chamber of length \( L \) and radius \( a \). Spherical and chemically-inert particles are injected with combustion gases from the sidewall. Furthermore, it is assumed that the density of the gaseous phase is sufficiently smaller than the particle phase, thereby permitting the use of the Stokes approximation as performed by Elliott and Majdalani [13].

![Geometry of a solid rocket motor with inert headwall and embedded particles released from the sidewall.](image)

3. Biglobal equations

As with past biglobal analysis, the fundamental equations are normalized using standard terminologies, where coordinates are normalized by the chamber radius and velocities by a constant injection speed [5, 13]. The instantaneous parameters are then decomposed, leading to the first order perturbation, \( \bar{M} = M + \bar{m} \). Here \( M \) is the mean flowfield, or bulk fluid parameter, and \( \bar{m} \) is the hydrodynamic wave. With the mean flowfield known, the \( O(M) \) terms are omitted and to reduce the system the second order \( O(\bar{m}^2) \) terms are neglected. The biglobal Ansatz is then applied, namely \( \bar{m} = m(r,z)\exp\left[i(q\theta - aw)\right] \), and the resulting two-phase biglobal stability equations are obtained:
\[
\left( \frac{\partial}{\partial r} + r^{-1} \right) u_r + \left( iqr^{-1} \right) u_\theta + \left( \frac{\partial}{\partial z} \right) u_z = 0, \quad (1)
\]

\[
\begin{aligned}
&\left( U_r \frac{\partial}{\partial r} + iq U_r r^{-1} + U_u \frac{\partial}{\partial z} + \frac{X_p}{S} - \varepsilon \left[ \frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r} - (1 + q^2) r^{-2} + \frac{\partial^2}{\partial z^2} \right] \right) u_r \\
&+ \left( 2iqr r^{2} - 2U_p r^{-1} + r^{-1} \frac{\partial U_u}{\partial \theta} \right) u_\theta + \left( \frac{\partial U_r}{\partial z} \right) u_z + \left( \frac{\partial}{\partial r} \right) p \frac{X_p}{S} u_{rp} = (i\omega) u_r,
\end{aligned} \quad (2)
\]

\[
\begin{aligned}
&\left( \frac{\partial U_u}{\partial r} + U_p r^{-1} - 2iqr r^{2} \right) u_r + \left( U_r \frac{\partial}{\partial r} + U_u r^{-1} + iq U_u r^{-1} + r^{-1} \frac{\partial U_u}{\partial \theta} + U_z \frac{\partial}{\partial z} + \frac{X_p}{S} \right) u_\theta \\
&- \varepsilon \left[ \frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r} - (1 + q^2) r^{-2} + \frac{\partial^2}{\partial z^2} \right] u_\theta + \left( \frac{\partial U_r}{\partial z} \right) u_z + \left( \frac{\partial}{\partial r} \right) p \frac{X_p}{S} u_{r\theta} = (i\omega) u_\theta,
\end{aligned} \quad (3)
\]

\[
\begin{aligned}
&\left( \frac{1}{r} U_{r\theta} + \frac{\partial U_{r\theta}}{\partial r} + \frac{\partial U_{r\theta}}{\partial z} \right) x_p + U_{r\theta} \frac{\partial x_p}{\partial r} + X_p \frac{\partial u_{r\theta}}{\partial r} + \left( \frac{1}{r} X_p - \frac{1}{r} \right) u_{r\theta} , \\
&+ \left( \frac{\partial X_p}{\partial \theta} + i q X_p \right) u_{r\theta} + U_{r\theta} \frac{\partial X_p}{\partial z} + X_p \frac{\partial u_{r\theta}}{\partial z} = (i\omega) x_p,
\end{aligned} \quad (4)
\]

\[
\begin{aligned}
&\left( \frac{\partial U_{r\theta}}{\partial r} + U_{r\theta} \frac{\partial}{\partial r} + U_{r\theta} \frac{\partial}{\partial z} + S^{-1} \right) u_{r\theta} + \left( \frac{\partial U_{r\theta}}{\partial z} \right) u_{r\theta} + \left( \frac{\partial U_{r\theta}}{\partial \theta} \right) u_{r\theta} r^{-1} - u_{r\theta} S^{-1} = (i\omega) u_{r\theta},
\end{aligned} \quad (5)
\]

\[
\begin{aligned}
&\left( U_{r\theta} \frac{\partial}{\partial r} + U_{r\theta} \frac{\partial}{\partial z} + S^{-1} \right) u_{r\theta} - u_{r\theta} S^{-1} = i\omega u_{r\theta},
\end{aligned} \quad (6)
\]

\[
\begin{aligned}
&\left( U_{r\theta} \frac{\partial}{\partial r} + U_{r\theta} \frac{\partial}{\partial z} + S^{-1} \right) u_{r\theta} + \left( \frac{\partial U_{r\theta}}{\partial \theta} \right) u_{r\theta} r^{-1} - u_{r\theta} S^{-1} = i\omega u_{r\theta}.
\end{aligned} \quad (7)
\]

4. Mean flow field
Following past work by Majdalani and Saad [14], the extended Taylor-Culick solution is used to represent the mean flowfield for the gaseous phase. The mean flowfield for the particle phase is chosen following Féralie and Casalis [4] with exception to the particle mass concentration. In their work they use a self-similar solution for particle mass concentration resulting in a PDE that accounts only for variation in the radial direction. Using what their generalized form we retain the axial variation and in so doing determine a semi-analytical solution to the particle mass concentration.

4.1. Particle mass concentration with axial variation
Assuming steady-state conditions and axisymmetric properties for the particle mass concentration, the normalized particle continuity equation becomes

\[
U_{r\theta} \frac{\partial X_p}{\partial r} + U_{r\theta} \frac{\partial X_p}{\partial z} + \left( \frac{\partial U_{r\theta}}{\partial r} + U_{r\theta} \frac{\partial}{r} + \frac{\partial U_{r\theta}}{\partial z} \right) X_p = 0. \quad (9)
\]

Equation (9) can be rewritten using the known characteristic similarity functions using
\[ J(r) \frac{\partial X_p(r,z)}{\partial r} + zG(r) \frac{\partial X_p(r,z)}{\partial z} + \left[ J'(r) + \frac{J(r)}{r} + G(r) \right] X_p(r,z) = 0 \text{ ,} \quad (10) \]

where at the sidewall, where \( r = 1 \), the particle mass concentration is assumed to smoothly transition to a maximum concentration, \( X_0 \), at a chamber length, \( z_0 \), as noted in Eq. (11)

\[
X_p(1, z) = X_{p,w}(z) = \begin{cases} \frac{1}{2} X_0 \left[ 1 - \cos \left( \frac{\pi z}{z_0} \right) \right] ; & 0 \leq z < z_0 \\ X_0 ; & z \geq z_0 \end{cases}.
\]

Dividing Eq. (10) through by \( J(r) \) and letting \( R(r) = G(r) / J(r) \), we find

\[
\frac{\partial X_p(r,z)}{\partial r} + zR(r) \frac{\partial X_p(r,z)}{\partial z} + \left[ J'(r) + \frac{1}{r} + R(r) \right] X_p(r,z) = 0.
\]

In this form we find that Eq. (12) can be turned into an ODE using the method of characteristics. To this end, we allow \( K(r) = X_p(r,z) \) and choose \( z = z(r) \) to satisfy the method. Differentiating with respect to \( r \) we obtain

\[
K'(r) = \frac{\partial X_p}{\partial r} + z'(r) \frac{\partial X_p}{\partial z}.
\]

Upon inspection of Eq. (12) we find that the first two terms collapse into \( K'(r) \), when \( z'(r) = zR(r) \) is chosen. The result is a separable ODE which has an initial condition at the wall, \( z(1) = z_0 \), and renders

\[
z(r) = z_0 \exp \int_1^r \frac{G(x)}{J(x)} \, dx.
\]

The original equation now collapses into

\[
K'(r) + \left[ J'(r) + \frac{1}{r} + R(r) \right] K(r) = 0.
\]

Given that \( K(1) = X_p(1,z_0) = X_{p,w}(z_0) \), the general solution becomes

\[
K(r) = K(1) \exp \left\{ - \int_1^r \left[ \frac{J'(x)}{J(x)} + \frac{1}{x} + R(x) \right] dx \right\} = X_{p,w}(z_0) \frac{J(1)}{rJ(r)} \exp \left[ - \int_1^r R(x) dx \right].
\]

With Eq. (14) substituted into Eq. (16), we find the following semi-analytical solution

\[
X_p(r,z) = X_{p,w} \left[ J(1) \frac{J(r)}{rJ(r)} \right] I(r) \quad \text{where} \quad I(r) = \exp \left[ - \int_1^r \frac{G(x)}{J(x)} \, dx \right].
\]

5. Results and discussion

As performed in past studies [5, 15-18], Chebyshev collocation methods are utilized for discretization followed by the selection of judicious boundary conditions [13]. The result is a well-posed eigenvalue problem, which is solved to find the characteristic eigenmode frequencies and the eigensolutions corresponding to the fluctuating parameters. Figures 2–3 illustrate these results for the simulated SRM under investigation. The stabilizing effects of particles are shown in Figure 1 along with a possible evolution of eigenmodes, which compares favorably with the findings of Féraïlle and Casalis [4]. Conversely, Figure 2 illustrates the destabilizing effect of particles, where arrows are used to indicate the evolution of the eigenmodes.
Figure 2. Possible evolution of two-phase eigenmodes. Illustration of stabilizing effects when $Re = 10^3$, $S = 10^{-1}$, $\phi = 0.1$, $q = 0$, and $u_h = 0.0$.

Figure 3. Possible evolution of two-phase eigenmodes. Illustration of destabilizing effects when $Re = 10^3$, $S = 10^{-3}$, $\phi = 1.0$, $q = 0$, and $u_h = 0.0$. 
The effects of varying the chamber length from the headwall, $z_0$, where the particle mass concentration smoothly transitions to its maximum were also investigated. The findings, which are shown in Figure 4, agree with experimental data obtained from the VECLA facility by Avalon et al. [8]. We find that when the transition reaches its maximum at or after $z_0 = 5$ the range of unstable eigenmodes increases. This is the location where the experimental studies indicate flow breaks down. Avalon et al. [8] also find further flow disturbances after a distance equivalent to $z_0 = 10$ in the present study. We find that after this critical chamber length the eigenmodes are significantly amplified, more than three times that of values found for smaller values of $z_0$.

6. Conclusions
This work details the development of a semi-analytical solution for the particle mass concentration in two-phase rocket flows. The solution is added to a biglobal stability framework, which is used to understand the effects of particles in particle-laden solid rocket motors. The particles are found to have both stabilizing and destabilizing effects on the flowfield, mostly attributed to large particle-to-gas injection speeds. Results are also found that show amplification of instabilities when the chamber length, $z_0$, is increased to and beyond lengths where flow breakdown occurs according to Avalon et al. [8]. Ongoing analysis of eigenmodes involves the addition of headwall injection to simulate solid rocket motors with reactive headwall and hybrid rocket engines.

In future work, considerations will be made to vary particle properties such as reactivity, regression, size, and shape. In addition, compressibility effects will be investigated using a merger between this framework and the framework by Akiki et al. [17, 18]. Other areas of interest include sensitivity to noncircular cross-sections, swirling mean flow, and code parallelization to increase efficiency of simulations.

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