THE THUE-MORSE SUBSTITUTIONS AND SELF-SIMILAR GROUPS AND ALGEBRAS

LAURENT BARTHOLDI, JOSÉ MANUEL RODRÍGUEZ CABALLERO, AND TANBIR AHMED

Abstract. We introduce self-similar algebras and groups closely related to the Thue-Morse sequence, and begin their investigation by describing a character on them, the “spread” character.

1. Introduction

Fix an alphabet $X = \{x_0, \ldots, x_{q-1}\}$. The Thue-Morse substitution is the free monoid morphism $\theta : X^* \to X^*$ given by

$$\theta(x_i) = x_ix_{i+1} \ldots x_{q-1}x_0 \ldots x_{i-1},$$

and the Thue-Morse word $W_q \in X^\omega$ is the limit of all words $\theta^n(x_0)$. For example, if $q = 2$ then $\theta(x_0) = x_0x_1$ and $\theta(x_1) = x_1x_0$ and $W_2 = x_0x_1x_0x_1x_0x_0x_1 \ldots$ is the classical, ubiquitous Thue-Morse sequence, see [1,6].

We construct some self-similar algebraic objects — groups and associative algebras — and report on a curious connection between them and the Thue-Morse substitution.

Fix an alphabet $A = \{a_0, \ldots, a_{q-1}\}$. Recall that a self-similar group is a group $G$ endowed with a group homomorphism $\phi : G \to G \wr A \Sigma A$, the decomposition: every element of $G$ may be written, via $\phi$, as an $A$-tuple of elements of $G$ decorating a permutation of $A$. Likewise, a self-similar algebra is an associative algebra $\mathcal{A}$ endowed with an algebra homomorphism $\phi : \mathcal{A} \to M_q(\mathcal{A})$ also called the decomposition: every element of $\mathcal{A}$ may be written as an $A \times A$ matrix with entries in $\mathcal{A}$. For more details see [3,9].

We insist that self-similarity is an attribute of a group or algebra, and not a property: it is legal to consider for $G$ or $\mathcal{A}$ a free group (respectively algebra), and then the decomposition $\phi$ may be defined at will on $G$ or $\mathcal{A}$’s generators. There will then exist a maximal quotient (called the injective quotient) of $G$ or $\mathcal{A}$ on which $\phi$ induces an injective decomposition. This is the approach we follow in defining our self-similar group.

Consider the free group $F = \langle x_0, \ldots, x_{q-1} \rangle$, the alphabet $A = \mathbb{Z}/q$, and define $\phi : F \to F \wr A \Sigma A$ by

$$\phi(x_0) = \langle x_0, \ldots, x_{q-1} \rangle(j \mapsto j + 1), \quad \phi(x_i) = \langle 1, \ldots, 1 \rangle(j \mapsto j + 1) \text{ for all } i \geq 1.$$
Here and below we denote by $\langle g_0, \ldots, g_{q-1}\rangle \pi$ the element of $F \wr \mathfrak{S}_A$ with decorations $g_i$ on the permutation $\pi$. We denote by $G_q$ the injective quotient of $F$, with self-similarity structure still written $\phi$. Note that it is a proper quotient; for example, the image of $x_1$ has order $q$ in $G_q$.

There is a standard construction of a self-similar algebra from a self-similar group, by mapping decorated permutations to monomial matrices. Fix a commutative ring $k$, consider the free associative (tensor) algebra $T = k \langle x_0, \ldots, x_{q-1} \rangle$, and define $\phi : T \to M_q(T)$ by

$$\phi(x_0) = \begin{pmatrix} 0 & \cdots & 0 & x_{q-1} \\ x_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & x_{q-2} & 0 \end{pmatrix}, \quad \phi(x_i) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$  

We denote by $\mathcal{A}_q$ the injective quotient of $T$, with self-similarity structure still written $\phi$. Our main result is a description of a natural character, the “spread”, on $\mathcal{A}_q$; see Section 3.1; roughly speaking, it measures the number of non-zeros in matrix rows or columns:

**Theorem A.** The “spread” character on $\mathcal{A}_q$ has image $\mathbb{Z}[1/q] \cap \mathbb{R}_+$.  

The proof crucially uses the fact that the decomposition of $G_q$ admits a partial splitting defined using the Thue-Morse endomorphism $\theta$; the same holds for $\mathcal{A}_q$. This is embodied in the following Lemma, proved in the next section:

**Lemma 1.1.** For all $w \in F$ we have $\phi(\theta(w)) = \langle w, \gamma(w), \ldots, \gamma^{q-1}(w) \rangle$, where $\gamma : F \to F$ is the automorphism permuting cyclically the generators $x_i \mapsto x_{i+1 \mod q}$.  

We conclude with some variants of the construction, and in particular relations to iterated monodromy groups of rational functions in one complex variable.  

## 2. The groups

As sketched in the introduction, a self-similar group is a group $G$ endowed with a homomorphism $\phi : G \to G \wr A \mathfrak{S}_A$, the decomposition. The range of $\phi$ is the permutational wreath product of $G$ with $A$; its elements may be represented as permutations of $A$ with a decoration in $G$ on each strand. We write $\phi(g) = \langle g_0, \ldots, g_{q-1} \rangle \pi$.

Starting from the free group $F = \langle x_0, \ldots, x_{q-1} \rangle$ and the alphabet $A = \{a_0, \ldots, a_{q-1}\}$, we define $\phi : F \to F \wr A \mathfrak{S}_A$ by

$$\phi(x_0) = \langle x_0, \ldots, x_{q-1} \rangle(j \mapsto j + 1), \quad \phi(x_i) = \langle 1, \ldots, 1 \rangle(j \mapsto j + 1),$$

turning $F$ into a self-similar group. Write $K_0 = 1$ and $K_{n+1} = \phi^{-1}(K_n A)$; these form an ascending sequence of normal subgroups of $F$, and $G := F / \bigcup_n K_n$ is again a self-similar group, but now on which the map induced by $\phi$ is injective. We christen the group $G$ just constructed the $q$th Thue-Morse group. The decompositions may be written, using permutations, as

$$\phi(x_0) = \begin{pmatrix} x_1 & x_0 & x_{q-1} \end{pmatrix}, \quad \phi(x_i) = \begin{pmatrix} \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 1 & \ddots & \ddots \end{pmatrix}.$$  

Note that in the injective quotient $G_q$ the generators $x_1, \ldots, x_{q-1}$ coincide and have order $q$. We thus have a presentation

$$G_q = \langle x_0, x_1 \mid x_1^q, [(x_0 x_1^{-1})^q, (x_1^{-1} x_0)^q], \ldots \rangle,$$
where producing an explicit presentation of the group is beyond our current goals, but could be done following the lines of [2].

It is straightforward to prove Lemma 2.1 for generator \( x_i \), we have \( \phi(\theta(x_i)) = \langle x_i, x_{i+1}, \ldots, x_{i-1} \rangle = \langle x_i, \gamma(x_i), \ldots, \gamma^{q^{-1}}(x_i) \rangle \), so

\[
\phi(\theta(w)) = \langle w, \gamma(w), \ldots, \gamma^{q^{-1}}(w) \rangle \quad \text{for all } w \in F.
\]

A self-similar group \( G \) is called contracting if there exists a finite subset \( N \subseteq G \) with the following property: for every \( g \in G \) there exists \( n \in \mathbb{N} \), such that if one iterates the decomposition at least \( n \) times on \( g \) then all entries belong to \( N \). The minimal admissible such \( N \) is called the nucleus.

**Lemma 2.1.** The Thue-Morse group \( G_q \) is contracting with \( N = \{ x_0^{\pm 1}, x_1^{\pm 1} \} \).

**Proof.** It suffices to check contraction on words in \( N^2 \), and this is direct. \( \square \)

Let \( G \) be a self-similar group, and consider an element \( g \in G \). Iterating \( n \) times the map \( \phi \) on \( g \) yields a permutation of \( A^n \) decorated by \( \# A^n \) elements. The element \( g \) is called bounded if only a bounded number of these decorations are non-trivial, independently of \( n \). The group \( G \) itself is called bounded if all its elements are bounded; by an easy argument, it suffices to check this property on generators of \( G \). It is classical [5] that if \( G \) is bounded and finitely generated then it is contracting.

### 2.1. Characters.

Recall that a character \( \chi : G \to \mathbb{C} \) on a group is a function that is normalized (\( \chi(1) = 1 \)), central (\( \chi(gh) = \chi(hg) \) for all \( g, h \in G \)) and positive semidefinite (\( \sum_{j=1}^n \lambda_j g_j^{-1} \bar{\lambda}_j \geq 0 \) for all \( g_j \in G, \bar{\lambda}_j \in \mathbb{C} \)). A model example of character are the “fixed points”: if \( G \) acts on a measure space \( (X, \mu) \), set \( \chi(g) = \mu(\{ x \in X : g(x) = x \}) \). By the Gelfand-Naimark-Segal construction, every character may be written as \( \chi(g) = \langle \xi, \pi(g)\xi \rangle \) for some unitary representation \( \pi : G \to \mathcal{U}(\mathcal{H}) \) and some unit vector \( \xi \in \mathcal{H} \).

Let now \( G \) be self-similar, with decomposition \( \phi : G \to G \wr A \mathcal{S}_A \). A character \( \chi \) will be called self-similar if there exists a positive semidefinite kernel \( k(\cdot, \cdot) \in \mathbb{C}^{A \times A} \) such that

\[
(\# A) \chi(g) = \sum_{a \in A} k(a, \pi(a)) \chi(g_a) \quad \text{whenever } \phi(g) = \langle g_a \rangle \pi.
\]

We also note the following easy property of characters:

**Lemma 2.2.** If \( G \) is a contracting, self-similar group, then every self-similar character on \( G \) is determined by its values on the nucleus. If moreover \( G \) is bounded and finitely generated, then every self-similar character on \( G \) is determined by the kernel \( k \).

**Proof.** For each element \( g \in G \), write the linear relation imposed on \( \chi(g) \) by self-similarity of the character \( \chi \). Substituting sufficiently many times, \( \chi(g) \) may be expressed in terms of \( \chi \restriction N \).

If \( G \) is bounded, then furthermore the nucleus may be decomposed as \( N = N_0 \sqcup N_1 \) with the property that for every \( g \in N_0 \), all decorations of \( g \) are eventually trivial, while if \( g \in N_1 \), then a single decoration \( g' \) of \( g \) is in \( N_1 \) and all the others are in \( N_0 \). Clearly \( \chi \restriction N_0 \) is determined by \( k \), while for \( g \in N_1 \) we obtain a linear relation \( \chi(g) = \chi(g')/\# A + C_g \) with \( C_g \) depending only on \( k \); this linear system is non-degenerate, yielding a unique solution for \( \chi \restriction N_1 \). \( \square \)
Let us check that $G_q$ is bounded. For the generators $x_1, \ldots, x_{q-1}$ this is obvious, since all their decorations are trivial starting from level $n = 1$. Then $x_0$ has a single decoration which is $x_0$ itself on top of the $x_1, \ldots, x_{q-1}$, so in fact for all $n \in \mathbb{N}$ there are at most $q$ non-trivial decorations in the $n$-fold decomposition of $x_0$.

Note that every self-similar group acts on a $\#A$-regular rooted tree, as follows. The group fixes the empty sequence $\varepsilon$. To determine the action of $g \in G$ on a word $v = v_1 v_2 \ldots v_n$, compute $\phi(g) = \langle g_a \rangle \pi$; then define recursively $g(v) = \pi(v_1) g_{v_1}(v_2 \ldots v_n)$.

This action extends naturally to the boundary of the rooted tree, which is identified with the space of infinite sequences $A^\omega$. This space comes naturally equipped with the Bernoulli measure $\mu$, assigning mass $1/\#A$ to each of the elementary cylinders $C_{v,a} = \{v \in A^\omega : v_1 = a\}$, and $G$ acts by measure-preserving transformations. It is easy to see that the constant kernel $(k(a,b) = 1/\#A$ for all $a,b)$ induces the trivial self-similar character $\chi(g) = 1$, and that the identity kernel $(k(a,b) = \delta_{a=b})$ induces the fixed-point self-similar character $\chi(g) = \mu\{v \in A^\omega : g(v) = v\}$.

Recall that every self-similar group $G$ admits an injective quotient, on which the decomposition $\phi$ induces an injection $G \hookrightarrow G /_A \mathfrak{S}_A$. The group $G$ also admits a faithful quotient, defined as the quotient of $G$ by the kernel of the natural map to $\mathfrak{S}_A^{\omega}$ given by the action defined above; it is the largest self-similar quotient of $G$ that acts faithfully on $A^\omega$. Clearly the faithful quotient is a quotient of the injective quotient, but they need not coincide.

It is easy to see that, for $G_q$, the injective and faithful quotients coincide, using the contraction property and the fact that the action on $A^\omega$ is faithful on the nucleus.

3. The algebras

We fix once and for all a commutative ring $k$. We are particularly interested in the example $k = \mathbb{F}_q$.

As in the case of groups, we start by considering the free associative (tensor) algebra $T = k\langle x_0, \ldots, x_{q-1} \rangle$, and define $\phi : T \to M_q(T)$ by

$$
\phi(x_0) = \begin{pmatrix} 0 & \cdots & 0 & x_{q-1} \\
 & & & \\
x_0 & & 0 \\
 & \ddots & \ddots & \\
 & & \ddots & \\
 & & & 0 \\
 & & & x_{q-2} \\
 & & & 0 \\
\end{pmatrix}, \quad \phi(x_i) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\
 & & & \\
1 & \ddots & \ddots & \\
 & \ddots & \ddots & \\
 & & \ddots & \\
 & & & 0 \\
\end{pmatrix}.
$$

Write $J_0 = 0$ and $J_{n+1} = \phi^{-1}(M_q(J_n))$; these form then an ascending sequence of ideals in $T$, and $A_q := T / \bigcup_{n} J_n$ is a self-similar algebra, on which the map induced by $\phi$ is injective.

The construction of $A_q$ from $G_q$ should be transparent: both algebraic objects have the same generating set, and if $\phi(g) = \langle g_a \rangle \pi$ in $G_q$, then the decomposition $\phi(g)$ in $A_q$ is a monomial matrix with permutation $\pi$ and non-zero entries $g_a$.

It may be convenient to extend $A_q$ into a $^*$-algebra, namely an algebra $B_q$ equipped with an anti-involution $x \mapsto x^*$. This may easily be done by extending $T$.
to \( kF \), the group ring of \( F \), and extending the decomposition by

\[
\phi(x_0^{-1}) = \begin{pmatrix} 0 & x_0^{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & x_{q-2}^{-1} \\ x_{q-1}^{-1} & 0 & \cdots & 0 \end{pmatrix}, \quad \phi(x_i^{-1}) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \]

We then have a natural group homomorphism \( G_q \to B_q^* \) given by \( x_i \mapsto x_i \) on the generating set. In particular, \( B_q \) is a quotient of the group ring \( kG_q \). A presentation of \( B_q \) begins as

\[ B_q = \langle x_0^\pm 1, x_1^q - 1, (x_0x_1^{-1})^q - 1, (x_1^{-1}x_0)^q - 1, \ldots \rangle; \]

we see in particular that \( B_q \) is a proper quotient of \( kG_q \), since in \( kG_q \) the elements \( (x_0x_1^{-1})^q - 1 \) and \( (x_1^{-1}x_0)^q - 1 \) commute while in \( B_q \) their product vanishes, being a product of two matrices each with a single non-zero entry. As in the case of groups, a presentation of \( A_q \) and of \( B_q \) could be computed following the techniques in [3], but this is beyond our purposes.

We naturally extend the Thue-Morse endomorphism \( \theta \) to \( T \); and note then, similarly to Lemma 1.1, the easy

Lemma 3.1. We have

\[ \phi(\theta(w)) = \begin{pmatrix} w & 0 & \cdots & 0 \\ 0 & \gamma(w) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma^{-1}(w) \end{pmatrix}, \]

where \( \gamma \) is the endomorphism of \( T \) permuting cyclically the generators \( x_i \mapsto x_{i+1} \text{ mod } q \). \( \square \)

A self-similar algebra \( A \) is called contracting if there exists a finite-rank submodule \( N \leq A \) with the following property: for every \( s \in A \) there exists \( n \in \mathbb{N} \), such that iterating the decomposition at least \( n \) times on \( s \) gives a matrix with all entries in \( N \). The minimal admissible such \( N \) is called the nucleus.

Lemma 3.2. The Thue-Morse algebras \( A_q \) and \( B_q \) are contracting, with respective nuclei \( k\{x_0, x_1\} \) and \( k\{x_0^\pm 1, x_1^\pm 1\} \).

Proof. It suffices to check contraction on monomials in \( N^2 \), and this is direct. \( \square \)

Let \( A \) be a self-similar algebra, and consider an element \( x \in A \). Iterating \( n \) times the map \( \phi \) on \( x \) yields an \( A^n \times A^n \)-matrix with entries in \( A \). The element \( x \) is called row-bounded if only a bounded number of entries are non-trivial on each row of that matrix, independently of \( n \) and the row; and is called column-bounded if the same property holds for columns. The algebra \( A \) itself is called bounded if all its elements are bounded. Evidently, the product of row-bounded elements in row-bounded, and the same holds for column-bounded elements; so it suffices, to prove that \( A \) is bounded, to check that property on its generators. The same argument as in the case of groups shows that row-bounded or column-bounded self-similar algebras are contracting.

It is again easy to see that the algebras \( A_q \) and \( B_q \) are bounded. This will play a major role in the computations below.
3.1. **Characters.** We begin by introducing some concepts. A character on \( k \) is a semigroup homomorphism \( \chi : (k, \cdot) \rightarrow \mathbb{C} \) satisfying \( \chi(1) = 1 \) and \( \chi(0) = 0 \). Recall that the group of units in \( \mathbb{F}_q \) is cyclic; so may be embedded in \( \mathbb{C}^\times \) by mapping a generator to a primitive \( (q - 1) \)th root of unity. The trivial character, mapping all non-zero elements to 1, is also a valid choice.

By characters we think of extensions to a group ring \( kG \) of Brauer characters, rather than algebra homomorphisms. For our purposes, the following definition suffices:

**Definition 3.3.** A character on a \( k \)-self-similar algebra \( \mathcal{A} \) is a map \( \chi : \mathcal{A} \rightarrow \mathbb{C} \) satisfying, for some character \( \chi_0 \) on \( k 

\begin{enumerate}
\item \( \chi(1) = 1 \);
\item \( \chi(\lambda s) = \chi_0(\lambda)\chi(s) \) for all \( \lambda \in k, s \in \mathcal{A} \);
\item \( \chi(x^* x) \geq 0 \) for all \( x \in \mathcal{A} \), if \( \mathcal{A} \) is a \(*\)-algebra.
\end{enumerate}

\( \triangle \)

Note in particular that we do not require \( \chi(xy) = \chi(x)\chi(y) \) (this holds only for “linear characters”) nor \( \chi(x + y) = \chi(x) + \chi(y) \) (this would be meaningless if \( k \) has positive characteristic), and we also do not require \( \chi(xy) = \chi(yx) \) (this holds only for “diagonalizable elements”).

A character \( \chi \) on \( \mathcal{A} \) is called self-similar if there is a character \( \chi_0 \) on \( k \) and a positive semidefinite kernel \( k(\cdot, \cdot) \in \mathbb{C}^{\mathcal{A} \times \mathcal{A}} \) such that

\[
q \cdot \chi(s) = \sum_{i,j=0}^{q} k(i,j)\chi(\phi_i(s)_{i,j}).
\]

We also note the following easy property of characters:

**Lemma 3.4.** If \( \mathcal{A} \) is a contracting, self-similar algebra, then every self-similar character on \( \mathcal{A} \) is determined by its values on the nucleus. If moreover \( \mathcal{A} \) is row- or column-bounded, then every self-similar character on \( \mathcal{A} \) is determined by the kernel \( k \).

\( \square \)

We concentrate on two specific characters, which are both self-similar, with trivial character \( \chi_0(\lambda) = 1 - \delta_{\lambda = 0} \), and determined (via Lemma 3.4) respectively by the kernels \( k(i,j) = \delta_{i = j} \) and \( k(i,j) = 1 \). We denote the first character by \( \chi_f \) since it measures in some sense the fixed points of an element, and the second one by \( \chi_s \) since it measures in some sense the “spread” of an element. For ease of reference, the “spread” character is characterized by

\[
q \cdot \chi_s(\lambda s) = \sum_{i,j=0}^{q} \chi_s(\phi_i(s)_{i,j}) \text{ for all } \lambda \in k^\times.
\]

3.2. **The “spread” character.** We embark in the proof of Theorem [A] which will occupy this whole subsection.

The “spread” character is in fact tightly connected to the boundedness property of \( \mathcal{A} \). In the case of \( \mathcal{B}_q \), or more generally self-similar algebras whose generators decompose as monomial matrices, the recursion formula of \( \chi_s \) implies \( \chi_s(x_0) = \chi_s(x_1) = 1 \), and in fact in \( \mathcal{B}_q \) we have \( \chi_s(x) = 1 \) for any monomial \( x \in G_q \).

It follows that \( \chi_s \) may be related to the growth of languages in \( (A \times A)^* \): for each \( x \in \mathcal{A} \), set

\[
L_x = \{(u,v) \in A^k \times A^k \mid \phi^k(x)_{u,v} \in k^\times \cup k^\times x_0 \cup k^\times x_1\}.
\]
Lemma 3.5. For all $x \in \mathcal{A}$, the language $L_x$ is related to the “spread” character $\chi_s(x)$ as follows: there is a constant $C$ such that

$$\#((A \times A)^k \cap L_x) = q^k \chi_s(x) - C$$

for all $k$ large enough.

Proof. This follows from a slight refinement of the contraction property: in fact, for every $x \in \mathcal{A}$, if one iterates sufficiently many times $\phi$ on $x$ then the resulting matrix (of size $q^k \times q^k$) has entries in $k \times k \mathbb{x}_0 \cup k \mathbb{x}_1$, and the language $L_x$ counts those entries that are not trivial. On the other hand, the “spread” character also counts (up to normalizing by a factor $q^k$) the number of non-trivial entries. From then on, increasing $k$ multiplies the number of words in $L_x$ by $q$ so the relationship between the growth of $L_x$ and $\chi_s(x)$ remains the same. \hfill \Box

Note that we could have considered a large number of different other languages: counting the number of entries $(u, v) \in A^k \times A^k$ such that the $(u, v)$-coefficient of $\phi^k(x)$ is, at choice,

- a scalar in $\mathcal{A}$;
- a non-zero element in $\mathcal{A}$;
- an element not in the augmentation ideal $\langle x_i - 1 \rangle$ of $\mathcal{A}$;
- a monomial in $\mathcal{A}$;
- an invertible element of $\mathcal{A}$;
- a unitary element of $\mathcal{A}$.

All these choices would yield essentially equivalent languages, with comparable growth.

Lemma 3.6. For all integers $k \geq 1$, the “spread” character satisfies

$$\chi_s(1 - x_0^q) = 2/q^{k-1}, \quad \chi_s(1 - \gamma^i(x_0 \cdots x_{q-1})q^k) = 2/q^k.$$

Proof. We compute recursively some values of $\chi_s$. First, $\chi_s(x_1) = 1$ since $\phi(x_1)$ is a permutation matrix. Then $\chi_s(x_0) = 1$ since self-similarity of $\chi_s$ yields $q \chi_s(x_0) = \chi_s(x_0) + q - 1$. We next note $\chi_s(1 - x_0) = \chi_s(1 - x_1) = 2$; indeed self-similarity yields $q \chi_s(x_0) = 2q = q \chi_s(x_1)$.

Next, $\phi(x_0^q) = \langle x_0 \cdots x_{q-1}, x_1 \cdots x_{q-1}x_0, \ldots, x_{q-1}x_0 \cdots x_{q-2} \rangle$, and $\phi(x_0 \cdots x_{q-1}) = \langle x_0, \ldots, x_{q-1} \rangle$ and similarly for its cyclic permutations; so self-similarity yields

$$q \chi_s(1 - \gamma^i(x_0 \cdots x_{q-1})) = 2q, \quad q \chi_s(1 - x_0^q) = 2q$$

so $\chi_s(1 - \gamma^i(x_0 \cdots x_{q-1})) = \chi_s(1 - x_0^q) = 2$.

This is the beginning of induction: for $k \geq 1$, the matrix $\phi(x_0^{q^{k+1}})$ is diagonal, with diagonal entries $\gamma^i(x_0 \cdots x_{q-1})q^k$, and $\phi(\gamma^i(x_0 \cdots x_{q-1})q^k)$ is also diagonal, with diagonal entries $x_0^{q^k}, \ldots, x_{q-1}^{q^k}$; so self-similarity yields

$$q \chi_s(1 - x_0^{q^{k+1}}) = \sum_{i=0}^{q-1} \chi_s(1 - \gamma^i(x_0 \cdots x_{q-1})q^k),$$

$$q \chi_s(1 - (x_0 \cdots x_{q-1})q^k) = \chi_s(1 - x_0^{q^k}) + q(q - 1) \chi_s(1 - x_0^{q^k}).$$

Now $x_1^q = 1$ so the last term vanishes because $k \geq 1$, and we get $\chi_s(1 - x_0^{q^{k+1}}) = \chi_s(1 - \gamma^i(x_0 \cdots x_{q-1})q^k) = \chi_s(1 - x_0^{q^k})/q$. \hfill \Box
Consider next the map $\sigma : T \times \cdots \times T \to T$ given by

$$\sigma(s_0, \ldots, s_{q-1}) = \theta(s_0) + x_1 \theta(s_1) + \cdots + x_{q-1} \theta(s_{q-1}).$$

Recalling that $\gamma$ is the automorphism of $T$ permuting cyclically all generators, we get

$$\phi(\sigma(s_0, \ldots, s_{q-1})) = \begin{pmatrix} s_0 & \gamma(s_{q-1}) & \cdots & \gamma^{q-1}(s_1) \\ s_1 & \gamma(s_0) & \cdots & \gamma^{q-1}(s_2) \\ \vdots & \vdots & \ddots & \vdots \\ s_{q-1} & \gamma(s_{q-2}) & \cdots & \gamma^{q-1}(s_0) \end{pmatrix}.$$ 

We are ready to prove Theorem A. Define subsets $\Omega_n$ of $T$ by

$$\Omega_0 = \{0, 1 - \gamma^i(x_0 \cdots x_{q-1})^{\frac{q}{k}} \text{ for all } i, k\},$$

and finally $\Omega = \bigcup_{n \geq 0} \Omega_n$.

**Lemma 3.7.** For all $x \in \Omega$ and all $i$ the matrix $\phi(x)$ is diagonal and $\chi_x(s) = \chi_x(\gamma^i(x))$.

**Lemma 3.8.** For all $s_0, \ldots, s_{q-1} \in \Omega$ we have

$$\chi_x(\sigma(s_0, \ldots, s_{q-1})) = \chi_x(s_0) + \cdots + \chi_x(s_{q-1}).$$

**Proof.** This follows directly from the form of $\phi(\sigma(s_0, \ldots, s_{q-1}))$ given above, and the fact that $\chi_x$ is $\gamma$-invariant on $\Omega$. \hfill $\Box$

**Proof of Theorem A.** Since $H_q$ is contracting, every element $s \in A$ decomposes in finitely many steps into elements of the nucleus; and $\chi_x$ takes values in $\mathbb{Z}[1/q] \cap \mathbb{R}_+$ on the nucleus; so $\chi_x(A)$ is contained in $\mathbb{Z}[1/q] \cap \mathbb{R}_+$.

On the other hand, by Lemma 3.6 the values of $\chi_x$ include all $2/q^k$, and Lemma 3.8 its values form a semigroup under addition. It follows (considering separately $q$ even and $q$ odd) that all fractions of the form $i/q^k$ with $i, k \geq 0$ are in the range of $\chi_x$. \hfill $\Box$

### 4. Variants

Essentially the same methods apply to numerous other examples; we have concentrated, here, on the one with the closest connection to the Thue-Morse sequence.

Here is another example we considered: write the alphabet $A = \{a_0, \ldots, a_{q-1}\}$, and define $\phi : F \to \mathcal{F}_A \mathcal{G}_A$ by

$$\phi(x_0) = \langle x_0, \ldots, x_{q-1} \rangle(a_i \mapsto a_{i-1} \mod q), \quad \phi(x_1) = \langle 1, \ldots, 1 \rangle(a_0 \leftrightarrow a_1),$$

or in terms of matrices

$$\phi(x_0) = \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots \\ x_0 & \ddots & \ddots & \ddots & 0 \end{pmatrix}, \quad \phi(x_1) = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & 1 \end{pmatrix}.$$
If furthermore one applies the automorphism that inverts every generator (noting that the \(x_i\) are involutions for \(i \geq 1\)), we may define an injective self-similar group \(H_q\), isomorphic to the above, by

\[
\phi(x_0) = \langle x_0^{-1}, \ldots, x_{q-1}^{-1} \rangle(a_i \mapsto a_{i-1} \mod q), \quad \phi(x_i) = \langle 1, \ldots, 1 \rangle(a_0 \mapsto a_i).
\]

We now note that \(H_q\) is a contracting “iterated monodromy group”. As such, it possesses a limit space — a topological space equipped with an expanding self-covering, whose iterated monodromy group is isomorphic to \(H_q\). Note that \(H_2\) and \(G_2\) are isomorphic. It is tempting to try to “read” the Thue-Morse sequence, and in particular the Thue-Morse word, within the dynamics of the self-covering map.

4.1. Iterated monodromy groups. Let \(f\) be a rational function, seen as a self-map of \(\mathbb{P}^1(\mathbb{C})\), and write \(P = \{f^n(z) : n \geq 1, f'(z) = 0\}\) the post-critical set of \(f\). For simplicity, assume that \(P\) is finite. Choose a basepoint \(* \in \mathbb{P}^1(\mathbb{C})\backslash P\), and write \(F = \pi_1(\mathbb{P}^1(\mathbb{C})\backslash P, *)\), a free group of rank \(#P - 1\).

The choice of a family of paths \(\lambda_x : [0, 1] \to \mathbb{P}^1(\mathbb{C})\backslash P\) from * to \(x \in f^{-1}(*)\) for all choices of \(x\) naturally leads to a self-similar structure on \(F\), following [7]: the decomposition of \(\gamma \in F\) has as permutation the monodromy action of \(F\) on \(f^{-1}(*)\), and the \(\text{deg}(f)\) elements of \(F\) are all \(\lambda_x \# f^{-1}(\gamma) \# \lambda^{-1}_x\), with \# denoting concatenation of paths. The faithful quotient of \(F\) is called the iterated monodromy group of \(G\).

**Proposition 4.1.** The Thue-Morse group \(H_q\) is the iterated monodromy group of a degree-\(q\) branched covering of the sphere.

**Proof.** This follows from the general theory of [4]. The branched covering, and its iterated monodromy group, may be explicitly described as follows.

Consider as post-critical set \(\{0, \infty, \zeta^0, \ldots, \zeta^{q-2}\}\) for the primitive \((q - 1)\)th root of unity \(\zeta = \exp(2\pi i/(q - 1))\). Put the basepoint * inside the unit disk, in such a way that it sees \(\zeta^0, \zeta^1, \ldots, \zeta^{q-2}, 0, \infty\) in cyclic CCW order. Put the preimages of * at * and points \(*_i\) inside the unit disk but very close to \(\zeta^i\). As connections between * and its preimages choose paths \(\ell_i\) as straight lines. Consider as generators \(g_x\) a straight path from * to \(x\), following by a small CCW loop around \(x\), and back, in the order mentioned above.

The lift of each \(g_x\) will be two homotopic paths exchanging * and \(*_i\) (all other lifts are trivial) and the lifts of \(g_x\) will be \(g_0\) and a straight path from \(*_i\) to \(\zeta_i\) encircling it once CCW before coming back. It is clear that we have defined a branched covering of the sphere with the appropriate recursion.

**Conjecture 4.2.** The branched covering described above is isotopic to a rational map of degree \(q\).
We could verify this conjecture for small $q$; the maps corresponding to $q \leq 5$ are

\[ f_2 \approx \frac{1}{z - 0.5z^2}, \]
\[ f_3 \approx \frac{0.128775 + 0.0942072i}{z + (-1.74702 + 0.285702i)z^2 + (0.831347 - 0.190468i)z^3}, \]
\[ f_4 \approx \frac{0.0232438 + 0.0757918i}{z + (-2.67804 + 1.10938i)z^2 + (2.37852 - 1.93187i)z^3 + (-0.694865 + 0.89421i)z^4}, \]
\[ f_5 \approx \frac{0.00877156 + 0.0526634i}{z + (-3.22614 + 2.0417i)z^2 + (3.13076 - 5.12089i)z^3 + (-0.677772 + 4.35662i)z^4 + (-0.245783 - 1.22944i)z^5}. \]

For $q = 2$, when the groups $H_2$ and $G_2$ agree, it would be particularly interesting to relate the Thue-Morse word $W_2$ with the geometry of the Julia set of $f_2$. Here is a graph approximating this Julia set; the path $W_2$ may be traced in it, and may be seen to explore neighbourhoods of the large Fatou regions:

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GEORG-AUGUST UNIVERSEITAT ZU GOTTINGEN
E-mail address: laurent.bartholdi@gmail.com

UNIVERSITY OF TARTU, TARTU
E-mail address: jose.manuel.rodriguez.caballero@ut.ee

UNIVERSITÉ DU QUÉBEC À MONTRÉAL, QUÉBEC
E-mail address: tanbir@gmail.com