MOMENTS OF SPINOR L-FUNCTIONS AND SYMPLECTIC KLOOSTERMAN SUMS

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Abstract. We compute the second moment of spinor L-functions at central points of Siegel modular forms on congruence subgroups of large prime level \( N \) and give applications to non-vanishing.

1. Introduction

For the analytical theory of modular forms on congruence subgroups in \( \text{GL}_2(\mathbb{Z}) \), spectral summation formulas such as the Petersson formula are a basic tool. A primary component is a sum over Kloosterman sums and many applications rely on a careful estimation of the latter. For Siegel cusp forms, Kitaoka \cite{Kitaoka} introduced an equivalent to Petersson’s formula that was extended in \cite{Kocik} to include congruence subgroups. In this case, however, the off-diagonal terms are very complex and contain generalized Kloosterman sums that run over matrices in \( \text{Sp}_4(\mathbb{Z}) \). So far, literature on these sums is limited.

The aim of this article is to evaluate spectral averages of second moments of spinor \( L \)-functions for Siegel congruence groups of large prime level by means of the Kitaoka-Petersson formula. The core of this computation is the manipulation of symplectic Kloosterman sums which may be of independent interest.

To state our results, we fix some notation. For a natural number \( N \) let

\[
\Gamma_0^{(2)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) \mid C \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mod N \right\}
\]

denote the Siegel congruence group of level \( N \) and let \( \mathbb{H}_2 \) be the Siegel upper half plane consisting of all symmetric 2-by-2 complex matrices whose imaginary parts are positive definite. Let \( S_k^{(2)}(N) \) denote the space of Siegel cusp forms on \( \Gamma_0^{(2)}(N) \) of weight \( k \). For \( F, G \in S_k^{(2)}(N) \), we define the Petersson inner product by

\[
\langle F, G \rangle = \int_{\Gamma_0^{(2)}(N) \backslash \mathbb{H}_2} F(Z) \overline{G(Z)} \left( \det Y \right)^k \frac{dX dY}{(\det Y)^3}.
\]

Any \( F \in S_k^{(2)}(N) \) has a Fourier expansion

\[
F(Z) = \sum_{T \in \mathcal{S}} a_F(T) \left( \det T \right)^{\frac{k}{2} - \frac{3}{4}} e(\text{tr}(TZ)),
\]

with Fourier coefficients \( a_F(T) \), where \( \mathcal{S} \) is the set of symmetric, positive definite, half integral matrices \( T \) with integral diagonal. If \( F \) is an eigenform of the Hecke algebra, the Fourier coefficients are real-valued. For such eigenforms of even weight

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Let $L(s, F)$ denote the spinor $L$-function, normalized so that its critical strip is $0 < \Re s < 1$. This is a degree 4 $L$-function. Furthermore, we set

$$w_{F,N} := \frac{\pi^{1/2}}{4} (4\pi)^{3-2k} \Gamma(k-3/2) \Gamma(k-2) \frac{a_F(I)^2}{||F||^2},$$

where $I$ is the 2-by-2 identity matrix. These ”harmonic” weights appear naturally in the Kitaoka-Petersson formula. Due to our non-normalization of the inner product in (1), the denominator of $w_{F,N}$ implicitly contains a factor $[\text{Sp}_4(\mathbb{Z}) : \Gamma_0(N)] = N^3 \prod_{p | N} (1 + p^{-1})(1 + p^{-2})$.

More precisely, they are of size $N^{-3}$ on average, since according to [4, p. 37] it holds that

$$\sum_{F \in B_k^{(2)}(N)} w_{F,N} = 1 + \mathcal{O}(N^{-1}k^{-2/3}).$$

In addition, the weights $w_{F,N}$ are related to central values of $L$ functions. This remarkable conjecture is due to Böcherer and was recently proven by [5, Theorem 2 & Remark 6]. If $F$ is not a Saito-Kurokawa lift, and $\pi_F = \oplus_v \pi_{F,v}$ is the automorphic representation of $\text{GSp}_4(\mathbb{A})$ associated to $F$, it holds that

$$w_{F,N} = \frac{2^{5+s+t} \Gamma(2k-4) L(1/2, F)L(1/2, F \times \chi_{-4})}{N^3 \Gamma(2k-1) L(1, \pi_F, \text{Ad})},$$

where $s = -1$ if $F$ is a weak Yoshida lift and 0 otherwise, and $t$ counts how many $\pi_{F,v}$ for $p | N$ are of type Vlb, cf. [4, Theorem 1.12 & p. 10]. The adjoint $L$-function $L(s, \pi_F, \text{Ad})$ is of degree 10.

Let $q_1, q_2 \ll N^\varepsilon$ be two coprime fundamental discriminants (possibly 1) and denote by $\chi_{q_i}$ the character which maps $x$ to the Kronecker symbol $(\frac{x}{q_i})$.

**Theorem 1.** Let $N \equiv 3 \mod 4$ be a large prime, $k \geq 8$ and $B_k^{(2)}(N)$ denote a Hecke basis of $\Gamma_0^{(2)}(N)$. Then

$$\sum_{F \in B_k^{(2)}(N)} w_{F,N} L(1/2, F \times \chi_{q_1}) L(1/2, F \times \chi_{q_2}) = \text{main term} + \mathcal{O}_{q_1, q_2, k}(N^{-\alpha + \varepsilon}),$$

where the main term is the residue at $s = t = 0$ of the expression (22) and $\alpha = \frac{1}{2}$ for $k \geq 12$, $\alpha = \frac{5}{12}$ for $k = 10$ and $\alpha = \frac{3}{4}$ for $k = 8$.

In particular, if $q_1 = q_2 = 1$, the main term equals

$$\frac{4}{3} L(1, \chi_{-4})^2 P_3(\log N)$$

for a certain monic polynomial $P_3$ of degree 3 depending on $q_1, q_2$ and $k$.

If $\{q_1, q_2\} \in \{1, -4\}$, the main term equals

$$2 L(1, \chi_{-4})^2 P_2(\log N)$$

for a certain monic polynomial $P_2$ of degree 2 depending on $q_1, q_2$ and $k$.

If $\{q_1, q_2\}$ are two coprime integers different from 1 and -4, the main term equals

$$4 L(1, \chi_{q_1}) L(1, \chi_{-4q_1}) L(1, \chi_{q_2}) L(1, \chi_{-4q_2}) L(1, \chi_{q_1q_2})$$

The requirement $N \equiv 3 \mod 4$ ensures that $L(s, F)$ has conductor $N^2$ for all newforms with $w_{F,N} \neq 0$ that are not Saito-Kurokawa lifts. This allows to apply an approximate functional equation in the proof of Theorem 1.
For large weights and the full modular group, i.e. \( N = 1 \), Blomer [2] shows a very similar result and the proof of Theorem 1 is based on his work. While obtaining a uniform estimate in weight \( k \) and level \( N \) is principally possible, this requires deriving a Petersson formula for newforms. This can be achieved by constructing an explicit orthogonal basis of oldforms and then applying Möbius inversion to sieve these forms out (cf. [10] for the \( \text{SL}_2(\mathbb{Z}) \) case).

The main difficulty of proving Theorem 1 is treating the off-diagonal contribution in the Kitaoka-Petersson formula. This term is a sum over Bessel functions and symplectic Kloosterman sums whose "moduli" run over integral 2-by-2 matrices with all entries divisible by \( N \). Consequently, we decompose each Kloosterman sum into two parts, separating a Kloosterman sum of modulus \( N I \) which is straightforward to handle. After applying Poisson summation, we see that the sum vanishes unless a congruence condition is fulfilled. In this way, only matrices in \( \text{GO}_2(\mathbb{Z}) = \mathbb{R}_{>0} \cdot \text{O}(2) \cap \text{Mat}_2(\mathbb{Z}) \) survive as possible moduli for the remaining Kloosterman sums. This corresponds to the case of large weight in [2] and the remaining term can be computed in exactly the same way. In contrast to Blomer, who uses special features of Bessel functions, we manipulate symplectic exponential sums and evaluate congruences. Hence, this work can be seen as a non-archimedean version of [2], where the analysis of oscillatory integrals is replaced - in disguise - by its p-adic analogue.

In view of Böcherer's conjecture, Theorem 1 even evaluates a fourth moment of central values and a degree 16 L-function. Indeed, the contribution of the Saito-Kurokawa lifts in (5) is small. If \( F, N \) is related to central \( L \)-values of \( f \), i.e. by [4, Theorem 3.12] we have that

\[
 w_{F,N} = \frac{3 \cdot 2^8 \pi^5 L(1, \chi_{-4})^2 \Gamma(2k-4)}{N^3 \Gamma(2k-1)} \frac{L(1/2, f \chi_{-4})}{L(3/2, f)L(1, f, \text{Ad})}.
\]

By applying the convexity bound for central \( L \)-values, we see that the contribution of Saito-Kurokawa lifts in (5) is \( \mathcal{O}(N^{-5/4+\epsilon}). \)

Let \( B_k^{(2)*}(N) \) denote a Hecke basis of the space orthogonal to Saito-Kurokawa lifts. Then, by applying (5), Cauchy-Schwarz and (7), we get:

**Corollary 2.** For \( k \geq 8 \) and a sufficiently large prime \( N \equiv 3 \pmod{4} \), it holds that

\[
 \sum_{F \in B_k^{(2)*}(N)} \frac{1}{L(1, \pi_f, \text{Ad})} \gg \frac{N^3}{(\log N)^2}.
\]

In particular, if \( L(1, \pi_f, \text{Ad}) \) has no zeros in \( |s - 1| \ll N^{-\epsilon} \), then \( N^3\epsilon \) forms \( F \in B_k^{(2)*}(N) \) satisfy \( w_{F,N} \neq 0 \) and thus \( L(1/2, F)L(1/2, F \chi_{-4}) \neq 0 \).

Moreover, we get the following quadruple non-vanishing result:

**Corollary 3.** Let \( q_1 \) and \( q_2 \) be any two coprime fundamental discriminants and let \( N \) be sufficiently large. Then there exists a newform \( F \in B_k^{(2)}(N) \) that is not a SK lift such that

\[
 L(1/2, F)L(1/2, F \chi_{-4})L(1/2, F \chi_{q_1})L(1/2, F \chi_{q_2}) \neq 0.
\]

**Notation and conventions.** We use the usual \( \epsilon \)-convention and all implied constants may depend on \( \epsilon \). A term is negligible, if it is of size \( \mathcal{O}(N^{-100}) \). By \( [\cdot, \cdot, \cdot] \), \( (\cdot) \)
we refer to the least common multiple respectively the greatest common divisor of two integers. Furthermore, we set \( \ell := k - 3/2 \).

## 2. Spinor zeta function and Saito-Kurokawa lifts

For a Siegel cusp form \( F \in S_k^{(2)}(N) \) of even weight \( k \) that is an eigenform of the Hecke algebra with eigenvalues \( \lambda_p \) and Satake parameters \( \alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p} \) (satisfying \( \alpha_{0,p}^2 \alpha_{1,p}^2 \alpha_{2,p} = 1 \)) at primes \( p \), the spinor \( L \)-function is defined by a degree 4 Euler product

\[
L(s, F) = \prod_p \left( 1 - \frac{\alpha_{0,p}}{p^s} \right)^{-1} \left( 1 - \frac{\alpha_{0,p} \alpha_{1,p}}{p^s} \right)^{-1} \left( 1 - \frac{\alpha_{0,p} \alpha_{2,p}}{p^s} \right)^{-1} \left( 1 - \frac{\alpha_{0,p} \alpha_{1,p} \alpha_{2,p}}{p^s} \right)^{-1}
\]

for \( \Re s \) sufficiently large. As in \([2]\), we normalize all \( L \) functions to have \( 0 < \Re s < 1 \) as the critical strip. This corresponds to a linear shift \( s \mapsto s + k - 3/2 \) in comparison with \([1]\) and many other sources. The Dirichlet expansion of \( L(s, F) \) is given by \([1]\) p. 69

\[
L(s, F) = \zeta_N(2s + 1) \sum_m \frac{\lambda_F(m)}{m^s},
\]

where \( \zeta_N(s) = \zeta(s) \prod_{p \mid N} (1 - p^{-s}) \). Furthermore, we need the following formula from \([1]\) Theorem 4.3.16

\[
L(s, F \times \chi_q) a_F(I) = L(s + 1/2, \chi_q) L(s + 1/2, \chi_{-4q}) \sum_m a_F(mI) \chi_q(m) m^{-s}
\]

with \( l = a = 1, \eta = \chi = \text{trivial} \). We denote by

\[
r(n) = r_q(n) = \frac{\chi_q(n)}{n^{1/2}} \sum_{d \mid n} \chi_{-4}(d)
\]

the Dirichlet coefficients of \( L(s + 1/2, \chi_q) L(s + 1/2, \chi_{-4q}) \). If \( q = 1 \) the latter is the Dedekind zeta function \( \zeta_{\mathbb{Q}(i)}(s + 1/2) \).

The space \( S_k^{(2)}(N) \) contains a subspace of lifts from elliptic Hecke cusp forms \( f \) of weight \( 2k - 2 \) and level \( N \). For odd, squarefree \( N \), there exists a Hecke equivariant isomorphism between the two spaces, cf. \([9]\)

\[
L(s, F) = \zeta_N(s - 1/2) \zeta_N(s + 1/2) L(s, f).
\]

A key ingredient for the proof of Theorem \([1]\) is the approximate functional equation for \( L(s, f) \). If \( N \equiv 3 \pmod{4} \) and \( F \) is a Siegel modular newform for \( \Gamma_0^{(2)}(N) \) that is not a lift, then \( L(s, F) \) has conductor \( N^2 \) in all cases where \( w_{F,N} \neq 0 \), cf. \([2]\) p. 16 and \([4]\) p. 13. For such \( F \), \( L(s, F) \) has the following meromorphic continuation and functional equation proved by \([1]\) Theorem 3.1]

\[
\Lambda(s, F) = N^s L_{\infty}(s, F) L(s, F) = N^s \Gamma_{\mathbb{C}}(s + 1/2) \Gamma_{\mathbb{C}}(s + k - 3/2) L(s, f) = \Lambda(1 - s, F),
\]

where \( \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s) \). For such \( F \) we get in a standard way \([6]\) Theorem 5.3]

by \([10]\) the following approximate function:

\[
a_F(I) L(1/2, F \times \chi_q) = 2 \sum_{n,m} r_q(n) a_F(mI) s \chi_q(m) \frac{W \left( \frac{nm}{N|q|^2} \right)}{(nm)^{1/2}}
\]
with
\[ W(x) = \frac{1}{2\pi i} \int_{(2)} \frac{L_x(s + 1/2)}{L_x(1/2)} (1 - s^2)x^{-s} \frac{ds}{s}. \]

By shifting the contour, we see that the integral satisfies for all \( A > 0 \) the bound
\[ W(x) \ll_A (1 + x)^{-A}. \]

3. Kitaoka-Petersson Formula

The primary tool in the proof of Theorem 1 is a spectral summation of Petersson type for Siegel cusp forms. For the full modular group, it was proved in [7] and later extended in [8] to include congruence subgroups. We quote their results and introduce some notation. Let \( \Lambda \) denote all symmetric, integral 2-by-2 matrices. A major role plays a generalized Kloosterman sum
\[ K(Q, T; C) = \sum_D e(\text{tr}(AC^{-1}Q + C^{-1}DT)), \]

where \( Q, T \in \mathcal{S}, C \in \text{Mat}_2(\mathbb{Z}) \), the sum runs over matrices
\[ \left\{ D \in \text{Mat}_2(\mathbb{Z}) \mod CA \mid \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) \right\}, \]

and \( A \) is any matrix such that \( \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) \). The cardinality of \( (15) \) depends only on the elementary divisors of \( C \), since
\[ K(Q, T; U^{-1}CV^{-1}) = K(UQU^T, VTV; C) \quad \text{for } U, V \in \text{GL}_2(\mathbb{Z}), \]

and for \( C = \begin{pmatrix} c_1 & c_2 \\ c_1c_2 & c_2 \end{pmatrix} \) one has \( |K(Q, T; C)| \leq c_1^2c_2 \leq \det C^{3/2}. \)

For a real, diagonalizable matrix \( P \) with positive eigenvalues \( c_1^2, c_2^2 \), we write
\[ J_k(P) = \int_0^{\pi/2} J_k(4\pi c_1 \sin \theta)J_k(4\pi c_2 \sin \theta) \sin \theta \, d\theta, \]
where \( J_k(x) \) denotes the Bessel function of weight \( k \). For \( J_k(x) \), we have the simple bounds
\[ J_k(x) \ll 1 \quad \text{and} \quad J_k(x) \ll x^k \]
for all \( x > 0, k > 2 \).

For two matrices \( P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4 \end{pmatrix} \in \mathcal{S}, S = \begin{pmatrix} s_1 & s_2/2 \\ s_2/2 & s_4 \end{pmatrix} \in \mathcal{S} \) and \( c \in \mathbb{N} \) we define a "Salié" sum
\[ H^\pm(P, S; c) = \delta_{s_4=p_4} \sum_{d_1 \pmod{c}}^{#} \sum_{d_2 \pmod{c}} e \left( \frac{d_1s_3d_2^2 + d_2s_2d_2 + s_2d_2 + d_4p_1 + d_3s_1 + p_2s_2}{c} \right). \]

This sum is relatively easy to handle, and by applying the well-known bound for Gauss sums
\[ \sum_{x \pmod{c}} e \left( \frac{ax^2 + bx}{c} \right) \ll (a, c)^{1/2} c^{1/2} \]
for the \( d_2 \) sum, and estimating the \( d_1 \) sum trivially, we get \( |H^\pm(P, S; c)| \ll c^{3/2}(c, s_4)^{1/2}. \)
For $Q \in \mathcal{S}$, we define a Poincaré series

$$P_Q(Z) = \sum_{\gamma \in \Gamma_x \backslash \mathcal{V}_{k}^{(2)}(N)} J(\gamma, Z)^{-k} e(\text{tr}(Q\gamma Z)) = \sum_{T \in \mathcal{S}} h_Q(T)(\det T)^{\frac{k}{2} - \frac{d}{4}} e(\text{tr}(TZ)),$$

where $\Gamma_x = \left\{ \left( \begin{array}{cc} S & I \\ I & \end{array} \right) \mid S \in \mathcal{A} \right\}$, $J(\gamma, z) = \det(CZ + D)$ for $\gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$. Then, we have by [3, Proposition 2.1] that

$$\langle F, P_Q \rangle = 8c_k(\det Q)^{-\frac{k}{2} + \frac{d}{8}} a_F(Q), \quad \text{with } c_k = \frac{1}{4} \pi^{1/2}(4\pi)^{3-2k} \Gamma(k-3/2) \Gamma(k-2),$$

for $F \in S_k^{(2)}(N)$. By computing $\langle P_T, P_Q \rangle$ for $T, Q \in \mathcal{S}$ it follows

$$\tag{18} 8c_k \left( \frac{\det T}{\det Q} \right)^{\frac{k}{2} - \frac{d}{4}} \sum_{F \in H_k^{(2)}(N)} \frac{a_F(T)a_F(Q)}{\|F\|^2} = h_Q(T)(\det T)^{\frac{k}{2} - \frac{d}{4}}.$$

The Fourier coefficients of the Poincaré series, $h_Q(T)$, have been computed in [3] :

**Lemma 4.** It holds for $T, Q \in \mathcal{S}$ and even $k \geq 6$ that

$$h_Q(T)(\det T)^{\frac{k}{2} - \frac{d}{4}} = \delta_{Q, T} \# \text{Aut}(T)$$

$$+ \left( \frac{\det T}{\det Q} \right)^{\frac{k}{2} - \frac{d}{4}} \sum_{s \in \mathcal{S}} \sum_{c > 1} \sum_{U, V} \sum_{N \in C} \left( -1 \right)^k \frac{2\pi}{c^{3/2}s^{1/2}} H^\pm(UQU^T, V^{-1}TV^{-T}, c)J_s \left( \frac{4\pi \sqrt{\det(TQ)}}{cs} \right),$$

$$+ 8\pi^2 \left( \frac{\det T}{\det Q} \right)^{\frac{k}{2} - \frac{d}{4}} \sum_{\text{det } C \neq 0} \frac{K(Q, T; C)}{|\text{det } C|^{3/2}} J_{\epsilon}(\text{det } C^{-1}QC^{-T}),$$

where the sum over $U, V \in \text{GL}_2(\mathbb{Z})$ in the second term on the right hand side runs over matrices

$U = \left( \begin{array}{cc} * & * \\ u_3 & u_4 \end{array} \right) / (\pm), \quad V = \left( \begin{array}{cc} v_1 & * \\ v_3 & u_4 \end{array} \right), \quad (u_3 u_4)Q \left( \begin{array}{cc} u_3 \\ u_4 \end{array} \right) = (-v_3 v_1)T \left( \begin{array}{cc} -v_3 \\ v_1 \end{array} \right) = s,$

$Q \sim T$ means equivalence in the sense of quadratic forms and $\text{Aut}(T) = \{ U \in \text{GL}_2(\mathbb{Z}) \mid U^T TU = T \}$. The sums are absolutely convergent for $k \geq 6$.

In the last term, [7] and [3] have the constant $1/2\pi^4$ instead of $8\pi^2$. As pointed out by [2, p. 7], this is incorrect. As in [7] and [3], we refer to the first term on the right side as diagonal term, the second as rank 1 and the third as rank 2 case.

4. Symplectic Kloosterman sums

The aim of this section is to decompose the modulus of a symplectic Kloosterman sum of the type $K(\mu_1 I, \mu_2 I; NC)$ with $\mu_1, \mu_2, N \in \mathbb{Z}$. Kitaoka [7] Lemma 1-3 shows how to decompose such a Kloosterman sum for the case that $C$ is diagonal and in combination with [16], this is sufficient in most cases. However, we want to avoid non-diagonal terms in the second argument of the Kloosterman sum, hence, we need to adjust Kitaoka’s proof slightly. To simplify notation, we set $\Gamma := \text{Sp}_4(\mathbb{Z})$.

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2There is a minor typo in [3, Proposition 2.1], there should be an additional factor of 2 on the right hand side. The reason is that the original proof from Klingen [3, Section 6] uses a different definition for $\Gamma_x$ that differs from the definition in [3] (that corresponds to ours) by a factor 2.
**Lemma 5.** Set $c := \det C$ and assume $(c,N) = 1$. Choose integers $s,t$ with $sN + tc = 1$ and set $X = tc \cdot C^{-1}$. Let $Q,T \in \mathcal{I}$. Then

$$K(Q,T,N C) = K(XQX^T,T,N)K(s^2 Q,T,C),$$

Proof. It holds that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ if and only if $A^T D - C^T B = I$ and $A^T C$ and $B^T D$ are symmetric. Since $A^T C$ symmetric implies that $AC^{-1}$ is symmetric, it holds that,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \Leftrightarrow \begin{pmatrix} C^T A & C^T B - s A^T D \\ N^C & X D \end{pmatrix}, \begin{pmatrix} NA & NB - X^T A^T D \\ s D \end{pmatrix} \in \Gamma,$$

cf. [7] Proof of Lemma 1. Consequently, we can show that the map

$$D \mod N C A \rightarrow \left( X D \mod N A, s D \mod C A \right)$$

$$\left\{ D \mod N C A \left| \begin{pmatrix} * & * \\ N C & D \end{pmatrix} \in \Gamma \right. \right\} \rightarrow \left\{ D \mod N A \left| \begin{pmatrix} * & * \\ N I & D \end{pmatrix} \in \Gamma \right. \right\} \rightarrow \left\{ D \mod C A \left| \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma \right. \right\}$$

is bijective. This works exactly as in the proof of [7] Lemma 2, since $NC = CN$. By $CX + sNI = I$, we obtain

$$\begin{aligned}
\tr(A(NC)^{-1} Q + (NC)^{-1} DT) &= \tr((X^T C^T A + sNA) N^{-1} C^{-1} Q + N^{-1} C^{-1} (C XD + sND) T) \\
&= \tr((X^T C^T A N^{-1} C^{-1} Q + sAC^{-1} Q + N^{-1} X DT + sC^{-1} DT) \\
&= \tr((X^T C^T A N^{-1} (sNI + XC) C^{-1} Q + sA(sNI + XC) C^{-1} Q) \\
&+ \tr(N^{-1} X DT + sC^{-1} DT) \\
&= \tr((C^T A N^{-1} X Q X^T + N^{-1} X DT) + \tr(NA C^{-1} s^2 Q + C^{-1} s DT) \\
&+ \tr(sX^T C^T A C^{-1} Q) + \tr(sAXQ).)
\end{aligned}$$

Since $sX^T C^T A C^{-1} Q = stcA C^{-1} Q = sAXQ$ is symmetric and integral, we conclude

$$\begin{aligned}
\tr(A(NC)^{-1} Q + (NC)^{-1} DT) &\equiv \tr((C^T A N^{-1} X Q X^T + N^{-1} X DT) + \tr(NA C^{-1} s^2 Q + C^{-1} s DT) \pmod 1.
\end{aligned}$$

If the modulus is $pI$ for a prime $p$, a symplectic Kloosterman sum simplifies as follows:

**Lemma 6.** Let $p$ be a prime, $Q = \begin{pmatrix} q_1 & q_2/2 \\ q_2/2 & q_4 \end{pmatrix}$ and $T = \begin{pmatrix} t_1 & t_2/2 \\ t_2/2 & t_4 \end{pmatrix}$. Then

$$K(Q,T; p I) = \sum_{d_1,d_2,d_4 \pmod p} e\left( \frac{\delta(d_1 q_1 - d_2 q_2 + d_1 q_4) + t_1 d_1 + t_2 d_2 + d_4 t_4}{p} \right),$$

where $\delta = d_1 d_4 - d_2^2$. 

□
The congruence \((\ref{eqn:congruence})\) equals \((\ref{eqn:congruence3})\), it follows for \(C = NI\) that \(d_2 = d_3\) and the sum runs over all \(d_1, d_2, d_4 \mod N\) that fulfill \(p \nmid d\). Let \(\delta\) be an integer such that \(\delta^2 \equiv 1 \mod N\).

Setting \(A = \delta \begin{pmatrix} d_4 & -d_2 \\ -d_2 & d_1 \end{pmatrix}\), it holds that \(A^T C\) is symmetric and \(B := (A^T D - I_2)N^{-1} \in M_2(\mathbb{Z})\).

To conclude this section, we present a lemma that counts the number of solutions of a congruence that arises in the proof of Theorem 1.

**Lemma 7.** Let \(N \equiv 3 \mod 4\) be a prime and \(h_1, h_2, c_1, c_2, c_4 \in \mathbb{Z}\) such that \(4c_1 c_4 - c_2^2 \equiv 0 \mod N\). Let \(L(c_1, c_2, c_4, h_1, h_2)\) denote the number of solutions \(d_1, d_2, d_4 \mod N\) of

\[
\begin{align*}
(19) & \quad h_1 \equiv a(d_1 + d_4) \quad \pmod{N} \\
(20) & \quad (d_1 d_4 - d_2^2) h_2 \equiv b(d_4 c_1 - d_2 c_2 + d_1 c_4) \quad \pmod{N} \\
(21) & \quad 0 \neq d_1 d_4 - d_2^2 \quad \pmod{N},
\end{align*}
\]

where \(a, b\) are arbitrary integers coprime to \(N\). Then

\[L(c_1, c_2, c_4, h_1, h_2) = \delta_{h_1=h_2=0} (\mod N) \delta_{c_1=c_4, c_2=0} (\mod N) N^2 + O(N).\]

**Proof.** First, we compute the left hand side for \(h_1 \equiv h_2 \equiv 0 \mod N\). It follows by \((\ref{eqn:congruence1})\) that \(d_4 \equiv -d_1 \mod N\) and by \((\ref{eqn:congruence3})\) therefore \(d_1 (c_1 - c_4) + c_2 d_2 \equiv 0 \mod N\).

For \(c_1 \equiv c_4\) and \(c_2 \equiv 0\) \(\mod N\), this congruence holds for arbitrary \(d_1, d_2\). In addition, congruence \((\ref{eqn:congruence3})\) requires that \(d_1^2 \neq d_2^2 \mod N\). Since only \(2N\) pairs \(d_1, d_2\) fulfill \(d_1^2 \equiv d_2^2 \mod N\), there are \(N^2 - 2N\) solutions for \(d_1, d_2, d_4 \mod N\) satisfying all three congruences. On the other hand, for \(c_1 \neq c_4\) or \(c_2 \neq 0\) \(\mod N\), there are less than \(N + 1\) solutions, since choosing \(d_1\) already fixes \(d_2\) and vice versa.

Next, we show that for fixed \(h_1 \neq 0\) or \(h_2 \neq 0\) and arbitrary fixed \(c_1, c_2, c_4\), we have \(L(h_1, h_2, c_1, c_2, c_4) \leq N + 1\). By \((\ref{eqn:congruence1})\) it holds that \(d_1 \equiv \delta h_1 - d_1 \mod N\). Without loss of generalization, we can assume \(a \equiv 1\) and \(b \equiv 1 \mod N\). Then, \((\ref{eqn:congruence3})\) equals

\[h_2 (-d_1^2 - d_2^2) + h_2 h_1 d_1 \equiv d_1 (c_4 - c_1) - d_2 c_2 + c_1 h_1 \mod N.\]

For \(h_2 \equiv 0\), this gives \(0 \equiv (c_4 - c_1) d_1 - c_2 d_2 + c_1 h_1\). Since \(h_1 \neq 0\) and either \(c_4 - c_1, c_1\) or \(c_2\) is \(\neq 0 \mod N\), choosing \(d_1\) fixes \(d_4\) and vice versa. For \(h_2 \neq 0\) we get

\[
(d_1 - \overline{2}(c_4 - c_1 - h_1))^2 + (d_2 - \overline{2} h_2 c_2)^2 \\
= h_1 c_1 + h_2 d_1 + (2(c_4 - c_1 - h_1))^2 + (2h_2 c_2)^2 \mod N.
\]

The congruence \(x^2 + y^2 \equiv n \mod N\) has \(N + 1\) solutions for \(x, y \mod N\) if \(N \nmid n\), and 1 solution if \(N \mid n\), namely \((0, 0)\). \(\square\)

---

\(^3(C, D)\) is primitive, if there exists \(U = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \text{GL}_4(\mathbb{Z})\).
5. Proof of Theorem 1

The proof follows [2] closely. For newforms that are not Saito-Kurokawa lifts, we use the functional approximation (13) obtaining

\[ w_{F,N}L(1/2, F \times \chi_{q_1})L(1/2, F \times \chi_{q_2}) = 4c_k \sum_{n_1, n_2} r_{q_1}(n_1) r_{q_2}(n_2) \frac{\chi_{q_1}(m_1) \chi_{q_2}(m_2)}{(n_1 n_2 m_1 m_2)^{1/2}} \times \frac{n_1 m_1}{N|q_1|^2} W_n \left( n_2 m_2 \right) \frac{a_F(m_1 I) a_F(m_2 I)}{\|F\|^2}. \]

For notational simplicity we write \( r_1, r_2 \) instead of \( r_{q_1}, r_{q_2} \) from now on. Both the right hand side and the left hand side make also sense for oldforms and lifts, but they do not have to coincide in this case. We write \( a_F(n) \) for the coefficients of \( L(F, s) \). Then, by (10) the right hand side of (13) equals

\[ 4w_{F,N} \sum_{n_1, n_2} a_F(n_1) \chi_{q_1}(n_1) a_F(n_2) \chi_{q_2}(n_2) \frac{n_1 m_1}{N|q_1|^2} W_n \left( n_2 m_2 \right) \frac{a_F(m_1 I) a_F(m_2 I)}{\|F\|^2}. \]

For the \( O(1) \) oldforms which are not lifts the left-hand side is \( O(N^{-3}) \) and the right side is \( O(N^{-14/5}) \) since \( \sum_{n \leq x} a_F(n) \ll x^{3/5+\varepsilon} \) for \( F \) not a Saito-Kurokawa lift, cf. [11 Lemma 1].

For the \( O(N) \) lifts, the left hand side is \( O(N^{-\frac{3}{4}+\varepsilon}) \) by (11) and the convexity bound for \( GL_2 \) L-functions. For the right side, we use Mellin inversion and (12), obtaining that

\[ \sum_n a_F(n) \chi_\ell(n) \frac{n}{n_1^{1/2}} W_n(n/N) = \frac{1}{2\pi i} \int_{(2)} L(s, \chi_\ell)(1 - \chi_\ell(N) N^{-s}) L(s + 1, \chi_\ell) \times (1 - \chi_\ell(N) N^{-s-1}) L(s + 1/2, f \times \chi_\ell) \frac{L(\ell, s + 1/2)}{L(\ell, 1/2)} N^s (1 - s^2) \frac{ds}{s}, \]

where \( f \) is the elliptic cusp form in \( S_{2k-2}(N) \) corresponding to \( F \). The \( (1 - s^2) \) term cancels with the pole from the zeta function for \( q = 1 \) and hence we can shift the contour to \( \Re s = \varepsilon \). By applying the convexity bound, the remaining integral contributes \( O(N^{-5/4+\varepsilon}) \).

Altogether, we obtain

\[ \sum_{F \in B_k^{(2)}(\mathbb{N})} w_{F,N}L(1/2, F \times \chi_{q_1})L(1/2, F \times \chi_{q_2}) = 4c_k \sum_{n_1, n_2} r_{q_1}(n_1) r_{q_2}(n_2) \frac{\chi_{q_1}(m_1) \chi_{q_2}(m_2)}{(n_1 n_2 m_1 m_2)^{1/2}} \times \frac{n_1 m_1}{|q_1|^2 N} W_n \left( n_2 m_2 \right) \frac{a_F(m_1 I) a_F(m_2 I)}{\|F\|^2} + O(N^{-5/4+\varepsilon}). \]

By (13), the main term on the right hand side of the previous display equals

\[ 4 \sum_{n_1, n_2} \frac{r_{q_1}(n_1) r_{q_2}(n_2)}{(n_1 n_2 m_1 m_2)^{1/2}} \frac{W_n \left( \frac{n_1 m_1}{|q_1|^2 N} \right) W_n \left( \frac{n_2 m_2}{|q_2|^2 N} \right) \chi_{q_1}(m_1) \chi_{q_2}(m_2)}{n_1 m_1 n_2 m_2} \frac{1}{8} N^{k-3/2} h_{m_2 I}(m_1 I). \]
The Fourier coefficients \( h_{m_1 l}(m_1 I) \) are given by Lemma 4. The diagonal term equals

\[
4 \sum_{n_1, n_2, m} r_1(n_1) r_2(n_2) \frac{\chi_1(m_1) \chi_2(m_2) W(n_1 m_1 \sqrt{t} N) W(n_2 m_2 \sqrt{t} N)}{(n_1 n_2 m)^{1/2}}.
\]

By double Mellin inversion, this equals

\[
\frac{4}{(2\pi i)^2} \int_{(2)} \int_{(2)} L(s + 1, \chi_{q_1}) L(t + 1, \chi_{q_2}) \frac{L(s + 1/2, \chi_{q_1})}{L_x(1/2)} \frac{L(t + 1/2, \chi_{q_2})}{L_x(1/2)} (1 - s)^2 (1 - t)^2 N^s N^t |q_1|^{2s} |q_2|^{2t} \frac{ds \, ds}{st}.
\]

We shift the \( s \)-contour to \( \Re s = -1 + \epsilon \) picking up a pole at \( s = 0 \) of order 1 or 2. For the former, we shift the \( t \)-contour to \( \Re t = -1 + \epsilon \) picking up a pole of order \( t = 0 \) of order at most 3. The remaining integral contributes \( O(N^{-2+\epsilon}) \).

For the latter, we shift the \( t \)-contour \( \Re t = -1 + \epsilon \), picking up a pole at \( t = -s \) (since \( q_1, q_2 = 1 \) that can only happen if \( q_1 = q_2 = 1 \)) and a pole at \( t = 0 \). The remaining integral is \( O(N^{-2+\epsilon}) \); the latter is the main term and the pole at \( t = -s \) equals

\[
-4 \frac{\Gamma(s + 1) \zeta_Q(i)}{2\pi i} \int_{-1+\epsilon} \Gamma(1 - s) \zeta_Q(i)(1 - s) \frac{\Gamma(k - 1 + s) \Gamma(k - 1 - s)}{\Gamma(k - 1)^2} (1 - s^2)^2 \frac{ds}{s^2}.
\]

It is not necessary to simplify this term further, since it will cancel with another term from the rank 2 contribution.

Next, we consider the rank 1 contribution. There are \( s^\epsilon \) choices for \( U, V \) and it must hold \( [m_1, m_2] \mid s \). By (14) and trivial estimation, the rank 1 case is bounded by

\[
N^\epsilon \sum_{m_1, m_2 \in \mathcal{N} \cap \mathbb{Z}^+} \sum_{s, c} (Nc, [m_1 m_2]s)^{1/2} \left| J_t \left( \frac{4\pi m_1 m_2}{|m_1 m_2| Ncs} \right) \right|.
\]

We write \( d = (m_1, m_2) \). By (17), the contribution of \( \frac{d}{cs} \ll N^{1-\epsilon} \) is negligible. Hence, the previous display is bounded by

\[
\ll N^\epsilon \sum_{d, c \in \mathcal{N}^{1+\epsilon}} d^{-3/2 + N^{1/2+\epsilon}} \sum_{(d, N) = 1} d^{-3/2} \ll N^{-1/2+\epsilon}.
\]

It remains to treat the rank 2 contribution

\[
\frac{4\pi^2}{n_1, n_2, m_2} \sum_{n_1, n_2, m_2} r_1(n_1) r_2(n_2) W \left( \frac{n_1 m_1 \sqrt{t} N}{|q_1 N|} \right) W \left( \frac{n_2 m_2 \sqrt{t} N}{|q_2 N|} \right) \chi_1(m_1) \chi_2(m_2)
\]

\[
\times \sum_{\det C \neq 0} \frac{K(m_2 I, m_1 I; N C)}{N^3 |\det C|^{1/2}} \int_{\mathcal{J}_t} \left( \frac{m_1 m_2 C^{-1} C^{-T}}{N^2} \right) d\theta.
\]

By the decay of \( W \) we can truncate the \( n_1, m_1, n_2, m_2 \) sums at \( n_1 m_1, n_2 m_2 \ll N^{1+\epsilon} \) at costs of a negligible error. Recall that

\[
\mathcal{J}_t \left( \frac{m_1 m_2 C^{-1} C^{-T}}{N^2} \right) = \int_0^{\pi/2} \prod_{i=1}^2 J_t \left( \frac{4\pi \sqrt{m_1 m_2 s_i \sin \theta}}{N} \right) \sin \theta \, d\theta
\]
with $s_1^2$ and $s_2^2$ denoting the eigenvalues of $C^{-1}C^{-T}$. If either $s_1$ or $s_2$ is small, say $\ll N^{-\frac{1}{1+\epsilon}}$ for $\gamma \in \mathbb{R}$, then by applying (17) and trivial estimation, we get that (24) is $\ll N^{-\gamma+\epsilon}$. If both $s_1, s_2 \gg N^{-\frac{1}{1+\epsilon}}$, then the eigenvalues of $C^T C$ and consequently $\text{tr}(C^T C)$ are $\ll N\left(\frac{1}{1+\epsilon}\right)^2$. Such $C$ are contained in

$$
\left\{ M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in M_{2 \times 2}(\mathbb{Z}) \mid 0 \neq \det M \ll N^{\frac{1}{1+\epsilon}}, |m_i| \ll N^{\frac{1}{1+\epsilon}} \text{ for } 1 \leq i \leq 4 \right\}
$$

which we denote by $C(\gamma)$. For notational simplicity, we set $c := |\det C|$. To apply Poisson summation, we complete the $n_1, m_1, n_2, m_2$ sum at a negligible error and split $m_1, m_2$ in residue classes modulo $N[c, q_1]$ and $N[c, q_2]$. This way, display (23) equals

$$
4\pi^2 \sum_{n_1, n_2} \frac{r_1(n_1)r_2(n_2)}{(n_1n_2)^{1/2}} \sum_{C \in C(\gamma)} \sum_{\mu_1, \mu_2 \mod N[c, q_1]} x_{q_1}(\mu_1) x_{q_2}(\mu_2) \frac{K(\mu_2 I, \mu_1 I; NC)}{N^{1/2}} \\
\times \sum_{m_1, m_2 \mod N[c, q_1]} W\left(\frac{n_1 m_1}{N}\right) W\left(\frac{n_2 m_2}{N}\right) J_{\ell}\left(\frac{m_1 m_2 C^{-1} C^{-T}}{N^2}\right) + O(N^{-\gamma+\epsilon}).
$$

Poisson summation yields for the main term

$$
4\pi^2 \sum_{n_1, n_2} \frac{r_1(n_1)r_2(n_2)}{(n_1n_2)^{1/2}} \sum_{C \in C(\gamma)} \sum_{\mu_1, \mu_2 \mod N[c, q_1]} x_{q_1}(\mu_1) x_{q_2}(\mu_2) \frac{K(\mu_2 I, \mu_1 I; NC)}{N^{5/2} c^{3/2}} \\
\times \sum_{h_1, h_2 \in \mathbb{Z}} \Psi_{n_1, n_2}(NC; h_1, h_2) e\left(-\frac{\mu_1 h_1}{N[c, q_1]} - \frac{\mu_2 h_2}{N[c, q_2]}\right),
$$

(24)

where $\Psi_{n_1, n_2}(NC; h_1, h_2)$ is

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} W\left(\frac{n_1 x_1}{N}\right) W\left(\frac{n_2 x_2}{N}\right) J_{\ell}\left(\frac{x_1 x_2 C^{-1} C^{-T}}{N^2}\right) e\left(-\frac{x_1 h_1}{N[c, q_1]} + \frac{x_2 h_2}{N[c, q_2]}\right) dx_1 dx_2.
$$

Substituting $x_1$ by $N x_1$ and $x_2$ by $N x_2$, this integral simplifies to

$$
N \int_{\mathbb{R}} \int_{\mathbb{R}} W\left(\frac{n_1 x_1}{N}\right) W\left(\frac{n_2 x_2}{N}\right) J_{\ell}\left(\frac{x_1 x_2 C^{-1} C^{-T}}{N^2}\right) e\left(-\frac{x_1 h_1}{N[c, q_1]} + \frac{x_2 h_2}{N[c, q_2]}\right) dx_1 dx_2,
$$

which we will denote by $N \tilde{\Psi}_{n_1, n_2}(C; h_1, h_2)$. By applying partial summation sufficiently often with respect to $x_1$ and $x_2$ (integrating the last term and differentiating the rest), we can truncate the $h_1, h_2$ sum at $h_1, h_2 \ll N^{\frac{1}{1+\epsilon}}$ at a negligible error.

We choose $s, t \in \mathbb{Z}$ with $sN + tc = 1$ and $(s, q_2) = 1$. Then by Lemma 5 the Kloosterman sum decomposes into

$$
K(\mu_2 I, \mu_1 I, NC) = K(\mu_2 t^2 c^{-1} C^{-T}, \mu_1 I, NI) K(s^2 \mu_2 I, \mu_1 I, C).
$$

The first Kloosterman sum on the right hand side equals by Lemma 8

$$
\sum_{d_1, d_2, d_4 \mod N} e\left(\frac{\mu_2 t^2 d_4 c_1 - d_2 c_2 + d_4 c_4 + \mu_1 (d_1 + d_4)}{N}\right),
$$
where \( cC^{-1}C^{-T} = \begin{pmatrix} c_1 & c_2/2 \\ c_2/2 & c_4 \end{pmatrix} \) with \( c_1, c_2, c_4 \in \mathbb{Z}, \delta = d_1d_4 - d_2^2 \).

Next, we split \( \mu_i \mod N[c, q_i] \) into \( \nu_i \mod N \) and \( \gamma_i \mod [c, q_i] \), i.e. write \( \mu_i = \nu_i[c, q_i] + \gamma_i N \). Then \( \chi_{q_i}(\mu_i) = \chi_{q_i}(N\gamma_i) \). In this way, display (24) equals

\[
4\pi^2 \sum_{n_1, n_2} r_1(n_1) r_2(n_2) \sum_{C \in \mathbb{C}(\gamma)} \frac{1}{|C|^3/2} \sum_{h_1, h_2 \in \mathbb{Z}} \Psi'_{n_1, n_2}(C; h_1, h_2) \\
\times \frac{\chi_{q_1}(N\gamma_1) \chi_{q_2}(N\gamma_2) K(s_1, I, N, \gamma_1 I, C)}{N} \left( -\frac{\gamma_1 h_1}{[c, q_1]} - \frac{\gamma_2 h_2}{[c, q_2]} \right) \\
\times \sum_{d_1, d_2, d_3 \mod N} \sum_{\nu_1 \mod N} e \left( \nu_1([c, q_1](d_1 + d_4) - h_1) \right) \\
\times \sum_{\nu_2 \mod N} e \left( \nu_2([c, q_2] \bar{t}_2 \delta(d_4 c_1 - d_2 c_2 + d_1 c_4) - h_2) \right).
\]

Hence, if \( h_1 \neq [c, q_1](d_1 + d_4) \mod N \) or \( h_2 \delta \neq \bar{t}^2[c, q_2](d_4 c_1 - d_2 c_2 + d_1 c_4) \mod N \) the sum vanishes. The number of solutions to these congruences are counted in Lemma [7]. The case \( h_1 = h_2 = 0 \) and \( c_1 = c_4, \epsilon = 0 \) forms the main term. This latter condition is equivalent to \( C \in \mathbb{G}_2(\mathbb{Z}) \), cf. [2] p. 15], where

\[
\mathbb{G}_2(\mathbb{Z}) = \left\{ \left\{ \frac{x}{y}, \frac{y}{x} \right\} \mid (x, y) \in \mathbb{Z}^2 \neq \{(0, 0)\} \right\}.
\]

Since all entries of \( C \) as well as \( h_1, h_2 \) are each bounded by \( O\left(N^{1+\gamma}\right) \), the last display equals

\[
= 4\pi^2 \sum_{n_1, n_2 \leq N^\gamma} r_1(n_1) r_2(n_2) \sum_{C \in \mathbb{G}_2(\mathbb{Z})} \frac{1}{|C|^3/2} \\
\times \frac{\chi_{q_1}(N\gamma_1) \chi_{q_2}(N\gamma_2) K(s_1, I, N, \gamma_1 I, C)}{N} \\
\times \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{W(n_1 x_1) W(n_2 x_2)}{\sqrt{x_1 x_2}} J_\ell \left( x_1 x_2 C^{-1} C^{-T} \right) dx_1 dx_2 + O\left(N^{-1+\frac{4-6\epsilon}{10}}\right).
\]

We choose \( \gamma \) such that both error terms \( O(N^{-\gamma+\epsilon}) \) and \( O(N^{-1+\frac{4-6\epsilon}{10}}) \) are of the same size, i.e. \( \gamma = \frac{1}{10} \). Applying [2] Lemma 4, we see that the sum vanishes for \( q_1 \neq 1 \) or \( q_2 \neq 1 \). Since matrices \( C \in \mathbb{G}_2(\mathbb{Z}) \) fulfill \( C^{-1} = C^T c^{-1} \), the main term of the previous display equals

\[
8\pi^2 \sum_{n_1, n_2} r_1(n_1) r_2(n_2) \sum_{\gamma \in \mathbb{Z}^{[i]}} \phi(\gamma) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{\pi/2} \frac{W(n_1 x_1) W(n_2 x_2)}{\sqrt{x_1 x_2}} J_\ell \left( \frac{x_1 x_2}{|\gamma|^2} \right) dx_1 dx_2.
\]

By (14) and (17) we can complete the \( \gamma \)-sum at a negligible error. Blomer [2] p. 15] shows that the completed sum equals

\[
\frac{4}{2\pi i} \int_{-1+\epsilon} \zeta_Q(i)(s+1)^i \zeta_Q(i)(1-s) \frac{\Gamma(k-1+s)\Gamma(k-1-s)\Gamma(s+1)\Gamma(1-s)}{\Gamma(k-1)^2} (1-s^2)^2 ds \frac{ds}{s^2}
\]

which cancels with the residue \( s = -t \) from the rank 0 case.
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