A LOCAL-TO-GLOBAL WEAK (1,1) TYPE ARGUMENT AND APPLICATIONS TO FOURIER INTEGRAL OPERATORS

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Abstract. In this work we provide a criterion for the global weak (1,1) type of integral operators which are known to be locally uniformly of weak (1,1) type. As an application, we establish the global weak (1,1) type for a class of Fourier integral operators. While the local result is known from the work of Tao [33] for Fourier integral operators of order $-(n-1)/2$, we give natural sufficient conditions in order to extend it to the corresponding global estimate.

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1. Introduction

1.1. Local-to-global argument. Many of the integral operators arising in harmonic analysis and partial differential equations are operators $K$ with kernels defined via

$$Kf(x) = \int_{\mathbb{R}^n} K(x, y, x - y)f(y)dy, \quad f \in C_0^\infty(\mathbb{R}^n).$$

That is the case of multipliers (convolution operators), pseudo-differential operators and Fourier integral operators. On an Euclidean sub-manifold $Y \subset \mathbb{R}^n$ (just for

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simplicity), Fourier integral operators can be written in the form

$$T u(x) = \int_Y \int_{\mathbb{R}^n} e^{i\phi(x,y,\theta)} a(x, y, \theta) u(y) d\theta dy, \ u \in C^\infty_0(Y).$$  \hspace{1cm} (1.2)

The integral kernel of (1.2) is an oscillatory integral (see for instance, Hörmander [14] and Duistermaat and Hörmander [7]) and the understanding of its singular support allows us to investigate the mapping properties of (1.2). In particular, in this work we are interested in assuring the weak type (1,1) of integral operators of the form (1.1) and its applications to Fourier integral operators.

Due to its microlocal nature, when studying the regularity properties of Fourier integral operators, these are presented in local criteria. This means that the symbol $a(x, y, \theta)$ is assumed to be compactly supported in $(x, y)$, and decay conditions on the symbol $a$ and non-degeneracy conditions on the phase are imposed in order to guarantee that $T$ admits bounded extensions

$$T^{(1)} : H^1_{\text{comp}} \to L^1_{\text{loc}}, \ T^{(2)} : L^p_{\text{comp}} \to L^p_{\text{loc}}, \ T^{(3)} : L^1_{\text{comp}} \to L^{1,\infty}_{\text{loc}},$$  \hspace{1cm} (1.3)

which means that for any compact subset $K \subset \mathbb{R}^n$, the operator $T_K^{(j)} := \chi_K T^{(j)} \chi_K$ admits bounded extensions like

$$T_K^{(1)} : H^1 \to L^1, \ T_K^{(2)} : L^p \to L^p, \ T_K^{(3)} : L^1 \to L^{1,\infty},$$  \hspace{1cm} (1.4)

with $\chi_K$ being the characteristic function of $K$.

It was observed by the second author and Sugimoto in [26] (with early ideas traced back from [28]), that, for example, if we further know the global $L^2$-boundedness of $T$ and that the endpoint local boundedness (from the Hardy space $H^1$ into $L^1$) is uniform with respect to the translation of localised regions, then one can obtain the global $L^p$-boundedness, or the global $H^1 \to L^1$ boundedness of $T$ if its kernel $K(x, y, z)$ is finely controlled on a set away from its singular support.

The main goal of this work is to prove that the same philosophy applies if the endpoint local boundedness (from $L^1$ into $L^{1,\infty}$) is uniform with respect to the translation of localised regions. So, by following the terminology in [26], we call to these type of results local-to-global boundedness argument.

So, to introduce our main theorem, let us define the notion of locally uniformly weak (1,1) type operator.

**Definition 1.1.** We say that $\mathcal{K}$ admits a locally weak (1,1) type extension (which we also denote by $\mathcal{K}$) if for any compact subset $B \subset \mathbb{R}^n$, the localised operator $\chi_B \mathcal{K} \chi_B$ is of weak (1,1) type (that is $\chi_B \mathcal{K} \chi_B : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ admits a bounded extension). Moreover, we say that $\mathcal{K}$ admits a uniformly locally weak (1,1) type extension, if

$$\sup_{h \in \mathbb{R}^n} \| \chi_{B_h} \mathcal{K} \chi_{B_h} \|_{\mathcal{B}(L^1, L^{1,\infty})} < \infty,$$  \hspace{1cm} (1.5)

for the translated set $B_h := h + B$ of any compact subset $B \subset \mathbb{R}^n$.

Now, we present our main result.

**Theorem 1.2.** Let us assume that the integral operator $\mathcal{K}$ defined in (1.1) admits a bounded extension from $L^2(\mathbb{R}^n)$ to itself, and a uniformly locally weak (1,1) type
extension. Let us assume that exists a real-valued measurable function \( (x, y, z) \mapsto H(x, y, z) \) on \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \), for which the following properties are valid,

(A1). There exists \( d > 0 \) and \( k \in \mathbb{R}, k \geq n \), such that
\[
\sup_{H(x,y,z) \geq d} |H(x,y,z)^k K(x,y,z)| < \infty.
\] (1.6)

(A2). Let us define for a.e. \( z \in \mathbb{R}^n \),
\[
\tilde{H}(z) := \inf_{x,y \in \mathbb{R}^n} H(x,y,z).
\]
There exist constants \( A, A_0 > 0 \) such that \( \tilde{H}(z) \geq A_0 |z| \), when \( |z| \geq A \).

(A3). For some \( b, b_0 > 0 \) we have
\[
\tilde{H}(z) \leq b_0 \tilde{H}(z - z') \text{ whenever } \tilde{H}(z) \geq b |z'|.
\]

Then \( K \) admits an extension of weak \((1,1)\) type.

Theorem 1.2 is an analogy of the local-to-global type argument proved by the second autor and Sugimoto in [26]. Indeed, it was proved in [26] that the hypothesis (A1) (with \( k > n \)), (A2), (A3), the existence of a bounded extension of \( K \) in \( L^2(\mathbb{R}^n) \), and the existence of a \( H^1_{\text{comp}}-L^1_{\text{loc}} \)-bounded extension of \( K \), imply the existence of a global \( H^1-L^1 \)-bounded extension of \( K \) (and also of global \( L^p \)-bounded extensions of \( K \), for any \( 1 < p < \infty \)). The present theorem gives a global weak \((1,1)\) type extension of that result.

1.2. Application to Fourier integral operators. It was proved by Tao in [33], that a Fourier integral operator on \( \mathbb{R}^n \), with a non-degenerate phase function and a standard symbol of order \( -(n - 1)/2 \) is locally of weak \((1,1)\) type. Here, we use our main Theorem 1.2 to extend the local result in [33] to a global type boundedness result. We will give the precise statement in Theorem 1.3. Before presenting Theorem 1.3 let us make a short introduction to the local estimates of Fourier integral operators.

1.3. Local theory. Historical comments. Let \( X \) and \( Y \) be paracompact manifolds of dimension \( n \). Let \( T \in I^\mu(X, Y; \Lambda) \) be a Fourier integral operator associated with the canonical relation \( \Lambda \) (see Hörmander [14] and Duistermaat and Hörmander [7]). Because, microlocally we can assume \( X \) and \( Y \) to be open subsets of \( \mathbb{R}^n \), and that \( T \) has the integral form
\[
Tu(x) = \int_Y \int_{\mathbb{R}^n} e^{i\phi(x,y,\theta)} a(x,y,\theta) u(y) d\theta dy, \quad u \in C_0^\infty(Y),
\] (1.7)
the canonical relation \( \Lambda \), that is a conic Lagrangian submanifold of the cotangent bundle \( T^*(X \times Y) \setminus \{0\} \), can be locally parametrised as the set of points
\[
\Lambda_\phi := \{(x,y,d_x\phi,d_y\phi) : d_\phi(x,y,\theta) = 0\}. \tag{1.8}
\]
The symplectic structure of \( \Lambda_\phi \) is determined by the symplectic 2-form \( \omega \) on \( X \times Y \), \( \omega = \sigma_X \oplus -\sigma_Y \), where \( \sigma_X \) and \( \sigma_Y \) are the canonical symplectic forms on \( T^*X \) and
$T^*Y$, respectively. Then $T \in I^\mu(X, Y; \Lambda)$ means that the (amplitude) symbol $a$ verifies estimates of the type

$$
|\partial_{x,y}^\alpha a(x, y, \theta)| \leq C_{\alpha, \beta}(1 + |\theta|)^{\mu - |\alpha|},
$$

locally uniformly in $(x, y)$. We will assume that $\Lambda$ is a local canonical graph, which means that in a suitable microlocal change of coordinates $T$ in (1.7) can be written as

$$
Pu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x\cdot \xi - \varphi(y, \xi))} b(x, y, \xi)u(y)dyd\xi, \quad u \in C_0^\infty(\mathbb{R}^n),
$$

with the phase $\phi$ being non-degenerate, that is

$$
|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C_0 > 0,
$$
on the support of the symbol $b$, which satisfies the estimates

$$
|\partial^\beta_{x,y} \partial^\alpha_\theta b(x, y, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{\mu - |\alpha|}.
$$

In this case we have the parametrisation $\Lambda_\phi = \{(\nabla_\xi \varphi, \xi, y, \nabla_y \varphi)\}$. The mapping properties of Fourier integral operators was always an area of intensive research, and the chronological order for the local results can be summarised as follows.

- $T \in I^\mu(X, Y; \Lambda)$ admits a $L^2_{\text{comp}}$-$L^2_{\text{loc}}$-bounded extension when $\mu \leq 0$ (Hörmander [14], Eskin [10]);
- $T \in I^\mu(X, Y; \Lambda)$ admits a $L^p_{\text{comp}}$-$L^p_{\text{loc}}$-bounded extension when $\mu \leq -(n - 1)[1/p - 1/2]$, $1 < p < \infty$ (Seeger, Sogge and Stein [29]);
- $T \in I^\mu(X, Y; \Lambda)$ admits a $H^1_{\text{comp}}$-$L^1_{\text{loc}}$-bounded extension when $\mu \leq -\frac{n - 1}{2}$ (Seeger, Sogge and Stein [29]);
- $T \in I^\mu(X, Y; \Lambda)$ admits a locally weak $(1, 1)$ type extension when $\mu \leq -\frac{n - 1}{2}$ (Tao [33]).

The local $L^p$-estimate in Seeger, Sogge and Stein [29] is sharp if for example $T$ is an elliptic operator, and $d\pi_{X \times Y}|_\Lambda$ has full rank equal to $2n - 1$ anywhere, where $\pi_{X \times Y} : T^*X \setminus \{0\} \times T^*Y \setminus \{0\} \to X \times Y$ is the canonical projection.

While several generalisations are possible, when working in a variety of problems arising in hyperbolic differential equations and other evolution problems in its basic form, a Fourier integral operator, in terms of the Fourier transform, can be described as

$$
Af(x) = \int_{\mathbb{R}^N} e^{2\pi i \Psi(x, \xi)} a(x, \xi) \hat{f}(\xi) d\xi, \quad f \in C_0^\infty(\mathbb{R}^n),
$$

with $\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx$ the Fourier transform of $f$, where the symbol $a(x, \xi)$ compactly supported in $x$, belonging to the class $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^N)$, and the phase function $\Psi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}))$, is positively homogeneous of degree 1 in $\xi \neq 0$, satisfying the non-degeneracy condition, i.e.

$$
|\partial^\beta_x \partial^\alpha_\xi a(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m - |\alpha|}, \quad \det(\partial_x \partial_\xi \Psi(x, \xi)) \neq 0, \quad (x, \xi) \in \text{supp}(a).
$$

When $Y = \mathbb{R}^n$ in (1.7) (which is not restrictive just changing the support of the amplitude symbol), $A$ in (1.13) is a special case of $T$ in (1.7) that arises when $N = n$, and $\phi(x, y, \theta) = \phi(x, \theta) - y \cdot \theta$, and the symbol $a(x, \theta) = a(x, y, \theta)$ is independent of $y$. 
What is crucial (see e.g. Stein [31, Page 424]) for these operators are the (local) canonical transformations from $T^*({\mathbb R}^n) \setminus \{0\}$ to itself determined by $\Psi$ and $\phi$. For the first, it is the mapping $(\partial_x \Psi, \xi) \mapsto (x, \partial_x \Psi)$, while for the second is $(y, -\partial_y \phi) \mapsto (x, \partial_x \phi)$, defined on the set where $\partial_y \phi \equiv 0$. What is important here, is that any operator of the type (1.7) can be written as finite sums, modulo terms of less order, as finite sums of operators of the kind (1.13), associated to the same canonical transformations. In this reduction, $m = \mu - \frac{N-n}{2}$, where $m$ is the order of (1.13) and $\mu$ is the order of (1.7).

In view of the aspects mentioned before, the local mapping properties for Fourier integral operators can be summarised as follows.

- $A$ admits a $L^2_{\text{comp}} - L^2_{\text{loc}}$-bounded extension when $m \leq 0$, (Hörmander [14], Eskin [10]);
- $A$ admits a $L^p_{\text{comp}} - L^p_{\text{loc}}$-bounded extension when $m \leq -(n-1)|1/p - 1/2|$, $1 < p < \infty$ (Seeger, Sogge and Stein [29]);
- $A$ admits a $H^1_{\text{comp}} - L^1_{\text{loc}}$-bounded extension when $m \leq -\frac{n-1}{2}$ (Seeger, Sogge and Stein [29]);
- $A$ admits a locally weak $(1,1)$ type extension when $m \leq -\frac{n-1}{2}$ (Tao [33]).

1.4. Global theory. Now, we recall some results about the global boundedness of Fourier integral operators.

- Let $\phi(x,y,\xi)$ and $a(x,y,\xi)$ be $C^\infty$-functions, and let
  \[
  D(\phi) := \begin{pmatrix}
  \partial_x \partial_y \phi & \partial_x \partial_\xi \phi \\
  \partial_x \partial_\eta \phi & \partial_x \partial_\xi \phi
  \end{pmatrix}.
  \]
  Assume that $|\det D(\phi)| \geq C > 0$. Also assume that every entry of the matrix $D(\phi)$, $a(x,y,\xi)$ and all their derivatives are bounded. Then operator $T$ defined by (1.7) (with $Y = \mathbb{R}^n$) is $L^2(\mathbb{R}^n)$-bounded (Asada and Fujiwara [2]). It is important to mention that for the operator $P$ defined in (1.10), the conditions above are reduced to a global version of the local graph condition (1.11) and the growth conditions $|\partial_y^\alpha \partial_\xi^\beta \phi(y,\xi)| \leq C_{\alpha\beta} \quad (\forall |\alpha + \beta| \geq 2, |\beta| \geq 1)$,
  \[
  |\partial_x^2 \partial_y^\alpha \partial_\eta^\beta a(x,y,\xi)| \leq C_{\alpha\beta\gamma} \quad (\forall \alpha, \beta, \gamma),
  \]
  for all $x, y, \xi \in \mathbb{R}^n$.
- Other types of growth conditions were introduced by the second author and Sugimoto in [21] to obtain the global $L^2$-boundedness for operators with phase functions and such result was then used to show global smoothing estimates for dispersive equations in a series of papers [22], [24] and [25].
- Improvements to the result of Asada and Fujiwara [2] were done by Kumano-go [16] who also showed the same conclusion under weaker conditions on the phase function, namely for
  \[
  |\partial_y^\alpha \partial_\xi^\beta (\phi(y,\xi) - y \cdot \xi)| \leq C_{\alpha\beta} |\xi|^{1-|\beta|} \quad (\forall \alpha, \beta),
  \]
  with applications to the global $L^2$ estimates for solutions to Cauchy problems of strictly hyperbolic equations.
- As for the global $L^p$-boundedness, the following conclusion was obtained by the second author and Coriasco [4, 5]. Let $\phi(y,\xi)$ and $a(x,y,\xi)$ be $C^\infty$-functions. Assume that $\phi(y,\xi)$ is positively homogeneous of order 1 for large...
ξ and satisfies (1.11). Also assume that
\[
\left| \partial_y^\alpha \partial_\xi^\beta \varphi(y, \xi) \right| \leq C_{\alpha\beta} \langle y \rangle^{1-|\alpha|} \langle \xi \rangle^{1-|\beta|} \quad (\forall \alpha, \beta),
\]
\[
\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi) \right| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1-|\alpha|} \langle y \rangle^{m_2-|\beta|} \langle \xi \rangle^{-(n-1)|1/p-1/2|-|\gamma|} \quad (\forall \alpha, \beta, \gamma),
\]
hold for all \(x, y, \xi \in \mathbb{R}^n\), and that \(a(x, y, \xi)\) vanishes around \(\xi = 0\). Then \(P\) in (1.10) is \(L^p(\mathbb{R}^n)\)-bounded, for every \(1 < p < \infty\), provided that \(m_1 + m_2 \leq -n|1/p-1/2|\) is required for amplitude functions in space variables. It is also shown in [5] that this order of decay is in general sharp: otherwise it is possible to construct an example of an operator that is not globally bounded on \(L^p(\mathbb{R}^n)\). Related global \(L^p\)-boundedness theorems were obtained by Miyachi [18], Beals [3] and Sugimoto [32]. We also refer the reader to [4] and references therein for details.

- Finally, the local-to-global argument of the second author and Sugimoto in [26] allowed to obtain the following global \(L^p\)-estimate (see Theorem 1.2 in [26]). Let \(\varphi(y, \xi)\) and \(a(x, y, \xi)\) be \(C^\infty\)-functions. Assume that \(\varphi(y, \xi)\) is positively homogeneous of order 1 for large \(\xi\) and satisfies (1.11). Also assume that
\[
\left| \partial_y^\alpha \partial^\beta (y \cdot \xi - \varphi(y, \xi)) \right| \leq C_{\alpha\beta} \langle \xi \rangle^{1-|\beta|} \quad (\forall \alpha, |\beta| \geq 1),
\]
hold for all \(x, y, \xi \in \mathbb{R}^n\). Then operator \(P\) defined in (1.10) with the amplitude \(a\) of order \(\mu \leq -(n-1)|1/p-1/2|\) admits a \(L^p(\mathbb{R}^n)\)-bounded extension, for every \(1 < p < \infty\). For \(p = 1\), and the amplitude \(a\) of order \(\mu \leq -(n-1)/2\), \(P : H^1 \to L^1\) admits a bounded extension.

In this paper we prove the global boundedness Theorem 1.3 for Fourier integral operators of the type
\[
Af(x) = \int_{\mathbb{R}^n} e^{2\pi i \Psi(x, \xi)} a(x, \xi) \hat{f}(\xi) d\xi, \quad f \in C^\infty_0(\mathbb{R}^n),
\]
(1.15)
associated to a standard symbol of order \(m\), without the assumption that \(a\) will be compactly supported in the spatial variables \(x\).

**Theorem 1.3.** Let \(A\) be the Fourier integral operator defined in (1.15), with the real-valued phase function \(\Psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})\), positively homogeneous of order 1 in \(\xi\), satisfying the non-degeneracy condition
\[
|\det(\partial_x \partial_\xi \Psi(x, \xi))| \geq C_0 > 0, \quad \xi \neq 0,
\]
(1.16)
and with \(\Psi\) satisfying the decay estimates
\[
|\partial_x^\alpha \partial_\xi^\beta (\Psi(x, \xi) - x \cdot \xi)| \leq C_{\alpha, \beta} |\xi|^{1-|\alpha|}, \quad \xi \neq 0, \quad (\forall \alpha, \beta).
\]
(1.17)
Let \(a(x, \xi)\) be a symbol of order \(m \leq -(n-1)\). Then \(A\) admits a global extension of weak \((1,1)\) type.

**Remark 1.4.** If the Fourier integral operator \(A\) in Theorem 1.3 has order
\[
m \leq -(n-1) \left\lfloor \frac{1}{p} - \frac{1}{2} \right\rfloor,
\]
with $1 < p < \infty$, then from Theorem 1.2 of [26], it follows that $A$ admits a bounded extension on $L^p(\mathbb{R}^n)$, see Remark 3.4 for details. So, the main contribution of Theorem 1.3 is the endpoint case $p = 1$.

Remark 1.5. Now, we mention some other references on the local and global boundedness properties of Fourier integral operators. In [23] and Dos Santos Ferreira and Staubach [8] weighted properties of Fourier integral operators were considered. Also, we refer to Rodríguez-López and Staubach [20] for global mapping properties of rough Fourier integral operators, and to [27] for $L^p$-estimates for Fourier integral operators with complex phase functions. In particular, in [28] one can find an earlier overview of local and global properties of Fourier integral operators with real and complex phase functions. The $L^p$-boundedness of bilinear Fourier integral operators has been also investigated, see e.g. Hong, Lu, Zhang [13] and references therein.

2. THE GLOBAL WEAK (1,1) TYPE OF INTEGRAL OPERATORS

2.1. Proof of the local-to-global argument. For our further analysis in the proof of Theorem 1.2 we require the following remarks.

Remark 2.1. The norm on $L^{1,\infty}(\mathbb{R}^n)$ is invariant under translations. This means, if $\tau_h f := f(\cdot + h)$ (observe that $\tau_h^* f = f(\cdot - h) = \tau_{-h} f$) with $h \in \mathbb{R}^n$, then
\[
\|f(\cdot + h)\|_{L^{1,\infty}} := \sup_{\lambda > 0} \lambda \|\{x : |f(x + h)| > \lambda\}\|
= \sup_{\lambda > 0} \lambda \|\{x - h : |f(x)| > \lambda\}\|
= \sup_{\lambda > 0} \lambda \| - h + \{x : |f(x)| > \lambda\}\|
= \sup_{\lambda > 0} \lambda \|\{x : |f(x)| > \lambda\}\|
= : \|f\|_{L^{1,\infty}}.
\]

Remark 2.2. In view of the equality $\chi_{B_r} = \tau_h \chi_B \tau_h^*$, with $\{\tau_h : h \in \mathbb{R}^n\}$ the standard translation group on $\mathbb{R}^n$, and since the norms in $L^1$ and $L^{1,\infty}$ are translation invariant, $\mathcal{K}$ is uniformly locally weak (1,1) type if and only if $\chi_B(\tau_h^* \mathcal{K} \tau_h) \chi_B$ is of weak (1,1) type for any compact set $B \subset \mathbb{R}^n$ and the $(L^1, L^{1,\infty})$-operator norms are bounded in $h \in \mathbb{R}^n$.

Remark 2.3. For any $h \in \mathbb{R}^n$, let us define
\[
H_h(x, y, z) := H(x - h, y - h, z), \quad (x, y, z) \in \mathbb{R}^{2n}.
\]

We remark that conditions (A1), (A2) and (A3) in Theorem 1.2 are invariant in $h \in \mathbb{R}^n$ in the sense that we have
\[
\sup_{H_h(x, y, z) \geq d} |H_h(x, y, z)^k K_h(x, y, z)| = \sup_{H(x, y, z) \geq d} |H(x, y, z)^k K(x, y, z)|,
\]
\[
\widehat{H}_h(z) = \inf_{x, y \in \mathbb{R}^n} H_h(x, y, z) = \widehat{H}(z).
\]

Remark 2.4. Note that the operator $\tau_h^* \mathcal{K} \tau_h$ is given by
\[
\tau_h^* \mathcal{K} \tau_h u(x) = \int_{\mathbb{R}^n} K_h(x, y, x - y) u(y) dy, \quad K_h(x, y, z) = K(x + h, y + h, z). \quad (2.1)
\]
Proof of Theorem 1.2. Let us introduce the following sets
• \( \Delta_r := \{(x, y, z) : H(x, y, z) \geq r\} \), \( \Delta_{br} := \{(x, y, z) : \tilde{H}(z) \geq r\} \).
The following is Lemma 2.2 of [26, Page 897].

Lemma 2.5. Let \( r \geq 1 \). Then there exists a constant \( c > 0 \) independent of \( r \) such that \( \mathbb{R}^n \setminus \Delta_{br} \subset \{z : |z| < cr\} \). Moreover, for all \( r > 0 \),
\[
(\mathbb{R}^n \setminus \Delta_{br}) \cap \{z : |z| \geq A\} \subset \{z : |z| < (2b/A_0) r\}.
\]
We divide the proof in two steps:
• Step 1. First, we will prove that for \( f \) with \( \text{supp}(f) \subset \{x \in \mathbb{R}^n : |x| \leq r\} \), and \( r \geq 1 \), there exists \( C > 0 \), independent of \( h \) and \( r \) such that
\[
\|\tau_h^*K\eta f\|_{L^1,\infty(\Delta_{br})} \leq C\|f\|_{L^1(\mathbb{R}^n)}.
\] (2.2)

• Step 2. We will apply the atomic decomposition for functions in \( L^1 \) (as in Abu-Shammala and Torchinsky [1]). Indeed, decomposing \( f \) (essentially) in atoms \( a_j \) we will prove the uniform estimate \( \|Ka_j\|_{L^1,\infty} \leq C \), and consequently, we will obtain the global weak \((1,1)\) type of \( \mathcal{K} \). In order to prove the uniform estimate \( \|Ka_j\|_{L^1,\infty} \leq C \), on atoms we will combine Step 1 and Lemma 2.5. Also, in order to use the \( L^2 \)-boundedness of \( \mathcal{K} \) in the proof of the estimate \( \|Ka_j\|_{L^1,\infty} \leq C \), we will make a sub-atomic decomposition of any atom \( a_j \) using Lemma 5.1 in Keel and Tao [15].

Now, we continue with this program.
• Step 1: Global estimate in the non-degenerate zone \( \Delta_{br} \) for \( r \geq 1 \). Let \( h \in \mathbb{R}^n \). Suppose that \( \text{supp}(f) \subset \{x \in \mathbb{R}^n : |x| \leq r\} \). Then we can claim that there exists \( C > 0 \), independent of \( h \) and \( r \) such that
\[
\|\tau_h^*K\eta f\|_{L^1,\infty(\Delta_{br})} \leq C\|f\|_{L^1(\mathbb{R}^n)}.
\] (2.3)
First we consider the case \( h = 0 \). For \( x \in \Delta_{br} \) and \( |y| \leq r \), we have \( \tilde{H}(x) \geq br \) by the definition of \( \Delta_{br} \), and hence we also have \( \tilde{H}(x) \geq b|y| \). Then from (A3) with \( b = b_0 \), we obtain
\[
br \leq H(x) \leq b\tilde{H}(x - y) \leq bH(x, y, x - y)
\]
which implies \( (x, y, x - y) \in \Delta_r \) and \( H(x, y, x - y)^{-1} \leq b\tilde{H}(x)^{-1} \). Then we have
\[
|\mathcal{K}f(x)| \leq b^k \tilde{H}(x)^{-k} \int_{|y| \leq r} |H(x, y, x - y)^k K(x, y, x - y)f(y)| dy
\]
\[
\leq b^k \tilde{H}(x)^{-k}\|H(x, y, z)^k K(x, y, z)\|_{L^\infty(\Delta_r)}\|f\|_{L^1},
\]
for \( x \in \Delta_{br} \). Consequently,
\[
\|\mathcal{K}f\|_{L^1,\infty(\Delta_{br})} := \sup_{\lambda > 0} \lambda \{|x \in \Delta_{br} : |\mathcal{K}f(x)| > \lambda|\}
\]
\[
\leq \sup_{\lambda > 0} \lambda \{|x \in \Delta_{br} : b^k \tilde{H}(x)^{-k}\|H(x, y, z)^k K(x, y, z)\|_{L^\infty(\Delta_r)}\|f\|_{L^1} > \lambda|\}.
\]
\[ \leq \sup_{\lambda > 0} \lambda\{ x \in \tilde{A}_{br} : \tilde{H}(x)^{-k} > \frac{\lambda}{b^k \| H(x, y, z)^k K(x, y, z)\|_{L^\infty(\Delta_r)} \| f \|_{L^1}} \} \]
\[ \lesssim \| \tilde{H}^{-k} \|_{L^\infty(\tilde{A}_{br})} \| H(x, y, z)^k K(x, y, z)\|_{L^\infty(\Delta_r)} \| f \|_{L^1}. \]

Hence, by the monotonicity $\Delta_{br} \subset \Delta_b$ and $\tilde{\Delta}_r \subset \tilde{\Delta}_1 (r \geq 1)$, we have
\[ \| Kf \|_{L^\infty(\tilde{A}_{br})} \leq b^k \| \tilde{H}(x)^{-k} \|_{L^\infty(\tilde{A}_{br})} \| H(x, y, z)^k K(x, y, z)\|_{L^\infty(\Delta_r)} \| f \|_{L^1} \]
\[ \leq b^k \| \tilde{H}(x)^{-k} \|_{L^\infty(\tilde{A}_b)} \| H(x, y, z)^k K(x, y, z)\|_{L^\infty(\Delta_1)} \| f \|_{L^1} \]
\[ \leq C \| f \|_{L^1}, \]

where we have assumed that $\| \tilde{H}(x)^{-k} \|_{L^\infty(\tilde{A}_b)} < \infty$ and we have used (A1) with $d = 1$. To prove this inequality let us consider the cases $k \geq n$ and $k < n$, separately. If (A1) holds with $k = n$ we can estimate
\[ \| \tilde{H}(z)^{-n} \|_{L^\infty(\tilde{A}_b)} \leq \| \tilde{H}(z)^{-n} \|_{L^\infty(\tilde{A}_b \cap \{|z| \leq A\})} + \| \tilde{H}(z)^{-n} \|_{L^\infty(\tilde{A}_b \cap \{|z| > A\})} \]
\[ \leq b^{-n} \| 1 \|_{L^1(|z| \leq A)} + A_0^{-n} \| |z|^{-n} \|_{L^\infty(|z| \geq A)} \]
\[ \leq C. \]

Indeed, observe that
\[ \| |z|^{-n} \|_{L^\infty(|z| \geq A)} \leq \sup_{\lambda > 0} \lambda \{ z : |z|^{-n} > \lambda \} = \sup_{\lambda > 0} \lambda \{ z : |z| \leq \lambda^{-\frac{1}{n}} \} \lesssim 1. \]

Observe that if (A2) holds with $k > n$, we can estimate
\[ \| \tilde{H}(z)^{-k} \|_{L^\infty(\tilde{A}_b)} \leq \| \tilde{H}(z)^{-k} \|_{L^\infty(\tilde{A}_b \cap \{|z| \leq A\})} + \| \tilde{H}(z)^{-k} \|_{L^\infty(\tilde{A}_b \cap \{|z| > A\})} \]
\[ \leq b^{-k} \| 1 \|_{L^1(|z| \leq A)} + A_0^{-k} \| |z|^{-k} \|_{L^1(|z| \geq A)} \]
\[ \leq C. \]

For general $h \in \mathbb{R}^n$, we apply the same argument for $K_h$ in (2.1) (see Remark 2.3) and
\[ H_h(x, y, z) = H(x - h, y - h, z) \]

instead of $K$ and $H$, respectively.

- Step 2. We will apply the atomic decomposition for functions in $L^1$ (see Abu-Shammala and Torchinsky [1]). Indeed, $f \in L^1(\mathbb{R}^n)$ if and only if $f$ can be written as
\[ f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x) + \lambda \chi J(x), \text{ a.e. } x \in \mathbb{R}^n, \quad (2.4) \]
where $J = [0,1]^n$, $\sum_j |\lambda_j| < \infty$, $\lambda := \int f(g)dg$, and the $a_j$’s are $q_j$-atoms with $1 < q_j \leq 2$, which means that

1. $a_j$ is supported in an interval $I_j$ of $\mathbb{R}^n$.
2. $a_j$ satisfies the cancellation property:

$$\int_{I_j} a_j(g)dg = 0.$$

3. We have

$$|I_j| \left( \frac{1}{|I_j|} \int_{I_j} |a_j(g)|^{q_j}dg \right)^{\frac{1}{q_j}} \leq 1.$$

Moreover

$$\|f\|_{L^1} \sim |\lambda| + \inf \sum_j |\lambda_j|. \tag{2.5}$$

So, we can estimate for $f \in L^1$,

$$\|Kf\|_{L^{1,\infty}} \leq \sum_j |\lambda_j| \|Ka_j\|_{L^{1,\infty}} + |\lambda| \|K\chi_{J_0}\|_{L^{1,\infty}}. \tag{2.6}$$

Note that the translation-invariance of the $L^{1,\infty}$-norm implies

$$\|K\chi_{J_0}\|_{L^{1,\infty}} = \|K\tau_h^*\chi_{J_0}\|_{L^{1,\infty}} = \|\tau_hK\tau_h^*\chi_{J_0}\|_{L^{1,\infty}} \leq C.$$ 

Indeed,

$$\|\tau_hK\tau_h^*\chi_{J_0}\|_{L^{1,\infty}} \leq \sup_{h' \in \mathbb{R}^n} \|\tau_{h'}K\tau_{h'}^*\chi_{J_0}\|_{L^{1,\infty}} \leq \sup_{h' \in \mathbb{R}^n} \|\tau_{h'}K\tau_{h'}^*\|_{L^{1,1}} \|\chi_{J_0}\|_{L^1}$$

$$\leq C \sup_{h' \in \mathbb{R}^n} \|\chi_{J_0}\|_{L^1} = C,$$

with $C = \sup_{h' \in \mathbb{R}^n} \|\tau_{h'}K\tau_{h'}^*\|_{L^{1,1}} < \infty$. Now, we are going to prove the estimate

$$\|Ka_j\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C,$$

with some constant $C > 0$ for all atoms $a_j$. By doing an appropriate translation, it is further reduced to the estimate (here $h = h(j)$)

$$\|\tau_h^*K\tau_hg\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C, \quad g \in \mathcal{A}_r,$$

where $\mathcal{A}_r$ is the set of all functions $g$ on $\mathbb{R}^n$ such that

$$\text{supp } g \subset B_r = \{x \in \mathbb{R}^n : |x| \leq r\}, \quad \int_{B_r} g(x)dx = 0,$$

$$|B_r| \left( \frac{1}{|B_r|} \int_{B_r} |g(x)|^{q_j}dx \right)^{\frac{1}{q_j}} \leq 1. \tag{2.6}$$

Suppose that $g \in \mathcal{A}_r$ with $r \geq 1$. Then we split $\mathbb{R}^n$ into two parts $\tilde{\Delta}_{br}$ and $\mathbb{R}^n \setminus \tilde{\Delta}_{br}$. For the part $\tilde{\Delta}_{br}$, we have by Step 1 that

$$\|\tau_h^*K\tau_hg\|_{L^{1,\infty}(\tilde{\Delta}_{br})} \leq C\|g\|_{L^1}.$$
In view of (2.6) we can estimate the $L^1$-norm of $g$ as follows,

$$\|g\|_{L^1} \leq \left( \int_{B_r} |g(x)|^{q_j} \, dx \right)^{\frac{1}{q_j}} |B_r|^{\frac{1}{q_j}} = \left( \int_{B_r} |g(x)|^{q_j} \, dx \right)^{\frac{1}{q_j}} |B_r| \frac{q_j - 1}{q_j} \leq |B_r| \frac{q_j - 1}{q_j} \leq 1.$$ 

So, we have the uniform estimate

$$\|\tau_h^* \mathbf{K} \tau_t g\|_{L^1, \infty(\Delta_{br})} \leq C.$$ 

For the degenerate part $\mathbb{R}^n \setminus \bar{\Delta}_{br}$, again let us apply a sub-atomic decomposition to any atom $g$.

Indeed, by following Lemma 5.1 of Keel and Tao [15], $g \in L^1(\mathbb{R}^n)$ can be written as

$$g(x) = \sum_{j \in \mathbb{Z}} \tilde{\lambda}_j \chi_j(x), \ a.e. \ x,$$ 

(2.7)

such that $\sum_j |\tilde{\lambda}_j| \leq \|g\|_{L^1}$, where the $\chi_j$’s are bounded functions with

$$\|\chi_j\|_{L^\infty} \leq 2^{-j}$$ 

(2.8)

and supported in measurable sets $I_j$ with measure $|I_j| \leq 2^j$. Observe that $g$ is supported in $B_r$ and then $g(x) = \sum_j \tilde{\lambda}_j \chi_j(x) 1_{B_r}$, so that any function $\chi_j$ can be considered also supported in $I_j \subset B_r$.

So, we can estimate

$$\|\mathbf{K} g\|_{L^1, \infty(\mathbb{R}^n \setminus \Delta_{br})} \leq \sum_j |\tilde{\lambda}_j| \|\mathbf{K} \chi_j\|_{L^1, \infty(\mathbb{R}^n \setminus \Delta_{br})} \leq \sup_{j \geq 0} \|\mathbf{K} \chi_j\|_{L^1, \infty(\mathbb{R}^n \setminus \Delta_{br})} \cdot \|g\|_{L^1}.$$ 

We claim that for any $j$ we have a uniform estimate

$$\|\mathbf{K} \chi_j\|_{L^1, \infty(\mathbb{R}^n \setminus \Delta_{br})} \lesssim 1.$$ 

Let $j \geq 0$. Define $r_j := 2^{j+1}$, so that $|I_j| \leq 2^j = r_j^n$. Observe that $r_j \geq 1$. Let us decompose $\mathbb{R}^n \setminus \bar{\Delta}_{br}$ into two pieces:

$$(\mathbb{R}^n \setminus \bar{\Delta}_{br}) \cap \bar{\Delta}_{br'} \text{ and } (\mathbb{R}^n \setminus \bar{\Delta}_{br}) \cap (\mathbb{R}^n \setminus \bar{\Delta}_{br'}).$$

Because any $\chi_j$ has support inside of the support of $g$, that is $I_j \subset B_r$, from Step 1 we have

$$\|\mathbf{K} \chi_j\|_{L^1, \infty((\mathbb{R}^n \setminus \Delta_{br}) \cap \Delta_{br'})} \leq \|\mathbf{K} \chi_j\|_{L^1, \infty(\Delta_{br'})} \leq C\|\chi_j\|_{L^1} \leq C\|\chi_j\|_{L^\infty} \leq C.$$ 

On the other hand, using the inequality

$$\|\mathbf{K} \chi_j\|_{L^1, \infty((\mathbb{R}^n \setminus \Delta_{br}) \cap (\mathbb{R}^n \setminus \Delta_{br'}))} \leq \|\mathbf{K} \chi_j\|_{L^1, \infty(\mathbb{R}^n \setminus \Delta_{br'})}$$

by Lemma 2.5 and the Schwarz inequality, we have

$$\|\mathbf{K} \chi_j\|_{L^1, \infty(\Delta_{br'})} \leq \|\mathbf{K} \chi_j\|_{L^1((\mathbb{R}^n \setminus \Delta_{br'}))} \leq \|1\|_{L^2(|x| < br')} \|\mathbf{K} \chi_j\|_{L^2(\mathbb{R}^n)}.$$
where we have used the assumption that \( K \) is \( L^2 \)-bounded and (2.8) in the last inequality. Indeed, note that from (2.8) we have

\[
r_j^{n/2} \|\mathcal{K} x_j\|_{L^2(\mathbb{R}^n)} \leq r_j^{n/2} \|x_j\|_{L^\infty(\mathbb{R}^n)} |\bar{I}_j|^\frac{1}{2} \leq 2^{j} \times 2^{-j} \times 2^{\frac{j}{2}} = 1.
\]

For \( j < 0 \), let us define \( r_j := 2^\frac{j}{2} \). Note that \( |\bar{I}_j| = r_j^n \leq 1 \). We split \( \mathbb{R}^n \) into the parts \( I = \{ |z| \geq A \} \) and \( \{ z : |z| \leq A \} \). For the bounded part \( |z| \leq A \) we have the estimate

\[
\|\mathcal{K} x_j\|_{L^1(\{ |z| \leq A \})} \leq \sup_{|\bar{I}_j| \leq 1} \left\| \int_{\bar{I}_j} \mathcal{K} \chi x_j \right\|_{L^1} \lesssim \sup_j \|x_j\|_{L^1} \leq 1.
\]

Now, we will consider the zone where \( |z| \geq A \). Using Lemma 2.5 we have

\[
\|\mathcal{K} x_j\|_{L^1(\{ |z| \geq A \}) \cap (\mathbb{R}^n \setminus \bar{\Delta}_{b/\rho_j})} \leq \|\chi_{\bar{I}_j} \|_{L^1(\{ |z| \leq (2b/A_0)\rho_j \})} \|\mathcal{K} x_j\|_{L^2} \leq C r_j^{n/2} \|x_j\|_{L^2(\mathbb{R}^n)} \leq C.
\]

On the other hand, observe that supp \( \chi_j \subset \bar{I}_j \subset B_r \), from which we deduce that the sum over \( j \) in the decomposition of \( g \) into the \( \chi_j \)'s runs over \( r_j \lesssim r \), and consequently \( \bar{\Delta}_{b/\rho_j} \subset \bar{\Delta}_{\rho_r} \) for some \( c_0 > 0 \). We deduce that

\[
\|\mathcal{K} x_j\|_{L^1(\{ |z| \geq A \}) \cap \bar{\Delta}_{b/\rho_j}} \leq \|\mathcal{K} x_j\|_{L^1(\bar{\Delta}_{\rho_r})} \leq \|x_j\|_{L^1} \leq 1,
\]

in view of Step 1. Now, in order to finish the proof, suppose that we have an atom \( g \in A_r \) with \( r \leq 1 \). Then we split the whole space \( \mathbb{R}^n \) into the parts \( \Delta_b \) and \( \mathbb{R}^n \setminus \Delta_b \). For the part \( \Delta_b \), we have by Step 1 with \( r = 1 \) and the inclusion supp \( f \subset B_r \subset B_1 \), that

\[
\|\tau_h^* \mathcal{K} \tau_h g\|_{L^1(\Delta_b)} \leq C \|g\|_{L^1} \leq C.
\]

For the degenerate part \( \mathbb{R}^n \setminus \Delta_b \), we have by Lemma 2.5 that

\[
\|\tau_h^* \mathcal{K} \tau_h g\|_{L^1(\mathbb{R}^n \setminus \Delta_b)} \leq \|\tau_h^* \mathcal{K} \tau_h g\|_{L^1(\{ |x| < c \})} \leq C \|g\|_{L^1} \leq C,
\]

where we have used the fact that \( \mathcal{K} \) is uniformly locally weak \((1,1)\) type. Thus the proof of Theorem 1.2 is complete. \( \square \)

2.2. Application to pseudo-differential operators. A first, but very important example of integral operators where the local-to-global argument in Theorem 1.2 can be applied is the case of pseudo-differential operators. We analyse it as follows.

Corollary 2.6. Let us consider an amplitude pseudo-differential operator

\[
Qu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i (x-y) \xi} a(x, y, \xi) d\xi u(y) dy, \quad f \in C_0^\infty(\mathbb{R}^n).
\]  

(2.9)

Let \( a(x, y, \xi) \) be a symbol satisfying the estimates

\[
|\partial^\alpha_{x,y} \partial^\beta_{\xi} a(x, y, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{\mu - |\alpha|},
\]

(2.10)
for some $\mu \in \mathbb{R}$. Assume that $A$ admits a bounded extension from $L^2(\mathbb{R}^n)$ to itself, and a uniformly locally weak $(1,1)$ type extension. Then $Q$ admits a global extension of weak $(1,1)$ type.

**Proof.** Observe that $Q$ has the Schwartz kernel
\[
K(x, y, x - y) = \int_{\mathbb{R}^n} e^{2\pi i (x - y) \cdot \xi} a(x, y, \xi) d\xi. \tag{2.11}
\]
It was proved in Lemma 3.5 of [26] that the function $H(x, y, z) := |z|$, satisfies the hypothesis (A1), (A2) and (A3) in Theorem 1.2, if the symbol $a(x, y, \xi)$ satisfies (2.10). So, we deduce the existence of a weak $(1,1)$ type extension of $Q$. $\square$

3. **Proof of Theorem 1.3**

In this section we apply the local-to-global argument in Theorem 1.2 to study the weak $(1,1)$ type of Fourier integral operators of the form
\[
Af(x) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot \xi} a(x, \xi) \hat{f}(\xi) d\xi, \quad f \in C_0^\infty(\mathbb{R}^n). \tag{3.1}
\]
We will prove in Theorem 3.1 an equivalent statement of Theorem 1.3 for Fourier integral operators of the type
\[
Af(x) = \int_{\mathbb{R}^n} e^{2\pi i (x \cdot \xi + \Psi(x, \xi))} a(x, \xi) \hat{f}(\xi) d\xi, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{3.2}
\]
associated to a standard symbol of order $m$, without the assumption that $a$ will be compactly supported in the spatial variables $x$.

**Theorem 3.1.** Let $A$ be the Fourier integral operator defined in (3.2), with the real-valued phase function $x \cdot \xi + \Psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, positively homogeneous of order 1 in $\xi$, satisfying the non-degeneracy condition
\[
|\det(\partial_x \partial_\xi (x \cdot \xi + \Psi(x, \xi)))| \geq C_0 > 0, \quad \xi \neq 0, \tag{3.3}
\]
and with $\Psi$ satisfying the decay estimates
\[
|\partial_\xi^\beta \partial_\xi^\alpha \Psi(x, \xi)| \leq C_{\alpha, \beta}|\xi|^{1-|\alpha|}, \quad \xi \neq 0, \quad (\forall \alpha, \beta). \tag{3.4}
\]
Let $a(x, \xi)$ be a symbol of order $m \leq -\frac{n-1}{2}$. Then $A$ admits a global extension of weak $(1,1)$ type.

Before presenting the proof of Theorem 3.1 let us note that (3.1) is a particular case of a Fourier integral operator of the general type
\[
Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i \phi(x, y, \theta)} a(x, y, \theta)u(y)dyd\theta, \quad u \in C_0^\infty(Y), \tag{3.5}
\]
with phase function
\[
\phi(x, y, \theta) := (x - y)\theta + \Phi(x, y, z), \tag{3.6}
\]
whose kernel is given by
\[ K(x, y, x - y) = \int_{\mathbb{R}^n} e^{i((x-y) \cdot \xi + \Phi(x,y,z))} a(x, y, \xi) d\xi. \tag{3.7} \]

The following is Lemma 3.1 of [26].

**Lemma 3.2.** Assume that \( a = a(x, y, \xi) \) is a smooth function satisfying estimates of the type
\[ |\partial_\beta^\alpha x y \partial^\alpha \xi a(x, y, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|}, \tag{3.8} \]
and assume also that \( \Phi(x, y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}) \) is a real-valued function and that
\[ |\partial_\beta^\alpha x y \partial^\alpha \xi \partial^\gamma \xi \Phi(x, y, \xi)| \leq C_{\alpha, \beta, \gamma} |\xi|^{-|\alpha|}, \xi \neq 0, \tag{3.9} \]
for all \( \gamma \in \mathbb{N}^n_0 \) with \( |\gamma| = 1 \). Let
\[ H(x, y, z) := \inf_{\xi \in \mathbb{R}^n} |z + \nabla_\xi \Phi(x, y, \xi)|, \]
and let
\[ \tilde{H}(z) := \inf_{x,y \in \mathbb{R}^n} H(x, y, z) = \inf_{x,y,\xi \in \mathbb{R}^n} |z + \nabla_\xi \Phi(x, y, \xi)|. \]

Then \( K(x, y, z) \) defined by (3.7) satisfies assumptions (A1)–(A3) in Theorem 1.2.

**Proof of Theorem 3.1.** With the notation in Lemma 3.2, observe that \( \Psi(x, \xi) = \Phi(x, y, \xi) \). Because \( \Psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}) \) satisfies estimates of the kind
\[ |\partial_\beta^\alpha \xi a(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{1 - |\alpha|}, \xi \neq 0, \tag{3.9} \]
and consequently the inequalities
\[ |\partial_\beta^\alpha \xi \partial^\beta \xi a(x, \xi)| \leq C_{\alpha, \beta, \gamma} |\xi|^{-|\alpha|}, \xi \neq 0, \tag{3.9} \]
for \( |\gamma| = 1 \), the kernel defined via
\[ K(x, y, x - y) = \int_{\mathbb{R}^n} e^{i((x-y) \cdot \xi + \Phi(x,y,z))} a(x, \xi) d\xi, \tag{3.10} \]
verifies the hypothesis in Lemma 3.2, and so the assumptions (A1)–(A3) in Theorem 1.2. Now, if we follow the argument in Tao [33], we can say that \( \chi_K A \chi_K \) is of weak type \((1,1)\) for any compact set \( K \subset \mathbb{R}^n \) and its operator norm is bounded by a constant depending only on \( n, K \) and quantities
\[ M_\ell = \sum_{|\alpha| + |\beta| + |\gamma| \leq \ell} \sup_{x, \xi \in \mathbb{R}^n} |\partial^\alpha \xi \partial^\beta \xi a(x, \xi)(\xi)^{(n-1)/2 + |\gamma|}|, \]
\[ N_\ell = \sum_{|\beta| \leq \ell} \sup_{x, y, \xi \in \mathbb{R}^n} |\partial^\beta \xi \partial^\gamma \xi (x \cdot \xi + \Psi(x, \xi))(\xi)^{-(1-|\gamma|)}|, \]
with some large \( \ell \). This analysis shows that \( A \) is locally uniformly weak type \((1,1)\). In order to finish the proof, we need to show that \( A \) is bounded on \( L^2 \). By the duality
argument it is equivalent to proving that its adjoint $A^*$ is $L^2$-bounded. Observe that the kernel of $A^*$ is given by

$$K(y, x, y - x) = \int_{\mathbb{R}^n} e^{i2\pi((x-y)\cdot\xi - \varphi(y, \xi))} \overline{a(y, \xi)} d\xi = \int_{\mathbb{R}^n} e^{i2\pi((x-x)\cdot\xi - \varphi(y, \xi))} \overline{a(y, \xi)} d\xi.$$ 

So, $A^*$ is a Fourier integral operator of the form (1.10), with a symbol of order zero. Let us deduce the $L^2$-boundedness of $A^*$ from the following result (see Theorem 1.2 in [26]).

**Proposition 3.3.** Let $P$ be a Fourier integral operator as in (1.10). Let $\varphi(y, \xi)$ and $a(x, y, \xi)$ be $C^\infty$-functions. Assume that $\varphi(y, \xi)$ is positively homogeneous of order 1 for large $\xi$ and satisfies (1.11). Also assume that

$$\left| \partial_y^\alpha \partial_\xi^\beta (y \cdot \xi - \varphi(y, \xi)) \right| \leq C_{\alpha \beta} (\xi)^{1-|\beta|} \quad (\forall \alpha, |\beta| \geq 1),$$

$$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi) \right| \leq C_{\alpha \beta \gamma} (\xi)^{-(n-1)/p - 1/2 - |\gamma|} \quad (\forall \alpha, \beta, \gamma),$$

hold for all $x, y, \xi \in \mathbb{R}^n$. Then operator $P$ admits a $L^2(\mathbb{R}^n)$-bounded extension.

Now, observe that $A^*$ is a Fourier integral operator of the form (1.10) with $\varphi(y, \xi) = y \cdot \xi + \Psi(y, \xi)$ and symbol independent of $y$. Applying Proposition 3.3 with $p = 2$, and observing that

$$\left| \partial_y^\alpha \partial_\xi^\beta (y \cdot \xi - \varphi(y, \xi)) \right| = \left| \partial_y^\alpha \partial_\xi^\beta (\Psi(y, \xi)) \right| \leq C_{\alpha \beta} (\xi)^{1-|\beta|} \quad (\forall \alpha, |\beta| \geq 1),$$

for $|\xi|$ large enough, and that $\varphi$ satisfies (1.11), that is

$$| \det \partial_y \partial_\xi \varphi(y, \xi) | = | \det \partial_y \partial_\xi (y \cdot \xi + \Psi(y, \xi)) | \geq C_0 > 0,$$

(3.11) in view of (3.3), we conclude the $L^2$-boundedness of $A^*$. The proof of Theorem 3.1 is complete. \hfill \Box

**Remark 3.4.** Let $A$ be the Fourier integral operator defined in (3.2), with the real-valued phase function $x \cdot \xi + \Psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, positively homogeneous of order 1 at $\xi \neq 0$, satisfying the non-degeneracy condition

$$| \det(\partial_x \partial_\xi (x \cdot \xi + \Psi(x, \xi))) | \geq C_0 > 0,$$

(3.12) and with $\Psi$ satisfying the decay estimates

$$| \partial_\xi \partial_\xi \Psi(x, \xi) | \leq C_{\alpha \beta} (\xi)^{1-|\alpha|}, \, \xi \neq 0, \, (\forall \alpha, \beta).$$

(3.13)

Let $a(x, \xi)$ be a symbol of order $m \leq -(n - 1)/p < \frac{1}{2}$. In view of Proposition 3.3 the adjoint $A^*$ is bounded on $L^p$, for all $1 < p < \infty$, and by the duality argument, $A$ is also bounded on $L^p$ for all $1 < p < \infty$.

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