Solving the contact problem when strengthening the slab with a beam using discontinuous functions

E A Kobelev, N K Lukashevic
Department of structural mechanics, Saint Petersburg State University of Architecture and Civil Engineering, 4, 2-aya Krasnoarmeiskaya St., St. Petersburg 190005, Russia

E-mail: evgeny.kobelev@gmail.com

Abstract. Problems of contact interaction of structures and their parts have a wide range of applications in construction. In particular, this problem arises during the reconstruction of existing buildings, when strengthen of floor slabs is required to increase their bearing capacity. In this case, the contact of the slab with the strengthening beams occurs along a continuous line of a given shape with a contact zone unknown in advance. In this paper, with the aim of constructing high-accuracy numerical-analytical solutions for the contact problem while keeping some series members, the mathematical apparatus of the theory of generalized functions is widely used. To solve the structurally nonlinear problem of strengthening the floor slab with an elastic beam, a step-by-step loading method is used. At each step, the deflection functions, the length of the contact segment, the value of the reaction along the contact segment are determined and the working scheme of the plate-beam reinforcement system is specified from the minimum functional condition of the potential energy of the contacting elastic elements. Using the proposed numerical-analytical approach, we obtained and analyzed solutions to the problem of one-sided support of a slab to an elastic gain beam. Due to the two-dimensional approximation of the function of a slab deflection by discontinuous functions, the proposed approach shows satisfactory convergence, stability and accuracy of the solution while keeping some terms of the series.

1. Introduction

The problems of calculating constructively nonlinear systems with unilateral constraints with an unknown contact zone have a wide range of applications in engineering practice. For example, in deformation and technological seams there can be both zones of closure and opening of the joint, with various combinations of external loads. A similar problem can occur with contact interaction of a structure with a base or with other structural elements. In this case, it is often the state of the contact area that is determining when assessing the stress-strain state, strength and reliability of structures and constructions [1-10].

Of particular interest are the problems of continuum contact along a line or surface of a given shape. This problem arises during the reconstruction of existing and old buildings, when strengthening of floor slabs is required to increase their bearing capacity under load. In this case, the contact of the slab with the strengthening beams occurs along a continuous line or surface of a given shape with a contact zone unknown in advance. In numerical calculations, the continuous contact problem is usually reduced to finite-dimensional problems with discrete unilateral constraints. Numerous studies have been devoted to solving the contact problem based on various schemes of the finite element
method, among which can be noted [11-18]. Meanwhile, when solving contact problems, it is often necessary to construct a generalized solution in a numerical-analytical form. For this purpose, the mathematical apparatus of generalized functions can be used [19-21].

Both iterative [11-14] and step-by-step [15-18] methods can be used to calculate structurally nonlinear systems. The positive side of step-by-step algorithms is that they can be used to obtain results at each loading stage. In addition, step-by-step calculation schemes are more effective under difficult contact conditions and the nature of loading, where the solution to the problem depends on the loading history [17, 18]. Constructive non-linearity in a step-by-step process will manifest itself in a sequential change in the working circuits of the system depending on the load value (resizing the contact zones and the separation of boundary surfaces interacting bodies). It is assumed that between successive stages of loading, the nature of the deformation of the system is linear.

In this paper, based on the apparatus of generalized functions and the method of step-by-step analysis, we propose to use a variational approximation technique [20] for constructing numerical-analytical solution of the contact problem when strengthening the floor slab with an elastic beam. The original problem is replaced by an equivalent problem for generalized functions, and a numerical-analytical approximation to the solution of the contact problem in the class of piecewise-continuous generalized functions is sought. To solve the structurally nonlinear problem of strengthening the floor slab with an elastic beam, a step-by-step loading method is used. At each step, the deflection functions, the length of the contact segment, the value of the reaction along the contact segment are determined and the working scheme of the plate – beam strengthen system is specified from the minimum functional condition of the potential energy of the contacting elastic elements.

Using the proposed numerical-analytical approach, we obtained and analyzed solutions to the problem of one-sided support of a slab along a line to an elastic gain beam. The reliability of the results of the calculations is confirmed by comparing them with the solution obtained by the finite element method. Moreover, Due to the two-dimensional approximation of the function of a slab deflection by discontinuous functions, the proposed approach shows satisfactory convergence, stability and accuracy of the solution while keeping some terms of the series. It can be used to solve contact problems when strengthening floor slabs with beams.

2. Statement and solution of the problem
Let’s consider the contact problem of interaction of two linearly elastic structures when a rectangular slab is strengthening along the line \( x = x_1 \) by the beam with stiffness \( EI \). Dividing the slab and beam along the contact line of the structures, we write the equilibrium equations for each of them

\[
D \nabla^4 w = p(x, y) - r(y) \delta(x - x_1), \quad EI \frac{\partial^2 w}{\partial y^4} = r(y) \delta(x - x_1),
\]

where \( \delta(x - x_i) \) is the Dirac Delta function; \( H(y - y_i) = H(y - y_1) - H(y - y_2) \) is step function that takes into account the length of the contact line between the slab and the beam; \( H(y - y_i) \) is Heaviside step function; \( y_i \) is the ordinate of the contact line; \( i = 1, 2 \).

To solve a constructively nonlinear contact problem, we use a step-by-step method of loading. First, we define the deflection function of the slab \( w_i(x, y) \) under the action of a given load and find the maximum deflection of the slab on the contact line. Let’s assign a reduction factor for the load

\[
K_L = \max_i w_i(x_i, y_i) / A,
\]

here \( A \) is possible gap between the slab and the beam along the contact line. Let’s determine the initial load level \( p_0 = p / K_L \) and set the loading step as \( \Delta p = \beta p \). Then on the first stage load is \( p_1 = p_0 + \alpha \Delta p \), where \( \alpha, \beta \) are some constants.
First, let's solve the equation $D^4 w_i = p_i(x,y)$, and then calculate the deflection function of the slab along the line $x = x_i$. If $w_i(x_i,y) \geq \Delta_i$, we find a segment of the contact, which we replace approximately with the contact at the point $A(x_i,y)$. Due to the smallness of its length.

Then the differential equation of the slab takes the form

$$D^4 w_i = p_i(x,y) - R \delta_x \delta_y^3,$$  \hspace{1cm} (2)

Here $R$ is unknown concentrated beam reaction at the point of contact with the slab.

We will solve the equation $D^4 w_R = \delta_x \delta_y^3$, by the method of variational approximations [20]. To do this, we first represent the desired deflection function of the slab in the form

$$w_R^{(1)} = \chi(y) \psi(x),$$  \hspace{1cm} (3)

where $\chi(y) = C_1\frac{y^3}{3!} + C_2\frac{y^2}{2!} + C_3y + C_4 + \frac{(y-y_0)^3}{\alpha EI 3!}H(y-y_0)$; $C_i$ – the constant of integration satisfying the boundary conditions at $y=0$; $y=b$, $i=1, 2, 3, 4$.

First, substituting the expression (3) into equation (2), we find a function $\psi(x)$ and calculate the discrepancy

$$q_i = \delta_x \delta_y^3 - D^4 w_R^{(1)} = \delta_x \delta_y^3 - D^4[\chi(y) \psi(x)].$$

In the second iteration, the solution of the equation $D^4 w_R^{(2)} = q_i$ is found as $w_R^{(2)} = \psi(x)\Psi(y)$. On the third iteration the solution is searched for in the form $w_R^{(3)} = \Psi(y)\Psi(x)$ and so on.

Then the deflection function $w_R$ is defined as the sum

$$w_R = \sum_{n=1}^{M} w_R^{(n)}.$$

Then we determine the deflections at the points of interest on the contact line and compare them with the corresponding deflections calculated in the previous iteration. If the condition is met

$$\left| w_j^{(m)} - w_j^{(m-1)} \right| \leq \varepsilon,$$  \hspace{1cm} (4)

where $\varepsilon$ is specified calculation accuracy, then the procedure for determining the deflection function is terminated.

Otherwise, the process that implements alternate refinement of approximating functions is performed until the solution converges with the specified accuracy. At the same time, the functions obtained in the previous iteration are used as the basis for each subsequent iteration.

The deflection function of the beam from the action of the reaction of the slab, conditionally concentrated at the contact point, at the first iteration is a solution of the equation $EI\dddot{v}_i = R \delta_y^3$. Its solution is sought in the form $v_i = R v(y)$, where the function $v(y)$ is constructed similarly to $\chi(y)$.

The unknown reaction $R$ is determined from the condition of compatibility of movements at the contact point $w_i(x_i,y) = v_i(y)$ by the formula

$$R = \frac{w_i^0(x_i,y)}{w_R(x_i,y) + v(y)}.$$
Since the nature of the stress distribution along the contact line is not yet known, let's assume in the first approximation that \( R \approx r_1 \bar{y}_1 \), where \( \bar{y}_1 = y^{(1)}_1 - y^{(1)}_1 \) is the length of the contact segment. Its length is determined from the condition \( R = 0 \)

The differential equation of bending of a slab loaded by evenly distributed load \( p_1 \) over the slab area and a uniformly distributed intensity load \( r_1 \) along a line \( x = x_1 \) on a segment \( y^{(1)}_1 \leq y \leq y^{(1)}_2 \) has the form

\[
D^4 \bar{w}_1 = p_1 - r_1 \delta_y H^{(1)}_{yy},
\]

where \( H^{(1)}_{yy} = H(y - y^{(1)}_1) - H(y - y^{(1)}_2) \); \( y^{(1)}_1 = y_0 - \bar{y}_1 / 2 \); \( y^{(1)}_2 = y_0 + \bar{y}_1 / 2 \).

The solution to this equation is sought in the form \( \bar{w}_1 = w_1^0 - r_1 w_r \), where \( w_r \) is unknown slab deflection function from the load \( r_1 \) to construct \( w_r \), we define the basis function of the first level

\[
\rho(y) = \rho_0(y) + \frac{1}{EI} \left[ (y - y^{(1)}_1)^4 \cdot H(y - y^{(1)}_1) - (y - y^{(1)}_2)^4 \cdot H(y - y^{(1)}_2) \right]
\]

and perform the procedure of the method of variational approximations

\[
w^{(1)}_r = \rho(y) \psi(x) \rightarrow w^{(1)}_r = \psi(x) \Phi^{(1)}_{yy}(y) \rightarrow w^{(1)}_r = \Phi^{(1)}_{yy}(y) \Psi(x).
\]

Under condition (4), the function \( w_r \) is defined similarly to \( w_g \).

The value of \( r_1 \) is found from the equality of the deflections of the slab and beam on the contact segment \( \bar{w}_1(x_1, y)H^{(1)}_{yy} = \psi(\psi)H^{(1)}_{yy} \) provided that \( \partial E = 0 \) and \( \partial^2 E > 0 \). Here \( E \) is the functional total deformation energy that represents the difference of the potential deformation energy of the system slab-beam and the work of external forces applied to each elastic element of the system \( E = E_s + E_b \), where

\[
E_s = \frac{D}{2} \int_0^a \int_0^a (w^{*}_{xx} + w^{*}_{yy})^2 \, dx \, dy - \int_0^a p_1(x, y) \bar{w}_1(x, y) \, dx \, dy;
\]

\[
E_b = \frac{EI}{2} \int_0^a (v^{*}_{yy})^2 \, dy - \int_0^a r_1(y) \psi(\psi)H^{(1)}_{yy} \, dy.
\]

If the slab rests on the beam through the gasket, then the length of the contact segment is known, and the function of the beam support is represented as

\[
r(y) = \sum_{m=1}^M r_m s_m(y),
\]

where \( s_m(y) \) is spline functions on the contact segment \( y_1 \leq y \leq y_2 \).

In the general case, when it is required to determine the length of the contact segment, the beam support function is sought in the form

\[
r(y) = \sum_{n=1}^M r_n H^{(n)}_{yy}(y).
\]

3. Algorithm
The algorithm for calculating structural-nonlinear systems when strengthening the slab with a beam by the method of variational approximations is constructed as the following nested cycles:

- the cycle of loading steps;
• determination of the length of the contact piece;
• calculation of the beam support function on the contact segment;
• refinement of the design scheme;
• the construction of deflection functions by the method of variational approximations.

A rational choice of approximating functions with a high degree of accuracy describing the irregular parameters of the considered constructively nonlinear systems while retaining some series members, combined with a step-by-step loading method, allowed us to develop an effective algorithm and compile a strength calculation program for analyzing the stress-strain state of these systems with unilateral constraints.

Due to the two-dimensional approximation of the function of a slab deflection by discontinuous functions, the proposed approach shows satisfactory convergence, stability and accuracy of the solution obtained by numerical analytical method.

4. Results and discussion

Based on the method of variational approximations using the proposed algorithm, numerical solutions are obtained for the problem of contact interaction of the slab and the strengthening beams. A square steel plate $E = 2.1 \cdot 10^8$ kN/m$^2$, $\mu = 0.3$ with a size $a = b = 1.2$ m and thickness of 0.01 m (figure 1 a), hinged along the contour, is loaded over the entire area with a uniformly distributed load $p = 10$ kN/m$^2$.

The slab is strengthened with three beams $A$, $B$ and $C$ along the lines $x_B = a/3$, $x_A = a/2$ and $x_C = 2a/3$.

In the middle of the span between the slab and the beams, there are initial gaps, respectively, $\Delta_A = 0.00438$ m; $\Delta_A_1 = \Delta_A_2 = 0.00425$ m.

First, as shown in figure 1 b and c, the slab is loaded on the left and right thirds of the span with a uniformly distributed load $\Delta q = 10$ kN/m$^2$, subject to the conditions $0 \leq x \leq a/3$; $2a/3 \leq x \leq a$; $0 \leq y \leq b$ (variant 1). Then the slab is loaded sequentially only on the left third of the span at $0 \leq x \leq a/3$; $0 \leq y \leq b$ by uniformly distributed load with step $\Delta q = 5$ kN/m$^2$ to the level $2\Delta q$ (variant 2) and finally to $9\Delta q$ (variant 3).

![Figure 1. Calculation scheme of the slab with strengthening beams.](image)

The main results of the calculation of the slab strengthened with beams for the three specified loading options are shown in table 1.
Since the system under consideration is structurally non-linear, with these loading options, the slab is based on various sections of the beams, which leads to a change in the design scheme. The involvement of linear unilateral constraints in the bending of the slab is reflected in table 1, with the “+” sign means the contact of the beam in the support area of the slab, and “–” its absence.

### Table 1. Results of calculation of the slab with strengthening beams.

| Name and the x, y coordinates of the plate points | Variant 1 | Variant 2 | Variant 3 |
|------------------------------------------------|-----------|-----------|-----------|
|                                                | Contact   | Mx, kNm/m | My, kNm/m | Contact   | Mx, kNm/m | My, kNm/m | Contact   | Mx, kNm/m | My, kNm/m |
| B1                                             | 0.7 m     | – 0.633   | 0.592     | + 0.551   | 0.437     | + 0.236   | – 0.168   |          |
| B                                              | a/3       | b/2       |           |           |           |           |           |           |
| B2                                             | 0.5 m     | – 0.633   | 0.592     | + 0.551   | 0.437     | + 0.236   | – 0.168   |          |
| A1                                             | 0.7 m     | + 0.465   | 0.513     | + 0.383   | 0.437     | – 0.491   | 0.396     |          |
| A                                              | a/2       | b/2       |           |           |           |           |           |           |
| A2                                             | 0.5 m     | + 0.465   | 0.513     | + 0.383   | 0.437     | – 0.491   | 0.396     |          |
| C1                                             | 0.7 m     | – 0.633   | 0.592     | + 0.551   | 0.515     | – 0.632   | 0.508     |          |
| C                                              | 2a/3      | b/2       |           |           |           |           |           |           |
| C2                                             | 0.5 m     | – 0.633   | 0.592     | + 0.551   | 0.515     | – 0.632   | 0.508     |          |

Under the action of a given uniformly distributed load $p$, the slab pivotally supported along the contour receives initial deflections and forces. In the first version of stepwise loading, the slab rests only on the middle strengthening beam. At load levels of the left third of the span from $2\Delta q$ (variant 2) to $7\Delta q$, the slab is supported on all three beams. With a load of $8\Delta q$ on the left third of the span, the middle beam is completely turned off from work. Starting from a loading level of $9\Delta q$ (variant 3) and higher, the slab rests only on the left extreme beam $B$. Thus, under different loading conditions, a given system has four different calculation schemes, which leads to a significant change in the components of the stress-strain state of the slab (see table 1). This circumstance must be taken into account, for example, when solving the problem of strengthening floor slabs during the reconstruction of buildings and structures.

### 5. Conclusion

Based on the application of the apparatus of generalized functions, a mathematical model is created to solve the contact problem in the calculation of elastic slabs strengthening with beams. To construct a numerical-analytical solution, the method of variational approximations is used.

A rational choice of approximating functions with a high degree of accuracy describing the irregular parameters of the considered constructively nonlinear systems while retaining some series members, combined with a step-by-step loading method, allowed us to develop an effective algorithm and compile a strength calculation program for analyzing the stress-strain state of these systems with unilateral constraints.

Numerical experiments performed according to the developed program showed that it is characterized by simplicity of input of initial data, significantly fewer unknowns, as well as calculation time, and at the same time, due to analytical dependencies, higher accuracy of calculations in
comparison with computing complexes that implement numerical methods of strength calculation of these building structures.

References
[1] Laursen T A 2002 Computational Contact and Impact Mechanics (Berlin: Springer) p 454
[2] Wriggers P 2006 Computational contact mechanics (Berlin-Heidelberg: Springer) p 521
[3] Tolstikov V V 2006 Vestnik MGSU 2 123–132
[4] Kravchuk A S 2009 Applied mathematics and mechanics 3 492–502
[5] Kolosova G S, Lalin V V and Kolosova A V 2013 Magazine of civil engineering 5 76–85
[6] Bukhartsev V N and Vu Man Khuan 2013 Magazine of civil engineering 1 57–64
[7] Lukashevich A A 2014 Advanced materials research 941 2264–2267
[8] Barboteu M, Danan D and Sofonea M 2016 Journal of Applied Mathematics and Mechanics 96(4) 408–428
[9] Lukashevich A A, Lukashevich N K and Timohina E I 2018 IOP Conf. Ser.: Mater. Sci. Eng. 463 042054
[10] Lukashevich A A 2019 Magazine of civil engineering 5 167–178
[11] Puso M A and Laursen T A 2004 Computer Methods in Applied Mechanics and Engineering 193 601–29
[12] Wriggers P, Schroder J and Schwarz A 2013 Computational Mechanics 52 837–47
[13] Galanin M P, Lukin V V, Rodin A S and Stankevich I V 2015 Computational Mathematics and Mathematical Physics 55 8 1393–406
[14] Sofonea M and Souleiman Y 2017 Mathematics and Mechanics of Solids 22 324–42
[15] Wriggers P, Rust W T and Reddy B D 2016 Computational Mechanics 58 1039–50
[16] Ignatyev A V, Ignatyev V A and Gamzatova E A 2018 Izvestiya vuzov. Stroitelstvo 8 5–14
[17] Lukashevich A A 2018 Magazine of civil engineering 5 149–159
[18] Lukashevich A A and Lukashevich N K 2019 IOP Conf. Ser.: Mater. Sci. Eng. 687 033024
[19] Akimov P A and Mozgaleva M L 2013 International Journal for Computational Civil and Structural Engineering 9(1) 34–41
[20] Kobelev E A 2018 Bulletin of civil engineers 6 30–36
[21] Zolotov A B, Akimov P A, Sidorov V N and Mozgaleva M L (2010) Numerical and Analytical Methods of Structural Analysis (Moscow: ASV) p 336