ON SOME SHARP SPECTRAL INEQUALITIES FOR SCHRÖDINGER OPERATORS ON SEMI-AXIS

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ABSTRACT. In this paper we obtain sharp Lieb-Thirring inequalities for a Schrödinger operator on semi-axis with a matrix potential and show how they can be used to other related problems. Among them are spectral inequalities on star graphs and spectral inequalities for Schrödinger operators on half-spaces with Robin boundary conditions.

1. INTRODUCTION

Let us consider a self-adjoint Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - V,$$  \hfill (1.1)

where $V$ is a real-valued function. If the potential function $V$ decays rapidly enough, then the spectrum of the operator $H$ typically is absolutely continuous on $[0, \infty)$. If $V$ has a non-trivial positive part, then $H$ might have finite or infinite number of negative eigenvalues $\{-\lambda_n(H)\}$. If the number of negative eigenvalues is infinite, the point zero is the only possible accumulating point. The inequalities

$$\sum_n \lambda_n^\gamma \leq \frac{R_{\gamma,d}}{(2\pi)^d} \int \int_{\mathbb{R}^{2d}} (|\xi|^2 - V(x))_+^\gamma d\xi dx \leq L_{\gamma,d} \int \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} dx$$ \hfill (1.2)

are known as Lieb-Thirring bounds. Here and in the following, $V_\pm = (|V| \pm V)/2$ denote the positive and negative parts of the function $V$.

It is known that the inequality (1.2) holds true with some finite constants if and only if $\gamma \geq 1/2$, $d = 1$; $\gamma > 0$, $d = 2$ and $\gamma \geq 0$, $d \geq 3$. There are examples showing that (1.2) fails for $0 \leq \gamma < 1/2$, $d = 1$ and $\gamma = 0$, $d = 2$.

Almost all the cases except for $\gamma = 1/2$, $d = 1$ and $\gamma = 0$, $d \geq 3$ were justified in the original paper of E.H.Lieb and W.Thirring [LT]. The critical case $\gamma = 0$, $d \geq 3$ is known as the Cwikel-Lieb-Rozenblum inequality, see [Cw, L, Roz]. It was also proved in [Fe, LY, Con] and very recently by R. Frank [Fr] using Rumin’s approach. The remaining case $\gamma = 1/2$, $d = 1$ was verified by T.Weidl in [W1].

The sharp value of the constants $R_{\gamma,d} = 1$ in (1.2) are known for the case $\gamma \geq 3/2$ in all dimensions and it was first proved in [LT] and [AizL] for $d = 1$ and later in

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for any dimension. In this case

\[ L_{\gamma,d} = L_{\gamma,d}^0 := (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)^\gamma_+ d\xi. \]

The only other case where the sharp value of the constant \( R_{\gamma,d} \) is known is the case \( R_{1/2,1} = 2 \).

In this paper we consider a one-dimensional systems of Schrödinger operators acting in \( L^2(\mathbb{R}_+, \mathbb{C}^N) \), \( \mathbb{R}_+ = (0, \infty) \), defined by

\[ H \varphi(x) = \left( -\frac{d^2}{dx^2} \otimes I - V(x) \right) \varphi(x), \quad \varphi'(0) - S \varphi(0) = 0, \quad (1.3) \]

where \( I \) is the \( N \times N \) identity matrix, \( V \) is a Hermitian \( N \times N \) matrix-function and \( S \) is a \( N \times N \) Hermitian matrix.

Assuming that the potential \( V \) generates only a discrete negative spectrum, we denote by \( \{-\lambda_n\} \) the negative eigenvalues of \( H \).

One of the main results of this paper is the following

**Theorem 1.1.** Let \( \text{Tr} \, V^2 \in L^1(\mathbb{R}_+) \), \( V \geq 0 \). Then the negative spectrum of the operator \( H \) defined in (1.3) is discrete and the following Lieb-Thirring inequality for its eigenvalues \( \{-\lambda_n\} \) holds

\[
\frac{3}{4} \lambda_1 \text{Tr} \, S + \frac{1}{2} (2 \kappa_1 - N) \lambda_1^{3/2} + \sum_{n=2}^{\infty} \kappa_n \lambda_n^{3/2} \leq \frac{3}{16} \int_0^\infty \text{Tr} \, V^2(x) \, dx + \frac{1}{4} \text{Tr} \, S^3, \quad (1.4)
\]

where \( \kappa_n \) is the multiplicity of the eigenvalue \( -\lambda_n \).

**Examples.**

1. Let \( V \equiv 0 \) and \( N = 1 \). Then the boundary value problem

\[-\varphi''(x) = -\lambda \varphi(x), \quad \varphi'(0) - \sigma \varphi(0) = 0, \quad \sigma < 0,\]

has only one \( L^2 \)-solution

\[ \varphi(x) = C \, e^{-\sqrt{\lambda} x}, \quad -\sqrt{\lambda} = \sigma. \]

In this case the inequality (1.4) becomes saturated, \( \frac{3}{4} \sigma^3 - \frac{1}{2} \sigma^3 \leq \frac{1}{4} \sigma^3 \).

2. Let \( N = 2 \), \( V \equiv 0 \) and

\[ S = \begin{pmatrix} \sigma & 0 \\ 0 & -\alpha \sigma \end{pmatrix}, \quad \sigma < 0 \]
2a) If $\alpha \geq 0$ then the boundary value problem \eqref{1.3} has one negative eigenvalue $-\lambda$ of multiplicity one satisfying the identity $-\sqrt{\lambda} = \sigma$. In this case $2\kappa_1 - N = 0$ and the inequality \eqref{1.4} becomes
\[
3\lambda \text{Tr} \mathcal{S} = 3\lambda \sigma (1 - \alpha) \leq (1 - \alpha^3) \sigma^3 = \text{Tr} \mathcal{S}^3,
\]
or
\[
3(\alpha - 1) \leq \alpha^3 - 1.
\]
which holds true for any $\alpha \geq 0$.

2b) If $-1 < \alpha < 0$, then the problem \eqref{1.3} has two eigenvalues satisfying $-\sqrt{\lambda_1} = \sigma$ and $-\sqrt{\lambda_2} = -\alpha \sigma$ and \eqref{1.4} is reduced to
\[
3(\alpha - 1) - 4\alpha^3 \leq \alpha^3 - 1.
\]

2c) Finally, if $\alpha = -1$, then $-\sqrt{\lambda_1} = \sigma$ is of multiplicity $\kappa_1 = 2$ and \eqref{1.4} becomes identity.

Note that if $\text{Tr} \mathcal{S}^3 \leq 0$, then the inequality \eqref{1.4} implies
\[
\frac{3}{4} \lambda_1 \text{Tr} \mathcal{S} + \frac{1}{2} (2\kappa_1 - N) \lambda_1^{3/2} + \sum_{n=2}^{\infty} \kappa_n \lambda_n^{3/2} \leq \frac{3}{16} \int_0^\infty \text{Tr} V^2(x) \, dx. \tag{1.5}
\]

The latter allows us to use the standard Aizenman-Lieb arguments \cite{AizL} and derive

**Corollary 1.2.** Let $\text{Tr} \mathcal{S}^3 \leq 0$, $V \geq 0$ and $\text{Tr} V^{\gamma + 1/2}(x) \in L^1(0, \infty)$. Then for any $\gamma \geq 3/2$ we have
\[
\frac{\mathcal{B}(\gamma - 3/2, 2)}{\mathcal{B}(\gamma - 3/2, 5/2)} \frac{3}{4} \lambda_1^{1/2} \text{Tr} \mathcal{S} + \frac{1}{2} (2\kappa_1 - N) \lambda_1^2 + \sum_{n=2}^{\infty} \kappa_n \lambda_n^\gamma \leq L_{\gamma,1}^d \int_0^\infty \text{Tr} (V(x))^{\gamma + 1/2} \, dx,
\]
where by $\mathcal{B}(p, q)$ we denote the classical Beta function
\[
\mathcal{B}(p, q) = \int_0^1 (1 - t)^{q-1} t^{p-1} \, dt.
\]

**Corollary 1.3.** If $\mathcal{S} = 0$, then \eqref{1.3} can be identified with the Neumann boundary value problem and we obtain
\[
\frac{1}{2} (2\kappa_1 - N) \kappa_1 \lambda_1^2 + \sum_{n=2}^{\infty} \kappa_n \lambda_n^\gamma \leq L_{\gamma,1}^d \int_0^\infty \text{Tr} (V(x))^{\gamma + 1/2} \, dx, \quad \gamma \geq 3/2.
\]
Remark.

Note that in the scalar case \( N = 1 \) we obtain

\[
\frac{1}{2} \lambda_i^2 + \sum_{n=2}^{\infty} \lambda_n^2 \leq L_{cl,1}^\gamma \int_0^\infty V^{\gamma+1/2}(x) \, dx, \quad \gamma \geq 3/2, \tag{1.6}
\]

which means that the semi-classical inequality holds true for all eigenvalues starting from \( n = 2 \) and that in the latter inequality the Neumann boundary condition affects only the first eigenvalue.

If \( V \geq 0 \) is a diagonal \( N \times N \) matrix-function, then the operator \( \mathcal{H} \) could be interpreted as a Schrödinger operator on a star graph with \( N \) edges; the matrix \( \mathcal{S} \) describes a vertex coupling without the Dirichlet component [Ku]. In such a case we obtain:

**Theorem 1.4.** Let \( V \geq 0 \) be a diagonal \( N \times N \) matrix-function and let \( \mathcal{S} \) be a Hermitian matrix. Then the operator \( (1.3) \) can be identified with a Schrödinger operator on a star graph with \( N \) semi-infinite edges and its negative spectrum satisfies the inequality \( (1.4) \).

If both \( V \geq 0 \) and \( \mathcal{S} \) are diagonal \( N \times N \) matrices, then the negative spectrum of the operator \( \mathcal{H} \) is the union of the eigenvalues from each channel and we obtain

**Theorem 1.5.** Let \( V \geq 0 \), \( \text{Tr} V^2 \in L^1(0, \infty) \) and let \( V \) and \( \mathcal{S} \) be diagonal \( N \times N \) matrices with entries \( v_j \) and \( \sigma_j \), \( j = 1, \ldots, N \), respectively. Then the negative eigenvalues of the operator \( \mathcal{H} \) defined in \( (1.3) \), satisfy the inequality

\[
\frac{3}{4} \sum_{j=1}^{N} \lambda_j \sigma_j + \frac{1}{2} \sum_{j=1}^{N} \lambda_j^{3/2} + \sum_{j=1}^{N} \sum_{n=2}^{\infty} \lambda_j^{3/2} \lambda_n \leq \frac{3}{16} \int_0^\infty \text{Tr} V^2(x) \, dx + \frac{1}{4} \text{Tr} \mathcal{S}^3, \tag{1.7}
\]

where \(-\lambda_j \lambda_n\) are negative eigenvalues of operators \( h_j \) defined by

\[
h_j \psi(x) = \frac{d^2}{dx^2} \psi(x) - v_j(x) \psi(x), \quad \psi'(0) - \sigma_j \psi(0) = 0.
\]

Remark. Note that the inequality \( (1.7) \) is much more precise than \( (1.4) \) due to the diagonal structure of the operator \( \mathcal{H} \). In \( (1.7) \) all \( N \) first eigenvalues generated by each channel are affected by the Robin boundary conditions, whereas in \( (1.4) \) only the first one, see Example 2b).

Finally we give an example how our results could be applied for spectral estimates of multi-dimensional Schrödinger operators.

Let \( \mathbb{R}^d_+ = \{ x = (x_1, x') : x_1 > 0, x' \in \mathbb{R}^{d-1} \} \) and let \( H \) be a Schrödinger operator in \( L^2(\mathbb{R}^d_+) \) with the Neumann boundary conditions

\[
H \psi = -\Delta \psi - V \psi = -\lambda \psi, \quad \frac{\partial}{\partial x_1} \psi(0, x') = 0. \tag{1.8}
\]

The following result could be obtained by a “lifting” argument with respect to dimension, see [L], [LT]:
Theorem 1.6. Let $V \geq 0$ and $V \in L^{\gamma + d/2}$, $\gamma \geq 3/2$. Then for the negative eigenvalues $\{-\lambda_n\}$ of the operator (1.8) we have

$$\sum_n \lambda_n^\gamma \leq L_{\gamma,d}^{cl} \int_{\mathbb{R}^d} V^{\gamma + d/2}(x) \, dx + \frac{1}{2} L_{\gamma,d-1}^{cl} \int_{\mathbb{R}^{d-1}} \mu_1^{\gamma+(d-1)/2}(x') \, dx' \leq 2 L_{\gamma,d}^{cl} \int_{\mathbb{R}^d} V^{\gamma + d/2} \, dx. \quad (1.9)$$

Here $\mu_1(x')$ is the ground state energy for the operator $-d^2/dx_1^2 - V(x_1, x')$ in $L^2(\mathbb{R}^d)$ with the Neumann boundary condition at zero.

Remark. A similar inequality could be obtained by extending the operator (1.8) to the whole space $L^2(\mathbb{R}^d)$ with the symmetrically reflected potential. However, applying then the known Lieb-Thirring inequalities, we would have the constant $2^{\gamma+d/2}$ instead of 2 in (1.9).

2. SOME AUXILIARY RESULTS

In this Section we assume that the matrix-function $V$ is compactly supported, $\text{supp } V \subset [a, b]$ for some $a, b : 0 < a < b < \infty$ and adapt the arguments from [BL] to the case of semiaxis.

We begin with stating a well-known fact concerning the ground state of the operator (1.3).

Lemma 2.1. Let $-\lambda < 0$ be the ground state energy of the operator $\mathcal{H}$ and let $\varphi(x) = \{\varphi_k\}_{k=0}^N$ be a $L^2(\mathbb{R}_+, \mathbb{C}^N)$-vector-function satisfying the equation

$$\mathcal{H}\varphi(x) = -\frac{d^2}{dx^2}\varphi(x) - V(x)\varphi(x) = -\lambda \varphi(x), \quad \varphi'(0) - \mathcal{S}\varphi(0) = 0, \quad (2.1)$$

and such that the $2N$ vector $(\varphi(0), \varphi'(0))$ is not trivial. Then $\varphi(x) \neq 0$, $x \in \mathbb{R}_+$, and the ground state energy multiplicity is at most $N$.

Proof. Suppose that $\varphi(x_0) = 0$ for some $x_0 > 0$. Consider the continuous function

$$\bar{\varphi}(x) = \begin{cases} \varphi(x), & x < x_0 \\ 0, & x \geq x_0. \end{cases}$$

This function is non-trivial, belongs to the Sobolev space $H^1(\mathbb{R}_+, \mathbb{C}^N)$ and satisfies

$$\int_{\mathbb{R}_+} (|\dot{\varphi}|^2 - (V\varphi, \varphi)_{\mathbb{C}^N}) \, dx = \int_0^{x_0} (|\dot{\varphi}|^2 - (V\varphi, \varphi)_{\mathbb{C}^N}) \, dx = \int_0^{x_0} (-\varphi'' - V\varphi, \varphi)_{\mathbb{C}^N} \, dx = -\lambda \int_0^{x_0} |\varphi|^2 \, dx = -\lambda \int_{\mathbb{R}_+} |\bar{\varphi}|^2 \, dx.$$

Therefore $\bar{\varphi}$ minimizes the closed quadratic form associated with $\mathcal{H}$. Thus by the variational principle $\bar{\varphi}$ belongs to the domain of $\mathcal{H}$ and solves the Cauchy problem pointwise. However, since $\bar{\varphi}(x) = 0$ for $x \geq x_0$ it also solves the backward Cauchy
problem with zero initial data at $x_0$ and by uniqueness must vanish everywhere. This contradicts the non-triviality of $\tilde{\varphi}$ for $x < x_0$. □

Similarly to [BL] let us introduce a (not necessary $L^2$) fundamental $N \times N$-matrix-solution $M(x)$ of the equation (2.1), where $-\lambda$ is the ground state energy for the operator $\mathcal{H}$, so $M$ satisfies the equation

$$-\frac{d^2}{dx^2}M(x) - V(x)M(x) = -\lambda M(x), \quad M'(0) - \mathcal{S}M(0) = 0. \quad (2.2)$$

Denoting $M(0) = A$ and $M'(0) = B$, $B - \mathcal{S}A = 0$, we shall always assume that the matrix $A$ is invertible.

By using Lemma 2.1 we obtain that the matrix-function $M(x)$ is invertible for any $x \in \mathbb{R}_+$ and thus we can consider

$$F(x) = M'(x)M^{-1}(x). \quad (2.3)$$

**Lemma 2.2.** The matrix function $F(x)$ satisfies the following properties:

- $F(x)$ is Hermitian for any $x \in \mathbb{R}_+$.
- $F(x)$ is independent of the choice of the matrices $A, B$, satisfying the equation $B - \mathcal{S}A = 0$ and $F(0) = B A^{-1} = \mathcal{S}$.
- $F$ satisfies the matrix Riccati equation

$$F'(x) + F^2(x) + V(x) = \lambda \mathbb{I}. \quad (2.4)$$

**Proof.** From the Wronskian identity

$$\frac{d}{dx}W(x) := \frac{d}{dx}\left(M^*(x)M'(x) - (M^*(x))'M(x)\right) = 0$$

we obtain

$$W(x) = M^*(x)M'(x) - (M^*(x))'M(x) = \text{const.}$$

Since $M(0) = A$ and $M'(0) = B$, using the fact that $\mathcal{S}$ is Hermitian we find

$$W(0) = M^*(0)M'(0) - (M^*(0))'M(0) = A^*(B A^{-1} - (A^*)^{-1}B^*)A = A^*(\mathcal{S} - \mathcal{S}^*)A = 0.$$

Thus

$$W(x) = M^*(x)M'(x) - (M^*(x))'M(x) = 0.$$

Multiplying the latter identity by $M^{-1}$ from the right and by $(M^{-1})^*$ from the left we obtain $F(x) = F^*(x)$. Moreover

$$F' + F^2 = (M' M^{-1})' + (M' M^{-1})^2 = M''M^{-1} - M' M^{-1}M' M^{-1} + M' M^{-1}M' M^{-1} = (\lambda - V) M M^{-1} = \lambda \mathbb{I} - V.$$
Next, we analyze the behavior of the matrices $F(x)$ and their eigenvalues and eigenvectors as $x \to \infty$. For $x > b$ any solution of the differential equation (2.2) can be written as

$$M(x) = \cosh(\sqrt{\lambda}(x - b))M(b) + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x - b))M'(b)$$

$$= \left( \cosh(\sqrt{\lambda}(x - b))I + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x - b))F'(b) \right) M(b). \quad (2.5)$$

With the help of this representation we show

**Lemma 2.3.** For all $x \geq b$ it holds $F(x) = f(x, F(b))$, where

$$f(x, \mu) = \sqrt{\lambda} \frac{\sqrt{\lambda} \tanh(\sqrt{\lambda}(x - b)) + \mu}{\sqrt{\lambda} + \mu \tanh(\sqrt{\lambda}(x - b))}. \quad (2.6)$$

**Proof.** In view of (2.5) we have

$$M'(x) = \left( \sqrt{\lambda} \sinh(\sqrt{\lambda}(x - b))I + \cosh(\sqrt{\lambda}(x - b))F(b) \right) M(b),$$

$$(M(x))^{-1} = (M(b))^{-1} \left( \cosh(\sqrt{\lambda}(x - b))I + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x - b))F(b) \right)^{-1}.$$

It remains to insert these expressions in the definition $F(x) = M'(x)(M(x))^{-1}$ and to apply the spectral theorem for the Hermitian matrix $F(b)$. □

Note that $f(x, \mu)$ is strictly monotone in $\mu$. As a direct consequence of Lemma 2.3 we conclude, that the eigenvectors of the matrix $F(x)$ are independent of $x$ for $x \geq b$ as vectors in $\mathbb{C}^N$. Moreover, the eigenvalues of $F$ may or may not depend on $x$ outside the support of $V$ depending on if they correspond to growing or decaying solutions.

**Corollary 2.4.** Each eigenvalue $\mu_k$ of $F(b)$ gives rise to a continuous eigenvalue branch $\mu_k(x) = f(x, \mu_k(b))$. In particular, we have

$$\mu_k(x) = -\sqrt{\lambda} \quad \text{iff} \quad \mu_k(b) = -\sqrt{\lambda},$$

and

$$\lim_{x \to \infty} \mu_k(x) = \sqrt{\lambda} \quad \text{iff} \quad \mu_k(b) \neq -\sqrt{\lambda}.$$

The limit in the last expression is achieved exponentially fast.

**Remark.** There is a one-to-one correspondence between the $\kappa_1$-dimensional space of ground states for $H$ and a $\kappa_1$-dimensional eigenspace of $F(b)$ corresponding to the eigenvalue $-\sqrt{\lambda}$. Indeed, since $M(x)$ is a fundamental system of the solutions of the Cauchy problem (2.1) and $F(b)$ is invertible, any particular solution $\varphi$ of...
\textbf{(2.1)} can be represented as \( \varphi(x) = F(x)(F(b))^{-1} \nu \) with some \( \nu \in \mathbb{C}^N \). Hence, by \textbf{(2.5)}

\[
\varphi_{\nu}(x) = \cosh(\sqrt{\lambda}(x-b)) \nu + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x-b)) F(\nu) \\
= \frac{1}{2\sqrt{\lambda}} e^{\sqrt{\lambda}(x-b)} \left( \sqrt{\lambda} \nu + F(\nu) \right) - \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}(x-b)} \left( \sqrt{\lambda} \nu - F(\nu) \right). \tag{2.7}
\]

This function becomes an \( L^2 \)-eigenfunction of \( \mathcal{H} \), if and only if \( F(b) \nu = -\sqrt{\lambda} \nu \).

\section{Proofs of the Main Results}

\textbf{Proof of Theorem 1.1.}

Let now \(-\lambda_1 \) be the ground state energy of the operator \( \mathcal{H} \) with multiplicity \( \kappa_1 \leq N \), let \( M_1(x) \) be a fundamental system of solutions corresponding the eigenvalue \(-\lambda_1 \) and \( F_1 = M_1' M_1^{-1} \). We consider the operator

\[ Q_1 = \frac{d}{dx} \otimes I - F_1(x) \]

and its adjoint

\[ Q_1^* = -\frac{d}{dx} \otimes I - F_1(x) \]

in \( L^2(\mathbb{R}^+, \mathbb{C}^N) \). Using Riccati’s equation \textbf{(2.4)} we obtain the following factorization of the original operator \( \mathcal{H} \)

\[ Q_1^* Q_1 = -\frac{d^2}{dx^2} \otimes I + F_1'(x) + (F_1(x))^2 = \mathcal{H} + \lambda_1 I. \]

Consider

\[ Q_1 Q_1^* = -\frac{d^2}{dx^2} \otimes I - V(x) - 2F_1'(x) + \lambda_1 I = \mathcal{H} - 2F_1'(x) + \lambda_1 I. \]

Note that non-zero eigenvalues of \( Q_1^* Q_1 \) and \( Q_1 Q_1^* \) are same. However, while the vector-eigenfunctions \( \varphi \) defined in \textbf{(2.7)} satisfy the boundary conditions

\[ \varphi'(0) - \mathcal{S}\varphi(0) = 0, \]

the vector-eigenfunctions of \( Q_1 Q_1^* \) satisfy the Dirichlet boundary condition at \( 0 \).

Indeed, if \( \varphi \) is a vector-eigenfunction of \( Q_1 Q_1^* \) satisfying \( \varphi'(0) - \mathcal{S}\varphi(0) = 0 \) then \( \psi = Q_1 \varphi \) is an eigenfunction of \( Q_1 Q_1^* \) and

\[ \psi(0) = (Q_1 \varphi)(0) = \varphi'(0) - F_1(0)\varphi(0) = 0. \]

Next, let us verify that the kernel \( \text{ker} Q_1^* \) is trivial, and consequently, \( 0 \notin \text{spec}(Q_1 Q_1^*) \). Indeed, assume for a moment that there is a non-trivial vector-function \( \psi \) satisfying the Dirichlet boundary conditions at \( x = 0 \) and such that

\[ Q_1 Q_1^* \psi = 0. \tag{3.1} \]

Then

\[ (Q_1 Q_1^* \psi, \psi) = \|Q_1^* \psi\|^2 = 0. \]
However, $Q_1^* \phi = 0$ if and only if $\psi'(x) = F(x)\psi(x)$ for all $x \in \mathbb{R}_+$ and, in particular, $\psi'(0) = F(0)\psi(0) = 0$. Since $\psi$ satisfies the equation (3.1) together with $\psi(0) = \psi'(0) = 0$ we obtain that $\psi \equiv 0$.

Hence, the negative spectra of $\mathcal{H}$ and $\mathcal{H} - 2F_1'$ coincide except for the spectral value of the ground state energy, which does not belong to the spectrum of $\mathcal{H} - 2F_1'$ anymore. We emphasize that even in the case of a $\kappa_1$-fold degenerate ground state $-\lambda_1 = -\lambda_2 = \ldots = -\lambda_{\kappa_1}$ of $\mathcal{H}$, this commutation method removes all these eigenvalues $-\lambda_1, -\lambda_2, \ldots, -\lambda_{\kappa_1}$.

Therefore the spectral problem for the operator (1.3) is reduced to the operator in $L^2(\mathbb{R}_+)$

$$\mathcal{H}_1 \psi = \left( -\frac{d^2}{dx^2} \otimes I - V(x) - 2F_1' \right) \psi = -\lambda \psi \quad \psi(0) = 0.$$ 

Let us extend $V$ by zero to the negative semi-axis. Using then the variational principle we can apply the well-known Lieb-Thirring inequalities for 1D Schrödinger operators with matrix-valued potentials (see [LW1], [BL]) and obtain

$$\sum_{n=2}^\infty \kappa_n \frac{\lambda_n^{3/2}}{2} \leq \frac{3}{16} \int_0^\infty \text{Tr} \left( V(x) + 2F_1'(x) \right)^2 dx$$

$$= \frac{3}{16} \int_0^\infty \text{Tr} \left( V^2(x) + 4F_1'(x)(V(x) + F_1'(x)) \right) dx.$$ 

Using the Riccati equation (2.4), the fact that the matrix $\lim_{x \to \infty} F(x)$ has the eigenvalue $-\sqrt{\lambda_1}$ of multiplicity $\kappa_1$ and the eigenvalue $\sqrt{\lambda_1}$ of multiplicity $N - \kappa_1$ and that $F(0) = \mathcal{S}$, we finally arrive at

$$\sum_{n=2}^\infty \kappa_n \frac{\lambda_n^{3/2}}{2} \leq \frac{3}{16} \int_0^\infty \text{Tr} \left( V^2(x) + 4F_1'(x)(\lambda_1 - F_2^2(x)) \right) dx$$

$$= \frac{3}{16} \int_0^\infty \text{Tr} V^2(x) dx + \frac{3}{4} \lambda_1 \text{Tr} F_1(x) \bigg|_0^\infty - \frac{1}{4} \text{Tr} F_2^3(x) \bigg|_0^\infty$$

$$= \frac{3}{16} \int_0^\infty \text{Tr} V^2(x) dx + \frac{3}{4} \lambda_1 \left( -\kappa_1 \sqrt{\lambda_1} + (N - \kappa_1) \sqrt{\lambda_1} - \text{Tr} \mathcal{S} \right)$$

$$- \frac{1}{4} \left( -\kappa_1 \sqrt{\lambda_1} \frac{3}{2} + (N - \kappa_1) \lambda_1^{3/2} - \text{Tr} \mathcal{S}^3 \right)$$

$$= \frac{3}{16} \int_0^\infty \text{Tr} V^2(x) dx - \frac{1}{2} (2\kappa_1 - N) \lambda_1^{3/2} - \frac{3}{4} \lambda_1 \text{Tr} \mathcal{S} + \frac{1}{4} \text{Tr} \mathcal{S}^3.$$ 

Finally using standard arguments we can consider the closure of the latter inequality from the class of compactly supported potentials to the class $L^2(\mathbb{R}_+, \mathbb{C}^N \times \mathbb{C}^N)$.

The proof of Theorem 1.1 is complete.

**Proof of Corollary 1.2**
Let us denote by $\lambda_n = \lambda_n(V)$ the eigenvalues of the Schrödinger operator with the potential $V$. Then by using the variational principle and the inequality (1.5) we find that for any $\gamma > 3/2$

$$
B(\gamma - 3/2, 2) \frac{3}{4} \text{Tr} \lambda_1^{\gamma - 1/2}(V)
$$

$$
+ B(\gamma - 3/2, 5/2) \left( \frac{1}{2} (2\kappa_1 - N) \lambda_1^\gamma(V) + \sum_{n=2}^\infty \kappa_n \lambda_n^\gamma(V) \right)
$$

$$
= \int_0^\infty \left( \frac{3}{4} \text{Tr} \mathcal{S} (\lambda_1(V) - t)_+ 
$$

$$
+ \frac{1}{2} (2\kappa_1 - N) (\lambda_1(V) - t)^{3/2}_+ + \sum_{n=2}^\infty \kappa_n (\lambda_n(V) - t)^{3/2}_+ \right) t^{\gamma - 5/2} dt
$$

$$
\leq \int_0^\infty \left( \frac{3}{4} \text{Tr} \mathcal{S} (\lambda_1((V - t)_+) 
$$

$$
+ \frac{1}{2} (2\kappa_1 - N) (\lambda_1((V - t)_+)^{3/2}_+ + \sum_{n=2}^\infty \kappa_n (\lambda_n((V - t)_+)^{3/2}_+ 
$$

$$
\leq \frac{3}{16} \int_0^\infty \int_0^\infty \text{Tr} (V(x) - t)^2_+ t^{\gamma - 5/2} dt dx
$$

$$
= B(\gamma - 3/2, 3) \frac{3}{16} \int_0^\infty \text{Tr} V^{\gamma + 1/2}(x) dx.
$$

Dividing by $B(\gamma - 3/2, 5/2)$ and noting that

$$
\frac{3}{16} \frac{B(\gamma - 3/2, 3)}{B(\gamma - 3/2, 5/2)} = L_{\gamma + 1/2, 1}^{cl}
$$

we complete the proof.

**Proof of Theorem 1.6**

Let $\{\mu_j(x')\}$ be eigenvalues of the Neumann problem for the Schrödinger operator

$$
- \frac{d^2}{dx^2_1} \psi(x_1, x') - V(x_1, x') \psi(x_1, x') = -\mu(x') \psi(x_1, x')
$$

considering $x'$ as a parameter.

For any $\gamma \geq 3/2$ and $d \geq 1$ let us apply the operator version of the Lieb-Thirring inequality (see [LWII]) with respect to $\mathbb{R}^{d-1}$ and obtain

$$
\sum_n \lambda_n^\gamma \leq L_{\gamma, d-1}^{cl} \int_{\mathbb{R}^{d-1}} \sum_j \mu_j^{\gamma+(d-1)/2}(x') dx'.
$$
By using (1.6) we find
\[
\sum_j \mu_j \gamma^{+(d-1)/2}(x') \leq \frac{1}{2} \mu_1 \gamma^{+(d-1)/2}(x') + L_{\gamma+(d-1)/2,1}^{cl} \int_0^\infty V \gamma^{d/2}(x_1, x') \, dx_1 \\
\leq 2 L_{\gamma+(d-1)/2,1}^{cl} \int_0^\infty V \gamma^{d/2}(x_1, x') \, dx_1.
\]

Noticing that
\[
L_{\gamma,d-1}^{cl} L_{\gamma+(d-1)/2,1}^{cl} = L_{\gamma,d}^{cl}
\]
we obtain the proof.

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REFERENCES

[AizL] M.Aizenman and E.H. Lieb, On semi-classical bounds for eigenvalues of Schrödinger operators, Phys. Lett. 66A (1978), 427–429.

[BL] R. Benguria and M. Loss, A simple proof of a theorem by Laptev and Weidl, Math. Res. Lett. 7 (2000), no. 2-3, 195–203.

[Con] J.G. Conlon, A new proof of the Cwikel-Lieb-Rosenbljum bound, Rocky Mountain J. Math., 15 (1985), 117–122.

[Cw] M. Cwikel, Weak type estimates for singular values and the number of bound states of Schrödinger operators, Trans. AMS, 224 (1977), 93–100.

[DLL] J. Dolbeault, A. Laptev and M. Loss, Lieb-Thirring inequalities with improved constants, JEMS, 10 (2008), 1121–1126.

[Fe] C. L. Fefferman, The uncertainty principle, Bull. Amer. Math. Soc. 9 (1983), no. 2, 129–206.

[Fr] R.L. Frank, Cwikel’s theorem and the CLR inequality, accepted by JST.

[HLW] D. Hundertmark, A. Laptev and T. Weidl, New bounds on the Lieb-Thirring constants, Inv. Math., 140 (2000), 693–704.

[Ku] P. Kuchment, Quantum graphs: I. Some basic structures, Waves in Random Media 14 (2004), S107–128.

[L] A. Laptev, Dirichlet and Neumann Eigenvalue Problems on Domains in Euclidean Spaces, J. Funct. Anal., 151 (1997), 531–545.

[LW1] A. Laptev and T. Weidl, Sharp Lieb-Thirring inequalities in high dimensions, Acta Mathematica, 184 (2000), 87–111.

[LW2] A. Laptev and T. Weidl, Recent results on Lieb-Thirring inequalities, Journées “Équations aux Dérivées Partielles” (La Chapelle sur Erdre, 2000), Exp. No. XX, Univ. Nantes, Nantes, 2000.

[LY] P. Li and S.-T. Yau, On the Schrödinger equation and the eigenvalue problem, Comm. Math. Phys., 88 (1983), 309–318.

[L] E.H Lieb, The Number of Bound States of One-Body Schrödinger Operators and the Weyl Problem, Proceedings of the Amer. Math. Soc. Symposia in Pure Math., 36 (1980), 241–252.
E.H. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Studies in Math. Phys., Essays in Honor of Valentine Bargmann, Princeton, (1976), 269–303.

G.V. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Dokl. AN SSSR, 202 (1972), 1012–1015, Izv. VUZov, Matematika, 1 (1976), 75–86.

T. Weidl, *On the Lieb-Thirring constants \( L_{\gamma,1} \) for \( \gamma \geq 1/2 \), Comm. Math. Phys., 178 (1996), 135-146.

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