Dynamics of $\mathcal{B}$-free systems generated by Behrend sets. I

by

STANISŁAW KASJAN (Toruń), MARIUSZ LEMAŃCZYK (Toruń) and SEBASTIAN ZUNIGA ALTERMAN (Turku)

Dedicated to the memory of Professor Andrzej Schinzel

1. Introduction

1.1. General overview and motivations. In this paper we mainly study subshifts $\left(X_1, S\right)$ of the full shift $\left(\{0,1\}^\mathbb{Z}, S\right)$, where $1 \in \mathcal{F} \subset \mathbb{Z} \setminus \{0\}$ is symmetric ($x \in \mathcal{F}$ implies $-x \in \mathcal{F}$) and satisfies the following conditions:

\begin{enumerate}
  \item $\mathcal{F}$ is closed under taking divisors
  \item the natural density $d(\mathcal{F}) := \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} 1_{\mathcal{F}}(n)$ of $\mathcal{F}$ exists and equals zero.
\end{enumerate}

Let us explain the first condition. It is not hard to see that for each $\mathcal{F}$ satisfying [1] there is a subset $\mathcal{B} \subset \mathbb{N} \setminus \{1\}$ such that $\mathcal{F} = \mathcal{F}_\mathcal{B}$, where $\mathcal{F}_\mathcal{B}$ denotes the set of $\mathcal{B}$-free numbers, i.e., of numbers with no divisor in $\mathcal{B}$. If we assume that $\mathcal{B}$ is primitive, that is, no two members of $\mathcal{B}$ divide one another, then $\mathcal{F} = \mathcal{F}_\mathcal{B}$ and such a $\mathcal{B}$ is unique. Classically (see [13] Section V, §4), for

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[1] Given $y \in \{0,1\}^\mathbb{Z}$, we denote by $(X_y, S)$ the subshift generated by $y$, i.e. $X_y := \{S^j y : j \in \mathbb{Z}\}$, where $S$ stands for the left shift, $S((z_n)_{n \in \mathbb{Z}}) = (z_{n+1})_{n \in \mathbb{Z}}$. We also consider $y$’s which are only one-sided, i.e. $y \in \{0,1\}^{\mathbb{N} \cup \{0\}}$. Then $X_y$ means the subshift obtained from the symmetrized $y$: $y(0) = 0$ and $y(-n) = y(n)$. For example, if $A \subset \mathbb{N} = \{1,2,\ldots\}$ then $X_{1_A}$ in fact means $X_{1_{A \cup \{-A\}}}$.

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each primitive (which is our standing assumption from now on) and infinite
set \( \mathcal{B} \), we have

\[
0 = d(\mathcal{B}) \leq \overline{d}(\mathcal{B}) \leq \frac{1}{2},
\]

where \( d, \overline{d} \) stand for the lower and upper density, respectively (for the density
notions, see Section 2.1), while

\[
\delta(\mathcal{B}) = 0,
\]

where \( \delta \) stands for the logarithmic density. Moreover,

\[
\sum_{b \in \mathcal{B}} \frac{1}{b \log b} < +\infty.
\]

Given a primitive subset \( \mathcal{B} \), the corresponding subshift \( (X_{1, \mathcal{B}}, S) \) is
called a \( \mathcal{B} \)-free subshift. Together with \( \mathcal{B} \)-free subshifts, we will also con-
sider subshifts \( (X_{\mathcal{B}}, S) \), called \( \mathcal{B} \)-admissible, where \( X_{\mathcal{B}} \) consists of all 0-1-
sequences whose support misses at least one residue class modulo \( b \), for
any \( b \in \mathcal{B} \). As \( \mathcal{F}_{\mathcal{B}} \) misses the zero residue class modulo all \( b \in \mathcal{B} \), we have

\[
X_{1, \mathcal{F}_{\mathcal{B}}} \subset X_{\mathcal{B}},
\]

the latter subshift being obviously hereditary \(^{(2)}\). Often, additional assump-
tions will be put on \( \mathcal{B} \). Throughout the paper we say that the set \( \mathcal{B} \) is coprime
if any two distinct elements of \( \mathcal{B} \) are coprime \(^{(3)}\). The correspond-
ing \( \mathcal{B} \)-free subshift is then called coprime \( \mathcal{B} \)-free.

Following \(^{[14]}\), if \( \mathcal{F} = \mathcal{F}_{\mathcal{B}} \) satisfies \(^{(2)}\) then \( \mathcal{B} \) is called a Behrend set and
the corresponding \( \mathcal{B} \)-free subshifts will be called Behrend subshifts. Condition \(^{(2)}\) hence indicates that we intend to study the dynamics of some
sparse sets. In fact, it is not only density as in \(^{(2)}\) or the logarithmic density
as in \(^{(4)}\) that vanish. Actually, when \( \mathcal{B} \) is Behrend, the upper Banach
density \( BD^*(\mathcal{F}_{\mathcal{B}}) \) of \( \mathcal{F}_{\mathcal{B}} \) equals 0 (see \(^{[10]}\)), and also \( BD^*(\mathcal{B}) = 0 \) (see
Corollary 3.7). Such sparse sets are interesting from the point of view of so
called non-conventional ergodic theorems (along subsequences) first pointed
out in \(^{[28]}\) (see also more recent articles \(^{[12, 23, 27]}\)).

However, there are at least three more direct reasons which make Behrend
\( \mathcal{B} \)-free subshifts of special interest in dynamics. Indeed, in general, the
complement of \( \mathcal{F} = \mathcal{F}_{\mathcal{B}} \) is a set of multiples, \( \mathbb{Z} \setminus \mathcal{F} = \mathcal{M}_{\mathcal{B}} := \bigcup_{b \in \mathcal{B}} b\mathbb{Z} \). So
firstly, \(^{(2)}\) means that the \( \mathcal{B} \)-free subshift \( (X_{\eta}, S) \) with \( \eta = 1_{\mathcal{F}_{\mathcal{B}}} \) captures
information about a “typical” natural number (since the density of \( \mathcal{M}_{\mathcal{B}} \) is
now 1). If we can obtain interesting dynamical results, we can count on proving
interesting facts in number theory; see also \(^{[14]}\) for the number-theoretic

\(^{(2)}\) A subshift \( (X, S) \) with \( X \subset \{0, 1\}^\mathbb{Z} \) is hereditary if whenever \( x \in X \) and \( y \leq x \)
(coordinatewise) we have \( y \in X \).

\(^{(3)}\) Clearly, this is much stronger than the condition that the greatest common divisor
of all elements of \( \mathcal{B} \) is 1.
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Secondly, it is not hard to see that if $B$ is a Behrend set, then the all zero sequence $0^\mathbb{Z}$ belongs to $X_B$ and it is known that the Behrend subshifts have only one invariant measure (the Dirac measure $\delta_{0^\mathbb{Z}}$) [10]. That is, from the dynamical point of view, these are uniquely ergodic models of the one-point system (in particular, topological entropy of such systems is zero). This makes them trivial from the ergodic theory point of view but we will see that the dynamics of Behrend subshifts are rich and complex from the topological dynamics point of view. Finally, Behrend subshifts seem to be crucial to understand the general theory of $B$-free subshifts in the proximal case (see footnote 21 for the definition of proximality). As a matter of fact, even though the sets of multiples have been studied in number theory for about 100 years, dynamically they have been investigated for the first time in Sarnak’s celebrated article [30] concerning (among other problems) the square-free system (given by $B = \{p^2 : p \in \mathbb{P}\}$) ($\mathbb{P}$ stands for the set of primes). This system, similarly to the Behrend case, is proximal but obviously it does not satisfy (2). The latter implies that the corresponding $B$-free system has positive entropy and is taut (see Section 2 for this notion). It follows that the square-free system has plenty of interesting invariant measures which makes possible an analysis using ergodic theory tools (for the general theory of $B$-free systems see [11, 10, 22]). Behrend subshifts are proximal, and among proximal $B$-free systems they are characterized by their zero entropy property (cf. Corollary 1.6). Building on some recent progress of the theory of $B$-free subshifts in [17, 21] and using [10], it is noticed in [20] that for each proximal $B$-free system there exists a (unique) taut $B'$-free subsystem such that $F_{B'} \subset F_B$ and the density of the difference of these two sets vanishes, which “justifies” the conclusion that $(X_{B'}, S)$ is “relatively Behrend” over $(X_B, S)$ (in fact, the hereditary closures of these two systems have the same sets of invariant measures). We should add that the dynamics in the taut case is much better understood since it leads to the theory of hereditary subshifts. Hence, to understand

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(1) According to [14, Section 1.3, p. 36], Erdős in 1979 wrote “It seems very difficult to obtain a necessary and sufficient condition that if $a_1 < a_2 < \cdots$ is a sequence of integers then almost all integers $n$ should be a multiple of one of the $a_i$'s.” From the dynamical point of view we have no problem to characterize the Behrend subshifts: these are precisely those $B$-free subshifts which are proximal and have zero entropy (see e.g. [9] for the definition of entropy).

(3) This is due to the fact that once (2) holds then, by (1), the upper Banach density of $F$ is also zero.

(6) A square-free system is a special instance of a coprime $B$-free system. It is not hard to see that in the coprime case we obtain a Behrend subshift if and only if $\sum_{b \in B} 1/b = \infty$. Furthermore, the latter (in the coprime case) is equivalent to zero entropy.

(7) Given a subshift $(X_y, S)$ with $X_y \subset \{0, 1\}^\mathbb{Z}$ by its hereditary closure we mean the smallest hereditary (cf. footnote 2) subshift $\tilde{X}_y$ containing $X_y$. 
the “relative Behrend” case, whence the general proximal case, it seems reasonable to first understand the possible dynamics of the Behrend subshifts themselves.

In what follows, there will be no special referring to illustrate an importance of the above three reasons and the paper is simply focused on showing how rich the dynamics of Behrend sets can be and what kind of consequences we can derive from it.

1.2. Toward results. Following [24], the (symmetrized) set of prime numbers with 1 added to it can be seen as a $\mathcal{B}$-free set (take $\mathcal{B} = \{pq : p, q \in \mathbb{P}\}$) with $\mathcal{B}$ clearly Behrend. So, remembering our convention in footnote 1, the subshift $(X_{1,\mathbb{P},(1)}, S)$ is Behrend. It is now rather standard to notice that this subshift is conjugate to the subshift of prime numbers $(X_{1,\mathbb{P}}, S)$ (9), so in what follows we consider this more natural subshift.

One can ask how big this system is, and as noticed (10) in [10], even the fact that this subshift is uncountable follows from the validity of an “unprovable” old conjecture, namely Dickson’s conjecture. Recalling that the entropy of the corresponding subshift is zero, one can ask what the complexity (11) of the prime numbers subshift is. As T. Tao [32] noticed to us, there is an upper bound for the complexity of $1_\mathbb{P}$ and, assuming the $k$-tuple Hardy–Littlewood conjecture (12), also a lower bound. Throughout, we denote by $\log$ the natural logarithm.

**Theorem 1.1.** We have

$$cpx_{1_\mathbb{P}}(N) \leq (4 + o(1)) \frac{N}{\log N}$$

(8) If we take $\mathcal{B} = \{pq : p \neq q \in \mathbb{P}\}$ then $F_{\mathcal{B}}$ equals the (symmetrized) support of the von Mangoldt function $\Lambda$. Other classical sets can be obtained similarly by considering the set of $k$-almost prime numbers, i.e. $\mathcal{B} = \mathbb{P}_k := \{p_1 \ldots p_k : p_i$ are primes for $i = 1, \ldots, k\}$; see also Section 2.5.

(9) If $y, z \in \{0, 1\}^\mathbb{Z}$ are isolated points in $X_y$ and $X_z$, respectively, and $y, z$ are asymptotic, that is, they differ only on finitely many coordinates, then the subshifts $(X_y, S)$ and $(X_z, S)$ are conjugate. Indeed, the map $S^i y \mapsto S^i z$ ($i \in \mathbb{Z}$) is uniformly continuous: Given $\varepsilon > 0$, select $N > 0$ so that $d(S^i y, S^i z) < \varepsilon/3$ for all $|i| > N$; then choose $0 < \delta < \varepsilon/3$ so that in the $\delta$-neighbourhood of $S^k y$, $k = -N, \ldots, N$, there are no points of the form $S^\ell y$ with $\ell \neq k$. It follows that if $d(S^i y, S^j y) < \delta$ then $d(S^i z, S^j z) < \varepsilon$.

(10) This observation is due to A. Schinzel.

(11) Given a subshift $(X, S)$, we say that a block appears in $X$ if there is $y \in X$ such that the block appears in $y$. By the complexity of $(X, S)$ we mean the function $N \mapsto cpx_X(N)$, where $cpx_X(N)$ stands for the number of blocks of length $N$ appearing in $X$. When $X = X_y$, then we write $cpx_y(N)$ for $cpx_{X_y}(N)$. If the entropy is zero, the growth of the complexity function must be subexponential; cf. footnote 18.

(12) The Hardy–Littlewood conjecture could be replaced by Dickson’s conjecture in Theorem 1.1; see Remark 5.1.
and
\begin{equation}
(2 + o(1))^{\log N} \leq \text{cpx}_{X_p}(N) \leq (4 + o(1))^{\log N}
\end{equation}
when \( N \to \infty \). If the Hardy–Littlewood conjecture is true then
\begin{equation}
(2 + o(1))^\frac{N}{\log N} \leq \text{cpx}_{\mathbb{1}_p}(N)
\end{equation}
when \( N \to \infty \).

We will provide Tao’s proof (private correspondence) of the above theorem in Section 5.1.

By proving \( X_{\mathbb{1}_p} \subset X_{\mathbb{1}_{p^2}} \) (Proposition 5.9) and using Theorem 1.1 we obtain a lower bound \((2 + o(1))^n/\log n \leq \text{cpx}_{\mathbb{1}_{p^2}}(n)\) for the subshift \((X_{\mathbb{1}_{p^2}}, \mathcal{S})\) of semi-primes, conditionally on the Hardy–Littlewood conjecture. Moreover, by deriving some consequences of Dickson’s conjecture, we prove in Sections 5.2 and 5.3 some other estimates, namely:

**Theorem 1.2.** We have
\begin{equation}
\text{cpx}_{\mathbb{1}_{p^2}}(N) \leq (4 + o(1))^{\frac{23N \log^2 \log N}{(\log 4)(\log N)}}.
\end{equation}
If Dickson’s conjecture is true then
\begin{equation}
(2 + o(1))^\frac{N \log \log N}{2 \log N} \leq \text{cpx}_{\mathbb{1}_{p^2}}(N)
\end{equation}
when \( N \to \infty \).

A natural question arises how fast the complexity in a general Behrend \( \mathcal{B} \)-free subshift can grow. We will show that any subexponential growth rate of this function can be realized by a Behrend \( \mathcal{B} \)-free subshift which is even \( \mathcal{B} \)-admissible.

**Theorem 1.3.** Let \( \rho : \mathbb{N} \to \mathbb{N} \) be a function such that \( \rho(n)/n \searrow 0 \) as \( n \to +\infty \). There exists a Behrend set \( \mathcal{B} \subset \mathbb{P} \) such that \( X_\eta = X_\mathcal{B} \). Moreover, \( \limsup_{n \to \infty} \text{cpx}_\eta(n)/2^{\rho(n)} \geq 1 \).

As one of our motivations is to prove the uncountability of certain subshifts, one could hope that if the complexity grows “almost” exponentially fast, then the subshift must be uncountable (if the entropy of a subshift is positive, then it must be uncountable), but Cyr and Kra [6] constructed countable subshifts with arbitrarily fast subexponential complexity. Their construction is, however, based on richness of periodic points in some countable systems. This is not our case, as a general \( \mathcal{B} \)-free subshift has only one minimal (see footnote 19) subsystem [10], and therefore we deal with

\footnote{We seek here unconditional proofs. To the best of our knowledge there is no unconditional proof of the uncountability of \( X_{\mathbb{1}_p} \) and the present paper does not change the status quo.}
(special) uniquely ergodic subshifts whose cardinality is not clear. One more attempt to prove uncountability (of a Behrend $\mathcal{B}$-free subshift) could be based on the equality in the inclusion (6) because, as noticed in [10], $\mathcal{B}$-admissible subshifts [14] are always uncountable. This approach works for Behrend sets considered in Theorem 1.3 but for $\mathcal{B} = \{pq : p, q \in \mathbb{P}\}$ we fail again, because whereas the subshift $(X_n \cup \{1\}, S)$ has isolated points (15), $\mathcal{B}$-admissible subshifts have no isolated points (see Section 3.3 for details). It is also worth noticing that $\mathcal{B}$-admissible subshifts always have “large” complexity, namely, it is at least the left-hand side of inequality (8) in Theorem 1.1:

**Proposition 1.4.** For each $\mathcal{B}$, $\text{cpx}_{X_{\mathcal{B}}}(n) \geq (2 + o(1)) \frac{n \log n}{\log \log n}$.

Returning to the case $X_\eta = X_{\mathcal{B}}$, note that the set $\{0, 2\}$ is always $\mathcal{B}$-admissible, so once we have $X_\eta = X_{\mathcal{B}}$ then (since now $X_\eta$ has no isolated points) the corresponding set of $\mathcal{B}$-free numbers will contain infinitely many pairs $(t, t + 2)$ of twin $\mathcal{B}$-free numbers (16).

Since the $\mathcal{B}$-admissible subshifts seem to be especially interesting from the number theory point of view, it is natural to ask whether these subshifts can be transitive, i.e. have points whose orbits are dense (17). The Chinese Remainder Theorem tells us that this is the case when $\mathcal{B}$ is coprime but a kind of surprise is that this is the only possibility for the existence of a “dense” admissible configuration:

**Theorem 1.5.** For each $\mathcal{B}$, the subshift $(X_{\mathcal{B}}, S)$ is transitive if and only if $\mathcal{B}$ is coprime.

This theorem has a number of consequences. As we have already noticed, the most prominent example of a (proximal) $\mathcal{B}$-free system is the square-free

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(14) As shown in [10], Behrend $\mathcal{B}$-admissible subshifts still have zero entropy. In fact, $\delta_{0*}$ is the only invariant measure.

(15) Given a subshift $(X, S)$, by $\text{Isol}(X)$ we denote its set of isolated points. This set is open, countable and $S$-invariant. Hence, $X_0 := X \setminus \text{Isol}(X)$ defines a subshift $(X_0, S)$ which is a subsystem of $(X, S)$. If we consider a particular situation when $X = X_u$, then either $u$ is not an isolated point and then $(X_u)_0 = X_u$, or $u$ is isolated (then so are all points from its orbit), and then $(X_u)_0 = X_u \setminus \{S^n u : n \in \mathbb{Z}\}$. Note also that $u$ is an isolated point if and only if there is a block $B$ that appears in $u$ only finitely many times. Note that for the prime numbers subshift there are isolated points because configurations like $\{2, 3\}$ or $\{3, 5, 7\}$ appearing in $\mathbb{P}$ are not $\mathbb{P}$-admissible and can appear in $\eta$ only finitely many times. Now, Dickson’s conjecture can be formulated as $(X_{1_{\mathbb{P}}})_0 = X_{\mathbb{P}}$ and the present theorem can be viewed as an instance of validity of Dickson’s conjecture for a Behrend set.

(16) For sets $\mathcal{B} \subseteq \mathbb{P}$ such that $\mathbb{P} \setminus \mathcal{B}$ has at most two elements, the result of Bennett [2] on Pillai’s type equation $a^x - b^y = c$ implies that there are at most two pairs of twin $\mathcal{B}$-free numbers.

(17) The transitivity of a topological system $(X, T)$ is equivalent to the condition that for any non-empty open sets $U, V \subset X$, there is $n \in \mathbb{Z}$ such that $U \cap T^n V \neq \emptyset$. 

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The natural class beyond it is the class of so called Erdős \( B \)-free systems – here \( B \) is said to satisfy the Erdős condition if it is infinite, coprime and \( \sum_{b \in B} 1/b < \infty \). While this seems to be just a technical condition, Theorem 1.5 and [17, 10] yield a dynamical characterization of the Erdős case:

**Corollary 1.6.**

(i) \( B \) is Erdős if and only if \( X_\eta = X_B \) and \( h(X_\eta, S) > 0 \).

(ii) \( B \) is Behrend if and only if \( (X_\eta, S) \) is proximal and \( h(X_\eta, S) = 0 \).

We recall that also hereditary closures \( (\tilde{X}_\eta, S) \) of subshifts (see footnote 7) are quite often transitive (see [10]); more precisely, the transitivity is equivalent to the condition that if a block \( B \) appears in \( \eta \) then a certain block (coordinatewise) bigger than \( B \) reappears in \( \eta \) infinitely often. In particular, if \( \eta \) is recurrent then \( (\tilde{X}_\eta, S) \) is transitive. An important generalization of the Erdős property of \( B \), complementary to the concept of Behrend set, is tautness [14] (see Section 2 for more details). We find that for each taut \( B \), the subshift \( (\tilde{X}_\eta, S) \) is transitive (this follows from the recurrence of \( \eta \), which is a consequence of Theorem 2.2 below and the fact that \( \eta \) is quasi-generic for the Mirsky measure [10]).

**Corollary 1.7.** Assume that \( B \) is taut and infinite. If \( B \) is not Erdős then \( \tilde{X}_\eta \subsetneq X_B \).

**Corollary 1.8.** Assume that \( B \) is taut and we have \( X_\eta = X_C \) for some \( C \). Then \( B \) is Erdős and \( B = C \).

**Corollary 1.9.** Assume that \( B \) is infinite and \( (X_\eta, S) \) is minimal \(^{19}\). Then \( X_\eta \subsetneq \tilde{X}_\eta \subsetneq X_B \).

2. **Necessary facts from the theory of \( B \)-free systems.** We recall that throughout we assume that \( B \subset \mathbb{N} \) is primitive and that \( 1 \notin B \) (unless otherwise stated).

**2.1. Densities.** Given a subset \( A \subset \mathbb{N} \), its lower density is defined as

\[
\underline{d}(A) = \liminf_{n \to \infty} \frac{1}{n} |A \cap [1, n]|.
\]

When \( \liminf \) is replaced by \( \limsup \), we speak about the upper density \( \overline{d}(A) \). If the lower density equals the upper density of \( A \) then we say that \( A \) has

\(^{18}\) Given a dynamical system \((X, T)\), we denote by \( h(X, T) \) its topological entropy. We recall that for a subshift \((X_\eta, S)\), we have \( h(X_\eta, S) = \lim_{n \to \infty} \frac{1}{n} \log \text{cpx}_\eta(n) \); see footnote 11.

\(^{19}\) Minimal systems are those in which each orbit is dense.
density \( d(A) = d(A) = \overline{d}(A) \). Similarly, we speak about logarithmic densities, in particular, the \textit{logarithmic density} of \( A \) is defined as the limit
\[
\delta(A) := \lim_{n \to \infty} \frac{1}{\log n} \sum_{A \ni j \leq n} \frac{1}{j}
\]
when it exists. Finally, we define the \textit{upper Banach density} of \( A \) as
\[
BD^*(A) := \limsup_{n \to \infty} \frac{1}{n} \max_{m \in \mathbb{N}} |A \cap [m, m + n]|.
\]
It is known (see e.g. [13, Chap. V, §2, Lemma 1]) that if \( A \) has density then it has logarithmic density and
\[
\delta(A) = d(A) \leq BD^*(A)
\]
(the right-hand inequality follows easily).

\textbf{2.2. Introduction to the theory of \( \mathcal{B} \)-free subshifts.} We set \( \eta = \eta_{\mathcal{B}} := \mathbb{1}_{\mathcal{F}_{\mathcal{B}}} \in \{0, 1\}^\mathbb{Z} \) and \( X_\eta := \{S^n \eta : n \in \mathbb{Z}\} \), where \( S \) is the left shift on \( \{0, 1\}^\mathbb{Z} \), and we define a \( \mathcal{B} \)-free subshift as \((X_\eta, S)\). By definition, \( \mathcal{B} \)-free subshifts are transitive. The dynamics of these subshifts varies in a significant way depending on the arithmetic properties of \( \mathcal{B} \): indeed, it varies \((20)\) from proximality \((21)\) to minimality. Both these dynamical properties have arithmetic characterizations: the proximality of \((X_\eta, S)\) is equivalent to \( \mathcal{B} \) containing an infinite coprime subset \([10]\) (it is also equivalent to the all-zero sequence belonging to \(X_\eta [10]\)), while the minimality is equivalent to \( \mathcal{B} \) not containing a rescaled copy of an infinite coprime set \([16]\). Moreover, \((X_\eta, S)\) is minimal if and only if it is a Toeplitz system; in fact, it is equivalent to \( \eta \) itself being a Toeplitz sequence \((22)\) \([18]\).

By the Davenport–Erdős theorem (see e.g. [14 Thm. 0.2]), the set \( \mathcal{F}_{\mathcal{B}} \) has logarithmic density which is equal to its upper density and
\begin{equation}
\delta(\mathcal{F}_{\mathcal{B}}) = \lim_{M \to \infty} d(\mathcal{F}_{\{b \in \mathcal{B} : b \leq M\}}).
\end{equation}
Moreover, if \((N_k)\) is any sequence “realizing” the upper density:
\begin{equation}
\lim_{k \to \infty} \frac{1}{N_k} |\mathcal{F}_{\mathcal{B}} \cap [1, N_k]| = \overline{d}(\mathcal{F}_{\mathcal{B}}),
\end{equation}

\textit{(20)} As proved in [10], each \( \mathcal{B} \)-free system has a unique minimal subset. Proximality corresponds to the smallest possible minimal subset (a fixed point), while minimality corresponds to the largest possible minimal subset.

\textit{(21)} A topological dynamical system \((X, T)\) is \textit{proximal} if for any \( x, y \in X \) there is a sequence \((q_n)\) such that \( d(T^{q_n} x, T^{q_n} y) \to 0 \). Such a system necessarily has a fixed point which is the unique minimal subset.

\textit{(22)} That is, for each \( n \in \mathbb{Z} \) there is \( k_n \in \mathbb{N} \) such that \( \eta(n) = \eta(n + jk_n) \) for each \( j \in \mathbb{Z} \). See e.g. [8] for the theory of Toeplitz systems.
then $\eta$ is generic (23) along $(N_k)$ for the Mirsky measure $\nu_\eta$ (see [10]), which is an invariant measure for the subshift $(X_\eta, S)$. In both classes, proximal and minimal, the entropy can be positive and also zero. In fact, using [21, 17], the following has been noticed in [20]:

**Theorem 2.1** ([20]). If $(X_\eta, S)$ is proximal, then

$$h(X_\eta, S) = \nu_\eta(C_{\{0\},\emptyset}) \log 2,$$

where $C_{\{0\},\emptyset} := \{y \in X_\eta : y(0) = 1\}$.

Note that by the definition of genericity and (13), we have $\nu_\eta(C_{\{0\},\emptyset}) = d(\{n \geq 1 : \eta(n) = 1\})$.

(14) if $B = B_1 \cup B_2$ is Behrend then either $B_1$ or $B_2$ is Behrend.

As the all-zero sequence $0^\infty$ is in $X_\eta$, the corresponding Behrend $B$-free subshift $(X_\eta, S)$ is proximal. Hence, each Behrend set contains an infinite coprime subset and, in view of Theorem 2.1, the entropy of the corresponding $B$-free subshift is zero. In fact, as shown in [10], $\nu_\eta$ is just the Dirac measure $\delta_0$ at the fixed point, and it is the only $S$-invariant measure on $X_\eta$. Remaining in the proximal case, at the other extreme, we have $B$ which are called Erdős, i.e. sets which are infinite, coprime and thin, i.e., $\sum_{b \in B} \frac{1}{b} < +\infty$. As for coprime sets $B$ we have

$$d(F_B) = \nu_\eta(C_{\{0\},\emptyset}) = \prod_{b \in B} \left(1 - \frac{1}{b}\right) > 0$$

(see [11]; see also the earlier article [30]), in the Erdős case, the entropy is positive by Theorem 2.1. A prominent example in this class is the square-free system for which $B = \{p^2 : p \in \mathbb{P}\}$, studied first by Sarnak [30]. Also, in the minimal case, the entropy can be zero [10] or positive [19].

Another classical arithmetic notion in the context of $B$-free sets is that of tautness. Namely, $B$ is taut if for each $b \in B$ the logarithmic density of $F_B$ is strictly smaller than the logarithmic density of $F_{B \setminus \{b\}}$. Each thin set $B$ is taut [10], Behrend sets are not taut (24) so in particular $\sum_{b \in B} \frac{1}{b} = +\infty$

(23) Given a topological system $(X, T)$ and a $T$-invariant measure $\nu$, a point $x \in X$ is called generic for $\nu$ along $(n_k)$ if $\lim_{k \to \infty} \frac{1}{n_k} \sum_{j \leq n_k} f(T^j x) = \int_X f \, d\nu$ for each $f \in C(X)$.

(24) Since no one-element set is Behrend, by [14], Behrend sets are not taut. In fact, tautness is characterized by the fact that $B$ (which is always assumed to be primitive) does not contain a rescaled copy of a Behrend set; see [11].
for Behrend sets, whereas Erdős sets are. Also, as noticed by A. Dymek (see [16]), all $\mathcal{B}$ which yield minimal $\mathcal{B}$-free systems are necessarily taut. We will use the following dynamical characterization of tautness:

**Theorem 2.2** ([17, 21]). $\mathcal{B}$ is taut if and only if $\text{supp}\, \nu_\eta = X_\eta$.

### 2.3. $\mathcal{B}$-free systems as model set systems.

In this section we look at $\mathcal{B}$-free systems from the viewpoint of the theory of model sets; see e.g. [1]. Assume that $\mathcal{B} = \{b_1, b_2, \ldots\}$. Let

$$H := \{(n, n, \ldots) : n \in \mathbb{Z}\} \subset \prod_{k=1}^{\infty} \mathbb{Z}/b_k\mathbb{Z}.$$  

Then $H$ is a compact Abelian group with Haar measure $m_H$. Consider also the translation $T(h_1, h_2, \ldots) = (h_1 + 1, h_2 + 1, \ldots)$ on $H$. This translation is ergodic with respect to $m_H$. Denote

$$W := \{h \in H : h_k \neq 0 \text{ for each } k \in \mathbb{Z}\}.$$  

The *window* $W$ gives us a natural way of coding points $h \in H$ according to the visits of the consecutive elements $T^nh$ either in $W$ or in $W^c$. Formally, let $\varphi : H \to \{0, 1\}^\mathbb{Z}$ be defined by

$$\varphi(h)(n) := \begin{cases} 0 & \text{if } (\exists k \geq 1) \ b_k | n + h_k, \\ 1 & \text{otherwise.} \end{cases}$$  

It is then not hard to see that $\eta = \varphi(0, 0, \ldots)$. From [10], we obtain:

- The Mirsky measure $\nu_\eta$ is equal to $\varphi_*(m_H)$, the image of $m_H$ via $\varphi$.
- $\mathcal{B}$ is Behrend if and only if $m_H(W) = 0$.

We also recall that tautness can be characterized in terms of the properties of the window. Namely, the following has been proved in [16]:

- $\mathcal{B}$ is taut if and only if the window $W$ is Haar regular, i.e. $\text{supp}(m_H|W) = W$.

### 2.4. $\mathcal{B}$-admissible subshifts.

To each $\mathcal{B}$, we can associate another natural subshift $(X_\mathcal{B}, S)$ called the $\mathcal{B}$-admissible subshift,

$$X_\mathcal{B} := \{x \in \{0, 1\}^\mathbb{Z} : \text{supp } x \text{ is $\mathcal{B}$-admissible}\},$$

where $\text{supp } x := \{i \in \mathbb{Z} : x_i = 1\}$ is the support of $x$. We recall that $A \subset \mathbb{Z}$ is $\mathcal{B}$-admissible if for each $b \in \mathcal{B}$, $|A \mod b| < b$. By definition, $b\mathbb{Z}$ is disjoint from $\mathcal{F}_\mathcal{B}$ for every $b \in \mathcal{B}$, whence $X_\eta \subset X_\mathcal{B}$.

It is easy to see that $X_\mathcal{B}$ is hereditary, i.e. if $x \in X_\mathcal{B}$ and $y \leq x$ (coordinatewise) then $y \in X_\mathcal{B}$. Therefore,

$$X_\eta \subset \tilde{X}_\eta \subset X_\mathcal{B},$$

where $\tilde{X}_\eta$ is the hereditary closure of $X_\eta$, i.e. the smallest hereditary subshift containing $X_\eta$. As proved in [11], in the Erdős case, $X_\eta = X_\mathcal{B}$ (in fact, this equality was first proved by Sarnak [30] in the square-free case).
**Remark 2.3.** In general, the three subshifts in \[16\] are different. However, if \( \mathcal{B} \) is finite and coprime then \( \tilde{X}_\eta = X_\mathcal{B} \). Indeed, this follows for example from the equivalence of (ii) and (iii) in Proposition 2.5 of \[11\] which holds in the finite, coprime case \((\text{25})\).

The following observation tells us that in the family of all \( \mathcal{B} \)-admissible subshifts there exists the smallest element.

**Proposition 2.4.** For each \( \mathcal{B} \), we have \( X_P \subset X_\mathcal{B} \).

*Proof.* Suppose that \( A \subset \mathbb{Z} \) is \( P \)-admissible but is not \( \mathcal{B} \)-admissible. It follows that, for some \( b \in \mathcal{B} \) and for each \( r = 0, 1, \ldots, b - 1 \), there exists \( j_r \in \mathbb{Z} \) such that \( r + j_r b \in A \). Take any \( p | b \) and note that \( \{ r + j_r b : r = 0, 1, \ldots, b - 1 \} \) contains all residue classes mod \( p \), a contradiction. \( \blacksquare \)

**Remark 2.5.** In fact, the above argument uses only the following elementary observation: if for every \( b \in \mathcal{B} \), we choose \( 1 < c_b | b \), then \( X_{\{c_b : b \in \mathcal{B}\}} \subset X_\mathcal{B} \).

### 2.5. Examples of Behrend sets.
A natural source of examples of Behrend sets is provided by non-thin coprime sets, namely (cf. (15)):

\[ \text{(17) A coprime set } \mathcal{B} \text{ is Behrend if and only if } \sum_{b \in \mathcal{B}} \frac{1}{b} = +\infty. \]

In particular, all subsets \( \mathcal{P} \subset \mathbb{P} \) satisfying \( \sum_{p \in \mathcal{P}} \frac{1}{p} = +\infty \) are Behrend.

To see other examples, recall that the function \( \Omega : \mathbb{N} \rightarrow \mathbb{N} \) counts, given \( n \in \mathbb{N} \), the number of prime divisors (with multiplicity) of \( n \). For \( k = 0, 1, 2, \ldots \), denote by \( \mathbb{P}_k \) the set of \( k \)-almost prime numbers \((26)\), that is,

\[ \mathbb{P}_k := \{ n \in \mathbb{Z} : \Omega(n) = k \}. \]

Hence \( \mathbb{P}_0 = \{1, -1\} \), \( \mathbb{P}_1 = \mathbb{P} \cup (-\mathbb{P}) \), i.e. the set of primes, \( \mathbb{P}_2 \) is the set of semi-primes, etc. It is not hard to see that, for each \( k \geq 1 \), we have

\[ \mathcal{F}_{\mathbb{P}_k} = \bigcup_{\ell < k} \mathbb{P}_\ell. \]

All the sets \( \bigcup_{\ell < k} \mathbb{P}_\ell \) have zero density, so all the sets \( \mathbb{P}_k \), \( k \geq 1 \), are Behrend sets. In particular,

\[ X_{\bigcup_{\ell < k} \mathbb{P}_\ell} \subset X_{\mathbb{P}_k}. \]

Moreover, the non-isolated points of \( X_{\mathbb{P}_k} \) belong to \( X_{\mathbb{P}_k} \); see Remark 3.5

\((25)\) A simple argument was pointed out by J. Kulaga-Przymus: For each \( n \in \mathbb{Z} \), \( \eta(m + n) = 0 \) if and only if \( m \in \bigcup_{b \in \mathcal{B}} (b\mathbb{Z} - n) = \bigcup_{b \in \mathcal{B}} (b\mathbb{Z} - (n \text{ mod } b)) \); since, by the Chinese Remainder Theorem, \( (n \text{ mod } b)_{b \in \mathcal{B}} \) realizes any configuration \( (r_b)_{b \in \mathcal{B}} \) of residue classes mod \( b \), \( b \in \mathcal{B} \), the claim follows.

\((26)\) The number of \( k \)-almost prime numbers less than \( n \) is \( (1 + o_k(1)) \frac{n \log \log n}{(k-1)! \log n} \); see \(85\)
For example, the semi-primes-admissible subshift contains the subshift of primes, i.e. $X_{\mathbb{P} \cup (-\mathbb{P})}$.

Note also the result below which, for a general Behrend set $\mathcal{B}$, yields new Behrend sets by an iterative procedure.

**Proposition 2.6.** The set $\{bc : b, c \in \mathcal{B}, b \neq c\}$ is Behrend whenever $\mathcal{B}$ is Behrend.

**Proof.** If $F \subset \mathbb{N}$ then we denote by $\text{spec}(F)$ the set of all prime divisors of elements of $F$.

To prove our claim, first note that if $\mathcal{B}$ is Behrend, then for each finite set $Q \subset \mathbb{P}$, the set $\mathcal{B}' := \{b \in \mathcal{B} : \text{spec}(b) \cap Q = \emptyset\}$ is also Behrend. Indeed, we have $\mathcal{B} = \mathcal{B}' \cup \bigcup_{q \in Q}(\mathcal{B} \cap q\mathbb{Z})$. Since the union is finite, in view of (14) at least one of the sets in the union is Behrend but no one of the sets $\mathcal{B} \cap q\mathbb{Z}$ is Behrend.

Let us now select a finite set $S \subset \mathcal{B}$ such that $\mathcal{M}_S$ has density close to 1 (here we use (12)). Let $\mathcal{B}'$ be the set of $b \in \mathcal{B}$ which are coprime to all elements of $S$; this set is Behrend in view of the first part of the proof. Choose a finite set $S' \subset \mathcal{B}'$ such that $\mathcal{M}_{S'}$ has density close to 1. The elements from $S$ are coprime to the elements of $S'$, whence the density of $\mathcal{M}_S \cdot S'$ is equal to the product of the densities of $\mathcal{M}_S$ and $\mathcal{M}_{S'}$ (see [10, Lemma 4.21]), so it is close to 1. As $S \cdot S' \subset \{bc : b, c \in \mathcal{B}, b \neq c\}$, the result follows. ■

3. Transitivity of $\mathcal{B}$-admissible subshifts. Proof of Theorem [1.5]. Let us first discuss a certain reduction of the problem of transitivity of $(X_{\mathcal{B}}, S)$. Given two finite sets $A, B \subset \mathbb{Z}$, set

$$C_{A,B} := \{x \in X_{\mathcal{B}} : x(n) = 1 \text{ for each } n \in A, x(n) = 0 \text{ for each } n \in B\}.$$  

If $A$ and $B$ are not disjoint then $C_{A,B} = \emptyset$. Note that $C_{A,\emptyset} \neq \emptyset$ if and only if $A$ is $\mathcal{B}$-admissible, and furthermore $C_{A,B} \neq \emptyset$ if and only if $C_{A,\emptyset} \neq \emptyset$ by the definition of $\mathcal{B}$-admissibility. We have

$$S^{-n}C_{A,B} = C_{A+n,B+n}, \quad C_{A,B} \cap C_{A',B'} = C_{A \cup A',B \cup B'},$$  

so in order for this last set to be non-empty, we must have $(A \cup A') \cap (B \cup B') = \emptyset$. Note also that if we aim at showing that $S^{-n}C_{A,B} \cap C_{A',B'} \neq \emptyset$, equivalently that $C_{(A+n) \cup A',(B+n) \cup B'} \neq \emptyset$, we have to show that $C_{(A+n) \cup A',\emptyset} \neq \emptyset$ and that the sets $(A + n) \cup A'$ and $(B + n) \cup B'$ are disjoint, the latter holding if $n$ is large enough. As the transitivity of $(X_{\mathcal{B}}, S)$ is equivalent to “for each finite, disjoint $A, B \subset \mathbb{Z}$, $A', B' \subset \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such

(27) Note that this set is primitive if $\mathcal{B}$ is coprime; in general, we consider the primitive basis of this set.
that $S^{-n}C_{A,B} \cap C_{A',B'} \neq \emptyset$, using the above observations, we obtain the following:

**Lemma 3.1.** $(X_{\mathcal{B}}, S)$ is transitive if and only if for each finite, $\mathcal{B}$-admissible sets $A, A' \subset \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such that $(A + n) \cup A'$ is $\mathcal{B}$-admissible.

**Proof of sufficiency in Theorem 1.5.** Fix finite, $\mathcal{B}$-admissible sets $A_1, A_2 \subset \mathbb{Z}$. By Lemma 3.1 we need to show that there is $n \in \mathbb{Z}$ such that
\[(A_1 + n) \cup A_2 \text{ is } \mathcal{B}\text{-admissible.}\]

Let $M := |A_1| + |A_2|$ and let $b_1, \ldots, b_k$ be all elements of $\mathcal{B}$ which are $\leq M$. We only need to check admissibility with respect to $b_1, \ldots, b_k$. Let $r_{1,j}$ and $r_{2,j}$ be missing residue classes mod $b_j$ in $A_1$ and $A_2$, respectively (perhaps there are more residue classes missing, of course). By the Chinese Remainder Theorem, we can find $n \in \mathbb{Z}$ such that
\[r_{1,j} + n = r_{2,j} \text{ mod } b_j, \quad j = 1, \ldots, k.\]

Now, in the sets $A_1 + n$ and $A_2$ the class $r_{2,j}$ is missing mod $b_j$, whence (19) holds. $\blacksquare$

### 3.1. Proof of the necessity in Theorem 1.5

**Lemma 3.2.** Assume that $M \in \mathbb{N}$ and $q_1, \ldots, q_N \in \mathbb{Z}$ for some $N \in \mathbb{N}$. Then, for any natural numbers $c_1, \ldots, c_M > NM$, there exists $q \in \mathbb{Z}$ such that
\[\bigcup_{i=1}^{M} (c_i \mathbb{Z} + q) \cap \{q_1, \ldots, q_N\} = \emptyset.\]

**Proof.** The density of the set
\[\bigcup_{j=1}^{N} \bigcup_{i=1}^{M} (c_i \mathbb{Z} - q_j)\]
does not exceed $N \sum_{i=1}^{M} \frac{1}{c_i} < 1$ as $c_i > NM$ for every $i$. Now, it is enough to take $q$ outside of this set. $\blacksquare$

To complete the proof of Theorem 1.5, it is enough to prove the following result.

**Theorem 3.3.** Let $\mathcal{C} \subset \mathbb{N}$ be primitive. Assume that for any finite $\mathcal{C}$-admissible sets $A_1, A_2 \subset \mathbb{Z}$ there exists $m \in \mathbb{Z}$ such that $A_1 \cup (A_2 + m)$ is $\mathcal{C}$-admissible. Then $\mathcal{C}$ is coprime.

**Proof.** Assume that $\mathcal{C}$ is not coprime and let $b \in \mathcal{C}$ be the smallest element such that $\gcd(a, b) > 1$ for some $a \in \mathcal{C}, b \neq a$. Fix such an $a$. Let
\[(20) \quad T = \{(r, t) : r \in \{1, \ldots, \lcm(a, b)\}, \quad t \in \{0, \ldots, r - 1\}, \gcd(r, t) \in \mathcal{F}_{\mathcal{C} \setminus \{a\}}\} \quad (28).\]
We define the number

\[ \Sigma_T = \sum_{(r,t) \in T} \sigma_0(\gcd(r,t)), \]

where, given a number \( m \), \( \sigma_0(m) \) denotes the number of divisors of \( m \).

For any \((r,t) \in T\), let \( q_{r,t} \in r\mathbb{Z} + t \) be an element such that \( q_{r,t} = \gcd(r,t)p_{r,t} \) and \( p_{r,t} \) is a prime number not dividing \( a \) and such that \( p_{r,t} > |T| \cdot \Sigma_T \). The existence of \( q_{r,t} \) is a consequence of the Dirichlet Theorem (applied to \( \frac{r}{\gcd(r,t)}\mathbb{Z} + \frac{t}{\gcd(r,t)} \)).

Let

\[ \mathcal{C}_0 = \{ c \in \mathcal{C} \setminus \{a\} : c | q_{r,t} \text{ for some } (r,t) \in T \}. \]

Any element \( c \in \mathcal{C} \setminus \{a\} \) dividing \( q_{r,t} \) for some \((r,t) \in T\) is equal to \( lp_{r,t} \) for a divisor \( l \) of \( \gcd(r,t) \). Indeed, if \( c | q_{r,t} = p_{r,t} \gcd(r,t) \) then, as \( \gcd(r,t) \in \mathcal{F}_{\mathcal{C}\setminus\{a\}} \), \( c \) is not coprime to \( p_{r,t} \). Since \( p_{r,t} \) is a prime, it follows that \( p_{r,t} | c \) and \( l := \frac{c}{p_{r,t}} | \gcd(r,t) \). Thus

\[ |\mathcal{C}_0| \leq \Sigma_T \]

and, as \( p_{r,t} > |T| \cdot \Sigma_T \),

\[ c > |T| \cdot \Sigma_T \quad \text{for every } c \in \mathcal{C}_0. \]

By Lemma 3.2 applied to \( \{q_1, \ldots, q_N\} = \{q_{r,t} : (r,t) \in T\} \) and \( \{c_1, \ldots, c_M\} = \mathcal{C}_0 \), thanks to (23) and (24), there exists \( q \in \mathbb{Z} \) such that

\[ (\bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q)) \cap \{q_{r,t} : (r,t) \in T\} = \emptyset. \]

Let \( n = \max \{q_{r,t} : (r,t) \in T\} \).

We claim that if \( r \leq \text{lcm}(a,b) \) is a natural number and \( t \in \mathbb{Z} \) then the implication

\[ [-n,n] \cap (r\mathbb{Z} + t) \subseteq \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q) \quad \Rightarrow \quad \gcd(r,t) \in \mathcal{M}_{\mathcal{C}'} \]

holds for \( \mathcal{C}' = \mathcal{C} \) and for \( \mathcal{C}' = \mathcal{C} \setminus \{a\} \). Indeed, assume that

\[ [-n,n] \cap (r\mathbb{Z} + t) \subseteq \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q). \]

Without loss of generality we can assume that \( t \in \{0, \ldots, r-1\} \). If \( (r,t) \notin T \) then \( \gcd(r,t) \in \mathcal{M}_{\mathcal{C}\setminus\{a\}} \subseteq \mathcal{M}_{\mathcal{C}'} \). It remains to consider the case \((r,t) \in T\).

Then \( q_{r,t} \in [-n,n] \cap (r\mathbb{Z} + t) \) by the choice of \( n \). Then (25) and (27) yield

\[ q_{r,t} \in \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0} c\mathbb{Z}. \]

If \( \mathcal{C}' = \mathcal{C} \setminus \{a\} \), this leads to a contradiction with the definition of \( \mathcal{C}_0 \). So we must have \( \mathcal{C}' = \mathcal{C} \) and we conclude that \( a | q_{r,t} = \gcd(r,t)p_{r,t} \). Since \( p_{r,t} \)
does not divide $a$, we get $a \mid \gcd(r, t)$, thus $\gcd(r, t) \in \mathcal{M}_C'$, for $\mathcal{C}' = \mathcal{C}$. The claim (26) follows.

We set
\[
A_1 = [-n, n] \setminus \left( \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q) \right),
\]
\[
A_2 = [-n, n] \setminus \left( (a\mathbb{Z} + 1) \cup \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0; c \neq a} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q) \right).
\]
Clearly, the sets $A_1$, $A_2$ are $\mathcal{C}$-admissible.

We are going to prove that $A_1 \cup (A_2 + m)$ is not $\mathcal{C}$-admissible for any $m \in \mathbb{Z}$. For contradiction, suppose that $m \in \mathbb{Z}$ is such that $A_1 \cup (A_2 + m)$ is $\mathcal{C}$-admissible. Then there exists $t \in \mathbb{Z}$ such that $a\mathbb{Z} + t$ is disjoint from $A_1 \cup (A_2 + m)$. Then $(a\mathbb{Z} + t) \cap A_1 = \emptyset$, equivalently
\[
(a\mathbb{Z} + t) \cap [-n, n] \subseteq \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q),
\]
and hence $c \mid \gcd(a, t)$ for some $c \in \mathcal{C}$ by (26). Since $\mathcal{C}$ is primitive, $c = a$ and $a \mid t$. Consequently, $a\mathbb{Z} + t = a\mathbb{Z}$.

Since $a\mathbb{Z} \cap (A_2 + m) = \emptyset$, it follows that $(a\mathbb{Z} - m) \cap A_2 = \emptyset$, equivalently
\[
(a\mathbb{Z} - m) \cap [-n, n] \subseteq (a\mathbb{Z} + 1) \cup \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0; c \neq a} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q).
\]
If we have
\[
(a\mathbb{Z} - m) \cap [-n, n] \subseteq \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0; c \neq a} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q)
\]
then, by (26), $c \mid \gcd(a, m)$ for some $c \in \mathcal{C}' \setminus \{a\}$. This leads to a contradiction, since $\mathcal{C}$ is primitive. Thus $(a\mathbb{Z} - m) \cap (a\mathbb{Z} + 1) \neq \emptyset$, hence
\[
(28) \quad a \mid m + 1.
\]
Since $\gcd(a, b) > 1$, we conclude that
\[
(29) \quad b \text{ does not divide } m.
\]

Since $A_1 \cup (A_2 + m)$ is $\mathcal{C}$-admissible, there exists $t' \in \mathbb{Z}$ such that $b\mathbb{Z} + t'$ is disjoint from $A_1 \cup (A_2 + m)$. Then $(b\mathbb{Z} + t') \cap A_1 = \emptyset$ and, as above, we prove that $b\mathbb{Z} + t' = b\mathbb{Z}$. Moreover, $b\mathbb{Z} \cap (A_2 + m) = \emptyset$, which is equivalent to
\[
(30) \quad (b\mathbb{Z} - m) \cap [-n, n] \subseteq (a\mathbb{Z} + 1) \cup \bigcup_{c \in \mathcal{C}' \setminus \mathcal{C}_0; c \neq a} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q).
\]
Furthermore, the numbers $jb - m$ are not in $a\mathbb{Z} + 1$ for $j = 1, \ldots, L - 1$ with $L := \text{lcm}(a, b)/b$. Indeed, if $jb - m = ax + 1$ then, by (28), we obtain $a \mid jb$, which is impossible for $j = 1, \ldots, L - 1$. Moreover, $\text{lcm}(a, b)\mathbb{Z} - m \subset a\mathbb{Z} + 1,$
again by (28). It follows that
\[(31) \quad (b\mathbb{Z} - m) \setminus (a\mathbb{Z} + 1) = \bigcup_{k=1}^{L-1} (\text{lcm}(a, b)\mathbb{Z} - m + kb).\]

Indeed, let \(x \in (b\mathbb{Z} - m) \setminus (a\mathbb{Z} + 1).\) Then \(b \mid x + m.\) There exist \(t \in \mathbb{Z}\) and \(k \in \{0, 1, \ldots, L - 1\}\) such that
\[
\begin{align*}
x + m/b &= tL + k.
\end{align*}
\]
Then \(x = t\text{lcm}(a, b) + kb - m\) as \(L = \frac{\text{lcm}(a, b)}{b}.\) Observe that \(x - 1\) is not divisible by \(a\) by assumption, hence \(k \neq 0\) because of (28). The right-hand side in (31) is obviously contained in \(b\mathbb{Z} - m,\) and if \(t\text{lcm}(a, b) - m + kb \in a\mathbb{Z} + 1,\) then \(a \mid kb,\) which we have already noticed to be impossible.

By (31) and (30), for any \(k \in \mathbb{Z}\) not divisible by \(L,\) we have
\[
(32) \quad (\text{lcm}(a, b)\mathbb{Z} - m + kb) \cap [-n, n] \subseteq \bigcup_{c \in \mathcal{C} \setminus \{a\}} c\mathbb{Z} \cup \bigcup_{c \in \mathcal{C}_0} (c\mathbb{Z} + q).
\]

Since \(L > 1,\) there exists \(k \in \mathbb{Z}\) not divisible by \(L\) such that
\[
(33) \quad |kb - m| \leq b.
\]

By (26) (applied to \(r = \text{lcm}(a, b), t = kb - m\) and \(\mathcal{C}' = \mathcal{C} \setminus \{a\}\)) and (32), there exists \(c \in \mathcal{C} \setminus \{a\}\) such that
\[
c \mid \text{gcd}(\text{lcm}(a, b), kb - m).
\]
In particular, \(c \mid kb - m.\) By (29), \(kb - m \neq 0.\) Then \(c \leq b\) by (33). If \(c = b\) then \(b \mid m\) and we have a contradiction with (29). Thus \(c < b.\) Assume that \(\text{gcd}(a, c) = 1.\) Then, as \(c \mid \text{lcm}(a, b),\) we get \(c \mid b,\) again a contradiction, since \(\mathcal{C}\) is primitive. Thus \(\text{gcd}(a, c) > 1,\) a contradiction with the choice of \(b,\) as \(a \neq c < b.\)

3.2. Some consequences

**Proximality and characterization of the Erdős case.** It has already been noticed in [20] that whenever \((X_\eta, S)\) is proximal we have
\[
(34) \quad h(X_\eta, S) = d \log 2,
\]
where \(d\) stands for the density of 1 in \(\eta,\) i.e., \(d = \nu_\eta(C_{\{0,1\}}).\) We recall that in the Erdős case and in the Behrend case, the corresponding \(\mathcal{B}\)-free subshifts are proximal.

**Proof of Corollary 1.6.** (i) The “only if” part is clear. If \(X_\eta = X_\mathcal{B},\) then by definition, \((X_\eta, S)\) is transitive and the claim follows from Theorem 1.5 and (34).

(ii) follows from (34).
Tautness and hereditary closure. We recall that the tautness of $B$ is equivalent to the Mirsky measure $\nu_\eta$ having full support \cite{17, 21}. Hence, if $B$ is taut then each block appearing in $\eta$ reappears infinitely often. Recall also that $(X_\eta, S)$ is transitive if and only if for each block $B$ appearing in $\eta$ there is $B' \geq B$ also appearing in $\eta$ such that $B'$ reappears infinitely many times in $\eta$; see \cite{10} Proposition 3.17. It follows that if $B$ is taut then its hereditary closure is transitive.

Proof of Corollary 1.7. This follows directly from Theorem 1.5.

Of course all non-coprime thin sets $B$ satisfy the above assumption so their hereditary closure is a proper subshift of $X_B$; cf. \cite{10} where this is shown for the case of abundant numbers.

Proof of Corollary 1.8. By Theorem 1.5, $C$ is coprime.

(a) If $C$ is Behrend then the entropy of $(X_C, S)$ is zero, and so is $h(X_\eta, S)$. But $(X_\eta, S)$ is proximal (since the all zero sequence is in $X_C = X_\eta$), so its entropy is the density of 1 in $\eta$, which means that $B$ is Behrend, so it is not taut, a contradiction.

(b) If $C$ is finite then the measure of maximal density in $X_C$ is periodic (it is the Mirsky measure given by $C = \{c_1, \ldots, c_m\}$). It must then be equal to $\nu_\eta$. This periodic measure in $X_C$ does not have full support, so $\nu_\eta$ does not have full support in $X_\eta$, which contradicts the tautness of $B$.

(c) We now know that $C$ is Erdős. It follows that $X_C = X_{\eta_C}$, where $\eta_C = 1_{F_C}$, and, by our assumption,

$$X_{\eta_C} = X_{\eta}.$$ 

Since now $C$ and $B$ are both taut, the assertion follows from \cite{10}.

Minimal and three subshifts

Proof of Corollary 1.9. Clearly, $X_\eta$ is not minimal (it has a fixed point). Moreover, $(X_\eta, S)$ is still transitive, since $(X_\eta, S)$ is a Toeplitz system by \cite{10} (in fact, by \cite{17}, $\eta$ itself is a Toeplitz sequence), while $(X_B, S)$ is not transitive.

3.3. Isolated points

PROPOSITION 3.4. For any $B$, the $B$-admissible subshift $(X_B, S)$ has no isolated points.

Proof. First note that if $y \in X_B$ and $\text{supp}(y)$ is infinite then we can approximate $y$ by $y^{(n)} \in X_B$, where $y^{(n)}$ arises from $y$ by changing one 1 into 0 at a position $k_n$, where $k_n \rightarrow \infty$. It follows that such a $y$ is not isolated.

So the problem is to show that finite support sequences are not isolated points in $X_B$. To see this we need to show that if $A \subset \mathbb{Z}$ is finite and $B$-admissible, then there exists $m$ arbitrarily large such that $A \cup \{m\}$ is still
\(\mathcal{B}\)-admissible. Let \(\mathcal{B}' = \{b_1, \ldots, b_n\} = \{b \in \mathcal{B} : b \leq |A| + 1\}\). It is not hard to see that if we take \(a \in A\) and consider \(m = a + xb_1 \ldots b_n\) then \(A \cup \{m\}\) is still \(\mathcal{B}'\)-admissible. For \(b \in \mathcal{B} \setminus \mathcal{B}'\), we have \(b > |A| + 1\), and so \(A \cup \{m\}\) misses some residue class modulo \(b\). Therefore, \(A \cup \{m\}\) is \(\mathcal{B}\)-admissible. ■

In contrast to \((X_{\mathcal{B}}, S)\), the \(\mathcal{B}\)-free subshift \((X_\eta, S)\) often has isolated points. Recall that if we consider \(\mathcal{B} = \{pq : p, q \in \mathbb{P}\}\) (the set of semi-primes), then \(X_\eta\) is the subshift of prime numbers. Hence \(X_{\mathbb{P} \cup \{1\}} \subset X_{\{pq : p, q \in \mathbb{P}\}}\). The subshift on the right-hand side is uncountable (all admissible subshifts are uncountable [10]) and it is an open problem whether the subshift on the left-hand side is uncountable. We might ask whether these two subshifts are equal, but this is not the case, as the subshift on the right has no isolated points by Proposition 3.4, while the subshift on the left clearly has such points. Indeed, in \(\eta = \mathbb{1}_{\mathbb{P} \cup \{1\}}\) there are blocks that reappear only finitely many times (e.g. the block \(29\) 11 has this property); see [10].

**Remark 3.5.** For each (primitive) set \(\mathcal{B}\), we have

\[(X_{\mathbb{1}_{\mathcal{B}\cup\{1\}}})_0 \subset X_{\mathcal{B}}.\]

Indeed, if a block \(C\) appears in \(y \in (X_{\mathbb{1}_{\mathcal{B}\cup\{1\}}})_0\), then it appears infinitely often in \(\mathbb{1}_{\mathcal{B}}\). Then the sets \(\text{supp}(C) + k\) appear as the supports of the block \(C\) for infinitely many \(k\). However, if \(|k|\) is large enough then \(\text{supp}(C) + k\) is \(\mathcal{B}\)-admissible, since \(\mathcal{B}\) is primitive (if \(\text{supp}(C) = \{b_{i_1}, \ldots, b_{i_r}\} \subset \mathcal{B}\) is not \(\mathcal{B}\)-admissible then for some \(b \in \mathcal{B}\), \(\text{supp}(C) \mod b\) gives all residue classes mod \(b\). If for some \(k \geq 1\), also \(\text{supp}(C) + k \subset \mathcal{B}\), then for some \(1 \leq j \leq r\) we must have \(b_{i_j} + k = 0 \mod b\), that is, \(b | b_{i_j} + k \in \mathcal{B}\); by primitivity, \(b_{i_j} + k = b\), so \(k\) stays bounded).

### 3.4. Minimal complexity of \(\mathcal{B}\)-admissible subshifts

As we have already mentioned, \(\mathcal{B}\)-admissible subshifts are uncountable (this also follows from Proposition 3.4). Another observation (which is a consequence of Proposition 2.4) is that the complexity of such subshifts is always superpolynomial, in fact, Proposition 1.4 holds:

**Proof of Proposition 1.4.** As \(X_{\mathbb{P}} \subseteq X_{\mathcal{B}}\) (Proposition 2.4), it is enough to prove the assertion for \(\mathcal{B} = \mathbb{P}\). Let us first observe that, for any \(N\), the set \((N, 2N] \cap \mathbb{P}\) is \(\mathbb{P}\)-admissible (and so every subset of it is \(\mathbb{P}\)-admissible). Indeed, denote the intersection \((N, 2N] \cap \mathbb{P}\) as \(p_1 < \cdots < p_k\). If now \(p > N\) then the cardinality of this set is strictly smaller than \(p\). So take \(p \leq N\). Then the zero residue class modulo \(p\) is not encountered in \(\{p_1, \ldots, p_k\}\). It follows that the set \(\{p_1, \ldots, p_k\}\) is \(\mathbb{P}\)-admissible.

\(29\) We denote a 0-1-sequence \((x_1, \ldots, x_n)\) by \(x_1 \ldots x_n\).
Moreover (see (51)),

\[(36) \quad |[N+1, 2N] \cap \mathbb{P}| = \pi(2N) - \pi(N) = \frac{N}{\log N}(1 + o(1)).\]

It follows that the lower bound of the number of \(\mathbb{P}\)-admissible blocks of length \(N\) is 
\[2^{\frac{N}{\log N}(1 + o(1))} = (2 + o(1))^{\frac{N}{\log N}}. \]  

**Remark 3.6.** Note that the argument used in the proof of Proposition 1.4 can be applied in the general context. Recall that \(\mathcal{B}\) is always assumed to be primitive. Let 
\[e_N := |(N, 2N] \cap \mathcal{B}|, \quad N \geq 1.\]

Note that the set \((N, 2N] \cap \mathcal{B}\) is always \(\mathcal{B}\)-admissible since, whenever \(b \leq N\), \(b\) cannot divide any number in \((N, 2N] \cap \mathcal{B}\) by primitivity. Hence, we obtain
\[(37) \quad \text{cpx}_{X_{\mathcal{B}}} (n) \geq 2^{e_n}.\]

This argument applied to the set \(\mathbb{P}_k\) of \(k\)-almost prime numbers yields
\[\text{cpx}_{\mathbb{P}_k} (n) \geq 2^{\frac{n(n \log n)^{k-1}}{(k-1)! \log n (1+o(1))}}, \quad n \geq 1.\]

However, this lower bound of the complexity via the density of the set \(\mathcal{B}\) itself is rather weak for sets \(\mathcal{B}\) which are very sparse. If \(\mathcal{B} = \{p^2 : p \in \mathbb{P}\}\) then \(e_n/n \to 0\), while the entropy of \(X_{\mathcal{B}}\) is positive (as \(\mathcal{B}\) is Erdős).

Remembering that we consider only sets \(\mathcal{B}\) which are primitive, the method which we used above yields the following classical result:

**Corollary 3.7.** Each Behrend set \(\mathcal{B}\) has upper Banach density zero.

**Proof.** Let us call an interval \([M, M+N]\) “good” if \(N \leq M\). Suppose that \(BD^*(\mathcal{B}) = c > 0\). Let us notice that for \(\varepsilon > 0\) small enough if for an interval \([M, M+N]\) we have \(|[M, M+N] \cap \mathcal{B}| \geq (1-\varepsilon)cN\) then dividing \([M, M+N]\) into small consecutive intervals of length \(x\) (with \(x \to \infty\) when \(N \to \infty\)), we find that at least one interval \([M + \ell_0 x, M + (\ell_0 + 1)x]\) with \(\ell_0 \geq 2\) is “good” and
\[|[M + \ell_0 x, M + (\ell_0 + 1)x] \cap \mathcal{B}| \geq (1-\varepsilon)\frac{c}{2} x.\]

As now \([M + \ell_0 x, M + (\ell_0 + 1)x] \cap \mathcal{B}\) is \(\mathcal{B}\)-admissible (since \(\mathcal{B}\) is primitive), the subshift \(X_{\mathcal{B}}\) contains at least \(2^{(1-\varepsilon)\frac{c}{2}} \mathcal{B}\)-admissible blocks of length \(x\), and clearly the entropy of the subshift \((X_{\mathcal{B}}, S)\) is at least \((1-\varepsilon)\frac{c}{2} > 0\), which is in contradiction with the fact that Behrend \(\mathcal{B}\)-admissible subshifts are of entropy zero \([10]\).  

\(^{(30)}\) Another “dynamical” proof can be obtained by making use of two observations: (i) \(\mathcal{B} \subset \mathcal{F}_{\{bc : b, c \in \mathcal{B}, b \neq c\}}\) for each set \(\mathcal{B}\); (ii) the set \(\{bc : b, c \in \mathcal{B}, b \neq c\}\) is Behrend if \(\mathcal{B}\) was; see Proposition 2.6. Then, we finish the proof in the same way, noticing that \(\mathcal{F}_{\{bc : b, c \in \mathcal{B}, b \neq c\}}\) has upper Banach density 0 in view of \([10]\).
Remark 3.8. The above proof also shows that in order to optimize the lower bound of the complexity in Remark 3.6 instead of numbers \( e_N \), it is better to consider the numbers \( |[M_n, M_n + N_n] \cap B| \), where \((M_n, N_n)\) is a sequence “realizing” the upper Banach density of \( B \).

4. Behrend set which yields a \( B \)-free subshift that coincides with the \( B \)-admissible shift: Proof of Theorem 1.3

Lemma 4.1. Let \( m \in \mathbb{N} \). Assume that \( C \) is a finite coprime set containing at least \( m \) elements greater than \( m \). Then every \( C \)-admissible block of length \( m \) appears in \( \eta_C = 1_{F_C} \).

Proof. Let \( C' = \{ c \in C : c \leq m \} \) and \( C'' = \{ c \in C : c > m \} \). Let \( B = x_1 \ldots x_m \) be an admissible block. By the admissibility of \( B \), for every \( c \in C' \) there exists \( r_c \in \{1, \ldots, m\} \) such that \( x_j = 0 \) for every \( j = 1, \ldots, m \) satisfying \( j \equiv r_c \mod c \).

We define \( r_c \in \{0,1,\ldots,m\} \) for \( c \in C'' \) in the following way. We set \( r_c = 0 \) for every \( c \in C'' \) in the case \( B = 11 \ldots 1 \). Otherwise, we can choose \( r_c \in \{1, \ldots, m\} \) in such a way that
\[
\{ j \in \{1, \ldots, m\} : x_j = 0 \} = \{ r_c : c \in C'' \}.
\]
This is possible since \( |C''| \geq m \).

Note that \( x_{r_c} = 0 \) whenever \( r_c \geq 1 \), for every \( c \in C \).

By the Chinese Remainder Theorem there exists \( n \in \mathbb{N} \) such that
\[
(38) \quad n \equiv -r_c \mod c
\]
for every \( c \in C \). We show that \( \eta_C[n + 1, n + m] = B \).

If \( x_j = 0 \) then \( j = r_c \) for some \( c \in C'' \subseteq C \). Then \( c \mid n + j \) by (38) and \( \eta_C[n + j] = 0 \).

Assume that \( \eta_C[n + j] = 0 \), that is, \( c \mid n + j \) for some \( c \in C \). By (38), this is equivalent to \( j \equiv r_c \mod c \). If \( c \in C' \) then \( x_j = 0 \) by the choice of \( r_c \). If \( c \in C'' \) then \( c > m \) and since \( 1 \leq j \leq m \), \( 0 \leq r_c \leq m \), it follows that \( r_c = j \geq 1 \), thus \( x_j = 0 \). ■

Now, we turn to the proof of Theorem 1.3

Proof of Theorem 1.3. We fix \( 0 < \varepsilon < 1 \). We construct sequences \( 2 = N_0 < N_1 < \cdots \) and \( M_1 < M_2 < \cdots \) of numbers and a sequence \( \{2\} = B_0 \subset B_1 \subset \cdots \) of finite sets of prime numbers such that for every \( l \in \mathbb{N} \cup \{0\} \),

(i) \( l \) \( N_l \geq l + 1 \),
(ii) \( l \) \( B_l \subseteq [1, N_l] \cap \mathbb{P} \),
(iii) \( l \) \( B_l \) contains at least \( l + 1 \) elements greater than \( l + 1 \),
(iv) \( l \) \( \prod_{p \in B_l} \left(1 - \frac{1}{p}\right) \leq \varepsilon^l \),
(v) \( l \) \( B_l \cap [1, N_{l-1}] = B_{l-1} \) if \( l > 0 \),
(vi)$_l$ every $\mathcal{B}_l$-admissible block of length $l$ appears in $\eta_l[1, N_l]$, where
\[ \eta_l = \frac{1}{\mathcal{F}_{\mathcal{B}_l}}. \]
(vii)$_l$ $N_{l-1} < M_l < N_l$ if $l > 0$,
(viii)$_l$ $\text{lcm}(\mathcal{B}_l) | M_{l+1} - N_l$ and $\frac{\rho(M_{l+1} - N_l)}{M_{l+1} - N_l} \leq d(\mathcal{F}_{\mathcal{B}_l})$ if $l > 0$.

Observe that with $N_0 = 2$ and $\mathcal{B}_0 = \{2\}$ conditions (i)$_0$–(viii)$_0$ are satisfied. Assume that $l \geq 0$ and $2 = N_0 < N_1 < \cdots < N_l$, $M_1 < \cdots < M_l$ and $\{2\} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_l$ have been defined and conditions (i)$_l$–(viii)$_l$ are satisfied. We will define $M_{l+1}$, $\mathcal{B}_{l+1}$ and $N_{l+1}$.

Since, by (iii)$_l$, $\mathcal{B}_l$ is a finite set of prime numbers containing at least $l+1$ elements greater than $l+1$, it follows by Lemma 4.1 that every $\mathcal{B}_l$-admissible block of length $l+1$ appears in $\eta_l$. Let $M_{l+1} > N_l$ be a number large enough that every $\mathcal{B}_l$-admissible block of length $l+1$ appears in $\eta_l[1, M_{l+1}]$. Since $\rho(n)/n \searrow 0$ as $n \to +\infty$ and $d(\mathcal{F}_{\mathcal{B}}) > 0$, we can choose $M_{l+1}$ such that condition (vii)$_{l+1}$ is satisfied.

There exist prime numbers $p_1 < \cdots < p_T$ greater than $M_{l+1}$ and such that
\[ \prod_{i=1}^{T} \left( 1 - \frac{1}{p_i} \right) \leq \varepsilon. \] (39)

We can also assume that $T \geq l + 2$.

We set $N_{l+1} = p_T$ and $\mathcal{B}_{l+1} = \mathcal{B}_l \cup \{p_1, \ldots, p_T\}$. Then condition (vii)$_{l+1}$ is satisfied. Observe that
\[ [1, M_{l+1}] \cap \mathcal{B}_{l+1} = \mathcal{B}_l. \] (40)

We show that conditions (i)$_{l+1}$–(viii)$_{l+1}$ hold. Indeed:

(i)$_{l+1}$ Clear by the construction since $N_{l+1} > M_l \geq N_l$ and $N_l \geq l + 1$ by the induction hypothesis (i)$_l$.

(ii)$_{l+1}$ Follows by the choice of $\mathcal{B}_{l+1}$ and $N_{l+1}$.

(iii)$_{l+1}$ Follows since $T \geq l + 2$ and $p_1 > M_{l+1} \geq N_l \geq l + 1$.

(iv)$_{l+1}$ We write
\[ \prod_{p \in \mathcal{B}_{l+1}} \left( 1 - \frac{1}{p} \right) = \prod_{p \in \mathcal{B}_l} \left( 1 - \frac{1}{p} \right) \cdot \prod_{i=1}^{T} \left( 1 - \frac{1}{p_i} \right). \]

The assertion follows from the induction hypothesis (iv)$_l$ and (39).

(v)$_{l+1}$ Follows by (40) and (ii)$_{l+1}$ since $M_{l+1} > N_l$.

(vi)$_{l+1}$ Let $B$ be a $\mathcal{B}_{l+1}$-admissible block of length $l+1$. Then $B$ is $\mathcal{B}_l$-admissible, hence $B$ appears in $\eta_l[1, M_{l+1}]$ by the choice of $M_{l+1}$. Moreover,
\[ \eta_l[1, M_{l+1}] = \eta_{l+1}[1, M_{l+1}] \]

by (40). Consequently, $B$ appears in $\eta_{l+1}[1, N_{l+1}]$ as $N_{l+1} > M_{l+1}$.

Conditions (vii)$_{l+1}$ and (viii)$_{l+1}$ were discussed before.
We set $\mathcal{B} = \bigcup_{l=1}^{\infty} \mathcal{B}_l$. Observe that $[1, N_l] \cap \mathcal{B} = \mathcal{B}_l$ by (v)$_{l+1}$ and thus
\begin{equation}
\eta[1, N_l] = \eta[1, N_l] \quad \text{for every } l \in \mathbb{N}.
\end{equation}

Assume that $B$ is a $\mathcal{B}$-admissible block of length $l$. By condition (vi)$_l$, $B$ appears in $\eta[1, N_l]$, hence it appears in $\eta$ by (41).

By the construction, $\mathcal{B} \subseteq \mathbb{P}$. Moreover, $\mathcal{B}$ is Behrend because $Y_p \in \mathcal{B}$ \hspace{1em} 1 \hspace{1em} \frac{1}{p} = 0$ thanks to (iv)$_l$, $l \geq 0$.

It remains to prove the statement on the complexity. By the construction of the set $\mathcal{B}$ (see (40)) we see that
\begin{equation}
\mathcal{F}_\mathcal{B} \cap [N_l + 1, M_{l+1}] = \mathcal{F}_{\mathcal{B}_l} \cap [N_l + 1, M_{l+1}].
\end{equation}

As $\text{lcm}(\mathcal{B}_l) \parallel M_{l+1} - N_l$ by (viii)$_l$, we have
\begin{equation}
|\mathcal{F}_{\mathcal{B}_l} \cap [N_l + 1, M_{l+1}]| = (M_{l+1} - N_l) d(\mathcal{F}_{\mathcal{B}_l}).
\end{equation}
By (42), (43) and (viii)$_l$ we get
\begin{equation}
|\mathcal{F}_\mathcal{B} \cap [N_l + 1, M_{l+1}]| \geq \rho(M_{l+1} - N_l).
\end{equation}
Every subset of the set $\mathcal{F}_\mathcal{B} \cap [N_l + 1, M_{l+1}]$ is $\mathcal{B}$-admissible, thus there are at least $2^{\rho(M_{l+1} - N_l)} \mathcal{B}$-admissible blocks and each of them appears in $\eta$. It follows that $\text{cpx}_\eta(M_l - N_l) \geq 2^{\rho(M_l - N_l)}$.

5. Complexity of the subshifts of primes and semi-primes

5.1. Complexity of the subshift of primes – proof of Theorem 1.1

Let us start with the upper bounds.

Let $N$ be a fixed positive integer and consider $n \geq \sqrt{N} + 1$. Set
\begin{equation}
B_n = B_{n,N} := (1_p(n+1), \ldots, 1_p(n+N)) \in \{0,1\}^N.
\end{equation}
We denote by
\begin{equation}
O_{n,N} := \{1 \leq i \leq N : 1_p(n+i) = 1\}
\end{equation}
the support of $B_n$. Note also that
\begin{equation}
O_{n,N} \subseteq \{1 \leq i \leq N : p \nmid n+i \hspace{1em} (\forall p \leq \sqrt{N})\}
\end{equation}
\hspace{2em} = \{1 \leq i \leq N : i \not\equiv -n \text{ mod } p, \forall p \leq \sqrt{N}\}
\hspace{2em} =: S_{n,N}.

The set $S_{n,N}$ depends solely on the residue classes of $-n$ modulo $p$ for $p \leq \sqrt{N}$; indeed, if we set $\mathbb{P} \cap [1, \sqrt{N}] = \{p_1, \ldots, p_k\}$ then for any $n$, there are $0 \leq m_j < p_j$ such that
\begin{align*}
S_{n,N} = \{1 \leq i \leq N : i \not\equiv m_j \text{ mod } p_j, 1 \leq j \leq k\} &=: C_N(m_1, \ldots, m_k) \\
&=: C(-n \text{ mod } p_1, \ldots, -n \text{ mod } p_k).
\end{align*}
Therefore, each set $S_{n,N}$, $n \geq \sqrt{N}+1$, must be one of the sets $C_N(m_1, \ldots, m_k)$ and the number of the latter sets is

$$\prod_{j=1}^{k} p_j = \prod_{p \leq \sqrt{N}} p.$$ 

This and (47) imply that the support of each $B_{n,N}$, $n \geq \sqrt{N}+1$, is a subset of one of the sets $C_N(m_1, \ldots, m_k)$.

Suppose that there exists $K = K_N$ such that for any $0 \leq m_j < p_j$, $j = 1, \ldots, k$, we have $|C_N(m_1, \ldots, m_k)| \leq K$. Hence, each of the sets $C_N(m_1, \ldots, m_k)$ has at most $2^K$ subsets. It follows that the supports of the blocks $B_{n,N}$, $n \geq \sqrt{N}+1$, can give at most $(\prod_{p \leq \sqrt{N}} p) \cdot 2^K$ subsets. Therefore,

$$A_N := |\{B_{n,N} : n \in \mathbb{N}\}| = |\{\text{blocks of length } N \text{ appearing in } X_{1_p}\}|$$

$$\leq |\{B_{n,N} : n \leq \sqrt{N}\}| + |\{B_{n,N} : n > \sqrt{N}\}|$$

$$\leq \sqrt{N} + \left( \prod_{p \leq \sqrt{N}} p \right) \cdot 2^K.$$

Now, by the Prime Number Theorem,

$$\prod_{p \leq \sqrt{N}} p = \exp\left( \sum_{p \leq \sqrt{N}} \log p \right) = \exp(\sqrt{N} (1 + o(1))).$$

Further, $K$ is just an upper bound for the maximal number of elements in the set $\{1, \ldots, N\}$ after deleting from it a particular residue class $0 \leq m_j < p_j$ for all $j = 1, \ldots, k$ (that is, for all $p \leq \sqrt{N}$).

In the language of Large Sieve (see (87)), we define $C \subset [1, N]$, $\omega(p) = 1$ for $p \leq \sqrt{N}$ and $\omega(p) = 0$ for all $p > \sqrt{N}$. Therefore, whenever $R < \sqrt{N}$, we have $\sigma(R) = \sum_{q \leq R} \frac{\mu^2(q)}{\varphi(q)}$ (remembering that $\mu^2$ is the indicator of the set of square-free numbers, and $\varphi$ stands for the Euler function). However, it is classical that $\sum_{q \leq R} \frac{\mu^2(q)}{\varphi(q)} > \log R$ (in fact, it is $= \log R + O(1)$; see (88)). If we now set $R = \sqrt{N}/\log N$, we obtain

$$|C_N(m_1, \ldots, m_k)| \leq \frac{2N}{\log N} (1 + o(1)) =: K_N.$$

Finally, by (48) and (49), we obtain

$$A_N \leq \sqrt{N} + \exp(\sqrt{N} (1 + o(1))) 2^{\frac{2N}{\log N} (1 + o(1))}$$

$$= (o(1) + 1)(\exp(\sqrt{N} (1 + o(1))) 2^{\frac{2N}{\log N} (1 + o(1))})$$

$$= 2^{\frac{\sqrt{N}}{2} (1 + o(1)) + \frac{2N}{\log N} (1 + o(1))} = 2^{2\frac{2N}{\log N} (1 + o(1))} = (4 + o(1))^{N/\log N}$$
and the upper bound in (7) follows. As the support of each $\mathbb{P}$-admissible block of length $N$ is a subset of one of the sets $C_N(m_1, \ldots, m_k)$, the upper bound in (8) follows from (49) and (50).

Let us turn now to the lower bounds. The lower bound in (8) follows by Proposition 1.4. On the other hand, by the Prime Number Theorem the number of primes in $[N + 1, 2N]$ is

$$\pi(2N) - \pi(N) = \left(\frac{2N}{\log 2N} - \frac{N}{\log N}\right)(1 + o(1)) = \frac{N}{\log N}(1 + o(1)).$$

Consider $k$ primes in $[N + 1, 2N]$, $p_1 < \cdots < p_k$. Then $k \leq N$ and the set $\{p_1, \ldots, p_k\}$ is $\mathbb{P}$-admissible (see the arguments in Remark 3.6). Moreover, assuming the Hardy–Littlewood conjecture (31), for any $X$ large enough we have

$$\sum_{n \leq X} \mathbb{1}_\mathbb{P}(n + p_1) \cdots \mathbb{1}_\mathbb{P}(n + p_k) \sim \mathcal{C} \frac{X}{\log^k X},$$

where $\mathcal{C} = \mathcal{C}(p_1, \ldots, p_k)$ is positive, since we consider the admissible case.

Moreover, by [26, Cor. 3.14 & Eq. (3.46)], for $i \not\in \{p_1, \ldots, p_k\}$,

$$\sum_{n \leq X} \mathbb{1}_\mathbb{P}(n + p_1) \cdots \mathbb{1}_\mathbb{P}(n + p_k) \mathbb{1}_\mathbb{P}(n + i) \leq 2^{k+1}(k+1)! \frac{X}{\log^{k+1} X} (\mathcal{C}_i + o(1)),$$

where $\mathcal{C}_i = \mathcal{C}(p_1, \ldots, p_k, i) \geq 0$, regardless of whether or not $\{p_1, \ldots, p_k, i\}$ is admissible. Therefore (for brevity, we denote $\mathbb{1}_\mathbb{P}(n + p_1) \cdots \mathbb{1}_\mathbb{P}(n + p_k)$ by $\chi_{p_1, \ldots, p_k}(n)$),

$$\sum_{n \leq X; \mathbb{1}_\mathbb{P}(n+i)=0 \forall i \in [N+1,2N] \setminus \{p_1, \ldots, p_k\}} \chi_{p_1, \ldots, p_k}(n)$$

$$= \sum_{n \leq X} \chi_{p_1, \ldots, p_k}(n) - \sum_{n \leq X; \mathbb{1}_\mathbb{P}(n+i)=1 \text{ for some } i \in [N+1,2N] \setminus \{p_1, \ldots, p_k\}} \chi_{p_1, \ldots, p_k}(n)$$

$$= \sum_{n \leq X} \chi_{p_1, \ldots, p_k}(n) - \sum_{i \in [N+1,2N] \setminus \{p_1, \ldots, p_k\}} \sum_{n \leq X} \chi_{p_1, \ldots, p_k}(n) \mathbb{1}_\mathbb{P}(n+i)$$

$$\geq \mathcal{C} \frac{X}{\log^k X}(1 + o(1)) - N 2^{k+1}(k+1)! \frac{X}{\log^{k+1} X} \left(\max_{i \in [N+1,2N] \setminus \{p_1, \ldots, p_k\}} \mathcal{C}_i + o(1)\right).$$

(31) See [15, Theorem X]. The symbol $\sim$ means that two expressions are of the same order, that is, $f(X) \sim g(X)$ whenever $\frac{f(X)}{g(X)} = 1 + o(1)$. 
The above inequality follows by (52) and (53). The right-hand side is positive for $X$ large enough. Thus, for $X$ large enough there exists $n^*$ such that $n^* + p_1, \ldots, n^* + p_k$ are all primes, while the $n^* + i$ for $i \in [N + 1, 2N] \setminus \{p_1, \ldots, p_k\}$ are not.

We have just shown that there is an injection from the set $T_N$ of tuples $\{p_1, \ldots, p_k\} \subset \{N + 1, \ldots, 2N\}$ to the set $A_N$ of blocks of length $N$ appearing in $X_1^P$. Therefore, by (51), $2 | T_N | = 2^{N \log N (1 + o(1))} = (2 + o(1))^{N \log N} \leq |A_N|$ and (9) follows.

Remark 5.1. The Hardy–Littlewood conjecture is used to prove the existence of an injection $T_N \rightarrow A_N$, in the notation from the proof above. More precisely, the argument is that every finite $\mathbb{P}$-admissible block appears in $1^P$. The same follows from Dickson’s conjecture, so the Hardy–Littlewood conjecture can be replaced by Dickson’s conjecture in the formulation of Theorem 1.1.

5.2. Consequences of Dickson’s conjecture: complexity for the subshift of semi-primes (a lower bound in Theorem 1.2). In this section we collect some consequences of Dickson’s conjecture, first mentioned in [7]. Below we formulate the conjecture following [33].

Consider a finite family $\Phi = \{f_i(x) = a_i + b_i x : i = 1, \ldots, k\}$ of linear polynomials with integer coefficients $a_i$ and $b_i, b_i \geq 1$ (for $i = 1, \ldots, k$). We say that $\Phi$ satisfies Dickson’s condition if

$$[(\forall p \in \mathbb{P})(\exists y \in \mathbb{Z})] \prod_{i=1}^k f_i(y) \neq 0 \mod p.$$  

Remark 5.2. Note that if $A \subset \mathbb{Z}$ is finite, then $A$ is $\mathbb{P}$-admissible if and only if the family $\{x - a : a \in A\}$ satisfies Dickson’s condition.

Conjecture 1 (Dickson’s conjecture). If $\Phi$ satisfies Dickson’s condition, then there exist infinitely many natural numbers $m$ such that all the numbers $f_1(m), \ldots, f_k(m)$ are primes.

For future use let us prove the following technical lemma.

Lemma 5.3. Let $\Phi = \{f_i(x) = a_i + b_i x : i = 1, \ldots, k\}$, where $a_i, b_i \in \mathbb{Z}, b_i \geq 1$, for $i = 1, \ldots, k$. Assume that

(a) $\gcd(a_i, b_i) = 1$ for $i = 1, \ldots, k$,

(b) $|\{i \in \{1, \ldots, k\} : p \nmid b_i\}| < p$ for every $p \in \mathbb{P}$.

Then $\Phi$ satisfies Dickson’s condition.

Proof. Fix $p \in \mathbb{P}$ and let $\{i_1, \ldots, i_l\} = \{i \in \{1, \ldots, k\} : p \nmid b_i\}$. If $l = 0$, that is, $p | b_i$ for $i = 1, \ldots, k$, then in view of (a), $p \nmid a_i$ for $i = 1, \ldots, k$, so $p$ does not divide $\prod_{i=1}^k (a_i + b_i z)$ for any $z \in \mathbb{Z}$. From now on we assume that
Denote $b = \text{lcm}(b_1, \ldots, b_l)$. Since, by assumption (b), $l < p$, there exists $a' \in \mathbb{Z}$ such that

$$a' \not\equiv a_{i_j} \frac{b}{b_{i_j}} \mod p \quad \text{for } j = 1, \ldots, l.$$  

(54)

As $\gcd(b, p) = 1$, there exists $y \in \mathbb{Z}$ such that

$$by \equiv -a' \mod p.$$  

(55)

Suppose that $p | a_i + b_i y$ for some $i \in \{1, \ldots, k\}$. If $p | b_i$ then, by assumption (a), $p \nmid a_i$, a contradiction. Thus $i = i_j$ for some $j \in \{1, \ldots, l\}$. We have

$$b_{i_j} y \equiv -a_{i_j} \mod p.$$  

Multiplying by $b/b_{i_j}$ we obtain

$$by \equiv -a_{i_j} \frac{b}{b_{i_j}} \mod p.$$  

In view of (55), $p \mid (a' - a_{i_j} \frac{b}{b_{i_j}})$, a contradiction with (54). We have shown that $p$ does not divide $\prod_{i=1}^{k}(a_i + b_i y)$.

It is known that Dickson’s conjecture [1] is equivalent to the following stronger statement.

**CONJECTURE 2** (Dickson’s conjecture*). Consider finite families $\Phi = \{f_i(x) = a_i + b_i x : i = 1, \ldots, k\}$ and $\Gamma = \{g_j(x) = c_j + d_j x : j = 1, \ldots, k'\}$ with $a_i, b_i, c_j, d_j \in \mathbb{Z}$, $b_i, d_j \geq 1$, for $i = 1, \ldots, k$, $j = 1, \ldots, k'$. Assume moreover that $\Phi \cap (-\Gamma \cup \Gamma) = \emptyset$, that is, $\pm g_j(x) \notin \Phi$ for every $j = 1, \ldots, k'$. If $\Phi$ satisfies Dickson’s condition, then there exist infinitely many natural numbers $m$ such that all the numbers $f_1(m), \ldots, f_k(m)$ are primes and all the numbers $g_1(m), \ldots, g_{k'}(m)$ are composite.

The non-trivial implication “Conjecture [1] ⇒ Conjecture [2]” follows from the arguments of A. Schinzel [31] (the proof of $H \Rightarrow C_{13}$). For the convenience of the reader we provide an alternative proof.

**LEMMA 5.4.** Suppose that $a_i, b_i, c, d \in \mathbb{Z}$, $b_i, d \geq 1$, where $i = 1, \ldots, k$ and $\gcd(c, d) = 1$. Assume that the polynomial $a_i + b_i x$ is not divisible by $c + dx$ for any $i = 1, \ldots, k$. Then there exist infinitely many $z \in \mathbb{N}$ such that $c + dz$ is a prime and it does not divide $\prod_{i=1}^{k}(a_i + b_i z)$.

**Proof.** By the Dirichlet Theorem, $c + dz$ is a prime for infinitely many integers $z$. Assume for a contradiction that for almost every such $z$,

$$c + dz \mid \prod_{i=1}^{k}(a_i + b_i z).$$  

(32) In the ring $\mathbb{Z}[x]$.
It follows that there exist \( i \in \{1, \ldots, k\} \) and an infinite increasing sequence \((z_j)\) of integers such that \( c + dz_j \) is prime and it divides \( a_i + b_i z_j \) for every \( j \). Since

\[
\frac{a_i + b_i z_j}{c + dz_j} \to \frac{b}{d} \quad \text{as} \quad j \to \infty,
\]

the sequence \( (\frac{a_i + b_i z_j}{c + dz_j}) \) (of integers) is eventually constant, whence \( c + dx \) divides \( a_i + b_i x \), a contradiction. The assertion follows. \( \blacksquare \)

**Lemma 5.5.** Assume that a family \( \Phi = \{f_i(x) = a_i + b_i x : i = 1, \ldots, k\} \) satisfies Dickson’s condition and \( g(x) = c + dx \) for some \( c, d \in \mathbb{Z}, d \geq 1 \). Assume that \( \pm g(x) \notin \Phi \). Then, for any \( L \in \mathbb{N} \), there exists a finite family \( \Psi = \{-c'_j + x : j = 1, \ldots, t\} \) where \( t \in \mathbb{N} \cup \{0\} \) and \( c_1', \ldots, c_t' \in \mathbb{Z}, c_1', \ldots, c_t' \geq L \), such that the family \( \Phi \cup \Psi \) satisfies Dickson’s condition, whereas the family \( \Phi \cup \Psi \cup \{g(x)\} \) does not.

**Proof.** If \( \Phi \cup \{g(x)\} \) does not satisfy Dickson’s condition, then we set \( \Psi = \emptyset \). This is always the case if \( c \) and \( d \) are not coprime. From now on we assume that \( \gcd(c, d) = 1 \) and \( \Phi \cup \{g(x)\} \) satisfies Dickson’s condition.

Clearly, \( \gcd(a_i, b_i) = 1 \) for \( i = 1, \ldots, k \), hence \( \pm g(x) \notin \Phi \) the polynomial \( g(x) \) does not divide any of \( f_i(x) \). By Lemma 5.4 there exists \( z \in \mathbb{N} \) such that \( p = c + dz \) is a prime which is greater than any of \( b_i, i = 1, \ldots, k \), greater than \( d \), and such that

\[
(56) \quad \prod_{i=1}^{k} (a_i + b_i z) \not\equiv 0 \mod p.
\]

Since \( \Phi \cup \{g(x)\} \) satisfies Dickson’s condition, the set

\[
R = \{r \in \mathbb{Z} : a_i + b_i r \not\in p\mathbb{Z} \text{ for } i = 1, \ldots, k \text{ and } c + dr \not\in p\mathbb{Z}\}
\]

is non-empty. As \( p = c + dz \),

\[
(57) \quad z \notin R.
\]

Clearly, \( R \) is \( p \)-periodic. Let \( R' \subset R \) be a set of representatives of \( R \mod p \). Then (by (57))

\[
(58) \quad |R'| \leq p - 1.
\]

Given a prime \( q \leq k + p \), let \( y_q \in \mathbb{Z} \) be such that \( (33) \)

\[
(59) \quad (c + dy_q) \prod_{i=1}^{k} (a_i + b_i y_q) \not\equiv 0 \mod q.
\]

(\(33\)) The existence of \( y_q \) follows by Dickson’s condition.
For \( r' \in R' \) choose (34)

\[
(60) \quad c'_{r'} \in \left( (r' + p\mathbb{Z}) \setminus \bigcup_{q \in \mathcal{P}, q \leq k + p, q \neq p} (y_q + q\mathbb{Z}) \right) \cap [L, +\infty).
\]

We claim that

\[
(61) \quad \text{the family } \{f_i(x) : i = 1, \ldots, k\} \cup \{-c'_{r'} + x : r' \in R'\}
\]

satisfies Dickson’s condition.

Indeed, we need to prove that for every prime \( q \) there exists \( y \in \mathbb{Z} \) such that

\[
(62) \quad \prod_{i=1}^{k} f_i(y) \prod_{r' \in R'} (y - c'_{r'}) \not\equiv 0 \mod q.
\]

Let \( q \in \mathcal{P} \). If \( q \leq k + p \) and \( q \neq p \) then \( \prod_{i=1}^{k} (a_i + b_i y_q) \not\equiv q\mathbb{Z} \) by (59) and \(-c'_{r'} + y_q \not\equiv q\mathbb{Z} \) for \( r' \in R' \) by (60). We set \( y = y_q \) in this case.

If \( q = p \), then \( z - c'_{r'} \not\equiv 0 \mod p\mathbb{Z} \) since \( c'_{r'} \in R \) (in view of (60)) while \( z \not\in R \) (see (57)). By (56), \( \prod_{i=1}^{k} (a_i + b_i z) \not\equiv 0 \mod p \) and we can set \( y = z \).

Assume now that \( q > k + p \). As \( q \) is coprime to \( b_i, i = 1, \ldots, k \), there exists \( b'_i \in \mathbb{Z} \) such that \( b'_i b_i \equiv 1 \mod q \) for \( i = 1, \ldots, k \). Since \( q > k + p \geq |\Phi| + |R'| \) (see (58)), there exists \( y \in \mathbb{Z} \) such that \( y \not\equiv c'_{r'} \mod q \) for every \( r' \in R' \) and \( y \not\equiv -b'_i a_i \mod q \) for \( i = 1, \ldots, k \). Then

\[
\prod_{i=1}^{k} f_i(y) \prod_{r' \in R'} (y - c'_{r'}) \not\equiv 0 \mod q
\]

and the claim (61) follows.

It remains to prove that the family \( \{f_i(x) : i = 1, \ldots, k\} \cup \{-c'_{r'} + x : r' \in R'\} \cup \{c + dx\} \) does not satisfy Dickson’s condition. Let \( y \in \mathbb{Z} \). By the definition of the set \( R \), if \( y \not\in R \) then \( p | (c + dy) \prod_{i=1}^{k} f_i(y) \). Otherwise \( y \equiv r' \mod p \) for some \( r' \in R' \), hence \( p | y - c'_{r'} \).

**Lemma 5.6.** Consider finite families \( \Phi = \{f_i(x) = a_i + b_i x : i = 1, \ldots, k\} \) and \( \Gamma = \{g_j(x) = c_j + d_j x : j = 1, \ldots, k'\} \) with \( a_i, b_i, c_j, d_j \in \mathbb{Z}, b_i, d_j \geq 1, \) for \( i = 1, \ldots, k \) and \( j = 1, \ldots, k' \), and let \( N \in \mathbb{N} \). Assume moreover that \( \Phi \cap (-\Gamma \cup \Gamma) = \emptyset \). If \( \Phi \) satisfies Dickson’s condition, then there exists a finite family \( \Phi' \) of degree-1 polynomials with integer coefficients such that \( \Phi' \) satisfies Dickson’s condition, \( \Phi \subseteq \Phi' \) and \( \Phi' \cup \{g_j(x)\} \) does not satisfy Dickson’s condition for any \( j = 1, \ldots, k' \). Moreover, \( \Phi' \setminus \Phi \) consists of monic polynomials of the form \( x - c \) with \( c > N \).

**Proof.** By induction on \( 0 \leq j \leq k' \) we construct finite families \( \Phi_j \) of polynomials of degree 1 with integer coefficients such that \( \Phi_0 = \Phi \) and

(34) The set on the right-hand side of (60) is non-empty, since the arithmetic progressions involved have pairwise coprime periods.
(i) \( \Phi_{j-1} \subseteq \Phi_j \),
(ii) \( \Phi_j \) satisfies Dickson’s condition,
(iii) \( \Phi_j \cap (-\Gamma \cup \Gamma) = \emptyset \),
(iv) \( \Phi_j \cup \{ g_j(x) \} \) does not satisfy Dickson’s condition,
for \( j = 1, \ldots, k' \). Assume that \( \Phi_0, \ldots, \Phi_j \) have been constructed and \( 0 \leq j < k' \). We apply Lemma \([5.5]\) to \( \Phi = \Phi_j \) and \( g(x) = g_{j+1}(x) \) taking \( L \geq N \) large enough to guarantee that \( \pm g_t(x) \notin \Psi \) for every \( t = 1, \ldots, k' \). We set \( \Phi_{j+1} = \Phi_j \cup \Psi \) and then conditions (i)\(_{j+1}\)-(iv)\(_{j+1}\) are satisfied. We set \( \Phi' = \Phi_{k'} \). Then \( \Phi' \cup \{ g_{k'}(x) \} \) does not satisfy Dickson’s condition. To see that the same holds for the remaining \( j \), note that already a subfamily \( \Phi_j \cup \{ g_j(x) \} \) does not satisfy Dickson’s condition. ■

**Proposition 5.7.** Assume that \( M < N \) and \( A \subseteq [M + 1, N] \) is a \( \mathbb{P} \)-admissible set of integers. There exists a finite set \( A' \subseteq \mathbb{Z} \cap [N + 1, +\infty) \) such that \( A \cup A' \) is \( \mathbb{P} \)-admissible, whereas \( A' \cup A \cup \{ i \} \) is not, for every \( i \in [M + 1, N] \setminus A \).

**Proof.** It is enough to apply Lemma \([5.6]\) to \( \Phi = \{ x - a : a \in A \} \) and \( \Gamma = \{ x - i : i \in [M + 1, N] \setminus A \} \).

**Lemma 5.8.** Conjecture \([1]\) \( \Rightarrow \) Conjecture \([2]\)

**Proof.** Let \( \Phi = \{ f_i(x) = a_i + b_i x : i = 1, \ldots, k \} \) and \( \Gamma = \{ g_j(x) = c_j + d_j x : j = 1, \ldots, k' \} \) be as in the formulation of Conjecture \([2]\) Let \( \Phi' \) be as in Lemma \([5.6]\). For every \( j \), the family \( \Phi' \cup \{ g_j(x) \} \) does not satisfy Dickson’s condition, and thus there exist primes \( p_1, \ldots, p_{k'} \) such that

\[
p_j \mid g_j(y) \prod_{f \in \Phi'} f(y) \quad \text{for every } y \in \mathbb{Z}
\]

for every \( j = 1, \ldots, k' \). Assuming Conjecture \([1]\) the set

\[
Y_1 := \{ y \in \mathbb{Z} : f(y) \in \mathbb{P} \text{ for } f(x) \in \Phi' \}
\]

is infinite. By (63), \( p_j \mid g_j(y) \) for every \( j = 1, \ldots, k' \) provided

\[y \in Y_2 := Y_1 \setminus \{ y \in Y_1 : \{ p_1, \ldots, p_{k'} \} \cap \{ f(y) : f(x) \in \Phi' \} \neq \emptyset \}\]

and \( Y_2 \) is cofinite in \( Y_1 \). As the polynomials \( g_j(x) \) are not constant, all the numbers \( g_j(y), j = 1, \ldots, k' \), are composite for infinitely many \( y \in Y_2 \). At the same time, \( f_1(y), \ldots, f_k(y) \) are prime for \( y \in Y_2 \subseteq Y_1 \), because \( f_1(x), \ldots, f_k(x) \in \Phi' \). ■

Now, we turn to the first consequence of Dickson’s conjecture.

**Proposition 5.9.** Assuming Dickson’s conjecture, for every \( k, N \in \mathbb{N} \) and every set \( A \subseteq [1, N] \cap \mathbb{P}_k \), there exists \( n \in \mathbb{N} \) such that

\[
[n + 1, n + N] \cap \mathbb{P}_{k+1} = \{ n + a : a \in A \}.
\]

In particular, \( X_{1\mathbb{P}_k} \subseteq X_{1\mathbb{P}_{k+1}} \) for every \( k \in \mathbb{N} \).
Proof. For \( N \in \mathbb{N} \) we denote by \( Q_N \) the primorial \( N\# \) of \( N \), that is,
\[
Q_N = \prod_{p \in \mathbb{P}, p \leq N} p.
\]

For \( k \in \mathbb{N} \) and \( q \in \mathcal{M}_{\mathbb{P}_k} \) choose \( m_q^{(k)} \in \mathbb{P}_k \) such that \( m_q^{(k)} \mid q \). If \( q \in \mathcal{F}_{\mathbb{P}_k} \), we set \( m_q^{(k)} = q \). Observe that \( m_q^{(k)} = q \) for \( q \in \mathbb{P}_k \). Now, fix \( N, k \in \mathbb{N} \). Given \( 1 \leq q \leq N \), we define a linear polynomial with integer coefficients
\[
h_{q,N}^{(k)}(x) = \frac{q}{m_q^{(k)}} + \frac{Q_k^N}{m_q^{(k)}} x.
\]

Observe that
\[
h_{q,N}^{(k)}(x) = 1 + \frac{Q_k^N}{q} x \quad \text{if} \quad q \in \mathcal{F}_{\mathbb{P}_{k+1}} = \mathcal{F}_{\mathbb{P}_k} \cup \mathbb{P}_k.
\]

Let
\[
\Phi = \{h_{q,N}^{(k)}(x) : q \in A \cup (\mathcal{F}_{\mathbb{P}_k} \cap [1, N])\},
\]
\[
\Gamma = \{h_{q,N}^{(k)}(x) : q \in \mathcal{M}_{\mathbb{P}_k} \cap [1, N] \setminus A\}.
\]

By (65), it follows that \( \Phi \) satisfies Dickson’s condition \((35)\).

Note that the polynomial \( h_{q,N}^{(k)}(x) \) determines \( q \): we have
\[
q = \frac{Q_k^N h_{q,N}^{(k)}(0)}{h_{q,N}^{(k)}(1) - h_{q,N}^{(k)}(0)}.
\]

It follows that \( \Phi \cap (-\Gamma \cup \Gamma) = \emptyset \).

Assuming Conjecture \([1] \) and so (in view of Lemma \([5.8] \) Conjecture \([2] \) for infinitely many \( y \in \mathbb{Z} \),
\[
h_{q,N}^{(k)}(y) \in \mathbb{P} \iff h_{q,N}^{(k)}(x) \in \Phi \iff q \in A \cup \mathcal{F}_{\mathbb{P}_k} \quad \text{for} \quad q \in [1, N].
\]

Fix such a \( y \in \mathbb{Z} \). We claim that
\[
m_q^{(k)} h_{q,N}^{(k)}(y) = q + Q_k^N y \in \mathbb{P}_{k+1} \iff q \in A \quad \text{for} \quad q \in [1, N].
\]

Indeed, if \( q \in A \subseteq \mathbb{P}_k \), then \( m_q^{(k)} = q \in \mathbb{P}_k \), and since \( h_{q,N}^{(k)}(y) \in \mathbb{P} \) by (66), \( m_q^{(k)} h_{q,N}^{(k)}(y) \in \mathbb{P}_{k+1} \). If \( q \notin A \), then one of the following conditions holds:

(i) \( q \in \mathbb{P}_k \setminus A \),
(ii) \( q \in \mathcal{F}_{\mathbb{P}_k} \),
(iii) \( q \in \mathcal{M}_{\mathbb{P}_{k+1}} \).

In case (i), \( h_{q,N}^{(k)}(x) \in \Gamma \), thus \( h_{q,N}^{(k)}(y) \) is composite and \( m_q^{(k)} h_{q,N}^{(k)}(y) = q h_{q,N}^{(k)}(y) \in \mathcal{M}_{\mathbb{P}_{k+2}} \). In case (ii), \( m_q^{(k)} = q \) and \( h_{q,N}^{(k)}(x) \in \Phi \), hence \( h_{q,N}^{(k)}(y) \in \mathbb{P} \).

(35) Because \( y \nmid 1 + ay \) for any \( a, y \in \mathbb{Z} \) such that \( |y| \geq 2 \).
and \( q h_{q,N}^{(k)}(y) \in \mathcal{F}_{P_{k+1}} \). If (iii) holds, then \( h_{q,N}^{(k)}(x) \in \Gamma, \) \( h_{q,N}^{(k)}(y) \) is composite, hence \( m_q^{(k)} h_{q,N}^{(k)}(y) \in \mathcal{M}_{P_{k+2}} \). In each of the cases (i)–(iii), \( m_q^{(k)} h_{q,N}^{(k)}(y) \notin P_{k+1} \).

It follows by (67) that \( n = Q_N^k y \) satisfies condition (64).

The following proposition will be applied to obtain a lower bound for the complexity of \( \mathcal{P}_2 \).

**Proposition 5.10.** Assuming Dickson’s conjecture, for any \( M, N \in \mathbb{N} \) such that \( \sqrt{N} \leq M < N, \) \( N - M < \sqrt{N} \) and \( \lfloor \sqrt{N} \rfloor \notin P \) \((36)\) and for every subset
\[ A \subseteq \mathbb{P}_2 \cap [M + 1, N] \]
there exists \( n \in \mathbb{N} \) such that
\[ (68) \quad \mathbb{P}_2 \cap [n + M + 1, n + N] = \{ n + a : a \in A \}. \]

**Proof.** First note that
\[ (69) \quad q \in \mathbb{P}_2 \cap [M + 1, N] \implies q \text{ has a prime divisor } p > \sqrt{N}. \]
Indeed, assume for a contradiction that \( M < p_1 p_2 \leq N \) for some primes \( p_1, p_2 \leq \sqrt{N}. \) Then \( \sqrt{N} \geq p_1 > M/\sqrt{N} \geq \sqrt{N} - 1 \) because \( M > N - \sqrt{N} \). This means that \( \lfloor \sqrt{N} \rfloor = p_1 \in \mathbb{P}, \) contrary to our assumption. Let
\[ Q = \sqrt{N}\# := \prod_{p \in \mathbb{P}, p \leq \sqrt{N}} p. \]
Given a number \( q \in \mathcal{M}_{\mathbb{P}_2}, \) let \( m_q \) be the minimal prime divisor of \( q. \) Then \( m_q \leq \sqrt{N} \) for \( q \in \mathcal{M}_{\mathbb{P}_2} \cap [M + 1, N]. \)
Let
\[ h_q(x) = \frac{q}{m_q} + \frac{Q^2}{m_q} x \]
for \( q \in \mathcal{M}_{\mathbb{P}_2} \cap [M + 1, N]. \)
For \( q \in \mathbb{P} \cap [M + 1, N], \) we set
\[ h_q(x) = q + Q^2 x. \]

Let
\[ \Phi = \{ h_q(x) : q \in A \cup (\mathbb{P} \cap [M + 1, N]) \}, \]
\[ \Gamma = \{ h_q(x) : q \in [M + 1, N] \setminus (A \cup \mathbb{P}) \}. \]
We claim that \( \Phi \) satisfies assumptions (a) and (b) of Lemma 5.3, hence it satisfies Dickson’s condition.

(a) If \( q \in A, \) then \( q/m_q \) is a prime and \( q/m_q > \sqrt{N} \) thanks to (69). Hence \( q/m_q \) is coprime to \( Q^2/m_q \). Clearly, \( q \) is coprime to \( Q^2 \) for \( q \in \mathbb{P} \cap [M + 1, N], \) because \( q > M \geq \sqrt{N}. \) Thus (a) of Lemma 5.3 is satisfied.
(b) If $p \leq \sqrt{N}$, then $p \mid Q^2$ and for every $q \in A$, as $m_q \in \mathbb{P}$, $p \mid \frac{Q^2}{m_q}$. If $p > \sqrt{N}$ then $p > |\Phi|$, because $\sqrt{N} > N - M \geq |\Phi|$ by assumption. Thus (b) is satisfied. The claim follows.

Observe that the polynomial $h_q(x)$ determines $q$: $q = \frac{Q^2h_q(0)}{h_q(1) - h_q(0)}$. Therefore $\Phi \cap (-\Gamma \cup \Gamma) = \emptyset$.

Assuming Conjecture 1 and so (in view of Lemma 5.8) Conjecture 2, for infinitely many $y \in \mathbb{Z}$, for $q \in [M + 1, N]$, we have

\begin{equation}
q + Q^2y = m_q h_q(y) \in \mathbb{P} \quad \text{for } q \in A.
\end{equation}

Moreover,

\begin{equation}
q + Q^2y = h_q(y) \in \mathbb{P} \quad \text{for } q \in \mathbb{P} \cap [M + 1, N].
\end{equation}

Finally, since $h_q(y)$ is composite for $q \in [M + 1, N] \setminus (A \cup \mathbb{P})$,

\begin{equation}
q + Q^2y = m_q h_q(y) \in \mathcal{M}_{\mathbb{P}_3} \quad \text{for } q \in [M + 1, N] \setminus (A \cup \mathbb{P}).
\end{equation}

We have shown that for $q \in [M + 1, N]$, $q + Q^2y \in \mathbb{P}_2$ if and only if $q \in A$. It follows that $n = Q^2y$ satisfies (68). ■

**Corollary 5.11.** Assuming Dickson’s conjecture,

\begin{equation}
\text{cpx}_{1_{\mathbb{P}_2}}(n) \geq (2 + o(1)) \frac{n \log \log n}{2 \log n}.
\end{equation}

**Proof.** By (89),

\begin{equation}
|\mathbb{P}_2 \cap [n^2 - n + 2, n^2]|
\end{equation}

\begin{equation}
= \left( \frac{n^2 \log \log(n^2)}{\log(n^2)} - \frac{(n^2 - n + 1) \log \log(n^2 - n + 1)}{\log(n^2 - n + 1)} \right)(1 + o(1))
\end{equation}

\begin{equation}
= \left( \frac{n \log \log n}{2 \log n} \right)(1 + o(1)).
\end{equation}

By Proposition 5.10, if $n \notin \mathbb{P}$, every subset of $\mathbb{P}_2 \cap [n^2 - n + 2, n^2]$ is the support of a block of length $n - 1$ appearing in $X_{1_{\mathbb{P}_2}}$ and the assertion follows. ■

### 5.3. Complexity for the subshift of semi-primes (an upper bound in Theorem 1.2).

The proof in this case is different from that in the prime case in §5.1. Let

\[ B_{n,N} := (1_{\mathbb{P}_2}(n + 1), \ldots, 1_{\mathbb{P}_2}(n + N)), \quad C_N := \{B_{n,N} : n \geq 0\}, \]

\[ O_{n,N} := \{1 \leq i \leq N : 1_{\mathbb{P}_2}(n + i) = 1\}. \]
Observe that
\begin{equation}
|O_{n,N}| = \sum_{n/p \leq n+N, \, p \in \mathbb{P}} 1 \leq \sum_{n/m \leq n+N, \, \omega(m) = 2} 1 + \sum_{\sqrt{n} < p \leq \sqrt{n+N}} 1,
\end{equation}
where \( \omega : n \mapsto \sum_p \nu_p \). If we suppose that \( n > N \), the first sum in (75) can be bounded by means of (92). Furthermore, as
\begin{equation}
\sum_{\sqrt{n} < p \leq \sqrt{n+N}} 1 \leq \sum_{\sqrt{n} < p \leq \sqrt{n+N}} 1,
\end{equation}
we can use the Brun–Titchmarsh theorem (91) to derive
\begin{equation}
|O_{n,N}| \leq \frac{23N \log \log N}{\log N} (1 + o(1)) + \frac{4\sqrt{N}}{\log N}
= \frac{23N \log \log N}{\log N} (1 + o(1)).
\end{equation}
Thus, (76) exhibits a bound for \( |O_{n,N}| \) that is independent of \( n \), as long as \( n > N \).

On the other hand, let \( L = L_N = \max_{n \geq N} |O_{n,N}| \). We have
\begin{equation}
|\{B_{n,N} : n > N\}| \leq \sum_{k=0}^L \binom{N}{k},
\end{equation}
where the right-hand side corresponds to the number of blocks of length \( N \) with at most \( L \) coordinates equal to 1 (the remaining coordinates being 0’s). Hence, by (93), and (76) (recalling that the function \( (0, N/e] \ni t \mapsto (N/t)^t \) is increasing), we conclude that for \( N \) large enough,
\begin{equation}
|\{B_{n,N} : n > N\}| \leq \left( \frac{eN}{L} \right)^L \leq \left( \frac{e(\log N)(1 + o(1))}{23 \log \log N} \right)^{23N \log \log N} (1 + o(1))
\leq \left( \frac{e(\log N)(1 + o(1))}{23 \log \log N} \right)^{23N \log \log N} (1 + o(1))
\leq \frac{23N \log \log N}{(\log 4)(\log N)} (1 + o(1)) \log \left( \frac{e(\log N)(1 + o(1))}{23 \log \log N} \right)
\leq \frac{23N \log^2 \log N}{(\log 4)(\log N)} (1 + o(1)).
\end{equation}
Finally, we deduce from (78) that
\begin{equation}
|C_N| \leq |\{B_{n,N} : n \leq N\}| + |\{B_{n,N} : n > N\}|
\leq N + 4 \frac{23N \log^2 \log N}{(\log 4)(\log N)} (1 + o(1)) \leq (4 + o(1)) \frac{23N \log^2 \log N}{(\log 4)(\log N)},
\end{equation}
which gives the claimed upper bound.
Remark 5.12. A similar approach could have been used for §5.1. Indeed, by the Brun–Titchmarsh theorem (91), the maximum number of coordinates 1 that we can distribute inside a prime indicator vector of length $N$ is $2N/\log N$. Therefore, we obtain the following upper bound for the number of possible patterns of length $N$ of the indicator of the primes (cf. (48)):

$$|A_N| \leq \left(\frac{e \log N}{2}\right)^{\frac{2N}{\log N}} = 4^{\frac{2N \log \log N}{(\log 4)(\log N)}} (1+o(1))$$

$$\leq (4 + o(1))^{\frac{2N \log \log N}{(\log 4)(\log N)}},$$

which is worse than the one in (8).

6. Appendix. In this appendix we collect some classical facts from number theory, which we repeatedly apply in the paper. Here $p$ is always assumed to be prime.

Primes. The Prime Number Theorem [26, §6.2] asserts that

$$\pi(N) := \sum_{p < N} 1 = (1 + o(1)) \frac{N}{\log N},$$

or equivalently

$$\prod_{p < N} p = \exp((1 + o(1))N).$$

As a consequence, if $p_n$ denotes the $n$th prime then (see [26, §6.2.1, Ex. 5])

$$p_n = (1 + o(1))n \log n.$$  

Moreover, the prime number estimates of Chebyshev and Mertens [26, §2.2], weaker than the Prime Number Theorem, suffice to state that

$$\sum_{p < N} \frac{1}{p} = (1 + o(1)) \log \log N,$$

as well as Mertens’ Theorem

$$\prod_{p < X} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log X} (1 + o(1)).$$

Sieves. One of the main tools in the theory of sieves is the Large Sieve [3].

The Large Sieve – Analytic Form. Consider the trigonometric polynomial

$$S(x) = \sum_{n=M}^{M+N} a_n e^{2\pi inx}$$

of length $N$, where $a_n \in \mathbb{C}$. Let $\{x_1, \ldots, x_R\}$ be a set of $R$ reals that are
\(\delta\)-well-spaced (i.e., \(\|x_i - x_j\| := \min_{\ell \in \mathbb{Z}} |(x_i - x_j) - \ell| \geq \delta > 0\) for \(1 \leq i \neq j \leq R\)). Then

\[
\sum_{j=1}^{R} \|S(x_j)\|^2 \leq (N + 1/\delta) \sum_{n=M+1}^{M+N} |a_n|^2.
\]

A direct application of the above result is the following.

**The Large Sieve Inequality.** Let \(C\) be a set of integers contained in \([M + 1, M + N]\) which avoids \(\omega(p)\) residue classes modulo \(p\) for each prime, and let \(R > 0\). Then

\[
|C| \leq \frac{N + R^2}{\sigma(R)},
\]

where \(\sigma(R) = \sum_{q \leq R} \mu^2(q) \prod_{p \mid q} \frac{\omega(p)}{p - \omega(p)}\).

Closely related to sieve theory is the following result (refer to the Selberg sieve [5, §4]):

\[
\sum_{q \leq N} \frac{\mu^2(q)}{\varphi(q)} = (1 + o(1)) \log N.
\]

This result relies upon a convolution identity. See [29, 36] for further details.

A weaker result, but sufficient for the estimate in (50), is that \(\sum_{q \leq N} \frac{\mu^2(q)}{\varphi(q)} > \log N\); see [26, (3.18)].

**Numbers that are products of a fixed number of primes.** By [26, §7.4], for fixed \(k\) we have

\[
\sum_{q < N} 1_{p_k}(q) = (1 + o_k(1)) \frac{N \log^{k-1} \log N}{(k-1)! \log^{k-1} N};
\]

as a consequence, by summation by parts we have

\[
\sum_{q < N} \frac{1_{p_2}(q)}{q} = \frac{1}{2} (\log^2 \log N + o(1)).
\]

**Uniform bounds on primes and semi-primes in intervals.** By means of the Large Sieve, one can find uniform bounds for almost primes. The Brun–Titchmarsh theorem [25, Thm. 2] asserts in particular that for all \(X > 0\) and \(Y > 1\),

\[
\pi(X + Y) - \pi(X) < \frac{2Y}{\log Y}.
\]

Tudesq [34] gives the following uniform bound related to semi-primes:

\[
\sum_{X < m \leq X + Y \atop \omega(m) = 2} 1 \leq \frac{23Y \log \log Y}{\log Y} (1 + o(1))
\]

provided that \(2 \leq Y \leq X\).
Refer to [4, 34] for further discussion of uniform bounds for almost primes in intervals in arithmetic progressions.

**Partial sums of binomial coefficients.** Let $N$ be a natural number. Then for any integer $0 \leq L \leq N$, we have

\begin{equation}
\sum_{k=0}^{L} \binom{N}{k} \leq \sum_{k=0}^{L} \frac{N^k}{k!} = \sum_{k=0}^{L} \frac{L^k}{k!} \frac{N^k}{L^k} \leq \frac{N^L}{L^L} \sum_{k=0}^{\infty} \frac{L^k}{k!} = (\frac{eN}{L})^L.
\end{equation}

**Added in February 2023.** In the recent preprint [33], Tao and Ziegler proved (unconditionally) that the subshift $X_{\mathbb{Z}}$ is uncountable.

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Stanisław Kasjan, Mariusz Lemańczyk
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
87-100 Toruń, Poland
E-mail: skasjan@mat.umk.pl
mlem@mat.umk.pl

Sebastian Zuniga Alterman
Department of Mathematics and Statistics
University of Turku
20014 Turku, Finland
E-mail: szualt@utu.fi