Thin shell quantization in Weyl spacetime

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Abstract

We study the problem of quantization of thin shells in a Weyl–Dirac theory by deriving a Wheeler–DeWitt equation from the dynamics. Solutions are found which have interpretations in both cosmology and particle physics.

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1 Introduction

Phase transitions and spontaneous symmetry breaking play a leading role in physics and account for a large number of phenomena. In cosmology they suggest improvements to the standard cosmological model, while in particle physics, they give faithful accounts of the behaviour and relationship of the fundamental interactions.

In cosmology, in particular, the mechanism which fosters the transition from false to true vacuum can account for the change from a de Sitter to a Friedmann–Robertson–Walker universe and can be enacted in a variety of ways. We can have first–order phase transitions [1] or higher–order transitions [2], but the aim always is to realize a transition from a totally symmetric spacetime, where the matter content is due to quantum fields in a polarized vacuum state, to our asymmetric world where we observe four fundamental interactions and three families of particles.

The Glashow–Weinberg–Salam model [3] and the Guth’s inflation [4] are the starting points of a plethora of models whose number is then restricted by the accuracy of sky and ground–based observations.

Symmetry breaking also is relevant in quantum gravity where it can give insight into topological changes and is useful in dealing with quantum gravitational fluctuations. Such fluctuations may induce a minimum length, for example, thus introducing an additional source of uncertainty in physics. Furthermore, if the geometry is subject to quantum fluctuations, these give rise to a spacetime foam at the Planck scale [5]. By understanding the quantum evolution of this foam and by the definition of its Hamiltonian structure, the problem of quantum gravity could become more tractable, if not completely solvable.

The breaking of some geometrical symmetry can be considered also classically from a dynamical point of view. In [6], the breaking of conformal invariance has been used in order to construct bubbles in a Weyl spacetime.

These can be associated with microscopic particles [7], but can be as well considered in a cosmological context. Their classical dynamics is discussed in [8]. We study the issue of their quantization in this work.

The problem of the quantization of (false-) vacuum bubbles, using the formalism of thin shells in general gelativity, was first studied by Berezin et al. [9] and then by several authors [10]. The different results obtained depend upon the different ways used to construct the Hamiltonian structure. Recently, Zloshchastiev [11] has shown that quantizing a conservation law is equivalent to introducing a Hamiltonian in a minisuperspace in the spirit of the Wheeler–De Witt quantization. Accordingly, it is possible to quantize thin shells directly from the equations of motion, if the time does not appear explicitly, thus avoiding the problems of time–slicing and time gauge.

In this note, we apply the approach of [11] to the Weyl–Dirac model discussed in [6],[8]. In Sect.2, we outline the main features of the classical model. Sect.3 is devoted to the quantization of the model along the lines of the Wheeler–De Witt approach. Several solutions are given. In Sect.4, the results are discussed.
2 The classical model.

Bubbles with an interior de Sitter geometry can be constructed dynamically by means of the Gauss–Mainardi–Codazzi (GMC) formalism, assuming a conformally–invariant exterior geometry. Specifically, one can use a Weyl–Dirac theory of the form

\[ I_D = \int \left[ -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \beta^2 R + 6 \beta,_{\mu} \beta^\mu - \lambda \beta^4 \right] \sqrt{-g} d^4 x. \]  

(1)

Here \( f_{\mu\nu} = \kappa_{\nu,\mu} - \kappa_{\mu,\nu} \) is the electromagnetic field and \( \beta \) is a nonminimally coupled scalar field. The equations of motion derived from action (1) are

\[ \Box_{\nu} f^{\mu\nu} = 0, \]  

(2)

which are the usual Maxwell equations, and

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2\beta^2} E_{\mu\nu} + I_{\mu\nu} + \frac{1}{2} \lambda g_{\mu\nu} \beta^2 \equiv T_{\mu\nu}, \]  

(3)

which represent the Einstein equations where, \( E_{\mu\nu} \) is the Maxwell tensor, and

\[ I_{\mu\nu} = \frac{2}{\beta} (\Box_{\nu} \mu \beta - g_{\mu\nu} \Box_\alpha \beta) - \frac{1}{\beta^2} (4 \beta,_{\mu} \beta,_{\nu} - g_{\mu\nu} \beta,_{\alpha} \beta,^\alpha), \]  

(4)

is the stress–energy tensor of the scalar field. \( \Box_{\nu} \) is the Riemannian covariant derivative. The units are such that \( c^3 (16\pi G)^{-1} = 1 \).

As shown below, the scalar field \( \beta \) provides the surface tension of the bubbles.

By breaking conformal symmetry in a spherical region of Weyl spacetime, it is possible to construct a stable bubble of standard Riemannian geometry (Minkowski, de Sitter or anti–de Sitter spacetimes) that can represent either an elementary particle or an entire universe, depending on the scale of the parameters. The most general spherically symmetric line element is given by

\[ ds^2 = -e^{\nu(r,t)} dt^2 + e^{\mu(r,t)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(5)

The exterior and interior geometries are distinguished by writing \( t_{E,I}, \nu_{E,I} \) and \( \mu_{E,I} \) in the exterior and interior spacetimes \( V^{E,I} \), respectively. The GMC formalism can be used to connect the two different spacetime regions separated by a hypersurface \( \Sigma \). In this formalism, one introduces the spherically symmetric intrinsic metric

\[ ds^2_\Sigma = -d\tau^2 + R^2(\tau) (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(6)

and imposes suitable conditions on the scalar field \( \beta \) in order to obtain the equation of motion of the thin shell. Actually, the condition \( \beta = \beta_0 \) in \( V^I \), where \( \beta_0 \) is a constant, together with the boundary condition that \( \beta \) is continuous across \( \Sigma \) for all \( t \), require that
\[ \beta \text{ be a constant with respect to the intrinsic time } \tau \text{ of the thin shell defined at } r = R(\tau). \]

That is,

\[
\left. \frac{d\beta}{d\tau} \right|_{r=R} = \beta_t X_E + \beta' \dot{R} = 0, \quad (7)
\]

where \( X_E \equiv dt/d\tau \) and the prime and dot denote differentiation with respect to \( r \) and \( \tau \), respectively. All the mathematical details of the model can be found in [6],[7],[8].

As the Birkhoff theorem is violated in Weyl spacetime, it also is possible to obtain a spherically symmetric solution depending on time [8]. This situation is extremely interesting in cosmology since \( R(t) \) can assume the role of the scale factor in a bubbly universe which is undergoing a phase transition after a symmetry breaking.

Solving the field equations, one finds the exact solutions for the interior and external geometries. One obtains, for the interior metric,

\[
e^{-\mu_I} = 1 + \frac{1}{6} \lambda \beta_0^2 r^2 = e^{\nu_I}, \quad (8)
\]

while in Weyl spacetime, one gets

\[
e^{\nu_E} = -\frac{\beta^2}{\beta^2(1 + \frac{1}{6} \lambda \beta_0^2 r^2)} \quad (9)
\]

and

\[
e^{\mu_E} = -\frac{1 + \frac{1}{6} \lambda \beta_0^2 r^2}{\gamma^2 \beta_0^2 r^4}, \quad (10)
\]

where \( \gamma \) is an arbitrary constant with dimension \((\text{length})^{-1}\). If one assumes that the sign of the constant \( \lambda \) does not change during the formation of the bubble, then the sign must be taken to be negative to ensure the correct signature of the metric (see [4] and [8] for the conventions used here) so the Riemannian geometry is a de Sitter spacetime. Since the electromagnetic field vanishes in \( V^I \), the interior stress–energy tensor reduces to

\[
T^I_{\mu \nu} = \frac{1}{2} \lambda g_{\mu \nu} \beta_0^2, \quad (11)
\]

and one recovers an effective cosmological constant.

These solutions can then be used to determine the equation of motion for \( r = R(\tau) \) in the frame comoving with the thin shell where the proper time is \( \tau \). In this case, the equation of motion acquires the simple form [8]

\[
\dot{R}^2 = \frac{\alpha^2 \left(1 - R^2/R^2_{eq}\right)}{\left(R^2_{eq}/R^2 - 1\right)^2 - \alpha^2}, \quad (12)
\]

where \( \alpha^2 = \gamma^2 \beta_0^2 R^4_{eq} \). From \( \dot{R} = 0 \), one obtains the condition

\[
R_{eq} = \frac{1}{\beta_0} \sqrt{\frac{6}{|\lambda|}}. \quad (13)
\]
It follows, from Eq.(13), that the size of the shell is governed by the value of the scalar field inside it and by the parameter $\lambda$. In other words, it is the effective cosmological constant
\[ \Lambda = \frac{1}{2} \lambda \beta_0^2, \] (14)
which rules the size of the bubble, as it must be in any first–order inflationary model [1].

Eq.(12) can be solved in several interesting cases [8]. Bubbles which are created in the Weyl vacuum with initial radii near either of the endpoints of the interval $0 < R < R_{eq}/\sqrt{2}$ exist indefinitely with a finite radius or else collapse to $R = 0$ while bubbles that are created in the Weyl vacuum, with an initial radius greater than $R_{eq}$, appear to be unstable in all cases, except for the trivial case $R = R_{eq}$.

3 The quantum model

Turnig now to the problem of quantization of the above dynamical model, we restrict ourselves to a minisuperspace approach, following [11], where an immediate quantization of conservation laws is given.

We can start from the pointlike Lagrangian
\[ \mathcal{L} = \frac{1}{2} m_{pl} \dot{R}^2 - \frac{m_{pl}}{2} \left[ \frac{\alpha^2 (R^2/R_{eq}^2 - 1)}{\left( R_{eq}^2/R^2 - 1 \right)^2 - \alpha^2} \right], \] (15)
which is defined on the tangent space $\mathcal{T}Q \equiv \{ R, \dot{R} \}$, which is just a one–dimensional minisuperspace (the only variable is $R$). Here $m_{pl}$ is the Planck mass. Eq.(12) is easily obtained as a first order integral of dynamics. The classical solutions are those in [8]. Let us now define the canonical momentum
\[ \dot{R}^2 = \frac{\pi^2}{m_{pl}^2}. \] (16)

Eqs.(15) and (16) yield the Hamiltonian
\[ \mathcal{H} = \frac{\pi^2}{2m_{pl}} + \frac{m_{pl}}{2} \left[ \frac{\alpha^2 (R^2/R_{eq}^2 - 1)}{\left( R_{eq}^2/R^2 - 1 \right)^2 - \alpha^2} \right]. \] (17)

We can then set the energy function related to the Lagrangian equal to zero, i.e.
\[ E_L = \frac{\partial \mathcal{L}}{\partial \dot{R}} \dot{R} - \mathcal{L} = 0. \] (18)

This implies that $\mathcal{H} = 0$ on the trajectories (12) and $\mathcal{H}$ can then be interpreted as the Hamiltonian constraint of the Wheeler–De Witt approach. By a canonical quantization
of the momentum \( (\ref{eq:10}) \), i.e. \( \pi = -i\partial_R \), we get the Wheeler–De Witt equation

\[
H|\Psi \rangle = 0 \quad \Rightarrow \quad \left\{ \frac{\partial^2}{\partial R^2} + m_{pl}^2 \frac{\alpha^2(1 - R^2/R_{eq}^2)}{(R_{eq}^2/R^2 - 1 - \alpha^2)} \right\} |\Psi \rangle = 0. \tag{19}
\]

By setting

\[
x = \frac{R}{R_{eq}}, \quad a = m_{pl} R_{eq} \alpha,
\]

Eq.\( (\ref{eq:19}) \) becomes

\[
\Psi'' + \left[ \frac{a^2(1 - x^2)x^4}{(1 - x^2)^2 - \alpha^2 x^4} \right] \Psi = 0,
\]

where primes indicate differentiation with respect to \( R \), and the ket–vector \( |\Psi \rangle \) has been replaced with the functional \( \Psi \) for simplicity.

At this point, an important remark is necessary. The quantization of thin shells can be obtained from several effective Lagrangians so that the corresponding quantum theory is not uniquely defined. This fact implies that the Wheeler–De Witt equation can be achieved in many different ways due, for example, to the choice of factor ordering and momentum operator). In our case, following \([11],[13],[14]\), we have done the simplest choices. Furthermore, we are going to take into account the semiclassical limit so that a more complicate approach is not necessary for the following discussion of solutions.

The solutions of Eq.\( (\ref{eq:21}) \) give the probability amplitude to get bubbles of a given size, mass and energy. The term inside the square brakets is the superpotential of the model which defines the classical/quantum boundary of the minisuperspace. It separates, in principle, the Euclidean from the Lorentzian zones. In the first case, the wavefunction is under the superpotential barrier and has an exponential behaviour. In the latter case, it has an oscillating behaviour and is over or above the barrier as in the case of standard quantum tunneling. The separation of these zones is given by the zeros of the superpotential. In our case, they are \( R = \pm R_{eq} \) and \( R = 0 \).

Approximate solutions can be found from the analysis of Eq.\( (\ref{eq:21}) \). The asymptotic behaviours are particularly interesting.

For \( |x| << 1 \), we have

\[
\Psi'' + a^2 x^4 \Psi = 0, \tag{22}
\]

whose solution is a superposition of Bessel functions \( Z_\nu(z) \) given by

\[
\Psi = \sqrt{x} Z_{1/6} \left( \frac{ax^3}{3} \right) \sim x^{2/3}. \tag{23}
\]

We note that, in this case, the Bessel function is \( Z_\nu = J_\nu \) since we must have \( |\Psi| \sim 0 \) for \( x \rightarrow 0 \). This condition would not be satisfied for \( Z_\nu = Y_\nu, H_\nu^{(1)} \), because \( Z_\nu \rightarrow \infty \) for \( x \rightarrow 0 \).
In the opposite case $|x| \gg 1$, Eq. (21) becomes
\[
\Psi'' + \left[ \frac{a^2 x^2}{(\alpha^2 - 1)} \right] \Psi = 0, \tag{24}
\]
with the solution
\[
\Psi(x) = \sqrt{x} Z_{1/4} \left( \frac{ax^2}{2\sqrt{\alpha^2 - 1}} \right), \tag{25}
\]
for $\alpha > 1$. In the limit $|x| \to +\infty$, we get
\[
\Psi(x) \approx \left[ \frac{2(\alpha^2 - 1)^{1/4}}{(\pi ax)^{1/2}} \right] \exp \left[ \pm i \left( \frac{ax^2}{2\sqrt{\alpha^2 - 1}} - \frac{3\pi}{4} \right) \right], \tag{26}
\]
which is clearly an oscillating solution. For $\alpha < 1$, the asymptotic solution is an instanton of the form
\[
\Psi(x) \approx \left[ \frac{2(|\alpha^2 - 1|)^{1/4}}{(\pi ax)^{1/2}} \right] \exp \left[ \pm \left( \frac{ax^2}{2\sqrt{\alpha^2 - 1}} - \frac{3\pi}{4} \right) \right], \tag{27}
\]
which rapidly diverges or converges to zero.

For $|x| \approx 1$, we find
\[
\Psi'' + 2(x - 1)\Psi = 0, \tag{28}
\]
and the solution, in this case, is
\[
\Psi(x) \approx \sqrt{x - 1} Z_{1/3} \left( \frac{3x^{3/2}}{\sqrt{2}} \right), \tag{29}
\]
and converges to zero for $x \sim 1$.

Finally, the case $\alpha = 1$ is best discussed starting from Eq. (21) directly. We find
\[
\Psi'' + a^2 \frac{(1 - x^2)x^4}{1 - 2x^2} \Psi = 0, \tag{30}
\]
which reduces to
\[
\Psi'' + a^2 x^4 \Psi = 0, \quad \text{for } x \ll 1, \tag{31}
\]
\[
\Psi'' + \frac{1}{2} a^2 x^4 \Psi = 0, \quad \text{for } x \gg 1, \tag{32}
\]
\[
\Psi'' + 2a^2 (x - 1) \Psi = 0, \quad \text{for } x \approx 1. \tag{33}
\]
Eqs. (31) and (32) are of a well-known type with solutions
\[
\Psi(x) = \sqrt{x} Z_{1/6} \left( \frac{a}{3} x^3 \right), \quad \text{and} \quad \Psi(x) = \sqrt{x} Z_{1/6} \left( \frac{a}{3\sqrt{2}} x^3 \right), \tag{34}
\]
respectively. Eq. (33) requires some standard transformations and has the solution
\[
\Psi(x) = \left[ 2a^2 (x - 1) \right]^{3/2} Z_{-3/4} \left( \frac{8}{3} a(x - 1)^{3/2} \right). \tag{35}
\]
4 Discussion

By looking at the above solutions, we immediately realize that we have oscillatory behaviours if we are well outside the superpotential barrier and instantonic solutions if we are near the singularities (i.e. the zeros) of the superpotential. This fact can be interpreted as the well known scheme of nucleation “from nothing” where quantum bubbles are produced in a Lorentzian region. In our case, deSitter bubbles are produced by breaking Weyl’s conformal symmetry.

If \( R(t) \) is interpreted as the scale factor of a bubbly universe, then \( \Psi(R) \) may be connected, in quantum cosmology, to the probability to obtain specific classes of cosmological models, which, in our case, are de Sitter universes, as requested by the inflationary paradigm. However, for \( \Psi \to 0 \), this information is lost. We must, however keep in mind that we are not in a “full” quantum gravity regime, where one expects \( \Psi \) to be exactly a probability amplitude. Here, we can only say that \( \Psi \) is related to the probability amplitude since the quantization scheme adopted, as all schemes developed in quantum gravity up to now, do not yield a Hilbert space.

We also see that the size of the bubble, i.e. its being larger or smaller than \( R_{eq} \), determines its survival and, then, the probability to give rise to a classical universe. It is important to remember that \( R_{eq} \) depends on the inverse value of an effective cosmological constant and is therefore related to the energy of the vacuum from which the bubbles are produced. In other words, the effective cosmological constant determines the dynamics. Moreover, the oscillatory behaviours of the wave function well fit Vilenkin’s no boundary conditions to obtain classical trajectories in the minisuperspace [13]. An oscillating wavefunction is capable of selecting first integrals of motion from which classical cosmological models are derived, in the semiclassical approximation of canonical quantum gravity [15]. In the present case, the quantum production of de Sitter bubbles gives a non–null probability to obtain initial conditions for inflation. In summary, the breaking of Weyl symmetry provides a standard of length in Riemannian geometry and, in addition, the initial conditions favourable to the production of observable universes.

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