Simple classical model of spin membrane particle

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Abstract

In this paper I developed a classical model of elementary particle that is associated with a membrane of finite size, surrounded by non-linear electromagnetic field. The form of local interaction which lead to bounded states of finite masses, charges and spins was constructed. To do this I added Kaluza-Klein Lagrangian on the surface, which is associated with extended particle, to quadratic in field potentials term (a la tachyon mass). This ensures that Lagrangian remains an even function of the field, but spontaneous symmetry breaking leads to nontrivial soliton-like solutions. I assumed that the particle has axial symmetry and that the surface has only one degree of freedom, i.e. is a disk with the radius determined from the equations of motion. The solution of system of two non-linear partial differential equations for the field potentials was obtained numerically by different methods. Several solutions with increasing orders of leading field multipoles and disk radius were obtained, and masses, electric charges and spins were calculated. In the framework of this model the ratio of a charge square to a double spin, i.e. the fine structure constant, which do not depend on parameters of the model, was calculated.

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1 Introduction

Physicists handle the laws of nature in two ways: they either explain them using more fundamental hypotheses or just postulate them. The values of electric charges and spins of elementary particles are still almost postulated in the framework of quantum relativistic field theory, whereas their masses became a subject
of non-trivial calculations a long time ago. Wineberg-Salam model predicted not only existence and characteristics of intermediate bosons but allowed to calculate electroweak processes in tree-like approach. However, the calculation of radiation corrections is limited by the absence of experimental data on Higgs bosons’ masses. Also further unifications of interactions such as supersymmetry \[1\] and great unification \[2\] were not supported by clear experimental observations so far, and electrodynamics are still leading in calculation precision of stationary states. Nevertheless electrodynamics can not be considered as an ideal model for the lepton world because of two problems: divergences in both classical and quantum theories, and the absence of calculations of some dimensionless quantities such as fine structure constant and lepton mass ratios.

Non-linear extensions of electrodynamics were first proposed by Kaluza \[3\], Klein \[4\] as well as by Born and Infeld \[5\]. Non-linearity leads to non-singular solution in case of point-like particle \[6\], but the particle mass, charge and magnetic moment remain arbitrary. To change this situation one can introduce a fundamental length into the model while preserving the local character of interaction, and construct a non-singular solution with finite size field sources.

In the present paper I propose an extended particle model, which consists of only one real vector electromagnetic field. The model is based on bilinear Maxwell Lagrangian

\[ L_M = \frac{1}{8\pi} \int (E^2 - B^2) \, dV, \]

(1)

and quadroilinear Kaluza-Klein Lagrangian (self-interaction of vector field)

\[ L_{K-K} = \frac{1}{8\pi \beta^2} \int (E \cdot B)^2 \, dV, \]

(2)

where constant \( \beta \) has dimension of field strength.

The key assumption of this work is proportionality of the four-current on some surface associated with the particle to the four-potential of the field. Therefore the gauge invariance is broken on the surface. In order to generate this dependence in the form of Lagrange equations I add quadratic in potentials term on the surface to the Lagrangian

\[ L_{\text{surface}} = -\frac{1}{4\pi l} \int (p^2 - A^2) \, dS, \]

(3)

where \( l \) is a fundamental length and \( p \) and \( A \) are scalar and vector potentials. The Lagrangian remains an even function of the field, and to provoke spontaneous symmetry breaking the sign of the last term must be the same as the sign used in the theory of vector tachyon field.

The solution of the problem was found by using approximation of infinite number of degrees of freedom by the finite number. It appears that non-linearity of Kaluza-Klein interaction on the one hand and the competition of surface and
volume energy on the other hand lead to a spectrum of non-trivial stationary states with the finite values of all observables. This result was obtained by assuming an axial symmetry of the whole system, and that the surface is a disk with only one degree of freedom — disk radius. The description of the field by only two degrees of freedom has already led to a set of stationary solutions. Then more field degrees of freedom were added (up to 1000) so that solutions achieved an asymptotic behavior.

An important feature of the presented classical model is that the charges, magnetic moments and spins are no longer parameters of the Lagrangian, but arise due to a spontaneous symmetry breaking. These observables are proportional to some powers of two fundamental parameters of the Lagrangian $\beta$ and $l$. To compare predictions of the model with experiment one can construct several dimensionless parameters, which do not depend on $\beta$ and $l$. Therefore, in the following text I set these two parameters to unity without loss of generality. For each state I calculate Lande coefficient and the ratio of a charge square to a double spin $e^2/(2s)$, i.e. the fine structure constant. I also extract the ratios of masses, spins and magnetic moments of different states.

### 2 Lagrangian in ellipsoidal coordinates

Owing to an axial symmetry and particular choice of calibration the problem was reduced to solution of two non-linear partial differential equations for electric field potential $p$ and the only non-zero component of vector-potential $A = A_\phi$, which both depend on two spacial coordinates. I introduced Cartesian coordinates so that $z$ is perpendicular to the disk plane, and then did the following transformation to the orthogonal ellipsoidal coordinates

\[
x = \rho \cos (\phi), \tag{4}
\]

\[
y = \rho \sin (\phi), \tag{5}
\]

\[
z = R \frac{\bar{v}(v) \cos (\pi/2 \bar{w}(w))}{\sin (\pi/2 \bar{w}(w))}, \tag{6}
\]

\[
\rho = R \frac{\sqrt{1 - \bar{v}(v)^2}}{\sin (\pi/2 \bar{w}(w))},
\]

where $R$ is the disk radius. This transformation reduced an infinite space to a unit square $(w, v)$ (note that none of the functions depends on the third angular coordinate $\phi$). I have concentrated on the mirror symmetry case $L(-v) = L(v)$, and therefore evaluated all functions in the interval $0 \leq v \leq 1$. The transformation can be tuned by two monotonically increasing functions $\bar{w}(w)$ and $\bar{v}(v)$ ($0 \leq w, \bar{w}, \bar{v} \leq 1$). The simplest case is $\bar{w}(w) = w$, $\bar{v}(v) = v$, which we explore in subsection 3.1. The particular choice of functions $\bar{w}(w)$ and $\bar{v}(v)$ should not
affect the values of observables, but could be used to achieve better convergence
to the asymptotic solution. I have also introduced contravariant potentials
\[ \tilde{\varphi} = p R, \tilde{A} = A R h_\phi, \]
where \( h_\phi = \sqrt{g_{\phi\phi}}/R \) is dimensionless Lame coefficient, and "field strength"
\[ E_w = -\frac{d}{dw} \tilde{\varphi}, E_v = -\frac{d}{dv} \tilde{\varphi}, \]
\[ B_w = \frac{d}{dv} \tilde{A}, B_v = -\frac{d}{dw} \tilde{A}. \]

The boundary conditions on the axis \( v = 1 \) are \( \tilde{A} = 0 \) with finite \( E_v \), and
at infinity \( w = 0 \) are \( \tilde{p} = 0, \tilde{A} = 0 \). The boundary conditions in the disk plane
outside the disk itself \( v = 0 \) are \( E_v = 0 \) and \( B_w = 0 \). Now we can write Lagrangian
using new variables as
\[ L = L_p + L_A + L_K - K, L_p = L_{pv} + L_{ps}, L_A = L_{Av} + L_{As}, \]
where subscripts \( v \) and \( s \) denote volume and surface terms, and
\[ L_{pv} = -\frac{R}{2} \int_0^1 \int_0^1 (F^w E_w^2 + F^v E_v^2) dv dw, \]
\[ L_{Av} = \frac{R}{2} \int_0^1 \int_0^1 (G^w B_w^2 + G^v B_v^2) dv dw, \]
\[ L_{K - K} = \frac{1}{2R} \int_0^1 \int_0^1 K (B \cdot E)^2 dv dw, \]
\[ L_{ps} = \frac{R^2}{2} \int_0^1 S^p \tilde{\varphi}^2 dv, \]
\[ L_{As} = -\frac{R^2}{2} \int_0^1 S^A \tilde{A}^2 dv, \]
where
\[ F^w = h_v h_\phi / h_w, F^v = h_w h_\phi / h_v, \]
\[ G^w = 1/F^w, G^v = 1/F^v, \]
\[ K = 1/(h_v h_w h_\phi), (B \cdot E) = (B_w E_w + E_v B_v) \]
\[ S^p = h_v h_\phi, S^A = h_v/h_\phi. \]

Orbital contribution to the angular momentum is zero due to the axial symmetry \( E_\phi = 0 \). The z-component of the spin is determined by the integral
\[ s = R^2 \int_0^1 \int_0^1 (D_w \frac{d\varphi}{d\rho} h_w^2 + D_v \frac{dv}{d\rho} h_v^2) \tilde{A} h_\phi dv dw, \]
where
\[ D_w = E_w + B_w (B \cdot E) K/F^w, D_v = E_v + B_v (B \cdot E) K/F^v. \]
3 Numerical solution

In order to solve the model numerically, I approximated the fields by the finite number of degrees of freedom either by expansion of both potentials in series or by discretization of the unit square. The third method combines both approaches. In all methods the problem was reduced to a solution of the system of many non-linear algebraic equations. The system was solved by the iterative Newton’s tangent method for many variables. I supposed that the approximate asymptotic solution was found if solutions converged and the energy values did not change significantly with increasing number of variables. To choose initial approximation I have used the property of Kaluza-Klein interaction (generally speaking, of any Lagrangian with only quadratic and fourth-order dependence on the fields), that

\[ L_A + L_{K-K} = 0, \quad L_p + L_{K-K} = 0, \quad (17) \]

on exact solutions. To ensure positive values of energy \( H \) (in the following I use \( H = -L \)) one requires \( H_A = H_p = -H_{K-K} > 0 \).

3.1 Basic functions method

The simplest method, which allows to find qualitative solutions with only two field degrees of freedom, is basic functions method. In this method one expresses potentials as a finite series of normalized functions with unknown amplitudes, so that Lagrangian becomes a function of these amplitudes. Calculation of bilinear part of Lagrangian requires evaluation of relatively small number of integrals, whereas number of integrals required for quadrolinear terms is proportional to fourth power of basic functions number. Therefore the advantages of the method can be seen only for the small number of correctly chosen functions.

The natural choice of basic functions is solutions of Maxwell equations (fixing the choice of coordinate system so that \( \bar{w}(w) = w \) and \( \bar{v}(v) = v \)) with the first few multipole moments

\[ \tilde{p}_1 = \frac{\pi w}{2}, \quad (18) \]
\[ \tilde{A}_2 = \frac{(-\pi w + \sin (\pi w)) (1 - v^2)}{-1 + \cos (\pi w)}, \quad (19) \]
\[ \tilde{p}_3 = \frac{(4 + 2 \cos (\pi w)) \pi w - 6 \sin (\pi w)) \left(3/2 v^2 - 1/2\right)}{-1 + \cos (\pi w)}, \quad (20) \]

which were normalized at \( v = 0 \) and \( w = 1 \) to be \( \pi/2 \). The potentials become

\[ \tilde{p}(v, w) = \sum_{k=1}^{\infty} \tilde{p}_{2k-1}(v, w) a_{2k-1}, \quad (21) \]
\[ \tilde{A}(v, w) = \sum_{k=1}^{\infty} \tilde{A}_{2k}(v, w) a_{2k}, \]  

(22)

where \( a_1 = e/R, a_2 = \mu/R^2 \), \( 3/2 \), \( a_3, \ldots \) are unknown multipole amplitudes, \( e \) – charge, and \( \mu \) – magnetic moment.

Integration of bilinear part of energy leads to (for the first six multipoles):

\[ H_{pv} = -R/2 (\pi/2) (a_1^2 + 16/5 a_3^2 + 4096/729 a_5^2), \]

(23)

\[ H_{Av} = R/2 (\pi/2) (4/3 a_2^2 + 256/63 a_4^2 + 16384/2475 a_6^2), \]

(24)

\[ H_{ps} = R^2/2 (\pi/2)^2 (1/2 a_1^2 + 1/2 a_3^2 + 1/2 a_5^2 - 1/2 a_1 a_3 - 1/9 a_1 a_5 - 13/36 a_3 a_5), \]

(25)

\[ H_{As} = -R^2/2 (\pi/2)^2 (1/4 a_2^2 + 11/24 a_4^2 + 29/60 a_6^2 - 1/3 a_2 a_4 - 1/12 a_2 a_6 - 1/3 a_4 a_6). \]

(26)

Integration of quadrolinear part of energy is evaluated analytically over \( v \) and numerically over \( w \). An approximate values of first \( 3 \times 3 = 9 \) integrals are listed below

\[ H_{K = K} = 1/(2 R) (2.13 a_1^2 a_2^2 + 6.19 a_1^2 a_2 a_4 + 15.4 a_1^2 a_4^2 + 22.2 a_3^2 a_2^2 + 16.7 a_1 a_2 a_3 a_4 + 121 a_3^2 a_4^2 - 12.0 a_1 a_2 a_3 a_5 + 15.0 a_3 a_4 a_1 - 98.5 a_3^2 a_4 a_2). \]

(27)

To find a simplest solution with only one electric and one magnetic multipoles, we note that the bilinear parts of the energies \( H_p \) and \( H_A \) must be positive, which is achieved if

\[ 1.27 < R_1, R_2 < 3.40 \]

\[ 4.07 < R_3, R_4 < 5.64 \]

\[ 7.15 < R_5, R_6 < 8.72 \]

\[ 10.3 < R_7, R_8 < 11.8 \]

\[ 13.4 < R_9, R_{10} < 15.0 \]

(28)

where the subscripts denote the number of corresponding multipole. I considered values of radii which satisfy each line of eq. 28 separately in order to obtain solutions with minimal energy. First pair of multipoles is centrosymmetric electric field and dipolar magnetic moment. Therefore the energy \( H_{1-2} \) is a function of three variables: two multipole amplitudes \( a_1, a_2 \) and the radius \( R \). From eqs. 23–27 one gets:

\[ H_{1-2} = -R/2 (1.57 a_1^2 - 2.09 a_3^2) + R^2/2 (1.23 a_1^2 - 0.617 a_5^2) - 2.13 a_1^2 a_2^2/(2R). \]

(29)

The values of these unknowns are determined from solution of the system of three algebraic equations, which are obtained from requiring that partial derivatives of the energy in respect to these variables are equal to zero. To obtain the
next state I chose two other degrees of freedom, which are quadrupole electric and octupole magnetic moments, and then solve analogous equations for $H_{3,4}$, etc.

Then the obtained solutions were used as a starting point for solution of Lagrange equations with 8 degrees of freedom in order to increase the precision. This procedure did not lead to a convergent process for the first pair of multipoles but did improve calculations for all others. The possible reason is that the determinant of the matrix of linearized equations does not have constant sign and can be close to zero, which lead to poor convergence. The physical reason of this problem is the areas with very strong electrical field where equation for magnetic field becomes hyperbolic rather than elliptic.

The results of calculation for the first three pairs of leading multipoles are listed in Table 1 ($m = H$).

| Leading multipoles | Included multipoles | $R$ | $m$ | $e$ |
|---------------------|---------------------|-----|-----|-----|
| 1, 2                | 1..2                | 2.81| 3.57| 3.25|
| 1, 2                | 1..6                | -   | -   | -   |
| 3, 4                | 3..4                | 5.03| 0.431| -   |
| 3, 4                | 1..8                | 5.49| 0.366| 1.14|
| 5, 6                | 5..6                | 8.05| 0.337| -   |
| 5, 6                | 1..10               | 8.05| 0.923| 2.05|

Table 1

The table shows that as expected from eqs. 28 the disk size increases with increasing number of leading multipoles. Further increase of number of multipoles must be accompanied by simultaneous addition of new basic functions with non-Maxwellian $v$-dependence for lower multipoles, which would lead to significant increase of number of integrals and calculation time.

### 3.2 Lattice approach

An alternative method is to define the potentials on some lattice and to assume linear interpolation between lattice points. Due to local character of interactions, presence of first derivatives only and linear interpolation assumption the energy depends only on the values of potentials at the neighboring sites, so that each algebraic equation contains limited number of variables (not more than 18), which does not increase with increasing number of lattice points. Therefore calculation time increases slower than the cube of number of points.

Linear interpolation and integration by trapezium rule works well only for slowly varying functions. To increase smoothness of potentials I used specific choice of function $v(v)$, which "stretched" oscillations of the potentials near the
disk axis. In particular I used $\bar{v}(v) = 3/2v - 1/2v^3$ and $\bar{w}(w) = w$. The results of calculation for different number of lattice points are listed in Table 2.

| Leading multipoles | Lattice size | $R$  | $m$  | $e$  |
|--------------------|--------------|------|------|------|
| 3,4                | 10 × 10      | 6.01 | 1.40 | 3.55 |
| 3,4                | 12 × 12      | 5.98 | 1.43 | 3.68 |
| 3,4                | 15 × 15      | 5.96 | 1.47 | 3.81 |
| 3,4                | 20 × 20      | 5.96 | 1.52 | 3.91 |
| 3,4                | $\infty \times \infty$ | 5.96 | 1.60 | 4.04 |
| 5,6                | 12 × 12      | 9.88 | 1.83 | 3.92 |
| 5,6                | 14 × 14      | 9.62 | 1.58 | 3.72 |
| 5,6                | 18 × 18      | 9.38 | 1.41 | 3.69 |
| 5,6                | 24 × 24      | 9.23 | 1.32 | 3.68 |
| 5,6                | $\infty \times \infty$ | 9.05 | 1.22 | 3.68 |
| 7,8                | 12 × 12      | 15.0 | 3.61 | 5.89 |
| 7,8                | 14 × 14      | 14.2 | 2.44 | 4.40 |
| 7,8                | 18 × 18      | 13.4 | 1.70 | 3.96 |
| 7,8                | 24 × 24      | 12.8 | 1.40 | 3.90 |
| 7,8                | $\infty \times \infty$ | 12.1 | 1.20 | 3.88 |

Table 2

I used solutions of the previous section as a starting point for the lattice method. At the end of each part of the table extrapolated values for the infinite lattice are listed. For leading multipoles 1,2 solutions was found for lattice 6×6 only. For leading multipoles 3,4 stable solutions exist for a large range of number of degrees of freedom, and observables quickly achieve their asymptotic values. Convergence gets worse with increasing order of leading multipole because higher order multipoles have more oscillations on the disk surface. In particular potentials of 3,4 multipole have two extremums, 5,6 — 3 extremums and 7,8 — 4 extremums.

### 3.3 Combined method

To improve precision of calculations and the consistency of two previous methods I divided space into two parts: for large distances from the disk ($w = 0..3/4$) I used basic functions method with $2 \times 5 = 10$ functions, and in the vicinity of the disk ($w = 3/4..1$) I used lattice approach. The results of combined method are listed in Table 3.
| Leading multipoles | Lattice size | $R$  | $m$  | $e$  |
|-------------------|-------------|------|------|------|
| 3,4               | 12 × 3      | 5.97 | 1.41 | 3.63 |
| 3,4               | 16 × 4      | 5.95 | 1.47 | 3.77 |
| 3,4               | 20 × 5      | 5.96 | 1.51 | 3.88 |
| 3,4               | 28 × 7      | 5.96 | 1.55 | 3.92 |
| 3,4               | ∞ × ∞/4     | 5.97 | 1.57 | 3.94 |
| 3,4               | ∞ × ∞       | 5.96 | 1.60 | 4.04 |
| 5,6               | 16 × 4      | 9.31 | 1.37 | 3.70 |
| 5,6               | 20 × 5      | 9.36 | 1.49 | 3.91 |
| 5,6               | 28 × 7      | 9.21 | 1.35 | 3.77 |
| 5,6               | 32 × 8      | 9.15 | 1.27 | 3.65 |
| 5,6               | 40 × 10     | 9.13 | 1.25 | 3.63 |
| 5,6               | ∞ × ∞/4     | 9.06 | 1.19 | 3.56 |
| 5,6               | ∞ × ∞       | 9.05 | 1.22 | 3.68 |
| 7,8               | 16 × 4      | 13.7 | 1.96 | 4.13 |
| 7,8               | 20 × 5      | 13.1 | 1.55 | 3.87 |
| 7,8               | 28 × 7      | 12.6 | 1.32 | 3.92 |
| 7,8               | 32 × 8      | 12.5 | 1.27 | 3.91 |
| 7,8               | 40 × 10     | 12.4 | 1.21 | 3.90 |
| 7,8               | ∞ × ∞/4     | 12.1 | 1.10 | 3.90 |
| 7,8               | ∞ × ∞       | 12.1 | 1.20 | 3.88 |

Table 3

The last lines of each part of this table repeats the asymptotic results of the previous section. The results of lattice and combined methods are in good agreement with each other. However, combined method is 10 times faster and therefore gives more precise results. The results for $\bar{v} = 2v - v^2$ and $40 \times 10$ lattice differ from Table 3 only by a few percent even for 7,8 multipole.

4 Conclusions

I proposed a new concept of extended particle that is associated with a membrane of finite size. Charges and currents on this surface are chosen to be proportional to the corresponding potentials of the non-linear Kaluza-Klein electromagnetic field, so that the gauge invariance is broken on this surface. Then I assumed the axial and mirror symmetry of the system and existence of the only one surface degree of freedom: radius of the disk. Number of field degrees of freedom was also chosen to be finite. Described set of assumptions allowed me to develop
a technique for iterative calculation of the first three stationary states of the model. Then I demonstrated that increasing number of degrees of freedom (up to 1000) leads to the asymptotic solutions of Lagrange equations. Several states with increasing order of leading electric and magnetic multipoles and disk radii were constructed numerically using this method. Each state can be potentially associated with an elementary particle.

The key result of the paper is that the values of all observables are finite. The masses were found to decrease weakly with increasing order of leading multipoles. The charges, magnetic moments and spins were also calculated from the model rather than postulated. It was not obvious that charges and spins of different states are identical or that their ratios are rational numbers. Calculations have shown that they are equal for the first three states within 10-20%:

| Leading multipoles | $R$ | $m$ | $e$ | $s$ | $1/\alpha$ | $\mu/\mu_B$ |
|---------------------|-----|-----|-----|-----|------------|------------|
| 3,4                 | 6.0 | 1.6 | 4.0 | 12  | 1.4        | 1.0        |
| 5,6                 | 9.1 | 1.2 | 3.6 | 11  | 1.5        | 1.3        |
| 7,8                 | 12.1| 1.1 | 3.9 | 12  | 1.5        | 1.1        |

Table 4

More detailed and extensive calculations are needed to make more precise conclusion. From the other side, the question about equality of charges and spins can be resolved by discovering some hidden symmetry of the described or modified interaction. It is also possible to change non-linearity to Born-Infeld type without adding any new parameters to the model.

Masses, charges, magnetic moments and spins are proportional to some powers of the two fundamental constants of the model with dimensions of length and field strength. However, there are two observables which do not depend on these constants — Lande coefficient ($\mu/\mu_B$) and the ratio of charge square to a double spin, which is equal to the fine structure constant if one assumes that the spin is $\hbar/2$. Lande coefficient was found about 1, and the fine structure constant — more than 100 times larger than expected. Therefore, calculated values of these parameters do not allow to associate discovered stationary states with charged leptons. Another interpretations (quarks) would require development of a calculation technique for bound states probably with non-additive charges and spins.

The presented method allows to modify calculations for different non-linearity and/or boundary conditions on the surface without adding new model parameters. This will preserve the predictive ability of the approach and the existence of finite values of the observables.
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