In this paper we will give a short presentation of the quantum Lévy-Khinchin formula and of the formulation of quantum continual measurements based on stochastic differential equations, matters which we had the pleasure to work on in collaboration with Prof. Holevo. Then we will begin the study of various entropies and relative entropies, which seem to be promising quantities for measuring the information content of the continual measurement under consideration and for analysing its asymptotic behaviour.

1 A quantum Lévy-Khinchin formula

The theory of measurements continuous in time in quantum mechanics (quantum continual measurements) started with the description of counting experiments and of situations in which an observable is measured imprecisely, but with continuity in time; both formulations are based on the notions of instrument and of positive operator valued measure. Soon after we succeeded in unifying the two approaches, Holevo realized that some quantum analogue of infinite divisibility was involved and thus started a search of a quantum Lévy-Khinchin formula, a review is given in refs. while a different approach is presented in refs. A review is given in refs. while a different approach is presented in refs.

Let $\mathcal{H}$ be a complex separable Hilbert space, $T(\mathcal{H})$ be the trace-class on $\mathcal{H}$ and $S(\mathcal{H})$ be the set of statistical operators. We denote by $\mathcal{L}(\mathcal{H}_1;\mathcal{H}_2)$ the space of linear bounded operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ and set $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H};\mathcal{H})$. $\langle a, \tau \rangle = \text{Tr}\{a\tau\}$, $\tau \in T(\mathcal{H})$, $a \in \mathcal{L}(\mathcal{H})$; $\|\tau\|_1 = \text{Tr}\{\sqrt{\tau^*\tau}\}$.

An instrument is a map-valued $\sigma$-additive measure $\mathcal{N}$ on some measurable space $(\mathcal{Y}, \mathcal{E})$; the maps are from $T(\mathcal{H})$ into itself, linear, completely positive and normalized in the sense that $\text{Tr}\{\mathcal{N}(\mathcal{Y})[\tau]\} = \text{Tr}\{\tau\}$.

The formulation of continual measurements given by Holevo is based on analogies with the Lévy processes and it is less general, but more fruitful, than the one initiated by our group and based on the generalized stochastic
processes. In order to simplify the presentation, we will only consider the case of one-dimensional processes. Let \( Y \) be the space of all real functions on the positive time axis starting from zero, continuous from the right and with left limits, and let \( B_a^b, 0 \leq a \leq b \), be the \( \sigma \)-algebra generated by the increments \( y(t) - y(s), a \leq s \leq t \leq b \). A time homogeneous instrumental process with independent increments (i-process) is a family \( \{ N_b^a; 0 \leq a \leq b \} \), where \( N_b^a \) is an instrument on \( (Y, B_a^b) \) such that

\[
N_b^a + t \left( E \right) = N_b^a(E) \quad \text{for arbitrary } b \geq a, \quad t \in \mathbb{R}_+, \quad E \in B_a^b,
\]

where \( E = \{ y : T_t y \in E \} \), \( (T_t y)(s) = y(s+t) \), and such that

\[
N_b^a(E) \circ N_c^b(F) = N_c^a(E \cap F), \quad 0 \leq a \leq b \leq c, \quad E \in B_a^b, \quad F \in B_b^c. \tag{1}
\]

Every i-process is determined by its finite-dimensional distributions, which have the structure

\[
\mathcal{N}_{t_0}^{t_p}(y(\cdot) : y(t_1) - y(t_0) \in B_1, \ldots, y(t_p) - y(t_{p-1}) \in B_p) = \mathcal{N}_{t_p - t_{p-1}}(B_p) \circ \cdots \circ \mathcal{N}_{t_1 - t_0}(B_1), \tag{2}
\]

where \( 0 \leq t_0 < t_1 < \cdots < t_p, B_1, \ldots, B_p \in B(\mathbb{R}) \), and

\[
\mathcal{N}_t(B) = \mathcal{N}_a^{a+t}(y(\cdot) : y(a+t) - y(a) \in B) \tag{3}
\]

is independent of \( a \) by the time homogeneity. The instrument \( \mathcal{N}_t \) completely determines the i-process and it is completely characterized by its Fourier transform (characteristic function) \( \int e^{iky} \mathcal{N}_t(dy) \); Eq. (1) and the continuity assumption

\[
\lim_{t \downarrow 0} ||\mathcal{N}_t(U_0) - \mathbb{1}|| = 0, \quad \text{for every neighbourhood } U_0 \text{ of } 0, \tag{4}
\]

imply that this characteristic function is of the form \( \exp\{tK(k)\} \), \( K(k) \in L(T(\mathcal{H})) \). The quantum Lévy-Khinchin formula is the complete characterization of the generator \( K \). The structure of \( K \) can be written in different equivalent ways and here we give an expression which is particularly convenient for reformulating the theory of the continual measurements in terms of stochastic differential equations, as illustrated in the next section.

The quantum Lévy-Khinchin formula for the generator \( K \) is: \( \forall \tau \in T(\mathcal{H}), \forall h, g \in \mathcal{H}, \forall k \in \mathbb{R}, \forall h, g \in \mathcal{H}, \)

\[
K(k)[\tau] = \mathcal{L}[\tau] + ikc\tau - \frac{1}{2} \varphi_{2}(R\tau + \tau R^*) + \int_{\mathbb{R}} \left[ (e^{ikz} - 1)J[z](z) - ikz\varphi_{2}(z)\tau \right] \mu(dz), \tag{5}
\]

imply that this characteristic function is of the form \( \exp\{tK(k)\} \), \( K(k) \in L(T(\mathcal{H})) \).
where \( c \in \mathbb{R}, \; r \in \mathbb{R}, \; \varphi_2(z) = \frac{b^2}{b^2 + z^2}, \; b > 0, \)
\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2,
\]
\[
\mathcal{L}_0[\tau] = -i[H, \tau] + \frac{1}{2} \sum_{j=1}^{\infty} \left( [L_j \tau, L^*_j] + [L_j, \tau L^*_j] \right),
\]
\[
\mathcal{L}_1[\tau] = \frac{1}{2} \left( [R \tau, R^*] + [R, \tau R^*] \right),
\]
\[
\mathcal{L}_2[\tau] = -\frac{1}{2} J^* J \tau - \frac{1}{2} \tau J^* J + \text{Tr}_{z^2} \{ J \tau J^* \},
\]
\[
\mathcal{J}[h|g](z) = \sum_{n=1}^{\infty} \frac{\nu(dz \times \{n\})}{\mu(dz)} |(Jh)(z, n) + h\rangle \langle (Jg)(z, n) + g|,\]
\[
R, H, L_j \in \mathcal{L}(\mathcal{H}), \; H = H^*, \; \sum_{j=1}^{\infty} L_j^* L_j \in \mathcal{L}(\mathcal{H}) \; \text{(strong convergence)}, \; \mathbb{R}_* = \mathbb{R}\setminus\{0\}, \nu \text{ is a } \sigma\text{-finite measure on } \mathbb{R}_* \times \mathbb{N} \text{ and } \mu(dz) = \sum_{n=1}^{\infty} \nu(dz \times \{n\});\]
\[
\int_{\mathbb{R}_*} \varphi_1(z) \mu(dz) = \sum_{n=1}^{\infty} \int_{\mathbb{R}_*} \varphi_1(z) \nu(dz \times \{n\}) < +\infty,
\]
with \( \varphi_1(z) = \frac{z^2}{b^2 + z^2} \). Note that \( \varphi_1(z) + \varphi_2(z) = 1 \) and that \( \mu \) is a Lévy measure on \( \mathbb{R}_* \). Finally, \( J \in \mathcal{L}(\mathcal{H}; L^2(\mathcal{H})) \), where \( L^2_v = L^2(\mathbb{R}_* \times \mathbb{N}, \nu) \), \( L^2(\mathcal{H}) = L^2(\mathbb{R}_* \times \mathbb{N}; \nu; \mathcal{H}) \simeq L^2_v \otimes \mathcal{H} \). The fact that the operators \( H, R, L_j, J \) are bounded is due to the assumption \( (7) \), which is therefore a strong restriction from a physical point of view.

It is convenient to introduce also the characteristic functional of the whole i-process as the solution of the equation: \( \forall a \in \mathcal{L}(\mathcal{H}), \forall \tau \in \mathcal{T}(\mathcal{H}), \)
\[
\langle a, \mathcal{G}_t(k)[\tau] \rangle = \langle a, \tau \rangle + \int_0^t \langle a, \mathcal{K}(k(s)) \circ \mathcal{G}_s(k)[\tau] \rangle \; ds,
\]
where \( k(t) \) is a real function, continuous from the left with right limits; let us call it a test function. By taking \( k(t) = \kappa \mathbb{1}_{[0,T]}(t) \), we get \( \mathcal{G}_T(k) = \exp\{t\mathcal{K}(\kappa)\} \) and, similarly, by taking a more general step function for \( k \) we get the Fourier transform of the finite-dimensional distributions \( (12) \), so that \( \mathcal{G}_t \) completely characterizes the i-process.

The operators \( \mathcal{U}(t) = \exp\{t\mathcal{L}\} = \mathcal{G}_t(0) = \mathcal{N}_t(\mathbb{R}), \) \( t \geq 0, \) form a completely positive quantum dynamical semigroup. We fix an initial state \( \varrho \in \mathcal{S}(\mathcal{H}) \) and set \( \eta_t = \mathcal{U}(t)[\varrho]; \eta_t \) is called the a priori state at time \( t \) because it represents the state of the system at time \( t \), when no selection is done on the basis of the results of the continual measurement. The a priori states satisfy the master equation
\[
\frac{d}{dt} \eta_t = \mathcal{L}[\eta_t], \quad \eta_0 = \varrho.
\]
2 Stochastic differential equations

An alternative useful formulation of quantum continual measurements is based on stochastic differential equations (SDE’s); it was initiated for the basic cases by Belavkin\(^{13}\) by using analogies with the classical filtering theory. The general SDE’s corresponding to the Lévy-Khinchin formula\(^5\) were studied in refs.\(^{14}\).

2.1 Output signal and reference probability

Let \(W\) be a one-dimensional standard continuous Wiener process and \(N(dz \times dt)\) be a random Poisson measure on \(\mathbb{R}_+ \times \mathbb{R}_+\) of intensity \(\mu(dz)dt\), independent of \(W\). The two processes are realized in a complete standard probability space \((\Omega, \mathcal{F}, Q)\) with the filtration of \(\sigma\)-algebras \(\{\mathcal{F}_t, t \geq 0\}\), which is the augmentation of the natural filtration of \(W\) and \(N\); we assume also \(\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t\). It is useful to introduce the compensated process

\[
\tilde{N}(dz \times dt) = N(dz \times dt) - \mu(dz)dt.
\]

In all the SDE’s such as Eqs. \((15), (17), (18), (19), (34)\), the presence of integrals with respect either to the jump process \(N\) or to the compensated processes \(\tilde{N}\) or \(\tilde{N}\) (see \((28)\)) is due to problems of convergence of the stochastic integrals which arise when infinitely many small jumps are present (the case \(\int_{\mathbb{R}_+} \mu(dz) = +\infty\)).

Now, by using \(W\), \(N\) and all the ingredients entering the Lévy-Khinchin formula\(^5\), we are able to construct various random quantities which allow us to reexpress in a different form the i-process of the previous section. Firstly, let us introduce the real process

\[
Y(t) = ct + rW(t) + \int_{\mathbb{R}_+ \times (0,t]} \varphi_1(z)zN(dz \times ds) + \int_{\mathbb{R}_+ \times (0,t]} \varphi_2(z)z\tilde{N}(dz \times ds),
\]

which, under the reference probability \(Q\), is a generic Lévy process; \(Y\) will represent the output process of the continual measurement introduced in the previous section. In the following we shall need the quantity

\[
\Phi_t(k) = \exp\left\{i \int_0^t k(s)dY(s)\right\}
\]

and its stochastic differential

\[
d\Phi_t(k) = \Phi_t(k) \left\{ \int_{\mathbb{R}_+} \left( e^{ik(t)}z - 1 - i k(t) \varphi_2(z)z \right) \mu(dz) + i ck(t)
\right.
\]

\[
- \frac{1}{2} r^2 k(t)^2 dt + i r k(t)dW(t) + \int_{\mathbb{R}_+} \left( e^{ik(t)}z - 1 \right) \tilde{N}(dz \times dt). \]

4
In Table 1 we summarize the rules of stochastic calculus for $W$ and $N$, which have been used in computing $d\Phi_t(k)$ and which shall be used to compute all the stochastic differentials in the rest of the paper.

|       | $dt$ | $dW(t)$ | $N(dz \times dt)$ |
|-------|------|---------|-------------------|
| $dt$  | 0    | 0       | 0                 |
| $dW(t)$ | 0   | $dt$    | 0                 |
| $N(dz \times dt)$ | 0   | 0       | $\delta(z - z') N(dz \times dt)$ |

$$f\left(X + \int_{\mathbb{R}} C(z) N(dz \times dt)\right) - f(X) = \int_{\mathbb{R}_+} \left[ f\left(X + C(z)\right) - f(X) \right] N(dz \times dt)$$

2.2 A linear SDE and the instruments

Let us consider now the linear SDE for $\sigma_t \in \mathcal{T}(\mathcal{H})$, $\sigma_t \geq 0$: $\forall a \in \mathcal{L}(\mathcal{H})$,

$$\langle a, \sigma_t \rangle = \langle a, \emptyset \rangle + \int_0^t \langle a, \mathcal{L}[\sigma_s] \rangle ds + \int_0^t \langle a, R\sigma_s + \sigma_s R^* \rangle dW(s)$$

$$+ \int_{\mathbb{R}_+ \times (0,t]} \langle a, \mathcal{J}[\sigma_s](z) - \sigma_s \rangle \tilde{N}(dz \times ds). \quad (18)$$

We call the $\sigma_t$ non normalized a posteriori states (nnap states); the reason will be clarified in the following. The coefficient of the jump term should be written as $\mathcal{J}[\sigma_s^{-}](z) - \sigma_s^{-}$, with the following meaning: when there is a jump of $N$, i.e. when $N(dz \times ds) = 1$, the nnap state before the jump $\sigma_s^{-}$ is transformed into the state after the jump $\sigma_s^{+} = \mathcal{J}[\sigma_s^{-}](z)$; however, we prefer to simplify the notation and not to write the superscripts “minus”. Similar considerations apply to all the other SDE’s.

By using Table 1 to differentiate $\Phi_t(k)\langle a, \sigma_t \rangle$, we get

$$d(\Phi_t(k)\langle a, \sigma_t \rangle) = \Phi_t(k) \left\{ \langle a, \mathcal{K}(k(t))[\sigma_t] \rangle dt + \langle a, i\sigma_t R^* \sigma_t \rangle dW(t) + \int_{\mathbb{R}_+} \langle a, e^{ik(t)z} \mathcal{J}[\sigma_t](z) - \sigma_t \rangle \tilde{N}(dz \times dt) \right\} \quad (19)$$

and by taking the expectation we see that the terms with $dW$ and $\tilde{N}$ disappear and that the resulting equation is the same as Eq. (12), which defines $\mathcal{G}$. Therefore, we have

$$\langle a, \mathcal{G}_t(k)[\emptyset] \rangle = E_Q \left[ \Phi_t(k)\langle a, \sigma_t \rangle \right], \quad (20)$$

an equation showing that $Y(t)$ and $\sigma_t$ completely determine the characteristic functional of the continual measurement and, so, the whole i-process. In
particular, by taking $k = 0$ we obtain that the expectation value of the nnap states gives the a priori states: $E_Q [\langle a, \sigma_t \rangle] = \langle a, \eta_t \rangle$.

### 2.3 The physical probability and the a posteriori states

Let us now study the norm of the nnap states: $\|\sigma_t\|_1 = \langle 1_l, \sigma_t \rangle = \text{Tr}\{\sigma_t\}$. By taking the trace of Eq. (18) we get

$$d \|\sigma_t\|_1 = \|\sigma_t\|_1 \left\{ m(t) dW(t) + \int_{\mathbb{R}} \left[ I_t(z) - 1 \right] N(\text{d}z \times \text{d}t) \right\},$$

where

$$m(t) = \langle R + R^*, \rho_t \rangle, \quad I_t(z) = \langle 1_l, J[\rho_t](z) \rangle = \langle J(z), \rho_t \rangle, \quad (22)$$

$$\langle g| J(z) h \rangle = \sum_{n=1}^{\infty} \frac{\nu(\text{d}z \times \{n\})}{\mu(\text{d}z)} \langle (Jg)(z, n) + g((Jh)(z, n) + h \rangle, \quad \forall g, h \in \mathcal{H}, \quad (23)$$

$$\rho_t = \begin{cases} \|\sigma_t\|_1^{-1} \sigma_t & \text{if } \|\sigma_t\|_1 > 0 \\ \varrho & \text{otherwise} \end{cases} \quad (24)$$

The operators $\rho_t$ belong to $\mathcal{S}(\mathcal{H})$ and will be called \textit{a posteriori states}, as explained below. Note the common initial state: $\eta_0 = \sigma_0 = \rho_0 = \varrho$. It is possible to show that $\|\sigma_t(\omega)\|_1$ is a martingale and that it can be used as a local density with respect to $Q$ to define a new probability $P_\varrho$ on $(\Omega, \mathcal{F})$, the \textit{physical probability}, by

$$P_\varrho(\text{d}\omega)\Big|_{\mathcal{F}_t} = \|\sigma_t(\omega)\|_1 Q(\text{d}\omega)\Big|_{\mathcal{F}_t}, \quad \text{or} \quad \frac{P_\varrho(\text{d}\omega)}{Q(\text{d}\omega)}\Big|_{\mathcal{F}_t} = \|\sigma_t(\omega)\|_1. \quad (25)$$

By taking $a = 1_l$ in (20) and by using the new physical probability we can write

$$\langle 1_l, G_t(\cdot)|a \rangle = E_{P_\varrho}[\Phi_t(\cdot)]. \quad (26)$$

This equation shows that the Fourier transform of all the probabilities involved in the continual measurement is given by the characteristic functional of the process $Y(t)$ under the probability $P_\varrho$. It is this fact which substantiates the interpretation of $P_\varrho$ as the physical probability and of $Y(t)$ as the output process.

It is possible to prove that under the physical probability $P_\varrho$

$$\hat{W}(t) = W(t) - \int_0^t m(s) \text{d}s \quad (27)$$
is a standard Wiener process and \( \tilde{N}(dz \times dt) \) is a point process of stochastic intensity \( I_t(z)\mu(dz)dt \); we set

\[
\tilde{N}(dz \times dt) = N(dz \times dt) - I_t(z)\mu(dz)dt.
\]

The typical properties of the trajectories of the output signal can be visualized in a particularly simple manner when \( \int_{\mathbb{R}} \varphi(z)z\mu(dz) < +\infty \); in this case we can write

\[
Y(t) = Y_{cbv}(t) + r\tilde{W}(t) + \int_{\mathbb{R} \times (0,t]} zN(dz \times ds)
\]

where \( \int_{\mathbb{R} \times (0,t]} zN(dz \times ds) \) is the jump part, with jumps of amplitude \( z \) and intensity \( I_s(z)\mu(dz)ds \), \( r\tilde{W}(t) \) is a continuous part proportional to a Wiener process and

\[
Y_{cbv}(t) = t\left(c - \int_{\mathbb{R}_+} \varphi(z)z\mu(dz)\right) + \int_0^t m(s)ds
\]

is a continuous part with bounded variation.

By rewriting Eq. (20) with the new probability, we have

\[
\langle a, G_t(k)[\rho] \rangle = \mathbb{E}_{P_\rho}[\Phi_t(k)(a, \rho_t)] .
\]

Because \( G \) is the Fourier transform of all the finite-dimensional distributions and these distributions determine the whole i-process, this last equation is equivalent to: \( \forall a \in \mathcal{L}(\mathcal{H}), \forall t \geq 0, \forall E \in \mathcal{B}_0 \),

\[
\langle a, N_0^t(E)[\rho] \rangle = \int_{\{\omega \in \Omega : Y(\omega) \in E\}} \langle a, \rho_t(\omega) \rangle P_\rho(d\omega).
\]

This equation shows that \( \rho_t \) is the state conditioned on the trajectory of the output observed up to time \( t \) and \( \rho_t \) has indeed the meaning of a posteriori state at time \( t \): the state we must attribute to the system under a selective measurement up to \( t \). By taking \( k = 0 \) into Eq. (31) or \( E = \mathcal{Y} \) into Eq. (32), we get

\[
\langle a, \eta_t \rangle = \mathbb{E}_{P_\rho}[\langle a, \rho_t \rangle],
\]

i.e. the a posteriori states \( \rho_t(\omega) \) with the physical probability \( P_\rho(d\omega) \) realize a demixture of the a priori state \( \eta_t \). Finally, by differentiating the definition (24) of the a posteriori states, we get the SDE

\[
\langle a, \rho_t \rangle = \langle a, \varphi \rangle + \int_0^t \langle a, R\rho_s + \rho_s R^* - m(s)\rho_s \rangle d\tilde{W}(s) \\
+ \int_{\mathbb{R} \times (0,t]} \langle a, j(\rho_s; z) - \rho_s \rangle \tilde{N}(dz \times ds) + \int_0^t \langle a, L[\rho_s] \rangle ds ,
\]

\[
j(\tau; z) = (\text{Tr} \{ J[\tau](z) \})^{-1} J[\tau](z) , \quad \tau \in \mathcal{S}(\mathcal{H}) ;
\]

Eq. (34) holds under the physical probability \( P_\rho \).
3 Entropies and information

3.1 Quantum and classical entropies

In quantum measurement theory both quantum states and classical probabilities are involved and, so, quantum and classical entropies are relevant.

For \( x, y \in \mathcal{T}(\mathcal{H}), \) \( x \geq 0, \) \( y \geq 0, \) we introduce the functionals, with values in \([0, +\infty)\),

\[
S_q(x) = -\Tr\{x \ln x\}, \quad S_q(x|y) = \Tr\{x \ln x - x \ln y\};
\]

if \( x, y \in \mathcal{S}(\mathcal{H}), \) \( S_q(x) \) is the von Neumann entropy and \( S_q(x|y) \) is the quantum relative entropy. The von Neumann entropy can be infinite only if the Hilbert space is infinite dimensional and it is zero only on the pure states, while the quantum relative entropy can be infinite even when the Hilbert space is finite dimensional and it is zero only if the two states are equal.

A first quantum entropy of interest is the a priori entropy \( S_q(\eta) \), which at time zero reduces to the entropy of the initial state \( S_q(\eta_0) = S_q(\rho) \).

On the other hand, a classical entropy is the relative entropy (or Kullback-Leibler informational divergence) of the physical probability \( P_\theta \) with respect to the reference probability measure \( Q \):

\[
I_t(P_\theta|Q) = \mathbb{E}_{P_\theta} \left[ \ln \frac{P_\theta(\omega)}{Q(\omega)} \right] = \mathbb{E}_Q \left[ ||\sigma_t||_1 \ln ||\sigma_t||_1 \right],
\]

Let us note that \( I_t(P_\theta|Q) \geq 0, \) \( I_0(P_\theta|Q) = 0 \) and that \( I_t(P_\theta|Q) \) is non decreasing, as one sees by computing its time derivative:

\[
\frac{d}{dt} I_t(P_\theta|Q) = \mathbb{E}_{P_\theta} \left[ \ln \frac{P_\theta(\omega)}{Q(\omega)} \right] = \mathbb{E}_Q \left[ ||\sigma_t||_1 \ln ||\sigma_t||_1 \right] 
\]

If we consider two different initial states \( \rho^\alpha \) and \( \rho \), with supp \( \rho^\alpha \subseteq \text{supp} \rho \), we can introduce the quantum relative entropy \( S_q(\eta^\alpha|\eta) \) and the classical \( P_{\rho^\alpha}|P_\rho \)-relative entropy \( I_t(P_{\rho^\alpha}|P_\rho), \)

\[
I_t(P_{\rho^\alpha}|P_\rho) = \mathbb{E}_{P_{\rho^\alpha}} \left[ \ln \frac{P_{\rho^\alpha}(\omega)}{P_\rho(\omega)} \right] = \mathbb{E}_Q \left[ ||\sigma^\alpha_t||_1 \ln ||\sigma^\alpha_t||_1 \right].
\]

Here and in the following \( P_{\rho^\alpha}, \sigma^\alpha_t, \rho^\alpha_t, \eta^\alpha_t, m^\alpha(t), \mathcal{I}^\alpha_t(z) \) are defined by starting from \( \rho^\alpha \) as \( P_\theta, \sigma_t, \rho_t, \eta_t, m(t), \mathcal{I}_t(z) \) are defined by starting from \( \rho \).

Let us stress the different behaviour in time of the two relative entropies; this discussion will be relevant later on. The quantum one starts from \( S_q(\rho^\alpha|\rho) \) at time zero and it is non increasing

\[
S_q(\eta^\alpha_t|\eta_t) = S_q(\mathcal{U}(t-s) [\eta^\alpha_s] | [\mathcal{U}(t-s) | \eta_s]) \leq S_q(\eta^\alpha_s|\eta_s), \quad t > s; \quad (40)
\]

this statement follows from the Uhlmann monotonicity theorem (ref. Theor. 5.3). The classical relative entropy starts from zero at time zero and it is non
the purity

The a posteriori entropy and purity vanish if and only if the a posteriori states $E$ takes the initial value $\Omega$. However, both relative entropies have the same bounds:

$$0 \leq S_q(\rho^n|\eta) \leq S_q(\rho^n|\eta)\leq S_q(\rho^n|\eta).$$

The first statement is clear [see Eq. (40)]. The second one too is a consequence of the Uhlmann monotonicity theorem, as can be seen by considering the “observation channel” $\Lambda: \mathcal{L}(\mathcal{H}) \to L^\infty(\Omega, \mathcal{F}_t, Q)$ with predual $\Lambda^*: \rho \to P_\rho \in L^1(\Omega, \mathcal{F}_t, Q)$ (in ref. [15] see p. 138, Theor. 5.3 and the discussions at pgs. 9 and 151).

### 3.2 Entropies and purity of the states

When one is studying the properties of an instrument, a relevant question is whether the a posteriori states are pure or not and, if not pure, how to measure their “degree of mixing”. Ozawa called quasi-complete an instrument which sends every initial pure state into pure a posteriori states. A first measure of purity of the a posteriori states is the a posteriori entropy $\mathbb{E}_{P_\rho} [S_q(\rho)]$, which takes the initial value $\mathbb{E}_{P_\rho} [S_q(\rho)] = \mathbb{S}_q(\rho)$. A related quantity, simpler to study, is the a posteriori purity (or linear entropy)$\rho(t) = \mathbb{E}_{P_\rho} [\text{Tr} \{\rho_t (I - \rho_t)]\}, \quad \rho(0) = \text{Tr} \{\rho (I - \rho)\}.$

The a posteriori entropy and purity vanish if and only if the a posteriori states are almost surely pure and one has $p(t) \leq \mathbb{E}_{P_\rho} [S_q(\rho)]$.

By the rules of stochastic calculus (Table 1) we get the time derivative of the purity

$$\frac{d}{dt} \rho(t) = \dot{\rho}_1(t) - \dot{\rho}_2(t) - \dot{\rho}_3(t),$$

$$\dot{\rho}_1(t) = 2 \sum_{j=1}^\infty \mathbb{E}_{P_\rho} \left[\text{Tr} \left\{\rho_t L_j^* L_j \rho_t - \rho_t^{1/2} L_j^* L_j \rho_t^{1/2}\right\}\right],$$

$$\dot{\rho}_2(t) = \mathbb{E}_{P_\rho} \left[\text{Tr} \left\{\rho_t^{1/2} (R + R^* - m(t)) \rho_t (R + R^* - m(t)) \rho_t^{1/2}\right\}\right] \geq 0,$$

$$\dot{\rho}_3(t) = \int_{\mathbb{R}_+} \mathbb{E}_{P_\rho} \left[\text{Tr} \left\{I_t(z) (\rho_t^2 z^2 - 2J[\rho_t^2](z) + I_t(z)\rho_t^2\right\}\right] \mu(dz)$$

$$= \int_{\mathbb{R}_+} \mathbb{E}_{P_\rho} \left[I_t(z) \left\{\rho_t^{1/2} \mathcal{H}(z) \rho_t^{1/2} - I_t(z)\rho_t^{1/2}\right\}\mu(dz)\right.$$
Then, one can check the following points.

(a) If $\rho_t$ is almost surely a pure state, then one has $\dot{p}_1(t) \geq 0$, $\dot{p}_2(t) = 0$, $\dot{p}_3(t) = -\int_{\mathbb{R}^*} E_{P_\tau} \left[ \text{Tr} \left\{ j(\rho_t; z) - j(\rho_t; z^2) \right\} I(z) \right] \mu(dz) \leq 0$.

(b) The a posteriori states are almost surely pure for all pure initial states (i.e. the measurement is quasi complete) if and only if the following conditions hold:

\begin{itemize}
  \item \textbf{C1.} $L_0[\cdot] = -i[H, \cdot]$;
  \item \textbf{C2.} $j(\tau; z)$ is a pure state ($\mu$-almost everywhere) for all pure states $\tau$ or, equivalently, in $\mathcal{H}$ $(Jh)(z, n) = (Jh)(z)$, $\forall h \in \mathcal{H}$.
\end{itemize}

(c) Under the same conditions one has $\dot{p}_1(t) = 0$, $\dot{p}_3(t) \geq 0$ for any initial state; as $\dot{p}_2(t) \geq 0$ always, the purity decreases monotonically.

The properties of the purity have also been used to find sufficient conditions (among which there is the quasi-completeness property) so that the long time limit of the a posteriori purity will vanish for every initial state; note that in a finite dimensional Hilbert space this is equivalent to the vanishing of the limit of the a posteriori entropy.

Differentiating the a posteriori entropy demands long computations involving an integral representation of the logarithm (ref. p. 51) and the rules of stochastic calculus. We get

$$
\frac{d}{dt} E_{P_\tau} \left[ S_q(\rho_t) \right] = E_{P_\tau} \left[ D_1(\rho_t) - D_2(\rho_t) - D_3(\rho_t) \right],
$$

(48)

where, $\forall \tau \in \mathcal{S}(\mathcal{H})$,

\begin{align*}
D_1(\tau) &= \sum_j \text{Tr} \left\{ (L_j^* L_j \tau - L_j \tau L_j^*) \ln \tau \right\}, \\
D_2(\tau) &= \int_0^{+\infty} du \left\{ \frac{u}{u+\tau} \right\} \left\{ (R + R^*) \ln \frac{\tau}{u+\tau} \right\} \\
&\quad \times (R + R^* - \text{Tr} \left\{ (R + R^*) \tau \right\}) + \frac{\tau}{(u+\tau)^2} \left[ R, R \right] \frac{\tau}{u+\tau} R^* \\
&\quad - \left[ \frac{\tau}{u+\tau}, R \right] \frac{\tau}{u+\tau} R^*,
\end{align*}

(49)

\begin{align*}
D_3(\tau) &= \int_{\mathbb{R}^*} \mu(dz) \left( \text{Tr} \left\{ -J[\tau \ln \tau](z) \right\} - \text{Tr} \left\{ J[\tau](z) \right\} S_q(j(\tau; z)) \right).
\end{align*}

(50)

From the time derivative of the a posteriori entropy we have the following results.

(i) When $\tau$ is a pure state, $D_1(\tau) = 0$ if $\sum_j \text{Tr} \left\{ \tau L_j^* (\mathbb{1} - \tau) L_j \right\} = 0$ and $D_1(\tau) = +\infty$ otherwise.
(ii) $D_2(\tau) \geq 0$ for any state $\tau$. When $\tau$ is a pure state $D_2(\tau) = 0$.

(iii) Under condition $C_2$ one has $D_3(\tau) \geq 0$ for any state $\tau$.

(iv) When $\tau$ is a pure state, $D_3(\tau) \leq 0$ in general and $D_3(\tau) = 0$ if condition $C_2$ holds.

Statements (i) and (iv) are easy to verify, while the proof of (iii) requires arguments introduced in Section 3.3 and will be given there. In order to study $D_2(\tau)$ we need the spectral decomposition of $\tau$: $\tau = \sum_k \lambda_k P_k$, with $k \neq r \Rightarrow \lambda_k \neq \lambda_r$; by inserting this decomposition into Eq. (50) we get

$$D_2(\tau) = \frac{1}{2} \sum_k \lambda_k \text{Tr} \left\{ [P_k (R + R^*) - \text{Tr} \{(R + R^*)\tau\}] P_k^2 \right\}$$
$$+ \frac{1}{2} \sum_{k \neq r} \text{Tr} \{P_k (R + R^*) P_r (R + R^*) P_k\} \frac{\lambda_k \lambda_r}{\lambda_k - \lambda_r} \ln \frac{\lambda_k}{\lambda_r},$$

(52)

which implies statement (ii).

3.3 Mutual entropies and amount of information

A basic concept in classical information theory is the mutual entropy (information). For two nonindependent random variables it is the relative entropy of their joint probability distribution with respect to the product of the marginal distributions and it is a measure of how much information the two random variables have in common. The idea of mutual entropy can be introduced also in a quantum context, when tensor product structures are involved. Ohya used the quantum mutual entropy in order to describe the amount of information correctly transmitted through a quantum channel $\Lambda^*$ from an input state $\varrho$ to the output state $\Lambda^* \varrho$. The starting point is the definition of a “compound state” which describes the correlation of $\varrho$ and $\Lambda^* \varrho$; it depends on how one decomposes the input state $\varrho$ in elementary events (orthogonal pure states).

The mutual entropy of the state $\varrho$ and the channel $\Lambda^*$ is then defined as the supremum over all such decompositions of the relative entropy of the compound state with respect to the product state $\varrho \otimes \Lambda^* \varrho$ (ref. 15, pp. 33–34, 139).

We want to generalize these ideas to our context, where we have not only a quantum channel $\mathcal{U}(t)$, but also a classical output with probability law $P_\varnothing$; let us note that $\sigma_t$ contains the a posteriori states and the probability law and that it can be identified with a state on $L(H) \otimes L^\infty(\Omega, \mathcal{F}, Q)$. Firstly, we define a compound state $\Sigma_t$ describing the correlation between the initial state $\varrho$ and the map state $\sigma_t$. Let $\varrho = \sum_\alpha w_\alpha \varrho^\alpha$ be a decomposition of the initial state into orthogonal pure states (an extremal Shatten decomposition); if $\varrho$ has degenerate eigenvalues, this decomposition is not unique. With the
represented by a non negative random trace-class operator  

\[\hat{\text{entropy}}\] in a von Neumann algebra (ref. 15 Chapt. 5).

Then, with some computations, we obtain the following relations:

\[\sigma_i = \sum_\alpha w_\alpha \sigma_i^\alpha, \quad \rho_i = \sum_\alpha w_\alpha \frac{\|\sigma_i^\alpha\|_1}{\|\sigma_i\|_1} \rho_i^\alpha, \quad \eta_i = \sum_\alpha w_\alpha \eta_i^\alpha, \quad P_\rho = \sum_\alpha w_\alpha P_{\rho^\alpha}, \quad \sum_\alpha w_\alpha \rho_i^\alpha(\omega)P_{\rho^\alpha}(d\omega)\mid_{F_i} = \rho_i(\omega)P_\rho(d\omega)\mid_{F_i} . \quad (53)\]

The compound state \(\Sigma_i\) will be a state on the von Neumann algebra \(\mathfrak{A} = \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) \otimes L^\infty(\Omega, F, Q) \equiv M_1 \otimes M_2 \otimes M_3\); a normal state \(\Sigma\) on \(\mathfrak{A}\) is represented by a non negative random trace-class operator \(\hat{\Sigma}\) on \(\mathcal{H} \otimes \mathcal{H}\) such that \(\int_\Omega \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} \{\hat{\Sigma}(\omega)\} Q(d\omega) = 1: \Sigma(A) = \int_\Omega \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} \{\hat{\Sigma}(\omega)A(\omega)\} Q(d\omega), A \in \mathfrak{A}\). The relative entropy of the state \(\Sigma\) with respect to another state \(\Pi\) with representative \(\hat{\Pi}\) is given by

\[S(\Sigma|\Pi) = \int_\Omega \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} \{\hat{\Sigma}(\omega) \left(\ln \hat{\Sigma}(\omega) - \ln \hat{\Pi}(\omega)\right)\} Q(d\omega) ; \quad (54)\]

this formula is consistent with the general Araki-Uhlmann definition of relative entropy in a von Neumann algebra (ref. 15 Chapt. 5).

We introduce the compound state \(\Sigma_i\) on \(\mathfrak{A}\) by giving its representative \(\sum_\alpha w_\alpha \hat{\sigma}_i^\alpha \otimes \sigma_i^\alpha\) and we consider the different possible product states which can be constructed with its marginal: \(\Pi_i = \Sigma_i\mid_{M_1} \otimes \Sigma_i\mid_{M_2} \otimes \Sigma_i\mid_{M_3}\) with representative \(\|\sigma_i\|_1 \hat{\sigma} \otimes \eta_i\), \(\Pi_i^1 = \Sigma_i\mid_{M_1} \otimes \Sigma_i\mid_{M_2 \otimes M_3}\) with representative \(\sum_\alpha w_\alpha \|\sigma_i^\alpha\|_1 \hat{\sigma}_i^\alpha \otimes \eta_i\), \(\Pi_i^2 = \Sigma_i\mid_{M_1 \otimes M_2} \otimes \Sigma_i\mid_{M_3}\) with representative \(\sum_\alpha w_\alpha \hat{\sigma}_i^\alpha \otimes \eta_i^\alpha\). The different mutual entropies, i.e. the relative entropies of \(\Sigma_i\) with respect to the different product states, are the object of interest. We can call \(S(\Sigma_i|\Pi_i)\) the mutual input/output entropy: this is a new informational quantity, which could be extended also to generic measurements represented by instruments. First of all, from Corollary 5:20 of ref. 15, we obtain the chain rule

\[S(\Sigma_i|\Pi_i) = S(\Sigma_i|\Pi_i^1) + S(\Sigma_i|\Pi_i^2) ; \quad i = 1, 2, 3 . \quad (55)\]

Then, with some computations, we obtain the following relations:

\[S(\Pi_i^1|\Pi_i) = \mathbb{E}_{P_\rho} [S_\eta(\rho_t|\eta_t)] = S_\eta(\eta_t) - \mathbb{E}_{P_\rho} [S_\eta(\rho_t)] , \quad (56)\]

\[S(\Pi_i^2|\Pi_i) = \sum_\alpha w_\alpha I_t(P_{\rho^\alpha}|P_\rho) = \sum_\alpha w_\alpha I_t(P_{\rho^\alpha}|Q) - I_t(P_{\rho^\alpha}|Q) , \quad (57)\]

\[S(\Pi_i^3|\Pi_i) = \sum_\alpha w_\alpha S_\eta(\eta_t^\alpha|\eta_t) = S_\eta(\eta_t) - \sum_\alpha w_\alpha S_\eta(\eta_t^\alpha) ; \quad (58)\]
\[ S(\Sigma_t|\Pi_t) = S(\Pi_t^2|\Pi_t) + \sum_{\alpha} w_\alpha \mathbb{E}_{P_\alpha} [S_q(\rho_t^\alpha|\rho_t)] \]
\[ = S(\Pi_t^2|\Pi_t) + \mathbb{E}_{P_\alpha} [S_q(\rho_t)] - \sum_{\alpha} w_\alpha \mathbb{E}_{P_\alpha} [S_q(\rho_t^\alpha)] , \] (59)
\[ S(\Sigma_0|\Pi_t) = \sum_{\alpha} w_\alpha \mathbb{E}_{P_\alpha} [S_q(\rho_t^\alpha|\eta_t)] = S_q(\eta_t) - \sum_{\alpha} w_\alpha \mathbb{E}_{P_\alpha} [S_q(\rho_t^\alpha)] , \] (60)
\[ S(\Sigma_t|\Pi_t^2) = S(\Pi_t^2|\Pi_t) + \sum_{\alpha} w_\alpha \mathbb{E}_{P_\alpha} [S_q(\rho_t^\alpha|\eta_t^\alpha)] \]
\[ = S(\Pi_t^2|\Pi_t) + \sum_{\alpha} w_\alpha S_q(\eta_t^\alpha) - \sum_{\alpha} w_\alpha \mathbb{E}_{P_\alpha} [S_q(\rho_t^\alpha)] ; \] (61)
\[ S(\Sigma_t|\Pi_t) = S(\Pi_t^2|\Pi_t) + S_q(\eta_t) - \sum_{\alpha} w_\alpha \mathbb{E}_{P_\alpha} [S_q(\rho_t^\alpha)] . \] (62)

The initial values are
\[ S(\Sigma_0|\Pi_0) = S(\Sigma_0|\Pi_0^3) = S(\Sigma_0|\Pi_0^3) = S(\Pi_0^3|\Pi_0) = S_q(\varnothing) , \]
\[ S(\Sigma_0|\Pi_0^3) = S(\Pi_0^3|\Pi_0) = S(\Pi_0^3|\Pi_0) = 0 . \] (63)

The quantity \( S(\Pi_t^2|\Pi_t) = \mathbb{E}_{P_\alpha} [S_q(\rho_t|\eta_t)] \) is the \textit{a posteriori relative entropy}\(^{[10]}\) because Eq. (62) can be interpreted by saying that \( \{ P_q(\omega), \rho_t(\omega) \} \) is a demixture of the a priori state \( \eta_t \), such a relative entropy is a measure of how much such a demixture is fine. Let us observe that, for \( s \leq t \),
\[ \mathbb{E}_{P_\alpha} [S_q(\rho_t|\eta_t)] = \mathbb{E}_{P_\alpha} [S_q(\rho_t|\mathcal{U}(t-s)[\rho_s])] + \mathbb{E}_{P_\alpha} [S_q(\mathcal{U}(t-s)[\rho_s]|\eta_t)] . \] (64)

It follows that the variation in time of the a posteriori entropy is the sum of two competing contributions of opposite sign:
\[ \Delta \mathbb{E}_{P_\alpha} [S_q(\rho_t|\eta_t)] = \mathbb{E}_{P_\alpha} [S_q(\rho_t + \Delta t|\mathcal{U}(\Delta t)[\rho_t])] \]
\[ + \{ \mathbb{E}_{P_\alpha} [S_q(\mathcal{U}(\Delta t)[\rho_t]|\mathcal{U}(\Delta t)[\eta_t])] - \mathbb{E}_{P_\alpha} [S_q(\rho_t|\eta_t)] \} . \] (65)

The first term is clearly positive and represents an information gain due to the process of demixture induced by the measurement. The second term is negative, once again as a consequence of the Uhlmann monotonicity theorem, and represents an information loss due to the partial lack of memory of the initial state induced by the dissipative part of the dynamics.

The quantity \( S(\Pi_t^2|\Pi_t) = \sum_{\alpha} w_\alpha I_t(P_\alpha|P_\varnothing) \) has been introduced by Ozawa\(^{[18]}\) for a generic instrument under the name of \textit{classical amount of information}. By the discussion in Section 6.4 and eqs. 11 and 12, one obtains that this quantity is non decreasing and bounded:
\[ 0 \leq S(\Pi_t^2|\Pi_t) = \sum_{\alpha} w_\alpha I_t(P_\alpha|P_\varnothing) \leq \sum_{\alpha} w_\alpha S_q(\varnothing^\alpha|\varnothing) = S_q(\varnothing) . \] (66)
The supremum over all extremal Shatten decompositions of \( S(\Pi^3_t|\Pi_t) = \sum_{\alpha} w_\alpha S_q(\eta^\alpha_t|\eta_t) \) is Ohya’s “mutual entropy of the input state \( \rho \) and the channel \( \mathcal{U}(t) \); by (40) \( S(\Pi^3_t|\Pi_t) \) is non-increasing and by Theor. 1.19 of ref. 15 it is bounded by
\[
0 \leq S(\Pi^3_t|\Pi_t) = \sum_{\alpha} w_\alpha S_q(\eta^\alpha_t|\eta_t) \leq \min \{ S_q(\rho), S_q(\eta_t) \} .
\] (67)

For general instruments Ozawa \(^{18} \) introduced an entropy defect, which he called the amount of information; it measures how much the a posteriori states are purer than the initial state (or less pure, when this quantity is negative). In the case of continual measurements it is defined by
\[
I_t(\rho) = S_q(\rho) - \mathbb{E}_{\rho_t} \left[ S_q(\rho_t) \right].
\] (68)

If an equilibrium state exists, \( \eta_{eq} \in \mathcal{S}(\mathcal{H}) \) and \( \mathcal{L}[\eta_{eq}] = 0 \), by (56) we have \( S_q(\eta_{eq}) \geq I_t(\eta_{eq}) = \mathbb{E}_{\rho_{eq}} \left[ S_q(\rho_t|\eta_{eq}) \right] \geq 0 \). For a quasi-complete continual measurement one has
\[
S_q(\rho) \geq I_t(\rho) \geq S(\Pi^3_t|\Pi_t) \geq 0, \quad I_t(\rho) \geq I_s(\rho), \quad t \geq s.
\] (69)

The first statement was proved by Ozawa \(^{18} \) for a generic quasi-complete instrument, while the second one follows from the first one by using conditional expectations. \(^{10} \) We have \( I_t(\rho) - I_s(\rho) = \mathbb{E}_{\rho_s} \left[ S_q(\rho_t|\eta_{eq}) \right] - \mathbb{E}_{\rho_s} \left[ S_q(\rho_t|\eta_{eq}) \right] \); but \( S_q(\rho_s) - \mathbb{E}_{\rho_s} \left[ S_q(\rho_t|\eta_{eq}) \right] \) is the amount of information at time \( t \) when the initial time is \( s \) and the initial state is \( \rho_s \); and, so, it is non-negative for a quasi-complete measurement. From the monotonicity of \( I_t(\rho) \) one obtains that the time derivative of \( \mathbb{E}_{\rho_s} \left[ S_q(\rho_t) \right] \) is negative and this holds in particular at time zero for any choice of the initial state and also for \( R = 0 \). This proves the statement (iii) of Section 5.2.

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