Non-Analytical Angular Dependence of the Upper Critical Magnetic Field in a Quasi-One-Dimensional Superconductor†

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We have derived the so-called gap equation, which determines the upper critical magnetic field, perpendicular to conducting chains of a quasi-one-dimensional superconductor. By analyzing this equation at low temperatures, we have found that the calculated angular dependence of the upper critical magnetic field is qualitatively different than that in the so-called effective mass model. In particular, our theory predicts a non-analytical angular dependence of the upper critical magnetic field, \( H_{c2}(0) - H_{c2}(\alpha) - \alpha^{3/2} \), when magnetic field is close to some special crystallographic axis and makes an angle \( \alpha \) with it. We discuss possible experiments on the superconductor (DMET)\(_2\)I\(_3\) to discover this non-analytical dependence.

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Upper critical magnetic field, which corresponds to destruction of superconductivity in type II superconductors, is known to be one of the most fundamental properties of the superconducting state. The first calculations of the upper critical magnetic field were done in the framework of the phenomenological Ginzburg–Landau (GL) theory (see, e.g., [1, 2]) before the creation of the Bardeen–Cooper–Schrieffer (BCS) microscopic theory of superconductivity. Later, it was shown [3] that the GL theory is a limiting case of the BCS theory at \( T_c - T \ll T_c \) and the upper critical magnetic fields were calculated [4] at \( T_c - T \ll T_c \) for superconductors with anisotropic electron spectra, where \( T_c \) is superconducting transition temperature in the absence of a magnetic field. Using the microscopic Gor’kov equations, the upper critical field was calculated for a 3D isotropic superconductor at zero temperature [5] and at arbitrary temperatures [6]. As for superconductors with anisotropic electron spectra, the common belief is that we can apply the results [4], obtained at \( T_c - T \ll T_c \), at any temperature, including \( T \ll T_c \). The results [4] are usually called the effective mass (EM) model.

The main goal of our Letter is to show that the shape and topology of the Fermi surface (FS) play a crucial role in determination of angular dependence of the upper critical magnetic field at low temperatures. We consider a quasi-one-dimensional (Q1D) superconductor, which is characterized by two open slightly corrugated sheets of the FS. By using the Gor’kov equations [3], we derive the so-called gap equation, determining the upper critical magnetic field, perpendicular to conducting chains in a Q1D superconductor. As a result, we obtain a rather complicated integral equation, which we numerically solve at \( T \ll T_c \). Our numerical analysis of this integral equation shows that the EM model cannot be applied to Q1D case at \( T \ll T_c \) even at a qualitative level. Our main finding is that we predict non-analytical angular dependence of the upper critical magnetic field, \( H_{c2}(0) - H_{c2}(\alpha) - \alpha^{3/2} \), in the case, where magnetic field is close to some special crystallographic axis and makes an angle \( \alpha \) with it. This fact is in a sharp disagreement with the common belief, based on the results of the EM model, that \( H_{c2}(0) - H_{c2}(\alpha) \) has to be proportional to \( \alpha^2 \). Our second finding is that superconducting nuclei (i.e., solutions of the gap integral equation) are not of an exponential shape. We show that they decay very slowly and change their signs with distance. It is important that the above described phenomena are novel and due to quasi-classical effects of an electron motion in a magnetic field along open sheets of the Q1D FS in a single Brillouin zone. They are different from quantum effects of an electron motion in the extended Brillouin zone, considered in [7, 8]. Moreover, for discovery of non-analytical angular dependence, we need different experimental conditions than for investigation of the so-called Reentrant superconductivity [7–11]. We propose to investigate effects, suggested in the Letter, in the Q1D superconductor (DMET)\(_2\)I\(_3\), where the upper critical magnetic fields have been recently measured along all three principal directions [12]. It has also been pointed out [12] that superconductivity in the above mentioned compound is very far from the Reentrant superconducting regime [7], in contrast to superconductivity in (TMTSF)\(_2\)X materials [7–11].

Let us consider a superconductor with the following Q1D electron spectrum,
non-analytical angular dependence

\[ \delta \varepsilon^\pm(p) = \pm \nu_F (p_x \pm p_y) \]
\[ -2t_y \cos(p_y) - 2t_z \cos(p_z), \]  
\[ H = (0, H \cos \alpha, H \sin \alpha), \]
\[ A = (0, H_x \sin \alpha, -H_x \cos \alpha), \]

perpendicular to its conducting chains. (Here, + (−) stands for right (left) sheet of the Q1D FS (1), \( t_y \gg t_z \) are electron hopping integrals along \( a \), and \( a_c \) crystallographic axes; \( \nu_F \) and \( p_F \) are the Fermi velocity and Fermi momentum, respectively; \( \hbar = 1 \).)

To determine electron wave functions in the mixed representation, \( \Psi^\pm(x,y,z) \), where

\[ \Psi^\pm(x,y,z) = \exp(ip_Fx) \exp(ip_y) \exp(ip_z) \Psi^\pm(x,y,z), \]

we use the so-called Peierls substitution method, \( p_x \pm p_y \rightarrow -id/dx, \ p_y \rightarrow p_y - eA_y/c, \ p_z \rightarrow p_z - eA_z/c \). As a result, we obtain the following electron Hamiltonian in the presence of a magnetic field:

\[ \hat{H} = \mp i \nu_F \frac{d}{dx} - 2t_y \cos(p_y) - \frac{\omega_x}{\nu_F}, \]
\[ -2t_z \cos(p_z) + \frac{\omega_z}{\nu_F}, \]

where \( \omega_y = e\nu_FH_a \sin \alpha/c \) and \( \omega_z = e\nu_F \sin \alpha/c \).

In this Letter, we ignore quantum effects of an electron motion in a magnetic field in the extended Brillouin zone [7–11] and use the so-called eikonal approximation [3]. Note that we consider the case of small angles, \( \alpha \ll 1 \), where \( \omega_x \gg \omega_y \), which is important for non-analytical dependence of the upper critical field. As shown in [7], the quantum effects are small only at high enough temperature, where

\[ T \gg T^* = \frac{\omega_1}{4\pi} \ln(4t_1/\omega_1) \]  
(see Eq. (6) of [7]). Under condition (5), we can linearize the Hamiltonian (4) with respect to a magnetic field,

\[ \hat{H} = \mp i \nu_F \frac{d}{dx} - 2t_y \cos(p_y) - \frac{\omega_x}{\nu_F} \sin(p_y), \]
\[ -2t_z \cos(p_z) + \frac{\omega_z}{\nu_F} \sin(p_z). \]

It is important that the corresponding Schrödinger equation for wave functions in the mixed representation,

\[ \hat{H} \Psi^\pm(x,y,z) = \delta \varepsilon \Psi^\pm(x,y,z), \]  

\[ \Psi^\pm(x,y,z) = \exp(\pm i \delta \varepsilon x/\nu_F) \exp[\pm i \phi_y(p,y)] \]
\[ \times \exp[\pm i \phi_z(p,z)], \]

where

\[ \phi_y(p,y) = \frac{2t_y}{\nu_F} \cos(p_y) + \frac{\omega_y}{\nu_F} \sin(p_y), \]
\[ \phi_z(p,z) = \frac{2t_z}{\nu_F} \cos(p_z) + \frac{\omega_z}{\nu_F} \sin(p_z). \]

Since the electron spectrum and wave functions are known, the corresponding finite temperatures Green functions can be determined by means of the standard procedure [13]:

\[ G^\pm (r, r_1) = \frac{-i \text{sgn}(\omega_n)}{\nu_F} \sum_{p_x,p_y,p_z} \exp[\pm ip_F(x-x_1)] \]
\[ \times \exp[ip_y(y-y_1)] \exp[ip_z(z-z_1)] \times \exp[\mp \nu_F(x-x_1)] \]
\[ \times \exp[\pm i \cdot 2t_y \cos(p_y) - \frac{\omega_y}{\nu_F}(x-x_1)/\nu_F] \]
\[ \times \exp[\pm i \cdot 2t_z \cos(p_z) - \frac{\omega_z}{\nu_F}(z-z_1)/\nu_F] \]
\[ \times \exp[\mp i \nu_F \sin \alpha/c \alpha/\nu_F] \times \exp[\mp i \nu_F \sin \alpha/c \alpha/\nu_F]. \]

The so-called gap equation, determining superconducting transition temperature in the presence of magnetic field (2), can be derived by using the Gor’kov equations for non-uniform superconductivity [14]. As a result, we obtain:

\[ \Delta(x) = \frac{2}{\nu_F} \int_{|x-x_1|<d} \frac{2\pi Tdx_1}{\nu_F \sinh(2\pi T |x-x_1|)/\nu_F} \]
\[ \times J_0 \left[ \frac{2t_1 \omega_1}{\nu_F^2} (x^2-x_1^2) \right] \Delta(x_1), \]

where \( g \) is an effective electron coupling constant, \( d \) is a cutoff distance. Here, we rewrite Eq. (11) in more convenient way:

\[ \Delta(x) = \frac{2}{\nu_F} \int_{|x-x_1|<d} \frac{2\pi Tdx_1}{\nu_F \sinh(2\pi T |x-x_1|)/\nu_F} \]
\[ \times J_0 \left[ \frac{2t_1 \omega_1}{\nu_F^2} (x^2-x_1^2) \right] \Delta(x_1), \]

\[ T \gg T^* = \frac{\omega_1}{4\pi} \ln(4t_1/\omega_1) \]  
conditions different from Eq. (5).
\[ \Delta(x) = \frac{g}{2} \int_{|z| < d} \frac{2\pi Td\zeta}{v_F \sinh(2\pi T|z|/v_F)} \]
\[ \times J_0 \left[ \frac{2t_1 \omega_z}{v_F} \right] = 0. \]

(13)

If we take into account that
\[ \frac{1}{g} = \int_{d}^{\infty} \frac{2\pi Td\zeta}{v_F \sinh(2\pi Tz/v_F)}, \]

then we can rewrite Eq. (13) in the following way:
\[ \Delta(x) = \frac{g}{2} \int_{d}^{\infty} \frac{2\pi Td\zeta}{v_F \sinh(2\pi Tz/v_F)} \]
\[ \times J_0 \left[ \frac{2t_1 \omega_z}{v_F} \right] = 0. \]

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(15)

\[ \tau = \frac{T_c - T}{T_c} \]

(16)

\[ \zeta(n) \] is the Riemann zeta function [17]. To find the GL slope of the upper critical magnetic field, perpendicular the conducting chains, we need to determine the lowest energy level of the Schrödinger-like GL equation (15). As a result, we obtain
\[ H_{c2}(\alpha) = \frac{\sin^2 \alpha}{2} + \frac{\cos^2 \alpha}{2}, \]

(18)

where
\[ H_{c2}(0, T) = \frac{\phi_0}{2\pi \xi_T^z} (T_c - T), \]

(19)

\[ H_{c2}(\pi/2, T) = \frac{\phi_0}{2\pi \xi_T^z} (T_c - T), \]

(20)

\[ H_{c2}(\pi, T) = \frac{\phi_0}{2\pi \xi_T^z} (T_c - T), \]

(21)

(Here, \( \phi_0 = \pi \hbar e/\zeta \) is the flux quantum and \( \xi_x, \xi_y, \) and \( \xi_z \) are the coherence lengths along the \( a_x, a_y, \) and \( a_z \) axes, respectively.) Note that above we use the following relationship:
\[ \int_{0}^{\infty} \frac{\zeta^2 dz}{\sinh(z)} = \frac{7\zeta(3)}{2}, \]

(17)
Note that experimental value of the parameter \( \beta \) in (DMET)\(_2\)I\(_3\) superconductor is estimated as \( \beta = 10 \) \([12]\]. Below, we analyze Eqs. (20), (21) numerically by solving the gap integral Eq. (20) under the condition (21) for \( \beta = 10 \). Let us first consider the case \( \alpha = 0 \), where magnetic field is applied along \( a \) axis. A typical solution of Eq. (20), which is called superconducting nucleus, in this case is shown in Fig. 1. As seen from Fig. 1, in our case superconducting nucleus changes its sign and slowly decays in space, in contrast to the exponential solution of Eq. (15) in the effective mass model \([4]\].

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In Fig. 3, we plot the calculated angular dependence of the upper critical magnetic field, normalized on the corresponding result (18) of the EM model, \([H_{c2}(\alpha) - H_{c2}(0)]/[H_{c2}^{EM}(\alpha) - H_{c2}^{EM}(0)]\). As it follows from Fig. 3, the maximum deviations from the EM model occur at low angles and in the vicinity of some angle \( \alpha = 5^\circ \). At low angles, the calculated in the Letter upper critical magnetic field exhibits different angular dependence than that in the EM model (18), as discussed above. To clarify the nature of the mini-

\[\text{Fig. 1. Typical solution of Eqs. (20) and (21) for } \alpha = 0. \text{ It is an oscillatory slow decaying function of the } x \text{ coordinate, in contrast to the exponential solution of Eq. (15) in the effective mass model [4].}\]

\[\text{Fig. 2. (Points) Angular dependence of the upper critical magnetic field at } T = 0, H_{c2}(\alpha) - H_{c2}(0), \text{ calculated from Eqs. (20), (21). This dependence is well fitted by (solid line) the function } -\alpha^{3/2}. \text{ The dashed line is the effective mass model result given by Eq. (18), where } H_{c2}(\alpha) - H_{c2}(0) \sim -\alpha^2.\]

\[\text{Fig. 3. Normalized angular dependence of the upper critical magnetic field calculated by means of Eqs. (20) and (21) (see the main text).}\]

\[\text{Fig. 4. Angular dependence of the calculated difference } H_{c2}(\alpha) - H_{c2}^{EM}(\alpha) \text{ of the upper critical magnetic field given by Eqs. (20) and (21) from that given by the effective mass model given by Eq. (18).}\]
moment in Fig. 3 at $\alpha = 5^\circ$, we plot the difference, $H_{c2}(\alpha) - H_{c2}^{\text{EM}}(\alpha)$, in Fig. 4. As seen from Fig. 4, the maximum difference corresponds to $\alpha = 5.6^\circ$—angle, which we relate to the following theoretical value:

$$\alpha^* = \arctan(1/\beta) = \arctan(1/10) = 5.71^\circ. \quad (22)$$

Note that, under condition (22), both Bessel functions in Eq. (20) have the same arguments and some kind of resonance appears. We suggest to measure experimentally the position of the peak in the angular dependence $H_{c2}(\alpha) - H_{c2}^{\text{EM}}(\alpha)$ to carefully determine the ratio $\beta = \frac{t_z a_z}{t_y a_y}$ from Eq. (22).

To summarize, we have shown that the EM model [4] is not adequate to describe the upper critical magnetic field in superconductors with anisotropic electron spectra at low temperatures. For the case of a Q1D superconductor, we have found non-analytical angular dependence of the upper critical magnetic field, $H_{c2}(\alpha) - H_{c2}^{\text{EM}}(\alpha)$, where a magnetic field is perpendicular to conducting axis, $a_y$, and makes angle $\alpha$ with axis $a_x$. In addition, some angular resonance is predicted for “magic” direction of a magnetic field (22). We suggest testing the above mentioned predictions of the Letter on the Q1D superconductor $(\text{DMET})_2\text{I}_3$, where the upper critical magnetic fields along the main crystallographic axes have been recently measured [12]. In our opinion, unconventional shapes of superconducting nuclei as well as the non-analytical angular behavior of the upper critical field, found in the Letter, may reflect the existence of an unusual vortex lattice in Q1D superconductors. Therefore, we also suggest experimental studies of the vortex lattice at magnetic fields, corresponding to small values of angle $\alpha$ in Eq. (2).

Let us prove that the $(\text{DMET})_2\text{I}_3$ superconductor satisfies the condition of validity of our theory,

$$T_\text{c} \gg T \gg T_\ast, \quad (23)$$

at experimentally used lowest temperature, $T = 0.05$ K, where $T_\ast$ is given by Eq. (5) and $T_\text{c} = 0.5$ K [12]. If we take from [12] the typical experimental values, $H_{c2}^\text{EM} = 0.2$ T, $v_F = 0.4 \times 10^7$ cm/s, $a_z = 15.8$ A, $t_z = 1$ K, we obtain $T_\ast = 0.006$ K. Therefore, we conclude that the suggested in the Letter theory is applicable to the superconductor $(\text{DMET})_2\text{I}_3$ at the lowest experimental temperature [12]. Note that, for neglecting the quantum corrections [7, 8] and, thus, the Reentrant Superconductivity effects [7–11], it is also important that $4t_z/\omega_z = 27 \gg 1$, as has been already mentioned in [12].

We point out that in a geometry, considered in the Letter, experiments were performed in [18] in the superconductor (TMTSF)$_2\text{ClO}_4$ in low magnetic fields, $H \ll H_{c2}$, to demonstrate another phenomenon—the so-called lock-in effect. To avoid lock-in effect [18], the experiments, suggested by us, have to be performed at magnetic fields, which satisfy the condition $H = H_{c2}^\text{EM} \sin \alpha \gg H_{c2}^\ast$ [18]. Although $H_{c2}^\ast$ is not known in the superconductor $(\text{DMET})_2\text{I}_3$, it is clear that in this typical type-II superconductor $H_{c2}^\text{EM} \ll H_{c2}^\ast = 0.02$ T, which shows that the above mentioned condition is presumably satisfied.

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REFERENCES

1. A. A. Abrikosov, *Fundamentals of the Theory of Metals* (Elsevier Science, Amsterdam, 1988).
2. M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996).
3. L. P. Gor’kov, Sov. Phys. JETP 9, 1364 (1959).
4. L. P. Gor’kov and T. K. Melik-Barkhudarov, Sov. Phys. JETP 18, 1031 (1964).
5. L. P. Gor’kov, Sov. Phys. JETP 37, 593 (1960).
6. N. R. Werthamer, E. Helfand, and P. C. Hohenberg, Phys. Rev. 147, 295 (1966).
7. A. G. Lebed, JETP Lett. 44, 114 (1986).
8. The Physics of Organic Superconductors and Conductors, Ed. by A. G. Lebed (Springer, Berlin, 2008).
9. N. Dupuis, G. Montambaux, and C. A. R. Sa de Melo, Phys. Rev. Lett. 70, 2613 (1993).
10. N. Dupuis and G. Montambaux, Phys. Rev. B 49, 8993 (1994).
11. A. G. Lebed, Phys. Rev. Lett. 107, 087004 (2011).
12. P. Dhakal, H. Yoshino, J. I. Oh, et al., Phys. Rev. B 83, 014505 (2011).
13. A. A. Abrikosov, L. P. Gor’kov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).
14. A. G. Lebed and K. Yamaji, Phys. Rev. Lett. 80, 2697 (1998).
15. R. A. Klemm, A. Luther, and M. R. Beasley, Phys. Rev. B 12, 877 (1975).
16. A. M. Clogston, Phys. Rev. Lett. 9, 266 (1962); B. S. Chandrasekhar, Appl. Phys. Lett. 1, 7 (1962).
17. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1994).
18. P. A. Mansky, G. Danner, and P. M. Chaikin, Phys. Rev. B 52, 7554 (1995).