Approximation algorithms for the random-field Ising model

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Abstract

Approximating the partition function of the ferromagnetic Ising model with general external fields is known to be #BIS-hard in the worst case, even for bounded-degree graphs, and it is widely believed that no polynomial-time approximation scheme exists. This motivates an average-case question: are there classes of instances for which polynomial-time approximation schemes exist? We investigate this question for the random field Ising model on graphs with maximum degree $\Delta$. We establish the existence of fully polynomial-time approximation schemes and samplers with high probability over the random fields if the external fields are IID Gaussians with variance larger than a constant depending only on the inverse temperature and $\Delta$. The main challenge comes from the positive density of vertices at which the external field is small. These regions, which may have connected components of size $\Theta(\log n)$, are a barrier to algorithms based on establishing a zero-free region, and cause worst-case analyses of Glauber dynamics to fail. The analysis of our algorithm is based on percolation on a self-avoiding walk tree.

1 Introduction

Recent years have seen the development of a rich interplay between statistical physics, computational complexity, and algorithm design. One central question is the extent to which phase transitions for discrete statistical mechanics models are related to the tractability of associated computational problems. In this paper we are primarily interested in approximate counting and sampling; see Section 1.4 for formal definitions. Results concerning these tasks have traditionally focused on (i) establishing algorithmic tractability in so-called ‘high temperature’ (weakly correlated) regimes, and (ii) establishing the failure of certain algorithmic techniques in ‘low temperature’ (strongly correlated) regimes. Very recently, positive algorithmic results have been obtained at low temperatures [19, 4, 20], and occasionally even at all temperatures [21, 6, 18, 2].

Many of these algorithmic results have been shown using a detailed probabilistic and physical understanding of the corresponding statistical mechanics problems. The intuition gained from this understanding is typically restricted to specific classes of graphs, e.g., lattices or specific models of random graphs. One of the challenges for algorithm design is to go beyond these restricted classes of graphs, and this can lead to situations in which the notions of ‘high temperature’, ‘low temperature’, and ‘phase transition’ are unclear.

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This loss of intuition is present when one asks about the average-case complexity of a well-known #BIS-hard problem, the ferromagnetic Ising model with general vertex-dependent external fields. Towards understanding this situation, in this paper we consider the Ising model with vertex-dependent random external fields \( h_x, x \in V(G) \), where the \( h_x \) are independent and identically distributed (IID) centered Gaussian. This model, primarily studied in statistical physics on the integer lattice \( \mathbb{Z}^d \), is known as the random field Ising model. A great deal of interest in the random field Ising model has arisen because it behaves differently than the zero-field model with \( h_x \equiv 0 \). To briefly describe this, recall that the zero-field model undergoes a phase transition on \( \mathbb{Z}^d \) when \( d \geq 2 \): if \( \beta > 0 \) is small, then correlations decay exponentially and there is a unique infinite volume Gibbs measure. On the other hand, if \( \beta \) is large, then correlations do not decay, and multiple Gibbs measures exist. The surprising phenomena is that this picture changes for the random field Ising model: on \( \mathbb{Z}^d \) for \( d \geq 3 \) there is still a phase transition if the variance of the random fields \( h_x \) is not too large [7], but on \( \mathbb{Z}^2 \) there is no phase transition if the variance is non-zero [1]. In fact, in recent breakthroughs, it was shown that on \( \mathbb{Z}^2 \), \( d \geq 3 \), correlations decay exponentially throughout the high-temperature regime [12]. It is natural to wonder if there are algorithmic counterparts to these physical phenomena.

While the phase transition phenomena of the previous paragraph concerned small variances, the random field Ising model also exhibits interesting properties (so-called Griffiths singularities) from the point of view of physics when the variance of the external fields is not small. See Section 1.1. This regime is also terra incognita from an algorithmic point of view, and we focus in this paper on the large variances.

It is straightforward, see Section 2, to design algorithms for the random field Ising model if one assumes \( |h_x| \) is uniformly large (depending on the inverse temperature \( \beta \) and on the maximum degree of the graphs being considered): these large external fields put the system in an effectively high temperature situation. If, however, \( |h_x| \) can be small for some vertices \( x \), then for large inverse temperature \( \beta \), highly correlated subsets of spins may appear — there can be large ‘low temperature’ islands in a ‘high temperature’ sea. Our main result shows that if \( |h_x| \) is typically large then for typical realizations of the external fields, the computational tasks of approximate counting and sampling are tractable. We will discuss our proof strategy and highlight the barriers faced by other methods after pausing to give precise formulations of our results.

**Definition 1.** Let \( G = (V, E) \) be a finite graph, \( h: V \to \mathbb{R} \), and \( \beta \in \mathbb{R} \). The Ising model on \( G \) with inverse temperature \( \beta \) and external fields \( h \) is the probability distribution on \( \{\pm 1\}^V \) given by

\[
P_{G,\beta,h}(\sigma) = \frac{e^{-H_{G,\beta,h}(\sigma)}}{Z_{G,\beta,h}}, \quad Z_{G,\beta,h} = \sum_{\sigma \in \{\pm 1\}^V} e^{-H_{G,\beta,h}(\sigma)},
\]

where the Hamiltonian \( H_{G,\beta,h} \) is the function

\[
-H_{G,\beta,h}(\sigma) = \beta \sum_{xy \in E} \sigma_x \sigma_y + \sum_{x \in V} h_x \sigma_x.
\]

A random field Ising model has external fields \( h \) that are independent random variables with prescribed distributions. We use \( \mathbb{P} \) to denote the law of the random external fields. In this paper we will be concerned with random external fields that are typically large, which we will model by centered Gaussians with large variance. Our main theorem in this setting is the following.

**Theorem 2.** For every \( \Delta \geq 2 \), \( \beta \in \mathbb{R} \), there exists \( H = H(\Delta, \beta) \) large enough so that for random field Ising models with inverse temperature \( \beta \) and external fields distributed as independent...
\( N(0, H) \) random variables the following holds. For every graph \( G \) of maximum degree \( \Delta \) on \( n \) vertices, with probability \( 1 - o(1) \) over the choice of random fields, there exists an FPTAS for \( Z_{G,\beta,h} \) and a polynomial-time sampling scheme.

**Remark 1.** The failure probability (over the randomness of the fields) tends to 0 with the size of the graph \( n \). We can make this failure probability arbitrarily small: to achieve failure probability \( \delta \) requires a factor polynomial in \( 1/\delta \) in the running time of the algorithm.

**Remark 2.** The external fields being Gaussian does not play a role in our proof. Theorem 2 applies more generally to independent external fields with distributions with the property that
\[
P(|h_x| < c) \leq p.
\]
for \( c = |\beta|\Delta + \log \Delta + c_1 \) and \( p = \frac{c^2}{\sqrt{2\pi}} \) for large enough constant \( c_1 \) and small enough constant \( c_2 \).
In particular, since \( P_{h \sim N(0,\sigma^2)}(|h| \leq c) \leq \sqrt{\frac{2c}{\pi \sigma}} \), we see that \( H = \Omega(\beta^2 \Delta^6) \) suffices in the case of Gaussian external fields and \( \beta \) bounded away from 0.

**Remark 3.** We can efficiently check whether a given instance of the external fields satisfy the required conditions in the following sense: given \( \epsilon > 0 \) it takes time polynomial in \( n \) and \( 1/\epsilon \) to both output an approximation of the partition function and to check the conditions that guarantee the \( \epsilon \)-relative accuracy of the approximation. See Proposition 14. We emphasize however, that with probability \( 1 - o(1) \), a single instance satisfies these conditions for all choices of \( \epsilon \).

**Remark 4.** Theorem 2 extends in a straightforward manner to the setting of edge-specific inverse temperatures \( \beta_{xy} \) provided all \( \beta_{xy} \) are bounded in absolute value by a fixed \( \beta > 0 \). We consider a single inverse temperature for notational simplicity.

**Remark 5.** Theorem 2 also applies in the presence of boundary conditions, which arise naturally in our analysis. We define boundary conditions formally in Section 1.4.

The key mechanism behind the proof of Theorem 2 is that large external fields cause the system to rapidly decorrelate, as spins tend to align with their external field. We formalize this decorrelation by generalizing a disagreement percolation argument due to Camia, Jiang, and Newman [10]. The resulting notion of correlation decay is similar to, but somewhat weaker than, strong spatial mixing. See Section 3. This correlation decay property is sufficiently strong to enable a recursive analysis on a self-avoiding walk (SAW) tree as was pioneered by Weitz [35]. Weitz’s method for approximately counting weighted independent sets in bounded degree graphs is now known as the ‘method of correlation decay’, and has found numerous applications in approximate counting and sampling and in proving Gibbs uniqueness for spin models on \( \mathbb{Z}^d \), e.g., [5, 29, 23, 27, 28]. The analysis of correlation decay algorithms involves proving strong spatial mixing on the SAW tree, usually by means of a contraction argument or a monotonicity argument with respect to boundary conditions.
What is new in our approach is proving a form of strong spatial mixing on the appropriate SAW tree by a probabilistic argument based on percolation theory (more precisely, based on disagreement percolation [31, 32]).

The proof of Theorem 2 is fairly robust and can be generalized to apply to graphs where the maximum degree is not necessarily bounded. To illustrate this, we use similar ideas to analyze the random field Ising model on sparse Erdős-Rényi random graphs. Recall that a graph drawn from \( G(n,p) \) (an **Erdős-Rényi random graph**) is defined as a graph on \( n \) vertices \( \{v_1, \ldots, v_n\} \) where each potential edge \( \{v_i, v_j\} \), \( i \neq j \) independently included with probability \( p \).
**Theorem 3.** For every $\Delta > 1$ and $\beta \in \mathbb{R}$, there exists $H$ large enough so that the following holds. With $G \sim G(n, \Delta/n)$ and independent random external fields distributed as $N(0, H)$, with probability $1 - o(1)$ over the random graph and random fields there exists an FPTAS and polynomial-time sampling scheme for the random field Ising model on $G$ at inverse temperature $\beta$.

**Remark 6.** Theorem 3 applies more generally to independent external fields with distributions with the property that

$$P(|h_x| < c) \leq p,$$

for $c = c_1(\beta|\Delta + \log \Delta + 1)$ and $p = \left(\frac{1}{\Delta}\right)^{c_2}$ for constants $c_1, c_2$.

### 1.1 Background

An important challenge for understanding the relative complexity of approximate counting was raised by Dyer, Goldberg, Greenhill, and Jerrum in [14]: how difficult is it to approximately count independent sets in bipartite graphs? This problem, (approximate) #BIS, occupies a central place in the analysis of approximate counting algorithms [17, 16, 24]. The most relevant fact for us is the #BIS-hardness of approximately computing the partition function of the ferromagnetic Ising model with general vertex-dependent external fields on bounded-degree bipartite graphs [9].

For #BIS-hardness, allowing general external fields is necessary: it is a classic result that there are efficient approximation algorithms for the ferromagnetic Ising model with no external fields or with consistent external fields (all non-negative or all non-positive) for all values of $\beta \geq 0$ of the inverse temperature [21, 17]. The novelty of Theorem 2 is that it allows for inconsistent external fields. Standard approaches for analyzing the Glauber dynamics of the Ising model do not seem capable of proving Theorem 2 — see Remark 7 below. Note that if $\beta$ is taken sufficiently small, then standard high temperature methods already apply [36], and thus the most interesting case of our theorem is when $\beta$ is large.

Another approach to approximation is based on zero-freeness of the partition function, either via Barvinok’s method [3] or cluster expansion methods [19]. The main barrier to applying these methods in the case of the random field Ising model is the phenomena of Griffiths singularities [33]. These singularities arise in spin systems with random Hamiltonians; in our case the randomness is contained in the external field. When the underlying graph is the integer lattice $\mathbb{Z}^d$, the existence of rare (but arbitrarily large) regions of atypical behaviour for the random field is widely believed to lead to thermodynamic functions being infinitely differentiable but not analytic [34, 15, 33]. The non-analyticity of limiting quantities rules out the existence of zero-free regions in finite volumes.

### 1.2 Future Directions

Our algorithm is based on Weitz’s method of correlation decay on a computational tree. A natural question is whether Markov chain-based algorithms can provide a similar guarantee. See Remark 7 for indications this may be a subtle question. Our thresholds are certainly improvable, and the tractability for more moderate values of external field is unknown, as is tractability in the presence of correlated external fields.

The difference in behaviour for the RFIM in $d = 2$ and $d \geq 3$ suggests that the design of approximate counting algorithms in the presence of weak disorder is a subtle task, and hence an interesting challenge for future research. Another interesting direction is to develop algorithms for problems that contain ‘high temperature’ islands in a ‘low temperature’ sea, i.e., with the roles of high and low temperature in the present paper exchanged. Our result does not rely on ferromagnetism, and it is a good question whether one can obtain a stronger result—such as one that works for more moderate external fields — in the ferromagnetic $\beta > 0$ regime.
We end this section by indicating a motivating connection (and potential future direction) between the results of this paper and \#BIS that does not pass through any formal reductions as in [9, 17]. A difficulty in investigating the complexity of \#BIS is that it is unclear which instances are hard. For a single random bipartite graph (balanced or not), the low temperature behaviour of independent sets is well understood (see, e.g., [26]): independent sets typically consist of significantly more vertices on one side of the bipartition than the other. Thus, in the search for a hard instance one may be tempted to treat single random bipartite graphs as gadgets, and to assemble many gadgets together by adding edges between the gadgets in a bipartite manner. If the density of added edges is low enough to avoid disrupting the behaviour of individual gadgets, then in the low temperature regime the resulting graph heuristically behaves like a ferromagnetic Ising model. The external field reflects if the constituent graphs are balanced \((h = 0)\) or not \((h \neq 0)\). Working directly with the Ising model with an inconsistent magnetic field allows for us to search for hard instances while bypassing the technicalities that would be present in making the preceding discussion precise.

1.3 Organization of the paper

In Section 2, we give approximate sampling and counting algorithms in the case that all external fields are large with probability one. In Section 3, we prove our main theorem (Theorem 2). In Section 4, we prove Theorem 3, the extension of our main result to random graphs. Finally, in Section 5, we show that our work generalizes one of the main theorems of [10] to infinite graphs of max-degree \(\Delta\). In Appendix A, we give details of the SAW tree construction and recursion used by the algorithms, and in Appendix B, we write out the algorithms explicitly.

1.4 Preliminaries and notation

Approximate counting and sampling. A fully polynomial-time approximation scheme (FPTAS) for a function \(Z(G)\) is a deterministic algorithm that given a graph \(G\) and a tolerance \(\epsilon > 0\) outputs a number \(\hat{Z}\) such that \(e^{-\epsilon}Z \leq \hat{Z} \leq e^{\epsilon}Z\), with running time polynomial in \(1/\epsilon\) and \(|V(G)|\). A polynomial time sampling scheme for a distribution \(\mu_G\) is a randomized algorithm that, given \(G\) and a tolerance \(\epsilon > 0\) outputs a sample from a distribution \(\hat{\mu}\) such that \(\|\hat{\mu} - \mu_G\|_{TV} \leq \epsilon\), with running time polynomial in \(1/\epsilon\) and \(|V(G)|\).

Notation. Throughout we implicitly restrict our attention to connected graphs, as all of the algorithmic tasks we consider factor over connected components. We let \(\mathcal{G}_\Delta\) denote the set of graphs with maximum degree at most \(\Delta\), and write \(\deg(v)\) for the degree of a vertex \(v\). Given a graph \(G = (V, E)\), we denote by \(d(v, w)\) the distance between vertices \(v, w\) on the graph, i.e., the length \(\ell\) of the shortest path \(v_0, \ldots, v_\ell\) with \(v_0 = v\) and \(v_\ell = w\), and \((v_i, v_{i+1}) \in E\). For a set \(S \subset V\), let \(d(v, S) := \min_{w \in S} d(v, w)\). For a vertex \(v\) let \(N(v)\) denote the set of neighbors of \(v\). Let \(N(v, \ell)\) denote the set of vertices at distance exactly \(\ell\) in a graph \(G\).

The Ising model with boundary condition \(\tau \in \{\pm 1\}^V\) on \(B \subset V\) is defined by the formulas in (1.1) but with the restriction that \(\sigma\) is a spin configuration that agrees with \(\tau\) on \(B\). We write \(p_{\tau, \beta, h}^G\) for the law of this model. Given an Ising model with fixed \(\tau, \beta, h\), and letting \(\sigma' \in \{\pm 1\}^\Lambda\) for a subset \(\Lambda \subset V\), let \(p_{\sigma'}^\tau\) be the marginal probability of spin 1 at vertex \(v\) conditioned on \(\sigma_\Lambda = \sigma'\), i.e.,

\[
p_{\sigma'}^\tau := p_{\tau, \beta, h}^G(\sigma_v = 1|\sigma_\Lambda = \sigma').
\] (1.2)

In (1.2) and above we have written \(\sigma_\Lambda = (\sigma_x)_{x \in \Lambda}\) to denote the spins at the vertices \(A \subset \Lambda\).

For functions \(f, g: \mathbb{R} \to \mathbb{R}\) we write \(f = O(g)\) if there exists \(C > 0\) such that \(|f(x)| \leq C|g(x)|\) for all \(x\) large enough, and \(f = \Omega(g)\) if \(g = O(f)\).
2 Deterministic large external fields

In this section we give approximate sampling and counting algorithms in the case that \(|h_x| \geq c(\beta, d)|\) for all \(x \in V\). This case in which all external fields are large with probability 1 provides some intuition for the main result by indicating how the presence of large fields facilitates correlation decay. However, the simple proof we provide here does not work without a uniform bound on the external fields, see Remark 7 below. Define

\[
M(\Delta, h, \beta) = \left| \frac{1}{1 + e^{-2\beta \Delta - 2h}} - \frac{1}{1 + e^{2\beta \Delta - 2h}} \right|.
\]  

(2.1)

The quantity \(M\), and particularly upper bounds on \(M\), will be important for our analysis in this and subsequent sections.

**Lemma 4.** For any \(\beta \in \mathbb{R}, \Delta \geq 0, \) and \(\epsilon > 0\), if \(|h| \geq |\Delta| + \frac{1}{2} \log \left( \frac{1}{\epsilon} \right)\), then \(M(\Delta, h, \beta) < \epsilon\).

**Proof.** Consider the terms \(\frac{1}{1 + e^{-2\beta \Delta - 2h}}\) and \(\frac{1}{1 + e^{2\beta \Delta - 2h}}\). If \(h \geq |\Delta| + \frac{1}{2} \log \left( \frac{1}{\epsilon} \right)\), then both terms are \(\frac{1}{1 + \epsilon}\), and if \(h \leq -(|\Delta| + \frac{1}{2} \log \left( \frac{1}{\epsilon} \right))\), then both terms are \(\frac{1}{1 + \epsilon} = \frac{1}{1 + \epsilon} < \epsilon\). Since both terms are bounded between 0 and 1, \(M(\Delta, h, \beta) \leq \frac{1}{1 + \epsilon} < \epsilon\) follows. \(\Box\)

The following monotonicity property follows by differentiating in \(\Delta\).

**Lemma 5.** For fixed \(h, \beta\), \(M(\Delta, h, \beta)\) is non-decreasing in \(\Delta\).

The following bounds the influence of boundary conditions on the marginal probability at \(v\).

**Lemma 6.** Let \(v \in V, \Lambda \subset V\) be a set not containing \(v\), and let \(\sigma_\Lambda, \tau_\Lambda \in \{\pm 1\}^\Lambda\). Then

\[|p_v^{\sigma_\Lambda} - p_v^{\tau_\Lambda}| \leq M(\deg(v), h_v, \beta).\]

**Proof.** We first prove the result when \(\Lambda\) contains all the neighbors of \(v\). In this case, considering only the relevant part of the Hamiltonian and temporarily abbreviating \(\sigma = \sigma_\Lambda\) on the right-hand side,

\[p_v^{\sigma_\Lambda} = \frac{e^{\sum_{y \in N(v)} \beta \sigma_y + h_v}}{e^{\sum_{y \in N(v)} \beta \sigma_y + h_v} + e^{\sum_{y \in N(v)} -\beta \sigma_y - h_v}}.\]

The maximum and minimum possible values of this are \(\frac{1}{1 + e^{-2\beta \deg(v) - 2h_v}}\) and \(\frac{1}{1 + e^{2\beta \deg(v) - 2h_v}}\). Hence \(|p_v^{\sigma_\Lambda} - p_v^{\tau_\Lambda}|\) is bounded by \(M(\deg(v), h_v, \beta)\). For general \(\Lambda\), by conditioning on the value of \(\sigma\) on \(N(v)\), we can write \(p_v^{\sigma_\Lambda}\) as a weighted average of \(p_v^{\sigma_{N(v)}}\), so the lemma follows in this case as well. \(\Box\)

This bound allows us to prove rapid mixing of Glauber dynamics via the method of path coupling [8]. This in turn gives us randomized polynomial-time approximate counting and sampling algorithms when the external field is uniformly large. We recall that the Glauber dynamics are a time-homogeneous Markov chain \((\sigma(n))_{n \in \mathbb{N}}\) that evolves by uniformly selecting a vertex \(x \in V\), and then updating the spin at \(x\) according to the marginal distribution at \(x\) conditioned on the spins of its neighbors, i.e., \(P(\sigma(n + 1) = 1) = p_{x, \beta, h}^\tau(\sigma_x = 1|\sigma_{V \setminus \{x\}} = \sigma(n)_{V \setminus \{x\}})\). The other spins are unchanged in this step.

**Theorem 7.** Fix \(\Delta \in \{2, 3, \ldots\}\). If \(|h_x| \geq h_0(\Delta, \beta) = \Delta|\beta| + \frac{1}{2} \log \Delta\), then the mixing time of the Glauber dynamics for the Ising model on \(G \in \mathcal{G}_\Delta\) with \(|V(G)| = n\) is \(O(n \log n)\).
Proof. Fix $G \in \mathcal{G}_{\Delta}$. As described above, in a single step of the Glauber dynamics we pick $x \in V(G)$ uniformly at random and then update the spin $\sigma_x$ conditioned on the spins of $N(x) = \{y \in V \mid \{x, y\} \in E(G)\}$.

Consider two configurations $\sigma, \sigma'$ that disagree only at vertex $y$. We couple two copies of the Glauber dynamics starting from $\sigma(0) = \sigma$ and $\sigma'(0) = \sigma'$ respectively by picking the same vertex $x$ to update and updating to the same spin with as high probability as possible. We analyze how the Hamming distance between the two configurations changes in a single step of the chain.

(i) If $x = y$, the vertex at which $\sigma$ and $\sigma'$ disagree, then with probability one $\sigma(1) = \sigma'(1)$, and the Hamming distance decreases by 1. This occurs with probability $1/n$.

(ii) If $d(x, y) > 1$, then both chains see the same boundary conditions and so make the same update. The Hamming distance does not change.

(iii) If $x \in N(y)$, then the two chains see different boundary conditions. By Lemmas 5 and 6 the difference in the probability of updating to +1 is at most $M(\Delta, h_v, \beta)$, and this is an upper bound on the expected change in Hamming distance. This occurs with probability $\Delta/n$.

Let $\delta(\sigma, \sigma')$ be the expected change in Hamming distance between $\sigma$ and $\sigma'$ after one step of the coupled chains. Since $|h_v| \geq h_0(\Delta, \beta)$, Lemma 4 and the considerations above give $$\delta(\sigma, \sigma') \leq -\frac{1}{n} [1 - \Delta M(\Delta, h_0, \beta)] < 0.$$ The theorem follows by path coupling, see, e.g., [22, Corollary 14.7].

Remark 7. Analyzing the Glauber dynamics without a uniform lower bound on $|h_x|$ would require addressing the fact that the dynamics are not contractive at each step. To see this, suppose that $h_x \in \{-h, 0, h\}$, and note $M(\Delta, 0, \beta) \approx 1$ if $\beta$ is large. In this case when the dynamics act on vertices with $h_x = 0$ there is no contraction. This reflects the fact that $\beta$ is above the uniqueness threshold for the Ising model on the $\Delta$-regular tree with no external field.

For some types of disordered systems, rigorous results that rule out exponential relaxation have been obtained in infinite volume, see, e.g., [11].

3 Random field Ising model

In this section we prove Theorem 2. We will that show strong spatial mixing, a strong form of correlation decay, holds with high probability on a tree when the typical value of $|h_x|$ is large enough. Recall that $p^\sigma_v$ denotes the probability that $\sigma_v = 1$ under boundary conditions $\sigma$.

Definition 8. Let $G = (V, E)$ be a graph, and let $v \in V$ be a vertex. We say that strong spatial mixing (SSM) with rate $\alpha(\cdot)$ and min-distance $\ell_0$ holds for $v$ if for any $\Lambda \subset V$ and any two configurations $\sigma_\Lambda, \tau_\Lambda \in \{\pm 1\}^\Lambda$, $$|p^\sigma_v - p^\tau_v| \leq \alpha(d(v, \Lambda'))$$ whenever $d(v, \Lambda') \geq \ell_0$, where $\Lambda' \subseteq \Lambda$ is the subset on which $\sigma_\Lambda$ and $\tau_\Lambda$ differ.

The standard definition of strong spatial mixing with rate $\alpha$ corresponds to taking $\ell_0 = 0$. Taking $\ell_0$ non-zero is a weaker condition. The preceding definition is partly inspired by Camia, Jiang, and Newman [10], who obtained a certain non-uniform spatial mixing result on $\mathbb{Z}^d$. For algorithmic purposes a uniform spatial mixing result is necessary, and we establish such a result in Lemma 11 using ideas similar to those of [10]. The cost of uniformity is that we obtain SSM with min-distance...
Let $G = (V \cup \partial V, E)$ be a finite graph. Define (inhomogeneous, independent) site percolation with probabilities $p_x \in [0, 1]$, for each $x \in V$, as the following process. Let $T \in \{0, 1\}^{V \cup \partial V}$, where $(T_x)_{x \in \partial V}$ are given boundary conditions on $\partial V$ and $(T_x)_{x \in V}$ are independent Bernoulli random variables with $\mathbb{P}(T_x = 1) = p_x$. We denote the law of $(T_x)_{x \in V}$ by $P_p$.

In the preceding definition the boundary condition is implicit in the notation $P_p$; we will explicitly highlight the boundary condition in what follows. As is standard in percolation, for disjoint $A, B \subset V$, we write $A \leftrightarrow B$ if there exists a path $v_0, v_1, \ldots, v_d$ with $v_0 \in A$ and $v_d \in B$, such that $T_{v_i} = 1$ for each $0 \leq i \leq d$.

Lemma 10 (cf. [10, Lemma 5]). Given an Ising model on a connected graph $G = (V \cup \partial V, E)$, let $A \subset V$, and let $\eta, \xi$ be two boundary conditions on $\partial V$. Let $P_p$ be the law of a site percolation $T$ with boundary condition $T_x = 1[\eta_x \neq \xi_x]$ for $x \in \partial V$, and $p_x = M(\deg(x), h_x, \beta)$ for all other vertices. Then

$$d_{TV}(p^\eta_{\beta,h}(\sigma_A \in \cdot), p^\xi_{\beta,h}(\sigma_A \in \cdot)) \leq P_p(\partial V \leftrightarrow A).$$

Proof. Order the vertices of $V = \{x_1, x_2, \ldots\}$ in such a way that $x$ precedes $y$ in the ordering if $d(x, \partial V) < d(y, \partial V)$. We couple draws $\sigma^{(1)}, \sigma^{(2)}, S_x$ by drawing $\sigma_x^{(1)}, \sigma_x^{(2)}, S_x$ sequentially according to an exploration process. $S$ will be a site percolation process with boundary condition given by

$$S_x = 1[\eta_x \neq \xi_x] \text{ when } x \in \partial V.$$

For $t \in \mathbb{N}_0$, let $W_t$ denote the set of sites explored up to and including time $t$, and let $V_t := \{x \in W_t : S_x = 1\}$. We now inductively define the explored set.

- Let $W_0 := \partial V$.

- For each $t \geq 0$, reveal the first unexplored site $x$ (according to the chosen ordering) that is adjacent to $V_t$. Note that this eventually exhausts the graph since $G$ is connected. We set the values of $\sigma_x^{(i)}$ to have the correct marginal distributions, and to be a maximal coupling. More precisely, let $U_x$ be a independent uniform random variable in $[0, 1]$, let $\nu^{(1)} = \eta, \nu^{(2)} = \xi$, and let

$$\sigma_x^{(i)} = \begin{cases} 1, & U_x \leq p^\eta_{\beta,h}(\sigma_x = 1|\sigma_{W_t} = \sigma_{W_{t+1}}^{(i)}) \\ -1, & \text{otherwise}, \end{cases}$$

$$S_x = 1[\sigma_x^{(1)} \neq \sigma_x^{(2)}].$$

Then let $W_{t+1} := W_t \cup \{x\}$. 

3.1 Disagreement percolation and spatial mixing

Lemma 10 below relates the distance between marginal distributions to the probability of disagreement percolation from the boundary to the region of interest. First, we define the relevant notion of a percolation process.

Definition 9. Let $G = (V \cup \partial V, E)$ be a finite graph. Define (inhomogeneous, independent) site percolation from the boundary to the region of interest. First, we define the relevant notion of a percolation process.
Note that conditioned on \( \sigma^{(i)}_{W_i}, i = 1, 2 \), we have \( \sigma^{(1)}_x \neq \sigma^{(2)}_x \) with probability at most \( M(\deg(x), h_x, \beta) \), and as a result the site percolation process \( T \) with the same boundary condition as \( S \) stochastically dominates \( S \). With this coupling, \( \sigma^{(1)}_A \neq \sigma^{(2)}_A \) only if \( \partial V \leftrightarrow A \) in \( S \); by stochastic domination this is at most the probability that \( \partial V \leftrightarrow A \) in \( T \).

We now use Lemma 10 to show how an assumption that the external field is typically large results in a strong spatial mixing property on trees. We quantify typically large by requiring the following condition on the external field distribution \( h \) (for a parameter \( h_0 \) to be specified):

\[
P(|h| < h_0) \leq \frac{1}{16\Delta^2}.
\]  

(3.1)

**Lemma 11.** Let \( G \) be a tree with max degree \( \Delta \) and root vertex \( v \). Let \( h_0 \) be such that \( M(\Delta, h_0, \beta) < \Delta^{-2} \). Suppose \( h_x \) are such that along each path from any \( w \) to \( v \), the \( h_x \) are independent and satisfy (3.1). Then with probability at least \( 1 - \delta \) over the \( h \), there is a \( c_1 > 0 \) such that strong spatial mixing holds with rate

\[
\alpha(\ell) = e^{-c_1 \ell}
\]

for \( v \) and for min-distance \( \ell_0 := \log_2 \left( \frac{1}{2\Delta} \right) \).

**Remark 8.** We can take \( c_1 = -\frac{1}{2} \log(M(\Delta, h_0, \beta)\Delta^2) \). Examining the proof shows that the right-hand side of (3.1) can be improved to \( O \left( \frac{1}{\Delta^c} \right) \) for any \( \epsilon > 0 \) at the cost of a tighter bound on \( M(\Delta, h_0, \beta) \), by replacing \( \frac{h}{2} \) in (3.2) by \( (1 - O(\epsilon))\ell \).

Proof. First, we fix \( \ell \geq \ell_0 \) and consider the case when \( \sigma_A \) and \( \tau_A \) disagree at exactly 1 vertex \( w \) at distance \( \ell \) from \( v \). There is a unique path \( \gamma_{wv} \) joining \( w \) and \( v \). If \( \gamma_{wv} \) contains another vertex \( w' \in \Lambda \) besides \( w \), then the probability of the spin at \( u \) is conditionally independent of the spin at \( w \) given the spin at \( w' \), so \( p^u_{\sigma} = p^u_{\tau} \). Otherwise, we apply Lemma 10: letting \( p_x = M(\deg(x), h_x, \beta) \) for \( x \in V \setminus \Lambda \), we have

\[
|p^w_{\sigma} - p^w_{\tau}| \leq |P_p(w \leftrightarrow v) \leq \prod_{u \in \gamma_{wv}} M(\Delta, h_u, \beta),
\]

since percolation occurs only if all sites on the path \( \gamma_{wv} \) have value 1; the second inequality is by Lemma 5. In the general case, by changing the vertices at distance \( \ell \) one at a time, we get that

\[
|p^u_{\sigma} - p^u_{\tau}| \leq \sum_{w : d(v, w) = \ell} \prod_{w \in \gamma_{wv}} M(\Delta, h_u, \beta).
\]

To estimate this we next establish that most \( h_u \) are large. Let \( p = P(|h| < h_0) \). Using the upper bound \( p \leq \frac{1}{16\Delta^2} \) from (3.1) and the Chernoff-Hoeffding bound we obtain that, with high probability, most of the \( h_u \)'s along a path are large.

\[
P\left( |\{u \in \gamma_{wv} : |h_u| \geq h_0 \}| \leq \frac{\ell}{2} \right) \geq 1 - e^{-\frac{\ell}{2} \log \left( \frac{1}{h} \right) + \frac{\ell}{2} \log \left( \frac{1}{h_0} \right)} \geq 1 - \left( \frac{1}{\Delta} \right)^{\ell} \geq 1 - 2^\ell \left( \frac{1}{\Delta} \right)^\ell \geq 1 - \left( \frac{1}{\Delta} \right)^{2^\ell}.
\]

(3.2)
Under this event and by the hypothesis that \( M(\Delta, h_0, \beta) < \Delta^{-2} \), we have that there exists \( c_1 > 0 \) such that
\[
\prod_{u \in \gamma_{uv}} M(\Delta, h_u, \beta) \leq \left( \frac{e^{-2c_1}}{\Delta^2} \right)^{\ell/2} = e^{-\ell c_1}.
\]
In particular, \( c_1 = \frac{1}{2} \log(M(\Delta, h_0, \beta)\Delta^2) \) works. Hence, doing a union bound over all paths, we get that with probability at least \( 1 - 2^{-\ell} \),
\[
|p^\sigma_v - p^\tau_v| \leq e^{-\ell c_1}.
\]
for any \( \sigma, \tau \) disagreeing in \( \Lambda' \), where \( d(\Lambda, \Lambda') = \ell \). Now taking a union bound over \( \ell \geq \ell_0 \), this holds for all \( \ell \geq \ell_0 \) with probability
\[
1 - \sum_{\ell=\ell_0}^{\infty} 2^{-\ell} \geq 1 - 2 \cdot 2^{-\ell_0} \geq 1 - \delta.
\]
where the last inequality uses the definition of \( \ell_0 \).

### 3.2 Strong spatial mixing on the SAW tree

Our next corollary will have nearly the same conclusion as Lemma 11, but it concerns a specific tree of self-avoiding walks (SAW tree). To prepare for this, we recall the construction of the SAW tree for Ising models, see [25, Appendix A] or [36], which follows Weitz’s original construction for the hard-core model [35]. As this construction is well-known and clear expositions exist in the literature, we will be somewhat brief.

A self-avoiding walk of length \( k \), \( (v_i)_{i=0}^{k} \), is a sequence of adjacent vertices, each \( v_i \) distinct. The set of self-avoiding walks started at a fixed vertex \( v \) has a natural rooted tree structure: the root is the length 0 walk consisting of \( v_0 = v \) alone, and the children of length \( k \) self-avoiding walk \( (v_i)_{i=0}^{k} \) are the length \( k + 1 \) self-avoiding extensions \( (v_i)_{i=0}^{k+1} \) with \( v_{k+1} \) adjacent to \( v_k \). Call this tree \( \hat{T}_v \). The self-avoiding walk tree \( T_v \) rooted at \( v \) is obtained from \( \hat{T}_v \) as follows. To each vertex \( (v_i)_{i=0}^{k} \) in \( \hat{T}_v \) append additional leaf vertices \( w_1, \ldots, w_j \), one for each \( v_{k+1} \in V, v_{k+1} \neq v_{k-1} \), such that \( (v_i)_{i=1}^{k+1} \) is not self-avoiding, i.e., \( v_{k+1} \) completes a cycle of length at least three. The tree \( T_v \) is finite, and one obtains an Ising model on \( T_v \) by taking the external field at \( (v_i)_{i=0}^{k} \) to be \( h_{v_k} \).

The key result is then that if one defines a boundary condition \( \tau \) on the leaves of \( T_v \) correctly (i.e., \( \tau_w \in \{-1, +1\} \) for each leaf \( w \), see Appendix A or [25, 36] for details of the construction), then the distribution of \( \sigma_v \) on \( T_v \) with boundary condition \( \tau \) is identical to the distribution of \( \sigma_v \) on \( G \):

**Lemma 12.** There is a choice of spins \( \tau_w \) for the leaves \( w \) of \( T_v \) such that the marginal distribution of the spin at the root is precisely the same as the marginal distribution of \( \sigma_v \) on the graph \( G \).

The exact way in which the boundary condition \( \tau \) is determined will not play a role in what follows, so we will not discuss this in the main text. The preceding construction generalizes to the situation in which there is a boundary condition \( \xi \) for the Ising model on \( G \): in this case the corresponding spins for the Ising model on \( T_v \) are fixed to agree with \( \xi \). Lemma 12 holds in this more general setting as well.

**Corollary 13.** Let \( G \) be a graph with max degree \( \Delta \) on \( n \) vertices. Suppose that the distribution of \( h_x \) satisfies (3.1) for \( h_0 \) such that \( M(\Delta, h_0, \beta) < \Delta^{-2} \). Then there exists a constant \( c_1 > 0 \) so that with probability \( 1 - o(1) \) over the realization of the external fields, for every vertex \( v \in V \) the SAW tree \( T_v \) at \( v \) satisfies strong spatial mixing with rate
\[
\alpha(\ell) = e^{-c_1 \ell}
\]
for \( v \) and for min-distance \( \ell_0 := \frac{\log n}{c_1} \).
Proof. Note that in the SAW tree rooted at \( v \) the value of \( h_x \) is repeated at some vertices, but there are no repetitions along any path to the root \( v \), because this path corresponds to a self-avoiding walk. Therefore Lemma 11 applies to the SAW tree rooted at \( v \). To obtain the corollary, apply the result of Lemma 11 with \( \delta \) taken to be \( \frac{\lambda}{n} \) for each vertex \( v \in V(G) \). The result follows by a union bound over all vertices of \( G \), choosing \( \delta = o(1) \), and decreasing \( c_1 > 0 \) if necessary.

3.3 Proof of Theorem 2

We use Weitz’s approach to approximate counting [35]. In brief, to approximate the partition function of a graph in \( G_\Delta \) with arbitrary boundary conditions it suffices to approximate the marginal of any vertex of any graph in \( G_\Delta \) with arbitrary boundary conditions by writing the partition function as a telescoping product; see Appendix A for the calculation. Similarly, given the ability to approximate marginals, one can sample by setting one spin at a time according to its marginal and updating the boundary conditions. Both the algorithms for counting and for sampling are written out in Appendix B.

Recall, see [35] or [29, Theorem 2.8] that Weitz proved that for any two-state spin system, strong spatial mixing (SSM) on the \( \Delta \)-regular tree implies the existence of an FPTAS on all graphs of degree at most \( \Delta \). In the following, we briefly recall this algorithm and its analysis. We will see that weaker notion of SSM at distance \( \ell_0 \) is sufficient to carry out the analysis.

By the SAW tree construction discussed in Section 3.2, to compute the marginal distribution of the spin at a fixed vertex \( v \) it would suffice to compute the marginal distribution of the spin at the root of \( T_v \). Given the tree structure, this is a recursive computation. The running time, however, is exponential in the depth of the recursion, which could be as large as \( n \). Weitz observed that one can truncate the tree \( T_v \) at logarithmic depth if correlations decay exponentially fast in the depth of the tree, as the analysis of the recursive computation will not be sensitive to the value of the spins at large distances. The running time of the recursion on this truncated tree is linear in the size of the tree, which is polynomial in \( n \).

To make this precise for the hard-core model, Weitz proved that when \( \lambda < \lambda_c(\Delta) \), SSM with rate \( \alpha(t) = e^{-\Omega(t)} \) holds on the SAW tree of a graph of maximum degree \( \Delta \). Consequently, to obtain an \( \epsilon/n \)-approximate evaluation of the marginal of the root of the SAW tree, one can truncate the SAW tree at depth \( \ell = O(\log(n/\epsilon)) \) (see [35, Section 5]). The running time of this algorithm is polynomial in \( n \).

Proof of Theorem 2. Given the discussion above and Lemma 12, it is clear that Corollary 13 suffices for verifying that validity of the algorithm in the setting of the RFIM: for a fixed \( 0 < \delta < 1 \), we have SSM at distance \( \ell_0 = \frac{\log n}{\alpha} \). Hence by taking \( \ell' = \max\{\ell, \ell_0\} \) and truncating at depth \( \ell' \) we obtain the desired polynomial-time algorithm.

Lastly, we check that the event on which the algorithm correctly outputs the desired approximation can be identified in polynomial time as was described in Remark 3.

Proposition 14. Fix \( \epsilon > 0 \). Given \( h \), there is a polynomial time algorithm in \( 1/\epsilon \) and \( n \) that determines if the output of the algorithm from Theorem 2 is an \( \epsilon \)-approximation to the partition function of the Ising model with external fields \( h \).

Proof. For each \( v \in V \), it takes polynomial time to construct the SAW tree \( T_v \) to depth \( \ell' = \max\{\ell, \ell_0\} \), where \( \ell \) is the constant from the proof of Theorem 2 above. Constructing the SAW tree for each \( v \) to this depth thus takes polynomial time as well. Each tree has polynomially many leaves, and for each leaf to check that the path from root to leaf has at least half of its vertices with external field at least \( h_0 \) in magnitude takes linear time.
4 Random field Ising model on random graphs

This section establishes Theorem 3. The arguments are similar to those that established Theorem 2, and we focus our exposition on the new aspects.

A graph drawn from $\mathcal{G}(n, p)$ (an Erdős-Rényi random graph) is defined as a graph on $n$ vertices \{v₁, ..., vₙ\} where each pair of vertices \{vᵢ, vⱼ\}, $i \neq j$ forms an edge independently with probability $p$. We will take $p = \Delta/n$, so that the average degree of a vertex is $p(n-1) \approx \Delta$. However, the maximum degree of $\mathcal{G}(n, \Delta/n)$ is $\Theta\left(\frac{\log n}{\log \log n}\right)$ with high probability.

The key observation that allows us to apply our results in this setting is that an Erdős-Rényi random graph has bounded connective constant with high probability [30, 28].

Lemma 15 ([30, in Proof of Theorem 1.2]). Let $\Delta > 1$. Suppose $G$ is distributed as $\mathcal{G}(n, \Delta/n)$, and let $\gamma, \nu > 0$. With probability at least $1 - n^{-\nu}$, for all $\ell \geq \frac{\nu+2}{\log(1+\gamma/2)} \log n$ and all vertices $v$, the SAW tree $T_v$ satisfies

$$|N(v, \ell)| \leq \left|\Delta(1 + \gamma/2)\right|^\ell, \quad \sum_{d=1}^{\ell} |N(v, d)| \leq \frac{\Delta}{\Delta - 1} \left|\Delta(1 + \gamma/2)\right|^{\ell}.$$

The following lemma is the analogue of Corollary 13. The key additional idea is to classify a vertex as bad if its degree is large, and that large degrees occur with low probability in $\mathcal{G}(n, p)$. Note that degrees of vertices are not independent, so we instead use the Chernoff–Hoeffding inequality on the edge count.

Lemma 16. Let $\Delta > 1$ and $G \sim \mathcal{G}(n, \Delta/n)$ be a random graph on $V$. There are constants $c₃, c₄, c₅$ such that the following hold. Fix a vertex $v \in V$. Let $c₁, c₂$ be any large enough constants. Let $h₀$ be such that $M\left(e^{c₃\Delta}, h₀, \beta\right) \leq \frac{e^{-c₁}}{e^{c₃}}$ and $\mathbb{P}(hₜ < h₀) \leq \left(\frac{1}{\Delta}\right)^{c₂}$. Then with probability $\geq 1 - \delta$ over $G$ and the realization of the external field $h$, the following hold.

(i) The SAW tree $T_v$ at $v$ satisfies strong spatial mixing with rate $\alpha(\ell) = e^{-c₁\ell/2}$ for $v$ and for min-distance $\ell₀ = \frac{\log(\frac{\Delta}{c₃})}{c₂}$.

(ii) For all $\ell \geq \ell₀$, on the SAW tree $T_v$, $\sum_{d=1}^{\ell} |N(v, d)| \leq \left(c₄\Delta\right)^{\ell/2}$.

This holds for the SAW trees at all vertices with the same rate and min-distance $\ell₀ = \frac{\log(\frac{\Delta}{c₃})}{c₂}$.

Proof. Call a path $v₀, \ldots, vℓ$ on the SAW tree starting at $v₀ = v$ bad if one of the following holds:

(i) At least $\frac{1}{4} \ell$ of the vertices $v₀, \ldots, vℓ₋₁$ have degree greater than $c₃\Delta$.

(ii) At least $\frac{1}{4} \ell$ of the values $|h_{v_j}|$ satisfy $|h_{v_j}| > h₀$.

The first step in the proof is to rule out the existence of bad paths with high probability by a union bound argument. To this end, we first bound the probability (over the randomness in $G$ and $h$) that a fixed sequence $v₀, \ldots, vℓ$ of vertices is a bad path. Since each edge is included in $G$ with probability $\Delta/n$, the probability of this fixed path being a path in the SAW tree is $(\Delta/n)^{\ell}$.

Next we bound the probability of event (i). Given a fixed subset $S$ of $\{v₀, \ldots, vℓ₋₁\}$ with $\lfloor \ell/4 \rfloor$ vertices, we bound the probability that all its vertices have degree greater than $e^{c₂c₃}\Delta$. For this to happen, there must be at least $\frac{e^{c₂c₃}\Delta}{2} \cdot \frac{\ell}{4}$ edges in $S \times (V \setminus S)$, which has cardinality $O(\ell n)$. By the Chernoff-Hoeffding bound, the probability of this is $e^{-\Omega(\Delta c₂ c₃)}$ for large enough $c₃$. By a union bound over appropriate subsets of $S$ (less than $2^\ell$ in number), the probability of (i) is still $e^{-\Omega(\Delta c₂ c₃)}$. 12
The probability of event (ii) is bounded exactly as in Lemma 11: letting \( p = \left( \frac{1}{2\Delta} \right)^{e_2c_3} \), it is bounded by \( p^{O(\ell)} \leq (2\Delta)^{-\Omega(e_2c_3\ell)} \). Thus, the probability of a fixed sequence \( v_0, \ldots, v_\ell \) being a bad path is

\[
\left( \frac{\Delta}{n} \right)^\ell \cdot \left( e^{-\Omega(\Delta e_2c_3\ell)} + (2\Delta)^{-\Omega(e_2c_3\ell)} \right) \leq \frac{1}{2} \left( \frac{1}{n} \right)^\ell \cdot e^{-c_3\ell}.
\]

for large enough \( c_3, c_5 \).

By a union bound over possible paths of length \( \ell \), of which there are most \( n^{\ell} \), the probability that a bad path of length \( \ell \geq \ell_0 \) exists is at most \( e^{-c_3\ell_0} \). This is at most \( \frac{\delta}{2} \) when \( \ell_0 = \log \left( \frac{2}{\delta} \right) / c_2 \).

Next, observe that we can choose \( c_4 \) sufficiently large so that by Lemma 15, the probability that \( \sum_{d=1}^{\ell} |N(v, d)| \leq (c_4\Delta)^{\ell/2} \) for all \( \ell \geq \ell_0 \) is at least \( 1 - \frac{\delta}{4} \). (We apply Lemma 15 directly when \( \Delta \geq 2 \); otherwise to avoid the \( \frac{1}{\Delta} \) factor, we note that \( \sum_{d=1}^{\ell} |N(v, d)| \) is stochastically dominated by its value when \( \Delta = 2 \).) Let \( E \) be the event that there are no bad paths of length \( \ell \geq \ell_0 \), and \( \sum_{d=1}^{\ell} |N(v, d)| \leq (c_4\Delta)^{\ell/2} \) for each \( \ell \geq \ell_0 \). By a union bound, \( E \) has probability at least \( 1 - \delta \).

If a path from \( v \) to \( w \) is not bad, then letting \( \ell = d(v, w) \), at least \( \frac{\ell}{2} \) of the vertices \( u \) on the path satisfy \( \deg(u) \leq e^{e_2c_3} \Delta \) and \( |h_u| \leq h_0 \). For any such \( u \), \( M(\deg(u), h_0, \beta) \leq e^{-c_1 \Delta^2} \) by Lemmas 4 and 5. As in the proof of of Lemma 11, let \( \gamma_{uv} \) denote the unique path from \( w \) to \( v \). Then, we have that if \( \sigma_x, \tau_x \) are boundary conditions differing only at \( w \), then

\[
\mathbb{P}(\sigma_x \neq \tau_x | \sigma_A, \tau_A) = \prod_{w \in \gamma_{uv}} M(\deg(u), h_{\partial w}, \beta) \leq \left( \frac{e^{-c_1}}{c_4 \Delta^2} \right)^{\ell/2} = \frac{e^{-c_3 \ell/2}}{c_4^{\ell/2} \Delta^\ell}.
\]

Fix \( \ell \geq \ell_0 \). On the event \( E \), by changing the vertices at distance \( \ell \) one at a time, we have

\[
|p^{\sigma_A}_v - p^{\tau_A}_v| \leq \sum_{w: d(v, w) = \ell} \sum_{u \in \gamma_{uw}} M(\deg(u), h_u, \beta) \leq (c_4\Delta)^{\ell/2} \frac{e^{-c_3 \ell/2}}{c_4^{\ell/2} \Delta^\ell} \leq e^{-c_3 \ell/2},
\]

and we obtain the same conclusion for all \( v \) on the event that no bad paths exist starting from any \( v \). Finally, by replacing \( \delta \) by \( \frac{\delta}{n} \) and union-bounding over all vertices, there are no bad paths in the SAW tree at \( v \) for each vertex \( v \). \( \square \)

**Proof of Theorem 3.** Theorem 3 follows from Lemma 16 after noting the high-probability bound on the neighborhood of a vertex \( v \) given by Lemma 15. The recursive computation of the marginals on the SAW tree still runs in polynomial time because the size of the \( \ell' \)-neighborhood of \( v \) in the SAW tree \( T_v \) is \( \sum_{d=1}^{\ell'} |N(v, d)| \leq (c_4\Delta)^{\ell'/2} \), which is polynomial in all parameters. \( \square \)

### 5 Non-uniform spatial mixing on infinite graphs

The following non-uniform spatial mixing result generalizes [10, Theorem 6]. In this section we work in the context of infinite graphs; we always fix boundary conditions on sets \( B \) such that \( V \setminus B \) is finite.

**Theorem 17** (cf. [10, Theorem 6]). Consider the RFIM with IID Gaussian external fields. Let \( c_2 > 0 \). There exists \( c_1(\Delta, \beta, c_2) \) such that for \( \text{Var}(h_x) \geq c_1(\Delta, \beta, c_2) \), for any \( A \) and \( B \) such that \( V \setminus B \) is finite, for almost all realizations \( h \),

\[
\sup_{\eta, \xi \in \{\pm 1\}^B} d_{TV}(p^{\tau_h \eta}_\beta, \sigma_A \in \cdot, p^{\tau_h \xi}_\beta, \sigma_A \in \cdot) \leq \sum_{x \in \partial A, y \in \partial B} c_3(x, h) e^{-c_2 d(x, y)}.
\]
where
\[(\tau \wedge \eta)(x) = \begin{cases} 
\tau(x), & x \in V \setminus B \\
\eta(x), & \text{otherwise},
\end{cases}\]
and similarly for \(\xi\). Here, for a set \(A \subset V\), \(\partial A \subset V \setminus A\) denotes the subset of \(A\) whose neighbors are not all contained in \(A\).

Recall the inhomogenous site percolation processes \(P_p\) with \(p_x = M(\deg(x), h_x, \beta)\) introduced in Lemma 10.

**Lemma 18** (cf. [10, Lemma 6]). Consider the measure \(P_p\) averaged over the randomness in \(h\), \(\overline{P}_p(\cdot) = \int_{\mathcal{H}} P_p(\cdot) \mathbb{P}(dh)\) where \(\mathbb{P}\) is the law of \(h\). Let \(c_2 > \log 2\). There exists \(h_0 = h_0(\Delta, \beta, c_2)\) so that when \(\mathbb{P}(|h_u| < h_0) \leq p := \frac{e^{-2c_2}}{1\Delta^2}\),
\[\overline{P}_p(x \leftrightarrow y) \leq 4e^{-c_2d(x, y)}\]
Proof. Choose \(h_0 = |\beta|\Delta + \log(\Delta) + c_2\), so by Lemma 4, \(M(\Delta, h_0, \beta) \leq \frac{e^{-2c_2}}{\Delta^2}\). Consider a path \(\gamma\) from \(x\) to \(y\) of length \(\ell := d(x, y)\). Then
\[\overline{P}_p(\{\forall u \in \gamma, S_u = 1\}) = \prod_{u \in \gamma} M(\deg(u), h_u, \beta). \tag{5.1}\]
As in (3.2), by the Chernoff bound, with high probabilty, most of the \(h_u\)’s for \(u \in \gamma\) are large:
\[\mathbb{P}
\left(|\{u \in \gamma : |h_u| \geq h_0\}| \geq \frac{\ell}{2}\right) \geq 1 - e^{-\left(\frac{1}{2}\log\left(\frac{1/2}{p}\right) + \frac{1}{2}\log\left(1/2\right)\right)\ell}
\geq 1 - 2p^{1/2} \geq 1 - \left(\frac{e^{-c_2}}{\Delta}\right)^\ell\]
Under this event,
\[\prod_{u \in \gamma} M(\Delta, h_u, \beta) \leq \left(\frac{e^{-2c_2}}{\Delta^2}\right)^{\ell/2} = \frac{e^{-c_2\ell}}{\Delta^{\ell}}\]
Hence, by breaking up (5.1) into two bad events,
\[\overline{P}_p(\{\forall u \in \gamma, S_u = 1\}) = \mathbb{P}
\left(|\{u \in \gamma : |h_u| \geq h_0\}| < \frac{\ell}{2}\right)
+ \overline{P}_p
\left(|\{u \in \gamma, S_u = 1\}| \left|\{u \in \gamma : |h_u| \geq h_0\}\right| \geq \frac{\ell}{2}\right)
\leq \frac{e^{-c_2\ell}}{\Delta^{\ell}} + \frac{e^{-c_2\ell}}{\Delta^{\ell}} = \frac{2e^{-c_2\ell}}{\Delta^{\ell}}\]
There are at most \(\Delta^j\) paths of length \(j\), so taking a union bound over all paths gives
\[\overline{P}_p(x \leftrightarrow y) \leq \sum_{j=\ell}^{\infty} \Delta^j \frac{2e^{-c_2j}}{\Delta^j} \leq 4e^{-c_2\ell}\]
Given Lemma 18, the proof of Theorem 17 is exactly the same as in [10], after noting that the neighborhood of a vertex grows at most exponentially.

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Proof of Theorem 17. By Lemma 18 applied to $c_2 \leftarrow 2c_2 + \log(\Delta)$,
\[
\sum_{y \in V} e^{c_2 d(x,y)} \mathbf{P}_p(x \leftrightarrow y) \leq \sum_{y \in V} e^{c_2 d(x,y)} 4e^{-2c_2 d(x,y) \Delta - d(x,y)} < \infty,
\]
where we use the fact that the number of vertices at distance $\ell$ from $x$ is at most $\Delta^{\ell}$. Expanding $\mathbf{P}_p$ as an integral over $h$ and using the Fubini-Tonelli theorem,
\[
\int_{\mathbb{R}^V} \sum_{y \in V} e^{c_2 d(x,y)} P_p(x \leftrightarrow y) \mathbb{P}(dh) < \infty,
\]
which implies
\[
\sum_{y \in V} e^{c_2 d(x,y)} P_p(x \leftrightarrow y) < \infty
\]
for almost all $h$, and
\[
e^{c_2 d(x,y)} P_p(x \leftrightarrow y) < c_3(x, h)
\]
for almost all $h$. Using Lemma 10, we get
\[
\sup_{\eta, \xi \in \{\pm 1\}^B} d_{TV}(p_{\beta, h}^{\tau \wedge \eta}(\sigma_A \in \cdot), p_{\beta, h}^{\tau \wedge \xi}(\sigma_A \in \cdot)) \leq P_p(A \leftrightarrow B)
\]
\[
\leq \sum_{x \in \partial A, y \in \partial B} P_p(x \leftrightarrow y)
\]
\[
\leq \sum_{x \in \partial A, y \in \partial B} c_3(x, h) e^{-c_2 d(x,y)}. \quad \Box
\]

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References

[1] Michael Aizenman and Jan Wehr. Rounding effects of quenched randomness on first-order phase transitions. *Communications in Mathematical Physics*, 130(3):489–528, 1990.

[2] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 1–12, 2019.

[3] Alexander Barvinok. *Combinatorics and Complexity of Partition Functions*, volume 9. Springer, 2016.

[4] Alexander Barvinok and Guus Regts. Weighted counting of solutions to sparse systems of equations. *Combinatorics, Probability and Computing*, 28(5):696–719, 2019.
[5] Mohsen Bayati, David Gamarnik, Dimitriy Katz, Chandra Nair, and Prasad Tetali. Simple deterministic approximation algorithms for counting matchings. In Proceedings of the thirty-ninth annual ACM symposium on Theory of computing, pages 122–127, 2007.

[6] Christian Borgs, Jennifer Chayes, Tyler Helmuth, Will Perkins, and Prasad Tetali. Efficient sampling and counting algorithms for the Potts model on $\mathbb{Z}^d$ at all temperatures. In Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, pages 738–751, 2020.

[7] Jean Bricmont and Antti Kupiainen. Phase transition in the 3d random field Ising model. Communications in Mathematical Physics, 116(4):539–572, 1988.

[8] Russ Bubley and Martin Dyer. Path coupling: A technique for proving rapid mixing in Markov chains. In Proceedings 38th Annual Symposium on Foundations of Computer Science, pages 223–231. IEEE, 1997.

[9] Jin-Yi Cai, Andreas Galanis, Leslie Ann Goldberg, Heng Guo, Mark Jerrum, Daniel Štefankovič, and Eric Vigoda. #BIS-hardness for 2-spin systems on bipartite bounded degree graphs in the tree non-uniqueness region. Journal of Computer and System Sciences, 82(5):690–711, 2016.

[10] Federico Camia, Jianping Jiang, and Charles M Newman. A note on exponential decay in the random field Ising model. Journal of Statistical Physics, 173(2):268–284, 2018.

[11] F Cesi, Christian Maes, and F Martinelli. Relaxation of disordered magnets in the Griffiths' regime. Communications in Mathematical Physics, 188(1):135–173, 1997.

[12] Jian Ding, Jian Song, and Rongfeng Sun. A new correlation inequality for Ising models with external fields. arXiv preprint arXiv:2107.09243, 2021.

[13] Jian Ding and Jiaming Xia. Exponential decay of correlations in the two-dimensional random field Ising model. Inventiones Mathematicae, 224(3):999–1045, 2021.

[14] Martin Dyer, Leslie Ann Goldberg, Catherine Greenhill, and Mark Jerrum. The relative complexity of approximate counting problems. Algorithmica, 38(3):471–500, 2004.

[15] Jürg Fröhlich and John Z Imbrie. Improved perturbation expansion for disordered systems: beating Griffiths singularities. Communications in mathematical physics, 96(2):145–180, 1984.

[16] Andreas Galanis, Daniel Stefankovic, Eric Vigoda, and Linji Yang. Ferromagnetic Potts model: Refined #BIS-hardness and related results. SIAM Journal on Computing, 45(6):2004–2065, 2016.

[17] Leslie Ann Goldberg and Mark Jerrum. The complexity of ferromagnetic Ising with local fields. Combinatorics, Probability and Computing, 16(1):43–61, 2007.

[18] Tyler Helmuth, Matthew Jenssen, and Will Perkins. Finite-size scaling, phase coexistence, and algorithms for the random cluster model on random graphs. arXiv preprint arXiv:2006.11580, 2020.

[19] Tyler Helmuth, Will Perkins, and Guus Regts. Algorithmic Pirogov–Sinai theory. Probability Theory and Related Fields, 176(3):851–895, 2020.

[20] Jeroen Huijben, Viresh Patel, and Guus Regts. Sampling from the low temperature Potts model through a Markov chain on flows. arXiv preprint arXiv:2103.07360, 2021.
[21] Mark Jerrum and Alistair Sinclair. Polynomial-time approximation algorithms for the Ising model. *SIAM Journal on computing*, 22(5):1087–1116, 1993.

[22] David A Levin and Yuval Peres. *Markov Chains and Mixing Times*, volume 107. American Mathematical Soc., 2017.

[23] Liang Li, Pinyan Lu, and Yitong Yin. Correlation decay up to uniqueness in spin systems. In *Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms*, pages 67–84. SIAM, 2013.

[24] Jingcheng Liu, Pinyan Lu, and Chihao Zhang. The complexity of ferromagnetic two-spin systems with external fields. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2014)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2014.

[25] Jingcheng Liu, Alistair Sinclair, and Piyush Srivastava. Fisher zeros and correlation decay in the Ising model. *Journal of Mathematical Physics*, 60(10):103304, 2019.

[26] Elchanan Mossel, Dror Weitz, and Nicholas Wormald. On the hardness of sampling independent sets beyond the tree threshold. *Probability Theory and Related Fields*, 143(3):401–439, 2009.

[27] Ricardo Restrepo, Jinwoo Shin, Prasad Tetali, Eric Vigoda, and Linji Yang. Improved mixing condition on the grid for counting and sampling independent sets. *Probability Theory and Related Fields*, 156(1-2):75–99, 2013.

[28] Alistair Sinclair, Piyush Srivastava, Daniel Štefankovič, and Yitong Yin. Spatial mixing and the connective constant: optimal bounds. *Probability Theory and Related Fields*, 168(1-2):153–197, 2017.

[29] Alistair Sinclair, Piyush Srivastava, and Marc Thurley. Approximation algorithms for two-state anti-ferromagnetic spin systems on bounded degree graphs. In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete algorithms*, pages 941–953, 2012.

[30] Alistair Sinclair, Piyush Srivastava, and Yitong Yin. Spatial mixing and approximation algorithms for graphs with bounded connective constant. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 300–309. IEEE, 2013.

[31] Jacob van den Berg. A uniqueness condition for Gibbs measures, with application to the 2-dimensional Ising antiferromagnet. *Communications in Mathematical Physics*, 152(1):161–166, 1993.

[32] Jacob van den Berg and Jeffrey E Steif. Percolation and the hard-core lattice gas model. *Stochastic Processes and their Applications*, 49(2):179–197, 1994.

[33] ACD van Enter. Griffiths singularities. *Modern Encyclopedia of Mathematical Physics*, 2007.

[34] Henrique Von Dreifus, Abel Klein, and J Fernando Perez. Taming Griffiths’ singularities: infinite differentiability of quenched correlation functions. *Communications in mathematical physics*, 170(1):21–39, 1995.

[35] Dror Weitz. Counting independent sets up to the tree threshold. In *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pages 140–149, 2006.

[36] Jinshan Zhang, Heng Liang, and Fengshan Bai. Approximating partition functions of the two-state spin system. *Information Processing Letters*, 111(14):702–710, 2011.
A SAW tree and recursion

In this section, we first describe how the Ising model on a graph $G$ translates to an Ising model on the SAW tree $T_v$ defined in Section 3.2, and then show how to compute the marginal probabilities for $\sigma_v$ in $T_v$.

**Ising model on the SAW tree.** Given an Ising model on a graph $G = (V, E)$ with inverse temperature $\beta$ and external fields $h$, we obtain an Ising model on $T_v$ by taking the same inverse temperature $\beta$ and taking the external field at $(v_i)^k_{i=0}$ to be $h_{v_k}$. If we are given boundary conditions $\tau$ on a set of vertices $\partial V$, then we take the boundary condition at $(v_i)^k_{i=0}$ to be $\tau_{v_k}$ whenever $v_k \in \partial V$.

Fix a lexicographic order on the vertices $V$; this induces an order on the edges incident to any fixed vertex. For the vertices $\gamma$ in $T_v \setminus \hat{T}_v$ (those representing paths with a cycle), we instead assign them the following boundary condition:

$$\tau_\gamma = \begin{cases} 1, & \text{if the edge closing the cycle is larger than the edge starting the cycle in } \gamma \\ -1, & \text{otherwise.} \end{cases}$$

**Marginal probabilities on the SAW tree.** For simplicity of notation, let $p = p_{T_v, \beta, h}$ denote the Ising model on the $T_v$. For convenience, we work with the occupation ratio

$$R_v := \frac{p(\sigma_v = -1)}{p(\sigma_v = +1)},$$

Recalling the definition of Gibbs measure, we have

$$R_v = e^{-2h_v} \frac{p'(\sigma_v = -1)}{p'(\sigma_v = +1)} = e^{-2h_v} \prod_{i=1}^{\deg(v)} \frac{p'_{u_i}(\sigma_v = -1)}{p'_{u_i}(\sigma_v = +1)},$$

where $p'$ is the Gibbs measure after removing $h_v$ (the external field at vertex $v$), and $u_i$ are the child vertices of $v$. The measure $p'_{u_i}$ is the Gibbs measure defined on the subtree by removing all other subtrees except the one rooted at vertex $u_i$. Note that this still includes the root vertex $v$. We define $p''_{u_i}$ as the Gibbs measure on the subtree rooted at $u_i$, excluding $v$. Then

$$\frac{p'_{u_i}(\sigma_v = -1)}{p'_{u_i}(\sigma_v = +1)} = e^\beta \frac{p''_{u_i}(\sigma_{u_i} = -1)}{p''_{u_i}(\sigma_{u_i} = +1)} + e^{-\beta} \frac{p''_{u_i}(\sigma_{u_i} = +1)}{p''_{u_i}(\sigma_{u_i} = +1)} = e^{2\beta R_{u_i} + 1} \frac{R_{u_i} + e^{2\beta}}{R_{u_i} + e^{2\beta}}.$$

Using $R_v = \frac{p_v}{1-p_v}$ and $p_v = \frac{1}{1+R_v}$, we can write this in terms of $p_v := p(\sigma_v = 1)$,

$$p_v = \frac{1}{1 + e^{-2h_v} \prod_{i=1}^{d} \frac{e^{2\beta R_{u_i} + 1}}{R_{u_i} + e^{2\beta}}} = \frac{1}{1 + e^2 h_v \prod_{i=1}^{d} \frac{e^{2\beta (1-p_{u_i}) + p_{u_i}}}{(1-p_{u_i}) + e^{2\beta} p_{u_i}}}.$$

(A.1)

This equation provides a recursive method for computing marginal probabilities. Given boundary conditions $\tau$ on $\partial V$, we set $p_v = 1$ or 0 according to whether $\tau_v = 1$ or $\tau_v = -1$, and then work our way up to the root vertex.
B Algorithms

We explicitly write out the algorithms for approximate sampling (Algorithm 1) and computation of $Z_{G,\beta,h}$ (Algorithm 2). These algorithms work for both max-degree $\Delta$ graphs in Theorem 2 and $G(n,\Delta/n)$ graphs in Theorem 3, in the appropriate regime and with the appropriate constants. For the sampling algorithm (Algorithm 1), we repeat the following: estimate the marginal probabilities for an unfixed vertex, use it to sample the spin for the vertex, and then add that value to the boundary conditions. For estimation of $Z_{G,\beta,h}$ (Algorithm 2), to see that the product $e^{-H_{\beta,h}(\sigma)} \prod_{i=1}^{n} r_i$ gives the right answer, note that if $p^*_v$ are the actual probabilities,

$$r_i^* := \begin{cases} \frac{1}{p^*_v}, & \sigma_i = 1 \\ \frac{1}{1-p^*_v}, & \sigma_i = -1 \end{cases} = p(\sigma'_i = \sigma_i | \sigma'_j = \sigma_j \text{ for } j < i) = \frac{p(\sigma'_j = \sigma_j \text{ for } j \leq i - 1)}{p(\sigma'_j = \sigma_j \text{ for } j \leq i)}.$$  

Then we have a telescoping product

$$e^{-H_{\beta,h}(\sigma)} \prod_{i=1}^{n} r_i^* = Z_{G,\beta,h} \cdot p(\sigma) \prod_{i=1}^{n} p(\sigma'_j = \sigma_j \text{ for } j \leq i - 1) / p(\sigma'_j = \sigma_j \text{ for } j \leq i) = Z_{G,\beta,h}.$$  

With appropriate choice of $c$, we can ensure that for each $i$, with probability at least $1 - \delta/n$, that $p_{v_i} \in [p^*_v e^{-\epsilon/n}, p^*_v e^{\epsilon/n}]$ and $r_i \in [r_i^* e^{-\epsilon/n}, r_i^* e^{\epsilon/n}]$. Then with probability at least $1 - \delta$, the estimate will be contained in $r_i \in [Z_{G,\beta,h} e^{\epsilon}, Z_{G,\beta,h} e^{-\epsilon}]$.

Algorithm 1 Approximate sampling from RFIM

**Input:** Random field Ising model $(G, \beta, h)$, failure probability $\delta$, accuracy $\epsilon$.

**Output:** Approximate sample

1: Order the vertices $v_1, \ldots, v_n$.
2: for $i = 1 \rightarrow n$ do
3: Construct the SAW tree at $v_i$, $T_{v_i}$.
4: Set boundary conditions $\tau'_w = 1$ in $T_{v_i}$ for all $w$ such that $d(v_i, w) > c \max \{ \log \left( \frac{n}{\epsilon} \right), \log \left( \frac{n}{\delta} \right) \}$ for an appropriately large constant $c$. Set $p_w = 1$ for these $w$. Note that arbitrary boundary conditions can be chosen.
5: Set boundary conditions corresponding to $\sigma_j$, $1 \leq j < i$ in $T_{v_i}$.
6: Use recursion (A.1) to compute $p_{v_i}$.
7: Set

$$\sigma_i = \begin{cases} 1, & \text{with probability } p_{v_i} \\ -1, & \text{with probability } 1 - p_{v_i} \end{cases}.$$  

8: end for
9: return $(\sigma_1, \ldots, \sigma_n)$. 

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Algorithm 2 Approximation of partition function for RFIM

**Input:** Random field Ising model on $G$, failure probability $\delta$, accuracy $\epsilon$ (where desired multiplicative accuracy is $e^\epsilon$).

**Output:** Approximation of partition function $Z_{G,\beta,h}$

1: Order the vertices $v_1, \ldots, v_n$.
2: for $i = 1 \rightarrow n$ do
3:     Construct SAW tree at $v_i$, $T_{v_i}$.
4:     Set boundary conditions $\tau'_w = 1$ in $T_{v_i}$ for all $w$ such that $d(v_i, w) > c \max \{\log \left(\frac{n}{\epsilon}\right), \log \left(\frac{n}{\delta}\right)\}$ for an appropriately large constant $c$. Set $p_w = 1$ for these $w$. \hspace{1em} ▷ Note that arbitrary boundary conditions can be chosen.
5:     Set boundary conditions corresponding to $\sigma_j, 1 \leq j < i$ in $T_{v_i}$.
6:     Use recursion (A.1) to compute $p_{v_i}$.
7:     if $p_{v_i} \geq \frac{1}{2}$ then
8:         Set $\sigma_i = 1$ and $r_i = \frac{1}{p_{v_i}}$.
9:     else
10:        Set $\sigma_i = -1$ and $r_i = \frac{1}{1-p_{v_i}}$.
11:     end if
12: Include $\sigma_i$ as boundary condition in $G'$.
13: end for
14: return $e^{-H_{\beta,h}(\sigma)} \prod_{i=1}^{n} r_i$. 