A Riemann–Hilbert approach to the Akhiezer polynomials

BY YANG CHEN1,* AND ALEXANDER R. ITS2

1Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, USA
2Department of Mathematical Sciences, Indiana University Purdue University Indiana, 402 North Blackford Street, Indianapolis, IN 46202-3216, USA

In this paper, we study those polynomials, orthogonal with respect to a particular weight, over the union of disjoint intervals, first introduced by N. I. Akhiezer, via a reformulation as a matrix factorization or Riemann–Hilbert problem. This approach complements the method proposed in a previous paper, which involves the construction of a certain meromorphic function on a hyperelliptic Riemann surface. The method described here is based on the general Riemann–Hilbert scheme of the theory of integrable systems and will enable us to derive, in a very straightforward way, the relevant system of Fuchsian differential equations for the polynomials and the associated system of the Schlesinger deformation equations for certain quantities involving the corresponding recurrence coefficients. Both of these equations were obtained earlier by A. Magnus. In our approach, however, we are able to go beyond Magnus’ results by actually solving the equations in terms of the Riemanni $\Theta$-functions. We also show that the related Hankel determinant can be interpreted as the relevant $\tau$-function.

Keywords: orthogonal polynomials; Riemann–Hilbert problems; Akhiezer polynomials

1. Introduction

The Chebyshev polynomials are those monic polynomials characterized by the property that $\max|\pi_n(x)|$, $x \in [-1,1]$ is as small as possible. Indeed, it is also known that $\pi_n$ is orthogonal with respect to $1/(\pi \sqrt{1-x^2})$ over $[-1,1]$. The polynomials $\pi_n$, the Chebyshev polynomials of the first kind, which satisfy constant coefficients three-term recurrence relations, can be thought of as the ‘hydrogen atom’ model of those polynomials orthogonal over $[-1,1]$. These play a fundamental role in the large $n$ asymptotics of the Bernstein–Szegö polynomials which are orthogonal with respect to a ‘deformed’ Chebyshev weight, $p(x)/\sqrt{1-x^2}$ over $[-1,1]$, where $p(x)$ is strictly positive, absolutely

* Author and address for correspondence: Department of Mathematics, Imperial College, 180 Queen’s Gates, London SW7 2BZ, UK (ychen@ic.ac.uk).

One contribution of 15 to a Theme Issue ‘30 years of finite-gap integration’.

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continuous and satisfies the Szegő condition (Szegő 1975)
\[
\int_{-1}^{1} \ln p(x) \frac{dx}{\sqrt{1-x^2}} > -\infty.
\]

Many years ago, Akhiezer (1960), Akhiezer & Tomchuk (1961) and Tomchuk (1964) considered a generalization of the Chebyshev polynomials, where the interval of orthogonality is a union of disjoint intervals henceforth denoted as
\[
E := (\beta_0, \alpha_1) \cup (\beta_1, \alpha_2) \cup \cdots \cup (\beta_{g}, \beta_{g+1}).
\]

For comparison with those of Akhiezer, we assume here \( \beta_0 = -1 \), and \( \beta_{g+1} = 1 \). For later convenience, when the end points become independent variables, we shall adopt the convention,
\[
(\alpha_1, \alpha_2, \ldots, \alpha_g, \beta_0, \beta_1, \ldots, \beta_{g+1}) \to (\delta_1, \delta_2, \ldots, \delta_g, \delta_{g+1}, \delta_{g+2}, \ldots, \delta_{2g+2}).
\]

Let
\[
w(z) := \frac{i}{\pi} \sqrt{\frac{\Pi_{j=1}^{g} (z - \alpha_j)}{\Pi_{j=0}^{g} (z - \beta_j)}}
\]
be defined in the \( \mathbb{C}P^1 \setminus E \). The multi-interval analogue of the Chebyshev weight
\[
w_{+}(t) = \frac{1}{\pi} \sqrt{\frac{\Pi_{j=1}^{g} (t - \alpha_j)}{(\beta_{g+1} - t)(t - \beta_0)\Pi_{j=1}^{g} (t - \beta_j)}} > 0, \quad t \in E,
\]
and is obtained from the continuation \( w(z) \) to the top of the cut, \( E \). The generalized Chebyshev or Akhiezer polynomials \( P_n \) are monic polynomials orthogonal with respect to \( w_{+} \), i.e.
\[
\int_{E} P_{m}(x) P_{n}(x) w_{+}(x) \, dx = h_{n}\delta_{m,n},
\]
where \( h_{n} \) is the square of the \( L^2 \) norm.

In the construction of the Bernstein–Szegő asymptotics over \( E \), for polynomials orthogonal with respect to the weight \( p(t)w_{+}(t) \), where \( p \) is an absolutely continuous positive function, exact information on \( P_n \) would be required. This would entail the solution of the hydrogen atom problem in the multiple interval situation. In the case of two intervals, \( [-1, \alpha] \cup [\beta, 1] \), \( P_n \) was constructed by Akhiezer with an innovation which we would now recognize as the Baker–Akhiezer function, associated with the discrete Schrödinger equation, namely the three-term recurrence relations, where the degree of the polynomials \( n \) is the ‘coordinates’ and \( z \) is spectral variable. Akhiezer based his construction on the conformal mapping of a doubly connected domain, with the aid of the Jacobian elliptic functions, as a demonstration for his students, the applications of elliptic functions (Akhiezer 1990). It is not at all clear how the conformal mapping could be adapted to handle the situation when there are more than two intervals. In the early 1960s, Akhiezer and also with Tomchuk published several very short and very deep papers regarding the Bernstein–Szegő asymptotics. Akhiezer and Tomchuk gave a description of \( P_n \) and \( Q_n \) (the second solution of the recurrence relations) with the aid of theory of hyperelliptic integrals in terms of a cerian Abelian integral of the third kind. However, certain unknown points on Riemann surface appear in this representation, later circumvented in Chen & Lawrence (2002).
In a recent work of Magnus (1995), a general class of semi-classical orthogonal polynomials, which includes the Akhiezer polynomials \( P_n \), was introduced and shown that these polynomials satisfy a certain system of linear Fuchsian equations. It was also demonstrated there that the recurrence coefficients, as functions of the natural parameters of the semi-classical weights, obey the nonlinear Schlesinger equations, i.e. the differential equations describing the isomonodromy deformations of the Fuchsian systems.

In this paper, we will study the Akhiezer polynomials \( P_n \) using the Riemann–Hilbert approach introduced in the theory of orthogonal polynomials in Fokas et al. (1992). This will allow us to exploit the well-developed Riemann–Hilbert and algebro-geometric schemes of the Soliton theory (Manakov et al. 1980; Faddeev & Takhtajan 1987; Belokolos et al. 1994)—with certain important technical modifications though, and not only re-derive the previous results of Chen & Lawrence (2002) and Magnus (1995), but also unite them in a single approach and produce further facts concerning the Akhiezer polynomials. Specifically, in addition to the derivation of Magnus’ equations, we will solve them in terms of the multidimensional \( \Theta \)-functions, and we will identify the corresponding Hankel determinant with the relevant \( \tau \)-function, i.e. with one of the central objects associated with an integrable system, in our case, with the Magnus–Schlesinger equation. It should also be mentioned that part of our \( \Theta \)-formulae, namely the ones describing the recurrence coefficients and the related Baker–Akhiezer function, reproduce the known expressions obtained in the late 1970s (the works of I. Krichever, D. Mumford, S. Novikov and M. Salle) for the finite-gap discrete Schrödinger operators which were then intensively studied in connection with the periodic Toda lattice (see the pioneering paper of Flaschka & McLaughlin (1976) and also Manakov et al. (1980), Faddeev & Takhtajan (1987) for more on the history of the subject).

We would like to think of our paper as a tribute to the pioneering works of N. I. Akhiezer which laid the foundation for the construction, in the 1970s, of the algebro-geometric method in the theory of integrable systems, whose modern ‘Riemann–Hilbert’ version we are using here.

2. Riemann–Hilbert problem

According to the classical theory of orthogonal polynomials, the monic \( P_n \) (with \( P_0 = 1 \) and \( P_{-1} = 0 \)) and the polynomials of the second kind,

\[
Q_n(z) := \int_E \frac{P_n(z) - P_n(t)}{z - t} w_+(t) \, dt
\]

(2.1)

of degree \( n - 1 \) are linearly independent solutions of the second-order difference equation,

\[
zv_n(z) = v_{n+1}(z) + b_{n+1}v_n(z) + a_n v_{n-1}(z).
\]

(2.2)

Following the general scheme of Fokas et al. (1992) (see also Bleher & Its 1999; Deift et al. 1999), let us introduce the 2×2 matrix \( Y_n(z) \) to be defined
for \( n = 1, 2, \ldots \) and \( z \in \mathbb{C} \) as follows:

\[
Y_n(z) = \begin{pmatrix}
P_n(z) & \int_E \frac{P_n(t)w_+(t)}{z-t} \, dt \\
\frac{P_{n-1}(z)}{h_n-1} & \frac{1}{h_{n-1}} \int_E \frac{P_{n-1}(t)w_+(t)}{z-t} \, dt
\end{pmatrix}
\]

\[
= \begin{pmatrix}
P_n(z) & \psi(z)P_n(z) - Q_n(z) \\
\frac{P_{n-1}}{h_{n-1}} & \psi(z)P_{n-1}(z) - Q_{n-1}(z)
\end{pmatrix},
\]

(2.3)

where

\[
\psi(z) := \int_E \frac{w_+(t)}{z-t} \, dt = \frac{\Pi_{i=1}^q(z - \alpha_i)}{\Pi_{j=0}^{q+1}(z - \beta_i)} = -\pi w(z).
\]

Proposition 2.1. The function \( Y_n(z) \) satisfies the following conditions.

RH1. \( Y_n(z) \) is analytic in \( \mathbb{C} \setminus E \).

RH2. \( Y_{n-}(z) = Y_{n,+}(z) \begin{pmatrix} 1 & 2\pi i w_+(z) \\ 0 & 1 \end{pmatrix}, \ z \in E \setminus \{ \beta_j \}_{j=0}^{g+1} \).

RH3. \( Y_n(z)z^{-n\sigma_3} \to I, \ z \to \infty \).

RH4. \( Y_n(z) = \hat{Y}^{(\beta_j)}_n(z) \begin{pmatrix} \sqrt{z - \beta_j} & 0 \\ \frac{1}{b_j} & \frac{1}{\sqrt{z - \beta_j}} \end{pmatrix}, \ z \in \mathcal{U}_0, 0 \leq j \leq g + 1 \),

where \( \mathcal{U}_0 \) denotes a neighbourhood of a point \( Z_0 \). The matrix-valued function \( \hat{Y}^{(\beta_j)}_n(z) \) is holomorphic in \( \sqrt{z - \beta_j} \) and \( b_j \) is defined by the equation

\[
w(z) = (z - \beta_j)^{-1/2} b_j \frac{1}{\pi} (1 + O(z - \beta_j)).
\]

(2.5)

We shall also assume that the branch of \( \sqrt{z - \beta_j} \) is defined by the condition

\[
0 < \arg(z - \beta_j) < 2\pi, \ \text{if} \ j \leq g \ \text{and} \ -\pi < \arg(z - \beta_{g+1}) < \pi.
\]

In addition, we assert that

\[
\det \hat{Y}^{(\beta_j)}_n(\beta_j) = 1 \neq 0.
\]

Proof. Using the basic properties of the Cauchy integrals and the Plemelj formulae, we directly verify that \( Y_n(z) \) satisfies RH1–RH2. To check property RH3, it is enough to note that owing to the orthogonality condition (1.5), we
have (cf. Fokas et al. 1992; Chen & Lawrence 2002)

\[
\int_E \frac{P_n(t)w_+(t)}{z-t} \, dt = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int_E P_n(t)w_+(t)t^k \, dt = \frac{h_n}{z^{n+1}} + O\left(\frac{1}{z^{n+2}}\right), \quad z \to \infty.
\]

To prove RH4, we observe that the matrix product

\[
Y_n(z) = \begin{pmatrix}
\frac{1}{\sqrt{z-\beta_j}} & 0 \\
-\frac{1}{b_j} & \sqrt{z-\beta_j}
\end{pmatrix}
\]

is bounded near \(\beta_j\) (the singular terms in the first column cancel out), and hence the function \(Y_n^{(\beta_j)}(z)\) defined by equation RH4 is indeed holomorphic in \(\sqrt{z-\beta_j}\). To complete the proof of the proposition, we only need to establish equation (2.6). To this end, we note that we have already established RH1–RH4 but short of equation (2.6). One can see, however, that RH1–RH4 already yield an even stronger statement. Namely, we claim that

\[
\det Y_n(z) \equiv 1. \quad (2.7)
\]

Indeed, the (scalar) function \(\det Y_n(z)\) is holomorphic in \(\mathbb{C}P^1 \setminus E\), has no jumps across \(E\) and removable singularities at the end points of \(E\); moreover, it approaches 1 as \(z \to \infty\). By the Liouville theorem, equation (2.7) follows. Equation (2.6) is a direct consequence of equation (2.7). The proposition is proven.

**Remark 2.1.** Equation (2.7) can be also derived using the first line of (2.3) and the Christoffel–Darbooux formula,

\[
\det Y_n(z) = \frac{1}{h_{n-1}} \int_E \frac{P_n(z)P_{n-1}(t) - P_{n-1}(z)P_n(t)}{z-t} \, dt
\]

\[
= \int_E \sum_{k=0}^{n-1} \frac{1}{h_k} P_k(z)P_k(t)w_+(t) \, dt
\]

\[
= \int_E K_n(z, t)w_+(t) \, dt = P_0(z)h_0 = 1, \quad (2.8)
\]

or from the recurrence relations,

\[
\det Y_n(z) = \frac{1}{h_{n-1}} (Q_n(z)P_{n-1}(z) - P_n(z)Q_{n-1}(z)) = 1. \quad (2.9)
\]

**Remark 2.2.** \(Y_n(z)\) also depend on \(\{\delta_j : 1 \leq j \leq 2g+2\}\).

**Proposition 2.2.** Conditions RH1–RH4 define the function \(Y_n(z)\) uniquely.

**Proof.** If \(\tilde{Y}_n(z)\) is another function that satisfies RH1–RH4, then \(X_n(z) := \tilde{Y}_n(z)Y_n^{-1}(z)\) is holomorphic for \(z \in \mathbb{C}P^1 \setminus \{\beta_j : 0 \leq j \leq g+1\}\). Furthermore,
for $z \in \mathcal{U}_{\beta_j}$

$$Y_{n}^{-1}(z) = \left( \begin{array}{cc} \frac{1}{\sqrt{z - \beta_j}} & 0 \\ -\frac{1}{b_j} & \sqrt{z - \beta_j} \end{array} \right) \hat{Y}_{n}^{(\beta_j)^{-1}}(z),$$

(2.10)

where $\hat{Y}_{n}^{(\beta_j)^{-1}}(z)$ is holomorphic (see equation (2.6)) in $\sqrt{z - \beta_j}$. This implies

$$X_{n}(z) = O(1), \quad z \sim \beta_j,$$

(2.11)

which in turn implies $X_{n}(z)$ is holomorphic for $z \in \mathbb{C}P^1$ and $X_{n}(z) = I$, for all $z \in \mathbb{C}P^1$.

The conditions RH1–RH4 constitute the Riemann–Hilbert problem whose unique solution is given by equation (2.3), due to proposition 2.1.

The Riemann–Hilbert problem RH1–RH4 together with the equation

$$P_{n}(z) = (Y_{n}(z))_{11},$$

(2.12)

will be used as an alternative definition of the Akhiezer polynomials. In addition, note that the asymptotic condition RH3 can be extended to the full Laurent series,

$$Y_{n}(z) = \left( I + \sum_{k=1}^{\infty} \frac{m_k(n)}{z^k} \right) z^{n\sigma_3}, \quad |z| > 1,$$

(2.13)

and from (2.3) we have

$$m_1(n) = \begin{pmatrix} p_1(n) & h_n \\ 1/h_{n-1} & -p_1(n) \end{pmatrix},$$

(2.14)

where $p_1(n)$ is the coefficient of $z^{n-1}$ of $P_{n}(z)$. Taking into account the recurrence relations (2.2), we have

$$a_n = \frac{h_n}{h_{n-1}},$$

and

$$b_{n+1} = p_1(n) - p_1(n + 1),$$

and the following relations supplementing (2.12)

$$h_n = (m_1(n))_{12},$$

(2.15)

$$a_n = (m_1(n))_{12}(m_1(n))_{21},$$

(2.16)

$$b_{n+1} = (m_1(n))_{11} - (m_1(n + 1))_{11}.$$  (2.17)

Therefore, all the basic ingredients of the theory of polynomials $P_{n}(z)$ (including the polynomials themselves) can be obtained directly from the solution $Y_{n}(z)$ of the Riemann–Hilbert problem.

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Remark 2.3. In the \textit{a priori} setting of the Riemann–Hilbert problem RH1–RH4, the condition RH3 can be replaced by the following weaker one:

\[
\text{RH4. } Y_n(z) \begin{pmatrix}
\frac{1}{\sqrt{z - \beta_j}} & 0 \\
-\frac{1}{b_j} & \sqrt{z - \beta_j}
\end{pmatrix} = O(1), \quad z \sim \beta_j, 0 \leq j \leq g + 1.
\] (2.18)

3. Differential equations

Having obtained equations (2.12)–(2.17) which represent orthogonal polynomials $P_n(z)$ and the corresponding norm and recurrence coefficients in terms of the solution $Y_n(z)$ of the Riemann–Hilbert problem RH1–RH4, we can now use the powerful techniques of the Soliton theory. Specifically in this and §§4 and 5, we will apply a certain modification of the standard Zakharov–Shabat dressing method (e.g. Manakov et al. 1980) to obtain the relevant differential and difference equations for the Akhiezer polynomials. The modification needed is caused by the presence of the condition RH4. This condition indicates the relation of the problem under consideration to the theory of Fuchsian systems. Indeed, our derivations will be close to the Zakharov–Shabat scheme and to the constructions of the Jimbo–Miwa–Ueno monodromy theory (Jimbo et al. 1981; see also Its (1986) were both methods are unified in a single general Riemann–Hilbert formalism).

To describe the change of $Y_n(z)$ with respect to $z$ for a fixed $n$, it is advantageous to transform the Riemann–Hilbert problem satisfied by $Y_n(z)$ into a form where jump matrix has constant entries. To this end, put

\[
\Phi_n(z) = Y_n(z) \begin{pmatrix}
1 & 0 \\
0 & w^{-1}(z)
\end{pmatrix} \begin{pmatrix}
\sqrt{2\pi i} & 0 \\
0 & 1/\sqrt{2\pi i}
\end{pmatrix}.
\] (3.1)

A direct computation shows that

\[
\Phi_{n,-}(z) = \Phi_{n,+}(z) \begin{pmatrix}
1 & -1 \\
0 & -1
\end{pmatrix}.
\] (3.2)

To specify the behaviour of the new function near the end points of the set $E$, let us observe that the new (constant !) jump matrix admits the following spectral representation,

\[
\begin{pmatrix}
1 & -1 \\
0 & -1
\end{pmatrix} = P^{-1} \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} P,
\]

where

\[
P = \begin{pmatrix}
0 & 1 \\
2 & -1
\end{pmatrix}.
\]

This implies that the function

\[
\Phi^{(\beta_j)}(z) := \begin{pmatrix}
\sqrt{z - \beta_j} & 0 \\
0 & 1
\end{pmatrix} P
\]

Phil. Trans. R. Soc. A (2008)
satisfies the jump condition (3.2) in the neighbourhood of $\beta_j$. Indeed, assuming $z \in \mathcal{U}_\beta \cap E$, we find,

$$[\Phi^{(\beta)}(z)]^{-1} \Phi^{(\beta)}(z) = P^{-1} \begin{pmatrix} 1 & 0 \\ \sqrt{z-\beta_j} \frac{1}{b_j} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{z-\beta_j}} & 0 \\ 0 & 1 \end{pmatrix} P \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}. $$

Hence, the matrix-valued function

$$\Phi_n(z)[\Phi^{(\beta)}(z)]^{-1}$$

has no jump across $E$ and therefore is holomorphic in the punctured neighbourhood $\mathcal{U}_\beta \setminus \{\beta_j\}$. Observe in addition that in the product,

$$\left( \frac{\sqrt{z-\beta_j}}{2\pi i b_j} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{w^{-1}(z)} \end{array} \right) \left( \begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array} \right) \left( \frac{1}{\sqrt{z-\beta_j}} & 0 \\ 0 & 1 \end{array} \right), \quad (3.3)$$

the negative powers of $\sqrt{z-\beta_j}$ cancel out.

Therefore, we conclude that the product $\Phi_n(z)[\Phi^{(\beta)}(z)]^{-1}$ is in fact holomorphic in the whole neighbourhood $\mathcal{U}_\beta$. Similar is also true for the matrix product

$$\Phi_n(z)[\Phi^{(\alpha)}(z)]^{-1} \equiv \Phi_n(z) \left[ \begin{pmatrix} 1 & 0 \\ \sqrt{z-\alpha_j} & 1 \end{pmatrix} \right]^{-1},$$

in the neighbourhood $\mathcal{U}_\alpha$ of the end point $\alpha_j$. Here we shall assume that the branch of $\sqrt{z-\alpha_j}$ is defined by the condition,

$$-\pi < \arg(z-\alpha_j) < \pi.$$

In summary, $\Phi_n(z)$ solves the following Riemann–Hilbert problem:

$\Phi_1$. $\Phi_n(z)$ is holomorphic for $z \in \mathbb{C} \setminus E$,

$\Phi_2$. $\Phi_{n,-}(z) = \Phi_{n,+}(z) \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, $z \in E$,

$\Phi_3$. $\Phi_n(z) = \left( I + O\left( \frac{1}{z} \right) \right) z^{n} \begin{pmatrix} n & 0 \\ 0 & -n + 1 \end{pmatrix} \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 0 & \sqrt{\frac{\pi i}{2}} \end{pmatrix}$, $z \to \infty$,

$\Phi_4$. $\Phi_n(z) = \Phi^{(\beta)}_n(z) \begin{pmatrix} \sqrt{z-\beta_j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, $z \in \mathcal{U}_\beta$,

$\Phi_5$. $\Phi_n(z) = \Phi^{(\alpha)}_n(z) \begin{pmatrix} 1 & 0 \\ \sqrt{z-\alpha_j} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, $z \in \mathcal{U}_\alpha$,
where \( \hat{\Phi}_n^{(\beta_j)}(z) \) and \( \hat{\Phi}_n^{(\alpha_j)}(z) \) are holomorphic in the neighbourhoods of the points \( \beta_j \) and \( \alpha_j \), respectively. Moreover, the matrices \( \hat{\Phi}_n^{(\beta_j)}(\beta_j) \) and \( \hat{\Phi}_n^{(\alpha_j)}(\alpha_j) \) are invertible. In fact,

\[
\hat{\Phi}_n^{(\beta_j)}(\beta_j) = \hat{Y}_n^{(\beta_j)}(\beta_j) \begin{pmatrix} \sqrt{\pi i/2} & 0 \\ 0 & \frac{1}{b_j} \sqrt{\pi i/2} \end{pmatrix},
\]

and

\[
\hat{\Phi}_n^{(\alpha_j)}(\alpha_j) = Y_n(\alpha_j) \begin{pmatrix} 0 & \sqrt{\pi i/2} \\ -\frac{1}{a_j} \sqrt{\pi i/2} & 0 \end{pmatrix},
\]

where \( \alpha_j \) is defined by the equation (cf. (2.5))

\[
w(z) = (z - \alpha_j)^{1/2} a_j \frac{i}{\pi} (1 + O(z - \alpha_j)). \tag{3.4}
\]

We want to emphasize, that unlike the case of the \( Y \)-Riemann–Hilbert problem, in the case of the \( \Phi \)-Riemann–Hilbert problem the left multipliers \( \hat{\Phi}_n^{(\beta_j)}(z) \) and \( \hat{\Phi}_n^{(\alpha_j)}(z) \) are holomorphic with respect to \( z \).

**Remark 3.1.** From \( \Phi_1-\Phi_5 \), it follows (independent of (3.1)) that

\[
det \Phi_n(z) = \frac{1}{w(z)}. \tag{3.5}
\]

Consider now, the logarithmic derivative of \( \Phi_n(z) \)

\[
A(z, n) := \frac{d\Phi_n(z)}{dz} \phi_n^{-1}(z). \tag{3.6}
\]

Since all the right matrix multipliers in the r.h.s. of \( \Phi_2-\Phi_5 \) are constant matrices, \( A(z, n) \) enjoys the following properties:

**A1.** \( A(z, n) \) is holomorphic for \( z \in \mathbb{C} \setminus \{\alpha_j, \beta_j\} \),

\[
A(z, n) = \begin{pmatrix} n & 0 \\ 0 & -n + 1 \end{pmatrix} + O\left(\frac{I}{z^2}\right), z \to \infty,
\]

**A2.** \( A(z, n) = \begin{pmatrix} n & 0 \\ 0 & -n + 1 \end{pmatrix} \frac{1}{z} + O\left(\frac{I}{z^2}\right) \), \( z \to \infty \),

\[
A(z, n) = \frac{1}{2} \hat{\Phi}_n^{(\beta_j)}(\beta_j) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{z - \beta_j} \hat{\Phi}_n^{(\beta_j)}(\beta_j) + O(1), \quad z \sim \beta_j,
\]

**A3.** \( A(z, n) = \frac{1}{2} \hat{\Phi}_n^{(\beta_j)}(\beta_j) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{z - \beta_j} \hat{\Phi}_n^{(\beta_j)}(\beta_j) + O(1), \quad z \sim \beta_j,
\]

**A4.** \( A(z, n) = \frac{1}{2} \hat{\Phi}_n^{(\beta_j)}(\beta_j) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{z - \beta_j} \hat{\Phi}_n^{(\beta_j)}(\beta_j) + O(1), \quad z \sim \beta_j.\)
By virtue of the Liouville theorem, it follows that
\[
A(z, n) = \sum_{j=0}^{g+1} \frac{B_j(n)}{z - \beta_j} + \sum_{j=1}^{g} \frac{A_j(n)}{z - \alpha_j},
\]
where
\[
B_j(n) := \frac{1}{2} \Phi_n^{(\beta_j)(\beta_j)}(1, 0, 0) \Phi_n^{(\beta_j)^{-1}}(\beta_j) = \frac{1}{2} \hat{Y}_n^{(\beta_j)}(\beta_j) (1, 0, 0) \hat{Y}_n^{(\beta_j)^{-1}}(\beta_j),
\]
and
\[
A_j(n) := -\frac{1}{2} \Phi_n^{(\alpha_j)(\alpha_j)}(1, 0, 0) \Phi_n^{(\alpha_j)^{-1}}(\alpha_j) = -\frac{1}{2} Y_n(\alpha_j) (1, 0, 0) Y_n^{-1}(\alpha_j).
\]
Note also
\[
\sum_{j=0}^{g+1} B_j(n) + \sum_{j=1}^{g} A_j(n) = \begin{pmatrix} n & 0 \\ 0 & -n + 1 \end{pmatrix}.
\]
Using (2.3) and RH4 gives
\[
\hat{Y}_n^{(\beta_j)}(\beta_j) = \begin{pmatrix} Q_n(\beta_j)/b_j & b_j P_n(\beta_j) \\ \frac{Q_{n-1}(\beta_j)}{b_j h_{n-1}} & b_j P_{n-1}(\beta_j)/h_{n-1} \end{pmatrix}.
\]

We conclude this section by recording the linear matrix differential equation with Fuchsian singularities at \(\{\alpha_j, \beta_j\}\) mentioned in the abstract
\[
\frac{d\Phi_n(z)}{dz} = A(z, n)\Phi_n(z),
\]
with \(A(z, n)\) defined by (3.7)–(3.9). Furthermore, using the second line of (2.3), the matrix-valued residues are expressed in terms of the evaluations of the polynomials at the branch points
\[
B_j(n) = \frac{1}{2} \begin{pmatrix} Q_n(\beta_j) P_{n-1}(\beta_j)/h_{n-1} & -Q_n(\beta_j) P_n(\beta_j) \\ Q_{n-1}(\beta_j) P_{n-1}(\beta_j)/h_{n-1}^2 & -Q_{n-1}(\beta_j) P_n(\beta_j)/h_{n-1} \end{pmatrix},
\]
and
\[
A_j(n) = \frac{1}{2} \begin{pmatrix} P_n(\alpha_j) Q_{n-1}(\alpha_j)/h_{n-1} & -Q_n(\alpha_j) P_n(\alpha_j) \\ Q_{n-1}(\alpha_j) P_{n-1}(\alpha_j)/h_{n-1}^2 & -P_{n-1}(\alpha_j) Q_n(\alpha_j)/h_{n-1} \end{pmatrix}.
\]
Note that from (3.8) and (3.9), it follows that
\[
\text{tr } B_j(n) \equiv \frac{1}{2h_{n-1}} (Q_{n-1}(\beta_j) P_n(\beta_j) - Q_n(\beta_j) P_{n-1}(\beta_j)) = 1/2,
\]
and
\[
\text{det } B_j(n) = 0,
\]
and
\[
\text{tr } A_j(n) \equiv -\frac{1}{2h_{n-1}} (Q_n(\alpha_j) P_{n-1}(\alpha_j) - P_n(\alpha_j) Q_{n-1}(\alpha_j)) = -1/2,
\]
and
\[
\text{det } A_j(n) = 0.
\]
We note that this leads to a discrete analogue of the ‘Wronskian’ relation,

\[ P_{n-1}(z)Q_n(z) - P_n(z)Q_{n-1}(z) = h_{n-1}, \]

which, of course, can be independently derived from the recurrence relations.

As has already been mentioned in §1, equation (3.11), even for more general weights of the type \( Q_j(t - \delta_j)^{k_j} \), was first obtained in Magnus (1995). In Magnus (1995), the Riemann–Hilbert problem is not used explicitly; rather, the author analyses directly the monodromy properties of the function \( Y_n(z) \), i.e. the approach of Magnus (1995) is based more on the ideas of Jimbo et al. (1981) than of Manakov et al. (1980). It is also worth mentioning that our approach can be extended to the general semi-classical weights without any serious modifications.

4. Derivatives with respect to the branch points

In this section, we determine differentiation formulae for \( F_n(z) \) with respect to \( \{\alpha_j, \beta_j\} \). First, let us consider the logarithmic derivative of \( F_n(z) \) with respect to a particular \( \beta_j \)

\[ V_j(z) := \frac{\partial \Phi_n(z)}{\partial \beta_j} \Phi_n^{-1}(z), \quad (4.1) \]

and note that \( V_j(z) \) has the following properties:

V1. \( V_j(z) \) is holomorphic for \( z \in \mathcal{C} \setminus \{\beta_j\} \),

V2. \( V_j(z) = O(1/z), \quad z \to \infty \),

\[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{z - \beta_j} \frac{\Phi_n^{(\beta_j)}(\beta_j)}{\Phi_n^{(\beta_j)}(\beta_j) - 1} + O(1). \quad (4.2) \]

By comparing with (3.8) and again invoking the Liouville theorem, we conclude that

\[ V_j(z) = -\frac{B_j(n)}{z - \beta_j}, \quad (4.3) \]

which implies

\[ \partial_{\beta_j} \Phi_n(z) = -\frac{B_j(n)}{z - \beta_j} \Phi_n(z). \quad (4.4) \]

A similar analysis gives

\[ \partial_{\alpha_j} \Phi_n(z) = -\frac{A_j(n)}{z - \alpha_j} \Phi_n(z). \quad (4.5) \]

5. Difference equation

Consider the ‘difference logarithmic derivative’

\[ U_n(z) := \Phi_{n+1}(z)\Phi_n^{-1}(z) \equiv Y_{n+1}(z)Y_n^{-1}(z). \]

Taking into account that all the right matrix multipliers in the r.h.s. of RH1–RH4 are constant with respect to \( n \), we conclude that \( U_n(z) \) is an entire
Moreover, from the asymptotics (2.13), we have that
\[ U_n(z) = \left( I + \frac{m_1(n+1)}{z} \right) z^{\sigma_1} \left( I - \frac{m_1(n)}{z} \right) + O\left( \frac{1}{z} \right) \]
\[ = \left( I + \frac{m_1(n+1)}{z} \right) \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & m_1(n) \\ 0 & 0 \end{pmatrix} + O\left( \frac{1}{z} \right) \]
\[ = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + m_1(n+1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} m_1(n) + O\left( \frac{1}{z} \right), \quad z \to \infty. \]

Appealing once again to the Liouville theorem, we conclude that \( U_n(z) \) is linear function in \( z \) defined by the equations
\[ U_n(z) = \begin{pmatrix} z + (m_1(n+1))_{11} - (m_1(n))_{11} & -(m_1(n))_{12} \\ (m_1(n+1))_{21} & 0 \end{pmatrix} = \begin{pmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{pmatrix}, \]
where in the last equation we have taken into account (2.15)–(2.17). To summarize, the difference equation for the function \( \Phi_n(z) \) reads
\[ \Phi_{n+1}(z) = \begin{pmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{pmatrix} \Phi_n(z). \] (5.1)

Of course, equation (5.1) is just the matrix form of the basic recurrence equation (2.2). Nevertheless, we gave its Riemann–Hilbert derivation to emphasize the ‘master’ role of the Riemann–Hilbert problem RH1–RH4 in our analysis.

### 6. Schlesinger equations and the Hankel determinant

With the unified notation mentioned in §1, we write
\[ A(z, n) = \sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j}, \] (6.1)
and the correspondence
\[ (A_1(n), \ldots, A_g(n), B_0(n), \ldots, B_{g+1}(n)) \to (C_1(n), \ldots, C_g(n), C_{g+1}(n), \ldots, C_{2g+2}(n)). \] (6.2)

Note that \( C_j(n) \) depend on \( \delta_j \)'s. We of course have
\[ \partial_z \Phi_n(z) = \sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j} \Phi_n(z), \] (6.3)
\[ \partial_{\delta_k} \Phi_n(z) = - \frac{C_k(n)}{z - \delta_k} \Phi_n(z). \] (6.4)

Applying \( \partial_z \) on (6.4) gives
\[ \partial_z \partial_{\delta_k} \Phi_n(z) = \frac{C_k(n)}{(z - \delta_k)^2} \Phi_n - \frac{C_k(n)}{z - \delta_k} \sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j} \Phi_n, \] (6.5)
and $\partial_{\delta_k}$ on (6.3) gives
\[
\partial_{\delta_k} \partial_z \Phi_n(z) = \frac{C_k(n)}{(z - \delta_k)^2} \Phi_n + \sum_{j=1}^{2g+2} \frac{\partial_{\delta_k} C_j(n)}{z - \delta_j} \Phi_n - \left( \sum_{j=1}^{2g+2} \frac{C_j(n)}{z - \delta_j} \right) C_k(n) \Phi_n.
\]  
(6.6)

Since $\partial_z \partial_{\delta_k} \Phi_n = \partial_{\delta_k} \partial_z \Phi_n$ and $\det \Phi_n \neq 0$, we get
\[
\sum_{j=1}^{2g+2} \frac{\partial_{\delta_k} C_j(n)}{z - \delta_k} = \sum_{j=1}^{2g+2} \left[ C_j(n), C_k(n) \right]_{\delta_j - \delta_k} \left( \frac{1}{z - \delta_j} - \frac{1}{z - \delta_k} \right).
\]  
(6.7)

We now send $z$ to a particular $\delta_j$ in (6.7), with $j \neq k$ and find by equating residues,
\[
\partial_{\delta_k} C_j(n) = \left[ C_j(n), C_k(n) \right]_{\delta_j - \delta_k}, \quad j \neq k.
\]  
(6.8)

If $j = k$, then a similar calculation gives
\[
\partial_{\delta_k} C_k(n) = -\sum_{l \neq k} \left[ C_l(n), C_k(n) \right]_{\delta_l - \delta_k}.
\]  
(6.9)

Equations (6.8) and (6.9) are the Schlesinger equations satisfied by $C_j(n)$. This is the equation first derived for the general semi-classical orthogonal polynomials in Magnus (1995). We are now going to move beyond the results of Magnus (1995) and show that the corresponding $\tau$-function can be identified with the Hankel determinant associated with the weight $w_+(t)$. To this end, we first recall Jimbo–Miwa–Ueno definition of the $\tau$-function.

Let $\Omega^{(1)}$ be the one form,
\[
\Omega^{(1)}(\delta_1, \ldots, \delta_{2g+2}) := \sum_{1 \leq j < k \leq 2g+2} \text{tr}(C_j(n)C_k(n)) \frac{d\delta_j - d\delta_k}{\delta_j - \delta_k}
\]  
\[
= \sum_{1 \leq j < k \leq 2g+2} \text{tr}(C_j(n)C_k(n)) \text{d} \ln |\delta_j - \delta_k|,
\]  
(6.10)

then it can be verified (Jimbo et al. 1981) using the Schlesinger equations that
\[
d\Omega^{(1)} = 0,
\]  
(6.11)

which implies that, locally, $\Omega^{(1)}$ is an exact form. The $\tau$-function of the completely integrable system of partial differential equations (6.8) and (6.9) is then defined by the relation
\[
\Omega^{(1)} = \text{d} \ln \tau_n(\delta_1, \ldots, \delta_{2g+2}).
\]  
(6.12)

In the theory of orthogonal polynomials, the Hankel determinant,
\[
D_n[w_+] := \det \left( \int_E t^j k^{n-1} w_+(t) \text{d}t \right)_{j,k=0}
\]  
(6.13)

has two other equivalent expressions,
\[
D_n[w_] = \frac{1}{n!} \int_E \ldots \int_E \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{l=1}^{n} w_+(x_l) \text{d}x_l = \prod_{j=0}^{n-1} h_j.
\]  
(6.14)
It is to be expected from the structure of the Riemann–Hilbert formulation that, $D_n$ considered as a function of $\{\delta_j\}_{j=1}^{2g+2}$, is the $\tau$-function for this problem. To understand this, we require the derivatives of $h_n$ with respect to $\delta_k$. To begin with, we use that

$$\partial_{\delta_k} F_n(z) = -\frac{C_k(n)}{z-\delta_k} \Phi_n(z) \quad (6.15)$$

must be compatible with

$$\Phi_{n+1}(z) = \begin{pmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{pmatrix} \Phi_n(z). \quad (6.16)$$

This results in

$$\begin{pmatrix} z - b_{n+1} & -h_n \\ h_n & 0 \end{pmatrix} = \begin{pmatrix} C_k(n) & C_k(n+1) \\ 0 & z - \delta_k \end{pmatrix} \begin{pmatrix} z - b_{n+1} & -h_n \\ h_n & 0 \end{pmatrix} = \begin{pmatrix} -\partial_{\delta_k} b_{n+1} & -\partial_{\delta_k} h_n \\ -(1/h_n)\partial_{\delta_k} \ln h_n & 0 \end{pmatrix}, \quad (6.17)$$

which holds for all $z \in \mathbb{C}P^1 \setminus \{\delta_1, \ldots, \delta_{2g+2}\}$. Putting $z = \infty$ in (6.17), gives

$$\begin{pmatrix} C_k^{11}(n) - C_k^{11}(n+1) & C_k^{12}(n) \\ -C_k^{21}(n+1) & 0 \end{pmatrix} = \begin{pmatrix} -\partial_{\delta_k} b_{n+1} & -\partial_{\delta_k} h_n \\ -(1/h_n)\partial_{\delta_k} \ln h_n & 0 \end{pmatrix},$$

which implies, among others,

$$\partial_{\delta_k} h_n = -C_k^{12}(n). \quad (6.18)$$

**Lemma 6.1.** Let the asymptotic expansion of $A(z, n)$ about $z = \infty$ be

$$A(z, n) = \sum_{k=0}^{\infty} A_k(n) z^{-k-1}, \quad (6.19)$$

where

$$A_k(n) := \sum_{j=1}^{2g+2} C_j(n) \delta_j^k(n). \quad (6.20)$$

Then the first two $A_k(n)$ are

$$A_0(n) = \begin{pmatrix} n & 0 \\ 0 & 1-n \end{pmatrix}, \quad (6.21)$$

$$A_1(n) = \begin{pmatrix} 0 & 0 \\ 0 & c_1 \end{pmatrix} + m_1(n) \begin{pmatrix} n-1 & 0 \\ 0 & -n \end{pmatrix} - \begin{pmatrix} n & 0 \\ 0 & 1-n \end{pmatrix} m_1(n), \quad (6.22)$$

where $c_1 = \sum_{j=1}^{g} (\beta_j - \alpha_j)$. 

*Phil. Trans. R. Soc. A* (2008)
Proof. Putting (3.1) into (3.11), we find

\[
\frac{d}{dz} Y_n(z) + Y_n(z) \begin{pmatrix} 0 & 0 \\ 0 & - \frac{d}{dz} \ln w(z) \end{pmatrix} = A(z, n) Y_n(z). \tag{6.23}
\]

Expansion of (6.23) in \( Z^{-1} \) gives the desired results.

**Theorem 6.1.** The Hankel determinant is the \( \tau \)-function of the Magnus–Schlesinger equations.

Proof. We start by equating the residues of (6.17) at \( z=\delta_j \). This gives

\[
U_n(\delta_j) C_j(n) = C_j(n+1) U_n(\delta_j),
\]

or

\[
C_j(n+1) = U_n(\delta_j) C_j(n) U_n^{-1}(\delta_j), \tag{6.24}
\]

where

\[
U_n(z) := \begin{pmatrix} z - b_{n+1} - h_n & -h_n \\ 1/h_n & 0 \end{pmatrix},
\]

\[
U_n^{-1}(z) = \begin{pmatrix} 0 & h_n \\ -1/h_n & z - b_{n+1} \end{pmatrix}. \tag{6.25}
\]

A simple calculation shows that

\[
U_n^{-1}(z) U_n(z') = \begin{pmatrix} 1 & 0 \\ z - z' & 1 \end{pmatrix} = I + z - z' \sigma_-, \tag{6.26}
\]

where \( \sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Now,

\[
d \ln \tau_n = \sum_k \partial_{\delta_k} \ln \tau_n \, d\delta_k, \tag{6.27}
\]

where (from (6.10)),

\[
\partial_{\delta_j} \ln \tau_n = \sum_{k(\neq j)} \frac{\text{tr} C_j(n) C_k(n)}{\delta_j - \delta_k}, \tag{6.28}
\]
We also note here some useful identities

\[ \text{which leads to} \]

\[
\partial_{\delta_j} \ln \frac{\tau_{n+1}}{\tau_n} = \sum_{k(\neq j)} \frac{\text{tr}(C_j(n+1)C_k(n+1) - C_j(n)C_k(n))}{\delta_j - \delta_k}
\]

\[ = \sum_{k(\neq j)} \frac{\text{tr}(U_n(\delta_j)C_j(n)U_n^{-1}(\delta_j)U_n(\delta_k)C_k(n)U_n^{-1}(\delta_k) - C_j(n)C_k(n))}{\delta_j - \delta_k}
\]

\[ = \sum_{k(\neq j)} \frac{\text{tr}\left[U_n^{-1}(\delta_k)U_n(\delta_j)C_j(n)U_n^{-1}(\delta_j)U_n(\delta_k)C_k(n) - C_j(n)C_k(n)\right]}{\delta_j - \delta_k}
\]

\[ = \sum_{k(\neq j)} \frac{\text{tr}\left[(I - \frac{\delta_j - \delta_k}{\hbar} \sigma_-)C_j(n)(I + \frac{\delta_j - \delta_k}{\hbar} \sigma_-)C_k(n) - C_j(n)C_k(n)\right]}{\delta_j - \delta_k}
\]

\[ = \sum_{k(\neq j)} \frac{\text{tr}\left[(I - \frac{\delta_j - \delta_k}{\hbar} \sigma_-)C_j(n)C_k(n)(I + \frac{\delta_j - \delta_k}{\hbar} \sigma_-) - C_j(n)C_k(n)\right]}{\delta_j - \delta_k}
\]

\[ (6.29) \]

A calculation shows that the term [...] in (6.29) is

\[ \frac{\delta_j - \delta_k}{\hbar} (C_j(n)\sigma_-C_k(n) - \sigma_- C_j(n)C_k(n)) - \left(\frac{\delta_j - \delta_k}{\hbar} \right)^2 \sigma_- C_j(n)\sigma_- C_k(n). \]

We also note here some useful identities

\[ \text{tr}(C_j(n)\sigma_-C_k(n) - \sigma_- C_j(n)C_k(n)) = C_j^{12}(n)(C_k^{11}(n) - C_k^{22}(n)) \]

\[ - C_k^{12}(n)(C_j^{11}(n) - C_j^{22}(n)), \]

and

\[ \text{tr}(\sigma_- C_j(n)\sigma_- C_k(n)) = C_j^{12}(n)C_k^{12}(n). \]

Therefore,

\[ \partial_{\delta_j} \ln \frac{\tau_{n+1}}{\tau_n} = \frac{1}{\hbar} \sum_{k(\neq j)} \left( C_j^{12}(n)(C_k^{11}(n) - C_k^{22}(n)) - C_k^{12}(n)(C_j^{11}(n) - C_j^{22}(n)) \right) \]

\[ - \frac{1}{\hbar^2} \sum_{k(\neq j)} (\delta_j - \delta_k)C_j^{12}(n)C_k^{12}(n). \]

\[ (6.30) \]

To simplify the r.h.s. of (6.30) we note, from (6.20)–(6.22)

\[ \sum_j C_j^{12}(n) = 0, \]

\[ \sum_j (C_j^{11}(n) - C_j^{22}(n)) = 2n - 1, \]

\[ \sum_j \delta_j C_j^{12}(n) = -2n\hbar. \]
Using these, and \( \sum_{k(\neq j)} f_k = -f_j + \sum j_kf_k \), the r.h.s. of (6.30) becomes
\[
\frac{C_{12}^j(n)}{h_n} \sum_k (C_{11}^j(n) - C_{22}^j(n)) + \frac{C_{12}^j(n)}{h_n^2} \sum_k \delta_k C_{12}^j(n) = -\frac{C_{12}^j(n)}{h_n}.
\]
Finally, using (6.18),
\[
\partial_{\delta_j} \ln \frac{\tau_{n+1}}{\tau_n} = -\frac{C_{12}^j(n)}{h_n} = \partial_{\delta_j} \ln h_n.
\]
Summing over \( n \) from 0 to \( N-1 \) we conclude that \( \tau_N \) is a constant multiple of \( D_N \), where the constant is independent of \( \{\delta_j\}_{j=1}^{2g+2} \). Since the \( \tau \)-function is defined up to such a constant, we can assume that the constant is unity
\[
\tau_N(\delta_1, \ldots, \delta_{2g+2}) = D_N[w_+].
\]

**Remark 6.1.** It is worth mentioning that equation (6.24) follows also (by putting \( z = \delta_j \)) from the equation
\[
A(z, n + 1) U_n(z) - U_n(z) A(z, n) = \frac{\partial U_n(z)}{\partial z},
\]
which, in turn, is the compatibility condition of the basic Fuchsian equation (3.11) and the difference equation (5.1). This is the matrix form of the so-called Freud equation which in principle can be written for any semi-classical polynomials (see Gammel & Nuttall 1982; Magnus 1995 and also Fokas et al. 1992). In the physical language, this is the ‘discrete string equation’ corresponding to the weight \( w_+(t) \). More precisely, equation (6.33) is the (discrete) Lax representation of the Freud equation which manifests its integrability from the algebraic point of view: linear equations (3.11) and (5.1) form a Lax pair for the Freud equation (cf. Fokas et al. 1990, 1992).

### 7. Nonlinear difference equations

As explained in remark 6.1, the matrix equation (6.33) should lead to the nonlinear difference equations for the recurrence coefficients, following the *genre* of the Freud equations for the Akhiezer polynomials. To this end, we rewrite (6.24) elementwise, by first specializing \( \delta_j \) to \( \alpha_j \) and second to \( \beta_j \). This will produce six difference equations, relating polynomial evaluations at the branch points and the recurrence coefficients. For later convenience, we introduce four quantities
\[
\begin{align*}
    r_{n}^{(\alpha)} &= \frac{1}{2h_{n-1}} P_{n}(\alpha_{j}) Q_{n-1}(\alpha_{j}), \\
    r_{n}^{(\beta)} &= \frac{1}{2h_{n-1}} P_{n}(\beta_{j}) Q_{n-1}(\beta_{j}), \\
    R_{n}^{(\alpha)} &= \frac{1}{2h_{n}} P_{n}(\alpha_{j}) Q_{n}(\alpha_{j}), \\
    R_{n}^{(\beta)} &= \frac{1}{2h_{n}} P_{n}(\beta_{j}) Q_{n}(\beta_{j}).
\end{align*}
\]
Thus by specializing to $\alpha_j$, $C_j(n)$ becomes

$$
\begin{pmatrix}
 r_n^{(\alpha)} & -h_n R_n^{(\alpha)} \\
 R_n^{(\alpha)} / h_n - r_n^{(\alpha)} - 1/2
\end{pmatrix},
$$

where we have taken into account that the trace of the above is $-1/2$. In component form (6.24) is equivalent to

$$
r_{n+1}^{(\alpha)} + r_n^{(\alpha)} + \frac{1}{2} = R_n^{(\alpha)} (\alpha_j - b_{n+1}),
$$

$$
a_{n+1} R_{n+1}^{(\alpha)} - a_n R_{n-1}^{(\alpha)} = (b_{n+1} - \alpha_j) \left( R_n^{(\alpha)} (b_{n+1} - \alpha_j) + 2r_n^{(\alpha)} + \frac{1}{2} \right).
$$

Note that out of the four possible equations, the 21 element is a tautology and the 11 and 22 elements are equivalent. Similarly, specializing to $\beta_j$, $C_j(n)$ becomes

$$
\begin{pmatrix}
 r_n^{(\beta)} + \frac{1}{2} & -h_n R_n^{(\beta)} \\
 R_n^{(\beta)} / h_n - r_n^{(\beta)}
\end{pmatrix},
$$

where the trace of the above is $1/2$. In component form (6.24) becomes

$$
r_{n+1}^{(\beta)} + r_n^{(\beta)} + \frac{1}{2} = R_n^{(\beta)} (\beta_j - b_{n+1}),
$$

$$
a_{n+1} R_{n+1}^{(\beta)} - a_n R_{n-1}^{(\beta)} = (\beta_j - b_{n+1}) \left( R_n^{(\beta)} (\beta_j - b_{n+1}) - 2r_n^{(\beta)} - \frac{1}{2} \right).
$$

In addition to these, we have

$$
a_n R_n^{(\alpha)} R_{n-1}^{(\alpha)} = r_n^{(\alpha)} \left( r_n^{(\alpha)} + \frac{1}{2} \right),
$$

$$
a_n R_n^{(\beta)} R_{n-1}^{(\beta)} = r_n^{(\beta)} \left( \frac{1}{2} + r_n^{(\beta)} \right),
$$

since $\det C_j(n) = 0$. Equations (7.1)–(7.6) are the difference equations mentioned above. We should be able to eliminate $r_n^{(\alpha)}$, $r_n^{(\beta)}$, $R_n^{(\alpha)}$ and $R_n^{(\beta)}$ from these to obtain nonlinear difference equations involving only $a_n$ and $b_n$. These equations are also discussed in Magnus (1995).

### 8. The $\sigma_1$ Riemann–Hilbert problem

In this section, we shall solve the Riemann–Hilbert problem RH1–RH4 for the Akhiezer polynomials in terms of the $\Theta$-functions. To this end, we will need a further transformation of the Riemann–Hilbert problem satisfied by $\Phi_n(z)$ to the so-called $\sigma_1$ problem, which first appeared in the theory of algebro-geometric solutions of integrable PDEs (see Its 1984; Belokolos et al. 1994).
We note that since the matrices \( \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \) and \( \sigma_1 \) have the same simple spectrum, they must be similar. Indeed, we have
\[
\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1.
\]

Therefore, if we define
\[
\psi_n(z) := \begin{pmatrix} \frac{1}{\sqrt{2\pi i}} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} \phi_n(z) \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}
= \begin{pmatrix} \frac{1}{\sqrt{2\pi i}} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} Y_n(z) \begin{pmatrix} 1 & 0 \\ 0 & 1/w(z) \end{pmatrix} \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 1/\sqrt{2\pi i} & -1/\sqrt{2\pi i} \end{pmatrix},
\]
then the jump matrix of the new function becomes \( \sigma_1 \). The left diagonal constant matrix multiplier is introduced to normalize the asymptotic behaviour of the function \( \psi_n(z) \) at \( z = \infty \)
\[
\begin{pmatrix} \frac{1}{\sqrt{2\pi i}} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} \begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix} \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 0 & -\sqrt{\frac{\pi i}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}
= \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix}.
\]

Taking also into account that
\[
\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},
\]

Phil. Trans. R. Soc. A (2008)
we can reformulate the Riemann–Hilbert problem in terms of \( \Psi_n(z) \), as follows.

**\( \Psi_1 \).** \( \Psi_n(z) \) is holomorphic for \( z \in C \setminus E \).

**\( \Psi_2 \).** \( \Psi_{n-}(z) = \Psi_{n+}(z)\sigma_1, \ z \in E \).

**\( \Psi_3 \).** \( \Psi_n(z) = \left( I + O\left( \frac{1}{z} \right) \right) \left( \begin{array}{cc} n & 0 \\ 0 & -n+1 \end{array} \right), z \to \infty \).

**\( \Psi_4 \).** \( \Psi_n(z) = \hat{\Psi}^{(\beta_j)}(z) \left( \begin{array}{cc} \sqrt{z - \beta_j} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \\
= \hat{\Psi}^{(\beta_j)}(z)(z-\beta_j) \left( \begin{array}{cc} 1/2 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \right) \). 

**\( \Psi_5 \).** \( \Psi_n(z) = \hat{\Psi}^{(\alpha_j)}(z) \left( \begin{array}{cc} 1/\sqrt{z - \alpha_j} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \\
= \hat{\Psi}^{(\alpha_j)}(z)(z-\alpha_j) \left( \begin{array}{cc} -1/2 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \right) \),

where \( \hat{\Psi}^{(\alpha_j)}(z) \) is holomorphic in the neighbourhood of \( z = \alpha_j \) and \( \det \hat{\Psi}^{(\alpha_j)}(\alpha_j) \neq 0 \), i.e.

\[
\hat{\Psi}^{(\alpha_j)}(z) = \sum_{k=0}^{\infty} \Psi^{(\alpha_j)}(z-\alpha_j)^k, \ \det \Psi^{(\alpha_j)}_{n0} \neq 0.
\]

Similarly, \( \hat{\Psi}^{(\beta_j)}(z) \) is holomorphic in the neighbourhood of \( z = \beta_j \) and \( \det \hat{\Psi}^{(\beta_j)}(\beta_j) \neq 0 \), i.e.

\[
\hat{\Psi}^{(\beta_j)}(z) = \sum_{k=0}^{\infty} \Psi^{(\beta_j)}(z-\beta_j)^k, \ \det \Psi^{(\beta_j)}_{n0} \neq 0.
\]

It is also worth noticing that the matrix products

\[
\left( \begin{array}{cc} \sqrt{z - \beta_j} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right),
\]

and

\[
\left( \begin{array}{cc} 1/\sqrt{z - \alpha_j} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right),
\]

have an exact \( \sigma_1 \)-jump matrix in the respective neighbourhoods.

*Phil. Trans. R. Soc. A* (2008)
Remark 8.1. From $Ψ_1$–$Ψ_5$, it follows (independent of (8.1)) that
\[ \det Ψ_n(z) = \frac{i}{\pi w(z)}. \] (8.3)

Remark 8.2. The function $Ψ_n(z)$, in terms of $P_n(z)$ and $Q_n(z)$ is given as
\[ Ψ_n(z) = \frac{1}{2\pi i} \begin{pmatrix} i\pi w(z)P_n(z) - Q_n(z) & i\pi w(z)P_n(z) + Q_n(z) \\ \frac{w(z)}{h_{n-1}w(z)} & \frac{w(z)}{h_{n-1}w(z)} \end{pmatrix}, \] (8.4)
and all the properties listed in $Ψ_1$–$Ψ_5$ can be deduced from this representation. It is worth emphasizing here that our approach does not require this formula. Our logic is: the initial Riemann–Hilbert problem for $Y_n(z)$, quite generally posed, is transformed via (8.1) to the $σ_1$ problem which in turn leads to equations (8.2) and (8.3) by the completely general principles of the Riemann–Hilbert problem.

Let us now solve the $σ_1$ problem defined by $Ψ_1$–$Ψ_5$, however, without any reference to (8.4). The philosophy we adopt here is similar to that in the asymptotic analysis of orthogonal polynomials via the Riemann–Hilbert problem (cf. Bleher & Its 1999; Deift et al. 1999): we simply ‘forget’ the explicit formulae involving polynomials.

Introduce the genus $g$ Riemann surface $\mathcal{R}$ defined by
\[ y^2 = (z - β_0)(z - β_{g+1}) \prod_{j=1}^{g} (z - α_j)(z - β_j), \]
and let $Ψ_n(P)$, where $P = (z, y) \in \mathcal{R}$ be the vector Baker–Akhiezer function determined by the following conditions.

BA1. $Ψ_n(P)$ is meromorphic on $\mathcal{R}\setminus \{∞\}$ with the pole divisor,
\[ (Ψ_n(P)) = - \sum_{j=1}^{g} α_j. \]

BA2. The behaviour of $Ψ_n$ at $∞$ is specified by the equations,
\[ Ψ_n(P) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(\frac{1}{z}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P \to ∞^+, \]
\[ Ψ_n(P) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(\frac{1}{z}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad P \to ∞^-. \]

In other words, $∞^+$ is a pole of order $n$ and $∞^-$ is a zero of order $n-1$. Here as usual, $∞$ means
\[ P \to ∞^+ \iff z \to ∞, \quad y \to ±z^{g+1}. \]

Let $π: \mathcal{R} \to \mathbb{CP}^1$ be the projection,
\[ π(P) = z, \quad P = (z, y), \]
and $*: \mathcal{R} \to \mathcal{R}^*$ be the involution,
\[ P \to P^* = (z, -y) \quad \text{if } P = (z, y). \]
The main observation (cf. Its 1984; Belokolos et al. 1994) is that the matrix function,

\[ \Psi_n(z) := (\Psi_n(P), \Psi_n(P^*)) , \]

where \( \pi(P) = z \) and \( P \to \infty^+ \) as \( z \to \infty \), solves the RH problem \( \Psi_1 - \Psi_5 \).

(i) Indeed, \( \Psi_1 \) is satisfied by construction since (8.5) defines \( \Psi_n(z) \) uniquely as an analytic function on \( \mathbb{CP}^1 \setminus E \).

(ii) If \( z \to E \) from the \('+\)' side (or from above the cut), then

\[ P \to (z, y_+(z)) = P_+ , \]

\[ P^* \to (z, -y_+(z)) = (z, y_-(z)) = P_- . \]

If \( z \to E \) from the \('-\)' side, then

\[ P \to (z, y_-(z)) = P_- , \]

\[ P^* \to (z, -y_-(z)) = (z, y_+(z)) = P_+ . \]

Hence,

\[ \Psi_n^-(z) = (\Psi_n^-(P_+), \Psi_n^+(P_+)) , \]

\[ \Psi_n^+(z) = (\Psi_n^+(P_+), \Psi_n^-(P_+)) , \]

and it follows that

\[ \Psi_n^-(z) = \Psi_n^+(z) \sigma_1 , \quad z \in E , \]

and therefore \( \Psi/2 \) is satisfied.

(iii) We have by construction, \( z \to \infty \) implies \( P \to \infty^+ \) and \( P^* \to \infty^- \). Therefore from BA2,

\[ \Psi_n(z) = \left( I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix} , \]

which shows that \( \Psi/3 \) is satisfied.

(iv) The function \( \Psi(P) \) is analytic in the neighbourhood of \( P = \beta_j \) as a point of the Riemann surface \( \mathcal{R} \). The local parameter at the point \( \beta_j \) is the square root of \( z - \beta_j \). Therefore, in the neighbourhood of \( P = \beta_j \) we have,

\[ \Psi_n(P) = \sum_{k=0}^{\infty} \psi_{jk}(z - \beta_j)^{k/2} , \]

\[ \Psi_n(P^*) = \sum_{k=0}^{\infty} (-1)^k \psi_{jk}(z - \beta_j)^{k/2} , \]

so that

\[ \Psi_n(z) = \left( \sum_{k=0}^{\infty} \psi_{jk}(z - \beta_j)^{k/2}, \sum_{k=0}^{\infty} (-1)^k \psi_{jk}(z - \beta_j)^{k/2} \right) . \]
This in turn implies that the function $\Psi_n^{(\beta_j)}(z)$ defined by the equation $\Psi 4$ is a holomorphic function of $z$. Indeed, we have

$$
\Psi_n^{(\beta_j)}(z) = \Psi_n(z) \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)^{-1} \left( \begin{array}{c} 1 \\ \sqrt{z - \beta_j} \end{array} \right) 
\begin{array}{c} \\
0 \\
1 \\
\end{array}
$$

\begin{align*}
&= \frac{1}{2} \left( \sum_{k=0}^{\infty} [\psi_{jk} - (-1)^k \psi_{jk}](z - \beta_j)^{k-1/2}, \sum_{k=0}^{\infty} [\psi_{jk} + (-1)^k \psi_{jk}](z - \beta_j)^{k/2} \right) \\
&= \left( \sum_{l=0}^{\infty} \psi_{j2l+1}(z - \beta_j)^l, \sum_{l=0}^{\infty} \psi_{j2l}(z - \beta_j)^l \right).
\end{align*}

(v) Since $P=\alpha_j$ is a simple pole of $\Psi(P)$, the Taylor series (8.7) and (8.8) should be replaced by the Laurent series,

$$
\Psi_n(P) = \sum_{k=-1}^{\infty} \phi_{jk}(z - \alpha_j)^{k/2},
$$

$$
\Psi_n(P^*) = \sum_{k=-1}^{\infty} (-1)^k \phi_{jk}(z - \alpha_j)^{k/2}.
$$

The rest of the arguments is literally the same as in the $\beta$-case, and we have that the function $\Psi_n^{(\alpha_j)}(z)$ defined by the equation $\Psi 5$ is holomorphic at $z=\alpha_j$.

Our final observation is that already established properties imply (8.3) (cf. our Riemann–Hilbert proof of (2.7)) and hence the inequalities,

$$
\det \Psi_n^{(\alpha_j)}(\alpha_j) \neq 0, \quad \det \Psi_n^{(\beta_j)}(\beta_j) \neq 0.
$$

We now come to the $\Theta$-formula for $\Psi_n(P)$. First, we assemble here for this purpose some facts about the Riemann surface $\mathcal{R}$ realized as a two-sheet covering of the $z$-plane in the usual way and with the first homology basis shown in figure 1. Let

$$
\{d\omega_j\}_{j=1}^{q}, \quad \int_{\alpha_j} d\omega_k = \delta_{jk},
$$

be a set of normalized Abelian differentials of the first kind. As is usual for a hyperelliptic curve, we shall choose the differentials $d\omega_j$ according to the equations,

$$
d\omega_j = \sum_{k=1}^{q} (A^{-1})_{jk} \frac{z^{g-k}}{y} \, dz,
$$

$$
A_{jk} = \int_{\alpha_k} \frac{z^{g-j}}{y} \, dz.
$$

The invertibility of the matrix $A$ is a (relatively simple) classical result. We refer the reader to a monograph (Farkas & Kra 1980) for the basic general facts concerning the theory of functions on the Riemann surfaces (see also ch. 1 of

Phil. Trans. R. Soc. A (2008)
Belokolos et al. (1994)). Let us also introduce the normalized Abelian differential of the third kind, having its only poles at $N G_j$, 

$$d\Omega(P) = \frac{z^g + \lambda_{g-1} z^{g-1} + \cdots + \lambda_0}{y} \, dz,$$

with vanishing $a$ period:

$$\int_{a_j} d\Omega = 0, \quad j = 1, \ldots, g.$$

The above $g$ conditions uniquely determine (Farkas & Kra 1980) the coefficients, $\{\lambda_j\}_{j=0}^{g-1}$. Put

$$\Omega(P) = \int_{\beta_{g+1}}^P d\Omega.$$

One easily deduces,

$$\Omega(P) = \pm \left( \ln z - \ln C(E) + O\left(\frac{1}{z}\right) \right), \quad P \to \infty^\pm, \quad (8.9)$$

where

$$C(E) = \exp \left( - \int_{\beta_{g+1}}^{\infty^+} \left( z^g + \sum_{j=0}^{g-1} \frac{\lambda_j z^j}{y(z)} - \frac{1}{z} \right) \, dz \right). \quad (8.10)$$

(We recall that $\beta_{g+1} = 1$.) Finally, the Riemann $\Theta$-function of $g$-complex variables $s \in \mathbb{C}^g$, is defined with the aid of the period matrix

$$B_{jk} := \int_{b_k} d\omega_j,$$

as follows:

$$\Theta(s) \equiv \Theta(s; B) := \sum_{t \in \mathbb{Z}^g} \exp(\ i\pi(t, Bt) + 2\pi i(t, s)).$$

Here is the fundamental periodic property of the $\Theta$-function

$$\Theta(s + n + Bm) = e^{-\pi i(Bm,m)} \Theta(s), \quad (8.11)$$

and the obvious symmetry relation

$$\Theta(-s) = \Theta(s).$$
Observe now that BA1–BA2 imply the following properties on the components of $\Psi_n(P)$.

$$\Psi_{n1}(P)$$ is meromorphic on $\Re\{\pm, \infty\}$,

$$(\Psi_{n1}(P)) = -\sum_{j=1}^{g} \alpha_j,$$

$$\Psi_{n1}(P) = z^n + O(z^{n-1}), \quad P \to \infty^+, \quad \Psi_{n1}(P) = O(z^{-n}), \quad P \to \infty^-.$$ (8.12)

Similarly for $\Psi_{n2}(P)$,

$$\Psi_{n2}(P) = z^{n+1} + O(z^{-n}), \quad O \to \infty^-, \quad \Psi_{n2}(P) = O(z^{n-1}), \quad P \to \infty^+.$$ (8.13)

By standard technique of the algebro-geometric method (e.g. Belokolos et al. 1994), we get

$$\Psi_{n1}(P) = e^{nQ(P)} \frac{\Theta\left(\int_{\beta_{g+1}}^{P} d\omega + nL - D\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\omega - D\right)} \frac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\omega + nL - D\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\omega - D\right)} C^n(E),$$

$$\Psi_{n2}(P) = e^{(n-1)Q(P)} \frac{\Theta\left(\int_{\beta_{g+1}}^{P} d\omega + (n-1)L - D\right)}{\Theta\left(\int_{\beta_{g+1}}^{P} d\omega - D\right)} \frac{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\omega + D\right)}{\Theta\left(\int_{\beta_{g+1}}^{\infty^+} d\omega - (n-1)L + D\right)} C^{(1-n)}(E),$$

where

$$L_j = \frac{1}{2\pi i} \int_{b_j} d\Omega,$$

$$D_j = \sum_{k=1}^{g} \int_{\beta_{g+1}}^{\alpha_k} d\omega_j + C_j = 2 \sum_{k=1}^{g} \int_{\beta_{g+1}}^{\alpha_k} d\omega_j,$$

and $C_j$ form the vector of the Riemann constants (see again Farkas & Kra 1980; Belokolos et al. 1994). Indeed, by the Riemann theorem (e.g. Farkas & Kra 1980), the first $\Theta$-functions in the denominators have zeros exactly at the points $\alpha_j$; the front exponential factors provide the needed asymptotic behaviour at $\infty^\pm$; the first $\Theta$-functions in the numerators, by virtue of the periodicity property (8.11), ensure the single-valuedness; and, finally, the $P$-independent $\Theta$-factors together with the back exponential factors provide the needed normalizations ad $\infty^\pm$ (cf. (8.12) and (8.13)). We also assume that we choose the same path between $\beta_{g+1}$ and $P$ for all the integrals involved.$^1$

$^1$ Alternatively, one can choose for each integral its own path. In this case though the paths must not intersect the basic cycles.

*Phil. Trans. R. Soc. A*(2008)
The formulae above can be simplified. To this end, we observe that

$$\int_{\beta_{g+1}}^{\alpha_k} d\omega_j = \frac{1}{2} \delta_{jk} + \frac{1}{2} \sum_{l=1}^{k} B_{jl},$$  \hspace{1cm} (8.14)$$

where the path of integration from $\beta_{g+1}$ to $\alpha_k$ lies on the upper plane of the upper sheet. Therefore, moduli the lattice periods,

$$D_j = 1 + \sum_{k=1}^{g} B_{jk}(g - k + 1).$$

In other words, the vector $D$ belongs to the lattice $\mathbb{Z}^g + B\mathbb{Z}^g$ and hence (property (8.11) again) can be dropped from the above formulae for $\Psi_n(P)$. This yields the following simplified $\Theta$-representation for $\Psi_n(P)$:

$$\Psi_{n1}(P) = e^{n\Omega(P)} \frac{\Theta\left(\int_{\beta_{g+1}}^{P} d\omega + nL\right)}{\Theta\left(\int_{\beta_{g+1}}^{P} d\omega\right)} = \Theta\left(\int_{\beta_{g+1}}^{\infty} d\omega + nL\right) C^n(E),$$

$$\Psi_{n2}(P) = e^{(n-1)\Omega(P)} \frac{\Theta\left(\int_{\beta_{g+1}}^{P} d\omega + (n-1)L\right)}{\Theta\left(\int_{\beta_{g+1}}^{P} d\omega\right)} = \Theta\left(\int_{\beta_{g+1}}^{\infty} d\omega - (n-1)L\right) C^{(1-n)}(E).$$

We conclude the $\Theta$-function solution of the Akhiezer Riemann–Hilbert problem by noticing the following equation for the vector $L$ of the $b$-periods of the integral $\Omega(P)$:

$$L = \text{res}_{P=\infty^+}(\omega d\Omega(P)) + \text{res}_{P=\infty^-}(\omega d\Omega(P)) = -\int_{\beta_{g+1}}^{\infty^+} d\omega + \int_{\beta_{g+1}}^{\infty^-} d\omega = -2 \int_{\beta_{g+1}}^{\infty} d\omega.$$  \hspace{1cm} (8.16)$$

The equation is just the classical Riemann bilinear identity (e.g. Farkas & Kra 1980 or Belokolos et al. 1994) applied to the pair of the Abelian integrals $\omega(P)$ and $\Omega(P)$.

**Remark 8.3.** Using equation (8.14), one can check directly, with the help of the periodic condition (8.11), that the $\Theta$-function,

$$\Theta\left(\int_{\beta_{g+1}}^{P} d\omega\right),$$

has the points $\alpha_j$ as its zeros.

**Remark 8.4.** The reader should not be misled by the formal possibility to diagonalize simultaneously the jump matrices of the Riemann–Hilbert problem $\Psi_1$–$\Psi_5$ (which all are equal to $\sigma_1$) and by apparently following from this conclusion that the problem can be reduced to the scalar one on the complex plane and hence solved without any use of the $\Theta$-functions. The obstructions come from the end points $\alpha_j$, $\beta_j$ and from the point at infinity, where the function $\Psi_n(z)$ must have the singularities specified by equations $\Psi_5$, $\Psi_4$ and $\Psi_3$, respectively. These singularities can be alternatively described as the addition jump conditions posed on the small circles around the end points and on the big
circle around the infinity. The relevant jump matrices are

\[
\begin{pmatrix}
\frac{1}{\sqrt{z - \alpha_j}} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
\sqrt{z - \beta_j} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
n & 0 \\
z & 0 & -n + 1
\end{pmatrix},
\]

respectively. Posed in this form, the \( \sigma_1 \) Riemann–Hilbert problem becomes the regular one—no singularities different from the jumps are prescribed. At the same time, the additional jump matrices depend on \( z \) and the whole new set of jump matrices cannot be simultaneously diagonalized. The only way to circumvent this obstacle, and not to use the \( \Theta \)-functions, is equation (8.4) which indeed gives an explicit representation of the solution of the \( \sigma_1 \) Riemann–Hilbert problem in terms of the elementary functions and their contour integrals. The \( \Theta \)-function representation (8.15) for the solution \( \Psi_n(z) \) obtained in this article has an important advantage comparing to (8.4). It reveals the nature of the dependence of \( \Psi_n(z) \), and hence of the Akhiezer polynomials themselves (see equation (9.1)), on the number \( n \), as \( n \) varies over the whole range \( 1 \leq n \leq \infty \) (see Chen & Lawrence (2002) for more on the use of the \( \Theta \)-representations in the analysis of the Akhiezer polynomials). Simultaneously, the comparison of equations (8.4) and (8.15) might, perhaps, be used to derive some new non-trivial identities for the hyperelliptic \( \Theta \)-functions.

**Remark 8.5.** Up to a trivial diagonal gauge transformation, the matrix function \( \Psi_n(z) \) satisfies the same Fuchsian equation (3.11) that is satisfied by the function \( \Phi_n(z) \). Note that the corresponding monodromy group is very simple; indeed, it has just one generator, the matrix \( \sigma_1 \). Once again, the reader might be wondering about the appearance of the highly non-trivial \( \Theta \)-functional formulae in the description of the function \( \Psi_n(z) \) which gives the solution of the corresponding inverse monodromy problem. Similar to the previous remark, the explanation comes from the fact that the solution \( \Psi_n(z) \), in addition to the given monodromy group, must exhibit the local behaviour at the singular points indicated by the conditions \( \Psi_3 - \Psi_5 \). This situation is typical in the theory of the finite-gap solutions of integrable PDEs\(^2\) (e.g. Jimbo et al. 1981; Belokolos et al. 1994).

### 9. A list of the \( \Theta \)-formulae

In this section, we give formulae expressing the polynomial \( P_n(z) \), recurrence coefficients \( a_n, b_n \), the square of the weighted \( L^2 \) norm \( h_n \) and the Hankel determinant in terms of the \( \Theta \)-functions. The expressions will be derived as simple corollaries of equations (8.5) and (8.15) representing the solution \( \Psi_n(z) \) of the Riemann–Hilbert problem \( \Psi_1 - \Psi_5 \) in terms of the \( \Theta \)-functions.

\(^2\) Another example of an apparently simple but non-trivially solved inverse monodromy problem can also be found in the theory of integrable PDEs. It is provided by the multi-soliton Baker–Akhiezer function whose monodromy group is just trivial. Of course, the formulae in this case are simpler than the finite-gap ones—they do not contain the \( \Theta \)-functions. At the same time, the answer is still rather complicated; in fact, it involves degenerated \( \Theta \)-functions corresponding to the singular curves of genus zero.
From (8.1), it follows that (see also (8.4))

\[ P_n(z) = (Y_n(z))_{11} = (\Psi_n(z))_{11} + (\Psi_n(z))_{12}. \]

This together with (8.5) and (8.15) leads to the following \( \Theta \)-representation of the Akhiezer polynomials:

\[
P_n(z) = \frac{\Theta \left( nL + \int_{\beta_{y+1}}^{z} d\omega \right) e^{n\Omega(z)} + \Theta \left( nL - \int_{\beta_{y+1}}^{z} d\omega \right) e^{-n\Omega(z)} }{\Theta \left( \int_{\beta_{y+1}}^{z} d\omega \right)} \times \frac{\Theta \left( \int_{\beta_{y+1}}^{\infty} d\omega \right)}{\Theta \left( \int_{\beta_{y+1}}^{\infty} d\omega + nL \right)} C^n(E),
\]

where all the hyperelliptic integrals are taken in the upper sheet of the curve \( \mathcal{R} \) (and along the same path).

**Remark 9.1.** It is a simple but an instructive exercise to check directly, using equation (8.14), the similar equation for the integral \( \Omega(P) \), i.e.

\[ \Omega(\alpha_k) = \pi i + \pi i \sum_{j=1}^{k} L_j, \]

and, once again, the periodicity property of the \( \Theta \)-function, that the r.h.s. of (9.1) is indeed a polynomial.

To evaluate the quantities \( a_n, b_n \) and \( h_n \), we shall use the relation

\[
\psi_1 = \begin{pmatrix} 1 & 0 \\ \sqrt{2\pi i} & 0 \\ 0 & \sqrt{\frac{2}{\pi i}} \end{pmatrix} m_1 \begin{pmatrix} \sqrt{2\pi i} & 0 \\ 0 & \frac{\pi i}{2} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \kappa \end{pmatrix}, \quad n > 1,
\]

between the first matrix coefficients, \( \psi_1 \) and \( m_1 \), of the Laurent series

\[ \Psi_n(z) = \left( I + \sum_{k=1}^{\infty} \frac{\psi_k(n)}{z^k} \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n+1} \end{pmatrix}, \quad |z| > 1, \]

and

\[ Y_n(z) = \left( I + \sum_{k=1}^{\infty} \frac{m_k(n)}{z^k} \right) z^{n\sigma_3}, \quad |z| > 1, \]

respectively. In (9.2), the parameter \( k \) is defined via the expansion,

\[ w(z) = \frac{i}{\pi z} \left( 1 + \frac{k}{z} + \cdots \right), \]

and the matrix,

\[
\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},
\]

**Phil. Trans. R. Soc. A** (2008)
should be added to the r.h.s. if \( n = 1 \). Combining equation (9.2) with the formula (2.15), we obtain that

\[
\eta_n = 2(\psi_1(n))_{12}.
\]

On the other hand, let us introduce the coefficient matrix \( c_{jk}, j, k = 1, 2 \) by the relations (cf. (8.12) and (8.13))

\[
\psi_{n1}(P) = z^n + c_{11}z^{n-1} + O(z^{n-2}), \quad P \to \infty^+,
\]

(9.3)

\[
\psi_{n1}(P) = c_{12}z^{-n} + O(z^{-n-1}), \quad P \to \infty^-,
\]

(9.4)

and

\[
\psi_{n2}(P) = z^{-n+1} + c_{22}z^{-n} + O(z^{-n-1}), \quad O \to \infty^-,
\]

(9.5)

\[
\psi_{n2}(P) = c_{21}z^{n-1} + O(z^{n-2}), \quad P \to \infty^+.
\]

(9.6)

Then, it is obvious that

\[
\psi_1(n)_{jk} = c_{jk},
\]

(9.7)

and, in particular, we arrive to the equation

\[
\eta_n = 2c_{12}.
\]

(9.8)

The coefficient \( c_{12} \), in its turn, can be immediately evaluated from the \( \Theta \)-formula (8.15) by letting \( P \to \infty^- \). In fact, we have

\[
c_{12} = C^{2n}(E) \frac{\Theta\left(\int_{\beta_{y+1}}^{\infty} d\omega - nL\right)}{\Theta\left(\int_{\beta_{y+1}}^{\infty} d\omega + nL\right)}.
\]

(9.9)

Taking into account the Riemann bilinear relation (8.16), we can represent the formula for \( \eta_n \) in the following final form:

\[
\eta_n = 2C^{2n}(E) \frac{\Theta\left((n + \frac{1}{2})L\right)}{\Theta\left((n - \frac{1}{2})L\right)}, \quad n = 1, 2, \ldots,
\]

(9.10)

\[
h_0 := 1.
\]

An important direct consequence of this equation is the explicit \( \Theta \)-functional representation for determinant of the \((n+1) \times (n+1)\) Hankel matrix

\[
D_{n+1}[w_+] = \prod_{j=0}^{n} h_j = 2^n(C(E))^{n(n+1)} \frac{\Theta\left((n + \frac{1}{2})L\right)}{\Theta\left((n - \frac{1}{2})L\right)}
\]

\[
= 2^n(C(E))^{n(n+1)} \frac{\Theta\left(2n + 1, \int_{\beta_{y+1}}^{\infty} d\omega\right)}{\Theta\left(\int_{\beta_{y+1}}^{\infty} d\omega\right)}.
\]

(9.11)
A similar use of the remaining equations in (9.2), (9.7) and (2.16) and (2.17) leads at once to the $\Theta$-representations of the recurrence coefficients $a_n$ and $b_n$

\[
a_n = \begin{cases} 
\frac{\Theta\left(\frac{3}{2}L\right)}{2C^2(E)} & \text{if } n = 1 \\
\frac{\Theta\left(\frac{1}{2}L\right)}{C^2(E)} & \frac{\Theta\left(\left(n + \frac{1}{2}\right)L\right)\Theta\left(\left(n - \frac{3}{2}\right)L\right)}{\Theta^2\left(\left(n - \frac{1}{2}\right)L\right)} & \text{if } n > 1
\end{cases}
\]  

(9.12)

and

\[
b_n = \frac{1}{2} \sum_{j=1}^{g} (\beta_j - \alpha_j)
= \sum_{j=1}^{g} A^{-1}_{j1} \left[ \frac{\Theta'_j\left(\left(n - \frac{1}{2}\right)L\right)}{\Theta\left(\left(n - \frac{1}{2}\right)L\right)} - \frac{\Theta'_j\left(\left(n - \frac{3}{2}\right)L\right)}{\Theta\left(\left(n - \frac{3}{2}\right)L\right)} - 2 \frac{\Theta'_j\left(\frac{1}{2}L\right)}{\Theta\left(\frac{1}{2}L\right)} \right].
\]  

(9.13)

Here,

\[
\Theta'_j(s) := \frac{\partial \Theta(s)}{\partial s_j}.
\]

Equations (9.1), (9.10)–(9.13) were previously obtained in Chen & Lawrence (2002) by a direct analysis of Akhiezer’s function defined as the sum $i\pi w(z)P_n(z) - Q_n(z)/(w(z))$ (cf. (8.4)). In Chen & Lawrence (2002), it was also shown that the above formulae allow one to identify the quantity $C(E)$ as the transfinite diameter of the set $E$. We remind readers that in our approach, $C(E)$ appears as a first non-trivial coefficient in the asymptotic expansion of the Abelian integral $\Omega(P)$ (see (8.9) and (8.10)). Finally, we should note that equations (8.15), (9.12) and (9.13), as the equations describing the eigenfunctions and the coefficients of a finite-gap discrete Schrodinger operator, have already been known (e.g. Krichever 1978) in the theory of the periodic Toda lattice.

Y.C. would like to thank the Department of Mathematics, University of Wisconsin-Madison, for the kind hospitality in hosting him and the EPSRC for an oversea travel grant that made this endeavour possible. A.R.I. was supported in part by NSF grant DMS-0099812 and by Imperial College of the University of London via an EPSRC grant. The final part of this project was done when he was visiting Institut de Mathématique de l’ Université de Bourgogne, and the support during his stay there is gratefully acknowledged.

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