Medium effects in the pion-pole mechanism
($\gamma\gamma \rightarrow \pi^0 \rightarrow \nu_R\bar{\nu}_L(\nu_L\bar{\nu}_R)$) of neutron star cooling

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Abstract

Nuclear medium effects in the neutrino cooling of neutron stars through the reaction channel $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu_R\bar{\nu}_L(\nu_L\bar{\nu}_R)$ are incorporated. Throughout the paper we discuss different possibilities of right-handed neutrinos, massive left-handed neutrinos and standard massless left-handed neutrinos (reaction is then allowed only with medium modified vertices). It is demonstrated that multi-particle effects suppress the rate of this reaction channel in the dense hadron matter by $6 - 7$ orders of magnitude that does not allow to decrease existing experimental upper limit on the corresponding $\pi^0\nu\bar{\nu}$ coupling. Other possibilities of the manifestation of the given reaction channel in different physical situations, e.g. in the quark color superconducting cores of the most massive neutron stars, are also discussed. We demonstrate that in the color-flavor-locked superconducting phase for temperatures $T \lesssim (0.1 \div 10)$ MeV (depending on the effective pion mass and the decay width) the process is feasibly the most efficient neutrino cooling process, although the absolute value of the reaction rate is rather small.
1 Introduction

Many years ago Pontecorvo and Chiu and Morrison[1] suggested that the process $\gamma\gamma \to \nu\bar{\nu}$ might play an important role as a mechanism for stellar cooling. Gell-Mann[2] subsequently showed that this process is forbidden in a local (V-A) theory. However, it can occur at the one-loop level which has been computed by Levine[3] for an intermediate-boson (V-A) theory, and the stellar energy loss rate through $\gamma\gamma \to \nu\bar{\nu}$ was found to be smaller than the rates for competing processes (pair annihilation $e^+e^- \to \nu\bar{\nu}$ and photo-neutrino production $\gamma e \to e\nu\bar{\nu}$). This result is not modified when the cross section of the above process is computed in the standard model, as was shown by Dicus[4]. Only for very peculiar neutrino coupling to photons or unnaturally large neutrino masses this reaction overwhelms the result of the standard model[5].

There is still another possibility proposed by Fischbach et al. [6], where the reaction $\gamma\gamma \to \nu\bar{\nu}$ could be significant. This is the case when the process is mediated by a pseudoscalar resonance and the latter decays into $\nu\bar{\nu}$ due to the existence of right-handed neutrinos or due to new interactions beyond the standard model. It was assumed that in astrophysical conditions only the pion resonance could be important (next in mass not strange $\eta$-resonance is too heavy in standard conditions) and the process was termed the pion-pole mechanism. Thus the process which we will continue to discuss in this paper is $\gamma\gamma \to \pi^0 \to \nu\bar{\nu}$.

Of course, if the temperature is high enough, and on the other hand, the pion dispersion relation in matter allows for the quasiparticle spectrum branch, there appears a significant number of thermally equilibrated pion quasiparticles. Then the process $\pi^0 \to \nu\bar{\nu}$ may also be important. In this process the initial thermally equilibrated pion is on its mass-shell modified in the matter. In the process $\gamma\gamma \to \pi^0 \to \nu\bar{\nu}$ the initial reaction states contain no pion, the virtual pion only transfers the interaction from thermally equilibrated photons in the $\gamma\gamma$ annihilation process.
to produced $\nu\bar{\nu}$. As we will see below, the process $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$ has an output of the energy $Q$ varying with the temperature as $Q \propto T^n$ where the power $n$ changes with the temperature typically from $n = 3$ for rather high temperature ($T$ is still much smaller than the pion mass $m_\pi = 140$ MeV) to $n = 11$ for low temperatures. The process $\pi^0 \rightarrow \nu\bar{\nu}$ having essentially larger phase space volume (one particle in the initial state) yields however the exponentially suppressed output of the energy, $Q \propto T^{3/2} e^{-m_\pi/T}$ at $T < m_\pi$ since the initial particle is massive, in difference with photons. Concentrating on the discussion of the rate of the $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$ reaction channel we shall also compare it with the rate of the competing $\pi^0 \rightarrow \nu\bar{\nu}$ process and other relevant processes.

Temperatures of the order of $10^{-60}$ MeV are expected in interiors of proto-neutron stars formed in supernova (SN) explosions during the early cooling phase of the proto-neutron star evolution, according to existing numerical simulations[7]. It is exactly in such a situation the pion-pole mechanism is the most effective ($T \sim m_\pi/a$, $a \sim 2 \div 50$). Indeed, in Ref.[8] the mechanism was applied to the case of supernova SN1987A and it was found to be a quite important process even if the pion partial production rate of neutrinos ($\Gamma(\pi^0 \rightarrow \nu_R\bar{\nu}_L(\nu_L\bar{\nu}_R))$) is many orders of magnitude smaller than the presently accepted value of the experimental upper limit. In case of $\nu_L\bar{\nu}_R$ with the vacuum vertex the process rate is proportional to the squared neutrino mass. The result [8] was criticized by Raffelt and Seckel [9] basing on the fact that in the calculation of Ref.[8] it was used the vacuum value for the total pion width, $\Gamma^{\text{vac}}_\pi$, which is a tiny quantity, $\Gamma^{\text{vac}}_\pi \simeq 0.58 \cdot 10^{-7} m_\pi$. They argued that in supernova cores pion states are damped mostly by nucleon absorption rather than by the free decay so that the width in the medium ($\Gamma^{\text{med}}_\pi$) is much larger than in vacuum and the pion-pole mechanism should be strongly suppressed. Besides, in the medium the process may go on massless left-handed neutrinos through the intermediate nucleon particle – nucleon hole states, i.e. as $\gamma\gamma \rightarrow \pi^0 \rightarrow nn^{-1} \rightarrow \nu_L\bar{\nu}_R$ reaction channel. The processes $\pi^0_{\text{virt}} \rightarrow nn^{-1} \rightarrow \nu_L\bar{\nu}_R$ on virtual pions have been discussed in [10].
Without any doubt the neutrino emission from the dense hadronic component in neutron stars is the subject of strong modifications due to collective effects in the nuclear matter [11], and it is interesting to quantitatively know how much the pion-pole mechanism is influenced by the properties of the dense medium and even, if the process is strongly suppressed, what is the main effect causing this suppression. In this work we compute the cooling rate due to the neutrino emission through the pion-pole mechanism in the reaction \( \gamma \gamma \rightarrow \pi^0 \rightarrow \nu_R \bar{\nu}_L \) (or \( \nu_L \bar{\nu}_R \)) for the conditions of proto-neutron stars including the effects of dense medium in the pion polarization operator (Section 2). Then we discuss other relevant reaction channels. In order to clarify their connection to the process \( \gamma \gamma \rightarrow \pi^0 \rightarrow \nu_R \bar{\nu}_L \) (or \( \nu_L \bar{\nu}_R \)) we review in the Appendix the optical theorem formalism, see the discussion in [11], that allows to calculate consistently the reaction rates including complicated medium effects. Continuing to study possible physical situations where the pion pole cooling mechanism could be important we consider in Section 3 a possible consequence of the pion pole mechanism in the case of neutron stars with the color superconducting quark cores. We argue that in the color-flavor-locked superconducting phase for temperatures \( T \lesssim (0.1 \div 10) \) MeV the process is feasibly the most efficient neutrino cooling process, although the absolute value of the reaction rate is rather small. Then we draw our conclusion.

2 Emissivity of the process \( \gamma \gamma \rightarrow \pi^0 \rightarrow \nu \bar{\nu} \) from the hadronic matter

The cross section of the process \( \gamma \gamma \rightarrow \pi^0 \rightarrow \nu_R \bar{\nu}_L (\nu_L \bar{\nu}_R) \) in vacuum is given by

\[
\sigma_{\pi}^{\text{vac}}(s) = \frac{8\pi(s^2/m_\pi^4)\Gamma(\pi^0 \rightarrow \gamma \gamma)\Gamma(\pi^0 \rightarrow \nu \bar{\nu}) F(s)}{(s - m_\pi^2)^2 + (m_\pi \Gamma_{\pi}^{\text{vac}})^2},
\]

where \( F(s) \) is an unknown function of \( s \equiv (\text{c.m. energy})^2 \), representing the product of the vertex function for the off-shell processes \( \gamma \gamma \rightarrow \pi^0 \) and \( \pi^0 \rightarrow \nu \bar{\nu} \), constrained
to $F(s = m_{\pi}^2) = 1$. Following Ref.[6] we may assume $F(s) \simeq 1$ off mass-shell, that seems to be a reasonable approximation for the energies and momenta that we are dealing with. $\Gamma(\pi^0 \rightarrow \gamma \gamma)$ is the partial width of the pion decay into two photons. Here we assume that $\Gamma(\pi^0 \rightarrow \gamma \gamma)$ is approximately equal to the total pion width ($\Gamma_{\pi}^{\text{vac}}$), for which we use the experimental value. The pion partial width into neutrinos $\Gamma(\pi^0 \rightarrow \nu \bar{\nu})$ has the following experimental upper limit $\Gamma(\pi^0 \rightarrow \nu \bar{\nu})/\Gamma_{\pi}^{\text{vac}} < 8.3 \times 10^{-7}$ [12]. In (1) one recognizes the free pion propagator modulus squared $|D_{\pi}^{\text{vac}}|^2$ entering squared matrix element of the reaction under consideration.

Strictly speaking, in application to the nuclear medium all the terms in (1) should be modified. E.g., the partial width $\Gamma(\pi^0 \rightarrow \gamma \gamma)$ may increase with the temperature [13]. The pion partial width into neutrinos $\Gamma(\pi^0 \rightarrow \nu \bar{\nu})$ is determined by the corresponding $\pi^0 - \nu \bar{\nu}$ coupling. The squared vertex is averaged over the neutrino–antineutrino phase-space volume, see Appendix. If we knew the explicit form of the coupling we could explicitly calculate $\Gamma(\pi^0 \rightarrow \nu \bar{\nu})$ and its temperature dependence. However the main modification comes from the change of the pion propagator in dense nuclear medium due to the pion pole. Therefore, below we consider only this modification. Then, one should replace in (1) the free pion propagator $D_{\pi}^{\text{vac}}$ by the in-medium one

$$D_{\pi}^{R} = \frac{1}{[\omega^2 - m_{\pi}^2 - k^2 - \Pi_{\pi}^{R}(w, k, \rho, Y, T)]},$$

(2)

where $\Pi_{\pi}^{R}(w, k, \rho, Y, T)$ is the total retarded polarization operator of the neutral pion, dependent on the pion energy $\omega$, momentum $k$, baryon (nucleon in our case) density $\rho$, isotopic composition $Y = Z/(N + Z)$, and the temperature $T$.

If photons are in thermal equilibrium, the energy loss rate of the process $\gamma \gamma \rightarrow \pi^0 \rightarrow \nu R \bar{\nu}_L(\nu L \bar{\nu}_R)$ occurring in dense interior of the proto-neutron star, at which the energy is converted into neutrino pairs, is given by

$$Q_{\pi}^{\text{med}} = \frac{4}{(2\pi)^6} \int \frac{d^3 k_1}{\exp(\omega_1/k_B T) - 1} \int \frac{d^3 k_2}{\exp(\omega_2/k_B T) - 1} (\omega_1 + \omega_2) v_{\text{rel}} \sigma_{\pi}^{\text{med}},$$

(3)
where $\omega_1$, $\omega_2$ are the photon energies, $\vec{k}_1$ and $\vec{k}_2$ are their momenta, $\theta$ is the angle between photons, $v_{\text{rel}}$ is the relative-velocity factor

$$v_{\text{rel}} = \left| (\vec{k}_1 \cdot \vec{k}_2 - \omega_1 \omega_2) / \omega_1 \omega_2 \right| = 1 - \cos \theta,$$

and $\sigma_{\pi}^{\text{med}}$ is the medium dependent cross section of the process given by (1) with $D_{\pi}^{\text{vac}}$ replaced by $D_{\pi}^{R,0}$. Eq. (3) is the straightforward generalization of the result [6, 8] obtained with the help of the replacement $\sigma_{\pi}^{\text{vac}} \rightarrow \sigma_{\pi}^{\text{med}}$.

The energy loss rate (3) can be presented in the following form

$$Q_{\text{med}} \sim 4.21 \times 10^{22} \left( \frac{\Gamma(\pi^0 \rightarrow \gamma \gamma)}{m_{\pi}} \right) \left( \frac{\Gamma(\pi^0 \rightarrow \nu \bar{\nu})}{m_{\pi}} \right) T_{9}^{\text{I}}(\tau), \text{ erg}/(\text{cm}^3 \cdot \text{s}),$$

(5)

$m_{\pi} = 140$ MeV, $T_{9} = T/10^9 K$, $\tau = (k_B T/m_{\pi})$ and $I(\tau)$ is given by the dimensionless integral

$$I(\tau) = \int_{0}^{\infty} \frac{x_1^4 dx_1}{e^{x_1} - 1} \int_{0}^{\infty} \frac{x_2^4 dx_2}{e^{x_2} - 1} \left( \frac{2x_1 x_2 u \tau}{2x_1 x_2 u \tau^2 - 1 - m_{\pi}^{-2} \text{Re} \Pi_{\pi}^{R,0})^2 + (m_{\pi}^{-1} \Gamma_{\pi})^2} \right)^{1/2},$$

(6)

where $\kappa = \frac{2}{x_1 x_2} (m_{\nu}/k_B T)^2$, $m_{\nu}$ is the neutrino mass, $x_{1,2} = \omega_{1,2}/k_B T$, $u = 1 - \cos \theta$, $m_{\pi}^{-1} \Gamma_{\pi} = m_{\pi}^{-1} \Gamma_{\pi}^{\text{vac}} - m_{\pi}^{-2} \text{Im} \Pi_{\pi}^{R,0}$. Further we assume that $m_{\nu}/k_B T \ll 1$ and then put the lower limit ($\kappa$) of the integral in the variable $u$ equal to zero, that corresponds to using $m_{\nu} = 0$ in all the phase space calculations. However we take $m_{\nu} \neq 0$ into account evaluating the $\Gamma(\pi^0 \rightarrow \nu L \bar{\nu}_R)$ width. For the further convenience we assume that $\text{Re} \Pi_{\pi}^{\text{vac}}$ and $\text{Im} \Pi_{\pi}^{R,0}$ are the real and imaginary parts of $\Pi_{\pi}^{R,0}(w, k, \rho, Y, T)$ describing only the strong interaction processes. Therefore we separated in (6) the value $\Gamma_{\pi}^{\text{vac}}$ which is due to the electromagnetic and the weak interaction. With $I(\tau)$ replaced to $I^{\text{vac}}(\tau)$ (putting $\text{Re} \Pi_{\pi}^{R,0}$ and $\text{Im} \Pi_{\pi}^{R,0}$ to zero) we reproduce the results of Refs.[6, 8]. One easily finds the corresponding asymptotic expression

$$I \sim I^{\text{vac}}(\tau \ll 0.1) \simeq 23040 \zeta(6) \zeta(5) \left[ 1 + 96 \tau^2 \zeta(7) / \zeta(5) \right],$$

(7)

$\zeta$ is the Riemann function, $\zeta(5) \simeq 1.037$, $\zeta(6) \simeq 1.017$, $\zeta(7) \simeq 1.008$. In order to get (7) we dropped $\Gamma_{\pi}$ and $\text{Re} \Pi^{R}$ and expanded (6) in the value $2x_1 x_2 u \tau^2 \ll 1$. Since
typical values \( x_1 \sim x_2 \sim 6 \) in the resulting integral, the limit expression is valid for \( \tau \ll 0.1 \). As we checked numerically the limit (7) is actually achieved only at \( \tau \lesssim 0.01 \), if \( \Gamma_\pi \) is as small as \( \Gamma_\pi^{\text{vac}} \). Using Eq. (7) and the evaluation of \( \Gamma(\pi^0 \to \gamma\gamma) \) we obtain

\[
Q^{\text{vac}}(\tau \lesssim 0.01) \simeq 0.6 \cdot 10^{20} \left( m_\pi^{-1} \Gamma(\pi^0 \to \nu\bar{\nu}) \right) T_9^{11}, \text{ erg/(cm}^3 \cdot \text{s}) \tag{8}
\]

at small temperatures.

When the temperature increases the denominator becomes to be near the pole. Then dividing and multiplying \( I(\tau) \) by \( \Gamma_\pi \) and using the corresponding presentation of the \( \delta \)-function we roughly estimate

\[
I \left( \left( \frac{m_\pi}{10 \Gamma_\pi} \right)^{1/4} \gtrsim \tau \gtrsim a^{-1} \right) \sim \frac{0.6(e^{-1/(2\tau)|\ln(2\tau)|} + 1)e^{-1/(2\tau)}}{\tau^8 m_\pi^{-1} \Gamma_\pi}, \ a \sim 10 \div 10^2. \tag{9}
\]

From the latter estimate we recognize the resonant character of the rate. \( I^{\text{vac}}(\tau \sim 0.1) \) is \( \sim 10^9 \) times larger compared with \( I^{\text{vac}}(\tau = 0) \) for \( \Gamma_\pi \sim \Gamma_\pi^{\text{vac}} \).

For higher temperatures assuming \( 2x_1 x_2 u\tau^2 \gg 1 \) we estimate

\[
I \left( \tau \gg \left( \frac{m_\pi}{10 \Gamma_\pi} \right)^{1/4} \right) \simeq 12 \zeta(3) \zeta(4)/\tau^4, \ \zeta(3) \simeq 1.202, \ \zeta(4) \simeq 1.082, \tag{10}
\]

that produces \( Q^{\text{vac}} \simeq Q^{\text{med}} \simeq 2.6 \cdot 10^{29} \left( m_\pi^{-1} \Gamma(\pi^0 \to \nu\bar{\nu}) \right) T_9^7, \text{ erg/(cm}^3 \cdot \text{s}) \).

In the so called ”standard scenario” of the neutron star cooling one considers the modified Urca process as the most efficient process. The emissivity of the modified Urca process is estimated [14] with the help of the free pion propagator as \( Q^{\text{MU}} \sim 10^{21} T_9^8 \text{ erg/(cm}^3 \cdot \text{s}) \). Comparing it with (5), (9) we see that for \( T \gtrsim 10 \text{ MeV} \) the process under consideration would be much more efficient process than the modified Urca process, if the pion width and the mass in medium were not essentially changed compared to the vacuum values. However in reality the width \( \Gamma_\pi^{\text{vac}} \) is replaced to a much larger medium value. Also the pion mass is modified by the polarization effect (thereby the pion pole begins to manifest itself at \( T \gtrsim m_\pi^{\text{eff}}/a \), rather than at \( T \gtrsim m_\pi/a \)). Thus one may expect that with taking into account of the medium
effects the estimation (7) is not essentially modified whereas the value (9) must be significantly suppressed mainly due to the suppression of the width.

Techniques for the description of collective effects in dense hadronic matter have been developed in the last decades[15, 16]. E.g., the medium effects appearing in the pion propagator have been discussed in detail in Ref.[17] (see Appendix B of that work). For the temperatures and pion energies with which we are concerned the main modification of the pion polarization operator is due to the density dependence rather than the temperature dependence and it is, thereby, a reasonable approximation to put \( T = 0 \) in \( \Pi^R_{\pi^0}(\omega, k, \rho, Y, T) \). Depending on the values of the typical pion energy and momentum different terms can be important in the pion polarization operator. At small pion energies \( \omega/m_\pi \ll 1 \) and for typical momenta \( k \sim p_{FN} \), \( p_{FN} \) is the Fermi momentum of the nucleon, the nucleon particle - hole contribution is attractive and the dominant one, what results in the softening of the virtual pion mode with increase of the density and leads to the possibility of the pion condensation at \( \rho > \rho_c > \rho_0 \), cf. [15, 16, 17]. In our case \( \omega > k \), as follows from the reaction kinematics, and the nucleon particle - hole contribution is minor. Then the main terms in the pion polarization operator are the \( \Delta \)-particle - nucleon hole part and the regular part related to more complicated intermediate states. Thus, \( \Pi^R_{\pi^0} \simeq \Pi^R_{\Delta} + \Pi^R_{\text{reg}} \), and the partial contributions are given by [17]:

\[
\text{Re} \Pi^R_{\Delta}(\omega, k, \rho, Y, T = 0) \simeq - \frac{B_0 \Gamma(g'_{\Delta})}{\tilde{\omega}_{\Delta}(t) - \omega^2}, \tag{11}
\]

in units \( \hbar = c = 1 \), \( B_0 \simeq 2.0 \ m_\pi (\rho/\rho_0) \Gamma_{\pi N \Delta}^2 k^2 \tilde{\omega}_\Delta(t) \), \( \rho \) is expressed in units \( \rho_0 \), \( \rho_0 \simeq 0.5 \ m_\pi^3 \) is the saturation nuclear density, the form factor \( \Gamma_{\pi N \Delta}^2 \approx \Gamma_{\pi NN}/\beta \approx 1/\beta \) for the rather small momenta of our interest, \( \beta \simeq 1 + 0.23 k^2/m_\pi^2 \) is an empirical factor taking into account a contribution of the high-lying nucleon resonances. The nucleon-\( \Delta \) isobar correlation factor \( \Gamma(g'_{\Delta}) \) is given by

\[
\Gamma(g'_{\Delta}) \simeq \left[ 1 + \frac{C}{\tilde{\omega}_{\Delta}(t) - \omega^2} \right]^{-1}, \tag{12}
\]
with $C \simeq 0.9 \, m_\pi^2 (\rho/\rho_0) \Gamma_{\pi N}^2$ and

$$\tilde{\omega}_\Delta(t) \simeq 2.1 \, m_\pi \left( 1 + \frac{2.1 \, m_\pi}{2m_N^*} \right) + \frac{t}{2m_N^*}, \quad t = k^2 - \omega^2, \quad (13)$$

$m_N^*(\rho)$ is the effective mass of the nucleon quasiparticle,

$$Im \, \Pi^R_\Delta(\omega, k, \rho, Y, T = 0) \simeq -\frac{2\omega B_0 \Gamma^2 (g'_\Delta) \gamma_0 k^3}{[\tilde{\omega}_\Delta^2 (t) - \omega^2]^2}, \quad (14)$$

$\gamma_0 k^3$ takes into account the $\Delta$-isobar width, with an empirical value $\gamma_0 \simeq 0.08 \, m_\pi^2 \Gamma_{\pi N}^2$. In the numerical evaluations below we for simplicity assume $\tilde{\omega}_\Delta^2 (t) \gg \omega^2$ that is a reasonable approximation for the conditions we are dealing with.

The regular part of the pion polarization operator is yet more model dependent. Its value is recovered with the help of the pionic atom data, cf. [18], and a procedure of going off mass-shell. We present it in the following form, cf. [17],

$$Re \, \Pi^R_{reg}(\omega, k, \rho, Y, T = 0) \simeq \left( -0.25 \, \omega^2 + 0.25 \, m_\pi^2 + 0.5 \, k^2 \right) \frac{\rho}{\rho_0}, \quad (15)$$

$$Im \, \Pi^R_{reg}(\omega, k, \rho, Y, T = 0) \simeq -0.15 \, m_\pi^2 \left( \frac{\rho}{\rho_0} \right)^2 \xi - \frac{0.09 (\rho/\rho_0)^2 k^2 \xi}{(1 + 0.38 (\rho/\rho_0) - 0.047 (\rho/\rho_0)^2 \xi)^2}, \quad (16)$$

The factor $\xi$ which we inserted in these expressions compared to those of [17] takes into account asymmetry of the isotopic composition of the proto-neutron star matter. Near the pion mass-shell, with $\xi = 4Y(1-Y)$ we approximately describe results given by the potentials I-III used in [18] for the pion atoms and with $\xi = 1$, the results for the potential IV. For the value $Y \simeq 0.4$ typical for initial stage of proto-neutron star cooling in both mentioned cases one can put $\xi \simeq 1$. The real and imaginary terms shown above are the ones which enter Eq.(6), being responsible for the density effects, where we put $\xi \simeq 1$. We also used that $\Pi_{\pi^0}^R(\omega, k, \rho, Y, T) = \frac{1}{2} \left[ \Pi_{\pi^-}^R(\omega, k, \rho, Y, T) + \Pi_{\pi^+}^R(-\omega, -k, \rho, Y, T) \right]$, and the linear in $\omega$ terms entering $Re \Pi_{\pi^\pm}^R$ for $Y \neq 1/2$ do not contribute to $Re \Pi_{\pi^0}^R$. 

9
We computed the energy loss rate including medium effects in the pion polarization operator. In Figure 1 we present the results for the ratio of the energy loss rate ($Q^{\text{med}}$) due to the pion-pole mechanism calculated with medium effects taken into account in the pion polarization operator to the one ($Q^{\text{vac}}$) computed with $Re \Pi^R = 0$ and the vacuum pion decay width (cf. (8)), for three values of the nuclear matter density: $\rho = (0.5; 1; 2) \rho_0$. We used $m^*_N \simeq (0.9; 0.85; 0.7) m_N$ for those densities and put $m_\pi \simeq 140$ MeV. The ratio $Q^{\text{med}}/Q^{\text{vac}}$ is depicted in Figure 1 as a function of temperature. This ratio does not depend on an unknown value of $\Gamma(\pi^0 \rightarrow \nu \bar{\nu})$. We see that the nuclear matter effects decrease the output of the energy typically by six to seven orders of magnitude depending on the temperature and the density, in agreement with the expectations of Raffelt and Seekel [9]. We also show in Table 1 the results for the dimensionless integral of Eq.(6).

Qualitatively, the reasons for the enormous decrease of the energy loss rate are
related to the strong pion absorption in nuclear matter, as has been predicted in
Ref.[9]. The total pion width at appropriate pion energies and momenta grows
up to tens of MeV with the density, that is orders of magnitude larger than the
vacuum contribution $\Gamma^{\text{vac}}_\pi$. Other reason is the energy-momentum dependence of
the real part of the pion polarization operator. This is also not a small change
because the mechanism is totally dependent on the resonant behavior, and outside
the resonance the contribution sharply decreases. We do not need to be on the
top of the resonance, however we cannot be too far away either. Medium effects
push the typical pion energy to a larger value ($\text{Re}\Pi^R(\omega, k) > 0$ at pion energies and
momenta in the region of the pole).

The ratio $\Gamma(\pi^0 \rightarrow \nu\bar{\nu})/\Gamma(\pi^0 \rightarrow \gamma\gamma)$ in many models of elementary particles
beyond the standard one is proportional to the neutrino mass ($m_\nu$). In Ref.[8] a
limit on the neutrino mass was obtained due to the strong constraint on $\Gamma(\pi^0 \rightarrow \nu\bar{\nu})$.
However, with medium effects included into consideration no astrophysical limit
on $m_\nu$ better than the existent ones can be obtained. A nonzero neutrino mass
induces other mechanisms much more efficient than the one generated by the pion
resonance in this situation, and these mechanisms provide then tighter constraints
on the neutrino masses (see, for instance, Ref.[19]).

Above we discussed only the role of the reaction channel $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$. But
there are other competing reaction channels. The relation between all these pro-
cesses becomes to be clear if one applies the so called optical theorem formalism for

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$T$(MeV) & 10 & 20 & 30 & 40 & 50 & 60 \\
\hline
$\rho = 0.5\rho_0$ & 468277 & 605757 & 157430 & 41569 & 12907 & 4713 \\
$\rho = \rho_0$ & 115894 & 131020 & 38552 & 11663 & 4149 & 1722 \\
$\rho = 2\rho_0$ & 25594 & 22432 & 8165 & 3044 & 1293 & 621 \\
\hline
\end{tabular}
\caption{Numerical results for the dimensionless integral $I(\tau)$ of Eq.(6).}
\end{table}
the calculation of the reaction rates, see discussion in the Appendix.

First, if the pion can be described within the quasiparticle approximation in some region of its energy and momentum, one has a contribution of the process $\pi^0 \to \nu \bar{\nu}$ whose rate is given by,

\[ Q^{\pi^0\nu\bar{\nu}} \sim \left( \frac{\Gamma(\pi^0 \to \nu \bar{\nu})}{m_\pi} \right) n_\pi m_\pi^* \]

\[ n_\pi = \frac{1}{\nu_\pi^3} \left( \frac{m_\pi^* k_B T}{2 \pi} \right)^{3/2} e^{-m_\pi^*/(k_B T)}. \]

Here we used the dispersion relation $\omega^2 = (m_\pi^*)^2 + v_\pi^2 k^2$ for small momenta $k \sim k_B T$ typical in this reaction. The effective mass $m_\pi^* \sim m_\pi$ and the velocity $v_\pi < 1$ are calculated according to eqs. (11) – (15). The effects of the partial $\Gamma(\pi^0 \to \gamma\gamma)$ width are not present in this process.

Second, the presence of finite width of the virtual pion $\text{Im} \Pi_{\pi^0 \neq 0}$ due to the strong interaction means also a finite contribution of the reaction $\pi^0_{\text{virt}} \to \nu \bar{\nu}$ that does not need $\gamma\gamma$ and pion quasiparticle states, but relates to the corresponding nucleon states, cf. [10, 11] (it is clearly seen after the cut of the in-medium pion Green function describing propagation of the in-medium pion). Presence of the imaginary part of the nucleon-hole term of the pion propagator (with the full vertex) in appropriate energy-momentum region would lead to a contribution of the processes $N \to N\nu\bar{\nu}$, $NN \to NN\nu\bar{\nu}$ going via the virtual pion, the later couples $N$ with $\nu\bar{\nu}$. The presence of the imaginary part of the regular term in the pion polarization operator corresponds to more involved multi-nucleon and multi-pion states.

Third, the processes as $\gamma\gamma \to \pi^0 \to nn^{-1} \to \nu_L \bar{\nu}_R$ ($n^{-1}$ is the neutron hole) with ordinary left-handed massless neutrinos are possible leading to substantially larger contribution to the emissivity than that related to the massive left-handed neutrinos in the reaction $\gamma\gamma \to \pi^0 \to \nu_L \bar{\nu}_R$. The reaction channel $\gamma\gamma \to \pi^0 \to nn^{-1} \to \nu_L \bar{\nu}_R$ may also lead to substantially larger contribution to the emissivity than that related to the right-handed neutrinos (depending on the value of the $\Gamma(\pi^0 \to \nu_{R\bar{L}})$) in
the reaction $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu_R\bar{\nu}_L$ which we have considered. Besides, the process $\pi^0 \rightarrow nn^{-1} \rightarrow \nu_L\bar{\nu}_R$ going on the left-handed massless neutrinos is possible going on the thermal pion quasiparticle. The latter rate can be calculated in complete analogy to that computed for the massive pseudo-Goldstone (photon) mode in [20]. The processes involving pion but going via $nn^{-1} - \nu\bar{\nu}$ coupling (with usual $V-A$ coupling of $N$ to $\nu_L\bar{\nu}_R$) exist only due to the medium modification of the vertices. Another possibility, which will be discussed below, is that due to the breaking of the Lorentz invariance in matter there might appear two pion decay coupling constants, the temporal one, $f_T$, and the space one, $f_S$,[21, 22]. Their finite difference would also lead to a contribution to the $\pi^0$ coupling with the left-handed neutrinos. Although all these relevant processes are related to each other, they are characterized by quite different phase-space volumes, kinematics and/or vertices. Having selected the $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$ channel among others, we considered the enhancement of the rate due to the particular kinematics of this process (massless particles in initial states). But, as we have shown, the presence of not as small pion width in the hadron matter significantly suppresses the rate.

Concluding this section we once again stress that although the temperature dependence of the energy output in the process $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$ is quite different from those for the other processes, where also enters the $\pi\nu\bar{\nu}$ coupling, the absolute value of the rate is proved to be strongly suppressed due to the presence of not as small pion width in the dense and heated nucleon matter. Thus, due to the width effects, the process has a minor role in the cooling of the dense hadron matter.
3 Emissivity of the process $\gamma\gamma \to \pi^0 \to \nu\bar{\nu}$ from the color-flavor-locked phase

There is another interesting possibility in connection with the process under consideration. As has been recently shown, the interiors of the most massive neutron stars may contain dense quark cores which are high temperature color superconductors with the critical temperature $T_c \simeq 0.6\Delta_q \lesssim 50$ MeV, where $\Delta_q$ is the pairing gap between colored quarks, see the review papers [23, 24]. Also the possibility of self-bounded strange quark stars, being the diquark condensates, is not excluded, see [25]. The diquark condensates may exist in different phases. The most symmetric phase of dense quark matter is the so called the color-flavor-locked phase. This phase becomes to be energetically preferable in the large density limit. The neutrino processes are significantly suppressed in this phase due to presence of the large diquark gap and absence of the electrons [26, 27]. On the other hand, it was shown, cf. [28, 24], that this phase contains low-lying excitations with the pion, kaon and $\eta, \eta'$ quantum numbers ($m^*_\pi, m^*_K, m^*_\eta, m^*_{\eta'}$ are in the range $(1-100)$ MeV, all in-medium masses are indicated by $m^*$). Excitation spectra, as they are calculated in the mentioned works, contain no widths effects, the $i$-meson Green function is given by $D_i = \frac{1}{(\omega^2 - k^2/3 - m_i^*)}$. However, in any case there is a width contribution at least from the process $\pi^0 \rightarrow \gamma\gamma$. Then $-Im\Pi R \simeq \Gamma(\gamma\gamma \rightarrow \pi^0) \sim \Gamma^{vac}(\gamma\gamma \rightarrow \pi^0)$. Although more involved effects may also simulate the corresponding width terms, one may expect that the width effects are, nevertheless, rather suppressed due to the nature of the superconducting phase with the large gap. In this situation the pion pole mechanism could become an efficient cooling mechanism. Also other meson poles may essentially contribute.

For the process under consideration the emissivity of the superconducting
medium \(Q^{SC}\) is given by expressions similar to Eqs. (3) to (5). We take
\[
| D_{x_0}^R |^2 = \frac{1}{(\omega^2 - k^2/3 - m_{\pi^0}^2)^2 + (Im\Pi)^2} \tag{18}
\]
and replace it into (1) instead of (2). Other quantities in (1) are assumed to be unchanged. Then the energy loss rate is presented as
\[
Q^{SC} \simeq 4.2 \times 10^{22} \frac{\Gamma(\pi^0 \to \gamma \gamma)}{m_{\pi}} \left( \frac{\Gamma(\pi^0 \to \nu \bar{\nu})}{m_{\pi}} \right) T_9^{11} \left( \frac{m_{\pi}}{m_{\pi}^*} \right)^4 I^{SC}(\bar{\tau}), \quad \text{erg} \, \text{cm}^{-3} \cdot \text{s}^{-1} \tag{19}
\]
now with the extra factor \((m_{\pi}^*/m_{\pi})^4\) and the integral \(I(\tau)\) replaced by
\[
I^{SC}(\bar{\tau}) = \int_0^\infty \frac{x_1^4 \, dx_1}{e^{x_1} - 1} \int_0^\infty \frac{x_2^4 \, dx_2}{e^{x_2} - 1} \times \int_0^\kappa \frac{du(x_1 + x_2)u^3}{(2\tau^2|\bar{\tau}|^2 - (x_1 + x_2)^2 + x_1 x_2 u)^2 + (m_{\pi}^*/2Im\Pi_{\pi^0}^R)^2}, \tag{20}
\]
\(\bar{\tau} = T/m_{\pi}^*\). Again we further take \(\kappa = 0\). For \(\bar{\tau} \ll 0.1\) we get
\[
I^{SC}(\tau \ll 0.1) \simeq 23040 \zeta(6)\zeta(5) \left[ 1 + 92\tau^2 \left( \frac{7\zeta(8)}{23\zeta(6)} + \frac{\zeta(7)}{\zeta(5)} \right) \right], \tag{21}
\]
that not essentially differs from the estimation given by Eq.(7), \(\zeta(8) = 1.004\). The value (21) is independent on \(m_{\pi}^*\) for \(\tau \ll 0.1\), and \(Q^{SC} \propto T^{11}(m_{\pi}^*)^{-4}\), \(Q^{SC}/Q^{vac} \sim (m_{\pi}/m_{\pi}^*)^4\), cf. (8). For \([m_{\pi}^2/(-10^2Im\Pi_R)]^{1/4} \geq \bar{\tau} \geq 1/a\) we roughly estimate \(I^{SC} \sim 0.1|\bar{\tau} - m_{\pi}^*^{2/2Im\Pi_R}|^{-1/4} e^{-1/(2\tau)}|\ln(2\bar{\tau})| + 1\) \(e^{-1/(2\tau)}\), \(a \sim 10^2\) for \(-Im\Pi_R \sim \Gamma^{vac}\), and \(Q^{SC} \propto T^3(m_{\pi}^*)^6 e^{-m_{\pi}^*/(2T)} / (-Im\Pi_R)\). For still higher temperatures \(I^{SC} \simeq 8/\bar{\tau}^4\) and \(Q^{SC} \propto T^7\) being almost independent on \(m_{\pi}^*\), \(Q^{SC} \simeq 1.3 \cdot 10^{29}(m_{\pi}^{-1}\Gamma(\pi^0 \to \nu \bar{\nu})) T_9^7\, \text{erg} / (\text{cm}^3 \cdot \text{s})\). Interpolation estimation being roughly valid for all temperatures produces
\[
I^{SC} \simeq \frac{24299(1 + 120\bar{\tau}^2 e^{-10\bar{\tau}})}{1 + 2880\bar{\tau}^4} + \frac{0.1(e^{-1/(2\bar{\tau})}|\ln(2\bar{\tau})| + 1)e^{-1/(2\bar{\tau})}}{\bar{\tau}^8|\bar{\tau} - m_{\pi}^*^{2/2Im\Pi_R}|}. \tag{22}
\]
The emissivity \((Q^{SC})\) computed with the above interpolation formula (Eq.(22)) and for the total pion width given by the vacuum one is shown in Fig.2. The value \(\Gamma(\pi^0 \to \nu \bar{\nu})\) is assumed to be equal to its experimental upper limit. This figure
serves as a guide for the numerical calculation of Eq.(20), showing the very fast increase of the emissivity as we approach temperatures near the pion pole one and demonstrating a slow increase as we go to temperatures above the effective pion mass scale. The interpolation formula and the full numerical calculation of the emissivity agree with high accuracy for small $T$ (compared to $m_\pi^*$) and differ by a factor $\lesssim 2$ for $T \approx m_\pi^*$.

In Figs. 3, 4 we show the numerically calculated emissivity of the process under consideration for the superconducting media, $Q^{SC}$, in the temperature range from 5 to 50 MeV for $m_\pi^* = 10$ MeV and $m_\pi^* = 50$ MeV, respectively. These curves were obtained assuming $\Gamma(\pi^0 \rightarrow \nu \bar{\nu})$ equal to its experimental upper limit. In the given temperature interval (from 5 to 50 MeV) the emissivity curves are scaled up for $m_\pi^* = 50$ MeV compared to $m_\pi^* = 10$ MeV in accordance with above estimations of $I^{SC}$ and interpolation equation (22). The pole asymptotic manifests itself even in the case when the width is rather suppressed (solid curves). It happens for $T < (15 \div 20)$ MeV in case $m_\pi^* = 10$ MeV and in the whole temperature interval (from 5 to 50 MeV) in case $m_\pi^* = 50$ MeV. For $m_\pi^* = 10$ MeV solid, dash and dotted curves reach the high temperature asymptotic for $T > 20$ MeV. For $m_\pi^* = 50$ MeV the solid curve achieves the high temperature asymptotic for $T > 50$ MeV. The smaller is the width the more pronounced is the pole mechanism. This is clearly seen if we compare Fig.2 (the result of the interpolation formula computed for the vacuum width) with Figs. 3, 4 where we assumed that more involved effects may simulate a width larger than the vacuum one.

Compared to the hadron case, in the color-flavor-locked phase of the superconducting quark medium the effective pion mass is much smaller and the width is suppressed. In this case the resonance effect appears with a larger amplitude (due to very small width) and at smaller temperatures. Thus we argue that for the star core in the color-flavor-locked phase at $T < T^{SC}_{opac}$, i.e. beyond the neutrino opacity regime, in absence of any other efficient cooling reaction channels the pion pole
mechanism could become an efficient cooling mechanism even if $\Gamma(\pi^0 \to \nu \bar{\nu})$ being
by many orders of magnitude below the value determined by the modern experimental upper limit. Here the value of $T_{^{\text{SC}}_{\text{opac}}}$ is different from that for the usual proto-neutron star being determined by the condition that the neutrino mean free path in the quark core becomes to be comparable with the size of the core. In the neutrino opacity regime, $T > T_{^{\text{SC}}_{\text{opac}}}$, the reaction $\pi^0 \to \nu \bar{\nu}$ delays the neutrino transport from the quark core to the hadron shell. To come to these conclusions we used only that the effective pion mass in the CFL phase is essentially smaller than the vacuum one, the pion width is small and the pion momentum is shifted by $k \to k/\sqrt{3}$. In some relevant temperature interval for the width $\Gamma_{\pi} \sim \Gamma_{\pi}^{\text{vac}}$ and for not too small effective pion mass the rate $Q_{^{\text{SC}}}$ may even exceed the emissivity of the most efficient direct Urca process being $Q_{^{\text{DU}}} \sim 10^{39}(T/10 \text{ MeV})^6 \text{ erg}/(\text{cm}^3 \cdot \text{s})$ for the non-superfluid (hadron) neutron star matter. Also at very low temperatures the emissivity of the process decreases with the temperature according to the power low ($Q_{^{\text{SC}}} \propto T^{11}$) rather than exponentially as other relevant processes in the CFL phase. Therefore in spite of a low absolute value of the rate at such low temperatures, the process is the dominating process.

Now we may compare the emissivity of the process $\gamma\gamma \to \pi^0 \to \nu \bar{\nu}$ with the estimation for the emissivity (17) of the process $\pi^0 \to \nu \bar{\nu}$ in the quark matter discussed in [26, 21, 22]. The emissivity of the later process is given by Eq.(17) with $v_{\pi}^2 = 1/3$ for the color-flavor-locked phase. We see that the ratio (19) to (17) is larger than unit in a wide temperature interval of our interest, $k_B T \lesssim 7 \text{ MeV}$ for $m_{\pi}^* \sim 70 \text{ MeV}$ and $k_B T \lesssim 0.2 \text{ MeV}$ for $m_{\pi}^* \sim 10 \text{ MeV}$. In this estimation we again used that the total pion width is almost exhausted by the $\gamma\gamma$ decay, $\Gamma(\gamma\gamma \to \pi^0) \simeq -\text{Im}\Pi^R$ and, as before, we took $\Gamma(\gamma\gamma \to \pi^0) \simeq \Gamma_{\pi}^{\text{vac}}$. If the width $\Gamma_{\pi}$ were larger as the consequence of the presence of some processes which were not considered up to now, then the pole contribution could be more suppressed (see dotted, dash and solid curves in Figs 3, 4) and the process $\gamma\gamma \to \pi^0 \to \nu \bar{\nu}$ would be relevant only for
low temperatures $T \lesssim (0.1 \div 1) \text{MeV}$.

Notice that the ratio of the reaction rates for $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$ and $\pi^0 \rightarrow \nu\bar{\nu}$ does not depend on the value of $\Gamma(\pi^0 \rightarrow \nu\bar{\nu})$ and it always becomes larger than unit with the decrease of the temperature (however the critical temperature when the rate reaches the unit depends on the values of the parameters).

Refs [21, 22] found an interesting possibility that in the color-flavor-locked phase the process $\pi^0 \rightarrow \nu\bar{\nu}$ is allowed also in the standard model with left-handed neutrinos even if the neutrino mass is zero. The important point that was noticed is that the temporal and thermal components of the pion decay constant need not be the same in the dense medium due to the violation of the Lorentz invariance. The calculation of [21] yields

$$
\Gamma(\pi^0 \rightarrow \nu\bar{\nu}) = \frac{1}{4\pi v_\pi^2} G_F^2 (f_T - f_S)^2 k_B T m^*_\pi m_\pi \\
\simeq 0.6 \cdot 10^{-20} \left( \frac{\mu_q}{10 \text{MeV}} \right)^2 \frac{m^*_\pi}{10 \text{MeV}} \left( \frac{k_B T}{10 \text{MeV}} \right) m_\pi,
$$

(23)

$\mu_q$ is the quark chemical potential. We see that for $m^*_\pi \sim k_B T \sim 10 \text{MeV}$ the value (23) is by three-four orders of magnitude smaller than the value of the experimental upper limit for $\Gamma(\pi^0 \rightarrow \nu\bar{\nu})$ for the right-handed neutrinos. Therefore both possibilities should be studied.

Ref. [21] also evaluated the value $\Gamma(\gamma\gamma \rightarrow \pi^0)$ for the corresponding decay in the color-flavor-locked phase. This estimation only by the factor $\sim 1$ differs from the value $\Gamma(\gamma\gamma \rightarrow \pi^0) \simeq \Gamma^\text{vac}_\pi$ which we used in our estimations.

4 Conclusion

In conclusion, (i) we evaluated the cooling rate of the proto-neutron stars via the pion pole mechanism taking into account the nuclear medium effects and we showed that with the width effects included this mechanism gives no constraints on the corresponding $\pi^0 \rightarrow \nu_R\bar{\nu}_L(\nu_L\bar{\nu}_R)$ decay width. (ii) We also discussed possible con-
tribution of this mechanism to the cooling of different astrophysical systems, as is the case of neutron stars with dense quark cores. We found that the process $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$ is proved to be the most efficient process in the color-flavor-locked superconducting core at temperatures $T \lesssim (0.1 \div 10)$ MeV depending on the effective pion mass and the pion decay width. Depending on what the future tells us about $\Gamma(\pi^0 \rightarrow \nu\bar{\nu})$ and on the possibility of the color superconductivity in neutron stars we believe that these questions deserve a further more detailed study (perhaps even further studying of a dependence on medium effects of all the terms appearing in the cross section of $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu_R\bar{\nu}_L(\nu_L\bar{\nu}_R)$).

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A Optical theorem formalism

In [29] and then in [30], see [11] for a review, there was developed the optical theorem formalism in terms of full non-equilibrium Green functions to calculate the reaction rates including finite particle widths and other in-medium effects. Applying this approach, e.g., to the antineutrino–neutrino production we can express the transition probability in a direct reaction in terms of the evolution operator $S$,

$$
\frac{dW_{X \rightarrow \bar{\nu}\nu}^{tot}}{dt} = \frac{d\Omega^3_{\bar{\nu}}d\Omega^3_\nu}{(2\pi)^6} \frac{\omega_{\nu}\omega_{\bar{\nu}}}{4} \sum_{\{X\}} \left< 0 \left| S^\dagger |\bar{\nu}\nu + X> <\bar{\nu}\nu + X| S |0> \right. \right>,
$$

(24)

where we presented explicitly the phase-space volume of $\bar{\nu}\nu$ states; lepton occupations of given spin are put equal to zero for $\nu$ and $\bar{\nu}$ which are supposed to be radiated.
directly from the system. The bar denotes statistical averaging. The summation goes over the complete set of all possible intermediate states \( \{X\} \) constrained by the energy-momentum conservation. Making use of the smallness of the weak coupling, we expand the evolution operator as 

\[
S \approx 1 - i \int_{-\infty}^{+\infty} T \{ V_{W}(x) S_{\text{nucl}}(x) \} dx_0,
\]

where \( V_{W} \) is the Hamiltonian of the weak interaction in the interaction representation, \( S_{\text{nucl}} \) is the part of the \( S \) matrix corresponding to the nuclear interaction, and \( T\{...\} \) is the chronological ordering operator. After substitution into (24) and averaging over the arbitrary non-equilibrium state of a nuclear system, there appear chronologically ordered \( (G^{-+}) \), anti-chronologically ordered \( (G^{++}) \) and disordered \( (G^{+-} \) and \( G^{-+}) \) exact Green functions.

In graphical form the general expression for the probability of the neutrino and anti-neutrino production is as follows

\[
\nu \quad \bar{\nu} \quad \nu \quad \bar{\nu}
\]

representing the sum of all closed diagrams \( (-i\Pi^{-+}) \) containing at least one \( (-+) \) exact Green function. The latter quantity is especially important. Various contributions from \( \{X\} \) can be classified according to the number \( N \) of \( G^{-+} \) lines in the diagram

\[
\frac{dW_{\nu\bar{\nu}}^{\text{tot}}}{dt} = \frac{d^3q_\nu d^3q_{\bar{\nu}}}{(2\pi)^6 4 \omega_\nu \omega_{\bar{\nu}}} \left( \nu \quad \bar{\nu} \quad \nu \quad \bar{\nu} \quad \nu \quad \bar{\nu} \quad \nu \quad \bar{\nu} \quad \nu \right).
\]

Being expressed in terms of the exact Green functions, each diagram in (26) represents a whole class of perturbative diagrams of any order in the interaction strength and in the number of loops. This procedure suggested in [29] is actually very helpful especially if the quasiparticle approximation holds for the intermediate fermions.
or bosons. Then contributions of specific processes contained in a closed diagram can be made visible by cutting the diagrams over the (+−), (−+) lines. In the framework of the quasiparticle approximation for the non-relativistic fermions

\[ G_F^{+} = 2\pi i \nu_F \delta(\varepsilon + \mu - \varepsilon^0 - \text{Re} \Sigma^R(\varepsilon + \mu, \vec{p})) \] (\(n_F\) are fermionic occupations, for equilibrium \(n_F = 1/[\exp((\varepsilon - \mu_F)/T) + 1]\)), and the cut eliminating the energy integral thus requires clear physical meaning. In this way one establishes the correspondence between closed diagrams and usual Feynman amplitudes although in the general case of finite fermion width the cut has only a symbolic meaning. Next advantage is that in the quasiparticle approximation any extra \(G_F^{+}\), since it is proportional to \(n_F\), brings a small \((T/\varepsilon_F)^2\) factor to the emissivity of the process. Dealing with small temperatures one can restrict by the diagrams of the lowest order in \((G_F^{+}G_F^{-})\), not forbidden by energy-momentum conservations, putting \(T = 0\) in all \(G^{++}\) and \(G^{−−}\) Green functions. For the relativistic bosons with the non-conserving number, in the quasiparticle approximation \(G_B^{+} = -2\pi i \nu_B \delta(\omega^2 - m_B^2 - k^2 - \text{Re} \Pi^R(\omega, k))\) (\(n_B\) are bosonic occupations, for equilibrium \(n_B = 1/[\exp(\omega/T) - 1]\)). In a wide temperature and energy-momentum region \(\text{Im} \Pi^R\) is not small and the quasiparticle approximation is then not valid for the boson Green functions whereas the region, where \(\text{Im} \Sigma^R\) is small and the quasiparticle approximation is valid for the fermion Green functions, is usually much wider.

If one is interested only in the processes related to the \(\nu \bar{\nu}\) coupling with the \(\pi^0\), the hatched block in (25) is reduced to the exact \(D_{\pi^0}^{-+}\) Green function. This Green function satisfies the exact Dyson equation. Only if in a specific region of the pion energies and momenta (\(\omega\) and \(k\)) the pion Green function \(D_{\pi^0}^{-+}\) can be approximated by the \(\delta\)-function, integrating over this region one may use the quasiparticle approximation for the pion. In such a way one usually calculates the rate of the \(\pi^0 \rightarrow \nu \bar{\nu}\) process. Certainly, in other regions of pion energies and momenta the pion Green function contains the width relating to different channels of the pion decay. E.g. the polarization operator of the \(\pi^0\) contains the ”−+”
γγ loop. This term corresponds to the contribution $D_{\pi^0} G_{\gamma^+} G_{\gamma^-} D_{\pi^0}$. Within the quasiparticle approximation to the γ one cuts the $G_{\gamma^+} G_{\gamma^-}$ lines and gets in this way the contribution of the $\gamma\gamma \rightarrow \pi^0 \rightarrow \nu\bar{\nu}$ process which we discuss in this paper. When considering right-handed neutrinos we do not know the explicit expression for the $\pi^0 \rightarrow \nu\bar{\nu}$ vertex. Several different expressions can be used. Therefore we will express the result of the integration in the $\nu\bar{\nu}$ states in (24) via the phenomenological value of the width $\Gamma(\pi^0 \rightarrow \nu\bar{\nu})$ for which there exists the experimental upper limit.

If we knew the coupling we could present an explicit calculation, as one usually does for the left-handed neutrinos.

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Figure 2: Emissivity $Q^{SC}$ in erg/(cm$^3 \cdot$ s) computed with the interpolation formula of Eq.(22) for $m^*_\pi = 10$ and 50 MeV. The curves were obtained assuming the total width $\Gamma_\pi \simeq \Gamma_\pi^{vac}$. 
Figure 3: Emissivity $Q^{SC}$ in $erg/(cm^3 \cdot s)$ as function of temperature for $-Im\Pi^R/m^*_\pi \simeq 1$ MeV (solid line), $-Im\Pi^R/m^*_\pi \simeq 0.1$ MeV (dashed line) and $-Im\Pi^R/m^*_\pi \simeq 0.01$ MeV (dotted line) for $m^*_\pi = 10$ MeV.
Figure 4: The same as in Fig. 4 but for \( m^* \pi = 50 \text{ MeV} \).