FACTORS OF HYPERCONTRACTIONS

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Abstract. In this article, we study a class of contractive factors of \( m \)-hypercontractions for \( m \in \mathbb{N} \). We find a characterization of such factors and this is achieved by finding explicit dilation of these factors on some weighted Bergman spaces. This is a generalization of the work done in [14].

1. Introduction

The structure of a commuting \( n \)-tuple of isometries (\( n \geq 2 \)) is complicated compare to that of a single isometry due to von Neumann and Wold (cf. [19]). Not much is known except the BCL representation for an \( n \)-tuple of isometries with product being a pure isometry (see [6, 7, 8, 9, 12, 16, 17] and references therein), that is for an \( n \)-tuple of isometries \( (V_1, \ldots, V_n) \) on \( \mathcal{H} \) with
\[
\bigcap_{k \geq 0} V_1^k V_2^k \cdots V_n^k \mathcal{H} = \{0\}.
\]
The structure theorem of such isometries also reveals all possible isometric factors of a pure isometry [9]. Following this, the analysis of finding factors has been extended further to the case of contractions, recently. A characterization of contractive factors of a pure contraction is obtained, by Sarkar, Sarkar and the second author of this article, in [14] and subsequently in [21] for general contractions. More specifically, it is shown that for a contraction \( T \) on a Hilbert space \( \mathcal{H} \), the following are equivalent:

(i) \( T = T_1 T_2 \) for some commuting contractions \( T_1 \) and \( T_2 \) on \( \mathcal{H} \);

(ii) there exist a triple \((\mathcal{E}, U, P)\) consisting of a Hilbert space \( \mathcal{E} \), a unitary \( U \) and an orthogonal projection \( P \), a pair of commuting unitaries \((W_1, W_2)\) on a Hilbert space \( \mathcal{R} \) with \( W = W_1 W_2 \) and a joint \((M_{z^*} \oplus W^*, M_{z^*} \oplus W_1^*, M_{z^*} \oplus W_2^*)\)-invariant subspace \( \mathcal{Q} \) of \( H^2_\mathbb{D}(\mathcal{D}) \oplus \mathcal{R} \) such that
\[
T_1 \cong P_{\mathcal{Q}} (M_{\Phi} \oplus W_1) |_{\mathcal{Q}}, \quad T_2 \cong P_{\mathcal{Q}} (M_{\Psi} \oplus W_2) |_{\mathcal{Q}}, \quad T \cong P_{\mathcal{Q}} (M_z \oplus W) |_{\mathcal{Q}}
\]
where \( \Phi(z) = (P + z P^\perp) U^* \) and \( \Psi(z) = U (P^\perp + z P) \) for all \( z \) in the unit disc \( \mathbb{D} \).

Moreover, \((M_{\Phi} \oplus W_1) (M_{\Psi} \oplus W_2) = (M_{\Psi} \oplus W_2) (M_{\Phi} \oplus W_1) = M_z \oplus W \). In the case of a pure contraction \( T \), the Hilbert space \( \mathcal{R} = \{0\} \) and therefore all the direct summands disappear. It is also worth mentioning here that the key to obtain such a characterization is an explicit Ando type dilation result and it is motivated by a recent technique of finding explicit dilation found

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in [13]. It is then natural to ask the following question: How to characterize contractive factors of \(m\)-hypercontractions? In this article, we answer this question and obtained a complete description for a class of contractive factors of \(m\)-hypercontractions. Our characterization for contractive factors of \(m\)-hypercontractions induces a similar characterization of factors for subnormal operators and, for \(m = 1\), it recovers the characterization obtained in [14] and [21]. To describe these results, we develop some background materials next.

For a Hilbert space \(E\) and \(n \in \mathbb{N}\), the \(E\)-valued weighted Bergman space over the unit disc, denoted by \(A^2\), is defined as
\[
A^2_{n}(E) = \{ f \in \mathcal{O}(D, E) : f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k, \| f \|^2_n = \sum_{k=0}^{\infty} (w_{n,k})^{-1}\| \hat{f}(k) \|_E^2 < \infty \},
\]
where the sequence of weights \(\{w_{n,k}\}_{k \geq 0}\) is given by
\[
(1 - x)^{-n} = \sum_{k=0}^{\infty} w_{n,k}x^k, \quad (|x| < 1).
\]
It is also a reproducing kernel Hilbert space with kernel
\[
K_n(z, w) = (1 - z\bar{w})^{-n}I_E \quad (z, w \in D).
\]
For the base case \(n = 1\), the space \(A^2_1(E)\) is known as the Hardy space over the unit disc which we denote by \(H^2(E)\) and denote the corresponding kernel, known as the Szegö kernel, by
\[
S(z, w) = (1 - z\bar{w})^{-1}I_E \quad (z, w \in D).
\]
If \(E = \mathbb{C}\), then we denote simply by \(A^2\) the \(\mathbb{C}\)-valued weighted Bergman space over the unit disc. The notion of \(m\)-hypercontractions \((m \in \mathbb{N})\), introduced by Agler in his seminal paper [2], is defined as follows. A bounded linear operator \(T\) on \(H\) is an \(m\)-hypercontraction if it satisfies
\[
K_n^{-1}(T, T^*) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^k T^{*k} \geq 0,
\]
for \(n = 1, m\). In addition, if \(T^{*n} \to 0\) in the strong operator topology then \(T\) is said to be a pure \(m\)-hypercontraction. It is important to note that the positivity \(K_n^{-1}(T, T^*) \geq 0\) for \(n = 1, m\) also implies all the intermediate positivity, that is \(K_n^{-1}(T, T^*) \geq 0\) for all \(n = 1, \ldots, m\) ([18]). This shows that if \(T\) is an \(m\)-hypercontraction then it is also an \(n\)-hypercontraction for all \(n = 1, \ldots, m\). The defect operators and the defect spaces of an \(m\)-hypercontraction \(T\) on \(H\) are defined by
\[
D_{n,T} = \left( K_n^{-1}(T, T^*) \right)^{\frac{1}{2}} \quad \text{and} \quad D_{n,T} = \overline{\text{ran}} D_{n,T}, \quad (1 \leq n \leq m)
\]
respectively. The Bergman shift \(M_z\) on \(A^2_m(E)\), defined by
\[
(Mzf)(w) = w f(w) \quad (f \in A^2_m(E), w \in D),
\]
is a pure \(m\)-hypercontraction. In fact, by [2], the Bergman shifts are model of pure \(m\)-hypercontractions. To be more precise, Agler proves the following characterization result.
Theorem 1.1. (cf. [2]) If $T$ is an $m$-hypercontraction on a Hilbert space $H$ then
\[ T \cong P_Q(M_z \oplus W)|_Q, \]
where $W$ is a unitary on a Hilbert space $\mathcal{R}$, $Q$ is a $(M^*_z \oplus W^*)$-invariant subspace of $A^2_m(\mathcal{D}_{m,T}) \oplus \mathcal{R}$ and $\mathcal{D}_{m,T}$ is the defect space of $T$ as in (1.1). In addition, if $T$ is pure then the Hilbert space $\mathcal{R} = \{0\}$.

There are now several different approaches to this result and to its multivariable generalization for different domains in $\mathbb{C}^n$ (see [1], [3], [4], [10], [11], [18], and [20]).

Now coming back to the context of this article, we denote by $F_m(H)$ the class of contractive factors of $m$-hypercontractions on a Hilbert space $H$ which we characterize in this paper. The class is defined as follows.

Definition 1.2. For $m \in \mathbb{N}$ and a Hilbert space $H$, a pair of operators $(T_1, T_2)$ is said to be an element of $F_m(H)$ if

(i) $T_1$ and $T_2$ are commuting contractions, and

(ii) for all $i = 1, 2$, $K^{-1}_{m-1}(T, T^*) - T_i K^{-1}_{m-1}(T, T^*) T_i^* \geq 0$ where $T = T_1 T_2$ and $K_0(T, T^*) = I_H$.

The positivity condition in (ii) is equivalent to the Szegö positivity of the commuting operator tuple
\[ \mathcal{T}_i = (T, \ldots, T, T_i) \]
for all $i = 1, 2$. Here for an $n$-tuple of commuting contraction $\mathcal{T} = (T_1, \ldots, T_n)$, the Szegö positivity of $\mathcal{T}$ is defined as
\[ S_n^{-1}(\mathcal{T}, \mathcal{T}^*) = \sum_{F \subset \{1, \ldots, n\}} (-1)^{|F|} \mathcal{T}_F \mathcal{T}_F^*, \]
where for $F \subset \{1, \ldots, n\}$, $\mathcal{T}_F = \prod_{i \in F} T_i$. For $m = 1$, the condition (ii) follows from (i). For that reason, $F_1(H)$ is the class of all commuting contractive operator pairs on $H$. For $(T_1, T_2) \in F_m(H)$ we show that their product contraction $T = T_1 T_2$ is an $m$-hypercontraction on $H$. In other words, for any $m \in \mathbb{N}$, $F_m(H)$ contains contractive factors of $m$-hypercontractions on $H$. In particular, this also provides a sufficient condition for the product of a pair of commuting contractions $(T_1, T_2)$ on $H$ to be an $m$-hypercontraction and the sufficient condition is simply that $(T_1, T_2) \in F_m(H)$. This sufficient condition is not necessary as we find counterexamples. The goal of this article is to describe the class of contractive factors $F_m(H)$ of $m$-hypercontractions, completely. One such explicit descriptions we obtain is as follows. For a Hilbert space $\mathcal{E}$, a bounded analytic function $\Phi : \mathbb{D} \to \mathcal{B}(\mathcal{E})$ is a $\mathcal{B}(\mathcal{E})$-valued Schur function on $\mathbb{D}$ if
\[ \sup_{z \in \mathbb{D}} \|\Phi(z)\| \leq 1. \]

If $T$ is a $m$-hypercontraction on a Hilbert space $H$, then the following are equivalent:

(i) $T = T_1 T_2$ for some $(T_1, T_2) \in F_m(H)$;
(ii) there exist a pair of commuting unitaries \((W_1, W_2)\) on a Hilbert space \(\mathcal{R}\) with \(W = W_1 W_2\) and a pair of \(\mathcal{B}(\mathcal{E})\)-valued Schur functions on \(\mathbb{D}\)

\[ \Phi(z) = (P + z P^\perp)U^* , \quad \Psi(z) = U(P^\perp + z P) \quad (z \in \mathbb{D}) \]

\(\Phi\) corresponding to a triple \((\mathcal{E}, U, P)\) consisting of a Hilbert space \(\mathcal{E}\), a unitary \(U\) on \(\mathcal{E}\) and an orthogonal projection \(P\) in \(\mathcal{B}(\mathcal{E})\) such that \(Q\) is a joint \((M_z^* \oplus W^*, M_{\Phi}^* \oplus W_1^*, M_{\Psi}^* \oplus W_2^*)\)-invariant subspace of \(A_m^2(\mathcal{E}) \oplus \mathcal{R}\) and

\[ T_1 \cong P_Q(M_{\Phi} \oplus W_1)|_Q, \quad T_2 \cong P_Q(M_{\Psi} \oplus W_2)|_Q, \quad T \cong P_Q(M_z \oplus W)|_Q. \]

Furthermore, if \(T\) is a pure \(m\)-hypercontraction then the Hilbert space \(\mathcal{R} = \{0\}\).

This in turn provides a similar factorization result for subnormal operators. The above factorization result is obtained by finding a suitable and explicit dilation of commuting contractive operator triples, of the form \((T_1, T_2, T_1 T_2)\) for \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\), on some weighted Bergman space. At the same time, the explicit dilation of triples relies on a Douglas type dilation of \(m\)-hypercontractions and a commutant lifting technique originally found in [14].

The plan of the paper is as follows. Section 2 contains Douglas type dilation for \(m\)-hypercontractions. We study different properties of \(\mathcal{F}_m(\mathcal{H})\) in Section 3. In Section 4, we find a suitable explicit dilation for the class of factors in \(\mathcal{F}_m(\mathcal{H})\). This is then used to obtain several factorization results in Section 5. In the last section, we find examples of factors of \(m\)-hypercontractions on \(\mathcal{H}\) which are not an element of \(\mathcal{F}_m(\mathcal{H})\).

2. Douglas Type Dilation for Hypercontractions

In this section, we find a Douglas type dilation and therefore the model for \(m\)-hypercontractions as in Theorem 1.1 which is required to obtain dilation of factors of hypercontractions. Our explicit construction of Douglas type dilation for \(m\)-hypercontractions seems to be new. We believe that this may be known to experts in the area. But, we include the construction of such explicit dilation for completeness.

Recall that a contraction \(T\) on \(\mathcal{H}\) is a \(m\)-hypercontraction if for all \(n = 1, \ldots, m,\)

\[ K_n^{-1}(T, T^*) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^k T^{*k} \geq 0. \]

Also for all \(n = 1, \ldots, m,\) \(n\)-th order defect operator and defect space are

\[ D_{n,T} = K_n^{-1}(T, T^*)^{1/2} \quad \text{and} \quad \mathcal{D}_{n,T} = \overline{\text{ran}} D_{n,T}, \]

respectively. The sequence of weights \(\{w_{n,k}\}_{k=0}^{\infty}\) given by

\[ (1 - x)^{-n} = \sum_{k=0}^{\infty} w_{n,k} x^k, \quad (|x| < 1, n \in \mathbb{N} \cup \{0\}) \]

play a crucial role in what follows and we invoke a lemma from [2] which describe certain relationship of these weights for different values of \(n\).

**Lemma 2.1** (cf [2]). Let \(\{w_{n,k}\}_{k \geq 0, n \geq 0}\) be as above. Then for all \(n, k \geq 1,\)

\[ w_{n,k} - w_{n,k-1} = w_{n-1,k}. \]
For a fixed $1 \leq n \leq m$, consider the orthonormal basis $\{\psi_{n,k}(z) = \sqrt{w_{n,k}}z^k\}_{k=0}^{\infty}$ for the weighted Bergman space $A^2_n$. Then the kernel function of $A^2_n$ is given by

$$K_n(z,w) = (1 - zw)^{-n} = \sum_{k=0}^{\infty} \psi_{n,k}(w)\overline{\psi_{n,k}(z)} \quad (z,w \in \mathbb{D}).$$

We set, for $r \geq 0$,

$$f^{(n)}_r(z,w) := \sum_{k=r}^{\infty} \psi_{n,k}(z)K^{-1}_n(z,w)\overline{\psi_{n,k}(w)} \quad (z,w \in \mathbb{D}).$$

Then it can be easily seen that $f^{(n)}_0 \equiv 1$ and

$$f^{(n)}_r(z,w) = 1 - \sum_{k=0}^{r-1} \psi_{n,k}(z)K^{-1}_n(z,w)\overline{\psi_{n,k}(w)}, \quad (r \geq 1)$$

and consequently, $f^{(n)}_r$ is a polynomial for all $r \geq 0$. As a result, using polynomial calculus, we define

$$f^{(n)}_r(T,T^*) := 1 - \sum_{k=0}^{r-1} w_{n,k}T^kK^{-1}_n(T,T^*)T^{*k}, \quad (r \geq 0, 1 \leq n \leq m)$$

for any $m$-hypercontraction $T$ on $\mathcal{H}$. These operators are used to study the canonical dilation map $\Pi_{m,T} : \mathcal{H} \to A^2_m(D_T)$ defined by

$$(\Pi_{m,T}h)(z) = D_{m,T}(I_{\mathcal{H}} - zT^*)^{-m}h, \quad (h \in \mathcal{H}, z \in \mathbb{D})$$

for any $m$-hypercontraction $T$ on $\mathcal{H}$. The next proposition shows that the operator $\Pi_{m,T}$ is a contraction and it is analogous to Proposition 10 in [4] for the case when $T$ is a pure $m$-hypercontraction.

**Proposition 2.2.** In the above setting, we have the following:

(i) For any $1 \leq n \leq m$, the sequence $\{f^{(n)}_r(T,T^*)\}_{r=0}^{\infty}$ is a decreasing sequence of positive operators.

(ii) $\|\Pi_{m,T}h\|^2 = \|h\|^2 - \lim_{r \to \infty} \langle f^{(m)}_r(T,T^*)h, h \rangle \quad (h \in \mathcal{H}).$
that for $r \geq 0$ and $1 \leq n \leq m$,

$$f_r^{(n)}(T, T^*) = 1 - \sum_{k=0}^{r-1} w_{n,k} T^k K_n^{-1}(T, T^*) T^{*k}$$

$$= 1 - \sum_{k=0}^{r-1} w_{n,k} T^k \left( K_n^{-1}(T, T^*) - T K_n^{-1}(T, T^*) T^* \right) T^{*k}$$

$$= 1 - w_{n,0} K_n^{-1}(T, T^*) - \sum_{k=1}^{r-1} (w_{n,k} - w_{n,k-1}) T^k K_n^{-1}(T, T^*) T^{*k}$$

$$+ w_{n,r-1} T^r K_n^{-1}(T, T^*) T^{*r}$$

(2.2)

$$f_r^{(n-1)}(T, T^*) = w_{n,r-1} T^r K_n^{-1}(T, T^*) T^{*r} + w_{n,r-1} T^r K_n^{-1}(T, T^*) T^{*r}.$$ 

Since $w_{n,r-1} T^r K_n^{-1}(T, T^*) T^{*r} \geq 0$, we conclude that $f_r^{(n)}(T, T^*) \geq f_r^{(n-1)}(T, T^*)$ for all $r \geq 0$ and for all $n = 1, \ldots, m$. As a result, we also have

$$f_r^{(n)}(T, T^*) \geq f_r^{(n-1)}(T, T^*) \geq \cdots \geq f_r^{(1)}(T, T^*) = T^r T^{*r} \geq 0.$$ 

This proves that $\{f_r^{(n)}(T, T^*)\}_{r=0}^{\infty}$ is a decreasing sequence of positive operators. The proof of (ii) is verbatim with the proof of Proposition 10 in [4].

By the above result, we denote the strong operator limit of the sequence $\{f_r^{(n)}(T, T^*)\}_{r=0}^{\infty}$ and its range as

$$Q_{n,T}^2 := \text{SOT} \quad q \lim_{r \to \infty} f_r^{(n)}(T, T^*) \quad Q_{n,T} = \overline{\text{ran}Q_{n,T}} \quad (1 \leq n \leq m).$$

It should be noted that if $T$ is a pure $m$-hypercontraction then

$$\text{SOT} \quad q \lim_{r \to \infty} f_r^{(n)}(T, T^*) = \text{SOT} \quad q \lim_{r \to \infty} f_r^{(m-1)}(T, T^*) = \cdots = \text{SOT} \quad q \lim_{r \to \infty} f_r^{(1)}(T, T^*) = 0.$$ 

This can derived from the identity (2.2) and from the fact that $w_{n,r-1} T^r K_n^{-1}(T, T^*) T^{*r} \to 0$ in the strong operator topology (see Lemma 2.11 in [2]). Thus the canonical dilation map $\Pi_{m,T}$ is an isometry if and only if $T$ is a pure $m$-hypercontraction. The intertwining property of $\Pi_{m,T}$, that is $\Pi_{m,T} T^* = M_z^* \Pi_{m,T}$ where $M_z$ is the shift on $A_m^2(D_{m,T})$, is evident from the definition of $\Pi_{m,T}$.

Before we present the main theorem of this section, we recall a well-known factorization result due to Douglas.

**Lemma 2.3.** (cf. [12]) Let $A$ and $B$ be two bounded linear operators on a Hilbert space $\mathcal{H}$. Then there exists a contraction $C$ on $\mathcal{H}$ such that $A = BC$ if and only if

$$AA^* \leq BB^*.$$ 

The explicit construction of Douglas type dilation for $m$-hypercontractions is given next.

**Theorem 2.4.** If $T \in \mathcal{B}(\mathcal{H})$ is an $m$-hypercontraction, then there exist a Hilbert space $\mathcal{R}$, an isometry $\Pi_T : \mathcal{H} \to A_m^2(D_{m,T}) \oplus \mathcal{R}$ and a unitary $W$ on $\mathcal{R}$ such that

$$\Pi_T T^* = (M_z^* \oplus W^*) \Pi_T.$$ 

In particular,
\[ T \cong P_Q(M_z \oplus W)|_Q, \]
where \( Q = \operatorname{ran} \Pi_T \) is the \((M_z \oplus W)^*\)-invariant subspace of \( A^2(D_{m,T}) \oplus R \).

**Proof.** Let \( Q_{n,T} \) be the positive operator as in (2.3) for all \( 1 \leq n \leq m \). By induction on \( n \), we prove that
\[ TQ_{n,T}^2 T^* = Q_{n,T}^2 \hspace{1mm} (n = 1, \ldots, m). \]
It is easy to see that it holds for \( n = 1 \). Then we assume that the identity holds for some \( n \) with \( 1 \leq n < m \). Thus by the assumption \( f_r^{(n)}(T, T^*) - T f_r^{(n)}(T, T^*) T^* \to 0 \) in the strong operator topology as \( r \to \infty \). Now,
\[
\begin{align*}
&f_{r+1}^{(n+1)}(T, T^*) - T f_r^{(n+1)}(T, T^*) T^* \\
&= I - TT^* - K_{n+1}^{-1}(T, T^*) + \sum_{k=0}^{r-1} (w_{n+1,k} - w_{n+1,k+1}) T^{k+1} K_{n+1}^{-1}(T, T^*) T^{*(k+1)} \\
&= I - TT^* - K_{n+1}^{-1}(T, T^*) - \sum_{k=0}^{r-1} w_{n,k} T^{k+1} K_{n+1}^{-1}(T, T^*) T^{*(k+1)} \\
&= I - TT^* - (K_n^{-1}(T, T^*) - T K_n^{-1}(T, T^*) T^*) \\
&\hspace{2cm} - \sum_{k=0}^{r-1} w_{n,k} T^{k+1} (K_n^{-1}(T, T^*) - T K_n^{-1}(T, T^*) T^*) T^{*(k+1)} \\
&= I - \sum_{k=0}^{r} w_{n,k} T^{k} K_n^{-1}(T, T^*) T^{*k} - (TT^* - \sum_{k=0}^{r} w_{n,k} T^{k+1} K_n^{-1}(T, T^*) T^{*(k+1)}) \\
&= f_{r+1}^{(n)}(T, T^*) - T f_{r+1}^{(n)}(T, T^*) T^*.
\end{align*}
\]
Consequently by the induction hypothesis, \( f_{r+1}^{(n+1)}(T, T^*) - T f_{r+1}^{(n+1)}(T, T^*) T^* \to 0 \) in the strong operator topology as \( r \to \infty \). This in turn implies that
\[ TQ_{n+1,T}^2 T^* = Q_{n+1,T}^2. \]
Thus we have proved that \( TQ_{n,T}^2 T^* = Q_{n,T}^2 \) for all \( n = 1, \ldots, m \). In particular since \( TQ_{m,T}^2 T^* = Q_{m,T}^2 \), by Lemma 2.3 there exists an isometry \( X^* \) on \( Q_{m,T} \) such that
\[ (2.4) \hspace{1mm} X^*Q_{m,T} = Q_{m,T}T^*. \]
Let \( W^* \) on \( R \supseteq Q^{(m)} \) be the minimal unitary extension of \( X^* \). Then, by Proposition 2.2, the map \( \Pi_T : H \to A^2_m(D_{m,T}) \oplus R \) defined by
\[ \Pi_T h = (\Pi_{m,T} h, Q_{m,T} h), \hspace{1mm} (h \in H) \]
is an isometry and it also satisfies
\[ \Pi_T T^* = (M_z \oplus W)^* \Pi_T. \]
Here the intertwining property follows from (2.4). Therefore, $Q = \text{ran} \Pi_T$ is a $(M_z \oplus W)^*$-invariant subspace of $A^2_m(D_{m,T}) \oplus \mathcal{R}$ and we have

$$T \cong P_Q(M_z \oplus W)|Q.$$  

This completes the proof. 

\section{The class $\mathcal{F}_m(\mathcal{H})$}

The class of contractive factors $\mathcal{F}_m(\mathcal{H})$ and its basic properties are studied in this section. To begin with we recall the definition of the class $\mathcal{F}_m(\mathcal{H})$. A commuting pair of contractions $(T_1, T_2)$ on $\mathcal{H}$ is an element of $\mathcal{F}_m(\mathcal{H})$ if $K_{m-1,1}^{-1}(T, T^*) - T_i K_{m-1,1}^{-1}(T, T^*) T_i^* \geq 0$ for all $i = 1, 2$ where $T = T_1 T_2$.

For $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$ with $T = T_1 T_2$, we fix the following notations for the rest of the article:

\begin{equation}
D_{n,T,T_i}^2 = K_{m-1,1}^{-1}(T, T^*) - T_i K_{m-1,1}^{-1}(T, T^*) T_i^* \quad \text{and} \quad D_{n,T,T_i} = \text{ran} D_{n,T,T_i}^2 \quad (n \in \mathbb{N}, i = 1, 2).
\end{equation}

With the above notation, we have the following useful identity

\begin{align}
D_{n,T,T_i}^2 - T D_{n,T,T_i}^2 T^* &= K_{m-1,1}^{-1}(T, T^*) - T_i K_{m-1,1}^{-1}(T, T^*) T_i^* - T_i (K_{m-1,1}^{-1}(T, T^*) - T K_{m-1,1}^{-1}(T, T^*) T_i^*) T_i^* \\
&= K_{m-1,1}^{-1}(T, T^*) - T_i K_{m-1,1}^{-1}(T, T^*) T_i^* \\
&= D_{n+1,T,T_i}^2
\end{align}

for all $n \geq 0$. Next we show an intermediate positivity type result.

\begin{lemma}
If $(T_1, T_2) \in \mathcal{F}_m(\mathcal{H})$ then $(T_1, T_2) \in \mathcal{F}_n(\mathcal{H})$ for all $1 \leq n \leq m$.
\end{lemma}

\begin{proof}
It is enough to show that $D_{n,T,T_i}^2 \geq 0$ for all $n = 1, \ldots, m$ and for all $i = 1, 2$. We only consider the case $i = 1$ as it is symmetrical for $i = 2$. By the hypothesis $D_{m,T,T_1}^2 \geq 0$ and $D_{1,T,T_1}^2 \geq 0$. To show $D_{(m-1),T,T_1}^2 \geq 0$, we assume $m \geq 2$ and consider the sequence $\{a_r\}_{r=0}^\infty$, corresponding to a fixed $h \in \mathcal{H}$, defined as

$$a_r = \langle T^r D_{(m-1),T,T_1}^2 T^{*r} h, h \rangle \quad (r \geq 0).$$

Then for any $r \geq 0$, using (3.6), we have

$$a_r - a_{r+1} = \langle T^r(D_{(m-1),T,T_1} - T D_{(m-1),T,T_1}^2 T^{*r} h, h) \rangle = \langle T^r D_{m,T,T_1}^2 T^{*r} h, h \rangle \geq 0.$$

Thus $\{a_r\}_{r=0}^\infty$ is a decreasing sequence. Also since

$$\left| \sum_{r=0}^N a_r \right| = \left| \sum_{r=0}^N T^r(D_{(m-2),T,T_1}^2 - T D_{(m-2),T,T_1}^2) T^{*r} h, h \right|$$

$$= \left| \sum_{r=0}^N (D_{(m-2),T,T_1}^2 - T D_{(m-2),T,T_1}^2) T^{*(r+1)} h, h \right|$$

$$\leq 2\|h\|^2 \|D_{(m-2),T,T_1}^2\|,$$
If \( T_1, T_2 \in \mathcal{F}_m(\mathcal{H}) \), then \( T_1T_2 \) is an \( m \)-hypercontraction.

**Proof.** Let \( T = T_1T_2 \). The proof is obvious for \( m = 1 \). For \( m \geq 2 \) note that

\[
K_m^{-1}(T, T^*) = K_m^{-1}(T, T^*) - TK_m^{-1}(T, T^*)T^* \\
= (K_m^{-1}(T, T^*) - T_1K_m^{-1}(T, T^*)T_1^*) + T_1(K_m^{-1}(T, T^*) - T_2K_m^{-1}(T, T^*)T_2^*)T_1^* \\
\geq 0.
\]

This completes the proof. ■

The converse of this lemma is not true as we find counterexamples in last section of this article. This suggests that \( \mathcal{F}_m(\mathcal{H}) \) does not contain all the factors of \( m \)-hypercontractions. Before going further, we consider elementary examples of elements in \( \mathcal{F}_m(\mathcal{H}) \). These examples are based on a triple \((\mathcal{E}, U, P)\) consists of a Hilbert space \( \mathcal{E} \), a unitary operator \( U \) on \( \mathcal{E} \) and an orthogonal projection \( P \) in \( \mathcal{B}(\mathcal{E}) \). For such a triple, the \( \mathcal{B}(\mathcal{E}) \)-valued analytic functions

\[
\Phi(z) = (P + zP^1)U^*, \quad \text{and} \quad \Psi(z) = U(P^1 + zP) \quad (z \in \mathbb{D})
\]

are easily seen to be Schur functions on \( \mathbb{D} \), that is they are in the unit ball of the Banach algebra \( H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}) \) consists of bounded \( \mathcal{B}(\mathcal{E}) \)-valued analytic functions on \( \mathbb{D} \). It is easy to see that

\[
\Phi(z)\Psi(z) = \Psi(z)\Phi(z) = zI_{\mathcal{E}} \quad (z \in \mathbb{D}).
\]

We refer to \( \Phi, \Psi \) as **canonical pair of Schur functions** on \( \mathbb{D} \) corresponding to the triple \((\mathcal{E}, U, P)\). We claim that the commuting pair of multiplication operators \((M_\Phi, M_\Psi)\) on \( A_m^2(\mathcal{E}) \) is an element of \( \mathcal{F}_m(A_m^2(\mathcal{E})) \). Indeed, if \( \mathcal{E}_1 = \text{ran}P \) and \( \mathcal{E}_2 = \text{ran}P^\perp \) then \( \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \). With respect to the above decomposition of the co-efficient space, we have \( A_m^2(\mathcal{E}) = A_m^2(\mathcal{E}_1) \oplus A_m^2(\mathcal{E}_2) \) and

\[
\begin{align*}
K_{m-1}(M_z, M_z^*) - M_\Psi K_{m-1}(M_z, M_z^*)M_\Phi^* \\
= K_{m-1}(M_z, M_z^*) - \begin{bmatrix} I_{A_m^2(\mathcal{E}_1)} & 0 \\ 0 & M_z \otimes I_{\mathcal{E}_2} \end{bmatrix} (I \otimes U^*)K_{m-1}(M_z, M_z^*) (I \otimes U) \begin{bmatrix} I_{A_m^2(\mathcal{E}_1)} & 0 \\ 0 & M_z^* \otimes I_{\mathcal{E}_2} \end{bmatrix} \\
= K_{m-1}(M_z, M_z^*) - \begin{bmatrix} I_{A_m^2(\mathcal{E}_1)} & 0 \\ 0 & M_z \otimes I_{\mathcal{E}_2} \end{bmatrix} K_{m-1}(M_z, M_z^*) \begin{bmatrix} I_{A_m^2(\mathcal{E}_1)} & 0 \\ 0 & M_z^* \otimes I_{\mathcal{E}_2} \end{bmatrix} \\
= \begin{bmatrix} 0 & K_m(M_z \otimes I_{\mathcal{E}_2}, M_z^* \otimes I_{\mathcal{E}_2}) \end{bmatrix} \geq 0,
\end{align*}
\]

as \( M_z \otimes I_{\mathcal{E}_2} \) on \( A_m^2(\mathcal{E}_2) \) is an \( m \)-hypercontraction. Similarly, we have

\[
K_{m-1}(M_z, M_z^*) - M_\Psi K_{m-1}(M_z, M_z^*)M_\Phi \geq 0.
\]
This proves the claim. In fact we will see below that any pair \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\) with \(T_1 T_2\) is pure dilates to such a pair \((M_\mathcal{F}, M_\mathcal{H})\) on some \(A^2_m(\mathcal{E})\), and therefore they serve as a model for a class of factors of pure \(m\)-hypercontractions.

4. Dilation of factors

Our main concern is to propose a model for the class \(\mathcal{F}_m(\mathcal{H})\) of factors of \(m\)-hypercontractions. This is achieved by finding an explicit dilation of a triple of commuting contractions \((T_1, T_2, T_1 T_2)\) on some weighted Bergman space, where \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\). We say an \(n\)-tuple of commuting contractions \((T_1, \ldots, T_n)\) on \(\mathcal{H}\) dilates to a commuting \(n\)-tuple of operators \((R_1, \ldots, R_n)\) on \(\mathcal{K}\) if there is an isometry \(\Pi : \mathcal{H} \to \mathcal{K}\), such that

\[
\Pi S^*_i = R^*_i \Pi \quad (i = 1, \ldots, n).
\]

The map \(\Pi\) is often refer as the dilation map.

We prove a lemma which will be the key to the dilation results obtained in this section. This is analogous to Theorem 2.1 in [14]. Let \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\). Since \(T = T_1 T_2\) is an \(m\)-hypercontraction, recall the canonical dilation map \(\Pi_{m,T} : \mathcal{H} \to A^2_m(\mathcal{D}_{m,T})\) defined by

\[
(\Pi_{m,T} h)(z) = D_{m,T}(I - z T^*)^{-m} h \quad (h \in \mathcal{H}, z \in \mathbb{D}),
\]

such that \(\Pi_{m,T} T^* = M^*_z \Pi_{m,T}\). If \(V : \mathcal{D}_{m,T} \to \mathcal{E}\) is an isometry for some Hilbert space \(\mathcal{E}\), then the map

\[
\Pi_V := (I \otimes V) \Pi_{m,T} : \mathcal{H} \to A^2_m(\mathcal{E})
\]

also intertwines with \(T^*\) and \(M^*_z\) on \(A^2_m(\mathcal{E})\), that is \(\Pi_V T^* = M^*_z \Pi_V\).

**Lemma 4.1.** With the above notation, if \(\mathcal{D}\) is a Hilbert space and if

\[
U_i = \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} : \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,i}) \to \mathcal{E} \oplus (\mathcal{D} \oplus \mathcal{D}_{m,T,i}) \quad (i = 1, 2)
\]

is a unitary operator such that for all \(h \in \mathcal{H},\)

\[
U_i(VD_{m,T} h, 0_{\mathcal{D}}, D_{m,T,i} T^* h) = (VD_{m,T} T^*_i h, 0_{\mathcal{D}}, D_{m,T,i} h), \quad (i = 1, 2)
\]

then the \(\mathcal{B}(\mathcal{E})\)-valued Schur function \(\Phi_i(z) = A^*_i + z C^*_i B^*_i \quad (z \in \mathbb{D})\), transfer function corresponding to the unitary \(U^*_i\), satisfies

\[
\Pi_V T^*_i = M^*_i \Pi_V,
\]

for all \(i = 1, 2\).

**Proof.** Since

\[
\begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} \begin{bmatrix} VD_{m,T} h \\ (0, D_{m,T,i} T^* h) \end{bmatrix} = \begin{bmatrix} VD_{m,T} T^*_i h \\ (0, D_{m,T,i} h) \end{bmatrix}, \quad (h \in \mathcal{H}, i = 1, 2)
\]

we have

\[
A_i VD_{m,T} h + B_i (0, D_{m,T,i} T^* h) = VD_{m,T} T^*_i h, \quad C_i VD_{m,T} h = (0, D_{m,T,i} h),
\]

for all \(h \in \mathcal{H}\) and \(i = 1, 2\). Simplifying further, we get

\[
VD_{m,T} T^*_i = A_i VD_{m,T} + B_i C_i VD_{m,T} T^*
\]
Therefore, we get \( \Pi V \) defined by
\[ \langle M_\Phi^i, \Pi_V h, z^n \eta \rangle = \langle (I \otimes V) D_{m,T}(1 - zT^*)^{-n} h, (A_i^* + zC_i^*B_i^*) (z^n \eta) \rangle \]
\[ = \langle (A_i V D_{m,T} + B_i C_i V D_{m,T} T^*) T^n h, \eta \rangle \]
\[ = \langle V D_{m,T} T_i^* (T^n h), \eta \rangle \]
\[ = \langle \Pi_V T_i^* h, z^n \eta \rangle, \quad (i = 1, 2). \]

This leads us to define isometries
\[ U : \{ D_{m,T,T_1} h \oplus D_{m,T,T_2} T_1^* h : h \in \mathcal{H} \} \to \{ D_{m,T,T_2} h \oplus D_{m,T,T_1} T_2^* h : h \in \mathcal{H} \} \]

\begin{equation}
U(D_{m,T,T_2} h, D_{m,T,T_1} T_1^* h) = (D_{m,T,T_2} h, D_{m,T,T_1} T_1^* h), \quad (h \in \mathcal{H})
\end{equation}

and \( V : D_{m,T} \to D_{m,T,T_1} \oplus D_{m,T,T_2} \) defined by
\begin{equation}
V(D_{m,T} h) = (D_{m,T,T_1} h, D_{m,T,T_2} T_1^* h) \quad (h \in \mathcal{H}).
\end{equation}

We are now ready to prove the explicit dilation result for the pure case.

**Theorem 4.2.** Let \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\) be such that \(T = T_1 T_2\) is a pure contraction. Then there exist a triple \((\mathcal{E}, U, P)\) consists of a Hilbert space \(\mathcal{E}\), a unitary \(U\) on \(\mathcal{E}\) and a projection \(P\) in \(\mathcal{B} (\mathcal{E})\) and an isometry \(\Pi : \mathcal{H} \to A_m^2(\mathcal{E})\) such that
\[ \Pi T_1^* = M_\Phi^1 \Pi, \quad \Pi T_2^* = M_\Psi^1 \Pi, \quad \text{and} \quad \Pi T^* = M_z^\ast \Pi \]
where \(\Phi\) and \(\Psi\) are the \(\mathcal{B}(\mathcal{E})\)-valued canonical Schur functions on \(\mathbb{D}\) corresponding to the triple \((\mathcal{E}, U, P)\) given by
\[ \Phi(z) = (P + zP^\perp) U^* \quad \text{and} \quad \Psi(z) = U(P^\perp + zP) \]
for all \(z \in \mathbb{D}\).

In particular, \(Q = \text{ran}\Pi\) is a joint \((M_\Phi^1, M_\Psi^1, M_z^\ast)\)-invariant subspace of \(A_m^2(\mathcal{E})\) such that
\[ T_1 \cong P_Q M_\Phi |_Q, \quad T_2 \cong P_Q M_\Psi |_Q \quad \text{and} \quad T \cong P_Q M_z |_Q. \]

**Proof.** We first consider the isometry \(U\) as in (4.7) and by adding an infinite dimensional Hilbert space \(\mathcal{D}\), if necessary, we extend it to a unitary on \(\mathcal{E} := (\mathcal{D} \oplus D_{m,T,T_1}) \oplus D_{m,T,T_2}\). We continue to denote the unitary by \(U\), and therefore we have a unitary \(U : \mathcal{E} \to \mathcal{E}\) which satisfies
\[ U(0_D, D_{m,T,T_1} T_2^* h, D_{m,T,T_2} h) = (0_D, D_{m,T,T_1} h, D_{m,T,T_2} T_1^* h), \quad (h \in \mathcal{H}). \]
Also we view the isometry $V$ in (4.8), as an isometry $V : D_{m,T} \to E$ defined by

$$V(D_{m,T}h) = (0_D, D_{m,T,T_1}h, D_{m,T,T_2}T_1^*h) \quad (h \in H).$$

Since $T$ is a pure $m$-hypercontraction, then the canonical dilation map $\Pi_{m,T} : H \to A_m^2(D_{m,T})$ is an isometry, and as a result

$$\Pi_V = (I \otimes V)\Pi_{m,T} : H \to A_m^2(E)$$

is also an isometry. The isometry $\Pi_V$ will be the dilation map in this context.

To complete the proof of the theorem, we construct unitaries which satisfy the hypothesis of Lemma 4.1. To this end, we consider the inclusion maps $\iota_1 : D \oplus D_{m,T,T_1} \to E$ and $\iota_2 : D_{m,T,T_2} \to E$ defined by

$$\iota_1(h, k_1) = (h, k_1, 0) \quad \text{and} \quad \iota_2(k_2) = (0, 0, k_2), \quad (h \in D, k_1 \in D_{m,T,T_1}, k_2 \in D_{m,T,T_2}).$$

We also consider the orthogonal projection $P = \iota_2\iota_1^*$. Then it is easy to check that

$$\begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} : E \oplus (D \oplus D_{m,T,T_1}) \to E \oplus (D \oplus D_{m,T,T_1}),$$

and

$$\begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} : E \oplus D_{m,T,T_2} \to E \oplus D_{m,T,T_2}$$

are unitary. The unitary

$$U_1 := \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} = \begin{bmatrix} UP & U\iota_1 \\ \iota_1^* & 0 \end{bmatrix} : E \oplus (D \oplus D_{m,T,T_1}) \to E \oplus (D \oplus D_{m,T,T_1}),$$

satisfies

$$U_1 \begin{bmatrix} VD_{m,T}h \\ D_{m,T,T_1}T_1^*h \end{bmatrix} = \begin{bmatrix} UP & U\iota_1 \\ \iota_1^* & 0 \end{bmatrix} \begin{bmatrix} VD_{m,T}h \\ D_{m,T,T_1}T_1^*h \end{bmatrix} = \begin{bmatrix} U(0_D, D_{m,T,T_1}T_2^*T_1^*h, D_{m,T,T_2}T_1^*h), D_{m,T,T_1}h) \\ (0_D, D_{m,T,T_1}h) \end{bmatrix}$$

$$= \begin{bmatrix} (0_D, D_{m,T,T_1}T_2^*T_1^*h, D_{m,T,T_2}T_1^*h), D_{m,T,T_1}h) \\ (0_D, D_{m,T,T_1}h) \end{bmatrix} = \begin{bmatrix} VD_{m,T}T_1^*h \\ (0_D, D_{m,T,T_1}h) \end{bmatrix},$$

for all $h \in H$. Subsequently, a similar computation also shows that the unitary

$$U_2 := \begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} : E \oplus D_{m,T,T_2} \to E \oplus D_{m,T,T_2},$$

satisfies

$$U_2(VD_{m,T}h, D_{m,T,T_2}T^*h) = (VD_{m,T}T_2^*h, D_{m,T,T_2}h)$$

for all $h \in H$. The proof now follows by appealing Lemma 4.1 for the unitaries $U_1$ and $U_2$. ■
Remark 4.3. The converse of the above theorem is also true. That is, if \((T_1, T_2, T)\) is a triple of commuting contractions on \(\mathcal{H}\) and if \((T_1, T_2, T)\) dilates to \((M_\Phi, M_\Psi, M_z)\) on \(A_m^2(\mathcal{E})\) for some Hilbert space \(\mathcal{E}\) where \(\Phi\) and \(\Psi\) are \(\mathcal{B}(\mathcal{E})\)-valued canonical Schur functions on \(\mathbb{D}\) corresponding to a triple \((\mathcal{E}, U, P)\), then \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\) and \(T = T_1T_2\). This follows immediately from the fact that \((M_\Phi, M_\Psi, M_z) \in \mathcal{F}_m(A_m^2(\mathcal{E}))\).

Having obtained the explicit dilation for the pure case, we now drop the pure assumption and find dilation for the general case.

Theorem 4.4. Let \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\) with \(T = T_1T_2\). Then there exist a triple \((\mathcal{E}, U, P)\) consists of a Hilbert space \(\mathcal{E}\), a unitary \(U\) on \(\mathcal{E}\) and an orthogonal projection \(P\) in \(\mathcal{B}(\mathcal{H})\), a Hilbert space \(\mathcal{R}\), a pair of commuting unitaries \((W_1, W_2)\) on a Hilbert space \(\mathcal{R}\) with \(W = W_1W_2\) and an isometry \(\Pi : \mathcal{H} \rightarrow A_m^2(\mathcal{E})\) such that

\[
\Pi T_1^* = (M_\Phi \oplus W_1)^*\Pi, \quad \Pi T_2^* = (M_\Psi \oplus W_2)^*\Pi \quad \text{and} \quad \Pi T^* = (M_z \oplus W)^*\Pi
\]

where \(\Phi\) and \(\Psi\) are the \(\mathcal{B}(\mathcal{E})\)-valued canonical Schur function on \(\mathbb{D}\) corresponding to the triple \((\mathcal{E}, U, P)\) given by

\[
\Phi(z) = (P + zP^\perp)U^* \quad \text{and} \quad \Psi(z) = U(P^\perp + zP)
\]

for all \(z \in \mathbb{D}\).

In particular, \(\mathcal{Q} = \text{ran}\Pi\) is a joint \((M_\Phi^* \oplus W^*, M_\Psi^* \oplus W_1^*, M_z^* \oplus W_2^*)\)-invariant subspace of \(A_m^2(\mathcal{E}) \oplus \mathcal{R}\) such that

\[
T_1 \cong P_\mathcal{Q}(M_\Phi \oplus W_1)|_\mathcal{Q}, \quad T_2 \cong P_\mathcal{Q}(M_\Psi \oplus W_2)|_\mathcal{Q} \quad \text{and} \quad T \cong P_\mathcal{Q}(M_z \oplus W)|_\mathcal{Q}.
\]

Proof. Let \((\mathcal{E}, U, P)\) be as in Theorem 4.2 and let \(V\) be as in 4.3. Then by the same way as it is done in the proof of Theorem 4.2 we have

\[
(4.10) \quad \Pi V T_1^* = M_\Phi^*\Pi V, \quad \Pi V T_2^* = M_\Psi^*\Pi V \quad \text{and} \quad \Pi V T^* = M_z^*\Pi V,
\]

where \(\Phi(z) = (P + zP^\perp)U^*\) and \(\Psi(z) = U(P^\perp + zP)\) for all \(z \in \mathbb{D}\), \(\Pi V = (I \otimes V)\Pi_{m,T}\) and \(\Pi_{m,T} : \mathcal{H} \rightarrow A_m^2(D_{m,T}), h \mapsto D_{m,T}(I - zT^*)^{-m}h\) is the canonical dilation map. However, note that \(\Pi V\) is not an isometry in general. To make it an isometry we follow the construction done in Theorem 2.4.

Let \(Q_{m,T}\) be the positive operator defined in (2.3) by taking strong operator limit of the decreasing sequence of positive operators \(\{f_r^{(m)}(T, T^*)\}_{r=0}^\infty\) where

\[
f_r^{(m)}(T, T^*) = 1 - \sum_{k=0}^{r-1} w_{m,k} T^k K^{-1}_m(T, T^*) T^* k \quad (r \geq 0).
\]

It also follows from the proof of Theorem 2.4 that

\[
TQ_{m,T}^2 T^* = Q_{m,T}^2.
\]

We claim here that \(Q_{m,T}^2 \geq T_i Q_{m,T} T_i^*\) for all \(i = 1, 2\). We prove the inequality for \(i = 1\) as the proof is similar for \(i = 2\). To this end, it is enough to show that

\[
f_r^{(m)}(T, T^*) - T_1 f_r^{(m)}(T, T^*) T_1^* \geq 0
\]
for all $r \geq 0$. For a fixed $r \geq 0$, we use induction on $m$ to establish it. Since $f_r^{(1)}(T, T^*) = T^r T^{*r}$, it is easy to see that the inequality holds for $m = 1$. We assume that for some $1 \leq n < m$, $f_r^{(n)}(T, T^*) - T_1 f_r^{(n)}(T, T^*) T_1^* \geq 0$. Then

$$f_r^{(n+1)}(T, T^*) - T_1 f_r^{(n+1)}(T, T^*) T_1^* = 1 - T_1 T_1^* + \sum_{k=0}^{r-1} w_{n+1,k} T^k (T_1 K_{n+1}^{-1}(T, T^*) T_1^* - K_{n+1}^{-1}(T, T^*)) T^{*k}$$

$$= Y_{n+1} - T_1 Y_{n+1} T_1^*,$$

where

$$Y_{n+1} = 1 - \sum_{k=0}^{r-1} w_{n+1,k} T^k (K_{n+1}^{-1}(T, T^*) - TK_{n+1}^{-1}(T, T^*)) T^{*k} \geq 0.$$
We conclude the section with a remark which is similar to the pure case.

**Remark 4.5.** The converse of the above theorem is also true. Naturally, this follows from the fact that \((M_\Phi \oplus W_1, M_\Psi \oplus W_2) \in \mathcal{F}_m(A^2_m(\mathcal{E}) \oplus \mathcal{R})\).

## 5. Factorization of Hypercontractions

Combining the dilation results, Theorem 4.2 and Theorem 4.4 obtained in the previous section with Remark 4.3 and Remark 4.5, we get the following immediate characterization of factors in the class \(\mathcal{F}_m(\mathcal{H})\).

**Theorem 5.1.** Let \((T_1, T_2)\) be a pair of contractions on \(\mathcal{H}\). Then the following are equivalent:

(i) \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\);

(ii) there exist a pair of commuting unitaries \((W_1, W_2)\) on a Hilbert space \(\mathcal{R}\) with \(W = W_1W_2\) and \(\mathcal{B}(\mathcal{E})\)-valued canonical Schur functions

\[
\Phi(z) = (P + zP^\perp)U^* \quad \text{and} \quad \Psi(z) = U(P^\perp + zP) \quad (z \in \mathbb{D})
\]

corresponding to a triple \((\mathcal{E}, U, P)\) consisting of a Hilbert space \(\mathcal{E}\), a unitary \(U\) and an orthogonal projection \(P\) in \(\mathcal{B}(\mathcal{E})\) such that \(\mathcal{Q}\) is a joint \((M^{*} \oplus W^*, M_{\Phi}^{*} \oplus W_{1}^{*}, M_{\Psi}^{*} \oplus W_{2}^{*})\)-invariant subspace of \(A^{2}_{m}(\mathcal{E}) \oplus \mathcal{R}\),

\[
T_1 \cong P_{\mathcal{Q}}(M_{\Phi} \oplus W_1)|_{\mathcal{Q}}, T_2 \cong P_{\mathcal{Q}}(M_{\Psi} \oplus W_2)|_{\mathcal{Q}}, \text{ and } T \cong P_{\mathcal{Q}}(M_z \oplus W)|_{\mathcal{Q}}.
\]

In particular, if \(T_1T_2\) is a pure contraction then the Hilbert space \(\mathcal{R} = \{0\}\).

It is now clear that the above theorem is obtained by realizing a factor \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\) on the dilation space \(A^2_m(\mathcal{E}) \oplus \mathcal{R}\) of \(T = T_1T_2\). However, one would expect to realize \((T_1, T_2)\) on the canonical dilation space of \(T\) as in Theorem 2.4.

To this end, we first consider \((T_1, T_2) \in \mathcal{F}_m(\mathcal{H})\) with \(T = T_1T_2\) is a pure contraction. Let \(\Pi_V\) be the dilation map as in Theorem 4.2 that is

\[
\Pi_V T_1^* = M^{*}_{\Phi} \Pi_V, \Pi_V T_2^* = M^{*}_{\Psi} \Pi_V \quad \text{and} \quad \Pi_V T^* = M^{*}_{z} \Pi_V
\]

and, by (4.9),

\[
\Pi_V = (I \otimes V)\Pi_{m,T},
\]

where \(\Pi_{m,T}\) is the isometric canonical dilation map corresponding to the pure \(m\)-hypercontraction \(T\) and \(V : \mathcal{D}_{m,T} \to \mathcal{E}\) is an isometry. Then, by the from of \(\Pi_V\), the above intertwining relations yield

\[
\Pi_{m,T} T_1^* = (I \otimes V^*)(M_{\Phi}^{*}(I \otimes V))\Pi_{m,T} \quad \text{and} \quad \Pi_{m,T} T_2^* = (I \otimes V^*)(M_{\Psi}^{*}(I \otimes V))\Pi_{m,T}.
\]

Set

\[
\bar{\Phi}(z) := V^{*}\Phi(z)V \quad \text{and} \quad \bar{\Psi}(z) := V^{*}\Psi(z)V, \quad (z \in \mathbb{D}).
\]

Then \(\bar{\Phi}\) and \(\bar{\Psi}\) are \(\mathcal{B}(\mathcal{D}_{m,T})\)-valued Schur functions on \(\mathbb{D}\) such that

\[
\Pi_{m,T} T_1^* = M^{*}_{\Phi} \Pi_{m,T}, \Pi_{m,T} T_2^* = M^{*}_{\Psi} \Pi_{m,T}.
\]
Theorem 5.2. Let $T$ be a pure $m$-hypercontraction on $\mathcal{H}$. Then the following are equivalent.

(i) $T = T_1T_2$ for some $(T_1, T_2) \in F_m(\mathcal{H})$;
(ii) there exist $\mathcal{B}(D_{m,T})$-valued Schur functions

$$\Phi(z) = V^*(P + zP^+)U^*V, \quad \text{and} \quad \Psi(z) = V^*U(P^+ + zP)V \quad (z \in \mathbb{D})$$

for some Hilbert space $\mathcal{E}$, isometry $V : D_{m,T} \to \mathcal{E}$, unitary $U : \mathcal{E} \to \mathcal{E}$ and projection $P$ in $\mathcal{B}(\mathcal{E})$ such that $Q$ is a joint $(M^*_\Phi \oplus W_1^*, M^*_\Psi \oplus W_2^*)$-invariant subspace of $A^2_m(D_{m,T}) \oplus \mathcal{R}$,

$$P_Q(M_z \oplus W)|Q = P_Q(M_{\Phi \tilde{\Phi}} \oplus W)|Q = P_Q(M_{\Psi \tilde{\Psi}} \oplus W)|Q,$$

and

$$T_1 \cong P_Q(M_\Phi |Q), \quad T_2 \cong P_Q(M_\Psi |Q).$$

We also have the following analogous result for general $m$-hypercontractions.

Theorem 5.3. Let $T$ be an $m$-hypercontraction on $\mathcal{H}$. Then the following are equivalent.

(i) $T = T_1T_2$ for some $(T_1, T_2) \in F_m(\mathcal{H})$;
(ii) there exist a commuting pair of unitaries $(W_1, W_2)$ on a Hilbert space $\mathcal{R}$ with $W = W_1W_2$ and $\mathcal{B}(D_{m,T})$-valued Schur functions

$$\Phi(z) = V^*(P + zP^+)U^*V, \quad \text{and} \quad \Psi(z) = V^*U(P^+ + zP)V \quad (z \in \mathbb{D})$$

for some Hilbert space $\mathcal{E}$, isometry $V : D_{m,T} \to \mathcal{E}$, unitary $U : \mathcal{E} \to \mathcal{E}$ and projection $P$ in $\mathcal{B}(\mathcal{E})$ such that $Q$ is a joint $(M^*_\Phi \oplus W_1^*, M^*_\Psi \oplus W_2^*)$-invariant subspace of $A^2_m(D_{m,T}) \oplus \mathcal{R}$,

$$P_Q(M_z \oplus W)|Q = P_Q(M_{\Phi \tilde{\Phi}} \oplus W)|Q = P_Q(M_{\Psi \tilde{\Psi}} \oplus W)|Q,$$

and

$$T_1 \cong P_Q(M_\Phi \oplus W_1)|Q, \quad T_2 \cong P_Q(M_\Psi \oplus W_2)|Q.$$
(ii) for each $m \in \mathbb{N}$, there exist a commuting pair of unitaries $(W_{1,m}, W_{2,m})$ on a Hilbert space $\mathcal{R}_m$ with $W_m = W_{1,m} W_{2,m}$ and $\mathcal{B}(\mathcal{D}_{m,T})$-valued Schur functions

$$\Phi_m(z) = V_m^*(P_m + zP_m^\perp)U_m^*V_m, \quad \text{and} \quad \Psi_m(z) = V_m^*U_m(P_m^\perp + zP_m)V_m \quad (z \in \mathbb{D})$$

for some Hilbert space $\mathcal{E}_m$, isometry $V_m : \mathcal{D}_{m,T} \rightarrow \mathcal{E}_m$, unitary $U_m : \mathcal{E}_m \rightarrow \mathcal{E}_m$ and projection $P_m$ in $\mathcal{B}(\mathcal{E}_m)$ such that $Q_m$ is a joint $(M_{\Phi_m}^* \oplus W_{1,m}^*, M_{\Psi_m}^* \oplus W_{2,m}^*)$-invariant subspace of $A_m^2(\mathcal{D}_{m,T}) \oplus \mathcal{R}_m$.

$$P_{\mathcal{Q}_m}(M_z \oplus W_m)|_{\mathcal{Q}_m} = P_{\mathcal{Q}_m}(M_{\Phi_m} \oplus \tilde{\Phi}_m \oplus W_m)|_{\mathcal{Q}_m} = P_{\mathcal{Q}_m}(M_{\Psi_m} \oplus \tilde{\Psi}_m \oplus W_m)|_{\mathcal{Q}_m},$$

and

$$T_1 \cong P_{\mathcal{Q}_m}(M_{\Phi_m} \oplus W_{1,m})|_{\mathcal{Q}_m}, \quad T_2 \cong P_{\mathcal{Q}_m}(M_{\Psi_m} \oplus W_{2,m})|_{\mathcal{Q}_m}.$$  

6. Examples and concluding remark

In this section, we find an example of a pair of commuting $2 \times 2$ contractive matrices such that their product is a 2-hypercontraction but the pair fails to belong in $\mathcal{F}_2(\mathbb{C}^2)$.

**Example:** For a real number $0 < r \leq 1$, consider a $2 \times 2$ matrix $T_r := \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$. Then by a direct calculation, it can be checked that $T_r$ is a 2-hypercontraction if and only if $r^2 \leq \frac{1}{2}$.

Also for strictly positive real numbers $a$ and $b$, consider the matrix $S = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Then $S$ is an invertible matrix and $S$ commutes with $T_r$ for any $r$. Thus, for $r \leq \frac{1}{\sqrt{2}}$, $T_r S^{-1}$ and $S$ are factors of the 2-hypercontraction $T_r$. On the other hand, again by a simple direct calculation, we have

$$K^{-1}_1(T_r, T_r^*) - SK^{-1}_1(T_r, T_r^*)S^* = \begin{bmatrix} (1 - r^2)(1 - a^2) - b^2 & -ab \\ -ab & 1 - a^2 \end{bmatrix}.$$  

Also note that $S$ is a contraction if and only if $b \leq 1 - a^2$. So for the particular choice $r = \frac{1}{\sqrt{2}}$, $a = \frac{1}{\sqrt{2}}$ and $b = \frac{1}{2}$, we see that $T_r$ is a 2-hypercontraction, $S$ and $T_r S^{-1}$ are contractions and

$$K^{-1}_1(T_r, T_r^*) - SK^{-1}_1(T_r, T_r^*)S^* = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is not a positive matrix. Therefore for such a particular choice, the contractions $T_r S^{-1}$ and $S$ are factors of the 2-hypercontraction $T_r$, but $(T_r S^{-1}, S) \notin \mathcal{F}_2(\mathbb{C}^2)$.

The above example shows that $\mathcal{F}_m(\mathcal{H})$ does not contain all the contractive factors of $m$-hypercontractions on $\mathcal{H}$ and the present article characterise a subclass of contractive factors of $m$-hypercontractions, namely $\mathcal{F}_m(\mathcal{H})$. We conclude the paper with the following natural question: How to characterize all the factors of $m$-hypercontractions?

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References

1. J. Agler, *The Arveson extension theorem and coanalytic models*, Integral Equations Operator Theory 5 (1982), 608-631.
2. J. Agler, *Hypercontractions and subnormality*, J. Operator Theory 13 (1985), 203-217.
3. C. Ambrozie, *Commutative dilation theory*, Operator Theory (2015), 1093-1124.
4. C. Ambrozie, M. Englis and V. Muller, *Operator tuples and analytic models over general domains in $\mathbb{C}^n$*, J. Operator Theory 47 (2002), no. 2, 287-302.
5. S. Barik, B.K. Das, K. Haria and J. Sarkar, *Isometric dilation and von Neumann inequality for a class of tuples in the polydisc*, Tran. Amer. Math. Soc. 372 (2019), 1429-1450.
6. H. Bercovici, R.G. Douglas and C. Foias, *Canonical models for bi-isometries*, A panorama of modern operator theory and related topics, 177-205, Oper. Theory Adv. Appl., 218, Birkhauser/Springer Basel AG, Basel, 2012.
7. H. Bercovici, R.G. Douglas and C. Foias, *Bi-isometries and commutant lifting*, Characteristic functions, scattering functions and transfer functions, 51-76, Oper. Theory Adv. Appl., 197, Birkhauser Verlag, Basel, 2010.
8. H. Bercovici, R.G. Douglas and C. Foias, *On the classification of multi-isometries*, Acta Sci. Math.(Szeged) 72 (2006), 639-661.
9. C.A. Berger, L.A. Coburn and A. Lebow, *Representation and index theory for $C^*$-algebras generated by commuting isometries*, J.Funct. Anal. 27 (1978) 4199.
10. R.E. Curto and F.H. Vasilescu, *Standard operator models in the polydisc*, Indiana Univ. Math. J. 42 (1993), no. 3, 791-810.
11. R.E. Curto and F.H. Vasilescu, *Standard operator models in the polydisc, II*, Indiana Univ. Math. J. 44 (1995), no. 3, 727-746.
12. B.K. Das, R. Debnath and J. Sarkar, *Commuting isometries and joint invariant subspace*, arXiv:1711.00769.
13. B.K. Das and J. Sarkar, *Ando dilations, von Neumann inequality, and distinguished varieties*, J. Funct. Anal. 272 (2017), 2114-2131.
14. B.K. Das, S. Sarkar and J. Sarkar, *Factorization of Contraction*, Advances in Mathematics 322 (2017), 186-200.
15. R.G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. 17 (1996), 413-415.
16. K. Guo and R. Yang, *The core function of submodules over the bidisk*, Indiana Univ. Math. J. 53 (2004), 205-222.
17. W. He, Y. Qin and R. Yang, *Numerical invariants for commuting isometric pairs*, Indiana Univ. Math. J. 64 (2015), 1-19.
18. V. Muller and F.-H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. 117 (1993), 979--989.
19. B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, Amsterdam, 1970.
20. A. Olofsson, *Parts of Adjoint weighted shifts*, J. Operator Theory 74 (2015), no. 2, 249-280.
21. Haripada Sau, *And dilations for a pair of commuting contractions: two explicit constructions and functional models*, arXiv:1710.11368.

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