Convergence of delay differential equations driven by fractional Brownian motion

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Abstract. In this note, we prove an existence and uniqueness result of solution for stochastic differential delay equations with hereditary drift driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. Then, we show that, when the delay goes to zero, the solutions to these equations converge, almost surely and in $L^p$, to the solution for the equation without delay. The stochastic integral with respect to the fractional Brownian motion is a pathwise Riemann–Stieltjes integral.

0. Introduction

Consider the stochastic differential equation on $\mathbb{R}^d$

$$
X^r(t) = \eta(0) + \int_0^t b(s, X^r) \, ds + \int_0^t \sigma(s, X^r(s-r)) \, dW^H_s, \quad t \in (0, T],
$$

$$
X^r(t) = \eta(t), \quad t \in [-r, 0].
$$

(1)

Here, $r$ denotes a strictly positive time delay, $W^H = \{W^H_j, j = 1, \ldots, m\}$ are independent fractional Brownian motions with Hurst parameter $H > 1/2$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $b(s, X^r)$, the hereditary term, depends on the path $\{X^r(u), -r \leq u \leq s\}$, while $\eta : [-r, 0] \rightarrow \mathbb{R}^d$ is a smooth function. We call (1) a delay differential equations with hereditary drift driven by a fractional Brownian motion, and to the best of our knowledge this problem has not been considered before in the wide literature on stochastic differential equations.

As usual in this field, we have to specify how we intend the stochastic integral in (1), being its definition not unique. Since $H > 1/2$, we can define the integral with respect to fractional Brownian motion using a pathwise approach. Indeed, if we have a stochastic processes $\{u(t), t \geq 0\}$ whose trajectories are $\lambda$-Hölder continuous with $\lambda > 1 - H$, then the Riemann–Stieltjes integral $\int_0^T u(s) \, dW^H_s$ exists for each trajectory (see Young [9]). Using the techniques introduced by Young [9] and the $p$-variation norm, Lyons [4] began the study of integral equations driven by functions with bounded
p-variation, with $p \in [1, 2)$. Then, Zahle [10] introduced a generalized Stieltjes integral using the techniques of fractional calculus. The integral is expressed in terms of fractional derivative operators, and it coincides with the Riemann–Stieltjes integral $\int_0^T f \, dg$ when the functions $f$ and $g$ are Hölder continuous of orders $\lambda$ and $\beta$, respectively, with $\lambda + \beta > 1$. Using this Riemann–Stieltjes integral, Nualart and Rascanu [7] obtained the existence and uniqueness of solution for a class of integral equations without delay, and they also proved that the solution is bounded on a finite interval.

In our paper, using also the Riemann–Stieltjes integral, we will first prove the existence and uniqueness of a solution to Eq. (1), extending the results in Nualart and Rascanu [7]. Then, we will study the convergence of the solutions of these equations when the delay $r$ tends to zero and the drift coefficient $b$ depends on $(s, X^r(s))$. As occurs for the Brownian motion case, we are able to prove that the solution to the delay equation converges, almost surely and in $L^p$, to the solution of the equation without the delay. All along the paper, we will prove first our results for deterministic equations, and then we will easily apply them pathwise to fractional Brownian motion.

There are many references on stochastic systems with delay (see for instance [5]), but the literature about stochastic differential equations with delay driven by a fractional Brownian motion is scarce. In a previous paper [2], we obtain the existence and uniqueness of solution and the smoothness of the density when $H > 1/2$ under strong hypothesis, using only techniques of the classical stochastic calculus. That approach is unfortunately not suitable for further investigation, like the presence of an hereditary drift and the convergence when the delay tends to zero. Using rough path analysis, Neukirch et al. [6] considered the case $H > 1/3$. Recently, León and Tindel [3] are studying the existence of solution and its regularity when $H > 1/2$.

The structure of the paper is as follows: in the next section, we state the main results of our paper. In Sect. 2, we give some useful estimates for Lebesgue integrals and for Riemann–Stieltjes ones. Section 3 is devoted to obtain the existence, uniqueness and boundedness for the solution for deterministic equations. Section 4 contains the study of the convergence of the deterministic equations. In Sect. 5, we apply the results of the previous sections to stochastic equations driven by fractional Brownian motion, and we give the proofs of our main theorems. Finally, in Sect. 6 we recall a couple of technical results.

We will denote by $C_\alpha$ a constant that will change from line to line.

1. Main results

Let $\alpha \in (\frac{1}{2}, 1)$ and $r > 0$. We will denote by $W_0^{\alpha, \infty}(-r, T; \mathbb{R}^d)$ the space of measurable functions $f : [0, T] \to \mathbb{R}^d$ such that

$$
\|f\|_{\alpha, \infty(r)} := \sup_{t \in [-r, T]} \left( |f(t)| + \int_{-r}^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} \, ds \right) < \infty.
$$