CENTRAL EXTENSIONS OF CURRENT ALGEBRAS

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Abstract. The second cohomology group of Lie algebras of kind $L \otimes U$ with
trivial coefficients is investigated, where $L$ admits a decomposition with one-
dimensional root spaces and $U$ is an arbitrary associative commutative algebra
with unit. This paper gives a unification of some recent results of C. Kassel
and A. Haddi and provides a determination of central extensions of certain
modular semisimple Lie algebras.

Introduction

The main result of this paper (Theorem 1.3) gives the expression of the second
cohomology group of a Lie algebra $L \otimes U$ with trivial coefficients (denoted by
$H^2(L \otimes U)$), where $L$ belongs to the class of Lie algebras referred to as Block algebras
(for the definition, see below), and $U$ is a commutative associative algebra with
unit, through the first-order cyclic cohomology $HC^1(U)$ and the dual $U^*$. Since
the classical Lie algebras, Zassenhaus algebras $W_1(n)$, and infinite-dimensional Witt
algebras $W$ are Block algebras, we obtain as corollaries the values of $H^2(L \otimes U)$
in all these cases. In the second section of this paper the cohomology groups
$H^2(W_1(n) \otimes U)$ are computed by a different method, based on the presentation
of $W_1(n)$ as a deformation of tensor product of $W_1(1)$ and divided power algebra.
Notice that $W_1(\infty)$ is not a Block algebra and hence this case is not covered by
the previous theorem. The third section contains an application of the preceding
results to the computation of central extensions of certain modular semisimple Lie
algebras.

Note that earlier C. Kassel [9] and A. Haddi [8] computed the second homology
group $H_2(L \otimes U)$, where $L$ is classical or $W$ and its subalgebra $W_1$ respectively.
Their method is quite different from ours and is based on the idea of universal
central extensions. Our main theorem may be considered as a generalization and
unification of Kassel's and Haddi's results.

All algebras are defined over a field $F$ of characteristic $p \neq 2, 3$, if the other-
wise condition is not stated. Remember that $U$ denotes an arbitrary associative
commutative algebra with a unit 1.

1. The main theorem and corollaries

Let $T$ be a torus in a Lie algebra $L$. Assume that there exists a root space
decomposition of $L$ with respect to the $T$-action:
Assume, moreover, that all root spaces are one-dimensional: \( L_\alpha = \mathbb{F}e_\alpha \) and \( \alpha([L_\alpha, L_{-\alpha}]) \neq 0 \) for all \( \alpha \in R \). Set \( h_\alpha = [e_{-\alpha}, e_\alpha] \), \( [e_\alpha, e_\beta] = N_{\alpha, \beta}e_{\alpha+\beta} \), \( N_{\alpha, \beta} \in \mathbb{F} \).

The next assumption concerning the structure of \( L \) is \( [L, L] = L \). This is equivalent to \( T = \sum \mathbb{F}h_\alpha \). Let \( \{h_\alpha | \alpha \in B\} \), \( B \subseteq R \), be a basis in \( T \).

Lie algebras \( L \) satisfying all these properties will be called Block algebras. In [1] R. E. Block showed that all finite-dimensional Block algebras over a perfect field of characteristic \( p > 5 \) with \( Z(L) = 0 \) are direct sums of simple classical Lie algebras and Albert-Zassenhaus algebras.

In the sequel, \( L \) will denote a Block algebra except in Lemma 1.1 and Lemma 3.1, where \( L \) is arbitrary.

Our cohomological notions and notations are standard (cf. [7] for Lie algebra cohomology and [5] for cyclic cohomology). When we consider the Lie algebra cohomology, we use the standard cochain complex. Notice that since \( U \) is commutative, the first-order cyclic cohomology \( HC^1(U) \) is exactly the module of all skew-symmetric functions \( F \in (U \otimes U)^\ast \) such that

\[
F(uv, w) + F(wu, v) + F(vw, u) = 0.
\]

If \( \mathcal{D} \) is a subalgebra in \( \text{Der}(U) \), the module of \( \mathcal{D} \)-invariant cyclic cohomology \( HC^1(U)^\mathcal{D} \) consists of cocycles satisfying

\[
F(D(u), v) + F(u, D(v)) = 0
\]

for all \( D \in \mathcal{D} \).

Recall that the Lie algebra structure on \( L \otimes U \) is defined by \([x \otimes u, y \otimes v] = [x, y] \otimes uv\).

Now compute \( H^2(L \otimes U) \).

The following lemma is probably one of the most useful results for computation of Lie algebra cohomology groups.

**Lemma 1.1.** (Cf. [7, Theorem 1.5.2]). Let \( L \) be a Lie algebra, and let \( T \) be a torus in \( L \) such that there exists a root space decomposition (1). Then for each cohomology class \([\Phi] \in H^k(L)\) there exists a representative cocycle \( \Phi \) such that \( \Phi(L_{\alpha_1}, \ldots, L_{\alpha_k}) = 0 \) if \( \alpha_1 + \cdots + \alpha_k \neq 0 \). \( \square \)

**Lemma 1.2** \( H^2(\mathfrak{sl}(2) \otimes U) \cong HC^1(U) \).

If we choose a basis \( \{e, f, h\} \) in \( \mathfrak{sl}(2) \) with multiplication table

\[
[h, e] = -\alpha e, \quad [h, f] = \alpha f, \quad [e, f] = h
\]

for certain \( \alpha \in F \backslash \{0\} \), then the basic cocycles may be chosen as follows:

\[
h \otimes u \wedge h \otimes v \mapsto -\alpha F(u, v),
\]
\[
e \otimes u \wedge f \otimes v \mapsto F(u, v),
\]
other combinations \( \mapsto 0 \)

where \( F \in HC^1(U) \).

**Proof.** Straightforward computations or reference to Kassel [9]. Perhaps it should only be remarked that the coboundary cyclic (= Hochschild) cohomology condition arises from the Jacobi identity. \( \square \)
Theorem 1.3. $H^2(L \otimes U) \cong ( \bigoplus_{a \in R} HC^1(U)_a ) / I \oplus ( \bigoplus_{a \in R} U^*_a ) / J$ where $HC^1(U)_a \cong HC^1(U), U^*_a \cong U^*$ for any $a \in R, I$ is the linear span of the following elements:

$$\alpha(h) F_a - \sum_{\beta \in B} \lambda_{a\beta} \beta(h) F_\beta, \quad \alpha(h) F_a - \beta(h) F_\beta, \quad N_{\beta, -\alpha - \beta} F_a + N_{\alpha, -\alpha - \beta} F_\beta$$

for all $h_a, h \in T$, where $h_a = \sum_{\beta \in B} \lambda_{a\beta} h_\beta$ is the linear expression of $h_a$ in terms of the basis $\{h_\beta | \beta \in B\}$, and for all $\alpha, \beta \in R$, where $F_\alpha$ is an arbitrary element in $HC^1(U)_\alpha$. $J$ is the linear span of the elements

$$N_{\beta, \alpha} G_\alpha + N_{\gamma, \alpha} G_\beta + N_{\alpha, \beta} G_\gamma, \quad \alpha, \beta, \gamma \in R, \quad \alpha + \beta + \gamma = 0;$$

$$G_{-\alpha} + G_\alpha, \alpha \in R; G_\alpha, \alpha \in B,$$

for all $G_\alpha \in U^*_\alpha$.

Basic cocycles may be chosen as follows:

1. $h_\alpha \otimes u \otimes h_\beta \otimes v \mapsto -\alpha(h_\beta) F_a(u, v)$,
2. $e_{-\alpha} \otimes u \otimes e_\alpha \otimes v \mapsto F_a(u, v) + G_a(uv)$,
3. $\text{other combinations} \mapsto 0$.

Proof. Obviously for any root $\alpha$ it is possible to find a complement $Z_\alpha$ of $\mathbb{F} h_\alpha$ in $T$ such that $[Z_\alpha, e_\alpha] = [Z_\alpha, e_{-\alpha}] = 0$. $T \oplus \mathbb{F} e_{-\alpha} \oplus \mathbb{F} e_\alpha$ is a subalgebra in $L$ which is isomorphic to an (essential) central extension of $\mathfrak{sl}(2)$. Since

$$H^2((T \oplus \mathbb{F} e_{-\alpha} \oplus \mathbb{F} e_\alpha) \otimes U) \cong H^2(Z_\alpha \otimes U) + H^2(\mathbb{F} h_\alpha \oplus \mathbb{F} e_{-\alpha} \oplus \mathbb{F} e_\alpha) \otimes U),$$

by Lemma 1.2 we get

$$H^2((T \oplus \mathbb{F} e_{-\alpha} \oplus \mathbb{F} e_\alpha) \otimes U) \cong C^2(Z_\alpha \otimes U) + HC^1(U).$$

Hence for arbitrary $\Phi \in Z^2(L \otimes U)$ and $a \in R$ we have

$$\Phi(h_\alpha \otimes u, h_\alpha \otimes v) = -\alpha(h_\alpha) F_a(u, v),$$

$$\Phi(e_{-\alpha} \otimes u, e_\alpha \otimes v) = F_a(u, v) + G_a(uv),$$

$$\Phi(h_\alpha \otimes U, Z_\alpha \otimes U) = 0,$$

for certain $F_a \in HC^1(U)$ and $G_a \in U^*$.

For any $h \in T$, $h - (\alpha(h)/\alpha(h_\alpha)) h_\alpha \in Z_\alpha$. Hence

$$\Phi(h_\alpha \otimes u, (h - (\alpha(h)/\alpha(h_\alpha))) h_\alpha) \otimes v) = 0$$

and

$$\Phi(h_\alpha \otimes u, h \otimes v) = (\alpha(h)/\alpha(h_\alpha)) \Phi(h_\alpha \otimes u, h_\alpha \otimes v) = -\alpha(h) F_a(u, v).$$

Now suppose that in the last equality $h_\alpha = \sum_{\beta \in B} \lambda_{a\beta} h_\beta$. Then

$$-\alpha(h) F_a(u, v) = \sum_{\beta \in B} \lambda_{a\beta} \Phi(h_\beta \otimes u, h \otimes v)$$

$$= -\sum_{\beta \in B} \lambda_{a\beta} \beta(h) F_\beta(u, v).$$

This implies the defining relations for $I$ of the first kind. Further, substituting in (5) $h = h_\beta$, we obtain (2) and (because of the skew-symmetry of $\Phi$) the defining relations for $I$ of the second kind:

$$\alpha(h_\beta) F_a(u, v) = \beta(h_a) F_\beta(u, v).$$
The cocycle equation \( d\Phi = 0 \) for the triple \( e_\alpha \otimes u, e_\beta \otimes v, e_\gamma \otimes w, \alpha + \beta + \gamma = 0 \), is equivalent to the relation \( N_{\alpha,\beta}F_\gamma(uw,w) + N_{\alpha,\gamma}G_{\beta}(uvw) + \) (simultaneous cyclic permutations of \((\alpha, \beta, \gamma)\) and \((u, v, w)\)) = 0. Substituting here \( v = w = 1 \), we obtain \( N_{\beta,\gamma}F_\alpha(vw,u) + N_{\alpha,\gamma}G_{\beta}(uvw) = 0 \) (which is equivalent to the defining relations for \( J \) of the first kind) and hence

\[
N_{\beta,\gamma}F_\alpha(vw,u) + N_{\alpha,\gamma}G_{\beta}(uvw) = 0.
\]

Substituting in the last equality \( w = 1 \) we get the defining relations for \( I \) of the third kind:

\[
N_{\beta,-\alpha-\beta}F_\alpha(v,u) + N_{-\alpha-\beta,\alpha}F_{\beta}(u,v) = 0.
\]

Since \( T \otimes 1 \) is a torus in \( L \otimes U \), by Lemma 1.1 we may assume

\[
\Phi(e_\alpha \otimes U, e_\beta \otimes U) = 0 \quad \text{if} \quad \alpha + \beta \neq 0,
\]

\[
\Phi(T \otimes U, e_\alpha \otimes U) = 0,
\]

which implies (4).

Now put \( \omega \in C^1(L \otimes U) \) as follows:

\[
\omega(h_\alpha \otimes u) = G_\alpha(u), \quad \alpha \in B; \quad \omega(e_\alpha \otimes U) = 0, \quad \alpha \in R.
\]

Replacing \( \Phi \) by \( \Phi - d\omega \), we obtain a cocycle which satisfies all previous equalities and, moreover, \( G_\alpha = 0, \alpha \in B \). Finally, the relation \( G_\alpha + G_{-\alpha} = 0 \) obviously follows from the skew-symmetry of \( \Phi \).

It remains to prove only that \( \Phi \) described in the theorem is really a cocycle and that all such cocycles are independent. This is achieved by an easy argument which mainly repeats the previous one and is left to the reader. \( \square \)

We keep the symbol \( g \) for classical Lie algebras, that is, for Lie algebras obtained from integral forms of complex classical simple Lie algebras by tensoring on \( F \). Let \( R \) be a root system of \( g \), \( B \) be a basis, \( \{h_\alpha, \alpha \in B; e_\alpha, \alpha \in R\} \) be a Chevalley basis of \( g \), and let \( \langle \cdot, \cdot \rangle \) denotes a scalar product in \( R \).

**Corollary 1.4** [9]. \( H^2(g \otimes U) \cong HC^1(U) \).

Basic cocycles may be chosen as follows:

\[
(6) \quad h_\alpha \otimes u \wedge h_\beta \otimes v \mapsto -\frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle\langle \beta,\beta \rangle} F(u,v), \quad \alpha, \beta \in B,
\]

\[
e_{-\alpha} \otimes u \wedge e_\alpha \otimes v \mapsto \frac{1}{\langle \alpha,\alpha \rangle} F(u,v), \quad \alpha \in R,
\]

other combinations \( \mapsto 0 \),

where \( F \in HC^1(U) \).

**Proof.** There is the following property of the structure constants:

\[
(7) \quad \frac{N_{\alpha,\beta}}{\langle \gamma,\gamma \rangle} = \frac{N_{\beta,\gamma}}{\langle \alpha,\alpha \rangle} = \frac{N_{\alpha,\gamma}}{\langle \beta,\beta \rangle} \quad \text{if} \quad \alpha + \beta + \gamma = 0
\]

(cf. [4, Chapter VIII, §2, Exercise 4]). This property and the defining relations for \( J \) of the first kind implies

\[
G_{-\alpha-\beta} = -\frac{\langle \alpha,\alpha \rangle}{\langle \alpha + \beta,\alpha + \beta \rangle} G_{\alpha} - \frac{\langle \beta,\beta \rangle}{\langle \alpha + \beta,\alpha + \beta \rangle} G_{\beta}, \quad \alpha, \beta, \alpha + \beta \in R.
\]
But $G_\alpha = 0$ for all $\alpha \in B$ and, consequently, $G_\alpha = 0$ for all negative roots $\alpha$. The defining relations for $J$ of the second kind yield $G_{-\alpha} = -G_\alpha$ and hence $G_\alpha = 0$ for all positive roots $\alpha$.

Further, (7) together with the defining relations for $I$ of the third kind gives
\[
\langle \alpha, \alpha \rangle F_\alpha = (\beta, \beta) F_\beta, \quad \alpha, \beta \in R.
\]
Hence
\[
F_\alpha = \frac{1}{\langle \alpha, \alpha \rangle} F, \quad \alpha \in R,
\]
for certain $F \in HC^1(U)$. Since for $\alpha, \beta \in R$, $\alpha(h_\beta) = 2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle$, the right-hand side of formula (2) takes the form
\[
-\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} F(u, v)
\]
which agrees with (6). Finally, it is easy to check that under condition (8) the defining relations for $I$ of the first and second kinds always holds. □

Now recall the definition of the modular Zassenhaus algebra $W_1(n)$. It may be defined in two ways. Firstly as an algebra with basis $\{f_\alpha | \alpha \in \mathbb{F}(p^n)\}$ and Lie bracket given by
\[
[f_\alpha, f_\beta] = (\beta - \alpha)f_{\alpha+\beta}.
\]
Secondly, as an algebra with basis $\{e_i | -1 \leq i \leq p^n - 2\}$ and bracket
\[
[e_i, e_j] = \left( \begin{pmatrix} i + j + 1 \\ i \end{pmatrix} - \begin{pmatrix} i + j + 1 \\ j \end{pmatrix} \right) e_{i+j}.
\]
There are some related infinite-dimensional algebras. If we assume that the indices in formula (9) run over all integer, then we obtain the infinite-dimensional Witt algebra $W$ (considered over a field of characteristic zero). If we assume that the indices in formula (10) run from $-1$ to infinity, we obtain the infinite-dimensional algebra $W_1(\infty)$ (considered over a field of positive characteristic). Obviously $W_1(n)$, $n < \infty$, and $W$ are Block algebras.

**Corollary 1.5.**

(i) $(p > 3, \ n < \infty)$ $H^2(W_1(n) \otimes U) \cong H^2(W_1(n)) \otimes U^*$.

(ii) [8] $(p = 0)$ $H^2(W \otimes U) \cong H^2(W) \otimes U^*$.

**Proof.** (i). Choose a basis $\{f_\alpha | \alpha \in \mathbb{F}(p^n)\}$. We have (in terms of Block algebras): $h_\alpha = 2f_\alpha$ for all $\alpha \in R = \mathbb{F}(p^n) \backslash \{0\}$, $\alpha(f_0) = \alpha$, $N_{\alpha, \beta} = \beta - \alpha$. Apply Theorem 1.3. The defining relations for $I$ of the second and third kinds give $\alpha F_\alpha = \beta F_\beta$ and $(\alpha + 2 \beta)F_\alpha + (2\alpha + \beta)F_\beta = 0$. These equalities evidently imply $F_\alpha = 0$ for all $\alpha$. The defining relations for $J$ give
\[
(\beta - \alpha)G_{-\alpha - \beta} - (\alpha + 2 \beta)G_\alpha + (2\alpha + \beta)G_\beta = 0, \quad G_{-\alpha} = -G_\alpha, \quad G_1 = 0.
\]
Then we proceed, as in Block’s paper [2, p. 1449], to obtain $G_\alpha = (\alpha - \alpha^3)G$ for certain $G \in U^*$. Together with a result of Block [2, Theorem 5.1] this completes the proof.

(ii) Quite analogously. □

Corollaries 1.4 and 1.5 show that the “cyclic cohomology” summand in our main formula (Theorem 1.3) arises from the “classical” properties of Block algebras, while the “dual” summand arises from “Zassenhaus” (“nonclassical”) ones. Of course, Theorem 1.1 may be applied to compute $H^2(L \otimes U)$ for other classes of Block algebras, e.g. Albert-Zassenhaus or Virasoro (both modular and nonmodular).
2. Central Extensions of $W_1(n) \otimes U$

Define a Lie algebra $\mathcal{L}(U, D)$, where $D \in \text{Der}(U)$, as the vector space $W_1(1) \otimes U$ with Lie bracket given by $\{x, y\} = [x, y] + \Psi(x, y)$ where $[\cdot, \cdot]$ is the ordinary Lie bracket in $W_1(1) \otimes U$ and

$$\Psi(e_i \otimes u, e_j \otimes v) = \begin{cases} e_{p-1} \otimes vD(u) - uD(v), & i = j = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Now the notion of divided power algebra $O_1(n)$ is necessary. It is defined as an associative commutative algebra over a field of characteristic $p > 1$ with basis $\{x^i\} \; 0 \leq i < p^n$ and multiplication $x^i x^j = \binom{i+j}{j} x^{i+j}$ (the case $n = \infty$ is also included). It is known that $O_1(n)$ is isomorphic to the truncated polynomial algebra $O_n = \mathbb{F}[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$. We reserve the special notation $\delta$ for the derivation of $O_1(n)$ defined by $\delta(x^i) = x^{i-1}$.

The following remarkable observation is due to M. I. Kuznetsov [10]: $W_1(n)$ is isomorphic to a certain deformation of $W_1(1) \otimes O_1(n-1)$. Namely, $W_1(n)$ is isomorphic to the Lie algebra $\mathcal{L}(O_1(n-1), -\partial)$. This isomorphism is given by $e_i \otimes x^k \mapsto e_{pk + i}$ (the $e_i$'s on the left-hand side belong to $W_1(1)$, while the ones on the right-hand side belong to $W_1(n)$).

**Theorem 2.1.** $H^2(\mathcal{L}(U, D)) \cong \{G \in U^* | D \circ G = 0\}$.

The basic cocycles may be chosen as follows:

\begin{equation}
(11) \quad e_i \otimes u \wedge e_j \otimes v \mapsto \delta_{i+j,0}(-1)^i G(vD(u)) + \delta_{i+j,p}(-1)^i G(uD(v)).
\end{equation}

**Proof.** It is a routine task to verify that cochains determined by formula (11) are cocycles, and we leave it again to the reader.

Let $\Phi \in Z^2(\mathcal{L}(U, D))$. Since $e_0 \otimes F$ is a torus in $\mathcal{L}(U, D)$ and the corresponding root spaces are $\mathcal{L}_i = e_i \otimes U$, by Lemma 1.1 we may assume that

\begin{equation}
(12) \quad \phi(e_i \otimes U, e_j \otimes U) = 0 \quad \text{if } i + j \neq 0 \text{ or } p.
\end{equation}

The cocycle equation is equivalent to the two following equalities:

\begin{equation}
(13) \quad N_{i,j} \Phi(e_{i+j} \otimes uv, e_k \otimes w) \\
+ \text{ (simultaneous cyclic permutations of } (i, j, k)) \\
\quad \text{and } (u, v, w) = 0, \quad \{i, j, k\} \neq \{-1, -1, 2\},
\end{equation}

and

\begin{equation}
(14) \quad \Phi(e_{-1} \otimes v, e_1 \otimes uw) - \Phi(e_{-1} \otimes u, e_1 \otimes vw) \\
- \Phi(e_2 \otimes w, e_{p-2} \otimes vD(u) - uD(v)) = 0.
\end{equation}

In (13), $k = -1, i + j = p + 1$:

$$\Phi(e_i \otimes u, e_{j-1} \otimes vw) + \Phi(e_{i-1} \otimes uw, e_j \otimes v) = 0.$$  

Substitute here $w = 1$:

\begin{equation}
(15) \quad \Phi(e_i \otimes u, e_{j-1} \otimes v) = -\Phi(e_{i-1} \otimes u, e_j \otimes v), \quad i + j = p + 1.
\end{equation}

Substituting the last equality in the previous one, we obtain

$$\Phi(e_{i-1} \otimes u, e_j \otimes vw) = \Phi(e_{i-1} \otimes uw, e_j \otimes v).$$
Substitute here \( v = 1 \):

\[(16) \quad \Phi(e_{i-1} \otimes u, e_j \otimes w) = \Phi(e_{i-1} \otimes uw, e_j \otimes 1), \quad i + j = p + 1.\]

(15) and (16) imply

\[(17) \quad \Phi(e_i \otimes u, e_j \otimes v) = (-1)^i G(uv), \quad i + j = p,\]

for certain \( G \in U^* \).

In (14), substitute \( v = 1 \):

\[(18) \quad \Phi(e_{-1} \otimes u, e_1 \otimes w) = \Phi(e_{-1} \otimes 1, e_1 \otimes uw) - G(wD(u)).\]

In (13), set \( i = -1, j = 0, k = 1, w = 1 \):

\[(19) \quad \Phi(e_0 \otimes u, e_0 \otimes v) = \Phi(e_{-1} uv, e_1 \otimes 1) - \Phi(e_{-1} \otimes u, e_1 \otimes v).\]

Substituting the previous equality in the last one, we obtain \( \Phi(e_0 \otimes u, e_0 \otimes v) = -G(uD(v)) \). Then the skew-symmetry of \( \Phi \) implies \( -G(uD(v)) - G(vD(u)) = 0 \) which is equivalent to \( D \circ G = 0 \).

Now put \( \omega \in C^1(\mathfrak{L}(U, D)) \) as follows: \( \omega(e_0 \otimes u) = \Phi(e_{-1} \otimes 1, e_1 \otimes u) \); \( \omega(e_i \otimes u) = 0, i \neq 0 \). Then \( d\omega(e_{-1} \otimes 1, e_1 \otimes u) = \Phi(e_{-1} \otimes 1, e_1 \otimes u) \). Hence if we take \( \Phi - d\omega \) instead of \( \Phi \), we obtain a cocycle satisfying all previous equalities and, moreover,

\[(20) \quad \Phi(e_{-1} \otimes 1, e_1 \otimes u) = 0.\]

Collecting (12) and (17)-(20), we obtain formula (11).

It remains to prove that if \( \Phi = d\omega \), where \( \Phi \) is determined by formula (11) for an appropriate \( G \), and \( \omega \in C^1(\mathfrak{L}(U, D)) \), then \( G = 0 \). But this is obvious:

\[G(u) = \Phi(e_2 \otimes u, e_{p-2} \otimes 1) = d\omega(e_2 \otimes u, e_{p-2} \otimes 1) = 0.\]

**Corollary 2.2.** \((p > 3)\) \( H^2(W_1(\infty) \otimes U) = 0 \).

Joint proof of Corollaries 1.5(i) and 2.2. According to the above-mentioned observation by Kuznetsov, \( \mathfrak{L}(U \otimes \mathcal{O}_1(n-1), -1 \otimes \partial) \cong W_1(n) \otimes U \), and the isomorphism being given by \( e_1 \otimes u \otimes x^k \mapsto e_{p+k-1} \otimes u \).

Let \( n < \infty \). By Theorem 2.1

\[H^2(W_1(n) \otimes U) \cong \{ G \in (U \otimes \mathcal{O}_1(n))^* | G(U \otimes \langle 1, x, x^2, \ldots, x^{n^2-2} \rangle) = 0 \} \cong U^*.\]

Rewriting (11) in our case, we have

\[
e_{p+k-1} \otimes v \wedge e_{p+1} \otimes u \mapsto -\left(\frac{k+l-1}{k-1}\right) \delta_{i+j,0} (-1)^i G(uv \otimes x^{k+l-1}) + \left(\frac{k+l}{k}\right) \delta_{i+j,p} (-1)^i G(uv \otimes x^{k+l}), \quad i, j = -1, 0, 1, \ldots, p - 2.
\]

Direct easy calculations show that it is exactly the formula

\[e_i \otimes u \wedge e_j \otimes v \mapsto \delta_{i+j,p} (-1)^i G(uv).\]

Let \( n = \infty \). Since \( \partial(\mathcal{O}_1(\infty)) = 0_1(\infty) \), by Theorem 2.1 \( H^2(W_1(\infty) \otimes U) = 0 \).
Remark. The expression of basic cocycles of $H^2(W_1(n) \otimes U)$ in terms of the basis \{\epsilon_i\} corresponds to the one for $H^2(W_1(n))$ found by Dzhumadil’daev [6].

3. CENTRAL EXTENSIONS OF MODULAR SEMISIMPLE LIE ALGEBRAS

According to [3], Lie algebras of type $g \otimes O_n + 1 \otimes \mathfrak{D}$ and $W_1(n) \otimes O_n + 1 \otimes \mathfrak{D}$, where $\mathfrak{D}$ is a subalgebra of Der$(O_n)$ such that $O_n$ does not contain $\mathfrak{D}$-invariant ideals, are one of the simplest but quite typical examples of modular semisimple Lie algebras. Preceding results allow us to determine their central extensions.

Lemma 3.1. Let $L + \mathfrak{D}$ be a semidirect product of Lie algebras, where $L$ is an ideal with $[L, L] = L$. Then $H^2(L + \mathfrak{D}) \cong H^2(L) \otimes + H^2(\mathfrak{D})$.

Proof. We may identify in an obvious manner $C^2(L + \mathfrak{D})$ with $C^2(L) + (L \otimes \mathfrak{D})^* + C^2(\mathfrak{D})$. Let us see how this identification affects cocycles and coboundaries. Let $\Phi \in C^2(L + \mathfrak{D})$ and $\Phi = \Psi + \Upsilon + \Xi$ be the corresponding decomposition. The cocycle equation $d\Phi = 0$ is equivalent to the following relations:

\[
d\Psi(x, y, z) = 0, \\
\Upsilon([x, y], D) = \Psi([x, D], y) + \Psi(x, [y, D]), \\
\Upsilon([x, D_1], D_2) - \Upsilon([x, D_2], D_1) - \Upsilon(x, [D_1, D_2]) = 0, \\
d\Xi(D_1, D_2, D_3) = 0,
\]

for all $x, y, z \in L$ and $D, D_1, D_2, D_3 \in \mathfrak{D}$. The second relation shows that $\Psi \in Z^2(L) \cong \{\Psi \in Z^2(L) | \Psi([x, D], y) + \Psi(x, [y, D]) \in B^2(L) \forall D \in \mathfrak{D}\}$ and, since $[L, L] = L$, $\Upsilon$ is fully determined by $\Psi$. Direct verification shows that the third relation follows from the two previous ones. So we have $Z^2(L + \mathfrak{D}) \cong Z^2(L) \otimes Z^2(\mathfrak{D})$. Obviously $B^2(L + \mathfrak{D}) \cong B^2(L) + B^2(\mathfrak{D})$. The last two isomorphisms imply the assertion of the lemma. \hfill $\square$

Corollaries 1.4, 1.5, and the preceding lemma immediately imply the following proposition.

Proposition 3.2. (i) $H^2(g \otimes U + 1 \otimes \mathfrak{D}) \cong HC^1(U) \otimes H^2(\mathfrak{D})$.

(ii) ($p > 3$, $n < \infty$), $H^2(W_1(n) \otimes U + 1 \otimes \mathfrak{D}) \cong \{G \in U^* | D \circ G = 0 \forall D \in \mathfrak{D}\} + H^2(\mathfrak{D})$. \hfill $\square$

We conclude with the computation of the first-order cyclic cohomology of the truncated polynomial algebra $O_n$. We use multi-index notation: $\Gamma_n := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n | 0 \leq \alpha_i < p\}$, $x^n = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\epsilon_i$ denotes $(0, \ldots, 1, \ldots, 0)$.

Proposition 3.3. $\dim HC^1(O_n) = (n - 1)p^n + 1$. Each cocycle is of the form

\[
x_1^{\alpha} \wedge x_1^{\beta} \mapsto \sum_{i=1}^{n} \alpha_i \lambda_i (\alpha + \beta - \epsilon_i)
\]

where $\lambda_i(\alpha) \in \mathbb{F}$ are parameters satisfying the relations

\[
\sum_{i=1}^{n} \alpha_i \lambda_i (\alpha - \epsilon_i) = 0
\]

for any $\alpha \in \Gamma_n$. 

\[
(21)
\]

\[
(22)
\]
Proof. From the computational point of view it is more convenient to consider cyclic homology. Since $O_n$ is commutative, $HC_1(O_n) \cong \Omega^1_{O_n}/dO_n$, where $\Omega^1_{O_n}$ is the module of Kähler differentials (cf. [5, 9]). It is easy to see that each element $udv$, $u, v \in O_n$, may be represented as a sum of elements of kind $x^\alpha dx^\varepsilon$. Moreover, we have

\begin{equation}
0 = dx^\alpha = \sum_{i=1}^{n} \alpha_i x^{\alpha - \varepsilon_i} dx_i
\end{equation}

for each $\alpha \in \Gamma_n$. So there are $np^n$ generators of the vector space $HC_1(O_n)$, and $p^n - 1$ linear relations between them (if $\alpha = 0$, (23) gives a trivial equality). Hence $\dim HC_1(O_n) = (n - 1)p^n + 1$.

Because of the duality isomorphism $HC^1(O_n) \cong HC_1(O_n)^*$, the first assertion of the lemma holds.

Further, it is easy to verify that the cochains, defined by (21) and (22), are really cocycles. Since there are $np^n$ parameters and $p^n - 1$ relations between them, the dimension equality implies that all cocycles are taken into account. □

Given a concrete subalgebra $D$, it is easy to compute $HC^1(U)^D$. In the general case we are able to give a “lower bound” only:

**Proposition 3.4.** $\dim HC^1(O_n)^{\text{Der}(O_n)} = np^{n-1}$.

Each $\text{Der}(O_n)$-invariant cocycle is determined by formula (21), where

\begin{equation}
\lambda_i(\alpha) = 0 \quad \text{if } \alpha_i \neq p - 1.
\end{equation}

**Proof.** $\text{Der}(O_n)$ has as a basis $\{x^\gamma \partial/\partial x_j | \gamma \in \Gamma_n, 1 \leq j \leq n\}$. Writing the $x^\gamma \partial/\partial x_j$-invariance condition for the cocycle determined by formula (21), we get

\begin{equation}
\sum_{i=1}^{n} (\alpha_i \beta_j - \alpha_j \beta_i) \lambda_i(\alpha + \beta + \gamma - \varepsilon_i - \varepsilon_j) = 0.
\end{equation}

Setting here $\beta = \varepsilon_j$ and $\gamma = 0$, we obtain $\alpha_j \lambda_j(\alpha - \varepsilon_j) = 0$, which is equivalent to (24). Obviously, the last condition implies (22) and (25). □

In particular, $HC^1(O_1)^D = HC^1(O_1)$ for any $D \subseteq \text{Der}(O_1)$.

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**Note added in proof.** Recently A. Haddi (Comm. Algebra 20 (1992), 1145–1166) has obtained the general formula expressing the homology group $H_2(L \otimes A)$ in terms of $H_2(L)$ and $HC_1(A)$ for arbitrary Lie algebra $L$ in the case of characteristic zero.

**References**

1. R. E. Block, On the Mills-Seligman axioms for Lie algebras of classical type, Trans. Amer. Math. Soc. 121 (1966), 378–392.
2. , On the extensions of Lie algebras, Canad. J. Math. 20 (1968), 1439–1450.
3. , Determination of the differentiability simple rings with a minimal ideal, Ann. of Math. 90 (1969), 433–459.
4. N. Bourbaki, Groupes et algèbres de Lie, Chapitres VII, VIII, Hermann, Paris, 1975.
5. P. Cartier, Homologie cyclique: rapport sur les travaux récents de Connes, Karoubi, Loday, Quillen, ... Astérisque 121–122 (1985), 123–148.
6. A. S. Dzhumadil’daev, Central extensions of the Zassenhaus algebra and their irreducible representations, Math. USSR-Sb. 54 (1986), 457–474.
7. D. B. Fuchs, *Cohomology of infinite-dimensional Lie algebras*, Consultants Bureau, New York, 1986.

8. A. Haddi, *Détermination des extensions centrales des algèbres de Kac-Moody*, C. R. Acad. Sci. Paris 306 (1988), 691–694.

9. C. Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra*, J. Pure Appl. Algebra 34 (1984), 265–275.

10. M. I. Kuznetsov, *The modular simple Lie algebras with a solvable maximal subalgebra*, Math. USSR-Sb. 30 (1976), 68–76.

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ERRATUM AND ADDENDUM TO “CENTRAL EXTENSIONS OF CURRENT ALGEBRAS”

PASHA ZUSMANOVICH

Regrettably, the author has spotted, with almost two decades delay, errors in [Z1].

In [Z1 Proposition 3.3], the dimension of the first order cyclic cohomology \( HC^1(O_n) \) of the reduced polynomial algebra \( O_n = K[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \) over a field \( K \) of characteristic \( p > 0 \), is specified incorrectly.

The correct value can be easily obtained as follows. For \( n = 1 \), we have \( \dim HC^1(O_1) = 1 \) (see, for example, [L, Corollary 5.4.17]), the basic cocycle being

\[
\alpha(x^i, x^j) = \begin{cases} 
  i, & \text{if } i + j = p \\
  0, & \text{otherwise.}
\end{cases}
\]

Now, from the fact that

\( O_n \cong O_{n-1} \otimes O_1 \),

and the Künneth formula for the cyclic cohomology (see formula (3) below), we get by induction \( \dim HC^1(O_n) = np^{n-1} \).

[Z1 Proposition 3.4] is wrong. The correct statement is given below. Recall that for an associative commutative algebra \( A \), its derivation algebra \( \text{Der}(A) \) acts on \( HC^1(A) \) by the formula

\[
(D\varphi)(a, b) = \varphi(D(a), b) + \varphi(a, D(b))
\]

for any \( D \in \text{Der}(A), \varphi \in HC^1(A), a, b \in A \).

**Proposition 1.**

\[
HC^1(O_n)^{\text{Der}(O_n)} = \begin{cases} 
  HC^1(O_1), & n = 1 \\
  0, & n > 1
\end{cases}
\]

**Proof.** In the case \( n = 1 \), it is easy to check that the cocycle (1) is invariant under the action of \( x^k \frac{d}{dx} \) for any \( 0 \leq k < p \).

**Lemma.** Let \( A, B \) be two associative commutative algebras with unit. Then

\[
HC^1(A \otimes B)^{\text{Der}(A \otimes B)}
\]

\[
\simeq \left( HC^1(A)^{\text{Der}(A)} \otimes (B^*)^{\text{Der}(B)} \right) \oplus \left( (A^*)^{\text{Der}(A)} \otimes HC^1(B)^{\text{Der}(B)} \right).
\]

**Proof.** By the Künneth exact sequence for the cyclic cohomology (see [L §4.4.10]),

\[
HC^1(A \otimes B) \simeq \left( HC^1(A) \otimes B^* \right) \oplus \left( A^* \otimes HC^1(B) \right),
\]

each cyclic 1-cocycle on \( A \otimes B \) being represented as the linear span of cocycles of the form

\[
(a \otimes b) \lor (a' \otimes b') \mapsto \varphi(a, a')\beta(bb') + \alpha(aa')\psi(b, b'),
\]

where \( \varphi \in HC^1(A), \psi \in HC^1(B), \alpha \in A^*, \beta \in B^*, \) and \( a, a' \in A, b, b' \in B \).
Similarly, by the Künneth formula for the Hochschild cohomology (derivations are just Hochschild cocycles of order 1),

\[ \text{Der}(A \otimes B) \simeq \left( \text{Der}(A) \otimes B \right) \oplus \left( A \otimes \text{Der}(B) \right), \]

each derivation of \( A \otimes B \) being represented as the linear span of derivations of the form \( D \otimes R_b + R_a \otimes F \), where \( D \in \text{Der}(A) \), \( F \in \text{Der}(B) \), \( a \in A \), \( b \in B \), and \( R_a \) denotes the multiplication by an element \( u \) in the respective algebra.

Using these explicit isomorphisms, we get that the result of the action of the derivation \( D \otimes 1_B \), where \( D \in \text{Der}(A) \) and \( 1_B \) is the unit of \( B \), on a cocycle of the form \((a, b')\), is equal to

\[ (D \varphi)(a, a') \otimes \beta(bb') + \alpha(D(aa')) \otimes \psi(b, b'), \]

and hence

\[ HC^1(A \otimes B)^{\text{Der}(A) \otimes 1_B} \simeq \left( HC^1(A)^{\text{Der}(A)} \otimes B^* \right) \oplus \left( (A^*)^{\text{Der}(A)} \otimes HC^1(B) \right). \]

Similarly,

\[ HC^1(A \otimes B)^{1_A \otimes \text{Der}(B)} \simeq \left( HC^1(A) \otimes (B^*)^{\text{Der}(B)} \right) \oplus \left( A^* \otimes HC^1(B)^{\text{Der}(B)} \right). \]

Taking intersection of the right-hand sides in the last two formulas, we arrive at the right-hand side of the isomorphism claimed. It is straightforward to check that each cocycle of the form \((4)\), where \( \varphi \in HC^1(A)^{\text{Der}(A)} \), \( \psi \in HC^1(B)^{\text{Der}(B)} \), \( \alpha \in (A^*)^{\text{Der}(A)} \), \( \beta \in (B^*)^{\text{Der}(B)} \), is invariant under the action of \( \text{Der}(A \otimes B) \).

**Remark.** Similar, but much more involved arguments could be used to prove the analogous statements for not necessarily commutative algebras, for algebras without unit, and for the cyclic cohomology of higher degrees. As we are interested here solely in application to the central extensions of modular semisimple Lie algebras, as specified below, we will not go into detail.

Continuation of the proof of Proposition 1. The case \( n > 1 \) follows from \((2)\), the lemma just proved, and the obvious fact that \( (\text{Der}(O_n))(O_n) = O_n \) for any \( n \).

These errors do not affect other statements in [Z1].

A more general, then those presented in [Z1], formula for the second cohomology with trivial coefficients of the current Lie algebra, allows us to obtain results more general than [Z1, Proposition 3.2]. Namely, for modular semisimple Lie algebras which are represented as the semidirect sum \((S \otimes O_n) \in \mathcal{D}\), where \( S \) is a simple Lie algebra, and \( \mathcal{D} \) is a Lie subalgebra of \( \text{Der}(O_n) \), we have:

**Proposition 2.**

\[ H^2((S \otimes O_n) \in \mathcal{D}) \simeq \left( H^2(S) \otimes (O_n^*)^\mathcal{D} \right) \oplus \left( \mathcal{B}(S) \otimes HC^1(O_n)^\mathcal{D} \right) \oplus H^2(\mathcal{D}). \]

Here \( \mathcal{B}(S) \) denotes the space of symmetric invariant bilinear forms on \( S \).

**Proof.** This is an immediate application of [Z1, Lemma 3.1] (which, in its turn, is a simple consequence of the Hochschild–Serre spectral sequence) and a cohomological version of [Z2, Theorem 0.1].

It is interesting to compare this situation with a somewhat dual one in [DZ, §6]. There, we have a similar formula for the space of so-called commutative 2-cocycles on semisimple modular Lie algebras of the form \((S \otimes O_n) \in \mathcal{D}\), but we do not bother with the cyclic cohomology, as the corresponding direct summand vanishes due to vanishing of its first tensor factor – a certain skew-symmetric analog of \( \mathcal{B} \).
References

[DZ] A. Dzhumadil’daev and P. Zusmanovich, *Commutative 2-cocycles on Lie algebras*, submitted to J. Algebra; arXiv:0907.4780v2.

[L] J.-L. Loday, *Cyclic Homology*, 2nd ed., Springer, 1998.

[Z1] P. Zusmanovich, *Central extensions of current algebras*, Trans. Amer. Math. Soc. 334 (1992), 143–152; arXiv:0812.2625.

[Z2] ———, *The second homology group of current Lie algebras*, Astérisque 226 (1994), 435–452; arXiv:0808.0217.

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