On Elliptic Differential Operators with Shifts
II. The Cohomological Index Formula

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Introduction

This paper is a continuation of [1], where we have studied a general class of
(pseudo)differential operators with nonlocal coefficients, referred to as op-

erators with shifts, and obtained a local index formula (i.e., a formula expressing
the index as the integral of a differential form explicitly determined by the
principal symbol of the operator) for matrix elliptic operators of this kind. In
the present paper we finish the business by establishing a cohomological in-
dex formula of Atiyah–Singer type for elliptic differential operators with shifts
acting between section spaces of arbitrary vector bundles. The key step is the
construction of closed graded traces on certain differential algebras over the
symbol algebra for this class of operators.

We do not formally assume the reader to be familiar with [1] as far as
definitions are concerned but freely use the results obtained there. We also do
not reproduce the discussion of general motivations for this research, which,
as well as the bibliography, can be found in [1].

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1 Elliptic operators with shifts

1.1 Pseudodifferential operators with shifts

The group \( \Gamma \). Let \( M \) be a compact oriented Riemannian manifold without boundary, and let \( \Gamma \) be a countable dense subgroup of a Lie group \( \overline{\Gamma} \) of orientation-preserving isometries of \( M \). The natural action of \( \overline{\Gamma} \) on functions
on $M$ will be denoted by $T$, so that

$$[T_g u](x) = u(g^{-1}(x)), \quad x \in M.$$  

We assume that $\Gamma$ satisfies the following two conditions:

1. (*Polynomial growth.*) The group $\Gamma$ is finitely generated, and the number of distinct elements of $\Gamma$ representable by words of length $\leq k$ in some finite system of generators grows at most polynomially in $k$.

   In what follows, we fix some system of generators and denote by $|g|$ the minimum length of words representing $g \in \Gamma$.

2. (*Diophantine property.*) Let $\text{fix}(g)$ be the set of fixed points of $g \in \Gamma$. The estimate

$$\text{dist}(g(x), x) \geq C|g|^{-N} \text{dist}(x, \text{fix}(g))$$

holds for some $N, C > 0$ and for all $x \in M$ and $g \in \Gamma$. Here $\text{dist}(x, \text{fix}(g))$ is the Riemannian distance between $x$ and the set $\text{fix}(g)$, and by convention we set $\text{dist}(x, \text{fix}(g)) = 1$ if $\text{fix}(g)$ is empty.

**Matrix operators.** Matrix pseudodifferential operators with shifts, i.e., $\Psi DO$ with shifts acting on vector functions on $M$, can be described as follows. (For more detail, see [1], where also further bibliographical references can be found.) A matrix $\Psi DO$ of order $m$ with shifts has the form

$$D = \sum_{g \in \Gamma} T_g D_g,$$  

where $D_g$ is a classical $\Psi DO$ of order $m$ on $M$ and the operators $D_g$ rapidly decay as $|g| \to \infty$ in the natural Fréchet topology on the set of $m$th-order $\Psi DO$.

**Operators on sections of vector bundles.** Pseudodifferential operators with shifts acting on sections of vector bundles are an easy generalization of matrix operators. To define them, one should localize into neighborhoods where the bundles are trivial. The only difference with the case of pseudodifferential operators without shifts is that our operators are no longer local, so we cannot localize into a neighborhood of the diagonal; hence two neighborhoods, instead of one, in the subsequent argument. Let $E$ and $F$ be finite-dimensional complex vector bundles on $M$. A linear operator

$$D: C^\infty(M, E) \longrightarrow C^\infty(M, F)$$

is called an $m$th-order $\Psi DO$ with shifts if for any trivializations of $E$ and $F$ over some neighborhoods $U_E, U_F \subset M$, respectively, and any functions
$\varphi \in C_0^\infty(U_E)$ and $\psi \in C_0^\infty(U_F)$ the operator $\psi D\varphi$ is an $m$th-order matrix
ΨDO with shifts of the form (1).

We point out that no action of $\Gamma$ on the bundles $E$ and $F$ is needed in this
definition.

The linear space of $m$th-order pseudodifferential operators [2] with shifts
will be denoted by $\Psi^m(E, F)_\Gamma$. If $E$, $F$ and $H$ are three vector bundles on $M$,
then the multiplication of operators induces a well-defined bilinear mapping

$$\Psi^m(E, F)_\Gamma \times \Psi^{m'}(F, H)_\Gamma \rightarrow \Psi^{m+m'}(E, H)_\Gamma.$$  

Just as for matrix operators, one readily proves that an $m$th-order $\Psi$DO with
shifts is a continuous operator of order $m$ in the Sobolev spaces of sections of
$E$ and $F$.

1.2 Symbol, ellipticity, and Fredholm property

Symbol: the matrix case. First, let us recall what happens in case the
bundles $E$ and $F$ are trivial.

For the $n \times n'$ matrix operator [1], the symbol is defined by the formula

$$\sigma(D) = \sum_{g \in \Gamma} T_{\partial g} \sigma(D_g): L^2(S^*M, \mathbb{C}^n) \rightarrow L^2(S^*M, \mathbb{C}^{n'}),$$

where the codifferential

$$\partial g: S^*M \rightarrow S^*M$$

is the map induced by $g$ (it acts as $g$ along the base and as $((dg)^*)^{-1}$ in the
fibers of $S^*M$).

Symbol: the general case. If the operator [2] is a usual pseudodifferential
operator, then its symbol is a bundle homomorphism $\pi^*E \rightarrow \pi^*F$, where
$\pi: S^*M \rightarrow M$ is the natural projection. For pseudodifferential operators with
shifts, which are highly nonlocal, this is no longer the case, and their symbols
are defined as homomorphisms of section spaces of the bundles $\pi^*E$ and $\pi^*F$
rather than of the bundles themselves.

Definition 1. The symbol of the operator [2] is the operator

$$\sigma(D): L^2(S^*M, \pi^*E) \rightarrow L^2(S^*M, \pi^*F)$$

such that for any trivializations of $E$ and $F$ over some neighborhoods $U_E, U_F \subset
M$, respectively, and any functions $\varphi \in C_0^\infty(U_E)$ and $\psi \in C_0^\infty(U_F)$ the operator
$\psi \sigma(D) \varphi$ is the symbol of the operator $\psi D\varphi$.  

3
One can readily verify that the symbol of a ΨDO with shifts is well defined. The space of symbols of ΨDO with shifts acting between section spaces of vector bundles $E$ and $F$ will be denoted by $C^\infty(S^*M, \text{Hom}(E, F))_\Gamma$. For $E = F$, we use the notation $C^\infty(S^*M, \text{End}(E))_\Gamma$, and for scalar symbols write $C^\infty(S^*M)_\Gamma$, just as in the first part of the paper. A generalization of the argument given there shows that $C^\infty(S^*M, \text{End}(E))_\Gamma$ is a local subalgebra of the $C^*$-algebra $\mathcal{B}L^2(S^*M, E)$. Hence if a symbol
\[ \sigma \in C^\infty(S^*M, \text{Hom}(E, F))_\Gamma \]
is invertible (as an operator in $L^2$), then one necessarily has
\[ \sigma^{-1} \in C^\infty(S^*M, \text{Hom}(F, E))_\Gamma. \]

**Definition 2.** An operator $D \in \Psi^m(E, F)_\Gamma$ is said to be elliptic if its symbol $\sigma(D)$ is invertible.

As usual, one has the finiteness theorem.

**Theorem 3** (the finiteness theorem). An operator $D \in \Psi^m(E, F)_\Gamma$ is Fredholm if and only if its symbol is invertible.

## 2 The index theorem

In this section we obtain a cohomological index formula for elliptic operators $D \in \Psi^m(E, F)_\Gamma$. First, we shall introduce the elements that occur in this formula.

### 2.1 Some objects associated with the group $\Gamma$

We represent the group $\Gamma$ as the disjoint union
\[ \Gamma = \bigsqcup_{g_0} \langle g_0 \rangle \]
of conjugacy classes and arbitrarily fix an element, $g_0$, in each conjugacy class $\langle g_0 \rangle$. In what follows, the symbol $g_0$ is invariably used to denote this fixed representative. By $C_{g_0}$ we denote the centralizer of $g_0$ in $\Gamma$:
\[ C_{g_0} = \{ h \in \Gamma : h g_0 h^{-1} = g_0 \}. \]

This is a closed Lie subgroup of $\Gamma$. For each $g \in \langle g_0 \rangle$, consider the set $\overline{\Gamma}_{g_0, g}$ of elements $h \in \Gamma$ conjugating $g_0$ with $g$, that is, satisfying
\[ h g_0 h^{-1} = g. \]
Clearly, $\Gamma_{g_0, g}$ is a left coset of $C_{g_0}$ in $\Gamma$ and, as such, has a well-defined normalized Haar measure $dh$ induced by that on $C_{g_0}$.

If the group $\Gamma$ acts on a compact manifold $X$, then by $X_g$ we denote the set of fixed points of an element $g \in \Gamma$. This is a $C^\infty$ submanifold of $X$ consisting of finitely many components (possibly of various dimensions).

### 2.2 The Todd class

The Todd class $\operatorname{Td}(TM \otimes \mathbb{C}; \Gamma)$ of the complexified tangent bundle of $M$ with respect to the action of $\Gamma$ is an element of the group $\prod_{g_0} H^{ev}(M_{g_0}, \mathbb{C})$. (The product is taken over representatives of all conjugacy classes in $\Gamma$.) The $g_0$th component of the Todd class is defined by the formula

$$\operatorname{Td}(TM \otimes \mathbb{C}; \Gamma)(g_0) = \frac{\operatorname{Td}(T^*M_{g_0} \otimes \mathbb{C})}{{\operatorname{ch}}_{\lambda}(NM_{g_0} \otimes \mathbb{C})}(g_0) \in H^{ev}(M_{g_0}, \mathbb{C})$$

(This form was apparently first introduced by Atiyah and Singer in [2]; following Baum and Connes [3], we refer to it as the "Todd class.")

Let us make some explanations concerning this formula. The numerator is the usual Todd class of the complexified tangent bundle of $M_{g_0}$. Next, $\lambda_{-1}(NM_{g_0})$ is the (virtual) vector bundle

$$\lambda_{-1}(NM_{g_0}) = \Lambda^{even}(NM_{g_0}) - \Lambda^{odd}(NM_{g_0})$$

composed of the exterior powers of $NM_{g_0}$, and $\operatorname{ch}_{\lambda}(NM_{g_0} \otimes \mathbb{C})(g_0)$ is the Chern character of the bundle $\lambda_{-1}(NM_{g_0}) \otimes \mathbb{C}$ localized at the element $g_0$. Recall that it is defined as follows. Since the mapping $g_0$ preserves the metric, it follows that the restriction of the differential $dg_0$ to the normal bundle $NM_{g_0}$ is a well-defined automorphism of this bundle. Let $\Omega$ be the curvature form of some $dg_0$-invariant connection on $\lambda_{-1}(NM_{g_0})$ (e.g., of the connection induced by the restriction of the Riemannian connection on $TM$ to $NM_{g_0}$). The localized Chern character

$$\operatorname{ch}_{\lambda}(NM_{g_0} \otimes \mathbb{C})(g_0) \in H^{ev}(M_{g_0}, \mathbb{C})$$

is defined as the cohomology class of the form

$$\operatorname{ch}_{\lambda}(NM_{g_0} \otimes \mathbb{C})(g_0) = \operatorname{tr} \left( dg_0^* \exp \left( -\frac{1}{2\pi i} \Omega \right) \right).$$

(Here $\operatorname{tr}$ stands for the trace in the fibers of a vector bundle.)

### 2.3 The Chern character of the symbol

Let the group $\Gamma$ act on a compact manifold $X$. 

5
Differential forms and graded traces over the algebra $C^\infty(X)_\Gamma$. Let $E \in \text{Vect}(X)$ be a vector bundle. By

$$\Lambda^*(X, \text{End } E)_\Gamma \subset B\mathcal{L}^2(X, \Lambda^*(X) \otimes E)$$

we denote the subalgebra of elements $A$ of the form

$$A = \sum_{g \in \Gamma} \omega_g,$$

where the $\omega_g$ have the following property: for any two functions $\psi$ and $\varphi$ with supports in neighborhoods where $E$ is trivialized, one has

$$\omega_g = T_g a_g,$$

where $T_g \omega := (g^*)^{-1} \omega$ and $a_g$ are some differential forms on $X$ rapidly decaying in the $C^\infty$ Fréchet topology as $|g| \to \infty$.

We define a mapping

$$\tau: \Lambda^*(X, \text{End } E)_\Gamma \longrightarrow \bigoplus_{g_0} \Lambda^*(X_{g_0})$$

(the sum is taken over representatives of all conjugacy classes in $\Gamma$) by setting

$$\tau \left( \sum_{g \in \Gamma} \omega_g, g_0 \right) = \sum_{g \in \langle g_0 \rangle} \int_{g_0 \cdot g} h^* h \omega \big|_{X_g} \big) dh.$$

This is well defined. Indeed, $g|_{X_g} = \text{id}$, and so the operator $\omega_g$ can be restricted to $X_g$, the restriction being an $\text{End } E$-valued differential form on $X_g$. The trace $\text{tr}$ in the last formula is the fiberwise trace in $\text{End } E$.

**Lemma 4.** The mapping $\tau$ is a graded trace on the algebra $\Lambda^*(X, \text{End } E)_\Gamma$ in the sense that

$$\tau([\omega_1, \omega_2]) = 0,$$

for all $\omega_1, \omega_2 \in \Lambda^*(X, \text{End } E)_\Gamma$, where $[\cdot, \cdot]$ is the supercommutator

$$[\omega_1, \omega_2] = \omega_1 \omega_2 - (-1)^{\text{deg } \omega_1 \text{deg } \omega_2} \omega_2 \omega_1.$$

**Chern character of projections.** Now we shall define the Chern character

$$\text{ch}: K_0(C^\infty(X, \text{End } E)_\Gamma) \longrightarrow \bigoplus_{g_0} H^{ev}(X_{g_0}, \mathbb{C})$$

(where $K_0(A)$ is the $K$-group of an operator algebra $A$). Let $p$ be a projection over the algebra $C^\infty(X, \text{End } E)_\Gamma$. (To make the subsequent formulas shorter,
we pretend that $p$ is a projection in the algebra $C^\infty(X,\text{End } E)_\Gamma$ itself rather than in a matrix algebra over it.) We take some connection
\[ \nabla_E : \Lambda^*(X, E) \longrightarrow \Lambda^*(X, E) \]
in the bundle $E$ and define a first-order differential operator with shifts,
\[ \nabla : \Lambda^*(X, E) \longrightarrow \Lambda^*(X, E), \quad (7) \]
by the formula
\[ \nabla = p\nabla_E p. \quad (8) \]
A straightforward computation shows that the following assertion is true.

**Lemma 5.** The operator
\[ \Omega \equiv \nabla^2 : \Lambda^*(X, E) \longrightarrow \Lambda^*(X, E) \]
belongs to $\Lambda^2(X, \text{End } E)_\Gamma$.

The noncommutative 2-form $\Omega$ is called the curvature form corresponding to the projection $p$ and the connection $\nabla_E$.

**Definition 6.** The Chern character of the class $[p] \in K_0(C^\infty(X, \text{End } E)_\Gamma)$ is the cohomology class
\[ \text{ch}_\Gamma[p] \in \bigoplus_{g_0} H^{ev}(X_{g_0}, \mathbb{C}) \]
of the differential form
\[ \text{ch}_\Gamma p := \tau(e^{-\Omega/2\pi i}) \in \bigoplus_{(g_0) \subset \Gamma} \Lambda^{ev}(X_{g_0}). \]

This is well defined. More precisely, the form $\text{ch}_\Gamma p$ is closed, and its cohomology class is independent of the choice of a connection in the bundle $E$ and is uniquely determined by the class of the projection $p$ in the $K$-group $K_0(C^\infty(X, \text{End } E)_\Gamma)$. The proof is based on the identity
\[ d\tau(A) = \tau([\nabla, A]), \]
where $A \in \Lambda^*(X, \text{End } E)_\Gamma$ is an arbitrary element such that $pA = A = Ap.$
Chern character of the symbol. Now let
\[ D : C^\infty(M, E) \longrightarrow C^\infty(M, F) \]
be an elliptic operator with shifts acting in sections of vector bundles on \( M \). To define the Chern character of the symbol \( \sigma(D) \), we introduce a projection and hence an element in \( K \)-theory associated with the symbol. To this end, we make use of the bundle
\[ 2B^*M = S(T^*M \oplus 1) \]
of unit spheres in the vector bundle \( T^*M \oplus 1 \) over \( M \).

Consider the projection \( p \) over the algebra \( C(2B^*M, \text{End}(E \oplus F))_\Gamma \) defined by the formula
\[ p(\xi \cos \psi, \sin \psi) = \frac{1}{2} \begin{pmatrix} (1 + \sin \psi) \text{id}_E & \sigma^{-1}(D)(\xi) \cos \psi \\ \sigma(D)(\xi) \cos \psi & (1 - \sin \psi) \text{id}_F \end{pmatrix}, \tag{9} \]
where \( \xi \) lies on the unit sphere in \( T^*M \), so that \( (\xi \cos \psi, \sin \psi) \) just lies on the unit sphere in \( T^*M \oplus 1 \).

Remark 7. Note that in general the projection \( p \) is only continuous but not infinitely differentiable at the points where \( \cos \psi = 0 \).

We set
\[ [\sigma(D)] \overset{\text{def}}{=} [p] \in K_0\left(C(2B^*M, \text{End}(E \oplus F))_\Gamma\right). \]

Note that
\[ K_0\left(C(2B^*M, \text{End}(E \oplus F))_\Gamma\right) = K_0\left(C^\infty(2B^*M, \text{End}(E \oplus F))_\Gamma\right), \]
since, as was already mentioned above, \( C^\infty(2B^*M, \text{End}(E \oplus F))_\Gamma \) is a dense local subalgebra of \( C(2B^*M, \text{End}(E \oplus F))_\Gamma \). Hence we obtain the cohomology class
\[ \text{ch}_\Gamma[\sigma(D)] \in \bigoplus_{g_0} H^{eu}(2B^*M_{g_0}; \mathbb{C}), \]
which will be called the Chern character of the symbol \( \sigma(D) \).

2.4 Index theorem

Now we are in position to state our main result.

Theorem 8. Let \( D \) be an elliptic operator with shifts on the manifold \( M \). Then the index of \( D \) is given by the formula
\[ \text{ind } D = \langle \text{ch}_\Gamma[\sigma(D)] \text{Td}(TM \otimes \mathbb{C}; \Gamma), [2B^*M; \Gamma] \rangle, \quad (10) \]
where

\[ [2B^*M; \Gamma] = \prod_{g_0} [2B^*M_{g_0}] \in \prod_{g_0} H_{ev}(2B^*M_{g_0}) \]

is the fundamental class and angle brackets denote the natural paring between cohomology and homology.

The proof involves extensive computations and goes by reduction to the local index formula obtained for elliptic operators with shifts in the first part of this paper.

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