BOSONIZATIONS OF $\hat{\mathfrak{sl}}_2$ AND INTEGRABLE HIERARCHIES

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Abstract. We construct embeddings of $\hat{\mathfrak{sl}}_2$ in lattice vertex algebras by composing the Wakimoto realization with the Friedan–Martinec–Shenker bosonization. The Kac–Wakimoto hierarchy then gives rise to two new hierarchies of integrable, non-autonomous, non-linear partial differential equations. A new feature of our construction is that it works for any value of the central element of $\hat{\mathfrak{sl}}_2$; that is, the level becomes a parameter in the equations.

1. Introduction

Vertex operators and vertex (operator) algebras are powerful tools for studying infinite-dimensional Lie algebras, their representations, and generalizations [LW, FK, B, FLM, K2, FB, LL]. For any even integral lattice $L$, one constructs the lattice vertex algebra $V_L$ associated to $L$. When $L$ is the root lattice of a finite-dimensional simply-laced Lie algebra $\mathfrak{g}$, this gives the Frenkel–Kac construction of a level one representation of the corresponding affine Kac–Moody algebra $\hat{\mathfrak{g}}$ (see [FK, K1, K2]).

In this paper we construct a different vertex operator realization of the affine Kac–Moody algebra $\hat{\mathfrak{sl}}_2$ of an arbitrary level $k$. We start with the Wakimoto realization of $\hat{\mathfrak{sl}}_2$, which can be viewed as an embedding of the associated affine vertex algebra in the vertex algebra generated by a pair of charged free bosons $a^+, a^-$ (also known as a $\beta\gamma$-system) and another free boson $b$ (which generates the Heisenberg algebra); see [W, F]. The Friedan–Martinec–Shenker bosonization of $a^+, a^-$ then gives us an embedding in a certain lattice vertex algebra $V_L$ (see [FMS, Wa, A]). The resulting realization of $\hat{\mathfrak{sl}}_2$ has appeared previously in [FS], and is also related to the ones in [FF, JMX]. Under some assumptions, we prove the uniqueness of this realization by classifying all such embeddings of $\hat{\mathfrak{sl}}_2$ in a lattice vertex algebra. We
then consider a twisted representation \( M \) of the vertex algebra \( V_L \) (cf. \([L, FLM, FFR, D, BK]\)), and obtain on \( M \) a representation of \( \hat{sl}_2 \) of level \( k \). This representation does not appear to exist elsewhere in the literature and may be of independent interest.

Let \( V \) be a highest-weight representation of an affine Kac–Moody algebra \( \hat{\mathfrak{g}} \), and \( \Omega_2 \in \text{End}(V \otimes V) \) be the Casimir operator that commutes with the diagonal action of \( \hat{\mathfrak{g}} \) (see \([KI]\)). Consider the equation

\[
\Omega_2(\tau \otimes \tau) = \lambda \tau \otimes \tau, \quad \tau \in V,
\]

where \( \lambda \in \mathbb{C} \) is a constant such that the equation holds when \( \tau \) is the highest-weight vector \( v \in V \). Then (1.1) holds for any \( \tau \) in the orbit of \( v \) under the Kac–Moody group associated to \( \hat{\mathfrak{g}} \) (see \([PK]\)). Equivalently, (1.1) is satisfied for all \( \tau \) such that \( \tau \otimes \tau \) is in the \( \hat{\mathfrak{g}} \)-submodule generated by \( v \otimes v \). In the case when \( V \) provides a vertex operator realization of \( \hat{\mathfrak{g}} \), such as the Frenkel–Kac construction, after a number of non-trivial changes of variables one can rewrite (1.1) as an infinite sequence of non-linear partial differential equations called the Kac–Wakimoto hierarchy \([KW]\). The action of \( \hat{\mathfrak{g}} \) allows one to construct some particularly nice solutions to these equations called solitons. For example, the Korteweg–de Vries and non-linear Schrödinger hierarchies are instances of Kac–Wakimoto hierarchies related to different realizations of \( \hat{sl}_2 \) (see \([K1]\)).

In this paper we investigate the hierarchy (1.1) arising from the Friedan–Martinec–Shenker bosonization of the Wakimoto realization of \( \hat{sl}_2 \), which we call the Wakimoto hierarchy. The Casimir operator \( \Omega_2 \) is replaced with one of the operators from the coset Virasoro construction, which still commutes with the diagonal action of \( \hat{sl}_2 \) (see \([GKO, KR]\)). We write the equations of the Wakimoto hierarchy explicitly as Hirota bilinear equations, and we find the simplest ones. This is done both in the untwisted case when \( V \) is a Fock space contained in \( V_L \), and the twisted case when \( V = M \) is a twisted representation of \( V_L \). The new phenomenon is that these are representations of \( \hat{sl}_2 \) of any level \( k \), so the level becomes a parameter in the equations of the Wakimoto hierarchy.

The paper is organized as follows. In Section 2 we construct explicitly the embedding of \( \hat{sl}_2 \) of level \( k \) in a lattice vertex algebra \( V_L \), and we prove a certain uniqueness property of this embedding. The untwisted Wakimoto hierarchy is investigated in Section 3. In Section 4 we determine the action of \( \hat{sl}_2 \) on a twisted representation \( M \) of \( V_L \), and study the corresponding twisted Wakimoto hierarchy. Throughout the paper, we work over the field of complex numbers.
2. Wakimoto realization and its FMS bosonization

We assume the reader is familiar with the basic definitions and examples of vertex algebras, and refer to [FLM, K2, FB, LL] for more details. Let us review the Wakimoto realization of $\hat{\mathfrak{sl}}_2$ of level $k$ from [W] in the formulation of [F]. Consider a pair of charged free bosons $a^+, a^-$ (also known as a $\beta\gamma$-system) and a free boson $b$ with the only non-zero OPEs given by:

$$ a^+(z)a^-(w) \sim \frac{1}{z-w}, \quad a^-(z)a^+(w) \sim -\frac{1}{z-w} $$

and

$$ b(z)b(w) \sim \frac{2k}{(z-w)^2}. $$

Then we have a representation of $\hat{\mathfrak{sl}}_2$ of level $k$ defined by:

$$ e(z) = a^-(z), \quad h(z) = -2:a^+(z)a^-(z): + b(z), \quad f(z) = -2:a^+(z)^2a^-(z): + k\partial_z a^+(z) + :a^+(z)b(z):, $$

where the normal ordering of several terms is from the right.

Recall that the boson-boson correspondence, or Friedan–Martinec–Shenker (FMS) bosonization, provides a realization of the charged free bosons $a^+, a^-$ in terms of fields in a lattice vertex algebra. Namely, consider the lattice $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ with $(\alpha_1 | \alpha_2) = 0$ and $|\alpha_1|^2 = -|\alpha_2|^2 = 1$; then

$$ a^+(z) = e^{\alpha_1 + \alpha_2}(z), \quad a^-(z) = -:\alpha_1(z)e^{-\alpha_1 - \alpha_2}(z): $$

(see [FMS, Wa, A]). Notice that $|\alpha_1 + \alpha_2|^2 = 0$.

We apply FMS-bosonization to the Wakimoto realization of $\hat{\mathfrak{sl}}_2$, and we obtain an embedding in a lattice vertex algebra $V_L$ of the general form:

$$ e(z) = :\alpha(z)e^\delta(z):, \quad h(z) = \beta(z), \quad f(z) = :\gamma(z)e^{-\delta}:, $$

where $|\delta|^2 = 0$ and $\alpha(z), \beta(z), \gamma(z)$ are Heisenberg fields. We will classify all such embeddings, where the lattice $L$ contains $\delta$ and

$$ \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L = \text{span}_\mathbb{C}\{\alpha, \beta, \gamma, \delta\}. $$

Note that the rescaling

$$ e \mapsto \lambda e, \quad f \mapsto \frac{1}{\lambda} f, \quad h \mapsto h \quad (\lambda \in \mathbb{C}) $$

is an automorphism of $\mathfrak{sl}_2$. 
Theorem 2.1. Up to the rescaling (2.2), the above formulas (2.1) provide an embedding of $\hat{\mathfrak{sl}}_2$ of level $k$ in the lattice vertex algebra $V_L$ if and only if

$$\beta = k\delta + \alpha - \gamma, \quad |\delta|^2 = 0, \quad (\alpha|\gamma) = k + 1,$$

$$|\alpha|^2 = |\gamma|^2 = (\delta|\alpha) = -(\delta|\gamma) = 1.$$

Proof. We need to verify the following OPEs:

$$e(z)f(w) \sim \frac{h(w)}{z-w} + \frac{k}{(z-w)^2}, \quad h(z)h(w) \sim \frac{2k}{(z-w)^2},$$

$$h(z)e(w) \sim \frac{2e(w)}{z-w}, \quad h(z)f(w) \sim \frac{-2f(w)}{z-w},$$

$$e(z)e(w) \sim 0, \quad f(z)f(w) \sim 0.$$

We compute (where “h.o.t.” stands for higher order terms in $z-w$):

$$e(z)f(w) = :\alpha(z)e^{\delta}(z): \gamma(w)e^{-\delta}(w):$$

$$\sim \left( -\frac{(\alpha|\delta)\gamma(w) - (\gamma|\delta)\alpha(z)}{z-w} + \frac{(\alpha|\gamma) + (\delta|\alpha)(\delta|\gamma)}{(z-w)^2} \right) \times$$

$$\times \left( 1 + (z-w)\delta(w) + \text{h.o.t.} \right)$$

$$\sim \frac{((\alpha|\gamma) + (\delta|\alpha)(\delta|\gamma))\delta(w)}{z-w} - \frac{(\alpha|\delta)\gamma(w) + (\gamma|\delta)\alpha(w)}{z-w}$$

$$+ \frac{(\alpha|\gamma) + (\delta|\alpha)(\delta|\gamma)}{(z-w)^2}.$$ 

This implies

$$k = (\alpha|\gamma) + (\delta|\alpha)(\delta|\gamma),$$

$$h = \beta = k\delta - (\alpha|\delta)\gamma - (\gamma|\delta)\alpha.$$

Similar computations for $e(z)e(w)$ and $f(z)f(w)$ give

$$(\alpha|\alpha) = (\delta|\alpha)^2, \quad (\gamma|\gamma) = (\delta|\gamma)^2.$$

Now,

$$h(z)e(w) = h(z) :\alpha(w)e^{\delta}(w):$$

$$\sim \frac{(h|\delta):\alpha(w)e^{\delta}(w):}{z-w} + \frac{(h|\alpha)}{(z-w)^2}$$

tells us that

$$(h|\delta) = 2, \quad (h|\alpha) = 0,$$

and a nearly identical computation for $h(z)f(w)$ gives

$$(h|\gamma) = 0.$$
Then one checks
\[
(h|h) = (h|k\delta - (\alpha|\delta)\gamma - (\gamma|\delta)\alpha) = k(h|\delta) = 2k,
\]
as required. Expanding \((h|\delta) = 2\), we find
\[
2 = (h|\delta) = (k\delta - (\delta|\alpha)\gamma - (\gamma|\delta)\alpha) = -2(\delta|\alpha)(\delta|\gamma),
\]
i.e.,
\[
(\delta|\alpha)(\delta|\gamma) = -1.
\]
Similarly, from \((h|\alpha) = 0\) we get
\[
0 = (h|\alpha) = (k\delta - (\delta|\alpha)\gamma - (\gamma|\delta)\alpha) = k(\delta|\alpha)(\gamma|\alpha) - (\delta|\gamma)(\alpha|\alpha) = (\delta|\alpha)(k + 1 - (\alpha|\gamma)),
\]
which gives
\[
(\alpha|\gamma) = k + 1.
\]
Gathering all the lattice equations so far obtained,
\[
(\delta|\alpha)(\delta|\gamma) = -1, \quad (\alpha|\gamma) = k + 1,
\]
\[
(\alpha|\alpha) = (\delta|\alpha)^2, \quad (\gamma|\gamma) = (\delta|\gamma)^2,
\]
we notice that we are free to rescale \(\alpha \mapsto \lambda \alpha\) and \(\gamma \mapsto \frac{1}{\lambda} \gamma\), which allows us to fix \((\delta|\alpha) = 1\). This then immediately fixes all the other inner products and we obtain the desired result. \(\square\)

**Remark 2.2.** The above embedding of \(\hat{\mathfrak{sl}}_2\) in \(V_L\) is essentially equivalent to the “symmetric” \(W_2^{(2)}\) algebra of \([FS]\), to which they refer as the “three-boson realization” of \(\hat{\mathfrak{sl}}_2\).

We will expand fields \(\phi(z)\) in the standard way
\[
\phi(z) = \sum_{n \in \mathbb{Z}} \phi(n) z^{-n-1}
\]
and call the coefficients \(\phi(n)\) the modes of \(\phi(z)\). The lattice vertex algebra \(V_L\) is equipped with the standard action of the Virasoro algebra (see, e.g., \([K2]\)):
\[
L_n = \frac{1}{2} \sum_{i=1}^{\text{rank } L} \sum_{m \in \mathbb{Z}} :a^i_{(m)} b^i_{(n-m)}:,
\]
where \(\{a^i\}\) and \(\{b^i\}\) are dual bases of \(\mathfrak{h}\) with respect to the bilinear form \((\cdot|\cdot)\). The Virasoro central charge is equal the rank of \(L\).
Lemma 2.3. When the level \( k = c - 1 \neq -2 \) is not critical, we have
\[
L_n = \frac{c - 1}{4} \sum_{m \in \mathbb{Z}} : \delta(m) \delta(n-m) : + \frac{1}{2} \sum_{m \in \mathbb{Z}} : \delta(m)(\alpha - \gamma)(n-m) :
\]
\[
+ \frac{1}{4(c+1)} \sum_{m \in \mathbb{Z}} : (\alpha + \gamma)(m)(\alpha + \gamma)(n-m) :.
\]
This formula remains true for \( c = -1 \) if we remove the last term and set \( \gamma = -\alpha \).

Proof. The proof is straightforward, using that for \( c \neq -1 \) the Gram matrix \( G \) of \( (\cdot|\cdot) \) relative to the basis \( \{ \delta, \alpha, \gamma \} \) of \( h \), and its inverse are:
\[
G = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & c \\ -1 & c & 1 \end{pmatrix}, \quad G^{-1} = \frac{1}{2(c+1)} \begin{pmatrix} c^2 - 1 & c + 1 & -c - 1 \\ c + 1 & 1 & 1 \\ -c - 1 & 1 & 1 \end{pmatrix}.
\]
For \( c = -1 \), the pair of dual bases for \( h \) are: \( \{ \delta, \alpha \} \) and \( \{ -\delta + \alpha, \delta \} \). \( \square \)

Note that the Gram matrix associated to the lattice \( L \) has determinant \(-2k - 4\). Therefore, rank \( L = 3 \), unless the level is critical (i.e., \( k = -2 \)), in which case rank \( L = 2 \) and we can set \( \gamma = -\alpha \).

3. The untwisted Wakimoto hierarchy

In the previous section, we saw that the modes of
\[
c(z) = : \alpha(z) e^\delta(z) :, \\
f(z) = : \gamma(z) e^{-\delta}(z) :, \\
h(z) = k\delta(z) + \alpha(z) - \gamma(z)
\]
give a representation of the affine Kac–Moody algebra \( \hat{sl}_2 \) of level \( k \), where
\[
|\delta|^2 = 0, \quad |\alpha|^2 = |\gamma|^2 = (\delta|\alpha) = -(\delta|\gamma) = 1, \quad (\alpha|\gamma) = c := k + 1.
\]
Introduce the bosonic Fock space
\[
B = \mathbb{C}[x, y, t; q, q^{-1}],
\]
where
\[
x = (x_1, x_2, x_3, \ldots), \quad y = (y_1, y_2, y_3, \ldots), \quad t = (t_1, t_2, t_3, \ldots).
\]
The Heisenberg fields \( \alpha(z), \gamma(z) \) and \( \delta(z) \) acts on \( B \) as follows \( (n > 0) \):
\[
\alpha(n) = \partial x_n + c \partial y_n + \partial t_n, \quad \alpha(-n) = nx_n, \quad \alpha(0) = q \partial q, \\
\gamma(n) = c \partial x_n + \partial y_n - \partial t_n, \quad \gamma(-n) = ny_n, \quad \gamma(0) = -q \partial q, \\
\delta(n) = \partial x_n - \partial y_n, \quad \delta(-n) = nt_n, \quad \delta(0) = 0.
\]
By setting $q = e^\delta$, we identify $B$ as a subspace of $V_L$. Then $B$ is preserved by the actions of $\hat{\mathfrak{sl}}_2$ and Virasoro.

Introduce the Virasoro field

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

and the Casimir field

$$\Omega(z) = e(z) \otimes f(z) + f(z) \otimes e(z) + \frac{1}{2} h(z) \otimes h(z) - k \otimes L(z) - L(z) \otimes k.$$

**Proposition 3.1.** All modes of $\Omega(z)$ commute with the diagonal action of $\hat{\mathfrak{sl}}_2$, i.e.,

$$[\Omega(z), a(w) \otimes 1 + 1 \otimes a(w)] = 0, \quad a \in \mathfrak{sl}_2.$$

**Proof.** This follows from the observation that the modes of $\Omega(z)$ give rise to the coset Virasoro construction (see [GKO, KR]). □

Note that, up to adding a scalar multiple of the identity operator, $\Omega(1) = \text{Res}_z z \Omega(z)$ is the Casimir element of $\hat{\mathfrak{sl}}_2$ (see [K1], where it is denoted $\Omega^2$). The *Kac–Wakimoto hierarchy* [KW] is given by the equation $\Omega(1) (\tau \otimes \tau) = \lambda (\tau \otimes \tau)$, where $\tau$ is in a certain highest-weight module and $\lambda \in \mathbb{C}$ is a constant such that the equation holds when $\tau$ is the highest-weight vector. Instead of $\Omega(1)$, we will consider the operator $\Omega(0) := \text{Res}_z \Omega(z)$. Since the vector $1 \in B$ satisfies $\Omega(0)(1 \otimes 1) = 0$, we obtain the following.

**Corollary 3.2.** Every vector $\tau \in B$, such that $\tau \otimes \tau$ is in the $\hat{\mathfrak{sl}}_2$-submodule of $B \otimes B$ generated by $1 \otimes 1$, satisfies the equation

$$\Omega(0)(\tau \otimes \tau) = 0.$$  

(3.1)

We will call (3.1) the *untwisted Wakimoto hierarchy*. Our goal now is to compute explicitly the action of $\Omega(0)$ on $B \otimes B$. The main step is to simplify $e(z) \otimes f(z)$. We will use the shorthand notation

$$\phi' = \phi \otimes 1, \quad \phi'' = 1 \otimes \phi,$$

so for example, $x'_n = x_n \otimes 1$; and we identify

$$B \otimes B = \mathbb{C}[x', y', t', x'', y'', t''; (q')^{\pm 1}, (q'')^{\pm 1}].$$

Recall the elementary Schur polynomials defined by the expansion

$$\exp \left( \sum_{n=1}^{\infty} t_n z^n \right) = \sum_{m \in \mathbb{Z}} S_m(t) z^m.$$  

(3.2)
Explicitly, one has \( S_m(t) = 0 \) for \( m < 0 \), \( S_0(t) = 1 \), and

\[
S_m(t) = \sum_{m_1+2m_2+3m_3-\cdots=m} \frac{t_{m_1} t_{m_2} t_{m_3}}{m_1! m_2! m_3!} \cdots , \quad m \geq 1 .
\]

**Lemma 3.3.** With the above notation, one has

\[
\text{Res}_z e(z) \otimes f(z) = q'(q'')^{-1} \sum_{i,j \in \mathbb{Z}} S_i(2t) \alpha_i' \gamma_j'' S_{l-i-j-1}(-\tilde{\partial}_x + \tilde{\partial}_y) ,
\]

where \( 2t_n = t'_n - t''_n \) and \( \tilde{\partial}_{x_n} = \frac{1}{n}(\partial_{x'_n} - \partial_{x''_n}) \).

**Proof.** Note that

\[
e^{\pm \delta}(z) = z^{\pm \delta(0)} e^{\pm \delta} \exp \left( \pm \sum_{n > 0} \delta(-n) z^n / n \right) \exp \left( \pm \sum_{n > 0} \delta(n) z^{-n} / -n \right)
\]

\[
= q^{\pm 1} \exp \left( \pm \sum_{n > 0} t_n z^n \right) \exp \left( \pm \sum_{n > 0} (\tilde{\partial}_y - \tilde{\partial}_{x_n}) z^{-n} \right),
\]

where \( \tilde{\partial}_{x_n} = \frac{1}{n} \partial_{x_n} \). Then

\[
e(z) \otimes f(z) = :\alpha'(z) e^{\delta'}(z) \gamma''(z) e^{-\delta''}(z) : = :\alpha'(z) \gamma''(z) e^{\delta'-\delta''}(z) :.
\]

Expanding the exponentials in \( e^{\delta'-\delta''}(z) \), we obtain

\[
e^{\delta'-\delta''}(z) = q'(q'')^{-1} \sum_{l,m \in \mathbb{Z}} S_l(t' - t'') S_m(-\tilde{\partial}_{x'} + \tilde{\partial}_{y'} + \tilde{\partial}_{x''} - \tilde{\partial}_{y''}) z^{l+m}
\]

\[
= q'(q'')^{-1} \sum_{l,m \in \mathbb{Z}} S_l(2t) S_m(-\tilde{\partial}_{x'} + \tilde{\partial}_{y'}) z^{l+m},
\]

which completes the proof. \( \square \)

Following the procedure of the Japanese school [KM, DKM, DJKM] (see also [MJD, KR]), we will rewrite the untwisted Wakimoto hierarchy (3.1) in terms of Hirota bilinear equations. Let us recall their definition.

**Definition 3.4.** Given a differential operator \( P(\partial_x) \) and two functions \( f(x), g(x) \), we define the Hirota bilinear operator \( Pf \cdot g \) to be

\[
P f \cdot g = P(\partial_u) \left. (f(x + u) g(x - u)) \right|_{u=0}
\]

where \( x = (x_1, x_2, \ldots) \), \( \partial_x = (\partial_{x_1}, \partial_{x_2}, \ldots) \), etc.

As above, we will consider

\[
\tau \otimes \tau \in \mathbb{C}[x', y', t', x'', y'', t''; (q')^\pm 1, (q'')^\pm 1].
\]
We will make the change of variables
\[ x'_n = x_n + \bar{x}_n, \quad x''_n = x_n - \bar{x}_n, \]
\[ \partial x'_n = \frac{1}{2} (\partial x_n + \partial \bar{x}_n), \quad \partial x''_n = \frac{1}{2} (\partial x_n - \partial \bar{x}_n), \]
and similarly for \( y', y'' \) and \( t', t'' \). Then
\[ \tau \otimes \tau = \tau(x', y', t'; q') \tau(x'', y'', t''; q'') = \tau(x + \bar{x}, y + \bar{y}, t + \bar{t}; q') \tau(x - \bar{x}, y - \bar{y}, t - \bar{t}; q''). \]

In order to rewrite the equations in Hirota bilinear form, we recall the formula (see e.g. [KR]):
\[ (3.3) \quad P(\partial_u)\tau(x + \bar{x})\tau(x - \bar{x}) = P(\partial_u)\tau(x + \bar{x} + u)\tau(x - \bar{x} - u)|_{u=0} = Q(\partial_u)\tau(x + u)\tau(x - u)|_{u=0} = Q \tau \cdot \tau, \]
where
\[ Q(\partial_u) = P(\partial_u) \exp\left(\sum_{j=1}^{\infty} \bar{x}_j \partial u_j \right). \]

Using Lemma 3.3 and the above notation, we obtain:
\[ (3.4) \quad \text{Res}_z e(z) \tau \otimes f(z) \tau = q'(q'')^{-1} \sum_{i,j,l \in \mathbb{Z}} S_l(2\tilde{u}) :a'_{(i)} g''_{(j)} :S_{l-i-j-1}(-\tilde{\partial}_u + \tilde{\partial}_v) E \tau \cdot \tau, \]
where
\[ E = \exp\left(\sum_{j=1}^{\infty} \bar{x}_j \partial u_j + \bar{y}_j \partial v_j + \bar{t}_j \partial w_j \right), \]
and
\[ a'_{(i)} = \begin{cases} q' \partial q', & i = 0, \\ -i(x_i + \bar{x}_{-i}), & i < 0, \\ \frac{1}{2}(\partial x_i + \partial \bar{x}_i) + \frac{1}{2}(\partial y_i + \partial \bar{y}_i) + \frac{1}{2}(\partial t_i + \partial \bar{t}_i), & i > 0, \end{cases} \]
and
\[ g''_{(j)} = \begin{cases} -q'' \partial q'', & j = 0, \\ -j(y_{-j} - \bar{y}_{-j}), & j < 0, \\ \frac{1}{2}(\partial x_j - \partial \bar{x}_j) + \frac{1}{2}(\partial y_j - \partial \bar{y}_j) - \frac{1}{2}(\partial t_j - \partial \bar{t}_j), & j > 0. \end{cases} \]

Note that \( \alpha'_{(i)} = \alpha_{(i)} \otimes 1 \) and \( \gamma''_{(j)} = 1 \otimes \gamma_{(j)} \) commute, while \( a'_{(i)} \) and \( g''_{(j)} \) do not commute for \( i = -j \). As usual, the normally ordered product \( :a'_{(i)} g''_{(j)} : \) is defined by putting all partial derivatives to the right.
Similarly, by switching the single-primed and double-primed terms, we find:

\[
\text{Res}_z f(z)\tau \otimes e(z)\tau = (q')^{-1}q'' \sum_{i,j,l \in \mathbb{Z}} S_l(-2t) : a''_{(i)} g'_{(j)} : S_{l-i-j-1}(\tilde{\partial}_u - \tilde{\partial}_v) E \tau \cdot \tau ,
\]

where

\[
a''_{(i)} = \begin{cases}
    q'' \partial_{q''}, & i = 0, \\
    -i(x_i - \bar{x}_i), & i < 0, \\
    \frac{1}{2}(\partial_{x_i} - \partial_{u_i}) + \frac{1}{2}(\partial_{y_i} - \partial_{v_i}) + \frac{1}{2}(\partial_{t_i} - \partial_{w_i}), & i > 0,
\end{cases}
\]

and

\[
g'_{(j)} = \begin{cases}
    -q' \partial_{q'}, & j = 0, \\
    -j(y_j + \bar{y}_j), & j < 0, \\
    \frac{1}{2}(\partial_{x_j} + \partial_{u_j}) + \frac{1}{2}(\partial_{y_j} + \partial_{v_j}) - \frac{1}{2}(\partial_{t_j} + \partial_{w_j}), & j > 0.
\end{cases}
\]

The other terms in (3.1) are easy to compute. Recalling that

\[h_{(j)} = \alpha_{(j)} - \gamma_{(j)} + k\delta_{(j)}, \quad k = c - 1,\]

we get:

\[
\text{Res}_z h(z)\tau \otimes h(z)\tau = \sum_{j \in \mathbb{Z}} h'_{(j-1)} h''_{(j)} E \tau \cdot \tau ,
\]

where

\[
h'_{(i)} = \begin{cases}
    2q' \partial_{q'}, & i = 0, \\
    -i(x_{-i} + \bar{x}_{-i}) + i(y_{-i} + \bar{y}_{-i}) - ki(t_{-i} + \bar{t}_{-i}), & i < 0, \\
    \partial_{t_i} + \partial_{w_i}, & i > 0,
\end{cases}
\]

and

\[
h''_{(j)} = \begin{cases}
    2q'' \partial_{q''}, & j = 0, \\
    -j(x_{-j} - \bar{x}_{-j}) + j(y_{-j} - \bar{y}_{-j}) - kj(t_{-j} - \bar{t}_{-j}), & j < 0, \\
    \partial_{t_j} - \partial_{w_j}, & j > 0.
\end{cases}
\]
Finally, we observe that $\text{Res}_z L(z) = L_{-1}$ and apply Lemma 2.3 to find:

$$
\text{Res}_z L(z) \tau \otimes \tau = \frac{c-1}{4} \sum_{j \in \mathbb{Z}} d'_{(j)} d'_{(-1-j)} E \tau \cdot \tau \\
+ \frac{1}{2} \sum_{j \in \mathbb{Z}} d'_{(j)} (a'_{(-1-j)} - g'_{(-1-j)}) E \tau \cdot \tau \\
+ \frac{1}{4(c+1)} \sum_{j \in \mathbb{Z}} (a'_{(j)} + g'_{(j)}) (a'_{(-1-j)} + g'_{(-1-j)}) E \tau \cdot \tau ,
$$

(3.7)

where

$$
d'_{(j)} = \begin{cases} 
0, & j = 0, \\
-j(t_{-j} + \bar{t}_{-j}), & j < 0, \\
\frac{1}{2} (\partial x_j + \partial u_j) - \frac{1}{2} (\partial y_j + \partial v_j), & j > 0.
\end{cases}
$$

Then $\text{Res}_z \tau \otimes L(z) \tau$ is obtained by switching the single-primed and double-primed terms, where

$$
d''_{(j)} = \begin{cases} 
0, & j = 0, \\
-j(t_{-j} - \bar{t}_{-j}), & j < 0, \\
\frac{1}{2} (\partial x_j - \partial u_j) - \frac{1}{2} (\partial y_j - \partial v_j), & j > 0.
\end{cases}
$$

In this way, we have rewritten all terms from (3.1) as Hirota bilinear operators. We expand $\tau$ as

$$
\tau = \sum_{m \in \mathbb{Z}} \tau_m (x, y, t) q^m , \quad \tau_m \in \mathbb{C}[x, y, t] ;
$$

then

(3.8)

$$
\tau \otimes \tau = \sum_{m, n \in \mathbb{Z}} \tau_m (x', y', t') \tau_n (x'', y'', t'') (q')^m (q'')^n .
$$

Since the functions $\tau_m$ do not depend on any of the variables $\bar{x}_i, \bar{y}_i, \bar{t}_i, q'$ and $q''$, all coefficients in front of monomials in these variables give Hirota bilinear equations for $\tau_m$. Observe that, in order to get $(q')^m (q'')^n$ in (3.1), we need to apply $\text{Res}_z e(z) \otimes f(z)$ to the summand

$$
\tau_{m-1} (x', y', t') \tau_{n+1} (x'', y'', t'') (q')^{m-1} (q'')^{n+1}
$$

from (3.8). Similarly, $\text{Res}_z f(z) \otimes e(z)$ is applied to

$$
\tau_{m+1} (x', y', t') \tau_{n-1} (x'', y'', t'') (q')^{m+1} (q'')^{n-1}
$$

On the other hand, $\frac{1}{2} \text{Res}_z h(z) \otimes h(z)$ and $L_{-1} \otimes 1 + 1 \otimes L_{-1}$ have to be applied to

$$
\tau_m (x', y', t') \tau_n (x'', y'', t'') (q')^m (q'')^n .
$$
If we further specialize to $m = n = 0$, we get equations for the three functions $\tau_{-1}$, $\tau_0$ and $\tau_1$. We will find the simplest such equations after setting

$$x_i = y_i = t_i = \bar{x}_i = \bar{y}_i = \bar{t}_{i-1} = 0, \quad i \geq 2.$$ 

Then (3.4) reduces to

$$\text{Res}_z e(z) \tau_{-1} \otimes f(z) \tau_1 = \sum_{i,j \geq -1, i+j \leq -1} \lambda_i \lambda_j \partial_{u_1}^i \partial_{v_1}^j E_1 \tau_{-1} \cdot \tau_1,$$

where now $a'(0) = g''(0) = -1$ and

$$E_1 = \exp(x_1 \partial_{u_1} + y_1 \partial_{v_1}).$$

Similarly, from (3.5) we have

$$\text{Res}_z f(z) \tau_1 \otimes e(z) \tau_{-1} = \sum_{i,j \geq -1, i+j \leq -1} \lambda_i \lambda_j \partial_{u_1}^i \partial_{v_1}^j \partial_{u_1} E_1 \tau_1 \cdot \tau_{-1},$$

where $a''(0) = g'(0) = -1$. The remaining terms from (3.1) become zero when applied to $\tau_0 \otimes \tau_0$.

Note that for any polynomial $P$, we have

$$P(\partial_{u_1}, \partial_{v_1}, \partial_{w_1}) \tau_1 \cdot \tau_{-1} = P(-\partial_{u_1}, -\partial_{v_1}, -\partial_{w_1}) \tau_{-1} \cdot \tau_1.$$

Using this, from the coefficient of 1 in (3.1), we find

$$[x_1 y_1 (\partial_{u_1} - \partial_{v_1}) + (x_1 + y_1)] \tau_{-1} \cdot \tau_1 = 0.$$

Similarly, the coefficient of $x_1^2$ in (3.1) gives the equation

$$[x_1 y_1 (\partial_{u_1} - \partial_{v_1}) \partial_{u_1}^2 + (x_1 + y_1) \partial_{u_1}^2 + 2y_1 (\partial_{u_1} - \partial_{v_1}) \partial_{u_1} + 2 \partial_{u_1}] \tau_{-1} \cdot \tau_1 = 0.$$

4. THE TWISTED WAKIMOTO HIERARCHY

We now investigate the integrable hierarchy arising from a twisted representation of $\widehat{\mathfrak{sl}_2}$. Recall from Theorem 2.1 that the embedding of $\widehat{\mathfrak{sl}_2}$ of level $k$ in the lattice vertex algebra $V_L$, where $L$ is a lattice containing an element $\delta$ such that

$$\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L = \text{span}\{\alpha, \gamma, \delta\}$$

and

$$|\delta|^2 = 0, \quad |\alpha|^2 = |\gamma|^2 = (\delta|\alpha) = -(\delta|\gamma) = 1, \quad (\alpha|\gamma) = c := k + 1.$$
Explicitly, \( \hat{\mathfrak{sl}}_2 \) is realized in \( V_L \) as
\[
e = \alpha(-1)e^\delta, \quad f = \gamma(-1)e^{-\delta}, \quad h = k\delta + \alpha - \gamma.
\]
Observe that \( h \) has an order 2 isometry \( \sigma \) given by
\[
\sigma(\alpha) = \gamma, \quad \sigma(\gamma) = \alpha, \quad \sigma(\delta) = -\delta,
\]
which preserves the \( \hat{\mathfrak{sl}}_2 \) subalgebra described above. In fact, \( \sigma \) operates on \( \mathfrak{sl}_2 \) as the involution \( \sigma = \exp(\frac{\pi i}{2} \text{ad}_e + \text{ad}_f) \), which acts as
\[
\sigma(e) = f, \quad \sigma(f) = e, \quad \sigma(h) = -h.
\]
Composing the embedding \( \hat{\mathfrak{sl}}_2 \hookrightarrow V_L \) with any \( \sigma \)-twisted representation of \( V_L \), we will obtain a representation of \( \hat{\mathfrak{sl}}_2 \).

We refer the reader to [L, FLM, FFR, D, BK] for twisted modules over lattice vertex algebras. Here we will only need a special case. Recall first the \( \sigma \)-twisted Heisenberg algebra \( \hat{\mathfrak{h}}_\sigma \), spanned over \( \mathbb{C} \) by a central element \( I \) and elements \( a_{(m)} \) (\( a \in \mathfrak{h}, m \in \frac{1}{2}\mathbb{Z} \)) such that \( \sigma a = e^{-2\pi i m}a \) (see e.g. [KP, L, FLM]). The Lie bracket on \( \hat{\mathfrak{h}}_\sigma \) is given by
\[
[a_{(m)}, b_{(n)}] = m\delta_{m,-n}(a|b)I, \quad a, b \in \mathfrak{h}, \quad m, n \in \frac{1}{2}\mathbb{Z}.
\]
Let \( \hat{\mathfrak{h}}_\sigma^\geq \) (respectively, \( \hat{\mathfrak{h}}_\sigma^\leq \)) be the subalgebra of \( \hat{\mathfrak{h}}_\sigma \) spanned by all elements \( a_{(m)} \) with \( m \geq 0 \) (respectively, \( m < 0 \)). Consider the irreducible highest-weight \( \hat{\mathfrak{h}}_\sigma \)-module \( M = S(\hat{\mathfrak{h}}_\sigma^\leq) \), called the \( \sigma \)-twisted Fock space, on which \( I \) acts as the identity operator and \( \hat{\mathfrak{h}}_\sigma^\geq \) annihilates the highest-weight vector \( 1 \in M \).

We will denote by \( a_{(j)}^M \) the linear operator on \( M \) induced by the action of \( a_{(j)} \in \hat{\mathfrak{h}}_\sigma \), and will write the twisted fields as:
\[
Y^M(a, z) = \sum_{j \in \frac{1}{2}\mathbb{Z}} a_{(j)}^Me^{-j-1}z^{-j-1}, \quad a_{(j)}^M \in \text{End } M.
\]
One of the main properties of twisted fields is the \( \sigma \)-equivariance
\[
Y^M(\sigma a, z) = Y^M(a, e^{2\pi i z}) = \sum_{j \in \frac{1}{2}\mathbb{Z}} a_{(j)}^Me^{-2\pi ijz}e^{-j-1}z^{-j-1}.
\]
In our case, this means that when \( \sigma a = a \) the modes \( a_{(j)}^M \) are nonzero only for \( j \in \mathbb{Z} \). On the other hand, if \( \sigma a = -a \) the modes \( a_{(j)}^M \) are nonzero only for \( j \in \frac{1}{2} + \mathbb{Z} \). Note that the eigenspaces of \( \sigma \) on \( \mathfrak{h} \) are spanned by \( \delta, \alpha - \gamma \) and by \( \alpha + \gamma \).
We have the inner products

\[ \| \alpha \pm \gamma \|_2 = c_\pm := \frac{1 \pm c}{2}, \quad \left( \frac{\alpha - \gamma}{2} \right)_\delta = 1, \]

and all other inner products are zero. Then we can identify

\[ M = \mathbb{C}[x,t], \quad \text{where} \quad x = (x_1, x_2, x_3, \ldots), \quad t = (t_1, t_3, t_5, \ldots) \]

and the action of \( \hat{\mathfrak{h}}_\sigma \) is given by:

\[
\delta^M_{(j)} = \begin{cases} 
\partial_{t_{2j}}, & j \in (\frac{1}{2} + \mathbb{Z})_{>0}, \\
-jx_{-2j}, & j \in (\frac{1}{2} + \mathbb{Z})_{<0}, 
\end{cases}
\]

\[
\left(\frac{\alpha - \gamma}{2}\right)^M_{(j)} = \begin{cases} 
\partial_{x_{2j}}, & j \in (\frac{1}{2} + \mathbb{Z})_{>0}, \\
-j(t_{-2j} + c_{-x_{-2j}}), & j \in (\frac{1}{2} + \mathbb{Z})_{<0}, 
\end{cases}
\]

\[
\left(\frac{\alpha + \gamma}{2}\right)^M_{(j)} = \begin{cases} 
\partial_{x_{2j}}, & j \in \mathbb{Z}_{>0}, \\
-jc_{x_{-2j}}, & j \in \mathbb{Z}_{<0}, 
\end{cases}
\]

From here we obtain

\[
\alpha^M_{(j)} = \begin{cases} 
\partial_{x_{2j}}, & j \in (\frac{1}{2} \mathbb{Z})_{>0}, \\
-j(t_{-2j} + c_{-x_{-2j}}), & j \in (\frac{1}{2} + \mathbb{Z})_{<0}, \\
-jc_{x_{-2j}}, & j \in \mathbb{Z}_{<0}.
\end{cases}
\]

Recall that we also have twisted fields corresponding to \( e^{\pm \delta} \) (see [L, FLM]):

\[
Y^M(e^{\pm \delta}, z) = \exp \left( \mp \sum_{j \in (\frac{1}{2} + \mathbb{Z})_{<0}} \delta^M_{(j)} \frac{z^{-j}}{j} \right) \exp \left( \mp \sum_{j \in (\frac{1}{2} + \mathbb{Z})_{>0}} \delta^M_{(j)} \frac{z^{-j}}{j} \right)
\]

\[
= \exp \left( \mp \sum_{j \in (\frac{1}{2} + \mathbb{Z})_{>0}} x_{2j} z^j \right) \exp \left( \mp \sum_{j \in (\frac{1}{2} + \mathbb{Z})_{<0}} \partial_{t_{2j}} \frac{z^{-j}}{j} \right).
\]

By definition, these also satisfy the \( \sigma \)-equivariance (4.3). Then the embedding (4.1) allows one to extend \( Y^M \) to the generators of \( \hat{\mathfrak{sl}}_2 \).

**Lemma 4.1.** We have:

\[ Y^M(e, z) = Y^M(\alpha(-1)e^\delta, z) = : Y^M(\alpha, z)Y^M(e^\delta, z) : = - \frac{1}{2z} Y^M(e^\delta, z). \]

**Proof.** Recall that for \( a \in \mathfrak{h} \), we have

\[ a_{(0)} e^\delta = (a|\delta) e^\delta, \quad a_{(m)} e^\delta = 0, \quad m > 0. \]
It follows from (3.13) in [BK] that
\[ Y_M(a, z) Y_M(e^\delta, z) = \sum_{m=0}^{\infty} \left( \frac{p}{m} \right) z^{-m} Y_M(a_{(m-1)} e^\delta, z) = Y_M(a_{(-1)} e^\delta, z) + pz^{-1}(a|\delta) Y_M(e^\delta, z), \]
where \( a \in h \) and \( p \in \{0, \frac{1}{2}\} \) are such that \( \sigma a = e^{2\pi ip} a \).

Now for \( a = \alpha + \gamma \), we have \( p = 0 \) and
\[ Y_M(\alpha + \gamma, z) Y_M(e^\delta, z) = Y_M((\alpha + \gamma)_{(-1)} e^\delta, z). \]
Similarly, for \( a = \alpha - \gamma \), we have \( p = \frac{1}{2} \) and
\[ Y_M(\alpha - \gamma, z) Y_M(e^\delta, z) = Y_M((\alpha - \gamma)_{(-1)} e^\delta, z) + z^{-1} Y_M(e^\delta, z), \]
since \( (\alpha - \gamma|\delta) = 2 \). Adding these two equations gives the desired result.

The above lemma can also be used to express \( Y_M(f, z) \), since
\[ Y_M(f, z) = Y_M(e, e^{2\pi i z}) = Y_M(e, e^{-2\pi i z}) \]
by (1.2), (1.3). Recall that \( V_L \) has a Virasoro element \( \omega \) so that
\[ Y(\omega, z) = L_M(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \]
Explicitly, by Lemma 2.3 we have:
\[ \omega = \frac{c - 1}{4} \delta_{(-1)} \delta + \frac{1}{2} \delta_{(-1)} (\alpha - \gamma) + \frac{1}{4(c+1)} (\alpha + \gamma)_{(-1)} (\alpha + \gamma). \]

Then we have an action of the Virasoro algebra on the twisted module \( M \), given by
\[ Y_M(\omega, z) = L_M(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \]

**Lemma 4.2.** We have:
\[ L_n^M = \frac{c - 1}{4} \sum_{j \in \frac{1}{2} + \mathbb{Z}} :\delta^{M}_{ij} \delta^{M}_{(n-j)} : + \frac{1}{2} \sum_{j \in \frac{1}{2} + \mathbb{Z}} :\delta^{M}_{ij} (\alpha - \gamma)^{M}_{(n-j)} : + \frac{1}{4(c+1)} \sum_{i \in \mathbb{Z}} : (\alpha + \gamma)^{M}_{(i)} (\alpha + \gamma)^{M}_{(n-i)} : + \frac{1}{8} \delta_{n,0}. \]

**Proof.** Recall that for \( a, b \in h \), we have
\[ a_{(m)} b = \delta_{m,1} (a|b), \quad m \geq 0. \]
Then, as in the proof of Lemma 4.1 above,

\[ Y^M(a, z) Y^M(b, z) = Y^M(a, z)(-1) Y^M(b, z) \]

\[= \sum_{m=0}^{\infty} \left( \frac{p}{m} \right) z^{-m} Y^M(a_{(m)} b, z) \]

\[= Y^M(a_{(-1)} b, z) + \frac{p(p-1)}{2} z^{-2}(a|b), \]

where \( p \in \{0, \frac{1}{2}\} \) is such that \( \sigma a = e^{2\pi ip} a \). Now computing \( Y^M(\omega, z) \), the last term in the above equation will be nonzero only when \( a = \delta \) and \( b = \alpha - \gamma \), in which case \( p = \frac{1}{2} \) and \( (\delta|\alpha - \gamma) = 2 \). □

The twisted version of the Casimir field \( \Omega(z) \) from Sect. 3 is

\[ \Omega^M(z) = Y^M(e, z) \otimes Y^M(f, z) + Y^M(f, z) \otimes Y^M(e, z) \]

\[+ \frac{1}{2} Y^M(h, z) \otimes Y^M(h, z) - k \otimes L^M(z) - L^M(z) \otimes k. \]

Note that

\[ Y^M(e, z) \otimes Y^M(f, z) + Y^M(f, z) \otimes Y^M(e, z) \]

\[= Y^M(e, z) \otimes Y^M(e, e^{2\pi i} z) + Y^M(e, e^{-2\pi i} z) \otimes Y^M(e, z). \]

Therefore, when computing the coefficients in front of integral powers of \( z \) in \( \Omega^M(z) \), we can replace the first two terms,

\[ Y^M(e, z) \otimes Y^M(f, z) + Y^M(f, z) \otimes Y^M(e, z), \]

with

\[ 2Y^M(e, z) \otimes Y^M(e, e^{2\pi i} z). \]

**Theorem 4.3.** The modes of the above twisted fields \( Y^M(e, z) \), \( Y^M(f, z) \) and \( Y^M(h, z) \) provide \( M \) with the structure of an \( \hat{sl}_2 \)-module of level \( k \). The modes of \( \Omega^M(z) \) commute with the diagonal action of \( \hat{sl}_2 \) on \( M \otimes M \), i.e.,

\[ [\Omega^M(z), a^M_{(m)} \otimes 1 + 1 \otimes a^M_{(m)}] = 0, \quad a \in \mathfrak{sl}_2, m \in \frac{1}{2} \mathbb{Z}. \]

**Proof.** We observe that the commutator formula for the modes of twisted fields,

\[ [a^M_{(m)}, b^M_{(n)}] = \sum_{j=0}^{\infty} \binom{m}{j} (a(j)b)_{(m+n-j)}^M, \]

is just like the commutator formula in the vertex algebra itself,

\[ [a_{(m)}, b_{(n)}] = \sum_{j=0}^{\infty} \binom{m}{j} (a(j)b)_{(m+n-j)}, \]
provided that $a$ is an eigenvector of $\sigma$ (see e.g. [FLM, FFR, D]). However, in the modes $a^M_{(m)}$ of twisted fields, the index $m$ is allowed to be nonintegral.

We already know from Theorem 2.1 that

$$[a^M_{(m)}, b^M_{(n)}] = [a, b]_{(m+n)} + m\delta_{m,-n} (a|b) k,$$

where $(a|b) = \text{tr}(ab)$. Thus for $a \in \{h, e+f, e-f\}$, we have:

$$[a^M_{(m)}, b^M_{(n)}] = [a, b]_{(m+n)} + m\delta_{m,-n} (a|b) k,$$

where $m \in \mathbb{Z}$ for $a = e+f$ and $m \in \frac{1}{2} \mathbb{Z}$ for $a = h, e-f$. Since $\sigma$ is an inner automorphism of $\mathfrak{sl}_2$, by Theorem 8.5 in [K1] the above modes $a^M_{(m)}$ give a representation of the affine Kac–Moody algebra $\hat{\mathfrak{sl}}_2$.

The statement about $\Omega^M_{(z)}$ follows again from the commutator formula and Proposition 3.1.

Note that the vector $1 \in M$ satisfies

$$a^M_{(j)} 1 = (e^{\pm \delta})^M_{(j)} 1 = 0, \quad a \in \mathfrak{h}, \; j \in \frac{1}{2} \mathbb{Z}_{\geq 0}.$$

By Lemmas 3.3, 4.2, this implies

$$a^M_{(m)} 1 = L^M_{1} = 0, \quad a \in \mathfrak{sl}_2, \; n \geq 1,$$

while $e^M_{(0)} 1 = -\frac{1}{2}$ and $L^M_{0} 1 = \frac{1}{8}$. In particular,

$$\Omega^M_{(m)} (1 \otimes 1) = \text{Res}_z z^m \Omega^M(z)(1 \otimes 1) = 0, \quad n \geq 2.$$

Similarly to Corollary 3.2, we have the following.

**Corollary 4.4.** Every vector $\tau \in M$, such that $\tau \otimes \tau$ is in the $\hat{\mathfrak{sl}}_2$-submodule of $M \otimes M$ generated by $1 \otimes 1$, satisfies the equation

$$(4.4) \quad \Omega^M_{(2)} (\tau \otimes \tau) = 0.$$

We will call (4.4) the twisted Wakimoto hierarchy. As in Sect. 3 we will compute explicitly the action of $\Omega^M_{(2)}$ on $M \otimes M$. We use the same notation as before regarding primed and double-primed objects,

$$x'_n = x_n \otimes 1, \quad x''_n = 1 \otimes x_n, \quad \text{etc.},$$

and make the change of variables

$$x'_n = x_n + \bar{x}_n, \quad x''_n = x_n - \bar{x}_n,$$

$$\partial x'_n = \frac{1}{2} (\partial x_n + \partial \bar{x}_n), \quad \partial x''_n = \frac{1}{2} (\partial x_n - \partial \bar{x}_n).$$

Introduce the “reduced” Schur polynomials

$$R_m(t) = S_m(t_1, 0, t_3, 0, t_5, 0, \ldots),$$
where \( S_m(t) \) are the elementary Schur polynomials defined by (3.2).

Then we can compute the first term in the expression

\[
\Omega^M_{(2)} = 2 \text{Res}_z z^2 Y^M(e, z) \otimes Y^M(e, e^{2\pi i} z) \\
+ \frac{1}{2} \text{Res}_z z^2 Y^M(h, z) \otimes Y^M(h, z) - k \otimes L^M_1 - L^M_1 \otimes k.
\]

\[(4.5)\]

**Lemma 4.5.** We have:

\[
\text{Res}_z z^2 Y^M(e, z) \otimes Y^M(e, e^{2\pi i} z) = \\
\sum_{i,j,l \in \mathbb{Z}} (-1)^j R_l(2\bar{x})(\alpha_{i/2}^M)'(\alpha_{j/2}^M)'' R_{l-i-j+2}(-2\bar{\theta}_t) \\
- \frac{1}{2} \sum_{j,l \in \mathbb{Z}} R_l(2\bar{x})((\alpha_{j/2}^M)' + (-1)^j(\alpha_{j/2}^M)'' R_{l-j+2}(-2\bar{\theta}_t) \\
+ \frac{1}{4} \sum_{l \in \mathbb{Z}} R_l(2\bar{x}) R_{l+2}(-2\bar{\theta}_t),
\]

where \( 2\bar{x}_n = x'_n - x''_n \) and \( \bar{\theta}_n = \frac{1}{\pi}(\partial'_n - \partial''_n) \).

**Proof.** Using Lemma 4.1 we obtain

\[
Y^M(e, z) \otimes Y^M(e, e^{2\pi i} z) = :Y^M(\alpha', z)Y^M(\alpha'', e^{2\pi i} z)Y^M(e^{\delta' - \delta''}, z) + \\
- \frac{1}{2 z}(Y^M(\alpha', z) + Y^M(\alpha'', e^{2\pi i} z))Y^M(e^{\delta' - \delta''}, z) + \\
+ \frac{1}{4 z^2} Y^M(e^{\delta' - \delta''}, z).
\]

Then we expand

\[
Y^M(e^{\delta' - \delta''}, z) = \\
\exp \left( \sum_{j \in \left( \frac{1}{2} + z \right)_{>0}} (x'_j - x''_j) z^j \right) \exp \left( \sum_{j \in \left( \frac{1}{2} + z \right)_{>0}} (-\partial'_j + \partial''_j) \frac{z^{-j}}{\bar{j}} \right) \\
= \exp \left( \sum_{j \in \left( \frac{1}{2} + z \right)_{>0}} 2\bar{x}_j z^j \right) \exp \left( - \sum_{j \in \left( \frac{1}{2} + z \right)_{>0}} 2\bar{\theta}_j z^{-j} \right) \\
= \sum_{l,m \in \mathbb{Z}} R_l(2\bar{x}) R_m(-2\bar{\theta}_t) z^{(l-m)/2}.
\]

We finish the proof by finding the coefficient of \( z^{-3} \) in \( Y^M(e, z) \otimes Y^M(e, e^{2\pi i} z) \).

Now, as in Sect. 3, we can express the action of \( \Omega^M_{(2)} \) on \( \tau \otimes \tau \) in terms of Hirota bilinear operators using formula (3.3). The recipe is
while $\tau \otimes \tau$ gets replaced with $E \tau \cdot \tau$, where
\[
E = \exp \left( \sum_{j=1}^{\infty} \bar{x}_j \partial u_j + \sum_{i=0}^{\infty} \bar{t}_{2i+1} \partial w_{2i+1} \right),
\]
and, accordingly, $\partial_\bar{u}$ is replaced with $\partial_u$, while $\partial_\bar{\tau}$ is replaced with $\partial_w$.

Then $(\alpha_{(j)}^M)'$ becomes
\[
d_{(j)}' = \begin{cases} \frac{1}{2}(\partial_{x_{2j}} + \partial_{u_{2j}}), & j \in (\frac{1}{2}Z)_{>0}, \\ -j(t_{-2j} + \bar{\tau}_{-2j}) - jc_-(x_{-2j} + \bar{x}_{-2j}), & j \in (\frac{1}{2} + Z)_{<0}, \\ -jc_+(x_{-2j} + \bar{x}_{-2j}), & j \in Z_{<0}, \end{cases}
\]
while $(\alpha_{(j)}^M)''$ becomes
\[
d_{(j)}'' = \begin{cases} \frac{1}{2}(\partial_{x_{2j}} - \partial_{u_{2j}}), & j \in (\frac{1}{2}Z)_{>0}, \\ -j(t_{-2j} - \bar{\tau}_{-2j}) - jc_-(x_{-2j} - \bar{x}_{-2j}), & j \in (\frac{1}{2} + Z)_{<0}, \\ -jc_+(x_{-2j} - \bar{x}_{-2j}), & j \in Z_{<0}. \end{cases}
\]

Putting these together, we obtain from Lemma 4.5
\[
\text{Res}_{z^2} Y^M(e, z)\tau \otimes Y^M(e, e^{2\pi i} z)\tau
= \sum_{i,j,l \in \mathbb{Z}} (-1)^j R_l(2\bar{x}) :a_{(i/2)}' a_{(j/2)}'' : \tilde{R}_{l-i-j+2}(-2\tilde{\delta}_w) E \tau \cdot \tau \\
\quad - \frac{1}{2} \sum_{i,j,l \in \mathbb{Z}} R_l(2\bar{x}) (a_{(i/2)}' + (-1)^j a_{(j/2)}'') R_{l-j+2}(-2\tilde{\delta}_w) E \tau \cdot \tau \\
\quad + \frac{1}{4} \sum_{l \in \mathbb{Z}} R_l(2\bar{x}) R_{l+2}(-2\tilde{\delta}_w) E \tau \cdot \tau.
\]

Recall that, by (4.3),
\[
\gamma_{(j)}^M = (-1)^{2j} \alpha_{(j)}^M, \quad j \in \frac{1}{2} \mathbb{Z}.
\]

So, to compute the other terms in $\Omega_{(2)}^M$, we just need to replace $(\delta_{(j)}^M)'$ with
\[
d_{(j)}' = \begin{cases} \frac{1}{2}(\partial_{t_{2j}} + \partial_{w_{2j}}), & j \in (\frac{1}{2} + Z)_{>0}, \\ -j(x_{-2j} + \bar{x}_{-2j}), & j \in (\frac{1}{2} + Z)_{<0}, \end{cases}
\]
and $(\delta_{(j)}^M)''$ with
\[
d_{(j)}'' = \begin{cases} \frac{1}{2}(\partial_{t_{2j}} - \partial_{w_{2j}}), & j \in (\frac{1}{2} + Z)_{>0}, \\ -j(x_{-2j} - \bar{x}_{-2j}), & j \in (\frac{1}{2} + Z)_{<0}. \end{cases}
\]
Then we obtain:

\[ (4.7) \quad \text{Res}_z z^2 Y^M(h, z) \tau \otimes Y^M(h, z) \tau = \sum_{j \in \frac{1}{2} + \mathbb{Z}} h'_j h''_{(1-j)} E \cdot \tau , \]

where

\[ h'_j = kd'_j + 2a'_j, \quad h''_j = kd''_j + 2a''_j, \quad j \in \frac{1}{2} + \mathbb{Z}. \]

Finally, we get from Lemma 4.2

\[ (4.8) \quad L^M_1 \tau \otimes \tau = \frac{c - 1}{4} \sum_{j \in \frac{1}{2} + \mathbb{Z}} d'_j d'_{(1-j)} E \cdot \tau \]

\[ + \sum_{j \in \frac{1}{2} + \mathbb{Z}} d'_j a'_{(1-j)} E \cdot \tau \cdot \tau + \frac{1}{c+1} \sum_{i \in \mathbb{Z}} a'_i a'_{(1-i)} E \cdot \tau . \]

Similarly, \( \tau \otimes L^M_1 \tau \) is given by the same formula with all primes replaced with double primes.

This completes the rewriting of the twisted Wakimoto hierarchy in terms of Hirota bilinear operators. Then, as in Sect. 3, all coefficients in front of monomials in the variables \( \bar{x}_i, \bar{t}_i \) give Hirota bilinear equations for \( \tau \). We will find the simplest such equation by setting

\[ \bar{x}_i = x_{i+2} = \bar{t}_{2i-1} = t_{2i+1} = 0, \quad i \geq 1. \]

Then \( a'_j = a''_j = 0 \) for \( j < -1 \) and \( d'_m = d''_m = 0 \) for \( m < -\frac{1}{2} \). Thus we obtain:

\[ \Omega^M_{(2)}(\tau \otimes \tau) = 2 \sum_{i,j \geq -\frac{1}{2}} (-1)^j a'_{(i/2)} a''_{(j/2)} : R_{2-i-j}(-2\bar{\partial}_w) \tau \cdot \tau \]

\[ - \sum_{j=\pm 1, \pm 2} (a'_{(j/2)} + (-1)^j a''_{(j/2)}) R_{2-j}(-2\bar{\partial}_w) \tau \cdot \tau \]

\[ + \frac{1}{2} R_2(-2\bar{\partial}_w) \tau \cdot \tau + \frac{1}{2} \sum_{j=\pm \frac{1}{2}, \pm \frac{3}{2}} h'_j h''_{(1-j)} \tau \cdot \tau \]

\[ - \frac{k^2}{4} \sum_{j=\pm \frac{1}{2}, \pm \frac{3}{2}} (d'_j d'_{(1-j)} + d''_j d''_{(1-j)}) \tau \cdot \tau \]

\[ - k \sum_{j=\pm \frac{1}{2}, \pm \frac{3}{2}} (d'_j a'_{(1-j)} + d''_j a''_{(1-j)}) \tau \cdot \tau \]

\[ - \frac{k}{k+2} \sum_{i=-1,2} (a'_{(i)} a'_{(1-i)} + a''_{(i)} a''_{(1-i)}) \tau \cdot \tau . \]
Note that, after setting all $\bar{x}_i$ and $\bar{t}_i$ equal to zero, we have $a'_{(j)} = a''_{(j)}$ and $d'_{(j)} = d''_{(j)}$ for $j < 0$. Also $P(\partial_u, \partial_w)\tau \cdot \tau = 0$ for any odd polynomial $P$, i.e., such that $P(-\partial_u, -\partial_w) = -P(\partial_u, \partial_w)$. Let us assume, in addition, that $\tau$ is independent of $x_3, x_4, t_3$ and $t_5$. Then the first term in the above sum simplifies to:

$$\left[-\frac{8}{45} c_+^2 x_2^2 \partial_{w_1}^6 - \frac{1}{3} (t_1 + c_- x_1)^2 \partial_{w_1}^4 - 2(t_1 + c_- x_1) \partial_{x_1} \partial_{w_1}^2 - 4c_+ x_2 \partial_{x_2} \partial_{w_1}^2 - \frac{1}{2} (\partial_{x_1}^2 - \partial_{w_1}^2)\right].$$

The other terms of $\Omega^{M(2)}$ are easier to compute and add up to:

$$\frac{4}{3} c_+ x_2 \partial_{w_1}^4 + 2\partial_u \partial_{w_1} + \partial_{x_2} + \frac{1}{2} \partial_{w_1}^2$$

$$+ \frac{k^2}{8} (\partial_{t_1}^2 - \partial_{w_1}^2) + \frac{k}{2} (\partial_{t_1} \partial_{x_1} - \partial_{u_1} \partial_{w_1}) + \frac{1}{2} (\partial_{x_1}^2 - \partial_{u_1}^2)$$

$$- \frac{k^2}{8} (\partial_{t_1}^2 + \partial_{w_1}^2) - \frac{k}{2} (\partial_{t_1} \partial_{x_1} + \partial_{u_1} \partial_{w_1}).$$

Putting these together, we obtain that the coefficient in front of 1 in $\Omega^{M(2)}$ gives the Hirota bilinear equation

$$\left[-\frac{8}{45} c_+^2 x_2^2 \partial_{w_1}^6 - \frac{1}{3} ((t_1 + c_- x_1)^2 - 4c_+ x_2) \partial_{w_1}^4$$

$$- 2(t_1 + c_- x_1) \partial_{x_1} \partial_{w_1}^2 + \frac{1}{4} (2 - k^2) \partial_{w_1}^2 + (2 - k) \partial_{u_1} \partial_{w_1} + \partial_{x_2}\right] \tau \cdot \tau = 0.$$

We then employ the change of variables,

$$x_2 = t, \quad t_1 = x, \quad x_1 = y, \quad u = \log(\tau),$$

which allows us to write the above as the evolutionary equation,

$$u_t = \frac{8}{45} c_+^2 t^2 \left(u_{xxxxx} + 30u_{xxxx}u_{xx} + 60u_{xx}^3\right)$$

$$+ \frac{1}{3} \left((x + c_- y)^2 - 4c_+ t\right) (u_{xxxx} + 6u_{xx}^2) + \frac{1}{2} (k^2 - 2) u_{xx}$$

$$+ (k - 2) u_{xy} + 2 (x + c_- y) (u_{xy} + 2u_{xx}u_y).$$

Note that at the critical level, $k = -2$, we have $c_+ = 0, c_- = 1$, and the above equation becomes

$$u_t = \frac{1}{3} (x + y)^2 \left(u_{xxxx} + 6u_{xx}^2\right) + 2(x + y) (u_{xy} + 2u_{xx}u_y) + u_{xx} - 4u_{xy}.$$

Another reduction is obtained by assuming $u_y = 0$ and letting $y = 0$. In this case, we get an order six non-autonomous non-linear PDE,
resembling those found in [DJKM],
\[
    u_t = \frac{2}{45}(k + 2)^2 t^2 \left( u_{xxxxx} + 30u_{xxx}u_{xx} + 60u_{xx}^3 \right) \\
    + \frac{1}{3} \left( x^2 - 2(k + 2)t \right) \left( u_{xxx} + 6u_{xx}^2 \right) + \frac{1}{2}(k^2 - 2)u_{xx}.
\]

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References

[A] D. Adamović, A construction of admissible $A_1^{(1)}$-modules of level $-\frac{1}{8}$. J. Pure Appl. Algebra 196 (2005), 119–134.

[BK] B. Bakalov and V.G. Kac, Twisted modules over lattice vertex algebras. In: “Lie theory and its applications in physics V,” 3–26, World Sci. Publishing, River Edge, NJ, 2004; math.QA/0402315.

[B] R.E. Borcherds, Vertex algebras, Kac–Moody algebras, and the Monster. Proc. Nat. Acad. Sci. USA 83 (1986), 3068–3071.

[DJKM] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, Transformation groups for soliton equations. III. Operator approach to the Kadomtsev–Petviashvili equation. J. Phys. Soc. Japan 50 (1981), 3806–3812.

[DKM] E. Date, M. Kashiwara, and T. Miwa, Transformation groups for soliton equations. II. Vertex operators and $\tau$ functions. Proc. Japan Acad. Ser. A Math. Sci. 57 (1981), 387–392.

[D] C. Dong, Twisted modules for vertex algebras associated with even lattices. J. Algebra 165 (1994), 91–112.

[FS] B.L. Feigin and A.M. Semikhatov, $W_n^{(2)}$ algebras. Nuclear Phys. B 698 (2004), 409–449.

[FF] A.J. Feingold and I.B. Frenkel, Classical affine algebras. Adv. Math. 56 (1985), 117–172.

[FFR] A.J. Feingold, I.B. Frenkel, and J.F.X. Ries, Spinor construction of vertex operator algebras, triality, and $E_8^{(1)}$. Contemporary Math., 121, Amer. Math. Soc., Providence, RI, 1991.

[F] E. Frenkel, Wakimoto modules, opers and the center at the critical level. Adv. Math. 195 (2005), 297–404.

[FB] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves. Math. Surveys and Monographs, vol. 88, Amer. Math. Soc., Providence, RI, 2001; 2nd ed., 2004.

[FK] I.B. Frenkel and V.G. Kac., Basic representations of affine Lie algebras and dual resonance models. Invent. Math. 62(1) (1980/81), 23–66.

[FLM] I.B. Frenkel, J. Lepowsky, and A. Meurman, Vertex operator algebras and the Monster. Pure and Appl. Math., vol. 134, Academic Press, Boston, 1988.

[FMS] D. Friedan, E. Martinec, and S. Shenker. Conformal invariance, supersymmetry and string theory. Nuclear Phys. B 271 (1986), 93–165.
BOSONIZATIONS OF $\hat{\mathfrak{sl}}_2$ AND INTEGRABLE HIERARCHIES

[1] P. Goddard, A. Kent, and D. Olive, Virasoro algebras and coset space models. Phys. Lett. B 152 (1985), 88–92.

[2] N. Jing, K.C. Misra, and C. Xu, Bosonic realization of toroidal Lie algebras of classical types. Proc. Amer. Math. Soc. 137 (2009), 3609–3618.

[3] V.G. Kac, Infinite-dimensional Lie algebras. 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[4] V.G. Kac, Vertex algebras for beginners. University Lecture Series, vol. 10, Amer. Math. Soc., Providence, RI, 1996; 2nd ed., 1998.

[5] V.G. Kac and D.H. Peterson, 112 constructions of the basic representation of the loop group of $E_8$. In: “Symposium on anomalies, geometry, topology” (Chicago, Ill., 1985), 276–298, World Sci. Publ., Singapore, 1985.

[6] V.G. Kac and A.K. Raina, Bombay lectures on highest weight representations of infinite-dimensional Lie algebras. Advanced Ser. in Math. Phys., 2. World Sci. Publ., Teaneck, NJ, 1987.

[7] V.G. Kac and M. Wakimoto, Exceptional hierarchies of soliton equations. In “Theta functions—Bowdoin 1987, Part 1” (Brunswick, ME, 1987), 191–237, Proc. Sympos. Pure Math. 49, Amer. Math. Soc., Providence, RI, 1989.

[8] M. Kashiwara and T. Miwa, Transformation groups for soliton equations. I. The $\tau$ function of the Kadomtsev–Petviashvili equation. Proc. Japan Acad. Ser. A Math. Sci. 57 (1981), 342–347.

[9] J. Lepowsky, Calculus of twisted vertex operators. Proc. Nat. Acad. Sci. USA 82 (1985), 8295–8299.

[10] J. Lepowsky and H. Li, Introduction to vertex operator algebras and their representations. Progress in Math., 227, Birkhäuser Boston, Boston, MA, 2004.

[11] J. Lepowsky and R.L. Wilson, Construction of the affine Lie algebra $A_1^{(1)}$. Comm. Math. Phys. 62 (1978), 43–53.

[12] T. Miwa, M. Jimbo, and E. Date, Solitons. Differential equations, symmetries and infinite-dimensional algebras. Cambridge Tracts in Math., 135. Cambridge Univ. Press, Cambridge, 2000.

[13] D.H. Peterson and V.G. Kac, Infinite flag varieties and conjugacy theorems. Proc. Nat. Acad. Sci. USA 80 (1983), 1778–1782.

[14] M. Wakimoto, Fock representations of the affine Lie algebra $A_1^{(1)}$. Comm. Math. Phys. 104 (1986), 605–609.

[15] W. Wang, $W_{1+\infty}$ algebra, $W_3$ algebra, and Friedan–Martinec–Shenker bosonization. Comm. Math. Phys. 195 (1998), 95–111.

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