Constructing Indecomposable Motivic Cohomology Classes on Algebraic Surfaces

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Abstract

We describe a method to construct indecomposable classes in Bloch's higher Chow group $\text{CH}^2(X,1)$ on algebraic surfaces over $\mathbb{C}$ via transcendental methods and apply it to obtain examples on K3 surfaces and some surfaces of general type.

Key Words: Higher Chow groups, algebraic cycles, Deligne cohomology, variation of mixed Hodge structures.

1 Introduction

Let $X$ be a smooth algebraic surface over $\mathbb{C}$. It is well known that $K_0(X) \otimes \mathbb{Q} \cong \oplus CH^p(X) \otimes \mathbb{Q}$ as a consequence of Grothendieck’s Riemann-Roch theorem. S. Bloch has generalized this to higher algebraic K-theory in [4], see also [26]. For example one obtains

$$K_1(X) \otimes \mathbb{Q} \cong \bigoplus_p CH^p(X,1) \otimes \mathbb{Q}$$

The groups $CH^p(X, n)$ are called higher Chow groups.

The purpose of this paper is to give an explicit way to construct classes in $CH^2(X, 1)$ that are non-trivial modulo the image of the natural map

$$\gamma : \text{Pic}(X) \otimes \mathbb{C}^* \to CH^2(X, 1)$$
and modulo torsion. We call such classes \textit{indecomposable}. Cycles in \(CH^2(X, 1)\) can be constructed by finding curves \(Z_i\) in \(X\) with rational functions \(f_i\) on them that satisfy \(\sum \text{div}(f_i) = 0\) on \(X\). By general conjectures (see section 7), the cokernel of \(\gamma\) is expected to be a countable group on any smooth surface.

We will construct indecomposable elements in \(CH^2(X, 1)\) on general quartic K3-surfaces that contain a line and on some special quintic surfaces of general type. Other examples over the complex numbers have been constructed by M. Nori (unpublished) on abelian surfaces, by A. Collino in \(\text{(1)}\) on Jacobian varieties, and by C. Voisin and C. Oliva on K3 surfaces in the unpublished work \(\text{(13)}\). The first examples ever given were over number fields, by A. Beilinson in \(\text{(3)}\), other examples on products of modular curves are contained in \(\text{(18)}, \text{(29)}\) and probably at other places.

Our method consists of deforming the complex structure of the pair \((X, Z)\) and studying the variation of mixed Hodge structures associated to the open complements. The ideas of this technique in the case of ordinary Chow groups are similar to those developed in \(\text{(1)}\) and in \(\text{(14)}\), but eventually go back to the fundamental idea of Griffiths to show the non-triviality of cohomology classes on the general member of a family of varieties by showing that their derivatives are non-zero. The advantage of our method is that it is not restricted to surfaces with trivial canonical bundle and that it is very simple to apply in situations where some geometry is known, in particular if an explicit family or a degeneration to a singular configuration can be written down.

Let us now describe the contents of this paper. In chapter 2 we first recall the definition of Bloch’s higher Chow groups \(CH^p(X, n)\) from \(\text{(4)}\) together with some of their basic properties. Then we sketch the construction of Deligne-Beilinson cohomology \((\text{(2)}, \text{(17)})\) and compare various definitions for the Chern class maps

\[
c_{p,n} : CH^p(X, n) \to H^{2p-n}_D(X_{an}, \mathbb{Z}(p))
\]

\((X_{an}\) denotes the underlying analytic space, where we assume that \(X\) is defined over \(\mathbb{C}\)) due to Beilinson, Bloch, Gillet and others. In particular we recall the explicit integral that computes \(c_{2,1}\) and study the relation between the extension of mixed Hodge structures given by the Gysin sequence attached to the support of a given cycle \(Z \in CH^p(X, 1)\) and \(c_{p,1}(Z)\), using the description of Deligne cohomology of smooth, projective varieties as \(\text{Ext}^1\) in the category of mixed Hodge structures.

In chapter 3 we present the circle of ideas around the deformation theory of Deligne-Beilinson cohomology classes and the rigidity of Chern classes from higher Chow groups. This is essentially due to Bloch and Beilinson, see \(\text{(2)}\). It implies in particular that

\[
c_{p,n} : CH^p(X, n) \otimes \mathbb{Q} \to H^{2p-n}_D(X_{an}, \mathbb{Q}(p))
\]
has a countable image for \( n \geq 2 \) if \( X \) is smooth and proper over \( \mathbb{C} \). In the case \( n = 1 \) this is not true anymore, instead we can show the following:

**Proposition 1.1** Let \( X \) be a smooth and projective variety over \( \mathbb{C} \). Then for all \( p \) the image of \( c_{p,1} \) in \( H_P^{2p-1}(X, \mathbb{Q}(p))/H_g^{2p-1}(X) \otimes \mathbb{C}/\mathbb{Q}(1) \) is countable.

Here \( H^i(X) \subset H^{2i}(X, \mathbb{Z}(i)) \) denotes the set of Hodge classes. This seems to be a well known fact, but I could not find a proof in the literature, so I decided to include one here for the sake of completeness. A similar result holds in the case \( n = 0 \), stating that the Griffiths group has countable image in the intermediate Jacobian \( J^p(X) \) modulo the maximal abelian subvariety \( J^p_a(X) \), see [37].

In chapter 4 these results are applied to the study of \( CH^2(X,1) \) for an algebraic surface \( X \) over \( \mathbb{C} \). We prove the following result, which was communicated by H. Esnault:

**Proposition 1.2** Let \( X \) be smooth, projective over \( \mathbb{C} \). Then \( CH^2(X,1) \) decomposes iff \( \mathcal{F}^2_Z = \mathcal{F}^1_Z \wedge \mathcal{F}^1_Z \) and \( H^1(X, \mathcal{F}^2_Z) \otimes \mathbb{Q} = 0 \).

The sheaves \( \mathcal{F}^p_Z \) have holomorphic p-forms with log-poles and \( \mathbb{Z}(p) \)–periods as sections. A precise formulation can be found in the text. The proof uses Gersten-Quillen resolutions. Then we discuss the relations with Bloch’s conjecture on the Chow groups of zero cycles on surfaces mentioned already above and the result of Esnault-Levine ([13]) about the relation between the decomposability of \( CH^p(X,1) \) and the injectivity of the cycle maps \( c_{r,0} \) for \( d - p + 1 \leq r \leq d \), where \( d = \text{dim}(X) \).

In the remaining part of the chapter we prepare the setup of variations of mixed Hodge structures associated to a family of cycles \( Z_t \in CH^2(X_t,1) \). Let us explain the necessary deformation theory. Assume we look at a smooth, proper deformation \( f : X \rightarrow S \) of \( X \) with \( S \) a smooth and quasiprojective variety, a base point \( 0 \in S \) such that \( f^{-1}(0) = X \) and a normal crossing divisor \( Z \) in \( X \) containing out of two smooth components \( Z_1, Z_2 \) such that \( Z_1 \) and \( Z_2 \) (resp. \( Z_1 \cap Z_2 \)) are smooth of relative dimension one (resp. zero) over \( S \) and restrict to \( Z_1 \) and \( Z_2 \) over the central fiber. We get an exact diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & T_X(logZ) & \rightarrow & T_X(log(Z)) |_X & \rightarrow & f^*T_{S,0} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & T_X & \rightarrow & T_X |_X & \rightarrow & f^*T_{S,0} & \rightarrow & 0 \\
\end{array}
\]

The logarithmic Kodaira-Spencer map is defined as the coboundary map

\[ T_{S,0} \rightarrow H^1(X, T_X(logZ)) \]
Let us denote the image of $T_{S,0} \to H^1(X, T_X(\log Z))$ by $W(\log)$ and the further image in $H^1(X, T_X)$ by $W$.

Our main result then is:

**Theorem 1.3 (CRITERION FOR INDECOMPOSABILITY):**

Let $X$ be a smooth projective surface over $\mathbb{C}$. Assume we are given two smooth and connected curves $Z_1$ and $Z_2$ on $X$ intersecting transversally and nontrivial rational functions $f_i$ on $Z_i$ ($i = 1, 2$), such that $\text{div}(f_1) + \text{div}(f_2) = 0$ as a zero-cycle on $X$. Denote by $Z = Z_1 \otimes f_1 + Z_2 \otimes f_2$ the resulting cycle in $CH^2(X, 1) = H^1(X, \mathcal{K}_2)$ and suppose the following conditions hold:

1. $Z$ also defines a cycle in Bloch’s higher Chow group $CH^1(|Z|, 1)$ - again denoted by $Z$ - and as such is not equivalent to $Z_1 \otimes a_1 + Z_2 \otimes a_2$ with $a_1, a_2 \in \mathbb{C}^*$.
2. There exist a smooth, proper deformation $f : \mathcal{X} \to S$ with $S$ a smooth and quasiprojective variety, a base point $0 \in S$ such that $f^{-1}(0) = X$ and the following properties hold:
   a. The situation in (1) deforms together with $X$: There exists a normal crossing divisor $Z = Z_1 + Z_2 \subset \mathcal{X}$ with $Z|_{\mathcal{X}} = Z_1 + Z_2$, consisting out of two smooth components $Z_1, Z_2$ such that $Z_1$ and $Z_2$ (resp. $Z_1 \cap Z_2$) are smooth of relative dimension one (resp. zero) over $S$. Furthermore there exist rational functions $F_i$ on $Z_i$ such that their restriction to each fiber $X_t := f^{-1}(t)$ satisfy $\text{div}(F_1, t) + \text{div}(F_2, t) = 0$ as a zero-cycle in $X_t$ and therefore define classes $Z_t = Z_{1,t} \otimes F_{1,t} + Z_{2,t} \otimes F_{2,t}$ in $CH^2(X_t, 1)$ and in $CH^1(|Z_t|, 1)$ for all $t \in S$.
   b. The cup-product map
      $$H^0(X, \Omega^2_X(\log Z)) \otimes H^1(X, T_X(-Z)) \to H^1(X, \Omega^1_X)/ \oplus_i H^0(Z_i, \mathcal{O}_{Z_i})$$
      has no left kernel.
   c. If $W(\log) \subset H^1(X, T_X(\log Z))$ denotes the image of the logarithmic Kodaira-Spencer map in $H^1(X, T_X(\log Z))$, then $W(\log)$ contains the image of the natural map
      $$H^1(X, T_X(-Z)) \to H^1(X, T_X(\log Z))$$
   d. For $t$ outside a countable number of proper analytic subsets of $S$, $Z_{1,t}$ and $Z_{2,t}$ generate $\text{NS}(X_t) \otimes \mathbb{Q}$.

**Then:** $Z_t$ is non-torsion in $CH^2(X_t, 1)/\text{Pic}(X_t) \otimes \mathbb{C}^*$ for $t$ outside a countable number of proper analytic subsets of $S$.

Note that (2b) and (2d) together imply that the Picard number of $X_t$ is not maximal.
for \( t \) outside a countable number of proper analytic subsets of \( S \). Instead of the assumptions (2b) and (2c) we can also state a weaker assumption (3) to get a sharper result:

**VARIANT:**

Assume (1),(2a),(2d) of the above main theorem and additionally the following instead of (2b),(2c):

(3) If \( W(\log) \subset H^1(X, T_X(\log Z)) \) denotes the image of the logarithmic Kodaira-Spencer map in \( H^1(X, T_X(\log Z)) \), then the following map has no left kernel:

\[
H^0(X, \Omega^1_X(\log Z)) \otimes W(\log) \to H^1(X, \Omega^1_X(\log Z))
\]

**Then:** \( Z_t \) is non-torsion in \( CH^2(X_t, 1)/\text{Pic}(X_t) \otimes \mathbb{C}^* \) for \( t \) outside a countable number of proper analytic subsets of \( S \).

We prove both statements in chapter 5. In chapter 6 we apply this result and Nori’s connectivity theorem to study some examples:

**Example 1:**

**Theorem 1.4** Let \( X \subset \mathbb{P}^3 \) be a general hypersurface of degree \( d \geq 5 \). Then the Chern class map \( CH^2(X, 1) \otimes \mathbb{Q} \to H_D^3(X, \mathbb{Q}(2)) \) has image isomorphic to \( H_D^3(\mathbb{P}^3, \mathbb{Q}(2)) \cong \mathbb{C}/\mathbb{Q}(1) \).

This result should be seen as a generalization of the classical Noether-Lefschetz theorem:

**Theorem 1.5** (Noether-Lefschetz)

Let \( X \subset \mathbb{P}^3 \) be a general hypersurface of degree \( d \geq 4 \). Then \( CH^1(X) \otimes \mathbb{Q} \) is isomorphic to \( H_D^2(\mathbb{P}^3, \mathbb{Q}(1)) \cong \mathbb{Q} \).

Both results can be shown using Nori’s connectivity theorem [32] by the methods of [22]. This was also observed by S.Bloch, M.Nori and C.Voisin for example in [42]. We mention a slightly more general statement:

**Theorem 1.6** Let \( (Y, \mathcal{O}(1)) \) be a smooth and projective polarized variety of dimension \( n + h \), \( X \subset Y \) a general complete intersection of dimension \( n \) and multidegree \( (d_1, \ldots, d_h) \) with \( \min(d_i) \) sufficiently large. Furthermore assume that \( 1 \leq p \leq n \). Then:

\[
\text{Image}(CH^p(X, 1) \otimes \mathbb{Q} \to H_D^{2p-1}(X, \mathbb{Q}(p))) \\
\subset \text{Image}(H_D^{2p-1}(Y, \mathbb{Q}(p)) \to H_D^{2p-1}(X, \mathbb{Q}(p)))
\]
This result - already in the case of projective space - is somehow of negative nature, because it destroys some obvious conjectures about the image of higher Chern classes, see [42]. It leaves open the possibility to construct examples on quartic K3-surfaces, also done in the paper [43].

**Example 2:** We can show that the general quartic K3 surface $X$ containing a line has indecomposable $CH^2(X,1)$: Let $N$ be the irreducible component of smooth quartic surfaces containing a line such that some point $0 \in N$ corresponds to the Fermat quartic surface with the equation $X_0 = \{ x | x_0^4 + x_1^4 - x_2^4 - x_3^4 = 0 \}$. Our cycles are obtained as follows:

A quartic surface $X$ that contains a line $G$ also contains an elliptic pencil cut out by the residual elliptic curves of all hyperplane sections through $G$. For a finite number of elements $E$ in this pencil, 2 of the 3 intersections points $P_1, P_2, P_3$ of $E$ and $G$ have the property that $2(P_1 - P_2)$ is rationally equivalent to zero. We show this in one example by giving the explicit hyperelliptic map from $E$ to $\mathbb{P}^1$ ramified at those two points. Hence there is a rational function $f_1$ on $E$ with zero divisor $2(P_1 - P_2)$ and a rational function $f_2$ on $G$ with zero divisor $2(P_2 - P_1)$. This construction can be extended to an irreducible component $N$ of the Noether-Lefschetz locus of surfaces containing a line. We obtain a cycle $E_t \otimes f_{1,t} + G_t \otimes f_{2,t} \in CH^2(X_t,1)$ on a suitable covering $S$ of that component that respects the choices of ordering of the chosen points.

**Theorem 1.7** Let $X_t$ be a general member of this family. Then $CH^2(X_t,1)$ is not decomposable.

It is even true that the general quartic K3-surface has indecomposable $CH^2(X,1)$, and this can be proved by the same method as in the next example. The cycles used there were used also by C. Oliva and C. Voisin in [43] on quartics.

In order to verify the assumptions of the main theorem we make use of the Green-Gotzmann theorem and Griffiths’ description of cohomology groups of hypersurfaces via residues of differential forms, as described in [21]. Assumption (2d) will hold for $t$ general on such a component by Noether-Lefschetz theory.

With some more work, using a monodromy argument of H. Clemens which was also used by A. Collino in [10], one can probably prove infinite generation for $CH^2(X_t,1)$ of a general quartic hypersurface containing a line. Since this was also proved in [10] and [43] we refrain from presenting it here. Obviously the idea would be to study the monodromy around a countable set of loci on the parameter space of the surfaces $X_t$. 

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Example 3: One can even obtain some examples of general type: Look at the Shioda hypersurface of degree 5:

$$X = \{ x \in \mathbb{P}^3 \mid x_0 x_1^4 + x_1 x_2^4 + x_2 x_0^4 + x_3^5 = 0 \}$$

It has an automorphism $\sigma$ of order 65, given by

$$\sigma : (x_0 : x_1 : x_2 : x_3) \mapsto (\zeta^{16} x_0 : \zeta^{-4} x_1 : \zeta x_2 : x_3)$$

where $\zeta$ is a 65-th root of unity. Shioda proves that the Picard group of $X$ is of rank one. Let $Z_1 := X \cap H_3$ and $Z_2 = X \cap H_0$, where $H_i$ are the linear hyperplane sections $H_i = \{ x_i = 0 \}$. Then $Z_1 \cap Z_2$ intersect in two points (one with multiplicity 4), called $P$ and $Q$ and one can show that $52P$ and $52Q$ are rationally equivalent on both curves. $Z_2$ is not smooth, but we construct a deformation $(X_t)_{t \in \mathbb{C}}$ and curves $Z_{i,t}$ that are smooth for $t \neq 0$. Taking a maximal irreducible component $N$ of quintic surfaces such that it contains all $X_t$ and the cycle $Z$ deforms along a suitable covering $S$ of $N$, the assumptions of the theorem can be checked in the same way as in the previous example. We obtain therefore:

Theorem 1.8 A general member $X_t$ of the irreducible components of quintics that deform to the Shioda hypersurface $X$ and that preserve the given cycle $Z \in CH^2(X, 1)$ has indecomposable $CH^2(X_t, 1)$.

In the remaining chapter 7 we sketch some ideas around these problems and formulate some open problems. In particular we think that it would be very convenient to have a good theory for singular surfaces in order to get shorter proofs for indecomposability by degeneration methods.

2 Higher Chow Groups and Chern Classes

2.1 Bloch’s Higher Chow Groups

Let $X$ be a quasiprojective variety over a field $k$. Define

$$\Delta^n := \text{Spec}(k[T_0, ..., T_n]/\sum T_i = 1)$$

Then $\Delta^n \cong \mathbb{A}^n_k$ is affine n-space and by setting the coordinates $t_i = 1$ one obtains $(n+1)$ linear hypersurfaces in $\Delta^n$ called codimension one faces. By iterating this one gets codimension $(n-m)$-faces isomorphic to $\Delta^m$ for every $m < n$ inside of $\Delta^n$. These are parametrized by strictly increasing maps $\rho : \{1,...,m\} \rightarrow \{1,...,n\}$. 

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Higher Chow groups are defined as the homology groups of a chain complex. Let $Z^p(X, n) \subset Z^p(X \times \Delta^n)$ the subset of cycles of codimension $p$ that meet all faces $X \times \Delta^m$ again in codimension $p$ for $m < n$. Let $\partial_i : Z^p(X, n) \to Z^p(X, n - 1)$ be the restriction map to the $i-$th codimension one face for $i = 0, \ldots, n$ and let $\partial = \sum (-1)^i \partial_i$. Then the homology of the complex

$$\ldots \to Z^p(X, n + 1) \to Z^p(X, n) \to Z^p(X, n - 1) \to \ldots$$

at position $n$ is denoted by $CH^p(X, n)$ [4]. We will need the following facts about higher Chow groups:

1. There is also a cubical version: Here let $\Box^n := (\mathbb{P}^1 \setminus \{1\})^n$ with coordinates $t_i$ and codimension one faces obtained by setting $t_i = 0, \infty$. The rest of the definition is completely analogous except that one has to divide out degenerate cycles and it is known [4] that both complexes are quasiisomorphic.

2. The groups $CH^*(X, *)$ are covariant for proper maps and contravariant for flat maps.

3. If $W \subset X$ is a codimension $r$ subvariety, then one has localization

$$\ldots \to CH^*(X, n) \to CH^*(X \setminus W, n) \to CH^{*-r}(W, n - r) \to CH^*(X, n - 1) \to \ldots$$

4. $CH^*(X, 0) = CH^*(X)$ are the usual Chow groups.

5. If $X$ is smooth, there is a product [4]

$$CH^p(X, q) \otimes CH^r(X, s) \to CH^{p+r}(X, q + s)$$

which can be easily defined using the cubical version. Thus it is possible to define an action of correspondences on higher Chow groups.

6. There exist cycle classes to Deligne-Beilinson cohomology [3] in case $k$ is a field of characteristic zero: If we fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and denote by $X_{an}$ the associated complex analytic space then we have maps

$$c_{p,n} : CH^p(X, n) \to H^{2p-n}_{DR}(X_{an}, \mathbb{Z}(p))$$

They will be discussed in section (2.3).

7. There is a Riemann-Roch formula $K_n(X) \otimes \mathbb{Q} = \bigoplus_p CH^p(X, n) \otimes \mathbb{Q}$, see [4] and [27].

8. Suslin’s theorem [11]: For $k$ itself: $CH^n(Spec(k), n) = K_M^n(k)$ (Milnor K-theory).

9. If $X$ is smooth and proper then $CH^1(X, 1) = k^*$ and $CH^1(X, n) = 0$ for $n \geq 2$.

10. For $X$ smooth, we have $CH^p(X, 1) = H^{p-1}(X, K_p)$ where $K_p$ is Quillen’s K-theory Zariski sheaf associated to the presheaf $U \mapsto K_p(O(U))$. Remember Bloch’s
formula $CH^p(X) = H^p(X, K_p)$.

Proof: (for (10)) Consider the diagram

$$
\begin{array}{ccc}
Z^p(X, 2) & \rightarrow & Z^p(X, 1) & \rightarrow & Z^p(X) \\
\downarrow N & & \downarrow N & & \| \\
\oplus_{x \in X^{(p-2)}} K_2(k(x)) & \rightarrow & \oplus_{x \in X^{(p-1)}} k(x)^* & \rightarrow & Z^p(X)
\end{array}
$$

Here $N$ denotes the norm map. The inverse map is given by taking the graph of rational functions on codimension $(p - 1)$–subvarieties. To show that the maps are mutually inverse to each other one needs only to show that the graph of the norm of a cycle in $X \times \mathbb{A}^1$ is equivalent to the cycle modulo $Z^2(X, 2)$. This can be shown explicitly: First reduce to the case where $X$ is a point and then use the explicit formulas in [10]. □

Remark: By similar methods $CH^p(X, 2) \rightarrow H^{p-2}(X, K_p)$ is surjective and an isomorphism for $p = 2$.

### 2.2 Deligne-Beilinson Cohomology

Let $X$ be any scheme of finite type over $\mathbb{C}$. Then there exists a twisted duality theory in the sense of Bloch-Ogus consisting of Deligne homology and cohomology groups ([17], [19], [24])

$$H^i_D(X, \mathbb{Z}(j)), \quad H^i_l(X, \mathbb{Z}(j))$$

satisfying the axioms in [3]. We just mention the duality isomorphism

$$H^i_{D, Z}(X, \mathbb{Z}(j)) \cong H^i_{D, l}(Z, \mathbb{Z}(d - j))$$

($d = \dim(X)$) for $X$ smooth and $Z \subset X$ a closed subvariety. Very important for our purposes is also the weak purity statement under the same assumptions: The groups $H^i_{D, Z}(X, \mathbb{Z}(j))$ vanish for $i < 2r$ where $r$ denotes the codimension of $Z$. Recall that for $X$ smooth one has an exact sequence

$$0 \rightarrow \frac{H^{i-1}(X, \mathbb{C})}{F^j + H^{i-1}(X, \mathbb{Z})} \rightarrow H^i_D(X, \mathbb{Z}(j)) \rightarrow F^j \cap H^i(X, \mathbb{Z}) \rightarrow 0$$

where the notation $F^j \cap H^i(X, \mathbb{Z})$ denotes the set of all classes $\alpha \in H^i(X, \mathbb{Z})$ such that $\alpha \otimes \mathbb{C} \in F^j H^i(X, \mathbb{C})$. If $X$ is smooth and proper we have additionally for $i = 2j$:

$$0 \rightarrow J^j(X) \rightarrow H^{2j}_{D}(X, \mathbb{Z}(j)) \rightarrow F^j \cap H^{2j}(X, \mathbb{Z}) \rightarrow 0$$

where $J^j(X)$ is the intermediate Jacobian. For $i < 2j$ and $X$ smooth and proper, the group $F^j \cap H^i(X, \mathbb{Z})$ is equal to the torsion subgroup of $H^i(X, \mathbb{Z})$. By the isomorphism

$$\Ext^1_{\operatorname{MHS}}(\mathbb{Z}(-j), H^{i-1}(X, \mathbb{Z})) = \frac{H^{i-1}(X, \mathbb{C})}{F^j + H^{i-1}(X, \mathbb{Z})}$$
both of the statements above can be subsumed into the exactness of the sequence
\[ 0 \to \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(-j), H^{i-1}(X, \mathbb{Z})) \to H_D^i(X, \mathbb{Z}(j)) \to \text{Hom}_{\text{MHS}}(\mathbb{Z}(-j), H^i(X, \mathbb{Z})) \to 0 \]
for \( X \) smooth and proper. See [3].

2.3 Cycle Classes

Let \( k \) be a field of characteristic zero. Fix some embedding \( \sigma : k \hookrightarrow \mathbb{C} \) and let \( X_{an} \) be the associated complex analytic space. We give several definitions of cycle classes \( c_{p,n} : CH^p(X, n) \to H^{2p-n}_D(X_{an}, \mathbb{Z}(p)) \) which are in fact equivalent. There is Bloch’s definition

**Definition 2.1 ([5])**

This is somehow the most general definition since it only uses some functoriality and weak purity of Deligne cohomology and can also be applied to get cycle classes to étale cohomology. \( X \) is just assumed to be quasiprojective over \( k \). A definition for all \( c_{p,n} \) was given first by Beilinson

**Definition 2.2 ([2], [19], see [36] for a survey)**

Again \( X \) is quasiprojective over \( k \). The definition uses simplicial schemes and the axioms of Bloch and Ogus. The following definition is due to Deninger and Scholl

**Definition 2.3 ([12], [23])**

Let \( X \) be smooth and projective over \( k \). Here for \( n \geq 1 \) a cycle class
\[ c_{p,n} : CH^p(X, n) \otimes \mathbb{Q} \to \text{Ext}^1_{\mathbb{Q}-\text{MHS}}(\mathbb{Q}(-p), H^{2p-n-1}(X, \mathbb{Q})) \]
is defined by giving an explicit extension.

These definition coincide by [3] for the first two definitions and [12] for the first and third definition. If \( n = 1 \) we can give more detailed descriptions of the cycle class ([3], [26]):

Assume that \( X \) is smooth and projective over \( \mathbb{C} \) and that \( H^{2p-1}(X, \mathbb{Z}) \) is torsionfree (otherwise work over \( \mathbb{Q} \)). Then we have by property (10) of higher Chow groups that \( CH^p(X, 1) = H^{p-1}(X, \mathcal{K}_p) \) and a class in this group is given by \( Z = \sum Z_i \otimes f_i \) where the \( Z_i \) are integral subschemes of codimension \( (p-1) \) and \( f_i \) are rational functions
on each $Z_i$ with $\sum \text{div}(f_i) = 0$ as a cycle on $X$. Then write $\text{div}(f_i) = \partial \gamma_i$ for real analytic $(2d - 2p + 1)$- dimensional chains on $Z_i$. In fact define $\gamma_i = f_i^{-1}(u_i)$ with $u_i$ the standard path on the real axis from 0 to $\infty$. Let $\gamma = \bigcup \gamma_i$. We have $\partial \gamma = 0$ by $\sum \text{div}(f_i) = 0$ and even more $Z$ defines a class in $F^pH^{2p-1}_{[Z]}(X, \mathbb{Z})$ and therefore $\gamma$ has zero cohomology class in $H^{2p-1}(X, \mathbb{Z})$ (by torsion-freeness). Thus $\gamma = \partial \Gamma$ for some chain $\Gamma \subset X$ of dimension $2d - 2p + 2$. The rational maps $f_i : Z_i \to \mathbb{P}^1$ are such that $\gamma_i$ is the preimage of a path connecting 0 and $\infty$ on $\mathbb{P}^1$ and hence allows to choose a branch of logarithm on $Z_i \setminus \gamma_i$. The functional

$$\alpha \mapsto \sum_i \int_{Z_i - \gamma_i} \log(f_i) \cdot \alpha + (2\pi \sqrt{-1}) \int_{\Gamma} \alpha$$

defines a functional on $(2p - 2)$ forms on $X$ and therefore via Poincaré duality a class in $H^{2p-2}(X, \mathbb{C})$. The choices we made let it become well defined only in the quotient group $H^{2p-1}_{\mathbb{Z}}(X, \mathbb{Z}(p))$.

The definition via extensions can also be described partially by the following construction:

If we set $U = X \setminus \bigcup Z_i$, then one has a long exact sequence

$$H^{2p-2}_{[Z]}(X, \mathbb{Z}) \to H^{2p-2}(X, \mathbb{Z}) \to H^{2p-2}(U, \mathbb{Z}) \to H^{2p-1}_{[Z]}(X, \mathbb{Z}) \to H^{2p-1}(X, \mathbb{Z})$$

A cycle $Z \in CH^p(X, 1)$ gives rise to a class in $F^p \cap H^{2p-1}_{[Z]}(X, \mathbb{Z})$ and hence gives an extension

$$0 \to H^{2p-2}(X, \mathbb{Z})/Hg^{p-1,p-1}(X) \to E \to \mathbb{Z}(-p) \to 0$$

Here $Hg^{i,i} = F^i \cap H^{2i}(X, \mathbb{Z})$ with the meaning defined in [22] in particular $Hg^{1,1} = \text{NS}(X)$. This extension class is the image of $c_{p,1}(Z)$ under the map

$$\text{Ext}^1(\mathbb{Z}(-p), H^{2p-2}(X, \mathbb{Z})) \to \text{Ext}^1(\mathbb{Z}(-p), H^{2p-2}(X, \mathbb{Z})/Hg^{p-1,p-1}(X))$$

**Remark:** Once we believe in the existence of regulator maps also for the local situation, there is another way to describe the cycle classes for $n = 1$ ([13]):

Let $K_p$ the Quillen K-theory sheaf (see above) and $H^p_D(p)$ be the sheafified Deligne-Beilinson cohomology (with presheaf $U \mapsto H^p_D(U, \mathbb{Z}(p))$) both viewed as sheaves in Zariski topology. Then there is a Leray spectral sequence (as in [8]) arising from changing from Zariski to the analytic site

$$E^{p,q}_2(r) = H^p_{\text{Zar}}(X, H^q_D(r)) \Rightarrow H^{p+q}_{D,\text{an}}(X, \mathbb{Z}(r))$$

and we get an edge morphism $H^{p-1}(X, H^p_D(p)) \to H^{2p-1}_D(X, \mathbb{Z}(p))$ as a byproduct.

The existence of regulator maps on affine schemes implies a regulator map of sheaves

$$\text{reg} : K_p \to H^p_D(p)$$
The composition \( CH^p(X, 1) \to H^{p-1}(X, K_p) \to H^{p-1}(X, H^p_D(p)) \to H^{2p-1}_D(X, Z(p)) \) is also equivalent to the cycle classes defined above.

3 Deformations and Rigidity of Cycle Classes

Let \( Y \) be a reduced quasiprojective scheme over \( \mathbb{C} \) and \( A \) a ring with \( \mathbb{Q} \subset A \subset \mathbb{R} \). Let \( \epsilon_A \) be the map \( \epsilon_A : H^k_D(Y, A(p)) \to F^p \cap H^k(Y, A(p)) \).

Lemma 3.1 ([2], 1.6.6.1.)
(a) If \( p > \min(k, \dim(Y)) \), then \( \epsilon_A \equiv 0 \).
(b) If \( Y = X \times S \) with \( X \) smooth, projective and \( S \) smooth, affine and \( k < 2p - \dim(S) \), then also \( \epsilon_A \equiv 0 \).

Proof: In both cases \( F^p \cap H^k(Y, A(p)) = 0 \) by type reasoning.

Lemma 3.2 ([2], 1.6.6.2.)
Let \( X \) be projective, \( S \) smooth affine, \( Y = X \times S \) and given \( s_1, s_2 \in S \). Assume \( k \leq 2p - 2 \) and a class \( \alpha \in H^k_D(Y, A(p)) \) is given. Then:

\[
\alpha|_{X \times \{s_1\}} = \alpha|_{X \times \{s_2\}} \in H^k_D(X, A(p))
\]

Proof: \( S \) is an affine curve wlog. By the lemma above \( \epsilon_A(\alpha) = 0 \), d.h. \( \alpha \in \frac{H^{k-1}(Y, \mathbb{C})}{F^p \cap H^{k-1}(Y, A(p))} \). But Betti classes are rigid.

Corollary 3.3 ([2], 2.3.4.)
Let \( X \) be smooth, projective over \( \mathbb{C} \) and \( n \geq 2 \). Then the image of \( c_{p,n} : CH^p(X, n) \otimes \mathbb{Q} \to H^{2p-n}_D(X, \mathbb{Q}(p)) \) is countable.

Proof: There exists an algebraically closed, countable field \( L \subset \mathbb{C} \) such that \( X = X_0 \otimes_L \mathbb{C} \) for some \( L \)-variety \( X_0 \). Hence \( CH^p(X_0, n) \) is countable and therefore the image of \( c_{p,n} : CH^p(X_0, n) \otimes \mathbb{Q} \to H^{2p-n}_D(X_{an}, \mathbb{Q}(p)) \). It remains to show that the cycle classes from \( X \) and \( X_0 \) have the same image in \( H^{2p-n}_D(X_{an}, \mathbb{Q}(p)) \). Choose a smooth affine scheme \( S = Spec(R) \) for some \( L \)-algebra \( R \) and a \( L \)-rational point \( 0 \in S \), such that the geometric general fiber in \( X_0 \times S \) is given by \( X_{\mathbb{C}} \). Given a cycle \( Z \in CH^p(X, n) \otimes \mathbb{Q} \), there is a spreading \( Z \in CH^p(X_0 \times S, n) \otimes \mathbb{Q} \) with Deligne class \( \alpha \in H^{2p-n}_D(X_{an} \times S_{an}, \mathbb{Q}(p)) \). In particular this means that \( Z = Z|_X \), the restriction to the geometric general fiber.
By the lemma above \( \alpha|_X = \alpha|_{X_0} \) and hence \( c_{p,n}(Z) \) is also contained in the countable image of \( c_{p,n}(CH^p(X_0,n) \otimes \mathbb{Q}) \subset H^{2p-n}_D(X_{an},\mathbb{Q}(p)) \). \(
abla\)

Now consider the cases \( k = 2p - 1, 2p \). Let \( X \) be again smooth and projective over \( \mathbb{C} \) and \( S \) an affine, smooth complex curve with good compactification \( \overline{S} = S \cup \Sigma \). Then:

**Lemma 3.4**  
(a) If \( n = 1 \), then  
\[
F^p \cap H^{2p-1}(X \times S, A(p)) = H^{p-1,p-1}(X, A(p-1)) \oplus \bigoplus_\Sigma A(-1)
\]
where the summation ranges over all divisors at infinity.

(b) (see [16].) If \( n = 0 \), then  
\[
F^p \cap H^{2p}(X \times S, A(p)) = F^p \cap H^{2p}(X \times \overline{S}, A(p)) \subset H^{p,p}(X, A(p)) \oplus H^{p-1,p}(X, \mathbb{C}) \otimes H^0(S, \Omega^1_S(\log \Sigma)) \oplus H^{p-1,p}(X, \mathbb{C}) \otimes H^1(S, \mathcal{O}_S)
\]

**Proof:** In both cases \( F^{p+1} \cap H^k(X \times S, A(p)) = 0 \) and the weight \( k \) piece arises exactly from the compactification of \( X \times S \). This is the only weight in (b). The claim follows therefore from Künneth decomposition. (a) follows in the same way but there is only the weight \( k + 1 = 2p \). \(
\)

**Remark:** For \( k = 2p - 1 \), \( \epsilon(\alpha) \) is only determined by the residues around boundary divisors.

If \( k = 2p \), then \( \epsilon(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 \), with \( \alpha_1 \in H^{p,p}(X, A(p)) \) the cohomology class of a restriction to the general fiber. \( \alpha_2 \) and \( \alpha_3 \) are complex conjugate.

**Corollary 3.5** Let \( X \) be smooth, projective over \( \mathbb{C} \) and \( n = 1 \). Then the image of the truncated cycle class  
\[
c_{p,1}^{tr} : CH^p(X,1) \otimes \mathbb{Q} \to H^{2p-1}_D(X_{an}, \mathbb{Q}(p))/\text{Hg}^{p-1,p-1}(X) \otimes \mathbb{C}/\mathbb{Q}(1)
\]
is countable.

**Proof:** Let \( S \) be a smooth, affine and connected curve. Consider the product map  
\[
CH^{p-1}(X) \otimes CH^1(S,1) \to CH^p(X \times S,1)
\]
respectively its Deligne cohomology version  
\[
\gamma : H^{2p-2}_D(X_{an}, \mathbb{Z}(p-1)) \otimes H^1_D(S_{an}, \mathbb{Z}(1)) \to H^{2p-1}_D(X_{an} \times S_{an}, \mathbb{Z}(p))
\]
Therefore $H^{2p-1}_D(X_{\text{an}} \times S_{\text{an}}, \mathbb{Z}(p))/\text{Image}(\gamma)$ becomes equal to

\[
\frac{H^{2p-2}(X \times S, \mathbb{C}^*)}{F^p + Hg^{p-1,p-1}(X) \otimes \mathbb{C}^* + \text{other terms}}
\]

If we mod out by the image of $\gamma$, we get a restriction map (the other terms restrict to zero) for every $t \in S$:

\[
r_t : \frac{H^{2p-1}_D(X_{\text{an}} \times S_{\text{an}}, \mathbb{Z}(p))}{\text{Image}(\gamma)} \to H^{2p-1}_D(X_{\text{an}}, \mathbb{Z}(p))/Hg^{p-1,p-1}(X) \otimes \mathbb{C}^*
\]

with the important property that the truncated cycle class $c^{tr}_{p,1}$ factors through it. Therefore we again have rigidity: Given $\alpha \in H^{2p-1}_D(X_{\text{an}} \times S_{\text{an}}, \mathbb{Z}(p))/\text{Image}(\gamma)$ one has after tensoring with $\mathbb{Q}$: $\alpha|_{X \times \{s_1\}} = \alpha|_{X \times \{s_2\}}$ in $H^{2p-1}_D(X_{\text{an}}, \mathbb{Q}(p))/Hg^{p-1,p-1}(X) \otimes \mathbb{C}/\mathbb{Q}(1)$, since $H^{2p-1}_D(X_{\text{an}} \times S_{\text{an}}, \mathbb{Z}(p))/\text{Image}(\gamma)$ can be represented by Betti classes. Now the same argument as in the proof of Cor. 3.3. can be applied to deduce the countability. □

### 4 Theory of $CH^2(X, 1)$

#### 4.1 Decomposability

Let $X$ be a smooth and projective variety over an algebraically closed field of characteristic zero. There is a natural map

\[
\gamma : \text{Pic}(X) \otimes k^* \to CH^2(X, 1)
\]

which in the cubical version of higher Chow groups can be described by sending $D \otimes a$ to $D \times \{a\} \subset X \times \mathbb{P}^1$ for $D$ an integral subscheme and $a \in k$ and extending linearly. Alternatively the cup-product map $\mathcal{O}_X^* \otimes \mathcal{O}_X^* \to \mathcal{K}_2$ gives rise to the map

\[
H^1(X, \mathcal{O}_X^*) \otimes H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{K}_2)
\]

Equally consider the product map

\[
CH^1(X, 0) \otimes CH^1(X, 1) \to CH^2(X, 1)
\]

and it is an easy exercise to show that all three definitions coincide.

**Definition 4.1** We say that $CH^2(X, 1)$ is decomposable, if the map $\gamma$ has torsion cokernel. More generally we will say that $CH^p(X, 1)$ is decomposable if the map $CH^{p-1}(X) \otimes CH^1(X, 1) \to CH^p(X, 1)$ has torsion cokernel.

The cokernel is a birational invariant in that case. About the kernel we note that the composed map $\text{Pic}^0(X) \otimes k^* \to CH^2(X, 1) \to H^3_D(X_{\text{an}}, \mathbb{Z}(2))$ is zero, however we do not know in which way the map $\text{Pic}^0(X) \otimes k^* \to CH^2(X, 1)$ itself behaves.
4.2 Criteria for Decomposability

Let $X$ be as in the previous section and $k = \mathbb{C}$. Let us fix some notations:

Denote by $\mathcal{F}^p_Z$ the Zariski sheaf associated to the presheaf that associates to each open set $U$ the vector space of holomorphic $p-$forms with $\mathbb{Z}(p)-$periods and logarithmic poles along a desingularization of $X \setminus U$. One has an exact sequence

$$0 \to \mathcal{H}^{p-1}_{\text{DR}}(\mathbb{C}/\mathbb{Z}(p)) \to \mathcal{H}^p_D(p) \to \mathcal{F}^p_Z \to 0$$

Here $\mathcal{H}^{p-1}_{\text{DR}}(\mathbb{C}/\mathbb{Z}(p))$ is the sheaf associated to the presheaf $U \mapsto \mathcal{H}^{p-1}(U, \mathbb{C}/\mathbb{Z}(p))$ and $\mathcal{H}^p_D(p)$ is the sheafified Deligne-Beilinson cohomology as explained before. We then have $\mathcal{F}^1_Z = d\log \mathcal{O}_X^*$ and define $\mathcal{C} := \mathcal{F}^2_Z / \mathcal{F}^1_Z \wedge \mathcal{F}^1_Z$. One can show that the sheaves $\mathcal{H}^{p-1}_{\text{DR}}(\mathbb{C}/\mathbb{Z}(p))$, $\mathcal{H}^p_D(p)$ and $\mathcal{F}^p_Z$ admit Gersten-Quillen type resolutions which we will use below. The following is private communication by H. Esnault and partially explained in [14].

**Theorem 4.2** $CH^2(X, 1)$ decomposes if $\mathcal{C} = 0$ and $H^1(X, \mathcal{F}^2_Z) \otimes \mathbb{Q} = 0$.

**Proof:** Consider the surjective map $\alpha : \mathcal{K}_2 \to \mathcal{F}^1_Z \wedge \mathcal{F}^1_Z$ induced by the $d\log$ map. By assumption $H^1(X, \mathcal{F}^1_Z \wedge \mathcal{F}^1_Z) \otimes \mathbb{Q} = 0$ and therefore it is sufficient to show that $H^1(X, \mathcal{K}^0_2)$ decomposes where $\mathcal{K}^0_2 := \text{Ker}(\alpha)$. Gersten-Quillen resolutions for all three sheaves form a commutative diagram

$$\begin{array}{ccc}
\mathcal{K}^2_0 & \to & \mathcal{K}^0_2(\mathbb{C}(X)) \\
\downarrow & & \downarrow \\
\mathcal{K}_2 & \to & \mathcal{K}_2(\mathbb{C}(X)) \\
\downarrow & & \downarrow \\
\mathcal{F}^2_Z & \to & \mathcal{F}^2_Z(\mathbb{C}(X)) \\
\end{array}$$

where the third column is the direct sum of exact sequences

$$0 \to \mathbb{C}/\mathbb{Z}(1) \to \mathcal{C}(D)^* \xrightarrow{d\log} \mathcal{F}^1_Z(\mathbb{C}(D)) \to 0$$

Thus we have a surjective map $\text{Pic}(X) \otimes \mathbb{C}^* \to H^1(X, \mathcal{K}^0_2)$ and the assertion follows. \qed

**Remark:** The statement in the theorem holds also in the reverse direction. Bloch conjectures that for a complex algebraic surface with $p_g = 0$ the kernel of the Albanese map $CH^3_0(X) \to \text{Alb}(X)$ is zero. The conjecture holds if $X$ is not of general type by [7]. If $X$ is of general type $p_g = 0$ implies also $q(X) = 0$ and the conjecture has been verified for Godeaux- and Barlow surfaces for example. Note that if $p_g > 0$, then by [30] quite the contrary happens. If Bloch’s conjecture holds, it implies that $CH^2(X, 1)$ decomposes if $p_g(X) = 0$ by [3]. There is a more general statement in [13]:
Theorem 4.3 \([13]\)

Let \(X\) be a smooth algebraic variety over \(\mathbb{C}\). Suppose that the cycle maps

\[
c_{p,0} : CH^p(X) \to H^{2p}_D(X, \mathbb{Z}(p))
\]

are injective for \(d - s \leq p \leq d = \text{dim}(X)\) for some \(s \geq 0\). Then \(CH^p(X,1)\) is decomposable for \(0 \leq p \leq s + 1\).

4.3 Criteria for Non-Decomposability and Main Theorem

In this section assume for simplicity that \(X\) is an algebraic surface over \(\mathbb{C}\). To show that \(CH^2(X, 1)\) is not decomposable we use cycle class maps to Deligne cohomology. Note that the image of \(c_{1,2} : CH^2(X, 1) \to H^3_D(X, \mathbb{Z}(2))\) when restricted to \(\text{Pic}(X) \otimes \mathbb{C}^*\) is contained in \(\text{NS}(X) \otimes \mathbb{C}^*\) considered as a subgroup of \(H^3_D(X, \mathbb{Z}(2))\). Therefore it will be enough to show that there are classes not contained in that subgroup modulo torsion. In fact the image of \(CH^2(X, 1)/\text{Pic}(X) \otimes \mathbb{C}^*\) in \(H^3_D(X, \mathbb{Z}(2))/\text{NS}(X) \otimes \mathbb{C}^*\) is at most countable by Corollary (3.5).

In the last section we will construct examples and the following theorem provides the necessary technical tool. We use the following notation: If \(Z\) is a divisor with normal crossings on \(X\) and smooth components \(Z_i\), then let \(\Omega^p_X(\log Z)\) be the sheaf of holomorphic \(p\)-forms with log-poles along \(Z\) and \(T_X(\log Z)\) be the dual of \(\Omega^1_X(\log Z)\).

There is an exact sequence

\[
0 \to \Omega^1_X \to \Omega^1_X(\log Z) \to \bigoplus \mathcal{O}_{Z_i} \to 0
\]

and a commutative diagram where we define \(\mathcal{G} := \Omega^2_X(\log Z)/\Omega^2_X\):

\[
\begin{array}{ccc}
\Omega^2_X & \to & W_1 \Omega^2_X(\log Z) \to \bigoplus \Omega^1_{Z_i} \\
\| & & \downarrow \\
\Omega^2_X & \to & \Omega^2_X(\log Z) \to \mathcal{G} \\
& & \downarrow \\
& & \mathbb{C}^k = \mathbb{C}^k
\end{array}
\]

where \(\mathbb{C}^k\) is supported on the \(k\) points in the singular locus of \(Z\) consisting of the intersection points in \(Z_i \cap Z_j\). The cup product with the Kodaira-Spencer class \(H^0(X, \Omega^2_X) \otimes H^1(X, T_X) \to H^1(X, \Omega^1_X)\) and the natural map \(H^1(X, T_X(\log Z)) \to H^1(X, T_X)\) induce a commutative diagram

\[
\begin{array}{ccc}
H^0(X, \Omega^2_X) \otimes H^1(X, T_X(\log Z)) & \to & H^1(X, \Omega^1_X) \\
\downarrow & & \downarrow \\
H^0(X, \Omega^2_X(\log Z)) \otimes H^1(X, T_X(\log Z)) & \to & H^1(X, \Omega^1_X(\log Z))
\end{array}
\]
Let us explain the necessary deformation theory. Assume we look at a smooth, proper deformation $f : \mathcal{X} \to S$ of $X$ with $S$ a smooth and quasiprojective variety, a base point $0 \in S$ such that $f^{-1}(0) = X$ and a normal crossing divisor $Z$ in $\mathcal{X}$ containing out of two smooth components $Z_1, Z_2$ such that $Z_1$ and $Z_2$ (resp. $Z_1 \cap Z_2$) are smooth of relative dimension one (resp. zero) over $S$ and restrict to $Z_1$ and $Z_2$ over the central fiber. We get an exact diagram

$$
\begin{array}{cccc}
0 & \to & T_X(\log Z) & \to & T_X(\log(Z))|_X & \to & f^*T_{S,0} & \to & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_X & \to & T_X|X & \to & f^*T_{S,0} & \to & 0 \\
\end{array}
$$

The logarithmic Kodaira-Spencer map is defined as the coboundary map

$$T_{S,0} \longrightarrow H^1(X, T_X(\log Z))$$

Let us denote the image of $T_{S,0} \to H^1(X, T_X(\log Z))$ by $W(\log)$ and the further image in $H^1(X, T_X)$ by $W$.

Before stating the theorem let us describe one of the higher Chow groups of the projective (however reducible) variety $|Z|$, the support of $Z$. For simplicity we will assume that $|Z|$ consists of two smooth components $|Z| = Z_1 + Z_2$ with $k$ intersection points in $Z_1 \cap Z_2$.

**Proposition 4.4**. $CH^1(|Z|, 1) \cong H^3_{\overline{\partial}, |Z|}(X, \mathbb{Z}(2))$ and there is an exact sequence

$$0 \to (\mathbb{C}^*)^{\oplus 2} \to CH^1(|Z|, 1) \xrightarrow{\tau} \ker(Z^k \to \oplus \text{Pic}(Z_i)) \to 0$$

where $Z^k$ is supported on $Z_1 \cap Z_2$.

*Proof:* next section. \[\square\]

Let us denote the map $CH^1(|Z|, 1) \to \ker(Z^k \to \oplus \text{Pic}(Z_i))$ given in the proposition by $\tau$. Now we are ready to state the main result and a stronger variant of it:

**MAIN THEOREM (CRITERION FOR INDECOMPOSABILITY):**

Let $X$ be a smooth projective surface over $\mathbb{C}$. Assume we are given two smooth and connected curves $Z_1$ and $Z_2$ on $X$ intersecting transversally and nontrivial rational functions $f_i$ on $Z_i$ ($i = 1, 2$), such that $\text{div}(f_1) + \text{div}(f_2) = 0$ as a zero-cycle on $X$.

Denote by $Z = Z_1 \otimes f_1 + Z_2 \otimes f_2$ the resulting cycle in $CH^2(X, 1) = H^1(X, K_2)$ and suppose the following conditions hold:

1. $Z$ also defines a cycle in Bloch’s higher Chow group $CH^1(|Z|, 1)$ - again denoted by $Z$ - and as such is not equivalent to $Z_1 \otimes a_1 + Z_2 \otimes a_2$ with $a_1, a_2 \in \mathbb{C}^*$.

2. There exist a smooth, proper deformation $f : \mathcal{X} \to S$ with $S$ a smooth and quasiprojective variety, a base point $0 \in S$ such that $f^{-1}(0) = X$ and the following
properties hold:

(a) The situation in (1) deforms together with $X$: There exists a normal crossing divisor $Z = Z_1 + Z_2 \subset X$ with $Z|_X = Z_1 + Z_2$, consisting out of two smooth components $Z_1, Z_2$ such that $Z_1$ and $Z_2$ (resp. $Z_1 \cap Z_2$) are smooth of relative dimension one (resp. zero) over $S$. Furthermore there exist rational functions $F_i$ on $Z_i$ such that their restriction to each fiber $X_t := f^{-1}(t)$ satisfy $\text{div}(F_1,t) + \text{div}(F_2,t) = 0$ as a zero-cycle in $X_t$ and therefore define classes $Z_t = Z_{1,t} \otimes F_{1,t} + Z_{2,t} \otimes F_{2,t}$ in $CH^2(X_t, 1)$ and in $CH^1(|Z_t|, 1)$ for all $t \in S$.

(b) The cup-product map

$$H^0(X, \Omega^2_X(\log Z)) \otimes H^1(X, T_X(-Z)) \to H^1(X, \Omega^1_X)/ \oplus_i H^0(Z_i, \mathcal{O}_{Z_i})$$

has no left kernel.

(c) If $W(\log) \subset H^1(X, T_X(\log Z))$ denotes the image of the logarithmic Kodaira-Spencer map in $H^1(X, T_X(\log Z))$, then $W(\log)$ contains the image of the natural map

$$H^1(X, T_X(-Z)) \to H^1(X, T_X(\log Z))$$

(d) For $t$ outside a countable number of proper analytic subsets of $S$, $Z_{1,t}$ and $Z_{2,t}$ generate $\text{NS}(X_t) \otimes \mathbb{Q}$.

Then: $Z_t$ is non-torsion in $CH^2(X_t, 1)/\text{Pic}(X_t) \otimes \mathbb{C}^*$ for $t$ outside a countable number of proper analytic subsets of $S$.

VARIANT:

Assume (1), (2a), (2d) of the above main theorem and additionally the following instead of (2b), (2c):

(3) If $W(\log) \subset H^1(X, T_X(\log Z))$ denotes the image of the logarithmic Kodaira-Spencer map in $H^1(X, T_X(\log Z))$, then the following map has no left kernel:

$$H^0(X, \Omega^2_X(\log Z)) \otimes W(\log) \to H^1(X, \Omega^1_X(\log Z))$$

Then: $Z_t$ is non-torsion in $CH^2(X_t, 1)/\text{Pic}(X_t) \otimes \mathbb{C}^*$ for $t$ outside a countable number of proper analytic subsets of $S$.

Remark: The assumption that we have only two components is not necessary but simplifies the proof and suffices for the applications. (2b) and (2d) imply necessarily that $p_g(X) \geq 1$, since it follows from (2b) and (2d) that the Picard number of $X_t$ is not maximal for $t$ outside a countable number of analytic subsets of $S$. (2b) and (2c) imply (3) (see proof of the main theorem) and therefore the variant is the more general formulation.
5 Proof of the Main Theorem

5.1 Auxiliary Results in Hodge Theory

Let $S$ be a smooth complex variety.

Definition 5.1 (41)
A graded polarized $\mathbb{Z}$–variation of mixed Hodge structures ($\mathbb{Z}$–VMHS) on $S$ is a local system $\mathcal{V}$ on $S$ of $\mathbb{Z}$–modules of finite rank with the following data:

(a) An increasing filtration ... $\subset W_k \subset W_{k+1} \subset \mathcal{V} \otimes_\mathbb{Z} \mathbb{Q}$ by local systems over $\mathbb{Q}$.
(b) A decreasing filtration ... $\subset F_{p+1} \subset F_p \subset ... \subset F_0 = \mathcal{V} \otimes_\mathbb{Z} \mathcal{O}_S$ by holomorphic vector bundles.
(c) The flat connection $\nabla$ on $\mathcal{V}$ satisfies $\nabla F_p \subset \Omega^1_S \otimes F_{p-1}$.
(d) The $\mathcal{F}$–filtration induces on the local systems $\text{Gr}_k \mathcal{V} := W_k / W_{k-1}$ a pure variation of polarized Hodge structures on $S$.

The graded pieces of the connection we denote by $\nabla^p$:

$$\nabla^p : F_p / F_{p+1} \rightarrow \Omega^1_S \otimes F_{p-1} / F_p$$

In particular there are examples where such VMHS come from a geometric situation, for example in the situation of the theorem: Assume we are given an explicit smooth, proper deformation $f : \mathcal{X} \rightarrow S$ with $S$ a smooth complex variety and there exist a cycle $Z \in CH^2(\mathcal{X}, 1)$ with $Z|_X = Z$ consisting out of two smooth components $Z_1, Z_2$ intersecting transversally, such that $Z_1$ and $Z_2$ (resp. $Z_1 \cap Z_2$) are smooth of relative dimension one (resp. zero) over $S$ and such that its restriction to each fiber $X_t := f^{-1}(t)$ defines classes $Z_t \in H^1(X_t, \mathcal{K}_2)$. Then it is a result in [41] that the cohomology groups $H^2(X_t \setminus Z_t, \mathbb{Z})$ form an admissible $\mathbb{Z}$–VMHS over $S$.

If all fibers are algebraic surfaces, there are no holomorphic 3-forms, hence $\mathcal{F}^3 = 0$ and $\mathcal{F}^2$ is the holomorphic subbundle with fibers $H^0(X_t, \Omega^2_{X_t}(log Z_t))$. The graded piece $\nabla^2 : \mathcal{F}^2 \rightarrow \Omega^1_S \otimes \mathcal{F}^1 / \mathcal{F}^2$ can be described in the stalk at $0 \in S$ by the maps

$$\nabla^2 : H^0(X, \Omega^2_X(log Z)) \rightarrow H^1(X, \Omega^1_X(log Z)) \otimes W(log)^*$$

where $W(log)$ is the logarithmic Kodaira-Spencer image of $T_{S,0}$. Bringing $W(log)$ to the other side we get the cup-product maps

$$H^0(X, \Omega^2_X(log Z)) \otimes W(log) \rightarrow H^1(X, \Omega^1_X(log Z))$$
**Lemma 5.2** In this situation, assume that 
\( H^0(X, \Omega^2_X(logZ)) \otimes W(log) \to H^1(X, \Omega^1_X(logZ)) \) has no left kernel.

**Then:** \( F^2 \cap H^2(U_t, \mathbb{Q}) = 0 \) for \( t \) outside a countable number of analytic subsets of \( S \).

**Proof:** Assume there exists a nonzero class \( \lambda_0 \in F^2 \cap H^2(U_0, \mathbb{Q}) \) that is supported over some germ of an analytic subvariety \( S(\lambda_0) \subset S \) containing \( 0 \in S \), i.e. there exists a holomorphic section \( \Lambda \in \Gamma(S(\lambda_0), \mathcal{F}^2) \) such that \( \lambda_t = \Lambda|X_t \in F^2 \cap H^2(U_t, \mathbb{Q}) \) for all \( t \in S(\lambda_0) \). Let \( T_0 \subset T_{S,0} \) be the holomorphic tangent space to \( S(\lambda_0) \) at \( 0 \in S \). Since \( \lambda_0 \) is an integral class,

\[ \nabla^2(\lambda_0) \in \text{Hom}(T_{S(\lambda_0)}, 0, (\mathcal{F}^1/\mathcal{F}^2)_0) \]

satisfies \( \nabla^2(\lambda_0)(T_0) = 0 \). Therefore - if \( W_0 \subset W(log) \) denotes the subspace corresponding to \( T_0 \) under the logarithmic Kodaira-Spencer map - we have that

\[ \mathbb{C} \cdot \lambda_0 \otimes W_0 \mapsto 0 \in H^1(X, \Omega^1_X(logZ)) \]

i.e. \( \mathbb{C} \cdot \lambda_0 \) is contained in the left kernel of \( \nabla^2 \) with respect to \( W_0 \). By the assumption, however, this implies that \( W_0 \) is a proper subspace of \( W(log) \) and hence also \( T_0 \) is a proper subspace of \( T_{S,0} \). It follows that \( S(\lambda_0) \) is a proper subvariety of \( S \). The countability follows from the countability of the groups \( H^2(U_t, \mathbb{Q}) \). \( \Box \)

### 5.2 Proof of Proposition 3.4.

We use the notations and assumptions of the main theorem and let \( k \) be the number of intersection points of \( Z_1 \) and \( Z_2 \).

**Proof:** (of Prop. 3.4.)

Let \( X_0 := X \setminus Z_{\text{sing}} \) and \( U = X \setminus Z \), where \( Z = Z_1 \cup Z_2 \). By weak purity

\[ H^i_{D,|Z_{\text{sing}}|}(X, \mathbb{Z}(2)) = 0 \text{ for } i \leq 3. \]

Therefore \( H^2_{D}(X, \mathbb{Z}(2)) = H^1_{D}(X_0, \mathbb{Z}(2)) \) for \( i \leq 2 \) and \( H^3_{D}(X, \mathbb{Z}(2)) \subset H^3_{D}(X_0, \mathbb{Z}(2)) \). Note that \( H^1_{D,|Z_{\text{sing}}|}(X, \mathbb{Z}(2)) \cong \mathbb{Z}^k \). Also let

\[ K := \text{Ker}(H^1_{D,|Z_{\text{sing}}|}(X, \mathbb{Z}(2)) \to H^1_{D}(X, \mathbb{Z}(2))) \]

There is a diagram

\[
\begin{array}{cccc}
0 & \to & 0 & \to \\
\downarrow & & \downarrow & \\
H^2_{D}(U, \mathbb{Z}(2)) & \to & H^3_{D,|Z|}(X, \mathbb{Z}(2)) & \to \\
\downarrow & \downarrow & \downarrow & \downarrow \\
H^2_{D}(U, \mathbb{Z}(2)) & \to & H^3_{D,|Z|}(X_0, \mathbb{Z}(2)) & \to \\
\downarrow & & \downarrow & \\
K & = & K & \\
\end{array}
\]
Let $D_i = Z_i \setminus Z_{\text{sing}}$. The $D_i$ are smooth disjoint divisors in $X_0$ and one has an isomorphism $H^3_{D_i, \mathbb{Z}}(X, \mathbb{Z}(2)) \cong \oplus H^1_B(D_i, \mathbb{Z}(1))$. These groups can be computed via the exact sequences

$$0 \to \mathbb{C}^* \to H^1_D(D_i, \mathbb{Z}(1)) \to F^1 \cap H^1(D_i, \mathbb{Z}) \to 0$$

for $i = 1, 2$. The localization sequence

$$0 \to \mathbb{C}^* = H^1_D(Z_i, \mathbb{Z}(1)) \to H^1_D(D_i, \mathbb{Z}(1)) \to H^0_D(Z_{\text{sing}}, \mathbb{Z}(0)) \cong \mathbb{Z}^k \to H^2_D(Z_i, \mathbb{Z}(1))$$

identifies $F^1 \cap H^1(D_i, \mathbb{Z})$ with $\text{Ker}(\mathbb{Z}^k \to \text{Pic}(Z_i))$. Using the fact that $\text{Ker}(\oplus_i \text{Ker}(\mathbb{Z}^k \to \text{Pic}(Z_i)) \to K) = \text{Ker}(\mathbb{Z}^k \to \oplus_i \text{Pic}(Z_i))$ we have proved the exactness of

$$0 \to (\mathbb{C}^*)^\oplus \to H^3_{D, \mathbb{Z}}(X, \mathbb{Z}(2)) \to \text{Ker}(\mathbb{Z}^k \to \oplus \text{Pic}(Z_i)) \to 0$$

and it remains to show that $\text{CH}^1([Z], 1) \cong H^3_{D, \mathbb{Z}}(X, \mathbb{Z}(2))$. But the $D_i$ are smooth and hence $\text{CH}^1(D_i, 1) \cong H^1_B(D_i, \mathbb{Z}(1))$ and the localization sequence for $D_i \subset Z_i$ with complement $Z_{\text{sing}}$ gives an exact sequence

$$0 \to (\mathbb{C}^*) \cong \text{CH}^1(Z_i, 1) \to \text{CH}^1(D_i, 1) \to \text{CH}^0(Z_{\text{sing}}, 0) \cong \mathbb{Z}^k \to \text{CH}^1(Z_i) \to \ldots$$

and the claim follows in the same way as for $H^3_{D, \mathbb{Z}}(X, \mathbb{Z}(2))$. \hfill \Box

### 5.3 Proof of the Main Theorem

We use the notations as in the theorem.

**Proof of the main theorem:** By the proposition above we have a sequence

$$0 \to (\mathbb{C}^*)^\oplus \to \text{CH}^1([Z], 1) \xrightarrow{\tau} \text{Ker}(\mathbb{Z}^k \to \oplus \text{Pic}(Z_i)) \to 0$$

and assumption (1) is equivalent to $\tau(Z) \neq 0$. The complex

$$H^2_B(X, \mathbb{Z}(2)) \to H^2_B(U, \mathbb{Z}(2)) \to H^3_{D, \mathbb{Z}}(X, \mathbb{Z}(2)) \to H^3_B(X, \mathbb{Z}(2)) \to H^3_B(U, \mathbb{Z}(2))$$

has the subcomplex

$$H^1(X, \mathbb{C}^*) \to H^1(U, \mathbb{C}^*) \to (\mathbb{C}^*)^\oplus \to \text{NS}(X) \otimes \mathbb{C}^* \to 0$$

by assumption (2d). Here we have assumed that $Z_1$ and $Z_2$ generate $\text{NS}(X) \otimes \mathbb{Q}$, because by assumption (2d) we can always choose a general deformation of $X$ that satisfies this property and also the other assumptions of the theorem, since they are of generic nature. We get a diagram:

$$\begin{array}{cccc}
\text{CH}^2(U, 2) & \to & \text{CH}^1(Z, 1) & \to & \text{CH}^2(X, 1) & \to & \text{CH}^2(U, 1) \\
\downarrow & & \downarrow \tau & & \downarrow & & \downarrow \\
H^2_B(U, \mathbb{Z}(2)) & \to & \text{Ker}(\mathbb{Z}^k \to \oplus \text{Pic}(Z_i)) & \to & H^2_B(X, \mathbb{Z}(2)) & \to & H^3_B(U, \mathbb{Z}(2)) \\
\text{NS}(X) \otimes \mathbb{C}^* & \to & & & & & \\
\end{array}$$

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Since \( H^2_D(U,Z(2))/H^1(U,\mathbb{C}^*) = F^2 \cap H^2(U,\mathbb{Z}) \), it is enough to show that the image of \( \alpha : F^2 \cap H^2(U,\mathbb{Z}) \rightarrow \text{Ker}(\mathbb{Z}^k \rightarrow \oplus \text{Pic}(Z_i)) \) does not contain \( \tau(Z) \). Note that the kernel is a free group, so it will be enough to show that \( F^2 \cap H^2(U,\mathbb{Z}) = 0 \) for general \( t \) by the following argument.

**Claim:** The cup-product map \( H^0(X,\Omega^2_X(\log Z)) \otimes W(\log) \rightarrow H^1(X,\Omega^1_X(\log Z)) \) has no left kernel.

Taking this for granted we are finished with the proof by Lemma (5.2).

Hence it remains to prove the claim:

Look at the commutative diagram that exists by assumption (2c):

\[
\begin{array}{c}
H^0(X,\Omega^2_X(\log Z)) \otimes H^1(X,T_X(-Z)) \rightarrow H^1(X,\Omega^1_X)/ \oplus H^0(Z_i,\mathcal{O}_{Z_i}) \\
\downarrow \\
H^0(X,\Omega^2_X(\log Z)) \otimes W(\log) \rightarrow H^1(X,\Omega^1_X(\log Z))
\end{array}
\]

By assumption (2b) the upper map has no left kernel and therefore also the lower map, since \( H^1(X,\Omega^1_X)/ \oplus H^0(Z_i,\mathcal{O}_{Z_i}) \rightarrow H^1(X,\Omega^1_X(\log Z)) \) is an injection. \( \square \)

6 Examples

6.1 Sufficiently Ample General Complete Intersections in Arbitrary Varieties

The result that follows has several roots. First of all there is the Noether-Lefschetz theorem stated in the introduction, which says that a general hypersurface of degree \( d \geq 4 \) in \( \mathbb{P}^3 \) has \( \text{Pic}(X) \cong \mathbb{Z} \). Then there is the theorem of Green-Voisin ([20]), saying that a general hypersurface of degree \( d \geq 6 \) in \( \mathbb{P}^4 \) has torsion Abel-Jacobi invariants.

In 1989 we started to generalize this together with M. Green by using a sophisticated version of residue representations of differential forms via global sections of adjoint linear systems. But at the same time M. Nori came up with his fantastic connectivity theorem from [32], which gave a much shorter proof of the following result:

**Theorem 6.1** [22]

Let \((Y,\mathcal{O}(1))\) be a smooth and projective polarized variety of dimension \( n+h \), \( X \subset Y \) a general complete intersection of dimension \( n \) and multidegree \((d_1,\ldots,d_h)\) with \( \min(d_i) \) sufficiently large. Furthermore assume that \( 0 \leq p \leq n-1 \). Then:

\[
\text{Image}(CH^p(X) \otimes \mathbb{Q} \rightarrow H^p_D(X,\mathbb{Q}(p)))
= \text{Image}(CH^p(Y) \otimes \mathbb{Q} \rightarrow H^p_D(X,\mathbb{Q}(p)))
\]

modulo possibly the image of a certain subtorus of \( J^p(Y)_\mathbb{Q} \), which vanishes if the generalized Hodge conjecture holds.
As always general means for all points in the moduli space outside a countable set of proper analytic subvarieties, which is sometimes also called very general. The generalization to higher Chow groups can be proved with the same method (this was also observed by S.Bloch, M.Nori and C. Voisin and is mentioned in [42]):

**Theorem 6.2** Let \((Y, \mathcal{O}(1))\) be a smooth and projective polarized variety of dimension \(n + h\), \(X \subset Y\) a general complete intersection of dimension \(n\) and multidegree \((d_1, \ldots, d_h)\) with \(\min(d_i)\) sufficiently large. Furthermore assume that \(1 \leq p \leq n\). Then:

\[
\text{Image}(CH^p(X, 1) \otimes \mathbb{Q}) \rightarrow H^{2p-1}_D(X, \mathbb{Q}(p))
\]

\[
\subset \text{Image}(H^{2p-1}_D(Y, \mathbb{Q}(p)) \rightarrow H^{2p-1}_D(X, \mathbb{Q}(p))
\]

**Proof:** Let \(S := \prod_t \mathbb{P}(H^0(Y, \mathcal{O}_Y(d_t)))\) and \(f : B \rightarrow S\) the universal complete intersection. If \(g : T \rightarrow S\) is any smooth morphism, we write also \(g : B_T = B \times_S T \rightarrow B\) for the base change and \(A_T = Y \times T\). If \(Z_s\) is a cycle on \(X = X_s\) for \(s \in S\), we can find a cycle \(Z \in CH^p(B_T, 1)\) for some smooth morphism \(g : T \rightarrow S\) with the property that for some \(t \in g^{-1}(s)\) we have \(Z \cap X_t = Z_s\) and such that \(g\) is smooth and finite. By Nori’s theorem [32] we have for \(1 \leq p \leq n\) that the restriction homomorphism

\[
i^* : H^{2p-1}_D(A_T, \mathbb{Q}(p)) \rightarrow H^{2p-1}_D(B_T, \mathbb{Q}(p))
\]

is an isomorphism. Therefore there is a cohomology class \(\alpha \in H^{2p-1}_D(A_T, \mathbb{Q}(p))\) such that \(i^* \alpha = c_{p,1}(Z)\). The commutative diagram of restriction maps

\[
\begin{array}{ccc}
H^{2p-1}_D(A_T, \mathbb{Q}(p)) & \rightarrow & H^{2p-1}_D(B_T, \mathbb{Q}(p)) \\
\downarrow & & \downarrow \\
H^{2p-1}_D(Y, \mathbb{Q}(p)) & \rightarrow & H^{2p-1}_D(X_s, \mathbb{Q}(p))
\end{array}
\]

shows that \(\alpha|_{X_t}\) lies in the image of \(H^{2p-1}_D(Y, \mathbb{Q}(p))\) for every \(t\) with \(g(t) = s\) and therefore the theorem is proved. \(\square\)

**Corollary 6.3** Let \(X \subset \mathbb{P}^3\) be a general hypersurface of degree \(d \geq 5\). Then: The Chern class \(CH^2(X, 1) \otimes \mathbb{Q} \rightarrow H^6_D(X, \mathbb{Q}(2))\) has image isomorphic to \(H^6_D(\mathbb{P}^3, \mathbb{Q}(2)) \cong \mathbb{C}/\mathbb{Q}(1)\).

**Proof:** Let \(B_T\) be any smooth base change of the universal hyperplane section \(f : B \rightarrow S\) with \(S = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))\). By [33] we have \(H^6_D(\mathbb{P}^3 \times T, \mathbb{Q}(2)) \cong H^6_D(B_T, \mathbb{Q}(2))\) for \(d \geq 5\). This is the required equality in [22]. \(\square\)
6.2 K3 Surfaces

6.2.1 A Deformation of the Fermat Quartic

Here we give an explicit example satisfying the assumptions of the main theorem:
Consider the Fermat quartic surface
\[ X = \{ x_0^4 + x_1^4 - x_2^4 - x_3^4 = 0 \} \subset \mathbb{P}^3 \]
and the linear forms
\[ L_0 = x_1 - x_2, \quad L_1 = x_0 - x_3, \quad L_2 = x_1 + x_3, \quad L_3 = x_0 - x_2 \]
in \( \mathbb{P}^3 \). We look at the family of quartics
\[ F_t(x_0 : x_1 : x_2 : x_3) = x_0^4 + x_1^4 - x_2^4 - x_3^4 + 2tL_0L_1L_2L_3 \]
and define \( X_t := \{ x \in \mathbb{P}^3 \mid F_t(x) = 0 \} \).
The following line on \( X \) is important:
\[ G = \{ x_1 - x_2 = x_0 - x_3 = 0 \} \]
Planes containing it define a pencil of planes in \( \mathbb{P}^3 \):
\[ H_\lambda = \{ x_1 - x_2 - \lambda(x_0 - x_3) = 0 \} \]
The residual curves \( E_{\lambda,t} \) to the intersection of \( H_\lambda \) with \( X_t \) are elliptic curves and their equations can be computed as follows: The coordinate transformation \((x_0 : x_1 : x_2 : x_3) \mapsto ((x : y : z), \lambda)\), where
\[ x_0 = x + z, x_1 = y + \lambda z, x_2 = y - \lambda z, x_3 = x - z \]
defines a rational map (blowing up along \( G \)) and we obtain the new equations
\[ F_t(x, y, z, \lambda) = 8z(x^3 + \lambda y^3 + z^2(x + \lambda^3 y) + \lambda tzL_2L_3) =: 8zF''(x, y, z, \lambda) \]
where \( L_2(x, y, z, \lambda) = x + y + (\lambda - 1)z, L_3(x, y, z, \lambda) = x - y + (\lambda + 1)z \) and hence for the strict transforms we get the following equations in \( \mathbb{P}^2 \times \mathbb{P}^1 \):
\[ X_t = \{ (x, y, z, \lambda) \mid x^3 + \lambda y^3 + z^2(x + \lambda^3 y) + \lambda tzL_2(x, y, z, \lambda)L_3(x, y, z, \lambda) = 0 \} \]
defining an elliptic pencil with fibers \( E_{\lambda,t} \) for every \( t \).
The strict transform of \( G \) is \( G = \{ z = 0 \} \) and \( E_{\lambda,t} \cap G = \{ P_1 = (a : 1 : 0), P_2 = \ldots \} \).
compute the tangent lines \( T_i \) of \( E_{\lambda,t} \) at \( P_i \) we compute the partial derivatives

\[
\frac{\partial F'_i}{\partial x} = 3x^2 + z^2 + \lambda tz(L_2 + L_3)
\]

\[
\frac{\partial F'_i}{\partial y} = 3\lambda y^2 + \lambda^3 z^2 + \lambda tz(L_3 - L_2)
\]

\[
\frac{\partial F'_i}{\partial z} = 2z(x + \lambda^3 y) + \lambda t L_2(x, y, z, \lambda) L_3(x, y, z, \lambda) + \lambda tz((\lambda - 1)L_3 + (\lambda + 1)L_2)
\]

Therefore we get the equations of the tangent lines

\[
T_1 = \{3a^2 x + 3\lambda y + \lambda(t(a^2 - 1))z = 0\}
\]

\[
T_2 = \{3a^2 \zeta^2 x + 3\lambda y + \lambda(t(a^2 \zeta^2 - 1))z = 0\}
\]

\[
T_3 = \{3a^2 \zeta^4 x + 3\lambda y + \lambda(t(a^2 \zeta^4 - 1))z = 0\}
\]

with intersection \( T_1 \cap T_2 \cap T_3 = \{(-\lambda : t : 3)\} =: \{P_{\lambda,t}\} \). The projection from \( P_{\lambda,t} \) onto \( G = \{ z = 0 \} \) defines a hyperelliptic morphism onto \( G \) if and only if \( P_{\lambda,t} \in E_{\lambda,t} \).

This is one algebraic condition we will compute:

\[
0 = F''(P_{\lambda,t}) = t\lambda(\lambda - 1)F''(\lambda)
\]

with \( F''(\lambda) = \lambda(t^2 - 18t + 9) + 4t^2 - 6t - 18 \). Solving \( F''(\lambda) = 0 \) we get

\[
\lambda(t) = -\frac{4t^2 - 6t - 18}{t^2 - 18t + 9}
\]

### 6.2.2 Non-Trivial Cycles

If \( \lambda = \lambda(t) \), the hyperelliptic map of the preceding section \( E_{\lambda(t),t} \rightarrow G \) defines a rational function \( f_1 \) with zero divisor \( 2(P_1) - 2(P_2) \) as a cycle on \( E_{\lambda(t),t} \). There is always a rational function \( f_2 \) on \( G \) with zero divisor \( 2(P_2) - 2(P_1) \) as a quotient of 2 squares. Let \( Z := E_{\lambda(t),t} \otimes f_{1,t} + G \otimes f_{2,t} \in CH^2(X_t, 1) \). Note that the cycle \( Z_t \) is only defined over a covering of \( \mathbb{P}^1 \) since the choice of two of the 3 intersection points with an order is only determined up to permutations.

However in order to verify the assumptions in the main theorem, in particular (2c), we have to extend the base space of our deformation. Let \( N \subset \mathbb{P}^H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(4)) \) be an irreducible component of the Noether-Lefschetz locus of quartics containing a line and containing the deformation of the Fermat quartic in section (6.2.1). After some smooth base change \( g : S \rightarrow N \) we get a family of lines \( G_t \), elliptic curves \( E_t \) in \( X_t \), rational functions \( f_{1,t} \) on \( E_t \), \( f_{2,t} \) on \( G_t \) and cycles \( Z_t = E_t \otimes f_{1,t} + G_t \otimes f_{2,t} \in CH^2(X_t, 1) \).
well-defined on $S$ that extend the cycles $E_{X(t),t}$ and $G = G_0$ above. $S$ can be chosen in such a way that assumption (2a) of the criterion is satisfied. The cycles even define classes also denoted by $Z_t$ in $CH^1(|Z_t|, 1)$ that map to $CH^2(X_t, 1)$ under the natural map $CH^1(|Z_t|, 1) \to CH^2(X_t, 1)$.

$W(\log) \subset H^1(X, T_X(logZ))$ is a codimension one subspace, since the deformations of the pair $(X_t, Z_t)$ that preserve the cycle in $CH^2(X_t, 1)$ map generically finite onto the deformations of $X$ itself: on every generic elliptic fibration only a finite number of fibers will have rationally equivalent intersection points with a line, as we checked in the example above. Note that it is sufficient to check such an assertion at one point of the moduli space. Thus $W(\log) \to W$ is an isomorphism. $W$ is isomorphic to the tangent space to $N$ at the point $X$ by Noether-Lefschetz theory. The following diagram is therefore exact and commutative:

$$
0 \to \mathbb{C} \to H^1(X, T_X(logZ)) \to H^1(X, T_X) \to \mathbb{C}^2 \to 0
$$

$$
\bigcup W(\log) \cong W
$$

**Lemma 6.4** The cycle $Z_t = E_t \otimes f_{1,t} + G_t \otimes f_{2,t}$ in $CH^2(X_t, 1)/\text{Pic}(X_t) \otimes \mathbb{C}^*$ is nontrivial modulo torsion for $t$ outside a countable union of proper analytic subsets of $S$.

**Proof:** We have to verify the assumptions of the main theorem. To simplify the notation, we assume for the moment that $t = 0$ and let $E = E_0, G = G_0$ and $Z = Z_0$. $Z$ will also denote the effective cycle $E + G$. To check (1), it is sufficient to remark that $\tau(Z)$ corresponds to the element $(2, -2, 0)$ in $\text{Ker}(Z^3 \to \text{Pic}(E) \oplus \text{Pic}(G))$ and therefore does not decompose. Note that if we write $\tau(Z)$ as $(2, -2, 0)$, we have chosen an ordering of our curves $Z_1$ and $Z_2$ (as we did in the proof of prop. 3.4.), and then the map $\tau : CH^1(|Z|, 1) \to Z^3$ is given by $Z_1 \otimes h_1 + Z_2 \otimes h_2 \mapsto \text{div}(h_1) = -\text{div}(h_2)$. Assumption 2(a) is clear by construction.

To prove (2b), we look at the following graded Artinian ring $R_* = \oplus R_d$ with

$$
R_d = \frac{H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))}{J_{F,d}}
$$

In the example the Jacobian ideal $J_{F,d}$ is the $d$-th graded piece of the homogenous ideal generated by the monomials $x_i^3$. One has the following isomorphisms first proved by Griffiths (see [21], page 44):

$$
H^1(X, \Omega^1_X)_{pr} \cong R_4, \quad H^{0,2}(X) \cong R_8 \cong \mathbb{C}
$$

$$
H^1(X, T_X(-Z)) \cong R_3, R_0 \cong \mathbb{C}, R_1 \cong H^0(\Omega_X^2(logZ))
$$

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Further there is Macaulay duality: $R^*_k \cong R_{8-k}$, see op.cit.

In this language we have to prove that

$$R_1 \otimes R_3 \rightarrow R_4 / \mathbb{C} \cdot g$$

has no left kernel, where $g$ is a quartic polynomial representing the extra cohomology class of $G$ (note that the cohomology class of $Z = G + E$ is the hyperplane class and therefore not primitive). In other words we have to prove that

$$R_1 \rightarrow R^*_3 \otimes R_4 / \mathbb{C} \cdot g$$

is injective. After dualizing this means that

$$m : V / J_{F,A} \otimes R_3 \rightarrow R_7$$

is surjective, where $V \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$ is the codimension one linear subspace dual to $\mathbb{C} \cdot g$. $V$ contains $J_{F,A}$ and is therefore basepointfree. The surjectivity of $m$ then follows from the theorem of Green-Gotzmann ([21], pg.74), which has the following corollary:

Let $V \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$ be a linear basepointfree subspace of codimension $c$, then the map

$$m : V \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k - d)) \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$$

is surjective if $k \geq d + c$.

We apply this result for $k = 7, d = 4, c = 1$ thus proving (2b).

To prove (2c), consider the sheaf $G := \Omega^2_X(logZ)/\Omega^2_X$, which we can also denote by $\omega_Z$. One has an exact sequence

$$0 \rightarrow \Omega^1_G \oplus \Omega^1_E \rightarrow G \rightarrow \mathbb{C}^{\oplus 3} \rightarrow 0$$

which, since $G$ is rational, has the same $H^0$ sequence of vector spaces as the one coming from

$$0 \rightarrow \Omega^1_E \rightarrow \Omega^1_E(logG) \rightarrow \mathbb{C}^{\oplus 3} \rightarrow 0$$

and the coboundary maps are equal. Hence here $H^0(G) \cong H^0(E, \Omega^1_E(logG))$. The cycle $Z \in CH^1(|Z|, 1)$ defines a non-zero element $\alpha \in H^0(G)$ with $\alpha = \frac{1}{2\pi i} \frac{df_1}{f_1}$. There is an induced map

$$H^0(G) \otimes H^1(X, T_X(logZ)) \rightarrow H^1(E, \mathcal{O}_E)$$

(since $G$ is rational) and $W(log)$ equals the right annihilator of $\alpha$ under that map (this follows, because $W(log)$ is a codimension 1 subspace of $H^1(X, T_X(logZ))$ that annihilates $\alpha$ and therefore is equal to the annihilator, since the latter also has codimension 1).
To show (2c) it is sufficient to look at the following diagram (note that the right column is exact but not the left one):

\[
\begin{array}{ccc}
H^0(X, \Omega_X^2(\log Z)) \otimes H^1(X, T_X(-Z)) & \rightarrow & H^1(X, \Omega_X^1)/\mathbb{C}^2 \\
\downarrow & & \downarrow \\
H^0(G) \otimes H^1(X, T_X(\log Z)) & \rightarrow & H^1(E, \mathcal{O}_E)
\end{array}
\]

It shows that the image of \(H^0(X, \Omega_X^2(\log Z)) \otimes H^1(X, T_X(-Z)) \rightarrow H^1(E, \mathcal{O}_E)\) is zero, hence the image of \(H^1(X, T_X(-Z)) \rightarrow H^1(X, T_X(\log Z))\) annihilates \(\alpha\) and therefore is a subspace of \(W(\log)\). This finishes the proof of (2c).

To prove (2d), look at the multiplication map

\[
W \otimes H^1(X, \Omega_X^1) \rightarrow H^2(X, \mathcal{O}_X) \cong \mathbb{C}
\]

\(W\) annihilates the cycle classes of the curves \(G\) and \(E\) and the restriction to the quotient

\[
W \otimes \frac{H^1(X, \Omega_X^1)}{C[G] \oplus C[E]} \rightarrow \mathbb{C}
\]

is a perfect pairing by the Hodge Riemann bilinear relations. It follows that no class in \(H^1(X, \Omega_X^1)\) survives in a general deformation direction of \(W\) by the same argument as in the proof of lemma (5.2.). Therefore for a general surface \(X_t\) one has that \(NS(X_t) \otimes \mathbb{Q}\) is generated by the two elements \([G]\) and \([E]\). \(\square\)

**Remark:** With some more work, using a monodromy argument of H. Clemens which was also used by A. Collino in [10], one can probably prove infinite generation for the image of \(CH^2(X_t, 1)\) for a general quartic hypersurface containing a line. Since for other examples this was proved in [10] and [43] we refrain from presenting it here.

Obviously the idea would be to study the monodromy around a countable set of loci on the parameter space of the surfaces \(X_t\).

### 6.3 Examples of General Type

In the previous section we have studied some special quartic surfaces and have seen that one can apply the main theorem. Now we would like to show that one can do a similar construction on some quintic hypersurfaces \(X\) in \(\mathbb{P}^3\) and again apply the main theorem to get cycles that are indecomposable in \(CH^2(X, 1)\). Note that they have to be very special surfaces in view of the result (6.2).

To construct examples of general type, we look at the Shioda hypersurface of degree 5 from [37]: \(X = \{x \in \mathbb{P}^3 \mid x_0x_1^4 + x_1x_2^4 + x_2x_3^4 + x_3^5 = 0\}\). It has an automorphism \(\sigma\)
of order 65, given by \((x_0 : x_1 : x_2 : x_3) \mapsto (\zeta^{16}x_0 : \zeta^{-4}x_1 : \zeta x_2 : x_3)\) where \(\zeta\) is a 65-th root of unity. Shioda proves that the Picard group of \(X\) is of rank one, by showing that \(\sigma\) acts irreducibly on \(H^2_{\text{pr}}(X, \mathbb{Q})\).

Let us look at the 1-parameter family

\[ X_t = \{ x \in \mathbb{P}^3 \mid F_t(x) = x_0x_1^4 + x_1x_2^4 + x_2x_3^4 + x_3^5 + tx_3x_1^4 = 0 \} \]

for \(t \in \mathbb{P}^1\). Furthermore let \(H_i = \{ x_i = 0 \}\) be the coordinate hyperplanes and define the curves \(Z_{1,t} := X_t \cap H_3\) and \(Z_{2,t} := X_t \cap H_0\). Finally we set \(P := (0 : 1 : 0 : 0)\), \(Q := (0 : 0 : 1 : 0)\) and \(R := (1 : 0 : 0 : 0)\).

**Proposition 6.5** (a) \(Z_{1,t}\) is smooth for all \(t\), \(Z_{2,t}\) is smooth for \(t \neq 0, \infty\) and \(Z_{2,0}\) is a rational curve with a point of multiplicity 4 at \(P\).

(b) \(P, Q \in Z_{1,t} \cap Z_{2,t}\).

(c) for all \(t\): \(4P = 4Q\) in \(CH_0(Z_{2,t})\).

(d) for all \(t\): \(13P = 13Q\) in \(CH_0(Z_{1,t})\).

**Proof**: (a) We omit the subscripts \(t\), if there is no ambiguity. \(Z_1 = \{ (x_0 : x_1 : x_2) \in \mathbb{P}^2 \mid x_0x_1^4 + x_1x_2^4 + x_2x_3^4 = 0 \}\) independent of \(t\), which is smooth by the Jacobian criterion. \(Z_{2,t}\) depends on \(t\): \(Z_{2,t} = \{ (x_1 : x_2 : x_3) \in \mathbb{P}^2 \mid x_1x_2^4 + x_3^5 + tx_3x_1^4 = 0 \}\). The gradient is therefore \((x_3^5 + 4tx_3x_1^4 : 4x_3^4x_1 : 5x_3^4 + tx_1^4)\), which is nowhere zero for \(t \neq 0, \infty\). For \(t = 0\) one gets a 4-fold point at \(P\). \(Z_{2,0}\) is rational by the parametrization \((t_0 : t_1) \mapsto (0 : -t_0^3 : t_1^5 : t_1^4t_0)\). This proves (a).

(b) One computes \(Z_1 \cap Z_2 = X_t \cap H_0 \cap H_3 = \{ x \in \mathbb{P}^3 \mid x_0 = x_3 = 0, \quad x_1x_2^4 = 0 \}\). Hence \(Z_2 \cdot H_3 = Q + 4P\) as cycles on \(\mathbb{P}^3\) with multiplicity counted. (b) follows.

(c) We obtain \(Z_2 \cap H_1 = X_t \cap H_0 \cap H_1 = \{ x \in \mathbb{P}^3 \mid x_0 = x_1 = 0, \quad x_3^5 = 0 \}\). Hence \(Z_2 \cdot H_1 = 5Q\) as a cycle on \(\mathbb{P}^3\). But \(Z_2 \cdot H_3\) is rationally equivalent to \(Z_2 \cdot H_1\) on \(Z_2\). Together with (b), we obtain that \(Q + 4P = 5Q\) and hence \(4P = 4Q\) in \(CH_0(Z_{2,t})\) for all \(t\).

(d) \(Z_{1,t} \cap H_2 = X_t \cap H_3 \cap H_2 = \{ x \in \mathbb{P}^3 \mid x_2 = x_3 = 0, \quad x_0x_1^4 = 0 \}\). It follows that \(Z_1 \cdot H_2 = P + 4R\) as cycles. Finally we compute \(Z_1 \cdot H_1 = R + 4Q\) by symmetry. Together we get that \(Q + 4P = P + 4R = R + 4Q\) in \(CH_0(Z_1)\). Solving these equations gives that \(13P = 13Q = 13R\) in \(CH_0(Z_1)\).

**Corollary 6.6** For all \(t\), \(52P\) and \(52Q\) are rationally equivalent on both curves. This defines rational functions \(f_t\) on \(Z_t\) with \(\text{div}(f_t) = 52P - 52Q = -\text{div}(f_2)\) as a zero cycle on \(X_t\) for all \(t\) and hence a cycle \(Z_t \in CH^2(X_t, 1)\).

As in the previous example we now choose a maximal irreducible component \(N \subset \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(5)))\), containing the 1-parameter family above and such that the properties of the cycles \(Z_t\) extend along a suitable étale covering \(S\) of \(N\) and therefore
cycles $Z_t \in CH^2(X_t,1)$ are well defined for all $t \in S$. The assumptions of the main theorem can be checked in the same way as in the previous example:

**Theorem 6.7** For $t \in S$ general, $Z_t$ is indecomposable in $CH^2(X_t,1)$.

**Proof:** Again the assumptions (1),(2a) are easy to verify. To prove (2b) we use the same method as above. Assume $t = 0$ and look at the Shioda hypersurface. Here $K_X = O_X(1)$ and $K_X + Z = O_X(3)$. Again let $R_*$ denote the Griffiths' Jacobian ring. One has $H^0(X, K_X + Z) = R_3$ and $H^{1,1}_{pr} = R_6$. One also computes that $H^1(X, T_X(-Z)) = R_3$. Note that $Z_1$ and $Z_2$ have the same cohomology class. Therefore we have to show that $R_3 \otimes R_3 \to R_6$ has no left kernel, or dually that $R_6 \otimes R_3 \to R_9$ is surjective, which is obvious. This proves (2b). The proof of (2c) proceeds in the same way as above with the necessary modifications. Finally (2d) holds since it holds on the Shioda surface $X_0$ and therefore the Picard number is also one on a general surface parametrized by $S$. □

### 7 Further Remarks

First let us imitate a method of Bloch ([6]) to show what structure the groups that are involved should have. Note that $CH^2(X,1) \cong H^1(X, K_2)$ via the Gersten-Quillen resolution. Bloch’s method is to study $H^1(X, K_2)$ infinitesimally by giving an infinitesimal version of the sheaf $K_2$. Instead of $\mathbb{C}$ let the ground field be $k$ and look at the local $k$-algebra of dual numbers $k[\epsilon]/\epsilon^2$. Imitating Bloch’s construction in the case of $CH^2(X)$, it makes sense to speak of the formal tangent space $TH^1(X, K_2)$ to $H^1(X, K_2)$ by defining it as the group $H^1(X, K_2, \epsilon)$ where $K_{2,\epsilon}$ is this case given by the absolute differentials $\Omega^1_{X/k}$. This is also nicely explained in Murre’s lecture in [21], chapter V. From the exact sequence

$$0 \to O_X \otimes_k \Omega^1_{k/Q} \to \Omega^1_{X/Q} \to \Omega^1_{X/k} \to 0$$

we get a long exact sequence

$$H^1(X, O_X) \otimes_k \Omega^1_{k/Q} \to TH^1(X, K_2) \to H^1(X, \Omega^1_{X/k}) \to \ldots$$

$$\ldots \to H^2(X, O_X) \otimes_k \Omega^1_{k/Q} \to TCH^2(X) \to T\text{Alb}(X)$$

We see two principles, however without giving a correct proof: If $k = \mathbb{C}$ and $p_g(X) = 0$ or $k = \mathbb{Q}$, then the Albanese map should be injective (Bloch’s conjecture on $CH^2(X)$) and if $k = \mathbb{C}$ and $X$ has maximal Picard number, then the map $\text{Pic}(X) \otimes \mathbb{C}^* \to CH^2(X,1)$ should be surjective modulo torsion. But there is a caveat: In [34] there
is an example of a submotive of a Hilbert modular surface that has maximal Picard
number but indecomposable $CH^2(X, 1)$. Over number fields the situation seems to
be different, we refer to [34], [29] and [18]. I was told that also [29] is probably a
counterexample. See also the discussion in [33].

Another hope would be to give a direct computation of the Chern class map $c_{2,1}$ in
terms of integrals and to show that for some special value of $t$, the Deligne classes of
$Z_t$ are not torsion modulo $NS(X_t) \otimes \mathbb{C}^*$.

**Question:** Does a direct computation of the integral imply that already on the Shioda surface $X$ the cycle $Z$ is indecomposable? Note that $X$ is defined over $\mathbb{Q}$ and a positive answer would contribute to the study of the conjectures of Bloch and Beilinson.

It would also be nice to prove directly (without use of Deligne cohomology) that
$CH^2(X, 1)/Pic(X) \otimes \mathbb{C}^*$ is only countable, for example by Hilbert scheme methods.

This was suggested to me by C. Voisin.

Finally one would like to study higher Chow groups of degenerations of algebraic
varieties, for example the degeneration of a quartic K3 surface into a tetrahedron
of 4 planes as we described in section (6.2.1). We would like to know whether already a
general member of that one-parameter degeneration has an indecomposable
$CH^2(X_t, 1)$. This can probably be done by using another slight variant of our main
theorem and checking the assumptions on the variety $X_\infty$, by showing that there are
no two-forms of a certain logarithmic type. More general this should lead to a good
understanding of the mixed motive of a variety like the tetrahedron (consisting out of linear varieties with certain combinatorial data) in the sense of [23] and relating it
to the smooth case via deformation theory as developed in chapter 3.

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