Trading drift and fluctuations in entropic dynamics: quantum dynamics as an emergent universality class

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Abstract. Entropic Dynamics (ED) is a framework that allows the formulation of dynamical theories as an application of entropic methods of inference. In the generic application of ED to derive the Schrödinger equation for \( N \) particles the dynamics is a non-dissipative diffusion in which the system follows a “Brownian” trajectory with fluctuations superposed on a smooth drift. We show that there is a family of ED models that differ at the “microscopic” or sub-quantum level in that one can enhance or suppress the fluctuations relative to the drift. Nevertheless, members of this family belong to the same universality class in that they all lead to the same emergent Schrödinger behavior at the “macroscopic” or quantum level. The model in which fluctuations are totally suppressed is of particular interest: the system evolves along the smooth lines of probability flow. Thus ED includes the Bohmian or causal form of quantum mechanics as a special limiting case. We briefly explore a different universality class – a non-dissipative dynamics with microscopic fluctuations but no quantum potential. The Bohmian limit of these hybrid models is equivalent to classical mechanics. Finally we show that the Heisenberg uncertainty relation is unaffected either by enhancing or suppressing microscopic fluctuations or by switching off the quantum potential.

1. Introduction
Entropic Dynamics (ED) is a framework in which quantum theory is derived as an application of entropic methods of inference.\(^1\) As in any theory of inference, establishing the subject matter is the first and most crucial step; this amounts to a choice of microstates, that is, a choice of beables. Once that choice is made the dynamics is driven by entropy subject to constraints which reflect the information needed for making physical predictions \(^2\)–\(^5\).

ED naturally leads to an epistemic view of the quantum state \( \psi \) but with an added twist. Within an inferential framework it is not sufficient to merely interpret the probability \( |\psi|^2 \) as a state of knowledge.\(^2\) It is just as important to require that the dynamics, that is the updates of \( \psi \), be consistent with the established rules for updating probability distributions. This is where the method of maximum entropy and the Bayesian methods enters. Furthermore, the entropic dynamics must include both the unitary time evolution described by the Schrödinger equation and the collapse of the wave function during measurement. As a result the ED framework turns out to be very restrictive. In a fully entropic dynamics we do not postulate an underlying

\(^1\) The principle of maximum entropy as a method for inference can be traced to the pioneering work of E. T. Jaynes. For a pedagogical overview of Bayesian and entropic inference and further references see \(^1\).

\(^2\) For a review with references on the epistemic vs ontic interpretations of the quantum state see \(^6\).
mechanics with an action principle that operates at some deeper level. Instead, at the sub-quantum level there is only inference and at the quantum level the emergent dynamics is described by an action principle that is derived rather than posited.

There is a vast literature on the attempts to reconstruct quantum mechanics and it is inevitable that the ED approach will resemble them in one aspect or another. Indeed, to the extent that any of these approaches are successful they must sooner or later converge to the same Schrödinger equation. However, there are important differences. For example, the central concern with the notion of time makes ED significantly different from other approaches that are also based on information theory (such as e.g., [7–14]). And ED also differs from those approaches (see e.g., [15–21]) that aim to explain the emergence of quantum behavior as the effective statistical mechanics of some underlying sub-quantum mechanics which might possibly include some additional stochastic element. Indeed, ED makes no reference to any sub-quantum action principles whether classical, deterministic, or stochastic.

As stated above in ED inferences are carried out on the basis of information introduced in the form of constraints. In the particular case of the ED of \( N \) particles the microstates are the positions of the particles. The basic physical input — that the particles follow continuous trajectories — is implemented through \( N \) constraints, one for each particle. The multiple roles played by the corresponding Lagrange multipliers \( \alpha_n (n = 1 \ldots N) \) are by now well understood. The multipliers regulate the flow of time and they serve to unify the concept of mass with that of quantum fluctuations.

If these were the only constraints each particle would experience its own independent dynamics and the resulting motion would be an isotropic diffusion. It turns out that one can obtain more interesting forms of dynamics by imposing just one single additional constraint. This constraint acts on configuration space and involves a “drift potential,” an epistemic tool that plays a role somewhat analogous to that of a pilot wave. The drift potential contributes to the phase of the wave function, it correlates the motion of particles, and it causes such quintessential quantum effects as interference and entanglement. The corresponding multiplier \( \alpha' \) is not nearly as well understood and the purpose of this paper is to fill this gap [22].

We begin with a brief overview of ED following the presentation in [4]. We show that the role of the multiplier \( \alpha' \) is to control the relative magnitudes of drift and fluctuations. Our main result is simple: ED models with different values of \( \alpha' \) lead to the same Schrödinger equation. In other words, different “microscopic” or sub-quantum models can lead to the same emergent quantum behavior — they belong to the same “universality” class. The limit of large \( \alpha' \) deserves particular attention. In this limit fluctuations are suppressed, the drift motion prevails, and the particles tend to move along the smooth lines of probability flow. This means that ED includes the Bohmian form of quantum mechanics as a special limiting case [23–25].

We briefly explore one alternative family of ED models — a different universality class. At the microscopic level these models also describe particles that follow Brownian trajectories but without the non-local correlations induced by the quantum potential [26]. The multiplier \( \alpha' \) also acts to suppress microscopic fluctuations and the Bohmian, or large \( \alpha' \) limit, approaches classical mechanics. All these microscopic models lead to the same dynamics at the macroscopic level — the emergent dynamics is an essentially classical mechanics.

The value of the ED approach to quantum theory lies in part in the conceptual clarity it brings to issues of interpretation. In [26] the methods originally developed in the context of stochastic mechanics [27–30] were adapted to derive the Heisenberg uncertainty relation for position and momentum within the context of ED.

Finally, the fact that in ED we can enhance or suppress microscopic fluctuations relative to the drift and that we can switch on or off the quantum potential leads us to raise the question of how these changes affect the uncertainty relations. Are there potential violations of the uncertainty principle? We find that the uncertainty relations are unaffected by changes in the
microscopic fluctuations, even when suppressing them to the Bohmian limit, or by switching off the quantum potential.

2. Entropic Dynamics — a brief overview

We consider the ED of $N$ particles living in a flat Euclidean space $X$ with metric $\delta_{ab}$. In the ED framework the particles have definite positions $x_n^a$ and it is their unknown values that we wish to infer. (The index $n = 1 \ldots N$ labels the particle and $a = 1, 2, 3$ its spatial coordinates.) The position of the system in configuration space $X_N = X \times \ldots \times X$ will also be denoted $x^A$ where $A = (n, a)$.

The main dynamical assumption is that motion is continuous which means that it can be analyzed as a sequence of many infinitesimally short steps. Thus we first find the probability $P(x'|x)$ that the system takes a short step from $x^A$ to $x'^A = x^A + \Delta x^A$ and then we determine how such short steps accumulate. To find $P(x'|x)$ we maximize the (relative) entropy,

$$S[P, Q] = - \int dx' P(x'|x) \log \frac{P(x'|x)}{Q(x'|x)} ,$$

where we adopt the notation $dx' = d^3N x'$ and the prior distribution $Q(x'|x)$ describes the state of knowledge (of an ideally rational agent) before any information about the motion is taken into account. A priori we shall assume extreme ignorance which is expressed by a uniform distribution. Since the space $X_N$ is flat we can set $Q(x'|x) = 1$.

The physical information about the motion is introduced through constraints. The fact that particles take infinitesimally short steps from $x_n^a$ to $x_n^a = x_n^a + \Delta x_n^a$ is imposed through $N$ separate constraints,

$$\langle \Delta x_n^a \Delta x_n^b \rangle \delta_{ab} = \kappa_n , \quad (n = 1 \ldots N)$$

where $\kappa_n$ are constants. The $\kappa_n$’s are chosen to be constant to reflect the translational symmetry of the space $X$; they are $n$-dependent in order to accommodate non-identical particles; and eventually we take $\kappa_n \to 0$ to implement infinitesimally short steps.

There is one additional constraint that leads to correlations among the particles,

$$\langle \Delta x^A \rangle \partial_A \phi = \sum_{n=1}^N \langle \Delta x_n^a \rangle \frac{\partial \phi}{\partial x_n^a} = \kappa' ,$$

which introduces the “drift” potential $\phi$ and $\partial_A = \partial/\partial x^A = \partial/\partial x_n^a$. $\kappa'$ is another small but for now unspecified position-independent constant. Eq. (3) is a single constraint that acts on the $3N$-dimensional configuration space.

Maximizing the entropy (1) subject to (2), (3), and normalization leads to

$$P(x'|x) = \frac{1}{\zeta} \exp[- \sum_n (\frac{1}{2} \alpha_n \Delta x_n^a \Delta x_n^b \delta_{ab} - \alpha' \Delta x_n^a \frac{\partial \phi}{\partial x_n^a})] ,$$

where $\zeta$ is a normalization constant and $\alpha_n$ and $\alpha'$ are the Lagrange multipliers associated to (2) and (3). The limit of infinitesimally short steps $\kappa_n \to 0$ is achieved as $\alpha_n \to \infty$.

3. Strictly uniform non-normalizable priors are mathematically problematic. This difficulty can be avoided by adopting a physically reasonable normalizable prior. By “uniform” we actually mean any distribution that is essentially flat over macroscopic scales.

4. Elsewhere, in the context of particles with spin, we will see that the potential $\phi(x)$ can be given a natural geometric interpretation as an angular variable. Its integral over any closed loop is $\oint d\phi = 2\pi n$ where $n$ is an integer.
Already at this early stage we can see that ED exhibits a rather trivial symmetry: Imposing the constraint (3) with the pair \((\phi, \kappa')\) leads to the same transition probability \(P(x'|x)\) as a constraint with the pair \((\tilde{\phi}, \tilde{\kappa}')\) = \((C \phi, C \kappa')\) where \(C\) is some arbitrary constant. In previous work [4] we took advantage of the symmetry and rescaled with \(C = 1\). This amounts to setting \(\alpha' \phi \rightarrow \phi\) which eliminates \(\alpha'\). Here we wish to examine the effect of \(\alpha'\) for given \(\phi\) so we will keep it explicit.

To find how these short steps accumulate to produce a finite change we introduce a bookkeeping device — this is how the notion of time enters dynamics. As discussed in [2–5] entropic time is measured by the fluctuations themselves (see eq.(8) below) which leads to the choice

\[ \alpha_n = \frac{m_n}{\eta \Delta t}, \]  

where \(\Delta t\) is the interval over which the short step \(x \rightarrow x'\) occurs, the \(m_n\) are particle-specific constants that will be called “masses”, and \(\eta\) is a constant that fixes the units of time relative to those of length and mass. With this choice of \(\alpha_n\) a generic displacement can be expressed as an expected drift plus a fluctuation,

\[ \Delta x^A = b^A \Delta t + \Delta w^A, \]  

where \(b^A(x)\) is the drift velocity,

\[ \langle \Delta x^A \rangle = b^A \Delta t \quad \text{with} \quad b^A = \frac{\eta \alpha'}{m_n} \delta^{AB} \partial_B \phi = \eta \alpha' m^{AB} \partial_B \phi. \]  

\(m_{AB} = m_n \delta_{AB}\) is the “mass” tensor and \(m^{AB} = \delta_{AB}/m_n\) is its inverse. The fluctuations \(\Delta w^A\) satisfy,

\[ \langle \Delta w^A \rangle = 0 \quad \text{and} \quad \langle \Delta w^A \Delta w^B \rangle = \frac{\eta}{m_n} \delta^{AB} \Delta t = \eta m^{AB} \Delta t. \]  

Comparing equations (7) and (8) for short steps, as \(\Delta t \rightarrow 0\), we see that the fluctuations are much larger than the drift \((\Delta w^A \sim \Delta t^{1/2})\) while \((\Delta x^A \sim \Delta t)\) which leads to non-differentiable trajectories characteristic of a Brownian motion. They also show that for fixed \(\phi\) the effect of the multiplier \(\alpha'\) is to enhance or suppress the drift \(b^A \Delta t\) relative to the fluctuations \(\Delta w^A\).

Having introduced a convenient notion of time through (5), the accumulation of many short steps leads to a probability distribution \(\rho(x, t)\) in configuration space that obeys a Fokker-Planck equation (FP), [1–3]

\[ \partial_t \rho = - \partial_A (\rho v^A). \]  

In this equation \(v^A\) is the velocity of the probability flow in configuration space or current velocity. It is given by

\[ v^A = b^A + u^A \quad \text{where} \quad u^A = - \eta m^{AB} \partial_B \log \rho^{1/2} \]  

is called the osmotic velocity. The interpretation of \(u^A\) follows immediately from looking at its contribution to the probability flux,

\[ \rho u^A = - \frac{1}{2} \eta m^{AB} \partial_B \rho. \]  

This is recognized as the analogue of Fick’s law of diffusion with a diffusion tensor \(\eta m^{AB}/2\). Since both \(b^A\) and \(u^A\) are gradients the current velocity \(v^A\) is a gradient too,

\[ v^A = m^{AB} \partial_B \Phi \quad \text{where} \quad \Phi = \eta \alpha' \phi - \eta \log \rho^{1/2}. \]
will be called the *phase*. Thus, the phase also has a drift component and a diffusive or osmotic component.

The dynamics described by the FP equation (9) is a standard diffusion. To describe a “mechanics” we require that the diffusion be “non-dissipative” which is achieved by an appropriate readjustment of the constraint (3) after each step $\Delta t$. The net effect is that the drift potential $\phi$, or equivalently the phase $\Phi$, is promoted to a fully dynamical degree of freedom. The diffusion is said to be “non-dissipative” when the actual updating of $\Phi$ is implemented by imposing that a certain functional $\tilde{H}[\rho, \Phi]$ be conserved. In order to offset the entropic change $\rho \rightarrow \rho + \delta \rho$, one requires that $\Phi$ changes $\Phi \rightarrow \Phi + \delta \Phi$ in such a way that

$$
\dot{H}[\rho + \delta \rho, \Phi + \delta \Phi] = \dot{H}[\rho, \Phi].
$$

As shown in [4] the requirement that $\dot{H}$ be conserved for arbitrary choices of $\rho$ and $\Phi$ implies that the coupled evolution of $\rho$ and $\Phi$ is given by a conjugate pair of Hamilton’s equations,

$$
\partial_t \rho = \frac{\delta \dot{H}}{\delta \Phi} \quad \text{and} \quad \partial_t \Phi = -\frac{\delta \dot{H}}{\delta \rho}.
$$

The “ensemble” Hamiltonian $\dot{H}$ is chosen so that the first equation reproduces the FP equation (9). Then the second equation becomes a Hamilton-Jacobi equation (HJ). Further arguments from information geometry can then be invoked to fully specify the form of the functional $\dot{H}[\rho, \Phi]$ [4]. They suggest that the natural choice of $\dot{H}$ is

$$
\dot{H}[\rho, \Phi] = \int dx \rho \left[ \frac{1}{2} m^{AB} \partial_A \Phi \partial_B \Phi + V + \xi m^{AB} \frac{1}{\rho^2} \partial_A \rho \partial_B \rho \right].
$$

The first term in the integrand is the “kinetic” term that reproduces the FP equation (9). The second term represents the simplest non-trivial interaction, a potential energy that is linear in $\rho$ and introduces the standard potential $V(x)$. The third term, motivated by information geometry, is the trace of the Fisher information and is called the “quantum” potential. The parameter $\xi$ turns out to be crucial: it controls the relative contributions of the two potentials. When $\xi > 0$ we write $\xi = \hbar^2/8$. Thus $\xi$ defines the value of what we call Planck’s constant $\hbar$, and sets the scale that separates quantum from classical regimes. The case $\xi = 0$ will be addressed below; the case $\xi < 0$ leads to instabilities and will not be discussed further.

To conclude this brief review of ED we note that at this point the dynamics is fully specified by equations (14) and (15). Nothing prevents us however from combining $\rho$ and $\Phi$ into a single complex function,

$$
\Psi = \rho^{1/2} \exp(i \Phi/\hbar).
$$

Then the pair of Hamilton’s equations (14) can be tremendously simplified and written as a single complex linear equation,

$$
i \hbar \partial_t \Psi = -\frac{\hbar^2}{2} m^{AB} \partial_A \partial_B \Psi + V \Psi,
$$

which we recognize as the Schrödinger equation.

### 3. Trading drift and fluctuations

According to ED at the microscopic sub-quantum level the dynamics is very irregular. The particles perform a Brownian motion, eq.(6), with an expected drift and fluctuations given by (7) and (8). From these equations we see that the effect of $\alpha'$ is to enhance the drift relative to
the fluctuations. This means that different values of the multiplier $\alpha'$ correspond different types of microscopic dynamics.

We can study the sub-quantum effect of $\alpha'$ directly through eqs.(7) and (8). However, it may be more instructive to rescale $\eta$ and write $\eta = \tilde{\eta}/\alpha'$. Under such rescaling the $\alpha'$ dependence migrates from the drift to the fluctuations,

$$\langle \Delta x^A \rangle = \tilde{\eta} m^{AB} \partial_B \phi \Delta t \quad \text{and} \quad \langle \Delta w^A \Delta w^B \rangle = \frac{\tilde{\eta}}{\alpha'} m^{AB} \Delta t,$$

and we see that increasing $\alpha'$ at fixed $\tilde{\eta}$ has the effect of suppressing the fluctuations.

In contrast, at the “macroscopic” or quantum level the dynamics is very smooth. It is a non-dissipative diffusion described by Hamilton’s equations (14) or by the Schrödinger equation (17). The quantum dynamics is clearly independent of $\alpha'$ which means that there is a whole family of microscopic models — one could call it a universality class — that leads to the same emergent quantum dynamics.

From eq.(12) we have

$$v^A = m^{AB} \partial_B \Phi \quad \text{with} \quad \Phi = \tilde{\eta} \phi - \frac{\tilde{\eta}}{\alpha'} \log \rho^{1/2}.$$  

Here too we see that when we change $\alpha'$ the phase $\Phi$ remains unchanged but the relative contributions of the drift and osmotic components change.

**The Bohmian limit** For large $\alpha'$ the fluctuations are suppressed; osmotic or diffusion effects become negligible. As $\alpha'$ increases the particles follow smoother trajectories which resemble a Brownian motion only at increasingly shorter spatial scales. In the limit $\alpha' \to \infty$ we have

$$\Phi \to \tilde{\eta} \phi \quad \text{so that} \quad v^A \to b^A.$$  

The current and the drift velocities coincide and particles follow smooth trajectories that coincide with the lines of probability flow. This is exactly the kind of motion postulated by Bohmian mechanics [23–25].

But a word of caution is necessary. ED is driven by entropy; it is essentially non-causal and indeterministic. This is what led us to introduce probabilities and allowed us to write down a Fokker-Planck equation. Therefore the $\Delta t \to 0$ limit or, going back to (5), the $\alpha_n \to \infty$ limit, which is the limit that enforces the continuity of trajectories, must be taken at fixed $\alpha'$, before we consider the effect of $\alpha' \to \infty$. Only then, once we recognize that we are dealing with a tricky singular limit, can we claim that entropic dynamics includes Bohmian mechanics as a special limiting case. Thus, no matter how large the (fixed) value of $\alpha'$, entropic dynamics remains “entropic”. Even for large $\alpha'$ the dynamics is still driven by fluctuations and at sufficiently microscopic scales the expected motion is Brownian.

We must also emphasize that it is only with respect to the mathematical formalism that ED includes Bohmian mechanics as a special case. The philosophical differences constitute an unbridgeable gap. Bohmian mechanics attempts to provide an actual description of reality, a description of the ontology of the universe as it “really” is and as it “really” happens. In the Bohmian view the universe consists of real particles that have definite positions and their trajectories are guided by a real field, the wave function $\Psi$ [23].

On the other hand ED is a purely epistemic theory. Its pragmatic goal is less ambitious: to make the best possible predictions on the basis of very incomplete information. In ED the particles also have definite positions and its formalism includes a function $\Phi$ that behaves as a wave. But $\Phi$ is a tool for reasoning; it is not meant to represent anything real. There is no implication that the particles move the way they do because they are pushed around by a pilot
wave or by some stochastic force. In fact ED is silent on the issue of what causative power is responsible for the peculiar motion of the particles. What the probability $\rho$ and the phase $\Phi$ are designed to do is not to guide the particles but to guide our inferences. They guide our expectations of where and when to find the particles but they do not exert any causal influence on the particles themselves.

4. Another universality class and its Bohmian limit

ED as a non-dissipative diffusion is defined by Hamilton’s equations (14). The ensemble Hamiltonian, eq.(15), includes a parameter $\xi$ that regulates the strength of the quantum potential. Any non-zero value $\xi > 0$ yields a fully quantum mechanics, albeit with differing values of $\hbar$. We can also treat $\xi$ and $\hbar$ as independent parameters and set $\xi = 0$. This leads us to a qualitatively different theory — a different universality class.

At the microscopic level the particles follow irregular Brownian trajectories described by eqs.(6-8). For $\xi = 0$, $\alpha'$ has the same effect of enhancing the drift relative to the fluctuations, and therefore different values of $\alpha'$ correspond to different types of microscopic dynamics.

At the “macroscopic” level the emergent behavior is smooth. According to equations (14) and (15) for $\xi = 0$ the probability $\rho$ follows the gradient of $\Phi$,

$$\partial_t \rho = \frac{\delta \tilde{H}}{\delta \Phi} = -\partial_A (\rho v^A) \quad \text{with} \quad v^A = m^{AB} \partial_B \Phi , \quad (21)$$

and $\Phi$ evolves according to

$$\partial_t \Phi = -\frac{\delta \tilde{H}}{\delta \rho} = -\frac{1}{2} m^{AB} \partial_A \Phi \partial_B \Phi - V , \quad (22)$$

which we recognize as the classical Hamilton-Jacobi equation. Therefore the probability $\rho$ flows along the classical path. In fact, these are exactly the classical equations of motion in a Liouville representation. We conclude that the emergent macroscopic dynamics is essentially classical mechanics. There is, however, no implication that the particles themselves follow the classical paths. Indeed, at any instant of time the particles undergo the same fluctuations, eq.(18), that we would expect for any non-zero value of $\xi$.

For $\xi = 0$ ED is a hybrid theory; it resembles classical mechanics in some respects and quantum mechanics in others. Just as in quantum mechanics the particles follow Brownian paths and the dynamics is a non-dissipative diffusion; they even satisfy an uncertainty principle [26]. On the other hand, just as in classical mechanics, the probability flows according to paths described by the classical Hamilton-Jacobi equation. One can even combine $\rho$ and $\Phi$ into a single complex function $^6 \Psi = \rho^{1/2} \exp(i\Phi/\hbar)$, and write the coupled evolution of $\rho$ and $\Phi$ in terms of a single complex equation that resembles a Schrödinger equation,

$$i\hbar \partial_t \Psi_k = -\frac{\hbar^2}{2} m^{AB} \partial_A \partial_B \Psi_k + V \Psi_k + \frac{\hbar^2}{2} m^{AB} \partial_A \partial_B |\Psi_k| \Psi_k . \quad (23)$$

But this equation is not linear which means that a central feature of quantum behavior, the superposition principle, has been lost.

Within the family of microscopic models with $\xi = 0$ we can also take the “Bohmian” limit, $\alpha' \to \infty$. Increasing $\alpha'$ at fixed $\eta$ suppresses the fluctuations so the particles follow smoother

5 In the hybrid theory $\xi$ and $\hbar$ are independent parameters. $\xi$ is set to 0 and $\hbar$ is defined as the constant with the appropriate units of action that is needed to define a wave function $\Psi = \rho^{1/2} e^{i\Phi/\hbar}$.

6 $^6$
trajectories that increasingly approximate the lines of probability flow determined by the classical Hamilton-Jacobi equation (22). Therefore for \( \alpha' \to \infty \) the particles follow classical trajectories. We conclude that the \( \alpha' \to \infty \) limit of the hybrid theory is classical mechanics.

Here too a word of caution is needed. We say the emergent entropic dynamics in the \( \xi = 0 \) case is “essentially” classical mechanics. The point is that here too, entropic dynamics remains “entropic”. Even for very large \( \alpha' \), at sufficiently microscopic scales the expected motion remains Brownian.

To summarize: Within the universality class of quantum dynamics, suppressing microscopic fluctuations yields the Bohmian mechanics. Similarly, within the universality class of hybrid dynamics, suppressing microscopic fluctuations yields classical mechanics.

5. Momentum and its uncertainty relations

In the ED of particles there is one set of beables — their positions. This immediately raises questions about the nature of other observables: are they beables or are they created in the act of measurement? The case of momentum and the Heisenberg uncertainty relation is discussed in [26]; other related matters in [1,31].

In this paper we have seen that ED allows the construction of a variety of models that can differ both at the microscopic and at the macroscopic level — we can enhance or suppress microscopic fluctuations by tuning \( \alpha' \) and we can turn the quantum potential on and off by setting \( \xi > 0 \) or \( \xi = 0 \). Our goal here is to revisit the issue of momentum and its uncertainty relations to find out how they are affected by the choices of \( \alpha' \) and \( \xi \).

First, let us recall the notion of momentum. For simplicity we consider a single particle. Since it follows a non-differentiable trajectory it is clear that the classical momentum \( \Delta \frac{d\vec{x}}{dt} \) tangent to the trajectory cannot be defined. One can however introduce two notions of momentum. One is the usual quantum operator,

\[
\hat{p}^a = -i\hbar\delta^{ab}\partial_b ,
\]

which acts on the space of wave functions \( \Psi \) and generates infinitesimal translations. The other momentum is a local quantity associated to the current velocity, \( v^A = m^{AB}\partial_B \Phi \) given in eq.(19), which leads to

\[
p_A(x) = m_{AB}v^B(x) = \partial_A\Phi(x) .
\]

For a single particle this “local” or “current” momentum is

\[
p^a(x) = m v^a(x) = \delta^{ab}\partial_b\Phi(x) .
\]

As we see from (19), the local momentum can be decomposed into drift and osmotic components, \( p^a = p^a_d + p^a_o \) where

\[
p^a_d = mb^a = \delta^{ab}\tilde{\eta}\partial_b\phi ,
\]

\[
p^a_o = mu^a = -\delta^{ab}\frac{\tilde{\eta}}{\alpha'}\partial_b\log \rho^{1/2} .
\]

Two points deserve to be emphasized. First, the two notions of momentum, (24) and (26), are not unrelated. As shown in [5] the infinitesimal displacement of a functional \( f[\rho,\Phi] \) is given by its Poisson bracket with the “ensemble” momentum

\[
\hat{P}^a[\rho,\Phi] = \int d^3x \rho \delta^{ab}\partial_b\Phi = \langle p^a \rangle .
\]

The second point is that the local momentum \( p^a(x) \) is expressed in terms of the probability \( \rho \) and the drift potential \( \phi \). It is not a beable; it is an epistemic concept. It is not an attribute of the particle but of the wave function.
Next we revisit the uncertainty relation. We recall that the relation between expected values follows from
\[ \langle \partial_a \log \rho \rangle = 0 \Rightarrow \langle p_a^a \rangle = 0 \quad \text{and} \quad \langle p^a \rangle = \langle p_a^a \rangle . \] 
\[ \text{(30)} \]
Also, using \( \Psi = \rho^{1/2} \exp(i\Phi/\hbar) \), we have
\[ \langle \hat{p}^a \rangle = \int dx \, \Psi^* \frac{\hbar}{i} \delta_a \partial_a \Psi = \langle p^a \rangle . \]
\[ \text{(31)} \]
The corresponding covariances are related by
\[ \text{Cov}(\hat{p}^a, \hat{p}^b) = \langle \hat{p}^a \hat{p}^b \rangle - \langle \hat{p}^a \rangle \langle \hat{p}^b \rangle = \text{Cov}(p^a, p^b) + \frac{\hbar^2}{4} I^{ab} , \]
\[ \text{(32)} \]
where \( I^{ab} = \delta^{ac} \delta^{bd} I_{cd} \) and
\[ I_{cd} = \int dx \, \rho \partial_c \log \rho \partial_d \log \rho = \text{Cov}(\partial_c \log \rho, \partial_d \log \rho) , \]
\[ \text{(33)} \]
is the Fisher information matrix. The corresponding variances of \( \hat{p}^a \) and \( p^a \) are related by
\[ \text{Var}(\hat{p}^a) = \text{Var}(p^a) = \frac{\hbar^2}{4} \text{Var}(\partial^a \log \rho) . \]
\[ \text{(34)} \]
(No sum on repeated \( a \)'s.) The uncertainty relation is obtained by multiplying \( \text{(34)} \) by \( \text{Var}(x^b) \),
\[ \text{Var}(\hat{p}^a) \text{Var}(x^b) = \text{Var}(p^a) \text{Var}(x^b) + \frac{\hbar^2}{4} \text{Var}(\partial^a \log \rho) \text{Var}(x^b) . \]
\[ \text{(35)} \]
and invoking the Cauchy-Schwartz inequality,
\[ \text{Var}(A) \text{Var}(B) \geq \text{Cov}^2(A, B) , \]
\[ \text{(36)} \]
to get
\[ \text{Var}(\hat{p}^a) \text{Var}(x^b) \geq \text{Cov}^2(p^a, x^b) + \frac{\hbar^2}{4} \text{Cov}^2(\partial^a \log \rho, x^b) . \]
\[ \text{(37)} \]
Finally, an explicit calculation of the covariances on the right gives
\[ \text{Cov}(\hat{p}^a, x^b) = \frac{1}{2} \langle \hat{p}^a x^b + x^b \hat{p}^a \rangle - \langle \hat{p}^a \rangle \langle x^b \rangle = \text{Cov}(p^a, x^b) , \]
\[ \text{(38)} \]
and
\[ \text{Cov}(\partial^a \log \rho, x^b) = -\delta^{ab} . \]
\[ \text{(39)} \]
Substituting into \( \text{(37)} \) leads to
\[ \text{Var}(\hat{p}^a) \text{Var}(x^b) \geq \text{Cov}^2(\hat{p}^a, x^b) + \left( \frac{\hbar}{2} \right)^2 , \]
\[ \text{(40)} \]
which is the version of the uncertainty relation proposed originally by Schrödinger. The weaker version due to Heisenberg follows immediately,
\[ \text{Var}(\hat{p}^a) \text{Var}(x^b) \geq \left( \frac{\hbar}{2} \right)^2 \quad \text{or} \quad \Delta \hat{p}^a \Delta x^b \geq \frac{\hbar}{2} . \]
\[ \text{(41)} \]
The derivation above shows that we can set \( \xi = 0 \) and switch the quantum potential off without affecting either the momentum or the uncertainty relation. It also shows that the value of \( \alpha' \) never entered the argument. We can enhance or suppress microscopic fluctuations without affecting the momentum or the uncertainty relation. \( \alpha' \) can affect the relative drift and osmotic components of the local momentum, but not the local momentum itself.
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