Vortex-type equations on compact Riemann surfaces

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Abstract

In this paper, we prove a priori estimates for some vortex-type equations on compact Riemann surfaces. As applications, we recover existing estimates for the vortex bundle Monge-Ampère equation, prove an existence and uniqueness theorem for the Calabi-Yang-Mills equations on vortex bundles and get estimates for $J$-vortex equation. We prove an existence and uniqueness result relating Gieseker stability and the existence of almost Hermitian Einstein metrics, i.e., a Kobayashi-Hitchin type correspondence. We also prove Kählerness of the negative of the symplectic form which arises in the moment map interpretation of the Calabi-Yang-Mills equations in [9].

1 Introduction.

Let $\Sigma$ be a compact Riemann surface, and $L$ be a holomorphic line bundle over it. Let $\phi \in H^0(\Sigma, L)$ be not identically zero and $\Sigma$ be endowed with a metric whose associated $(1,1)$-form is $\chi$. We are interested in the following family of equations (for a Hermitian metric $h_{\alpha,t}$ on $L$) that depend on the parameters $\alpha \geq 0$ and $0 \leq t \leq 1$

$$i\Theta_{h_{\alpha,t}} = (d - |\phi|_{h_{\alpha,t}}^2)\frac{e_\alpha u_\alpha^{1-t} \chi + it k_\alpha \nabla^{(1,0)}\phi \wedge \nabla^{(0,1)}\phi^*}{a_\alpha + b_\alpha t |\phi|_{h_{\alpha,t}}^2 - c_\alpha t^2 |\phi|_{h_{\alpha,t}}^4},$$

where $d > 0, a_\alpha > 0, e_\alpha > 0, b_\alpha, c_\alpha, k_\alpha \geq 0$ are constants, $u_\alpha = \frac{a_\alpha (\Theta_{h_{\alpha,t}=0})}{e_\alpha (d - |\phi|_{h_{\alpha,t}=0}^2)} > 0$ is a function, and $\Theta_{h_{\alpha,t}}$ is the curvature of the metric $h_{\alpha,t}$. We choose a path in the $\alpha$-$t$ plane (lying in the region $\alpha \geq 0, 0 < t \leq 1$) which starts at $(\alpha_0, t_0)$ and ends at $(\alpha_1, t_1)$.

Suppose $h_{\alpha,t} = h_{\alpha,t=0} e^{-\psi_{\alpha,t}}$ solves the above equation. The following theorem proves $C^{2,\beta}$ a priori estimate on $\psi_{\alpha,t}$. From now onwards, we suppress the dependence of $a_\alpha, b_\alpha, c_\alpha, e_\alpha, k_\alpha$ on $\alpha$.

**Theorem 1.2** Suppose $a, b, c, e, k$ depend on $\alpha$ continuously and $b - cd \geq 0$. Then the following statements hold.

1. If $\|\psi_{\alpha,t}\|_{C^0} \leq C_\alpha$, then $\|\psi_{\alpha,t}\|_{C^{2,\beta}} \leq C$, where $C_\alpha$ is independent of $t$.

2. If $b - (k + ct) d \geq 0$, then $\psi_{\alpha,t} \geq -C'_\alpha$, where $C'_\alpha$ is independent of $t$.

Moreover, if we have $de > a$ and $i\Theta_0 = \chi$, then $\psi_{\alpha,t} \leq C''_\alpha$, where $C''_\alpha$ is independent of $t$.

In addition, if the hypotheses above hold for all $\alpha \in [\alpha_0, \alpha_1]$ where $\alpha_0, \alpha_1 > 0$, then $C_\alpha, C'_\alpha$ and $C''_\alpha$ depend only on $\alpha_0, \alpha_1$. 

1
The second fundamental form of the extension is \( \beta \) equations amount to solving the following vortex-CYM equation for a Hermitian metric \( X \) vector bundle over \( \alpha \) equations arising out of a vortex-type bundle. In more detail, let \( \alpha \) interpretation for them[9]. In [7], an openness result was proved for a special case of the CYM as an easier toy model of the more complicated KYM equations in general, is quite challenging because they are of order four. Taking cue from the positivity of the Ricci curvature of \( X, E, L \) the moduli space of triples (of a polarised manifold with a holomorphic vector bundle over it) and proved existence, non-existence, and uniqueness results for the same. Solving the CP\(^2\) Proposition 1.4 following result.

In [1,2,3], the authors introduced the Kähler-Yang-Mills (KYM) equations to parametrise the moduli space of triples \( (X, E, L) \) (of a polarised manifold with a holomorphic vector bundle over it) and proved existence, non-existence, and uniqueness results for the same. Solving the KYM equations in general, is quite challenging because they are of order four. Taking cue from the Calabi volume conjecture, Pingali [7] proposed to study the Calabi-Yang-Mills (CYM) equations as an easier toy model of the more complicated KYM equations and provided a moment map interpretation for them[9]. In [7], an openness result was proved for a special case of the CYM equations arising out of a vortex-type bundle. In more detail, let \( X = \Sigma \times \mathbb{C}P^1 \). An action of \( SU(2) \) on \( X \) can be defined as follows: \( SU(2) \) acts trivially on \( \Sigma \) and in the standard manner on \( \mathbb{C}P^1 = SU(2)/U(1) \). We now follow the calculations in [7,3]. Let \( E \) be a rank-2 holomorphic vector bundle over \( X \) defined as an extension:

\[
0 \to \pi_1^*L \to E \to \pi_2^*\mathcal{O}(2) \to 0.
\]

The second fundamental form of the extension is \( \beta = \pi_1^*\phi \otimes \pi_2^*\xi \), where \( \xi = \frac{\sqrt{2\pi}dz}{(1+|z|^2)^2} \otimes d\bar{z} \).

Let \( \tau > 0 \) be a constant and \( \omega_{FS} = \frac{1dz\wedge d\bar{z}}{(1+|z|^2)^2} \) be the Fubini-Study metric on \( \mathbb{C}P^1 \). Denote by \( \Omega = \pi_1^*\omega_\Sigma + \frac{\tau}{\pi} \omega_{FS} \), an \( SU(2) \)-invariant Kähler form on \( X \) where \( \int_\Sigma \omega_\Sigma = vol(\Sigma) \) is fixed, by \( H \), an \( SU(2) \)-invariant hermitian metric on \( E \), and by \( \Theta \) the curvature of \( H \). Then for this case the CYM equations amount to solving the following vortex-CYM equation for a Hermitian metric \( h \) on \( L \):

\[
i\Theta_h = \left( \frac{\tau - |\phi|^2}{2} \right) - \frac{4f + \frac{i\tau}{\pi(2\pi)^2} \nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^*}{4 + \frac{r\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) + \frac{r\alpha}{2(2\pi)^2}|\phi|^2_h - \frac{\alpha}{4(2\pi)^2}|\phi|_h^4},
\]

where \( \alpha \geq 0 \) satisfies \( 8 + \frac{2r\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) > 0 \). In [1,3], \( \Theta_h \) is the curvature of the metric \( h \) on the line bundle \( L \). In [7], Pingali proved the set of \( \alpha \geq 0 \) satisfying

\[
8 + \frac{2r\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) > 0
\]

for which there exists a smooth form \( \Omega_\alpha > 0 \) and a smooth metric \( H_\alpha \) such that the vortex-CYM equation is satisfied, contains \( \alpha = 0 \) and is open. Our first application of Theorem 1.2 is the following result.

**Proposition 1.4** A smooth solution of the vortex-CYM equation exists and is unique among all \( SU(2) \)-invariant solutions when \( \alpha \) satisfies \( 8 + \frac{2r\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) > 0 \).
In [9], Pingali gave the moment map interpretation of Calabi-Yang-Mills equations. For the vortex bundle ansatz, we prove the following.

**Theorem 1.5** The negative of the symplectic form is Kähler whenever the Ricci curvature of $f$ is positive, $\tau < \frac{8}{3}$ and $\alpha$ is small.

In [3], the symplectic form is Kähler but here the symplectic form is not always Kähler. This phenomenon is happening probably because of the fact that the openness argument in [7] is not the standard integration-by-parts argument. The details and the proof are in sub-section 5.

In a different development [8], Pingali introduced the vector bundle Monge-Ampère (vbMA) equation motivated by a desire to study stability conditions involving higher Chern forms. The vbMA equation for a metric $g'$ on a holomorphic vector bundle $F$ over a compact complex $n$-dimensional manifold $Y$ is:

$$\left(\frac{i\Theta_{g'}}{2\pi}\right)^n = \eta Id,$$

where $\Theta_{g'}$ is the curvature of the Chern connection of $(F, g')$ and $\eta$ is a given volume form. The vbMA equation for vortex-type bundles akin to above, was studied in [8]. The a priori estimates proved in [8] follow as a direct corollary of Theorem 1.2 (as indicated in Subsection 3.3).

Finally, using some results in [8] and Theorem 1.2, we prove an existence and uniqueness result (Theorem 4.2) for vortex-type bundles over a product of a Riemann surface and the sphere, relating Gieseker stability and almost Hermitian Einstein metrics. In more detail, let $E$ be a holomorphic vector bundle of rank $r$ over a compact Kähler manifold $X$ of dimension $n$. Suppose $\omega$ is an integral form and therefore defines a line bundle $L'$ on $X$. The almost Hermitian Einstein equation is

$$[e^{(\frac{i}{2\pi}R_E+k\omega I_E)}Td_X]^{TOP} = (const)I_E \frac{\omega^n}{n!},$$

where $Td_X$ is the harmonic representative (with respect to $\omega$) of the Todd class. The constant (const) is calculated by taking the trace and integrating on both sides.

$$\chi(X, E \otimes L^k) := \int Tr([e^{(\frac{i}{2\pi}R_E+k\omega I_E)}Td_X]^{TOP}) = (const)rVol(X).$$

So Equation (1.7) becomes

$$[e^{(\frac{i}{2\pi}R_E+k\omega I_E)}Td_X]^{TOP} = \frac{\chi(X, E \otimes L^k)}{rVol(X)} I_E \frac{\omega^n}{n!}.$$ 

In [5], Leung claimed a general existence result for Equation (1.7) for large $k$. Our result is not subsumed by Leung’s claim because we provide an effective lower bound on $k$, and our result is equivariant in the sense that Gieseker stability only needs to be checked for $SU(2)$-invariant subbundles. Moreover, we prove uniqueness in the space of $SU(2)$-invariant solutions. The precise statement and proof of Theorem 4.2 is in Section 4.

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2 Proof of Theorem 1.2

In this section, we prove our main Theorem 1.2. Firstly, we have the following lemma which is useful in proving the desired estimates.

Lemma 2.1 If \( b - cd \geq 0 \), then \( |\phi|^{2}_{h_{t,\alpha}} \leq d \), for \( 0 \leq t \leq 1 \).

**Proof.** We have the following identity

\[
\partial \bar{\partial} \phi^{2}_{h_{t,\alpha}} = - \Theta_{h_{t,\alpha}} |\phi|^{2}_{h_{t,\alpha}} + \nabla^{(1,0)}_{t,\alpha} \phi \wedge \nabla^{(0,1)}_{t,\alpha} \phi^{*}.
\]  
(2.2)

At the maximum point of \( |\phi|^{2}_{h_{t,\alpha}} \) (say \( p \)), we have \( i \partial \bar{\partial} |\phi|^{2}_{h_{t,\alpha}} \leq 0 \) and \( \nabla_{t,\alpha} |\phi|^{2}_{h_{t,\alpha}} = 0 \). Therefore \( \nabla_{t,\alpha} \phi(p) = 0 \) since \( \phi \) is not identically zero. That gives us \( i \Theta_{h_{t,\alpha}}(p) \geq 0 \). Now we can write the denominator of equation 1.1 as \( a + t|\phi|^{2}_{h_{t,\alpha}}(b - ct|\phi|^{2}_{h_{t,\alpha}}) \). Under the hypothesis, it is clear that \( d - |\phi|^{2}_{h_{t,\alpha}}(p) \geq 0 \). Hence \( |\phi|^{2}_{h_{t,\alpha}}(x) \leq |\phi|^{2}_{h_{t,\alpha}}(p) \leq d \). □

From now onwards, we suppress the dependence of \( \psi_{t,\alpha} \) on \( t \) and \( \alpha \).

**Lemma 2.3** If \( \|\psi\|_{C^{1}} \leq C \), then \( \|\psi\|_{C^{2,\beta}} \leq C \).

**Proof.** Using the hypothesis and lemma 2.1, we see that the right-hand side of 1.1 is uniformly bounded in \( C^{0} \). Therefore, by \( L^{p} \) regularity of elliptic equations, \( \psi \) is bounded uniformly in \( W^{2,p} \) for all large \( p \). Using the Sobolev embedding theorem, we see that \( \|\psi\|_{C^{1,\beta}} \leq C \). Thus the right-hand side is in \( C^{0,\beta} \). Now using Schauder estimates, we are done. □

The following lemma completes the proof of the first part of 1.2

**Lemma 2.4** If \( \|\psi\|_{C^{0}} \leq C \), then \( \|\psi\|_{C^{1}} \leq C \).

**Proof.** To arrive at a contradiction, we assume that there exists a sequence \( \psi_{n} \) (corresponding to \( t_{n} \)) such that \( |d\psi_{n}| = \|d\psi_{n}(p_{n})\| = M_{n} \to \infty \). We may assume \( p_{n} \to p \) (up to some subsequence). Now choose large enough \( n \) so that \( p_{n}, p \) lie in a coordinate ball \( B \) centered at \( p \) with coordinate \( z \) (with \( z = 0 \) corresponding to \( p \)). Define \( \tilde{\psi}_{n}(\tilde{z}) = \psi_{n}(p_{n} + \tilde{z}^{\pi}_{n}) \). Now \( |d\tilde{\psi}_{n}| = 1 = |d\tilde{\psi}_{n}|(0) \). Note that

\[
\frac{\partial \tilde{\psi}_{n}}{\partial \tilde{z}} = \frac{1}{M^{\pi}_{n}} \frac{\partial \psi_{n}}{\partial z}, \quad \frac{\partial \tilde{\psi}_{n}}{\partial \tilde{z}^{\pi}_{n}} = \frac{1}{M^{\pi}_{n}} \frac{\partial \psi_{n}}{\partial z}.
\]

Now using 2.5 and 1.1, we get

\[
i \Theta_{0} + i \frac{\partial^{2} \tilde{\psi}_{n}}{\partial z \partial \tilde{z}} dz \wedge d\tilde{z} = (d - |\phi|^{2}_{h_{n}}) \frac{e^{u_{n} - t_{n}u_{n}} + it_{n}(k^{(1,0)} + \nabla^{(0,1)}_{n}) \phi \wedge \nabla^{(0,1)}_{n} \phi^{*}}{a + bt_{n}|\phi|^{2}_{h_{n}} - ct_{n}|\phi|^{4}_{h_{n}}}.
\]

We abuse notation from this point onwards and denote the functions \( \frac{\chi_{\pi_{z}z^{\pi}_{z}dz}}{dz \partial \tilde{z}^{\pi}_{n}} = \frac{\Phi_{n}}{dz \partial \tilde{z}^{\pi}_{n}} \) by \( \chi \) and \( \frac{\nabla^{(1,0)}_{n} \phi \wedge \nabla^{(0,1)}_{n} \phi^{*}}{dz \partial \tilde{z}^{\pi}_{n}} \) by \( \nabla^{(1,0)}_{i} \phi \wedge \nabla^{(0,1)}_{i} \phi^{*} \).

So the above equation becomes

\[
\frac{\chi}{M^{2}_{n}} + \frac{\partial^{2} \tilde{\psi}_{n}}{\partial \tilde{z} \partial \tilde{z}^{\pi}_{n}} = (d - |\phi|^{2}_{h_{n}}) \frac{e^{u_{n} - t_{n}u_{n}} + it_{n}(k^{(1,0)} + \nabla^{(0,1)}_{n}) \phi \wedge \nabla^{(0,1)}_{n} \phi^{*}}{a + bt_{n}|\phi|^{2}_{h_{n}} - ct_{n}|\phi|^{4}_{h_{n}}},
\]

(2.6)
Now the denominator in (2.6) can be written as $a + t_n\phi^2_{h_n}(b - ct_n|\phi|^2_{h_n})$ and this shows that the denominator is bounded below by $a$ because $t_n\phi^2_{h_n} \leq |\phi|^2_{h_n} \leq d \leq \frac{b}{c}$. On a coordinate ball $B_r(0)$ in the $\tilde{z}$ coordinates, we have $|\tilde{d}\psi_n| \leq 1$. Using (2.6) we conclude that $|\tilde{\Delta}\psi_n| \leq C$ on $B_r(0)$. Therefore, by interior $L^p$ regularity and the Sobolev embedding, we see that $\|\psi_n\|_{C^{1,\beta}(B_{0.5r}(0))} \leq C$. Thus by the interior Schauder estimates $\|\tilde{\psi}_n\|_{C^{2,\beta}(B_{0.5r}(0))} \leq C$. Suppose $\|\tilde{\psi}_n\|_{C^{2,\beta}(B_{0.5r}(0))} \leq C_r$ for some fixed $\beta > 0$. For every fixed $r$, a subsequence of $\tilde{\psi}_n$ converges in $C^{2,\gamma}(B_{0.5r}(0))$ to a function $\tilde{\psi}_r$ for a fixed $\gamma < \beta$. Choosing a diagonal subsequence, we may assume that for all $r$, we have a single function $\tilde{\psi}$. Now it is easy to see using (2.6) that $\frac{\partial^2 \tilde{\psi}}{\partial z \partial \bar{z}} \geq 0$. But a subharmonic function on $\mathbb{C}$ cannot be bounded above unless it is a constant. Hence $\tilde{\psi}$ is a constant. But this contradicts the fact that $|\tilde{d}\psi|(0) = 1$. Hence $|\nabla \psi| \leq C$, thus implying a $C^1$ estimate. \hfill \Box

To prove the second part of theorem 1.2, that is the $C^0$ estimate, we need the following form of the Green representation formula. Let $G$ be a Green function of the background metric $\chi$ such that $-C[1 + |\ln(d_\chi(P,Q))|] \leq G(P,Q) \leq 0$. Then any function $f$ satisfies the following equation

$$f(Q) = \frac{\int f}{\chi} + \int _{\Sigma} G(P,Q) i\partial\bar{\partial} f(P). \quad (2.7)$$

Now we prove a lower bound on $\psi$.

**Lemma 2.8** If $b - (k + ct)d \geq 0$, then the function $\psi$ satisfies $\psi \geq -C$, where $C$ is independent of $t$.

**Proof.** Firstly note that $|\phi|^2_h \leq d$. That is $|\phi|^2_{h_0} \leq e^{\psi + d'}$ (where $e^{d'} = d$). Therefore

$$\int \psi \chi \geq \int \ln(|\phi|^2_{h_0}) \chi - \int d' \chi. \quad (2.9)$$

Now

$$\psi(P) \geq \frac{\int \ln(|\phi|^2_{h_0}) \chi}{\chi} - d' + \int_{\Sigma} G(Q,P) i\partial\bar{\partial} \psi(Q)$$

$$\psi(P) \geq \frac{\int \ln(|\phi|^2_{h_0}) \chi}{\chi} - d' + \int_{\Sigma} G(Q,P) i\Theta_h(Q) - \int G(Q,P) \chi. \quad (2.10)$$

From (1.4) we get

$$i\Theta_h \leq d \frac{eu^{1-t} \chi + itk \nabla(1.0) \phi \wedge \nabla(0.1) \phi^*}{a + tb|\phi|^2_h - ct^2|\phi|^2_h - d} = dct^2|\phi|^2_h$$

$$\Rightarrow i\Theta_h \leq d \frac{eu^{1-t} \chi + itk \nabla(1.0) \phi \wedge \nabla(0.1) \phi^*}{a + t|\phi|^2_h (b - ctd)}.$$

Now $b - ctd \geq 0$ because $b - cd \geq 0$. Therefore using (2.2), we have

$$i\Theta_h(a + t|\phi|^2_h (b - ctd)) \leq deu^{1-t} \chi + iktd\bar{\partial} \phi|\phi|^2_h + iktd|\phi|^2_h \Theta_h$$

$$\Rightarrow i\Theta_h(a + t|\phi|^2_h (b - ctd - kd)) \leq deu^{1-t} \chi + iktd\bar{\partial} \phi|\phi|^2_h.$$

We need the hypothesis for the following inequality to hold.

$$i\Theta_h a \leq i\Theta_h(a + t|\phi|^2_h (b - ctd - kd)) \leq deu^{1-t} \chi + iktd\bar{\partial} \phi|\phi|^2_h.$$
\[ \Rightarrow G(Q, P) i\Theta h \geq G(Q, P) \frac{deu^{1-t} \chi + ikt\bar{\partial}\partial|\phi|_h^2}{a}. \] (2.11)

Using (2.10) and (2.11) we get

\[ \psi(P) \geq \int \frac{\ln(|\phi|_{h_0}^2)\chi}{\chi} - d' - \int G(Q, P)\chi + \int G(Q, P) \frac{deu^{1-t} \chi + ikt\bar{\partial}\partial|\phi|_h^2}{a} \]

Now using Green representation formula (2.7) we get

\[ \Rightarrow \psi(P) \geq \int \frac{\ln(|\phi|_{h_0}^2)\chi}{\chi} - d' - \int G(Q, P)\chi + \int G(Q, P) \frac{deu^{1-t} \chi + ikt\bar{\partial}\partial|\phi|_h^2}{a}(P) \geq - C. \]

Hence, we are done. \[ \square \]

Now we prove an upper bound on \( \psi \).

**Lemma 2.12** If \( de > a \) and \( i\Theta_0 = \chi \), then \( \psi \leq C \), where \( C \) is independent of \( t \).

**Proof.** Suppose \( \psi \) achieves its maximum value at a point \( p \). Then at that point, we have \( i\bar{\partial}\partial\psi \leq 0 \), and \( \partial\psi = 0 = \bar{\partial}\psi \).

Now from equation (1.1) we have

\[ i\Theta_0 = \chi \geq (d - |\phi|_{h_0}^2) \frac{eu^{1-t} \chi}{a + bt|\phi|_{h_0}^2 - ct^2|\phi|^4_h}. \] (2.13)

If the upper bound does not hold and suppose there exists a sequence \( \psi_n(p_n) \to \infty \) (with \( p_n \to q \)) then \( |\phi|_{h_n}^2(q) \to 0 \). Hence from (2.13) we get

\[ 1 \geq \frac{de}{a} u^{1-t} \geq \frac{de}{a} \frac{a^{1-t}}{(ed)^{1-t}} = \frac{(de)^t}{a^t} > 1. \]

We have a contradiction since \( 0 < t \leq 1 \). Hence \( \psi \leq C \). \[ \square \]

This completes the proof of the theorem 1.2.

### 3 Three Applications.

In this section, we apply our result in three cases.

#### 3.1 Calabi Yang Mills Equations.

##### 3.1.1 A Priori Estimates For Calabi Yang Mills Equations.

In [7], Pingali considered the Calabi-Yang-Mills equations (Theorem 1.2). If we rewrite the equation (3.29) of [7], then it becomes the following one:

\[ i\Theta_h = \left( \frac{\tau - |\phi|_{h_0}^2}{2} \right) \frac{4f}{4 + \frac{2\pi^2}{(2\pi)^2} \phi \wedge \nabla^{1,0} \phi} + \frac{i\alpha}{2(2\pi)^2} \phi \wedge \nabla^{0,1} \phi^* \] (3.1)

We consider the following continuity path with parameter \( \alpha \):

\[ i\Theta_{h_{\alpha}} = \left( \frac{\tau - |\phi|_{h_{\alpha}}^2}{2} \right) \frac{4f}{4 + \frac{2\pi^2}{(2\pi)^2} \phi \wedge \nabla^{1,0} \phi} + \frac{i\alpha}{2(2\pi)^2} \phi \wedge \nabla^{0,1} \phi^*. \] (3.2)
Here the only difference is the curvature of the initial metric. The curvature of the initial metric is $i\Theta_0 = (\frac{\tau - |\phi|^2_{h_0}}{2})\omega_\Sigma$ (which can be seen from equation 3.8 of [7]). In [7], Pingali proved the set of $\alpha \geq 0$ satisfying

$$8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) > 0,$$

for which there exists a smooth form $\Omega_\alpha > 0$ and a smooth metric $H_\alpha$ such that the vortex-CYM equation is satisfied, contains $\alpha = 0$ and is open. So we can assume that our path starts at $\alpha_0 > 0$. If we compare the above equations with the equations we considered with $t = 1$, then $a = 8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}), b = \frac{\tau_\alpha}{(2\pi)^2}, c = \frac{\alpha}{2(2\pi)^2}, d = \tau, e = 4, k = \frac{\alpha}{2(2\pi)^2}$. We see that $a > 0, d > 0, e > 0, b, c, k \geq 0$. Now $b - cd = \frac{\tau_\alpha}{2(2\pi)^2} \geq 0$ and $b - (k + c)d = 0$. This shows that all the estimates hold except for the upper bound one because here the curvature of the initial metric is not of the form, we considered. However, the following lemma proves the upper bound estimate.

**Lemma 3.3** If $h_\alpha = h_0 e^{-\psi_\alpha}$ solves $\mathcal{L}_\omega$ for $\alpha_0 < \alpha \leq \alpha_1$ where $\alpha_1$ satisfies $8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) > 0$, then $\psi_\alpha \leq C$, where $C$ is independent of $\alpha$. Here, $h_0$ denotes the metric corresponding to $\alpha = 0$.

**Proof.** Suppose $\psi$ achieves its maximum value at a point $p$. Then at that point, we have $i\partial\bar{\partial}\psi \leq 0$, and $\partial\psi = 0 = \bar{\partial}\psi$. Now from equation 3.2 we have

$$i\Theta_0 \geq \frac{4f}{8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) + \frac{\alpha}{2(2\pi)^2}|\phi|_{h_\alpha}^2}.$$

(3.4)

If the upper bound does not hold and suppose there exists a sequence $\psi_n(p_n) \to \infty$ (with $p_n \to q$) then $|\phi|_{h_n}^2(q) \to 0$. Hence from 3.3 we get

$$i\Theta_0(q) \geq \frac{4f}{8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2})}\frac{\tau - |\phi|_{h_n}^2}{2} \geq \frac{4f}{8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2})}\frac{\tau f}{2} \geq \frac{4f}{8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2})} \geq 1 \geq \frac{2e}{a}$$

Now $\frac{2e}{a} > 1$ because $2\lambda - \frac{\tau}{2} < 0$, which follows from [7]. So we have a contradiction. Hence $\psi \leq C$. So we now have closedness and hence existence for all those $\alpha$ satisfying $8 + \frac{2\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) > 0$.

### 3.1.2 Uniqueness Of Solutions Of Calabi-Yang-Mills Equations.

We prove that for all those $\alpha$ for which 3.1 has a solution is essentially unique among all $SU(2)$-invariant solutions. Our proof of uniqueness is as follows.

Let $h_1$ be the solution arising from 3.1 that is

$$i\Theta_{h_1} = \frac{\tau - |\phi|_{h_1}^2}{2} + \frac{4f}{4 + \frac{\tau_\alpha}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) + \frac{\alpha}{2(2\pi)^2}|\phi|_{h_1}^2}.$$
Let $h_2$ denote any other solution of equation \textit{3.1}. We run the continuity path backward with continuity parameter $\beta$ starting with $h_2$:

$$i \Theta_h = \frac{\tau - |\phi|_h^2}{2} \frac{4f + \frac{i\beta}{(2\pi)^2} \nabla_h^{1,0}\phi \wedge \nabla_h^{0,1}\phi^*}{4 + \frac{\beta}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) + \frac{\beta}{(2\pi)^2}|\phi|_h^2 - \frac{\beta}{(2\pi)^2}|\phi|_h^4}. \quad (3.5)$$

Denote $\tilde{T} \subset [0, \alpha]$ the set of $\beta$ such that \textit{3.5} has a solution. This is non-empty because $\alpha \in \tilde{T}$. The proof of openness in [7], shows that $\tilde{T} \subset [0, \alpha]$ is open. We prove that there exists a “small” $\beta_0 \in [0, \alpha]$ such that \textit{3.1} has a unique solution for $\beta_0$. That is, there exists a unique smooth $h$ satisfying $|\phi|_h^2 \leq \tau$ and

$$i \Theta_h = \frac{\tau - |\phi|_h^2}{2} \frac{4f + \frac{i\beta_0}{(2\pi)^2} \nabla_h^{1,0}\phi \wedge \nabla_h^{0,1}\phi^*}{4 + \frac{\beta_0}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) + \frac{\beta_0}{(2\pi)^2}|\phi|_h^2 - \frac{\beta_0}{(2\pi)^2}|\phi|_h^4}.$$

This implies that the two continuity path \textit{3.5} and \textit{3.2} intersect at $\beta_0$.

**Lemma 3.6** There exists a number $\beta_0 \in (0, \alpha]$ depending only on $\lambda, \tau, \alpha$ such that there is a unique smooth $h$ satisfying $|\phi|_h^2 \leq \tau$ and the following equation

$$i \Theta_h = \frac{\tau - |\phi|_h^2}{2} \frac{4f + \frac{i\beta_0}{(2\pi)^2} \nabla_h^{1,0}\phi \wedge \nabla_h^{0,1}\phi^*}{4 + \frac{\beta_0}{(2\pi)^2}(2\lambda - \frac{\tau}{2}) + \frac{\beta_0}{(2\pi)^2}|\phi|_h^2 - \frac{\beta_0}{(2\pi)^2}|\phi|_h^4}.$$

**Proof.** Let $h_1$ be the solution coming from the forward path \textit{3.2} and $h_2$ be the solution coming from the backward path \textit{3.5}, satisfying $|\phi|_h^2 \leq \tau$. We define a function $g$ to satisfy $h_2 = h_1 e^{-g}$. Let $h_s = h_1 e^{-sg} = h_s h_1^{1-s}$, where $0 \leq s \leq 1$. It is easy to see that

$$|\phi|_{h_s}^2 = (|\phi|_{h_1}^{2(1-s)})(|\phi|_{h_2}^2) \leq \tau.$$

Let $I_s = m + l \beta_0 |\phi|_{h_s}^2 - q \beta_0 |\phi|_{h_s}^4$ and $J_s = \frac{uf + iv h_0 \nabla_h^{1,0}\phi \wedge \nabla_h^{0,1}\phi^*}{I_s}$, where $m = 8 + \frac{2\pi \beta_0}{(2\pi)^2}(2\lambda - \frac{\tau}{2})$, $l = \frac{\beta_0}{(2\pi)^2}, q = \frac{\beta_0}{(2\pi)^2}, u = 4, v = \frac{1}{(2\pi)^2}$.

So $i \Theta_{h_s} = (\tau - |\phi|_{h_s}^2)J_s$.

By assumption $i \Theta_{h_1} = (\tau - |\phi|_{h_1}^2)J_1$ and $i \Theta_{h_2} = (\tau - |\phi|_{h_2}^2)J_2$.

Now

$$i \bar{\partial} \partial g = i \Theta_{h_2} - i \Theta_{h_1} = \int_0^1 ds \frac{d}{ds} (i \Theta_{h_s}) = \int_0^1 ds \frac{d}{ds} ((\tau - |\phi|_{h_s}^2)J_s) = \int_0^1 ds (g |\phi|_{h_s}^2 J_s + (\tau - |\phi|_{h_s}^2) \frac{dJ_s}{ds}) \quad (3.7)$$

We now calculate $\frac{dJ_s}{ds}$.
\[
\frac{dJ_s}{ds} = \frac{iv\beta_0 \frac{d}{ds} (\nabla_{h_s}^1 \phi \wedge \nabla_{h_s}^0 \phi^*)}{I_s} - \frac{uf + iv\beta_0 \nabla_{h_s}^1 \phi \wedge \nabla_{h_s}^0 \phi^*}{I_s} (-lg|\phi|^2_{h_s} + 2qg|\phi|^4_{h_s}) \beta_0
\]

\[
= \frac{iv\beta_0 \frac{d}{ds} (\partial \bar{\partial} \bar{\partial}(-g|\phi|^2_{h_s}) - g\Theta_{h_s}|\phi|^2_{h_s} + |\phi|^2_{h_s} \partial \bar{\partial}g)}{I_s} - \frac{uf + iv\beta_0 \nabla_{h_s}^1 \phi \wedge \nabla_{h_s}^0 \phi^*}{I_s} (-lg|\phi|^2_{h_s} + 2qg|\phi|^4_{h_s}) \beta_0
\]

Putting \( \frac{dJ_s}{ds} \) in (3.4) we have

\[
i\partial \bar{\partial}g = \int_0^1 ds \{ g|\phi|^2_{h_s} J_s + (\tau - |\phi|^2_{h_s}) \frac{iv\beta_0 (-\partial|\phi|^2_{h_s} \wedge \bar{\partial}g - \partial g \wedge \bar{\partial}|\phi|^2_{h_s} - g\partial \bar{\partial}|\phi|^2_{h_s} - g\Theta_{h_s}|\phi|^2_{h_s})}{I_s}\}
\]

\[
- \int_0^1 ds \{ (\tau - |\phi|^2_{h_s}) \frac{uf + iv\beta_0 \nabla_{h_s}^1 \phi \wedge \nabla_{h_s}^0 \phi^*}{I_s} (-lg|\phi|^2_{h_s} + 2qg|\phi|^4_{h_s}) \beta_0\}
\]

\[
= \int_0^1 ds \{ g|\phi|^2_{h_s} J_s + (\tau - |\phi|^2_{h_s}) \frac{iv\beta_0 (-\partial|\phi|^2_{h_s} \wedge \bar{\partial}g - \partial g \wedge \bar{\partial}|\phi|^2_{h_s} - g\nabla_{h_s}^1 \phi \wedge \nabla_{h_s}^0 \phi^*)}{I_s}\}
\]

\[
- \int_0^1 ds \{ (\tau - |\phi|^2_{h_s}) \frac{uf + iv\beta_0 \nabla_{h_s}^1 \phi \wedge \nabla_{h_s}^0 \phi^*}{I_s} (-lg|\phi|^2_{h_s} + 2qg|\phi|^4_{h_s}) \beta_0\} \tag{3.8}
\]

We know that \( g \geq -C_1 \), where \( C_1 \) depends on \( \lambda, \tau, \alpha \).

Now define \( \bar{g} = g(\gamma + |\phi|^2_{h_s}) \), where \( \gamma > 1 \) is a large constant (depending only on \( \lambda, \tau, \alpha \)) to be chosen later on and \( h \) is defined as

\[
h = \int_0^1 h_s ds = h_1 \int_0^1 e^{-sg} ds.
\]

It follows that \( |\phi|^2_{h_s} \leq \tau \). If maximum of \( \bar{g} \) occurs at \( p \), then

\[
\partial \bar{g}(p) = 0 = \bar{\partial} \bar{g}(p).
\]

Therefore

\[
\partial g(p)(\gamma + |\phi|^2_{h_s})(p) = -g(p)\partial(|\phi|^2_{h_s})(p)
\]

\[
\implies \partial g(p) = -\frac{g(p)\partial(|\phi|^2_{h_s})(p)}{\gamma + |\phi|^2_{h_s}(p)}. \tag{3.9}
\]

And similarly

\[
\bar{\partial} g(p) = -\frac{g(p)\bar{\partial}(|\phi|^2_{h_s})(p)}{\gamma + |\phi|^2_{h_s}(p)}. \tag{3.10}
\]
Moreover, \( i \partial \bar{\partial} g(p) \leq 0 \), i.e.,

\[
0 \geq (\gamma + |\phi_h^2(p)|) i \partial \bar{\partial} g(p) + i \partial g(p) \wedge \bar{\partial} |\phi_h^2(p)| + i \partial |\phi_h^2(p)| \wedge \bar{\partial} g(p) + g(p)(-i\Theta_h(p)|\phi_h^2(p)| + i\nabla_{h,0}^0 \phi(p) \wedge \nabla_{h,0}^{0,1} \phi^*(p)).
\]

Using (3.9, 3.10) and \( \partial |\phi_h^2| \wedge \bar{\partial} |\phi_h^2| = |\phi_h^2|^2 \nabla_{h}^{1,0} \phi \wedge \nabla_{h}^{0,1} \phi^* \) we have

\[
0 \geq (\gamma + |\phi_h^2(p)|) i \partial \bar{\partial} g(p) - \frac{2ig(p)}{\gamma + |\phi_h^2(p)|^2} |\phi_h^2(p)| \nabla_{h}^{1,0} \phi(p) \wedge \nabla_{h}^{0,1} \phi^*(p) + ig(p) \nabla_{h}^{1,0} \phi(p) \wedge \nabla_{h}^{0,1} \phi^*(p) - ig(p) \Theta_h(p)|\phi_h^2(p)|
\]

\[
\Rightarrow 0 \geq i \partial \bar{\partial} g(p) - \frac{2ig(p)}{\gamma + |\phi_h^2(p)|^2} |\phi_h^2(p)| \nabla_{h}^{1,0} \phi(p) \wedge \nabla_{h}^{0,1} \phi^*(p)
\]

\[
+ \frac{ig(p)}{\gamma + |\phi_h^2(p)|^2} \Theta_h(p)|\phi_h^2(p)|
\]

\[
\Rightarrow 0 \geq i \partial \bar{\partial} g(p) + \frac{ig(p)(\gamma - |\phi_h^2(p)|)}{\gamma + |\phi_h^2(p)|^2} \nabla_{h}^{1,0} \phi(p) \wedge \nabla_{h}^{0,1} \phi^*(p)
\]

\[
- \frac{g(p)(\tau - |\phi_h^2(p)|) u f(p)}{\gamma + |\phi_h^2(p)|^2} (m + l\beta_0|\phi_h^2(p)| - q\beta_0|\phi_h^2(p)|) |\phi_h^2(p)|
\]

\[
- \frac{g(p)(\tau - |\phi_h^2(p)|) iv \beta_0}{(m + l\beta_0|\phi_h^2(p)| - q\beta_0|\phi_h^2(p)|)(\gamma + |\phi_h^2(p)|^2)} \nabla_{h}^{0,1} \phi(p) \wedge \nabla_{h}^{0,1} \phi^*(p).
\]

Now using (3.8) and suppressing the dependence on \( p \), we have

\[
\Rightarrow 0 \geq \int_0^1 ds \left\{ \frac{iv \beta_0( - \partial |\phi_h^2| \wedge \bar{\partial} g - \partial g \wedge \bar{\partial} |\phi_h^2| - g \nabla_{h,0}^1 \phi \wedge \nabla_{h,0}^{0,1} \phi^*) }{I_s} \right\}
\]

\[
- \int_0^1 ds \left\{ \frac{- u f + iv \beta_0 \nabla_{h,0}^{1,0} \phi \wedge \nabla_{h,0}^{0,1} \phi^* }{I_s^2} ( - l g|\phi_h^2| + 2 q g|\phi_h^2| \beta_0 ) \right\}
\]

\[
+ \frac{ig(\gamma - |\phi_h^2|)}{(\gamma + |\phi_h^2|^2)^2} \frac{g(\tau - |\phi_h^2|) iv \beta_0 |\phi_h^2|}{(m + l\beta_0|\phi_h^2| - q\beta_0|\phi_h^2|)(\gamma + |\phi_h^2|^2)} \nabla_{h}^{0,1} \phi \wedge \nabla_{h}^{0,1} \phi^*
\]

\[
- \frac{g(\tau - |\phi_h^2|) u f}{(\gamma + |\phi_h^2|^2)(m + l\beta_0|\phi_h^2| - q\beta_0|\phi_h^2|)} |\phi_h^2|.
\]

(3.13)
Using \(3.9\) and \(3.10\) and \(J_s = \frac{u f + iv \beta_0 \nabla_{h_s}^0 \phi \wedge \nabla_{h_s}^0 \phi^*}{I_s}\), we get the following inequality

\[
\Rightarrow 0 \geq \int_0^1 ds \left\{ \frac{g|\phi|_h^2 u f}{I_s} - \frac{(\tau - |\phi|^2_h) u f}{I_s} \right\} \left( -l g|\phi|^2_h + 2 q g|\phi|^4_h \beta_0 \right) + \int_0^1 ds \left\{ \frac{g|\phi|_h^2 iv \beta_0}{I_s} - \frac{iv \beta_0 (\tau - |\phi|^2_h) l g|\phi|^2_h + 2 q g|\phi|^4_h}{I_s} \right\} \nabla_{h_s}^0 \phi \wedge \nabla_{h_s}^0 \phi^* 
\]

\[
+ \int_0^1 ds \left\{ \frac{iv \beta_0 g(\tau - |\phi|^2_h)}{I_s} (\phi^2_h \wedge \phi^2_h + \phi^2_h \wedge \phi^2_h) \right\} 
\]

\[
+ \frac{ig(\gamma - |\phi|^2_h)}{(\gamma + |\phi|^2_h)^2} \left\{ \frac{g(\tau - |\phi|^2_h) iv \beta_0}{(\gamma + |\phi|^2_h)(m + l \beta_0 |\phi|^2_h - q \beta_0 |\phi|^4_h)} \right\} \nabla_{h_s}^0 \phi \wedge \nabla_{h_s}^0 \phi^*
\]

\[
(3.14)
\]

\[
\Rightarrow 0 \geq \int_0^1 ds \left\{ \frac{g|\phi|_h^2 u f}{I_s} + \frac{\tau - |\phi|^2_h}{I_s} u f (l - 2 q g|\phi|^2_h) \right\} - \frac{g(\tau - |\phi|^2_h) u f}{(\gamma + |\phi|^2_h)(m + l \beta_0 |\phi|^2_h - q \beta_0 |\phi|^4_h)} |\phi|^2_h 
\]

\[
+ \int_0^1 ds \left\{ \frac{iv \beta_0 g(2|\phi|^2_h - \tau)}{I_s} + \frac{iv \beta_0 g(\tau - |\phi|^2_h)}{I_s}(l - 2 q g|\phi|^2_h) \right\} \nabla_{h_s}^0 \phi \wedge \nabla_{h_s}^0 \phi^* 
\]

\[
+ \int_0^1 ds \left\{ \frac{iv \beta_0 g(\tau - |\phi|^2_h)}{I_s} (\phi^2_h \wedge \phi^2_h + \phi^2_h \wedge \phi^2_h) \right\} 
\]

\[
+ \frac{ig(\gamma - |\phi|^2_h)}{(\gamma + |\phi|^2_h)^2} \left\{ \frac{g(\tau - |\phi|^2_h) iv \beta_0}{(\gamma + |\phi|^2_h)(m + l \beta_0 |\phi|^2_h - q \beta_0 |\phi|^4_h)} \right\} \nabla_{h_s}^0 \phi \wedge \nabla_{h_s}^0 \phi^*.
\]

\[
(3.15)
\]

The following equations describe the relationship between \(\nabla_{h_s}^0 \phi\) and \(\nabla_{h_s}^0 \phi^*\).

\[
\nabla_{h_s}^0 \phi \wedge \nabla_{h_s}^0 \phi^* = (\nabla_{h_s}^0 \phi - \partial \ln(\int_0^1 e^{-tg} dt) - s \partial g \phi) \wedge \nabla_{h_s}^0 \phi^* e^{-sg}(\int_0^1 e^{-tg} dt)^{-1})
\]

\[
= \frac{e^{-sg}}{\int_0^1 e^{-tg} dt} (\nabla_{h_s}^0 \phi ((s) - s) \partial g \phi) \wedge \nabla_{h_s}^0 \phi^* (\nabla_{h_s}^0 \phi^* + ((s) - s) \partial g \phi^*)
\]

\[
= \frac{e^{-sg}}{\int_0^1 e^{-tg} dt} \nabla_{h_s}^0 \phi \wedge \nabla_{h_s}^0 \phi^* \left( 1 + \frac{(s - (s)) g|\phi|^2_h}{\gamma + |\phi|^2_h} \right)^2,
\]

where \(s = \frac{\int_0^1 e^{-sg} ds}{\int_0^1 e^{-sg} ds} \leq 1\).

Note that

\[
\left| \frac{(s - (s)) g|\phi|^2_h}{\gamma + |\phi|^2_h} \right| \leq \frac{2}{\gamma} |g| |\phi|^2_h \int_0^1 e^{-tg} dt \leq \frac{2 \pi}{\gamma} |1 - e^{-g}| \leq \frac{2 \pi}{\gamma} (1 + e^C_1).
\]

We argue by contradiction. Assume that

\[
g(p) \geq 0.
\]
We see that $l - 2q|\phi_{h,s}^2| = \frac{\partial}{(2\pi)^2} \left( \tau - |\phi_{h,s}^2| \right) \geq 0.$

Using 3.16 and 3.18 the inequality 3.15 becomes the following

$$0 \geq \int_0^1 ds \left\{ \frac{g|\phi_{h,s}^2|uf}{m + l\beta_0 \tau} + \frac{(\tau - |\phi_{h,s}^2|)uf\beta_0 g|\phi_{h,s}^2|(l - 2q|\phi_{h,s}^2|)}{(m + l\beta_0 \tau)^2} \right\} \frac{g\tau uf|\phi_{h,s}^2|}{(\gamma + |\phi_{h,s}^2|)(m - q\beta_0 \tau^2)}$$

$$+ \int_0^1 ds \left\{ \frac{i\nu \beta_0 g}{I_s}(2|\phi_{h,s}^2| - \tau) + \frac{i\nu \beta_0 g|\phi_{h,s}^2|}{I_s} \left( \frac{l - 2q|\phi_{h,s}^2|}{(m + l\beta_0 \tau)^2} \right) \frac{e^{-sg}}{\int_0^1 e^{-t\sigma dt}} \nabla_{h,s}^{0,1} \phi \right\} \left( 1 + \frac{(s - \langle s \rangle) g|\phi_{h,s}^2|}{\gamma + |\phi_{h,s}^2|} \right)^2$$

$$+ \int_0^1 ds \left\{ \frac{i\nu \beta_0 g}{I_s}(\tau - |\phi_{h,s}^2|) \frac{e^{-sg}}{\int_0^1 e^{-t\sigma dt}} \nabla_{h,s}^{0,1} \phi \right\} \left( 1 + \frac{(s - \langle s \rangle) g|\phi_{h,s}^2|}{\gamma + |\phi_{h,s}^2|} \right)^2$$

$$+ \int_0^1 ds \left\{ \frac{i\nu \beta_0 g}{I_s}(\tau - |\phi_{h,s}^2|) \frac{e^{-sg}}{\int_0^1 e^{-t\sigma dt}} \nabla_{h,s}^{0,1} \phi \right\} \left( 1 + \frac{(s - \langle s \rangle) g|\phi_{h,s}^2|}{\gamma + |\phi_{h,s}^2|} \right)^2$$

(3.19)

Now

$$\int_0^1 \left| \partial_{\phi_{h,s}} \phi_{h,s} \right| ds$$

$$\leq \int_0^1 \left| \partial_{\phi_{h,s}} \phi_{h,s} \right| \omega_{\Sigma} ds$$

$$\leq \int_0^1 \left| \phi_{h,s} \phi_{h,s} \right| \nabla_{h,s}^{0,1} \phi \right\} \omega_{\Sigma} ds$$

$$\leq \tau \int_0^1 \left| \nabla_{h,s}^{0,1} \phi \right| \omega_{\Sigma} ds$$

$$\leq \tau \int_0^1 \left| \nabla_{h,s}^{0,1} \phi \right| \omega_{\Sigma} ds$$

Using 3.16 we have

$$\leq \tau \left| \nabla_{h,s}^{0,1} \phi \right| \left| \nabla_{h,s}^{0,1} \phi \right| \int_0^1 \left( 1 + \frac{(s - \langle s \rangle) g|\phi_{h,s}^2|}{\gamma + |\phi_{h,s}^2|} \right) \sqrt{\int_0^1 e^{-t\sigma dt}} \omega_{\Sigma} ds.$$
We can write 3.20 as

Using Cauchy-Schwarz inequality, we have

Putting 3.17, we get

and similarly

Using Cauchy-Schwarz inequality, we have

and similarly

We can write 3.21 as

where

and

We recall that \( l - 2q|\phi|_{h_n}^2 = \frac{\beta_0}{(2\pi)^2}(\tau - |\phi|_{h_n}^2) \geq 0 \). Now we can choose \( \gamma \) (depending only on \( \tau, \lambda, \alpha \)) large enough so that

\( A \geq 0 \).

implies that 3.21 is bounded and 3.21, 3.22 implies that

\( i\int_0^1 ds\left\{ \frac{\nu I_s\beta_0(t - |\phi|_{h_n}^2)}{I_s(\gamma + |\phi|_{h_n}^2)}(\partial\phi|_{h_n}^2 \wedge \bar{\partial}\phi|_{h_n}^2 + \partial\phi|_{h_n}^2 \wedge \bar{\partial}\phi|_{h_n}^2) \right\} \) is bounded. Now

\[ iB\nabla_h^{1,0}\phi \wedge \nabla_h^{0,1}\phi^* + i\int_0^1 ds\left\{ \frac{\nu I_s\beta_0(t - |\phi|_{h_n}^2)}{I_s(\gamma + |\phi|_{h_n}^2)}(\partial\phi|_{h_n}^2 \wedge \bar{\partial}\phi|_{h_n}^2 + \partial\phi|_{h_n}^2 \wedge \bar{\partial}\phi|_{h_n}^2) \right\} \]
is positive when \( \beta_0 = 0 \). So for small \( \beta_0 \), we have the following.

\[
i B \nabla^1_{\bar{h}} \phi \wedge \nabla^1_{\bar{h}} \phi^* + i \int_0^1 ds \frac{v \beta_0 (\tau - |\phi|^2_{\bar{h}})}{s (\gamma + |\phi|^2_{\bar{h}})} (\partial |\phi|^2_{\bar{h}} \wedge \bar{\partial} |\phi|^2_{\bar{h}} + \bar{\partial} |\phi|^2_{\bar{h}} \wedge \partial |\phi|^2_{\bar{h}}) \geq 0. \tag{3.26}
\]

Since the line bundle is of degree 1, either \( \phi(p) \neq 0 \) or \( |\nabla \phi|(p) \neq 0 \). This implies that \( g(p) \leq 0 \), which contradicts 3.18. Therefore \( g \leq 0 \). The same argument applied to a point of minimum of \( g \) shows that \( g \geq 0 \) hence \( g = 0 \) showing uniqueness for small \( \beta_0 \).

The inequality 4.65 in [8] is not true. As remarked in the introduction, the gap can be fixed using the method done in this paper.

We now complete the proof of uniqueness.

**Lemma 3.27** If there exists a \( \beta_0 \in [0, 1] \) such that \( h_{\beta_0} = \tilde{h}_{\beta_0} \) then \( h_1 = h_2 \).

**Proof.** Let \( T \subset [\beta_0, 1] \) be the set of all \( \alpha \) such that \( h_{\alpha} = \tilde{h}_{\alpha} \). Then \( T \) satisfies the following.

(1) It is non-empty : \( \beta_0 \in T \)

(2) It is open : The proof of openness (see [7]) and the Inverse Function Theorem of Banach manifolds shows that locally the solution is unique and hence \( T \) is open.

(3) It is closed : The \textit{a priori} estimates show that \( T \) is closed.

Therefore \( T = [\beta_0, 1] \). \( \square \)

### 3.2 J-Vortex Equation

In [6], Takahashi introduced \( J \)-equation on holomorphic vector bundles and came up with the \( J \)-vortex equation (6.21 in [6]). The continuity path (6.28 in [6]) in [6] is

\[
F_{h_t} = \omega_{\Sigma} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \psi_t = 2(1 - |\phi|^2_{h_t}) \frac{su^{1-t} \omega_{\Sigma} + \sqrt{-1}c^{2t} D_{h_t}^t \phi D_{h_t}^t \phi^*}{(4c' r_2 + 2c' t |\phi|^2_{h_t} - 1 + 4c' (4c' r_2 + 2c' t |\phi|^2_{h_t} - 1))},
\]

where \( u = \frac{1}{\alpha (1 - |\phi|^2_{h_t})} \) and \( \alpha = \frac{2^s}{(4c' r_2 - 1 + 4c') (4c' r_2 - 1)} \). We replaced \( c \) by \( c' \) in the continuity path to avoid confusion in notation. This is exactly the type of equations that we considered. Here \( a = (4c' r_2 - 1)(4c' r_2 - 1 + 4c' r_2), b = 8c'^2, c = 4c'^2, e = 2s > 0, k = \frac{2^c}{2^e}, d = 1 \), where \( c' = \frac{2^r + 1 + 2s (2r + 1)}{2^r + 4(2r + 1)} \), \( r_1, r_2 \) are positive integers and \( s \) is a positive real number. Now \( b - cd = 4c'^2 > 0 \) and \( b - (k + ct) d = 4c'^2 (1 - t) + 4c'^2 - \frac{2^c}{2^e} \geq 0 \) and \( de > a \) which follows from lemma 6.17 in [6]. So the \textit{a priori} estimates for this equations follow.

### 3.3 Vector Bundle Version Of The Monge Ampère Equation

In [8], Pingali considered the vector bundle version of the Monge Ampère equation. The continuity path (4.20) in [8] is

\[
i \Theta_{h_t} = (1 - |\phi|^2_{h_t}) \mu u^{1-t} \omega_{\Sigma} + it \nabla^{(1, 0)}_t \phi \wedge \nabla^{(0, 1)}_t \phi^* \frac{2r_2 + t |\phi|^2_{h_t}}{(2 + 2r_2 - t |\phi|^2_{h_t})},
\]

where \( u = \frac{1}{\alpha (1 - |\phi|^2_{h_t})} \) and \( \alpha = \frac{-2r}{2r (2 + 2r_2)}, \mu = 2(2r_1 r_2 + r_1 + r_2) \). This is exactly the type of equations that we considered. Here \( a = 2r_2 (2 + 2r_2), b = 2, c = 1, d = 1, e = \mu, k = 1 \) and \( b - c d > 0, b - (k + ct) d \geq 0, d e > a \) since \( r_1 > r_2 \). So the \textit{a priori} estimates for this equations follow.
4 Gieseker Stability and Almost Hermitian Einstein Metric

Consider a genus-$g$ compact Riemann surface $\Sigma$ endowed with a metric whose $(1,1)$ form $\omega_\Sigma = i\Theta_0$, where $\Theta_0$ is the curvature of a metric $h_0$ on a degree 1 line bundle $L$. Let $\mathbb{CP}^1$ be endowed with the Fubini-Study metric $\omega_{FS}$ which is the curvature of a metric $h_{FS}$ on $\mathcal{O}(1)$.

Consider the rank-2 vector bundle

$$E = \pi_1^*((r_1 + 1)L) \otimes \pi_2^*(r_2\mathcal{O}(2)) \oplus \pi_1^*(r_1L) \otimes \pi_2^*((r_2 + 1)\mathcal{O}(2)),$$

where $r_1, r_2 \geq 2$ and $\pi_1, \pi_2$ are projections from $\Sigma \times \mathbb{CP}^1$ to $\Sigma$ and $\mathbb{CP}^1$ respectively. Endow $E$ with a holomorphic structure arising from the second fundamental form $\beta$ just like in Section [11].

Now we calculate $[e^{\frac{1}{2\pi}R_E + k\omega_{I_E}}]_{TOP}$ for the bundle $E$ over the manifold $\Sigma \times \mathbb{CP}^1$ with $\omega = \frac{\tau}{2}\omega_\Sigma + 2\omega_{FS}$, where $\tau$ is an even integer. We know that

$$Td(\Sigma \times \mathbb{CP}^1) = 1 + \frac{c_1(\Sigma)}{2} + \frac{c_1(\mathbb{CP}^1)}{2} + \frac{c_1(\Sigma) \wedge c_1(\mathbb{CP}^1)}{4},$$

$$e^{\frac{1}{2\pi}R_E + k\omega_{I_E}} = (1 + k\omega)I_E + \frac{i}{2\pi}(1 + k\omega) \wedge R_E + \frac{1}{2(2\pi)^2}(iR_E)^2 + \tau k^2 \omega_\Sigma \wedge \omega_{FS}I_E.$$

Therefore

$$[e^{\frac{1}{2\pi}R_E + k\omega_{I_E}}]_{TOP} = \frac{c_1(\Sigma) \wedge c_1(\mathbb{CP}^1)}{4}I_E + \frac{kc_1(\Sigma) \wedge \omega}{2}I_E + \frac{k\omega_\Sigma \wedge \omega}{2}I_E + \frac{g_1(\Sigma) \wedge R_E}{2(2\pi)^2} + \frac{i}{2\pi}k\omega \wedge R_E + \frac{i}{2(2\pi)}c_1(\Sigma) \wedge R_E.$$

Here we note some equalities. $c_1(\Sigma) = \alpha \omega_\Sigma$, $c_1(\mathbb{CP}^1) = 2\omega_{FS}$, $c_1(S) = (r_1 + 1)\omega_\Sigma + 2r_2\omega_{FS}$, $c_1(E) = (2r_1 + 1)\omega_\Sigma + (4r_2 + 2)\omega_{FS}$, $ch_2(E) = 2\{(r_1 + 1)r_2 + r_1(r_2 + 1)\}\omega_\Sigma \wedge \omega_{FS}$, $ch_2(S) = 2r_2(r_1 + 1)\omega_\Sigma \wedge \omega_{FS}$

where $\alpha = 2 - 2g, S = \pi_1^*((r_1 + 1)L) \otimes \pi_2^*(r_2\mathcal{O}(2))$.

We now recall the definition of Gieseker stability.

**Definition 4.1** Let $E$ be a rank $r$ holomorphic vector bundle over a projective variety $X$ with ample line bundle $L'$, $E$ is called Gieseker stable if for any nontrivial coherent subsheaf $S$ of $E$, we have

$$\frac{\chi(X, S \otimes L^k)}{\text{rank}S} < \frac{\chi(X, E \otimes L^k)}{\text{rank}E}$$

for large enough $k$.

**Theorem 4.2** If $k$ satisfies $k(\tau - 2) + (\alpha - 1) + 2(r_1 - r_2) > 0$ and $k\tau + \alpha > 0$, then the following are equivalent.

(1) $E$ is Gieseker stable.

(2) There exists an almost Hermitian Einstein metric on $E$.

Moreover, the solution is unique among all $SU(2)$-invariant solutions.

**Proof.** 2 $\Rightarrow$ 1 follows from [3]. We only prove 1 $\Rightarrow$ 2. To this end, we only need to use the Gieseker stability assumption for the $SU(2)$-invariant subbundle $S = \pi_1^*((r_1 + 1)L) \otimes \pi_2^*(r_2\mathcal{O}(2))$.

The assumption reads as follows.

$$2\chi(X, S \otimes L^k) < \chi(X, E \otimes L^k). \quad (4.3)$$
Now
\[ \chi(X, E \otimes L^k) = \int Tr([e^{(\frac{i}{2\pi}R_E + k\omega_I E)}]^{TOP}) \]

\[ = \int Tr\left( \frac{c_1(S) \wedge c_1(\mathbb{CP}^1)}{4} I_E + \frac{k\omega}{2} I_E + \frac{k\omega \wedge R_E}{2} I_E + \frac{i\omega \wedge R_E}{2(2\pi)} c_1(S) \wedge R_E \right) \]

\[ + \int Tr\left( \frac{i}{2(2\pi)} c_1(\mathbb{CP}^1) \wedge R_E + k^2 \tau \omega \wedge \omega_{FS} I_E + \frac{(i R_E)^2}{2(2\pi)^2} \right). \]

Hence Inequality 4.3 becomes
\[ 2 \int k \omega \wedge c_1(S) + \int c_1(S) \wedge c_1(S) + \int c_1(\mathbb{CP}^1) \wedge c_1(S) + 2 \int ch_2(S) \]
\[ < \int k \omega \wedge c_1(E) + \frac{1}{2} \int c_1(S) \wedge c_1(E) + \frac{1}{2} \int c_1(\mathbb{CP}^1) \wedge c_1(E) + \int ch_2(E) \]
\[ \Rightarrow 2k \int (r_2\tau + 2(r_1 + 1))\omega \wedge \omega_{FS} + \int 2\alpha r_2 \omega \wedge \omega_{FS} + \int 2(r_1 + 1)\omega \wedge \omega_{FS} + 2 \int ch_2(S) \]
\[ < k \int ((2r_2 + 1)\tau + 2(2r_1 + 1))\omega \wedge \omega_{FS} + \frac{1}{2} \int \alpha(4r_2 + 2)\omega \wedge \omega_{FS} + \frac{1}{2} \int 2(2r_1 + 1)\omega \wedge \omega_{FS} + \int ch_2(E) \]
\[ \Rightarrow 0 < k(\tau - 2) \int \omega \wedge \omega_{FS} + (\alpha - 1 + 2r_1 - 2r_2) \int \omega \wedge \omega_{FS} \]
\[ \Rightarrow k(\tau - 2) + (\alpha - 1 + 2(r_1 - r_2)) > 0. \] (4.4)

Next we write the almost Hermitian Einstein equation for this bundle. The equation is
\[ \frac{1}{2} \left( \frac{i}{2\pi} R_E + k\omega I_E \right)^2 + \left( \frac{i}{2\pi} R_E + k\omega I_E \right) \wedge \left( \frac{c_1(S) + c_1(\mathbb{CP}^1)}{2} \right) + \frac{c_1(S) \wedge c_1(\mathbb{CP}^1)}{4} \]
\[ = \frac{\tau}{Vol(X)} \omega \wedge \omega_{FS} I_E \]
\[ \Rightarrow \left( \frac{i}{2\pi} R_E + k\omega I_E + \frac{c_1(S) + c_1(\mathbb{CP}^1)}{2} \right)^2 = \frac{\tau}{Vol(X)} \chi(X, E \otimes L^k) \omega \wedge \omega_{FS} I_E \]
\[ \Rightarrow \left( \frac{i}{2\pi} R_E + k\omega I_E + \frac{c_1(S) + c_1(\mathbb{CP}^1)}{2} \right)^2 = \frac{\tau}{Vol(X)} \left( \int \alpha \omega \wedge \omega_{FS} + \int 2\alpha k \omega \wedge \omega_{FS} + \int k \tau \omega \wedge \omega_{FS} \right) \omega \wedge \omega_{FS} I_E \]
\[ + \frac{\tau}{Vol(X)} \left( \int k(\tau(2r_2 + 1) + 2(2r_1 + 1))\omega \wedge \omega_{FS} + \int \alpha(2r_2 + 1)\omega \wedge \omega_{FS} + \int \omega \wedge \omega_{FS} I_E \right) \]
\[ + \frac{\tau}{Vol(X)} \left( \int 2(2r_1 + 1)\omega \wedge \omega_{FS} + \int 2k^2 \tau \omega \wedge \omega_{FS} + \int ch_2(E)\omega \wedge \omega_{FS} I_E \right). \]

Now using \( \int \omega \wedge \omega_{FS} = \frac{Vol(X)}{\tau} \), we have
\[ \Rightarrow \left( \frac{i}{2\pi} R_E + k\omega I_E + \frac{c_1(S) + c_1(\mathbb{CP}^1)}{2} \right)^2 = \left\{ \alpha + 2\alpha k + k \tau + k(\tau(2r_2 + 1) + 2(2r_1 + 1)) \right\} \omega \wedge \omega_{FS} I_E \] (4.5)
\[ \{\alpha(2r_2 + 1) + 2r_1 + 1 + 2k^2\tau + 2(r_1(r_2 + 1) + r_2(r_2 + 1))\}\omega_\Sigma \wedge \omega_{FS}I_E. \]

The term \( \frac{1}{2\pi} R_E + k\omega I_E + \frac{c_1(\Sigma) + c_1(\text{CP}^1)}{2} \) equals \( \frac{1}{2\pi} R_E + k\omega I_E + \frac{\omega_\Sigma + \omega_{FS}}{2} \).

But \( \frac{1}{2\pi} R_E + k\omega I_E + \frac{\omega_\Sigma + \omega_{FS}}{2} \) is the curvature of the bundle \( E \otimes (\frac{k\pi}{E} L \otimes 2kO(1)) \otimes (\frac{\phi}{E} L \otimes O(1)) \).

A small calculation shows that \( E \otimes (\frac{k\pi}{E} L \otimes 2kO(1)) \otimes (\frac{\phi}{E} L \otimes O(1)) \) equals \( \pi_1^*(R_1 + 1) \otimes \pi_L^*(R_2O(2)) \otimes \pi_2^*(R_1 \otimes \pi_2^*(R_2 + 1)O(2)) \), where \( R_1 = r_1 + \frac{k\pi + \alpha}{2}, R_2 = r_2 + k + \frac{1}{2} \).

Now \( R_1 - R_2 > 0 \implies k(\tau - 2) + (\alpha - 1) + 2(r_1 - r_2) > 0 \).

We invoke two theorems of [8]. Theorem 1.4 of [8] is as follows:

**Theorem 4.6** Let \((L, h_0)\) be a holomorphic line bundle over a compact Riemann surface \(M\) such that its curvature \(\Theta_0\) defines a Kähler form \(\omega_\Sigma = i\Theta_0\) on \(M\). Assuming the degree(\(L\)) is equal to 1, \(r_1, r_2 \geq 2\) are integers and \(\phi \in H^0(M, L)\) which is not identically zero, the following are equivalent.

1. Stability : \(r_1 > r_2\).
2. Existence : There exists a smooth metric \(h\) on \(L\) such that the curvature \(\Theta_h\) of its Chern connection \(\nabla_h\) satisfies the Monge-Ampère vortex equation.

\[
i\Theta_h = (1 - |\phi_h|^2) \frac{\mu \omega_\Sigma + i\nabla_{h_0}^{1,0} \phi \wedge \nabla_{h_0}^{0,1} \phi^*}{(2r_2 + |\phi_h|^2)(2 + 2r_2 - |\phi_h|^2)}, \tag{4.7} \]

where \(\mu = 2(r_2(r_1 + 1) + r_1(r_2 + 1))\) and \(\phi^*\) is the adjoint of \(\phi\) with respect to \(h\) when \(\phi\) is considered as an endomorphism from the trivial line bundle to \(L\).

Moreover, if a solution \(h\) to 4.7 satisfying \(|\phi_h|^2 \leq 1\) exists, then it is unique.

Theorem 4.2 of [8] is:

**Theorem 4.8** Suppose there is a smooth metric \(h\) on \(L\) satisfying

\[ |\phi_h|^2 \leq 1, \]

and solving the following equation.

\[
i\Theta_h = (1 - |\phi_h|^2) \frac{\xi + i\nabla_{h_0}^{1,0} \phi \wedge \nabla_{h_0}^{0,1} \phi^*}{(2r_2 + |\phi_h|^2)(2 + 2r_2 - |\phi_h|^2)}, \]

where \(\xi > 0\) is given \((1,1)\)-form on \(\Sigma\) satisfying

\[ \int_\Sigma \xi = 2(r_1(r_2 + 1) + r_2(r_1 + 1)). \]

Then there is a smooth Griffiths positively curved metric \(H\) on the vortex bundle \(V\) whose curvature \(\Theta\) satisfies the \(\nu\text{bMA}\) equation: \((i\Theta)^2 = \pi_1^*\xi \wedge \pi_2^*\omega_{FS}I_E\).

Equation 1.34 implies that \(R_1 > R_2\). Using the theorems stated above, we have a solution of 1.35.

Therefore, there exists an almost Hermitian Einstein metric on \(E\).

Uniqueness follows from the aforementioned theorems. \(\square\)
5 Kählerness of the symplectic form.

In \[9\], Pingali gave the moment map interpretation of the Calabi-Yang-Mills equations. Let \((M, \omega)\) be an \(n\)-complex dimensional compact Kähler manifold such that \([\omega] = [c_1(\tilde{L}, h)]\) for some hermitian holomorphic line bundle \((\tilde{L}, h)\) satisfying \(\int \omega^n = 1\). Let \((E, \tilde{h})\) be a Hermitian holomorphic vector bundle of rank \(r\). The Calabi-Yang-Mills equations (as given in \[9\]) on vortex bundle are

\[
\sqrt{-1} \Theta_B \wedge \omega_{\phi}^{n-1} = -\lambda' \omega_{\phi}^n \text{Id} \\
\omega_{\phi}^n(1 + \alpha' \lambda'^2) - \eta' = 2 \alpha' \text{ch}_2(B)n(n-1)\omega_{\phi}^{n-2},
\]

where \(\lambda'\) is a topological constant, \(\eta'\) is an \((n, n)\) form, \(B\) is a connection and \(\omega_{\phi} = \omega + \sqrt{-1} \partial \bar{\partial} \phi\), \(\phi\) is a function on the manifold.

If we put \(r = 2, n = 2\) in \((5.1)\) we get the following.

\[
\sqrt{-1} \Theta_B \wedge \omega_{\phi} = -\frac{\lambda'}{2} \omega_{\phi}^2 \text{Id} \\
\omega_{\phi}^2(1 + \alpha' \lambda'^2) - \eta' = 2 \alpha' \text{ch}_2(B).
\]

If we replace \(\lambda'\) by \(-2\lambda\), \(\alpha'\) by \(-\frac{\alpha}{2 + 4\alpha \lambda'^2}\) and \(\frac{\eta'}{1 + 4\alpha \lambda'^2}\) by \(\eta\) in \((5.2)\), then it becomes

\[
\sqrt{-1} \Theta_B \wedge \omega_{\phi} = \lambda \omega_{\phi}^2 \text{Id} \\
\omega_{\phi}^2 + \alpha \text{ch}_2(B) - \eta = 0.
\]

Vortex bundle is a rank 2 vector bundle over the manifold \(\Sigma \times \mathbb{P}^1\), where \(\Sigma\) is a Riemann Surface. The Calabi-Yang-Mills equations (as given in \[7\]) on vortex bundle are

\[
\sqrt{-1} \Theta_{\alpha} \wedge \Omega_{\alpha} = \lambda \Omega_{\alpha}^2 \text{Id} \\
\Omega_{\alpha}^2 + \alpha \text{ch}_2(E, H_{\alpha}) - \eta = 0,
\]

where \(\Omega_{\alpha} > 0\) is a smooth form and \(H_{\alpha}\) is a smooth metric on the vortex bundle.

Now the equations \((5.3)\) resemble like \((5.4)\).

The symplectic form on the infinite dimensional manifold \(\mathcal{A}_{E}^{1,1} \times \mathcal{A}^{1,1}\) is

\[
\frac{(2\pi)^{n+1}}{(\sqrt{-1})^{n-1}} \omega_{\alpha'}(a_E \oplus a_{\tilde{L}}, b_E \oplus b_{\tilde{L}}) = -N \alpha' \int_M \text{tr}(a_E \wedge b_E)n\Theta_{L}^{n-1} \\
- N \alpha' \int_M (\text{tr}(\Theta_E a_E)b_{\tilde{L}}n(n-1)\Theta_{L}^{n-2} + a_{\tilde{L}} \text{tr}(\Theta_E b_E)n(n-1)\Theta_{L}^{n-2}) \\
- N \alpha' \lambda' \int_M n\Theta_{L}^{n-1} \text{tr}(a_E)b_{\tilde{L}} - N \alpha' \lambda' \int_M n\Theta_{L}^{n-1} a_{\tilde{L}} \text{tr}(b_E) + N \int_M a_{\tilde{L}} \wedge b_{\tilde{L}} \wedge n\Theta_{L}^{n-1} \\
+ N \left(-\alpha' \int_M \text{tr}(\Theta_E^2)n \left(\frac{n-1}{2}\right)a_{\tilde{L}} \wedge b_{\tilde{L}} \wedge \Theta_{L}^{n-3} - \lambda' \alpha' \int_M \left(\frac{n}{2}\right)\text{tr}(\Theta_E)\Theta_{L}^{n-2} a_{\tilde{L}} \wedge b_{\tilde{L}}\right),
\]

where \(\mathcal{A}_{E}^{1,1}\) is the space of smooth unitary integrable connections on a vector bundle \(E\) and \(\mathcal{A}^{1,1}\) is the space of smooth integrable unitary connections on \(\tilde{L}\). The tangent space at \(A \in \mathcal{A}_{E}^{1,1}\) consists of
skew-hermitian endomorphism valued 1-forms whose $(0, 1)$ part is $d_A^{0,1}$ closed (may also be identified with $d_A^{0,1}$ closed endomorphism valued $(0, 1)$ forms). The tangent spaces at $A \in A^{1,1}$ consists of $(0, 1)$ forms $a^{0,1}$ satisfying $\partial a^{0,1} = 0$.

For $n = 2$, the symplectic form is the following.

$$ \frac{(2\pi)^3}{\sqrt{-1}} \Omega_{\alpha'}(a_E \oplus a_L, b_E \oplus b_L) = -2N\alpha' \int_M tr(a_E \wedge b_E)\Theta_L $$

$$ - 2N\alpha' \int_M (tr(\Theta_E a_E) b_L + a_L tr(\Theta_E b_E)) $$

$$ - 2N\alpha' \lambda' \int_M \Theta_L tr(a_E) b_L - 2N\alpha' \lambda' \int_M \Theta_L a_L tr(b_E) $$

$$ + N \left( -\lambda' \alpha' \int_M tr(\Theta_E a_L \wedge b_L) + 2N \int_M a_L \wedge b_L \wedge \Theta_L \right). $$

We want to check whether this is Kähler or not for the vortex bundle ansatz. In [9], the almost complex structure is mentioned. The elements of $A^{1,1}_E$ (where $E$ is the vortex bundle) is of the form

$$ a_E = \begin{bmatrix} (\delta A_{h_1})^{0,1} & \delta \beta \\ 0 & (\delta A_{g_2})^{0,1} \end{bmatrix} $$

The elements of $A^{1,1}$ is of the form $a_L = \pi^*_\Sigma \xi$, where $\xi$ is a $(0, 1)$ form on $\Sigma$. We want to check whether $\sqrt{-1} \Omega_{\alpha'}(a_E \oplus a_L, a_E^* \oplus a_L^*)$ is positive or negative.

$$ (2\pi)^3 \Omega_{\alpha'}(a_E \oplus a_L, a_E^* \oplus a_L^*) = -2N\alpha' \int_{\Sigma \times P^1} tr(a_E \wedge a_E^*)\sqrt{-1}\Theta_L $$

$$ - 2N\alpha' \int_{\Sigma \times P^1} (tr(\sqrt{-1}\Theta_E a_E a_L^* + a_L \Theta_E a_E^*)) $$

$$ - 2N\alpha' \lambda' \int_{\Sigma \times P^1} \sqrt{-1}\Theta_L tr(a_E) a_L^* - 2N\alpha' \lambda' \int_{\Sigma \times P^1} \sqrt{-1}\Theta_L a_L tr(a_E^*) $$

$$ + N \left( -\lambda' \alpha' \int_{\Sigma \times P^1} tr(\sqrt{-1}\Theta_E) a_L \wedge a_L^* + 2N \int_{\Sigma \times P^1} a_L \wedge a_L^* \wedge \sqrt{-1}\Theta_L \right). $$

We follow the calculations in [7]. We have $\sqrt{-1}\Theta_L = \pi^*_\Sigma \omega_\Sigma + \frac{1}{2} \pi^*_\Sigma \omega_{FS}$ (we will omit the pullback in the following calculations) and

$$ \Theta_E = \begin{bmatrix} \Theta_{h_1} - \beta \wedge \beta^* & \nabla^{(1,0)} \beta \\ -\nabla^{(0,1)} \beta^* & \Theta_{g_2} - \beta^* \wedge \beta \end{bmatrix}. $$

We now calculate the followings.

$$ tr(a_E \wedge a_E^*) $$

$$ = (\delta A_{h_1})^{0,1} \wedge (\delta A_{h_1})^{0,1} + \delta \beta \wedge (\delta \beta)^* + (\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} $$

$$ tr(\Theta_E a_E) $$

$$ = \Theta_{h_1} \wedge (\delta A_{h_1})^{0,1} - \beta \wedge \beta^* \wedge (\delta A_{h_1})^{0,1} - \nabla^{(0,1)} \beta^* \wedge \delta \beta + \Theta_{g_2} \wedge (\delta A_{g_2})^{0,1} - \beta^* \wedge \beta \wedge (\delta A_{g_2})^{0,1} $$
\[ \text{tr}(\Theta_E a_E^\dagger) \]
\[ = \Theta_{h_1} \wedge (\delta A_{h_1})^{0,1} - \beta \wedge \beta^* \wedge (\delta A_{h_1})^{0,1} + \nabla^{(1,0)} \beta \wedge (\delta \beta)^* + \Theta_{g_2} \wedge (\delta A_{g_2})^{0,1} - \beta^* \wedge \beta \wedge (\delta A_{g_2})^{0,1} \]
\[ (5.12) \]

We now calculate the terms of 5.8.

We have
\[ \sqrt{-1} \Theta_L \wedge \text{tr}(a_E) \wedge a_E^\dagger = -\frac{4}{\tau} \omega_{FS} \wedge \bar{a}_E \wedge (\delta A_{h_1})^{0,1} - \frac{4}{\tau} \omega_{FS} \wedge \bar{a}_E \wedge (\delta A_{g_2})^{0,1} \]
\[ (5.13) \]
\[ \sqrt{-1} \Theta_L \wedge a_L \wedge \text{tr}(a_E) = -\frac{4}{\tau} \omega_{FS} \wedge (\delta A_{h_1})^{0,1} \wedge a_L - \frac{4}{\tau} \omega_{FS} \wedge (\delta A_{g_2})^{0,1} \wedge a_L \]
\[ (5.14) \]

and
\[ \text{tr}(a_E \wedge a_E^\dagger) \wedge \sqrt{-1} \Theta_L \wedge \text{tr}(a_E) \wedge a_E^\dagger = -\sqrt{-1} \beta \wedge \beta^* \wedge (\delta A_{h_1})^{0,1} + \omega_{\Sigma} \wedge \delta \beta \wedge (\delta \beta)^* + (\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} \wedge (\omega_{\Sigma} + \frac{4}{\tau} \omega_{FS}) \]
\[ (5.15) \]

Using 5.11 and \( \sqrt{-1} \Theta_{g_2} = \sqrt{-1} \Theta_{f_2} + 2 \omega_{FS} \), we have
\[ \text{tr}(\sqrt{-1} \Theta_E a_E) a_L \]
\[ = -\sqrt{-1} \beta \wedge \beta^* \wedge (\delta A_{h_1})^{0,1} \wedge a_L - \sqrt{-1} \nabla^{0,1} \beta^* \wedge \delta \beta \wedge a_L + 2 \omega_{FS} \wedge (\delta A_{g_2})^{0,1} \wedge a_L \]
\[ (5.16) \]

Using 5.12 and \( \sqrt{-1} \Theta_{g_2} = \sqrt{-1} \Theta_{f_2} + 2 \omega_{FS} \), we have
\[ a_L \wedge \text{tr}(\sqrt{-1} \Theta_E a_E^\dagger) \]
\[ = -\sqrt{-1} \beta \wedge \beta^* \wedge a_L \wedge (\delta A_{h_1})^{0,1} + \sqrt{-1} a_L \wedge \nabla^{1,0} \beta \wedge (\delta \beta)^* + 2 a_L \wedge \omega_{FS} \wedge (\delta A_{g_2})^{0,1} \]
\[ (5.17) \]

Using \( \sqrt{-1} \Theta_{g_2} = \sqrt{-1} \Theta_{f_2} + 2 \omega_{FS} \), we have
\[ \text{tr}(\sqrt{-1} \Theta_E) \wedge a_L \wedge \bar{a}_L = 2 \omega_{FS} \wedge a_L \wedge \bar{a}_L \]
\[ (5.18) \]
If we put \(5.15, 5.16, 5.17, 5.18, 5.19, 5.19\) and \(\sqrt{-1} \Theta_L = \omega_{\Sigma} + \frac{4}{\tau} \omega_{FS}\) in \(5.10\), then we get

\[
\frac{8N\alpha'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\delta A_{h_1})^{0,1} \wedge (\delta A_{h_1})^{0,1} + 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \omega_{\Sigma} \wedge (\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} \\
+ \frac{8N\alpha'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} + 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \sqrt{-1} \beta \wedge \beta^* (- \bar{a}_L \wedge (\delta A_{h_1})^{0,1} - (\delta A_{h_1})^{0,1} \wedge a_L) \\
- 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \sqrt{-1} \beta \wedge (\bar{a}_L \wedge (\delta A_{g_2})^{0,1} - (\delta A_{g_2})^{0,1} \wedge a_L) \\
+ 4N\alpha' \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\bar{a}_L \wedge (\delta A_{g_2})^{0,1} + (\delta A_{g_2})^{0,1} \wedge a_L) + \frac{8N\alpha' \lambda'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\bar{a}_L \wedge (\delta A_{h_1})^{0,1} + (\delta A_{h_1})^{0,1} \wedge a_L) \\
+ \frac{8N\alpha' \lambda'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\bar{a}_L \wedge (\delta A_{g_2})^{0,1} + (\delta A_{g_2})^{0,1} \wedge a_L) \\
+ 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\bar{a}_L \wedge a_L) - \frac{8N}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\bar{a}_L \wedge a_L) + 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \omega_{\Sigma} \wedge (\delta \beta)^* \wedge \delta \beta \\
+ 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \sqrt{-1} (\nabla^{0,1} \beta^* \wedge \delta \beta \wedge \bar{a}_L) + \nabla^{1,0} \beta \wedge (\delta \beta)^* \wedge a_L) \\
(5.19)
\]

Using \(\nabla^{0,1} \beta^* \wedge \delta \beta \wedge \bar{a}_L = \sqrt{-1} (\delta \phi) \omega_{FS} \wedge \bar{a}_L \wedge \nabla^{0,1} \phi^* \), \(\nabla^{1,0} \beta \wedge (\delta \beta)^* \wedge a_L = \sqrt{-1} \omega_{FS} \wedge \nabla^{1,0} \phi \wedge a_L\) and \((\delta \beta) \wedge (\delta \beta)^* = |\delta \phi|^2 \sqrt{-1} \omega_{FS}\), \(5.19\) becomes

\[
\int_{\Sigma \times \mathbb{P}^1} \frac{8N\alpha'}{\tau} - \frac{2N\alpha'|\phi|_h^2}{\tau} - \frac{8N\alpha' \lambda'}{\tau}) \omega_{FS} \wedge (\delta A_{h_1})^{0,1} \wedge (\delta A_{h_1})^{0,1} + 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \omega_{\Sigma} \wedge (\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} \\
+ \int_{\Sigma \times \mathbb{P}^1} \frac{8N\alpha'}{\tau} + \frac{2N\alpha'|\phi|_h^2}{\tau} + 4N\alpha' + \frac{8N\alpha' \lambda'}{\tau}) \omega_{FS} \wedge (\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} \\
+ \int_{\Sigma \times \mathbb{P}^1} \frac{2N\alpha'|\phi|_h^2}{\tau} + \frac{8N\alpha' \lambda'}{\tau}) \omega_{FS} \wedge |a_L| + (\delta A_{h_1})^{0,1} |^2 + \int_{\Sigma \times \mathbb{P}^1} (- \frac{4N\alpha'}{\tau} - \frac{8N\alpha' \lambda'}{\tau})) \omega_{FS} \wedge |a_L| - (\delta A_{g_2})^{0,1} |^2 \\
- 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} |\delta \phi|^2 \sqrt{-1} \omega_{FS} \wedge \omega_{\Sigma} + \int_{\Sigma \times \mathbb{P}^1} (- \frac{4N\alpha'|\phi|_h^2}{\tau} + 4N\alpha' + 2N\alpha' \lambda' - \frac{8N}{\tau}) \omega_{FS} \wedge |a_L| \wedge a_L \\
+ \frac{2N\alpha'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\frac{1}{K} \bar{a}_L - K(\delta \phi) \nabla^{1,0} \phi^2 - \frac{1}{K^2} \bar{a}_L \wedge a_L - K^2 |\delta \phi|^2 \nabla^{1,0} \phi \wedge \nabla^{0,1} \phi^*) \\
+ \int_{\Sigma \times \mathbb{P}^1} \frac{2N\alpha'|\phi|_h^2}{\tau} \omega_{FS} \wedge |a_L| - (\delta A_{g_2})^{0,1} |^2 ,
(5.20)\]
where $K$ is a constant to be chosen later. So the symplectic form is the following.

\[
\sqrt{-1}(2\pi)^3 \Omega_{\alpha'}(a_E \oplus a_L, \alpha^*_E \oplus \alpha^*_L)
\]

\[
= \int_{\Sigma \times \mathbb{P}^1} \left( \frac{8N\alpha'}{\tau} - \frac{2N\alpha' |\phi|^2_h}{\tau} - \frac{8N\alpha' \lambda'}{\tau} \right) \omega_{FS} \wedge (\sqrt{-1})(\delta A_{h_1})^{0,1} \wedge (\delta A_{h_2})^{0,1}
\]

\[
+ \int_{\Sigma \times \mathbb{P}^1} \left( \frac{8N\alpha'}{\tau} + \frac{2N\alpha' |\phi|^2_h}{\tau} + 4N\alpha' + \frac{8N\alpha' \lambda'}{\tau} \right) \omega_{FS} \wedge (\sqrt{-1})(\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1}
\]

\[
+ \int_{\Sigma \times \mathbb{P}^1} \left( \frac{2N\alpha' |\phi|^2_h}{\tau} + \frac{8N\alpha' \lambda'}{\tau} \right) \omega_{FS} \wedge (\sqrt{-1})\bar{\omega}_L + (\delta A_{h_2})^{0,1} \]

\[
+ \int_{\Sigma \times \mathbb{P}^1} (4N\alpha' - \frac{8N\alpha' \lambda'}{\tau}) \omega_{FS} \wedge (\sqrt{-1})\bar{\omega}_L - (\delta A_{g_2})^{0,1} \]

\[
+ 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \left( \frac{\delta \phi|^2_h}{\tau} \omega_{FS} \wedge \omega_{\Sigma} + \int_{\Sigma \times \mathbb{P}^1} \left( -\frac{4N\alpha' |\phi|^2_h}{\tau} + 4N\alpha' + 2N\alpha' \lambda' - \frac{8N}{\tau} - \frac{2N\alpha'}{K^2 \tau} \right) \omega_{FS} \wedge (\sqrt{-1})\bar{\omega}_L \wedge a_L
\]

\[
+ \frac{2N\alpha'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\sqrt{-1}) \left( \frac{1}{K\bar{\omega}_L} - K(\delta \phi) \nabla^{1,0} \phi \right) - \frac{2N\alpha'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge |\delta \phi|^2_h K(\delta \phi) \nabla^{1,0} \phi \wedge \nabla^{0,1} \phi^*
\]

\[
+ \int_{\Sigma \times \mathbb{P}^1} \frac{2N\alpha' |\phi|^2_h}{\tau} \omega_{FS} \wedge (\sqrt{-1})\bar{\omega}_L - (\delta A_{g_2})^{0,1} \]

\[
+ 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \omega_{\Sigma} \wedge (\sqrt{-1})(\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1}.
\]

(5.21)

Here, we want to remind ourselves that we made the substitution $\lambda' = -2\lambda < 0$ and $\alpha' = -\frac{\alpha}{2+4\lambda^2\alpha} \leq 0$ to compare the Calabi-Yang-Mills equations. From [1], we have

\[
\lambda = \frac{\tau}{8} + \frac{c_1(L)\pi}{2\nu(\Sigma)}
\]

(5.22)

and

\[
0 < c_1(L) < \frac{\tau \nu(\Sigma)}{4\pi}.
\]

(5.23)

Now

\[
\frac{2N\alpha' |\phi|^2_h}{\tau} + \frac{8N\alpha' \lambda'}{\tau}
\]

\[
= \frac{N\alpha'}{\tau} (2|\phi|^2_h - 2\tau - \frac{c_1(L)\pi}{2\nu(\Sigma)}) \geq 0.
\]

(5.24)

In the second line we substituted $\lambda'$ by $-2\lambda$ and used \[5.22\]. The term \[5.24\] is non-negative because $\alpha' \leq 0$, $|\phi|^2_h \leq \tau$ and \[5.23\].

Also

\[
- 4N\alpha' - \frac{8N\alpha' \lambda'}{\tau}
\]

\[
= N\alpha'( -2 + \frac{8c_1(L)\pi}{\tau \nu(\Sigma)}) \geq 0
\]

(5.25)

In the second line we substituted $\lambda'$ by $-2\lambda$ and used \[5.22\]. The term \[5.25\] is non-negative because $\alpha' \leq 0$ and \[5.23\].
Now using the inequality \(|a \pm b|^2 \leq 2|a|^2 + 2|b|^2\), (5.24) becomes

\[
\sqrt{-1}(2\pi)^3\Omega_\alpha'(a_E \oplus a_L, a_E' \oplus a_L) \\
\leq \int_{\Sigma \times \mathbb{P}^1} \left( \frac{8N\alpha'}{\tau} + \frac{2N\alpha'\Delta h^2}{\tau} + \frac{8N\alpha'\lambda'}{\tau}\right) \omega_{FS} \wedge (\sqrt{-1})(\delta A_{h_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} \\
+ \int_{\Sigma \times \mathbb{P}^1} \left( \frac{2N\alpha'|\Delta h^2}{\tau}\right) \omega_{FS} \wedge (\omega_{\Sigma} - K^2(\sqrt{-1})\nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^*) + 2N\alpha' \int_{\Sigma \times \mathbb{P}^1} \omega_{\Sigma} \wedge (\sqrt{-1})(\delta A_{g_2})^{0,1} \wedge (\delta A_{g_2})^{0,1} \\
+ \int_{\Sigma \times \mathbb{P}^1} (-4N\alpha' + 2N\alpha'\lambda') - \frac{8N}{\tau} \cdot \frac{2N\alpha'}{K^2\tau} \omega_{FS} \wedge (\sqrt{-1})\omega_L \wedge a_L \\
+ \frac{2N\alpha'}{\tau} \int_{\Sigma \times \mathbb{P}^1} \omega_{FS} \wedge (\sqrt{-1})\frac{1}{K^2\omega_{FS} - K(\delta \phi)\nabla^{1,0}\phi^*} + \int_{\Sigma \times \mathbb{P}^1} \frac{2N\alpha'|\Delta h^2}{\tau} \omega_{FS} \wedge (\sqrt{-1})\omega_L - (\delta A_{g_2})^{0,1}^2.
\]  

(5.26)

Now

\[
\frac{8N\alpha'}{\tau} + \frac{2N\alpha'|\Delta h^2}{\tau} + \frac{8N\alpha'\lambda'}{\tau} \\
= \frac{N\alpha}{\tau(2 + 4\lambda^2\alpha)} (-8 - 2|\Delta h^2| + 2\tau + \frac{8c_1(L)\pi}{2\text{vol}(\Sigma)}) \\
\leq \frac{N\alpha}{\tau(2 + 4\lambda^2\alpha)} (-8 + 2\tau + \tau) < 0.
\]  

(5.27)

In the second line we substituted \(\lambda'\) by \(-2\lambda\), used 5.22 substituted \(\alpha'\) by \(-\frac{\alpha}{2+4\lambda^2\alpha}\) and in the last line, we used 5.23 The last line holds whenever \(\tau < \frac{8}{5}\).

Also

\[
\frac{8N\alpha'}{\tau} + \frac{2N\alpha'|\Delta h^2}{\tau} - 4N\alpha' - \frac{8N\alpha'\lambda'}{\tau} \\
= \frac{N\alpha}{2 + 4\lambda^2\alpha} (-\frac{8}{\tau} - \frac{2|\Delta h^2|}{\tau} + 2 - \frac{8c_1(L)\pi}{\text{vol}(\Sigma)}) \\
< \frac{N\alpha}{2 + 4\lambda^2\alpha} (-\frac{8}{\tau} + 2) < 0.
\]  

(5.28)

In the second line, we substituted \(\lambda'\) by \(-2\lambda\), used 5.22 substituted \(\alpha'\) by \(-\frac{\alpha}{2+4\lambda^2\alpha}\). In the last line, we used 5.23 The last inequality holds whenever \(\tau < 4\).

First note that \(\frac{7}{8} < \lambda < \frac{7}{4}\) because of 5.23.
Then
\[ -4N\alpha' + 2N\alpha'\lambda' - \frac{8N}{\tau} - \frac{2N\alpha'}{K^2\tau} \]
\[ = N(-4\alpha' - \frac{\tau\alpha'}{2} - \frac{2\alpha'c_1(L)\pi}{vol(\Sigma)} - \frac{8}{\tau} - \frac{2\alpha'}{K^2\tau}) \]
\[ < N(-4\alpha' - \frac{\tau\alpha'}{2} - \frac{8}{\tau} - \frac{2\alpha'}{K^2\tau}) \]
\[ = \frac{N}{\tau(2 + 4\lambda^2\alpha)}(-16 - 32\lambda^2\alpha + \alpha(4\tau + \tau^2 + \frac{2}{K^2})) \]
\[ = \frac{N}{2\tau(2 + 4\lambda^2\alpha)}(-32 + 8\lambda\alpha + \tau^2\alpha + \frac{4\alpha}{K^2}) < 0. \]

In the second line, we used \(\lambda' = -2\lambda\) and 5.22. In the third line, we used 5.23. In the fourth line, we used \(\alpha' = -\frac{\sqrt{\lambda^2}\alpha}{2}\). In the fifth line, we used \(\frac{\tau}{8} < \lambda\). The last line holds whenever 
\[-32 + 8\lambda\alpha + \tau^2\alpha + \frac{4\alpha}{K^2} < 0. \]

Let \(\xi = (e^{-s|\phi|_h^2})\sqrt{-1}\nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^*\). Let us take
\[ w = \frac{\sqrt{-1}\nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^*}{f}. \]

Then
\[ \partial\bar{\partial}\xi 
= s^2e^{-s|\phi|_h^2}\partial|\phi|_h^2 \wedge \bar{\partial}|\phi|_h^2w - se^{-s|\phi|_h^2}\bar{\partial}\partial|\phi|_h^2w - se^{-s|\phi|_h^2}\partial w \wedge \bar{\partial}|\phi|_h^2 \]
\[ - se^{-s|\phi|_h^2}\bar{\partial}|\phi|_h^2 \wedge \partial w + e^{-s|\phi|_h^2}\bar{\partial}\partial w \]

Suppose \(\xi\) achieves its maximum value at a point \(p, q\). Then \(\sqrt{-1}\partial\bar{\partial}\xi(q) \leq 0\) and \(\partial\xi(q) = 0 = \bar{\partial}\xi(q)\).

We suppress the dependence on \(q\), now onwards. So
\[ - se^{-s|\phi|_h^2}\partial|\phi|_h^2w + e^{-s|\phi|_h^2}\partial w = 0 \]
\[ \implies \partial w = sw\partial|\phi|_h^2. \]

Similarly, we have-
\[ \bar{\partial}w = sw\bar{\partial}|\phi|_h^2. \]

Now
\[ 0 \geq \sqrt{-1}\partial\bar{\partial}\xi \]
\[ \implies 0 \geq -s^2we^{-s|\phi|_h^2}\sqrt{-1}\partial|\phi|_h^2 \wedge \bar{\partial}|\phi|_h^2 - swe^{-s|\phi|_h^2}\sqrt{-1}\partial\bar{\partial}|\phi|_h^2 + e^{-s|\phi|_h^2}\sqrt{-1}\partial\bar{\partial}w \]
\[ \implies 0 \geq -s^2w|\phi|_h^2f + sw|\phi|_h^2f\left(\frac{\tau - |\phi|_h^2}{2}\right)\left(\frac{4}{I} + \frac{\alpha w}{2(2\pi)^2I}ight) - sw^2f + \sqrt{-1}\partial\bar{\partial}w \]
\[ \implies 0 \geq f(-s^2|\phi|_h^2 + s|\phi|_h^2\left(\frac{\tau - |\phi|_h^2}{2}\right)\frac{\alpha}{2(2\pi)^2I} - s)w^2 + \frac{2s|\phi|_h^2f(\tau - |\phi|_h^2)}{I}w + \sqrt{-1}\partial\bar{\partial}w \]

24
In the second line, we used \(5.32\) and \(5.33\). In the third line, we used \(\partial |\phi|^2_h \wedge \bar{\partial} |\phi|^2_h = |\phi|^2_h \nabla^{1.0} \phi \wedge \nabla^{0.1} \phi^*\), \(\partial \bar{\partial} |\phi|^2_h = -\Theta_h |\phi|^2_h + \nabla^{1.0} \phi \wedge \nabla^{0.1} \phi^*\), \(5.30\) and \(5.1\), where
\[
I = 4 + \frac{\tau\alpha}{(2\pi)^2} (2\lambda - \frac{\tau}{\gamma}) + \frac{\tau\alpha}{2(2\pi)^2} |\phi|^2_h - \frac{\alpha}{4(2\pi)^2} |\phi|^4_h.
\]
Now to calculate \(\sqrt{-1} \partial \bar{\partial} w\), we need to choose good coordinates and trivialisations. We choose \((z, e)\) such that it is normal for \(f\) at the point \(q\) and normal for \(h\) at the point \(q\) with the properties that \(\frac{\partial h}{\partial z}(q) = 0 = \frac{\partial h}{\partial z^*}(q)\). Suppose, in this coordinate \(f = \tilde{f} dz \wedge d\bar{z}\). The following calculations are done at the point \(q\).

First, we calculate
\[
w = \sqrt{-1} \nabla^{1.0} \phi \wedge \nabla^{0.1} \phi^* = \frac{f}{\tilde{f} dz \wedge d\bar{z}} \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} = \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}}
\]
\[
A^{1.0} \text{ can be taken to be } 0 \text{ at the point } q. \text{ Now, using }5.35, \text{ we have}
\]
\[
\partial \bar{\partial} w = \frac{1}{f} \frac{\partial}{\partial z} \left( \frac{\partial \phi}{\partial z} \frac{\partial \phi^*}{\partial \bar{z}} \right) + \phi \bar{A}^{1.0} \frac{\partial \phi^*}{\partial z} + \phi A^{1.0} \bar{\phi} \frac{\partial \phi^*}{\partial \bar{z}} + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right)
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[
= \frac{1}{f} \left[ \bar{\partial} \left( \phi \frac{\partial \phi}{\partial \bar{z}} \right) + \bar{\partial} \left( \bar{\phi} \frac{\partial A^{1.0}}{\partial \bar{z}} \right) + \bar{\partial} \left( \phi A^{1.0} \frac{\partial \phi}{\partial \bar{z}} \right) + \frac{\partial \phi \partial \phi^*}{\partial z \partial \bar{z}} + A^{1.0} \phi \frac{\partial \phi^*}{\partial z} \bar{\partial} \bar{\partial} \left( \frac{1}{f} \right) \right]
\]
\[ \partial A^{1,0}(q) = 0 \because \frac{\partial^2 h}{\partial^2 z} = 0. \] Now substituting 5.36 in 5.34 we get

\[ 0 \geq f(-s^2|\phi_h|^2 + s|\phi_h^2|(\tau - |\phi_h^2|) - s)w^2 + \frac{2s|\phi_h^2|f(\tau - |\phi_h^2|)w}{I} \]

\[ + \sqrt{-1} \frac{\partial}{\partial z}(\frac{\partial}{\partial z})(\frac{\partial}{\partial \bar{z}})(\frac{\partial}{\partial z})d\bar{z} + \bar{f}w^2 \]

\[ + f|\phi_h^2|\bar{f}w\left\{\left(\frac{1}{2I}\right)^2(\tau - |\phi_h^2|)\right\} \]

\[ \implies 0 \geq \sqrt{-1} \frac{\partial}{\partial z}(\frac{\partial}{\partial z})(\frac{\partial}{\partial \bar{z}})(\frac{\partial}{\partial z})d\bar{z} + \bar{f}w^2 \]

\[ + f(-s^2|\phi_h|^2 + s|\phi_h^2|(\tau - |\phi_h^2|) - s)w^2 + \frac{2s|\phi_h^2|f(\tau - |\phi_h^2|)w}{I} \]

\[ + \frac{\partial}{\partial z}(\frac{\partial}{\partial z})(\frac{\partial}{\partial \bar{z}})(\frac{\partial}{\partial z})d\bar{z} + \bar{f}w^2 \]

\[ \implies 0 \geq \sqrt{-1} \frac{\partial}{\partial z}(\frac{\partial}{\partial z})(\frac{\partial}{\partial \bar{z}})(\frac{\partial}{\partial z})d\bar{z} + \bar{f}w^2 \]

\[ + f\left(2s|\phi_h^2|(\tau - |\phi_h^2|) \right) I \]

\[ + \bar{f}\left(2(\tau - |\phi_h^2|) \right) I \]

\[ - \frac{s}{2} + f(\tau - |\phi_h^2|) \frac{\alpha s}{2(2\pi)^2} + \bar{f}|\phi_h^2|^2 \frac{I + (\tau - |\phi_h^2|)\alpha}{2(2\pi)^2} \]

\[ (5.37) \]

We can ignore the term \( \frac{\partial}{\partial z}(\frac{\partial}{\partial z})(\frac{\partial}{\partial \bar{z}})(\frac{\partial}{\partial z})d\bar{z} + \bar{f}w^2 \) in the above maximum principle, whenever Ricci curvature of \( f \) is positive. We can choose \( s \) small negative number and \( \alpha \) small enough such that the coefficient of \( w^2 \) is positive. This implies that \( \xi \) is bounded above and the bound does not depend on \( f \). So we have

\[ \sqrt{-1}\nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^* < C_1 \implies \frac{4}{\omega_Y} \frac{\sqrt{-1}\nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^*}{\omega_Y} < 1, \]

\[ (5.38) \]

where \( C_1 \) is a constant and \( V = 4 + \frac{\tau_0}{2(2\pi)^2}(2\lambda - \frac{\tau}{2}) + \frac{\tau_2^2}{2(2\pi)^2} \). Now if we choose \( K = \frac{2}{\sqrt{\omega_Y}} \), then we have

\[ \omega_Y = K^2 \frac{\sqrt{-1}\nabla^{1,0}\phi \wedge \nabla^{0,1}\phi^*}{\omega_Y} \]

\[ (5.39) \]

Using 5.27, 5.28, 5.29, 5.30, we see that 5.26 is negative whenever \( \tau < \frac{3}{4} \) and \( \alpha \) is small enough such that the coefficient of \( w^2 \) is positive in 5.37 and \( \tau < 8 \lambda + \tau^2 \alpha + \frac{4}{3} \beta < 0 \) for \( K = \frac{2}{\sqrt{\omega_Y}} \).

Using 5.26 is negative whenever \( \tau < \frac{3}{4} \) and \( \alpha \) is small enough such that the coefficient of \( w^2 \) is positive in 5.37 and \( \tau < 8 \lambda + \tau^2 \alpha + \frac{4}{3} \beta < 0 \) for \( K = \frac{2}{\sqrt{\omega_Y}} \).

This implies that \( -\sqrt{-1}(2\pi)^3\Omega^\lambda(a_E \oplus a_L, a_E \oplus a_L) \) is positive whenever the \( \tau, \alpha \) satisfies the above conditions and Ricci curvature of \( f \) is positive.

References

[1] L. Alvarez-Consul, M. Garcia-Fernandez, and O. Garcia-Prada. “Coupled equations for Kähler metrics and Yang-Mills connections.” Geom. Top. 17, 2731-2812 (2013).

[2] L. Alvarez-Consul, M. Garcia-Fernandez, and O. Garcia-Prada. “Gravitating vortices, cosmic strings, and the Kähler-Yang-Mills equations.” Comm. Math. Phys. 351 (2017), 361-385.
[3] L. Alvarez-Consul, M. Garcia-Fernandez, O. Garcia-Prada, and V. Pingali. “Gravitating vortices and the Einstein-Bogomolnyi equations”. Math. Ann. (2020), https://doi.org/10.1007/s00208-020-01964-z

[4] O. García-Prada. “Invariant connections and vortices.” Commun.Math. Phys., 156 (1993) 527-546.

[5] Leung, Naichung Conan. “Einstein type metrics and stability on vector bundles.” J. Differential Geom. 45 (1997), no. 3, 514–546.

[6] Takahashi, Ryosuke. “J-equation on holomorphic vector bundles.” 2021. arXiv:2112.00550

[7] V. Pingali. “Representability of Chern-Weil forms.” Math. Zeit., 288 (1-2) (2018) 629-641.

[8] V. Pingali. “A vector bundle version of the Monge-Ampère Equation.” Adv. Math. 360, 106921 (2020).

[9] V. Pingali. “Quillen metrics and perturbed equations.” Lett. Math. Phys. 110 (202), 1861-1875.

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