Relations among topological solitons

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Abstract

We clarify relations among topological solitons in various dimensions: a domain wall, non-Abelian vortex, magnetic monopole and Yang-Mills instanton, together with a (non-Abelian) sine-Gordon soliton, baby Skyrmion (lump), and Skyrmion. We construct a composite configuration consisting of a domain wall, vortex, magnetic monopole and Yang-Mills instanton (wall-vortex-monopole-instanton) using the effective theory technique or moduli approximation. Removing some solitons from such a composite, we obtain all possible composite solitons in the form of solitons within a soliton, including all the previously known configurations, yielding relations among topological solitons.
I. INTRODUCTION

Topological solitons and instantons are ubiquitous in nature: they appear and play significant roles in quantum field theories [1–7], supersymmetric field theories [8–10], cosmology [11–13], QCD [14], and various condensed matter systems [15] such as helium-3 superfluids [16], superfluids [17], Josephson junctions in superconductors [18], nonlinear media [19], and Bose-Einstein condensates (BECs) of ultracold atomic gasses [20]. Formations and dynamics of topological solitons have been discussed in various contexts. Among them, topological defects are ubiquitously formed during phase transitions (the Kibble-Zurek mechanism) [11, 21], and thus have been studied extensively in cosmology and condensed matter physics. Dynamics of topological solitons are mostly studied numerically. When exact or approximate soliton solutions are available, their low-energy dynamics can be analytically described by the moduli approximation [22] (for solitons with supersymmetry, see Ref. [23]).

Topological solitons can be classified into topological defects, topological textures, and topological gauge structures (instantons). Topological defects are formed due to the aforementioned Kibble-Zurek mechanism when spontaneous symmetry breaking (SSB) \( G \rightarrow H \) of gauge or global symmetry \( G \) occurs. Nontrivial homotopy groups \( \pi_{n-1}(G/H) \) of the order parameter manifold (OPM) \( G/H \) admit the existence of topological defects of codimension \( n \) (particle-like in an \( n \) dimensional space \( \mathbb{R}^d \)), which can be surrounded by \( S^{n-1} \) at the boundary of \( \mathbb{R}^n \). One of the remarkable features of topological defects is the fact that the unbroken symmetry \( H \) is partially or fully recovered around their cores. On the other hand, topological textures are classified by a homotopy group \( \pi_n(G/H) \) as a map from the whole \( n \)-dimensional space \( \mathbb{R}^n \) to the OPM \( G/H \) with one point compactification of the base space \( \mathbb{R}^n \) to \( S^n \), which can be justified by the assumption that the boundary is mapped to the same point. Gauge structures imply nontrivial pure gauge field configurations without SSB classified by a homotopy group \( \pi_{n-1}(G) \) of the gauge group \( G \) on an \( n \)-dimensional space \( \mathbb{R}^n \). Taking into account their dimensionality up to four space-time dimensions, there are following eight topological solitons, as summarized in Table I:

Yang-Mills instantons are solutions to self-dual Yang-Mills equations [24] which are of codimension four (point-like in the Euclidean space \( \mathbb{R}^4 \)) and can be particle-like in \( d = 5 + 1 \) dimensions. The (framed) moduli space of a single \( SU(N) \) instanton is \( \mathbb{R}^4 \times \mathbb{R}^+ \times \frac{SU(N)}{SU(N-2)\times U(1)} \). For multi-instanton configurations and their moduli space, the Atiyah-
| Topological solitons          | Type    | Codim | SSB    | Homotopy | Example                         | Single soliton moduli |
|-------------------------------|---------|-------|--------|----------|---------------------------------|-----------------------|
| Yang-Mills instantons        | Gauge   | 4     | Non    | $\pi_3(G)$ | $\pi_3[SU(N)] \simeq \mathbb{Z}$ | $\mathbb{R}^4 \times \mathbb{R}^+ \times \frac{SU(N)}{SU(N-2)\times U(1)}$ |
| Magnetic monopoles            | Defect  | 3     | Gauge  | $\pi_2(G/H)$ | $\pi_2 \left[ \frac{SU(N)}{U(1)^{N-1}} \right] \simeq \mathbb{Z}^{N-1}$ | $\mathbb{R}^3 \times U(1)$ |
| Vortices                      | Defect  | 2     | Gauge  | $\pi_1(G/H)$ | $\pi_1[U(N)] \simeq \mathbb{Z}$ | $\mathbb{R}^2 \times \mathbb{C}P^{N-1}$ |
| Kinks, Domain walls           | Defect  | 1     | Global | $\pi_0(G/H)$ | $\pi_0(\mathbb{Z}_N) \simeq \mathbb{Z}_N$ | $\mathbb{R} \times U(N)$ |
| Skyrmions                     | Texture | 3     | Global | $\pi_3(G/H)$ | $\pi_3[SU(N)] \simeq \mathbb{Z}$ | $\mathbb{R}^3 \times \frac{SU(N)}{SU(N-2)\times U(1)}$ |
| Baby Skyrmions(Lumps)         | Texture | 2     | Global | $\pi_2(G/H)$ | $\pi_2[\mathbb{C}P^{N-1}] \simeq \mathbb{Z}$ | $\mathbb{R}^2 \times U(1) \left( \mathbb{R}^2 \times \mathbb{C}^* \right)$ |
| Sine-Gordon solitons          | Texture | 1     | Global | $\pi_1(G/H)$ | $\pi_1[U(N)] \simeq \mathbb{Z}$ | $\mathbb{R} \times \mathbb{C}P^{N-1}$ |
| Hopfions                      | Texture | 3     | Global | $\pi_3(S^2)$ | $\pi_3(S^2) \simeq \mathbb{Z}$ | $\mathbb{R}^3 \times U(1)$ |

TABLE I. “Type” implies the types of topological solitons classified into topological gauge structures, defects, or textures, and “Codim” implies codimensions on which soliton configurations depend. Topological gauge structures imply topological solitons in pure gauge theories in an $n$-dimensional space $\mathbb{R}^n$, classified by a homotopy group $\pi_{n-1}(G)$ of a map from the boundary $S^{n-1}$ of $\mathbb{R}^n$ to a gauge group $G$. Topological defects arise in SSB $G \to H$ of either gauge or global symmetry $G$, classified by a homotopy group $\pi_{n-1}(G/H)$ of a map from the boundary $S^{n-1}$ to the OPM $G/H$. Topological textures are classified by a homotopy group $\pi_n(G/H)$ of a map from the whole $n$-dimensional space $\mathbb{R}^n$ to the OPM $G/H$ with one point compactification of the base space $\mathbb{R}^n$ to $S^n$ (justified by the assumption that the boundary is mapped to the same point). “Example” gives typical examples, and “Single soliton moduli” denotes moduli space of a single soliton in such examples.

Drinfeld-Hitchin-Manin (ADHM) construction is available [25]. Instantons have important applications to determine non-perturbative effects in quantum field theories in four spacetime dimensions, in particular, with supersymmetry [26].

’t Hooft-Polyakov magnetic monopoles are particle-like topological defects in $d = 3 + 1$ [27] (see Refs. [3, 28] as a review) and instanton-like in $d = 2 + 1$. The simplest SSB $SU(2) \to U(1)$ admits a nontrivial second homotopy group $\pi_2[SU(2)/U(1)] \simeq \mathbb{Z}$, supporting monopoles. This can be generalized to an $SU(N)$ gauge group maximally broken by the adjoint Higgs field as $SU(N) \to U(1)^{N-1}$, with $\pi_2[SU(N)/U(1)^{N-1}] \simeq \mathbb{Z}^{N-1}$. In the Bogomol’nyi-Prasad-Sommerfield (BPS) limit, analytic solutions are available [29], for which the Nahm construction offers solutions of multiple BPS monopoles and their moduli space...
The moduli spaces of two monopoles and well-separated multiple monopoles are available. There are several alternative methods such as the Donaldson’s rational map. When the $SU(N)$ gauge symmetry is not maximally broken by the adjoint Higgs field, there remains a partially unbroken non-Abelian gauge symmetry in the vacuum. In this case, monopoles are non-Abelian monopoles. In high-energy phenomenology and cosmology, grand unified theories (GUTs) always admit GUT monopoles (the monopole problem) because of $\pi_2[G/(SU(3) \times SU(2) \times U(1))] \simeq \mathbb{Z}$. Vortices or cosmic strings are topological defects of codimension two, which are string-like in $d = 3 + 1$, particle-like in $d = 2 + 1$, and instanton-like in $d = 1 + 1$ (see Refs. [2, 13] as a review). The simplest case is Abrikosov-Nielsen-Olesen (ANO) vortices (magnetic flux tubes) supported by $\pi_1[U(1)] \simeq \mathbb{Z}$ when a $U(1)$ gauge symmetry is spontaneously broken relevant for superconductors. Global analogues are global vortices having logarithmically divergent tension in infinite space, examples of which are given by axion strings in cosmology, and superfluid vortices in superfluids or atomic BECs in condensed matter physics. In an SSB of both local and global symmetries, semilocal strings are known. Some time ago, non-Abelian vortices containing non-Abelian magnetic fluxes inside them were discovered, and since then they have been extensively studied in various contexts (see Refs. [8–10] as a review). The moduli space of a single $U(N)$ vortex is $\mathbb{C} \times \mathbb{C}P^{N-1}$, and thus the low-energy effective world-sheet theory is a two-dimensional $\mathbb{C}P^{N-1}$ model, explaining similarities between four dimensional gauge theories and two-dimensional sigma models. Multiple vortex solutions and their moduli space are available by the moduli matrix method and half-ADHM construction. The moduli space metric can be calculated for well-separated vortices yielding the low-energy dynamics, while coincident vortices can be described group theoretically. These vortices can be generalized to non-Abelian semilocal strings and those of arbitrary gauge groups. In high energy phenomenology, similar non-Abelian vortices are present in high density QCD (color superconductors) (see Ref. [14] as a review, and Refs. [47] for recent developments) relevant for neutron star cores and in two-Higgs doublet model (2HDM). On the other hand, electroweak Z-strings in the Standard Model also have non-Abelian magnetic fluxes but they are nontopological and unstable.

Finally, the simplest topological defects are domain walls or kinks of codimension one (see Refs. [2, 4, 7] as a review). While the $\phi^4$ kink is the simplest example, one of the more
interesting examples is the $\mathbb{C}P^1$ domain wall (or those in a $U(1)$ gauge theory with two complex Higgs scalar fields) [50]. These are relevant for ferromagnets and supersymmetric theories. These can be generalized to $\mathbb{C}P^{N-1}$ domain walls (or those in a $U(1)$ gauge theory with $N$ complex Higgs scalar fields) [51]. The moduli space of a single kink is $\mathbb{R} \times U(1)$ if masses are nondegenerate (for degenerate masses [52]). Further generalizations can be found as Grassmann domain walls (or a $U(N)$ gauge theory) [53, 54], those with generic $U(1)$ charges [55], and those in nonlinear sigma models with the other target spaces [56]. Among all the cases, a particularly important case in this paper is given by a non-Abelian domain wall [52, 57, 58] that carries non-Abelian moduli $\mathbb{R} \times U(N)$, and thus its low-energy effective theory is a $U(N)$ chiral Lagrangian (principal chiral model), or the Skyrme model if a four derivative term is taken into account [58].

Apart from instantons and topological defects, another class of topological solitons is given by topological textures such as Skyrmions. Skyrmions [59] are topological textures of the codimension three characterized by $\pi_3(SU(2)) \simeq \mathbb{Z}$ in the chiral Lagrangian of pions with a four derivative (Skyrme) term, which were proposed to describe baryons [60] (see Refs. [2, 7] as a review). The moduli space of a single $SU(N)$ Skyrmion is $\mathbb{R}^3 \times \frac{SU(N)}{SU(N-2) \times U(1)}$. While analytic solutions are not available, there are some proposals to give approximate configurations, such as the Atiyah-Manton construction based on Yang-Mills instantons [61] and rational map ansatz [62, 63].

Baby Skyrmions or planar Skyrmions are $2 + 1$ dimensional analogues of Skyrmions characterized by the second homotopy group $\pi_2(S^2) \simeq \mathbb{Z}$, typically present in a $O(3)$ sigma model with a potential term and a four derivative term [64] (see Refs. [2, 7] as a review). They are string-like in $d = 3+1$, particle-like in $d = 2+1$ and instanton-like in $d = 1+1$. The case without a four-derivative term was known earlier as lumps or sigma model instantons [65]. The moduli space of a single lump (baby Skyrmion) is $\mathbb{C} \times \mathbb{C}^* (\mathbb{C} \times U(1))$. In condensed matter physics, these solitons are simply called Skyrmions and are relevant in various systems such as quantum Hall effects [66] and ferromagnets. In particular there are great interests in chiral magnets because of experimental finding of a Skyrmion lattice in chiral magnets [67] (see Refs. [68] for analytic studies of Skyrmions and their lattice in chiral magnets). The same Skyrmions can be present also in chiral liquid crystals [69]. The moduli space of a single Skyrmion is only $\mathbb{C}$.

Finally, the lowest dimensional analogues of Skyrmions are sine-Gordon solitons [70] (and
double sine-Gordon solitons [71]), characterized by the first homotopy group \( \pi_1[U(1)] \simeq \mathbb{Z} \) (see Refs. [2, 7] as a review). These solitons are of codimension one: planar in \( d = 3 + 1 \), string-like in \( d = 2 + 1 \), and particle-like in \( d = 1 + 1 \). They are similar to domain walls but are distinct from them in the sense that the vacua far from solitons are identical as the other Skyrmions, in contrast to domain walls separating distinct vacua. In cosmology, a sine-Gordon soliton is attached to an axion string (if the domain wall number is one) [37]. Double sine-Gordon solitons also appear in various context such as 2HDM in high energy physics [48]. In condensed matter physics, (double) sine-Gordon solitons are present in a Josephson junction (an insulator sandwiched by two superconductors) [18, 72]. A lattice of (double) sine-Gordon solitons is the ground state in chiral magnets (in a certain parameter region), which is called a chiral soliton lattice [73] (see Ref. [74] as a review). Recently, in high energy physics, such chiral soliton lattices are found to be realized as the ground states in QCD under rapid rotation [75] or in the strong magnetic field [76]. Among various extensions of sine-Gordon solitons, particularly important ones in this study are non-Abelian sine-Gordon solitons supported by \( \pi_1[U(N)] \simeq \mathbb{Z} \) [77, 78]. In QCD, a single non-Abelian sine-Gordon soliton can be bounded by global non-Abelian strings [14, 79] and is stretched between two chiral non-Abelian strings [80]. The moduli space of a single \( U(N) \) sine-Gordon soliton is \( \mathbb{R} \times \mathbb{C}P^{N-1} \), and thus the low energy theory is the \( \mathbb{C}P^{N-1} \) model, as the case of non-Abelian vortices. Such solitons can exist in a non-Abelian extension of a Josephson junction (two color superconductors separated by an insulator or a domain wall) [81, 82]. More interestingly, a chiral soliton lattice of non-Abelian sine-Gordon solitons is found to be the ground state of QCD under rapid rotation in a wide range of the parameter region [83] (instead of Abelian chiral soliton lattices [75]).

The last topological textures are Hopfions characterized by a Hopf map \( \pi_3(S^2) \simeq \mathbb{Z} \) [84] (see Refs. [7, 85] as a review). Hopfions are present in an \( O(3) \) sigma model with a four derivative term, called the Faddeev-Skyrme model. Typically these solitons are string-like and are linked [86]. In condensed matter physics, Hopfions were suggested in \(^3\)He superfluids [87] and superconductors [88], and have been experimentally confirmed in a spinor BEC (but are unstable without a four derivative term) [89], liquid crystals [90], and magnetic materials [91].

As we have seen, various topological solitons appear in diverse subjects. It is, however, usually the case that each soliton is studied individually, and a unified understanding is
yet to be clarified. The purpose of this paper is to present connections among all kinds of topological solitons with a help of composite solitons in the form of “solitons within a soliton,” where a daughter soliton is trapped inside a mother soliton, as schematically shown in Fig. 1.\textsuperscript{1} Thus, the mother soliton should have larger world-volume dimensions (less codimensions) than the daughter soliton (see Appendix A for the exceptional cases in which mother and daughter solitons have the same dimensions where a single daughter soliton must be split into fractional solitons). In general, the mother soliton possesses moduli parameters or collective coordinates, that is, a set of solutions with the same energy contains free parameters. Then, the low-energy effective world-volume theories of the mother solitons can be constructed by the moduli approximation [22] (with supersymmetry for BPS solitons [23]), where the moduli are promoted to fields on the world-volume. It is usually a nonlinear sigma model whose target space is the moduli space.

Let the codimension of the daughter soliton be $n$ in the bulk so that the daughter soliton is surrounded by $S^{n-1}$ as A in Fig. 1. When it is absorbed into the mother soliton C, it can be expressed by a soliton (at B) in the effective world-volume theory of the mother soliton

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{soliton_diagram}
\caption{A relation of topological solitons. When a daughter “A” of codimensions $n$, surrounded by $S^{n-1}$ in the bulk, is absorbed into a mother “C” it becomes “B”. The effective theory of the moduli fields of C admits a topological soliton B surrounded by $S^{m-1}$, which has the same topological charge with A in the bulk. Thus, A and B can be identified.}
\end{figure}

\textsuperscript{1} There are also other types of composite solitons: solitons ending on a soliton or soliton junctions, such as a non-Abelian vortex ending on a monopole [92], electroweak $Z$-string ending on a Nambu monopole [49], axion domain wall(s) ending on an axion string [37], and axial (chiral) domain walls(s) ending on an axial vortex [14, 79, 80]. In addition, vortices ending on a domain wall or stretched between domain walls, and domain wall junctions, which can be BPS, are known. We do not consider these in this paper.
In this case, the daughter soliton inside the mother soliton is surrounded by $S^{m-1}$, which is the intersection of $S^{n-1}$ at B and the world-volume of the mother soliton, where $m$ can be expressed by $n$ and the codimension $c$ of the mother soliton as $m = n - c$. The topological charge of the daughter soliton is characterized by $\pi_{n-1}$ in the bulk if it is a defect or gauge structure, or by $\pi_n$ if it is a texture. It should be unchanged at A and B. In the world-volume theory of the mother soliton, the topological charge is characterized by $\pi_{m-1}$ if it is a defect, or by $\pi_m$ if it is a texture.

All previously known examples are listed in the next section as a review. One of typical examples is given by confined monopoles (the second in a summary in the next section): In the Higgs phase, magnetic fluxes emanating from the monopole are squeezed into flux tubes. If they are squeezed into two flux tubes, stable configuration of a monopole attached by two vortices from the both sides. In a certain situation, those vortices are non-Abelian vortices. These two vortices can be regarded as a single non-Abelian vortex with non-Abelian magnetic fluxes directed in two opposite directions. In this case, the monopoles can be represented by a kink in the effective world-sheet theory of a single non-Abelian vortex, which is the $\mathbb{CP}^{N-1}$ model [98]. This vortex monopole configuration has important applications to the correspondence quantum effects in the two dimensional $\mathbb{CP}^{N-1}$ model and dimensional gauge theory [99].

In this paper, as a configuration including all the composite solitons in the form of solitons within a soliton summarized in the next section, we construct a configuration consisting of four different solitons: a wall-vortex-monopole-instanton schematically drawn in Fig. 2, which is the most general and unique configuration. Relations among topological solitons deduced from this configuration are summarized in Table II. Further hidden relations which can be obtained from Table II are summarized in Tables III and IV (derivations are explained in subsequent sections). We conjecture all possible relations among topological solitons can be obtained from this configuration. In our setup, Yang-Mills instantons, magnetic monopoles, non-Abelian vortices, and non-Abelian domain walls are primary solitons — those who can live in the bulk. The others, Skyrmions, baby Skyrmions (lumps), (non-Abelian) sine-Gordon solitons, $\mathbb{CP}^{N-1}$ lumps, $\mathbb{CP}^{N-1}$ domain walls, and $U(1)^{N-1}$ global vortices are all secondary solitons — those who can live only as daughter solitons inside a

On the other hand, a defect in the effective theory of the mother soliton represents a defect ending on the mother defect. Examples are given by D-brane solitons [53, 93–95] (see also [96]). Higher dimensional generalizations can be found in Ref. [97]. We do not consider these cases in this paper.
FIG. 2. A schematic picture of the configuration of the wall-vortex-monopole-instanton in $d = 4 + 1$. The codimensional direction $x^1$ of the wall is not shown. The black box, blue sheet, green rod, and red circle denote the non-Abelian domain wall, non-Abelian vortex, magnetic monopole, and Yang-Mills instanton, respectively. YM, NA, SG and w.v. denote Yang-Mills, non-Abelian, sine-Gordon and world-volume, respectively.

mother soliton. All relations among the secondary solitons can be deduced from Table II as discussed in subsequent sections. Throughout the paper, we use the term “Yang-Mills instantons” as codimensions four-objects in four Euclidean space in $d = 4 + 1$ space-time.

This paper is organized as follows. In Sec. II, we summarize all known composite solitons in the form of solitons within a soliton. In Sec. III, we present our model, $U(N)$ gauge theory coupled with Higgs scalar fields. In Sec. IV, we construct the wall-vortex-monopole-instanton configuration in the effective field theory technique. In Sec. V, we obtain various secondary relations deduced from the configuration. Section VI is devoted to a summary and discussion. In Appendices, we summarize solitons within a soliton which we do not use in this paper. In Appendix A, we summarize composite solitons in which mother and daughter solitons have the same dimensions. In Appendix B, we summarize composite solitons in compactified spaces $\mathbb{R}^{d-1} \times S^1$ with twisted boundary conditions.
### Daughters

| Mothers                          | Moduli          | U(N) NA vortex | \(SU(N)\) monopole | SU(N) YM instanton |
|----------------------------------|-----------------|----------------|---------------------|-------------------|
| U(N) NA wall                     | U(N)            | U(N) NA SG soliton | U\(^{(1)}\)^{N-1} global vortex | SU(N) Skyrmion |
| U(N) NA vortex                   | \(\mathbb{C}P^{N-1}\) | –             | \(\mathbb{C}P^{N-1}\) wall | \(\mathbb{C}P^{N-1}\) lump |
| \(\frac{SU(N)}{U(1)^{N-1}}\) monopole | U(1)        | –             | –                    | SG soliton        |

**TABLE II.** Relations among all kinds of solitons. Mother solitons can host daughter solitons inside them. The rightmost column: An \(SU(N)\) Yang-Mills instanton becomes an \(SU(N)\) Skyrmion, \(\mathbb{C}P^{N-1}\) lump, and \((U(1)^{N-1}\) coupled) sine-Gordon soliton inside a non-Abelian domain wall, non-Abelian vortex and monopole, respectively. The middle column: An \(SU(N)\) monopole becomes \(U(1)^{N-1}\) global vortex and \(\mathbb{C}P^{N-1}\) kink inside a non-Abelian domain wall and non-Abelian vortex, respectively. The leftmost column: A non-Abelian vortex becomes a non-Abelian sine-Gordon soliton inside a non-Abelian vortex.

### Daughters

| Mothers                          | Moduli          | \(U(1)^{N-1}\) global vortex | SU(N) Skyrmion |
|----------------------------------|-----------------|------------------------------|---------------|
| U(N) NA SG soliton               | \(\mathbb{C}P^{N-1}\) | \(\mathbb{C}P^{N-1}\) wall | \(\mathbb{C}P^{N-1}\) lump |
| \(U(1)^{N-1}\) global vortex     | U(1)            | –                            | SG soliton    |

**TABLE III.** Hidden relations (1) among solitons deduced from Table II.

### Daughter

| Mother                          | Moduli          | \(\mathbb{C}P^{N-1}\) lump (baby skyrmion) |
|---------------------------------|-----------------|--------------------------------------------|
| \(\mathbb{C}P^{N-1}\) wall     | U(1)            | SG soliton                                 |

**TABLE IV.** Hidden relation (2) among solitons deduced from Table II.
II. SOLITONS WITHIN A SOLITON: A REVIEW

There are already several known such composite solitons in the form of solitons within a soliton in the literature, deducing relations among topological solitons. In this section, let us summarize all known composite solitons in the form of solitons within a soliton. In the first subsection, we summarize composite solitons consisting of two different solitons, which we call composite solitons of 2-generations. Then, in the second subsection, we summarize composite solitons of 3-generations.

A. 2-generations (mother and daughter)

We use a rule for names of composite solitons as follows – Mother daughter.

1. Domain-wall baby-Skyrmions (lumps), or domain-wall vortices:

The simplest example of composite solitons is a domain-wall baby-Skyrmion.

(a) $\mathbb{C}P^1$ lumps (baby Skyrmions, or semilocal vortices) are sine-Gordon solitons inside a $\mathbb{C}P^1$ wall [100] (see also [101]). Theoretically, this setup provides a physical proof for the lower dimensional analogue of the Atiyah-Manton construction [102]. Physically, this configuration is realized in two condensed matter systems. The first is “domain-wall Skyrmions” in chiral magnets [103] (see also [104]). (In the condensed matter community, baby Skyrmions are simply called Skyrmions.) See also Ref. [105] for domain-wall instantons in $d = 1 + 1$ chiral magnets. The second is Josephson vortices (also called fluxons) in a Josephson junction [18, 72]. A Josephson junction can be regarded as a heavy tension limit of a domain wall in a $U(1)$ gauge theory coupled with two complex scalar fields [100]. The domain wall separates two vacua with only one of the complex scalar fields developing a VEV, and thus the wall is an insulator region of the Josephson junction. Then, vortices in the bulk are absorbed into the wall to become Josephson vortices (fluxons), whose dynamics can be described as sine-Gordon solitons in the domain wall effective theory.

(b) $\mathbb{C}P^{N-1}$ lumps are $U(1)^{N-1}$ coupled sine-Gordon solitons inside $\mathbb{C}P^{N-1}$ walls [106].
The aforementioned Josephson junction of two superconductors can be generalized to multi-layered $N$ parallel Josephson junctions. This case can be described by $N - 1$ parallel domain walls in the $\mathbb{C}P^{N-1}$ model [106]. A single Josephson vortex (fluxson) can be described in each wall, while the interaction between Josephson vortices in different domain walls can be also studied [106].

(c) Non-Abelian vortices are non-Abelian sine-Gordon solitons inside a non-Abelian domain wall [81]. A non-Abelian domain wall [52, 57, 58] can describe a non-Abelian Josephson junction (of two color superconductors) once a Josephson term is introduced. The effective theory of the non-Abelian domain wall is then a $U(N)$ chiral Lagrangian with a pion mass or non-Abelian sine-Gordon model [81, 82]. Then, non-Abelian vortices [39] absorbed into the domain wall become non-Abelian Josephson vortices described as a non-Abelian sine-Gordon soliton [77, 78].

Similar (but different) configurations that cannot be regarded as solitons within a soliton can be found.  

2. Vortex monopoles:

The second simplest example of composite solitons is a vortex monopole.

(a) Monopoles are $\mathbb{C}P^{N-1}$ kinks inside a non-Abelian vortex (confined monopoles) [98]. This vortex monopole configuration (as well as vortex instanton configuration given below) has important applications to the correspondence of quantum effects in the two dimensional $\mathbb{C}P^{N-1}$ model and four dimensional gauge theory [99].

(b) Non-Abelian monopoles on multiple non-Abelian vortices can be also realized [112]. Mathematically, this provides a physical realization of the Donaldson’s rational map [33].

(c) This correspondence can be generalized to $SO, USp$ groups [113].

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3 Similar configurations can be found as axion domain wall(s) ending on an axion string [37], axial (chiral) domain wall(s) ending on an axial (chiral) vortex in QCD [14, 79, 107], and wall-vortex composites in GUT [108], 2HDMs [48], the Gerogi-Machacek model [109], and supersymmetric theories [110]. These cases are not solitons within a soliton except for some special cases in which a vortex is attached by two domain walls of the same tension. The wall decays quantum mechanically [111].
These can be obtained from vortex instantons given below by the Scherk-Schwarz dimensional reduction \cite{114} or an $S^1$ compactification with a twisted boundary condition \cite{115}. There are similar (but different) configurations that cannot be regarded as soliton within a soliton.\footnote{As similar configurations, there are vortices ending on monopoles in QCD \cite{116}, GUTs \cite{117}, supersymmetric QCD \cite{92,117}, and high density QCD \cite{118}. A Nambu monopole in the Standard Model \cite{38,49,119} is attached by a single $Z$-string, similar to the above. On the other hand, a Nambu monopole in 2HDMs \cite{120} is attached by two non-Abelian $Z$-strings of the same tension, similar to vortex monopoles in the topic of this paper. However, it cannot be regarded as a kink on a single $Z$-string since magnetic fluxes spread out spherically from the monopole. See further Refs. \cite{121} for similar objects. These cases are not solitons within a soliton except for some special cases in which a monopole is attached by vortex strings of the same tension.}

3. Domain-wall monopoles:

Monopoles are $U(1)^{N-1}$ global vortices inside a non-Abelian domain wall \cite{82}. We call them Josephson monopoles. These can be obtained from domain-wall instantons given below by the Scherk-Schwarz dimensional reduction \cite{114} or an $S^1$ compactification with a twisted boundary condition.

4. Vortex instantons:

Yang-Mills instantons are $CP^{N-1}$ lumps inside a non-Abelian vortex \cite{99,115,122}. Fractional instantons and calorons are transformed to vortex monopoles with the twisted boundary condition \cite{115}. This composite soliton can be applied for correspondence between the vortex counting and instanton counting \cite{123}.

5. Domain-wall instantons:

Yang-Mills instantons are Skyrmions inside a non-Abelian domain wall \cite{58} (see also \cite{82,124}). We also call these instantons Josephson instantons. Originally, the term “domain-wall Skyrmion” was used for this configuration, but in this paper we call it a “domain-wall instanton” because of the rule of names (mother-daughter). This configuration provides a proof of the Atiyah-Manton construction of Skyrmions \cite{61}. The situation is also similar to holographic QCD \cite{125}.

6. Vortex Skyrmions:

Skyrmions are sine-Gordon solitons inside a global vortex in a Skyrme model with a twisted mass term \cite{126}. Similar configurations can be found in a BEC-Skyrme model (Skyrme model with a BEC motivated potential term) \cite{127}.
7. **Domain-wall Skyrmions and sine-Gordon Skyrmions**:

The final examples are domain-wall Skyrmions and sine-Gordon Skyrmions.

(a) Skyrmions are baby Skyrmions (lumps) inside a “non-Abelian” $S^2$ domain wall \cite{128} (see also Refs. \cite{129}). This configuration can be generalized to higher dimensions (called matryoshka Skyrmions) \cite{130}. Similar configurations can be found in a BEC-Skyrme model \cite{131}. This configuration provides a translational (or Donaldson’s type) rational map ansatz of $SU(2)$ Skyrmions \cite{62}. Physically, domain wall Skyrmions were discussed in a chiral soliton lattice in QCD \cite{132}.

(b) Skyrmions are $\mathbb{C}P^{N-1}$ lumps inside a non-Abelian sine-Gordon soliton \cite{78}. This configuration provides a translational (or Donaldson’s type) rational map ansatz of $SU(N)$ Skyrmions \cite{63}.

8. **Lump-string Hopfions**:

Hopfions are sine-Gordon solitons inside a $\mathbb{C}P^1$ lump-string. These are all known examples of composite solitons consisting of two generations. Some cases are BPS states preserving a fraction of supersymmetries if one embeds the theories to supersymmetric theories \cite{133}.

B. **3-generations (grandmother, mother and daughter)**

In addition, there are a few examples of three generations, that is, composite solitons of three different dimensions. We use a rule for names of composite solitons as grandmother-mother-daughter.

1. **Vortex-monopole-instanton**:

Yang-Mills instantons inside a monopole inside a non-Abelian vortex \cite{134} (see also Refs. \cite{124, 135}). Instantons are sine-Gordon solitons inside the monopole and lumps inside the vortex at the same time, while the monopole is a kink inside the vortex.

2. **Wall-vortex-instanton**:

Yang-Mills instantons inside a vortex inside a non-Abelian domain wall \cite{82}. Instantons are lumps inside a vortex and Skyrmions inside the domain wall at the same time, while the vortex is a non-Abelian sine-Gordon soliton inside the domain wall.
3. Wall-monopole-instanton:

Yang-Mills instantons inside a monopole inside a non-Abelian domain wall [82]. Instantons are sine-Gordon solitons inside a monopole and Skyrmions inside the domain wall at the same time, while the monopole is a kink inside the vortex. This configuration can be obtained as the Scherk-Schwarz dimensional reduction [114] of the wall-vortex-instanton.

As already mentioned, in the following sections, we construct a configuration of four generations as a configuration including all the aforementioned composite solitons in the form of solitons within a soliton of two or three generations: a wall-vortex-monopole-instanton schematically drawn in Fig. 2.

III. THE MODEL: NON-ABELIAN GAUGE THEORY IN THE HIGGS PHASE

The theory that we consider is a $U(N)$ gauge theory coupled with Higgs scalar fields in the Higgs phase in $d = 4 + 1$ dimensions with the following matter contents: a $U(N)$ gauge field $A_\mu(x)$, two $N$ by $N$ charged complex scalar fields $H(x) = (H_1(x), H_2(x))$, and a real neutral adjoint $N$ by $N$ scalar field $\Sigma(x)$. The $U(N)$ gauge (color) symmetry acts on fields as

$$A_\mu \rightarrow gA_\mu g^{-1} + ig\partial_\mu g^{-1}, \quad H \rightarrow gH, \quad \Sigma \rightarrow g\Sigma g^{-1}, \quad g \in U(N)_C. \quad (1)$$

The Lagrangian is given as follows:

$$\mathcal{L} = -\frac{1}{4g^2}\text{tr} F_{\mu\nu}F^{\mu\nu} + \frac{1}{g^2}\text{tr} (D_\mu\Sigma)^2 + \text{tr}|D_\mu H|^2 + \mathcal{L}_J - V \quad (2)$$

where $V$ is the potential term

$$V = \frac{g^2}{4}\text{tr}(HH^\dagger - v^21_N)^2 + \tr|\Sigma H - HM|^2, \quad (3)$$

and $D_\mu$ is the covariant derivative, given by $D_\mu H = \partial_\mu H - iA_\mu H$ and $D_\mu \Sigma = \partial_\mu \Sigma - i[A_\mu, \Sigma]$, $g$ is the gauge coupling constant that we take common for the $U(1)$ and $SU(N)$ factors of $U(N)$, $v$ is a real constant representing the vacuum expectation value of $H$, and $M$ is a $2N$ by $2N$ mass matrix for $H$ given below. Apart from the Josephson term $\mathcal{L}_J$, the model is a truncation of the bosonic part of $\mathcal{N} = 2$ supersymmetric theory (with eight supercharges) in $d = 4 + 1$ [9].
In the massless case $M = 0$, the flavor symmetry is the maximum $SU(2N)$. This is explicitly broken by the mass matrix $M$ that we take

$$M = \text{diag}(m_{1N} + \Delta M, -m_{1N} - \Delta M), \quad \Delta M = \text{diag}(m_1, m_2, \cdots, m_N)$$

(4)

with a real mass $m$ and real mass shifts $m_a$ much smaller than $m$: $m_a \ll m$. For $m \neq 0$ with $\Delta M = 0$, the flavor symmetry is $SU(N)_L \times SU(N)_R \times U(1)_{L-R}$, given by

$$H_1 \rightarrow H_1 U_L e^{i\alpha}, \quad H_2 \rightarrow H_2 U_R e^{-i\alpha}, \quad U_{L,R} \in SU(N)_{L,R}, \quad e^{i\alpha} \in U(1)_{L-R},$$

(5)

while for $\Delta M \neq 0$ with non-degenerate mass perturbation $m_a \neq m_b$ for $a \neq b$, the flavor symmetry is further explicitly broken to $U(1)^{N-1}_L \times U(1)^{N-1}_R \times U(1)_{L-R}$. In this paper, we consider the non-degenerate masses. Without loss of generality, we can assume $m_r > m_{r+1}$.

In the Lagrangian in Eq. (2), $\mathcal{L}_J$ consists of scalar couplings that we call the Josephson interactions

$$\mathcal{L}_J = \mathcal{L}_{J,1} + \mathcal{L}_{J,2},$$

(6)

$$\mathcal{L}_{J,1} = -\gamma \text{tr}(H_1^\dagger H_2 + H_2^\dagger H_1)$$

(7)

$$\mathcal{L}_{J,2} = -\sum_{r=1}^{N-1} \frac{\beta_r^2}{v^2} [\text{tr}(H_1 X_r H_2^\dagger) + \text{tr}(H_2 X_r H_1^\dagger)]$$

(8)

where $X_r = X_r^\dagger$ ($r = 1, \cdots, N-1$) are elements of $SU(N)$ algebra into which $\sigma_1$ is embedded as a diagonal submatrix. Here, $\mathcal{L}_{J,1}$ gives a Josephson interaction of two color superconductors separated by a domain wall, given below.

The vacuum structures of the model are as follows. In the massless case $m = 0$, $\Delta M = 0$, $\gamma = 0$, $\beta_r = 0$, the vacuum can be taken without the lost of generality as

$$H = (v 1_N, 0_N), \quad \Sigma = 0_N$$

(9)

by using the $SU(2N)$ flavor symmetry. The unbroken symmetry is $SU(N)_{C+L} \times SU(N)_{R} \times U(1)$, in which the factor $SU(N)_{C+L}$ is the color-flavor locked (global) symmetry. The moduli space of vacua is the complex Grassmann manifold $[136]$

$$Gr_{2N,N} \simeq \frac{SU(2N)}{SU(N) \times SU(N) \times U(1)}.$$  

(10)

In the massive case, $m \neq 0$ but still $\Delta M = 0$, the above vacua are split into the following two disjoint vacua

$$H = (v 1_N, 0_N), \quad \Sigma = +m 1_N : SU(N)_{C+L},$$

$$H = (0_N, v 1_N), \quad \Sigma = -m 1_N : SU(N)_{C+R}$$

(11)
with the unbroken color-flavor locked (global) symmetries \( g = U_L \) and \( g = U_R \), respectively. These vacua are color-flavor locked vacua that can be interpreted as non-Abelian color superconductors. With the non-degenerate mass deformation \( \Delta M \neq 0 \), each vacuum in Eq. (11) is shifted to

\[
H = (v'1_N, 0_N), \quad \Sigma = +m1_N + \Delta M : U(1)^{N-1}_{C+L}, \\
H = (0_N, v'1_N), \quad \Sigma = -m1_N - \Delta M : U(1)^{N-1}_{C+R},
\]

(12)

where \( v' \) is shifted from \( v \).

In the following sections, we often work in the strong coupling (nonlinear sigma model) limit \( g \rightarrow \infty \) for explicit calculations. In this limit, we have the constraints

\[
HH^\dagger = v^21_N, \quad \Sigma = v^{-2}MH^\dagger, \quad A_\mu = \frac{i}{2}v^{-2}[H\partial_\mu H^\dagger - (\partial_\mu H)H^\dagger],
\]

(13)

and the model is reduced to the Grassmann sigma model with the target space given in Eq. (10) together with a potential term, known as the massive (twisted-mass deformed) Grassmann sigma model [137]. In this limit, vortices reduce to Grassmann sigma model lumps.

In this paper, we consider the following hierarchical symmetry breakings:

\[
m \gg \gamma \gg \Delta m_r \gg \beta_r
\]

wall vortex monopole instanton

(14)

with \( \Delta m_r \equiv m_r - m_{r+1} \) \( (r = 1, 2, \ldots, N - 1) \). The second line denotes topological solitons that form when turning on the corresponding parameters. In the following, we turn on these parameters from the left to right gradually.

IV. WALL-VORTEX-MONOPOLE-INSTANTON

In this section, we construct the wall-vortex-monopole-instanton configuration in the moduli approximation, by gradually turning of hierarchical parameters in Eq. (14). In each subsection we construct the wall, vortex, monopole, and instanton with turning on \( m, \gamma, \Delta m_r, \) and \( \beta_r \) in Eq. (14), respectively.
A. The first hierarchy: non-Abelian domain wall

We first turn on the mass \( m \) in the hierarchy in Eq. (14). In the sigma model limit, a non-Abelian domain wall solution interpolating between the two vacua in Eq. (11) perpendicular to the coordinate \( x^1 \) can be given by \([52, 53, 57, 58]\)

\[
H = H_{\text{wall}}(x^1) = \frac{v}{\sqrt{1 + e^{\mp 2m(x^1 - X^1)}}} \left( 1_N, e^{\mp m(x^1 - X^1)} U \right),
\]

(15)

with \( \Sigma \) and \( A_1 \) in Eq. (13). The width of the wall is \( m^{-1} \). Here, \( X^1 \) is the position (translational modulus) of the domain wall in the coordinate \( x^1 \) and \( U \) contains group-valued moduli \( U \in U(N) \):

\[
(X^1, U) \in \mathcal{M}_{\text{wall}} \simeq \mathbb{R} \times U(N).
\]

(16)

The effective theory of the non-Abelian domain wall can be constructed by using the moduli approximation \([22, 23]\); First, we promote the moduli parameters \( X^1 \) and \( U \) to moduli fields \( X^1(x^i) \) and \( U(x^i) \), respectively \((i = 0, 2, 3, 4)\) on the world-volume of the domain wall, and then perform integration over the codimension \( x^1 \). We thus obtain the effective theory given by \([52, 57, 58]\):

\[
L_{\text{wall}} = \int dx^1 \mathcal{L}(H = H_{\text{wall}}(x^1; X^1(x^i), U(x^i)))
= \frac{v^2}{2m} \partial_i X^1 \partial^i X^1 - f^2 \pi \text{tr} (U^\dagger \partial_i U U^\dagger \partial^i U), \quad f^2 \equiv \frac{v^2}{4m},
\]

(17)

which is a \( U(N) \) chiral Lagrangian, or principal chiral model. If we calculate the next leading order of the derivative expansion of the effective theory, we would obtain the Skyrme term \([58]\).

B. The second hierarchy: non-Abelian vortex trapped inside the wall

Now we turn on the Josephson interaction \( \gamma \) as the second largest parameter in the hierarchy in Eq. (14), so that the domain wall becomes the Josephson junction. Let us evaluate the effect of \( \gamma \) perturbatively in the domain wall effective theory in Eq. (17) provided that \( \gamma \) is small enough not to affect the domain wall solution at the leading order. We thus find the pion mass term \([81]\)

\[
\Delta L_{\text{wall}, J} = \int dx^1 \mathcal{L}_{J,1}(H = H_{\text{wall}}(x^1; X^1(x^i), U(x^i)))
= -m^2(\text{tr} U + \text{tr} U^\dagger), \quad m^2 \equiv \frac{\pi \gamma}{2m}.
\]

(18)
This potential term lifts the $U(N)$ vacuum manifold, leaving the unique vacuum $U = 1_N$ as the case of the usual chiral Lagrangian.

When the non-Abelian vortex is placed parallel to the non-Abelian Josephson junction (domain wall), it is absorbed into the junction to minimize the total energy. The resulting configuration can be described as a non-Abelian sine-Gordon soliton in the $U(N)$ chiral Lagrangian in Eq. (17) with the mass term in Eq. (18). A non-Abelian sine-Gordon soliton (perpendicular to the $x^2$ coordinate) is given by [77, 78]:

$$U = U_{\text{vortex}}(x^2) = V \text{diag}(u(x^2), 1, \cdots, 1)V^\dagger = 1_N + (u - 1)\phi\phi^\dagger$$

with $m''^2 = \frac{m'^2}{f^2} = \frac{2\pi\gamma}{v^2}$. The width of the soliton is $m''^{-1} \sim v/\sqrt{\gamma}$, and the tension of the soliton is $T_{SG} = 8m''$. Here $X^2$ is translational modulus in the coordinate $x^2$, and $V$ are orientational moduli taking a value in $\mathbb{C}P^{N-1} \simeq SU(N)_{\mathbb{C}} \times U(1)_{\mathbb{C}}$. Therefore, the moduli of the non-Abelian sine-Gordon soliton are

$$\mathcal{M}_{\text{vortex}} \simeq \mathbb{R} \times \mathbb{C}P^{N-1}$$

that coincide with the moduli of the non-Abelian vortex in the bulk, except for one translation modulus fixed to be the position $X^1$ of the wall. It was also shown in Ref. [81] from the flux matching that this is precisely a non-Abelian vortex.

The effective theory of the sine-Gordon soliton with the world-volume $x^\alpha$ ($\alpha = 0, 3, 4, 5$) can be also obtained by the moduli approximation [78]:

$$\mathcal{L}_{\text{vortex}} = \int dx^2 \mathcal{L}_{\text{wall}}(U = U_{\text{vortex}}(x^2; X^\alpha, \phi(x^\alpha)))$$

$$= C_X \partial_\alpha X^2 \partial^\alpha X^2 + C_\phi \left[ \partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\phi^\dagger \partial^\alpha \phi) \right]$$

with the constants $C_X = \frac{f^2 T_{SG}}{2} = \sqrt{2\pi\gamma}v^2$ and $C_\phi = \frac{f^2 T_{SG}}{m''^2} = \sqrt{\frac{2}{\pi}\frac{v^4}{\gamma m^2}}$. This is the $\mathbb{C}P^{N-1}$ model written in terms of the homogeneous coordinates $\phi$. The four derivative correction to the vortex effective theory was obtained for a non-Abelian vortex in the bulk [138], but we do not need it in our study.

C. The third hierarchy: monopole trapped inside the vortex inside the wall

Now let us turn on $\Delta m_r$ (in $\Delta M$) in the hierarchy in Eq. (14). First, $\Delta M$ induces a twisted mass in the $U(N)$ chiral Lagrangian as the domain wall effective theory in Eq. (17).
This potential can be obtained by the Scherk-Schwarz dimensional reduction \[82\]:

\[
V_{\text{wall}} = \frac{v^2}{4m} \text{tr} \left( [\Delta M, U]^\dagger [\Delta M, U] \right),
\]

(22)

implying that diagonal \(U\) has lower energy. The sine-Gordon soliton (vortex in the bulk) in Eq. (19) should be either of \(N\) diagonal embedding.

In fact, the vortex effective theory in Eq. (21) is also modified by the twisted mass \(\Delta M\) as \[82\]:

\[
V_{\text{vortex}} = C_\phi \left[ (\phi^\dagger \Delta M \phi)^2 - \phi^\dagger (\Delta M)^2 \phi \right].
\]

(23)

The Lagrangian in Eq. (21) with this potential term is known as the massive \(\mathbb{C}P^{N-1}\) model. For non-degenerate mass deformation \(\Delta M\), this potential admits \(N\) discrete vacua

\[
\phi^T_a = v(0, \ldots, 0, 1, 0, \ldots), \quad a = 1, \ldots, N
\]

(24)

where only the \(a\)-component is nonzero. These correspond to embedding of \(u\) into diagonal elements in the solution \(U\) in Eq. (19).

For later use, it is often enough to consider a \(\mathbb{C}P^1\) submanifold. The vortex effective theory of the \(r\)-th \(\mathbb{C}P^1\) submanifold parametrized by a homogeneous coordinate \(u\) in \(\phi = \frac{1}{\sqrt{1+|u|^2}}(0, \ldots, 0, 1, u, 0 \ldots)\), where only \(r\)-th and \((r+1)\)-th components are nonzero, can be written as

\[
\mathcal{L}_{\text{vortex,CP}^1} = C_X \partial_a X^2 \partial^a X^2 + C_\phi \left[ \partial_a u^* \partial^a u - \delta m_r^2 |u|^2 \right].
\]

(25)

with \(\delta m_r \equiv m_{r+1} - m_r\). The vacua in the vortex theory, \(u = 0\) (the north pole) and \(u = \infty\) (south pole) of the target space \(\mathbb{C}P^1\), correspond to embedding of \(u\) to the upper-left and lower-right elements of the non-Abelian sine-Gordon solution \(U\) in Eq. (19), respectively.

The Lagrangian (23) admits \(N - 1\) multi-kink solutions \[51, 53\], where the constituent kink connecting the \(r\)-th and \((r+1)\)-th vacua has the mass \(E_{\text{kink},r}\) \((r = 1, \ldots, N - 1)\) which is proportional to the mass of a monopole \(E_{\text{monopole},r}\) \[98, 99, 112\]:

\[
E_{\text{kink},r} = C_\phi \delta m_r, \quad E_{\text{monopole},r} = 4\pi g^2 \delta m_r.
\]

(26)

When the vortex goes to the bulk outside the wall where the vortex becomes BPS, the Kähler moduli \(C_\phi\) in the vortex effective theory in Eq. (23) is replaced by \(\frac{4\pi g}{g^2}\). Accordingly, the energy of kinks on the vortex becomes \(E_{\text{kink},r} = E_{\text{monopole},r} = \frac{4\pi g}{g^2} \delta m_r\). Then, one can
confirm the kinks on the vortex represent the monopoles, and from the BPS properties, they carry correct monopole charges. When the vortex gets back into the wall, the energy becomes the first in Eq. (26) different from the monopole energy in the bulk [the second in Eq. (26)], but still carrying the same monopole charges.

A single monopole solution can be constructed by restricting ourselves to the \( r \)-th \( \mathbb{CP}^1 \) submanifold in Eq. (25). It is a domain wall interpolating the two vacua \( u = 0 \) and \( u = \infty \) [50] in the vortex effective theory (25). We place it in the \( x^3 \)-coordinate as

\[
u = \nu_{\text{monopole}}(x^3) = e^{\mp \delta m_r (x^3 - X^3)} + i \varphi, \tag{27}\]

where \( \mp \) represents a monopole and an anti-monopole with the width \( 1/\delta m_r \). Here, \( X^3 \) and \( \varphi \) are moduli parameters representing the position in the \( x^3 \)-coordinate and \( U(1) \) phase of the (anti-)monopole. The moduli of the monopole is then

\[ \mathcal{M}_{\text{monopole}} \simeq \mathbb{R} \times U(1) \tag{28} \]

coinciding with the monopole moduli except for two translational moduli fixed to the positions \( X^1 \) and \( X^2 \) of the domain wall and the vortex.

Let us construct the effective theory of the single monopole-string by promoting the moduli \( X^3 \) and \( \varphi \) to fields \( X^3(x^m) \) and \( \varphi(x^m) \) \((m = 0, 4)\) [22, 23] on the monopole string [134]:

\[
\mathcal{L}_{r-\text{th monopole}} = \int dx^3 \mathcal{L}_{\text{vortex,CP}^1}(u = \nu_{\text{monopole}}(x^3; X^3(x^m), \varphi(x^m))) \\
= \frac{C_{\phi}}{2 \delta m_r} [(\partial_m X^3)^2 + (\partial_m \varphi)^2] \tag{29}\]

which is a free theory, a sigma model with the target space \( \mathbb{R} \times U(1) \).

D. The fourth hierarchy: instanton inside the monopole inside the vortex inside the wall

Finally, we turn on the smallest parameters \( \beta_r \) in the hierarchy in Eq. (14). The term proportional to the parameter \( \beta_r \) perturbatively induces the following deformation term to the effective Lagrangian on the non-Abelian domain wall in Eq. (17):

\[
\Delta \mathcal{L}_{\text{wall,} \beta} = \int dx^4 \mathcal{L}_{J,2}(H = H_{\text{wall}}(x^1; X^i(x^i), U(x^i))) \\
= \sum_r \frac{\pi \beta_r^2}{2 m^p} \text{tr} [X_r(U + U^\dagger)]. \tag{30}\]

Next, this term induces the following deformation term to the effective Lagrangian of the non-Abelian vortex:

\[ \Delta L_{\text{vortex},\beta} = \int dx^2 \Delta L_{\text{wall},\beta}(U = U_{\text{vortex}}(x^2, X^2(x^a), \phi(x^a))) \]

\[ = - \sum_r \frac{4\pi \beta_r^2}{m^2} (\phi^r X_r \phi). \]  

This is known as a momentum map (or a D-term in supersymmetric theories). It reduces in the \( r \)-th \( \mathbb{C}P^1 \) submanifold to

\[ \Delta L_{\text{vortex},\beta} = - \frac{4\pi \beta_r^2}{m^2} \frac{1}{1 + |u|^2} \left( \begin{array}{c} 1 \\ u^* \end{array} \right) \sigma_1 \left( \begin{array}{c} 1 \\ u \end{array} \right) = - \frac{4\pi \beta_r^2}{m^2} \frac{u + u^*}{1 + |u|^2}. \]  

Finally, this deformation induces the deformation term in the monopole effective action. By considering a single monopole-string, the effective Lagrangian is given by

\[ \Delta L_{\text{r-th monopole},\beta} = \int dx^3 \Delta L_{\text{vortex},\beta}(u = u_{\text{monopole}}(x^3, X^3(x^m), \varphi(x^m))) \]

\[ = - C_r \cos \varphi \equiv - V_{\text{monopole}} \]

\[ C_r \equiv \frac{\pi^2 \beta_r^2}{2\delta m_r m^2} = \frac{\pi \beta_r^2 v^2}{4\delta m_r \gamma}. \]  

(33)

We thus arrive at the monopole effective Lagrangian summarized as

\[ \mathcal{L}_{\text{r-th monopole},\beta} = \frac{C_\phi}{2\delta m_r} [(\partial_m X^3)^2 + (\partial_m \varphi)^2 - D_r \cos \varphi], \]

\[ D_r \equiv \frac{2\delta m_r C_r}{C_\phi} = \frac{\pi^{3/2} m \beta_r^2}{2\sqrt{2} v \sqrt{\gamma}}. \]  

(34)

This is nothing but the sine-Gordon model.

We can consider multi-monopoles (multi-walls in the \( \mathbb{C}P^{N-1} \) model). In that case, we will obtain a \( U(1)^{N-1} \) coupled sine-Gordon model \[106\] in which the interaction of sine-Gordon solitons on different domain walls was studied. In our purpose, a single monopole is enough.

The monopole effective action in Eq. (34) admits sine-Gordon solitons. As the simplest, a single soliton solution is given by

\[ \varphi = \varphi_{\text{instanton}}(x^4) = 4 \arctan \exp \sqrt{\frac{D_r}{2}} (x^4 - X^4) + \pi. \]  

(35)

The width of the soliton is \( \Delta x^4 \sim 1/\sqrt{D_r} \sim \sqrt{\gamma}^{1/4}/\sqrt{m \beta_r} \). This carries a lump charge in the \( \mathbb{C}P^1 \) model as the vortex effective action \[100\] :

\[ T_{\text{lump}} = \int d^2 x \frac{i(\partial_i u^* \partial_j u - \partial_j u^* \partial_i u)}{(1 + |u|^2)^2} = 2\pi k \]  

(36)
with $k = 1 \in \pi_2(\mathbb{C}P^1)$, which is also the topological charge $\pi_2(\mathbb{C}P^{N-1})$ of the whole $\mathbb{C}P^{N-1}$. The lump charge in the vortex induces the instanton charge in the bulk, and thus lumps in the vortex effective action correspond to Yang-Mills instantons in the bulk \cite{99, 115}.

Combining all discussions together, we finally reach the configuration in Fig. 2.

V. RELATIONS AMONG ALL TOPOLOGICAL SOLITONS

A. Subconfigurations

Once we obtain the composite state of the wall-vortex-monopole-instanton, we can obtain various configurations consisting of a fewer number of solitons by removing some solitons from it.

Taking limits to send various parameters to zero, we obtain subconfigurations as schematically shown in Fig. 3. We start from (c) wall-vortex-monopole-instanton. We first remove one soliton from (c). By taking the limit $m \to 0$, $\gamma \to 0$, or $\delta m \to 0$, we can remove the domain wall, vortex, or monopole to reach (f) the vortex-monopole-instanton, (b) wall-monopole-instanton, and (d) wall-vortex-instanton, respectively. Next, we remove two solitons from (c). By taking the limit $\delta m, \gamma \to 0$, $m, \delta m \to 0$, or $m, \gamma \to 0$, we reach (a) the wall-instanton, (g) vortex-instanton, or (e) monopole-instanton, respectively. Only (e) is unstable as it is, but it can be stable if we make a closed loop \cite{124}. We also can remove instantons from each configuration in the limit $\beta_a \to 0$.

B. Hidden relations

All possible relations among topological solitons (domain walls, vortices, monopoles and Yang-Mills instantons) obtained thus far are summarized in Table II. Mother solitons can host solitons with less world-volume dimensions or equivalently of more codimensions as daughters. The first row of the table shows that a $U(N)$ domain wall can host a $U(N)$ non-Abelian vortex, $SU(N)/U(1)^{N-1}$ monopole and $SU(N)$ instanton, and in the domain wall effective theory, which is the $U(N)$ chiral Lagrangian, these solitons are realized as a $U(N)$ non-Abelian sine-Gordon soliton, $U(1)^{N-1}$ global vortex and $SU(N)$ Skyrmion, respectively. In the second row, a vortex can host a monopole and instanton as a $\mathbb{C}P^{N-1}$ kink and $\mathbb{C}P^{N-1}$ lump, respectively in the vortex world-volume theory, that is the $\mathbb{C}P^{N-1}$ model. Finally in
FIG. 3. Removing constituent solitons. Starting from (c) the wall-vortex-monopole-instanton, we can remove the domain wall, vortex, or monopole in the limits $\gamma \to 0$, $\delta m \to 0$, respectively to reach (f) the vortex-monopole-instanton, (b) wall-monopole-instanton or (d) wall-vortex-instanton, respectively. We can remove two solitons in the limits $\delta m, \gamma \to 0$, $m, \delta m \to 0$ or $m, \gamma \to 0$ to reach (a) the wall-instanton, (g) vortex-instanton or (e) monopole-instanton, respectively. We also can remove instantons from each configuration in the limit $\beta_a \to 0$. 
FIG. 4. Hidden relations. (a) Within a domain wall, an $SU(N)$ Skyrmion at A turns to a $\mathbb{C}P^{N-1}$ lump at B inside a $U(N)$ non-Abelian sine-Gordon soliton (that is a non-Abelian vortex in the bulk point of view), and to a sine-Gordon soliton at C in a $U(1)^{N-1}$ global vortex (that is a monopole in the bulk point of view). Within a non-Abelian vortex, a $\mathbb{C}P^{N-1}$ lump at B becomes a sine-Gordon soliton at C inside a $\mathbb{C}P^{N-1}$ kink. (b) A $U(1)^{N-1}$ global vortex (that is a monopole in the bulk point of view) becomes a $\mathbb{C}P^{N-1}$ kink inside a $U(N)$ non-Abelian sine-Gordon soliton.

The third row, a magnetic monopole-string can host an $SU(N)$ Yang-Mills instanton as a sine-Gordon soliton in the monopole world-volume effective theory which is the sine-Gordon model.

Table II contains further hidden relations among topological solitons.

If one places an instanton inside a domain wall as A in Fig. 4(a), it is a Skyrmion in the domain wall effective theory as denoted above. Then, if this Skyrmion moves into B on a non-Abelian vortex, it becomes a $\mathbb{C}P^{N-1}$ lump in the vortex effective theory. This relation implies that a $U(N)$ non-Abelian sine-Gordon in the $U(N)$ chiral Lagrangian can host an $SU(N)$ Skyrmion as a $\mathbb{C}P^{N-1}$ lump in its world-volume effective theory which is the $\mathbb{C}P^{N-1}$ model [78]. This corresponds to the first row and second column in Table III.

If the Skyrmion at A moves to C on a monopole Fig. 4(a) directly without passing through a vortex, it becomes a sine-Gordon soliton. This implies that a $U(1)^{N-1}$ global vortex in the $U(N)$ chiral Lagrangian can host an $SU(N)$ Skyrmion as a sine-Gordon soliton in its world-volume effective theory which is a sine-Gordon model. This corresponds to the second row and second column in Table III. The case of $N = 2$ was constructed in Ref. [126]. This relation for general $N$ is not studied yet.
We also can consider a monopole as a daughter soliton instead of the instanton. As in Fig. 4(b), if a monopole inside a domain wall is trapped into a non-Abelian vortex inside the domain wall as indicated by the arrow in the figure, it becomes a $\mathbb{C}P^{N-1}$ kink. This implies that a $U(N)$ non-Abelian sine-Gordon in the $U(N)$ chiral Lagrangian can host a $U(1)^{N-1}$ global vortex (which is a monopole in the bulk point of view) as a $\mathbb{C}P^{N-1}$ kink in its world-volume effective theory which is the $\mathbb{C}P^{N-1}$ model. This corresponds to the first row and first column in Table III. This relation is not studied yet.

Finally, going back to Fig. 4(a), if the instanton in the bulk (that is a $\mathbb{C}P^{N-1}$ lump (or baby Skyrmion) inside the non-Abelian vortex) at B moves to C on a monopole in the bulk (that is a $\mathbb{C}P^{N-1}$ kink in the non-Abelian vortex), it becomes a sine-Gordon soliton, see Table IV. This relation implies that a $\mathbb{C}P^{N-1}$ domain wall in the massive $\mathbb{C}P^{N-1}$ model can host a $\mathbb{C}P^{N-1}$ lump as a sine-Gordon soliton [100, 106]. In the literature, this relation is known as a domain wall (baby) Skyrmion in field theory, attracting a lot of attention recently in condensed matter physics because of experimental confirmations in chiral magnets [103] (see also [104]).

VI. SUMMARY AND DISCUSSION

By using the effective field theory technique or the moduli approximation, we have constructed the wall-vortex-monopole-instanton configuration in Fig. 2 as the most general composite soliton in the form of solitons within a soliton. It reduces to all the previously known composite solitons, providing relations among various topological solitons in various dimensions summarized in Tables II, III and IV. We have conjectured that this exhausts all possible relations among topological solitons.

We have considered non-degenerate masses $m_a \neq m_b$ if $a \neq b$, giving rise to Abelian domain walls in the $\mathbb{C}P^{N-1}$ model on a non-Abelian vortex as well as ’t Hooft-Polyakov monopoles. If we consider partially degenerate masses, the $\mathbb{C}P^{N-1}$ domain walls [52] and monopoles on the vortex [112] are non-Abelian. In this case, instantons as sine-Gordon solitons on the monopole-string may be non-Abelian sine-Gordon solitons.

Among all major topological solitons in Table I, we have not considered Hopfions. Hopfions inside a non-Abelian vortex was studied before [135], but it is unclear to what it corresponds in the bulk.
As one of the most important applications, one can consider quantum effects, or non-perturbative effects in different dimensions. Instanton corrections to monopoles correspond to vortex corrections to kinks through instanton vortex configurations. This gives a correspondence between a four dimensional $U(N)$ gauge theory and a two dimensional $\mathbb{C}P^{N-1}$ model. Hopefully, this relation can be generalized to other mother-daughter relations completed in this paper.

Electrically charged solitons or dyonic extensions can be also studied. As for dyonic extensions of individual solitons, dyonic instantons [139], dyons [140], dyonic vortices [141], Q-lumps [142] and Q-kinks [50] are known. Dyonic extensions of composite solitons are not well studied except for few examples: domain wall networks and strings stretched between walls [133, 143].

We have not studied fermions in this paper. When fermions are coupled to the system, topological solitons are often accompanied by fermion zero modes, such as on a kink [144] and a vortex [145]. The existence of such fermions is ensured by the index theorem. For instance, the index theorem and fermion zero modes were studied for non-Abelian vortices in supersymmetric $U(N)$ gauge theory [39] and in dense QCD [146]. However, they have not been well understood for composite solitons such as solitons within a soliton, and thus remain as one of future problems.

Probably, a more elegant framework to systematically understand composite solitons is offered by higher-form symmetry and higher group [147]. In fact, mathematical structures of the axion electrodynamics admitting composite solitons were recently clarified in terms of higher-form symmetries and higher groups [148] (see also Refs. [149]). Mathematical structures behind the configuration in the present study are yet to be clarified.

Finally, we hope that our most general configuration itself has a direct application in condensed matter physics. See Refs. [150] for composite solitons in helium-3 superfluids.

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Appendix A: Mother and daughter of the same dimensions

As composite solitons in the form of solitons within a soliton, mother and daughter solitons can have the same dimensions in the following cases. In these cases, a single daughter soliton must be split into a set of fractional solitons.

1. Domain-wall sine-Gordon solitons:

The double sine-Gordon model admits a false (metastable) vacuum in addition to the true vacuum in a certain parameter region [71] (see also Refs. [48, 73]). In this case, a single soliton profile has double peaks which can be identified with a molecule of two constituents. Each constituent is an (unstable) domain wall or anti-domain wall separating the true and false vacua, carrying half sine-Gordon soliton winding. They are linearly confined by the false vacuum energy present between them. In the triple sine-Gordon model, a single sine-Gordon soliton is split into three constituents with 1/3 sine-Gordon topological charges [14].

2. Vortex baby-Skyrmions (or vortex lump):

In the baby Skyrme model (\(O(3)\) or \(\mathbb{C}P^1\) model) with the easy plane potential, a single baby Skyrmion is split into a pair of global vortex and anti-global vortex with half baby-Skyrmion topological charges, constituting a half baby-Skyrmion molecule [151]. Each half baby-Skyrmion (lump) can be interpreted as an Ising spin inside a global (anti-)vortex. Half baby-Skyrmions (lumps) are sometimes called merons. For a \(\mathbb{C}P^{N-1}\) model with a generalization of the easy plane potential, a single baby Skyrmion is split into a set of \(N U(1)^{N-1}\) global-vortex Skyrmions with 1/N lump (\(\pi_2\)) topological charges [152].

A \(U(1)\) gauged \(\mathbb{C}P^1\) model (without a Skyrme term) or \(U(1)^{N-1}\) gauged \(\mathbb{C}P^{N-1}\) model are also known, for which vortices are local, and configurations are BPS, and thus can be embedded into a supersymmetric theory [153]. This can be applied to the stabilization of semilocal strings [154]. See Ref. [155] for the case with a Skyrme term.

3. Monopole Skyrmions:

In a Skyrme model with the potential term admitting \(S^2\) vacua, a single \(SU(2)\) Skyrmion is split into a pair of a global monopole and an anti-global monopole with
half-Skyrmion topological charges, constituting a half Skyrmion molecule \[156\]. Each half Skyrmion can be interpreted as an Ising spin inside a global (anti-)monopole. An \(SU(2)\) gauged version is also known, for which monopoles are local (‘t Hooft-Polyakov type) \[157\]. It is also called a Skyrmmed monopole.

**Appendix B: Solitons on compactified spaces with twisted boundary conditions**

Topological solitons on compactified spaces \(\mathbb{R}^{d-1} \times S^1\) with twisted boundary conditions along \(S^1\) have half topological charges together with topological charges in \(\mathbb{R}^{d-1}\). These solitons are also composite solitons in the form of solitons within a soliton.

1. **Monopole instantons:**

   Yang-Mills instantons on \(\mathbb{R}^3 \times S^1\) are called calorons. The situation with twisted boundary condition is known as calorons with a nontrivial holonomy along \(S^1\) \[158\]. A single \(SU(2)\) instanton can be decomposed into a pair of a monopole string and an anti-monopole string winding along \(S^1\) with half instanton charges. A bion is a pair of a monopole-string with \(+1/2\) instanton charge and an anti-monopole string with \(-1/2\) instanton charge, and has been considered for applications to confinement and mass gap \[159\]. Similarly, an \(SU(N)\) instanton is decomposed into \(N\) monopole strings with \(1/N\) instanton charges.

2. **Vortex Skyrmions:**

   Skyrmions on \(\mathbb{R}^2 \times S^1\) with twisted boundary conditions are considered with \[160\] and without \[161\] the Skyrme term. A single \(SU(2)\) Skyrmion can be decomposed into a pair of a global vortex string and an anti-global vortex string winding along \(S^1\) with half Skyrme (\(\pi_3\)) charges. A bion is a pair of a global vortex-string with \(+1/2\) Skyrme charge and an anti-global vortex string with \(-1/2\) Skyrme charge \[161\]. Likewise, a single \(SU(N)\) Skyrmion is decomposed into \(N\) \(U(1)^{N-1}\) global-vortex Skyrmions with \(1/N\) Skyrme charges \[161\]. Applications of bions in this case are yet to be clarified. This relation can be embedded inside a non-Abelian domain wall, reproducing the aforementioned monopole instantons \[82\].

3. **Domain-wall lumps:**
(a) $\mathbb{CP}^{N-1}$ lumps on $\mathbb{R}^1 \times S^1$ with twisted boundary conditions were first found in Ref. [115] (see also Refs. [162]). See Ref. [163] for baby Skyrmions for the case with the Skyrme term. A single $\mathbb{CP}^1$ lump can be decomposed into a pair of a domain wall and an anti-domain wall winding along $S^1$ with half lump ($\pi_2$) charges. Similarly a single $\mathbb{CP}^{N-1}$ lump can be decomposed into a set of $N$ domain walls with $1/N$ lump ($\pi_2$) charges. A bion is a pair of a domain wall with $+1/2$ lump charge and an anti-domain wall with $-1/2$ lump charge, and similar for the $\mathbb{CP}^{N-1}$ model [164], which has been extensively applied to resurgence theory of the $\mathbb{CP}^{N-1}$ model. This relation can be embedded inside a non-Abelian vortex reproducing the aforementioned monopole instantons [115]. Recently, Monte Carlo simulations have been performed [165] in which fractional lumps and bions have been observed.

(b) Grassmann lumps on $\mathbb{R}^1 \times S^1$ with twisted boundary conditions were first found in Refs. [166]. Bions and application to the resurgence theory can be found in Refs. [167].

4. **Lump-string Hopfions**: Hopfions on $\mathbb{R}^2 \times S^1$ were studied in Ref. [168].

[1] R. Rajaraman, *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland Personal Library, 1987).

[2] N. S. Manton and P. Sutcliffe, *Topological solitons*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2004).

[3] Y. M. Shnir, *Magnetic Monopoles*, Text and Monographs in Physics (Springer, Berlin/Heidelberg, 2005).

[4] T. Vachaspati, *Kinks and domain walls: An introduction to classical and quantum solitons* (Cambridge University Press, 2010).

[5] M. Dunajski, *Solitons, instantons, and twistors*, Oxford Graduate Texts In Mathematics (Oxford University Press, U.S.A., 2010).

[6] E. J. Weinberg, *Classical solutions in quantum field theory: Solitons and Instantons in High
Energy Physics, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2012).

[7] Y. M. Shnir, Topological and Non-Topological Solitons in Scalar Field Theories (Cambridge University Press, 2018).

[8] D. Tong, in Theoretical Advanced Study Institute in Elementary Particle Physics: Many Dimensions of String Theory (2005) arXiv:hep-th/0509216; Annals Phys. 324, 30 (2009), arXiv:0809.5060 [hep-th].

[9] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, J. Phys. A 39, R315 (2006), arXiv:hep-th/0602170.

[10] M. Shifman and A. Yung, Rev. Mod. Phys. 79, 1139 (2007), arXiv:hep-th/0703267; Super-symmetric solitons, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2009).

[11] T. W. B. Kibble, J. Phys. A 9, 1387 (1976); Phys. Rept. 67, 183 (1980).

[12] A. Vilenkin, Phys. Rept. 121, 263 (1985); M. Hindmarsh and T. Kibble, Rept. Prog. Phys. 58, 477 (1995), arXiv:hep-ph/9411342; T. Vachaspati, L. Pogosian, and D. Steer, Scholarpedia 10, 31682 (2015), arXiv:1506.04039 [astro-ph.CO].

[13] A. Vilenkin and E. S. Shellard, Cosmic Strings and Other Topological Defects (Cambridge University Press, 2000).

[14] M. Eto, Y. Hirono, M. Nitta, and S. Yasui, PTEP 2014, 012D01 (2014), arXiv:1308.1535 [hep-ph].

[15] N. D. Mermin, Rev. Mod. Phys. 51, 591 (1979).

[16] G. E. Volovik, The Universe in a helium droplet, International Series of Monographs on Physics (Oxford Scholarship Online, 2009).

[17] B. V. Svistunov, E. S. Babaev, and N. V. Prokof’ev, Superfluid States of Matter, Cambridge Monographs on Mathematical Physics (CRC Press, 2015).

[18] A. V. Ustinov, Solitons in Josephson Junctions: Physics of Magnetic Fluxons in Superconducting Junctions and Arrays (Wiley-VCH, 2015).

[19] L. Pismen, Vortices in Nonlinear Fields: From Liquid Crystals to Superfluids, from Non-Equilibrium Patterns to Cosmic Strings, International Series of Monographs on Physics (Clarendon Press, 1999); Y. M. Bunkov and H. Godfrin, Topological Defects and the Non-Equilibrium Dynamics of Symmetry Breaking Phase Transitions (NATO Science Series)
(Springer, Dordrecht, 2000).

[20] Y. Kawaguchi and M. Ueda, Phys. Rept. 520, 253 (2012).

[21] W. H. Zurek, Nature 317, 505 (1985); Phys. Rept. 276, 177 (1996), arXiv:cond-mat/9607135.

[22] N. S. Manton, Phys. Lett. B 110, 54 (1982).

[23] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D 73, 125008 (2006), arXiv:hep-th/0602289.

[24] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, Phys. Lett. B 59, 85 (1975).

[25] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin, Phys. Lett. A 65, 185 (1978); E. Corrigan and P. Goddard, Annals Phys. 154, 253 (1984).

[26] N. Seiberg and E. Witten, Nucl. Phys. B 426, 19 (1994), [Erratum: Nucl.Phys.B 430, 485–486 (1994)], arXiv:hep-th/9407087; Nucl. Phys. B 431, 484 (1994), arXiv:hep-th/9408099; N. Dorey, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, Phys. Rept. 371, 231 (2002), arXiv:hep-th/0206063; N. A. Nekrasov, Adv. Theor. Math. Phys. 7, 831 (2003), arXiv:hep-th/0206161.

[27] G. ’t Hooft, Nucl. Phys. B 79, 276 (1974); A. M. Polyakov, JETP Lett. 20, 194 (1974).

[28] J. Preskill, Ann. Rev. Nucl. Part. Sci. 34, 461 (1984); E. J. Weinberg and P. Yi, Phys. Rept. 438, 65 (2007), arXiv:hep-th/0609055; K. Konishi, Lect. Notes Phys. 737, 471 (2008), arXiv:hep-th/0702102.

[29] E. B. Bogomolny, Sov. J. Nucl. Phys. 24, 449 (1976); M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[30] W. Nahm, Phys. Lett. B 90, 413 (1980).

[31] M. F. Atiyah and N. J. Hitchin, Phys. Lett. A 107, 21 (1985).

[32] G. W. Gibbons and N. S. Manton, Phys. Lett. B 356, 32 (1995), arXiv:hep-th/9506052.

[33] S. K. Donaldson, Commun. Math. Phys. 96, 387 (1984).

[34] P. Goddard, J. Nuyts, and D. I. Olive, Nucl. Phys. B 125, 1 (1977); E. J. Weinberg, Nucl. Phys. B 167, 500 (1980); Nucl. Phys. B 203, 445 (1982); R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and H. Murayama, Nucl. Phys. B 701, 207 (2004), arXiv:hep-th/0405070.

[35] C. P. Dokos and T. N. Tomaras, Phys. Rev. D21, 2940 (1980); G. Lazarides and Q. Shafi, Phys. Lett. 94B, 149 (1980); J. Preskill, Phys. Rev. Lett. 43, 1365 (1979).

[36] A. A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957); H. B. Nielsen and P. Olesen, Nucl. Phys.
B 61, 45 (1973).

[37] A. Vilenkin and A. E. Everett, Phys. Rev. Lett. 48, 1867 (1982); M. Kawasaki and K. Nakayama, Ann. Rev. Nucl. Part. Sci. 63, 69 (2013), arXiv:1301.1123 [hep-ph].

[38] T. Vachaspati and A. Achucarro, Phys. Rev. D 44, 3067 (1991); A. Achucarro and T. Vachaspati, Phys. Rept. 327, 347 (2000), arXiv:hep-ph/9904229.

[39] A. Hanany and D. Tong, JHEP 07, 037 (2003), arXiv:hep-th/0306150; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Nucl. Phys. B 673, 187 (2003), arXiv:hep-th/0307287.

[40] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. Lett. 96, 161601 (2006), arXiv:hep-th/0511088; M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, Phys. Rev. D 74, 065021 (2006), arXiv:hep-th/0607070; M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi, and W. Vinci, Phys. Rev. Lett. 98, 091602 (2007), arXiv:hep-th/0609214.

[41] T. Fujimori, G. Marmorini, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D 82, 065005 (2010), arXiv:1002.4580 [hep-th].

[42] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D 84, 125030 (2011), arXiv:1105.1547 [hep-th].

[43] M. Eto, T. Fujimori, S. Bjarke Gudnason, Y. Jiang, K. Konishi, M. Nitta, and K. Ohashi, JHEP 11, 042 (2010), arXiv:1009.4794 [hep-th].

[44] M. Shifman and A. Yung, Phys. Rev. D 73, 125012 (2006), arXiv:hep-th/0603134; M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, Phys. Rev. D 76, 105002 (2007), arXiv:0704.2218 [hep-th].

[45] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi, and W. Vinci, Phys. Lett. B 669, 98 (2008), arXiv:0802.1020 [hep-th]; M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, T. Nagashima, M. Nitta, K. Ohashi, and W. Vinci, JHEP 06, 004 (2009), arXiv:0903.4471 [hep-th].

[46] A. P. Balachandran, S. Digal, and T. Matsuura, Phys. Rev. D 73, 074009 (2006), arXiv:hep-ph/0509276; E. Nakano, M. Nitta, and T. Matsuura, Phys. Rev. D 78, 045002 (2008), arXiv:0708.4096 [hep-ph]; Prog. Theor. Phys. Suppl. 174, 254 (2008), arXiv:0805.4539 [hep-ph]; M. Eto and M. Nitta, Phys. Rev. D 80, 125007 (2009), arXiv:0907.1278 [hep-ph]; M. Eto, E. Nakano, and M. Nitta, Phys. Rev. D 80, 125011 (2009), arXiv:0908.4470 [hep-ph]; M. Eto, M. Nitta, and N. Yamamoto, Phys. Rev. Lett. 104, 161601 (2010), arXiv:0912.1352 [hep-ph];
M. G. Alford, S. K. Mallavarapu, T. Vachaspati, and A. Windisch, Phys. Rev. C 93, 045801 (2016), arXiv:1601.04656 [nucl-th].

[47] M. G. Alford, G. Baym, K. Fukushima, T. Hatsuda, and M. Tachibana, Phys. Rev. D 99, 036004 (2019), arXiv:1803.05115 [hep-ph]; C. Chatterjee, M. Nitta, and S. Yasui, Phys. Rev. D 99, 034001 (2019), arXiv:1806.09291 [hep-ph].

[48] G. R. Dvali and G. Senjanovic, Phys. Rev. Lett. 71, 2376 (1993), arXiv:hep-ph/9305278; M. Eto, M. Kurachi, and M. Nitta, Phys. Lett. B 785, 447 (2018), arXiv:1803.04662 [hep-ph]; JHEP 08, 195 (2018), arXiv:1805.07015 [hep-ph]; M. Eto, Y. Hamada, and M. Nitta, (2021), arXiv:2111.13345 [hep-ph].

[49] Y. Nambu, Nucl. Phys. B 130, 505 (1977); T. Vachaspati, Phys. Rev. Lett. 68, 1977 (1992), [Erratum: Phys.Rev.Lett. 69, 216 (1992)].

[50] E. R. C. Abraham and P. K. Townsend, Phys. Lett. B 291, 85 (1992); Phys. Lett. B 295, 225 (1992); M. Arai, M. Naganuma, M. Nitta, and N. Sakai, Nucl. Phys. B 652, 35 (2003), arXiv:hep-th/0211103; , 299 (2003), arXiv:hep-th/0302028.

[51] J. P. Gauntlett, D. Tong, and P. K. Townsend, Phys. Rev. D 64, 025010 (2001), arXiv:hep-th/0012178; D. Tong, Phys. Rev. D 66, 025013 (2002), arXiv:hep-th/0202012.

[52] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D 77, 125008 (2008), arXiv:0802.3135 [hep-th].

[53] Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. Lett. 93, 161601 (2004), arXiv:hep-th/0404198; Phys. Rev. D 70, 125014 (2004), arXiv:hep-th/0405194; Phys. Rev. D 71, 065018 (2005), arXiv:hep-th/0405129; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, Phys. Rev. D 71, 125006 (2005), arXiv:hep-th/0412024.

[54] A. Hanany and D. Tong, Commun. Math. Phys. 266, 647 (2006), arXiv:hep-th/0507140.

[55] M. Eto, Y. Isozumi, M. Nitta, K. Ohta, N. Sakai, and Y. Tachikawa, Phys. Rev. D 71, 105009 (2005), arXiv:hep-th/0503033.

[56] M. Arai, S. Lee, and S. Shin, Phys. Rev. D 80, 125012 (2009), arXiv:0908.3713 [hep-th]; M. Arai and S. Shin, Phys. Rev. D 83, 125003 (2011), arXiv:1103.1490 [hep-th]; B.-H. Lee, C. Park, and S. Shin, Phys. Rev. D 96, 105017 (2017), arXiv:1708.05243 [hep-th]; M. Arai, A. Golubtsova, C. Park, and S. Shin, Phys. Rev. D 97, 105012 (2018), arXiv:1803.09275 [hep-th].

[57] M. Shifman and A. Yung, Phys. Rev. D 70, 025013 (2004), arXiv:hep-th/0312257.
[58] M. Eto, M. Nitta, K. Ohashi, and D. Tong, Phys. Rev. Lett. 95, 252003 (2005), arXiv:hep-th/0508130.
[59] T. H. R. Skyrme, Proc. Roy. Soc. Lond. A 260, 127 (1961); Nucl. Phys. 31, 556 (1962).
[60] E. Witten, Nucl. Phys. B 223, 433 (1983).
[61] M. F. Atiyah and N. S. Manton, Phys. Lett. B 222, 438 (1989); Commun. Math. Phys. 153, 391 (1993).
[62] C. J. Houghton, N. S. Manton, and P. M. Sutcliffe, Nucl. Phys. B 510, 507 (1998), arXiv:hep-th/9705151.
[63] T. A. Ioannidou, B. Piette, and W. J. Zakrzewski, J. Math. Phys. 40, 6353 (1999).
[64] B. M. A. G. Piette, B. J. Schroers, and W. J. Zakrzewski, Nucl. Phys. B 439, 205 (1995), arXiv:hep-ph/9410256; Z. Phys. C 65, 165 (1995), arXiv:hep-th/9406160.
[65] A. M. Polyakov and A. A. Belavin, JETP Lett. 22, 245 (1975).
[66] Z. F. Ezawa, Quantum Hall Effects Recent Theoretical and Experimental Developments, 3rd Edition (World Scientific, 2013) p. 928.
[67] S. Muhlbaier, B. Binz, F. Jonietz, C. Pfleiderer, A. R. A. Neubauer, R. Georgii, and P. Boni, SCIENCE 323, 915 (2009); X. Yu, Y. Onose, N. Kanazawa, J. Park, J. Han, Y. Matsui, N. Nagaosa, and Y. Tokura, Nature 465, 901 (2010); J. H. Han, J. Zang, Z. Yang, J.-H. Park, and N. Nagaosa, Phys. Rev. B 82, 094429 (2010), arXiv:1006.3973 [cond-mat.other]; S.-Z. Lin, A. Saxena, and C. D. Batista, Phys. Rev. B 91, 224407 (2015), arXiv:1406.1422.
[68] B. Barton-Singer, C. Ross, and B. J. Schroers, Commun. Math. Phys. 375, 2259 (2020), arXiv:1812.07268 [cond-mat.str-el]; B. Schroers, in 11th International Symposium on Quantum Theory and Symmetries (2019) arXiv:1910.13907 [cond-mat.mes-hall]; C. Ross, N. Sakai, and M. Nitta, JHEP 02, 095 (2021), arXiv:2003.07147 [cond-mat.mes-hall].
[69] D. Foster, C. Kind, P. J. Ackerman, J.-S. B. Tai, M. R. Dennis, and I. I. Smalyukh, Nature Phys. 15, 655 (2019).
[70] J. K. Perring and T. H. R. Skyrme, Nucl. Phys. 31, 550 (1962).
[71] C. A. Condat, R. A. Guyer, and M. D. Miller, Phys. Rev. B 27, 474 (1983); V. A. Gani, A. M. Marjaneh, A. Askari, E. Belendryasova, and D. Saadatmand, Eur. Phys. J. C 78, 345 (2018), arXiv:1711.01918 [hep-th].
[72] A. Ustinov, Physica D: Nonlinear Phenomena 123, 315 (1998), annual International Conference of the Center for Nonlinear Studies.
[73] Y. Togawa, T. Koyama, K. Takayanagi, S. Mori, Y. Kousaka, J. Akimitsu, S. Nishihara, K. Inoue, A. Ovchinnikov, and J.-i. Kishine, Phys. Rev. Lett. 108, 107202 (2012); Y. Togawa, Y. Kousaka, K. Inoue, and J.-i. Kishine, Journal of the Physical Society of Japan 85, 112001 (2016); A. A. Tereshchenko, A. S. Ovchinnikov, I. Proskurin, E. V. Sinitsyn, and J.-i. Kishine, Phys. Rev. B 97, 184303 (2018); J. Chovan, N. Papanicolaou, and S. Komineas, Phys. Rev. B 65, 064433 (2002); C. Ross, N. Sakai, and M. Nitta, JHEP 12, 163 (2021), arXiv:2012.08800 [cond-mat.mes-hall].

[74] J. ichiro Kishine and A. Ovchinnikov, Solid State Physics, 66, 1 (2015).

[75] X.-G. Huang, K. Nishimura, and N. Yamamoto, JHEP 02, 069 (2018), arXiv:1711.02190 [hep-ph]; K. Nishimura and N. Yamamoto, JHEP 07, 196 (2020), arXiv:2003.13945 [hep-ph].

[76] T. Brauner and N. Yamamoto, JHEP 04, 132 (2017), arXiv:1609.05213 [hep-ph].

[77] M. Nitta, Nucl. Phys. B 895, 288 (2015), arXiv:1412.8276 [hep-th].

[78] M. Eto and M. Nitta, Phys. Rev. D 91, 085044 (2015), arXiv:1501.07038 [hep-th].

[79] A. P. Balachandran and S. Digal, Phys. Rev. D 66, 034018 (2002), arXiv:hep-ph/0204262; M. Nitta and N. Shiiki, Phys. Lett. B 658, 143 (2008), arXiv:0708.4091 [hep-ph]; E. Nakano, M. Nitta, and T. Matsuura, Phys. Lett. B 672, 61 (2009), arXiv:0708.4092 [hep-ph]; M. Eto, E. Nakano, and M. Nitta, Nucl. Phys. B 821, 129 (2009), arXiv:0903.1528 [hep-ph]; M. Eto, Y. Hirono, and M. Nitta, PTEP 2014, 033B01 (2014), arXiv:1309.4559 [hep-ph].

[80] M. Eto and M. Nitta, Phys. Rev. D 104, 094052 (2021), arXiv:2103.13011 [hep-ph].

[81] M. Nitta, Nucl. Phys. B 899, 78 (2015), arXiv:1502.02525 [hep-th].

[82] M. Nitta, Phys. Rev. D 92, 045010 (2015), arXiv:1503.02060 [hep-th].

[83] M. Eto, K. Nishimura, and M. Nitta, (2021), arXiv:2112.01381 [hep-ph].

[84] L. D. Faddeev and A. J. Niemi, Nature 387, 58 (1997), arXiv:hep-th/9610193.

[85] E. Radu and M. S. Volkov, Phys. Rept. 468, 101 (2008), arXiv:0804.1357 [hep-th].

[86] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. 81, 4798 (1998), arXiv:hep-th/9808129; Proc. Roy. Soc. Lond. A 455, 4305 (1999), arXiv:hep-th/9811077; M. Kobayashi and M. Nitta, Phys. Lett. B 728, 314 (2014), arXiv:1304.6021 [hep-th].

[87] G. Volovik and V.P.Mineev, JETP 46, 401 (1977).

[88] E. Babaev, L. D. Faddeev, and A. J. Niemi, Phys. Rev. B 65, 100512 (2002), arXiv:cond-mat/0106152; F. N. Rybakov, J. Garaud, and E. Babaev, Phys. Rev. B 100, 094515 (2019), arXiv:1807.02509 [cond-mat.supr-con].
[89] Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. 100, 180403 (2008), [Erratum: Phys.Rev.Lett. 101, 029902 (2008)], arXiv:0802.1968 [cond-mat.other]; Y. Kawaguchi, M. Kobayashi, M. Nitta, and M. Ueda, Prog. Theor. Phys. Suppl. 186, 455 (2010), arXiv:1006.5839 [cond-mat.quant-gas]; D. Hall, M. Ray, and K. Tiurev et. al., Nature Phys 12, 478 (2016); T. Ollikainen, A. Blinova, M. Möttönen, and D. S. Hall, Phys. Rev. Lett. 123, 163003 (2019), arXiv:1908.01285 [cond-mat.quant-gas].

[90] B. G.-g. Chen, P. J. Ackerman, G. P. Alexander, R. D. Kamien, and I. I. Smalyukh, Phys. Rev. Lett. 110, 237801 (2013); P. Ackerman, J. van de Lagemaat, and I. Smalyukh, Nat. Comm. 6, 6012 (2015); P. Ackerman and I. Smalyukh, Nat. Mater. 16, 426 (2017); Phys. Rev. X 7, 011006 (2017); J.-S. Tai, P. Ackerman, and I. Smalyukh, PNAS 115, 921 (2018); 365.

[91] N. Kent, N. Reynolds, and D. Raftrey et al., Nat Commun 12, 1562 (2021).

[92] R. Auzzi, S. Bolognesi, J. Evslin, and K. Konishi, Nucl. Phys. B 686, 119 (2004), arXiv:hep-th/0312233; M. Eto, L. Ferretti, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, Nucl. Phys. B 780, 161 (2007), arXiv:hep-th/0611313.

[93] J. P. Gauntlett, R. Portugues, D. Tong, and P. K. Townsend, Phys. Rev. D 63, 085002 (2001), arXiv:hep-th/0008221; M. Shifman and A. Yung, Phys. Rev. D 67, 125007 (2003), arXiv:hep-th/0212293.

[94] R. Auzzi, M. Shifman, and A. Yung, Phys. Rev. D 72, 025002 (2005), arXiv:hep-th/0504148.

[95] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D 79, 045015 (2009), arXiv:0810.3495 [hep-th].

[96] M. Bowick, A. De Felice, and M. Trodden, JHEP 10, 067 (2003), arXiv:hep-th/0306224.

[97] S. B. Gudnason and M. Nitta, Phys. Rev. D 91, 045018 (2015), arXiv:1412.6995 [hep-th].

[98] D. Tong, Phys. Rev. D 69, 065003 (2004), arXiv:hep-th/0307302.

[99] M. Shifman and A. Yung, Phys. Rev. D 70, 045004 (2004), arXiv:hep-th/0403149; A. Hanany and D. Tong, JHEP 04, 066 (2004), arXiv:hep-th/0403158.

[100] M. Nitta, Phys. Rev. D 86, 125004 (2012), arXiv:1207.6958 [hep-th]; M. Kobayashi and M. Nitta, Phys. Rev. D 87, 085003 (2013), arXiv:1302.0989 [hep-th].

[101] A. E. Kudryavtsev, B. M. A. G. Piette, and W. J. Zakrzewski, Nonlinearity 11, 783 (1998), arXiv:hep-th/9709187; R. Auzzi, M. Shifman, and A. Yung, Phys. Rev. D 74, 045007 (2006), arXiv:hep-th/0606060; P. Jennings and P. Sutcliffe, J. Phys. A 46, 465401 (2013),
arXiv:1305.2869 [hep-th]; V. Bychkov, M. Kreshchuk, and E. Kurianovych, Int. J. Mod. Phys. A 33, 1850111 (2018), arXiv:1603.06310 [hep-th].

[102] P. M. Sutcliffe, Phys. Lett. B 283, 85 (1992); G. N. Stratopoulos and W. J. Zakrzewski, Z. Phys. C 59, 307 (1993).

[103] R. Cheng, M. Li, A. Sapkota, A. Rai, A. Pokhrel, T. Mewes, C. Mewes, D. Xiao, M. De Graef, and V. Sokalski, Phys. Rev. B 99, 184412 (2019); S. Lepadatu, Phys. Rev. B 102, 094402 (2020); T. Nagase et al., Nature Commun. 12, 3490 (2021), arXiv:2004.06976 [cond-mat.mtrl-sci]; K. Yang, K. Nagase, and Y. Hirayama et.al., Nat Commun 12, 6006 (2021).

[104] S. K. Kim and Y. Tserkovnyak, Phys. Rev. Lett. 119, 047202 (2017), arXiv:1701.08273 [cond-mat.mes-hall].

[105] M. Hongo, T. Fujimori, T. Misumi, M. Nitta, and N. Sakai, Phys. Rev. B 101, 104417 (2020), arXiv:1907.02062 [cond-mat.mes-hall].

[106] T. Fujimori, H. Iida, and M. Nitta, Phys. Rev. B 94, 104504 (2016), arXiv:1604.08103 [cond-mat.supr-con].

[107] A. P. Balachandran and S. Digal, Int. J. Mod. Phys. A 17, 1149 (2002), arXiv:hep-ph/0108086.

[108] T. W. B. Kibble, G. Lazarides, and Q. Shafi, Phys. Rev. D 26, 435 (1982); A. E. Everett and A. Vilenkin, Nucl. Phys. B 207, 43 (1982).

[109] C. Chatterjee, M. Kurachi, and M. Nitta, Phys. Rev. D 97, 115010 (2018), arXiv:1801.10469 [hep-ph].

[110] A. Ritz, M. Shifman, and A. Vainshtein, Phys. Rev. D 70, 095003 (2004), arXiv:hep-th/0405175; S. Bolognesi, J. Phys. A 42, 195404 (2009), arXiv:0710.5198 [hep-th].

[111] J. Preskill and A. Vilenkin, Phys. Rev. D 47, 2324 (1993), arXiv:hep-ph/9209210.

[112] M. Nitta and W. Vinci, Nucl. Phys. B 848, 121 (2011), arXiv:1012.4057 [hep-th].

[113] M. Eto, T. Fujimori, S. B. Gudnason, Y. Jiang, K. Konishi, M. Nitta, and K. Ohashi, JHEP 12, 017 (2011), arXiv:1108.6124 [hep-th].

[114] J. Scherk and J. H. Schwarz, Nucl. Phys. B 153, 61 (1979).

[115] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D 72, 025011 (2005), arXiv:hep-th/0412048.

[116] Y. Nambu, Phys. Rev. D 10, 4262 (1974); S. Mandelstam, Phys. Lett. B 53, 476 (1975); Phys. Rept. 23, 245 (1976).
[117] P. Langacker and S.-Y. Pi, Phys. Rev. Lett. 45, 1 (1980); R. Auzzi, S. Bolognesi, and J. Evslin, JHEP 02, 046 (2005), arXiv:hep-th/0411074; M. Cipriani, D. Dorigoni, S. B. Gudnason, K. Konishi, and A. Michelini, Phys. Rev. D 84, 045024 (2011), arXiv:1106.4214 [hep-th]; C. Chatterjee and K. Konishi, JHEP 09, 039 (2014), arXiv:1406.5639 [hep-th].

[118] A. Gorsky, M. Shifman, and A. Yung, Phys. Rev. D 83, 085027 (2011), arXiv:1101.1120 [hep-ph]; M. Eto, M. Nitta, and N. Yamamoto, Phys. Rev. D 83, 085005 (2011), arXiv:1101.2574 [hep-ph].

[119] M. Eto, K. Konishi, M. Nitta, and Y. Ookouchi, Phys. Rev. D 87, 045006 (2013), arXiv:1211.2971 [hep-th].

[120] M. Eto, Y. Hamada, M. Kurachi, and M. Nitta, Phys. Lett. B 802, 135220 (2020), arXiv:1904.09269 [hep-ph]; JHEP 07, 004 (2020), arXiv:2003.08772 [hep-ph]; M. Eto, Y. Hamada, and M. Nitta, Phys. Rev. D 102, 105018 (2020), arXiv:2007.15587 [hep-th].

[121] M. Hindmarsh and T. W. B. Kibble, Phys. Rev. Lett. 55, 2398 (1985); T. W. B. Kibble and T. Vachaspati, J. Phys. G 42, 094002 (2015), arXiv:1506.02022 [astro-ph.CO]; B. Kleihaus, J. Kunz, and Y. Shnir, Phys. Lett. B 570, 237 (2003), arXiv:hep-th/0307110; Phys. Rev. D 68, 101701 (2003), arXiv:hep-th/0307215; Y. Ng, T. W. B. Kibble, and T. Vachaspati, Phys. Rev. D 78, 046001 (2008), arXiv:0806.0155 [hep-th].

[122] T. Fujimori, M. Nitta, K. Ohta, N. Sakai, and M. Yamazaki, Phys. Rev. D 78, 105004 (2008), arXiv:0805.1194 [hep-th].

[123] T. Fujimori, T. Kimura, M. Nitta, and K. Ohashi, JHEP 12, 110 (2015), arXiv:1509.08630 [hep-th]; JHEP 06, 028 (2012), arXiv:1204.1968 [hep-th].

[124] M. Nitta, Nucl. Phys. B 885, 493 (2014), arXiv:1311.2718 [hep-th].

[125] H. Hata, T. Sakai, S. Sugimoto, and S. Yamato, Prog. Theor. Phys. 117, 1157 (2007), arXiv:hep-th/0701280.

[126] S. B. Gudnason and M. Nitta, Phys. Rev. D 94, 025008 (2016), arXiv:1606.00336 [hep-th].

[127] S. B. Gudnason and M. Nitta, Phys. Rev. D 90, 085007 (2014), arXiv:1407.7210 [hep-th]; Phys. Lett. B 747, 173 (2015), arXiv:1407.2822 [hep-th].

[128] M. Nitta, Phys. Rev. D 87, 025013 (2013), arXiv:1210.2233 [hep-th]; S. B. Gudnason and M. Nitta, Phys. Rev. D 89, 085022 (2014), arXiv:1403.1245 [hep-th].

[129] A. E. Kudryavtsev, B. M. A. G. Piette, and W. J. Zakrzewski, Phys. Rev. D 61, 025016 (2000), arXiv:hep-th/9907197; S. B. Gudnason and M. Nitta, Phys. Rev. D 89, 025012.
(2014), arXiv:1311.4454 [hep-th].

[130] M. Nitta, Nucl. Phys. B 872, 62 (2013), arXiv:1211.4916 [hep-th].

[131] S. B. Gudnason and M. Nitta, Phys. Rev. D 98, 125002 (2018), arXiv:1809.01025 [hep-th].

[132] S. Chen, K. Fukushima, and Z. Qiu, Phys. Rev. D 105, L011502 (2022), arXiv:2104.11482 [hep-ph].

[133] M. Eto, Y. Isozumi, M. Nitta, and K. Ohashi, Nucl. Phys. B 752, 140 (2006), arXiv:hep-th/0506257.

[134] M. Nitta, Phys. Rev. D 87, 066008 (2013), arXiv:1301.3268 [hep-th].

[135] M. Nitta, Int. J. Mod. Phys. A 28, 1350172 (2013), arXiv:1206.5551 [hep-th].

[136] K. Higashijima and M. Nitta, Prog. Theor. Phys. 103, 635 (2000), arXiv:hep-th/9911139.

[137] M. Arai, M. Nitta, and N. Sakai, Prog. Theor. Phys. 113, 657 (2005), arXiv:hep-th/0307274.

[138] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, and N. Sakai, Prog. Theor. Phys. 128, 67 (2012), arXiv:1204.0773 [hep-th].

[139] N. D. Lambert and D. Tong, Phys. Lett. B 462, 89 (1999), arXiv:hep-th/9907014.

[140] B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).

[141] B. Collie, J. Phys. A 42, 085404 (2009), arXiv:0809.0394 [hep-th]; M. Eto and Y. Murakami, JHEP 03, 078 (2015), arXiv:1412.7892 [hep-th].

[142] R. A. Leese, Nucl. Phys. B 366, 283 (1991); E. Abraham, Phys. Lett. B 278, 291 (1992).

[143] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, and N. Sakai, Phys. Rev. D 76, 125025 (2007), arXiv:0707.3267 [hep-th].

[144] R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976).

[145] R. Jackiw and P. Rossi, Nucl. Phys. B 190, 681 (1981).

[146] S. Yasui, K. Itakura, and M. Nitta, Phys. Rev. D 81, 105003 (2010), arXiv:1001.3730 [math-ph]; T. Fujiwara, T. Fukui, M. Nitta, and S. Yasui, Phys. Rev. D 84, 076002 (2011), arXiv:1105.2115 [hep-ph].

[147] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, JHEP 02, 172 (2015), arXiv:1412.5148 [hep-th].

[148] Y. Hidaka, M. Nitta, and R. Yokokura, Phys. Lett. B 808, 135672 (2020), arXiv:2006.12532 [hep-th]; JHEP 01, 173 (2021), arXiv:2009.14368 [hep-th]; Phys. Lett. B 823, 136762 (2021), arXiv:2107.08753 [hep-th]; (2021), arXiv:2108.12564 [hep-th].

[149] Y. Hidaka, Y. Hirono, M. Nitta, Y. Tanizaki, and R. Yokokura, Phys. Rev. D 100, 125016
(2019), arXiv:1903.06389 [hep-th]; Y. Hidaka, M. Nitta, and R. Yokokura, Phys. Lett. B 803, 135290 (2020), arXiv:1912.02782 [hep-th].

[150] G. E. Volovik, J. Exp. Theor. Phys. 131, 11 (2020), arXiv:1912.05962 [cond-mat.other]; G. E. Volovik and K. Zhang, Phys. Rev. Res. 2, 023263 (2020), arXiv:2002.07578 [hep-ph].

[151] J. Jaykka and M. Speight, Phys. Rev. D 82, 125030 (2010), arXiv:1010.2217 [hep-th]; M. Kobayashi and M. Nitta, Phys. Rev. D 87, 125013 (2013), arXiv:1307.0242 [hep-th]; J. Low Temp. Phys. 175, 208 (2014), arXiv:1307.1345 [cond-mat.quant-gas]; P. Leask, Phys. Rev. D 105, 025010 (2022), arXiv:2111.02217 [hep-th].

[152] Y. Akagi, Y. Amari, S. B. Gudnason, M. Nitta, and Y. Shnir, JHEP 11, 194 (2021), arXiv:2107.13777 [hep-th].

[153] B. J. Schroers, Phys. Lett. B 356, 291 (1995), hep-th/9506004; Nucl. Phys. B 475, 440 (1996), hep-th/9603101; J. M. Baptista, Commun. Math. Phys. 261, 161 (2006), arXiv:math/0411517; M. Nitta and W. Vinci, J. Phys. A 45, 175401 (2012), arXiv:1108.5742 [hep-th].

[154] M. Eto, M. Nitta, and K. Sakurai, JHEP 10, 048 (2016), arXiv:1608.03516 [hep-th].

[155] A. Samoilenka and Y. Shnir, Phys. Rev. D 97, 045004 (2018), arXiv:1712.00161 [hep-th].

[156] S. B. Gudnason and M. Nitta, Phys. Rev. D 91, 085040 (2015), arXiv:1502.06596 [hep-th].

[157] K. Arthur and D. H. Tchrakian, Phys. Lett. B 378, 187 (1996), arXiv:hep-th/9601053; B. Kleihaus, D. H. Tchrakian, and F. Zimmerschied, J. Math. Phys. 41, 816 (2000), arXiv:hep-th/9907035; D. Y. Grigoriev, P. M. Sutcliffe, and D. H. Tchrakian, Phys. Lett. B 540, 146 (2002), arXiv:hep-th/0206160.

[158] K.-M. Lee, Phys. Lett. B 426, 323 (1998), arXiv:hep-th/9802012; K.-M. Lee and C.-h. Lu, Phys. Rev. D 58, 025011 (1998), arXiv:hep-th/9802108; T. C. Kraan and P. van Baal, Nucl. Phys. B 533, 627 (1998), arXiv:hep-th/9805168; Phys. Lett. B 435, 389 (1998), arXiv:hep-th/9806034.

[159] M. Unsal, Phys. Rev. D 80, 065001 (2009), arXiv:0709.3269 [hep-th]; M. M. Anber and E. Poppitz, JHEP 06, 136 (2011), arXiv:1105.0940 [hep-th]; P. C. Argyres and M. Unsal, JHEP 08, 063 (2012), arXiv:1206.1890 [hep-th].

[160] D. Harland and R. S. Ward, JHEP 12, 093 (2008), arXiv:0807.3870 [hep-th].

[161] M. Nitta, JHEP 08, 063 (2015), arXiv:1503.06336 [hep-th].

[162] F. Bruckmann, Phys. Rev. Lett. 100, 051602 (2008), arXiv:0707.0775 [hep-th]; W. Bren-
del, F. Bruckmann, L. Janssen, A. Wipf, and C. Wozar, Phys. Lett. B 676, 116 (2009), arXiv:0902.2328 [hep-th]; D. Harland, J. Math. Phys. 50, 122902 (2009), arXiv:0902.2303 [hep-th]; F. Bruckmann and T. Sulejmanpasic, Phys. Rev. D 90, 105010 (2014), arXiv:1408.2229 [hep-th].

[163] D. Harland and R. S. Ward, Phys. Rev. D 77, 045009 (2008), arXiv:0711.3166 [hep-th].

[164] G. V. Dunne and M. Unsal, JHEP 11, 170 (2012), arXiv:1210.2423 [hep-th]; G. V. Dunne and M. Ünsal, Phys. Rev. D 87, 025015 (2013), arXiv:1210.3646 [hep-th]; T. Misumi, M. Nitta, and N. Sakai, JHEP 06, 164 (2014), arXiv:1404.7225 [hep-th]; J. Phys. Conf. Ser. 597, 012060 (2015), arXiv:1412.0861 [hep-th]; T. Fujimori, S. Kamata, T. Misumi, M. Nitta, and N. Sakai, JHEP 02, 190 (2019), arXiv:1810.03768 [hep-th].

[165] T. Misumi, T. Fujimori, E. Itou, M. Nitta, and N. Sakai, PoS LATTICE2019, 015 (2019), arXiv:1911.07398 [hep-lat]; T. Fujimori, E. Itou, T. Misumi, M. Nitta, and N. Sakai, JHEP 08, 011 (2020), arXiv:2006.05106 [hep-th].

[166] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, Phys. Rev. D 73, 085008 (2006), arXiv:hep-th/0601181; M. Eto, T. Fujimori, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, Nucl. Phys. B 788, 120 (2008), arXiv:hep-th/0703197.

[167] T. Misumi, M. Nitta, and N. Sakai, PTEP 2015, 033B02 (2015), arXiv:1409.3444 [hep-th]; G. V. Dunne and M. Unsal, JHEP 09, 199 (2015), arXiv:1505.07803 [hep-th].

[168] M. Kobayashi and M. Nitta, Nucl. Phys. B 876, 605 (2013), arXiv:1305.7417 [hep-th].