Modular forms constructed from moduli of elliptic curves, with applications to explicit models of modular curves

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Abstract. These are the lecture notes from my portion of a mini-course for the summer school “Building Bridges 3” that was held in Sarajevo during July 2016. My lectures covered the Katz definition of modular forms, a family of forms defined from this perspective and their relation to Eisenstein series, and methods of finding explicit models of modular curves. The treatment is purely expository, and the results are mostly standard, although a few points of view may not be as widely known as they deserve to be.

1. Lecture 1

In the previous lectures in this summer school, we have considered modular forms as holomorphic functions $f(\tau)$ for $\tau \in \mathcal{H}$, with the $q$-expansion (when $f$ is a newform) encoding an associated Galois representation that we have used as a black box.

We now want to describe the connection between modular forms and the modular curves, such as $\mathcal{X}(N)$, parametrizing elliptic curves with level structure. A word of caution: this parametrization of elliptic curves by the points of the modular curve $\mathcal{X}(N)$ is completely different from the arithmetic parametrization of a single elliptic curve over $\mathbb{Q}$ as a quotient of the Jacobian of $\mathcal{X}_0(N)$.

Recall our fundamental congruence subgroups of $\text{SL}(2, \mathbb{Z})$:

$$\text{SL}(2, \mathbb{Z}) = \Gamma(1) > \Gamma_0(N) > \Gamma_1(N) > \Gamma(N),$$

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod{N} \},$$

$$\Gamma_1(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0, \quad a \equiv d \equiv 1 \pmod{N} \},$$

$$\Gamma(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b \equiv c \equiv 0, \quad a \equiv d \equiv 1 \pmod{N} \}. $$

The connection between modular forms and elliptic curves arises by associating, to each value $\tau \in \mathcal{H}$, an elliptic curve $E_{\tau}$, which is analytically $E_{\tau} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

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Suppose $\tau, \tau' \in \mathcal{H}$ are related via an element $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma(1)$, so that $\tau' = \gamma \tau = (a\tau + b)/(c\tau + d)$. It then follows that $E_\tau \cong E_{\tau'}$, via

$$z + \mathbb{Z} + \mathbb{Z}\tau \in E_\tau \iff z' = \frac{z}{c\tau + d} + \mathbb{Z} + \mathbb{Z}\tau' \in E_{\tau'}.$$

This basically expresses, in terms of the elliptic curves $E_\tau$ and $E_{\tau'}$, the standard fact that the lattices $\mathbb{Z} + \mathbb{Z}\tau$ and $\mathbb{Z} + \mathbb{Z}\tau'$ are homothetic if and only if $\tau'$ and $\tau$ are related by an element $\gamma \in \Gamma(1)$. For equivalence under the smaller congruence subgroup $\Gamma(N)$, we have the following more precise statement.

**Proposition 1.1.** Let $\tau, \tau' \in \mathcal{H}$. Then $\tau$ and $\tau'$ are related by an element $\gamma \in \Gamma(N)$ if and only if there is an isomorphism $\phi : E_\tau \to E_{\tau'}$ between the corresponding elliptic curves, such that

$$\phi(1/N) = 1/N, \quad \phi(\tau/N) = \tau'N.$$

The above is of course shorthand for saying $\phi(1/N + \mathbb{Z} + \mathbb{Z}\tau) = 1/N + \mathbb{Z} + \mathbb{Z}\tau'$ and $\phi(\tau/N + \mathbb{Z} + \mathbb{Z}\tau) = \tau'/N + \mathbb{Z} + \mathbb{Z}\tau'$.

From the above, we deduce that the quotients $Y(1) = \Gamma(1)\backslash \mathcal{H}$ and $Y(N) = \Gamma(N)\backslash \mathcal{H}$ parametrize, respectively, isomorphism classes of elliptic curves over $\mathbb{C}$, and isomorphism classes of triples $(E, P, Q)$ where $E$ is an elliptic curve over $\mathbb{C}$, and $\{P, Q\}$ is a basis for the $N$-torsion $E[N]$ that is symplectic, in the sense that $e_N(P, Q) = \exp(2\pi i/N)$ for the Weil pairing. The $\Gamma(N)$-orbit of a point $\tau \in \mathcal{H}$ corresponds to the isomorphism class of the triple $(\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), 1/N, \tau/N)$.

One says that $Y(1)$ and $Y(N)$ are *moduli spaces* parametrizing the moduli of elliptic curves, respectively without or with a basis for the $N$-torsion. We leave it to the reader to look up or determine as an exercise the moduli problems that are parametrized by the quotients $Y_0(N) = \Gamma_0(N)\backslash \mathcal{H}$ and $Y_1(N) = \Gamma_1(N)\backslash \mathcal{H}$.

It turns out to be much better to work with a compactification, the modular curve $X(N)$, of $Y(N)$, which one can think of as adding the cusps to $Y(N)$.

With the cusps included, one can view the space of modular forms $\mathcal{M}_k(\Gamma(N)) = H^0(X(N), \mathcal{L}^k)$ as being the space of holomorphic sections of the $k$th power of a line bundle $\mathcal{L}$ on $X(N)$, at least for $N \geq 3$ to avoid issues of elliptic elements (which would arise if we used, say, $X_0(N)$). This point of view makes the required behavior of modular forms at cusps automatic, once we require holomorphy at the cusps that have been added to obtain $X(N)$. We will follow up on the interpretation of modular forms as sections of line bundles in Lecture 2.

In this lecture, we will focus instead on interpreting modular forms based on the moduli of elliptic curves parametrized by $X(N)$. The precise algebraic formulation is due to N. Katz; see for example Section 2.1 of [Kat76]. We shall be somewhat cavalier with the precise definition, and simply state the following.

**Definition 1.2.** Let $k \geq 1$ be an integer. A Katz modular form of weight $k$ on $\Gamma(N)$ is a “nice” function $f(E, P, Q, \omega)$ satisfying the homogeneity property

$$f(E, P, Q, c\omega) = c^{-k}f(E, P, Q, \omega).$$
Informally, the domain of definition of $f$ is tuples $(E,P,Q,ω)$, where $E$ is an elliptic curve, the pair $(P,Q)$ is a symplectic basis for the $N$-torsion $E[N]$, and $ω ∈ H^0(E,Ω^1)$ is a global 1-form on $E$. The precise definition allows the argument $E$ in the tuple to be a generalized elliptic curve scheme $E/S$ over a base $S$, and brings in compatibility conditions under change of base; over $C$, these compatibility conditions amount to holomorphy on $H$ and at the cusps (this is what we mean by a “nice” function). In these lectures, however, we will pretend to consider only tuples defined over $C$. In this context, the choice of $ω$, up to a complex scalar, determines the homothety class of the lattice $L$ of periods of $E$, namely $L = \{ f_0, ω \mid γ ∈ H_1(E,Z) \}$, and $E \cong C/L$. This ties in with the perspective seen elsewhere, of modular forms on $Γ(1)$ as functions of lattices. In the context of these lectures, we will simply pass between modular forms and their Tate models (1.5).

**Example 1.3.** In case of level $N = 1$, we can dispense with specifying the $P$ and $Q$ in the tuples above. Then the Eisenstein series (restricted in this example to even weight $k ≥ 4$) is given in Katz and traditional form as

$$G_k(E,ω) = \sum_{0 ≠ γ ∈ H_1(E,Z)} \left[ ∫_γ ω \right]^{-k},$$

$$G_k(τ) = \sum_{0 ≠ m+nτ ∈ Z+Zτ} (m+nτ)^{-k} = ∑' (m+nτ)^{-k},$$

where as usual $∑'$ denotes a sum where we omit terms that look like $0^{-k}$ and are hence meaningless.

We admit that calling (1.5) a Katz form is not quite fair, because one really wants to have an algebraic construction of the value on the tuple. So we point out that $G_4$ and $G_6$ can be obtained from the following algebraic construction: starting from the elliptic curve $E$, its short Weierstrass form $y^2 = x^3 + ax + b$ (over, say, a field not of characteristic 2 or 3) is uniquely determined up to changing $(x,y)$ into $(x',y') = (c^{-2}x,c^{-3}y)$ with $c ≠ 0$; this transforms $(a,b)$ into $(a',b') = (c^{-4}a,c^{-6}b)$. However, if one has access not only to $E$ but also to the global differential $ω$ on $E$, then one can normalize the choice of coordinates to obtain $ω = dx/2y$. In this context, the coefficients $a$ and $b$ in the normalized Weierstrass equation are Katz modular forms of level $Γ(1)$ of weights 4 and 6, and are in fact simple multiples of $G_4$ and $G_6$, respectively. This can be seen via the parametrization of a complex elliptic curve by $x = ϕ(z) = z^{-2} + ⋯$, $y = (1/2)ϕ'(x) = -z^{-3} + ⋯$, and the usual differential equation for $ϕ$. We can also identify the higher weight Eisenstein series on $Γ(1)$ as coefficients in the Laurent expansion of $ϕ$, which can be defined purely algebraically over a field of characteristic zero (in terms of the completion of the local ring of $E$ at the origin $O$, which allows us to integrate the formal power series of $ω$ and obtain an “analytic” uniformizer $z$ in this completed local ring).

We wish to generalize the abovementioned principle to $Γ(N)$, and to define a wide family of Katz-style modular forms, which when evaluated on a tuple $(E,P,Q,ω)$ are given as coefficients in the Laurent series of certain elements in the function field of $E$, which analytically can be viewed as elliptic functions with
respect to $\mathbb{Z} + \mathbb{Z} \tau$. This family of modular forms will include all the Eisenstein series on $\Gamma(N)$. We therefore begin by recalling the definition of the relevant Eisenstein series.

**Definition 1.4.** For $i,j \in \mathbb{Z}$, let $\alpha = (i/N, j/N)$, which we will usually view as an element of $\mathbb{Q}^2/\mathbb{Z}^2$; sometimes, by abuse of notation, we will identify $\alpha$ with the torsion point $(i+j\tau)/N = iP+jQ \in E[N]$, all of which depends of course on a varying $\tau \in \mathcal{H}$, or equivalently on the corresponding tuple $(E = E_\tau, P, Q, \omega)$. We then define the Eisenstein series of arbitrary weight $k$, with parameter $\alpha$, by

\begin{equation}
G_{k,\alpha}(\tau) = \sum_{m,n \in \mathbb{Z}}' (m + n\tau + i/N + (j/N)\tau)^{-k} = \sum_{\ell \in \mathbb{Z} + \mathbb{Z}\tau} (\ell + \alpha)^{-k}.
\end{equation}

In the above, the notation $\sum'$ means we omit the term with $\ell + \alpha = 0$, if it is present in the sum (which is essentially only when $i = j = 0$). For $k > 2$, the sum in (1.6) converges absolutely and uniformly for $\tau$ in any compact set, and yields a modular form of weight $k$ on $\Gamma(N)$.

It is traditional to modify the definition to make sense of the lower weights $k \in \{1, 2\}$ by Hecke’s summation method \cite{Hecke27}:

\begin{equation}
G_{k,\alpha}(\tau, s) = \sum_{\ell \in \mathbb{Z} + \mathbb{Z}\tau} (\ell + \alpha)^{-k} |\ell + \alpha|^{-2s},
\end{equation}

\begin{equation}
G_{k,\alpha}(\tau) = G_{k,\alpha}(\tau, 0) \text{ after analytic continuation in } s.
\end{equation}

This yields the same Eisenstein series as before for $k > 2$. For $k = 2$, it turns out that $G_{2,\alpha}(\tau)$ is not quite holomorphic, but is the sum of $-\pi/(\text{Im } \tau)$ and a holomorphic function of $\tau$; so to obtain a holomorphic weight 2 Eisenstein series one must consider a difference such as $G_{2,\alpha} - G_{2,0}$. Reassuringly, for weight $k = 1$, $G_{1,\alpha}$ is indeed holomorphic. We will not consider the case $k = 0$ in these lectures, but elsewhere in this summer school the series $G_{0,0}(\tau, s)$ plays a major role via the Kronecker limit formula.

We need a few more preliminaries before we can construct the general family of modular forms that we promised above. In the meantime, to whet the reader’s appetite, let us give an ad hoc Katz-style interpretation of the Eisenstein series of weights 2 and 3. Take a nonzero $\alpha$, and view it also as a nonzero $N$-torsion point on the elliptic curve $E_\tau$. Then its coordinates on the Weierstrass model are $(x_\alpha, y_\alpha) = (\varphi(\alpha), (1/2)\varphi'(\alpha))$, so we obtain from the series for $\varphi$ and $\varphi'$ that

\begin{equation}
\varphi(\alpha) = \sum_{m,n} [(\alpha + m + n\tau)^{-2} - (m + n\tau)^{-2}] = G_{2,\alpha}(\tau) - G_{2,0}(\tau),
\end{equation}

\begin{equation}
\varphi'(\alpha) = -2\sum_{m,n} (\alpha + m + n\tau)^{-3} = -2G_{3,\alpha}(\tau).
\end{equation}

This interprets the holomorphic weight 2 form $G_{2,\alpha} - G_{2,0}$ as the $x$-coordinate of a torsion point on the Weierstrass model (once normalized by the choice of global differential); similarly, the form $G_{3,\alpha}$ is essentially the $y$-coordinate. The identification $\varphi(\alpha) = G_{2,\alpha}(\tau) - G_{2,0}(\tau)$ is however slightly more delicate than written above, since convergence issues prevent us from simply expanding the sum over $(m, n)$ in the first line above. The end result is correct, however, using techniques similar to the construction in Definition 3.5 below.
It is also interesting to consider the slope \( \lambda = (y_\beta - y_\alpha)/(x_\beta - x_\alpha) \) through two Katz torsion points in the Weierstrass model; this has a natural interpretation as a Katz modular form. Here we assume that \( \alpha, \beta \neq 0 \) and that \( \alpha + \beta \neq 0 \). The line in question through the points \((x_\alpha, y_\alpha)\) and \((x_\beta, y_\beta)\) also passes through the point \((x_\gamma, y_\gamma)\) on the elliptic curve, with \( \alpha + \beta + \gamma = 0 \), by the addition law on the elliptic curve. It is immediate that this slope \( \lambda \) is essentially the sum \( G_1,\beta - G_3,\alpha)/(G_2,\beta - G_2,\alpha) \), so it transforms under \( \Gamma(N) \) the same way as a modular form of weight 1. The question is whether the quotient \( \lambda \) (when viewed as a function of \( \tau \)) is holomorphic or merely meromorphic on \( \mathcal{H} \) and the cusps. However, it is known from the formulas for the addition law on a Weierstrass curve that \( \lambda^2 = x_\alpha + x_\beta + x_\gamma \), and this last sum is a genuine modular form (being a linear combination of \( G_2 \)'s), so \( \lambda \) cannot have any poles. We will show later in this lecture that \( \lambda \) itself is essentially the sum \( G_{1,\alpha} + G_{1,\beta} + G_{1,\gamma} \); this is roughly equivalent to a classical formula for the Weierstrass \( \zeta \) function, but we will present the argument differently below.

As our final preparatory comment on Eisenstein series, we point out that one can bypass Hecke’s analytic continuation in \( s \) by adopting a different point of view on Eisenstein series of weights 1 and 2. Instead of defining a single \( G_{k,\alpha} \), it turns out that one can write down convergent series for certain linear combinations \( \sum_m \alpha m G_{k,\alpha} \), with good convergence for all \( k \geq 1 \). The technique is as follows [KMT12].

**Definition 1.5.** Consider a finite number of \( \alpha \in (1/N)\mathbb{Z}^2 \), and attach to each \( \alpha \) a coefficient \( m_\alpha \in \mathbb{C} \), with the properties

\[
\sum_\alpha m_\alpha = 0, \quad \sum_\alpha m_\alpha \alpha = 0.
\]

Write \( D = \sum_\alpha m_\alpha [\alpha] \) for the formal linear combinations of symbols \([\alpha]\). We then define

\[
G_{k,D}(\tau) = \left[ \sum_\alpha m_\alpha G_{k,\alpha}(\tau, s) \right] = \sum_{\ell \in \mathbb{Z} + \mathbb{Z} \tau} \left( \sum_\alpha m_\alpha (\ell + \alpha)^{-k} \right).
\]

The latter sum converges in the given order \( \sum_\ell (\sum_\alpha') \), since condition (1.9) implies that \( \sum_\ell m_\alpha (\ell + \alpha)^{-k} = O(\ell^{-k-2}) \), which implies good convergence of the sum over \( \ell \) for all \( k > 0 \). Note that if the coefficients \( m_\alpha \) are integers, we can view the formal sum \( D \) as a divisor on the elliptic curve \( E_\tau \). In that setting, (1.9) says that \( D \) is a principal divisor on \( E_\tau \), and that the preimages \( \alpha \) of the points \( \alpha + \mathbb{Z} + \mathbb{Z} \tau \in E_\tau \) are chosen so that their sum (in \( \mathbb{C} \), not just in \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \)) is exactly zero.

We can now describe the construction of our family of Katz modular forms.

**Definition 1.6.** Take a finite number of \( \alpha \in N^{-1}\mathbb{Z}^2 \) and coefficients \( m_\alpha \in \mathbb{Z} \), satisfying condition (1.9) above. Write \( D = \sum_\alpha m_\alpha [\alpha] \) for the formal sum. For \( k \geq 1 \), we define functions \( f_{k,D} \) and \( g_{k,D} \) of the tuple \((E, P, Q, \omega)\) by the following procedure (which works only if \( E \) is defined over a field \( K \) of characteristic zero):

1. Associate to each \( \alpha = (i/N, j/N) \) the point \( P_\alpha = iP + jQ \) on the elliptic curve \( E \), as usual, and form the associated divisor (which by abuse of notation will also be called \( D \)) as above. Thus \( D = \sum_\alpha m_\alpha P_\alpha \), and we will ignore the distinction between \( \alpha \) and \( P_\alpha \) whenever it suits us for the exposition. It follows that \( D \) is a principal divisor on \( E \), and there exists
an element $\phi_D \in K(E)$ (the function field of $E$) with $\text{div} \phi_D = D$. The
values $f_{k,D}(E, P, Q, \omega)$ and $g_{k,D}(E, P, Q, \omega)$ will be constructed out of the
Laurent expansion of $\phi_D$ and its logarithmic derivative $d\phi_D/\phi_D$ at the
origin $O$ of the elliptic curve $E$.

(2) The Laurent expansion of $\phi_D$ at $O$ needs to be expressed in terms of a
uniformizer, i.e., a local coordinate near $O$ which vanishes there. Since
we are in characteristic zero, we can integrate the global form $\omega$ to define
an “analytic uniformizer” $z$ at $O$, with $dz = \omega$. More precisely, start with
an “algebraic uniformizer” $t \in K(E)$ at the origin $O$; for example, if a
Weierstrass form of $E$ is $y^2 = x^3 + ax + b$, then one choice of algebraic
uniformizer is $t = x/y$. Then the completed local ring of $E$ at $O$ is
$\hat{O}_{E, O} = K[[t]]$. In this situation, the expansion of $\omega$ at $O$ can be written
as $\omega = (c_0 + c_1t + c_2t^2 + \cdots)dt$, with $c_0 \neq 0$. Then define $z \in \hat{O}_{E, O}$ by
$z = \int \omega = c_0t + c_1t^2/2 + c_2t^3/3 + \cdots$. It follows that in fact
$\hat{O}_{E, O} = k[[z]]$ as well.

(3) Normalize $\phi_D$, which is unique up to a factor in $K^*$, by requiring that its
Laurent expansion at the origin $O$ be of the form $\phi_D = z^n(1 + f_1z + f_2z^2 +
\cdots)$, where $n$ is the multiplicity of $O$ in $D$ (this is the sum of $m_\alpha$
over all those $\alpha$ that map to $O$ in the curve). Similarly, and without need
for normalizing $\phi_D$, consider the logarithmic differential $d\phi_D/\phi_D$, and its
expansion in terms of $z$ at $O$: $d\phi_D/\phi_D = (n/z + g_1 + g_2z + g_3z^2 + \cdots)dz$.

(4) The value of $f_{k,D}$ at our tuple is then the coefficient $f_k$ above, while the
value of $g_{k,D}$ is the coefficient $g_k$ above.

**Proposition 1.7.** The functions $f_{k,D}$ and $g_{k,D}$ defined above are Katz modular
forms, whose value on the standard tuple $(E_\tau, 1/\tau, \tau/\tau, dz)$ give modular forms in
$M_k(\Gamma(N))$. Moreover, when we evaluate $g_{k,D}$ at the standard tuple associated to
$\tau \in \mathcal{H}$, the value is $g_{k,D}(E_\tau, 1/\tau, \tau/\tau, dz) = -G_{k,D}$.

**Proof.** For the full proof, see Sections 2 and 3 of [KM12]. We note that in
arbitrary characteristic, it is still possible to define Katz modular forms from the
Laurent series coefficients of $\Phi_D$ and its logarithmic differential, when expanded in
terms of the algebraic uniformizer $t$, which can be chosen sufficiently canonically.

Let us sketch a proof of the assertion that $g_{k,D} = -G_{k,D}$, thereby exhibiting
Eisenstein series as special cases of this construction. We first identify the function
$\phi_D$ (up to a constant factor, which disappears in the logarithmic differential).
Morally, we would like to write down directly the desired function with zeros and
poles as predicted by the (translates of the) $\{\alpha\}$:

$$\phi_D(z) = \prod_{\ell \in \mathbb{Z} + \mathbb{Z}_\tau} \left[ \prod_{\alpha'} \left( 1 - \frac{z}{\ell + \alpha} \right)^{m_\alpha} \right],$$

(1.11)

where in the above, the notation $\prod_{\alpha'}$ means that if $\ell + \alpha = 0$, then we include the
factor $z^{m_\alpha}$ instead. (The above product is somewhat different from the usual construction of
$\phi_D = \prod_{\alpha} \sigma(z - \alpha)^{m_\alpha}$ as a product of shifted Weierstrass $\sigma$-functions.)

The question is whether the product in (1.11) is really doubly periodic with
respect to $\mathbb{Z} + \mathbb{Z}_\tau$. The conditions (1.9) ensure that the above product over $\ell$
converges well. Then the identity $1 - \frac{z}{\ell + \alpha} = (1 - \frac{z}{\ell})/(1 - \frac{z}{\ell + \alpha})$, plus the good
convergence of the products $\prod_{\ell} \prod_{\alpha}$ of each factor on the right hand side, tells us
that $\phi_D(z + w) = C_w \phi_D(z)$ for $w \in \mathbb{Z} + \mathbb{Z}_\tau$, and some appropriate constant $C_w$ for
each \( w \). It turns out however that the \( C_w \) are 1, which follows by comparing the logarithmic differential of the product in (1.11):

\[
\frac{d\phi_D}{\phi_D} = \left[ \sum_{\ell \in \mathbb{Z} + \mathbb{Z} \tau} \left( \sum_{\alpha} \frac{m_\alpha}{z - \ell - \alpha} \right) \right] dz,
\]

which agrees with the logarithmic derivative from the “correct” product of \( \sigma \)-functions, as presented in Theorem 2.8 of [KM12]. Once again, the sum over \( \ell \) has good convergence, and we can expand \( m_\alpha/(z - \ell - \alpha) = -m_\alpha((\ell + \alpha)^{-1} + (\ell + \alpha)^{-2}z + \cdots) \) for the pairs with \( \ell + \alpha \neq 0 \); the terms with \( \ell + \alpha = 0 \) contribute \( n/z \). Combining this, we get the expansion of \( d\phi_D/\phi_D \) in terms of the coefficients \( \{g_k\} \), and we immediately identify each \( g_k \) as the negation of the desired Eisenstein series. \( \square \)

2. Lecture 2

This lecture will discuss explicit models for modular curves. At first, we will work primarily with the function field of \( X_0(N) \) when we discuss the modular equation, but in the second part of the lecture, we will view modular forms on \( \Gamma(N) \) primarily as sections of line bundles on the modular curve \( X(N) \). We assume some familiarity with Riemann-Roch spaces, but not necessarily with line bundles and their connection with projective embeddings, which we will discuss informally once we start using those concepts.

We will work entirely over \( \mathbb{C} \), but, as in the first lecture, the reader is encouraged to picture how most of our constructions actually take place over a number field, viewed as a subfield of \( \mathbb{C} \). This is the main, but not the only, source of the arithmetic subtlety captured by modular forms. The background to this is that modular curves such as \( X(N) \) and \( X_0(N) \) have a rich structure in arithmetic geometry, so that rational points on these curves (over a number field \( K \)) correspond to interesting elliptic curves defined over \( K \). Having access to good models of modular curves is also useful in a number of algorithmic applications, such as the Schoof–Elkies–Atkin algorithm for counting points on an elliptic curve over a finite field. The supreme arithmetic application of modular curves is in their relation to the Galois representations attached to Hecke eigenforms, as extensively illustrated in other lectures from this summer school. For a cuspidal eigenform \( f \) of weight 2 on \( \Gamma_1(N) \), say, the mod \( \ell \) Galois representation \( \rho_{f,\ell} \) can be realized inside the \( \ell \)-torsion points of the Jacobian variety of the modular curve \( X_1(N) \), and the \( \ell \)-adic Galois representation can be assembled out of the \( \ell^n \) torsion points of the Jacobian, for varying \( n \). It is thus of interest to be able to find explicit algebraic equations for modular curves and their Jacobians.

These explicit equations can even help with finding explicit models for the Galois representations \( \rho_{f,\ell} \) if the weight of \( f \) is greater than 2: in that case, the Galois representation is realized in an étale cohomology group of a modular curve with respect to a nonconstant system of coefficients. This is less amenable to direct computation, but it turns out that our given \( f \) is in fact congruent modulo \( \ell \) to a Hecke eigenform \( g \) of weight 2 but of level \( \Gamma_1(N\ell) \), so that the mod \( \ell \) representations of \( f \) and \( g \) are the same. So, subject to increasing the level, this reinforces the usefulness of having access to explicit models for modular curves and for working with their Jacobians. This approach is used in the work of Couveignes-Edixhoven [EC11].
and their students to give algorithms to compute explicit Galois representations attached to forms of higher weight.

Now that the reader is, we hope, sufficiently motivated to find models of modular curves, we address the issue of precisely how we can represent a smooth projective algebraic curve $X$, such as $X(N)$. Broadly speaking, one can view such a curve algebraically, via a model for its function field $\mathbb{C}(X)$, or geometrically, via an embedding of the curve $X$ in some projective (or other explicit) space $\mathbb{P}^n$. From the algebraic point of view, the field $\mathbb{C}(X)$ is of transcendence degree 1 over $\mathbb{C}$, so one chooses a transcendental element $x \in \mathbb{C}(X)$, and considers the finite extension $\mathbb{C}(X)$ of $\mathbb{C}(x)$. This is the same as considering a finite map of curves $X \to \mathbb{P}^1$, where $x$ is the coordinate on $\mathbb{P}^1$, so $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(x)$. Let $y \in \mathbb{C}(X)$ be a primitive element for the field extension, so $\mathbb{C}(X) = \mathbb{C}(\mathbb{P}^1)[y]$. Then the elements $x, y$ generate the function field $\mathbb{C}(X)$, and they satisfy a polynomial equation $f(x, y) = 0$. The geometric meaning of this is that $X$ is birationally equivalent to the plane curve with affine equation $f(x, y) = 0$. However, the plane curve in question will usually have singularities (including at infinity, once one moves to the projective plane), and working directly with that plane model can be delicate. What one usually does is to work with the extension $\mathbb{C}(X)/\mathbb{C}(\mathbb{P}^1)$ using algorithms analogous to those used for computing in a number field $\mathbb{Q}(\alpha)/\mathbb{Q}$: there, one computes the integral closure $R$ of $\mathbb{Z}$ in $\mathbb{Q}(\alpha)$ to find the ring of integers, and represents fractional ideals of $R$ as $\mathbb{Z}$-lattices of rank $[\mathbb{Q}(\alpha): \mathbb{Q}]$. In the function field case, one has to consider integral closures over both $\mathbb{C}[x]$ (the analog of $\mathbb{Z}$ here) and a ring such as $\mathbb{C}[1/x]$, in order to get a handle on the points of $X$ lying above $\infty \in \mathbb{P}^1$. In this lecture, we will use the algebraic point of view to describe a model for $X_0(N)$, where the map to $\mathbb{P}^1$ is the natural projection to $X(1)$, the transcendental element generating $\mathbb{C}(X(1))$ is the usual $j$-function, and the polynomial that we called $f(x, y)$ above is in fact the modular polynomial $\Phi_N(j, j')$.

As for the geometric point of view, one can run the range between two extremes. On the one hand, one can represent $X$ as a curve in $\mathbb{P}^2$ or $\mathbb{P}^3$, so the equations of $X$ involve few variables, but can be of high degree. On the other hand, one can take an embedding of $X$ arising from a line bundle of moderately large degree; this yields an embedding of $X$ into a projective space $\mathbb{P}^n$ with $n$ moderately large (but still comparable to the genus $g$ of $X$), however with the benefit that the equations for $X$ now have low degree and a simpler structure. We will illustrate this second approach later for $X(N)$, where the line bundle in question is the one whose sections are modular forms of a given weight on $\Gamma(N)$. That will require us to review a few constructions in algebraic geometry, so we will postpone it, and start with the more concrete approach of using the modular equation to get models for $X_0(N)$.

We thus proceed to study the modular curve $X_0(N)$, which parametrizes pairs of elliptic curves connected by an isogeny whose kernel is cyclic of order $N$. Over $\mathbb{C}$, one can always analytically bring this situation to the map $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}N\tau) \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Equivalently, there are two maps from $X_0(N)$ to $X(1)$, the first sending $\tau \in \Gamma_0(N)\setminus\mathcal{H}$ to $\tau \in \Gamma(1)\setminus\mathcal{H}$, and the second sending $\tau$ to $N\tau \in \Gamma(1)\setminus\mathcal{H}$. The resulting map $X_0(N) \to X(1) \times X(1)$, induced by the map $\tau \mapsto (\tau, N\tau)$ from $\mathcal{H}$ to $\mathcal{H} \times \mathcal{H}$, is a birational equivalence between $X_0(N)$ and its image. Concretely, we can use the $j$-function as a coordinate on $X(1)$ to identify $X(1)$ with $\mathbb{P}^1$. We then obtain that the function field of $X_0(N)$ is generated by the two modular functions.
appear (parametrizing different sublattices of \( \mathbb{Z} \))

representative: coset into single cosets, where we know each \( \Gamma(1) \)-coset contains an upper triangular matrix. The situation with Hecke operators, one considers the decomposition of a double coset into single cosets, where we know each \( \Gamma(1) \)-coset contains an upper triangular matrix.

This means effectively that if one fixes \( \tau \) and hence a value \( j(\tau) \), then the roots in \( y \) of the polynomial \( \Phi_N(j(\tau), y) \) are the values \( y = j((a\tau + b)/d) \) for those \((a, b, d)\) that appear (parametrizing different sublattices of \( \mathbb{Z} + \mathbb{Z}\tau \)) of cyclic index \( N \); note that one of the values of \((a, b, d)\) is \((N, 0, 1)\), which corresponds to the root \( y = j(N\tau) \). We thus conclude that

\[
\sum_{k, \ell} c_{k, \ell} j(\tau)^k y^\ell = \Phi_N(j(\tau), y) = \prod_{\text{the same } a, b, d} [y - j((a\tau + b)/d)].
\]

It follows that the coefficients \( c_{k, \ell} \) for fixed \( \ell \) and varying \( k \) are obtained when one expresses the \( \ell \)-th symmetric polynomial in the \( \{j((a\tau + b)/d)\} \) as a polynomial in \( j(\tau) \). This is possible because this symmetric polynomial is a modular function that is \( \Gamma(1) \) invariant (due to the double coset in (2.2)), and its only pole is at the cusp \( \infty \). We can compute the \( q \)-expansion of this \( \ell \)-th symmetric polynomial from the \( q \)-expansion of \( j(\tau) = q^{-1} + 744 + \cdots \in \mathbb{Z}[q] \), and then identify the resulting series in \( q \) as a polynomial \( \sum_k c_{k, \ell} j(\tau)^k \). In carrying out this calculation, one uses \( j((a\tau + b)/d) = q^{-a/d} - 744 + \cdots \), where \( q = \exp((2\pi i)/d) \). In fact, the calculation takes place over \( \mathbb{Z}[\zeta_N] \), which contains all the \( \zeta_d \). The invariance of everything under \( \Gamma(1) \) implies however that the final result is invariant under any Galois automorphism of \( \mathbb{Q}(\zeta_N) \), which shows that the coefficients in the final answer all belong to \( \mathbb{Z} \).

The coefficients of \( \Phi_N \) are notoriously large, and the birational plane model for \( X_0(N) \) given by the equation \( \Phi_N(j, j') = 0 \) is rather singular, but this model is still quite useful in explicit computations. It should be pointed out that there are now better ways to compute the modular polynomial, namely, by interpolation.

The degree of \( \Phi_N \) is known (e.g., for \( N \) prime, it is \( N + 1 \)), and one knows that \( \Phi_N(x, y) = \Phi_N(y, x) \), because the dual morphism to a cyclic \( N \)-isogeny is again cyclic of degree \( N \). It follows that it is enough to generate enough points \((j_0, j_0')\) on the curve \( \Phi_N(j, j') = 0 \), in order to obtain enough values to solve for the coefficients...
The articles [Eng09] and [BLST12] do this respectively for collections of points \((j_\alpha, j'_\alpha) \in \mathbb{C}^2\) or \((j_\alpha, j'_\alpha) \in \mathbb{F}_p^2\), by taking a suitable collection of isogenous pairs of elliptic curves over \(\mathbb{C}\) or \(\mathbb{F}_p\). In the latter setting, one gets equations for the \(c_{k,t} \mod p\), which one can combine for various \(p\) to obtain the true value over \(\mathbb{Z}\).

We now move on to the second approach outlined in the introduction of finding equations for modular curves. As promised, we begin with an informal overview of the needed prerequisites from algebraic geometry: line bundles on (as always, smooth projective) algebraic curves. For pedagogical reasons, we continue to work over \(\mathbb{C}\), to allow the reader to visualize the situation in the analytic category, not just algebraically.

**Definition 2.2.** A complex line bundle \(\mathcal{L}\) on an algebraic curve \(X\) is a choice, for each point \(p \in X\), of a one-dimensional complex vector space \(\mathcal{L}_p\), in a way that “varies holomorphically” with \(p\).

Concretely, this means that one can cover \(X\) by open sets \(\{U_i\}\) such that, for each \(U_i\), the totality of vector spaces \(\mathcal{L}_{U_i} = \{\mathcal{L}_p \mid p \in U_i\}\) is isomorphic to the product \(U_i \times \mathbb{C}\). This means that there is an isomorphism (of two-dimensional complex manifolds) \(\psi_i : \mathcal{L}_{U_i} \to U_i \times \mathbb{C}\), where a vector \(v \in \mathcal{L}_p\) is mapped to \(\psi_i(v) = (p, c)\) for some \(c \in \mathbb{C}\), and the map sending \(v\) to \(c\) is a \(\mathbb{C}\)-linear isomorphism between \(\mathcal{L}_p\) and \(\mathbb{C}\); hence the set \(\mathcal{L}_p \subset \mathcal{L}_{U_i}\) is isomorphic to \(\{p\} \times \mathbb{C} \subset U_i \times \mathbb{C}\).

Whenever \(U_i \cap U_j \neq \emptyset\), these two different identifications of \(\mathcal{L}_p\) can be compared via a homomorphic nowhere vanishing transition function \(\varphi_{i,j} : U_i \cap U_j \to \mathbb{C}^*\), where \(\psi_j(\varphi_{i,j}^{-1}(p, c)) = (p, \varphi_{i,j}(p)c)\).

Conversely, given a covering \(\{U_i\}\) of \(X\) by open sets, and a collection of transition functions \(\varphi_{i,j}\) (which need to be compatible on intersections \(U_i \cap U_j \cap U_k\)), then one can glue the line bundles \(\{U_i \times \mathbb{C}\}\) together, using the \(\varphi_{i,j}\), to obtain a line bundle \(\mathcal{L}\) on \(X\).

The key concept that will matter to us is that of a holomorphic section of a line bundle \(\mathcal{L}\) on \(X\). This generalizes holomorphic functions on \(X\), which are sections of the trivial line bundle \(X \times \mathbb{C}\).

**Definition 2.3.** Let \(\mathcal{L}\) be a line bundle on \(X\). A (holomorphic) section \(s\) of \(\mathcal{L}\) is a function \(s : X \to \mathcal{L}\), such that for every \(p \in X\), we have \(s(p) \in \mathcal{L}_p\). In terms of the local isomorphisms \(\psi_i : \mathcal{L}_{U_i} \to U_i \times \mathbb{C}\), requiring \(s\) to be holomorphic means that for \(p \in U_i\), \(\psi_i(s(p)) = (p, f_i(p))\) with \(f_i : U_i \to \mathbb{C}\) a holomorphic function. The resulting “values in local coordinates” \(f_i\) of the section \(s\) will then be compatible in the sense that for \(p \in U_i \cap U_j\), we have \(f_j(p) = \varphi_{i,j}(p)f_i(p)\).

The set of holomorphic sections is written \(H^0(X, \mathcal{L})\); it is a finite-dimensional vector space, that we can always identify with a Riemann-Roch space, as we will discuss presently. We can also consider meromorphic sections of \(\mathcal{L}\), which the reader should have no trouble defining. Although \(H^0(X, \mathcal{L})\) can be zero, there are always nonzero meromorphic sections of \(\mathcal{L}\).

We now describe the relation with Riemann-Roch spaces. Recall that for a divisor \(D = \sum n_p p\) on \(X\), the Riemann-Roch space \(L(D)\) is the set of function field elements \(f \in \mathbb{C}(X)\) that satisfy \(\text{div} f + D \geq 0\); in other words, for each of the (finitely many) \(p\) in the support of \(D\), we have \(v_p(f) \geq -n_p = -v_p(D)\). Here the valuation \(v_p\) gives the order of the zero of \(f\) at \(p\) (or of the pole, if \(v_p(f) < 0\)); for convenience, we set \(v_p(0) = +\infty\). We remark that the valuation \(v_p\) also makes sense
for a meromorphic section \( s \) of a line bundle \( \mathcal{L} \), where it will be written \( v_{p,\mathcal{L}}(s) \). Namely, suppose \( p \in U_i \) for one of the open sets of the cover, where \( s \) is represented by the function \( f_i \). Then \( v_{p,\mathcal{L}}(s) = v_p(f_i) \). This is independent of the choice of \( U_i \) containing \( p \), since \( \varphi_{i,j}(p) \neq 0 \) whenever \( p \in U_i \cap U_j \).

**Proposition 2.4.** Let \( D \) be a divisor on \( X \). Then there exists a line bundle \( \mathcal{L}_D \) with the property that a meromorphic section \( s \) of \( \mathcal{L}_D \) can be identified with a meromorphic function \( \phi_s \in \mathcal{C}(X) \) on \( X \), but with a modified valuation: \( v_{p,\mathcal{L}_D}(s) = v_p(\phi_s) + v_p(D) \). Thus \( s \in H^0(X, \mathcal{L}_D) \) if and only if for every \( p \in X \), we have \( v_{p,\mathcal{L}_D}(s) \geq 0 \), which corresponds precisely to \( \phi_s \in L(D) \); note that it is possible to have \( H^0(X, \mathcal{L}_D) = L(D) = 0 \).

Conversely, every line bundle \( \mathcal{L} \) on \( X \) is isomorphic to a line bundle \( \mathcal{L}_D \) for some \( D \), which is unique up to equivalence of divisors (by principal divisors of rational functions in \( \mathcal{C}(X) \)).

**Proof.** Every divisor \( D \) is locally principal, in the sense that there exists an open cover \( \{U_i\} \) of \( X \), each with an analytic function \( u_i \) satisfying \( (\text{div } u_i)|_{U_i} = D|_{U_i} \). (The restriction \( D|_{U_i} \) of a divisor \( D \) can be thought of as the intersection \( D \cap U_i \), i.e., the restriction includes only those points of \( D \) that belong to \( U_i \).) Then construct \( \mathcal{L}_D \) by gluing the \( U_i \times \mathcal{C} \) along the transition functions \( \varphi_{i,j} = u_j/u_i \). A holomorphic (respectively, meromorphic) section \( s \) of \( \mathcal{L}_D \) thus corresponds to a collection \( \{f_i\} \) of holomorphic (respectively, meromorphic) functions on each \( U_i \), satisfying \( f_j = (u_j/u_i)f_i \). Every section \( s \) corresponds to the unique \( \phi_s \) that is obtained by gluing together the functions \( f_i/u_i \). So locally, \( f_i = \phi_s \cdot u_i \). Recall that for \( p \in U_i \), we have \( v_{p,\mathcal{L}_D}(s) = v_p(f_i) \); this yields the desired relation between the valuations of \( s \) and \( \phi_s \).

For the converse, let \( \mathcal{L} \) be given, choose any nonzero meromorphic section \( s_0 \) of \( \mathcal{L} \), and let \( D = \sum_p v_{p,\mathcal{L}}(s_0) \cdot p = \text{div}(\mathcal{L}) \cdot s_0 \) be the divisor of \( s_0 \), viewed as a section of \( \mathcal{L} \). Then we can identify any other meromorphic section \( s \) of \( \mathcal{L} \) with the meromorphic function \( \phi_s = s/s_0 \in \mathcal{C}(X) \); note that although the values of \( s \) and \( s_0 \) at a point \( p \) belong to \( \mathcal{L}_p \), their ratio is canonically an element of \( \mathcal{C} \) (at least, away from the poles of \( \phi_s \)). This identifies \( \mathcal{L} \) with \( \mathcal{L}_D \); incidentally, the section \( s_0 \) of \( \mathcal{L} \) corresponds to the collection of functions \( \{f_i\} = \{u_i\} \) which give a section of \( \mathcal{L}_D \). In terms of \( \mathcal{L} \), the bijection between \( L(D) \) and \( H^0(X, \mathcal{L}) \) identifies \( \phi \in L(D) \) with \( \phi \cdot s_0 \in H^0(X, \mathcal{L}) \). Finally, if we make a different choice of meromorphic section \( s_1 \) instead of \( s_0 \) at the start, this modifies \( D \) by the principal divisor \( \text{div}(s_1/s_0) \). \( \Box \)

The next important notion in our overview is the degree of a line bundle.

**Definition 2.5.** Let \( \mathcal{L} \) be a line bundle on the Riemann surface \( X \). We say that \( \deg \mathcal{L} = d \) if one meromorphic section \( s \) of \( \mathcal{L} \) vanishes at exactly \( d \) points, counting multiplicities, and subtracting any multiplicities of poles. Thus \( \deg \mathcal{L} = \deg \text{div}(\mathcal{L}) = \sum_p v_{p,\mathcal{L}}(s) \), and this degree does not depend on the choice of \( s \), since all other choices are of the form \( sf \) with \( f \in \mathcal{C}(X) \), with moreover \( \deg \text{div } f = 0 \).

Equivalently, if \( \mathcal{L} \cong \mathcal{L}_D \), then \( \deg \mathcal{L} = \deg D \).

A basic consequence of Riemann-Roch is that if \( X \) has genus \( g \), and \( \deg \mathcal{L} \geq 2g - 1 \), then \( \dim H^0(X, \mathcal{L}) = \deg \mathcal{L} + 1 - g \). Another consequence is that if \( \deg \mathcal{L} \geq 2g \), then \( \mathcal{L} \) is base point free, which means that for every \( p \in X \), there exists a holomorphic section \( s \in H^0(X, \mathcal{L}) \) with \( s(p) \neq 0 \).

We are now ready to discuss some aspects of the relation between line bundles on a curve \( X \), and maps from \( X \) to a projective space.
Definition 2.6. Let \( \mathcal{L} \) be a base point free line bundle on \( X \). Take a basis \( \{ s_0, s_1, \ldots, s_n \} \) for \( H^0(X, \mathcal{L}) \) (more generally, we only need a basis for a base point free subspace of \( H^0(X, \mathcal{L}) \)). The associated map from \( X \) to the projective space \( \mathbb{P}^n \) is given by
\[
(2.4) \quad \varphi : X \rightarrow \mathbb{P}^n, \quad \varphi(p) = [s_0(p) : s_1(p) : \cdots : s_n(p)].
\]
Note in the above that, as usual, the values \( s_i(p) \) all belong to the one-dimensional vector space \( \mathcal{L}_p \), but that the proportions between their values make enough sense for us to get the projective coordinates of a point in \( \mathbb{P}^n \). The reason for requiring \( \mathcal{L} \) to be base point free is to ensure that we never map a point \( p \) to the invalid projective point \([0 : 0 : \cdots : 0]\).

Example 2.7. Let \( X \) be an elliptic curve, say for definiteness with affine equation \( y^2 = x^3 + 3141x + 5926 \), and let \( O \in X \) be the point at infinity. Consider the line bundles \( \mathcal{L}_{3O} \) and \( \mathcal{L}_{4O} \). We can identify \( H^0(X, \mathcal{L}_{3O}) \) with the Riemann-Roch space \( \mathcal{L}_{3O} \), which has the basis \( \{ 1, x, y \} \). The resulting map from \( X \) to the projective plane is the usual one; it sends the affine point \( p \) to the projective point \([1 : x(p) : y(p)]\), while the point \( O \) is sent to \([0 : 0 : 1]\). One can see this by “continuity”, because of the Laurent series \( x = t^{-2} + \cdots \) and \( y = t^{-3} + \cdots \) in terms of a uniformizer \( t \) at \( O \), so as our point “approaches” \( O \), its projective coordinates \([1 : t^{-2} + \cdots : t^{-3} + \cdots] = [t^3 : t + \cdots : 1 + \cdots] \) “approach” \([0 : 0 : 1]\). A less informal way to see this is to remember that the sections in \( H^0(X, \mathcal{L}_{3O}) \) corresponding to \( 1, x, y \) are in fact everywhere holomorphic, when viewed as sections of the line bundle, and to work with a trivialization of \( \mathcal{L}_{3O} \) in a neighborhood of \( O \); this corresponds to the transition function \( y^{-3} \) sending the local coordinate functions \( 1, x, y \) to \( 1/y, x/y, 1 \) near \( O \).

As for the line bundle \( \mathcal{L}_{4O} \), take the basis \( \{ s_0, s_1, s_2, s_3 \} \) of \( H^0(X, \mathcal{L}_{4O}) \), corresponding to the basis \( \{ 1, x, y, x^2 \} \) of \( L(4O) \). The resulting map \( p \mapsto [s_0(p) : s_1(p) : s_2(p) : s_3(p)] \in \mathbb{P}^3 \) embeds \( X \) as the intersection of the two quadric surfaces \( s_1^2 - s_0s_3 = 0 \) and \( s_2^2 - s_1s_3 - 3141s_0s_1 - 5926s_0^2 = 0 \).

In the above example, the image of the genus 1 curve \( X \) under the projective embedding given by \( \mathcal{L}_{4O} \) is described by quadrics (i.e., by polynomials of degree 2). This is a special case of the following general theorem, due independently to Fujita [Fuj77] and Saint-Donat [SD72a, SD72b], building on results of Castelnuovo and Mumford:

Theorem 2.8. If \( X \) has genus \( g \) and \( \deg \mathcal{L} \geq 2g + 2 \), then the map to projective space given by \( \mathcal{L} \) is an embedding of \( X \), and the image is defined by quadrics; more precisely, the homogeneous ideal of vanishing of the image of \( X \) in projective space is generated by its degree 2 elements.

We now finally come to the application of all this to modular curves. We first review how modular forms of weight \( k \) on \( \Gamma(N) \), with \( N \geq 3 \), are sections of a particular line bundle \( \mathcal{L}_k \) on \( X(N) \); this result holds in fact for any subgroup of \( \Gamma(1) \), but the advantage of the group \( \Gamma(N) \) is that it has no elliptic points for \( N \geq 3 \), and all its cusps are moreover regular; it follows that \( \mathcal{L}_k \cong \mathcal{L}_1^{\otimes k} \), so that \( \deg \mathcal{L}_k = k \deg \mathcal{L}_1 \). In the presence of elliptic points or irregular cusps, the degree of \( \mathcal{L}_k \) is slightly more delicate; see for example the discussion of the divisor of a modular form in Chapter 2 of [Sh94].
To define the line bundle $L_k$ on $X(N)$, we depart from our previous description in terms of an open cover, and instead obtain $L_k$ as the quotient of the trivial bundle on $\mathcal{H}$ by a nontrivial action of $\Gamma(N)$. To be precise, we need to consider the extended upper half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$, with the topology given in Chapter I of Shi94, so as to correctly deal with the cusps; we will however ask for the reader’s indulgence, and gloss over this important point from here on. The idea is that (holomorphic) sections of the trivial bundle $\mathcal{H} \times \mathbb{C}$ are precisely holomorphic functions $f : \mathcal{H} \to \mathbb{C}$. We define an action of $\Gamma(1)$ on the line bundle $\mathcal{H} \times \mathbb{C}$, in such a way that sections invariant under a subgroup $\Gamma$ of $\Gamma(1)$ are precisely the modular forms of weight $k$ on $\Gamma$. One can then see that the desired action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ on a pair $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ is given by

$$
(2.5) \quad \gamma \cdot (\tau, z) = \left( \frac{a\tau + b}{c\tau + d}, \frac{(c\tau + d)k}{z} \right).
$$

We then define $L_k$ to be the resulting line bundle on $X(N) = \Gamma(1) \backslash \mathcal{H}$ (where we apologize one last time about the cusps) whose total space is $\Gamma(N) \backslash (\mathcal{H} \times \mathbb{C})$. It follows that

$$
(2.6) \quad H^0(X(N), L_k) = M_k(\Gamma(N));
$$

it turns out to be slightly more convenient to use the full space of modular forms than to restrict to cusp forms, which would correspond to sections of $L_k$ that vanish at all cusps.

We can now use a basis $\{f_0, \ldots, f_n\}$ for $M_k(\Gamma(N))$ to obtain a projective embedding of $X(N)$ into $\mathbb{P}^n$, at least when $\deg L_k \geq 2g + 2$, with $g$ the genus of $X(N)$. Similarly to (2.4), this sends $\tau \in \Gamma(N) \backslash \mathcal{H}$ to $[f_0(\tau) : \cdots : f_n(\tau)] \in \mathbb{P}^n$; the value in projective space is independent of the representative $\tau$ chosen, provided all the $f_i$ are evaluated at the same $\tau$.

**Proposition 2.9.** Let $N \geq 3$. For the resulting curve $X(N)$, the line bundle $L_2$ has degree $\deg L_2 \geq 2g + 2$, and hence gives rise to a projective embedding of the modular curve with image described by quadrics. Knowing the equations of the resulting modular curve is equivalent to knowing the multiplication map $M_2(\Gamma(N)) \times M_2(\Gamma(N)) \to M_4(\Gamma(N))$.

**Proof.** We can relate $L_2$ to the canonical line bundle $\Omega^1$ of holomorphic 1-forms on $X(N)$. It is standard that $H^0(X(N), \Omega^1) \cong \mathcal{S}_2(\Gamma(N))$ (in fact, this works for any group $\Gamma$), by identifying a cusp form $f(z)$ with the differential form $f(z)dz$, which is now invariant under $\Gamma(N)$. The reason that the corresponding modular forms are cuspidal can be seen in terms of the local coordinate at infinity $q^{1/N} = \exp(2\pi i z/N)$. Since $2\pi i N^{-1}dz = q^{-1/N}d(q^{1/N})$, the $q$-expansion of $f$ must start with $c_1q^{1/N} + c_2q^{2/N} + \cdots$ for $f(z)dz$ to avoid a pole at $q = 0$, i.e., at $z = \infty$; a similar condition holds at the other cusps. A modular form which does not have to be cuspidal thus corresponds to a meromorphic section of $\Omega^1$, with a possible simple pole at each cusp of $X(N)$. This means that $L_2 \cong \Omega^1 \otimes L_{\text{cusps}}$, where $\text{cusps}$ is the divisor of the cusps. In particular,

$$
(2.7) \quad \deg L_2 = \deg \Omega^1 + \deg(\text{cusps}) = (2g - 2) + c,
$$

where $c$ is the number of cusps of $X(N)$. But one knows that $c \geq 4$ once $N \geq 3$. Thus $\deg L_2 \geq 2g + 2$, as desired, and the projective embedding given by a basis of $M_2(\Gamma(N))$ is described by quadrics.
Let \( \{f_0, \ldots, f_n\} \) be a basis for \( \mathcal{M}_2(\Gamma(N)) \). Suppose we wish to determine the quadrics that vanish on the image of \( X(N) \), since these generate the homogeneous ideal of the projective curve. The presence of such a quadric of the form \( q(T_0, \ldots, T_n) = \sum_{i,j} c_{i,j}T_iT_j \), in terms of the homogeneous coordinates \([T_0 : \cdots : T_n]\), corresponds to the identity of modular forms \( \sum_{i,j} c_{i,j}f_if_j = 0 \in \mathcal{M}_4(\Gamma(N)) \). Hence, to find the generators of our homogeneous ideal, it is enough to know how to multiply \( f_if_j \) for every pair \( (i,j) \), and how to find linear relations between these elements of \( \mathcal{M}_4(\Gamma(N)) \).

We remark that the same result holds for \( X_1(N) \) with \( N \geq 5 \). Let us therefore compute equations for \( X_1(5) \) as an example. This is not extremely interesting, since the genus of \( X_1(5) \) is zero, but it illustrates the above theorem. In this situation, we have \( \dim \mathcal{M}_2(\Gamma_1(5)) = 3 \), and a basis is \( \{f, g, h\} \) with

\[
\begin{align*}
  f &= 1 + 60q^3 - 120q^4 + \cdots, \\
  g &= q + 6q^3 - 9q^4 + \cdots, \\
  h &= q^2 - 4q^3 + 12q^4 + \cdots.
\end{align*}
\]

One also knows that \( \mathcal{M}_4(\Gamma_1(5)) \) is 5-dimensional, and its elements are determined by knowing their \( q \)-expansions up to and including the \( q^4 \) term. One then computes \( f^2 - 1 + 120q^3 - 240q^4 + \cdots, fg = q + 6q^3 + 51q^4 + \cdots \), and so forth, all of which belong to \( \mathcal{M}_4(\Gamma_1(5)) \). One then obtains the equation

\[
g^2 - fh - 4gh - 16h^2 = 0,
\]

which means that we have identified \( X_1(5) \) with the conic in the projective plane given by the equation \( U^2 - TV - 4UV - 16V^2 = 0 \). The conic contains the rational point \([T : U : V] = [1 : 0 : 0]\) (namely, the point \( q = 0 \) corresponding to the cusp \( \infty \)), so we can identify the curve \( X_1(5) \) with \( \mathbb{P}^1 \) over \( \mathbb{Q} \), not just over \( \mathbb{C} \).

We point out that describing the multiplication map \( \mathcal{M}_2(\Gamma(N)) \times \mathcal{M}_2(\Gamma(N)) \to \mathcal{M}_4(\Gamma(N)) \) can be done by interpolation, which ties in with our earlier description of finding the modular equation \( \Phi_N \) by interpolation. Namely, suppose that we take a large number (“\( L \)”) of points \( \tau_1, \ldots, \tau_L \in X(N) \), where we require \( L > \deg \mathcal{L}_4 \). Then a modular form \( f \in \mathcal{M}_4(\Gamma(N)) \) is completely determined by its values at these \( L \) points, because if \( g \) were a different form agreeing with \( f \) at \( \tau_1, \ldots, \tau_L \), then the difference \( f - g \) would be a nonzero section of the line bundle \( \mathcal{L}_4 \), so \( f - g \) could only vanish at \( \deg \mathcal{L}_4 \) points, contradicting the fact that it vanishes at \( \tau_1, \ldots, \tau_L \). In that case, the basis \( \{f_0, \ldots, f_n\} \) of \( \mathcal{M}_4(\Gamma(N)) \), as well as all products \( f_if_j \), can be represented by their values at \( \tau_1, \ldots, \tau_L \). Thus we can identify \( f_i \) by its vector of values \( (f_i(\tau_1), \ldots, f_i(\tau_L)) \), and carry out multiplication into \( \mathcal{M}_4 \) componentwise in order to find the quadrics that vanish on the image of \( X(N) \). This is equivalent to finding the quadrics on \( \mathbb{P}^n \) that vanish on (i.e., interpolate through) the projective points \( P_1, \ldots, P_L \), which are the images of \( \tau_1, \ldots, \tau_L \) under the projective embedding. Concretely,

\[
P_j = [f_0(\tau_j) : \cdots : f_n(\tau_j)]
\]

so we have reversed our viewpoint. Whereas we previously fixed \( f_i \) and represented it as a function by its values at varying \( \tau_j \), we now fix \( \tau_j \) and represent it as a projective point by the values of the various \( f_i \) at that point. It turns out \cite{KM07} that this is an effective way to represent the curve if one is interested in computing with its Jacobian; this is called “Representation B” in that article. We also note
that the points $\tau_1, \ldots, \tau_L$ do not actually have to be distinct; representing modular forms by their $q$-expansions up to degree $q^L$, as we did in the example above, is a way of evaluating the forms at the divisor $L \cdot \infty$. We note that the family of Katz modular forms from Lecture 1 gives an ample supply of modular forms that can be easily evaluated at points, since evaluating at a $\tau_j$ can be carried out algebraically by evaluating on a tuple $(E, P, Q, \omega)$.

3. Exercises

The following exercises were distributed to students at the summer school.

**Exercise 0.** Give the argument that evaluating a weight $k$ Katz modular form on the tuple $(E, \tau, 1/N, \tau/N, dz)$ defines a function $f(\tau)$ that transforms like a usual weight $k$ modular form.

**Exercise 1.**

a) Use SageMath or Magma (or any other software) to find the modular polynomial $\Phi_2(X, Y)$ from the identity
\[ \Phi_2(X, j(\tau)) = (X - j(2\tau))(X - j(\tau/2))(X - j((\tau + 1)/2)), \]
by comparing $q$-expansions.

b) Find $\Phi_2(X, Y)$ in a different way, by finding enough pairs $(j, j')$ of $j$-invariants of elliptic curves that are 2-isogenous to interpolate $\Phi_2$ through these points. (What is the degree of $\Phi_2$ in each of $X$ and $Y$, and how many points are needed? I suggest taking curves $E : y^2 = x(x - 1)(x - \lambda)$ for a few values of $\lambda \in \mathbb{Q}$, and all their quotients by cyclic 2-torsion subgroups.)

c) Question to think about later: can you find, e.g., $\Phi_5(X, Y)$ by finding its reduction modulo many primes? This involves finding for each $p$ a number of pairs of 5-isogenous elliptic curves over $\mathbb{F}_p$ and interpolating through the corresponding $(j, j') \bmod p$.

**Exercise 2.** In this exercise, you may assume that $N \geq 3$ is prime if you like, but see if you can do the general case too.

a) What is the index $[\Gamma(1) : \Gamma(N)]$? What is the degree $d$ of the map $\pi : X(N) \to X(1)$ between modular curves?

b) What is the ramification of $\pi$ at the cusps? Use this to find the number $c$ of cusps of $X(N)$.

c) Let $L$ be the line bundle on $X(N)$ whose sections give $\mathcal{M}_1(\Gamma(N))$. Show that $\deg L = d/12$. (Hint: $\Delta(z) \in S_{12}(\Gamma(1)) \subset S_{12}(\Gamma(N))$.)

d) Find the genus of $X(N)$ in terms of $d$ and $c$. (Hint: consider the line bundles $L^2$, whose sections are $\mathcal{M}_2(\Gamma(N))$, and $\Omega^1 \cong L^2(-\text{cusps})$, whose sections are $\mathcal{S}_2(\Gamma(N))$.)

**Exercise 3.** Let $X$ be the (projective model of the) curve $y^3 = x^4 + x + 2$. Its points are the affine points satisfying the above equation, plus one point $P_0$ at infinity where the rational function $x$ has a pole of order 3, and $y$ has a pole of order 4. This is a $C_{3,4}$ curve; generally, $C_{a,b}$ curves are a nice source of examples. (However, if $|a - b| \geq 2$, then the plane model of a $C_{a,b}$ curve is singular at infinity.)

a) Compute the Riemann-Roch spaces $L(kP_0)$ for $k \leq 15$, and deduce that $X$ has genus 3, either from Riemann-Roch or by any other method you like.

b) For each of $k = 6, 7, 8$, consider the resulting map of $X$ to projective space, and give generators for the ideal describing the image.
Exercise 4. Our goal is to find equations for $X = X_1(11)$, which has genus 1.

a) Use SageMath or Magma to find $q$-expansions of a basis for $M_2(\Gamma_1(11))$, ordered as an echelon basis in terms of the order of vanishing at the cusp $\infty$ (i.e., $q = 0$).

b) The projective embedding of $X$ given by $M_2(\Gamma_1(11))$ has too large dimension for a human-readable model of the curve. Instead, obtain a smaller embedding by restricting to a subset $V \subset M_2(\Gamma_1(11))$ defined by imposing a certain order of vanishing at the cusp $\infty$. This means that, viewing $M_2(\Gamma_1(11)) = H^0(X, \mathcal{L})$ for a suitable line bundle $\mathcal{L}$ on $X$, your space $V$ will be $H^0(X, \mathcal{L}(-k\infty))$ for some $k$. Thus the line bundle you will consider will have degree $(\deg \mathcal{L}) - k$.

Suggestion: Take $(\deg \mathcal{L}) - k = 3$ or 4. This will produce either one cubic equation in $\mathbb{P}^2$, or two quadric equations in $\mathbb{P}^3$.

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