QUANTUM COHERENT ATOMIC TUNNELING BETWEEN
TWO TRAPPED BOSE-EINSTEIN CONDENSATES

A. Smerzi 1, S. Fantoni 1,2, S. Giovanazzi 1 and S. R. Shenoy 2
1) International School of Advanced Studies, via Beirut 2/4, I-34014, Trieste, Italy
2) International Centre for Theoretical Physics, I-34100, Trieste, Italy

Abstract

We study the coherent atomic tunneling between two zero-temperature Bose-Einstein condensates (BEC) confined in a double-well magnetic trap. Two Gross-Pitaevskii equations for the self-interacting BEC amplitudes, coupled by a transfer matrix element, describe the dynamics in terms of the inter-well phase-difference and population imbalance. In addition to the anharmonic generalization of the familiar ac Josephson effect and plasma oscillations occurring in superconductor junctions, the non-linear BEC tunneling dynamics sustains a self-maintained population imbalance: a novel "macroscopic quantum self-trapping effect".

PACS numbers: 03.75.Fi, 74.50.+r, 05.30.Jp, 32.80.Pj
The recent experimental observation of the Bose-Einstein condensation (BEC) in a dilute gas of trapped atoms [1,2], has generated much interest in the properties of this new state of matter. A quite fascinating possibility is the observation of new quantum phenomena on macroscopic scales, related with the superfluid nature of the condensate. In fact, broken symmetry arguments show that the condensate atoms can be described by a common, "macroscopic" one-body wave function \( \Psi(\vec{r}, t) = \sqrt{\rho} e^{i\theta} \) (the order parameter), with \( \rho \) the condensate density. For a weakly interacting BEC, the order parameter obeys a non-linear Schroedinger, or Gross-Pitaevskii equation (GPE) [3]:

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + [V_{\text{ext}}(\vec{r}) + g_0 \|\Psi\|^2]\Psi
\]  

(1)

where \( V_{\text{ext}} \) is the external potential and \( g_0 = \frac{4\pi\hbar^2a}{m} \) is the inter-atomic scattering pseudo-potential, with \( a, m \) the atomic scattering length and mass respectively.

The GPE has been successfully applied to investigate the collective mode frequencies of a trapped BEC in the linear regime [4], the relaxation times of monopolar oscillations [5], and, because of the non-linear self-interaction, it could also induce chaotic behavior in dynamical quantum observables [5,6].

The existence of a macroscopic quantum phase (difference) was dramatically demonstrated recently [2]. A far off-resonant intense laser sheet divided a trapped condensate, creating a large barrier in between. Switching off the double-well trap, the two released condensates overlapped, producing a robust "two-slit" atomic interference pattern, clear signature of phase coherence over a macroscopic scale (\( \sim 10^{-2}\text{cm} \)). The non-destructive detection of phase-differences between two trapped BEC could be achieved by lowering the intensity of the laser sheet. This allows for atomic tunneling through the barrier, and the detection of Josephson-like current-phase effects [2,7–9]. In superconductor Josephson junctions (SJJ), phase coherence signatures include a \( dc \) external voltage producing an \( ac \) current, or the "plasma" oscillations of an initial charge imbalance [10,11]. For neutral superfluid He II, voltage drives, tunnel junctions, or capacitive charges are absent. The only accessible Josephson analogue [12] involves two He II baths connected by a sub-micron orifice, at
which vortex phase-slips \cite{13} support a chemical potential (height) difference, through the Josephson frequency relation.

Although the trapped BEC is also a neutral-atom Bose system, its population can be monitored by phase-contrast microscopy; the double-well curvatures and barrier heights can be tailored by the position and the intensity of the laser sheet partitioning the magnetic trap \cite{2}. We note that the chemical potential between the two condensates depends both on the zero-point energy difference, that acts like an external "dc" SJJ voltage, and on the non-linear interaction that, through an initial population imbalance, acts like a capacitive SJJ charging energy. Thus, we suggest that the BEC tunnel junction can show the analogues of the familiar Josephson effects in superconductor junctions, with the ability to tailor traps and the atomic interaction compensating for electrical neutrality.

In this Letter we study the atomic tunneling at zero temperature between two non-ideal, weakly linked BEC in a (possibly asymmetric) double well trap. This induces a coherent, oscillating flux of atoms between wells, that is a signature of the macroscopic superposition of states in which the condensates evolves. The dynamics is governed by two Gross-Pitaevskii equations for the BEC amplitudes, coupled by a transfer matrix element (Josephson tunneling term). Analogues of the superconductor Josephson effects such as the ac effects and "plasma" oscillations are predicted. We also find that the non-linearity of the dynamic tunneling equations produces a novel self-trapping effect.

Consider a double-well magnetic trap 1,2 as in Fig.(1). This system can be described by a two-state model:

\begin{align}
   i\hbar \frac{\partial \psi_1}{\partial t} &= (E^0_1 + U_1 N_1)\psi_1 - K\psi_2 \\
   i\hbar \frac{\partial \psi_2}{\partial t} &= (E^0_2 + U_2 N_2)\psi_2 - K\psi_1
\end{align}

with uniform amplitudes \( \psi_{1,2} = \sqrt{N_{1,2}} e^{i\theta_{1,2}} \), where \( N_{1,2}, \theta_{1,2} \) are the number of particles and phases in the trap 1,2 respectively, and \( K \) is the coupling matrix element \cite{4}. The parameters \( E^0_{1,2}, U_{1,2} \) and \( K \) can be determined from appropriate overlap integrals of the time-independent GPE eigenfunctions \( \Phi_{1,2} \) of the isolated traps, as outlined later. The total
number of atoms $N_T = N_1 + N_2$ is constant, but we stress that a coherent phase description, i.e. the existence of definite phases $\theta_{1,2}$, implies that the phase fluctuations ($\simeq \frac{1}{\sqrt{N_{1,2}}}$) must be small, giving $N_{1,2} > N_{min} \simeq 10^3$, say.

We note that the BEC inter-trap tunneling Eq.s(2) are similar in form to models of single-trap atomic-level transitions [8], or polaron hopping in semiclassical approximation [16], although they describe quite different physics.

In terms of the phase-difference $\phi = \theta_2 - \theta_1$ and fractional population difference $-1 < z = \frac{N_1 - N_2}{N_T} < 1$, Eq.s(2) become ($\hbar = 1$):

\begin{equation}
\dot{z} = -\sqrt{1 - z^2} \sin \phi
\end{equation}

\begin{equation}
\dot{\phi} = \Lambda z + \frac{z}{\sqrt{1 - z^2}} \cos \phi + \Delta E
\end{equation}

where the time has been rescaled as $2Kt \to t$. The dimensionless parameters are:

\begin{equation}
\Delta E = (E_1^0 - E_2^0)/(2K) + (U_1 - U_2)N_T/(4K)
\end{equation}

\begin{equation}
\Lambda = (U_1 + U_2)N_T/(4K)
\end{equation}

For two symmetric traps, $E_1^0 = E_2^0$ ($\Delta E = 0$), $U_1 = U_2 = U$, and $\Lambda = UN_T/2K$. In the following we will assume a positive scattering length $a$ ($\Lambda > 0$); note, however, that Eq.s(3) are invariant under the transformation $\Lambda \to -\Lambda$, $\phi \to -\phi + \pi$, $\Delta E \to \Delta E$.

The $z, \phi$ variables are canonically conjugate, with $\dot{z} = -\frac{\partial H}{\partial \phi}$, $\dot{\phi} = \frac{\partial H}{\partial z}$ and the Hamiltonian is given by:

\begin{equation}
H = \frac{\Lambda}{2} z^2 - \sqrt{1 - z^2} \cos \phi + \Delta E z
\end{equation}

In a simple mechanical analogy, $H$ describes a non-rigid pendulum, of tilt angle $\phi$ and length proportional to $\sqrt{1 - z^2}$, that decreases with the ”angular momentum” $z$.

The inter-trap tunneling current is given by:

\begin{equation}
I = \frac{z N_T}{2} = I_0 \sqrt{(1 - z^2)} \sin \phi; \quad I_0 = KN_T
\end{equation}

The detailed analysis of Eq.s(3) with exact analytical solutions will be presented elsewhere [17]; here we consider three regimes.
1) *Non-interacting limit.* For symmetric wells and negligible interatomic interactions \((\Lambda \to 0)\), Eq.s(2) can be solved exactly, yielding Rabi-like oscillations in the population of each trap with a frequency:

\[
\omega_R = 2K
\]  

(7)
as studied in [8]. However, the ideal Bose gas limit is not accessible experimentally.

2) *Linear regime.* In the linear limit \(|z| << 1, |\phi| << 1\) Eq.s(3) become:

\[
\dot{z} \simeq -\phi
\]  

(8)
\[
\dot{\phi} \simeq (\Lambda + 1)z
\]  

(9)

These describe the small amplitude oscillations of the pendulum analogue, with a sinusoidal \(z(t)\) with a frequency:

\[
\omega_L = \sqrt{2UN_TK + 4K^2}
\]  

(10)
The BEC oscillations of population should show up as temporal oscillations of phase-contrast patterns [2], or other probes of atomic population [1].

Linearizing Eq.s(3) in \(z(t)\) only, we obtain:

\[
\dot{z} \simeq -\sin\phi
\]  

(11a)
\[
\dot{\phi} \simeq \Delta E + (\Lambda + \cos\phi)z
\]  

(11b)
\[
I \simeq I_0\sin\phi
\]  

(11c)

For large trap asymmetries with \(\Delta E >> (\Lambda + \cos(\phi))z\), we have: \(\phi = \phi(0) + \Delta Et\), giving an oscillating \(z(t)\) with frequency:

\[
\omega_{ac} \simeq E_1^0 - E_2^0
\]  

(12)

where an "ac" current \(I(t)\) is produced by the "dc" trap asymmetry \(\Delta E\). It is simple to show that a small oscillation in the laser position, or in its intensity \((K \to K(1 + \delta\sin(\omega_0t)), \delta << 1)\), will result in a dc inter-trap current of non-zero time average <
\( I(t) \approx \delta \times \sin(\omega_0 t) \sin(\omega_{ac} t) \neq 0 \), at resonant match \( \omega_0 = \omega_{ac} \). This is the analogue of the Shapiro effect \[10\] in superconductor junctions.

In SJJ, the current of Cooper pairs \( N_{1,2} \) is \( I_{SJJ} = -2e(\dot{N}_1 - \dot{N}_2) = 2eE_J \sin \phi \), and the Josephson frequency relation for the relative phase is \( \dot{\phi} = \Delta \mu = 2eV + (N_1 - N_2)E_c \), for a dc applied voltage \( V \) and a junction capacitance \( C \), with \( E_c = (2e)^2/2C \) \[10\]. These SJJ equations can be directly compared with the BEC Eq.s(3). It is then clear that the ac Josephson frequency \( \omega_{ac} = 2eV \) and the Josephson ”plasma” frequency \[10,11\] \( \omega_p = \sqrt{E_cE_J} \) are the analogue of Eq.(12) and Eq.(10) respectively. Note however that, for SJJ, \( \omega_p \) is independent of the barrier cross section \( A \), since \( E_J \approx A \) and \( E_c \approx C^{-1} \approx A^{-1} \), while the BEC has \( \omega_L \approx A^{1/2} \) since \( K \approx A \), and the bulk energy \( UN_T \) is approximately \( A \)-independent.

3) Non-linear regime. A numerical solution of Eq.s(3) yields non-sinusoidal oscillations, that are the anharmonic generalization of the Josephson effects. Moreover, an additional novel non-linear effect occurs in the BEC: a self-locked population imbalance.

Fig.(2) shows solutions of Eq.s(3) with initial conditions \( z(0) = 0.6, \phi(0) = 0 \) and illustrative parameters \( \Lambda = 1, 8, 9.99, 10, 11 \) respectively. The sinusoidal oscillations around \( z = 0 \) became anharmonic as \( \Lambda \) increases, Fig.(2a,b), and with a precursor slowing down, Fig.(2c), there is a critical transition for \( \Lambda = \Lambda_c = 10 \), dashed line in Fig(2d). Then for \( \Lambda = 11 \) the population in each trap oscillates around a non-zero time averaged \( \langle z(t) \rangle \neq 0 \), solid line in Fig.(2d). In the non-rigid pendulum analogy, this corresponds to an initial angular momentum \( z(0) \) sufficiently large to swing the pendulum bob over the \( \phi = \pi \) vertical orientation, with a non-zero \( \langle z(t) \rangle \) average angular momentum corresponding to the rotatory motion. This critical behavior depends on \( \Lambda_c = \Lambda_c[z(0),\phi(0)] \), as can be easily found from the energy conservation constraint. In fact from Eq.(5), the value \( z(t) = 0 \) is inaccessible at any time if:

\[
\Lambda > 2 \left( \frac{\sqrt{1 - z(0)^2} \cos(\phi(0)) + 1}{z(0)^2} \right) \tag{13}
\]

The full dynamical behavior of Eq.(3) is summarized in Fig(3), that shows the \( z - \phi \) phase portrait with constant energy lines. There are energy minima along \( z = 0 \) at \( 2n\pi \), and
"running" solutions $< \dot{\phi}(t) > \neq 0$ with $< z(t) > \neq 0$, moving along the sides of these wells. The vertical points $\phi = (2n + 1)\pi$, that would be isolated unstable points for the rigid pendulum, now support oscillations of restricted range, as a consequence of non-rigidity, i.e. non-linearity.

The self-trapping of an initial BEC population imbalance, seen in Fig.(2d) and Fig.(3) arises because of the inter-atomic interaction in the Bose gas (non-linear self-interaction in GPE). It has a quantum nature, involving the coherence of a macroscopic number of atoms. It differs from single polaron trapping of an electron in a medium [16] and from gravitational effects on He II baths [12]. It can be considered as a novel "macroscopic quantum self-trapping" (MQST).

Non-linear effects like MQST are unobservable in SJJ. The requirement that the chemical potential difference $\mu_1 - \mu_2 \simeq (N_1 - N_2)E_c$ must be less than the quasiparticle gap $2\Delta_{qp}$ (to prevent a jump off the Josephson I-V branch) implies a stringent constraint in imbalances $|z| < (2\Delta_{qp}/E_c N_T) \simeq 10^{-12}$ for typical parameters. For the BEC, the requirement that tunneling does not access excitation energies is much less restrictive. As an example, let consider two weakly linked condensates of $N_T \simeq 10^4$ atoms, confined in two symmetric spherical traps with frequency $\omega_0 \simeq 100 \text{ Hz}$. Evaluating Eq.s(17) below with a simple variational wave function [18], we have $E^0 \simeq 0.5 \text{ } nK$, $UN_T \simeq 3 \text{ } nK$, and from an estimation of the excitation gap we obtain the constraint $|z| \leq 0.5$. Taking $K \simeq 0.1 \text{ } nK$, we have $\Lambda \simeq 10$, close to the onset of the critical behavior. Increasing the number of particles or the trap frequencies, the value of $\Lambda$ can increase, making MQST observable. Typical frequency oscillations are $\omega_L \simeq KHz$ for $N_T \simeq 10^6$, that should be compared with the plasma frequency of SJJ that are [11] of the order of $\omega_p \simeq GHz$.

We now outline the derivation of the parameters $E_0, U, K$.

To this purpose we note that in the barrier region the modulus of the order parameter in the GPE is exponentially small. This allows us to look for an eigenstate of GPE of the
form:

\[ \Psi = \psi_1(t)\Phi_1(x) + \psi_2(t)\Phi_2(x) \]  

(14)

where \( \Phi_1, \Phi_2 \) are the ground state solutions for isolated traps with \( N_1 = N_2 = N_T/2 \).

Replacing Eq. (14) in the GPE Eq. (1) with the conditions:

\[ \int \Phi_1 \Phi_2 d\vec{r} \simeq 0 \]  

(15)

and

\[ \int \|\Phi_1\|^2 d\vec{r} = \int \|\Phi_2\|^2 d\vec{r} = 1 \]  

(16)

we obtain:

\[ E_{1,2}^0 = \int \left[ \frac{\hbar^2}{2m} |\nabla \Phi_{1,2}|^2 + \Phi_{1,2}^4 V_{ext} \right] d\vec{r} \]  

(17a)

\[ U_{1,2} = g_0 \int \Phi_{1,2}^4 d\vec{r} \]  

(17b)

\[ K = - \int \left[ \frac{\hbar^2}{2m} (\nabla \Phi_1 \nabla \Phi_2) + \Phi_1 \Phi_2 V_{ext} \right] d\vec{r} \]  

(17c)

At finite temperature the interaction between the normal component of the Bose gas with the condensate should be included, and the parameters become temperature dependent. Such corrections are small for temperatures smaller than excitation energies. High density BEC could induce quasiparticle/collective mode scattering with finite lifetime of the coherent oscillations \([5]\); phase diffusion could induce phase coherence collapse and revival \([19]\). These effects deserve further studies.

In conclusion, the BEC coherent atomic tunneling in a double-well trap induces non-linear population oscillations that are a generalization of the sinusoidal Josephson effects familiar in superconductors. A novel population imbalance occurs for parameters beyond critical values: a macroscopic quantum self-trapping effect.

Discussions with S. Raghavan and useful references from L. Bonci and G. Williams are acknowledged.
REFERENCES

[1] M.H.Anderson, J.R.Ensher, M.R.Matthews, C.E.Wieman and E.A.Cornell, Science 269 (1995) 198; K.B.Davis et al., Phys.Rev.Lett.75, (1995) 3969; C.C.Bradley et al., Phys.Rev.Lett.75 (1995) 1687

[2] M.R. Andrews, C.G. Townsend, H.-J. Miesner, D.S. Durfee, D.M. Kurn, and W. Ketterle Science Vol. 275 (1997) 637

[3] L. P. Pitaevskii, Sov. Phys. JETP 13, 451 (1961); E.P.Gross, Nuovo Cimento 20, 454 (1961); J.Math.Phys.4 (1963)4704

[4] S. Stringari, Phys.Rev.Lett.77 (1996) 2360; M.Edwards et al. Phys.Rev.Lett.77 (1996) 1671

[5] A.Smerzi and S.Fantoni, Phys.Rev.Lett. 78 (1997) 3589

[6] Yu.Kagan, E.L.Surkov and G.V.Shlyapnikov Phys.Rev.A55 (1997)18R

[7] J. Javanainen, Phys.Rev.Lett. 57 (1986)3164; M.W.Jack, M.J.Collet and D.F.Walls, Phys.Rev.A 54 (1996)R4625

[8] J.Ruostekoski and D.F.Walls ”Non destructive optical measurement of the relative phase between two Bose condensate”, cond-mat/9703190

[9] F. Dalfovo, L. Pitaevskii and S. Stringari, Phys. Rev. A54 (1996) 4213

[10] L. Solymar ”Superconductive Tunneling and Applications”, Chapman and Hall Ltd, London

[11] A.J.Dahm et al. Phys.Rev.Lett 20 (1968) 859

[12] P. W. Anderson, Rev. of Mod. Phys. 38 (1966) 298

[13] O.Avenel and E. Varoquaux, Phys.Rev.Lett. 55 (1985) 2704

[14] R.P.Feynman ”Statistical Mechanics”, Frontiers in Physics, Addison-Wesley.
[15] M. Gaida and K. Rzazewski, Phys. Rev. Lett. 78 (1997) 2686

[16] V. M. Kenkre and D. K. Campbell, Phys. Rev. B 34, 4959 (1986).

[17] S. Raghavan, A. Smerzi, S. Fantoni and S. R. Shenoy (unpublished)

[18] G. Baym and C. J. Pethick, Phys. Rev. Lett. 76 (1996) 6

[19] A. Imamoglu, M. Lewenstein and L. You, Phys. Rev. Lett. 78 (1996) 2511
FIGURES

FIG. 1. The double well trap for two Bose-Einstein condensates with $N_{1,2}$ and $E_{0,1,2}$ the number of particles and the zero-point energies in the trap 1, 2 respectively.

FIG. 2. Fractional population imbalance $z(t)$ versus rescaled time, with initial conditions $z(0) = 0.6$, phase difference $\phi(0) = 0$, and $\Lambda = 1$ (a), $\Lambda = 8$ (b), $\Lambda = 9.99$ (c), $\Lambda = 10$ (dashed line, d), $\Lambda = 11$ (solid line, d).

FIG. 3. Constant energy lines in a phase-space plot of population imbalance $z$ versus phase difference $\phi$. Bold solid-line: $z(0) = 0.6$, $\phi(0) = 0$, $\Lambda = 1, 8, 10, 11, 20$. Solid line: $z(0) = 0.6$, $\phi(0) = \pi$, $\Lambda = 0, 1, 1.2, 1.5, 2$. 
Potential Energy vs. Distance

Fig. 1
Fig. 2
