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Some aspects of universality in Efimov physics

M Gattobigio\(^1\) and A Kievsky\(^2\)

\(^1\)Universitè de Nice-Sophia Antipolis, Institut Non-Linéaire de Nice, CNRS, 1361 route des Lucioles, 06560 Valbonne, France
\(^2\)Istituto Nazionale di Fisica Nucleare, Largo B. Pontecorvo 3, 56127 Pisa, Italy
E-mail: mario.gattobigio@inln.cnrs.fr

Abstract. The Efimov effect is the appearance of an infinite tower of bound states, \(E_3^n\), that accumulate in a geometrical way at zero energy in systems of three-identical bosons having an infinite large two-body scattering length \(a\). In the zero-range limit the geometrical factor is \(E_3^{n+1}/E_3^n = e^{-2\pi/s_0}\) with \(s_0 \approx 1.00624\) an universal number. The Efimov effect is the signature of the discrete scale invariance (DSI) present in these systems close to the unitary limit (1/\(a\) = 0). The DSI constrains also the spectrum in \(N > 3\) systems. In this contribution the two-level tree structure observed in bosons systems with \(N \geq 4\) will be discussed using potential models.

1. Introduction
The theoretical prediction of the Efimov effect \([1, 2]\) and its experimental verification \([3, 4, 5, 6]\) can be indicated as the starting points of an intense program of research in the sector of few-boson systems having large two-body scattering length \(a\). This new field of research is today called Efimov physics; it refers to the investigation of shallow states in which some universal aspects have been discovered. At the two-body level, a large scattering length implies the appearance of a shallow dimer (a bound-state of two particles) with energy \(E_2 \approx \hbar^2/ma^2\). The natural value of the binding energy of two particles should be around \(\langle r^2 \rangle \approx a^2/2\) and the cross section is \(\sigma \approx 8\pi a^2/(1 + a^2k^2)\), with \(k\) the modulus of the wave vector. These relations are approximate for \(a \gg r_0\) but they are exact in the zero-range limit. In particular, in this limit, the two-body system presents a continuous scale invariance (CSI). When the two-body scattering length is scaled by a positive real number \(a \rightarrow ua\), the mentioned observables scale with the corresponding powers: \(E_2 \rightarrow u^{-2}E_2\), \(\langle r^2 \rangle \rightarrow u^2\langle r^2 \rangle\) and \(\sigma \rightarrow u^2\sigma\). At the level of three particles the CSI is broken and, in the three-body \(L = 0\) state, the residual symmetry is the DSI. The physics is invariant under the rescaling \(r \rightarrow \Lambda^nr\), with the scaling constant usually written as \(\Lambda = e^{\pi/s_0}\).

One question that naturally arises is in which way the \(N\)-boson spectrum, governed by the DSI, evolves as the number of bosons increases. The case of four bosons has been investigated with great detail in Refs. \([7, 8, 9, 10]\). The main conclusions of those studies have been that the four-boson spectrum has a tree structure of two levels, \(E_4^{n,0}\) and \(E_4^{n,1}\) attached to each \(E_3^n\) level. The first one is deep and the second one is shallow, very close to the three-boson threshold given by \(E_3^n\). Moreover, the following ratios have universal character, \(E_4^{n,0}/E_3^n \approx 4.611\) and \(E_4^{n,1}/E_3^n \approx 1.002\). However, it should be noticed that only the \(E_4^{0,0}\) and \(E_4^{0,1}\) states are true
bound states: all the other levels appear as resonances embedded in the three-boson continuum. Few studies exist for $N > 4$, as discussed in Refs. [11, 12]; using potential models and solving the Schrödinger equation, in Ref. [11] universal ratios have been estimated at the unitary limit up to $N = 13$, whereas in Ref. [12] the spectrum of atomic helium clusters, up to $N = 6$, has been studied in the $(1/a, \kappa)$ plane with $\kappa = \text{sign}(E) |E|/(\hbar^2/m) |^{1/2}$ and $E$ the energy of the system. Based on these findings, in the present contribution we propose a general formula to describe the tree structure of a boson systems having a large two-body scattering length.

2. The $N$-boson system with large scattering length

In a sequence of papers, V. Efimov derived the three-boson spectrum in the zero-range limit. It can be expressed in a parametric form as follows

$$
\frac{E_3^n}{(\hbar^2/ma^2)} = \tan^2 \xi \\
\kappa_{n} a = e^{(n-n^*)\pi/s_0} \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi},
$$

(1)

with $\kappa_{n}$ the wave number corresponding to the energy of the $n^*$-level at the unitary limit; it is referred to as the three-body parameter. The function $\Delta(\xi)$ is an universal function and its parametrization can be found for example in [13]. According to the above equation, the knowledge of the scattering length $a$ and of the three-body parameter $\kappa_{n}$ completely determines the spectrum of the three-boson system. In a given system, $a$ is related to the specific interaction between constituents and to this respect can be thought as fixed by nature. However, with the progress in manipulating trapped-ultracold atoms, the interaction between atomic species can be varied, for instance by means of magnetic fields (Feshbach resonances). Accordingly, the value of $a$ can be varied and eventually it can take values around the unitary limit. In this way the Efimov physics can be studied.

Zero-range interactions do not describe real systems. In order to match the result of the zero-range theory with experiments and/or numerical calculations, one must take into account finite-range corrections. To this respect equation (1) has to be modified in order to consider this kind of corrections. In Refs. [15, 16] this problem was studied using potential models and, from those studies, the following modification to equation (1) has been proposed:

$$
\frac{E_3^n}{E_2} = \tan^2 \xi \\
\kappa_{3}^{n} a + \Gamma_n = e^{-\Delta(\xi)/2s_0} \frac{e^{-\pi/s_0}}{\cos \xi}.
$$

(2)

The differences between the above formula and equation (1) are the following. In the first row of equation (2) it appears the two-body binding energy whereas in equation (1) we have its zero-range limit. In the second row each level has been identified with the wave number $\kappa_{3}^{n}$ corresponding to the energy at the unitary limit. The DSI is to some extent disrupted by range corrections and the ratio between the wave numbers of two consecutive levels $\kappa_{3}^{n+1}/\kappa_{3}^{n} \to e^{-\pi/s_0}$ only for large values of $n$. For low $n$-values this relation is only approximately verified and, therefore, it is convenient to introduce a three-body parameter for each energy level. However, the main modification in the above equation is the introduction of the finite-size scaling parameter $\Gamma_n$ in which the finite-range character of the interaction is absorbed.

Studies of Efimov physics has been pursued in Ref. [12] using potential models. The atomic two-helium system is taken as a reference system. To this end an attractive two-body gaussian (TBG) potential has been defined

$$
V(r) = V_0 e^{-r^2/\sigma_0^2}.
$$

(3)
with range \( r_0 = 10 a_0 \) (with \( a_0 \) the atomic length) and strength \( V_0 \) fixed to reproduce the values of \( a \) given by one of the widely used He-He potentials, the LM2M2 interaction [17]. With the strength \( V_0 = -1.2343566 \) K, the LM2M2 low-energy data are closely reproduced, \( E_2 = -1.303 \) mK, \( a = 189.42 a_0 \), and the effective range \( r_s = 13.80 a_0 \). To explore the \((a^{-1}, \kappa)\) plane, the strength \( V_0 \) of the potential has been varied to cover an extended range of \( a \) values. The results up to \( N = 6 \) bosons are shown in figure 1. The tree structure of the levels is evident. Starting with \( N = 4 \), the systems present a two-level structure, one deep state with energy \( E_0^N \) and one shallow state with energy \( E_1^N \). This level structure can be described with the following extension of equation (2)

\[
\frac{E_N^m}{E_2} = \tan^2 \xi \\
\kappa_N^m a + \Gamma_N^m = \frac{e^{-\Delta(\xi)/2a_0}}{\cos \xi},
\]

with \( m = 0, 1 \). Another form of equation (4) is the following

\[
y(\xi) = \kappa_N^m a + \Gamma_N^m,
\]

which puts in evidence the linear relation between \( a \) and \( y(\xi) = e^{-\Delta(\xi)/2a_0}/\cos \xi \) for fixed values of \( \kappa_N^m \) and \( \Gamma_N^m \). These values can be extracted from the calculations by making a fit: at each value of \( a \), using the corresponding strength of the TBG potential, the \( N \)-boson energy is computed. Then the angle \( \xi \) is determined from first row of equation (4) and, using the universal function \( \Delta(\xi) \) we obtain the value of \( y(\xi) \); at the end we can verify the linear relation between \( a \) and \( y \).

The values of \( \kappa_N^m \) and \( \Gamma_N^m \) are given in table 1 for \( N = 4, 5, 6 \). The corresponding straight lines for both cases, \( m = 0 \) (left panel) and \( m = 1 \) (right panel) are given in figure 2. The solid lines

**Figure 1.** (Color online) Energy levels of \( N \)-cluster of bosons as a function of the scattering length. The point \( \lambda = 1 \) corresponds to the case of Helium, and \( \ell \) is the van der Waals length of Helium.
### Table 1. Values of $\kappa_N^m$ and $\Gamma_N^m$ as a function of $N$ obtained from a fit of the calculated values using equation (4).

| $N$ | $\kappa_N^0$ [$a_0^{-1}$] | $\Gamma_N^0$ | $\kappa_N^1$ [$a_0^{-1}$] | $\Gamma_N^1$ |
|-----|----------------|------------|----------------|------------|
| 4   | 0.1185         | 0.642      | 0.0511         | 0.555      |
| 5   | 0.1955         | 0.740      | 0.1240         | 0.772      |
| 6   | 0.2770         | 0.817      | 0.2071         | 0.969      |

represent equation (5) whereas the circles are the calculations. The linear behavior is evident. An interesting aspect of the figure is the tendency of the lines to cross in a single point, one for $m = 0$ and one for $m = 1$. This is a consequence of the dependence of the parameters $\kappa_N^m$ and $\Gamma_N^m$. In fact these parameters follows a linear relation with $N$ as it is discussed in Ref. [18].

### 3. Conclusions

In the present contribution different aspects of universal behavior in $N$-boson systems having large two-body scattering lengths have been discussed. In first place the extension of the zero-range energy equations has been discussed. These equations have to be modified due to the finite-range character of the interaction forces [18]. In fact experiments and calculations using potential models cannot be describe directly by the zero-range theory. It has been shown that the main modification is the introduction of a shift in the parametric form of the energy equations.

![Figure 2](image_url)  

**Figure 2.** Same data set as figure 1 represented using equation (5). The left panel corresponds to the ground states, $m = 0$, while the right panel corresponds to the excited states, $m = 1$. 
The shift however maintains the linear behavior between the scattering length \(a\) and the universal function \(y(\xi)\) defined in equation (5). A further remarkable characteristic of the finite-range theory proposed in equation (4) is that the same universal function \(\Delta(\xi)\), defined in the three-body system, equation (1), governs the \(N\)-boson spectrum.

Moreover, we have observed the tree structure of two levels starting from \(N = 4\). Here we have discussed the lowest branch \((n = 0)\) of the structure which corresponds to true bound states. States corresponding to branches with higher values of \(n\) appear embedded in the continuum of the \((N - 1)\)-boson system and they are difficult to study using potential models.

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