ON THE GENERATING HYPOTHESIS IN NONCOMMUTATIVE
STABLE HOMOTOPY

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Abstract. Freyd’s Generating Hypothesis is an important problem in topology with deep
structural consequences for finite stable homotopy. Due to its complexity some recent work
has examined analogous questions in various other triangulated categories. In this short
note we analyze the question in noncommutative stable homotopy, which is a canonical
generalization of finite stable homotopy. Along the way we also discuss Spanier–Whitehead
duality in this extended setup.

Introduction

In (finite) stable homotopy theory the Spanier–Whitehead category of finite spectra, de-
denoted by $\text{SW}^f$, is a central object of study. Roughly speaking, it is constructed by formally
inverting the suspension functor on the category of finite pointed CW complexes and it is
a triangulated category. Its counterpart in the noncommutative setting is the triangulated
noncommutative stable homotopy category, denoted by $\text{NSH}$. This triangulated category
was constructed by Thom [24] (see also [7]) building upon earlier works of Rosenberg [19],
Schochet [21], Connes–Higson [5], Dădărlat [6] and Houghton-Larsen–Thomsen [12] amongst
others. The triangulated category $\text{NSH}$ is a canonical generalization of $\text{SW}^f$. It comes in a
mysterious package carrying vital information about bivariant homology theories on the cat-
egory of separable $C^*$-algebras. Triangulated category structures also play an important
role in the theory of $C^*$-algebras; for instance, the work of Meyer–Nest on the Baum–Connes
conjecture via localization of triangulated categories has had tremendous impact [17].

Freyd stated the following Generating Hypothesis in [11] (Chapter 9):

Conjecture (Freyd). The object $((S^0, \ast), 0)$ is a graded generator in $\text{SW}^f$, where $(S^0, \ast)$ is
the pointed 0-sphere ($\ast$ being the basepoint).

An alternative formulation of the Generating Hypothesis asserts that for any two finite
spectra $X, Y$ the canonical homomorphism

$$\Phi : \text{SW}^f(X, Y) \rightarrow \text{Hom}_{\pi_*(S)}(\pi_*(X), \pi_*(Y))$$

is injective. Here $S$ denotes the sphere spectrum and $\text{Mod}(\pi_*(S))$ denotes the category of
right modules over the graded commutative ring $\pi_*(S)$. The conjecture has some interesting
formulations and generalizations [11, 8, 13, 3, 23]. It remains an open problem at the time
of writing this article. However, it has spurred a lot of stimulating research. Analogues
of the Generating Hypothesis have been addressed in several other contexts, such as the

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stable module category of a finite group algebra \([1, 4]\), the derived category of a ring \([15, 14]\),
equivariant stable homotopy \([2]\), and so on. By Proposition 9.7 of \([10]\) (see also Corollary 3.2
of \([13]\)) the injectivity of the map \(\Phi\) in \(\text{SW}^f\) automatically implies its bijectivity. If true, the
Generating Hypothesis would reduce the task of understanding the stable homotopy classes
of maps between finite pointed CW complexes to a more tractable algebraic problem, i.e.,
understanding the module category of \(\pi_\ast(S)\).

A straightforward generalization of the Generating Hypothesis to the noncommutative
setting would predict that the canonical map \(\Phi' : \text{NSH}(A, B) \to \text{Hom}_{\pi_\ast(C)}(\pi_\ast(A), \pi_\ast(B))\)
in \(\text{NSH}\) is injective. We call it the Na"ive Generating Hypothesis in \(\text{NSH}\). This question is
motivated by the algebraization problem of noncommutative stable homotopy. This assertion
generalizes the following Cogenerating Hypothesis in finite stable homotopy: The canonical
map \(\Phi : \text{SW}^f(Y, X) \to \text{Hom}_{\pi_\ast(S)}(\pi_\ast(X), \pi_\ast(Y))\) is injective, where \(\pi_\ast(\cdot)\) denotes the stable
cohomotopy functor. Observe that \((\text{SW}^f)^{\text{op}}\) sits inside \(\text{NSH}\) as a full triangulated subcategory.
In \(\text{SW}^f\) there is a contravariant duality functor \(D : \text{SW}^f \to \text{SW}^f\) with a natural isomorphism
\[\text{Id}_{\text{SW}^f} \cong D \circ D\] that satisfies the property
\[\text{SW}^f(X \wedge Z, Y) \cong \text{SW}^f(X, DZ \wedge Y).\]
Here \(\wedge\) denotes the smash product of spectra. This phenomenon in \(\text{SW}^f\) is called Spanier–
Whitehead duality. Using it one can see that the category \(\text{SW}^f\) is self-dual and that the
Cogenerating Hypothesis is actually equivalent to the Generating Hypothesis. In the first
part of this article (Section 1) we show that the Na"ive Generating Hypothesis in noncom-
mutative stable homotopy fails to hold; however, our result is not applicable to Freyd’s
Generating Hypothesis in finite stable homotopy. More precisely, we show

**Theorem.** The canonical map \(\Phi' : \text{NSH}(A, B) \to \text{Hom}_{\pi_\ast(C)}(\pi_\ast(A), \pi_\ast(B))\) in \(\text{NSH}\) is not
injective in general.

**Remark.** Our arguments below exploit the fact that noncommutative stable homotopy of
stable \(C^\ast\)-algebras agrees with their E-theory naturally. Thus the above result should be
viewed as a failure of the Generating Hypothesis in bivariant E-theory.

Spanier–Whitehead duality is a peculiar property of finite stable homotopy with many
interesting consequences. It is a natural question to ask whether noncommutative stable
homotopy also possesses this property. Our answer to this question is

**Theorem.** The Spanier–Whitehead duality functor \(D\) on \(\text{SW}^f\) does not extend to \(\text{NSH}\).

In the second part of the paper (Section 2) we explain the issues that the naïve extrapo-
lation of Freyd’s Generating Hypothesis suffers from. The main problem is the size of \(\text{NSH}\)
and we provide a modified formulation by restricting our attention to a suitable subcate-
cgory, denoted by \(\text{NSH}^f\) (see Question 2.2 entitled Matrix Generating Hypothesis in \(\text{NSH}^f\)).
This modified formulation also implies Freyd’s Generating Hypothesis in finite stable ho-
motopy. Our modification does not correct the failure of Spanier–Whitehead duality in the
noncommutative setting. We believe that it is an intrinsic feature of noncommutative stable
homotopy that needs no remedy.

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1. Naïve Generating Hypothesis in $\mathbf{NSH}$

Let $\mathbf{SC}'$ denote the category of separable $C^*$-algebras and $*$-homomorphisms. Recall from [21] that there is a stable homotopy functor $\pi_* : \mathbf{SC}' \rightarrow \mathbf{Mod}(\pi_*(C))$ in noncommutative topology, which factors through $\mathbf{NSH}$ giving rise to the canonical map

$$\Phi' : \mathbf{NSH}(A, B) \rightarrow \text{Hom}_{\pi_*(C)}(\pi_*(A), \pi_*(B)).$$

Here $\text{Mod}(\pi_*(C))$ denotes the category of right modules over the ring $\pi_*(C)$.

1.1. Failure of the injectivity of $\Phi'$. It is known that on the category of stable $C^*$-algebras noncommutative stable homotopy agrees with bivariant E-theory [5, 6], i.e., $\mathbf{NSH}(A, B) \cong E_0(A, B)$ and $\pi_*(A) \cong E_*(A)$. In addition, on the category of nuclear $C^*$-algebras one has $E_0(A, B) \cong \text{KK}_0(A, B)$ and $E_*(A) \cong K_*(A)$ (see, for instance, the paragraph preceding Section 6 in [6]). It follows from the Universal Coefficient Theorem in $\text{KK}$-theory [20] that there is a natural short exact sequence of abelian groups in the bootstrap class

$$0 \rightarrow \text{Ext}^1(K_*(\Sigma A), K_*(B)) \rightarrow \text{KK}_*(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B)) \rightarrow 0,$$

which splits unnaturally.

**Theorem 1.1.** The canonical map $\Phi' : \mathbf{NSH}(A, B) \rightarrow \text{Hom}_{\pi_*(C)}(\pi_*(A), \pi_*(B))$ is not injective in general.

**Proof.** Let us choose judiciously $A = C(X, x) \hat{\otimes} \mathbb{K}$ and $B = C(Y, y) \hat{\otimes} \mathbb{K}$ ($\mathbb{K}$ being the $C^*$-algebra of compact operators) in such a manner that $\text{Ext}^1(K_*(\Sigma A), K_*(B))$ is non-zero (with every abelian group in $[4]$ finitely generated). This can be easily achieved by choosing finite pointed CW complexes $(X, x)$ and $(Y, y)$, such that their K-theory groups contain non-zero torsion. Then $\mathbf{NSH}(A, B) \cong \text{KK}_0(A, B)$ due to the stability and nuclearity of all the $C^*$-algebras in sight. One has the following commutative diagram

$$\begin{array}{ccc}
\mathbf{NSH}(A, B) & \xrightarrow{\Phi'} & \text{Hom}_{\pi_*(C)}(\pi_*(A), \pi_*(B)) \\
\cong & & \text{Hom}(K_*(A), K_*(B)). \\
\text{KK}_0(A, B) & \xrightarrow{\text{Hom}(K_*(A), K_*(B))} & \end{array}$$

Here the left vertical map is an isomorphism as argued before and the right vertical map is an injection. Now the map $\text{KK}_0(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B))$ is not injective due to the non-vanishing of the Ext$^1$-term in (4), whence $\Phi'$ cannot be injective. \qed

1.2. Failure of Spanier–Whitehead duality. The Spanier–Whitehead category is a tensor triangulated category under smash product of finite spectra. Thom proved in Theorem 3.3.7 of [21] that $\mathbf{NSH}$ is a tensor (under maximal $C^*$-tensor product $\hat{\otimes}$) triangulated category. The sphere spectrum $\mathbb{S}$ (resp. $(\mathbb{C}, 0)$) turns out to be the tensor unit in $\mathbf{SW}^\mathbb{F}$ (resp. $\mathbf{NSH}$).

**Lemma 1.2.** $\mathbf{NSH}(M_2(A), \mathbb{C}) \cong 0$ for all $A \in \mathbf{NSH}$. 3
Using the previous Lemma 1.2, we conclude that
\[ \text{Set} A_{DM} \]

**Proof.** Suppose one could extend the functor \( \sigma \) [22]. The constructions in ibid. involve \( \in \) \( n \) \( \square \) (see, for instance, Section 3 of [16]).

**Remark 1.4.** In (equivariant) bivariant K-theory there are positive results in this direction [22]. The constructions in ibid. involve \( \sigma \)-\( C^* \)-algebras.

### 2. Modified Matrix Generating Hypothesis

We called the Generating Hypothesis in the previous section na\( \text{i} \)\( \ve \) for the following reasons:

1. In \( \text{SW}^f \) the objects are finite pointed CW complexes, whereas in \( \text{NSH} \) they are arbitrary pointed compact metrizable spaces. The presence of finite matrix algebras in the noncommutative analogue of \( \text{SW}^f \) is non-negotiable, since they form natural building blocks of noncommutative CW complexes [9] [13]. However, \( K \) can be expressed as \( \lim \) \( M_n(\mathbb{C}) \) in \( \text{SC}^* \), i.e., it can be regarded as a countable inverse limit of noncommutative or fat points. Thus \( K \) is not indispensable.

2. We did not take finite matrix algebras into consideration as test objects for the Generating Hypothesis, which would be the correct way to think about the Generating Hypothesis in this situation [3].

Let us first address issue number (1). We already observed that \( \text{NSH} \) is a tensor triangulated category under \( \circ \), where the tensor structure generalizes the smash product of finite spectra.

**Definition 2.1.** We define the category of noncommutative finite spectra, denoted by \( \text{NSH}^f \), to be the smallest thick tensor triangulated subcategory of \( \text{NSH} \) generated by \( M_n(\mathbb{C}) \) for all \( n \in \mathbb{N} \). It is the noncommutative analogue of the Spanier–Whitehead category \( \text{SW}^f \).

**Question 2.2** (Matrix Generating Hypothesis in \( \text{NSH}^f \)). Let \( f \in \text{NSH}^f(A, B) \) be a morphism, such that \( \text{NSH}^f(M_n(\mathbb{C}), f) : \text{NSH}^f(M_n(\mathbb{C}), (A, i)) \rightarrow \text{NSH}^f(M_n(\mathbb{C}), (B, i)) \) is the zero morphism for all \( n \in \mathbb{N} \) and \( i \in \mathbb{Z} \). Is \( f \) itself the zero morphism in \( \text{NSH}^f(A, B) \)?
The above formulation uses $M_n(\mathbb{C})$ for all $n \in \mathbb{N}$ as test objects and hence addresses issue number \ref{2}. The full subcategory of $\mathbb{NSH}^f$ consisting of commutative $C^*$-algebras is equivalent to the opposite category of $\mathbb{SW}^f$ via the functor $(X, x) \mapsto C(X, x)$. Due to the self-duality of $\mathbb{SW}^f$ one can also view it sitting inside $\mathbb{NSH}^f$ via the duality functor $D$.

**Proposition 2.3.** The Matrix Generating Hypothesis in $\mathbb{NSH}^f$ implies Freyd’s Generating Hypothesis in $\mathbb{SW}^f$.

**Proof.** Since the Cogenerating Hypothesis is equivalent to the Generating Hypothesis in $\mathbb{SW}^f$, it suffices to show that the former is a consequence of the Matrix Generating Hypothesis. Let $A = C(X, x)$ and $B = C(Y, y)$ be separable commutative $C^*$-algebras. Now suppose $f \in \mathbb{NSH}^f(A, B) \cong \mathbb{SW}^f((Y, y), (X, x))$ is a morphism that induces the zero morphism $\pi_*(f) : \pi_*(A) \cong \pi_*(X, x) \to \pi_*(Y, y) \cong \pi_*(B)$. We need to show that $f$ itself is the zero morphism in $\mathbb{NSH}^f(A, B) \cong \mathbb{SW}^f((Y, y), (X, x))$. Since $B$ is commutative, arguing as in Lemma \ref{2} we deduce that $\mathbb{NSH}^f(M_n(\mathbb{C}), (B, i)) = 0$ for all $n > 1$ and $i \in \mathbb{Z}$. It follows that $\mathbb{NSH}^f(M_n(\mathbb{C}), f) = 0$ for all $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, whence by the Matrix Generating Hypothesis $f = 0$. \hfill \Box

**Remark 2.4.** Notice that Proposition \ref{2} is no longer applicable to $\mathbb{NSH}^f$. Indeed, the $C^*$-algebra $\mathbb{K}$ does not belong to $\mathbb{NSH}^f$ anymore as it is not finitely built. If we were to include $\mathbb{K}$ in the generating set of Definition \ref{2}, then the corresponding predictable Matrix + Compact Generating Hypothesis would once again be falsified by Proposition \ref{1}.

Let us reiterate that our results in Section \ref{1} do not invalidate Freyd’s Generating Hypothesis in finite stable homotopy. The failure of the injectivity of $\Phi'$ in the bigger category $\mathbb{NSH}$ crucially exploited the presence of genuinely noncommutative $C^*$-algebras. We hope that the generalized perspective will shed some light on the original problem.

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