IMPROVED WELL-POSEDNESS RESULTS FOR THE MAXWELL-KLEIN-GORDON SYSTEM IN 2D

HARTMUT PECHER
Fakultät für Mathematik und Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20, 42119 Wuppertal, Germany
(Communicated by Yachun Li)

Abstract. The local well-posedness problem for the Maxwell-Klein-Gordon system in Coulomb gauge as well as Lorenz gauge is treated in two space dimensions for data with minimal regularity assumptions. In the classical case of data in $L^2$-based Sobolev spaces $H^s$ and $H^l$ for the electromagnetic field $\phi$ and the potential $A$, respectively, the minimal regularity assumptions are $s > \frac{1}{2}$ and $l > \frac{3}{4}$, which leaves a gap of $\frac{1}{2}$ and $\frac{1}{4}$ to the critical regularity with respect to scaling $s_c = l_c = 0$. This gap can be reduced for data in Fourier-Lebesgue spaces $\hat{H}^{s,r}$ and $\hat{H}^{l,r}$ to $s > \frac{1}{16}$ and $l > \frac{9}{8}$ for $r$ close to 1, whereas the critical exponents with respect to scaling fulfill $s_c \to 1$, $l_c \to 1$ as $r \to 1$. Here $\|f\|_{\hat{H}^{s,r}} := \|\langle \xi \rangle^s \hat{f}\|_{L^{r'}_{\xi}}$, $1 < r \leq 2$, $\frac{1}{r} + \frac{1}{r'} = 1$. Thus the gap is reduced for $\phi$ as well as $A$ in both gauges.

1. Introduction and main results. Consider the Maxwell-Klein-Gordon system

$$D_\mu D^\mu \phi = m^2 \phi$$
$$\partial^\nu F_{\mu\nu} = j_\mu,$$

where $m > 0$ is a constant and

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$D_\mu \phi := \partial_\mu + i A_\mu \phi$$
$$j_\mu := \text{Im}(\phi D_\mu \phi) = \text{Im}(\phi \partial_\mu \phi) + |\phi|^2 A_\mu.$$

Here $F_{\mu\nu} : \mathbb{R}^{2+1} \to \mathbb{R}$ denotes the electromagnetic field, $\phi : \mathbb{R}^{2+1} \to \mathbb{C}$ a scalar field and $A_\nu : \mathbb{R}^{2+1} \to \mathbb{R}$ the connection field. We use the notation $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where we write $(x^0, x^1, x^2) = (t, x^1, x^2)$ and also $\partial_0 = \partial_t$ and $\nabla = (\partial_1, \partial_2)$. Roman indices run over 1, 2 and greek indices over 0, 1, 2 and repeated upper/lower indices are summed. Indices are raised and lowered using the Minkowski metric $\text{diag}(-1, 1, 1)$.

The Maxwell-Klein-Gordon system describes the motion of a spin 0 particle with mass $m$ self-interacting with an electromagnetic field.

We are interested in local well-posedness (LWP) results, so we have to choose an appropriate gauge. In this paper we consider the Coulomb gauge $\partial_0 A^0 = 0$ and the Lorenz gauge $\partial_\mu A^\mu = 0$. Our aim is to minimize the regularity assumptions for the
Cauchy data. If we consider data in classical $L^2$-based $H^s$-spaces we improve the result in Lorenz gauge and also slightly in Coulomb gauge. In order to improve the results further with respect to scaling we use data in Fourier-Lebesgue spaces $\hat{H}^{s,r}$ (cf. the definition below) with $1 < r < 2$ and obtain an improvement for $r$ close to 1.

Let us make some historical remarks. In space dimension $n = 3$, with Coulomb gauge and Cauchy data $\phi(0) = \phi_0 \in H^s$ , $(\partial_t \phi)(0) \in H^{s-1}$ , $A_j(0) = a_{0j} \in H^1$ , $(\partial_t A_j)(0) = b_{0j} \in H^{l-1}$ , Klainerman and Machedon [12] proved global well-posedness in energy space and above ($s > \frac{1}{2}$) . They detected that the nonlinearities fulfill a null condition. The global well-posedness result was improved by Keel, Roy and Tao [11] to the condition $s = l > \frac{3}{2}$ . LWP for low regularity data was shown by Cuccagna [3] for $l = s > \frac{1}{2}$ . Selberg [17] remarked that a smallness assumption in [3] can be removed. Machedon and Sterbenz [14] proved LWP even in the almost critical range $s > \frac{1}{2}$ , but had to assume a smallness assumption on the data.

In the case $n = 2$ and with Coulomb gauge Czubak and Pikula [4] obtained LWP if either $1 \geq s = l > \frac{1}{2}$ or $s = \frac{5}{8} + \epsilon$ , $l = \frac{3}{4} + \epsilon$ , where $\epsilon > 0$ is arbitrary. Their methods are crucial for our results.

Another paper that is fundamental for our results is due to Seilberg and Tesfahun [18]. In Lorenz gauge and $n = 3$ they proved global well-posedness for finite energy data $\phi_0 \in H^1$ , $\phi_1 \in L^2$ , $F_{\mu\nu}(0) = F_{\mu\nu}^0 \in L^2$ , $\partial_\nu \Phi(0) = a_{0\nu} \in H^1$ , $(\partial_t A_\nu)(0) = b_{0\nu} \in L^2$ . The solution fulfills $\phi \in C^0([0,T],H^s) \cap C^1([0,T],L^2)$ , $F_{\mu\nu} \in C^0([0,T],L^2)$ . The potential possibly loses some regularity compared to the data, which however is of minor interest. In Lorenz gauge the situation is more delicate, because the nonlinearity $Im(\phi \bar{\partial}_\nu \phi)$ has no null structure. The author [15] obtained LWP for less regular data, namely $\phi_0 \in H^s$ , $\phi_1 \in H^{s-1}$ , $F_{\mu\nu}^0 \in H^{s-1}$ , $\nabla a_{0\nu} \in H^{s-1}$ , $b_{0\nu} \in H^{s-1}$ , with $s > \frac{3}{4}$ , where $\phi \in C^0([0,T],H^s) \cap C^1([0,T],H^{s-1})$ , $F_{\mu\nu} \in C^0([0,T],H^{s-1}) \cap C^1([0,T],H^{s-2})$ . $A_\nu$ loses regularity compared to $a_{0\nu}$ and $b_{0\nu}$ , we only obtain $A_\mu = A_{\mu}^{inh} + A_{\mu}^{inh}$ with $\nabla A_{\mu}^{inh} , \partial_\nu A_{\mu}^{inh} \in C^0([0,T],H^s)$ . $A_{\mu}^{inh}$ is of minor interest. In Lorenz gauge the situation is more delicate, because the nonlinearity $Im(\phi \bar{\partial}_\nu \phi)$ has no null structure. The author [15] obtained LWP for less regular data, namely $\phi_0 \in H^s$ , $\phi_1 \in H^{s-1}$ , $F_{\mu\nu}^0 \in H^{s-1}$ , $\nabla a_{0\nu} \in H^{s-1}$ , $b_{0\nu} \in H^{s-1}$ , with $s > \frac{3}{4}$ , where $\phi \in C^0([0,T],H^s) \cap C^1([0,T],H^{s-1})$ , $F_{\mu\nu} \in C^0([0,T],H^{s-1}) \cap C^1([0,T],H^{s-2})$ . $A_\nu$ loses regularity compared to $a_{0\nu}$ and $b_{0\nu}$ , we only obtain $A_\mu = A_{\mu}^{inh} + A_{\mu}^{inh}$ with $\nabla A_{\mu}^{inh} , \partial_\nu A_{\mu}^{inh} \in C^0([0,T],H^s)$ . $A_{\mu}^{inh}$ is of minor interest.

As can be easily seen the minimal regularity for LWP predicted by scaling is $s_c = \frac{3}{8} - 1$ , so that for $n = 3$ there is a gap of $\frac{1}{4}$ both in Coulomb and Lorenz gauge. The author [16] in Lorenz gauge closed this gap up to the endpoint in the sense of scaling, if Cauchy data are given in Fourier-Lebesgue spaces $\hat{H}^{s,r}$ instead of standard $L^2$-based Sobolev spaces $H^s$ . Here we define

$$\|f\|_{\hat{H}^{s,r}} = \|\langle \xi \rangle^s \hat{f}(\xi)\|_{L^r},$$

where $\frac{1}{r} + \frac{1}{2} = 1$ , for $1 < r < 2$ . More precisely, given Cauchy data $\phi_0 \in \hat{H}^{s,r}$ , $\phi_1 \in \hat{H}^{s-1,r}$ , $F_{\mu\nu}^0 \in \hat{H}^{s-1,r}$ , $\nabla a_{0\nu} \in \hat{H}^{s-1,r}$ , $b_{0\nu} \in \hat{H}^{s-1,r}$ , where $s > \frac{3}{2r} - \frac{1}{2}$ , there exists a local solution $\phi \in C^0([0,T],\hat{H}^{s,r}) \cap C^1([0,T],\hat{H}^{s-1,r})$ , $\nabla F_{\mu\nu} , \partial_\nu F_{\mu\nu} \in C^0([0,T],\hat{H}^{s-2,r})$ relative to a potential $A_\mu \in C^0([0,T],\hat{H}^{L,r}) \cap C^1([0,T],\hat{H}^{L-1,r})$ , where $l > \frac{3}{2} - 1$ . In the limit $r \to 1$ ($r > 1$) the condition reduces to $s > 1$ , $l > 2$ , which is optimal up to the endpoint.

Cauchy data in Fourier-Lebesgue spaces were previously considered among others by Grünrock [8] and Grünrock-Vega [10] for the KdV and the modified KdV equation. They were also used by Grünrock [9] in order to prove almost optimal LWP results for wave equations with quadratic nonlinearity for $n = 3$ . For $n = 2$ this was proven by Grigoryan-Nahmod [6] for wave equations with nonlinear terms
that we obtain the equivalent system

\[ L = \text{Lorenz gauge for data in standard Fourier transform with respect to space and time by } \alpha \]

which fulfill a null condition. These results rely on an adaptation of bilinear estimates by Foschi-Klainerman [5] in the classical \( L^2 \)-case.

In the present paper we consider exclusively the case \( n = 2 \) in Coulomb as well as Lorenz gauge for data in standard \( L^2 \)-based Sobolev spaces as well as Fourier- \LWP \ for data with minimal regularity assumptions.

Let us first consider the system (1.1),(1.2) in Coulomb gauge. It is well-known that we obtain the equivalent system

\[ \Delta A_0 = -Im(\phi \partial_t \phi) + |\phi|^2 A_0, \quad (1.5) \]

\[ \Box A_j = Im(\phi \partial_j \phi) + |\phi|^2 A_j - \partial_j \partial_t A_0, \quad (1.6) \]

\[ \Box \phi = -2iA^j \partial_j \phi + 2iA_0 \partial_0 \phi + i(\partial_t A_0) \phi + A^\mu A_\mu \phi + m^2 \phi, \quad (1.7) \]

\[ \partial^j A_j = 0. \quad (1.8) \]

We consider the Cauchy problem for this system with data

\[ A_j(0) = a_j \in \tilde{H}^{l,r}, \quad (\partial_t A_j)(0) = b_j \in \tilde{H}^{l-1,r}, \quad (1.9) \]

\[ \phi(0) = \phi_0 \in H^{s,r}, \quad (\partial_t \phi)(0) = \phi_1 \in \tilde{H}^{s-1,r}, \quad (1.10) \]

which fulfill the compatibility condition

\[ \partial^j a_j = \partial_j b_j = 0. \quad (1.11) \]

The natural scaling transformation for the Maxwell-Klein-Gordon system is given by

\[ A_\mu(t,x), \phi(t,x) \rightarrow \lambda A_\mu(\lambda t, \lambda x), \lambda \phi(\lambda t, \lambda x) \]

for \( \lambda > 0 \). Since \( \| \lambda f(\lambda x) \|_{\tilde{H}^s} = \lambda^{s - \frac{n}{2} + 1} \| f \|_{\tilde{H}^s} \), we conclude that \( s_c = \frac{n}{2} - 1 \) is critical, i.e., \( \LWP \) is expected for \( s \geq s_c \) at best and ill-posedness for \( s < s_c \). More generally we also obtain for \( 1 < r \leq 2 \) : \( \| \lambda f(\lambda x) \|_{\tilde{H}^{s,r}} = \lambda^{s - \frac{n}{2} + 1} \| f \|_{\tilde{H}^{s,r}} \), so that \( s_c = \frac{n}{2} - 1 \).

Before formulating our main result we introduce some notation. We denote the Fourier transform with respect to space and time by \( \hat{\cdot} \). \( u \lesssim v \) is defined by \( |\hat{u}| \lesssim |\hat{v}| \), \( \Box = \partial_t^2 - \Delta \) is the d’Alembert operator, \( a \pm := a \pm \epsilon \) for a sufficiently small \( \epsilon > 0 \), so that \( a + + a + a > a \), and \( \langle \cdot \rangle := (1 + |\cdot|^2)^\frac{1}{2} \).

Let \( \Lambda^s \) be the multiplier with symbol \( \langle \xi \rangle^s \). Similarly let \( D^a \) and \( D^0 \) be the multipliers with symbols \( |\xi|^a \) and \( |\tau| - |\xi|^a \), respectively.

**Definition 1.1.** Let \( 1 \leq r \leq 2 \), \( s, a \in \mathbb{R} \). The Fourier-Lebesgue space \( \tilde{H}^{s,r} \) is the completion of the Schwartz space \( S(\mathbb{R}^2) \) with norm \( \| f \|_{\tilde{H}^{s,r}} = \| \langle \xi \rangle f(\xi) \|_{L^{r'}} \), where \( r' \) is the dual exponent to \( r \), and \( \tilde{H} \) denotes the homogeneous space. The wave-Sobolev spaces \( H_{r,b}^{s} \) are the completion of the Schwartz space \( S(\mathbb{R}^{1+2}) \) with norm

\[ \| \phi \|_{H_{r,b}^{s}} = \| \langle \xi \rangle^s (|\tau| - |\xi|^b \hat{\varphi}(\tau,\xi)) \|_{L^{r'}} \],

where \( r' \) is the dual exponent to \( r \). We also define \( H_{s,b}^r[0,T] \) as the space of the restrictions of functions in \( H_{r,b}^s \) to \( [0,T] \times \mathbb{R}^2 \). Similarly we define \( X_{r,b,\pm} \) with norm

\[ \| \phi \|_{X_{r,b,\pm}} := \| \langle \xi \rangle^s (\tau \pm |\xi|^b \hat{\varphi}(\tau,\xi)) \|_{L^{r'}} \]

and \( X_{r,b,\pm}[0,T] \). \( H_{r,b}^s \) and \( X_{r,b,\pm} \) are the corresponding homogeneous spaces, where \( \langle \xi \rangle \) is replaced by \( |\xi| \). In the case \( r = 2 \) we denote \( H_{s,b}^2 = H_{s,b} \) and \( X_{s,b,\pm} = X_{s,b,\pm} \). For brevity we denote \( \| u \|_{X_{r,b}} = \| u \|_{X_{r,b,-}} + \| u \|_{X_{r,b,+}} \).
Our main result is the following theorem.

**Theorem 1.1.** Let $1 < r \leq 2$. Assume that $s - 1 \leq l \leq s$, $s > \frac{13}{8r} - \frac{5}{16}$, $l > \frac{7}{4r} - \frac{s}{8}$ and $2s - l > \frac{3}{2r}$, $2l - s > \frac{2}{r} - \frac{5}{4}$. Then the Maxwell-Klein-Gordon system (1.5) - (1.11) is locally well-posed in the sense that there exist $T > 0$ and $b > \frac{1}{r}$, such that there exists a unique solution

$$A_0 \in C^0([0, T], \dot{H}^{0+}) \cap \dot{H}^{\min(2s, s+1)} \cap C^1([0, T], \dot{H}^{0+} \cap \dot{H}^{\min(2s-1, s)}) ,$$

$$A_j \in X^r_{l, s} \cap [0, T] + X^r_{l, s} \cap [0, T], \phi \in X^r_{l, s} \cap [0, T] + X^r_{l, s} \cap [0, T] .$$

This solution satisfies

$$A_j \in C^0([0, T], \dot{H}^{l/r}) \cap C^1([0, T], \dot{H}^{l-1/r}) , \phi \in C^0([0, T], \dot{H}^{s,r}) \cap C^1([0, T], \dot{H}^{s-1,r}) .$$

The solution depends continuously on the data and persistence of higher regularity pertains.

**Remark 1.** 1. In the case $r = 2$ we obtain $s > \frac{1}{2}$, $l > \frac{1}{2}$ (and $s - 1 \leq l \leq s$, $2s - l > \frac{1}{2}$, $2l - s > -\frac{1}{2}$). This is $\frac{1}{2}$ away from the critical value $s_c = 0$, hence improving the result of [4].

2. In the case $r > 1$, but close to 1, the conditions are $s > \frac{21}{16}$, $l > \frac{9}{8}$, $2s - l > \frac{3}{2}$, $2l - s > \frac{3}{4}$, whereas $s_c \to 1$ as $r \to 1$. The gap shrinks to $\frac{5}{16}$.

3. It is not at all clear, whether in dimension $n = 2$ the gap disappears as in the case $n = 3$ (cf. [16]).

Now the most important observation was (cf. e.g. [17]) that this system can be rewritten in the following form, which involves null forms.

$$\Delta A_0 = -I m(\phi D_j \phi) + |\phi|^2 A_0 ,$$

(1.12)

$$\Delta \phi A_0 = -D^j I m(\phi D_j \phi) + D^j (|\phi|^2 A_j) ,$$

(1.13)

$$\Box A_j = 2R^k D^{-1} Q_{jk} (Re \phi, Im \phi) + P(|\phi|^2 A_j) =: N_j(A_j, \phi) ,$$

(1.14)

$$\Box \phi = -i Q_{jk} (\phi, D^{-1} (R^k A^k - R^k A^j)) + 2i A_0 \partial_k \phi + i (\partial_i A_0) \phi + A^0 A_\phi \phi + m^2 \phi =: M(A, \phi) ,$$

(1.15)

$$\partial^j A_j = 0 ,$$

where $R_k = D^{-1} \partial_k$ is the Riesz transform, $P$ denotes the projection onto the divergence-free vector fields given by $PX_j = R^k (R_j X_k - R_k X_j)$ and

$$Q_{jk}(u, v) = \partial_j u \partial_k v - \partial_k u \partial_j v$$

(1.16)

denotes the null form.

In the classical case $r = 2$ we follow the arguments by Czubak-Pikula [4], especially when handling the elliptic equations. The bilinear estimates for the null form are by use of Klainerman-Machedon [12] reduced to standard bilinear estimates which are contained in the very convenient paper [1]. The cubic estimates are obtained by using bilinear estimates twice. In the general case $1 < r \leq 2$, especially for $r$ close to 1, we handle the null forms by using estimates from [5], as already Grünrock [8] and Grigoryan-Nahmid [6] did before. It is also essential to use a result by Grigoryan-Tanguay [7] for bilinear estimates in $H^s_{l, b}$-spaces. Finally, in order to optimize the result in the general case $1 < r \leq 2$ we interpolate between the extreme cases $r = 1+$ and $r = 2$.

In Lorenz gauge our main result is given in the following theorem.
Theorem 1.2. Let $1 < r \leq 2$, Assume that $s - 1 \leq l \leq s$ and $s > \frac{13}{8} - \frac{5}{16}$, $l > \frac{7}{4} - \frac{5}{8}, 2s - l > \frac{3}{2}, 2l - s > \frac{5}{4}$. Let initial data be given such that $A_\mu(0) = a_\mu \in \tilde{H}^{1,r}$, $(\partial_\mu A_\nu)(0) = b_\mu \in \tilde{H}^{l-1,r}$, such that the problem satisfies \[\phi(0) = \phi_0 \in \tilde{H}^{s,r}, (\partial_\mu \phi)(0) = \phi_0 \in \tilde{H}^{s-1,r},\] which fulfill the compatibility condition $b_\mu = \partial^\nu a_\nu$. Then there exist $T > 0, b > \frac{1}{2}, b' > \frac{1}{2} + \frac{1}{2T}$, such that the problem $(1.1,1.2)$ in Lorenz gauge $\partial^\mu A_\mu = 0$ has a unique solution

\[\phi \in X_{s,b,+}^r[0,T] + X_{s,b,-}^r[0,T], A_\mu \in X_{l,b,+}^r[0,T] + X_{l,b,-}^r[0,T].\]

This solution satisfies

\[\phi \in C^0([0,T], \tilde{H}^{s,r}) \cap C^1([0,T], \tilde{H}^{s-1,r}), A_\mu \in C^0([0,T], \tilde{H}^{l,r}) \cap C^1([0,T], \tilde{H}^{l-1,r}).\]

Corollary 1.1. In the classical case $r = 2$ assume $s - 1 \leq l \leq s + 1, s > \frac{1}{2}, l > \frac{1}{2}, 2s - l > \frac{5}{4}, 2l - s > -\frac{1}{4}$. Then the system $(1.1,1.2)$ in Lorenz gauge $\partial^\mu A_\mu = 0$ with data $a_\mu \in H^1, b_\mu \in H^{l-1}, \phi_0 \in H^s, \phi_1 \in H^{s-1}$, which fulfill $b_\mu = \partial^\nu a_\nu$, has a unique local solution

\[\phi \in X_{s,b,+}^r[0,T] + X_{s,b,-}^r[0,T], A_\mu \in X_{l,b,+}^r[0,T] + X_{l,b,-}^r[0,T],\]

where $b > \frac{1}{2}, b' > \frac{1}{4}$. The solution satisfies

\[\phi \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1}), A_\mu \in C^0([0,T], H^l) \cap C^1([0,T], H^{l-1}).\]

Remark 2. 1. The result of this Corollary improves the former result of the author [15] from $s > \frac{3}{2}$ to $s > \frac{1}{2}$. This is $\frac{1}{2}$ away from the scaling critical exponent $s_c = 0$.

2. In the case $r = 1+$ we have to assume in Theorem 1.2 the conditions $s > \frac{21}{16}$, $l > \frac{9}{8}, 2s - l > \frac{3}{2}, 2l - s > -\frac{1}{4}$, and $s - 1 \leq l \leq s + 1$. Similarly as in Coulomb gauge this reduces the gap to the scaling critical exponent $s_c$, which converges to $1$ as $r \to 1$.

We start by reformulating the system $(1.1,1.2)$ in Lorenz gauge

\[\partial^\mu A_\mu = 0 \tag{1.17}\]

as follows:

\[\Box A = \partial^\nu \partial_\nu A_\mu - \partial^\nu (\partial^\mu A_\nu - F_{\mu \nu}) = -\partial^\nu F_{\mu \nu} = -j_\mu,\]

thus (using the notation $\partial = (\partial_0, \partial_1,\ldots, \partial_n)$):

\[\Box A = -\text{Im}(\phi \overline{\partial_0^\mu \phi}) - A|\phi|^2 =: N(A, \phi) \tag{1.18}\]

and

\[m^2 \phi = D_\mu D^\mu \phi = \partial_\mu \partial^\mu \phi - iA_\mu \partial^\mu \phi - i\partial_\mu (A^\mu \phi) - A_\mu A^\mu \phi\]

\[= \Box \phi - 2iA^\mu \partial_\mu \phi - A_\mu A^\mu \phi\]

thus

\[\Box = \Box - m^2 = 2iA^\mu \partial_\mu \phi + A_\mu A^\mu \phi =: M(A, \phi). \tag{1.19}\]

In order to prove the theorem we start by reformulating the well-known fact that the term $A^\mu \partial_\mu \phi$ has null structure whereas the term $\text{Im}(\phi \overline{\partial_0^\mu \phi})$ seems to fulfill no null condition. The null forms are then handled similarly as in Coulomb gauge as well as the cubic terms.
2. Preliminaries. We start by collecting some fundamental properties of the solution spaces. We rely on [8]. The spaces $X^r_{s,b}$, where $1 < r < \infty$, are Banach spaces with $S$ as a dense subspace. The dual space is $X^r_{-s,-b}$, where $\frac{1}{r} + \frac{1}{s} = 1$.

The complex interpolation space is given by

$$
(X^{r_0}_{s_0,b_0}, \ldots, X^{r_1}_{s_1,b_1})[\theta] = X^{r}_{s,b},
$$

where $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}$, $b = (1 - \theta)b_0 + \theta b_1$. Similarly the dual space of $H^r_{s,b}$ is $H^{r'}_{-s,-b}$ and

$$
(H^{r_0}_{s_0,b_0}, H^{r_1}_{s_1,b_1})[\theta] = H^{r'}_{s,b}.
$$

If $u = u_+ + u_-$, where $u_{\pm} \in X^r_{s,b,\pm}[0,T]$, then $u \in C^0([0,T], H^s)$, if $b > \frac{1}{r}$.

The "transfer principle" is the following proposition, which is well-known in the case $r = 2$, also applies for general $1 < r < \infty$ (cf. [6], Prop. A.2 or [8], Lemma 1).

We denote $\|u\|_{L^p_t(L^q_x)} := \|\hat{u}\|_{L^p_t(L^q_x)}$.

**Proposition 2.1.** Let $1 \leq p, q \leq \infty$. Assume that $T$ is a bilinear operator which fulfills

$$
\|T(e^{\pm i\xi^D} f_1, e^{\pm i\xi^D} f_2)\|_{L^p_t(L^q_x)} \lesssim \|f_1\|_{H^{1-r},r} \|f_2\|_{H^{r'},r'}
$$

for all combinations of signs $\pm_1, \pm_2 \in \{+1, -1\}$, then for $b > \frac{1}{r}$ the following estimate applies:

$$
\|T(u_1, u_2)\|_{L^p_t(L^q_x)} \lesssim \|u_1\|_{H^r_{s_1,b}} \|u_2\|_{H^{r'}_{s_2,b}}.
$$

The following local well-posedness theorem is an obvious generalization of [8], Thm. 1.

**Theorem 2.1.** Let $N_\pm(u, v) := N_\pm(u_+, u_-, v_+, v_-)$ and $M_\pm(u, v) := M_\pm(u_+, u_-, v_+, v_-)$ be multilinear functions. Assume that for given $s, l \in \mathbb{R}$, $1 < r < \infty$ there exist $b, b' > \frac{1}{r}$ such that the estimates

$$
\|N_\pm(u, v)\|_{X^r_{s,b} \times \ldots} \leq \omega_1(\|u\|_{X^r_{s,b}}, \|v\|_{X^r_{s,b}})
$$

and

$$
\|M_\pm(u, v)\|_{X^r_{l,b'} \times \ldots} \leq \omega_2(\|u\|_{X^r_{s,b}}, \|v\|_{X^r_{s,b}})
$$

are valid with nondecreasing functions $\omega_j$, where $\|u\|_{X^r_{s,b}} := \|u_+\|_{X^r_{s,b,\pm}} + \|u_-\|_{X^r_{s,b,\pm}}$.

Then there exist $T = T(\|u_{0\pm}\|_{\dot{H}^{r-\gamma}}, \|v_{0\pm}\|_{\dot{H}^{r'}}) > 0$ and a unique solution $(u_+, u_-, v_+, v_-) \in X^r_{s,b,\pm}[0,T] \times X^r_{s,b,\pm}[0,T] \times X^r_{l,b',\pm}[0,T] \times X^r_{l,b',\pm}[0,T]$ of the Cauchy problem

$$
\partial_t u_{\pm} \pm i\Lambda u = N_\pm(u, v) \quad \partial_t v_{\pm} \pm i\Lambda v = M_\pm(u, v)
$$

$$
u_{\pm}(0) = u_{0\pm} \in \dot{H}^{s,r}, \quad v_{\pm}(0) = v_{0\pm} \in \dot{H}^{l,r}.
$$

This solution is persistent and the mapping data upon solution $(u_{0\pm}, u_{0-, v_{0+}, v_{0-}}) \mapsto (u_+, u_-, v_+, v_-) : \dot{H}^{s,r} \times \dot{H}^{s,r} \times \dot{H}^{l,r} \times \dot{H}^{l,r} \rightarrow X^r_{s,b,\pm}[0,T] \times X^r_{s,b,\pm}[0,T] \times X^r_{l,b',\pm}[0,T] \times X^r_{l,b',\pm}[0,T]$ is locally Lipschitz continuous for any $T_0 < T$. 
3. **MKG in Coulomb gauge.** In a standard way we rewrite the system (1.14), (1.15) as a first order (in t) system. Defining \( A_{j,\pm} = \frac{1}{2} (A_j \pm (i\Lambda)^{-1}\partial_t A_j), \) \( \phi_{\pm} = \frac{1}{2} (\phi \pm (i\Lambda)^{-1}\partial_t \phi), \) so that \( A_j = A_{j,+} + A_{j,-}, \) \( \partial_t A_j = i\Lambda (A_{j,+} - A_{j,-}), \) \( \phi = \phi_+ + \phi_-), \) \( \partial_t \phi = i\Lambda (\phi_+ - \phi_-) \) the system transforms to

\[
\begin{align*}
&(i\partial_t \pm \Lambda)A_{j,\pm} = -A_j \mp (2\Lambda)^{-1}M_j(A_j, \partial_t A_j, \phi, \partial_t \phi), \\
&(i\partial_t \pm \Lambda)\phi_{\pm} = -\phi \mp (2\Lambda)^{-1}N(A_j, \partial_t A_j, \phi, \partial_t \phi).
\end{align*}
\]

(3.1) (3.2)

The initial data transform to

\[
A_{j,\pm}(0) = \frac{1}{2} (a_j \pm (i\Lambda)^{-1}b_j) \in \tilde{H}^{r,r}, \quad \phi_{\mp}(0) = \frac{1}{2} (\phi_0 \pm (i\Lambda)^{-1}\phi_1) \in \tilde{H}^{r,r}.
\]

This transformation allows to apply Theorem 2.1. For the wave equations one has to estimate

\[
\|\Box A_j\|_{H^{r-1,\alpha'-1+}}
\]

and

\[
\|\Box \phi\|_{H^{r-1,\alpha'-1+}}.
\]

(3.3) (3.4)

We obtain the necessary estimates for (3.3) in Lemma 3.22 and Lemma 3.24, and for (3.4) in Lemma 3.21, Lemma 3.23 and Corollary 3.4.

Let us first consider the elliptic equations (1.12) and (1.13). The equation (1.13) is easier to handle. We prove that it is solved by \( B_0 = \partial_t A_0 \in C^0([0,T], H^r). \) Defining \( A_0(t) := \int_0^t B_0(s)ds + a_0, \) where \( a_0 \) is the solution of the following variational problem at \( t = 0: \)

\[
\int_{\mathbb{R}^2} |\nabla A_0|^2 + |D_0\phi|^2 \, dx \to \text{min}
\]

in \( H^1(\mathbb{R}^2). \) The following Lemma was shown by Czubak-Pikula [4], Lemma 3.1 and Lemma 3.2.

**Lemma 3.1.** If \( B_0 \) solves (1.13), then \( A_0(t) \) solves (1.12) in the sense of tempered distributions for every \( t \in [0,T]. \) The solution of (1.12) is unique in \( \tilde{H}^{r} \cap H^1. \)

The regularity of \( A_0 \) and \( B_0 \) and estimates for \( A_0(t) \) and \( B_0(t) \) are studied in Lemma 3.9, Cor. 3.2 and Lemma 3.10.

We start by estimating the null forms. The proof of the following bilinear estimates relies on estimates given by Foschi and Klainerman [5]. We first treat the case \( r > 1, \) but close to 1.

**Lemma 3.2.** Let \( 1 < r \leq 2. \) Assume \( 0 \leq \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \geq \frac{1}{r} \) and \( b > \frac{1}{r}. \) Let

\[
q_{12}(u, v) := Q_{12}(D^{-1}u, D^{-1}v),
\]

where the null form \( Q_{12} \) is given by (1.16). The following estimate applies

\[
\|q_{12}(u, v)\|_{H^{r',0}} \lesssim \|u\|_{H^{r_1,\beta}} \|v\|_{H^{r_2,\beta}}.
\]

**Proof.** Because we use inhomogeneous norms it is obviously possible to assume \( \alpha_1 + \alpha_2 = \frac{1}{r}. \) Moreover, by interpolation we may reduce to the case \( \alpha_1 = \frac{1}{r}, \alpha_2 = 0. \)

The left hand side of the claimed estimate equals

\[
\|F(q_{12}(u, v))\|_{L^{r_1}_t} = \int q_{12}(\eta, \eta - \xi) \hat{u}(\lambda, \eta) \hat{v}(\tau - \lambda, \xi - \eta) d\lambda d\eta \|_{L^{r_1}_t}.
\]

(3.5)

Let now \( u(t, x) = e^{\pm iD} u_0^{\pm 1}(x), \) \( v(t, x) = e^{\pm iD} v_0^{\pm 2}(x) \), so that

\[
\hat{u}(\tau, \xi) = c\delta(\tau \mp 1 |\xi|) \hat{u}_0^{\pm 1}(\xi), \quad \hat{v}(\tau, \xi) = c\delta(\tau \mp 2 |\xi|) \hat{v}_0^{\pm 2}(\xi).
\]
This implies
\[
\|F(q_{12}(u,v))\|_{L_{\xi}^{r_1}}
= c^2 \int q_{12}(\eta, \xi - \eta) u_0^+(\eta) v_0^+ (\xi - \eta) \delta(\tau - \lambda_\eta) d\lambda d\eta \|_{L_{\xi}^{r_1}}
= c^2 \int q_{12}(\eta, \xi - \eta) u_0^+(\eta) v_0^+ (\xi - \eta) \delta(\tau - \lambda_\eta) d\eta \|_{L_{\xi}^{r_1}}.
\]
By symmetry we only have to consider the elliptic case \( \pm_1 = \pm_2 = + \) and the hyperbolic case \( \pm_1 = +, \pm_2 = - \).

**Elliptic case.** We obtain by [5], Lemma 13.2:
\[
|q_{12}(\eta, \xi - \eta)| \leq \frac{|\eta_1(\xi - \eta)_2 - \eta_2(\xi - \eta)_1|}{|\eta| |\xi - \eta|} \lesssim \frac{|\xi|^{\frac{1}{2}} (|\eta| + |\xi - \eta| - |\xi|)^{\frac{1}{2}}}{|\eta|^{\frac{1}{2}} |\xi - \eta|^{\frac{1}{2}}.}
\]
By Hölder’s inequality we obtain
\[
\|F(q_{12}(u,v))\|_{L_{\xi}^{r_1}}
\lesssim \|\int |\xi|^{\frac{1}{2}} |\xi - \eta| |\xi|^{\frac{1}{2}} \delta(\tau - |\eta| - |\xi - \eta|) |\hat{u}_0^+(\eta)| |\hat{v}_0^+(\xi - \eta)| d\eta\|_{L_{\xi}^{r_1}}
\lesssim \sup_{\tau, \xi} \|D^{\frac{1}{2}} u_0^+\|_{L_{\xi}^{r_1}} \|\hat{v}_0^+\|_{L_{\xi}^{r_1}}
\]
where
\[
I = |\xi|^{\frac{1}{2}} |\xi - \eta|^{\frac{1}{2}} \left( \int \delta(|\tau - |\eta| - |\xi - \eta|)|\eta|^{-1 - \frac{r}{2}} |\xi - \eta|^{-\frac{r}{2}} d\eta \right)^{\frac{1}{r}}.
\]
We want to prove \( \sup_{\tau, \xi} I \lesssim 1 \). By [5], Lemma 4.3 we obtain
\[
\int \delta(|\tau - |\eta| - |\xi - \eta|)|\eta|^{-1 - \frac{r}{2}} |\xi - \eta|^{-\frac{r}{2}} d\eta \sim A |\tau|^B,
\]
where \( A = \max(1 + \frac{r}{2}, \frac{3}{2}) - 1 - r = -\frac{r}{2} \) and \( B = 1 - \max(1 + \frac{r}{2}, \frac{3}{2}) = -\frac{r}{2} \). Using \( |\xi| \leq |\tau| \) this implies
\[
I \lesssim |\xi|^{\frac{1}{2}} |\tau| - |\xi|^{\frac{1}{2}} |\tau| - |\xi|^{\frac{1}{2}} \lesssim 1.
\]

**Hyperbolic case.** We start with the following bound (cf. [5], Lemma 13.2):
\[
|q_{12}(\eta, \xi - \eta)| \leq \frac{|\eta_1(\xi - \eta)_2 - \eta_2(\xi - \eta)_1|}{|\eta| |\xi - \eta|} \lesssim \frac{|\xi|^{\frac{1}{2}} (|\eta| + |\xi - \eta| - |\xi|)^{\frac{1}{2}}}{|\eta|^{\frac{1}{2}} |\xi - \eta|^{\frac{1}{2}}},
\]
so that similarly as in the elliptic case we have to estimate
\[
I = |\xi|^{\frac{1}{2}} |\tau| - |\xi|^{\frac{1}{2}} \left( \int \delta(|\tau - |\eta| + |\xi - \eta|)|\eta|^{-1 - \frac{r}{2}} |\xi - \eta|^{-\frac{r}{2}} d\eta \right)^{\frac{1}{r}}.
\]
In the subcase \(|\eta| + |\xi - \eta| \leq 2|\xi| \) we apply [5], Prop. 4.5 and obtain
\[
\int \delta(|\tau - |\eta| + |\xi - \eta|)|\eta|^{-1 - \frac{r}{2}} |\xi - \eta|^{-\frac{r}{2}} d\eta \sim |\xi|^{A_1} |\xi - |\tau||^B,
\]
where in the subcase \( 0 \leq \tau \leq |\xi| \) we obtain \( A = \max(\frac{r}{2}, \frac{3}{2}) - 1 - r = \frac{1}{2} - r \) and \( B = 1 - \max(\frac{r}{2}, \frac{3}{2}) = -\frac{1}{2} \). This implies
\[
I \lesssim |\xi|^{\frac{1}{2}} |\tau| - |\xi|^{\frac{1}{2}} |\xi|^{\frac{r}{2} - 1} |\tau| - |\xi|^{\frac{r}{2}} \lesssim 1.
\]
Similarly in the subcase $-|\xi| \leq \tau \leq 0$ we obtain $A = \max(1 + \frac{r}{2}, \frac{3}{2}) - 1 = -\frac{r}{2}$, 
$B = 1 - \max(1 + \frac{r}{2}, \frac{3}{2}) = -\frac{r}{2}$, which implies 
\[ I \sim |\xi|^\frac{1}{2} ||\tau| - |\xi||^\frac{1}{2} ||\tau| - |\xi||^{\frac{3}{2}} = 1. \]

In the subcase $|\eta| + |\xi - \eta| \geq 2|\xi|$ we obtain by [5], Lemma 4.4:
\[ \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-\frac{1}{2}} |\xi - \eta|^{-\frac{5}{2}} d\eta 
\sim ||\tau| - |\xi||^{-\frac{1}{2}} ||\tau| + |\xi||^{-\frac{1}{2}} \int_2^\infty (|\xi| x + \tau)^{-r} (|\xi| x - \tau)^{1-r} (x^2 - 1)^{-\frac{1}{2}} dx \]
\[ \sim ||\tau| - |\xi||^{-\frac{1}{2}} ||\tau| + |\xi||^{-\frac{1}{2}} \int_2^\infty (x + |\xi|)^{-r} (x - |\xi|)^{1-r} (x^2 - 1)^{-\frac{1}{2}} dx \cdot |\xi|^{1-r}. \]

We remark that in fact the lower limit of the integral can be chosen as 2 by inspection of the proof in [5]. The integral converges, because $|\tau| \leq |\xi|$ and $r > 1$. This implies the bound
\[ I \lesssim |\xi|^\frac{1}{2} ||\tau| - |\xi||^{\frac{1}{2}} ||\tau| + |\xi||^{-\frac{1}{2}} |\xi|^{\frac{3}{2}} \lesssim 1. \]

Summarizing we obtain
\[ ||q_{12}(u,v)||_{H^{s,b}_{r,-1,0}} \lesssim ||D^{\frac{1}{2}} v_{0}^{\pm 1} ||_{L^{r'}} ||v_{0}^{\pm 2} ||_{L^{r'}}. \]

By the transfer principle Prop. 2.1 we obtain the claimed result. \hfill \Box

An immediate consequence is the following corollary.

**Corollary 3.1.** Assume $1 \leq s \leq l + 1$, $l \geq \frac{1}{r}$ and $b > \frac{1}{r}$. Then the following estimate applies:
\[ \|Q_{jk}(\phi, D^{-1}(R_{j} A_{k} - R_{k} A_{j}))\|_{H^{s,b}_{r,-1,0}} \lesssim \|\phi\|_{H^{s,b}_{r,1}} \sum_{j} \|A_{j}\|_{H^{1,b}.} \]

**Proof.** After application of the fractional Leibniz rule we obtain by Lemma 3.2 the result as follows:
\[ \|Q_{12}(\phi, D^{-1}v)\|_{H^{s,b}_{r,-1,0}} = \|q_{12}(D\phi, v)\|_{H^{s,b}_{r,-1,0}} \lesssim \|D\phi\|_{H^{s,b}_{r,-1,0}} \|v\|_{H^{s,b}_{r,1}} \lesssim \|\phi\|_{H^{s,b}_{r,1}} \|v\|_{H^{s,b}_{r,1}}, \]
if $l \geq \frac{1}{r}$. \hfill \Box

**Lemma 3.3.** Assume $1 \leq l \leq s + \frac{1}{2}$, $2 - l > \frac{3}{2r}$ and $b > \frac{1}{r}$. Then the following estimate pertain:
\[ \|D^{-1} Q_{jk}(Re \phi, Im \psi)\|_{H^{s,b}_{r,-1,0}} \leq \|\phi\|_{H^{s,b}_{r,1}} \|\psi\|_{H^{s,b}_{r,1}}. \]

**Proof.** If we apply the elementary estimate (cf. (13))
\[ Q_{jk}(u,v) \lesssim D(D^{\frac{1}{2}} u D^{\frac{1}{2}} v), \]
we reduce to
\[ \|uv\|_{H^{s,b}_{r,-1,0}} \lesssim \|u\|_{H^{s,-\frac{1}{2},b}} \|v\|_{H^{s,-\frac{1}{2},b}}. \]
If $1 \leq l \leq s + \frac{1}{2}$ the fractional Leibniz rule reduces this to
\[ \|uv\|_{H^{s,b}_{r,0}} \lesssim \|u\|_{H^{s,l+\frac{1}{2},b}} \|v\|_{H^{s,-\frac{1}{2},b}} \]
This follows from Lemma 3.4 if $2s - l > \frac{3}{2r}$. \hfill \Box
Lemma 3.4. Let $1 < r \leq 2$ and $\alpha_1, \alpha_2 \geq 0$. The estimate
\[ \|uv\|_{H_{\alpha,b}^{r}} \lesssim \|u\|_{H_{\alpha_1,b_1}^{r}} \|v\|_{H_{\alpha_2,b_2}^{r}} \]
applies in the following cases:
1. $\alpha_1 + \alpha_2 > \frac{2}{r} , b_1, b_2 \geq 0 , b_1 + b_2 > \frac{1}{r} $,
2. $\alpha_1 + \alpha_2 > \frac{3}{2r} , b_1, b_2 > \frac{1}{2r} $ and $b_1 + b_2 > \frac{3}{2r} $.

Proof. This follows from [7], Prop. 3.1 by summation over the dyadic pieces. □

Lemma 3.5. Let $1 < r \leq 2$ , $0 \leq \alpha_1, \alpha_2$ and $\alpha_1 + \alpha_2 \geq \frac{1}{r} + b , b > \frac{1}{r} $. Then the following estimate applies:
\[ \|uv\|_{H_{\alpha,b}^{r}} \lesssim \|u\|_{H_{\alpha_1,b_1}^{r}} \|v\|_{H_{\alpha_2,b_2}^{r}} . \]

Proof. We may assume $\alpha_1 = \frac{1}{r} + b , \alpha_2 = 0$ . We apply the “hyperbolic Leibniz rule” (cf. [1], p. 128):
\[ ||\tau| - |\xi|| \lesssim ||\rho| - |\eta|| + ||\tau - \rho| - |\xi - \eta|| + b_{\pm}(\xi, \eta) , \]
(3.5)
where
\[ b_{+}(\xi, \eta) = |\eta| + |\xi - \eta| - |\xi| , \quad b_{-}(\xi, \eta) = |\xi| - |\eta| - |\xi - \eta| . \]

Let us first consider the term $b_{\pm}(\xi, \eta)$ in (3.5). Decomposing $uv = u_{+}v_{+} + u_{+}v_{-} + u_{-}v_{+} + u_{-}v_{-}$, where $u_{\pm}(t) = e^{\pm i\xi D}f , v_{\pm}(t) = e^{\pm i\xi D}g$ , we use
\[ \hat{u}_{\pm}(\tau, \xi) = c\delta(\tau \mp |\xi|) \hat{f}(\xi) , \quad \hat{v}_{\pm}(\tau, \xi) = c\delta(\tau \mp |\xi|) \hat{g}(\xi) \]
and have to estimate
\[ \| \int b_{\pm}^{b}(\xi, \eta)\delta(\tau - |\eta| \mp |\xi - \eta|)\hat{f}(\xi)\hat{g}(\xi - \eta)\, d\eta \|_{L_{t}^{r'}} \]
\[ = \| \int ||\tau| - |\xi||^{b}\delta(\tau - |\eta| \mp |\xi - \eta|)\hat{f}(\xi)\hat{g}(\xi - \eta)\, d\eta \|_{L_{t}^{r'}} \]
\[ \lesssim \sup_{\tau, \xi} I \| \frac{D^{b_{\pm}}}{b_{\pm}} f \|_{L_{t}^{r'}} \| \frac{D^{b_{\pm}}}{b_{\pm}} g \|_{L_{t}^{r'}} . \]

Here we used Hölder’s inequality, where
\[ I = ||\tau| - |\xi||^{b}(\int \delta(\tau - |\eta| \mp |\xi - \eta|)|\eta|^{-1 + br} d\eta)^{\frac{1}{b}} . \]

In order to obtain $I \lesssim 1$ we first consider the elliptic case $\pm_{1} = \pm_{2} = +$ and use [5], Prop. 4.3. Thus
\[ I \sim ||\tau| - |\xi||^{b} ||\tau - |\xi||^{\frac{A}{B}} \]
with $A = \max(1 + br, \frac{3}{2}) - (1 + br) = 0$ and $B = 1 - \max(1 + br, \frac{3}{2}) = -br$ .

Next we consider the hyperbolic case $\pm_{1} = + , \pm_{2} = -$ .
First we assume $|\eta| + |\xi - \eta| \leq 2|\xi|$ and use [5], Prop. 4.5 which gives
\[ \int \delta(\tau - |\eta| + |\xi - \eta|)|\eta|^{-1 + br} d\eta \sim |\xi|^{A} |\xi| - |\tau||^{B} , \]
where $A = \frac{3}{2} - (1 + br) = \frac{1}{2} - br , B = 1 - \frac{3}{2} = -\frac{1}{2}$ , if $0 \leq \tau \leq |\xi|$ , so that
\[ I \sim ||\tau| - |\xi||^{b} ||\tau - |\xi||^{\frac{A}{B}} \lesssim 1 . \]

If $-|\xi| \leq \tau \leq 0$ we obtain $A = \max(1 + br, \frac{3}{2}) - (1 + br) = 0$ , $B = 1 - \max(1 + br, 2) = -br$ , which implies $I \lesssim 1$ .
Next we assume $|\eta| + |\xi - \eta| \geq 2|\xi|$, use [5], Lemma 4.4 and obtain
\[
I \sim ||r| - |\xi||^6 (\int \delta(r - |\eta| - |\xi - \eta|)|\eta|^{-1-br} d\eta)^{\frac{1}{2}}
\]
\[
\sim ||r| - |\xi||^6 (||r| - |\xi||^{-\frac{1}{2}}||r| + |\xi||^{-\frac{1}{2}}) \int_2^\infty (|\xi|x + \tau)^{-br}(|\xi|x - \tau)(x^2 - 1)^{-\frac{1}{2}} dx)^{\frac{1}{2}}
\]
\[
\sim ||r| - |\xi||^6 (||r| - |\xi||^{-\frac{1}{2}}||r| + |\xi||^{-\frac{1}{2}}) \cdot \int_2^\infty (x + \frac{\tau}{|\xi|})^{-br}(x - \frac{\tau}{|\xi|})(x^2 - 1)^{-\frac{1}{2}} dx \cdot |\xi|^{1-br})^{\frac{1}{2}}.
\]
This integral converges, because $\tau \leq |\xi|$ and $b > \frac{1}{2}$. This implies
\[
I \lesssim ||r| - |\xi||^{-\frac{1}{2}}||r| + |\xi||^{-\frac{1}{2}}|\xi|^{\frac{1}{2} - b} \lesssim 1,
\]
using $|\tau| \leq |\xi|$.

By the transfer principle we obtain
\[
\|B^b_{\pm}(u, v)\|_{H^s_{0,0}} \lesssim \|u\|_{H^s_{\frac{1}{2}, b}} \|v\|_{H^s_{0, b}}.
\]
Here $B^b_{\pm}$ denotes the operator with Fourier symbol $b_{\pm}$.

Consider now the term $||\rho| - |\eta||$ (or similarly $||r| - |\xi| - |\eta||$) in (3.5). We have to prove
\[
\|uD^b_{-} v\|_{H^s_{0,0}} \lesssim \|u\|_{H^{s+1}_{1,b}} \|v\|_{H^s_{\alpha, b}},
\]
which is implied by
\[
\|uv\|_{H^s_{0,0}} \lesssim \|u\|_{H^{s+1}_{1,b}} \|v\|_{H^s_{\alpha, b}}.
\]
This results from Lemma 3.4, because $\alpha_1 + \alpha_2 \geq 1 + b > \frac{2}{r}$, which completes the proof.

**Lemma 3.6.** Let $1 < r \leq 2$. Assume $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 > \frac{1}{r}$, $b > \frac{1}{2}$. Then the following estimate applies:
\[
\|uv\|_{H^s_{0,r}} \lesssim \|u\|_{H^{s+1}_{1,b}} \|v\|_{H^s_{\alpha, b}}.
\]

**Proof.** Interpolation between the estimates in Lemma 3.4 and Lemma 3.5 implies the result.

Let us now consider the cubic nonlinearities.

**Lemma 3.7.** Let $1 < r \leq 2$. If $1 \leq s \leq l + 1$, $l > \frac{13}{8} - \frac{1}{2}$, $2l - s > \frac{7}{4} - 1$ and $b > \frac{1}{r}$, the following estimate applies:
\[
\|A^j \phi\|_{H^s_{r-1,0}} \lesssim \|A^j \|_{H^s_{1,b}} \|A\|_{H^s_{\alpha, b}} \|\phi\|_{H^s_{r,b}}.
\]

**Proof.** We obtain
\[
\|uv\|_{H^s_{r-1,0}} \lesssim \|u\|_{H^s_{1,b}} \|v\|_{H^s_{r_{m}, 2}} \lesssim \|u\|_{H^s_{1,b}} \|v\|_{H^s_{1,b}} \|w\|_{H^s_{1,b}}.
\]
For the first estimate we need $m + 1 \geq s \geq 1$ and $1 - s + m > \frac{3}{2r}$, if we use Lemma 3.4, thus $m > \frac{3}{2r} - 1$. For the second estimate we apply Lemma 3.6, which requires $l \geq m$ and $2l - m > \frac{7}{4}$. This implies the conditions $2l - (s - 1) > \frac{7}{4} \Leftrightarrow 2l - s > \frac{7}{4} - 1$ and $l > \frac{7}{8} + \frac{3}{8} = \frac{1}{2}$.

**Lemma 3.8.** Let $1 < r \leq 2$. Assume $1 \leq l \leq s + 1$, $s > \frac{13}{8} - \frac{1}{2}$, $2s - l > \frac{7}{4} - 1$ and $b > \frac{1}{r}$. Then
\[
\|\phi\|_{H^s_{1,b}} \lesssim \|\phi\|_{H^s_{1,b}} \|\phi\|_{H^s_{\alpha, b}} A_j \|A\|_{H^s_{\alpha, b}}.
\]
Proof. We obtain
\[ \|uvw\|_{H^{-1,0}} \lesssim \|u\|_{H^{-s}} \|v\|_{H^{-s}} \lesssim \|u\|_{H^{-s}} \|v\|_{H^{-s}} \|w\|_{H^{-s}}. \]
For the first estimate we need \( m + 1 \geq l \geq 1 \) and \( m > \frac{3}{2} - 1 \), if we use Lemma 3.4. For the second estimate we apply Lemma 3.6, which requires \( s \geq m \) and \( 2s - m > \frac{5}{2} \). This implies the conditions \( 2s - (l - 1) > \frac{7}{4} \Leftrightarrow 2s - l > \frac{7}{4} - 1 \) and \( s > \frac{7}{4r} + \frac{3}{4r} - \frac{1}{2} \).

We come now to the estimates for the elliptic part and start with \( B_0 \).

**Lemma 3.9.** 1. Assume \( s > 1 \), \( \phi \in C^0([0,T], H^s) \), \( A_j \in C^0([0,T], H^{s-1}) \).

Assume \( B_0 \) solves
\[ \Delta B_0 = -\partial^j Im(\phi \overline{\partial_j \phi}) + \partial^j (|\phi|^2 A_j). \]
Then \( B_0 \in C^0([0,T], H^s) \) for \( 0 < \sigma \leq s \) and
\[ \|B_0\|_{L^\infty((0,T), H^s)} \lesssim \|\phi\|_{L^\infty((0,T), H^s)}^2 (1 + \sum_j \|A_j\|_{L^\infty((0,T), H^{s-1})}). \]

2. Assume \( \frac{1}{2} < s \leq 1 \), \( A_j \in C^0([0,T], L^2) \). Then \( B_0 \in C^0([0,T], H^s) \) for \( 0 < \sigma < 2s - 1 \) and
\[ \|B_0\|_{L^\infty((0,T), H^s)} \lesssim \|\phi\|_{L^\infty((0,T), H^s)}^2 (1 + \sum_j \|A_j\|_{L^\infty((0,T), L^2)}). \]

**Proof.** 1. We have to prove for \( 0 < \sigma \leq s \):
\[ \|\phi \overline{\partial_j \phi}\|_{H^{-s-1}} \lesssim \|\phi\|^2_{H^s}. \] (3.6)

For \( s > 1 \) the Sobolev multiplication law (SML) ([19], Proposition 3.15) implies
\[ \|\phi \overline{\partial_j \phi}\|_{H^{-s-1}} \lesssim \|\phi\|_{H^s} \|\partial_j \phi\|_{H^{-s-1}} \lesssim \|\phi\|^2_{H^s}, \]

and by the homogeneous version of SML we obtain for \( s \geq \frac{1}{2} \):
\[ \|\phi \overline{\partial_j \phi}\|_{H^{-s-1-\sigma}} \lesssim \|\phi\|_{H^{s-\frac{1}{2}}} \|\partial_j \phi\|_{H^{-s-\frac{1}{2}}} \lesssim \|\phi\|^2_{H^s}. \]

This implies (3.6). Next we have to prove for \( 0 < \sigma \leq s \)
\[ \|\phi\|^2 A_j\|_{H^{-s-1}} \lesssim \|\phi\|^2_{H^s} \|A_j\|_{H^{-s-1}}. \] (3.7)

The SML implies for \( s > 1 \):
\[ \|\phi\|^2 A_j\|_{H^{-s-1}} \lesssim \|\phi\|^2_{H^s} \|A_j\|_{H^{-s-1}} \lesssim \|\phi\|^2_{H^s} \|A_j\|_{H^{-s-1}} \]

and by the homogeneous SML:
\[ \|\phi\|^2 A_j\|_{H^{-s-1}} \lesssim \|\phi\|^2_{H^s} \|A_j\|_{H^s} \lesssim \|\phi\|^2_{H^s} \|A_j\|_{H^s} \lesssim \|\phi\|^2_{H^s} \|A_j\|_{H^s}, \]

which implies (3.7).

2. Next we prove (3.6) for \( \frac{1}{2} < s \leq 1 \) and \( 0 < \sigma < 2s - 1 \). If the frequencies of \( \partial_j \phi \) are large, we obtain by the homogeneous SML
\[ \|\phi \overline{\partial_j \phi}\|_{H^{s-1}} \lesssim \|\phi\|_{H^{1-s}} \|\partial_j \phi\|_{H^{s-1}} \lesssim \|\phi\|_{H^s} \|\partial_j \phi\|_{H^{s-1}} \]

and
\[ \|\phi \overline{\partial_j \phi}\|_{H^{s-2-\sigma}} \lesssim \|\phi\|_{H^{s-\frac{1}{2}}} \|\partial_j \phi\|_{H^{s-1}} \lesssim \|\phi\|_{H^s} \|\partial_j \phi\|_{H^{s-1}}. \]

If the frequencies of \( \partial_j \phi \) are small we obtain
\[ \|\phi \overline{\partial_j \phi}\|_{H^{s-1}} \lesssim \|\phi\|_{H^{s-\sigma}} \|\partial_j \phi\|_{L^2} \lesssim \|\phi\|_{H^s} \|\partial_j \phi\|_{H^{s-1}}. \]
2. If

\[ \text{Proof.} \]

We only have to recall Lemma 3.11.

Corollary 3.2.

Next we prove (3.7). We obtain

\[ \| | \phi^2 A_j \|_{\dot{H}^{2- \sigma}} \leq \| | \phi^2 \|_{\dot{H}^{2- \sigma}} \cdot \| A_j \|_{L^2} \leq \| | \phi^2 \|_{\dot{H}^{2- \sigma}} \cdot \| A_j \|_{L^2} \leq \| | \phi^2 \|_{\dot{H}^{2- \sigma}} \cdot \| A_j \|_{L^2} \].

Corollary 3.2. Let \( s \) and \( \sigma \) be as in Lemma 3.9. Then \( A_0 \in C^0([0,T], \dot{H}^\sigma) \) for \( \sigma \leq 1 \) and

\[ \| A_0 \|_{L^\infty((0,T), \dot{H}^\sigma)} \lesssim T \sum_j \| A_j \|_{L^\infty((0,T), \dot{H}^\sigma)} + \| A_0 \|_{\dot{H}^\sigma}. \]

Proof. We only have to recall \( A_0(t) := \int_0^t B_0(s) ds + a_0 \), and \( a_0 \) as the solution of the variational problem considered before Lemma 3.1 belongs to \( \dot{H}^1 \).

It is possible to improve the regularity of \( A_0 \) by using that \( A_0 \) solves the elliptic equation (1.12) by Lemma 3.1.

Lemma 3.10. 1. If \( s > 1 \) and \( \phi \in C^0([0,T], \dot{H}^\sigma) \cap C^1([0,T], \dot{H}^{s-1}) \) then \( A_0 \in C^0([0,T], \dot{H}^{s+1}) \) and the following estimate applies:

\[ \| A_0 \|_{L^\infty((0,T), \dot{H}^{s+1})} \lesssim \| \phi \|_{L^\infty((0,T), \dot{H}^\sigma)} \cdot \| \partial_t \phi \|_{L^\infty((0,T), \dot{H}^{s-1})} + \| \phi \|_{L^\infty((0,T), \dot{H}^{1+ \sigma})} \cdot \| A_0 \|_{L^\infty((0,T), \dot{H}^{\sigma+1})}. \]

2. If \( \frac{1}{2} < s \leq 1 \) and \( A_0 \in C^0([0,T], \dot{H}^\sigma) \) for some \( 0 < \sigma < 2s - 1 \), then \( A_0 \in C^0((0,T), \dot{H}^\sigma) \) for \( 1 < \sigma < 2s \), and the following estimate applies:

\[ \| A_0 \|_{L^\infty((0,T), \dot{H}^\sigma)} \lesssim \| \phi \|_{L^\infty((0,T), \dot{H}^\sigma)} \cdot \| \partial_t \phi \|_{L^\infty((0,T), \dot{H}^{s-1})} + \| \phi \|_{L^\infty((0,T), \dot{H}^{1+ \sigma})} \cdot \| A_0 \|_{L^\infty((0,T), \dot{H}^{\sigma+1})}. \]

Proof. 1. By the elliptic equation (1.12) we obtain

\[ \| A_0 \|_{\dot{H}^{s+1}} \lesssim \| \phi \|_{\dot{H}^{s-1}} + \| \phi^2 A_0 \|_{\dot{H}^{s-1}}. \]

By the SML we obtain

\[ \| \phi \|_{\dot{H}^{s-1}} \lesssim \| \partial_t \phi \|_{\dot{H}^{s-1}} \]

and

\[ \| \phi^2 A_0 \|_{\dot{H}^{s-1}} \lesssim \| A_0 \|_{\dot{H}^{s+1}} \cdot \| \phi \|_{\dot{H}^{s-1}} \lesssim \| A_0 \|_{\dot{H}^{\sigma+1}} \cdot \| \phi \|_{\dot{H}^\sigma}. \]

2. This can be proven similarly as for Lemma 3.9.

Next we address the bilinear terms involving \( A_0 \) and \( \partial_t A_0 \).

Lemma 3.11. Let \( 1 < r \leq 2 \), \( s > 1 \) and \( b > \frac{1}{r} \).

1. If \( s \leq 2 \) the following estimate applies:

\[ \| A_0 \partial_t \phi \|_{\dot{H}^{s-1,b}} \lesssim \| A_0 \|_{L^\infty((0,T), \dot{H}^{s+1} \cap \dot{H}^{\sigma+1})} \cdot \| \partial_t \phi \|_{\dot{H}^{s-1,b}}. \]

2. If \( s > 2 \) the following inequality applies:

\[ \| A_0 \partial_t \phi \|_{\dot{H}^{s-1,b}} \lesssim \| A_0 \|_{L^\infty((0,T), \dot{H}^{s-1} \cap \dot{H}^{\sigma+1})} \cdot \| \partial_t \phi \|_{\dot{H}^{s-1,b}}. \] (3.8)
Proof. 1. We want to use the fractional Leibniz rule and estimate by Hölder and Young:

\[ \| A_0 \Lambda^{s-1} \partial_t \phi \|_{H^s_0} = \| A_0 \Lambda^{s-1} \partial_t \phi \|_{L^2 \tau} \]

\[ \lesssim \| \hat{A}_0 \|_{L^2} \| \hat{\tau} \| \| \hat{\phi} \|_{L^\infty} \]

\[ \lesssim \| \hat{A}_0 \|_{L^2} \| \partial_t \phi \|_{H^{s-1,0}}. \]

Now for high frequencies of \( A_0 \) we obtain

\[ \| \hat{A}_0 \|_{L^2} \lesssim \| \hat{\xi} \|_{L^2} \| \hat{\phi} \|_{L^\infty} \]

\[ \lesssim \| A_0 \|_{L^2} \lesssim \| A_0 \|_{L^\infty} \lesssim \| A_0 \|_{\hat{H}^{1+}} \]

and for low frequencies of \( A_0 \) we obtain

\[ \| \hat{A}_0 \|_{L^2} \lesssim \| \hat{\xi} \|_{L^2} \| \hat{\phi} \|_{L^\infty} \]

\[ \lesssim \| A_0 \|_{L^\infty} \lesssim \| A_0 \|_{\hat{H}^{1+}}. \]

Thus we obtain

\[ \| A_0 \Lambda^{s-1} \partial_t \phi \|_{H^s} \lesssim \| A_0 \|_{L^\infty} \| \partial_t \phi \|_{H^{s-1,0}}. \]

Next we estimate \( \| \Lambda^{s-1} A_0 \partial_t \phi \|_{H^s_0} \). For low frequencies we obtain as before

\[ \| \Lambda^{s-1} A_0 \partial_t \phi \|_{H^s_0} \lesssim \| A_0 \|_{L^\infty} \| \partial_t \phi \|_{H^{s-1,0}} \]

and for high frequencies of \( A_0 \) we obtain by Hölder and Young:

\[ \| \Lambda^{s-1} A_0 \partial_t \phi \|_{L^2 \tau} \]

\[ \lesssim \| \hat{\xi} \|^{2-s} \| \hat{\phi} \|^{2+s} \| A_0 \|_{L^2} \| \hat{\tau} \| \| \hat{\phi} \|_{L^\infty} \]

\[ \times \| \hat{\xi} \|^{s-1} \| \hat{\phi} \| \| \hat{\tau} \|_{L^\infty} \]

\[ \lesssim \| A_0 \|_{L^\infty} \| \partial_t \phi \|_{H^{s-1,0}}. \]

because \( 1 + \frac{1}{\tau} = \frac{2-s}{2} + \frac{s-1}{2} + \frac{1}{\tau} \). By the fractional Leibniz rule we obtain the claimed estimate.

2. If \( s > 2 \) we handle the integration with respect to \( \tau \) as in 1. We obtain for high frequencies of \( A_0 \):

\[ \| \Lambda^{s-1} A_0 \partial_t \phi \|_{L^2 \tau} \lesssim \| \hat{\xi} \|^{2-s} \| \hat{\phi} \|^{2+s} \]

\[ \times \| \hat{\tau} \| \| \hat{\phi} \|_{L^\infty} \]

\[ \lesssim \| A_0 \|_{L^\infty} \| \partial_t \phi \|_{H^{s-1,0}}. \]

and

\[ \| A_0 \partial_t \Lambda^{s-1} \phi \|_{L^2 \tau} \lesssim \| \hat{\xi} \|^{s-1} \| \hat{\phi} \| \| \hat{\tau} \| \| \hat{\phi} \|_{L^\infty} \]

\[ \lesssim \| A_0 \|_{L^\infty} \| \partial_t \phi \|_{H^{s-1,0}}. \]

so that the fractional Leibniz rule implies (3.8). Low frequencies of \( A_0 \) are handled as in part 1.

\[ \square \]

Lemma 3.12. Let \( 1 < r \leq 2 \) and \( s > 1 \). Then

\[ \| (\partial_t A_0) \phi \|_{H^{s-1,0}} \lesssim \| \partial_t A_0 \|_{L^\infty} \| \phi \|_{H^{s-1,0}}. \] (3.9)
Proof. We handle the $\tau$-integration as in the previous lemma. We apply the fractional Leibniz rule and estimate first for high frequencies of $\partial_t A_0$:

$$
\|(A^{s-1} \partial_t A_0) \phi\|_{L^2_{\xi'}} \lesssim \|A^{s-1} \partial_t A_0\|_{L^2_{\xi}} \|\langle \xi \rangle^{-s} \|_{L^2_{\xi}} \|\langle \xi \rangle^{s} \phi\|_{L^2_{\xi'}} \lesssim \|\partial_t A_0\|_{H^{s-1}} \|\hat{A}^s \phi\|_{L^2_{\xi'}}
$$

and for low frequencies of $\partial_t A_0$:

$$
\|(A^{s-1} \partial_t A_0) \phi\|_{L^2_{\xi'}} \lesssim \|\langle \xi \rangle^{0-} \|_{L^2_{\xi'}} \|\langle \xi \rangle^{0+} \partial_t A_0\|_{L^2_{\xi}} \|\hat{\phi}\|_{L^2_{\xi'}} \lesssim \|\partial_t A_0\|_{H^{0+}} \|\hat{\phi}\|_{L^2_{\xi'}}.
$$

Moreover we obtain for high frequencies of $\partial_t A_0$:

$$
\|\langle \partial_t A_0 \rangle A^{s-1} \phi\|_{L^2_{\xi'}} \lesssim \|\langle \xi \rangle^{0-} \|_{L^2_{\xi'}} \|\langle \xi \rangle^{0+} \partial_t A_0\|_{L^2_{\xi}} \|\hat{\phi}\|_{L^2_{\xi'}} \lesssim \|\partial_t A_0\|_{H^{0+}} \|\hat{A}^s \phi\|_{L^2_{\xi'}};
$$

whereas for low frequencies of $\partial_t A_0$ we obtain

$$
\|\langle \partial_t A_0 \rangle A^{s-1} \phi\|_{L^2_{\xi'}} \lesssim \|\langle \xi \rangle^{0-} \|_{L^2_{\xi'}} \|\langle \xi \rangle^{0+} \partial_t A_0\|_{L^2_{\xi}} \|\hat{A}^{s-1} \phi\|_{L^2_{\xi'}} \lesssim \|\partial_t A_0\|_{H^{0+}} \|\hat{A}^{s-1} \phi\|_{L^2_{\xi'}}.
$$

Finally we treat the cubic term involving $A_0$.

Lemma 3.13. Let $1 < r \leq 2$, $s > 1$ and $b > \frac{1}{r}$. Then the following estimate applies:

$$
\|A_b^2 \phi\|_{H^{s-1,0}_{\tau}} \lesssim \|A_0\|^2_{L^\infty((0,T),H^{0s+\frac{1}{2}+})} \langle \phi\|_{H^{s,b}_{\tau}}.
$$

Proof. By Young’s and Hölder’s inequality we obtain

$$
\|A_b^2 \phi\|_{L^2_{\xi'}} \lesssim \|A_0^2\|_{L^2_{\xi'}} \|\langle |\xi| \rangle^{-b} \|_{L^2_{\xi'}} \|\langle |\xi| \rangle^b \phi\|_{L^2_{\xi'}} \lesssim \|
$$

where $\frac{1}{\eta} = \frac{1}{r} + \frac{1}{2}$ and $\frac{1}{p} = \frac{1}{2} + \cdot$. An application of the fractional Leibniz rule implies the claimed result.

In the following we prove the necessary estimates in the case of data in $L^2$-based Sobolev spaces.

The following bilinear estimates for wave-Sobolev spaces were proven in [1], Lemma 7.

Proposition 3.1. Let $s_0, s_1, s_2 \in \mathbb{R}$, $b_0, b_1, b_2 \geq 0$. Assume that

$$
b_0 + b_1 + b_2 > \frac{1}{2},
$$

$$
s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2),
$$

$$
s_0 + s_1 + s_2 > 1 - \min_{i \neq j}(b_i + b_j),
$$

$$
s_0 + s_1 + s_2 > 1 - \min_i b_i.
$$
where the last two inequalities are not both equalities. Then the following estimate applies:

$$\|uv\|_{H^{s_0, b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}}.$$  

If $b_0 < 0$, this remains true provided we additionally assume $b_0 + b_1 > 0$, $b_0 + b_2 > 0$ and $s_1 + s_2 > -b_0$.

**Corollary 3.3.** If $b_0 \geq 0$, $b_1, b_2 > \frac{1}{2}$ the following assumptions are sufficient:

$$s_0 + s_1 + s_2 > 1 - (b_0 + s_1 + s_2), \quad s_0 + s_1 + s_2 \geq \frac{3}{4}, \quad \min(s_i + s_j) \geq 0,$$

where the last two inequalities are not both equalities.

**Lemma 3.14.** Assume $s \leq l + 1$, $l > \frac{1}{4}$, $s > \frac{1}{2}$. The following estimate applies for $\epsilon > 0$ sufficiently small:

$$\|Q_{12}(u, D^{-1}v)\|_{H^{s-1,-\frac{1}{4}+2\epsilon}} \lesssim \|u\|_{X^{s,2+\epsilon,\pm_1}} \|v\|_{X^{s,2+\epsilon,\pm_2}}.$$  

**Proof.** By [13] we obtain the estimate

$$\|Q_{12}(u, D^{-1}v)\|_{H^{s-1,-\frac{1}{4}+2\epsilon}} \lesssim \|D^\frac{1}{2}D u D^{-\frac{1}{2}}v + D^\frac{1}{2}(D^\frac{1}{2}D u D^{-\frac{1}{2}}v) + D^\frac{1}{2}(D^\frac{1}{2}u D^\frac{1}{2}D^{-\frac{1}{2}}v)\|_{H^{s-1,-\frac{1}{4}+2\epsilon}}.$$

(3.10)

For high frequencies of $v$ we have to prove the following estimates:

$$\|uv\|_{H^{s-\frac{1}{4}+2\epsilon}} \lesssim \|u\|_{H^{s-\frac{1}{4}+2\epsilon}} \|v\|_{H^{s,2+\epsilon}},$$

$$\|uv\|_{H^{s-\frac{1}{4}+2\epsilon}} \lesssim \|u\|_{H^{s-\frac{1}{4}+2\epsilon}} \|v\|_{H^{s,2+\epsilon}},$$

$$\|uv\|_{H^{s-\frac{1}{4}+2\epsilon}} \lesssim \|u\|_{H^{s,2+\epsilon}} \|v\|_{H^{s,2+\epsilon}}.$$  

We apply Prop. 3.1. Using the notation of this proposition we require in all these estimates $s_0 + s_1 + s_2 = l + \frac{1}{2} > \frac{3}{4}$, thus $l > \frac{1}{4}$. The first estimate requires $l + \frac{1}{2} > 1 - (s - \frac{1}{2} + l + \frac{1}{2})$, thus $2l + s > \frac{1}{2}$, which is fulfilled. For the second estimate we need $l + \frac{1}{2} > 1 - (\frac{1}{2} - s + l + \frac{1}{2})$, thus $2l + s > -\frac{1}{2}$. For the third estimate we need the condition $l + \frac{1}{2} > 1 - (\frac{1}{2} - s + l + \frac{1}{2}) \Leftrightarrow l > \frac{1}{4}$. Here is the point where it is important to have $A_j \in H^{l,\frac{1}{2}+\epsilon}$ instead of $H^{l,\frac{1}{2}+\epsilon}$, which would lead to the condition $l > \frac{1}{4}$.

For low frequencies of $v$ we use the elementary estimate

$$Q_{12}(u, D^{-1}v) \lesssim D^\frac{1}{2}((D^\frac{1}{2}u)v),$$

so that we reduce to

$$\|uv\|_{H^{s-\frac{1}{4}+2\epsilon}} \lesssim \|u\|_{H^{s-\frac{1}{4}+2\epsilon}} \|v\|_{H^{l,\frac{1}{2}+\epsilon}}.$$  

This however is clear, because $\|v\|_{H^{l,\frac{1}{2}+\epsilon}} \sim \|v\|_{H^{N,\frac{1}{2}+\epsilon}}$ for arbitrary $N$.  

**Lemma 3.15.** Assume $s \geq l$, $s > \frac{1}{2}$ and $4s - l > \frac{7}{4}$. The following estimate applies for $\epsilon > 0$ sufficiently small:

$$\|D^{-1}Q_{12}(Re \phi, Im \psi)\|_{H^{l-1,-\frac{1}{4}+2\epsilon}} \lesssim \|\phi\|_{X^{s,\frac{1}{2}+\epsilon,\pm_1}} \|\psi\|_{X^{s,\frac{1}{2}+\epsilon,\pm_2}}.$$
Proof. We apply (3.10). Combining this with the elementary estimate (cf. [13])

\[ Q_{12}(u, v) \lesssim D(D^2 u D^2 v), \quad (3.11) \]

we obtain

\[ Q_{12}(u, v) \lesssim D^2 + 2e D^2 + 2e (D^2 u D^2 v) + D^2 (D^2 D^2 u D^2 v) + D^2 (D^2 u D^2 D^2 v). \]

For high frequencies of the product we have to prove the following estimates:

\[ \|uv\|_{H^\epsilon} \lesssim \|u\|_{H^\epsilon} + \|v\|_{H^\epsilon} + \|uv\|_{H^\epsilon} \]
\[ \|uv\|_{H^\epsilon} \lesssim \|u\|_{H^\epsilon} + \|v\|_{H^\epsilon} + \|uv\|_{H^\epsilon} \]

Now apply Prop. 3.1. We require for the first estimate \( \frac{5}{4} - l + 2s - 1 > \frac{3}{4} \), thus \( 2s - l > \frac{1}{2} \), which is true by our assumptions \( s \geq l \) and \( s > \frac{1}{2} \). Moreover we need \( 2s - l + \frac{1}{2} > 1 - (2s - 1) \leftrightarrow 4s - l > \frac{1}{2} \) as assumed. For the second estimate we need \( 2s - l + \frac{1}{2} > \frac{3}{4} \) \( \Rightarrow 2s - l > \frac{1}{2} \), which is true, and \( 2s - l + \frac{1}{2} > 1 - (\frac{3}{2} - l + s - \frac{1}{2}) \), thus \( 3s - 2l > \frac{1}{2} \), which applies, because \( s \geq l \) and \( s > \frac{1}{2} \). For low frequencies of the product we use the elementary estimate (3.11), so that we reduce to

\[ \|uv\|_{H^\epsilon} \lesssim \|u\|_{H^\epsilon} + \|v\|_{H^\epsilon} + \|uv\|_{H^\epsilon} \]

This however is clear, because \( N \) may be chosen arbitrarily. \( \square \)

Lemma 3.16. The following estimate applies for \( s \geq 0 \):

\[ \|A_0 \partial_\phi\|_{H^{-1}} \lesssim \|A_0\|_{H^{max(s, 1, 1+\gamma)\cap H^{0+}}} \|\partial_\phi\|_{H^{s-1}}. \]

Proof. 1. Let \( s \leq 1 \) . Then by SML for \( s < 1 \):

\[ \|D^{1-s} u v\|_{L^2} \lesssim \|D^{1-s} u\|_{H^{s-\epsilon}} \|v\|_{H^{s-\epsilon}} \quad \|D^{1-s} u v\|_{L^2} \lesssim \|D^{1-s} u\|_{H^{s-\epsilon}} \|v\|_{H^{s-\epsilon}}, \]

\[ \|D^{1-s} v\|_{L^2} \lesssim \|u\|_{L^\infty} \|D^{1-s} u v\|_{L^2} \lesssim \|u\|_{H^{1+\gamma} \cap H^{0+}} \|v\|_{H^{s-\epsilon}}, \]

\[ \|u v\|_{L^2} \lesssim \|u\|_{L^\infty} \|v\|_{L^2} \lesssim \|u\|_{H^{1+\gamma} \cap H^{0+}} \|v\|_{L^2}, \]

where the last estimate suffices for \( s = 1 \), thus

\[ \|u v\|_{H^{1-s}} \lesssim \|u\|_{H^{1+\gamma} \cap H^{0+}} \|v\|_{H^{s-1}}, \]

so that by duality we obtain

\[ \|u v\|_{H^{s-1}} \lesssim \|u\|_{H^{1+\gamma} \cap H^{0+}} \|v\|_{H^{s-1}}. \]

2. Let \( 1 < s \leq 2 \) . Similarly as before we obtain

\[ \|D^{s-1} u v\|_{L^2} \lesssim \|D^{s-1} u\|_{H^{2-s+\epsilon}} \|v\|_{H^{2-s+\epsilon}} \lesssim \|u\|_{H^{1+}} \|v\|_{H^{s-1}}, \]

\[ \|u D^{s-1} v\|_{L^2} \lesssim \|u\|_{L^\infty} \|D^{s-1} v\|_{L^2} \lesssim \|u\|_{H^{1+\gamma} \cap H^{0+}} \|v\|_{H^{s-1}}, \quad (3.12) \]

thus

\[ \|u v\|_{H^{s-1}} \lesssim \|u\|_{H^{1+\gamma} \cap H^{0+}} \|v\|_{H^{s-1}}. \]

3. For \( s > 2 \) we use

\[ \|D^{s-1} u v\|_{L^2} \lesssim \|D^{s-1} u\|_{L^2} \|v\|_{L^\infty} \lesssim \|D^{s-1} u\|_{L^2} \|v\|_{H^{s-1}}, \]

thus combining this with (3.12) :

\[ \|u v\|_{H^{s-1}} \lesssim \|u\|_{H^{max(s, 1+\gamma)\cap H^{0+}}} \|v\|_{H^{s-1}}. \quad (3.13) \]

\( \square \)
Lemma 3.17. The following estimate applies for \( s > 0 \):
\[
\| (\partial_t A_0) \phi \|_{H_t} \leq \| \partial_t A_0 \|_{H^{\max\left(s-0.1+H^{0.1}\right)}} \| \phi \|_{H^s}.
\]

Proof. 1. If \( s \leq 1 \) we obtain by SML:
\[
\| uv \|_{L^1} \leq \| uv \|_{L^1} \leq \| u \|_{H^{0.1}} \| v \|_{H^{0.1}} \leq \| u \|_{H^{0.1}} \| v \|_{H^1}.
\]
2. If \( s > 1 \) we obtain
\[
\| D^{s-1} u v \|_{L^2} \leq \| D^{s-1} u \|_{L^2} \| v \|_{L^\infty} \leq \| u \|_{H^{s-1}} \| v \|_{H^1},
\]
\[
\| u D^{s-1} v \|_{L^2} \leq \| u \|_{H^{0.1}} \| D^{s-1} v \|_{H^{0.1}} \leq \| u \|_{H^{0.1}} \| uv \|_{H^{1.1}},
\]
thus
\[
\| uv \|_{H^{s-1}} \leq \| u \|_{H^{\max\left(s-0.1+H^{0.1}\right)}} \| v \|_{H^s}.
\]
In the next three lemmas we address the cubic nonlinearities.

Lemma 3.18. Assume \( s > \frac{1}{2} \). Then the following estimate applies:
\[
\| A_0^2 \phi \|_{H^{s-1.1}} \leq \| A_0 \|_{L^\infty((0,T),H^{0.1})} \| \phi \|_{H^{0.1}}.
\]

Proof. In the case \( s \leq 1 \) we obtain
\[
\| A_0^2 \phi \|_{H^{s-1.1}} \leq \| A_0 \|_{L^\infty((0,T),H^{0.1})} \| \phi \|_{H^{0.1}} \leq \| A_0 \|_{L^\infty((0,T),H^{1.1})} \| \phi \|_{H^{1.1}},
\]
and in the case \( s > 1 \):
\[
\| A_0^2 \phi \|_{H^{s-1.1}} \leq \| A_0 \|_{L^\infty((0,T),H^{0.1})} \| \phi \|_{H^{0.1}} \leq \| A_0 \|_{L^\infty((0,T),H^{1.1})} \| \phi \|_{H^{1.1}}.
\]
These bounds are more than sufficient for our claim.

Lemma 3.19. Assume \( s > \frac{1}{2} \), \( l > \frac{1}{4} \) and \( 2l-s > -\frac{1}{4} \). Then the following estimate applies:
\[
\| A_j A^j \phi \|_{H^{s-0.1}} \leq \| A_j \|_{H^{0.1}} \| \phi \|_{H^{0.1}}.
\]

Proof. 1. Let \( s \leq \frac{1}{2} \). We use Prop. 3.1 twice and obtain
\[
\| A_j A^j \phi \|_{H^{s-0.1}} \leq \| A_j A^j \|_{H^{0.1}} \| \phi \|_{H^{0.1}} \leq \| A_j \|_{H^{0.1}} \| \phi \|_{H^{0.1}}.
\]
The conditions in Prop. 3.1 are easily checked for the first estimate, whereas the second estimate requires \( 2l + \frac{1}{4} > \frac{3}{4} \), which is also fulfilled.

2. If \( s > \frac{3}{4} \) we obtain
\[
\| A_j A^j \phi \|_{H^{s-0.1}} \leq \| A_j A^j \|_{H^{s-0.1}} \| \phi \|_{H^{s-0.1}} \leq \| A_j \|_{H^{s-0.1}} \| \phi \|_{H^{s-0.1}},
\]
where the first estimate requires \( s > \frac{3}{4} \) and the last estimate \( 2l-s > -\frac{1}{4} \) and \( 2l-s+1 > 1 - 2l \) implies \( 4l-s > 0 \), which is true under our assumptions.

Lemma 3.20. If \( s > \frac{1}{4} \), \( l > \frac{1}{4} \) and \( 2s-l > -\frac{1}{4} \) the following estimate applies:
\[
\| A_j \|_{H^{s-0.1}} \leq \| A_j \|_{H^{s-0.1}} \| \phi \|_{H^{s-0.1}}.
\]
Proof. 1. Let $l \leq \frac{3}{4}$ . We use Prop. 3.1 twice and obtain

$$
\|A_j|\phi|^2\|_{H^{l-1,-\frac{1}{4}++}} \lesssim \|A_j\|_{H^{l+\frac{3}{4}++}} \|\phi|^2\|_{H^{l-\frac{1}{4},0}} \lesssim \|A_j\|_{H^{l,\frac{3}{4}++}} \|\phi\|^2_{H^{l,\frac{1}{4}++}}.
$$

For the first estimate we require $\frac{3}{4} > 1 - (\frac{1}{4} + l - \frac{1}{4})$ if $l > \frac{1}{4}$ and for the last estimate $2s + \frac{1}{4} > \frac{3}{4}$ if $s > \frac{1}{4}$, which also implies $2s + \frac{1}{4} > 1 - 2s$.

2. Let $l > \frac{3}{4}$. Similarly we obtain

$$
\|A_j|\phi|^2\|_{H^{l-1,-\frac{1}{4}++}} \lesssim \|A_j\|_{H^{l+\frac{3}{4}++}} \|\phi|^2\|_{H^{l-1,0}} \lesssim \|A_j\|_{H^{l,\frac{3}{4}++}} \|\phi\|^2_{H^{l,\frac{1}{4}++}}.
$$

Prop. 3.1 implies both estimates, where the last estimate requires $2s - l > -\frac{1}{4}$. □

Hereafter we interpolate between the obtained bi- and trilinear estimates for $r = 1+$ and for $r = 2$.

Lemma 3.21. Let $1 < r \leq 2$ , $s \leq l + 1$ , $s > \frac{1}{r}$ and $l > \frac{3}{2r} - \frac{1}{2}$ . The following estimate applies:

$$
\|Q_{12}(\phi, D^{-1}A)\|_{H^{r-1,-\frac{1}{4}++}} \lesssim \|A\|_{X^{r,\frac{1}{4}++}} \|\phi\|_{X^{r,\frac{1}{4}++}}.
$$

Proof. We apply Lemma 3.2 for $r = 1+$ , which shows

$$
\|Q_{12}(\phi, D^{-1}A)\|_{H^{r-1,0}} \lesssim \|A\|_{X^{r,\frac{1}{4}++}} \|\phi\|_{X^{r,\frac{1}{4}++}}
$$

provided $1 \leq s_1 \leq l_1 + 1$ and $l_1 \geq 1$ . Lemma 3.14 implies

$$
\|Q_{12}(\phi, D^{-1}A)\|_{H^{r-1,-\frac{1}{4}++}} \lesssim \|A\|_{X^{2s,\frac{1}{4}++}} \|\phi\|_{X^{2s,\frac{1}{4}++}}
$$

provided $\frac{1}{2} \leq s_2 \leq l_2 + 1$ and $l_2 > \frac{1}{4}$ . Bilinear interpolation completes the proof. □

Lemma 3.22. Let $1 < r \leq 2$ , $s > \frac{3}{2r} - \frac{1}{2}$ , $l > \frac{3}{2r} - \frac{1}{2}$ , $l \leq s + \frac{1}{r} - \frac{1}{2}$ and $2s - l > \frac{3}{2r}$ . Then the following estimate applies:

$$
\|D^{-1}Q_{12}(Re \phi, Im \psi)\|_{H^{r-1,-\frac{1}{4}++}} \lesssim \|\phi\|_{X^{r,\frac{1}{4}++}}^2.
$$

Proof. By Lemma 3.3 we obtain for $r = 1+$ the estimate

$$
\|D^{-1}Q_{12}(Re \phi, Im \psi)\|_{H^{r-1,0}} \lesssim \|\phi\|_{X^{r,\frac{1}{4}++}} \|\psi\|_{X^{r,\frac{1}{4}++}},
$$

provided $l_1 \geq 1$ , $l_1 \leq s_1 + \frac{1}{r}$ and $2s_1 - l_1 > \frac{3}{2}$ so that we may assume especially $s_1 > \frac{5}{4}$ . By Lemma 3.15 the following estimate applies:

$$
\|D^{-1}Q_{12}(Re \phi, Im \psi)\|_{H^{2s+1,-\frac{1}{4}++}} \lesssim \|\phi\|_{X^{2s,\frac{1}{4}++}} \|\psi\|_{X^{2s,\frac{1}{4}++}},
$$

if $l_2 \leq l_2$ , $s_2 > \frac{1}{r}$ and $4s_2 - l_2 > \frac{7}{4}$ . The last condition is strengthened by $2s_2 - l_2 > \frac{3}{2}$ , which we assume. We also restrict to $l_2 > \frac{1}{4}$ , recalling that we needed this assumption in previous lemmas already. Therefore by bilinear interpolation the result follows as one easily checks. □

Lemma 3.23. Let $1 < r \leq 2$ . Assume $s \leq l + 1$ , $s > \frac{3}{2r} - \frac{1}{2}$ , $l > \frac{7}{4r} - \frac{5}{8}$ and $2l - s > \frac{1}{2} - \frac{3}{2}$ . Then the following estimate applies:

$$
\|A_j A^j \phi\|_{H^{r-1,-\frac{1}{4}++}} \lesssim \|A_j\|_{H^{r,\frac{1}{4}++}} \|\phi\|_{H^{r,\frac{1}{4}++}}.
$$
Lemma 3.12 and Lemma 3.17 gives the second inequality. Similarly Lemma 3.13
and Lemma 3.18 imply the last inequality. □

Lemma 3.24. Let \( 1 < r \leq 2 \). Assume \( l \leq s + 1 \), \( s > \frac{7}{4r} - \frac{3}{8} \), \( l > \frac{3s}{2r} - \frac{1}{2} \) and
\( 2s - l > \frac{2}{r} - \frac{3}{2} \). The following estimate applies:

\[
\| A_j \| H^{-1,-\frac{1}{2}+} \supseteq \| A_j \| H^{r,1,-\frac{1}{2}+},
\]

Proof. We apply Lemma 3.8 in the case \( r = 1+ \) and obtain

\[
\| A_j \| H^{r,1,-\frac{1}{2}+} \supseteq \| A_j \| H^{r,1,-\frac{1}{2}+},
\]

assuming \( 1 \leq l \leq s + 1 \), \( s > \frac{9}{8} \) and \( 2s - l > \frac{3}{4} \). Lemma 3.23 implies

\[
\| A_j \| H^{r,1,-\frac{1}{2}+} \supseteq \| A_j \| H^{r,1,-\frac{1}{2}+},
\]

if \( \frac{1}{2} < l \leq s + 1 \), \( s > \frac{1}{2} \) and \( 2s - l > -\frac{1}{2} \). Trilinear interpolation implies the result. □

Lemma 3.25. Assume \( 1 < r \leq 2 \), \( s > \frac{1}{r} \), \( b > \frac{1}{r} \). The following estimates apply:

\[
\| A_0 \partial_t \| H^{-1,-1,0} \supseteq \| A_0 \| L^\infty((0,T), H^{\max(s-1,0)+} \cap H^{0+}) \| \partial_t \| H^{-1,1,0},
\]

\[
\| (\partial_t A_0) \| H^{-1,-1,0} \supseteq \| (\partial_t A_0) \| L^\infty((0,T), H^{\max(s-1,0)+} \cap H^{0+}) \| \| \partial_t \| H^{-1,1,0},
\]

\[
\| A_0^2 \| H^{-1,-1,0} \supseteq \| A_0 \| L^\infty((0,T), H^{\max(s-1,0)+} \cap H^{0+}) \| \| \partial_t \| H^{-1,1,0},
\]

Proof. Using bilinear interpolation between the estimates in Lemma 3.11 and
Lemma 3.16 we obtain the first inequality, interpolation between the estimates in
Lemma 3.12 and Lemma 3.17 gives the second inequality. Similarly Lemma 3.13
and Lemma 3.18 imply the last inequality. □

Combining Lemma 3.25, Lemma 3.9, Lemma 3.10, and Cor. 3.2 we obtain

**Corollary 3.4.** There exists a polynomial bound for \( \| A_0 \partial_t \phi \| H_{s}^{-1,0} \) in terms of
\( \| \phi \| H_{s}^{-1,0} \) and \( \| A_j \| H_{s}^{-1,0} \). A similar bound applies for \( \| (\partial_t A_0) \phi \| H_{s}^{-1,0} \) and also
\( \| A_0^2 \phi \| H_{s}^{-1,0} \).

**Proof of Theorem 1.1.** This is an application of Theorem 2.1 to the system (3.1)
and (3.2). The necessary estimates were given in Lemma 3.21 - Lemma 3.24 and
Corollary 3.4. We just have to check the assumptions. The most restrictive
condition on \( l \) is \( l > \frac{7}{4r} - \frac{5}{8} \) (cf. Lemma 3.23). Thus the condition
\( 2s - l > \frac{3}{2r} \) (cf. Lemma 3.22) implies \( s > \frac{13}{8r} - \frac{5}{16} \). Moreover we assumed \( s > \frac{3}{2r} - \frac{1}{4} \) (cf. Lemma
3.22), which is weaker, and \( 2l - s > \frac{2}{r} - \frac{5}{8} \). The remaining conditions are easily
seen to be weaker. □
4. MKG in Lorenz gauge. We can reformulate the system (1.1),(1.2) in Lorenz gauge
\[ \partial^\nu A_\nu = 0 \]
as follows:
\[ \Box A_\mu = \partial^\nu \partial_\nu A_\mu = \partial^\nu (\partial_\nu A_\mu - F_{\mu \nu}) = -\partial^\nu F_{\mu \nu} = -j_\mu, \]
thus (using the notation \( \partial = (\partial_0, \partial_1, \ldots, \partial_n) \)):
\[ \Box A = -Im(\Phi(\partial_\mu \phi) - A|\phi|^2) =: N(A, \phi) \] (4.1)
and
\[ m^2 \phi = D_\mu D^\mu \phi = \partial_\mu \partial^\mu \phi - i A_\mu \partial^\mu \phi - i \partial_\mu (A^\mu \phi) - A_\mu A^\mu \phi = \Box \phi - 2i A^\mu \partial_\mu \phi - A_\mu A^\mu \phi \]

thus (using the notation \( \Box = \partial^2 \))
\[ (\Box - m^2) \phi = 2i A^\mu \partial_\mu \phi + A_\mu A^\mu \phi =: M(A, \phi). \] (4.2)

As in the case of the Coulomb gauge we may also consider the equivalent first order (in \( t \)) system. Defining \( A_\pm := \frac{1}{2} (A \pm i \Lambda)^{-1} A_t \), so that \( A = A_+ + A_- \) and \( A_t = i \Lambda (A_+ - A_-) \), and \( \phi_\pm := \frac{1}{2} (\phi \pm i \Lambda)^{-1} \phi_t \), so that \( \phi = \phi_+ + \phi_- \) and \( \phi_t = i \Lambda (\phi_+ - \phi_-) \).
We transform (4.1),(4.2) into
\[ (i \partial_t \pm i \Lambda) \phi_\pm = - (\pm 2 \Lambda)^{-1} M(A, \phi) - (m^2 + 1) \phi \] (4.3)
\[ (i \partial_t \pm D) A_\pm = - (\pm 2 \Lambda)^{-1} N(A, \phi) - A_\pm. \] (4.4)

We follow Selberg-Tesfahun [18] and split the spatial part \( A = (A_1, A_2) \) of the potential into divergence-free and curl-free parts and a smoother part:
\[ A = A^{df} + A^{cf} + \Lambda^{-2} A, \] (4.5)
where
\[ A^{df} = \Lambda^{-2} (\partial_2 (\partial_1 A_2 - \partial_2 A_1), -\partial_1 (\partial_1 A_2 - \partial_2 A_1)) \]
\[ A^{cf} = -\Lambda^{-2} \nabla(\nabla \cdot A). \]

Using (4.5) we write
\[ A^\alpha \partial_\alpha \phi = (-A_0 \partial_t \phi + A^{cf} \cdot \nabla \phi) + A^{df} \cdot \nabla \phi + \Lambda^{-2} A \cdot \nabla \phi. \] (4.6)

We have (with \( R_j := \Lambda^{-1} \partial_j \)):
\[ A^{df} \cdot \nabla \phi = (\Lambda^{-2} \partial_2 (\partial_1 A_2 - \partial_2 A_1)) \partial_1 \phi - (\Lambda^{-2} \partial_1 (\partial_1 A_2 - \partial_2 A_1)) \partial_2 \phi \]
\[ = -Q_{12} (\Lambda^{-2} (\partial_1 A_2 - \partial_2 A_1), \phi) \]
\[ = -Q_{12} (\Lambda^{-1} (R_1 A_2 - R_2 A_1), \phi). \] (4.7)

Next we use the Lorenz gauge, \( \partial_t A_0 = \nabla \cdot A \), to write
\[ A^{cf} \cdot \nabla \phi = -\Lambda^{-2} \partial^i (\partial_t A_0) \partial_i \phi = -\partial_i (\Lambda^{-1} R^i A_0) \partial_t \phi. \]

We can also write
\[ A_0 \partial_t \phi = -\Lambda^{-2} \partial_t \partial^i A_0 \partial_t \phi + \Lambda^{-2} A_0 \partial_t \phi = -\partial_i (\Lambda^{-1} R^i A_0) \partial_t \phi + \Lambda^{-2} A_0 \partial_t \phi. \]

Combining the above identities, we get
\[ -A_0 \partial_t \phi + A^{cf} \cdot \nabla \phi = Q_{10} (\Lambda^{-1} R^i A_0, \phi) - \Lambda^{-2} A_0 \partial_t \phi, \] (4.8)
where we define the null form
\[ Q_{10}(u, v) = -\partial_t (u \partial_t v + \partial_t u \partial_t v). \]
We may ignore the factor $R^i$. By the definition of the null form we obtain with $D_i := \frac{\partial}{\partial x_i}$ with symbol $\xi_i$:

$$Q_{00}(\Lambda^{-1}A_0, \phi) = \sum_{\pm_1, \pm_2} (\pm_1)(\pm_21) \left(-A_{0, \pm_1}(\pm_2 D_i)\phi_{\pm_2} + (\pm_1 D_i)\Lambda^{-1}A_{0, \pm_1}\Lambda\phi_{\pm_2}\right).$$

Its symbol is given by

$$\sum_{\pm_1, \pm_2} (\pm_1)(\pm_21) \left(-\mp_2(\xi - \eta)i) \mp_1 \eta_i\eta^{-1}(\xi - \eta)\right).$$

This symbol is bounded by

$$\left|\frac{(\pm_2(\xi - \eta)i)|\eta|}{\langle \eta \rangle} + \frac{(\pm_1\eta)i|\xi - \eta|}{\langle \eta \rangle} + \left|\frac{\xi - \eta}{\langle \eta \rangle}\right| + \left|\frac{\xi - \eta}{\langle \eta \rangle}\right| + |\eta| \left|\frac{\xi - \eta - |\xi - \eta|}{\langle \eta \rangle}\right|^\alpha \lesssim \frac{\xi - \eta \ltimes |\eta|}{\langle \eta \rangle} \zeta(\pm_1, \mp_2(\xi - \eta)) + \left|\frac{\xi - \eta}{\langle \eta \rangle}\right| + 1 =: S_1 + S_2 + S_3. \quad (4.9)$$

If we use the estimate for the angle in Lemma 4.1 below (for $\alpha, \beta, \gamma = \frac{1}{2}$) for the term $S_1$ we obtain by (4.9) the following estimate:

$$Q_{00}(\Lambda^{-1}u, v) \lesssim D^\frac{3}{2}(\Lambda^{-\frac{3}{2}}uDv) + D^\frac{1}{2}(uD^\frac{1}{2}v) + (D^\frac{3}{2}\Lambda^{-\frac{3}{2}}u)Dv + D^\frac{3}{2}uD^\frac{3}{2}v$$

$$+ \Lambda^{-\frac{1}{2}}u(D^\frac{3}{2}Dv) + u(D^\frac{3}{2}D^\frac{3}{2}v) + \Lambda^{-1}uAv + uv. \quad (4.10)$$

The estimate for the angle in the following Lemma was proven in [2], Lemma 7:

**Lemma 4.1.** Let $\alpha, \beta, \gamma \in [0, \frac{1}{2}]$, $\tau, \lambda \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^2$, $\xi, \eta \neq 0$. Then the following estimate applies for all signs $\pm_1, \pm_2$:

$$\zeta(\pm_1\xi, \pm_2\eta) \lesssim \left(\frac{(\tau + |\lambda| - |\xi + \eta|)}{\min(|\xi, \langle \eta \rangle|)}\right)^\alpha + \left(\frac{(-\tau \pm_1 |\xi|)}{\min(|\xi, \langle \eta \rangle|)}\right)^\beta + \left(\frac{(-\lambda \pm_2 |\eta|)}{\min(|\xi, \langle \eta \rangle|)}\right)^\gamma.$$

Similarly as in Lemma 3.2 we can also estimate the (modified) nullform $q_{00}(u, v)$.

**Lemma 4.2.** Assume $0 \leq \alpha_1, \alpha_2$ , $\alpha_1 + \alpha_2 \geq \frac{1}{2}$ and $b > \frac{1}{2}$. The following estimate applies

$$\|q_{00}(u, v)\|_{H^b_{\alpha_1, \alpha_2}} \lesssim \|u\|_{X^r_{\alpha_1, b, \pm_1}}\|v\|_{X^r_{\alpha_2, b, \pm_2}}.$$

where

$$q_{00}^r(u, v) := -u(D^{-1}\partial_x v) \pm (D^{-1}\partial_x u)v.$$

**Proof.** Again we may reduce to the case $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = 0$. Arguing as in the proof of Lemma 3.2 we use in the elliptic case the estimate (cf. [5], Lemma 13.2):

$$|q_{00}(\eta, \xi - \eta)| \lesssim \left(\frac{|\eta| + |\xi - \eta| - |\xi|}{\min(|\eta|, |\xi| - |\xi|)}\right)^\frac{1}{2}.$$

In the case $|\eta| \leq |\xi - \eta|$ we obtain

$$I = \|\tau| - |\xi|\|^\frac{1}{2} \left(\int \delta(\tau - |\eta| - |\xi - \eta|)|\eta|^{-1}\frac{|\eta|}{\gamma}d\eta\right)^\frac{1}{2}$$

$$\sim \|\tau - |\xi|\|^{\frac{1}{2}}|\gamma| \gamma| |\tau - |\xi|\|^{\frac{1}{2}} = 1,$$

because $A = \max(1 + \frac{r}{2} - \frac{3}{2}) - 1 - \frac{5}{2} = 0$ and $B = 1 - \max(1 + \frac{r}{2} - \frac{3}{2}) - \frac{5}{2} = -\frac{5}{2}.$

In the case $|\eta| \geq |\xi - \eta|$ we obtain

$$I = \|\tau| - |\xi|\|^{\frac{1}{2}} \left(\int \delta(\tau - |\eta| - |\xi - \eta|)|\eta|^{-1}|\xi - \eta|^{-\frac{1}{2}}d\eta\right)^\frac{1}{2}$$
\[ \sim ||\tau| - |\xi| ||^{1/4} |\tau|^{1/4} ||\tau| - |\xi| ||^{1/4} (1 + \log \frac{|\tau|}{||\tau| - |\xi||})^{1/7}, \]

where \( A = \text{max}(1, \frac{3}{7}, \frac{3}{2}) - 1 = \frac{1}{2} - \frac{3}{7} \) and \( B = \frac{1}{2} \), so that

\[ I \lesssim ||\tau| - |\xi| ||^{1/4} |\tau|^{1/4} ||\tau| - |\xi| ||^{-1/4} \lesssim 1. \]

In the hyperbolic case we obtain by [5], Lemma 13.2:

\[ |q_{0j}(\eta, \xi - \eta)| \lesssim |\xi|^{1/2} \frac{(|\xi| - ||\eta| - |\eta - \xi||)^{1/2}}{|\eta|^{1/2} |\xi - \eta|^{1/2}} \]

and argue exactly as in the proof of Lemma 3.2. The proof is completed as before. \( \square \)

**Corollary 4.1.** Let \( 1 < r \leq 2 \). Assume \( s \leq l + 1 \), \( l \geq \frac{1}{r} \), \( s \geq 1 \), \( b > \frac{1}{r} \). The following estimate applies

\[ ||q_{0j}(u, Dv)||_{H^l_{s-1,0}} \lesssim \|u\|_{X^r_{l,b,\pm 1}} \|v\|_{X^r_{b,\pm 2}}. \]

The last term in (4.8) and \( S_1 \) and \( S_2 \) in (4.9) are easily treated by Lemma 3.4 in the following lemma.

**Lemma 4.3.** Let \( 1 < r \leq 2 \), \( s \geq 0 \), \( l > \frac{1}{r} - \frac{3}{2} \), \( s \leq l + 1 \) and \( b > \frac{1}{r} \). Then the following estimates apply:

\[ ||\Lambda^{-1}u\Lambda v||_{H^r_{s-1,0}} \lesssim \|u\|_{H^r_{l,b}} \|\Lambda v||_{H^r_{s-1,b}}, \]
\[ ||uv||_{H^r_{s-1,0}} \lesssim \|u\|_{H^r_{l,b}} \|\Lambda v||_{H^r_{s-1,b}}. \]

**Proof.** By Lemma 3.4 this is satisfied, if \( 1 < r \leq 2 \), \( s \geq 1 \), \( l \geq \frac{3}{2} - 1 \), \( s - 1 \leq l \) and \( b > \frac{1}{r} \). We use this result for \( r = 1+ \). By Sobolev this estimate is certainly true in the case \( r = 2 \), \( b = \frac{1}{r} \), if \( l > 0 \) and \( s - 1 \leq l \). By interpolation between the cases \( r = 1+ \) and \( r = 2 \) we obtain the result. \( \square \)

Next we estimate the null form \( Q_{0j} \) in the case \( r = 2 \).

**Lemma 4.4.** Assume \( s > \frac{1}{2} \), \( l > \frac{1}{4} \) and \( 2l - s > -\frac{1}{2} \). Then the following estimate applies:

\[ ||Q_{0j}(\Lambda^{-1}u, v)||_{H^{l-1,-\frac{1}{2}+}} \lesssim ||u||_{X^{l+\frac{1}{4},\pm 1}} ||Dv||_{X^{s-1,\frac{1}{4}+,\pm 2}}. \]

**Proof.** We use the estimate (4.10). This reduces the claimed estimate to the following eight inequalities:

\[ ||uv||_{H^{l-1,0+}} \lesssim ||u||_{H^{l+\frac{1}{4}+,\pm \frac{1}{2}}} ||v||_{H^{l-1,\frac{1}{4}+}} \]
\[ ||uv||_{H^{l-1,0+}} \lesssim ||u||_{H^{l+\frac{1}{4}+,\pm \frac{1}{2}}} ||v||_{H^{l-1,\frac{1}{4}+}} \]
\[ ||uv||_{H^{l-1,-\frac{1}{2}+}} \lesssim ||u||_{H^{l+\frac{1}{4}+,\pm \frac{1}{2}}} ||v||_{H^{l-1,\frac{1}{4}+}} \]
\[ ||uv||_{H^{l-1,-\frac{1}{2}+}} \lesssim ||u||_{H^{l+\frac{1}{4}+,\pm \frac{1}{2}}} ||v||_{H^{l-1,\frac{1}{4}+}} \]
\[ ||uv||_{H^{l-1,-\frac{1}{2}+}} \lesssim ||u||_{H^{l+1,\frac{1}{4}+,\pm \frac{1}{2}}} ||v||_{H^{l-1,\frac{1}{4}+}} \]
\[ ||uv||_{H^{l-1,-\frac{1}{2}+}} \lesssim ||u||_{H^{l+1,\frac{1}{4}+,\pm \frac{1}{2}}} ||v||_{H^{l-1,\frac{1}{4}+}} \]

(4.19) and (4.20) are easily proven by Sobolev. For the other estimates we apply Prop. 3.1. Using the notation in this proposition we require in all cases: \( s_0 + s_1 + \)
\[ s_2 = l + \frac{1}{2} > \frac{3}{4} \iff l > \frac{1}{4}. \]

(4.13): \( l + \frac{1}{2} > 1 - (l + \frac{1}{2}) - (s - 1) \iff 2l + s > 1 \), which is true in our case \( s > \frac{1}{2} \) and \( l > \frac{1}{4} \).

(4.14) is true if \( 2l + s > 1 \).

(4.15): This is the point where it is important to consider \( u \in H^{l,b} \) with \( b > \frac{3}{4} \) instead of the standard choice \( b = \frac{1}{2} + \). Because of \( b_1 = \frac{1}{4} + \) we only need \( l + \frac{1}{2} > 1 - (1 - s) - (s - 1) - \frac{1}{4} \iff l > \frac{1}{4} \) (instead of \( l > \frac{1}{2} \)).

(4.16) and (4.17) require no additional assumptions.

(4.18): This requires the condition \( l + \frac{1}{2} > 1 - (1 - s) - l \iff 2l - s > -\frac{1}{2} \).

**Lemma 4.5.** Let \( 1 < r \leq 2 \), \( s > \frac{1}{r} \), \( l > \frac{3}{2r} - \frac{1}{2} \), \( 2l - s > \frac{1}{r} - 1 \). Then the following estimate applies:

\[ \|Q_{0j}(\Lambda^{-1}u, v)\|_{H^{r}_{s-1, \frac{1}{r}++}} \lesssim \|u\|_{X^{r}_{s, \frac{1}{r}++}} \|v\|_{X^{r}_{s, \frac{1}{r}++}}. \]

**Proof.** We remark, that by (4.9) we have proven that \( l > 1 \) instead of the standard choice \( l = \frac{1}{4} \). Let \( 1 < r \leq 2 \), \( s > \frac{1}{r} \), \( l > \frac{3}{2r} - \frac{1}{2} \), \( 2l - s > \frac{1}{r} - 1 \). Then the following estimate applies:

\[ \|Q_{0j}(\Lambda^{-1}u, v)\|_{H^{r}_{s-1, \frac{1}{r}++}} \lesssim \sum_{j \in \mathbb{Z}} q_{0j}^+(u, Dv) + \Lambda^{-1}u Dv + uv. \]

We interpolate between Cor. 4.1 (and Lemma 4.3) for the case \( r = 1+ \), which requires \( s \geq 1 \) and \( l \geq 1 \) as well as \( s - 1 \leq l \), which implies \( 2l - s > 2l - (l + 1) = l - 1 \geq 0 \), and Lemma 4.4 for \( r = 2 \), which requires \( s > \frac{1}{2} \), \( l > \frac{1}{2} \) and \( 2l - s > -\frac{1}{2} \). This completes the proof, as one easily checks.

**Lemma 4.6.** Let \( 1 < r \leq 2 \), \( b > \frac{1}{r} \), \( s \geq l \), \( l \geq 1 \) and \( 2s - l > \frac{3}{2r} \). Then

\[ \|\text{Im}(\phi \overline{\partial \phi})\|_{H^{r}_{s-1, \frac{1}{r}++}} \lesssim \|\phi\|_{H^{r}_{s, \frac{1}{r}++}}. \]

**Proof.** An application of the fractional Leibniz rule and Lemma 3.4 implies the result.

**Lemma 4.7.** Assume \( s \geq l \), \( s \geq \frac{1}{2} \) and \( 2s - l > \frac{3}{4} \). The following estimate applies:

\[ \|\text{Im}(\phi \overline{\partial \phi})\|_{H^{r}_{s-1, \frac{1}{r}++}} \lesssim \|\phi\|_{H^{r}_{s, \frac{1}{r}++}} \|\partial \phi\|_{H^{r}_{s-1, \frac{1}{r}++}}. \]

**Proof.** We use Prop. 3.1, which requires \( 2s - l > \frac{3}{4} \) and \( s \geq \frac{1}{2} \). This implies also \( 4s - l > \frac{7}{2} \), which is also required.

**Lemma 4.8.** Let \( 1 < r \leq 2 \), \( s \geq l \), \( l > \frac{3}{2r} - \frac{1}{2} \), \( 2s - l > \frac{3}{2r} \). The following estimate applies:

\[ \|\text{Im}(\phi \overline{\partial \phi})\|_{H^{r}_{s-1, \frac{1}{r}++}} \lesssim \|\phi\|_{H^{r}_{s, \frac{1}{r}++}} \|\partial \phi\|_{H^{r}_{s-1, \frac{1}{r}++}}. \]

Remark: The assumptions of the lemma imply \( s > \frac{3}{2r} - \frac{1}{4} \).

**Proof.** By Lemma 4.6 for \( r = 1+ \) we obtain

\[ \|\text{Im}(\phi \overline{\partial \phi})\|_{H^{r}_{s-1, \frac{1}{r}++}} \lesssim \|\phi\|_{H^{r}_{s, \frac{1}{r}++}} \|\partial \phi\|_{H^{r}_{s-1, \frac{1}{r}++}} \]

for \( s_1 \geq l_1 \geq 1 \), \( 2s_1 - l_1 > \frac{3}{4} \), which implies \( s_1 > \frac{5}{4} \), and Lemma 4.7 implies

\[ \|\text{Im}(\phi \overline{\partial \phi})\|_{H^{r_{l_2-1, \frac{1}{r}++}}} \lesssim \|\phi\|_{H^{r_{l_2, \frac{1}{r}++}}} \|\partial \phi\|_{H^{r_{l_2-1, \frac{1}{r}++}}}, \]

provided \( s_2 \geq l_2 \) and \( 2s_2 - l_2 > \frac{3}{4} \). We may also assume \( l_2 > \frac{1}{4} \), because this condition is required for other estimates already, which implies automatically \( s_2 > \frac{1}{2} \). Bilinear interpolation between these inequalities implies the claimed result.
Proof of Theorem 1.2. We apply Theorem 2.1 to the system (4.3) and (4.4). Using (4.6), (4.7) and (4.8) the necessary estimates for the term $A^\mu \partial_\mu \phi$ were given in Lemma 3.21, Lemma 4.5 and Lemma 4.3. Moreover we need Lemma 4.8, Lemma 3.24 and Lemma 3.23. We just have to check the assumptions. The most restrictive condition on $l$ is $l > \frac{7}{4} \frac{r}{8} - \frac{5}{8}$ (cf. Lemma 3.23). Thus the condition $2s - l > \frac{3}{8} \frac{r}{r}$ (cf. Lemma 4.8) is only compatible, if $s > \frac{13}{8} \frac{r}{r} - \frac{5}{16}$. Moreover we assumed $2l - s > \frac{2}{r} - \frac{5}{4}$. The remaining conditions are weaker.

Acknowledgements. I thank the referees for many remarks which were very helpful to improve the paper.

REFERENCES

[1] P. d’Ancona, D. Foschi and S. Selberg, Product estimates for wave-Sobolev spaces in 2+1 and 1+1 dimensions, Contemp. Math., 526 (2010), 125–150.
[2] P. d’Ancona, D. Foschi and S. Selberg Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system, J. EMS, 9 (2007), 877–898.
[3] S. Cuccagna, On the local existence for the Maxwell-Klein-Gordon system in $\mathbb{R}^{3+1}$, Commun. Partial Differ. Equ., 24 (1999), 851–867.
[4] M. Czubak and N. Pikula, Low regularity well-posedness for the 2D Maxwell-Klein-Gordon equation in the Coulomb gauge. Commun. Pure Appl. Anal., 13 (2014), 1669–1683.
[5] D. Foschi and S. Klainerman, Bilinear space-time estimates for homogeneous wave equations. Ann. Sc. ENS., 33 (2000), 211–274.
[6] V. Grigoryan and A. Nahmod, Almost critical well-posedness for nonlinear wave equation with $Q_{\mu\nu}$ null forms in 2D. Math. Res. Lett., 21 (2014), 313–332.
[7] V. Grigoryan and A. Tanguay, Improved well-posedness for the quadratic derivative nonlinear wave equation in 2D, J. Math. Anal. Appl., 475 (2019), 1578–1595.
[8] A. Grünrock, An improved local well-posedness result for the modified KdV equation., Int. Math. Res. Not., 61 (2004), 3287–3308.
[9] A. Grünrock, On the wave equation with quadratic nonlinearities in three space dimensions., Hyperbolic Differ. Equ., 8 (2011), 1–8.
[10] A. Grünrock and L. Vega, Local well-posedness for the modified KdV equation in almost critical $H^s$ -spaces., Trans. Amer. Mat. Soc., 361 (2009), 5681–5694.
[11] M. Keel, T. Roy and T. Tao, Global well-posedness of the Maxwell-Klein-Gordon equation below the energy norm, Discrete Cont. Dyn. Syst., 30 (2011), 573–621.
[12] S. Klainerman and M. Machedon, On the Maxwell-Klein-Gordon equation with finite energy, Duke Math. J., 74 (1994), 19–44.
[13] S. Klainerman and S. Selberg, Bilinear estimates and applications to nonlinear wave equations, Commun. Contemp. Math., 4 (2002), 223–295.
[14] M. Machedon and J. Sterbenz, Almost optimal local well-posedness for the (3+1)-dimensional Maxwell-Klein-Gordon equations, J. AMS, 17 (2004), 297–359.
[15] H. Pecher, Low regularity local well-posedness for the Maxwell-Klein-Gordon equations in Lorenz gauge, Adv. Differ. Equ., 19 (2014), 359–386.
[16] H. Pecher, Almost optimal local well-posedness for the Maxwell-Klein-Gordon system in Fourier-Lebesgue spaces, Commun. Pure Appl. Anal., 19 (2020), 3303–3321.
[17] S. Selberg, Almost optimal local well-posedness of the Maxwell-Klein-Gordon equations in 1+4 dimensions, Commun. Partial Differ. Equ., 27 (2002), 1183–1227.
[18] S. Selberg and A. Tesfahun, Finite-energy global well-posedness of the Maxwell-Klein-Gordon system in Lorenz gauge, Commun. Partial Differ. Equ., 35 (2010), 1029–1057.
[19] T. Tao, Multilinear weighted convolutions of $L^2$-functions, and applications to non-linear dispersive equations, Amer. J. Math., 123 (2001), 839–908.

Received December 2020; revised May 2021.
E-mail address: pecher@math.uni-wuppertal.de