ENDPOINT RESULTS FOR THE RIESZ TRANSFORM OF THE ORNSTEIN-UHLENBECK OPERATOR

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ABSTRACT. In this paper we introduce a new atomic Hardy space $X^1(\gamma)$ adapted to the Gauss measure $\gamma$, and prove the boundedness of the first order Riesz transform associated with the Ornstein-Uhlenbeck operator from $X^1(\gamma)$ to $L^1(\gamma)$. We also provide a new, short and almost self-contained proof of its weak-type $(1,1)$.

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1. INTRODUCTION

For $x \in \mathbb{R}^n$, let $d\gamma(x) = \pi^{-n/2}e^{-|x|^2}dx$ be the Gauss measure and denote with $\mathcal{L}$ the Ornstein-Uhlenbeck operator, i.e. the closure on $L^2(\gamma)$ of the operator given by

$$\frac{-1}{2}\Delta + x \cdot \nabla$$

on the space $C_0^\infty$ of smooth and compactly supported functions. It is well known that $\mathcal{L}$ is self-adjoint. We denote by $\nabla \mathcal{L}^{-1/2}$ its first order Riesz transform, which can be defined on $L^2(\gamma)$ via the spectral theorem (see Section 1.1 below).

For every $p \in (1, \infty)$, the operator $\nabla \mathcal{L}^{-1/2}$ extends to a bounded operator on $L^p(\gamma)$, but this fails when $p = 1$ (see e.g. [13] or [25]). This motivates the interest in boundedness results involving $L^1(\gamma)$, which we call endpoint results, for this operator. Concerning boundedness properties from $L^1(\gamma)$, the following result is well known:

THEOREM 1.1. $\nabla \mathcal{L}^{-1/2}$ is of weak type $(1,1)$, i.e. bounded from $L^1(\gamma)$ to $L^{1,\infty}(\gamma)$.

The proof of this result when $n = 1$ is due to Muckenhoupt [21]; in arbitrary dimension to Fabes, Gutiérrez and Scotto [4]. A new proof of this fact, shorter but still rather involved, was given by Pérez and Soria [23] who used related results of Pérez [22] and Menárguez, Pérez and Soria [20].

The question of finding a subspace of $L^1(\gamma)$ mapped by $\nabla \mathcal{L}^{-1/2}$ into $L^1(\gamma)$ has been considered more recently. In the pioneering paper [12], Mauceri and Meda introduced an atomic Hardy space $H^1(\gamma)$ adapted to the Gauss measure and studied boundedness properties of certain singular integral operators associated with $\mathcal{L}$ from this space to $L^1(\gamma)$. Among other results, they proved that the imaginary powers $\mathcal{L}^{iu}$ and the adjoint Riesz transform $\mathcal{L}^{-1/2}\nabla^s$ are bounded from $H^1(\gamma)$ to $L^1(\gamma)$. A few years later, however, the same authors and Sjögren [14] proved that, though the Riesz transform $\nabla \mathcal{L}^{-1/2}$ is bounded from $L^\infty$ to the dual of $H^1(\gamma)$ in any dimension, it is bounded from $H^1(\gamma)$ to $L^1(\gamma)$ if and only if $n = 1$. The problem of finding an appropriate subspace of $L^1(\gamma)$ mapped boundedly to $L^1(\gamma)$ by $\nabla \mathcal{L}^{-1/2}$ was addressed by Portal [24] who introduced a new Gaussian Hardy space $h^1(\gamma)$ and proved that $\nabla \mathcal{L}^{-1/2}$ is bounded from $h^1(\gamma)$ to $L^1(\gamma)$. The Hardy space $h^1(\gamma)$ is defined equivalently either by conical square functions or by a maximal function.

Portal’s proof hinges on a theory of tent spaces for the Gauss measure developed by the same author and Maas and Van Nerven [10]. Though the tent spaces introduced in [10] admit an atomic decomposition and Portal’s space is defined as a retract of a tent
space via a Calderón reproducing formula, an explicit atomic characterization of \( h^1(\gamma) \) is not provided in [24]. This is our main motivation to explore a different approach to the problem, which we present in the first part of this paper. Indeed, atomic decompositions are a useful tool to prove boundedness of linear operators: in many circumstances (see e.g. [11, 19]), it is enough to check that an operator maps atoms boundedly in some target space \( Y \) to extend it to a bounded operator from the whole atomic space to \( Y \). Inspired by the work of Mauceri, Meda and Vallarino [16] for the Riesz transforms on certain noncompact manifolds of infinite volume, we introduce a new atomic Gaussian Hardy space \( X^1(\gamma) \), strictly contained in the space \( H^1(\gamma) \) of Mauceri and Meda, and we prove

**Theorem 1.2.** \( \nabla L^{-1/2} \) is bounded from \( X^1(\gamma) \) to \( L^1(\gamma) \).

In the second part of the paper we provide a new proof of the weak type \((1,1)\) of \( \nabla L^{-1/2} \) (Theorem 1.1) shorter and simpler than those appearing up to now in the literature. This is obtained by suitably combining some ideas of Pérez and Soria [23] with some techniques introduced by García-Cuerva, Mauceri, Sjögren and Torrea [6] and the same authors and Meda [5]. Except for an elementary result [5, Lemma 4.4] and the theory developed in [6] for “local” Calderón-Zygmund operators, which can be considered an adaptation of the classical Calderón-Zygmund theory to the Gaussian setting in a certain neighbourhood of the diagonal of \( \mathbb{R}^n \times \mathbb{R}^n \), our proof is self-contained.

In the remaining of this section, we fix the notation and introduce the Riesz transform and some spectral multipliers of \( L \) which will be of use. The definition of \( X^1(\gamma) \) and the boundedness of \( \nabla L^{-1/2} \) from \( X^1(\gamma) \) to \( L^1(\gamma) \) is the object of Section 2, while the new proof of the weak type \((1,1)\) of \( \nabla L^{-1/2} \) occupies Section 3. Further details are given at the beginning of these two sections.

### 1.1. Integral kernels.

The \( L^2(\gamma) \)-spectrum of the Ornstein-Uhlenbeck operator \( L \) is the set of nonnegative integers \( \{0, 1, \ldots\} \), and its eigenfunctions are (tensor product of) Hermite polynomials. Its spectral resolution \( \{\mathcal{P}_k\}, k = 0, 1, \ldots \) is the family of orthogonal projectors of \( L^2(\gamma) \) onto the subspaces generated by the Hermite polynomials. It is also well known that \( L \) is the infinitesimal generator of the Mehler semigroup \( e^{-tL} \), whose kernel \( M_t \) with respect to the Lebesgue measure is\[^1\]

\[
M_t(x, y) = \frac{1}{2\pi^{n/2}} \exp\left(\frac{-|t(z-x-y)|^2}{2} \right).
\]

We refer the reader e.g. to [25] for further details.

For every \( z \in \mathbb{C} \), with a slight abuse of notation, we define

\[
L_z = \sum_{k=1}^{\infty} k z^k \mathcal{P}_k, \quad \text{Dom}(L_z) = \left\{ f \in L^2(\gamma) : \sum_{k=1}^{\infty} k^{2\Re z} \|\mathcal{P}_k\|_2^2 < \infty \right\}.
\]

If \( \Re z < 0 \), \( L_z \) is bounded on \( L^2(\gamma) \) and \( \text{Dom}(L_z) = L^2(\gamma) \). If \( \Re z \geq 0 \), observe that \( C_c^{\infty} \subset \text{Dom}(L_z) \) by the decomposition \( L_z = L_z^{-N} L^N \) where \( N = \lfloor \Re z \rfloor + 1 \).

Let \( \Pi_0 \) be the orthogonal projection

\[
\Pi_0 : L^2(\gamma) \to \ker(L^1) = \left\{ f \in L^2(\gamma) : \int f \, d\gamma = 0 \right\}.
\]

\[^1\]Given a bounded operator \( T \) on \( L^2(\gamma) \), we say that a distribution \( K_T \) on \( \mathbb{R}^n \times \mathbb{R}^n \) is its Schwartz kernel with respect to the Lebesgue measure if

\[
Tf(x) = \int_{\mathbb{R}^n} K_T(x, y) f(y) \, dy
\]

for a.e. \( x \in \mathbb{R}^n \).
In terms of the spectral resolution \( (\mathcal{P}_k) \) of \( \mathcal{L} \), \( \Pi_0 = I - \mathcal{P}_0 \), since
\[
\mathcal{P}_0 : L^2(\gamma) \to \text{ker}(\mathcal{L}) = C, \quad \mathcal{P}_0 f = \int f d\gamma.
\]
Observe moreover that \( \text{Ran}(\mathcal{L}) \) is closed, since \( \mathcal{L} \) is closed and has spectral gap. Thus \( \text{ker}(\mathcal{L})^\perp = \text{Ran}(\mathcal{L}) \). We shall denote the space \( \Pi_0 L^2(\gamma) \) also by \( L_0^2(\gamma) \). Observe that
\[
\mathcal{L} \mathcal{L}^{-1} f = \Pi_0 f \quad \forall f \in L^2(\gamma), \quad \mathcal{L}^{-1} \mathcal{L} f = \Pi_0 f \quad \forall f \in \text{Dom}(\mathcal{L}),
\]
and in particular
\[
(1.2) \quad \mathcal{L} \mathcal{L}^{-1} f = f \quad \forall f \in L_0^2(\gamma), \quad \mathcal{L}^{-1} \mathcal{L} f = f \quad \forall f \in \text{Dom}(\mathcal{L}) \cap L_0^2(\gamma).
\]
For every \( b \in \mathbb{R} \setminus \mathbb{N} \), the kernel of the operator \( \mathcal{L}^b \) with respect to the Lebesgue measure is
\[
K_{\mathcal{L}^b}(x, y) = \frac{1}{\Gamma(-b)} \int_0^\infty t^{-b-1} \left( M_t(x, y) - \frac{\pi^{-n/2} e^{-|y|^2}}{2^n} \right) dt
\]
where we used the change of variables \( t = -\log r \). See e.g. [6, 7]. In particular, for every \( j = 1, \ldots, n \), the kernel of the operator \( \nabla \mathcal{L}^{1/2} \) is
\[
K_{\nabla \mathcal{L}^{1/2}}(x, y) = -\frac{\pi^{n/2} e^{|x|^2 - |y|^2}}{2^n} \int_0^1 \frac{(-\log r)^{-3/2}}{(1 - r^2)^{(n+2)/2}} (r x - y) e^{-\frac{|x - r y|^2}{1 - r^2}} dr,
\]
while the kernel of the Riesz transform associated with \( \mathcal{L} \), i.e. the operator \( \nabla \mathcal{L}^{-1/2}, \) is
\[
K_{\nabla \mathcal{L}^{-1/2}}(x, y) = -\frac{2}{\pi^{(n+1)/2}} \int_0^1 \frac{r^{-1/2} e^{-t}}{(1 - e^{-2t})^{(n+2)/2}} (e^{-t} x - y) e^{-\frac{|x - r y|^2}{1 - r^2}} dt
\]
again by the change of variables \( t = -\log r \). Both the kernels in (1.3) and (1.4) are with respect to the Lebesgue measure.

All throughout the paper, we shall use the letters \( c \) and \( C \) to denote constants, not necessarily equal at different occurrences. For any quantity \( A \) and \( B \), we write \( A \lesssim B \) by meaning that there exists a constant \( c > 0 \) such that \( A \leq c B \). If \( A \lesssim B \) and \( B \lesssim A \), we write \( A \approx B \).

2. The Hardy space \( X^1(\gamma) \)

In a recent series of papers Mauceri, Meda and Vallarino [15–18] developed a theory of Hardy-type spaces on certain noncompact manifolds of infinite volume, to obtain endpoint estimates for imaginary powers and Riesz transforms associated with the Laplace-Beltrami operator of the manifold. Though in a rather different context, we shall adapt their Hardy spaces to the Gaussian setting (thus of finite volume) and to the Ornstein-Uhlenbeck operator.

The atoms we shall use are classical atoms supported in (dilations of) “hyperbolic” balls, which will be called admissible. We inherit such atoms and terminology from [12]. When talking about balls, we always mean Euclidean balls. If \( B \) is a ball, \( c_B \) will stand for its center and \( r_B \) for its radius. For every positive integer \( k \) and ball \( B \), we shall write \( k B \) to denote the ball with same center \( c_B \) and radius \( k r_B \).

**Definition 2.1.** We call admissible ball a ball \( B \) of center \( c_B \) and radius \( r_B \leq \min(1, 1/|c_B|) \). The family of all admissible balls will be denoted by \( \mathcal{B}_1 \).
DEFINITION 2.2. Let $\Omega$ be a bounded open set and $K$ be a compact set.

- We denote by $q^2(\Omega)$ the space of all functions $u \in L^2(\Omega)$ such that $\mathcal{L}u$ is constant on $\Omega$, and by $q^2(K)$ the space of functions on $K$ which are the restriction to $K$ of a function in $q^2(\Omega)$ for some bounded open $\Omega' \supset K$;
- we denote by $h^2(\Omega)$ the space of all functions $u \in L^2(\Omega)$ such that $\mathcal{L}u = 0$ on $\Omega$, and by $h^2(K)$ the space of functions on $K$ which are the restriction to $K$ of a function in $h^2(\Omega')$ for some bounded open $\Omega' \supset K$.

The spaces $h^2(\Omega)^\perp$ and $q^2(\Omega)^\perp$ are the orthogonal complements of $h^2(\Omega)$ and $q^2(\Omega)$ in $L^2(\Omega, \gamma)$, respectively. The spaces $h^2(K)^\perp$ and $q^2(K)^\perp$ are the orthogonal complements in $L^2(K, \gamma)$.

We now introduce the atomic Gaussian Hardy space $X^1(\gamma)$. The reader should compare our definitions with those of [16].

DEFINITION 2.3. An $X^1$-atom is a function $a \in L^2(\gamma)$, supported in a ball $B \in \mathcal{B}_1$, such that

(i) $\|a\|_{L^2(\gamma)} \leq \gamma(B)^{-1/2}$,
(ii) $a \in q^2(B)^\perp$.

DEFINITION 2.4. The Hardy space $X^1(\gamma)$ is the space

$$X^1(\gamma) := \{f \in L^1(\gamma) : f = \sum_j \mu_j a_j, \ a_j \text{ X}^1\text{-atom} , (\mu_j) \in \ell^1\}$$

endowed with the norm

$$\|f\|_{X^1(\gamma)} := \inf \{\|(\mu_j)\|_{\ell^1} : f = \sum_j \mu_j a_j, \ a_j \text{ X}^1\text{-atom}\}.$$ 

If $B \in \mathcal{B}_1$, the functions in $q^2(\bar{B})$ will be referred to as (Gaussian) quasi-harmonic functions on $B$.

Observe that the space $X^1(\gamma)$ is strictly contained in the Hardy space $H^1(\gamma)$ introduced by Mauceri and Meda [12]. Indeed, the atoms defining $H^1(\gamma)$ are supported on admissible balls and satisfy property (i) of Definition 2.3, but have only zero integral, a much weaker condition than (ii) of the same definition. In this sense, the space $X^1(\gamma)$ may be inserted in the framework of the theory developed by Mauceri and Meda [12] for the Gauss measure or more generally by Carbonaro and the same authors [2] in the setting of metric measure spaces. However, it is worth mentioning that our understanding of the space $X^1(\gamma)$ is still far from being complete and it will be the object of further investigations.

Our proof of Theorem 1.2 follows the same order of ideas of [17, Theorem 5.3].

2.1. Support preservation on atoms. A key point of the proof of [17, Theorem 5.3] is that the inverse of the Laplace-Beltrami operator preserves the support of atoms. In the following proposition, we prove that $\mathcal{L}^{-1}$ (suitably defined, recall (1.1)) shares the same behaviour on $X^1$-atoms. Its proof will occupy the remainder of this subsection.

PROPOSITION 2.5. For every $X^1$-atom $a$ supported in an admissible ball $B$, $\text{supp} \mathcal{L}^{-1} a \subseteq \bar{B}$ and

$$\|\mathcal{L}^{-1} a\|_{L^2(\gamma)} \leq r_B^2 \gamma(B)^{-1/2}.$$ 

For every ball $B$, in the same spirit of [17], we introduce two operators $\mathcal{L}_B$ and $\mathcal{L}_{B,\text{Dir}}$, defined as the restriction of $\mathcal{L}$ (in the distributional sense) to

$$\text{Dom}(\mathcal{L}_B) := \{f \in \text{Dom}(\mathcal{L}) : \text{supp} f \subseteq \bar{B}\},$$
$$\text{Dom}(\mathcal{L}_{B,\text{Dir}}) := \{f \in W^{1,2}_0(B, \gamma) : \mathcal{L} f \in L^2(B, \gamma)\}$$

respectively. Here $W^{1,2}_0(B, \gamma)$ denotes the closure of $C^\infty_c(B)$ with respect to the graph norm of the gradient $\nabla$ on $L^2(B, \gamma)$. We shall also use the space $W^{2,2}_0(B, \gamma)$ which is the closure of $C^\infty_c(B)$ with respect to the graph norm of $\mathcal{L}$. Notice that, since $\gamma$ and $\gamma^{-1}$
are bounded on any compact set, $W^{2,2}_0(B, \gamma) = W^{2,2}_0(B)$ as vector spaces, with equivalent norms.

It is well known (cf. [9, Theorem 10.13]) that $\mathcal{L}_{B, \text{Dir}}$ has purely discrete spectrum. We denote by $\lambda_\text{Dir}^1(B)$ its first eigenvalue.

We begin by proving the following proposition. Its proof is essentially the same as [18, Proposition 3.5], but avoids to use the existence of global quasi-harmonic functions. We include all the details for the ease of the reader.

**Lemma 2.6.** Let $B$ be a ball. Then

1. both $h^2(B)$ and $q^2(B)$ are closed in $L^2(B)$;
2. $\mathcal{L}$ is a Banach space isomorphism between $\text{Dom}(\mathcal{L}_B)$ and $h^2(\bar{B})^\perp$;
3. $h^2(B)^\perp = h^2(\bar{B})^\perp$;
4. $q^2(B)^\perp = q^2(\bar{B})^\perp$.

**Proof.** In the whole proof, $B$ will be a fixed ball.

1. Let $\psi_B \in C_c^\infty \cap L^2_0(\gamma)$ be such that $\psi_B|_B \equiv 1_B$. Then

   $$q^2(B) = h^2(B) \oplus C(\mathcal{L}^{-1}\psi_B)|_B$$

and thus it is enough to prove that $h^2(B)$ is closed, since it is a subspace of $q^2(B)$ of codimension one. Now let $(v_k)$ be a sequence in $h^2(B)$ converging to $v$ in $L^2(B)$. Then $\mathcal{L}v_k$ converges to $\mathcal{L}v$ in the sense of distributions in $B$, thus $\mathcal{L}v = 0$ in $B$ and hence $v \in h^2(B)$.

2. We first show that $\mathcal{L}(\text{Dom}(\mathcal{L}_B)) \subseteq h^2(\bar{B})^\perp$. Let then $f \in \text{Dom}(\mathcal{L}_B)$, and $v \in h^2(B)$. Let $\tilde{v}$ be a smooth function with compact support which is harmonic in an open neighbourhood of $\bar{B}$ and satisfies $\tilde{v}|_B = v$. Then

   $$\int_{\bar{B}} \mathcal{L}f \, d\gamma = \int_{\mathbb{R}^n} \tilde{v} \mathcal{L}f \, d\gamma = \int_{\mathbb{R}^n} \mathcal{L}\tilde{v} f \, d\gamma = 0$$

since $\text{supp} f \subset B$ and $\mathcal{L}\tilde{v}$ is zero in a neighbourhood of $\bar{B}$.

3. We now prove that $\mathcal{L}$ maps $\text{Dom}(\mathcal{L}_B)$ onto $h^2(\bar{B})^\perp$. In order to do this, let $v \in h^2(\bar{B})^\perp$ and let $\tilde{v}$ be the extension of $v$ to a null function outside $\bar{B}$. Let $f = \mathcal{L}^{-1} \tilde{v}$.

   By definition, $f \in \text{Dom}(\mathcal{L})$. Since the constant function 1 is in $h^2(\bar{B})$, moreover, $\int \tilde{v} \, d\gamma = \int v \, d\gamma = 0$; thus $\tilde{v} \in L^2_0(\gamma)$ and by (1.2), $\mathcal{L}f = \tilde{v}$.

   Let now $\phi \in C_c^\infty(\mathbb{R}^n) \cap L^2_0(\gamma)$. Then, by (1.2), $\mathcal{L}\mathcal{L}^{-1} \phi = \phi$ which is identically zero on a neighbourhood of $\bar{B}$. Therefore, $\mathcal{L}\mathcal{L}^{-1} \phi \in h^2(B)$ and hence

   $$\langle \phi, f \rangle_{L^2(\gamma)} = (\mathcal{L}\mathcal{L}^{-1} \phi, f)_{L^2(\gamma)} = (\mathcal{L}^{-1} \phi, \mathcal{L}f)_{L^2(\gamma)} = \int_B (\mathcal{L}^{-1} \phi) v \, d\gamma = 0.$$

   This implies that there exists a constant $c$ such that $f = c$ on $\bar{B}^c$. Thus, $g := f - c$ is such that $g \in \text{Dom}(\mathcal{L})$, $\text{supp} g \subset \bar{B}$ and $\mathcal{L}g = \mathcal{L}f = \tilde{v}$.

   We have proved that $\mathcal{L}$ is a bijection between $\text{Dom}(\mathcal{L}_B)$ and $h^2(\bar{B})^\perp$. Since $\mathcal{L}$ is continuous from $\text{Dom}(\mathcal{L}_B)$ to $h^2(\bar{B})^\perp$, its inverse is also continuous by the closed graph theorem, and $\mathcal{L}$ is then a Banach space isomorphism.

3. We use a simple adaptation of [18, Theorem 3.4, (ii) $\Rightarrow$ (i)]. The obvious inclusion $h^2(\bar{B}) \subseteq h^2(B)$ leads to $h^2(B)^\perp \subseteq h^2(\bar{B})^\perp$.

As for the converse inclusion, observe first that, if $W^{2,2}(\mathbb{R}^n)_B$ denotes the functions in $W^{2,2}(\mathbb{R}^n)$ with support in $\bar{B}$, then $W^{2,2}(\mathbb{R}^n)_B = W^{2,2}_0(B)$ by [3, Chapter 5.5, Theorem 2]
(or [1, Theorem 5.29]). Moreover, since $B$ is bounded and has finite measure, $\text{Dom}(\mathcal{L}_B) = \text{Dom}(\Delta_B)$. Therefore 

\begin{equation}
\text{Dom}(\mathcal{L}_B) = \text{Dom}(\Delta_B) = W^{2,2}(\mathbb{R}^n)_h = W^{0,2}_{0}(B) = W^{2,2}_0(B, \gamma).
\end{equation}

Let now $v \in h^2(\bar{B})$ and $\bar{v}$ be the extension of $v$ which vanishes on $\bar{B}$. By (2) there exists $f \in \text{Dom}(\mathcal{L}_B)$ such that $\mathcal{L}f = \bar{v}$, and by (2.1) there exists a sequence $(\phi_k) \subset C_c(\mathbb{R}^n)$ converging to $f$ in the graph norm of $\mathcal{L}$. Thus, if $g \in h^2(B)$ and $\tilde{g}$ is its trivial extension to $\mathbb{R}^n$, 

\begin{equation}
\int_B v g \, d\gamma = (\tilde{v}, \tilde{g}) = (\mathcal{L}f, \tilde{g}) = \lim_k (\mathcal{L}\phi_k, \tilde{g}) = \lim_k (\phi_k, \mathcal{L}\tilde{g})
\end{equation}

which vanishes because $\mathcal{L}\tilde{g} = 0$ on $B$. Thus, $v \in h^2(B)^\perp$.

(4) We prove that, for every ball $B$, $q^2(B) = q^2(B)$. The inclusion $\subseteq$ follows easily, since the obvious inclusion $q^2(B) \subseteq q^2(B)$ leads to 

\begin{equation}
q^2(B) \subseteq q^2(B) = q^2(B),
\end{equation}

the last equality being true by (1).

To prove the converse inclusion $\supseteq$, let $v \in q^2(B)$ so that $\mathcal{L}v = c$ on $B$ for some constant $c$. Let $g \in L^2_0(\gamma)$ be such that $g = c$ on a neighbourhood of $\bar{B}$. Let $q = \mathcal{L}^{-1}g$, so that $\mathcal{L}q = g$ by (1.2), $q \in q^2(B)$ and $v - q \in h^2(B)$. By (3) and (1) 

\begin{equation}
h^2(B) = h^2(B) = h^2(B),
\end{equation}

and thus there exists a sequence $(h_k) \subseteq h^2(B)$ such that $h_k \to v - q$ in $L^2(B)$, and then $h_k + q \to v$ in $L^2(B)$. Therefore $v \in q^2(B)$.

\section*{Lemma 2.7. Let $B$ be a ball. Then}

1. $\mathcal{L}_B \subset \mathcal{L}_{B,\text{Dir}}$,

2. $\text{Ran}(\mathcal{L}_B) = h^2(B)^\perp$.

\textbf{Proof.} We adopt the same strategy of [17, Proposition 3.1 (i)].

1. Let $f \in \text{Dom}(\mathcal{L}_B)$. Then 

\[ \|\nabla f\|_{L^2(\gamma)} \leq \|f\|_{L^1(\gamma)} \|\mathcal{L}f\|_{L^1(\gamma)} < \infty \]

and since $\text{supp} f \subseteq \bar{B}$, $f \in W^{1,2}(B)$. Since $f = 0$ on the complement of $\bar{B}$ and the boundary of $B$ is smooth, the trace of $f$ on the boundary of $B$ is zero. Thus $f \in W^{1,2}_0(B)$ by a classical result (see e.g. [3, Chapter 5.5, Theorem 2]). Thus $\text{Dom}(\mathcal{L}_B) \subset \text{Dom}(\mathcal{L}_{B,\text{Dir}})$.

2. First of all, $\text{Ran}(\mathcal{L}_B)$ is closed in $L^2(B)$, since it is closed in $L^2(\gamma)$, because $\mathcal{L}$ has spectral gap and is closed. Thus, to prove the inclusion $\supseteq$ of (2) it suffices to show that $\text{Ran}(\mathcal{L}_B)^\perp \subseteq h^2(B)$.

Let $g \in \text{Ran}(\mathcal{L}_B)^\perp$. Then 

\[ 0 = \int_B (\mathcal{L}g) \, d\gamma = \langle \gamma \psi, \mathcal{L}g \rangle \quad \forall \psi \in C_c(\mathbb{R}^n) \]

in the sense of distributions on $B$. Hence $\mathcal{L}g = 0$ on $B$, namely $g \in h^2(B)$.

We finally prove the inclusion $\subseteq$. Since $h^2(B) = h^2(B)$ by Lemma 2.6, (3), it is enough to prove that $\text{Ran}(\mathcal{L}_B)$ is orthogonal to $h^2(B)$. Let then $f \in \text{Dom}(\mathcal{L}_B)$, $g \in h^2(B)$ and let $\tilde{g}$ be any extension of $g$ to all $\mathbb{R}^n$, such that $\tilde{g} \in \text{Dom}(\mathcal{L})$. Thus, since $\text{supp}(\mathcal{L}f) \subseteq \bar{B}$ 

\[ (\mathcal{L}_B f, g)_{L^2(B, \gamma)} = (\mathcal{L}_B f, \tilde{g})_{L^2(\gamma)} = (f, \mathcal{L}\tilde{g})_{L^2(\gamma)} = 0 \]

because $\text{supp} f \subseteq \bar{B}$ and $\mathcal{L}\tilde{g}$ vanishes on a neighbourhood of $\bar{B}$. \hfill $\Box$
Proof of Proposition 2.5. Let \( a \) be an \( X^1 \)-atom. By Lemmata 2.6, (4) and 2.7, (2) we get
\[
a \in q^2(B) = q^2(B) \subset h^2(B) \subset \text{Ran}(\mathcal{L}_B).
\]
Therefore, there exists \( f \in \text{Dom}(\mathcal{L}_B) \) such that \( \mathcal{L}_{B,\text{Dir}}f = \mathcal{L}_B f = a \), the first equality by Lemma 2.7, (1). Thus \( \text{supp}(\mathcal{L}^{-1}_{B,\text{Dir}}a) = \text{supp} f \subseteq B \). Moreover, \( \mathcal{L}^{-1}_{B,\text{Dir}}a \). Thus
\[
\|\mathcal{L}^{-1}_{B,\text{Dir}}a\|_2 = \|\mathcal{L}^{-1}_{B,\text{Dir}}a\|_2 \leq \frac{1}{\lambda^\gamma_{\text{Dir}}(B)} \|\gamma(B)\|^{1/2}.
\]
It then remains to estimate \( \lambda^\gamma_{\text{Dir}}(B) \). Recall that on \( \mathbb{R}^n \) we have the usual Faber-Krahn inequality for the Laplacian
\[
\lambda_1(B) \geq C|B|^{-2/n}
\]
where \( \lambda_1(B) \) is the first eigenvalue of the Dirichlet Laplacian (see e.g. [9, (14.5)]). Then, by the minmax principle [9, Theorem 10.18] and the equivalence of the Lebesgue measure and \( \gamma \) on \( B \)
\[
\lambda^\gamma_{\text{Dir}}(B) = \inf_{\phi \in C^\infty_c(\gamma)} \int_B |\nabla \phi|^2(y) \gamma(y) \, dx \geq c \inf_{\phi \in C^\infty_c(B)} \int_B |\nabla \phi|^2(y) \, dx = c\lambda_1(B)
\]
for some \( c > 0 \), independent of \( B \in \mathcal{B}_1 \). Then, by (2.4)
\[
\lambda^\gamma_{\text{Dir}}(B) \geq c\lambda_1(B) \geq C|B|^{-2/n} \geq cr_B^{-2}
\]
since \( |B| \approx r_B^n \). This together with (2.3) completes the proof. \( \square \)

2.2. Proof of Theorem 1.2.

Lemma 2.8. For every ball \( B \in \mathcal{B}_1 \) and every \( f \in L^1(\gamma) \) with \( \text{supp} f \subseteq B \),
\[
\|\nabla \mathcal{L}^{1/2}f\|_{L^1((4B)^c, \gamma)} \lesssim r_B^{-2} \|f\|_{L^1(B, \gamma)}.
\]

Proof. By (1.3)
\[
\|\nabla \mathcal{L}^{1/2}f\|_{L^1((4B)^c, \gamma)} \lesssim \int_{(4B)^c} \int_B \int_0^1 \frac{|x-y|^{1/2}}{(1-r^2)^{3/2}} \, dr \, |f(y)| \, d\gamma(y) \, dx
\]
\[
= \int_B I(y) |f(y)| \, d\gamma(y)
\]
where for \( y \in B \)
\[
I(y) = \int_0^1 \frac{1}{(1-r^2)^{3/2}} \int_{(4B)^c} |x-y|^{1/2} \, dx \, dr.
\]
The proof will then be complete if we can show \( I(y) \lesssim r_B^{-2} \) for every \( y \in B \). We split \( I(y) \) into \( I_1(y) + I_2(y) \) according to the splitting \( (0, 1) = (0, 1/2) \cup (1/2, 1) \). Thus
\[
I_1(y) \lesssim \int_0^{1/2} \frac{1}{(1-r^2)^{3/2}} \int_{(4B)^c} |x-y|^{1/2} \, dx \, dr.
\]
We make the change of variables \( x-y = \nu \) in the inner integral and then extend the integration domain to \( \mathbb{R}^n \). This yields
\[
I_1(y) \lesssim \int_0^{1/2} \frac{1}{(1-r^2)^{3/2}} \int_{\mathbb{R}^n} |\nu + (r^2-1)y|^{1/2} \, d\nu \, dr.
\]
Now observe that, since \( |\nu| \leq |c_B| + r_B \leq 2/r_B \) by the admissibility condition of the ball \( B \),
\[
|\nu + (r^2-1)y| \leq r|\nu| + |y| \leq r|\nu| + \frac{2}{r_B} \leq \frac{|\nu| + 1}{r_B}
\]
Proof of Theorem 1.2. We follow the same line as [17, Theorem 5.3] to prove that

\[ \sup \{ \| \nabla \mathcal{L}^{-1/2} a \|_{L^1(\gamma)} : a \text{ is an } X^1 \text{-atom} \} < \infty. \]

Since \( \nabla \mathcal{L}^{-1/2} \) is of weak type \((1,1)\), this implies the boundedness \( X^1(\gamma) \to L^1(\gamma) \) by a classical argument [8, p. 95].

Let \( a \) be an \( X^1 \)-atom supported in an admissible ball \( B \). Since

\[ \| \nabla \mathcal{L}^{-1/2} a \|_{L^1(\gamma)} = \| \nabla \mathcal{L}^{-1/2} a \|_{L^1(I_2(\gamma))} + \| \nabla \mathcal{L}^{-1/2} a \|_{L^1(I_1(\gamma))} \]

it is enough to estimate the two summands separately. First, by Cauchy-Schwarz

\[ \| \nabla \mathcal{L}^{-1/2} a \|_{L^1(I_2(\gamma))} \leq \gamma(4B)^{1/2} \| \nabla \mathcal{L}^{-1/2} a \|_{L^2(I_2(\gamma))} \leq \| a \|_2 \gamma(4B)^{1/2} \leq C \]

since \( r_B \leq 1 \), and hence

\[ I_1(y) \leq \frac{C}{r_B} \int_0^{1/2} \frac{1}{(-\log r)^{3/2}} \int_{\mathbb{R}^n} (|v| + 1)e^{-|v|^2} \, dv \, dr \leq \frac{C}{r_B}. \]

Therefore, \( a \text{ fortiori}, I_1(y) \leq r_B^{-2} \). Before looking at \( I_2(y) \), we observe that for every \( r \in (1/2, 1) \)

\[ |rx - y| \leq |x - ry| + (1 - r^2)|y|, \]

since \( rx - y = r(x - ry) - (1 - r^2)y \). Hence

\[ I_2(y) \leq \int_1^{1/2} \frac{1}{(1 - r^2)^{n/2+2}} \int_{\{y \leq r\}} \left( \frac{|x - ry|}{\sqrt{1 - r^2}} \right) \, dx \, dr. \]

By using the inequalities \( se^{-s^2} \leq e^{-s^2/2} \) for \( s > 0 \) and \( e^{-s^2} \leq e^{-s^2/2} \), we get

\[ I_2(y) \leq \int_1^{1/2} \frac{1 + \sqrt{1 - r^2}|y|}{(1 - r^2)^{n/2+2}} \int_{\{y \leq r\}} \frac{|r - x|}{\sqrt{1 - r^2}} \, dx \, dr. \]

We now separate the cases when \( r_{B,Y} \geq 1 \) and \( r_{B,Y} < 1 \), where (see [12, Lemma 7.1] for the notation)

\[ r_{B,Y} = r_B/|2|y|). \]

If \( r_{B,Y} \geq 1 \), by [12, Lemma 7.1, (i) and (iii)]

\[ I_2(y) \leq \int_1^{1/2} \frac{1 + \sqrt{1 - r^2}|y|}{(1 - r^2)^{n/2+2}} \varphi \left( \frac{r_B}{\sqrt{1 - r^2}} \right) \, dr \]

which yields, after the change of variables \( r_B/\sqrt{1 - r^2} = s \),

\[ I_2(y) \leq \frac{1}{r_B} \int_0^{\infty} (s + r_B|y|) \varphi(s) \, ds \leq \frac{1}{r_B} \int_0^{\infty} (s + 1) \varphi(s) \, ds = \frac{C}{r_B}, \]

since \( r_B|y| \leq C \).

If \( r_{B,Y} < 1 \), we split \((1/2, 1) = (1/2, 1 - r_{B,Y}) \cup (1 - r_{B,Y}, 1) \) and \( I_2(y) = I_2^1(y) + I_2^2(y) \) accordingly. By [12, Lemma 7.1, (ii)], \( I_2^2(y) \) can be treated exactly as we did in the case \( r_{B,Y} \geq 1 \), so we concentrate on \( I_2^1(y) \) only. By the change of variable \( x - ry = v \) in the inner integral, we get

\[ I_2^1(y) \leq \int_0^{1 - r_{B,Y}} \frac{1}{r_B} \int_{\mathbb{R}^n} e^{-|v|^2} \, dv \, dr \]

\[ \leq \int_0^{1 - r_{B,Y}} \frac{1}{r_B} \int_{\mathbb{R}^n} \frac{|y|}{(1 - r^2)} \, dr \leq \frac{1}{r_B} + \frac{|y|}{\sqrt{r_{B,Y}}} \leq \frac{1}{r_B^2} \]

since \( |y| \leq 1/r_B \) and by the definition of \( r_{B,Y} \).

\[ \square \]
where we used the boundedness of $\nabla \mathcal{L}^{-1/2}$ on $L^2(\gamma)$, the size property of $\alpha$ and the local doubling property of $\gamma$. As for the second summand, we write
\[
\nabla \mathcal{L}^{-1/2} a = \nabla \mathcal{L}^{1/2} \mathcal{L}^{-1} a
\]
by the spectral theorem. By Proposition 2.5, $\text{supp} \mathcal{L}^{-1} a \subseteq \bar{B}$. Therefore, by Lemma 2.8, Cauchy-Schwarz inequality and Proposition 2.5 respectively
\[
\|\nabla \mathcal{L}^{-1/2} a\|_{L^1((4B)\gamma)} \leq r_B^{-2}\|\mathcal{L}^{-1} a\|_{L^1(B, \gamma)} \leq r_B^{-2} \gamma(B)^{1/2} \|\mathcal{L}^{-1} a\|_{L^2(\gamma)} \leq C.
\]
The proof is complete. \hfill \Box

3. Weak type $(1,1)$

Since $\gamma$ is locally, but not globally doubling, it is a standard procedure to split $\mathbb{R}^n \times \mathbb{R}^n$ as the union of a neighbourhood of the diagonal and of its complement, and to split accordingly the kernels of the operators. Thus, for $\delta > 0$ we define
\[
N_{\delta} := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq \frac{\delta}{1 + |x| + |y|} \right\}, \quad G := N_{\delta}^c.
\]
We shall call both $N_1$ and $N_2$ the local regions and $G$ the global region, in analogy with [6]. We shall also fix once and for all a smooth function $\chi$ such that
\[
\chi_{N_1} \leq \chi \leq \chi_{N_2}, \quad |\nabla_x \chi(x, y)| + |\nabla_y \chi(x, y)| \leq \frac{C}{|x - y|} \quad \text{for} \ x \neq y,
\]
and define
\[
K_{\nabla \mathcal{L}^{-1/2}, \text{loc}} := \chi K_{\nabla \mathcal{L}^{-1/2}}, \quad K_{\nabla \mathcal{L}^{-1/2}, \text{glob}} := K_{\nabla \mathcal{L}^{-1/2}} - K_{\nabla \mathcal{L}^{-1/2}, \text{loc}}.
\]
We shall denote the operators with kernel $K_{\nabla \mathcal{L}^{-1/2}, \text{loc}}$ and $K_{\nabla \mathcal{L}^{-1/2}, \text{glob}}$ by $\nabla \mathcal{L}^{-1/2}_\text{loc}$ and $\nabla \mathcal{L}^{-1/2}_\text{glob}$ respectively. Of course
\[
\nabla \mathcal{L}^{-1/2} = \nabla \mathcal{L}^{-1/2}_\text{loc} + \nabla \mathcal{L}^{-1/2}_\text{glob}.
\]
Therefore, in order to prove the weak type $(1,1)$ of $\nabla \mathcal{L}^{-1/2}$ it will be enough to prove the weak type $(1,1)$ of both $\nabla \mathcal{L}^{-1/2}_\text{loc}$ and $\nabla \mathcal{L}^{-1/2}_\text{glob}$. The proof for $\nabla \mathcal{L}^{-1/2}_\text{loc}$ (Proposition 3.7) is rather standard, since by a general result (see [6, Theorem 2.7]) this can be reduced to proving weak type $(1,1)$ boundedness of some classical Calderón-Zygmund operators. The key proof is then that concerning $\nabla \mathcal{L}^{-1/2}_\text{glob}$.

To do this, we prove that $K_{\nabla \mathcal{L}^{-1/2}, \text{glob}}$ is controlled by a kernel $\overline{M}$ which arises naturally from the global part of the Mehler maximal operator. This idea is not completely new, as it comes from the paper [23] of Pérez and Soria. Their proof is based on the following facts: (1) providing a kernel $\overline{K}$ equivalent to the Mehler maximal kernel in the global region [20, Proposition 2.1] (2) proving that $\overline{K}$ is the kernel of an operator of weak type $(1,1)$ [20, Theorem 2.3], and (3) proving that the kernel of the Riesz transform is controlled by $\overline{K}$ in the global region [23, Proposition 2.2]. Though we follow the same order of ideas, the kernel $\overline{M}$ that we obtain (Proposition 3.3) controls only from above the Mehler maximal kernel, except in a certain region (see Remark 3.4) where they are equivalent. This greatly simplifies the proofs, for the weak type $(1,1)$ of the operator associated with $\overline{M}$ can be easily deduced (Lemma 3.5) by a kernel obtained by García-Cuerva, Mauceri, Meda, Sjögren and Torrea [5]. Finally, we prove that $\overline{M}$ controls also the kernel of the Riesz transform in the global region (Proposition 3.8). Our proofs use a useful rescaling of the Mehler kernel introduced by García-Cuerva, Mauceri, Sjögren and Torrea in [6].

We begin by fixing the notation and obtaining some elementary results that will be used later on. Then, in Subsection 3.1 we shall show that the kernel $\overline{M}$ arises naturally from
the study of the Mehler maximal operator in the global region, and prove the weak type 
(1, 1) of its associated operator. Finally, in Subsection 3.2 we shall prove Theorem 1.1 by 
proving the weak type (1, 1) of both $\nabla L^{-1/2}_g$ and $\nabla L^{-1/2}_l$.

For $x, y \in \mathbb{R}^n$ set

$$\alpha := |x - y||x + y|, \quad \beta := \frac{|x - y|}{|x + y|}, \quad \eta(x, y) := e^{-\frac{|x|^2}{2} + \frac{|y|^2}{2} - \frac{|x - y||x + y|}{2}}.$$ 

We also set $\theta = \theta(x, y)$ to be the angle between $x$ and $y$, and $\theta'$ the angle between $y - x$ and $y + x$. Observe that $\beta < 1$ if and only if $(x, y) > 0$. The results contained in 
the following lemma will be used all throughout the remainder of paper. Though their proofs 
are elementary, we provide all the details.

**Lemma 3.1.** Let $(x, y) \in \mathbb{R}^n$. Then

1. if $(x, y) \in G$ and $\beta < 1$, then $\alpha \geq 1/4$.
2. if $(x, y) \in G$, then $|x - y| \geq \frac{1}{2}(1 + |x|)^{-1}$.
3. $|x + y| \geq |x| \sin \theta$. In particular, $\alpha \geq |x|^2 \sin^2 \theta$.
4. $-|x|^2 + |y|^2 - |x - y||x + y| \leq 0$.
5. $-|x|^2 + |y|^2 - \frac{|x+y||x-y|}{2} = \frac{-2|x|^2|y|^2 \sin^2 \theta}{|x-y||x+y|(1+\cos \theta)}$.

**Proof.** To prove (1), first assume $|x| + |y| \leq 1$. Then

$$|x - y||x + y| \geq |x - y|^2 \geq \frac{1}{(1 + |x| + |y|)^2} \geq \frac{1}{4}.$$ 

Since $\beta < 1$, $|x + y| \geq |x|$ and $|x + y| \geq |y|$. Observe moreover that the function $t \mapsto t/(1 + t)$ is increasing. Thus, if $|x| + |y| > 1$,

$$|x + y||x - y| \geq \frac{|x + y|}{1 + |x| + |y|} \geq \frac{1}{2} \frac{|x| + |y|}{1 + |x| + |y|} \geq \frac{1}{4}.$$ 

The proof of (2) is shown in [5, pg. 225]. The point (3) holds since $|x| \sin \theta$ is the length 
of the projection of $x \pm y$ on the hyperplane orthogonal to $y$. To be more explicit,

$$|x + y|^2 - |x|^2 \sin^2 \theta = |x|^2 \cos^2 \theta + |y|^2 + 2|x||y| \cos \theta = (|x| \cos \theta + |y|)^2 \geq 0.$$ 

As for (4), just observe that

$$-|x|^2 + |y|^2 - |x - y||x + y| = (y + x, y - x) - |x - y||x + y| \leq 0$$ 

by Hölder’s inequality. Equivalently, one can see (4) as a consequence of (5), which is just 
a computation. \hfill \square

### 3.1. The Mehler Maximal Operator

It is well known that the Mehler maximal operator $M^*$, namely the operator with kernel

$$M^*(x, y) := \sup_t M_t(x, y)$$

with respect to the Lebesgue measure, is of weak type $(1, 1)$. See, for example, [20] 
and [5]. Here, we provide a different proof of the weak type $(1, 1)$ of its global part 
(Proposition 3.3 below), from which the following kernel arises naturally.

**Definition 3.2.** Define

$$\overline{M}(x, y) := e^{||x|^2 - |y|^2}\left(\frac{|x + y|}{|x - y|}\right)^{n/2} e^{-\frac{|x|^2}{2} + \frac{|y|^2}{2} - \frac{|x - y||x + y|}{2}} \Psi(x, y) \chi_G(x, y),$$ 

where

$$\Psi(x, y) = \max\left(1, \frac{1}{\alpha^{n/2}}\right).$$

Though the following result plays no role in the proof of Theorem 1.1, we provide its 
proof for it highlights the origin of the kernel $\overline{M}$. 


Proposition 3.3. For every \((x, y) \in G\), \(M^*(x, y) \lesssim M(x, y)\).

Proof. First of all, we perform the change of variable
\[
\tau(s) := \log \frac{1+s}{1-s}
\]
introduced in [5]. Then
\[
M^*(x, y) = \sup_{0 < s < 1} M_{\tau(s)}(x, y).
\]
An easy computation shows that
\[
M_{\tau(s)}(x, y) = (1 + s)^n e^{-\frac{1}{2} \frac{|y|^2}{s} + \frac{1}{2} \frac{|x+y|^2}{s} + \frac{1}{2} \frac{|x-y|^2}{s}}.
\]
Thus, \(M^*(x, y) = e^{\frac{1}{2} |x|^2 |y|^2} \eta(x, y) \sup_{0 < s < 1} (1 + s)^n e^{-\frac{1}{2} \frac{|x+y|^2}{s} + \frac{1}{2} \frac{|x-y|^2}{s} - 2|x||y|} \).

We now make the substitution \(s/\beta = \sigma\) in the supremum, and get
\[
M^*(x, y) = 2^{-n} e^{\frac{1}{2} |x|^2 |y|^2} \left(\frac{|x+y|}{|x-y|}\right)^{n/2} \eta(x, y) \sup_{0 < \sigma < 1/\beta} \frac{(1 + \sigma \beta)^n}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)},
\]
where
\[
\varphi(\sigma) := \sigma + \frac{1}{\sigma} - 2 = (\sigma - 1)^2 \sigma.
\]

It remains then to estimate the supremum. The first observation is that the contribution of the term \((1 + \sigma \beta)^n\) can be neglected, since \(1 \leq (1 + \sigma \beta)^n \leq 2^n\) for \(\sigma \in (0, 1/\beta)\). Thus
\[
\sup_{0 < \sigma < 1/\beta} \frac{(1 + \sigma \beta)^n}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)} \approx \sup_{0 < \sigma < 1/\beta} \frac{1}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)}.
\]

If \(\beta < 1\), we have \(\alpha \geq 1/4\) by Lemma 3.1 (1). Thus
\[
\sup_{0 < \sigma < 1/\beta} \frac{1}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)} \leq \sup_{0 < \sigma < \infty} \frac{1}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)} \leq C.
\]

Let now \(\beta \geq 1\), and observe that the function
\[
\sigma \mapsto \frac{1}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)}
\]
is increasing in the interval \((0, \sigma_0)\) and decreasing in \((\sigma_0, \infty)\), where
\[
\sigma_0 = \frac{\sqrt{4n^2 + \alpha^2} - 2n}{\alpha}.
\]

Therefore
\[
\sup_{0 < \sigma < 1/\beta} \frac{1}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)} \leq \frac{1}{\sigma_0^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)} \lesssim \max \left(1, \frac{1}{\sigma_0^{n/2}}\right).
\]

In other words, we have proved that (see also Remark 3.4 below)
\[
\sup_{0 < \sigma < 1/\beta} \frac{1}{\sigma^{n/2}} e^{-\frac{1}{4} \alpha \varphi(\sigma)} \lesssim \Psi(x, y)
\]
and this completes the proof. \(\square\)

Remark 3.4. If \((x, y) \in G\) and \(\beta < 1\), then \(\alpha \geq 1/4\) by Lemma 3.1, (1). Thus, \(1 \leq \Psi(x, y) \leq 2^n\) for every \((x, y) \in G\). In this case then \(M\) controls from above and below the kernel \(M^*\), with absolute constants. This was first shown in [20, Proposition 2.1].
As stated above, we now prove the weak type \((1,1)\) of the operator whose kernel is \(\overline{M}\). By Proposition 3.3, this implies the weak type \((1,1)\) of the global part of the Mehler maximal operator \(\mathcal{M}_{\text{glob}}^*\), which is the operator with kernel \(M^*(1-\chi)\) with respect to the Lebesgue measure.

**Lemma 3.5.** The operator with kernel \(\overline{M}(x,y)\) with respect to the Lebesgue measure is of weak type \((1,1)\). In particular, \(\mathcal{M}_{\text{glob}}^*\) is of weak type \((1,1)\).

**Proof.** We only prove that, for \((x,y) \in G,\)
\[
\overline{M}(x,y) \lesssim e^{[x^2-y^2](1+|x|)^n \wedge (|x| \sin \theta)^{-n}}
\]
or, equivalently, that
\[
\left(\frac{|x+y|}{|x-y|}\right)^{n/2} \eta(x,y) \Psi(x,y) \lesssim (1+|x|)^n \wedge (|x| \sin \theta)^{-n}.
\]
The conclusion will then follow by \([5, \text{Lemma 4.4}]\).

We first consider the inequality involving \((1+|x|)^n\). We consider the cases \(\Psi(x,y) = 1\) and \(\Psi(x,y) = 1/\alpha^{n/2}\) separately.

1. If \(\Psi(x,y) = 1\), then by Lemma 3.1, (4) it is enough to prove that
\[
(3.5) \quad \left(\frac{|x+y|}{|x-y|}\right)^{n/2} \lesssim (1+|x|)^n.
\]
If \(|y| \leq 2|x|\) then by Lemma 3.1, (2) we get
\[
\frac{|x+y|}{|x-y|} \leq \frac{|x| + |y|}{|x-y|} \leq |x|(1+|x|) \leq (1+|x|)^2.
\]
If instead \(|y| > 2|x|\), we have
\[
|x-y| \geq |y|-|x| \geq |y|/2, \quad |x-y| \geq |y|-|x| \geq |x|
\]
so that
\[
\frac{|x+y|}{|x-y|} \leq \frac{|x|}{|x-y|} + \frac{|y|}{|x-y|} \leq C
\]
and hence a fortiori (3.5) holds.

2. If \(\Psi(x,y) = 1/\alpha^{n/2}\)
\[
\left(\frac{|x+y|}{|x-y|}\right)^{n/2} \eta(x,y) \Psi(x,y) = \frac{\eta(x,y)}{|x-y|^n} \leq \frac{1}{|x-y|^n} \lesssim (1+|x|)^n,
\]
again by Lemma 3.1, (4) and (2).

We then concentrate on the inequality involving \((|x| \sin \theta)^{-n}\). We again consider the cases \(\Psi(x,y) = 1\) and \(\Psi(x,y) = 1/\alpha^{n/2}\) separately.

1'. Let \(\Psi(x,y) = 1\), and observe that the function \(0 \leq u \rightarrow u^{n/2} e^{-u}\) is bounded. Thus, by Lemma 3.1 (5)
\[
\left(\frac{|x+y|}{|x-y|}\right)^{n/2} \eta(x,y) = \left(\frac{|x+y|}{|x-y|}\right)^{n/2} e^{-2|x-y|/|x+y| \sin \theta} \leq \left(\frac{|x+y|^2(1+\cos \theta')}{2|x|^2|y|^2 \sin^2 \theta}\right)^{n/2}
\]
\[
= C(|x| \sin \theta)^{-n} \left(\frac{|x+y|^2(1+\cos \theta')}{|y|^2}\right)^{n/2}.
\]
Therefore, it remains only to prove that
\[
\frac{|x+y|^2(1+\cos \theta')}{|y|^2} \leq C.
\]
If $|x| \leq 2|y|$ this is straightforward. Otherwise, note that

$$|x + y|^2(1 + \cos \theta') = g_\theta(|x|^2/|y|^2),$$

where

$$g_\theta(t) = (1 + t + 2\sqrt{t} \cos \theta) \left( 1 + \frac{1 - t}{\sqrt{(1 + t)^2 - 4t \cos^2 \theta}} \right).$$

Finally, observe that the functions $g_\theta$ are bounded on $(4, \infty)$ uniformly in $\theta$.

2'. If $\Psi = 1/\alpha^{n/2}$, observe that

$$\left( \frac{|x + y|}{|x - y|} \right)^{\alpha/2} \eta(x, y) \Psi(x, y) \leq \frac{1}{|x - y|^\alpha} \lesssim \frac{1}{(|x| \sin \theta)^\alpha}$$

by Lemma 3.1, (4) and (3). This completes the proof. \hfill \Box

### 3.2. Proof of Theorem 1.1

As already said, we treat separately the local and the global part of $\nabla L_1^{-1/2}$. By means of (3.2), Theorem 1.1 will be a consequence of Propositions 3.7 and 3.8 below.

In order to treat the local part $\nabla L_1^{-1/2}$, we shall need the following lemma.

**Lemma 3.6.** Let $\mu, \nu \geq 0$ be such that $\mu > \nu + 1$. Then, for every $(x, y) \in N_2$, $x \neq y$

$$R_{\mu, \nu}(x, y) := \int_0^1 \frac{|r x - y|^\nu}{(1 - r^2)^{\alpha \mu/2}} e^{-r |x - y|^2} \, dr \leq \frac{C}{|x - y|^n \mu - \nu - 2}.$$ 

**Proof.** Assume $(x, y) \in N_2$ and $x \neq y$. Observe that

$$|r x - y|^2 \geq |x - y|^2 - 2(1 - r)|x||x - y| \geq |x - y|^2 - 4(1 - r)$$

where the last inequality holds since for all $(x, y) \in N_2$

$$|x||x - y| \leq \frac{2|x|}{1 + |x| + |y|} \leq 2.$$

Thus

$$R_{\mu, \nu}(x, y) \lesssim \int_0^1 \frac{1}{(1 - r^2)^{\alpha \mu/2}} e^{-r |x - y|^2} \, dr \lesssim \int_0^1 \frac{1}{(1 - r^2)^{\alpha \mu/2}} e^{-r |x - y|^2} \, dr$$

and by performing the change of variable $|x - y|^2/(1 - r) = t$ we get

$$R_{\mu, \nu}(x, y) \leq \frac{C}{|x - y|^n \mu - \nu - 2} \int_0^\infty t^{(n + \mu - \nu - 4)/2} e^{-ct} \, dt \leq \frac{C}{|x - y|^n \mu - \nu - 2},$$

where the last inequality holds since by assumption

$$(n + \mu - \nu - 4)/2 > (n - 3)/2 \geq -1.$$ \hfill \Box

**Proposition 3.7.** For every $j = 1, \ldots, n$, $\nabla L_1^{-1/2}$ is of weak type $(1, 1)$.

**Proof.** Let $(x, y) \in N_2$, $x \neq y$. Observe that by (1.4) and Lemma 3.6,

$$|K_{\nabla L_1^{-1/2}}(x, y)| \lesssim R_{3, 1}(x, y) \chi(x, y) \lesssim |x - y|^{-n}$$

and

$$|\nabla K_{\nabla L_1^{-1/2}}(x, y)| + |\nabla_y K_{\nabla L_1^{-1/2}}(x, y)| \lesssim (R_{3, 0}(x, y) + R_{5, 2}(x, y) + R_{3, 1}(x, y)|x - y|^{-1}) \chi(x, y) \lesssim |x - y|^{-(n + 1)}$$

and
for every \( j = 1, \ldots, n \). Therefore, the conclusion follows by [6, Theorem 2.7]. \( \square \)

**Proposition 3.8.** For every \( j = 1, \ldots, n \) and \((x, y) \in G\)

\[
(3.6) \quad |K_{\nabla x^{-1/2}}(x, y)| \lesssim \mathcal{M}(x, y).
\]

In particular, \( \nabla x^{-1/2} \) is of weak type \((1, 1)\) for every \( j = 1, \ldots, n \).

**Proof.** Let \((x, y) \in G\), and observe first that for every \( j = 1, \ldots, n \)

\[
|K_{\nabla x^{-1/2}}(x, y)| \lesssim \int_0^\infty \frac{e^{-t}}{(1 - e^{-2t})^{(n+2)/2}} \frac{|e^{-t}x - y|}{\sqrt{1 - e^{-2t}}} \, dt =: R(x, y)
\]

since \( t \geq (1 - e^{-2t})/2 \) for every \( t \geq 0 \). With the change of variables \( t = \tau(s) \) (recall (3.3)) in the integral defining the kernel \( R \),

\[
|K_{\nabla x^{-1/2}}(x, y)| \lesssim \int_0^1 \frac{1}{s^{(n+3)/2}} |(1 - s)x - (1 + s)y| e^{-\frac{(1-s)(1+s)}{2}} \, ds
\]

where we used that \( 1 + s \geq 1 \) for every \( s \in (0, 1) \). Now make the change of variables \( s/\beta = \sigma \) in the integral, which gives

\[
\int_0^1 \frac{1}{s^{(n+3)/2}} |(1 - s)x - (1 + s)y| e^{-\frac{1}{\beta}a(\sigma/\beta)} \, ds
\]

\[
= \frac{1}{\beta^{n/2}} \int_0^{1/\beta} \frac{1}{\sigma^{(n+3)/2}} \frac{|(1 - \sigma \beta)x - (1 + \sigma \beta)y|}{\sqrt{\beta}} e^{-\frac{1}{\beta}a(\sigma)} \, d\sigma.
\]

Observe moreover that

\[
\frac{|(1 - \sigma \beta)x - (1 + \sigma \beta)y|}{\sqrt{\beta}} = \frac{|(x - y) - \sigma \beta(x + y)|}{\sqrt{\beta}} \leq \frac{|x - y| + \sigma |x - y|}{\sqrt{\beta}} = (1 + \sigma)\sqrt{\alpha}.
\]

Therefore, we proved that for every \((x, y) \in G\)

\[
|K_{\nabla x^{-1/2}}(x, y)| \lesssim e^{\frac{|x|^2}{\beta}} \left( \left\| \frac{x + y}{x - y} \right\| \right)^{n/2} \eta(x, y) \sqrt{\alpha} \int_0^{1/\beta} \frac{(1 + \sigma)}{\sigma^{(n+3)/2}} e^{-\frac{1}{\beta}a(\sigma)} \, d\sigma.
\]

It remains to prove that, if \((x, y) \in G\),

\[
(3.7) \quad \sqrt{\alpha} \int_0^{1/\beta} \frac{(1 + \sigma)}{\sigma^{(n+3)/2}} e^{-\frac{1}{\beta}a(\sigma)} \, d\sigma \lesssim \Psi(x, y).
\]

Observe first that

\[
\sqrt{\alpha} \int_0^{1/\beta} \frac{1 + \sigma}{\sigma^{(n+3)/2}} e^{-\frac{1}{\beta}a(\sigma)} \, d\sigma
\]

\[
\leq \sqrt{\alpha} \sup_{0 < \sigma < 1/\beta} \left( \frac{1}{\sigma^{n/2}} e^{-\frac{1}{\beta}a(\sigma)} \right)^{1-\frac{1}{\beta}} \int_0^{1/\beta} \frac{(1 + \sigma)}{\sigma^{2}} e^{-\frac{1}{\beta}a(\sigma)} \, d\sigma
\]

\[
\lesssim \Psi(x, y)^{1-\frac{1}{\beta}} \sqrt{\alpha} \int_0^{1/\beta} \frac{(1 + \sigma)}{\sigma^{2}} e^{-\frac{1}{\beta}a(\sigma)} \, d\sigma.
\]

The last inequality holds by (3.4). We now split the integral as

\[
\int_0^{1/\beta} \frac{1 + \sigma}{\sigma^{2}} e^{-\frac{1}{\beta}a(\sigma)} \, d\sigma = \int_0^{\min(1,1/\beta)} \cdots d\sigma + \int_{\min(1,1/\beta)}^{1/\beta} \cdots d\sigma,
\]
where we mean that the second integral is identically zero if $\beta \geq 1$. Since $\varphi$ is invertible in $(0,1)$ and $(1, \infty)$, it is invertible in both the integrals above, so that by the change of variables $\alpha \varphi(\sigma) = t$ we get

\begin{equation}
\sqrt{\alpha} \int_0^{\min(1,1/\beta)} \frac{1 + \sigma}{\sigma^2} e^{-\frac{t}{\beta} \alpha \varphi(\sigma)} d\sigma \leq \frac{1}{\sqrt{\alpha}} \int_0^\infty \frac{1}{1 - \sigma(t)} e^{-\frac{t}{\beta}} dt 
\end{equation}

while

\begin{equation}
\sqrt{\alpha} \int_{\min(1,1/\beta)}^{1/\beta} \frac{1 + \sigma}{\sigma^2} e^{-\frac{t}{\beta} \alpha \varphi(\sigma)} d\sigma \leq \frac{C}{\sqrt{\alpha}} \int_0^\infty \frac{1}{\sigma_+(t) - 1} e^{-\frac{t}{\beta}} dt,
\end{equation}

where

$$
\sigma_-(t) = 1 - \frac{\sqrt{t^2 + 4at} - t}{2a}, \quad \sigma_+(t) = 1 + \frac{\sqrt{t^2 + 4at} + t}{2a}.
$$

It is not hard to see that

$$1 - \sigma_-(t) = \frac{\sqrt{t^2 + 4at} - t}{2a} \geq C \min \left(1, \frac{\sqrt{t}}{\sqrt{\alpha}}\right) = \frac{C}{\sqrt{\alpha}} \min (\sqrt{\alpha}, \sqrt{t}),$$

by the inequality $\sqrt{1+z} - 1 \geq C \min(z, \sqrt{z})$. In other words,

$$\frac{1}{1 - \sigma_-(t)} \lesssim \sqrt{\alpha} \max \left(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{t}}\right).$$

Moreover

$$\sigma_+(t) - 1 = \frac{t + \sqrt{t^2 + 4at}}{2a} \geq 2 \frac{\sqrt{t}}{\sqrt{\alpha}}.$$

Therefore, from (3.8)

$$\sqrt{\alpha} \int_{\min(1,1/\beta)}^{\min(1,1/\beta)} \frac{1 + \sigma}{\sigma^2} e^{-\frac{t}{\beta} \alpha \varphi(\sigma)} d\sigma \lesssim \frac{1}{\sqrt{\alpha}} \int_0^\infty \frac{1}{1 - \sigma(t)} e^{-\frac{t}{\beta}} dt \lesssim \int_0^\infty \max \left(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{t}}\right) e^{-\frac{t}{\beta}} dt \lesssim \max \left(\frac{1}{\sqrt{\alpha}}, 1\right),$$

and from (3.9)

$$\sqrt{\alpha} \int_{\min(1,1/\beta)}^{1/\beta} \frac{1 + \sigma}{\sigma^2} e^{-\frac{t}{\beta} \alpha \varphi(\sigma)} d\sigma \lesssim \frac{1}{\sqrt{\alpha}} \int_0^\infty \frac{1}{\sigma_+(t) - 1} e^{-\frac{t}{\beta}} dt \lesssim C.$$

The proof of (3.6) is now complete. The weak type $(1,1)$ of $\nabla \varphi_{-1/2}$ is then a consequence of the straightforward observation that $|K_{\nabla \varphi_{-1/2}}| \leq |K_{\nabla \varphi_{-1/2}}| \chi_G$.

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