Jump detection in Besov spaces via a new BBM formula. Applications to Aviles-Giga type functionals

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Abstract

Motivated by the formula, due to Bourgain, Brezis and Mironescu,

\[ \lim_{\varepsilon \to 0^+} \int \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^q}{|x-y|^q} \rho_\varepsilon(x-y) \, dx \, dy = K_{q,N} \|\nabla u\|_{L^q}^q, \]

that characterizes the functions in \( L^q \) that belong to \( W^{1,q} \) (for \( q > 1 \)) and \( BV \) (for \( q = 1 \)), respectively, we study what happens when one replaces the denominator in the expression above by \( |x-y| \). It turns out that, for \( q > 1 \) the corresponding functionals “see” only the jumps of the \( BV \) function. We further identify the function space relevant to the study of these functionals, the space \( BV^{1/q}_q \), as the Besov space \( B_{1,q}^{1/q} \). We show, among other things, that \( BV^{1/q}_q(\Omega) \) contains both the spaces \( BV(\Omega) \cap L^\infty(\Omega) \) and \( W^{1/q,q}_1(\Omega) \). We also present applications to the study of singular perturbation problems of Aviles-Giga type.

1 Introduction

Bourgain, Brezis and Mironescu introduced in [5] a new characterization of the spaces \( W^{1,q}(\Omega) \), \( q > 1 \), and \( BV(\Omega) \) using certain double integrals involving radial mollifiers \( \{\rho_\varepsilon\} \) (see [5] for the precise assumptions). In the case of a domain \( \Omega \subset \mathbb{R}^N \) with Lipschitz boundary, the so called “BBM formula” states that for any \( u \in L^q(\Omega) \) (\( q > 1 \)):

\[ \lim_{\varepsilon \to 0^+} \int \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^q}{|x-y|^q} \rho_\varepsilon(x-y) \, dx \, dy = K_{q,N} \|\nabla u\|_{L^q}^q, \]

with the convention that \( \|\nabla u\|_{L^q} = \infty \) if \( u \notin W^{1,q} \). For the case \( q = 1 \) the expression in (1.1) characterizes the \( BV \)-space (the latter result in its full strength is due to Dávila [11]). For

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2010 Mathematics Subject Classification. Primary 46E35.
further developments in this direction see [8, 16, 17, 20, 21]. In particular, for the simplest choice of

\[
\rho_\varepsilon(z) = \begin{cases} \\
\frac{1}{|z|} \frac{1}{\mathcal{L}^N(B_1(0))} & z \in B_\varepsilon(0) \\
0 & z \in \mathbb{R}^N \setminus B_\varepsilon(0) \\
\end{cases},
\]  

(1.2)

we may rewrite (1.1) in the cases \( q > 1 \) and \( q = 1 \), respectively, as

\[
\lim_{\varepsilon \to 0^+} \int \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} \frac{|u(x) - u(y)|^q}{|x - y|^q} \, dy \, dx = \mathcal{L}^N(B_1(0)) \| \nabla u \|_{L^q}^q, \\
\lim_{\varepsilon \to 0^+} \int \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} \frac{|u(x) - u(y)|}{|x - y|} \, dy \, dx = \mathcal{L}^N(B_1(0)) \| Du \|. 
\]  

(1.3)

(1.4)

We are interested in a related formula to (1.3), that is obtained when we replace \( |x - y|^q \) by \( |x - y| \) in the denominator (for \( q > 1 \)). We shall see in our main result Theorem 1.1 that the resulting formula is very different from the one in (1.4): it involves only the “jump part” of the gradient. We denote the space consisting of the functions for which the resulting expression is bounded by \( BV^q \). It turns out, as we shall explain below, that this space is closely related to the Besov Space \( B^{1/q}_{q,\infty} \).

A related, but different phenomenon was investigated by Ponce and Spector in [17]: for another variation on the BBM-formula they obtained a limit where the singular part of \( Du \) appears (i.e., the sum of the jump and Cantor parts).

In order to state our results we shall need some definitions.

**Definition 1.1.** Given an open set \( \Omega \subset \mathbb{R}^N \), a real number \( q \geq 1 \) and a function \( u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d) \) define:

\[
\bar{A}_{u,q}(\Omega) := \sup_{\varepsilon \in (0,1)} \int \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} \frac{|u(y) - u(x)|^q}{|y - x|} \, dy \, dx, \\
\]  

(1.5)

and the infinitesimal version of this quantity:

\[
\hat{A}_{u,q}(\Omega) := \limsup_{\varepsilon \to 0^+} \int \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} \frac{|u(y) - u(x)|^q}{|y - x|} \, dy \, dx. \\
\]  

(1.6)

**Remark 1.1.** It is clear that for any open \( \Omega \subset \mathbb{R}^N \) and any \( u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d) \) we have

\[
\hat{A}_{u,q}(\Omega) \leq \bar{A}_{u,q}(\Omega). \\
\]  

(1.7)

Moreover, if \( u \in L^q(\Omega, \mathbb{R}^d) \) then

\[
\bar{A}_{u,q}(\Omega) < \infty \quad \text{if and only if} \quad \hat{A}_{u,q}(\Omega) < \infty. \\
\]  

(1.8)

Clearly,

\[
\hat{A}_{u,q}(\mathbb{R}^N) = \limsup_{\varepsilon \to 0^+} \int \int_{B_1(0) \times \mathbb{R}^N} \frac{1}{\varepsilon |z|} \left| u(x + \varepsilon z) - u(x) \right|^q \, dx \, dz \\
\bar{A}_{u,q}(\mathbb{R}^N) := \sup_{\varepsilon \in (0,1)} \int \int_{B_1(0) \times \mathbb{R}^N} \frac{1}{\varepsilon |z|} \left| u(x + \varepsilon z) - u(x) \right|^q \, dx \, dz. \\
\]  

(1.9)
Using the quantities $\hat{A}_{u,q}$, $\hat{A}_{u,q}$ we can now define the space $BV^q(\Omega, \mathbb{R}^d)$:

**Definition 1.2.** Given an open set $\Omega \subset \mathbb{R}^N$, a real number $q \geq 1$ and a function $u \in L^q(\Omega, \mathbb{R}^d)$ we say that $u \in BV^q(\Omega, \mathbb{R}^d)$ if

$$\hat{A}_{u,q}(\Omega) < \infty.$$  \hfill (1.10)

Clearly, for $u \in L^q(\Omega, \mathbb{R}^d)$ we have $u \in BV^q(\Omega, \mathbb{R}^d)$ if and only if

$$\hat{A}_{u,q}(\Omega) < \infty.$$  \hfill (1.11)

Moreover, $BV^q(\Omega, \mathbb{R}^d)$ becomes a Banach space when equipped with the norm

$$\|u\|_{BV^q(\Omega, \mathbb{R}^d)} := \left(\hat{A}_{u,q}(\Omega)\right)^{\frac{1}{q}} + \|u\|_{L^q(\Omega, \mathbb{R}^d)}.$$  \hfill (1.12)

Next, given a function $u \in L^q_{loc}(\Omega, \mathbb{R}^d)$ we say that $u \in BV^q_{loc}(\Omega, \mathbb{R}^d)$ if for every open $\Omega' \subset \subset \Omega$ we have $u \in BV^q(\Omega', \mathbb{R}^d)$.

**Remark 1.2.** By the “BBM formula” we have $BV^1_{loc}(\Omega, \mathbb{R}^d) = BV^1(\Omega, \mathbb{R}^d)$ and in the case of a domain $\Omega$ with Lipschitz boundary, also $BV^1(\Omega, \mathbb{R}^d) = BV(\Omega, \mathbb{R}^d)$.

In our main result, Theorem 1.1. we prove an explicit formula for $\hat{A}_{u,q}(\Omega)$ when $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$. This formula justifies the name we have chosen for the space $BV^q$.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^N$ be an open set with bounded Lipschitz boundary and let $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$. Then, for every $q > 1$ we have $u \in BV^q(\Omega, \mathbb{R}^d)$ and

$$\hat{A}_{u,q}(\Omega) = C_N \int_{\Omega} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x).$$  \hfill (1.13)

with the dimensional constant $C_N > 0$ defined by

$$C_N := \frac{1}{N} \int_S |z_1| d\mathcal{H}^{N-1}(z),$$  \hfill (1.14)

where we denote $z := (z_1, \ldots, z_N) \in \mathbb{R}^N$.

**Remark 1.3.** Note the **big difference** between the case $q > 1$ and $q = 1$. Indeed, by (1.4) for $BV^1 = BV$ the analog of (1.13) is

$$\hat{A}_{u,1}(\Omega) = L^N(B_1(0)) K_{1,N} \|Du\|_{(\Omega)},$$

that is, for $q = 1$ we see the **full** BV-seminorm, not just the “jump part”!

Our next result deals with functions in $W^{\frac{1}{q}, q}$:

**Theorem 1.2.** Given an open set $\Omega \subset \mathbb{R}^N$, $q \geq 1$ and a function $u \in W^{\frac{1}{q}, q}(\Omega, \mathbb{R}^d)$ we have $u \in BV^q(\Omega, \mathbb{R}^d)$, and if in addition $q > 1$, then $\hat{A}_{u,q}(\Omega) = 0$. Moreover, the embedding $W^{\frac{1}{q}, q}(\Omega, \mathbb{R}^d) \subset BV^q(\Omega, \mathbb{R}^d)$ is continuous.

Next we recall the definition of the Besov Spaces $B^{s}_{q, \infty}$ with $s \in (0, 1)$:
**Theorem 1.3.** Given \( q \geq 1 \) and \( s \in (0, 1) \), we say that \( u \in L^q(\mathbb{R}^N, \mathbb{R}^d) \) belongs to the Besov space \( B^s_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d) \) if

\[
\sup_{\rho \in (0,\infty)} \left( \sup_{|h| \leq \rho} \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^q}{\rho^s} \, dx \right) < \infty. \tag{1.15}
\]

Moreover, for every open \( \Omega \subset \mathbb{R}^N \) we say that \( u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d) \) belongs to Besov space \( (B^s_{q,\infty})_{\text{loc}}(\Omega, \mathbb{R}^d) \) if for every compact \( K \subset \subset \Omega \) there exists \( u_K \in B^s_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d) \) such that \( u_K(x) = u(x) \) for every \( x \in K \).

The next result clarifies the relation between the space \( BV^q \) and Besov spaces:

**Proposition 1.1.** For \( q > 1 \) we have:

\[
BV^q(\mathbb{R}^N, \mathbb{R}^d) = B^{1/q}_{1,\infty}(\mathbb{R}^N, \mathbb{R}^d). \tag{1.16}
\]

Moreover for every open \( \Omega \subset \mathbb{R}^N \) and \( q > 1 \) we have:

\[
BV^q_{\text{loc}}(\Omega, \mathbb{R}^d) = \left( B^{1/q}_{1,\infty} \right)_{\text{loc}}(\Omega, \mathbb{R}^d). \tag{1.17}
\]

We should mention that (1.16) of Proposition 1.1 can be deduced from a more general result, obtained independent by Brasseur in [7], that characterizes the Besov spaces \( B^s_{p,\infty}(\mathbb{R}^N) \) via a BBM-type formula, for all values of \( s \in (0, 1) \) and \( p \in [1, \infty) \).

**Remark 1.4.** Similar results hold also for more general mollifiers than in (1.2), e.g., of the form \( \rho_\varepsilon(x) = \rho(|x|/\varepsilon) \), where \( \rho(t) \) is a nonnegative function on \([0, \infty)\) with compact support, such that \( \text{ess inf}_{(0,\delta)} \rho \geq \alpha \) for some \( \alpha, \delta > 0 \) and \( \int_{\mathbb{R}^N} \rho(|x|) \, dx = 1 \). We did not investigate more general families of radial mollifiers \( \{ \rho_\varepsilon(x) \} \) as in [5].

In [6] Bourgain, Brezis and Mironescu introduced a new space, that they called \( B \), that contains the spaces \( BV, W^{1,q} \) for \( q \geq 1 \) and \( BMO \). Moreover, they introduced a proper subspace \( B_0 \subset B \), such that \( B_0 \) contains \( W^{1,q} \) for \( q \geq 1 \) as well as \( VMO \). For every \( u \in B \) they defined the seminorm \( |u|_B \) and its infinitesimal version \( [u](\Omega) \). The precise definitions are given bellow in Definition 2.4. Our next result deals with the relations between the \( BV^q \) spaces and the spaces \( B \) and \( B_0 \):

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and \( q \geq 1 \). Then for every \( u \in BV^q(\Omega, \mathbb{R}^d) \) we have

\[
|u|_{B(\Omega, \mathbb{R}^d)} \leq N^{\frac{N+1}{2q}} (\hat{A}_{u,q}(\Omega))^{\frac{q}{2}} \tag{1.18}
\]

and

\[
[u](\Omega) \leq N^{\frac{N+1}{2q}} (\hat{A}_{u,q}(\Omega))^{\frac{q}{2}}. \tag{1.19}
\]

Moreover, if in addition \( \mathcal{L}^N(\Omega) < \infty \) then \( BV^q(\Omega, \mathbb{R}^d) \subset B(\Omega, \mathbb{R}^d) \) with continuous embedding. In particular, if \( \hat{A}_{u,q}(\Omega) = 0 \) then \( u \in B_0(\Omega, \mathbb{R}^d) \).
We now turn to the role of $BV^q$-spaces in the study of singular perturbation problems. In various applications one is led to study the $\Gamma$-limit, as $\varepsilon \to 0^+$, of the Aviles-Giga functional $I_{\varepsilon}^{(2)}$, defined for scalar functions $\psi$ by

$$I_{\varepsilon}^{(2)}(\psi) := \int_\Omega \left\{ \varepsilon |\nabla^2 \psi|^2 + \frac{1}{\varepsilon} \left( 1 - |\nabla \psi|^2 \right)^2 \right\} \, dx$$  \hspace{1cm} (1.20)

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain.

A generalization of (1.20) to any $p > 1$ is:

$$I_{\varepsilon}^{(p)}(\psi) := \int_\Omega \left( \varepsilon^{p-1} |\nabla^2 \psi|^p + \frac{1}{\varepsilon} \left( 1 - |\nabla \psi|^2 \right)^{\frac{p}{p-1}} \right) = \int_\Omega \left( \varepsilon |\nabla^2 \psi|^p + \left( 1 - |\nabla \psi|^2 \right)^{\frac{p}{p-1}} \right).$$  \hspace{1cm} (1.21)

It is clear that the functional $(\Gamma - \limsup_{\varepsilon \to 0^+} I_{\varepsilon}^{(p)}(\psi))$, calculated in the strong $W^{1,q}$ topology, can be finite only if $|\nabla \psi|^2 = 1$ for a.e. $x \in \Omega$, i.e., if we define:

$$A_0 := A_0(\Omega, q) := \left\{ \psi \in W^{1,q}(\Omega) : |\nabla \psi(x)|^2 = 1 \text{ for a.e. } x \in \Omega \right\},$$

$$A := A(\Omega, q) := \left\{ \psi \in W^{1,q}(\Omega) : (\Gamma - \limsup_{\varepsilon \to 0^+} I_{\varepsilon}^{(p)}(\psi)) < \infty \right\}.$$  \hspace{1cm} (1.22)

then clearly $A \subset A_0$. Note that the set $A$ consists of functions with discontinuous gradients. The natural space of discontinuous functions is $BV$ space. It turns out that we have $A_{BV} \subset A \subset A_0$, where $A_{BV} := A_0 \cap \{ \psi : \nabla \psi \in BV \}$. However, Ambrosio, De Lellis and Mantegazza showed in [1] that $A_{BV} \subsetneq A$ in the special case of the energy (1.20) when $N = 2$. On the other hand, as shown by Camillo de Lellis and Felix Otto in [13], for the energy (1.20) the set $A$ is contained in a certain space of functions that still inherits some good geometric measure theoretical properties of $BV$ space.

A lower bound for (1.20) when $N = 2$ was found by Aviles and Giga in [4], by Jin and Kohn in [15], and by Ambrosio, De Lellis and Mantegazza in [1]. A matched upper bound, in the case $\nabla \psi \in BV$, was found independently by Conti and De Lellis [9] and Poliakovsky [18]. These results imply that for the particular case $\nabla \psi \in BV$ and $N = 2$, the $\Gamma$-limit functional of (1.20), calculated in the strong $W^{1,q}$-topology, is

$$\tilde{I}_0(\psi) := \begin{cases} \frac{1}{3} \int_{J_{\nabla^p}} |\nabla \psi^+(x) - \nabla \psi^-(x)|^3 \, d\mathcal{H}^{N-1}(x) & \text{if } |\nabla \psi| = 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$  \hspace{1cm} (1.23)

This results can also be generalized to show that, up to a multiplicative constant, the energy (1.23) is also the $\Gamma$-limit of functional (1.21). Indeed, the lower bound for (1.21) can be obtained analogously to that for (1.20), using Hölder inequality instead of Cauchy-Schwarz and the matched upper bound can be obtained as a special case of a more general result, obtained in [19]. However, as we already mentioned, we have $A_{BV} \neq A$ for problem (1.20) and thus the question of the value of the $\Gamma$-limit in the case $\nabla \psi \notin BV$ is still open.

We also recall that De Lellis showed in [12] that for $N = 3$ and $\nabla \psi \in BV$, the functional (1.23) is not lower semicontinuous in the $L^1$-topology and thus cannot by the $\Gamma$-limit of (1.20).
In the particular case of the functional (1.20) with $N = 2$ we propose here a candidate for the set $A$, namely the set \( \{ \psi : \Omega \to \mathbb{R} : \nabla \psi \in BV^3, |\nabla \psi| = 1 \} \) (where $BV^3$ is the case $q = 3$ of the space $BV^q$). Indeed, by Theorem 1.1 and (1.23), when $N = 2$, $|\nabla \psi| = 1$ and $\nabla \psi \in BV$, the $\Gamma$-limit of the functional (1.20) equals \( \frac{1}{3c^2} \hat{A}_{\nabla \psi, 3}(\Omega) \). Therefore, it is natural to conjecture that \( \frac{1}{3c} \hat{A}_{\nabla \psi, 3}(\Omega) \) is the $\Gamma$-limit also in the case $\nabla \psi \notin BV$, and more specifically that $A = \{ \psi : \Omega \to \mathbb{R} : \nabla \psi \in BV^3, |\nabla \psi| = 1 \}$. We have an analogous conjecture for the functional (1.21), with a different constant multiplying $\hat{A}_{\nabla \psi, 3}(\Omega)$. An additional suppor for this conjecture is provided by the fact that the example constructed by Ambrosio, De Lellis and Mantegazza in [1], of a function $\psi \in A \setminus A_{BV}$, turns out to satisfy $\psi \in BV^3$ (as it can be easily verified).

Our next result provides a (non-sharp) upper bound for a more general energy than the one in (1.21):

**Theorem 1.4.** Given an open set $\Omega \subset \mathbb{R}^N$, let $\Omega_0 \subset \subset \Omega$ be a compactly embedded open subset and $\psi \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R})$ be such that $|\nabla \psi(x)| = 1$ for a.e. $x \in \Omega$. Let $\eta \in C^1_c(\mathbb{R}^N, \mathbb{R})$ be a nonnegative function such that $\supp \eta \subset B_1(0)$ and $\int_{\mathbb{R}^N} \eta(z)dz = 1$. For every $x \in \Omega$ and every $0 < \varepsilon < \text{dist}(x, \partial \Omega)$ define

\[
\psi_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y - x}{\varepsilon}\right) \psi(y)dy = \int_{\mathbb{R}^N} \eta(z)\psi(x + \varepsilon z)dz. \tag{1.24}
\]

Assume in addition that $\nabla \psi(x) \in BV^q_{\text{loc}}(\Omega, \mathbb{R}^N) \cap BV^p_{\text{loc}}(\Omega, \mathbb{R}^N)$ for some $q > 1$ and $p \geq 2$. Then we have:

\[
\lim_{\varepsilon \to 0^+} \left( \int_{\Omega_0} \varepsilon^{q-1} |\nabla^2 \psi_\varepsilon(x)|^q dx + \frac{1}{\varepsilon} \left( 1 - |\nabla \psi_\varepsilon(x)|^2 \right)^{\frac{q}{2}} dx \right) \leq \left( \int_{\mathbb{R}^N} |z|^{\frac{1}{q-1}} |\nabla \eta(z)|^{\frac{q}{q-1}} dz \right)^{q-1} A_{\nabla \psi,q}(\Omega_0) + \left( \int_{\mathbb{R}^N} |z|^{\frac{p}{p-2}} |\eta(z)|^{\frac{p}{p-2}} dz \right)^{\frac{p-2}{p}} A_{\nabla \psi,p}(\Omega_0), \tag{1.25}
\]

where

\[
A_{\nabla \psi,p}(\Omega_0) := \lim_{\varepsilon \to 0^+} \int_{\Omega_0} \int_{B_\varepsilon(x)} \frac{1}{\varepsilon^N} \frac{|\nabla \psi(y) - \nabla \psi(x)|^p}{|y - x|} dydx. \tag{1.26}
\]

In particular, if $\nabla \psi(x) \in BV^3_{\text{loc}}(\Omega, \mathbb{R}^N)$ then:

\[
\frac{3}{\sqrt{4}} \lim_{\varepsilon \to 0^+} \left( \int_{\Omega_0} |\nabla^2 \psi_\varepsilon(x)| \left| 1 - |\nabla \psi_\varepsilon(x)|^2 \right| dx \right) \leq \lim_{\varepsilon \to 0^+} \left( \int_{\Omega_0} \varepsilon^2 |\nabla^2 \psi_\varepsilon(x)|^3 dx + \int_{\Omega_0} \frac{1}{\varepsilon} \left( 1 - |\nabla \psi_\varepsilon(x)|^2 \right)^{\frac{3}{2}} dx \right) \leq D_\eta A_{\nabla \psi,3}(\Omega_0) = D_\eta \lim_{\varepsilon \to 0^+} \left( \int_{B_1(0)} \int_{\Omega_0} \frac{1}{\varepsilon |z|} |\nabla \psi(x + \varepsilon z) - \nabla \psi(x)|^3 dxdz \right), \tag{1.27}
\]

where the constant $D_\eta$ is given by

\[
D_\eta := \left( \int_{\mathbb{R}^N} |z|^{\frac{3}{2}} |\nabla \eta(z)|^{\frac{3}{2}} dz \right)^{2} + \left( \int_{\mathbb{R}^N} |z|^{2} |\eta(z)|^3 dz \right)^{\frac{4}{3}}. \tag{1.28}
\]
As a direct consequence of the last Theorem we extend the previously known result about
the boundedness of the $\Gamma - \limsup$ for the energy in $[1.21]$ when $p = 3$ from the case $\nabla \psi \in BV$ (see [19]) to the case $\nabla \psi \in BV^3$:

**Corollary 1.1.** Given an open set $\Omega \subset \mathbb{R}^N$, let $\psi \in W^{1,\infty}_{loc}(\Omega, \mathbb{R})$ be such that $|\nabla \psi(x)| = 1$ for a.e. $x \in \Omega$ and $\nabla \psi(x) \in BV^3_{loc}(\Omega, \mathbb{R}^N)$. Then, for every compactly embedded open subset $\Omega' \subset \subset \Omega$ and every $q \geq 1$ we have, $\psi \in A(\Omega', q)$, with $A(\Omega', q)$ given by

$$A(\Omega', q) := \left\{ \psi \in W^{1,q}(\Omega') : (\Gamma - \limsup_{\varepsilon \to 0^+} I^3_{\varepsilon}(\psi) \right\} < +\infty \text{ (calculated in the } W^{1,q} \text{ topology)} \right\}, \tag{1.29}$$

where $I^3_{\varepsilon}$ is given by $[1.21]$ with $p = 3$. Moreover, we have

$$(\Gamma - \limsup_{\varepsilon \to 0^+} I^3_{\varepsilon}(\psi)) \leq CA_{\nabla \psi, A}(\Omega') \tag{1.30}$$

for some constant $C > 0$.

**Remark 1.5.** We do not know whether one can get a global and sharp “improved” version of Corollary 1.1 with $\Omega' = \Omega$ and with the constant $C := \frac{1}{2^{4N}CN}$ in (1.30). This is the sharp constant for the energy (1.21) with $p = 3$ and $N = 2$ in the particular case where $\nabla \psi \in BV$.

The paper is organized as follows. Section 2 is devoted to definitions and properties of the spaces $BV^q$. In subsection 2.1 we present some additional definitions and generalized versions of some of the results stated above. In subsection 2.2 we give the proofs of our main results about the spaces $BV^q$. In Section 3 we give the proof of Theorem 1.4, which is an application of the spaces $BV^q$ to the study of energies of Avies-Giga type. The proofs of Proposition 2.3 and Lemma 2.1 are given in the Appendix B. For the convenience of the reader, in Appendix A we states some known results on $BV$ functions, that we need for the proof.

### Acknowledgments

The research was supported by the Israel Science Foundation (Grant No. 999/13). I thank Itai Shafrir for some interesting discussions and Petru Mironescu for very helpful suggestions that helped me improve an earlier version of the manuscript.

## 2 Properties of the space $BV^q$

### 2.1 Some additional definitions and results

First we introduce local versions of the quantity $\hat{A}_{u,q}$ that are related to the space $BV^q_{loc}$:

**Definition 2.1.** Given a compact set $\overline{U} \subset \subset \Omega$ let

$$A_{u,q}(U) := \limsup_{\varepsilon \to 0^+} \int_{B_1(0)} \int_{U \setminus B_\varepsilon(0)} \frac{1}{\varepsilon |z|} |u(x + \varepsilon z) - u(x)|^q \, dx \, dz$$

$$= \limsup_{\varepsilon \to 0^+} \int_{U \setminus B_\varepsilon(x)} \frac{1}{\varepsilon N} \frac{|u(y) - u(x)|^q}{|y - x|} \, dy \, dx. \tag{2.1}$$
For an open set $\Omega \subset \mathbb{R}^N$ let
\[ A_{u,q}(\Omega) := \sup_{K \subset \subset \Omega} A_{u,q}(K). \tag{2.2} \]

**Remark 2.1.** It is clear that for any open $\Omega \subset \mathbb{R}^N$, any $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d)$ and for any compactly embedded open set $\Omega_0 \subset \subset \Omega$ we have
\[ \hat{A}_{u,q}(\Omega_0) \leq A_{u,q}(\Omega_0) \leq A_{u,q}(\Omega) \leq \hat{A}_{u,q}(\Omega). \tag{2.3} \]

**Remark 2.2.** Clearly, given an open set $\Omega \subset \mathbb{R}^N$, $q \geq 1$ and a function $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d)$ we have $u \in BV^q_{\text{loc}}(\Omega, \mathbb{R}^d)$ if and only if for every compact subset $K \subset \subset \Omega$ we have $A_{u,q}(K) < \infty$.

Next we define the following quantities, that are closely related to $\hat{A}_{u,q}$:

**Definition 2.2.** Given a compact set $\overline{U} \subset \subset \Omega$ let
\[ B_{u,q}(U) := \limsup_{\varepsilon \to 0^+} \sup_{k \in S^{N-1}} \int_{\overline{U}} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)|^q \, dx. \tag{2.4} \]

Next, given an open set $\Omega \subset \mathbb{R}^N$ define
\[ B_{u,q}(\Omega) := \sup_{K \subset \subset \Omega} B_{u,q}(K). \tag{2.5} \]

Finally, set
\[ \hat{B}_{u,q}(\mathbb{R}^N) := \limsup_{\varepsilon \to 0^+} \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)|^q \, dx. \tag{2.6} \]

The following result is known; for the convenience of a reader we will give its proof in the Appendix.

**Lemma 2.1.** For any $q > 1$, a function $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$ belongs to $B^{q/\infty}_{1,q}(\mathbb{R}^N, \mathbb{R}^d)$ if and only if $\hat{B}_{u,q}(\mathbb{R}^N) < \infty$. Moreover, for any open $\Omega \subset \mathbb{R}^N$, a function $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d)$ belongs to $(B^{1/\infty}_{q,\text{loc}}(\Omega, \mathbb{R}^d))$ if and only if for every compact $K \subset \subset \Omega$ we have $B_{u,q}(K) < \infty$.

Then Proposition 2.1 is a part of the following statement:

**Proposition 2.1.** For every open set $\Omega \subset \mathbb{R}^N$, every $q \geq 1$ and $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d)$ we have
\[ \frac{A_{u,q}(\Omega)}{\mathcal{L}^N(B_1(0))} \leq B_{u,q}(\Omega) \leq 2^{N+q} \frac{A_{u,q}(\Omega)}{\mathcal{L}^N(B_1(0))}. \tag{2.7} \]

Moreover, if $u \in L^q(\mathbb{R}^N, \mathbb{R}^d)$ then
\[ \frac{\hat{A}_{u,q}(\mathbb{R}^N)}{\mathcal{L}^N(B_1(0))} \leq \hat{B}_{u,q}(\mathbb{R}^N) \leq 2^{N+q} \frac{\hat{A}_{u,q}(\mathbb{R}^N)}{\mathcal{L}^N(B_1(0))}. \tag{2.8} \]

In particular, for $q > 1$ we have:
\[ BV^q(\mathbb{R}^N, \mathbb{R}^d) = B^{1/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^d) \quad \text{and} \quad BV^q_{\text{loc}}(\Omega, \mathbb{R}^d) = (B^{1/q}_{q,\text{loc}}(\Omega, \mathbb{R}^d)). \tag{2.9} \]

Proposition 2.1 will be deduced from Lemma 2.3 below.

The next theorem is a generalization of Theorem 1.1.
Theorem 2.1. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in BV_{loc}(\Omega, \mathbb{R}^d) \cap L^\infty_{loc}(\Omega, \mathbb{R}^d)$. Then, for every $q > 1$ we have $u \in BV^q_{loc}(\Omega, \mathbb{R}^d)$ and for every compact set $K \subset \subset \Omega$ such that $\|Du\|(\partial K) = 0$ we have

$$A_{u,q}(K) = C_N \int_{J_u \cap K} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x),$$

(2.10)

where $C_N$ is defined in (14). Moreover, if in addition $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$, then for every $q > 1$ we have

$$A_{u,q}(\Omega) = C_N \int_{J_u} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x).$$

(2.11)

Finally, if $\Omega$ is an open set with a bounded Lipschitz boundary and $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$ then we have $u \in BV^q(\Omega, \mathbb{R}^d)$ for every $q > 1$ and

$$\hat{A}_{u,q}(\Omega) = C_N \int_{J_u} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x) = A_{u,q}(\Omega).$$

(2.12)

The next proposition is an easy consequence of the definitions; the details are left to the reader.

Proposition 2.2. For every open set $\Omega \subset \mathbb{R}^N$, two real numbers $q_2 > q_1 \geq 1$ and $u \in L^\infty(\Omega, \mathbb{R}^d)$ we have

$$\hat{A}_{u,q_2}(\Omega) \leq 2^{q_2-q_1} \|u\|_{L^\infty(\Omega, \mathbb{R}^d)}^{q_2-q_1} A_{u,q_1}(\Omega), \quad \hat{A}_{u,q_2}(\Omega) \leq 2^{q_2-q_1} \|u\|_{L^\infty(\Omega, \mathbb{R}^d)}^{q_2-q_1} \hat{A}_{u,q_1}(\Omega),$$

$$A_{u,q_2}(\Omega) \leq 2^{q_2-q_1} \|u\|_{L^\infty(\Omega, \mathbb{R}^d)}^{q_2-q_1} A_{u,q_1}(\Omega) \quad \text{and} \quad B_{u,q_2}(\Omega) \leq 2^{q_2-q_1} \|u\|_{L^\infty(\Omega, \mathbb{R}^d)}^{q_2-q_1} B_{u,q_1}(\Omega).$$

(2.13)

In particular, for every open set $\Omega \subset \mathbb{R}^N$ and any two real numbers $q_2 > q_1 \geq 1$ we have $BV_{loc}^{q_1}(\Omega, \mathbb{R}^d) \cap L^\infty_{loc}(\Omega, \mathbb{R}^d) \subset BV_{loc}^{q_2}(\Omega, \mathbb{R}^d)$ and $BV^{q_1}(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d) \subset BV^{q_2}(\Omega, \mathbb{R}^d)$.

Remark 2.3. If $\Omega \subset \mathbb{R}^N$ is an open set, $D \subset \mathbb{R}^N$ is a Borel set and $\chi_D$ is the characteristic function of $D$, i.e.,

$$\chi_D(x) := \begin{cases} 1 & x \in D, \\ 0 & x \notin D, \end{cases}$$

(2.14)

then clearly for every $q \geq 1$ we have:

$$\hat{A}_{\chi_D,q}(\Omega) = \hat{A}_{\chi_D,1}(\Omega), \quad \hat{A}_{\chi_D,q}(\Omega) = \hat{A}_{\chi_D,1}(\Omega),$$

$$A_{\chi_D,q}(\Omega) = A_{\chi_D,1}(\Omega), \quad B_{\chi_D,q}(\Omega) = B_{\chi_D,1}(\Omega).$$

(2.15)

In particular, $\chi_D \in BV_{loc}^{q_1}(\Omega, \mathbb{R}^d)$ if and only if $D$ has a locally finite perimeter. Moreover, if in addition $\mathcal{L}^N(D) < \infty$ then we have $\chi_D \in BV^{q_1}(\Omega, \mathbb{R}^d)$ if and only if $D$ has finite perimeter.

In the special case $N = 1$, i.e., when the domain $\Omega$ is an interval, there exists a classical notion of a space of functions of bounded $q$-variation (see e.g., Kolyada and Lind [14] and the references therein). This space, denoted by $V_q(\Omega, \mathbb{R}^d)$, was first considered by Wiener [22] (for $q = 2$). Below we recall the definition of $V_q(\Omega, \mathbb{R}^d)$ and also define its a.e.-equivalent version that we denote by $\hat{V}_q(\Omega, \mathbb{R}^d)$. 


**Definition 2.3.** Given an interval $I \subseteq \mathbb{R}$ (open, closed, bounded or unbounded) denote for every $n \in \mathbb{N}$,

$$
\Pi_n(I) := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 < x_2 < \ldots < x_n < x_{n+1}, \ x_1 \in I, \ x_{n+1} \in I\}.
$$

For any function $f : I \to \mathbb{R}^d$ defined everywhere in $I$ and for every $q \geq 1$ let

$$
v_{q,I}(f) := \sup_{n \in \mathbb{N}} \left( \sup_{(x_1, \ldots, x_{n+1}) \in \Pi_n(I)} \left( \sum_{k=1}^{n} |f(x_{k+1}) - f(x_k)|^q \right)^{\frac{1}{q}} \right). \quad (2.16)
$$

We shall say that $f \in V_q(I, \mathbb{R}^d)$ if $v_{q,I}(f) < \infty$. Next, for a measurable $\mathbb{R}^d$-valued function $f$, defined a.e. in $I$, and $q \geq 1$ let

$$
\hat{v}_{q,I}(f) := \inf \left\{ v_{q,I}(g) : g : I \to \mathbb{R}^d, \ g(x) = f(x) \text{ a.e. in } I \right\}. \quad (2.17)
$$

We shall say that such $f$ belongs to the space $\hat{V}_q(I, \mathbb{R}^d)$ if $\hat{v}_{q,I}(f) < \infty$. Evidently, if $\hat{v}_{q,I}(f) < \infty$ then $f \in L^\infty(I, \mathbb{R}^d)$ and moreover, if $v_{q,I}(f) < \infty$ then $f$ is bounded everywhere.

The next Proposition is concerned with the relation between the spaces $\hat{V}_q([a, b], \mathbb{R}^d)$ and $BV^q((a, b), \mathbb{R}^d)$:

**Proposition 2.3.** For every $q \geq 1$ and every $a < b \in \mathbb{R}$, if a measurable function $f : (a, b) \to \mathbb{R}^d$ defined a.e. in $(a, b)$ belongs to the space $\hat{V}_q([a, b], \mathbb{R}^d)$, then $f \in BV^q((a, b), \mathbb{R}^d)$. Moreover, we have:

$$
\hat{A}_{f,q}((a, b)) \leq 4 (\hat{v}_{q,[a,b]}(f))^q. \quad (2.18)
$$

I.e. the space $\hat{V}_q([a, b], \mathbb{R}^d)$ is continuously embedded in $BV^q((a, b), \mathbb{R}^d)$.

The proof of Proposition 2.3 is given in the Appendix.

**Remark 2.4.** By Proposition 2.3 we have $\hat{V}_q([a, b], \mathbb{R}^d) \subset BV^q((a, b), \mathbb{R}^d)$. While for $q = 1$ it is well known that the two spaces coincide, the inclusion is strict when $q > 1$. Indeed, while $\hat{V}_q([a, b], \mathbb{R}^d) \subset L^\infty((a, b), \mathbb{R}^d)$, by Theorem 1.2 we have $W^{1,q}_{\mathcal{P}}((a, b), \mathbb{R}^d) \subset BV^q((a, b), \mathbb{R}^d)$ and it is well known that for $q > 1$, $W^{1,q}_{\mathcal{P}}((a, b), \mathbb{R}^d) \setminus L^\infty((a, b), \mathbb{R}^d) \neq \emptyset$.

### 2.2 Proofs of the main results for the space $BV^q$

We begin with two technical Lemmas that are used in the proof of Proposition 2.1.

**Lemma 2.2.** Let $\Omega \subset \mathbb{R}^N$ be an open set, $q \geq 1$ and let $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d)$. Then, for every open $\Omega_1 \subset \subset \Omega_2 \subset \subset \Omega$, for every $h_1 \in \mathbb{R}^N$ such that $0 < |h_1| \leq \text{dist}(\Omega_1, \mathbb{R}^N \setminus \Omega_2)$ and every $h_2 \in \mathbb{R}^N$ such that $0 < |h_2| \leq \text{dist}(\Omega_2, \mathbb{R}^N \setminus \Omega)$, we have

$$
\int_{\Omega_1} \frac{1}{|h_1 + h_2|} \left| u(x + (h_1 + h_2)) - u(x) \right|^q dx \leq \left( 2^{q-1} \left( \frac{|h_2|}{|h_1 + h_2|} \int_{\Omega_2} \frac{1}{|h_2|} \left| u(x + h_2) - u(x) \right|^q dx + \frac{|h_1|}{|h_1 + h_2|} \int_{\Omega_1} \frac{1}{|h_1|} \left| u(x + h_1) - u(x) \right|^q dx \right) \right).
$$

(2.19)
In particular, for every \( h \in S^{N-1} \), \( 0 < \varepsilon_1 \leq \text{dist}(\Omega_1, \mathbb{R}^N \setminus \Omega_2) \) and \( 0 < \varepsilon_2 \leq \text{dist}(\Omega_2, \mathbb{R}^N \setminus \Omega) \), we have

\[
\int_{\Omega} \frac{1}{\varepsilon_1 + \varepsilon_2} |u(x + (\varepsilon_1 + \varepsilon_2)h) - u(x)|^q \, dx \leq 2^{q-1} \max \left\{ \int_{\Omega_2} \frac{1}{\varepsilon_2} |u(x + \varepsilon_2 h) - u(x)|^q \, dx, \int_{\Omega_1} \frac{1}{\varepsilon_1} |u(x + \varepsilon_1 h) - u(x)|^q \, dx \right\}. \tag{2.20}
\]

**Proof.** By the triangle inequality and the convexity of \( g(s) := |s|^q \) we have

\[
\int_{\Omega_1} \frac{1}{h_1 + h_2} |u(x + (h_1 + h_2)) - u(x)|^q \, dx = \int_{\Omega_1} \frac{1}{h_1 + h_2} \left( |u(x + (h_1 + h_2)) - u(x + h_1)| + |u(x + h_1) - u(x)| \right)^q \, dx \leq \int_{\Omega_1} \frac{2^{q-1}}{|h_1| + |h_2|} |u(x + (h_1 + h_2)) - u(x + h_1)|^q \, dx + \int_{\Omega_1} \frac{2^{q-1}}{|h_1| + |h_2|} |u(x + h_1) - u(x)|^q \, dx \leq 2^{q-1} \left( \frac{|h_2|}{|h_1| + |h_2|} \int_{\Omega_2} \frac{1}{|h_2|} |u(x + h_2) - u(x)|^q \, dx + \frac{|h_1|}{|h_1| + |h_2|} \int_{\Omega_1} \frac{1}{|h_1|} |u(x + h_1) - u(x)|^q \, dx \right). \]

In particular, for every \( h \in S^{N-1} \), \( 0 < \varepsilon_1 \leq \text{dist}(\Omega_1, \mathbb{R}^N \setminus \Omega_2) \) and \( 0 < \varepsilon_2 \leq \text{dist}(\Omega_2, \mathbb{R}^N \setminus \Omega) \) we have

\[
\int_{\Omega_1} \frac{1}{\varepsilon_1 + \varepsilon_2} |u(x + (\varepsilon_1 + \varepsilon_2)h) - u(x)|^q \, dx \leq 2^{q-1} \left( \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \int_{\Omega_2} \frac{1}{\varepsilon_2} |u(x + \varepsilon_2 h) - u(x)|^q \, dx + \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \int_{\Omega_1} \frac{1}{\varepsilon_1} |u(x + \varepsilon_1 h) - u(x)|^q \, dx \right) \leq 2^{q-1} \max \left\{ \int_{\Omega_2} \frac{1}{\varepsilon_2} |u(x + \varepsilon_2 h) - u(x)|^q \, dx, \int_{\Omega_1} \frac{1}{\varepsilon_1} |u(x + \varepsilon_1 h) - u(x)|^q \, dx \right\}. \]

\[\square\]

**Lemma 2.3.** Let \( \Omega \subset \mathbb{R}^N \) be an open set, \( q \geq 1 \) and let \( u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d) \). Then, for every open \( \Omega_1 \subset \Omega_2 \subset \subset \Omega \), \( k \in S^{N-1} \) and \( \varepsilon \) satisfying

\[
0 < \varepsilon \leq \min \left\{ \text{dist}(\Omega_1, \mathbb{R}^N \setminus \Omega_2), \text{dist}(\Omega_2, \mathbb{R}^N \setminus \Omega) \right\}, \tag{2.21}
\]

we have

\[
\int_{\Omega_1} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)|^q \, dx \leq \frac{2^{N+q}}{C^N(B_1(0))} \int_{B_1(0)} \int_{\Omega_2} \frac{1}{\varepsilon|z|} |u(x + \varepsilon z) - u(x)|^q \, dx \, dz. \tag{2.22}
\]

Moreover, if \( u \in L^q(\mathbb{R}^N, \mathbb{R}^d) \) then

\[
\int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)|^q \, dx \leq \frac{2^{N+q}}{C^N(B_1(0))} \int_{B_1(0)} \int_{\mathbb{R}^N} \frac{1}{\varepsilon|z|} |u(x + \varepsilon z) - u(x)|^q \, dx \, dz. \tag{2.23}
\]
In particular,
\[
\frac{A_{u,q}(\Omega_1)}{\mathcal{L}^N(B_1(0))} \leq B_{u,q}(\Omega_1),
\]
(2.24)
and
\[
B_{u,q}(\Omega_1) \leq 2^{N+q} \frac{A_{u,q}(\Omega_2)}{\mathcal{L}^N(B_1(0))}
\]
(2.25)
(see Definitions 2.1 and 2.2). Moreover, if \( u \in L^q(\mathbb{R}^N, \mathbb{R}^d) \) then
\[
\frac{\hat{A}_{u,q}(\mathbb{R}^N)}{\mathcal{L}^N(B_1(0))} \leq \hat{B}_{u,q}(\mathbb{R}^N),
\]
(2.26)
and
\[
\hat{B}_{u,q}(\mathbb{R}^N) \leq 2^{N+q} \frac{\hat{A}_{u,q}(\mathbb{R}^N)}{\mathcal{L}^N(B_1(0))}.
\]
(2.27)

Proof. Inequalities (2.24) and (2.26) are clear from the definitions. Next, by (2.19) we have:
for every \( k \in \mathbb{R}^N \setminus \{0\} \) such that \( |k| \leq 1 \), for every \( z \in \mathbb{R}^N \) such that \( |z - \frac{1}{2}k| < \frac{1}{2} |k| \), \( \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega \) and \( \varepsilon \) satisfying (2.21) there holds
\[
\int_{\Omega_1} \frac{1}{|k|} \left| u(x + \varepsilon k) - u(x) \right|^q dx \leq 2^{q-1} \frac{|z|}{|k|} \int_{\Omega_2} \frac{1}{\varepsilon |z|} \left| u(x + \varepsilon z) - u(x) \right|^q dx + \frac{2^{q-1} |k - z|}{|k|} \int_{\Omega_1} \frac{1}{\varepsilon |k - z|} \left| u(x + \varepsilon (k - z)) - u(x) \right|^q dx.
\]
(2.28)
Since the inequality \( |z - \frac{1}{2}k| < \frac{1}{2} |k| \) implies the inequalities \( |z| < |k| \) and \( |k - z| < |k| \), we have by (2.28):
\[
\int_{\Omega_1} \frac{1}{|k|} \left| u(x + \varepsilon k) - u(x) \right|^q dx \leq \frac{2^{q-1}}{\mathcal{L}^N(\{z \in \mathbb{R}^N : |z - \frac{1}{2}k| < \frac{1}{2} |k|\})} \int_{\{z \in \mathbb{R}^N : |z - \frac{1}{2}k| < \frac{1}{2} |k|\}} \int_{\Omega_2} \frac{1}{\varepsilon |z|} \left| u(x + \varepsilon z) - u(x) \right|^q dx dz
\]
\[
+ \frac{2^{q-1}}{\mathcal{L}^N(B_1(0))} \int_{\{z \in \mathbb{R}^N : |z - \frac{1}{2}k| < \frac{1}{2} |k|\}} \int_{\Omega_1} \frac{1}{\varepsilon |k - z|} \left| u(x + \varepsilon (k - z)) - u(x) \right|^q dx dz
\]
\[
\leq \frac{2^{N+q-1}}{|k|^N \mathcal{L}^N(B_1(0))} \int_{\{z \in \mathbb{R}^N : |z| < |k|\}} \int_{\Omega_2} \frac{1}{\varepsilon |z|} \left| u(x + \varepsilon z) - u(x) \right|^q dx dz
\]
\[
+ \frac{2^{N+q-1}}{|k|^N \mathcal{L}^N(B_1(0))} \int_{\{z \in \mathbb{R}^N : |z - k| < |k|\}} \int_{\Omega_1} \frac{1}{\varepsilon |k - z|} \left| u(x + \varepsilon (k - z)) - u(x) \right|^q dx dz
\]
\[
= \frac{2^{N+q}}{|k|^N \mathcal{L}^N(B_1(0))} \int_{\Omega_1} \int_{\Omega_2} \frac{1}{|k||z|} \left| u(x + \varepsilon |k|z) - u(x) \right|^q dx dz,
\]
(2.29)
and (2.22) follows. Moreover, if \( u \in L^q(\mathbb{R}^N, \mathbb{R}^d) \) then taking the supremum of (2.22) over all \( \Omega_1 \subset \subset \Omega_2 \subset \subset \mathbb{R}^N \) we deduce (2.23).
Finally, from (2.29) we deduce that for every \( 0 < \rho \leq 1 \) we have
\[
\limsup_{\epsilon \to 0^+} \left( \sup_{\{k \in \mathbb{R}^N : |k| = \rho\}} \left( \int_{B_1(0)} \frac{1}{\epsilon^p} |u(x + \epsilon k) - u(x)|^q \right) \right) \leq \frac{2^{N+q}}{L^N(B_1(0))} \limsup_{\epsilon \to 0^+} \left( \int_{B_1(0)} \int_{\mathbb{R}^N} \frac{1}{\epsilon^p} |u(x + \epsilon z) - u(x)|^q \right) (2.30),
\]
which clearly implies (2.25). Moreover, if \( u \in L^q(\mathbb{R}^N, \mathbb{R}^d) \) then by (2.23) we have:
\[
\limsup_{\epsilon \to 0^+} \left( \sup_{\{k \in \mathbb{R}^N : |k| = 1\}} \left( \int_{\mathbb{R}^N} \frac{1}{\epsilon^p} |u(x + \epsilon k) - u(x)|^q \right) \right) \leq \frac{2^{N+q}}{L^N(B_1(0))} \limsup_{\epsilon \to 0^+} \left( \int_{B_1(0)} \int_{\mathbb{R}^N} \frac{1}{\epsilon^p} |u(x + \epsilon z) - u(x)|^q \right) (2.31),
\]
which clearly implies (2.27).

The next Proposition is a key ingredient in the proof of Theorem 2.1.

**Proposition 2.4.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and let \( W(a, b) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be a nonnegative continuously differentiable function, which satisfies \( W(a, a) = 0 \) and \( W(a, b) = W(b, a) \) for every \( a, b \in \mathbb{R}^d \). Let \( u \in BV_{loc}(\Omega, \mathbb{R}^d) \cap L^\infty_{loc}(\Omega, \mathbb{R}^d) \). Then, for every compact set \( K \subset \subset \Omega \) such that \( ||D\nu||(\partial K) = 0 \) and any vector \( k \in \mathbb{R}^N \) we have
\[
\lim_{t \to 0^+} \frac{1}{t} \int_K W(u(x + tk), u(x)) dx = \int_{J_u \cap K} W(u^+, u^-) |k \cdot \nu(x)| d\mathcal{H}^{N-1}(x). (2.32)
\]
In particular, for \( q > 1 \) we have
\[
\lim_{t \to 0^+} \frac{1}{t} \int_K |u(x + tk) - u(x)|^q dx = \int_{J_u \cap K} |u^+(x) - u^-(x)|^q |k \cdot \nu(x)| d\mathcal{H}^{N-1}(x), (2.33)
\]
and
\[
A_{u,q}(K) = \lim_{t \to 0^+} \int_{B_1(0)} \int_K \frac{1}{t|z|} |u(x + tz) - u(x)|^q dx dz
\]
\[
= C_N \left( \int_{J_u \cap K} |u^+(x) - u^-(x)|^q d\mathcal{H}^{N-1}(x) \right), (2.34)
\]
with \( C_N \) defined in (1.14).

**Proof.** Let \( \eta(z) \in C_c^\infty(\mathbb{R}^N, \mathbb{R}) \) be a radial function such that \( \eta \geq 0 \), \( \text{supp} \eta \subset B_1(0) \) and \( \int_{\mathbb{R}^N} \eta(z) dz = 1 \). For every \( x \in \Omega \) and every \( 0 < \epsilon < \text{dist}(x, \partial \Omega) \) define
\[
u(\epsilon) := \frac{1}{\epsilon^N} \int_{\mathbb{R}^N} \eta \left( \frac{y - x}{\epsilon} \right) u(y) dy = \int_{\mathbb{R}^N} \eta(z) u(x + \epsilon z) dz = \int_{B_1(0)} \eta(z) u(x + \epsilon z) dz. (2.35)
\]
Then, following definition \( A.2 \) we have
\[
\lim_{\epsilon \to 0^+} u_{\epsilon}(x) = \bar{u}(x) := \begin{cases} \bar{u}(x) & x \in \Omega \setminus J_u \\ \frac{1}{2}(u^+(x) + u^-(x)) & x \in J_u \end{cases} \quad \text{for } \mathcal{H}^{N-1}-a.e. \ x \in \Omega. (2.36)
\]
Moreover, since there exist two open sets $U_1 \subset \subset U_2 \subset \subset \Omega$ such that $K \subset \subset U_1$ and $u \in L^\infty(U_2, \mathbb{R}^d)$, we deduce that there exist constants $M > 0$ and $\varepsilon_0 > 0$, such that
\begin{equation}
\begin{cases}
|u_\varepsilon(x)| \leq M & \forall x \in U_1, \forall \varepsilon \in (0, \varepsilon_0), \\
|\bar{u}(x)| \leq M & \text{for } \mathcal{H}^{N-1} - \text{a.e. } x \in U_1.
\end{cases}
\tag{2.37}
\end{equation}

Then, denoting for any $s \in [0, 1]$, $x \in \Omega$ and $k \subset \mathbb{R}^N$
\[ P_t(u_\varepsilon, x, s, k) = s u_\varepsilon(x + t k) + (1 - s) u_\varepsilon(x), \]
using the Dominated Convergence Theorem, the Fundamental Theorem of Calculus and finally
\[ (2.35), \] we get for small $t > 0$,
\begin{align*}
I_t := \frac{1}{t} \int_K W\left(u(x + t k), u(x)\right) dx &= \frac{1}{t} \lim_{\varepsilon \to 0^+} \int_K W\left(u_\varepsilon(x + t k), u_\varepsilon(x)\right) dx = \\
\frac{1}{t} \lim_{\varepsilon \to 0^+} \int_K \int_0^1 \nabla_a W\left(P_t(u_\varepsilon, x, s, k), u_\varepsilon(x)\right) \cdot (u_\varepsilon(x + t k) - u_\varepsilon(x)) ds dx = \\
\lim_{\varepsilon \to 0^+} \int_0^1 \int_K \nabla_a W\left(P_t(u_\varepsilon, x, s, k), u_\varepsilon(x)\right) \cdot \left( \int_{\mathbb{R}^N} \left\{ \eta\left(\frac{y - x - t k}{\varepsilon}\right) - \eta\left(\frac{y - x}{\varepsilon}\right) \right\} u(y)dy \right) ds dx.
\end{align*}
\tag{2.38}

Next, by \[ (2.38), \] the Fundamental Theorem of Calculus, Fubini theorem and integration by parts we obtain,
\begin{align*}
I_t &= \\
- \lim_{\varepsilon \to 0^+} \int_K \int_0^1 \nabla_a W\left(P_t(u_\varepsilon, x, s, k), u_\varepsilon(x)\right) \cdot \frac{1}{\varepsilon^{N+1}} \left( \int_{\mathbb{R}^N} \left\{ \int_0^1 k \cdot \nabla \eta\left(\frac{y - x - \tau t k}{\varepsilon}\right) d\tau \right\} u(y)dy \right) ds dx \\
= \lim_{\varepsilon \to 0^+} \int_0^1 \int_K \nabla_a W\left(P_t(u_\varepsilon, x, s, k), u_\varepsilon(x)\right) \cdot \frac{1}{\varepsilon^N} \left( \int_{\mathbb{R}^N} \eta\left(\frac{y - x - \tau t k}{\varepsilon}\right) d\left[ Du(y) \cdot k \right] \right) ds dx d\tau.
\end{align*}
\tag{2.39}

By \[ (2.39), \] using Fubini Theorem, we deduce for small $t > 0$,
\begin{align*}
I_t &= \\
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \left\{ \int_0^1 \int_K \int_0^1 \eta\left(\frac{y - x - \tau t k}{\varepsilon}\right) \nabla_a W\left(P_t(u_\varepsilon, x, s, k), u_\varepsilon(x)\right) ds dx d\tau \right\} \cdot d\left[ Du(y) \cdot k \right] \\
= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N} \left\{ \int_0^1 \int_K \int_0^1 \eta\left(\frac{x - y - t k}{\varepsilon}\right) \nabla_a W\left(P_t(u_\varepsilon, y, s, k), u_\varepsilon(y)\right) ds dy d\tau \right\} \cdot d\left[ Du(x) \cdot k \right].
\end{align*}
\tag{2.40}

Performing a change of variables on the r.h.s. of \[ (2.40), \] using Fubini theorem and denoting for short
\[ y = y(\varepsilon, x, z, \tau, k) = x - \varepsilon z - \tau t k, \]
\tag{2.41}
we infer

\[
I_t = \lim_{\varepsilon \to 0^+} \frac{1}{0} \int \eta (z) \left\{ \int \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) ds \right\} \cdot d \left[ Du(x) \cdot k \right] dz d\tau
\]

\[
= O \left( \| Du \| \left( \cup_{\tau \in [0,1]} (\partial K + \tau tk) \right) \right) +
\]

\[
\lim_{\varepsilon \to 0^+} \frac{1}{0} \int \eta (z) \left\{ \int \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) ds \right\} \cdot d \left[ Du(x) \cdot k \right] dz d\tau
\]

\[
= O \left( \| Du \| \left( \cup_{\tau \in [0,1]} (\partial K + \tau tk) \right) \right) +
\]

\[
\lim_{\varepsilon \to 0^+} \frac{1}{0} \int \eta (z) \left\{ \int \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) dz ds \right\} \cdot d \left[ Du(x) \cdot k \right] d\tau
\]

Using the easy to check fact that

\[
\int \left[ D_{\tau} \cap J_u \right] d \left[ Du(x) \cdot k \right] = \int \left[ D_{\tau} \cap J_u \right] \left[ u^+(x) - u^-(x) \right] |k \cdot \nu(x)| dH^{N-1}(x) = 0 \text{ for a.e. } \tau \in (0, 1),
\]

where \( D_{\tau} := (J_u + \tau tk) \cup (J_u - (1 - \tau)tk) \),

we decompose (2.42) as:

\[
I_t = O \left( \| Du \| \left( \cup_{\tau \in [0,1]} (\partial K + \tau tk) \right) \right) +
\]

\[
\lim_{\varepsilon \to 0^+} \frac{1}{0} \int \left( \int \eta (z) \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) dz ds \right) \cdot d \left[ Du(x) \cdot k \right] d\tau
\]

\[
+ \lim_{\varepsilon \to 0^+} \frac{1}{0} \int \left( \int \eta (z) \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) dz ds \right) \cdot d \left[ Du(x) \cdot k \right] d\tau
\]

\[
= O \left( \| Du \| \left( \cup_{\tau \in [0,1]} (\partial K + \tau tk) \right) \right) +
\]

\[
\lim_{\varepsilon \to 0^+} \frac{1}{0} \int \left( \int \eta (z) \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) dz ds \right) \cdot d \left[ Du(x) \cdot k \right] d\tau
\]

\[
+ \lim_{\varepsilon \to 0^+} \frac{1}{0} \int \left( \int \eta (z) \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) dz ds \right) \cdot d \left[ Du(x) \cdot k \right] d\tau
\]

\[
+ \lim_{\varepsilon \to 0^+} \frac{1}{0} \int \left( \int \eta (z) \nabla_a W \left( P_t(u_{\varepsilon}, y, s, k), u_{\varepsilon}(y) \right) dz ds \right) \cdot d \left[ Du(x) \cdot k \right] d\tau.
\]
On the other hand, by (2.35) we obtain that for \( \mathcal{H}^{N-1} \)-a.e. \( x \in \Omega \setminus J_u \), for every \( z \in \mathbb{R}^N \) and for every small \( \varepsilon > 0 \) we have

\[
|u_\varepsilon(x-\varepsilon z) - \tilde{u}(x)| = \left| \int_{\mathbb{R}^N} \eta(y) \left( u(x-\varepsilon z + \varepsilon y) - \tilde{u}(x) \right) dy \right| = \left| \int_{\mathbb{R}^N} \eta(y+z) \left( u(x+y) - \tilde{u}(x) \right) dy \right|
\]

\[
= \left| \int_{B(1+|z|)(0)} \eta(y+z) \left( u(x+y) - \tilde{u}(x) \right) dy \right| \leq \left( \sup_{y \in \mathbb{R}^N} |\eta(y)| \right) \left( \int_{B(1+|z|)(0)} |u(x+y) - \tilde{u}(x)| dy \right).
\]

Then, by the definition of the approximate limit, for every \( z \in \mathbb{R}^N \) we deduce

\[
\lim_{\varepsilon \to 0^+} u_\varepsilon(x-\varepsilon z) = \tilde{u}(x) = \tilde{u}(x) \quad \text{for } \mathcal{H}^{N-1} \text{-a.e. } x \in \Omega \setminus J_u
\]

(2.46)

(where \( \tilde{u}(x) \) was defined in (2.36)). In particular, by (2.46) for every small \( t > 0 \) and for every \( \tau \in (0,1) \) we have:

\[
\lim_{\varepsilon \to 0^+} u_\varepsilon(x-\varepsilon z - \tau t k) = \tilde{u}(x - \tau t k) \quad \text{for } \mathcal{H}^{N-1} \text{-a.e. } x \in (K + \tau t k) \setminus (J_u + \tau t k) \quad (2.47)
\]

and

\[
\lim_{\varepsilon \to 0^+} u_\varepsilon \left( x - \varepsilon z - \tau t k + t k \right) = \tilde{u} \left( x + (1 - \tau)t k \right) \quad \text{for } \mathcal{H}^{N-1} \text{-a.e. } x \in (K + \tau t k) \setminus (J_u - (1 - \tau)t k). \quad (2.48)
\]

Then, using (2.47), (2.48), (2.37), Dominated Convergence and (2.43), yields

\[
\lim_{\varepsilon \to 0^+} \int_0^1 \left( \int_{((K+\tau t k)\cap J_u)\setminus D_\varepsilon} \left\{ \int_{\mathbb{R}^N} \eta(z) \nabla_a W \left( P_t(u_\varepsilon, y, s, k), u_\varepsilon(y) \right) ds \right\} \cdot d\left[ Du(x) \cdot k \right] \right) d\tau =
\]

\[
\int_0^1 \int_{((K+\tau t k)\cap J_u)\setminus D_\varepsilon} \left\{ \int_{\mathbb{R}^N} \nabla_a W \left( s \tilde{u}(x+(1-\tau)t k)+(1-s)\tilde{u}(x-\tau t k), \tilde{u}(x-\tau t k) \right) ds \right\} \cdot d\left[ Du(x) \cdot k \right] d\tau
\]

\[
= \int_0^1 \int_{(K+\tau t k)\cap J_u} \left\{ \int_{\mathbb{R}^N} \nabla_a W \left( s \tilde{u}(x+(1-\tau)t k)+(1-s)\tilde{u}(x-\tau t k), \tilde{u}(x-\tau t k) \right) ds \right\} \cdot d\left[ Du(x) \cdot k \right] d\tau.
\]

(2.49)

Similarly, using (2.47), (2.48), (2.37) and Dominated Convergence yields

\[
\lim_{\varepsilon \to 0^+} \int_0^1 \left( \int_{((K+\tau t k)\cap (J_u \cup D_\varepsilon))} \left\{ \int_{\mathbb{R}^N} \eta(z) \nabla_a W \left( P_t(u_\varepsilon, y, s, k), u_\varepsilon(y) \right) ds \right\} \cdot d\left[ Du(x) \cdot k \right] \right) d\tau =
\]

\[
\int_0^1 \int_{((K+\tau t k)\cap (J_u \cup D_\varepsilon))} \left\{ \int_{\mathbb{R}^N} \nabla_a W \left( s \tilde{u}(x+(1-\tau)t k)+(1-s)\tilde{u}(x-\tau t k), \tilde{u}(x-\tau t k) \right) ds \right\} \cdot d\left[ Du(x) \cdot k \right] d\tau.
\]

(2.50)
On the other hand, since the set \((K + \tau k) \cap D_\tau\) is \(\mathcal{H}^{N-1}\) \(\sigma\)-finite, by Theorem A.2 we have

\[
\int_{((K + \tau k) \cap D_\tau) \setminus J_u} d|Du(x) \cdot k| = 0,
\]

and in particular,

\[
\lim_{\varepsilon \to 0^+} \int_0^1 \left( \int_{((K + \tau k) \cap D_\tau) \setminus J_u} \left\{ \int_{\mathbb{R}^N} \eta(z) \nabla_a W \left( P_t(u_\varepsilon, y, s, k), u_\varepsilon(y) \right) dz \right\} \cdot d \left[ Du(x) \cdot k \right] \right) d\tau = 0.
\]

Thus, inserting (2.49), (2.50) and (2.52) into (2.44) yields

\[
I_t = O \left( \left\| Du \right\| \left( \bigcup_{\tau \in [0,1]} (\partial K + \tau k) \right) \right) + \int_0^1 \int_0^1 \left\{ \int_{(K + \tau k) \setminus J_u} \nabla_a W \left( s \bar{u} \left( x + (1-\tau)k \right) + (1-s)\bar{u} \left( x - \tau k \right) \right) ds \right\} \cdot d \left[ Du(x) \cdot k \right] d\tau + \int_0^1 \int_0^1 \left\{ \int_{(K + \tau k) \setminus (J_u \cup D_\tau)} \nabla_a W \left( s \bar{u} \left( x + (1-\tau)k \right) + (1-s)\bar{u} \left( x - \tau k \right) \right) ds \right\} \cdot d \left[ Du(x) \cdot k \right] d\tau.
\]

Then, using again (2.51) in (2.53) we get

\[
I_t = O \left( \left\| Du \right\| \left( \bigcup_{\tau \in [0,1]} (\partial K + \tau k) \right) \right) + \int_0^1 \int_0^1 \left\{ \int_{K + \tau k} \nabla_a W \left( s \bar{u} \left( x + (1-\tau)k \right) + (1-s)\bar{u} \left( x - \tau k \right) \right) ds \right\} \cdot d \left[ Du(x) \cdot k \right] d\tau.
\]

On the other hand, by Theorem 3.108 and Remark 3.109 from [2] we deduce that

\[
\begin{align*}
\lim_{\rho \to 0^+} & \int_0^1 \left( \left| \bar{u}(x + \rho sk) - \bar{u}(x) \right| + \left| \bar{u}(x - \rho sk) - \bar{u}(x) \right| \right) ds = 0, \\
\mathcal{H}^{N-1} \text{ a.e. in } & \Omega \setminus J_u \\
\lim_{\rho \to 0^+} & \int_0^1 \left( \left| \bar{u}(x + \rho sk) - u^+(x) \right| + \left| \bar{u}(x - \rho sk) - u^-(x) \right| \right) ds = 0, \\
\mathcal{H}^{N-1} \text{ a.e. in } & \left\{ x \in J_u : k \cdot \nu(x) > 0 \right\} \\
\lim_{\rho \to 0^+} & \int_0^1 \left( \left| \bar{u}(x + \rho sk) - u^-(x) \right| + \left| \bar{u}(x - \rho sk) - u^+(x) \right| \right) ds = 0, \\
\mathcal{H}^{N-1} \text{ a.e. in } & \left\{ x \in J_u : k \cdot \nu(x) < 0 \right\}.
\end{align*}
\]
Thus, since $\|Du\|(\partial K) = 0$, by (2.54), the first equation in (2.55), Dominated Convergence and the properties $W(a, b) = W(b, a)$, $W(a, a) = 0$ and $\nabla W(a, a) = 0$ (since $W \geq 0$), we obtain

$$
\lim_{t \to 0^+} I_t = \lim_{t \to 0^+} \frac{1}{t} \int_W \left( u(x + t k), u(x) \right) dx = \int \int_{J_a \cap K} \left\{ \int_0^1 \nabla_a W \left( s \bar{u}(x + (1 - \tau)t k) + (1 - s)\bar{u}(x - \tau t k), \bar{u}(x - \tau t k) \right) ds \right\} \cdot d \left[ Du(x) \cdot k \right] d\tau.
$$

Then by inserting the second two equations in (2.55) into (2.56) and using Dominated Convergence and Theorem A.2 we deduce:

$$
\lim_{t \to 0^+} I_t = \int \int_{J_a \cap K} \left\{ \int_0^1 \nabla_a W \left( s u^+(x) + (1 - s)u^-(x), u^-(x) \right) \cdot (u^+(x) - u^-(x)) \left( \max \left\{ k \cdot \nu(x), 0 \right\} \right) d\nu d\mathcal{H}^{N-1}(x) + \int \int_{J_a \cap K} \nabla_a W \left( s u^-(x) + (1 - s)u^+(x), u^+(x) \right) \cdot (u^-(x) - u^+(x)) \left( \max \left\{ -k \cdot \nu(x), 0 \right\} \right) d\nu d\mathcal{H}^{N-1}(x). \right.
$$

Then, using the Fundamental Theorem of Calculus in (2.57) gives

$$
\lim_{t \to 0^+} I_t = \int_{J_a \cap K} W \left( u^+(x), u^-(x) \right) \left( \max \left\{ k \cdot \nu(x), 0 \right\} \right) d\mathcal{H}^{N-1}(x) + \int_{J_a \cap K} W \left( u^-(x), u^+(x) \right) \left( \max \left\{ -k \cdot \nu(x), 0 \right\} \right) d\mathcal{H}^{N-1}(x) = \int_{J_a \cap K} W \left( u^+(x), u^-(x) \right) |k \cdot \nu(x)| d\mathcal{H}^{N-1}(x). \tag{2.58}
$$

The desired estimate (2.32) follows immediately from (2.58) and (2.33) is deduced from the particular case $W(a, b) = |a - b|^q$. Moreover, for any compact set $K \subset \subset \Omega$, we can choose $\Omega_1 \subset \subset \Omega$ such that $K \subset \subset \Omega_1$ and then for every small $t > 0$ we clearly have

$$
0 \leq \frac{1}{t} \int_K \left| u(x + t k) - u(x) \right|^q dx \leq 2^{q-1} \|u\|_{L^\infty(K)}^{q-1} \int_K \frac{1}{t} \left| u(x + t k) - u(x) \right| dx \leq 2^{q-1} \|u\|_{L^\infty(K)}^{q-1} \|u\|_{BV(\Omega_1)}. \tag{2.59}
$$
Thus by dominated convergence we get

$$A_{u,q}(K) = \lim_{t \to 0^+} \left( \int_{B_1(0)} \int_K \frac{1}{t|z|} |u(x + tz) - u(x)|^q \, dx \, dz \right)$$

$$= \left( \int_{B_1(0)} \left( \int_{J_u \cap K} |u^+(x) - u^-(x)|^q \frac{z}{|z|} \cdot \nu(x) \, d\mathcal{H}^{N-1}(x) \right) \, dz \right) = \left( \int_{B_1(0)} \frac{|z_1|}{|z|} \, dz \right) \left( \int_{J_u \cap K} |u^+(x) - u^-(x)|^q \, d\mathcal{H}^{N-1}(x) \right)$$

$$= \left( \frac{1}{N} \int_{S^{N-1}} |z_1| \, d\mathcal{H}^{N-1}(z) \right) \left( \int_{J_u \cap K} |u^+(x) - u^-(x)|^q \, d\mathcal{H}^{N-1}(x) \right), \quad (2.60)$$

and (2.34) follows. \hfill \square

**Proof of Theorem 2.7** Identities (2.10) and (2.11) follow from Proposition 2.4. For every $k \in S^{N-1}$, every open $\Omega_1 \subset \Omega$ such that $u \in BV(\Omega_1, \mathbb{R}^d) \cap L^\infty(\Omega_1, \mathbb{R}^d)$, every $K \subset \subset \Omega_1$ and $0 < t < \text{dist}(K, \mathbb{R}^N \setminus \Omega_1)$ we have

$$0 \leq \frac{1}{t} \int_K |u(x + tk) - u(x)|^q \, dx \leq 2^{q-1} \|u\|_{L^\infty(K)}^{q-1} \int_K \frac{1}{t} |u(x + tk) - u(x)| \, dx \leq 2^{q-1} \|u\|_{L^\infty(K)}^{q-1} \|u\|_{BV(\Omega_1)}. \quad (2.61)$$

Therefore, we obtain $BV_{loc}(\Omega, \mathbb{R}^d) \cap L^\infty_{loc}(\Omega, \mathbb{R}^d) \subset BV^q_{loc}(\Omega, \mathbb{R}^d)$.

Finally, if $\Omega$ is an open set with bounded Lipschitz boundary and $u \in BV(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$, then we can extend the function $u(x)$ to all of $\mathbb{R}^N$ in such a way that $u \in BV(\mathbb{R}^N, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^d)$ and $\|Du\|(\partial \Omega) = 0$. Next in the case of bounded $\Omega$ clearly, we have

$$A_{u,q}(\Omega) \leq \hat{A}_{u,q}(\Omega) \leq A_{u,q}(\overline{\Omega}). \quad (2.62)$$

Thus, since $\|Du\|(\partial \Omega) = 0$, combining (2.62) together with (2.10) and (2.11) yields $A_{u,q}(\Omega) = \hat{A}_{u,q}(\Omega) = A_{u,q}(\overline{\Omega})$, and in particular, $u \in BV^q(\Omega, \mathbb{R}^d)$. On the hand, if $\Omega$ is unbounded consider a strictly increasing positive sequence $R_n \uparrow \infty$, such that $\|Du\|(\partial \Omega \cup \partial B_{R_n}(0)) = 0$. Then, similarly to (2.61) we have

$$A_{u,q}(\Omega) \leq \hat{A}_{u,q}(\Omega) \leq A_{u,q}(\overline{\Omega} \cap \overline{B}_{R_{n+2}}(0)) + A_{u,q}(\mathbb{R}^N \setminus \overline{B}_{R_{n+1}}(0))$$

$$\leq A_{u,q}(\overline{\Omega} \cap \overline{B}_{R_{n+2}}(0)) + \frac{1}{L^N(B_1(0))} B_{u,q}(\mathbb{R}^N \setminus \overline{B}_{R_{n+1}}(0))$$

$$\leq A_{u,q}(\overline{\Omega} \cap \overline{B}_{R_{n+2}}(0)) + \frac{2^{q-1}}{L^N(B_1(0))} \|u\|_{L^\infty(\mathbb{R}^N)}^{q-1} \|u\|_{BV(\mathbb{R}^N \setminus \overline{B}_{R_n}(0))}. \quad (2.63)$$

Thus letting $n$ tend to $\infty$ in (2.63) and using (2.10) and (2.11) again yields

$$A_{u,q}(\Omega) = \hat{A}_{u,q}(\Omega) = \lim_{n \to \infty} A_{u,q}(\overline{\Omega} \cap \overline{B}_{R_{n+2}}(0)),$$

that completes the proof. \hfill \square
The next Lemma contains the main ingredient of the proof of Theorem 1.2.

**Lemma 2.4.** For any open set $\Omega \subset \mathbb{R}^N$, $q > 1$ and $u \in W^{1,q}_0(\Omega, \mathbb{R}^d)$ we have $u \in BV^q(\Omega, \mathbb{R}^d)$.

Moreover,

$$
\tilde{A}_{u,q}(\Omega) \leq \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+1}} dy dx \tag{2.64}
$$

and

$$
\tilde{A}_{u,q}(\Omega) = 0. \tag{2.65}
$$

**Proof.** For every $\varepsilon \in (0, 1)$ we have

$$
\infty > T = \int_\Omega \left( \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+1}} dy \right) dx \geq \int_\Omega \left( \int_{B_\varepsilon(x) \cap \Omega} \frac{|u(x) - u(y)|^q}{|x - y|^{N+1}} dy \right) dx \\
\geq \int_\Omega \left( \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} |u(x) - u(y)|^q dy \right) dx. \tag{2.66}
$$

In particular, we deduce (2.64). Next, by (2.66) we infer

$$
\limsup_{\varepsilon \to 0^+} \left( \int_\Omega \left( \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} |u(x) - u(y)|^q dy \right) dx \right) \leq
\limsup_{\varepsilon \to 0^+} \left( \int_\Omega \left( \int_{B_\varepsilon(x) \cap \Omega} |u(x) - u(y)|^q dy \right) dx \right) \leq \int_\Omega \left( \int_{\Omega} |u(x) - u(y)|^q dy \right) dx < \infty. \tag{2.67}
$$

On the other hand, dominated convergence implies that

$$
\limsup_{\varepsilon \to 0^+} \left( \int_\Omega \left( \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} |u(x) - u(y)|^q dy \right) dx \right) = 0.
$$

Plugging the above in (2.67) yields

$$
\limsup_{\varepsilon \to 0^+} \left( \int_\Omega \left( \int_{B_\varepsilon(x) \cap \Omega} \frac{1}{\varepsilon^N} |u(x) - u(y)|^q dy \right) dx \right) = 0, \tag{2.68}
$$

and (2.65) follows. \qed

We recall below the definitions of the spaces $B$ and $B_0$ from [6].

**Definition 2.4.** For every $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and every $\varepsilon > 0$ consider the $\varepsilon$-cube:

$$
Q_\varepsilon(x) := \left\{ z = (z_1, \ldots, z_N) \in \mathbb{R}^N : |z_j - x_j| < \frac{\varepsilon}{2} \right\}. \tag{2.69}
$$

Then, for any open set $\Omega \subset \mathbb{R}^N$ and any $\varepsilon > 0$ denote by $\mathcal{R}_\varepsilon(\Omega)$ the set of all collections of disjoint $\varepsilon$-cubes $\{Q_\varepsilon(x_j)\}_{j=1}^m$ contained in $\Omega$ with $m \in \left[ 0, \frac{1}{\varepsilon^{N-1}} \right]$, such that $\bigcup_{j=1}^m Q_\varepsilon(x_j) \subseteq \Omega$.
and $Q_\varepsilon(x_k) \cap Q_\varepsilon(x_j) = \emptyset$ whenever $k \neq j$. Furthermore, for every small $\varepsilon > 0$ and every $u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^d)$ define

$$[u]_\varepsilon(\Omega) := \sup \left\{ \sum_{j=1}^m \left( \frac{\varepsilon^{N-1}}{(\mathcal{L}^N(Q_\varepsilon(x_j)))^2} \right) \int_{Q_\varepsilon(x_j)} \left| u(y) - u(x) \right| dy dx : \{Q_\varepsilon(x_j)\}_{j=1}^m \in \mathcal{R}_\varepsilon(\Omega) \right\},$$

(2.70)

$$|u|_{B(\Omega, \mathbb{R}^d)} := \sup_{\varepsilon \in (0, 1)} [u]_\varepsilon(\Omega),$$

(2.71)

and

$$[u](\Omega) := \limsup_{\varepsilon \to 0^+} [u]_\varepsilon(\Omega).$$

(2.72)

Define the spaces

$$B(\Omega, \mathbb{R}^d) := \{ u \in L^1(\Omega, \mathbb{R}^d) : |u|_{B(\Omega, \mathbb{R}^d)} < \infty \} = \{ u \in L^1(\Omega, \mathbb{R}^d) : |u|(\Omega) < \infty \},$$

$$B_0(\Omega, \mathbb{R}^d) := \{ u \in L^1(\Omega, \mathbb{R}^d) : |u|(\Omega) = 0 \} \subset B(\Omega, \mathbb{R}^d).$$

Then, $B(\Omega, \mathbb{R}^d)$ is a normed linear space with the norm

$$\|u\|_{B(\Omega, \mathbb{R}^d)} := |u|_{B(\Omega, \mathbb{R}^d)} + \|u\|_{L^1(\Omega, \mathbb{R}^d)},$$

(2.74)

and $B_0(\Omega, \mathbb{R}^d)$ is a closed subspace of $B(\Omega, \mathbb{R}^d)$.

**Lemma 2.5.** For any open set $\Omega \subset \mathbb{R}^N$, $q \geq 1$, $u \in L^q(\Omega, \mathbb{R}^d)$, $\varepsilon > 0$, an integer $m \in \left[ 0, \frac{1}{2N} \right]$ and arbitrary $m$ points \( \{x_j\}_{j=1}^m \subset \Omega \), such that $\bigcup_{j=1}^m Q_\varepsilon(x_j) \subset \Omega$ and $Q_\varepsilon(x_k) \cap Q_\varepsilon(x_j) = \emptyset$ for $k \neq j$, we have

$$\sum_{j=1}^m \varepsilon^{N-1} \left( \frac{1}{(\mathcal{L}^N(Q_\varepsilon(x_j)))^2} \right) \int_{Q_\varepsilon(x_j)} \int_{Q_\varepsilon(x_j)} \left| u(y) - u(x) \right| dy dx \leq N^\frac{N+1}{2N} \left( \int_{\Omega_{B(\varepsilon)} \cap \Omega} \frac{1}{(\varepsilon')^N} \frac{1}{|y-x|} \frac{1}{|y-x|} |u(y) - u(x)|^q dy dx \right)^{\frac{1}{q}},$$

(2.75)

where $\varepsilon' := \varepsilon \sqrt{N}$.

**Proof.** By Hölder inequality, we have

$$\left( \frac{1}{(\mathcal{L}^N(Q_\varepsilon(x_j)))^2} \right) \int_{Q_\varepsilon(x_j)} \int_{Q_\varepsilon(x_j)} \left| u(y) - u(x) \right| dy dx \leq$$

$$\left( \frac{1}{(\mathcal{L}^N(Q_\varepsilon(x_j)))^2} \right) \int_{Q_\varepsilon(x_j)} \int_{Q_\varepsilon(x_j)} \left| u(y) - u(x) \right|^q dy dx \right)^{\frac{1}{q}} =$$

$$\left( \frac{1}{\varepsilon^{2N}} \int_{Q_\varepsilon(x)} \int_{Q_\varepsilon(x)} \left| u(y) - u(x) \right|^q dy dx \right)^{\frac{1}{q}} \leq \left( \frac{\varepsilon^{N}}{\varepsilon^{N-1}} \int_{Q_\varepsilon(x)} \int_{Q_\varepsilon(x)} \frac{1}{\varepsilon^{N}} \frac{1}{|y-x|} \frac{1}{|y-x|} |u(y) - u(x)|^q dy dx \right)^{\frac{1}{q}}.$$ 

(2.76)
On the other hand, by the Hölder’s inequality (on finite sums) we have
\[
\left( \sum_{j=1}^{m} s_j^\frac{q}{q-1} \right)^{\frac{q-1}{q}} \leq m^{\frac{q-1}{q}} \left( \sum_{j=1}^{m} s_j \right)^{\frac{1}{q}}.
\] (2.77)

Therefore, by (2.76) and (2.77) we have
\[
\sum_{j=1}^{m} \varepsilon^{N-1} \left( \frac{1}{\mathcal{L}^N(Q_{\varepsilon}(x_j))} \right)^2 \int_{Q_{\varepsilon}(x_j)} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N-1}} \frac{1}{|y-x|} |u(y) - u(x)| \, dy \, dx
\]
\[
\leq \sum_{j=1}^{m} \varepsilon^{N-1} \left( \frac{\sqrt{N}}{\varepsilon^{N-1}} \int_{Q_{\varepsilon}(x_j)} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N}} \frac{1}{|y-x|} |u(y) - u(x)|^q \, dy \, dx \right)^{\frac{1}{q}}
\]
\[
\leq \varepsilon^{N-1} m^{\frac{q-1}{q}} \left( \sum_{j=1}^{m} \frac{\sqrt{N}}{\varepsilon^{N-1}} \int_{Q_{\varepsilon}(x_j)} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N}} \frac{1}{|y-x|} |u(y) - u(x)|^q \, dy \, dx \right)^{\frac{1}{q}}
\]
\[
= \left( \varepsilon^{N-1} m \right)^{\frac{q-1}{q}} N^{\frac{1}{q}} \left( \sum_{j=1}^{m} \int_{Q_{\varepsilon}(x_j)} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N}} \frac{1}{|y-x|} |u(y) - u(x)|^q \, dy \, dx \right)^{\frac{1}{q}}. (2.78)
\]

By (2.78) and our assumption \( m \leq \frac{1}{\varepsilon^{N-1}} \) it follows that
\[
\sum_{j=1}^{m} \varepsilon^{N-1} \left( \frac{1}{\mathcal{L}^N(Q_{\varepsilon}(x_j))} \right)^2 \int_{Q_{\varepsilon}(x_j)} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N-1}} \frac{1}{|y-x|} |u(y) - u(x)| \, dy \, dx
\]
\[
\leq N^{\frac{1}{q}} \left( \sum_{j=1}^{m} \int_{Q_{\varepsilon}(x_j)} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N}} \frac{1}{|y-x|} |u(y) - u(x)|^q \, dy \, dx \right)^{\frac{1}{q}}. (2.79)
\]

Since for every \( x \in Q_{\varepsilon}(x_j) \) we have \( Q_{\varepsilon}(x_j) \subset B(\varepsilon \sqrt{N}) (x) \cap \Omega \), we get from (2.79) that
\[
\sum_{j=1}^{m} \varepsilon^{N-1} \left( \frac{1}{\mathcal{L}^N(Q_{\varepsilon}(x_j))} \right)^2 \int_{Q_{\varepsilon}(x_j)} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N-1}} \frac{1}{|y-x|} |u(y) - u(x)| \, dy \, dx
\]
\[
\leq N^{\frac{1}{q}} \left( \sum_{j=1}^{m} \int_{Q_{\varepsilon}(x_j)} \frac{1}{\varepsilon^{N}} \frac{1}{|y-x|} |u(y) - u(x)|^q \, dy \, dx \right)^{\frac{1}{q}} =
\]
\[
N^{\frac{N+1}{Nq}} \left( \sum_{j=1}^{m} \int_{B(\varepsilon \sqrt{N}) (x) \cap \Omega} \frac{1}{(\varepsilon \sqrt{N})^N} \frac{1}{|y-x|} |u(y) - u(x)|^q \, dy \, dx \right)^{\frac{1}{q}}. (2.80)
\]
Since \( \bigcup_{j=1}^{m} Q_\varepsilon(x_j) \subset \Omega \) and \( Q_\varepsilon(x_k) \cap Q_\varepsilon(x_j) \) whenever \( k \neq j \), by (2.80) we finally obtain
\[
\sum_{j=1}^{m} \varepsilon^{N-1} \left( \frac{1}{\mathcal{L}^N(Q_\varepsilon(x_j))} \right)^2 \int_{Q_\varepsilon(x_j)} \int_{Q_\varepsilon(x_j)} |u(y) - u(x)| \, dy \, dx \\
\leq N^{\frac{N+1}{2q}} \left( \int_{\bigcup_{j=1}^{m} Q_\varepsilon(x_j)} \int_{B(\varepsilon \sqrt{N}) \cap \Omega} \frac{1}{(\varepsilon \sqrt{N})^N} \frac{|u(y) - u(x)|^q}{|y-x|} \, dy \, dx \right)^{\frac{1}{q}} \\
\leq N^{\frac{N+1}{2q}} \left( \int_{\Omega} \int_{B(\varepsilon \sqrt{N}) \cap \Omega} \frac{1}{(\varepsilon \sqrt{N})^N} \frac{|u(y) - u(x)|^q}{|y-x|} \, dy \, dx \right)^{\frac{1}{q}}. \tag{2.81}
\]

From the above we can now deduce the main results about \( BV^q \)-spaces as stated in the Introduction.

**Proof of Proposition 2.1.** Follows from Lemma 2.3.

**Proof of Theorem 1.2.** For \( q = 1 \) it is well known. On the other hand, for \( q > 1 \) the results follow from Lemma 2.4.

**Proof of Theorem 1.3.** Follows from Lemma 2.5 and Definition 2.4.

### 3 An application to Aviles-Giga type energies: proof of Theorem 1.4

The main ingredient needed for the proof of Theorem 1.4 is given by the next Lemma.

**Lemma 3.1.** Let \( \Omega, \Omega_0 \subset \mathbb{R}^N \) be two open sets such that \( \Omega_0 \subset \subset \Omega \). Let \( q > 1 \) and \( \psi \in W^{1,\infty}_{loc}(\Omega, \mathbb{R}) \) be such that \( |\nabla \psi(x)| = 1 \) for a.e. \( x \in \Omega \) and \( \nabla \psi(x) \in BV^q_{loc}(\Omega, \mathbb{R}^N) \). For \( \eta \in C^\infty_c(\mathbb{R}^N, [0, \infty)) \) satisfying \( \text{supp} \eta \subset B_1(0) \) and \( \int_{\mathbb{R}^N} \eta(z) \, dz = 1 \), every \( x \in \Omega \) and every \( 0 < \varepsilon < \text{dist}(x, \partial \Omega) \) define
\[
\psi_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta(x) \frac{y - x}{\varepsilon} \psi(y) \, dy = \int_{\mathbb{R}^N} \eta(z) \psi(x + \varepsilon z) \, dz. \tag{3.1}
\]

Then,
\[
\limsup_{\varepsilon \to 0^+} \int_{\Omega_0} \varepsilon^{q-1} |\nabla^2 \psi_\varepsilon(x)|^q \, dx \leq \left( \int_{\mathbb{R}^N} |z|^\frac{q}{q-1} |\nabla \eta(z)|^\frac{q}{q-1} \, dz \right)^{q-1} A_{\nabla \psi,q}(\bar{\Omega}_0). \tag{3.2}
\]
Moreover, if \( q \geq 2 \) then
\[
\limsup_{\varepsilon \to 0^+} \int_{\Omega_0} \frac{1}{\varepsilon} \left( 1 - |\nabla \psi_\varepsilon(x)|^2 \right)^{\frac{q}{2}} \, dx \leq \left( \int_{\mathbb{R}^N} |z|^\frac{q}{q-2} |\eta(z)|^\frac{q}{q-2} \, dz \right)^{\frac{q-2}{q}} A_{\nabla \psi,q}(\bar{\Omega}_0). \tag{3.3}
\]
Proof. For every $x \in \Omega_0$ and small enough $\varepsilon > 0$ we have

$$\nabla \psi_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y-x}{\varepsilon}\right) \nabla \psi(y)dy = \int_{\mathbb{R}^N} \eta(z) \nabla \psi(x + \varepsilon z)dz,$$

(3.4)

and

$$\varepsilon \nabla^2 \psi_\varepsilon(x) := -\frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \nabla \eta\left(\frac{y-x}{\varepsilon}\right) \nabla \psi(y)dy = -\int_{\mathbb{R}^N} \nabla \eta(z) \nabla \psi(x + \varepsilon z)dz.$$

(3.5)

By (3.5),

$$\int_{\Omega_0} \varepsilon^{-1} |\nabla^2 \psi_\varepsilon(x)|^q dx = \int_{\Omega_0} \frac{1}{\varepsilon^N} \left|\int_{\mathbb{R}^N} \eta\left(\frac{y-x}{\varepsilon}\right) \nabla \psi(y)dy\right|^q dx = \int_{\Omega_0} \frac{1}{\varepsilon^N} \left|\int_{\mathbb{R}^N} \nabla \eta(z) \nabla \psi(x + \varepsilon z)dz\right|^q dx = \int_{\Omega_0} \frac{1}{\varepsilon^N} \left|\int_{\mathbb{R}^N} \nabla \eta(z) \nabla \psi(x + \varepsilon z - \nabla \psi(x))dz\right|^q dx. \quad (3.6)$$

From (3.6) and Hölder inequality we finally deduce that

$$\int_{\Omega_0} \varepsilon^{-1} |\nabla^2 \psi_\varepsilon(x)|^q dx \leq \left(\int_{\mathbb{R}^N} \frac{1}{|z|^{N+1}} |\nabla \eta(z)|^{N+1} dz\right)^{1-q} \left(\int_{B_1(0)} \frac{1}{\varepsilon^N} \left|\int_{\Omega_0} \frac{1}{|z|} \left|\nabla \psi(x + \varepsilon z - \nabla \psi(x)\right|^q dx dz\right)^{q-1} \left(\int_{\mathbb{R}^N} \left|\nabla \psi(x + \varepsilon z - \nabla \psi(x)\right|^q dx dz\right), \quad (3.7)$$

and (3.2) follows.

On the other hand, since $|\nabla \psi|^2 = 1$ a.e. in $\Omega$ we may write

$$\int_{\Omega_0} \frac{1}{\varepsilon^N} \left(1 - |\nabla \psi_\varepsilon(x)|^2\right)^{\frac{q}{2}} dx = \int_{\Omega_0} \frac{1}{\varepsilon^N} \left(1 - \int_{\mathbb{R}^N} \eta(z) \nabla \psi(x + \varepsilon z)dz\right)^{\frac{q}{2}} dx = \int_{\Omega_0} \frac{1}{\varepsilon^N} \left(\int_{\mathbb{R}^N} \eta(z) |\nabla \psi(x + \varepsilon z)|^2 dz - \int_{\mathbb{R}^N} \eta(z) \nabla \psi(x + \varepsilon z)dz\right)^{\frac{q}{2}} dx. \quad (3.8)$$

By elementary computations we find for every $x \in \Omega_0$,

$$\int_{\mathbb{R}^N} \eta(z) |\nabla \psi(x + \varepsilon z)|^2 dz - \int_{\mathbb{R}^N} \eta(z) \nabla \psi(x + \varepsilon z)dz = \int_{\mathbb{R}^N} \eta(z) \nabla \psi(x + \varepsilon z) - \int_{\mathbb{R}^N} \eta(y) \nabla \psi(x + \varepsilon y)dy dz = \int_{\mathbb{R}^N} \eta(z) |\nabla \psi(x + \varepsilon z) - \nabla \psi(x) - \int_{\mathbb{R}^N} \eta(y) (\nabla \psi(x + \varepsilon y) - \nabla \psi(x))dy dz = \int_{\mathbb{R}^N} \eta(z) |\nabla \psi(x + \varepsilon z) - \nabla \psi(x)|^2 - \left(\int_{\mathbb{R}^N} \eta(z) (\nabla \psi(x + \varepsilon z) - \nabla \psi(x))dz\right)^2. \quad (3.9)$$
Plugging (3.9) in (3.8), and then applying Hölder inequality (using \(q \geq 2\)) yields

\[
\int_{\Omega_0} \frac{1}{\varepsilon} \left(1 - |\nabla \psi_\varepsilon(x)|^2 \right)^{\frac{q}{2}} dx \leq \int_{\Omega_0} \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^N} \eta(z) \left| \nabla \psi(x + \varepsilon z) - \nabla \psi(x) \right|^2 dz \right)^{\frac{q}{2}} dx =
\]

\[
\int_{\Omega_0} \frac{1}{\varepsilon} \left( \int_{\mathbb{R}^N} |z|^\frac{q}{2} \eta(z) \left| \nabla \psi(x + \varepsilon z) - \nabla \psi(x) \right|^2 dz \right)^{\frac{q}{2}} dx \leq \left( \int_{\mathbb{R}^N} |z|^\frac{q}{2} \eta(z) \left| \nabla \psi(x + \varepsilon z) - \nabla \psi(x) \right|^2 dz \right)^{\frac{q}{2}} \left( \int_{\mathbb{B}_1(0)} \frac{1}{\varepsilon} \left| \nabla \psi(x + \varepsilon z) - \nabla \psi(x) \right|^q dxdz \right). \tag{3.10}
\]

Passing to the limit \(\varepsilon \to 0^+\) in (3.10) gives immediately (3.3).

**Proof of Theorem 1.4.** Inequality (1.25) follows from Lemma 3.1. Next by Hölder inequality we have:

\[
\int_{\Omega_0} \varepsilon^2 \left| \nabla^2 \psi_\varepsilon(x) \right|^3 dx + \int_{\Omega_0} \frac{1}{\varepsilon^2} \left| 1 - |\nabla \psi_\varepsilon(x)|^2 \right|^\frac{3}{2} dx =
\]

\[
\frac{1}{\varepsilon} \int_{\Omega_0} \frac{1}{3} \left( \sqrt[3]{3} |\varepsilon \nabla^2 \psi_\varepsilon(x)| \right)^3 dx + \frac{1}{\varepsilon} \int_{\Omega_0} \frac{2}{3} \left( \frac{3}{4} |1 - |\nabla \psi_\varepsilon(x)|^2| \right)^\frac{3}{2} dx \geq \int_{\Omega_0} \frac{1}{\varepsilon} \left( \sqrt[3]{3} |\varepsilon \nabla^2 \psi_\varepsilon(x)| \right) \left( \frac{3}{4} |1 - |\nabla \psi_\varepsilon(x)|^2| \right) dx = \frac{3}{\sqrt[4]{4}} \int_{\Omega_0} \left| \nabla^2 \psi_\varepsilon(x) \right| \left| 1 - |\nabla \psi_\varepsilon(x)|^2 \right| dx. \tag{3.11}
\]

Thus we deduce the first inequality in (1.27). On the other hand, the second inequality in (1.27) is just a special case of (1.25) for \(q = p = 3\).

**A Appendix: Some known results on BV-spaces**

In what follows we present some known definitions and results on BV-spaces; some of them were used in the previous sections. We rely mainly on the book [2] by Ambrosio, Fusco and Pallara.

**Definition A.1.** Let \(\Omega\) be a domain in \(\mathbb{R}^N\) and let \(f \in L^1(\Omega, \mathbb{R}^m)\). We say that \(f \in BV(\Omega, \mathbb{R}^m)\) if the following quantity is finite:

\[
\int_{\Omega} |Df| := \sup \left\{ \int_{\Omega} f \cdot \text{div} \varphi \, dx : \varphi \in C_c^1(\Omega, \mathbb{R}^{m \times N}), \, |\varphi(x)| \leq 1 \forall x \right\}.
\]

**Definition A.2.** Let \(\Omega\) be a domain in \(\mathbb{R}^N\). Consider a function \(f \in L^1_{loc}(\Omega, \mathbb{R}^m)\) and a point \(x \in \Omega\).

i) We say that \(x\) is an approximate continuity point of \(f\) if there exists \(z \in \mathbb{R}^m\) such that

\[
\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} |f(y) - z| \, dy}{\rho^N} = 0.
\]
In this case we denote $z$ by $\tilde{f}(x)$. The set of approximate continuity points of $f$ is denoted by $G_f$.

ii) We say that $x$ is an approximate jump point of $f$ if there exist $a, b \in \mathbb{R}^m$ and $\nu \in S^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} |f(y) - \chi(a, b, \nu)(y)| \, dy}{\rho^N} = 0,$$

where $\chi(a, b, \nu)$ is defined by

$$\chi(a, b, \nu)(y) := \begin{cases} b & \text{if } \nu \cdot y < 0, \\ a & \text{if } \nu \cdot y > 0. \end{cases}$$

The triple $(a, b, \nu)$, uniquely determined, up to a permutation of $(a, b)$ and a change of sign of $\nu$, is denoted by $(f^+(x), f^-(x), \nu_f(x))$. We shall call $\nu_f(x)$ the approximate jump vector and we shall sometimes write simply $\nu(x)$ if the reference to the function $f$ is clear. The set of approximate jump points is denoted by $J_f$. A choice of $\nu(x)$ for every $x \in J_f$ determines an orientation of $J_f$. At an approximate continuity point $x$, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

**Theorem A.1** (Theorems 3.69 and 3.78 from [2]). Consider an open set $\Omega \subset \mathbb{R}^N$ and $f \in BV(\Omega, \mathbb{R}^m)$. Then:

i) $\mathcal{H}^{N-1}$-a.e. point in $\Omega \setminus J_f$ is a point of approximate continuity of $f$.

ii) The set $J_f$ is $\sigma$-$\mathcal{H}^{N-1}$-rectifiable Borel set, oriented by $\nu(x)$. I.e., the set $J_f$ is $\mathcal{H}^{N-1}$-$\sigma$-finite, there exist countably many $C^1$ hypersurfaces $\{S_k\}_{k=1}^\infty$ such that $\mathcal{H}^{N-1} \left( J_f \setminus \bigcup_{k=1}^\infty S_k \right) = 0$, and for $\mathcal{H}^{N-1}$-a.e. $x \in J_f \cap S_k$, the approximate jump vector $\nu(x)$ is normal to $S_k$ at the point $x$.

iii) $[(f^+ - f^-) \otimes \nu_f](x) \in L^1(J_f, d\mathcal{H}^{N-1})$.

**Theorem A.2** (Theorems 3.92 and 3.78 from [2]). Consider an open set $\Omega \subset \mathbb{R}^N$ and $f \in BV(\Omega, \mathbb{R}^m)$. Then the distributional gradient $Df$ can be decomposed as a sum of two Borel regular finite matrix-valued measures $\mu_f$ and $D^jf$ on $\Omega$,

$$Df = \mu_f + D^jf,$$

where

$$D^jf = (f^+ - f^-) \otimes \nu_f \mathcal{H}^{N-1} \cup J_f$$

is called the jump part of $Df$ and

$$\mu_f = (D^af + D^cf)$$

is a sum of the absolutely continuous and the Cantor parts of $Df$. The two parts $\mu_f$ and $D^jf$ are mutually singular to each other. Moreover, $\mu_f(B) = 0$ for any Borel set $B \subset \Omega$ which is $\mathcal{H}^{N-1}$-$\sigma$-finite.
B  Appendix: Proof of Proposition 2.3

Lemma B.1. For every \( q \geq 1 \), if a measurable function \( f : \mathbb{R} \to \mathbb{R}^d \) defined a.e. in \( \mathbb{R} \) belongs to the space \( \tilde{V}_q(\mathbb{R}, \mathbb{R}^d) \), then \( f \in BV_{\text{loc}}(\mathbb{R}, \mathbb{R}^d) \). Moreover, we have:

\[
\bar{A}_{f,q}(\mathbb{R}) \leq 4 (v_{q,\mathbb{R}}(f))^q. \tag{B.1}
\]

Proof. First, assume that \( f : \mathbb{R} \to \mathbb{R}^d \) is defined everywhere in \( \mathbb{R} \) and satisfies \( v_{q,\mathbb{R}}(f) < \infty \). Then by (1.5) we have:

\[
\bar{A}_{f,q}(\mathbb{R}) = \sup_{\varepsilon \in (0,1)} \left( \int_{\mathbb{R}} \int_{x-\varepsilon}^{x+\varepsilon} \frac{|f(y) - f(x)|^q}{\varepsilon |y-x|} dy \right) = \sup_{\varepsilon \in (0,1)} \left( \int_{-1}^{1} \int_{\mathbb{R}} \frac{|f(x+\varepsilon z) - f(x)|^q}{|z|} dx dz \right) \leq \sup_{\varepsilon \in (0,1)} \left\{ \lim_{n \to \infty} \int_{-1}^{1} \sum_{k=0}^{n} \frac{1}{\varepsilon |z|} \left( \int_{(2k+1)\varepsilon |z|}^{(2k+2)\varepsilon |z|} |f(x+\varepsilon z) - f(x)|^q dx \right) \right\}.
\]

Denoting \( J_m = (m \varepsilon |z|, (m+1) \varepsilon |z|) \), we get from (B.2) that

\[
A_{f,q}(\mathbb{R}) \leq \sup_{\varepsilon \in (0,1)} \left\{ \lim_{n \to \infty} \int_{-1}^{1} \sum_{k=0}^{n} \left( \sup_{x \in J_{2k}} |f(x+\varepsilon z) - f(x)|^q + \sup_{x \in J_{-(2k+2)}} |f(x+\varepsilon z) - f(x)|^q \right) dz \right\} \leq 4(v_{q,\mathbb{R}}(f))^q. \tag{B.3}
\]

In the general case we have, by (B.3), for every \( g : \mathbb{R} \to \mathbb{R}^d \) (defined everywhere on \( \mathbb{R} \)) satisfying \( g(x) = f(x) \) a.e. in \( \mathbb{R} \)

\[
\bar{A}_{f,q}(\mathbb{R}) = \bar{A}_{g,q}(\mathbb{R}) \leq 4 (v_{q,\mathbb{R}}(g))^q. \tag{B.4}
\]

Thus, taking infimum of the r.h.s. of (B.4) over all such \( g \)'s we finally deduce (B.1).

Proof of Proposition 2.3. Let \( g : [a, b] \to \mathbb{R}^d \) be defined everywhere in \([a, b]\), satisfying \( g(x) = f(x) \) a.e. in \([a, b]\) and \( v_{q,[a,b]}(g) < \infty \). Consider \( \tilde{g} : \mathbb{R} \to \mathbb{R}^d \) defined by

\[
\tilde{g}(x) := \begin{cases} 
  g(x) & \forall x \in [a, b], \\
  g(a) & \forall x \in (-\infty, a), \\
  g(b) & \forall x \in (b, \infty).
\end{cases} \tag{B.5}
\]
By the definition of $v_{q,I}$ in (2.16) we clearly have

$$v_{q,R}(g) = v_{q,[a,b]}(g).$$  \hspace{1cm} (B.6)

Combining (B.3) with (B.6) we obtain

$$A_{f,q}((a,b)) = A_{g,q}((a,b)) = A_{g,q}((a,b)) \leq A_{g,q}(R) \leq 4(v_{q,R}(g))^q = 4(v_{q,[a,b]}(g))^q. \hspace{1cm} (B.7)$$

Taking the infimum of the r.h.s. of (B.7) over all $g$’s as above we finally deduce (2.18) and that $f \in BV^q((a,b), \mathbb{R}^d)$.

**Proof of Lemma 2.1.** We have,

$$\sup_{\rho \in (0, \infty)} \left( \sup_{|h| = \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^s} |u(x+h) - u(x)| \right)^q dx \right) \leq \sup_{\rho \in (0, \infty)} \left( \sup_{|h| = \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^s} |u(x+h) - u(x)| \right)^q dx \right)$$

$$= \sup_{\rho \in (0, \infty)} \left( \sup_{t \in (0, \rho]} \left( \sup_{|h| = t} \int_{\mathbb{R}^N} \left( \frac{1}{t^s} |u(x+h) - u(x)| \right)^q dx \right) \right)$$

$$\leq \sup_{\rho \in (0, \infty)} \left( \sup_{t \in (0, \rho]} \left( \sup_{|h| = t} \int_{\mathbb{R}^N} \left( \frac{1}{t^s} |u(x+h) - u(x)| \right)^q dx \right) \right) =$$

$$= \sup_{\rho \in (0, \infty)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^s} |u(x+\rho k) - u(x)| \right)^q dx \right). \hspace{1cm} (B.8)$$

In particular, for $s = \frac{1}{q}$, by (B.8) we deduce

$$\sup_{\rho \in (0, \infty)} \left( \sup_{|h| = \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^s} |u(x+h) - u(x)| \right)^q dx \right) =$$

$$= \sup_{\rho \in (0, \infty)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho} |u(x+\rho k) - u(x)|^q dx \right). \hspace{1cm} (B.9)$$

On the other hand by the triangle inequality and the convexity of $g(s) := |s|^q$ for every $\delta > 0$ we have,

$$\sup_{\rho \in (0, \delta)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho} |u(x+\rho k) - u(x)|^q dx \right) \leq \sup_{\rho \in (0, \infty)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho} |u(x+\rho k) - u(x)|^q dx \right) \leq$$

$$\sup_{\rho \in (0, \delta)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho} |u(x+\rho k) - u(x)|^q dx \right) + \sup_{\rho \in [\delta, \infty)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho} |u(x+\rho k) - u(x)|^q dx \right)$$

$$\leq \sup_{\rho \in (0, \delta)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho} |u(x+\rho k) - u(x)|^q dx \right) + \frac{2^{q-1}}{\delta} \sup_{\rho \in [\delta, \infty)} \left( \sup_{k \in \mathbb{R}^{N-1}} (|u(x+\rho k)|^q + |u(x)|^q) dx \right)$$

$$= \sup_{\rho \in (0, \delta)} \left( \sup_{k \in \mathbb{R}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho} |u(x+\rho k) - u(x)|^q dx \right) + \frac{2^{q-1}}{\delta} \|u\|_{L^q(\mathbb{R}^N, \mathbb{R}^d)}^q. \hspace{1cm} (B.10)$$

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Therefore, by (B.9) and (B.10) we have:

\[
\sup_{\varepsilon \in (0, \delta)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)|^q \, dx \right) \leq \sup_{\rho \in (0, \infty)} \left( \sup_{|h| \leq \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^{1/q}} |u(x + h) - u(x)|^q \right) \, dx \right)
\]

\[
\leq \sup_{\varepsilon \in (0, \delta)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)|^q \, dx + \frac{2^q}{\delta} \|u\|_{L^q(\mathbb{R}^N, \mathbb{R}^d)}^q \right). \tag{B.11}
\]

Thus by (B.11) we clearly obtain that if \( u \in L^q(\mathbb{R}^N, \mathbb{R}^d) \) then

\[
\sup_{\rho \in (0, \infty)} \left( \sup_{|h| \leq \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^{1/q}} |u(x + h) - u(x)|^q \right) \, dx \right) < \infty \quad \text{if and only if}
\]

\[
\limsup_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)|^q \, dx \right) < \infty. \tag{B.12}
\]

So we proved that \( u \in L^q(\mathbb{R}^N, \mathbb{R}^d) \) belongs to \( B^{1/q}_{q, \infty}(\mathbb{R}^N, \mathbb{R}^d) \) if and only if we have \( \hat{B}_{u,q}(\mathbb{R}^N) < \infty \).

Next, given open \( \Omega \subset \mathbb{R}^N \) let \( u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^d) \) and \( K \subset \subset \Omega \) be a compact set. Moreover, consider an open set \( U \subset \mathbb{R}^N \) such that we have the following compact embedding:

\( K \subset \subset U \subset \overline{U} \subset \subset \Omega. \)

Then, assuming \( u \in (B^{1/q}_{q, \infty})_{\text{loc}}(\Omega, \mathbb{R}^d) \) implies existence of \( \hat{u} \in B^{1/q}_{q, \infty}(\mathbb{R}^N, \mathbb{R}^d) \) such that \( \hat{u}(x) = u(x) \) for every \( x \in U \), that gives

\[ B_{u,q}(K) = B_{\hat{u},q}(K) \leq \hat{B}_{u,q}(\mathbb{R}^N) < \infty. \]

On the other hand, if we assume

\[ B_{u,q}(\overline{U}) < +\infty, \tag{B.13} \]

then define

\[ \hat{u}(x) = \begin{cases} 
\eta(x)u(x) & \forall x \in U \\
0 & \forall x \in \mathbb{R}^N \setminus U,
\end{cases} \tag{B.14} \]

where \( \eta(x) \in C^\infty_c(U, [0, 1]) \) is some cut-off function such that \( \eta(x) = 1 \) for every \( x \in K \). Thus in particular \( \hat{u}(x) = u(x) \) for every \( x \in K \) and so, in order to complete the proof, we need just to show that \( \hat{u} \in B^{1/q}_{q, \infty}(\mathbb{R}^N, \mathbb{R}^d) \). Thus by (B.12) it is sufficient to show:

\[
\limsup_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |\hat{u}(x + \varepsilon k) - \hat{u}(x)|^q \, dx \right) < \infty. \tag{B.15}
\]
However, since $|\eta| \leq 1$, $\text{supp}\, \eta \subset U$ and $\eta$ is smooth we have:

\[
\limsup_{\varepsilon \to 0^+} \left( \sup_{k \in \mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon} |\hat{u}(x + \varepsilon k) - \hat{u}(x)|^q \, dx \right) = \\
\limsup_{\varepsilon \to 0^+} \left( \sup_{k \in \mathbb{S}^{N-1}} \int_{U} \frac{1}{\varepsilon} |\eta(x + \varepsilon k)u(x + \varepsilon k) - \eta(x)u(x)|^q \, dx \right) = \\
\limsup_{\varepsilon \to 0^+} \left( \sup_{k \in \mathbb{S}^{N-1}} \int_{U} \frac{1}{\varepsilon} \eta(x + \varepsilon k)(u(x + \varepsilon k) - u(x)) + (\eta(x + \varepsilon k) - \eta(x))u(x) \right)^q \, dx \\
\leq \limsup_{\varepsilon \to 0^+} \left( \sup_{k \in \mathbb{S}^{N-1}} \int_{U} \frac{2^{q-1}}{\varepsilon} \left( |\eta(x + \varepsilon k)(u(x + \varepsilon k) - u(x))|^q + |(\eta(x + \varepsilon k) - \eta(x))u(x)|^q \right) \, dx \right) \\
\leq 2^{q-1} \limsup_{\varepsilon \to 0^+} \left( \sup_{k \in \mathbb{S}^{N-1}} \int_{U} \frac{1}{\varepsilon} \left( |u(x + \varepsilon k) - u(x)|^q \right) \, dx \right) + \\
2^{q-1} \limsup_{\varepsilon \to 0^+} \left( \varepsilon^{q-1} \sup_{k \in \mathbb{S}^{N-1}} \int_{U} \frac{1}{\varepsilon} \left( |\eta(x + \varepsilon k) - \eta(x)|^q \right) \, dx \right) \\
\leq 2^{q-1} B_{u,q}(U) + 2^{q-1} \left( \int_{U} |u(x)|^q \, dx \right) \|\nabla \eta\|_{L^\infty}^q < \infty. \quad (B.16)
\]

\[\square\]

References

[1] L. Ambrosio, C. De Lellis and C. Mantegazza, \textit{Line energies for gradient vector fields in the plane}, Calc. Var. PDE \textbf{9} (1999), 327–355.

[2] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs. Oxford University Press, New York, 2000.

[3] P. Aviles and Y. Giga, \textit{A mathematical problem related to the physical theory of liquid crystal configurations}, Proc. Centre Math. Anal. Austral. Nat. Univ. \textbf{12} (1987), 1–16.

[4] P. Aviles and Y. Giga, \textit{On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields}, Proc. Roy. Soc. Edinburgh Sect. A \textbf{129} (1999), 1–17.

[5] J. Bourgain, H. Brezis, P. Mironescu, \textit{Another look at Sobolev spaces}, J.L. Menaldi, et al. (Eds.), Optimal Control and Partial Differential Equations, A volume in honour of A. Benssoussan’s 60th birthday, IOS Press (2001), 439–455.

[6] J. Bourgain, H. Brezis, P. Mironescu, \textit{A new function space and applications}, J. Eur. Math. Soc. (JEMS) \textbf{17} (2015), 2083–2101.

[7] J. Brasseur, \textit{A Bourgain-Brezis-Mironescu characterization of higher order Besov-Nikol’skii spaces}, preprint, https://arxiv.org/abs/1610.05162
[8] H. Brezis and H.-M. Nguyen, *The BBM formula revisited*, Rend. Accad. Lincei **27** (2016), 515–533.

[9] S. Conti and C. De Lellis, *Sharp upper bounds for a variational problem with singular perturbation*, Math. Ann. **338** (2007), 119–146.

[10] S. Conti, I. Fonseca, G. Leoni, *A $\Gamma$-convergence result for the two-gradient theory of phase transitions*, Comm. Pure Appl. Math. **55** (2002), 857–936.

[11] J. Dávila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), 519–527

[12] C. De Lellis, *An example in the gradient theory of phase transitions*, ESAIM Control Optim. Calc. Var. **7** (2002), 285–289 (electronic).

[13] C. De Lellis and F. Otto, *Structure of entropy solutions to the eikonal equation*, J. Eur. Math. Soc. **5** (2003), 107–145.

[14] V. I. Kolyada and M. Lind, *On functions of bounded $p$-variation*, J. Math. Anal. Appl. **356** (2009), 582-604.

[15] W. Jin and R.V. Kohn, *Singular perturbation and the energy of folds*, J. Nonlinear Sci. **10** (2000), 355–390.

[16] G. Leoni and D. Spector, *Characterization of Sobolev and BV spaces*, J. Funct. Anal. **261** (2011), 2926–2958.

[17] G. Leoni and D. Spector, *Corrigendum to "Characterization of Sobolev and BV spaces" [J. Funct. Anal. 261 10 (2011) 2926–2958]*, J. Funct. Anal. **266** (2014), 1106–1114.

[18] A. Poliakovsky, *Upper bounds for singular perturbation problems involving gradient fields*, J. Eur. Math. Soc. **9** (2007), 1–43.

[19] A. Poliakovsky, *A general technique to prove upper bounds for singular perturbation problems*, J. Anal. Math. **104** (2008), 247–290.

[20] A. Ponce, *A new approach to Sobolev spaces and connections to Gamma-convergence*, Calc. Var. Partial Differential Equations **19** (2004), 229–255.

[21] A. Ponce and D. Spector, *On formulae decoupling the total variation of BV functions*, Nonlinear Anal. **154** (2017), 241-257.

[22] N. Wiener *The quadratic variation of a function and its Fourier coefficients* J. Math. Phys. **3** (1924), 72–94