A FOURIER RESTRICTION THEOREM
BASED ON CONVOLUTION POWERS

XIANGHONG CHEN

(Communicated by Alexander Iosevich)

Abstract. We prove a Fourier restriction estimate under the assumption that certain convolution power of the measure admits an $r$-integrable density.

Introduction

Let $F$ be the Fourier transform defined on the Schwartz space by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle} f(x) dx$$

where $\langle \xi, x \rangle$ is the Euclidean inner product. We are interested in Borel measures $\mu$ defined on $\mathbb{R}^d$ for which $F$ maps $L^p(\mathbb{R}^d)$ boundedly to $L^2(\mu)$, i.e.

$$\|\hat{f}\|_{L^2(\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \forall f \in \mathcal{S}(\mathbb{R}^d).$$

(1)

Here "$\lesssim$" means the left-hand side is bounded by the right-hand side multiplied by a positive constant that is independent of $f$.

If $\mu$ is a singular measure, then such a result can be interpreted as a restriction property of the Fourier transform. Such restriction estimates for singular measures were first obtained by Stein in the 1960's. If $\mu$ is the surface measure on the sphere, the Stein-Tomas theorem \[12\], \[13\] states that (1) holds for $1 \leq p \leq \frac{2(d+1)}{d+3}$.

Mockenhaupt \[10\] and Mitsis \[9\] have shown that Tomas’s argument in \[12\] can be used to obtain an $L^2$-Fourier restriction theorem for a general class of finite Borel measures satisfying

$$|\hat{\mu}(\xi)|^2 \lesssim \|\xi\|^{-\beta}, \forall \xi \in \mathbb{R}^d,$$

(2)

$$\mu(B(x, r)) \lesssim r^{\alpha}, \forall x \in \mathbb{R}^d, r > 0,$$

(3)

where $0 < \alpha, \beta < d$; they showed that (1) holds for $1 \leq p < p_0 = \frac{4(d-\alpha)+2\beta}{4(d-\alpha)+\beta}$. Bak and Seeger \[2\] proved the same result for the endpoint $p_0$ and further strengthened it by replacing the $L^{p_0}$-norm with the $L^{p_0,2}$-Lorentz norm.

It is well known that if $\mu$ is the surface measure on a compact $C^\infty$ manifold, then the sharpness can be tested by some version of Knapp's homogeneity argument. See e.g. the work by Iosevich and Lu \[6\], who proved that if $\mu$ is the surface measure on a compact hypersurface and if $F : L^{p_0} \to L^2(\mu)$, $p_0 = \frac{2(d+1)}{d+3}$, then the Fourier decay assumption (2) is satisfied with $\alpha = d - 1$. For general measures satisfying...
(2) and (3), there is no Knapp argument available to prove the sharpness of $p_0$. Here we show that indeed for certain measures the restriction estimate (1) holds in a range of $p$ beyond the range given above. This will follow from a restriction estimate based on an assumption on the $n$-fold convolution $\mu^{*n} = \mu * \cdots * \mu$.

**Theorem 1.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$, let $1 \leq r \leq \infty$ and assume that $\mu^{*n} \in L^r(\mathbb{R}^d)$. Let $1 \leq p \leq \frac{2n}{2n-1}$ if $r \geq 2$, and $1 \leq p \leq \frac{n r^*}{n r^* - 1}$ if $1 \leq r \leq 2$, and let $1 \leq q \leq \frac{r^*}{n r^*}$. Then

$$\left\| \hat{f} \right\|_{L^q(\mu)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}, \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Applying Theorem 1 with $n = 2$, $r = \infty$, one obtains the following.

**Corollary 1.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^1$ such that $\mu * \mu \in L^\infty(\mathbb{R}^1)$. Then (1) holds for $1 \leq p \leq 4/3$.

**Remarks.** (i) It is not easy to construct measures supported on lower dimensional sets for which Corollary 1 applies. Remarkably, Körner showed by a combination of Baire category and probabilistic argument that there exist “many” Borel probability measures $\mu$ supported on compact sets of Hausdorff dimension $1/2$ so that $\mu * \mu \in C_c(\mathbb{R}^1)$.

(ii) In Corollary 1 since $\mu * \mu$ satisfies (3) with $\alpha = 1$, $\mu$ satisfies (3) with $\alpha = 1/2$ (see Proposition 4). Suppose $\mu$ is supported on a compact set of Hausdorff dimension $\gamma$. It follows that $\gamma \geq 1/2$ (cf. [14], Proposition 8.2). Furthermore, if $\gamma < 1$, then $\beta$ and $\alpha$ in (2) and (3) cannot exceed $\gamma$ (cf. [14], Corollary 8.7).

(iii) Under the above situation, since $\alpha, \beta \leq \gamma$, the range of $p$ in (1) obtained from (10), (9), (2) is no larger than $1 \leq p \leq \frac{6 - 4 \epsilon}{5 - 6 \epsilon}$ where $\epsilon = \gamma - 1/2$, while Corollary 1 gives the range $1 \leq p \leq 4/3$, which is an improvement if $\gamma < 2/3$. However, we do not know of any example of such a measure $\mu$ with $\beta$ (and $\alpha$) close to $\gamma$.

(iv) Suppose $\mu$ is as in Corollary 1 and is supported on a compact set of Hausdorff dimension $1/2$. By Theorem 1 the restriction estimate (4) holds for $1 \leq p \leq 4/3, 1 \leq q \leq p/2$. By dimensionality considerations (see Proposition 2 and Proposition 4), these are all the possible exponents $1 \leq p, q \leq \infty$ for which (4) holds.

**Proof of Theorem 1.** The proof proceeds in a similar spirit as in [11], [4]. Fix a nonnegative function $\phi \in C_c^\infty(\mathbb{R}^d)$ that satisfies $\int_{\mathbb{R}^d} \phi(\xi) d\xi = 1$. Let $\phi_\epsilon(\xi) = \epsilon^{-d} \phi(\xi/\epsilon)$ and $\mu_\epsilon(\xi) = \phi_\epsilon * \mu(\xi) = \int_{\mathbb{R}^d} \phi_\epsilon(\xi - \eta) d\mu(\eta)$. Since $\mu_\epsilon$ converges weakly to $\mu$, we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^q \mu_\epsilon(\xi) d\xi = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^q d\mu(\xi)$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. Thus it suffices to show that

$$\left\| \hat{f} \right\|_{L^q(\mu)} \leq C \| f \|_{L^p(\mathbb{R}^d)},$$

where $C$ is a constant independent of $f$ and $\epsilon$. By Hölder’s inequality, we may assume $q = \frac{p^*}{n r^*}$. Set $s = \frac{p^*}{n}$. Note that by our assumption on the range of $p$, $s \geq 2, q \geq 1$. By duality, we need to prove that

$$\left( \int_{\mathbb{R}^d} |\hat{g} \mu_\epsilon(x)|^{n s} dx \right)^{1/n s} \leq C \left( \int_{\mathbb{R}^d} |g(\xi)|^{q'} \mu_\epsilon(\xi) d\xi \right)^{1/q'}$$

for all $g \in \mathcal{S}(\mathbb{R}^d)$.
for all bounded Borel functions $g$. By the Hausdorff-Young inequality,
\[
\left( \int_{\mathbb{R}^d} |\hat{g\mu}_e(x)|^{ns} \, dx \right)^{1/s} = \left( \int_{\mathbb{R}^d} |\hat{g\mu}_e^n(x)|^s \, dx \right)^{1/s} \\
\leq \left( \int_{\mathbb{R}^d} |g\mu_e \ast \cdots \ast g\mu_e(\xi)|^{s'} \, d\xi \right)^{1/s'} \\
= \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^{(n-1)d}} G(\xi, \eta) M_\epsilon(\xi, \eta) \, d\eta \, d\xi \right)^{1/s'},
\]
where $\eta = (\eta_1, \ldots, \eta_{d-1})$, $\eta_0 \equiv \xi$,
\[
G(\xi, \eta) = g(\eta_{n-1}) \prod_{j=1}^{n-1} g(\eta_{j-1} - \eta_j), \\
M_\epsilon(\xi, \eta) = \mu_\epsilon(\eta_{n-1}) \prod_{j=1}^{n-1} \mu_\epsilon(\eta_{j-1} - \eta_j).
\]
Now by Hölder’s inequality for the inner integral,
\[
\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^{(n-1)d}} G(\xi, \eta) M_\epsilon(\xi, \eta) \, d\eta \, d\xi \right)^{1/s'} \\
\leq \left( \int_{\mathbb{R}^d} \left( G(\xi, \eta) M_\epsilon(\xi, \eta) \right)^{s'/q} \, d\xi \right)^{1/s'}.
\]
Applying Hölder’s inequality again, this is bounded by
\[
\|\mu_\epsilon^*\|_r^{1/q} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^{(n-1)d}} |G(\xi, \eta)|^{q'} M_\epsilon(\xi, \eta) \, d\eta \, d\xi \right)^{1/q} \left( \int_{\mathbb{R}^d} \left( |G(\xi, \eta)|^{q'} M_\epsilon(\xi, \eta) \right)^{(q/q')'} \, d\xi \right)^{1/q'}
\]
\[
= \|\mu_\epsilon^*\|_r^{1/q} \int_{\mathbb{R}^d} |g(\xi)|^{q'} \mu_\epsilon(\xi) \, d\xi \left( \int_{\mathbb{R}^d} |g(\xi)|^{q'} \mu_\epsilon(\xi) \, d\xi \right)^{(q/q')'}
\]
\[
\leq \|\mu_\epsilon^*\|_r^{1/q} \left( \int_{\mathbb{R}^d} |g(\xi)|^{q'} \mu_\epsilon(\xi) \, d\xi \right)^{(q/q')'},
\]
where we have used Young’s inequality in the last line. Since $\frac{1}{s'} - \frac{1}{q'} = \frac{1}{q}$, we obtain (5) after taking the $n$th root.

**APPENDIX**

For the sake of completeness, we include the proofs of the claims made in the remarks. Similar results can be found in [10] and [9].

**Proposition 1.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$. If $\mu^\alpha$ satisfies (3) with $0 \leq \alpha \leq d$, then $\mu$ satisfies (3) with exponent $\alpha/n$.

**Proof.** Assume to the contrary that given $k$, $\mu(B_{r_k}) \geq kr_k^{n/n}$ for some ball $B_{r_k}$ with radius $r_k > 0$. Let $B_{nr_k}^* = B_{r_k}^* + \cdots + B_{r_k}^*$ be the $n$-fold Minkowski sum; then
\[
\mu^\alpha(B_{nr_k}^*) \geq \mu(B_{r_k})^n \geq k^n r_k^\alpha.
\]
On the other hand, since $\mu^\alpha$ satisfies (3),
\[
\mu^\alpha(B_{nr_k}^*) \lesssim (nr_k)^\alpha \lesssim r_k^\alpha, \quad \forall r_k.
\]
Letting $k \to \infty$, we obtain a contradiction.
Proposition 2. Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ supported on a compact set of Hausdorff dimension $0 \leq \gamma < d$; then
\[ \|\hat{\mu}\|_s = \infty, \forall 0 < s < \frac{2d}{\gamma}. \]

Proof. Assume to the contrary that $\|\hat{\mu}\|_s < \infty$ for some $2 < s < 2d/\gamma$. Then
\[ \int_{B(0,R)} |\hat{\mu}(\xi)|^2 d\xi \leq \left( \int_{B(0,R)} |\hat{\mu}(\xi)|^s d\xi \right)^{2/s} \lesssim R^{-2d/s}. \]
This decay in $R \to \infty$ implies $\gamma \geq 2d/s$ (cf. [14], Corollary 8.7). Since $2d/s > \gamma$, we obtain a contradiction. \hfill \square

For the endpoint $s = \frac{2d}{\gamma}$ we have

Proposition 3. Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ supported on a compact set $K$. Suppose $d/2 \leq \gamma < d$ and there exists $C \geq 1$ so that
\[ C^{-1} r^\gamma \leq \mu(B(x,r)) \leq C r^\gamma \]
for all $x \in K$ and $0 < r < 1$. Then $\|\hat{\mu}\|_\frac{2d}{\gamma} = \infty$.

Proof. Assume to the contrary that $\|\hat{\mu}\|_{2d/\gamma} < \infty$. Let $\tilde{\mu}$ be the reflection of $\mu$, i.e. $\tilde{\mu}(A) = \mu(-A)$ for Borel sets $A$. Then $\mu \ast \tilde{\mu} = |\hat{\mu}|^2 \in L^{d/\gamma}$. By the Hausdorff-Young inequality, this implies $\mu \ast \tilde{\mu} \in L^{(d/\gamma)'}$, and hence
\[ \mu \ast \tilde{\mu}(B(0,\epsilon)) \lesssim \|\mu \ast \tilde{\mu}\|_{L^{(d/\gamma)'}(B(0,\epsilon))} \epsilon^\gamma. \]
On the other hand, by the upper regularity assumption in (6) we can find $N_\epsilon$ many disjoint balls $B_j$ of radius $\epsilon/2$ centered in $K$ with $N_\epsilon \gtrsim \epsilon^{-\gamma}$. Since the difference set $B_j - B_j \subset B(0,\epsilon)$, we have
\[ \mu \ast \tilde{\mu}(B(0,\epsilon)) \gtrsim \sum_{j=1}^{N_\epsilon} \mu(B_j)^2 \gtrsim N_\epsilon \epsilon^{2\gamma} \gtrsim \epsilon^\gamma, \]
where we have used the lower regularity assumption in (6) in the second inequality. Comparing this with (7) and noticing that $\|\mu \ast \tilde{\mu}\|_{L^{(d/\gamma)'}(B(0,\epsilon))} \to 0$ as $\epsilon \to 0$, we obtain a contradiction. \hfill \square

Proposition 4. Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ supported on a compact set of Hausdorff dimension $0 \leq \gamma \leq d$. If (4) holds with $1 \leq p, q \leq \infty$, then $q \leq \frac{2d}{\gamma} p'$.

Proof. Given $\epsilon > 0$, by Billingsley’s lemma (cf. [3], Proposition 4.9) there exist $x_0 \in \mathbb{R}^d$ and $r_k \to 0$ such that $\mu(B(x_0,r_k)) \gtrsim r_k^{\gamma+\epsilon}, \forall k$. For our purpose, we may assume $x_0 = 0$. Pick a bump function $\phi$ at 0 and let $\tilde{f} = \phi(\cdot/r_k)$ in (4); we obtain $r_k^{(\gamma+\epsilon)/q} \lesssim r_k^{d/p'}, \forall k$. Comparing the powers then gives the desired result. \hfill \square

Additional remarks

(i) After the submission of this paper, Hambrook and Laba posted a preprint [5] in which they provide examples of Cantor-type measures for which the range obtained from [2] is sharp.

(ii) If $\mu$ is at Corollary 1 with compact support, then by Proposition 3 it cannot have lower regularity as in (6) of degree 1/2.
(iii) As pointed out by the referee, Corollary 1 also follows from
\[ \| f \mu * g \mu \|_{L^p(\mathbb{R}^d)} \leq \| \mu * \mu \|_p^{1/p} \| f \|_{L^p(\mu)} \| g \|_{L^p(\mu)}, \]
which can be obtained by interpolating the cases \( p = 1, p = \infty \). See also Bak and McMichael \[1\], and Iosevich and Roudenko \[7\].

ACKNOWLEDGEMENT

The author would like to thank Andreas Seeger for suggesting this problem and for a simplification of the original proof of the theorem which used a generalized coarea formula.

REFERENCES

[1] Jong-Guk Bak and David McMichael, *Convolution of a measure with itself and a restriction theorem*, Proc. Amer. Math. Soc. 125 (1997), no. 2, 463–470, DOI 10.1090/S0002-9939-97-03569-7. MR1350932 (97d:42011)

[2] Jong-Guk Bak and Andreas Seeger, *Extensions of the Stein-Tomas theorem*, Math. Res. Lett. 18 (2011), no. 4, 767–781. MR2831841 (2012b:42014)

[3] Kenneth Falconer, *Fractal geometry*, Mathematical foundations and applications, John Wiley & Sons Ltd., Chichester, 1990. MR1102677 (92j:28008)

[4] Charles Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), 9–36. MR0257819 (41 #2468)

[5] K. Hambrook, I. Laba, *On the sharpness of Mockenhaupt’s restriction theorem*, Geom. Funct. Anal. 23 (2013), no. 4, 1262–1277. MR3077913

[6] Alex Iosevich and Guozhen Lu, *Sharpness results and Knapp’s homogeneity argument*, Canad. Math. Bull. 43 (2000), no. 1, 63–68, DOI 10.4153/CMB-2000-009-7. MR1749949 (2001g:42023)

[7] Alex Iosevich and Svetlana Roudenko, *A universal Stein-Tomas restriction estimate for measures in three dimensions*, Additive number theory, Springer, New York, 2010, pp. 171–178, DOI 10.1007/978-0-387-68361-4_12. MR2744755 (2012b:42030)

[8] Thomas Körner, *On a theorem of Saeki concerning convolution squares of singular measures* (English, with English and French summaries), Bull. Soc. Math. France 136 (2008), no. 3, 439–464. MR2415349 (2009d:42006)

[9] Themis Mitsis, *A Stein-Tomas restriction theorem for general measures*, Publ. Math. Debrecen 60 (2002), no. 1-2, 89–99. MR1882456 (2003b:42026)

[10] G. Mockenhaupt, *Salem sets and restriction properties of Fourier transforms*, Geom. Funct. Anal. 10 (2000), no. 6, 1579–1587, DOI 10.1007/PL00001662. MR1810754 (2001m:42026)

[11] Walter Rudin, *Trigonometric series with gaps*, J. Math. Mech. 9 (1960), 203–227. MR0116177 (22 #6972)

[12] Peter A. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. 81 (1975), 477–478. MR0358216 (50 #10681)

[13] Peter A. Tomas, *Restriction theorems for the Fourier transform*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, Proc. Sympos. Pure Math., XXXV, Amer. Math. Soc., Providence, R.I., 1979, pp. 111–114. MR545245 (81d:42029)

[14] Thomas H. Wolff, *Lectures on harmonic analysis*, with a foreword by Charles Fefferman and preface by Izabella Laba, edited by Laba and Carol Shubin; University Lecture Series, vol. 29, American Mathematical Society, Providence, RI, 2003. MR2003254 (2004e:42002)

Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706

E-mail address: xchen@math.wisc.edu