ON THE PARTIAL DERIVATIVES OF DRINFELD MODULAR FORMS OF ARBITRARY RANK

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Abstract. In this paper, we obtain an analogue of the Serre derivation acting on the product of spaces of Drinfeld modular forms which also generalizes the differential operator introduced by Gekeler in the rank two case. We further introduce a finitely generated algebra $M_r$ containing all the Drinfeld modular forms for the full modular group and show its stability under the partial derivatives. Moreover, we obtain the transcendence of the non-zero values of the generators of $M_r$ and certain Drinfeld modular forms at CM points over the rational function field.

1. Introduction

Let $M_k(\Gamma)$ be the $\mathbb{C}$-vector space of elliptic modular forms of weight $k$ for a congruence subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ and $G_2$ be the false Eisenstein series of weight 2 normalized so that the first coefficient of its Fourier expansion is one. The Serre derivation is the differential operator $D_k : M_k(\Gamma) \to M_{k+2}(\Gamma)$ given by

$$D_k f := \frac{d}{dz} f - kG_2 f.$$ 

Naturally, one expects to generalize the Serre derivation to Hilbert modular forms or Siegel modular forms. However, although there are several differential operators defined by using the partial derivatives of Hilbert modular forms (see [13, 29, 37, 38, 44]) which preserve modularity, there is no analogue of Serre derivation for this setting. On the other hand, to carry the notion of the Serre derivation for Siegel modular forms of degree $g$, Yang and Yin required the existence of a symmetric $g \times g$ matrix $G$ consisting of functions on the Siegel upper half plane and satisfying a certain functional equation [42 Thm. 2.9]. Nonetheless, Hofman and Kohnen [28] showed that such a matrix $G$ never exists and thus the generalization of $D_k$ also could not be established for Siegel modular forms. For more details and explicit list of references, we refer the reader to aforementioned articles.

Unlike the case of classical setting briefly mentioned above, for Drinfeld modular forms whose construction is strikingly analogous to elliptic modular forms in the rank two case, there is a generalization of the Serre derivation to more general objects and constructing it will be one of the main themes of the present paper. In particular, we introduce a multi-linear operator in Theorem 1.1 generalizing the Serre derivation defined by Gekeler [18 Sec. 8] for
Drinfeld modular forms of rank two to the setting of Drinfeld modular forms of arbitrary rank.

To state our results, we firstly introduce some background. Let $X$ be a smooth projective geometrically connected algebraic curve over $\mathbb{F}_q$ and $K$ be its function field. We fix a closed point ‘$\infty$’ of $X$ and let $A$ be the ring of functions in $K$ regular away from $\infty$. We define the degree function $\deg : A \to \mathbb{Z}$ by

$$\deg(a) := (\text{order of the pole of } a \text{ at } \infty) \cdot (\text{degree of } \infty \text{ over } \mathbb{F}_q), \quad a \in A$$

and extend it to $K$ canonically. Consider the absolute value $|\cdot|$ with respect to the place $\infty$ given by $|c| = q^{\deg(c)}$ for $c \in K$ and let $K_\infty$ be the completion of $K$ with respect to $|\cdot|$. Moreover we let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of $K_\infty$ with respect to the unique extension of $|\cdot|$ to $K_\infty$. To make things more accessible to the reader, we note that when $X$ is the projective line over $\mathbb{F}_q$ and ‘$\infty$’ is the point at infinity, we have $K = \mathbb{F}_q(\theta)$ where $\theta$ is a variable over $\mathbb{F}_q$ and $A = \mathbb{F}_q[\theta]$. Furthermore, $\deg(a)$ is given by the degree of $a \in A$ as a polynomial in $\theta$.

For $r \geq 2$, we define the Drinfeld period domain $\Omega^r$ by

$$\Omega^r := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{K_\infty\text{-rational hyperplanes}\}$$

and identify each element $\mathbf{z} \in \Omega^r$ by $\mathbf{z} := (z_1, \ldots, z_r)^{\mathbb{C}_\infty} \in \mathbb{C}_\infty^r$ so that $z_r = 1$ and, by definition, the entries $z_1, \ldots, z_r$ are $K_\infty$-linearly independent. By construction, the $A$-module $A^r\mathbf{z} \subset \mathbb{C}_\infty^r$ generated by $z_1, \ldots, z_r$ over $A$ forms an $A$-lattice of rank $r$ and hence, due to Drinfeld, there exists a unique Drinfeld $A$-module, which we denote by $\phi^r$, corresponding to $A^r\mathbf{z}$ (see §2.1 for more details). We note that $\Omega^r$ has a connected rigid analytic space structure whose admissible open subsets and open coverings can be explicitly described (see [14], [26, Sec. 4], [36] and [15, Sec. 4] for more details). Let $\{\Omega^r_n\}_{n=1}^\infty$ be such a covering of $\Omega^r$ (see §3 for its explicit definition). Then we call $f : \Omega^r \to \mathbb{C}_\infty$ a rigid analytic function if, for each $n \in \mathbb{Z}_{\geq 1}$, its restriction to $\Omega^r_n$ is the uniform limit of rational functions which do not have any pole in $\Omega^r_n$.

The study of Drinfeld modular forms for the rank two setting was initiated by Goss, in the 1980s, in his PhD thesis (see also [24]) and continued to be developed in following years by Gekeler (see [18], [20] and [21]). In [4], based on the results of Pink in [35] and Häberli in his PhD thesis [26], Basson, Breuer and Pink generalized the theory of Drinfeld modular forms to the arbitrary rank. They define Drinfeld modular forms, algebraically, to be global sections of particular ample invertible sheaves on the compactification of the Drinfeld moduli space and analytically to be rigid analytic functions on $\Omega^r$ satisfying a holomorphy condition at infinity and a certain automorphy condition with respect to an arithmetic subgroup of GL$_r(K)$. They further studied the graded $\mathbb{C}_\infty$-algebra of Drinfeld modular forms for GL$_r(A)$ as well as the $\mathbb{C}_\infty$-vector space of Drinfeld modular forms of any given weight and type (see [21, Sec. 17]). We mention that the work of Pellarin [33] provides another direction of generalization of the theory via the study of vectorial Drinfeld modular forms and their deformations. For more details on the history of Drinfeld modular forms as well as the contribution of Gekeler to the higher rank case such as in [17], the reader can refer to [4, [3, Sec. 7] and [34, Sec. 1.1].

From now on in this section, let us concentrate on the case $A = \mathbb{F}_q[\theta]$. For $1 \leq i \leq r$, we define the $i$-th coefficient form $g_i : \Omega^r \to \mathbb{C}_\infty$, a Drinfeld modular form of weight $q^i - 1$ and type 0 for GL$_r(A)$, which sends each $\mathbf{z} \in \Omega^r$ to the $i$-th coefficient of $\phi^r$. Note that when $i = r$, $g_r$ is indeed a nowhere-vanishing Drinfeld cusp form. Furthermore, Gekeler [22]
introduced a Drinfeld cusp form $h_r : \Omega^r \rightarrow \mathbb{C}_\infty$ of non-zero type which plays an essential role in the present work (see §3 for more details). For any rigid analytic function $\mathfrak{h} : \Omega^r \rightarrow \mathbb{C}_\infty$, let $\partial_i(\mathfrak{h})$ be the partial derivative of $\mathfrak{h}$ with respect to $z_i$ and for each $1 \leq j \leq r - 1$, we consider the map $E^{[j]}_r : \Omega^r \rightarrow \mathbb{C}_\infty$ given by

$$
E^{[j]}_r(z) := \frac{1}{g_r(z)} \partial_j(g_r)(z), \quad z = (z_1, \ldots, z_r)^t \in \Omega^r.
$$

Note that, when $r = 2$, $E^{[1]}_2$ serves as a function field analogue of the false Eisenstein series of weight 2 (see [18 Sec. 8]). For $k \in \mathbb{Z}$, we further define the operator $D_{j,k}$ by

$$
D_{j,k}(\mathfrak{h}) := \partial_j(\mathfrak{h}) + k\mathfrak{h}E^{[j]}_r.
$$

Let $M^m_k(\Gamma)$ be the $\mathbb{C}_\infty$-vector space of Drinfeld modular forms of weight $k \in \mathbb{Z}_{\geq 0}$ and type $m \in \mathbb{Z}/(q - 1)\mathbb{Z}$ for a congruence subgroup $\Gamma$ of $GL_r(A)$ (see §3 for the explicit definition). Our first result, restated as Theorem 5.45 later, shows the existence of a differential operator acting on the product of the $\mathbb{C}_\infty$-vector spaces of Drinfeld modular forms.

**Theorem 1.1.** For each $1 \leq i \leq r - 1$, let $\Gamma_i \leq GL_r(A)$ be a congruence subgroup, $k_i \in \mathbb{Z}_{\geq 0}$ and $m_i \in \mathbb{Z}/(q - 1)\mathbb{Z}$. Consider the operator $D_{(k_1, \ldots, k_{r-1})}$ on $M^{m_1}_{k_1}(\Gamma_1) \times \cdots \times M^{m_{r-1}}_{k_{r-1}}(\Gamma_{r-1})$ defined by

$$
D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1}) := det \begin{pmatrix}
D_{1,k_1}(f_1) & \cdots & D_{r-1,k_1}(f_1) \\
\vdots & & \vdots \\
D_{1,k_{r-1}}(f_{r-1}) & \cdots & D_{r-1,k_{r-1}}(f_{r-1})
\end{pmatrix}.
$$

Then the following statements hold.

(i) $D_{(k_1, \ldots, k_{r-1})}$ is a $\mathbb{C}_\infty$-multilinear derivation.

(ii) $D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1}) \in M^{m_1 + \cdots + m_{r-1} + 1}_{k_1 + \cdots + k_{r-1} + r}(\cap_{i=1}^{r-1} \Gamma_i)$.

Motivated by Theorem 1.1, we call $D_{(k_1, \ldots, k_{r-1})}$ the Serre derivation. To prove Theorem 1.1, we first study the functional equations of $E^{[1]}_r, \ldots, E^{[r-1]}_r$ and $D_{j,k}(f)$ provided that $f$ is a Drinfeld modular form of weight $k$, and combine such a study with our results following from the Legendre relation, which is the function field analogue of the classical Legendre relation for elliptic curves (see Proposition 5.23). Later on, we analyze the limiting behavior of $D_{j,k}(f)(z)$ in Proposition 5.44 whenever $z$ is an element of $\Omega^r$ satisfying certain conditions. The key step in this analysis is to investigate the partial derivatives of the coefficient forms which we will state after defining some rigid analytic functions in what follows.

Let $\exp_{\phi} : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ be the exponential function of $\phi^z$. We consider the period matrix of $\phi^z$ given as

$$
P_z := \begin{pmatrix}
-z_1 & F_{\tau}^{\phi^z}(z_1) & \cdots & F_{\tau,r-1}^{\phi^z}(z_1) \\
\vdots & \vdots & & \vdots \\
-z_r & F_{\tau}^{\phi^z}(z_r) & \cdots & F_{\tau,r-1}^{\phi^z}(z_r)
\end{pmatrix} \in GL_r(\mathbb{C}_\infty)
$$

where for each $1 \leq j \leq r - 1$, $F_{\tau,j}^{\phi^z} : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is the unique entire function satisfying the functional equation

$$
F_{\tau,j}^{\phi^z}(\theta z) - \theta F_{\tau,j}^{\phi^z}(z) = \exp_{\phi^z}(z)^q, \quad z \in \mathbb{C}_\infty.
$$
Note that $F_r, \ldots, F_{r-1}$ are indeed quasi-periodic functions of $\phi^r$, which play the analogous role of the quasi-periodic functions for elliptic curves (see §2 for more details). For each $1 \leq i \leq r - 1$ and $1 \leq j \leq r$, consider
\[ L_{ij} : \Omega^r \to \mathbb{C}_\infty \]
\[ z \mapsto P_z^{(i,j)} \]
where $P_z^{(i,j)}$ is the $(i, j)$-cofactor of the period matrix $P_z$. We finally let $(-\theta)^{1/(q-1)}$ be a fixed $(q - 1)$-st root of $-\theta$ and define the Carlitz period, analogous to $2\pi i$ in the classical setting (see [41, Sec. 2.5]), by
\[ \tilde{\pi} := \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in \mathbb{C}_\infty^\times. \]

Our next theorem, restated as Theorem 5.19 later, is motivated by our analysis in §4 on the Gauss-Manin connection and the Kodaira-Spencer map for Drinfeld $A$-modules.

**Theorem 1.3.** For each $1 \leq i, j \leq r - 1$ and $z \in \Omega^r$, the following identities hold.

(i) $\partial_j(h_r)(z) = -h_r(z)E^{[j]}_r(z)$.

(ii) $\partial_j(g_i)(z) = E^{[j]}_r(z)g_i(z) + \tilde{\pi}^{q^{i+1}+q^{r-1}}h_r(z)L_{ij}(z) - 1$.

When $r = 2$, Theorem 1.3 was established by Gekeler in [18, Sec. 9] and [20, (8.4)] by using the “k/12 formula for Drinfeld modular forms” and an analysis of the behavior at a neighborhood of infinity of the function $h_2$. Our method, however, follows a completely different path, which also provides a new proof in the rank two case. More precisely, we study the work of Pellarin in [31, Sec. 8] on the deformation of vector-valued modular forms and combine it with the theory of Drinfeld modules over the Tate algebra developed in [1]. Later on, we use the explicit relations between the Eisenstein series and the coefficients of the exponential and the logarithm series of Drinfeld $A$-modules to complete the proof (see §5.1 for more details).

An immediate corollary of Theorem 5.19 which follows from an explicit algebraic relation between the values of $h_r$ and $L_{ij}$ (see (5.24)) yields a different characterization of the $h$-function of Gekeler than given in [22, Sec. 3.7].

**Corollary 1.4.** We have
\[ D_{(q_1, \ldots, q_{r-1})}(g_1, \ldots, g_{r-1})(z) = \tilde{\pi}^{q^{i+1}+q^{r-1}}h_r(z). \]

Our next goal is to introduce a particular $\mathbb{C}_\infty$-algebra invariant under the partial derivatives. More precisely, for $1 \leq i \leq r - 1$, we set
\[ g_i^{\text{new}} := \tilde{\pi}^{1-q^i}g_i \]
and for $1 \leq j \leq r$, we let
\[ E_{ij} := \tilde{\pi}^{q^{i+1}+q^{r-1}-q^j}h_r L_{ij}. \]

We define the $\mathbb{C}_\infty$-algebra $\mathcal{M}_r$ by
\[ \mathcal{M}_r := \mathbb{C}_\infty[g_i^{\text{new}}, E_{ij} | 1 \leq i \leq r - 1, 1 \leq j \leq r]. \]

Our result, later stated as Theorem 6.3, can be given as follows.

**Theorem 1.5.** The $\mathbb{C}_\infty$-algebra $\mathcal{M}_r$ is stable under the partial derivatives $\partial_1, \ldots, \partial_{r-1}$ and strictly contains the graded $\mathbb{C}_\infty$-algebra of Drinfeld modular forms for $\text{GL}_r(A)$. 
Remark 1.6. We emphasize that when $r = 2$, $E_{11}$ is the false Eisenstein series of weight 2 defined in [18, Sec. 8] by Gekeler and $E_{12}$ is a scalar multiple of $h_2$. Hence, $M_2$ is the $\mathbb{C}_\infty$-algebra of Drinfeld quasi-modular forms in the rank two setting, defined by Bosser and Pellarin, containing the graded $\mathbb{C}_\infty$-algebra of Drinfeld modular forms for $\text{GL}_2(A)$ and is stable not only under the first derivative but also under higher derivatives (see [7]). Although it is still not clear how to define quasi-modular forms explicitly in the higher rank setting, due to Theorem 1.5, it is reasonable to expect that $M_r$ is the ring of such forms. Moreover, to determine the transcendence degree of $M_r$ as a $\mathbb{C}_\infty$-algebra, it would be interesting to investigate a suitable adaptation of the strategy used in the work of Bertrand and Zudilin on Siegel modular forms [3, Thm. 1]. We hope to tackle these problems in the near future.

We say that $z \in \Omega^r$ is a CM point if the endomorphism ring of $\phi^r$ is a free $A$-module of rank $r$. Generalizing the method used in [12, Sec. 6], we obtain the following result, restated in Theorem 6.5 and Theorem 6.11, on the transcendence of the non-zero values of some explicitly defined non-constant functions at CM points. We remark that when $r = 2$, it was obtained by Chang in [9] and [10].

**Theorem 1.7.** Let $z \in \Omega^r$ be a CM point. For $1 \leq i \leq r - 1$ and $1 \leq j \leq r$, if $g_{i,\text{new}}(z)$ and $E_{ij}(z)$ are non-zero, then they are transcendental over $\overline{K}$. Moreover, let $f \in \overline{K}[g_{1,\text{new}}, \ldots, g_{r-1,\text{new}}, h_r]$ be a Drinfeld modular form of non-zero weight for $\text{GL}_r(A)$. Then $f(z)$ is either zero or transcendental over $\overline{K}$.

The outline of the paper is as follows. In §2, we introduce basic properties of Drinfeld $A$-modules as well as the de Rham module attached to them. In §3, we explain some details on Drinfeld modular forms and provide several examples including the $h$-function of Gekeler. In §4, we investigate the Gauss-Manin connection and the Kodaira-Spencer map for Drinfeld $A$-modules of arbitrary rank. Note that our results in this section hold not only when $A = \mathbb{F}_q[\theta]$ but also for the general setting of Drinfeld $A$-modules. In §5, we obtain Theorem 1.1 and Theorem 1.3 after analyzing the deformation of vector-valued Drinfeld modular forms whose study was initiated by Pellarin. Finally in §6, we provide a proof for Theorem 1.5 and Theorem 1.7.

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2. Preliminaries and Background

In this section, we introduce the necessary background for Drinfeld $A$-modules, biderivations, quasi-periodic functions and the de Rham module for Drinfeld $A$-modules. Our exposition is mainly based on [19]. For more details, we refer the reader to [25, Sec.3 and 4], [21] or [41, Sec. 2].
2.1. Drinfeld $A$-modules. For any $\mathbb{F}_q$-algebra $L \subseteq \mathbb{C}_\infty$ containing $K$, we define the non-commutative ring $L[[\tau]]$ of power series in $\tau$ subject to the condition
\[
\tau z = z^q \tau, \quad z \in L.
\]
Moreover, we let $L[\tau] \subset L[[\tau]]$ be the ring of polynomials in $\tau$ with coefficients in $L$. It has an action on $L$ given by
\[
\sum c_i \tau^i \cdot z := \sum c_i z^{q^i} \in L.
\]
Let $r \in \mathbb{Z}_{\geq 1}$. A Drinfeld $A$-module $\phi$ of rank $r$ over $L$ is an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to L[[\tau]]$ given by
\[
\phi_a := a + g_{1,a} \tau + \cdots + g_{r,\deg(a),a} \tau^r \deg(a)
\]
such that $g_{r,\deg(a),a} \neq 0$ for any $a \in A$. For each $1 \leq i \leq r \deg(a)$, we call $g_{i,a}$ the $i$-th coefficient of $\phi_a$. Note that when $A = \mathbb{F}_q[\theta]$, since $\phi$ can be uniquely defined by the image of $\phi_0$, we simply call $g_{i,\theta}$ the $i$-th coefficient of $\phi$.

For each $\phi$, there exists a unique series $\exp_\phi = \sum_{i \geq 0} \alpha_i \tau^i \in \mathbb{C}_\infty[[\tau]]$, called the exponential series of $\phi$, satisfying $\alpha_0 = 1$ and
\[
\exp_\phi a = \phi_a \exp_\phi.
\]
Moreover it induces an $\mathbb{F}_q$-linear entire function $\exp_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $\exp_\phi(z) = \sum_{i \geq 0} \alpha_i z^i$ for each $z \in \mathbb{C}_\infty$. Similarly, one can have the logarithm series $\log_\phi$ of $\phi$ which is the formal inverse of $\exp_\phi$ and given by the infinite series $\log_\phi = \sum_{i \geq 0} \beta_i \tau^i \in \mathbb{C}_\infty[[\tau]]$ satisfying $\beta_0 = 1$ and
\[
a \log_\phi = \log_\phi \phi_a.
\]
We call an $A$-module strongly discrete if its intersection with any ball in $\mathbb{C}_\infty$ of finite radius is finite. Moreover, an $A$-lattice $\Lambda$ of rank $r$ is a strongly discrete projective $A$-module of rank $r$ in $\mathbb{C}_\infty$. Consider the function $e_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by
\[
e_\Lambda(z) := z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right), \quad z \in \mathbb{C}_\infty.
\]
For each $a \in A$, one can define a map $\phi_a^\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ so that the following diagram commutes:
\[
\begin{array}{cccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_\infty & \xrightarrow{e_\Lambda} & \mathbb{C}_\infty & \longrightarrow & 0 \\
\downarrow a & & \downarrow a & & \downarrow \phi_a^\Lambda & & \downarrow \phi_a^\Lambda & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_\infty & \xrightarrow{e_\Lambda} & \mathbb{C}_\infty & \longrightarrow & 0
\end{array}
\]
Indeed, for any Drinfeld $A$-module $\phi$ of rank $r$ over $\mathbb{C}_\infty$, there exists a unique $A$-lattice of rank $r$, called the period lattice of $\phi$, so that the above diagram is commutative. Moreover, due to Drinfeld [14], we know that the association $\Lambda \to \phi^\Lambda$ defines a bijection between the set of $A$-lattices of rank $r$ in $\mathbb{C}_\infty$ and the set of Drinfeld $A$-modules of rank $r$ over $\mathbb{C}_\infty$. Furthermore, we call each non-zero element of $\Lambda$ a period of $\phi^\Lambda$.

We now define the endomorphism ring $\text{End}(\phi)$ of $\phi$ to be the subring of $\mathbb{C}_\infty[[\tau]]$ given by
\[
\text{End}(\phi) := \{v \in \mathbb{C}_\infty[[\tau]] \mid v \phi_a = \phi_a v, \quad a \in A\}.
\]
By Drinfeld [14], $\text{End}(\phi)$ is a projective $A$-module whose $A$-module structure is given by the left multiplication with $\phi_a$ for each $a \in A$. Moreover it is commutative and finitely generated.
whose rank is less than or equal to $r$. Furthermore, we call $\phi$ a CM Drinfeld $A$-module if the dimension of $\text{End}(\phi) \otimes_A K$ over $K$ is $r$.

We finally introduce an $A$-lattice invariant which will be fundamental for the present work. Let $\Lambda$ be an $A$-lattice of rank $r$. We define the Eisenstein series of weight $k \in \mathbb{Z}_{\geq 0}$ for $\Lambda$ to be the infinite sum given by

$$
Eis_k(\Lambda) := \sum_{\lambda \in \Lambda \atop \lambda \neq 0} \frac{1}{\lambda^k} \in \mathbb{C}_\infty, \quad \text{for } k \geq 1
$$

and set $Eis_0(\Lambda) := -1$. Note that when $k$ is not divisible by $q - 1$, we have $Eis_k(\Lambda) = 0$. Moreover for any $z \in \mathbb{C}_\infty$, we get \cite{[2]} Lem. 3.4.10 (see also \cite{[18]} (2.8))

$$
\frac{z}{\exp_{\phi^A}(z)} = 1 - \sum_{k=1}^{\infty} Eis_k(\Lambda) z^k.
$$

Furthermore we have the following relation between the Eisenstein series and the coefficients of the logarithm series $\log_{\phi^A} = \sum_{i \geq 0} \beta_i \tau^i$ \cite{[2]} Lem. 3.4.13 (see also \cite{[18]} (2.9)):}

$$
Eis_{q^k-q^j}(\Lambda) = -\beta_{q^{k-j}}, \quad j, k \geq 0.
$$

\section{The de Rham module.} In this subsection, we focus on the de Rham module of a Drinfeld $A$-module which has been developed over the years by the contribution of Anderson, Deligne, Gekeler and Yu. We refer the reader to \cite{[19]} and \cite{[43]} for further details.

Throughout this subsection, we will fix a Drinfeld $A$-module $\phi$ over $\mathbb{C}_\infty$. Let $B$ be a commutative $\mathbb{C}_\infty$-algebra which contains $\mathbb{C}_\infty$ and consider the non-commutative ring $B[\tau]$ of polynomials in $\tau$ with coefficients in $B$. We equip $B[\tau]$ with the $A$-bimodule structure so that the left action of $A$ is given by the left multiplication in $B$ and the right action of $A$ is given by the right multiplication with $\phi_a$ for any $a \in A$.

\begin{definition}
A map $\eta : A \to B[\tau] \tau$ is called a biderivation if
(i) $\eta$ is $\mathbb{F}_q$-linear,
(ii) $\eta_{ab} = a\eta_b + \eta_a \phi_b$ for any $a, b \in A$, where we set $\eta_c := \eta(c)$ for $c \in A$.

The set of all biderivations is denoted by $D(\phi, B)$. Let $m \in B[\tau]$. The map $\eta^{(m)} : A \to B[\tau] \tau$ given by

$$
\eta_a^{(m)} := am - m\phi_a
$$

forms a biderivation and will be called inner. Moreover, we call $\eta^{(m)}$ strictly inner if $m \in B[\tau] \tau$ and let

$$
D_{si}(\phi, B) := \{\eta^{(m)} \mid m \in B[\tau] \tau\}
$$

be the $B$-module of the strictly inner biderivations. Furthermore, we say that a biderivation $\eta$ is reduced (resp. strictly reduced) if $\text{deg}_\tau(\eta) \leq r \text{deg}(a)$ (resp. $\text{deg}_\tau(\eta_a) < r \text{deg}(a)$) and we denote the $B$-module of reduced biderivations by $D_r(\phi, B)$ (resp. $D_{sr}(\phi, B)$). Note that the notion of reducedness for a biderivation does not depend on $a \in A$ \cite{[19]} (2.12)(iii). We have the following decomposition of $D(\phi, B)$:

$$
D(\phi, B) = D_{sr}(\phi, B) \oplus B\eta^{(1)}(1) \oplus D_{si}(\phi, B).
$$

We define the de Rham module $H_{DR}(\phi, B)$ of $\phi$ by the quotient $B$-module

$$
H_{DR}(\phi, B) := D(\phi, B)/D_{si}(\phi, B) = \{[\eta] \mid \eta \in D(\phi, B)\}$$

where $[\eta]$ is the equivalence class of $\eta$ in $H_{DR}(\phi, B)$ and hence by (2.5), we obtain the decomposition

$$H_{DR}(\phi, B) = H_1(\phi, B) \oplus H_2(\phi, B)$$

where $H_1(\phi, B)$ is the free $B$-module of rank one generated by the class of $\eta^{(1)}$ and $H_2(\phi, B)$ is the set of equivalence classes of strictly reduced biderivations. Moreover, by [20, 2.12], each equivalence class in $H_{DR}(\phi, B)$ has a unique reduced representative.

Assume now that $B = \mathbb{C}_\infty$. Let $a$ be a non-constant element of $A$ and $\eta$ be a biderivation. Then, by [20, (2.2)], there exists a unique entire function $F_\eta^\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ satisfying

$$F_\eta^\phi(a) - aF_\eta^\phi(z) = \eta_a(z).$$

Moreover, we know that $F_\eta^\phi$ is independent of the choice of $a \in A \setminus \mathbb{F}_q$. We call such unique function the quasi-periodic function corresponding to $\eta$. As an illustration, one can see that the function $F_{\eta^{(1)}}^\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $F_{\eta^{(1)}}^\phi(z) = z - \exp_\phi(z)$ is the quasi-periodic function corresponding to $\eta^{(1)}$.

## 3. Drinfeld Modular Forms

In this section, we review the notion of Drinfeld modular forms of arbitrary rank for general $A$. We refer the reader to [4] for related details.

For any integer $r \geq 2$, let $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ be the projective space of dimension $r - 1$ with coefficients in $\mathbb{C}_\infty$ whose elements are represented by the column vectors $\eta = (y_1, \ldots, y_r)^t \in \mathbb{C}_\infty^r$. We define the Drinfeld period domain

$$\Omega^r = \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{K_\infty\text{-rational hyperplanes}\}$$

and identify any of its elements as $z = (z_1, \ldots, z_r)^t \in \mathbb{C}_\infty^r$ whose entries are $K_\infty$-linearly independent and normalized so that $z_r = 1$.

For later use, we now briefly explain the rigid analytic structure of $\Omega^r$. Set $|\eta|_\infty := \max_{i=1}^r |y_i|$. Let $H$ be a $K_\infty$-rational hyperplane in $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. We choose a unimodular linear form

$$\ell_H(X_1, \ldots, X_r) := a_1 X_1 + \cdots + a_r X_r \in K_\infty[X_1, \ldots, X_r]$$

to be the defining equation of $H$, that is, $\max_{1 \leq i \leq r} |a_i| = 1$. Then setting $\ell_H(z) := a_1 z_1 + \cdots + a_r z_r$, we see that $|\ell_H(z)|$ is well-defined for any $z \in \Omega^r$. For each $n \geq 1$, we define

$$H_n := \{w \in \mathbb{P}^{r-1}(\mathbb{C}_\infty) \mid |\ell_H(w)| < q^{-n} |w|_\infty\}$$

and set

$$\Omega^r_n := \{z \in \Omega^r \mid |\ell_H(z)| \geq q^{-n} |z|_\infty \text{ for any } K_\infty\text{-rational hyperplane } H\}.$$ 

In fact, $\Omega^r_n$ is a finite intersection of sets of the form $\mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus H_n$ for some $K_\infty$-rational hyperplane $H$ (see [2, Thm. 2.6.12]). Moreover, $\{\Omega^r_n\}_{n=1}^\infty$ forms an admissible covering of $\Omega^r$ [4, Prop. 3.4] (cf. [40, Prop. 1]).

For any $\gamma = (a_{ij}) \in \text{GL}_r(K_\infty)$, define

$$\gamma \cdot z := \left(\begin{array}{c} a_{11} z_1 + \cdots + a_{1r} z_r \\
 a_{r1} z_1 + \cdots + a_{rr} z_r \end{array}\right) \in \Omega^r.$$ 

We further define

$$j(\gamma, z) := a_{r1} z_1 + \cdots + a_{rr} z_r \in \mathbb{C}_\infty^*.$$
We call a subgroup $\Gamma \subset \text{GL}_r(K)$ arithmetic if $\Gamma \cap \text{GL}_r(A)$ has finite index in both $\Gamma$ and $\text{GL}_r(A)$. A rigid analytic function $f : \Omega^r \to \mathbb{C}_\infty$ is called a weak modular form of weight $k \in \mathbb{Z}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ for $\Gamma$ if it satisfies

$$f(\gamma \cdot z) = j(\gamma, z)^k \det(\gamma)^{-m}f(z), \quad \gamma \in \Gamma, \quad z \in \Omega^r.$$ 

Consider the map $\iota : K^{r-1} \to \text{GL}_r(K)$ given by

$$\iota : (a_2, \ldots, a_r) \mapsto \begin{pmatrix} a_2 & \cdots & a_r \\ 1 & \cdots & 1 \end{pmatrix}.$$ 

For an arithmetic subgroup $\Gamma$, we define $\Gamma_\iota := \Gamma \cap \iota(K^{r-1})$ and say that $f : \Omega^r \to \mathbb{C}_\infty$ is $\Gamma_\iota$-invariant if $f(\gamma \cdot z) = f(z)$ for all $\gamma \in \Gamma_\iota$. Note that any weak modular form $f$ of weight $k$ and type $m$ for $\Gamma$ is automatically $\Gamma_\iota$-invariant. We define the subgroup $\Lambda_\Gamma := \iota^{-1}(\Gamma_\iota) \subset K^{r-1}$ and for $z = (z_1, \ldots, z_r)^r \in \Omega^r$, we set $\tilde{z} = (z_2, \ldots, z_r)^r$. Note that $\Lambda_\Gamma \cap A^{r-1}$ is of finite index in $A^{r-1}$ and $\Lambda_\Gamma$. Hence

$$\Lambda_\Gamma \tilde{z} := \{z_2 a_2 + \cdots + z_r a_r \mid (a_2, \ldots, a_r) \in \Lambda_\Gamma\} \subset \mathbb{C}_\infty$$

forms a discrete subgroup of $\mathbb{C}_\infty$ [4, Sec. 4]. Now considering $\Lambda_\Gamma \tilde{\pi} \tilde{z} \subset \mathbb{C}_\infty$ which is constructed by the left multiplication of elements of $\Lambda_\Gamma \tilde{z}$ by $\tilde{\pi}$, we also define

$$u_\Gamma(z) := \exp_{\Lambda_\Gamma \tilde{\pi} \tilde{z}}(\tilde{\pi} z_1)^{-1} \in \mathbb{C}_\infty^r,$$

where we set

$$\exp_{\Lambda_\Gamma \tilde{\pi} \tilde{z}}(z) := z \prod_{\lambda \in \Lambda_\Gamma \tilde{\pi} \tilde{z}, \lambda \neq 0} \left(1 - \frac{z}{\lambda}\right), \quad z \in \mathbb{C}_\infty.$$

By [4, Prop. 5.4] we know that for any $\Gamma_\iota$-invariant rigid analytic function $f : \Omega^r \to \mathbb{C}_\infty$, for each $n \in \mathbb{Z}$, there exists a unique rigid analytic function $f_n : \Omega^{r-1} \to \mathbb{C}_\infty$ such that the series

(3.2) $$\sum_{n \in \mathbb{Z}} f_n(\tilde{z}) u_\Gamma(z)^n$$

converges to $f(z)$ on some neighborhood of infinity and its admissible subsets (see [4, Def. 4.12] for the explicit definition of a neighborhood of infinity). We call the infinite sum given in (3.2) the $u_\Gamma$-expansion of $f$. Note that when $r = 2$, $f_n$ is a constant in $\mathbb{C}_\infty$ for each $n$.

For any $\delta \in \text{GL}_r(K)$, we know that if $f$ is a weak modular form of weight $k$ and type $m$ for $\Gamma$, then the rigid analytic function $f_{[k,m][\delta]} : \Omega^r \to \mathbb{C}_\infty$ given by $f_{[k,m][\delta]}(z) := j(\delta, z)^{-k} \det(\delta)^{-m}f(\delta \cdot z)$ is a weak modular form of weight $k$ and type $m$ for the arithmetic subgroup $\delta^{-1} \Gamma \delta$. Moreover, we call $f$ a Drinfeld modular form of weight $k$ and type $m$ for $\Gamma$ if the function $f_n$ in the $u_{\delta^{-1} \Gamma \delta}$-expansion of $f_{[k,m][\delta]}$ for all $\delta \in \text{GL}_r(K)$ is identically zero when $n < 0$, in other words, $f$ is holomorphic at infinity with respect to $(\delta^{-1} \Gamma \delta)^r$. Furthermore, we call $f$ a Drinfeld cusp form of weight $k$ and type $m$ for $\Gamma$, if, in addition, $f_0 \equiv 0$ in the $u_{\delta^{-1} \Gamma \delta}$-expansion of $f_{[k,m][\delta]}$ for each $\delta \in \text{GL}_r(K)$. We define $\mathcal{M}_k^m(\Gamma)$ to be the $\mathbb{C}_\infty$-vector space spanned by all Drinfeld modular forms of weight $k$ and type $m$ for $\Gamma$.

In what follows, we give some examples of Drinfeld modular forms.

**Example 3.3.**

(i) Let $z = (z_1, \ldots, z_r)^r \in \Omega^r$ and $L$ be a finitely generated projective $A$-module of rank $r$ in $K^r$ whose elements can be viewed as row vectors. Consider the stabilizer of $L$ in $\text{GL}_r(K)$, which is also an arithmetic subgroup, by

$$\Gamma_L := \{\gamma \in \text{GL}_r(K) \mid (\ell_1, \ldots, \ell_r) \gamma \in L \quad \text{for all} \quad (\ell_1, \ldots, \ell_r) \in L\}. $$
Observe that when $L = A^r$, we have $\Gamma_L = \text{GL}_r(A)$. Moreover note that the set 
$$Lz := \{z_1\ell_1 + \cdots + z_r\ell_r \mid (\ell_1, \ldots, \ell_r) \in \mathcal{L}\} \subset \mathbb{C}_\infty$$
forms an $A$-lattice of rank $r$. Let $k$ be a positive integer divisible by $q - 1$ and $z \in \Omega^r$. It is shown in [4] Prop. 13.3 that the map $\text{Eis}_k : \Omega^r \to \mathbb{C}_\infty$ sending $z \mapsto \text{Eis}_k(Lz)$, where $\text{Eis}_k(Lz)$ is given in (2.11), is a Drinfeld modular form of weight $k$ and type 0 for $\Gamma_L$.

(ii) Let $\phi^L$ be the Drinfeld $A$-module corresponding to $Lz$ and consider a non-constant element $a \in A$. Set 
$$\phi^L_a := a + g_{1,a}(z)\tau + \cdots + g_{r,a}(z)\tau^r \text{deg}(a).$$
Then, for any $1 \leq i \leq \text{deg}(a)$, the function $g_{i,a} : \Omega^r \to \mathbb{C}_\infty$ mapping $z \mapsto g_{i,a}(z)$ is a Drinfeld modular form of weight $q^i - 1$ and type 0 for $\Gamma_L$ [4, Prop. 15.12]. Such functions are called $\textit{coefficients forms}$. In addition, we remark that $g_{r,a}$ is indeed a nowhere-vanishing Drinfeld cusp form on $\Omega^r$. Furthermore, when $A = \mathbb{F}_q[\theta]$, we denote the Drinfeld $A$-module corresponding to $A^r z$ by $\phi^z$ and call $g_i := g_{i \theta}^A z$ the $i$-\textit{th coefficient form}. Moreover, setting $g_{0,a} := a$, we have [21, pg. 251] (see also [18, (2.10)])

$$\text{a \ \text{Eis}_{q^i - 1}(Lz) = \sum_{k=0}^i \text{Eis}_{q^k - 1}(Lz)(g_{i-k,a}^L z)^{q^k}.} \tag{3.4}$$

(iii) Let $A = \mathbb{F}_q[\theta]$. We now review the definition of one fundamental example of Drinfeld cusp forms which was introduced by Gekeler [22]. Let $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{F}_q^r \setminus \{(0, \ldots, 0)\}$. We call $\mu$ \textit{monic} if $\mu_k = 1$ for the largest subscript $k$ satisfying $\mu_k \neq 0$. For any $z = (z_1, \ldots, z_r)^{tr} \in \Omega^r$, let 
$$\text{Eis}_\mu(z) := \sum_{(a_1, \ldots, a_r) \in K^r \atop (a_1, \ldots, a_r) \equiv \theta^{-1} \mu \pmod{A^r}} \frac{1}{a_1 z_1 + \cdots + a_r z_r}.$$ 
Then the $h$-\textit{function} $h_r : \Omega^r \to \mathbb{C}_\infty$ of Gekeler is defined by 

$$h_r(z) := \frac{1-q^r}{\pi} (-\theta)^{1/(q-1)} \prod_{\mu \in \mathbb{F}_q^r \text{monic}} \text{Eis}_\mu(z).$$

We note that $h_r$ is a nowhere-vanishing Drinfeld cusp form of weight $q^r - 1/(q - 1)$ and type 1 for $\text{GL}_r(A)$. In other words, for each $\gamma \in \text{GL}_r(A)$, we have 

$$h_r(\gamma \cdot z) = \det(\gamma)^{-1} j(\gamma, z)^{q^r-1} h_r(z). \tag{3.5}$$

Furthermore, by [22, Thm. 3.8], one can also get 

$$g_r(z) = \hat{\pi}^{q^r-1} (-1)^{r-1} h_r(z)^{q-1}. \tag{3.6}$$

4. \textbf{THE GAUSS-MANIN CONNECTION AND THE KODAIRA-SPENCER MAP}

In this section, we will analyze the Gauss-Manin connection and the Kodaira-Spencer map for Drinfeld $A$-modules of arbitrary rank introduced by Gekeler in [21, Sec. 6]. We emphasize that an explicit study of these maps for the rank 2 case has been done by Gekeler
Let $\mathcal{H}$ be the ring of rigid analytic functions on $\Omega^r$. We set $\phi$ to be the Drinfeld $A$-module considered over $\mathcal{H}$ associated to the $A$-lattice $A^r \mathbf{z}$ where $\mathbf{z} = (z_1, \ldots, z_r)^\text{T} \in \Omega^r$ varies. More precisely, for any non-constant $a \in A$, we set

$$\phi_a := a + g_{1,a} \tau + \cdots + g_{r, \deg(a), a} \tau^r \deg(a)$$

where for each $1 \leq i \leq r \deg(a)$, $g_{i,a} : \Omega^r \to \mathbb{C}_\infty$ is the coefficient form defined in Example 3.3(ii) sending $\mathbf{z} \mapsto g_{i,a}^\text{T} \mathbf{z}(\mathbf{z})$.

We denote the $\mathbb{C}_\infty$-vector space of $\mathbb{C}_\infty$-derivations of $\mathcal{H}$ by $\text{Der}_{\mathbb{C}_\infty}(\mathcal{H})$. If $f = \sum f_i \tau^i \in \tau \mathcal{H}[\tau]$, then we set $\mathcal{D}(f) := \sum \mathcal{D}(f_i) \tau^i$ for any $\mathcal{D} \in \text{Der}_{\mathbb{C}_\infty}(\mathcal{H})$. Now let $\eta \in D(\phi, \mathcal{H})$ be given. Then the map $\nabla_\mathcal{D} : D(\phi, \mathcal{H}) \to \tau \mathcal{H}[\tau]$ defined by $(\nabla_\mathcal{D}(\eta))_a := \mathcal{D}(\eta_a)$ has indeed image in $D(\phi, \mathcal{H})$. Moreover, if $\eta \in D_{\text{ad}}(\phi, \mathcal{H})$ then so is $\nabla_\mathcal{D}(\eta)$. Hence the map

$$\nabla_\mathcal{D} : H_{\text{DR}}(\phi, \mathcal{H}) \to H_{\text{DR}}(\phi, \mathcal{H})$$

$$[\eta] \mapsto [\nabla_\mathcal{D}(\eta)]$$

is well-defined. Furthermore, since, for any $g \in \mathcal{H}$ and $[\eta] \in H_{\text{DR}}(\phi, \mathcal{H})$, we have

(4.1) $$\nabla_\mathcal{D}(g[\eta]) = \mathcal{D}(g)[\eta] + g \nabla_\mathcal{D}(\eta)$$

we call the collection of the maps $\{\nabla_\mathcal{D} | \mathcal{D} \in \text{Der}_{\mathbb{C}_\infty}(\mathcal{H})\}$ the Gauss-Manin connection for $\phi$.

Our next goal is to introduce some special functions which play a fundamental role for our work and study some of their properties. For any $1 \leq i \leq r - 1$, recall the $A$-derivation $\partial_i \in \text{Der}_{\mathbb{C}_\infty}(\mathcal{H})$ defined by $\partial_i = \frac{\partial}{\partial z_i}$ and consider the map $E^{[i]}_r : \Omega^r \to \mathbb{C}_\infty$ given by

$$E^{[i]}_r(\mathbf{z}) := \frac{1}{g_{r, \deg(a), a}(\mathbf{z})} \partial_i(g_{r, \deg(a), a})(\mathbf{z}), \ \mathbf{z} \in \Omega^r.$$

Since $g_{r, \deg(a), a}(\mathbf{z}) \neq 0$, $E^{[i]}_r$ is a rigid analytic function. Moreover, by [21 Lem. 6.7], $E^{[i]}_r$ is independent of the choice of any non-constant element $a \in A$.

**Remark 4.2.** When $A = \mathbb{F}_q[\theta]$, the functions $E^{[1]}_r, \ldots, E^{[r-1]}_r$ are studied and their relations with quasi-periodic functions are determined in [12].

The following proposition was implicitly used in an earlier version of the paper and its statement as well as its proof become more complete in the present version thanks to suggestions and comments of Ernst-Ulrich Gekeler.

**Proposition 4.3.** For any $1 \leq i, j \leq r - 1$, let $\mathcal{D} := \partial_i \circ \partial_j \in \text{Der}_{\mathbb{C}_\infty}(\mathcal{H})$ be a homogeneous linear differential operator of degree 2. Let $\mathcal{L}$ be a finitely generated projective $A$-module of rank $r$ in $K^r$ as in Example 3.3. For $k \in \mathbb{Z}_{\geq 1}$ and any non-constant $a \in A$, if $f$ is either a coefficient form $g_{k,a}^\text{T} \mathbf{z}$ or an Eisenstein series of weight $q^k - 1$, then $\mathcal{D}(f) \equiv 0$.

**Proof.** Recall the notation used in §2.1 and §3. Suppose first that $f(\mathbf{z}) = \text{Eis}_{q^{k-1}}(\mathcal{L} \mathbf{z})$. Then, we have

$$(\partial_j f)(\mathbf{z}) = - \sum_{(\ell_1, \ldots, \ell_r) \in \mathcal{L}} \frac{\ell_j}{(\ell_1 z_1 + \cdots + \ell_r z_r)q^k}.$$

This implies that the derivation $\partial_j$ vanishes on $\partial_j f$ and it finishes the proof. On the other hand, if $f = g_{k,a}^\text{T} \mathbf{z}$, then the proposition follows from applying the previous argument and
considering (3.4) which provides a recursive formula for the coefficient forms in terms of Eisenstein series of weight \( q^\mu - 1 \) for positive integers \( \mu \).

Noting the definition of the functions \( E_r^{[1]}, \ldots, E_r^{[r-1]} \), one can see that our next lemma is immediate from Proposition 4.3.

**Lemma 4.4.** For any \( 1 \leq i, j \leq r - 1 \), we have
\[
\partial_j(E_r^{[i]})(z) = -E_r^{[j]}(z)E_r^{[i]}(z).
\]

Let \( \eta^{[0]} \in H_{\text{DR}}(\phi^2, \mathcal{H}) \) be the equivalence class of the biderivation \( \eta^{in} : A \rightarrow \tau B[\tau] \) given by \( \eta^{in}_a := \phi_a - a \).

**Lemma 4.5.** For each \( 1 \leq j \leq r - 1 \), the biderivation \( \eta^{sj} : A \rightarrow \tau B[\tau] \) sending \( a \mapsto (\nabla_{\partial_j} - E_r^{[j]})\eta^{in}_a \) is strictly reduced.

**Proof.** We follow the idea of the proof of [21, Prop. 7.7]. Note that
\[
\nabla_{\partial_j}(\eta^{[0]}) = \sum_{k \leq r \text{deg}(a)} \partial_j(g_k, a)\tau^i = \partial_j(g_r, a)\tau^{r \text{deg}(a)} + \text{lower degree terms in } \tau.
\]

Thus, by the definition of \( E_r^{[j]} \), \( (\nabla_{\partial_j} - E_r^{[j]})\eta^{in}_a \) has \( \tau \)-degree strictly lower than \( r \text{deg}(a) \), which finishes the proof.

For \( 1 \leq j \leq r - 1 \), we now let \( \eta^{[j]} \in H_{\text{DR}}(\phi, \mathcal{H}) \) be the equivalence class of the biderivation given in Lemma 4.5.

The next theorem is to determine the image of \( \eta^{[j]} \) under the map \( \nabla_{\partial_i} \).

**Theorem 4.6.** For each \( 1 \leq i, j \leq r - 1 \), we have
\[
(i) \quad \nabla_{\partial_i}(\eta^{[0]}) = E_r^{[i]}\eta^{[0]} + \eta^{[i]},
(ii) \quad \nabla_{\partial_i}(\eta^{[j]}) = -E_r^{[j]}\eta^{[i]}.
\]

**Proof.** The first part follows from the definition of \( \eta^{[i]} \). We prove the second identity. To prove the second part, note, by Proposition 4.3, that
\[
(4.7) \quad \nabla_{\partial_i} \circ \nabla_{\partial_j}(\eta^{in}_a) = 0.
\]

On the other hand, using (4.1), Lemma 4.4, (4.7) and the first part, we have
\[
\nabla_{\partial_i}(\eta^{sr}_a) = \nabla_{\partial_i}((\nabla_{\partial_j} - E_r^{[j]})\eta^{in}_a)
\]
\[
= \nabla_{\partial_i} \circ \nabla_{\partial_j}(\eta^{in}_a) + E_r^{[i]}E_r^{[j]}\eta^{in}_a - E_r^{[j]}E_r^{[i]}\eta^{in}_a
\]
\[
= E_r^{[i]}E_r^{[j]}\eta^{in}_a - E_r^{[j]}E_r^{[i]}\eta^{in}_a + \eta^{sr}_a
\]

as desired.

Recall from Section 2.1 that we can decompose the de Rham module
\[
H_{\text{DR}}(\phi, \mathcal{H}) = H_1(\phi, \mathcal{H}) \oplus H_2(\phi, \mathcal{H})
\]
so that \( H_1(\phi, \mathcal{H}) \) is the free \( \mathcal{H} \)-module of rank one generated by the class of \( \eta^{(1)} \) and \( H_2(\phi, \mathcal{H}) \) is the set of equivalence classes of strictly reduced biderivations. We now consider the
set $\text{Hom}_F(H_1(\phi, H), H_2(\phi, H))$ of $H$-module homomorphisms from $H_1(\phi, H)$ to $H_2(\phi, H)$. Following Gekeler, we define the Kodaira-Spencer map $KS$ by

$$KS : \text{Der}_{C_\infty}(H) \to \text{Hom}_F(H_1(\phi, H), H_2(\phi, H))$$

$$\mathcal{D} \mapsto \pi_D$$

where

$$\pi_D : H_1(\phi, H) \hookrightarrow H_{DR}(\phi, H) \xrightarrow{\nabla} H_{DR}(\phi, H) \twoheadrightarrow H_2(\phi, H)$$

so that the last arrow is the projection from $H_{DR}(\phi, H)$ to $H_2(\phi, H)$. It is known that $KS$ is a well-defined map [21 Sec. 6].

We have the following corollary obtained by using Lemma 4.5 and Theorem 4.6 as well as the construction of $KS$.

**Corollary 4.8.** For $1 \leq i \leq r - 1$, we have

$$KS(\partial_i) : H_1(\phi, H) \to H_2(\phi, H)$$

given by $KS(\partial_i)(\eta^{[0]}) = \eta^{[i]}$.

### 5. Derivatives of Drinfeld modular forms when $A = \mathbb{F}_q[\theta]$

In this section, we will investigate a higher rank analogue of the Serre derivation when $A$ is the polynomial ring. Throughout this section, we set $A = \mathbb{F}_q[\theta]$ where $\mathbb{F}_q$ is the finite field with $q$ elements and $\theta$ is a variable over $\mathbb{F}_q$. Let $K = \mathbb{F}_q(\theta)$, the fraction field of $A$. Set $|\cdot|$ to be the $\infty$-adic norm normalized so that $|\theta| = q$ and let $K_\infty = \mathbb{F}_q((1/\theta))$ be the completion of $K$ with respect to $|\cdot|$. We also let $C_\infty$ be the completion of a fixed algebraic closure of $K_\infty$.

#### 5.1. The Tate algebra and Anderson generating functions.

Let $t$ be a variable over $C_\infty$. We define the Tate algebra $\mathbb{T}$ as the subring of power series in $C_\infty[[t]]$ satisfying a certain condition:

$$\mathbb{T} := \{ \sum_{i \geq 0} c_i t^i \in C_\infty[[t]] \mid |c_i| \to 0 \text{ as } i \to \infty \}. $$

For any $g = \sum_{j \geq 0} g_j t^j \in \mathbb{T}$, we define the Gauss norm of $g$ by

$$\|g\| := \sup\{|c_j| \mid j \in \mathbb{Z}_{\geq 0}\}.$$  

Note that $\mathbb{T}$, equipped with the Gauss norm, forms a Banach algebra. Furthermore, for any $i \in \mathbb{Z}$ and $g \in \mathbb{T}$ as above, we define the $i$-th fold twist of $g$ to be $g^{(i)} := \sum_{j \geq 0} g_j^i t^j \in \mathbb{T}$.

We now briefly discuss the theory of Drinfeld $A[t]$-modules over $\mathbb{T}$ introduced by Anglès, Pellarin, and Tavares Ribeiro [11]. Let $\mathbb{F}_q[t]$ (resp. $A[t]$) be the ring of polynomials in $t$ with coefficients in $\mathbb{F}_q$ (resp. $A$). A Drinfeld $A[t]$-module $\psi$ of rank $r \in \mathbb{Z}_{\geq 1}$ over $\mathbb{T}$ is an $\mathbb{F}_q[t]$-algebra homomorphism $\psi : A[t] \to \mathbb{T}[\tau]$ given by

$$\psi_\theta := \theta + g_1 \tau + \cdots + g_r \tau^r, \quad g_r \neq 0.$$ 

For each $\psi$, there exists a unique exponential series $\exp_\psi := \sum_{i \geq 0} \gamma_i t^i \in \mathbb{T}[[t]]$ satisfying $\gamma_0 = 1$ and for each $a \in A[t]$

$$\exp_\psi a = \psi_a \exp_\psi. \quad (5.1)$$
It induces an everywhere convergent function \(\exp_\phi : \mathbb{T} \to \mathbb{T}\) given by \(\exp_\phi(g) = \sum_{i \geq 0} \gamma_i g^{(i)}\) for any \(g \in \mathbb{T}\). Moreover, there exists a unique logarithm series \(\log_\phi := \sum_{i \geq 1} \ell_i \tau^i \in \mathbb{T}[[\tau]]\) of \(\psi\) which is the formal inverse of \(\exp_\psi\) satisfying \(\ell_0 = 1\) and for each \(a \in A[t]\)

\[
(5.2) \quad a \log_\psi = \log_\psi \psi_\alpha.
\]

It is clear that any Drinfeld \(A\)-module \(\phi\) over \(\mathbb{C}_\infty\) can be also considered as a Drinfeld \(A[t]\)-module over \(\mathbb{T}\) and hence one can consider the obvious extension \(\exp_\phi : \mathbb{T} \to \mathbb{T}\) (resp. \(\log_\phi : \mathbb{T} \to \mathbb{T}\)) of the exponential function (resp. the logarithm function) of \(\phi\). For more details, we refer the reader to [23].

Now we return to the theory of Drinfeld \(A\)-modules over \(\mathbb{C}_\infty\) and let \(\phi\) be a Drinfeld \(A\)-module over \(\mathbb{C}_\infty\). For any \(z \in \mathbb{C}_\infty\), we define the Anderson generating function \(s_\phi(z, t)\) of \(\phi\) by the infinite series

\[
s_\phi(z, t) := \sum_{i=0}^{\infty} \exp_\phi(\frac{z}{\theta^{i+1}}) t^i \in \mathbb{C}_\infty[[t]].
\]

By using the properties of the exponential series, it can be easily seen that \(s_\phi(z, t) \in \mathbb{T}\). Moreover, by [32, Sec. 4.2], we have the following series expansion

\[
(5.3) \quad s_\phi(z, t) = \sum_{j=0}^{\infty} \alpha_j z^{q^j} \frac{1}{\theta q^j - t} \in \mathbb{T}
\]

where \(\alpha_j\) is the \(j\)-th coefficient of \(\exp_\phi\). This indeed implies that as a function of \(t\), \(s_\phi(z, t)\) has poles at \(t = \theta^i\) for each \(i \in \mathbb{Z}_{\geq 0}\) with the residue \(\text{Res}_{t=\theta^i} s_\phi(z, t) = -\alpha_i z^{q^i}\).

The next proposition, due to Pellarin, is to derive a useful relation between \(s_\phi(z, t)\) and certain quasi-periodic functions.

**Proposition 5.4.** [32, Sec. 4.2.2] For each \(1 \leq k \leq r - 1\), let \(F^{\psi}_{r_k}\) be the quasi-periodic function corresponding to the biderivation \(\delta_k : A \to \tau \mathbb{C}_\infty[\tau]\) defined by \(\theta \mapsto \tau^k\). Then we have \(F^{\psi}_{r_k}(z) = s_\phi^{(k)}(z, t)|_{t=\theta}\).

We finish this subsection by introducing a special element in \(\mathbb{T}\). We define the Anderson-Thakur element \(\omega(t)\) by

\[
\omega(t) := (-\theta)^{1/(q-1)} \prod_{i=0}^{\infty} \left(1 - \frac{t}{\theta q^i}\right)^{-1} \in \mathbb{T}^\times.
\]

As a function of \(t\), it has a pole at \(t = \theta\) and furthermore we have

\[
(5.5) \quad -\text{Res}_{t=\theta} \omega(t) = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} = \pi.
\]

5.2. Partial derivatives of the coefficient forms and the \(h\)-function. For any integer \(r \geq 2\), recall that \(\Omega^r = \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{K_\infty\text{-rational hyperplanes}\}\) is the Drinfeld upper half plane which can be identified as the set of elements \(z = (z_1, \ldots, z_r)^{tr} \in \mathbb{C}_\infty^r\) whose entries are \(K_\infty\)-linearly independent and normalized so that \(z_r = 1\). Throughout this subsection, to ease the notation we set \(\text{Eis}_{q^{r-1}}(z) := \text{Eis}_{q^{r-1}}(A^r(z))\).

Let \(\phi^z\) be the Drinfeld \(A\)-module of rank \(r\) over \(\mathbb{C}_\infty\) corresponding to the \(A\)-lattice of rank \(r\) generated by the entries of \(z = (z_1, \ldots, z_r)^{tr} \in \Omega^r\) over \(A\) which is given by

\[
\phi^z_\theta := \theta + g_1(z)\tau + \cdots + g_{r-1}(z)\tau^{r-1} + g_r(z)\tau^r.
\]
For each \( z \in \Omega^r \), \( k \in \mathbb{Z}_{\geq 1} \) and \( 1 \leq j \leq r \), we define
\[
E_{z,k}^{[j]}(t) := \sum_{a_1, \ldots, a_r \in A} \sum_{a_1, \ldots, a_r \not\equiv 0} \frac{a_j(t)}{(a_1 z_1 + \cdots + a_r z_r)^k} \in \mathbb{T}.
\]
Note that \( E_{z,k}^{[j]}(t) \) converges in \( \mathbb{T} \) with respect to the Gauss norm \( ||\cdot|| \).

Let \( Z \) be an indeterminate over \( \mathbb{C}_\infty \). We now introduce a power series \( T(Z) \in \mathbb{T}[[Z]] \) studied by Pellarin in [31, Sec. 8] as follows. For any element \( f = \sum_{i \geq 0} a_i Z^i \in \mathbb{T}[[Z]] \) and \( j \in \mathbb{Z}_{\geq 0} \), we let \( \tau^j(f) := \sum_{i \geq 0} a_i^{(j)} Z^{q^j} \). Considering the exponential series \( \exp_{\phi} = \sum_{i \geq 0} \alpha_i(z) r^i \) denoted as \( \exp_{\phi}^* \), we have the following extension \( \exp_{\phi} : \mathbb{T}[[Z]] \to \mathbb{T}[[Z]] \) of \( \exp_{\phi}^* \) still denoted as \( \exp_{\phi}^* \), and given by \( \exp_{\phi}^*(f) := \sum_{i \geq 0} \alpha_i(z) r^i(f) \). We further set the coefficients \( c_i(z) \in \mathbb{T} \) so that
\[
T(Z) := \sum_{i = 0}^{\infty} c_i(z) Z^i := \exp_{\phi}^*(Z)^{-1} \exp_{\phi}^*(Z/(\theta - t)) \in \mathbb{T}[[Z]].
\]

We continue with investigating a certain relation between \( c_i(z) \) and the coefficients of the exponential and the logarithm series of \( \phi \).

**Lemma 5.6.** Let \( \log_{\phi^*} = \sum_{i = 0}^{\infty} \beta_i(z) r^i \) be the logarithm series of \( \phi^* \). Then for each \( i \geq 0 \), we have
\[
c_{q^i - 1}(z) = \frac{\beta_i(z)}{\theta - t} + \frac{\alpha_1(z) \beta_{i - 1}(z) q^{i - 1}}{\theta q - t} + \cdots + \frac{\alpha_{i - 1}(z) \beta_1(z) q^{i - 1}}{\theta q^{i - 1} - t} + \frac{\alpha_i(z)}{\theta q^i - t}.
\]

**Proof.** Recall that \( \E_{i=0}(z) = -1 \) and for \( k \in \mathbb{Z}_{\geq 1} \), \( \E_{k}(z) \) is the \( k \)-th Eisenstein series given by
\[
\E_{k}(z) = \sum_{a_1, \ldots, a_r \in A} \frac{1}{a_1 z_1 + \cdots + a_r z-r + a_r} \in \mathbb{C}_\infty.
\]

Then by [22], we obtain
\[
Z T(Z) = \exp_{\phi^*}(Z/\theta - t) \frac{Z}{\exp_{\phi}^*(Z)}
\]
\[
= \left( \frac{1}{\theta - t} Z + \frac{\alpha_1(z)}{\theta q - t} Z q^{i} + \cdots \right) \left( 1 - \sum_{k \geq 1} \E_{k}(z) Z^k \right)
\]
\[
= \sum_{i = 0}^{\infty} c_i(z) Z^{i + 1}.
\]

After comparing the coefficients of \( Z^{q^i} \) in \((5.7)\), we see that
\[
c_{q^i - 1}(z) = \frac{\E_{i=0}(z)}{\theta - t} - \frac{\alpha_1(z) \E_{i=0}(z)}{\theta q - t} - \cdots - \frac{\alpha_{i-1}(z) \E_{i-1}(z)}{\theta q^{i-1} - t} + \frac{\alpha_i(z)}{\theta q^i - t}.
\]

On the other hand, by [23], we know that \( \E_{i=0}(z) = -\beta_{k-j}(z)^{q^j} \) for \( k, j \geq 0 \). Hence \((5.8)\) becomes
\[
c_{q^i - 1}(z) = \frac{\beta_i(z)}{\theta - t} + \frac{\alpha_1(z) \beta_{i-1}(z) q^{i}}{\theta q - t} + \cdots + \frac{\alpha_{i-1}(z) \beta_1(z) q^{i-1}}{\theta q^{i-1} - t} + \frac{\alpha_i(z)}{\theta q^i - t}
\]
as desired. \( \square \)
The next proposition is fundamental to prove the main result of this subsection.

**Proposition 5.9.** For any $i \geq 1$, we have
\[
(\theta^q - t)c_{q-1}(z) + g_1(z)^{q-1}c_{q-1}(z) + \cdots + g_{i-1}(z)^q c_{q-1}(z) + g_i(z)c_0(z) = 0.
\]
In particular,
\[
g_i(z) = (t - \theta)((\theta^q - t)c_{q-1}(z) + g_1(z)^{q-1}c_{q-1}(z) + \cdots + g_{i-1}(z)^q c_{q-1}(z)).
\]

**Proof.** Set $C := \exp_{q, s} \circ \frac{1}{q-t} \circ \log_{q, s} \in \mathbb{T}[[\tau]]$. By Lemma 5.6, we have
\[
C = \left(\frac{1}{\theta - t} + \frac{\alpha_1(z)}{\theta^q - t} \tau + \frac{\alpha_2(z)}{\theta^q - t} \tau^2 + \cdots \right)(1 + \beta_1(z)\tau + \beta_2(z)\tau^2 + \cdots)
\]
\[
= \frac{1}{\theta - t} + \left(\frac{\beta_1(z)}{\theta - t} + \frac{\alpha_1(z)}{\theta^q - t} \tau + \frac{\beta_2(z)}{\theta - t} + \frac{\alpha_1(z)\beta_1(z)^q}{\theta^q - t} + \frac{\alpha_2(z)}{\theta^q - t} \tau^2 + \cdots \right)
\]
\[
\quad + \left(\frac{\beta_1(z)}{\theta - t} + \frac{\alpha_1(z)\beta_1(z)^q}{\theta^q - t} + \cdots + \frac{\alpha_{i-1}(z)\beta_1(z)^{q-1}}{\theta^q - t} + \frac{\alpha_i(z)}{\theta^q - t} \tau^i + \cdots \right)
\]
\[
= \sum_{i=0}^{\infty} c_{q-1}(z)\tau^i.
\]

Since $\phi^z$ can be also considered as a Drinfeld $A[t]$-module over $\mathbb{T}$, using (5.2) and (5.10), we obtain
\[
1 = C \circ \phi^z_{q, t}
\]
\[
= (c_0(z) + c_{q-1}(z)\tau + c_{q-2}(z)\tau^2 + \cdots)((\theta - t) + g_1(z)\tau + g_2(z)\tau^2 + \cdots + g_r(z)\tau^r)
\]
\[
= (\theta - t)c_0(z) + ((\theta^q - t) - c_{q-1}(z) + g_1(z)c_0(z))\tau + \cdots +
\]
\[
((\theta^q - t)c_{q-1}(z) + g_1(z)^{q-1}c_{q-1}(z) + \cdots + g_{i-1}(z)^q c_{q-1}(z) + g_i(z)c_0(z))\tau^i + \cdots.
\]

Thus the first part of the proposition follows from comparing $\tau^i$-th coefficients in (5.11). The second part of the proposition is a consequence of the first part and the fact that $c_0(z) = (\theta - t)^{-1}$ which can be easily deduced from (5.11). \qed

**Remark 5.12.** Let $C$ be as in the proof of Proposition 5.9. Using (5.11), it is clear that $\phi^z_{q, t} \circ C = 1$. Moreover, by using (5.10), for each $i \geq 1$, one can obtain
\[
(\theta - t)c_{q-1}(z) + g_1(z)c_{q-1}(z)^{(1)} + \cdots + g_{i-1}(z)c_1(z)^{(i-1)} + g_i(z)c_0(z)^{(i)} = 0.
\]

We remark that one can provide another proof for [12, Thm. 3.2] by using the identity (5.13) and [12, Prop. 3.18]. We leave the details to the reader.

Following the notation in [12, Sec. 3], for each $1 \leq i \leq r$, we set $s_i(z; t)$ to be the Anderson generating function $s_i(z; t) := s_{\phi^z}(z_i, t)$ and define the matrix
\[
F(z, t) := \begin{pmatrix}
s_1(z, t) & \cdots & s_1^{(r-1)}(z, t) \\
\vdots & \ddots & \vdots \\
s_r(z, t) & \cdots & s_r^{(r-1)}(z, t)
\end{pmatrix} \in \text{Mat}_r(\mathbb{T}).
\]

By [12, Prop. 3.4], $F(z, t)$ is indeed an element in $\text{GL}_r(\mathbb{T})$. For each $1 \leq i, j \leq r$, we set $L_{ij}(z, t)$ to be the $(i, j)$-cofactor of $F(z, t)$. 
Lemma 5.15. We have
\begin{equation}
(5.16) \begin{pmatrix}
\mathcal{E}_{z,1}^{[1]}(t) & \ldots & \mathcal{E}_{z,1}^{[r]}(t) \\
\mathcal{E}_{z,q}^{[1]}(t) & \ldots & \mathcal{E}_{z,q}^{[r]}(t) \\
\vdots & \vdots & \vdots \\
\mathcal{E}_{z,q-r+1}^{[1]}(t) & \ldots & \mathcal{E}_{z,q-r+1}^{[r]}(t)
\end{pmatrix}
= - \frac{q^{r-1}}{\pi^{r-1}} h_r(z) \begin{pmatrix}
c_0(z) & c_0(z)^{(1)} \\
c_{q-1}(z) & \ldots & c_0(z) \\
\vdots & & \vdots \\
c_{q^{r-1}-1}(z) & c_{q^{r-2}-1}(z)^{(1)} & \ldots & c_0(z)^{(r-1)}
\end{pmatrix}
\begin{pmatrix}
\tilde{L}_{11}(z, t) & \ldots & \tilde{L}_{r1}(z, t) \\
\tilde{L}_{12}(z, t) & \ldots & \tilde{L}_{r2}(z, t) \\
\vdots & \vdots & \vdots \\
\tilde{L}_{1r}(z, t) & \ldots & \tilde{L}_{rr}(z, t)
\end{pmatrix}.
\end{equation}

Proof. Using Pellarin’s result [31, Lem. 8.3 and Thm. 8.8] (see also [12, Thm. 3.11]), for any $0 \leq j \leq r-1$, one can have
\begin{equation}
(5.17) \begin{pmatrix}
\mathcal{E}_{z,q}^{[1]}(t) & \ldots & \mathcal{E}_{z,q}^{[r]}(t)
\end{pmatrix}
\begin{pmatrix}
s_1(z, t) \\
\vdots \\
s_r(z, t)
\end{pmatrix} = -c_{q^{j-1}}(z).
\end{equation}

On the other hand, by [12, Thm. 3.2], we have
\begin{equation}
(5.18) \begin{pmatrix}
\mathcal{E}_{z,1}^{[1]}(t) & \ldots & \mathcal{E}_{z,1}^{[r]}(t)
\end{pmatrix}
\mathcal{F}(z, t) = -(c_0(z), 0, \ldots, 0).
\end{equation}

Thus, using (5.17) and (5.18) as well as applying suitable twisting operator, we obtain
\begin{equation}
\begin{pmatrix}
\mathcal{E}_{z,1}^{[1]}(t) & \ldots & \mathcal{E}_{z,1}^{[r]}(t) \\
\mathcal{E}_{z,q}^{[1]}(t) & \ldots & \mathcal{E}_{z,q}^{[r]}(t) \\
\vdots & \vdots & \vdots \\
\mathcal{E}_{z,q-r+1}^{[1]}(t) & \ldots & \mathcal{E}_{z,q-r+1}^{[r]}(t)
\end{pmatrix}
\mathcal{F}(z, t) = - \begin{pmatrix}
c_0(z) & c_0(z)^{(1)} \\
c_{q-1}(z) & \ldots & c_0(z) \\
\vdots & & \vdots \\
c_{q^{r-1}-1}(z) & c_{q^{r-2}-1}(z)^{(1)} & \ldots & c_0(z)^{(r-1)}
\end{pmatrix}.
\end{equation}

Now the proposition follows from the right multiplication of both sides of above by the inverse of $\mathcal{F}(z, t)$ and using [12, Prop. 3.4].

For each $1 \leq i \leq r-1$ and $1 \leq j \leq r$, recall the function $L_{ij} : \Omega^r \to \mathbb{C}_\infty$ which sends $z \in \Omega^r$ to the $(i, j)$-cofactor of the period matrix $P_z$. It can be easily seen by using Proposition 5.4 that $L_{ij}$ is a rigid analytic function on $\Omega^r$. Finally, recall from §3, the $h$-function of Gekeler $h_r : \Omega^r \to \mathbb{C}_\infty$.

Theorem 5.19. For $1 \leq i, j \leq r-1$, the following identities hold.
(i) $\partial_j(h_r)(z) = -h_r(z) E_{r}^{[i]}(z)$.
(ii) $\partial_j(g_r)(z) = E_{r}^{[i]}(z) g_r(z) + \pi^{q^{i+1}+q^{i-1}} h_r(z) L_{j(i+1)}(z)$.

Proof. We have
\begin{align*}
E_{r}^{[i]}(z) & = \frac{\partial_j(g_r)}{g_r(z)} = -\frac{h_r(z)^{q-2} \partial_j(h_r)(z)}{h_r(z)^{q-1}} = -\frac{\partial_j(h_r)(z)}{h_r(z)}.
\end{align*}
which implies the first part. We now prove the second part. Set

\[ G(z, t) := (\theta^q t - t) \mathcal{E}^{[j]}_{z,q^i}(t) + \sum_{k=1}^{i-1} \mathcal{E}^{[j]}_{z,q^k}(t) g_{i-k}(z) q^k. \]

Observe from (5.16) that

\[ g(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial t^n} G(z, t) \bigg|_{t=\theta}. \]

Moreover, by [12, Thm. 5.4(i), Cor. 5.9] and Proposition 5.4, we have

\[ (5.20) \qquad g_i(z) = (\theta^q t - t) \text{Eis}_{q^i-1}(z) + \sum_{k=1}^{i-1} \text{Eis}_{q^k-1}(z) g_{i-k}(z) q^k. \]

Since the characteristic of \( \mathbb{C}_\infty \) is \( p \), applying the operator \( \partial_j \) to both sides of (5.20) implies that

\[ (5.21) \qquad \partial_j(g_i)(z) = G(z, \theta). \]

Moreover, by [12] Thm. 5.4(i), Cor. 5.9 and Proposition 5.4 we have

\[ E^{[j]}(z) = -\frac{\pi^q}{\omega(t)} h_r(z) \hat{L}_{j1}(z, t)_{t=\theta} \]

and

\[ \text{Res}_{t=\theta} \hat{L}_{j(i+1)}(z, t) = \hat{L}_{j(i+1)}(z). \]

Thus by (5.3), (5.21) and the above calculation, we have

\[ \partial_j(g_i)(z) = G(z, \theta) \]
Lemma 5.22. Let \( m(\cdot) = \frac{q^{k-1} - 1}{\omega(t)} \left( \tilde{L}_{j(i+1)}(z) + \frac{g_i(z)}{t - \theta} \tilde{L}_{j1}(z) \right) \big|_{t = \theta} \).
\[
= E_{r}^{[i]}(z)g_i(z) + \pi^{q^{i+1} + q^{r-1}} h_r(z) L_{j(i+1)}(z)
\]
as desired. \( \square \)

5.3. The differential operator \( D_{i, \ldots, k-1} \). In this subsection, we show the existence of a differential operator acting on the product of the \( \mathbb{C}_\infty \)-vector spaces of Drinfeld modular forms, which also generalizes the notion of the Serre derivation due to Gekeler to the higher rank setting.

Throughout this subsection, we let \( r \geq 2 \), \( z = (z_1, z_2, \ldots, z_r)^{\text{tr}} \in \Omega^r \) and hence \( \tilde{z} = (z_2, \ldots, z_r)^{\text{tr}} \in \Omega^{r-1} \). Recall that there is an action of \( \text{GL}_r(A) \) on the Drinfeld upper half plane \( \Omega^r \) given by
\[
\gamma \cdot z := \left( \frac{a_{11}z_1 + \cdots + a_{1r}z_r}{a_{r1}z_1 + \cdots + a_{rr}z_r}, \ldots, \frac{a_{(r-1)1}z_1 + \cdots + a_{(r-1)r}z_r}{a_{rr}z_1 + \cdots + a_{rr}z_r}, 1 \right)^{\text{tr}} \in \Omega^r, \quad \gamma = (a_{\mu \nu}) \in \text{GL}_r(A)
\]
and we let
\[
j(\gamma, z) := a_{r1}z_1 + \cdots + a_{r(r-1)}z_{r-1} + a_{rr}.
\]
For \( 1 \leq j, l \leq r - 1 \), set
\[
c_{jl}^\gamma(z) := c_{jl}^\gamma - c_{jr}^\gamma z_l
\]
where \( c_{jl}^\gamma \) (resp. \( c_{jr}^\gamma \)) is the \((j, l)\)-cofactor (resp. \((j, r)\)-cofactor) of \( \gamma \).

Lemma 5.22. Let \( f \in M_k^m(\Gamma) \) for some arithmetic subgroup \( \Gamma \leq \text{GL}_r(A) \), \( k \in \mathbb{Z}_{\geq 0} \) and \( m \in \mathbb{Z}/(q - 1)\mathbb{Z} \). For any \( 1 \leq i, j \leq r - 1 \), the following identities hold.

(i) For any \( \gamma \in \text{GL}_r(A) \), \( E_r^{[i]}(\gamma \cdot z) = j(\gamma, z) \det(\gamma)^{-1} \left( \sum_{l=1}^{r-1} E_r^{[i]}(z)c_{il}^\gamma(z) + c_{ir}^\gamma \right) \).

(ii) For any \( \gamma \in \Gamma \), we have
\[
\partial_i(f)(\gamma \cdot z) = j(\gamma, z)^{k+1} \det(\gamma)^{-m} \left( \sum_{l=1}^{r-1} \partial_l(f)(z)c_{il}^\gamma(z) - kc_{ir}^\gamma f(z) \right).
\]

(iii) Define the operator \( D_{i, k} \) given by \( D_{i, k}(f) = \partial_i(f) + kE_r^{[i]}f \). For any \( \gamma \in \Gamma \), we have
\[
D_{i, k}(f)(\gamma \cdot z) = j(\gamma, z)^{k+1} \det(\gamma)^{-m} \sum_{l=1}^{r-1} D_{i, k}(f)(z)c_{il}^\gamma(z).
\]

(iv) For any \( \gamma \in \text{GL}_r(A) \), \( (h_rL_{j(i+1)})(\gamma \cdot z) = j(\gamma, z)^{q^i} \det(\gamma)^{-1} \left( \sum_{l=1}^{r-1} (h_rL_{j(i+1)})(z)c_{il}^\gamma(z) \right) \).

Proof. The first two identities basically follow from the same method used in [12, Sec. 4]. The third part is a consequence of (i) and (ii) as well as the fact that \( \partial_i \) is a \( \mathbb{C}_\infty \)-linear derivation on the set of rigid analytic functions on \( \Omega^r \). Finally, (iv) follows from using Theorem 5.19(ii) and (iii). \( \square \)

Proposition 5.23. For any \( \gamma \in \text{GL}_r(A) \), define \( \mathfrak{C}^\gamma(z) := (c_{jl}^\gamma(z)) \in \text{Mat}_{r-1}(\mathbb{C}_\infty) \). Then

(i) \( \det(\mathfrak{C}^\gamma(z)) = \det(\gamma)^{-q^i} j(\gamma, z) \).

(ii) \( \mathfrak{C}^\gamma^{-1}(\gamma \cdot z)^{-1} = \mathfrak{C}^\gamma(z) \).
**Proof.** Consider the matrix \( \mathcal{R} \) given by

\[
\mathcal{R} := \begin{pmatrix}
L_{12} & \cdots & \cdots & L_{1r} \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
L_{(r-1)2} & \cdots & \cdots & L_{(r-1)r}
\end{pmatrix} \in \text{Mat}_{r-1}(\mathcal{H}).
\]

Writing the inverse of the period matrix \( P_z \) in terms of its cofactors and comparison of entries imply that

\[ (-1)^{r+1} \det(\mathcal{R}(z)) = \det(P_z)^{r-2}(-z_r) = -\det(P_z)^{r-2}. \]

Moreover, \([12, (3.4)]\) and Proposition \([5.4]\) imply that \( \det(P_z) = -\pi^{-(q+\cdots+q^{r-1})}h_r(z)^{-1} \) and hence we obtain

\[
(5.24) \quad \det((h_r \mathcal{R}(z))) = \pi^{-(q+\cdots+q^{r-1})(r-2)}h_r(z)
\]

where \( (h_r \mathcal{R})(z) \) is the matrix whose \((i, j)\)-th entry is the \((i, j)\)-th entry of \( \mathcal{R}(z) \) multiplied by \( h_r(z) \). On the other hand, by Lemma \([5.22]\)(iv), we have

\[
(5.25) \quad (h_r \mathcal{R})(\gamma \cdot z) = C^\gamma(z) \det(\gamma)^{-1} \begin{pmatrix}
j(\gamma, z)^q(h_rL_{12})(z) & \cdots & \cdots & j(\gamma, z)^{q^{-1}}(h_rL_{1r})(z) \\
\vdots & & & \vdots \\
j(\gamma, z)^q(h_rL_{(r-1)2})(z) & \cdots & \cdots & j(\gamma, z)^{q^{-1}}(h_rL_{(r-1)r})(z)
\end{pmatrix}.
\]

Taking the determinant of both sides of \((5.25)\) and using \((3.3)\) and \((5.24)\), we get

\[
(5.26) \quad \det(h_r \mathcal{R})(\gamma \cdot z) = \det(C^\gamma(z)) \det(\gamma)^{-(r-1)}j(\gamma, z)^{q+\cdots+q^{r-1}}\pi^{-(q+\cdots+q^{r-1})(r-2)}h_r(z)
\]

\[= \pi^{-(q+\cdots+q^{r-1})(r-2)} \det(\gamma)^{-1}j(\gamma, z)^{q^{r-1}}h_r(z). \]

Thus the first part follows from comparing two identities in \((5.26)\). To prove the second part, replacing \( \gamma \) with \( \gamma^{-1} \) in \((5.25)\), we obtain

\[
(h_r \mathcal{R})(\gamma^{-1} \cdot z) = C^{\gamma^{-1}}(z) \det(\gamma^{-1})^{-1} \begin{pmatrix}
j(\gamma^{-1}, z)^q(h_rL_{12})(z) & \cdots & \cdots & j(\gamma^{-1}, z)^{q^{-1}}(h_rL_{1r})(z) \\
\vdots & & & \vdots \\
j(\gamma^{-1}, z)^q(h_rL_{(r-1)2})(z) & \cdots & \cdots & j(\gamma^{-1}, z)^{q^{-1}}(h_rL_{(r-1)r})(z)
\end{pmatrix}.
\]
Now replacing \( z \) with \( \gamma \cdot z \) above, we obtain
\[
(5.27)
(h_r \mathcal{R})(z)
= C^{-1}(\gamma \cdot z) \det(\gamma^{-1})^{-1} \times
\begin{pmatrix}
    j(\gamma^{-1}, \gamma \cdot z)^0(h_r L_{12})(\gamma \cdot z) & \cdots & j(\gamma^{-1}, \gamma \cdot z)^q(h_r L_{1r})(\gamma \cdot z) \\
    \vdots & \ddots & \vdots \\
    j(\gamma^{-1}, \gamma \cdot z)^0(h_r L_{(r-1)2})(\gamma \cdot z) & \cdots & j(\gamma^{-1}, \gamma \cdot z)^q(h_r L_{(r-1)r})(\gamma \cdot z)
\end{pmatrix}
\]
\[
= C^{-1}(\gamma \cdot z) C^\top(z) \det(\gamma^{-1})^{-1} \det(\gamma)^{-1}
\begin{pmatrix}
    (h_r L_{12})(z) & \cdots & (h_r L_{1r})(z) \\
    \vdots & \ddots & \vdots \\
    (h_r L_{(r-1)2})(z) & \cdots & (h_r L_{(r-1)r})(z)
\end{pmatrix}
\]
\[
= C^{-1}(\gamma \cdot z) C^\top(z) (h_r \mathcal{R})(z)
\]
where the second equality follows from the fact that \( j(\gamma^{-1}, \gamma \cdot z)j(\gamma, z) = 1 \) (see [2 Lem. 3.1.3]) and (5.25). Since the matrix \((h_r \mathcal{R})(z)\) is invertible and, by the first part, \(\det(C^{-1}(\gamma \cdot z)) = \det(C^\top(z))^{-1} = \det(\gamma)^{-1} \det(j(\gamma, z) \cdot \gamma^{-1})\) is an invertible element in \(H\), we conclude the proof by multiplying both sides of (5.27) with \(C^{-1}(\gamma \cdot z)^{-1}\) from the left and then multiplying the resulting identity with \((h_r \mathcal{R})(z)^{-1}\) from the right.

The next lemma is to deduce a useful relation among the entries of \(C^\top(z)\) and the cofactors of \(\gamma^{-1}\) whenever \(\gamma \in \text{GL}_r(A)\).

**Lemma 5.28.** For any \(\gamma = (a_{ij}) \in \text{GL}_r(A)\), we have
\[
\sum_{\ell=1}^{r-1} c^\gamma_{i\ell}(z)c^\gamma_{\ell r} = -j(\gamma, z) \det(\gamma)^{-1} c^\gamma_{ir}.
\]

**Proof.** Let \(f \in M^n_k(\text{GL}_r(A))\) for some \(k \in \mathbb{Z}_{\geq 0}\) and \(m \in \mathbb{Z}/(q-1)\mathbb{Z}\). Then we have
\[
(5.29)
f(\gamma \cdot z) = j(\gamma, z)^k \det(\gamma)^{-m} f(z).
\]
To ease the notation, for each \(1 \leq i \leq r - 1\), we first set \(\partial_i := a_{i1}z_1 + \cdots + a_{ir}z_r\). Applying \(\partial_i\) to both sides of (5.29), we have
\[
\partial_i(f)(\gamma \cdot z) = \partial_i(j(\gamma, z) - a_{ri}\partial_1) + \cdots + \partial_{r-1}(f)(\gamma \cdot z)(a_{r(i-1)}j(\gamma, z) - a_{ri}\partial_{r-1})
\]
\[
= j(\gamma, z)^k \det(\gamma)^{-m} \partial_i(f)(z) + kj(\gamma, z)^{k+1} \det(\gamma)^{-m} f(z)a_{ri}.
\]
Note, by [2 Lem. 3.1.3], that \(j(\gamma, z) = j(\gamma^{-1}, \gamma \cdot z)^{-1}\). Thus, a simple calculation implies that \(a_{ij} = \det(\gamma)c^\gamma_{ij}^{-1}\) and \(\partial_i = (\gamma \cdot z)_i j(\gamma^{-1}, \gamma \cdot z)^{-1}\) where \((\gamma \cdot z)_i\) is \(i\)-th entry of \(\gamma \cdot z\). Hence (5.30) becomes
\[
\partial_i(f)(\gamma \cdot z)(c^\gamma_{i1} - c^\gamma_{ir}^{-1}(\gamma \cdot z)_1) + \cdots + \partial_{r-1}(f)(\gamma \cdot z)(c^\gamma_{ir(r-1)} - c^\gamma_{ir}^{-1}(\gamma \cdot z)_{r-1})
\]
\[
= j(\gamma^{-1}, \gamma \cdot z)^{-k-1} \det(\gamma)^{-m-1} \partial_i(f)(z) + kj(\gamma^{-1}, \gamma \cdot z)^{-k} \det(\gamma)^{-m} f(z)c^\gamma_{ir}^{-1}.
\]
We now obtain
Furthermore, for any $1 < \delta < q$ and $r < n$, we have
\begin{equation}
\partial_i(f)(\gamma \cdot z)(c_i^r - c_{i+1}^r(\gamma \cdot z)) + \cdots + \partial_{r-1}(f)(\gamma \cdot z)(c_{i(r-1)}^r - c_{i+1}^r(\gamma \cdot z)) = j(\gamma, z)^k \det(\gamma)^{-m-1} \partial_i(f)(z) + k j(\gamma, z)^k \det(\gamma)^{-m} f(z) c_i^r.
\end{equation}

On the other hand, Proposition 5.23(ii) and (5.31) imply the identity
\begin{equation}
\partial_i(f)(\gamma \cdot z) = j(\gamma, z)^k+1 \det(\gamma)^{-m-1} \partial_i(f)(z) + k j(\gamma, z)^k \det(\gamma)^{-m} f(z) \sum_{\ell=1}^{r-1} c_{i\ell}(z) c_{i\ell}^r.
\end{equation}

Now assume further that $f$ is nowhere-vanishing whose existence is known by Example 3.3(iii). Therefore, using Lemma 5.22(ii) and (5.32), we get
\begin{equation}
\sum_{\ell=1}^{r-1} c_{i\ell}(z) c_{i\ell}^r = -j(\gamma, z) \det(\gamma)^{-1} c_i^r
\end{equation}
as desired.

Let $\mathcal{Y}$ be an admissible subdomain of $\Omega^r$ and $f : \mathcal{Y} \to \mathbb{C}_\infty$ be a rigid analytic function. We define
\begin{equation}
\|f\|_{\mathcal{Y}} := \sup_{w \in \mathcal{Y}} |f(w)|.
\end{equation}

Recall the admissible covering $\{\Omega_n^r\}_{n=1}^\infty$ of $\Omega^r$ given in §3. For each $n \geq 1$, we set $\|f\|_n := \|f\|_{\Omega_n^r}$. Furthermore, for any $z \in \Omega_n^r$ and $\delta > 0$, we consider the closed polydisk $D(z, \delta)$ centered at $z$ and of radius $\delta|z|_\infty$ given by
\begin{equation}
D(z, \delta) := \{w \in \mathbb{P}^{r-1}(\mathbb{C}_\infty) \mid |z - w|_\infty \leq \delta|z|_\infty\}.
\end{equation}

Our next lemma provides an estimation of the size of the partial derivatives for rigid analytic functions on $\Omega_n^r$.

**Lemma 5.33.** Let $f : \Omega_n^r \to \mathbb{C}_\infty$ be a rigid analytic function. Then for each $z \in \Omega_n^r$ and $0 < \delta < q^{-n}$, we have
\begin{equation}
D(z, \delta) \subset \Omega_n^r.
\end{equation}
Furthermore, for any $1 \leq j \leq r - 1$, we have
\begin{equation}
\|\partial_j f\|_{D(z, \delta)} \leq \delta^{-1}|z|_\infty - 1 \|f\|_n.
\end{equation}

**Proof.** Let us denote $z = (z_1, \ldots, z_r)^{\text{tr}} \in \Omega^r_n$ and $w = (w_1, \ldots, w_r)^{\text{tr}} \in D(z, \delta)$. Then for any $K_\infty$-rational hyperplane $H$ with $\ell_H(X_1, \ldots, X_r) = a_1 X_1 + \cdots + a_r X_r$, we have
\begin{equation}
|\ell_H(w)| = |a_1 w_1 + \cdots + a_r w_r|
\end{equation}
\begin{equation}
= |a_1(w_1 - z_1) + \cdots + a_r(w_r - z_r) + a_1 z_1 + \cdots + a_r z_r|
\end{equation}
\begin{equation}
= |\ell_H(z)|
\end{equation}
\begin{equation}
\geq q^{-n}|z|_\infty
\end{equation}
where the third and forth equality follow from the facts that
\begin{equation}
|a_1(w_1 - z_1) + \cdots + a_r(w_r - z_r)| \leq \max_{1 \leq i \leq r} \{ |a_i||w_i - z_i| \} \leq \delta|z|_\infty < q^{-n}|z|_\infty
\end{equation}
and
\begin{equation}
|a_1 z_1 + \cdots + a_r z_r| = |\ell_H(z)| \geq q^{-n}|z|_\infty.
\end{equation}
It follows that $\mathbf{w} \in \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus H_n$. Since $\Omega_n^r$ is a finite intersection of sets of the form $\mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus H_n$ for some $K_\infty$-rational hyperplane $H$, we have $D(z, \delta) \subset \Omega_n^r$. Moreover, since $f$ is a rigid analytic function on $\Omega_n^r$, its restriction to $D(z, \delta)$ can be expressed as

$$f|_{D(z, \delta)} = \sum_{I=(i_1, \ldots, i_r)\in\mathbb{Z}_{>0}^r} C_I(X_1 - z_1)^{i_1} \cdots (X_r - z_r)^{i_r},$$

where $C_I \in \mathbb{C}_\infty$ and $C_I(\delta|z_\infty)^{|I|} \to 0$ as $|I| := i_1 + \cdots + i_r \to \infty$. Thus,

$$\|\partial_j f\|_{D(z, \delta)} = \sup_{w \in D(z, \delta)} \left\{ \left| \sum_{I=(i_1, \ldots, i_r)\in\mathbb{Z}_{>0}^r} i_j C_I(w_1 - z_1)^{i_1} \cdots (w_r - z_r)^{i_r} \right| \right\} \leq \delta^{-1} \sup_{w \in D(z, \delta)} \left\{ \left| \sum_{I=(i_1, \ldots, i_r)\in\mathbb{Z}_{>0}^r} i_j C_I(w_1 - z_1)^{i_1} \cdots (w_r - z_r)^{i_r} \right| \right\} \leq \delta^{-1} \sup_{|w|_\infty} ||f||_n,$$

where the last inequality follows from the fact that $D(z, \delta) \subset \Omega_n^r$. 

For any $z = (z_1, z_2, \ldots, z_r)^{tr} \in \Omega^r$, we set

$$|z|_{\text{im}} := \inf \{|z_1 - a| : a = a_2 z_2 + \cdots + a_r z_r, \ a_2, \ldots, a_r \in K_\infty\}.$$ 

Moreover, we let $\phi^\delta$ be the Drinfeld $A$-module corresponding to the $A$-lattice generated by the entries of $\hat{z} = (z_2, \ldots, z_r)^{tr} \in \Omega^{r-1}$ over $A$ given by

$$\phi^\delta_\theta := \theta + \hat{g}_1(\hat{z}) \tau + \cdots + \hat{g}_{r-1}(\hat{z}) \tau^{r-1}$$

and let $\mathfrak{L}_{ij} : \Omega^{r-1} \to \mathbb{C}_\infty$ be the rigid analytic function sending $\hat{z} \in \Omega^{r-1}$ to the $(i, j)$-cofactor of the period matrix $P_{\hat{z}}$ of $\phi^\delta$.

Before stating our next lemma, for each $w \in \Omega^{r-1}$, we let $J_w$ to be the set of elements of the form $\mathbf{z} = (z_1, w)^{tr} \in \Omega^r$ so that $|z_1| = |z|_{\text{im}}$. Recall also that $|z|_\infty = \max_{i=1}^r |z_i|$.

**Lemma 5.34.** For any $1 \leq j \leq r - 1$ and $\mathbf{w} \in \Omega^{r-1}$, we have

$$\lim_{|\mathbf{z}|_\infty \to \infty} E_{r-1}^{|j|}(\mathbf{z}) q^{r-1} < \infty.$$

**Proof.** By [12], Cor. 5.9], we know that

$$E_r^{|j|}(\mathbf{z}) = \tilde{\pi}^{q^j + \cdots + q^{r-1}} h_r(\mathbf{z}) L_j(\mathbf{z}).$$

By [12], Rem. 3.22, we have

$$\lim_{|\mathbf{z}|_\infty \to \infty} \exp_{\phi^\delta}(z_k \frac{\theta^k}{|\theta|^{k+1}}) = \exp_{\phi^\delta}(z_k \theta^k), \ k = 1, \ldots, r, \ \mu \in \mathbb{Z}_{\geq 0}.$$

By [20], Sec. 4], we have $u_{GL_r(A)}(\mathbf{z}) = \tilde{\pi}^{-1} \exp_{\phi^\delta}(z_1)^{-1}$. Recall the Anderson generating function $s_{\phi^\delta}(z, t)$ of $\phi^\delta$ for any $z \in \mathbb{C}_\infty$ defined in §5.1. Since $h_r$ is a Drinfeld cusp form, by Proposition 5.4 (5.35) and (5.36), for $1 \leq \nu \leq r - 2$, it suffices to show that

$$\lim_{|\mathbf{z}|_\infty \to \infty} \exp_{\phi^\delta}(z_1)^{-q^{r-1}} \sum_{\ell \geq 0} \exp_{\phi^\delta}(z_1 \frac{\theta^\ell}{|\theta|^\ell}) q^{r-1+\nu} \theta^{q^{r-1} t} = \lim_{|\mathbf{z}|_\infty \to \infty} \tilde{\pi}^{q^{r-1}} u_{GL_r(A)}(\mathbf{z}) q^{r-1}(s_{\phi^\delta}(z_1, t)_{t=\theta}) q^{r-1} < \infty.$$
Firstly, by [4] Prop. 4.7(d), we have \( \lim_{z \to \infty, z \in \mathbb{Z}_w} \exp_{\phi^k}(z_1) \) = 0. Now let \( \ell \in \mathbb{Z}_{\geq 0} \). A simple modification of the proof of [12] Prop. 3.24 implies that

\[
\lim_{z \to \infty, z \in \mathbb{Z}_w} \frac{\exp_{\phi^k}(z_1/\theta^{\ell+1})\theta^{\ell+1}}{\exp_{\phi^k}(z_1)} = 0
\]

uniformly in \( \ell \). On the other hand, using (5.1) recursively yields

\[
\frac{\exp_{\phi^k}(z_1/\theta^{\ell+1})\theta^{\ell+1}}{\exp_{\phi^k}(z_1)} = \frac{\theta^{q-1}\ell}{\exp_{\phi^k}(z_1)} \left( F_{1,\nu} \exp_{\phi^k}(z_1/\theta^\ell)^{q-1} + \cdots + F_{r-1,\nu} \exp_{\phi^k}(z_1/\theta^\ell)^q \right) + G_{1,\nu} \exp_{\phi^k}(z_1/\theta^{\ell+1})^{q-1} + \cdots + G_{r-1,\nu} \exp_{\phi^k}(z_1/\theta^{\ell+1})^q
\]

where for each \( \nu \), \( F_{1,\nu}, \ldots, F_{r-1,\nu}, G_{1,\nu}, \ldots, G_{r-1,\nu} \) are elements in the \( K \)-algebra generated by \( \tilde{g}_1(z), \ldots, \tilde{g}_{r-2}(z) \) and \( g_{r-1}(z) \). Thus, considering (5.39) by using (5.38) implies (5.37) and thus finishes the proof of the lemma. \( \square \)

For any \( n \in A \setminus \{0\} \), we define a principal congruence subgroup \( \Gamma(n) \) to be the kernel of the projection \( \text{GL}_r(A) \to \text{GL}_r(A/\mathfrak{n}A) \). Moreover, we call \( \Gamma \subseteq \text{GL}_r(A) \) a congruence subgroup if it contains \( \Gamma(n) \) for some non-zero \( n \). Note that every congruence subgroup of \( \text{GL}_r(A) \) is an arithmetic subgroup [2 Sec. 4.1]. Furthermore, by [20 Sec. 4], we have

\[
u_{\Gamma(n)}(z) = n^{-1} \pi^{-1} \exp_{\phi^k}(n^{-1}z_1)^{-1}.
\]

**Lemma 5.41.** For any non-zero \( n \in A \), \( 2 \leq j \leq r-1 \) and \( k \in \mathbb{Z}_{\geq 1} \), we have

\[
\partial_j(u_{\Gamma(n)}(z)^k) = -kE_{r-1}^{[j-1]}(z)u_{\Gamma(n)}(z)^k - k\pi^{-1+q+\cdots+q^{r-2}}h_{r-1}(z)u_{\Gamma(n)}(z)^k \times 
\left( L_{j}^{(1)}(z_1, t)|_{t=\theta} + \cdots + L_{j}^{(r-2)}(z_1, t)|_{t=\theta} \right).
\]

**Proof.** Set \( \exp_{\phi^k} = \sum_{i \geq 0} \tilde{\alpha}_i(z)\tau^i \). We also consider the convention that \( \tilde{\alpha}_i(z) = 0 \) when \( i < 0 \). By [3 (3.4)], we have

\[
\tilde{\alpha}_i(z) = \frac{1}{\theta^{q^i} - \theta} \left( \tilde{g}_i(z) + \sum_{\mu=1}^{i-1} \tilde{g}_\mu(z)\tilde{\alpha}_{i-\mu}(z)^{q^\mu} \right).
\]

On the other hand, by Theorem 5.19(ii), we obtain

\[
\partial_j(\tilde{\alpha}_i(z)) = \frac{1}{\theta^{q^i} - \theta} \left( E_{r-1}^{[j]}(z)\tilde{g}_i(z) + \pi^{-1+q+\cdots+q^{r-2}}h_{r-1}(z)L_{j}^{(i+1)}(z) \right) + 
\frac{1}{\theta^{q^i} - \theta} \sum_{\mu=1}^{i-1} \left( E_{r-1}^{[j]}(z)\tilde{g}_\mu(z) + \pi^{-1+q+\cdots+q^{r-2}}h_{r-1}(z)L_{j}^{(\mu+1)}(z) \right) \tilde{\alpha}_{i-\mu}(z)^{q^\mu}
\]

\[
= E_{r-1}^{[j]}(z)\tilde{\alpha}_i(z) + \pi^{-1+q+\cdots+q^{r-2}}h_{r-1}(z) \sum_{\mu=1}^{r-2} \frac{L_{j}^{(\mu+1)}(z)\tilde{\alpha}_{i-\mu}(z)^{q^\mu}}{\theta^{q^\mu} - \theta}
\]
where the last equality follows from (5.12) and fact that $\mathfrak{L}_{j+1} \equiv 0$ if $\mu \geq r - 1$. Thus, we get

\begin{equation}
\sum_{i \geq 0} \partial_j(\tilde{\alpha}_i)(\tilde{z})(n^{-1}z_1)^{q^i} = E_{r-1}^{[j-1]}(\tilde{z}) \exp_{\phi^k}(n^{-1}z_1) + \pi^{q^{r-1} + q^{r-2}} h_{r-1}(\tilde{z}) \times
\end{equation}

\[
\left( \mathfrak{L}_{j+1}(\tilde{z}) \sum_{n \geq 0} \tilde{\alpha}_n(\tilde{z}) q^n (n^{-1}z_1)^{q^{n+1}} + \cdots + \right.
\]

\[
\mathfrak{L}_{j+1}(\tilde{z}) \sum_{n \geq 0} \tilde{\alpha}_n(\tilde{z}) q^{r+2-n} (n^{-1}z_1)^{q^{r+2-n}}
\]

\[
= E_{r-1}^{[j-1]}(\tilde{z}) \exp_{\phi^k}(n^{-1}z_1) + \pi^{q^{r-1} + q^{r-2}} h_{r-1}(\tilde{z}) \left( \mathfrak{L}_{j+1}(\tilde{z}) s_{\phi^k}^{(1)}(n^{-1}z_1, t) \right)_{t=\theta}
\]

\[
+ \cdots + \mathfrak{L}_{j+1}(\tilde{z}) s_{\phi^k}^{(r-2)}(n^{-1}z_1, t) \right)_{t=\theta}
\]

where the last equality follows from (5.33). It can be now easily seen that (5.40) and (5.43) imply the desired equality.

**Proposition 5.44.** Let $f \in M_k^n(\Gamma(n))$ for some non-zero element $n \in A$. Then for any $1 \leq j \leq r - 1$ and $w \in \Omega^{-1}$, we have

\[
\lim_{\substack{z \in \mathfrak{C}_n \atop |z| \to \infty}} D_{j,k} (f)(z)^{q^{r-1}} = \lim_{\substack{z \in \mathfrak{C}_n \atop |z| \to \infty}} (\partial_j (f)(z) + kE_r^{[j]}(z)f(z))^{q^{r-1}} < \infty.
\]

**Proof.** Since $f$ is a Drinfeld modular form for $\Gamma(n)$, there exists a unique rigid analytic function $\mathfrak{F}_i : \Omega^{-1} \to \mathbb{C}_\infty$ for each $i \in \mathbb{Z}_{\geq 0}$ such that

\[
f(z) = \sum_{i \geq 0} \mathfrak{F}_i(z) u_{\Gamma(n)}(z)^i
\]

whenever $z$ is in some neighborhood of infinity. When $j = 1$, the proposition follows immediately since $D_1(f)$ is $\Gamma(n)$-invariant and $f$ is a Drinfeld modular form. We assume that $2 \leq j \leq r - 1$. Applying $\partial_j$ to both sides of above, using Lemma 5.41 and the definition of the operator $D_{j,k}$, we obtain

\[
D_{j,k}(f)(z) = \sum_{i \geq 0} (\partial_j \mathfrak{F}_i)(z) u_{\Gamma(n)}(z)^i + \sum_{i \geq 1} \mathfrak{F}_i(z) \partial_j u_{\Gamma(n)}(z)^i + kf(z) E_r^{[j]}(z)
\]

\[
= \sum_{i \geq 0} (\partial_j \mathfrak{F}_i)(z) u_{\Gamma(n)}(z)^i - kE_r^{[j]}(z) \sum_{i \geq 1} \mathfrak{F}_i(z) u_{\Gamma(n)}(z)^i
\]

\[
- kn \pi^{q^{r-1} + q^{r-2}} h_{r-1}(\tilde{z}) \sum_{i \geq 1} \mathfrak{F}_i(z) u_{\Gamma(n)}(z)^i \left( \mathfrak{L}_{j+1}(\tilde{z}) s_{\phi^k}^{(1)}(n^{-1}z_1, t) \right)_{t=\theta}
\]

\[
+ \cdots + \mathfrak{L}_{j+1}(\tilde{z}) u_{\Gamma(n)}(z) s_{\phi^k}^{(r-2)}(n^{-1}z_1, t) \right)_{t=\theta} + kf(z) E_r^{[j]}(z).
\]

It can be seen from the proof of Lemma 5.34 that, for $k = 1, \ldots, r - 2$, we have

\[
\lim_{\substack{z \in \mathfrak{C}_n \atop |z| \to \infty}} u_{\Gamma(n)}(z)^{q^{r-1}}(s_{\phi^k}^{(1)}(n^{-1}z_1, t)_{t=\theta})^{q^{r-1}} < \infty.
\]
On the other hand, since $f$ is a Drinfeld modular form, by [4] Prop. 4.7(d), Lem. 5.1, we have
\[ \lim_{z \in \mathbb{C}_w} \sum_{i \geq 1} \mathcal{F}_i(z) u_{\Gamma}(z)^i = 0, \]
and using [4, Lem. 5.1] and Lemma 5.3 we further get
\[ \lim_{z \in \mathbb{C}_w} \sum_{i \geq 0} \sum (\partial_j \mathcal{F}_i)(z) u_{\Gamma}(z)^i < \infty. \]
Thus, by Lemma 5.34 and the above discussion, we obtained the desired boundedness. \hfill \Box

We are now ready to prove the main result of this subsection.

**Theorem 5.45.** For each $1 \leq i \leq r - 1$, let $f_i \in M^{m_i}_{k_i}(\Gamma_i)$ for some congruence subgroup $\Gamma_i \leq \text{GL}_r(A)$, $k_i \in \mathbb{Z}_{\geq 0}$ and $m_i \in \mathbb{Z}/(q - 1)\mathbb{Z}$. We further let $\Gamma' := \cap_{i=1}^{r} \Gamma_i$. Set $\mathfrak{f} := k_1 + \cdots + k_{r-1} + r$ and $m := m_1 + \cdots + m_{r-1} + 1$. Consider the operator $D_{(k_1, \ldots, k_{r-1})}$ on $M^{m_1}_{k_1}(\Gamma_1) \times \cdots \times M^{m_{r-1}}_{k_{r-1}}(\Gamma_{r-1})$ defined by
\[
D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1}) = \det \begin{pmatrix}
D_{1,k_1}(f_1) & \cdots & D_{r-1,k_1}(f_1) \\
\vdots & & \vdots \\
D_{1,k_{r-1}}(f_{r-1}) & \cdots & D_{r-1,k_{r-1}}(f_{r-1})
\end{pmatrix}.
\]
The following statements hold.

(i) $D_{(k_1, \ldots, k_{r-1})}$ is a $\mathbb{C}_\infty$-multilinear derivation. In other words, for any $f_i, \tilde{f}_i \in M^{m_i}_{k_i}(\Gamma_i)$ and $f_j \in M^{m_j}_{k_j}(\Gamma_j)$, we have
\[
D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_i, \tilde{f}_i, f_{i+1}, \ldots, f_{r-1}) = D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_i, f_{i+1}, \ldots, f_{r-1}) + D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, \tilde{f}_i, f_{i+1}, \ldots, f_{r-1})
\]
and
\[
D_{(k_1, \ldots, k_i+k_j, \ldots, k_{r-1})}(f_1, \ldots, f_{i-1}, f_i, f_j, f_{i+1}, \ldots, f_{r-1}) = f_j D_{(k_1, \ldots, k_i, \ldots, k_{r-1})}(f_1, \ldots, f_i, f_{i+1}, \ldots, f_{r-1}) + f_i D_{(k_1, \ldots, k_j, \ldots, k_{r-1})}(f_1, \ldots, f_{i-1}, f_j, f_{i+1}, \ldots, f_{r-1}).
\]

(ii) $D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1}) \in M^{m}_{\mathfrak{f}}(\Gamma')$.

**Proof.** To prove part (i), by Lemma 5.22(iii), observe for $1 \leq \mu \leq r - 1$ that $D_{\mu,k_i}(f_i + \tilde{f}_i) = D_{\mu,k_i}(f_i) + \mathcal{D}_{\mu,k_i}(\tilde{f}_i)$ and $D_{\mu,k_i+k_j}(f_i,f_j) = f_j D_{\mu,k_i}(f_i) + f_i D_{\mu,k_j}(f_j)$. Thus the desired equalities follow from the sum property of determinants.

We now prove the second part. Note that using Lemma 5.22(iii) and Proposition 5.23(i), for any $\gamma \in \Gamma'$, we obtain
\[
D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})(\gamma \cdot z) = \det(\gamma)^{-m-r+2j} \gamma(z)^{t-1} \det(\mathcal{C}^\gamma(z)) D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})(z)
\]
\[
= \det(\gamma)^{-m} j(\gamma, z)^{t} D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})(z).
\]
Therefore $D_{r}(f_1, \ldots, f_{r-1})(z)$ is a weak modular form of weight $\mathfrak{f}$ and type $m$ for $\Gamma'$.

Let $\delta \in \text{GL}_r(A)$ and consider the congruence subgroup $\bar{\Gamma} := \delta^{-1} \Gamma' \delta \leq \text{GL}_r(A)$. There exists a non-zero element $n \in A$ such that $\Gamma(n) \subseteq \bar{\Gamma}$. We claim that the function
\[
D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})|_{t,m}[\delta](z) = j(\delta, z)^{-t} \det(\delta)^{m} D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})(\delta \cdot z)
\]
is holomorphic at infinity with respect to \( \Gamma(n) \). Once the claim is established, we conclude that \( D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})|_{t,m}[\delta] \) is holomorphic at infinity with respect to \( \Gamma \) by using [4 Prop. 5.14]. By [4 Prop. 6.3(c)], this will conclude the the proof of the theorem.

Firstly, for any \( 1 \leq j \leq r - 1 \), set

\[
(5.46) \quad f_j(\delta \cdot z) = j(\delta, z)^{k_j} \det(\delta)^{-m_j} g_j(z)
\]

where \( g_j \) is a Drinfeld modular form of weight \( k_j \) and type \( m_j \) for \( \delta^{-1} \Gamma_j \delta \leq \text{GL}_r(A) \) (see [4 Prop. 6.6]). Using (5.32), for each \( 1 \leq i \leq r - 1 \), one can obtain

\[
(5.47) \quad \partial_i(f_j)(\delta \cdot z) = j(\delta, z)^{k_j} \det(\delta)^{-m_j} \left( j(\delta, z) \det(\delta)^{-1} \sum_{l=1}^{r-1} \partial_l(g_j)(z) c_{il}^\delta(z) + k_j g_j(z) \sum_{k=1}^{r-1} c_{ik}^\delta(z) c_{kr}^{\delta^{-1}} \right)
\]

\[
= j(\delta, z)^{k_j+1} \det(\delta)^{-m_j-1} \sum_{l=1}^{r-1} \partial_l(g_j)(z) c_{il}^\delta(z) - k_j j(\delta, z)^{k_j+1} \det(\delta)^{-m_j-1} g_j(z) c_{ir}^{\delta}
\]

\[
= j(\delta, z)^{k_j+1} \det(\delta)^{-m_j-1} \sum_{l=1}^{r-1} \partial_l(g_j)(z) c_{il}^\delta(z) - k_j j(\delta, z) \det(\delta)^{-1} f_j(\delta \cdot z) c_{ir}^{\delta}
\]

where the second equality follows from Lemma 5.28 and the third equality follows from (5.46). Thus, Lemma 5.22(1) and (5.47) imply that

\[
(5.48) \quad D_{i,k_j}(f_j)(\delta \cdot z) = (\partial_i(f_j) + k_j f_j E_r^{[i]})(\delta \cdot z) = j(\delta, z)^{k_j+1} \det(\delta)^{-m_j-1} \sum_{l=1}^{r-1} D_{i,k_j}(g_j)(z) c_{il}^\delta(z).
\]

Hence, by Proposition 5.23(i) and (5.48), we obtain

\[
D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})|_{t,m}[\delta] = \det \begin{pmatrix} D_{1,k_1}(g_1) & \cdots & D_{r-1,k_1}(g_1) \\ \vdots & \ddots & \vdots \\ D_{1,k_{r-1}}(g_{r-1}) & \cdots & D_{r-1,k_{r-1}}(g_{r-1}) \end{pmatrix}.
\]

Since, for each \( 1 \leq i \leq r - 1 \), \( g_i \) is a Drinfeld modular form for \( \Gamma(n) \), the above discussion implies that \( D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})|_{t,m}[\delta] \) is a weak modular form of weight \( t \) and type \( m \) for \( \Gamma(n) \). Therefore, for any \( z \in \Omega^r \) in some neighborhood of infinity, there exists \( n_0 \in \mathbb{Z} \) and for each \( n \geq n_0 \), a unique rigid analytic function \( F_n : \Omega^{r-1} \to \mathbb{C} \) such that

\[
D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})|_{t,m}[\delta](z) = \lim_{n \geq n_0} F_n(z) u_{\Gamma(n)}(z).
\]

Assume that \( n_0 < 0 \) and let \( w \in \Omega^{r-1} \). Then Proposition 5.42 implies

\[
\infty > \lim_{z \in \mathbb{C}[w]} D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})|_{t,m}[\delta](z) q^{r-1} = \lim_{z \in \mathbb{C}[w]} \sum_{n \geq n_0} F_n(z) q^{r-1} u_{\Gamma(n)}(z) q^{r-1} \to \infty
\]

where the fact that the right hand side approaches to the infinity follows from our assumption on \( n_0 \). This gives us a contradiction and hence \( n_0 \) must be a non-negative integer. Thus \( D_{(k_1, \ldots, k_{r-1})}(f_1, \ldots, f_{r-1})|_{t,m}[\delta] \) is holomorphic at infinity with respect to \( \Gamma(n) \).

\[ \square \]

**Remark 5.49.** We highlight the work of Pellarin [30 Sec. 6.1] providing a multi-linear differential operator by using the partial derivatives of Hilbert modular forms which may be compared with our construction.
6. THE $\mathbb{C}_\infty$-ALGEBRA $\mathcal{M}_r$

In this section we analyze certain properties of the $\mathbb{C}_\infty$-algebra $\mathcal{M}_r$ introduced in §1 and determine the transcendence of non-zero values of some explicitly defined functions at special points. Our methods mainly rely on the results obtained in §5 as well as [12, Sec. 6]. Throughout this section, we continue to use the notation given in §5 as well as [12, Sec. 6].

6.1. Some properties of $\mathcal{M}_r$. For the convenience of the reader, we recall the definition of $\mathcal{M}_r$. To begin with, for $1 \leq i \leq r - 1$, we set

$$g_i^{\text{new}} = \tilde{\pi}^{1-q^i} g_i$$

and moreover, for $1 \leq j \leq r$, we let

$$E_{ij} = \tilde{\pi}^{q^1 + \cdots + q^{r-1} - q^{j-1}} h_r L_{ij}.$$ 

We define the $\mathbb{C}_\infty$-algebra $\mathcal{M}_r$ generated by the following rigid analytic functions

$$\mathcal{M}_r = \mathbb{C}_\infty[g_i^{\text{new}}, E_{ij} | 1 \leq i \leq r - 1, 1 \leq j \leq r].$$

Remark 6.1. Analyzing a subset of generators of $\mathcal{M}_r$, we can explain the reason for the normalization used in the definition of $E_{ij}$ as follows. Generalizing the definition of integral forms given in [39, Def. 3.5], we define an arithmetic function of rank 2 to be a rigid analytic function $g : \Omega^2 \to \mathbb{C}_\infty$ which has a $u$-expansion with coefficients in $\mathbb{K}$. Moreover, for arbitrary $r \geq 3$, we say that a rigid analytic function $g : \Omega^r \to \mathbb{C}_\infty$ is an arithmetic function of rank $r$ if it has a $u$-expansion of the form $g = \sum_{i \geq 0} g_i u^i$ for some arithmetic functions $g_i$ of rank $r - 1$ where $i \in \mathbb{Z}_{\geq 0}$. By [39, Thm. 1.1] and [12, Thm. 1.1, (4.2)], we know that for each $1 \leq i \leq r - 1$, $g_i^{\text{new}}$ and $E_{11}$ are arithmetic functions of rank $r$. Furthermore, for each $2 \leq j \leq r$, by using the identity

$$E_{1j}(z) = \tilde{\pi}^{-1} \partial_1 (g_1^{\text{new}}(z)) - E_{11}(z) g_1^{\text{new}}(z)$$

which follows from [12, Thm 1.1] and Theorem [5.19(ii)], one can see that $E_{1j}$ is also an arithmetic function of rank $r$. This observation suggests our normalization for $E_{1j}$. Moreover, as studied in the next subsection, we will obtain certain transcendence results for special values of not only these functions discussed above, but also for each $E_{ij}$.

Lemma 6.2. Let $z \in \Omega^r$. For each $1 \leq i, j, l \leq r - 1$, we have

$$\partial_l(h_r L_{j(i+1)})(z) = -E_r^{[j]}(z) h_r(z) L_{l(i+1)}(z).$$

Proof. Observe that we have

$$\tilde{\pi}^{q^1 + \cdots + q^{r-1}} \partial_l(h_r L_{j(i+1)})(z) = (\partial_l \circ \partial_j)(g_i)(z) - \partial_l(E_r^{[j]} g_i)(z)

= E_r^{[l]}(z) E_r^{[j]}(z) g_i(z) - E_r^{[l]}(z) E_r^{[j]}(z) g_i(z)

- \tilde{\pi}^{q^1 + \cdots + q^{r-1}} E_r^{[j]}(z) h_r(z) L_{l(i+1)}(z)

= -\tilde{\pi}^{q^1 + \cdots + q^{r-1}} E_r^{[j]}(z) h_r(z) L_{l(i+1)}(z)$$

where the first equality follows from Theorem [5.19(iii)] and the second equality follows from Proposition [4.3] and Lemma [4.3]. This calculation finishes the proof.

Now we are ready to prove the main result of this subsection.
Theorem 6.3. The $\mathbb{C}_\infty$-algebra $\mathcal{M}_r$ is stable under the partial derivatives $\partial_1, \ldots, \partial_{r-1}$ and strictly contains the graded $\mathbb{C}_\infty$-algebra of Drinfeld modular forms for $GL_r(A)$.

Proof. Using [12] Thm. 1.1, Cor. 5.9, for each $1 \leq i \leq r-1$, we have that $E_{i1} = \tilde{\pi}^{-1}F_{r}[i]$. Hence the stability of $E_{i1}$ under $\partial_i$ for $1 \leq l \leq r-1$ follows from Lemma 4.4. Similarly, the stability of the functions $g_{i}^w$ and $E_{i(i+1)}$ follow from Theorem 5.19 and Lemma 6.2 concluding the proof of the first part. The identity [5.24] implies that $h_r \in \mathcal{M}_r$. Hence the proof of the second part of the theorem is completed by using [1] Thm. 17.6.

6.2. Transcendence of special functions at CM points. Throughout this subsection, we let $r \geq 2$ and $z = (z_1, \ldots, z_r) \in \Omega^r$ be a CM point, that is the Drinfeld $A$-module $\phi^z$ corresponding to the $A$-lattice $A^r z$ is a CM Drinfeld $A$-module. Moreover, we choose an element $w_{\phi} \in \mathbb{C}^\infty_\phi$ so that

$$g_r(z)w_1^{1-q_r} = 1.$$  

For any $w \in \mathbb{C}^\infty_\phi$, let $\phi^{wz}$ be a CM Drinfeld $A$-module defined over $\overline{K}$ with the period lattice $A^r w_1 z$ for some $w \in \mathbb{C}^\infty_\phi$ and $z' \in \Omega^r$. Consider $P_{wz}$ to be its period matrix as in [12]. We set $\overline{K}(P_{wz})$ to be the field generated by the entries of $P_{wz}$ over $\overline{K}$. We express $wz' := (w_1, \ldots, w_{r-1}, w) \in \mathbb{C}^\infty_\phi$ and recall the quasi-periodic functions $F_{wz}^{wz'}, \ldots, F_{wz}^{wz', r}$ defined in Proposition 5.4. Setting $F_{wz}^{wz'}(z) := \exp(\phi(z) - z)$ for $z \in \mathbb{C}_\phi$, by [8] (3.12), (3.13)] we know, for $0 \leq k \leq r-1$, that $F_{\tau^k}(w_i) = L_{ik}(w, F_{\tau^k}^{wz'}(w), \ldots, F_{\tau^{r-1}}^{wz'}(w))$ where $L_{ik}(X_0, \ldots, X_{r-1}) \in \overline{K}[X_0, \ldots, X_{r-1}]$ is a homogeneous polynomial of total degree 1. We set

$$P(X_0, \ldots, X_{r-1}) := \begin{pmatrix} L_{10} & \cdots & L_{1(r-1)} \\ \vdots & \ddots & \vdots \\ L_{r0} & \cdots & L_{r(r-1)} \end{pmatrix} \in \text{Mat}_r(\overline{K}[X_0, \ldots, X_{r-1}])$$

and note that $P(w, F_{\tau}^{wz'}(w), \ldots, F_{\tau^{r-1}}^{wz'}(w)) = P_{wz}$. For $1 \leq i, j \leq r$, we set

$$Q_{ij}(X_0, \ldots, X_{r-1}) := \text{the } (i, j) \text{ cofactor of } P(X_0, \ldots, X_{r-1}) \in \overline{K}[X_0, \ldots, X_{r-1}].$$

We further define

$$D(X_0, \ldots, X_{r-1}) := \det(P(X_0, \ldots, X_{r-1})) \in \overline{K}[X_0, \ldots, X_{r-1}].$$

Note that

$$Q_{ij}(w, F_{\tau}^{wz'}(w), \ldots, F_{\tau^{r-1}}^{wz'}(w)) = L_{ij}(wz')$$

and by [12] (3.8)] we have

$$D(w, F_{\tau}^{wz'}(w), \ldots, F_{\tau^{r-1}}^{wz'}(w)) = \det(P_{wz'}) = -\frac{\tilde{\pi}}{q^{-1}(-1)^{r-1}g_r(wz')}.$$  

The following algebraically independent set plays a crucial role in this subsection.

**Proposition 6.4.** Let

$$\mathcal{S} := \{ \frac{w}{\tilde{\pi}}, \frac{F_{\tau}^{wz'}(w)}{\tilde{\pi}}, \ldots, \frac{F_{\tau^{r-1}}^{wz'}(w)}{\tilde{\pi}} \}.$$  

Then $\mathcal{S}$ is an algebraically independent set over $\overline{K}$.  

Proof. By [11] Thm. 1.2.2, we know that
\[ \text{tr. deg}_K \mathcal{K}(P_{w^r}) = r. \]
On the other hand, by [8] (3.12), (3.13)], we know that the set
\[ S := \{ w, F_{\tau}^{\phi}(w), \ldots, F_{\tau}^{\phi}(w) \} \]
generates \( \mathcal{K}(P_{w^r}) \) over \( K \). Thus, these two results imply that \( S \) forms a transcendence basis for \( \mathcal{K}(P_{w^r}) \) over \( K \). On the other hand, by [12] (3.8)], we have
\[ \mathcal{K}(S) = \mathcal{K} (\mathcal{G} \cup \{ \tilde{\pi} \} ). \]
Since \( \text{tr. deg}_K \mathcal{K}(P_{w^r}) = |\mathcal{G}| = r \), it remains to show that \( \tilde{\pi} \) is algebraic over \( \mathcal{K}(\mathcal{G}) \). Let
\[ f_1(Y) := D \left( \frac{w}{\tilde{\pi}}, \frac{F^{\phi}(w)}{\tilde{\pi}}, \ldots, \frac{F_{\tau}^{\phi}(w)}{\tilde{\pi}} \right) Y^{r-1} + (-1)^{\frac{r-1}{2}} = - \left( \frac{Y}{\tilde{\pi}^{-\frac{1}{q-1}}/(1)} \right)^{r-1} + (-1)^{\frac{r-1}{2}} \in \mathcal{K}(\mathcal{G})[Y]. \]
Then \( f_1(Y) \) is a non-zero polynomial with coefficients in \( \mathcal{K}(\mathcal{G}) \) such that \( f_1(\tilde{\pi}) = 0 \), which implies that \( \tilde{\pi} \) is algebraic over \( \mathcal{K}(\mathcal{G}) \).

For \( 1 \leq i \leq r - 1 \), let \( J_i : \Omega^r \rightarrow \mathbb{C}_\infty \) be the function given by
\[ J_i(z) := \frac{g_i(z)_{(q^{r}-1)/(q^{\gcd(i,r)}-1)}}{g_r(z)_{(q^{r}-1)/(q^{\gcd(r,r)}-1)}} \]
so that it is well-defined due to the nowhere-vanishing of \( g_r(z) \). By [16] and [27] Prop. 6.2, \( J_i(z) \in \mathcal{K} \) for each \( 1 \leq i \leq r - 1 \).

Let \( \mathcal{K}[g_1^{\text{new}}, \ldots, g_{r-1}^{\text{new}}, h_r] \) be the \( \mathcal{K} \)-algebra of Drinfeld modular forms over \( \mathcal{K} \) for \( \text{GL}_r(A) \). Our next theorem concerns the transcendence of the non-zero values of any non-constant function in \( \mathcal{K}[g_1^{\text{new}}, \ldots, g_{r-1}^{\text{new}}, h_r] \) at CM points.

For the convenience of our later use, for any \( \alpha, \beta \in \mathbb{C}_\infty^\times \), we write \( \alpha \sim \beta \) if \( \alpha/\beta \in \mathcal{K} \).

**Theorem 6.5.** Let \( f \in \mathcal{K}[g_1^{\text{new}}, \ldots, g_{r-1}^{\text{new}}, h_r] \) be a Drinfeld modular form of weight \( \ell \) and type \( m \) for \( \text{GL}_r(A) \). If \( f(z) \neq 0 \), then
\[ f(z) \sim \left( \frac{w^r_{\phi}}{\tilde{\pi}} \right)^{\ell}. \]
In particular, \( f(z) \) is transcendental over \( \mathcal{K} \).

Proof. First we show that for \( 1 \leq i \leq r \), if \( g_i^{\text{new}}(z) \neq 0 \), then \( g_i^{\text{new}}(z) \notin \mathcal{K} \) by using an idea of Chang [91] Thm. 2.2.1]. By [27] Prop. 6.2, the Drinfeld \( A \)-module
\[ \phi_{\theta}^{w_{\phi}z} = w_{\phi}^{q^{\ell}} = \theta + J_1(z)^{q^{\ell}-1} \tau + \cdots + J_{r-1}(z)^{q^{\ell}-1} \tau^{r-1} + \tau^r \]
is defined over \( \mathcal{K} \) with the period lattice \( A^r w_{\phi} z \). On the other hand, we have
\[ \phi_{\theta}^{w_{\phi}z} = w_{\phi}^{q^{\ell}} = \theta + g_1(z)w_{\phi}^{q^{\ell}} \tau + \cdots + g_{r-1}(z)w_{\phi}^{1-q^{r}} \tau^{r-1} + \tau^r. \]
By comparing the $\tau$-coefficients, for $1 \leq i \leq r$, we have $g_i(z) \sim w^q 
abla^{-1}$ which implies that

$$g_i^{\text{new}}(z) \sim \left(\frac{w}{\pi}\right)^q i^{-1}.$$

By Proposition 6.4 we know that the set

$$\mathcal{S} = \left\{ \frac{w}{\pi}, \frac{F_{q^2}w^\pi}{\pi}(w_{\phi}), \ldots, \frac{F_{q^r}w^\pi}{\pi}(w_{\phi}) \right\}$$

is algebraically independent over $\overline{K}$ and particularly $w_{\phi}/\pi$ is transcendental over $\overline{K}$. The desired result now follows. Now let $f \in \overline{K}[g_1^{\text{new}}, \ldots, g_r^{\text{new}}, h_r]$ be a Drinfeld modular form of weight $\ell$ and type $m$ for $\text{GL}_r(A)$. Then we can express $f$ as

$$f = \sum_{e=(\epsilon_1, \ldots, \epsilon_r) \in \mathbb{Z}_{\geq 0}^r, \epsilon_r \equiv \ell \pmod{q-1}} c_e \left( g_1^{\text{new}} \right)^{\epsilon_1} \cdots \left( g_1^{\text{new}} \right)^{\epsilon_r} h_r, \ c_e \in \overline{K}.$$

By (3.6) and (6.7), we have

$$h_r(z) \sim \left(\frac{w}{\pi}\right)^{q^r-1}/(q-1).$$

Since $\epsilon_1(q-1) + \cdots + \epsilon_r(q^r-1) = \ell$, again by (6.7) and (6.8), if $f(z) \not\equiv 0$, then $f(z) \sim \left(\frac{w}{\pi}\right)^{\ell}$. The last assertion follows from the proof of [12, Prop. 6.2] (see also [11, Thm. 1.2.2]).

**Remark 6.9.** In this remark, we briefly explain how Theorem 6.5 can be interpreted in terms of the terminology in [6]. Let $L$ be a finite separable extension of $K$. Following [6, Sec. 3], we call a subfield $K^+$ of $L$ *totally real* if the infinite place $\infty$ of $K$ splits completely in $K^+$. Moreover, we say that $L$ is a *CM field* if every place of its maximal totally real subfield lying over $\infty$ has a unique extension to $L$. Let $J_L$ be the set of all embeddings from $L$ into $\mathbb{C}_\infty$ and $I_L$ be the free abelian group generated by $J_L$. Then, by [6, Thm. 1.3.1], there exists a bi-additive pairing $\mathcal{P}_L : I_L \times I_L \to \mathbb{C}_\infty^\times/\overline{K}^\times$ which serves as an analogue of Shimura’s period symbols in the function field setting. Let $\phi$ be a CM Drinfeld $A$-module defined over $\overline{K}$. If $L := \text{End}(\phi) \otimes_A K$ is separable over $K$, then it is a CM field by [25, Prop. 4.7.17].

Note that the map

$$\Xi : \text{End}(\phi) \to \mathbb{C}_\infty$$

$$\sum_{i=0}^m c_i \tau^i \mapsto c_0$$

induces an embedding from $L$ into $\mathbb{C}_\infty$. By abuse of the notation, we still denote it by $\Xi : L \to \mathbb{C}_\infty$. Then, by the proof of [6, Thm. 8.5.2], for any non-zero element $w$ in $\text{Ker}(\exp_{\phi})$, we have

$$w \sim p_L(\Xi, \Xi),$$

where $p_L(\Xi, \Xi)$ is any representative of $\mathcal{P}_L(\Xi, \Xi)$ in $\mathbb{C}_\infty^\times$. In particular, Theorem 6.5 can be restated as follows: Let $f \in \overline{K}[g_1^{\text{new}}, \ldots, g_r^{\text{new}}, h_r]$ be a Drinfeld modular form of weight $\ell$ and type $m$ for $\text{GL}_r(A)$ and $z \in \Omega^r$ be a CM point. If $f(z) \not\equiv 0$, then

$$f(z) \sim \left(\frac{p_L(\Xi, \Xi)}{\pi}\right)^\ell,$$
where $L_z := \text{End}(\phi^z) \otimes_A K$.

The next proposition is crucial for proving the transcendence of $E_{ij}$ at CM points and [12 Prop. 6.2] can be seen as a special case of the following result when $(i, j, \ell) = (1, 1, 1)$.

**Proposition 6.10.** Let $\phi^{wz'}$ be the CM Drinfeld $A$-module defined over $K$ of rank $r$ whose $r$-th coefficient is 1 with period lattice $A'wz'$ for some $w \in \mathbb{C}_\infty^\times$ and $z' \in \Omega^r$. Assume that for some $1 \leq i \leq r - 1$ and $1 \leq j \leq r$, $L_{ij}(wz') \neq 0$. Then for all $\ell \in \mathbb{Z}_{\geq 0}$, $\Omega_{ij\ell} := (\frac{w}{\pi})^\ell \left( \frac{L_{ij}(wz')}{\pi} \right) \in \mathbb{C}_\infty^\times$ is transcendental over $K$.

**Proof.** By Proposition 6.4, the field $K(\mathfrak{S})$ is of transcendence degree $r$ over $K$ with a transcendence basis

$$\mathfrak{S} = \{ \frac{w}{\pi}, \frac{F_{r}^{\phi^{wz'}}(w)}{\pi}, \ldots, \frac{F_{r-1}^{\phi^{wz'}}(w)}{\pi} \}.$$  

We first claim that $\Omega_{ij\ell}$ is algebraic over $K(\mathfrak{S})$. Let

$$f_2(Y) := D\left( \frac{w}{\pi}, \frac{F_{r}^{\phi^{wz'}}(w)}{\pi}, \ldots, \frac{F_{r-1}^{\phi^{wz'}}(w)}{\pi} \right)^{r-2} Y^{r-1}$$

$$- (-1)^{\frac{(r-2)(q-r)}{q-1}} \left( \frac{w}{\pi} \right)^{\ell(1)} \left( \frac{w}{\pi} \right)^{\ell(r-1)} \left( \frac{L_{ij}(wz')}{\pi} \right)^{r-1} \in K(\mathfrak{S})[Y].$$

Then $f_2(Y)$ is a non-zero polynomial with coefficients in $K(\mathfrak{S})$ such that $f_2(\Omega_{ij\ell}) = 0$, which implies that $\Omega_{ij\ell}$ is algebraic over $K(\mathfrak{S})$ as desired. Now we claim that the set $\mathfrak{S}$ is algebraically dependent over $K(\Omega_{ij\ell})$. Let

$$f_3(X_0, \ldots, X_{r-1}) := D(X_0, \ldots, X_{r-1})^{r-2} \Omega_{ij\ell}^{-1}$$

$$- (-1)^{\frac{(r-2)(q-r)}{q-1}} \left( X_0^{\ell} Q_{ij}(X_0, \ldots, X_{r-1}) \right)^{r-1} \in K(\Omega_{ij\ell})[Y].$$

Since $\Omega_{ij\ell} \neq 0$, $D(X_0, \ldots, X_{r-1})^{r-2} \Omega_{ij\ell}^{-1}$ is a non-zero homogeneous polynomial of degree $r(r - 2)$. On the other hand, $\left( X_0^{\ell} Q_{ij}(X_0, \ldots, X_{r-1}) \right)^{r-1}$ is a non-zero homogeneous polynomial of degree $(\ell + r - 1)(r - 1)$. Therefore $f_3$ is not identically zero as it is formed by two non-zero homogeneous polynomials of different total degree. Finally, since

$$f_3\left( \frac{w}{\pi}, \frac{F_{r}^{\phi}(w)}{\pi}, \ldots, \frac{F_{r-1}^{\phi}(w)}{\pi} \right) = f_2(\Omega_{ij\ell}) = 0,$$

we conclude that $\mathfrak{S}$ is algebraically dependent over $K(\Omega_{ij\ell})$. In particular,

$$\text{tr. deg}_{K(\Omega_{ij\ell})} K(\mathfrak{S} \cup \{ \Omega_{ij\ell} \}) < |\mathfrak{S}| = r.$$

Hence $\text{tr. deg}_{K(\Omega_{ij\ell})} K(\mathfrak{S} \cup \{ \Omega_{ij\ell} \}) = \text{tr. deg}_{K(\mathfrak{S} \cup \{ \Omega_{ij\ell} \})} K(\mathfrak{S} \cup \{ \Omega_{ij\ell} \}) - \text{tr. deg}_{K(\Omega_{ij\ell})} K(\mathfrak{S} \cup \{ \Omega_{ij\ell} \}) > 0$ which gives the desired result. \hfill \Box

Now we are ready to prove the transcendence of the value of $E_{ij}$ at a CM point. Note that [12 Thm. 6.3] is a special case of the following theorem in the case $(i, j) = (1, 1)$.  


Theorem 6.11. For $1 \leq i \leq r - 1$ and $1 \leq j \leq r$, if $E_{ij}(z) \neq 0$, then it is transcendental over $\overline{K}$.

Proof. By [27, Prop. 6.2], the Drinfeld $\mathcal{A}$-module $\phi^{w_{\phi}z} = w_{\phi} \phi z w_{\phi}^{-1}$ can be given as in (6.6) which is defined over $\overline{\mathcal{A}}$ with the period lattice $w_{\phi}z \mathcal{A}$. Since $g_r(z) = w_{\phi}^{-1}$, we have $h_r(z) \sim (w_{\phi}/\pi)^{1+\cdots+q^{-1}}$. By the uniqueness of the function $F_{\tau r}^{\phi}$, we get $F_{\tau r}^{\phi}(X) = w_{\phi}^{q_{r}} F_{\tau r}^{\phi}(X/w_{\phi})$ for any $X \in \mathbb{C}_\infty$. In particular, one has $w_{\phi}^{q_{r}} F_{\tau r}^{\phi}(z) = F_{\tau r}^{\phi}(w_{\phi}z)_{i}$ for any $1 \leq i \leq r$ and $1 \leq j \leq r - 1$. Thus, we obtain $L_{ij}(z) = w_{\phi}^{-1}(z)_{1+q+\cdots+q^{-1}} L_{ij}(w_{\phi}z)$. Then

$$E_{ij}(z) = \pi^{q+\cdots+q_{r}-1-q_{r}-1} h_r(z) L_{ij}(z)$$

$$\sim \pi^{q+\cdots+q_{r}-1-q_{r}-1} \frac{w_{\phi}^{j+\cdots+q_{r}}}{w_{\phi}^{1+q+\cdots+q_{r}-1-q_{r}}} L_{ij}(w_{\phi}z)$$

$$\sim \left(\frac{w_{\phi}}{\pi}\right)^{q_{j}} \left(\frac{L_{ij}(w_{\phi}z)}{\pi}\right)$$

which is transcendental over $\overline{K}$ by Proposition 6.10. \qed

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