A three-field formulation of the Poisson problem with Nitsche approach

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Abstract

We modify a three-field formulation of the Poisson problem with Nitsche approach for approximating Dirichlet boundary conditions. Nitsche approach allows us to weakly impose Dirichlet boundary condition but still preserves the optimal convergence. We use the biorthogonal system for efficient numerical computation and introduce a stabilisation term so that the problem is coercive on the whole space. Numerical examples are presented to verify the algebraic formulation of the problem.

Contents

1 Introduction \hfill 1
2 A Three-field Formulation for Poisson Problem \hfill 2
3 Finite Element Discretisation \hfill 4
4 Algebraic Formulation \hfill 6
5 Numerical Examples \hfill 9
6 Conclusion \hfill 10
References \hfill 10

1 Introduction

The finite element method is a powerful and efficient method to handle complicated geometries and impose the associated boundary conditions. However, in some cases, the treatment of the Dirichlet-type boundary
conditions compromise the stability and accuracy of the standard finite element method [9].

In order to relax the Dirichlet boundary condition constraint, we need to modify the standard finite element approach. Generally, we can do this by imposing the Dirichlet boundary condition as a penalty term [11, 12]. One of such methods is Nitsche’s method [12], which imposes the Dirichlet boundary condition weakly in the formulation without the need of a Lagrange multiplier. Moreover, compared to other penalty method, Nitsche’s method adds the consistency, symmetry and stability terms so that this method can achieve optimal convergence. There are so many applications of Nitsche’s method in many areas, such as elasticity [3], interface problems [6], potential flows [8] and plasticity [13].

In this article, we modify a mixed finite element method, based on the three-field formulation [7], with Nitsche approach to solve a Poisson problem. A similar three-field formulation, known as Hu-Washizu formulation, is popular in linear elasticity field [10]. The three-field formulation allows us to apply a biorthogonal system which leads to a very efficient finite element method. In order to overcome the difficulty of coercivity condition, we introduce a stabilisation term [7] of the associated bilinear form so that it is coercive on the whole space.

The structure of the article is as follows. In the next section we recall the Nitsche formulation for Poisson problem and introduce a three-field formulation with this approach. We modify the three-field formulation to include a stabilisation term. We introduce the finite element approximation and prove the well-posedness condition in Section 3. We then show the algebraic formulation and a priori error estimate in Section 4. Two numerical examples are presented in Section 5. Finally, a short conclusion is written in Section 6.

2 A Three-field Formulation for Poisson Problem

Sobolev Spaces

Let \( V = H^1(\Omega) \) and \( L = [L^2(\Omega)]^2 \). The Sobolev spaces \( H^k(S) \) for \( S \subset \Omega \) or \( S \subset \Gamma \), and \( k \geq 0 \) are defined in the standard way [5]. We introduce the space \( H^{-1/2}(\Gamma) \), the dual space of \( H^{1/2}(\Omega) \), with the norm

\[
\|\mu\|_{-1/2,\Gamma} = \sup_{z \in H^{1/2}(\Gamma)} \frac{\langle \mu, z \rangle}{\|z\|_{1/2,\Gamma}}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing. For functions \( v \in H^1(\Omega) \) with \( \Delta v \in L^2(\Omega) \), it holds \( \| \frac{\partial u}{\partial n} \|_{-1/2, \Gamma} \leq C (\| u \|_1 + \| \Delta v \|_0) \).

We will also introduce the mesh-dependent norms
\[
\| v \|_{1/2, h}^2 = \sum_e h_e \| v \|_{0,e}^2 \quad \text{for } v \in H^1(\Omega),
\]
\[
\| z \|_{-1/2, h}^2 = \sum_e h_e \| z \|_{0,e}^2 \quad \text{for } z \in L^2(\Gamma),
\]
and for these norms it holds
\[
\langle v, z \rangle \leq \| v \|_{1/2, h} \| z \|_{-1/2, h} \quad \text{for } (v, z) \in H^1(\Omega) \times L^2(\Gamma).
\]

For the rest of the article, we denote
\[
\| u \|_{1, h} = \| u \|_1 + \| u \|_{1/2, h} \quad \text{for } u \in H^1(\Omega).
\]

## Nitsche Formulation for the Poisson Problem

The mixed formulation is obtained by introducing \( \sigma = \nabla u \). Given \( f \in L^2(\Omega) \), the (Nitsche) minimisation problem can be written as
\[
\begin{align*}
\operatorname{argmin}_{(u, \sigma) \in V \times L} & \quad \frac{1}{2} \| \sigma \|_{0, \Omega}^2 + \frac{\alpha}{2} \| u - g_D \|_{1/2, h}^2 - \langle \sigma \cdot \mathbf{n}, v - g_D \rangle - \int_\Omega f v \, dx \\
& \quad \text{for } (v, \tau) \in V \times L,
\end{align*}
\]

We write a variational equation for \( \sigma = \nabla u \) using the Lagrange multiplier space \( M = L \) to obtain the saddle-point problem of the minimisation problem \( \text{(2)} \). The saddle point formulation is to find \( (u, \sigma, \varphi) \in V \times L \times M \) such that
\[
\begin{align*}
\tilde{a} \left[ (u, \sigma), (v, \tau) \right] + b \left[ (v, \tau), \varphi \right] &= \ell (v, \tau), \quad (v, \tau) \in V \times L, \\
b \left[ (u, \sigma), \psi \right] &= 0, \quad \psi \in M,
\end{align*}
\]
where
\[
\begin{align*}
\tilde{a} \left[ (u, \sigma), (v, \tau) \right] &= \int_\Omega \sigma \cdot \tau \, dx + \alpha \langle u, v \rangle_{1/2, h} - \langle \sigma \cdot \mathbf{n}, v \rangle - \langle \tau \cdot \mathbf{n}, u \rangle, \\
b \left[ (u, \sigma), \psi \right] &= \int_\Omega (\sigma - \nabla u) \psi \, dx, \\
\ell (v, \tau) &= \int_\Omega f v \, dx - \langle \tau \cdot \mathbf{n}, g_D \rangle + \alpha \langle g_D, v \rangle_{1/2, h},
\end{align*}
\]
where \( \langle \cdot, \cdot \rangle \) denotes duality pairing between \( H^{1/2}(\Omega) \) and \( H^{-1/2}(\Gamma) \).
3 Finite Element Discretisation

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the polygonal domain $\Omega$. We use the standard linear finite element space $V_h \subset H^1(\Omega)$ defined on the triangulation $\mathcal{T}_h$, where

$$V_h := \{ v \in C^0(\Omega) : v|_T \in P_1(T), T \in \mathcal{T}_h \}.$$

The finite element space for the gradient of the solution is $L^2_h$. Let $\{\rho_1, \rho_2, \ldots, \rho_N\}$ be the finite element basis for $V_h$. Starting with the standard basis for $V_h$, we construct a space $Q_h$ spanned by the basis $\{\mu_1, \mu_2, \ldots, \mu_N\}$ so that the basis functions of $V_h$ and $Q_h$ satisfy the biorthogonality condition

$$\int_{\Omega} \rho_i \mu_j \, dx = c_j \delta_{ij}, \quad c_j \neq 0, \quad 1 \leq i, j \leq N,$$

where $\delta_{ij}$ is the Kronecker symbol, and $c_j$ a scaling factor. Therefore, the sets of basis functions of $V_h$ and $Q_h$ form a biorthogonal system.

The basis functions of $Q_h$ are constructed locally on a reference element $\hat{T}$ so that the basis functions of $V_h$ and $Q_h$ have the same support, and in each element the sum of all the basis functions of $Q_h$ is one \cite{[10]}. We let $M_h = [Q_h]^2$, thus our problem is to find $(u_h, \sigma_h, \varphi_h) \in V_h \times L_h \times M_h$ such that

$$\tilde{a} [(u_h, \sigma_h), (v_h, \tau_h)] + b [(v_h, \tau_h), \varphi_h] = \ell (v_h, \tau_h), \quad (v_h, \tau_h) \in V_h \times L_h,$$

$$b [(u_h, \sigma_h), \psi_h] = 0, \quad \psi_h \in M_h.$$

(3)

To show that the saddle-point problem has a unique solution, we need to show that the following well-posedness conditions are satisfied.

1. The linear form $\ell (\cdot)$, the bilinear forms $\tilde{a} [\cdot, \cdot]$ and $b [\cdot, \cdot]$ are continuous on the spaces in which they are defined.

2. The bilinear form $\tilde{a} [\cdot, \cdot]$ is coercive on the kernel space $K_h$ defined as

$$K_h = \{(u_h, \sigma_h) \in V_h \times L_h : b [(u_h, \sigma_h), \psi_h] = 0, \text{ for all } \psi_h \in M_h \}.$$

3. The bilinear form $b [\cdot, \cdot]$ satisfies the inf-sup condition

$$\inf_{\psi_h \in M_h} \sup_{(v_h, \tau_h) \in V_h \times L_h} \frac{b [(v_h, \tau_h), \psi_h]}{\|v_h, \tau_h\|_{V_h \times L_h} \|\psi_h\|_{0, \Omega}} \geq \gamma, \quad \gamma > 0.$$

The mesh-dependent norm for the product space $V_h \times L_h$ is defined by

$$\|u_h, \sigma_h\|^2_{V_h \times L_h} = \|u_h\|^2_{1,h} + \|\sigma_h\|^2_{0,\Omega}, \quad (u_h, \sigma_h) \in V_h \times L_h.$$
With the introduction of $M_h$, the bilinear form $\tilde{a} [\cdot, \cdot]$ is not coercive on the kernel subspace $K_h \subset V_h \times L_h$. Thus, we need to modify the bilinear form $\tilde{a} [\cdot, \cdot]$ so that it is coercive on the kernel space $K_h$ or even the whole space $V_h \times L_h$. In this article, we modify the bilinear form $\tilde{a} [\cdot, \cdot]$ by adding a stabilisation term so that it is coercive on the whole space $V_h \times L_h$.

\[
\begin{align*}
a [(u_h, \sigma_h), (v_h, \tau_h)] &= r \int_{\Omega} \sigma_h \cdot \tau_h \, dx + (1 - r) \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx \\
&\quad + \alpha \langle u_h, v_h \rangle_{1/2, h} - \langle \sigma_h \cdot n, v_h \rangle - \langle \tau_h \cdot n, u_h \rangle,
\end{align*}
\]

for $0 < r < 1$.

We use the following inverse estimate result [7] to show the continuity condition of $\ell (\cdot)$ and also continuity and coercivity condition of the bilinear form $a [\cdot, \cdot]$,

\[
C_I \left\| \frac{\partial v_h}{\partial n} \right\|_{-1/2, h} \leq \| \nabla v_h \|_{0, \Omega} \quad \text{for} \quad v_h \in V_h. \quad (4)
\]

The continuity of the linear form $\ell (\cdot)$, and the bilinear forms $a [\cdot, \cdot]$ and $b [\cdot, \cdot]$ then follows from the Cauchy-Schwarz inequality, the duality pairing (1) and the inverse estimate (4).

For the coercivity condition, using the inverse estimate (4) and the following Poincare-Friedrichs inequality,

\[
\|u_h\|_{1, \Omega}^2 = \|u_h\|_{0, \Omega}^2 + \|\nabla u_h\|_{0, \Omega}^2 \leq (c^2 + 1) \|\nabla u_h\|_{0, \Omega}^2,
\]

we can write

\[
\begin{align*}
|a [(u_h, \sigma_h), (u_h, \sigma_h)]| &= r \|\sigma_h\|_{0, \Omega}^2 + (1 - r) \|\nabla u_h\|_{0, \Omega}^2 + \alpha \|u_h\|_{1/2, h}^2 - 2 \langle \sigma_h \cdot n, u_h \rangle, \\
&\geq r \|\sigma_h\|_{0, \Omega}^2 + (1 - r) \|\nabla u_h\|_{0, \Omega}^2 - 2 \|\sigma_h \cdot n\|_{-1/2, h} \|u_h\|_{1/2, h} + \alpha \|u_h\|_{1/2, h}^2, \\
&\geq r \|\sigma_h\|_{0, \Omega}^2 + (1 - r) \|\nabla u_h\|_{0, \Omega}^2 - \left( \frac{1}{\varepsilon} \|\sigma_h \cdot n\|_{-1/2, h}^2 + \varepsilon \|u_h\|_{1/2, h}^2 \right) + \alpha \|u_h\|_{1/2, h}^2, \\
&\geq \left( r - \frac{1}{\varepsilon C_I} \right) \|\sigma_h\|_{0, \Omega}^2 + (1 - r) \|\nabla u_h\|_{0, \Omega}^2 + (\alpha - \varepsilon) \|u_h\|_{1/2, h}^2, \\
&\geq \left( r - \frac{1}{\varepsilon C_I} \right) \|\sigma_h\|_{0, \Omega}^2 + \frac{1 - r}{\varepsilon^2 + 1} \|u_h\|_{1/2, h}^2 + (\alpha - \varepsilon) \|u_h\|_{1/2, h}^2, \\
&\geq C \|(u_h, \sigma_h)\|_{V_h \times L_h}^2,
\end{align*}
\]

where $C$ is the minimum of $\left( r - \frac{1}{\varepsilon C_I} \right)$, $\frac{1 - r}{\varepsilon^2 + 1}$ and $(\alpha - \varepsilon)$. We also require $\frac{1}{C_I} < r \varepsilon < \varepsilon < \alpha$ and $0 < r < 1$. From this point forward, we use constant $C$ as a mesh-independent generic constant.

Now the inf-sup condition for the bilinear form $b [\cdot, \cdot]$ can be shown as in [7]. Thus we have proved the following theorem.
4 Algebraic Formulation

Theorem 1. The saddle point problem \( [3] \) with stabilised \( a[·,·] \) has a unique solution \((u_h, \sigma_h, \varphi_h) \in V_h \times L_h \times M_h \). The solution also satisfies \( \| (u_h, \sigma_h) \|_{V_h \times L_h} + \| \varphi_h \|_{0, \Omega} \leq C \| f \|_{0, \Omega} \).

4 Algebraic Formulation

In order to present an algebraic formulation of the problem, we use \((x_u, x_\sigma, x_\varphi)\) for the vector representation of the solution \((u_h, \sigma_h, \varphi_h)\) as elements in \(V_h \times L_h \times M_h\). Let \(S, D, A, B, C\) and \(M\) be the matrices associated with bilinear forms \(\int_\Omega \nabla u_h \cdot \nabla v_h\ dx, \int_\Omega \tau_h \cdot \varphi_h\ dx, \int_\Gamma (\sigma_h \cdot n) u_h\ ds, \int_\Omega \nabla v_h \cdot \varphi_h\ dx, \sum_\Gamma \int_\epsilon u_h v_h\ ds, \) and \(\int_\Omega \sigma_h \cdot \tau_h\ dx\), respectively. For the right hand side, we write \(f_1\) and \(f_2\) to represent the discrete forms of \(\int_\Omega f v_h\ dx v_h,\ ds + \alpha \langle g_D, v_h \rangle_{1/2, \Omega}\) and \(\langle \tau_h \cdot n, g_D \rangle\), respectively. Then the algebraic formulation of the problem is

\[
\begin{bmatrix}
(1 - r) S + \alpha C & -A & -B \\
-A^T & rM & D \\
-B^T & D & 0
\end{bmatrix}
\begin{bmatrix}
x_u \\
x_\sigma \\
x_\varphi
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
-f_2 \\
0
\end{bmatrix}, \tag{6}
\]

where the first two equations of \( [6] \) correspond to first equation of \( [3] \) with stabilised \( a[·,·], \) by setting \( \sigma_h = 0 \) and \( v_h = 0 \), respectively. After statically condensing out degrees of freedom associated with \( \sigma_h \) and \( \phi_h \) in \( [6] \), we arrive at the following system

\[ K x_u = F \]

where

\[ K = (1 - r) S + \alpha C - AD^{-1}B^T - BD^{-1}A^T + rBD^{-1}MD^{-1}B^T, \]

\[ F = f_1 - BDf_2. \]

Due to the choice of a biorthogonal system, matrix \( D \) is diagonal. As a result, the statically condensed system matrix is sparse.

We introduce two projections \( P_h : L^2(\Omega) \to Q_h \) and \( P_h^r : L^2(\Omega) \to V_h \) as follows for \( v \in L^2(\Omega) \).

\[
\int (P_h v - v) \cdot \mu_h\ dx = 0, \quad \mu_h \in Q_h, \quad \int (P_h^r v - v) \cdot \varphi_h\ dx = 0, \quad \varphi_h \in V_h.
\]

They satisfy the following estimates for \( u \in H^1(\Omega) \) :

\[
\| P_h u - u \|_{0, \Omega} \leq Ch \| u \|_{1, \Omega}, \quad \| P_h^r u - u \|_{0, \Omega} \leq Ch \| u \|_{1, \Omega}. \tag{8}
\]

Using this projection, our problem is to find \( u_h \in V_h \) such that,

\[ A(u_h, v_h) = L(v_h), \quad v_h \in V_h \tag{9} \]
where

\[ A(u_h, v_h) = \int_\Omega P_h(\nabla u_h) \cdot P_h(\nabla v_h) \, dx + \alpha \langle u_h, v_h \rangle_{1/2,h} \]

\[ - \int_\Gamma (P_h(\nabla u_h) \cdot n) v_h \, ds - \int_\Gamma (P_h(\nabla v_h) \cdot n) u_h \, ds, \]

\[ L(v_h) = \int_\Omega f v_h \, dx - \int_\Gamma (P_h(\nabla v_h) \cdot n) g_D \, ds + \alpha \langle g_D, v_h \rangle_{1/2,h}. \]

We also introduce two mesh-dependent norms

\[ \|u_h\|^2 = \|u_h\|^2_{1,h} + \|P_h(\nabla u_h)\|^2_{0,\Omega}, \quad u_h \in V_h, \]

\[ \|u\|^2 = \|u\|^2_{1,h} + \|P_h(\nabla u)\|^2_{0,\Omega} + \|\nabla u \cdot n\|^2_{-1/2,h}, \quad u \in H^2(\Omega), \]

so that

\[ |A(u, v_h)| \leq \|u\|_h \|v_h\|_h, \quad u \in V \text{ and } v_h \in V_h. \]

We get the following estimate combining the interpolation estimate of Lemma 3.4 of [9].

\[ \inf_{v_h \in V_h} \|u - v_h\|_h \leq Ch \|u\|_{2,\Omega}. \]

We then have the following theorem.

**Theorem 2.** Let \( u_h \in V_h \) be the solution to the problem [9]. Suppose that \( u \in H^2(\Omega) \) is the solution to the problem [3] then

\[ \|u - u_h\|_h \leq Ch \|u\|_{2,\Omega}. \]

**Proof:** From the coercivity [5] and continuity condition [11],

\[ \alpha \|u_h - v_h\|^2_h \leq A(u_h - v_h, u_h - v_h), \]

\[ = A(u - v_h, u_h - v_h) - A(u, u_h - v_h) + A(u_h, u_h - v_h), \]

\[ = A(u - v_h, u_h - v_h) - A(u, u_h - v_h) + L(u_h - v_h), \]

\[ \leq \|u - v_h\|_h \|u_h - v_h\|_h + L(u_h - v_h) - A(u, u_h - v_h). \]

Using \( w_h = u_h - v_h \) and divide both sides by \( \|w_h\|_h \), we get

\[ \alpha \|u_h - v_h\|_h \leq \|u - v_h\|_h + \frac{L(w_h) - A(u, w_h)}{\|w_h\|_h}. \]

Following exactly as in the proof of Strang’s second lemma [4] we get

\[ \|u - u_h\|_h \leq C \left( \inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{L(w_h) - A(u, w_h)}{\|w_h\|_h} \right). \]
The first term on the right hand side of (13) can be estimated using (12). For the second term of (13), recall that $f = -\Delta u$ so we can write the numerator as follows.

$$
\int_{\Omega} f w_h \, dx - \int_{\Omega} P_h (\nabla u) \cdot P_h (\nabla w_h) \, dx + \int_{\Gamma} (P_h (\nabla u) \cdot n) w_h \, ds \\
= - \int_{\Omega} \Delta u w_h \, dx - \int_{\Omega} P_h (\nabla u) \cdot P_h (\nabla w_h) \, dx + \int_{\Gamma} (P_h (\nabla u) \cdot n) w_h \, ds \\
= \int_{\Omega} \nabla u \cdot \nabla w_h \, dx - \int_{\Gamma} (\nabla u \cdot n) w_h \, ds \\
- \int_{\Omega} P_h (\nabla u) \cdot P_h (\nabla w_h) \, dx + \int_{\Gamma} (P_h (\nabla u) \cdot n - \nabla u \cdot n) w_h \, ds \\
= \int_{\Omega} \nabla u \cdot \nabla w_h \, dx - \int_{\Omega} P_h (\nabla u) \cdot P_h (\nabla w_h) \, dx \\
+ \int_{\Gamma} (P_h (\nabla u) \cdot n - \nabla u \cdot n) w_h \, ds.
$$

\[(14)\]

We can estimate the first term of (14) using approximation property of $P_h^*$ (8) as

$$
\int_{\Omega} \nabla u \cdot (\nabla w_h - P_h (\nabla w_h)) \, dx = \int_{\Omega} (\nabla u - P_h^* (\nabla u)) \cdot (\nabla w_h - P_h (\nabla w_h)) \, dx, \\
\leq \|\nabla u - P_h^* (\nabla u)\|_{0, \Omega} \|\nabla w_h - P_h (\nabla w_h)\|_{0, \Omega}, \\
\leq Ch \|u\|_{2, \Omega} \|\nabla w_h - P_h (\nabla w_h)\|_{0, \Omega}.
$$

We can estimate the second term of (14) using approximation property of $P_h$ (8) as

$$
\int_{\Omega} (\nabla u - P_h (\nabla u)) \cdot P_h (\nabla w_h) \, dx \leq \|\nabla u - P_h (\nabla u)\|_{0, \Omega} \|P_h (\nabla w_h)\|_{0, \Omega}, \\
\leq Ch \|u\|_{2, \Omega} \|P_h (\nabla w_h)\|_{0, \Omega}.
$$

We can estimate the third term of (14) as

$$
\int_{\Gamma} (P_h (\nabla u) \cdot n - \nabla u \cdot n) w_h \, ds \leq \|P_h (\nabla u) \cdot n - \nabla u \cdot n\|_{-1/2, h} \|w_h\|_{1/2, h}, \\
\leq Ch \|u\|_{2, \Omega} \|w_h\|_{1/2, h}.
$$
where
\[ \| P_h (\nabla u) \cdot n - \nabla u \cdot n \|_{-1/2,h} \leq C h \| u \|_{2,\Omega}, \]
follows from the approximation property of \( P_h \) \[\text{[11]}\]. Combining all estimates concludes the proof.

5 Numerical Examples

In this section, we show two numerical examples to verify the convergence rate of our approach. We compute the error in \( L^2 \)-norm and the rate of convergence for \( u \) and \( \sigma \). We also compute the error in \( H^1 \)-norm and the rate of convergence for \( u \). We will use pure Dirichlet boundary conditions for all our examples.

Example 1

We consider the exact solution
\[ u = xy (1 - x) (1 - y), \]
for the first example. The errors for this example with pure Dirichlet boundary conditions are shown in Table 1.

Table 1: Discretisation errors with pure Dirichlet boundary conditions for example 1

| elem | \[ \| u - u_h \|_{0,\Omega} \] error rate | \[ \| u - u_h \|_{1,h} \] error rate | \[ \| \sigma - \sigma_h \|_{0,\Omega} \] error rate |
|------|-----------------|-----------------|-----------------|
| 8    | 3.74e-02        | 1.98e-01        | 1.73e-01        |
| 32   | 8.89e-03        | 2.0742          | 1.09e-01        |
| 128  | 1.92e-03        | 2.2083          | 5.53e-02        |
| 512  | 4.37e-04        | 2.1364          | 2.76e-02        |
| 2048 | 1.04e-04        | 2.0752          | 1.37e-02        |
| 8192 | 2.52e-05        | 2.0392          | 6.85e-03        |

Example 2

We consider the exact solution
\[ u = e^{x^2 + y^2} + y^2 \cos (xy) + x^2 \sin (xy), \]
for our second example. The errors for this example with pure Dirichlet boundary conditions are shown in Table 2.
Table 2: Discretisation errors with pure Dirichlet boundary conditions for example 2

| elem | $\|u - u_h\|_{0,\Omega}$ | $\|u - u_h\|_{1,h}$ | $\|\sigma - \sigma_h\|_{0,\Omega}$ |
|------|-----------------|------------------|------------------|
|      | error           | rate             | error            | rate             | error            | rate             |
| 8    | 7.36e-01        | 4.23e+00         | 2.32e+00         |
| 32   | 1.50e-01        | 2.2902           | 1.0110           | 8.56e-01         | 1.4381           |
| 128  | 3.12e-02        | 2.2694           | 1.0258           | 2.93e-01         | 1.5457           |
| 512  | 6.83e-03        | 2.1915           | 1.0231           | 1.00e-01         | 1.5494           |
| 2048 | 1.57e-03        | 2.1179           | 1.0149           | 3.45e-02         | 1.5384           |
| 8192 | 3.76e-04        | 2.0661           | 1.0085           | 1.20e-02         | 1.5263           |

From Tables 1 and 2 we can see that the rate of convergence of errors for $u$ in $L^2$-norm and $(1,h)$-norm is 2 and 1, respectively, while the rate of convergence of errors for $\sigma$ in $L^2$-norm is 1.5. These results are very similar to the result from the three-field formulation for Poisson problem with same examples.

6 Conclusion

In this article, we describe a mixed finite element method to solve Poisson equation based on Nitsche’s method. We add a stabilisation term so that our bilinear form is coercive on the whole space. From numerical examples, we can observe that the error and rate of convergence is very similar to our previous three-field formulation for Poisson problem. Thus we can conclude that this approach works well as an alternative to the standard formulation.

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References

[1] J. P. Aubin. Approximation of elliptic boundary-value problems, volume XXVI of Pure and Applied Mathematics. Wiley-Interscience, New York, 1972.
[2] I. Babuška. The finite element method with penalty. *Mathematics of Computation*, 27(122):221–228, 1973.

[3] R. Becker, E. Burman, and P. Hansbo. A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity. *Computer Methods in Applied Mechanics and Engineering*, 198(41):3352 – 3360, 2009.

[4] D. Braess. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press, Cambridge, UK, 3rd edition edition, 2007.

[5] P. G. Ciarlet. *The finite element method for elliptic problems*. North Holland, Amsterdam, 1978.

[6] P. Hansbo. Nitsche’s method for interface problems in computational mechanics. *GAMM-Mitteilungen*, 28(2):183–206, 2005.

[7] M. Ilyas and B. P. Lamichhane. A stabilised mixed finite element method for the poisson problem based on a three-field formulation. In *Proceedings of the 12th Biennial Engineering Mathematics and Applications Conference, EMAC-2015*, volume 57 of *ANZIAM J.*, pages C177–C192, September 2016.

[8] A. Johansson, M. Garzon, and J. A. Sethian. A three-dimensional coupled Nitsche and level set method for electrohydrodynamic potential flows in moving domains. *Journal of Computational Physics*, 309:88 – 111, 2016.

[9] M. Juntunen and R. Stenberg. Nitsche’s method for general boundary conditions. *Mathematics of Computation*, 78:1353–1374, 2009.

[10] B. P. Lamichhane, A. T. McBride, and B. D. Reddy. A finite element method for a three-field formulation of linear elasticity based on biorthogonal systems. *Computer Methods in Applied Mechanics and Engineering*, 258:109–117, 2013.

[11] B.P. Lamichhane and B.I. Wohlmuth. A quasi-dual Lagrange multiplier space for Serendipity Mortar finite elements in 3D. *M²AN*, 38:73–92, 2004.

[12] J. Nitsche. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 36(1):9–15, Jul 1971.

[13] T. J. Truster. A stabilized, symmetric Nitsche method for spatially localized plasticity. *Computational Mechanics*, 57(1):75–103, Jan 2016.
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