KNOT FLOER HOMOLOGY OF FIBRED KNOTS AND FLOER HOMOLOGY OF SURFACE DIFFEOMORPHISMS

PAOLO GHIGGINI AND GILBERTO SPANO

Abstract. We prove that the Knot Floer homology group of a fibred knot of genus $g$ in the Alexander grading $1 - g$ is isomorphic to a version of the fixed point Floer homology of an area-preserving representative of the monodromy.

Contents

1. Introduction 2
2. Preliminaries 4
  2.1. Open book decompositions 4
  2.2. Heegaard Floer homology via open books 5
  2.3. Knot Floer homology via open book decompositions 8
  2.4. Fixed point Floer homology 10
  2.5. Periodic Floer homology 12
  2.6. Periodic Floer homology for fibered knots 13
3. The chain map from $HFK$ to $PFK$ 14
  3.1. Review of the chain map $\Phi$ 15
  3.2. $\Phi$ respects the filtrations 17
4. Proof of the isomorphism 20
  4.1. Some algebra 21
  4.2. The exact triangle in fixed point Floer cohomology 24
  4.3. The exact triangle in Heegaard Floer homology 32
    4.3.1. Heegaard diagrams 32
    4.3.2. The chain map $i$ 33
    4.3.3. The chain map $l$ 35
    4.3.4. Sketch of the proof of the exact triangle 35
  4.4. Comparing the double cones 37
    4.4.1. Seidel’s exact sequence revisited 37
    4.4.2. The first square 41
    4.4.3. The second square 44
    4.4.4. The homotopy of homotopies 45
  4.5. The induction 47
    4.5.1. Initialization 47
    4.5.2. The inductive step 49
5. Applications 49
References 58
1. Introduction

Knot Floer homology [49, 55] is a family of abelian groups — or vector spaces; here we will work over a field of characteristic two — $\widehat{HFK}(Y,K,i)$ associated to any oriented null-homologous knot $K$ in an oriented three-manifold $Y$. If $g$ is the minimal genus of a Seifert surface of $K$, then $\widehat{HFK}(Y,K,i) = 0$ if $|i| > 0$ and $\widehat{HFK}(Y,K,i) \neq 0$ for $i = -g, g$ by [50, 42]. Moreover, $\widehat{HFK}(Y,K,-g)$ has rank one if and only if $K$ is fibred by [24, 41]. See also [30].

The power of knot Floer homology, which is not at all limited to the results mentioned above, comes from its connections to many areas of low-dimensional topology, but its topological meaning is obscured by the complexity of its definition. In this article we try to shed some light on it by relating the knot Floer homology of a fibered knot to the dynamics of its monodromy. This is a partial result in the direction of a relative version of the isomorphism between Heegaard Floer homology and embedded contact homology; see [10].

A fibred knot $K \subset Y$ gives rise to an open book decomposition of $Y$ with binding $K$, fibre $S$ and monodromy $\varphi$; see Section 2.1. The monodromy is well defined up to isotopy relative to the boundary of $S$. We will fix an area form on $S$ and assume that $\varphi$ is an area-preserving diffeomorphism. Then we can associate to it a Floer homology group $HF^\#(\varphi)$, which is an intermediate version between the Floer homology groups $HF(\varphi, +)$ and $HF(\varphi, -)$ considered by Seidel in [61]. In [9] Colin, Ghiggini and Honda defined an embedded contact homology group $ECH^\#(T_\varphi)$, which is conjecturally isomorphic to $\widehat{HFK}(Y,K)$ when $T_\varphi$ is the complement of a tubular neighbourhood of $K$. The group $HF^\#(\varphi)$ is isomorphic to the subgroup of $ECH^\#(T_\varphi)$ generated by the Reeb orbits with algebraic intersection one with a fibre.

Let $\overline{K}$ and $\overline{Y}$ denote $K$ and, respectively, $Y$ with the reversed orientation. If $Y = S^3$, then $(\overline{Y}, \overline{K}) = (Y, m(K))$, where $m(K)$ denotes the mirror of $K$. The main result of this paper is the following:

**Theorem 1.1.** Let $K \subset Y$ be a fibered knot with associated monodromy $\varphi: S \to S$. Then there exists an isomorphism of vector spaces over $\mathbb{Z}/2\mathbb{Z}$

$$\widehat{HFK}(\overline{Y}, \overline{K}; -g + 1) \cong HF^\#(\varphi).$$

This isomorphism is induced by an open-close map

$$\Phi_*^\#: \widehat{HFK}(\overline{Y}, \overline{K}; -g + 1) \to HF^\#(\varphi)$$

which is a simplified version of the open-close map $\Phi_*: \widehat{HF}(Y) \to \widehat{ECH}(Y)$ defined in [11]. The proof that $\Phi_*^\#$ is an isomorphism is by induction on the length of a minimal factorisation of $\varphi$ as a product of Dehn twists and, unlike the proof of the isomorphism between Heegaard Floer homology and embedded contact homology, does not require the construction of an inverse map. The initial step of the induction, for $\varphi = id$, is an explicit computation. The inductive step relies on the comparison between two exact triangles: Ozsváth and Szabó’s surgery exact triangle for knot Floer homology [49, Theorem 8.2] and Seidel’s exact triangle for Dehn twists in the context of fixed point Floer homology [61, Theorem 4.2]. We give the first complete proof of the latter
for exact symplectomorphisms of closed surfaces, a result which can be of independent interest.

More precisely, an essential circle $L$ embedded in $S$ induces a knot in $Y$ via the open book decomposition. We denote by $Y^+$ and $Y_0$ the manifolds obtained by $+1$– and, respectively, $0$–surgery on $Y$ along $L$ with surgery coefficient computed with respect to the framing induced by the page. The knot $K \subset Y$ induces knots $K^+ \subset Y^+$ and $K_0 \subset Y_0$. Moreover, $K^+$ is the binding of an open book decomposition of $Y^+$ with page $S$ and monodromy $\varphi \circ \tau_L^{-1}$, where $\tau_L$ is a positive Dehn twist around $L$.

Seidel’s exact sequence for fixed point Floer homology, in this context, is

$$\begin{align*}
&HF(\varphi(L), L) 
\quad \xrightarrow{HF^2(\varphi)}
\quad \xrightarrow{HF^2(\varphi \circ \tau_L^{-1})}
\end{align*}$$

while Ozsváth and Szabó’s exact sequence for knot Floer homology is

$$\begin{align*}
&\widehat{HFK}(Y_0; K_0; -g + 1) 
\quad \xrightarrow{\widehat{HFK}(K, Y; -g + 1)}
\quad \xrightarrow{\widehat{HFK}(K^+, Y^+; -g + 1)}
\end{align*}$$

It is easy to see, using the surface decomposition formula in sutured Floer homology [30, Theorem 1.3], that $\widehat{HFK}(Y_0; K_0; -g + 1) \cong HF(\varphi(L), L)$. Then we will prove that the diagram

$$\begin{align*}
&\widehat{HFK}(Y_0; K_0; -g + 1) 
\quad \xrightarrow{\widehat{HFK}(K, Y; -g + 1)}
\quad \xrightarrow{\widehat{HFK}(K^+, Y^+; -g + 1)}
\end{align*}$$

commutes. For a suitable choice of $L$, one between $\varphi$ and $\varphi \circ \tau_L^{-1}$ has a factorisation in Dehn twists which is shorter than the other’s, and thus the inductive step will follow from the Five Lemma.

In the last section we will give some applications of Theorem (1.1). In particular we show that, under certain hypotheses, the dimension of $\widehat{HFK}(Y, K, -g + 1)$ detects the minimal number $F_{\text{min}}(\varphi)$ of fixed points that an area-preserving non-degenerate representative of $\varphi$ may have.

**Theorem 1.2.** Let $K \subset Y$ be a genus $g > 0$ fibered knot and let $\varphi$ be a representative of the monodromy of its complement. Assume that either

1. $Y$ is a rational homology sphere or
(2) the mapping class \([\varphi]\) is irreducible in the sense of the Nielsen–Thurston classification.

Then:

\[
\dim \left( \hat{HFK}(Y, K; -g + 1) \right) = \begin{cases} 
F_{\min}(\varphi) - 1 & \text{if } \varphi \sim \text{id}; \\
F_{\min}(\varphi) + 1 & \text{otherwise.}
\end{cases}
\]

As a consequence we obtain that if \(K\) is a fibered knot satisfying the hypotheses of last theorem then

\[
\dim \left( \hat{HFK}(Y, K; -g + 1) \right) \geq 1
\]

with equality if and only if the mapping class of monodromy of its complement admits an area-preserving non-degenerate representative with no fix points. This implies the following corollary.

**Corollary 1.3.** L-space knots in \(S^3\) admit a representative of the monodromy with no interior fixed points.

Another application of our main theorem gives a link between Heegaard Floer homology and the geometry of three-manifolds. Let \(K^n\) be the branched locus of the \(n\)-th branched cover \(Y^n(K)\) of \(Y\) over \(K\): if \((K, S, \varphi)\) is an open book decomposition of \(Y\) then \((K^n, S, \varphi^n)\) is an open book decomposition of \(Y^n(K)\). Combining Theorem 1.1 with a result of Fel’shtyn [21] we recover the following result, that was already proven in [39] by Lipshitz, Ozsváth and Thurston using bordered Floer homology techniques.

**Corollary 1.4.** If \((K, S, \varphi)\) is an open book decomposition of \(Y\) the growth rate for \(n \to \infty\) of the dimensions of \(\hat{HFK}(Y^n(K), K^n; -g + 1)\) coincides with the largest dilatation factor among all the pseudo–Anosov components of the canonical representative of the mapping class \([\varphi]\).

Another interesting consequence of Theorem 1.1 is about algebraic knots and comes from an analogous result of McLean about fix point Floer homology (see [46, Corollary 1.4]).

**Corollary 1.5.** Let \(K \subset S^3\) be the 1–component link of an isolated singularity of an irreducible complex polynomial \(f\) with two variables. Then

\[
\min \left\{ n > 0 \mid \dim \left( \hat{HFK}(S^3^n(K), K^n; -g + 1) \right) \neq 1 \right\} = \mathfrak{m}(f)
\]

where \(\mathfrak{m}(f)\) is the multiplicity of \(f\) in \(0\).

Similar results have recently been proved by Ni in [43, 44].

### 2. Preliminaries

**2.1. Open book decompositions.** Let \(K\) be an oriented knot in a closed oriented 3–manifold \(Y\). Recall that \(K\) is fibred if there exists a neighborhood \(U \cong K \times D^2\) of \(K\) and a fibration \(Y \setminus U \to S^1\) that extends the fibration

\[
U \setminus K \cong K \times (D^2 \setminus \{0\}) \to S^1; \quad (x, z) \mapsto \frac{z}{|z|}.
\]
This implies that there exists an oriented surface $S$ with boundary and an orientation preserving diffeomorphism $\varphi: S \to S$ such that $\varphi|_{\partial S}$ is the identity and $Y \setminus U$ is diffeomorphic to the mapping torus

$$T_\varphi = \frac{\mathbb{R} \times S}{\langle t + 1, x \rangle \sim (t, \varphi(x))}.$$ 

The images of the fibers $S_t := \{t\} \times S$ under this diffeomorphism extend canonically to Seifert surfaces for $K$. The triple $(K, S, \varphi)$ is often called an open book decomposition of $Y$ with binding $K$, page $S$ and monodromy $\varphi$. We will refer to the genus of $S$ as to the genus of the open book or the genus of the knot $K$.

Let $Y$ be a closed oriented 3-manifold endowed with an open book decomposition $(K, S, \varphi)$. We parametrise a collar neighborhood $A \cong [0, 2] \times S^1$ of $\partial S$ by $\{(y, \theta) \in [0, 2] \times [0, 2\pi]\}$ so that $\partial S$ corresponds to $\{2\} \times [0, 2\pi]$. On $S$ we fix once and for all a Liouville form $\lambda$, such that, on $A$, it can be written as $y\,d\theta$ and denote $\omega = d\lambda$. We also assume, without loss of generality, that the monodromy satisfies

(2.1) \quad $\varphi^* \lambda - \lambda = d\chi$

for a compactly supported function $\chi: S \to \mathbb{R}$ (see [9, Lemma 9.3.2]) and, moreover, for every $(y, \theta) \in A$,

(2.2) \quad $\varphi(y, \theta) = (y, \theta + f(y))$

for a function $f \in C^\infty([0, 2])$ such that

- $f(2) = 0$,
- $f(y) = 3\varepsilon$ for some $\varepsilon \in \left(0, \frac{\pi}{g}\right)$ when $y \in [0, 1]$, and
- $f'(y) < 0$ when $y \in (1, 2)$.

2.2. Heegaard Floer homology via open books. As shown in [25] by Honda, Kazez and Matić, to an open book decomposition of $Y$ it is possible to associate a special kind of Heegaard diagram for $Y$. Let us recall the slightly different construction given by Colin, Honda and the first author in [11].

If $S$ is a surface of genus $g$, a basis of arcs of $S$ is a set $a = \{a_1, \ldots, a_{2g}\}$ of smooth properly embedded arcs such that $S \setminus \{a_1, \ldots, a_{2g}\}$ is a topological disk. Given the open book decomposition $(K, S, \varphi)$, the surface

$$\Sigma := S_{\frac{1}{2}} \sqcup_{\partial S_0} \overline{S_0}$$

is a Heegaard surface for $Y$, where $\overline{S_0}$ denotes $S_0$ with the orientation reversed. Fix a basis of arcs $\{a_1, \ldots, a_{2g}\}$ for $S_0$, let $\{\tilde{a}_1, \ldots, \tilde{a}_{2g}\}$ be a copy of the basis in $S_{\frac{1}{2}}$ and define the set of closed curves $\alpha = \{\alpha_1, \ldots, \alpha_{2g}\} \subset \Sigma$ by

$$\alpha_i = \tilde{a}_i \sqcup_{\partial a_i} a_i,$$

conveniently smoothed near $\partial S_0$.

Consider now the set of arcs $\{\tilde{b}_1, \ldots, \tilde{b}_{2g}\}$ in $S_{\frac{1}{2}} \subset \Sigma$ obtained by modifying the $\tilde{a}_i$'s by a small isotopy relative to the boundary such that, for every $i \in \{1, \ldots, 2g\}$:

- $\tilde{a}_i \cap \tilde{b}_i = \{c_i, c_i', c_i''\}$, where $c_i''$ is in the interior of $S_{\frac{1}{2}}$, $c_i$ and $c_i'$ are on the boundary and all intersections are transverse,
if we orient $a_i$ and give $b_i$ the orientation induced from that of $a_i$, then an oriented basis of $T_{c_i}b_i$ followed by an oriented basis of $T_{c_i}a_i$ yields an oriented basis of $T_{c_i}\Sigma$, 

• $a_i \cap b_j = \emptyset$ for $j \neq i$ and

• in a neighborhood of $\partial S_{1/2} \subset \Sigma$, $b_i$ is a smooth extension to $S_{1/2}$ of $\varphi(a_i) \subset S_0$.

Then we define the set of curves $\beta = \{\beta_1, \ldots, \beta_{2g}\} \subset \Sigma$ by

$$\beta_i = b_i \cup \partial \varphi(a_i) \varphi(a_i).$$

Figure 1. A basis of arcs in $\overline{S_{1/2}}$ and its perturbation.

If we choose a basepoint $w \in S_{1/2}$ outside of the thin strips given by the isotopies from the $a_i$’s to the $b_i$’s, then $(\Sigma, \alpha, \beta, w)$ is a pointed Heegaard diagram for $Y$ compatible with $(K, S, \varphi)$. However, we prefer to work with a Heegaard diagram where the orientation of the Heegaard surface matches the orientation of $S_0$, and therefore we will consider the Heegaard diagram $(\overline{\Sigma}, \alpha, \beta, z)$ for $\overline{Y}$. This Heegaard diagram is clearly weakly admissible: see [25].

Let $\mathfrak{S}_k$ denote the symmetric group with $k$ elements. We recall that the Heegaard Floer chain complex $\widehat{CF}(\Sigma, \alpha, \beta, w)$ is defined as the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the $2g$-tuples of intersection points $x = \{x_1, \ldots, x_{2g}\}$ such that $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_{2g}$.

To define the Heegaard Floer differential we will use Lipshitz’s cylindrical reformulation from [36] with the conventions of [11]. A generator $x = \{x_1, \ldots, x_{2g}\}$ of $\widehat{CF}(\Sigma, \alpha, \beta, w)$ can be identified with the set of $2g$ chords $[0, 1] \times \Sigma$. We endow $\mathbb{R} \times [0, 1] \times \Sigma$ with the symplectic form

$$ds \wedge dt + \omega_{\Sigma}.$$
where $s$ and $t$ are the coordinates of $\mathbb{R}$ and $[0, 1]$ respectively and $\omega_\Sigma$ is an area form on $\Sigma$ which restricts to the area form $\omega$ on $S$, and choose an *admissible* almost complex structure $J$ (see [11, Definition 4.2.1]).

For every $i \in \{1, \ldots, 2g\}$, call $L_{\alpha_i}$ and $L_{\beta_i}$ the Lagrangian submanifolds $\mathbb{R} \times \{1\} \times \alpha_i$ and, respectively, $\mathbb{R} \times \{0\} \times \beta_i$ of $\mathbb{R} \times [0, 1] \times \Sigma$. Define moreover $L_{\alpha} = \bigsqcup_{i=1}^{2g} L_{\alpha_i}$ and $L_{\beta} = \bigsqcup_{i=1}^{2g} L_{\beta_i}$.

Let $(F, j)$ be a compact (possibly disconnected) Riemann surface with two sets of punctures $\mathbf{p}^+ = \{p_1^+, \ldots, p_k^+\}$ and $\mathbf{p}^- = \{p_1^-, \ldots, p_k^-\}$ on $\partial F$ such that

(i) every connected component of $F$ has nonempty boundary, and

(ii) every connected component of $\partial F$ contains at least one element of $\mathbf{p}^+$ and one of $\mathbf{p}^-$, and

(iii) the elements of $\mathbf{p}^+$ and $\mathbf{p}^-$ alternate along $\partial F$.

Let $\hat{F}$ denote $F$ with the sets of punctures $\mathbf{p}^+$ and $\mathbf{p}^-$ removed.

**Definition 2.1.** Let $\mathbf{x} = \{x_1, \ldots, x_k\}$ and $\mathbf{y} = \{y_1, \ldots, y_k\}$ be two $k$-tuple ($k \leq 2g$) of points in $\Sigma$ with $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ and $y_i \in \alpha_i \cap \beta_{\sigma'(i)}$ for some permutations $\sigma, \sigma' \in \mathcal{S}_k$. A degree-$k$ *multisection* of $\mathbb{R} \times [0, 1] \times \Sigma$ from $\mathbf{x}$ to $\mathbf{y}$ is a $J$-holomorphic map

$$u : (\hat{F}, j) \longrightarrow (\mathbb{R} \times [0, 1] \times \Sigma, J)$$

satisfying the following conditions:

1. $(\hat{F}, j)$ is a punctured Riemann surface as above;
2. $u(\partial \hat{F}) \subset L_{\alpha} \cup L_{\beta}$ and each connected component of $\partial \hat{F}$ is mapped to a different $L_{\alpha_i}$ or $L_{\beta_i}$;
3. $\lim_{w \to p_i^+} u_\mathbb{R}(w) = +\infty$ and $\lim_{w \to p_i^-} u_\mathbb{R}(w) = -\infty$ where $u_\mathbb{R}$ is the component of $u$ to $\mathbb{R}$;
4. near $p_i^+$ (respectively $p_i^-$), $u$ is asymptotic to the strip over $[0, 1] \times \{x_i\}$ (respectively $[0, 1] \times \{y_i\}$).

For a $J$-holomorphic map $u$ as above, we define $n_w(u)$ as the algebraic intersection number between the image of $u$ and the $J$-holomorphic section $\mathbb{R} \times [0, 1] \times \{w\}$. By positivity of intersection $n_w(u) \geq 0$.

We define $\hat{\mathcal{M}}_1(\mathbf{x}, \mathbf{y})$ as the set of equivalence classes (modulo reparametrisations and $\mathbb{R}$-translations) of holomorphic multisections $u : (\hat{F}, j) \longrightarrow (\mathbb{R} \times [0, 1] \times \Sigma, J)$ of degree $2g$ from $\mathbf{x}$ to $\mathbf{y}$ such that:

1. $n_w(u) = 0$ and
2. $u$ has Lipshitz’s $ECH$-type index 1 (see [11, Section 4] for details).

Note that, for multisections of $\mathbb{R} \times [0, 1] \times \Sigma$, having $ECH$-type index 1 is equivalent to being embedded and having Fredholm index 1.

The Heegaard Floer differential (in the hat version) is then defined by

$$\partial^{\text{HF}}(\mathbf{x}) = \sum_y \#_2 \hat{\mathcal{M}}_1(\mathbf{x}, \mathbf{y}, J)y,$$

where the sum is taken over the set of generators of $\hat{\mathcal{C}}F(\Sigma, \alpha, \beta, w)$ and $\#_2$ denotes the cardinality modulo 2. The homology of $\hat{\mathcal{C}}F(\Sigma, \alpha, \beta, w)$ does not depend on the various choices and is denoted by $\hat{HF}(\Sigma)$. 
In [11, Section 4] it was shown that \( \hat{HF}(Y) \) can be computed using only the part of the diagram \((\Sigma, \alpha, \beta, w)\) contained in \(S_0\). To recall the construction, observe first that if \(u\) is a connected degree \(k\) multisection with \(n_w(u) = 0\) and a positive end at a chord associated to \(c_i\) or \(c'_i\), then \(k = 1\) and \(u\) is a trivial strip over that chord, i.e. a reparametrisation of either \(R \times [0, 1] \times \{c_i\}\) or \(R \times [0, 1] \times \{c'_i\}\).

Let \(CF'(S, a, \varphi(a))\) be the submodule of \(\hat{CF}(\Sigma, \alpha, \beta, w)\) generated by the \(2g\)-tuples of intersection points contained in \(S_0\) and endow it with the restriction of \(\partial\). \(CF'(S, a, \varphi(a))\) is a subcomplex of \((\hat{CF}(\Sigma, \alpha, \beta, w), \partial)\).

Let \(R \subset CF'(S, a, \varphi(a))\) be the subspace generated by all elements \(x - x'\) where there exists \(i\) such that \(x = \{x_1, \ldots, c_i, \ldots, x_{2g}\}\) and \(x' = \{x_1, \ldots, c'_i, \ldots, x_{2g}\}\). Then define

\[
\text{CF}(S, a, \varphi(a)) = CF'(S, a, \varphi(a))/R.
\]

In [11, Subsection 4.9] it was proved, with a slightly different terminology, that \(R\) is a subcomplex of \(CF'(S, a, \varphi(a))\), and therefore the quotient \(\text{CF}(S, a, \varphi(a))\) is a chain complex with the induced differential. We will call \(\hat{HF}(S, a, \varphi(a))\) its homology.

**Theorem 2.2.** (see [11, Theorem 4.9.4])

\[
\hat{HF}(S, a, \varphi(a)) \cong \hat{HF}(Y).
\]

### 2.3. Knot Floer homology via open book decompositions

In this section we show how the hat version of knot Floer homology can be computed using the diagram \((S, a, \varphi(a))\). To obtain this, however, it is necessary to be slightly more careful in the construction of the diagram. Let \(\varepsilon\) be the same constant as in Equation 2.2. We assume that, for every \(i \in \{1, \ldots, 2g\},\)

1. \(\{c_i, c'_i\} = a_i \cap \partial S \subset \{2\} \times (0, \varepsilon),\) and
2. \(a_i \cap [1, 2] \times \partial S = [1, 2] \times \{c_i, c'_i\}\).

These conditions can be achieved by a Hamiltonian isotopy of the arcs \(a_i\).

If \(z = (1, 2\varepsilon),\) then \((\Sigma, \alpha, \beta, z, w)\) is a a doubly pointed Heegaard diagram for \((K, Y)\). Let \(Y_0(K)\) be the zero surgery on \(Y\) along \(K\). Following Ozsváth and Szabó [49], given

![Figure 2. A diagram compatible with the knot in a neighborhood of \(\partial S_0\).](image-url)
a generator $x$ of $\widehat{CF}(\Sigma, \alpha, \beta, w)$, its Alexander degree $A(x)$ with respect to $S_0$ is the integer

$$A(x) = \frac{1}{2} \langle c_1(g_w(x)), [\tilde{S}_0] \rangle$$

where $g_w(x)$ is the $Spin^c$–structure on $Y_0(K)$ determined by $x$ and $w$ and $\tilde{S}_0$ is the surface $S_0 \setminus A$ capped off in $Y_0(K)$. By standard computations using periodic domains (see for example [48, Section 7.1], and also [5, Lemma 6.1] for a similar computation) one can check that

$$A(x) = -g + \# (x \cap (S_0 \setminus A)).$$

Moreover the positivity of intersection between holomorphic curves in dimension 4 implies that, if $u$ is a multisection from $x$ to $y$, then:

$$A(x) - A(y) = n_z(u) \geq 0.$$

For this reason the Alexander grading $A$ induces a filtration on $\widehat{CF}(\Sigma, \alpha, \beta, w)$. The knot Floer homology complex of $(Y, K)$ is the associated graded complex

$$\left( \widehat{CFK}(\Sigma, \alpha, \beta, z, w), \partial_K \right) := \bigoplus_{i \in \mathbb{Z}} \left( \widehat{CFK}(\Sigma, \alpha, \beta, z, w, i), \partial_i \right),$$

where $\widehat{CFK}(\Sigma, \alpha, \beta, z, w, i)$ is the subspace of $\widehat{CF}(\Sigma, \alpha, \beta, w)$ generated by the $2g$-tuples of intersection points $x$ with $A(x) = i$ and

$$\partial_i : \widehat{CFK}(\Sigma, \beta, \alpha, z, w, i) \to \widehat{CFK}(\Sigma, \alpha, \beta, z, w, i)$$

is the restriction to $\widehat{CFK}(\Sigma, \beta, \alpha, z, w, i)$ of the component of $\partial$ that preserves the Alexander degree. The resulting homology

$$\widehat{HFK}(Y, K) = \bigoplus_{i \in \mathbb{Z}} \widehat{HFK}(Y, K; i) = \bigoplus_{i \in \mathbb{Z}} \check{H}_*(\widehat{CFK}(\Sigma, \alpha, \beta, z, w, i), \partial_i)$$

is the knot Floer homology of $K$ in $Y$.

The restriction to generators and holomorphic curves that are contained in the $S_0$ part of the diagram and the quotient (2.3) are compatible with the Alexander grading.

The result is the chain complex

$$\left( \widehat{CFK}(S, \alpha, \varphi(\alpha), z), \partial_K \right) = \bigoplus_{i \in \mathbb{Z}} \left( \widehat{CFK}(S, \alpha, \varphi(\alpha), z, i), \partial_i \right)$$

whose homology is isomorphic to $\widehat{HFK}(Y, K)$. The proof is the same of that of [11, Theorem 4.9.4]. The base point $z$ will often be dropped from the notation, as we did for $w$, because it is placed in the region $A$ where all our diagrams will have the standard form described above.
2.4. Fixed point Floer homology. Let $M$ be a closed manifold endowed with a symplectic form and let $\psi: M \rightarrow M$ be a symplectomorphism. If $\psi$ satisfies suitable properties (see below for some of the details), one can define the fixed point Floer homology $HF(\psi)$ of $\psi$ as the homology of a finite dimensional chain complex whose generators are the fixed points of $\psi$. This homology was defined by Floer ([22]) in the case $\psi$ is Hamiltonian isotopic to $id_M$ and by Dostoglou and Salamon ([18]) in the general case. In general $HF(\psi)$ is an invariant of the Hamiltonian isotopy class of $\psi$. However, if $\dim(M) = 2$, Seidel showed in [59] that $HF(\psi)$ is in fact a topological invariant of the mapping class $[\psi]$.

Fix now $(K, S, \varphi)$ as in Section 2.1. To define the fixed point Floer homology of $\varphi$, in order to get a finitely generated chain complex, one needs first to perturb the infinite family of fixed points given by $\varphi_{|_{\partial S}}$. One way to do this is to compose $\varphi$ with a small rotation along $\partial S$: this gives two versions $HF(\varphi, +)$ and $HF(\varphi, -)$ of Floer homologies of $\varphi$, which correspond to choosing a positive or, respectively, negative rotation along $\partial S$ (with respect to the orientation induces by $S$). See for example [61] for details. In this section we introduce an intermediate version of fixed point Floer homology $HF^+(\varphi)$.

Let $Fix(\varphi)$ be the set of fixed points of $\varphi$. There is a natural identification between $Fix(\varphi)$ and the set of closed orbits of period one of the vector field $\partial_t$ in the mapping torus $T_{\varphi}$: to a fixed point $x$ corresponds the periodic orbit $\gamma_x$ through $x$. Up to Hamiltonian isotopy, we can suppose that $\varphi$ has only non-degenerate fixed points in the interior of $S$. We recall that a fixed point $x$ is non-degenerate if $\det(1 - d_x \varphi) \neq 0$. This implies, in particular, that $\varphi$ has finitely many points in the interior of $S$. Recall that a non degenerate fixed point $x$, and the corresponding orbit $\gamma_x$, are elliptic if $d_x \varphi$ has complex conjugated eigenvalues and is positive or, respectively, negative hyperbolic if the eigenvalues of $d_x \varphi$ are positive or, respectively, negative real.

The boundary $\partial S$ is a Morse-Bott circle of degenerate fixed points and, correspondingly, $\partial T_{\varphi}$ is a Morse-Bott torus of degenerate orbits. We will describe here only the effect of the Morse-Bott perturbation of $\partial T_{\varphi}$, referring the reader to Bourgeois [7] for a more complete discussion of the subject and to Colin, Ghiggini and Honda [9] for one more adapted to the situation at hand.

The Morse–Bott perturbation takes place in the mapping torus of an extension of $\varphi$ to a larger surface $\widehat{S}$ that we now describe (cf. [11, Section 2]). Let $(y, \vartheta) \in [1, 2] \times S^1$ and $f \in C^\infty([1, 2])$ be the system of coordinates and, respectively, the function introduced in Section 2.1. We define $\widehat{S} := S \cup ([2, 4] \times S^1)$ where $\partial S$ is glued to $\{2\} \times S^1$ and extend the coordinates $(y, \vartheta)$ to $[2, 4] \times S^1$ in the natural way. We extend also $f$ to a function $f \in C^\infty([1, 4])$ with $f'(y) < 0$ for all $y \in (1, 4)$ and $\frac{\pi}{2} < f(4) < 0$. Then we extend $\varphi$ to a symplectomorphism (still denoted by $\varphi$) of $\widehat{S}$ by setting $\varphi(y, \vartheta) = (y, \vartheta + f(y))$ for all $y \in [1, 4]$. Let $\widehat{T}_{\varphi}$ be the mapping torus of $(\widehat{S}, \varphi)$. Observe that the properties of $f$ imply that $\varphi$ has no periodic orbits of period smaller or equal to $2g$ crossing a page in the region $\{y \in (2, 4]\}$. As for the fibers of $T_{\varphi}$, we denote $\widehat{S}_t := \{t\} \times \widehat{S} \subset \widehat{T}_{\varphi}$.

We also extend $\omega$ to $\widehat{S}$ using the Liouville structure and let $\omega_v$ denote the closed two-form on $\widehat{T}_{\varphi}$ which restricts to $\omega$ on every fiber and such that $i_{\vartheta} \omega_v = 0$. Then $(\omega_v, dt)$ is a stable Hamiltonian structure on $\widehat{T}_{\varphi}$ with Reeb vector field $\partial_t$. The goal of the Morse-Bott perturbation is to replace $(\omega_v, \partial_t)$ with a new stable Hamiltonian
structure \((\tilde{\omega}_v, dt)\) with Reeb vector field \(\tilde{R}\) such that all periodic orbits of the flow of \(R\) of period at most \(2g\) are nondegenerate. The first return map of the flow of \(\tilde{R}\) is a symplectomorphism \(\tilde{\varphi}\) of \((S, \omega)\) which is Hamiltonian isotopic to \(\varphi\).

The Morse–Bott perturbation can be made with support in the neighborhood \(\{y \in (1, 3)\}\) of \(\partial T_\varphi\). Before the perturbation each of the boundary parallel tori \(T_{y_0} := \{(t, y, \theta) \in \tilde{T}_\varphi \mid y = y_0\}\), for \(y_0 \in [1, 4]\), is linearly foliated by orbits of \(\partial_t\) and \(T_2 = \partial T_\varphi\) is the only one that is foliated by orbits with period smaller or equal to \(2g\). After the perturbation the only periodic orbits with period smaller than or equal to \(2g\) and crossing the region \(\{y \in [1, 4]\}\) are two period–1 orbits \(e\) and \(h\) contained in \(\partial T_\varphi\). The tori \(T_y\) which are contained in the support of the perturbation are not foliated by trajectories of \(\tilde{R}\). On the other hand, a new family of tori foliated by trajectories of \(\tilde{R}\) is created (see Figure 3) and each of these tori bounds a solid torus in \(\tilde{T}_\varphi\) with core curve \(e\).

**Figure 3.** The dynamics before and after the Morse–Bott perturbation of the torus \(\partial T_\varphi\). Both pictures represent the annulus \(\{y \in [0, 3]\}\) in \(\hat{S}_0\) (the left side has to be identified with the corresponding right side). Each flow line represents an invariant subset of \(\hat{S}_0\) under the first return map of \(\partial_t\) or \(\tilde{R}\) and the arrows give the direction in which any point is mapped. In particular, on both annuli, each closed non-singular flow line is the intersection between the page and an embedded torus foliated by flow trajectories of the corresponding vector field.

**Remark 2.3.** The reason for which the two orbits above are called \(e\) and \(h\) is that the first is elliptic and the second is (positive) hyperbolic. We will denote \(p_e\) and \(p_h\) the corresponding fixed points of \(\tilde{\varphi}\).

**Remark 2.4.** The form \(\tilde{\omega}_v\) is exact. In fact \(\omega_v\) is exact because Equation 2.1 allows us to construct a primitive by interpolating between \(\beta\) and \(\varphi^*\beta\), and \(\tilde{\omega}_v = \omega_v + dH \wedge dt\) for a function \(H: \tilde{T}_\varphi \to \mathbb{R}\).

We define \(CF^\#(\varphi)\) as the vector space generated over \(\mathbb{Z}/2\mathbb{Z}\) by \(\text{Fix}(\tilde{\varphi}) \setminus \{p_e\}\). We endow it with a differential as follows. Consider the symplectic fibration \((\mathbb{R} \times \tilde{T}_\varphi, \Omega)\) over \(\mathbb{R} \times S^1\) with symplectic form \(\Omega = ds \wedge dt + \tilde{\omega}_v\) and fiber \(\tilde{S}\), and endow it with an \(\mathbb{R}\)–invariant \(\Omega\)–tame almost complex structure \(J\) such that \(J(\partial_s) = \tilde{R}\) and \(J(T\tilde{S}) = T\tilde{S}\). If \(x_+, x_-\) are fixed points of \(\tilde{\varphi}\), let \(\mathcal{M}(x_+, x_-, J)\) be the moduli space of \(J\)–holomorphic
sections \( u : \mathbb{R} \times S^1 \to \mathbb{R} \times \hat{T}_\varphi \) such that
\[
\lim_{s \to \pm \infty} u(s, \cdot) = \gamma_{x_{\pm}}.
\]

To each \( J \)-holomorphic cylinder \( u \) is associated a Fredholm operator \( D_u \) of index \( \text{ind}(u) \). Call \( M_k(x, y, J) \) the subset of \( M(x, y, J) \) with \( \text{ind}(u) = k \). We define the differential \( \partial^k \) on \( CF^k(\varphi) \) by
\[
\partial^k(x) = \sum_{y \in \text{Fix}(\varphi) \setminus \{pc\}} \#_2 \overline{M}_1(x, y, J)y
\]
where \( \#_2 \) denotes the cardinality modulo 2 and \( \overline{M}_1(x, y, J) \) is \( M_1(x, y, J) \) quotiented by the \( \mathbb{R} \)-action given by translations in the \( \mathbb{R} \)-direction. The map \( \partial^k \) is well defined because exactness of the form \( \tilde{\omega}_\nu \) (see Remark 2.4) implies the compactness of the moduli spaces \( \overline{M}_1(x, y, J) \) and the finiteness of the sum. Moreover it is a differential because every holomorphic cylinder in \( \mathbb{R} \times \hat{T}_\varphi \) with positive end at \( e \) is a trivial cylinder over \( e \); see [9, Section 7].

The sharp version of the fixed point Floer homology of \( \varphi \) is then
\[
HF^k(\varphi) := H_*\bigl(CF^k(\varphi), \partial^k\bigr).
\]
The argument of [59, Section 3] can be easily adapted to show that \( HF^k(\varphi) \) is an invariant of the mapping class of \( \varphi \) in the mapping class group of \( (S, \partial S) \).

**Remark 2.5.** The definition of \( HF^k \) given here is inspired by the definition of \( ECH^k \) in [9, Section 7]. An equivalent definition has been given by A. Kotelsky in [33].

### 2.5. Periodic Floer homology

The construction of symplectic Floer homology can be generalized to consider orbits of \( \varphi \) of higher period: the resulting invariant is the periodic Floer homology of \( \varphi \). We will recall the definition of two versions of this homology defined when \( \varphi \) is the monodromy of a fibered knot \( K \) in a three-manifold \( Y \): the first, denoted \( \text{PFH}_{2g}(T_\varphi) \), is defined in this subsection and is an invariant of \( Y \) (cf. [12]); the second version, denoted \( \text{PFH}^k_1(T_\varphi) \), for \( i = -g, \ldots, g \), will be defined in the next subsection and is an invariant of \( (Y, K) \) (cf. [10, Section 10]). For the details we refer the reader to [11, Section 3] and [28].

Consider the stable Hamiltonian structure \((\tilde{\omega}_\nu, dt)\) on \( \hat{T}_\varphi \) with Reeb vector field \( \hat{R} \) introduced in the previous section. An orbit set (or multiorbit) in \( T_\varphi \) is a formal finite product \( \gamma = \prod_i \gamma_i^{k_i} \), where \( \gamma_i \) is a simple (i.e. embedded) orbit of the flow of \( \hat{R} \) and \( k_i \in \mathbb{N} \) is the multiplicity of \( \gamma_i \) in \( \gamma \), with \( k_i \in \{0, 1\} \) whenever \( \gamma_i \) is hyperbolic.

We will denote by \( P \) the set of simple orbits \( \gamma \) of \( \hat{R} \) in \( T_\varphi \) and \( O_d \) the set of orbit sets \( \gamma = \prod_i \gamma_i^{k_i} \) with \( \gamma_i \in P \) and
\[
\langle \gamma, S_0 \rangle = \sum_i k_i \langle \gamma_i, S_0 \rangle = d.
\]

Given \( d \in \mathbb{Z} \), we define \( \text{PFC}_d(T_\varphi) \) to be the vector space over \( \mathbb{Z}/2\mathbb{Z} \) generated by the orbit sets in \( O_d \). Observe that we are considering orbit sets for the perturbed Reeb vector field \( \hat{R} \), so that both orbits \( e \) and \( h \) can appear in the generators. Remark moreover that, since \( \langle \delta, S_0 \rangle > 0 \) for any orbit in \( P \), we have \( \text{PFC}_d(T_\varphi) = 0 \) whenever \( d < 0 \), and \( \text{PFC}_0(T_\varphi) \cong \mathbb{Z}/2\mathbb{Z} \) is generated by the empty orbit set.
Let $J$ be an almost complex structure as in the previous section. Given orbit sets $\gamma_-$ and $\gamma_+$ in $O_d$, let $M(\gamma_+, \gamma_-, J)$ be the moduli space of degree $d$, possibly disconnected, $J$-holomorphic multisections of $\mathbb{R} \times \hat{T}_\varphi$ which are positively asymptotic to $\gamma_+$ and negatively asymptotic to $\gamma_-$. Here convergence to $\gamma_+$ and $\gamma_-$ has to be understood in the sense of ECH; see [27, Definition 1.2]. We call $J$ and $\partial \gamma_+$ and $\partial \gamma_-$ half integer, where $(2.6)$

$\partial(\gamma_+) = \sum_{\gamma_- \in O_d} \# \bar{M}_1(\gamma_+, \gamma_-, J) \gamma_-,$

where $\bar{M}_1(\gamma_+, \gamma_-, J)$ is the quotient of $M(\gamma_+, \gamma_-, J)$ by the $\mathbb{R}$-action given by translations in the $\mathbb{R}$-direction. The resulting homology

$PFH_d(T_\varphi) := H_*(PF_{d}(T_\varphi), \partial)$

is the periodic Floer homology of $\varphi$ (or $T_\varphi$).

The following theorem is one of the main results of [11, 12] and an important step toward the isomorphism between Heegaard Floer homology and embedded contact homology. However we will not need it in the present article.

**Theorem 2.6.** Let $(K, S, \varphi)$ be an open book decomposition of genus $g \geq 1$ of $Y$. Then there is an isomorphism

$HF(Y) \cong PFH_{2g}(T_\varphi),$

2.6. Periodic Floer homology for fibered knots. In this section define the *sharp-version* of periodic Floer homology $PFH^s(T_\varphi)$ associated to an open book decomposition. The construction is analogous to the construction of $ECH^s$ from [10, Section 7].

**Definition 2.7.** For any generator $\gamma = e^k \prod_j \gamma_j^{k_j}$ of $PF_{d}(T_\varphi)$, with $\gamma_j \neq e$ for all $j$, we define $\gamma := \prod_j \gamma_j^{k_j}$. The *Alexander degree* of $\gamma$ with respect to $S_0$ is the integer, or half integer,

$A(\gamma) := \langle \gamma, S_0 \rangle - \frac{d}{2} = \frac{d}{2} - k.$

The most interesting case is when $d = 2g$. If $\gamma \in O_{2g}$, then $A(\gamma) \in \{-g, \ldots, g\}$ and $A(\gamma) = -g$ if and only if $\gamma = e^{2g}$.

**Lemma 2.8.** if $\gamma_+$ and $\gamma_-$ are two orbit sets in $O_d$ such that $M(\gamma_+, \gamma_-, J) \neq \emptyset$, then $A(\gamma_+) \geq A(\gamma_-)$.

**Proof.** Let $u: (\hat{F}, j) \to \mathbb{R} \times \hat{T}_\varphi$ be a $J$-holomorphic curve multisection in $M(\gamma_+, \gamma_-, J)$ and let $\hat{F}_0$ be a connected component of $F$ with a positive puncture asymptotic to $e$, or a multiple cover of $e$. If $u|_{\hat{F}_0}$ is not a cover of a trivial cylinder on $e$, then the projection to $\hat{T}_\varphi$ of the end of $u$ which is asymptotic to $e$ must approach $e$ with nonzero winding number with respect to the longitude induced by $\partial T_\varphi$: this is a consequence of Lemma 5.3.2 of [9] and the fact that $\partial T_\varphi$ is a negative Morse-Bott torus. However, for topological reasons, this is not possible for a map with value in $\hat{T}_\varphi$ and thus $u|_{\hat{F}_0}$ is a cover of a trivial cylinder on $e$. This implies that the total multiplicity of $e$ in $\gamma_-$ is at least equal to the total multiplicity in $\gamma_+$. □
The previous lemma shows that the Alexander degree induces a filtration on $PFC_{2g}(T_\varphi)$ called the Alexander filtration.

Now we recall the sharp-version of periodic Floer homology, which is similar to the sharp-version of embedded contact homology defined in [9, Section 7] and generalises the sharp-version of fixed point Floer homology defined in Section 2.4. Given $d \geq 0$, we denote by $O_d^\#$ the subset of $O_d$ consisting of orbit sets which do not contain $e$. Then we define $PFC_d^\#(T_\varphi)$ as the vector space generated over $\mathbb{Z}/2\mathbb{Z}$ by the orbit sets in $O_d^\#$.

By Lemma 2.8 we can identify $PFC_d^\#(T_\varphi)$ to the quotient of the complex of $PFC_d(T_\varphi)$ by the subcomplex generated by the orbit sets containing $e$, and therefore it carries an induced differential $\partial^\#$. We denote $PFH^\#_d(T_\varphi) = H^*(PFC_d(T_\varphi), \partial^\#)$.

From Lemma 2.8 it follows that the graded complex of $PFC_{2g}(T_\varphi)$ associated to the Alexander filtration is isomorphic to

$$\bigoplus_{i = -g}^g PFC_{i+g}(T_\varphi).$$

Combining the isomorphism $PFH^\#_i(T_\varphi) \cong ECH^\#_i(T_\varphi)$ from [11, Section 3.6], the isomorphism between $PFH^\#_i(T_\varphi)$ and the sutured contact homology of the knot complement from [9, Theorem 10.3.2] and [14, Conjecture 1.5], we obtain the following conjecture.

**Conjecture 2.9.** If $K \subset Y$ is a fibered knot with monodromy $\varphi$, then

$$\hat{HFK}(Y, K, i) \cong PFH^\#_{i+g}(T_\varphi).$$

Note that, dropping the requirement that $K$ is fibered and denoting by $N$ the complement of a tubular neighbourhood of $K$, one can still define $ECH^\#_i(N)$ and in that case it is expected that $\hat{HFK}(Y, K, i)$ is isomorphic to $ECH^\#_{i+g}(N)$.

**Remark 2.10.** By Hutchins and Sullivan [28], there is an identification of chain complexes

$$(PFC^\#_i(T_\varphi), \partial^\#) = (CF^\#(\varphi), \partial^\#)$$

inducing a canonical isomorphism between $PFH^\#_i(T_\varphi)$ and $HF^\#(\varphi)$.

3. **The Chain Map from HFK to PFK**

The aim of this section is to define chain maps

$$\Phi^\#: \hat{CFK}(S, a, \varphi(a), z, i) \to PFC^\#_{i+g}(T_\varphi)$$

for any $i \in \{-g, \ldots, g\}$. In particular, for $i = -g + 1$, by Remark 2.10, we get a chain map

$$\Phi^\#: \hat{HFK}(S, a, \varphi(a), z, i) \to CF^\#(\varphi).$$

In the next section we will prove that $\Phi^\#$ induces an isomorphism in homology. In order to define the maps $\Phi^\#$ we will first review the chain map

$$\Phi: \hat{CF}(S, a, \varphi(a)) \to PFC_{2g}(T_\varphi)$$
defined in [11] by Colin, Ghiggini and Honda.

We will then show that $\Phi$ respects the Alexander filtrations induced by $K$ on both sides and define the maps $\Phi^g_i$ as the maps induced by $\Phi$ between the homogeneous summands of the graded complexes, taking into account the identification of $\bigoplus_{i=-g}^g \text{PFC}^g_{i+g}(T_\varphi)$ with the graded complex of $\text{PFC}_{2g}(T_\varphi)$.

3.1. Review of the chain map $\Phi$. In this section we recall the definition of the chain map

$$\Phi: \widehat{CF}(S, a, \varphi(a)) \to \text{PFC}_{2g}(T_\varphi)$$

introduced in [11]. We refer the reader to sections 5 and 6 of [11] for details. Moreover, only for this section, we will regard the mapping torus of $(S, \varphi)$ as $T_\varphi = S \times [0, 2] \sim (\varphi(x), 0)$, and analogously for the mapping torus $\widehat{T}_\varphi$ of $(\widehat{S}, \varphi)$. We denote $\pi: \mathbb{R} \times \widehat{T}_\varphi \to S^1 \cong [0, 2]/0 \sim 2$ the fibration naturally extending to the $\mathbb{R}$-component the fibration $\widehat{T}_\varphi \to S^1$ with fibre $\widehat{S}$ defined by $(x, t) \mapsto t$.

Define now the subset $B_+^c := [2, \infty) \times (1, 2)$ of $\mathbb{R} \times [0, 2]/0 \sim 2$ with all the corners smoothed and let $B_+ = (\mathbb{R} \times S^1) \setminus B_+^c$. Note that $B_+$ is biholomorphic to $\mathbb{D}^2 \setminus \{0, 1\}$. Then we define the cobordism

$$W_+ = \pi^{-1}(B_+)$$

and a map $\pi_{B_+}: W_+ \to B_+$ by restricting $\pi$ to $W_+$. By construction $\pi_{B_+}: W_+ \to B_+$ is a fibration with fibre $\widehat{S}$. We will continue to indicate by $\pi_R$ the restriction to $W_+$ of the projection $\mathbb{R} \times \widehat{T}_\varphi \to \mathbb{R}$.

Consider the Morse-Bott perturbed stable Hamiltonian structure $(\tilde{\omega}_\varphi, dt)$ on $\widehat{T}_\varphi$ with Reeb vector field $\tilde{R}$ defined in Section 2.4. The 2–form $\Omega = ds \wedge dt + \tilde{\omega}_\varphi$ on $\mathbb{R} \times \widehat{T}_\varphi$ is symplectic and induces, by restriction, a symplectic form $\Omega_+$ on $W_+$. Endowed $W_+$ with this symplectic form, $\pi_{B_+}: W_+ \to B_+$ can be seen as a symplectic fibration. The
symplectic connection on $W_{\pm}$ given by the $\Omega_+\!$-orthogonal of the tangent space of the fibers is spanned by $\partial_k$ and $R$.

Fix a basis of arcs $a = \{a_1, \ldots, a_{2g}\}$ in $S$ as in Section 2.3 and extend each $a_i$ to a segment $\tilde{a}_i$ in $\tilde{S}$ straight until it meets $\partial \tilde{S}$. If $f \in C^\infty([1, 4])$ is the function defined in last section, we assume that $|f(4)|$ is small enough to ensure that $\tilde{a}_i \cap \tilde{\varphi}(\tilde{a}_j) \cap \{y \in (2, 4]\} = \emptyset$ for any $i, j$. The proof of the following lemma is an immediate application of the implicit function theorem.

**Lemma 3.1.** If the Morse-Bott perturbation is sufficiently small, then the intersection points between the arcs $\tilde{a}_i$ and $\tilde{\varphi}(\tilde{a}_i)$ are in natural bijection with the intersection points between $a_i$ and $\varphi(a_j)$ for all $i$ and $j$. Moreover this bijection induces an isomorphism between the chain complex $\widehat{CF}(S, a, \varphi(a))$ and the chain complex $\widehat{CF}(\tilde{S}, \tilde{a}, \tilde{\varphi}(\tilde{a}))$.

Given a $k$-tuple of intersection points $x = (x_1, \ldots, x_k)$ between $a$ and $\varphi(a)$, we denote $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k)$ the corresponding $k$-tuple of intersection points between $\tilde{a}$ and $\tilde{\varphi}(\tilde{a})$.

Take a copy of $\tilde{a} = \{\tilde{a}_1, \ldots, \tilde{a}_{2g}\}$ in $\pi_{B+}^{-1}(3, 1)$ and call $L^+_a$ the trace of the parallel transport of $\tilde{a}$ along $\partial B_+$ using the symplectic connection; then $L^+_a$ is Lagrangian and

\[
\begin{align*}
L^+_a \cap \{s \geq 3, t = 0\} &= \{s \geq 3\} \times \{t = 0\} \times \tilde{\varphi}(\tilde{a}); \\
L^+_a \cap \{s \geq 3, t = 1\} &= \{s \geq 3\} \times \{t = 1\} \times \tilde{a}.
\end{align*}
\]

Note that $L^+_a$ has $2g$ connected components $L^+_{a_i}$, one for each component $\tilde{a}_i$ of $\tilde{a}$.

The map $\tilde{\varphi}$ is given as a chain map

$$
\Phi: \widehat{CF}(S, a, \varphi(a)) \longrightarrow PFC_{2g}(T_\varphi)
$$

defined by counting certain *multisections* of the symplectic fibration $(W_+, \pi_{B+})$ with Lagrangian boundary condition $L^+_a$.

Let $J_+$ be a suitable almost complex structure on $W_+$ which is compatible with $\Omega_+$ and cylindrical for $s > 3$ and $s < 1$. (See [11, section 5], and in particular Remark 5.3.10 for the Morse-Bott perturbation.) Let $(\hat{F}, j)$ be a compact (possibly disconnected) Riemann surface with two sets of punctures $p = \{p_1, \ldots, p_l\}$ in the interior and $q = \{q_1, \ldots, q_k\}$ in the boundary of $F$ such that

(i) every connected component of $F$ contains at least an element of $p$ and a connected component of $\partial F$, and

(ii) every connected component of $\partial F$ contains at least an element of $q$.

We will set $\hat{F} = F \setminus \{p \cup q\}$.

**Definition 3.2.** Let $\tilde{\xi} = \{\tilde{x}_1, \ldots, \tilde{x}_k\}$ be a $k$-tuple $(k \leq 2g)$ of $\tilde{a} \cap \tilde{\varphi}(\tilde{a})$ and $\gamma = \prod_j \gamma^m_j \in \mathcal{O}_k$. A degree $k$ multisection of $(W_+, J_+)$ from $\tilde{\xi}$ to $\gamma$ is a holomorphic map

$$
u: (\hat{F}, j) \longrightarrow (W_+, J_+)
$$

where $(\hat{F}, j)$ is a Riemann surface as above, and $u$ is such that

1. $u(\partial \hat{F}) \subset L^+_a$ and maps each connected component of $\partial \hat{F}$ to a different $L^+_{a_i}$;
2. $\lim_{w \to q_i} \pi_\mathbb{R} \circ u(w) = +\infty$ and $\lim_{w \to p_i} \pi_\mathbb{R} \circ u(w) = -\infty$,
3. near $q_i$, $u$ is asymptotic to a strip over $[0, 1] \times \{\tilde{x}_i\}$;
4. near each $p_i$, $u$ is asymptotic to a cylinder over a multiple of some $\gamma_j$ so that the total multiplicity of $\gamma_j$ over all the $p_i$ is $m_i$. 

In practice, holomorphic multisections in $W_+$ interpolate between multisections in $\mathbb{R} \times [0, 1] \times \hat{S}$ and $\mathbb{R} \times \hat{T}_\varphi$. Moreover, in [11, Section 5] the authors define an ECH index for holomorphic multisections of $W_+$ which interpolates between the ECH-type index for holomorphic curves in $\mathbb{R} \times [0, 1] \times \hat{S}$ and the ECH index for holomorphic curves in $\mathbb{R} \times \hat{T}_\varphi$. As in [11], we will refer to the image of the connected components of $\hat{F}$ as to the irreducible components of $u$. We say that $u$ is irreducible if $\hat{F}$ is connected.

For $x = (x_1, \ldots, x_{2g})$ intersection points between $a$ and $\varphi(a)$, and $\gamma \in O_{2g}$, we denote by

$$\mathcal{M}_0(\tilde{x}, \gamma, J_+)$$

the space of ECH index 0, degree 2g multisections $(W_+, J_+)$ from $\tilde{x}$ to $\gamma$.

Recall the chain complex $CF'(S, a, \varphi(a))$ defined in Section 2.2 as a precursor of $\hat{CF}(S, a, \varphi(a))$. We define a map $\Phi': CF'(S, a, \varphi(a)) \to PFC_{2g}(T_\varphi)$ as

$$\Phi'(x) = \sum_{\gamma \in O} \#_2 \mathcal{M}_0(\tilde{x}, \gamma, J_+) \gamma.$$ 

In the following theorem we summarize some of the results about $\Phi$ proved in [11].

**Theorem 3.3.** The following hold:

1. $\Phi'$ is a chain map, and
2. $\Phi'$ maps the subspace $R$ defined in Section 2.2 to zero.

The first statement is [11, Proposition 6.2.2] together with Lemma 3.1. The second statement is [11, Proposition 6.2.2].

**Corollary 3.4.** The map $\Phi'$ induces a well defined chain map

$$\Phi: \hat{CF}(S, a, \varphi(a)) \to PFC_{2g}(T_\varphi).$$

By [12, Theorem 1.0.1], the map $\Phi$ induces an isomorphism in homology. We will not need this result.

3.2. $\Phi$ respects the filtrations. The aim of this section is prove the following theorem.

**Theorem 3.5.** $\Phi$ is filtered with respect to the Alexander filtrations induced by $K$ on $\hat{CF}(S, a, \varphi(a))$ and by $K$ on $PFC_{2g}(T_\varphi)$.

Before proving the theorem we introduce some notation. Let $D \subset \hat{S}$ be the disc which is invariant under the monodromy $\varphi$ and passes through the base point $z$. Then $D$ contains the fixes point $p_\varepsilon$ and, if $\varepsilon$ in the definition of $\varphi$ is small enough, $\hat{\gamma} \cap \varphi(\hat{\alpha}) \cap D = \hat{\alpha} \cap \varphi(\hat{\alpha}) \cap \{ y \in [1, 3] \}$; i.e., $D$ contains all the intersection points in Figure 2. Let $\mathcal{C} = \partial D$ with the induced boundary orientation. Since $\mathcal{C}$ is invariant under the monodromy, it gives rise to a torus $\mathcal{T} \subset \hat{T}_\varphi$ which is foliated by trajectories of the flow of $\hat{R}$. Finally we denote $\Sigma = (\mathbb{R} \times \mathcal{T}) \cap W_+ \cong B_+ \times \mathcal{C}$. If we project $L_a \cap \Sigma$ to $\mathcal{T}$ we obtain $2g$ segments $\sigma = \{ \sigma_1, \ldots, \sigma_{2g} \}$ which are tangent to $\hat{R}$.

**Lemma 3.6.** Let $u: (\hat{F}, J) \to (W_+, J_+)$ be a multisection of $(W_+, J_+)$ from $\tilde{x}$ to $\gamma$. Then

$$A(x) - A(\gamma) = \langle u(\hat{F}), \mathbb{R} \times \{ t_0 \} \times \mathcal{C} \rangle$$

for any $t_0 \in [0, 1]$.
Proof. We recall that all intersection points from Figure 2 are contained in \( D \) and therefore, by the reinterpretation of the Alexander grading given in Equation (2.5), we have
\[
\mathcal{A}(\mathbf{x}) = -g + \langle [0, 1] \times \tilde{x}, \{t_0\} \times (\tilde{S} \setminus D) \rangle
\]
for any \( t_0 \in [0, 1] \). Also, since \( \gamma \) is the only periodic orbit of \( \tilde{\gamma} \) with period at most \( 2g \) intersecting \( \{t_0\} \times D \), we have
\[
\mathcal{A}(\mathbf{\gamma}) = -g + \langle \gamma, \{t_0\} \times (\tilde{S} \setminus D) \rangle.
\]

Let \( \tilde{u} : \hat{F} \to W_+ \) be the continuous extension of the holomorphic multisection \( u : \hat{F} \to W_+ \) where \( \hat{F} \) is obtained by performing a real blow up of \( \hat{F} \) at the punctures and \( W_+ \) is obtained, roughly speaking, by adding \( \{-\infty\} \times \tilde{T}_\varphi \) and \( \{+\infty\} \times [0, 1] \times \tilde{S} \) to \( W_+ \) (see [11, Section 5.4.2]).

Consider the two surfaces
\[
M_{+\infty} = \{+\infty\} \times \{t_0\} \times (\tilde{S} \setminus D) \quad \text{and} \quad M_{-\infty} = \{-\infty\} \times \{t_0\} \times (\tilde{S} \setminus D),
\]
and define the closed surface
\[
M := M_{-\infty} \cup ([-\infty, +\infty] \times \{t_0\} \times (\mathcal{C} \cup \partial \tilde{S})) \cup M_{+\infty}.
\]
Since \( 0 = [M] \in H_2(\hat{W}_+; \mathbb{Z}) \), we have:
\[
0 = \langle \tilde{u}(\hat{F}), M \rangle = -\langle \tilde{u}(\hat{F}), M_{+\infty} \rangle + \langle \tilde{u}(\hat{F}), \mathbb{R} \times \{t_0\} \times (\mathcal{C} \cup \partial \tilde{S}) \rangle + \langle \tilde{u}(\hat{F}), M_{-\infty} \rangle = -\langle \mathcal{A}(\mathbf{x}) + g \rangle + \langle u(\hat{F}), \mathbb{R} \times \{t_0\} \times \mathcal{C} \rangle + \langle \mathcal{A}(\mathbf{\gamma}) + g \rangle,
\]
where we used the fact that \( u(\hat{F}) \) is disjoint from \( B_+ \times \partial \tilde{S} \subset \partial W_+ \) and has no ends intersecting \( \mathcal{C} \).

Proof of Theorem 3.5. Let \( u : \hat{F} \to W_+ \) be a multisection of \((W_+, J_+)\) from \( x \) to \( \gamma \). By Lemma 3.6 it remains to prove that
\[
\langle u(\hat{F}), \mathbb{R} \times \{t_0\} \times \mathcal{C} \rangle \geq 0.
\]
We can assume without loss of generality that the intersection of the projection of \( u(\hat{F}) \) to \( \hat{T}_\varphi \) with \( \mathcal{T} \) consists of a finite collections of closed curves and arcs with endpoints in \( \sigma \) because, by construction, a neighbourhood of \( \mathcal{T} \) in \( \hat{T}_\varphi \) is foliated by invariant tori for the flow of \( \hat{R} \) and replacing \( \mathcal{T} \) with a nearby one does not change the arguments.

Let \( c_u \) be the intersection of the projection of \( u(\hat{F}) \) to \( \hat{T}_\varphi \) with \( \mathcal{T} \) closed by adding segments in \( \sigma \). We orient \( c_u \) by requiring that, at the intersection points between the projection of \( u(\hat{F}) \) with \( \mathcal{T} \), the coorientation of \( \mathcal{T} \) followed by the orientation of \( c_u \) gives the orientation of the projection of \( u(\hat{F}) \). Then one can verify that
\[
\langle u(\hat{F}), \mathbb{R} \times \{t_0\} \times \mathcal{C} \rangle_{W_+} = \langle c_u, \{t_0\} \times \mathcal{C} \rangle_{\mathcal{T}}.
\]

The homology group \( H_1(\mathcal{T}; \mathbb{Z}) \) is freely generated by classes \( \lambda \), represented by \( \{t_0\} \times \mathcal{C} \), and \( \mu \), represented by a curve isotopic to \( e \) and intersecting each \( \tilde{S}_t \) in \( \tilde{S}_t \setminus S_t \), such that \( \langle \mu, \lambda \rangle = 1 \). Then \( [c_u] = k\mu \) because \( c_u \) is disjoint from \( \tilde{S}_t \setminus S_t \), and therefore \( \langle c_u, \mu \rangle = 0 \).

We form a curve \( \mathcal{I} \subset \mathcal{T} \) by connecting the endpoints of a sufficiently long trajectory of the flow of \( \hat{R} \) with a short segment disjoint from \( c_u \). Then, by the positivity of
intersection between the projection of \( u(\hat{F}) \) and \( \hat{R} \), we have \( \langle c_u, t \rangle \geq 0 \). Since \( t = l\lambda + m\mu \) with \( l, m > 0 \), we obtain \( \langle c_u, t \rangle = kl \), from which we deduce that \( k \geq 0 \). \( \square \)

We observe that the chain map \( \Phi^\sharp \) can be defined explicitly as

\[
\Phi^\sharp(x) = \sum_{\gamma \in \mathcal{P} \setminus \{e\}} #2\mathcal{M}_0(\tilde{x}, \gamma e^{2g-1}, J_+) \gamma
\]

for all \( x \) with \( A(x) = 1 - g \).

Lemma 3.7. Let \( a \) and \( b \) be bases of arcs for \( S \). Then there is a commutative diagram

\[
\begin{array}{ccc}
\widehat{HFK}(S, a, \varphi(a); -g + 1) & \xrightarrow{\Phi^\sharp_*} & HF^\sharp(\varphi) \\
\downarrow s_* & & \downarrow = \\
\widehat{HFK}(S, b, \varphi(b); -g + 1) & \xrightarrow{\Phi^\sharp_*} & HF^\sharp(\varphi)
\end{array}
\]

where \( s_* \) is the isomorphism of knot Floer homology for changing of basis.

Proof. Since two bases of arcs are always related by a sequence of arc slides, we assume without loss of generality that \( b \) is obtained from \( a \) by sliding \( a_1 \) over \( a_2 \). This means that \( b \) is as in the picture. It is important that we choose \( \theta^\pm_i \) close enough to the boundary so that every intersection point on \( a_i, b_i, \varphi(a_i) \) and \( \varphi(b_i) \) is between \( \theta^+_i \) and \( \theta^-_i \) and not between \( \theta^\pm_i \) and the boundary.

We define \( B_\delta = D^2 \setminus \{\pm 1, \pm i\} \) and \( E_\delta = B_\delta \times S \). Let \( \ell_1, \ldots, \ell_4 \) be the connected components of \( \partial B_\delta \) numbered in counterclockwise order starting from the arc between 1 and \( i \). On \( \partial E_\delta \) we consider the Lagrangian submanifold \( L = \ell_1 \times a \sqcup \ell_2 \times b \sqcup \ell_3 \times \varphi(b) \sqcup \ell_4 \times \varphi(a) \).
\( \ell_4 \times \phi(a) \). The map \( \Phi: \widehat{CFK}(S, a, \varphi(a); -g + 1) \to \widehat{CFK}(S, b, \varphi(b); -g + 1) \) is defined by counting index zero, degree \( 2g \) embedded multisections of \( E_a \) with boundary on \( L \) which are asymptotic to \( \Theta^+ = (\theta_{1}^+, \ldots, \theta_{2g}^+) \) at \( i \) and to \( \varphi(\Theta^-) = (\varphi(\theta_{1}^-), \ldots, \varphi(\theta_{2g}^-)) \) at \(-i\).

This map induces an isomorphism in homology because for every \( 2g \)-tuple of intersection points \( y_a \) between \( a \) and \( \varphi(a) \) there is a unique closest \( 2g \)-tuple of intersection points \( y_b \) and a unique small area, index zero multisection of \( E_a \) which is asymptotic to \( y_a, \Theta^+, y_b \) and \( \varphi(\Theta^-) \). On the fibre this multisection projects to a \( 2g \)-tuple of small fish-tail shaped quadrilaterals.

Next, we define \( B_{h,z} = D^2 \setminus \{0, 1, z, \bar{z}\} \) for \( z \in \partial D^2 \) with \( \Im(z) > 0 \) and \( E_{h,z} \) as the total space of a fibration over \( B_{h,z} \) with fibre \( S \) and monodromy \( \varphi \). Let \( \ell_1, \ell_2, \ell_3 \) the connected component of \( \partial B_{h,z} \) numbered counterclockwise starting from 1 and define the Lagrangian submanifold \( L = L_1 \sqcup L_2 \sqcup L_3 \) where \( L_1 = \ell_1 \times a, L_3 = \ell_3 \times \varphi(a) \) and \( L_2 \) is the parallel transport of \( b \) over \( \ell_3 \).

We define the chain homotopy \( h: \widehat{CFK}(S, a, \varphi(a); -g + 1) \to \widehat{CF}(\varphi) \) by counting index \(-1\) embedded degree \( 2g \) multisections in the family of fibrations \( E_{h,z} \to B_{h,z} \), for \( z \in \partial D^2 \cap \{ \Im(z) > 0 \} \), which have boundary on \( L \) and are asymptotic to \( \Theta^+ \) at \( z \) and \( \varphi(\Theta^-) \) at \( \bar{z} \).

The moduli space of index zero multisections of \( E_{h,z} \to B_{h,z} \) (with varying \( z \)) is a one dimensional moduli space which has boundary degenerations which correspond to \( \partial \circ h + h \circ \partial \), degenerations for \( z \to 1 \) which correspond to \( \Phi^i \circ s \) and degenerations for \( z \to -1 \) which, we claim, correspond to \( \Phi^i \).

As \( z \to -1 \), the surface \( B_{h,z} \) degenerates towards a nodal surface \( B_{h,-1} = B'_{h,-1} \cup B''_{h,-1} \) where \( B'_{h,-1} = D^2 \setminus \{0, 1\} \) and \( B''_{h,-1} = D^2 \setminus \{\pm i\} \). The two sides of the node are \(-1 \in B'_{h,-1} \) and \( 1 \in B''_{h,-1} \). The fibration also splits as \( E_{h,-1} = E'_{h,-1} \sqcup E''_{h,-1} \), where \( B'_{h,-1} \) is the total space of a fibration with fibre \( S \) and monodromy \( \varphi \) over \( B'_{h,-1} \) and \( E''_{h,-1} = B''_{h,-1} \times S \). On \( \partial E''_{h,-1} \) there is the Lagrangian submanifold \( L' \) obtained by parallel transport of \( a \) over \( \partial B''_{h,-1} \). We denote \( \ell_+ = \partial D^2 \cap \{ \Re > 0 \} \) and \( \ell_- = \partial D^2 \cap \{ \Re < 0 \} \). On \( \partial E''_{h,-1} \) there is the Lagrangian submanifold \( L'' = \ell_+ \times a \sqcup \ell_- \times b \). The count of multisections of \( E''_{h,-1} \) with boundary on \( L' \) gives \( \Phi^i: \widehat{CFK}(\varphi(a), a; 1 - g) \to \widehat{CF}(\varphi) \).

Each multisection counted in \( \widehat{CF}(\varphi) \) intersect the the fibre over \(-1\) into a \( 2g \)-tuple \( z \) of points in \( L \) which are not too close to the boundary of the fibre. On the other hand, it is easy to check on the diagram \((S, a, b)\) that for any \( 2g \)-tuple of points \( z \) on \( a \) which are not too close to the boundary there is a unique degree \( 2g \) multisection of \( E''_{h,-1} \) with boundary in \( L'' \) and asymptotic to \( \Theta^+ \) at \( i \) and \( \Theta^- \) at \(-i \). This proves the last claim.

\[ \square \]

4. PROOF OF THE ISOMORPHISM

To prove Theorem 1.1 we will show that the chain map

\[ \Phi^i: \widehat{CFK}(S, a, \varphi(a); -g + 1) \to \widehat{CF}(\varphi) \]

given by Theorem 3.5 is an isomorphism.

The section is organized as follows. We will first recall some algebraic tools about exact sequences that will be used later. We will then prove the existence of the exact
triangles in fixed point Floer homology (already known by Seidel, cf. [61, Theorem 4.2]) and in knot Floer homology (cf. [49, Theorem 8.2]). We will then prove the commutativity of the diagram (1.3) and, finally, finish the proof of the fact that $\Phi^*\#\ast$ is an isomorphism by proving the initial step of the induction.

4.1. Some algebra. In this subsection we recall some results that will be useful to show the existence of the two exact sequences (1.1) and (1.2) and the commutativity of the diagram (1.3).

**Lemma 4.1.** Let $(A, \partial_A)$, $(B, \partial_B)$ and $(C, \partial_C)$, be chain complexes, $f: A \to B$ and $g: B \to C$ chain maps and $H: A \to C$ a chain homotopy from $g \circ f$ to 0. If

$$H_* \left( A \oplus B \oplus C, \begin{pmatrix} \partial_A & 0 & 0 \\ f & \partial_B & 0 \\ H & g & \partial_C \end{pmatrix} \right) = 0$$

then there exists a linear map $d: H_*(C) \to H_*(A)$ such the triangle

$$\begin{array}{c}
H_*(A) \\
\downarrow f_* \\
H_*(B) \\
\downarrow g_* \\
H_*(C) \\
\downarrow d
\end{array}$$

is exact.

**Proof.** The homology of the cone

$$\text{Cone}(f) := \left( A \oplus B, \begin{pmatrix} \partial \\ f \\ H \end{pmatrix} \right)$$

fits in a long exact sequence

$$H_*(A) \xrightarrow{f_*} H_*(B) \xrightarrow{(i_2)_*} H_*(\text{Cone}(f)) \xrightarrow{(p_1)_*} H_*(C)$$

where $i_2$ and $p_1$ are the inclusion and, respectively, the projection of the corresponding summand. Moreover condition (4.2) implies that

$$(4.4) \quad \text{Cone}(f) \xrightarrow{(H,g)} C$$

induces an isomorphism in homology. Putting everything together, we obtain a commutative diagram

$$\begin{array}{c}
\cdots \\
\downarrow f_* \\
H_*(B) \\
\downarrow g_* \\
H_*(C) \\
\downarrow d \\
H_*(A) \\
\downarrow f_* \\
\cdots
\end{array}$$

and the exactness of (4.3) implies that also the bottom line is exact. □
Lemma 4.2. Let $A, B, C, A', B'$ and $C'$ be chain complexes fitting in the diagram

![Diagram](image)

where:

1. $H$ and $H'$ are chain homotopies from $g \circ f$ and, respectively, $g' \circ f'$ to $0$;
2. $R$ is a chain homotopy from $f \circ a$ to $b \circ f'$;
3. $S$ is a chain homotopy from $g \circ b$ to $c \circ g'$;
4. $T$ is a map such that $(c \circ H' + S \circ f') + (H \circ a + g \circ R) = T \circ \partial + \partial \circ T$;
5. the homologies of the two iterated cones

\[
\begin{pmatrix}
A \oplus B \oplus C, \\
\begin{pmatrix}
f & \partial & 0 \\
H & g & \partial
\end{pmatrix}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
A' \oplus B' \oplus C', \\
\begin{pmatrix}
f' & \partial' & 0 \\
H' & g' & \partial'
\end{pmatrix}
\end{pmatrix}
\]

are trivial.

Then there exist linear maps $d$ and $d'$ such that the following diagram

\[
\begin{array}{ccccccccccc}
\ldots & \overset{d'}{\longrightarrow} & H_*(A') & \overset{f'_*}{\longrightarrow} & H_*(B') & \overset{g'_*}{\longrightarrow} & H_*(C') & \overset{d'}{\longrightarrow} & \ldots \\
\overset{a_*}{\circlearrowleft} & & \overset{b_*}{\circlearrowleft} & & \overset{c_*}{\circlearrowleft} & & & & & \\
\ldots & \overset{d}{\longrightarrow} & H_*(A) & \overset{f_*}{\longrightarrow} & H_*(B) & \overset{g_*}{\longrightarrow} & H_*(C) & \overset{d}{\longrightarrow} & \ldots
\end{array}
\]

commutes and the two rows are long exact sequences.

Proof. Observe first that the commutativity of the two squares in Diagram (4.7) comes from the assumptions (2) and (3). The existence of the linear maps $d$ and $d'$ and the exactness of the rows follows from Lemma 4.1. It remains to show the commutativity of the third square. Consider the quasi-isomorphisms

\[(H, g): \text{Cone}(f) \longrightarrow C \quad \text{and} \quad (H', g'): \text{Cone}(f') \longrightarrow C'
\]

and the chain map

\[
\begin{pmatrix}
a & 0 \\
R & b
\end{pmatrix} : \text{Cone}(f') \longrightarrow \text{Cone}(f).
\]
Naturality of mapping cones implies that the following diagram of long exact sequences commutes. Moreover the diagram commutes because \((T,S): \text{Cone}(f') \to C\) is a chain homotopy between \(c \circ (H',g')\) and \((H,g) \circ \left(\begin{array}{cc} a & 0 \\ R & b \end{array}\right)\). Hence the lemma follows. \(\square\)

**Lemma 4.3.** Suppose we have a diagram as in Lemma 4.2. We assume moreover that \(A = A_+ \oplus A_-\) and decompose \(a = (a_+, a_-), f = f_+ + f_-\) and \(H = H_+ + H_-\). If there is a map \(\alpha_-: A' \to A_\) such that \(a_- = \partial \circ \alpha_- + \alpha_- \circ \partial\), then there exist maps \(\tilde{R}\) and \(\tilde{T}\) such that the diagram satisfies the properties of Diagram (4.6).
Proof. It is evident, by restriction, that $H_+$ is a homotopy between $g \circ f_+$ and 0. We define $\tilde{R} = R + f_- \circ \alpha_-$ and $\tilde{T} = T + H_+ \circ \alpha_-$. Then we compute
\[
\partial \circ \tilde{R} + \tilde{R} \circ \partial' = \partial \circ R + R \circ \partial' + \partial \circ f_- \circ \alpha_- + f_- \circ \alpha_- \circ \partial \\
= \partial \circ R + R \circ \partial' + f_- \circ (\partial \circ \alpha_- + \alpha_- \circ \partial') \\
= f \circ a + b \circ f' + f_- \circ a_- = f_+ \circ a_+ + b \circ f'
\]
and
\[
\partial \circ \tilde{T} - \tilde{T} \circ \partial = \partial \circ T - T \circ \partial + \partial \circ H_- \circ \alpha_- + H_- \circ \alpha_- \circ \partial' \\
= \partial \circ T - T \circ \partial + (\partial \circ H_- + H_- \circ \partial) \circ \alpha_- + H_- \circ (\partial \circ \alpha_- + \alpha_- \circ \partial') \\
= c \circ H' + S \circ f' + g \circ R + H \circ a + g \circ f_- \circ \alpha_- + H_- \circ a_- \\
= c \circ H' + S \circ f' + g \circ \tilde{R} + H_+ \circ a_+.
\]

\[\square\]

4.2. The exact triangle in fixed point Floer cohomology. In [61] Seidel sketches the existence of an exact triangle that encodes the behaviour of fixed point Floer cohomology (in the ± versions) under the composition by a Dehn twist. In this section we prove the exactness of an analogous exact triangle for $HF^\pm$. The proof is inspired by that for Seidel’s exact triangle in Lagrangian Floer cohomology from [62] and works also for the ± versions.

Fix a compact Liouville surface $(S, \lambda)$ with genus $g \geq 1$ and boundary $\partial S \cong S^1$ and an exact symplectomorphism $\varphi : S \to S$. We denote $\omega = d\lambda$. If $L \subset S \setminus \partial S$ is an exact Lagrangian submanifold, i.e. a closed essential embedded curve such that $\lambda|_L = 0 \in H^1_{dR}(L)$, we denote by $\tau_L : S \to S$ the positive Dehn twist along $L$, which is an exact symplectomorphism. If $L'$ is another exact Lagrangian submanifold, we denote by $HF(L, L')$ the Lagrangian Floer cohomology of $(L, L')$.

Suppose (without loss of generality) that $L$ and $L'$ intersect transversely. The Floer chain complex $CF(L, L')$ is the $\mathbb{Z}/2\mathbb{Z}$-vector space freely generated by the intersection point $x \in L \cap L'$. To define the boundary, we consider the symplectic fibration
\[(\mathbb{R} \times [0, 1] \times S, ds \wedge dt + \omega) \xrightarrow{\pi} (\mathbb{R} \times [0, 1], ds \wedge dt)\]
endowed with a compatible almost complex structure $J$ such that $J(\partial_s) = \partial_t$ and $J(TS) = TS$. Given $x_+, x_- \in L \cap L'$, let $\mathcal{M}(x_+, x_-, J)$ be the moduli space of $J$-holomorphic sections $u : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1] \times S$ of $\pi$ such that
\begin{enumerate}
  \item $\lim_{s \to \pm \infty} \pi \circ u(s, t) = x_{\pm}$,
  \item $u(s, 0) \in L$ for all $s \in \mathbb{R}$, and
  \item $u(s, 1) \in L'$ for all $s \in \mathbb{R}$.
\end{enumerate}

As in the fixed point case, to each $J$-holomorphic section $u$ is associated a Fredholm operator $D_u$ of index $\text{ind}(u)$. Call $\mathcal{M}_k(x, y, J)$ the subset of maps in $\mathcal{M}(x, y, J)$ with $\text{ind}(u) = k$. Define then a differential on $CF(L, L')$ by
\[
\partial x_+ = \sum_{y \in L \cap L'} \#_2(\overline{\mathcal{M}}_1(x_+, x_-, J)) x_-
\]
where again $\mathcal{M}_1(x_+, x_-, J)$ denotes the quotient of $\mathcal{M}_1(x_+, x_-, J)$ by the $\mathbb{R}$-action given by translations in the $\mathbb{R}$-direction. The resulting homology

$$HF(L, L') := H_*(CF(L, L'), \partial)$$

is the Lagrangian Floer cohomology of $L$ and $L'$. The definition of Lagrangian Floer cohomology using sections is (tautologically) equivalent to the usual definition using time-dependent almost complex structures.

**Theorem 4.4.** There is an exact triangle

$$HF(\varphi(L), L) \xrightarrow{\iota_*} HF^2(\varphi) \xrightarrow{\lambda_*} HF^2(\varphi \circ \tau_{L}^{-1}).$$

In comparing this result with Theorem 4.2 of [61], the reader should keep in mind the different conventions.

Before giving the proof of the Theorem we describe the maps $\iota$ and $\lambda$, which are defined in terms of Gromov invariants associated to exact Lefschetz fibrations over surfaces. We refer the reader to Seidel’s book [58] for the general theory of these invariants and to [61] and [62] for further details about the construction of $\iota$ and $\lambda$ and other closely related maps.

Let $B$ be an oriented surface (possibly with boundary) with a set $\Gamma$ of marked points and consider a Lefschetz fibration $E \xrightarrow{\pi} B \setminus \Gamma$ with smooth fiber a surface $S$ and without critical points above $\partial B$. Fix a symplectic form $\Omega$ on $E$ of the form $\Omega = \theta + \pi^*\kappa$ with $\kappa$ a symplectic form on $B$ and $\theta$ an exact two-form on $E$ which is positive on each fiber. Assume that $\Omega$ is exact and let $\Upsilon$ be a primitive of $\Omega$ which restricts to a Liouville form on every fiber.

If $B$ is without boundary, one can endow $B$ and $E$ with suitable almost complex structures and define the Gromov invariant $G(E, \pi)$ by counting (Fredholm) index–0 pseudo-holomorphic sections of $\pi$. If $B$ has non-empty boundary we fix an exact Lagrangian boundary condition on $\partial E$: this consists in a Lagrangian submanifold $Q \subset \pi^{-1}(\partial B)$ such that $\pi|_Q: Q \to \partial B$ is a locally trivial fibration and $[\Upsilon|_Q] = 0 \in H^1_{dR}(Q)$. Observe that there is a canonical symplectic connection on $E$ given by the $\Omega$-orthogonal of the tangent space of the fibers, and an exact Lagrangian submanifold $L$ on a fiber above $\partial B \setminus \Gamma$ induces, by parallel transport, an exact Lagrangian boundary condition on the corresponding connected component of $\partial B \setminus \Gamma$.

To this data it is possible to associate a relative Gromov invariant $G_{rel}(E, \pi, Q)$, which counts index–0 pseudo-holomorphic sections of $\pi$ with boundary in $Q$.

**Remark 4.5.** In the literature (and in particular in Seidel’s work) the Gromov invariants $G(E, \pi)$ and $G_{rel}(E, \pi, Q)$ are usually denoted by $\Phi(E, \pi)$ and $\Phi_{rel}(E, \pi, Q)$ respectively. We prefer to change the notation here to avoid possible confusions with the map $\Phi$ from Heegaard Floer homology to periodic homology.

(1) Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. To define $\iota: CF(\varphi(L), L) \to CF^2(\varphi)$, consider the surface $B_i = \mathbb{D} \setminus \{0, 1\}$ and the symplectic fibration $\pi_i: E^*_i \to B_i$ with fiber $S$ and monodromy $\varphi$. On $B_i$ we fix a positive strip-like and at 1 and a negative cylindrical
end at 0, i.e. holomorphic identifications of a punctured neighbourhood of 1 in $B_{\varepsilon}$ with $(0, +\infty) \times [0, 1]$ and of a punctured neighbourhood of 0 with $(-\infty, 0) \times S^1$. We fix also a trivialisation of $\pi_1$ over the strip-like end, which means an identification of the preimage of the strip-like end with $(0, +\infty)$ equipped with a product symplectic form. We equip the boundary of $E^\varphi_\varepsilon$ with an exact Lagrangian boundary condition $Q_L$ which restricts to $(L \times (0, +\infty)\times \{1\}) \cup (\varphi(L) \times (0, +\infty) \times \{0\})$ on the strip-like end. Such exact Lagrangian boundary condition is obtained as the trace of the parallel transport of $L \times \{c\} \times \{1\}$ for some $c > 0$.

Let $\mathcal{M}(x, \gamma)$ be the moduli space of $J$-holomorphic sections $u: B_{\varepsilon} \to E^\varphi_{\varepsilon}$ with boundary on $Q_L$, positively asymptotic to $x \in \varphi(L) \cap L$ and negatively asymptotic to a periodic orbit $\gamma$ corresponding to a fixed point of $\varphi$. We denote by $M_k(x, \gamma)$ the subset of $\mathcal{M}(x, \gamma)$ consisting of sections of Fredholm index $k$. For a generic choice of almost complex structure, $M_0(x, \gamma)$ is a transversely cut out compact manifold of dimension zero, and we define

$$\iota(x) = \sum_{\gamma \neq e} \# M_0(x, \gamma) \gamma.$$
Remark 4.6. Observe that $B_\lambda \cong S^2 \setminus \{N, S\}$, where punctured neighborhoods of $N$ and $S$ (the north and the south pole respectively) are identified with the cylindrical–like ends 0 and $\infty$ respectively. In the rest of the paper we will often and implicitly consider $E^\varphi_\lambda$ as a fibration over $S^2 \setminus \{N, S\}$.

(3) We want describe also another Gromov invariant that will be useful later. Consider the Lefschetz fibration $\pi_L : E_L \to B = \mathbb{D} \setminus \{-1\}$ with a unique singular fiber over 0 with vanishing cycle $L$. If $L'$ is another exact Lagrangian submanifold $S$, we can define an exact Lagrangian boundary condition $Q_{L'}$ in $\partial E_L$ by parallel transport of $L'$ such that the count of index zero $J$-holomorphic sections $u : B \to E_L$ defines an element $\overline{G}_L(L') \in CF(L', \tau_L(L'))$.

Lemma 4.7. The maps $\iota$ and $\lambda$ are chain maps and $\overline{G}_L(L')$ defines an element in $HF(L', \tau_L(L'))$.

Proof. The lemma follows from standard degeneration arguments for one-dimensional moduli spaces and an argument similar to the proof of Theorem 3.5. \qed

Example 4.8. If $L' = L$ then

$$0 = \overline{G}_L(L) \in HF(L, \tau_L(L)) \cong HF(L, L).$$

See [60, Example 3.1].

Lemma 4.9. There is a chain homotopy

$$\mathcal{H} : CF(\varphi(L), L) \to HF^2(\varphi \circ \tau_L^{-1})$$

from $\lambda \circ \iota$ to the 0-map.

Proof. For every $t \in (0, 1)$, fix the marked point $z_t = (\sqrt{2t}, \sqrt{2t}) \in B_t$ and consider the family of Lefschetz fibrations $(E^\varphi_t, \pi^H_t)$ over $B_t$ with monodromy $\tilde{\varphi} \circ \tau_L^{-1}$ around 0 and a unique critical point over $z_t$ with vanishing cycle $L$. On the preimages of $\partial B_t$ we consider the Lagrangian submanifolds $Q^H_t$ obtained by parallel transport of $L$. The homotopy $\mathcal{H}$ is defined by counting pairs $(t_0, u)$, where $t_0 \in (0, 1)$ and $u$ is an index $-1$ $J$-holomorphic section of the Lefschetz fibration $(E^\varphi_{t_0}, \pi^H_{t_0})$ with boundary on $Q^H_{t_0}$ for a generic almost complex structure $J$. 
The family of Riemann surfaces with marked points \((B, \mathcal{Z})\) can be compactified by adding two-level buildings
\[
B^H = B_i \sqcup (S^2 \setminus \{\{N\}, \{S\}\})
\]
with a marked point \(\mathcal{Z} \in S^2 \setminus \{\{N\}, \{S\}\}\) and
\[
B^H_+ = \left( B_i \setminus \{\left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \} \right) \sqcup \left( \mathbb{D} \setminus \{\left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \} \right)
\]
with the marked point \(\mathcal{Z} = 0 \in \mathbb{D} \setminus \{\left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \}\). To the degenerations of \((B, \mathcal{Z})\) correspond, by pull back, degenerations of \(E^H\).

![Figure 8](image) The homotopy from \(B^H_-(\text{left})\) to \(B^H_+\) \(\text{right}\). \((B^H_+, \mathcal{Z}_0)\) is pictured in the middle.

Thus \((E^H_-, \pi^H_-) = (E^\varphi_0 \sqcup E^\varphi_1, \pi_0 \sqcup \pi_1)\), and therefore counting \(J\)-holomorphic sections of \((E^H_-, \pi^H_-)\) gives \(\lambda \circ \iota\). On the other hand, \((E^H_+, \pi^H_+) \cong (E_L \sqcup E_L^{\varphi \tau^{-1}_L}, \pi_L \sqcup \pi)\). Since the Lagrangian boundary condition on the component \(E_L\) is induced by \(L\), Example 4.8 implies the relative Gromov invariant associated to \((E^H_+, \pi^H_+)\) is trivial, and therefore the total count of \(J\)-holomorphic sections of \((E^H_+, \pi^H_+)\) is zero.

We assume, without loss of generality, that \(L\) is disjoint from all fixed point of \(\varphi \circ \tau^{-1}_L\) and intersects \(\varphi(L)\) transversely. We fix an orientation on \(L\) and orient \(\varphi(L)\) accordingly. We say that an intersection point \(x \in L \cap \varphi(L)\) is positive if the orientation of \(T_xL\), followed by the orientation of \(T_x\varphi(L)\), gives the orientation of \(T_xS\). We say that a fixed point of \(\varphi\) is positive hyperbolic if the eigenvalues of \(d_x\varphi\) are real and positive, and negative hyperbolic if they are real and negative.

**Lemma 4.10.** We can choose a Hamiltonian isotopy representative of \(\varphi\) so that for every intersection point \(x \in L \cap \varphi(\tau^{-1}_L(x))\) there is a neighbourhood \(U_x\) of \(x\) in \(S\) such that:

1. no fixed point of \(\varphi \circ \tau^{-1}_L\) is contained in \(U_x\),
2. exactly one fixed point of \(\varphi\) is contained in \(U_x\), and it is hyperbolic with the same sign as \(x\), and
3. if \(x\) and \(x'\) are distinct intersection points, then \(U_x\) and \(U_{x'}\) are disjoint.
Then the graphs of \( f \) preimage of \((y,t)\) for all intersection points between \( L \) and \( L' \) such that \( N' = \psi(N) \). We choose \( N \) small enough that no fixed point of \( \psi \) is contained in \( N \) or \( N' \).

We fix open connected neighbourhoods \( V \) of \( y \) and \( V' \) of \( x \) in \( L \) and choose symplectic identifications \( N \cong L \times (-\varepsilon,\varepsilon) \) and \( N' \cong L' \times (-\varepsilon,\varepsilon) \) such that \( \psi(V \times (-\varepsilon,\varepsilon)) = V' \times (-\varepsilon,\varepsilon) \). We parametrise \( L \) as \([0,2\pi]/0 \sim 2\pi\) so that \( 0 < x < y < 2\pi \), \( V = (y-\varepsilon,y+\varepsilon) \), and \( V' = (x-\varepsilon,x+\varepsilon) \).

We have two possibilities for \( \psi|_{V \times (-\varepsilon,\varepsilon)} \). If \( \theta \) is the coordinate on \( L \) and \( t \) is the coordinate on \((-\varepsilon,\varepsilon)\), they are:

1. \( \psi(\theta,t) = (x-t,\theta-y) \), and
2. \( \psi(\theta,t) = (x+t,y-\theta) \).

The first case corresponds to a positive intersection point between \( L \) and \( \psi(L) \) and the second case corresponds to a negative intersection point.

We represent the positive Dehn twist around \( L \) by the map

\[
\tau_L(\theta,t) = (\theta + f(t),t),
\]

where of course \( \theta + f(t) \) is to be understood modulo \( 2\pi \). If we want the proof to work for all intersection points between \( L \) and \( L' \) at the same time, we need that the parametrisations of \( L \) associated to different intersection points differ only by a translation, so that the Dehn twists is represented by Equation (4.10) in the neighbourhood of every intersection point.

First we consider case (1). We are looking for \((\theta_0, t_0) \in (x-\varepsilon,x+\varepsilon) \times (-\varepsilon,\varepsilon)\) such that \( \tau_L(\theta_0,t_0) \in (y-\varepsilon,y+\varepsilon) \times (-\varepsilon,\varepsilon) \) and \( \psi(\tau_L(\theta_0,t_0)) = (\theta_0,t_0) \). We have \( \psi(\tau_L(\theta_0,t_0)) = (x-t_0,\theta_0 + f(t_0) - y) \) and therefore we have to solve the system

\[
\begin{align*}
\theta_0 &= x - t_0, \\
t_0 &= \theta_0 + f(t_0) - y \\
|\theta_0 - y + f(t_0)| &< \varepsilon
\end{align*}
\]

The last inequality is automatic from the second equation, provided that the solution satisfies \( t_0 \in (-\varepsilon,\varepsilon) \).

From the two equations we obtain

\[
f(t_0) = y - x + 2t_0.
\]

We define the function \( g(t) = y - x + 2t \) and observe that, if we choose \( \varepsilon \) small enough,

- \( f(-\varepsilon) = 0 \) and \( g(-\varepsilon) = y - x - 2\varepsilon > 0 \),
- \( f(\varepsilon) = 2\pi \) and \( g(\varepsilon) = y - x + 2\varepsilon < 2\pi \).

Then the graphs of \( f \) and \( g \) must cross, and since we can arrange \( f \) to be linear on the preimage of \((y-x-2\varepsilon,y-x+2\varepsilon)\), we can assume that they cross at a single point \( t_0 \). We set \( \theta_0 = x - t_0 \), and therefore \((\theta_0,t_0)\) is the unique fixed point of \( \psi \circ \tau_L \) in \( U' \times (-\varepsilon,\varepsilon) \). The linearisation of \( \psi \circ \tau_L \) at \((\theta_0,t_0)\) is \( \begin{pmatrix} 0 & -1 \\ 1 & f'(t_0) \end{pmatrix} \). The determinant is 1 and the trace is \( f'(t_0) > 2 \), and therefore the eigenvalues are positive real. Then \((\theta_0,t_0)\) is a positive hyperbolic fixed point.
Now we consider case (2). Since \( \psi(\tau_L(\theta, t)) = (x + t, y - \theta - f(t)) \), we need to solve the system
\[
\begin{aligned}
\theta &= x + t \\
t &= y - \theta - f(t) \\
|x - \theta - f(t)| &< \varepsilon.
\end{aligned}
\]
As in case (1), solving this system is equivalent to solving the equation \( f(t) = y - x - 2t \).
We define \( g(t) = y - x - 2t \) and observe that
\[
\begin{align*}
\bullet \quad f(-\varepsilon) &= 0 \quad \text{and} \quad g(-\varepsilon) > 0, \\
\bullet \quad f(\varepsilon) &= 2\pi \quad \text{and} \quad g(\varepsilon) < 2\pi.
\end{align*}
\]
This gives a fixed point \( (\theta_0, t_0) \) as before. The linearisation of \( \psi \circ \tau_L \) at \( (\theta_0, t_0) \) is
\[
\begin{pmatrix}
0 & 1 \\
-1 & -f(t_0)
\end{pmatrix}.
\]
This is the negative of the matrix from the previous case, and therefore its eigenvalues are negative real. Then \( (\theta_0, t_0) \) is a negative hyperbolic fixed point. \( \Box \)

The next step of the proof of theorem 4.4 involves a quite standard argument in Floer homologies (see for example [62] and [48, Section 9]). Consider the \textit{iterated cone} \( \mathcal{C}, \partial\mathcal{C} \) where \( \mathcal{C} := (\text{CF}(\varphi(L), L) \oplus \text{CF}^2(\varphi) \oplus \text{CF}^2(\varphi \circ \tau_L^{-1})) \) and
\[
\partial\mathcal{C} := \begin{pmatrix}
\partial_{\mathcal{C}}^{\text{CF}(\varphi(L), L)} & 0 & 0 \\
0 & \partial_{\mathcal{C}}^{\text{CF}^1(\varphi)} & 0 \\
\mathcal{H} & \lambda & \partial_{\mathcal{C}}^{\text{CF}^1(\varphi \circ \tau_L^{-1})}
\end{pmatrix}.
\]
If we show that \( H_*(\mathcal{C}, \partial\mathcal{C}) = \{0\} \) then Theorem 4.4 comes then from Lemma 4.1.

In the symplectic fibrations \( W_* \xrightarrow{\pi} B_* \) we fix a fibrewise symplectic form \( \Omega_* \), which induces a splitting \( TW_* = T^vW_* \oplus T^hW_* \). The fibrations \( W_* \xrightarrow{\pi_*} B_* \) is \textit{non-negatively curved} if \( \Omega_*|_{T^hW_*} \geq 0 \) with respect to the orientation of \( T^hW_* \) induced by the orientation of \( B_* \). The \textit{energy} of a section \( u \): \( B_* \rightarrow W_* \) is \( E(u) = \int_{B_*} u^*\Omega_* \). A section \( u \): \( B_* \rightarrow W_* \) is called \textit{horizontal} if \( du(TB_*) = T^hW_* \). An almost complex structure \( J \) is \textit{horizontal} if it preserves both distributions. Horizontal sections are always \( J \)-holomorphic for a horizontal almost complex structure \( J \). If \( W_* \xrightarrow{\pi_*} B_* \) is non-negatively curved and \( u \) is a \( J \)-holomorphic section, then \( E(u) \geq 0 \), and if \( E(u) = 0 \) implies that \( u \) is horizontal. The symplectic fibrations \( E^\varphi_* \rightarrow B_* \) and \( E^\varphi_* \rightarrow B_* \) are non-negatively curved, and a section \( u \) is horizontal if and only if \( E(u) = 0 \). This is immediate for \( E^\varphi_* \rightarrow B_* \), and follows from [62, Subsection 3.3] for \( E^\varphi_* \rightarrow B_* \). By [62, Lemma 2.27] (see also [57, Lemma 3.2]), a horizontal section of Fredholm index zero is regular (i.e. the linearised Cauchy-Riemann operator is surjective) if and only if the corresponding fixed point of \( \varphi \) is nondegenerate.

Given \( \delta \geq 0 \), we denote by
\[
\partial^\varphi_{\delta} := \begin{pmatrix}
\partial^{\text{CF}(\varphi(L), L)}_{\delta} & 0 & 0 \\
\iota_{\delta} & \partial^{\text{CF}^1(\varphi)}_{\delta} & 0 \\
\mathcal{H}_{\delta} & \lambda_{\delta} & \partial^{\text{CF}^1(\varphi \circ \tau_L^{-1})}_{\delta}
\end{pmatrix}.
\]
the component of \( \partial^\varphi_{\delta} \) that counts only those holomorphic sections \( u \) counted by \( \partial^\varphi \) that have energy \( E(u) < \delta \).
If we show that there exists \( \delta \) such that \( \partial^C - \partial^C_\delta \) counts only holomorphic sections \( u \) with energy \( E(u) \geq 2\delta \) and \( H_x(C, \partial^C_\delta) = \{0\} \), then by Lemma 2.31 of [62] it follows that \( H_x(C, \partial^C) = \{0\} \). To show this, we analyse the low-energy contributions to the components of \( \partial^C \).

If \( \delta \) is sufficiently small, \( \partial^C\{\psi(L)\} = \partial^C\{\psi\} = \partial^C\{\psi \circ \tau_L^{-1}\} \) and \( \mathcal{H}_\delta = 0 \) because no horizontal section contributes to that maps. For \( \delta \) small enough, \( \lambda_\delta \) counts horizontal sections of \( W_{\psi_\lambda} \rightarrow B_\lambda \), which correspond to the fixed points of \( \psi \circ \tau_L^{-1} \), and therefore \( \lambda_\delta = \lambda_0 \).

The map \( \iota_\delta \) is more involved, because it doesn’t necessarily count horizontal sections. In fact horizontal sections of \( W_{\psi_\delta} \rightarrow B_\delta \) with boundary on \( Q_L \) correspond to fixed points of \( \psi \) which are also intersection points between \( L \) and \( \varphi(L) \). If \( x \in L \cap \varphi(L) \), we denote by \( \gamma_x \) the orbit corresponding to the fixed point \( p_x \) of \( \varphi \) close to \( x \) which was constructed in Lemma 4.10.

**Lemma 4.11.** If \( \delta \) and \( N \) are small enough and \( u: B_\delta \rightarrow E^\mathbb{R}_\delta \) is a \( J \)-holomorphic section with boundary on \( Q_L \) and energy \( E(u) < \delta \), then \( u \in \mathcal{M}(x, \gamma_x) \) for some \( x \in L \cap \varphi(L) \).

**Proof.** The proof is based on the two following two facts:

1. up to Hamiltonian isotopy, all orbits of \( \varphi \) and all intersection points in \( L \cap \varphi(L) \) have distinct action, and
2. for every intersection point \( x \in L \cap \varphi(L) \), the action of \( \gamma_x \) can be made arbitrarily close to the action of \( x \) by a suitable choice of the parameters in the definition of the Dehn twist.

(i) If \( p \) is a fixed point of \( p \) and \( H: S \rightarrow \mathbb{R} \) is a function with \( d^p_H = 0 \) and Hamiltonian flow \( \varphi^H_t \), then \( p \) is a fixed point of \( \varphi^H_t \circ \varphi \). If \( \alpha^x_{\varphi^H_t \circ \varphi}(x) \) and \( \alpha_{\varphi^H_t \circ \varphi}(p) \) denote the action of \( p \) as a fixed point of \( \varphi \) and of \( \varphi^H_t \circ \varphi \) respectively, then \( \alpha_{\varphi^H_t \circ \varphi}(p) = \alpha^x_{\varphi^H_t \circ \varphi}(x) - tH(p) \).

(ii) By exactness, it is enough to compute the energy of one (smooth) section between \( x \) and \( \gamma_x \). If we trivialise the fibration \( W^\mathbb{R} \rightarrow B_\delta = D^2 \setminus \{0,1\} \) over \( D^2 \setminus [0,1] \) and project to the fibre \( S \), we obtain a correspondence between smooth sections from \( x \) to \( \gamma_x \) and boundary on \( Q_L \) with maps from a triangle to \( S \) with vertices on \( x, \phi^{-1}(x) \) and \( p_x \) (in counterclockwise order) such that the edge between \( x \) and \( \phi^{-1}(x) \) is in \( L \), and the edge between \( x \) and \( p_x \) is the image under \( \varphi \) of the edge between \( \phi^{-1}(x) \) and \( p_x \). Moreover, the energy of a section is equal to the area covered by the triangle.

We can find a triangle as above which is embedded and contained in \( N \) if the edge \( x \) and \( p_x \) is contained in \( \mathcal{N} \cap \varphi(\mathcal{N}) \). Then the energy of any section \( u \in \mathcal{M}(x, \gamma_x) \) is not larger than the area of \( \mathcal{N} \), which can be made arbitrarily small by reducing \( \varepsilon \). \( \square \)

**Lemma 4.12.** If \( x \in L \cap \varphi(L) \) is also a hyperbolic fixed point of \( \varphi \) and the signs as intersection point and as fixed point are the same, then the horizontal section \( u_x \) over \( x \) has Fredholm index \( \text{ind}(u_x) = 0 \).

**Proof.** By Proposition 11.13 in [58], Theorem 9 in [17] and the additivity of the index, we have \( \text{ind}(u_x) = \mu(x) - \mu(\gamma_x) \), where \( \mu(x) \) is the Conley-Zehnder index of the intersection point \( x \), \( \mu(\gamma_x) \) is the Conley-Zehnder index of the orbit \( \gamma_x \) corresponding to \( x \), and they are computed with respect to trivialisations which extend to a trivialisation of \( u^*xT^uW^\mathbb{R}_\psi \).

Let \( \xi^- \) and \( \xi^+ \) the stable and unstable directions of \( \varphi \) at \( x \), respectively. They give sub-bundles \( \Xi^\pm \) of \( u^*xT^uW^\mathbb{R}_\psi \). The Conley-Zehnder index \( \mu(\gamma_x) \) is equal to the Maslov
index of $\Xi_{\pm}$. The sub-bundle $T^nQ_L = TQ_L \cap T^nW^e_x \subset T^nW^c_x|_{\partial B_l}$ is always transverse to $\Xi_{\pm}$. On the strip-like end we close it by the shortest clockwise path from $\varphi(L)$ to $L$. This path does not intersect $\xi_{\pm}$ because the sign of $x$ as intersection point is the same as fixed point. The Conley Zehnder index $\mu(x)$ is equal to the Maslov index of the closure of $T^nQ_L$, which is homotopic to $\Xi_{\pm}$. Then $\mu(x) = \mu(\gamma_x)$. □

**Proposition 4.13.** If $\delta$ is small enough, $\iota_{\delta}(x) = \gamma_x$.

**Proof.** By Lemma 4.11, $\iota_{\delta}(x) = \#M_0(x, \gamma_x \gamma_x)$. For the moment, let us assume that $x$ is also a fixed point of $\varphi$. Then $M(x, \gamma_x)$ contains only the horizontal section $u_x$ because $E(u_x) = 0$ and by exactness all holomorphic sections in $M(x, \gamma_x)$ have the same energy. By Lemma 4.12 $\text{ind}(u_x) = 0$ and moreover $u_x$ is regular (i.e. its linearised Cauchy-Riemann operator is surjective) by Lemma 2.27 of [62]. Then $\#M_0(x, \gamma_x) = 1$ when $x$ is a fixed point of $\phi$, which happens when $L$ is parametrised so that $x$ and $\phi(\tau^{-1}_L(x))$ are antipodal. There are obstructions to obtain this for every intersection point in $L \cap \varphi(L)$, and therefore we use a deformation argument as in the proof of Proposition 3.4 in [62]. For every point $x \in L \cap \varphi(L)$ we define a family of symplectomorphisms $\varphi_t$ by changing the parametrisation of $L$ such that $\varphi_1 = \varphi$ and $\varphi_0(x) = x$. To this family of symplectomorphisms we associate a family of symplectic fibrations $W^*_L \to B_L$. Let $\gamma_x^{(i)}$ be the periodic orbit corresponding the fixed point of $\varphi_i$ close to $x$ constructed in Lemma 4.10; in particular $\gamma_x^{(1)} = \gamma_x$. We define the parametric moduli space $M_{\text{par}}(x, \gamma_x)$ consisting of pairs $(u, t)$ where $t \in [0, 1]$ and $u: B \to W^*_L$ asymptotic to $x$ and $\gamma_x^{(i)}$. For a generic choice of almost complex structures $M_{\text{par}}(x, \gamma_x)$ is a 1-dimensional manifold. Moreover a sequence $(u_n, t_n) \in M_{\text{par}}(x, \gamma_x)$ with $t_n \to t_\infty \in [0, 1]$ cannot break into a multi-level building by action reasons. Then $\partial M_{\text{par}}(x, \gamma_x) = M(x, \gamma_x^{(1)}) - M(x, \gamma_x^{(0)})$ and therefore $\#M(x, \gamma_x) = 1$. □

We have obtained that $\partial_C^C = \begin{pmatrix} 0 & 0 & 0 \\ t_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \end{pmatrix}$, and therefore $H_*(C, \partial_C^C) = 0$. From this it follows that $H_*(C, \partial_C)$, and therefore we have proved Theorem 4.4.

4.3. **The exact triangle in Heegaard Floer homology.** The aim of this subsection is to prove the exact sequence 1.2. Although we have no claim of originality on that exact sequence, we need to recast its proof in the language of symplectic fibrations.

**Proposition 4.14.** Given a genus $g$ open book decomposition $(K, S, \varphi)$ of a 3–manifold $Y$ and a closed exact Lagrangian $L \subset S$, there exists an exact triangle

\begin{equation}
\label{exact-triangle}
\text{HFK}(Y_0, K_0; -g + 1) \xrightarrow{i_*} \text{HFK}(Y, K; -g + 1) \xrightarrow{l_*} \text{HFK}(Y^+, K^+; -g + 1)
\end{equation}

4.3.1. **Heegaard diagrams.** If $(K, S, \varphi)$ is an open book decomposition for $Y$, then $(K^+, S, \varphi \circ \tau^{-1}_L)$ is an open book decomposition for $Y^+$. 


Definition 4.15. If \( L \subset S \) is a nonseparating curve, a basis of arcs \( \mathbf{a} = \{a_1, \ldots, a_{2g}\} \) for \( S \) is compatible with \( L \) if:

1. \( a_1 \) intersects transversely \( L \) in a single point;
2. \( a_i \cap L = \emptyset \) for \( i > 1 \);
3. \( a_2 \) is homotopic to \( \gamma + L \) in \( S \), where \( \gamma \) is a connected component of \( a_1 \setminus L \).

It is easy to check that any nonseparating curve \( L \subset S \) admits a compatible basis of arcs for \( S \). Fix \( L \) and a compatible basis of arcs \( \mathbf{a} \).

Let \( \tau_L \) be the composition of a Dehn twist along \( L \) with support in a thin neighborhood \( \mathcal{N}(L) \) of \( L \) and of a small Hamiltonian diffeomorphism \( h \) of \( S \). Consider the set of arcs \( \mathbf{a}' := \{a'_1, \ldots, a'_{2g}\} \) where \( a'_i = \tau^{-1}_L(a_i) \) for every \( i \). Obviously \( \mathbf{a}' \) is a basis for \( S \) and, assuming that \( \mathcal{N}(L) \) is thin enough, \( a'_i \) is Hamiltonian isotopic to \( a_i \) for \( i > 1 \). We choose \( h \) so that \( \tau^{-1}_L|_{\partial S} \) is a small positive rotation and \( a'_i \cap a_i \) consists of a single point for \( i > 1 \) and in two points, contained in \( \mathcal{N}(L) \), for \( i = 1 \) (see Figure 10).

We define also a set of arcs \( \mathbf{\tilde{a}} := \{\tilde{a}_1, \ldots, \tilde{a}_{2g}\} \) where \( \tilde{a}_1 := L \) and, for \( i = 2, \ldots, 2g \), \( \tilde{a}_i \) is the result of a perturbation of \( a'_i \) under a small Hamiltonian isotopy that rotates \( \partial S \) by a small positive angle and such that \( \tilde{a}_i \cap a'_i \) consists of a single point \( \Theta_{\mathbf{a}'_i, \mathbf{\tilde{a}}_i} \). We also define \( \Theta_{\mathbf{a}'_1, \mathbf{\tilde{a}}_1} \) to be the only intersection point of \( a'_1 \) with \( \tilde{a}_1 \).

With slight abuse of notation we will call \( \varphi(a') \) the set of curves image of \( \mathbf{a}' \) under \( \varphi \circ h^{-1} \), so that \( a_i \cap \varphi(a'_i) \cap \partial S \) consists of two points for each \( i \in \{1, \ldots, 2g\} \).

We have then three diagrams \((S, a', \varphi(a'), z)\), \((S, a, \varphi(a'), z)\) and \((S, \tilde{a}, \varphi(a'), z)\). The first diagram is diffeomorphic to \((S, a, \varphi(a), z)\), and therefore represents \((Y, K)\). The second diagram is diffeomorphic to \((S, a, (\varphi \circ \tau^{-1}_L)(a), z)\), and therefore represents \((\overline{Y}, \overline{K})\). The third diagram represents \((\overline{Y}_0, \overline{K}_0)\).

Remark 4.16. If \( A \) is the neighborhood of \( \partial S \) defined in Subsection 2.3, we can assume that \( L \cup \varphi(L) \subset S \setminus A \). It follows that the only component of any generator \( x \) of \( \mathcal{CFK}(S, \tilde{a}, \varphi(a'), z, -g + 1) \) that lies in \( S \setminus A \) has to belong to \( \tilde{a}_1 = L \). Similarly, if \( x \) is a generator of \( \mathcal{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1) \), the only component of \( x \) lying in \( S \setminus A \) has to belong to \( \tilde{a}_1 \cap \varphi(\tilde{a}_1) \).

4.3.2. The chain map \( i \). Let \((E_i, \pi_i)\) be the trivial symplectic fibration over \( B_i := \mathbb{D} \setminus \{1, -1, i\} \) with fiber \( S \). Identify \( \partial \mathbb{D} \) with \([0, 2\pi]/(0 \sim 2\pi)\) in the standard way. Endow \((E_i, \pi_i)\)

Figure 9. An example of basis of arcs compatible with \( L \).
Figure 10. An abstract representation of the curves $a_1$ and $a'_1$ near $\tilde{a}_1 = L$. In the rest of the section we will assume that the intersections among the curves are as in the picture.

Figure 11. The fibration $(E_i, \pi_i)$ with its Lagrangian boundary condition.

with the Lagrangian boundary condition $Q_i$ given by the symplectic parallel transport along $(0, \frac{\pi}{2}), (\frac{\pi}{2}, \pi), (\pi, 2\pi)$ of copies of $\bar{a}, a'$ and $\varphi(a')$ respectively (cf. Figure 11). The chain map

$$i: \widehat{CFK}(S, \bar{a}, \varphi(a'), z, -g + 1) \to \widehat{CFK}(S, a', \varphi(a'), z, -g + 1)$$

is defined on the generators $x$ by

$$i(x) = \sum_y \#_2 M_0^i(x, y) y$$

where the sum is taken over all the generators $y$ of $\widehat{CF}(S, a', \varphi(a'))$, and $M_0^i(x, y)$ is the moduli space of index 0, embedded, degree $2g$ holomorphic multisections $u$ of $(E_i, \pi_i)$ with Lagrangian boundary condition $Q_i$, asymptotic to $x$ at 1, to $\Theta_{a'}, \bar{a} = \{\Theta_{a'_1, \bar{a}'_1}, \ldots, \Theta_{a'_{2g}, \bar{a}'_{2g}}\}$ at $i$ and to $y$ at $-1$, and such that $\text{Im}(u) \cap (B_i \times \{z\}) = \emptyset$.

Observe that the triviality of the intersection with $B_i \times \{z\}$ implies that $\mathcal{A}(y) = -g + 1$, and therefore $y$ is a generator of $\widehat{CFK}(S, a', \varphi(a'), z, -g + 1)$, if $M_0^i(x, y) \neq \emptyset$. 
4.3.3. The chain map \( l \). Let \((E_l, \pi_l)\) be a Lefschetz fibration over 
\[ B_l := \mathbb{D} \setminus \{1, -1\} \]
with fiber \( S \) and only one critical point \( x_0 \) over \( z = 0 \) with vanishing cycle \( L \). We

![Figure 12](image_url)

**Figure 12.** The fibration \((E_l, \pi_l)\) with its Lagrangian boundary condition. Crossing the dotted ray applies \( \tau_L \) to the fibers.

trivialize \( \pi_l \) over the ray \( \{\text{Re}(\zeta) = 0, \text{Im}(\zeta) > 0\} \) so that we can assume that the monodromy \( \tau_L \) around \( z \) acts on the fibers when crossing that ray from the first standard quadrant of \( \mathbb{C} \) to the second. As at the beginning of this section, \( \tau_L \) denotes the composition of the positive Dehn twist with support in a thin neighborhood of \( L \) with a small Hamiltonian perturbation that maps \( a'_i \) to \( a_i \) for every \( i > 1 \).

Endow \((E_l, \pi_l)\) with the Lagrangian boundary condition \( Q_l \) given by parallel transport of a copy of \( \varphi(a') \) along \( (\pi, 2\pi) \) and of a copy of \( a' \) along \( (0, \pi) \). By our choice for the trivialization, \( Q_l \) can be identified with a copy of \( a' \) inside each fiber over \( (0, \frac{\pi}{2}) \) and a copy of \( a \) inside each fiber over \( (\frac{\pi}{2}, \pi) \).

The chain map
\[
l : \widehat{CFK}(S, a', \varphi(a'), z, -g + 1) \to \widehat{CFK}(S, a, \varphi(a'), z, -g + 1)
\]
is defined on the generators by
\[
l(x) = \sum_y \#_2M_0^l(x, y)y,
\]
where the sum is taken over all the generators \( y \) of \( \widehat{CF}(S, a, \varphi(a')) \) and \( M_0^l(x, y) \) is the moduli space of embedded index 0 degree 2g holomorphic multisections \( u \) of \((E_l, \pi_l)\) with Lagrangian boundary condition \( Q_l \), asymptotic to \( x \) at 1 and to \( y \) at \(-1\), and such that \( \text{Im}(u) \cap (B_l \times \{z\}) = \emptyset \). The triviality of the intersection with \( B_l \times \{z\} \) implies that \( A(y) = -g + 1 \), and therefore \( y \) is a generator of \( \widehat{CFK}(S, a, \varphi(a'), z, -g + 1) \), if \( M_0^l(x, y) \neq \emptyset \).

4.3.4. Sketch of the proof of the exact triangle. The exactness of the triangle in (4.11) can be proved using the same argument as in the proof of the exact surgery triangle in Heegaard Floer homology. Here we recast the proof in the language of symplectic fibrations.

**Lemma 4.17.** There is a homotopy
\[
\mathcal{H}' : \widehat{CFK}(S, \bar{a}, \varphi(a'), z, -g + 1) \to \widehat{CFK}(S, a, \varphi(a'), z, -g + 1)
\]
between \( l \circ i \) and zero.

**Proof.** Consider the path \( \{z_t\}_{t \in (-1,1)} \) in \( B_i \) defined by

\[
\begin{align*}
    z_t &= \begin{cases} 
        (t, 0) & \text{for } t \in (-1, 0] \\
        (\frac{\sqrt{2t}}{2}, \frac{\sqrt{2t}}{2}) & \text{for } t \in [0, 1) 
    \end{cases}
\end{align*}
\]

and smoothed near 0. Let \((E^H_t, \pi^H_t)\) be the Lefschetz fibration over \( B_i \) with a unique critical point over \( z_t \) with associated vanishing cycle \( L \). The one-parameter family of punctured Riemann surfaces \( B^H_t = (B_i, z_t) \) can be compactified by adding \((B_+^i, z_+^i)\), where \( B_+ = (B_i \setminus \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}) \cup (\mathbb{D} \setminus \{(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\}) \) and \( z_+ = (0, 0) \in \text{int}(\mathbb{D} \setminus \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}) \), as \( t \to +1 \) and \((B_-^i, z_-^i)\), where \( B_- = B_i \sqcup B_i \) and \( z_- = (0, 0) \in \text{int}(B_i) \), as \( t \to -1 \).

We endow these Lefschetz fibrations with Lagrangian boundary conditions \( Q^H_t \) induced by the one represented in the picture in the middle of Figure 13 (we assume that the fibrations are trivialised in the complement of the dotted half-line so that \( \tau_L \) acts when one crosses it.) Then we define \( \mathcal{H}' \) by counting pairs \((t, u)\) where \( t \in (-1, 1) \) and \( u \) is a pseudoholomorphic multisection of degree 2 of \((E^H_t, \pi^H_t)\) of index \(-1\) with boundary on \( Q^H_t \).

The degenerations of \((B_i, z_t)\) correspond, by pull back, to degenerations \((E^H_\pm^i, \pi^H_\pm^i)\) of \((E^H_t, \pi^H_t)\). The relative Gromov invariant of \((E^H_t, \pi^H_t)\) is \( l \circ i \), while the relative Gromov invariant of \((E^H_\pm^i, \pi^H_\pm^i)\) is trivial by Example 4.8. \( \Box \)

**Proof of Proposition 4.14.** Since the end of the proof goes pretty much as in the proofs of Theorem 4.4 and of the exact triangle in Lagrangian Floer homology ([62, Section 3]), we will leave some details to the reader. The key point is to study the small energy components of the maps \( i \) and \( l \).
If \( a'_1 \) is close enough to \( a_1 \cup \tilde{a}_1 \) and, for \( i > 1 \), \( a'_i \) and \( a_i \) are close enough to \( \tilde{a}_i \), for any \( j \in \{1, \ldots, 2g\} \) we have evident bijections

\[
\begin{align*}
(a'_1 \cap \varphi(a'_j)) \quad &\xrightarrow{L} \quad (\tilde{a}_1 \cap \varphi(a'_j)) \quad \cup \quad (a_1 \cap \varphi(a'_j)) \\
\quad \quad x'_1 \quad &\mapsto \quad \tilde{x}_1 \quad \text{or} \quad \quad x_1
\end{align*}
\]

(where \( \tilde{x}_1 \in (\tilde{a}_1 \cap \varphi(a'_j)) \) and \( x_1 \in (a_1 \cap \varphi(a'_j)) \)) and

\[
(a_i \cap \varphi(a'_j)) \quad \mapsto \quad (a'_i \cap \varphi(a'_j)) \quad \mapsto \quad (a_i \cap \varphi(a'_j))
\]

where the image of any point is the closest among all the elements of the corresponding codomain. These induce an injection

\[
i_0: \overline{CFK}(S, \tilde{a}, \varphi(a'), z, -g + 1) \quad \mapsto \quad \overline{CFK}(S, a', \varphi(a'), z, -g + 1)
\]

and a quotient

\[
l_0: \overline{CFK}(S, a', \varphi(a'); -g + 1) \quad \mapsto \quad \overline{CFK}(S, a, \varphi(a); -g + 1)
\]

\[
\begin{cases}
(x_1, x_2, \ldots, x_{2g}) & \text{if } f(x'_1) \in (a_1 \cap \varphi(a'_j)) \\
0 & \text{if } f(x'_1) \in (\tilde{a}_1 \cap \varphi(a'_j)).
\end{cases}
\]

Reasoning as in [62, Subsection 3.2], one can check that \( i_0 \) can be expressed by counting holomorphic degree 2g multisectons of \( (E_i, \pi_i, Q_i) \) and that, taking \( \tilde{a}_i \) close enough to \( a'_i \) outside a small neighborhood of \( a_1 \), the energy of these multisectons can be made arbitrarily small. We have then a decomposition \( i = i_0 + i' \) where \( i' \) counts higher energy sections.

Similarly, if \( a_i \) is close enough to \( a'_i \) outside a small neighborhood of \( \tilde{a}_1 \), \( l_0 \) coincides with the lower energy component of \( l \), giving a decomposition \( l = l_0 + l' \) where \( l' \) counts higher energy sections (cf. Section 3.3 of [62] and, in particular, Lemma 3.8 for a description of the lower energy sections that appear in the definition of \( l \)). Since \( l_0 \circ i_0 = 0 \), again Lemma 2.31 of [62] and Lemma 4.1 above imply Proposition 4.14.

4.4. **Comparing the double cones.** Applying the main result of Section 3 we obtain a chain map \( \Phi^z \) that allows us to compare two of the three terms of the two exact triangles in Heegaard Floer and symplectic homologies as in Diagram (1.3). To proceed with our strategy for the proof of Theorem 1.1, we first need to define a chain map inducing the isomorphism on the first column of (1.3) that behaves well with respect to the Lefschetz fibrations framework. The main difficulty is that we defined the chain maps inducing the exact sequence (4.11) by counting degree 2g holomorphic multi-sectons, and those inducing the exact sequence (4.9) by counting holomorphic sectons. As a first stem, we need to put both exact sectons on equal footing.

4.4.1. **Seidel’s exact sequence revisited.** Let us consider the diagram \( (S, \tilde{a}, \varphi(\tilde{a}), z) \). Although it is not a diagram of the type considered in Section 2.2, we can still define chain complex \( \overline{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) \) in the same way. Note, however, that we should be careful to choose for the definition a Liouville form on \( S \) for which \( L \) is an exact Lagrangian submanifold; this is always possible, as long as \( L \) is nonseparating.
If $A$ is as in Section 2.3 then $(\tilde{a}_2 \cap \varphi(\tilde{a}_2)) \cap A = \{c_2, c_2', d\}$ where $c_2, c_2' \subset \partial S$. Up to moving the base point $w$ and, possibly, changing the compatible basis of arcs, we can assume that

\begin{equation}
(4.14) \quad a_2 \cap A \text{ is the first arc that one encounters when moving from } w \text{ in the direction of } \partial S.
\end{equation}

This assumption implies that $\tilde{a}_2 \cap A$ intersects $\varphi(\tilde{a}_i)$ if and only if $i = 2$. Moreover, if $x = (x_1, \ldots, x_{2g}) \in \widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$, then $x_1 \in \tilde{a}_1 \cap \varphi(\tilde{a}_1)$ because only one intersection point is outside of $A$. Thus we have a splitting of vector spaces

\begin{equation}
(4.15) \quad \widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1) = \widehat{CFK}^{-}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1) \oplus \widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)
\end{equation}

where $\widehat{CFK}^{-}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ is the subspace generated by the $2g$-tuples of intersection points $(x_1, x_2, \ldots, x_{2g})$ with $x_2 \in \{c_2, c_2'\}$ and $\widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ is the subspace generated by the $2g$-tuples of intersection points $(x_1, x_2, \ldots, x_{2g})$ with $x_2 = d$, both with the usual identifications $c_i \sim c_i'$.

**Lemma 4.18.** The following hold:

1. $\widehat{CFK}^{-}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ and $\widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ are subcomplexes of $\widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$, and
2. the chain maps

\begin{align*}
&j_{-} : CF(\varphi(\tilde{a}_1), \tilde{a}_1) \to \widehat{CFK}^{-}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1), \quad j_{-}(x) = (x, c_2, c_3, \ldots, c_{2g}) \\
&\text{and} \\
&j_{+} : CF(\varphi(\tilde{a}_1), \tilde{a}_1) \to \widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1), \quad j_{+}(x) = (x, d, c_3, \ldots, c_{2g})
\end{align*}

induce isomorphisms in homology.

**Proof.** The fact that $\widehat{CFK}^{-}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ is a subcomplex of $\widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ is a direct consequence of the fact that if $u$ is an irreducible component of a holomorphic curve counted in the definition of the Heegaard Floer differential that has a positive end at some $c_i$ or $c_i'$ then $u$ is a trivial strip. To see that $\widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ is also a subcomplex, we observe first that if $x = (x_1, d, x_3, \ldots, x_{2g})$ is a generator of $\widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$, then there are two index 1 holomorphic curves in $\mathbb{R} \times [0, 1] \times S$ with positive ends at $(\tilde{x}_1, d)$ and negative ends at $(x_1, c_2)$ or $(x_1, c_2')$. These two curves project over $S$ to the two shaded annuli in Figure 14 (cf. [47, Lemma 9.4] for a description of similar holomorphic curves). Moreover it is not difficult to see from Condition (4.14) that these are the only two holomorphic curves that appear in the expression for $\partial x$ that have negative limit not contained in $\widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$. The identification $c_2 \sim c_2'$ and the fact that we work with $\mathbb{Z}/2$-coefficients imply then that $\partial x \in \widehat{CFK}^{+}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$.

A part from the pair of canceling holomorphic curves described above, the projection to $S$ of the other holomorphic curves that appear in the definition of the differential $\partial$ of $\widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ do not cross $\partial A$. Then we have then a splitting of the differential of $\widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ as $\partial = \partial_0 + \partial_1$, where
The projections to $S$ of the two index 1 holomorphic curves from $(x_1,d)$ to $(x_1,c_2)$ or $(x_1,c_2')$. For simplicity we avoided to draw the curves $\tilde{a}_i$ and $\varphi(\tilde{a}_i)$ for $i > 2$.

- $\partial_0$ is defined counting pseudo-holomorphic multisections of the form $u_0 \sqcup u_0'$ where $u_0$ is an index 1 pseudo-holomorphic section counted in the definition of the differential of $CF(\varphi(\tilde{a}_1), \tilde{a}_1)$ and $u_0'$ is a $2g-1$-tuple of trivial sections over the components of the generators contained in $A$;
- $\partial_1$ is defined counting pseudo-holomorphic multisections of the form $u_1 \sqcup u_1'$ where $u_1$ is an index 1 degree $2g-1$ pseudo-holomorphic multisection that projects to $A$ and $u_1'$ is a trivial section over a point in $\tilde{a}_1 \cap \varphi(\tilde{a}_1)$.

Since the point in $\tilde{a}_2 \cap \varphi(\tilde{a}_2)$ cannot interact with anything else (besides the two cancelling curves described above), the splitting of the differential gives isomorphisms of chain complexes

$$\widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1) \cong \text{CF}(\varphi(L), L) \otimes \widehat{CFK}(S, \{\tilde{a}_3, \ldots, \tilde{a}_{2g}\}, \{\varphi(\tilde{a}_3), \ldots, \varphi(\tilde{a}_{2g})\}, z, 1 - g).$$

Now, observe that the $\widehat{CFK}(S, \{\tilde{a}_3, \ldots, \tilde{a}_{2g}\}, \{\varphi(\tilde{a}_3), \ldots, \varphi(\tilde{a}_{2g})\}, z, 1 - g)$ is isomorphic to the knot Floer complex of a fibred knot in the bottom Alexander degree, and therefore its homology is generated by the class of $(c_3, \ldots, c_{2g})$. Then the chain maps $j_\pm$ induce isomorphisms in homology by Künneth’s formula.

$\square$
We define a map
\[ \tilde{\iota} : \widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) \to \mathcal{C}F^\sharp(\varphi) \]
by counting embedded holomorphic multisections of \( E^\varphi_{\tilde{a}} \to B_{\tilde{a}} \) with boundary on the Lagrangian boundary condition \( Q_{\tilde{a}} \) obtained by parallel transport of \( \tilde{a} \) and which are asymptotic to a generator \( x \) of \( \hat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) \) at the boundary puncture and to the multiorbit \( \gamma e^{2g-1} \) for a generator \( \gamma \) of \( \mathcal{C}F^\sharp(\varphi) \) at the interior puncture. The reason why \( \tilde{\iota} \) is a chain map is that it is defined in essentially the same way of \( \Phi^\sharp \), and therefore the arguments given in Section 3.2 apply also here. Let \( \tilde{\iota}^\pm : \hat{CFK}^\pm(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) \to \mathcal{C}F^\sharp(\varphi) \) be the restrictions of \( \tilde{\iota} \).

**Lemma 4.19.** The diagram
\[
\begin{array}{ccc}
\hat{CFK}_-(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) & \xrightarrow{\tilde{\iota}_-} & \mathcal{C}F^\sharp(\varphi) \\
\downarrow{j_-} & & \downarrow{i} \\
\mathcal{C}F(\varphi(L), L) & & \mathcal{C}F(\varphi(L), L)
\end{array}
\]
(4.16)
commutes.

**Proof.** As in Lemma 6.2.3 of [11], if \( u \) is an irreducible component of a multisection of \( (E^\varphi_{\tilde{a}}, \pi_i) \) with positive end to some \( c_i \), then \( u \) is a holomorphic section with negative end at \( e \). □

We define also a map \( \tilde{\lambda} : \mathcal{C}F^\sharp(\varphi) \to \mathcal{C}F^\sharp(\varphi \circ \tau_L^{-1}) \) by counting holomorphic multisections of degree \( 2g \) of \( E^\varphi_{\tilde{a}} \to B_\lambda \) which are asymptotic to \( \gamma_+ e^{2g-1} \) and \( \gamma_- e^{2g-1} \) for \( \gamma_+ \) and \( \gamma_- \) generators of \( \mathcal{C}F^\sharp(\varphi) \) and \( \mathcal{C}F^\sharp(\varphi \circ \tau_L^{-1}) \) respectively.

**Lemma 4.20.** \( \tilde{\lambda} = \lambda \).

**Proof.** By lemma 5.3.2 of [9], the holomorphic curves used to define \( \tilde{\lambda} \) consist of the union of a holomorphic section of \( E^\varphi_\lambda \to B_\lambda \) between \( \gamma_+ \) and \( \gamma_- \) with \( 2g - 1 \) copies of the trivial cylinder over \( e \). □

With obvious modifications to Lemma 4.9 one can define a chain homotopy
\[ \tilde{H} : \widehat{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) \to \mathcal{C}F^\sharp(\varphi \circ \tau_L^{-1}) \]
between \( \tilde{\lambda} \circ \tilde{\iota} \) and the zero map. Let \( \tilde{H}^\pm \) denote the restriction of \( \tilde{H} \) to the subcomplexes \( \hat{CFK}^\pm(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) \). The proof of the next lemma is the same as the proof of Lemma 4.19.
Lemma 4.21. The diagram

\[
\begin{array}{c}
\CFK_-(S, \bar{a}, \varphi(\bar{a}), z, 1-g) \\
\downarrow j_- \\
\CF(\varphi(L), L)
\end{array} \xrightarrow{H_-} \begin{array}{c}
\CF^\varphi(\varphi \circ \tau_L^{-1}) \\
\uparrow \mathcal{H}
\end{array}
\]

(4.17)

commutes.

We can then form a double cone of the maps \(\tilde{\iota}_-\) and \(\tilde{\lambda}\), which is

\[
\CFK_-(S, \bar{a}, \varphi(\bar{a}), z, 1-g) \oplus CF^\varphi(\varphi) \oplus CF^\varphi(\varphi \circ \tau_L^{-1})
\]

with differential

\[
\begin{pmatrix}
\partial & 0 & 0 \\
\tilde{\iota}_- & \partial & 0 \\
\tilde{H}_- & \tilde{\lambda} & \partial
\end{pmatrix}
\]

Lemma 4.22. The double cone of the maps \(\tilde{\iota}_-\) and \(\tilde{\lambda}\) is acyclic.

Proof. By Lemmas 4.19, 4.20 and 4.21 the double cone of \(\iota\) and \(\lambda\) defined in Section 4.2 is a subcomplex of the double cone of \(\tilde{\iota}_-\) and \(\tilde{\lambda}\). Double cones are naturally filtered complexes because the differential is a lower triangular matrix. Moreover the inclusion induces an isomorphisms on the homology of the associated graded complexes by Lemma 4.18. This implies that the inclusion induces an isomorphisms between the homologies of the double cones by a standard algebraic trick. The double cone of \(\iota\) and \(\lambda\) is acyclic, and therefore the double cone of \(\tilde{\iota}_-\) and \(\tilde{\lambda}\) is also acyclic. \(\square\)

4.4.2. The first square. Let now \((E_\Upsilon, \pi_\Upsilon)\) be the trivial symplectic fibration with basis \(B_\Upsilon = \mathbb{D} \setminus \{1, -i, -1\}\) and fiber \(S\). Endow \((E_\Upsilon, \pi_\Upsilon)\) with the Lagrangian boundary condition \(Q_\Upsilon\) given by the symplectic parallel transport over \((0, \pi), (\pi, \frac{3\pi}{2})\) and, respectively, \((\frac{3\pi}{2}, 2\pi)\) of copies of \(\bar{a}, \varphi(\bar{a})\) and, respectively, \(\varphi(a')\).

Figure 15. The fibration \((E_\Upsilon, \pi_\Upsilon)\) with its Lagrangian boundary condition.

Let \(\Upsilon: \CFK(S, \bar{a}, \varphi(a'), z, -g + 1) \rightarrow \CFK(S, \bar{a}, \varphi(\bar{a}), z, -g + 1)\)
be the chain map defined on the generators $x$ by
\[ \Upsilon(x) = \sum_y \#_2 \mathcal{M}_0^T(x, y) \]
where the sum is taken over all the generators $y$ of $\overline{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ and $\mathcal{M}_0^T(x, y)$ is the moduli space of index 0, embedded degree 2g holomorphic multisections $u$ of $(E_Y, \pi_Y)$ with Lagrangian boundary condition $Q_Y$, asymptotic to $x$ at 1, to $\varphi(\Theta_{a', \tilde{a}})$ at $-i$ and to $y$ at $-1$ and such that $\text{Im}(u) \cap (B_Y \times \{z\}) = \emptyset$. We denote by $\Upsilon_\pm: \overline{CFK}(S, \tilde{a}, \varphi(a'), z, -g + 1) \to \overline{CFK}_\pm(S, \tilde{a}, \varphi(\tilde{a}), z, -g + 1)$ the components of $\Upsilon$.

**Lemma 4.23.** In homology $\Upsilon_-$ induces an isomorphism and $\Upsilon_+$ the trivial map.

**Proof.** Let $\Sigma = S \cup_0 \overline{S}$ be the double of $S$. We complete $\tilde{a}, \varphi(a')$ and $\varphi(\tilde{a})$ in $S$ to collections of curves $\tilde{a}, a'$ and $\tilde{a}$ as explained in Subsection 2.2, with the caveat that $\tilde{a}_1 = \tilde{a}_1 = L$ and $\tilde{a}_1 = \varphi(\tilde{a}_1)$, and only the arcs $\tilde{a}_i$ and $\varphi(\tilde{a}_i)$ for $i = 2, \ldots, 2g$ are completed in $\overline{S}$.

$(\Sigma, \tilde{a}, a', \tilde{a})$ is a Heegaard diagram for $Y_0$: in fact we can slide $\tilde{a}_2$ over $\tilde{a}_1$ and isotope the resulting curve so that it intersects $\tilde{a}'_1$ in only one point, and intersects no other $\tilde{a}'$-curve. Then we can destabilise the diagram and remove those two curves. In the resulting diagram $\tilde{a}'_2$ becomes isotopic to $\varphi(L)$. One can check that this is a Heegaard diagram for $Y_0$ compatible with a broken fibration over $S^1$ with one critical point and vanishing cycle $L$ (cf. Lekili [34] for a similar construction).

$(\Sigma, \tilde{a}, \tilde{a}, \beta')$ is a Heegaard diagram for $Y_0 \#(S^2 \times S^1)$, where the connected sum is performed away from the knot $K_0$. In fact one can handleslide $\tilde{a}_2$ over $\tilde{a}_1$ and $\tilde{a}_2$ over $\tilde{a}_1$ and, by condition (3) of Definition 4.15, the resulting curves are both isotopic to copies of $L$ in $\overline{S}$. They give a copy of $S^2 \times S^1$ and, after removing it, we remain with the same Heegaard diagram we obtained above after the stabilisation. Finally, $(\Sigma, \beta', \beta)$ is a Heegaard diagram for $(S^2 \times S^1)^{(2g-1)}$.

Then $(\Sigma, \tilde{a}, \beta', \beta)$ is the triple Heegaard diagram for a cobordism from $Y_0$ to $Y_0 \#(S^2 \times S^1)$ obtained by a single 2-handle addition because $\beta'_i$ is isotopic to $\tilde{a}_i$ for $i = 2, \ldots, 2g$. Moreover, the knot to which the handle is attached is an unknot which is geometrically unlinked with $K_0$. In fact, the longitude is $\tilde{a}_1$, which, in the Heegaard diagram for $Y_0$ obtained after destabilising $(\Sigma, \tilde{a}, a')$, is isotopic to $\beta'_2$, which bounds in $Y_0$ a disc disjoint from $K_0$.

The same arguments which prove Theorem 2.2 also show that the map $\Upsilon$ coincides with the map $F: \overline{HFK}(Y_0, K_0, 1 - g) \to \overline{HFK}(Y_0 \#(S^2 \times S^1), K_0, 1 - g) \cong \overline{HFK}(Y_0, K_0, 1 - g) \otimes \overline{HF}(S^2 \times S^1)$ defined by the doubly pointed triple Heegaard diagram $(\Sigma, \tilde{a}, \beta', \beta, w, z)$. The group $\overline{HF}(S^2 \times S^1)$ is generated by homogeneous elements $\theta_{op}$ and $\theta_{bot}$, and and a direct computation (on a simpler triple Heegaard diagram for the same cobordism, using the invariance of triangle maps) shows that $F(x) = x \otimes \theta_{bot}$.

To prove the lemma it remains only to identify $\overline{HF}_-(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g)$ with $\overline{HF}(Y_0, K_0, 1 - g) \otimes \langle \theta_{bot} \rangle$. For that we use Heegaard Floer homology with twisted coefficients. We recall that $\overline{HF}(S^2 \times S^1) \cong \mathbb{Z}/2\mathbb{Z}$, the universal coefficient map $\overline{HF}(S^2 \times S^1) \to \overline{HF}(S^2 \times S^1)$ is injective and its image is $\langle \theta_{bot} \rangle$. Then, if $\mathbb{Z}/2\mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}/2\mathbb{Z}$.
\[ \mathbb{Z}/2\mathbb{Z}[H^1(S^2 \times S^1)] \] is endowed with the \( H^1(\tilde{Y}_0\#(S^2 \times S^1)) \)-module structure induced by the map \( H^1(\tilde{Y}_0\#(S^2 \times S^1)) \rightarrow H^1(S^2 \times S^1) \), Künneth's formula gives that
\[ \tilde{HFK}(\tilde{Y}_0\#(S^2 \times S^1)), K_0, 1 - g; \mathbb{Z}/2\mathbb{Z}[t, t^{-1}]) \cong \tilde{HFK}(\tilde{Y}_0, K_0, 1 - g) \otimes HF(S^2 \times S^1) \] and the image of the universal coefficient map
\[ \tilde{HFK}(\tilde{Y}_0\#(S^2 \times S^1)), K_0, 1 - g; \mathbb{Z}/2\mathbb{Z}[t, t^{-1}]) \rightarrow \tilde{HFK}(\tilde{Y}_0\#(S^2 \times S^1)), K_0, 1 - g) \] is \( \tilde{HFK}(\tilde{Y}_0, K_0, 1 - g) \otimes \langle \theta_{\text{-bot}} \rangle \). Now it is easy to see that for this twisted coefficient system, all holomorphic curves contributing to the differential of \( \tilde{CFK}(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g; \mathbb{Z}/2\mathbb{Z}[t, t^{-1}]) \) are counted with coefficient 1, except for the two curves in Figure 14, one of which is counted with coefficient 1 and the other with coefficient \( t \). Then \( \tilde{HFK}(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g; \mathbb{Z}/2\mathbb{Z}[t, t^{-1}]) \cong HF - \tilde{HFK}(S, \tilde{a}, \varphi(\tilde{a}), z, 1 - g) \). This concludes the proof. \( \square \)

**Lemma 4.24.** There exists a chain homotopy
\[ R: \tilde{CFK}(S, \tilde{a}, \varphi(a'); -g + 1) \rightarrow CF^2(\varphi) \]
between \( \tilde{\tau} \circ \Upsilon \) and \( \Phi^2 \circ i \) that makes the following diagram commute in homology:
\[ \begin{array}{ccc}
\tilde{CFK}(S, \tilde{a}, \varphi(a'), z, -g + 1) & \xrightarrow{i} & \tilde{CFK}(S, a', \varphi(a'), z, -g + 1) \\
\Upsilon & \xrightarrow{R} & \Phi^2 \circ i \\
\tilde{CFK}(S, \tilde{a}, \varphi(a), z, -g + 1) & \xrightarrow{\tilde{\iota}} & CF^2(\varphi)
\end{array} \]

**Proof.** Let \( \{w_t\}_{t \in (-1, 1)} \) be the path in \( \mathbb{D} \setminus \{1, i\} \) defined by \( w_t = te^{i\pi/4} \) for \( t \in (-1, 1) \). Consider the one-parameter family of punctured Riemann surfaces \( B^R_t := \mathbb{D} \setminus \{1, i, w_t\} \) and let \( \pi_t^R: E^R_t \rightarrow B^R_t \) be the symplectic fibration over \( B^R_t \) with fiber \( \tilde{S} \) and monodromy \( \tilde{\varphi} \) around \( w_t \). We endow these fibrations with Lagrangian boundary conditions \( Q^R_t \) induced by the one represented in the picture in the middle of Figure 16. The homotopy \( R \) is defined by counting pairs \((t, u)\) where \( t \in (-1, 1) \) and \( u \) is a degree \( 2g \) multisection of \( \pi_t^R: E^R_t \rightarrow B^R_t \) of index \(-1\) with boundary on \( Q^R_t \) which are asymptotic to a generator of \( \tilde{CFK}(S, \tilde{a}, \varphi(a'); -g + 1) \) at 1, to a generator of \( CF^2(\varphi) \) at \( w_t \) and to \( \Theta_{a', \tilde{a}} \) or \( \varphi(\Theta_{a', \tilde{a}}) \) at \( i \), depending on the choice of a trivialization of the fibration that determines the action of \( \tilde{\varphi} \) (in the picture in the middle of Figure 16 we represented, in the usual way, the trivialization that induces limits at \( \Theta_{a', \tilde{a}} \)).

The family \( B^R_t \) can be compactified by adding two-level buildings
\[ B^R_+ = (\mathbb{D} \setminus \{1, i, e^{i\pi/4}\}) \cup (\mathbb{D} \setminus \{0, e^{i\pi/4}\}) \]
as \( t \to +1 \) and
\[ B^R_- = (\mathbb{D} \setminus \{1, i, -e^{i\pi/4}\}) \cup (\mathbb{D} \setminus \{e^{i\pi/4}, 0\}) \]
as \( t \to -1 \). These degenerations correspond, by pull back, to degenerations \( \pi_t^R: E^R_t \rightarrow B^R_\pm \) of \( \pi_t^R: E^R_t \rightarrow B^R_\pm \). Thus, after reparametrisation, \( (E^R_+, \pi^R_+) = (E_T, \pi_T) \cup (E^R_T, \pi_t) \) and the corresponding count of multisections gives \( \tilde{\tau} \circ \Upsilon \). Similarly, \( (E^R_-, \pi^R_-) = (E_i, \pi_i) \cup (W_+, \pi_{B_+}) \) and the corresponding count of multisections gives \( \Phi^2 \circ i \). \( \square \)
4.4.3. The second square.

**Lemma 4.25.** There exists a chain homotopy

\[
\hat{S} : \tilde{CFK}(S, a', \varphi(a'), z, -g + 1) \rightarrow CF^\natural(\varphi \circ \tau_{L}^{-1})
\]

from \(\tilde{\lambda} \circ \hat{\Phi}^z\) to \(\hat{\Phi}^z \circ l\) that makes the following diagram commute in homology:

\[
\begin{array}{ccc}
\tilde{CFK}(S, a', \varphi(a'); -g + 1) & \xrightarrow{l} & \tilde{CFK}(S, a', \varphi(a'); -g + 1) \\
\Phi^z & \downarrow \hat{\lambda} & \Phi^z \\
CF^\natural(\varphi) & \xrightarrow{S} & CF^\natural(\varphi \circ \tau_{L}^{-1}).
\end{array}
\]

**Proof.** Let \(\{w_t\}_{t \in (-1, 0)}\) be the path in \(\mathbb{D}\) defined by \(w_t = te^{i\pi/4}\). Consider the one-parameter family of punctured Riemann surfaces \(B_t^S = \mathbb{D} \setminus \{1, w_t\}\) endowed with the marked point \(j = 0\). Let \(\pi_t^S : E_t^S \rightarrow E_t^S\) be the Lefschetz fibration over \(B_t^S\) with regular fiber \(\tilde{S}\), one critical point over \(j\) with vanishing cycle \(L\) and monodromy \(\tilde{\varphi} \circ \tau_{L}^{-1}\) around \(w_t\). We endow these Lefschetz fibrations with the Lagrangian boundary conditions \(Q_t^S\) induced by the one represented in the picture in the middle of Figure 17.

The homotopy \(S\) is defined by counting pairs \((t, u)\) where \(t \in (-1, 0)\) and \(u\) is a degree \(2g\) multisection of \(\pi_t^S : E_t^S \rightarrow E_t^S\) of index \(-1\), with boundary on \(Q_t^S\), and which are asymptotic to a generator of \(\tilde{CFK}(S, a', \varphi(a'); -g + 1)\) at \(1\) and to a generator of \(CF^\natural(\varphi \circ \tau_{L}^{-1})\) at \(w_t\).

The family \(B_t^S\) can be compactified by adding two-level buildings

\[
B_0^S = B_t \sqcup (S^2 \setminus \{N, S\})
\]

with \(j \in S^2 \setminus \{N, S\}\) as \(t \to 0\), and (after reparametrisation)

\[
B_t^S = B_t \sqcup B_t
\]

with \(j \in B_t\), as \(t \to -1\). Again, these degenerations correspond, by pull-back, to degenerations \(\pi_0^S : E_0^S \rightarrow B_0^S\) and \(\pi_\natural^S : E_\natural^S \rightarrow B_\natural^S\) of \(\pi_t^S : E_t^S \rightarrow B_t^S\). It is not difficult to
see that $(E_{B}^{S}, \pi_{B}^{S}) \cong (W_{+}, \pi_{B_{+}}) \cup (E_{0}^{S}, \pi_{-})$ and the corresponding count of multisections gives $\tilde{\lambda} \circ \Phi$. Similarly, $(E_{0}^{S}, \pi_{0}^{S}) = (E_{l}, \pi_{l}) \cup (W_{+}, \pi_{B_{+}})$ and the corresponding count of multisections gives $\Phi \circ l$.

4.4.4. The homotopy of homotopies.

Lemma 4.26. There exists a homotopy $T: \widehat{\text{CFK}}(S, \tilde{a}, \varphi(a'); -g + 1) \to \text{CF}^{\sharp}(\varphi \circ \tau_{L}^{-1})$ from $(\mathcal{H}' \circ Y + \tilde{\lambda} \circ \mathcal{R})$ to $(\Phi \circ \mathcal{H} + S \circ i)$.

Proof. Let $\Delta = \{(s, t) \in \mathbb{R}^{2} : 0 < s < 1, 0 < t < s\}$ be an open two-dimensional symplex. For $(s, t) \in \Delta$ we choose points $\tilde{z}_{s} = se^{i\pi/4}$ and $\tilde{w}_{t} = te^{i\pi/4}$ and define the two-parameter famility of punctured Riemann surfaces $E_{T}^{s, t}$ endowed with the Lagrangian boundary condition $Q_{T}^{s, t}$ induced by the labelling in Figure 18.

The Deligne–Mumford style compactification of the family $\{B_{T}^{s, t} \}_{(s, t) \in \Delta}$ can be represented as a pentagon and the edges and the vertexes of its boundary are associated with the degenerations represented in Figure 18:

(AB) These degenerations occur when $s, t \to -1$. The limit configurations, after reparametrisation, are $B_{l}^{S} \sqcup B_{i}$ and therefore yield the term $S \circ i$. 

![Figure 17](image-url)
Figure 18. The compactification of the family \( \{ B_{s,t}^T \}_{(s,t) \in \Delta} \). The points \( \mathfrak{w}_t \) and \( \mathfrak{z}_s \) are free to move over the leaning dashed line.

(BC) These degenerations occur when \( s - t \to 0 \). The limit configurations are \( B_\lambda \sqcup B_t^R \) and therefore they yield the term \( \tilde{\lambda} \circ R \).

(CD) These degenerations occur when \( s, t \to 1 \). The limit configuration, after reparametrisation, is \( B_t^H \sqcup B_T \) and therefore yields the term \( \bar{H} \circ \Upsilon \).

(DE) These degenerations occur when \( s \to 1 \) but \( t \) remains in \( (0, 1) \). One of the components of the limit configurations is the base of the fibration of Example 4.8, and therefore the contribution of these degenerations is zero.

(EA) These degenerations occur when \( t \to -1 \) and \( s \) remains in \( (0, 1) \). The limit configurations, after reparametrisation, are \( B_\perp \sqcup B_t^{H'} \) and therefore they yield the term \( \Phi^\sharp \circ \mathcal{H}' \).

\[ \square \]
From Lemma 4.26, Lemma 4.3 and Lemma 4.2 we get the main ingredient for the proof of Theorem 1.1.

**Theorem 4.27.** The diagram (1.3) commutes.

**4.5. The induction.** As recalled in Subsection 4.3.1, the Dehn-Lickorish-Humphries theorem states that, up to isotopy, any diffeomorphism $\varphi$ of $S$ can be decomposed as

$$\varphi = \tau_n \circ \cdots \circ \tau_1$$

for some $n \in \mathbb{N}$, where $\tau_i$ is a positive or negative Dehn twist along some nonseparating curve $L_i$. In this subsection we use Diagram (1.3) to finish the proof of Theorem 1.1. We will proceed by induction on $n$.

**4.5.1. Initialization.** The base case for the induction is when $\varphi$ is Hamiltonian isotopic to the identity. We construct $\hat{\varphi}$ as the time-one flow of a Hamiltonian function $\hat{H} : \hat{S} \to \mathbb{R}$. We fix a function $H : S \to \mathbb{R}$ satisfying the following properties.

- In $A$ it depends only on the coordinate $y$, and moreover:
  - $\frac{\partial H}{\partial y} > 0$ in $\{ y \in (2, 3] \}$,
  - $\{ y = 0 \} = \partial S$ is a Morse-Bott circle of minima for $H$,
  - $\frac{\partial H}{\partial y} \leq 0$ and $\frac{\partial^2 H}{\partial y^2} \leq 0$ in $\{ y \in [0, 2) \}$,
  - $\{ y = 1 \}$ is a Morse-Bott circle of minima for $\frac{\partial H}{\partial y}$, and
  - $\frac{\partial H}{\partial y}$ is small in absolute value near $y = 0$.

- In $\hat{S} \setminus A$ it is a Morse function with a unique maximum $e_+$ and $2g$ saddles $h_1, \ldots, h_{2g}$, and its differential is small.

The conditions on $A$ are chosen so that the Hamiltonian flow of $H$ produces the finger move of Figure 2. The Function $\hat{H}$ is obtained by perturbing the Morse-Bott circle of minima at $y = 0$ into a minimum $e$ and a saddle $h$.

The chain complex $CF^\#(\varphi)$ is generated by $e_+, h_1, \ldots, h_{2g}$ and holomorphic cylinders contributing to the differential of $CF^\#(\varphi)$ correspond, by Morse-Bott theory, to negative gradient flow lines between generators. This is a fairly elementary instance of Morse-Bott perturbation, where the correspondence between holomorphic cylinders and Morse flow-lines can be worked out explicitly. In fact the projection of a holomorphic cylinder in $\mathbb{R} \times \hat{T}_\varphi$ to $\hat{S}$ satisfied the Floer equation, and for a suitably small Hamiltonian function (and in absence of $J$-holomorphic spheres, which is the case for $\hat{S}$) the solutions of the Floer equation are in bijection with the flow lines between critical points. Then the differential in $CF^\#(\varphi)$ is

$$\partial^\# h = h, \quad \partial^\# e_+ = \cdots = \partial^\# h_{2g} = 0.$$

Next we choose a convenient basis of arcs $a_1, \ldots, a_{2g}$ of $\hat{S}$. First we extend $A = [0, 3] \times S_1$ inside $\hat{S}$ to an annulus $\hat{A} = [-1, 3] \times S_1$ so that no critical point of $\hat{H}$ is contained in $[-1, 0] \times S_1$. We define the arcs such that

- $a_i \cap (\hat{S} \setminus \hat{A})$ is the unstable manifold of the critical point $h_i$ for all $i = 1, \ldots, 2g$,
- in $\hat{A} \setminus A$ the arcs come close together,
- $a_i \cap A = \{ \theta = \theta_i \}$ for some $\theta_i \in S^1$. 


We also assume that the distance between the arcs in $A$ is smaller than the size of the finger move and that all intersection points between $a$ and $\tilde{\varphi}a$ are contained in $A$ except for $h_1, \ldots, h_{2g}$.

We denote by $h_i$ the equivalence class of $(c_1, \ldots, c_{i-1}, h_i, c_{i+1}, \ldots, c_{2g})$ for $i = 1, \ldots, 2g$, which are evidently generators of $\widehat{HFK}(S, a, \varphi(a), z, 1-g)$, and by $H$ the subspace they generate. We denote also by $C_0$ the subspace of $\widehat{HFK}(S, a, \varphi(a), z, 1-g)$ generated by all other generators.

**Lemma 4.28.** $H$ and $C_0$ are subcomplexes of $\widehat{CFK}(S, a, \varphi(a), z, 1-g)$. The differential on $H$ is trivial and $C_0$ is acyclic.

**Proof.** The intersection points $c_j$ and $c_j'$ cannot appear at the positive end of a nontrivial component of a holomorphic curve contributing to $\partial h_i$. This implies that any such curve must consist of a nontrivial section with boundary on $a_i$ and $\varphi(a_i)$, which however must pass through the point $z$. This shows that $\partial h_i = 0$, and therefore $H$ is a subcomplex on which the differential is trivial.

To show that $C_0$ is also a subcomplex, we observe that no holomorphic curve contributing to the differential in $\widehat{CFK}(S, a, \varphi(a), z, 1-g)$ can have a nontrivial component with a negative end at $h_i$ because the projection of that component to $\hat{S}$ should cover a region which intersect $\partial \hat{S}$. Then the only possibility left is the union of a trivial section over $h_i$ with a multisection with negative ends at intersection points of the form $c_j$ or $c_j'$ and whose projection to $\hat{S}$ is contained in $\hat{A}$. One can see that such a curve is either an union of trivial sections, or must cross the basepoint $z$: the portion of the diagram in $\hat{A}$ is nice (in the sense of Sarkar and Wang [56]) and therefore the $J$-holomorphic curves contributing to the differential correspond to empty bigons and rectangles in the diagram. One can readily check that bigons must cross $z$ and rectangles cannot have two diagonally opposed vertices in $\{c_1, c'_1, \ldots, c_{2g}, c'_{2g}\}$.

When $g = 1$ the diagram $(S, a, \varphi(a), z)$ describes the fibred knot in $(S^2 \times S^1) \# (S^2 \times S^1)$ which is denoted by $B(0,0)$ in [49, Section 9] so, in general, it describes $B(0,0)^{\#g} \subset (S^2 \times S^1)^{\#2g}$. By Proposition 9.2 [49, Proposition 9.2] and the Künneth formula for knot Floer homology [49, Theorem 7.1] then $\dim \widehat{HFK}(S, a, \varphi(a), z, 1-g) = 2g = \dim H$, from which we deduce that $C_0$ is acyclic. \qed

Let $\Phi^\pm_0$ be the “low energy part” or $\Phi^\pm$. The low energy $J$-holomorphic multisections contributing to $\Phi^\pm$ which have $(c_1, \ldots, c_{i-1}, h_i, c_{i+1}, \ldots, c_{2g})$ at the positive end consist the union of the horizontal section over $h_i$ and $2g - 1$ sections from $c_i$ to $e$. By construction $h_i$ is a positive intersection point between $a_i$ and $\tilde{\varphi}(a_i)$, and therefore the trivial section over $h_i$ has index zero by Lemma 4.12 and is regular by Lemma 2.27 of [62]. The low energy sections from $c_i$ to $e$ are obtained by Morse-Bott perturbation of horizontal sections over $c_i$ of the fibration with monodromy $\varphi$.

Then $\Phi^\pm_0(h_i) = h_i$, which implies, by Gauss elimination, that the composition

$$H \to \widehat{CFK}(a, \tilde{\varphi}(a), z, 1-g) \xrightarrow{\Phi} \widehat{CF^\pm} \to \langle h_1, \ldots, h_{2g} \rangle$$

is an isomorphism, where the first map is the inclusion and the last map is the projection. The first and the last maps induce isomorphisms in homology by Lemma 4.28 and
Equation (4.20). This proves that $\Phi^\sharp$ is an isomorphism when $\varphi$ is Hamiltonian isotopic to the identity.

4.5.2. The inductive step. Now we assume that $\varphi = \tau_{L_1}^{s_1} \circ \ldots \circ \tau_{L_n}^{s_n}$ where $L_i$ is a non-separating simple closed curve in $S$ and $s_i \in \{+1, -1\}$. Denote $\psi = \tau_{L_1}^{s_1} \circ \ldots \circ \tau_{L_{n-1}}^{s_{n-1}}$. The inductive hypothesis is that

$$\Psi^\sharp: \widehat{HFK}(S,a,\psi(a),z,1-g) \to HF^\sharp(\psi)$$

is an isomorphism. If $s_n = -1$, then $\varphi = \psi \circ \tau_{L_n}^{-1}$, and therefore we have the commutative diagram of exact sequences

\[
\begin{array}{ccc}
HF(\varphi(L),L) & \xrightarrow{l_*} & \widehat{HFK}(S,a,\psi(a),a,1-g) \\
\downarrow{(\Upsilon_\pm)_*} & & \downarrow{\Phi^\sharp_*} \\
HF(\varphi(L),L) & \xrightarrow{l_*} & \widehat{HFK}(S,a,\psi(a),a,1-g) \\
\downarrow{\delta} & & \downarrow{\Phi^\sharp_*} \\
HF^\sharp(\varphi) & \xrightarrow{\lambda_*} & HF^\sharp(\psi).
\end{array}
\]

The first vertical map in the diagram is an isomorphism by Lemma 4.23 and the third vertical map is an isomorphism by the inductive hypothesis, and therefore by the five lemma the middle vertical map is also an isomorphism. If $s_n = +1$, then $\psi = \varphi \circ \tau_{L_n}^{-1}$ and the argument is similar.

5. Applications

Knot Floer homology is a powerful invariant of knots and it is an interesting question to understand what kind informations one can extract from it. From this point of view Theorem 1.1 has interesting consequences.

Let $\varphi: S \to S$ be a diffeomorphism and let $\varphi_c$ be the canonical Nielsen–Thurston representative of the mapping class $[\varphi]$. Recall that, by the Thurston’s classification of surfaces homeomorphisms, $\varphi_c$ may be pseudo-Anosov, periodic or reducible (see [63], [20] and [15] for details). When, as in our case, $\partial S \cong S^1$ it is convenient to see $\varphi_c$ as a homeomorphism of a subsurface $S \setminus A$ of $S$, where $A$ is a collar of $\partial S$, such that $\varphi_c$ is isotopic relative to $\partial(S \setminus A)$ to $\varphi|_{S \setminus A}$.

The canonical representative $\varphi_c$ of $[\varphi]$ defines a canonical rotation angle $\vartheta_c \in [0, 2\pi)$ that corresponds to the rotation defined by $\varphi_c|_{\partial (S \setminus A)}$. If $\varphi|_{\partial S} = id$, the “difference” near the boundary between $\varphi$ and $\varphi_c$ is then encoded by the fractional Dehn twist coefficient of $\varphi$.

**Definition** 5.1. Let $S$ be a surface with boundary $\partial S \cong S^1$ and $\varphi: S \to S$ a surface diffeomorphism such that $\varphi|_{\partial S} = id$. Fix a collar $A \cong (0,1] \times S^1$ of $\partial S = \{1\} \times S^1$ and
perturb $\varphi$ by an isotopy relative to $\partial S$ in a way that $\varphi|_{(S\setminus A)} = \varphi_c$. The fractional Dehn twist coefficient $t_c(\varphi)$ of $\varphi$ is the winding number of the arc $\varphi(\{(1 - y, 1) | y \in [0, 1]\})$.

Observe in particular that
\begin{equation}
2\pi t_c(\varphi) = \vartheta_c + 2\pi k_{\varphi}
\end{equation}
where $k_{\varphi} \in \mathbb{Z}$ is the sum with signs of the full boundary parallel Dehn twists of $\varphi$.

For the first application of Theorem 1.1 we are interested in fibered knots whose monodromy $\varphi$ has canonical representative $\varphi_c$ that is irreducible.

**Theorem 5.2.** Let $(K, S, \varphi)$ be an open book decomposition of $Y$ with $\varphi_c$ irreducible. Denote $F_{\min}([\varphi])$ the minimum number of fixed points that an area-preserving non-degenerate representative of $[\varphi]$ may have. Then:

\[
\dim \left( \widehat{HF}K(Y, K; -g + 1) \right) = \begin{cases} 
F_{\min}([\varphi]) - 1 & \text{if } \varphi_c = \text{id} \text{ and } t_c(\varphi) = 0; \\
F_{\min}([\varphi]) + 1 & \text{otherwise}.
\end{cases}
\]

Before giving the proof we need to recall some of the ideas behind an analogous result for symplectic homology.

**Lemma 5.3.** If the canonical representative of $\varphi$ is irreducible and $k_{\varphi} = 0$, then:
\begin{equation}
\dim (HF(\varphi)) = F_{\min}([\varphi]).
\end{equation}

This result was essentially proven by Gautschi (in the periodic case) and Cotton-Clay (in the pseudo-Anosov case). We refer the reader to [23] and [15] for the details and for some definitions that we will not recall. Observe that we did not specify the version $\pm$ of the Floer homology: this is because the behavior of $\varphi|_{\partial S}$ is assumed to coincide with the one prescribed by $t_c(\varphi)$ (cf. Remark 5.6 below).

We also recall the following result (see Cotton-Clay [16, Theorem 1.1]).

**Lemma 5.4.** If the canonical representative of $\varphi$ is irreducible and $k_{\varphi} = 0$, then:
\begin{equation}
F_{\min}([\varphi]) = \sum \left| \text{ind}(\eta) \right|
\end{equation}
where the sum is over the set of Nielsen classes of $\varphi$ (i.e. the set of free homotopy classes of oriented paths $S^1 \hookrightarrow T_{\varphi}$), $\text{ind}(\eta) := \sum_{x \in \text{Fix}(\varphi) \cap \eta} \varepsilon(x)$ and $\varepsilon(x) := \text{sign}(\det(1 - d_x \varphi))$ denotes the Lefschetz sign of $x$.

We remark that lemmas 5.3 and 5.4 can be generalized to the irreducible case (see Theorem 4.16 of [15] and Theorem 1.1 of [16]) but to prove Theorem 5.2 we will only use the statements for irreducible maps plus some partial results of [15] about boundary parallel Dehn twists. In the rest of this section, if $C \subset S$ is a set of fix points all in the same Nielsen class, we will refer to this class as to the Nielsen class of $C$.

The two main ingredients to show (5.2) and (5.3) are the following:
(I.1) the canonical representative of an irreducible diffeomorphism minimizes the number of fixed points in its mapping class;
(I.2) if $\langle \partial x, y \rangle \neq 0$, then $x$ and $y$ are in the same Nielsen class and $\varepsilon(x) \neq \varepsilon(y)$.

The first is a classical result in fixed point theory (see Jang–Guo [29]), while the second is an obvious consequence of the definitions and the fact that the Floer differential inverts Lefschetz signs (see for example [59, §2]).
If \([\varphi] \neq [id]\) is periodic then in [23, §3] Gautschi shows that the canonical representative \(\varphi_c\) of \([\varphi]\) is symplectic and monotone: the monotonicity condition was introduced by Seidel in [59] to ensure the compactness of the moduli spaces of holomorphic curves that come into play in the definition of \(HF\) (even if we will not recall the definition, we underline that condition (2.1) implies monotonicity). Moreover Gautschi shows that all the fixed points of \(\varphi_c\) are in different Nielsen classes and have Lefschetz signs +1 (see [23, Proposition 9]): (I.2) implies then that the Floer differential of \(CF(\varphi)\) vanishes and (I.1) gives equations (5.2) and (5.3).

If \([\varphi]\) is pseudo-Anosov the argument is a bit more complicate because in general \(\varphi_c\) is not smooth (and in particular not symplectic). On the other hand in [15, §3] Cotton–Clay shows that if \(\varphi\) is symplectic and non-degenerate then is also weakly monotone: the last condition was defined by Cotton Clay as a weaker version of Seidel’s monotonicity, but which is still sufficient to ensure the good definition of \(HF(\varphi)\). In particular, if \(\varphi_{\text{min}} \in [\varphi]\) is any symplectic non-degenerate representative with exactly \(F_{\text{min}}([\varphi])\) fixed points, then \(HF(\varphi_{\text{min}})\) is well defined and, by invariance of \(HF\):

\[
(5.4) \quad \dim (HF(\varphi)) = \dim (HF(\varphi_{\text{min}})) \leq F_{\text{min}}([\varphi]).
\]

His strategy to show (5.2) consists then in smoothing the singularities of \(\varphi_c\) to obtain a special symplectic representative \(\varphi_{\text{sm}}\) of \([\varphi]\) for which the differential of \(CF(\varphi_{\text{sm}})\) vanishes. This gives

\[
(5.5) \quad \dim (HF(\varphi)) = \dim (HF(\varphi_{\text{sm}})) = \#\text{Fix}(\varphi_{\text{sm}}) \geq F_{\text{min}}([\varphi]),
\]

which, in combination with (5.4), implies Equation (5.2).

The special symplectic representative \(\varphi_{\text{sm}}\) (which, by what we just said, has exactly \(F_{\text{min}}([\varphi])\) fixed points) is obtained as follows. By [6] and [29], it is known that any two fixed (smooth or singular) points of the (canonical) pseudo-Anosov homeomorphism \(\varphi_c\) are in different Nielsen classes. In order to get \(\varphi_{\text{sm}}\), in [15, §3] Cotton–Clay perturbs \(\varphi_c\) in a neighborhood of all the singular fixed point. If \(\varpi\) is one of these singularities and has \(p\) prongs (cf [15, §3.2]), the perturbation for \(\varphi_{\text{sm}}\) produces either:

1. one elliptic fixed point if \(\varpi\) is a rotated singularity, i.e. if the prongs are cyclically permuted by a non trivial permutation;
2. \(p - 1\) positive hyperbolic fixed points if \(\varpi\) is an unrotated singularity, i.e. if the permutation of the prongs is trivial.

It is important to underline the fact that all the fixed points of \(\varphi_{\text{sm}}\) described in (i) and (ii) are in the same Nielsen class of \(\varpi\).

To deal with the boundary (that we assume connected here) of the surface, Cotton–Clay applies in [15, §4.2] the ideas of [29, §2.1] and proceeds as follows. One starts with a closed surface \(\overline{S}\) and a pseudo-Anosov homeomorphism \(\overline{\varphi}_c\) of \(\overline{S}\) with a (hyperbolic) fixed point \(\varpi\) such that \(\overline{S} \setminus \{\varpi\} \cong \text{int}(S)\) and \(\overline{\varphi}_c|_{\overline{S}\setminus\{\varpi\}}\) is isotopic to \(\varphi_c|_{\text{int}(S)}\). In order to get \(\varphi_{\text{sm}}\) one smooths all the singular fixed points of \(\overline{\varphi}_c\) to obtain a symplectomorphism \(\overline{\varphi}_{\text{sm}} \sim \overline{\varphi}_c\): the smoothing is as above for all the singular fixed points different from \(\varpi\), while the latter is smoothed in a slightly different way that gives:

1. one elliptic fixed point \(\varpi_e\) if \(\varpi\) was rotated;
2. one elliptic fixed point \(\varpi_e\) plus \(p\) positive hyperbolic fixed points if \(\varpi\) was unrotated with \(p\) prongs.
In either case one removes a $\mathcal{N}_{x}$-invariant open neighborhood $\mathcal{N}(x)$ of $x$ (not containing other fixed points) and obtains the special symplectic representative $\varphi_{sm} := \varphi_{sm}|_{\mathcal{N}(x)}$ that we were looking for. It is again important to remark that all the fixed points in both (i') and (ii') are in the same Nielsen class of $\varphi$, which in turn is different from the Nielsen class of any other fixed point of $\varphi_{sm}$.

Now, by the separation of the Nielsen classes of the fixed points of $\varphi$, the construction above implies that two fixed points $x$ and $y$ of $\varphi_{sm}$ are in the same Nielsen class if and only if they come from the perturbation of type (ii) (or (ii')) of one unrotated singularity (or, respectively, puncture): in this case $x$ and $y$ are both positive hyperbolic, so that $\varepsilon(x) = \varepsilon(y) = -1$. Then (I.2) implies that the differential of $CF(\varphi_{sm})$ is 0, so that

$$\dim (HF(\varphi)) = \dim (HF(\varphi_{sm})) = \#\text{Fix}(\varphi_{sm})$$

and

$$\mathcal{F}_{min}(\varphi) = \text{Fix}(\varphi_{sm}) = \sum_{\eta} \left| \sum_{x \in \text{Fix}(\varphi_{sm}) \cap \gamma_{\eta} x} \varepsilon(x) \right| = \sum_{\eta} |\text{ind}(\eta)|$$

giving Equation (5.3).

**Remark 5.5.** For simplicity, in the rest of the paper we will talk about special symplectic representatives also referring to periodic diffeomorphisms by setting simply $\varphi_{sm} := \varphi$. This makes sense since by Gautschi $\varphi$ is symplectic, monotone and, if $\varphi \neq id$, also non-degenerate.

**Remark 5.6.** As said before, $\varphi$ uniquely defines a canonical rotation angle $\vartheta$ for $\partial S$. On the other hand we do not have a similar well defined rotation angle $\vartheta_{sm}$ for $\varphi_{sm}$, since it depends on the choice of a Hamiltonian isotopy near $\partial S$. As remarked by Cotton-Clay in Section 4.2 of [15], if $\vartheta_{c} \neq 0$ (which occurs either when $\varphi_{c} \neq id$ is periodic or when $\varphi_{c}$ is pseudo-Anosov with a rotated puncture) then a natural choice is $\vartheta_{sm} = \vartheta_{c}$. On the other hand if $\vartheta_{c} = 0$, in order to ensure non-degeneracy we have to choose a (positive or negative) small non-trivial rotation angle $\vartheta_{sm}$ and consequently perturb $\varphi_{c}$ to get a special representative $\varphi_{sm}$ that satisfies $\varphi_{sm}|_{\partial S} = \vartheta_{sm}$. We then follow Cotton-Clay’s conventions and choose $-1 \ll \vartheta_{sm} < 0$ if $t_{c}(\varphi) > 0$ and $0 < \vartheta_{sm} \ll 1$ if $t_{c}(\varphi) \leq 0$. As we will see later, this choice minimizes the number of fixed points of $\varphi_{sm}$ near the boundary. Observe that, once these conventions fixed, we do not need to specify a $\pm$–version for $HF(\varphi_{sm})$.

Now we briefly recall what happens when $\varphi_{c}$ is reducible. In this case there exists a $\varphi_{c}$–invariant collection $C$ of essential circles such that $S \setminus C = \{A_{1}, \ldots, A_{l}\}$ where, if $k_{i}$ is the smallest integer for which $\varphi_{c}^{k_{i}}$ maps $A_{i}$ to itself, then $\varphi_{c}^{k_{i}}|_{A_{i}}$ is either periodic or pseudo-Anosov. We will refer to $\{A_{1}, \ldots, A_{l}\}$ as to the canonical decomposition of $S$ induced by $\varphi_{c}$, which is well defined only up to isotopy. Cotton-Clay defines then component-wise the special symplectic representative $\varphi_{sm}$ of $\varphi$, in a way that, if $k_{i} = 1$, then $\varphi_{sm}|_{A_{i}}$ coincides with the special representative $(\varphi_{c}|_{A_{i}})_{sm}$ of the corresponding type (periodic or pseudo-Anosov) as defined above. In particular $\vartheta_{sm}$ depends only on the component of $S \setminus C$ containing $\partial S$.

**Remark 5.7.** The reason for which the argument to show (5.2) can not be directly used when $\varphi_{c}$ is reducible is that in this case a non-degenerate symplectic representative
of \([\varphi_c]\) is not necessarily weakly monotone: in particular we can not a priori ensure that there exists a representative \(\varphi_{\min}\) realizing \(F_{\min}(\varphi)\) for which \(HF(\varphi_{\min})\) is well defined.

**Definition 5.8.** Given a surface diffeomorphism \(\varphi: S \to S\), a subset \(F \subset S\) is called \(\varphi\)-invariant if \(\varphi(F) \subset F\). Two \(\varphi\)-invariant sets \(F_0\) and \(F_1\) are \(\varphi\)-related if there exists a path \(c\): \(([0,1],0,1) \to (S,F_0,F_1)\) such that \(\varphi \circ c \simeq c\) through maps \(([0,1],0,1) \to (S,F_0,F_1)\).

Observe that two fixed points of \(\varphi\) are in the same Nielsen class if and only if they are \(\varphi\)-related.

**Lemma 5.9.** Let \(x \in \text{Fix}(\varphi_{\sm})\) and \(A \subset S\) be the component of the canonical decomposition of \(S\) induced by \(\varphi_c\) that contains \(\partial S\). Assume that \(k_{\varphi} = 0\). Then \(x\) is \(\varphi_{\sm}\)-related to \(\partial S\) if and only if either:

1. \(\varphi_c|_A = \text{id}\) and \(x\) is in the same Nielsen class of \(\text{Fix}(\varphi_{\sm}|_A)\) or
2. \(\varphi_c|_A\) is pseudo-Anosov and \(x\) is one of the positive hyperbolic fixed points of case (ii').

**Proof.** This comes directly from Cotton-Clay’s adaptation to \(\varphi_{\sm}\) of the work of Jiang and Guo [29] about \(\varphi_c\). We prove the lemma for \(\varphi_c\) reducible: the irreducible case is a direct consequence.

Let \(A' \neq A\) be some component of the canonical decomposition of \(S\) induced by \(\varphi_c\) having has some boundary component \(C\) in common with \(A\). The discussion in Section 4.3 of [15] implies that a fixed point of \(\varphi_{\sm}|_{A'}\) is \(\varphi_{\sm}\)-related to some point of \(A\) if and only if \(\varphi_c|_{A'} = \text{id}\), \(\varphi_c|_{A'}\) is pseudo-Anosov, the boundary component \(C\) of \(A'\) comes from an unrotated puncture and \(x\) is one of the hyperbolic fixed points in (ii'). This is in particular the only case in which a point \(x \in \text{Fix}(\varphi_{\sm}|_{S\setminus A})\) can be \(\varphi_{\sm}\)-related to \(\partial S\). On the other hand the condition \(\varphi_c|_A = \text{id}\) implies that \(x\) is \(\varphi_{\sm}\)-related to the whole \(A\), which in turn is obviously \(\varphi_{\sm}\)-related to \(\partial S\). This gives the first case.

Assume now that \(x \in A\) and \(\varphi_c|_A \neq \text{id}\). If \(\varphi_c|_A\) is periodic then Lemma 1.2 of [29] implies that no fixed point in \(\text{int}(A)\) is \(\varphi_{\sm}\)-related to \(\partial S\). If \(\varphi_c|_A\) is pseudo-Anosov and \(\partial S\) comes from a rotated puncture then it is easy to see that the hyperbolic fixed points of case (ii') are \(\varphi_{\sm}\)-related to \(\partial S\): on the other hand Lemma 2.2 of [29] implies that these are the only fixed points for which the statement can be true (the reader should be aware of the fact that in [29] the hyperbolic fixed points of case (ii') are assumed to belong to \(\partial S\)).

The main step to prove Theorem 5.2 is the following result.

**Proposition 5.10.** Let \((K,S,\varphi)\) be an open book of genus \(g > 0\). Let \(A \subset S\) be the component of the canonical decomposition of \(S\) induced by \(\varphi_c\) that contains \(\partial S\). Then:

\[
\dim \left( HF^\pm(\varphi) \right) = \begin{cases} 
\dim \left( HF(\varphi) \right) - 1 & \text{if } A = S, \ \varphi_c = \text{id} \text{ and } t_c(\varphi) = 0; \\
\dim \left( HF(\varphi) \right) + 1 & \text{otherwise},
\end{cases}
\]

where the \(\pm\)-version of \(HF(\varphi)\) is the one specified by \(\varphi_{\sm}\) as in Remark 5.6.

**Proof.** Observe first Equation (5.2) implies that in general

\[
\dim(\text{HF}^\pm(\varphi)) = \dim(\text{HF}(\varphi)) \pm 1
\]
where the contribution $\pm 1$ is given by the generator $p_h \in \partial S$. To prove the result we need to study the interaction under the Floer differential between $p_h$ and the other generators of $CF^2(\varphi)$.

Fix a neighborhood $N \cong (0, 2] \times S^1$ of $\partial S = \{2\} \times S^1$ on which we fix the usual coordinates $(y, \vartheta)$. Set $N_0 := \{y \in (0, 1]\}$ and $N_1 := \{y \in (1, 2]\}$. We can then assume that

$$\varphi = \tau_\varphi \circ \varphi_{sm}$$

where:

- the only fixed points of $\varphi_{sm}\vert_N$ are the (possible) hyperbolic fixed points obtained as in the smoothing of type (ii') above, in which case they are all contained in $\text{int}(N_0)$ (observe that these fixed points can occur only if $\vartheta_c = 0$);

- $\varphi_{sm}(y, \vartheta) = (y, \vartheta + \vartheta_{sm})$ for every $y \in [0, 1]$ where, according to Remark 5.6, we set:
  
  (C.1) if $\vartheta_c \neq 0$ then $\vartheta_{sm} = \vartheta_c$;
  
  (C.2) if $\vartheta_c = 0$ and $t_c(\varphi) > 0$ then $-1 \ll \vartheta_{sm} < 0$;
  
  (C.3) if $\vartheta_c = 0$ and $t_c(\varphi) \leq 0$ then $0 < \vartheta_{sm} \ll 1$;

- $\tau_\varphi$ is a symplectomorphism with support in $N_1$ that interpolates between $\varphi_{sm}\vert_{\partial S \mid N_1}$ and $\id_{\partial S}$ and realizes $t_c(\varphi)$: more precisely, $\tau_\varphi\vert_{N_1}$ is of the form $\tau_\varphi(y, \vartheta) = (y, \vartheta - g(\vartheta))$

for some $g \in C^\infty([1, 2])$ such that $g(1) = g'(1) = 0$ and

$$g(2) = 2\pi t_c(\varphi) + (\vartheta_{sm} - \vartheta_c) = 2\pi k_\varphi + \vartheta_{sm}$$

where $k_\varphi$ is as in Equation (5.1);

- in order to minimize the number of Dehn twists in $N_1$ and, at the same time, make $\partial T_\varphi$ into a negative Morse–Bott torus, we require that:

  (a) if $t_c(\varphi) \geq 0$ then $g$ is monotone increasing;

  (b) if $t_c(\varphi) < 0$ there exists $0 < \nu \ll 1$ such that $g$ is monotone decreasing in $(1, 2-\nu)$ and monotone increasing in $(2-\nu, 2]$; $g(2-\nu) = 2\pi k_\varphi$ and $g(y) \leq 2\pi k_\varphi + \vartheta_{sm}$ for all $y < 2-\nu$ (observe that if $t_c(\varphi) < 0$ then $k_\varphi \leq 0$ and $\vartheta_{sm} \geq 0$, so that the function $\vartheta \mapsto \vartheta_{sm} - g(\vartheta)$ has a maximum in $2-\nu$ of value $\vartheta_{sm} + 2\pi |k_\varphi|$).

As explained in Subsection 2.4, the result $\hat{\varphi}$ of a Morse–Bott perturbation of $\varphi$ near $\partial S$ gives two fixed points $p_c$ and $p_h$ that are in the same Nielsen class of $\varphi\vert_{\partial S} = \id$ (here we identify the Nielsen classes of $T_\varphi$ and $T_{\hat{\varphi}}$ via the obvious diffeomorphism). By the property (I.2) above, since $\varepsilon(p_h) = -1$, to prove the proposition it is enough to show that $\partial S$ is not $\varphi$–related to any other $x \in \text{Fix}(\varphi)$ with $\varepsilon(x) = +1$. We will do that distinguishing the different cases given by (C.1)–(C.3) and (a) and (b).

(C.1-a): The set of fixed points of $\varphi\vert_N$ can be decomposed into a collection of boundary parallel circles

$$\text{Fix}(\varphi\vert_N) = \partial S \sqcup_{k_\varphi} C_i$$

where $C_i := \{y = y_i\}$ with $y_i \in (1, 2)$ such that $g(y_i) = \vartheta_{sm} + 2i\pi$. Assume that there exists $x \in \text{Fix}(\varphi\vert_{\text{int}(S)})$ which is $\varphi$–related to $\partial S$ via a path $c: ([0, 1], x, \partial S) \to (S, x, \partial S)$.

If $k_\varphi = 0$ then $\text{Fix}(\varphi\vert_N) = \partial S$, so that $x \in S \setminus N$. Since $\varphi\vert_N \sim \id$, the restriction of $c$ to $S \setminus \text{int}(N)$ would give a $\varphi_{sm}$–relation between $x$ and $\partial(S \setminus \text{int}(N))$, which would contradict Lemma 5.9 (the condition $\vartheta_c \neq 0$ implies that $\vartheta_c\vert_A \neq \id$).
If $k_{\varphi} > 0$, the argument to compute the contribution to $\dim(HF(\varphi))$ of each $C_i$ is again due to Gautschi and Cotton-Clay and works essentially as follows. Lemma 4.8 and Corollary 4.9 of [15] imply that each circle of the decomposition above has an its own Nielsen class in $T_{\varphi}$ (in [15] these components are called of type Ib). A Morse–Bott perturbation near $C_t$ gives two new generators of $CF(\varphi)$ that, by what we just said, are in their own Nielsen class and, by Morse–Bott theory, there exist exactly two index 1 holomorphic cylinders between these two fixed points. It follows that their contribution increase by 2 the total dimensions of $HF(\varphi)$ and, by definition, also of $HF^2(\varphi)$. Summing up over $i$ and adding the contribution of $p_h$ (which also has an its own Nielsen class) we get the result.

(C.1-b): The set of fixed points of $\varphi|_N$ can again be decomposed into a collection of boundary parallel circles

$$\text{Fix}(\varphi|_N) = \partial S \sqcup \bigcup_{i=1}^{k_{\varphi}} C_i$$

where $C_i = \{ y = y_i \}$ with $y_i \in \{1, 2\}$ such that $g(y_i) = \vartheta_{sm} - 2i\pi$. Observe that, if $\nu$ is as in the description of the case (b), then $y_{k_{\varphi}} < 2 - \nu$.

As in the previous case, each $C_i$ has an its own Nielsen class for $i = 1, \ldots, k_{\varphi} - 1$ and each of these circles contributes by 2 to both $\dim(HF(\varphi))$ and $\dim(HF^2(\varphi))$. We need then to show that the contribution of $C_{k_{\varphi}} \sqcup \partial S$ to $\dim(HF^2(\varphi))$ is +1. First we observe that $C_{k_{\varphi}}$ is $\varphi$-related to $\partial S$ via a path of the form \{$(y_{k_{\varphi}} + t(2 - y_{k_{\varphi}}), \vartheta_0)| t \in [0, 1]$\} for some $\vartheta_0$. A Morse–Bott perturbation of $C_{k_{\varphi}}$ gives two fixed points $p_h'$ and $p_e'$ that are hyperbolic and, respectively, elliptic. The situation here is the same as the one considered in Colin–Ghiggini–Honda [9, §8-9] and Wendl [64, §4.2]: their computations directly apply here and we get that $\partial p_e' = p_h$ and $\partial p_h' = 0$ (here we use the fact that $p_e$ is not a generator of $CF^2(\varphi)$). Since $p_h'$ is not $\varphi$-related to other fixed points it contributes by +1 to $\dim(HF^2(\varphi))$.

(C.2): The conditions $\vartheta_e = 0$ and $t_e(\varphi) > 0$ imply $k_{\varphi} = 1$. Now we can decompose:

$$\text{Fix}(\varphi|_N) = P \sqcup \partial S \sqcup \bigcup_{i=1}^{k_{\varphi} - 1} C_i$$

where $P = \{ x_1, \ldots, x_p \}$ is the set of (possible) positive hyperbolic fixed points given by the perturbation of type (ii) above (here $p$ is the number of prongs of the pre-smoothed puncture) and $C_i = \{ y = y_i \}$ is such that $g(y_i) = \vartheta_{sm} + 2i\pi$.

Again each circle of the decomposition has an its own Nielsen class and the proof of the result works as above.

(C.3-a): The conditions $\vartheta_e = 0$ and $t_e(\varphi) = 0$ imply $k_{\varphi} = 0$, so that

$$\text{Fix}(\varphi|_N) = P \sqcup \partial S$$

where $P$ is as above. We are then in the situation of Lemma 5.9, which gives only two cases in which $\partial S$ can be $\varphi$-related to other fixed points.

The first possibility is when $\varphi|_A = id$, so that $P = \emptyset$. For this case we reason as in Lemma 4.14 of [15] and we assume that $\varphi|_A$ is the time 1 Hamiltonian flow of a small Morse–Smale function on $A$. If $A = S$ we can choose a Morse–Smale function with only one maximum and no minima: this situation has been treated in Section 4.5.1 and gives the special case of the statement. If $A \neq S$ we can choose a Morse–Smale function with no maxima or minima, so that the
fixed points of $\varphi_{sm}|_A$ are all positive hyperbolic. On the other hand, by Lemma 5.9, the only fixed points of $\varphi_c$ in the same Nielsen class of $\partial S$ are the fixed points in the interior of $A$ and the (possible) positive hyperbolic fixed points of case (ii') coming from a pseudo-Anosov component $A' \neq A$ that abuts $A$. In any case all these fixed points are positive hyperbolic and by (I.2) they can not interact with $p_h$ under the Floer differential.

The second possibility given by Lemma 5.9 is that $\varphi|_A$ is pseudo-Anosov, in which case $\partial S$ is in the same Nielsen class of the points of $P$ (if $P \neq \emptyset$). Again these points are all positive hyperbolic and can not interact with $p_h$ under the Floer differential.

**(C.3-b):** The conditions $\vartheta_c = 0$ and $t_c(\varphi) < 0$ imply $k \varphi \precsim 0$, so that

$$\text{Fix}(\varphi|_N) = P \sqcup \partial S \sqcup \bigcup_{i=1}^{k \varphi} C_i$$

where $C_i = \{ y = y_i \}$ is such that $g(y_i) = \vartheta_{sm} - 2i\pi$. Again $P$ and each $C_i$ have a their own Nielsen class and the proof of the result goes as in case (C.1-b).

\[\Box\]

**Proof of Theorem 5.2.** We show that the result holds for $HF^\sharp(\varphi)$: Theorem 1.1 and the fact that $\widehat{HF}(Y, K, -g + 1) \cong \widehat{HF}(\overline{Y}, \overline{K}, -g + 1)$ will imply the theorem. By last proposition and Lemma 5.3, it is enough to check that $p_h$ and the boundary parallel Dehn twists equally contribute to $F_{\min}(\varphi)$ and $\dim(\widehat{HF}^\sharp(\varphi))$. This can be easily done in each of the different cases considered in the proof of Proposition 5.10, using Lemma 5.4 and the fact that if a circle of fixed points has an its own Nielsen class then the Poincaré–Birkhoff fixed point theorem implies that it contributes by $+2$ to $F_{\min}(\varphi)$.

As explained in Remark 5.7, we can not directly generalize the argument above to reducible case. Still, in some case the last theorem can be generalized to reducible maps.

**Theorem 5.11.** Let $Y$ be a rational homology sphere and $(K, S, \varphi)$ an open book decomposition of $Y$ with $g(S) \geq 1$. Then:

$$\dim \left( \widehat{HF}(Y, K; -g + 1) \right) = F_{\min}(\varphi) + 1.$$

**Proof.** If $Y$ is a homology 3–sphere any symplectic and non-degenerate representative of $[\varphi]$ is also monotone: this follows directly either from the definition of monotonicity given in [59] or from Lemma 3.2 of [16]. In particular a symplectomorphism $\varphi_{min}$ realizing $F_{\min}(\varphi)$ is monotone and, by definition, Equation (5.4) holds also in this case.

On the other hand Proposition 4.9 of [16] implies that there exists a representative $\varphi_{sm}$ of $[\varphi]$ such that $\#\text{Fix}(\varphi_{sm})$ equals the rank of a twisted version $HF(\varphi_{sm}; Q(\mathbb{Z}/(2\mathbb{Z})[M]))$ of the symplectic homology of $\varphi_{sm}$. Here the twisted coefficients are taken over the field of fractions of the group ring $\mathbb{Z}/(2\mathbb{Z})[M]$ where $M$ is a given subgroup of $\ker(\varphi_* - id) \subset H_1(T_{\varphi}, \mathbb{R})$ (see [16, §3.1] for the details). If $Y$ is a homology sphere then $\ker(\varphi_* - id) = \{0\}$, $HF(\varphi_{sm}; Q(\mathbb{Z}/(2\mathbb{Z})[M]))$ reduces to the standard $HF(\varphi_{sm})$ and Equations (5.5) and (5.2) also hold.

The proof of the theorem is then analogue to that of Theorem 5.2. Observe that we do not need to care about the special case of Theorem 5.2 because the condition
\[ \ker(\varphi_* - id) = \{0\} \] implies that if \( A \subset S \) is a non-contractible component on which the canonical representative of \([\varphi]\) is the identity map then \( A \) is a boundary-parallel annulus, on which the monodromy can be perturbed to give only a circle of fixed points that can be treated as in the proof of the aforementioned theorem. \( \square \)

A direct consequence of Proposition 5.10 and the last two theorems is the following result.

**Corollary 5.12.** If \((K, S, \varphi)\) is an open book decomposition of a 3–manifold \(Y\) with \(g(S) > 0\) then

\[
\dim \left( \widehat{HFK}(Y, K; -g + 1) \right) \geq 1.
\]

Moreover, if \([\varphi]\) is irreducible or \(Y\) is a rational homology sphere then the equality holds if and only if \([\varphi]\) admits a symplectic non-degenerate representative with no fixed points in \(\text{int}(S)\).

We remark that Equation (5.7) has recently been proved by Baldwin and Shea Vela-Vick in [4].

Last corollary has an interesting consequence about the topology of \(L\)–space knots. We recall that an \(L\)–space is a rational homology sphere \(Y\) such that the rank of \(\widehat{HF}(Y)\) coincides with the number of elements of \(H_1(Y, \mathbb{Z})\). An \(L\)–space knot is a knot in \(S^3\) that admits a non-trivial surgery to an \(L\)–space.

In [54], Ozsváth and Szabó proved that if \(K\) is an \(L\)–space knot then

\[
\dim \left( \widehat{HFK}(S^3, K, i) \right) \leq 1
\]

for every \(i\). In particular if \(K\) is an \(L\)–space knot the inequality (5.7) is sharp and the following holds.

**Corollary 5.13.** Let \(K \subset S^3\) be an \(L\)–space knot whose complement fibers with fiber \(S\) and monodromy \(\varphi\). Then \([\varphi]\) admits a symplectic non-degenerate representative with no fixed points.

Theorem 1.1 has also interesting applications in the study of the geometric and topological informations that one can extract from the family of knot Floer homologies \(\widehat{HFK}(Y^n(K), K^n)\), where \(K^n\) is the branched locus of the \(n\)-th branched cover \(Y^n(K)\) of \(Y\) over \(K\). If \((K, S, \varphi)\) is an open book decomposition of \(Y\) then \((K^n, S, \varphi^n)\) is an open book decomposition of \(Y^n(K)\), so that, by Theorem 1.1, we have

\[
\widehat{HFK}(Y^n(K), K^n; -g + 1) \cong HF^\sharp(\varphi^n).
\]

We remark that, while Heegaard Floer homology of double branched covers has been studied in different situations (like in Manolescu–Owens [45] and Ozsváth–Szabó [53]), the case of higher branched covers appear less often in the literature.

From Theorem 1.1 of Fel’shtyn [21], we can recover the following result, already proven by Lipshitz, Ozsváth and Thurston using bordered Floer homology techniques (cf. Corollary 4.2 and Proposition 3.18 of [39] and Theorem 14 of [38]).
Corollary 5.14. Given a collection of vector spaces \( \{A_n\}_{n=1}^{\infty} \), define the growth rate of (the dimensions of) \( A_n \) by

\[ \mathcal{GR}(A_n) := \limsup_{n \to \infty} \left( \frac{\dim(A_n)}{n} \right)^{\frac{1}{n}}. \]

If \( (K, S, \varphi) \) is an open book decomposition of \( Y \), then

\[ \mathcal{GR} \left( \widehat{HF}(Y^n(K), K^n; -g + 1) \right) = \lambda_{[\varphi]} \]

where \( \lambda_{[\varphi]} \) is the largest dilatation factor among all the pseudo–Anosov components of the canonical representative of the mapping class \( [\varphi] \).

By classical results about 3–dimensional geometry and topology (see for example [20]) we get the following nice relation between Heegaard Floer homology and 3–dimensional geometry (cf. Cotton-Clay [15, Corollary 1.7]).

Corollary 5.15. If \( K \subset Y \) is a fibered knot then \( \dim \left( \widehat{HF}(K^n, Y^n(K); -g + 1) \right) \) grows exponentially if and only the JSJ decomposition of \( Y \setminus K \) has some hyperbolic component.

Another interesting consequence of Theorem 1.1 and Proposition 5.10 is about algebraic knots and comes from an analogous result of McLean about fixed point Floer homology (see [46, Corollary 1.4]).

Corollary 5.16. Let \( K \subset S^3 \) be the 1–component link of an isolated singularity of an irreducible complex polynomial \( f \) with two variables. Then

\[ \min \left\{ n > 0 \mid \dim \left( \widehat{HF}(Y^n(K), K^n; -g + 1) \right) \neq 1 \right\} = m(f) \]

where \( m(f) \) is the multiplicity of \( f \).

REFERENCES

[1] N. A’Campo: La fonction zêta d’une monodromie; Comment. Math. Helv. 50 (1975), 233–248.
[2] C. Abbas: An introduction to compactness results in symplectic field theory; Springer, Heidelberg, 2014.
[3] V. I. Arnold: Some remarks on symplectic monodromy of Milnor fibrations; The Floer Memorial Volume, Progress in Mathematics, vol 133 (1995), Birkhäuser Basel, 99–104.
[4] J. A. Baldwin, D. Shea Vela-Vick: A note on the knot Floer homology of fibered knots; preprint, arXiv:1801.06563v1.
[5] J. Baldwin, D.S. Vela-Vick, V. Vértesi: On the equivalence of Legendrian and Transverse Invariants in Knot Floer Homology; Geom. Topol. 17 (2013), 905–924.
[6] J. S. Birman, M. E. Kidwell: Fixed points of pseudo-Anosov diffeomorphisms of surfaces; Adv. Math. 46 (1982), 217–220.
[7] F. Bourgeois: A Morse-Bott approach to contact homology; Sympl. and Cont. Top.: Interactions and Perspectives, Fields Inst. Comm. 35, AMS, Providence, RI (2003), 55–77.
[8] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, E. Zehnder: Compactness results in symplectic field theory, Geom. Topol. 7 (2003), 799–888.
[9] V. Colin, P. Ghiggini, K. Honda: Embedded contact homology and open book decompositions; arXiv:1008.2734v5, 2010.
[10] V. Colin, P. Ghiggini, K. Honda: Equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions; PNAS 108(20):8100–8105, 2011.
[11] V. Colin, P. Ghiggini, K. Honda: The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions I; arXiv:1208.1074v1, 2012.
[12] V. Colin, P. Ghiggini, K. Honda: The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions II; arXiv:1208.1077, 2012.
[13] V. Colin, P. Ghiggini, K. Honda: The equivalence of Heegaard Floer homology and embedded contact homology III: from hat to plus; arXiv:1208.1526, 2012.
[14] V. Colin, P. Ghiggini, K. Honda, M. Hutchings: Sutures and contact homology I; Geom. Topol. 15(3) (2011), 1749–1842.
[15] A. Cotton-Clay: Symplectic Floer homology of area-preserving surface diffeomorphisms; Geom. Topol. 13 (2009), 2619–2674.
[16] A. Cotton-Clay: A sharp bound on fixed points of surface symplectomorphisms in each mapping class; preprint (2010), arXiv:1009.0760.
[17] D. Dragnev: Fredholm theory and transversality for noncompact pseudoholomorphic maps in symplectizations; Comm. Pure Appl. Math. 57 (2004), no. 6, 726-763.
[18] S. Dostoglou, D. A. Salamon: Self-Dual Instantons and Holomorphic Curves; Ann. of Math. 139 (1994), no. 3, 581–640.
[19] B. Farb, D. Margalit: A primer on mapping class groups; Princeton Mathematical Series, PMS49, 2011.
[20] A. Fathi, F. Laudenbach, V. Poénaru: Travaux de Thurston sur les Surfaces; Astérisque 66 (1979), Soc. Math. France.
[21] A. Fel’shtyn: The growth rate of symplectic Floer homology; J. Fixed Point Theory Appl. 12 (2012), 93–119.
[22] A. Floer: Symplectic Fixed Points and Holomorphic Spheres; Comm. Math. Phys. 120 (1989), no. 4, 575–611.
[23] R. Gautschi: Floer homology of algebraically finite mapping classes; J. Symp. Geom. 1(4) (2002), 715–765.
[24] P. Ghiggini: Knot Floer homology detects genus-one fibred knots; Amer. J. Math. 130 (2008), no. 5, 1151–1169.
[25] K. Honda, W. H. Kazez, G. Matić: On the contact class in Heegard Floer homology; J. Diff. Geom. 83 (2009), 289–311.
[26] M. Hutchings: The embedded contact homology index revisited; New perspectives and challenges in symplectic field theory, 263–297, CRM Proc. Lecture Notes, 49, AMS, 2009.
[27] M. Hutchings: An index inequality for embedded pseudoholomorphic curves in symplectizations; J. Eur. Math. Soc. 4 (2002), no. 4, 313–361.
[28] M. Hutchings, M. Sullivan: The periodic Floer homology of a Dehn twist; Algebr. Geom. Topol. 5 (2005), 301–354.
[29] B. Jiang, J. Guo: Fixed points of surface diffeomorphisms; Pac. J. Math. 160(1) (1993), 67–89.
[30] A. Juhász: Floer homology and surface decompositions; Geom. Topol 12(1) (2008), 299–350
[31] A. Juhász: Holomorphic discs and sutured manifolds; Algebr. Geom. Topol. 6 (2006), 1429–1457.
[32] A. Juhász: Cobordisms of sutured manifolds and the functoriality of link Floer homology; Adv. Math. 299 (2016), 940–1038.
[33] A. Kotelskiy: Comparing homological invariants for mapping classes of surfaces; preprint, arXiv:1702.04071.
[34] Y. Lekili: Heegaard Floer homology of broken fibrations over the circle; Adv. Math. 244 (2013), 268–302.
[35] F. Lin: The surgery exact triangle in Pin(2)-monopole Floer homology; arXiv:1504.01993v2 (2016).
[36] R. Lipshitz: A cylindrical reformulation of Heegaard Floer homology; Geom. Topol. 10 (2006), 955–109.
[37] R. Lipshitz: Heegaard Floer Homologies: lecture notes; arxiv.org/abs/1411.4540 (2014).
[38] R. Lipshitz, P. Ozsváth, D. Thurston: Bimodules in bordered Heegaard Floer homology; Geom. Topol. 19 (2015), 525–724; arXiv:1003.0598v4.
[39] R. Lipshitz, P. Ozsváth, D. Thurston: A faithful linear-categorical action of the mapping class group of a surface with boundary; J. Eur. Math. Soc. 15 (2013), 1279–1307; arXiv.org:1012.1032v2.
[40] G. Liu and G. Tian: On the equivalence of multiplicative structures in Floer homology and quantum homology; Acta Math. Sin. 15 (1999), 53–80.
[41] Y. Ni: Knot Floer homology detects fibred knots; Invent. Math. 170 (2007), no. 3, 577–608.
[42] Y. Ni: *Link Floer homology detects the Thurston norm*; Geom. Topol. 13 (2009), no. 5, 2991–3019.
[43] Y. Ni: *A note on knot Floer homology and fixed points of monodromy*, preprint (2021), arXiv:2106.03884.
[44] Y. Ni: *Knot Floer homology and fixed points*, preprint (2022), arXiv:2201.10546.
[45] C. Manolescu, B. Owens: *A Concordance Invariant from the Floer Homology of Double Branched Covers*; International Mathematics Research Notices (2007), rnm077.
[46] M. McLean: *Floer Cohomology, Multiplicity and the Log Canonical Threshold*; preprint (2016), arXiv:1608.07541v1.
[47] P. Ozsváth, Z. Szabó: *Holomorphic disks and topological invariants for closed three–manifolds*; Ann. of Math. 159(2) (2004), 1027–1158.
[48] P. Ozsváth, Z. Szabó: *Holomorphic disks and three-manifolds invariants: Properties and applications*; Ann. of Math. 159(2) (2004), 1159–1245.
[49] P. Ozsváth, Z. Szabó: *Holomorphic disks and knot invariants*; Adv. Math. 186 (2004), 58–116.
[50] P. Ozsváth, Z. Szabó: *Holomorphic disks and genus bounds*; Geom. Topol. 8 (2004), 311–334.
[51] P. Ozsváth, Z. Szabó: *Heegaard Floer homologies and contact structures*; Duke Math. J. 129 (2005), 39–61.
[52] P. Ozsváth, Z. Szabó: *Holomorphic triangles and invariants for smooth four-manifolds*; Adv. Math. 202 (2006), no. 2, 326–400.
[53] P. Ozsváth, Z. Szabó: *On the Heegaard Floer homology of branched double-covers*; Adv. Math. 194 (2005), no. 1, 1–33.
[54] P. Ozsváth, Z. Szabó: *On knot Floer homology and lens space surgeries*; Topology 44 (2005), no. 6, 1281–1300.
[55] J. Rasmussen: *Floer homology and knot complements*; Ph.D. Thesis; 2003; arXiv:math.GT/0306378.
[56] S. Sarkar and J. Wang: *An algorithm for computing some Heegaard Floer homologies*; Ann. of Math. (2) 171 (2010), no. 2, 1213–1236.
[57] P Seidel: *Lectures on four-dimensional Dehn twists*, Symplectic 4-manifolds and algebraic surfaces, Lecture Notes in Math., Springer, Berlin, 2008.
[58] P. Seidel: *Fukaya categories and Picard-Lefschetz theory*, Zurich Lect. in Adv. Math., European Math. Soc., Zürich, 2008.
[59] P. Seidel: *Symplectic Floer homology and the mapping class group*; Pac. J. Math. 206 (2002), no. 1, 219–229.
[60] P. Seidel: *Vanishing cycles and mutation*; European Congress of Mathematics (Barcelona), Birkhäuser (2002), 65–85.
[61] P. Seidel: *More about vanishing cycles and mutation*; Symplectic Geometry and Mirror Symmetry, World Scientific (2001), 429–465.
[62] P. Seidel: *A long exact sequence for symplectic Floer cohomology*; Topology 42 (2003), 1003–1063.
[63] W. P. Thurston: *On the geometry and dynamics of diffeomorphisms of surfaces*; Bull. Amer. Math. Soc. 19 (1988), no. 2, 417–431.
[64] C. Wendel: *Finite energy foliations on overtwisted contact manifolds*; Geom. Top. 12 (2008), no. 105, 531–616.

Laboratoire de Mathématiques Jean Leray, CNRS and Université de Nantes
Email address: paolo.ghiggini@univ-nantes.fr

Email address: gilbertospano.math@gmail.com