Nature vs. Nurture in Complex and Not-So-Complex Systems

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Abstract. Understanding the dynamical behavior of many-particle systems both in and out of equilibrium is a central issue in both statistical mechanics and complex systems theory. One question involves “nature vs. nurture”: given a system with a random initial state evolving through a well-defined stochastic dynamics, how much of the information contained in the state at future times depends on the initial condition (“nature”) and how much on the dynamical realization (“nurture”)? We discuss this question and present both old and new results for low-dimensional Ising spin systems.

Keywords: heritability, persistence, aging, damage spreading, Ising spin dynamics

1 Introduction

The nonequilibrium dynamics of both thermodynamic and complex systems (the intersection of these two sets is nonempty) remains an area of intensive research, and a host of open problems remains. The most extreme case of nonequilibrium dynamics occurs after a deep quench, in which a system in equilibrium at a very high temperature is instantaneously cooled to a very low temperature, after which it evolves according to a well-defined dynamics corresponding to that low temperature. The extreme case of a deep quench is the instantaneous cooling of a system from infinite to zero temperature. The subsequent zero-temperature dynamics consists of the system’s running “downhill” in energy (or uphill in survival probability, if one is dealing with a biological system) to some local or global minimum (or maximum).

Determining the state of such a system at long times, given both the initial state and the subsequent dynamics, is a difficult — and generally unsolved — problem, even for relatively simple systems. In this paper, we will review progress on this question for Ising spin systems, both homogeneous and disordered, in one and two dimensions. We will see that even in 2D the problem is far from simple, with open questions remaining even for — in fact, especially for — the uniform ferromagnet. However, recent progress has been made, and the insights gained

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may be useful in understanding dynamical properties of more interesting — and possibly complex — systems.

## 2 Types of Long-Time Behavior

For concreteness we consider an Ising spin system on the infinite lattice $\mathbb{Z}^d$; that is, at each site $x \in \mathbb{Z}^d$ we assign a binary variable $\sigma_x = \pm 1$. We restrict our attention to models in which the spin-spin couplings are nearest-neighbor. The most basic question one might ask is whether, after a deep quench, the dynamics eventually settles down to a fixed state, or whether some or all spins continue to flip forever.

The notion of equilibration of an infinite system after a finite time contains some subtleties, which we will address in the next section. But without addressing these subtleties, we can pose the question in a precise way: does the spin configuration have a limit as $t \to \infty$? Equivalently, for every $x$, does $\sigma_x$ flip infinitely often or only finitely many times? (Note that even for those systems in which the latter is true, it will generally not be the case that there exists some finite time $T_0$ after which every spin has stopped flipping. This is discussed further in the next section.) From this perspective, it is useful to distinguish among three classes of dynamical system: a system is type $F$ if every spin flips only finitely many times; type $I$ if every spin flips infinitely often; and type $M$ (for “mixed”) if some spins flip infinitely often and others do not [1]. The overall spin configuration has a limit only of the system is type $F$.

Determining which class a system belongs to is generally a nontrivial problem; in fact, the answer remains unknown even for uniform ferromagnets (and antiferromagnets) in $\mathbb{Z}^d$ for $d \geq 3$. There does exist some numerical work, however, suggesting that these might be type $I$ for $d = 3$ and 4, but type $F$ — or possibly $M$ — for $d \geq 5$ [2].

However, some progress has been made. There exist proofs that uniform ferromagnets or antiferromagnets in one dimension (on $\mathbb{Z}$) and in two dimensions (on $\mathbb{Z}^2$) are type $I$. Moreover, in any dimension on a lattice where each site has an odd number of nearest neighbors, they are type $F$ [3]. Work has also been done on two-dimensional “slabs”: that is, systems that are infinite in two dimensions but consist of a finite number of layers in the third. Here the system can be either type $F$ or $M$, depending both on the number of layers and on the boundary conditions (free or periodic) in the third (finite) direction. For details, see [4].

It was further proved in [3] that all models with continuous disorder, in which the spin-spin couplings are chosen from a common distribution with finite mean, belong to class $F$ in all dimensions and on all types of lattice. These include ordinary Edwards-Anderson spin glasses [5] and random ferromagnets. We ignore here systems with continuous disorder in which the distribution has infinite mean, and refer the interested reader to [16].

Another class of systems comprises the so-called $\pm J$ spin glass models, where each coupling independently takes on the value $+J$ or $-J$ with equal probability.
It was shown that in one dimension these are type $\mathcal{I}$, and in two dimensions (again on $\mathbb{Z}^2$) type $\mathcal{M}$ \cite{ref1}. And once again, on any lattice regardless of dimension where each site has an odd number of neighbors, they are type $\mathcal{F}$.

Results exist also for systems with more exotic coupling distributions; we refer the interested reader to \cite{ref1}. We now turn to the next question, which is our main interest here: what can be learned about the state of a system at a finite time $t$ after a deep quench. The answer, not surprisingly, depends on which class the system belongs to, but as we shall see, in most cases one is forced to undertake numerical simulations to gain insight.

### 3 Local Equilibration, Local Non-Equilibration, and Chaotic Size Dependence

How might one think about equilibration in an infinite system, even one of type-$\mathcal{F}$, given that at any finite time some spins still have not reached their final state? It was proposed in \cite{ref2} that this problem could be understood in the sense of local equilibration: choose a region of fixed size surrounding the origin, and ask whether, after a finite time, domain walls cease to sweep across the region, overturning the spins within. This timescale $\tau(L)$ is expected to increase without bound as $L$ goes to infinity (and in general will also depend on the choice of initial condition, dynamics, lattice type and dimensionality, and possibly other factors); but the idea is that as long as $\tau(L) < \infty$ for any $L < \infty$, no matter how large, then we can say that the system undergoes local equilibration. Any system of type-$\mathcal{F}$ obviously undergoes local equilibration. Types $\mathcal{I}$ and $\mathcal{M}$ do not, and we say that these systems experience local nonequilibration (LNE) \cite{ref2}.

LNE can be of two types. Even though the configuration in a given finite region never settles down, one can still ask whether, if one averages over all dynamical realizations, the dynamically averaged configuration settle down to a limit at large times. Or does even this averaged configuration not settle down?

The first possibility (a limit of the dynamically averaged configuration) can be thought of as “weak LNE”, while the second (no limit) is referred to as chaotic time dependence (CTD) \cite{ref2}. As shown in \cite{ref2}, weak LNE implies a complete lack of predictability (nurture “wins” — after some time, the dynamics wipes out information about the initial state), while CTD implies that some amount (which can be quantified) of predictability remains (nature wins).

So a study of nature vs. nurture provides a great deal of information on a number of central dynamical issues concerning classes of dynamical systems. We now review both older and more recent results for different Ising-like spin systems, both homogeneous and disordered.

### 4 Nature vs. Nurture in 1D Random Ferromagnets and Spin Glasses

Because type-$\mathcal{F}$ models always equilibrate locally, one can simply compare the final state of a spin with its initial state over many dynamical trials to deter-
mine whether initial information has been fully retained, partially retained, or completely lost. This can be quantified by introducing a type of dynamical order parameter, denoted $q_D$, that in some ways serves an analogue of the (equilibrium) Edwards-Anderson order parameter $q_{EA}$.

Let $\sigma^t$ denote the (infinite-volume) spin configuration at time $t$ given a specific initial configuration $\sigma^0$ and dynamical realization $\omega$ (for notational convenience, the dependence of $\sigma^t$ on $\sigma^0$ and $\omega$ is suppressed). We want to study, for fixed $\sigma^0$ (and, if the model is disordered, fixed coupling realization $J$), this quantity averaged over all dynamical realizations up to time $t$; denote such an average by $\langle \cdot \rangle_t$. One then needs to study the resulting quantity averaged over all initial configurations and coupling realizations. Denoting the latter averages (with respect to the joint distribution $P_{J,\sigma^0} = P_J \times P_{\sigma^0}$) by $E_{J,\sigma^0}$, we define

$$q_D = \lim_{t \to \infty} q_t = \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \langle (\langle \sigma_x \rangle_t)^2 \rangle = E_{J,\sigma^0} (\langle (\sigma_x)^2 \rangle_t)$$

(1)

and $\Lambda_L$ is a $d$-dimensional cube of side $L$ centered at the origin. The equivalence of the two formulas for $q_t$ follows from translation-ergodicity.

The order parameter $q_D$ measures the extent to which $\sigma^\infty$ is determined by $\sigma^0$ rather than by $\omega$. It was proved in [3] that for the 1D random ferromagnet and/or spin glass with continuous disorder, $q_D = 1/2$. What this means is that, for a.e. $J$ and $\sigma^0$, precisely half of the $x$’s in $\mathbb{Z}$ have $\sigma^\infty_x$ completely determined by $\sigma^0$ with the other $\sigma^\infty_x$’s completely undetermined by $\sigma^0$.

We turn now to the more difficult case of type-I systems. It seems somewhat counterintuitive that models with continuous disorder, in particular random ferromagnets and spin glasses, whose equilibrium thermodynamics are much more difficult to ascertain than those of uniform ferromagnets, are (at least in some cases) easier to analyze in the context of nature vs. nurture.

### 5 Persistence and Heritability in Low-Dimensional Uniform Ferromagnets

The nature vs. nurture question is intimately related to older notions of persistence [9], defined as the fraction of spins that are unchanged from their initial values at time $t$. This was found to decay as a power law in a number of systems, in particular uniform ferromagnets and Potts models in low dimensions, and the associated decay exponent $\theta_p$ is known as the “persistence exponent”.

In a similar manner, one can define a “heritability exponent” [10] as follows: prepare two Ising systems with the same initial configuration but then allow them to evolve independently using zero-temperature Glauber dynamics. The spin overlap between these “twin” copies, with the same initial condition but two different dynamical realizations, was found (after averaging over many trials and different initial conditions) to decay as a power law in time [10]. This spin overlap, which we refer to as the “heritability”, is essentially the same as
The exponent $\theta_h$ associated with the power-law decay of heritability is the “heritability exponent”.

Heritability defined in this way is in some sense the opposite of “damage spreading” [11,12,13]: the latter involves starting with two slightly different initial configurations and letting them evolve with the same dynamical realization. The extent of the spread of the initial difference throughout the system is then measured.

The persistence and heritability exponents can be computed exactly in the 1D uniform Ising ferromagnet. It was shown in [14,15] that $\theta_p = 3/8$ for this system. On the other hand, it can be shown that $\theta_h = 1/2$, as discussed in [10], by using the mapping to the voter model and coalescing random walks (see, e.g., [15,16]).

While the persistence and heritability exponents differ in one dimension, they may be identical in the 2D uniform ferromagnet, where numerical simulations yield $\theta_p = 0.21 \pm 0.02$ [217] and $\theta_h = 0.22 \pm 0.02$ [10]. Whether the two exponents are exactly the same, or simply close but not identical, remains to be understood.

6 Positive temperature

Does the preceding discussion have anything to say about what happens at nonzero temperature? Here one needs to study the behavior of positive temperature Gibbs states and the local order parameter, rather than that of single spin configurations. Construction of the appropriate dynamical measures, analysis of their evolution, and relation to pure state structure are extensively discussed in [8]. Here we mention only a few relevant results.

The categorization into types $I$, $F$, and $M$ is specifically tailored to zero temperature and needs to be modified at positive temperature. In the latter case, one can still define local equilibration, in the sense that, on any finite lengthscale, the system equilibrates into a pure state after a finite time (depending on all of the usual culprits), in the sense that interfaces cease to move across the region after that time. If finite regions exist without a corresponding finite equilibration timescale, then LNE occurs.

A main result of [8] is relevant to spin glasses in particular: if only a single pair, or countably many pairs (including a countable infinity) of pure states exists (with fixed $J$), and these all have nonzero Edwards-Anderson (EA) order parameter [5], then LNE occurs. A corollary is that if LNE does not occur, and the limiting pure states have nonzero EA order parameter, then there must exist an uncountable infinity of pure states, with almost every pair having overlap zero.

One consequence of these results is that LNE occurs at positive temperature (with $T < T_c$) in the 2D uniform ferromagnet and (presumably) random Ising ferromagnets for $d < 5$. Because the number and structure of pure states at positive temperature in Ising spin glasses is unknown for $d \geq 3$ (and, from a rigorous point of view, unproved even for $d = 2$), occurrence of LNE there remains an open question.
7 Open problems

The behavior of homogeneous and disordered Ising spin systems in one and two dimensions is now relatively well understood. Beyond that, however, most questions remain open. Do uniform ferromagnets belong to class $F$, $I$, or $M$ in dimension three and higher? If $F$, what is the value of $q_D$? If not, is weak LNE or CTD displayed, and what is the value of the heritability exponent?

The relationships among heritability, persistence, and damage spreading form an interesting set of open problems as well. Are the heritability and persistence exponents the same in the 2D ferromagnet on a square lattice, and if so, why? What about higher dimensions and other models? It would be interesting to study these relations in two and higher dimensions and work out the connections between these different but related quantities.

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References

1. Newman, C.M., Stein, D.L.: Zero-Temperature Dynamics of Ising Spin Systems Following a Deep Quench: Results and Open Problems. Physica A 279, 159–168 (2000).
2. Stauffer, D.: Ising Spinodal Decomposition at $T = 0$ in One to Five Dimensions. J. Phys. A 27, 5029–6032 (1994).
3. Nanda, S., Newman, C.M., Stein, D.L.: Dynamics of Ising Spin Systems at Zero Temperature. In: Minlos, R., Shlosman, S., Suhov, Y. (eds.) On Dobrushin's Way (from Probability Theory to Statistical Physics). Amer. Math. Soc. Transl. vol. 198 pp. 183–194 (2000).
4. Damron, M., Kogan, H., Newman, C.M., Sidoravicius, V.: Coarsening in 2D Slabs. http://arxiv.org/abs/1303.2605
5. Edwards, S. and Anderson, P.W.: Theory of Spin Glasses. J. Phys. F 5, 965–974 (1975).
6. Nanda, S., Newman, C.M.: Random Nearest Neighbor and Influence Graphs on $\mathbb{Z}^d$. Random Struct. Alg. 15, 262–278 (1999).
7. Gandolfi, A., Newman, C.M., Stein, D.L.: Zero-Temperature Dynamics of $\pm J$ Spin Glasses and Related Models. Commun. Math. Phys. 214, 373–387 (2000).
8. Newman, C.M., Stein, D.L.: Equilibrium Pure States and Nonequilibrium Chaos. J. Stat. Phys. 94, 709–722 (1999).
9. Derrida, B., Bray, A.J., Godreche, C.: Nontrivial Exponents in the Zero Temperature Dynamics of the 1D Ising and Potts Models. J. Phys. A, Math. Gen. 27, L357–L361 (1994).
10. Ye, J., Machta, J., Newman, C.M., Stein, D.L.: Nature vs. Nurture: Predictability in Zero-Temperature Ising Dynamics. Submitted to Phys. Rev. Lett.
11. Creutz, M.: Deterministic Ising Dynamics. Ann. Phys. 167, 62–72 (1986).
12. Stanley, H.E., Stauffer, D., Kertesz, J., Hermann, H.: Dynamics of Spreading Phenomena in Two-Dimensional Ising Models. Phys. Rev. Lett. 59, 2326–2328 (1987).
13. Grassberger, P.: Damage Spreading and Critical Exponents for Model A Ising Dynamics. Physica A 214,547–559 (1995).
14. Derrida, B., Hakim, V., Pasquier, V.: Exact First-Passage Exponents of 1D Domain Growth: Relation to a Reaction-Diffusion Model. Phys. Rev. Lett. 75, 751–754 (1995).
15. Derrida, B., Hakim, V., Pasquier, V.: Exact Exponent for the Number of Persistent Spins in the Zero-Temperature Dynamics of the One-Dimensional Potts Model. J. Stat. Phys. 85, 763–797 (1996).
16. Fontes, L.R., Isopi, M., Newman, C.M., Stein, D.L.: Aging in 1D Discrete Spin Models and Equivalent Systems. Phys. Rev. Lett. 87, 110201-1–110201-4 (2001).
17. Jain, S.: Zero-Temperature Dynamics of the Weakly Disordered Ising Model. Phys. Rev. E 59, R2493–R2495 (1999).