Disentangled Topological Numbers by a Purification of Entangled Mixed States for Non-Interacting Fermion Systems

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We argue that the entanglement Chern number proposed recently is invariant under the adiabatic deformation of a gapped many-body groundstate into a disentangled/purified one, which implies a partition of the Chern number into subsystems (disentangled Chern number). We generalize the idea to another topological number, the $\mathbb{Z}_2$ Berry phase for a system with particle-hole symmetry, and apply it to a groundstate in a weak topological phase where the Chern number vanishes but the groundstate nevertheless has edge states. This entanglement Berry phase is especially useful for characterizing random systems with nontrivial edge states.

A quantum many-body groundstate can be regarded as a mixed state if a system is divided into several pieces and some of them are traced out. This enables us to define the entanglement entropy and spectrum (Hamiltonian)\textsuperscript{14,15}. These are widely accepted as new tools to characterize quantum many-body states. Recently, the entanglement spectrum has been successfully applied to topological insulators through the study of edge states along the fictitious boundaries between a partition of a system\textsuperscript{16–18}. This may be reflected by the surprising universality of the bulk-edge correspondence\textsuperscript{18}.

Instead of a partition with definite boundaries, an extensive partition has been introduced: it has been argued that the entanglement spectrum of a bulk subsystem can be gapless, at which a topological phase transition occurs\textsuperscript{15,16}. Not only the spectrum of the entanglement Hamiltonian but also the corresponding eigenstates are useful to characterize the topological phases of the original model. We have shown that, taking an extensive but asymmetric partition, if the entanglement Hamiltonian is still gapped, we can define the entanglement Chern number\textsuperscript{17}. The entanglement spectrum and entropy of the bipartition reflect how the states in subsystems are entangled in the groundstate wavefunction. On the other hand, the entanglement topological numbers clarify the topological properties of the groundstate of a fermion system, and let $\lambda_s,\lambda_x$ be a partition of the total system $A + \bar{A}$. The various kinds of partitioning for a single pure bulk groundstate reveal its topological properties in different environments.

In this paper, we argue that the entanglement Chern number and its generalization to other topological numbers are invariant under the adiabatic deformation to make the subsystems disentangled. In other words, the original entangled state is eventually modified to the single tensor product of the two subsystems. In this sense, the entanglement topological numbers may be called topological numbers for a disentangled groundstate or simply disentangled topological numbers attached to subsystems.

This also has the meaning of a partition of the topological numbers. The entanglement entropy and spectrum of the bipartition reflect how the states in subsystems are entangled in the groundstate wavefunction. On the other hand, the entanglement topological numbers clarify the property that remains if the entanglement is eliminated or disentangled. This process of disentanglement may be considered as a purification of the mixed state. After describing the general idea and the validity of the entanglement Chern number, we discuss the entanglement Berry phases applied to the weak topological (WT) phase\textsuperscript{19,20} which is topologically nontrivial in spite of a vanishing Chern number.

Schmidt decomposition: Let $|G\rangle$ be a many-body groundstate of a fermion system, and let $A$ and $\bar{A}$ be a partition of the total system $A + \bar{A}$. Then, $|G\rangle$ can be Schmidt-decomposed into

$$|G\rangle = \sum_{s,x} D_{sx} |\Phi_s \rangle \otimes |\Phi_x \rangle,$$

where $|\Phi_s \rangle$ and $|\Phi_x \rangle$ are, respectively, orthonormal basis states for $A$ and $\bar{A}$. The normalization of $|G\rangle$ requires $\langle G | G \rangle = \sum_{s,x} D_{sx} D_{sx}^* = \text{tr} DD^\dagger = 1$. The singular value decomposition for $D$, $D = U \Lambda V^\dagger$, where $U_{st}$ and $V_{xt}$ are unitary matrices and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m, 0, \ldots, 0)$ with $\lambda_t > 0$, leads to

$$|G\rangle = \sum_{t} \lambda_t |\Phi_t \rangle \otimes |\Phi_t \rangle,$$

where $|\Phi_t \rangle = \sum_s |\Phi_s \rangle U_{st}$ and $|\Phi_t \rangle = \sum_x |\Phi_x \rangle V_{xt}^*$. The number $m$ of nonzero eigenvalues is called the Schmidt number. The normalization condition for $|G\rangle$ is $\sum_{t} \lambda_t^2 = 1$.

Reduced density matrix: The density matrix of the pure state $|G\rangle$ is $\rho^{\text{tot}} = |G\rangle \langle G|$. Tracing out $\bar{A}$ and $A$, we have the reduced density matrix $\rho \equiv \text{tr}_{\bar{A}} \rho^{\text{tot}}$ in subsystem $A$ and its complementary density matrix $\bar{\rho} \equiv \text{tr}_A \rho^{\text{tot}}$ in subsystem $\bar{A}$. The reduced density matrix is an ensemble averaged over the eigenstates of the reduced density matrix $\rho$ and can be written as $\rho = \sum_{s} \rho_s |\psi_s \rangle \langle \psi_s|$, where $|\psi_s \rangle$ is a pure state and $\rho_s = |\langle \psi_s | \psi_s \rangle|$. The reduced density matrix $\rho$ satisfies the following properties:

- **Positivity:** $\rho_{ij} \geq 0$ for all $i, j$.
- **Normalization:** $\text{tr} \rho = 1$.
- **Trace invariance:** $\text{tr} \rho = \text{tr} \bar{\rho}$.
- **Entropy:** $S(\rho) = -\text{tr} \rho \log \rho$.

These properties are essential for characterizing the topological phases of the system. The reduced density matrix $\rho$ can be further decomposed into a disentangled density matrix $\rho^{\text{tot}} = \sum_{s} \rho_s |\psi_s \rangle \langle \psi_s|$, where $|\psi_s \rangle$ is a pure state and $\rho_s = |\langle \psi_s | \psi_s \rangle|$. The disentangled density matrix $\rho^{\text{tot}}$ satisfies the following properties:

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subsystem $\tilde{A}$ such that

$$\rho = \sum_{s,t} |\Phi_s\rangle\langle D D^\dagger s| \langle \Phi_t| = \sum_\ell |\Phi_\ell\rangle \lambda^2_\ell \langle \Phi_\ell|,$$

$$\tilde{\rho} = \sum_{x,y} |\bar{\Phi}_x\rangle\langle (D D^\dagger)^\ast x| \langle \bar{\Phi}_y| = \sum_\ell |\bar{\Phi}_\ell\rangle \lambda^2_\ell \langle \bar{\Phi}_\ell|.$$  \hspace{1cm} (3)

The same $\lambda^2_\ell$ appear in these equations because $D D^\dagger$ and $D D^\dagger$ have the same eigenvalues except for the zero eigenvalue.

**Non-interacting fermion system:** Let $H$ be a Hamiltonian defined on a lattice by

$$H = \sum_{i,j} c_i^\dagger h_{ij}^{\text{tot}} c_j,$$  \hspace{1cm} (4)

where $i,j$ denote some internal degrees of freedom as well as the sites. Let $\Psi_{jn}$ be the $j$th component of the $n$th eigenstate of the Hamiltonian $h^{\text{tot}}$, $\sum_j h_{ij}^{\text{tot}} \Psi_{jn} = \sum_m \Psi_{im} E_{mn}$, where $E$ is the energy eigenvalues $E = \text{diag}(\epsilon_1, \epsilon_2, \cdots)$. The orthogonality and completeness of the eigenstates are expressed by $\sum_j \Psi_{jn}^\dagger \Psi_{jn} = \delta_{mn}$, $\sum_m \Psi_{jn}^\dagger \Psi_{jn} = \delta_j$, or simply $\Psi^\dagger \Psi = \Psi^\dagger = \mathbb{I}$, if $\Psi$ is regarded as a matrix. Let us define the normal mode operator $a_n = \sum_j (\Psi^\dagger)^j n c_j = \sum_j c_j \Psi^\dagger_j$. Then, the Hamiltonian is diagonal, $H = \sum_n c_n d_n^\dagger d_n$, and the groundstate is given by $|G\rangle = \prod_{n \leq n_F} d_n^\dagger |0\rangle$, where $n_F$ is a state index below which all the states are occupied.

Let us discuss $\rho$ and $\tilde{\rho}$ in Eq. (3) for a non-interacting fermion system. To this end, assume that all the sites and/or internal degrees of freedom are divided into two subsystems $A$ and $\tilde{A}$, which have $N_A$ and $N_{\tilde{A}}$ dimensions, respectively. They are denoted by $a, b \in A$ and $\tilde{a}, \tilde{b} \in \tilde{A}$ with $a, b = 1, 2, \cdots, N_A$ and $\tilde{a}, \tilde{b} = 1, 2, \cdots, N_{\tilde{A}}$. Define

$$\rho = e^{-H}/Z, \quad \tilde{\rho} = e^{-\tilde{H}}/\tilde{Z},$$  \hspace{1cm} (5)

where $Z = \text{tr}_A e^{-H}$ and $\tilde{Z} = \text{tr}_{\tilde{A}} e^{-\tilde{H}}$. For the time being, we restrict our discussion to $\rho$. Since the entanglement Hamiltonian for a non-interacting fermion system is also non-interacting, we set $H = \sum_{a,b} c^\dagger_a h_{ab} c_b$, where $a, b \in A$. Let us diagonalize the Hamiltonian $h$ as $\sum_{a,b} h_{ab} b_n = \sum_m \psi_{mn} e_{mn}$, where $E = \text{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{N_A})$. Then, introducing the normal mode operator $f_n^\dagger = \sum_a (\psi^\dagger)^a n c_a = \sum_a c_a \psi^\dagger_n$, we have $H = \sum_n \epsilon_n f_n^\dagger f_n$. $\rho$ is now written as

$$\rho = \frac{e^{-\sum_n \epsilon_n f_n^\dagger f_n}}{\prod_n (1 + e^{-\epsilon_n})} = \prod_{n=1}^N \left[ \frac{1}{1 + \xi_n} \right],$$  \hspace{1cm} (6)

where $|1_n\rangle$ and $|0_n\rangle$ are, respectively, the occupied and vacant states of the $n$th fermion defined by $f_n^\dagger|0_n\rangle = 0$ and $|1_n\rangle = f_n^\dagger|0_n\rangle$, and $\xi_n$ is the Fermi distribution function

$$\xi_n = \frac{1}{e^{\epsilon_n} + 1}.$$  \hspace{1cm} (7)

$\xi$, as well as $\epsilon$, is often called the entanglement spectrum for convenience. To rewrite $\rho$ in Eq. (3) in the form of Eq. (4), let us define the many-fermion state in the occupation number representation $|\ell\rangle = |\ell_1, \cdots, \ell_n, \cdots, \ell_{N_A}\rangle$ with the occupation number of the $n$th fermion $l_n = 0, 1$ and

$$\lambda^2_\ell = \prod_{n=1}^{N_A} (1 - \xi_n)^{-l_n} \xi_n^{l_n}.$$  \hspace{1cm} (8)

Then, $\rho$ in Eq. (3) is now expressed as $\rho = \sum_\ell |\ell\rangle \lambda^2_\ell |\ell\rangle$.

**Correlation matrix:** The two-point correlation matrix is useful as an alternative to the entanglement Hamiltonian. Noting the relation $c_a^\dagger c_j = \sum_{m,n} \Psi^\ast_{in} d_m^\dagger \Psi_{jn} a_m$, we see that the one-particle correlation function is given by

$$C_{ij} = \langle G | c_i^\dagger c_j | G \rangle = P_{ji},$$  \hspace{1cm} (9)

where $P_{ji} = \sum_{n \leq n_F} \Psi_{jn}^\dagger \Psi_{in} = \sum_{n \leq n_F} \Psi_{jn}^\dagger (\Psi^\dagger)^n |0\rangle$ is the projection operator to the groundstate. We can now define the correlation matrices in subsystems $A$ and $\tilde{A}$ as follows by simply restricting the sites and/or internal degrees of freedom in $A$ or $\tilde{A}$:

$$C_{ab} = C_{ba} = P_{ba}, \quad \bar{C}_{\bar{a}b} = C_{\bar{a}b} = P_{\bar{b}a}.$$  \hspace{1cm} (10)

Alternatively, noting the relationship $c_{\bar{a}}^\dagger c_b = \sum_{n,m} \psi_{n_m}^\ast \psi_{b_m} f_n^\dagger f_m$, we obtain

$$C_{ab} = \frac{\langle G | c^\dagger_{\bar{a}} c_b | G \rangle}{\text{tr} |G\rangle} = \frac{\text{tr} |G\rangle}{\text{tr} |G\rangle} c_{\bar{a}}^\dagger c_b = \sum_{n,m} \psi_n^\ast \psi_m (\text{tr}_A \rho f_n^\dagger f_m) \equiv \langle \Xi \Xi^\dagger \rangle_{ba},$$  \hspace{1cm} (11)

where $\Xi$ is the diagonal matrix $\Xi = \text{diag}(\xi_1, \xi_2, \cdots, \xi_{N_A})$. Thus, the $\xi$ are the eigenvalues of $C_{\bar{a}}$.

The complementary reduced density matrix in a fermionic representation and the correlation matrix are calculated similarly. Solving the one-particle eigenvalue equation $\sum_{\bar{a},b} h_{\bar{a}b} \psi_{\bar{a}b} = \sum_m \psi_{m\bar{a}} \epsilon_{mn}$, where $\tilde{E} = \text{diag}(\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_{N_{\tilde{A}}})$, the entanglement Hamiltonian $H = \sum_{\bar{a},b} c_{\bar{a}}^\dagger h_{\bar{a}b} c_b$ can be expressed in terms of the normal mode operator $\bar{f}_n = \sum_{\bar{a}} (\tilde{\psi})^\dagger_{n\bar{a}} c_{\bar{a}} = \sum_{\bar{a}} c_{\bar{a}} \tilde{\psi}_{n\bar{a}}$, we have $H = \sum_n \tilde{\epsilon}_n \bar{f}_n^\dagger \bar{f}_n$. $\rho$ is now written as

$$\rho = \frac{e^{-\sum_n \tilde{\epsilon}_n \bar{f}_n^\dagger \bar{f}_n}}{\prod_n (1 + e^{-\tilde{\epsilon}_n})} = \prod_{n=1}^N \left[ \frac{1}{1 + \tilde{\xi}_n} \right],$$  \hspace{1cm} (6)

where $|1_n\rangle$ and $|0_n\rangle$ are, respectively, the occupied and vacant states of the $n$th fermion defined by $\bar{f}_n^\dagger|0_n\rangle = 0$ and $|1_n\rangle = \bar{f}_n^\dagger|0_n\rangle$, and $\tilde{\xi}_n$ is the Fermi distribution function

$$\tilde{\xi}_n = \frac{1}{e^{\tilde{\epsilon}_n} + 1}.$$  \hspace{1cm} (7)
where the extra eigenvalues \( \xi_{N_A+1}, \ldots, \xi_{N_B} \) are restricted to 0 and 1. We conclude that \( \mathcal{C} \) for the larger subsystem \( \mathcal{A} \) has the same eigenvalues as \( \mathcal{C} \) for the smaller subsystem \( \mathcal{A} \) plus extra trivial eigenvalues of 0 or 1.

\[
\begin{bmatrix}
1 & 1 \\
\vline & \\
1 & 0
\end{bmatrix}
\]  
\[
\begin{bmatrix}
0 & \Delta \\
\vline & \\
0 & 0
\end{bmatrix}
\]

\[
|G\rangle = \sum \lambda_i |\Phi_i \rangle \otimes |\bar{\Phi}_i \rangle \\
|G\rangle = |\Phi_i \rangle \otimes |\bar{\Phi}_i \rangle
\]

**FIG. 1:** Schematic illustration of a disentanglement deformation. The spectra of \( \xi \) and \( \bar{\xi} \) are identical except for at 0 and 1 (\( \varepsilon = \infty \) and \(-\infty\), respectively), and hence the larger system inevitably has extra eigenvalues of 0 and 1. If subsystem \( \mathcal{A} \) includes generic eigenvalues \( 0 < \xi < 1 \), (i) the other one also includes the same generic eigenvalues, (ii) each reduced density matrix satisfies the grand canonical ensemble with a finite weight given by Eq. (1) and therefore, (iii) the groundstate \( |G\rangle \) is entangled in the sense that it is composed of multiple tensor products of the wavefunctions of subsystems \( \mathcal{A} \) and \( \mathcal{A} \). If we can deform the spectrum in the left panel into the spectrum in the right one (as if we could take the “zero temperature limit”), \( |G\rangle \) can be a single tensor product. This process can be considered as a purification of the mixed state to the pure state, which results in a disentanglement of the groundstate wavefunction.

Suppose that the spectrum of the entanglement Hamiltonian \( \xi \) has a gap at \( \xi = 1/2 \), as illustrated in Fig. 1 and that we calculate some topological numbers of the upper bands. In a generic spectrum, the groundstate is a linear combination of the tensor product in the Schmidt decomposition as in Eqs. (1) and (2). Then, suppose that we deform the spectrum adiabatically, making the gap larger, and that we eventually reach an extreme spectrum with \( \Delta = 1 \), i.e., all states have \( \xi = 1 \) or 0. In this case, the largest eigenvalue, \( \lambda_1 \), in the singular value decomposition \( \{\lambda_k\} \) becomes \( \lambda_1 = 1 \) and the other eigenvalues are 0, implying that the groundstate is a single tensor product. Therefore, this adiabatic process can be considered as a disentanglement deformation of the groundstate wave function between subsystems \( \mathcal{A} \) and \( \mathcal{A} \). From the point of view of the reduced density matrix, the process is considered as a purification of the mixed state to the pure state. On the other hand, topological numbers calculated using the eigenstates of \( \mathcal{C} \) and \( \mathcal{C} \) are expected to be invariant in this process since the gap between the upper and lower bands never closes. Therefore, such topological numbers, referred to as entanglement topological numbers, reveal the topological properties that are invariant even if the entanglement between \( \mathcal{A} \) and \( \mathcal{A} \) is eliminated. In this sense, they may alternatively be called topological numbers of a disentangled groundstate or simply disentangled topological numbers. If the groundstate can be represented by a single tensor product such that \( |G\rangle = |\Phi_1 \rangle \otimes |\bar{\Phi}_1 \rangle \), a topological number, such as the first Chern number or the Berry phase of \( |G\rangle \), is the sum of the topological numbers of \( |\Phi_1 \rangle \) and \( |\bar{\Phi}_1 \rangle \). This is indeed possible since \( \lambda_1 = 1 \) in Eq. (2). Otherwise, for generic nonintegral \( \lambda_i \), it may be difficult to define integral topological numbers simultaneously for \( |G\rangle \), \( |\Phi_1 \rangle \), and \( |\bar{\Phi}_1 \rangle \).

This also implies that a set of entanglement topological numbers for \( \mathcal{A} \) and \( \mathcal{A} \) may be referred to as a partition of a topological number, provided that the bulk gap of \( h^\text{tot} \) remains open in the disentanglement deformation. This can be checked by the natural sum rule that the topological number of the groundstate is the sum of the two entanglement topological numbers. Note that assuming a finite gap for the entanglement Hamiltonian is in contrast to the case with edge states for a bipartition, where gapless modes of the entanglement Hamiltonian mainly contribute to the entanglement entropy.

**Translational invariant system:** So far, we have used the subscripts \( i, j \) for some internal degrees of freedom as well as the sites. We next consider a system with translational invariance. To this end, we replace \( i, j \rightarrow i\alpha, j\beta \), where \( i, j \) and \( \alpha, \beta \) denote the sites and species, respectively. On the \( N^d \) lattice in \( d \) dimensions with the periodic boundary condition, the fermion operator is now denoted by \( c_{\alpha}(j) \) and its Fourier transformation is \( c_{\alpha}(j) = \frac{1}{\sqrt{N}} \sum_{k} e^{i j k} c_k(\alpha) \), where \( V = N^d \) and \( k_\mu = 2\pi/N \times \text{integer} \). For a translationally invariant system, the Hamiltonian given by Eq. (1) becomes

\[
h^\text{tot} \rightarrow h_{ij}^\text{tot} = h^\text{tot}_{i\alpha,j\beta} = h_{\alpha\beta}(i-j) \text{ and its Fourier transformation is given by } h^\text{tot}(i-j) = \frac{1}{V} \sum_{k} e^{i j k} h^\text{tot}(k)\text{.}
\]

Then, the total Hamiltonian is separated into \( k \) sectors, \( H = \sum_{k} \sum_{\alpha,\beta} c^\dagger_{\alpha,k}(j) h^\text{tot}_{\alpha\beta}(k) c_{\beta,k}(j) \). The Schrödinger equation for a given \( k \) is given by \( \sum_{\beta} h^\text{tot}_{\alpha\beta}(k) \Psi_{\beta,k}(\alpha) = \sum_{\alpha} \Psi_{\alpha,k}(\alpha) E_{\alpha}(k) \). We assume that the groundstate is insulating and that the fermions are occupied up to the \( n_F \)-th band, \( |G\rangle = \prod_{\alpha} \prod_k d^\dagger_{\alpha,k}(0) \), where the normal mode operators are defined by \( d_{\alpha,k}(j) = \sum_{\alpha} c_k(\alpha) \Psi_{\alpha,k}(\alpha) \).

The correlation matrix in Eq. (9) is then

\[
C_{\alpha\beta}(j, j') = \frac{1}{V} \sum_k e^{i j' k} P_{\beta\alpha}(k),
\]

where \( P_{\alpha\beta}(k) = \sum_{n \leq n_F} \Psi_{\beta,n}(\alpha) \Psi_{\alpha,n}^*(\alpha) \) is the projection operator to the groundstate at a fixed \( k \).

**Example 1: Entanglement Chern number:** A typical example of the entanglement Chern number is the entanglement spin Chern number\([22]\) for the Kane-Mele model\([23]\). The Hamiltonian \( h^\text{tot} \) is given by a \( 4 \times 4 \) matrix due to the spin and the bipartite lattice. The Rashba term mixes the spins, so that it is basically impossible to define the spin Chern number simply in the momentum space\([21,22]\). However, projecting the \( 4 \times 4 \) \( P_{\alpha\beta}(k) \) matrix in Eq. (13) into each spin sector \( \sigma = \uparrow, \downarrow \) such that \( P_{\alpha\beta} \rightarrow P_{\sigma} P_{\alpha\beta} P_{\sigma} \), where \( P_{\sigma} \) stands for the projection to spin \( \sigma \), we have successfully computed the set of entanglement Chern numbers \( (c_1, c_2) \), which indeed describes the spin Hall phase when \( (c_1, c_2) = (\pm 1, -1) \).\([22]\) Although
spin is not conserved in a topological insulator in general and the time-reversal symmetry guarantees the vanishing of the Chern number, disentanglement between the spins implies a nontrivial entanglement spin Chern number, which justifies the existence of nontrivial spin edge states characterizing the phase. The advantage of the topological characterization here is that the idea of the disentanglement/purification of the mixed state to the pure state is simply extended to correlated electrons with an interaction.

Example 2: Entanglement Berry phase: The Berry phase here means a winding number for a one-dimensional system. We apply it to a two-dimensional system, based on the method in Refs. to study the nontrivial edge states in a WT phase. Consider an $N \times N$ square lattice with the periodic boundary condition. Let $A$ be a subsystem with $n_A$ ladders. The remaining subsystem is denoted as $\tilde{A}$ and is composed of $n_{\tilde{A}} = N - n_A$ ladders. For the partitions shown in Figs. 2(a) and 2(b), we set $A = X$ and $A = Y$, respectively. Let us consider the case $A = X$. Since the translational invariance in the $y$ direction is broken, we regard $j_y$ as the species, and thus the correlation matrix is denoted by

$$C_{j_y \alpha, j_y' \beta}(J_x, J_x') = \frac{1}{N} \sum_{k_x} e^{i k_x (J_x' - J_x)} C_{j_y \alpha, j_y' \beta}(k_x),$$

(14)

where

$$C_{j_y \alpha, j_y' \beta}(k_x) = \frac{1}{N} \sum_{k_y} e^{i k_y (j_y' - j_y)} P_{\beta \alpha}(k),$$

(15)

When $j_y$ and $j_y'$ are restricted within $1 \leq j_y, j_y' \leq n_X$ ($n_X$), the above correlation matrix is $C$ ($\tilde{C}$). We assume that the eigenvalues $\xi_\alpha(k_x)$ of $C_{j_y \alpha, j_y' \beta}(k_x)$ have a spectral gap, as shown in Fig. 4. This is possible in general for an extremely asymmetric partition with $X \ll \tilde{X}$. Let $\psi_\alpha(k_x) = (\psi_{j_y \alpha, j_1}(k_x), \psi_{j_y \alpha, j_2}(k_x), \ldots)$ be the set of eigenstates with eigenvalues $\xi_\alpha(k_x) > 1/2$. Then, the entanglement Berry phase for $X$ is calculated by $\gamma_X = \text{Im} \log \prod_{k_x} U_x(k_x)$, where the U(1) link variable is defined as $U_x(k_x) = \text{det} \psi_\uparrow(k_x) \psi_\downarrow(k_x + \delta k_x)$ with $\delta k_x = \frac{2\pi}{N}$ a unit of the discrete momentum. Likewise, solving the eigenvalue equation for $\tilde{C}$ and/or choosing $R = Y$, we obtain the other entanglement Berry phases $\gamma_{\tilde{X}}$, $\gamma_Y$, $\gamma_{\tilde{Y}}$. The sets of $(\gamma_X, \gamma_{\tilde{X}})$ and $(\gamma_Y, \gamma_{\tilde{Y}})$ are considered as a real-space partition of the conventional Berry phases $\gamma_x(k_y)$ and $\gamma_{\tilde{y}}(k_x)$ as discussed below. While the latter are already partitioned into each $k_y$ and $k_x$ for the pure model, the former partition is advantageous when we study disordered systems.

In what follows, we consider an anomalous Hall effect model of the Wilson-Dirac type with an anisotropic Wilson term. The pure model is defined by

$$h^{\text{tot}}(k) = t \sigma_1 \sin k_x + t \sigma_2 \sin k_y + \sigma_3 \left[ m - b_x (1 - \cos k_x) - b_y (1 - \cos k_y) \right].$$

(16)

This model has particle-hole symmetry and its ground-state is characterized by the Chern number. In case of an anisotropic Wilson term, an interesting $c = 0$ phase appears, which has edge states. This phase has been referred to as the WT phase. In Figs. 3(a) and 3(b), we show the spectrum of the system on a cylinder belonging to the WT phase, in which edge states can be seen only in (a). The question is their stability: Unless there are specific reasons, such states are expected to be unstable against, for example, disorder or interactions.

For a model with particle-hole symmetry, the Berry phase can serve as a $Z_2$ topological invariant. Furthermore, with translational invariance, the conventional Berry phase (winding number along $k_x$) $\gamma_x(k_y)$ can be computed. Then, at a certain $k_y$ where the particle-hole symmetry is enhanced to chiral symmetry, $\gamma_x$ is quantized as 0 or $\pi$. The Berry phase $\gamma_x = \pi$ for a periodic system is a topological invariant for a zero-energy state localized at the end of a finite chain. This state forms, in turn, an edge state at the boundary parallel to the $y$-axis. Namely, from the $k_y$-resolved Berry phase $\gamma_x(k_y)$, we can predict the edge states at the boundary parallel to the $y$-axis. The Berry phase $\gamma_x(k_y)$ corresponding to Fig. 3(a) becomes $\pi$ at $k_y = 0$, $\pi$. However, if the system breaks translational symmetry, $\gamma_x(k_y)$ is no longer defined. This is a part of our motivation for proposing the entanglement Berry phase.

Let us start with a model with translational symmetry. In Figs. 3(c) and 3(d), we show the entanglement spectra for the symmetric partition $n_A = n_{\tilde{A}} = N/2$. It turns out that the edge states in the real space are well simulated by the entanglement spectrum in a cylindrical partition. To study the stability of these states in (a) and (c), we calculate the entanglement Berry phase for a minimum subsystem $X$ with $n_X = 1$ and its complement $\tilde{X}$ with $n_{\tilde{X}} = N - 1$, whose spectra are given in Figs. 3(c) and 3(f), respectively. These are indeed gapped, and the entanglement Berry phase is therefore well defined. We obtain $(\gamma_X, \gamma_{\tilde{X}}) = (\pi, \pi)$ numerically. This is in sharp contrast to the trivial $c = 0$ state of the model with $(\gamma_X, \gamma_{\tilde{X}}) = (0, 0)$ and to the $c = 1$ state with $(\gamma_X, \gamma_{\tilde{X}}) = (\pi, 0)$ or $(0, \pi)$.
Finally, we study the same model with impurities. We can define the Berry phase even in such a model by imposing the twisted boundary condition and by using the twist angle \((\phi_x, \phi_y)\) instead of the momentum \((k_x, k_y)\). We then obtain \(\gamma_x = 0 \mod 2\pi\) for \(\phi_y = 0\). This is expected because the edge states that cross the zero energy at \(k_y = 0\) and \(\pi\) in the pure model are no longer distinguished by \(k_y\), and hence \(\gamma_x = 0 = \pi + \pi \mod 2\pi\) is observed, even if the edge states remain. Figure 3(g) shows the entanglement spectrum for \(Y\) under the symmetric partition \(n_Y = N/2\). At \(\phi_y = 0\) we see that even with disorder, the four states seem degenerate near the zero energy, which may originate from the zero-energy edge states of the pure model. Here, the entanglement Berry phase for this state plays a crucial role when discussing the stability of the WT phase. Let us calculate the entanglement Berry phase for a minimum subsystem \(X\) with \(n_X = 1\) and its complement \(\bar{X}\) with \(n_{\bar{X}} = N - 1\). The spectrum of \(X\) is displayed in Fig. 3(h). Note that it is indeed gapped, and the computed Berry phase is \((\gamma_X, \gamma_{\bar{X}}) = (\pi, \pi)\) even with disorder. These entanglement Berry phases imply that if one divides the system into two pieces \(X\) and \(\bar{X}\) and regard them as two one-dimensional chains, each subsystem has each edge states. The natural sum rule \(0 = \gamma_x = \gamma_X + \gamma_{\bar{X}} = \pi + \pi \mod 2\pi\) indeed holds. In other words, a partitioning of the Berry phase enables us to observe the Berry phase \(\pi\).

To summarize, we have argued that the entanglement topological numbers are invariant under the disentanglement of an entangled groundstate and that they are topological numbers attached to disentangled subsystems. In this sense, the entanglement topological numbers serve as a partitioning of the topological numbers. We have introduced the entanglement Berry phase to show the stability of the edge states in the WT phase.

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