Shortest Paths of Bounded Curvature for the Dubins Interval Problem

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Abstract

The Dubins interval problem aims to find the shortest path of bounded curvature between two targets such that the departure angle from the first target and the arrival angle at the second target are constrained to two respective intervals. We propose a new and a simple algorithm to this problem based on the minimum principle of Pontryagin.

1. Introduction

Path planning problems involving Dubins vehicles have received significant attention in the literature due to their applications involving unmanned vehicles [1-6]. A Dubins vehicle [7] is a vehicle that travels at a constant speed and has a lower bound on the radius of curvature at any point along its path. The basic problem of finding a shortest path for a vehicle from a point at \((x_1, y_1)\) with heading \(\theta_1\) to a point at \((x_2, y_2)\) with heading \(\theta_2\) was solved by Dubins in [7], and later by authors in [8,9] using Pontryagin’s minimum principle [10]. This article considers a generalization of this standard problem called the Dubins Interval Problem and is stated as follows: Given two targets located at \((x_1, y_1)\) and \((x_2, y_2)\), respectively, on a plane, a closed interval \(\Theta_1\) of departure angles from target 1, and a closed interval \(\Theta_2\) of arrival angles at target 2, find a departure angle \(\theta_1 \in \Theta_1\), an arrival angle \(\theta_2 \in \Theta_2\) and a path from \((x_1, y_1, \theta_1)\) to \((x_2, y_2, \theta_2)\) such that the radius of curvature at any point in the path is lower bounded by \(\rho\) and the length of the path is a minimum (refer to Fig. 1).

The Dubins interval problem arises while lower bounding Traveling Salesman Problems (TSPs) involving Dubins vehicles [11]. In [11], the lower bounding problem was posed as a generalized TSP where the cost of traveling between any two nodes requires one to solve the Dubins interval problem. The Dubins interval problem was solved using calculus and some monotonicity properties of the optimal paths in [11]. In this article, we give a simple and a direct algorithm using Pontryagin’s minimum principle [10]. Similar to the solution to the standard Dubins problem, we characterize the optimal paths to our problem. Therefore, an optimal solution can be obtained by simply comparing the length of few candidate solutions. Apart from applications to vehicle routing problems with motion constraints, solutions to the Dubins interval problem may be of independent interest.

2. Notations

The interval \(\Theta_k\) at target \(k\) is defined as \(\Theta_k = [\theta_k^{\min}, \theta_k^{\max}] \subseteq [0, 2\pi]\) with \(\theta_k^{\min} < \theta_k^{\max}\) for \(k = 1, 2\). Given an initial configuration \((x_1, y_1, \theta_1)\) and a final configuration \((x_2, y_2, \theta_2)\), L.E. Dubins [7] showed that the

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shortest path for a vehicle to travel between the two configurations subject to the minimum turning radius ($\rho$) constraint must consist of at most three segments where each segment is a circle of radius $\rho$ or a straight line. In particular, if a curved segment of radius $\rho$ along which the vehicle travels in a counterclockwise (clockwise) rotational motion is denoted by $L(R)$, and the segment along which the vehicle travels straight is denoted by $S$, then the shortest path is one of $RSR$, $RSL$, $LSL$, $RLR$ and $LRL$ or a degenerate form of these paths. For example, the degenerate forms of $RSL$ are $S$, $L$, $R$, $RS$, $SL$ and $RL$. We also subscript a curved segment in some places ($L_\psi$ or $R_\psi$) to indicate the angle of turn ($\psi$) in the curved segment.

3. Main Result

**Theorem 1.** Any shortest path which is $C^1$ and piecewise $C^2$ of bounded curvature between the two targets with the departure angle $\theta_d \in \Theta_1 = [\theta_1^{\min}, \theta_1^{\max}]$ at target 1 and the arrival angle $\theta_a \in \Theta_2 = [\theta_2^{\min}, \theta_2^{\max}]$ at target 2 must be one of the following or a degenerate form of these:

**Case 1.** $S$ or $L_\psi$ or $R_\psi$ or $L_\psi R$ or $R_\psi L$ with $\psi > \pi$.

**Case 2.** Both departure and arrival angles belong to the boundaries of the intervals:

(a) $\theta_d = \theta_1^{\max}$ and $\theta_a = \theta_2^{\max}$ and the path is $LSR$.
(b) $\theta_d = \theta_1^{\max}$ and $\theta_a = \theta_2^{\min}$ and the path is either $LSL$ or $LR_\psi L$ with $\psi > \pi$.
(c) $\theta_d = \theta_1^{\min}$ and $\theta_a = \theta_2^{\min}$ and the path is $RSL$.
(d) $\theta_d = \theta_1^{\min}$ and $\theta_a = \theta_2^{\max}$ and the path is either $RSR$ or $RL_\psi R$ with $\psi > \pi$.

**Case 3.** The departure angle is either $\theta_1^{\max}$ or $\theta_1^{\min}$ and the arrival angle is an interior point in $[\theta_2^{\min}, \theta_2^{\max}]$:

(a) $\theta_d = \theta_1^{\max}$ and $\theta_2^{\min} < \theta_a < \theta_2^{\max}$ and the path is either $LS$ or $LR_\psi$ with $\psi > \pi$.
(b) $\theta_d = \theta_1^{\min}$ and $\theta_2^{\min} < \theta_a < \theta_2^{\max}$ and the path is either $RS$ or $RL_\psi$ with $\psi > \pi$.

**Case 4.** The departure angle is an interior point in $[\theta_1^{\min}, \theta_1^{\max}]$ and the arrival angle is either $\theta_2^{\max}$ or $\theta_2^{\min}$:
(a) $\theta_1^{\text{min}} < \theta_d < \theta_1^{\text{max}}$ and $\theta_a = \theta_2^{\text{max}}$ and the path is either SR or $L\psi R$ with $\psi > \pi$.

(b) $\theta_1^{\text{min}} < \theta_d < \theta_1^{\text{max}}$ and $\theta_a = \theta_2^{\text{min}}$ and the path is either SL or $R\psi L$ with $\psi > \pi$.

Remark: Note that in cases 1, 3 and 4, the departure and arrival angles are implicitly specified by each of the paths. For example, if $L\psi R$ with $\psi > \pi$ exists between the two targets, as the length of the segment $L$ is equal to the length of the segment $R$, the departure and arrival angles are simply specified by geometry (we will later discuss this in the proofs; refer to Fig. 2). Similarly, if $\theta_d = \theta_1^{\text{max}}$ (case 3(a)), the arrival angle at target 2 is determined by the $LS$ or $LR$ paths. The only remaining part would be to check if the arrival angle at target 2 lies in the interval $[\theta_2^{\text{min}}, \theta_2^{\text{max}}]$. If it does, then the corresponding path is a candidate for an optimal solution to the Dubins interval problem.

4. Proof

Let $v_o$ be the speed of the vehicle, and $u(t)$ denote the control input for the vehicle at time $t$. Let $x(t), y(t), \theta(t)$ denote the position and angle coordinates of the vehicle as a function of time on a plane. The Dubins interval problem can be formulated as an optimal control problem as follows:

$$\min_{u(t) \in [-1, 1]} \int_0^T 1 dt$$

subject to

$$\frac{dx}{dt} = v_o \cos \theta,$$
$$\frac{dy}{dt} = v_o \sin \theta,$$
$$\frac{d\theta}{dt} = \frac{u}{\rho},$$

and the following boundary conditions:

$$x(0) = x_1, x(T) = x_2,$$
$$y(0) = y_1, y(T) = y_2,$$

$$\theta_1^{\text{min}} - \theta(0) \leq 0,$$
$$\theta(0) - \theta_1^{\text{max}} \leq 0,$$
$$\theta_2^{\text{min}} - \theta(T) \leq 0,$$
$$\theta(T) - \theta_2^{\text{max}} \leq 0.$$

Let the adjoint variables associated with $p(t) = (x(t), y(t), \theta(t))$ be denoted as $\Lambda(t) = (\lambda_x(t), \lambda_y(t), \lambda_\theta(t))$. The Hamiltonian associated with above system is defined as:

$$H(\Lambda, p, u) = 1 + v_o \cos \theta \lambda_x + v_o \sin \theta \lambda_y + \frac{u}{\rho} \lambda_\theta,$$
and the differential equations governing the adjoint variables are defined as:

\[
\begin{align*}
\frac{d\lambda_x}{dt} &= 0, \\
\frac{d\lambda_y}{dt} &= 0, \\
\frac{d\lambda_\theta}{dt} &= v_o \sin \theta \lambda_x - v_o \cos \theta \lambda_y.
\end{align*}
\]  
(10)

Applying the fundamental theorem of Pontryagin \[10\] to the above problem, we obtain the following: If \(u^*\) is an optimal control to the Dubins interval problem, then there exists a non-zero adjoint vector \(\Lambda(t)\) and \(T > 0\) such that \(p(t), \Lambda(t)\) being the solution to the equations in (2) and (10) for \(u(t) = u^*(t)\), the following conditions must be satisfied:

- \(\forall t \in [0, T], H(\Lambda, p, u^*) \equiv \min_{u \in [-1, 1]} H(\Lambda, p, u).\)
- \(\forall t \in [0, T], H(\Lambda, p, u^*) \equiv 0.\)
- Suppose \(\alpha_1, \alpha_2, \beta_1, \beta_2\) are the Lagrange multipliers corresponding to the boundary conditions in (5)-(8) respectively. Then, we have,

\[
\begin{align*}
\alpha_1, \alpha_2, \beta_1, \beta_2 &\geq 0, \\
\alpha_1(\theta_1^{\min} - \theta(0)) &= 0, \\
\alpha_2(\theta(0) - \theta_1^{\max}) &= 0, \\
\beta_1(\theta_2^{\min} - \theta(T)) &= 0, \\
\beta_2(\theta(T) - \theta_2^{\max}) &= 0, \\
\lambda_\theta(T) &= \beta_2 - \beta_1, \\
\lambda_\theta(0) &= \alpha_1 - \alpha_2.
\end{align*}
\]  
(11)-(17)

Given a departure angle at target 1 and an arrival angle at target 2, the following facts are known for the basic Dubins problem in \[8\], \[9\]. We will use them in our proofs later.

**Fact 1.** Consider any point \(P\) on an optimal path which is either an inflexion point of the path (point joining two curved segments or a point joining a curved segment and a straight line) or any point on a straight line segment of the path. Suppose the vehicle crosses this point at time \(t \in [0, T]\). Then, \(\lambda_\theta(t) = 0\).

**Fact 2.** All the points of an optimal path where \(\lambda_\theta(t) = 0\) lie on the same straight line.

**Fact 3.** Given an optimal path, let the times \(t_1, t_2\) be such that \(0 \leq t_1 < t_2 \leq T\), \(\lambda_\theta(t_1) = \lambda_\theta(t_2) = 0\) and \(\lambda_\theta(t) \neq 0\) for \(t_1 < t < t_2\). Then, the segment of the optimal path between the times \(t_1\) and \(t_2\) is an arc of length greater than \(\pi \rho\).

**Fact 4.** For any \(t \in [0, T]\), the optimal control \(u^*(t) = -\text{sign}(\lambda_\theta(t))\) if \(\lambda_\theta(t) \neq 0\).
Now, we will use the conditions from the Pontryagin’s minimum principle and the above facts to solve the Dubins interval problem.

**Lemma 1.** Let $\lambda_\theta(0) = \lambda_\theta(T) = 0$. Then the optimal path for the Dubins interval problem must be either $S$ or $L_\psi$ or $R_\psi$ or $L_\psi R_\psi$ or $R_\psi L_\psi$ with $\psi > \pi$.

*Proof.* An optimal path can just be a straight line from fact [2] From fact [3] a curved segment of length greater than $\pi\rho$ can satisfy the boundary conditions $\lambda_\theta(0) = \lambda_\theta(T) = 0$. In addition, any path containing three curved segments and satisfying the boundary conditions must have the length of each curved segment (between any two inflexion points) greater than $\pi\rho$; however, as shown in [9], such a path cannot be optimal. Therefore, an optimal path may consist of either one or two curved segments with the length of each segment greater than $\pi\rho$.

If an optimal path consists of exactly two curved segments, then there are three points where $\lambda_\theta(t)$ becomes zero for $t \in [0, T]$. From fact [2] all these three points must lie on the same straight line. Therefore, the length of the first curved segment must be equal to the length of the second curved segment as shown in Fig. 2 (in this case, $\theta(0) = \theta(T)$).

![Figure 2. A RS path with the boundary values of $\lambda_\theta$ equal to 0.](image)

**Lemma 2.** Let $\lambda_\theta(0) < 0$ and $\lambda_\theta(T) = 0$. Then the optimal path for the Dubins interval problem must be either $LS$ or $LR_\psi$ with $\psi > \pi$.

*Proof.* If $\lambda_\theta(0) < 0$ and $\lambda_\theta(T) = 0$, then the first segment of the path must be $L$ (fact [4]) and $(x_2, y_2)$ must be an inflexion point or lie on a straight line segment of the path (fact [1]). Path $LSR_\psi$ or $LSL_\psi$ with $\psi > 0$ is not possible because this path would violate fact [2] unless the length of the straight line is equal to 0. $LRL$ is also not possible because the length of the $R$ segment and the last $L$ segment would each be greater than $\pi\rho$ which then cannot be optimal [9]. Therefore, the possible candidates are $LS$ or $LR_\psi$ with $\psi > \pi$.

The following result also follows using the same arguments as in the above Lemma.

**Lemma 3.** Let $\lambda_\theta(0) = 0$ and $\lambda_\theta(T) < 0$. Then the optimal path for the Dubins interval problem must be either $SL$ or $R_\psi L$ with $\psi > \pi$.

**Lemma 4.** Let $\lambda_\theta(0) = 0$ and $\lambda_\theta(T) > 0$. Then the optimal path for the Dubins interval problem must be either $SR$ or $L_\psi R$ with $\psi > \pi$. 
Proof. If $\lambda_\theta(0) = 0$ and $\lambda_\theta(T) > 0$, then $(x_1, y_1)$ must be an inflexion point or lie on a straight line segment of the path (fact 1) and the last segment of the path must be $R$ (fact 4). Using similar arguments as in Lemma 2, we can conclude that the only possible candidates are $SR$ or $L_\psi R$ with $\psi > \pi$.

The following result also follows using the same arguments as in the above Lemma.

**Lemma 5.** Let $\lambda_\theta(0) > 0$ and $\lambda_\theta(T) = 0$. Then the optimal path for the Dubins interval problem must be either $RS$ or $RL_\psi$ with $\psi > \pi$.

We now prove cases 1 and 2(a) of the main theorem. Each of the other cases can be shown using the same approach.

**Lemma 6.** If $\theta_d = \theta_1^{\max}$ and $\theta_a = \theta_2^{\max}$, then the optimal path must be $LSR$ or a degenerate form of $LSR$ or belong to case 1.

**Proof.** If $\theta_d = \theta_1^{\max}$, from equation (12), $\alpha_1 = 0$ since $\theta_{\min}^1 - \theta_1^{\max} \neq 0$. Therefore, using equation (17), $\lambda_\theta(0) = -\alpha_2 \leq 0$. Similarly, $\theta_a = \theta_2^{\max}$, from equation (14), $\beta_1 = 0$ since $\theta_{\min}^2 - \theta_2^{\max} \neq 0$. Therefore, using equation (16), $\lambda_\theta(T) = \beta_2 \geq 0$.

- If $\lambda_\theta(0) < 0$ and $\lambda_\theta(T) > 0$, from fact 4, the first segment of the path must be $L$ and the last segment of the path must be $R$.
- If $\lambda_\theta(0) < 0$ and $\lambda_\theta(T) = 0$, from Lemma 2, the optimal path must be either $LS$ or $LR_\psi$ with $\psi > \pi$.
- If $\lambda_\theta(0) = 0$ and $\lambda_\theta(T) > 0$, from Lemma 4, the only possible candidates are $SR$ or $L_\psi R$ with $\psi > \pi$.
- If $\lambda_\theta(0) = 0$ and $\lambda_\theta(T) = 0$, we get case 1.

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