INVERTIBILITY OF TOEPLITZ OPERATORS WITH POLYANALYTIC SYMBOLS

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Abstract. For a class of continuous functions including complex polynomials in \( z, \bar{z} \), we show that the corresponding Toeplitz operator on the Bergman space of the unit disc can be expressed as a quotient of certain differential operators with holomorphic coefficients. This enables us to obtain several nontrivial operator theoretic results about such Toeplitz operators, including a new criterion for invertibility of a Toeplitz operator for a class of harmonic symbols.

1. Introduction

Throughout \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) will denote the unit disc and \( A^2(\mathbb{D}) \) will denote the Bergman space of square integrable holomorphic functions on \( \mathbb{D} \) with respect to the normalized Lebesgue measure. Also, by \( H(\mathbb{D}) \) will denote the Frechet space of all holomorphic functions on \( \mathbb{D} \), while \( H(\mathbb{D}) \) denotes the set of all holomorphic functions defined on a neighbourhood of \( \mathbb{D} \), given a bounded measurable function \( g \in L^\infty(\mathbb{D}) \), we will denote by \( T_g : A^2(\mathbb{D}) \to A^2(\mathbb{D}) \) the corresponding Toeplitz operator. Recall its definition: \( T_g(f) = P(gf) \), where \( P : L^2(\mathbb{D}) \to A^2(\mathbb{D}) \) denotes the orthogonal projection. Let \( \phi \in H(\mathbb{D})[\bar{z}] \) be a polyanalytic function on \( \mathbb{D} \). We are interested in the question of invertibility of the corresponding Toeplitz operator \( T_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D}) \). More generally we are interested in determining dimensions of its kernel and cokernel. Similar problem in the setting of the Hardy space \( H^2(\mathbb{D}) \) is well understood. Indeed, recall that a well-known theorem of Coburn asserts that given any \( g \in L^\infty(\mathbb{D}) \), then the corresponding Toeplitz operator \( T_g : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \) is either injective, or its conjugate \( T_g^* \) is injective. We will recall the following refinement of the Coburn's theorem for symbols that are continuous up to the boundary of \( \mathbb{D} \) (such symbols will be the object of our main interest.)

Theorem 1.1. \((\mathbb{D})\). Let \( g \in C(\mathbb{D}) \). then \( T_g : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \) is a Fredholm operator if and only if \( g \) does not vanish on \( \partial \mathbb{D} \). Let \( n \) be the the winding number of \( g(\partial \mathbb{D}) \) around 0. If \( n > 0 \), then \( T_g : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \) is onto with \( n \)-dimensional kernel. If \( n < 0 \) then \( T_g \) is injective with \( n \)-dimensional cokernel. Finally \( T_g \) is invertible if \( n = 0 \).

We recall the following well-known partial analogue of this statement in the Bergman space setting (see for example [SZ1], Theorem 24). It provides the full description of all Fredholm operators of the form \( T_g, g \in C(\mathbb{D}) \).

Lemma 1.1. If \( g \in C(\mathbb{D}) \), then \( T_g : A^2(\mathbb{D}) \to A^2(\mathbb{D}) \) is a Fredholm operator if and only if \( g \) does not vanish on \( \partial \mathbb{D} \), in this case its index equals to minus of the the winding number of \( g(\partial \mathbb{D}) \) around 0.

The full analogue of Theorem \((\mathbb{D})\) in the Bergman space setting fails even for harmonic functions: Sundberg and Zheng [SZ] constructed an example \( g = \bar{z} + \phi, \phi \in H(\mathbb{D}) \) such that \( T_g \) is not invertible while the winding number of \( g(\partial \mathbb{D}) \) around 0 is 0.

Determining dimensions of \( \ker(T_g), \text{coker}(T_g) \) for general classes of harmonic symbols \( g \) (in the Bergman setting) is a fundamental problem, full solution to which in seems to be out of reach at the
moment. We will also recall that for real harmonic function \( h, T_h \) is invertible if \( h(\mathbb{D}) \) is bounded away from zero by Mcdonald-Sundberg [MS].

Let us start by recalling results of Sundberg and Zheng [SZ] in more detail. They made a crucial observation that

\[ T_z(f) = \frac{1}{z^2} \int_0^z w f'(w) dw. \]

Based on this it is easy to deduce that given \( g = \bar{z} + \phi \) then \( T_g(f) = 0 \) if and only if \( f \) satisfies the following first order differential equation

\[ (1 + z\phi(z))f'(z) = -2\phi(z) + z\phi'(z))f(z). \]

Thus \( T_g \) is invertible iff

\[ \text{res}_w \frac{2\phi(z) + z\phi'(z)}{1 + z\phi(z)} \in \mathbb{Z}_{<0}. \]

This observation led Sundberg and Zheng to a construction of a rational function \( \phi(z) \) with poles outside \( \overline{\mathbb{D}} \), such that \( g = \bar{z} + \phi(z) \) has the property that \( T_g \) is a Fredholm operator of index 0, but \( \ker(T_g) \) (hence \( \text{coker}(T_g) \)) is nontrivial. Moreover 0 is an isolated element of the spectrum of \( T_g \) [SZ, Theorem 2.3, Lemma 2.2].

We will use the following notation/convention to state our main results. Given an \( n \)-th order polyanalytic function \( \phi(z) = \sum_{i=0}^n a_i(z) z^i, a_i \in H(\mathbb{D}) \), we will define the following holomorphic function \( \tilde{g} \) as follows:

\[ \tilde{g}(z) = \sum a_i(z) z^{n-i}. \]

The crucial relation between \( \phi \) and \( \tilde{\phi} \) is that \( \phi(z) = z^{-n} \tilde{\phi}(z) |_{\partial \mathbb{D}} \). Therefore it follows from the argument principle that (assuming \( \phi \in C(\overline{\mathbb{D}}) \) if the winding number of \( \phi(\partial \mathbb{D}) \) around 0 is \( m \), then \( \tilde{\phi} \) has \( n - m \) zeros on \( \mathbb{D} \).

Now we will state our main results. Given an \( n \)-th order polyanalytic function \( \phi(z) = \sum_{i=0}^n a_i(z) z^i, a_i \in H(\mathbb{D}) \), we will define the following \( n \)-th order differential operator

\[ D_{\phi} = \prod_{i=2}^{n+1} (zD + i) a_0(z) + \sum_{i=1}^n \prod_{k=i+2}^{n+1} (zD + k) D^i a_i(z), D = \frac{\partial}{\partial z}. \]

The following is the key result.

**Lemma 1.2.** Let \( f, g \in A^2(\mathbb{D}) \). Put \( \phi(z) = \sum_{i=1}^n a_i(z) z^i, a_i(\mathbb{D}) \in H^{\infty}(\mathbb{D}) \). Then \( T_{\phi}(f) = g \) if and only if \( \prod_{i=2}^{n+1} (zD + i)(g) = D_{\phi}(f) \). In particular \( T_{\phi}(f) = 0 \) if and only if \( D_{\phi}(f) = 0 \).

This result allows us to transfer the problems about the kernel of Toeplitz operators with polyanalytic symbols (in particular questions about their invertibility) to the problems about existence of solutions of holomorphic ordinary differential equations. The key for proving the above result will be to explicitly realize Toeplitz operators with polyanalytic symbols as a fraction of differential operators with analytic coefficients. This enables us to embed the algebra generated of Toeplitz operators with polyanalytic symbols into a skew field of analytic differential operators on \( D \). As an immediate corollary we obtain the following.

**Proposition 1.1.** The algebra generated by all Toeplitz operators with polyanalytic symbols has no zero divisors.
Theorem 1.2. Let $\phi \in H^\infty(D)[z]$ be an $n$-th order polyanalytic function. Then the kernel of $T_\phi$ is at most \(n\)-dimensional. If $\phi \in H^\infty(D)[z]$ and $\tilde{\phi}$ is nowhere vanishing on $\overline{D}$, then $T_\phi$ is surjective with \(n\)-dimensional kernel. If $\tilde{\phi}$ has a zero $w$ on $D$ such that
\[
\text{res}_w(\frac{\partial\tilde{\phi}}{\partial z}\tilde{\phi}^{-1}) \notin \mathbb{Z}_{\geq n+1},
\]
then the kernel of $T_\phi$ is at most $n-1$-dimensional. If $\phi \in H^\infty(D)[z]$ and $\tilde{\phi}$ has a single zero $w$ on $D$ and no zeroes on $\partial D$, then $T_\phi$ is onto if and only if the above condition holds, in which case $\ker T_\phi$ is $(n-1)$-dimensional.

Our next result provides invertibility criterion for Toeplitz operators $T_\phi$ where $\phi$ is an $n$-th order harmonic function such that $\tilde{\phi}$ has a zero with multiplicity $n$ in $D$.

Theorem 1.3. Let $0 \neq w \in D, \psi \in H^\infty(D)$ and $\phi = (z - w)^n\psi(z) + (\overline{z} - w^{-1})^n$. Then Toeplitz operator $T_\phi$ is injective if the following equation has no roots in $\mathbb{Z}_{\geq 0}$:
\[
\prod_{i=1}^{n}(\lambda + i + 1) + \psi(w)(-w)^n(n! + \sum_{i=1}^{n}\frac{n!^2}{i!(n-i)!}\lambda \cdot (\lambda - i + 1)) = 0.
\]
Moreover, $\dim \ker(T_\phi)$ equals to the number of distinct roots of the above equation in $\mathbb{Z}_+$ if $1 + (-z)^n\psi(z)$ has no zeroes on $\overline{D}$. In particular, if $\psi(w)(-w)^n \notin \mathbb{Q}_{< -1}$ and $(-z)^n\psi(z)$ is not equal to $-1$ on $\overline{D}$, then $T_\phi$ is invertible. Thus if $\psi \in H(D)$ such that $(-z)^n\psi(z) \neq -1$ on $D$ and $(-z)^n\psi(z)(\overline{D})$ is a starlike domain around 0, then $T_\phi$ is invertible for any $w$.

Finally, let us recall the following version of a question of Douglas about invertibility of Toeplitz operators for harmonic symbols in the Bergman space setting.

Question 1. Let $\phi(z) \in C(\overline{D})$ be a nowhere vanishing harmonic function in a neighbourhood of $\overline{D}$, then is $T_\phi$ invertible? More generally, is the spectrum of $T_\phi$ a subset of $\phi(\overline{D})$?

Remark that while given a nowhere vanishing harmonic $\phi$ as above, then the winding number of $\phi|_{\partial D}$ around 0 is 0 (hence $T_\phi$ has index 0), inverse of this statement is certainly not true: There are examples of vanishing harmonic functions on $D$ with winding number on the boundary around 0 being 0 (see [ZZ]). To the best of our knowledge there are no negative answers known in the existing literature to the above version of Douglas’s question. We will show in Corollary [ZZ] that Douglas’s question has an affirmative answer for harmonic polynomials of the form $\phi = \bar{z} + f(z)$, where $f(z)$ is a quadratic polynomial. Note however that even for quadratic $f(z)$ $\phi(D)$ need not be a subset of the spectrum of $T_\phi$ as shown in [[ZZ], Theorem 4.1]. For a linear $f(z)$ the spectrum of $T_\phi$ does equal to $\phi(\overline{D})$ [[ZZ], Theorem 3.1].
2. The differential operator $D_\phi$

Recall the following well known formula

$$T_z(z^n) = \begin{cases} 0 & k > n \\ \frac{n-k+1}{n+1}z^{n-k} & n \geq k. \end{cases}$$

This easily implies that for any polynomial $f \in \mathbb{C}[z]$, we have

$$T_z(f) = \prod_{i=2}^{k+1}(zD + i)^{-1}D^k(f),$$

Where $D : H(\mathbb{D}) \to H(\mathbb{D})$ denotes the differentiation operator, and $zD + i : H(\mathbb{D}) \to H(\mathbb{D}), i > 0$ are understood as invertible differential operators.

In particular, $T_z = (zD+2)^{-1}D$ which is equivalent to the following formula from Sundberg-Zheng [SZ]

$$T_zf = \frac{1}{z^2} \int_0^z w f'(w)dw.$$  

Proof of Lemma [1,2]. Since $\Lambda = \prod_{i=2}^{n+1}(zD + i) : H(\mathbb{D}) \to H(\mathbb{D})$ is an injective linear operator, it suffices to check that $\Lambda(T_k)(f) = \Lambda(g)$. Hence we need to show that

$$\Lambda(T_{a_i z^i})(f) = \prod_{k=i+2}^{n+1} (zD + i)D^i(a_i(z)f)$$

It suffices to check this equality for $f = z^m, m \geq 0$. But this is immediate from the above discussion. \hfill \Box

Next we will compute the first two leading terms of $D_\phi$, i.e. coefficients in front of $D^n, D^{n-1}$. Clearly the leading term of $D_\phi$ is $\phi D^n = \sum_{i=0}^n a_i(z)z^{n-i}D^n$. Recall that the following commutator relation holds in the ring of differential operators

$$Dg(z) - g(z)D = g'(z), \quad g(z) \in H(D).$$

Using this relation we easily obtain the following expansion in terms of powers of $D$

$$\prod_{k=1}^m (zD + b_k) = z^m D^m + (m(m-1)/2 + \sum b_k) z^{m-1}D^{m-1} + \cdots, b_k \in H(\mathbb{D}).$$

Thus

$$\prod_{k=1}^m (zD + b_k)\psi = \psi z^m D^m + ((m(m-1)/2 + \sum b_k) z^{m-1}\psi + mz^m \psi')D^{m-1} + \cdots, \psi \in H(\mathbb{D}).$$

Our differential operator is $D_\phi$ is

$$\prod_{i=2}^{n+1}(zD + i)a_0(z) + D^na_n(z) + (zD + n + 1)D^{n-1}a_0(z) + \cdots + \prod_{i=3}^{n+1}(zD + i)Da_1(z).$$

Therefore the coefficient in front of $D^{n-1}$ is

$$n(n+1)a_0(z)z^{n-1} + nz^n a'_0(z) + na'_n(z) + (a_{n-k-1}(z) \sum_{k<n-1} (n+1)(k+1) + na'_{n-k}(z))z^k.$$
Hence the differential equation is
\[ \sum_{k=0}^{n} a_k z^{n-k} = 0. \]

Moreover, a classical theorem of Fuchs' [11] asserts that if the equation in the essential part at \( w \) is said to have regular singularity at \( w \) if the space of solutions is at most \( n \)-dimensional, thus \( \dim \ker D_\phi \leq n \) for any \( n \)-th order polyanalytic function \( \phi \). Now suppose that \( \phi \in H(\mathbb{D})[\bar{z}] \) is of order \( n \) such that \( \tilde{\phi} \) has no zeroes on \( \mathbb{D} \). Thus the index of \( T_\phi \) is \( n \), on the other hand \( \dim \ker T_\phi \leq n \). Therefore \( \dim \ker T_\phi = n \) and \( \dim \ker T_\phi = 0 \).

The rest of the proof will proceed by observing that the differential equation \( D_\phi(y) = 0 \) has regular singularity at \( w \), and then applying the Frobenius method provides the corresponding indicial equation in \( \lambda \), where \( \lambda \) is the smallest nonzero power of \((z-w)\) appearing in the Taylor expansion of a nontrivial solution \( y \) at \( w \). Indeed, recall that an \( n \)-th order differential equation \( \sum a_i(z)y^{(n)} = 0 \) is said to have regular singularity at \( w \) if \((z-w)^{-1}a_i(z)\) is holomorphic in a neighbourhood of \( w \). For such an equation let \( a_i \) be the value of \((z-w)^{-1}a_i(z)\) at \( w \). We will refer to \( \sum a_i D_i \) as the essential part at \( w \) of the differential operator \( \sum a_i(z)D^i \). Then the indicial equation of the above differential equation is \( \sum_{i=0}^{n} \lambda \cdots (\lambda - i + 1) a_i = 0 \). Recall that the dimension of the space of holomorphic solutions around \( w \) equals to the number of distinct roots of the indicial equation in \( \mathbb{Z}_+ \). Moreover, a classical theorem of Fuchs’ [11] asserts that if \( a_i(z) \) are holomorphic in a neighbourhood of \( \mathbb{D} \) then for each such root \( \lambda \in \mathbb{Z}_+ \) there is a holomorphic solution around \( \mathbb{D} \) with order of vanishing at \( w \) equalling \( \lambda \).

Now in the setting of Theorem 1.2 it follows that the equation \( (z-w)^{n-1}D_\phi(y) = 0 \) has a regular singularity at \( w \), and the corresponding indicial equation is
\[ \lambda(\lambda - 1) \cdots (\lambda - n + 2)(\lambda - n + 1 + \text{res}_w \left( \frac{(n+1)\tilde{\phi'} - \tilde{\phi} z}{\tilde{\phi}} \right) = 0, \]
which gives \( \lambda = \text{res}_w(\tilde{\phi}^{-1} \tilde{\phi'}) - 2 \) and \( \lambda = 0, 1, \cdots, n-2 \). Thus we are done by Fuchs’ theorem.

Now we will show Theorem 1.3. We want to show that the differential operator \( D_\phi \) for \( \phi(z) = (z-w)^n \psi(z) + (\bar{z} - w^{-1})^n \) has regular singularity at \( w \), and then compute its essential part. Then we will obtain the desired indicial equation by evaluating the essential part on \((z-w)^\lambda \) and setting it to equal 0. Recall that
\[ (zD + 2)(T_{\bar{z}} - w^{-1}) = -w^{-1}((z-w)D + 2), \quad (zD + i)T_{\bar{z}} = T_{\bar{z}}(zD + i - 1). \]

Hence
\[ (zD + i)(T_{\bar{z}} - w^{-1}) = (T_{\bar{z}} - w^{-1})(zD + i - 1) - w^{-1}. \]

Now we can obtain the following recursive equality
\[ (zD + n + 1)(T_{\bar{z}} - w^{-1})^n = (T_{\bar{z}} - w^{-1})(zD + n)(T_{\bar{z}} - w^{-1})^{n-1} - w^{-1}(T_{\bar{z}} - w^{-1})^{n-1}. \]
which yields
\[(zD + n + 1)(T_z - w^{-1})^n = -(n - 1)w^{-1}(T_z - w^{-1})^{n-1} + (T_z - w^{-1})^{n-1}(-w^{-1}((z - w)D + 2))\]
Hence we have the following recursive formula (put \(L = ((z - w)D + 2)\) for brevity)
\[
\prod_{i=2}^{n+1}(zD + i)(T_z - w^{-1})^n = (-w^{-1})\prod_{i=2}^{n}(zD + i)(T_z - w^{-1})^{n-1}(L + (n - 1)).
\]
Finally, we have
\[
\prod_{i=2}^{n+1}(zD + i)(T_z - w^{-1})^n = (-w^{-1})\prod_{i=1}^{n}(L + (i - 1)).
\]
The latter has regular singularity at \(w\), and evaluated on \((z - w)^\lambda\) gives
\[(-1)^nw^{-n}\prod_{i=1}^{n}(\lambda + i + 1).\]
Next we will compute the essential part at \(w\) of the differential operator \(\prod_{i=2}^{n+1}(zD + i)(z - w)^n\psi(z)\).
Suffices to compute this for \(\prod_{i=2}^{n+1}(zD + i)(z - w)^n\). On the other hand the essential part at \(w\) of differential operator \(D^i(z-w)^n\) is 0 unless \(n = i\). So the desired essential part is that of \((zD)^n(z-w)^n\), which is equal to the essential part of \(w^nD^n(z-w)^n\). Now recall a well-known equality in the algebra of differential operators
\[D^n(z-w)^n = (z-w)^nD^n + \sum_{i=0}^{n-1} \frac{n!^2}{i!(n-i)!} (z-w)^i D^i\]
Hence the essential part is \(\psi(w)\sum_{i=0}^{n} \frac{n!^2}{i!(n-i)!} D^i\). Finally, we obtain the sought after indicial equation is
\[\prod_{i=1}^{n}(\lambda + i + 1) + \psi(w)(-w)^n\sum_{i=0}^{n} \frac{n!^2}{i!(n-i)!} \lambda \cdots (\lambda - i + 1) = 0.\]
Now the desired result follows by Fuchs’ theorem just as in the proof of Theorem 1.2.

Finally, it suffices to see that the above equation has no solutions in \(\mathbb{N}\) for \((-w)^n\psi(w) \notin \mathbb{Q}_{< -1}\).
The letter follows from an easy fact that
\[\prod_{i=1}^{n}(\lambda + i + 1) > \sum_{i=0}^{n} \frac{n!^2}{i!(n-i)!} \lambda \cdots (\lambda - i + 1)\]
for all \(n, \lambda \in \mathbb{N}\).

\[\square\]

**Remark 2.1.** Let \(\phi\) as above be such that the winding number around 0 of \(\phi(\partial \mathbb{D})\) is 0. Then for such generic \(\phi\), the corresponding Toeplitz operator is invertible. Let \(w_1, \ldots, w_n\) be zeros of \(\phi\) on \(\mathbb{D}\). It follows that \(T_{\phi}\) is not invertible if and only if equation \(D_{\phi}f = 0\) has a nontrivial solution in \(A^2(\mathbb{D})\).
Let \(y' = A\phi\) be the matrix form of this equation. For generic such \(\phi\) it follows that this equation has regular singularities at \(w_i\). Put \(A_i = res_{w_i}A\). Thus generically, distinct eigenvalues of \(A_i\) do not differ by integers. Let \(M_1, \ldots, M_n\) be the monodromy matrices around \(w_1, \ldots, w_n\) respectively. Then \(M_i\) is conjugate to \(\exp(2\pi iA_i)\), hence it has an eigenvalue 1 with multiplicity \(n - 1\). Then existence of such a solution implies that matrices \(M_1, \ldots, M_n\) have a simultaneous eigenvector with eigenvalue 1. But generically this does not hold.
Let us explicitly write down the differential equation \( D_\phi y = 0 \) of \( n = 2 \). For computational simplicity we will consider harmonic functions
\[
\phi = a_2 \bar{z}^2 + a_1 \bar{z} + a_0(z), \quad a_1, a_2 \in \mathbb{C}, a_0(z) \in H(D).
\]
So \( \tilde{\phi} = a_2 + a_1 z + a_0 z^2 \). The corresponding differential equations is
\[
(6 + z^2 D^2 + 6zD)(a_0 y) + a_2 (D^2 y) + a_1 (zD + 3) Dy = 0,
\]
which simplifies to
\[
\tilde{\phi} y'' + (3(\tilde{\phi}') - z^2 a_0') y' + (3(\tilde{\phi}'') - 2z^2 a_0'') y = 0.
\]

We will end by explicitly working out invertibility criteria for \( T_\phi \) for certain relatively simple harmonic functions \( \phi \).

**Corollary 2.1.** Let \( \phi(z) = az^n(z-w)^m + (\bar{z} - w^{-1})^m \) with \( a \in \mathbb{C}, w \in \mathbb{D} \setminus \{0\} \) and \( n, m \in \mathbb{N} \). Then \( T_\phi \) is invertible if and only if \( |a| < |w|^{-n} \). Let \( \phi(z) = \bar{z} + a + bz + cz^2 \) with \( a, b, c \in \mathbb{C} \). Then the spectrum of \( T_\phi \) is a subset of \( \phi(\mathbb{D}) \).

**Proof.** If \( |a| > 1 \) then the index of \( T_\phi \) is nonzero. Indeed, as \( \tilde{\phi} = w^{-n}(z-w)^m(1 + a(-w)^n z^{n+m}) \), it has \( m \) roots in \( \mathbb{D} \) (hence \( T_\phi \) has index 0) if and only if \( |a| < |w|^{-n} \). Now suppose that \( |a| < |w|^{-n} \). Then \( T_\phi \) is invertible by Theorem 1.3.

Finally let \( \phi = \bar{z} + a + bz + cz^2 \). It suffices to show that \( T_\phi \) is invertible if \( T_\phi \) has index . Thus \( \tilde{\phi} = 1 + z(a + bz + cz^2) \) has exactly one zero \( w \in \mathbb{D} \). Then we may write \( \phi = -w^{-1}(\bar{z} - w^{-1}) + (z - w) \psi \), where \( \psi \) is a linear function. Then \( (-z)\psi(z)(\mathbb{D}) \) is a starlike domain around 0. Hence by Theorem 1.3 \( T_\phi \) is invertible.

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