Asymptotic density for k-almost primes

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Abstract

Landau’s well known asymptotic formula

\[ N_k(x) := |\{ n \leq x : \Omega(n) = k\} | \sim \left( \frac{x}{\log x} \right) \frac{(\log \log x)^{k-1}}{(k-1)!} (x \to \infty), \]

which also holds for

\[ \pi_k(x) := |\{ n \leq x : \omega(n) = k\} |, \]

is known to be fairly poor for \( k > 1 \), and when \( k \) is allowed to tend to infinity with \( x \), the study of \( N_k(x) \) and \( \pi_k(x) \) becomes very technical [1, Chapter II.6, § 6.1, p.200]. I hope to show that the method described below provides not only a more accurate approach, but rather increases in its asymptotic accuracy as \( k \) tends to infinity.

1 Introduction

Landau’s formula holds when \( k = o(\log \log x) \), and also when \( k = (1+o(1)) \log \log x \), but it does not hold in general. Selberg proved that

\[ N_k(x) = G \left( \frac{k-1}{\log \log x} \right) \frac{x(\log \log x)^{k-1}}{(k-1)! \log x} \left( 1 + O_R \left( \frac{k}{(\log \log x)^2} \right) \right) \]

where

\[ G(z) = \frac{1}{\Gamma(z+1)} \prod_p \left( 1 - \frac{1}{p} \right)^{-1} \left( 1 - \frac{z}{p} \right)^z \]

uniformly for \( 1 \leq k \leq R \log \log x \), where \( R \) is a positive real, and \( R < 2 \) [2, Chapter 7.4, Theorem 7.19, p.232].

Since \( G(0) = G(1) = 1 \), Landau’s asymptotic is in agreement with the above, but owing to the erratic behaviour of \( G(z) \) for \( R > 2 \), this definition of \( N_k(x) \) becomes impractical in the study of \( N_k(x) \) for \( k > 2 \log \log x \).

What follows then, is a study of the behaviour of \( N_k(x) \) when \( k \) is allowed to tend to infinity with \( x \).
2 Outline

![Figure 1](image1.png)  ![Figure 2](image2.png)

2.1 Initial observations

Figure 1 shows the well-known Poisson distribution of $N_k(x)$ at $x = 5000$ for $1 < k < 20$, where the solid line is the actual value, and the dotted line is Landau’s estimate, which appears to be fairly accurate. Figure 2 however, shows $N_k(2^k x)$, and whereas Landau’s estimate tends to zero, the actual value can be seen to approach a maximum as $k \to \infty$, and remain at that value.

2.2 Elementary analysis

Where $p_n$ is the $n$th prime, the following holds for $N_k(x \cdot p_1^{k-1})$ for $x \geq p_1$.

For $k = 1$, it is clear that $N_k(p_n \cdot p_1^{k-1}) = n$. In fact, for any $k$, it can be said that $N_k(p_1 \cdot p_1^{k-1}) = 1$, and $N_k(p_2 \cdot p_1^{k-1}) = 2$, but since $p_3 - p_2 > 1$, for $k > 1$, $N_k(p_3 \cdot p_1^{k-1}) > 3$, since $p_1 \cdot p_1 < p_3$.

For $k \geq 2$, $N_k(p_3 \cdot p_1^{k-1}) = 4$, since $N_k(p_3 \cdot p_1^{k-1})$ can be deconstructed thus:

$$p_1^{k-2} \times \begin{cases} p_1 \cdot p_3 \\ p_1 \cdot p_2 \\ p_2 \cdot p_2 \\ p_1 \cdot p_1 \end{cases}$$

Similarly, for $k \geq 3$, $N_k(p_4 \cdot p_1^{k-1}) = 6$, since $N_k(p_3 \cdot p_1^{k-1})$ may be represented as follows:

$$p_1^{k-3} \times \begin{cases} p_1 \cdot p_1 \cdot p_4 \\ p_2 \cdot p_2 \cdot p_2 \\ p_1 \cdot p_1 \cdot p_3 \\ p_1 \cdot p_2 \cdot p_2 \\ p_1 \cdot p_1 \cdot p_2 \\ p_1 \cdot p_1 \cdot p_1 \end{cases}$$

The representative sequence for $\limsup (k \to \infty) N_k(p_n \cdot p_1^{k-1})$ begins $1, 2, 4, 6, 10, 13, 18 \ldots$ at $n = 1, 2, 3, 4 \ldots$ for $k = 1, 1, 2, 3, 4, 4, 5 \ldots$ respectively.
(i.e. for \( k \geq 4, N_k(p_5 \cdot p_1^{k-1}) = 10 \), and for \( k \geq 4, N_k(p_6 \cdot p_1^{k-1}) = 13 \), etc.).

It may therefore be said that the prime factors (counted with multiplicity) are distributed cumulatively at \( N_k(2^{k-1}x) \). The same is also true of course for \( N_k(2^kx) \), and more accurate approximations of the asymptotics of \( k \)-almost primes may be derived as a result.

This may be more clearly seen by arranging the set of natural numbers \( \mathbb{N} \) into columns, where each column \( k \) is the set of all products of exactly \( k \) not-necessarily-distinct primes (Figures 3 and 4). Figure 3 shows that any given prime, when multiplied by a power of 2, aside from the obvious of moving into the next column, eventually plateaus. Figure 4 shows the defining boundary for the plateau; namely when power of 3 is introduced into a column \( k \).

Figures 5 and 6 are graphic representations of Figures 3 and 4, with \( \Omega \) running up the \( y \)-axis, and \( k \) running along the \( x \)-axis, where the dotted lines in Figure 5 show the pattern of the doubling primes, and Figure 6 shows the approximate asymptotics. An approximate estimate for the growth of the power-3 bounding curve for low \( k \) is \( \alpha (e+1)^{k/e} \), where \( \alpha \) is the Alladi-Grinstead Constant as shown by the solid black line in Figure 6. The asymptotes of \( 2^{k-1}p_0 \) (dotted lines) are simply translations of \( -\alpha (e+1)^{(-k/e)} - k \). It may be of interest to note here that the primes in column \( k = 1 \) do not lie on these lines.

\[
\begin{array}{cccccccc}
& k=1 & k=2 & k=3 & k=4 & k=5 & k=6 & k=7 & k=8 \\
W_{18} & 61 & 51 & 76 & 140 & 272 & 544 & 1088 & 2176 \\
W_{17} & 59 & 49 & 75 & 136 & 270 & 540 & 1080 & 2160 \\
W_{16} & 53 & 46 & 70 & 135 & 264 & 528 & 1056 & 2112 \\
W_{15} & 47 & 40 & 63 & 126 & 252 & 504 & 1008 & 2016 \\
W_{14} & 43 & 38 & 60 & 120 & 240 & 480 & 960 & 1920 \\
W_{13} & 39 & 34 & 52 & 104 & 208 & 416 & 832 & 1664 \\
W_{12} & 35 & 30 & 48 & 96 & 192 & 384 & 768 & 1536 \\
W_{11} & 31 & 26 & 42 & 84 & 168 & 336 & 672 & 1344 \\
W_{10} & 27 & 22 & 38 & 76 & 152 & 304 & 608 & 1216 \\
W_{9} & 23 & 18 & 34 & 68 & 136 & 272 & 544 & 1088 \\
W_{8} & 19 & 14 & 28 & 56 & 112 & 224 & 448 & 896 \\
W_{7} & 15 & 10 & 20 & 40 & 80 & 160 & 320 & 640 \\
W_{6} & 11 & 6 & 12 & 24 & 48 & 96 & 192 & 384 \\
W_{5} & 7 & 3 & 6 & 12 & 24 & 48 & 96 & 192 \\
W_{4} & 3 & 1 & 2 & 4 & 8 & 16 & 32 & 64 \\
W_{3} & 1 & 0 & 1 & 2 & 4 & 8 & 16 & 32 \\
\end{array}
\]
3 Approximations for asymptotics

A sensible starting point to finding the asymptotics of $N_k(x)$ would be to first find the limit for $N_k(2^{k-1}x)$, and then work backwards. As can be seen in figure 2, the global maximum of Landau’s estimate (at approximately $k = 6$) approaches the actual limit, albeit prematurely. Landau’s formula may then be approximated (and I suggest, improved upon) with the following amendments, which holds for some constant $c$:

$$\lim_{k \to \infty} N_k(x \cdot 2^{k-1}) \sim \frac{cx \log(\log(cx))^{\log(c) + \log(c)^{1/\pi}}}{\log(cx)(\log(c) + \log(c)^{1/\pi})!}$$

Figure 7 shows a plot of the actual values for $2 < x < 5000$ for $1 < k < 9$. The dotted line shows the limit for $N_k(2^{k-1}x)$, which is reached at approximately $k = 25$. The value for $c$ at this point is approximately $e^{e^2}$, and for want of a more accurate value being found, this is the one that I shall use from here on.

By taking approximate numerical values for $c$, that align with $k = 1, 2, 3, \ldots$, and plotting against a CDF with equation

$$\frac{1}{2} \text{erfc}\left(-\frac{x - \mu}{\sqrt{2}\sigma}\right) - 2\Gamma\left(\frac{x - \mu}{\sigma}, \alpha\right)$$

where $\mu$ and $\sigma$ both $= 2\gamma + \frac{1}{2}$, and $\alpha = \gamma - \frac{1}{2}$; where erfc is the complementary error function, $\Gamma$ is Owen’s $T$-Function, and $\gamma$ is Euler’s constant; figure 8 is arrived at. All that then remains is to apply the CDF to the above formula for
$N_k(2^{k-1}x)$, and rearrange for $N_k(x)$.

I realise that the method described may not sit comfortably with most readers, and I would certainly prefer to furnish such an explanation with a rigorous proof, but not being a number theorist, I can offer only estimation and approximation based on testing. That mathematics may be treated as a science in this age of technology, where numerical experimentation has been possible, I can but offer an informed guess as to the values of these constants. It has seemed natural to me to employ universally recognised constants, largely because experimentation has show that purely numerical values lie extremely close to the ones given, but of course they may just as easily be replaced with numerical values until a proof justifying their use has been found.

3.1 Estimations for asymptotics of k-almost primes

Compiling the above estimates into workable formulae, we have

$$N_k(x) := | \{ n \leq x : \Omega(n) = k \} | \sim \Re\left( \frac{e^{e+1} \alpha y \log \log(e^{e+1} \alpha y)^\beta}{\beta! \log(e^{e+1} \alpha y)} \right)$$

where $y = \frac{x}{2^{k-1}}$,

or simply

$$\Re\left( \frac{2^{1-k} \alpha e^{1+x} \log(1 + e + \log(2^{1-k} \alpha x))^\beta}{\beta!(1 + e + \log(2^{1+\alpha x}))} \right)$$

where $\gamma$ is Euler’s constant, $\beta = 1 + e + \log \alpha + (1 + e + \log \alpha)^{1/\pi}$,

and $\alpha = \frac{1}{2} \text{erfc}\left( - \frac{k - (2e^\gamma + \frac{1}{4})}{(2e^\gamma + \frac{1}{4})\sqrt{2}} \right) - 2T\left( - \frac{k}{(2e^\gamma + \frac{1}{4}) - 1}, e^\gamma - \frac{1}{4} \right)$

where erfc is the complementary error function and T is the Owen T-function. In integral form, $\alpha =$

$$\frac{1}{\pi} \int_{-3+8e^\gamma/(\sqrt{(1+8e^\gamma)})}^{\infty} e^{-t^2} \ dt + \int_{0}^{1/4} - e^{-3 - 8e^\gamma(1+t^2)/(2(1+8e^\gamma)^2)} \frac{e^{-(3 - 8e^\gamma)^2(1+t^2)/(2(1+8e^\gamma)^2)}}{1 + t^2} \ dt$$
As $k$ tends to infinity, $\alpha$ tends to 1, and we are left with the following:

$$\lim_{k \to \infty} N_k(x \cdot 2^{k-1}) \sim \frac{e^{x+1} x \log \log(e^{x+1} x)^\beta}{\log(e^{x+1} x)^\beta!},$$

where $\beta = \log(e^{x+1}) + \log(e^{x+1})^{1/\pi}$

as stated above. For $k \leq 3$, improvements to the above can certainly be made, but as $k \to \infty$, the formulae above, as far as has been tested, seem to be fairly accurate.

### 4 Numerical results

Figure 9 shows a comparison of results for $0 < x < 10^4$, $1 < k < 9$, where the actual values are shown as a solid black line, Landau’s estimate is shown as a dashed line, and the CDF estimate is shown as the dotted line. Table 1 shows a comparison of results (rounded to the nearest integer) for $N_k(10^7)$ for $1 < k < 20$.

| $k$ | Landau | CDF   | Actual |
|-----|--------|-------|--------|
| 1   | 620 421| 586 778| 664 579|
| 2   | 1 724 734| 2 390 994| 1 904 324|
| 3   | 2 397 331| 2 694 223| 2 444 359|
| 4   | 2 221 480| 2 082 840| 2 050 696|
| 5   | 1 543 897| 1 325 485| 1 349 779|
| 6   | 858 389| 753 332| 774 078|
| 7   | 397 712| 399 691| 409 849|
| 8   | 157 945| 203 132| 207 207|
| 9   | 54 885| 100 418| 101 787|
| 10  | 16 953| 48 728| 49 163|
| 11  | 4 713| 23 335| 23 448|
| 12  | 1 191| 11 059| 11 068|
| 13  | 276| 5 194| 5 210|
| 14  | 59| 2 418| 2 406|
| 15  | 12| 1 115| 1 124|
| 16  | 2| 509| 510|
| 17  | 0| 229| 233|
| 18  | 0| 102| 102|
| 19  | 0| 44| 45|
| 20  | 0| 19| 21|

*Table 1*
5 Notes

5.1 Observations on the apparent connection between the bounds of $N_k(2^{k-1}x)/\pi(x)$ and $\sqrt{\text{Li}(x)}$

Figure 10 shows actual values of $N_k(2^{k-1}x)/\pi(x)$ (where $\pi(x)$ is the prime counting function) for $0 < x < 1000$, $2 < k < 9$ against a plot of $\sqrt{\text{Li}(x)}$ (dotted line), while figure 11 shows the same against a plot of $\sqrt{\text{Li}(x)}$ (dashed line), and a plot of $2\text{Li}(x)^{1/3}$. It may be interesting to note that the asymptotic $\limsup(k \to \infty) N_k(2^{k-1}x)/\pi(x)$ is approximately $\sqrt{\text{Li}(x)}$ until it intersects with $2\text{Li}(x)^{1/3}$, after which, $\limsup(k \to \infty) N_k(2^{k-1}x)/\pi(x)$ appears to "cling" to this, until approximately $x = 2000$. The reason for this is unknown, but as seen in figure 12, the curve

$$\frac{cx \log \log(cx)^{\beta}}{\log(cx)^{\beta!} \text{R}(x)}$$

where $\beta = \log(c) + \log(c)^{1/\pi}$, $c$ is $e^{e+1}$ and $\text{R}$ is the Riemann counting function, intersects, and becomes greater than $\sqrt{\text{Li}(x)}$ if $c$ is replaced with any greater a
Also of interest to note is that the greatest value that \( c \) can take is 64, before the curve begins to tend to zero. At \( c = 64 \), the curve intersects \( \sqrt{\text{Li}(x)} \) at \( x = \text{Li}^{-1}(64) \) (where \( \text{Li}^{-1} \) is the inverse Logarithmic Integral), precisely at the point where \( 2\text{Li}(x)^{1/3} \) intersects \( \sqrt{\text{Li}(x)} \). From this, one may tentatively conjecture that \( \sqrt{\text{Li}(x)} \) plays some part in the limiting asymptote for \( N_k(2^{k-1}x)/\pi(x) \). Figures 13 through 15 show Landau’s (13), CDF (14) and actual (15) values of the same format as above, all plotted against the curve

\[
\frac{e^{c+1}x \log \log(e^{c+1}x)^3}{\log(e^{c+1}x)\beta! \, R(x)}
\]

which is shown as a dotted line.
6 Appendix

6.1 Mathematica code

cdf[k_, x_] :=
Re[N[
(2^-k E^(1 + E) x Log[1 + E + Log[2^-k x (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])]^(1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])])^(1/(Pi))]
(1 + E^EulerGamma)] + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])])^(1/(Pi))]
(1 + E + Log[2^-k x (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])] + (1 + E + Log[1/2 (Erfc[(1 + 8 E^EulerGamma - 4 k)/(Sqrt[2]
(1 + 8 E^EulerGamma)]) + 4 OwenT[(1 + 8 E^EulerGamma - 4 k)/(1 + 8 E^EulerGamma),
1/4 - E^EulerGamma)])]))]
1/(Pi))])
]

landau[k_, x_] := N[(x Log[Log[x]]^(-1 + k))/((-1 + k)! Log[x])]

actual[k_, x_] := N[Sum[1, ## & @@ Transpose[{#, Prepend[Most[#, 1], PrimePi@Prepend[Prime[First[#]]^(1 - k) Rest@FoldList[Times, x, Prime@First[#]/Prime@Most[#]], x^(1/k)]}]] &@Table[Unique[], {k}]]

References

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