Tangent and cotangent lifts and graded Lie algebras associated with Lie algebroids

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November 21, 2018

Abstract. Generalized Schouten, Frölicher-Nijenhuis and Frölicher-Richardson brackets are defined for an arbitrary Lie algebroid. Tangent and cotangent lifts of Lie algebroids are introduced and discussed and the behaviour of the related graded Lie brackets under these lifts is studied. In the case of the canonical Lie algebroid on the tangent bundle, a new common generalization of the Frölicher-Nijenhuis and the symmetric Schouten brackets, as well as embeddings of the Nijenhuis-Richardson and the Frölicher-Nijenhuis bracket into the Schouten bracket, are obtained.

Keywords: Lie algebroid, Schouten bracket, Frölicher-Nijenhuis bracket, Poisson structure, tangent lift, cotangent lift

MSC 1991: 17B70, 53C15, 17B66, 58F05

0 Introduction.

There are three main ingredients of the presented paper: to study natural graded Lie brackets of tensor fields associated with Lie algebroids over a manifold, procedures of lifting these tensor fields to tensor fields over derived manifolds (like lifting tensor fields on $M$ to tensor fields on $TM$), and the behaviour of the brackets under the lifting procedures. All this is done in the setting of an arbitrary Lie algebroid. We tried to make the presentation as systematic as possible, but still most of the presented results seem to be new in the literature (the most important can be found in Sections 5-7). We work coordinate-free as long as it is reasonable, but for the readers convenience we provide also detailed calculations in local coordinates.

Graded Lie algebras and Lie super-algebras play an important role in the contemporary geometry and mathematical physics. Fundamental examples are, defined in a natural way, graded extensions of the Lie bracket of vector fields: the Schouten (Schouten-Nijenhuis) bracket and the Frölicher-Nijenhuis bracket. Another example is the Lie algebra of a Lie group. A Lie algebroid is a common generalization of the notion of tangent bundle and the Lie algebra of a Lie group [20]. Lie algebroids have the basic ingredients which make possible usual constructions of differential geometry, like exterior derivative, Lie derivative, contractions, Schouten bracket, etc. (cf. [22]).

In the first section of the paper, we introduce natural analogues of the Schouten, Frölicher-Nijenhuis and Nijenhuis-Richardson brackets in the framework of an arbitrary Lie algebroid.
An important example of a Lie algebroid is provided by the cotangent bundle of a Poisson manifold. In Section 2, we present basic facts concerning Lie brackets related to Poisson structures, like Poisson bracket of functions, brackets of 1-forms, and their extensions.

Definitions of tangent and cotangent lifts of a Lie algebroid are given in Section 3, and in Section 4 we study tangent lifts of associated tensor fields. The results of Section 5 show the behaviour of the introduced graded brackets under the vertical and tangent lifts and generalize results of [31] and [10]. Similar questions for the cotangent lift, which seems to be less functorial, are studied in Section 6.

All above is applied in the last section to the case of the canonical Lie algebroid – the tangent bundle $TM$, in relation with our previous work [10]. Among other lifts, there is an embedding of the Frölicher-Nijenhuis bracket on a manifold $M$ into a graded Lie bracket of differential forms introduced earlier by the first author [9], into the Frölicher-Nijenhuis bracket on $T^*M$ (discovered by Dubois-Violette and Michor [7] and regarded as a part of a ‘common generalization’ of the Frölicher-Nijenhuis and the symmetric Schouten brackets), and embeddings of the Nijenhuis-Richardson and the Frölicher-Nijenhuis bracket on $M$ into the Schouten bracket on $T^*M$.

1. Graded Lie brackets of a Lie algebroid.

Let $M$ be a manifold and let $\tau: E \rightarrow M$ be a vector bundle. By $\Phi(\tau)$ we denote the graded exterior algebra generated by sections of $\tau$, $\Phi(\tau) = \bigoplus_{k \in \mathbb{Z}} \Phi^k(\tau)$, where $\Phi^k(\tau) = \Gamma(M, \wedge^k E)$ for $k \geq 0$ and $\Phi^k(\tau) = \{0\}$ for $k < 0$. The dual vector bundle we denote by $\pi:E^* \rightarrow M$. For $\tau = \tau_M:TM \rightarrow M$, we get the graded algebra of multivector fields on $M$, and for $\tau = \pi_M:T^*M \rightarrow M$ the graded algebra of differential forms on $M$.

Let us consider a Lie algebroid structure on $\tau:E \rightarrow M$ with the Lie bracket $[,]$ and the anchor $\alpha_E:E \rightarrow TM$. The Lie bracket $[,]$ of sections of $E$ induces the well-known generalization of the standard calculus of differential forms and vector fields ([20], [21], [22]).

The exterior derivative $d_\tau: \Phi^k(\pi) \rightarrow \Phi^{k+1}(\pi)$ is defined by the formula

$$d_\tau \mu(X_1, \ldots, X_{k+1}) = \sum_{i} (-1)^{i+1} \alpha_\tau(X_i)(\mu(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) + \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),$$

(1.1)

where $X_i \in \Phi^1(\tau)$ and the hat over a symbol means that this is to be omitted. For $X \in \Phi^k(\tau)$, the contraction $i_X: \Phi^p(\pi) \rightarrow \Phi^{p-k}(\pi)$ is defined in the standard way and the Lie differential operator

$$\mathcal{L}_X: \Phi^p(\pi) \rightarrow \Phi^{p-k+1}(\pi)$$

is defined by the graded commutator

$$\mathcal{L}_X = i_X \circ d_\tau - (-1)^k d_\tau \circ i_X.$$

The following theorem contains a list of well-known formulae.

**Theorem 1.** Let $\mu \in \Phi^k(\pi)$, $\nu \in \Phi(\pi)$ and $X, Y \in \Phi^1(\tau)$. We have

1. $d_\tau \circ d_\tau = 0$,
2. $d_\tau(\mu \wedge \nu) = d_\tau(\mu) \wedge \nu + (-1)^k \mu \wedge d_\tau(\nu)$,
3. $i_X(\mu \wedge \nu) = i_X(\mu) \wedge \nu + (-1)^k \mu \wedge i_X(\nu)$,
4. $\mathcal{L}_X(\mu \wedge \nu) = \mathcal{L}_X \mu \wedge \nu + \mu \wedge \mathcal{L}_X \nu$,
5. $\mathcal{L}_X \circ \mathcal{L}_Y = \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]}$,
6. $\mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X,Y]}$.

The last formula can be generalized in the following way (cf. [24], [16]).
Theorem 2. For \( X \in \Phi^{k+1}(\tau) \) and \( Y \in \Phi^{l+1}(\tau) \) there is a unique \( Z \in \Phi^{k+l+1}(\tau) \) (denoted by \([X,Y]\)) such that
\[
\mathcal{L}_Y \circ i_X - (-1)^{(k+1)l} i_X \circ \mathcal{L}_Y = -i_Z.
\]

In particular, we have for \( X \in \Phi^{1}(\tau) \) and \( f \in \Phi^{0}(\tau) = C^\infty(M) \)
\[
[X,f] = \alpha_\tau(X)(f).
\]
The bracket
\[
[ , ] : \Phi^{k+1}(\tau) \times \Phi^{l+1}(\tau) \rightarrow \Phi^{k+l+1}(\tau)
\]
is an extension of the Lie algebroid bracket on \( \Phi^{1}(\tau) \) and it is called the generalized Schouten bracket of the Lie algebroid. It defines a graded Lie algebra structure on \( \Phi(\tau) \) with the shifted grading \( \Phi(\tau) = \oplus_{k \in \mathbb{Z}} \Phi^{(k)}(\tau) \), where \( \Phi^{(k)}(\tau) = \Phi^{k+1}(\tau) \).

Moreover, for \( X \in \Phi^{(k)}(\tau) \), the adjoint action \( \text{ad}_X = [X, ] \) is a graded derivation of rank \( k \) of the exterior algebra, i.e.,
\[
[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{k} Y \wedge [X, Z] \quad (1.2)
\]
for \( Y \in \Phi^{l}(\tau) \).

The fact, that the generalized Schouten bracket is a graded Lie bracket, means that it is

1. graded skew-symmetric, i.e., for \( X \in \Phi^{(k)}(\tau) \), \( Y \in \Phi^{(l)}(\tau) \), we have
\[
[X, Y] = -(-1)^{kl}[Y, X]; \quad (1.3)
\]

2. satisfies the graded Jacobi identity, i.e., for \( X, Y \) as before and for \( Z \in \Phi^{(m)}(\tau) \), we have
\[
(-1)^{km}[[X, Y], Z] + (-1)^{lk}[[Y, Z], X] + (-1)^{ml}[[Z, X], Y] = 0. \quad (1.4)
\]

The Jacobi identity can be written in the form
\[
[X, [Y, Z]] - (-1)^{kl}[Y, [X, Z]] = [[X, Y], Z], \quad (1.5)
\]
which means that the graded commutator
\[
[\text{ad}_X, \text{ad}_Y] = \text{ad}_X \circ \text{ad}_Y - (-1)^{kl} \text{ad}_Y \circ \text{ad}_X
\]
equals \( \text{ad}_{[X,Y]} \).

From (1.2), it follows that for \( X_1, \ldots, X_k, Y_1, \ldots, Y_l \in \Phi^{1}(\tau) \), we have
\[
[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_k] = \sum_{i,j} (-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l \quad (1.6)
\]
(see [16], [24]).

There is also a symmetric Schouten bracket which extends the Schouten bracket on \( \Phi^{0}(\tau) \oplus \Phi^{1}(\tau) \) to symmetric multisections. This bracket is defined by the following formula, similar to (1.6) (see [7])
\[
[X_1 \vee \cdots \vee X_k, Y_1 \vee \cdots \vee Y_k] = \sum_{i,j} [X_i, Y_j] \vee X_1 \vee \cdots \vee \hat{X}_i \vee \cdots \vee X_k \vee Y_1 \vee \cdots \vee \hat{Y}_j \vee \cdots \vee Y_l. \quad (1.7)
\]

The symmetric Schouten bracket provides the symmetric algebra \( \Phi_s(\tau) = \oplus_{k \in \mathbb{Z}} \Phi^{(k)}(\tau) \), \( \Phi_s^k(\tau) = \Gamma(M, S^k E) \), with the abstract Poisson algebra structure ([2]).

In the case of the canonical skew-symmetric and symmetric Schouten brackets ([28], [25]).
Before we pass to other brackets, let us introduce the space $\Phi_k^i(\tau) = \Gamma(M, \wedge^k E \otimes \wedge^l E^*)$ of tensor fields of mixed type. Of course, we can identify $\Phi_k^i(\tau)$ and $\Phi_k^1(\tau)$. For $K \in \Phi_k^1(\tau)$, we define in the natural way the contraction

$$i_K: \Phi^n(\pi) \to \Phi^{n+k-1}(\pi).$$

For simple tensors $K = \mu \otimes X$, where $\mu \in \Phi_k^1(\tau)$, $X \in \Phi_1^i(\tau)$, we put

$$i_K \nu = \mu \wedge i_X \nu. \quad (1.8)$$

The corresponding Lie differential is defined by the formula

$$\mathcal{L}_K = i_K \circ d + (-1)^k d \circ i_K$$

and, in particular,

$$\mathcal{L}_{\mu \otimes X} = \mu \wedge \mathcal{L}_X + (-1)^k \mathcal{L}_\mu \wedge i_X. \quad (1.9)$$

This definition is compatible with the previous one in the case of $K \in \Phi_0^0(\tau) = \Phi^1(\tau)$.

The contraction (insertion) $i_K$ can be extended to an operator

$$i_K: \Phi^n(\pi) \to \Phi^{n+k-1}(\pi)$$

by the formula

$$i_K(\mu \otimes X) = i_K(\mu) \otimes X.$$

**THEOREM 3. The bracket**

$$[\cdot, \cdot]_{N-R}^k: \Phi_k^1(\pi) \times \Phi_k^1(\pi) \to \Phi_k^{k+1}(\pi),$$

given by the formula

$$[K, L]_{N-R}^k = i_K L - (-1)^{kl} i_L K, \quad (1.10)$$

defines a graded Lie algebra structure on the graded space $\Phi_1^i(\pi) = \oplus_{k \in \mathbb{Z}}\Phi_k^i(\pi)$, where $\Phi_k^i(\pi) = \Phi_k^i(\tau)$.

For simple tensors $\mu \otimes X \in \Phi_k^1(\pi)$ and $\nu \otimes Y \in \Phi_1^i(\pi)$, the following formula holds

$$[\mu \otimes X, \nu \otimes Y]_{N-R}^k = \mu \wedge i_X \nu \otimes Y + (-1)^k i_Y \mu \wedge \nu \otimes X. \quad (1.11)$$

**PROOF:** The bracket is obviously skew-symmetric. Now, let $K \in \Phi_k^i(\pi)$, $L \in \Phi_1^j(\pi)$ and $N \in \Phi_1^m(\pi)$. It is a simple and standard task to verify that

$$i_{[K, L]} = i_K \circ i_L - (-1)^{kl} i_L \circ i_K. \quad (1.12)$$

From (1.12) we get

$$[K, [L, N]_{N-R}^{N-R}]_{N-R}^k = i_K(i_L N - (-1)^l i_N L) - (-1)^{kl+n} i_L(i_N \circ i_N - (-1)^l i_N \circ i_L)K =$$

$$= i_K i_L N - (-1)^l i_K i_N L - (-1)^{kl+n} i_L i_N K + (-1)^{kl+n} i_L i_N i_L K$$

and, after easy calculations,

$$(-1)^{kn} [K, [L, N]_{N-R}^{N-R}]_{N-R}^k + (-1)^{kl} [L, [N, K]_{N-R}^{N-R}]_{N-R}^k + (-1)^{nl} [N, [K, L]_{N-R}^{N-R}]_{N-R}^k = 0.$$

The bracket $[\cdot, \cdot]_{N-R}^k$ is called the generalized Nijenhuis-Richardson bracket.

**Remark.** The generalized Nijenhuis-Richardson bracket is a purely vector bundle bracket and does not depend on any additional Lie algebroid structure. For $\tau = \tau_M$ we get the classical Nijenhuis-Richardson bracket of vector-valued forms ([23], [24], [26]).

The **generalized Frölicher-Nijenhuis bracket** is defined for simple tensors $\mu \otimes X \in \Phi_k^1(\pi)$ and $\nu \otimes Y \in \Phi_1^i(\pi)$ by the formula

$$[[\mu \otimes X, \nu \otimes Y]_{W-N}^k =$$

$$= \mu \wedge \nu \otimes [X, Y] + \mu \wedge \mathcal{L}_X \nu \otimes Y - \mathcal{L}_Y \mu \wedge \nu \otimes X + (-1)^k (d_r \mu \wedge i_X \nu \otimes Y + i_Y \mu \wedge d_r \nu \otimes X) =$$

$$= (\mathcal{L}_{\mu \otimes X} \nu) \otimes Y - (-1)^k (\mathcal{L}_{\nu \otimes Y} \mu) \otimes X + \mu \wedge \nu \otimes [X, Y]. \quad (1.13)$$

**THEOREM 4. The formula (1.13) defines a graded Lie bracket $[\cdot, \cdot]_{W-N}^k$ on the graded space $\Phi_1^i(\pi) = \oplus_{k \in \mathbb{Z}}\Phi_k^i(\pi)$**. Moreover,

$$\mathcal{L}_{[K, L]}_{W-N}^k = \mathcal{L}_K \circ \mathcal{L}_L - (-1)^{kl} \mathcal{L}_L \circ \mathcal{L}_K \quad (1.14)$$

for $K \in \Phi_k^1(\pi)$, $L \in \Phi_1^i(\pi)$. 

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Proof: The right-hand side in formula (1.13) is linear with respect to \( \mu, \nu, X, Y \), and for \( f \in C^\infty(M) \) we have

\[
[f \otimes X, \nu \otimes Y]^{F-N} = [\mu \otimes f X, \nu \otimes Y]^{F-N}.
\]

It follows that this formula defines actually a graded skew symmetric bracket

\[
[\cdot, \cdot]: \Phi^k_1(\pi) \times \Phi^k_1(\pi) \to \Phi^{k+1}_1(\pi).
\]

Standard calculations show that (1.14) holds and we use it to verify the Jacobi identity. For \( \mu \otimes X, \nu \otimes Y \) as above and \( \theta \otimes Z \in \Phi^m_1(\pi) \), we have

\[
[\theta \otimes Z, [\mu \otimes X, \nu \otimes Y]^{F-N}]^{F-N} = \left( -1 \right)^{kl} \left( \mathcal{L}_{\theta \otimes Z} \mathcal{L}_{\mu \otimes X} \nu \otimes Y - \mathcal{L}_{\theta \otimes Z} \mathcal{L}_{\nu \otimes Y} \mu \otimes X + \mathcal{L}_{\theta \otimes Z} (\mu \wedge \nu) \otimes [X, Y] - \left( -1 \right)^{k+l} \mathcal{L}_{\nu \otimes Y} \mathcal{L}_{\mu \otimes X} \theta \right) \otimes Z + \theta \wedge \mathcal{L}_{\mu \otimes X} \nu \otimes [Z, Y] - \left( -1 \right)^{k+l} \theta \wedge \mathcal{L}_{\nu \otimes Y} \mu \otimes [Z, Y].
\]

Now, with some effort and with the use of (1.14), one checks that

\[
\left( -1 \right)^{m} [\theta \otimes Z, [\mu \otimes X, \nu \otimes Y]^{F-N}]^{F-N} + \left( -1 \right)^{k} [\mu \otimes X, [\nu \otimes Y, \theta \otimes Z]^{F-N}]^{F-N} + \left( -1 \right)^{k} [\nu \otimes Y, [\theta \otimes Z, \mu \otimes X]^{F-N}]^{F-N} = 0.
\]

In the case of the canonical Lie algebroid (the tangent bundle), we obtain the classical Frölicher-Nijenhuis bracket on vector-valued forms ([8], [13], [7], [23]). Let us make a remark that, since, in general, the correspondence \( K \to \mathcal{L}_K \) is not injective, the identity (1.14) cannot be used as a definition of the generalized Frölicher-Nijenhuis bracket.

All the introduced graded Lie algebra structures have their purely algebraic versions. Lie algebroids correspond (depending on authors) to pseudo-Lie algebras, Lie-Rinehart algebras, Lie-Cartan pairs etc. ([16], [29]).

2. Graded Lie brackets on Poisson manifolds.

Let us consider a Poisson structure on a manifold \( M \), i.e., a bi-vector field \( P \in \Theta^2(\pi_M) \) such that \( [P, P] = 0 \). The tensor \( P \) defines a Lie algebra structure on \( C^\infty(M) \) by the Poisson bracket

\[
\{f, g\}_P = i_P(df \wedge dg).
\]

It is known ([3], [12], [18], [19]) that \( P \) induces also a Lie algebroid structure on the cotangent bundle \( \pi_M: T^\ast M \to M \). The anchor map of this structure is the corresponding to \( P \) morphism of vector bundles \( \tilde{P}: T^\ast M \to TM \). The Lie bracket is given by the formula

\[
[\mu, \nu]_P = \mathcal{L}_{\mathcal{L}_P(\mu)} \nu - \mathcal{L}_{\mathcal{L}_P(\nu)} \mu - \mathcal{L}_{\mathcal{L}_P(\mu \wedge \nu)}
\]

and the Lie algebroid exterior derivative \( d_{\pi_M}: \Phi(\pi_M) \to \Phi(\pi_M) \) is given by the Schouten bracket

\[
d_{\pi_M}(X) = [P, X]
\]

([2], [16], [11]).

The Lie algebroid structure on \( T^\ast M \) induces a generalized Schouten bracket \([\cdot, \cdot]_P\) of differential forms – the Koszul-Schouten bracket and a generalized Frölicher-Nijenhuis bracket \([\cdot, \cdot]^{F-N}_P\) of multivector valued 1-forms, described in the previous section.

The vector bundle morphism \( \tilde{P}: T^\ast M \to TM \) is a morphism of Lie algebroids and, consequently, the induced mapping

\[
\Lambda_P = \bigwedge_{k} \tilde{P}: \Phi(\pi_M) \to \Phi(\pi_M), \quad \bigwedge_{k} \tilde{P}: \bigwedge_{k} T^\ast M \to \bigwedge_{k} TM,
\]

\[
\Lambda_P(\mu_1 \wedge \cdots \wedge \mu_k) = \tilde{P}(\mu_1) \wedge \cdots \wedge \tilde{P}(\mu_k)
\]

gives us a homomorphism of the corresponding Schouten brackets ([17], [14], [15]).
Theorem 5.

(a) The mapping $\Lambda_P$ is a homomorphism between the Koszul-Schouten and the Schouten brackets

$$
\Lambda_P: (\Phi(\pi_M), [\cdot, \cdot]_P) \to (\Phi(\tau_M), [\cdot, \cdot]).
$$

It is an isomorphism if and only if $\tilde{P}$ is invertible, i.e., if and only if $P$ defines a symplectic structure.

(b) If $\tilde{P}$ is invertible, then the formula

$$
F_P(\mu \otimes X) = \Lambda_P(\mu) \otimes \Lambda_P^{-1}(X)
$$

defines an isomorphism of the Frölicher-Nijenhuis brackets

$$
F_P: (\Phi_k(\pi_M), [\cdot, \cdot]^{F-N}) \to (\Phi_k(\tau_M), [\cdot, \cdot]^{F-N}_P).
$$

Proof: Part (a) is essentially due to Koszul ([17]). Part (b) is obvious, because $\tilde{P}$ is an isomorphism of Lie algebroids and, consequently, respects the Lie algebroid brackets, exterior derivatives, contractions, Lie differentials and, finally, the Frölicher-Nijenhuis brackets.

Beside the homomorphism $\Lambda_P$ of graded commutative algebras, we can consider a module valued graded derivation of degree $-1$, introduced by Michor [23]:

$$
R_P: \Phi^k(\pi_M) \to \Phi^{k-1}_1(\pi_M),
$$

characterized by the properties

$$
R_P|_{\Phi^0(\pi_M)} = 0, \quad R_P|_{\Phi^1(\pi_M)} = \tilde{P},
$$

and

$$
R_P(\mu \wedge \nu) = R_P(\mu) \wedge \nu + (-1)^k \mu \wedge R_P(\nu),
$$

(2.4)

for $\mu \in \Phi^k(\pi_M)$, so that for 1-forms $\mu_1, \ldots, \mu_k$ we have

$$
R_P(\mu_1 \wedge \cdots \wedge \mu_k) = \sum_i (-1)^{i+1} \mu_1 \wedge \cdots \wedge \hat{\mu}_i \wedge \cdots \wedge \mu_k \otimes \tilde{P}(\mu_i).
$$

(2.5)

We can define $R_P$ in a more intrinsic way. Let $\tilde{\mu}: \Lambda^{k-1} TM \to T^*M$ be the bundle morphism corresponding to $\mu \in \Phi^k(\pi_M)$. The morphism $R_P(\mu): \Lambda^{k-1} TM \to TM$ corresponding to $R_P(\mu) \in \Phi^{k-1}_1(\pi_M)$ is defined by the formula

$$
R_P(\mu) = \tilde{P} \circ \tilde{\mu}.
$$

Let us define the hamiltonian map $H_P = R_P \circ d$ and the total hamiltonian map $G_P = \Lambda_P \circ d$. We have

$$
H_P: \Phi^k(\pi_M) \to \Phi^k_1(\pi_M)
$$

and

$$
G_P: \Phi^k(\pi_M) \to \Phi^{k+1}(\tau_M).
$$

For $f \in C^\infty(M)$, $H_P(f) = G_P(f) = \tilde{P}(df) = -[P, f]$ is the hamiltonian vector field corresponding to $f$. For a fixed $P$, we shall write also $R_P$, $H_P$, and $G_P$, instead of $R_P(\mu)$, $H_P(\mu)$ and $G_P(\mu)$. The Koszul-Schouten bracket may be written now in the form

$$
[\mu, \nu]_P = i_{H_P} \nu - (-1)^k L_{R_P} \nu
$$

(2.6)

for $\mu \in \Phi^k(\pi_M)$ ([1], [9]).

There is another graded Lie bracket on $\Phi(\pi_M)$, which extends directly the Poisson bracket $\{ \cdot, \cdot \}_P$ of functions.
Theorem 6 ([9]). On the graded space $\Phi(\pi_M) = \oplus_{k \in \mathbb{Z}} \Phi^k(\pi_M)$, the formula
\[
\{\mu, \nu\}_p = \mathcal{L}_{\mathcal{H}_\mu} \nu + d\mathcal{L}_{R_\mu} \nu
\]
(2.7)
defines a graded Lie algebra structure such that on functions $\{,\}_p$ is the Poisson bracket and $\{d\mu, \nu\}_p = d\{\mu, \nu\}_p$. Moreover, for $g_0, \ldots, g_k, f_0, \ldots, f_l \in C^\infty(M)$, we have
\[
\{g_0 d g_1 \wedge \cdots \wedge d g_k, f_0 d f_1 \wedge \cdots \wedge d f_l\}_p =
\{g_0, f_0\} d g_1 \wedge \cdots \wedge d g_k \wedge d f_1 \wedge \cdots \wedge d f_l
\]
\[
- g_0 \sum_{i > 0, j} (-1)^{i+j} d\{g_i, f_j\} \wedge d g_1 \wedge \cdots \wedge \widehat{d g_i} \wedge \cdots \wedge d g_k \wedge d f_1 \wedge \cdots \wedge \widehat{d f_j} \wedge \cdots \wedge d f_l
\]
\[
- (-1)^k f_0 \sum_{j > 0, i} (-1)^{i+j} d\{g_i, f_j\} \wedge d g_0 \wedge \cdots \wedge \widehat{d g_i} \wedge \cdots \wedge d g_k \wedge d f_1 \wedge \cdots \wedge \widehat{d f_j} \wedge \cdots \wedge d f_l
\]
\[
- \sum_{i, j > 0} (-1)^{i+j} \{g_i, f_j\} d g_0 \wedge \cdots \wedge \widehat{d g_i} \wedge \cdots \wedge d g_k \wedge d f_0 \wedge \cdots \wedge \widehat{d f_j} \wedge \cdots \wedge d f_l. \quad (2.8)
\]

Remark. The bracket (2.7) is similar to brackets introduced by Michor ([23]) and Cabras and Vinogradov ([6]). Their brackets, however, are either not skew-symmetric or do not satisfy the Jacobi identity, so that they are not graded Lie brackets. These brackets define a proper graded Lie bracket on co-exact forms which coincides with (2.7) projected to co-exact forms. The following theorem shows, how the Poisson bracket (2.7) is related to other graded Lie brackets associated with the Poisson structure.

Theorem 7 ([9]).
(a) The exterior derivative
\[
d: (\Phi(\pi_M), \{,\}_p) \to (\Phi(\pi_M), [\cdot,\cdot]_p)
\]
is a homomorphism of the Koszul-Schouten bracket into the bracket (2.7), i.e.,
\[
[d\mu, d\nu]_p = d\{\mu, \nu\}_p.
\]
The bracket $\{,\}_p$ may be then understood as an ‘integral’ of the Koszul-Schouten bracket.
(b) The Hamiltonian map
\[
\mathcal{H}_P: (\Phi(\pi_M), \{,\}_p) \to (\Phi_1(\pi_M), [\cdot,\cdot]^{F-N})
\]
is a homomorphism of $\{,\}_p$ into the Frölicher-Nijenhuis bracket:
\[
\mathcal{H}_P(\{\mu, \nu\}_p) = [\mathcal{H}_P(\mu), \mathcal{H}_P(\nu)]^{F-N}.
\]
(c) The total Hamiltonian map
\[
\mathcal{G}_P: (\Phi(\pi_M), \{,\}_p) \to (\Phi(\pi_M), [\cdot,\cdot])
\]
is a homomorphism of $\{,\}_p$ into the Schouten bracket:
\[
\mathcal{G}_P(\{\mu, \nu\}_p) = [\mathcal{G}_P(\mu), \mathcal{G}_P(\nu)].
\]
(d) If $\widetilde{P}$ is invertible, i.e., if $P$ defines a symplectic form $\omega$, then the isomorphism
\[
\Lambda_P: \Phi(\pi_M) \to \Phi(\tau_M)
\]
transports $\{,\}_p$ to an ‘integral’ $\{,\}_\omega$ of the Schouten bracket.
3. Tangent and cotangent lifts of Lie algebroids.

Let $\tau: E \to M$ be a vector bundle with a Lie algebroid structure and let $\alpha_{\tau}: E \to TM$ be the anchor map. Let $P$ be the corresponding linear Poisson structure on the dual bundle $\pi: E^* \to M$. Linearity of this Poisson structure means that the bracket of functions which are linear on fibers is linear on fibers ([22], [4], [5]). There is a one-to-one correspondence $X \to \iota(X)$ between sections of $E$ and linear functions on $E^*$: $\iota(\mu) = \langle X(\pi(\mu)), \mu \rangle$. The relation between the Lie algebroid and the Poisson bracket is given by

$$\iota([X, Y]) = \{\iota(X), \iota(Y)\}_P.$$  \hfill (3.1)

For each $X \in \Phi^1(\tau)$, we define a vector field $\mathcal{G}(X)$ on $E^*$ putting

$$\mathcal{G}(X) = \bar{P}(\iota(X)) = H_P(\iota(X)) = -[P, \iota(X)].$$

The vector field $\mathcal{G}(X)$ is projectable and

$$\alpha_{\tau}(X) = \pi_* \mathcal{G}(X), \text{ i.e., } \pi^*(\alpha_{\tau}(X)(f)) = \{\iota(X), \pi^*f\}.$$  \hfill (3.2)

The tangent space $TE^*$ carries the structure of a double vector bundle ([27], [20], [21])

$$\begin{array}{ccc}
TE^* & \xrightarrow{\tau_{E^*}} & TM \\
\tau E^* \downarrow & & \downarrow \tau_M \\
E^* & \xrightarrow{\pi} & M
\end{array}.$$  \hfill (3.3)

There is a natural lift of the Poisson tensor $P$ on $E^*$ to a Poisson tensor $d_{\tau}P$ on $TE^*$ ([5], [10]) which is linear with respect to both vector bundle structures on $TE^*$. It follows, that we have two bundles with Lie algebroid structures - dual bundles with respect to the horizontal and vertical bundle structures in (3.3).

The bundle dual to $\tau_{E^*}: TE^* \to E^*$ is $\pi_{E^*}: T^*E^* \to E^*$ with the canonical pairing. The bundle dual to $\tau: TE \to TM$ can be identified with the bundle $\tau_T: TE \to TM$. The pairing is the tangent pairing obtained from the canonical pairing $\langle ., . \rangle: E \times_M E^* \to \mathbb{R}$ by applying the tangent functor ([30], [10]).

The Lie algebroid structures on $\pi_{E^*}: T^*E^* \to E^*$ and $\tau_T: TE \to TM$ are called the tangent and cotangent lifts of the Lie algebroid structure on $E$.

Let us see all this in local coordinates. Let $(x^a)$ be a local coordinate system on $M$ and let $e_1, \ldots, e_m$ be basis of local sections of $E$. The dual basis of local sections of $E^*$ we denote by $e^1, \ldots, e^m$. The corresponding coordinates in $E^*$ we denote by $(x^a, \xi_i)$ and the coordinates in $E$ by $(x^a, y^i)$, i. e., $\iota(e_i) = \xi_i$ and $\iota(e^*\iota) = y^i$. Since $P$ is a linear Poisson structure, we have

$$P = \frac{1}{2} \varepsilon_{ij}^k (x) \xi_k \partial_{\xi_j} \wedge \partial_{\xi_i} + \delta_i^a (x) \partial_{x^a} \wedge \partial_{x^i},$$  \hfill (3.4)

where $\varepsilon_{ij}^k, \delta_i^a$ depend on $x$ only and where we clearly use the summation convention. The anchor map is given by

$$\alpha_{\tau}(y^i e_i(x)) = y^i \delta_i^a (x) \partial_{x^a}$$  \hfill (3.5)

and the Lie bracket by

$$[e_i, e_j] = \varepsilon_{ij}^k e_k,$$  \hfill (3.6)

so that

$$[f^i e_i, g^j e_j] = \left( f^i g^j \varepsilon_{ij}^k \frac{\partial g^k}{\partial x^a} - g^i \delta_i^a \frac{\partial f^j}{\partial x^a} \right) e_k.$$  \hfill (3.7)

The exterior derivative $d_{\tau}$ is determined by

$$d_{\tau}(f) = \delta_i^a \frac{\partial f}{\partial x^a} e^{*i}, \quad d_{\tau}(e^{*k}) = \frac{1}{2} \varepsilon_{ij}^k e^{*i} \wedge e^{*j}.$$  \hfill (3.8)
Let us introduce the adopted coordinate systems 

\[(x^a, \dot{x}^b) \text{ in } TM,\]
\[(x^a, y^i, \dot{x}^b, \dot{y}^i) \text{ in } TE,\]

and \((x^a, \xi_i, \dot{x}^b, \xi_j) \text{ in } TE^*\).

In these coordinates, we have (5), (10):

\[
d_T = \ell_0^i(x)\xi_k \partial_{\xi_i} \wedge \partial_{\xi_i} + \frac{1}{2} \ell_0^j(x)\xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \ell_0^i \partial_{\xi_i} \wedge \partial_{x^a} + \frac{1}{2} \ell_0^b(x)\xi_k \partial_{\xi_i} \wedge \partial_{\xi_j}.
\]

For sections \(dx^a\) and \(d\xi_i\) of the Lie algebroid bundle \(\pi_{E^*}: T^* E^* \to E^*\), we have \(\iota(dx^a) = \dot{x}^a\), \(\iota(d\xi_i) = \dot{\xi}_i\), and

\[
\begin{align*}
[dx^a, dx^b] &= 0, \\
[d\xi_i, dx^a] &= \frac{\partial \xi_i}{\partial x^b}(x) dx^b, \\
[d\xi_i, d\xi_j] &= \ell_{ij}^k(x) d\xi_k + \frac{\partial \xi_i}{\partial x^b}(x)\xi_k \frac{\partial \xi_j}{\partial x^a}(x) dx^a.
\end{align*}
\]

The anchor map is given by

\[
\begin{align*}
\alpha_{\pi_{E^*}}(dx^a) &= -\delta_i^a(x) \partial_{\xi_i}, \\
\alpha_{\pi_{E^*}}(d\xi_i) &= \ell_i^j(x) \partial_{x^a} + \ell_{ij}^k(x)\xi_k \partial_{\xi_j},
\end{align*}
\]

and the exterior derivative \(d\pi_{E^*}\) on \(\Phi(\tau_{E^*})\) is determined by

\[
\begin{align*}
d\pi_{E^*}(f) &= \left( \delta_i^a \frac{\partial f}{\partial x^a} + \ell_{ij}^k \xi_k \frac{\partial f}{\partial \xi_j} \right) \partial_{\xi_i} - \delta_i^a \frac{\partial f}{\partial \xi_i} \partial_{x^a}, \\
d\pi_{E^*}(\partial_{x^a}) &= -\frac{\partial \xi_k}{\partial x^a} \partial_{\xi_i} \wedge \partial_{x^b} - \frac{1}{2} \frac{\partial \xi_k}{\partial x^a}(x) \partial_{\xi_i} \wedge \partial_{\xi_j}, \\
d\pi_{E^*}(\partial_{\xi_k}) &= \frac{1}{2} \ell_{ij}^k \partial_{\xi_i} \wedge \partial_{\xi_j}.
\end{align*}
\]

Now, let us turn to the Lie algebroid bundle \(T\tau: TE \to TM\) and its dual \(\tau\pi: TE^* \to TM\). We choose a basis \((\bar{e}_i, \dot{e}_j)\) of local sections of \(T\tau: TE \to TM\), such that \(\iota(\bar{e}_i) = \dot{\xi}_i\) and \(\iota(\dot{e}_j) = \dot{\xi}_j\), and a similar basis \((\bar{e}^i, \dot{e}^j)\) of local sections of \(\tau\pi: TE^* \to TM\), such that \(\iota(\bar{e}^i) = y^i\), \(\iota(\dot{e}^j) = \dot{y}^j\). In the chosen bases, section \(\bar{e}_i\) is dual to \(\dot{e}^i\) and \(\dot{e}_j\) is dual to \(e^j\), with respect to the pairing between \(TE\) and \(TE^*\).

The anchor map \(\alpha_T: TE \to TM\) is given by

\[
\begin{align*}
\alpha_T(\bar{e}_i) &= \delta_i^a(x) \partial_{x^a} + \frac{\partial \delta_i^a}{\partial x^b}(x)\dot{x}^b \partial_{x^a}, \\
\alpha_T(\dot{e}_j) &= \delta_j^a(x) \partial_{x^a},
\end{align*}
\]

and the Lie bracket by

\[
\begin{align*}
[\bar{e}_i, \bar{e}_j] &= 0, \\
[\bar{e}_i, \dot{e}_j] &= \ell_{ij}^k(x)\bar{e}_k, \\
[\dot{e}_i, \dot{e}_j] &= \ell_{ij}^k(x)\dot{e}_k + \frac{\partial \ell_{ij}^k}{\partial x^a}(x)\dot{e}^a \bar{e}_k.
\end{align*}
\]

The exterior derivative is determined by
\[ d_{\tau}(f) = \left( \delta^a_i \frac{\partial f}{\partial x^a} + \frac{\partial \delta^a_i}{\partial x^b} \frac{\partial f}{\partial x^b} \right) \tilde{e}^a_i + \delta^i_a \frac{\partial f}{\partial x^a} \tilde{e}^a_i, \]

\[ d_{\tau}(e^* k) = \frac{1}{2} e^*_{ij}(x) \tilde{e}^* i \wedge \tilde{e}^* j, \]

\[ d_{\tau}(e^{*k}) = e^{*}_{ij}(x) \tilde{e}^{* i} \wedge \tilde{e}^{* j} + \frac{1}{2} \frac{\partial e^*_{ij}}{\partial x^a}(x) \tilde{x}^a \tilde{e}^{* i} \wedge \tilde{e}^{* j}. \] (3.15)

To make our presentation more complete, we end this section with a simple theorem on iterated lifts.

**Proposition.** Let \( \tau: E \rightarrow M \) be a Lie algebroid. Then the Lie algebroid structures provided by tangent-cotangent, cotangent-tangent and cotangent-cotangent lifts are canonically isomorphic.

**Proof:** The tangent and cotangent lifts of the algebroid correspond to the tangent lift \( d_{\tau} P \) of the Poisson structure \( P \) on the dual bundle \( \pi: E^* \rightarrow M \). Thus the iterated lifts correspond to the tangent of the tangent Poisson structure, i.e., the Poisson structure \( d_{\tau} d_{\tau} P \) on \( TTE^* \). Since \( d_{\tau} P \) is linear with respect to both vector bundle structures on \( T^* E^* \), the Poisson structure \( d_{\tau} d_{\tau} P \) is linear with respect to the three vector bundle structures on \( TTE^* \).

The Lie algebroid on the bundle \( \pi_{TE^*}: T^* TE^* \rightarrow TE^* \), dual to \( \tau_{TE^*}: TTE^* \rightarrow TE^* \), is, by definition, the cotangent lift of the tangent lift of the Lie algebroid and, at the same time, the cotangent lift of the cotangent lift of the Lie algebroid.

The bundle dual to \( T\pi_{E^*}: T^* E^* \rightarrow TE^* \) is the bundle \( T\pi_{TE^*}: T^* TE^* \rightarrow TE^* \) and the corresponding Lie algebroid structure is the tangent lift of the tangent lift of the algebroid \( \tau: E \rightarrow M \).

There is well known canonical isomorphism (see Section 7) \( \kappa_{E^*} \) between \( \tau_{TE^*}: T^* TE^* \rightarrow TE^* \) and \( T\pi_{TE^*}: T^* TE^* \rightarrow TE^* \), and the Poisson structure \( d\tau_{\pi_{TE^*}} P \) is invariant with respect to it (cf. [10]). It follows, that the dual bundles are isomorphic and this isomorphism is an isomorphism of Lie algebroid structures.

\[ \square \]

**4. Tangent lifts of sections of vector bundles associated with a Lie algebroid.**

In [31] (see also the references there) there are described different types of lifts of tensor fields from a manifold to the tangent and cotangent bundles. In [10], the tangent lifts were studied from the categorial point of view and were applied to Poisson geometry (Courant in [4], [5] did the first steps in that direction). All these lifts are related to the canonical Lie algebroid structure in the tangent bundle \( \tau_M: TM \rightarrow M \). Here we show, how the lifting procedures can be generalized to the case of an arbitrary Lie algebroid.

Let us recall first lifting processes in the case of the canonical Lie algebroid. For a function \( f \in C^\infty(M) \), we define the vertical and complete lifts \( v_{\tau}(f), d_{\tau} f \in C^\infty(TM) \):

\[ v_{\tau}(f) = \tau^*_M(f), \quad d_{\tau} f(v) = \langle v, df \rangle, \] (4.1)

i.e., \( v_{\tau}(f) \) is the pull-back and \( d_{\tau} f \) is the exterior derivative \( df \), regarded as a function on \( TM \). The vertical and complete lifts \( \tau^*_M(X), d_{\tau} X \) of a vector field \( X \) on \( M \) can be defined as unique vector fields on \( TM \) such that

\[ \tau^*_M(X)(v_{\tau}(f)) = 0, \]

\[ \tau^*_M(X)(d_{\tau} f) = d_{\tau} X(v_{\tau}(f)) = \tau^*_M(X(f)), \]

\[ d_{\tau} X(d_{\tau} f) = d_{\tau} X(f) \] (4.2)

(see [31], [10]).
In local coordinates \(((x^a)\) on \(M\) and \((\dot{x}^a, \dot{x}^b)\) on \(TM\), we have
\[
\nu_T(f)(x, \dot{x}) = f(x), \quad d_T(f)(x, \dot{x}) = \frac{\partial f}{\partial x^a}(x)\dot{x}^a, \tag{4.3}
\]
and
\[
\nu_T(f^a\partial_{x^a}) = f^a\partial_{x^a}, \quad d_T(f^a\partial_{x^a}) = f^a\partial_{x^a} + \frac{\partial f^a}{\partial x^b}\dot{x}^b\partial_{x^a}. \tag{4.4}
\]

Let \((\tau_j: E_j \to M)\) be a family of vector bundles. We are going to define the complete and vertical lifts of sections of the bundle \(\otimes M E_j \to M\) to sections of the bundle \(\otimes_T M E_j \to TM\).

In the case of a section \(X\) of a vector bundle \(\tau: E \to M\), we can proceed in the following way. The section \(X\) defines a linear function \(\iota(X)\) on \(E^*\). The functions \(d_T(\iota(X))\) and \(\nu_T(\iota(X))\) on \(TE^*\) are linear with respect to the vector bundle structure \(T\pi: TE^* \to TM\). It follows that these functions define sections of the dual bundle \(T\pi: TE \to TM\). We denote by \(\mathcal{T}(X)\) the section which corresponds to \(d_T(\iota(X))\) and by \(\mathcal{V}(X)\) the section which corresponds to \(\nu_T(\iota(X))\).

We can extend this procedure to the case of sections of the bundle \(\otimes M E_j \to M\). For every such section \(X\), we define a multilinear function \(\tilde{X}: \times M E^*_j \to \mathbb{R}\) by
\[
\tilde{X}(\mu_1, \ldots, \mu_k) = i_X(\mu_1 \wedge \cdots \wedge \mu_k).
\]
In the case of \(X \in C^\infty(M)\) understood as a section of the 0-tensor product, we put \(\tilde{X} = X\).

**Theorem 8.** For every section \(X \in \Gamma(M, \otimes M E_j)\), there are, uniquely determined, sections \(\mathcal{V}(X), \mathcal{T}(X) \in \Gamma(M, \otimes_T M E_j)\) such that
\[
(a) \quad \mathcal{V}(X) = \nu_T(\tilde{X}),
(b) \quad \mathcal{T}(X) = d_T(\tilde{X}) \tag{4.5}
\]
(we identify \(T(x M E_j)\) and \(\times_T M E_j\)).

In particular,
\[
\mathcal{V}(f) = \nu_T(f) \quad \text{and} \quad \mathcal{T}(f) = d_T f \quad \text{for } f \in C^\infty(M),
\]
and
\[
\mathcal{V}(e_{1,i_1} \otimes \cdots \otimes e_{k,i_k}) = \tilde{e}_{1,i_1} \otimes \cdots \otimes \tilde{e}_{k,i_k}, \quad \mathcal{T}(e_{1,i_1} \otimes \cdots \otimes e_{k,i_k}) = \sum_j \tilde{e}_{1,i_1} \otimes \cdots \otimes \tilde{e}_{j,i_j} \otimes \cdots \otimes \tilde{e}_{k,i_k},
\]
where \((e_{j,i})\) is a basis of local sections of \(\tau_j\).

**Proof:** Let \((e_{j,i})\) be a basis of local sections of \(\tau_j\) and let \((x^a, y^j), (\dot{x}^a, \dot{y}^j)\) be the corresponding coordinate system in \(E_j, E^*_j\), \(j = 1, \ldots, k\). These coordinate systems induce a coordinate system \((x^a, \xi_1, i_1, \ldots, \xi_k, i_k)\) on \(\times M E^*_j\). In the introduced coordinate systems and bases,
\[
X = f^{i_1 \cdots i_k} e_{1,i_1} \otimes \cdots \otimes e_{k,i_k}
\]
and
\[
\tilde{X} = f^{i_1 \cdots i_k} \xi_{1,i_1} \cdots \xi_{k,i_k}. \tag{4.6}
\]
In the adopted coordinates \((x^a, \xi_1, i_1, \ldots, \xi_k, i_k, \dot{x}^a, \dot{\xi}_{1,i_1}, \ldots, \dot{\xi}_{k,i_k})\) on \(\times_T M E^*_j\), the vertical lift \(\nu_T(\tilde{X})\) looks formally like (4.6). It follows that \(\nu_T(\tilde{X}) = \mathcal{V}(\tilde{X})\), where
\[
\mathcal{V}(X) = f^{i_1 \cdots i_k} \tilde{e}_{1,i_1} \otimes \cdots \otimes \tilde{e}_{k,i_k}. \tag{4.7}
\]
The complete lift $d_T(\tilde{X})$ is given in local coordinates by (cf. (4.3))

$$d_T(\tilde{X}) = \frac{\partial f_{i_1\ldots i_k}}{\partial x^a} \tilde{x}_{i_1, i_1} \cdot \ldots \cdot \tilde{x}_{i_k, i_k} + \sum_j f_{i_1\ldots i_k} \xi_{j, i_j} \cdot \ldots \cdot \xi_{k, i_k}$$

and, consequently,

$$\tau(X) = \frac{\partial f_{i_1\ldots i_k}}{\partial x^a} \xi_1, i_1 \cdot \ldots \cdot \xi_{i_k, i_k} + \sum_j f_{i_1\ldots i_k} \epsilon_{j, i_j} \cdot \ldots \cdot \epsilon_{i_k, i_k}. \quad (4.8)$$

Let us notice that both lifts commute with permutations and, consequently, lifts of skew-symmetric and symmetric tensors are skew-symmetric and symmetric, respectively.

There is another (equivalent) way to introduce the vertical and complete lifts. Again, we begin with the case of a section $X: M \to E_i$ of $\tau$. Applying the tangent functor to $X$, we obtain $TX: TM \to TE$. Since $\tau \circ X = \text{id}_M$, we have $T \tau \circ T X = \text{id}_M$, i.e., $TM$ is a section of $T \tau: TE \to TM$. It is easy to check that $TM = \tau(X)$. In order to get a similar construction of $T(X)$ for $X \in \Gamma(M, \otimes^i_M E_i)$, we observe first that applying the tangent functor to the tensor product map $\otimes: \times^i_M E_i \to \otimes^i_M E_i$, we get a bundle morphism with respect to the tangent bundle structures

$$T \otimes: T \times^i_M E_i \simeq \times^i_M T E_i \to T \otimes^i_M E_i.$$ 

This morphism is multilinear and, consequently, defines a linear morphism

$$\otimes_T: \otimes^i_M T E_i \to T \otimes^i_M E_i.$$ 

We have also the dual morphism of the dual bundles

$$\otimes^T: T \otimes^i_M E_i^* \to \otimes^i_M T E_i^*.$$ 

If we replace $E_i$ by $E_i^*$, we obtain

$$\otimes^T: T \otimes^i_M E_i \to \otimes^i_M T E_i.$$ 

The complete lift $T(X)$ of a section $X: M \to \otimes^i_M E_i$ can be now defined by

$$T(X) = \otimes^T \circ TX.$$ 

For the vertical lift $V(X)$, we provide a different approach which makes use of the structure of a double vector bundle on $TE$:

$$\begin{matrix}
TE & \xrightarrow{T \tau} & TM \\
\tau E & \downarrow \quad & \quad \downarrow \tau_M \\
E & \xrightarrow{\tau} & M
\end{matrix}$$

Since $\tau_E \ (T \tau)$ is a vector bundle morphism with respect to the horizontal (vertical) vector bundle structures, we have two kernel subbundles of these projections:

$$V_T E = \ker \tau_E \text{ is a subbundle of } T \tau: TE \to TM,$$

$$V_r E = \ker T \tau \text{ is a subbundle of } T \tau: TM \to E.$$
It is known that there are canonical vector bundle isomorphisms
\[ V_T E \simeq TM \times_M E, \quad V_\tau \simeq E \times_M E, \]
which can be extended to isomorphisms of tensor products
\[ TM \times_M \otimes^i_M E_i \simeq \otimes^i_T M V_T E_i \subset \otimes^i_TM TE_i, \]
\[ E \times_M \otimes^k_M E \simeq \otimes^k_T M V_\tau E \subset \otimes^k_TE. \] (4.9)

With these identifications, we can define vertical lifts \( \mathcal{V}_T \) and \( \mathcal{V}_\tau \) of sections of \( \otimes^i_M E_i \) and \( \otimes^k_M E \) respectively. In local coordinates, we have
\[
\begin{align*}
\mathcal{V}_T(f^{i_1 \ldots i_k}e_{i_1} \otimes \cdots \otimes e_{i_k}) &= f^{i_1 \ldots i_k} v_{i_1} \otimes \cdots \otimes v_{i_k}, \\
\mathcal{V}_\tau(f^{j_1 \ldots j_k}e_{j_1} \otimes \cdots \otimes e_{j_k}) &= f^{j_1 \ldots j_k} \partial_{y_{j_1}} \otimes \cdots \otimes \partial_{y_{j_k}}.
\end{align*}
\] (4.10)

The vertical lift \( \mathcal{V}_T \) coincides with \( \mathcal{V} \), the vertical lift \( \mathcal{V}_\tau \) will be used in the definition of cotangent lifts.

**Theorem 9.** Let \( \tau_i: E_i \rightarrow M \) and \( \tau'_j: E'_j \rightarrow M \) be vector bundles over \( M \), and let \( X \in \Gamma(M, \otimes^i_M E_i) \) or \( X = f \in C^\infty(M), Y \in \Gamma(M, \otimes^k_M E'_j) \). Then,
\[
\begin{align*}
(a) \quad & \mathcal{V}(X \otimes_M Y) = \mathcal{V}(X) \otimes_M \mathcal{V}(Y), \\
(b) \quad & \mathcal{T}(X \otimes_M Y) = \mathcal{T}(X) \otimes_M \mathcal{V}(Y) + \mathcal{V}(X) \otimes_M \mathcal{T}(Y).
\end{align*}
\] (4.11)

**Proof:** Using the canonical projections
\[
\begin{align*}
\left( \otimes^i_M E^*_i \right) \times_M \left( \otimes^k_M E^*_j \right) & \rightarrow \otimes^i_M E^*_i \\
\left( \otimes^i_M E^*_i \right) \times_M \left( \otimes^k_M E^*_j \right) & \rightarrow \otimes^k_M E^*_j,
\end{align*}
\]
and
\[
\begin{align*}
\left( \otimes^i_M E^*_i \right) \times_M \left( \otimes^k_M E^*_j \right) & \rightarrow \otimes^i_M E^*_i \\
\left( \otimes^i_M E^*_i \right) \times_M \left( \otimes^k_M E^*_j \right) & \rightarrow \otimes^k_M E^*_j,
\end{align*}
\]
we can consider \( \tilde{X} \) and \( \tilde{Y} \) as functions on \( \left( \otimes^i_M E^*_i \right) \times_M \left( \otimes^k_M E^*_j \right) \). We have then
\[
X \otimes_M Y = \tilde{X} \cdot \tilde{Y}.
\]

The statements of the theorem are now direct consequences of the following properties of the vertical and complete lifts of functions (cf. [10]):
\[
\begin{align*}
\nu_T(\tilde{X} \cdot \tilde{Y}) &= \nu_T(\tilde{X}) \cdot \nu_T(\tilde{Y}), \\
d_T(\tilde{X} \cdot \tilde{Y}) &= d_T(\tilde{X}) \cdot \nu_T(\tilde{Y}) + \nu_T(\tilde{X}) \cdot d_T(\tilde{Y}).
\end{align*}
\]
\]

**Corollary.** Let \( \tau: E \rightarrow M \) be a vector bundle.
(a) The vertical lift
\[ \mathcal{V}: \Phi(\tau) \rightarrow \Phi(T\tau) \]
is a homomorphism of graded commutative algebras:
\[ \mathcal{V}(X \land Y) = \mathcal{V}(X) \land \mathcal{V}(Y). \]
(b) The complete lift
\[ \mathcal{T}: \Phi(\tau) \rightarrow \Phi(T\tau) \]
is a graded \( \mathcal{V} \)-derivation of degree 0:
\[ \mathcal{T}(X \land Y) = \mathcal{T}(X) \land \mathcal{V}(Y) + \mathcal{V}(X) \land \mathcal{T}(Y). \]
5. Tangent lifts and graded Lie brackets.

Throughout this section $\tau : E \to M$ is a vector bundle of a Lie algebroid with the anchor $\alpha : E \to \mathcal{T}M$. We shall prove theorems describing the behaviour of contractions, Lie differentials, graded Lie brackets, etc., with respect to the tangent lifts.

**Theorem 10.** Let $X \in \Phi^1(\tau)$. Then,

(a) $\alpha_{\tau^*}(\mathcal{V}(X)) = \nu_{\tau}(\alpha_\tau(X))$,
(b) $\alpha_{\tau^*}(\mathcal{T}(X)) = d\nu(\alpha_{\tau}(X)).$

**Proof:** (a) With the notation of Section 3, we have, in view of (3.13),

$$\alpha_{\tau^*}(\mathcal{V}(f^i e_i)) = \alpha_{\tau^*}(f^i \overline{e}_i) = f^i \alpha_{\tau^*}(\overline{e}_i) = f^i \delta_0^a \partial_{\bar{z}^a}.$$ 

On the other hand, it follows from (4.4) that

$$\nu_{\tau}(\alpha_{\tau^*}(f^i e_i)) = \nu_{\tau}(f^i \delta_0^a \partial_{\bar{z}^a}) = f^i \delta_0^a \partial_{\bar{z}^a}.$$

(b) From (3.13) and (4.8), we get

$$\alpha_{\tau^*}(\mathcal{T}(f^i e_i)) = \alpha_{\tau^*} \left( \frac{\partial f^i}{\partial x^a} \tilde{\partial}_{\bar{z}^a} + f^i \tilde{e}_i \right) = \frac{\partial f^i}{\partial x^a} \alpha_{\tau^*}(\overline{e}_i) + f^i \alpha_{\tau^*}(\tilde{e}_i) = \frac{\partial f^i}{\partial x^a} \delta_0^a \partial_{\bar{z}^a} + f^i \delta_0^a \partial_{\bar{z}^a}.$$

On the other hand, from (4.4), we have

$$d\nu(\alpha_{\tau^*}(f^i e_i)) = d\nu(f^i \delta_0^a \partial_{\bar{z}^a}) = f^i \delta_0^a \partial_{\bar{z}^a} + \frac{\partial}{\partial x^a} \left( f^i \delta_0^a \partial_{\bar{z}^a} \right)$$

and the theorem follows. \[\square\]

The behaviour of the Schouten bracket with respect to vertical and complete lifts is described by the following theorem.

**Theorem 11.** Let $X, Y \in \Phi(\tau)$. Then,

(a) $\{\mathcal{V}(X), \mathcal{V}(Y)\} = 0$,
(b) $\{\mathcal{V}(X), \mathcal{T}(Y)\} = [\mathcal{T}(X), \mathcal{V}(Y)] = \mathcal{V}([X, Y])$,
(c) $[\mathcal{T}(X), \mathcal{T}(Y)] = \mathcal{T}([X, Y])$, i.e., the complete lift is a homomorphism of the Schouten brackets.

**Proof:** (a) We have for the tangent Lie algebroid ((3.13), (3.14)) $\alpha_{\tau^*}(\overline{e}_i) = \delta_0^a \partial_{\bar{z}^a}$ and $[\overline{e}_i, \overline{e}_j] = 0$. Since $\mathcal{V}(f^i e_i) = f^i \overline{e}_i$, we get $[\mathcal{V}(X), \mathcal{V}(Y)] = 0$.

(b) The proof goes by induction with respect to the degree of $X$ and $Y$. Let us take first $X \in \Phi^1(\tau)$ and $Y = f \in C^\infty(M)$. It follows from Theorem 10 and (4.2) that

$$[\mathcal{V}(X), \mathcal{T}(f)] = \alpha_{\tau^*}(\mathcal{V}(X))(d\nu f) = \nu_{\tau^*}(\alpha_{\tau}(X))(d\nu f) = \nu_{\tau^*}(\alpha_{\tau}(X)(f)) = \mathcal{V}([X, f]).$$

Similarly, $[\mathcal{T}(X), \mathcal{V}(f)] = \mathcal{V}([X, f])$.

Assume now that $X, Y \in \Phi^1(\tau)$. The tangent lift $\{, \}_{\mathcal{T}}$ of the Poisson bracket $\{, \}_P$ has the following property (cf. [4], [5], [10])

$$\nu_{\tau}(\{f, g\}_P) = \{d\nu f, \nu_{\tau}(g)\}_{\mathcal{T}} = \{\nu_{\tau}(f), d\nu g\}_{\mathcal{T}}.$$

Since $\iota([X, Y]) = \{\iota(X), \iota(Y)\}_P$, we get by definitions of the lifts $\mathcal{V}, \mathcal{T}$ ((4.5)) that

$$\iota([\mathcal{V}(X), \mathcal{T}(Y)]) = \{\iota(\mathcal{V}(X)), \iota(\mathcal{T}(Y))\}_{\mathcal{T}} = \{\nu_{\tau}(\iota(X)), d\nu(\iota(Y))\}_{\mathcal{T}} = \nu_{\tau}(\{\iota(X), \iota(Y)\}_P) = \nu_{\tau}(\iota([X, Y])) = \iota(\mathcal{V}([X, Y])).$$
It follows that \([V(X),T(Y)] = V([X,Y])\) and that
\[
[T(X), V(Y)] = -[V(Y), T(X)] = -V([X,Y]) = V([Y,X]).
\]

Now, it remains to show that if (b) holds for \(X \in \Phi^k(\tau),\ Y \in \Phi^j(\tau)\), then it holds also for \(X\) and \(Y \land Z,\ Z \in \Phi^1(\tau)\). We get from (a), from Corollary to Theorem 9, and from (1.2) that
\[
[V(X), T(Y \land Z)] = [V(X), T(Y) \land V(Z) + V(Y) \land T(Z)] = \\
= [V(X), T(Y)] \land V(Z) + (-1)^{(k-1)} T(Y) \land [V(X), V(Z)] + [V(X), V(Y)] \land T(Z) + \\
+ (-1)^{(k-1)} V(Y) \land [V(X), T(Z)] = V([X,Y]) \land V(Z) + (-1)^{(k-1)} V(Y) \land V([X,Z]) = \\
= V([X,Y] \land Z + (-1)^{(k-1)} Y \land [X,Z]) = V([X,Y \land Z]).
\]

The equality \([T(X), V(Y \land Z)] = V([X,Y \land Z])\) follows in an analogous way.

(c) As in (b), we check the equality for \(X \in \Phi^1(\tau)\) and \(Y = \partial f \in C^\infty(M)\). We have in this case
\[
[T(X), T(f)] = \alpha_\tau(T(X))(d_\tau f) = d_\tau(\alpha_\tau(X))(d_\tau f) = \\
d_\tau(\alpha_\tau(X)(f)) = d_\tau([X,f]) = T([X,f]).
\]

If \(X,Y \in \Phi^1(\tau)\), then
\[
\alpha([T(X), T(Y)]) = \{\alpha(T(X)), \alpha(T(Y))\} d_\tau sp = \{d_\tau \alpha(X), d_\tau \alpha(Y)\} d_\tau sp = \\
= d_\tau \{\alpha(X), \alpha(Y)\} p = d_\tau(\alpha([X,Y])) = \alpha(T([X,Y])).
\]

Now, the proof proceeds like in the part (b).

The following theorem contains a collection of formulae describing the behaviour of inserting operators, exterior derivatives, and Lie derivatives under the tangent lifts.

**Theorem 12.** Let \(\mu \in \Phi(\pi)\) and \(X \in \Phi^1(\tau)\). Then,
\[
(1) \quad a) \quad i_{V(X)} V(\mu) = 0, \\
\quad b) \quad i_{V(X)} T(\mu) = i_{T(X)} V(\mu) = V(i_X \mu), \\
\quad c) \quad i_{T(X)} T(\mu) = T(i_X \mu), \\
(2) \quad a) \quad d_\tau T(V(\mu)) = V(d_\tau \mu), \\
\quad b) \quad d_\tau T(\mu) = T(d_\tau \mu), \\
(3) \quad a) \quad L_{V(X)} V(\mu) = 0, \\
\quad b) \quad L_{V(X)} T(\mu) = L_{T(X)} V(\mu) = V(L_X \mu), \\
\quad c) \quad L_{T(X)} T(\mu) = T(L_X \mu).
\]

**Proof:** (1) As in the proof of Theorem 8, we choose a basis \((e_i)\) of local sections of \(\tau: E \to M\). We have then \(X = f^j e_j, \mu = \sum_{i_1 < \cdots < i_k} \mu_{i_1 \cdots i_k} e^{i_1} \land \cdots \land e^{i_k}, V(X) = f^j \tau_j, \) and
\[
V(\mu) = \sum_{i_1 < \cdots < i_k} \mu_{i_1 \cdots i_k} \tau^{i_1} \land \cdots \land \tau^{i_k}.
\]

Since the basis \((\epsilon^*_i, \tau_j)\) is dual (with respect to the tangent pairing) to the basis \((\tau_i, \epsilon_j)\), we have \(i_{V(X)} V(\mu) = 0\). With
\[
T(X) = \frac{\partial f^j}{\partial x^i} \epsilon^*_i \tau_j + f^j \epsilon_j,
\]
we get
\[ i_{\tau(X)} \mathcal{V}(\mu) = \sum_{i_1 < \ldots < i_k} (-1)^{j+1} \mu_{i_1 \ldots i_k} f^j e^{i_1} \wedge \cdots \wedge e^{i_j} \wedge \cdots \wedge e^{i_k} = \mathcal{V} \left( \sum_{i_1 < \ldots < i_k} (-1)^{j+1} \mu_{i_1 \ldots i_k} f^j e^{i_1} \wedge \cdots \wedge e^{i_j} \wedge \cdots \wedge e^{i_k} \right) = \mathcal{V}(i_{\tau} \mu). \]

Similarly, we obtain for
\[ \mathcal{T}(\mu) = \sum_{i_1 < \ldots < i_k} \mu_{i_1 \ldots i_k} e^{i_1} \wedge \cdots \wedge e^{i_j} \wedge \cdots \wedge e^{i_k} + \sum_{i_1 < \ldots < i_k} \frac{\partial \mu_{i_1 \ldots i_k}}{\partial x^a} e^{i_1} \wedge \cdots \wedge e^{i_j} \wedge \cdots \wedge e^{i_k} \]
that \( i_{\tau(X)} \mathcal{T}(\mu) = \mathcal{V}(i_{\tau} \mu). \)

Finally,
\[ i_{\tau(X)} \mathcal{T}(\mu) = \sum_{i_1 < \ldots < i_k} (-1)^{j+1} \frac{\partial \mu_{i_1 \ldots i_k}}{\partial x^a} f^j e^{i_1} \wedge \cdots \wedge e^{i_j} \wedge \cdots \wedge e^{i_k} + \sum_{i_1 < \ldots < i_k} \mu_{i_1 \ldots i_k} \partial f^j \wedge \cdots \wedge e^{i_1} \wedge \cdots \wedge e^{i_j} \wedge \cdots \wedge e^{i_k} = \mathcal{T} \left( \sum_{i_1 < \ldots < i_k} (-1)^{j+1} \mu_{i_1 \ldots i_k} f^j e^{i_1} \wedge \cdots \wedge e^{i_j} \wedge \cdots \wedge e^{i_k} \right) = \mathcal{T}(i_{\tau} \mu). \]

(2) For \( \mu \in \Phi^k(\pi), \nu \in \Phi(\pi), \) we have
\[ \mathcal{V}(d_{\tau}(\mu \wedge \nu)) = \mathcal{V}(d_{\tau}(\mu) \wedge \nu + (-1)^k \mu \wedge d_{\tau}(\nu)) = \mathcal{V}(d_{\tau}(\mu)) \wedge \mathcal{V}(\nu) + (-1)^k \mathcal{V}(\mu) \wedge \mathcal{V}(d_{\tau}(\nu)). \]

On the other hand,
\[ d_{\tau}(\mathcal{V}(\mu \wedge \nu)) = d_{\tau}(\mathcal{V}(\mu)) \wedge \mathcal{V}(\nu) + \mathcal{V}(d_{\tau}(\mu)) \wedge \mathcal{V}(\nu) + (-1)^k \mathcal{V}(\mu) \wedge d_{\tau}(\mathcal{V}(\nu)). \]

It follows, that in order to prove (a), it is enough to consider the case of \( \mu \) being a function and \( \mu = e^{s_k}. \) For \( \mu = f \in C^\infty(M) \) and \( X \in \Phi^1(\tau), \) we have, using (1) and Theorem 10,
\[ \langle \mathcal{V}(d_{\tau} f), \mathcal{T}(X) \rangle = i_{\tau(X)} \mathcal{V}(d_{\tau} f) = \mathcal{V}(i_X d_{\tau} f) = \mathcal{V}(\alpha_{\tau}(X)(f)) = d_{\tau}(\alpha_{\tau}(X))(\nu_{\tau}(f)) = \alpha_{\tau}(\mathcal{T}(X))(\mathcal{V}(f)) = \langle d_{\tau}(\mathcal{V}(f)), \mathcal{T}(X) \rangle. \]

We have also
\[ \langle \mathcal{V}(d_{\tau} f), \mathcal{V}(X) \rangle = 0 = \nu_{\tau}(\alpha_{\tau}(X))(\nu_{\tau}(f)) = \alpha_{\tau}(\mathcal{V}(X))(\mathcal{V}(f)) = \langle d_{\tau}(\mathcal{V}(f)), \mathcal{V}(X) \rangle. \]

Since \( \mathcal{V}(X), \mathcal{T}(X) \) generate sections of the \( T\tau\)-bundle, it follows that \( d_{\tau}(\mathcal{V}(f)) = \mathcal{V}(d_{\tau} f). \)

Similarly, if \( \mu = e^{s_k}, \) then
\[ \mathcal{V}(d_{\tau}(\mu)) = \mathcal{V}(\frac{1}{2} e^{s_i} \wedge e^{s_j}) = \frac{1}{2} e^{s_i} \wedge e^{s_j} = d_{\tau}(e^{s_k}) = d_{\tau}(\mathcal{V}(\mu)). \]
To prove (b), we show first, using (a), that
\[ \mathcal{T}(d_r(\mu \land \nu)) = \mathcal{T}(d_r(\mu) \land \nu + (-1)^k \mu \land d_r(\nu)) = \]
\[ = \mathcal{T}(d_r(\mu)) \land \mathcal{V}(\nu) + \mathcal{V}(d_r(\mu)) \land \mathcal{T}(\nu) + (-1)^k \mathcal{T}(\mu) \land \mathcal{V}(d_r(\nu)) + (-1)^k \mathcal{V}(\mu) \land \mathcal{T}(d_r(\nu)) = \]
\[ = \mathcal{T}(d_r(\mu)) \land \mathcal{V}(\nu) + d_r(\mathcal{T}(\nu)) \land \mathcal{T}(\nu) + (-1)^k \mathcal{T}(\mu) \land d_r(\mathcal{V}(\nu)) + (-1)^k \mathcal{V}(\mu) \land d_r(\mathcal{T}(\nu)). \]
On the other hand,
\[ d_r(\mathcal{T}(\mu \land \nu)) = d_r(\mathcal{T}(\mu) \land \mathcal{V}(\nu) + \mathcal{V}(\mu) \land \mathcal{T}(\nu)) = \]
\[ = d_r(\mathcal{T}(\mu)) \land \mathcal{V}(\nu) + (-1)^k \mathcal{T}(\mu) \land d_r(\mathcal{V}(\nu)) + d_r(\mathcal{T}(\nu)) \land \mathcal{T}(\nu) + (-1)^k \mathcal{V}(\mu) \land d_r(\mathcal{T}(\nu)). \]
This shows that it is enough to prove (b) for \( \mu = f \) and \( \mu = e^{\ast k} \). Let us take \( \mu = f \in C^\infty(M) \) and \( X \in \Phi^1(\tau) \). We have
\[ \langle \mathcal{T}(d_r f), \mathcal{T}(X) \rangle = i_{\mathcal{T}(X)} \mathcal{T}(d_r f) = d_r \langle i_X d_r f \rangle = d_r \langle \alpha_r(X)(f) \rangle = d_r \langle \alpha_r(X) \rangle(d_r f) = \]
\[ = d_r(\mathcal{T}(f)) = (d_r(\mathcal{T}(f)), \mathcal{T}(X)) \]
and
\[ \langle \mathcal{T}(d_r f), \mathcal{V}(X) \rangle = i_{\mathcal{V}(X)} \mathcal{T}(d_r f) = \nu_r(i_X d_r f) = \alpha_r(\mathcal{V}(X))(d_r f) = \langle d_r(\mathcal{T}(f)), \mathcal{V}(X) \rangle. \]
It follows that \( \mathcal{T}(d_r f) = d_r(\mathcal{T}(f)). \)
For \( \mu = e^{\ast k} \), we have
\[ \mathcal{T}(d_r(e^{\ast k})) = \mathcal{T}\left( \frac{1}{2} \partial^k \partial x^i \wedge e^{\ast j} \right) = c^k_{ij} e^{\ast i} \wedge \partial x^j + \frac{1}{2} \frac{\partial^k e^{\ast i}}{\partial x^j} \partial x^j \wedge \partial x^j = d_r(\mathcal{T}(e^{\ast k})). \]
(3) Since \( L_X = i_X \circ d_r + d_r \circ i_X \), we get, using (1) and (2), that
\[ L_{\mathcal{V}(X)} \mathcal{T}(\mu) = i_{\mathcal{V}(X)} d_r(\mathcal{T}(\mu)) + d_r i_{\mathcal{V}(X)} \mathcal{T}(\mu) = i_{\mathcal{V}(X)} \mathcal{T}(d_r(\mu)) + d_r i_{\mathcal{V}(X)} \mu = \]
\[ = \mathcal{V}(i_X d_r(\mu) + d_r i_X(\mu)) = \mathcal{V}(L_X \mu). \]
The rest of the proof is analogous. \( \blacksquare \)

Remark. Sections \( \mathcal{T}(X) \) span the \( \mathcal{T}-\)bundle over each point except for the zero section of \( TM \). It follows that in the proof above we can use the continuity argument, instead of considering the case of sections \( \mathcal{V}(X) \).

**Theorem 13.** Let \( K, L \in \Phi^1(\pi) \). Then,
(a) \[ [\mathcal{V}(K), \mathcal{V}(L)]^{N-R} = 0, \]
(b) \[ [\mathcal{V}(K), \mathcal{L}(L)]^{N-R} = [\mathcal{T}(K), \mathcal{V}(L)]^{N-R} = \mathcal{V}([K, L]^{N-R}), \]
(c) \[ [\mathcal{T}(K), \mathcal{L}(L)]^{N-R} = [\mathcal{T}(K), [L]^{N-R}], \]
it. e., the complete lift is a homomorphism between Nijenhuis-Richardson brackets on \( \Phi^1(\pi) \) and \( \Phi^1(\tau) \).

**Proof:** The theorem follows immediately from (1.11) and Theorem 12 (1). Indeed, for \( K = \mu \otimes X \in \Phi^1(\pi), L = \nu \otimes Y \), we have
\[ [\mathcal{V}(K), \mathcal{V}(L)]^{N-R} = [\mathcal{V}(\mu) \otimes \mathcal{V}(X), \mathcal{V}(\nu) \otimes \mathcal{V}(Y)]^{N-R} = \]
\[ = \mathcal{V}(\mu) \wedge i_{\mathcal{V}(X)} \mathcal{V}(\nu) \otimes \mathcal{V}(Y) + (-1)^k i_{\mathcal{V}(Y)} \mathcal{V}(\mu) \wedge \mathcal{V}(\nu) \otimes \mathcal{V}(X) = 0. \]
Similarly,
\[ [\mathcal{V}(K), \mathcal{L}(L)]^{N-R} = [\mathcal{V}(\mu) \otimes \mathcal{V}(X), \mathcal{T}(\nu) \otimes \mathcal{V}(Y) + \mathcal{V}(\nu) \otimes \mathcal{T}(Y)]^{N-R} = \]
\[ = \mathcal{V}(\mu) \wedge i_{\mathcal{V}(X)} \mathcal{T}(\nu) \otimes \mathcal{V}(Y) + \mathcal{V}(\nu) \otimes \mathcal{T}(\nu) \otimes \mathcal{V}(Y) + (-1)^k i_{\mathcal{V}(Y)} \mathcal{V}(\mu) \wedge \mathcal{T}(\nu) \otimes \mathcal{V}(X) + (-1)^k i_{\mathcal{T}(Y)} \mathcal{V}(\mu) \wedge \mathcal{V}(\nu) \otimes \mathcal{V}(X) = \]
\[ = \mathcal{V}(\mu \otimes X \nu \otimes Y) + (-1)^k i_{\mathcal{V}(Y)} \mathcal{V}(\mu) \wedge \mathcal{V}(\nu) \otimes X = \mathcal{V}([\mu \otimes X, \nu \otimes Y]^{N-R}). \]
(c) can be proved by analogous calculations. \( \blacksquare \)

We have also similar formulae for the Frölicher-Nijenhuis brackets.
Theorem 14. Let $K, L \in \Phi_1(\pi)$. Then,

(a) $[V(K), V(L)]^{F-N} = 0$,
(b) $[V(K), T(L)]^{F-N} = [T(K), V(L)]^{F-N} = V([K, L]^{F-N})$,
(c) $[T(K), T(L)]^{F-N} = T([K, L]^{F-N})$, i.e., the complete lift $T$ is a homomorphism between Frölicher-Nijenhuis brackets on $\Phi_1(\pi)$ and $\Phi_1(T\pi)$.

Proof: We make use of formulae of Theorem 11 and Theorem 12. First, let us recall that in the local coordinates introduced in Section 4, we have

\[
\tau^{\alpha} = \tau^{\alpha} + (1)^{k}(d_{\tau}^{\alpha} \mu \nabla Y + i_{Y} \mu \nabla X),
\]

so that

\[
[V(K), V(L)]^{F-N} = V(\mu) \nabla \nabla (X, Y) + V(\mu) \nabla \nabla (Y) - \mathcal{L}_{Y} V(\mu) \nabla \nabla (X) + \nabla \nabla (Y) + \nabla \nabla (X) = 0.
\]

Similarly,

\[
[V(K), T(L)]^{F-N} = [V(\mu) \nabla \nabla (X), T(Y)]^{F-N} = V(\mu) \nabla \nabla (X) + T(Y) + \nabla \nabla (Y) = V(\mu) \nabla \nabla (Y) + T(Y) + \nabla \nabla (X) = V(\mu) \nabla \nabla (X, Y) + \nabla \nabla (Y) + \nabla \nabla (X) = 0.
\]

The remaining part of the proof consists of analogous easy calculations and we skip them.

\section{Cotangent lifts and graded Lie brackets.}

We shall study cotangent lifts of tensor fields associated with a Lie algebroid. They are, however, defined only for certain types of tensor fields and they do not have so nice functorial properties as the tangent lifts. On the other hand, they can be very useful, as we shall see later. Let $\tau: E \to M$ be a vector bundle with a Lie algebroid structure, let $\alpha_{\cdot}: E \to TM$ be the anchor map of this structure and let $P$ be the corresponding linear Poisson structure on the dual bundle $\pi: E^{*} \to M$. We define the (cotangent) complete lift

\[ G: \Phi^{1}(\tau) \to \Phi^{1}(\tau_{E^{*}}) \]

by the formula

\[ G(X) = \bar{P}(\alpha(X)) = H_{P}(\iota(X)) = -[P, \iota(X)]. \]

In the local coordinates introduced in Section 4, we have

\[ G(f^{*}e_{i}) = \left(f^{*}e_{i}^{k} \xi_{k} - \frac{\partial f^{i}}{\partial x^{a}} \delta^{a}_{i} \xi_{j} \right) \partial_{\xi_{j}} + f^{*} \delta_{i}^{a} \partial_{x^{a}}. \]
For $\mu \in \Phi^k(\pi)$, we have the vertical lift $\mathcal{V}_\pi(\mu) \in \Phi^k(\tau_E)$ (compare with (4.7)). In local coordinates,

$$\mathcal{V}_\pi(\sum_{i_1<\cdots<i_k} \mu_{i_1\cdots i_k} e^{*i_1} \cdots e^{*i_k}) = \sum_{i_1<\cdots<i_k} \mu_{i_1\cdots i_k} \partial_{\xi_{i_1}} \cdots \partial_{\xi_{i_k}}.$$  \hspace{1cm} (6.2)

**Theorem 15.** Let $\mu, \nu \in \Phi(\pi)$ and let $Y, X \in \Phi^1(\tau)$. Then,

(a) $\mathcal{V}_\pi(\mu \wedge \nu) = \mathcal{V}_\pi(\mu) \wedge \mathcal{V}_\pi(\nu)$;
(b) $[\mathcal{V}_\pi(\mu), \mathcal{V}_\pi(\nu)] = 0$;
(c) $[\iota(X), \mathcal{V}_\pi(\mu)] = -\mathcal{V}_\pi(\iota X \mu)$;
(d) $[P, \mathcal{V}_\pi(\mu)] = \mathcal{V}_\pi(\delta_X \mu)$;
(e) $[\mathcal{G}(X), \mathcal{V}_\pi(\mu)] = \mathcal{V}_\pi(L_X \mu)$;
(f) $[\mathcal{G}(X), \iota(Y)] = \mathcal{G}([X, Y])$;
(g) $[\mathcal{G}(X), \pi(Y)] = \iota([X, Y])$,

where brackets on the left-hand sides are classical Schouten brackets and brackets on the right-hand sides are Lie algebroid brackets.

**Proof:** (a) and (b) follow immediately from (6.2).

(c) Since $\text{ad}_{\iota(X)}$ and $i_X$ are graded derivations of degree -1 and $\mathcal{V}_\pi$ is, in view of (a), a homomorphism, both sides of (c) are graded derivations of degree -1 with respect to $\mu$. It follows by induction that it is enough to consider the case of $\mu = e^X$. We have for $\mu = e^X$, $X = f^i e_i$,

$$[\iota(X), \mathcal{V}_\pi(e^X)] = [f^i \xi_i, \partial_{\xi_i}] = -f^k = -\mathcal{V}_\pi(\iota X e^X).$$

(d) Similarly as in (c), we observe first that both sides of (d) are graded derivations of degree 1 with respect to $\mu$. It follows then that it is enough to prove (d) for $\mu \in C^\infty(M)$ and $\mu = e^X$. Let $X \in \Phi^1(\tau)$. The formulae (3.2) and (c) imply that

$$i_{\delta(\lambda X)}[P, \mathcal{V}_\pi(f)] = [[P, \pi^* f], \iota(Y)] = -[[P, \iota(Y)], \pi^* f] = [\mathcal{G}(Y), \pi^* f] = \pi^* (\alpha_X)(f)) = \mathcal{V}_\pi(\iota \nu d_X f) = -[\iota(Y), \mathcal{V}_\pi(d_X f)] = i_{\delta(\nu Y)} \mathcal{V}_\pi(d_X f).$$

We conclude that $[P, \mathcal{V}_\pi(f)] = \mathcal{V}_\pi(\iota \nu d_X f)$.

For $\mu = e^X$, we have in local coordinates (with $P$ as in (3.4))

$$[P, \mathcal{V}_\pi(e^X)] = [P, \partial_{\xi_i}] = \left[ \frac{1}{2} \sum_{i,j} \sigma_{ij} \partial_{\xi_i} \wedge \partial_{\xi_j} + \delta^a_i \partial_{\xi_i} + \partial_{\xi_i} \wedge \partial_{\xi_i} \right] = \frac{1}{2} \sum_{i,j} \sigma_{ij} \partial_{\xi_i} \wedge \partial_{\xi_j}.$$

On the other hand,

$$\mathcal{V}_\pi(d_X e^X) = \mathcal{V}_\pi \left( \frac{1}{2} \sum_{i,j} \sigma_{ij} e^{*i} \wedge e^{*j} \right) = \frac{1}{2} \sum_{i,j} \sigma_{ij} \partial_{\xi_i} \wedge \partial_{\xi_j}$$

and (d) follows.

(e) According to (c) and (d), we have

$$\mathcal{V}_\pi(L_X \mu) = \mathcal{V}_\pi(\delta_X \iota_X \mu) + \mathcal{V}_\pi(\iota_X \delta_X \mu) = [P, \mathcal{V}_\pi(\iota_X \mu)] - [\iota(X), \mathcal{V}_\pi(\delta_X \mu)] = -[P, [\iota(X), \mathcal{V}_\pi(\mu)]] - [\iota(X), [P, \mathcal{V}_\pi(\mu)]] = -[P, [\iota(X), \mathcal{V}_\pi(\mu)]] - [\mathcal{G}(X), \mathcal{V}_\pi(\mu)].$$

(f) Using the graded Jacobi identity and the formula $[P, P] = 0$, we get

$$[\mathcal{G}(X), \mathcal{G}(Y)] = [[P, \iota(X)], [P, \iota(Y)]] = [P, [[P, \iota(X)], \iota(Y)]] = [P, \iota([X, Y])] = -[P, \iota([X, Y])] = \mathcal{G}([X, Y]).$$

(g) We have

$$[\mathcal{G}(X), \iota(Y)] = -[\iota(X), \iota(Y)] = \{\iota(X), \iota(Y)\} = \iota([X, Y]).$$

\[\blacksquare\]
Theorem 16. The formulae
\[ J(\mu \otimes X) = -\iota(X)V_\pi(\mu) \] (6.3)
and
\[ G(\mu \otimes X) = [P, J(\mu \otimes X)] = G(X) \wedge V_\pi(\mu) - \iota(X)V_\pi(d_\pi \mu), \] (6.4)
for simple tensors \( \mu \otimes X \in \Phi_1(\pi) \), define linear mappings
\[ J, G : \Phi_1(\pi) \to \Phi(\tau_{E^*}) \]
which coincide on \( \Phi^0_1(\pi) \) with already defined \(-\iota \) and \( G \) respectively.

Proof: The right-hand side of (6.3) is linear with respect to \( X \) and \( \mu \) and for \( f \in C^\infty(M) \) we have
\[ J((f \mu) \otimes X) = -\iota(X)V_\pi(f \mu) = -\pi^*(f\mu(X)V_\pi(\mu) = -\iota(fX)V_\pi(\mu) = J(\mu \otimes (fX)). \]

Consequently, \( J \) is well defined. Since (6.4) is the composition of \( J \) and the Schouten bracket with \( P \), also \( G \) is well defined. Of course, we have, for \( X \in \Phi^1(\pi), \)
\[ J(X) = -\iota(X), \quad G(X) = -[P, \iota(X)]. \]

We call \( G(K) \) the dual complete lift of \( K \in \Phi_1(\pi) \). We shall show that \( J \) and \( G \) are homomorphisms of graded Lie algebras.

Theorem 17. The mapping
\[ J : \Phi_1(\pi) \to \Phi(\tau_{E^*}) \]
is an injective homomorphism of the Nijenhuis-Richardson bracket into the Schouten bracket:
\[ J([K, L]^{N-R}) = [J(K), J(L)]. \]
In particular, in the case of the canonical Lie algebroid, we get an embedding of the Nijenhuis-Richardson bracket on \( M \) into the Schouten bracket on \( T^*M \).

Proof: Let \( K = \mu \otimes X \in \Phi^k_1(\pi) \) and \( L = \nu \otimes Y \in \Phi_1(\pi) \). It follows from Theorem 15 ((a), (b),(c)) and from (1.5) that
\[ [J(K), J(L)] = [\iota(X)V_\pi(\mu), \iota(Y)V_\pi(\nu)] = [\iota(X)V_\pi(\mu), \iota(Y)] \wedge V_\pi(\nu) + \iota(Y)[\iota(X)V_\pi(\mu), V_\pi(\nu)] = \iota(X)V_\pi(\mu) \wedge V_\pi(\nu) - \iota(Y)V_\pi(\mu) \wedge V_\pi(\nu) = \iota(X)V_\pi(\mu \wedge i_X \nu) + \iota(Y)V_\pi(\mu \wedge i_Y \nu) = J(\mu \wedge i_X \nu \otimes Y + (-1)^k i_Y \mu \wedge \nu) = J([K, L]^{N-R}). \]

The injectivity follows from the local form of \( J \) on \( \Phi^k_1(\pi) \):
\[ J \left( \sum_{i_1 < \cdots < i_k \cdot j} \mu^{j}_{i_1 \cdots i_k}(x) e^{x_{i_1}} \wedge \cdots \wedge e^{x_{i_k}} \otimes e_j \right) = - \sum_{i_1 < \cdots < i_k \cdot j} \mu^{j}_{i_1 \cdots i_k}(x) \xi_j \partial_{\xi_{i_1}} \wedge \cdots \wedge \partial_{\xi_{i_k}}. \]

It is evident that \( J(K) = 0 \) if and only if \( K = 0 \).
Theorem 18. The dual complete lift

\[ \mathcal{G}: \Phi_1(\pi) \to \Phi(\tau_{\mathcal{E}}) \]

is a homomorphism of the Frölicher–Nijenhuis bracket into the Schouten bracket:

\[ \mathcal{G}([K, L]^{F-N}) = [\mathcal{G}(K), \mathcal{G}(L)]. \]

Proof: Let \( K = \mu \otimes X \in \Phi_1(\pi) \) and \( L = \nu \otimes Y \in \Phi_1(\pi) \). Using the graded Jacobi identity and Theorem 15, we get three identities:

\[ [\mathcal{G}(X) \wedge \mathcal{V}_\pi(\mu) \mathcal{G}(Y) \wedge \mathcal{V}_\pi(\nu)] = [\mathcal{G}(X), \mathcal{G}(Y)] \wedge \mathcal{V}_\pi(\mu) \wedge \mathcal{V}_\pi(\nu) + \]
\[ + \mathcal{G}(X) \wedge [\mathcal{V}_\pi(\mu), \mathcal{G}(Y)] \wedge \mathcal{V}_\pi(\nu) + \mathcal{G}(Y) \wedge \mathcal{V}_\pi(\mu) \wedge [\mathcal{G}(X), \mathcal{V}_\pi(\nu)] = \]
\[ = \mathcal{G}([X, Y]) \wedge \mathcal{V}_\pi(\mu \wedge \nu) - \mathcal{G}(X) \wedge \mathcal{V}_\pi(\mathcal{L}_Y \mu \wedge \nu) + \mathcal{G}(Y) \wedge \mathcal{V}_\pi(\mu \wedge \mathcal{L}_X \nu), \quad (1) \]

\[ [\mathcal{G}(X) \wedge \mathcal{V}_\pi(\mu), \iota(Y) \mathcal{V}_\pi(\mathcal{d}_\tau \nu)] = \]
\[ = (-1)^k [\mathcal{G}(X), \iota(Y)] \wedge \mathcal{V}_\pi(\mu) \wedge \mathcal{V}_\pi(\mathcal{d}_\tau \nu) + \mathcal{G}(X) \wedge [\mathcal{V}_\pi(\mu), \iota(Y)] \wedge \mathcal{V}_\pi(\mathcal{d}_\tau \nu) + \]
\[ + (-1)^k \iota(Y) \mathcal{V}_\pi(\mu) \wedge [\mathcal{G}(X), \mathcal{V}_\pi(\mathcal{d}_\tau \nu)] = \]
\[ = (-1)^k \iota([X, Y]) \mathcal{V}_\pi(\mu \wedge \mathcal{d}_\tau \nu) - (-1)^k \mathcal{G}(X) \wedge \mathcal{V}_\pi(\mathcal{I}_Y \mu \wedge \mathcal{d}_\tau \nu) + (-1)^k \iota(Y) \mathcal{V}_\pi(\mu \wedge \mathcal{L}_X \mathcal{d}_\tau \nu), \quad (2) \]

\[ [\iota(X) \mathcal{V}_\pi(\mathcal{d}_\tau \mu), \iota(Y) \mathcal{V}_\pi(\mathcal{d}_\tau \nu)] = \]
\[ = \iota(X) [\mathcal{V}_\pi(\mathcal{d}_\tau \mu), \iota(Y)] \wedge \mathcal{V}_\pi(\mathcal{d}_\tau \nu) + \iota(Y) \mathcal{V}_\pi(\mathcal{d}_\tau \mu) \wedge [\iota(X), \mathcal{V}_\pi(\mathcal{d}_\tau \nu)] = \]
\[ = (-1)^k \iota(X) \mathcal{V}_\pi(\mathcal{I}_Y \mathcal{d}_\tau \mu \wedge \mathcal{d}_\tau \nu) - \iota(Y) \mathcal{V}_\pi(\mathcal{d}_\tau \mu \wedge \mathcal{I}_X \mathcal{d}_\tau \nu). \quad (3) \]

Hence,

\[ [\mathcal{G}(K), \mathcal{G}(L)] = [\mathcal{G}(X) \wedge \mathcal{V}_\pi(\mu) - \iota(X) \mathcal{V}_\pi(\mathcal{d}_\tau \mu), \mathcal{G}(Y) \wedge \mathcal{V}_\pi(\nu) - \iota(Y) \mathcal{V}_\pi(\mathcal{d}_\tau \nu)] = \]
\[ = \mathcal{G}([X, Y]) \wedge \mathcal{V}_\pi(\mu \wedge \nu) - \iota([X, Y]) \mathcal{V}_\pi(\mathcal{d}_\tau \mu \wedge \nu + (-1)^k \mu \wedge \mathcal{d}_\tau \nu) + \mathcal{G}(Y) \wedge \mathcal{V}_\pi(\mu \wedge \mathcal{L}_X \nu) - \]
\[ - \iota(Y) \mathcal{V}_\pi(\mathcal{d}_\tau \mu \wedge \mathcal{I}_X \mathcal{d}_\nu + (-1)^k \mu \wedge \mathcal{L}_X \mathcal{d}_\tau \nu) - \mathcal{G}(X) \wedge \mathcal{V}_\pi(\mathcal{L}_Y \mu \wedge \nu) + \]
\[ + \iota(X) \mathcal{V}_\pi(\mathcal{L}_Y \mathcal{d}_\tau \mu \wedge \nu + (-1)^k \mathcal{I}_Y \mathcal{d}_\tau \mu \wedge \mathcal{d}_\nu) + (-1)^k \mathcal{G}(Y) \wedge \mathcal{V}_\pi(\mathcal{d}_\tau \mu \wedge \mathcal{I}_X \nu) + \]
\[ + (-1)^k \mathcal{G}(X) \wedge \mathcal{V}_\pi(\mathcal{I}_Y \mu \wedge \mathcal{d}_\tau \nu) = \mathcal{G}(\mu \wedge \nu \otimes [X, Y]) + \mathcal{G}(\mu \wedge \mathcal{L}_X \nu \otimes Y) - \mathcal{G}(\mathcal{L}_Y \mu \wedge \nu \otimes X) - (-1)^k \mathcal{G}(X) \mathcal{V}_\pi(\mathcal{d}_\tau \mathcal{I}_Y \mu \wedge \mathcal{d}_\tau \nu) + \]
\[ + (-1)^k \mathcal{G}(Y) \wedge \mathcal{V}_\pi(\mathcal{d}_\tau \mu \wedge \mathcal{I}_X \nu) + (-1)^k \mathcal{G}(X) \wedge \mathcal{V}_\pi(\mathcal{I}_Y \mu \wedge \mathcal{d}_\tau \nu) = \]
\[ = \mathcal{G}(\mu \wedge \nu \otimes [X, Y]) + \mathcal{G}(\mu \wedge \mathcal{L}_X \nu \otimes Y) - \mathcal{G}(\mathcal{L}_Y \mu \wedge \nu \otimes X) + (-1)^k \mathcal{G}(\mathcal{d}_\tau \mu \wedge \mathcal{I}_X \nu \otimes Y) + \]
\[ + (-1)^k \mathcal{G}(\mathcal{I}_Y \mu \wedge \mathcal{d}_\tau \nu \otimes X) = \mathcal{G}([K, L]^{F-N}). \]

7. The case of the canonical Lie algebroid.

It is obvious that the canonical Lie algebroid, including the standard differential calculus on a manifold, is of special interest. It is exactly the case, when the corresponding linear Poisson on the dual bundle is symplectic. Let us suppose that the linear Poisson structure \( P \) on \( E^* \) is nondegenerate, i. e., the mapping \( \tilde{P}: T^* E^* \to T E^* \) is an isomorphism of vector bundles. The inverse morphism \( \tilde{\omega} = \tilde{P}^{-1} \) defines a symplectic form \( \omega \) on \( E^* \):

\[ \langle P, \mu \wedge \nu \rangle = \langle \tilde{P}(\mu) \wedge \tilde{P}(\nu), \omega \rangle. \]
We have in local coordinates (as in (3.4))

$$P = \frac{1}{2} \varepsilon_{ij}^k (x) \xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \delta_i^a (x) \partial_{\xi_i} \wedge \partial_{x^a}.$$  

Since $P$ is nonegative, the anchor map $\alpha_r : E \to TM$, given by

$$\alpha_r (y^i e_i (x)) = y^i \delta_i^a (x) \partial_{x^a},$$

is an isomorphism of vector bundles. The anchor map is a homomorphism of Lie brackets and it follows that $\alpha_r$ defines an isomorphism between the Lie algebroid structure on $E$ and the canonical Lie algebroid structure on $TM$. The dual (Poisson) bundle $(E^*, P)$ can be identified with $(T^* M, P_M)$ - the cotangent bundle with the canonical Poisson structure. In the following, we consider the case of canonical structures only.

Let $(x^a)$ be a local coordinate system on $M$. We choose $(e_a = \partial_{x^a})$ and $(e^* a = dx^a)$ as local bases of sections of $\tau_M : TM \to M$ and $\pi_M : T^* M \to M$, respectively. We denote by $(x^a, \dot{x}^b)$ and $(\dot{x}^a, p_b)$, instead of $(x^a, y^i)$ and $(x^a, \xi_i)$, the adopted coordinate systems. In these coordinate systems, the canonical Poisson bivector field is given by

$$P = \partial_{p_a} \wedge \partial_{x^a},$$  

the anchor map $\alpha_{\tau M} : TM \to TM$ is the identity, and $d_{\tau M} = d$ is the usual exterior derivative. The formulae (3.6) and (3.8) take now the form

$$[\partial_{x^a}, \partial_{x^b}] = 0, \quad d(f) = \frac{\partial f}{\partial x^a} dx^a, \quad d(dx^a) = 0. \quad (7.2)$$

As in the previous sections, we can consider the tangent and cotangent lifts of the canonical Lie algebroid structure on $\tau_M : TM \to M$ to Lie algebroid structures of bundles $T_{\tau M} : TTM \to TM$ and $\pi_{TT^*} : T^* T^* M \to T^* M$. On the other hand, there are canonical Lie algebroid structures of bundles $\tau_{TM} : TTTM \to TM$ and $\tau_{TT^*} : TTT^* M \to TTT^* M$. In the following, we discuss relations between these Lie algebroids.

There is the well-known isomorphism $\kappa_M$ of vector bundles (cf. [30], [10]),

$$TTM \xrightarrow{\kappa_M} \tau_{TTM} \xrightarrow{\tau_{TTM}} TTM \xrightarrow{T_{TM}} TM \xrightarrow{T_{TM}} T^* M \xrightarrow{T_{TM}} TTTM.$$  

As a mapping of manifolds, $\kappa_M$ is an involution: $\kappa_M \circ \kappa_M = \text{id}$. As in Section 3., we introduce the basis $(\partial_{x^a}, \partial_{\dot{x}^b})$ of local sections of $T_{\tau M}$. The morphism $\kappa_M$ is characterized by

$$\kappa_M \circ \partial_{x^a} = \partial_{\dot{x}^a}, \quad \kappa_M \circ \partial_{\dot{x}^b} = \partial_{x^b} \quad \text{(7.4)}$$

and it can be extended to isomorphisms of tensor bundles

$$\kappa^r_M : \otimes^r_{\tau_{TTM}} TTM \to \otimes^r_{\tau_{TTM}} TTM$$

by

$$\kappa^r_M (X_1 \otimes_{TTM} \cdots \otimes_{TTM} X_r) = \kappa_M (X_1) \otimes_{TTM} \cdots \otimes_{TTM} \kappa_M (X_r).$$

It follows that $\kappa^r_M$ can be restricted to skew-symmetric and symmetric tensors:

$$\kappa^r_M (\bigwedge^r_{\tau_{TTM}} TTM) = \bigwedge^r_{\tau_{TTM}} TTM, \quad \kappa^r_M (\bigwedge^r_{\tau_{TTM}} TTM) = \bigwedge^r_{\tau_{TTM}} TTM.$$  

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For dual vector bundles $\nabla\pi_M: \mathbb{T}\pi^*M \to \mathbb{T}M$ and $\pi\nabla_M: \mathbb{T}^*\pi^*M \to \mathbb{T}M$, we have the dual isomorphism $\alpha_M$

$$\begin{array}{ccc}
\mathbb{T}\pi^*M & \xrightarrow{\alpha_M} & \mathbb{T}^*\pi^*M \\
\mathbb{T}\pi_M & \downarrow & \mathbb{T}\pi_M \\
\mathbb{T}M & \xrightarrow{\pi} & \mathbb{T}M
\end{array}$$ (7.5)

and, consequently, isomorphisms of tensor bundles

$$\alpha_r^\pi_M: \otimes^r\pi\nabla^*M \to \otimes^r\pi\nabla\pi^*M, \quad \alpha_r^\pi_M: \bigwedge^r\pi\nabla^*M \to \bigwedge^r\pi\nabla\pi^*M.$$ (7.6)

The induced mapping $\alpha_M: \Phi^1(\mathbb{T}\pi^*M) \to \Phi^1(\pi\nabla\pi^*M)$ is given by

$$\alpha_M(\tilde{d}p_a) = dx^a, \quad \alpha_M(\tilde{d}p_b) = dx^b.$$ (7.7)

There are canonical lifts $v_T$ (vertical) and $d_T$ (complete) of multivector fields and forms on $M$ to multivector fields and forms on $TM$ (cf. [31], [10] and (4.1), (4.2)), which are strictly related to $V$ and $T$.

**Theorem 19.**

(1) For $X \in \Gamma(M, \bigotimes^r\tau_M)$, we have

$$\kappa^r_M(v_T(X)) = \mathcal{V}(X), \quad \kappa^r_M(d_T(X)) = \mathcal{T}(X).$$ (7.8)

(2) For $\mu \in \Gamma(M, \bigotimes^r\pi_M)$, we have

$$v_T(\mu) = \alpha^r_M(\mathcal{V}(\mu)), \quad d_T(\mu) = \alpha^r_M(\mathcal{T}(\mu)).$$ (7.9)

There are analogous formulae for skew-symmetric and symmetric tensors, as well as for tensors of type $(r, s)$.

**Proof:** Since

$$v_T(X \otimes Y) = v_T(X) \otimes v_T(Y), \quad d_T(X \otimes Y) = d_T(X) \otimes v_T(Y) + v_T(X) \otimes d_T(Y)$$

(cf. [30], [10]) and

$$\mathcal{V}(X \otimes Y) = \mathcal{V}(X) \otimes \mathcal{V}(Y), \quad \mathcal{T}(X \otimes Y) = \mathcal{T}(X) \otimes \mathcal{V}(Y) + \mathcal{V}(X) \otimes \mathcal{T}(Y)$$

(Theorem 9), it is sufficient to consider the case $r = 1$.

Let $X = f^a \partial_{x^a}$. Then ((4.6) and (4.7)), $\mathcal{V}(X) = f^a \partial_{x^a}$ and $\mathcal{T}(X) = f^a \partial_{x^a} + \frac{\partial f^a}{\partial x^b} \partial^b \partial_{x^a}$. It follows from (7.4) that

$$\mathcal{T}(X) = \kappa_M(f^a \partial_{x^a} + \frac{\partial f^a}{\partial x^b} \partial^b \partial_{x^a}) = \kappa_M(d_T(X)), \quad \mathcal{V}(X) = \kappa_M(f^a \partial_{x^a}) = v_T(X).$$

Similar arguments prove (7.9). \[ \blacksquare \]

Let us remark that the above theorem can be proved in a coordinate-free way. First, we make use of the following identities (cf. [10]):

$$d_T\tilde{X} = d_T\mathcal{X} \circ \alpha_M, \quad v_T\tilde{X} = v_T\mathcal{X} \circ \alpha_M, \quad d_T\tilde{\mu} = d_T\mathcal{\mu} \circ \kappa_M, \quad v_T\tilde{\mu} = v_T\mathcal{\mu} \circ \kappa_M.$$ (7.10)
Then, using the pairings
\[ \langle \cdot, \cdot \rangle : TT\!M \times TM \to \mathbb{R}, \quad \langle \cdot, \cdot \rangle' : TT\!M \times TM \to \mathbb{R} \]
and the equality
\[ \langle v, \alpha_M(w) \rangle = \langle \kappa_M(v), w \rangle', \]
we get
\[ \langle V(X), w \rangle' = V(\tilde{X})(w) = \nu_T(\tilde{X})(w) = \nu_T(X)(\alpha_M(w)) = \langle \nu_T(X), \alpha_M(w) \rangle = \langle \kappa_M(\nu_T(X)), w \rangle'. \]
Hence, \( \kappa_M(V(X)) = \nu_T(X) \), etc.

**Theorem 20.** The isomorphism \( \kappa_M \) of vector bundles is an isomorphism of the canonical Lie algebroid structure on \( \tau_M : TT\!M \to TM \) and the tangent Lie algebroid structure on \( T\tau_M : TT\!M \to TM \).

The dual isomorphism \( \alpha_M \) of vector bundles is an isomorphism of linear Poisson structures \( (TT^*\!M, d_T P_M) \) and \( (T^*\!TM, P_{TM}) \). Moreover,
\[ \alpha_M \circ d_T \tau_M = d \circ \alpha_M. \]  

**Proof:** For a section \( X : M \to E \), we have \( T(X) = TX \). It follows that
\[ T(\alpha_T \circ X) = T(\alpha_T \circ X) = T \alpha_T \circ T(X). \]

We have then, in view of Theorem 19 and Theorem 10, that
\[ T \alpha_T \circ T(X) = T(\alpha_T \circ X) = \kappa_M(d_T(\alpha_T \circ X)) = \kappa_M \circ \alpha_T \tau_T. \]

Since sections \( T(X) \) span the \( T\tau \)-bundle (except for the zero sections), we obtain, by the continuity argument, that, for a Lie algebroid \( \tau : E \to M \) with the anchor \( \alpha_T \), we have (cf. [22])
\[ \kappa_M \circ \alpha_T \tau_T = T \alpha_T. \]  

Since the anchor of the canonical Lie algebroid structure is the identity on \( TM \), we get from (7.13) that the anchor \( \alpha_T \tau_M \) of the tangent Lie algebroid structure is \( \kappa_M^{-1} \). As in the beginning of this section, we obtain that \( \kappa_M \) is an isomorphism of Lie algebroids.

The remaining part of the theorem follows directly from (7.11).

**Corollary 21.** Let \( X, Y \in \Phi(\tau_M) \) or \( X, Y \in \Phi_1(\pi_M) \) and let \( \{ \cdot, \cdot \} \) stays for the Schouten or Nijenhuis-Richardson or Frölicher-Nijenhuis bracket. Then,
\[ \{ \nu_T(X), \nu_T(Y) \} = 0; \]
\[ \{ \nu_T(X), d_T(Y) \} = \{ d_T(X), \nu_T(Y) \} = \nu_T[ X, Y ]; \]
\[ \{ d_T(X), d_T(Y) \} = d_T[ X, Y ]. \]  

We have also all the formulae of Theorem 12 with \( V \) replaced by \( \nu_T \), \( T \) replaced by \( d_T \), and the bracket of the tangent Lie algebroid (Theorems 11, 13, 14). The formulae (7.14) are then simple consequences of Theorem 20 and Theorem 19.

Since \( P_M \) is invertible, it follows from Theorem 5 that the mapping
\[ A_{P_M} : \Phi(\pi_T^* \! M) \to \Phi(\tau_T^* \! M) \]
is an isomorphism between the Koszul-Schouten and the Schouten brackets. Since $P_M$ is skew-symmetric, for the dual map $\Lambda^*_{P_M}$, we have $\Lambda^*_{P_M}(\mu) = (-1)^k\Lambda^*_{P_M}(\mu)$ with $\mu \in \Phi^k(\pi_{T^*M})$. Therefore it defines an antihomomorphism of these brackets. We have also the well-known formula

$$\Lambda^*_{P_M}(d\mu) = [P, \Lambda^*_{P_M}(\mu)]. \quad (7.15)$$

**Theorem 22.** For $\mu \in \Phi^k(\pi_M), X \in \Phi^1(\pi_M)$, we have

1. $\Lambda^*_{P_M}(\pi^*_M(\mu)) = \mathcal{V}_{\pi_M}(\mu)$,
2. $\Lambda^*_{P_M}(\iota(X)\pi^*_M(\mu)) = -\mathcal{J}(\mu \otimes X)$,
3. $\Lambda^*_{P_M}(d(\iota(X)\pi^*_M(\mu))) = -\mathcal{G}(\mu \otimes X)$

**Proof:**

1. Let $\mu = \mu_1 \wedge \cdots \wedge \mu_k$, where $\mu_i \in \Phi^1(\pi_M)$. Since

$$\Lambda^*_{P_M}(\pi^*_M(\mu)) = (-1)^k\widetilde{P}_M(\pi^*_M(\mu_1)) \wedge \cdots \wedge \widetilde{P}_M(\pi^*_M(\mu_k))$$

and

$$\mathcal{V}_{\pi_M}(\mu) = \mathcal{V}_{\pi_M}(\mu_1) \wedge \cdots \wedge \mathcal{V}_{\pi_M}(\mu_k),$$

it is sufficient to prove (1) in the case $k = 1$. For $\mu = \mu_a dx^a$, we have

$$\widetilde{P}_M(\mu_a dx^a) = \widetilde{P}_M(\mu_a dx^a) = -\mu_a \partial_p = -\mathcal{V}_{\pi_M}(\mu_a dx^a)$$

(cf. (4.1) and (7.1)).

2. It follows from (1) and (7.1) that

$$\Lambda^*_{P_M}(\iota(X)\pi^*_M(\mu)) = \iota(X)\Lambda^*_{P_M}(\pi^*_M(\mu)) = \iota(X)\mathcal{V}_{\pi_M} = -\mathcal{J}(\mu \otimes X).$$

3. $\Lambda^*_{P_M}(d(\iota(X)\pi^*_M(\mu))) = [P, \Lambda^*_{P_M}(\iota(X)\pi^*_M(\mu))] = -[P, \mathcal{J}(\mu \otimes X)] = -\mathcal{G}(\mu \otimes X)$.

The next theorems provide us with homomorphisms of graded Lie brackets related to certain cotangent lifts.

**Theorem 23.** The mapping

$$\mathcal{J}^*: \Phi_1(\pi_M) \ni \mu \otimes X \mapsto \mathcal{J}^*(\mu \otimes X) = \iota(X)\pi^*_M(\mu) \in \Phi(\pi_{T^*M})$$

defines an injective homomorphism of the Frölicher-Nijenhuis bracket of vector-valued forms on $M$ into the extended Poisson bracket $\{,\}$ of differential forms on $T^*M$ defined by the Poisson structure $P_M$ (cf. (2.7)).

**Proof:** Let $\mu \otimes X, \nu \otimes Y \in \Phi_1(\pi_M)$. We can always take $\mu, \nu$ such that $d\mu = d\nu = 0$, since such vector-valued forms span $\Phi_1(\pi_M)$. In such cases,

$$[\mu \otimes X, \nu \otimes Y]^{F-N} = \mu \wedge \nu \otimes [X, Y] + \mu \wedge \mathcal{L}_X \nu \otimes Y - \mathcal{L}_Y \mu \wedge \nu \otimes X \quad (7.16)$$

and, locally,

$$\mu = d\mu_1 \wedge \cdots \wedge d\mu_k, \nu = d\nu_1 \wedge \cdots \wedge d\nu_l, \mu_i, \nu_j \in C^\infty(M).$$

Let us denote $g_i = \pi^*_M(\mu_i), f_j = \pi^*_M(\nu_j)$ and $g_0 = \iota(X), f_0 = \iota(Y)$. Then

$$\mathcal{J}^*(\mu \otimes X) = g_0 d\mu_1 \wedge \cdots \wedge d\mu_k, \mathcal{J}^*(\nu \otimes Y) = f_0 d\mu_1 \wedge \cdots \wedge d\mu_l.$$
Since \( \{f_j, g_i\} = 0 \) for \( i, j > 0 \), we get, using (2.8),

\[
\{J^*(\mu \otimes X), J^*(\nu \otimes Y)\} = \{\iota(X), \iota(Y)\} d\gamma_1 \wedge \cdots \wedge d\gamma_k \wedge df_1 \wedge \cdots \wedge df_1 +
\]

\[
\iota(X) \sum_{i>0} (-1)^i d(\iota(Y), g_i) \wedge df_1 \wedge \cdots \wedge df_1 -
\]

\[
-(-1)^k \iota(Y) \sum_{j>0} (-1)^j d(\iota(X), f_j) \wedge df_1 \wedge \cdots \wedge df_1 =
\]

\[
= \iota([X, Y]) \pi_M^*(\mu \wedge \nu) - \sum_i \iota(X) d\gamma_1 \wedge \cdots \wedge d(\iota(Y), g_i) \wedge \cdots \wedge \pi_M^*(\nu) \wedge df_1 \wedge \cdots \wedge df_1.
\]

(7.17)

Since

\[
d(\iota(Y), g_i) = d(\iota(Y), \pi_M^*(\mu_i)) = d(\pi_M^* L_Y \mu_i) = \pi_M^* L_Y (d\mu_i)
\]

we finally get from (7.17)

\[
\{J^*(\mu \otimes X), J^*(\nu \otimes Y)\} = \iota([X, Y]) \pi_M^*(\mu \wedge \nu) - \iota(X) \pi_M^*(L_Y \mu \wedge \nu) + \iota(Y) \pi_M^*(\mu \wedge L_X \nu) = J^*([\mu \otimes X, \nu \otimes Y]^{F^{-N}}).
\]

Thus \( J^* \) defines a homomorphism of the brackets. The injectivity follows directly from the form of \( J^* \) in local coordinates.

**Remark.** It is well known that the mapping

\[
\iota: S(\tau_M) \to \Phi(\pi_{T*M}),
\]

\[
\iota(X_1 \vee \cdots \vee X_k) = \iota(X_1) \cdots \iota(X_k)
\]

for \( X_i \in \Phi^1(\tau_M) \), is an injective homomorphism of the symmetric Schouten bracket on \( M \) into the Poisson bracket on \( T^*M \). Thus we can consider (\( \Phi(\pi_{T*M}), \{\ , \} \)) - the extended Poisson bracket of differential forms on \( T^*M \) - as a common generalization of the Frölicher-Nijenhuis and the symmetric Schouten bracket (cf. [7]).

**Theorem 24.**

(1) The mapping

\[
\mathcal{H}: \Phi(\pi_M) \to \Phi(\pi_{T^*M}): \mu \otimes X \mapsto H_{P_M}(J^*(\mu \otimes X))
\]

is an injective homomorphism of the Frölicher-Nijenhuis bracket on \( M \) and the Frölicher-Nijenhuis bracket on \( T^*M \).

(2) The mapping

\[
\mathcal{G}: \Phi(\pi_M) \to \Phi(\tau_{T^*M})
\]

is an injective homomorphism of the Frölicher-Nijenhuis bracket on \( M \) into the Schouten bracket on \( T^*M \).

**Proof:** (1) By Theorem 7 b and Theorem 23, \( \mathcal{H} \) is a homomorphism of the Frölicher-Nijenhuis brackets. The local form of \( \mathcal{H} \) shows that it is injective:

\[
\mathcal{H} \left( \sum_{a_1 < \cdots < a_k, a} f_{a_1 \cdots a_k}^a \, dx^{a_1} \wedge \cdots \wedge dx^{a_k} \otimes \partial_{x^a} \right) =
\]

\[
= \sum_{a_1 < \cdots < a_k, a} f_{a_1 \cdots a_k}^a \left( dx^{a_1} \wedge \cdots \wedge dx^{a_k} \otimes \partial_{x^a} - \sum_i (-1)^i d\gamma_i \wedge dx^{a_1} \wedge \cdots \wedge \widehat{dx^{a_i}} \wedge \cdots \wedge dx^{a_k} \otimes \partial_{x^a} \right)
\]

\[
= \sum_{a_1 < \cdots < a_k, a} \frac{\partial f_{a_1 \cdots a_k}^a}{\partial x^b} \partial_{a_i} \left( dx^{a_1} \wedge \cdots \wedge dx^{a_i} \otimes \partial_{p_{a_i}} + \sum_i (-1)^i dx^b \wedge dx^{a_i} \wedge \wedge \widehat{dx^{a_i}} \wedge \cdots \wedge dx^{a_k} \otimes \partial_{p_{a_i}} \right).
\]

(7.18)
(2) We have already proved that \( \mathcal{G} \) is a homomorphism (Theorem 18, but it follows also from Theorem 7c and Theorem 23). Locally, we have

\[
\mathcal{G} \left( \sum_{a_1 < \cdots < a_k, a} f_{a_1, \ldots, a_k} \, dx^{a_1} \wedge \cdots \wedge dx^{a_k} \otimes \partial_x^a \right) = \]

\[
= - \sum_{a_1 < \cdots < a_k, a} f_{a_1, \ldots, a_k} \partial_{x^a} \wedge \partial_{p_{a_1}} \wedge \cdots \wedge \partial_{p_{a_k}} + \sum_{a_1 < \cdots < a_k, a, b} \frac{\partial f_{a_1, \ldots, a_k}}{\partial x^b} p_b \partial_{p_{a_1}} \wedge \cdots \wedge \partial_{p_{a_k}}
\]

and the injectivity follows.

Remarks. The mapping \( \mathcal{H} \), extended by the mapping

\[
\mathcal{H} : S(\pi_M) \rightarrow \Phi_1(\pi_{T^*M}),
\]

is the "common generalization" of the Frölicher-Nijenhuis and Schouten brackets as defined by Dubois-Violette and Michor ([7]). The second part of the theorem shows that we can define the classical Frölicher-Nijenhuis bracket on \( M \) by means of the classical Schouten bracket on \( T^*M : [K, L]^{P-N} \) is the only element of \( \Phi_1(\pi_M) \) such that \( \mathcal{G}([K, L]^{P-N}) \) is the Schouten bracket \( \mathcal{G}(K), \mathcal{G}(L) \). In particular, for \( N \in \Phi^1(\pi_M) \) we have, that \([N, N]^{P-N} = 0\) if and only if \( \mathcal{G}(N) \) is a Poisson tensor on \( T^*M \). It is easy to verify that the Poisson structure defined by \( \mathcal{G}(N) \) is linear and, consequently, corresponds to an algebroid structure on the tangent bundle \( \pi_M : TM \rightarrow M \). This suggests a new approach to the theory of Poisson-Nijenhuis manifolds which will be developed in a separate publication.

Conclusions.

We defined natural graded Lie brackets associated with Lie algebroids, their lifts, and we studied relations between the brackets and the lifted brackets. All the constructions seem to be canonical, generalizing some known results, but we also obtained interesting results on the classical level, as for example embeddings of the Nijenhuis-Richardson and the Frölicher-Nijenhuis bracket over \( M \) into the Schouten bracket over \( T^*M \). However, the analysis presented in this paper is far from being complete. We did not discuss the canonical homomorphism between tangent Lie algebroid on \( T_T : TE \rightarrow TM \) and the canonical algebroid on \( \tau_E : TE \rightarrow E \). We skipped also the problem of iterated lifts. In Section 3, we mentioned that some iterated lifts of an algebroid are canonically isomorphic. The existence of these isomorphisms provokes the question, whether iterated lifts of sections can be identified. For example, one could expect that \( \mathcal{G} \circ T(K) \) can be identified with \( T \circ \mathcal{G}(K) \). In a forthcoming publication we shall discuss identities of this kind, as well as other observations and applications of the presented results.

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