Extended Phase Space Thermodynamics of Black Holes in Massive Gravity

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We study the extended phase space thermodynamics of black holes in massive gravity. Particularly, we examine the critical behaviour of this black hole using the extended phase space formalism. Extended phase space in a sense that in which the cosmological constant should be treated as a thermodynamic pressure and its conjugate variable as a thermodynamic volume. In this phase space, we derive the black hole equation of state, the critical pressure, the critical volume and the critical temperature at the critical point. We also derive the critical ratio of this black hole. Moreover, we derive the black hole reduced equation of state in terms of the reduced pressure, the reduced volume and the reduced temperature. Furthermore, we examine the Ehrenfest equations of black holes in massive gravity in the extended phase space at the critical point. We show that the Ehrenfest equations are satisfied of this black hole and the black hole encounters a second order phase transition at the critical point in the said phase space. This is re-examined by evaluating the Pregogine-Defay ratio (Π). We determine the value of this ratio is Π = 1. The outcome of this study is completely analogous to the nature of liquid-gas phase transition at the critical point. This investigation also further gives us the profound understanding between the black hole of massive gravity with the liquid-gas systems.

Keywords: P − V Criticality, Massive gravity, Extended phase-space

1. Introduction

An excited field of research in recent times on black hole (BH) thermodynamics particularly in anti de-Sitter (AdS) space is due to the seminal work of Hawking and Page[1] in which the authors first examined certain type of phase transitions occur between small and large black holes in case of Schwarzschild-AdS spacetime. Another interesting feature of a Van-der-Waals liquid-gas system was studied by Chamblin et al.[2,3] for a spherically symmetric Reissner-Nordström-AdS (RN-AdS) BH. In which the authors demonstrated that there exists first order phase

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transition in case of RN-AdS BH. The critical behaviour of this BH has been studied in details there. Significantly, this structure is modified by Kubizňák & Mann [9] by studying the $P - V$ criticality of a RN-AdS BH. They have used the extended phase-space formalism. This is a formalism in which the cosmological constant could be treated as a thermodynamic pressure and its conjugate variable as a thermodynamic volume [5,6,7,8]. In this phase-space, the authors reviewed the critical behaviour of a spherically symmetric charged AdS spacetime. They compared the BH equation of state with the Van-der Waal’s liquid-gas system. They also determined the critical constants and critical ratio of this charged AdS BH. Finally, they computed the critical exponents using mean field theory and which coincides with those of the Van-der Waal’s liquid-gas system.

Extended phase space (EPS) thermodynamics proved that the Smarr like relation and the first law is satisfied for this kind of BH (by using the scaling argument). It implies that the conventional phase-space is quite different from extended phase-space. Where the extra “PdV” term is present. It should be noted that in EPS the ADM mass of the BH could be identified as the total heat i.e. enthalpy of the system rather than internal energy. This also indicates that normal phase-space is quite different from EPS. Another interesting feature of EPS is that thermodynamic volume should be satisfied the reverse isoperimetric inequality [8].

In this work, we wish to apply the above mentioned formalism for a BH in massive gravity. In massive gravity theory, the massive graviton (due to Lorentz symmetry breaking) is playing the key role. This BH has a scalar charge due to this massive graviton and it is asymptotically AdS. We have examined the $P-V$ criticality of this BH in massive gravity in the extended phase-space: in which the cosmological constant should be treated as a thermodynamic variable and its conjugate variable as a thermodynamic volume. Moreover, we derive the BH thermodynamic equation of state in analogy with the Van-der-Waal’s equation of state. At the inflection point, we compute the critical pressure, the critical volume and the critical temperature. Using these constants, we compute the critical ratio. Moreover, we derive the reduced equation of state in terms of the reduced pressure, the reduced volume and the reduced temperature.

The organization of the paper is as follows. In Sec. (2), we have given the brief introduction of BHs in massive gravity in AdS space. In Sec. (3), we have studied the critical behaviour in extended phase space. In Sec. (4), we have described the Ehrenfest scheme for $P-V$ criticality in the extended phase space. Finally, we have given the conclusions in Sec. (5).

2. A brief introduction of BHs in Massive Gravity in AdS Space
Here we would like to give a brief introduction of BHs in massive gravity in the AdS space. In Einstein’s general theory of relativity, the graviton has no mass but in theory of massive gravity the graviton has mass. That’s why, it is called massive
gravity. On the other hand, massive gravity is a kind of gravity which modified
the Einstein’s general theory of gravity. It is endowed with a mass which is called
massive graviton. The idea of massive gravity was first initiated in 1939 by Fierz
and Pauli. In this theory, they proposed that a massive 2-spin propagates on a
background of the flat spacetime. They found that the graviton has five degrees of
freedom. Afterwards, Boulware and Deser proved that Fierz-Pauli’s prescription
sustains from ghosts when one taking into account the non-linear extension. In 2010,
de Rham, Gabadadze and Tolley proposed that a massive gravity theory which is a
free from ghost i.e. “ghost free gravity”. Now it is popularly known as dRGT theory
in the literature. There exists other massive gravity models in the literature such as
DGP model and BH T model. For a detail review on the “Massive Gravity”
one could see the Ref. and Ref. [See also 25, 26].

What is interesting in the massive gravity is that it explains the “accelera-
tion of the universe” which does not require any dark energy or any cosmologi-
cal constant parameter. In 2016, in the first history of science the gravita-
tional waves have been observed by the LIGO team in which the graviton mass is restricted to $m_g < 1.2 \times 10^{-22} eV/c^2$.

There are several interesting theories of massive gravity. Among them one is a
special theory of massive gravity in which the Lorentz symmetry is broken which
is popularly known as “Lorentz symmetry breaking theory”. The reasons behind
this symmetry breaking is that the spacetime contains a scalar field which is called
Goldstone fields. Such fields are coupled to gravity through non-derivative coupling.
When the Lorentz symmetry is broken spontaneously, the massive graviton gets the
mass which is quite analogous to Higgs mechanism.

We do not repeat here the derivation for this kind of BHs and the action for
massive gravity theory which could be found in the Ref. The metric for
such type of BHs could be written as

$$ds^2 = -\mathcal{Y}(r)dt^2 + \frac{dr^2}{\mathcal{Y}(r)} + r^2 d\Omega^2,$$

(1)

where,

$$\mathcal{Y}(r) = 1 - \frac{2M}{r} - \frac{\gamma Q^2}{r^\lambda} - \frac{\Lambda}{3} r^2.$$

(2)

and $d\Omega^2$ is the metric on unit sphere in two dimensions. Where $Q$ denotes a scalar
charge related to massive gravity. The parameter $\gamma$ may have the values $\pm 1$ leads
to different types of geometry and the value of the integration constant $\lambda > 0$.

When $\gamma = 1$, the BH solution is quite analogous to the Schwarzschild-AdS BH
which has a single horizon. For the parameter $\lambda < 1$, the function $\mathcal{Y}(r)$ “domi-
nates at large distances” thus the ADM mass parameter become diverges. Whereas for
$\lambda > 1$, the BH solution becomes Schwarzschild-AdS type. Therefore throughout the
work we will take the value of $\lambda > 1$.

When $\gamma = -1$ and $\lambda = 2$, one obtains the BH solution which is quite similar to
the RN-AdS BH which has two horizons.
3. \( P - V \) criticality of BHs in massive gravity in AdS space

In this section, we would like to study the critical behaviour of BHs in massive gravity by using the extended phase-space formalism. Let us now put \(-\frac{\Lambda}{3} = \frac{1}{\ell^2}\) for AdS case. Then the metric function becomes

\[
\mathcal{Y}(r) = 1 - \frac{2M}{r} - \frac{\gamma Q^2}{r^\lambda} + \frac{r^2}{\ell^2}.
\]  

(3)

The outer horizon location can be determined from the condition \(\mathcal{Y}(r_+) = 0\). The quantity \(M\) denotes the ADM mass of the BH which could be identified as enthalpy of the system in extended phase-space. The parameter \(Q\) denotes the total charge of the BH.

Since we are dealing with extended phase-space in which the thermodynamic pressure \((P)\) is equal to the negative cosmological constant \((\Lambda)\) divided by \(8\pi\) (where \(G = c = k_B = h = 1\)) i.e.

\[
P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi \ell^2}.
\]  

(4)

and the corresponding thermodynamic volume can be written as

\[
V = \left(\frac{\partial M}{\partial P}\right)_{S,Q}.
\]  

(5)

This volume for BHs in massive gravity in AdS space is

\[
V = \frac{4}{3} \pi r_+^3.
\]  

(6)

where \(r_+\) corresponds to the outer horizon radius.

The entropy of BH can be defined as

\[
S = \frac{A}{4}.
\]  

(7)

where the area of BH is given by

\[
A = 4\pi r_+^2.
\]  

(8)

Now the BH temperature is calculated to be

\[
T = \frac{\mathcal{Y}''(r)}{4\pi} = \frac{1}{4\pi r_+} \left( 1 + 8\pi P r_+^2 + \frac{\gamma (\lambda - 1) Q^2}{r_+^\lambda} \right).
\]  

(9)

The first law of thermodynamics in the extended phase-space becomes

\[dM = TdS + \Phi dQ + VdP.
\]  

(10)

where \(\Phi\) is the electric potential given by

\[
\Phi = -\frac{\gamma Q}{r_+^{\lambda - 1}}.
\]  

(11)

Thus the Smarr-Gibbs-Duhem relation\(^\text{[21]}\) becomes

\[
M = H = 2TS - 2PV + \frac{\lambda}{2} Q \Phi.
\]  

(12)
Now we are ready to derive the BH equation of state in the extended phase-space.
Therefore we would like to determine the critical constants at the critical point.
The critical point occurs at the point of inflection. Using Eq. (9), one could write the BH equation of state as
\[ P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^4} - \frac{\gamma(\lambda - 1)Q^2}{8\pi r_+^{\lambda + 2}}. \]
(13)
where \( r_+ = \left(\frac{3V}{4\pi}\right)^{1/3} \). In terms of specific volume \( v = 2r_+ \) the above equation can be re-written as
\[ P = \frac{T}{v} - \frac{1}{2\pi v^2} - \frac{2^{\lambda - 1}\gamma(\lambda - 1)Q^2}{\pi v^{\lambda + 2}}. \]
(14)
The critical point could be determined from the following conditions
\[ \frac{\partial P}{\partial v} = \frac{\partial^2 P}{\partial v^2} = 0. \]
(15)
Using this criterion one obtains the critical values
\[ P_c = \frac{\lambda}{2\pi(\lambda + 2)[\gamma(1 - \lambda)(1 + \lambda)(2 + \lambda)2^{\lambda - 1}Q^2]^\frac{1}{\lambda}}, \]
(16)
\[ v_c = \left[\gamma(1 - \lambda)(1 + \lambda)(2 + \lambda)2^{\lambda - 1}Q^2\right]^\frac{1}{\lambda}, \]
(17)
\[ T_c = \frac{\lambda}{\pi(\lambda + 1)[\gamma(1 - \lambda)(1 + \lambda)(2 + \lambda)2^{\lambda - 1}Q^2]^\frac{1}{\lambda}}. \]
(18)
The critical thermodynamic volume could be obtain from the critical radius as
\[ V_c = \frac{4}{3}\pi r_+^3 = \frac{\pi}{6}\left[\gamma(1 - \lambda)(1 + \lambda)(2 + \lambda)2^{\lambda - 1}Q^2\right]^\frac{3}{\lambda}. \]
(19)
Now the critical ratio is computed to be
\[ Z_c = \frac{P_c v_c}{T_c} = \frac{(1 + \lambda)}{2(2 + \lambda)}. \]
(20)
This is a universal ratio. For RN-AdS BH, this ratio is \( \frac{3}{8} \). Moreover, the Van-der-Waal’s constants are
\[ a = \frac{9\lambda}{8\pi(1 + \lambda)}, \]
(21)
\[ b = \frac{\left[\gamma(1 - \lambda)(1 + \lambda)(2 + \lambda)2^{\lambda - 1}Q^2\right]^\frac{1}{\lambda}}{3}. \]
(22)
It implies that the presence of the charge parameter reduces the effective volume smaller because of \( b \) is proportional to \( Q \).

Therefore “the law of corresponding states” could be written as
\[ 2(2 + \lambda)\Theta = (1 + \lambda)\Phi \left[\sigma + \left(1 + \frac{2\lambda}{1}\right)\frac{1}{\Phi^2}\right] - \frac{2\lambda}{1}\frac{1}{\Phi^{\lambda + 1}}. \]
(23)
where \( \Theta, \Phi \) and \( \sigma \) could be defined as
\[ \Theta = \frac{T}{T_c}, \Phi = \frac{v}{v_c}, \sigma = \frac{P}{P_c}. \]
(24)
and these quantities $\Theta$, $\Phi$ and $\sigma$ are called the reduced temperature, the reduced volume and the reduced pressure respectively. Thus the above equation is called the \textit{reduced equation of state} or \textit{the law of corresponding states}.

Now we should compute the specific heat to determine the local thermodynamic stability of the BH. There are two specific heats in BH thermodynamics as well as in classical thermodynamics. Namely, the specific heat at constant volume and the specific heat at constant pressure. They are defined as

$$C_V = T \left( \frac{\partial S}{\partial T} \right)_V. \quad (25)$$

and

$$C_P = T \left( \frac{\partial S}{\partial T} \right)_P. \quad (26)$$

First, we should calculate the specific heat at constant volume. To determine it we first define the free energy which is calculated to be

$$F = \frac{r_+}{2} \left[ 1 - 2\pi r_+ T - \frac{\gamma Q^2}{r_+^2} \right]. \quad (27)$$

From the above equation we can easily compute the entropy which is defined to be

$$S = - \left( \frac{\partial F}{\partial T} \right)_V = \pi r_+^2. \quad (28)$$

Since the entropy $S$ is independent of $T$ thus one gets

$$C_V = 0. \quad (29)$$

The temperature in terms of entropy and pressure could be defined as

$$T (S, P) = \frac{1}{4\sqrt{\pi S}} \left[ 1 + 8PS + \frac{\gamma(\lambda - 1)\pi^2 Q^2}{S^{\frac{1}{2}}} \right]. \quad (30)$$

To determine the local thermodynamic stability one should calculate the specific heat at constant pressure which is turned out to be

$$C_P = 2S \left[ \frac{S^{\frac{1}{2}} + 8PS^{\frac{1}{2}} + \frac{\gamma(\lambda - 1)\pi^2 Q^2}{S^{\frac{1}{2}}}}{8PS^{\frac{1}{2}} - S^{\frac{1}{2}} - \frac{\gamma(\lambda - 1)(\lambda + 1)\pi^2 Q^2}{S^{\frac{1}{2}}}} \right]. \quad (31)$$

For local thermodynamic stability it requires that $C_P > 0$ and the specific heat $C_P$ diverges at

$$8PS^{\frac{1}{2}} - S^{\frac{1}{2}} - \frac{\gamma(\lambda - 1)(\lambda + 1)\pi^2 Q^2}{S^{\frac{1}{2}}} = 0. \quad (32)$$

i.e. precisely at the critical point. It must be noted that the specific heat that $C_P$ but not $C_V$ diverges at the critical point. It could be observed from the above calculation.
To determine the global thermodynamic stability of the BH one must compute the Gibbs free energy which is determined to be

\[ G = G(T, P) = \frac{r_+}{4} \left[ 1 - \frac{8\pi P}{3} r_+^2 - \frac{\gamma(\lambda + 1)Q^2}{r_+^\lambda} \right]. \] (33)

where \( r_+ \) is a function of temperature and pressure, \( r_+ = r_+(P, T) \) by using the equation of state Eq. (13). In terms of BH temperature the above equation can be re-expressed as

\[ G = \frac{r_+}{3} \left[ 1 - \pi r_+ T - \frac{\gamma(\lambda + 2)Q^2}{2r_+^\lambda} \right]. \] (34)

Solving the equation \( G = 0 \), one obtains the critical temperature

\[ T_c = \frac{1}{\pi r_+^\lambda + \frac{\gamma(\lambda + 2)Q^2}{2r_+^\lambda}}. \] (35)

The value of \( r_+ \) can be eliminated by using Eq. (9). Although it is a very difficult task for this BH but it is easy for \( Q = 0 \). In this case \( r_+ = \sqrt{\frac{3}{8\pi P}} \) and \( T_c = T_{HP} = \sqrt{\frac{8P}{35}} \). This is called the Hawking-Page phase transition temperature. For \( T > T_c \), the BH will be stable one where the Gibbs’s free energy is minimum. The Gibbs’s free energy has been studied in detail in Ref. [20].

4. Ehrenfest equations for BHs in massive gravity in the Extended Phase Space

Phase transition is a fascinating topic of research in BH thermodynamics. In order to understand the nature of phase transitions whether it is first order phase transition or second order phase transition at the critical point in the extended phase space we should examine the Clausius-Clapeyron equations and Ehrenfest equations [24].

If the Clausius-Clapeyron equations are satisfied that means there should exist the first order phase transition and if the Ehrenfest equations are satisfied that implies there should exist second order phase transition. In this section we would try to understand what type of phase transition should occur for BHs in massive gravity?

For analytical check of Ehrenfest equations for BHs in massive gravity in the extended phase space first we should consider the comparison between BH thermodynamic parameters with the classical thermodynamic parameters i.e.

\[ V = Q \] (36)

\[ P = -\Phi \] (37)

This nice comparison of Ehrenfest equations first introduced in BH thermodynamics by Banerjee et al. [22,23] as

\[ -\left( \frac{\partial \Phi}{\partial T} \right)_S = \frac{C_{\Phi 2} - C_{\Phi 1}}{TQ(\alpha_2 - \alpha_1)} = \frac{\Delta C_\Phi}{TQ \Delta \alpha}. \] (38)

\[ -\left( \frac{\partial \Phi}{\partial T} \right)_Q = \frac{\alpha_2 - \alpha_1}{K_{T2} - K_{T1}} = \frac{\Delta \alpha}{\Delta K_T}. \] (39)
where $\alpha$ and $K_T$ are defined as
\begin{align}
\alpha &= \frac{1}{Q} \left( \frac{\partial Q}{\partial T} \right)_\Phi, \\
K_T &= \frac{1}{Q} \left( \frac{\partial Q}{\partial \Phi} \right)_T
\end{align}
which are analogous to the volume expansion coefficient and the isothermal compressibility.

The Ehrenfest equations in classical thermodynamics which could be found in any standard text book of thermodynamics are
\begin{align}
\left( \frac{\partial P}{\partial T} \right)_S &= \frac{C_{P2} - C_{P1}}{TV(\alpha_2 - \alpha_1)} = \frac{\Delta C_P}{TV\Delta\alpha}, \\
\left( \frac{\partial P}{\partial T} \right)_V &= \frac{\alpha_2 - \alpha_1}{K_{T2} - K_{T1}} = \frac{\Delta\alpha}{\Delta K_T}.
\end{align}
where $\alpha$ and $K_T$ are
\begin{align}
\alpha &= \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P, \\
K_T &= -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T
\end{align}
$\alpha$ is the volume expansion coefficient and $K_T$ is the isothermal compressibility coefficient. Now our motivation here is that the above classical Ehrenfest equations could be implemented in the BH thermodynamics particularly in the extended phase space because this formalism gives us the specific heat of BHs at constant pressure.

Now we are ready to compute the relevant thermodynamic quantities in the extended phase space for massive gravity. First we need to compute the BH temperature which is found to be from Eq. (30)
\begin{equation}
T = \frac{1}{4\sqrt{\pi S}} \left[ 1 + 8PS + \frac{\gamma(\lambda - 1)\pi^2 Q^2}{S^{\frac{3}{2}}} \right].
\end{equation}
The other relevant quantities are
\begin{align}
C_P &= T \left( \frac{\partial S}{\partial T} \right)_P = 2S \frac{\left[ S^{\frac{3}{2}} + 8PS^{1 + \frac{\lambda}{2}} + \gamma(\lambda - 1)\pi^2 Q^2 \right]}{\left[ 8PS^{1 + \frac{\lambda}{2}} - S^{\frac{3}{2}} - \gamma(\lambda - 1)(\lambda + 1)\pi^2 Q^2 \right]}, \\
\alpha &= \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P = \frac{12\sqrt{\pi S^{(\lambda + 1)/2}}}{\left[ 8PS^{1 + \frac{\lambda}{2}} - S^{\frac{3}{2}} - \gamma(\lambda - 1)(\lambda + 1)\pi^2 Q^2 \right]},
\end{align}
Now using the following identity
\begin{equation}
\left( \frac{\partial P}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P \left( \frac{\partial T}{\partial P} \right)_V = -1.
\end{equation}
one obtains the expression for the coefficient of isothermal compressibility as
\[ K_T = \frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = \frac{24S^{1+\lambda^2}}{8PS^{1+\lambda^2} - S^{2+\lambda^2} - \gamma(\lambda - 1)(\lambda + 1)\pi^2Q^2} \]. \quad (50)

From the above calculation we observed that the denominator in the expressions of \( C_P \), \( \alpha \) and \( K_T \) are same. It indicates that the volume expansion coefficient and isothermal compressibility coefficient are singular at the critical point as the \( C_P \) does.

Now we would like to check the validity of the Ehrenfest equations at the critical point. Using the expression of volume expansion one could find
\[ \alpha V = \left( \frac{\partial V}{\partial T} \right)_P = \frac{C_P}{T} \left( \frac{\partial V}{\partial S} \right)_P. \quad (51) \]

Thus the R.H.S of Eq. (42) at the critical point becomes
\[ \frac{\Delta C_P}{TV\Delta \alpha} = \left[ \left( \frac{\partial S}{\partial V} \right)_P \right]_c. \quad (52) \]
which is computed to be
\[ \frac{\Delta C_P}{TV\Delta \alpha} = \frac{\sqrt{\pi}}{2\sqrt{S_c}}. \quad (53) \]
Interestingly this is independent of both the parameters \( \gamma \) and \( \lambda \). Using Eq. (46), one could compute the L.H.S of Eq. (42) as
\[ \left[ \left( \frac{\partial P}{\partial T} \right)_V \right]_c = \frac{\sqrt{\pi}}{2\sqrt{S_c}}. \quad (54) \]

From the Eq. (53) and the Eq. (54) we observed that the first Ehrenfest equation is satisfied at the critical point in the extended phase space for massive gravity.

Analogously using the volume expansion coefficient and isothermal compressibility coefficient, one obtains
\[ \frac{\Delta \alpha}{\Delta K_T} = \left[ \left( \frac{\partial P}{\partial T} \right)_V \right]_c = \frac{\sqrt{\pi}}{2\sqrt{S_c}}. \quad (55) \]
This implies that the second Ehrenfest equation is satisfied at the critical point.

From the above calculation we can conclude that both Ehrenfest equations are satisfied at the critical point in the extended phase space for massive gravity. Now we compute the Prigogine-Defay (PD) ratio using Eq. (52) and Eq. (55) as
\[ \Pi = \frac{\Delta C_P \Delta K_T}{TV(\Delta \alpha)^2} = 1. \quad (56) \]
The Prigogine-Defay \( \Pi = 1 \) and Ehrenfest equations are satisfied means that the phase transition in the extended phase space for massive gravity is a second

\(^a\)This ratio was first suggested by Prigogine and Defay. That’s why it is called PD ratio.
order one. This is an interesting result for this BH. This result is also compatible with the nature of liquid-gas phase transition in the extended phase space and at the critical point. Moreover, this result is strengthening the analogy between the massive gravity and the liquid-gas system.

5. Conclusion
We demonstrated the extended phase space thermodynamics of BHs in massive gravity. Particularly, we have examined the critical behaviour for this BH by using the extended phase space procedure. In this procedure one may treated the cosmological constant as a dynamical pressure and its conjugate variable as a thermodynamic volume. We derived the BH entropy, the BH temperature, the BH equation of state, the first law of thermodynamics, the Smarr formula, the Gibbs free energy and finally the law of corresponding states. We have found that the complete BH thermodynamic system which is quite analogous to the liquid-gas system.

We also derived the specific heat at constant pressure, the coefficient of volume expansion and the coefficient of isothermal compressibility. Moreover we derived the Pregogine-Defay ratio and we showed that the value of this ratio is equal to one. Which indicates that the phase transition is second order one. Furthermore using the analogy between the thermodynamic state variables and the BH parameters i.e. $V \leftrightarrow Q$ and $P \leftrightarrow \Phi$, we showed that in the extended phase space the Ehrenfest equations are satisfied for this BH. The consequences of this study gives us further very intense understanding between the BH of massive gravity in the extended phase space and the liquid-gas systems.

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