TWO NEW NORMAL FORMS FOR POLYNOMIAL ENDOMORPHISMS OF THE PROJECTIVE LINE WITH APPLICATIONS TO POSTCRITICALLY FINITE MAPS

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Abstract. We explore two normal forms for polynomial endomorphisms of the projective line. The first is a normal form for degree 3 polynomials in terms of the multipliers of the fixed points. This normal form allows for an enumeration of all $K$-rational conjugacy classes in the moduli space of degree 3 polynomials. The second normal form is for polynomials of arbitrary degree with $n$ critical points. As an application, we give an algebraic proof of Thurston transversality in the periodic case for bicritical polynomials of any degree.

Let $f(z) \in \mathbb{Q}(z)$ be a rational function of degree $d \geq 2$, considered as an endomorphism of $\mathbb{P}^1$. Define the $n$-th iterate of $f$ recursively as $f^n(z) = f(f^{n-1}(z))$, with $f^0(z) = z$. Since the dynamical behavior of $f$ is preserved by the action of $\text{PGL}_2$, we may consider the set of equivalence classes of degree $d$ rational endomorphisms of $\mathbb{P}^1$ under $\text{PGL}_2$ conjugation. We denote this moduli space as $\mathcal{M}_d$, and denote by $\mathcal{P}_d \subset \mathcal{M}_d$ the moduli space of degree $d$ polynomials [15, 16]. Recall that $f \in \mathcal{P}_d$ if it has a totally ramified fixed point. For our purposes, we say a family of maps $f_t(z)$ provides a normal form if each choice of parameter value determines a different conjugacy class of $\mathcal{M}_d$.

Define a critical point of $f$ to be a point with ramification index at least 2. When the forward orbits of all the critical points are finite, we say the map is post-critically finite (PCF). Thurston’s rigidity theorem [4] says that any specified behavior of the critical points of a PCF map will be realized by only finitely many conjugacy classes of rational maps, excepting Lattès maps. Furthermore, in many cases, the equations defining these maps by critical orbit relations intersect transversely [4].

At this point there are already a number of interesting directions to pursue. One direction involves enumerating all PCF maps defined over a given field. Ingram proves that the set of PCF maps in $\mathcal{P}_d$ is a set of bounded height and, hence, finite for any given field of definition [9]. He goes on to calculate a specific bound for the coefficients of a degree 3 PCF polynomial in monic centered form and enumerate all possibilities over $\mathbb{Q}$. However, in choosing to use monic centered form, $z^d + a_{n-2}z^{d-2} + \cdots + a_1 z + a_0$, as his starting point, he did not find all PCF polynomials defined over $\mathbb{Q}$ since not every rational PCF degree 3 polynomial is conjugate to a polynomial in monic centered form with $a_i \in \mathbb{Q}$. Tobin [18] improves on his classification with her normal form for bicritical polynomials, but also omits a case due to the same type of issue with conjugation changing the field of definition. As a work in progress, Manes-Tobin cover this last case and all degree 3 PCF polynomials defined over $\mathbb{Q}$ will be enumerated. Lukas-Manes-Yap [10] find all degree 2 PCF rational maps defined over $\mathbb{Q}$ using a normal form from Manes-Yasafuku [11] involving invariants of the moduli space derived from the multipliers of the fixed points. This allows them to use the bound on the multipliers of a PCF map from Benedetto-Ingram-Jones-Levy [3] to enumerate all possibilities without
the problem of omission encountered by Ingram and Tobin when bounding the coefficients. A second direction is providing algebraic proofs of transversality as in Epstein, Silverman, Hutz-Towsley [5, 16, 8]. The typical approach is to find a suitably nice normal form where the Jacobian of the intersection can be shown to be non-zero modulo a suitably chosen prime. The feasibility of these proofs relies heavily on being able to find a suitable prime for the normal form and being able to calculate the appropriate intersection Jacobian for that normal form.

With this motivation in mind, we provide two new normal forms for polynomials. Theorem 1.1 presents a normal form for degree 3 polynomials in terms of the moduli space invariants. This normal form would provide a way to complete the classification of degree 3 PCF polynomials utilizing the bound from Benedetto-Ingram-Jones-Levy; however, this was completed by Anderson-Manes-Tobin [2] while our work was underway, so is not undertaken here. In Theorem 1.4, we prove that the idea behind the normal form from Manes-Yasafuku [11], the fixed points being “equal” to their multipliers, is only possible for certain classes of degree 3 rational maps. In Section 2, Theorem 2.2 generalizes Tobin’s bicritical normal form to any number of critical points. In Section 3, the normal form for bicritical polynomials of any degree $d \geq 2$ of Theorem 3.1 is used to prove transversality when the critical points are both periodic in Theorem 3.3.

The organization of the article is as follows. In Section 1 we recall the construction of the multiplier invariants and present a normal form for degree 3 polynomial in terms of these invariants. In Section 2 we prove the normal form for polynomials of arbitrary degree with $n$ critical points. In Section 3 we give an algebraic proof of transversality for periodic critical points for bicritical polynomials of arbitrary degree.

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1. Normal Form for Degree 3 Polynomials

Define a fixed point of $f$ to be a point $P \in \overline{\mathbb{Q}}$ such that $f(P) = P$, and the multiplier at $P$ to be $\lambda_P := f'(P)$. Again, we may make a linear change of variables to calculate $\lambda_{\infty}$ as needed. Define the $\sigma$-invariants of $f$ to be the elementary symmetric polynomials evaluated on the set of fixed point multipliers (with multiplicity). Because the set of fixed point multipliers is preserved under conjugation, these $\sigma$-invariants are the same for every element of a conjugacy class, i.e., are invariants of $\mathcal{M}_d$.

Note that Milnor [12] gave a normal form for degree 2 rational maps in terms of the multipliers themselves. However, such a form may not utilize the smallest possible field of definition since for $f$ defined over $\mathbb{Q}$, the multipliers could be defined over an extension field. The $\sigma$-invariants are always defined over the field of moduli so a normal form based on the $\sigma$-invariants has the smallest possible field of definition.

**Theorem 1.1.** Define a map $\phi : \mathbb{A}^2 \to \mathcal{P}_3$, from the affine plane to the moduli space of degree 3 polynomials as follows
as desired. Now assume \((\sigma_1, \sigma_3)\) we can say

\[ \text{Proposition 8}, \]

we have a totally ramified fixed point at \(z = \sigma_1 \) and the second form at \(z = 0\). The final three cases are clearly polynomials. Now let \((\sigma_1, \sigma_3) \in \mathbb{A}^2\). We next show that in each case, the \(\sigma\)-invariants of \(\phi(\sigma_1, \sigma_3)\) are given by \(\{\sigma_1, \sigma_2, \sigma_3, 0\}\). First, assume \(\sigma_1, \sigma_3 \notin \mathcal{C}\). From \([7, \text{Proposition 8}]\), we can say \(\sigma_2 = 2\sigma_1 - 3\). We use Sage to compute that the \(\sigma\)-invariants of \(\phi(\sigma_1, \sigma_3)\) are given by

\[
\{\sigma_1, 2\sigma_1 - 3, \sigma_3, 0\},
\]
as desired. Now assume \((\sigma_1, \sigma_3) \in \mathcal{C}\) so that, in particular, we can say \(\sigma_3 = \frac{1}{27}(-4\sigma_1^3 + 36\sigma_1^2 - 81\sigma_1 + 54)\). We next compute the \(\sigma\)-invariants of \(\phi(\sigma_1, \sigma_3)\). Since \((\sigma_1, \sigma_3) \in \mathcal{C}\), we take the remainder of each \(\sigma_i\) on division by the defining polynomial of \(\mathcal{C}\) to obtain

\[
\{\sigma_1, 2\sigma_1 - 3, \frac{1}{27}(-4\sigma_1^3 + 36\sigma_1^2 - 81\sigma_1 + 54), 0\},
\]
as desired. The last three cases are easily verified.

Now it is easy to show that \(\phi\) is one-to-one. Assume there is \((\sigma_1', \sigma_3') \in \mathbb{A}^2\) such that \(\phi(\sigma_1, \sigma_3) = \phi(\sigma_1', \sigma_3')\). Combining this assumption with the argument of the preceding paragraph yields

\[
\{\sigma_1, \sigma_2, \sigma_3, 0\} = \{\sigma_1', \sigma_2', \sigma_3', 0\},
\]
so that we conclude \(\sigma_1 = \sigma_1'\) and \(\sigma_3 = \sigma_3'\). To prove that \(\phi\) is onto, we use the fact from Fujimura-Nishizawa [6] that the set of fixed point multipliers, and thus the set of \(\sigma\)-invariants, uniquely determines the conjugacy class of a cubic polynomial. In particular, given \([f] \in \mathcal{P}_3\) with \(\sigma\)-invariants \(\{\sigma_1, \sigma_2, \sigma_3, 0\}\), we conclude that \(\phi(\sigma_1, \sigma_3) \in [f]\). It follows that \(\phi\) is a bijection and the proposition is proved.

We note that the curve \(\mathcal{C}\) corresponds to the resultant of the numerator and denominator of the first form of the image of \(\phi\). In other words, if \((\sigma_1, \sigma_3) \in \mathcal{C}\), then there is a common root between the numerator and denominator so that the degree decreases and the map is no longer an element of \(\mathcal{P}_3\). Similarly, computing the resultant of the second form shows that the degree of this form decreases when \(\sigma_1 \in \{3, 6\}\) or \(\sigma_3 = 0\), requiring the last three cases to cover all of \(\mathcal{P}_3\).

In the first two cases of Theorem [11], the representative given by \(\phi\) has the unique property that the three non-zero fixed point multipliers are three of its fixed points. This property mirrors that of the normal form given in Manes-Yasufuku [11]. Recall that there is a unique

\[
\phi(\sigma_1, \sigma_3) = \begin{cases} 
\frac{(12-2\sigma_1)z^3+(2\sigma_1^2-15\sigma_1+18)z^2+2\sigma_1^3-33^3}{-32^3+9+3\sigma_1^2z^2-(18\sigma_1-27)z+4\sigma_1^3+12\sigma_1+33^3+9} & (\sigma_1, \sigma_3) \notin \mathcal{C} \\
\frac{(9\sigma_1-27)z^3}{(-2\sigma_1-6)z^3-(2\sigma_1^2-33\sigma_1+45)z^2+(6\sigma_1^2-15\sigma_1-9\sigma_3+54)z+24\sigma_3-4\sigma_1^3} & (\sigma_1, \sigma_3) \in \mathcal{C}, \\
\begin{cases} 
z^3 & (\sigma_1, \sigma_3) = (6, 0) \\
z^3 + z & (\sigma_1, \sigma_3) = (3, 1) \\
z^3 + \frac{3}{2}z & (\sigma_1, \sigma_3) = (\frac{3}{2}, 0),
\end{cases}
\end{cases}
\]

where \(\mathcal{C}\) is the curve \(4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54 = 0\). The map \(\phi\) is a bijection.

Proof. First, we verify that \(\phi\) actually maps into \(\mathcal{P}_3\). We can compute in Sage that the first form has a totally ramified fixed point at \(z = \frac{2}{3}\sigma_1 - 1\) and the second form at \(z = 0\). The final three cases are easily verified. So that we conclude \(\mathcal{C}\) is onto, we use the fact from Fujimura-Nishizawa [6] that the set of fixed point multipliers, and thus the set of \(\sigma\)-invariants, uniquely determines the conjugacy class of a cubic polynomial. In particular, given \([f] \in \mathcal{P}_3\) with \(\sigma\)-invariants \(\{\sigma_1, \sigma_2, \sigma_3, 0\}\), we conclude that \(\phi(\sigma_1, \sigma_3) \in [f]\). It follows that \(\phi\) is a bijection and the proposition is proved.

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element of PGL₂ that moves three points in \( \mathbb{P}^1(\mathbb{C}) \) to three other points. Since the form \([11]\) is for \( \mathcal{M}_2 \), where every map has exactly three fixed points, there is a unique conjugation to a map whose fixed points and fixed point multipliers coincide. One might hope there is a more general normal form defined over the field of moduli for \( \mathcal{M}_3 \) with the property that the fixed-point multipliers equal the fixed points. However, as Theorem 1.4 clarifies, this is not the case.

**Definition 1.2.** We say that \( f \) is in fixed-point multiplier form if \( f \) has at least three distinct fixed points and all fixed points are equal to their multipliers. We say that \( f \) is in partial fixed-point multiplier form if either

1. \( f \) has two or less fixed points and all fixed points are equal to their multipliers.
2. \( f \) has four fixed points and at least three of the fixed points are equal to their multipliers.

**Lemma 1.3.** Let \( x_1, x_2, x_3, x_4 \) be distinct algebraic numbers different from 1 which form a \( \text{Gal}(\overline{K}/K) \) invariant set and satisfy

\[
\sum_{i=1}^{4} \frac{1}{1-x_i} = 1.
\]

Then there is a unique degree 3 rational function defined over \( K \) in fixed-point multiplier form with fixed points \( \text{Fix}(f) = \{ x_1, x_2, x_3, x_4 \} \). Furthermore, every degree 3 rational function defined over \( K \) with 4 distinct fixed points in fixed-point multiplier form can be obtained in this way.

We adapt the proof of Hutz-Tepper [7, Theorem 6], which proves that specifying the four fixed points and their multipliers uniquely determines a degree 3 rational function. However, the statement in the cited paper only assumes the multipliers are defined as a set, whereas their proof requires knowing which multiplier is attached to which fixed point.

**Proof.** The values \( x_1, x_2, x_3, x_4 \) are affine, so the point at infinity is not a fixed point. We dehomogenize and denote the rational map as \( F \). We can write \( F \) in the form

\[
F(z) = z - \frac{p(z)}{q(z)} = z - \frac{a \prod_{i=1}^{4} (z-x_i)}{az^3 + bz^2 + cz + d}.
\]

Computing

\[
F'(z) = 1 - \frac{p'(z)}{q(z)} - \frac{p(z)q'(z)}{q(z)^2}
\]

and evaluating at the fixed points \( \{ x_1, x_2, x_3, x_4 \} \), we need to have

\[
x_i = F'(x_i) = 1 - \frac{p'(z)}{q(z)}, \quad i = 1, 2, 3, 4.
\]

This gives the system of equations (linear in \( a, b, c, d \)) defined by

\[
(1-x_i)q(x_i) - p'(x_i) = 0, \quad i = 1, 2, 3, 4.
\]

As in Hutz-Tepper, this system has a unique solution for \( (a, b, c, d) \) and hence for \( f \). To see that this solution is defined over \( K \) recall that we have assumed \( \{ x_1, x_2, x_3, x_4 \} \) is \( \text{Gal}(\overline{K}/K) \) invariant. In particular, applying any element of Galois to the system \((11)\), fixes the system. In other words, fixes the solution \( (a, b, c, d) \). Hence, the solution, and \( f \), is defined over \( K \).
For the final statement, assume $f$ is in fixed-point multiplier form defined over $K$ with 4 distinct fixed points $\text{Fix}(f) = \{x_1, x_2, x_3, x_4\}$. Then the first dynatomic polynomial is defined over $K$ so that $\text{Fix}(f)$ is $\text{Gal}(\overline{K}/K)$ invariant. Furthermore, since the fixed points are equal to their multipliers, they must satisfy the classical relation \cite[Theorem 12.4]{[13]}:

$$\sum_{i=1}^{4} \frac{1}{1 - x_i} = 1.$$ 

\[\Box\]

**Theorem 1.4.** Let $f$ be a degree 3 rational function defined over $K$. Then there exists a $\rho \in \text{PGL}_2(\overline{K})$ such that $f^\rho$ is in partial fixed-point multiplier form and defined over $K$ if and only if the following conditions are satisfied:

1. There are at least $\min(\# \text{Fix}(f), 3)$ distinct fixed point multipliers.
2. One of the following:
   a. $f$ has a $\text{Gal}(\overline{K}/K)$ invariant set of $\min(\# \text{Fix}(f), 3)$ distinct fixed points.
   b. The automorphism group of $f$ has order 2. Moreover, let $\alpha$ be the nontrivial automorphism, and $\alpha(x) \neq x$ where $x$ is some fixed point of $f$. Then, for any element $\sigma \in \text{Gal}(\overline{K}/K)$, $\sigma$ is either fixed $x$ or $\sigma(x) = \alpha(x)$.
   c. $f^\rho$ is in fixed-point multiplier form and is one of the maps in Lemma 1.3.

**Proof.** We first recall that the map $f$ is defined over $K$ if and only if $\sigma(f) = f$ for all $\sigma \in \text{Gal}(\overline{K}/K)$. (Hilbert Theorem 90, \cite[Exercise 1.12]{[14]}).

Condition (1) avoids the trivial obstruction of there not being enough distinct values for up to three distinct fixed points to be equal to their multipliers. So we focus on condition (2).

Throughout the proof we may assume without loss of generality that the point at infinity is not a fixed point. We dehomogenize and denote the fixed points as $x_i$, $1 \leq i \leq \# \text{Fix}(f)$ and the rational map as $F$.

We first prove the conditions imply the existence of the desired form. Assume first that condition (1) and (2a) hold. Let $k = \min(\# \text{Fix}(f), 3)$. If $k = \# \text{Fix}(f)$, then we may add arbitrary distinct rational points to the set of fixed points so that we have a Galois invariant set $\{z_1, z_2, z_3\}$ of distinct points that we wish to move via conjugation. The target set $\{t_1, t_2, t_3\}$ is the set of fixed point multipliers plus an arbitrary choice of distinct rational points. There is a unique $\rho \in \text{PGL}_2(\overline{K})$ so that $f^\rho$ satisfies

$$F^\rho(\rho^{-1}(x_i)) = \rho^{-1}(x_i) = F'(\rho^{-1}(x_i)), \quad 1 \leq i \leq \# \text{Fix}(f)$$

$$\rho^{-1}(z_i) = t_i, \quad \# \text{Fix}(f) < i \leq 3.$$ 

We will see that $\rho \in \text{PGL}_2(K)$ so that $f^\rho$ is defined over $K$. Write

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Then we need to solve the system of equations

$$az_i + b = t_i(cz_i + d), \quad 1 \leq i \leq 3.$$ 

Note that when $z_i$ is the fixed point $x_i$, then $t_i = F'(x_i)$. We know that (up to scaling) there is a unique solution $(a, b, c, d)$. Since the set of fixed points is Galois invariant and any
additional points are rational, applying any element of Galois to this system of equations leaves the system fixed and, thus, its solutions unchanged. Therefore, \((a, b, c, d)\) are defined over \(K\).

Assume now condition (11) is satisfied and condition (2a) is not. Then \(f\) has 4 fixed points of which there is not a size 3 subset that is Galois invariant and at least 3 of the multipliers are distinct. If there are exactly 3 distinct multipliers, then we can construct an automorphism of \(f\) by fixing the two fixed points with distinct multipliers and exchanging the two fixed points with the same multiplier. This induces a unique nontrivial \(\alpha \in \text{PGL}_2\) which is an order 2 automorphism of \(f\), and we are in condition (2b). Let \(\rho \in \text{PGL}_2(\overline{K})\) such that \(\rho(x_i) = F'(x_i)\) for \(i = 1, 2, 3\). We know \(\alpha(x_i) = x_i\) for \(i = 1, 2\) and \(\alpha(x_3) = x_4\). Note that \(\rho(x_4) = F'(x_3)\). Let \(\sigma \in \text{Gal}(\overline{K}/K)\). Then \(\sigma(f^\rho)\) either fixes \(f\) or exchanges the fixed points \(\sigma(x_3) = \sigma(x_4)\). In particular, \(f^\rho\) and \(\sigma(f^\rho)\) are degree 3 rational functions with 4 distinct fixed points with the same set of associated multipliers. So by (the corrected) Hutz-Tepper [7, Theorem 6] we have \(\sigma(f^\rho) = f^\rho\). In particular, \(f^\rho\) is defined over \(K\).

If instead all four multipliers are distinct, then we are in condition (2c).

For the other direction, without loss of generality assume that \(f\) is in partial fixed-point multiplier form and defined over \(K\). Since \(f\) is defined over \(K\), the fixed point equation (first dynatomic polynomial) is defined over \(K\) and is Galois invariant. If \(f\) does not have a set of \(\min(\#\text{Fix}(f), 3)\) distinct fixed points that are Galois invariant, then there must be a 4th fixed point \(x_4\) and a \(\sigma \in \text{Gal}(\overline{K}/K)\) so that \(\sigma(x_i) = x_i\) for \(i \in \{1, 2, 3\}\), say \(\sigma(x_4) = x_3\). If \(F'(x_4) = F'(x_3)\), then \(f\) has a nontrivial automorphism and we are in condition (2b). Otherwise, we calculate

\[
x_4 = \sigma(x_3)
= \sigma(F'(x_3)) \quad \text{(because } f \text{ is in partial fixed-point multiplier form with } x_i = F(x_i) = F'(x_i) \text{ for } i = 1, 2, 3)
= F'(\sigma(x_3)) \quad \text{(because } F \text{ is defined over } K)
= F'(x_4)
\]

so that \(f\) is, in fact, in fixed-point multiplier form and we are in condition (2a).

We exhibit examples of each of the conditions (2a, 2b, 2c). Note that (2b) is mutually exclusive to (2c) since we cannot have an automorphism if the multipliers are all distinct.

(2a) \(F(z) = \frac{2z^2}{-z^2 + 4z - 2}\) with fixed points \(\{0, 1 + i, 1 - i\}\) in fixed-point multiplier form.

(2b) \(F(z) = \frac{18z^3}{-11z^3 + 57z^2 + 7z + 25}\) in partial fixed-point multiplier form with fixed points \(\{-1, 5, 0, -5/11\}\) and respective multipliers \(\{-1, 5, 0, 5\}\) and a nontrivial automorphism group of order 2.

(2c) \(F(z) = \frac{-3z^2 + z^2 - 2z - 2}{z^3 - 3z^2 + 3z - 4}\) with fixed points \(\{-i, i, 1 - i, 1 + i\}\) in fixed-point multiplier form.

Let \(L\) be the field of definition of the fixed points of \(f\), called the first dynatomic field. We call the Galois Group \(\text{Gal}(L/K)\) the first dynatomic Galois group. In the proof of Theorem 1.4, the main idea is to use the existence of a Galois-invariant subset that contains three fixed points, so it is no surprise that if \(f\) has a conjugate in partial fixed-point multiplier form, then we gain some control on the first dynatomic Galois group of \(f\). We are also able to count how many different conjugates of \(f\) are in partial fixed-point multiplier form and
defined over $K$. Corollary 1.7 provides a convenient method to examine whether a conjugate of a rational map for partial fixed-multiplier form exists.

**Corollary 1.5.** Let $f$ be a rational map defined over $K$. Assume $\text{Aut}(f)$ is trivial. Then, there are $\rho \in \text{PGL}_2(K)$ such that $f^\rho$ is in partial fixed-point multiplier form if and only if $\rho$ is defined over $K$. Moreover, we have the following:

1. $\rho$ is unique if and only if $f^\rho$ is in fixed-point multiplier form or the first dynatomic Galois group is isomorphic to $S_3$, the symmetric group of three letters, or $\mathbb{Z}_3$, the order 3 abelian group.
2. There are exactly two distinct $\rho \in \text{PGL}_2(K)$ such that $f^\rho$ is in partial fixed-point multiplier form if and only if $f$ has four fixed points, and one of the following is true:
   (a) the first dynatomic Galois group is isomorphic to a group of order 2;
   (b) $f$ has two distinct fixed points with the same multiplier.
3. There are exactly four distinct $\rho \in \text{PGL}_2(K)$ such that $f^\rho$ is in partial fixed-point multiplier form if and only if the first dynatomic Galois group is trivial and $f$ has four fixed points.

**Proof.**

(1) Suppose that there is a unique $\rho \in \text{PGL}_2(K)$ such that $f^\rho$ is in partial fixed-point multiplier form defined over $K$. If $f^\rho$ is in fixed-point multiplier form, then we are done. Otherwise, Theorem 1.4 implies that there must exist a Galois-invariant set that contains three fixed points of $f$. This Galois-invariant set is unique since, by hypothesis, there is a unique $\rho$. If $f$ has only three fixed points, then $f^\rho$ is definitely in fixed-point multiplier form. Thus, we assume $f$ has four fixed points for the following proof. Obviously, the fixed point not in the Galois invariant set must be defined over $K$. Moreover, the first dynatomic Galois group cannot be trivial, which implies there are four distinct such Möbius transformations $\rho$, so it can only beomorphic to $\mathbb{Z}_2$.

(2) It is enough to show that the existence of two such $\rho$ implies the conditions. Since we assume $\text{Aut}(f)$ is trivial, Theorem 1.4 implies there must exist a Galois-invariant set of three fixed points. Since there are two choices of $\rho$, there are two Galois invariant sets. If condition (2b) holds, then we are done. Otherwise, there are four distinct multipliers. Moreover, the dynatomic Galois group cannot be trivial, which will implies there are four distinct such Möbius transformations $\rho$, so it can only beomorphic to $\mathbb{Z}_2$.

(3) The remaining case.

**Corollary 1.6.** Let $f$ be a rational map defined over $K$, and let $L = K(f(x) - x)$ be the first dynatomic field of $f$. Assume $\text{Aut}(f)$ is of order 2. Then, there are $\rho \in \text{PGL}_2(K)$ such that $f^\rho$ is in partial fixed-point multiplier form if and only if the first dynatomic group $\text{Gal}(L/K)$ is isomorphic to the trivial group, $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the group of order 2 and $f$ has four fixed points. In particular, there always exist two distinct such $\rho$.

**Proof.** Condition (2b) of Theorem 1.4 implies that if (up to renaming) swap $x_1$ and $x_2$ or $x_3$ and $x_4$ where $x_1, x_2, x_3$ and $x_4$ are fixed points of $f$ and the nontrivial automorphism
switches $x_3$ and $x_4$. Therefore, the first dynatomic Galois group can be embedded into $\mathbb{Z}_2 \times \mathbb{Z}_2$. When $f$ has a nontrivial automorphism, $f$ does not have a conjugate in fixed-point multiplier form. Since there are only three distinct multipliers, there are only two choices for sets of three fixed points that give enough multipliers, so only two mobius transformations $\rho$ conjugate $f$ to a fixed-point multiplier form.

**Corollary 1.7.** Let $f$ be a degree 3 rational map written as $\frac{p(x)}{q(x)}$ for some polynomials $p$ and $q$ defined over $K$. We further suppose that $f$ has at least three distinct fixed point multipliers. Then, there exists some $\rho \in \text{PGL}_2(K)$ such that $f^\rho$ is in partial fixed-point multiplier form if and only if one of the following is true.

1. Some conjugate of $f$ passes the criteria in Lemma 1.3.
2. The degree of $q(x)$ is equal to or less than 2.
3. $p(x) - xq(x)$ is of degree 4 and has at least one zero in $K$ with $\text{char}(K) \neq 2$.
4. $f$ has a unique nontrivial automorphism.

**Proof.** Assume there exists $\rho \in \text{PGL}_2(K)$ such that $f^\rho$ is in partial fixed-point multiplier form. If $q(x)$ has degree equal to or less than 2, then infinity is a fixed point of $f$. In particular, $f$ has either only three fixed points or the three finite fixed points form a Galois-invariant set.

Now, we can assume $q$ has degree 3. If $f$ is in fixed-point multiplier form, then we are in the case of Lemma 1.3. If $f$ is not in fixed-point multiplier form, then we must either have $\text{Aut}(f) = 2$ or $f$ has a Galois-invariant set containing three fixed points. Note that all the fixed points of $f$ are finite, so they are the solutions of $p(x) - xq(x) = 0$. Clearly, if there are three roots forming a Galois invariant set, then one of the roots must be in $K$, or there are only three solutions to $p(x) - xq(x) = 0$. If there are only three roots, then one of the roots is a double root. The double root may be not defined on $K$ if the root is ramified over $K$. Since $p(x) - xq(x)$ is a degree 4 polynomial defined over $K$ and $\text{char}(K) \neq 2$, $p(x) - xq(x)$ will not have an irreducible ramified factor. Therefore, the double root must be in $K$.

Conversely, (1) and (4) are proved in Theorem 1.4. Both (2) and (3) imply there is a Galois-invariant set of three fixed points of $f$.

To find a rational map that has no conjugate defined over the field of moduli with the property that the fixed points are the multipliers, we simply take one that contradicts the assumptions of Theorem 1.4, for example $\frac{x^3 + x + 1}{x^2 + 1}$ has neither a Galois invariant set of three fixed points, a nontrivial automorphism, nor is in fixed-point multiplier form.

**Remark 1.8.** Note that the set of maps which cannot be conjugated to partial fixed-point multiplier form defined over $K$ is dense in the moduli space $\mathcal{M}_3$. This is because conditions (1, 2, and 4) from Corollary 1.7 are closed conditions and condition (3) is contained in a (type II) thin set.

### 2. n-critical Normal Form

We prove a generalization to $n$-critical points of the normal form for bicritical maps given in Tobin [18]. This normal form will be useful when discussing generalizations of the algebraic proof of Thurston’s transversality for bicritical polynomials in Section 3. First, we define some notation.
Definition 2.1. We define
\[ \sum_{j_0 \ldots j_n = 0}^{k_0 \ldots k_n} \] to be \( n + 1 \) nested sums. That is,
\[ \sum_{j_0 \ldots j_n = 0}^{k_0 \ldots k_n} := \sum_{j_0 = 0}^{k_0} \sum_{j_1 = 0}^{k_1} \ldots \sum_{j_n = 0}^{k_n} \]

For the following proposition, any sum with lower and upper bounds omitted is a sum from 0 to \( n - 2 \). That is,
\[ \sum_l := \sum_{l=0}^{n-2} \]

Theorem 2.2. Let \( g \in K[z] \) be a degree \( d \) polynomial with \( n \) critical points. There exist \( k_0, \ldots, k_{n-2} \in \mathbb{N} \), with \( n - 1 \leq \sum_l k_l \leq d - 2 \) and a, c, \( \gamma_0, \ldots, \gamma_{n-2} \in \overline{K} \) such that \( g \) is conjugate to
\[ a \left( \frac{d!}{(d - \sum_l k_l - 1)!} \sum_{j_0 \ldots j_{n-2} = 0}^{k_0 \ldots k_{n-2}} \prod_{i=0}^{n-2} (-\gamma_i)^{k_i-j_i} \left( \frac{k_i}{j_i} \right) \frac{z^{d+\sum_l j_l - k_l}}{d+\sum_l j_l - k_l} \right) + c. \]

Proof. Let \( g \in K[z] \) be a polynomial with \( n \) critical points \( \xi_0, \ldots, \xi_{n-1} \). Conjugate by \( \phi(z) = \frac{z - \xi_0}{z - \xi_{n-1}} \in \text{PGL}_2(\overline{K}) \), which sends the critical points \( \xi_0 \) and \( \xi_{n-1} \) to 1 and 0, respectively. Then, \( f(z) = g^{\phi(z)} \) has critical points \( 0, \gamma_0 = 1, \gamma_1, \ldots, \gamma_{n-2} \). Let \( d - \sum_l k_l \) be the ramification index of 0 and let \( k_0 + 1, \ldots, k_{n-2} + 1 \) be the ramification indices of \( \gamma_0, \ldots, \gamma_{n-2} \), respectively. We can then write the derivative \( f'(z) \) as
\[ f'(z) = \alpha z^{d-\sum_l k_l - 1} \prod_{i=0}^{n-2} (z - \gamma_i)^{k_i} \]
for some \( \alpha \in \overline{K} \). Then,
\[ f(z) = \alpha \int z^{d-\sum_l k_l - 1} \sum_{j_0 \ldots j_{n-2} = 0}^{k_0 \ldots k_{n-2}} \prod_{i=0}^{n-2} (-\gamma_i)^{k_i-j_i} \left( \frac{k_i}{j_i} \right) z^{j_i} \, dz \]
\[ = \alpha \int z^{d-\sum_l k_l - 1} \sum_{j_0 \ldots j_{n-2} = 0}^{k_0 \ldots k_{n-2}} z^{\sum_j j_i} \prod_{i=0}^{n-2} (-\gamma_i)^{k_i-j_i} \left( \frac{k_i}{j_i} \right) \, dz \]
\[ = \alpha \sum_{j_0 \ldots j_{n-2} = 0}^{k_0 \ldots k_{n-2}} \prod_{i=0}^{n-2} (-\gamma_i)^{k_i-j_i} \left( \frac{k_i}{j_i} \right) \int z^{d+\sum_l j_l - k_l - 1} \, dz \]
\[ = \alpha \sum_{j_0 \ldots j_{n-2} = 0}^{k_0 \ldots k_{n-2}} \prod_{i=0}^{n-2} (-\gamma_i)^{k_i-j_i} \left( \frac{k_i}{j_i} \right) z^{d+\sum_l j_l - k_l} + c. \]
We can then make the substitution

\[ \alpha = a \left( \frac{d!}{(d - \sum k_l - 1)!} \right) \]

so that

\[ f(z) = a \left( \frac{d!}{(d - \sum k_l - 1)!} \right) \sum_{j_0...j_{n-2}=0}^{k_0...k_{n-2}} \prod_{i=0}^{n-2} (-\gamma_i)^{k_l - j_i} \left( \begin{array}{c} k_i \\ j_i \end{array} \right) \frac{z^{d+\sum j_l - k_l}}{d + \sum j_l - k_l} \) + c. \]

Additionally, since \( k_0 + 1, \ldots k_{n-2} + 1 \) are the ramification indices of the critical points, \( \forall l \) \( k_l + 1 \geq 2 \) so that \( k_l \geq 1 \). Since \( d - \sum k_l \) is the ramification index of 0,

\[ \sum_{l} k_l \leq d - 2. \]

Combining these two inequalities, we have

\[ n - 1 \leq \sum_{l} k_l \leq d - 2. \]

\[ \square \]

Applying Theorem 2.2 to a particular case leads to a specific normal form as demonstrated in the next example.

**Example 2.3.** Consider the family of polynomials \( f_{a,c,\gamma}(z) \) of degree 4 with 3 critical points, \( \gamma_0, \gamma_1, \gamma_2 \), each of ramification index 2.

Note that the proof of Theorem 2.2 conjugates so that \( \gamma_0 = 1 \) and \( \gamma_2 = 0 \). For clarity, we relabel \( \gamma_1 = \gamma \). The proof of Theorem 2.2 also shows that \( k_i = e_{\gamma_i}(f_{a,c,\gamma}) - 1 \), where \( e_{\gamma_i}(f_{a,c,\gamma}) \) is the ramification index of \( f_{a,c,\gamma} \) at \( \gamma_i \). Hence, we have that \( k_0 = e_{\gamma_0}(f_{a,c,\gamma}) - 1 = 1 \) and \( k_1 = e_{\gamma_1}(f_{a,c,\gamma}) - 1 = 1 \). Now, we substitute with \( d = 4 \), \( n = 3 \), \( k_0 = 1 \), \( k_1 = 1 \), \( \gamma_0 = 1 \), and \( \gamma_1 = \gamma \).

\[ f_{a,c,\gamma}(z) = a \left( \frac{4!}{4 - \sum k_l - 1!} \sum_{j_0,j_1=0}^{1} \prod_{i=0}^{1} (-\gamma_i)^{k_l - j_i} \left( \begin{array}{c} k_i \\ j_i \end{array} \right) \frac{z^{4+\sum j_l - \sum k_l}}{4 + \sum j_l - \sum k_l} \right) + c. \]

Since \( \forall i, k_i = 1 \), and \( \sum_{l} k_l = 2 \),

\[ f_{a,c,\gamma}(z) = a \left( 4! \sum_{j_0,j_1=0}^{1} (-\gamma_i)^{1-j_i} \left( \begin{array}{c} 1 \\ j_i \end{array} \right) \frac{z^{2+j_0+j_1}}{2 + j_0 + j_1} \right) + c. \]
We can bring the $4!$ into the sum and use the fact that $j_i \leq k_i = 1$ to simplify $\frac{4!}{2^{j_0+j_1}}$ as $\prod_{l=0, l \neq j_0 + j_1}^{2} 2 + l$:
\[
f_{a,c,\gamma} = a \left( \sum_{j_0,j_1=0}^{1,1} (-1)^{1-j_0}(-\gamma)^{1-j_1} \left( \prod_{l=0, l \neq j_0 + j_1}^{2} 2 + l \right) z^{2+j_0+j_1} \right) + c
\]
\[
= a \sum_{j_0=0}^{1} (-1)^{1-j_0}(-\gamma) \left( \prod_{l=0, l \neq j_0}^{2} 2 + l \right) z^{2+j_0} - (-1)^{1-j_0} \left( \prod_{l=0, l \neq j_0+1}^{2} 2 + l \right) z^{3+j_0} \right) + c
\]
\[
= a \left( (3 \cdot 4)z^2 - (2 \cdot 4)z^3 - \gamma(2 \cdot 4)x^3 + (2 \cdot 3)z^4 \right)
\]
\[
= a(6z^4 - 8(1 + \gamma)z^3 + 12\gamma z^2) + c
\]

3. An Algebraic Proof of Thurston’s Transversality

Recall that two curves intersect transversally if the determinant of the matrix of partial
derivatives (the Jacobian) of the two curves does not equal 0 at all points of intersection.

To prove the main result of this section, we rely on the description of bicritical maps given
by Tobin [18]. We first state some results from [18] which we will use without proof.

Lemma 3.1 ([18 Proposition 4.0.2]). Let $g \in K[z]$ be a bicritical polynomial of degree $d \geq 3$.
Then $g$ is conjugate to a map $f_{a,c}(z) = aB_{d,k} + c$, where $B_{d,k}$ a single-cycle, normalized Belyi
map of combinatorial type $(d; d - k, k + 1, d)$ and $a, c \in K$. Moreover, $f_{a,c}(z)$ has affine
critical points $\{0, 1\}$.

Note that the proof of [18 Proposition 4.0.2] implies that the ramification index of 0 with
respect to $f_{a,c}(z) = aB_{d,k} + c$ is $d - k$, while the ramification index of 1 is $k + 1$. That 0 is
a critical point implies a ramification index of at least 2, so that $k \leq d - 2$. Similarly, the
ramification index of 1 must be at least 2, so that $1 \leq k \leq d - 2$.

Lemma 3.2 ([18 Proposition 4.0.4]). Let $f_0 \neq f_1 \in K[z]$ with $f_0(z) = a_0B_{d,k_0} + c_0$ and
$f_1(z) = a_1B_{d,k_1} + c_1$. The polynomials $f_0$ and $f_1$ are conjugate if and only if $k_0 + k_1 = d - 1$,
$a_0 = a_1$, and $c_1 = 1 - a_0 - c_0$.

Combining Lemma 3.2 with the inequality $1 \leq k \leq d - 2$, we note that $f_{a,c}(z) = aB_{d,k} + c$
can be chosen such that $1 \leq k \leq \left\lfloor \frac{d-2}{2} \right\rfloor$.

This description of bicritical polynomials allows us to find primes which reduce bicritical polynomials
nicely, allowing us to prove the main theorem of this section.

Theorem 3.3. Let $B_d^{\text{crit}}$ be the moduli space of bicritical polynomials of degree $d$ with marked
critical points. Points in $B_d^{\text{crit}}$ are equivalence classes of triples $(f, c_1, c_2)$, where $f \in \mathbb{C}[z]$ is
a bicritical polynomial of degree $d$ with critical points $c_1$ and $c_2$. For integers $n, m \geq 1$ let
\[
C_{d,0,n} = \{ f \in B_d^{\text{crit}} | \text{0 is periodic with } f^n(0) = 0 \}
\]
\[
C_{d,1,m} = \{ f \in B_d^{\text{crit}} | \text{1 is periodic with } f^m(1) = 1 \}.
\]
The curves $C_{d,0,m}$ and $C_{d,1,n}$ intersect transversely.

Proof. We begin by proving transversality for $f_{a,c}(z) \in \mathbb{Q}[z]$ and then show the proof can
be easily modified for any finite extension. By Lemma 3.1, we may write any bicritical polynomial as
\[
f_{a,c}(z) = aB_{d,k} + c.$
where we can use [1, Proposition 3.1] to write
\[ B_{d,k} = \sum_{i=0}^{k} (-1)^{k-i} \left[ \prod_{j=0, j \neq i}^{k} (d-j) \left( \frac{1}{(k-i)!i!} \right) \right] z^{d-i}. \]

Note that since \( k \leq \left\lfloor \frac{d-2}{2} \right\rfloor \), every term in the product \( \prod_{j=0}^{k} (d-j) \) is greater than or equal to \( d - \frac{d-2}{2} = \frac{d}{2} + 1 \). A theorem from Sylvester [17] states that the product of \( m \) consecutive integers greater than \( m \) is divisible by a prime greater than \( m \). In particular, there exists some prime \( p > k + 1 \) such that \( p \) divides \( \prod_{j=0}^{k} (d-j) \). Since a multiple of \( p \) can occur only once in an interval of length \( k + 1 \leq p \), we know \( p \) divides exactly one term in the product. So there exists some \( 0 \leq r \leq k \) such that \( p \mid (d-r) \).

Note that since \( (k-r) \) and \( r \) are both less than \( p \), we have \( p \nmid (k-r)!r! \), implying \( ((k-r)!r!)^{-1} \in \mathbb{F}_p \). Thus, for our choice of \( p \), we have shown that \( B_{d,k} \) reduces to a monomial.

Now we can reduce \( f_{a,c} \) using the reduction of \( B_{d,k} \):
\[ f_{a,c}(z) \equiv asz^{tp} + c \pmod{p}, \]
where
\[ s \equiv (-1)^{k-r} \prod_{j=0, j \neq r}^{k} (d-j) \cdot \frac{1}{(k-r)!r!} \pmod{p} \]
and \( tp = d-r \) for some \( t \in \mathbb{N} \). Since the critical points of \( f \) in this bicritical normal form are 0 and 1, given periods \( m, n \geq 1 \), the intersection of
\[ f_{a,c}^m(0) = 0 \quad \text{and} \quad f_{a,c}^n(1) - 1 = 0 \]
gives the locus of PCF bicritical polynomials with 0 periodic of period \( m \) and 1 periodic of period \( n \). It remains to be shown that these curves intersect transversely. To prove transversality, we compute the Jacobian of the two curves and show that it cannot be 0 mod \( p \) at the points of intersection. We can explicitly compute the partial derivatives of \( f_{a,c}^m(0) \) and \( f_{a,c}^n(1) \) as follows:
\[ \frac{\partial}{\partial a} (f_{a,c}^m(0)) \equiv \frac{\partial}{\partial a} (as(f_{a,c}^{m-1}(0))^{tp} + c) \]
\[ \equiv s(f_{a,c}^{m-1}(0))^{tp} + astp(f_{a,c}^{m-1}(0))^{tp-1} \frac{\partial}{\partial a} \left( f_{a,c}^{m-1}(0) \right) \]
\[ \equiv s(f_{a,c}^{m-1}(0))^{tp} \pmod{p} \]
\[ \frac{\partial}{\partial c} (f_{a,c}^m(0)) \equiv \frac{\partial}{\partial c} (as(f_{a,c}^{m-1}(0))^{tp} + c) \]
\[ \equiv astp(f_{a,c}^{m-1}(0))^{tp-1} \frac{\partial}{\partial c} \left( f_{a,c}^{m-1}(0) \right) + 1 \]
\[ \equiv 1 \pmod{p}. \]
Similarly,
\[
\frac{\partial}{\partial a} \left( f_{a,c}^n(1) \right) \equiv s(f_{a,c}^{n-1}(1))^{\text{tp}} \pmod{p}
\]
\[
\frac{\partial}{\partial c} \left( f_{a,c}^n(1) \right) \equiv 1 \pmod{p}.
\]
Thus, the Jacobian is given by
\[
J(a, c) \equiv \det \begin{pmatrix} \frac{1}{s(f_{a,c}^{m-1}(0))^{\text{tp}}} & \frac{1}{s(f_{a,c}^{n-1}(1))^{\text{tp}}} \\ s(f_{a,c}^{n-1}(1))^{\text{tp}} & s(f_{a,c}^{m-1}(0))^{\text{tp}} \end{pmatrix} \pmod{p}.
\]
Since \( f_{a,c}^n(1) - 1 = 0 \), we have \( s(f_{a,c}^{n-1}(1))^{\text{tp}} + c = 1 \), so that
\[
(f_{a,c}^{n-1}(1))^{\text{tp}} = \frac{1-c}{as}
\]
and, similarly,
\[
(f_{a,c}^{m-1}(0))^{\text{tp}} = -\frac{c}{as}.
\]
It follows that
\[
J(a, c) \equiv s(f_{a,c}^{n-1}(1))^{\text{tp}} - f_{a,c}^{m-1}(0)^{\text{tp}}
\]
\[
\equiv s \left( \frac{1-c}{as} + \frac{c}{as} \right) \pmod{p}.
\]
Notice that if \( a \equiv 0 \pmod{p} \), the relations \( as(f_{a,c}^{n-1}(1))^{\text{tp}} + c \equiv 1 \pmod{p} \) and \( as(f_{a,c}^{m-1}(0))^{\text{tp}} + c \equiv 0 \pmod{p} \) would imply that \( c \equiv 1 \) and \( c \equiv 0 \pmod{p} \), which is clearly a contradiction, so we conclude that at a point \( (a, c) \) of intersection of the two curves,
\[
J(a, c) \not\equiv 0 \pmod{p}.
\]
To extend to any finite extension \( K \) of \( \mathbb{Q} \), we quotient by a prime lying above \( p \). \( \square \)

3.1. Failure to Extend to \( n \) critical points. Using the normal form in Theorem 2.2, we might hope to provide algebraic proofs of transversality for polynomials with 3 or more critical points. Unfortunately, the next two examples show that choosing \( p \) as in the bicritical case ultimately fails.

The first example shows the importance of the results on conjugacy from [18] in the bicritical case. For polynomials with 3 or more critical points, we can no longer assume that \( 1 \leq k_i \leq \left\lfloor \frac{d-2}{2} \right\rfloor \) Hence, we cannot always find a prime for which the polynomial reduces nicely.

Example 3.4. Consider the family of polynomials \( f_{a,c,\gamma} \) with \( d = 10, n = 3, k_0 = 7, k_1 = 1, \gamma_0 = -1, \) and \( \gamma_1 = \gamma \)
Using the normal form and the notation from in Theorem 2.2

\[
f_{a,c,\gamma}(z) = a \left( \frac{10!}{7!} \sum_{j_0,j_1=0}^{7,1} \prod_{i=0}^{1} (-\gamma_i)^{k_i-j_i} \left( \frac{x^{2+j_0+j_1}}{2+j_0+j_1} \right) \right) + c
\]

\[
= a \left( \sum_{j_0,j_1=0}^{7,1} \prod_{i=0}^{1} (-\gamma_i)^{k_i-j_i} \left( \prod_{l=0,l\neq j_0+j_1}^{8} 2 + l \right) x^{2+j_0+j_1} \right) + c.
\]

Note that the product

\[
\prod_{l=0,l\neq j_0+j_1}^{8} 2 + l
\]

will always be nonzero for any prime \( p > 10 \) and will always be 0 for any prime \( p \leq 5 \). We must then try to reduce \( f_{a,c,\gamma} \) modulo \( p = 7 \). Values of \( j_0 \) and \( j_1 \) that will make the previous product non zero modulo 7 are those such that \( j_0 + j_1 = 5 \), which results in two solutions, \( j_0 = 5 \) and \( j_1 = 0 \) or \( j_0 = 4 \) and \( j_1 = 1 \). In both of these cases, however, we have

\[
7 | \binom{k_0}{j_0}
\]

as \( \binom{k_0}{j_0} = \binom{7}{5} \) or \( \binom{7}{4} \), giving

\[
f_{a,c,\gamma} \equiv c \pmod{7}.
\]

There is, therefore, no prime for which the reduction \( f_{a,c,\gamma} \) is useful for proving transversality.

If we assume that \( 1 \leq k_i \leq \left\lceil \frac{d-2}{2} \right\rceil \), then we can once again apply Sylvester’s theorem to guarantee a prime for which the reduction is sufficiently nice. However, as the next example shows, this is not enough to be able to prove transversality.

**Example 3.5.** Consider the family \( f_{a,c,\gamma} \) with \( d = 4 \), \( n = 3 \), \( k_0 = 1 \), \( k_1 = 1 \), \( \gamma_0 = -1 \), and \( \gamma_1 = \gamma \)

From **Example 2.3**

\[
f_{a,c,\gamma}(z) = a(6z^4 - 8(1 + \gamma)z^3 + 12\gamma z^2) + c.
\]

Clearly, we must reduce by \( p = 3 \) to get

\[
f_{a,c,\gamma} \equiv a(1 + \gamma)x^3 + c \pmod{3}.
\]

We compute the Jacobian \( J(a,c,\gamma) \) as

\[
J(a,c,\gamma) = \det \begin{pmatrix}
1 & 1 & 1 \\
(1 + \gamma)(f^{m-1}(0))^3 & (1 + \gamma)(f^{n-1}(1))^3 & (1 + \gamma)(f^{k-1}(\gamma))^3 \\
a(f^{m-1}(0))^3 & a(f^{n-1}(1))^3 & a(f^{k-1}(\gamma))^3
\end{pmatrix}
\]

\[
= a(1 + \gamma) \det \begin{pmatrix}
1 & 1 & 1 \\
(f^{m-1}(0))^3 & (f^{n-1}(1))^3 & (f^{k-1}(\gamma))^3 \\
(f^{m-1}(0))^3 & (f^{n-1}(1))^3 & (f^{k-1}(\gamma))^3
\end{pmatrix},
\]

which is zero as the second and third rows are equal.
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