Bilinear Bäcklund transformations and Lax pair for the Boussinesq equation

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Abstract: Hirota’s bilinear approach is a very effective method to construct solutions for soliton systems. In terms of this method, the nonlinear equations can be transformed into linear equations, and can be solved by using perturbation method. In this paper, we study the bilinear Boussinesq equation and obtain its bilinear Bäcklund transformation. Starting from this bilinear Bäcklund transformation, we also derive its Lax pair and test its integrability.

Keywords: Hirota method, D-operator, Bäcklund transformation, Lax pair

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1 Introduction

The integrability and the searching for explicit solution of nonlinear equations are always important and forefront research topics\textsuperscript{1-4}. Hirota’s bilinear approach is an effective method to construct solutions for soliton systems by now\textsuperscript{5-7}. Meanwhile, the Lax pair and Bäcklund transformation are also very important in discussing nonlinear evolution equations\textsuperscript{8-12}. Similar to nonlinear partial differential equations, one can also use the appropriate method to obtain the bilinear Bäcklund transformation, the Lax pair, the Miura transformation and so forth for bilinear equations\textsuperscript{5,13-15}. 
The purpose of the present paper is to present our results on the bilinear approach to the Boussinesq equation. We will obtain its bilinear Bäcklund transformation and Lax pair.

The paper is organized as follows. In the next section, we will obtain a new bilinear equation by eliminating a variable from the original Boussinesq equation, and the corresponding nonlinear partial differential equation. In section 3, we construct a bilinear Bäcklund transformation for the bilinear Boussinesq equation system. In section 4, we will be devoted to the construction of lax pair for the equation. Final section contains our discussion and conclusion.

2 New equations from the Boussinesq equation system

In this section, we start from the bilinear Boussinesq equation system to obtain a new bilinear equation. By means of appropriate transformation and the properties of D operator, we also get a new nonlinear differential equation. The Boussinesq equation system can be written as

\[
\begin{align*}
(D_{m}^{4} + 3D_{2m}^{2})f \cdot f &= 0, \\
(D_{m}D_{m+3} - 3D_{1}D_{2}D_{m} - \frac{1}{4}D_{4}D_{m})f \cdot f &= 0,
\end{align*}
\]

where \( m \neq 3k, \quad k \in \mathbb{Z}^+, \quad D_{k}D_{m} = D_{k}D_{m}.

From Eq. (2.1), one can get the system of Boussinesq equations

\[
\begin{align*}
(D_{x}^{4} + 3D_{t_{2}}^{2})f \cdot f &= 0, \\
(D_{x}D_{t_{4}} - 4D_{x}^{3}D_{t_{2}})f \cdot f &= 0.
\end{align*}
\]

By taking

\[
\begin{align*}
g &= f_{t_{2}}, \\
h &= f_{t_{2}t_{2}}.
\end{align*}
\]

and using the following formula

\[
\begin{align*}
D_{t_{2}}^{2}f \cdot f &= 2(f_{t_{2}t_{2}}f - f_{t_{2}}f) = 2(hf - gf), \\
D_{x}^{2}D_{t_{2}}f \cdot f &= 2g_{xxx}f - 6g_{xx}f_{x} + 6g_{xx}f_{x} - 2f_{xxx}g = 2D_{x}^{3}g \cdot f,
\end{align*}
\]

we can eliminate \( t_{2} \) from Eq.(2.2) and get the bilinear differential equation

\[
\begin{align*}
(D_{x}^{4}f \cdot f + 6(hf - gf) &= 0, \\
D_{x}D_{t_{4}}f \cdot f - 8D_{x}^{3}g \cdot f &= 0, \\
D_{x}D_{t_{4}}g \cdot f - 4D_{x}^{3}h \cdot f &= 0.
\end{align*}
\]

If we take

\[
\begin{align*}
u &= 2(\ln f)_{xx}, \\
v &= \frac{g}{f}, \\
w &= \frac{h}{f},
\end{align*}
\]

2
and take advantage of the following property of $D$–operator
\[
\begin{align*}
\frac{D^4 f}{f} &= u_{xx} + 3u^2, \\
\frac{D^2 D^1 f}{f} &= 2(\ln f)_{xt_4}, \\
\frac{D^3 g}{f} &= v_{xxx} + 3uv_x, \\
\frac{D^1 D^2 f}{f} &= v_{t_4x} + 2v(\ln f)_{xt_4}, \\
\frac{D^3 h}{f} &= w_{xxx} + 3uw_x.
\end{align*}
\] (2.7)

We can obtain the nonlinear form of Eqs. (2.5)
\[
\begin{align*}
\begin{cases}
u_{xx} + 3u^2 + 6(w - v) = 0, \\
\partial^{-1}u_{t_4} - 8v_{xxx} + 3uv_x) = 0, \\
v_{t_4x} + 2v(\ln f)_{xt_4} - 4(w_{xxx} + 3uw_x) = 0.
\end{cases}
\end{align*}
\] (2.8)

After eliminating $\partial^{-1}u_{t_4}$ from the above equations, one can get the nonlinear evolution equation
\[
\begin{align*}
\begin{cases}
u_{xx} + 3u^2 + 6(w - v) = 0, \\
v_{t_4x} + 8v(\ln f)_{xt_4} - 4(w_{xxx} + 3uw_x) &= 0.
\end{cases}
\end{align*}
\] (2.9)

## 3 Bilinear Bäcklund Transformation

In this section, we will discuss the bilinear Bäcklund transformation of the Boussinesq equations
\[
\begin{align*}
\begin{cases}
(D_x^4 + 3D_{x}^2) f \cdot f &= 0, \\
(D_x D_{t_4} - 4D_x^2 D_{t_4}) f \cdot f &= 0.
\end{cases}
\end{align*}
\] (3.1)

To begin with, we consider the first equation of the Eqs. (3.1). By means of the general method to obtain bilinear Bäcklund transformation, one have
\[
P_1 = [(D_x^4 + 3D_{x}^2) f \cdot f]g^2 - f^2[(D_x^4 + 3D_{x}^2) g \cdot g]
= [(D_x^4 f \cdot f)g^2 - f^2(D_x^4 g \cdot g)] + 3[(D_x f \cdot f)g^2 - f^2(D_x g \cdot g)]
= 2D_x(D_x^2 f \cdot g)(gf) + 6D_x(D_x^2 f \cdot g)(D_x g \cdot f) + 6D_{t_4}(D_{t_4}^2 f \cdot g)(gf)
\] (3.2)

According to
\[
D_{t_4}(D_x^2 f \cdot g)(gf) = D_x(D_{t_4} D_x^2 f \cdot g)(gf) - D_x(D_{t_4} f \cdot g)(D_x g \cdot f),
\] (3.3)

Eq. (3.2) become
\[
P_1 = 6D_{t_4}[(D_{t_4} + aD_x^2) f \cdot g](gf) - 6aD_x(D_{t_4} D_x f \cdot g)(gf) - 6D_x(D_{t_4} f \cdot g)(D_x g \cdot f)
+ 6D_x(D_x^2 f \cdot g)(D_x g \cdot f) + 2D_x(D_x^2 f \cdot g)(gf)
\] (3.4)

\[
= 6D_{t_4}[(D_{t_4} + aD_x^2) f \cdot g](gf) + 6D_x[D_x^2 + aD_{t_4}] f \cdot g)(gf) + 6D_x(D_{t_4} f \cdot g)(D_x g \cdot f)
+ 6D_x(D_x^2 f \cdot g)(D_x g \cdot f) + 2D_x[D_x^2 + aD_{t_4}] f \cdot g)(gf).
\]
We can deduce from the above equation that it is necessary that $P_1 = 0$

It is obvious that as $a = \pm 1$, $(D_{t_z} + aD_x^2)f \cdot g = 0$ is equivalent to $(D_x^2 + aD_{t_z})f \cdot g = 0$.

Thereafter, we can get

$$\begin{align*}
\left\{ \begin{array}{l}
(D_{t_z} + aD_x^2)f \cdot g = 0, \\
(D_x^3 - 3aD_{t_z}D_x)f \cdot g = 0, \\
a = \pm 1.
\end{array} \right.
\end{align*}
\tag{3.5}$$

In a similar way, by analysing the second equation of the Eqs. (3.1), we have

$$
P_2 = [(D_xD_{t_z} - 4D_x^3D_{t_z})f \cdot f][g]^2 - f^2\left[\left((D_xD_{t_z} - 4D_x^3D_{t_z})g \cdot g\right) - 4\left[(D_x^2D_{t_z}f \cdot f)g^2 - f^2(D_x^2D_{t_z}g \cdot g)\right]\right]
= 2D_x(D_{t_z}f \cdot g)(g f) - 8D_{t_z}(D_x^3f \cdot g)(g f) - 24D_x(D_xD_{t_z}f \cdot g)(D_xg \cdot f).
\tag{3.6}
$$

By now, we consider the following property of the $D$ operator,

$$
\begin{align*}
\exp(\varepsilon D_x)\exp(\delta_1 D_x + \delta_2 D_{t_z})f \cdot g)(\exp(\delta_1 D_x - \delta_2 D_{t_z})f \cdot g) &= \exp(\delta_2 D_{t_z})\exp(\delta_1 D_x + \varepsilon D_x)f \cdot g)(\exp(\delta_1 D_x - \varepsilon D_x)f \cdot g).
\tag{3.7}
\end{align*}
$$

Consider the coefficient of the term $\varepsilon \delta_2 \delta_1^2$ in Eq.(3.7),

$$
\begin{align*}
LHS &= 2D_x\left[\frac{1}{2}(D_x^2D_{t_z}f \cdot g)(g f) - \frac{3}{8}(D_x^3f \cdot g)(D_{t_z}f \cdot g) + \frac{3}{8}(D_xD_{t_z}f \cdot g)(D_xf \cdot g)\right]
= 2D_x\left[\frac{1}{2}(D_x^2D_{t_z}f \cdot g)(g f) + \frac{3}{8}(D_x^3f \cdot g)(D_{t_z}g \cdot f) - (D_xD_{t_z}f \cdot g)(D_xg \cdot f)\right],
\tag{3.8}
\end{align*}
$$

$$
\begin{align*}
RHS &= 2D_{t_z}\left[\frac{1}{2}(D_x^2f \cdot g)(g f) - \frac{3}{8}(D_x^3f \cdot g)(D_xf \cdot g) + \frac{3}{8}(D_x^3f \cdot g)(D_xg \cdot f)\right]
= 2D_{t_z}\left[\frac{1}{2}(D_x^2f \cdot g)(g f) - \frac{1}{2}(D_x^3f \cdot g)(D_xg \cdot f)\right].
\tag{3.9}
\end{align*}
$$

From the first equation of the Eq. (3.5) $(D_{t_z} + aD_x^2)f \cdot g = 0$, we can get

$$
D_x(D_x^2f \cdot g)(D_{t_z}f \cdot g) = 0.
\tag{3.10}
$$

From Eqs. (3.8), (3.9), (3.10), we have

$$
D_x(D_xD_{t_z}f \cdot g)(D_xg \cdot f) = \frac{1}{2}D_x(D_x^2D_{t_z}f \cdot g)(g f) + \frac{1}{2}D_{t_z}(D_x^2f \cdot g)(D_xg \cdot f) - \frac{1}{2}D_{t_z}(D_x^2f \cdot g)(g f).
\tag{3.11}
$$

After substituting Eq. (3.11) into Eq. (3.6), we obtain

$$
P_2 = 2D_x(D_{t_z}f \cdot g)(g f) - 8D_{t_z}(D_x^3f \cdot g)(g f) - 12D_x(D_x^2D_{t_z}f \cdot g)(g f) - 12D_{t_z}(D_x^2f \cdot g)(D_xg \cdot f) + 12D_x(D_x^2f \cdot g)(g f)
= 2D_x[(D_{t_z} - 6D_x^2D_{t_z})f \cdot g)(g f) + 4D_{t_z}(D_x^2f \cdot g)(g f) + 12aD_{t_z}(D_{t_z}f \cdot g)(D_xf \cdot g).
\tag{3.12}
$$

According to the realtions

$$
D_{t_z}(D_{t_z}f \cdot g)(D_xg \cdot f) = D_x(D_x^2f \cdot g)(g f) - D_{t_z}(D_xD_{t_z}f \cdot g)(g f),
\tag{3.13}
$$

$$
(D_x^3 - 3aD_{t_z}D_x)f \cdot g = 0,
\tag{3.14}
$$
Eq. (3.12) can be rewritten as
\[ P_2 = 2D_x[(D_t^4 - 6D_x^2D_{t^2})f \cdot g](gf) + 12aD_{t^2}(D_tD_xf \cdot g)(gf) + 12aD_x(D_t^2f \cdot g)(gf) - 12aD_{t^2}(D_xD_{t^2}f \cdot g)(gf) \]
\[ = 2D_x[(D_{t^4} - 6D_x^2D_{t^2} + 6aD_{t^2}^2)f \cdot g](gf). \] (3.15)

As \( P_2 = 0 \), we can get
\[ (D_{t^4} - 6D_x^2D_{t^2} + 6aD_{t^2}^2)f \cdot g = 0. \] (3.16)

By combining Eqs. (3.5) and (3.16), one have
\[
\begin{cases}
(D_{t^4} + aD_{t^2}^2)f \cdot g = 0, \\
(D_x^3 - 3aD_tD_x)f \cdot g = 0, & a = \pm 1, \\
(D_{t^4} - 6D_x^2D_{t^2} + 6aD_{t^2}^2)f \cdot g = 0,
\end{cases}
\] (3.17)

Therefore, we obtain the bilinear Bäcklund transformation of the Bousinessq equation. One can know the relations of the solutions of the Bousinessq equations or get other solutions from this transformation.

### 4 The Lax pair of the Boussinesq equation

In this section, we will obtain the Lax pair of the Boussinesq equation through their Bäcklund transformation in §3. To begin with, we transform the bilinear Bäcklund transformation
\[
\begin{cases}
(D_{t^4} + aD_{t^2}^2)f \cdot g = 0, \\
(D_x^3 - 3aD_tD_x)f \cdot g = 0, & a = \pm 1, \\
(D_{t^4} - 6D_x^2D_{t^2} + 6aD_{t^2}^2)f \cdot g = 0,
\end{cases}
\] (4.1)
to nonlinear form. Let \( a = 1 \), and
\[ \psi = \frac{f}{g}, \quad u = 2(\ln g)_{xx} \quad w = \frac{g_{t^2}}{g}, \quad p = \frac{g_{t^2}t^2}{g}. \] (4.2)

According to the relations
\[
\begin{align*}
\frac{D_{t^2}f}{g^2} &= \psi_{t^2}, \\
\frac{D_tg}{g^2} &= \psi_t, \\
\frac{D_xD_{t^2}f}{g^2} &= \psi_{xt^2} + 2w_x \psi, \\
\frac{D_x^2f}{g^2} &= \psi_{xx} + u \psi, \\
\frac{D_{t^4}f}{g^2} &= \psi_{xxx} + 3u \psi_x, \\
\frac{D_{t^2}D_{t^2}f}{g^2} &= \psi_{xxt^2} + u \psi_{t^2} + 4w_x \psi_x, \\
\frac{D_{t^2}^2f}{g^2} &= \psi_{t^2t^2} + 2(p - w^2) \psi.
\end{align*}
\] (4.3)
one can get the nonlinear differential equations by eliminating \( t_2 \) from Eqs. (4.1)

\[
\begin{cases}
\psi_{xxx} + 3u\psi_x - 3\psi_{xt_2} - 6w_x\psi = 0, \\
\psi_{t_2} + \psi_{xx} + u\psi_x = 0, \\
\psi_{tt_2} - 6\psi_{xxt_2} - 6u\psi_{t_2} - 24w_x\psi_x + 6\psi_{t_2}t_2 + 12(p - w^2)\psi = 0.
\end{cases} \tag{4.4}
\]

In order to get the Lax pair of the above equations, we suppose

\[
\psi_1 = \psi_x, \quad \psi_2 = \psi_{xx}, \quad \phi = \psi_{t_2} \tag{4.5}
\]

The Lax matrix \( U, V \) or Lax pair will satisfy the relations

\[
\begin{cases}
\Psi_x = U\Psi, \\
\Psi_{t_4} = V\Psi. \tag{4.6}
\end{cases}
\]

One can get that

\[
\phi_x = -\psi_{xxx} - u_x\psi - u\psi_x \\
= \frac{3}{4}u\psi_x + \frac{3}{4}(u_x - 2w_x)\psi - u_x\psi - u\psi_x \\
= \frac{1}{4}u\psi_x - \frac{1}{4}(u_x + 6w_x)\psi \\
= \frac{1}{2}u\psi_1 - \frac{1}{4}(u_x + 6w_x)\psi, \tag{4.7}
\]

\[
(\psi_2)_x = -\frac{3}{4}u\psi_x - \frac{3}{4}(u_x - 2w_x)\psi \\
= -\frac{3}{2}u\psi_1 - \frac{3}{4}(u_x - 2w_x)\psi.
\]

Therefore, one can obtain the Lax matrix \( U \) in Eq. (4.6) as

\[
U = \begin{pmatrix}
0 & \frac{1}{4}(u_x + 6w_x) & \frac{1}{4}u & 0 \\
0 & 0 & 1 & 0 \\
0 & -\frac{1}{4}(u_x - 2w_x) & -\frac{3}{2}u & 0
\end{pmatrix}.
\]

In a similar way, we can get the following equation from the second equation of the Eqs. (4.4)

\[
\psi_{t_2} = -\psi_{tt_2} - 2w_x\psi - u\psi_{t_2}. \tag{4.8}
\]

We can also get the equation from the first equation of the Eqs. (4.4)

\[
\psi_{xxt_2} = -\psi_{tt_2} - 2w_x\psi - u\psi_{t_2}. \tag{4.8}
\]

We can also get the equation from the first equation of the Eqs. (4.4)

\[
\psi_{xxt_2} = \frac{1}{4}\psi_{xxxx} + u_x\psi_x + u\psi_{xx} - 2w_x\psi - 2w_x\psi_x \\
= \frac{1}{4}(\frac{3}{4}u\psi_x - \frac{3}{4}(u_x - 2w_x)\psi)_x + \psi_{xx} + (u_x - 2w_x)\psi_x - 2w_x\psi \\
= \frac{1}{2}u\psi_{xx} + \frac{1}{4}(u_x - 6w_x)\psi_x - \frac{1}{2}(u_x + 6w_x)\psi_x. \tag{4.9}
\]

Combining Eqs.(4.8), (4.9), we have

\[
\psi_{t_2t_2} = -\frac{1}{2}u\psi_{xx} - \frac{1}{4}(u_x - 6w_x)\psi_x + \frac{1}{4}(u_x - 2w_x)\psi - u\psi_{t_2}. \tag{4.10}
\]

Taking advantage of Eqs. (4.4) and the above equations, we get the following
more relations

\[ \psi_{t_4} = 24u_x \psi_x - 12(w_{xx} + (p - u^2)) \psi - 12\psi_{t_2} \]
\[ = 24u_x \psi_x - 12(w_{xx} + (p - u^2)) \psi - 6u_{xx} - 3(u_x - 6u_x) \psi - 3(u_{xx} - 2w_{xx}) \psi + 12u \psi_{t_2} \]
\[ = 6u_{xx} + 3(u_x + 2w_x) \psi_x - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi + 12u \psi_{t_2} \]
\[ = 6u_{xx} + 3(u_x + 2w_x) \psi_x - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi + 12u \psi_{t_2} \]

\[ (\psi_1)_{t_4} = 6u_{xx} \psi_{xx} + 6w_{xx} + 3(u_x + 2w_x) \psi_x + 3(u_x + 2w_x) \psi_{xx} - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi \]
\[ -3(u_{xx} + 2w_{xx} + 4(p - u^2)) \psi \]
\[ = 6u_{xx} + 12u \psi_{t_2} \]
\[ = 3(3u_x + 2w_x) \psi_x - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi_x - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi_x \]
\[ + 12u \psi_{t_2} \]

\[ (\psi_2)_{t_4} = (9u_{xx} + 6w_{xx}) \psi_{xx} + (9u_x + 6w_x) \psi_{xx} - 6(u_{xx} + 12(p - u^2)) \psi_x \]
\[ - (3u_x + 2w_x) \psi_{xx} - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi_x \]
\[ - 3(u_{xx} + 6w_{xx} + 12(p - w^2)) \psi_x + 12u \psi_{t_2} \]
\[ = (9u_{xx} + 6w_{xx}) \psi_{xx} + (9u_x + 6w_x) \psi_{xx} - 6(u_{xx} + 12(p - u^2)) \psi_x \]
\[ - (3u_x + 2w_x) \psi_{xx} - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi_x \]
\[ + 12u \psi_{t_2} \]

\[ \phi_{t_4} = -\psi_{xx} - u_{t_1} \psi - w_{t_4} \]
\[ = -3(3u_{xx} + 2u_{xx} + u_x + 4(p - w^2)) \phi_x - 3(u_{xx} + 2w_{xx} + 8(p - w^2)) \psi_x - 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi_x \]
\[ + 12u \phi_{t_2} \]
\[ = -3(3u_{xx} + 2u_{xx} + u_x + 4(p - w^2)) \phi_x - 3(u_{xx} + 2w_{xx} + 8(p - w^2)) \psi_x + 12u \phi_{t_2} \]
\[ + 3(u_{xx} + 2w_{xx} + 4(p - w^2)) \psi_x \]
\[ + 4u(p - w^2) + 5w_{xx}) \psi - 12(u_{xx} + u^2) \phi \]

From which, we can get another Lax matrix \( V \) as,

\[ V = (v_{ij}) \quad i, j = 1, 2, 3, 4. \] (4.11)
where
\[ v_{11} = -12(u_{xx} + u^2), \]
\[ v_{12} = 3(u_{xxxx} + 2w_{xxx} + 4(p_{xx} - 2w_{xx}) - 11w_x^2 + \frac{11}{4}u_x^2 + \frac{5}{2}uw_{xx} + 6uw_x - \frac{1}{4}u_t + 4u(p - w^2) + 5uw_{xx}), \]
\[ v_{13} = 3(u_{xxx} + 2w_{xxx} + 8(p - w^2)_x + 6uw_x + 4uw_x), \]
\[ v_{14} = -3(3u_{xx} + 2w_{xx} + u^2 - 4(p - w^2)) \]
\[ v_{21} = 12u, \]
\[ v_{22} = -3(u_{xx} + 2w_{xx} + 4(p - w^2)), \]
\[ v_{23} = 3(u_x + 2w_x), \]
\[ v_{24} = 6u, \]
\[ v_{31} = 12u_x, \]
\[ v_{32} = -3(3u_{xx} + 2w_{xx} + 4(p - w^2)_x + \frac{5}{2}uw_x + 3uw_x), \]
\[ v_{33} = -3(u^2 + 4(p - w^2)), \]
\[ v_{34} = 3(3u_x + 2w_x), \]
\[ v_{41} = 12u_{xx}, \]
\[ v_{42} = -3(u_{xxxx} + 2w_{xxxx} + 4(p_{xx} - 2w_{xx}) - 11w_x^2 + \frac{21}{4}u_x^2 + \frac{5}{2}uw_{xx} + 6uw_x + 3uw_{xx}), \]
\[ v_{43} = -3(u_{xxx} + 2w_{xxx} + 8(p - w^2)_x + 7uw_x + 6uw_x), \]
\[ v_{44} = 3(3u_{xx} + 2w_{xx} - u^2 - 4(p - w^2)). \]

It can be verified that the Lax pair satisfy the zero curvature condition. The equations by eliminating \( t_2 \) form the Boussinesq equations is integrable.

5 Conclusions

In this paper, we have studied the Boussinesq equation from the viewpoint of Hirota bilinear method. We have derived the bilinear Backlund transformation and Lax pair of this equation. It is obvious that one can also use this method to discuss other bilinear differential equations or supersymmetric nonlinear differential equations.

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