Newtonian Self-Gravitation in the Neutral Meson System

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August 29, 2014

Abstract

We derive the effect of the Schrödinger–Newton equation, which can be considered as a non-relativistic limit of classical gravity, for a composite quantum system in the regime of high energies. Such meson-antimeson systems exhibit very unique properties, e.g. distinct masses due to strong and electroweak interactions. We find conceptually different physical scenarios due to lacking of a clear physical guiding principle which mass is the relevant one and due to the fact that it is not clear how the flavor wave-function relates to the spatial wave-function. There seems to be no principal contradiction. However, a nonlinear extension of the Schrödinger equation in this manner strongly depends on the relation between the flavor wave-function and spatial wave-function and its particular shape. In opposition to the CSL collapse models we find a change in the oscillating behavior and not in the damping of the flavor oscillation.

1 Introduction

The search for a theory that consistently combines quantum theory and gravitation is certainly one of the bigger challenges of contemporary theoretical physics. In the context of non-relativistic quantum mechanics, the problem is basically condensed to the question how quantum matter sources the gravitational field. As far as broadly accepted and experimentally tested principles are concerned, there is no unambiguous answer to this question.

One hypothesis that has been brought into the debate\textsuperscript{[1–4]} is that the gravitational interaction for non-relativistic quantum matter is described by a nonlinear extension of the Schrödinger equation, the Schrödinger–Newton equation

\[
\begin{align*}
\imath \hbar \partial_t \psi(t, \mathbf{r}) & = \left( -\frac{\hbar^2}{2m} \nabla^2 - Gm^2 \int \frac{d^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \right) \psi(t, \mathbf{r}),
\end{align*}
\]

originally proposed as a model for the localization of macroscopic quantum objects\textsuperscript{[5,6]}. The intuition behind such an equation is that the absolute-value squared of the wave-function corresponds to a mass density sourcing a Newtonian gravitational potential\textsuperscript{[7]}. The equation can also be shown to follow naturally as the non-relativistic limit of a semi-classical theory of gravity, i.e. a theory in which the gravitational field stays classical even at the fundamental level and quantum matter is coupled by the semi-classical Einstein equations

\[
R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \langle \Psi \mid \hat{T}_{\mu\nu} \mid \Psi \rangle,
\]

\textsuperscript{[1]}While such a mass-density interpretation would be incompatible with the instantaneous collapse and the perception of the wave-function as a pure probability density in the Copenhagen interpretation, a dynamical state-reduction as in collapse models\textsuperscript{[8]} allows for such an interpretation.
where \( \hat{T}^{\mu\nu} \) is the energy-momentum operator and the expectation value is taken in some quantum state \([4]\). One particularly charming aspect of the Schrödinger–Newton equation is that it most likely can be experimentally tested in the foreseeable future, e.g., with large molecules \([8–10]\) or with crystalline nanospheres \([11]\).

In this letter we want to consider neutral meson-antimeson systems that are typically produced at accelerator facilities. In particular we focus for the sake of simplicity on the neutral K-meson system, also dubbed kaons, however, all considerations hold for all meson-systems. These massive systems, that can also be produced even in entangled pairs, have shown to be a unique laboratory for precisions measurements of particle properties and fundamental principles in particle physics (e.g. discrete symmetries) as well as for testing fundamental principles in quantum physics such as superposition and entanglement (for an overview see e.g. ref. \([12]\)). For example, a violation of Bell’s inequality that is only due to the breaking of a discrete symmetry resulting in a tiny difference between matter and antimatter properties has been discovered \([13]\). Or due to the existence of two distinct measurement procedures, a special feature of kaons, the very working of a quantum eraser \([14,15]\) or Heisenberg's principle \([16]\) can in a novel way be demonstrated. Proposals how to test decoherence effects have been developed \([17,18]\) and put to experimental tests \([19–21]\). Models testing for Lorentz-symmetry violations or assuming intrinsic violations of the CPT symmetry induced by quantum gravity \([22]\) have been put to test for K-mesons \([19,23]\). Recently, also the prediction of collapse models were computed \([24,25]\).

At first sight neutral kaons, composite systems of a quark and an anti-quark, seem not to be good candidates to test for gravitational effects since the mass is very low, approximately half of a proton mass, however, the unique properties of these meson-antimeson systems—as witnessed by the above literature— makes it an interesting case to see whether conceptual contradictions can be derived. Let us here quote the famous Feynman lectures \([26]\), where Feynman writes after introducing K-mesons:

“If there is any place where we have a chance to test the main principles of quantum mechanics in the purest way—does the superposition of amplitudes work or doesn’t it?— this is it.”

The superposition of these two different mass eigenstates due to weak interaction exhibiting oscillations of the eigenstates of the strong interaction have been proven by now at many accelerator facilities. Moreover, since 1964 the unexpected breaking of the CP symmetry (C. . . charge conjugation, P. . . parity), a tiny difference between matter and antimatter properties, was discovered. One may immediately rise the questions:

Do both eigenstates of the mass-Hamiltonian couple independently to the gravitation field? Or is only the rest mass of the neutral K-meson the one relevant for any gravitational effect?

In this contribution we analyze in detail which options are conceptually possible and to which effects they may have on the flavor oscillation. We will briefly review the properties of the neutral kaon system in the second section. We then discuss different ways to implement the features of the neutral kaon into the Schrödinger–Newton equation in the third section. In the fourth section we derive the results for two scenarios under certain assumptions and then compare in the next section our result to the one for a specific collapse model. Then we discuss and conclude in the final section.

2 The neutral kaon system

Via strong interactions one has to distinguish between two different eigenstates labeled by the strangeness number \( S \), the kaon state \( | K^0 \rangle \) \((S = 1)\) and the anti-kaon \( | \bar{K}^0 \rangle \) \((S = -1)\). Neutral
kaons decay via the weak interaction giving rise to the following non-Hermitian Hamiltonian

\[
H = \left( \begin{array}{cc}
\langle K^0 | H^{(|\Delta S| = 0)} | K^0 \rangle & \langle K^0 | H^{(|\Delta S| = 2)} | K^0 \rangle \\
\langle K^0 | H^{(|\Delta S| = 2)} | K^0 \rangle & \langle K^0 | H^{(|\Delta S| = 0)} | K^0 \rangle 
\end{array} \right) = M - \frac{i}{2} \Gamma
\]

(3)

where both the mass matrix \(M\) and decay-matrix \(\Gamma\) are chosen to be Hermitian. \(H^{(|\Delta S| = 0)}\) describes the processes which conserve the strangeness number \(S\) and \(H^{(|\Delta S| = 2)}\) those which differ by two. The states diagonalising this Hamiltonian are denoted as mass eigenstates, namely the short- and long-lived states \(| K_S \rangle \) and \(| K_L \rangle \). If we assume \(CPT\) conservation (\(T\) time reversal), the two diagonal elements of \(M\) have to be equal and have to correspond to the rest mass \(m_K\). Analogously, the two diagonal elements of \(\Gamma\) have to be equal and to correspond to the total decay width \(\Gamma\) of \(K^0, \bar{K}^0\).

The complex eigenvalues of the Hamiltonian \(H\) are derived to

\[
\lambda_{S/L} = m_{S/L} - \frac{i}{2} \Gamma_{S/L} = m_K - \frac{i}{2} \Gamma + \sqrt{(M_{12} - \frac{i}{2} \Gamma_{12})(M_{12}^* - \frac{i}{2} \Gamma_{12}^*)}
\]

(4)

and, consequently the mass difference \(\Delta m := m_L - m_S\) and the decay width difference \(\Delta \Gamma := \Gamma_L - \Gamma_S\) are given by

\[
\Delta m = 2 \text{Re}\left\{\sqrt{(M_{12} - \frac{i}{2} \Gamma_{12})(M_{12}^* - \frac{i}{2} \Gamma_{12}^*)}\right\}
\]

\[
\Delta \Gamma = -4 \text{Im}\left\{\sqrt{(M_{12} - \frac{i}{2} \Gamma_{12})(M_{12}^* - \frac{i}{2} \Gamma_{12}^*)}\right\}.
\]

(5)

The mass difference \(\Delta m\) and the two decay widths \(\Gamma_S, \Gamma_L\) have been measured for all neutral meson systems \([27]\), however, only for neutral \(K\)-mesons the two decay widths differ greatly.

Thus the time evolution of the mass-Hamilton eigenstates is given by\(^2\)

\[
| K_S(t) \rangle = e^{-i m_S t} e^{-\frac{\Gamma_S}{2} t} | K_S(t = 0) \rangle, \quad | K_L(t) \rangle = e^{-i m_L t} e^{-\frac{\Gamma_L}{2} t} | K_L(t = 0) \rangle,
\]

(6)

preserving their identity in time. The strangeness states are connected via the following basis transformation

\[
| K^0 \rangle = \frac{(1 + \varepsilon) | K_S \rangle - (1 - \varepsilon) | K_L \rangle}{\sqrt{2(1 + |\varepsilon|^2)}}, \quad | \bar{K}^0 \rangle = \frac{(1 + \varepsilon) | K_S \rangle + (1 - \varepsilon) | K_L \rangle}{\sqrt{2(1 + |\varepsilon|^2)}}.
\]

(7)

where \(\varepsilon\) is the \(CP\) violating parameter that equals in a conventional phase choice to

\[
\varepsilon = \frac{(M_{12} - \frac{i}{2} \Gamma_{12}) - (M_{12}^* - \frac{i}{2} \Gamma_{12}^*)}{(M_{12} - \frac{i}{2} \Gamma_{12}) + (M_{12}^* - \frac{i}{2} \Gamma_{12}^*)}.
\]

(8)

To understand the difference between the dynamical parameters \(\Delta m, \Delta \Gamma\) and the \(CP\) violating parameter \(\varepsilon\) let us introduce two complex numbers \(X, Y\) for which it is straightforwardly to show that if we define

\[
\frac{Y}{X} = M_{12} - \frac{i}{2} \Gamma_{12}
\]

(9)

the equation \(X \cdot Y = M_{12}^* - \frac{i}{2} \Gamma_{12}^*\) holds. With that we find (up to non-physical sign changes)

\[
\Delta m = 2 \text{Re}\{Y\}
\]

\[
\Delta \Gamma = -4 \text{Im}\{Y\}
\]

\[
\varepsilon = \frac{1 - X^2}{1 + X^2}.
\]

(10)

\(^2\)In this section we use natural units, \(\hbar = c = 1\).
Obviously the values $\Delta m, \Delta \Gamma$ are independent of $\varepsilon$ in the sense that the value of $X$ does not influence the value of these dynamical parameters, however, the time evolution does depend on all three parameters as we show explicitly in the following.

The probabilities of finding a $K^0$ or a $\bar{K}^0$ after a certain time $t$ if a state $\ket{K^0}$ was produced at time $t = 0$ is consequently given by

\begin{align}
P(K^0 t; |K^0|) &= |\langle K^0 | K^0(t) \rangle|^2 = \frac{1}{4} \left( e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2 \cos(\Delta m t) \cdot e^{-\Gamma t} \right) \tag{11} \\
P(\bar{K}^0 t; |K^0|) &= |\langle \bar{K}^0 | K^0(t) \rangle|^2 = \frac{|1 - \varepsilon|^2}{4|1 + \varepsilon|^2} \left( e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2 \cos(\Delta m t) \cdot e^{-\Gamma t} \right) \tag{12}
\end{align}

Taking the difference we derive the time dependent asymmetry

\begin{align}
\frac{P(K^0 t; |K^0|) - P(\bar{K}^0 t; |K^0|)}{P(K^0 t; |K^0|) + P(\bar{K}^0 t; |K^0|)} &= \frac{2 \text{Re}\{\varepsilon\} \frac{\cos(\Delta m t)}{\cosh(\Delta \Gamma t)} + \frac{\cos(\Delta m t)}{1 + |\varepsilon|^2} \cosh(\Delta \Gamma t)}{1 + \frac{2 \text{Re}\{\varepsilon\} \cos(\Delta m t)}{1 + |\varepsilon|^2} \cosh(\Delta \Gamma t)}, \tag{13}
\end{align}

where for short times the oscillation is visible whereas in the long time limit the CP violation can be measured. In summary one observes a damped oscillation due to $\Delta m$ and the decay constants where the mass $m_K$ does not enter and for the huge time regime the difference of both probabilities reveals the tiny CP violation, namely $\frac{2 \text{Re}\{\varepsilon\}}{1 + |\varepsilon|^2} \approx 10^{-3}$.

Certainly, we can be interested in the kinematics of the neutral meson system, then the relevant Hamiltonian would be

\[ H_{\text{kin}} = m_{\text{inert}} c^2 - \frac{\hbar^2}{2m_{\text{inert}}} \nabla^2. \tag{14} \]

The inertial mass $m_{\text{inert}}$ could be considered as those of the composite K-meson, i.e. $m_K$, or one may assume that each mass-energy eigenstates $K_{S/L}$ exhibits a different kinetic/spatial wave-function due to its different decaying property, namely $m_{S/L}$. Note that $m_{S/L}$ alone has also contributions of $m_K$. Differently stated, considering the spatial wave-function of mesons:

**Do we have to consider only one unique wave-function or do we have to handle it as a two–state system?**

We will differentiate between those two scenarios in the following.

### 3 Schrödinger–Newton equation for the neutral kaon

The Schrödinger–Newton equation is based on the assumption that the wave-function sources a gravitational field as if the total mass of the particle would be smeared with the spatial probability density $|\psi|^2$. If the Schrödinger–Newton equation is applied to the kaon system this raises two questions:

- Which is the right mass that acts as the source of the gravitational field?
- How to describe the spatial wave-function of the neutral kaon?

Due to Newton the mass entering into the kinetic term of the Hamiltonian is the inertial mass. In the famous experiment with neutrons the authors of ref. [28] demonstrated that non-relativistic quantum matter can couple to the gravitational field in the following way:

\[ i\hbar \partial_t \psi(t, \mathbf{r}) = \left( -\frac{\hbar^2}{2m_{\text{inert}}} \nabla^2 + m_{\text{grav}}^{\text{passive}} \Phi_{\text{grav}} \right) \psi(t, \mathbf{r}). \tag{15} \]
Here $\Phi_{\text{grav}}$ is the gravitational potential, and $m_{\text{grav}}^{\text{passive}}$ is the passive gravitational mass, i.e. the coupling of matter to the gravitational field. If the weak equivalence principle holds, these two masses must be equal: $m_{\text{grav}}^{\text{passive}} = m_{\text{inert}}$. Let us remark here that so far in all situations that have been put to test by experiments, $\Phi_{\text{grav}}$ belongs to an external gravitational field.

In the framework of the Schrödinger–Newton equation, however, $\Phi_{\text{grav}}$ yields the gravitational self-interaction. With the assumptions underlying the Schrödinger–Newton equation the gravitational potential satisfies the Poisson equation

$$\nabla^2 \Phi_{\text{grav}} = 4\pi G m_{\text{active}}^{\text{grav}} |\psi(t, \mathbf{r})|^2.$$  \hfill (16)

The mass entering here as the source of the gravitational field is referred to as the active gravitational mass. Note that neither the equivalence principle nor any other fundamental principle of physics demands that this mass is equal to the inertial mass or passive gravitational mass.

Interestingly, equation (16) can be shown to follow from the semi-classical Einstein equations (2) as derived in refs. [4, 9]. The mass density on the right-hand side is then given by the expectation value of the non-relativistic limit of the energy-momentum operator,

$$m_{\text{active}}^{\text{grav}} |\psi|^2 = \frac{1}{c^2} \langle \psi | \hat{T}_{00} | \psi \rangle.$$  \hfill (17)

Following this logic, from a quantum field theoretical point of view the mass $m_{\text{active}}^{\text{grav}}$ would be the one appearing in the mass term of the field Lagrangian (after renormalization). But the kaon is a composite system and its mass is mainly binding energy of quarks.

Usually, one assumes that all three masses, inertial as well as active and passive gravitational mass, correspond to $m_K = (497.614 \pm 0.024) \text{ MeV } c^{-2}$, the measurable invariant mass of the neutral kaon. Whereas the flavor eigenstates $|K^0\rangle$, $|\bar{K}^0\rangle$ have equal mass (assuming CPT symmetry) with the value $m_K$, the long-lived and short-lived eigenstates $|K_S\rangle$, $|K_L\rangle$ of the strong and weak interaction Hamiltonian manifest a small mass difference $\Delta m = (3.483 \pm 0.006) \text{ MeV } c^{-2}$ that is at another energy scale, as discussed in the previous section.

So, does this mass difference $\Delta m$ show up in the Schrödinger–Newton equation, if so, for which of the three masses does it show up, and what would be the consequences?

Similar conceptual problems encounter for a spatial wave-function $\psi(\mathbf{r})$ of the non-relativistic neutral kaon. Again, one could simply consider a Schrödinger equation with one unique spatial wave-function that evolves with the invariant mass of the neutral kaon. But since the states $|K_S\rangle$ and $|K_L\rangle$ diagonalize the Hamiltonian, there should in principle be different wave-functions $\psi_S(\mathbf{r})$ and $\psi_L(\mathbf{r})$ evolving with the masses $m_S$ and $m_L$, respectively. Because of the difference in the free spreading of the wave-function due to the mass difference one would then expect additional flavor oscillations in space.

Therefore, we will in the following distinguish two scenarios and discuss their implications:

**Scenario 1: Unique spatial wave-function**

Let us first assume that we can treat $|K_S\rangle$ and $|K_L\rangle$ as if they would mix only in flavor space, while they are described by one unique spatial wave-function. The total wave-function is then

$$\psi_{\text{flavor}} \otimes \psi_{\text{space}} = (\alpha \psi_{S,\text{flavor}} + \beta \psi_{L,\text{flavor}}) \otimes \psi_{\text{space}},$$  \hfill (18)

where $\psi_{\text{space}}$ denotes the spatial wave-function and $\psi_{\text{flavor}}$ the flavor part. $\psi_{S,\text{flavor}}$ and $\psi_{L,\text{flavor}}$ denote a short-lived and long-lived kaon, respectively. $\psi_{\text{space}}$ satisfies the Schrödinger–Newton equation for the mass $m_{\text{inert}} = m_{\text{grav}}^{\text{passive}} = m_K$.

\(^3\)In the classical limit, of course, the active and passive gravitational mass are equal due to Newton’s third law.
If the active gravitational mass does not depend on the flavor part, the equation can be separated and the spatial wave-function will simply satisfy the Schrödinger–Newton equation for the kaon mass $m_K$, independent of its composition of $| K_S \rangle$ and $| K_L \rangle$.

If, on the other hand, $m_{\text{grav}}^{\text{active}}$ does depend on the flavor part, the most naive ansatz would be $m_{\text{grav}}^{\text{active}} = |\alpha|^2 m_S + |\beta|^2 m_L$. One then obtains the Schrödinger–Newton equation

\begin{equation}
\tag{19}
\begin{aligned}
\psi_{\text{space}}(t, r) &= \left( H_{\text{kin}}(m_K) - G m_K m_S |\alpha(t)|^2 \int d^3 r' \frac{|\psi_{\text{space}}(t, r')|^2}{|r - r'|} \right. \\
&\quad \left. - G m_K m_L |\beta(t)|^2 \int d^3 r' \frac{|\psi_{\text{space}}(t, r')|^2}{|r - r'|} \right) \psi_{\text{space}}(t, r).
\end{aligned}
\end{equation}

Since the mass difference $\Delta m = m_L - m_S$ is small, the effect is only a tiny modification of the already unmeasurable gravitational self-interaction.

**Scenario 2: Different spatial wave-functions**

Now let us consider the case assuming that $| K_S \rangle$ and $| K_L \rangle$ have different wave-functions also in space. Therefore, the total wave-function is given by

\begin{equation}
\tag{20}
\psi_{\text{flavor}} \otimes \psi_{\text{space}} = \alpha \psi_{S,\text{flavor}} \otimes \psi_{S,\text{space}} + \beta \psi_{L,\text{flavor}} \otimes \psi_{L,\text{space}}.
\end{equation}

Each of the spatial wave-functions will contribute to the total gravitational potential, and both wave-functions will see this same gravitational potential. Therefore, we get the following two Schrödinger–Newton equations:

\begin{equation}
\tag{21a}
\begin{aligned}
\psi_S(t, r) &= \left( H_{\text{kin}}(m_S) - G m_S^2 |\alpha(t)|^2 \int d^3 r' \frac{|\psi_S(t, r')|^2}{|r - r'|} \right. \\
&\quad \left. - G m_S m_L |\beta(t)|^2 \int d^3 r' \frac{|\psi_L(t, r')|^2}{|r - r'|} \right) \psi_S(t, r),
\end{aligned}
\end{equation}

\begin{equation}
\tag{21b}
\begin{aligned}
\psi_L(t, r) &= \left( H_{\text{kin}}(m_L) - G m_S m_L |\alpha(t)|^2 \int d^3 r' \frac{|\psi_S(t, r')|^2}{|r - r'|} \right. \\
&\quad \left. - G m_L^2 |\beta(t)|^2 \int d^3 r' \frac{|\psi_L(t, r')|^2}{|r - r'|} \right) \psi_L(t, r),
\end{aligned}
\end{equation}

where we write $\psi_{S,L}$ for the spatial wave-functions of the short and long lived contribution, respectively.

### 4 Resulting wave-function dynamics

In experimental situations, the kaon is usually not well-localized. Therefore, the wave-function is usually assumed to be a plane wave, in very good agreement with the experiment. However, to determine the effect of the Schrödinger–Newton equation, the localization of the wave-function must be taken into account. We will therefore model it by a spherically symmetric Gaussian:

\begin{equation}
\tag{22}
\psi^I(t, r; m, a) = \left( \frac{\pi a^2}{2} \right)^{-3/4} \left( 1 + \frac{i \hbar t}{m a^2} \right)^{-3/2} \exp \left( -\frac{r^2}{2a^2 (1 + \frac{i \hbar t}{m a^2})} \right).
\end{equation}
This is the solution of the free Schrödinger equation, where the width, \(a\), will be assumed to be large. In general, the Schrödinger–Newton dynamics disturb the Gaussian shape of the wavefunction. Since the gravitational interaction is very weak due to the large value of \(a\) and the small mass \(m\), we will approximate the wave-function appearing as the mass density in the gravitational potential by the free solution \((22)\). The approximated gravitational potential is then 
\[
\Phi_{\text{grav}} = -G m_{\text{grav}}^\text{active} f(t, r; m, a) \quad \text{with}
\]
\[
f(t, r; m, a) = \int d^3 r' \left| \frac{\psi(t, r'; m, a)}{|r - r'|} \right|^2 = \frac{1}{r} \text{erf} \left[ \frac{r}{a} \left( 1 + \frac{\hbar^2 t^2}{m^2 a^4} \right)^{-1/2} \right] \quad \text{(23)}
\]
The function \(f\) could now be expanded in terms of the mass, yielding
\[
f(t, r; m, a) = \frac{2 am}{\sqrt{\pi} \hbar t} + O(m^3), \quad \text{(24)}
\]
or in terms of time, yielding
\[
f(t, r; m, a) = \frac{\text{erf}(r/a)}{r} - \frac{\hbar^2}{\sqrt{\pi} m^2 a^5} \exp \left( -\frac{r^2}{a^2} \right) t^2 + O(t^4). \quad \text{(25)}
\]
Here, however, we choose an expansion around \(a = \infty\), which is justified in all usual experimental situations – which is why one usually assumes plane-wave solutions. This approximation yields
\[
f(t, r; m, a) = \frac{2}{\sqrt{\pi} a} \left( 1 - \frac{r^2}{3a^2} + \frac{r^4}{10a^4} \right) - \frac{\hbar^2 t^2}{\sqrt{\pi} m^2 a^5} + O(a^{-7}), \quad \text{(26)}
\]
which is time-independent up to order \(a^{-5}\).

**Scenario 1: Unique spatial wave-function**

In the case of equation \((19)\) we have
\[
\hat{i} \hbar \partial_t \psi = H_{\text{kin}}(m_K) \psi - G_{m_K} \left( |\alpha|^2 m_S + |\beta|^2 m_L \right) f(t, r; m_K, a) \psi. \quad \text{(27)}
\]
If we then write \(m_S = m, m_L = m + \Delta m\) and use the expansion \((26)\) we get
\[
\hat{i} \hbar \partial_t \psi = H_{\text{kin}}(m_K) \psi - \frac{2G_{m_K}}{\sqrt{\pi} a} \left( m_K + |\beta(t)|^2 \Delta m \right) \psi. \quad \text{(28)}
\]
This is time dependent only through the coefficient \(\beta\). We also used \((4)\) to replace \(m\) by \(m_K\) in the approximation.

**Scenario 2: Different spatial wave-function**

If we assume two different wave-functions, as in equations \((21)\), we can write them as
\[
\hat{i} \hbar \partial_t \psi_S = H_{\text{kin}}(m_S) \psi_S - G_{m_S}^2 |\alpha(t)|^2 f(t, r; m_S, a) \psi_S - G_{m_S} m_L |\beta(t)|^2 f(t, r; m_L, a) \psi_S \quad \text{(29a)}
\]
\[
\hat{i} \hbar \partial_t \psi_L = H_{\text{kin}}(m_L) \psi_L - G_{m_S} m_L |\alpha(t)|^2 f(t, r; m_S, a) \psi_L - G_{m_L} m_L |\beta(t)|^2 f(t, r; m_L, a) \psi_L. \quad \text{(29b)}
\]
If we then again write \(m_S = m, m_L = m + \Delta m\) and use the expansion \((26)\) we get
\[
\hat{i} \hbar \partial_t \psi_S = H_{\text{kin}}(m) \psi_S - \frac{2G m^2_{m_S}}{\sqrt{\pi} a} \left[ 1 + \frac{1}{m} + \frac{\hbar^2 t^2}{m^3 a^4} \right] |\beta(t)|^2 \Delta m \psi_S \quad \text{(30a)}
\]
\[
\hat{i} \hbar \partial_t \psi_L = H_{\text{kin}}(m_L) \psi_L - \frac{2G m^2_{m_L}}{\sqrt{\pi} a} \left[ 1 + \frac{1}{m} + \frac{\hbar^2 t^2}{m^3 a^4} \right] |\beta(t)|^2 \Delta m \psi_L \quad \text{(30b)}
\]
5 Gravity-induced energy shift

In the previous section we obtained the nonlinear Schrödinger equation which describes the dynamics of the kaon system in the presence of gravity, and therefore an approximation of the space-dependent Hamiltonian. The Hamiltonian that governs the flavor oscillations is modified by the energy shift due to the gravitational interaction. In order to calculate this energy shift, we must consider the expectation value of the Hamiltonian. This expectation value is proportional to
\[ \langle \psi^f | f(t, r; m, a) | \psi^f \rangle = \sqrt{\frac{2}{\pi}} \frac{\hbar^2 a^2}{m^2 a^4} \left( 1 + \frac{\hbar^2 t^2}{m^2 a^4} \right)^{-1/2}, \] (31)
where we approximated the wave-function by the solution of the free Schrödinger equation, as previously explained. For the mass \( m + \Delta m \) we can expand this and obtain up to first order in \( \Delta m \):
\[ \langle \psi^f | f(t, r; m + \Delta m, a) | \psi^f \rangle = \langle \psi^f | f(t, r; m, a) | \psi^f \rangle + \frac{\hbar^2 t^2}{m^3 a^4} \left( 1 + \frac{\hbar^2 t^2}{m^2 a^4} \right)^{-3/2} \Delta m. \] (32)

Scenario 1: Unique spatial wave-function

From equation (27) we get the energy shift
\[ \Delta E = -Gm_K \left( |\alpha|^2 m_S + |\beta|^2 m_L \right) \langle \psi^f | f(t, r; m_K, a) | \psi^f \rangle \]
\[ = -\sqrt{\frac{2}{\pi}} \frac{Gm_K}{m} \left( m + |\beta|^2 \Delta m \right) + \mathcal{O}(a^{-6}). \] (33)
Both states \( |K_S\rangle \) and \( |K_L\rangle \) obtain a constant energy shift, but these only yield a constant phase shift. The contribution proportional to \( \Delta m \), however, adds to the flavor oscillations (11). It acts like a shift of the mass difference:
\[ \Delta m \to (1 - \Delta_{SN}) \Delta m \] (34)
with
\[ \Delta_{SN} = \sqrt{\frac{2}{\pi}} \frac{Gm_K}{c^2 a}. \] (35)

Scenario 2: Different spatial wave-function

From equations (29) one obtains
\[ \Delta E_S = -\sqrt{\frac{2}{\pi}} \frac{Gm}{m} \left[ m + |\beta|^2 \Delta m \right] + \mathcal{O}(a^{-4}) \] (36a)
\[ \Delta E_L = -\sqrt{\frac{2}{\pi}} \frac{Gm}{m} \left[ m + \left( 1 + |\beta|^2 \right) \Delta m \right] + \mathcal{O}(a^{-4}), \] (36b)
where higher order terms in \( 1/a \) have been omitted. The shift in \( \Delta m \) therefore is twice the one before:
\[ \Delta m \to (1 - 2 \cdot \Delta_{SN}) \Delta m, \] (37)
where we assume that \( m \approx m_K \). Inserting the kaon mass, one finds that a large effect is only expected if the wave-function becomes close to or narrower than about \( 10^{-54} \) m, far below the Planck length. This result does not change for other meson types.
Let us remark here that the effect does depend on the particular shape of the spacial wave-function about which is not known much from the experiments. Thus dedicated experiments would help to handle in considering any model that modifies the Schrödinger equation.

In addition to the energy shift, the Schrödinger–Newton equation of course also leads to the usual localization of the wave-function as it has been described in [8], which is a very small effect due to the weakness of the gravitational self-interaction in the situation at hand (large wave-function, small masses).

6 Comparison to the CSL collapse model

Collapse models [7] predict the spontaneous collapse of the wave-function, in order to avoid the emergence of macroscopic superpositions. In their mass-dependent formulation, they claim that the collapse of any system’s wave-function depends on its mass. Recently, the most popular collapse model, the mass-proportional CSL (Continuous Spontaneous Localization) model was applied to the meson-antimeson systems [25]. Here, the crucial point was again to connect the spatial with the flavor wave-function part. The authors chose the sum of the kinetic contribution of the short and long-lived component. After a cumbersome computation solving the stochastic non-linear differential equation they found the following probability

\begin{equation}
P(K^0; |K^0|) = \frac{1}{4} \left( e^{-\Gamma_s t} + e^{-\Gamma_L t} + 2 \cos(\Delta m t) \cdot e^{-\Gamma t} \cdot \frac{e^{-\frac{m_0^2}{2 - 1/m_0^2}}}{\left( e^{\frac{m_0^2}{2 - 1/m_0^2}} \right)^t} \right)
\end{equation}

(38)

where the collapse rate \( \gamma \) and the coherence length \( r_C \) are parameters of the collapse model and \( m_0 \) is a reference mass. Thus, in opposition to the solution of the Schrödinger–Newton equation, the mass-dependent collapse effect leads to a damping proportional to \( \Delta m t^2 \) and has, consequently, to be compared to decoherence effects.

7 Summary and conclusions

The aim of this contribution was to investigate how Newtonian self-gravitation can be included in the standard framework to handle flavor oscillations of neutral meson systems. It is not straightforwardly physically intuitive which mass is the relevant one for the different terms in the Hamiltonian. Moreover, for any nonlinear extension of the Schrödinger evolution as in the case of the Schrödinger–Newton equation or spatial collapse models one has to assume a certain relation between the spatial and flavor wave-functions. We have considered two possible scenarios, a separable and entangled ansatz, and derived the effect of the Schrödinger–Newton equation under certain assumptions. We find the shift in energy is twice the one for the scenario of a unique wave-function of the two mass-energy eigenstates. It is worthy to note that if self gravitation or any form of collapse exists one would have via the spatial wave-function a certain control over the oscillations in flavor space.

Let us remark that the non-Hermitian part of the Hamiltonian was not considered or affected by the Schrödinger–Newton equation. This is consistent with the results in ref. [29] where the authors showed that the non-Hermitian part can be removed by changing the Schrödinger equation into a Lindblad equation by extending the Hilbert space. Also the \( CP \) violation in mixing was not affected by self-gravitation since it does not enter in the eigenvalues of the considered Hamiltonian. It is not clear what happens with direct \( CP \) violating effects, i.e. where the violation occurs on the amplitude level.

In principle, we found no conceptional contradictions between the Schrödinger–Newton equation and the treatment of kaons as a non-relativistic quantum system, however, attempting to include Schrödinger–Newton effects into the kaon system rises conceptional questions.
In particular, which are the relevant masses and how to treat the spatial wave-function. In all considered cases the effects within our assumptions were found to be irrelevant in practical situations, therefore no contradictions to already performed or envisaged experiments in elementary particle physics seem to exist.

Last but not least let us mention that neutrinos exhibit a similar oscillation feature in flavour space, hence the same conceptual problems apply to this case also.

Acknowledgements: Both authors want to gratefully thank the COST action MP1006 “Fundamental Problems in Quantum Physics” initiating this research. AG is supported by the John Templeton foundation (grant 39530). BCH gratefully acknowledges the Austrian Science Fund (FWF-P26783).

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