ENTIRE SUBSOLUTIONS OF MONGE-AMPÈRE TYPE EQUATIONS

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Abstract. In this paper, we consider the subsolutions of the Monge-Ampère type equations \( \det(D^2u + \alpha I) = f(u) \) in \( \mathbb{R}^n \). We obtain the necessary and sufficient condition of the existence of subsolutions.

1. Introduction. In this paper we consider the existence of subsolutions to the Monge-Ampère type equations which can be written in the general form

\[
\det[D^2u + A(x,u,Du)] = B(x,u,Du) \quad \text{in} \quad \Omega, \tag{1}
\]

where \( \Omega \) is a domain in \( \mathbb{R}^n \), \( D\) and \( D^2u \) are respectively the gradient vector and Hessian matrix of a function \( u \in C^2(\Omega) \), \( A \) is a given \( n \times n \) symmetric matrix function in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \) and \( B \) is a positive scalar valued function in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \). We use \((x,z,p)\) to denote points in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \), then \( A(x,z,p) \in \mathbb{R}^{n \times n} \) and \( B(x,z,p) \in \mathbb{R} \), here \( \mathbb{R}^{n \times n} \) is the \( n(n+1)/2 \) dimensional space of real symmetric \( n \times n \) matrices.

Equations (1) are elliptic with respect to \( u \) whenever \( D^2u + A(x,u,Du) > 0 \). At the same time, a solution \( u \in C^2(\Omega) \) of (1) is called an elliptic solution. We say a function \( u \in C^2(\Omega) \) is a subsolution of (1), if \( u \) satisfies

\[
\det[D^2u + A(x,u,Du)] \geq B(x,u,Du) \quad \text{in} \quad \Omega.
\]

In recent years, the Monge-Ampère type equations have attracted many interests. For a recent survey and the earlier history, see for instance [18]. When \( A \equiv 0 \), (1) reduce to the standard Monge-Ampère equations \( \det(D^2u) = B(x,u,Du) \) which were investigated in many works. For example, the existence of smooth solutions to the Dirichlet problem was settled by Caffarelli, Nirenberg and Spruck [3], Krylov [13] and Ivochkina [8]. When \( A \neq 0 \) in (1), if \( A \) is linear in, or independent of \( p \), we have very similar a priori estimates as the standard Monge-Ampère equations [6]. But if \( A \) is nonlinear in \( p \), the situation is very different. The interior a priori estimates for (1) are given in [20] for dimension two and [5] for all dimensions. Ma, Trudinger and Wang [16] got the interior \( C^2 \) estimates of solutions for (1) under
an analytical structure condition on the matrix $A$ and the interior regularity of solutions under a generalized target convexity condition. Trudinger and Wang [19] obtained the global $C^2$ estimates and the global regularity of the second boundary value problem for (1) under weaker conditions. But the results of $C^{2,\alpha}$ estimates are not many. Liu, Trudinger and Wang [14] established the interior $C^{2,\alpha}$ estimates of solutions for (1) with $A = A(x,p), B = B(x)$ under appropriate assumptions. Recently, Huang, Jiang and Liu [7] got the boundary $C^{2,\alpha}$ estimates of solutions with $A = A(x,p), B = B(x)$. For the existence of solutions, Jiang, Trudinger and Yang [10] proved the existence and uniqueness of classical solutions to the Dirichlet problem of (1). In this paper, we study the existence of subsolutions to the Monge-Ampère type equations

$$\det \frac{1}{2}(D^2 u + \alpha I) = f(u) \text{ in } \mathbb{R}^n,$$

where $\alpha \geq 0$ is a constant and $f$ is a positive function defined in $\mathbb{R}$.

For the nonlinear partial differential equations

$$F(D^2 u) = f(u) \text{ in } \mathbb{R}^n,$$

where $F$ is a real function defined in $\mathbb{R}^{n \times n}$, there have been many results about the existence of solutions. If $F$ is the Laplacian operator, i.e. $F(D^2 u) = \Delta u$, (3) has no positive solution if $f(u) = u^p, p > 1$, see Keller [12], Osserman [17], Loewner and Nirenberg [15] and Brezis [1]. In particular, Osserman [17] considered the necessary and sufficient condition under which the equation

$$\Delta u = f(u) \text{ in } \mathbb{R}^n$$

has a subsolution if $f$ is a positive nondecreasing continuous function. The following growth condition on $f$ at infinity,

$$\int_0^\infty \left( \int_0^\tau f(t) dt \right)^{-\frac{1}{2}} d\tau = \infty$$

is well known as Keller-Osserman condition, where we omit the lower limit to admit any positive constant. Capuzzo Dolcetta, Leoni and Vitolo [4] proved that

$$\mathcal{M}_{0,1}^+(D^2 u) = f(u) \text{ in } \mathbb{R}^n$$

has an entire viscosity subsolution $u \in C(\mathbb{R}^n)$ if and only if $f$ satisfies the Keller-Osserman condition (4). Here

$$\mathcal{M}_{\lambda,\Lambda}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

$0 < \lambda \leq \Lambda$ and $e_i = e_i(M), i = 1,2,\cdots,n$ are the eigenvalues of the matrix $M$. If in (3), $F$ is the Hessian operator, i.e. $F(D^2 u) = \sigma_k^2(\lambda(D^2 u))$, where $\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$ is the Hessian operator, Jin, Li and Xu [11] have proved that for $p > 1$ the equation

$$\sigma_k^\frac{1}{k}(\lambda(D^2 u)) = u^p \text{ in } \mathbb{R}^n$$

has no positive subsolution. But the method used in [11] cannot prove that $p > 1$ is optimal. Ji and Bao [9] proved that if $f$ is nonnegative, nondecreasing and continuous on $\mathbb{R}$, the Hessian equation

$$\sigma_k^\frac{1}{k}(\lambda(D^2 u)) = f(u) \text{ in } \mathbb{R}^n$$
has a positive subsolution \( u \in C^2(\mathbb{R}^n) \) if and only if
\[
\int_0^\infty \left( \int_0^\tau f^k(t) \, dt \right)^{-\frac{n+2}{n+1}} \, d\tau = \infty.
\] (6)

For the case \( f(u) = u^p \) in [9], (6) indicates that (5) has a positive subsolution if and only if \( 0 < p \leq 1 \) and then \( p > 1 \) in [11] is optimal. Bao, Ji and Li [2] also established a generalized Keller-Osserman condition on the existence and nonexistence for positive entire subsolutions of \( k\)-Yamabe type equations
\[
\sigma_k^k (\lambda(-A^u)) = u^{-\frac{n+2}{n+1}} f(u) \quad \text{in} \quad \mathbb{R}^n,
\]
where \( A^u \) is given by
\[
A^u = -\frac{2}{n-2} u^{-\frac{n+2}{n+1}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n+1}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n+1}} |\nabla u|^2 I,
\]
and \( I \) is the \( n \times n \) identity matrix.

In this paper, we shall establish the necessary and sufficient condition on the solvability for the subsolutions of (2). Let \( \alpha_0 \) be a positive constant. In the following, we always assume that \( \alpha_0 > \alpha \). The main results of this paper are the following.

**Theorem 1.1.** Let \( f(t) \) be a nondecreasing continuous function on \( \mathbb{R} \) and \( f(t) \geq \alpha_0 > \alpha \geq 0 \). There exists a subsolution \( u \) of (2) if and only if
\[
\int_0^\infty \left( \int_0^\tau f^n(t) \, dt \right)^{-\frac{1}{n+1}} \, d\tau = \infty.
\] (7)

In view of the impact of \( \alpha I \), different from the standard Monge-Ampère equation, we cannot directly integrate the equation on both sides in the proof of necessity. In this paper, by means of Lemma 2.2 and the binomial theorem, we overcome the impact of \( \alpha I \), see the necessity in the proof of Lemma 3.3. We need not only \( \phi'(r) > 0 \), but also \( \phi'(r) \geq (\alpha_0 - \alpha) r \), see Lemma 2.2.

From Theorem 1.1, we can get the following corollaries.

**Corollary 1.** If \( p > 0 \), then there exists a nonnegative subsolution of
\[
\det \frac{1}{n} (D^2 u + \alpha I) = u^p + \alpha_0 \quad \text{in} \quad \mathbb{R}^n
\]
if and only if \( p \leq 1 \).

Consider the equations with some constant \( k \geq 0 \),
\[
\det \frac{1}{n} (D^2 u + \alpha I) = e^{ku} \quad \text{in} \quad \mathbb{R}^n.
\] (8)

**Corollary 2.** Let \( 0 \leq \alpha < 1 \). For \( k = 0 \), there exists a subsolution of (8), and for \( k > 0 \), there exists no nonnegative subsolution of (8).

This paper is arranged as follows. In section 2, we give some basic results of radial functions. The main results will be proved in section 3.

2. Some basic results of radial functions. In this section, we collect some results for the radial functions. Let \( r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}. \)
Lemma 2.1. Let \( \varphi : [0, \infty) \to \mathbb{R} \) be a \( C^2 \) radial function with \( \varphi'(0) = 0 \). Then for \( u(x) = \varphi(r) \), we have \( u(x) \in C^2(\mathbb{R}^n) \), and the eigenvalues of \( D^2u + \alpha I \) are
\[
\chi(D^2u + \alpha I) = \begin{cases} 
\varphi''(r) + \alpha, & r \in (0, \infty), \\
\varphi''(0) + \alpha, & r = 0,
\end{cases}
\]
and so
\[
\det(D^2u + \alpha I) = \begin{cases} 
(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^{n-1}, & r \in (0, \infty), \\
(\varphi''(0) + \alpha)^n, & r = 0.
\end{cases}
\]

The proof of Lemma 2.1 is similar to the proof of Lemma 2.1 in [9]. Here we omit the proof.

From Lemma 2.1, we know that \( u(x) = \varphi(r) \) is a \( C^2 \) radial solution of (2) if and only if \( \varphi(r) \) satisfies
\[
(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^{n-1} = f^n(\varphi(r)). 
\tag{9}
\]

Lemma 2.2. Let \( f(t) \) be a continuous function on \( \mathbb{R} \), \( f(t) \geq \alpha_0 \) and \( \varphi(r) \in C^2[0, \infty) \) satisfy (9) with \( \varphi'(0) = 0 \), then \( \varphi'(r) \geq (\alpha_0 - \alpha)r > 0 \) and \( \varphi''(r) + \alpha > 0 \).

Proof. By (9), we know that
\[
((\varphi'(r) + \alpha r)^n)' = nr^{n-1}f^n(\varphi(r)).
\]
Noting that \( \varphi'(0) = 0 \) and integrating on \( r \), we have
\[
\varphi'(r) = \left( \int_0^r ns^{n-1}f^n(\varphi(s))ds \right)^\frac{1}{n} - \alpha r. 
\tag{10}
\]

Since \( f \geq \alpha_0 > \alpha \), we have
\[
\varphi'(r) \geq \left( \int_0^r \alpha_0^n ns^{n-1}ds \right)^\frac{1}{n} - \alpha r = (\alpha_0 - \alpha)r > 0.
\]
So by (9) \( \varphi''(r) + \alpha > 0 \). The lemma is proved.

Lemma 2.3. Let \( f(t) \) be a continuous function on \( \mathbb{R} \). Assume that \( \varphi(r) \in C^0[0, R) \cap C^1(0, R) \) satisfies (10). Then \( \varphi(r) \in C^2[0, R) \) satisfies (9) and \( \varphi'(0) = 0 \).

Proof. Firstly, we have
\[
\varphi'(0) = \lim_{r \to 0} \frac{\varphi(r) - \varphi(0)}{r - 0} = \lim_{r \to 0} \varphi'(\xi) = \lim_{\xi \to 0} \left( \left( \int_0^\xi ns^{n-1}f^n(\varphi(s))ds \right)^\frac{1}{n} - \alpha \xi \right) = 0,
\]
where \( \xi \in (0, r) \). Moreover,
\[
\lim_{r \to 0} \varphi'(r) = \lim_{r \to 0} \left( \left( \int_0^r ns^{n-1}f^n(\varphi(s))ds \right)^\frac{1}{n} - \alpha r \right) = 0 = \varphi'(0).
\]
So \( \varphi(r) \in C^1[0, R) \).
Secondly,
\[
\varphi''(0) = \lim_{r \to 0} \frac{\varphi'(r) - \varphi'(0)}{r - 0}
\]
\[
= \lim_{r \to 0} \left( \int_0^r ns^{n-1} f^n(\varphi(s))ds \right)^{\frac{1}{n}} - \alpha r
\]
\[
= \lim_{r \to 0} \left( \int_0^r ns^{n-1} f^n(\varphi(s))ds \right)^{\frac{1}{n}} - \alpha
\]
\[
= f(\varphi(0)) - \alpha.
\]
It is easy to see that \( \varphi(r) \in C^2(0, R) \) and for \( r \in (0, R) \), a direct calculation gives that
\[
\varphi''(r) = \left( \int_0^r ns^{n-1} f^n(\varphi(s))ds \right)^{\frac{1}{n-1}} - \alpha.
\]
Further,
\[
\lim_{r \to 0} \varphi''(r) = \lim_{r \to 0} \left( \int_0^r ns^{n-1} f^n(\varphi(s))ds \right)^{\frac{1}{n-1}} - \alpha
\]
\[
= \lim_{r \to 0} \left( \int_0^r ns^{n-1} f^n(\varphi(s))ds \right)^{\frac{n-1}{n}} - \alpha
\]
\[
= f(\varphi(0)) - \alpha
\]
\[
= \varphi''(0).
\]
Hence \( \varphi(r) \in C^2[0, R] \).

Finally, by (11), we know that \( \varphi \) satisfies (9). \( \square \)

From (10) and Lemma 2.3, we know that \( \varphi(r) \in C^2[0, R] \) and \( \varphi \) satisfies (9) with \( \varphi'(0) = 0 \) if and only if \( \varphi(r) \in C^0[0, R] \cap C^1(0, R) \) and \( \varphi \) satisfies (10).

Next we prove an existence result for (10).

**Lemma 2.4.** Let \( f(t) \geq \alpha_0 > \alpha \), \( f(t) \) be continuous and nondecreasing on \( \mathbb{R} \). Then for any constant \( a \), there exists a positive number \( R \) such that the equation (10) with the initial value
\[
\varphi(0) = a, \quad \varphi'(0) = 0
\]
has a solution \( \varphi \in C^2(0, R) \cap C^0[0, R] \).

**Proof.** As the proof of Lemma 2.3 in [9]. We define a functional \( F(\cdot, \cdot) \) on
\[
\mathcal{R} = [0, l] \times \{ \varphi \in C^2[0, l] : a - h < \varphi < a + h \}
\]
as
\[
F(r, \varphi) := \left( \int_0^r ns^{n-1} f^n(\varphi(s))ds \right)^{\frac{1}{n}} - \alpha r,
\]
where \( l \) and \( h \) are positive constants small enough. Then (10) can be rewritten as
\[
\varphi'(r) = F(r, \varphi).
\]

By Lemma 2.2, \( F > 0 \) for \( r > 0 \).

Define an Euler’s break line on \([0, l]\) as
\[
\left\{ \begin{array}{l}
\psi(0) = a, \\
\psi(r) = \psi(r_{i-1}) + F(r_{i-1}, \psi(r_{i-1}))(r - r_{i-1}), \quad r_{i-1} < r \leq r_i,
\end{array} \right.
\]
where \( 0 = r_0 < r_1 < \cdots < r_m = l \). Then \( \psi \in C^2[0, l] \). We claim that \( (r, \psi) \in \mathcal{R} \).
Indeed, for any \((r, \varphi) \in \mathcal{R}\), we have

\[
F(r, \varphi(r)) \leq \left( \int_{0}^{r} ns^{n-1} f^n(\varphi(s))ds \right)^{\frac{1}{n}} \\
\leq f(a + h) \left( \int_{0}^{r} ns^{n-1}ds \right)^{\frac{1}{n}} \\
= rf(a + h) \\
\leq lf(a + h).
\]

Then for the break line \(\psi\), we have

\[a - h \leq a \leq \psi(r) \leq a + ml^2 f(a + h), \quad r \in [0, l].\]

Thus if \(h\) is fixed, we can choose \(l\) sufficiently small such that

\[a - h \leq \psi(r) \leq a + h.\]

Next, we prove that the Euler’s break line \(\psi\) is an \(\varepsilon\)–approximation solution of (10). For this, we only need to prove that for any \(\varepsilon > 0\), we can choose points \(\{r_i\}_{i=1, \ldots, m}\) such that the break line satisfies

\[
\left| \frac{d \psi(r)}{dr} - F(r, \psi(r)) \right| < \varepsilon, \quad r \in [0, l].
\]

In fact, we see from (14) that

\[
\lim_{r \to 0} F(r, \varphi) = 0
\]

holds for any \(\varphi \in C^2[0, l], a - h \leq \varphi \leq a + h.\) So for any \(\varepsilon > 0\), there exists \(\bar{r} \in (0, l)\) such that for \(0 \leq r < \bar{r}\), we have

\[F(r, \psi) < \frac{\varepsilon}{2}.\]

Then

\[
\left| \frac{d \psi(r)}{dr} - F(r, \psi(r)) \right| = \left| F(r_{i-1}, \psi(r_{i-1})) - F(r, \psi(r)) \right| < \varepsilon.
\]

For \(\bar{r} \leq r \leq l,\)

\[
\left| \frac{d \psi(r)}{dr} - F(r, \psi(r)) \right| = \\
= \left| F(r_{i-1}, \psi(r_{i-1})) - F(r, \psi(r)) \right| \\
= \left| \left( \int_{0}^{r_{i-1}} ns^{n-1} f^n(\psi(s))ds \right)^{\frac{1}{n}} - \alpha r_{i-1} - \left( \int_{0}^{r} ns^{n-1} f^n(\psi(s))ds \right)^{\frac{1}{n}} + \alpha \right| \\
\leq \left| \left( \int_{0}^{r_{i-1}} ns^{n-1} f^n(\psi(s))ds \right)^{\frac{1}{n}} - \left( \int_{0}^{r} ns^{n-1} f^n(\psi(s))ds \right)^{\frac{1}{n}} \right| + \alpha |r - r_{i-1}| \\
\leq \left| \int_{r_{i-1}}^{r} ns^{n-1} f^n(\psi(s))ds \right| + |r - r_{i-1}| \\
\leq f(a + h) |r^n - r_{i-1}^n| + \alpha |r - r_{i-1}|.
\]

Since \(r^n\) is Lipschitz continuous on \(\bar{r}, l\), for the above \(\varepsilon\), there exists \(0 < \delta(\varepsilon) < \frac{\varepsilon}{2n}\) such that

\[|r^n - r_{i-1}^n| < \frac{\varepsilon}{2n f^n(a + h)}.
\]
for \( r', r'' \in [\bar{r}, l] \) and \(|r' - r''| < \delta(\varepsilon)\). Suppose \( r_1 = \bar{r} \) and
\[
\max_{2 \leq i \leq m} |r_i - r_{i-1}| < \min\{\bar{r}, \delta(\varepsilon)\},
\]
then we have (15).

The rest of the proof is similar to the proof of Lemma 2.3 in [9].

Then according to Lemma 2.3, we know that the solution \( \varphi(r) \) of (10) with the initial value \( \varphi(0) = a, \varphi'(0) = 0 \) is in \( C^2(0, R) \).

3. Proof of Theorem 1.1. In this section, we prove the main results. The proof of Theorem 1.1 includes the following three Lemmas.

Lemma 3.1. Let \( f(t) \) be a positive and nondecreasing continuous function on \( \mathbb{R} \). Assume that (9) has a solution \( \varphi(r) \in C^2(0, R) \) satisfying \( \varphi'(0) = 0 \), and \( \varphi(r) \to +\infty \) as \( r \to R \). Then if \( u \) is a subsolution of (2), we have \( u(x) \leq \varphi(|x|) \) in \( B_R = \{ x : |x| \leq R \} \).

Proof. Let \( v(x) = \varphi(r) = \varphi(|x|) \), then \( v(x) \) satisfies
\[
det(D^2v + \alpha I) = f^n(v), \quad |x| < R.
\]

Suppose on the contrary that \( u > v \) at some point. Then there exists some positive constant \( c_0 \) such that \( u - c_0 \) touches \( v \) from below at some interior point \( x_0 \in B_R \), i.e. \( u(x_0) - c_0 - v(x_0) = 0 \), and \( u - c_0 - v \leq 0 \) in \( B_R \). Since \( v(x) \to +\infty \) as \( |x| \to R \), we can choose some \( R' \in (0, R) \) such that \( u - c_0 - v \leq 0 \) in \( B_{R'}(x_0) \) and \( \sup_{\partial B_{R'}(x_0)} (u - c_0 - v) < 0 \).

Moreover, we have in \( B_{R'}(x_0) \),
\[
\begin{align*}
0 &= \sup_{B_{R'}(x_0)} (u - c_0 - v) = \sup_{\partial B_{R'}(x_0)} (u - c_0 - v) < 0,
\end{align*}
\]
which is a contradiction. Then we complete the proof of Lemma 3.1.

Lemma 3.2. If \( f(t) \) is a nondecreasing continuous function on \( \mathbb{R} \), and \( f(t) \geq \alpha_0 > \alpha \), then there exists a subsolution \( u \) of (2) for all \( x \in \mathbb{R}^n \) if and only if (10) has a solution \( \varphi \in C^2(0, +\infty) \) satisfying \( \varphi'(0) = 0 \) and \( \varphi(0) = a \) for any constant \( a \).

Proof. Sufficiency. If (10) has a solution \( \varphi \), then obviously \( v(x) = \varphi(r) = \varphi(|x|) \) is the desired subsolution of (2).

Necessity. Conversely, assume that there exists no such function \( \varphi(r) \) existing globally. By Lemma 2.4 and Lemma 2.3, (10) possesses a \( C^2 \) solution \( \varphi \) on some interval with \( \varphi(0) = a, \varphi'(0) = 0 \). Then there is a maximal interval \( [0, R) \) in which the solution exists. By Lemma 2.2, we know that \( \varphi'(r) > 0 \) for \( r \in (0, R) \), then \( \varphi(r) \to \infty \) as \( r \to R \). So by Lemma 3.1, any subsolution of (2) satisfies \( u(x) \leq \varphi(|x|) \) for \( |x| < R \). Particularly, we have \( u(0) \leq \varphi(0) = a \). But since \( a \) is arbitrary, we let \( a = u(0)/2 \), then it is a contradiction. Lemma 3.2 is proved.

Lemma 3.3. If \( f(t) \) is a nondecreasing continuous function on \( \mathbb{R} \), and \( f(t) \geq \alpha_0 > \alpha \), then (10) has a solution \( \varphi \in C^2(0, +\infty) \) satisfying \( \varphi'(0) = 0 \) if and only if (7) holds.
Proof. Sufficiency. Suppose there exists no such solution of (10). Just as the proof of Lemma 3.2, there is a $C^2$ solution $\varphi(r)$ of (10) with $\varphi'(0) = 0, \varphi(0) = 0$ on the maximal interval $[0, R]$ and $\varphi(r) \to \infty$ as $r \to R$. So $\varphi$ satisfies
\[(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^{n-1} = f^n(\varphi(r)) r^{n-1}. \tag{16}\]
Since $\varphi''(r) + \alpha > 0, \varphi'(r) > 0$, we have
\[(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^{n-1} \geq (\varphi''(r) + \alpha)(\varphi'(r))^{n-1} \geq \varphi''(r)(\varphi'(r))^{n-1}.\]
Therefore,
\[\varphi''(r)(\varphi'(r))^{n-1} \leq f^n(\varphi(r)) r^{n-1}.\]
Then
\[\varphi''(r)(\varphi'(r))^n \leq r^{n-1} f^n(\varphi(r)) \varphi'(r).\]
For $0 < r < R$, we know
\[\varphi''(r)(\varphi'(r))^n \leq R^{n-1} f^n(\varphi(r)) \varphi'(r).\]
Integrating on $r$ from 0 to $r$, we have
\[(\varphi'(r))^{n+1} \leq (n+1) R^{n-1} \int_0^r f^n(\varphi(s)) \varphi'(s) ds = (n+1) R^{n-1} \int_{\varphi(0)}^{\varphi(r)} f^n(t) dt.\]
Thus
\[\varphi'(r) \leq (n+1)^{\frac{1}{n+1}} R^{\frac{n-1}{n+1}} \left( \int_{\varphi(0)}^{\varphi(r)} f^n(t) dt \right)^{\frac{1}{n+1}}.\]
As a result,
\[\left( \int_{\varphi(0)}^{\varphi(r)} f^n(t) dt \right)^{-\frac{1}{n+1}} d\varphi \leq (n+1)^{\frac{1}{n+1}} R^{\frac{n-1}{n+1}} dr.\]
Noticing that $\varphi(0) = 0$ and $\varphi(R) = \infty$ and integrating on $r$ from 0 to $R$, we have
\[\int_0^\infty \left( \int_0^\tau f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau \leq (n+1)^{\frac{1}{n+1}} R^{\frac{n-1}{n+1}} < \infty.\]
So
\[\int_0^\infty \left( \int_0^\tau f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau < \infty.\]
which is a contradiction with (7).

Necessity. Suppose on the contrary, (7) does not hold, i.e.
\[\int_0^\infty \left( \int_0^\tau f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau < \infty.\]
Multiplying (16) by $\varphi'(r) + \alpha r$ on both sides, we know from $f > \alpha$ that
\[(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^n = r^{n-1} f^n(\varphi(r)) (\varphi'(r) + \alpha r) \geq r^{n-1} f^n(\varphi(r)) \varphi'(r) + \alpha r f^n(\varphi(r)) \tag{17}\]
From Lemma 2.2, we know that
\[\varphi(r) \geq \frac{1}{2}(\alpha_0 - \alpha r^2 + \varphi(0)).\]
Hence there exists $r_1 > 1$ such that $\varphi(r_1) > 0$. For $r > r_1 > 1$, by (17) we have
\[
(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^n \geq f^n(\varphi(r))\varphi'(r) + \alpha^{n+1}r^n.
\]
After integrating on $r$ from $r_1$ to $r$, we have
\[
(\varphi'(r) + \alpha r)^{n+1} - (\varphi'(r_1) + \alpha r_1)^{n+1}
\geq (n + 1) \int_{r_1}^{r} f^n(\varphi(s))\varphi'(s)ds + \alpha^{n+1}r^{n+1} - \alpha^{n+1}r_1^{n+1}.
\]
Thus
\[
(\varphi'(r) + \alpha r)^{n+1} \geq (n + 1) \int_{r_1}^{r} f^n(\varphi(s))\varphi'(s)ds + \alpha^{n+1}r^{n+1}
= (n + 1) \int_{\varphi(r_1)}^{\varphi(r)} f^n(t)dt + \alpha^{n+1}r^{n+1}.
\]
(18)
By binomial expansion theorem,
\[
(\varphi'(r) + \alpha r)^{n+1} = (\varphi'(r))^{n+1} + C_{n+1}^1 \alpha r(\varphi'(r))^n + C_{n+1}^2(\alpha r)^2(\varphi'(r))^{n-1} + \cdots + C_{n+1}^n(\alpha r)^n(\varphi'(r))^{n-1} + (\alpha r)^{n+1},
\]
where $C_{n+1}^m = \frac{(n+1)!}{m!(n-m+1)!}$, $m = 1, \ldots, n$. Due to Lemma 2.2, we know
\[
r \leq \frac{1}{\alpha_0 - \alpha} \varphi'(r).
\]
So
\[
(\varphi'(r) + \alpha r)^{n+1} \\
\leq (\varphi'(r))^{n+1} + C_{n+1}^1 \frac{\alpha}{\alpha_0 - \alpha}(\varphi'(r))^n + C_{n+1}^2 \left(\frac{\alpha}{\alpha_0 - \alpha}\right)^2 (\varphi'(r))^{n+1}
+ \cdots + C_{n+1}^n \left(\frac{\alpha}{\alpha_0 - \alpha}\right)^n (\varphi'(r))^{n+1} + (\alpha r)^{n+1}
\]
= $(\varphi'(r))^{n+1} \left[1 + C_{n+1}^1 \frac{\alpha}{\alpha_0 - \alpha} + C_{n+1}^2 \left(\frac{\alpha}{\alpha_0 - \alpha}\right)^2 + \cdots + C_{n+1}^n \left(\frac{\alpha}{\alpha_0 - \alpha}\right)^n \right] + (\alpha r)^{n+1},$
where
\[
C_0 = 1 + C_{n+1}^1 \frac{\alpha}{\alpha_0 - \alpha} + C_{n+1}^2 \left(\frac{\alpha}{\alpha_0 - \alpha}\right)^2 + \cdots + C_{n+1}^n \left(\frac{\alpha}{\alpha_0 - \alpha}\right)^n.
\]
Then by (18),
\[
(\varphi'(r))^{n+1} \geq \frac{n + 1}{C_0} \int_{\varphi(r_1)}^{\varphi(r)} f^n(t)dt.
\]
And thus
\[
\left(\int_{\varphi(r_1)}^{\varphi(r)} f^n(t)dt\right)^{-\frac{1}{n+1}} d\varphi \geq C_1 dr.
\]
where \( C_1 = \left( \frac{n+1}{\alpha_0} \right)^{\frac{1}{n+1}} \). Integrating on \( r \) from \( r_1 \) to \( r \), we know that

\[
C_1(r-r_1) \leq \int_{\phi(r_1)}^{\phi(r)} \left( \int_{\phi(r_1)}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau \\
\leq \int_{\phi(r_1)}^{\infty} \left( \int_{\phi(r_1)}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau. \tag{19}
\]

Since \( f \) is nondecreasing, then for \( \tau \geq 2\phi(r_1) \), we have

\[
\int_0^{\phi(r_1)} f^n(t) dt \leq \phi(r_1)f^n(\phi(r_1)) \leq \frac{\tau}{2} f^n \left( \frac{\tau}{2} \right) \leq \int_{\frac{\tau}{2}}^{\tau} f^n(t) dt.
\]

Therefore

\[
\int_{0}^{\tau} f^n(t) dt = \int_{0}^{\phi(r_1)} f^n(t) dt - \int_{\frac{\tau}{2}}^{\tau} f^n(t) dt - \int_{\phi(r_1)}^{\tau} f^n(t) dt \leq - \int_{\phi(r_1)}^{\tau} f^n(t) dt = \int_{\phi(r_1)}^{\tau} f^n(t) dt.
\]

As a result, from (19),

\[
C_1(r-r_1) \leq \int_{\phi(r_1)}^{\infty} \left( \int_{0}^{\phi(r_1)} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau \\
= 2 \int_{\phi(r_1)}^{\infty} \left( \int_{0}^{\phi(r_1)} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau.
\]

Letting \( r \to \infty \), we get a contradiction. \( \square \)

**Proof of Theorem 1.1.** By Lemma 3.2 and 3.3, we know that Theorem 1.1 is true. \( \square \)

**Proof of Corollary 1.** Let

\[
f(t) = \begin{cases} 
  t^p + \alpha_0, & t \geq 0, \\
  \alpha_0, & t < 0.
\end{cases}
\]

Then \( f \) satisfies the conditions in Theorem 1.1.

If \( 0 < p \leq 1 \), then

\[
\int_{0}^{\infty} \left( \int_{0}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau = \int_{0}^{\infty} \left( \int_{0}^{\tau} (t^p + \alpha_0)^n dt \right)^{-\frac{1}{n+1}} d\tau \\
\geq \int_{0}^{\infty} \left( \int_{0}^{\tau} (\tau^p + \alpha_0)^n dt \right)^{-\frac{1}{n+1}} d\tau \\
= \int_{0}^{\infty} \tau^{-\frac{1}{n+1}} (\tau^p + \alpha_0)^{-\frac{n}{n+1}} d\tau.
\]

Moreover,

\[
\lim_{\tau \to \infty} \tau^{\frac{np+1}{n+1}} \cdot \tau^{-\frac{1}{n+1}} (\tau^p + \alpha_0)^{-\frac{n}{n+1}} = 1.
\]

Since \( \frac{np+1}{n+1} \leq 1 \), the infinite integral \( \int_{0}^{\infty} \tau^{-\frac{1}{n+1}} (\tau^p + \alpha_0)^{-\frac{n}{n+1}} d\tau \) diverges. Therefore

\[
\int_{0}^{\infty} \tau^{-\frac{1}{n+1}} (\tau^p + \alpha_0)^{-\frac{n}{n+1}} d\tau = \infty,
\]
and then
\[ \int_{\tau}^{\infty} \left( \int_{0}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau = \infty. \]

If \( p > 1 \), then
\begin{align*}
\int_{\tau}^{\infty} \left( \int_{0}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau &= \int_{\tau}^{\infty} \left( \int_{0}^{\tau} (t^p + a_0)^n dt \right)^{-\frac{1}{n+1}} d\tau \\
&\leq \int_{\tau}^{\infty} \left( \int_{0}^{\tau} t^{np} dt \right)^{-\frac{1}{n+1}} d\tau < \infty.
\end{align*}

Thus by Theorem 1.1, we know that the corollary is true. \( \square \)

**Proof of Corollary 2.** For \( k = 0 \), then \( f(t) = e^{kt} = 1 \), hence
\[ \int_{\tau}^{\infty} \left( \int_{0}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau = \int_{\tau}^{\infty} \left( \int_{0}^{\tau} dt \right)^{-\frac{1}{n+1}} d\tau = \infty. \]

By Theorem 1.1, there exists a subsolution of (8).

For \( k > 0 \), let
\[ f(t) = \begin{cases} 
e{kt}, & t \geq 0, \\ 1, & t < 0. \end{cases} \]

Then \( f \) satisfies the conditions in Theorem 1.1. In addition,
\[ \left( \int_{0}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} = \int_{\tau}^{\infty} \left( \int_{0}^{\infty} e^{nkt} dt \right)^{-\frac{1}{n+1}} = n^{\frac{1}{n+1}} \sqrt{nk} \left( e^{nk\tau} - 1 \right)^{-\frac{1}{n+1}}. \]

So
\[ \int_{\tau}^{\infty} \left( \int_{0}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau = \int_{\tau}^{\infty} \left( \int_{0}^{\tau} e^{nkt} dt \right)^{-\frac{1}{n+1}} d\tau = n^{\frac{1}{n+1}} \sqrt{nk} \int_{\tau}^{\infty} \frac{1}{n^{\frac{1}{n+1}} \sqrt{e^{nk\tau} - 1}} d\tau. \]

Since
\[ \lim_{\tau \to +\infty} \frac{\tau^2}{(e^{nk\tau} - 1)^{\frac{1}{n+1}}} = \lim_{\tau \to +\infty} \left( \frac{\tau^{2(n+1)}}{e^{nk\tau} - 1} \right)^{\frac{1}{n+1}} = 0. \]

So the infinite integral \( \int_{\tau}^{\infty} \frac{1}{n^{\frac{1}{n+1}} \sqrt{e^{nk\tau} - 1}} d\tau \) converges. Therefore
\[ \int_{\tau}^{\infty} \left( \int_{0}^{\tau} f^n(t) dt \right)^{-\frac{1}{n+1}} d\tau < \infty. \]

By Theorem 1.1, there exists no nonnegative subsolution of (8). Then Corollary 2 is proved. \( \square \)

**Remark 1.** If \( \alpha \geq 1 \), the method we used is not applicable to (8).

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