Flow polynomials of a signed graph

Jianguo Qian*
School of Mathematical Sciences, Xiamen University
Xiamen, Fujian 361005, P.R. China

Abstract

In contrast to ordinary graphs, the number of the nowhere-zero group-flows in a signed graph may vary with different groups, even if the groups have the same order. In fact, for a signed graph $G$ and non-negative integer $d$, it was shown that there exists a polynomial $F_d(G, x)$ such that the number of the nowhere-zero $\Gamma$-flows in $G$ equals $F_d(G, x)$ evaluated at $k$ for every Abelian group $\Gamma$ of order $k$ with $\epsilon(\Gamma) = d$, where $\epsilon(\Gamma)$ is the largest integer $d$ for which $\Gamma$ has a subgroup isomorphic to $\mathbb{Z}_d^2$. We focus on the combinatorial structure of $\Gamma$-flows in a signed graph and the coefficients in $F_d(G, x)$. We first define the fundamental directed circuits for a signed graph $G$ and show that all $\Gamma$-flows (not necessarily nowhere-zero) in $G$ can be generated by these circuits. It turns out that all $\Gamma$-flows in $G$ can be evenly classified into $2^{\epsilon(\Gamma)}$-classes specified by the elements of order 2 in $\Gamma$, each class of which consists of the same number of flows depending only on the order of the group. This gives an explanation for why the number of $\Gamma$-flows in a signed graph varies with different $\epsilon(\Gamma)$, and also gives an answer to a problem posed by Beck and Zaslavsky. Secondly, using an extension of Whitney’s broken circuit theory we give a combinatorial interpretation of the coefficients in $F_d(G, x)$ for $d = 0$, in terms of the broken bonds. As an example, we give an analytic expression of $F_0(G, x)$ for a class of the signed graphs that contain no balanced circuit. Finally, we show that the sets of edges in a signed graph that contain no broken bond form a homogeneous simplicial complex.

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*email address: jgqian@xmu.edu.cn
1 Introduction

Nowhere-zero $\mathbb{Z}_k$-flows, or modular $k$-flows, in a graph was initially introduced by Tutte [18] as a dual problem to vertex-colouring of plane graphs. It has long been known that the number of nowhere-zero $\mathbb{Z}_k$-flows or more in general, nowhere-zero $\Gamma$-flows (flows with values in $\Gamma$) for an Abelian group $\Gamma$ of order $k$ is a polynomial function in $k$, which does not depend on the algebraic structure of the group [18]. An analog to $\mathbb{Z}_k$-flow is the integer $k$-flow or, simply $k$-flow. It is well known that a graph has a nowhere-zero $k$-flow if and only if it has a nowhere-zero $\mathbb{Z}_k$-flow [17]. In [15], Kochol showed that the number of nowhere-zero $k$-flows is also a polynomial in $k$, although not the same polynomial as that for nowhere-zero $\mathbb{Z}_k$-flows. For more topics related to nowhere-zero flows in graphs see also Brylawski and Oxley [5], Jaeger [12], Seymour [16] and Zhang [22].

The notion of the signed graphs was introduced by Harary [11] initially as a model for social networks. In comparison with flows in plane graphs or more generally, in graphs embedded on orientable surface, the definition of $\mathbb{Z}_k$-flows in signed graphs is naturally considered for the study of graphs embedded on non-orientable surface, where nowhere-zero $\mathbb{Z}_k$-flows emerge as the dual notion to local tensions [14].

In contrast to ordinary graphs, the problem of counting the nowhere-zero flows in a signed graph seems more complicated and there are relatively few results to be found in the literatures. Using the theory of counting lattice points in inside-out polytopes to signed graphs, Beck and Zaslavsky [1] showed that the number of the nowhere-zero $k$-flows in a signed graph is a quasi-polynomial of period two, that is, a pair of polynomials, one for odd values of $k$ and the other for even $k$. In the same paper, Beck and Zaslavsky also showed that there exists a polynomial $f(G, x)$ such that, for every odd integer $k$, the number of nowhere-zero $\Gamma$-flows in a signed graph $G$ equals $f(G, x)$ evaluated at $k$ for every Abelian group $\Gamma$ with $|\Gamma| = k$. This result was recently extended by DeVos, Rollová and ˇSˇamal [7] (available from arXiv) to general Abelian group: for any non-negative integer $d$, there exists a polynomial $f_d(G, x)$ such that the number of nowhere-zero $\Gamma$-flows in $G$ is exactly $f_d(G, x)$ evaluated at $n$ for every Abelian group $\Gamma$ with $\epsilon(\Gamma) = d$ and $|\Gamma| = 2^d n$, where $\epsilon(\Gamma)$ is the largest integer $d$ for which $\Gamma$ has a subgroup isomorphic to $\mathbb{Z}_d^2$. More recently, Goodall et. al. [10] (available from arXiv) gave an explicit expression of $f_d(G, x)$ in form of edge-subgraph expansions.

In this paper we focus on the combinatorial structure of $\Gamma$-flows in a signed graph $G$ and the coefficients in the polynomial $f_d(G, x)$. For con-
convenience, instead of working on \( f_d(G, x) \), we will work on the polynomial \( F_d(G, x) \) defined by \( F_d(G, x) = f_d(G, 2^{-d}x) \) and call \( F_d(G, x) \) the \( d \)-type flow polynomial, or simply, the flow polynomial of \( G \). It can be seen that \( F_d(G, x) \) evaluated at \( k \) is exactly the number of the nowhere-zero \( \Gamma \)-flows in \( G \) for every Abelian group \( \Gamma \) with \( \epsilon(\Gamma) = d \) and \( |\Gamma| = k \).

In the third section we introduce the fundamental directed circuits and the fundamental root circuit (a particular unbalanced circuit) in a signed graph \( G \). We show that every \( \Gamma \)-flow (not necessarily nowhere-zero) in \( G \) can be generated by these circuits, each of which is assigned with a proper \( \Gamma \)-flow. More specifically, the values of the flows assigned to the fundamental directed circuits are the elements in \( \Gamma \) while the value to the fundamental root circuit is an element of order 2 in the group \( \Gamma \). Therefore, all \( \Gamma \)-flows in \( G \) can be evenly classified into \( 2^{\epsilon(\Gamma)} \)-classes specified by the elements of order 2 in \( \Gamma \). Moreover, each class consists of the same number of flows, which depends only on the order of the group. This gives an explanation for why the number of the \( \Gamma \)-flows in a signed graph varies with different \( \epsilon(\Gamma) \) and, also gives an answer to a problem posed by Beck and Zaslavsky in [1]. Further, this result also yields an explicit expression of the polynomial \( F_d(G, x) \) obtained earlier by Goodall et. al.

In the fifth section we give a combinatorial interpretation of the coefficients in \( F_d(G, x) \) for \( d = 0 \). To this end, we apply Whitney’s broken circuit theory [19]. In the study of graph coloring, one significance of Whitney’s broken circuit theorem is that it gives a very nice ‘cancellation’ to reduce the terms in the chromatic polynomial (represented in the form of inclusion-exclusion) so that the remaining terms can not be cancelled out anymore and, therefore, yield a combinatorial interpretation for the coefficients of the polynomial [3, 4]. Using an extended form of the Whitney’s theorem given by Dohmen and Trinks [8], we show that \( F_0(G, x) \) is a polynomial with leading term \( x^{m-n} \) and with its coefficients alternating in signs. More specifically, the coefficient of \( (-1)^i x^{m-n-i}, i = 0, 1, \cdots, m-n \), is exactly the number of the sets consisting of \( i \) edges that contain no broken bond. As an example, we give an analytic expression of \( F_0(G, x) \) for a class of the signed graphs that contain no balanced circuit. Finally, we show that the broken bonds in a signed graph form a nice topological structure, that is, a homogeneous simplicial complex of top dimension \( m-n \). Thus, the coefficients of \( F_0(G, x) \) are the simplex counts in each dimension of the complex.
2 Preliminaries

Graphs in this paper may contain parallel edges or loops. For a graph \( G \), we use \( V(G) \) and \( E(G) \) to denote its vertex set and edge set, respectively. A signed graph is a pair \( (G, \Sigma) \), where \( \Sigma \subseteq E(G) \) and the edges in \( \Sigma \) are negative while the other ones are positive.

A circuit is a connected 2-regular graph. An unbalanced circuit in a signed graph \( (G, \Sigma) \) is a circuit in \( G \) that has an odd number of negative edges. A balanced circuit in \( (G, \Sigma) \) is a circuit that is not unbalanced. A subgraph of \( G \) is unbalanced if it contains an unbalanced circuit; otherwise, it is balanced. In particular, a subgraph without negative edges is balanced. A barbell is the union of two unbalanced circuits \( C_1, C_2 \) and a (possibly trivial) path \( P \) with end vertices \( v_1 \in V(C_1) \) and \( v_2 \in V(C_2) \), such that \( C_1 - v_1 \) is disjoint from \( P \cup C_2 \) and \( C_2 - v_2 \) is disjoint from \( P \cup C_1 \). We call \( P \) the barbell path of the barbell. A signed circuit is either a balanced circuit or a barbell.

Given a signed graph \( (G, \Sigma) \), switching at a vertex \( v \) is the inversion of the sign of each edge incident with \( v \). Two signed graphs are said to be switching-equivalent if one can be obtained from the other by a series of switchings. It is known \[14\] and easy to see that equivalent signed graphs have the same sets of unbalanced circuits and the same sets of balanced circuits. This means, in particular, that a balanced signed graph \( (G, \Sigma) \) is switching-equivalent to an ordinary graph \( G \).

Following Bouchet \[2\] we now introduce the notion of the half-edges so as to orient a signed graph: each negative edge of \( (G, \Sigma) \) is viewed as composed of two half-edges. An orientation of a negative edge \( e \) is obtained by giving each of the two half-edges \( h \) and \( h' \) a direction so that both \( h \) and \( h' \) point toward the end vertices of \( e \), called extroverted, or both \( h \) and \( h' \) point toward the inside of \( e \), called introverted.

In the following, we will use \( G \) simply to denote a signed graph if no confusion can occur. Let \( D \) be a fixed orientation of a signed graph \( G \) and \( \Gamma \) be an additive Abelian group. A map \( f : E(D) \rightarrow \Gamma \) is called a \( \Gamma \)-flow if the usual conservation law (Kirchhoff’s law) is satisfied, that is, for each vertex \( v \), the sum of \( f(e) \) over the incoming edges \( e \in E^-(v) \) at \( v \) (i.e., the edges and half-edges directed towards \( v \)) equals the sum of \( f(e) \) over the outgoing edges \( e \in E^+(v) \), i.e.,

\[
\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e).
\]
A flow \( f \) is called nowhere-zero if \( f(e) \neq 0 \) for each \( e \in E(D) \). It is well known that the number of nowhere-zero \( \Gamma \)-flows is independent of the orientation of \( G \). A signed graph is said to be \( \Gamma \)-flow admissible if it admits at least one nowhere-zero \( \Gamma \)-flow. It is clear that the property of ‘\( \Gamma \)-flow admissible’ is invariant under switching inversion.

# 3 Fundamental circuits in a signed graph

In this section we generalize the notion of fundamental circuits in graphs to signed graphs, which will play an important role in revealing the structural property of \( \Gamma \)-flows in signed graphs.

For a signed graph \( G \) and a set \( F \) of edges, we denote by \( G + F \) and \( G - F \) the subgraphs obtained from \( G \) by adding and deleting the edges in \( F \), respectively. Let \( E_N = \{e_0, e_1, e_2, \cdots, e_{m_N-1}\} \) be the set of all the negative edges of \( G \), where \( m_N = |E_N| \). In this section we always assume that \( G \) is unbalanced and, with no loss of generality, contains as few negative edges as possible in its switching equivalent class. Thus, \( E_N \neq \emptyset \) and \( G - E_N \) is connected [21].

Let \( T \) be a spanning tree of \( G - E_N \). Choose an arbitrary edge \( e_0 \) from \( E_N \) and call \( T_0 = T + e_0 \) a signed rooted tree of \( G \) with root edge \( e_0 \) (note that a signed rooted tree we defined here is not a real tree because it has a unique unbalanced circuit). Let \( T_0 = E(G) \setminus E(T_0) \). For any \( e \in T_0 \), it is clear that \( T_0 + e \) contains a unique signed circuit. We call this circuit a fundamental circuit and denote it by \( C_e \). We can see that, if \( e \in T_0 \setminus E_N \) then \( C_e \) is an ordinary circuit (a circuit without negative edge) and if \( e \in E_N \setminus \{e_0\} \) then \( C_e \) is a barbell or a balanced circuit with two negative edges \( e_0 \) and \( e \).

Given a fixed orientation \( D \) of \( G \), a fundamental directed circuit \( \overrightarrow{C_e} \) of \( G \) is the orientation of a fundamental circuit \( C_e \) such that the direction of \( e \) is the same as what it was in \( D \) and the directions of all other edges on \( \overrightarrow{C_e} \) coincide consistently with \( e \) along with \( C_e \). Under this orientation, it can be seen that if \( C_e \) is an ordinary circuit then an edge \( e' \) on \( \overrightarrow{C_e} \) is clockwise oriented if and only \( e \) is clockwise oriented, and if \( C_e \) is a balanced circuit or a barbell (with two negative edges \( e_0 \) and \( e \)), then the direction of the two negative edges are always opposite, that is, \( e_0 \) is extroverted if and only if \( e \) is introverted, see Figure 1.

For an fundamental circuit \( C_e \), let \( C_e^D \) be the orientation \( D \) restricted on
Figure 1. The edges $e_0, e_1, e_2$ are negative and $e$ is positive.

$C_e$. We associate with $C_e$ a function $f_e$ on $E(G)$ defined by

$$f_e(a) = \begin{cases} 
1, & \text{if } a \in \overrightarrow{C_e}; \\
-1, & \text{if } a \in C_e^D \setminus \overrightarrow{C_e}; \\
2, & \text{if } a \in \overrightarrow{C_e} \text{ and } a \text{ is on the barbell path of } C_e; \\
-2, & \text{if } a \in C_e^D \setminus \overrightarrow{C_e} \text{ and } a \text{ is on the barbell path of } C_e; \\
0, & \text{otherwise}
\end{cases}$$

for any $a \in E(D)$, where ‘a is on the barbell path of $C_e$’ means that $C_e$ is a barbell and $a$ is on the barbell path of $C_e$.

Form the above definition, it can be seen that $f_e(e) = 1$ for any $e \in \overrightarrow{T_0}$.

Let $C_0$ be the unique (un-balanced) circuit in $T_0$ (i.e., formed by $e_0$ and $T$). Choose an arbitrary vertex $v$ on $C_0$ and let $\overrightarrow{C_0}$ be the orientation of $C_0$ such that the direction of $e_0$ is extroverted and all other edges on $C_0$ are oriented so that $d^-(v) = 2, d^+(v) = 0$ and $d^-(u) = d^+(u) = 1$ for any vertex $u$ on $C_0$ other than $v$, where $d^-(v)$ and $d^+(v)$ are the in-degree and out-degree of $v$ on $\overrightarrow{C_0}$, respectively, see Figure 1. We call $\overrightarrow{C_0}$ the fundamental root circuit and associate it with a function $g$ on $E(G)$ defined by

$$g(e) = \begin{cases} 
1, & \text{if } e \in \overrightarrow{C_0}; \\
-1, & \text{if } e \in C_0^D \setminus \overrightarrow{C_0}; \\
0, & \text{otherwise}
\end{cases}$$

for any $e \in E(D)$.

For convenience, in the following we regard each $\Gamma$-flow, each function $f_e$ ($e \in \overrightarrow{T_0}$) and the function $g$ as $m$-dimensional vectors indexed by $e \in E(G)$. Let $S_G$ denote the class of all $\Gamma$-flows (not necessarily nowhere-zero) in $G$.  

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For a finite additive Abelian group $\Gamma$, let $\Gamma_2$ be the set of the elements of order 2 in $\Gamma$ (including the zero element). Recalling that $\epsilon(\Gamma)$ is the largest integer $d$ for which $\Gamma$ has a subgroup isomorphic to $\mathbb{Z}_2^d$, we have $|\Gamma_2| = 2^{\epsilon(\Gamma)}$.

**Theorem 3.1.** Let $\Gamma$ be an additive Abelian group and let $G$ be a connected unbalanced signed graph. Let $T$ be a spanning tree $T$ of $G$ consisting of positive edges and let $e_0 \in E_N$. Then

$$S_G = \{ \gamma g + \sum_{e \in T_0} \gamma_e f_e : \gamma \in \Gamma_2, \gamma_e \in \Gamma \}.$$  \hspace{1cm} (1)

**Proof.** It is clear that

$$\gamma g + \sum_{e \in T_0} \gamma_e f_e$$

is a $\Gamma$-flow for any $\gamma \in \Gamma_2$ and $\gamma_e \in \Gamma$. Let $f$ be an arbitrary $\Gamma$-flow in $G$. We need only prove that $f$ can be written as the combination (2).

Since a $\Gamma$-flow is independent of the orientation $D$, to simplify our discussion we make the following assumption:

**Assumption 1.** In orientation $D$, the direction of the root edge $e_0$ is extroverted while the directions of all other negative edges are introverted.

For each negative edge $e_i = u_iv_i \in E_N$, insert a new vertex $w_i$ into the middle of $e_i$ so that the two half edges of $e_i$ in $D$ become two ordinary directed edges $w_iu_i$ (with direction from $w_i$ to $u_i$) and $w_iv_i$ if $i = 0$, or $u_iw_i$ and $v_iw_i$ if $i \in \{1, 2, \ldots, m_N - 1\}$. We call $w_i$ the middle vertex of $e_i$.

Further, add a new vertex $w$ to $D$ and, for each middle vertex $w_i$, add the directed edge $e'_i = ww_i$ if $i = 0$ and the directed edge $e'_i = w_iw$ if $i \in \{1, 2, \ldots, m_N - 1\}$. The resulting graph, denoted by $D^w$, is a directed graph without negative edges, that is, $D^w$ is an ordinary directed graph. Further, for $i \in \{0, 1, 2, \ldots, m_N - 1\}$, assign the edge $e'_i$ with the value $2f(e_i)$. It clear that, except the possible $w$, the conservation law is satisfied at all the vertices in $D^w$ and therefore must be satisfied at $w$, either. As a result, we get a span of the $\Gamma$-flow $f$ to $D^w$ and denote it by $f^w$. Thus, by the conservation law at $w$, we have

$$f^w(e'_0) = \sum_{i=1}^{m_N-1} f^w(e'_i)$$

or equivalently,

$$2f(e_0) = \sum_{i=1}^{m_N-1} 2f(e_i) = 2 \sum_{e_i \in E_N^*} f(e_i),$$  \hspace{1cm} (3)
where $E_N^* = E_N \setminus \{e_0\} = \{e_1, e_2, \ldots, e_{m_N-1}\}$.

Further, we notice that, for any $\gamma \in \Gamma$, the solution of the equation $2x = 2\gamma$ (in $x$) over $\Gamma$ has the form $x = \gamma + \gamma_2$, where $\gamma_2$ is an element of order 2 (possibly the zero element), i.e., $\gamma_2 \in \Gamma_2$. Thus, (3) is equivalent to

$$f(e_0) = \gamma_2 + \sum_{e_i \in E_N^*} f(e_i), \quad (4)$$

where $\gamma_2 \in \Gamma_2$.

On the other hand, for any $e \in E_N^*$, by Assumption 1 and the definitions of $\overrightarrow{C_e}$ and $\mathbf{f_e}$, we have

$$f_e(e_0) = f_e(e) = 1. \quad (5)$$

In (2), we set $\gamma = \gamma_2$ and for $e \in \overrightarrow{T_0}$, set $\gamma_e = f(e)$. Let

$$f' = f - (\gamma_2 \mathbf{g} + \sum_{e \in T_0} \gamma_e f_e). \quad (6)$$

Then for any $e \in \overrightarrow{T_0}$, by the definition of the vector $\mathbf{g}$ we have $\gamma_2 \mathbf{g}(e) = 0$ since $e$ is not on $C_0$. This implies that $f'(e) = 0$ for any $e \in \overrightarrow{T_0}$ because $\gamma_e = f(e)$ and, as mentioned earlier, $f_e(e) = 1$. Further, by (4), (5) and (6) we have

$$f'(e_0) = f(e_0) - (\gamma_2 \mathbf{g}(e_0) + \sum_{e \in T_0} \gamma_e f_e(e_0))$$

$$= \gamma_2 + \sum_{e \in E_N^*} f(e) - (\gamma_2 \mathbf{g}(e_0) + \sum_{e \in T_0 \setminus E_N^*} \gamma_e f_e(e_0) + \sum_{e \in E_N^*} \gamma_e f_e(e_0))$$

$$= \sum_{e \in E_N^*} f(e) - \sum_{e \in E_N^*} \gamma_e f_e(e_0)$$

$$= \sum_{e \in E_N^*} f(e)(1 - f_e(e_0))$$

$$= 0,$$

where the third equality holds because $\mathbf{g}(e_0) = 1$ and $e_0 \notin C_e$ for any $e \in \overrightarrow{T_0} \setminus E_N^*$ and therefore, $f_e(e_0) = 0$; and the last two equalities hold because of (5) and $\gamma_e = f(e)$ for any $e \in E_N^*$.

The above discussion means that $f'$ evaluated at each edge outside of $T$ is zero. Thus, we must have $f' = 0$ (the vector of all zeros) because the values of $f'$ at the edges of $T$ are uniquely determined by that outside of $T$. In conclusion, $f$ is represented as the combination (2), which completes our proof. \qed
4 Classification of $\Gamma$-flows in a signed graph

From Theorem 3.1, we have known that all the $\Gamma$-flows in a connected unbalanced signed graph can be ‘generated’ by fundamental root circuit $\overrightarrow{C}_0$ and the fundamental directed circuits $\overrightarrow{C}_e, e \in \overrightarrow{T}_0$. This leads to the following classification of $\Gamma$-flows in a signed graph, which are specified by the elements of order 2 in $\Gamma$.

**Theorem 4.1.** Let $\Gamma$ be an additive Abelian group of order $k$ and let $G$ be a connected unbalanced signed graph. Let $T$ be a spanning tree $T$ of $G$ consisting of positive edges and let $e_0 \in E_N$.

1. The flows in $S_G$ are pairwise distinct and, therefore,
$$|S_G| = 2^{\epsilon(\Gamma)}k^{m-n};$$  \hspace{1cm} \text{(7)}

2. $S_G$ can be evenly classified into $|\Gamma_2|$ classes specified by the elements in $\Gamma_2$, i.e., $S_G = \bigcup_{\gamma \in \Gamma_2} S_G(\gamma)$ and $|S_G(\gamma)| = k^{m-n}$ for any $\gamma \in \Gamma_2$, where
$$S_G(\gamma) = \{\gamma g + \sum_{e \in \overrightarrow{T}_0} \gamma_e f_e : \gamma_e \in \Gamma\}. \hspace{1cm} \text{(8)}$$

**Proof.** 1). We need only prove that
$$\gamma g + \sum_{e \in \overrightarrow{T}_0} \gamma_e f_e = \gamma' g + \sum_{e \in \overrightarrow{T}_0} \gamma'_e f_e$$  \hspace{1cm} \text{(9)}
if and only if $\gamma = \gamma'$ and $\gamma_e = \gamma'_e$ for any $e \in \overrightarrow{T}_0$. For any $e \in \overrightarrow{T}_0$, by the definition of $g$ and $f_e$ we have $f_e(e) = 1, g(e) = 0$ and $f'_e(e) = 0$ for any $e' \in \overrightarrow{T}_0$ with $e' \neq e$. Thus, (9) implies that $\gamma_e f_e(e) = \gamma'_e f_e(e)$ and therefore, $\gamma_e = \gamma'_e$ for any $e \in \overrightarrow{T}_0$. Consequently, again by (9), we have $\gamma g = \gamma' g$ and therefore, $\gamma = \gamma'$.

2). Since the flows in $S_G$ are pairwise distinct, 2) follows directly. \hfill \Box

For a component $\omega$ of a signed graph $G$, denote
$$\beta(\omega) = \begin{cases} m(\omega) - n(\omega) + 1, & \text{if } \omega \text{ is balanced;} \\ m(\omega) - n(\omega), & \text{if } \omega \text{ is unbalanced}; \end{cases}$$  \hspace{1cm} \text{(10)}
where $m(\omega)$ and $n(\omega)$ are the number of edges and vertices in $\omega$, respectively. In general, we denote $\beta(G) = \sum \beta(\omega)$, where the sum is taken over all the components $\omega$ of $G$. Let $\kappa(G)$ be the number of unbalanced components and $F^*(G, \Gamma)$ be the number of $\Gamma$-flows (not necessarily nowhere-zero) in $G$. 
Corollary 4.2. Let $G$ be a signed graph and let $\Gamma$ be an additive Abelian group of order $k$. Then
\[
F^*(G, \Gamma) = 2^{\kappa(G)\epsilon(\Gamma)}k^{\beta(G)}.
\] (11)

Proof. If $G$ is not connected then $F^*(G, \Gamma) = \prod F^*(\omega, \Gamma)$, where the product is taken over all the components $\omega$ of $G$. We need only consider the case when $G$ is connected.

If $G$ is unbalanced then (11) follows directly from (7). Now assume that $G$ is balanced. Recall that a balanced signed graph is switching-equivalent to an ordinary graph. In this case it is known [13] that the number of $\Gamma$-flows (not necessarily nowhere-zero) in an ordinary graph is $k^{m-n+1}$, i.e., $F^*(G, \Gamma) = k^{m-n+1}$, where $m$ and $n$ are the numbers of edges and vertices in $G$, respectively. This agrees with (11) because $\kappa(G) = 0$ and $\beta(G) = m-n+1$ when $G$ is balanced. The proof is completed. \[\square\]

Remark 1. When $k$ (the order of $\Gamma$) is odd, Beck and Zaslavsky posed a problem (Problem 4.2, [1]): Is there any significance to $F^*(G, \Gamma)$ evaluated at even natural numbers? By Theorem 4.1 and Corollary 4.2 we can now give an answer to this problem. For simplicity, let’s consider the case when $G$ is connected and unbalanced. Since $k$ is odd, we have $\epsilon(\Gamma) = 0$ and therefore, $F^*(G, \Gamma) = k^{m-n}$. Thus, $F^*(G, \Gamma)$ evaluated at an even number $h$ equals $h^{m-n}$, which is exactly the number of the $\Gamma'$-flows in $G$ divided by $2^{\epsilon(\Gamma')}$ for any group $\Gamma'$ of order $h$. More specifically, by Theorem 4.1 $F^*(G, \Gamma)$ evaluated at $h$ equals the number of those $\Gamma'$-flows in $G$ which have the form
\[
f = \gamma g + \sum_{e \in T_0} \gamma_ee, \quad \gamma_\epsilon \in \Gamma',
\]
where $\gamma$ is an arbitrary fixed element of order 2 in $\Gamma'$ (in particular we may choose $\gamma = 0$). \[\square\]

For any $e \in E(G)$, the number of the $\Gamma$-flows in $G$ with value 0 at $e$ is clearly equal to $F^*(G-e, \Gamma)$. Notice that the flows counted by $F_d(G, x)$ are nowhere-zero. So by Corollary 4.2 and the principle of inclusion-exclusion, we get the following expression of $F_d(G, x)$ obtained by Goodall et. al.:

Corollary 4.3. [11] For any signed graph $G$ and non-negative integer $d$,
\[
F_d(G, x) = \sum_{F \subseteq E} (-1)^{|F|}2^{\kappa(G-F)}dx^{\beta(G-F)}.
\]
We note that, if $G$ is an ordinary graph then $\kappa(G - F) = 0$ for any $F \subseteq E(G)$. Therefore, Corollary 4.3 generalizes the corresponding result for ordinary graph \[9, 13\].

**Example.** By Corollary 4.3 if $G$ is the graph with two vertices joined by a negative edge and a positive edge then $F_d(G, x) = 2^d - 1$; if $G$ is the graph consisting of two negative loops at a vertex then $F_d(G, x) = 2^d x - 2^{d+1} + 1$; and if $G$ is the graph consisting of a negative loop and a positive loop at a vertex then $F_d(G, x) = (2^d - 1)(x - 1)$.

5 **Coefficients in $F_0(G, x)$**

In this section we will give a combinatorial interpretation of the coefficients in $F_0(G, x)$ for $d = 0$. We begin with the following extension of Whitney’s broken theorem given by Dohmen and Trinks.

**Lemma 5.1.** [8] Let $P$ be a finite linearly ordered set, $\mathcal{B} \subseteq 2^P \setminus \{\emptyset\}$ and $\Gamma$ be an additive Abelian group. Let $f : 2^P \rightarrow \Gamma$ be a mapping such that, for any $B \in \mathcal{B}$ and $A \supseteq B$,

$$f(A) = f(A \setminus \{B_{\text{max}}\}).$$

Then

$$\sum_{A \in 2^P} (-1)^{|A|} f(A) = \sum_{A \in 2^P \setminus \mathcal{B}^*} (-1)^{|A|} f(A),$$

where $B_{\text{max}}$ is the maximum element in $B$ and $\mathcal{B}^* = \{A : A \in 2^P, A \supseteq B \setminus \{B_{\text{max}}\} \text{ for some } B \in \mathcal{B}\}$.

We call $\mathcal{B}$ in Lemma 5.1 a **broken system** of $f$ and $B \setminus \{B_{\text{max}}\}$ a **broken set** for any $B \in \mathcal{B}$.

To apply Lemma 5.1 we need to define a broken system and broken sets for signed graphs. We follow the idea of the notion of ‘bonds’ introduced in [6, 21]. For a signed graph $G$ and $X \subseteq V(G)$, denote by $[X, X^C]$ the set of edges between $X$ and its complement $X^C$, by $G[X]$ the subgraph of $G$ induced by $X$, and by $E(X)$ the set of the edges in $G[X]$. A non-empty edge subset $B \subseteq E(G)$ is called a **cut** [6] or **improving set** [21] of $G$ if it has the form $B = [X, X^C] \cup E_X$, where $X \subseteq V(G)$ is non-empty and $E_X \subseteq E(X)$ is minimal to have $G[X] - E_X$ balanced. A cut is called a **bond** of $G$ if it is minimal. We note that, in the case when $G$ is balanced, we have $E_X = \emptyset$ by the minimality of $E_X$ and therefore, a bond is exactly a usual bond as in an
ordinary graph. In this sense, the notion ‘bond’ for signed graph is a very nice extension of that for ordinary graphs [13].

By the definition of the broken set, it is not difficult to see that if \( B \) is a bond then, for any \( e \in B \),
\[
\beta(G - B) = \beta(G - (B \setminus \{e\})).
\] (14)

On the other hand, by Corollary 4.3, we have
\[
F_0(G, x) = \sum_{F \subseteq E} (-1)^{|F|} x^{\beta(G - F)}.
\]

Thus, an edge subset of \( G \) is a broken set of \( F_0(G, x) \) if it has the form \( B \setminus \{B\_{\text{max}}\} \) for some \( B \subseteq E(G) \) such that, for any \( A \supseteq B \),
\[
\beta(G - A) = \beta(G - (A \setminus \{B\_{\text{max}}\})).
\] (15)

On the other hand, by (14), for any bond \( B \) we have
\[
\beta(G - B) = \beta(G - (B \setminus \{B\_{\text{max}}\})).
\]

Moreover, it is not difficult to see that, for any \( A \supseteq B \), (15) is satisfied by \( A \) and \( B \). Thus, \( B \setminus \{B\_{\text{max}}\} \) is a broken set of \( F_0(G, x) \) for any bond \( B \) and is called a broken bond of \( G \).

Let \( \mathcal{B} \) be the class of all the broken bonds of \( G \) and let
\[
\mathcal{B}^* = \{ F : F \in 2^{E(G)}, F \supseteq B \text{ for some } B \in \mathcal{B} \}.
\]

Then by Lemma 5.1, we have the following result immediately.

**Theorem 5.2.** For any signed graph \( G \) with a linear order \( \prec \) on \( E(G) \),
\[
F_0(G, x) = \sum_{F \in 2^{E(G)} \setminus \mathcal{B}^*} (-1)^{|F|} x^{\beta(G - F)}.
\] (16)

**Remark 2.** If \( G \) is balanced, then each broken bond is exactly a usual broken bond of an ordinary graph. In this case, (16) is still valid. Thus, Theorem 5.2 is a generalization of that for ordinary graph [13]. Further, in a very special case when an unbalanced signed graph \( G \) contains an edge whose removal leaves a balanced graph, the empty set is a broken bond and therefore, any set of edges (including the empty set) contains a broken bond. This case means that \( \mathcal{B}^* = 2^{E(G)} \) and thus, \( F_0(G, x) = 0 \), which coincides with an obvious fact that such \( G \) is not \( \Gamma \)-flow admissible when \( |\Gamma| \) is odd. □
Proposition 5.3. For any signed graph \( G \) and \( F \subseteq E(G) \), if \( F \) contains no broken bond then each component of \( G - F \) is unbalanced, unless \( G \) is balanced.

Proof. To the contrary suppose that one component \( \omega \) of \( G - F \) is balanced. Let \( B = [V(\omega), V(\omega)] \cup E_F \), where \( E_F \) is the set of edges in \( F \) whose two end vertices are both in \( \omega \). Then \( B \) is a bond since \( \omega \) is balanced and thus \( B \setminus \{B_{\text{max}}\} \) is a broken bond. Notice that \( B \setminus \{B_{\text{max}}\} \subset B \subseteq F \), which contradicts that \( F \) contains no broken bond.

Let \( \sigma(G) \) be the number of those edges \( e \) such that there is an edge \( e' \) with \( e < e' \) satisfying one of the following three conditions:
1). one of \( e \) and \( e' \) is a cut edge and \( G - \{e, e'\} \) has a balanced component;
2). \( \{e, e'\} \) is an edge cut and \( G - \{e, e'\} \) has a balanced component;
3). \( \{e, e'\} \) is contained in a component \( \omega \) of \( G \) and \( \omega - \{e, e'\} \) is balanced.

Corollary 5.4. Let \( G \) be an unbalanced, \( \Gamma \)-flow admissible (\( |\Gamma| \) is odd) signed graph with \( n \) vertices and \( m \) edges. Then for any linear order \( \prec \) on \( E(G) \),

\[
F_0(G, x) = a_0 x^{m-n} - a_1 x^{m-n-1} + a_2 x^{m-n-2} - \cdots + (-1)^{m-n} a_{m-n}, \tag{17}
\]

where, for any \( i \in \{0, 1, \cdots, m-n\} \), \( a_i \) is the number of the edge subsets of \( G \) having \( i \) edges and containing no broken bond as a subset. In particular,
1). \( a_i > 0 \) for every \( i = 0, 1, 2, \cdots, m-n \);
2). \( a_0 = 1 \);
3). \( a_1 = m - \sigma(G) \);

Proof. Let \( F \subseteq E(G) \) be an edge subset that contains no broken bond. Since \( G \) is unbalanced, so by Proposition 5.3, every component \( \omega \) of \( G - F \) is unbalanced. Thus, \( \beta(\omega) = m(\omega) - n(\omega) \) due to (11). Therefore,

\[
\beta(G - F) = \sum_{\omega} \beta(\omega) = m(G - F) - n(G - F) = m - n - |F|,
\]

where the sum is taken over all the components of \( G - F \). This equation means that the value of \( \beta(G - F) \) is determined uniquely by the number of edges in \( F \), as long as \( F \) contains no broken bond. So by Theorem 5.2, the coefficient of \((-1)^i x^{m-n-i}\) in \( F_0(G, x) \) counts exactly those edge subsets \( F \) which have \( i \) edges and contain no broken bond. Thus, (17) follows directly.

1). We first show that there is an edge set \( F \) with \( n \) edges that contains no broken bond. By the definition of the broken bond, an edge set \( F \) contains no broken bond if and only if \( E(G) \setminus F \) contains at least one edge from each
broken bond of $G$. Let $F^*$ be maximum such that $E(G) \setminus F^*$ contains at least one edge from each broken bond of $G$ (such $F^*$ clearly exists because $E(G) \setminus \emptyset$ does). Let $\omega$ be a component of $G - F^*$. Then by Proposition 5.3 $\omega$ contains at least one unbalanced circuit, say $C_u$. We claim that $\omega$ does not contain any other circuit.

Suppose to the contrary that $C$ is a circuit in $\omega$ with $C \neq C_u$. Since $C$ is a circuit, the property that $G - F^*$ contains at least one edge from each broken bond is still satisfied by $G - F^* - C_{\text{max}}$ because any bond containing $C_{\text{max}}$ must contain another edge $e$ on $C$ with, of course, $e \prec C_{\text{max}}$. This contradicts our assumption that $F^*$ is maximum. Our claim follows.

In a word, each component $\omega$ of $G - F^*$ contains exactly one unbalanced circuit and no any other circuit. This means that $m(\omega) = n(\omega)$ and therefore, $m(G - F^*) = n$, i.e., $|F^*| = m - n$. Thus, $a_{m-n} > 0$. Further, if an edge subset $F$ contains no broken bond then any subset of $F$ contains neither broken bond, which implies $a_i > 0$ for any $i$ with $0 \leq i \leq m - n$.

2). Since $G$ is flow-admissible, as pointed out in Remark 1, $G$ contains no edge whose removal leaves a balanced graph. This means that the empty set is not a broken bond. Thus, $a_0$ equals the number of the edge subsets of $G$ having 0 edges, that is, the unique empty set.

3). Now we consider the coefficient $a_1$. From the above discussion we see that $a_1$ equals the number of the edges that are not broken bond. On the other hand, an edge $e$ is a broken bond if there is $e'$ such that $B = \{e, e'\}$ is a bond and $e' = B_{\text{max}}$. By the definition of a bond, $B = \{e, e'\}$ must satisfy one of the above three conditions and, vice versa.

6 Applications

An ordinary graph can be viewed as a signed graph that contains no any unbalanced circuit. Oppositely, our first application is to consider a class of the signed graphs which are $\Gamma$-flow admissible for $|\Gamma|$ odd but contain no any balanced circuit.

For a tree $T$, let $G_T$ be the signed graph obtained from $T$ by replacing each of its end vertices (the vertices of degree 1) with an unbalanced circuit. It is clear that $G_T$ contains no balanced circuit.

Let $v_1, v_2, \ldots, v_p$ be the vertices in $T$ that have degree at least 3 and let $d_1, d_2, \ldots, d_p$ be their degrees, respectively. Choosing an arbitrary vertex $r$ of $T$ as the root, we get a rooted tree (here the ‘rooted tree’ is not the same
thing as the ‘signed rooted tree’ defined earlier). For a vertex \( v_i \) (with degree at least 3) and an edge \( e \) incident with \( v_i \), we call \( e \) the father of the family \( v_i \) if \( e \) is nearer to the root than other edges incident with \( v_i \) and call every edge other than the father a child of the family \( v_i \). In particular, we call the set of all the children of \( v_i \) the children class of \( v_i \) and denote it by \( C(v_i) \).

Let \( \prec \) be an ordering on \( E(G_T) \) such that no child is greater than its father and no edge on an unbalanced circuit is greater than one on \( T \). Let \( F \) be an edge set of \( G_T \) that contains no broken bond. By Corollary 5.2, \( F \) contribute \((-1)^{|F|}x^{m-n-|F|}\) to \( F_0(G_T, x) \), where \( m = |E(G_T)|, n = |V(G_T)| \).

On the other hand, we notice that \( F \) contains no broken bond if and only if \( F \) contains neither an edge from an unbalanced circuit nor a children class of a family. For any vertex \( v_i \), let \( F_i = F \cap C(v_i) \). In particular, let \( F_r = F \cap \{ e_r \} \), where \( e_r \) is the unique edge incident with the root \( r \). Thus, the contribution of \( F \) to \( F_0(G_T, x) \) can be specified as

\[
x^{m-n}(-1)^{|F_i|}x^{-|F_i|} \prod_{i=1}^{p}(-1)^{|F_i|}x^{-|F_i|}.
\]

(18)

On the other hand, we notice that \( m - n = (d_1 - 2) + (d_2 - 2) + \cdots + (d_p - 2) + 1 \). Rewrite (18) as

\[
(-1)^{|F_i|}x^{1-|F_i|} \prod_{i=1}^{p}(-1)^{|F_i|}x^{d_i-2-|F_i|}.
\]

This means that \((-1)^{|F_i|}x^{1-|F_i|}\) and \((-1)^{|F_i|}x^{d_i-2-|F_i|}\) could be regarded as the contribution of \( F \) restricted on \( \{ e_r \} \) and \( C(v_i) \), respectively. Since \( F \cap \{ e_r \} = \emptyset \) or \( F \cap \{ e_r \} = \{ e_r \} \), all the possible contributions of \( F \) restricted on \( \{ e_r \} \) can be represented as

\[
(-1)^{0}x^1 + (-1)^{1}x^1 = x - 1.
\]

In general, for any \( v_i \), since \( v_i \) has exactly \( d_i - 1 \) children, all the possible contributions of \( F \) restricted on \( C(v_i) \) equals

\[
x^{d_i-2} - \binom{d_i - 1}{1}x^{d_i-3} + \cdots + (-1)^{d_i-2}\binom{d_i - 1}{d_i - 2}.
\]

Thus, the total contributions of all \( F \) that contains no broken bond equals

\[
F_0(G_T, x) = (x - 1) \prod_{i=1}^{p}(x^{d_i-2} - \binom{d_i - 1}{1}x^{d_i-3} + \cdots + (-1)^{d_i-2}\binom{d_i - 1}{d_i - 2}).
\]

Our second application is to show that all the broken bonds in a signed graph form a nice topological structure, namely, the homogeneous simplicial
complex. A finite collection $S$ of finite sets is called a *simplicial complex* if $S \in S$ implying $T \in S$ for any $T \subseteq S$. A simplicial complex is *homogeneous* [20] or *pure* [3] if all the maximal simplices have the same dimension (cardinality). A classic example of homogeneous simplicial complex related to a graph is the broken-circuit complex [3, 4]. It was shown [20] that the class $\mathcal{B}(G)$ consisting of all the edge subsets of an ordinary graph $G$ that contain no broken circuit is a homogeneous simplicial complex of top dimension $|V(G)| - 1$ and, moreover, the coefficients of the chromatic polynomial of $G$ are the simplex counts in each dimension of $\mathcal{B}(G)$.

Let $\mathfrak{F}(G)$ be the class consisting of all the edge subsets of a signed graph $G$ that contain no broken bond.

**Corollary 6.1.** Let $G$ be a non-trivial signed graph with $n$ vertices, $m$ edges and with a linear order $\prec$ on $E(G)$. Then

1). $\mathfrak{F}(G)$ is a homogeneous simplicial complex, i.e., every simplex is a subset of some simplex of top dimension $m - n$;

2). An edge set $F$ is a simplex of top dimension $m - n$ of $\mathfrak{F}(G)$ if and only if $E(G) \setminus F$ contains at least one edge from each broken bond of $G$ and each component $G - F$ contains no but exactly one unbalanced circuit;

3). For each $i \in \{0, 1, 2, \ldots, m - n\}$, the coefficient $a_i$ in $F_0(G, x)$ is the number of the $i$-dimensional simplexes in $\mathfrak{F}(G)$.

**Proof.** 1). It is obvious that $\mathfrak{F}(G)$ is a simplicial complex. We prove that $\mathfrak{F}(G)$ is homogeneous.

Let $F$ be a set of edges that contains no broken bond. If $|F| = m - n$ then we are done. We now assume that $|F| < m - n$, i.e., $|E(G - F)| > n$. In this case, it can be seen that there is a component $\omega$ in $G - F$ which contains at least two circuits $C$ and $C'$. By Proposition 5.3 one of these two circuits, say $C$, is unbalanced. So by the same argument as that in Corollary 5.4 we can find an edge $e$ in $C'$ such that $G - F - e$ still contains an edge from each broken bond. Replacing $F$ by $F \cup \{e\}$, the assertion follows by repeating this procedure, until $|F| = m - n$.

2) and 3) follows directly by Corollary 5.4. \qed

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