NOTES ON THE BANACH–NEČAS–BABUŠKA THEOREM AND KATO’S MINIMUM MODULUS OF OPERATORS

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ABSTRACT. This note was prepared for a lecture given at Kyoto University (RIMS Workshop: “The State of the Art in Numerical Analysis: Theory, Methods, and Applications”, November 8–10, 2017). That lecture described the variational analysis of the discontinuous Galerkin time-stepping method for parabolic equations based on an earlier paper by the author [24]. I also presented the Banach–Nečas–Babuška (BNB) Theorem or the Babuška–Lax–Milgram (BLM) Theorem as the key theorem of our analysis. For proof of the BNB theorem, it is useful to introduce the minimum modulus of operators by T. Kato. This note presents a review of the proofs of Closed Range Theorem and BNB Theorem following the idea of Kato. Moreover, I present an application to BNB theorem to parabolic equations. The well-posedness is proved by BNB theorem. This note is not an original research paper. It includes no new results. This is a revised manuscript and several incorrect descriptions in the original version are fixed.

0. Notation

All functions and function spaces in this note are real-valued.

Letting $X$ be a Banach space with the norm denoted as $\| \cdot \|_X$, then the dual space of $X$, say, the set of all linear bounded functional defined on $X$ is denoted by $X'$. For $\varphi \in X'$, we write $\varphi(x) = \langle \varphi, x \rangle_{X',X} = \langle x, \varphi \rangle_{X',X}$ and call it the duality pairing between $X'$ and $X$. The norm of $X'$ is defined as

$$\| \varphi \|_{X'} \overset{\text{def.}}{=} \sup_{x \in X} \frac{\langle \varphi, x \rangle_{X',X}}{\| x \|_X} \quad (\varphi \in X').$$

It is well known that $X'$ forms a Banach space equipped with the norm $\| \cdot \|_{X'}$.

Letting $Y$ be a (possibly another) Banach space, the set of all bounded bilinear forms on $X \times Y$ is designated as $\mathcal{B}(X,Y)$. That is, if $b \in \mathcal{B}(X,Y)$, then $b(\cdot, y)$ is a linear functional on $X$ for a fixed $y \in Y$, then $b(x, \cdot)$ is a linear functional on $Y$ for a fixed $x \in X$, and

$$\| b \| \overset{\text{def.}}{=} \sup_{x \in X, y \in Y} \frac{b(x, y)}{\| x \|_X \| y \|_Y} < \infty.$$

For a subset $M$ of $X$, we set

$$M^\perp = \{ f \in X' \mid (f, x)_{X',X} = 0 \ (\forall x \in M) \}. $$
which is called the annihilator of $M$. The space $M^\perp$ is a closed subspace of $X'$. We write
\[
\text{dist}_X(x, M) = \inf_{z \in M} \|x - z\|_X \quad (x \in X).
\]

Let $T$ be an operator from $X$ into $Y$ with its domain $\mathcal{D}(T) \subset X$. $\mathcal{N}(T) = \{x \in \mathcal{D}(T) \mid Tx = 0\}$ is the null space (kernel) of $T$ and $\mathcal{R}(T) = \{Tx \in Y \mid x \in \mathcal{D}(T)\}$ is the range (image) of $T$. $\mathcal{N}(T)$ is a closed subspace of $X$ and $\mathcal{R}(T)$ is a subspace of $Y$. The set of all bounded linear operators of $X \to Y$ with their domain $X$ is denoted by $\mathcal{L}(X,Y)$: if $T \in \mathcal{L}(X,Y)$, then $T$ is a linear operator of $X \to Y$ with $\mathcal{D}(T) = X$ and
\[
\|T\|_{X,Y} \overset{\text{def.}}{=} \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X} < \infty,
\]
which is called the operator norm of $T$.

1. Introduction

1.1. Banach–Něčas–Babuška Theorem. The present note presents specific examination of the following theorem called the Banach–Něčas–Babuška (BNB) Theorem or the Babuška–Lax–Milgram (BLM) Theorem.

**Theorem 1.** Letting $V$ be a Banach space and letting $W$ be a reflexive Banach space, then, for any $a \in B(V,W)$, the following (i)–(iii) are equivalent.

(i) For any $L \in W'$, there exists a unique $u \in V$ such that
\[
a(u, w) = \langle L, w \rangle_{W',W} \quad (\forall w \in W).
\]

(ii)
\[
\exists \beta > 0, \quad \inf_{v \in V} \sup_{w \in W} \frac{a(v, w)}{\|v\|_V \|w\|_W} = \beta;
\]

(iii)
\[
\exists \beta_1, \beta_2 > 0, \quad \inf_{v \in V} \sup_{w \in W} \frac{a(v, w)}{\|v\|_V \|w\|_W} = \beta_1, \quad \inf_{w \in W} \sup_{v \in V} \frac{a(v, w)}{\|v\|_V \|w\|_W} = \beta_2.
\]

**Remark 2.** If (3) is satisfied, then we have $\beta_1 = \beta_2$. Moreover, the value of $\beta$ in (2a) agrees with $\beta_1 = \beta_2$ in (3).

**Remark 3.** Condition (2a) is expressed equivalently as
\[
\exists \beta > 0, \quad \sup_{w \in W} \frac{a(v, w)}{\|w\|_W} \geq \beta \|v\|_V \quad (\forall v \in V).
\]

Usually, (2a) is called the Babška–Brezzi condition or the inf-sup condition.

**Remark 4.** Condition (2b) is expressed equivalently as
\[
\sup_{v \in V} |a(v, w)| > 0 \quad (\forall w \in W, w \neq 0).
\]

**Remark 5.** The solution $u \in V$ of (1) satisfies
\[
\|u\|_V \leq \frac{1}{\beta} \|L\|_{W'}
\]
in view of (2a) and Remark 3.
Remark 6. If $V$ and $W$ are finite-dimensional and $\dim V = \dim W$, then (2a) implies (2b). See [10, Proposition 2.21].

Theorem 1 might be understood as a generalization of the following fundamental result, called the Lax–Milgram Theorem.

Theorem 7. Letting $a \in B(V, V)$, where $V$ is a Hilbert space, then we assume that a positive constant $\alpha$ exists such that

$$a(v, v) \geq \alpha \|v\|_V^2. \quad (4)$$

Then, for any $L \in V'$, there exists a unique $u \in V$ such that

$$a(u, w) = \langle L, w \rangle_{V', V} \quad (\forall w \in V).$$

This theorem was presented in [15, theorem 2.1]; the special case was presented earlier in [26]. It is interesting that the main aim of [15] is to resolve higher order parabolic equations by Hille–Yosida’s semigroup theory. It is described in [15] that

The following theorem is a mild generalization of the Fréchet–Riesz Theorem on the representation of bounded linear functionals in Hilbert space. [page 168]

The condition (4) is usually called the coercivity condition. If $W = V$, then (4) implies (3); Theorem 7 is a corollary of Theorem 1.

Theorem 1 has a long history.

- In 1962, Nečas [17, Théorème 3.1] proved that part “(iii) $\Rightarrow$ (i)” for the Hilbert case (i.e., the case where both $V$ and $W$ are Hilbert spaces) as a simple generalization of the Lax–Milgram theorem. Nečas described that\(^1\)

  Considérant les espaces complexes et les opérateurs différentiels elliptiques, le théorème de P. D. Lax and A. Milgram (cf. p. ex. L. Nirenberg [20]) paraît être très utile pour la méthode variationnelle d’abord nous en signlons une généralisation facile. [page 318]

He also described that (see [17, Théorème 3.2]) (2a) and

$$\mathcal{R}(A) \text{ is dense in } W'$$

implies (i) for the Hilbert case, where $A$ denotes the associating operator with $a(\cdot, \cdot)$; see (5) for the definition. Later, in 1967, Nečas [18, Théorème 6-3.1] proved that (2a) and

$$\exists c > 0, \sup_{v \in V} \frac{a(v, w)}{\|v\|_V} \geq c \|w\|_Z \quad (w \in W)$$

implies (i) for the Hilbert case, where $Z$ denotes a Banach space such that $W \subset Z$ (algebraically and topologically). See also [19]. I infer that Nečas noticed the part “(ii) $\Rightarrow$ (i)”.\(^1\)

- In 1968, Hayden [12, Theorem 1] proved that

$$(2a) \text{ and } \mathcal{N}(A) = \mathcal{N}(A') = \{0\} \Leftrightarrow (i)$$

for the Banach case, where $A'$ denotes the dual operator of $A$; see (6) for the definition.

\(^1\)In quotations below, we have adapted reference numbers for the list of references of this paper.
• In 1971, Babuška [1, theorem 2.1] stated the part “(iii) ⇒ (i)” for the Hilbert case. Babuška described that

The proof is adapted from Nečas [17] and Nirenberg [20]. We present this proof because we shall use a portion of it for proof of the next theorem. [page 323]

Later, Babuška–Aziz [2, Theorem 5.2.1] stated in 1972 the part “(ii) ⇒ (i)” for the Hilbert case. It is described that

This theorem is a generalization of the well known Lax–Milgram theorem. The theorem might be generalized easily to the case where $H_1$ and $H_2$ are reflexive Banach spaces. The method proof is an adaptation from [17] and [20] (see also Necas [18], p.294). [page 116]

• In 1972, Simader [25, Theorem 5.4] presented the part “(iii) ⇒ (i)” for $V = W^{m,p}_0(\Omega)$ and $W = W^{m,q}_0(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq m \in \mathbb{Z}$. The proof could be applied to the general reflexive Banach spaces $V$ and $W$. It is noteworthy that [25] is essentially an English translation of his dissertation in 1968.

• In 1974, Brezzi [6, Corollary 0.1] proved the part “(i)⇔(iii)” for the Hilbert case. It is described that

the results contained in theorem 0.1 and in corollary 0.1 are of classical type and that they might not be new. For instance part I)⇒III) of corollary 0.1 was used by Babuška [1]. [page 132]

• In 1989, Roșca [23, Theorem 3] proved the part “(i)⇔(ii)” for the Banach case and called it the Babuška–Lax–Milgram theorem.

• In 2002, Ern and Guermond presented the part “(i)⇔(ii)” as Theorem of Nečas in their monograph [9, §3.2]. Later, they named the part “(i)⇔(ii)” the Banach–Nečas–Babuška Theorem in an expanded version of [9]; see [10, §2.1]. It is described in [10] that

The BNB Theorem plays a fundamental role in this book. Although it is by no means standard, we have adopted the terminology “BNB Theorem” because the result is presented in the form below was first stated by Nečas in 1962 [17] and popularized by Babuska in 1972 in the context of finite element methods [2, p. 112]. From a functional analysis perspective, this theorem is a rephrasing of two fundamental results by Banach: the Closed Range Theorem and the Open Mapping Theorem. [page 84]

• I could find no explicit reference to the part “(ii) ⇔ (iii)”. However, it is known among specialists.

As for the naming of Theorem 1, I follow conventions in [10].

1.2. Operator version of Theorem 1. To elucidate Theorem 1 more deeply, it is useful to reformulate it using operators. Below, supposing that $V$, $W$, and $a$ are those described in Theorem 1, unless otherwise stated explicitly, then we introduce

\footnote{However, I was unable to find where proof of the theorem was given in [20].}

\footnote{In the article “Babuška–Lax–Milgram theorem” in Encyclopedia of Mathematics (http://www.encyclopediaofmath.org/), the part “(i)⇔(ii)” of Theorem 1 is called the Babuska–Lax–Milgram Theorem. (This article was written by I. Roșca.)}
A ∈ ℒ(V, W′) as

\[ a(v, w) = \langle Av, w \rangle_{W′, W} \quad (v ∈ V, w ∈ W). \] (5)

Then, (i) of Theorem 1 is interpreted as “the operator \( A : V \to W′ \) is bijective”. The dual (adjoint) operator \( A′ : W \to V′ \) of \( A \) is defined as

\[ \langle Av, w \rangle_{W′, W} = \langle v, A′w \rangle_{V′, V} \quad (v ∈ V, w ∈ W). \] (6)

Then we have \( A′ ∈ ℒ(W, V′) \). We introduce

\[ \mu(A) = \inf_{v ∈ V} \frac{\|Av\|_{W′}}{\|v\|_V} \quad \text{and} \quad \mu(A′) = \inf_{w ∈ W} \frac{\|A′w\|_{V′}}{\|w\|_W}, \]

which we will call the minimum modulus of operators (see Definition 21).

Because

\[
\begin{align*}
\mu(A) &= \inf_{v ∈ V} \frac{\|Av\|_{W′}}{\|v\|_V} = \inf_{v ∈ V} \frac{1}{\|v\|_V} \sup_{w ∈ W} \frac{\langle Av, w \rangle_{W′, W}}{\|w\|_W} = \inf_{v ∈ V} \sup_{w ∈ W} \frac{a(v, w)}{\|v\|_V \|w\|_W}, \\
\mu(A′) &= \inf_{w ∈ W} \frac{\|A′w\|_{V′}}{\|w\|_W} = \inf_{w ∈ W} \frac{1}{\|w\|_W} \sup_{v ∈ V} \frac{\langle v, A′w \rangle_{V′, V}}{\|v\|_V} = \inf_{w ∈ W} \sup_{v ∈ V} \frac{a(v, w)}{\|v\|_V \|w\|_W},
\end{align*}
\]

we have

\[ (2a) \iff \mu(A) > 0, \quad (3) \iff \mu(A) = \mu(A′) > 0. \]

Moreover,

\[ (2b) \iff \mathcal{N}(A′) = \{0\}. \]

Consequently, Theorem 1 is equivalent to the following theorem in view of (5).

**Theorem 8.** Letting \( V \) be a Banach space and letting \( W \) be a reflexive Banach space, then for any \( A ∈ ℒ(V, W′) \), the following (i)–(iii) are equivalent:

(i) \( A \) is a bijective operator of \( V \to W′ \);
(ii) \( \mu(A) > 0 \) and \( \mathcal{N}(A′) = \{0\} \); and
(iii) \( \mu(A) = \mu(A′) > 0 \).

In those expressions, \( A′ ∈ ℒ(W, V′) \) denotes the dual operator of \( A \) defined as (6).

As explained clearly in [10, §A.2], the proof of Theorem 8 is an application of

- Open Mapping Theorem (or Closed Graph Theorem),
- Closed Range Theorem.

In fact, in view of Open Mapping Theorem (or Closed Graph Theorem), it can be shown that

\[ \mu(A) > 0 \iff \mathcal{N}(A) = \{0\} \text{ and } \mathcal{R}(A) \text{ is closed.} \]

Then, combining this with Closed Range Theorem, we can prove Theorem 8. Particularly if \( \mathcal{N}(A) = \{0\} \) and \( \mathcal{N}(A′) = \{0\} \), then \( A^{-1} \) and \( (A′)^{-1} \) exist and

\[ \mu(A) = \|A^{-1}\|_{W′, V}, \quad \mu(A′) = \|(A′)^{-1}\|_{V′, W}. \]

Therefore, \( \mu(A) = \mu(A′) \) is nothing but the standard fact of

\[ \|A^{-1}\|_{W′, V} = \|(A′)^{-1}\|_{V′, W}. \]
1.3. Remarks on Closed Range Theorem. In standard textbooks of numerical analysis, we use Closed Range Theorem without proof. The proof is left as a black box. However, in my opinion, it is worth knowing how to prove Closed Range Theorem for researchers of numerical analysis. I would like to offer an approach. I recommend introducing the following quantities:

\[ \gamma(A) = \inf_{v \in V} \frac{\|Av\|_{W'}}{\text{dist}(v, N(A))} \quad \text{and} \quad \gamma(A') = \inf_{w \in W} \frac{\|A'w\|_{V'}}{\text{dist}(w, N(A'))} \]

instead of \( \mu(A) \) and \( \mu(A') \). Following Kato [14], we call this quantity \( \gamma(A) \) the reduced minimum modulus of \( A \) (see Definition 21). Indeed, we can prove:

- \( \mathcal{R}(A) \) is closed \( \Leftrightarrow \gamma(A) > 0 \) (Theorem 24 below);
- \( \gamma(A) = \gamma(A') \) (Theorem 28 below).

Particularly \( \mathcal{R}(A) \) is closed if and only if \( \mathcal{R}(A') \) is closed. This is a part of Closed Range Theorem. These are classical results by Kato [13]. The main objective of [13] is to develop the perturbation theory for eigenvalue problems of linear operators. To accomplish this main objective, Kato studied \( \gamma(A) \) and gave the proof of Closed Range Theorem for closed (possibly unbounded) operators. (The original theorem by S. Banach was formulated and proved for bounded operators; see [3, Theorems X.8, X.9].)

Kato’s proof of [13] was later generalized in [14]. A simple explanation can be found in [5].

It is noteworthy that the introduction of \( \gamma(A) \) was not originally Kato’s idea. Many researchers have introduced the same quantity. However, Kato realized the importance of this quantity and developed his theory using it as a key tool.

R. G. Bartle stated in Mathematical Review that

*The author introduces a constant \( \gamma(A) \), called the lower bound of \( A \), which is defined to be the supremum of all numbers \( \gamma \geq 0 \) such that \( \|Ax\| \geq \gamma \|\tilde{x}\|, x \in D(A) \), where \( \tilde{x} \) is the coset \( x + N(A) \) and \( \|\tilde{x}\| \) denotes the usual factor space norm in \( X/N(A) \). Others have considered this constant before [see the reviewer’s note, Ann. Acad. Sci. Fenn. Ser. A. I. no. 257 (1958); MR0104172], but this reviewer is not aware of any previous systematic use of \( \gamma(A) \).

[MR0107819]*

I believe that Kato’s proof includes an idea full of suggestion for the study of numerical analysis and that it is worthy of study for researchers of numerical analysis.

1.4. Application of Theorem 1. Nečas originally established Theorem 1, the part “(iii)⇒(i)” to deduce the well-posedness (the unique existence of a solution with a priori estimate) of higher-order elliptic equations in weighted Sobolev spaces. However, Theorem 1 plays a crucial role in the theory of the finite element method. Pioneering work was done for error analysis of elliptic problems (see [1], [2]). Moreover, active applications for the mixed finite element method are well-known: see [7], [4] and [10] for systematic study. Another important application is the well-posedness of parabolic equations (see [10, §6] for example). Although this later application is apparently unfamiliar, it is actually useful for studying the discontinuous Galerkin time-stepping method, as reported recently in [24].
1.5. Purpose and contents. This note has a dual purpose. The first is to review Kato’s proof of Closed Range Theorem using $\gamma(A)$ and to state the proof of Theorems 1 and 8. The second is to present the proof of the well-posedness of parabolic equations using Theorem 1. To clarify the variational characteristics of the method of analysis, we consider abstract evolution equations of parabolic type, where the coefficient might depend on the time.

The contents of this note are the following:

0. Notations
1. Introduction
2. Preliminaries
3. Kato’s minimum modulus of operators
4. Proof of Theorems 1 and 8
5. Application to evolution equations of parabolic type
   A. Proof of “(22) $\Rightarrow$ (24)"
   B. Comments on the revised version

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2. Preliminaries

We recall two fundamental results, Closed Graph Theorem and Hahn–Banach Theorem together with their consequences. Throughout this section, $X$ and $Y$ are assumed to be Banach spaces.

Lemma 9 (Closed Graph Theorem). Let $T$ be a linear operator of $X \to Y$. If $X = D(T)$ and

$$D(T) \text{ is complete under the norm } \|x\|_{D(T)} = \|x\|_X + \|Tx\|_Y,$$

then we have $T \in \mathcal{L}(X,Y)$.

An operator $T$ satisfying (7) is called a closed operator and $\|x\|_{D(T)}$ is called the graph norm of $T$. Closed Graph Theorem is also described as “a closed linear operator from $X$ into $Y$ with $D(T) = X$ is bounded”. Because $X$ and $Y$ are Banach spaces, (7) is equivalent to

$$x_n \in D(T), \quad x_n \to x \in X \text{ in } X \quad (n \to \infty) \implies x \in D(T). \quad (8)$$

A bounded operator is a closed operator; (7) and (8) are satisfied for $T \in \mathcal{L}(X,Y)$. In fact, $\|x\|_X$ and $\|x\|_{D(T)}$ are equivalent norms of $X$, because $\|x\|_{D(T)} = \|x\|_X + \|Tx\|_Y \leq (1 + \|T\|_{X,Y})\|x\|_X$. Therefore, $X$ is complete under $\|x\|_{D(T)}$, which implies (7).

Let us consider a linear operator $T : X \to Y$ such that $\mathcal{N}(T) = \{0\}$. Then, the inverse operator $T^{-1} : \mathcal{R}(T) \to X$ can be defined. If $T^{-1}$ is bounded, then

$$y_n \in D(T^{-1}) = \mathcal{R}(T), \quad y_n \to y \in Y \text{ in } Y \quad (n \to \infty) \implies y \in \mathcal{R}(T), \quad (9)$$
as just mentioned above. In other words, $\mathcal{R}(T)$ is a closed set in $Y$ if $T^{-1}$ is bounded. On the other hand, if $\mathcal{R}(T)$ is closed, (9) is satisfied. Therefore, we can apply Closed Graph Theorem to conclude that $T^{-1} \in \mathcal{L}(\mathcal{R}(T), X)$. As a result, we obtain the following lemma.

**Lemma 10.** Let $T$ be a linear operator from $X$ into $Y$ such that $\mathcal{N}(T) = \{0\}$. Then, we have $T^{-1} \in \mathcal{L}(\mathcal{R}(T), X)$ if and only if $\mathcal{R}(T)$ is closed.

**Remark 11.** I presented Lemma 10 as a corollary of Closed Graph Theorem. However, S. Banach proved the following proposition (see [3, Theorem X.10]): if $\mathcal{R}(T)$ is closed, then there exists a positive constant $m > 0$ such that, for any $y \in \mathcal{R}(T)$, we can take $x \in X$ satisfying $y = Tx$ and $\|x\|_X \leq m\|y\|_Y$.

**Lemma 12** (Hahn–Banach Theorem). Let $E$ be a vector space and let $p$ be a functional on $E$ such that

$$p(\lambda x) = \lambda p(x) \quad (x \in E, \lambda > 0);$$

$$p(x + y) \leq p(x) + p(y) \quad (x, y \in E).$$

Suppose that $G$ is a subspace (linear subset) of $E$ and that $g$ is a functional on $G$ satisfying

$$g(x) \leq p(x) \quad (x \in G).$$

Then, there exists a functional $\tilde{g}$ on $E$, which is called the extension of $g$ into $E$, such that

$$\tilde{g}(x) = g(x) \quad (x \in G), \quad \tilde{g}(x) \leq p(x) \quad (x \in E).$$

We present some useful results.

**Lemma 13.** Let $M$ be a subspace of $X$. Then, every $g \in M'$ admits an extension $\tilde{g} \in X'$ such that $\|\tilde{g}\|_{X'} = \|g\|_{M'}$.

**Proof.** Apply Hahn–Banach Theorem with $p(x) = \|g\|_{C^*} \|x\|_X$.

**Lemma 14.** For a subspace $M$ of $X$, we have

$$\text{dist}_{X'}(f, M^\perp) = \sup_{x \in M} \frac{|\langle f, x \rangle_{X', X}|}{\|x\|_X} \quad (f \in X'). \quad (10)$$

**Proof.** Letting $f \in X'$, and introducing the restriction $f_M = f|_M : M \to \mathbb{R}$ of $f$ into $M$, we have $f_M \in M'$. Then, (10) is expressed as $\text{dist}_{X'}(f, M^\perp) = \|f_M\|_{M'}$.

By Lemma 13, an extension $g \in X'$ of $f_M$ exists such that $\|g\|_{X'} = \|f_M\|_{M'}$. Set $h = f - g \in X'$. Consequently, we have $h \in M^\perp$ because $\langle f, x \rangle_{X', X} = \langle g, x \rangle_{X', X}$ for $x \in M$, which implies that $\text{dist}_{X'}(f, M^\perp) \leq \|f - h\|_{X'} = \|g\|_{X'} = \|f_M\|_{M'}$.

On the other hand, for any $h \in M^\perp$, we have $|\langle f, x \rangle_{X', X}| = |\langle f - h, x \rangle_{X', X}| \leq \|f - h\|_{X'} \|x\|_X$ for $x \in M$. Therefore, $\|f_M\|_{M'} \leq \text{dist}_{X'}(f, M^\perp)$.

**Lemma 15.** Let $C$ be an open convex subset with $0 \in C$ of $X$. Supposing that $x_0 \in X$ and $x_0 \notin C$, then there exists a $\varphi \in X'$ such that $\langle \varphi, x_0 \rangle_{X', X} = 1$ and $\langle \varphi, x \rangle_{X', X} < 1$ for $x \in C$.

**Proof.** Setting $G = \{tx_0 \mid t \in \mathbb{R}\}$, we introduce a functional $g$ on $G$ as

$$g(tx_0) = t.$$

We recall that the gauge of $C$ is given as

$$p(x) = \inf \{\alpha > 0 \mid \alpha^{-1} x \in C\}$$
and that it satisfies the following:

\[ \exists M > 0, \; 0 \leq p(x) \leq M \|x\|_X \quad (x \in X); \quad (11a) \]

\[ C = \{ x \in E \mid p(x) < 1 \}; \]

\[ p(\lambda x) = \lambda p(x) \quad (x \in X, \lambda > 0); \quad (11b) \]

\[ p(x + y) \leq p(x) + p(y) \quad (x, y \in X). \quad (11d) \]

Then, it is apparent that

\[ g(x) \leq p(x) \quad (x \in G). \quad (12) \]

Indeed, it is trivial if \( x = tx_0 \) with \( t \leq 0 \). If \( x = tx_0 \) with \( t > 0 \), then \( \alpha^{-1}tx_0 \in C \) implies \( \alpha > t \) because \( t'x_0 \) with \( t' \geq 1 \) cannot belong to \( C \). Therefore, (12) follows.

According to Hahn–Banach Theorem, there exists a functional \( \varphi \) defined on \( X \) such that \( \varphi(x) = g(x) \) for \( x \in G \) and \( \varphi(x) \leq p(x) \) for \( x \in X \). Using (11a), \( \varphi(x) \leq p(x) \leq M \|x\|_X \) and \( -\varphi(x) = \varphi(-x) \leq p(-x) \leq M \|x\|_X \) for \( x \in X \). Consequently, we have \( \varphi(x) = 0 \) for \( x \in \mathcal{Y} \). Moreover, we have \( \varphi(x_0) = \langle \varphi, x_0 \rangle_{\mathcal{Y}, X} = 1 \) and \( \varphi(x) = \langle \varphi, x \rangle_{\mathcal{Y}, X} < 1 \) for \( x \in C \).

We recall the proof of (11) to emphasize that \( C \) must be open and convex.

**Proof of (11a).** Because \( C \) is an open set, an \( r > 0 \) exists such that \( B_X(0, r) = \{ x \in X \mid \|x\|_X < r \} \subset C \). Then, \( p(x) \leq \inf \{ \alpha > 0 \mid \alpha^{-1}x \in B_X(0, r) \} = r^{-1} \|x\|_X \). \( \square \)

**Proof of (11b).** First, let \( x \in C \). Small \( \varepsilon > 0 \) exists such that \( (1 + \varepsilon)x \in C \) because \( C \) is open. Therefore, \( p(x) \leq \frac{1}{1 + \varepsilon} < 1 \). Conversely, let \( p(x) < 1 \) for \( x \in X \). Then, an \( \alpha \in (0, 1) \) exists such that \( \alpha^{-1}x \in C \). Therefore, \( x = \alpha(\alpha^{-1}x) + (1 - \alpha) \cdot 0 \in C \) because \( C \) is convex. \( \square \)

**Proof of (11c).** \( p(\lambda x) = \inf \{ \beta \lambda > 0 \mid \beta^{-1}x \in C \} = \lambda p(x) \). \( \square \)

**Proof of (11d).** We apply (11b) and (11c). Letting \( \varepsilon > 0 \) be arbitrary, then we have \( x/(p(x) + \varepsilon) \in C \) and \( y/(p(y) + \varepsilon) \in C \) because \( p(x/(p(x) + \varepsilon)) = p(x)/(p(x) + \varepsilon) < 1 \). Therefore, for any \( t \in [0, 1] \),

\[ \frac{tx}{p(x) + \varepsilon} + \frac{(1 - t)y}{p(y) + \varepsilon} \in C. \]

Choosing \( t = (p(x) + \varepsilon)/(p(x) + p(y) + 2\varepsilon) \), we obtain \( (x + y)/(p(x) + p(y) + 2\varepsilon) \in C \). This result implies that \( 1 > p((x+y)/(p(x) + p(y) + 2\varepsilon)) = p(x+y)/(p(x) + p(y) + 2\varepsilon) \), which means that \( p(x+y) < p(x) + p(y) + \varepsilon \). Letting \( \varepsilon \downarrow 0 \), we infer \( p(x+y) \leq p(x) + p(y) \).

**Lemma 16.** Let \( M \) be a closed convex subset with \( 0 \in M \) of \( X \). Supposing that \( x_0 \in X \) and \( x_0 \not\in M \), then there exists a \( \varphi \in \mathcal{Y}' \) such that \( \langle \varphi, x_0 \rangle_{\mathcal{Y}, X} > \langle \varphi, x \rangle_{\mathcal{Y}, X} \) for \( x \in M \).

**Proof.** Because \( M \) is closed, we have \( d = \text{dist}_X(x_0, M) > 0 \). Apply Lemma 15 to \( C = \{ x \in X \mid \text{dist}_X(x, M) < d/2 \} \) which is an open convex subset not containing \( x_0 \). \( \square \)

**Lemma 17.** Letting \( M \) be a subspace of \( X \) and supposing that \( x_0 \in X \) and \( x_0 \not\in M \) with \( d = \text{dist}_X(x_0, M) > 0 \), then there exists a \( \varphi \in \mathcal{Y}' \) such that \( \langle \varphi, x_0 \rangle_{\mathcal{Y}, X} = 1 \), \( \langle \varphi, x \rangle_{\mathcal{Y}, X} = 0 \) for \( x \in M \) and \( \|\varphi\|_{\mathcal{Y}'} \leq 1/d \).

**Remark 18.** We actually have \( \|\varphi\|_{\mathcal{Y}'} = 1/d \).
Proof of Lemma 17. We introduce \( M_0 = \{ tx_0 + y \mid t \in \mathbb{R}, \ y \in M \} \). This \( M_0 \) is a subspace of \( X \). Writing \( x \in M_0 \) as \( x = tx_0 + y \) with \( t \in \mathbb{R} \) and \( y \in M \), we have

\[
|t| \leq \frac{1}{d} \|x\|_X.
\]  

(13)

In fact, \( \|t^{-1}x\|_X = \|x_0 + t^{-1}y\|_X \geq d \) if \( t \neq 0 \) whereas (13) is trivial if \( t = 0 \). At this stage, we introduce a functional \( g \) on \( M_0 \) by

\[
g(x) = t \quad (x = tx_0 + y \in M_0).\]

By (13), we have \( g \in M_0^* \) and \( \|g\|_{M_0^*} \leq 1/d \). This \( g \) can be extended to \( X \) preserving the bound. The extension is denoted by \( \phi \). Then, it is apparent that \( \phi(x_0) = \langle \phi, x_0 \rangle_{X',X} = \langle \phi, 1 \rangle_{X',X} = 1 \), \( \phi(x) = \langle \phi, x \rangle_{X',X} = 0 \) for \( x = 0 \cdot x_0 + x \in M \), and \( \|\phi\|_{X'} \leq 1/d \).

Remark 19. Lemma 14 is taken from [14, Lemma IV.2.8]. Lemma 15 could be found in [5, Lemma 1.3]. Lemma 17 is taken from [14, Theorem III.1.22].

3. Kato’s Minimum Modulus of Operators

Letting \( V \) and \( W \) be Banach spaces as in §1, and noting particularly that \( W \) is reflexive, supposing that we are given \( A \in \mathcal{L}(V,W') \), then the dual operator \( A' \in \mathcal{L}(W',V) \) of \( A \) is given as \( (Av, w)'_{W',V} = \langle v, A'w \rangle_{V',V} \) for \( v \in V \), \( w \in W \).

If \( A \) is considered as an operator from \( V \) to \( W \), the reflexivity of \( W \) is not necessary in the following discussion. See Remark 34.

The following lemma is well known.

Lemma 20. We have

\[
\mathcal{N}(A') = \mathcal{R}(A)^\perp;
\]  

(14a)

\[
\mathcal{N}(A) = \mathcal{R}(A')^\perp;
\]  

(14b)

\[
\mathcal{R}(A') \subset \mathcal{N}(A)^\perp;
\]  

(14c)

\[
\mathcal{R}(A) \subset \mathcal{N}(A')^\perp.
\]  

(14d)

\( (W \text{ needs not to be reflexive.}) \)

Proof of (14a). Let \( w \in \mathcal{N}(A') \subset W \). For any \( v \in V \), we have \( (Av, w)'_{W',V} = \langle v, A'w \rangle_{V',V} = 0 \), which gives that \( w \in \mathcal{R}(A)^\perp \). Consequently \( \mathcal{N}(A') \subset \mathcal{R}(A)^\perp \).

The proof of \( \mathcal{N}(A') \supseteq \mathcal{R}(A)^\perp \) can be shown similarly.

\( \square \)

Proof of (14b). In fact, it is exactly the same as the previous proof.

\( \square \)

Proof of (14c). Letting \( f \in \mathcal{R}(A') \subset V' \), where \( f \) is expressed as \( f = A'w \) with \( w \in W \), then for any \( v \in \mathcal{N}(A) \), we have \( \langle v, f \rangle_{V',V} = \langle v, A'w \rangle_{V',V} = \langle Av, w \rangle_{W',W} = 0 \). Therefore, \( f \in \mathcal{N}(A)^\perp \) and \( \mathcal{R}(A') \subset \mathcal{N}(A)^\perp \).

\( \square \)

Proof of (14d). It is exactly the same as the previous proof.

\( \square \)

Relations \( \mathcal{R}(A') = \mathcal{N}(A)^\perp \) and \( \mathcal{R}(A) = \mathcal{N}(A')^\perp \) are not always true because, for example, \( \mathcal{N}(A)^\perp \) is always closed but \( \mathcal{R}(A') \) need not be closed. To derive the opposite inclusions to (14c) and (14d), we require some deeper consideration. We will use the quotient (factor) space

\[
\tilde{V} = V/\mathcal{N}(A) = \{ \tilde{v} = v - \mathcal{N}(A) \mid v \in V \}.
\]
which is a Banach space equipped with the norm

\[ \| \tilde{v} \|_{\tilde{V}} = \inf_{g \in N(A)} \| v - g \|_V = \text{dist}_V(v, N(A)). \]  

By consideration of this notion for \( v - 0 \in v - N(A) \), we have

\[ \| \tilde{v} \|_{\tilde{V}} \leq \| v \|_V \quad (v \in V). \]  

We introduce a linear operator \( \tilde{A} : \tilde{V} \to W' \) by setting

\[ \tilde{A} \tilde{v} = Av \quad (\tilde{v} = v - N(A) \in \tilde{V}). \]

The operator \( \tilde{A} \) is bounded and

\[ \mathcal{R}(\tilde{A}) = \mathcal{R}(A), \quad N(\tilde{A}) = \{ \tilde{0} \}. \]  

Therefore, the inverse \( \tilde{A}^{-1} \) exists, where \( D(\tilde{A}^{-1}) = \mathcal{R}(\tilde{A}) \). In view of Closed Graph Theorem (see Lemma 10 and Remark 11), \( \tilde{A}^{-1} \) is bounded if and only if \( \mathcal{R}(\tilde{A}) \) is closed. That is, we have

\[ \mathcal{R}(\tilde{A}) \text{ is closed} \iff \| \tilde{A}^{-1}f \|_{\tilde{V}} \leq \sup_{f \in \mathcal{R}(\tilde{A})} \| f \|_{W'} < \infty. \]  

Motivated by the observation above, we can present the following definition.

**Definition 21.** The minimum modulus of an operator \( T \) from a Banach space \( X \) to another Banach space \( Y \) is defined as

\[ \mu(T) = \inf_{x \in X} \frac{\| Tx \|_Y}{\| x \|_X}. \]

The reduced minimum modulus of \( T \) is defined as

\[ \gamma(T) = \inf_{x \in X} \frac{\| Tx \|_Y}{\| x \|_{\tilde{X}}} = \inf_{x \in X} \frac{\| Tx \|_Y}{\text{dist}_X(x, N(T))}, \]

where \( \tilde{X} \) denotes the quotient space \( \tilde{X} = X/N(T) \).

**Remark 22.** It is noteworthy that \( \gamma(T) = \infty \) if and only if \( Tx = 0 \) for all \( x \in X \). It is apparent that

\[ N(T) = \{ 0 \} \Rightarrow \gamma(T) = \mu(T). \]  

Moreover,

\[ \mu(T) > 0 \Rightarrow N(T) = \{ 0 \}. \]  

**Remark 23.** The quantity \( \gamma(T) \) was introduced into \[13, \S 3.2\] and called the lower-bound of \( T \). Actually, \( \gamma(T) \) was called the reduced minimum modulus of \( M \) in \[14, \S IV.5\]; it is described in \[14\] that the naming follows \[11\], where \( \mu(T) \) was defined.

It is apparent that

\[ \gamma(A) = \inf_{v \in V} \frac{\| Av \|_{W'}}{\| v \|_{\tilde{V}}} = \inf_{v \in V} \frac{\| Av \|_{W'}}{\text{dist}_V(v, N(A))}; \]

\[ \gamma(\tilde{A}) = \gamma(A); \]

\[ \gamma(\tilde{A}) = \| \tilde{A}^{-1} \|_{\mathcal{R}(\tilde{A}), \tilde{V}}. \]

Putting (16), (17), and (20) together, we have the following theorem.
Theorem 24 ([13, Lemma 322]). $\mathcal{R}(A)$ is closed if and only if $\gamma(A) > 0$. (W needs not to be reflexive.)

Remark 25. Theorem 24 might be understood as a “quantitative version” of the result of Banach described in Remark 11.

Remark 26. Theorem 24 could be found in [21, Theorems 5.17.3, 5.18.2].

Example 27. We give an example of $A$ whose range $\mathcal{R}(A)$ is not a closed set. Let $V = W = L^2(I)$ with $I = (0, 1)$. We introduce $A \in \mathcal{L}(V, W')$ by

$$\langle Av, w \rangle_{W', W} = \int_0^1 tv(t)w(t) \, dt.$$  

(Verify that $A$ is actually a bounded linear operator of $V \to W'$.) We consider $f \in W'$ defined as

$$\langle f, w \rangle_{W', W} = \int_0^1 w(t) \, dt.$$  

Then, we have $f \not\in \mathcal{R}(A)$. Indeed, if there is a $u_0 \in V$ such that $Au_0 = f$, this $u_0$ must satisfy $1 - tu_0 = 0$ a.e. $t \in I$. The “candidate” is given as $u_0 = 1/t$; however, $u_0 = 1/t$ cannot belong to $V$. Next, for $\varepsilon > 0$, we consider $f_\varepsilon \in W'$ and $u_\varepsilon \in V$ defined as

$$\langle f_\varepsilon, w \rangle_{W', W} = \int_\varepsilon^1 w(t) \, dt \quad \text{and} \quad u_\varepsilon = \begin{cases} 0 & (0 < t < \varepsilon) \\ 1/t & (\varepsilon \leq t < 1). \end{cases}$$  

Then, we have $Au_\varepsilon = f_\varepsilon$ and, hence, $f_\varepsilon \in \mathcal{R}(A)$. Moreover, we have $f \in \overline{\mathcal{R}(A)}$, because $\|f_\varepsilon - f\|_{W'} \to 0$ as $\varepsilon \to \infty$. Those imply that $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}$. Therefore, $\mathcal{R}(A)$ is not closed. (In the similar way, we can prove that $W' = \overline{\mathcal{R}(A)}$.)

On the other hand, because $\mathcal{N}(A) = \{0\}$, we estimate as

$$\gamma(A) \leq \lim_{\varepsilon \to 0} \frac{\|Au_\varepsilon\|_W}{\|u_\varepsilon\|_V} = 0,$$

which implies $\gamma(A) = 0$.

The following theorem plays a key role below.

Theorem 28 ([13, Lemma 334]). We have

$$\gamma(A) = \gamma(A').$$

Particularly $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A')$ is closed.

Proof. For abbreviation, we write $\gamma = \gamma(A)$ and $\gamma' = \gamma(A')$.

Step 1. We prove that $\gamma' \geq \gamma$. If $\gamma = \infty$, then we have $Av = 0$ for all $v \in V$. Therefore, $0 = \langle Av, w \rangle_{W', W} = \langle v, A'w \rangle_{V', V}$ for all $v \in V$ and $w \in W$, which implies $A'w = 0$ for all $w \in W$. Consequently, $\gamma' = \infty$. Therefore, we might assume that $0 < \gamma < \infty$, because $\gamma' \geq \gamma$ might be readily apparent if $\gamma = 0$. Letting $w \in W$, then $\mathcal{R}(A)$ is closed by theorem 24. Therefore, we can apply Lemmas 20 and 14 (for $X = W'$, $M = \mathcal{R}(A)$) to obtain

$$\text{dist}_W(w, \mathcal{N}(A')) = \sup_{f \in \mathcal{R}(A)} \frac{\|f, w\|_{W', W}}{\|f\|_{W'}} \quad (w \in W).$$
This, together with (15), implies that
\[ \|\tilde{w}\|_W = \sup_{f \in \mathcal{R}(A)} \frac{|\langle f, w \rangle_{W', W}|}{\|f\|_{W'}}. \tag{21} \]

Therefore, for a sufficiently small \( \varepsilon > 0 \), \( f \in \mathcal{R}(A) \) exists such that \( |\langle f, w \rangle_{W', W}| \geq (1 - \varepsilon)\|f\|_{W'}\|\tilde{w}\|_W \). \( f \) admits the representation \( f = Av \in \mathcal{R}(A) \) with \( v \in V \). Therefore, we deduce
\[ |\langle Av, w \rangle_{W', W}| \geq (1 - \varepsilon)\|Av\|_{W'}\|\tilde{w}\|_W \]
\[ \geq (1 - \varepsilon)\gamma \|\tilde{v}\|_V \|\tilde{w}\|_W. \]

Using \( |\langle Av, w \rangle_{W', W}| = |\langle v, A'w \rangle_{V', V}| \leq \|v\|_V \|A'w\|_{V'} \), we have
\[ \|v\|_V \|A'w\|_{V'} \geq (1 - \varepsilon)\gamma \|\tilde{v}\|_V \|\tilde{w}\|_W. \]

These inequalities remain valid if \( v \) is replaced by \( v - g \) for any \( g \in \mathcal{N}(A) \). Consequently, we have
\[ \|A'w\|_{V'} \inf_{g \in \mathcal{N}(A)} \|v - g\|_V \geq (1 - \varepsilon)\gamma \|\tilde{v}\|_V \|\tilde{w}\|_W. \]

Therefore, we obtain
\[ \gamma' \geq (1 - \varepsilon)\gamma. \]

Letting \( \varepsilon \downarrow 0 \), we deduce \( \gamma' \geq \gamma \).

Step 2. (a) We prove the opposite inequality \( \gamma' \leq \gamma \). Because the inequality is trivial if \( \gamma' = 0 \), we assume that \( \gamma' > 0 \). In general, we write \( B_X(r) \) to express the open ball in a Banach space \( X \) with center \( 0 \) and radius \( r > 0 \); \( B_X(r) = \{ x \in X \mid \|x\|_X < r \} \).

The closure of \( AB_V(1) = \{ Av \in W' \mid v \in B_V(1) \} \) in \( W' \) is denoted as \( K = AB_V(1) \), which is a convex closed subset in \( W' \) with \( 0 \in K \).

(b) We show that
\[ B_{W'}(\gamma') \subset K = AB_V(1). \tag{22} \]

To this purpose, we prove that
\[ f_0 \in \mathcal{R}(A), \ f_0 \notin K \quad \Rightarrow \quad \|f_0\|_{W'} \geq \gamma'. \tag{23} \]

In view of Lemma 16, there exists an \( \eta \in (W')' \) such that
\[ \langle \eta, f \rangle_{(W')', W'} < \langle \eta, f_0 \rangle_{(W')', W'} \quad (f \in K). \]

Because \( W \) is reflexive, there exists a \( w \in W \) such that
\[ \langle f, w \rangle_{W', W} < \langle f_0, w \rangle_{W', W} \quad (f \in K). \]

By considering \( -f \) instead of \( f \), we have
\[ \|f, w\|_{W', W} < \|f_0, w\|_{W', W} < \|f_0, w\|_{W', W} \quad (f \in K). \]

Letting \( 0 \neq v \in V \) and \( 0 < \varepsilon < 1 \) and setting \( \hat{v} = (1 - \varepsilon)v/\|v\|_V \in B_V(1) \), then by substituting \( f = Av \), we obtain
\[ (1 - \varepsilon)\frac{|\langle Av, w \rangle_{W', W}|}{\|v\|_V} = (1 - \varepsilon)\frac{|\langle v, A'w \rangle_{V', V}|}{\|v\|_V} \leq \|f_0, w\|_{W', W}. \]

Consequently,
\[ (1 - \varepsilon) \sup_{v \in V} \frac{|\langle v, A'w \rangle_{V', V}|}{\|v\|_V} \leq \|f_0, w\|_{W', W}. \]
By letting \( \varepsilon \downarrow 0 \),
\[
\|A'w\|_{W'} \leq \langle f_0, w \rangle_{W', W}.
\]
We can apply (21) to obtain
\[
\|A'w\|_{W'} \leq \|\tilde{w}\|_{\tilde{W}} \|f_0\|_{W'}.
\]
We know that \( \|A'w\|_{V'} \geq \gamma' \|\tilde{w}\|_{\tilde{W}} \) for any \( w \in W \). Combining these, we have \( \|f_0\|_{W'} \geq \gamma' \), which completes the proof of (23).

(c) The inclusion (22) implies that
\[
B_{\tilde{W}'}(\gamma') \subset A B_V(1).
\]
This is verified by a standard argument; we will mention the detail in Appendix A.

At this stage, letting \( 0 \neq v \in V \) and letting \( 0 < \varepsilon < 1 \), we set \( v^* = (1 - \varepsilon)\gamma'v / \|Av\|_{W'} \). Then, because \( \|Av^*\|_{W'} = (1 - \varepsilon)\gamma' < \gamma' \), we have \( Av^* \in A B_X(1) \). This implies that there exists a \( v^# \in B_V(1) \) satisfying \( Av^# = Av^* \) and \( v^# = v^* - g \) for any \( g \in N(A) \). We have
\[
1 > \|v^#\|_V = \frac{(1 - \varepsilon)\gamma'}{\|Av\|_{W'}} \|v - \alpha g\|_V,
\]
where \( \alpha = \|Av\|_{W'}/(1 - \varepsilon)\gamma' \). This gives that
\[
\|Av\|_{W'} > (1 - \varepsilon)\gamma'\|v - \alpha g\|_V \geq (1 - \varepsilon)\gamma'\|\tilde{v}\|_V.
\]
Because \( \varepsilon \) is arbitrary, we infer \( \gamma'\|\tilde{v}\|_V \leq \|Av\|_{W'} \), which implies that \( \gamma \geq \gamma' \). This completes the proof of Theorem 28. \( \square \)

Using this theorem, we can prove the following results.

**Theorem 29.** \( R(A) \supset N(A')^+ \) if \( R(A') \) is closed.

**Proof.** Letting \( f \in N(A')^+ \), then we prove \( f \in R(A) \) by presenting a contradiction: assume \( f \notin R(A) \). Because \( R(A') \) is closed, \( R(A) \) is also closed in view of Theorem 28. Therefore, we have \( d = \text{dist}_{W'}(f, R(A)) > 0 \) and can apply Lemma 17. Consequently, there exists an \( \eta \in (W')^\perp \) such that
\[
\langle \eta, f \rangle_{(W')^\perp, W'} = 1, \quad \langle \eta, g \rangle_{(W')^\perp, W'} = 0 \quad (g \in R(A)).
\]
Because \( W \) is reflexive, there exists a \( w \in W \) such that
\[
\langle f, w \rangle_{W', W} = 1, \quad \langle g, w \rangle_{W', W} = 0 \quad (g \in R(A)).
\]
By the second identity of (25), we have \( 0 = \langle Av, w \rangle_{W', W} = \langle v, A'w \rangle_{V', V} \) for any \( v \in V \), which implies that \( A'w = 0 \). Therefore \( w \in N(A) \). Because \( f \in N(A')^+ \), \( \langle f, w \rangle_{W', W} = 0 \). However, this contradicts to the first equality of (25). \( \square \)

**Theorem 30** ([13, Lemma 335]). \( R(A') \supset N(A)^+ \) if \( R(A) \) is closed.

**Proof.** Letting \( f \in N(A)^+ \), then we introduce a linear functional \( \phi_f \) on \( R = R(A) \) by setting \( \phi_f(Av) = \langle f, v \rangle_{V', V} \) for \( v \in V \), which is possible because \( \langle f, v \rangle_{V', V} = 0 \) for \( v \in N(A) \). The functional \( \phi_f \) is bounded. In fact, we have
\[
|\phi_f(Av)| = |\langle f, v \rangle_{V', V}| \leq \|f\|_{V'} \|v\|_V
\]
and \( v \) might be replaced by \( v - g \) with any \( g \in N(A) \). Consequently,
\[
|\phi_f(Av)| \leq \|f\|_{V'} \|v\|_V \leq \|f\|_{V'} \frac{1}{\gamma(A)} \|Av\|_{W'}.
\]
which implies that \( \|\hat{\phi}_f\|_{R'} = \sup_{\psi \in R} |\hat{\phi}_f(\psi)|/\|\psi\|_{W'} \leq \gamma(A)^{-1}\|f\|_{V'} \). By Hahn–Banach theorem, there exists a \( \hat{\phi}_f \in (W')' \) such that
\[
\langle \hat{\phi}_f, \psi \rangle_{W',W'} = \hat{\phi}_f(\psi) \quad (\psi \in R), \quad \|\hat{\phi}_f\|_{(W')'} \leq \gamma(A)^{-1}\|f\|_{V'}.
\]
Because \( W \) is reflexive, there exists a \( w \in W \) such that
\[
\langle \hat{\phi}_f, \psi \rangle_{W',W'} = \langle \psi, w \rangle_{W',W} \quad (\forall \psi \in W').
\]
Summing up, we deduce
\[
\langle Av, w \rangle_{W',W} = \hat{\phi}_f(Av) = \langle f, v \rangle_{V',V} \quad (v \in V).
\]
This relation implies the expression \( f = A'w : f \in \mathcal{R}(A') \). □

Now, we can prove the following well-known result called Closed Range Theorem.

**Corollary 31.** The following (i)–(iv) are equivalent:

(i) \( \mathcal{R}(A) \) is closed;
(ii) \( \mathcal{R}(A') \) is closed;
(iii) \( \mathcal{R}(A) = \mathcal{N}(A')^\perp \);
(iv) \( \mathcal{R}(A') = \mathcal{N}(A)^\perp \).

**Proof.** (i) ⇔ (ii): We have already verified this part. See Theorems 24 and 28.
(iv)⇒(ii): If \( \mathcal{R}(A') = \mathcal{N}(A)^\perp \), then \( \mathcal{R}(A') \) is closed because \( \mathcal{N}(A)^\perp \) is closed.
(ii)⇒(iv): Assuming that \( \mathcal{R}(A') \) is closed, then we can apply Lemma 20 and Theorem 30 to deduce \( \mathcal{R}(A') = \mathcal{N}(A)^\perp \).
(i)⇔(iii): It is exactly the same as that of the part “(ii)⇔(iv)”. □

**Remark 32.** In the discussion presented above, the boundedness of \( A \) plays no essential role. All the theorems and their proofs remain valid for a closed linear operator \( A \) if the dual operator \( A' \) is well-defined.

**Remark 33.** The original version of Closed Range Theorem could be found in [3, Theorems X.8, X.9].

**Remark 34.** In this section, we considered \( A \in \mathcal{L}(V,W') \) with the intention of applying results to the proof of Theorems 1 and 8. However, if we consider a linear densely defined closed operator \( T \) from a Banach space \( X \) to a Banach space \( Y \), we can prove the following results in exactly the same way. In particular, \( Y \) needs not to be reflexive.

- \( \mathcal{R}(T) \) is closed if and only if \( \gamma(T) > 0 \).
- \( \gamma(T) = \gamma(T') \).
- The following (i)–(iv) are equivalent:
  (i) \( \mathcal{R}(T) \) is closed;
  (ii) \( \mathcal{R}(T') \) is closed;
  (iii) \( \mathcal{R}(T) = \mathcal{N}(T')^\perp \);
  (iv) \( \mathcal{R}(T') = \mathcal{N}(T)^\perp \).

4. **Proof of Theorems 1 and 8**

It suffices to state the proof of Theorem 8 because Theorems 1 and 8 are equivalent through the relation (5).

**Proof of Theorem 8, the part “(i) ⇔ (iii)”**

\( A: \) bijective
\[ N(A) = \{0\}, \mathcal{R}(A) = W' \]
\[ \Rightarrow \mathcal{N}(A) = \{0\}, (A) = \{0\} \quad \text{(by Corollary 31)} \]
\[ \Rightarrow \mu(A) = \gamma(A), \gamma(A) > 0, \gamma(A') = \mu(A') \quad \text{(by (18))} \]
\[ \Rightarrow \mu(A) = \mu(A') > 0 \quad \text{(by Theorem 28)} \]
\[ \Rightarrow N(A) = \{0\}, N(A') = \{0\}, N(A') = \{0\} \quad \text{(by (18), (19))} \]
\[ \Rightarrow \mu(A) = \mu(A') > 0 \quad \text{(by Theorems 24, 28 and Corollary 31)} \]
\[ \Rightarrow A: \text{bijective.} \quad \square \]

**Proof of Theorem 8, the part “(ii) \iff (iii)”**.

\[ \mu(A) = \mu(A') > 0 \]
\[ \Rightarrow \mu(A) > 0, \mathcal{N}(A') = \{0\} \quad \text{(by (19))} \]
\[ \Rightarrow \mu(A) > 0, N(A) = \{0\}, \gamma(A') = \mu(A') \quad \text{(by (18), (19))} \]
\[ \Rightarrow \mu(A) > 0, \gamma(A) = \mu(A), \gamma(A') = \mu(A') \quad \text{(by (18))} \]
\[ \Rightarrow \mu(A) = \mu(A') > 0. \quad \square \]

5. **Application to Evolution Equations of Parabolic Type**

In this section, we present an application of Theorem 1 to evolution equations of parabolic type.

### 5.1. Example

We start with a concrete example. Letting \( J = (0, T) \) with \( T > 0 \), and supposing that \( \Omega \) is a Lipschitz domain in \( \mathbb{R}^d \), \( d \geq 1 \), we consider the initial-boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot \nu(x,t) \nabla u - \nabla \cdot (b(x,t)u) - c(x,t)u + F(x,t) \quad (x \in \Omega, \ t \in J), \quad (26a) \\
u(x,0) &= u_0(x) \quad (x \in \Omega), \quad (26c)
\end{align*}
\]

where \( \nu, b, c, F \) and \( u_0 \) are given functions.

Several frameworks and methods are used to establish the well-posedness (the unique existence of a solution with a priori estimate) of (26):

- Semigroup method ([22] for example);
- Variational method: Galerkin method based on compactness theorems ([8] and [27] for example);
- Variational method: Operator method ([16] for example).

As described, we present another variational method. To this end, we first derive a weak formulation of (26). For the time being, those \( \nu, b, c, F \) and \( u_0 \) are assumed to be suitably smooth as well as a solution \( u \). Set

\[ \mathcal{D} = \{ v = \tilde{v}|_{J \times \Omega} \mid \tilde{v} \in C^\infty(\mathbb{R} \times \mathbb{R}^d), \ \text{supp} \tilde{v} \subset J \times \Omega \}. \]
Multiplying both sides of (26a) by \( v \in \mathcal{D} \), integrating it in \( x \in \Omega \) and \( t \in J \) and using the boundary condition (26b), we obtain

\[
\int_J \int_\Omega (\partial_t u) v \, dx dt + \int_J \int_\Omega \left[ \nu(x,t) \nabla u \cdot \nabla v - b(x,t) u \cdot \nabla v + c(x,t) uv \right] \, dx dt = \int_J \int_\Omega F v \, dx dt. \tag{27}
\]

We introduce

\[
H = L^2(\Omega), \quad (\cdot, \cdot) = (\cdot, \cdot)_H = (\cdot, \cdot)_{L^2(\Omega)}, \quad \| \cdot \|_H = \| \cdot \|_{L^2(\Omega)},
\]

\[
V = H^1_0(\Omega), \quad (\cdot, \cdot)_V = (\nabla \cdot, \nabla \cdot)_{L^2(\Omega)}, \quad \| \cdot \|_V = \| \nabla \cdot \|_{L^2(\Omega)}
\]

and

\[
\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V', V} \quad \text{is the duality pairing between } V \text{ and } V'.
\]

Moreover, set

\[
a(t; w,v) = \int_\Omega \left[ \nu(x,t) \nabla w \cdot \nabla v - b(x,t) w \cdot \nabla v + c(x,t) w v \right] \, dx,
\]

\[
\langle f, v \rangle = \int_\Omega F v \, dx
\]

for \( w, v \in V \).

We make the following assumptions:

\[
\exists \nu_0 > 0, \quad \nu(x,t) \geq \nu_0 > 0 \quad (x \in \Omega, \ t \in J), \quad \nu \in L^\infty(\Omega); \tag{28a}
\]

\[
b \in L^\infty(\Omega \times J)^d, \quad c \in L^\infty(\Omega \times J); \tag{28b}
\]

\[
\exists c_0 > 0, \quad \frac{1}{2} \nabla \cdot b(x,t) + c(x,t) \geq c_0 > 0 \quad (x \in \Omega, \ t \in J). \tag{28c}
\]

Using (28a), (28b), (28c) and Poincaré inequality

\[
\| v \|_V \leq C_P \| v \|_{H^1(\Omega)} \quad (v \in V),
\]

one can prove that there exist positive constants \( M \) and \( \alpha \) which depend only on \( \nu_0, c_0, \| \nu \|_{L^\infty(\Omega)}, \| b \|_{L^\infty(\Omega \times J)^d}, \| c \|_{L^\infty(\Omega)} \) and \( C_P \) such that

\[
|a(t; w,v)| \leq M \| w \|_V \| v \|_V \quad (w, v \in V, \ t \in J),
\]

\[
a(t; v,v) \geq \alpha \| v \|_V^2 \quad (v \in V, \ t \in J).
\]

Therefore, for a.e. \( t \in J \), we can introduce a linear operator \( A(t) \) from \( V \) into \( V' \) as

\[
\langle A(t)w, v \rangle = a(t; w,v) \quad (w, v \in V, \ t \in J)
\]

satisfying

\[
\langle A(t)w, v \rangle \leq M \| w \|_V \| v \|_V \quad (w, v \in V, \ t \in J), \tag{29a}
\]

\[
\langle A(t)v, v \rangle \geq \alpha \| v \|_V^2 \quad (v \in V, \ t \in J). \tag{29b}
\]

As a result, (27) is expressed as

\[
\int_J \langle \partial_t u, v \rangle \, dt + \int_J \langle A(t)u, v \rangle \, dt = \int_J \langle f, v \rangle \, dt \quad (v \in \mathcal{D}).
\]
However, the initial condition (26c) is interpreted as
\[(u(0), v) = (u_0, v) \quad (v \in H).\]

At this stage, we introduce the following function spaces:
\[ \mathcal{X} = L^2(J; V) \cap H^1(J; V'), \quad \|u\|_\mathcal{X}^2 = \|u\|_{L^2(J; V)}^2 + \|u'\|_{L^2(J; V')}^2, \]
\[ \mathcal{Y}_1 = L^2(J; V), \quad \|v_1\|_{\mathcal{Y}_1}^2 = \|v_1\|_{L^2(J; V)}^2, \]
\[ \mathcal{Y} = \mathcal{Y}_1 \times H, \quad \|v\|_{\mathcal{Y}}^2 = \|v_1\|_{L^2(J; V)}^2 + \|v_2\|_H^2, \]
where
\[ L^2(J; V) = \{ v : J \to V \mid \|v\|_{L^2(J; V)} < \infty \}, \quad \|v\|_{L^2(J; V)} = \int_J \|v\|_V^2 \, dt, \]
\[ H^1(J; V') = \{ v : J \to V' \mid \|v\|_{H^1(J; V')} < \infty \}, \quad \|v\|_{H^1(J; V')} = \int_J (\|v\|_V^2 + \|v'\|_{V'}^2) \, dt. \]

It is noteworthy that \( \mathcal{D} \) is dense in \( \mathcal{Y}_1 \).

We can state the weak formulation of (26) as follows. Assuming
\[ f \in L^2(J; V'), \quad u_0 \in H, \tag{30} \]
we find \( u \in \mathcal{X} \) such that
\[
\int_J \left[ \langle u', v_1 \rangle + \langle A(t)u, v_1 \rangle \right] \, dt + (u(0), v_2)
= \langle f, v_1 \rangle \, dt + (u_0, v_2) \quad (\forall v = (v_1, v_2) \in \mathcal{Y}), \tag{31}
\]
where \( u' \) denotes \( du(t)/dt \). Alternatively, (31) is expressed formally as
\[ u' + A(t)u = f(t), \quad t \in J; \quad u(0) = u_0. \]

Remark 35. In (30), \( f \in L^2(J; V') \) is guaranteed by assuming \( F \in L^2(J; H) \). Moreover, \( u(0) \in H \) is well-defined; see Lemma 36.

5.2. Problem. We consider more general settings. Letting \( H \) and \( V \) be (real) Hilbert spaces such that \( V \subset H \) is dense with the continuous injection, then the inner product and norms are denoted as \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H, \quad \| \cdot \| = \| \cdot \|_H \) and \( \| \cdot \|_V \). The topological dual spaces \( H \) and \( V \) are denoted, respectively, by \( H' \) and \( V' \). As usual, we identify \( H \) with \( H' \) and consider the triple \( V \subset H \subset V' \).

Moreover, \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V', V} \) denotes duality pairing between \( V' \) and \( V \). Consider function spaces \( \mathcal{X}, \mathcal{Y}_1 \) and \( \mathcal{Y} \) as presented above.

Supposing that, for a.e. \( t \in J \), we are given a linear operator \( A(t) \) of \( V \to V' \) satisfying (29), where \( M \) and \( \alpha \) are positive constants independent of \( t \in J \). Without loss of generality, we assume that \( \alpha \leq 1 \leq M \). Given (30), we consider the abstract evolution equation of parabolic type (31).

The following result is called the trace theorem (see [8, theorem XVIII-1], [27, theorem 25.2], [28, theorem 41.15]).

Lemma 36. There exists a positive constant \( C_{\mathcal{Y}, T} \) depending only on \( T \) such that
\[ \max_{t \in J} \|v(t)\|_H \leq C_{\mathcal{Y}, T} \|v\|_{\mathcal{X}} \quad (v \in \mathcal{X}). \]
In other words, the space $\mathcal{X}$ is embedded continuously in the set of $H$-valued continuous functions on $\mathcal{J}$. Particularly, $u(0) \in H$ in (31) is well-defined.

The main result of this section is the following result, which is often called the Lions Theorem.

**Theorem 37.** Given (30), problem (31) admits a unique solution $u \in \mathcal{X}$ that satisfies

$$\|u\|_{\mathcal{X}} \leq C \left( \|f\|_{L^2(J;V')} + \|u_0\|_H \right),$$

where $C$ denotes a positive constant depending only on $M$ and $\alpha$.

To prove this theorem, it suffices to verify the following:

$$\exists \mu > 0, \quad \sup_{u \in \mathcal{X}, v \in \mathcal{Y}} \frac{B(u, v)}{\|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} = \mu; \quad (33a)$$

$$\exists \beta > 0, \quad \inf \sup_{u \in \mathcal{X}, v \in \mathcal{Y}} \frac{B(u, v)}{\|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} = \beta; \quad (33b)$$

$$v \in \mathcal{Y}, \quad (\forall u \in \mathcal{X}, B(u, v) = 0) \implies (v = 0). \quad (33c)$$

Subsequently, we can apply Theorem 1 to conclude a unique existence of the solution $u$. Moreover, the a priori estimate (32) is a readily obtainable consequence of (33b).

5.3. **Proof of Theorem 37.** We use the following auxiliary results. By virtue of (29), $A(t)$ is invertible for a.e. $t \in J$. Moreover, we have the following.

**Lemma 38.** (i) $\|A(t)^{-1}g\|_{V'} \leq \frac{1}{\alpha} \|g\|_{V'}$ for all $g \in V'$ and a.e. $t \in J$.

(ii) $\langle g, A(t)^{-1}g \rangle \geq \frac{\alpha}{M^2} \|g\|_{V'}$ for all $g \in V'$ and a.e. $t \in J$.

**Proof.** (i) For $g \in V$, set $v = A(t)^{-1}g \in V$. Then, $\langle g, A(t)^{-1}g \rangle = \langle A(t)v, v \rangle \geq \alpha \|v\|_{V'}^2$. However, $\|g\|_{V'} = \|A(t)^{-1}g\|_{V'} \leq \|g\|_{V'}$. Combining these, we have $\|A(t)^{-1}g\|_{V'} = \|v\| \leq (1/\alpha) \|g\|_{V'}$. (ii) (29a) implies $\|A(t)v\|_{V'} \leq M \|v\|_{V'}$ for $v \in V$. Now, set $v = A(t)^{-1}g \in V$ for $g \in V'$. Then, $\|g\|_{V'} = \sup \|g(w)\|_{V'} = \sup \|A(t)v\|_{V'} \leq M \|v\|_{V'}$. Combining this with $\langle g, A(t)^{-1}g \rangle \geq \alpha \|v\|_{V'}^2$, we obtain the desired inequality.

We introduce an alternate norm of $\mathcal{X}$ as

$$\|w\|_{\mathcal{X}}^2 = \int_J \|w'(t) + A(t)w(t)\|^2_{V'} \, dt + \|w(0)\|^2$$

for $w \in \mathcal{X}$.

**Lemma 39.** Two norms $\| \cdot \|_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}^{\alpha}$ are equivalent in $\mathcal{X}$. In particular,

$$\alpha \|w\|_{\mathcal{X}} \leq \|w\|_{\mathcal{X}} \leq C_{\max} \|w\|_{\mathcal{X}}$$

for $w \in \mathcal{X}$, where $C_{\max}^2 = 1 + M^2 + C_{_{\text{Tr,}T}}^2$.
Proof. For \( w \in \mathcal{X} \), we calculate as
\[
\|w\|_{\mathcal{X}}^2 = \int_J \left[ \sup_{v \in \mathcal{V}} \frac{\langle w' + Aw, v \rangle}{\|v\|_{\mathcal{V}}} \right]^2 dt + \|w(0)\|^2
\]
\[
= \int_J \|w'\|_{\mathcal{V}}^2 + \sup_{v \in \mathcal{V}} \left( \frac{\langle Aw, v \rangle}{\|v\|_{\mathcal{V}}} \right)^2 dt + \|w(0)\|^2
\]
\[
\leq \int_J \|w'\|_{\mathcal{V}}^2 + M \|w\|_{\mathcal{V}}^2 dt + C_{T,T}^2 \|w\|_{\mathcal{X}}^2
\]
\[
\leq (1 + M^2 + C_{T,T}^2) \|w\|_{\mathcal{X}}^2.
\]
and
\[
\|w\|_{\mathcal{X}}^2 \geq \int_J \left[ \|w'\|_{\mathcal{V}} + \frac{\langle Aw, w \rangle}{\|w\|_{\mathcal{V}}} \right]^2 dt \geq \int_J \|w'\|_{\mathcal{V}}^2 + \alpha \|w\|_{\mathcal{V}}^2 dt \geq \alpha^2 \|w\|_{\mathcal{X}}^2.
\]

The following lemma can be found in [8, Theorem 2, §XVIII-1] and [28, Theorem 41.15].

Lemma 40. For \( w, v \in \mathcal{X} \), we have
\[
\int_J \langle w', v \rangle dt = (w(T), v(T)) - (w(0), v(0)) - \int_J \langle v', w \rangle dt \quad (34a)
\]
and
\[
\int_J \langle w', w \rangle dt = \frac{1}{2} (\|w(T)\|^2 - \|w(0)\|^2) \geq -\frac{1}{2} \|w(0)\|^2. \quad (34b)
\]

Now we can state the following proof.

Proof of (33a). We apply the Cauchy–Schwarz inequality and Lemma 39 to obtain
\[
B(u, v) = \int_J \left[ \langle u', v_1 \rangle + \langle Au, v_1 \rangle \right] dt + \langle u(0), v_2 \rangle
\]
\[
= \int_J \langle u' + Au, v_1 \rangle dt + \langle u(0), v_2 \rangle
\]
\[
\leq \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}} \leq C_{\max} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}
\]
for \( u \in \mathcal{X} \) and \( v = (v_1, v_2) \in \mathcal{Y} \).

Proof of (33b). Let \( u \in \mathcal{X} \) be arbitrary. Set \( v_1 = A(t)^{-1}u' + u \in L^2(J; \mathcal{V}) \), \( v_2 = u(0) \in H \) and \( v = (v_1, v_2) \in \mathcal{Y} \). Using Lemma 38, we have
\[
\|v\|_{\mathcal{Y}}^2 = \int_J \|A^{-1}u' + u\|_{\mathcal{V}}^2 dt + \|u(0)\|^2
\]
\[
= \int_J \|A^{-1}(u' + Au)\|_{\mathcal{V}}^2 dt + \|u(0)\|^2
\]
\[
\leq \frac{1}{\alpha^2} \int_J \|u' + Au\|_{\mathcal{V}}^2 dt + \|u(0)\|^2 \leq \frac{1}{\alpha^2} \|u\|_{\mathcal{X}}^2.
\]
Moreover,
\[ B(u, v) = \int_J \langle u' + Au, A^{-1}u' + u \rangle \, dt + (u(0), u(0)) \]
\[ = \int_J \langle u' + Au, A^{-1}(u' + Au) \rangle \, dt + \|u(0)\|^2 \]
\[ \geq \frac{\alpha}{M^2} \int_J \|u' + Au\|^2 \, dt + \|u(0)\|^2 \]
\[ \geq \frac{\alpha}{M^2} \|u\|_{X}^2 \geq \frac{\alpha^2}{M^2} \|u\|_{X} \|v\|_{Y}. \]

Consequently, using Lemma 39, we obtain
\[ B(u, v) \geq \frac{\alpha^3}{M^2} \|u\|_X \|v\|_Y, \]
which implies (33b).

Proof of (33c). Assume that \( v = (v_1, v_2) \in Y \) satisfies \( B(u, v) = 0 \) for all \( u \in X \).
That is, we assume that
\[ \int_J [(u', v_1) + \langle Au, v_1 \rangle] \, dt + (u(0), v_2) = 0 \quad (\forall u \in X). \]  
(35)

For any \( \varepsilon > 0 \), we take \( u^* \in X \) such that \( u(0) = v_2 \) and \( u(t) = 0 \) for \( t \geq \varepsilon \).
Substituting \( u = u^* \) for (35), we have
\[ \int_0^\varepsilon [(u'^*, v_1) + \langle Au^*, v_1 \rangle] \, dt + \|v_2\|^2 = 0. \]

Because \( \varepsilon \) is arbitrarily chosen, we infer that \( v_2 = 0 \). Moreover, we have \( v_1 \in H^1(J; V') \). In fact, letting \( u = \tilde{u}\phi \in X \) with \( \tilde{u} \in V \) and \( \phi \in C_0^\infty(J; \mathbb{R}) \), we have
\[ \int_J \langle \phi' \tilde{u}, v_1 \rangle \, dt = -\int_J \langle A\tilde{u}, v_1 \rangle \, dt. \]

This result implies that
\[ \left\langle \int_0^T v_1 \phi' \, dt, \tilde{u} \right\rangle = \left\langle -\int_0^T A'v_1\phi \, dt, \tilde{u} \right\rangle. \]

Consequently, we deduce \( v_1' \in L^2(J; V') \) and \( v_1' = A(t)v_1 \).

Therefore, we can apply (34a) to obtain
\[ \int_J [\langle v_1', \psi \rangle + \langle A'v_1, \psi \rangle] \, dt = 0 \quad (\forall \psi \in C_0^\infty(J; V)). \]

Because \( C_0^\infty(J; V) \) is dense in \( L^2(J; V) \), this gives
\[ \int_J [\langle v_1', w \rangle + \langle A'v_1, w \rangle] \, dt = 0 \quad (\forall w \in L^2(J; V)). \]

Letting \( \tilde{w} \in V \) arbitrarily and substituting \( w = t\tilde{w} \) for (36), we have
\[ \int_J [\langle v_1', t\tilde{w} \rangle + \langle A'v_1, t\tilde{w} \rangle] \, dt = 0. \]

Again we apply (34a) to obtain
\[ -(T\tilde{w}, v_1(T)) + \int_J \langle (t\tilde{w})', v_1 \rangle + \int_J \langle A(t)(t\tilde{w}), v_1 \rangle \, dt = 0. \]
Choosing \( \hat{w} = v_1(T) \) and using (35), we obtain \( v_1(T) = 0 \).

At this stage, substituting \( w = v_1 \) for (36) and using (29b) and (34b), then we have
\[
-\frac{1}{2} \|v_1(T)\|^2 + \frac{1}{2} \|v_1(0)\|^2 + \alpha \int_0^J \|v_1\|^2 d t \leq 0.
\]
This result implies that \( v_1 = 0 \), which completes the proof. \( \Box \)

**Remark 41.** The case \( u_0 = 0 \) is described explicitly in [10].

**Appendix A. Proof of “(22) \( \Rightarrow \) (24)”**

We prove a more general lemma described below.

**Lemma 42.** Let \( T \) be a linear closed operator of a Banach space \( X \) to a (possibly another) Banach space \( Y \). Then,
\[
B_Y(r) \subset T \overline{B_X(1)} \text{ with } r > 0 \tag{37}
\]
implies that
\[
B_Y(r) \subset T B_X(1) \tag{38}
\]

Recall that \( B_Y(r) = \{ y \in Y \mid \| y \|_Y < r \} \) and \( T \overline{B_X(1)} \) denotes the closure of \( T B_X(1) = \{ T x \in Y \mid x \in B_X(1) \} \) in \( Y \). To show the lemma, we apply a standard argument usually used to prove Open Mapping Theorem or Closed Graph Theorem.

**Proof.** Assume that (37) is satisfied. Let \( \sigma > 0 \) be arbitrary. For the time being, we admit that
\[
B_Y(r) \subset T B_X(1 + \sigma). \tag{39}
\]
Then, for any \( 0 < r' < r \), choosing \( \sigma = r/r' - 1 > 0 \), we have
\[
B_Y(r') = \frac{r'}{r} B_Y(r) \subset \frac{r'}{r} T B_X(1 + \sigma) = T B_X(1).
\]
The relation (38) is a readily obtainable consequence of this relation.

We now verify that (39) is true; we will show that, for any \( y \in B_Y(r) \), there exists an \( x \in B_X(1 + \sigma) \) satisfying \( T x = y \).

As just remarked above, (37) gives
\[
B_Y(\lambda r) \subset T \overline{B_X(\lambda)} \tag{40}
\]
for any \( \lambda > 0 \).

Set \( \varepsilon = \sigma/(2 + \sigma) < 1 \). According to (37), there is a \( y_0 \in T B_X(1) \) satisfying \( \| y - y_0 \|_Y < \varepsilon r \). That is, there is a \( \xi_0 \in B_X(1) \) satisfying
\[
\| y - T \xi_0 \|_Y < \varepsilon r.
\]
Then, we apply (40) with \( \lambda = \varepsilon \). Because \( y - T \xi_0 \in B_Y(\varepsilon r) \), there is a \( \xi_1 \in B_X(\varepsilon) \) satisfying
\[
\| y - T \xi_0 - T \xi_1 \|_X < \varepsilon^2 r.
\]
Proceeding in this way, we can construct a sequence \( \{ \xi_n \}_{n \geq 0} \) in \( X \) with the properties
\[
\| y - T \xi_0 - T \xi_1 - \cdots - T \xi_n \|_Y < \varepsilon^{n+1} r, \quad \| \xi_n \|_X < \varepsilon^n.
\]
If we set $x_n = \xi_0 + \xi_1 + \cdots + \xi_n$, we have
\[
\|x_{n+m} - x_n\|_X \leq \sum_{j=n+1}^{n+m} \|\xi_j\|_X \leq \sum_{j=n+1}^{n+m} \varepsilon^j \leq \frac{\varepsilon^{n+1}}{1 - \varepsilon} \to 0 \quad (n \to \infty).
\]
Therefore, there exists an $x \in X$ satisfying $x_n \to x$ in $X$ as $n \to \infty$. Moreover, we have $\|y - Tx_n\|_Y < \varepsilon^{n+1}r \to 0$ as $n \to \infty$. This implies that $Tx = y$ because $T$ is closed. Finally,
\[
\|x\|_X \leq \sum_{n=0}^{\infty} \|\xi_n\|_X \leq \sum_{n=0}^{\infty} \varepsilon^n = \frac{1}{1 - \varepsilon} = 1 + \frac{\sigma}{2} < 1 + \sigma;
\]
therefore, we have $x \in B_X(1 + \sigma)$. This completes the proof of (39).
\[
\Box
\]

APPENDIX B. COMMENTS ON THE REVISED VERSION

(1) Open Mapping Theorem recalled in the original version (November 5, 2017) has been removed.
(2) Lemmas 16 and 42 have been added; they are used in the proof of Theorem 28.
(3) Proof of Theorem 28 has been corrected. Consequently, the theorems and their proofs in §3 remain valid for a closed linear operator $A$ if the dual operator $A'$ is well-defined. See Remark 32.
(4) Remark 34 has been added.
(5) Proof of (33b) has been modified. I believe that it is a new proof.

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