Discrete L’Hospital’s rule

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1 Introduction

The aim of this paper is to formulate discrete analog of L’Hospital’s rule and describe
some of its applications. While the usual L’Hospital’s rule is taught to all undergraduate
students studying calculus, its discrete analog apparently was not in the literature. Since
the L’Hospital’s rule proved to be very useful in many applications, one may think that its
discrete analog will also be useful. We start by stating the usual, well-known L’Hospital’s
rule, so that the reader could see the similarities in the formulation of this rule and its
discrete analog. Then we formulate this discrete analog. In Section 2 we prove the
discrete analog and illustrate it by some examples.

After this note has been written, the author learned that in [1], p.67, there is a result
of O.Stolz, which is the same as Theorem 2 below. Our proof is slightly different from
the proof in [1]. The application, given in our Example 2 is of interest in the theory of
the dynamical systems method ([2]).

A version of the usual L’Hospital’s rule is the following theorem.

**Theorem 1.** Assume that:

1) the functions $F$ and $G$ are continuously differentiable on the interval $I = (a, a+h)$,
where $h > 0$ and $a$ are real numbers, $f := F'$, $g := G'$, and

$$
\lim F = \lim G = \infty, \quad \lim := \lim_{x \to a, x > a}.
\tag{1.1}
$$

2)

$$
\lim \frac{f}{g} = L, \quad g(x) \neq 0 \quad \forall x \in I.
\tag{1.2}
$$

2000 Math subject classification: 26A24, 26D15

Key words: L’Hospital rule, inequalities
Then there exists the limit:
\[ \lim_{n \to \infty} \frac{F}{G} = L. \tag{1.3} \]

The proof of Theorem 1 can be found in any calculus text and does not need a reference.

Let us now formulate the discrete analog of the above theorem.

**Theorem 2.** Let \( f_j > 0 \) and \( g_j > 0 \) be sequences of numbers, \( F_n := \sum_{j=1}^{n} f_j \), \( G_n := \sum_{j=1}^{n} g_j \).

Assume that:
\[ \lim_{n \to \infty} F_n = \lim_{n \to \infty} G_n = \infty, \tag{1.4} \]
and
\[ \lim_{n \to \infty} \frac{f_n}{g_n} = L. \tag{1.5} \]

Then
\[ \lim_{n \to \infty} \frac{F_n}{G_n} = L. \tag{1.6} \]

The similarity of Theorems 1 and 2 is obvious.

**Remark:** One can write equation (1.5) as
\[ \lim_{n \to \infty} \frac{F_n - F_{n-1}}{G_n - G_{n-1}} = L. \tag{1.7} \]
Thus, the role of the derivative of \( F \) is played by the difference \( F_n - F_{n-1} \).

## 2 Proofs

Fix an arbitrary small \( \varepsilon > 0 \). Using assumption (1.5), find \( M := M(\varepsilon) \), such that
\[ L - \varepsilon < \frac{f_j}{g_j} < L + \varepsilon, \quad \forall j > M. \tag{2.1} \]

Denote \( F_{n,M} := \sum_{j=M}^{n} f_j \), and define \( G_{n,M} \) similarly. Using assumption (1.4), find \( N := N(\varepsilon) \), such that
\[ \frac{F_M}{F_{n,M}} < \varepsilon, \quad \frac{G_M}{G_{n,M}} < \varepsilon, \quad \forall n > N. \tag{2.2} \]

Now one gets:
\[ \frac{F_n}{G_n} = \frac{F_M + F_{n,M}}{G_M + G_{n,M}} = \frac{F_{n,M}}{G_{n,M}} \frac{1 + \frac{F_M}{F_{n,M}}}{1 + \frac{G_M}{G_{n,M}}}, \tag{2.3} \]
and
\[ 1 + \varepsilon_1 := \frac{1 - \varepsilon}{1 + \varepsilon} < \frac{1 + \frac{F_M}{F_{n,M}}}{1 + \frac{G_M}{G_{n,M}}} < \frac{1 + \varepsilon}{1 - \varepsilon} := 1 + \varepsilon_2, \tag{2.4} \]
where $\varepsilon_1 = O(\varepsilon)$ and $\varepsilon_2 = O(\varepsilon)$, as $\varepsilon \to 0$. Using assumption (1.5), one gets

$$L - \varepsilon \leq \min_{j \geq M} \frac{f_j}{g_j} \leq \frac{F_{nM}}{G_{nM}} \leq \max_{j \geq M} \frac{f_j}{g_j} \leq L + \varepsilon. \quad (2.5)$$

Since $\varepsilon > 0$ is arbitrarily small, equation (1.6) follows from relations (2.2)-(2.5). Theorem 2 is proved. \qed

Consider examples of applications of Theorem 2.

**Example 1.** By Theorem 2, one has

$$\lim_{n \to \infty} \frac{\sum_{j=1}^{n} j^{m}}{\sum_{j=1}^{n} j^{p}} = \lim_{n \to \infty} \frac{n^{m}/(1 + n^{m+1})}{n^{p}/(1 + n^{p+1})} = 1. \quad (2.6)$$

Assume that

$$0 < a_n < 1, \quad \lim_{n \to \infty} \frac{b_{n-1}}{a_n} = 0, \quad \sum_{n=1}^{\infty} a_n = \infty. \quad (2.7)$$

Note that assumptions (2.7) imply $\lim_{n \to \infty} b_n = 0$.

Using assumptions (2.7) one can apply Theorem 2 and conclude that

$$\lim_{n \to \infty} \sum_{k=1}^{n-1} b_k \prod_{j=k+1}^{n} (1 - a_j) = 0. \quad (2.8)$$

This result implies, that $\lim_{n \to \infty} g_n = 0$ under the assumptions (2.7), where $g_n$ is a sequence solving inequality (2.6).

Let us discuss in detail the application of Theorem 2 in this example.

From (2.6) by induction one gets:

$$g_{n+1} \leq (1 - a_n) g_n + b_n, \quad n = 1, 2, 3, \ldots. \quad (2.9)$$

Assumption (2.7) implies that

$$\lim_{n \to \infty} b_n = 0 \quad \text{and} \quad \lim_{n \to \infty} g_1 \prod_{j=1}^{n} (1 - a_j) = 0. \quad (2.10)$$

Let us write the term $J_k := \sum_{j=1}^{n-1} b_k \prod_{j=k+1}^{n} (1 - a_j)$ in the form: $J_k = \frac{\sum_{k=1}^{n-1} b_k \prod_{j=k+1}^{n} (1 - a_j)}{\prod_{j=1}^{n} (1 - a_j) - 1}$. We want to apply Theorem 2 in order to prove that

$$\lim_{n \to \infty} J_k = 0. \quad (2.11)$$
The denominator in $J_n$ tends to infinity. If the numerator in $J_n$ is bounded, then (2.11) follows. If this numerator tends to infinity, then one has assumption (1.4) satisfied. To check assumption (1.5) with $L = 0$, one calculates the limit:

$$
\lim_{n \to \infty} \frac{b_{n-1} \prod_{j=1}^{n-1} (1 - a_j)^{-1}}{\prod_{j=1}^{n} (1 - a_j)^{-1}[1 - (1 - a_n)]} = \lim_{n \to \infty} \frac{b_{n-1}(1 - a_n)}{a_n} = 0.
$$

At the last step assumption (2.7) was used. So, Theorem 2 yields the desired conclusion (2.11). The discussion of Example 2 is completed.

\[\square\]

References

[1] Fikhtengolts, G., Course of differential and integral calculus, vol.1, Fizmatgiz, Moscow, 1962.

[2] Ramm, A. G., Dynamical systems method for ill-posed equations with monotone operators, Comm. in Nonlinear Sci. and Numer. Simulation, 10, N2, (2005).