**TRiPoD**
(Temporal Relationalism incorporating Principles of Dynamics)

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**Abstract**

Temporal Relationalism is that there is no time for the universe as a whole at the primary level [1]. This can be implemented by a formulation of the following kind.

i) It is not to include any appended times – such as Newtonian time – or appended time-like variables.

ii) Time is not to be smuggled into the formulation in the guise of a label either.

To formulate this, we need to know what does remain available, which starts with the configurations $Q^A$. The collection of all possible values of the configuration of a physical system form the configuration space $\mathbf{q}$; $k := \text{dim}(\mathbf{q})$.

Then one implementation of ii) is for a label to be present but physically meaningless because it can be changed for any other (monotonically related) label without changing the physical content of the theory. E.g. at the level of the action, this is to be **Manifestly Reparametrization Invariant**:

$$S = 2 \int d\lambda \sqrt{T W}, \quad \text{for} \quad T := ||Q||_{M^2}/2 := M_{AA} \dot{Q}^A \dot{Q}^A/2.$$  \hfill (1)

Therein, $T$ is the the kinetic energy, $M$ the configuration space metric and $W = W(Q)$ is the potential factor.

**Arena 1)** Jacobi’s action principle [2, 3], which can be considered to be for temporally-relational spatially-absolute Mechanics. Here $T = ||\dot{q}||_{m^2}/2 = m_i \dot{q}^i \dot{q}^i/2$ so the configuration space metric $m$ is just the ‘mass matrix’ with components $m_{ij} \delta_{ij}$, and $W = E - V(q)$ for $V(q)$ the potential energy and $E$ is the total energy of the model universe.

**Arena 2)** Scaled 1-d RPM with translations trivially removed [4] is another case of Jacobi’s action principle. Here $T = ||\dot{\rho}||^2/2$ and $W = E - V(\rho)$ for $\rho$ the mass-weighted relative Jacobi coordinates with components $\rho^{iA}$ [5, 6, 4]. This has the advantage over 1) of being a relational whole-universe model.

**Arena 3)** Full GR can indeed also be formulated in this manner. This involves an increasing number of departures of formalism from the familiar Arnowitt–Deser–Misner (ADM) action [7], though all of these formalisms remain equivalent. For instance, the ADM lapse of GR is an appended time-like variable. It can however be removed in the Baierlein–Sharp–Wheeler (BSW) action [8, 9], and then the ADM shift can be replaced with various distinct but equivalent auxiliaries to match the rest of each relational formalism [10, 11, 12, 13]. Because this example strays outside of the current Article’s account of finite models, we do not elaborate on its details, though here the configurations are spatial 3-metrics and the configuration space metric is the inverse of the DeWitt supermetric [14], which is indefinite.

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In this Article, I often use particular sans serif indices for particular Principles of Dynamics (PoD) objects, then using primed versions when more than one index of a given kind is required. I use bold font for coordinate-free version in configuration space and other spaces familiar from the PoD, and underlines for spatial vectors. Particle number indices $I, J$ run from 1 to $N$, relative inter-particle cluster indices $A = 1$ to $N-1$, spatial dimension indices $i$ run from 1 to $d$. $m_I$ are the particles’ masses.
The implementation of Temporal Relationalism can be further upgraded as follows. It is a further conceptual advance to formulate one’s action and subsequent equations without use of any meaningless label at all. I.e. a Manifestly Parametrization Irrelevant formulation in terms of change $dQ^A$ rather than a Manifestly Reparametrization Invariant one in terms of a label-time velocity $dQ^A/d\lambda$.

However, it is better still to formulate this directly: without even mentioning any meaningless label or parameter, by use of how the preceding implementation is dual to a Configuration Space Geometry formulation. Indeed, Jacobi’s action principle is often conceived of geometrically [2, 3] terms rather than in its dual aspect as a timeless formulation. This formulation’s action now involves not kinetic energy $T$ but a kinetic arc element $ds$:

$$S = \sqrt{2} \int ds\sqrt{W}, \quad ds := ||dQ||_M = \sqrt{M_{AB}(Q)dQ^AdQ^B}. \quad (2)$$

The above two forms of Temporal Relationalism implementing actions are clearly equivalent to each other. See e.g. [2, 3, 8, 10, 11, 4, 13] for how the product-type actions (1, 2) are furthermore equivalent to the more familiar difference-type actions of Euler–Lagrange for Mechanics and of ADM for GR.

The main way in which actions implementing ii) work is that they necessarily imply primary constraints. I.e. relations between the momenta that are obtained without use of the equations of motion; see Sec 2.7 for more context. For (1), this implication is via the following well-known argument of Dirac [16]. An action that is reparametrization-invariant is homogeneous of degree 1 in the velocities. Thus the $k$ conjugate momenta are (by the above definition) homogeneous of degree 0 in the velocities. Therefore they are functions of at most $k - 1$ ratios of the velocities. So there must be at least one relation between the momenta themselves (i.e. without any use made of the equations of motion). But this is the definition of a primary constraint. In this manner, Temporal Relationalism acts as a constraint provider.

The constraint it provides has a purely quadratic form induced from [9] that of the action (2)$^2$

$$\text{CHRONOS} := N^A_{AB}P_AP_B/2 - W(Q) = 0 \quad (3)$$

for $N$ the inverse of the configuration space metric $M$. For instance, for Mechanics it is of the form $\varepsilon := ||p||_n + V(q) = E$, for $n = m^{-1}$, with components $\delta_{ij}/m_1$. This usually occurs in Physics under the name and guise of an energy constraint, though this is not the interpretation it is afforded in the relational approach. On the other hand, for GR, it is the well-known and similarly quadratic Hamiltonian constraint $\kappa$ – now containing the DeWitt supermetric itself – that arises at this stage as a primary constraint.

Reconciling timelessness for the universe as a whole at the primary level with time being apparent none the less in the parts of the universe that we observe, proceeds via Mach’s Time Principle [17]: that ‘time is to be abstracted from change’. I.e. a particular type of emergent time at the secondary level. This Machian position is particularly aligned with the second and third formulations of Temporal Relationalism ii), which are indeed in terms of change rather than velocity. Machian times are of the general form

$$t^{\text{em(Mach)}} = F[Q^A, dQ^A]. \quad (4)$$

More specifically, [9, 4, 18, 20] one is best served by adopting a conception of time along the lines of the astronomers’ ephemeris time [19]. This is from including a sufficient totality of locally relevant change.

A specific implementation of this comes from rearranging (3). This amounts to interpreting this not as an energy-type constraint but as an equation of time thus called $\text{CHRONOS}$ (after the primordial Greek God of Time). We shall see that this rearrangement is aligned with

$$* := \frac{\partial}{\partial t^{\text{em}(i)}} = \sqrt{2W} \frac{\partial}{\partial s} \frac{\partial}{\partial \lambda} \quad (5)$$

simplifying$^3$ the system’s momenta and equations of motion. Integrating up,

$$t^{\text{em}(j)} - t^{\text{em}(j)}(0) = \int ds/\sqrt{2W} \quad (\text{emergent Jacobi time}) \quad . \quad (6)$$

At the quantum level, (3) gives rise to

$$\text{CHRONOS}\psi = 0, \quad (7)$$

$^2$Purely’ here means in particular that there is no accompanying linear dependence on the momenta. At the level of conic sections and higher-dimensional quadratic forms, this means a ‘centred’ choice of coordinates, meaning that the conic section is centred about the origin.

$^3$Choosing time so that the equations of motion are simple was e.g. already argued for in [21]; see also Sec 3.4.
for $\Psi$ the wavefunction of the universe. This includes the time independent Schrödinger equation $\hat{H}\Psi = E\Psi$ for Mechanics and the well-known Wheeler–DeWitt equation [14, 22] $\hat{H}\Psi = 0$. Moreover, these equations occur in a situation in which one might expect time-dependent Schrödinger equations $\hat{H}\Psi = i\hbar\partial\Psi/\partial t$ for some notion of time $t$. Thus (7) exhibits a Frozen Formalism Problem [14, 23]. This is a well-known facet of the Problem of Time (PoT); the preceding steps which trace this back to the classical level and point to Machian resolutions are for now less well-known.

Moreover, the above suite of implementations of Temporal Relationalism can be extended to cover every physical formalism, rather than just at the level of the Lagrangian (or Jacobi) formulations. This is necessary in order for Temporal Relationalism to be tractable in tandem with the many other facets of the PoT [23, 24, 4, 25, 20]. Relevant parts of the standard PoD are recollected in Sec 2 and extended by completion of the square of Legendre transformations to include ‘anti-Routhians’. This Article concentrates on finite models, though its results readily carry over to field theory also (including in particular full GR) [12, 4, 20].

The idea now is to compose Temporal Relationalism with the other PoT facets (see [25] for a recent account with facets specifically laid out piecemeal out prior to composition). Reformulating the PoD to be Temporal Relationalism incorporating – TRiPoD – ensures that all subsequent considerations of other Problem of Time facets within this new paradigm do not violate the Temporal Relationalism initially imposed (Sec 3). This is a big issue because the PoT facets have been likened to the gates of an enchanted castle [32]: if one goes through one of them and then through another, one often finds oneself once again outside of the first one. In the current Article I reformulate the entirety of the PoD to ensure that one stays within Temporal Relationalism in one’s subsequent efforts to pass through further PoT facets’ gates. This is indicatory that a lot of work is required in order to not lose one’s program’s previous successes with a subset of the gates when one tries to extend that program to pass through more of the gates. On the other hand, that this can be done at all is encouraging as a sign of the non-impossibility of the whole (or at least local) PoT.

Thus I next pass from velocities $\dot{Q}^A$ to changes $\delta Q^A$ to align with equations (2), (4) and (5). Then Lagrangians are supplanted by Jacobi arc elements, Euler–Lagrange equations by Jacobi–Mach ones, and momentum requires redefining but actions remain unchanged. The TRiPoD version of the PoD is similar in spirit to Dirac’s introduction of multiple further notions of Hamiltonian to start to deal systematically with constrained systems, by appending diverse kinds of constraints with diverse kinds of Lagrange multipliers. A differential (d) version of the Hamiltonian is required to start to deal systematically with constrained systems, by appending diverse kinds of Lagrange multipliers. This gives rise to a variant of the Dirac approach based on a d-almost Hamiltonian subcase of d-anti Routhian. On the other hand, the forms of the constraints themselves are unaltered, as are the constraint algebraic structure and expression in terms of beables are also unaltered in the finite-model case. Finally, Hamilton–Jacobi theory remains unaltered.

This Article concludes in Sec 4 by briefly pointing out the TRi-Foliations [26] and TRi-canonical quantum mechanics extensions that complete the TRi program within which to tackle the rest of the PoT.

2 The Standard Principles of Dynamics

2.1 Lagrangians and Euler–Lagrange Equations

Begin by considering a for now finite second-order classical physical system [2, 27] expressed in terms of Lagrangian variables $Q^A, \dot{Q}^A$. All dynamical information is contained within the Lagrangian function $L(Q^A, \dot{Q}^A, t)$. The commonest form for this is $L = T - V(Q, t)$ for $T$ as in (1) but with the dot signifying $\partial/\partial t$ rather than $\partial/\partial \lambda$. Then apply the standard prescription of the Calculus of Variations to obtain the equations of motion such that the action $S = \int dt L$ is stationary with respect to $Q^A$. This approach considers the true motion between two particular fixed endpoints $e_1$ and $e_2$ along with the set of varied paths (subject to the same fixed endpoints) about this motion. This gives the Euler–Lagrange equations

$$\frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{Q}^A} \right\} - \frac{\partial L}{\partial Q^A} = 0. \quad (8)$$

These equations simplify in the below three special cases, two of which involve particular types of coordinates. Indeed, one major theme in the principles of dynamics is judiciously chosen a coordinate system with as many simplifying coordinates as possible.

1) Lagrange multiplier coordinates $m^M \subseteq Q^A$ are such that $L$ is independent of $\dot{m}^M$:

$$\frac{\partial L}{\partial \dot{m}^M} = 0. \quad (9)$$

4I subsequently use the acronym TRi more widely in this Article.

5To avoid confusion, note that ‘cyclic’ in ‘cyclic differential’ just means the same as ‘cyclic’ in cyclic velocity, rather than implying some particular kind of differential itself. Thus nothing like ‘exact differential’ or ‘cycle’ in algebraic topology – which in de Rham’s case is tied to differentials – is implied.

6Note that this Article concentrate on systems with no extra explicit time dependence (currently in the Lagrangian).
Then the corresponding Euler–Lagrange equation simplifies to
\[ \frac{\partial L}{\partial m} = 0. \tag{10} \]

2) Cyclic coordinates \( c^Y \subseteq Q^A \) are such that \( L \) is independent of \( c^Y \):
\[ \frac{\partial L}{\partial c^Y} = 0, \tag{11} \]
but features \( \dot{c}^Y \): the corresponding cyclic velocities. Then the corresponding Euler–Lagrange equation simplifies to
\[ \frac{\partial L}{\partial \dot{c}^Y} = \text{const}^Y. \tag{12} \]

3) The energy integral type simplification. If \( L \) is independent of the independent variable itself (usually \( t \)): \( \frac{\partial L}{\partial t} = 0 \), then one Euler–Lagrange equation may be supplanted by the first integral
\[ L - \dot{Q}^A \frac{\partial L}{\partial \dot{Q}^A} = \text{constant}. \tag{13} \]

Further suppose that 1)’s equations
\[ 0 = \frac{\partial L}{\partial m}(Q^O, \dot{Q}^O, m^M) \]
happen to be solvable for \( m^M \).

(Here \( Q^O \) denotes ‘other than multiplier coordinates’).
Then one can pass from \( L(Q^O, \dot{Q}^O, m^M) \) to a reduced \( L_{\text{red}}(Q^O, \dot{Q}^O) \); this is known as multiplier elimination.

### 2.2 Conjugate momenta

These are defined by
\[ P_A := \frac{\partial L}{\partial \dot{Q}^A}. \tag{15} \]
Explicit computation of this then gives the momentum–velocity relation
\[ P_A = M_B \dot{Q}^B. \tag{16} \]

As a first application of this, it permits rewriting some of the preceding simplifications. The preliminary condition (9) in deducing the multiplier condition is now \( \dot{P}^Y = 0 \). The cyclic coordinate condition (10) is now
\[ P^Y = \text{const}^Y, \tag{17} \]
whereas the energy integral (13) is
\[ L - Q^A P_A = \text{constant}. \tag{18} \]

### 2.3 Legendre transformations

Suppose one has a function \( F(x^W, v^V) \) and one wishes to use \( z^W = \partial F/\partial x^W \) as variables in place of the \( x^W \). To avoid losing information in the process, a Legendre transformation is required, by which one passes to a function
\[ G(z^W, v^V) = x^W z^W - F(x^W, v^V). \tag{19} \]

Legendre transformations are symmetric between \( x^W \) and \( z^W \): if one defines \( x^W := \partial G/\partial z^W \), the reverse passage is now to \( F(x^W, v^V) = x^W z^W - G(z^W, v^V) \).

In particular, in Mechanics if one’s function is a Lagrangian \( L(Q_A, \dot{Q}^A) \), one may wish to use some of the conjugate momenta \( P_A \) as variables in place of the corresponding velocities \( \dot{Q}^A \).

Example 1) Passage to the Routhian. Given a Lagrangian with cyclic coordinates \( c^Y \), \( L(Q^X, \dot{Q}^X, c^Y) \), then \( \partial L/\partial \dot{c}^Y := P_Y = \text{const}^Y \). Thus one may pass from \( L \) to the Routhian
\[ R(Q^X, \dot{Q}^X, p^c, t) := L(Q^X, \dot{Q}^X, c^Y, t) - P_Y \dot{c}^Y. \tag{20} \]

This amounts to treating the cyclic coordinates, as a package, differently from the non-cyclic ones.

Much of the motivation for the Routhian is that it is in some ways a useful trick to treat cyclic coordinates differently from the others. The usual setting for this is simplifying the Euler–Lagrange equations. This requires the cyclic velocities analogue of multiplier elimination, termed Routhian reduction, which requires being able to solve
\[ \text{const}^Y = \frac{\partial L}{\partial c^Y}(Q^X, \dot{Q}^X, c^Y) \] as equations for the \( c^Y \). \tag{21}
Moreover, unlike in the corresponding multiplier elimination, one does not just substitute these back into the Lagrangian. Rather, one additionally needs to apply the Legendre transformation in (20), thus indeed passing to a Routhian rather than just a reduced Lagrangian. If the above reduction can be performed, then one can furthermore use the status of the cyclic momenta as constants (17) to end any further involvement of the cyclic variables in the dynamical problem at hand. I.e. the corresponding part of the integration of equations of motion has thereby been completed. Note finally that the manoeuvre [2] from Euler–Lagrange’s action principle with no explicit $t$ dependence to Jacobi’s action principle is a subcase of Routhian reduction.

Example 2) Passage to the Hamiltonian. In general,

$$
H(Q^A, P_A, t) := P_A \dot{Q}^A - L(Q^A, \dot{Q}^A, t),
$$

which makes use of all the conjugate momenta. This Article then concentrates on the $t$-independent version. For instance, for the most common form of Lagrangian,

$$
H = ||P||^2/2 + V(Q, t).
$$

In the general case, the equations of motion are Hamilton’s equations,

$$
\dot{Q}^A = \partial H/\partial P_A, \quad \dot{P}_A = -\partial H/\partial Q^A.
$$

In the time-dependent case, these are supplemented by $-\partial L/\partial t = \partial H/\partial t$.

For the Lagrange multiplier coordinates, half of the corresponding Hamilton’s equations collapse to just

$$
\partial H/\partial m^M = 0
$$
in place of (10) On the other hand (13) becomes $H = \text{const}$ in the time-dependent case.

The Hamiltonian formulation is further motivated firstly by admitting a systematic treatment of constraints due to Dirac ([16, 28] and Sec 2.7). Secondly, it offers a more direct link to quantum theory. N.B. also that in this case (within for the range of theories considered in this Article) the Legendre transformation is from a geometrical perspective effectuating passage from the tangent bundle $T(q)$ to the cotangent bundle $T^*(q)$.

Example 3) The next Sec motivates consideration also of passage to the anti-Routhian

$$
A(Q^X, P^X, \epsilon^Y, t) := L(Q^X, \dot{Q}^X, \dot{\epsilon}^Y, t) - P_X \dot{Q}^X.
$$

Introducing $A$ is natural insofar as it completes the Legendre square whose other vertices are $L$, $H$ and $R$ (Fig 1). Returning to the motivation of the Routhian itself, passage to the anti-Routhian turns out to be a distinct useful trick involving treating cyclic coordinates differently from the others, but now under the diametrically opposite Legendre transformation. The smaller part of the new trick is in Sec 2.7, whilst the larger part has to await the further special case that Sec 3 centres upon.

Moreover, both of these tricks in general come at a price. The first part of this price is geometrical, namely that these now involve slightly more complicated mixed cotangent–tangent bundles over $q$. I.e. $T(q) \times T^*(c)$ for the Routhian case and $T^*(\tilde{q}) \times T(c)$ for the anti-Routhian case, where $c$ is the subconfiguration space of cyclic coordinates and $\tilde{q}$ is the complementary subconfiguration space of the $Q^X$.

![Figure 1](image-url) Figure 1: a) completes an elsewise well-known Legendre square by introducing an anti-Routhian that is the diametric opposite of the Routhian in terms of which variables it swaps by Legendre transformation. I.e. it makes the same cyclic-other packaging, but converts the other set’s velocities to momenta. b) The analogous Legendre square from Thermodynamics, where all 4 corners are well-known: internal energy $U$, enthalpy $H$, Gibbs free energy $G$ and Helmholtz free energy $A$.

### 2.4 Corresponding morphisms

The morphisms corresponding to $q$’s are the point transformations $\text{Point}(q)$: the coordinate transformations of $q$. These readily induce the transformation theory for Lagrangian variables. More generally, but not in this Article, one has the time-dependent extension $\text{Point}_t$. The transformation theory for Hamiltonian variables is in general more subtle. This in part reflects that $P_A \dot{Q}^A$ becomes involved due to featuring in the conversion from $L$ to $H$. Here starting from
Point, the momenta then follow suit by preserving $P_A \dot{Q}^A$ \[^2\]; these transformations still preserve $H$. On the other hand, starting from Point, induces gyroscopic corrections to $H$ \[^2\]; this illustrates that $H$ itself can change form. More general transformations which mix the $Q_A$ and the $P^A$ are additionally possible, though they are not as unrestrictedly general functions of their $2k$ arguments as Point’s transformations are as functions of their $k$ arguments. A first case are the transformations which preserve the Liouville 1-form

$$P_A dQ^A$$

that is clearly associated with $P_A \dot{Q}^A$. These can again be time-independent (termed scleronomous) or time-dependent in the sense of parametrization adjunction of $t$ to the $Q^A$ (termed rheonomous). Again, the former preserve $H$ whereas the latter induce correction terms \[^2\]. These are often known as contact transformations, so I denote them by Contact and Contact, respectively. Point, Point, Contact and Contact, form a diamond of subgroups.

More generally still, it turns out to be the integral of (27) that needs to be preserved. Correspondingly, at the differential level, this means that (27) itself is preserved up to an additive complete differential $dG$ for $G$ the generating function. In this generality, one arrives at the canonical transformations alias symplectomorphisms, once again in the form of a rheonomous group Can and a scleronomous subgroup Can. Whilst arbitrary canonical transformations do not permit explicit representation, the infinitesimal ones do. Finally, applying Stokes’ Theorem to the integral of (27) reveals a more basic invariant: the bilinear antisymmetric symplectic 2-form

$$dP_A \wedge dQ^A$$

(denoted by $\omega$ with components $\omega_{A\dot{A}'}$ to cover a wider generality of cases \[^3\]). Thus one arrives at brackets built from this, of which the Poisson bracket is the most important example.

Concentrating on the $t$-independent case that is central to this Article, for the Routhian, the morphisms are $\text{Point}(\dot{q}) \times \text{Can}(c)$, though the latter piece is usually ignored out of the $c^t$ being absent and the $P^A$ being constant. For the $t$-independent anti-Routhian, the morphisms are $\text{Can}(\dot{q}) \times \text{Point}(c)$. The above more complicated spaces are the second price to pay.

2.5 Bracket structures

The Poisson bracket $\{ , \}$ of quantities $F(Q^A, P_A)$ and $G(Q^A, P_A)$ is given by

$$\{F, G\} := \frac{\partial F}{\partial Q^A} \frac{\partial G}{\partial P_A} - \frac{\partial G}{\partial Q^A} \frac{\partial F}{\partial P_A}.$$  \hfill (29)

Then in terms of these, the equations of motion take the form

$$\{P_A, H\} = \dot{P}_A , \quad \{Q^A, H\} = \dot{Q}^A.$$  \hfill (30)

Furthermore, for any $f(Q^A, P_A, t)$, the total derivative $df/dt = \{f, H\} + \partial f/\partial t$, so if $f$ does not depend explicitly on $t$, the intuitive conserved quantity condition $0 = df/dt$ becomes

$$\{f, H\} = 0.$$  \hfill (31)

Phase space is the space $T^*(\mathcal{Q})$ of both the $Q^A$ and the $P_A$ as equipped with $\{ , \}$.

One can also at this stage go back and find an equivalent description to the Poisson bracket at the Lagrangian tangent bundle level rather than requiring the Hamiltonians cotangent bundle canonical variables $\{q, p\}$ to be defined in advance. The resulting notion is the Peierls bracket \[^2\]. A price to pay in using this rather than the Poisson bracket is that it involves rather more complicated mathematics in terms of Green’s functions. [Its explicit form is not required in this Article.]

Finally, the second part of Sec 2.3’s price to pay if one uses a Routhian or anti-Routhian is that the mixed cotangent–tangent bundle nature of the variables further implies in general the involvement of mixed Poisson–Peierls brackets.

2.6 Hamilton–Jacobi Theory

If one replaces $P_A$ by $\partial S/\partial Q^A$ in $H$, one obtains the Hamilton–Jacobi equation, whose most general form is

$$\partial S/\partial t + H(Q^A, \partial S/\partial Q^A, t) = 0.$$  \hfill (32)

This is to be solved as a p.d.e for the as-yet undetermined Hamilton’s principal function $S(Q^A, t)$. Then as a particular subcase, if the Hamiltonian is itself time-independent, one can use $S = \chi(Q^A) - Et$ as a separation ansatz. This then obeys

$$H(Q^A, \partial \chi/\partial Q^A) = E.$$  \hfill (33)
This is to be solved for Hamilton’s characteristic function \( \chi \). As well as sometimes being computationally useful, the Hamilton–Jacobi formulation is close to the semiclassical approximation to QM and the Semiclassical Approach to the Problem of Time and Quantum Cosmology.

[In the Routhian formulation, the constancy of \( p_\mathcal{E} \) – the sole momenta involved – precludes an analogue of Hamilton–Jacobi theory prior to further Legendre transformations. On the other hand, in the anti-Routhian formulation, \( \partial S/\partial t + A(Q^X, \partial S/\partial Q^X, \varepsilon^Y, t) = 0 \) makes sense in the \( t \)-dependent case, and \( A(Q^X, \partial S/\partial Q^X, \varepsilon^Y) = E \) in the \( t \)-independent case. However the current Article in any case bypasses use of this last equation.]

## 2.7 Hamiltonian formulation in the case with constraints

Passage from Lagrangian to Hamiltonian formulation can be nontrivial. The Legendre matrix

\[
\Lambda_{AB} := \frac{\partial^2 L}{\partial \dot{Q}^A \partial \dot{Q}^B} \quad (= \frac{\partial P_B}{\partial \dot{Q}^A})
\]  

(34)

– named by its association with the Legendre transformation – is in general non-invertible, so the momenta cannot be independent functions of the velocities.\(^7\) I.e. there are relations of the type \( c_C(Q^A, P_B) = 0 \) between the momenta. This is quite a general type\(^8\) type of constraint considered by Dirac. Indeed, the Euler–Lagrange equations can be rearranged to reveal the explicit presence of the Legendre matrix,

\[
\ddot{Q}^A \frac{\partial^2 L}{\partial \dot{Q}^A \partial \dot{Q}^B} \dot{Q}^B = \frac{\partial L}{\partial Q^A} - \dot{Q}^A \frac{\partial^2 L}{\partial Q^A \partial \dot{Q}^A}.
\]  

(35)

By this, its noninvertibility has additional significance as accelerations not being uniquely determined by \( Q^A, \dot{Q}^A \).\(^9\)

Constraints as arising from the above non-invertibility of the momentum–velocity relations are furthermore termed \textit{primary}, as opposed to those requiring input from the variational equations of motion which are termed \textit{secondary} \cite{16, 28}. I index these by \( P \) and \( S \) respectively. \( \mathcal{H} \) and \( \varepsilon \) illustrate that the primary-secondary distinction is artificial insofar as it is malleable by change of formalism.

Since the Routhian and anti-Routhian also arise by Legendre transformations, one may wonder at this stage whether these involve their own notions of Legendre matrix. In the case of the Routhian, \( \Lambda_{\mathcal{X}Y} := \frac{\partial^2 L}{\partial \dot{\mathcal{X}} \partial \dot{\mathcal{Y}}} = 0 \) by (12), so it is an uninteresting albeit entirely obstructive object. The corresponding expressions for acceleration are similarly entirely free of reference to the cyclic variables. On the other hand, the anti-Routhian’s \( \Lambda_{\mathcal{X}X} := \frac{\partial^2 L}{\partial \dot{\mathcal{X}} \partial \dot{\mathcal{X}}} \) is in general nontrivial, and one can base a theory of primary constraints on it rather than on the usual larger (34). The smaller anti-Routhian trick is then that the acceleration of \( Q^X \) is unaffected by the cyclic variables. I.e. one can take (35) again with \( X \) in place of \( A \) since the further terms involving the cyclic variables that arise from the chain rule are annihilated by (12).

Dirac also introduced the concept of \textit{weak equality} \( \approx \), i.e. equality up to additive functionals of the constraints. ‘Strong equality’, on the other hand, means equality in the usual sense.

For \( H \) one’s incipient or ‘bare’ Hamiltonian, one can additively append one’s formalism’s primary constraints using \textit{a priori} any functions \( f \) of the \( Q \) and \( P \) to form Dirac’s further ‘starred’ Hamiltonian \( H^* := H + f^P \mathcal{C} \). Dirac’s \textit{total Hamiltonian} is \( H_{\text{total}} := H + u^P \mathcal{C} \), where the \( u^P \) are now regarded as unknowns. One then begins to consider \( \mathcal{C} \) \( = \{ c_P, H \} + u^P \{ c_P, \mathcal{C} \} \approx 0 \) with the functions \( u^P \) now regarded as unknowns.

The \textit{Dirac algorithm} \cite{16} then permits five kinds of outcomes.

### Outcome 0) Inconsistencies.

- Outcome 1) Mere identities – equations that reduce to \( 0 \approx 0 \), i.e. \( 0 = 0 \) modulo the \( c_P \).
- Outcome 2) Equations independent of the unknowns \( u^P \), which constitutes an extra constraint and is secondary.
- Outcome 3) Demonstration that existing constraints are in fact second-class (see below).
- Outcome 4) Relations amongst the appending functions \( u^P \)’s themselves, which are a further type of equation termed ‘specifier equations’ (i.e. specifying restrictions on the \( u^P \)).

Lest 0) be unexpected, Dirac supplied a basic counterexample to PoD formulations entailing consistent theories. For \( L = x \), the Euler–Lagrange equations read \( 0 = 1 \).

---

\(^7\)In the case of the purely-quadratic kinetic term action, this is just \( M_{\mathcal{XX}} \).

\(^8\)I.e. these are scleronomous equality constraints, as opposed to the thus-excluded \textit{rheonomous} constraints. Inequality constraints are also excluded. Another term often used to describe constraints is \textit{holonomic}, meaning that it is integrable into the form \( f(Q^A, t) = 0 \). Note that there are two ways this can fail: nonintegrability or inequality constraints. Now, whilst Dirac’s treatment solely considers equality constraints, it says nothing about whether these are to be integrable or not, and hence are holonomic or the first sense of not so.

\(^9\)I remind the reader that this basic account of Physics restricts itself to no higher than second-order theories.
Suppose 2) occurs. Then defining \( Q = P + S \) indexing the constraints obtained so far, one can restart with a more general form form the problem,
\[
\dot{c}_Q = \{c_Q, H\} + u^P \{c_Q, c_P\} \approx 0 \quad .
\]
(36)
It may be necessary to do this multiple times [but clearly terminate if 0) occurs]. Suppose now that we have finished what solving there is to be made for the final problem’s \( u^P \). These are of the form \( u^P = U^P + V^P \) by the split into i) \( U^P \) the particular solution. ii) \( V^P = v^P V^P_Z \) for \( Z \) indexing the number of independent solutions in the ‘complementary function’: the general solution of the corresponding homogeneous system
\[
V^P \{c_C, c_P\} \approx 0 \quad .
\]
(37)
with \( v^A \) the totally arbitrary coefficients of the independent solutions indexed by \( A \). This done, Dirac also defined the ‘primed Hamiltonian’ \( H' := H + U^P \bar{c}_P \) These can be viewed as appendings by a determined mixture of free and fixed Lagrange multipliers.

A further classification of constraints is into first-class constraints (indexed by \( F \) which weakly close under the classical bracket (ab initio the Poisson bracket, but this can change during the procedure). Second-class constraints are then simply defined by exclusion as those constraints that fail to be first-class. First-class constraints use up 2 degrees of freedom each, and second-class constraints 1.

At least in the more standard theories of Physics, first-class secondary constraints arise from variation with respect to mathematically disjoint auxiliary variables. Furthermore, the effect of this variation is to additionally use up part of an accompanying mathematically coherent block that however only contains partially physical information.

Some constraints are regarded as gauge constraints; however in general exactly which constraints these comprise remains disputed. It is however agreed upon that second-class constraints are not gauge constraints; all gauge constraints use up two degrees of freedom. Dirac [16] conjectured a fortiori that all first-class constraints are gauge constraints,\(^\text{10}\) so that using up two degrees of freedom would then conversely imply being a gauge constraint. However, e.g. [28] contains a counterexample refuting this conjecture. One feature of Gauge Theory is an associated group of transformations that are held to be unphysical. The above-mentioned disjoint auxiliary variables are often in correspondence with such a group. Gauge-fixing conditions \( F_X \) may be applied to whatever Gauge Theory (though one requires the final answers to physical questions to be gauge-invariant).

Dirac finally defined \( H_{\text{Extended}} := H + u^P \bar{c}_P + u^S \bar{c}_S \).

Particularly with quantization in mind, first-class constraints tend to be relatively unproblematic, but second-class ones cause difficulties. Thus it is fortunate that there exist procedures for freeing one’s theory of second-class constraints. There are two directions one can take: the Dirac bracket method excises them, whereas the effective method extends the phase space with further auxiliary variables so as to ‘gauge-unfix’ second-class constraints into first-class ones.

Strategy A) Passage to the Dirac brackets. This replaces the incipient Poisson brackets with
\[
\{F, G\}^\ast := \{F, G\} - \{F, c_i\}\{c_i, c_P\}^{-1}\{c_P, G\} \quad .
\]
(38)
Here the \(-1\) denotes the inverse of the given matrix whose \( I \) indices index irreducibly [16, 28] second-class constraints. The classical brackets role played ab initio by the Poisson brackets is then taken over by the Dirac brackets. The version of Dirac brackets formed once no second-class constraints remain illustrates the concept of ‘final classical brackets’ forming a ‘final classical brackets algebra’ of constraints.

Strategy B) Second-class constraints can always in principle\(^\text{11}\) be handled by alternatively thinking of them as ‘already-applied’ gauge fixing conditions that can be recast as first-class constraints by adding suitable auxiliary variables. By doing this, a system with first- and second-class constraints extends to a more redundant description of a system with just first-class constraints.

Moreover, Strategies A) and B) each make clear that the first class to second class distinction is also formalism-dependent.

So, all in all, one then enters the set of constraints in one’s possession into the brackets in use to form a constraint algebraic structure. This may enlarge one’s set of constraints, or cause of one to adopt a distinct bracket. If inconsistency is evaded, the eventual output is an algebraic structure for all of a theory’s constraints.

Symbolically, the algebraic structure formed by the constraints is
\[
\{c_{F'}, c_{F''}\}_{\text{final}} = C^F_{FF'} c_{F''} \quad .
\]
(39)
In some cases of relevance – especially GR – the \( C^F_{FF'} \) are structure functions rather than a Lie algebra’s constants.

\(^\text{10}\)This is in Dirac’s sense of Gauge Theory [31, 16]: concerning data at a given time, so ‘gauge’ here means data-gauge. Contrast this with Bergmann’s perspective [33] that that Gauge Theory concerns whole paths (dynamical trajectories), so ‘gauge’ there means path-gauge.

\(^\text{11}\)To this incipient statement of [28], I add ‘locally’, because gauge-fixing conditions themselves in general are not global entities.
2.8 Observables or beables

It is then also natural to ask which quantities form zero brackets with a given closed algebraic structure of constraints,

$$\{c_c, \nu_B\} \quad ^\dagger = 0 \quad (40)$$

These entities are observables or beables\(^\text{12}\). In the case of Dirac observables\([31]\) – involving all the first-class constraints — these entities are more useful than just any other function(al)s of the \(Q^A\) and \(P_A\) through containing solely physical information. On the other hand, in the case of Kuchař observables\([32]\) involving all the first-class linear constraints,\(^\text{13}\) these entities are of middling usefulness. This is due to being Dirac data gauge invariant quantities, albeit not ascertained to commute with the non-linear \textit{chronos} constraint as well. Each such case itself closes as an algebraic structure associated with the corresponding constraint algebraic structure. N.B. that for investigation of these notions of observables or beables, whichever of Dirac’s Hamiltonians are equivalent. See e.g.\([33, 34, 30]\) for yet further notions of observables or beables.

2.9 Hamilton–Jacobi theory in the presence of constraints

If the system has Dirac-type constraints, \(c_C(Q^A; P_A)\), the corresponding Hamilton–Jacobi equation – whether (32) or an incipiently timeless (33) — would be supplemented by

$$c_C(Q^A, \partial \chi / \partial Q^A) = 0 \quad . \quad (41)$$

3 Temporal Relationalism incorporating Principles of Dynamics (TriPoD)

3.1 Jacobi elements and Jacobi–Mach equations

Consider now working in the absence of time at the primary level, as per Leibniz’s Time Principle. [This is taken to include an absence of label time, as per Temporal Relationalism ii.)] Restrict also for now to the finite second-order classical physical system. Now without time, there is no derivative with respect to time and thus no notion of velocity at the primary level. Thus one cannot use Lagrangian variables. However, Mach’s Time Principle points to the availability of change at the primary level, so it is Machian variables \(Q^A, \dot{Q}^A\) that supplant the Lagrangian ones \(Q^A, \dot{Q}^A\). Now all information is contained within the Jacobi arc element \(dJ(Q^A, \dot{Q}^A)\); this has supplanted the TRi time-independent Lagrangian \(L(Q^A, \dot{Q}^A)\) itself. The action \(S\) is itself an unmodified concept: it is already in TRiPoD form, albeit now additionally bearing the relation \(S = \int dJ\) to the TRiPoD formulation specific Jacobi arc element \(dJ\). Actions of this form are manifestly parametrization irrelevant, thus indeed implementing Leibniz’s Time Principle as per Sec 1. There is clearly also no primary notion of kinetic energy; this has been supplanted by \textit{kinetic arc element} \(ds\) which for purely second-order systems is given by (2).

Then \(dJ = \sqrt{2W}ds\) for \(W(Q)\) the usual potential factor. In this manner, the kinetic and Jacobi arc elements are simply related by a conformal transformation; In terms of \(dJ\), dynamics has been cast in the form of a \textit{geodesic principle} \([9]\), whereas in terms of \(ds\) it has been cast in the form of a \textit{parageodesic principle} \([15]\).

Then apply the Calculus of Variations to obtain the equations of motion such that \(S\) is stationary with respect to \(Q^A\). The particular form that this variation takes is further commented upon in Sec 3.6. Given the above set-up, I term the resulting equations of motion the \textit{Jacobi–Mach equations},

$$d \left\{ \frac{\partial dJ}{\partial Q^A} \right\} - \frac{\partial dJ}{\partial Q^A} = 0 \quad . \quad (42)$$

These supplant the Euler–Lagrange equations.

The Jacobi–Mach equations admit three corresponding simplified cases, as follows.

1) \textit{Lagrange multiplier coordinates} \(m^M \subseteq Q^A\) are such that \(dJ\) is independent of \(dm^M\):

$$\partial dJ / \partial dm^M = 0 \quad . \quad (43)$$

Then the corresponding Jacobi–Mach equation simplifies to

$$\partial dJ / \partial m^M = 0 \quad . \quad (44)$$

2) \textit{Cyclic coordinates} \(c^Y \subseteq Q^A\) are such that \(dJ\) is independent of \(c^M\):

$$\partial dJ / \partial c^Y = 0 \quad , \quad (45)$$

\(^{12}\)See \([30]\) for discussion of the distinction between these two concepts.

\(^{13}\)These are often but not always the same \([30]\) as the shuffle constraints of Sec 3.5.
3) The energy integral type simplification. dJ is independent of what was previously regarded as the independent variable t, by which one Jacobi–Mach equation may be supplanted by the first integral

\[ dJ = \frac{\partial dJ}{\partial dQ^A} dQ^A = \text{constant} \]  

Further suppose that 1)’s equations

\[ 0 = \frac{\partial dJ}{\partial m^M} (Q^O, dQ^O, m^M) \]  

happen to be solvable for \( m^M \). Then one can pass from \( dJ(Q^O, dQ^O, m^M) \) to a reduced \( dJ_{\text{red}}(Q^O, dQ^O) \): Jacobi–Mach multiplier elimination.

Note 1) More generally than for purely quadratic systems, the above formulation’s presentation as a geodesic principle corresponding to some notion of geometry continues to hold – for some more general notion of geometry (e.g. Finsler geometry). This more general case was pioneered by Synge (see [2] for a discussion).

Note 2) Configuration–change space and configuration–velocity space are different presentations of the same tangent bundle \( T(q) \).

Note 3) Formulation in terms of change \( dQ^A \) can be viewed as introducing a change covector. This is in the sense of inducing ‘change weights’ to principles of dynamics entities, analogously to how introducing a conformal factor attaches conformal weights to tensorial entities. For instance, ds and dJ are change covectors too, whereas S is a change scalar. Change scalars are entities which remain invariant under passing from PoD to TRiPoD, out of being already-TRi.

Note 4) Finally, an explicit computation of (42) gives

\[ \frac{\sqrt{2W}}{|dQ||M|} \left\{ \frac{\sqrt{2W}}{|dQ||M|} \right\} + \Gamma^A_{BC} \frac{\sqrt{2W}}{|dQ||M|} \frac{\sqrt{2W}}{|dQ||M|} = \Lambda^{AB} \frac{\partial W}{\partial Q^B} \]  

for \( \Gamma^A_{BC} \) the Christoffel symbols corresponding to \( M \).

3.2 TRiPoD’s formulation of momentum

This is

\[ P_A := \frac{\partial dJ}{\partial dQ^A} \]  

Furthermore, explicitly computing this for actions of type (2) gives the momentum–change relation

\[ P_A = M_{AA} \frac{\sqrt{2W}}{|dQ||M|} \]  

With (50) depending on ratios of changes alone, momentum is a change scalar.

3.3 TRiPoD’s Legendre transformation

One can now apply Legendre transformations that inter-convert changes \( dQ^A \) and momenta \( P_A \).

Example 1) Passage to the d-Routhian

\[ dR(Q^X, dQ^X, P^\gamma_c) := dJ(Q^X, dQ^X, d\gamma) - P_\gamma d\gamma \]  

d-Routhian reduction then requires being able to solve

\[ \text{const}_\gamma = \frac{\partial dJ}{\partial d\gamma} (Q^X, dQ^X, d\gamma) \]  

as equations for \( d\gamma \), followed by substitution into the above. One application of this is the passage from Euler–Lagrange type actions to the geometrical form of the Jacobi actions without ever introducing a parameter. See Sec 3.5 for another application.

Example 2) Passage to the d-anti-Routhian.

\[ dA(Q^X, P^X, d\gamma) = dJ(Q^X, dQ^X, d\gamma) - P^X dQ^X \]  

A subcase of this plays a significant role in the next Sec.

Example 3) Passage to the d-Hamiltonian.

\[ dH(Q^A, P_A) = P_A dQ^A - dJ(Q^A, dQ^A) \]  

The corresponding equations of motion are in this case d-Hamilton’s equations

\[ \frac{\partial dH}{\partial P_A} = dQ^A, \quad \frac{\partial dH}{\partial Q^A} = -dP_A. \]
3.4 The simpler case

As per the Introduction, firstly \texttt{CHRONOS} arises as a primary constraint (this term is redefined in TRiPoD terms in Sec 3.8, derivation cast in TRiPoD terms there too). Secondly, \texttt{CHRONOS} can be rearranged to obtain expression (6) for emergent time. Also note that (13) is the Jacobi–Mach formulation’s manifestation of \texttt{CHRONOS}: an equation of time rather than a statement of energy conservation.

Furthermore, adopting emergent Jacobi time simplifies the expression (51) for the momenta to

\[ P_A = M_{AB} Q^B , \]

and the form of the Jacobi–Mach equations of motion (49) to

\[ **Q^A + \Gamma^A_{BC} Q^B Q^C = N^{AB} \partial W / \partial Q^A . \]

The emergent Jacobi time thus has the desirable property of being a time with respect to which the equations of motion take a particularly simple form. In this manner, in the case of Mechanics the emergent Jacobi time is a recovery of the same entity that is usually modelled by Newtonian time but now on relational premises. In the GR case, it amounts to a recovery of GR proper time (including of cosmic time in the cosmological case) [20].

3.5 Configurational Relationalism and its TRiPoD form

The corresponding harder case involves \textit{Configurational Relationalism} as well. This second aspect of Background Independence covers both of the following.

a) \textit{Spatial Relationalism} [35]: no absolute space properties.

b) \textit{Internal Relationalism}: the post-Machian addition of not ascribing any absolute properties to any additional internal space that is associated with the matter fields.

These are notably different cases because b) holds at a fixed spatial point whereas a) move spatial points around. Configurational Relationalism is then axiomatized as follows.

i) One is to include no appended configurational structures either (spatial or internal-spatial metric geometry variables that are fixed-background rather than dynamical).

ii) Physics in general involves not only a \( q \) but also a \( g \) of transformations acting upon \( q \) that are taken to be physically redundant.

The usual Jacobi action principle itself is temporally-relational but spatially-absolute mechanics. On the other hand, Barbour and Bertotti [35] found a means of freeing Mechanics actions from absolute space. This involved using not \( \dot{q}^i \) but \( \dot{q}^i - A^i - \{ \mathbf{B} \times q^i \}^i \) which allows for a frame that is translating and rotating in an arbitrary time-dependent manner. Unfortunately, these corrections ruin Manifest Reparametrization Invariance of the Jacobi action. However [36, 11] the two implementations can be stacked together by use instead of \( \dot{q}^i - \dot{a}^i - \{ \mathbf{B} \times q^i \}^i \). One can then pass to a Jacobi–Mach version of this that stacks Configurational Relationalism with Temporal Relationalism’s geometric implementation that is dual to Manifest Parametrization Irrelevance [4, 12]. Here one uses \( d\dot{q}^i - da^i - \{ \mathbf{B} \times q^i \}^i \).

These stackings are moreover nontrivial due to requiring careful considerations of how to vary auxiliary variables that are not just Lagrange multipliers, as per Sec 3.6. The above suite of arbitrary frame corrections bear relation to gauge theory in Dirac’s data-gauge sense. Let us furthermore pass to the more general case of \( d_g Q^A := Q^A - \overrightarrow{g}_{d_g} Q^A \), where \( \overrightarrow{g}_{d_g} \) denotes an infinitesimal group action of a group \( g \) acting on \( q \), which action is held to be physically irrelevant. In particular, the GR case of this then involves the spatial diffeomorphisms (and so is trivial for minisuperspace). Electromagnetism, Yang–Mills theory and corresponding gauge theories gauge symmetries can be considered along the lines of the internal part of Configurational Relationalism.

The finite cases’ relational action is then

\[ S = \sqrt{2} \int d_s g \sqrt{W} , \quad d_s q := ||d_g Q||_M . \]

The conjugate momenta are now

\[ P_A := \frac{\partial dJ}{\partial dQ^A} = M_{AB} \sqrt{2W} ||d_g Q||_M d_g Q^A . \]

These obey one primary constraint per relevant notion of space point, interpreted as an equation of time (3), so it is purely quadratic in the momenta. They also obey \( G \) secondary constraints per relevant notion of space point from variation with respect to \( g^G \)

\[ 0 = \partial dJ / \partial dG = P_A \delta d_g G / \delta (\overrightarrow{g}_{d_g} Q^A) = SHUFFLE G ; \]

\[ 0 = \partial dJ / \partial dG = P_A \delta d_g G / \delta (\overrightarrow{g}_{d_g} Q^A) = SHUFFLE G ; \]

\[ 0 = \partial dJ / \partial dG = P_A \delta d_g G / \delta (\overrightarrow{g}_{d_g} Q^A) = SHUFFLE G ; \]
also note that these are linear in the momenta. Here also δ denotes variational derivative.

Next, denote the joint set of these constraints by \( c_{F} \), under the presumption that they are confirmed as first-class as per Sec 3.8. The indexing set designation assumes there is only one quadratic constraint, so all our examples’ \( F \) ranges over \( G \) plus one value indexing CHRONOS.

The arbitrary \( G \), arbitrary \( Q \) generalization of Barbour and Bertotti’s Best Matching method – also reformulated in Jacobi–Mach terms – then consists of the following steps.

Best Matching 1) Start with the ‘arbitrary \( G \) frame corrected’ action (59).

Best Matching 2) Next, extremizes over \( G \) in accord with the variational principle including variation with respect to \( dg^G \). This extremization produces the above linear constraint equation \( \mathcal{L} \Sigma G = 0 \).

Best Matching 3) In Machian variables \( \hat{Q}^A, d\hat{Q}^A \), this is to be solved for the \( dg^G \) themselves.

Best Matching 4) This solution is then to be substituted back into the action, in a further example of d-Routhian reduction. This produces a final \( G \)-independent expression.

Best Matching 5) Elevate this new action to be one’s primary starting point.

A further step, added later with gradually increasing detail \([9, 4, 12, 13]\) is that the classical Machian emergent time now takes the form

\[
\begin{align*}
\mu^{em(JBB)} - \mu^{em(JBB)}(0) & = E_{dg} \in G \left( \int \|dg^Q\|_M / \sqrt{2W(Q)} \right) .
\end{align*}
\]  

\[ JBB \] here stands for ‘Jacobi–Barbour–Bertotti’ and \( E \) denotes extremum.] This illustrates that, in the absence of being able to explicitly solve the Best Matching, one does not have an explicit solution for the classical Machian emergent time either. On the other hand, if one succeeds in carrying out Best Matching. Then \( dg \) is replaced by an extremal expression solely in terms of \( \hat{Q}^R, d\hat{Q}^R \) with tildes used to denote the reduced space quantities. By this, \( \mu^{em(JBB)} \) is expressed in terms of the reduced configuration space’s geometry: the tilded version of (6). The latter occurs nontrivially and solvably for all scaled 2-d RPM’s \([4]\) (simplest of which is the scaled relational triangle: Arena 5), as well as for pure-shape (i.e. scale free) RPM’s in both 1- and 2-d. On the other hand, the former occurs for full GR: Wheeler’s Thin Sandwich Problem \([8]\), which is listed as a second PoT facet \([23]\). Finally note the title of \([8]\): “three-dimensional geometry as carrier of information about time”. This clearly illustrates that Wheeler was already aware then of the time connotations of the thin sandwich subcase of the above ‘Best Matching’ working. This title is furthermore a clear illustration of ‘duality between geometry and Manifest Parametrization Irrelevance’ as an implementation of timelessness at the primary level leading to time being abstracted from change at the secondary level.

### 3.6 Free endpoint variation

Suppose a formulation’s auxiliary multiplier coordinate \( m \) is replaced by a cyclic velocity \( c \) \([36, 11]\) or cyclic differential \( dc \) \([4, 12]\). Then right-hand side zero of the multiplier equation is replaced by a constant in the corresponding cyclic equation. However, if the quantity being replaced is an entirely physically meaningless auxiliary, the meaningfulness of its values at the endpoint becomes nontrivial in the cyclic case. I.e. the appropriate type of variation is free end point (FEP) variation alias variation with natural boundary conditions \([37, 2]\).\(^{14}\) By this means it has enough freedom to impose more conditions than the more usual fixed-end variation does. In particular, it imposes 3 conditions per variation,

\[
\partial J / \partial g^G = dp_G , \quad \text{alongside} \quad p_G|_{\text{endpoint}} = 0 , \quad a = 1, 2 .
\]  

Case 1) If the auxiliaries \( g^G \) are multipliers \( m^G \), (63) reduces to \( p_G = 0 \), \( \partial L / \partial m^G = 0 \) and redundant equations. Thus the endpoint value terms automatically vanish in this case by applying the multiplier equation to the first factor of each. This is the case regardless of whether the multiplier is not auxiliary and thus standardly varied, or auxiliary and thus FEP varied. This is because this difference in status merely translates to whether or not the cofactors of the above zero factors are themselves zero or not. Thus the FEP subtlety in no way affects the outcome in the multiplier coordinate case. Thus the FeP subtlety remained unnoticed whilst following Dirac and ADM in encoding fundamental physics auxiliaries as multiplier coordinates.

Case 2) If the auxiliaries \( g^G \) are considered to be cyclic coordinates \( c^G \), (63) reduces to

\[ p_G|_{\text{endpoint}} = 0 \]  

alongside

\[ dp_G = 0 \Rightarrow p_G = C .
\]  

Then \( C \) is identified as 0 at either of the two endpoints (64). Being invariant along the curve of notions of space, it is therefore zero everywhere. Thus (65) and the definition of momentum give

\[ \partial J / \partial dc^G = 0 .
\]  

\(^{14}\)To be clear, this here refers to free value at the end-NoS rather than the also quite commonly encountered freedom of the end-NoS itself.
In conclusion, the above FEP working ensures that the cyclic and multiplier formulations of auxiliaries in fact give the same variational equation. Thus complying with Temporal Relationalism by passing from encoding one’s auxiliaries as multipliers to encoding them as cyclic velocities or differentials is valid without spoiling the familiar and valid physical equations.

Note that a similar working [11] establishes that for an auxiliary formulated in the cyclic manner, passage to the (d-)Routhian reproduces the outcome of multiplier elimination in the case of that same auxiliary entity being formulated as a multiplier.

3.7 TRi-morphisms and brackets. i.

Suppose there are no cyclic differentials to be kept. Then TRi morphisms carry down, except that specifically Point rather than Point, is involved. The Liouville 1-form (27) is already TRi and thus a change 1-form, whereas the symplectic 2-form (28) is already TRi and thus a change 2-form. The use of Can rather than Can, is also now imperative in the d-Hamiltonian formulation. Also Phase space carries down since all of $T^A$, $P_A$ and the Poisson bracket carry over.

3.8 dA-Hamiltonians, -phase space and -Dirac algorithms

The Legendre matrix encoding the non-invertibility of the momentum-velocity relations is now supplanted by the $d^{-1}$-Legendre matrix

$$d^{-1}L_{AB} := \partial^2 ds/\partial \dot{Q}^A \partial \dot{Q}^B \quad (= \partial P_B/\partial \dot{Q}^A) \quad (67)$$

encoding the non-invertibility of the momentum-change relation. N.B. here $d^{-1}$ denotes that this is a change vector.

Then the TRiPoD definition of primary constraint follows from this in parallel to how the usual definition of primary constraint follows from the Legendre matrix, with secondary constraint remaining defined by exclusion of primary constraints.

Example 1) Dirac’s argument for Reparameterization Invariance implying at least one primary constraint then takes the following TRiPoD form. A geometrical action is dually parametrization-irrelevant. Thus it is homogeneous of degree 1 in its changes. So each of its total of $k$ momenta are homogeneous of degree 0 in the changes. Thus these are functions of $k - 1$ independent ratios of changes. So there must be at least one relation between the momenta themselves without any use made of the equations of motion. But by definition, this is a primary constraint.

Then the specific form of the primary constraint that follows from actions of the form (2) is of course the same $CHRONOS$ that follows from actions of the form (1).

The next idea in building a TRiPoD version of Dirac’s general treatment of constraints is to append constraints to one’s incipient d-Hamiltonian not with Lagrange multipliers (which would break TRi) but rather with cyclic differentials. Thus a dA-Hamiltonian is formed; ‘A’ here stands for ‘almost’. The dA-Hamiltonian is a particular case of d-anti-Routhian. Note moreover that the dA-Hamiltonian dA symbol has an extra minus sign relative to the d-anti-Routhian dA symbol. This originates from the definition of Hamiltonian involving an overall minus sign where the definitions of Routhian and anti-Routhian have none. And which particular case? The one in which all the cyclic coordinates involved have auxiliary status and occur in best-matched combinations. [In the event of a system possessing physical as well as auxiliary cyclic velocities this would be a ‘partial’ rather than ‘complete’ Routhian, though this case does not further enter this Article.]

The equations of motion are then dA-Hamilton’s equations are

$$\partial dA/\partial P_A = \dot{Q}^A, \ \partial dA/\partial Q^A = -\dot{P}_A, \quad (68)$$

augmented by $\partial dA/\partial dG = 0$. Also note that Sec 2.7’s comment about using the anti-Routhian’s own Legendre matrix carries over to the d-anti-Routhian and thus also to the reidentification of a subcase of this as the dA-Hamiltonian.

Examples of the above appendings, paralleling Dirac’s treatment, are then firstly the starred dA-Hamiltonian $dA^* := dA + df^p c_p$ for arbitrary functions of $Q^A, P_A$ now represented as cyclic differentials $df(Q^A, P_A)$. Secondly, the total dA-Hamiltonian $dA_{\text{Total}} := dA + d\tilde{c}_p c_p$, where $d\tilde{c}_p$ are now unknown cyclic differentials.

The next issue to arise is the counterpart of the bracket expression in (36). Now the best-matched form of the action ensures the constraints are of the form $c(Q^A, P_A$ alone), because for these

$$\text{passage from } dQ^A \text{ to } P_A \text{ absorbs all the } dG. \quad (69)$$

In the case in hand, $c^C = c(Q^A, P_A$ alone) means that the chain-rule expansion

$$dc = \frac{\partial dc}{\partial Q^A} dQ^A + \frac{\partial dc}{\partial P_A} dP_A$$
applies, whence by (68)
\[ dc_C = \{ c_C, dA_{Total} \} = \{ c_C, dA \} + du^P \{ c_C, c_P \} . \]  

Next let this be solved for unknown cyclic differentials under the split \( du^P = dU^P + dV^P \) in direct parallel with Dirac’s: a cyclic differential particular solution \( dU^P \) and a cyclic differential complementary function \( dV^P = u^2 dV^P_Z \).

Then the \textit{primed} dA-Hamiltonian \( dA' := dA + dU^P c_P \) where \( dU^P \) are now the cyclic differential particular solution part of \( du^P \). Finally, the \textit{extended} dA-Hamiltonian \( dA_{Extended} := dA + du^P c_P + du^S c_S \).

Fig 2.b) exposes the significance via the usual cyclic velocity intermediate in Fig 2.a).

Phase is then replaced by A-Phase and dA-Phase; these are all types of bundle twice over (cotangent bundles and g bundles).

\[ \text{Figure 2: a) Almost-Hamiltonian subcase of Fig 1.a)’s Legendre square. b) then elevates this to fully TRIPoD form in terms of d-Legendre transformations that dually switch momenta and changes. These are between change covectors: dJ, dR, dA, dH, the information-preserving extra terms now being subsystem Liouville forms, which were always change covectors. Routhians go to d-Routhians (there is no need for ‘almost’ in this case since Routhians are already allowed to contain velocities, and so already include almost-Routhians as a subset). c) and d) exhibit the choices by which the total Hamiltonian, A-Hamiltonian and dA-Hamiltonian arise. The starred, primed and extended versions follow suit.} \]

For example, in Arena 5: the scaled relational triangle, \( dA_{Total} := dJ + dA' p_i + dB^i \xi_i \) (strictly \( + du^P P_i^A + du^V P_i^B \)). Here \( P_i \) is the zero total momentum constraint and \( L_i \) is the zero total angular momentum constraint. \( dJ \) is the differential of the instant, which is numerically equal to \( dt \) for Mechanics, albeit obtained from a relational perspective rather than assumed. On the other hand, there is no d-Hamiltonian to dA-Hamiltonian distinction for minisuperspace.

From only the Poisson bracket part acting on the constraints, it follows that the definition of first-class and second class remain unaffected. So are the Dirac bracket and the extension procedure.

Also Dirac’s algorithm is now supplanted by the dA-Dirac algorithm. The five cases this is capable of producing at each step, in any combination, are as follows.

Outcome (0), 1 and 3) are as before.
Outcome 2) Equations independent of the unknowns \( du^P \), which constitute an extra constraint and is secondary. [This independence perpetuates the Poisson bracket sufficing for action upon constraints.]
Outcome 4) Relations amongst the appending functions \( du^P \) themselves: specifier equations now specifying restrictions on the formulation’s auxiliary cyclic differentials.

The algebraic structure of the constraints is unaffected by passing to TRIPoD form: same constraints and same brackets for the purpose of acting on the constraints. This is other than a minor and physically inconsequential point of difference in formulation of the smearing in the field-theoretic case.

### 3.9 TRI-morphisms and brackets. ii.

Suppose there are now cyclic ordinals to be kept, or arising from the dA-Dirac procedure. Then the morphisms are a priori of the mixed type \( \text{Can}(T^*(q)) \times \text{Point}(g) \). Also the brackets are a priori of the mixed Poisson–Peierls type: Poisson as regards \( QA, PA \) and Peierls as regards \( dq^5 \).

Moreover, (69) has the following implications.

i) \( \text{Can}(T^*(q)) \times \text{Point}(g) \) act as just \( \text{Can}(T^*(q)) \) on the constraints.

ii) As far as the constraints are concerned, the brackets act as just Poisson brackets on \( QA, PA \). \textit{The physical part of the dA-Hamiltonian’s incipient bracket is just a familiar Poisson bracket}. This good fortune follows from the dA-Hamiltonian being a type of d-anti-Routhian, alongside its non-Hamiltonian variables absenting themselves from the constraints due to the best-matched form of the action.
3.10 TRi observables or beables

Much as how for investigation of these notions of observables or beables, whichever of Dirac’s Hamiltonians are equivalent, the same applies to A- and dA-variants. This includes definitions based on cases of (40) as well as the particular PDEs to solve for specific examples, since those are based on the same constraints and the same brackets. As per the end of the previous Subsec, there remains a minor point of difference in formalism of the smearing in the field-theoretic case.

Kuchař observables are further motivated in the relational approach by shuffle and chronos arising separately. See [30, 4] for a number of examples of Kuchař observables, including for arenas 2 and 5.

3.11 TRiPoD’s Hamilton–Jacobi Theory

This is a t-independent version, for a totally constrained dA-Hamiltonian. In such a case, only the constraints themselves feature. Then (69) implies that the form of this is no different from the usual PoD’s corresponding t-independent totally constrained Hamiltonian case. Thus the closeness to the Semiclassical Approach carries over from the standard PoD case to here.

The Hamilton–Jacobi formulation of relational theories thus consists of

\[
\text{chronos}(Q^A, \partial \chi / \partial Q^A) = 0,
\]

with allowed extra dependence on a meaningful constant such as \(E\) for Mechanics or \(\Lambda\) for (minisuperspace) GR. N.B. Hamilton’s characteristic function \(\chi\) is a change scalar. In the case of nontrivial \(g\), this is supplemented by

\[
\text{shuffle}_g(Q^X, \partial \chi / \partial Q^X) = 0.
\]

4 Conclusion

4.1 Three-legged summary

See Fig 3 for a summary of which Principles of Dynamics (PoD) structures and formulations require supplanting in order to be Temporal Relationalism implementing (TRi) and which are already satisfactory. The red leg is supplanted by the green leg whilst the blue leg is unaffected: the “already temporally relational” parts of the standard PoD. This mostly consists of change scalars. The new green leg is powered by the FENoS (free end notion of space) variation of Sec 3.6). The differential almost Hamiltonian formulation is as good at handling classical constraints as the Hamiltonian formulation itself, and bridges just as well to quantum theory. Its advantage is that it is TRi as well, so it has extra applicability to whole universes and other closed models. If the \(d g\) can be reduced out, many of the differences in the above figure collapse down to their forms: all \(A\)’s are lost, leaving a lesser number of entities still possessing a distinct \(d\) form. Finally, explicitly time-dependent structures such as time-dependent Lagrangians, Hamiltonians, point transformations and canonical transformations have no TRiPoD counterparts.

4.2 Further extensions of TRi

1) In the further Spacetime Construction working [13] (based in part on the earlier [10]), the Dirac Algorithm gives not only Constraint Closure but also the form of GR’s Hamiltonian constraint \(\mathcal{H}\), construction of GR-type spacetime and local Special Relativity.

Then one has to consider the following two extra facets of the PoT.

i) Spacetime Relationalism (of the unsplit but emergent spacetime) as per usual. There are to be no appended spacetime structures, in particular no indefinite background spacetime metrics. Fixed background spacetime metrics are also more well-known than fixed background space metrics. As well as considering a spacetime manifold \(\mathcal{M}\), consider also a \(g_S\) of transformations acting upon \(\mathcal{M}\) that are taken to be physically redundant. For GR, \(g_S = \text{Diff}(\mathcal{M})\).

ii) Foliation Independence of the split spacetime, for which it is now natural to consider a TRi version [26] using the TRi A-split rather than the ADM-split. This leads to a new Machian interpretation [13, 26] of GR’s Thin Sandwich [8] and of Kuchař’s universal kinematics [38].

2) One can also then consider TRi canonical quantum mechanics (TRiCQM) [20] e.g. along the lines of geometrical quantization [39]. One can then apply semiclassical approximations to pass to the intermediate case of TRi semiclassical quantum cosmology (TRiSQC).
Figure 3: The Temporal Relationalism incorporating Principles of Dynamics (TRiPoD). The powers of d displayed indicate the change tensoriality of each entity. ‘A’ stands for ‘almost’, meaning that auxiliary entities are treated differently but all physical entities are still treated the same way. Can denotes canonical transformations.

References

[1] See The Leibnitz–Clark Correspondence, ed. H.G. Alexander (Manchester 1956).
[2] C. Lanczos, The Variational Principles of Mechanics (University of Toronto Press, Toronto 1949).
[3] V.I. Arnol’d, Mathematical Methods of Classical Mechanics (Springer, New York 1978).
[4] E. Anderson, arXiv:1111.1472.
[5] See e.g. C. Marchal, Celestial Mechanics (Elsevier, Tokyo 1990).
[6] R.G. Littlejohn and M. Reinsch, Rev. Mod. Phys. 69 213 (1997).
[7] R. Arnowitt, S. Deser and C. Misner, in Gravitation: An Introduction to Current Research ed. L. Witten (Wiley, New York 1962), arXiv:gr-qc/0405109.
[8] R.F. Baierlein, D.H. Sharp and J.A. Wheeler, Phys. Rev. 126 1864 (1962).
[9] J.B. Barbour, Class. Quant. Grav. 11 2853 (1994).
[10] J.B. Barbour, B.Z. Foster and N. Ó Murchadha, Class. Quant. Grav. 19 3217 (2002), gr-qc/0012089.
[11] E. Anderson, Class. Quant. Grav. 25 175011 (2008), arXiv:0711.0288.
[12] E. Anderson, arXiv:1205.1256.
[13] E. Anderson and F. Mercati, arXiv:1311.6541.

[14] B.S. DeWitt, Phys. Rev. 160 1113 (1967).

[15] C.W. Misner, in Magic Without Magic: John Archibald Wheeler ed. J. Klauder (Freeman, San Francisco 1972).

[16] P.A.M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, New York 1964).

[17] E. Mach, Die Mechanik in ihrer Entwicklung, Historisch-kritisch dargestellt (J.A. Barth, Leipzig 1883). An English translation is The Science of Mechanics: A Critical and Historical Account of its Development Open Court, La Salle, Ill. 1960).

[18] E. Anderson, arXiv:1209.1266.

[19] G.M. Clemence, “Astronomical Time", Rev. Mod. Phys. 29 2 (1957).

[20] E. Anderson, “The Problem of Time between General Relativity and Quantum Mechanics", forthcoming book.

[21] C.W. Misner, K. Thorne and J.A Wheeler, Gravitation (Freedman, San Francisco 1973).

[22] J.A. Wheeler, in Battelle Rencontres: 1967 Lectures in Mathematics and Physics ed. C. DeWitt and J.A. Wheeler (Benjamin, New York 1968).

[23] K.V. Kuchař, in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics ed. G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992), reprinted as Int. J. Mod. Phys. Proc. Suppl. D20 3 (2011); C.J. Isham, in Integrable Systems, Quantum Groups and Quantum Field Theories ed. L.A. Ibort and M.A. Rodríguez (Kluwer, Dordrecht 1993), gr-qc/9210011.

[24] E. Anderson, in Classical and Quantum Gravity: Theory, Analysis and Applications ed. V.R. Frignanni (Nova, New York 2012), arXiv:1009.2157; Annalen der Physik, 524 757 (2012), arXiv:1206.2403.

[25] E. Anderson, arXiv:1409.4117.

[26] E. Anderson, “TRiFol (Temporal Relationalism incorporating Foliations)", forthcoming.

[27] H. Goldstein, Classical Mechanics (Addison-Wesley, Reading, Massachusetts 1980).

[28] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems (Princeton University Press, Princeton 1992).

[29] R. Peierls, Proc. R. Soc. Lond. A214 143 (1952); G. Esposito, G. Marmo and C. Stornaiolo, hep-th/0209147; B.S. DeWitt, The Global Approach to Quantum Field Theory. Volume 1 (Oxford University Press, New York 2003).

[30] E. Anderson, SIGMA 10 092 (2014), arXiv:1312.6073.

[31] P.A.M. Dirac, Rev. Mod. Phys. 21 392 (1949).

[32] K.V. Kuchař, in General Relativity and Gravitation 1992, ed. R.J. Gleiser, C.N. Kozamah and O.M. Moresci M (Institute of Physics Publishing, Bristol 1993), gr-qc/9304012.

[33] P.G. Bergmann, Phys. Rev. 124 274 (1961).

[34] C. Rovelli, Quantum Gravity (Cambridge University Press, Cambridge 2004).

[35] J.B. Barbour and B. Bertotti, Proc. Roy. Soc. Lond. A382 295 (1982).

[36] E. Anderson, J.B. Barbour, B.Z. Foster and N. Ó Murchadha, Class. Quant. Grav. 20 157 (2003), gr-qc/0211022.

[37] C. Fox, An Introduction to the Calculus of Variations (Oxford University Press, London 1950); R. Courant and D. Hilbert, Methods of Mathematical Physics Vol. 2 (John Wiley and Sons, Chichester 1989); U. Brechtken-Manderscheid Introduction to the Calculus of Variations (T.J. Press, Padstow, Cornwall 1991; English translation of 1983 German text).

[38] K.V. Kuchař, J. Math. Phys. 17 777; 792 (1976).

[39] C.J. Isham, in Relativity, Groups and Topology II ed. B. DeWitt and R. Stora (North-Holland, Amsterdam 1984).