Multiscale Proper Orthogonal Decomposition (mPOD) of TR-PIV data– a Case Study on Stationary and Transient Cylinder Wake Flows

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Abstract

Data-driven decompositions of Particle Image Velocimetry (PIV) measurements are widely used for a variety of purposes, including the detection of coherent features (e.g., vortical structures), filtering operations (e.g., outlier removal or random noise mitigation), data reduction and compression. This work presents the application of a novel decomposition method, referred to as Multiscale Proper Orthogonal Decomposition (mPOD, Mendez et al 2019) to Time-Resolved PIV (TR-PIV) measurement. This method combines Multiresolution Analysis (MRA) and standard Proper Orthogonal Decomposition (POD) to achieve a compromise between decomposition convergence and spectral purity of the resulting modes. The selected test case is the flow past a cylinder in both stationary and transient conditions, producing a frequency-varying Karman vortex street. The results of the mPOD are compared to the standard POD, the Discrete Fourier Transform (DFT) and the Dynamic Mode Decomposition (DMD). The mPOD is evaluated in terms of decomposition convergence and time-frequency localization of its modes. The multiscale modal analysis allows for revealing beat phenomena in the stationary cylinder wake, due to the three-dimensional nature of the flow, and to correctly identify the transition from various stationary regimes in the transient test case.

Keywords: Data-Driven Modal Analysis, Multi-scale Proper Orthogonal Decomposition, Time-Resolved Particle Image Velocimetry, Transient Turbulent Cylinder Wake Flow, Time Dependent Vortex Sheding.

1 Introduction and Motivation

Data-driven decompositions of PIV measurements are nowadays part of the standard toolbox for post-processing techniques used to extract knowledge from data. Initially developed for identifying coherent structures in turbulent flows (Berkooz et al. 1993, Lumley 1970, Sirovich 1987) and for reduced-order modeling (Holmes et al. 1997, 1996), these decompositions have found applications also as random noise removal (Raiola et al. 2015), image pre-processing tools (Mendez et al. 2017) and for validation of numerical simulations (Erik et al. 2007).

The scope of any decomposition is to describe the data as a linear combination of elementary information referred to as modes. Each mode has its own spatial structure and temporal evolution and is potentially capable of describing essential features of the data. The removal of irrelevant modes enables filtering and data compression, while the dominant modes offer a basis for constructing reduced models capable of significantly reducing the computational burdens in the simulations of large dynamical systems. Modal decompositions and reduced-order models
are thus fundamentals in the development of control laws (Berger et al., 2014, Brunton and Noack, 2015).

A mode can be seen as a projection of the dataset onto a particular basis element. By data-driven decomposition, one usually refers to algorithms that tailor their basis on the dataset at hand rather than pre-defining general-purpose bases such as in the discrete Fourier, Chebyshev, or Wavelet transforms. Although many decompositions fall in this category (see Amor et al. (2019), Benner et al. (2015), Rowley and Dawson (2017), Taira et al. (2017) for a review), the most common data-driven approaches can be broadly classified into two main classes: energy-based and frequency-based.

Energy-based decompositions arise from the Proper Orthogonal Decomposition (POD, Lumley (1970), Sirovich (1987, 1989)) and its variants. The POD provides the most energetic modes with no constraints on their frequency content. The spatial and the temporal structures of POD modes are eigenvectors of correlation matrices (in space and in time respectively), as dictated by the solution of a minimization problem that seeks the best approximation of the data for a given number of modes. Variants of the POD can be constructed from different choices of the inner product or in the use of different averaging procedures in the computation of the correlations. Examples of the first variants are proposed by Lumley and Poje (1997), Maurel et al. (2001), Rowley et al. (2004), where multiple quantities are involved in the inner product. Examples of the second variants are proposed by Citriniti and George (2000) and Towne et al. (2018), where the correlation matrix is computed in the frequency domain using time averaging over short windows, following the popular Welch’s periodogram method Welch (1967). It is worth noticing that this decomposition is well known in other fields as Empirical Orthogonal Functions (EOF) decomposition or Principal Component Analysis (PCA). A review of these formulations in climatology is provided by Hannachi et al. (2007) and Ghil (2002).

Frequency-based methods arise from the assumption that a linear dynamical system can represent the dataset. The most common formulation in the fluid mechanics community is the Dynamic Mode Decomposition (DMD) introduced by Schmid (2010) and Rowley et al. (2009), although analogous formulations were introduced in the late ‘80s in climatology under the names of Principal Oscillation Patterns (POP, Hasselmann (1988), von Storch and Xu (1990)) or Linear Inverse Modeling (LIM, Penland (1996), Penland and Magorian (1993)). The link between DMD, POP, and LIM is discussed by Schmid (2010) and Tu et al. (2014) and is briefly reviewed in this work.

Both the constraints of energy optimality and spectral purity can become unnecessarily extreme, as discussed by Mendez et al. (2019). Energy-based methods fail to separate phenomena that occur at widely different frequencies (scales) and yet have comparable energetic contributions. In this case, as different modes approach the limit of equal energetic contribution –that is repeated eigenvalues of the correlation matrices– the POD is non-unique, and the resulting modes become linear combinations of largely different phenomena. Frequency-based methods, on the other hand, face problems for datasets that are far from stationary and feature nonlinear phenomena such as frequency modulation, jitter, or impulsive events. A linear combination of complex exponentials cannot approximate any of these phenomena, and the failure of the underlying assumption of linear dynamics often results in the loss of decomposition convergence.

Significant effort has been placed in the development of hybrid methods, to complement the limitations of energy-based and frequency-based formulations, as recently discussed by Noack (2016). Examples of successful hybrid methods are the Spectral Proper Orthogonal Decomposition proposed by Sieber et al. (2016), the multiresolution Dynamic Mode Decomposition by Kutz et al. (2016), the Recursive Dynamic Mode Decomposition by Noack et al. (2016) or Cronos-Koopman analysis by Cammilleri et al. (2013). These methods propose ingenious combinations of POD and DMD of various degrees of complexity and are designed for data sets that are statistically stationary (hinging on the time-frequency duality of the POD) or have short-duration departures from fixed points (hinging on the linearization of the dynamics in the
DMD).

For datasets featuring transient evolution between different states and generally nonlinear dynamics, these methods lose their theoretical foundations and yield poor feature detection capabilities. The Multiscale Proper Orthogonal Decomposition (mPOD) proposed by Mendez et al. (2018b, 2019) is a hybrid decomposition that requires neither assumptions of stationary data nor linear dynamics. The mPOD combines Multi-resolution Analysis (MRA) via filter banks and standard POD to produce modes that are optimal within a certain range of frequencies (scales). The MRA is used to decompose the correlation matrix of the data into the contribution of different scales, each retaining non-overlapping portions of the frequency spectra and leading to a POD constrained within the corresponding frequency bandwidth. The set of POD of each scale is then used to assemble the mPOD basis, which is kept orthogonal by construction. A brief description of the decomposition is presented in Section 2.

In this work, two experimental test cases are considered to analyze the performance of the mPOD further and compare it to classical energy-based and frequency-based tools. These investigated cases consist of Time-Resolved Particle Image Velocimetry (TR-PIV) of the turbulent flow past a cylinder in stationary and transient conditions, with varying free stream velocity and hence varying frequency of the vortex shedding. While the stationary test case is one of the most classical paradigms for wake flows exhibiting large scale vortex shedding (see Williamson (1996) for a review), the transient configuration in turbulent conditions has received considerably less attention. Previous investigations on data-driven decomposition of the transient cylinder focus on the onset of the vortex shedding, both for reduced-order modeling (Murata et al., 2019, Noack et al., 2016, 2003, Pawar et al., 2019, Siegel et al., 2008) and flow control applications (Bergmann and Cordier, 2008, Grönskis et al., 2009, Rabault et al., 2019).

It is worth highlighting that the cylinder wake flow at the Reynolds number investigated ($Re = [4000 − 2600]$) is inherently three-dimensional, characterized by the interaction of quasi-two-dimensional structures with predominantly span-wise vorticity and longitudinal structures in the traverse plane (see Chen et al. (2017), Hussain and Hayakawa (1987), Wu et al., (1996)). A detailed experimental characterization of this interaction is out of the scope of this work (interested readers are referred to Huang et al. (2006), Sung and Yoo (2001), Zhang et al. (2000), Zhou et al. (2003)), which focuses on the assessment of the enhanced feature detection capabilities of the proposed mPOD.

The rest of the article is structured as follows. Section 2 briefly reviews the general formulation of energy-based and frequency-based methods, and introduces the mPOD. Section 3 presents the experimental setup and selected test cases, while Section 4 collects the results. Conclusions and perspectives are reported in Section 5.

2 The Data-Driven Decompositions: POD, DMD/OPD and mPOD

The dataset investigated in this work is a set of 2D TR-PIV velocity fields $\vec{u}(x_i, t_k) = (u(x_i, t), v(x_i, t))$ over a uniform grid $x_i \in \mathbb{R}^{nx \times ny}$, with $i$ a matrix linear index, and a temporal discretization $t_k = \{(k - 1)\Delta t\}_{k=1}^{k=n}$ with $\Delta t = 1/f_s$ the time step and $f_s$ the sampling frequency. In what follows, bold fonts are used to indicate matrices.

All the data-driven decompositions discussed in this work break a discrete dataset into a linear combination of modes. Each mode has a spatial structure $\vec{\phi}_r(x_i)$, a temporal structure $\psi_r(t_k)$, and an amplitude $\sigma_r$ that weights its importance. The sampled velocity field is thus expanded as

$$\vec{u}(x_i, t_k) = \sum_{r=1}^{R} C_r(x_i) \psi_r(t_k) = \sum_{r=1}^{R} \sigma_r \vec{\phi}_r(x_i) \psi_r(t_k),$$  (1)
where the contribution of each mode is defined as \( \sigma_r = \| \tilde{C}_r(x_i) \|_2 \) and the spatial structures are obtained via normalization \( \tilde{\phi}_k = \tilde{C}_r(x_i) / \sigma_r \).

The number of possible modes \( R \) – that is the number of non-null amplitudes \( \sigma_r \) – defines the rank of the dataset. A truncation of the summation to \( \tilde{R} < R \) modes, assuming that these are sorted in descending order of energy contribution, defines the best rank \( \tilde{R} \) approximation of the data for a specific decomposition. The convergence of the decomposition as a function of the number of modes included can be monitored in terms of relative \( L_2 \) error in both space and time:

\[
\mathcal{E}(\tilde{R}) = \frac{\| \tilde{u}(x_i, t_k) - \sum_{r=1}^{\tilde{R}} \sigma_r \tilde{\phi}_r(x_i) \psi_r(t_k) \|_2}{\| \tilde{u}(x_i, t_k) \|_2}
\]

(2)

The decompositions used in this work, namely the POD, the DMD/OPD, and the mPOD are hereinafter distinguished using the subscripts \( P, D \) and \( M \) respectively.

2.1 Energy-Based Formalism: the POD

The Proper Orthogonal Decomposition arises by minimizing the error in \( \mathcal{E}(\tilde{R}) \) \( \forall \tilde{R} \in [1, R] \) or, equivalently, by maximizing the energy contribution of every mode. Both optimization problems are solved (see Holmes et al. [1997] or Bishop [2011] in the framework of Principal Component Analysis, PCA) by taking the temporal structures as eigenvectors of the temporal correlation matrix. This matrix is defined as \( \mathbf{K}[k, n] = \langle \tilde{u}(x_i, t_k), \tilde{u}(x_i, t_n) \rangle_T \), with \( \langle \bullet \rangle_T \) the inner product in the time domain. Hence, the temporal structures of the POD satisfy the eigenvalue problem

\[
\mathbf{K} \psi_{P_r}(t_k) = \lambda_r \psi_{P_r}(t_k) \quad \forall r \in [1, R] \rightarrow \mathbf{K} = \sum_{r=1}^{R} \lambda_r \psi_{P_r}(t_k) \psi_{P_r}^T(t_k)
\]

(3)

having considered each temporal structure as a column vector, and having recalled, in the last summation, that the correlation matrix is symmetric and positive definite. The POD structures are, therefore, orthonormal.

Furthermore, one could show that \( \lambda_r = \sigma_{P_r}^2 \) and \( \mathcal{E}(\tilde{R}) = \sigma_{P_{\tilde{R}+1}} \). The normalization step in the calculation of the spatial structures in \( \mathbf{C}[i, n] \) is thus no longer necessary, and the corresponding spatial structures can be computed, hinging on the orthonormality of the temporal basis, using the inner product in the time domain, that is

\[
\tilde{\phi}_k(x_i) = \frac{1}{\sigma_r} \langle \tilde{u}(x_i, t_k), \psi_{P_r}(t_k) \rangle_T = \frac{1}{\sigma_r} \sum_{k=1}^{n_i} \tilde{u}(x_i, t_k) \psi_{P_r}(t_k).
\]

(4)

It is possible to show that these spatial structures are by construction also orthonormal and eigenvectors of the spatial correlation matrix \( \mathbf{C}[i, n] = \langle \tilde{u}(x_i, t_k), \tilde{u}(x_n, t_k) \rangle_s \), with \( \langle \bullet \rangle_s \) the inner product in the space domain. The computation of the POD modes from the eigenvalue decomposition of \( \mathbf{C} \) was originally presented by Lumley [1970] and is often referred to as standard POD. The computation of the spatial structures from the temporal ones via \( \mathbf{D} \) and \( \mathbf{C} \) was proposed by Sirovich [1991] and it is often referred to as snapshot POD.

The equivalence of the two approaches, first analyzed by Aubry et al. [1991], can be easily revealed by writing the decomposition as a factorization of the dataset matrix. This is the matrix \( \mathbf{D} = [d_1, d_2, \ldots, d_n] \subset \mathbb{R}^{n_s \times n_t} \), obtained by flattening each of the \( k \)th snapshot into a column vector \( d_k \subset \mathbb{R}^{n_s \times 1} \). Here \( n_s = 2n_t n_y \) denotes the dimension of the snapshot and the factor 2 accounts for the vector nature of the 2D velocity field. If the Euclidean inner product is used in both space and time, the POD is equivalent to a Singular Value Decomposition of \( \mathbf{D} \). For later convenience, we introduce it here as \( \mathbf{D} = \mathbf{\Phi}_P \Sigma_P \mathbf{\Psi}_P^T \), where \( \mathbf{\Phi}_P = [\phi_{P_1}[i], \ldots, \phi_{P_R}[i]] \subset \mathbb{R}^{n_s \times R} \) is the matrix of spatial structures, \( \Sigma_P = diag[\sigma_{P_1}, \ldots, \sigma_{P_R}] \) is a diagonal matrix containing the contribution of each mode, and \( \mathbf{\Psi}_P = [\psi_{P}[k], \ldots, \psi_{P_R}[k]] \subset \mathbb{R}^{n_t \times R} \) is the matrix of temporal
structures. In what follows, a tilde is used to denote basis matrices that are reduced, i.e. include only the columns corresponding to the first $\tilde{R}$ dominant modes.

### 2.2 Frequency-Based Formalism: The DMD and the OPD

In the Dynamic Mode Decomposition (DMD) and the Oscillatory Pattern Decomposition (OPD), the temporal structures are complex exponentials with complex frequencies, proposing, in essence, a discrete version of the Laplace transform similar to the Z-transform (see Oppenheim and Schafer 2009). The temporal structures are thus constructed as powers of a complex number $\lambda_r \in \mathbb{C}$. Starting the time discretization starts such that $t_0 = 0$ for $k = 1$, these structures can be written as follows:

$$\psi_{Df}(t_k) = \lambda_r^{k-1} = (|\lambda_r| e^{i\theta_r})^{(k-1)} = |\lambda_r|^{(k-1)} e^{i\theta_r(k-1)} = |\lambda_r|^{(k-1)} e^{i2\pi f_r \Delta t (k-1)} = |\lambda_r|^{(k-1)} e^{i2\pi f_r t_k},$$

(5)

where $\theta_r = \text{arg}(\lambda_r)$ is the argument of the complex number $\lambda_r$ and the oscillation frequency associated to each mode is $f_r = \theta_r/(2\pi \Delta t)$.

In the Discrete Fourier Transform (DFT), the temporal basis is obtained by choosing the temporal mean has been subtracted (see Chen et al. (2012)). The temporal structures are thus constructed as powers of complex exponentials with complex frequencies, proposing, in the Discrete Fourier Transform (DFT), the temporal basis is obtained by choosing the temporal mean has been subtracted (see Chen et al. (2012)). The temporal structures are thus constructed as powers of complex exponentials with complex frequencies, proposing, in the Discrete Fourier Transform (DFT), the temporal basis is obtained by choosing the temporal mean has been subtracted (see Chen et al. (2012)). The temporal structures are thus constructed as powers of complex exponentials with complex frequencies, proposing, in the Discrete Fourier Transform (DFT), the temporal basis is obtained by choosing the temporal mean has been subtracted (see Chen et al. (2012)).

In data-driven harmonic decomposition, aiming at approximating the dataset as a linear dynamical system, the complex numbers $\lambda$, are eigenvalues of the propagator $P$ that advances one snapshot $\bar{u}(x_i, t_k)$ to the following –that is $\bar{u}(x_i, t_{k+1}) = P \bar{u}(x_i, t_k)$. Flattening each snapshot into a column vector $d_k$, the linear dynamical system is defined as $d_{k+1} = PD_k$, with the propagator being a square real matrix $P \in \mathbb{R}^{n_s \times n_s}$.

Since the least square calculation of such propagator is computationally prohibitive in most real applications, the fitting of the linear dynamical system is carried out on a reduced space: the one spanned by the first $\tilde{R}$ POD modes of the dataset. Splitting the dataset matrix into two shifted portions, that is $D_1 = [d_1, \ldots, d_{n_1-1}]$ and $D_2 = [d_2, \ldots, d_{n_1}]$, the propagator $P$ and its reduced counterpart $\tilde{S}$ are:

$$D_2 = PD_1 \rightarrow \tilde{\Phi}_P D_2 \approx \tilde{\Phi}_P P \tilde{\Phi}_P \tilde{\Phi}_P D_1 \rightarrow \tilde{V}_2 \approx \tilde{S} \tilde{V}_1$$

(6)

where the $L_2$ approximation lies in the second step, i.e. in the assumption that $\tilde{\Phi}_P \tilde{\Phi}_P^T \approx I$ with $I$ the identity matrix of size $\tilde{R} \times \tilde{R}$.

The frequency-based formalism for data-driven decomposition is therefore based on the computation of the eigenvalues $\lambda_r$ of the reduced propagator $\tilde{S}$. In the classical DMD (Schmid 2010; Tu et al. 2014) this propagator is defined as follows:

$$D_2 = PD_1 \rightarrow P \approx D_2 D_1^+ = D_2 \tilde{\Psi}_P \tilde{\Sigma}_P^{-1} \tilde{\Phi}_P^T$$

$$\rightarrow \tilde{S} = \tilde{\Phi}_P^T P \tilde{\Phi}_P = \tilde{\Phi}_P^T D_2 \tilde{\Psi}_P \tilde{\Sigma}_P^{-1},$$

(7)

having introduced the Moore-Penrose inverse $D_1^+ = \Psi_P \Sigma_P^{-1} \tilde{\Phi}_P^T$. Related codes available at [https://github.com/mendezVKI/MODULO](https://github.com/mendezVKI/MODULO) Pre-print submitted to Meas Sci and Tech.
In the literature of Principal Oscillatory Patterns (PIP, Hasselmann (1988), Penland (1996), von Storch and Xu (1990)), the propagator is defined in terms of spatial covariance matrix $D_1 D_1^T$. Multiplying both sides by $D_1^T$ and inverting the spatial covariance matrix gives:

$$D_2 = PD_1 \rightarrow D_2 D_1^T = PD_1 D_1^T \implies P \approx D_2 D_1^T (D_1 D_1^T)^{-1}$$

(8)

The PIP introduced in Penland (1996) is based on the assumption that the POD of $D_1$ and $D_2$ share the same spatial structures and amplitude of the full dataset, and differ only in a shift of the temporal structures, that is $D_1 = \Phi_P \Sigma_P \Psi_{p1}$ with $\Psi_{p1} = \Psi_P(t_1, \ldots, t_{n_2} - 1)$ and $D_2 = \Phi_P \Sigma_P \Psi_{p2}$ with $\Psi_{p2} = \Psi_P(t_2, \ldots, t_{n_1})$. Introducing these ‘shifted’ PODs in (7) and projecting on the the reduced POD basis yields the following reduced propagator

$$\tilde{S} = \Phi_P P \Phi_P = \Sigma_P \Psi_{p1}^T \Psi_{p2} \Sigma_P^{-1},$$

(9)

noticing that $(D_1 D_1^T)^{-1} \approx \Phi_P \Sigma_P^{-2} \Phi_P^T$ if $\Phi_P \Phi_P^T \approx I$.

This propagator is computationally slightly less expensive than the one in (7) and is used in the Oscillatory Pattern Decomposition (Uruba 2012, 2015, Uruba and Procházka 2019) implemented in the Dantec’s software DynamicStudio (Ergin et al., 2014, Petersson et al., 2012). Noteworthy, if the shifted POD of $D_2$ is introduced in (7), the same reduced propagators are obtained. For the datasets analyzed in this work, no appreciable difference between the two formulations is observed, and only the results with the classical DMD using the propagator in (7) are presented. For alternative formulations of the reduced propagators, the reader is referred to Pan et al. (2015), Penland (1996), Schmid (2010), Tu et al. (2014), while an interesting higher-order extension of the linear propagator is proposed in Clainche and Vega (2017a,b, 2017a), Martinez et al. (2017). Finally, writing the diagonalization of the reduced propagator as $\tilde{S} = Q \Lambda Q^{-1}$, the (complex) spatial structure from these decompositions can be computed as $\Phi_P = \Phi_P Q$.

2.3 The Multiscale Proper Orthogonal Decomposition (mPOD)

The Multiscale Proper Orthogonal Decomposition (Mendez et al. 2018b, 2019) presents a compromise between the POD and the DMD/OPD by adding spectral constraints to the energy optimality of the POD.

These constraints consist of breaking the datasets into the contributions of various scales using Multi-Resolution Analysis (MRA), as commonly proposed in Wavelet theory (Mallat 2009), and in performing the POD for each scale.

A scale $m$ corresponds to a range of frequencies, say $[f_{m1}, f_{m2}]$, identified by a transfer function, here indicated as a row vector $H_m \in \mathbb{C}^{1 \times n_1}$. This transfer function is ideally unitary (‘all pass’) within the range of interest and null (‘all stop’) otherwise. In the formulation presented in Mendez et al. (2018b), the transfer function of a given scale is applied in an equal manner to all the spatial realizations: to this end, the transfer function is copied row-wise in order to create a matrix of appropriate size $H_m' \in \mathbb{C}^{n_2 \times n_1}$, later applied via direct matrix multiplication. Moreover, these transfer functions are taken as complementary, that is $\sum_m H_m' = 1$, and $H_i \odot H_j = 0 \forall i \neq j$, where $1 \in \mathbb{R}^{n_2 \times n_1}$ is the unitary matrix (‘all-pass’) and $\odot$ is the Shur (entry by entry) product.

The dataset decomposition into the contributions $D_m$ of $M$ scales is:

$$D = \sum_{m=1}^{M} D_m = \sum_{m=1}^{M} \left[ D \Psi_{m}^T \odot H_m' \right] \Psi_{m}$$

(10)

where $\Psi_{m} \in \mathbb{C}^{n_2 \times n_1}$ is the Fourier matrix $\Psi_{m}[i, j] = \exp(2 \pi i j n_1)$, with $i, j = [1, n_2]$, the over bar denotes complex conjugation. Note that right multiplication by $\Psi_{m}$ produce the row-wise Fourier transform of a matrix (i.e. in the time domain for $D$) while multiplication by

Related codes available at https://github.com/mendezVKI/MODULE. Pre-print submitted to Meas Sci and Tech.
its conjugation (i.e. inverse, since $\Psi F \Psi = I$) produce the row-wise inverse Fourier transform. The hat is used to denote the Fourier transform in time of the data ($\hat{D} = D \Psi F$) and that of each contribution ($\hat{D}_m = D_m \Psi F$).

Every contribution has its own POD, which could be used to construct suitable approximation of each scale:

$$D = \sum_{m=1}^{M} D_m = \sum_{m=1}^{M} \psi_p^{(m)} \sum_p \psi_p^{(m)T} = \sum_{m=1}^{M} D_m \psi_p^{(m)} \psi_p^{(m)T}. \quad (11)$$

The Multiscale POD is constructed by assembling the final mPOD temporal basis from all bases of all the scales. The mPOD algorithm saves computational time by operating the MRA on the correlation matrix and not on the dataset matrix. In particular, the correlation matrix is split into the contribution of all the scales using an appropriate filter bank, i.e.:

$$K = \sum_{m=1}^{M} K^{(m)} = \sum_{m=1}^{M} \Psi F \left[ \hat{K} \circ \mathcal{H}_m \right] \Psi F = \sum_{m=1}^{M} \left( \sum_{r=1}^{n_m} \lambda_r^{(m)} \psi_r^{(m)} \psi_r^{(m)T} \right) \quad (12)$$

where $\hat{K} = \overline{\Psi F} K \overline{\Psi F}$ is the 2D Fourier transform of the correlation matrix, $n_m$ the number of non-zero eigenvalues at each scale, and $\mathcal{H}_m = H_m^T H_m'$ an appropriate 2D transfer function.

The eigenvalue decomposition of each contribution is introduced on the right-hand side of (12). The filters preserve the symmetry of the correlation matrices at each scale and hence the orthogonality of their eigenvectors. Moreover, avoiding frequency overlapping between different contributions to the correlation matrix, it is possible to keep the eigenvectors of different scales mutually orthogonal (see Mendez et al. (2019) and hence orthogonal complements spanning the entire $\mathbb{R}^{n_t}$ space (that is $\sum_{m=1}^{M} n_m = n_t$). This is achieved by constructing the 2D transfer functions, including only ‘pure’ terms, that is, the correlation of a transfer function with itself. In wavelet terminology, this corresponds to performing a Wavelet decomposition of the correlation matrix $K$ and retaining only the ‘approximation’ and ‘diagonal details’.

More information on the MRA formulation via dyadic wavelet decomposition is available in Mendez et al. (2018a,b). In this work, the MRA is performed using a generalized filter bank, as in Mendez et al. (2019), constructed using standard FIR filters designed via windowing method (?).

The resulting decomposition lets the mPOD recover the energy optimality of the POD at the limit of a single scale (spanning the entire frequency range), and the spectral purity of the DMD at the limit of $n_t$ scales (each taking one frequency bin).

3 Experimental Set Up and Selected Test Case

The selected test case is the flow past a cylinder of $d = 5$ mm diameter and $L = 20$ cm length in transient conditions, with varying free stream velocity. The experiments were carried out in the L10 low-speed wind tunnel of the von Karman Institute, instrumented with a TR-PIV system from Dantec Dynamics. The tunnel has a cross-section of 20 cm $\times$ 20 cm and is equipped with a piezoresistive pressure transducer AMS5812 to monitor the pressure in the honeycomb chamber. A Laskin Nozzle PIVTEC45-M, operating with mineral oil Ondina Shell 91, is used to produce seeding particles of about 1.5 $\mu$m in diameter, injected in the intake manifold of the wind tunnel fan.

The particles are illuminated with a Photonics Industries DM20-527DH Nd:YLF laser offering 20 mJ/pulse at 1 kHz. The exposed scene of about 70 $\times$ 26 mm is recorded by a SpeedSense 9090 camera offering 7500 fps at a resolution of 1280 $\times$ 800 px. To extend the measurement duration, the sensor is cropped to 1280 $\times$ 720 px resulting in about $n_t = 13500$ double frames, sampled at a frequency of $f_s = 3 kHz$. At this frequency, the available light intensity is of the
order of 6 mJ/pulse. The full dataset covers about 4.5 seconds. A picture of the experimental set-up is shown in Figure 1a).

![Picture of the experimental set-up](image)

Figure 1: Fig a) Picture of the experimental set up during a TR-PIV measurement. Fig b) Instantaneous velocity field and normalized vorticity (that is scaled between $[-1,1]$). Fig c) time evolution of the free stream velocity $U_\infty$, taken from a point far from the cylinder surface.

The DynamicStudio software is used for acquisition and analysis. For the initial data processing, standard adaptive PIV (see [Theunissen (2010)]) is used with an initial interrogation area of 96 x 96 px and final size of 24 x 24 px, the vectors are calculated every 12 px resulting in an overlap of 50%. Before the mPOD analysis, the only filter applied is the universal outlier detection with a filter kernel of 3 x 3 vectors, as suggested by [Westerweel and Scarano (2005)].

An example of instantaneous velocity is shown in Figure 1b, together with the vorticity field normalized by the maximum and minimum values so as to have $\omega \in [-1,1]$. In the investigated test case, the velocity of the free stream $U_\infty$ evolves through two steady state conditions, namely from $U_\infty = 12.1 \pm 3\%$ to $U_\infty = 7.9 \pm 3\%$ m/s, as shown in Figure 1c). The transition between these is a smooth step of approximately 1s. The Reynolds number varies from $Re \approx 4000$ to $Re = 2600$ and the frequency of the vortex shedding varies from 450 Hz to 303 Hz, corresponding to a Strouhal number of about $St = f d/U_\infty \approx 0.19$ in both stationary regimes. The entire range of Reynolds number experienced during the test falls in the three-dimensional vortex shedding regime (see [Williamson (1996)]).

4 Results and Discussion

The analysis is divided into two parts: the stationary conditions and the transient conditions. In the stationary conditions, discussed in subsection 4.1 only the first portion of the dataset is considered. Referring to figure 1b), this covers a duration of $[0,1.33]$ and consists of $n_t = 4000$ snapshots. Since the PIV acquisition starts several minutes after the wind-tunnel has reached its stationary regime, this configuration can be seen as a fully developed stationary test case. In the transient conditions, discussed in subsection 4.2 the full set of $n_t = 13500$ snapshots is considered in the decomposition.

4.1 Analysis of the Steady State Condition

The mean flow field $U = \langle \bar{u} \rangle$ and the root mean square field $u_{rms} = \sqrt{\langle u'^2 \rangle + \langle v'^2 \rangle}$, with $(u',v')$ the mean shifted velocity components, are shown in Figure 2(a) and Figure 2(b) respectively.

Related codes available at [https://github.com/mendezVKI/MODULO](https://github.com/mendezVKI/MODULO) Pre-print submitted to Meas Sci and Tech.
The root-mean-square reaches a maximum of about $u_{\text{rms}} = 6.9 \text{m/s}$ at approximately 16mm from the cylinder axis. The flow statistics being well developed, the time average is subtracted before performing all the decompositions.

![Figure 2](image_url)

**Figure 2:** Mean velocity field (a) and root mean square (b) of the investigated test case. Both color-maps are in m/s while spatial units are in mm.

The key results from the DMD/OPD are collected in Figure 3. Figure 3a) shows the eigenvalue spectrum from the DMD/OPD, while 3b) and 3c) show the real and the imaginary part of the spatial structure associated with the dominant DMD/OPD mode. As stated earlier, no appreciable difference is observed between the propagators in (7) and (8), and hence only the results from the first are shown.

The plot of the eigenvalue spectra is shown in terms of modulus of the reduced propagator’s eigenvalue $\tilde{S}$ versus the associated frequency (only $f_n \geq 0$ are shown). The spectra is constructed for two choices of reduced POD basis, namely including $\tilde{R} = 10$ (red squares) or $\tilde{R} = 100$ POD modes (white circles) in (7) out of the $n_t = 4000$ available modes.

For both reduced POD bases, one eigenvalue with modulus $|\lambda_r| \approx 1$ is visible in the plots: this is linked to the dominant frequency in the flow produced by the vortex shedding. This eigenvalues, however, has a modulus slightly below unity ($|\lambda_r| = 0.9935$), possibly due to round-off errors, measurement noise, or insufficient sampling. As a result, this mode (as well as all the others) has a temporal structure that is exponentially decaying: the half-life of the dominant mode is of the order of 100 steps, i.e., about 3.3ms, while all the other vanishes faster.

The energy associated with the DMD modes $\sigma^2_D$ is hence of little interest and not shown. As discussed in Section 2.2, these are the correlation between the initial snapshot of the data and the computed spatial basis, but the vanishing nature of the modes and the normalization of temporal and spatial structures make them unrepresentative of their relative importance.

The major limitation of the DMD is thus convergence: regardless of the size of the reduced POD basis, the vanishing of the modes make the error in (2) of the order of 95% even if all the modes of the decomposition are included.

The spatial structure associated with the dominant DMD modes in Figure 3b)-c) features a periodic pattern with a stream-wise wavelength of the order of the cylinder diameter. The first large vortical structure of the wake is centered at a distance that of $\approx 16 \text{mm}$ from the cylinder. Real and imaginary parts are phase-shifted in space, due to the harmonic ‘traveling wave’ nature of the flow.

Related codes available at [https://github.com/mendezVKI/MODULO](https://github.com/mendezVKI/MODULO) Pre-print submitted to Meas Sci and Tech.
In order to identify the structures associated with harmonic (i.e., not decaying) modes for these stationary test cases, the results of a DFT is also considered in Figure 4. Figure 4a) shows the energy contributions of the DFT modes as a function of their associated frequency (only $f_n > 0$ is shown) while Figures 4b)-c) show the real and the imaginary parts of the dominant mode. The energy spectrum is narrow and centered around the dominant frequency of the vortex shedding, labeled in the plot, and in good agreement with the one identified by the DMD.

Although this mode is purely harmonic (with no decay), the real part of the spatial structures is similar to the DMD ones. The imaginary part, on the other hand, is not: while the classic overall spatial shift is also visible, the DFT mode displays a saddle point in place of the vortex core of the corresponding imaginary part in the DMD mode. Since the DMD and the DFT modes evolve at nearly the same frequency, this difference can only be due their different duration: the saddle pattern appears as more representative of the flow evolution in the entire dataset while the structure identified by the DMD mode vanishes within 1/5 of the observation time. The flow topology of the saddle pattern is described by Chen et al. (2017), Hussain and Hayakawa (1987) as the result of the interaction between phase-shifted counter-rotating vortices produced in the von Karman street (see Fig. 15 in Chen et al. (2017)). This is also the region of larger strain and thus turbulence production; not surprisingly, the saddle point is located close to the
region of larger root-mean-square velocity, as shown in Figure 2b).

To conclude the analysis of the stationary test case using state-of-the-art data-driven decomposition, the results for the POD, collected in Figure 5, are now considered. Figures 5a,b) show the spatial structure of the two dominant POD modes. These two modes appear strongly linked, with nearly identical temporal evolution and frequency content. The time-frequency evolution of the first, using Continuous Wavelet Transform, is shown in 5c); the CWT of the second, being almost identical, is not shown. The CWT is performed using MATLAB’s function cwt with a complex Morlet wavelet with bandwidth parameter $f_b = 5$ and center frequency $f_c = 1$ (Misiti et al., 2015) as a compromise between frequency and time resolution. The DFT of the associated temporal structures is shown in 5d), normalized with respect to the dominant frequency. Figure 5e) presents their phase portrait (Lissajous curves) produced by plotting one temporal structure versus the other.

Since the POD modes are real, the traveling wave pattern is described by two modes in quadrature, which is clearly highlighted by the circular pattern produced by the Lissajous curve. The regularity of these patterns shows that the phase shift between the modes is overall time-invariant, similarly to what occurs in a DFT mode. As a result, the spatial structures of these two modes appear qualitatively similar to the real and the imaginary counter parts of the dominant DFT mode.

Nevertheless, the Lissajous curve shows a strong amplitude modulation, as the radius of

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Figure 4: Spectra of the DFT amplitudes ($\text{diag}(\Sigma_D)$) the stationary test case with $U_\infty = 12.1 \text{m/s}$. The dominant frequency of the vortex shedding is clearly visible. The real and the imaginary parts of the associated spatial structure are shown in b) and c) respectively.

Related codes available at https://github.com/mendezVKI/MODULO. Pre-print submitted to Meas Sci and Tech.
Figure 5: Results from the POD analysis of the stationary test case. The spatial structures of the dominant POD modes are shown in (a) and (b). Figure (c) shows the time-frequency evolution of their temporal structures via CWT while Figure (d) shows their DFT. These modes have similar evolution, up to a $\pi/2$ phase difference, as shown by the phase portrait (Lissajous curve) in Figure (e).

The system trajectory varies in time between two extreme values, indicated in the figure with dashed red lines. Since the frequency spectra of these modes appear time-invariant and narrow in the frequency domain, as shown by the CWT in Figure (c), this modulation can only be explained by two mechanisms: 1) the contribution due to phenomena having broad frequency spectrum (e.g. noise of turbulence) 2) the competition between harmonic modes having frequency close to the dominant shedding frequency.

The distinction between these two mechanisms cannot be revealed either by the DFT, because its harmonic modes do not admit modulation in time, neither by the POD, because its modes have too broad frequency content. In particular, it is interesting to observe that the
DFT spectra of the temporal structures of the POD, shown in Figure 5d, is very similar to the DFT spectra of the entire dataset in Figure 4a. This is a direct consequence of the energy optimality, which allows these two POD modes to capture a significant amount of the dataset’s energy, at the cost of mixing in the same modes a large range of scales. As shown by the stem plot zooming around the dominant peak, these modes have both contributions from frequencies close to the dominant one and larger tails of smaller amplitudes that extends over a frequency range of about 300 Hz. It is worth highlighting that the observation time covers about 5330 periods of the dominant frequency, and hence these secondary peaks are unlikely due to windowing phenomena (i.e., spectral leakage, see [Harris 1978]).

Figure 6: (a) Contour of the modulus of the Fourier transform of the correlation matrix \( \hat{K} \) and (b) auto-correlation spectra (i.e. diagonal of \( \hat{K} \)) with the modulus of the transfer functions of the selected scales (\( \text{diag}(\Sigma_D) \)).

The results from the mPOD are now discussed. For the purpose of this work and to illustrate the flexibility of this decomposition, it is interesting to obtain modes that have cleaner spectra than the POD ones in order to analyze the origin of the shedding modulation previously described. On the other hand, it is also of interest to achieve this without significantly compromising the convergence of the POD. To this end, only three scales are selected, starting from the one that should isolate the vortex shedding. The selection of the frequency range of each scale is illustrated in Figure 6 along with the auto-correlation spectra of the dataset (shown in 6b), that is the diagonal of the 2D Fourier transform of the correlation matrix \( \hat{K} \) (shown in 6a).

The correlation spectra show an important frequency content at a range \(< 100 Hz\); this is not visible in the first two dominant POD modes, but it is strongly present in the third and fourth POD modes (not shown). The scale centered around the vortex shedding is identified by a band-pass filter \( H_1 \) with pass-band \( f \in [445, 455] Hz \), while the other two scales take the remaining larger (filter \( |H_C| = 1 \) in \( f \approx [0, 445] Hz \)) and finer (filter \( |H_3| = 1 \) in \( f \approx [455, 1500] Hz \)) scales. It is essential to observe that these spectral constraints only impose that a mode having frequency content in one scale does not have frequency content in others. Each scale is equipped with its POD, and the final mPOD basis is constructed by sorting these POD from various scales by energy contribution (associated eigenvalue).

All the filters are constructed using a windowing method with a Hamming window and kernels of size \( n_I = 2000 \). The filtering is performed on the temporal correlation matrix using Matlab’s function \textsc{imfilter} with replicating boundary conditions (‘\textsc{replicate}’ option). All the Matlab codes for the mPOD implementation are available online and extensively described by Ninni and Mendez (2020).

The first two mPOD modes are shown in figure 7; the third and fourth are shown in figure 8. Both pairs share the same frequency spectra as for the POD pairs and the figures follow the same structure of figure 5.

The first mPOD pair of modes arise from the scale \( H_1 \), that is the one centered around the...
dominant vortex shedding frequency. The spatial structures of both modes feature the saddle region in the wake flow, and their CWT shows the modulation of their evolution in time. Because of their narrow frequency bandwidth, this modulation can only be linked to the interaction of different harmonic modes close to the shedding, which results in a beating phenomenon. This phenomenon is well described in standard textbooks on mechanical vibrations (e.g., Rao (2010)) and results from the interference of two harmonics motion having a similar frequency. The phase portrait of these modes (Figure 7e) shows that the beating accounts for a significant portion of the modulation observed in the POD modes.

The second mPOD pair of modes in Figure 8 arise from the large scale $L$ and is characterized by a frequency content which extends over the entire allowed spectra. A finer frequency selection could be used to reduce the frequency bandwidth of these modes, but for the purposes of this work it suffices to observe that these are not stationary: comparing the CWT transform of these modes in Figure 8c) with the one of the first two in Figure 7 shows that their contributions are complementary– the mode pair $3-4$ is most active when the mode pair $1-2$ is not. This
intermittency results in the phase portraits that continuously spirals in and out (see Fig. 8c).

The spatial structures of these modes show the dominant vortical structure also observed in the harmonic decomposition. The mPOD describes, therefore, the dataset as a rather intermittent interplay of modes: the vortex shedding appears to slightly change its frequency content in time resulting in a temporal modulation and beating of the shedding amplitude mostly occurring at the dominant frequency detected by DFT and DMD. This intermittent effect is possibly linked to the three-dimensional nature of the flow and particularly to the interaction between the quasi two dimensional Karman street with span-wise structures, that are characterized by similar frequency content (Zhang et al., 2000, Zhou et al. 2003).

Modes belonging to the highest frequency portion $H_3$ have much lower energy and are not among the first ten mPOD modes – these are therefore not shown.

Finally, to conclude the analysis of the stationary test case, the decomposition convergence is discussed. Figure 9 shows the $L_2$ error over the full dataset (in this case, the first $n_t = 4000$ snapshots) as a function of the number of modes used in the approximation. The convergence
results of the DMD/OPD are not shown since this decomposition does not converge to the dataset. While the spectral constraints in the mPOD do not allow to reach the optimal convergence of the POD, the two decompositions reach comparable convergence for $\tilde{R} > 8$. In particular, it appears that the information contained in the first five POD modes (producing $\mathcal{E} = 0.204$) is distributed to the first six mPOD modes (producing $\mathcal{E} = 0.200$). This result shows that the mPOD allows for more degrees of freedom than the POD in the modal analysis while maintaining comparable convergence, contrary to harmonic methods. For comparison purposes, the plot also shows the convergence of the DFT in time, which is considerably poorer. This represents the limit at which the mPOD tend if the spectral constraints in the MRA are increased to the extreme of having only one harmonic per scale.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\linewidth]{figure9.png}
\caption{Decomposition error $\mathcal{E}$ in (2) as a function of the number of modes included in the summation (1) for the POD, the mPOD and the DFT of the stationary dataset at $U_\infty = 12.1 \text{m/s}$.}
\end{figure}

4.2 Analysis of the Transient Conditions

The full set of velocity fields is now considered, with free stream velocity evolving, as shown in Figure 10. Since the flow is in transient conditions, the time-averaged flow is of little significance, and it is therefore not removed from the dataset before performing POD and mPOD. For the DFT, the mean removal is only a matter of plotting purposes (since the mode at $f = 0$ is available on the basis), while for the DMD/OPD, the time-averaged flow removal turned out to have no impact on the presented results.

The eigenvalue spectra of the DMD/OPD with the same reduced propagator described in the previous case is shown in Figure 10. A dominant frequency of $f_d = 390.2 \text{Hz}$ is observed. This is somewhere in between the one linked to the vortex shedding in the first ($f_d \approx 450 \text{Hz}$) and the second ($f_d \approx 303 \text{Hz}$) stationary conditions. As for the previous case, this decomposition does not converge to the dataset. The associated spatial structures are, therefore, not shown.

The results of the DFT are shown in Figure 11. Figure 11a) shows the energy distribution as a function of the associated frequency. The frequencies of the vortex shedding in the two stationary test cases are visible and labeled. Figures 11b) and 11c) shows the real parts of the spatial structures associated with the dominant frequencies in the two shedding regimes.

These structures are similar: the first is associated with the stationary regime at $U_\infty = 12.1 \text{m/s}$ while the second is linked to the stationary regime at $U_\infty = 7.9 \text{m/s}$. The saddle pattern observed in the previous stationary test case is present in both, with a slight shift linked to the different free stream velocity in which they become more representative. On the other hand, since these modes are purely harmonic, it is not possible to infer any temporal information regarding their occurrence. Moreover, since these modes are obtained by projecting the entire dataset onto harmonics that are mostly present in different time intervals, a distortion is produced in the spatial structures if compared to the ones obtained in an entirely stationary
Figure 10: Eigenvalue spectra from the DMD analysis of the full dataset (transient conditions). The computed dominant frequency is somewhere in between the two expected ones.

Figure 11: (a) Spectra of the DFT amplitudes \((\text{diag}(\Sigma_D))\) for the transient test case, showing the two harmonics dominating in the two stationary regimes. The imaginary part of the spatial structures associated to the frequency of the shedding at \(U_\infty = 12.1\, \text{m/s}\) (that is \(f_d = 450\, \text{Hz}\)) is shown in (b) while the one associated to the frequency of the shedding at \(U_\infty = 7.9\, \text{m/s}\) (that is \(f_d = 303.6\, \text{Hz}\)) is shown in (c).

test case. This can be observed by comparing the imaginary part of the structure at Figure 11(b) with the one in Figure 4(c) where only the stationary case was considered.

The POD is now discussed. Since the temporal average is not removed, it is of interest to

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consider the first mode separately, which accounts for about 85.5% of decomposition convergence, i.e., $\mathcal{E}(2) = 15.5\%$ from (2). Its spatial and temporal structures are shown in Figure 12 and describe the large scale variation of the flow. In the general framework of a triple decomposition (Reynolds and Hussain, 1972), this mode could be seen as the base flow, i.e., the slowly varying contribution or ‘shift mode’ (Bourgeois et al. 2013, Noack et al. 2003, Tadmor et al. 2010). Nevertheless, since the POD sets no constraints on the frequency content of its modes, the temporal evolution exhibits additional higher frequencies.

Figure 12: Spatial structure (a) and temporal structure (b) of the dominant POD mode for the transient test case

The first pair of POD modes linked to the shedding frequencies (modes 2 and 3) are shown in Figure 13, following the same structure as Figures 5. These two POD modes account for an additional 11.54% of the decomposition convergence, and the error produced by an approximation including the first three POD modes is $\mathcal{E}(3) = 3.46\%$. These POD modes describe the evolution of the vortex shedding in the entire dataset, i.e., including both the stationary conditions and the transient. The CWT of their temporal structure shows how the frequency changes in time and evolves, interestingly, so as to keep the Strouhal number constant also during the transitory. The DFT of their temporal structure shows the two peaks and is, again, very close to the DFT of the entire dataset shown in Figure 11. As a result, the phase portrait is difficult to interpret, and so are the associated spatial structures that are, in fact, a combination of the ones to be expected for the different phases, namely from Figure 1c), the steady-state 1, the transitory, and the steady-state 2.

The results from the mPOD are now discussed. For the purpose of this test case, it is interesting to separate the modes that are associated with the two steady conditions and the ones associated with the transient. As with the previous test case, Figure 15a) shows the contour map of the modulus of the Fourier transformed transfer function $\hat{K}$ while Figure 15b) shows its diagonal together with the filter transfer function used to identify the scales. The dataset is partitioned into six scales. The first scale is focused on the slow variation of the flow, identified by the low pass filter with $\mathcal{H}_L$ with pass-band $f \in [0, 10] Hz$. The third scale is centered around the first peak, thanks to the band-pass filter $|\mathcal{H}_3| \approx 1 \forall f \in [290, 320] Hz$ while the fift is centered around the second peak, with $|\mathcal{H}_4| \approx 1 \forall f \in [430, 470] Hz$. The other scales take the remaining portion of the spectra, that is $|\mathcal{H}_1| \approx 1 \forall f \in [10, 290] Hz$ and $|\mathcal{H}_6| \approx 1 \forall f \in [470, 1500] Hz$.

The spatial structure and the temporal evolution of the first mPOD modes are shown in
Figure 13: Second and third POD modes for the transient test case. Figure structured as Figure 5.

Figure 13. This mode is analogous to the first POD mode in Figure 12 with a smoother temporal evolution as its frequency content is bounded to $f < 10\text{Hz}$. The remaining mPOD modes, obtained in pairs of equal energy, are shown in Figures 16, 17, and 18, keeping the usual structure of the figure as in the previous subsection.

The first pair of modes in Figures 16 is associated to the first stationary condition, with no significant contribution appearing outside the spectral bandwidth that characterizes the first vortex shedding. Interestingly, this mode has also no appreciable contribution for $t > 1.5\text{s}$, as shown by the CWT of its temporal structure in 16c): while the filters bound the spectral constraints of the mode, the diagonalization of the correlation matrix at each of the corresponding scales also allows for temporal localization. The associated spatial structures resemble more closely the ones observed in the stationary test case investigated in the previous section (see Figure 7a). Similarly, the second pair of modes in Figure 17 captures the vortex
Figure 14: Spatial structure (a) and temporal structure (b) of the dominant mPOD mode for the transient test case.

Figure 15: Same as Figure 6, but for the transient test case. The spectra is divided into six scales. \( H_L \) is associated with the large scale variation of the flow, \( H_{H2} \) and \( H_{H4} \) with the vortex shedding in the two steady states, while the remaining two scales cover the intermediate portions.

Shedding in the second steady-state condition, while Figure 18 allows analyzing the modes associated with the flow transient.

To conclude, Figure 19 shows the decomposition converge for the POD, the DFT, and the mPOD for the transient test case. Recalling that for this dataset the temporal average has not been removed, the decomposition convergence starts in the DFT from \( \mathcal{E}_F(1) = 18.7\% - \) that is the temporal average accounts for 81.3\% of the convergence. Since the spectrum has been here partitioned more than in the previous case, the loss of decomposition convergence of the mPOD over the POD is here slightly more pronounced. The information in the first three POD modes is now distributed over the first seven mPOD modes. Nevertheless, the mPOD allows for much better convergence than the DFT while enabling localization both in the frequency and in the time domain and hence– through a projection of the data– in space.
Figure 16: Same as Figure 13 but for the second and third mPOD modes.
Figure 17: Same as Figure 13 but for the fourth and fifth mPOD modes.
Figure 18: Same as Figure 13 but for the sixth and seventh mPOD modes.

Figure 19: Decomposition error $\mathcal{E}$ in (2) as a function of the number of modes included in the summation \( \mathcal{E} \) for the POD, the mPOD and the DFT of the transient test case.
5 Conclusions and Outlook

A brief review of data-driven decompositions was presented, including the recently developed Multiscale Proper Orthogonal Decomposition (mPOD). This decomposition blends Multiresolution Analysis (MRA) and POD, generalizing POD and DMD in a single formulation, which combines the energy optimality of the first and the spectral purity of the second. The Multiscale Proper Orthogonal Decomposition (mPOD) has been tested on Time-Resolved PIV measurements of the flow past a cylinder in stationary and in transitory conditions.

In stationary conditions, the dataset consists of 4000 snapshots sampled at a frequency of $3k\ Hz$ over an observation time of $1.33s$. The free stream velocity is kept constant at $U_\infty = 12.1m/s$ producing the canonical von Karman vortex shedding at a frequency of $f_d \approx 450\ Hz$. While the DMD algorithm correctly identifies the dominant frequency of the vortex shedding, the standard implementation does not converge to the dataset as all the computed modes vanish within a small fraction of the observation time. The DFT, on the other hand, allows detecting the dominant harmonic modes while still ensuring convergence, which is nevertheless rather poor and requires about 1000 modes to reach a decomposition error of approximately $E \approx 20\%$. On the other extreme, such decomposition error is reached by the POD using only three modes. These capture the overall spatial structures of the vortex shedding and reveal an amplitude modulation whose energy content is scattered across the whole range of discrete frequencies. The mPOD proved capable of separating the two sources of such modulation, namely the contribution of harmonic phenomena evolving at a similar frequency, most probably linked to three-dimensional effects on the wake flows, from those linked to other contributions with a broader spectrum. This result is achieved while ensuring a decomposition convergence comparable to the POD, with 7 modes sufficient to have a decomposition error of $E \approx 20\%$.

In the transient conditions, the dataset consists of 13500 snapshots sampled over an observation time of $4.5s$. The free stream velocity is varied from $U_\infty = 12.1m/s$ to $U_\infty = 7.9m/s$, producing a change of frequency in the vortex shedding from $f_d \approx 450Hz$ to $f_d \approx 300Hz$. Because of the nonlinearity of such transitory, the DMD is unable to detect the relevant frequencies and to converge to the dataset. While the DFT is able to identify the shedding frequencies and to converge to the data (yet requiring 1230 modes to achieve $E \approx 20\%$), no time localization is possible due to the infinite duration of its basis.

The optimal convergence of the POD results in the opposite problem, with only three modes describing the entire dataset and producing $E_P \approx 3.4\%$. This is achieved at the cost of assigning to a single mode (and its phase-shifted counterpart) the whole evolution of the vortex shedding. The corresponding spatial structures are consequently a mix of features from these different phases of the flow evolution, which becomes indistinguishable. The spectral constraints of the mPOD allow for distinguishing the various steps in the evolution of the dataset both in the time and the frequency domains, assigning different modes accordingly. Thanks to the MRA architecture, this is achieved with a minor loss in the decomposition convergence with respect to the optimal POD, with only $7mPOD$ modes required to achieve $E \approx 4\%$.

In conclusion, the results prove the enhanced feature detection capabilities of the mPOD on TR-PIV datasets and the significant enhancement advantages of combining energy optimality and spectral constraints. Future works are now aimed at testing different forms of MRA on the resulting mPOD, in implementing methods for the objective definition of the MRA scales and in developing a faster algorithm that hinges on the sparsity of the correlation matrices of each scale in the frequency domain.

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