Abstract

A new formulation of relativistic quantum mechanics is proposed in the framework of the rest-frame instant form of dynamics with its instantaneous Wigner 3-spaces and with its description of the particle world-lines by means of derived non-canonical predictive coordinates. In it we quantize the frozen Jacobi data of the non-local 3-center of mass and the Wigner-covariant relative variables in an abstract (frame-independent) internal space $h_1$ whose existence is implied by Wigner-covariance. The formalism takes care of the properties of both relativistic bound states and scattering ones. There is a natural solution to the relativistic localization problem. The non-relativistic limit leads to standard quantum mechanics but with a frozen Hamilton-Jacobi description of the center of mass. Due to the non-locality of the Poincare’ generators the resulting theory of relativistic entanglement is both kinematically non-local and spatially non-separable, properties absent in the non-relativistic limit.
I. INTRODUCTION

Atomic physics is an approximation to QED, in which the atoms are described as non-relativistic particles in quantum mechanics (QM) with a coupling to the electro-magnetic field of order $1/c$ [1–3]. For all the applications in which the energies involved do not cross the threshold of pair production, this description with a fixed number of particles is enough. Therefore atomic physics and the theory of entanglement are formulated in the absolute Euclidean 3-space and use Newton absolute time, namely they are formulated in Galilei space-time. The main drawback is that, due to the coupling to the electromagnetic field there is not a realization of the kinematical Galilei group connecting non-relativistic inertial frames. On the other hand, if we want to arrive at an understanding of relativistic entanglement, we must reformulate the theory in Minkowski space-time with a well defined realization of the kinematical Poincare’ group connecting relativistic inertial frames. This would lead to relativistic atomic physics as the quantization of a fixed number of classical relativistic charged scalar (or spinning) particles interacting with the classical electro-magnetic field.

In the papers in Refs.[4–7] it was shown that it is possible to describe any isolated relativistic system (particles, fields, strings, fluids) admitting a Lagrangian formulation (allowing one to define the energy-momentum tensor of the system) in arbitrary non-inertial frames in Minkowski space-time by means of parametrized Minkowski theories. The transition among different non-inertial frames (with their different clock synchronization conventions identifying the instantaneous, in general non-Euclidean, 3-spaces) is performed by frame-preserving diffeomorphisms, i.e. by suitable gauge transformations. As a consequence the freedom in the choice of the clock synchronization convention (needed to formulate a Cauchy problem for classical field theories) becomes a choice of gauge. If we restrict ourselves to inertial frames, the inertial rest frame is automatically selected as the only one which can be identified in an intrinsic geometric way: its instantaneous Euclidean 3-spaces are orthogonal to the conserved 4-momentum of the isolated system.

This allows us to define the rest-frame instant form of dynamics for arbitrary isolated systems: a complete exposition of all its properties has been done in Ref.[8] and extended to non-inertial rest frames in Ref.[9]. The study of relativistic collective variables replacing the non-relativistic center of mass leads to the description of the isolated system as a decoupled globally-defined non-local (and therefore un-observable) non-covariant canonical external (Newton-Wigner) center of mass carrying a pole-dipole structure (the invariant mass $M$ and the rest spin $\vec{S}$ of the system) and an external realization of the Poincare’ group with generators $P^\mu$, $J^{\mu\nu}$. $Mc$ and $\vec{S}$ are the energy and angular momentum of a unfaithful internal realization of the Poincare’ group, with generators $M c$, $\vec{P}_{(\text{int})}$, $\vec{J}_{(\text{int})} = \vec{S}$, $\vec{K}_{(\text{int})}$, built with the energy-momentum tensor of the system and acting inside the instantaneous Wigner 3-spaces where all the 3-vectors are Wigner covariant. The vanishing of the internal 3-momentum and of the internal Lorentz boosts eliminate the internal 3-center of mass inside the Wigner 3-spaces, so that at the end the isolated system is described only by Wigner-covariant canonical internal relative variables, and imply that the Fokker-Pryce covariant non-canonical 4-center of inertia has to be chosen as the inertial observer origin of the 3-coordinates inside each Wigner 3-space. The external 4-center of mass and the Fokker-Pryce 4-center of inertia are parametrized in terms of canonical non-covariant frozen Jacobi data $\vec{z}$, $\vec{h}$. 

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In particular, due to Refs. [10, 11], it was shown in Ref. [8] that it is possible to define the inertial rest-frame instant form of a semi-classical version of relativistic atomic physics in which the electric charges of the positive-energy scalar particles are Grassmann-valued (so that the Coulomb self-energies are regularized) and in which the electro-magnetic potential is in the radiation gauge (all the fields are transverse). Therefore the isolated system is composed of \( N \) positive-energy charged scalar particles with mutual Coulomb interaction plus a transverse electro-magnetic field. The effect of this (both ultraviolet and infrared) regularization is such that in the final Hamiltonian only the potentials coming from the one-photon-exchange Feynman diagrams appear, since all the radiative corrections and production diagrams are eliminated. Therefore, our particles describe consistently the semi-classical limit of a fixed-particle-number sector of some matter QFT. Moreover the main features of the treatment of relativistic bound states in the framework of QED are taken into account, since the Darwin potential is emerging from the Lienard-Wiechert solution [10].

The covariant world-lines of the particles are reconstructed in terms of the covariant non-canonical external Fokker-Pryce center of inertia, of the external 4-momentum and of the internal Wigner-covariant relative variables: they are covariant but not canonical, so that they correspond to the predictive coordinates of predictive mechanics [13]. Since they are not canonical, their quantum version are operators which do not commute so that, in general, the only covariant statements about them concern their expectation values in given quantum states.

In Ref. [14] we showed how to determine a collective variable associated with the internal 3-center of mass on the instantaneous 3-spaces, to be eliminated with the constraints \( \vec{K}_{\text{(int)}} \approx 0 \). Here \( \vec{K}_{\text{(int)}} \) is the Lorentz boost generator in the unfaithful internal realization of the Poincare’ group and its vanishing is the gauge fixing to the rest-frame conditions \( \vec{P}_{\text{(int)}} \approx 0 \).

Moreover in Ref. [8] it is shown that there is a canonical transformation which allows one to describe the isolated system of "\( N \) positive-energy charged scalar particles with mutual Coulomb interaction plus a transverse electro-magnetic field" as a set of \( N \) Coulomb-dressed charged particles interacting through a Coulomb plus Darwin potential plus a free transverse radiation field: these two subsystems are not mutually interacting (the internal Poincare’ generators are a direct sum of the two components) and are interconnected only by the rest-frame conditions \( \vec{P}_{\text{(int)}} \approx 0 \) and the elimination of the internal 3-center of mass with the gauge fixings \( \vec{K}_{\text{(int)}} \approx 0 \). Therefore in this framework with a fixed number of particles there is a way out from the Haag theorem, at least at the classical level.

\[ \text{1} \] The matter part of QED; at level of quark sub-constituent it is the QFT of the standard model of particle physics; for an atom of protons, neutrons and electrons it is an effective QFT.

\[ \text{2} \] As shown in Ref. [11], in this framework it is possible also to describe positive-energy spinning particles (with Grassmann-valued spin) and to identify the Salpeter potential instead of the Darwin one. The Grassmann-valued 3-spins of the particles are all defined in the same instantaneous 3-space, the Wigner hyper-planes, and therefore transform as Wigner spin-1 3-vectors. In Ref. [12] the positive-energy spinning particles are quantized in a special family of non-inertial frames.
After the canonical transformation the two "non-interacting" subsystems are only kinematically coupled by the rest-frame conditions $\vec{P}_{\text{int}} \approx 0$ and $\vec{K}_{\text{int}} \approx 0$ (3 pairs of second class constraints eliminating the spurious internal 3-center of mass): only internal relative variables survive with the exception of two collective variables of the radiation field, i.e. the constant of motion $p_\text{rad} = M_\text{rad} c$ (the energy of the radiation field) and $X_\tau = -\tau + \text{const.}$ (an internal time discriminating the various symplectic sub-manifolds of a surface of constant energy of the radiation field).

The two papers of Refs[8, 14] will be quoted as I and II and their formulas will be denoted (I.2.5) or (II.1.13).

As a consequence we now have a formalism which, for the first time, takes into account all the known aspects of relativistic kinematics and dynamics of point particles by means of 3+1 splittings of Minkowski space-time, parametrized Minkowski theories and the rest-frame instant form of dynamics. One still open problem is the possibility of defining a consistent relativistic statistical mechanics by evaluating the relativistic micro-canonical ensemble in the rest-frame instant form of dynamics. Maybe other formulations are possible, but they have not yet been developed.

We refer to Subsection F of Section I of paper I for a review of the other approaches to relativistic mechanics, in particular of those with first-class constraints, which were the precursors of the present formulation. However all these approaches suffered from some problems. For instance it was too complicated to get a Lagrangian description. See also the bibliography of the review part of Ref.[7].

Everyone of these approaches to relativistic mechanics tried to perform the quantization and to define a consistent relativistic QM. See Ref.[15] and its bibliography for the attempt to quantize the two-particle models with two first-class constraints. However in these models there was not a 3+1 splitting of Minkowski space-time. Instead the problem of the instantaneous 3-space (a space-like hyperplane) is present in the papers of Fleming in Ref.[16] (see also Refs.[17, 18]): however these papers did not succeeded in giving an acceptable description of the comparison of the dynamics on different space-like hyper-planes connected by Lorentz transformations.

All the previous attempts to define relativistic QM employ the so-called zeroth postulate of QM (see Zurek in Ref.[19]). According to it a composite system of two spatially separated subsystems is described by the tensor product of the Hilbert spaces of the subsystems. The notation $H_t = (H_1 \otimes H_2)_t = (H_\text{com} \otimes H_\text{rel})_t$ means that the quantum 2-body isolated system can be imagined to be constituted either by the two single particle subsystems with masses $m_1$ and $m_2$ or as the tensor product of a decoupled center-of-mass particle of mass $m = m_1 + m_2$ carrying an internal space with an internal relative motion of reduced mass $\mu = m_1 m_2/m$. The second description is implied by the separation of variables in the Schroedinger equation when the mutual interaction respects the Galilei covariance of the isolated system. The two descriptions are connected by a unitary transformation and correspond to different choices of bases in $H_t$.

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3 Among them $\hat{\pi}^{(12)3}$ is defined as a constant of motion, describing the relative motion of the matter subsystem with respect the radiation field subsystem.

4 Let us remark that in non-relativistic QM the Hilbert space $\otimes_{i=1}^N H_i$ for a N-body system could be replaced
The zeroth postulate, i.e. $\mathcal{H}_t = (\mathcal{H}_1 \otimes \mathcal{H}_2)_t$, is based on a notion of separability independent from the Galilei group, which instead is at the basis of the decomposition $\mathcal{H} = (\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}})_t$ emphasizing that the center-of-mass momentum is a constant of motion for an isolated system. This notion of separability goes back to Einstein (see the EPR paper [21] and Ref.[22]): according to him proper separability means that separate objects have their independent real states, since for him it should be possible to divide the world up into pieces about which statements can be made (realism). The EPR argument leads to the statement that non-relativistic QM is incomplete because realism and locality do not coexist. Here locality means the real state of one system remains unaffected by changes to a distant system (usually it is said that it is locality which fails in orthodox QM with collapse, even if in a benign way: it does not seem to make the testing of predictions for isolated systems impossible, due to the presence of the no-signalling theorem about the probabilities of the outcomes of measurements; QM remains empirically testable despite violating locality). The no-signalling theorem (ruling out the possibility of signalling using entangled states) saves QM from explicit non-locality conflicting with relativity.

Given these notions and two subsystems A and B, we can introduce the notions of a separable pure state $|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B$ and of an entangled non-separable state $|\Psi\rangle_{AB} = \sum_i \sqrt{p_i} |\bar{\phi}_i\rangle_A \otimes |\bar{\psi}_i\rangle_B$ ($\{|\bar{\phi}_i\rangle\}, \{|\bar{\psi}_i\rangle\}$ are orthonormal bases for subsystems A and B respectively and $p_i$ are the non-zero eigenvalues of the reduced density matrix of A). This is the starting point for the description of entanglement in non-relativistic QM and for its foundational problems connected to its probabilistic aspects and to its non-locality (see Refs.[23, 24] for a review).

Instead the attempts to define relativistic entanglement (see for instance Ref.[25]) usually start from quantum field theory (QFT) and always use the notion of separability in the form of the zeroth postulate. For a complete discussion of the state of the art and for the open problems caused by Lorentz transformations and massless particles see Refs. [23, 26].

The other source of problems in putting together QM and special relativity is the notion of localization. This is connected with the unusual properties of the non-covariant Newton-Wigner operator [27, 28] and of its eigenvalues (absence of sharp localization, an aspect of the non-locality present in special relativity with self-adjoint position operators) and with the connected problem of the instantaneous spreading of wave packets (the Hegerfeldt theorem [29, 39]). As clearly shown in Ref. [31] in local QFT there is a notion of localization deeply
different from *Newton-Wigner localization*. Even if the Reeh-Schlieder theorem [32] says that the vacuum is super-entangled (every state can be approximated with states obtained from the vacuum by applying local operators defined in bounded 4-regions (not 3-regions!) of space-time), the conclusion is that the Newton-Wigner position operator cannot be described by means of either local or quasi-local operators of algebraic QFT. The accepted consequence is that this operators is *not measurable*. Connected problems are the validity of micro-causality and the relevance of un-sharp observables to try to define a theory of measurement going beyond local QFT.

In this paper we propose a general scheme of quantization of relativistic positive-energy scalar particles induced by the rest-frame instant form of dynamics given by classical relativistic mechanics \(^5\) and we will discuss in Section VI which of the quoted problems are solved by our scheme.

The main result will be the non-validity of the zeroth postulate: the quantization can be done only in a Hilbert space \(\mathcal{H}_{\tau}\) admitting the presentation \(\mathcal{H}_{\tau} = (\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}})_{\tau}\) in each instantaneous Wigner 3-space and in which the evolution is parametrized in terms of the rest-frame time \(\tau\). Actually after a 3+1 splitting of Minkowski space-time it is not possible to define single-particle Hilbert spaces \(\mathcal{H}_1, \mathcal{H}_2 \ldots\) : our basic operators are the Jacobi data \(\hat{z}, \hat{h}\) and internal relative variables \(\hat{p}_a, \hat{\delta}_a, a = 1, \ldots, N - 1\). The single particle (predictive) operators \(\hat{x}_i^\mu, \hat{p}_i^\mu, i = 1, \ldots, N\), are derived non-canonical quantities built in terms of the previous operators and not independent variables like in the approaches considering the tensor product \((\mathcal{H}_1)_{x_1^\mu} \otimes (\mathcal{H}_2)_{x_2^\mu} \otimes \ldots\) of free Klein-Gordon quantum particles. While \(\mathcal{H}_{\tau} = (\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}})_{\tau}\) is the natural Hilbert space for the description of relativistic bound states (and also of scattering states described in terms of relative variables), in the tensor-product Hilbert space \((\mathcal{H}_1)_{x_1^\mu} \otimes (\mathcal{H}_2)_{x_2^\mu} \otimes \ldots\) there is no correlation among the times of the particles (their clocks are not synchronized) so that in most of the states there are some particles in the absolute future of the others. As a consequence the two types of Hilbert spaces lead to inequivalent descriptions.

Moreover the decoupled external non-covariant 4-center of mass is *not measurable* (it is non-local in the sense of Newton-Wigner localization) and evades Hegerfeldt’s theorem being described by frozen (non-evolving) Jacobi data.

In Section VI we will show what are the implications for relativistic entanglement: since the dynamics is described by relative variables in the Wigner 3-spaces, there is a *spatial non-separability* and a *non-locality* of kinematical origin besides the quantum non-locality.

The quantization scheme is defined initially for free particles and then extended to particles with action-at-a-distance mutual interactions. We will treat explicitly the two-body case and we will show that there is no problem in the extension to \(N\) particles.

In Subsection A of Section II we give a review of the rest-frame instant form of dynamics for isolated systems of relativistic positive-energy scalar particles living in the instantaneous Wigner 3-spaces. Then in Subsection B we study its non-relativistic limit. Finally in

\(^5\) In Refs.[12] a first attempt of quantization of relativistic mechanics in inertial and non-inertial frames (with the non-relativistic limit given in Ref.[33]) was done.
Subsection C we define an abstract internal space of relative variables independent of the orientation of the conserved 4-momentum of the isolated system: this is possible due to the Wigner covariance of the relative variables.

In Subsection A of Section III we revisit the non-relativistic QM of two particles, while in Subsection B we reformulate it in a form suitable to be extended to the relativistic level (which uses a Hamilton-Jacobi description of the decoupled center of mass).

In Section IV we introduce our quantization scheme. In Subsection A we emphasize that we do not quantize the non-covariant canonical 4-center of mass but only its frozen Jacobi data (it is a derived quantity); instead for the particles we can either quantize the relative variables or the original variables $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)$ but with the supplementary requirements $<\hat{\vec{P}}_{\text{(int)}}> = <\hat{\vec{K}}_{\text{(int)}}> = 0$ (in both case the particle world-lines are derived quantities). In Subsection B we describe the quantization of the relative variables, while in Subsection C we delineate the quantization before the elimination of the internal 3-center of mass.

In Section V we give examples of quantization of two-particle systems with action-at-a-distance interactions.

In Section VI we show which problems connected to relativistic localization are solved by our quantization scheme and its implications for relativistic entanglement. The new quantization scheme contains a non-locality and a spatial non-separability originating from the Lorentz signature of Minkowski space-time and from the properties of the Poincare’ group besides the standard quantum non-locality. These new features disappear in the non-relativistic limit due to the absolute nature of time and 3-space in Galileo space-time and due to the fact that Galilei boosts are interaction independent.

In the Conclusions we make some comments on the open problem of quantizing the free transverse radiation field with the added rest-frame requirements $<\hat{\vec{P}}_{\text{(int)}}> = <\hat{\vec{K}}_{\text{(int)}}> = 0$.

In Appendix A we give the form of the Darwin potential in the unequal mass case.

In Appendix B there is the quantization of two equal mass scalar particles with mutual Coulomb plus Darwin interaction by means of Weyl ordering.
II. REVIEW OF THE REST-FRAME INSTANT FORM OF DYNAMICS FOR RELATIVISTIC PARTICLES

In this Section we review the rest-frame instant form of dynamics for isolated systems developed in I using a two-particle system as an example. $\eta_{\mu\nu} = \epsilon (+- - -)$ is the flat metric ($\epsilon = \pm 1$ according to either the particle physics $\epsilon = 1$ or the general relativity $\epsilon = -1$ convention).

A. The Rest-Frame Instant Form

Let us consider an arbitrary inertial frame, centered on an inertial observer whose world-line is the time axis, in Minkowski space-time. If $P^\mu = Mc h^\mu = Mc \left( \sqrt{1 + \vec{h}^2}; \vec{h} \right)$ ($\vec{h} = \vec{v}/c = \vec{P}/Mc$ is an a-dimensional 3-velocity) is the conserved total 4-momentum of the isolated particle system in this inertial frame, the 3+1 splitting of Minkowski space-time associated with the inertial rest-frame instant form description of the isolated system has the instantaneous Wigner 3-spaces orthogonal to $P^\mu$ (the 3-vectors inside them are Wigner spin-1 3-vectors; the 3-metric inside the Euclidean Wigner 3-spaces is taken to be positive definite, i.e. $\delta_{rs}$ with signature $(+++)$, so that for the Wigner 3-vectors we have $V^r = V_r$). Their embedding in Minkowski space-time is

$$z^\mu_W(\tau, \vec{\sigma}) = Y^\mu(\tau) + \epsilon^\mu_r(\vec{h}) \sigma^r,$$

$$h^\mu = \epsilon^\mu_r(\vec{h}) = (\sqrt{1 + \vec{h}^2}; \vec{h}); \quad \epsilon^\mu_r(\vec{h}) = \left( h_r; \delta_r^i + \frac{h^i h_r}{1 + \sqrt{1 + \vec{h}^2}} \right), \quad (2.1)$$

where $Y^\mu(\tau) = Y^\mu(0) + h^\mu \tau = z^\mu_W(\tau, \vec{0})$ is the world-line of the external Fokker-Pryce 4-center of inertia with $\eta_{\mu\nu} \epsilon^\mu_A(\vec{h}) \epsilon^\nu_B(\vec{h}) = \eta_{AB}$.

In these rest frames there are only three notions of collective variables, which can be built by using only the Poincare’ generators (they are non-local quantities knowing the whole $\Sigma_\tau$): the canonical non-covariant Newton-Wigner 4-center of mass (or center of spin) $\tilde{x}^\mu(\tau)$, the non-canonical covariant Fokker-Pryce 4-center of inertia $Y^\mu(\tau)$ and the non-canonical non-covariant Møller 4-center of energy $R^\mu(\tau)$. All of them tend to the Newtonian center of mass in the non-relativistic limit.

As shown in I, these three variables can be expressed as known functions of the Lorentz-scalar rest time $\tau = c T_s = h \cdot \vec{x} = h \cdot Y = h \cdot R$, of canonically conjugate Jacobi data (frozen Cauchy data) $\vec{z} = Mc \vec{x}_{NW}(0)$ ($\{ z^i, h^j \} = \delta^{ij}$; $\vec{x}_{NW}(\tau)$ is the standard Newton-Wigner non-covariant 3-position, classical counterpart of the corresponding position operator [27]) and $\vec{h} = \vec{P}/Mc$, of the invariant mass $Mc = \sqrt{\epsilon P^2}$ of the system and of its rest spin $\vec{S}$:

1) the pseudo-world-line of the canonical non-covariant external 4-center of mass is
\[ \ddot{x}^\mu(\tau) = (\ddot{x}^\mu(\tau); \dddot{x}(\tau)) = \left( \sqrt{1 + \vec{h}^2} \left( \tau + \frac{\vec{h} \cdot \vec{z}}{M_c} \right); \frac{\vec{z}}{M_c} + \left( \tau + \frac{\vec{h} \cdot \vec{z}}{M_c} \right) \vec{h} \right) = \\
= z_W^\mu(\tau, \vec{\sigma}) = Y^\mu(\tau) + \left( 0; \frac{-\vec{S} \times \vec{h}}{M_c (1 + \sqrt{1 + \vec{h}^2})} \right), \quad (2.2) \]

so that we get \[ Y^\mu(0) = \left( \sqrt{1 + \vec{h}^2} \left( \tau + \frac{\vec{h} \cdot \vec{z}}{M_c} \right); \frac{\vec{z}}{M_c} + \frac{\vec{z} \cdot \vec{h}}{M_c (1 + \sqrt{1 + \vec{h}^2})} \right) \] (we have used \[ \vec{\sigma} = \frac{-\vec{S} \times \vec{h}}{M_c (1 + \sqrt{1 + \vec{h}^2})} \]).

2) the world-line of the non-canonical covariant external Fokker-Pryce 4-center of inertia is

\[ Y^\mu(\tau) = (\ddot{x}^\mu(\tau); \dddot{Y}(\tau)) = \left( \sqrt{1 + \vec{h}^2} \left( \tau + \frac{\vec{h} \cdot \vec{z}}{M_c} \right); \frac{\vec{z}}{M_c} + \left( \tau + \frac{\vec{h} \cdot \vec{z}}{M_c} \right) \vec{h} + \frac{\vec{S} \times \vec{h}}{M_c (1 + \sqrt{1 + \vec{h}^2})} \right) = \\
= z_W^\mu(\tau, \vec{0}); \quad (2.3) \]

3) the pseudo-world-line of the non-canonical non-covariant external Møller 4-center of energy is

\[ R^\mu(\tau) = (\ddot{x}^\mu(\tau); \dddot{R}(\tau)) = \left( \sqrt{1 + \vec{h}^2} \left( \tau + \frac{\vec{h} \cdot \vec{z}}{M_c} \right); \frac{\vec{z}}{M_c} + \left( \tau + \frac{\vec{h} \cdot \vec{z}}{M_c} \right) \vec{h} - \frac{\vec{S} \times \vec{h}}{M_c \sqrt{1 + \vec{h}^2} (1 + \sqrt{1 + \vec{h}^2})} \right) = \\
= z_W^\mu(\tau, \vec{\sigma_R}) = Y^\mu(\tau) + \left( 0; \frac{\vec{S} \times \vec{h}}{M_c \sqrt{1 + \vec{h}^2}} \right), \quad (2.4) \]

(we have used \[ \vec{\sigma_R} = \frac{\vec{S} \times \vec{h}}{M_c \sqrt{1 + \vec{h}^2}} \]).

While \( Y^\mu(\tau) \) is a 4-vector, \( \ddot{x}^\mu(\tau) \) and \( R^\mu(\tau) \) are not 4-vectors. See Ref.[5] for the Møller non-covariance world-tube around the Fokker-Pryce 4-vector identified by these collective variables. Their transformation properties under Poincare' transformations \( (a, \Lambda) \) can be deduced from those for \( \vec{h}, \vec{z} \) and \( \tau \) (see Appendix B of Ref.[15])

\[ h^\mu \rightarrow h'^\mu = \Lambda^\mu_\nu h^\nu, \]
\[ z^i \rightarrow z'^i = \left( \Lambda^i_j - \frac{\Lambda^i_\mu h^\mu}{\Lambda^\rho_j h^\rho} \Lambda^\rho_j \right) z^j + \left( \Lambda^i_\mu - \frac{\Lambda^i_\rho h^\rho}{\Lambda^\mu_j h^\mu} \Lambda^\mu_j \right) (\Lambda^{-1} a)^\mu; \]
\[ \tau \rightarrow \tau' + h_\mu (\Lambda^{-1} a)^\mu, \]
\[ \vec{h}' \cdot \vec{z}' = \vec{h} \cdot \vec{z} + \frac{\Lambda^i_\mu z^i}{\Lambda^\rho_j h^\rho}, \quad \text{for} \quad a^\mu = 0. \quad (2.5) \]
As said in I every isolated system (i.e. a closed universe) can be visualized as a decoupled non-covariant collective (non-local) pseudo-particle described by the frozen Jacobi data \( z, \vec{h} \) carrying a pole-dipole structure, namely the invariant mass \( Mc \) and the rest spin \( \vec{S} \) of the system, and with an associated external realization of the Poincare' group (the last term in the Lorentz boosts induces the Wigner rotation of the 3-vectors inside the Wigner 3-spaces):

\[
P^\mu = Mc h^\mu = Mc \left( \sqrt{1 + \vec{h}^2}; \vec{h} \right),
\]

\[
J^{ij} = z^i h^j - z^j h^i + \epsilon^{ijk} S^k, \quad K^i = J^{0i} = -\sqrt{1 + \vec{h}^2} z^i + \frac{(\vec{S} \times \vec{h})^i}{1 + \sqrt{1 + \vec{h}^2}}, \quad (2.6)
\]

satisfying the Poincare' algebra: \( \{P^\mu, P^\nu\} = 0, \{P^\mu, J^{\alpha\beta}\} = \eta^{\mu\alpha} P^\beta - \eta^{\mu\beta} P^\alpha, \{J^{\mu\nu}, J^{\alpha\beta}\} = C^{\mu\nu\alpha\beta} J^{\gamma\delta}, \quad C^{\mu\nu\alpha\beta} = \delta_\gamma^\mu \delta_\delta^\nu \eta^{\alpha\beta} + \delta_\gamma^\nu \delta_\delta^\mu \eta^{\alpha\beta} - \delta_\gamma^\alpha \delta_\delta^\beta \eta^{\mu\nu} - \delta_\gamma^\beta \delta_\delta^\alpha \eta^{\mu\nu}.

The universal breaking of Lorentz covariance is connected to this decoupled non-local collective variable and is irrelevant because all the dynamics of the isolated system lives inside the Wigner 3-spaces and is Wigner-covariant. Inside these Wigner 3-spaces the system is described by an internal 3-center of mass with a conjugate 3-momentum and by relative variables and there is an unfaithful internal realization of the Poincare' group: the internal 3-momentum, conjugate to the internal 3-center of mass \( \vec{S} \), must also vanish. As shown in I the only non-zero internal generators are the invariant mass \( Mc \) and the rest spin \( \vec{S} \) and the dynamics is re-expressed only in terms of internal Wigner-covariant relative variables. Moreover this construction implies that the time-like observer, at the origin of the 3-coordinates on the Wigner 3-spaces, must be identified with the Fokker-Pryce inertial observer as it was done in Eq.(2.1).

As shown in Eq.(4.2) of the second paper of Ref.[6], given the external realization (2.6) of the Poincare' generators the spatial part of the external Møller center of energy (2.4) is given by \( \vec{R}(0) = -\vec{K}/P^\alpha \). In that paper it is also shown that the Jacobi data \( z \) can be written in the form \( \vec{z} = Mc \vec{R} + \frac{Mc \vec{S} \times \vec{P}}{P^\alpha(Mc + P^\alpha)}, \) with \( \vec{S} = \vec{J} - \vec{z} \times \frac{\vec{P}}{Mc} \), and that this implies \( \vec{z} = -\frac{P^\alpha}{Mc} \vec{K} + \frac{\vec{P} \times \vec{R}}{P^\alpha(Mc + P^\alpha)} \vec{P} + \frac{\vec{S} \times \vec{B}}{Mc + P^\alpha} \). Eq.(2.2) then allows us to express the external 4-center of mass \( \vec{x}^\mu(\tau) \) in terms of the external Poincare' generators. The same can be done for \( Y^\mu(\tau) \) by using Eq.(2.3). Therefore the three collective variables of an isolated relativistic system are non-local quantities like the Poincare' generators.

As shown in I and in Ref.[5], in each Lorentz frame one has different pseudo-worldlines describing \( R^\mu \) and \( \vec{x}^\mu \): the canonical 4-center of mass \( \vec{x}^\mu \) lies in between \( Y^\mu \) and \( R^\mu \) in every (non rest)-frame. As discussed in Subsection IIF of paper I, this leads to the existence of the \( \text{Møller non-covariance world-tube} \), around the world-line \( Y^\mu \) of the covariant non-canonical Fokker-Pryce 4-center of inertia \( Y^\mu \). The invariant radius of the tube is

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6 As shown in Ref.[6] the three internal collective 3-variables (canonical \( \vec{q}_+(\tau) \), Fokker-Pryce \( \vec{y}(\tau) \), Møller \( \vec{R}_+(\tau) \)) coincide due to the rest-frame conditions: \( \vec{q}_+ \approx \vec{y} \approx \vec{R}_+ \)
\[ \rho = \sqrt{-W^2/p^2} = |\vec{S}|/\sqrt{P^2} \text{ where } (W^2 = -P^2 \vec{S}^2 \text{ is the Pauli-Lubanski invariant when } P^2 > 0). \] This classical intrinsic radius delimitates the non-covariance effects (the pseudo-world-lines) of the canonical 4-center of mass \( \tilde{x}^\mu \). They are not detectable because the Møller radius is of the order of the Compton wave-length: an attempt to test its interior would mean to enter in the quantum regime of pair production. Finally the Møller radius \( \rho \) is also a remnant of the energy conditions of general relativity in flat Minkowski spacetime [5] and is the classical background of the violation of the weak energy condition of the renormalized stress-energy tensor in QFT (the Epstein, Glaser, Jaffe theorem [34]).

The world-lines of the positive-energy particles are parametrized by Wigner 3-vectors \( \vec{\eta}_i(\tau) \), \( i = 1, 2, ..., N \), and are given by

\[ x^\mu_i(\tau) = z^\mu_W(\tau, \vec{\eta}_i(\tau)) = Y^\mu(\tau) + e^\mu_r(\tau) \eta^*_r(\tau). \] (2.7)

For \( N \) free particles we have the following form of the internal Poincaré’ generators \( (\vec{\kappa}_i(\tau) \) are the canonical momenta conjugate to \( \vec{\eta}_i(\tau), \{\vec{\eta}_i^a(\tau), \kappa^a_j(\tau)\} = \delta_{ij} \delta^{rs} \); the usual particle 4-momenta are the derived quantities \( p^\mu_i = h^\mu \sqrt{m^2_i c^2 + \vec{\kappa}^2_i} - e^\mu_r(\vec{h}) \kappa^r_i \) with \( e \rho_i^2 = m^2_i c^2 \)

\[ M c = \frac{1}{c} \mathcal{E}(\text{int}) = \sum_{i=1}^{N} \sqrt{m^2_i c^2 + \vec{\kappa}^2_i}, \]

\[ \vec{P}(\text{int}) = \sum_{i=1}^{N} \vec{\kappa}_i \approx 0, \]

\[ \vec{S} = \vec{J}(\text{int}) = \sum_{i=1}^{N} \vec{\eta}_i \times \vec{\kappa}_i, \]

\[ \vec{K}(\text{int}) = -\sum_{i=1}^{N} \eta^*_i \sqrt{m^2_i c^2 + \vec{\kappa}^2_i} \approx 0. \] (2.8)

Instead of using the real internal 3-center-of-mass and relative variables which can be obtained only with a non-linear non-point canonical transformation as shown in the Appendix of the third paper in Ref.[6], it is more convenient to use a naive linear point canonical transformation. Therefore we will use the following collective and relative variables which, written in terms of the masses \( m_i \) of the particles, make it easier to evaluate the non-relativistic limit \( (m = \sum_{i=1}^{N} m_i)\)

\[ \text{In the rest-frame the world-tube is a cylinder: in each instantaneous 3-space there is a disk of possible positions of the canonical 3-center of mass orthogonal to the spin. In the non-relativistic limit the radius \( \rho \) of the disk tends to zero and we recover the non-relativistic center of mass.} \]

\[ \text{The Møller radius of a field configuration (think to the radiation field studied in Section III of paper II) could be a candidate for a physical (configuration-dependent) ultraviolet cutoff in QFT [5].} \]
\[ m_i \eta_i + m_i \bar{\kappa}_i = \bar{P}_{(int)} = \sum_{i=1}^{N} \bar{\kappa}_i, \]

\[ \bar{\rho}_a = \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \bar{\eta}_i, \quad \bar{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Gamma_{ai} \bar{\kappa}_i, \quad a = 1, ..., N - 1, \]

\[ \bar{\eta}_i = \bar{\eta}_1 + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \Gamma_{ai} \bar{\rho}_a, \quad \bar{\kappa}_i = \frac{m_i}{m} \bar{\kappa}_1 + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \bar{\pi}_a, \quad (2.9) \]

with the following canonicity conditions \(^9\)

\[ \sum_{i=1}^{N} \gamma_{ai} = 0, \quad \sum_{i=1}^{N} \gamma_{ai} \gamma_{bi} = \delta_{ab}, \quad \sum_{a=1}^{N-1} \gamma_{ai} \gamma_{aj} = \delta_{ij} - \frac{1}{N}, \]

\[ \Gamma_{ai} = \gamma_{ai} - \sum_{k=1}^{N} \frac{m_k}{m} \gamma_{ak}, \quad \gamma_{ai} = \Gamma_{ai} - \frac{1}{N} \sum_{k=1}^{N} \Gamma_{ak}, \]

\[ \sum_{i=1}^{N} \frac{m_i}{m} \Gamma_{ai} = 0, \quad \sum_{i=1}^{N} \gamma_{ai} \Gamma_{bi} = \delta_{ab}, \quad \sum_{a=1}^{N-1} \gamma_{ai} \Gamma_{aj} = \delta_{ij} - \frac{m_i}{m}. \quad (2.10) \]

For \( N = 2 \) we have \( \gamma_{11} = -\gamma_{12} = \frac{1}{\sqrt{2}}, \quad \Gamma_{11} = \sqrt{2} \frac{m_1}{m}, \quad \Gamma_{12} = -\sqrt{2} \frac{m_2}{m} \).

Therefore in the two-body case, by introducing the notation \( \bar{\eta}_{12} = \bar{\eta}_1, \bar{\kappa}_{12} = \bar{\kappa}_1 = \bar{P}_{(int)} \), we have the following collective and relative variables

\[ \bar{\eta}_{12} = \frac{m_1}{m} \bar{\eta}_1 + \frac{m_2}{m} \bar{\eta}_2, \quad \bar{\rho}_{12} = \bar{\eta}_1 - \bar{\eta}_2, \]

\[ \bar{\kappa}_{12} = \bar{\kappa}_1 + \bar{\kappa}_2 \approx 0, \quad \bar{\pi}_{12} = \frac{m_2}{m} \bar{\kappa}_1 - \frac{m_1}{m} \bar{\kappa}_2, \]

\[ \bar{\eta}_i = \bar{\eta}_{12} + (-)^{i+1} \frac{m_i}{m} \bar{\rho}_{12}, \quad \bar{\kappa}_i = \frac{m_i}{m} \bar{\kappa}_{12} + (-)^{i+1} \bar{\pi}_{12}, \quad (2.11) \]

where we use the convention \( m_3 \equiv m_1 \).

The collective variable \( \bar{\eta}_{12}(\tau) \) has to be determined in terms of \( \bar{\rho}_{12}(\tau) \) and \( \bar{\pi}_{12}(\tau) \) by means of the gauge fixings \( \bar{\kappa}_{(int)} \) \( \equiv -M \bar{R}_+ \approx 0 \). For two free particles Eqs.(2.8) imply \( (\bar{\eta}_{12}(\tau) \approx 0 \) for \( m_1 = m_2 \))

\(^9\) Eqs.(2.9) describe a family of canonical transformations, because the \( \gamma_{ai} \)'s depend on \( \frac{1}{2}(N - 1)(N - 2) \) free independent parameters.
In the interacting case the rest-frame conditions \( \eta_{12} \approx 0 \) and the conditions eliminating the internal 3-center of mass \( \vec{K}_{(int)} \approx 0 \) will determine \( \eta_{12} \) in terms of the relative variables \( \vec{\rho}_{12}, \vec{\pi}_{12} \) in an interaction-dependent way.

Then the relative variables satisfy Hamilton equations with the invariant mass \( M(\vec{\rho}_{12}, \vec{\pi}_{12}) \) as Hamiltonian and the particle world-lines \( x_i^\mu(\tau) \) can be rebuilt [7].

The position of the two positive-energy particles in each instantaneous Wigner 3-space is identified by the intersection of the world-lines \( x_i^\mu(\tau) \rightarrow c \rightarrow \infty x_i(t) \) with \( Y_{i}^\mu(\tau) \) given in Eq.(2.3) in terms of \( \vec{z}, \vec{h} \) and \( \tau \). In the non-relativistic limit they identify the Newton trajectories \( x_{(n)i}(t) \). The covariant predictive world-lines \( x_i^\mu(\tau) \) depend on the relative position variables \( \vec{\rho}_{12} \): a) if the interaction among the particles is such that the relative position variables have a compact support when \( \tau \) varies (as happens with the classical analogue of bound states) the world-lines will be included in some finite time-like world-tube; b) instead, if the interactions describe the classical analogue of scattering states, the world-lines can diverge one from the other (cluster decomposition property). This qualitative description has to be checked in every system with a well defined action-at-a-distance interaction.

They turn out to have a non-commutative (predictive) associated structure since we have \( (f_i^r = \eta_i^r(\vec{\rho}_{12}(\tau), \vec{\pi}_{12}(\tau))) \)
\[
\{x_i^\mu(\tau), x_j^\nu(\tau)\} = \{Y^\mu(\tau), Y^\nu(\tau)\} - \{Y^\mu(\tau), e^\nu_s(\vec{h})\} f_j^s + \{Y^\nu(\tau), e^\mu_i(\vec{h})\} f_i^s + \\
e^\nu_s(\vec{h}) e^\mu_i(\vec{h}) \{f_i^s, f_j^s\} \neq 0,
\]

\[
\{Y^\alpha(\tau), Y^\beta(\tau)\} = \frac{z^i \sqrt{1 + \vec{h}^2}}{(Mc)^2} + \frac{\vec{S} \times \vec{h}}{(Mc)^2 (1 + \sqrt{1 + \vec{h}^2})},
\]

\[
\{Y^i(\tau), Y^j(\tau)\} = \frac{\epsilon^{ijk}}{(Mc)^2} \left[ \left( \vec{z} \times \vec{h} + \vec{S} \right) - \frac{h^k \vec{h} \cdot \vec{S}}{(1 + \sqrt{1 + \vec{h}^2})^2} \right],
\]

\[
\{Y^i(\tau), e^\alpha_s(\vec{h})\} = \frac{h^r \sqrt{1 + \vec{h}^2}}{Mc}, \quad \{Y^i(\tau), e^\alpha_i(\vec{h})\} = \frac{1}{Mc} (\delta^ir + h^i h^r),
\]

\[
\{Y^i(\tau), e^\alpha_s(\vec{h})\} = \frac{h^j h^r}{Mc},
\]

\[
\{Y^i(\tau), e^\alpha_i(\vec{h})\} = \frac{1}{Mc (1 + \sqrt{1 + \vec{h}^2})} \left( \delta^{ij} h^r + \delta^ir h^j + \frac{2 + \sqrt{1 + \vec{h}^2}}{1 + \sqrt{1 + \vec{h}^2}} h^i h^j h^r \right).
\]

Eqs.(2.3) have been used to get these results

In the free case Eqs.(2.13) imply \( \{f_i^s, f_j^s\} = \frac{(m_2^2 - m_1^2) c^2}{\sqrt{m_1^2 c^2 + \vec{p}_{12}^2} \sqrt{m_2^2 c^2 + \vec{p}_{12}^2} \sum_{k=1}^2 \sqrt{m_k^2 c^2 + \vec{p}_{12}^2}} \).

B. The Non-Relativistic Limit of the Rest-Frame Instant Form

Let us consider the non-relativistic limit of two positive-energy scalar free particles, following I, where the kinematics is described in Eq.(I-2.27) and the generators of the Galilei algebra are given in Eq.(I-2.28).

The particles are described by the Newtonian canonical variables \( \vec{x}_{(n)}^i, \vec{p}_{(n)}^i, i = 1, 2, \) or by the canonically equivalent center-of-mass and relative variables \( \vec{x}_{(n)}, \vec{p}_{(n)}, \vec{r}_{(n)}, \vec{q}_{(n)} \) (see Ref.[20] for the case of N particles)

\[
\vec{x}_{(n)} = \frac{1}{m} \sum_{i=1}^2 m_i \vec{x}_{(n)}^i, \quad \vec{p}_{(n)} = \sum_{i=1}^2 \vec{p}_{(n)}^i, \quad m = m_1 + m_2,
\]

\[
\vec{r}_{(n)} = \vec{x}_{(n)} - \vec{x}_{(n)}^2, \quad \vec{q}_{(n)} = \frac{1}{m} \left( m_2 \vec{p}_{(n)} - m_1 \vec{p}_{(n)}^2 \right),
\]

\[
\vec{x}_{(n)}^1 = \vec{x}_{(n)} + \frac{m_2}{m} \vec{r}_{(n)}, \quad \vec{x}_{(n)}^2 = \vec{x}_{(n)} - \frac{m_1}{m} \vec{r}_{(n)},
\]

\[
\vec{p}_{(n)}^1 = \frac{m_1}{m} \vec{p}_{(n)} + \vec{q}_{(n)}, \quad \vec{p}_{(n)}^2 = \frac{m_2}{m} \vec{p}_{(n)} - \vec{q}_{(n)}.
\]

The generators of the centrally extended Galilei algebra are (we have changed the sign of the Galilei boosts with respect to Refs.[35])
Let us remark that this property is preserved by the most general potential

\[ E_{\text{Galilei}} = \sum_{i=1}^{2} \frac{\vec{p}_i^2}{2m_i} + \frac{\vec{q}_i^2}{2\mu}, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \]

\[ \vec{P}_{\text{Galilei}} = \vec{p}(n) = \sum_{i=1}^{2} \vec{p}(n)_i, \]

\[ \vec{J}_{\text{Galilei}} = \sum_{i=1}^{2} \vec{x}(n)_i \times \vec{p}(n)_i = \vec{x}(n) \times \vec{p}(n) + \vec{S}(n), \quad \vec{S}(n) = \vec{r}(n) \times \vec{q}(n), \]

\[ \vec{K}_{\text{Galilei}} = t \vec{p}(n) - m \vec{x}(n), \]

\[ \{E_{\text{Galilei}}, \vec{K}_{\text{Galilei}}\} = \vec{P}_{\text{Galilei}}, \quad \{P^i_{\text{Galilei}}, K^j_{\text{Galilei}}\} = m \delta^{ij}, \quad \{K^i_{\text{Galilei}}, K^j_{\text{Galilei}}\} = 0, \]

\[ \{A^i, J^j_{\text{Galilei}}\} = \epsilon^{ijk} A^k, \quad \vec{A} = \vec{P}_{\text{Galilei}}, \vec{J}_{\text{Galilei}}, \vec{K}_{\text{Galilei}}. \] (2.16)

The main property of the Galilei algebra is that the presence of interactions changes the energy, \( E_{\text{Galilei}} \rightarrow \vec{E}_{\text{Galilei}} = E_{\text{Galilei}} + V(\vec{r}(n)) \) but not the Galilei boosts \(^{10}\).

Another property of the Galilei algebra, absent in the Poincare’ one, is that the energy generator is the sum of two distinct constants of motion: the center-of-mass energy \( E_{(n)\text{com}} = \frac{\vec{p}^2}{2m}, \vec{p} = \vec{p}(n) \), and the internal energy \( \epsilon(n) = \frac{\vec{q}^2}{2\mu} + V(\vec{r}(n)) \) \(^{11}\). This justifies the separation of variables in the Schroedinger equation. By comparison for two relativistic particles we have \( P^0 = \sqrt{M^2 c^2 + \vec{P}^2} \) with \( Mc = \sum_i \sqrt{m_i^2 c^2 + \vec{p}_{12}^2} + V(\vec{p}_{12}) \) or \( Mc = \sum_i \sqrt{m_i^2 c^2 + V(\vec{p}_{12}) + \vec{p}_{12}^2} \); \( P^0 \) is not a sum of two independent constants of motion \([7]\).

At the classical level the non-relativistic canonical transformation separating the center of mass from the relative variables is point both in the coordinate and in the momenta \(^{12}\). The non-relativistic point canonical transformation from the canonical basis \( \vec{x}(n)_i, \vec{p}(n)_i, i = 1, 2 \), to the one \( \vec{x}(n) = \sum_i \frac{m_i}{m} \vec{x}(n)_1 + \frac{m_i}{m} \vec{x}(n)_2, \vec{p}(n) = \vec{p}(n)_1 + \vec{p}(n)_2, \vec{r}(n) = \vec{x}(n)_1 - \vec{x}(n)_2, \vec{q}(n) = \frac{m_i}{m} \vec{p}(n)_2 - \frac{m_i}{m} \vec{p}(n)_2 \) can be obtained from the sequence of the two following canonical transformations connected with the identity \( e^{iS_2} e^{iS_1} \) with generating functions \( S_1 = \frac{m}{m} \vec{x}(n)_1 \cdot \vec{p}(n)_2 \) and \( S_2 = -\vec{x}(n)_2 \cdot \vec{p}(n)_1 \) \((m = m_1 + m_2)\).

\(^{10}\) This is the reason why there is no "No-Interaction Theorem" in Newtonian mechanics, so that Newtonian kinematics is trivial. However, this theorem reappears when we make a many-time reformulation of Newtonian mechanics \([36]\).

\(^{11}\) Let us remark that this property is preserved by the most general potential \( V(\vec{r}(n), \vec{q}(n), E_{(n)\text{com}}) \) admissible for an isolated two-particle system.

\(^{12}\) Its relativistic version on the Wigner hyper-plane for the internal motions is not point \([6, 7]\) (in absence of interactions it is point only in the momenta).
Also at the non-relativistic level the 2-body system can be presented as a decoupled particle, the external center of mass \( \vec{x}_{(n)}(t) \) with momentum \( \vec{p}_{(n)} \), of mass \( m \) in the absolute Euclidean 3-space carrying an internal space of relative variables \( (\vec{r}_{(n)}(t), \vec{q}_{(n)}(t)) \) with Hamiltonian \( H_{rel} = \frac{\vec{p}_{(n)}^2}{2m} \) and rest spin \( \vec{S}_{(n)} \).

The external center of mass is associated with an external realization of the Galilei group with generators \( E_{Galilei} = \frac{\vec{p}_{(n)}}{2m} + H_{rel}, \vec{P}_{Galilei} = \vec{p}_{(n)}, \vec{J}_{Galilei} = \vec{x}_{(n)} \times \vec{p}_{(n)} + \vec{S}_{(n)}, \vec{K}_{Galilei} = t \vec{p}_{(n)} - m \vec{x}_{(n)}(t) \).

The internal space can be identified with the rest frame \( (\vec{p}_{(n)} \approx 0) \) if we choose the origin of 3-coordinates in the external center of mass \( (\vec{x}_{(n)}(t) \approx 0) \): in it the particles variables are \( \vec{r}_{(n)}(t) = \vec{x}_{(n)}(t)|_{\vec{x}_{(n)}=\vec{p}_{(n)}=0}, \vec{r}_{(n)}(t) = \vec{p}_{(n)}|_{\vec{x}_{(n)}=\vec{p}_{(n)}=0} \) (they are the non-relativistic counterpart of the variables \( \vec{r}_{(n)}(\tau), \vec{q}_{(n)}(\tau) \) on the instantaneous Wigner 3-spaces). With this identification we get an unfaithful internal realization of the Galilei group with generators \( E_{Galilei} = H_{rel}, \vec{P}_{Galilei} = \vec{p}_{(n)} \approx 0 \) (the rest-frame conditions), \( \vec{J}_{Galilei} = \vec{S}_{(n)}, \vec{K}_{Galilei} = t \vec{p}_{(n)} - m \vec{x}_{(n)}(t) \approx 0 \) (the gauge fixings to the rest-frame conditions implying \( \vec{x}_{(n)}(t) \approx 0 \)).

Inside the internal space we have \( \vec{x}_{(n)}(t) \approx \vec{r}_{(n)}(t) = \frac{m}{m} \vec{r}_{(n)}(t), \vec{x}_{(n)}(t) \approx \vec{q}_{(n)}(t) = -\frac{m}{m} \vec{r}_{(n)}(t), \vec{p}_{(n)}(t) \approx \vec{r}_{(n)}(t), \vec{p}_{(n)}(t) \approx \vec{q}_{(n)}(t) = -\vec{r}_{(n)}(t) \) and we can introduce the following auxiliary variables (having an obvious relativistic counterpart) \( \vec{p}_{(n)}(t) = \vec{q}_{(n)}(t) = \vec{r}_{(n)}(t), \vec{p}_{(n)}(t) = \vec{r}_{(n)}(t), \vec{p}_{(n)}(t) = \vec{q}_{(n)}(t) \approx 0 \).

In the relativistic rest-frame instant form the two-particle system is described by

1) the external center-of-mass frozen Jacobi data \( \vec{z}, \vec{h} \), carrying the internal mass \( M \): \( c = \sum_{i=1}^{2} \sqrt{m_i c^2 + \vec{r}_{(i)}^2} \) and the spin \( \vec{S} = \sum_{i=1}^{2} \vec{q}_i \times \vec{r}_i \);

2) the two pairs of Wigner 3-vectors \( \vec{r}_i, \vec{q}_i, i = 1, 2, \) or by the canonically equivalent variables (2.11).

Since in the non-relativistic limit we have \( \vec{p} = \vec{p}_{(n)}, \vec{h} = \frac{\vec{p}_{(n)}}{M} \to_{c \to \infty} 0 \), implying \( h^\mu \to_{c \to \infty} \left( 1; 0 \right) \) and \( \epsilon_\mu^\nu (\vec{h}) \to_{c \to \infty} \left( 0; \delta_\nu^\nu \right) \), it turns out that \( \tau/c, \vec{x}_o/c, Y^o/c, R^o/c \) and \( \vec{x}_i/c \) all become the absolute Newton time \( t \).

Moreover we have the following results:

A) In the reference inertial system we get \( \vec{x}(\tau), \vec{Y}(\tau), \vec{R}(\tau) \to_{c \to \infty} \vec{x}_{NW}(t), \vec{y}_{(n)}(t) = \vec{x}_{NW}(t) \to_{c \to \infty} \vec{x}_{NW}(0) \). being \( \vec{x} = M c \vec{x}_{NW}(0) \to_{c \to \infty} 0 \) and \( \vec{h} \cdot \vec{z} \to_{c \to \infty} \vec{p}_{(n)} \cdot \left( \vec{x}_{(n)}(t) - \frac{\vec{p}_{(n)}}{m} t \right) = \vec{p}_{(n)} \cdot \vec{x}_{(n)}(0) \) (it is a Jacobi data of the non-relativistic theory).
B) In the inertial rest frame, \( \vec{p}_{(n)} \approx 0 \), we get \( \vec{\eta}_i(\tau) \to c\to\infty \vec{\eta}_i(t) \), \( \vec{r}_i(\tau) \to c\to\infty \vec{r}_i(t) \), \( \vec{x}_i(t) \to c\to\infty \vec{x}_i(t) \), \( \vec{\eta}_i(\tau) \to c\to\infty \vec{\eta}_i(t) \), \( \vec{p}_i(\tau) \to c\to\infty \vec{p}_i(t) \), \( p_i^0 \to c\to\infty m_i c \pm \frac{\vec{\pi}_i(t)}{2m_i} \).

The internal Poincare’ generators (2.8) have the limits (modulo the rest mass \( mc \) they are the internal Galilei generators)

\[
M c \to c\to\infty m c + \sum_{i=1}^{2} \frac{\vec{\pi}_i^2}{2m_i} / 2 \mu \approx m c + H_{rel},
\]

\[
\vec{P}_{(int)} \to c\to\infty \vec{r}_i(t) \approx 0,
\]

\[
\vec{S} \to c\to\infty \sum_{i=1}^{2} \vec{\eta}_i \times \vec{r}_i \approx \vec{\eta}_i \times \vec{\eta}_i = \vec{\eta}_i \times \vec{\eta}_i = \vec{S}_i(t),
\]

\[
\vec{K}_{(int)} \to c\to\infty - \sum_{i=1}^{2} m_i \vec{\eta}_i = -m \vec{\eta}_i \approx 0,
\]

while the limits of the external Poincare’ generators (2.6) are

\[
\vec{P} = \vec{p}_{(n)} = \vec{P}_{Galilei},
\]

\[
P^o \to c\to\infty m c + \frac{\vec{p}_i^0}{2m} + \sum_{i=1}^{2} \frac{\vec{\pi}_i^2}{2m_i} / 2 \mu \approx m c + E_{Galilei},
\]

\[
\vec{j} \to c\to\infty \vec{x}_i(\tau) \times \vec{p}_i + \vec{S}_i(t) = \vec{j}_{Galilei},
\]

\[
\vec{K}/c \to c\to\infty t \vec{p}_i(t) - m \vec{x}_i = \vec{K}_{Galilei},
\]

Therefore the non-relativistic limit of the rest-frame instant form leads to the following presentation of the Newton 2-body problem:

1) we have a decoupled external center of mass described by the canonical variables \( \vec{x}_i(t) \), \( \vec{p}_i(t) \), and carrying an internal space of relative variables coinciding with the non-relativistic rest center on the center of mass, \( \vec{\eta}_i(t) \approx 0 \) and \( \vec{r}_i(t) \approx 0 \) with the Hamiltonian \( H_{rel} \) and the rest spin \( \vec{S}_i(t) \);

2) in the internal space we have two pairs of variables \( \vec{\eta}_i(t), \vec{r}_i(t) \), or the canonically equivalent \( \vec{\eta}_i \approx 0, \vec{r}_i \approx 0, \vec{\eta}_i^2, \vec{r}_i^2 \), and, as a consequence from Eqs. (2.11) and (2.13) we have the following identifications

\[
\vec{\pi}_{12} = \vec{\eta}_1(\tau) - \vec{\eta}_2(\tau) \to c\to\infty \vec{\pi}_{12}(t) = \vec{\eta}_1(t) - \vec{\eta}_2(t) = \vec{\eta}(t),
\]

\[
\vec{p}_{12} = \frac{m_2}{m} \vec{r}_1(\tau) - \frac{m_1}{m} \vec{r}_2(\tau) \to c\to\infty \vec{p}_{12}(t) = \frac{m_2}{m} \vec{r}_1(t) - \frac{m_1}{m} \vec{r}_2(t) = \vec{q}(t),
\]

\[
\vec{x}_1(\tau) \to c\to\infty \vec{x}_1(t) = \vec{x}_1(t),
\]

\[
\vec{x}_2(\tau) \to c\to\infty \vec{x}_2(t) = \vec{x}_2(t).
\]

(2.20)
Let us remark that, while at the relativistic level the rest-frame world-lines (2.8) depend upon the 4-momentum $P^\mu$ of the external 4-center of mass (because it identifies the instantaneous Wigner 3-space in every inertial frame, being orthogonal to it), the non-relativistic trajectories $\bar{x}_{(n)}(t)$ do not depend upon $\bar{P}_{(n)}$, but only on $\bar{x}_{(n)}$ (the non-relativistic definitions of center of mass and relative variables do not mix coordinates and momenta).

C. The Abstract Internal Space of Relative Variables

In the fixed inertial frame chosen for the description of the isolated two-body system, to each value of its constant 4-momentum $P^\mu = M c \bar{h}^\mu$, i.e. to each value of the a-dimensional 3-velocity $\bar{h} = \bar{\eta}/c$, is associated a different rest-frame 3+1 splitting of Minkowski spacetime, whose Wigner hyper-planes $\Sigma_{\bar{h}}^{(\tau)}$ are orthogonal to the given $P^\mu$. In the chosen inertial system the natural rest frame, with Wigner 3-spaces $\Sigma_{\bar{h}}^{(\theta)}$, is associated with the 4-momentum $\bar{P} = M c (1; \bar{0})$, i.e. to $\bar{h} = 0$. Let us denote $\bar{\rho}_{12}^{(\bar{h})}(\tau)$ and $\bar{\pi}_{12}^{(\bar{h})}(\tau)$ the relative variables living inside $\Sigma_{\bar{h}}^{(\theta)}$.

Since we have $P^\mu = M c (\sqrt{1 + \bar{h}^2}; \bar{h}) = L^\mu_\nu (P, \bar{P}) \bar{P}_\nu = M c L^\mu_\nu (P, \bar{P})$ 13, we get $h^i \equiv L^i_\nu (P, \bar{P})$. Therefore, since $\bar{\rho}_{12}^{(\bar{h})}$ and $\bar{\pi}_{12}^{(\bar{h})}$ are Wigner spin-1 3-vectors transforming under Wigner rotations 14, we have that the 3-vectors inside $\Sigma_{\bar{h}}^{(\theta)}$ can be obtained from those inside $\Sigma_{\bar{h}}^{(0)}$ by means of the Wigner rotation $R^{\mu \nu}_{\alpha \beta} (L(P, \bar{P}), \bar{P}) = [L(\bar{P}, \bar{P}) L^{-1}(P, \bar{P}) L(P, \bar{P}) \bar{P}]^{\mu \nu} = [L^{-1}(P, \bar{P}) L(P, \bar{P})]^{\mu \nu} = \eta^{\mu \nu}$ associated to the Wigner boosts $L(P, \bar{P})$ sending $\bar{h} = 0$ into $\bar{h}$. As a consequence, we can make the identifications

$$
\bar{\rho}_{12}^{(\bar{h})}(\tau) = \bar{\rho}_{12}^{(\bar{0})}(\tau) \equiv \bar{\rho}_{12}(\tau),
\bar{\pi}_{12}^{(\bar{h})}(\tau) = \bar{\pi}_{12}^{(\bar{0})}(\tau) \equiv \bar{\pi}_{12}(\tau).
$$

Therefore, there is an abstract internal space of relative variables, living on an abstract Wigner 3-space $\Sigma_{\bar{h}} \equiv \Sigma_{\bar{h}}^{(\theta)}$, independent from the rest-frame foliation, i.e. independent from $\bar{h}$. Both the internal mass $M$ and the internal spin $\bar{S}$ depend only on these abstract relative variables living in an abstract Wigner 3-space $\Sigma_{\bar{h}}$: as a consequence there is a universal pole-dipole structure carried by the external center of mass.

---

13 The standard Wigner boost for time-like Poincare’ orbits is $L^\mu_\nu (P, \bar{P}) = \eta^{\mu}_\nu - 2 u^\mu(P) u_\nu(\bar{P}) - \frac{[u^\mu(P) + u^\nu(\bar{P})][u_\mu(P) + u_\nu(\bar{P})]}{1 + u^\nu(\bar{P})}$. We have $L^\mu_\nu (\bar{P}, \bar{P}) = \eta^{\mu}_\nu$.

14 To each Lorentz transformation $\Lambda^{\mu \nu}$ is associated the Wigner rotation $R^{\mu \nu}(\Lambda, P) = [L(\bar{P}, P) \Lambda^{-1} L(\bar{P}, P)]^{\mu \nu}$, with $R^{\mu \nu}(\Lambda, P) = 1$, $R^{\mu \nu}(\Lambda, P) = R^{\nu \mu}(\Lambda, P) = 0$, $R^{\mu \nu}(\Lambda, P) = (\Lambda^{-1})^\mu_j \frac{\Lambda_{\nu \mu} - u_\nu u_\mu}{1 + u^\nu(\bar{P})} + \frac{u^\nu(P)}{1 + u^\nu(\bar{P})} \frac{1}{1 + u^\nu(\bar{P}) (\Lambda^{-1})^\mu_j}$. 

19
These identifications can be done also for the internal 3-center-of-mass variables $\vec{\eta}_{12} \approx \vec{\eta}_{12}[\vec{\rho}_{12}, \vec{\pi}_{12}], \vec{\kappa}_{12} \approx 0$ before solving the Wigner-covariant constraints $\vec{P}(\text{int}) \approx 0, \vec{K}(\text{int}) \approx 0$ and therefore also for the variables $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)$.

This abstract internal relative space is carried by the external 3-center of mass, which is described by the Jacobi data $\vec{z}, \vec{h}$ (the time-independent Cauchy data). While the Jacobi data $\vec{z}, \vec{h}$, can be quantized independently from the eigenvalues of the internal mass operator $\hat{M} c$, the Newton-Wigner 3-position $\vec{x}_{NW} = \vec{z}/M c$ and the 3-momentum $\vec{P} = M c \vec{h}$, depend on these eigenvalues.
III. NON-RELATIVISTIC QUANTUM MECHANICS OF TWO PARTICLES

Let us review the standard QM description of a two-particle system in Galilei space-time with the notation of I.

A. Non-Relativistic Quantum Mechanics

In non-relativistic QM there is a Hilbert space $\mathcal{H}_t = (\mathcal{H}_1 \otimes \mathcal{H}_2)_t = (\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}})_t$ associated with each instant $t$ of the absolute Newton time: it contains wave functions $\psi_t(\vec{x}(n)_i)$ or $\psi_t(\vec{x}(n), \vec{r}(n))$ depending upon the 3-coordinates of the particles in the absolute Euclidean 3-space. The Galilei group acts in the Hilbert space $\mathcal{H}$, $\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}$, so that the time evolution can be described as a unitary transformation with parameter $t$ inside $\mathcal{H}$: in $\mathcal{H}$ we have the wave functions $\psi(t, \vec{x}(n)_i)$ or $\psi(t, \vec{x}(n), \vec{r}(n))$ connected by a unitary transformation.

By quantization of the sequence of the two canonical transformations (2.17) ($e^{i\ldots S_i} \rightarrow e^{i\hat{S}_i}, \hat{S}_i = \hat{S}_i^\dagger$) we get an explicit unitary transformation connecting the description $\mathcal{H}_1 \otimes \mathcal{H}_2$ to the one $\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}$. In $\mathcal{H}$ it corresponds to a change of basis: it sends the position basis $\psi_1(\vec{x}(n)_1) \psi_2(\vec{x}(n)_2)$ labeled by the eigenvalues of the maximal set $\hat{x}_{(n)_1}, \hat{x}_{(n)_2}$ of commuting operators to the position basis $\psi_{\text{com}}(\vec{x}(n)) \psi_{\text{rel}}(\vec{r}(n))$ labeled by the eigenvalues of the maximal set $\hat{x}_{(n)}, \hat{r}_{(n)}$ of commuting operators.

When there is an interaction between the particles of an isolated system, the separation of variables implies that the Schroedinger equation is written in the coordinate representation associated with the preferred basis in $\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}$ ($\vec{L} = \vec{x}(n) \times \vec{p}(n) + \vec{S}, \vec{S} = \vec{r}(n) \times \vec{q}(n)$)

$$i \frac{\partial}{\partial t} \psi(n)(t, \vec{x}(n), \vec{r}(n)) = \left( \frac{\vec{p}(n)^2}{2m} + \hat{H}_{\text{rel}} \right) \psi(n)(t, \vec{x}(n), \vec{r}(n)),$$

$$\frac{\vec{p}(n)^2}{2m} \psi(n)p(\vec{x}(n)) = \frac{\vec{p}^2}{2m} \psi(n)p(\vec{x}(n)), \quad \psi(n)p(\vec{x}(n)) = \text{const.} e^{i\vec{p}\vec{x}(n)};$$

$$\hat{H}_{\text{rel}} \phi(n)_{nlm}(\vec{r}(n)) = \epsilon(n)_{nlm} \phi(n)_{nlm}(\vec{r}(n)),$$

$$\hat{S}_3 \phi(n)_{nlm}(\vec{r}(n)) = m \phi(n)_{nlm}(\vec{r}(n)),$$

$$\hat{E}_{(n)n\vec{p}} = \frac{\vec{p}^2}{2m} + \epsilon(n)_n = E_{(n)\text{com}} \vec{p} + \epsilon(n)_n;$$

$^15$ $\hat{A}$ denotes the operator corresponding to the classical variable $A$. 

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Therefore the separation of variables implies that the Schroedinger equation can be replaced by two separate Schroedinger equations, one for the center of mass and one for the relative motion.

At the quantum level we have that the centrally extended Galilei group is implemented with a projective realization. A Galilei boost \( \vec{x} \rightarrow \vec{x} - \vec{v} t, t \rightarrow t \) in Galilei space-time (so that \( \vec{x}_n \rightarrow \vec{x}_n - \vec{v} t \) and \( \vec{r}_n \rightarrow \vec{r}_n \)) is implemented as a projective unitary transformation:

\[
\psi(t, \vec{x}_n, \vec{r}_n) \rightarrow e^{-i \frac{m}{2} \vec{v} \cdot \vec{x}_n + \frac{m}{2} \vec{v}^2 t} \psi(t, \vec{x}_n, \vec{r}_n) = \psi'(t, \vec{x}_n - \vec{v} t, \vec{r}_n).
\]

Therefore, in the presence of mutual interactions, the bases of Hilbert space corresponding to \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is not a natural one for isolated systems. Its use, for instance in the theory of entanglement, is a realistic one only in the free case (see also footnote 42 in the conclusion).

### B. The Hamilton-Jacobi Description of the Center of Mass as the Non-Relativistic Limit of the Rest-Frame Instant Form

Since we want to make a comparison of the non-relativistic limit of the rest-frame instant form of paper I with the standard non-relativistic theory, let us define the quantum dynamics in \( \mathcal{H} = \mathcal{H}_{com} \otimes \mathcal{H}_{rel} \) in the representation arising after the transition to the Hamilton-Jacobi form for the motion of the decoupled center of mass at the classical level. Given the Hamiltonian \( H = H_{com} + H_{rel} \), \( H_{com} = \frac{\vec{p}^2(n)}{2m} \), the transition to the Hamilton-Jacobi description of the center of mass is usually done with a time-dependent canonical transformation whose generating function is the solution \( \tilde{S}(t, \vec{x}_n(t), \vec{p}_n(0)) = \vec{p}_n \cdot \vec{x}_n(t) - \frac{\vec{p}^2_n}{2m} t \) (\( \vec{p}_n \) is time-independent being a constant of motion) of the Hamilton-Jacobi equation \( H_{com}(\vec{x}_n(t), \frac{\partial \tilde{S}}{\partial \vec{x}_n}) + \frac{\partial \tilde{S}}{\partial t} = 0 \). This canonical transformation can be implemented in the form \( e^{i \tilde{S}} \) if we choose the generating function \( S = -\frac{\vec{p}^2(n)}{2m} t = -H_{com} t \)
The Hamilton-Jacobi description is Eqs. (2.16) functions \( \psi \) into the frozen center of mass wave functions. As already said, the classical isolated system is represented as a decoupled frozen point particle, the center of mass \( \vec{x} \). Inertial frame where \( \vec{p} \) space can be identified with the rest frame description of the isolated system \( \vec{S} \). At the quantum level, the associated unitary transformation to get the center-of-mass wave functions \( \psi(t, \vec{x}) \) = \( e^{-i \vec{p} \cdot \vec{x}} \psi(0) \) for plane waves into the frozen wave functions \( \psi(p, \vec{x}(0)) = \text{const.} e^{i \vec{p} \cdot \vec{x}(0)} \) with the identification \( \vec{x} = \vec{x}(0) \).

In this basis, the Schrödinger equation of Eqs. (3.1) becomes

\[
\frac{i}{\partial t} \tilde{\psi}_n(\vec{x}(0)|t, \vec{r}(n)) = \hat{H}_{\text{rel}} \tilde{\psi}_n(\vec{x}(0)|t, \vec{r}(n)),
\]

\[
\tilde{\psi}_n(\vec{x}(0)|t, \vec{r}(n)) = \psi_n(p(\vec{x}(0)) \phi_n(t, \vec{r}(n)).
\]

If we go to the momentum basis for the frozen center of mass, we get that the wave functions

\[
\hat{p}_n \tilde{\psi}_n(\vec{p}(t)|t, \vec{r}(n)) = \tilde{\psi}_n(\vec{p}(t)|t, \vec{r}(n)) \phi_n(t, \vec{r}(n)), \quad \text{or} \quad \tilde{\psi}_n(\vec{p}(t)|t, \vec{r}(n)) = \hat{\delta}^3(\vec{p}(t) - \vec{k}(t)).
\]

\[
\frac{i}{\partial t} \tilde{\psi}_n(\vec{p}(t)|t, \vec{r}(n)) = \hat{H}_{\text{rel}} \tilde{\psi}_n(\vec{p}(t)|t, \vec{r}(n)).
\]
IV. QUANTIZATION OF RELATIVISTIC PARTICLES IN THE REST-FRAME
INSTANT FORM OF DYNAMICS

A. Quantization

Let us now study the quantization of the isolated 2-body problem in the rest-frame instant form.

We have to quantize the frozen Jacobi data \( \vec{z}, \vec{h} \), of the external 3-center of mass in the preferred momentum basis \( \vec{h} \) or \( \vec{P} = M c \vec{h} \), needed to define the foliations and the abstract internal relative space, and the relative variables \( \vec{\rho}_{12}, \vec{\pi}_{12} \) of the decoupled internal space, whose evolution in the rest time \( \tau = c T_s \) is governed by the internal mass \( M \), i.e. the energy of the internal Poincare’ group acting in the abstract Wigner 3-space \( \Sigma_\tau \).

The external canonical non-covariant 4-center of mass \( \vec{x}_\mu(\tau) \) (the function of \( \vec{z}/M c = \vec{x}_{NW}, \vec{h} \) and \( \tau \) given in Eq.(2.2)) and its conjugate momentum \( P_\mu = M c \sqrt{1 + \vec{h}^2}; \vec{h} \) are derived quantities. The evolution in \( \tau \) governed by \( M \) will be shown to imply an evolution of the external 4-center of mass \( \vec{x}_\mu(\tau) \) in terms of the time variable \( \vec{x}_0 \): consistently this evolution is governed by the Hamiltonian \( P_o = \sqrt{M^2 c^2 + \vec{P}^2} = M c \sqrt{1 + \vec{h}^2} \) (the energy of the external Poincare’ group) due to the relation \( \tau = c T_s = h \cdot \vec{x} \).

Also the covariant non-canonical world-lines \( x_i(\tau) = z_i(\tau, \vec{\eta}(\tau)) \) of the particles are derived quantities, which becomes non-commuting operators, as implied by Eqs.(2.14), depending on the Jacobi data of the external center of mass and on the relative variables.

For more complicated systems, like the ones of the standard semi-relativistic atomic physics (see papers I and II), for which we do not know how to solve the rest-frame conditions \( \vec{P}_{(int)} \approx 0, \vec{K}_{(int)} \approx 0 \), we must define a more general quantization scheme for the internal space including also the conjugate variables describing the internal 3-center of mass \( (\vec{\rho}_{12} \approx 0, \vec{\kappa}_{12} \approx 0 \) in the two-body case). Namely we have to quantize the Jacobi data for the external center of mass and the redundant variables \( \vec{\eta}(\tau), \vec{\kappa}_i(\tau) \) and then impose the quantum version of the second class constraints \( \vec{P}_{(int)} = \vec{K}_{(int)} \approx 0 \) with some prescription (the Gupta-Bleuler one if possible). Differently from the non-relativistic case, where \( \vec{K}_{(int)} \approx 0 \) becomes \( \vec{\kappa}_{12}(\tau) \approx 0 \), this is a non-trivial task.

The extension of the previous quantization to isolated N-body systems is automatic if we know the explicit form of the generators of the internal Poincare’ algebra and we use the relative variables of Eqs.(II-2.1). As said in I, this requires the knowledge of the energy-momentum tensor of the N-body system. This is known explicitly only for a limited number of systems [7, 8].

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\[16\] As shown in Refs.[5–7], before the reduction of the external 4-center of mass \( \vec{x}_\mu, P_\mu \) to the external 3-center of mass \( \vec{z}, \vec{h} \), the variables \( \tau = c T_s = h \cdot \vec{x} \) and \( \epsilon_s = \sqrt{c P^2} \approx M c \) are canonically conjugate variables. This is just the same situation like with the Galilei energy \( E \) and Newton time \( t \) in ordinary quantum mechanics, where to get the time-dependent Schroedinger equation 1) one sends \( E \) into the operator \( i \frac{\partial}{\partial \tau}; 2 \) one uses \( E = H \) to write \( i \frac{\partial}{\partial \tau} \psi = \hat{H} \psi \) with the Hamiltonian operator \( \hat{H} \) depending on the canonically conjugate particle variables.

\[17\] We will get a positive-energy Klein-Gordon equation for each eigenvalue \( M_n \) of the internal mass (like for a scalar positive-energy particle of mass \( M_n \)).
B. Quantization after the Elimination of $\vec{\eta}_{12}$

1. The Hilbert Space

To quantize we must consider a Hilbert space $\mathcal{H} = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}$ with the following constraints implying the use of wave functions $\Psi(\vec{h}|\tau, \vec{\rho}_{12})$ (i.e. in the center-of-mass momentum representation and in the coordinate representation for the relative variables):

1) $\mathcal{H}_{\text{com}}$ is the Hilbert space of a positive energy frozen 3-center-of-mass particle described by the quantum Jacobi data $\tilde{z}$, $\tilde{h}$. We must use the preferred $\tilde{h}$-basis in the momentum representation because it is needed for the kinematical definition of the rest frame, i.e. of the Wigner 3-space. Consistently with the frozen nature of the external 3-center of mass, instead of an evolution equation we have

$$\hat{\tilde{h}} \Psi_{\vec{k}}(\vec{h}|\tau, \vec{\rho}_{12}) = \vec{k} \Psi_{\vec{k}}(\vec{h}|\tau, \vec{\rho}_{12}),$$

or

$$\Psi_{\vec{k}}(\vec{h}|\tau, \vec{\rho}_{12}) = \delta^3(\vec{h} - \vec{k}) \phi(\tau, \vec{\rho}_{12}) \overset{\text{def}}{=} \psi_{\vec{k}}(\vec{h}) \phi(\tau, \vec{\rho}_{12}). \quad (4.1)$$

Let us assume that $\hat{\tilde{z}}$ (and therefore also $\hat{\tilde{z}}_{NW} = \hat{\tilde{z}}/\tilde{M}$) is a self-adjoint operator. Since this is a problematic assumption (see Section VI), the preferred $\tilde{h}$-basis is also useful to avoid facing these problems at this level. With $\tilde{z}$ self-adjoint we can go to the $\tilde{z}$-representation and use plane waves as elementary solutions for the external 3-center of mass: $\Psi_{\vec{k}}(\vec{z}|\tau, \vec{\rho}_{12}) = \text{const.} e^{i\vec{k} \cdot \vec{z}} \phi(\tau, \vec{\rho}_{12}) = \psi_{\vec{k}}(\vec{z}) \phi(\tau, \vec{\rho}_{12})$.

Let us remark that, as shown in Ref.\[12\], in the momentum representation we have

$$z^i \rightarrow i \hbar \frac{\partial}{\partial \vec{h}^i} - i \hbar \frac{\vec{h}^i}{1 + \vec{h}^2}$$

and the time-independent scalar product in this frozen Hilbert space has the form

$$< \Psi_1, \Psi_2 > = \int \frac{d^3\vec{h}}{2 \sqrt{1 + \vec{h}^2}} \psi^*_1(\vec{h}) \psi_2(\vec{h}). \quad (4.2)$$

It is a Lorentz scalar.

2) $\mathcal{H}_{\text{rel}}$ is the abstract internal rest-frame Hilbert space, corresponding to the abstract internal relative space on the abstract Wigner 3-space $\Sigma_{\tau}$, for the relative motions. Its scalar product is

$$< \phi_1, \phi_2 > = \int d^3\vec{\rho}_{12} \phi^*_1(\tau, \vec{\rho}_{12}) \phi_2(\tau, \vec{\rho}_{12}). \quad (4.3)$$

It is conserved in the time $\tau$ and Lorentz scalar \[12, 15\] with $\phi \in L^2(R^3)$.

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18 Strictly speaking the internal Poincare' groups, acts in $\mathcal{H}_{\text{rel}} = U^\infty_{\tau=-\infty} \mathcal{H}_{\text{rel},\tau}$. 

25
At the classical level we have that the external Poincare’ group has the generators given in Eq.(2.6) with \( M = M(\vec{\rho}_{12}, \vec{\pi}_{12}) \) and \( \vec{S} = \vec{\rho}_{12} \times \vec{\pi}_{12} \). In \( \mathcal{H} = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}} \) in the \( \vec{h} \)-basis they can be realized as the following Hermitean operators

\[
\hat{P}^o = \hat{M} c \sqrt{1 + \vec{h}^2}, \quad \hat{P} = \hat{M} \vec{c} \vec{h}, \\
\hat{J} = \vec{z} \times \vec{h} + \vec{S}, \\
\hat{K} = -\frac{1}{2} \left( \vec{z} \sqrt{1 + \vec{h}^2} + \sqrt{1 + \vec{h}^2} \vec{z} \right) + \frac{\vec{S} \times \vec{h}}{1 + \sqrt{1 + \vec{h}^2}},
\]

with \( \hat{M} \) a self-adjoint suitably ordered operator depending upon \( \hat{\rho}_{12}, \hat{\pi}_{12} \) and with \( \vec{S} = \vec{\rho}_{12} \times \vec{\pi}_{12} \). They satisfy the Poincare’ algebra: [\( \hat{P}^o, \hat{P}^\nu \)] = 0, [\( \hat{P}^o, \hat{J} \)] = 0, [\( \hat{P}^i, \hat{J}^j \)] = \( \hat{M} e^{ijk} \hat{K}^k \), [\( \hat{K}^i, \hat{J}^j \)] = \( \hat{M} \delta^{ij} \hat{P}^o \), [\( \hat{K}^i, \hat{K}^j \)] = \( -\hat{M} \delta^{ij} \hat{J}^k \). Therefore, as in Refs.[15], [12], there is a unitary realization of the external Poincare’ group.

2. The Relativistic Schroedinger Equation

As shown in Section III, in non-relativistic QM the Schroedinger equation can be split in two separate Schroedinger equations for the center of mass and the relative motion in the non-relativistic Hilbert space \( \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}} \) due to the special property of Galilei energy. In the center-of-mass Hamilton-Jacobi description of the 2-body system these two equations are given in Eqs.(3.4).

Also, in the rest-frame instant form of dynamics the quantum description of an isolated relativistic system is split in two parts:

A) A non-evolving 3-center of mass described by frozen Jacobi data, so that there is no relativistic Schroedinger equation as a counterpart of the first equation in Eqs.(3.4);

B) An internal space of relative motions with a \( \tau \)-evolution governed by the invariant mass \( \hat{M} \) (the energy generator of the internal Poincare’ group). This will lead to a Schroedinger equation for the \( \tau \)-evolution of the internal motion, which is the relativistic counterpart of the second of Eqs.(3.4). The eigenvalues \( M_n \) of the invariant mass are determined by the associated stationary Schroedinger equation, which will take also into account the internal spin.

For the \( \tau \)-evolution of the internal motion inside the Wigner 3-space we have the following Schroedinger equations in \( \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}} \) in the preferred \( \vec{h} \)-basis with frozen 3-center-of-mass wave function \( \psi_{\vec{h}}(\vec{h}) \), \( [\Psi_{\vec{h}}(\vec{h}\tau, \vec{\rho}_{12}) = \psi_{\vec{h}}(\vec{h}) \phi(\tau, \vec{\rho}_{12})] \)
\[ \hat{h} \Psi_{\hat{h}}(h|\tau, \vec{\rho}_{12}) = \hat{k} \Psi_{\hat{h}}(h|\tau, \vec{\rho}_{12}), \]

\[ i \frac{\partial}{\partial \tau} \Psi_{\hat{h}}(h|\tau, \vec{\rho}_{12}) = \hat{M}(\vec{\rho}_{12}, \vec{\pi}_{12}) c \Psi_{\hat{h}}(h|\tau, \vec{\rho}_{12}), \]

\[ \psi_{\hat{h}}(h) \left[ \left( i \frac{\partial}{\partial \tau} - \hat{M}(\vec{\rho}_{12}, \vec{\pi}_{12}) c \right) \phi(\tau, \vec{\rho}_{12}) \right] = 0, \quad (4.5) \]

where \( \hat{M}(\vec{\rho}_{12}, \vec{\pi}_{12}) \) is the operator defined by the quantization of classical models with either \( M(\vec{\rho}_{12}, \vec{\pi}_{12}) c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{V}_1(\vec{\rho}_{12}) + \vec{\pi}_{12}^2} \) or \( M(\vec{\rho}_{12}, \vec{\pi}_{12}) c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + V_2(\vec{\rho}_{12}) \) \( (V_1 \text{ or } V_2 \text{ are a-a-a-d potentials}) \)

Let us put \( \phi(\tau, \vec{\rho}_{12}) = e^{-i \vec{z} \cdot \vec{\rho}_{12}} \). If we can find the solutions \( \phi_{nlm}(\vec{\rho}_{12}) \) of the stationary equation in \( \mathcal{H}_{\text{rel}} \)

\[ \hat{M} e^2 \phi_{nlm}(\vec{\rho}_{12}) = \epsilon_n \phi_{nlm}(\vec{\rho}_{12}), \]

\[ \hat{S}_2^2 \phi_{nlm}(\vec{\rho}_{12}) = l (l + 1) \phi_{nlm}(\vec{\rho}_{12}), \]

\[ \hat{S}_3 \phi_{nlm}(\vec{\rho}_{12}) = m \phi_{nlm}(\vec{\rho}_{12}), \quad (4.6) \]

then we have the elementary solutions \( (\rho_n^u = \left( \epsilon_n \sqrt{1 + \vec{k}^2}; \epsilon_n c \vec{k} \right), M_n e^2 = \epsilon_n^{\text{def}} = m c^2 + \bar{\epsilon}_n \) with \( \bar{\epsilon}_n \to_{c \to \infty} \epsilon_{(n)n} \quad (20) \)

\[ \Psi_{\vec{k},nlm}(\vec{h} \, | \tau, \vec{\rho}_{12}) = \delta^3(\vec{h} - \vec{k}) (2\pi)^{-3/2} e^{-i M_n \cdot c \tau} \psi_{nlm}(\vec{\rho}_{12}), \quad (4.7) \]

In the \( \vec{z} \)-basis we have \( \Psi_{\vec{k},nlm}(\vec{z} \, | \tau, \vec{\rho}_{12}) = \text{const.} e^{i \vec{k} \cdot \vec{z}} (2\pi)^{-3/2} e^{-i M_n \cdot c \tau} \phi_{nlm}(\vec{\rho}_{12}) \).

The wave packets for the internal motion are \( \Psi_{\vec{k}}(\vec{z} \, | \tau, \vec{\rho}_{12}) = \text{const.} e^{i \vec{k} \cdot \vec{z}} \sum_{nlm} F_{nlm} (2\pi)^{-3/2} e^{-i M_n \cdot c \tau} \phi_{nlm}(\vec{\rho}_{12}) \).

The wave packets also on the external 3-center of mass are \( \Psi(\vec{z} \, | \tau, \vec{\rho}_{12}) = \int \frac{d^3 \vec{k}}{2 \sqrt{1 + \vec{k}^2}} G(\vec{k}) \Psi_{\vec{k}}(\vec{z} \, | \tau, \vec{\rho}_{12}) \). These last wave packets correspond to superpositions of different \( 3+1 \) rest-frame splittings. See Section VI for a discussion on the self-adjointness of \( \vec{z} \) and the status of these wave packets.

\[ ^{19} \text{In Ref.}[7] \text{ there is the evaluation of the internal Poincare' generators for the case in which the arbitrary potential } V_1(\vec{\rho}_{12}) \text{ is under the square root. For the more relevant case in which the potential } V_2(\vec{\rho}_{12}) \text{ is outside the square root the form of the internal Lorentz boosts is not known except for the Coulomb plus Darwin potential } V_2(\vec{\rho}_{12}, \vec{\pi}_{12}) \text{ of II (in this case the knowledge of the energy-momentum tensor of the system allows the determination). As shown in Ref.}[7] \text{ and in II, they induce potential-dependent terms in the internal Lorentz boosts, so that the solution } \vec{\eta}_{12} = \vec{\eta}_{12}(\vec{\rho}_{12}, \vec{\pi}_{12}) \text{ of the conditions } \vec{K}_{(int)} \approx 0 \text{ eliminating the internal 3-center of mass are potential dependent.} \]

\[ ^{20} \text{They are the non-relativistic energy levels of the the relative Hamiltonian } \hat{H}_{\text{rel}} \text{ resulting from the non-relativistic limit of } \hat{M} \text{ in Eq.(4.5). Let us remark that different relativistic theories (potential either inside or outside the square roots) can have the same non-relativistic potential as a limit.} \]
If we can find a complete set of solutions of Eqs.(4.6), then the generic solutions of Eqs.(4.5) will be the most general square-integrable superposition of center-of-mass plane waves and elementary solutions for the relative motion.

Let us remark that the non-relativistic limit of Eq.(4.7) is

$$e^{i\vec{k} \cdot \vec{z}} e^{-i M_n c \tau} \phi_{nlm}(\vec{p}_{12}) \to c \to \infty e^{i M_n c \vec{k} \cdot \vec{x}(0)} e^{-i (mc^2 + \epsilon_n t)} \phi_{nlm}(\vec{r}_n) =$$

$$e^{i \vec{p}_n \cdot \vec{x}(0)} e^{-im c^2 t} \left( \text{non-relativistic relative motion elementary solution} \right).$$

(4.8)

By comparison with Eq.(3.1) we see that it corresponds to a reformulation of non-relativistic quantum mechanics in a framework in which the non-relativistic center of mass is described in terms of the frozen Jacobi data \( \vec{k} = \vec{p}_n / M_n c \) and \( \vec{z} = M_n c \vec{x}(0) \).

3. The External 4-Center of Mass

To recover the motion of the external 4-center of mass, carrying the pole-dipole structure with mass \( M \) and spin \( \vec{S} \), we have to replace the frozen \( M \)-independent plane wave \( e^{i \vec{k} \cdot \vec{z}} \), living in \( H_{\text{com}} \), with a wave function \( \psi_{M_n}(\vec{x}^o, \vec{P}) \) knowing the levels \( M_n \) of the quantum invariant mass \( \hat{M} \) (the internal wave function \( \phi_{nlm}(\vec{r}_{12}) \) in \( H_{\text{rel}} \) takes care of the spin \( \vec{S} \)).

Its \( \vec{x}^o \)-evolution is governed by the external Poincare’ energy \( P^o = \sqrt{M_n^2 c^2 + \vec{P}^2} \) corresponding to the level. Therefore we have to introduce as many new auxiliary Hilbert spaces \( H_{\text{extcom}} \) as mass levels \( M_n \). For the \( \vec{x}^o \)-evolution we have as Schrödinger equation the positive-energy Klein-Gordon equation

$$i \frac{\partial}{\partial \vec{x}^o} \psi_{M_n}(\vec{x}^o, \vec{P}) = \sqrt{M_n^2 c^2 + \vec{P}^2} \psi_{M_n}(\vec{x}^o, \vec{P}),$$

(4.9)

This is equivalent to undoing the Hamilton-Jacobi transformation on the external center of mass independently for each level of the internal motion: the non-relativistic limit of Eqs.(4.9) is the first equation in Eqs.(3.1), because we have \( M_n \to c \to \infty m + O(c^{-1}) \), \( \sqrt{M_n^2 c^2 + \vec{P}^2} \to c \to \infty mc^2 + \frac{\vec{p}_n^2}{2m} \). The irrelevant phase factor \( e^{-im c^2} \) has to be omitted.

If we take into account both positive- and negative-energies for the external 4-center of mass, we have the Klein-Gordon equation in the \textit{preferred} momentum basis

$$ \left( \hat{\vec{P}}^2 - M_n^2 c^2 \right) \psi_{M_n}(P^\mu) = 0,$$

(4.10)

whose solutions are \( (\eta = \text{sign} P^o) \)

---

It is obtained from the Klein-Gordon equation by means of the Feshbach-Villars transformation [27, 37]. With both signs of energy the scalar product is the same as for a Klein-Gordon scalar particle of mass \( M_n \).
\[ \psi_{M_n}(P^\mu) = \text{const.} e^{-i P \cdot \tilde{x}} = \text{const.} e^{-i M_n \cdot c \tau} = e^{const. e^{-i (\eta \sqrt{M_n^2 c^2 + \vec{P}^2 \tilde{x}^2 - \vec{P} \cdot \tilde{x})}} e^{\rightarrow \infty, \eta = 1} e^{-i (mc^2 + \frac{\vec{p}^2}{2m} - \vec{P} \cdot \tilde{x})} = e^{-imc^2 t} \left( \text{non-relativistic center of mass plane wave} \right). \] (4.11)

Consistently \( \psi_{M_n}(P^\mu) \) coincides with the piece \( e^{-i M_n m \cdot c \tau} \) of Eq.(4.7) due to the relation \( \tau = \frac{c}{\hbar} \cdot \tilde{x} \). Therefore, the auxiliary Hilbert spaces \( \mathcal{H}_{\text{ext com}} \) are a byproduct of this relation: due to it every elementary solution (4.7) in \( \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}} \) with fixed \( \vec{h} \) and \( M_n \) and with the non-relativistic limit (4.8) contains a phase describing also a plane wave for the external 4-center of mass in \( \mathcal{H}_{\text{ext com}} \) as shown in Eq.(4.11) \(^{22}\).

In the preferred momentum representation the plane wave solution is \((2\pi)^{-3/2} \delta^3(\vec{P} - \eta \frac{M_n m}{c} \vec{h}) \delta(P^\mu - \eta \frac{M_n m}{c} \sqrt{1 + \vec{h}^2})\).

C. Quantization with \( \vec{h}_{12} \)

If we cannot solve the rest-frame conditions and the conditions for the elimination of the internal 3-center of mass, we must start with an unphysical internal Hilbert space \( \mathcal{H}_{\vec{h}_1} \otimes \mathcal{H}_{\vec{h}_2} = \mathcal{H}_{\vec{h}_{12}} \otimes \mathcal{H}_{\vec{p}_{12}} \) (its formal separability is unphysical) with a unphysical scalar product, write Eq.(4.5) in \( \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\vec{h}_1} \otimes \mathcal{H}_{\vec{h}_2} \) and then impose the 3 pairs of second class constraints \( \vec{P}_{(\text{int})} = \vec{k}_{12} \approx 0 \), \( \vec{K}_{(\text{int})} \approx 0 \) as restrictions on the states. Therefore, besides Eq.(4.5) with \( \hat{M} \) function of \( \vec{h}_i \) and \( \vec{k}_i \), \( i = 1, 2 \) there will be the 6 equations

\[ < \Phi_{\text{phys}} | \hat{P}_{(\text{int})} | \Phi_{\text{phys}} > = < \Phi_{\text{phys}} | \hat{K}_{(\text{int})} | \Phi_{\text{phys}} > = 0, \] (4.12)

which should lead to the identification of the physical Hilbert space \( \mathcal{H}_{\text{rel}} \) and of its physical scalar product. But the second set of conditions (4.12) are interaction-dependent, so that the quantization is non-trivial and could be unitarily inequivalent to the one of the previous Subsection. An open problem is whether Eqs.(4.12) can be replaced by conditions of the type \( \hat{A} | \Phi_{\text{phys}} > = 0 \) and \( 0 = < \Phi_{\text{phys}} | \hat{A}^\dagger \) corresponding to a generalized Gupta-Bleuer-like approach.

In this case, besides writing the quantum external Poincare’ algebra with \( \hat{M} \) and \( \hat{S} \) depending on the operators \( \hat{h}_i \) and \( \hat{k}_i \), one should check also the validity of the quantum internal Poincare’ algebra by using a suitable ordering.

Let us consider the case of two free particles as an example. From Eqs.(2.8) and (2.11) we have the following two forms of the internal Poincare’ generators with the Poincare’ algebra trivially satisfied (\( m_3 \equiv m_1 \))

\(^{22}\) This is the relativistic description, which should be used for the motion of an atom in atom interferometry instead of the effective Schroedinger equation of Ref.[38], obtained by extracting the positive-energy part of relativistic first-quantized wave equations like Klein-Gordon, Dirac or Proca, whose second quantization is assumed to describe an effective QFT for spin 0, \( \frac{1}{2} \) or 1 (two-level) atoms.
\[M c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \left(\frac{m_i}{m} \vec{\kappa}_{12} - (-)^i \vec{\pi}_{12}\right)^2} \approx \]
\[= \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2},\]

\[\vec{P}_{(int)} = \sum_{i=1}^{2} \vec{\kappa}_i = \vec{\kappa}_{12} \approx 0,\]

\[\vec{J}_{(int)} = \sum_{i=1}^{2} \vec{\eta}_i \times \vec{\kappa}_i = \vec{\eta}_{12} \times \vec{\kappa}_{12} + \vec{S} \approx \vec{S} = \vec{\rho}_{12} \times \vec{\pi}_{12},\]

\[\vec{K}_{(int)}^2 = -\sum_{i=1}^{2} \vec{\eta}_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} = -\vec{\eta}_{12} \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \left(\frac{m_i}{m} \vec{\kappa}_{12} - (-)^i \vec{\pi}_{12}\right)^2} + \]

\[+ \vec{\rho}_{12} \sum_{i=1}^{2} (-)^i \frac{m_{i+1}}{m} \sqrt{m_i^2 c^2 + \left(\frac{m_i}{m} \vec{\kappa}_{12} - (-)^i \vec{\pi}_{12}\right)^2} \approx \]

\[\approx -\vec{\eta}_{12} \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + \vec{\rho}_{12} \sum_{i=1}^{2} (-)^i \frac{m_{i+1}}{m} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} \approx 0.\]

\[(4.13)\]

The rest-frame conditions imply \(\vec{\kappa}_{12} \approx 0\), Eq. (2.12) for \(\vec{\eta}_{12}\) and Eqs. (2.13) for \(\vec{\eta}_i\), \(x_i^\mu\) and \(p_i^\mu\).

In the quantization without \(\vec{\eta}_{12}\) in \(\mathcal{H}_{rel}\) one uses the operators \(\hat{M} c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2}\) and \(\hat{S} = \vec{\rho}_{12} \times \vec{\pi}_{12}\). For the quantization of the derived quantities \(x_i^\mu(\tau)\) and \(p_i^\mu(\tau)\) one must start from Eqs. (2.14), as it will be done in Eq. (5.3) of the next Section.

Instead the quantization with \(\vec{\eta}_{12}\) is done in the unphysical Hilbert space \(\mathcal{H}_{\vec{\eta}_1} \otimes \mathcal{H}_{\vec{\eta}_2} = \mathcal{H}_{\vec{\eta}_{12}} \otimes \mathcal{H}_{\vec{\rho}_{12}}\) with scalar product \(\langle \tilde{\phi}_1, \tilde{\phi}_2 \rangle = \int d^3\eta_{12} d^3\rho_{12} \tilde{\phi}_1^* (\tau, \vec{\eta}_{12}, \vec{\rho}_{12}) \tilde{\phi}_2 (\tau, \vec{\eta}_{12}, \vec{\rho}_{12})\). In it we define the quantum operators corresponding to the internal generators (4.13) (for the boosts we use a symmetrical ordering) and we get that the quantum internal Poincare\' algebra is trivially satisfied.

It is still convenient to use as Hamiltonian \(\hat{M} c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} = \hat{H}_{rel}\), because it corresponds to a Hamilton-Jacobi description of the internal 3-center of mass with frozen Jacobi data \(\vec{\eta}_{12}, \vec{\kappa}_{12}\). Therefore, in the coordinate representation the Schrödinger equation (4.5) is replaced by the following one

\[i \frac{\partial}{\partial \tau} \tilde{\phi}(\tau, \vec{\eta}_{12}, \vec{\rho}_{12}) = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} \tilde{\phi}(\tau, \vec{\eta}_{12}, \vec{\rho}_{12}).\]

\[(4.14)\]

The energy eigenfunctions \(e^{-iE\tau} \phi_E\) satisfy \(\hat{H}_{rel} \phi_E = E \phi_E\) with \(E = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2}\) if \(\vec{\pi}\) is the eigenvalue of \(\vec{\pi}_{12}\). By inversion we have \(\vec{\pi}^2 = \frac{1}{4E} [E^2 - (m_1 + m_2)^2 c^2] [E^2 - (m_1 - m_2)^2 c^2]\).
However the physical Hilbert space is identified by the following conditions

\[
< \phi_{\text{phys}}|\hat{\eta}_{12}|\phi_{\text{phys}}> = 0, \\
< \phi_{\text{phys}}|\hat{\eta}_{12}|\phi_{\text{phys}}> = \frac{1}{2} < \phi_{\text{phys}}|\hat{p}_{12} \frac{m \sqrt{m_2^2 c^2 + \beta^2_{12}} - m \sqrt{m_1^2 c^2 + \beta^2_{12}} + \sqrt{m_1^2 c^2 + \beta^2_{12}} + \sqrt{m_2^2 c^2 + \beta^2_{12}}}{\sqrt{m_1^2 c^2 + \beta^2_{12}} + \sqrt{m_2^2 c^2 + \beta^2_{12}}}, \phi_{\text{phys}} > .
\]

(4.15)

Before studying the relativistic case let us look at the non-relativistic one.

1. The Non-Relativistic Case

After Eq.(2.17) we said that, given a two-body problem with \( E_{\text{Galilei}} = \frac{\hat{p}_2^2}{2m} + \frac{\hat{q}_2^2}{2\mu} + V(\vec{r}_n) = \frac{\hat{p}_n^2}{2m} + H_{\text{rel}} \) at the classical level, the identification of the non-relativistic internal space of relative variables can be done by adding the second class constraints \( \tilde{p}(n) \approx 0 \) (rest-frame condition) and \( \vec{x}(n) \approx 0 \) (elimination of the center of mass). As a consequence we get \( E_{\text{Galilei}} \approx H_{\text{rel}} \), i.e. a Hamilton-Jacobi description of the 3-center of mass with frozen Jacobi data. At the quantum level in the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}} \) of section III we quantize the frozen center-of-mass variables and we use the Hamiltonian \( \hat{\tilde{H}} = \hat{H}_{\text{rel}} = \frac{\hat{p}_n^2}{2\mu} + V(\vec{r}_n) \). In the coordinate representation the wave functions are \( \psi(t, \vec{x}(n), \vec{r}(n)) \). Let us restrict the Hilbert space to wave functions \( \phi_{\text{phys}}(t, \vec{x}(n), \vec{r}(n)) \) satisfying the requirements \(< \phi_{\text{phys}}|\hat{\tilde{H}}(n)|\phi_{\text{phys}} > = < \phi_{\text{phys}}|\hat{\tilde{\tilde{A}}}(n)|\phi_{\text{phys}} > = 0 \). If we define creation and annihilation operators \( \hat{a}^\dagger = \hat{\tilde{x}}(n) + i \frac{\hbar}{m_e} \hat{\tilde{p}}(n), \hat{a} = \hat{\tilde{x}}(n) - i \frac{\hbar}{m_e} \hat{\tilde{p}}(n) \), the wave functions \( \phi_{\text{phys}} \) are identified by the Gupta-Bleuler-like conditions \(< \phi_{\text{phys}}|\hat{\tilde{\tilde{A}}}(n)|\phi_{\text{phys}} > = 0, < \phi_{\text{phys}}|\hat{\tilde{a}} > = 0 \). In the coordinate representation this implies the following form of the wave functions: \( \phi_{\text{phys}}(t, \vec{x}(n), \vec{r}(n)) = N e^{-\frac{\beta^2_{12}}{2\sigma^2}} \phi(t, \vec{r}(n)) (\beta = \frac{\hbar^2}{m_e}) \). Their scalar product is \(< \phi_{\text{phys}1}, \phi_{\text{phys}2} >_{\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}} = \int d^3x_1 d^3r_1 \phi_{\text{phys}1}^*(t, \vec{x}_1(\vec{r}_1)) \phi_{\text{phys}2}(t, \vec{x}_1, \vec{r}_1) = N \int d^3r_1 \tilde{\phi}_1^*(t, \vec{r}_1) \tilde{\phi}_2(t, \vec{r}_1) = N < \phi_1, \phi_2 >_{\mathcal{H}_{\text{rel}}} \).

Therefore we have a reduction from the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}} \) to the Hilbert space \( \mathcal{H}_{\text{rel}} \) with Hamiltonian \( \hat{H}_{\text{rel}} \): we have only to reabsorb the factor \( e^{-\frac{\beta^2_{12}}{2\sigma^2}} \) in the normalization constant.

2. The Relativistic Case

In the relativistic case of two free particles the Gupta-Bleuler-like conditions are not convenient because the constraints \( \vec{\chi} = \hat{\eta}_{12} - \frac{\sum_{i=1}^{2} (-)^i m_i \sqrt{m_i^2 c^2 + \beta^2_{12}}}{\sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \beta^2_{12}}} \hat{p}_{12} \approx 0 \) do not have
vanishing Poisson bracket \( \{\chi^i, \chi^j\} \neq 0 \). One should replace the second class constraints \( \vec{\chi} \approx 0, \vec{\kappa}_{12} \approx 0 \), with their suitable combinations \( \vec{\chi} \approx 0, \vec{\phi} \approx 0 \), such that \( \{\vec{\chi}^i, \vec{\phi}^j\} = \delta^{ij} \), \( \{\vec{\chi}^i, \vec{\chi}^j\} = \{\vec{\phi}^i, \vec{\phi}^j\} = 0 \). Then, due to the resulting mixing of the canonical variables \( \vec{\eta}_{12}, \vec{\kappa}_{12}, \vec{\rho}_{12}, \vec{\pi}_{12} \), one should find a canonical transformation to a new base \( \vec{\eta}_{\hat{1}2}, \vec{\kappa}_{\hat{1}2}, \vec{\rho}_{\hat{1}2}, \vec{\pi}_{\hat{1}2} \), adapted to the second class constraints, i.e. such that \( \eta_{\hat{1}2}^i = \vec{\chi}^i \approx 0, \kappa_{\hat{1}2}^i = \vec{\phi}^i \approx 0 \). Presumably the new Hamiltonian would weakly be function only of \( \vec{\pi}_{\hat{1}2}^2 \). A Gupta-Bleuler-like quantization of the new canonical basis could then be done following the non-relativistic pattern.

Instead let us evaluate Eqs.(4.15) by using the energy eigenfunctions \( \phi_E \) of Eq.(4.14). We get \( <\phi_E|\hbar \frac{\partial}{\partial \vec{\eta}_{12}}|\phi_E> = 0 \) and \( <\phi_E|\left(\vec{\eta}_{12} - f(E)\vec{\rho}_{12}\right)|\phi_E> = 0 \), where \( f(E) = \sum_{i=1}^{2} (-)^i \frac{m_i}{\sqrt{m_i^2 c^2 + \vec{\pi}^2(E)}} \) with \( \vec{\pi}^2(E) = \frac{1}{4E^2} [E^2 - (m_1 + m_2)^2 c^2] [E^2 - (m_1 - m_2)^2 c^2] \).

For each value of \( E \) the conditions are satisfied by the following energy eigenfunctions \( \tilde{\phi}_E = e^{-\frac{(\vec{\eta}_{12} - f(E)\vec{\rho}_{12})^2}{2\beta^2}} \psi_E(\tau, \vec{\rho}_{12}) (\beta = \frac{\hbar^2}{m c}) \) with \( \hat{H}_{\text{rel}} \psi_E = \epsilon \psi_E \) and \( <\psi_{E_1}, \psi_{E_2} >_{\mathcal{H}_{\text{rel}}} = \delta(E_1 - E_2) \). As in the non-relativistic case we get \( <\phi_{\text{phys}E_1}, \phi_{\text{phys}E_2} >_{\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}} = N <\psi_{E_1}, \psi_{E_2} >_{\mathcal{H}_{\text{rel}}} = N \delta(E_1 - E_2) \). Therefore we can build the abstract physical Hilbert space \( \mathcal{H}_{\text{rel}} \) starting from its complete energy basis \( |\psi_{E}> \).

In presence of interactions the construction of the physical Hilbert space \( \mathcal{H}_{\text{rel}} \) is much more complex because the Hamiltonian \( \hat{H}_{\text{rel}} \) depends also on the relative position operator \( \hat{\rho}_{12} \). Again one has to start from the energy eigenfunctions.
V. EXAMPLES OF TWO-BODY SYSTEMS WITH MUTUAL ACTION-A-A-DISTANCE INTERACTION

In this Section we analyze the two classes of models with action-at-a-distance interaction of Refs. [7, 8].

A. Quantization of the Non-Trivial Interacting Two-Particle System of Ref.[7].

In Ref.[7] we introduced the rest-frame instant form of a class of positive-energy two-particle models with an arbitrary action-at-a-distance potential. They were defined by the following form of the internal Poincare’ generators (use Eq.(2.11) and $m_3 \equiv m_1$)

\[
M_c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \Phi(\vec{\rho}_{12}) \approx \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + \Phi(\vec{\rho}_{12}),
\]

\[
\vec{P}_{(int)} \approx \vec{\pi}_{12} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0,
\]

\[
\vec{J}_{(int)} \approx \vec{\eta}_1 \times \vec{\kappa}_1 + \vec{\eta}_2 \times \vec{\kappa}_2 = \vec{\eta}_{12} \times \vec{\kappa}_{12} + \vec{\rho}_{12} \times \vec{\pi}_{12} \approx \vec{\rho}_{12} \times \vec{\pi}_{12} = \vec{S},
\]

\[
\vec{K}_{(int)} \approx -\sum_{i=1}^{2} \vec{\eta}_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \Phi(\vec{\rho}_{12}) \approx -\vec{\eta}_{12} \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + \Phi(\vec{\rho}_{12}) + \vec{\rho}_{12} \sum_{i=1}^{2} (-)^i \frac{m_{i+1}}{m} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + \Phi(\vec{\rho}_{12}) \approx 0.
\]

The classical internal Poincare’ algebra closes only using the rest-frame condition $\vec{P}_{(int)} \approx 0$.

The elimination of the internal 3-center of mass is done with the conditions

\[
\vec{\eta}_{12} \approx \vec{\rho}_{12} \sum_{i=1}^{2} (-)^i \frac{m_{i+1}}{m} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + \Phi(\vec{\rho}_{12}), \quad \vec{\kappa}_{12} \approx 0.
\]

The orbit reconstruction is given by Eqs.(2.12) with $\vec{\pi}_{12}^2 \mapsto \vec{\pi}_{12}^2 + \Phi(\vec{\rho}_{12})$.

1. Quantization without $\vec{\eta}_{12}$

We have to quantize the Hamiltonian $M_c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + \Phi(\vec{\rho}_{12})$ together with the spin $\vec{S} = \vec{\rho}_{12} \times \vec{\pi}_{12}$.

See Ref.[40] for the definition of the pseudo-differential operators connected with the quantization of quantities like $\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}$. When also the potential $\Phi(\vec{\rho}_{12})$ appears under the square root, we follow Ref.[12]: in its Eq.(C7) the following definition was given

\footnote{See Ref. [7, 15] and its bibliography for the corresponding models with the two signs of the energy and with mass-shell constraints.}
\[ \sqrt{m^2 c^2 + H} = m c \sum_{n=0}^{\infty} c_n \left( \frac{\hat{h}}{m c} \right)^n \] where \( c_n \) are the coefficients of the Taylor expansion
\[ \sqrt{1 + x} = \sum_{n=0}^{\infty} c_n x^n. \]

In our model we have the operator \( \hat{H} = \hat{\mathbf{H}} \), which coincides with the Hamiltonian of the relative motion of a non-relativistic two-body problem with reduced mass \( \mu = \frac{1}{2} \). Therefore if a complete set of eigenfunctions of this problem is known \( (\hat{H} \psi_{nlm} = \epsilon_n \psi_{nlm}, \hat{S}^2 \psi_{nlm} = (s + 1) \psi_{nlm}, \hat{S}^3 \psi_{nlm} = m \psi_{nlm}) \), then the relativistic mass levels will be \( M_n c = \sum_{i} \sqrt{m_i^2 c^2 + \epsilon_n}. \)

The derived (non-commuting) single particle self-adjoint operators are obtained by quantizing Eq.(2.12) and \( Y^\mu(\tau) \) of Eq.(2.3) with a symmetric ordering
\[ \hat{x}_\mu^\mu(\tau) = \hat{Y}^\mu(\tau) + \frac{1}{2} \epsilon^\mu_\mu(\hat{h}) [(-1)^{i+1} \hat{\rho}_{12}^i - \frac{1}{2} (m_i^2 - m_j^2) c^2 \left( \hat{\rho}_{12}^i \sum_{j=1} m_j^2 c^2 + \hat{H} \right) ] + \frac{1}{2} \hat{\rho}_{12}^i \].
\[ \hat{p}_\mu^\mu = \hat{\hbar}^\mu \sqrt{m_i^2 c^2 + \hat{\pi}_{12}^2} + (-)^{i+1} \epsilon^\mu_\mu(\hat{h}) \hat{\pi}_{12}^i, \]
\[ \hat{Y}^o(\tau) = \frac{1}{2} \left( \hat{\mathcal{M}}_{(int)} c \right)^{-1} \left( \sqrt{1 + \hat{\hbar} \cdot \hat{x} + \hat{\zeta} \cdot \hat{\hbar} \sqrt{1 + \hat{\hbar}^2}} + \sqrt{1 + \hat{\hbar}^2} \tau, \right) \]
\[ \hat{Y}(\tau) = \left( \hat{\mathcal{M}}_{(int)} c \right)^{-1} \left( \hat{\zeta} + \frac{1}{2} \left( \hat{\hbar} \hat{\zeta} + \hat{\zeta} \cdot \hat{\hbar} \hat{\hbar} \right) + \hat{\hbar} \tau. \right). \]

Therefore \( \hat{x}_\mu^\mu(\tau) \) depends both on the quantum frozen Jacobi data \( \hat{\zeta}, \hat{\hbar} \), describing the external evolution, and on the quantum internal relative variables \( \hat{\rho}_{12}, \hat{\pi}_{12} \), describing the mutual particle interaction.

### 2. Quantization with \( \hat{\eta}_{12} \)

As a consequence we cannot check the quantum internal Poincare’ algebra: to do it we should need a form of the internal Poincare’ generators satisfying the Poincare’ algebra without using the rest-frame conditions.

It cannot be done until one finds the form of the boosts \( \hat{\mathcal{K}}_{(int)} \) so that the internal Poincare’ algebra closes without using the rest-frame condition \( \hat{\mathcal{P}}_{(int)} \approx 0 \).

### B. Quantization of the Two-Particle System with Coulomb plus Darwin Mutual Interaction of Ref.[8].

In Eq.(I-5.4) of I we found the following internal Poincare’ algebra for a system of two positive-energy charged scalar particles (with Grassmann-valued electric charges) with a mutual Coulomb plus Darwin potential
\[ \mathcal{E}_{(\text{int})} = M c^2 = c \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \frac{Q_1 Q_2}{4\pi |\bar{\eta}_1 - \bar{\eta}_2|} + V_{\text{DARWIN}}(\bar{\eta}_1(\tau) - \bar{\eta}_2(\tau); \bar{\kappa}_i(\tau)), \]

\[ \vec{P}_{(\text{int})} = \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0, \]

\[ \vec{J}_{(\text{int})} = \sum_{i=1}^{2} \bar{\eta}_i \times \vec{\kappa}_i, \]

\[ \vec{K}_{(\text{int})} = -\sum_{i=1}^{2} \bar{\eta}_i \left[ \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \frac{\vec{\kappa}_i \cdot \sum_{j \neq i} Q_i Q_j [\bar{\eta}_i \frac{1}{2} \mathcal{K}_{ij}(\bar{\kappa}_i, \bar{\eta}_i - \bar{\eta}_j) - 2\bar{A}_{\perp S j}(\bar{\kappa}_j, \bar{\eta}_i - \bar{\eta}_j)]}{2c \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} \right] - \frac{1}{2c} \sum_{i=1}^{2} \sum_{j \neq i} Q_i Q_j \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} \bar{\kappa}_i \mathcal{K}_{ij}(\bar{\kappa}_i, \bar{\kappa}_j, \bar{\eta}_i - \bar{\eta}_j) - \frac{1}{2c} \sum_{i=1}^{2} \sum_{j \neq i} Q_i Q_j \frac{d^3 \mathcal{S}_i(\bar{\kappa}_i - \bar{\eta}_i, \bar{\kappa}_i)}{4\pi c} - \frac{1}{2c} \sum_{i=1}^{2} \sum_{j \neq i} Q_i Q_j \int d^3 \sigma \frac{\vec{\mathcal{S}}_{\perp S i}(\bar{\kappa}_i - \bar{\eta}_i, \bar{\kappa}_i) \cdot \vec{\mathcal{S}}_{\perp S j}(\bar{\kappa}_j - \bar{\eta}_j, \bar{\kappa}_j)}{d^3 \sigma}\]

\[ \approx 0. \] (5.4)

with the form of the Darwin potential and of the Lienard-Wiechert quantities given in Appendix A.

**C. Quantization without \( \bar{\eta}_{12} \)**

By eliminating \( \vec{\kappa}_{12} \approx 0 \) and \( \bar{\eta}_{12} \) we get for the invariant mass

\[ M c = M c = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\pi}_{12}^2} + \frac{Q_1 Q_2}{4\pi |\vec{\rho}_{12}|} + V_{\text{DARWIN}}(\vec{\rho}_{12}, \vec{\pi}_{12}). \] (5.5)

The expression of the Darwin potential is given in Eq.(A7).

In Eq.(6.37) of Ref.[10] the following expression for the Darwin potential was obtained in the case of equal masses \( m_1 = m_2 = m \) (with \( m = m_1 + m_2 \mapsto 2m \)
\[
\tilde{V}_{\text{DARWIN}}(\vec{\rho}_{12}, \vec{\pi}_{12}) = \frac{Q_1 Q_2}{8\pi |\vec{\rho}_{12}|} \left( m^2 c^2 + \vec{\pi}_{12}^2 \right) \left[ m^2 c^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|} \right)^2 \right]^{-1} \\
- m^2 \left[ 3 \vec{\pi}_{12}^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|} \right)^2 \right] - 2 \vec{\pi}_{12}^2 \left[ \vec{\pi}_{12}^2 - 3 \left( \vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|} \right)^2 \right] - 2 \left[ \vec{\pi}_{12}^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|} \right)^2 \right] \\
- \left[ m^2 c^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|} \right)^2 \right] \left[ m^2 c^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|} \right)^2 \right]. 
\]

(5.6)

In Appendix B we obtain the Schrödinger equation corresponding to the total Hamiltonian given in Eq. (5.5). Because of the nontrivial momentum dependence in both its kinetic and potential energy portions, we carry out its quantization by using Weyl ordering [41]. A noteworthy result of this quantization is that not only do we obtain the expected nonlocal coordinate space form of the kinetic energy term, but the Coulomb term itself, in the context of the Darwin potential corresponding to Eq. (5.6), takes on a nonlocal coordinate space form. Only in the limit of small Compton wavelength does it recover its local coordinate space form.

The Weyl ordering of the order \( 1/c^2 \) Darwin potential below is also carried out. We demonstrate that the Weyl ordering leads to the hermitian ordering given at the beginning of Appendix B. At the order \( 1/c^2 \), where the Darwin potential for unequal masses becomes

\[ \tilde{V}_{\text{DARWIN}}(\vec{\rho}_{12}, \vec{\pi}_{12}) = \frac{Q_1 Q_2}{8\pi |\vec{\rho}_{12}|} \frac{\vec{\pi}_{12}^2 - \left( \vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|} \right)^2}{m_1 m_2 c^2} + O(c^{-4}), \]

(5.7)
as shown in Eq. (6.35) of Ref. [10], we recover the effective stationary Schrödinger equation used for relativistic bound states in Refs. [42–44].

With the methods of Appendix A we can study the two-body problem for positive-energy charged spinning particles [11].

1. Quantization with \( \vec{\eta}_{12} \)

Elsewhere, by using the Weyl ordering in, we will study the implementation of the quantum internal Poincaré algebra, the extended Schrödinger equation and its reduction to the previous results.
VI. IMPLICATIONS FOR RELATIVISTIC LOCALIZATION AND RELATIVISTIC ENTANGLEMENT

In this Section we will indicate which problems connected with localization are solved with our rest-frame formulation of RQM. Moreover we will delineate which are the implications for relativistic entanglement.

A. Relativistic Localization

In non-relativistic QM a wave function strictly localized in a finite volume at $t = 0$ will spread instantaneously to all the 3-space with infinite tails as shown in Ref.[45]. The position operator $\hat{x}$ is a self-adjoint operator with a continuous spectrum, whose distributional eigenfunctions corresponding to the localization associated to the eigenvalue $\vec{\xi}$ are $\psi_{\vec{\xi}}(\vec{x}) = \delta^3(\vec{x} - \vec{\xi})$. These wave functions are mutually orthogonal: $<\psi_{\vec{\xi}_1}, \psi_{\vec{\xi}_2}> = \delta^3(\vec{\xi}_1 - \vec{\xi}_2)$. Localization is invariant under the invariance group of Galilei space-time, the Galilei group. The uncertainty relations limit the sharpness with which a system’s position can be determined in certain circumstances. The only problems of non-locality are connected with entanglement, see for instance the EPR argument [21, 23, 24].

1. The Newton-Wigner Position Operator and the Hegerfeldt Theorem

In relativistic QM, in a fixed inertial frame, to a scalar positive-energy particle of mass $m$ is associated the self-adjoint Newton-Wigner (NW) position operator $\hat{x}$

$$\hat{x} = i\hbar \frac{d}{d\vec{P}} - \frac{i\hbar \vec{P}}{m^2 c^2 + \vec{P}^2},$$

in a Hilbert space with Lorentz-scalar product

$$<\psi_1, \psi_2> = \int \frac{d^3 P}{\sqrt{m^2 c^2 + \vec{P}^2}} \tilde{\psi}_1^*(\vec{P}) \tilde{\psi}_2(\vec{P}),$$

in the momentum representation. Its position eigenvectors at time $t = x^o/c = 0$, corresponding to the eigenvalue $\vec{\xi}$, are

$$\psi_{\vec{\xi}}(\vec{x}, 0) = (2\pi)^{-3} \int \frac{d^3 P}{(m^2 c^2 + \vec{P}^2)^{1/4}} e^{i\vec{P} \cdot (\vec{x} - \vec{\xi})},$$

with momentum representation $\tilde{\psi}_{\vec{\xi}}(\vec{P}) = (2\pi)^{-3/2} (m^2 c^2 + \vec{P}^2)^{1/4} e^{-i\vec{P} \cdot \vec{\xi}}$. They are orthogonal, $<\psi_{\vec{\xi}_1}, \psi_{\vec{\xi}_2}> = \delta^3(\vec{\xi}_1 - \vec{\xi}_2)$ but they are spread out in $\vec{x}$. Instead of a delta function like in non-relativistic QM, they are proportional to the Hankel functions of the first kind $H_5^{(1)}(\vec{x} - \vec{\xi}) \rightarrow |\vec{x}| \rightarrow \infty e^{-|\vec{x} - \vec{\xi}|/\lambda_m}$, where $\lambda_m = \hbar/mc$ is the Compton wavelength. Therefore there are

$$\tilde{\psi}_{\vec{\xi}}(\vec{P}) = \frac{(2\pi)^{-3/2}}{m^2 c^2 + \vec{P}^2} e^{-i\vec{P} \cdot \vec{\xi}}.$$
infinite tails governed by the Compton wavelength, even if at the classical level the associated Møller radius is zero.

This absence of sharp localization, due to the form of the scalar product and to the orthogonality requirement is an aspect of the non-locality present in special relativity with self-adjoint position operators.

This counterintuitive aspect of relativistic localization has the following two inter-related implications:

A) Newton-Wigner localization is not invariant under Lorentz boosts, consistently with the classical non-covariance of the 3-center of mass. If in the original inertial frame we have the Newton-Wigner eigenstate $\psi_{\vec{\xi}=0}(\vec{x})$ at $t=0$, in a moving frame the boosted wave function is a superposition of the Newton-Wigner eigenstates corresponding to every value of $\vec{\xi}$. This means that the probability density amplitude to be in a given eigenstate with eigenvalue $\vec{\xi}$ is frame-dependent: if it is sharply localized in one frame, it has infinite tails in a moving frame. Frame-independent objectivity of localization is lost.

B) Time evolution in a fixed inertial frame destroys sharp localization. At time $x^o = ct$ the Newton-Wigner eigenstate with eigenvalue $\xi = 0$ is

$$\psi(x^o, \vec{x}) = (2\pi)^{-3} \int \frac{d^3P}{(m^2 c^2 + \vec{P}^2)^{1/4}} e^{i(\vec{P} \cdot \vec{x} - \sqrt{m^2 c^2 + \vec{P}^2} x^o)} = \int d^3\xi G(x^o, \vec{\xi}) \psi_{\vec{\xi}}(0, \vec{x}),$$

where

$$G(x^o, \vec{\xi}) = (2\pi)^{-3} \int d^3P e^{i(\vec{P} \cdot \vec{\xi} - \sqrt{m^2 c^2 + \vec{P}^2} x^o)} \neq \delta^3(\vec{\xi}).$$

The form of $G$ is due to the branch points at $|\vec{P}| = \pm imc$. Infinite tails in $\vec{\xi}$ develop and there is an apparent violation of Einstein causality. $G(x^o, \vec{\xi})$ is non-zero everywhere for arbitrarily small $x^o$ and this implies the possibility of a non-local phenomenon.

This is the content of Hegerfeldt theorem [29, 39], which says that the requirement that the NW operator be a self-adjoint operator implies the instantaneous super-luminal spreading of wave packets: only at the level of wave packets with power tails could there be consistency with relativistic causality. As a consequence, the requirement of relativistic causality implies bad localization of the Newton-Wigner position, as already anticipated at the classical level with the non-covariance Møller world-tube for the relativistic canonical 3-center of mass. Since it is impossible to explore the interior of the Møller world-tube (i.e. distances less than the Compton wavelength of the isolated system) of the isolated system [5] without breaking manifest Lorentz covariance, this would be compatible with a non-self-adjoint Newton-Wigner position operator.

25 For a particle sharply localized at $\vec{\xi}$ the non-relativistic wave function is $\psi(\vec{x}) = \delta^3(\vec{x} - \vec{\xi}) = \psi_{\vec{\xi}}(\vec{x})$. Instead at the relativistic level we have $\psi(\vec{x}) = \int d^3\xi G(\vec{\xi}) \psi_{\vec{\xi}}(\vec{x})$ with $G(\vec{\xi}) = (2\pi)^{-3} \int d^3x \int d^3P (m^2 c^2 + \vec{P}^2)^{1/4} e^{i\vec{P} \cdot (\vec{\xi} - \vec{x})} \psi(\vec{x})$ for every wave function, also for those strongly peaked at some $\vec{\xi}_o$.

26 In Ref.[39] it is also noted that the theorem does not create any problem for the interpretation of the Dirac equation due to the presence of both positive- and negative-energy component as shown in Ref.[47]: but the same problems reappear if we restrict ourselves to the positive-energy sector.
As clarified in Refs.[39] with the hypotheses of the theorem (Hilbert space and positive energy) it is not yet possible to show that there is at least \textit{weak causality}, namely that Einstein causality holds only for the expectation values or the ensemble averages of a projection operator \( N(V) \) on a fixed 3-region \( V \). If the position operator is not self-adjoint, the operator \( N(V) \) is not a projector but a \textit{positive operator-valued measure} (POVM). But again the infinite tails spread too fast.

2. \textit{Quantum Field Theory}

In conclusion the localization problem in relativistic QM cannot be solved without taking into account quantum field theory (QFT). In Ref. [49] it is claimed that a \textit{relativistic QM of localizable particles does not exist and that only relativistic QFT makes sense} (the basic ontological objects are fields). It is argued that in QFT particle detection is an \textit{approximately local} measurement: for all practical purposes (FAPP) of phenomenology non strictly localized objects will appear as strictly localized (particles with localized mutual interactions) to local finite observers. According to Haag [50] the concept of position at a given time is not a meaningful attribute of the electron: rather it is an attribute of the interaction between the electron and a suitable detector.

Fraser [51] shows that the particle concept (as elementary quanta in Fock space) is meaningful only in the description of free fields in QFT. Till now in interacting systems there is no acceptable extension of this notion. The assumption that a particle is localizable is not used in this exposition. Therefore the notion of particle seems to be only an effective one to be used in perturbative QFT. Let us note that in perturbative QFT one uses Feynman diagrams as an intermediate tools to evaluate the S matrix. These diagrams describe interacting particles by using the momentum basis (they correspond to a Dirichlet problem and not to a Cauchy problem): in this way the problem of NW-localization is avoided. Instead a well-posed Cauchy problem is needed for predictability in classical field theory: only in this way (modulo integrability) can we use the existence and uniqueness theorem for partial differential equations. Only with the \textit{non factual} 3+1 splitting of Minkowski space-time and the \textit{non factual} definition of the global Poincare' generators of an isolated system is it possible define the instantaneous 3-spaces where to give the Cauchy data. In this way, at least at the classical level, it is possible to avoid the Haag theorem [52] preventing the existence of interpolating fields as shown in paper I.

\footnote{For such projectors there is \textit{Malament's theorem} [48] saying that the requirements of localizability, translation covariance, energy bounded below and microcausality imply that there is no chance that a particle will be detected in any local region.}

\footnote{As shown by Peres in Ref. [26] (see also Ref.[23]) POVM are complete sets of (in general non-commuting) positive operators (more general than projectors) describing \textit{detectors} used to describe the \textit{measurement of an observable}. If the density matrix \( \rho \) describes an \textit{emitter}, then the probability that the detector \( \mu \) is excited is \( Tr(\rho E_\mu) \). According to Peres the notion of particle has an operational meaning depending on the context of experiments: particles are what is registered by detectors localizing them (see Ref.[26] for a review of the localization of particles).}

\footnote{In the basic Wightman axioms there is the \textit{time-slice axiom} (primitive causality) saying that there should be a dynamical law which allows one to compute fields at an arbitrary time in terms of the fields in a small time slice \( O_{t,\epsilon} = \{ x || x^o - ct || < \epsilon \} \).}
See Ref.
[31] for standard localization scheme in spatial regions of perturbative local QFT (where "localized in" means "measurable in" and microcausality holds in 4-regions)  
and for the comparison with the NW-localization scheme of Refs.
[53] and [18] using the NW position operator which cannot be described neither with local nor quasi-local operators. See Ref.
[54] for the notion of unsharp observables (if a local operator is not measurable with local actions in a given 3-region) and for a criticism of the request of microcausality, because sharp spatial localization is an operationally meaningless idealization (it requires an infinite amount of energy with unavoidable pair production; the quantum nature of the constituents of the detectors should be taken into account,...).

Finally differently from perturbative QFT, in local algebraic QFT local (and quasi-local) operators are introduced having in mind that they can be used to describe phenomena and measurements confined in local bounded 4-regions of space-time with the vanishing of the commutator of local operators in disjoint space-like-separated 4-regions (causality) implying the independence of the disjoint measurements (no action-at-a-distance communication).

However the relevance of relativistic QM against these attacks from QFT, because in it particles are only effective nearly-localized entities. See for instance Ref. [55] where an approximate notion of effective localization in a 3-region G of radius L of the order of the particle Compton wavelength (it is an effective notion of NW-localization ) is given starting from an analogy with a solid-state system on length-scales which are large compared to the interatomic spacing.

Even if there is no agreement on the relevance of the notion of particle in QFT, particles are effective tools for phenomenology and for the S matrix. Moreover atomic and solid-state physics are specific sectors of certain QFT’s in which there is a wealth of situations in which particles (electrons, atomic nuclei,...) are strongly interacting and yet maintain their own particle character.

3. The New Relativistic Quantum Mechanics

The rest-frame instant form of relativistic QM developed in papers I and II and its quantization done in this paper leads to an effective theory for the description of relativistic atomic physics, and hopefully quantum optics, below the threshold of pair production. It can be interpreted as an approximation to QED in which the particle number is fixed, needed for going from quantum optics with non-relativistic two-level atoms [56], used in the experiments on non-relativistic entanglement where strictly speaking photons do not exist (only their polarization and not their world-line is described), to a relativistic theory in which both atoms and photons can coexist. It will allow one to arrive at a relativistic formulation of entanglement experiments with laser beams with a fixed number of photons.

\[\text{However when one introduces the spectrum condition, implying the positivity of the energy and that the velocity of light is the upper bound for the propagation of physical effects, one makes a non local statement on the global 4-momentum operator.}\]
The quantization of the rest-frame instant form of relativistic particle dynamics presented in this paper, which can be trivially extended from two to N particles (see Ref.[6] for the kinematics), has the following advantages with respect to other approaches to relativistic mechanics:

A) There is a complete solution to the problem of the *non-objectivity of localization*, i.e. the dependence of the particle position also from the simultaneity hyperplane (frame-dependence) of Refs.[16–18]; it is avoided by using the embedding of the Wigner 3-spaces (intrinsic rest frames) in Minkowski space-time with the dynamics described by Wigner-covariant relative 3-variables living in an abstract Wigner 3-space.

B) We use a modified NW-localization scheme: we do not quantize the canonical non-covariant NW 3-center of mass. Instead we quantize the non-covariant frozen Jacobi data of the external 3-center of mass $\hat{z}$, $\hat{h}$ in the frozen Hilbert space $\mathcal{H}_{com}$. Therefore we do not have evolving wave packets for the 3-center of mass so that we avoid the instantaneous spreading of wave packets of Hegerfeldt theorem. The Jacobi data $\hat{z}$ are more fundamental than the center-of-mass Newton-Wigner position $\hat{x}_{NW} = \hat{z}/M c$, because they do not depend explicitly on the internal mass which is quantized at the quantum level (so that the operators $\hat{x}_{NW n} = \hat{z}/M_n c$ depend upon the mass eigenvalue $M_n$).

C) We quantize the Wigner covariant relative 3-variables inside the Wigner 3-spaces with an abstract Hilbert space $\mathcal{H}_{rel}$. In it there will be instantaneous spreading in the $\tau$-evolution of initially localized (in the position relative variables) wave packets for the relative motion. This Hilbert space, like its non-relativistic counterpart, contains only relative variables with action-at-a-distance interactions (the mutual Coulomb interaction when the transverse electro-magnetic field is present), but, once initial Cauchy data are given on an initial instantaneous 3-space, then the evolution is compatible with Lorentz covariance (there is no violation of Einstein causality or superluminal signalling). The action of a Lorentz boost of the external Poincare’ group on $\mathcal{H}_{com} \otimes \mathcal{H}_{rel}$ induces a Wigner rotation (and not a $\tau$-evolution) of the relative variables and leads to a non-covariant transformation of the Jacobi data $\hat{z}$ according to Eq.(2.5).

D) In terms of the quantum Jacobi data and of the quantum invariant mass and rest-spin we can build the $\tau$-evolving position operators for the external Fokker-Pryce 4-center of inertia $\hat{Y}^\mu(\tau)$ (a 4-vector operator with non-commuting components), the external 4-center of mass $\hat{\mathcal{Z}}^\mu(\tau)$ (a pseudo-4-vector non-covariant operator whose components require a suitable ordering to be commuting) and the external Møller 4-center of energy $\hat{R}^\mu(\tau)$ (a non-covariant non-commuting pseudo-4-vector operator). One should study their mean value $\langle \phi | ... | \phi \rangle$ and see whether some form of Ehrenfest theorem holds for them. Since these collective variables are global non-local quantities, they cannot be localized with local means! This is our answer the NW-localization problem. Moreover, when the spatial region containing the particles on a simultaneity Wigner instantaneous 3-space has a radius bigger than the Møller radius of the particle configuration, then the classical energy density is everywhere positive definite (weak energy condition; classical version of the Epstein-Glaser-Jaffe theorem [34]).

In $\mathcal{H}_{com} \otimes \mathcal{H}_{rel}$ the property of the frozen Jacobi data $\hat{h}$ (or of the total 4-momentum $P^\mu$ of the isolated system) of being a constant of the motion is made explicit. Therefore, the description of the external non-covariant center of mass carrying a pole-dipole structure fits with the point of view that the isolated system is a closed universe. If we use the Wigner-Araki-Yanase theorem on the constants of motion in QM [57] (see also p. 421 of Ref.[23]),
it turns out that the conjugate variable, namely the Jacobi data $\vec{z}$ are not measurable quantities. Therefore the same is true for the external non-covariant 4-center of mass $\hat{\vec{x}}^{\mu}(\tau)$, the decoupled pseudo-particle carrying the pole-dipole structure. The conceptual problem is always the same: Who will measure the wave function of a closed universe?

This fact, together with the avoidance of the Hegerfeldt theorem due to the frozen nature of $\mathcal{H}_{\text{com}}$, leads to the two following open problems: a) do we take the Jacobi data $\hat{\vec{z}}$ self-adjoint?; b) if $\hat{\vec{z}}$ is chosen self-adjoint, is it meaningful to consider superpositions of center of mass wave functions with different eigenvalues of $\hat{\vec{h}}$ (one could introduce a superselection rule forbidding them like it has been proposed in canonical gravity [58, 59])? Actually a generic non-factorizable wave function in $\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}$ implies entanglement among different internal energy levels and among the different rest-frame 3+1 splittings associated with center-of-mass plane waves.

E) The problem of NW-localization of the individual particles has a different formulation, because the position 4-coordinates $x_i^{\mu}(\tau)$ parametrizing the world-lines and the 4-momenta $p_i^{\mu}(\tau)$ ($p_i^2 = m_i^2 c^2$) are derived quantities. At the classical level the world-lines are obtained with the orbit reconstruction of the 4-vectors $x_i^{\mu}(\tau)$ of Eq.(2.13). Therefore after quantization the information about the individual particles is hidden in the quantum operators $\hat{x}_i^{\mu}(\tau)$, $\hat{p}_i^{\mu}(\tau)$. The position operators $\hat{x}_i^{\mu}(\tau)$ have a non-commutative structure (implied by Eqs.(2.14)) already at fixed time $[\hat{x}_1(\tau), \hat{x}_2(\tau)] \neq 0$, $[\hat{x}_1(\tau), \hat{x}_2^{\nu}(\tau)] \neq 0$, $[\hat{x}_2(\tau), \hat{x}_2^{\nu}(\tau)] \neq 0$ for $N = 2$. Even if we have $[\hat{p}_i^{\mu}(\tau_1), \hat{p}_j^{\mu}(\tau_2)] = 0$, also the commutators $[\hat{x}_i(\tau), \hat{p}_j^{\nu}(\tau)]$ are probably non trivial. Have these non-commutative properties any connection with the existing non-commutative models for interactions and/or space-time structure? One should study a version of the Ehrenfest theorem adapted to the rest-frame relativistic QM for the recovering of the classical world-lines $x_i^{\mu}(\tau)$ from the mean values $\langle \phi | \hat{x}_i^{\mu}(\tau) | \phi \rangle$ on suitable quasi-classical states $\phi$.

As a consequence, the operators of two space-like separated particles do not satisfy micro-causality as happens in the NW-localization scheme of Refs.[18, 31, 53] but without implying superluminal signalling. This supports the criticism to the validity of the notion of local measurability associated to local algebras and to the associated notion of microcausality (or weak Einstein causality) of Ref.[54].

### B. Relativistic Entanglement

As we have seen the absence of absolute simultaneity due to the Lorentz signature of Minkowski space-time, the non-locality of Poincare’ generators, the non-covariance of the relativistic canonical center of mass and the presence of interactions in the Poincare’ boosts (absent in the Galilei boosts) identify the tensor product $\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}$, $\mathcal{H}_{\text{rel}} = \otimes_a \mathcal{H}_{\text{rel}a}$ as

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31 As a consequence of the canonical transformation of paper I, it turns out that in this description of the isolated system “charged particles with mutual Coulomb interaction plus a transverse electro-magnetic field” there is another hidden constant of the motion, namely the relative momentum $\vec{p}_{(12)3}$ of the particle subsystem with respect to the center of phase of the transverse electro-magnetic field. This implies that the relative variable $\vec{p}_{(12)3}$ is not measurable.

32 In Newton QM, by using $\mathcal{H}_1 \otimes \mathcal{H}_2$ we can apply the Ehrenfest theorem to both $\hat{x}_{(n)i}^{(n)}$ and $\hat{p}_{(n)i}^{(n)}$. 

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the relevant Hilbert space. The Hilbert space \( \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}} \) cannot be presented in the form of the Hilbert space \( (\mathcal{H}_1)_{x_1^0} \otimes (\mathcal{H}_2)_{x_2^0} \) of two free Klein-Gordon quantum particles, even if these two Hilbert spaces are isomorphic. In \( (\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}) \) there is a frozen external center-of-mass wave function\(^{33}\) and a \( \tau \)-independent scalar product in \( \mathcal{H}_{\text{rel}} \)\(^{34}\). In the Hilbert space \( (\mathcal{H}_1)_{x_1^0} \otimes (\mathcal{H}_2)_{x_2^0} \) there are two conserved currents implying that the scalar products in the Hilbert spaces \( (\mathcal{H}_i)_{x_i^0} \) are independent from the times \( x_i^0 \) as shown in Ref.[15], but there is no correlation between \( x_1^0 \) and \( x_2^0 \)\(^{35}\). The problem is that in the tensor product \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) the clocks of the two particles are not synchronized: there are states in which one particle is in absolute future of the other one, so that we cannot define a well-posed Cauchy problem.

One relevant point of the definition of relativistic rest-frame QM is that it selects a preferred bases for \( \mathcal{H}_{\text{com}} \), i.e. the momentum basis, because with each eigenvalue \( \vec{k} \) of \( \tilde{h} \) is associated an inertial 3+1 splitting of Minkowski space-time with the Euclidean instantaneous Wigner 3-spaces orthogonal to \( h^0 = (\sqrt{1 + \vec{h}^2}; \vec{h}) \). This preferred basis is therefore induced by the need of clock synchronization for the identification of the instantaneous 3-space: it is a consequence of Lorentz signature. Instead the selection of preferred bases in \( \mathcal{H}_{\text{rel}} \) has to be done with the methods of decoherence [24]. The derived momentum operators \( \hat{p}_\mu (\tau) \), needed for the description of the individual particles, will depend on the preferred basis of \( \mathcal{H}_{\text{com}} \) and on the chosen basis for \( \mathcal{H}_{\text{rel}} \). The same holds for the derived world-lines of the particles.

Let us remark that one could also study relativistic entanglement in the unphysical Hilbert space \( \mathcal{H}_\tilde{\eta}_1 \otimes \mathcal{H}_\tilde{\eta}_2 \otimes \mathcal{H}_\tilde{\eta}_3 \otimes \ldots \), where there is separability on the instantaneous Wigner 3-spaces. However this type of separability is then destroyed by the quantum version of the interaction-dependent second class constraints \( \tilde{P}_{(\text{int})} \approx 0, \tilde{K}_{(\text{int})} \approx 0 \). In the non-relativistic limit, where the interaction dependent terms are at order \( 1/c^2 \), this amounts to study the non-relativistic entanglement in the rest frame with the center of mass put in the origin of the coordinates.

In conclusion relativistic rest-frame QM has the following important kinematical properties induced by the absence of absolute simultaneity due to the Lorentz signature of Minkowski space-time and to the structure of the Poincaré’ group: non-locality of the collective relativistic variables and spatial non-separability. The fact that a relativistic composite

\(^{33}\) As said it can also be described by a Klein-Gordon center-of-mass wave function with its conserved current implying the independence of the external center-of-mass scalar product from \( \tilde{x}^0 \) in the auxiliary Hilbert spaces \( \mathcal{H}_{\text{ext,com}} \).

\(^{34}\) \( \mathcal{H}_{\text{rel}} \) can be thought as the reduction of the Hilbert space \( \left( \mathcal{H}_{\tilde{\eta}} \otimes \mathcal{H}_{\tilde{\eta}} \right) \) by means of the conditions \(< |\tilde{\tilde{P}}_{(\text{int})}| > = < |\tilde{\tilde{K}}_{(\text{int})}| > = 0 > .

\(^{35}\) In Ref.[15] there is also the quantization of the first-class constraints \( \epsilon \hat{p}_1^2 - m_1^2 c^2 \approx 0 \) after the introduction of suitable center-of-mass \( (x^\mu) \) and relative \( (r^\mu) \) variables in place of the positions \( x_1^\mu \)’s: in this way one gets a quantum model, adapted to the sum and the difference of the two constraints, with a Hilbert space \( \left( \mathcal{H}_1 \right)_{x^0} \otimes (\mathcal{H}_{\text{rel}})_{r^0} \) where there are conserved currents implying that the new scalar product is independent from the center-of-mass time \( x^0 \) and from the relative time \( r^0 \). As a consequence also the presentation \( (\mathcal{H}_1)_{x^0} \otimes (\mathcal{H}_{\text{rel}})_{r^0} \) (a precursor of the approach in this paper) is inequivalent to the one \( (\mathcal{H}_1)_{x_1^0} \otimes (\mathcal{H}_2)_{x_2^0} \) with its single-particle conserved currents.
system is never the tensor product of the elementary subsystems, but is described by the Hilbert space $\mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{rel}}$, implies an intrinsic spatial non-separability. It is induced by the clock synchronization problem, which is not present in Galilei space-time where time and space are separate absolute notions, so that the separability of the subsystems of a composite system is always assumed (the zeroth law of QM). However, as shown in Section III, non-relativistic QM can be presented in the same non-separable form as the rest-frame instant form of relativistic QM if we emphasize the role of the Galilei group in the separation of variables in the Schroedinger equation in presence of interactions.

Let us remark that if we do not succeed to solve the interaction-dependent constraints $\vec{K}_{(\text{int})} \approx 0$ (gauge fixings of the rest frame conditions $\vec{P}_{(\text{int})} \approx 0$), so that the internal 3-center of mass becomes an interaction-dependent function, $\vec{\eta}_a \approx \vec{\eta}_a + [\vec{P}_a, \vec{\pi}_a]$, of the relative degrees of freedom, we must work in the unphysical Hilbert space $\otimes_{i=1}^N \mathcal{H}_{\vec{\eta}_i}$ and then make a Gupta-Bleuler reduction to $\mathcal{H}_{\text{rel}}$ as said in Section IV. The formal separability in subsystems inside the Wigner 3-spaces of the unphysical Hilbert space is destroyed by the dependence upon the interaction of the constraints. It is only in the non-relativistic limit, where the solution of $\vec{K}_{(\text{int})} \approx 0$ is $\vec{\eta}_a(\tau) \approx 0$ independently from the interactions, that separability can be recovered (if wished) as shown in Section III.

Since the non-separable physics is completely contained in the relative variables of $\mathcal{H}_{\text{rel}}$, we can say that the absence of an absolute notion of simultaneity in special relativity induces a weak-relationist point of view: only relative motions are locally accessible because the globally defined center of mass motion cannot be locally determined. Therefore an isolated system (a closed universe) composed by subsystems of the type physical system + observer 1 + observer 2 + (particles of the experimental protocol) + environment must be analyzed in terms of relative variables after the separation of the global (not locally accessible) center of mass (being decoupled its non-covariance is irrelevant). In this respect there are some analogies with Rovelli’s relational QM [60] (all systems and observers are equivalent and all the observations are observer dependent), but Rovelli’s notions of locality and separability are completely different.

The previously described kinematical properties of non-locality and spatial non-separability derive from the choice (required by predictability) of the instantaneous 3-space with a clock synchronization convention which introduces a correlation among all the particles. Therefore this kinematical property is independent of the distances between the particles like the non-local aspects of quantum mechanics connected with the entanglement (the fake a-a-a-d implied by entanglement if we accept Einstein notion of reality). Therefore quantum non-locality is superimposed to already existing relativistic non-locality and spatial non-separability.

Let us remark that till now the approaches to relativistic entanglement have been based on Hilbert spaces of the type of tensor product of the constituents (the type of separability suggested by scattering theory but incompatible with relativistic bound states) trying to analyze it using group theoretical methods from the theory of representations of the Poincare’ group. See Refs. [26] for the attempts to define relativistic entanglement.

See Refs. [23, 24, 61] for the problems of entanglement, of what is a measurement and for the discussion on the interpretations of non-relativistic QM. For the role of decoherence see Refs.[19, 24, 62]. The implications of our relativistic version of entanglement for these problems and for the emergence of classical properties will be investigated elsewhere.
Finally, to include Maxwell equations and their quantization in the relativistic theory of entanglement (the great absent in non-relativistic entanglement), we must either use Fock states with fixed number of photons or make an eikonal approximation of classical Maxwell equations to introduce rays of light (in both cases we can use classical massless helicity 2 classical relativistic particles [63] and their first quantization adapted to the rest-frame instant form \(^{36}\)); see also Refs.[26, 66]. This will be needed to study relativistic teleportation, before facing the problem of gravity \(^{37}\) as in the proposed teleportation experiments between Earth and the Space Station [68].

\(^{36}\) For the positive-energy spinning particles in the rest-frame instant form see the Appendix of Ref.[12] and Ref.[64]. For positive-energy massless particle with helicity one (photons in the eikonal approximation of Maxwell equations with light rays) see Ref.[65].

\(^{37}\) See Refs.[67] for an attempt to formulate atom interferometry in the gravitational field of the Earth by assuming that atoms follow time-like geodetics.
VII. CONCLUSIONS

In this paper we propose a new quantization scheme for positive-energy relativistic particles in the inertial rest-frame instant form of dynamics. The isolated system of $N$ particles is visualized as a non-local decoupled 4-center of mass, described by canonical non-covariant frozen Jacobi data $\vec{z}$ and $\vec{h}$, carrying a pole-dipole structure, i.e. a rest mass $M c$ and a rest spin $\vec{S}$ functions of Wigner-covariant relative variables $\vec{\rho}_a$, $\vec{\pi}_a$, $a = 1, ..., N - 1$ lying in the instantaneous Wigner 3-spaces centered on the Fokker-Pryce 4-center of inertia. The internal 3-center of mass inside the Wigner 3-space is eliminated with the rest-frame condition avoiding a double counting of the center of mass. The Wigner 3-spaces are orthogonal to the conserved 4-momentum of the isolated system, but the internal relative variables are independent of its orientation due to their Wigner covariance (abstract frame-independent internal space). The particle world-lines are derived quantities described by non-canonical 4-vectors (predictive coordinates): a well defined (in general interaction-dependent) non-commutative structure emerges.

The non-relativistic limit of this relativistic QM reproduces the ordinary QM in the Hamilton-Jacobi description of the non-relativistic center of mass.

The quantization scheme is applied to two classes of models with mutual action-at-a-distance interaction among the particles. Besides scattering states also the known properties of relativistic bound states can be described by this quantization scheme. Included in Appendix B is the Weyl-ordered quantization of the classical two-body Hamiltonian including Coulomb plus Darwin interactions to all orders of $1/c^2$.

After a review of the known problems with the notion of relativistic localization in classical relativistic mechanics, in relativistic QM and in QFT, we emphasize that the only open problem in our quantization scheme is connected with the quantum Jacobi data $\vec{z}$: A) If we take them self-adjoint (like in non-relativistic QM), we may either allow superpositions of center-of-mass states or introduce superselection rules forbidding them; B) if we take them to be non-self-adjoint, we need to introduce a modified theory of measurement. The non-observability of the center-of-mass gives rise to these global problems, whose solution requires further study.

Then we study the properties of the relativistic entanglement implied by the new quantization scheme. It turns out to be qualitatively different from non-relativistic entanglement whose most relevant property is quantum non-locality whichever attitude one takes about the foundational interpretative problems. At the relativistic level the prominent properties are the kinematical non-locality and spatial non-separability induced by the non-local nature of the relativistic 4-center of mass and by the use of relative variables in the instantaneous Wigner 3-spaces and not quantum non-locality in the absolute Euclidean 3-space of Galilei space-time. Both properties are consequences of the Lorentz signature of Minkowski space-time and of the structure of the Poincare’ group whose generators are non-local quantities knowing the whole instantaneous 3-space (moreover with the Lorentz boosts interaction-dependent differently from the Galilei boosts). These properties of relativistic entanglement disappear as $1/c$ effects in the non-relativistic limit.

The future developments of the research will be:

A) The extention of the calculations of Appendix B for the quantization of charged
particles with mutual Coulomb plus Darwin interaction to include positive energy spin-one-half particles.

B) The standard quantization of the radiation field in the radiation gauge (see paper I), in the transverse Fock space \( \mathcal{H} \) with creation and annihilation operators \( \hat{a}_\lambda^\dagger(\vec{k}) \), \( \hat{a}_\lambda(\vec{k}) \), \( \lambda = 1, 2 \), followed to its reduction to the rest-frame instant form of dynamics. The physical reduced Fock space \( \mathcal{H}_{phys} \) has to be defined by imposing the conditions

\[
\langle \hat{P}_r \rangle = \frac{1}{c} \sum_{\lambda=1,2} \int d\vec{k} \hat{a}_\lambda^\dagger(\vec{k}) \hat{a}_\lambda(\vec{k}) = 0 \quad \text{and} \quad \langle \hat{K}_r \rangle = i \frac{1}{c} \sum_{\lambda,\lambda'=1,2} \int d\vec{k} \left[ \hat{a}_\lambda(\vec{k}) \hat{a}_\lambda^\dagger(\vec{k}) - \hat{a}_\lambda^\dagger(\vec{k}) \hat{a}_\lambda(\vec{k}) \right] e_\lambda(\vec{k}) \cdot \omega(\vec{k}) \frac{\partial \hat{F}_r(\vec{k})}{\partial k} = 0 \quad \text{[see Eq.(II-3.2); } d\vec{k} = d^3k/2\omega(\vec{k}) (2\pi)^3; \omega(\vec{k}) = |\vec{k}|].
\]

If \( \mathcal{H}_{phys} \) is well defined and can be explicitly constructed, this method would be a first definition of the quantization of the modulus-phase variables of II with the elimination of the un-observable global phase (only relative phases can be measured) described by the internal 3-center of mass (it is a 3-center of phase) of the field configuration on the instantaneous Wigner 3-spaces \(^{38}\). See the reviews of Refs. \cite{69} and Ref. \cite{3} for the obstruction to quantize angles and phases. If phase could be quantized, then we could quantize the relative variables of Eq.(II-3.10) with Hamiltonian \( M_{rad} = P_{rad} \) of Eq.(II-3.2)\(^{39}\) and to get the quantum theory defined in the Hilbert space \( \mathcal{H}_{com} \otimes \mathcal{H}_{Fock rel} \).

C) If the previous quantization of the transverse radiation field would work, then we could study the first quantization of the positive-energy particles with Coulomb plus Darwin mutual interaction together with a second quantized transverse radiation field in the rest-frame instant form, i.e. of the system obtained in I after the canonical transformation.

If the inverse (I-3.10) of the canonical transformation (I-3.6) \(^{40}\) would be unitarily implementable after this quantization, we would get a definition of positive-energy charged quantum particles with mutual Coulomb interaction coupled to a transverse (not radiation) electro-magnetic field in the radiation gauge. Therefore by construction we would get that this fixed-particle-number semi-classical approximation admits a quantum interaction picture description unitarily equivalent to a QM of mutually interacting dressed-particle system plus an "IN" second quantized free radiation field kinematically connected by the rest-frame conditions \(^{41}\).

D) Finally, as a preliminary step in the study of the properties of protocols like teleportation from the space station to an earth station requiring the theory of relativistic entanglement, we have to rephrase non-relativistic entanglement in the rest-frame instant form after the elimination of the center of mass so that the theory depends only on relative

---

\(^{38}\) For fermion fields, which must be Grassmann-valued to become anti-commuting fields after quantization, it is still an open problem how to eliminate the internal 3-center of mass, because action-angle variables cannot be defined for fermion fields.

\(^{39}\) Without fixing the gauge \( X_{rad}^\tau \approx \pm \tau \) where \( X_{rad}^\tau \) is the phase center conjugate to \( P_{rad}^\tau \), i.e. conjugated to the Hamiltonian.

\(^{40}\) It is neither a coordinate- nor momentum- point transformation.

\(^{41}\) A consequence of the clock synchronization convention needed to formulate a Cauchy problem for the isolated system.
The limitation of the approach is that to get an isolated system we must use a mixing of macroscopic and microscopic objects without knowing which is the "relevant effective" description of the macro-objects needed to describe the observers and their instruments. In the isolated system physical system + observer 1 + observer 2 + (particles of the experimental protocol) + environment the observers (measuring apparatuses or Alice and Bob) have to be described as quasi-classical systems. However the spatial non-separability implies that they must be described by relative variables which interconnect them with the microscopic physical system and with the environment. With macroscopic bodies the constraints $\vec{K}_{\text{int}}(\text{int}) \approx 0$ are probably dominated by the approximate solution $\vec{\eta}+ \approx 0$ (with corrections depending on the interactions; $\vec{\eta}+$ is the internal 3-center of mass) so that the use of the separable unphysical Hilbert space $\otimes_i \mathcal{H}_{\vec{\eta}_i}$ becomes an acceptable approximation.

\[ \text{variables} \]
Appendix A: Darwin Potential in the Unequal Mass Case

From Eq.(I-4.5) the Darwin potential has the following expression

\[
V_{\text{DARWIN}}(\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau); \tilde{\kappa}_i(\tau)) = \sum_{i \neq j}^{1,2} Q_i Q_j \left( \frac{\tilde{\kappa}_i \cdot \tilde{A}_{\perp S_2}(\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau), \tilde{\kappa}_j(\tau))}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2}} + \int d^3 \sigma \left[ \frac{1}{2} \left( \tilde{\pi}_{\perp S_1}(\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \tilde{\pi}_{\perp S_2}(\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) + \tilde{B}_{S_1}(\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \tilde{B}_{S_2}(\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) \right) + \frac{\tilde{\kappa}_i}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2}} \cdot \frac{\partial}{\partial \tilde{\eta}_i} \right) \left( \tilde{A}_{\perp S_1}(\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \tilde{\pi}_{\perp S_2}(\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) - \tilde{\pi}_{\perp S_1}(\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) \cdot \tilde{A}_{\perp S_2}(\tilde{\sigma} - \tilde{\eta}_j, \tilde{\kappa}_j) \right) \right].
\]

(A1)

with the following form of the Lienard-Wiechert fields [see Eqs. (I-2.51), (I-2.52) and (I-2.53)]

\[
\tilde{A}_{\perp S}(\tau, \tilde{\sigma}) = \sum_{i=1}^{2} Q_i \tilde{A}_{\perp S_i}(\tilde{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)),
\]

\[
\tilde{A}_{\perp S_i}(\tilde{\sigma} - \tilde{\eta}_i, \tilde{\kappa}_i) = \frac{1}{4\pi |\tilde{\sigma} - \tilde{\eta}_i|} \frac{1}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2}} \times \left[ \frac{\tilde{\kappa}_i \cdot (\tilde{\sigma} - \tilde{\eta}_i)}{|\tilde{\sigma} - \tilde{\eta}_i|^2} \frac{1}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2}} \right],
\]

(A2)

\[
\tilde{E}_{\perp S}(\tau, \tilde{\sigma}) = \tilde{\pi}_{\perp S}(\tau, \tilde{\sigma}) = -\frac{\partial \tilde{A}_{\perp S}(\tau, \tilde{\sigma})}{\partial \tau} = \sum_{i=1}^{2} Q_i \tilde{\pi}_{\perp S_i}(\tilde{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)) = \sum_{i=1}^{2} Q_i \frac{\tilde{\kappa}_i(\tau) \cdot \partial_\sigma}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)}} \tilde{A}_{\perp S_i}(\tilde{\sigma} - \tilde{\eta}_i(\tau), \tilde{\kappa}_i(\tau)) = -\sum_{i=1}^{2} Q_i \times \frac{1}{4\pi |\tilde{\sigma} - \tilde{\eta}_i(\tau)|^2} \left[ \frac{\tilde{\kappa}_i(\tau) \cdot \tilde{\eta}_i(\tau)}{|\tilde{\sigma} - \tilde{\eta}_i(\tau)|} \right] \frac{1}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)}} \frac{1}{\left[ m_i^2 c^2 + (\tilde{\kappa}_i(\tau) \cdot \frac{\tilde{\sigma} - \tilde{\eta}_i(\tau)}{|\tilde{\sigma} - \tilde{\eta}_i(\tau)|})^2 \right]^{3/2}} + \tilde{\sigma} - \tilde{\eta}_i(\tau) \left( \frac{\tilde{\kappa}_i^2(\tau)}{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)} - (\tilde{\kappa}_i(\tau) \cdot \frac{\tilde{\sigma} - \tilde{\eta}_i(\tau)}{|\tilde{\sigma} - \tilde{\eta}_i(\tau)|})^2 \right) \frac{1}{\sqrt{m_i^2 c^2 + \tilde{\kappa}_i^2(\tau)}} - 1 \right). \]

49
From Eq. (I-3.5) we get the following form of the function $\mathcal{K}_{ij}(\tau)$

$$
\mathcal{K}_{12}(\tau) = \int d^{3}\sigma \left[ \hat{A}_{\perp S1} \cdot \hat{\pi}_{\perp S2} - \hat{\pi}_{\perp S1} \cdot \hat{A}_{\perp S2} \right](\tau, \hat{\sigma}).
$$

The internal Poincare’ algebra closes without using the rest-frame condition $\vec{P}_{(\text{int})} \approx 0$.

By using Eq. (II-2.4) of II, the vanishing of the internal boost in Eq. (5.5) gives the following form of $\vec{\eta}_{i12}(\tau)$ ($\vec{\kappa}_{i} \approx (-)^{i+1} \vec{\pi}_{12}$)

$$
\vec{\eta}_{i12} = \left[ \sum_{i=1}^{2} \sqrt{m_{i}^{2} c^{2} + \vec{\pi}_{12}^{2}} + \frac{Q_{1} Q_{2}}{c} \left( \vec{\kappa}_{1} \cdot \left[ \frac{1}{2} \vec{\partial}_{\vec{\pi}_{12}} \mathcal{K}_{12}(\vec{\pi}_{12}, -\vec{\pi}_{12}, \vec{\rho}_{12}) - 2 \hat{A}_{\perp S2}(\vec{\rho}_{12}, -\vec{\pi}_{12}) \right] \right) + \frac{\vec{\kappa}_{2} \cdot \left[ \frac{1}{2} \vec{\partial}_{\vec{\pi}_{12}} \mathcal{K}_{12}(\vec{\pi}_{12}, -\vec{\pi}_{12}, \vec{\rho}_{12}) - 2 \hat{A}_{\perp S1}(\vec{\rho}_{12}, -\vec{\pi}_{12}) \right]}{2 \sqrt{m_{1}^{2} c^{2} + \vec{\pi}_{12}^{2}}} \right]^{-1} \times \left( -\vec{\rho}_{12} \left( \frac{m_{2}}{m} \sqrt{m_{1}^{2} c^{2} + \vec{\pi}_{12}^{2}} - \frac{m_{1}}{m} \sqrt{m_{2}^{2} c^{2} + \vec{\pi}_{12}^{2}} + \frac{m_{1} \vec{\pi}_{12} \cdot \left[ \frac{1}{2} \vec{\partial}_{\vec{\pi}_{12}} \mathcal{K}_{12}(\vec{\pi}_{12}, -\vec{\pi}_{12}, \vec{\rho}_{12}) - 2 \hat{A}_{\perp S2}(\vec{\rho}_{12}, -\vec{\pi}_{12}) \right]}{2 \sqrt{m_{2}^{2} c^{2} + \vec{\pi}_{12}^{2}}} \right) \right) - \frac{Q_{1} Q_{2}}{c} \left( m_{1} \vec{\pi}_{12} \cdot \left[ \frac{1}{2} \vec{\partial}_{\vec{\pi}_{12}} \mathcal{K}_{12}(\vec{\pi}_{12}, -\vec{\pi}_{12}, \vec{\rho}_{12}) - 2 \hat{A}_{\perp S1}(\vec{\rho}_{12}, -\vec{\pi}_{12}) \right] \right) + 2 m \sqrt{m_{1}^{2} c^{2} + \vec{\pi}_{12}^{2}} \right].
$$
By eliminating $\eta_{12}$ and $\kappa_{12} \approx 0$ we get the following form of the Darwin potential and of the Lienard-Wiechert quantities

$$
\tilde{V}_{\text{DARWIN}}(\rho_{12}, \bar{\eta}_{12}) = 
$$

$$
= Q_1 Q_2 \left( \frac{\bar{\eta}_{12}(\tau) \cdot \bar{A}_{LS2}(\rho_{12}(\tau), -\bar{\eta}_{12}(\tau))}{\sqrt{m_1^2 c^2 + \bar{\eta}_{12}^2(\tau)}} - \frac{\bar{\eta}_{12}(\tau) \cdot \bar{A}_{LS2}(\rho_{12}(\tau), \bar{\eta}_{12}(\tau))}{\sqrt{m_2^2 c^2 + \bar{\eta}_{12}^2(\tau)}} \right) + 
$$

$$
+ \int d^3\sigma \left[ \frac{m}{m_2} \frac{m_1}{m} \bar{\eta}_{12}(\tau) \cdot \tilde{\pi}_{LS1}(\sigma - \frac{m_2}{m} \rho_{12}(\tau), \bar{\eta}_{12}(\tau)) \cdot \tilde{\pi}_{LS2}(\sigma + \frac{m_1}{m} \rho_{12}(\tau), -\bar{\eta}_{12}(\tau)) \right] 
$$

$$
- \left( [\bar{\eta}_{12}(\tau) \cdot \frac{\partial}{\partial \rho_{12}}] \bar{\pi}_{LS1}(\sigma - \frac{m_2}{m} \rho_{12}(\tau), \bar{\eta}_{12}(\tau)) \right) \cdot \bar{A}_{LS2}(\sigma + \frac{m_1}{m} \rho_{12}(\tau), -\bar{\eta}_{12}(\tau)) 
$$

$$
+ \frac{m_1}{m} \frac{m_2}{m} \bar{\pi}_{LS1}(\sigma - \frac{m_2}{m} \rho_{12}(\tau), \bar{\eta}_{12}(\tau)) \cdot \bar{\pi}_{LS2}(\sigma + \frac{m_1}{m} \rho_{12}(\tau), -\bar{\eta}_{12}(\tau)) 
$$

$$
- \left( [\bar{\eta}_{12}(\tau) \cdot \frac{\partial}{\partial \rho_{12}}] \bar{\pi}_{LS2}(\sigma + \frac{m_1}{m} \rho_{12}(\tau), -\bar{\eta}_{12}(\tau)) \right) \cdot \bar{A}_{LS1}(\sigma - \frac{m_2}{m} \rho_{12}(\tau), \bar{\eta}_{12}(\tau)) 
$$

$$
+ \bar{\pi}_{LS1}(\sigma - \frac{m_2}{m} \rho_{12}, \bar{\eta}_{12}) \cdot \bar{\pi}_{LS2}(\sigma + \frac{m_1}{m} \rho_{12}, -\bar{\eta}_{12}) 
$$

$$
+ \bar{B}_{LS1}(\sigma - \frac{m_2}{m} \rho_{12}, \bar{\eta}_{12}) \cdot \bar{B}_{LS2}(\sigma + \frac{m_1}{m} \rho_{12}, -\bar{\eta}_{12}) \right) (\tau),
$$

(A7)
\[ \vec{A}_{\perp S1}(\vec{\rho}_{12}, \vec{\pi}_{12}) = \frac{1}{4\pi |\vec{\rho}_{12}|} \frac{1}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \]

\[ \vec{\pi}_{12} + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right) \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|^2} \frac{1}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \] \[ \frac{\vec{\pi}_{12} \cdot \left(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}\right)}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|^2} \left(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}\right) \frac{m_1^2 c^2 + \vec{\pi}_{12}^2}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \]

\[ \vec{A}_{\perp S2}(\vec{\rho}_{12}, -\vec{\pi}_{12}) = -\frac{1}{4\pi |\vec{\rho}_{12}|} \frac{1}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \]

\[ \vec{\pi}_{12} + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right) \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|^2} \frac{1}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \] \[ \frac{\vec{\pi}_{12} \cdot \left(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}\right)}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|^2} \left(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}\right) \frac{m_1^2 c^2 + \vec{\pi}_{12}^2}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \]

\[ \vec{A}_{\perp S2}(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}, -\vec{\pi}_{12}) = -\frac{1}{4\pi |\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \]

\[ \vec{\pi}_{12} + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right) \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|^2} \frac{1}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \] \[ \frac{\vec{\pi}_{12} \cdot \left(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}\right)}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|^2} \left(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}\right) \frac{m_2^2 c^2 + \vec{\pi}_{12}^2}{\sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \sqrt{m_2^2 c^2 + \left(\vec{\pi}_{12} \cdot \frac{\vec{\rho}_{12}}{|\vec{\rho}_{12}|}\right)^2}} \]
\[
\pi_{\perp s_1}(\rho_{12}, \pi_{12}) = \frac{1}{4\pi |\rho_{12}|^2} \left( \frac{\pi_{12} \cdot \rho_{12}}{|\rho_{12}|} \sqrt{m_1^2 c^2 + \rho_{12}^2} \right) + \\
+ \frac{\rho_{12}}{|\rho_{12}|} \left[ \frac{\pi_{12}^2 + \left( \pi_{12} \cdot \rho_{12} \right)^2}{\pi_{12}^2 - \left( \pi_{12} \cdot \rho_{12} \right)^2} \right] \left( -\frac{\sqrt{m_1^2 c^2 + \pi_{12}^2}}{m_1^2 c^2 + \left( \pi_{12} \cdot \rho_{12} \right)^2} - 1 \right) + \\
\pi_{\perp s_2}(\rho_{12}, -\pi_{12}) = -\frac{1}{4\pi |\rho_{12}|^2} \left( \frac{\rho_{12} \cdot \pi_{12}}{|\rho_{12}|} \sqrt{m_2^2 c^2 + \rho_{12}^2} \right) + \\
+ \frac{\rho_{12}}{|\rho_{12}|} \left[ \frac{\pi_{12}^2 + \left( \pi_{12} \cdot \rho_{12} \right)^2}{\pi_{12}^2 - \left( \pi_{12} \cdot \rho_{12} \right)^2} \right] \left( -\frac{\sqrt{m_2^2 c^2 + \pi_{12}^2}}{m_2^2 c^2 + \left( \pi_{12} \cdot \rho_{12} \right)^2} - 1 \right) + \\
+ \frac{\left( \pi_{12} \cdot \rho_{12} \right)^2}{\pi_{12}^2 - \left( \pi_{12} \cdot \rho_{12} \right)^2} \left( \frac{\sqrt{m_2^2 c^2 + \pi_{12}^2}}{m_2^2 c^2 + \left( \pi_{12} \cdot \rho_{12} \right)^2} - 1 \right),
\]
\[ \vec{\pi}_{1S1}(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}, \vec{\pi}_{12}) = -\frac{1}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|^2} \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|} \right) \sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} + \right. \\
+ \frac{\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|} \left[ \vec{\pi}_{12}^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|} \right)^2 \right] \left( \sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2} - 1 \right) + \right. \\
\left. \frac{\left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|} \right)^2 \sqrt{m_1^2 c^2 + \vec{\pi}_{12}^2}}{m_1^2 c^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|} \right)^2} \right]^3/2, \right. \\
\vec{\pi}_{1S2}(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}, -\vec{\pi}_{12}) = -\frac{1}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|^2} \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \right) \sqrt{m_2^2 c^2 + \vec{\pi}_{12}^2} + \right. \\
+ \frac{\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \left[ \vec{\pi}_{12}^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \right)^2 \right] \left( \sqrt{m_2^2 c^2 + \vec{\pi}_{12}^2} - 1 \right) + \right. \\
\left. \frac{\left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \right)^2 \sqrt{m_2^2 c^2 + \vec{\pi}_{12}^2}}{m_2^2 c^2 + \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \right)^2} \right]^3/2, \right. \\
\vec{\pi}_{1S}(\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}, \vec{\pi}_{12}) = -\frac{1}{4\pi |\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|^2} \left[ m_1^2 c^2 \vec{\pi}_{12} \times \frac{\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|} \right] \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}}{|\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|} \right)^2 \right]^3/2, \right. \\
\vec{\pi}_{1S}(\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}, -\vec{\pi}_{12}) = -\frac{1}{4\pi |\vec{\sigma} - \frac{m_2}{m} \vec{\rho}_{12}|^2} \left[ m_2^2 c^2 \vec{\pi}_{12} \times \frac{\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \right] \left( \vec{\pi}_{12} \cdot \frac{\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}}{|\vec{\sigma} + \frac{m_1}{m} \vec{\rho}_{12}|} \right)^2 \right]^3/2, \right. \\
(A8)
Appendix B: Weyl Ordering of the Invariant Mass of Two Equal Mass Particles with Mutual Coulomb plus Darwin Interaction

This appendix consists of two parts. In the first part we show that the Weyl ordered form obtained from the $O(1/c^2)$ classical Darwin interaction given in Eq. (5.7) is identical to the self adjoint version

$$Q_1Q_1 \frac{1}{8\pi m_1 m_2 c^2} \left[ \hat{\pi}_{12} \cdot \hat{\rho}_{12} + \hat{\pi}_{12} \cdot \hat{\rho}_{12} \right] \left[ \hat{\rho}_{12} \cdot \hat{\pi}_{12} \right].$$  \hspace{1cm} (B1)

In the second part of this appendix we develop the Weyl quantization of the exact equal mass Darwin Hamiltonian given in Eq. (5.6) including the kinetic and Coulomb portions. An unusual and unexpected result of that part of this appendix is that the usual local form of the Coulomb potential is the $c \rightarrow \infty$ limit of the complete and nonlocal Coulomb plus Darwin interactions.

1. Weyl Ordering, the operator $\frac{1}{|\hat{\rho}_{12}|}$ and the Weyl Ordered Darwin Operators

We wish to compare the Weyl ordered quantum operator corresponding to the $O(1/c^2)$ classical Darwin interaction to the standard hermitian form. Its classical form is from Eq. (5.7)

$$H_D = \frac{Q_1 Q_1}{8\pi m_1 m_2 c^2} \left( \hat{\pi}_{12}^2 \frac{1}{\hat{\rho}_{12}} + (\hat{\pi}_{12} \cdot \hat{\rho}_{12}) \frac{1}{\hat{\rho}_{12}^3} \right).$$  \hspace{1cm} (B2)

(For simplicity of notation in this part of the appendix we use for the hatted quantum operators the abbreviations $\hat{\pi} = \hat{\pi}_{12}, \hat{\rho} = \hat{\rho}_{12}$). In order to use the Weyl ordered product for the Coulomb potential we replace its singular form with

$$\frac{1}{\hat{\rho}} \rightarrow \frac{1}{\hat{\rho}},$$  \hspace{1cm} (B3)

where we define

$$\hat{\rho} = \sqrt{\hat{\rho}^2 + \varepsilon^2} = \sqrt{\hat{\rho}_x^2 + \hat{\rho}_y^2 + \hat{\rho}_z^2 + \varepsilon^2}.$$  \hspace{1cm} (B4)

This removes the singularity of this operator at the origin. For small $\varepsilon$, our results are independent of $\varepsilon$ and reproduce the known behaviors. Part of this follows from the form

$$\nabla^2 \frac{1}{\hat{\rho}} = \frac{1}{\hat{\rho}} \frac{d^2}{d\hat{\rho}^2} \frac{1}{\hat{\rho}} = -\frac{3\varepsilon^2}{\hat{\rho}^5} = -\frac{3\varepsilon^2}{(\hat{\rho}^2 + \varepsilon^2)^{5/2}},$$ \hspace{1cm} (B5)

of the Laplacian. This equation is a particular form, for infinitesimal $\varepsilon$, of the Poisson equation for a point charge

$$\nabla^2 \frac{1}{\hat{\rho}} = -4\pi \delta^3(\hat{\rho}).$$ \hspace{1cm} (B6)

To see this notice that

$$-3\varepsilon^2 \int \frac{d^3\rho}{(\hat{\rho}^2 + \varepsilon^2)^{5/2}} = -4\pi.$$ \hspace{1cm} (B7)
Thus,

$$\lim_{\varepsilon \to 0} \frac{3\varepsilon^2}{(\rho^2 + \varepsilon^2)^{5/2}} = 4\pi\delta(\rho).$$  \hspace{1cm} (B8)

Now, we return to the determination of the Weyl ordered Darwin form of the Hamiltonian. In rectangular coordinates the classical Darwin interaction Eq. (5.7) is

$$H_D = \frac{Q_1 Q_1}{8\pi m_1 m_2 c^2} \left[ \frac{1}{r} \pi_x^2 + \frac{1}{r} \pi_y^2 + \frac{1}{r^3} \rho_x \pi_x + \rho_y \pi_y + \rho_z \pi_z \right]^2. \hspace{1cm} (B9)$$

We examine $\frac{1}{r} \pi_z^2$ first. The others in the initial portion would be similarly treated. Let

$$\zeta^2 = \rho_z^2 + \rho_y^2 + \varepsilon^2. \hspace{1cm} (B10)$$

Then

$$\frac{1}{r} \pi_z^2 = \frac{1}{\zeta} \sum_{n=0}^{\infty} \left( \frac{-1/2}{n} \right) \frac{\rho_z^{2n}}{\zeta^{2n} \pi_z^{2n}}. \hspace{1cm} (B11)$$

The Weyl ordered quantum form of this is (see Eq. (2.41) in [46])

$$\left( \rho_z^{2n} \pi_z^{2m} \right)^W = \frac{1}{2^{2n}} \sum_{m=0}^{2n} \left( \frac{2n}{m} \right) \rho_z^{2n-m} \pi_z^m \rho_z^m. \hspace{1cm} (B12)$$

Use

$$\pi_z^m \rho_z^m = \rho_z^m \pi_z^m - 2i \rho_z^{m-1} \pi_z - m(m-1) \rho_z^{m-2}, \hspace{1cm} (B13)$$

and so

$$\left( \frac{1}{r} \pi_z^2 \right)^W = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{1}{(2\zeta)^{2n}} \left( \frac{-1/2}{n} \right) \sum_{m=0}^{2n} \left( \frac{2n}{m} \right) \rho_z^{2n-m} \pi_z^m \rho_z^m. \hspace{1cm} (B14)$$

Perform the inner summations and we obtain

$$\left( \frac{1}{r} \pi_z^2 \right)^W = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{1}{(\zeta)^{2n}} \left( \frac{-1/2}{n} \right) \left( \rho_z^{2n} \pi_z^2 - 2i \rho_z^{2n-1} \pi_z - 2n(2n-1) \rho_z^{2n-2} \right)$$

$$= \frac{1}{\pi_z^2} + i \rho_z \pi_z - \frac{1}{4r^3} + \frac{3}{4r^5}. \hspace{1cm} (B15)$$

By cyclic symmetry we thus have

$$\left( \frac{1}{r} \pi_x^2 + \frac{1}{r} \pi_y^2 + \frac{1}{r} \pi_z^2 \right)^W = \frac{1}{r^2} + i \frac{\vec{p}}{r^3} \cdot \vec{\pi} + \frac{3}{4r^3} - \frac{3}{4r} \frac{\varepsilon^2}{r^5} = \frac{1}{r^2} + i \frac{\vec{p}}{r^3} \cdot \vec{\pi} + \frac{3}{4r^3} \frac{\varepsilon^2}{4r^5}$$

$$\to \frac{1}{r^2} + i \frac{\vec{p}}{r^3} \cdot \vec{\pi} + \pi \delta^3(\vec{\rho}). \hspace{1cm} (B16)$$

On the other hand, we would obtain from the standard Hermitean form

$$\vec{\pi} \cdot \frac{1}{\rho} = \frac{1}{\rho} \vec{\pi}^2 - i \frac{\vec{p}}{r^3} \cdot \vec{\pi}. \hspace{1cm} (B17)$$

43 In this Appendix we use the $\hbar = 1$ convention.
Thus,

\[
\left(\frac{1}{\rho} \pi^2\right)^W = \frac{1}{\rho} \pi - \frac{2i\rho}{r^3} \cdot \pi + \pi \delta^3(\rho).
\] (B18)

The remaining part of the Weyl ordered Darwin interaction has the classical form of

\[
\frac{\left(\rho \pi_x + \rho_y \pi_y + \rho_z \pi_z\right)}{r^3} = \frac{\rho_x^2 \pi_x^2 + \rho_y^2 \pi_y^2 + \rho_z^2 \pi_z^2 + 2 \rho_x \rho_y \pi_x \pi_y + 2 \rho_y \rho_z \pi_z \pi_y + 2 \rho_x \rho_z \pi_x \pi_z}{r^3}.
\] (B19)

It is sufficient to examine the two terms

\[
\frac{\rho_x^2 \pi_x^2 + 2 \rho_x \rho_y \pi_x \pi_y}{r^3},
\] (B20)

and the rest we determine by cyclic symmetry.

Consider first \( \frac{\rho_x^2 \pi_x^2}{r^3} \).

The term that needs Weyl ordering is \( \rho_x^{2n+2} \pi_x^2 \). In analogy to above we find

\[
\left(\frac{\rho_x^2 \pi_x^2}{r^3}\right)^W = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{1}{(2\zeta)^{2n+2}} \binom{2n+2}{n} \binom{2n+2}{m} (\rho_x^{2n+2} \pi_x^2 - 2im \rho_x^{2n+1} \pi_x - m(m-1) \rho_x^{2n})
\]

\[
= \frac{\rho_x^2 \pi_x^2}{r^3} - \frac{2 \rho_x}{r^3} \pi_x + 3 \rho_x^3 \pi_x - \frac{1}{2r^3} + \frac{15 \rho_x^2}{4r^5} - \frac{15 \rho_x^4}{4r^7}.
\] (B21)

Including cyclic terms we find

\[
\left(\frac{\rho_x^2 \pi_x^2 + \rho_y^2 \pi_y^2 + \rho_z^2 \pi_z^2}{r^3}\right)^W
\]

\[
= \frac{1}{\zeta} \left(\frac{\rho_x^2 \pi_x^2 + \rho_y^2 \pi_y^2 + \rho_z^2 \pi_z^2}{r^3} - \frac{2}{r^3} \rho_x \cdot \pi + i \frac{3}{r^5} (\rho_x^3 \pi_x + \rho_y^3 \pi_y + \rho_z^3 \pi_z) + \frac{9}{4r^3} \right)
\]

\[
+ \frac{15 \rho_x^2}{4r^5} - \frac{15 (\rho_x^4 + \rho_y^4 + \rho_z^4)}{4r^7}.
\] (B23)

The next term we consider (here \( \zeta^2 = \rho_x^2 + \varepsilon^2 \))

\[
\frac{2 \rho_x \rho_y \pi_x \pi_y}{r^3} = \frac{2 \rho_x \rho_y \pi_x \pi_y}{(\rho_x^2 + \rho_y^2 + \zeta^2)^{3/2}} = \frac{1}{\zeta^3} \frac{2 \rho_x \rho_y \pi_x \pi_y}{(1 + (\rho_x^2 + \rho_y^2)/\zeta^2)^{3/2}}
\]

\[
= \frac{1}{\zeta^3} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{3}{n} \binom{n}{l} \rho_x^{2(n-l)+1} \rho_y^{2l+1}.
\] (B24)

Thus we need

\[
\left(\pi_x \rho_x^{2(n-l)+1} \pi_y \rho_y^{2m+1}\right)^W = \left(\pi_x \rho_x^{2(n-l)+1}\right)^W \left(\pi_y \rho_y^{2l+1}\right)^W
\]

\[
= \frac{1}{2^{2n+2}} \sum_{m=0}^{2(n-l)+1} \binom{2(n-l)+1}{m} \sum_{k=0}^{2l+1} \binom{2l+1}{k} \left[\rho_x^{2(n-l)+1} \pi_x - im \rho_x^{2(n-l)} \right] \left[\rho_y^{2l+1} \pi_y - ik \rho_y^{2l}\right].
\] (B25)
Performing the next inner sum we find
\[
\left( \frac{2\rho_x\rho_y\pi_x\pi_y}{r^3} \right)^W
= \frac{2}{\zeta^3} \sum_{n=0}^\infty \left( -\frac{3/2}{n} \right)^n \sum_{l=0}^n \left( \frac{n}{l} \right) \frac{1}{\zeta} \left[ \rho_x^{2(2n-l)} \rho_y^{2l} \rho_x \pi_x \rho_y \pi_y \right]
\]
\[
-i \frac{(2(n-l)+1)}{2} \rho_y \pi_y - i \frac{(2l+1)}{2} \rho_x \pi_x - \frac{(2(n-l)+1)(2l+1)}{4}.
\] (B26)

Performing the inner sums, we obtain
\[
\left( \frac{2\rho_x\rho_y\pi_x\pi_y}{r^3} \right)^W
= \frac{2}{\zeta^3} \sum_{n=0}^\infty \left( -\frac{3/2}{n} \right)^n \sum_{l=0}^n \left( \frac{n}{l} \right) \frac{1}{\zeta} \left[ \rho_x^{2(2n-l)} \rho_y^{2l} \rho_x \pi_x \rho_y \pi_y \right]
\]
\[
-i \frac{(2(n-l)+1)}{2} \rho_y \pi_y - i \frac{(2l+1)}{2} \rho_x \pi_x - \frac{(2(n-l)+1)(2l+1)}{4}.
\] (B26)

Adding the cyclic terms and we obtain
\[
\left( \frac{2\rho_x\rho_y\pi_x\pi_y + 2\rho_y\rho_x\pi_x\pi_y + 2\rho_x\rho_y\pi_x\pi_z + 2\rho_x\rho_z\pi_x\pi_z}{r^3} \right)^W
= \frac{2}{\zeta^3} \left( \rho_x \rho_y \pi_x \pi_y + \rho_z \rho_y \pi_x \pi_z + \rho_x \rho_z \pi_x \pi_z \right)
\]
\[
-i \left( \frac{1}{r^3} + \frac{3(\varepsilon^2 + \rho_x^2)}{r^5} \right) \rho_y \pi_y - i \left( \frac{1}{r^3} + \frac{3(\varepsilon^2 + \rho_x^2)}{r^5} \right) \rho_x \pi_x - i \left( \frac{1}{r^3} + \frac{3(\varepsilon^2 + \rho_x^2)}{r^5} \right) \rho_z \pi_z
\]
\[
+ \frac{3}{2r^3} - \frac{3\varepsilon^2}{r^3} - \frac{15(\rho_x \rho_y \rho_z^2 + \rho_y \rho_z \rho_x^2 + \rho_x \rho_z \rho_y^2)}{2r^7}.
\] (B28)

Let us combine this with Eq. (B50) and in addition to Eq. (B8) use
\[
2\pi\delta^3(\tilde{\rho}) = \lim_{\varepsilon \to 0} \frac{15\varepsilon^4}{4r^7}.
\] (B29)

We find
\[
\left( \frac{\rho_x^2 \pi_x^2 + \rho_y^2 \pi_y^2 + \rho_z^2 \pi_z^2 + 2\rho_x \rho_y \pi_x \pi_y + 2\rho_y \rho_z \pi_x \pi_y + 2\rho_x \rho_z \pi_x \pi_z}{r^3} \right)^W
= \frac{1}{\rho^3} \rho_i \rho_j \pi_i \pi_j - i \frac{1}{\rho^3} \tilde{\rho} \cdot \tilde{\pi} - i 4\pi\delta^3(\tilde{\rho}) \tilde{\rho} \cdot \tilde{\pi} + \pi\delta^3(\tilde{\rho})
\]
\[
\to \frac{1}{\rho^3} \rho_i \rho_j \pi_i \pi_j - i \frac{1}{\rho^3} \tilde{\rho} \cdot \tilde{\pi} - \pi\delta^3(\tilde{\rho})
\] (B30)
We note for comparison that
\[
\vec{\pi} \cdot \frac{1}{\rho^3} \vec{\rho} \cdot \vec{\pi} = \pi_i \rho_i \frac{1}{\rho^3} \rho_j \pi_j
\]
\[= \frac{1}{\rho^3} \pi_i \rho_i \pi_j + \frac{1}{\rho^3} \rho_j \pi_j \]
\[= \frac{1}{\rho^3} \rho_i \rho_j \pi_i \pi_j + \frac{i}{\rho^3} \vec{\rho} \cdot \vec{\pi} - \frac{3i \varepsilon^2}{\rho^5} \vec{\rho} \cdot \vec{\pi}
\]
\[\rightarrow \frac{1}{\rho^3} \rho_i \rho_j \pi_i \pi_j + \frac{i}{\rho^3} \vec{\rho} \cdot \vec{\pi}.
\]
(B31)
since the delta function in the third line kills the \(\vec{\rho}\). Thus
\[
\left(\vec{\pi} \cdot \frac{1}{\rho^3} \vec{\rho} \cdot \vec{\pi}\right)^W = \vec{\pi} \cdot \frac{1}{\rho^3} \vec{\rho} \cdot \vec{\pi} = -\frac{2i}{\rho^3} \vec{\rho} \cdot \vec{\pi} - \pi \delta^3(\vec{\rho})
\]
(B32)
The total Weyl Darwin terms combine to
\[
\left(\frac{1}{\rho^3} \rho^2 \pi_x^2 + \frac{1}{\rho^3} \rho^2 \pi_y^2 + \frac{1}{\rho^3} \rho^2 \pi_z^2\right)^W + \left(\frac{\rho_x^2 \pi_x^2 + \rho_y^2 \pi_y^2 + \rho_z^2 \pi_z^2 + 2 \rho_x \rho_y \pi_x \pi_y + 2 \rho_y \rho_z \pi_x \pi_z + 2 \rho_z \rho_x \pi_x \pi_z}{\rho^3}\right)^W
\]
\[= \frac{1}{\rho^2} + \frac{1}{\rho^3} \rho_i \rho_j \pi_i \pi_j.
\]
(B33)
which gives the same as the total \(O(1/c^2)\) hermitian Darwin interaction
\[
\frac{Q_1 Q_1}{8 \pi m_1 m_2 c^2} \left[\vec{\pi} \cdot \frac{1}{\rho} \vec{\pi} + \vec{\rho} \cdot \frac{1}{\rho^3} \vec{\rho} \cdot \vec{\pi}\right] = \frac{Q_1 Q_1}{8 \pi m_1 m_2 c^2} \left[\frac{1}{\rho} \vec{\pi}^2 + \frac{1}{\rho^3} \rho_i \rho_j \pi_i \pi_j\right].
\]
(B34)
Thus, we find that the difference in two treatments (Weyl and conventional) of the total quantum \(O(1/c^2)\) Darwin terms is zero, although the differences in the two treatments of the individual Darwin terms are not zero!
2. Weyl Quantization of the Classical Darwin Hamiltonian

From [10] Eqs. (6.35), (6.36), (6.37), for equal masses the Hamiltonian is

\[ M = P_{(\text{int})}^r = 2\sqrt{m^2 + \pi_1^2} + Q_1 Q_2 \frac{8\pi}{4\pi |\vec{\rho}_{12}|} \left[ \frac{\pi_1^2}{m^2 c^2 + (\pi_1 |\vec{\rho}_{12}|)^2} \right] \]

\[ + \frac{1}{(m^2 c^2 + \pi_1^2)(m^2 c^2 + (\pi_1 |\vec{\rho}_{12}|)^2)} \times (m^2 c^2 [3\pi_1^2 + (\pi_1 |\vec{\rho}_{12}|)^2] - \left[ 3\pi_1^2 + (\pi_1 |\vec{\rho}_{12}|)^2 \right]^2 + (\pi_1 |\vec{\rho}_{12}|)^2)^3 \right] \]

\[ = 2\sqrt{m^2 + \pi_1^2} + Q_1 Q_2 \frac{8\pi}{4\pi |\vec{\rho}_{12}|} \left[ m^2 c^2 + (\pi_1 |\vec{\rho}_{12}|)^2 \right] \left[ \left( \pi_1^2 + (\pi_1 |\vec{\rho}_{12}|)^2 \right)^2 - 2\pi_1^2 (\pi_1 |\vec{\rho}_{12}|)^2 \right] \]

\[ - 2\pi_1^2 (\pi_1 |\vec{\rho}_{12}|)^2 \left[ m^2 c^2 + (\pi_1 |\vec{\rho}_{12}|)^2 \right] \]  \]

\[ \text{(B35)} \]

Let us see how we can construct a corresponding self-adjoint quantum operator on position space wave functions using the Weyl - quantization procedure [41]

a. Weyl-quantization

Let \( K \) be a classical observable and a function of the relative variables \( \pi_{12} \) and \( \vec{\rho}_{12} \). The Weyl-quantization \( K^W \) of \( K(\vec{\rho}_{12}, \pi_{12}) \) is defined on a well behaved wave function \( \psi(\vec{\rho}_{12}) \) as

\[ K^W \psi(\vec{\rho}_{12}) = \frac{1}{(2\pi)^3} \int \int \exp(i(\vec{\rho}_{12} - \vec{\rho}')) \cdot \pi_{12}) K\left(\frac{\pi_{12} + \pi'}{2}, \pi_{12}\right) \psi(\vec{\rho}_1') d^3 \rho_{12} d^3 \pi_{12}. \]  \]

(B36)

In the case of functions that are dependent only on \( \vec{\rho}_{12} \) we have by doing the \( d^3 \pi_{12} \) integral

\[ K^W \psi(\vec{\rho}_{12}) = \frac{1}{(2\pi)^3} \int \int \exp(i(\vec{\rho}_{12} - \vec{\rho}')) \cdot \pi_{12}) K\left(\frac{\pi_{12} + \pi'}{2}, \pi_{12}\right) \psi(\vec{\rho}_1') d^3 \rho_{12} d^3 \pi_{12} \]

\[ = K(\vec{\rho}_{12}) \psi(\vec{\rho}_{12}). \]  \]

(B37)

The only term of this form in Eq. (B35) is the Coulomb term

\[ K^W_C(\vec{\rho}_{12}) = \frac{Q_1 Q_2}{4\pi |\vec{\rho}_{12}|} \psi(\vec{\rho}_{12}). \]  \]

(B38)

There is a sign error in Eq. (6.35) which effects Eq. (6.37) of that reference [10]. Here we correct (6.37).

In this appendix we use the \( \hbar = 1 \) convention.
For functions that are dependent only on $\tilde{\pi}_{12}$ we have

$$K^W \psi(\tilde{\rho}_{12}) = \frac{1}{(2\pi)^3} \int \int \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) K(\tilde{\pi}_{12}) \psi(\tilde{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}$$

$$= \int \tilde{K}(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \psi(\tilde{\rho}'_{12}) d^3 \rho_{12}, \quad (B39)$$

in which $\tilde{K}(\tilde{\rho}_{12} - \tilde{\rho}'_{12})$ is the Fourier transform of $K(\tilde{\pi}_{12})$.

$$\tilde{K}(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) = \frac{1}{(2\pi)^3} \int \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) K(\tilde{\pi}_{12}) d^3 \pi_{12}. \quad (B40)$$

The only term like that in the whole Hamiltonian (B35) is the kinetic piece

$$2\tilde{K}_T(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) = \frac{1}{(2\pi)^3} \int \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) 2\sqrt{m^2 + \tilde{\pi}_{12}^2} d^3 \pi_{12}. \quad (B41)$$

b. Lowest Order $1/c^2$ Expressions

From the second part of Eq. (B35) the first order $1/c^2$ Darwin terms are

$$D \equiv \frac{Q_1 Q_2}{8\pi} \left( \frac{\tilde{\pi}_{12}^2 + (\tilde{\pi}_{12} \cdot \tilde{\pi}_{12})^2}{m^2 c^2} \right). \quad (B42)$$

Our Weyl quantized version on a position space wave function is

$$D^w \psi(\tilde{\rho}_{12}) = \frac{1}{(2\pi)^3} \frac{Q_1 Q_2}{8\pi m^2 c^2} \int \int \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) \frac{2}{|\tilde{\rho}_{12} + \tilde{\rho}'_{12}|} \times (\tilde{\pi}_{12}^2 + (\tilde{\pi}_{12} \cdot \tilde{\pi}_{12})^2) \psi(\tilde{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}. \quad (B43)$$

in which

$$\tilde{n}_{12} = \frac{\tilde{\rho}_{12} + \tilde{\rho}'_{12}}{|\tilde{\rho}_{12} + \tilde{\rho}'_{12}|}. \quad (B44)$$

The first term from integration by parts is

$$\frac{1}{(2\pi)^3} \frac{Q_1 Q_2}{8\pi m^2 c^2} \int \int \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) \frac{2}{|\tilde{\rho}_{12} + \tilde{\rho}'_{12}|} \tilde{\pi}_{12}^2 \psi(\tilde{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}$$

$$= -\frac{1}{(2\pi)^3} \frac{Q_1 Q_2}{8\pi m^2 c^2} \int \int \left[ \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) (\tilde{\rho}'_{12})^2 \right] \frac{2}{|\tilde{\rho}_{12} + \tilde{\rho}'_{12}|} \psi(\tilde{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}$$

$$= -\frac{Q_1 Q_2}{8\pi m^2 c^2} \tilde{\pi} \cdot \left( -\frac{\tilde{\rho}_{12}}{|\tilde{\rho}_{12}|^3} + \frac{1}{|\tilde{\rho}_{12}|} \tilde{\rho} \right) \psi(\tilde{\rho}_{12}). \quad (B45)$$

The second term is

$$\frac{1}{(2\pi)^3} \frac{Q_1 Q_2}{8\pi m^2 c^2} \int \int \left[ \left( \tilde{n}_{12}, \tilde{n}_{12}, \tilde{n}_{12}, \tilde{n}_{12} \right) \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) \right] \frac{2}{|\tilde{\rho}_{12} + \tilde{\rho}'_{12}|} \psi(\tilde{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}$$

$$= -\frac{1}{(2\pi)^3} \frac{Q_1 Q_2}{8\pi m^2 c^2} \int \int \exp \left( i(\tilde{\rho}_{12} - \tilde{\rho}'_{12}) \cdot \tilde{\pi}_{12} \right) \partial,' \partial,' \left[ \left( \tilde{n}_{12}, \tilde{n}_{12}, \tilde{n}_{12} \right) \right] \frac{2}{|\tilde{\rho}_{12} + \tilde{\rho}'_{12}|} \psi(\tilde{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}$$

$$= -\frac{Q_1 Q_2}{8\pi m^2 c^2} \partial_1 \left[ \frac{\rho_{12}}{|\rho_{12}|^3} \psi(\tilde{\rho}_{12}) + \frac{\rho_{12} \rho_{12}}{|\rho_{12}|^3} \partial_1 \psi(\tilde{\rho}_{12}) \right]. \quad (B46)$$
Thus, the $O(1/c^2)$ Darwin contribution to the Schrödinger equation has the two form

$$D^w\psi(\vec{p}_{12}) = -\frac{Q_1 Q_2}{8\pi m^2 c^2} \frac{\delta_{rs}}{|\vec{p}_{12}|} \frac{\rho_{12r} \rho_{12s}}{|\vec{p}_{12}|^3} \partial_{\vec{p}_{12}} \psi(\vec{p}_{12}).$$

(B47)

This agrees (in notation of this paper) with known Darwin results (see [42], [43] and [44]). If we bring the remaining derivative through we obtain

$$D^w\psi(\vec{p}_{12}) = -\frac{Q_1 Q_2}{8\pi m^2 c^2} \left( \frac{\delta_{rs}}{|\vec{p}_{12}|} + \frac{\rho_{12r} \rho_{12s}}{|\vec{p}_{12}|^3} \right) \partial_{\vec{p}_{12}} \psi(\vec{p}_{12}).$$

(B48)

These two results arise from the Weyl ordering on the position wave function of the classical function Eq. (B42) In operator form they correspond to the operator forms of

$$\frac{Q_1 Q_2}{8\pi m^2 c^2} \left( \frac{1}{|\vec{p}_{12}|} \pi_{12} + \pi_{12} \cdot \vec{p}_{12} \frac{\hat{p}_{12}}{|\vec{p}_{12}|} \right),$$

$$\frac{Q_1 Q_2}{8\pi m^2 c^2} \left( \frac{1}{|\vec{p}_{12}|} \right) \left( \delta_{rs} + \hat{\eta} \hat{\eta}_s \right) \pi_{12r} \pi_{12s}.$$  

(B49)

c. Quantization of the Complete Expression Eq. (B35) - The Coulomb Potential as a Local $c \to \infty$ Limit

Before going on to the Weyl quantization of the exact expression we rearrange the first two portions of the Darwin term in the last line of Eq. (B35) to read

$$\frac{Q_1 Q_2}{8\pi |\vec{p}_{12}|} \left( \frac{1}{m^2 c^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2} \right) \left( \frac{3 m^2 c^2}{m^2 c^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2} \right) \times \left[ m^2 c^2 \left( \vec{\pi}_{12}^2 + (\vec{\rho}_{12} \cdot \vec{\pi}_{12})^2 \right) \right]$$

$$- 2 (\vec{\pi}_{12}^2 + m^2 c^2) ((\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2 + m^2 c^2) + 2 (\vec{\pi}_{12}^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2) m^2 c^2 + 2 m^4 c^4$$

$$= - \frac{Q_1 Q_2}{8\pi |\vec{p}_{12}|} \left( \frac{3 m^2 c^2}{m^2 c^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2} \right) + \frac{3 m^2 c^2}{m^2 c^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2}$$

$$- \frac{4 m^4 c^4}{(m^2 c^2 + \vec{\pi}_{12}^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2)}.$$

Our total classical Hamiltonian then becomes

$$M = 2 \sqrt{m^2 c^2 + \vec{\pi}_{12}^2} +$$

$$+ \frac{Q_1 Q_2}{8\pi |\vec{p}_{12}|} \left( \frac{3 m^2 c^2}{m^2 c^2 + \vec{\pi}_{12}^2} \right) + \frac{3 m^2 c^2}{m^2 c^2 + \vec{\pi}_{12}^2} - \frac{4 m^4 c^4}{m^2 c^2 + \vec{\pi}_{12}^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2}$$

$$- \frac{Q_1 Q_2}{8\pi |\vec{p}_{12}|} \left( \frac{3 m^2 c^2}{m^2 c^2 + \vec{\pi}_{12}^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2} \right)$$

$$- \frac{2 m^4 c^4}{m^2 c^2 + \vec{\pi}_{12}^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2} \sqrt{m^2 c^2 + \vec{\pi}_{12}^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2}.$$  

(B50)

Note that in this rearrangement, the local Coulomb potential is canceled and replaced by momentum dependent terms. Note, however, that in the non-relativistic limit ($c \to \infty$)
the momentum dependent potential energy terms in the first line reduces to the ordinary Coulomb term while the second line vanishes. Although Eq. (B35) has the advantage of seeing the lowest order expansion more clearly, the above shows that the exact expression does not have local Coulomb potentials except in the the non-relativistic limit \((c \to \infty)\).

Consider the simplest part
\[
\frac{Q_1 Q_2}{8\pi |\vec{\rho}_{12}| (m^2 c^2 + \vec{n}_{12}^2)}.
\]

The corresponding Weyl term would be
\[
K_1^W \psi(\vec{\rho}_{12}) = \frac{3m^2 c^2 Q_1 Q_2}{(2\pi)^3} \frac{1}{8\pi} \int \int \exp(i(\vec{\rho}_{12} - \vec{\rho}')) \cdot \vec{n}_{12} \frac{1}{|\vec{\rho}_{12} + \vec{\rho}'| (m^2 c^2 + \vec{n}_{12}^2)} \psi(\vec{\rho}') d^3 \rho' d^3 \pi_{12}.
\]

Perform the \(\pi_{12}\) integral to give
\[
\frac{1}{(2\pi)^3} \int d^3 \pi_{12} \exp(i(\vec{\rho}_{12} - \vec{\rho}') \cdot \vec{n}_{12}) \frac{1}{(m^2 c^2 + \vec{n}_{12}^2)} = \frac{1}{4\pi} \frac{\exp(-mc |\vec{\rho}_{12} - \vec{\rho}'|)}{|\vec{\rho}_{12} - \vec{\rho}'|}
\]
and so
\[
K_1^W \psi(\vec{\rho}_{12}) = \frac{3m^2 c^2 Q_1 Q_2}{16\pi^2} \int \frac{1}{|\vec{\rho}_{12} + \vec{\rho}'|} \frac{\exp(-mc |\vec{\rho}_{12} - \vec{\rho}'|)}{|\vec{\rho}_{12} - \vec{\rho}'|} \psi(\vec{\rho}') d^3 \rho'.
\]

It is a nonlocal term just as the kinetic energy is.

Note that we recover the local non-relativistic limit of this expression by using the form below for the Dirac delta function,
\[
\delta^3(\vec{\rho}'_{12} - \vec{\rho}_{12}) = \lim(c \to \infty) \frac{m^2 c^2 \exp\left[-mc |\vec{\rho}'_{12} - \vec{\rho}_{12}|\right]}{4\pi |\vec{\rho}'_{12} - \vec{\rho}_{12}|}.
\]

In that case
\[
K_1^W \psi(\vec{\rho}_{12}) \to \frac{3Q_1 Q_2}{8\pi |\vec{\rho}_{12}|},
\]
which agrees with the expectation from the \(c \to \infty\) limit of the corresponding expression in Eq. (B50).

More problematic is
\[
\frac{3Q_1 Q_2}{8\pi |\vec{\rho}_{12}| m^2 c^2 + (\vec{n}_{12} \cdot \vec{\rho}_{12})^2}.
\]

Its Weyl ordering is
\[
K_2^W \psi(\vec{\rho}_{12}) = \frac{3m^2 c^2 Q_1 Q_2}{(2\pi)^3} \frac{1}{8\pi} \int \int \exp(i(\vec{\rho}_{12} - \vec{\rho}')) \cdot \vec{n}_{12} \frac{1}{|\vec{\rho}_{12} + \vec{\rho}'| (m^2 c^2 + (\vec{n}_{12} \cdot \vec{\rho}_{12})^2)} \psi(\vec{\rho}') d^3 \rho' d^3 \pi_{12},
\]

Let us focus on the \(\pi_{12}\) integral
\[
I = \int \exp(i(\vec{\rho}_{12} - \vec{\rho}')) \cdot \vec{n}_{12} \frac{1}{(m^2 c^2 + (\vec{n}_{12})^2)} d^3 \pi_{12}.
\]
Let us divide $\vec{\pi}_{12} = \vec{\pi}_{12} \cdot (\hat{n}_{12}) \hat{n}_{12} + \vec{\pi}_{12 \perp}$. Then we obtain

$$I = \int \exp \left( i (\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot (\vec{\pi}_{12} \cdot (\hat{n}_{12}) \hat{n}_{12} + \vec{\pi}_{12 \perp}) \right) \frac{1}{(m^2 c^2 + (\vec{\pi}_{12} \cdot \hat{n}_{12})^2)^2} d^2 \vec{\pi}_{12 \perp} d (\vec{\pi}_{12} \cdot (\hat{n}_{12})).$$

(B60)

Perform the $d^2 \vec{\pi}_{12 \perp}$ integral and call $k = \vec{\pi}_{12} \cdot (\hat{n}_{12})$. Then, with

$$(2\pi)^2 \delta^2((\vec{\rho}_{12} - \vec{\rho}'_{12}) \perp) \equiv \int \exp \left( i (\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12 \perp} \right) d^2 \vec{\pi}_{12 \perp},$$

we have

$$I = (2\pi)^2 \delta^2((\vec{\rho}_{12} - \vec{\rho}'_{12}) \perp) \int_{-\infty}^{\infty} \frac{\exp \left( i (\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \hat{n}_{12} k \right) dk}{m^2 c^2 + k^2} = \frac{(2\pi)^3 \delta^2((\vec{\rho}_{12} - \vec{\rho}'_{12}) \perp)}{2mc} \exp(-mc((\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \hat{n}_{12})).$$

(B61)

and so

$$K^W_{2} \psi(\vec{\rho}_{12}) = \frac{3m^2 c^2 Q_1 Q_2}{(2\pi)^3} \int \int \exp \left( i (\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12} \right) \frac{2}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \times \frac{1}{(m^2 c^2 + (\vec{\pi}_{12} \cdot \hat{n}_{12})^2)^3} \psi(\vec{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}$$

$$= \frac{3mc Q_1 Q_2}{8\pi} \int \frac{\exp(-mc((\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \hat{n}_{12}))}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \delta^2((\vec{\rho}_{12} - \vec{\rho}'_{12}) \perp) \psi(\vec{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12}.$$ (B63)

We recover the non-relativistic limit by using the one dimensional expression for the delta function of

$$\delta((\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \hat{n}_{12}) = \lim(c \to \infty) \frac{mc}{2} \exp(-mc((\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \hat{n}_{12})).$$

(B64)

Thus in that limit

$$K^W_{2} \psi(\vec{\rho}_{12}) \rightarrow \frac{3Q_1 Q_2}{4\pi} \int \frac{\exp(-mc((\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \hat{n}_{12}))}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \delta((\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \hat{n}_{12}) \delta^2((\vec{\rho}_{12} - \vec{\rho}'_{12}) \perp) \psi(\vec{\rho}'_{12}) d^3 \rho_{12}$$

$$\rightarrow \frac{3Q_1 Q_2}{8\pi |\vec{\rho}_{12}|} \psi(\vec{\rho}_{12}),$$

(B65)

as expected.

The next term to Weyl transform in Eq.(B50) is

$$- \frac{Q_1 Q_2}{2\pi |\vec{\rho}_{12}|} \left[ \frac{m^4 c^4}{(m^2 c^2 + \vec{\pi}_{12}^2)(m^2 c^2 + (\vec{\pi}_{12} \cdot \vec{\rho}_{12})^2)} \right].$$

(B66)

The corresponding Weyl transform is

$$K^W_{2} \psi(\vec{\rho}_{12}) = -\frac{m^4 c^4 Q_1 Q_2}{(2\pi)^3} \int \int \frac{\exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \frac{1}{(m^2 + \pi_{12}^2)(m^2 + (\vec{\pi}_{12} \cdot \hat{n}_{12})^2)} \times \psi(\vec{\rho}'_{12}) d^3 \rho_{12} d^3 \pi_{12},$$

(B67)
We focus on the Fourier transform
\[
J = \frac{1}{(2\pi)^3} \int \exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12}) \frac{d^3\pi_{12}}{(m^2c^2 + \vec{\pi}_{12}^2)(m^2c^2 + (\vec{\pi}_{12} \cdot \hat{n}_{12})^2)}. \tag{B67}
\]
Let us recall that if
\[
\begin{align*}
\tilde{f}(\pi_{12}) &= \int \exp(-i\pi_{12} \cdot \vec{\rho}_{12}) f(\vec{\rho}_{12}) d^3\eta, \\
\tilde{g}(\pi_{12}) &= \int \exp(-i\pi_{12} \cdot \vec{\rho}_{12}) g(\vec{\rho}_{12}) d^3\eta',
\end{align*}
\tag{B68}
\]
then we obtain the convolution result of
\[
\frac{1}{(2\pi)^3} \int \exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \pi_{12}) \tilde{f}(\pi_{12}) \tilde{g}(\pi_{12}) d^3\pi_{12} = \int f(\vec{\rho}_{12}) g(\vec{\rho}_{12} - \vec{\rho}'_{12} - \vec{\rho}_{12}) d^3\eta. \tag{B69}
\]
Thus, with
\[
\begin{align*}
\frac{1}{(m^2c^2 + \vec{\pi}_{12}^2)} &= \frac{1}{4\pi} \int \exp(-i\pi_{12} \cdot \vec{\rho}_{12}) \frac{\exp(-mc|\vec{\rho}_{12}|)}{|\vec{\rho}_{12}|} d^3\pi_{12}, \\
\frac{1}{(m^2c^2 + (\vec{\pi}_{12} \cdot \hat{n}_{12})^2)} &= \int \exp(-i\pi_{12} \cdot \vec{\rho}_{12}) \frac{(2\pi)^3 \delta_1^2 (\vec{\rho}_{12})}{2m} \exp(-mc|\vec{\rho}_{12} \cdot \hat{n}_{12}'|),
\end{align*}
\tag{B70}
\]
we have
\[
\begin{align*}
J &= \frac{1}{(2\pi)^3} \int \exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \pi_{12}) \frac{1}{(m^2c^2 + \vec{\pi}_{12}^2)} \frac{1}{(m^2c^2 + (\vec{\pi}_{12} \cdot \hat{n}_{12})^2)} d^3\pi_{12} \\
&= \frac{1}{4\pi |\vec{\rho}_{12}|} \frac{(2\pi)^3 \delta_1^2 (\vec{\rho}_{12} - \vec{\rho}'_{12} - \vec{\rho}_{12})}{2mc} \exp(-mc|(\vec{\rho}_{12} - \vec{\rho}'_{12} - \vec{\rho}_{12}) \cdot \hat{n}_{12}|) d^3\rho_{12}'.
\end{align*}
\tag{B71}
\]
Hence,
\[
K^W_3 \psi(\vec{\rho}_{12}) = -\frac{m^4c^4 Q_1 Q_2}{(2\pi)^3 \pi} \int \int \frac{\exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \frac{1}{(m^2c^2 + \vec{\pi}_{12}^2)(m^2c^2 + (\vec{\pi}_{12} \cdot \hat{n}_{12})^2)} d^3\pi_{12} \\
\times \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\pi_{12} \\
= -\frac{m^3c^3 Q_1 Q_2}{8\pi^2} \int \frac{\exp(-mc|\vec{\rho}'_{12}|)}{|\vec{\rho}'_{12}|} \frac{\delta_1^2 (\vec{\rho}_{12} - \vec{\rho}'_{12} - \vec{\rho}_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \\
\times \exp(-mc|(\vec{\rho}_{12} - \vec{\rho}'_{12} - \vec{\rho}_{12}) \cdot \hat{n}_{12}|) \psi(\vec{\rho}_{12}) d^3\rho_{12}' d^3\rho_{12}. \tag{B72}
\]
Using the delta function expression in Eq. (B55) and (B64) we see that in the $c \to \infty$ limit the above becomes
\[
K^W_3 \psi(\vec{\rho}_{12}) \to -\frac{Q_1 Q_2}{2\pi |\vec{\rho}_{12}|}, \tag{B73}
\]
We point out that had we chosen not to make the rearrangement of Eq. (B35) in the quantization, a cancelation of the local Coulomb potential would still have taken place by the multiple derivatives (of the nonlocal Yukawa kernels) that come from the higher order momentum terms in the numerator. 46

d. Combined Non-local Weyl Ordered Hamiltonian

The second portion of the classical Darwin Hamiltonian is

\[
\frac{Q_1 Q_2}{4 \pi |\vec{p}_{12}|} \left\{ \frac{\vec{p}_{12}^2 - 3(\vec{p}_{12} \cdot \vec{\pi}_{12})^2}{(m^2 c^2 + \vec{p}_{12}^2)[m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2]} \right\}^{1/2} \quad \frac{m^2 c^2 + \vec{p}_{12}^2}{m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2}^{1/2} - \frac{3 (\vec{p}_{12}^2 + m^2 c^2)^{1/2}}{(m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2)^{1/2}} \\
+ \frac{m^2 c^2 (\vec{p}_{12}^2 + m^2 c^2)^{1/2}}{(m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2)^{3/2}} \quad \frac{3 m^2 c^2}{(m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2)^{1/2}} \\
- \frac{2 m^4 c^4}{(m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2)^{3/2}(m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2)^{3/2}}. \quad (B74)
\]

Each Weyl transform would involve a convolution. The first is

\[
K_{4}^{\psi}(\vec{p}_{12}) = -\frac{1}{(2 \pi)^3} \frac{2 Q_1 Q_2}{4 \pi} \int \int \frac{\exp(i(\vec{p}_{12} - \vec{\rho}_{12}) \cdot \vec{p}_{12})}{|\vec{p}_{12} + \vec{\rho}_{12}|} \times \frac{(\vec{p}_{12}^2 + m^2 c^2)^{3/2}}{(m^2 c^2 + (\vec{p}_{12} \cdot \vec{\pi}_{12})^2)^{3/2}} \psi(\vec{p}_{12}) d^3 \rho_{12} d^3 \pi_{12}. \quad (B75)
\]

As with \( K_{3}^{\psi}(\vec{p}_{12}) \) it involves a convolution

\[
\frac{1}{(2 \pi)^3} \int \exp(i(\vec{p}_{12} - \vec{\rho}_{12}) \cdot \vec{\pi}_{12}) \hat{f}(\vec{\pi}_{12}) \hat{g}(\vec{\pi}_{12}) d^3 \pi_{12} = \int f(\vec{\pi}_{12}) g(\vec{\rho}_{12} - \vec{\rho}_{12} - \vec{\rho}_{12}') d^3 \eta. \quad (B76)
\]

Now defining

\[
K_{T1}(\vec{\rho}_{12}) = \frac{1}{(2 \pi)^3} \int \exp(i \vec{\pi}_{12} \cdot \vec{\rho}_{12}) \left( \vec{\pi}_{12}^2 + m^2 c^2 \right)^{3/2} d^3 \pi_{12}, \\
K_{K1}(\vec{\rho}_{12}) = \frac{1}{(2 \pi)^3} \int \frac{\exp(i \vec{\pi}_{12} \cdot \vec{\rho}_{12})}{(m^2 c^2 + (\vec{\pi}_{12} \cdot \vec{\pi}_{12})^2)^{3/2}} d^3 \pi_{12}, \quad (B77)
\]

---

46 We point out that had we chosen not to make the rearrangement of Eq. (B35) in the quantization, a cancelation of the local Coulomb potential would still have taken place by the multiple derivatives (of the nonlocal Yukawa kernels) that come from the higher order momentum terms in the numerator.
we have
\[ K_4^W \psi(\vec{\rho}_{12}) = -\frac{2Q_1Q_2}{4\pi} \int \frac{K_{T1}(\vec{\rho}'_{12}) K_{K1}(\vec{\rho}_{12} - \vec{\rho}'_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\rho'_{12}. \] (B78)

The next portion is
\[ K_5^W \psi(\vec{\rho}_{12}) = \frac{3}{(2\pi)^3} \frac{2Q_1Q_2}{4\pi} \int \int \frac{\exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \times \frac{(|\vec{\pi}_{12}^2 + m^2c^2)^{1/2}}{(m^2c^2 + (\vec{\pi}_{12} \cdot \vec{n}_{12})^2)^{1/2}} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\pi_{12}. \] (B79)

Defining
\[ K_{K2}(\vec{\rho}_{12}) = \frac{1}{(2\pi)^3} \int \frac{\exp(i\vec{\pi}_{12} \cdot \vec{\rho}_{12})}{(m^2c^2 + (\vec{\pi}_{12} \cdot \vec{n}_{12})^2)^{1/2}} d^3\pi_{12}, \] (B80)
we have
\[ K_5^W \psi(\vec{\rho}_{12}) = \frac{6Q_1Q_2}{4\pi} \int \frac{K_T(\vec{\rho}'_{12}) K_{K2}(\vec{\rho}_{12} - \vec{\rho}'_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\rho'_{12}. \] (B81)

Following this term is
\[ K_6^W \psi(\vec{\rho}_{12}) = -\frac{2m^2c^2Q_1Q_2}{4\pi} \int \frac{K_T(\vec{\rho}'_{12}) K_{K1}(\vec{\rho}_{12} - \vec{\rho}'_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\rho'_{12}. \] (B82)

and its contribution is
\[ K_6^W \psi(\vec{\rho}_{12}) = -\frac{2m^2c^2Q_1Q_2}{4\pi} \int \frac{K_T(\vec{\rho}'_{12}) K_{K1}(\vec{\rho}_{12} - \vec{\rho}'_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\rho'_{12}. \] (B83)

The next term is
\[ K_7^W \psi(\vec{\rho}_{12}) = \frac{3m^2c^22Q_1Q_2}{(2\pi)^3} \int \int \frac{\exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \times \frac{1}{(\vec{\pi}_{12}^2 + m^2c^2)^{1/2}} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\pi_{12}. \] (B84)

and with
\[ K_{T2}(\vec{\rho}_{12}) = \frac{1}{(2\pi)^3} \int \frac{\exp(i\vec{\pi}_{12} \cdot \vec{\rho}_{12})}{(\vec{\pi}_{12}^2 + m^2c^2)^{1/2}} d^3\pi_{12}, \] (B85)
we have
\[ K_7^W \psi(\vec{\rho}_{12}) = -\frac{6m^2c^2Q_1Q_2}{4\pi} \int \frac{K_T(\vec{\rho}'_{12}) K_{K2}(\vec{\rho}_{12} - \vec{\rho}'_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\rho'_{12}. \] (B86)

The final term is
\[ K_8^W \psi(\vec{\rho}_{12}) = \frac{2m^4c^42Q_1Q_2}{(2\pi)^3} \int \int \frac{\exp(i(\vec{\rho}_{12} - \vec{\rho}'_{12}) \cdot \vec{\pi}_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \times \frac{1}{(\vec{\pi}_{12}^2 + m^2c^2)^{1/2}} \psi(\vec{\rho}_{12}) d^3\rho_{12} d^3\pi_{12}. \] (B87)
and it contributes

\[
K_8^W \psi(\vec{\rho}_{12}) = -\frac{4m^4 c^4 Q_1 Q_2}{4\pi} \int \frac{K_{T2}(\vec{\rho}'_{12}) K_{K1}(\vec{\rho}_{12} - \vec{\rho}'_{12} - \vec{\rho}''_{12})}{|\vec{\rho}_{12} + \vec{\rho}'_{12}|} \psi(\vec{\rho}_{12}) d^3 \rho_{12}. \tag{B88}
\]

Although in the \( c \to \infty \) limit each of the Weyl ordered terms is finite, they cancel altogether. Altogether, our Weyl order Hamiltonian is

\[
2K_T(\vec{\rho}_{12}) + \sum_{n=1}^{8} K_n^W (\vec{\rho}_{12}). \tag{B89}
\]
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