UNOBRU CtEDNESS OF FILLING SECANTS
AND THE GRUSON-PESKINE GENERAL PROJECTION THEOREM

ZIV RAN

ABSTRACT. We prove an unobstructedness result for deformations of subvarieties constrained by
intersections with another, fixed subvariety. We deduce smoothness and expected-dimension re-
sults for multiple-point loci of generic projections, mainly from a point or a line, or for fibres of
embedding dimension 2 or less.

The study of linear projections of a smooth projective variety \( X \subset \mathbb{P}^N \), and the closely re-
lated study of multisecant spaces of \( X \), have occupied projective geometers for generations (see
e.g. [14]). Though many pathologies are possible for multisecant spaces, projections from a
generic center \( \Lambda \), or equivalently the multisecant spaces containing \( \Lambda \), seem not to be afflicted by
them, at least when \( \Lambda \) is a point. This has led to the formulation of a folklore ‘generic projection
conjecture’ (about which we first heard from R. Lazarsfeld \( \text{ca.} \) 1990): the projection of \( X \) from a
generic point \( p \in \mathbb{P}^N \) has only the ‘expected’ singularities (see below). This statement is equiva-
 lent to one about families of secant lines to \( X \) satisfying contact conditions, filling up the ambient
\( \mathbb{P}^N \). After numerous partial results including [8], [2], [10], [3], the conjecture was recently proven
by Gruson and Peskine [5] (which the reader may consult for further introductory comments,
references, as well as examples and applications).

In this paper we will prove 3 results, each extending the Gruson-Peskine Generic Projection
Theorem:

- Theorem 4.1 is a version for arbitrary ambient spaces in place of \( \mathbb{P}^N \), in the form of a general
result about deformations of rational curves on varieties constrained by contact conditions with
a fixed subvariety: it is shown that these deformations are well-behaved (unobstructed, of the
expected dimension) provided they fill up the ambient space.

- Theorem 5.1 is a generic projection theorem in \( \mathbb{P}^N \) (though without contact conditions) for
projections from arbitrary-dimension centers and their fibres which have local embedding di-
mension 2, which includes all fibres in the case of projections from a generic line.

- Theorem 6.1 is a full generic projection theorem in \( \mathbb{P}^N \), contact conditions included, for the
case of curvilinear fibres (and arbitrary-dimension centers).

The difference between our approach and that of Gruson and Peskine is, in a word, cohomol-
ogy (albeit, of the most elementary kind). We will develop some deformation theory for secants
to \( X \), then apply the ‘filling’ hypothesis to show that obstructions- even obstruction groups- vanish.
Thus in outline, the proof of each of these theorems follows the same overall pattern:

\[\text{Date: February 4, 2014.}\]
\[1991 \ \text{Mathematics Subject Classification.} \ 14N05. \]
\[\text{Key words and phrases.} \ \text{multisecant lines, rational curves, generic projections, multiple points.} \]
\[\text{arxiv.org/1302.0824.}\]
- the secant or contact conditions are analyzed locally and encoded in certain secant or contact sheaves \( M \) on the secant plane or curve \( L \), which control the corresponding deformations;
- the filling hypothesis implies a certain generic spannedness property for \( M \);
- \( L \) is isomorphic to \( \mathbb{P}^\ell \), often even \( \mathbb{P}^1 \), the generic spannedness implies spannedness and the vanishing of higher cohomology, whence well-behaved deformations.

This note grew out of an attempt to understand the exciting result of Gruson and Peskine [5], viewed as a statement about multisecant lines which are mobile enough to go through a generic point of the ambient space. Though our proof is independent of [5] and the connection to it may be hidden, it is nonetheless fundamental. It occurs above all through the following heuristic idea gleaned from [5]:

**Uniformity principle:** The family of secants to \( X \) behaves in the same way at all points off \( X \)

For example: if \( L \) is postulated to contain a \( k \)-tuple \( Z \subset X \) and \( p \notin Z \), we may consider whether under infinitesimal motion of \( L \) as \( k \)-secant to \( X \), and \( p \) with it, \( p \) fills up (the tangent space of) the ambient projective space. The uniformity principle says that the answer is *independent* of the point \( p \). This is a shockingly powerful conclusion: it implies inter alia that if \( k \)-secants to \( X \) fill up the ambient space, they cannot all meet another variety of codimension > 1, and cannot meet \( X \) itself in > \( k \) points. This is quite close to saying that the filling \( k \)-secants of \( X \) are well-behaved.

A (future ?) Google-type search on ‘uniformity principle’ in Algebraic Geometry is likely to bring up prominently the fact that on a \( \mathbb{P}^1 \), a generically spanned vector bundle is spanned everywhere (parenthetically, such bundles also have no higher cohomology). Our idea is that the two uniformity principles are related, the first being ultimately a consequence of the second. Applying this idea to the bundles and sheaves occurring in the deformation theory of secants is the basis for our cohomological/deformation-theoretic approach outlined above.

The paper is organized as follows. In §1 we give some precise definitions of secants and their scheme structure, as well as the notion of ‘filling’ and its infinitesimal analogue, which requires a little care. In §2 we study finite schemes, especially curvilinear ones, and their deformation spaces. In §3 we study various sheaves related to deformations of secants, which are fundamental for our deformation-theoretic approach. Then in §§4,5,6 we prove the aforementioned 3 theorems, along with a few corollaries.

We thank Yeongrak Kim for helpful discussions and the referees for constructive comments and suggestions and stimulating questions that have greatly improved the paper. Hopefully, our indebtedness to the breakthrough work of Gruson and Peskine would be obvious to anyone.

**Conventions.** In this paper we work over \( \mathbb{C} \); sometimes for added generality, we will work in the complex-analytic category (the kind of analytic varieties we have in mind are open subsets of algebraic varieties). If \( X \) is an algebraic scheme or analytic space, and \( a \) is any negative number, the statements ‘\( X \) has dimension \( a \)’ and ‘\( X \) is empty’ are by definition equivalent. Similarly, if \( X \) has dimension \( n \geq 0 \) and \( m > n \) and \( Y \subset X \), the statements ‘\( Y \) is empty’ and ‘\( Y \) has codimension \( m \) in \( X \)’ are equivalent. We use \( \ell \) to denote the length of a module and \( \ell_z \) its local length at a point \( z \). A statement about a ‘general’ point \( y \) of an analytic variety \( Y \) is by definition true if it holds for all \( y \) in the complement of a nowhere dense analytic subset of \( Y \).

See [13], [6], [7] for some foundational results on Hilbert schemes and deformation theory. A more general setting based on the language of Lie theory is presented in [11] and [12].
1. Secants and Fillers

1.1. Secants. In this subsection, we will define secant loci set-theoretically. Let \( P \) be a nonsingular, quasi-projective variety and \( G \) a connected smooth open subset of a suitable Hilbert scheme of \( P \) parametrizing smooth connected complete unobstructed subvarieties \( L \subset P \), to be called flats. In applications, flats \( L \) will be mostly isomorphic to \( \mathbb{P}^m \) (often \( \mathbb{P}^1 \)). If in fact \( L \approx \mathbb{P}^1 \), it is well known that

\[
\dim(G) = -K_P.L + \dim(P) - 3.
\]

In any case, \( G \) comes equipped with a tautological family \( L \rightarrow G \).

Let \( X \) be a nonsingular, locally closed subvariety of \( P \), closed in a neighborhood of \( L \). Throughout, \( P \) and \( X \) will be considered fixed, though \( L \) will move. Let \( X[k] \) and \( X^{(k)} \) denote, respectively, the Hilbert scheme of length-\( k \) subschemes of \( X \), and the \( k \)-th symmetric product. There is a well-known cycle (or 'Hilb to Sym') morphism (see [6] §3.2 or [9], §1.2)

\[
\tau : X[k] \rightarrow X^{(k)}.
\]

Then \( X[k] \) is canonically stratified by closed subschemes \( X^{[k]} \) for all partitions \( (k) \) of weight \( k \), where \( X^{[k]} \) consists of the finite schemes \( z \) such that \( \tau(z) = \sum k_i p_i \), \( p_i \in X \) not necessarily distinct. Such \( z \) is said to be of cycle type \( (k) \), and properly so when the \( p_i \) are distinct. We have that \( X^{[k]} \subset X^{[m]} \) whenever the partition \( (m) \) is a refinement of \( (k) \), i.e. obtained by subdividing some blocks.

For a partition \( (k) = (k_1, \ldots, k_r) \), \( L \) is said to be \( (k) \)-secant to \( X \) if the schematic intersection \( L \cap X \) is of finite length and contains a subscheme of type \( (k) \), and proper as such if \( L \cap X \) itself is properly of type \( (k) \). \( L \) is said to be \( k \)-secant to \( X \) if the length of \( L \cap X \) is finite and at least \( k \) (equivalently, \( L \) is \((1^k)\)-secant, i.e. properly \((1)\)-secant for some partition \((l)\) of weight \( \sum l_i = l \geq k \).

Let

\[
\bar{S}^{(k)} \subset G \times X^{[k]}
\]

denote the locus of pairs \((L, z)\) where \( z \subset L \cap X \), and let \( S^{(k)} \) denote its projection to \( G \). Thus, \( S^{(k)} \subset G \) is the locus of flats that are \((k)\)-secant to \( X \). Note that \( \bar{S}^{(k)}, S^{(k)} \) are closed subsets of \( G \times X^{[k]}, G \) respectively and the map \( \bar{S}^{(k)} \rightarrow S^{(k)} \) is finite-to-one and bijective over the open subset parametrizing \( L \)s such that \( L \cap X \) has length \( k \). Moreover if either \( L \) is a smooth curve or \( k_i = 1 \) for all \( i \), or more generally if the intersections \( X \cap L \) are (locally) curvilinear schemes, then \( X^{[k]} \) has a well-defined structure of smooth closed subscheme of the smooth curvilinear Hilbert scheme \( P^d_0 \) (see [2] in a neighborhood of \( z \), hence \( \bar{S}^{(k)} \) is endowed with a scheme structure as the pullback of \( X^{[k]} \) by the natural map \( \mathcal{L}^{[k]}_G \rightarrow P^d \) where \( \mathcal{L}^{[k]}_G \) is the relative Hilbert scheme. Then \( S^{(k)} \) is endowed with the image scheme structure. Note that \( S^{(k)} \) is closed in \( G \) if \( X \subset P \) is. For \( L \in S^{(k)} \), the intersection \( L \cap X \) may have length greater than the weight of \( (k) \) and possibly \( \infty \).

If \( Y \subset P \) is an analytic submanifold of a tubular (analytic) neighborhood of a fibre \( L \), or germ of one, we denote by \( \bar{S}^{(k), Y} \) and \( S^{(k), Y} \) the appropriate loci (or schemes) corresponding to deformations of \( L \) within \( Y \).

One can similarly define a secant scheme \( \bar{S}^{L} \) postulating just the total length of \( L \cap X \), as the pullback of closed subscheme \( X^{[k]} \subset P^d \) by the natural map \( \mathcal{L}^{[k]}_G \rightarrow P^d \).
1.2. Fillers. The working method of this paper is to study filling families by infinitesimal methods, viz. the powerful tools of Grothendieck’s deformation theory. There is a certain subtlety involved here, because the filling notion is essentially of global character and meaningful mainly for (quasi-projective) varieties, while deformation theory applies directly to formal completions, hence to germs. That some care is required is illustrated by the fact that a map of point germs can be birational without being smooth at the unique (closed) point.

We are thus led to the notion of ‘infinitesimal filling’, which serves to bridge between filling notions and infinitesimal methods.

**Definition 1.1.** (i) Let $Y$ be a smooth analytic open subset of an irreducible projective variety $\bar{Y}$ and $\mathcal{L}/B$ a smooth flat family of closed subvarieties of $\bar{Y}$ contained in $Y$, parametrized by an irreducible analytic variety $B$, such that $\mathcal{L}$ is irreducible. The family is said to be filling if the natural projection $\pi_Y: \mathcal{L} \to Y$ is dominant, i.e. its image contains an analytic open set.

(ii) Let $Y$ be a smooth local germ along a smooth irreducible subvariety $L$ of a smooth quasi-projective variety $P$. A local deformation $L \to \text{Spec}(A)$ of $L$ within $Y$ is said to be infinitesimally filling for $Y$ if there exists an irreducible closed subscheme $S$ of $\text{Spec}(A)$, such that the map $(\mathcal{L}_S)_{\text{red}} \to Y$ induces a surjection on (Zariski) tangent spaces $T_p(\mathcal{L}_S)_{\text{red}} \to T_pY$ for a general point $p \in L$.

**Remark 1.2.** In Part (i) above, $Y$ could equal $\bar{Y}$ or be Zariski open in it. A typical non-algebraic $Y$ can be a tubular neighborhood of some fibre of $\mathcal{L}/B$.

**Remark 1.3.** To be clear, we are talking here about an arbitrary member of the $\mathcal{L}/B$ family through a general point of $Y$, which is tantamount to talking about an arbitrary point of a general fibre of a suitable morphism; we aim to invoke generic smoothness, which says that the morphism is smooth at such a point.

A link between the two types of ‘filling’ is the following.

**Lemma 1.4.** In the situation of Definition 1.1 (i), assume additionally that $\mathcal{L} \to B$ is proper. Then if $p \in Y$ is general and $L \to b$ is an arbitrary fibre of the filling family $\mathcal{L}/B$ going through $p$, then the germ of $L/B$ along $L \to b$, i.e. $L \times_b \text{Spec}(\mathcal{O}_{B,b})$, is infinitesimally filling, hence so is any larger family.

**Proof.** $\mathcal{L} \to B$ extends to a proper flat algebraic family $\hat{\mathcal{L}} \to \hat{B}$. By enlarging $B$, we may assume $\hat{B}$ coincides with the open subset of $\hat{B}$ corresponding to fibres contained in $Y$. Let $\hat{\mathcal{L}} \to \hat{\bar{Y}}$ be a desingularization. Then the fibre $F$ of $\hat{\mathcal{L}} \to \hat{\bar{Y}}$ over a general point $p \in Y \subset \hat{\bar{Y}}$ is smooth, i.e. for all $f \in F$, the derivative map $T_f\hat{\mathcal{L}} \to T_p\hat{\bar{Y}}$ is surjective. In particular, if $f$ is in the open subset of $F$ going into a point of $\mathcal{L}$, say $q \in (L \to b)$, then the latter derivative factors through $T_q\mathcal{L} \to T_pY$, so the latter is surjective as well. We may identify $q$ with $p$, so the infinitesimal filling property follows.

**Remark 1.5.** In applications, $\mathcal{L} \to B$ will be a smooth morphism but $B$, hence $\mathcal{L}$ as total space, may have arbitrary singularities.

There are analogues of the notions of filling and infinitesimally filling, and of their relation, as in the last Lemma, with respect to $\Lambda$, where ‘point’ is replaced by $\Lambda$, a subvariety moving in a smooth filling family on $Y$: the precise definition is the following

**Definition 1.6.** Notation as in Definition 1.1 let $\mathcal{L}_i \to B_i$ be proper and filling families, $i = 1, 2$. Let $B \subset B_1 \times B_2$ be a subset parametrizing pairs $(\Lambda, L)$ with $\Lambda \subset L$. Then the family of pairs parametrized
by $B$ is said to be filling with respect to $\Lambda$ if the projection $B \to B_1$ is dominant. The family is said to be infinitesimally filling with respect to $\Lambda$ at a given $L$ corresponding to $b_2 \in B$ projecting to $b_2$, the projection of Zariski tangent spaces $T_bB \to T_{b_1}B_1$ is surjective.

The following extension of Lemma 1.4 is proved in the same way:

**Lemma 1.7.** Notations as above, if $b \in B_1$ is general and $b \in B$ projecting to $b_1$ is arbitrary, then the map of Zariski tangent spaces $T_bB \to T_{b_1}B_1$ is surjective.

## 2. Curvilinear schemes

### 2.1. Basics.** Let $L$ be a nonsingular variety. Let $L[k], L_0[k] \subset L[k]$, and $L^{(k)}$ denote, respectively, the Hilbert scheme of length-$k$ subschemes of $L$, its open subset consisting of curvilinear schemes, i.e. those of local embedding dimension 1 or less (see below), and the $k$-th symmetric product. There is a well-known cycle (or 'Hilb to Sym') morphism

$$c : L[k] \to L^{(k)}.$$  

Then $L[k]$ is canonically stratified by closed subsets $L_0[k]$ for all partitions $(k)$ of weight $k$, where $L[k]$ consists of the finite schemes $z$ such that $c(z) = \sum k_i p_i$, $p_i \in X$ not necessarily distinct. Such $z$ is said to be of cycle type $(k)$, and properly so if the $p_i$ are distinct. The open subset $L_0[k] \subset L[k]$ is well-understood and has a canonical scheme structure.. We have that $L_0[k] \subset L^{[m]}$, whenever the partition $(m)$ is a refinement of $(k)$, i.e. obtained by subdividing some blocks. The set of curvilinear schemes properly of type $(k)$ is well known (compare 2.2 below) to be a smooth open subset $L_0[k] \subset L[k]$ of dimension $k(dim(L) - 1) + r$ where $(k) = (k_1 \geq \ldots \geq k_r > 0)$.

### 2.2. Embedded deformations.** A local curvilinear scheme $z$ of length $k \geq 2$ is isomorphic to $\text{Spec}(\mathbb{C}[x]/(x^k))$. It has unobstructed versal deformation space

$$\text{Def}(z) \cong T_z^1 \cong \mathcal{O}_z/(x^{k-1})$$

with basis $1, x, \ldots, x^{k-2}$. Identifying $\text{Def}(z)$ with the set of polynomials

$$\{x^k + b_{k-2}x^{k-2} + \ldots + b_0 : b_i \in \mathbb{C}\},$$

it is stratified by loci $D_{(k)}$ corresponding to partitions of weight $k$. If the partition $(k)$ is written as $(l_i^{e_i})$ with $(l_i)$ strictly decreasing, then $D_{(k)}$ can be identified with the following collection of factored polynomials:

$$D_{(k)} = \{ \prod_{i}^{e_i} (x-a_{ij})^{l_i} : a_{ij} \in \mathbb{C}, \sum_{ij} a_{ij} = 0 \}.$$  

Identifying, for fixed $i$, $\prod_{j} (x-a_{ij})$ with a point in $\text{Sym}^{e_i}(\mathbb{A}^1)$ and, in turn, with a point in $\mathbb{A}^{e_i}$ via the elementary symmetric functions $\sigma_r(a_{i,1}, \ldots, a_{i,e_i})$, $D_{(k)}$ is bijectively and birationally (but not isomorphically) parametrized by the subspace of $\prod_{i} \text{Sym}^{e_i}(\mathbb{A}^1)$ defined by $\sum_{i} \sigma_1(a_{i,1}, \ldots, a_{i,e_i}) = 0$.

When $z$ is embedded in a smooth variety, its ideal $\mathcal{I}_z$ has the form

$$\left(f_1 = x_1^k, f_2 = x_2, \ldots, f_n = x_n \right)\]
for a suitable regular system of parameters $x_1, \ldots, x_n$; first-order deformations of $Z$ take the form

$$(f_1 + b_1, \ldots, f_n + b_n), b_i = \sum_{j=0}^{k-1} b_{ij} x_i^j, i = 1, \ldots, n.$$  

The obstructions vanish, because $Z$ is a local complete intersection. The locus of deformations preserving the cycle type, or equivalently, locally trivial deformations, is the (smooth) subscheme given by the condition

$$b_{ij} = 0, j < k - 1.$$  

For a general, non-local curvilinear scheme $Z = \bigsqcup z_i$, the versal deformation and Hilbert scheme are the product of those for the $z_i$.

2.3. **Abstract deformations.** Up to a smooth factor, deformations of $Z$ on a smooth variety $L$ are the same as deformations of $z$ on a smooth curve-germ $C$ containing $Z$ as given in the above notation by $x_2 = \ldots = x_n = 0$. In fact, if $Z$ is properly of type $(k,)$, i.e. has multiplicity exactly $k_i$ at $p_i$, and $z_i$ denotes the part of $Z$ supported at $p_i$, then the versal deformation $\text{Def}(Z)$ of $Z$ as abstract scheme splits as $\bigsqcup \text{Def}(z_i)$ . There is a local classifying map from the Hilbert scheme of $L$:

$$(2) \quad \rho : L^{[k]} \to \text{Def}(Z).$$

On the level of 1st-order deformations, this corresponds to the natural surjective map

$$N_{Z,L} \to T_L^1$$

where $N_{Z,L} = \text{Hom}(I_{Z,L}, O_Z)$ is the lci normal bundle. Denoting the kernel of this map by $N_{Z,L}'$, the 'locally trivial' normal bundle, and identifying $T_L \otimes O_Z \simeq \text{Der}(O_L, O_Z)$, we have exact sequences

$$(3) \quad 0 \to T_{L/Z} \to T_L \otimes O_Z \to N_{Z,L}' \to 0,$$

$$0 \to N_{Z,L}' \to N_{Z,L} \to T_L^1 \to 0$$

where $T_{L/Z}$ is the derivations preserving $I_Z$. In particular, the morphism $\rho$ is smooth.

Consequently, if $Z$ is properly of type $(k,)$ then $L^{[k]}$ and $L^{[k]} = \sum k_i$ are smooth at $Z$. Moreover, if $L^{[m]} \subset L^{[k]}$ is any other stratum, necessarily containing $L^{[k]}$, containing $Z$, then the singularity of $L^{[m]}$ at $Z$ is the same up to a smooth factor as that of $C^{[m]}$ and the normalization of $L^{[m]}$ is smooth. Indeed the normalization in question is just $\prod(C^{(b_i)})$ where the $b_i$ are the multiplicities of distinct-size blocks of $(m)$, i.e. $(m) = (a_1 = \ldots = a_1 > a_2 = \ldots)$ with each $a_i$ occurring $b_i$ times. For instance, if $(k,)$ = $(3)$, $(m,)$ = $(2, 1)$ then $C^{[m]}$, hence $L^{[m]}$, is locally (smooth x ordinary cusp).

The identification of $\prod(C^{(b_i)})$ with the normalization of $C^{[m]}$ is immediate from the fact that $\prod(C^{(b_i)})$ is smooth and the map $\prod(C^{(b_i)}) \to C^{[m]}$ is finite and birational.

In the situation of Theorem[4,1] note that there is an induced map near $(L, Z)$

$$(4) \quad \mu : \tilde{S}_{k,Y} \to \text{Def}(Z)$$

such that $\tilde{S}_{(k),Y} = \mu^{-1}(D_{(k)})$. The theorem’s assertions are mostly covered by the statement that $\mu$ is a smooth morphism, which we will prove in [4]. The remaining assertions have to do with the case of intersections that are ‘excessive’, i.e. have length $> k$. To prepare this we next turn to nested pairs of curvilinear schemes.
In [12] one can find a description of the ‘tangent complex’ of any affine scheme like $Z$ as a
differential graded Lie algebra, whose deformation theory as such coincides with that of $Z$; similarly, when $Z$ is embedded in a smooth scheme $L$, the normal complex (or sheaf, when $Z$ is
a local complete intersection) $N_{Z/L}$ has the structure of ‘Lie atom’ whose deformation theory is
that of embedded deformations of $Z$ in $L$.

2.4. Pairs. We consider nested pairs of curvilinear schemes (cf. [13], §4.5 for general flag Hilbert
schemes).

**Lemma 2.1.** Let $C$ be a smooth curve and denote by $C^{(d,k)} \subset C^{(d)} \times C^{(k)}$ denote the locus of nested pairs
of schemes $z_d \leq z_k$ and by $C^{(d,k)} \subset C^{(d,k)}$ the sublocus where $z_d$ is of type $(d,k)$ where $(d,k)$ is a partition of
weight $d$. Then the projection $C^{(d,k)} \rightarrow C^{(d)}$ is a smooth morphism, hence the singularities of $C^{(d,k)}$ are
the same up to a smooth factor as those of $C^{(d)}$.

**Proof.** The projection in question is a base-change of the projection $C^{d,k} \rightarrow C^d$, so it suffices to
prove the latter morphism is smooth. For this we may work locally on $C$ and assume $C = \mathbb{A}^1, z_k = (x^k)$ (hence $z_d = (x^d)$). Denote by $V_d$ the space of monic polynomials of degree $< d$ in $x$, identified with the local Hilbert scheme of $(x^d)$ on $\mathbb{A}^1$, and let $V_{d,k} \subset V_d \times V_k$ denote the locus of
pairs $(h, g)$ such that

$$(x^d + h)(x^k + g).$$

Geometrically, $V_{d,k}$ is the space of pairs $(z_d \leq z_k)$ of cycles or subschemes on $\mathbb{A}^1$. This is obviously isomorphic to $V_d \times V_{k-d}$, via $x^k + g = (x^d + h)(x^{k-d} + h')$. In particular, $V_{d,k}$ and the
projection $V_{d,k} \rightarrow V_d$ are smooth. Consider the tangent space at the origin:

$$T_{d,k} = T_0(V_{d,k}) \subset V_d \oplus V_k.$$  (5)

This is a $k$-dimensional subspace. Though $V_{d,k}$ and $V_k$ are smooth and $k$-dimensional, the projection $p : V_{d,k} \rightarrow V_k$ is ramified. The image of the differential at the origin $dp : T_{d,k} \rightarrow V_k$ can be identified as the space of polynomials divisible by $x^{\min(d,k-d)}$ and therefore has dimension $\max(d/k - d)$. The projection $T_{d,k} \rightarrow V_d$ is surjective, which proves the Lemma.

For future reference, let $T_{d,k}^0$ denote the kernel of the composite surjection $T_{d,k} \rightarrow V_d \rightarrow
V_d/(x^{d-1})$. This $T_{d,k}^0$ is a subspace of codimension $d - 1$ in $T_{d,k}$. It corresponds to (embedded)
deformations of $(z_d \leq z_k)$ where $z_d$ deforms locally trivially.

The Lemma can be easily extended to the case of an arbitrary smooth ambient variety:

**Proposition 2.2.** Let $X$ be a smooth variety or complex manifold, and $X^{[d,k]}_0 \subset X^{[d]}_0 \times X^{[k]}_0$ be the locus
of nested curvilinear schemes. The $X^{[d,k]}_0$ and the morphism $X^{[d,k]}_0 \rightarrow X^{[d]}_0$ are smooth.

**Proof.** We can work locally on $X$ at a punctual pair $z_d \leq z_k$ and choose local parameters $x_1, ..., x_n$
so that

$$\mathcal{I}_{z_d} = (x_1^d, x_2, ..., x_n), \mathcal{I}_{z_k} = (x_1^k, x_2, ..., x_n).$$

Then deformations of these are given by

$$x_1 + h_1, x_2 + h_2, ..., x_n + h_n, h_i \in \sum_{j=0}^{d-1} \mathbb{C} x_i^j, i = 1, ..., n.$$
\[ x_1^k + g_1, x_2 + g_2, \ldots, x_n + g_n, g_i \in \sum_{j=0}^{k-1} C x_1^j, \quad i = 1, \ldots, n. \]

The compatibility condition on local (arbitrary-order) deformations is

\[ x_1^i + h_1 | x_1^k + g_1, h_1 \equiv g_i \mod x_1^i, i = 2, \ldots, n. \]

The corresponding first-order conditions are

\[ (h_1, g_1) \in T_{d, k}, h_1 \equiv g_i \mod x_1^i, i = 2, \ldots, n. \]

Canonically, we may identify these deformations as follows. Note the canonical map

\[ \rho : N_{z_k/X} \oplus N_{z_d/X} \to N_{z_k/X} \otimes \mathcal{O}_{z_d} = \text{Hom}(\mathcal{I}_{z_k/X}, \mathcal{O}_{z_d}) \]

given on the first summand by restriction \( \mathcal{O}_{z_k} \to \mathcal{O}_{z_d} \) and on the second summand by dualizing the inclusion \( \mathcal{I}_{z_k/X} \subset \mathcal{I}_{z_d/X} \). Clearly \( \rho \) is surjective. Then \( H^0(\ker(\rho)) \) is the canonical version of \( T_{d,k} \). Furthermore, \( \ker(\rho) \), identified with the mapping cone of \( \rho \) and right-shifted by 1, has the structure of semi-simplicial Lie algebra, equivalent to a Lie atom (see [11], [12]), and the deformation theory of this structure is the deformation theory of the pair \( (z_d, z_k) \). In particular, obstructions lie in \( H^1(\ker(\rho)) = 0 \). \( \square \)

3. The secant and contact sheaves and associated deformations

3.1. Non-excess case. Our purpose here is to prove

**Proposition 3.1.** Let \( L, X \) be closed submanifolds of a smooth variety or complex manifold, \( Y \) intersecting in a finite-length scheme \( Z \) with ideal \( \mathcal{I}_Z = \mathcal{I}_X + \mathcal{I}_L \). Then there subsheaves and Lie subatoms

\[ N^\text{ct} \subset N^s \subset N_{L/Y} = \text{Hom}(\mathcal{I}_L, \mathcal{O}_L) \]

called the contact and secant sheaves, respectively, whose formation commutes with passage to an open subset, which control deformation of \( L \) in \( Y \) inducing locally trivial (resp. flat) deformations of \( L \cap X \). The quotients \( N_{L/Y} / N^s, N_{L/Y} / N^\text{ct} \) are \( \mathcal{O}_Z \)-modules.

**Proof.** We define a the secant subsheaf

\[ N^s \subset N_{L/Y} = \text{Hom}(\mathcal{I}_L, \mathcal{O}_L) \]

to consist of those homomorphisms that are compatible with \( \mathcal{I}_X \), i.e. that map the subsheaf \( \mathcal{I}_L \cap \mathcal{I}_X \) to \( \mathcal{I}_X \mathcal{O}_L = \mathcal{I}_X / \mathcal{I}_X \cap \mathcal{I}_L \). Thus the secant sheaf consists of (the triple of horizontal arrows in) all commutative diagrams with exact columns

\[ \begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\mathcal{I}_X \cap \mathcal{I}_L & \to & \mathcal{I}_X / \mathcal{I}_X \cap \mathcal{I}_L \\
\downarrow & & \downarrow \\
\mathcal{I}_L & \to & \mathcal{O}_L \\
\downarrow & & \downarrow \\
\mathcal{I}_L / \mathcal{I}_X \cap \mathcal{I}_L & \to & \mathcal{O}_Z \\
0 & & 0
\end{array} \]
Whereas $N_{L/Y}$ parametrizes local, first-order flat deformations of $L$ in $Y$, the secant subsheaf corresponds to those deformations that induce a flat, i.e. length-preserving, deformation of the intersection scheme $Z$. Thus $H^0(N^a)$ is the Zariski tangent space at $(L,Z)$ to the scheme $D^a$ parametrizing such deformations, while obstructions to such deformations are in $H^1(N^a)$. Formally, $D^a$ can be defined as follows: let $D_1$ be the germ at $[L]$ Hilbert scheme of $Y$(see [13] §4.3), and let $D_2$ be the germ at $[Z]$ of the Hilbert scheme of $X$. Then $D^a \subset D_1 \times D_2$ is the zero-scheme of the natural map

$$\mathcal{I}_1 \to \mathcal{O}_2$$

where $\mathcal{I}_1$ is the universal ideal in $\mathcal{O}_Y$ and $\mathcal{O}_2$ is the universal length-$k$ quotient pulled back from $X^{[k]}$. A priori $D^a$ parametrizes pairs $(Z' \subset L')$ such that $Z'$ has the same length as $Z$ and $Z' \subset L' \cap X$ but by then semicontinuity, $Z' = L' \cap X$.

Assigning the triple of horizontal arrows in (9) to the bottom arrow yields the natural map

$$\text{res}_X : N^g \to N_{Z/X} = \text{Hom}(\mathcal{I}_{Z/X}, \mathcal{O}_Z) = \text{Hom}(\mathcal{I}_L/\mathcal{I}_L \cap \mathcal{I}_X, \mathcal{O}_p/(\mathcal{I}_X + \mathcal{I}_L))$$

which corresponds to the induced deformation of $Z$ as subscheme of $X$. We define the contact subsheaf $N^c \subset N^a$ as the kernel of the composite map

(10)

$$N^a \to N_{Z/X} \to T^1_Z.$$

The associated map $H^0(N^a) \to H^0(T^1_Z)$ is the derivative of the natural map of the deformation space $D^a$ above to the abstract versal deformation of $Z$. Thus, $N^c$ corresponds to the subscheme $D^c \subset D^a$ parametrizing deformations of $L$ which induce a locally trivial deformation of $Z$. This will mainly be of interest when $Z$ has embedding dimension at most 1 or 2. Note that $N^a$ and $N^c$, in any event, contain $\mathcal{I}_Z N_{L/p}$, hence $N/N^a, N/N^c$ are $\mathcal{O}_Z$-modules.

3.2. Curvilinear non-excess case. Our purpose here is to further analyze the secant and contact subsheaves introduced above, in the case of curvilinear schemes. Set

$$m = \dim(Y), c = \text{codim}(X,Y), a = \dim(L), k = \ell(Z), r = \#(\text{supp}(Z)),$$

where $\ell$ denotes length and $\#$ denotes cardinality. Note $c \geq a$. We will prove the following

Proposition 3.2. Notations as in Proposition 3.1, assume further that $Z$ is curvilinear. Then the respective colengths of the secant and contact sheaves are given by

(11)

$$\ell(N_{L/Y}/N^a) = k(c - a),$$

$$\ell(N^a/N^c) = k - r.$$

Proof. We work locally at a point where $Z$ has multiplicity $k_1 > 1$, as the case $k_1 = 1$ is similar and simpler. There, $Z$ is contained in a smooth curve $C \subset X$, which makes contact of order $k_1$ with $L$. We can write $C$ parametrically as

$$t \mapsto (y_1 = t, y_2 = t^{k_1}, y_3 = 0, ..., y_m = 0)$$

where $t$ is a coordinate on $C$, $y_1, ..., y_m$ are coordinates on $Y$, $y_1$ is a coordinate on $L$ and the $y_2$ coordinate is conormal to $L$, i.e. $y_2 = 0$ on $L$. Then we can find another set of local coordinates $x_1, ..., x_m$ on $Y$ so that $L$ is defined by $x_{a+1} = ... = x_m = 0$ and $\mathcal{I}_X$ is generated by

$$g_1 = x_1^{k_1} + x_{a+1} g_2 = x_{a+2}, ..., g_c = x_c.$$
Note that $\mathcal{I}_X \cap \mathcal{I}_L$ is generated by $x_{a+1}, ..., x_c$. Then $N_{L/Y} \otimes \mathcal{O}_Z$ is given locally by
\[
x_i \mapsto x_i + b_i, b_i \in \mathcal{O}_Z, i = a + 1, ..., m.
\]
The subsheaf $N^s \subset N_{L/Y}$ is the unique subsheaf containing $\mathcal{I}_Z N_{L/Y}$ defined by the conditions, locally at every point of $\text{supp}(Z)$,
\[
b_{a+1} = ... = b_c = 0 \in \mathcal{O}_Z
\]
while $N^c \subset N^s$ is the unique subsheaf containing $\mathcal{I}_Z N_{L/Y}$ defined by the additional condition in $\mathcal{O}_Z$:
\[
b_N \equiv 0 \mod x_1^{k_1-1}.
\]
From this explicit description the dimension counts (11) follow.

3.3. Curvilinear excess case. Here notations are as in [3.2] but we wish to consider motions of $L$ which keep just part of the (curvilinear) intersection $Z = X \cap L$.

**Proposition 3.3.** Notations as in Proposition 3.1, assume $L$ is a closed submanifold of a projective variety $P$ and that $Y$ is an analytic open subset of $P$. Assume further that $Z = X \cap L$ is curvilinear, properly of type $(k)$, let $(d.) \leq (k.)$ be a partition of weight $(d.)$, and let $W \subset Z$ be a subscheme properly of type $(d.)$. Then

(i) there exists a sheaf $N^s_{L,W}$ supported on $L$, endowed with a Lie atom structure, and whose formation commutes with localization, which controls flat deformations of the pair $(L \subset Y, W \subset L \times X)$;

(ii) there is a natural map $\gamma : N^s_{L,W} \rightarrow N_{L/Y}$ with finitely supported kernel and cokernel, such that if $H^1(\text{im}(\gamma)) = 0$ then the deformation space controlled by $N^s_{L,W}$ is smooth of its expected dimension, which is $h^0(N_{L/Y}) - (c - a)d$;

(iii) there is a subsheaf and subatom $N^c_{L,W} \subset N^s_{L,W}$ controlling deformations where $W$ deforms locally trivially and if $H^1(\gamma(N^c_{L,W})) = 0$ then the deformation space controlled by $N^c_{L,W}$ is smooth of its expected dimension, which is $h^0(N_{L/Y} - (c - a + 1)d + \#(\text{supp}(W)))$.

**Proof.** Let $D^s_{(d.)}$ as in [3.1] be the space of deformations ($W' \subset L'$) of ($W \subset L$) within $Y$ so that $W' \subset X$. Let $D^c_{(d.)} \subset D^s_{(d.)}$ be the subspace where $W$ deforms locally trivially, i.e. $W'$ is (abstractly) isomorphic to $W$. Locally at a point $p_1 \in W$, deformations of $L$ can be described as above by
\[
x_i \mapsto x_i + b_i, b_i \in \mathcal{O}_L, i = a + 1, ..., m.
\]
Note that local coordinates on $X$ are $x_1, x_{c+1}, ..., x_{m-1}$ Deformations of $W$ in $X$ are described by sections of the normal sheaf $N_{W/X}$, a free $\mathcal{O}_W$-module, or more concretely by
\[
x_1^{k_1} \mapsto x_1^{k_1} + h_1, x_i \mapsto x_i + h_i, i = c + 1, ..., m - 1, h_i \in \sum_{j=0}^{d-1} \mathbb{C}x_j.
\]
The compatibility of the two means (cf. [2.4])
\[
b_j \equiv 0 \mod \mathcal{I}_W, j = a + 1, ..., c;
\]
\[
b_j \equiv h_j \mod \mathcal{I}_W, j = c + 1, ..., m - 1;
\]
\[
x_1^{k_1} + b_m \equiv 0 \mod \mathcal{I}_W + (x_1^{d_1} + h_1)
\]
Identifying $\mathcal{O}_W$ with $V_{d_1}$, as in (2.4) the last condition means exactly that $(h_1, b_m) \in V_{d_1,k_1}$. The corresponding condition on first-order deformations is that $(h_1, b_m)$ should be tangent to $V_{d_1,k_1}$ at 0, i.e. (cf. §2.4 (5))

\[(h_1, b_m) \in T_0 V_{d_1,k_1} = T_{d_1,k_1}\]

where as we have seen, $T_{d_1,k_1}$ is a codimension-$d_1$ subspace of $V_{d_1} \oplus V_{a_1}$. We may use the second condition in (12) to eliminate $h_j, j = c + 1, ..., m - 1$. Note that $(h_1, h_{c+1}, ..., h + m - 1)$ may be identified with and element of $H^0(N_{W/X})$. Thus, the Zariski tangent space $T^s$ to $D_{(d)}$ can be identified with the subspace of $H^0(N_{L/Y}) \oplus H^0(N_{W/X})$ defined by the conditions

\[
\begin{align*}
  b_j &\equiv 0 \mod I_W, j = a + 1, ..., c; \\
  b_j &\equiv h_j \mod I_W, j = c + 1, ..., m - 1; \\
  (h_1, b_m) &\in T_{d_1,k_1}.
\end{align*}
\]

Canonically, as in §2.4 especially (8), letting $i_W : W \to L$ denote the inclusion, we have a canonical surjection of sheaves over $L$

\[
\rho_{L,W} : N_{L/Y} \oplus (i_W)_* N_{W/X} \to (i_W)_* i_W^* N_{L/Y}
\]

We set

\[
N_{L/W}^s = \ker(\rho_{L,W}).
\]

In the language of Lie-theoretic deformation theory ([11], [12]), the sheaf $N_{L/W}^s$ or equivalently the complex (14) (right-shifted by 1), has the structure of semi-simplicial Lie algebra (SELA), in fact one equivalent to a Lie atom, and the deformation theory of the pair $(L, W)$ where $L \subset Y, W \subset X \cap L$, is the deformation theory associated to the SELA structure. Thus, first-order deformations are in $H^0(N_{L/W}^s)$ while obstructions are in $H^1(N_{L/W}^s)$.

Then

\[
T^s = H^0(N_{L/W}^s)
\]

Applying the snake lemma to the diagram

\[
\begin{array}{cccccc}
  0 & \to & N_{L/W}^s & \to & N_{L/Y} \oplus (i_W)_* N_{W/X} & \to & (i_W)_* i_W^* N_{L/Y} \to 0 \\
  & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & \to & A & \to & N_{L/W}^s & \to & N_{L/Y} \to B \to 0
\end{array}
\]

we get an exact sequence

\[
0 \to A \to N_{L/W}^s \to N_{L/Y} \to B \to 0
\]

where $A, B$ are respectively the kernel and cokernel of the map

\[
N_{W/X} \to i_W^* N_{L/Y}.
\]

By an elementary computation from the explicit description above, we get

\[
\ell_{p_i}(A) = \min(d_i, k_i - d_i), B \simeq A \oplus (c - a)\mathcal{O}_W.
\]

Thus

\[
\ell(A) = \sum \min(d_i, k_i - d_i), \ell(B) = (c - a)d + \sum \min(d_i, k_i - d_i).
\]
We will denote the image of $N_{L,W}^s \rightarrow N_{L/Y}$ by $N_0^s$. Its colength is the length of $B$. Because $A$ has finite length, we have an exact sequence

$$0 \rightarrow H^0(A) \rightarrow T^s \rightarrow H^0(N_0^s) \rightarrow 0.$$  

(20)

This implies that if $N_0^s$ is ‘unobstructed’ in the sense that $H^i(N_0^s) = 0, \forall i > 0$, then $H^i(N_{L/Y}) = 0$ too and $T^s$ has its expected dimension, viz. $h^0(N_{L/Y}) - (c - a)d$.

Next, note the natural map

$$\tau : N_{L,W}^s \rightarrow (i_W)_*(T_1^W)$$

which by the above local computations is surjective. Set

$$N_{ct}^s = \ker(\tau)$$

(21)

As the mapping cone of $\tau$, this again has the structure of SELA [12], and the inclusion $N_{ct}^s \rightarrow N_{L,W}^s$ is a SELA homomorphism. Then

$$T^c = H^0(N_{ct}^s).$$

(22)

Thus we have an exact diagram

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & N_{ct}^s \rightarrow N_{L/Y} \oplus (i_W)_*N_{W/X}^s \rightarrow (i_W)_*i_W^*N_{L/Y} \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & N_{L,W}^s \rightarrow N_{L/Y} \oplus (i_W)_*N_{W/X}^s \rightarrow (i_W)_*i_W^*N_{L/Y} \rightarrow 0 \\
\downarrow & & \downarrow \\
(i_W)_*T_1^W = (i_W)_*T_1^W & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

(23)

where $N_{W/X}^s$ denotes the ‘locally trivial deformations’ subsheaf, i.e. the kernel of $N_{W/X} \rightarrow T_1^W$ or equivalently, the image of $T_X \otimes O_W \rightarrow N_{W/X}$. By an easy computation, we have an exact sequence

$$0 \rightarrow H^0(A^{ct}) \rightarrow T^c \rightarrow H^0(N_{ct}^s) \rightarrow 0.$$  

(24)

where $A^{ct} \subset A^s$ has local colength 0 if $\ell_p(A^c) < d_i$ and 1 otherwise, and the same for the colength of $N_{ct}^s \subset N_0^s$. Again it follows that if $H^i(N_0^s) = 0, \forall i > 0$, then $T^c$ has its expected dimension, viz $h^0(N_{L/Y}) - (c - a + 1)d + \#(\text{supp}W)$.

\[ \square \]

4. Secant Rational Curve Theorem

We will use the notation and definitions of [11]

**Theorem 4.1.** Let:

$Y$ be a smooth analytic open subset of an irreducible $m$-dimensional projective variety $P$, $X \subset Y$ a smooth closed $n$-dimensional subvariety, $y \in Y$ a general point;
\[ G \text{ be an analytic open subset of the Hilbert scheme of curves in } \mathbb{P} \text{ parametrizing nonsingular rational curves contained in } Y, L \to G \text{ the tautological family, } L \text{ a fibre of } L \text{ containing } y \text{ and } (k.) \text{ a partition of weight } k. \]

Assume that \( L \) is a proper \((k.)\)-secant to \( X \). Then:

(i) for any partition \((l.)\) refining \((k.)\) with exactly \( r \) nonzero blocks, \( S_{(l.),Y} \) has the expected dimension, namely

\[ m - 3 - K_Y.L - kc + r, c := m - n, \]

and its singularities at \( L \) are the same up to a smooth factor as those of of \( D_{(l.)} \) at \( L \cap X \); in particular, \( S_{k,Y} \) and \( S_{(k.),Y} \) are smooth at \( L \);

(ii) for any partition \((d.) \leq (k.)\) and subscheme \( W \subset L \cap X \) properly of type \((d.)\), \( \tilde{S}_{(d.),Y} \) is smooth near \((L, W)\) and the normalization of \( S_{(d.),Y} \) is smooth at \( L \);

(iii) if \( L \) is general among \((k.)\)-secants to \( X \), then each point in the support of \( L \cap X \) is general on some component of \( X \).

**Remark 4.2.** The number \( (25) \) is the expected dimension of \( S_{(l.),Y} \) because for a proper \((l.)\)-secant, it is the Euler characteristic of the relevant normal sheaf (atom), namely \( N^{ct} \), controlling contact-preserving deformations, while non-proper \((l.)\)-secants are ‘expected’ to be-and in fact are, by the following Corollary- specializations of proper ones.

**Corollary 4.3.** For the general memeber \( L \) of any component of \( S_{(m.),Y} \), \( L \cap X \) is properly of type \((m.)\).

**Proof of Corollary.** Dimension count. More specifically, let \((k.)\) be the unique partition such that \( L \cap X \) is properly of type \((k.)\). Then the partition \((k.)\) is obtained from \((m.)\) by a succession of

- removing a wall, i.e. uniting two blocks, thus preserving total weight while reducing the number of blocks;
- increasing the size of a block, thus increasing total weight and preserving cardinality of support.

Hence if \((m.) \neq (k.)\), then \( \langle \sum m_i \rangle c - \#(\text{blocks of}(m.)) < \langle \sum k_i \rangle c - \#(\text{blocks of}(k.)) \), so \( L \) cannot be general. \( \square \)

**Remark 4.4.** A typical non-algebraic situation covered by the Theorem is where \( Y \) is a tubular (analytic) neighborhood of a smooth complete rational curve in \( \mathbb{P} \), e.g. a projective space, and \( X \) is a union of some - not all- of the branches of an algebraic variety intersected with \( Y \).

**Proof of Theorem.** Let notations be as in the Theorem, and consider the secant and contact sheaves \( N^{ct} \subset N^\alpha \subset N_{L/Y} \) as in Proposition 3.1. Because \( y \in Y \) is general, the family of proper \((k.)\)-secants to \( X \) is filling for \( Y \), and Lemma 4.1 shows that \( N^{ct} \) is generated by global sections at \( y \). Therefore \( N^{ct} \) is generically spanned, hence \( N^{ct}, N^\alpha \) and \( N_{L/Y} \) are all direct sums of line bundles of of nonnegative degree. This implies first that \( S_{k,Y} \) is smooth at \( L \) with tangent space \( H^0(N^\alpha) \), and then that the local classifying morphism \( \kappa \) of \( S_{k,Y} \) to the abstract deformation space \( \text{Def}(z) \) is smooth at \( L \). Because \( S_{(l.),Y} = \kappa^{-1}(D_{(l.)}) \), assertion (i) follows.

Assertion (iii) follows from the fact that \( N^{ct} \) is globally generated.

For (ii) we will similarly use Proposition 3.3 (and its notations) in lieu of 3.1. Our assumptions imply that \( \text{im}(\gamma) \) is a direct sum of line bundles of nonnegative degree, hence again \( H^1(\text{im}(\gamma)) = 0 \) so we conclude as before the smoothness of \( \tilde{S}_{(d.),Y} \) at \((L, W)\) for all possible subschemes \( W \).
Because $W$ is properly of type $(d,)$, the fibre of the projection $\tilde{S}_{(d),Y} \to S_{(d),Y}$ through $(L, W)$ is the reduced point $[W]$, the projection is unramified, hence the normalization is smooth. □

**Remark 4.5.** As the referee points out, the argument above using Lemma 1.4 replaces an inductive argument used in a similar place by Gruson and Peskine in [5].

**Example 4.6.** As an obvious example, one can consider a $(2, d)$ complete intersection $X \subset \mathbb{P}^{n+2}, d \geq 3$. Is has an oversize family of $d$-secants, namely the lines in the quadric, which form a $(2n - 1)$-dimensional family. Of course, this family is not filling for $\mathbb{P}^{n+2}$.

**Example 4.7.** Consider a plane $X = \mathbb{P}^2 \subset \mathbb{P}^3$ and a transverse twisted cubic $L \subset \mathbb{P}^3$. Viewing things in $\mathbb{P}^5$, let $Y$ be a generic quadric containing $X \cup L$. This is easily checked to be smooth. The normal bundle $N_{L/Y}$ is of type $(4, 3, 3)$ and the secant bundle with respect to $X$ is of type $(3, 2, 2)$. The family of trisecant twisted cubics to $X$ in $Y$ is unobstructed of dimension 14.

**Example 4.8.** For the (smooth) projected Veronese $X \subset \mathbb{P}^4$, there is a unique trisecant line $L$ through a general point of $\mathbb{P}^4$ (see [5], §5). Each point of $L \cap X$ is general on $X$.

## 5. Planar fibres

**5.1. Statement.** The main difficulty in extending Theorem 4.1 to higher-dimensional secant flats and projections from (generic) higher-dimensional centers lies with the complexity of the finite schematic intersection $X \cap L$, which a priori is as ill-behaved as any finite scheme. This difficulty is still manageable when $X \cap L$ is locally planar, i.e. has embedding dimension 2 or less (e.g. $L$ itself is 2-dimensional). This is due to Fogarty’s theorem [4] about the smoothness of the Hilbert scheme of a smooth surface, and its consequence, Lemma 5.7 below, which shows that the secant sheaf $N^S$ has the expected colength in $N_L$. Accordingly, we are able to prove Theorem 5.1 below, which extends Theorem 4.1 essentially, for ambient space $\mathbb{P}^N$ and locally planar fibres, such as those which occur upon projection from a generic line $\Lambda$ (i.e. have the form $X \cap L$ where $L$ is a plane containing $\Lambda$). ‘Essentially’ means we are able to control the locus of fibres of given length but not those of given cycle type. Note, as a matter of terminology, that by ‘locus of fibres’ of a map we mean locus of point-images of fibres, a locus in the target of the map.

For a subvariety $X \subset \mathbb{P}^m$ and a linear $\lambda$-plane $\Lambda$ disjoint from $X$, we denote by $X_k^\Lambda \subset \mathbb{P}^{m-\lambda-1}$ the locus of fibres of length $k$ or more of the projection

$$\pi_\Lambda : X \to \mathbb{P}^{m-\lambda-1}.$$  

Thus, $X_k^\Lambda$ is the locus of $(\lambda + 1)$-planes containing $\Lambda$ and meeting $X$ in a scheme of length $k$ or more. We denote by $\pi_\Lambda, X_k^\Lambda$ the analogous objects corresponding to the generic $\Lambda$.

**Theorem 5.1.** Let $X \subset \mathbb{P}^m$ be an irreducible closed subvariety of codimension $c > \lambda \geq 0$. Then $X_k^\Lambda$ is smooth of codimension $k(c - \lambda - 1)$ in $\mathbb{P}^{m-\lambda-1}$, in a neighborhood of any point image of a fibre of length exactly $k$ that is disjoint from the singular locus of $X$ and has embedding dimension 2 or less.

**Remark 5.2.** Note that the smoothness assertion of the Theorem applies to any fibre of length exactly $k$, including non-reduced ones.

**Remark 5.3.** The local planarity hypothesis is of course automatic when $\lambda \leq 1$ with the case $\lambda = 0$ being due already to Gruson-Peskine [5]. Alzati [1] applied the Gruson-Peskine theorem to $k$-normality of codimension-3 subvarieties of $\mathbb{P}^m$. Subsequently, Alzati (pers. comm.) was able to
apply Theorem 5.1 together with the method of [1] to obtain a stronger result on $k$-normality, namely: a smooth, codimension-3 subvariety $X$ of degree $d$ and dimension 3 or more in $\mathbb{P}^m$ is $k$-normal provided

$$k \geq d + 1 + (m - 1)(m - 2)(m - 6)/3.$$  

**Corollary 5.4.** Notations as in Theorem 5.1, assume

$$\lambda < \min(c, c + 2 - n/3).$$  

Then all fibres of $\pi_\lambda$ are planar, and for any $k \geq 1$, the locus of (point fibres of) fibres of $\pi_\lambda$ of length $k$ is smooth of codimension $k(c - \lambda - 1)$ in $\mathbb{P}^{m-\lambda-1}$, in a neighborhood of any fibre of length exactly $k$ that is disjoint from the singular locus of $X$.

**Proof of Corollary.** It suffices to prove the planarity assertion. If projection from $\Lambda$ has a fibre of embedding dimension 3 or more at $x \in X$, then $\dim(\Lambda \cap T_x X) \geq 2$. An elementary dimension count shows that the locus of $\Lambda$ satisfying the latter incidence for some unspecified $x \in X$ is of dimension at most

$$4n - 6 + (\lambda - 2)(n + c - \lambda),$$

which is less than $(\lambda + 1)(n + c - \lambda) = \dim(G(\lambda, n + c))$ provided $\lambda < c + 2 - n/3$.

**Remark 5.5.** Notations as above, a similar dimension count shows that whenever

$$\lambda < \min(c, c + e - n/(e + 1)),$$

any fibre of the generic projection of $X$ from a $\lambda$-plane either has embedding dimension at most $e$ or it meets the singular locus of $X$.

\[\square\]

**Corollary 5.6.** In the situation of Theorem 5.1 or Corollary 5.4, if $X$ is smooth then

(i) for all $\lambda < \min(c, c + 2 - n/3)$, the locus of fibres of $\pi_\lambda$ of length exactly $k$ is smooth of codimension exactly $k(c - \lambda - 1)$ in $\mathbb{P}^{m-\lambda-1}$ or is empty;

(ii) if $\lambda = 1$, then the locus $X^1_k \subset \mathbb{P}^{m-2}$ of fibres of $\pi_1$ of length $k$ or more is of codimension $k(c - 2)$ or more, or is empty;

(iii) if $\lambda = 1$ and $k(c - 2) > m - 2$, then $X^1_k$ is empty;

(iv) if $n \leq 6$ and $\lambda < c$, then $X^c_k$ is of codimension $k(c - \lambda - 1)$ in $\mathbb{P}^{m-\lambda-1}$ and smooth outside of $X^c_{k+1}$.

**Proof.** (i) Trivial from the foregoing Corollary.

(ii) Planarity is automatic and the fibres of greater length have greater codimension.

(iii) The codimension is too big.

(iv) Immediate from (26).  

\[\square\]

5.2. **Proof of Theorem 5.1.** Our point of view again is that a fibre of $\pi_\lambda$ comes from a (line, plane) pair $(\Lambda, L)$ which is so mobile that $\Lambda$ is a generic $\mathbb{P}^1$ in $\mathbb{P}^n$ (such a pair may be said to be ‘infinitesimally filling with respect to $\Lambda$’). The proof of Theorem 5.1 is a simple consequence of the two lemmas that follow. Each of them is stated in somewhat greater generality than is required. The hard part is dealing with the local complexity of the finite scheme $L \cap X$. 

15
Lemma 5.7. Let $X \subset P$ be a locally closed embedding of smooth varieties of respective dimensions $v - c, v$. Let $L$ be a smooth closed subvariety of dimension $\lambda + 1$ in $P$ meeting $X$ in a scheme $Z$ of finite length $k$ and embedding dimension 2 or less, so that $X$ is closed in a neighborhood of $L$. Let $N^s \subset N_{L/P}$ be the secant subsheaf, parametrizing deformations of $L$ preserving the length of $X \cap L$ (cf. §3.7). Then the colength of $N^s$ in $N_{L/P}$ is $k(c - \lambda - 1)$.

Proof. Working locally at a point $z$ of $Z$, is will suffice to prove that the local colength of $N^s$ is $\ell_z(Z)(c - 2)$. Now $N^s$ is the kernel of the natural map

$$\text{Hom}(\mathcal{I}_L, \mathcal{O}_L) \rightarrow \text{Hom}(\mathcal{I}_L \cap I_X, \mathcal{O}_Z)$$

which factors through the surjection $\text{Hom}(\mathcal{I}_L, \mathcal{O}_L) \rightarrow \text{Hom}(\mathcal{I}_L, \mathcal{O}_Z)$. The latter clearly has length $\ell_z(Z)(v - \lambda - 1), v = \dim(P)$. Now we have an exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{I}_L/(\mathcal{I}_L \cap I_X), \mathcal{O}_Z) \rightarrow \text{Hom}(\mathcal{I}_L, \mathcal{O}_Z) \rightarrow \text{Hom}(\mathcal{I}_L \cap I_X, \mathcal{O}_Z)$$

and

$$\text{Hom}(\mathcal{I}_L/(\mathcal{I}_L \cap I_X), \mathcal{O}_Z) = \text{Hom}(\mathcal{I}_Z/I_X, \mathcal{O}_Z)$$

Now the latter has length $\ell_z(Z) \dim(X)$ due to the fact that $Z$ is a smooth point of the Hilbert scheme of $X$, since it has embedding dimension 2 (an easy consequence of Fogarty's theorem on smoothness of the Hilbert scheme of a smooth surface, see [4] or [6], Cor. 3.4). Therefore locally the image of

$$\text{Hom}(\mathcal{I}_L, \mathcal{O}_Z) \rightarrow \text{Hom}(\mathcal{I}_L \cap I_X, \mathcal{O}_Z)$$

hence also that of

$$\text{Hom}(\mathcal{I}_L, \mathcal{O}_L) \rightarrow \text{Hom}(\mathcal{I}_L \cap I_X, \mathcal{O}_Z)$$

is of length $\ell_z(Z)(v - \lambda - 1 - \dim(X)) = \ell_z(Z)(c - \lambda - 1)$. $\square$

In the following Lemma we consider a sheaf $M$ on $L$ (notations as above) such that $M \otimes \mathcal{O}_\Lambda$ admits a natural map from $N_{\Lambda/P^n}$. We will say that 'H$^0(M)$ moves infinitesimally with $\Lambda$' if, given any $v_1 \in H^0(M)$, there exists $v_2 \in H^0(N_{\Lambda/P^n})$ having the same image in $H^0(M \otimes \mathcal{O}_\Lambda)$.

Lemma 5.8. Let $X \subset \mathbb{P}^v$ be a smooth codimension-$c$ locally closed subvariety, and let $\Lambda \subset \mathbb{P}^v$ be a generic linear $\mathbb{P}^{\lambda}$. Suppose $L$ is an arbitrary linear $\mathbb{P}^{\lambda+1}$ in $\mathbb{P}^N$ containing $\Lambda$ and meeting $X$ in a scheme $Z$ of finite length $k$. Let $M \subset N_{L/P^v}$ be any coherent subsheaf of $L$-modules such that the support of $N_{L/P^v}/M$ is disjoint from $\Lambda$ and such that $H^0(M)$ deforms infinitesimally with $\Lambda$. Then we have $H^1(M(t)) = 0$ for all $t \geq -1$.

Proof. By assumption, $\Lambda$ moves as a generic $\mathbb{P}^{\lambda}$ of $\mathbb{P}^v$ as $X$ is fixed, (in particular $\Lambda \cap X = \emptyset$). It follows first that $M \otimes \mathcal{O}_\Lambda = (v - \lambda - 1)\mathcal{O}_\Lambda(1)$. Our assumption that $H^0(M)$ deforms infinitesimally with $\Lambda$ means that in the diagram

$$\begin{array}{ccc}
H^0(N_{\Lambda/P^v}) & \rightarrow & H^0(M) \\
\downarrow & & \rightarrow \\
H^0(M) & \rightarrow & H^0(M \otimes \mathcal{O}_\Lambda)
\end{array}$$

the image of the vertical map is contained in that of the horizontal map. Since the vertical map is obviously surjective, we conclude that the natural map

$$H^0(M) \rightarrow H^0(M \otimes \mathcal{O}_\Lambda)$$

is obviously surjective, we conclude that the natural map

$$H^0(M) \rightarrow H^0(M \otimes \mathcal{O}_\Lambda)$$
is surjective. This implies that $H^1(M(-1)) \to H^1(M)$ is injective (in fact, an isomorphism, as $H^1(M \otimes O_A) = 0$), and that $H^0(M(s)) \to H^0(M(s) \otimes O_A)$ is surjective for all $s \geq 0$, because the latter group is generated by $H^0(M \otimes O_A) \otimes H^0(O_A(s))$. Consequently, $H^1(M(s-1)) \to H^1(M(s))$ is injective for all $s \geq 0$, hence $H^1(M(s-1)) = 0$ for all $s \geq 0$. Actually, because $M \otimes O_A = (v - \lambda - 1)O_A(1)$, it follows similarly that $H^1(M(s-1)) = 0, \forall i > 0, s \geq 0$. □

Now Theorem 5.1 follows by putting together Lemma 5.7 for $L$ an $(\lambda + 1)$-plane in $P = \mathbb{P}^m = \mathbb{P}^n$ and Lemma 5.8 for $M = N^s$. Note $\Lambda$ being generic implies that the family $(L, \Lambda)$ is filling, hence by Remark 1.6, infinitesimally filling, with respect to $\Lambda$, which means precisely that $H^0(N^s)$ deforms infinitesimally with $\Lambda$, as in the hypothesis of Lemma 5.8.

Remark 5.9. It seems likely that a general locally planar secant $L \cap X$ of given length $k$ is actually reduced and perhaps even in uniform position or, if $X$ is non-degenerate, in general position within $L$. We have no proof of this. However in the curvilinear case, $L \cap X$ is reduced, as proved in the next section.

6. THE CURVILINEAR CASE

In this section we will prove a (full, with contact conditions) generic projection theorem for centers of any dimension and fibres which are curvilinear. For $X, \Lambda$ as above, we denote by $X^A_{(k)}$ the locus of (point images of) curvilinear fibres of type $(k)$ disjoint from the singular locus of $X$, and let $X^A_{(\lambda)}$ denote $X^A_{(k)}$ for $\Lambda$ generic of dimension $\lambda$.

Theorem 6.1. Let $X \subset \mathbb{P}^m$ be an irreducible closed subvariety of codimension $c \geq 2$. Let

$$\pi_{\lambda} : X \to \mathbb{P}^{m-\lambda-1}$$

be the projection from a general $\lambda$-plane $\Lambda \subset \mathbb{P}^m, \lambda < c$. Then for any $k_1 + \ldots + k_r = k \geq 1$, the Thom-Boardman locus $X^A_{(k)} \subset \mathbb{P}^{m-\lambda-1}$ of fibres of $\pi_{\lambda}$ of cycle type $(k_1, \ldots, k_r)$ is smooth of codimension $k(c - \lambda) - r$ in $\mathbb{P}^{m-\lambda-1}$, in a neighborhood of any curvilinear fibre properly of cycle type $(k_1, \ldots, k_r)$ that is disjoint from the singular locus of $X$. Moreover, for any partition $(l)$ refining $(k)$, the singularities of $X^A_{(l)}$ at $L$ are the same, up to a smooth factor, as those of the locus of divisors of type $(l)$ on $\mathbb{P}^1$ at a divisor properly of type $(k)$, and the normalization of $X^A_{(l)}$ is smooth.

Remark 6.2. In case $\lambda = 0$, curvilinearity is automatic and Theorem 6.1 reduces to the Grushon-Peskine theorem.

Proof of Theorem. The proof is essentially identical to that of Theorem 4.1, using Lemma 5.8 to substitute for the appropriate secant and contact sheaves being direct sums of line bundles of nonnegative degrees. □

Corollary 6.3. Notations as in Theorem 6.1 assume

$$\lambda < \min(c, c + 1 - n/2).$$

Then for any $k_1 + \ldots + k_r = k \geq 1$, the locus of fibres of $\pi_{\lambda}$ of proper cycle type $(k_1, \ldots, k_r)$ is smooth of codimension $k(c - \lambda) - r$ in $\mathbb{P}^{N-\lambda-1}$, in a neighborhood of any fibre properly of cycle type $(k_1, \ldots, k_r)$ that is disjoint from the singular locus of $X$; also, all these fibres are curvilinear.
Proof. Under assumption (28), Remark 5.5 shows that all fibres of \( \pi_\lambda \) disjoint from the singular locus of \( X \) are curvilinear, so Theorem 6.1 applies. \( \square \)

In particular, for \( X \) of dimension up to 4, all generic projections that are morphisms, i.e. with \( \lambda < c \), have only curvilinear fibres and good Thom-Boardman loci.

Example 6.4. If \( X \) is a smooth 3-fold in \( \mathbb{P}^7 \), its generic projection to \( \mathbb{P}^5 \) has a smooth double curve and no triple points. If \( X \) is a smooth 3-fold in \( \mathbb{P}^6 \), its generic projection to \( \mathbb{P}^4 \) has a double surface, which is smooth outside of a triple curve, which itself is smooth outside of a finite number of 4-fold points. In fact, these points are ordinary, i.e. come from reduced 4-tuples on \( X \). This follows from Corollary 6.3.

Remark 6.5. It seems likely that the ‘least common multiple’ of Theorems 6.1 and 4.1, as well as the corresponding generalization of Theorem 5.1 hold (curvilinear or locally planar intersections \( L \cap X \) in an arbitrary ambient space, \( L \) isomorphic to a projective space). The precise formulation and proof will have to await another occasion.

REFERENCES

1. A. Alzati, A new Castelnuovo bound for codimension three subvarieties, Arch. Math. 98 (2012), 219–227.
2. A. Alzati and G. Ottaviani, The theorem of Mather on generic projections in the setting of algebraic geometry, Man. math. 74 (1992), 391–412.
3. R. Beheshti and D. Eisenbud, Fibers of generic projections, Comp. math 146 (2010), 435–456.
4. J. Fogarty, Algebraic families on an algebraic surface, Amer. J. Math. 90 (1968), 511–521.
5. L. Gruson and C. Peskine, On the smooth locus of aligned Hilbert schemes. The k-secant lemma and the general projection theorem, Duke math. J. (2013), \texttt{arxiv.org/1010.2399}.
6. M. Lehn, Lectures on Hilbert schemes, CRM notes, Centre de Recherches Mathématiques, Montreal, 2004.
7. S. Lichtenbaum and M. Schlessinger, The cotangent complex of a morphism, Trans. AMS 128 (1967), 41–70.
8. J. Mather, Generic projections, Ann. Math. 98 (1973), 226–245.
9. Z. Ran, Structure of the cycle map for Hilbert schemes of families of nodal curves, 1–34, \texttt{http://arXiv.org/0903.3693}.
10. Z. Ran, The (dimension+2)-secant lemma, Invent. math 106 (1991), 65–71.
11. Z. Ran, Lie atoms and their deformation theory, Geometric and Functional Analysis 18 (2008), 184–221.
12. Z. Ran, Jacobi-Bernoulli cohomology and deformations of schemes and maps, C. Europ. J. Math. 10 (2012), 1541–1591.
13. E. Sernesi, Deformations of algebraic schemes, Grundl. d. math. Wiss., vol. 334, Springer International, Berlin, Heidelberg, 2006.
14. F.L. Zak, Tangents and secants of algebraic varieties, Transl. math. monog., vol. 127, Amer. math. soc., 1993.

Ziv Ran
University of California Mathematics Department
Big Springs Surge Facility
Riverside CA 92521
ziv.ran@ucr.edu

18