Planes of the form $b(X, Y)Z^n - a(X, Y)$ over a DVR

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Abstract
In this paper we extend an epimorphism theorem of D. Wright to the case of discrete valuation rings. We will show that if $(R, t)$ is a discrete valuation ring, $n \geq 2$ is an integer not divisible by the characteristic of the residue field $R/tR$, and $g \in R[X, Y, Z]$ is a polynomial of the form $g = b(X, Y)Z^n - a(X, Y)$ such that $R[X, Y, Z]/(g)$ is a polynomial algebra in two variables, then $g$ and $Z$ form a pair of variables in $R[X, Y, Z]$. We will also show that the result holds over any Noetherian domain containing $\mathbb{Q}$.

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1. Introduction

For a commutative ring $R$ with unity, let $R^{[n]}$ denote the polynomial ring in $n$ variables. An important question in affine algebraic geometry is the following epimorphism problem:

Question 1. Let $K$ be a field of characteristic 0. Let $g \in K[X, Y, Z] (= K[3])$ be such that $K[X, Y, Z]/(g) = K[2]$. Is then $K[X, Y, Z] = K[g][2]$?

While the problem is open in general, a few special cases have been investigated by Sathaye, Russell and Wright in [Sat76], [Rus76], [Wri78] and [RS79]; in some of these cases, Question 1 has an affirmative answer even when $K$ is a field of positive characteristic. In particular, they considered polynomials of the form $b(X, Y)Z^n - a(X, Y)$ and obtained affirmative answers when

1. $n = 1$, $K$ a field of characteristic 0 (A. Sathaye, [Sat76]),
2. $n = 1$, $K$ a field of any characteristic (P. Russell, [Rus76]).
In this paper we shall first show (see Theorem 4.5) that the above result (3) of D. Wright holds even when $K$ is not necessarily algebraically closed.

We now consider the corresponding question over a discrete valuation ring (to be abbreviated henceforth as DVR).

**Question 2.** Let $(R, t)$ be a DVR containing $\mathbb{Q}$ and $g \in R[X, Y, Z](= R^3)$ be such that $R[X, Y, Z]/(g) = R^2$. Is then $R[X, Y, Z] = R[g]^2$?

As shown by Bhatwadekar-Dutta in ([BD94b], section 4), this problem is closely related to the problem of $A^2$-fibration over a regular two-dimensional affine spot over a field of characteristic zero. Hence, one could explore Question 2 at least for polynomials like $g = b(X, Y)Z^n - a(X, Y)$ for which the corresponding Question 1 has been settled. For such polynomials, in view of the corresponding results over fields, one could extend the investigation of Question 2 even to the positive characteristic case.

The first investigation in this direction was made by Bhatwadekar-Dutta in ([BD94a]). They showed ([BD94a], Theorem 3.5) that Question 2 has an affirmative answer (in any characteristic) when $g = b(X, Y)Z^n - a(X, Y)$ with $t \nmid b(X, Y)$, thereby partially generalizing A. Sathaye’s theorem on linear planes over a field ([Sat76]).

The main aim of this paper is to show that Question 2 has an affirmative answer for polynomials of the form $g = bZ^n - a$ where $a, b \in R[X, Y]$ with $b \neq 0$ and $n$ is an integer $\geq 2$ such that $n$ is not divisible by the characteristic of $K$. Suppose that $R[X, Y, Z]/(g) = R^2$. Then $R[X, Y, Z] = R[g, Z][1]$, $R[X, Y] = R[a][1]$ and $b \in R[X_0]$ where $X_0 \in R[X, Y]$ and $K[X, Y] = K[X_0, a]$.

The proof of Bhatwadekar-Dutta’s theorem on linear planes over a DVR is highly technical. However, in the case of planes of the form $bZ^n - a$ with $n \geq 2$, the proof turns out to be much simpler due to the fact that $g$ is a variable along with $Z$.

Using theorems on residual variables of Bhatwadekar-Dutta ([BD93]), one can also see that the result for $n \geq 2$ holds over any Noetherian domain containing $\mathbb{Q}$. We shall prove (see Theorem 6.2):
Theorem B. Let \( R \) be a Noetherian domain containing \( \mathbb{Q} \). Let \( g \in R[X, Y, Z](= R^{[3]} \) be of the form \( g = bZ^n - a \) where \( a, b \in R[X, Y] \) and \( n \) is an integer \( \geq 2 \). Suppose that \( R[X, Y, Z]/(g) = R^{[2]} \). Then \( R[X, Y, Z] = R[g, Z][1] \) and \( R[X, Y] = R[a][1] \).

In fact Theorem 2.2 will show that the above result also holds over any Noetherian seminormal domain containing a field of characteristic \( p \geq 0 \), if \( p \nmid n \).

In section 2, we state some results which will be used subsequently; in section 3, we review the case \( n = 1 \); in sections 4 and 5 we prove our main results over a field and DVR respectively; and in section 6, we prove our main result for rings containing a field.

2. Preliminaries

Throughout this paper all rings will be commutative with unity. For a ring \( R \), we shall use the notation \( A = R^{[n]} \) to mean that \( A \) is isomorphic, as an \( R \)-algebra, to a polynomial ring in \( n \) variables over \( R \); the symbol \( R^* \) will denote the group of units of \( R \). For a prime ideal \( P \) of \( R \), \( k(P) \) will denote the residue field \( R_P/PR_P \). An integral domain \( R \) with field of fractions \( K \) is called seminormal if it satisfies the condition: an element \( a \in K \) will belong to \( R \) if \( a^2, a^3 \in R \).

We now state some results which will be used in our proofs. First we state the result of D. Wright ([Wri78], pg. 95) which we will generalize in sections 4–6.

Theorem 2.1. Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \). Let \( g \in k[X, Y, Z](= k^{[3]} \) be of the form \( bZ^n - a \) with \( b \neq 0 \) and \( n \) is an integer \( \geq 2 \) not divisible by \( p \). Suppose that \( k[X, Y, Z]/(g) = k^{[2]} \). Then there exist variables \( \tilde{X}, \tilde{Y} \) in \( k[X, Y] \) such that \( a = \tilde{Y}, b \in k[\tilde{X}] \) and \( k[X, Y, Z] = k[\tilde{X}, g, Z] \).

We now state a version of the Automorphism Theorem of Jung and van der Kulk ([Jum42] and [vdK53]) as presented in ([Wri78], Appendix, Theorems 2 and 3).

Theorem 2.2. Let \( k \) be a field and \( A = k[U, V](= k^{[2]} \). Let \( GA_2(k) \) denote the group of \( k \)-automorphisms of \( A \), \( AF_2(k) \) the subgroup of \( GA_2(k) \) defined by \( AF_2(k) = \{(U, V) \mapsto (\alpha_1U + \beta_1V + \gamma_1, \alpha_2U + \beta_2V + \gamma_2)| \alpha_i, \beta_i, \gamma_i \in k \text{ and } \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0 \} \), \( E_2(k) \) the subgroup of \( GA_2(k) \) defined by \( E_2(k) = \{(U, V) \mapsto (\alpha U + h(V), \beta V + \gamma)| \alpha, \beta \in k^*, \gamma \in k \text{ and } h(V) \in k[V] \} \) and \( BF_2(k) = AF_2(k) \cap E_2(k) \). Then \( GA_2(k) = AF_2(k) * BF_2(k) \). Moreover, if
\( \sigma \in GA_2(k) \) is of finite order, then there exists \( \tau \in GA_2(k) \) such that either \( \tau \sigma \tau^{-1} \in Af_2(k) \) or \( \tau \sigma \tau^{-1} \in E_2(k) \).

Now we state a result of A. Sathaye ([Sat76], Corollary 1) which we will use to prove Lemma 1.2.

**Theorem 2.3.** Let \( L|k \) be a separable field extension. Assume that there exist \( h \in k[X,Y] \) and \( f_i \in L[X,Y] \), \( 1 \leq i \leq s \), such that

1. \( L[X,Y]/(f_i) = L^{[1]} \) for each \( i \).
2. \((f_i, f_j)L[X,Y] = L[X,Y] \) for \( i \neq j \).
3. \( h = \prod_{i=1}^n f_i^{r_i}, \ r_i > 0 \).

Then there exist \( f \in k[X,Y], \lambda_i \in L^* \) and \( \mu_i \in L \) such that \( f_i = \lambda_i f + \mu_i \) for each \( i \), \( 1 \leq i \leq s \).

We will also use the following special case of the result ([Dut00], Theorem 7).

**Theorem 2.4.** Let \( k \) be a field, \( L \) a separable field extension of \( k \), \( A \) a UFD containing \( k \) and \( B \) an \( A \)-algebra such that \( B \otimes_k L = (A \otimes_k L)^{[1]} \). Then \( B = A^{[1]} \).

We will use the following version of a cancellation theorem due to Abhyankar-Eakin-Heinzer ([AEH72], Theorem 3.3).

**Theorem 2.5.** Let \( A \) be an affine domain over a field \( k \) such that \( k \) is algebraically closed in \( A \) and \( \text{tr.deg}_k(A) = 1 \). Suppose that \( B \) is another \( k \)-algebra such that \( A^{[n]} = B^{[n]} \) for some \( n \geq 1 \). Then either \( B = A \) or \( B \cong A = k^{[1]} \).

We now state a version of the Russell-Sathaye criterion ([RS79], Theorem 2.3.1) for a ring to be a polynomial algebra over a subring (see [BD94a], Theorem 2.6).

**Theorem 2.6.** Let \( R \subset A \) be integral domains with \( A \) being finitely generated over \( R \). Suppose that there exist primes \( p_1, p_2, \ldots, p_n \) in \( R \) such that for each \( i \), \( 1 \leq i \leq n \),

1. \( p_i \) remains prime in \( A \),
2. \( p_i A \cap R = p_i R \),
3. \( A[p_1, p_2, \ldots, p_n ] = R^{[1]} \) and
4. \( R/p_i R \) is algebraically closed in \( A/p_i A \).
Then $A = R^{[1]}$.

The following result from ([BD94a], 2.5) will enable us to apply Theorem 2.6.

**Lemma 2.7.** Let $R$ be an integral domain and $F \in R[X,Y](= R^{[2]})$ be such that $R[X,Y]/(F) = R^{[1]}$. Then $R[F]$ is algebraically closed in $R[X,Y]$.

Finally, we state a result on residual variables which will be our main tool to prove Theorem B. It comes as a direct consequence of Theorem 3.1, Theorem 3.2 and Remark 3.4 in [BD93].

**Theorem 2.8.** Let $R$ be a Noetherian domain such that either $R$ contains $\mathbb{Q}$ or $R$ is seminormal, $A$ be a polynomial algebra in $n$ variables over $R$ and $W_1, W_2, \ldots, W_{n-1} \in A$. Then the following are equivalent:

1. $A = R[W_1, W_2, \ldots, W_{n-1}]^{[1]}$.
2. $A \otimes_R k(P) = (R[W_1, W_2, \ldots, W_{n-1}] \otimes_R k(P))^{[1]}$ for every prime ideal $P$ of $R$.

3. Planes of the form $bZ - a$

We recall below the earlier result on linear planes over a DVR ([BD94a], Theorem 3.5).

**Theorem 3.1.** Let $(R, t)$ be a DVR and $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ - a$ where $a, b \in R[X,Y]$ and $b \notin tR[X,Y]$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R^{[g]}$.

We now show that the result can be generalized to the case of Dedekind domain in the following form.

**Theorem 3.2.** Let $R$ be a Dedekind domain and $g \in R[X,Y,Z](= R^{[3]})$ be of the form $g = bZ - a$ where $a, b \in R[X,Y]$ and the coefficients of $b$ generate the unit ideal of $R$. Suppose that $B = R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R^{[g]}$.

**Proof.** By Theorem 3.1, $R_m[X,Y,Z] = R_m[g]^{[2]}$ for each maximal ideal $m$ of $R$. Hence, by ([BCW77]), it follows that $R[X,Y,Z]$ is $R[g]$-isomorphic to the symmetric algebra $\text{Sym}_{R[g]}(P)$ for some finitely generated projective $R[g]$-module $P$ of rank two. Thus it is enough to show that $P$ is a free $R[g]$-module. Since $R[g]$ is a retract of $R[X,Y,Z]$, it is enough to show that $P \otimes_{R[g]} R[X,Y,Z]$ is a free $R[X,Y,Z]$-module. Note that since $R[X,Y,Z] \cong \text{Sym}_{R[g]}(P)$, we have $\Omega_{R[g]}(R[X,Y,Z]) = P \otimes_{R[g]} R[X,Y,Z]$. 

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Thus the proof will be complete if we show that the projective $R[X,Y,Z]$-module $Ω_R([R[X,Y,Z]])$ is actually free.

Now consider the exact sequence:

$$Ω_R([R[g]]) ⊗_{R[g]} ([R[X,Y,Z]]) → Ω_R([R[X,Y,Z]]) → Ω_{R[g]}([R[X,Y,Z]]) → 0.$$ 

Let $g_X$, $g_Y$ and $g_Z$ denote the partial derivatives of $g$ with respect to $X$, $Y$ and $Z$ respectively. Now note that $([g_X,g_Y,g_Z])R[X,Y,Z] = R[X,Y,Z]$. Since $dim R = 1$, by Suslin’s theorem (Sus77, 2.6), the unimodular row $[g_X,g_Y,g_Z]$ can be completed to an invertible matrix. Since $Ω_R([R[X,Y,Z]])$ is a free $R[X,Y,Z]$-module of rank three with basis $dX$, $dY$ and $dZ$, and since $Im (θ)$ is generated by $g_XdX + g_YdY + g_ZdZ$, it now follows that $Ω_{R[g]}([R[X,Y,Z]])(= Ω_R([R[X,Y,Z]])/Im (θ))$ is a free $R[X,Y,Z]$ module of rank two. This completes the proof.

**Remark 3.3.** Let $(R,t)$ be a DVR containing $\mathbb{Q}$ and let $g = bZ - a$ where $b = ty^2$ and $a = -Y - tY(X + X^2) - t^2X$. Then $R[X,Y,Z]/(g) = R[2]$ (see [BD94], Example 4.13). In this example, $t \mid b$; and it is not yet known whether $R[X,Y,Z] = R[2]$.

4. Planes of the form $bZ^n - a$ over a field

In this section we will show that Wright’s arguments in ([Wri78]) can be modified to show that his result (Theorem 2.1) can be extended over any field. We first prove a few auxiliary results (Lemmas 4.1 and 4.2), then consider the case when the field $k$ contains all $n$th roots of unity (Proposition 4.3) and finally show that Theorem 2.1 holds over any field (Theorem 4.5). We first record a result on $Aut_k(k^{[3]}$).

**Lemma 4.1.** Let $k$ be a field of characteristic $p ≥ 0$ and $σ$ a $k$-automorphism of $B = k^{[2]}$ of order $n$ such that $p ∤ n$. Suppose that $k$ contains all the $n$th roots of unity. Then there exist elements $U, V ∈ B$ and $α, β ∈ k^*$ such that $B = k[U, V]$, $σ(U) = αU$ and $σ(V) = βV$, where $α^n = β^n = 1$.

**Proof.** By Theorem 2.2 one can choose coordinates $U'$, $V'$ of $B$ such that either $σ ∈ E_2(k)$ or $σ ∈ A_{f_2}(k)$.

Case: $σ ∈ E_2(k)$.

In this case $σ(U') = αU' + μ$ and $σ(V') = βV' + f_1(U')$, where $α, β ∈ k^*$, $μ ∈ k$ and $f_1(U') ∈ k[U']$. Since $σ$ is of order $n$, we have $α^n = β^n = 1$. Note that if $α = 1$, then $U' = σ^n(U') = U' + nμ$ and hence $μ = 0$, as $p ∤ n$.

Set

$$U := \begin{cases} U' & \text{if } α = 1, \\ U' + \frac{μ}{α - 1} & \text{if } α ≠ 1. \end{cases}$$
Then $k[U', V'] = k[U, V']$, $\sigma(U) = \alpha U$ and $\sigma(V') = \beta V' + f(U)$ for some $f(U) \in k[U]$. We will now show that we can choose $g(U) \in k[U]$ such that $\sigma(V' + g(U)) = \beta(V' + g(U))$. Let $f(U) = \sum_{i=0}^{r} a_i U^i$.

First we show that for any $i$, $1 \leq i \leq r$, if $a_i \neq 0$, then $\alpha^i \neq \beta$. Suppose $\beta = \alpha^i$. Now, from the relation $V' = \sigma^n(V')$, we get

$$\beta^{n-1} f(U) + \beta^{n-2} f(\alpha U) + \cdots + f(\alpha^{n-1} U) = 0,$$

which implies that

$$\beta^{n-1} a_i + \beta^{n-2} \alpha^i a_i + \beta^{n-3} \alpha^{2i} a_i + \cdots + \alpha^{(n-1)i} a_i = 0,$$

i.e., $n \beta^{n-1} a_i = 0$, and hence $a_i = 0$ (as $p \nmid n$ and $\beta \neq 0$). Thus $\alpha^i \neq \beta$ if $a_i \neq 0$.

Now, for each $i$, $1 \leq i \leq r$, we define $b_i$ as follows:

$$b_i = \begin{cases} 
0 & \text{if } a_i = 0, \\
\frac{a_i}{(\beta - \alpha^i)} & \text{if } a_i \neq 0.
\end{cases}$$

Let $g(U) = \sum_{i=0}^{r} b_i U^i$ and set

$$V := V' + g(U).$$

Then $\sigma(V) = \beta V$. Thus $k[U', V'] = k[U, V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$.

**Case:** $\sigma \in A_{F_2}(k)$.

In this case $\sigma(U') = \alpha_1 U' + \beta_1 V' + \gamma_1$ and $\sigma(V') = \alpha_2 U' + \beta_2 V' + \gamma_2$ for some $\alpha_1, \beta_1, \gamma_1 \in k$ ($i = 1, 2$) with $\alpha_1 \beta_2 \neq \beta_1 \alpha_2$. Choose $\lambda \in k$ such that $(\alpha_1 - \lambda)(\beta_2 - \lambda) - \alpha_2 \beta_1 = 0$. Then $\lambda$ is an eigen value of the linear transformation $(X, Y) \mapsto (\alpha_1 X + \alpha_2 Y, \beta_1 X + \beta_2 Y)$ of $k^2$. Let $(\nu_1, \nu_2) \in k^2$, not both zero, be an eigen vector corresponding to the eigen value $\lambda$. Then we have

$$\alpha_1 \nu_1 + \alpha_2 \nu_2 = \lambda \nu_1,$$

$$\beta_1 \nu_1 + \beta_2 \nu_2 = \lambda \nu_2.$$

Therefore, $\sigma(\nu_1 U' + \nu_2 V') = \lambda(\nu_1 U' + \nu_2 V') + \mu$ where $\mu = \nu_1 \gamma_1 + \nu_2 \gamma_2$. Since $\sigma$ is of order $n$, we have $\lambda^n = 1$ and hence $\lambda \in k^*$. Thus we may choose $\nu_1, \nu_2 \in k$. Therefore, setting $U := \nu_1 U' + \nu_2 V'$, we have $\sigma(U) = \lambda U + \mu$ and hence $\sigma(V') = \kappa V' + h(U)$ for some $\kappa \in k^*$ and $h(U) \in k[U]$. Now, by taking $U$ and $V'$ to be the coordinates for $B$, the problem reduces to the previous case: $\sigma \in E_2(k)$.

Thus in both the cases we get $U, V \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U, V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$. This completes the proof. \qed
We now record a consequence of Sathaye’s result (Theorem 2.3).

**Lemma 4.2.** Let $k$ be a field, $B = k^{[2]}$ and $b \in B\backslash k$. Suppose that there exist a separable algebraic extension $E_{|k}$ and an element $X' \in B \otimes_k E$ such that $B \otimes_k E = E[X']$ and $b \in E[X']$. Then there exists $X \in B$ such that $b \in k[X]$, $B = k[X]^{[1]}$ and $E[X'] = E[X]$.

**Proof.** Without loss of generality, we assume $E_{|k}$ to be a finite Galois extension. Let $B = k[X_1, Y_1]$. Then $B \otimes_k E = E[X_1, Y_1] = E[X']$. Let $X' = \phi(X_1, Y_1)$. Interchanging $X_1$ and $Y_1$ if necessary, we may assume that the $X_1$-degree of $\phi(X_1, Y_1)$ is positive. Hence the leading coefficient of $X_1$ in $\phi(X_1, Y_1)$ is a non-zero element $\lambda \in E$ ([Abh77], Proposition 11.12, pg. 85). Let $X'' = X'/\lambda$.

Let $G = \{\sigma_i \mid i = 1, 2, \ldots, m\}$ be the group of $k$-automorphisms of $E_{|k}$. We extend each $\sigma \in G$ to a $B$-automorphism of $B \otimes_k E$. Let $\bar{k}$ be an algebraic closure of $k$ containing $E$ and $b = \prod_{i=1}^s (\lambda_i, X'' + \mu_i)^{n_i}$ be the prime decomposition of $b$ in $\bar{k}[X'']$, where $\lambda_i \in \bar{k}^*$, $\mu_i \in \bar{k}$ and $n_i \in \mathbb{N}$, $1 \leq i \leq s$. Since $\sigma(b) = b$ for each $\sigma \in G$, $b = \prod_{i=1}^s (\sigma(\lambda_i)\sigma(X'') + \sigma(\mu_i))^{n_i}$ is also a prime decomposition of $b$ in $\bar{k}[X'']$. This shows that for each $\sigma \in G$, $\exists \alpha \in \bar{k}$ and $\beta \in \bar{k}$ such that $\sigma(X'') = \alpha X'' + \beta$. Since $X''$ and $\sigma(X'')$ are both monic in $X_1$, it follows that $\alpha = 1$.

Since $X''$ is a variable of $B \otimes_k E$, we have $(B \otimes_k E)/\langle \sigma(X'') \rangle = E^{[1]}$ for each $\sigma \in G$. It is also easy to see that if $\sigma_i(X'') \neq \sigma_j(X'')$ for $\sigma_i, \sigma_j \in G$, then $\sigma_i(X'')$ and $\sigma_j(X'')$ are comaximal in $B \otimes_k \bar{k}$ and hence comaximal in $B \otimes_k E$.

Let $f_1, \ldots, f_t$ be the distinct elements of the set $\{\sigma(X'') | \sigma \in G\}$. Then, for each $i$, $1 \leq i \leq t$, there exists $m_i \in \mathbb{N}$ such that $\prod_{\sigma \in G} \sigma(X'') = \prod_{i=1}^t f_i^{m_i} \in B$, $(B \otimes_k E)/(f_i) = E^{[1]}$, and for $i \neq j$, $f_i$ and $f_j$ are comaximal in $B \otimes_k L$. Since $B = k^{[2]}$, applying Theorem 2.3, we get that for each $\sigma \in G$ there exist $\lambda \in E^*$ and $\mu \in E$ such that $\lambda \sigma(X'') + \mu \in B$. Fix $\sigma \in G$ and let $X = \lambda \sigma(X'') + \mu \in B$. Then $E[X''] = E[\sigma(X'')] = E[X]$ and $b \in E[X] \cap B$. Since $B = k^{[2]}$ and $X \in B$, we have $E[X] \cap B = k[X]$. Hence $b \in k[X] \subseteq B$. Now since $B = k^{[2]}$ and $B \otimes_k E = E[X'']^{[1]} = E[X]^{[1]}$, by Theorem 2.4 we see that $B = k[X]^{[1]}$. By construction, $E[X] = E[X''] = E[X']$. \hfill $\Box$

For convenience, we state below a result which follows from a lemma of A. Sathaye ([Sat76], Lemma 1).

**Lemma 4.3.** Let $k$ be a field and suppose $X'$ is a variable in $k[X_1, X_2, \ldots, X_n](= k^{[n]})$ which is comaximal with $X_1$. Then $X' = \alpha X_1 + \beta$ with $\alpha, \beta \in k, \alpha \neq 0$. 8
Proposition 4.4. Let $k$ be a field of characteristic $p \geq 0$ containing the $n^{th}$ roots of unity and $g \in k[X,Y,Z]$ be of the form $bZ^n - a$ where $a,b \in k[X,Y]$ with $a \neq 0$ and $n$ is an integer $\geq 2$ not divisible by $p$. Suppose that $B := k[X,Y,Z]/(g) = k[\ell]$. Then there exist variables $U,V$ in $B$ such that $V$ is the image of $Z$ in $B$, $b \in k[U]$ and $k[X,Y] = k[U,a] = k[\ell]$.

Proof. Let $\sigma$ be the $k$-automorphism of $B$ induced by the $k$-automorphism $\tilde{\sigma}$ of $k[X,Y,Z]$ defined by $\tilde{\sigma}((X,Y,Z)) = (X,Y,\omega Z)$ where $\omega$ is a primitive $n^{th}$ root of unity. Obviously, $\sigma$ has order $n$.

Since $B = k[\ell]$, by Lemma 4.1 there exist elements $U',V' \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U',V']$, $\sigma(U') = \alpha U''$ and $\sigma(V') = \beta V''$, where $\alpha^n = \beta^n = 1$. Let $\tilde{\sigma}$ be the image of $Z$ in $B$ and $A = k[X,Y][a/b]$. Then $\tilde{\sigma} = a/b$ and $B = A[\tilde{\sigma}] = k[X,Y][\tilde{\sigma}] = A \oplus A \oplus A \oplus \cdots \oplus A$ so that, for any $x \in B$, $\tilde{\sigma}/(x - \sigma(x))$. Thus $\tilde{\sigma}/(1 - \alpha)U'$ and $\tilde{\sigma}/(1 - \beta)V'$. But since $U'$ and $V'$ cannot have common (non-unit) factor and $\tilde{\sigma} \notin k^*$, we have either $\alpha = 1$ or $\beta = 1$. Interchanging $U'$ and $V'$ if necessary, we assume that $\alpha = 1$. Then the ring of invariants of $\sigma$ is $A = k[X,Y][a/b] = k[U',a/b](= k[\ell])$.

Note that $\sigma'$ is a unit multiple of $\tilde{\sigma}$. Thus $B = k[U',\tilde{\sigma}]$. Set $V := \tilde{\sigma}$.

Now we show that we can choose $U$ from $k[X,Y]$ such that $B = k[U,V]$. If $b \in k^*$, then $k[X,Y] = k[X,Y][a/b] = k[U',a/b]$, so that, in this case, we may set $U := U'$. We now consider the case $b \notin k^*$. Let $p_1,p_2,\ldots,p_m$ be the distinct irreducible factors of $b$ in $A(= k[\ell])$, and set $p_i := k[X,Y] \cap p_i A$. Note that for each $i = 1,2,\ldots,m$, both $b$ and $a/b \in k[X,Y] \cap bA \subseteq p_i$. This shows that $bZ^n - a \in k[X,Y,Z] \subseteq p_i[Z]$ which implies $ht p_i > 1$. Thus each $p_i$ is a maximal ideal of $k[X,Y]$. Let $L$ denote an algebraic closure of $k$, $L_i$ be a subfield of $k$ isomorphic to $k[X,Y]/p_i$, and let $L$ be the subfield of $k$ generated by the fields $L_1,L_2,\ldots,L_m$. Then $L$ is an algebraic extension of $k$ and $A/p_i A = (k[X,Y]/p_i)[\zeta_i] = L_i[\zeta_i]$ where $\zeta_i$ is the image of $a/b$ in $A/p_i A$. Since $p_i A \subseteq p_i A$, it follows that $\zeta_i$ is transcendental over $L_i$ and $p_i A$ is a prime ideal of $A$. As $ht p_i A = 1$ and $p_i A \neq 0$, we have $p_i A = p_i A$. This shows that $p_i$ are pairwise comaximal in $A$ and hence in $B$.

Let $g(\zeta_i)$ be the image of $U'$ in $A/p_i A = L_i[\zeta_i]$. Then $U' - g(a/b)$ is divisible by $p_i$ in $A \otimes_k L_i$. But $U' - g(a/b) = U' - g(V^n)$ is a variable in both $A \otimes_k L_i$ and $B \otimes_k L_i$. Hence $U' - g(a/b)$ is a constant multiple of $p_i$. Thus $A \otimes_k L_i = L_i[p_i,a/b]$, $B \otimes_k L_i = L_i[p_i,V]$, and for $i \neq j$, $(p_i,p_j)B \otimes_k L = B \otimes_k L$. Set $U := p_1$. Using Lemma 4.3 we have $p_i = \lambda_i U + \mu_i$ for $\lambda_i \in L^*$ and $\mu_i \in L$. So, we have $b \in L[U]$. This shows that $U$ is integral over $L[X,Y]$ and hence over $k[X,Y]$. As $U \in k[X,Y][a/b]$ and $k[X,Y]$ is a normal domain, we have $U \in k[X,Y]$. Since $L/k$ is faithfully flat, it follows that $B = k[U,V]$ with $U \in k[X,Y], V = \tilde{\sigma}$ and $b \in k[U]$. Now, the argument
in \([\text{Wri78}], \text{pg. 98}\) shows that \(k[X,Y] = k[U,a]\).

\[\text{Theorem 4.5.} \text{ Let } k \text{ be a field of characteristic } p \geq 0 \text{ and } g \in k[X,Y,Z] \text{ be of the form } bZ^n - a \text{ where } a,b \in k[X,Y] \text{ with } b \neq 0 \text{ and } n \text{ is an integer } \geq 2 \text{ not divisible by } p. \text{ Suppose that } B := k[X,Y,Z]/(g) = k[2] \text{ and identify } k[X,Y] \text{ with its image in } B. \text{ Then there exist variables } U,V \text{ in } B \text{ such that } V \text{ is the image of } Z \text{ in } B, U \in k[X,Y], b \in k[U], k[X,Y] = k[U,a] \text{ and } k[X,Y,Z] = k[U,g,Z].\]

\[\text{Proof.} \text{ Let } E \text{ be the field obtained by adjoining all the } n^{th} \text{ roots of unity to } k. \text{ Since } p \nmid n, E \text{ is a Galois extension over } k. \text{ By Proposition 4.3, we get variables } U' \text{ and } V' \text{ of } B \otimes_k E = k[X,Y,Z]/(g) = E[2] \text{ such that } V' \text{ is the image of } Z, b \in E[U'] \text{ and } E[X,Y] = E[U',a]. \text{ As } E[k] \text{ is separable, we have } k[X,Y] = k[a][1] \text{ by Theorem 2.4. If } b \in k[X,Y]\backslash k, \text{ then, by Lemma 4.2, we get } U \in k[X,Y] \text{ such that } k[X,Y] = k[U][1], b \in k[U] \text{ and } E[U] = E[U']. \text{ Since } E[k] \text{ is faithfully flat, } E[U',a] = E[U,a] \text{ and } k[U,a] \subseteq k[X,Y], \text{ we have } k[U,a] = k[X,Y]. \text{ If } b \in k, \text{ then we choose } U \text{ to be any complementary variable of } a \text{ in } k[X,Y]. \]

\[\text{From the relation } k[U,a] = k[X,Y], \text{ we have } k[X,Y,Z] = k[U,a,Z] = k[U,bZ^n - a, Z] = k[U,g,Z].\]

The relation \(k[X,Y,Z] = k[U,g,Z]\) shows that \(B\) is generated by the images of \(U\) and \(Z\). This completes the proof. \(\square\)

\[\text{Remark 4.6.} \text{ Theorem 4.5 does not hold if } p \mid n. \text{ Consider a field } k \text{ of characteristic } p > 0 \text{ and the polynomial } g = Z^s - Y - Y^{sp} \in k[Y,Z] \text{ where } p \nmid s \text{ and } e \geq 2. \text{ Then } k[Y,Z] = k[1] \text{ but } k[Y,Z] \neq k[g][1] \text{ (see [Abh77], Example 9.12, pg. 72). Using a result of E. Hamann ([Ham75], Theorem 2.6), it follows that } k[X,Y,Z] \neq k[g][2] \text{ although } \frac{k[X,Y,Z]}{(g)} = k[2].\]

5. Planes of the form \(bZ^n - a\) over a DVR

In this section we shall prove Theorem A. We first record two results on factorial domains.

\[\text{Lemma 5.1.} \text{ Let } R \text{ be a UFD with field of fractions } K. \text{ Let } U \in R[X,Y] \text{ be such that } K[X,Y] = K[U][1]. \text{ Then } K[U] \cap R[X,Y] \text{ is an inert ring of } R[X,Y] \text{ and } K[U] \cap R[X,Y] = R[W](= R[1]), \text{ where } W \text{ is an element of } R[X,Y] \text{ such that } K[W] = K[U].\]

\[\text{Proof.} \text{ Let } D = K[U] \cap R[X,Y]. \text{ Clearly, } D \text{ is an inert ring of } R[X,Y] \text{ and hence a UFD of transcendence degree one over } R. \text{ Therefore, by ([AEH72], Theorem 4.1), } D = R[W](= R[1]) \text{ for some } W \in R[X,Y]. \text{ Clearly, } K[W] = K[U]. \]
Lemma 5.2. Let $R$ be a UFD of characteristic $p \geq 0$ with field of fractions $K$ and $g \in R[X,Y,Z]$ (i.e., $R^3$) be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ with $b \neq 0$ and $n$ is an integer $\geq 2$ such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^2$. Then

(i) $R[a] = K[a] \cap R[X,Y]$.
(ii) $R[a]$ is an inert subring of $R[X,Y]$.
(iii) $tR[X,Y] \cap R[a] = tR[a]$ for every $t \in R$.

Proof. (i) By Theorem 4.5, $K[X,Y] = K[a]^{[1]}$ and by Lemma 5.1, $K[a] \cap R[X,Y] = R[W]$ for some $W \in R[X,Y]$ satisfying $K[a] = K[W]$. It then follows that $a = \lambda w + \mu$ where $\lambda, \mu \in R$. We claim that $\lambda \in R^*$. Suppose $\lambda \notin R^*$. Let $g$ be a prime factor of $\lambda$ and let $L$ denote the algebraic closure of the field of fractions of $R/qR$. Let $\bar{a}$ and $\bar{b}$ denote the images of $a$ and $b$ respectively in $L[X,Y]$. Then we would have $\bar{a}(= \mu) \in L$; in fact, as $L[X,Y,Z]/(g) = L[X,Y,Z]/(bZ^n - \bar{a}) = L^2$, we would have that $\bar{a}$ is a unit in $L$. Since $L[X,Y,] \ni L[X,Y,Z]/(bZ^n - \bar{a}) = L^2$, it would follow that $\bar{b} \in L^*$. But then, as $n \geq 2$, $L[X,Y,Z]/(bZ^n - \bar{a})$ would not be an integral domain, contradicting that $L[X,Y,Z]/(bZ^n - \bar{a}) = L^2$. Thus $\lambda \in R^*$ and hence $R[a] = R[W] = K[a] \cap R[X,Y]$.

(ii) and (iii) follow from (i).

We now prove Theorem A.

Theorem 5.3. Let $(R,t)$ be a DVR with residue field $k$ and let $p(\geq 0)$ be the characteristic of $k$. Let $g \in R[X,Y,Z]$ (i.e., $R^3$) be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ with $b \neq 0$ and $n$ is an integer $\geq 2$ such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^2$. Then $R[X,Y,Z] = R[g, Z]^{[1]}$, $R[X,Y] = R[a]^{[1]}$ and $b \in R[X_0]$ where $X_0 \in R[X,Y]$ and $K[X,Y] = K[X_0, a]$.

Proof. Let $K$ and $k$ denote, respectively, the field of fractions and the residue field of $(R,t)$. For any $f \in R[X,Y,Z]$, let $\bar{f}$ denote the image of $f$ in $k[X,Y,Z]$. By hypotheses, $K[X,Y,Z]/(bZ^n - a) = K^2$ and $k[X,Y,Z]/(bZ^n - \bar{a}) = k^2$. Hence, by Theorem 4.5, $K[X,Y] = K[a]^{[1]}$ and $K[X,Y,Z] = K[Z,bZ^n - a]^{[1]}$.

If $t \nmid b$, then, by Theorem 4.5, $k[X,Y,Z] = k[Z, \bar{g}]^{[1]}$ and $k[X,Y] = k[\bar{a}]^{[1]}$. Hence, by Theorem 2.26, we get $R[X,Y,Z] = R[g, Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.

We now consider the case $t \mid b$. Now $\bar{g} = \bar{a}$ so that

$$k[X,Y,Z]/(\bar{a}) = k[X,Y]/(\bar{a})^{[1]} = k[X,Y,Z]/(\bar{g}) = k^2.$$ 

Hence, by Theorem 2.25, $k[X,Y]/(\bar{a}) = k[\bar{a}]^{[1]}$. Therefore, by Lemma 2.7, we see that $k[\bar{a}]$ is algebraically closed in $k[X,Y]$. Since $t$ is prime in both
$R[a](= R^{[1]}$) and $R[X,Y]$, and since $a$ is a generic variable of $R[X,Y]$, using Theorem 2.6 we see that $R[X,Y] = R[a]^{[1]}$. By similar argument, we have $R[X,Y,Z] = R[g,Z]^{[1]}$.

Now, by Theorem 4.3, one can choose $U \in R[X,Y]$ such that $K[X,Y] = K[U,a]$ and $b \in K[U]$. By Lemma 5.1, $K[U] \cap R[X,Y] = R[X_0]$ for some $X_0 \in R[X,Y]$ satisfying $K[U] = K[X_0]$. Thus $b \in R[X_0]$ where $K[X_0,a] = K[U,a] = K[X,Y]$. Hence the result.

Note that, in the case $R$ is a $\mathbb{Q}$-algebra, the hypothesis in Theorem 5.3 regarding $n (p \not| n)$ is automatically satisfied. Thus, in particular, Theorem 5.3 holds when $R$ is a DVR containing $\mathbb{Q}$. In the next section we shall see a generalisation of this result (Theorem 6.2).

Remark 5.4. Note that, in the notation of Theorem 5.3, $X_0$ need not be a variable in $R[X,Y]$. Consider a DVR ($R,t$). Let $g = bZ^n - a$ where $a = -Y$ and $b = t^2X + tY^2$, and let $X_0 = tX + Y^2$. Then $R[X,Y,Z]/(g) = R^{[2]}$, $b \in R[X_0]$, $K[X,Y] = K[X_0,Y]$ but $R[X,Y] \neq R[X_0]^{[1]}$.

The following example shows that, without the hypothesis $p \not| n$, Theorem 5.3 need not hold even over a DVR of characteristic 0.

Example 5.5. Let $R = \mathbb{Z}_{(p)}$ where $p$ is a prime in $\mathbb{Z}$, $K = \mathbb{Q}t(R) = \mathbb{Q}$ and $k = R/pR = \mathbb{Z}/p\mathbb{Z}$. Let $a = Y^p + Y + pX$ and $g = Z^p - a \in R[X,Y,Z]$. Then $R[X,Y,Z] = R[g]^{[2]}$; in particular, $R[X,Y,Z]/(g) = R^{[2]}$. But $R[X,Y] \neq R[a]^{[1]}$.

Proof. Let $Z' = Z - Y$. Then $R[X,Y,Z] = R[X,Y,Z']$ and $g = Z^p - pf(Z',Y) - Y - pX$ for some $f \in R[Z',Y]$. Let $D = R[g,Z']$. We have $K[X,Y,Z] = K[g,Y,Z] = K[g,Z']^{[1]}$ and $k[X,Y,Z] = k[\tilde{g},X,Z'] = k[\tilde{g},Z']^{[1]}$ where $\tilde{g}$ denotes the image of $g$ in $k[X,Y,Z]$. Since $p$ is prime in $R$, $p$ is prime in both $R[X,Y,Z]$ and $D$. Hence, by Theorem 2.6, $R[X,Y,Z] = D^{[1]} = R[g]^{[2]}$. Let $\tilde{a}$ denote the image of $a$ in $k[X,Y]$. Since $k[\tilde{a}] = k[\tilde{Y} + Y^p]$ is not algebraically closed in $k[X,Y]$, $\tilde{a}$ is not a variable in $k[X,Y]$ and hence $a$ is not a variable in $R[X,Y]$. □

However the next result shows that Theorem 5.3 holds over any DVR ($R,t$) of characteristic 0 (without assuming that the characteristic of $R/tR$ does not divide $n$), if the element $a$ is such that $(R/tR)[\tilde{a}]$ is algebraically closed in $(R/tR)[X,Y]$.

Proposition 5.6. Let ($R,t$) be a DVR of characteristic 0 with residue field $k$ and $g \in R[X,Y,Z](= R^{[0]})$ be of the form $g = bZ^n - a$ where $a,b \in R[X,Y]$,
\[ b \neq 0 \text{ and } n \text{ is an integer } \geq 2. \] Suppose that \( R[X, Y, Z]/(g) = R^{[2]} \) and \( k[a] \) is algebraically closed in \( k[X, Y] \). Then \( R[X, Y] = R[a]^{[1]} \) and \( R[X, Y, Z] = R[Z, g]^{[1]} \).

Proof. We see that \( R[1/t][X, Y] = R[1/t][a]^{[1]} \) by Theorem 2.6. \( t \) is prime in both \( R[a] \) and \( R[X, Y] \), \( tR[X, Y] \cap R[a] = tR[a] \) by Lemma 5.2 and \( (R/tR)[a] \) is algebraically closed in \( (R/tR)[X, Y] \) by hypothesis. Hence, by Theorem 2.6, \( R[X, Y] = R[a]^{[1]} \). Let \( B := R[X, Y, Z]/(g) (= R^{[2]}) \) and denote the image of \( Z \) in \( B \) by \( \mathfrak{z} \). Then \( B/(\mathfrak{z}) = R[X, Y, Z]/(Z, bZ^n - a) = R[X, Y]/(a) = R^{[1]} \) and hence, by the generalized epimorphism theorem of Bhatwadekar (BhaSS, Theorem 3.7), we have \( B = R[\mathfrak{z}]^{[1]} \). Let \( C = R[Z] \). Identifying the image of \( Z \) in \( B \) with \( Z \) itself, we have \( C[X, Y]/(g) = C^{[1]} \). Since \( C \) is a normal domain of characteristic 0, again by Bhatwadekar’s result (BhaSS, Theorem 3.7), we have \( C[X, Y] = C[g]^{[1]} \), i.e., \( R[X, Y, Z] = R[g, Z]^{[1]} \).

In view of Example 5.5 and Proposition 5.6 we ask:

**Question 5.7.** Let \( (R, t) \) be a DVR of characteristic 0 such that the characteristic of the residue field is positive, say \( p \). Let \( g = bZ^m - a \in R[X, Y, Z] \) be such that \( R[X, Y, Z]/(g) = R^{[2]} \) where \( a, b(\neq 0) \in R[X, Y] \) and \( m \geq 1 \). Is then \( R[X, Y, Z] = R[g]^{[2]} \)?

**6. Planes of the form \( bZ^n - a \) over rings containing a field**

In this section we prove a generalized version of Theorem B (Theorem 6.2). The authors thank Neena Gupta for her observations on the earlier drafts of this paper which have resulted in the formulation of Theorem 6.2 in its present generality. We shall essentially follow the approach of Bhatwadekar in (BhaSS) and then apply the result on residual variables (Theorem 2.3). We first state a result which will be needed in the proof of Theorem 6.2.

**Lemma 6.1.** Let \( R \) be a Noetherian domain and let \( b \neq 0 \in R \). Then, for each non-zero prime ideal \( P \) of \( R \), there exists a discrete valuation ring \( V \) with maximal ideal \( \mathfrak{m}_V \) together with a homomorphism \( \phi : R \to V \) such that \( \phi(b) \neq 0, \phi^{-1}(\mathfrak{m}_V) = P \) and \( V/\mathfrak{m}_V \) is algebraic over \( k(P) \).

Proof. Let \( P \) be a non-zero prime ideal of \( R \) and let \( n \) be the height of \( P \). Since \( R \) is a Noetherian domain, there exists a prime ideal \( Q \) of \( R \) of height \( n - 1 \) such that \( Q \subseteq P \) and \( b \notin Q \). Let \( D = R/Q \) and \( p = P/Q \) be the image of \( P \) in \( D \). Let \( C \) be the normalisation of \( D \) and \( \mathcal{P} \) a prime ideal of \( C \) lying over \( p \). Set \( V := C_p \) and \( \mathfrak{m}_V := \mathcal{P}V \), the maximal ideal of the local ring \( V \). Since the height of \( p \) (and hence that of \( \mathcal{P} \)) is one, \( V \) is a DVR. Now let \( \phi \)
denote the composite map $R \twoheadrightarrow D(= R/Q) \twoheadrightarrow C \twoheadrightarrow V (= C_P)$. Clearly, $\phi^{-1}(m_V) = P$, $\phi(b) \neq 0$ and $V/m_V$ is algebraic over $k(P)$. 

We now prove the main result of this section.

**Theorem 6.2.** Let $R$ be a Noetherian domain containing a field of characteristic $p \geq 0$ and $g \in R[X,Y,Z](= R^{[3]}$) be of the form $hZ^n - a$ where $a, b \in R[X,Y]$, $b \neq 0$ and $n$ is an integer $\geq 2$ such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] \otimes_R k(P) = (R[g, Z] \otimes_R k(P))^{[1]}$ and $R[X,Y] \otimes_R k(P) = (R[a] \otimes_R k(P))^{[1]}$ for all $P \in Spec(R)$. Thus, if $R$ contains $\mathbb{Q}$ or if $R$ is seminormal, then $R[X,Y,Z] = R[g, Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.

**Proof.** Fix $P \in Spec(R)$. Let the images of $b, a$ and $g$ in $R[X,Y,Z] \otimes_R k(P)$ be $\bar{b}$, $\bar{a}$ and $\bar{g}$ respectively. Let $k$ denote $k(P)$. We show that $k[X,Y] = k[\bar{a}]^{[1]}$ and $k[X,Y,Z] = k[\bar{g}, Z]^{[1]}$.

If $ht P = 0$, we are done by Theorem 4.3. So we assume that $ht P = n \geq 1$. If $\bar{b} \neq 0$, then by Theorem 4.5 we are through. So we assume that $\bar{b} = 0$ (and hence $\bar{g} = \bar{a}$).

Using Lemma 6.1, we have a DVR $(V, \pi)$ with a homomorphism $\phi : R \twoheadrightarrow V$ such that $\phi(\bar{b}) \neq 0$, $\phi^{-1}(\pi) = P$ and $V/\pi$ is algebraic over $k(P)$. Note that $V[X,Y,Z]/(g) = V^{[2]}$ and hence, by Theorem 5.3, we have $V[X,Y,Z] = V[g, Z]^{[1]}$ and $V[X,Y] = V[a]^{[1]}$; in particular, $\frac{V}{\pi}[X,Y,Z] = \frac{V}{\pi}[\bar{g}, Z]^{[1]}$ and $\frac{V}{\pi}[X,Y] = \frac{V}{\pi}[\bar{a}]^{[1]}$. Now, since $\frac{k[X,Y]}{\bar{a}}[Z] = k^{[2]}$, by Theorem 2.5, we have $\frac{k[X,Y]}{\bar{a}}[Z] = k^{[1]}$. Since $V/\pi$ is algebraic over $k$ and since $\frac{V}{\pi}[X,Y] = \frac{V}{\pi}[\bar{a}]^{[1]}$, by $(\text{Gan79}, \text{Proposition 1.16})$, we have $k[X,Y] = k[\bar{a}]^{[1]}$ and hence $k[X,Y,Z] = k[\bar{g}, Z]^{[1]}$.

Thus $R[X,Y,Z] \otimes_R k(P) = (R[g, Z] \otimes_R k(P))^{[1]}$ and $R[X,Y] \otimes_R k(P) = (R[a] \otimes_R k(P))^{[1]}$ for all $P \in Spec(R)$.

Now, if $R$ is seminormal or contains $\mathbb{Q}$, then $R[X,Y,Z] = R[g, Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$ by Theorem 2.8.

**Remark 6.3.** (1) If $R$ is seminormal or $R$ contains $\mathbb{Q}$, then under the hypotheses of Theorem 6.2, one can show, by suitable reductions, that $R[X,Y,Z] = R[g, Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$, even when $R$ is non-Noetherian.

(2) If $R$ is a UFD, then under the hypotheses of Theorem 6.2, $b \in R[X_0]$ where $X_0 \in R[X,Y]$ and $K[X,Y] = K[X_0, a]$. This follows from the proof of Theorem 5.3.

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