Quasitriangular structures on cocommutative Hopf algebras

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March 31, 2022

Abstract

The article is devoted to the description of quasitriangular structures (universal R-matrices) on cocommutative Hopf algebras. It is known that such structures are concentrated on finite dimensional Hopf subalgebras.

In particular, quasitriangular structure on group algebra is defined by the pairs of normal inclusions of a finite abelian group and by invariant bimultiplicative form on it. The structure is triangular in the case of coinciding inclusions and skewsymmetric form.

The nonstandard λ-structure on the representation ring of finite group, corresponding to the triangular structure on group ring, is described.

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1 Introduction

The notion of quasitriangular Hopf algebra was introduced by V.G. Drinfeld in connection with the so-called quantum Yang-Baxter equation, which appears in different contexts of mathematical physics.

Quasitriangular Hopf algebra is a pair, which consists of a Hopf algebra and invertible element of its tensor square (universal R-matrix), satisfying some additional conditions.

From categorical point of view quasitriangular Hopf algebras are characterized by the fact that its representations categories (categories of modules having finite dimensional over ground field) are rigid and quasitensor.

The present article is devoted to the description of quasitriangular structures (universal R-matrices) on cocommutative Hopf algebras over algebraically closed field of characteristic zero. It is used that any quasitriangular structure are concentrated on some finite dimensional Hopf subalgebra \([15]\). Since finite dimensional cocommutative Hopf algebras over algebraically closed field of characteristic zero are group algebras, the special attention is devoted to the case of this class of Hopf algebras.

It is shown that quasitriangular structures on group algebra is defined by the pairs of normal inclusions of an finite abelian group and by invariant bimultiplicative form on it (theorem \([3]\)). The triangular structures corresponds to the case of coinciding inclusions and skewsymmetric form.

It is well known that the tensor structure on monoidal category allows to define external powers of any object and gives a \(\lambda\)-structure on its Grothendieck ring. In the last section the nonstandart \(\lambda\)-structures on the representation ring of finite group, corresponding to the triangular structures on group ring, is described (theorem \([4]\)). It is appears that these \(\lambda\)-structure depends only of central involution of the group, which is defined by skewsymmetric bimultiplicative form.

The work was partially supported by Russian Fund of Fundamental Research, grant n 96-01-00149.
2 Tensor categories

Let $k$ be a ground field. All categories and functors will be $k$-linear.

The monoidal category is a category $G$ with a bifunctor

$$\otimes : G \times G \rightarrow G \quad (X,Y) \mapsto X \otimes Y$$

which called tensor (or monoidal) product. This functor must be equipped with a functorial collection of isomorphisms (so-called associativity constraint)

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$$

for any $X,Y,Z \in G$

which satisfies to the following pentagon axiom:

$$(X \otimes \varphi_{Y,Z,W})\varphi_{X,Y,Z,W}(\varphi_{X,Y,Z} \otimes W) = \varphi_{X,Y,Z} \otimes W \varphi_{X,Y,Z,W}.$$ 

Consider two tensor products of objects $X_1,...,X_n$ from $G$ with an arbitrary arrangement of the brackets. The coherence theorem [13] states that the pentagon axiom implies the existence of a unique isomorphism between them, which is the composition of the associativity constraints. This fact allows us to omit brackets in the tensor products.

An object $1$ together with the functorial isomorphisms

$$\rho_X : X \otimes 1 \rightarrow X \quad \lambda_X : 1 \otimes X \rightarrow X$$

in a monoidal category $G$ is called a unit if $\lambda_1 = \rho_1$ and

$$\lambda_{X \otimes Y} = \lambda_X \otimes I : 1 \otimes X \otimes Y \rightarrow X \otimes Y, \quad \rho_X \otimes I = I \otimes \lambda_Y : X \otimes 1 \otimes Y \rightarrow X \otimes Y,$$

$$\rho_{X \otimes Y} = I \otimes \rho_Y : X \otimes Y \otimes 1 \rightarrow X \otimes Y$$

for any $X,Y \in G$.

It is easy to see that the unit object is unique up to isomorphism. We will suppose additionally that the endomorphisms ring $\text{End}_G(1)$ of the unite object coincides with the ground field.

A monoidal functor between monoidal categories $G$ and $H$ is a functor $F : G \rightarrow H$, which is equipped with the functorial collection of isomorphisms (the so-called monoidal structure)

$$F_{X,Y} : F(X \otimes Y) \rightarrow F(X) \otimes F(Y)$$

for any $X,Y \in G$. 

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for which
\[ F_{X,Y \otimes Z}(I \otimes F_{Y,Z}) = F_{X,Y,Z}(F_{X,Y} \otimes I) \]
for any objects \( X, Y, Z \in \mathcal{G} \).

A morphism \( f : F \to G \) of monoidal functors \( F \) and \( G \) is called \textit{monoidal} if
\[ G_{X,Y}f_{X,Y} = (f_X \otimes f_Y)F_{X,Y} \]
for any \( X, Y \in \mathcal{G} \).

The tensor product allows to correspond to any functor \( F : \mathcal{G} \to \mathcal{H} \) from monoidal category \( \mathcal{G} \) the collection of functors from the cartesian powers of the category \( \mathcal{G} \)
\[ F^{\otimes n} : \mathcal{G}^n \to \mathcal{H} \quad F^{\otimes n+1}(X_1, \ldots, X_{n+1}) = F((X_1 \otimes \ldots) \otimes X_{n+1}) \]
The collection of the endomorphisms algebras of tensor powers of a monoidal functor \( F \) can be equipped with the structure of cosimplicial complex \( E(F)_* = \text{End}(F^{\otimes *}) \).

The image of the coface map
\[ \partial_{n+1}^i : \text{End}(F^{\otimes n}) \to \text{End}(F^{\otimes n+1}) \quad i = 0, \ldots, n + 1 \]
of the endomorphism \( \alpha \in \text{End}(F^{\otimes n}) \) has the following specialization on the objects \( X_1, \ldots, X_{n+1} \):
\[ \partial_{n+1}^i(\alpha)_{X_1, \ldots, X_{n+1}} = \begin{cases} \phi_0(I_{X_1} \otimes \alpha_{X_2, \ldots, X_{n+1}})\phi_0^{-1}, & i = 0 \\ \phi_i(\alpha_{X_1, \ldots, X_i \otimes X_{i+1}, \ldots, X_{n+1}})\phi_i^{-1}, & 1 \leq i \leq n \\ \alpha_{X_1, \ldots, X_i} \otimes I_{X_{n+1}}, & i = n + 1 \end{cases} \]
here \( \phi_i \) is the unique isomorphism between \( F((X_1 \otimes \ldots) \otimes X_{n+1}) = F^{\otimes n+1}(X_1, \ldots, X_{n+1}) \) and \( F((X_1 \otimes \ldots) \otimes (X_i \otimes X_{i+1})) \otimes \ldots) \otimes X_{n+1}) \).

The specialization of the image of the coboundary map
\[ \sigma_{n-1}^i : \text{End}(F^{\otimes n}) \to \text{End}(F^{\otimes n+1}) \quad i = 0, \ldots, n - 1 \]
is
\[ \sigma_{n-1}^i(\alpha)_{X_1, \ldots, X_{n-1}} = \alpha_{X_1, \ldots, X_i, X_{i+1}, \ldots, X_{n-1}}. \]

Let us note that the monoidality of the automorphism \( f \in \text{Aut}(F) \) of monoidal functor \( F \) can be presented in the following form:
\[ \partial_1^1(f) = \partial_2^0(f)\partial_2^1(f). \]
Two monoidal functors will be called by *twisted forms* of each other if they are isomorphic as functors. It this case the differ only by the monoidal structure. The ratio of its monoidal structures is an automorphism \( \alpha \in Aut(F^{\otimes 2}) \) of the tensor square of one of these functors. The fact that the composition \( F_{X,Y} \alpha_{X,Y} \) is a monoidal structure on the functor \( F \) is equivalent to the condition:

\[
\partial^0_3(\alpha) \partial^2_3(\alpha) = \partial^2_3(\alpha) \partial^1_3(\alpha).
\]

(1)

So twisted forms of monoidal functors \( F \) correspond to the automorphisms \( \alpha \in Aut(F^{\otimes 2}) \), which satisfies (1). We will call this automorphisms by 2-cocycles of the monoidal functor \( F \) and by \( Z^2(F) \) we will denote the set of 2-cocyles of \( F \).

The twisted form of the functor \( F \) corresponding to 2-cocycle \( \alpha \in Z^2(F) \) will be denoted by \( F(\alpha) \).

Monoidal category \( G \) with the unite object is rigid if it is equipped by the contravariant equivalence of categories (\( \ast \)) : \( G \rightarrow G \) (dualization) together with the functorial collection of morphisms

\[
\kappa_X : 1 \rightarrow X \otimes X^*, \quad ev_X : X^* \otimes X \rightarrow 1,
\]

for which the compositions

\[
X \rightarrow^{\kappa \otimes I} X \otimes X^* \otimes X \rightarrow^{I \otimes ev} X,
\]

\[
X^* \rightarrow^{I \otimes \kappa} X^* \otimes X \otimes X^* \rightarrow^{ev \otimes I} X^*
\]

are identical.

It is easy to see that for any monoidal functor \( F \) between rigid categories the composition \( F((\ast)^*) \) is canonically isomorphic to \( F(\ast)^* \). The isomorphism can be presented as the composition

\[
F(X^*) \rightarrow^{I \otimes \kappa_{F(X)}} F(X^*) \otimes F(X) \otimes F(X)^* \rightarrow^{F_{X^*,X} \otimes I} F(X^* \otimes X) \otimes F(X)^* \rightarrow^{F_{ev_X} \otimes I} F(X)^*.
\]

The monoidal category \( G \) is quasitensor if it is equipped by the functorial collection of isomorphisms (commutativity constraint) \( c_{X,Y} : X \otimes Y \rightarrow Y \otimes X \), for which

\[
c_{X,Y \otimes Z} = (Y \otimes c_{X,Z})(c_{X,Y} \otimes Z) \quad c_{X \otimes Y,Z} = (c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}).
\]

Quasitensor category is tensor if \( c_{Y,X}c_{X,Y} = I \) for any \( X, Y \in G \).
Monoidal functor $F : \mathcal{G} \to \mathcal{H}$ between (quasi)tensor categories is (quasi)tensor if
$$F_Y X F(c_{X,Y}) = c_{F(X),F(Y)} F_{X,Y},$$
for any $X, Y \in \mathcal{G}$.

For the rigid quasitensor category the square of the dualization is isomorphic to the identity functor $\nu : Id \to (\quad)^{**}$. The isomorphism can be presented as the functorial composition
$$X \to I \otimes^{\kappa_X} X \otimes X^{**} \to c_{X,Y}^{**} I X \otimes X^{**} \to ev^1 X^{**}.$$

The morphism $\nu$ allows to define the automorphism $\gamma$ of identity functor $Id_\mathcal{G}$:
$$\gamma_X : X \to \nu^1 X \otimes X^{**} \to \nu_X X^{**} \to (\nu_X)^* X^{**} \to \nu^{-1}_X X.$$

Let us remark that the square of the dualization $(\quad)^{**}$ is a monoidal functor. The morphism $\nu$ is monoidal in the case of tensor category. In the general case it defines 2-cocycle $\sigma \in Z^2(id_\mathcal{G})$ of the identity functor
$$\sigma_{X,Y} : X \otimes Y \to \nu_X^{\otimes^{\nu_Y} X^{**} \otimes Y^{**}} \to (X \otimes Y)^{**} \to \nu^{-1}_{X \otimes Y} X \otimes Y.$$

The proof of the following proposition can be extracted from [16].

**Proposition 1**
1. $\sigma_{X,Y} = c_{Y,X} c_{X,Y}$.
   In particular $\sigma = 1$ for the tensor category.
2. $\sigma^2 = \partial(\gamma)$.

The isomorphism $\nu$ allows to define the trace of any endomorphism of any object. Let $f \in End_\mathcal{G}(X)$ is an endomorphism of the object $X$ of rigid quasitensor category $\mathcal{G}$. The trace of the endomorphism $f$ is an element of the ground field $Tr_X(f) \in k$, for which the composition
$$1 \to^{\kappa_X} X \otimes X^* \to^{\nu_X f \otimes I} X^{**} \otimes X^* \to^{ev_X^*} 1$$
coincides with $Tr_X(f) Id_1$.

The trace is additive by the definition:
$$Tr_X(f + g) = Tr_X(f) + Tr_X(g), \quad \forall f, g \in End_\mathcal{G}(X).$$

In the case of tensor category the trace is also multiplicative:
$$Tr_{X \otimes Y}(f \otimes g) = Tr_X(f) Tr_Y(g), \quad \forall f \in End_\mathcal{G}(X), g \in End_\mathcal{G}(Y).$$
The rank of the object $X$ is a trace of its identity morphism $rk(X) = Tr_X(I_X)$.

Using the morphism $\nu$ we can define an automorphism $\mu(F) = \nu_F^{-1} F(\nu_G)$ of any monoidal functor $F : \mathcal{G} \rightarrow \mathcal{H}$ between quasitensor categories which will be called Markov automorphism. Let us note that $\mu(F) = 1$, iff $F$ is a quasitensor functor.

**Proposition 2** Let $F : \mathcal{G} \rightarrow \mathcal{H}$ be monoidal functor between quasitensor categories and $f \in End_G(X)$. Then 1. $F(\sigma_G) = \partial(\mu(F)) \sigma_H$.

in particular $F(\sigma) = \partial(\mu(F))$, in the case of tensor category $\mathcal{H}$

2. $Tr_X(f) = Tr_{F(X)}(\mu(F)_X F(f))$.

Proof:

The first equation can be obtained by replacing $F(\nu_X)$ on $\nu_{F(X)} \mu(F)_X$ in the expression of $F(\sigma_{X,Y})$.

As was mentined above the object $F(X^*)$ is canonically isomorphic to $F(X)$ and this isomorphism identifies $F(\kappa_X)$ with $\kappa_{F(X)}$ and $F(ev_X)$ with $ev_{F(X)}$. Hence the composition

$$1 \rightarrow F(\kappa_X) F(X \otimes X^*) \rightarrow F(\nu_X f \otimes I) F(X^* \otimes X^*) \rightarrow F(ev_{X^*}) 1,$$

wich is equal to $F(Tr_X(f)) = Tr_X(f)$, is isomorphic to the following

$$1 \rightarrow \kappa_{F(X)} F(X) \otimes F(X^*) \rightarrow \nu_{F(X)} F(f) \otimes I F(X^* \otimes X^*) \rightarrow ev_{F(X)} 1.$$

Now it is enough to note that $F(\nu_X) = \nu_{F(X)} \mu(F)_X$. □

### 3 Quasitriangular Hopf algebras

This section contains the definitions and general properties of quasitriangular structures on the Hopf algebras.

The Hopf algebra is an associative unitary algebra $H$ with the algebra homomorphisms:

$$\Delta : H \otimes H \rightarrow H \quad \text{(coproduct),}$$

$$\varepsilon : H \rightarrow k \quad \text{(counite)}$$

and with the antihomomorphism of algebras:

$$S : H \rightarrow H \quad \text{(antipode),}$$
which satisfies to the following conditions:

\[(I \otimes \Delta)\Delta = (\Delta \otimes I)\Delta \quad \text{(coassociativity),}\]

\[(I \otimes \varepsilon)\Delta = (\varepsilon \otimes I)\Delta = I \quad \text{(counitarity),}\]

\[(I \otimes S)\Delta = (S \otimes I)\Delta = \varepsilon.\]

Here \(I\) is an identical map.

An element \(g \in H\) for which \(\Delta(g) = g \otimes g\) will be called grouplike. It is easy to verify that \(g \varepsilon(g) = 1\) and \(S(g) = g^{-1}\) for grouplike \(g\). The set of grouplike elements of Hopf algebra \(H\) form a group \(G(H)\) under multiplication.

Coproduct allows to define the structure of \(H\)-module on the tensor product \(M \otimes_k N\) of two \(H\)-modules:

\[h \ast (m \otimes n) = \Delta(h)(m \otimes n) \quad h \in H, m \in M, n \in N.\]

Coassociativity of coproduct implicates that the standard associativity constraint of vector spaces

\[\varphi: L \otimes (M \otimes N) \rightarrow (L \otimes M) \otimes N \quad \varphi(l \otimes (m \otimes n)) = (l \otimes m) \otimes n\]

is \(H\)-linear.

The counit defines \(H\)-module structure on the ground field \(k\) and the counit axiom means that this is a unite object.

Finally the antipode of Hopf algebra allows to define (left) \(H\)-module structure on the dual space \(M^* = \text{Hom}(M, k)\) of any (left) \(H\)-module \(M\):

\[h \ast l(x) = l(S(h)x), \quad \forall h, x \in H, l \in M^*.\]

It follows from the antipode axiom that for \(H\)-module \(M\) finite dimensional over ground field the Kazimir inclusion \(k \rightarrow M \otimes M^*\) and the evaluation map \(M^* \otimes M \rightarrow k\) are \(H\)-linear.

By another words the category \(\text{Rep}(H)\) of representations (that are (left) \(H\)-modules which are finite dimensional over ground field) of Hopf algebra \(H\) is rigid monoidal. It is easy to see that for any homomorphism of Hopf algebras \(f: H \rightarrow F\) (algebra homomorphism for which \(\Delta f = (f \otimes f)\Delta\)) the restriction functor \(f^*: \text{Rep}(F) \rightarrow \text{Rep}(H)\) is monoidal with trivial monoidal
structure. For example, the forgetful functor $\text{Rep}(H) \to k - \text{mod}$, which coincides with the restriction functor of the inclusion $k \to H$, is monoidal.

Accordingly to the previous section nontrivial monoidal structures on the restriction functor correspond to some invertible elements of its endomorphisms complex.

The cobar complex of Hopf algebra $H$ is a tensor algebra $\oplus H^\otimes n$ with coface maps given by

$$\partial^i_n : H^\otimes n \to H^\otimes(n-1),$$

$$\partial^i_n(h_1 \otimes \ldots \otimes h_n) = \begin{cases} 1 \otimes h_1 \otimes \ldots \otimes h_n, & i = 0 \\ h_1 \otimes \ldots \otimes \Delta(h_i) \otimes \ldots \otimes h_n, & 1 \leq i \leq n \\ h_1 \otimes \ldots \otimes h_n \otimes 1, & i = n + 1 \end{cases},$$

and codegeneration maps

$$\sigma^i_n(h_1 \otimes \ldots \otimes h_{n+1}) = h_1 \otimes \ldots \otimes \varepsilon(h_i) \otimes \ldots \otimes h_{n+1}).$$

The following proposition was proved in [4].

**Proposition 3** Endomorphisms complex of the restriction functor $f^* : \text{Rep}(F) \to \text{Rep}(H)$ coincides with the subcomplex $C_{F^\otimes 2}(\Delta(f(H)))$ of centralizers of the image of coproduct in the cobar complex of Hopf algebra $F$.

An isomorphism is given by the map

$$H^\otimes n \to \text{End}(F^\otimes n),$$

sending the element $x \in H^\otimes n$ to the endomorphism of multiplication by $x$.

In particular twisted forms of the restriction functor $f^* : \text{Rep}(F) \to \text{Rep}(H)$ correspond to the invertible elements $F \in C_{F^\otimes 2}(\Delta(f(H)))$, satisfying to the condition:

$$(1 \otimes F)(I \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes I)(F). \quad (2)$$

Hopf algebra is cocommutative if $t\Delta = \Delta$, where $t : H \otimes H \to H \otimes H$ denotes the permutation of tensor factors. Let us note that the antipode of cocommutative Hopf algebra is involutive: $S^2 = I$.

It is not hard to verify that finite dimensional cocommutative Hopf algebra over the algebraically closed field of characteristic zero is a group algebra of its grouplike elements [12].
The following notion can be regarded as a generalization of the cocommutativity. The pair \((H, R)\), which consists of the Hopf algebra \(H\) and the invertible element \(R \in H^\otimes 2\), is *quasitriangular* Hopf algebra if the following conditions are fulfilled:

\[
Rt\Delta(h) = \Delta(h)R \quad \text{for any} \quad h \in H, \tag{3}
\]

\[
(I \otimes \Delta)(R) = R_{13}R_{12}, \tag{4}
\]

\[
(\Delta \otimes I)(R) = R_{13}R_{23}. \tag{5}
\]

The element \(R\) will be called an *universal R-matrix*.

A quasitriangular Hopf algebra will be called *triangular* if

\[
RR_{21} = 1 \quad \text{(unitarity of the universal } R\text{-matrix)}. \tag{6}
\]

The proof of the following proposition is contained in the Drinfeld’s paper [7].

**Proposition 4** The universal \(R\)-matrix \(R\) in the quasitriangular Hopf algebra satisfies to the following conditions:

\[
(\varepsilon \otimes I)(R) = (I \otimes \varepsilon)(R) = 1,
\]

\[
(S \otimes I)(R) = (I \otimes S)(R) = R^{-1}, \quad (S \otimes S)(R) = R,
\]

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad \text{(quantum Yang-Baxter equation)}. \tag{7}
\]

The universal \(R\)-matrix allows to define quasitensor structure on the representations category \(\text{Rep}(H)\):

\[
c_{M,N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto R(n \otimes m).\tag{8}
\]

Let us note that the unitarity of the \(R\)-matrix is equivalent to the tensority of the corresponding structure.

Let \((H, R)\) and \((H', R')\) be two quasitriangular Hopf algebras and \(f : H \to H'\) is a homomorphism of Hopf algebras. It is easy to see that the twisted form of the restriction functor \(f^*\), corresponding to 2-cocycle \(F \in Z^2(f^*)\), is a quasitensor functor iff \(R'F = t(F)R\).

The element \(u = (S \otimes I)(R_{21})\) will be called *Markov element* of the universal \(R\)-matrix \(R\). Let us note that Markov automorphism of forgetful functor \(\text{Rep}(H) \to k - \text{mod}\) coincides with the multiplication by \(u^{-1}\).

The proof of the following proposition, which borrows from the Drinfeld’s paper [7], can be also concluded from the results of first article.
Proposition 5 Markov element \( u \) satisfies to the following conditions:
1) \( u \) is invertible, and \( u^{-1} = (S^{-1} \otimes S) (R_{21}) \),
2) \( S^2 = \text{Ad}(u) \), that is \( S^2(h) = uhu^{-1} \) for any \( h \in H \), in particular, \( u \) is central for cocommutative \( H \),
3) \( \Delta(u) = (R_{21}R)^{-1} (u \otimes u) = (u \otimes u) (R_{21}R)^{-1} \), in particular, \( u \) is group-like for unitary \( R \)-matrix \( R \),
4) \( \Delta(g) = (g \otimes g) \) and \( g = uS(u)^{-1} \).

4 Minimal quasitriangular Hopf algebras

This section contains the description of minimal quasitriangular Hopf algebras which belongs to Radford [13].

Any bivector \( R \in H^\otimes 2 \) provides two finite dimensional subspaces of \( H \)
\[
H_l = \{(I \otimes l)(R), \ l \in H^*\} \quad H_r = \{(l \otimes I)(R), \ l \in H^*\}.
\]
Here \( H^* \) denotes the space \( \text{Hom}(H, k) \) of linear functions on \( H \).

Let us remark that \( R \) lies in \( H_l \otimes H_r \) by the construction.
Moreover, the bivector \( R \) defines two bijective maps:
\[
\alpha : H_r^* \to H_l \quad \alpha(l) = (I \otimes l)(R),
\]
\[
\beta : H_l^* \to H_r \quad \beta(l) = (l \otimes I)(R),
\]
for which
\[
\alpha^* = \beta.
\]
Indeed, by the definition of dual map \( m(\alpha^*(l)) = l(\alpha(m)) \) for any \( l, m \in H^* \).

Hence
\[
m(\alpha^*(l)) = l(\alpha(m)) = (l \otimes m)(R) = m(\beta(l)), \quad \text{for any} \ l, m \in H^*.
\]

It is appear that the spaces \( H_l \) and \( H_R \), defined by an universal \( R \)-matrix, are Hopf subalgebras in \( H \).

Here we will need the notions of dual Hopf algebra and adjoint action.

It is easy to see that the dual maps to the product, coproduct and antipode are coproduct, product and antipode on the dual space \( H^* \) for finite dimensional Hopf algebra \( H \):
\[
\Delta^*_H = \mu_{H^*}, \quad \mu^*_H = \Delta_{H^*}, \quad i^*_H = \epsilon_{H^*}, \quad S^*_H = S_{H^*}.
\]
This Hopf algebra will be called dual.

Let us define the adjoint action of Hopf algebra $H$ on itself by setting $x^h = \sum (h) h_0 x S(h_1)$ for $x \in H$. This action induce also the action of $H$ on $H^*$:

$$l^h(x) = l(x^{S(h)}), \quad \forall l \in H^*, h, x \in H.$$  

We will call subspace in $H$ normal if it is invariant respectively the adjoint action.

**Theorem 1** For any universal $R$-matrix on the Hopf algebra $H$ subspaces $H_l, H_r$ are finite dimensional Hopf subalgebras of $H$, the map $\alpha$ is an antihomomorphism of algebras and homomorphism of coalgebras.

In the case of unitary $R$-matrix subalgebras $H_l$ and $H_r$ coincides and the map $\alpha$ satisfies to the condition

$$\alpha^*_R = S \alpha_R.$$  

If $H$ is cocommutative Hopf algebra, then the maps $\alpha$ and $\beta$ are $H$-invariant:

$$\alpha(l^h) = \alpha(l)^h, \quad \beta(l^h) = \beta(l)^h, \quad \forall h \in H, l \in H^*.$$  

In particular, $H_l, H_r$ are normal subalgebras.

Proof: Let $l$ and $m$ be two linear functions on $H$. Applying $I \otimes l \otimes m$ to both sides of the equation (4), we will receive that the product of any two elements from $H_l$ lies in $H_l$:

$$(I \otimes l)(R)(I \otimes m)(R) = (I \otimes l \otimes m)(R_{13}R_{12}) = (I \otimes l \otimes m)(I \otimes \Delta)(R).$$

The equality $(I \otimes \varepsilon)(R) = 1$ (proposition 3) means that the identity also lies in $H_l$. So $H_l$ is a subalgebra of $H$.

To verify that $H_l$ is a subcoalggebra of $H$ it is enough to apply $I \otimes l$ to both sides of the equation (3):

$$(\Delta \otimes l)(R) = (I \otimes I \otimes l)(\Delta \otimes I)(R) = (I \otimes l)(R_{13}R_{23}).$$

The fact that $H_r$ is a Hopf subalgebra can be proved analogously.
Antihomomorphity of the map $\alpha$ also follows from the equation (4). Let $l$ and $m$ be two linear functions from $H^*_r$. Then

$$\alpha(l \ast m) = (I \otimes l \ast m)(R) = (I \otimes l \otimes m)(I \otimes \Delta)(R) =$$

$$(I \otimes l \otimes m)(R_{13}R_{12}) = (I \otimes m)(R)(I \otimes l)(R) = \alpha(m)\alpha(l).$$

The equality (5) implices that the map $\alpha$ is a coalgebra homomorphism:

$$\Delta(\alpha_R(l)) = (\Delta \otimes l)(R) = (I \otimes I \otimes l)(\Delta \otimes I)(R) = (I \otimes l)(R_{13}R_{23}).$$

$$(I \otimes \Delta(l))t_{23}(R \otimes R) = (\alpha_R \otimes \alpha_R)(\Delta(l)),$$

where $t_{23}$ is a permutation of second and third tensor factors.

The unitarity condition $Rt(R) = 1$ and the equation $(I \otimes S)(R) = R^{-1}$ (proposition 4) implicate that $H_l$ coincides with $H_r$:

$$(I \otimes l)(R) = (l \otimes I)(t(R)) = (l \otimes I)(R^{-1}) = (lS \otimes I)(R).$$

Let us check the equality $\alpha^* = S\alpha$:

$$S\alpha(l) = (S \otimes l)(R) = (I \otimes l)(R^{-1}) =$$

$$(I \otimes l)(t(R)) = \alpha^*(l).$$

In the case of cocommutative Hopf algebra $H$ the condition (3) is equivalent to the $H$-invariance of universal $R$-matrix:

$$R\Delta(h) = \Delta(h)R \text{ for any } h \in H,$$

which is equivalent to the condition:

$$\sum_{(h)}(h_{(0)} \otimes 1)R(h_{(1)} \otimes 1) = \sum_{(h)}(1 \otimes S(h_{(0)}))R(1 \otimes h_{(1)}).$$

Applying $(1 \otimes l)$ to the previous equality, we will obtain $H$-invariance of the map $\alpha$:

$$\alpha(l^h) = \alpha(l)^h.$$  

$H$-invariance of $\beta$ can be verified analogously. □

So we can define by the universal $R$-matrix $R$ in the Hopf algebra $H$ the finite dimensional Hopf algebra $F(= H_l)$ and two Hopf algebra inclusions
i : F → H, j : F* → H. Moreover R coincides with the image (i ⊗ j)(C) of Kazimir element C ∈ F ⊗ F*. To characterize the image of the map i ⊗ j it is convinient to use the notion of quantum double.

Quantum double D(A) of (finite dimensional) Hopf algebra A is unique quasitriangular Hopf algebra satisfying to the following conditions:

1) D(A) contains A and A*op as subalgebras. Here A*op denotes the dual Hopf algebra with opposite multiplication.

2) Universal R-matrix R is the image of Kazimir element C ∈ A ⊗ A* under natural inclusion of vector spaces A ⊗ A* → D(A) ⊗ D(A).

3) The multiplication A ⊗ A*op → D(A) is a bijective.

The quasitriangular Hopf algebra (H, R) will be called minimal if its not contains proper quasitriangular subalgebras.

By means of quantum double the theorem can be stated in the following form.

**Corollary 1** For any minimal quasitriangular Hopf algebra (H, R) there are finite dimensional Hopf algebra F and the surjective homomorphism of quasitriangular algebras (D(F), R) → (H, R), which restrictions on F and F* are injective.

Let us cosider the case of triangular Hopf algebra in more detail. We will call universal R-matrix R nondegenerated if the corresponding map αR : H* → H is a bijective.

**Theorem 2** For any unitary universal R-matrix R in the Hopf algebra H there is a Hopf subalgebra F, such that R ∈ F⊗2 and R is nondegenerated on F.

Proof:

Let us denote by F the image of the map αR. By the construction the bivector R lies in F⊗2.

Let us show that αR decomposes into a product of the maps

\[ H^* \rightarrow i^* F^* \rightarrow^\alpha F \rightarrow^i H, \]

where i is a standard inclusion. Write αR as the composition of standard inclusion and projection:

\[ H^* \rightarrow^\pi F \rightarrow^i H. \]
Since by the theorem $\alpha^*_R = S\alpha_R$, the map $\pi$ differs from $i^*$ by an automorphism of Hopf algebra $F$. $\square$

For cocommutative algebra $H$ the finite dimensional Hopf algebra $F$ is commutative and cocommutative. It is follows from the description of cocommutative Hopf algebras over the algebraically closed fields of characteristic zero [12] that $F$ is a group algebra of finite abelian group.

## 5 Group algebras

Let us remind that our ground field $k$ is algebraically closed of characteristic zero.

**Theorem 3** Universal $R$-matrix in the group algebra $k[G]$ of a group $G$ is defined by the pair of normal inclusions $i, j : A \to G$ of finite abelian group, which induce the same $G$-module structure on $A$, and by nondegenerated bimultiplicative $G$-invariant form $\beta : \hat{A} \otimes \hat{A} : \to k^*$.

The universal $R$-matrix defined by this data have the following form:

$$R = \frac{1}{|A|^2} \sum_{a,b \in A} \sum_{\chi, \xi \in \hat{A}} \beta(\chi, \xi)\chi(a)\xi(b)(i(a) \otimes j(b)).$$

Unitary $R$-matrix corresponds to the case of coinciding inclusions $i = j$ and skewsymmetric form.

The Markov element $u$ of the unitary $R$-matrix, defined by the form $\beta$, is a central involution of the group $G$ and satisfies to the equation, which defines it uniquely:

$$\chi(u) = \beta(\chi, \chi) \quad \text{for any} \quad \chi \in \hat{A}. \quad (6)$$

Remark. Here $\hat{A}$ denotes the characters group $\text{Hom}(A, k^*)$ of the group $A$. Bimultiplicative form $\beta : \hat{A} \otimes \hat{A} : \to k^*$ is a homomorphism from tensor product to the group of nonzero elements of the ground field. Nondegeneracy of $\beta$ means that the map

$$\hat{A} : \to \hat{A} \quad \chi \mapsto \beta(\chi, ?)$$

is bijective.
Proof:
Let $R$ be a $R$-matrix in the group algebra $k[G]$. Accordingly to the theorem there are finite abelian group $A$ and two inclusions $i, j : A \to G$ with normal images such that the homomorphism of Hopf algebras $\alpha_R : k[G]^* \to k[G]$ decomposes into the product

$$k[G]^* \to^i k[A]^* \to^\alpha k[A] \to^j k[G],$$

where $\alpha$ is an isomorphism.

For finite abelian group $A$ any isomorphism of Hopf algebras $\alpha : k[A]^* \to k[A]$ is defined by the nondegenerated bimultiplicative form $\beta : \hat{A} \otimes \hat{A} \to k^*$. Indeed the Hopf algebra $k[A]^*$ coincides with the group Hopf algebra $k[\hat{A}]$ of the characters group of $A$. The isomorphism of group Hopf algebras $\alpha : k[\hat{A}] \to k[A]$ is defined by an isomorphism of groups $\hat{A} \to A$. It is enough to note that such isomorphisms are the same as nondegenerated bimultiplicative forms $\beta : \hat{A} \otimes \hat{A} \to k^*$.

By the theorem the inclusions $i, j : A \to G$, defined by unitary $R$-matrix, coincide, and isomorphism $\alpha : k[A]^* \to k[A]$ satisfies to the condition $\alpha^* = S\alpha$. The direct checking shows the equivalence of this condition to skewsymmetry of the corresponding form.

By the definition of Markov element:

$$u = \mu(S \otimes I)(R) = \frac{1}{\vert A \vert^2} \sum_{a,b \in A} \sum_{\chi, \xi \in \hat{A}} \beta(\chi, \xi)\chi(a)\xi(b)a^{-1}b =$$

$$\frac{1}{\vert A \vert^2} \sum_{a,c \in A} \sum_{\chi \in \hat{A}} \beta(\chi, \xi)\chi(\xi(a)c) = \frac{1}{\vert A \vert} \sum_{c \in A} \sum_{\chi \in \hat{A}} \beta(\chi, \chi)^{-1}\chi(c).$$

Now let us use the condition (3):

$$\xi(u) = \frac{1}{\vert A \vert} \sum_{c \in A} (\sum_{\chi \in \hat{A}} \beta(\chi, \chi)^{-1}\chi(c)) = \beta(\xi, \xi^{-1}) = \beta(\xi, \xi).$$

The fact that the Markov element is an involution follows from skewsymmetry of the form $\beta$. $\square$

Example.
Let $A$ be a cyclic group of order 2 with the generator $u$. There is unique nondegenerated skewsymmetric bimultiplicative form $\beta : \hat{A} \otimes \hat{A} \to k^*$. The
The corresponding $R$-matrix has the following form:

$$R_u = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u).$$

The representations category of the group $A$ with tensor structure, defined by this $R$-matrix, is equivalent to the tensor category $\mathcal{K}oz$, whose objects are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces with commutativity constraint, defined by Kozul rule of signs [6]:

$$x \otimes y \mapsto (-1)^{\bar{x}\bar{y}} y \otimes x$$

for homogeneous $x, y$. Here $\bar{x}$ denotes the degree of the element $x$. □

Let us note that for arbitrary unitary $R$-matrix $R$ in the group algebra $k[G]$ there is not tensor functor from the representations category of $G$ with tensor structure defined by the $R$ to the category of vector spaces. However the following fact take place.

**Proposition 6** For any tensor structure on the representations category $\mathcal{R}ep_k(G)$ of finite group $G$ there is a tensor functor to the category $\mathcal{K}oz$.

Proof:

Let $R$ be unitary $R$-matrix in $k[G]$, which defines the given tensor structure on the representation category $\mathcal{R}ep_k(G)$. Let $A$ be an abelian normal subgroup of $G$ and $\beta$ - nondegenerated skewsymmetric bimultiplicative form on the characters group $\hat{A}$, which corresponds to the $R$-matrix $R$.

Let us note that the Markov element $u$ of $R$-matrix $R$ belongs to the subgroup $A$. We can define bimultiplicative form $\beta_u$ on $\hat{A}$ as the composition

$$\beta_u : \hat{A} \otimes \hat{A} \to \langle u \rangle \otimes \langle \hat{u} \rangle \to k^*$$

of the restriction on the cyclic subgroup $\langle u \rangle$, generated by the element $u$, and unique nondegenerated skewsymmetric bimultiplicative form on $\langle \hat{u} \rangle$.

The forms $\beta, \beta_u$ are related as follows:

$$\beta_u(\chi, \chi) = \chi(u) = \beta(\chi, \chi) \quad \forall \chi \in \hat{A}.$$

Hence its ratio lies in the kernel of the map

$$\text{Hom}(\Lambda^2(\hat{A}), k^*) \to \text{Hom}(\hat{A}, k^*) \simeq A,$$
which sends skewsymmetric bimultiplicative form $\beta$ to the multiplicative form $l(\chi) = \beta(\chi, \chi)$. Let us note that for any abelian group $B$ the map $Hom(\Lambda^2(B), k^*) \to Hom(B, k^*)$ can be complemented to the exact sequence

$$H^2(B, k^*) \to Hom(\Lambda^2(B), k^*) \to Hom(B, k^*),$$

which first map sends the class of 2-cocycle $\gamma$ to the form $\beta(a, b) = \gamma(a, b)\gamma(b, a)^{-1}$ [3, 4]. In our case we can find the 2-cocycle $\gamma \in Z^2(A, k^*)$ such that

$$\beta(\chi, \xi)\gamma(\xi, \chi) = \beta_u(\chi, \xi)\gamma(\chi, \xi).$$

Let us define an element $F_\gamma \in k[G]^\otimes 2$ by

$$F_\gamma = \sum_{a,b \in A} \sum_{\chi,\xi \in \hat{A}} \gamma(\chi, \xi)\chi(a)\xi(b)a \otimes b.$$

By the construction it satisfies to the conditions:

$$Rt(F_\gamma) = R_u F_\gamma,$$

$$(u \otimes u)F_\gamma = F_\gamma(u \otimes u),$$

$$(1 \otimes F_\gamma)(I \otimes \Delta)(F_\gamma) = (F_\gamma \otimes 1)(\Delta \otimes I)(F_\gamma).$$

Hence it defines the tensor structure of the restriction functor

$$res^G_{\langle u \rangle} : Rep_k(G) \to Koz.$$

\[\square\]

6 Nonstandart $\lambda$-structures on the character rings of finite groups

Let us remind the definition of the $\lambda$-ring.

Let $A$ be a commutative ring without additive torsion. The set of maps $\lambda^i : A \to A \quad i \geq 0$, satisfying to the conditions

$$\lambda^0(x) = 1 \quad \lambda^1(x) = x \quad \lambda^i(x + y) = \sum_{s + t = i} \lambda^s(x)\lambda^t(y) \quad \text{for any} \quad x, y \in A,$$

(7)

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will be called \( \lambda \)-operations or \textit{external powers}. It is convinient to write the conditions (7) by means of generating series \( \lambda_t(x) = \sum_{i \geq 0} \lambda^i(x) \)

\[
\lambda_t(x + y) = \lambda_t(x) \lambda_t(y) \quad x, y \in A.
\]

The generating series is also useful to define another types of operations:

\( \sigma \)-operations or (\textit{symmetric powers}) \( \sigma^i : A \to A \quad i \geq 0 \), with generating series

\[
\sigma_t(x) = \lambda_{-t}(x)^{-1}
\]

and \textit{Adams operations} \( \psi^i : A \to A \quad i \geq 1 \), with generating series

\[
\psi_{-t}(x) = -i \frac{\lambda_t(x)}{\lambda_t(x)}.
\]

The generating series of these operations satisfies to the conditions:

\[
\sigma_t(x + y) = \sigma_t(x) \sigma_t(y) \quad \psi_t(x + y) = \psi_t(x) + \psi_t(y) \quad \text{for any} \quad x, y \in A.
\]

The ring \( A \) with the collection of \( \lambda \)-operations will be called \( \lambda \)-\textit{ring}, if its Adams operations are ring homomorphisms.

The tensor structure on the category of complex representations \( \mathcal{R}ep_k(G) \) of finite group \( G \), which corresponds to the unitary \( R \)-matrix \( R \), allows to define the structure of the \( \lambda \)-ring on the characters ring \( \mathcal{R}k(G) \) (the Grothendieck ring of the category \( \mathcal{R}ep_k(G) \)).

The action of the symmetric group \( S_n \) on the tensor power \( X \otimes^n \) of representation \( X \), defined by the tensor structure, provides the decomposition of \( X \otimes^n \) into the direct sum of isotypical components, labeled by the irreducible representations of \( S_n \). The \textit{external power} \( \Lambda^n(X) \) is an isotypical component \( \text{Hom}_{S_n}(\text{sign}, X \otimes^n) \) of the nontrivial one dimensional representation \( \text{sign} \).

The external power \( \lambda^n_R(X) \) of the class \( [X] \) of the representation \( X \) is by definition the class of its external power \( [\Lambda^n_R(X)] \). Using the additivity of external powers these operations can be extended to representations ring.

The \( \lambda \)-structure of representations ring can be expressed in more simple form by means of characters.

We can identify the representations ring with the subring (ring of characters) in the ring of complex functions of conjugate classes by corresponding to the representation the traces of elements of the group. Adams operations have the following form on the characters [2]:

\[
\psi^i(x)(g) = x(g^i) \quad \text{for any} \quad x \in \mathcal{R}C(G), g \in G.
\]

\[19\]
We will use the following proposition in the proof of the general statement of this article.

**Proposition 7** Let $G$ be an abelian $k$-linear tensor category over an algebraically closed fields $k$ of characteristic zero. Then for any monoidal automorphism $g$ of the identity functor of the category $G$

$$\text{Tr}(g_{\psi^k}(X)) = \text{Tr}(g^k_X) \quad \forall X \in R_k(G).$$

**Proof:**
Using the property $\psi^{nm} = \psi^n \psi^m$ of the Adams operations [9, 11] we can suppose that $k = p$ is a prime number.

Let us use the representation of Adams operation $\psi^p$ as a difference of cyclic operations $\psi^p = c_1^p - c_\varepsilon^p$, where $\varepsilon$ is a nontrivial degree $p$ root of unity [10, 1].

By the definition, the value of the cyclic operation, corresponding to degree $p$ root of unity $\varepsilon$, on the class of object $X$ is a class of the image of the projector on $p$-th tensor power $X^{\otimes p}$

$$c_\varepsilon^p(X) = \frac{1}{p} \sum_{i=0}^{p-1} \varepsilon^i \tau^i(X^{\otimes p}),$$

where $\tau$ is a cycle of length $p$ in the symmetric group $S_p$.

By the additivity of trace we have

$$\text{Tr}(g_{c_\varepsilon^p}(X)) = \frac{1}{p} \sum_{i=0}^{p-1} \varepsilon^i \text{Tr}(\tau^i g^{\otimes p}_X),$$

and using lemma 7.2 from [3] we can write $\text{Tr}_{X^{\otimes p}}(\tau^i g^{\otimes p}_X) = \text{Tr}_X(g^p_X)$. Hence the trace

$$\text{Tr}(g_{c_\varepsilon^p}(X)) = \text{Tr}_X(g^p_X) \frac{1}{p} \sum_{i=0}^{p-1} \varepsilon^i$$

equals zero, for nontrivial root of unity $\varepsilon$ and 1, if $\varepsilon = 1$. $\square$

**Theorem 4** The $\lambda$-structure of the representation ring $R_k(G)$ of finite group $G$, corresponding to the unitary $R$-matrix $R \in k[G]^{\otimes 2}$, depends only of its Markov element $u \in G$.

The Adams operations of this $\lambda$-structure have the following form:

$$\psi^k_u(\chi)(g) = \chi(u^{k+1}g^k), \quad \forall \chi \in R_k(G), g \in G.$$  \hspace{1cm} (8)
Proof:

Additivity of Adams operations allows to verify the equation (8) only for the characters of representations. Let \( \chi \) be the character of the representation \( X \in \mathcal{R}ep(G) \). By the definition \( \chi(g) \) coincides with the trace \( Tr_{\omega(X)}(g_X) \), where \( \omega : \mathcal{R}ep(G) \to k - \text{mod} \) is forgetful functor. By the proposition 2 we can replace \( Tr_{\omega(X)}(g_X) \) by \( Tr_X(u_Xg_X) \). In particular

\[
\psi^k_u(\chi)(g) = Tr_{\omega(X)}(g_{\psi^k_u(X)}) = Tr_{\psi^k_u(X)}((ug)_{\psi^k_u(X)}).
\]

By the proposition 3 the previous expansion coincides with the following

\[
Tr_X((ug)^k_X) = Tr_{\omega(X)}((u^{k+1}g^k)_X) = \chi(u^{k+1}g^k).
\]

\[\square\]

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