Clock-symmetric non-Hermitian second-order topological insulator

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Abstract. We consider a breathing Kagome lattice with complex hoppings by imposing \( \mathbb{Z}_3 \) clock symmetry in the complex-energy plane. It is a non-Hermitian generalization of the second-order topological insulator characterized by the emergence of topological corner states. We construct topological corner states with the \( \mathbb{Z}_3 \) clock-symmetric model. It is also shown that the model is realized in an electric circuit properly designed, where corner states are observed by impedance resonance. We also construct the \( \mathbb{Z}_4 \) and \( \mathbb{Z}_6 \) symmetric models on breathing square and honeycomb lattices, respectively.

1 Introduction

Higher order topological insulators and superconductors are generalization of topological insulators and superconductors \([1–14]\). They are prominent by the emergence of zero-energy corner states instead of gapless edge states. A typical example is given by the breathing Kagome lattice \([10]\), where three topological corner states emerge. The merit of this model is that we need only positive hopping terms, which enables us to realize it experimentally in various ways. Indeed, they are experimentally realized in acoustic \([15,16]\), photonic \([17,18]\) systems, solid-state surface \([19,20]\), and electric circuits \([21]\).

Non-Hermitian topological phases also attract much attention \([22–37]\). The eigenenergy is not necessary to be real in the non-Hermitian systems. There are also some generalization to non-Hermitian higher order topological insulators \([38–41]\). Non-Hermitian topological phases are realized in various systems such as photonic systems \([42–45]\), microwave resonators \([46]\), wave guides \([47]\), quantum walks \([48,49]\), cavity systems \([50]\), and electric circuits \([39,40,51–53]\).

Clock symmetry was proposed in the context of the clock-spin model \([54–56]\) to support parafermions. However, topological phases characterized by clock symmetry are yet to be explored. It is an interesting problem to construct models of topological insulators possessing clock symmetry. Clock symmetry requires that the energy spectrum forms clock-symmetric sets in complex plane. Such a system is necessarily non-Hermitian.

In this paper, we generalize the breathing Kagome second-order topological insulator model by imposing \( \mathbb{Z}_3 \) clock symmetry. This is a new type of non-Hermitian higher order topological insulator with the emergence of topological corner states. The energy spectrum is \( \mathbb{Z}_3 \) symmetric in the complex-energy plane as in the case of the \( \mathbb{Z}_3 \) clock-spin model. Furthermore, we present an electric circuit to implement the present model. We also construct the \( \mathbb{Z}_4 \) and \( \mathbb{Z}_6 \) clock-symmetric models with topological corner states on breathing square and honeycomb lattices.

This paper is composed as follows. In Sect. 2, we construct a second-order topological phase with \( \mathbb{Z}_3 \) clock symmetry based on non-Hermitian breathing Kagome model. We also discuss how to implement this model by an electric circuit. In Sects. 3 and 4, we make a similar construction of second-order topological phase with \( \mathbb{Z}_4 \) and \( \mathbb{Z}_6 \) clock symmetries based on the square and honeycomb lattices, respectively.

2 \( \mathbb{Z}_3 \) clock-symmetric model

The simplest clock symmetry is \( \mathbb{Z}_3 \) symmetry. It is represented by the clock operator \([56–58]\)

\[
\sigma = \text{diag.} \left( 1, \eta, \eta^2 \right),
\]

(1)

together with the shift operator \([56–58]\)

\[
\tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]

(2)

where

\[
\eta = e^{2\pi i/3}.
\]

(3)
Here, $\sigma$ and $\tau$ satisfy the $Z_3$ clock symmetric relations
\[
\sigma^3 = \tau^3 = 1, \quad \sigma\tau = \eta\tau\sigma.
\] (4)

In the $Z_3$ clock-symmetric model, the energy spectrum is composed of triplets [54, 55] $E_n^{(0,1,2)}$, $n = 0, 1, 2, \ldots$
\[
E_n^{(0,1,2)} = \varepsilon_n, \quad \eta\varepsilon_n, \quad \eta^2\varepsilon_n,
\] (5)
satisfying $\varepsilon_n + \eta\varepsilon_n + \eta^2\varepsilon_n = 0$. The system is necessarily non-Hermitian, because the eigen energies are complex except for zero-energy states. It also follows from Eq. (5) that only the zero-energy states form a set of degenerate states respecting the $Z_3$ clock symmetry.

We denote the zero-energy states as $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$. They satisfy
\[
\tau|\psi_0\rangle = |\psi_1\rangle, \quad \tau|\psi_1\rangle = |\psi_2\rangle, \quad \tau|\psi_2\rangle = |\psi_0\rangle,
\] (6)
\[
\sigma|\psi_0\rangle = |\psi_0\rangle, \quad \sigma|\psi_1\rangle = \eta|\psi_1\rangle, \quad \sigma|\psi_2\rangle = \eta^2|\psi_2\rangle.
\] (7)

These are key relations, because the matrix representations (2) and (1) are constructed from them, and hence, the $Z_3$ clock-symmetric relations (4) are derived. Namely, it is necessary and sufficient to examine Eqs. (6) and (7) for a triplet set of zero-energy states to show that they satisfy $Z_3$ clock symmetry.

### 2.1 Breathing Kagome lattice

The basic problem is the construction of a realistic $Z_3$ clock-symmetric model. We propose a model on the breathing Kagome lattice, whose structure in triangle geometry is illustrated in Fig. 1a2. The bulk Hamiltonian is defined by
\[
H = \begin{pmatrix}
0 & h_{12} & \eta h_{13} \\
h_{12}^* & 0 & \eta^2 h_{23} \\
\eta h_{13}^* & \eta^2 h_{23}^* & 0
\end{pmatrix},
\] (8)
with Eq. (3) for $\eta$ and
\[
h_{12} = t_a + t_be^{ik_s},
\] (9)
\[
h_{23} = t_a + t_be^{-i(k_s/2+\sqrt{3}k_s/2)},
\] (10)
\[
h_{13} = t_a + t_be^{-i(k_s/2-\sqrt{3}k_s/2)},
\] (11)
where we have introduced two hopping parameters $t_a$ and $t_b$, corresponding to the magenta link and the cyan link along the horizontal axis in Fig. 1a4. The hopping parameters along the other two triangle sides are given by $\eta t_a$ and $\eta^2 t_a$ for a magenta triangle, and $\eta t_b$ and $\eta^2 t_b$ for a cyan triangle. This model is non-Hermitian due to the presence of $\eta$.

A comment is in order. When the factor $\eta$ is absent, the breathing Kagome model is a typical model for the conventional second-order topological insulators [10] together with the emergence of topological corner states, where the Hamiltonian is Hermitian. The present generalization of the breathing Kagome lattice model provides us with a new type of non-Hermitian second-order topological insulators.

It is instructive to consider the following two limits. First, when $t_a = 0$, there are three isolated corners, as indicated by blue disks in Fig. 1a1. Apart from these corners, an edge is made of dimers, while the bulk is made of small-inverted triangles. We will soon see that this is a topological phase with three zero-energy corner states. On the other hand, when $t_b = 0$, all the sites are solely made of small-upward triangles, as shown in Fig. 1a3. We will see that this structure represents a trivial phase without corner states. In general, the system has nonzero $t_a$ and $t_b$ as in Fig. 1a2, but it is in the topological phase for $t_a \simeq 0$, while it is in the trivial phase for $t_b \simeq 0$.

#### 2.2 Clock symmetry

We analyze the symmetric properties of the Hamiltonian (8). First, it has the $Z_3$ clock symmetry [56].

![Fig. 1](image-url)

**Fig. 1** a Breathing Kagome lattice, b breathing square lattice, and c breathing honeycomb lattice. The hopping parameters are $t_a = 0$ for (a1–c1), $t_a t_b \neq 0$ for (a2–c2), and $t_b = 0$ for (a3–c3). There emerge three, four, and six corner states in the topological phase. When $t_a = 0$, they are isolated as marked by blue disks in (a1), (b1), and (c1), respectively. The unit cells are given in (a4), (b4), and (c4) with the hopping parameters indicated.
which implies the relation 

\[ \tau H(k) \tau^\dagger = \eta H(Rk), \]  

where \( R \) rotates the momentum by 120° as 

\[ R(k_x,0) = \left( \begin{array}{c} -k_x/2 \sqrt{3} k_y/2 \\ \end{array} \right), \]  

\[ R \left( \begin{array}{c} k_x/2 \sqrt{3} k_y/2 \\ \end{array} \right) = \left( \begin{array}{c} -k_x/2 \sqrt{3} k_y/2 \\ \end{array} \right), \]  

\[ R \left( \begin{array}{c} k_x/2 \sqrt{3} k_y/2 \\ \end{array} \right) = (k_x,0), \]  

making the energy spectrum have \( \mathcal{Z}_3 \) symmetry in the complex-energy plane as in Eq. (5). Second, it has reflection symmetry 

\[ H(k) = H^*(-k), \]  

making the energy spectrum have reflection symmetry between \( E \) and \( E^\ast \). It follows from these two properties that the energy spectrum has \( \mathcal{C}_3 \) symmetry in the complex-energy plane, which consists of the threefold rotational symmetry and three reflection symmetries. Moreover, it has a generalized chiral symmetry for a three-band model \[16\] 

\[ \sigma H \sigma^{-1} = H_1, \quad \sigma H_1 \sigma^{-1} = H_2, \]  

\[ H + H_1 + H_2 = 0, \]  

which implies the relation 

\[ E_n^{(0)} + E_n^{(1)} + E_n^{(2)} = 0. \]  

The Hamiltonian necessarily be non-Hermitian.

### 2.3 Topological number

Diagonalizing the Hamiltonian (8), we obtain the eigen function \( \psi_{\text{bulk}}(k_x,k_y) \) and the eigen energy \( E(k_x,k_y) \) for the bulk. We find it convenient to define a topological number for the bulk by the Berry phase as follows:

\[ Q = \frac{1}{2\pi i} \int_0^{2\pi} \langle \psi_{\text{bulk}}(k_x,0)| \partial_{k_y} |\psi_{\text{bulk}}(k_x,0) \rangle dk_x. \]  

We calculate it numerically, whose results are shown in Fig. 2a. We find \( Q = 1 \) for \(-1 < t_a/t_b < 1/2\), and \( Q = 0 \) for \( t_a/t_b < -1 \) and \( t_a/t_b > 2 \), while it continuously changes from 1 to 0 for \( 1/2 < t_a/t_b < 2 \). We show the topological number as a function of \( t_a/t_b \) in Fig. 2a.

As we see in the succeeding subsection, the analysis of the energy spectrum \( E(k_x,k_y) \) demonstrates that \( Q \) is quantized in the insulator phases as in Fig. 2b.

Actually, the topological number changes its value 

\[ Q \rightarrow Q + \alpha, \]  

if we make a gauge transformation.

\[ |\psi_{\text{bulk}}(k_x,0)\rangle \rightarrow e^{it\alpha} |\psi_{\text{bulk}}(k_x,0)\rangle. \]  

We have made a gauge choice to make \( Q = 0 \) in the trivial phase in accord with the bulk-boundary correspondence.

### 2.4 Energy spectrum

The notion of insulator and metal is generalized to the non-Hermitian Hamiltonian in two ways. One is a point-gap insulator [34,59], where \( |E| \) has a gap. The other is a line-gap insulator [34,59], where \( \text{Re}[E] \) or \( \text{Im}[E] \) has a gap. In our model, we adopt the definition of the point-gap insulator due to the presence of the \( \mathcal{Z}_3 \) clock symmetry.

We calculate the energy spectra of the Hamiltonian (8) numerically for the bulk, a nanoribbon, and a triangle. The main purpose is to show the emergence of zero-energy corner states in triangle geometry but no gapless edge states in nanoribbon geometry. Then, it follows from the topological number in Fig. 2a and the bulk-boundary correspondence that the Hamiltonian (8) describes a second-order topological insulator.

We show the numerical results of the energy spectrum in triangle geometry in Fig. 3. We clearly observe the \( \mathcal{Z}_3 \) clock symmetry in the complex-energy plane as in Fig. 3a and the emergence of isolated zero-energy states as in Fig. 3b, where the value of \( t_a/t_b \) is indicated by color.
To investigate the structure of the energy spectrum in more detail, we show $|E|$ as a function of $t_a/t_b$ in Fig. 2, $|E|$ as a function of the momentum $k$ in Fig. 4, and the complex-energy spectrum (Im[$E$], Re[$E$]) in Fig. 5, for several fixed values of $t_a/t_b$.

Let us focus on the bulk spectrum in Fig. 2b. Interestingly, we are able to obtain the energy spectrum analytically at the $\Gamma = (0, 0)$ point as

$$E(0, 0) = (t_a + t_b), \quad \eta(t_a + t_b), \quad \eta^2(t_a + t_b),$$

which are illustrated by cyan lines in the same figure, and at the $K = (4\pi/3, 0)$ and $K' = (-4\pi/3, 0)$ points as

$$E^3(\pm 4\pi/3, 0) = (t_a + t_b)(t_a - 2t_b)(2t_a - t_b),$$

which are illustrated by blue curves in the same figure. The point gap closes at the $K$ and $K'$ points for $t_a/t_b = 1/2, t_a/t_b = 2$, and at the $K, K'$ and $\Gamma$ points for $t_a/t_b = -1$. The agreement between the bulk spectrum obtained numerically and the energy spectrum analytically determined is reasonably good.

### 2.5 Bulk-edge correspondence and bulk-corner correspondence

The bulk-edge correspondence is an essential concept in ordinary topological insulators. It dictates that topological edge states emerge in a sample with edges when the system has a nontrivial topological number defined in the bulk. In the present system, accordingly, topological edge states emerge in nanoribbon geometry when the system is a first-order topological insulator. It is generalized to the bulk-corner correspondence in the second-order topological insulators. It dictates that topological corner states emerge in a sample with corners when the system has a nontrivial topological number defined in the bulk. In the present system, accordingly, topological corner states emerge in triangle geometry when the system is a second-order topological insulator.

### 2.6 Edge states

We compare the energy spectrum $|E|$ of a nanoribbon along the nanoribbon direction in Fig. 2c against the energy spectrum of the bulk in Fig. 2b as a function of $t_a/t_b$. They are almost identical with slight differences. In particular, there are no gapless states in the topological and trivial phases. However, the structure of the energy spectrum is rather different for the metallic phase ($1/2 \leq t_a/t_b < 2$).

We also make a similar comparison of the energy spectra $|E|$ in Fig. 4a2–h2 with Fig. 4a1–h1 as a function of the momentum. The absence of gapless states is clear in the topological and trivial phases.

We make a further comparison of the energy spectra but now in the (Im[$E$], Re[$E$]) plane as in Fig. 5a2–h2 and Fig. 5a1–h1. Here, the difference of the energy spectrum is clear for the metallic phase ($1/2 \leq t_a/t_b \leq 2$).
that this is the second-order topological insulator $k = 0$. We have used a nanoribbon with width 128 for the momentum of along the nanoribbon direction in and the vertical axis is $\text{Re}[E]$ of all, Fig. 5a2–h2, but all of them are gapped. 2). The emergence of edge states is clearly observed in Fig. 5a2–h2, but all of them are gapped.

### 2.7 Topological corner states

We now compare the energy spectra ($\text{Im}[E], \text{Re}[E]$) of the bulk, a nanoribbon, and a triangle in Fig. 5. First of all, $C_{3v}$ symmetry is manifested not only in the bulk (Fig. 5a1–h1) but also in the triangle (Fig. 5a3–h3). This is because the triangle respects $Z_3$ clock symmetry. The prominent feature is the emergence of zero-energy states in Fig. 5c3–e3. In particular, the zero-energy states in Fig. 5c3, d3 are the topological corner states, which are identical to the isolated zero-energy states, which are identical to the isolated zero-energy states in Fig. 3b and to the zero-energy states indicated by a magenta line in Fig. 2d. They are absent in the bulk spectrum (Fig. 5a1–h1) and the nanoribbon spectrum (Fig. 5a2–h2).

Consequently, the present model is a second-order topological insulator in the region $-1 < t_a/t_b < 1/2$, being characterized by the emergence of zero-energy corner states.

There are three zero-energy corner states $|\psi_i\rangle$, $i = 0, 1, 2$, which are expected to have $Z_3$ clock symmetry. To demonstrate this, it is necessary and sufficient that the wave functions satisfy the clock-symmetric relations (6) and (7). At a particular limit $t_a = 0$, we may represent the wave functions as

$$|\psi_0\rangle = (1, 0, 0)^t, \quad |\psi_1\rangle = (0, 1, 0)^t, \quad |\psi_2\rangle = (0, 0, 1)^t,$$

where we have picked up only three corner sites, since all the other components are zero. It is trivial to see that they satisfy Eqs. (6) and (7).

They must describe the $Z_3$ clock-symmetric relation even for $t_a \neq 0$, because the wave functions can be continuously connected to the state $t_a = 0$ as long as the system remains to be topological. We have numerically confirmed that they satisfy the relations (6) and (7) by obtaining numerically the wave functions. Therefore, they satisfy $Z_3$ clock symmetry.

### 2.8 Electric-circuit implementation

It is very hard to construct a model with $Z_3$ clock symmetry in condensed matter physics. However, it is possible to realize it by an electric circuit as in Fig. 6. A real hopping is represented by a condenser, while a complex hopping is by a set of a condenser and an operator amplifier as in Fig. 6a). Each node is grounded via an inductor as in Fig. 6b.

An electric circuit is governed by the Kirchhoff current law. By making the Fourier transformation with respect to time, the Kirchhoff current law is expressed as

$$I_a(\omega) = \sum_b J_{ab}(\omega) V_b(\omega), \quad (27)$$

where $I_a$ is the current between node $a$ and the ground, while $V_b$ is the voltage at node $b$. The matrix $J_{ab}(\omega)$ is called the circuit Laplacian. Once the circuit Laplacian is given, we can uniquely set up the corresponding electric circuit. By equating it with the Hamiltonian $H$ as $[60, 61]$

$$J_{ab}(\omega) = i\omega H_{ab}(\omega), \quad (28)$$

$$

Fig. 5 Complex-energy spectrum of the Hamiltonian (8) for various values of $t_a$ and $t_b$, where the horizontal axis is $\text{Im}[E]$ and the vertical axis is $\text{Re}[E]$. a1–h1 for the bulk, a2–h2 for a nanoribbon, and a3–h3 for a triangle. The values of $t_a$ and $t_b$ are the same as in Fig. 4. The numerical value on the horizontal axis in a1–h1 is the energy in unit of $t_b$. Color indicates the momentum of along the nanoribbon direction in a2–h2, where the red color indicates $k = \pi$ and the blue color indicates $k = 0$. We have used a nanoribbon with width 128 for a2–h2, and a triangle with size 16 for a3–h3. The green ellipses in a2–h2 and a3–c3 represent the edge states. The magenta dots in c3 and d3 represent the topological corner states. It is notable that there are no gapless edge states, but there are zero-energy corner states in the topological phase, indicating that this is the second-order topological insulator.
it is possible to simulate various topological phases of the Hamiltonian by electric circuits [14,39,40,60–68]. The relations between the parameters in the Hamiltonian and in the electric circuit are determined by this formula.

The circuit Laplacian is constructed as follows. To simulate the positive and negative hoppings in the Hamiltonian, we replace them with the capacitance and the inductance $1/i\omega L$, respectively. We note that $\sin k = (e^{ik} - e^{-ik})/2i$ represents an imaginary hopping in the tight-bind model. The imaginary hopping is realized by an operational amplifier [67].

We thus make the following replacements with respect to hoppings in the Hamiltonian to derive the circuit Laplacian: (i) $+X \rightarrow i\omega C_X$ for $X = t_a$ and $t_b$, where $C_X$ represents the capacitance whose value is $X$ [pF]. (ii) $-X \rightarrow 1/i\omega L_X$ for $X = t_a/2$ and $t_b/2$, where $L_X$ represents the inductance whose value is $X [\mu H]$.

We explicitly study the breathing Kagome lattice described by the Hamiltonian (8), which is decomposed as

$$H = H_1 + H_2,$$

with

$$H_1 = \begin{pmatrix} 0 & h_{12} & -\frac{1}{2}h_{13} \\ h_{12}^* & 0 & \eta h_{12} \\ -\frac{1}{2}h_{13}^* \eta^* h_{23} & \eta^* h_{12} & 0 \end{pmatrix},$$

and

$$H_2 = \frac{\sqrt{3}}{2} \begin{pmatrix} 0 & 0 & -ih_{13} \\ 0 & 0 & -ih_{23} \\ -ih_{13}^* & -ih_{23}^* & 0 \end{pmatrix},$$

where $H_1$ is Hermitian ($H_1^\dagger = H_1$), and $H_2$ is anti-Hermitian ($H_2^\dagger = -H_2$). It is necessary to construct imaginary hopping Hamiltonians

$$\frac{\sqrt{3}}{2}t_a \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

and

$$\frac{\sqrt{3}}{2}t_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{pmatrix},$$

for the magenta lines in Fig. 6a. They are constructed using operational amplifiers and resistors.

We review a negative impedance converter with current inversion based on an operational amplifier with resistors [67]. The voltage–current relation for the operational amplifier circuit is given by [67]

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{1}{R} \begin{pmatrix} -\nu & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

with $\nu = R_a/R_b$, where $R_a$ and $R_b$ are the resistances in an operational amplifier. We note that the resistors in the operational amplifier circuit are tuned to be $\nu = 1$ in the literature[67], so that the system becomes Hermitian, where the corresponding Hamiltonian represents a spin–orbit interaction.

In this paper, we use two negative impedance converters parallely connected with the opposite direction as in Fig. 6c. The circuit Laplacian due to these two converters is given by

$$\frac{1}{R} \left[ \begin{pmatrix} -\nu & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -\nu & 1 \end{pmatrix} \right] = \frac{1}{R} \begin{pmatrix} -\nu & 1 \\ -1 & 1 \end{pmatrix}.$$

It corresponds to the Hamiltonian

$$H = \frac{1}{i\omega R} \begin{pmatrix} 1 & -\nu & -1 \\ -\nu & 1 & -1 \\ -1 & -1 & -\nu \end{pmatrix}.$$

It is embedded in the $3 \times 3$ matrix as

$$H = \frac{1}{i\omega R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\nu & 1 & -1 \\ -1 & -\nu & 1 \\ -1 & -1 & -\nu \end{pmatrix},$$

where we have set

$$\sqrt{3} \frac{1}{2}t_a = \frac{1 - \nu}{\omega R},$$

with $\nu < 1$, and

$$H = \frac{1}{i\omega R} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\nu & 1 & -1 \\ -1 & -\nu & 1 \\ -1 & -1 & -\nu \end{pmatrix},$$

where we have set

$$\sqrt{3} \frac{1}{2}t_a = \frac{\nu - 1}{\omega R},$$

with $\nu > 1$. These matrices are different from Eqs. (32) and (33) by the diagonal terms. They are cancelled by adding a resistor (or $\nu > 1$) or an operational amplifier (for $\nu < 1$) with the amount of

$$\frac{1 - \nu}{i\omega R},$$

between a lattice site and the ground.
2.9 Impedance resonance

The zero-energy corner states are well observed by impedance resonance, which is defined \([62]\) by

\[
Z_{ab} = V_a / I_b = G_{ab},
\]

where \(G = J^{-1}\) is the Green function. It diverges at the frequency where the admittance is zero \((J = 0)\). Taking the nodes \(a\) and \(b\) at two corners, we show the impedance in topological, metallic, and trivial phases in Fig. 7a–c, respectively. A strong impedance peak is observed at the critical frequency \(\omega_0 \equiv 1/\sqrt{LC}\) only in the topological phase. It signals the emergence of three topological corner states. Because the circuit Laplacian is identical to the \(Z_3\) clock-symmetric Hamiltonian \((8)\), we may conclude that they are the \(Z_3\) clock-symmetric states.

3 \(Z_4\) clock-symmetric model

We proceed to a model with \(d = 4\). \(Z_4\) clock symmetry are represented by the shift operator[56–58]

\[
\gamma_1 \equiv \tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\]

and the clock operator[56–58]

\[
\gamma_2 \equiv \sigma = \text{diag}(1, i, -1, -i).
\]

Here, \(\tau\) and \(\sigma\) satisfy the \(Z_4\) clock symmetry relations

\[
\tau^4 = \sigma^4 = 1, \quad \tau \sigma = \eta \sigma \tau.
\]

In the \(Z_4\) clock-symmetric model, the energy spectrum is composed of quartets \(E_n^{(0,1,2,3)}:\)

\[
E_n^{(0,1,2,3)} = \varepsilon_n, \quad i \varepsilon_n, \quad -\varepsilon_n, \quad -i \varepsilon_n.
\]

The system is necessarily non-Hermitian, because the eigen energies are complex except for zero-energy states.

3.1 Zero-energy corner states

It follows from Eq. (46) that only the zero-energy states form a set of degenerate states respecting \(Z_4\) clock symmetry. We denote them as \(|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\) and \(|\psi_3\rangle\). They are characterized by the properties

\[
\tau |\psi_0\rangle = |\psi_1\rangle, \quad \tau |\psi_1\rangle = |\psi_2\rangle,
\]

\[
\tau |\psi_2\rangle = |\psi_3\rangle, \quad \tau |\psi_3\rangle = |\psi_0\rangle,
\]

\[
\sigma |\psi_0\rangle = i |\psi_0\rangle, \quad \sigma |\psi_1\rangle = i |\psi_1\rangle,
\]

\[
\sigma |\psi_2\rangle = -i |\psi_2\rangle, \quad \sigma |\psi_3\rangle = -i |\psi_3\rangle,
\]

from which the matrix representations (43) and (44) follow. Then, the \(Z_4\) clock symmetry relations (45) are verified. Namely, it is necessary and sufficient to examine Eqs. (47)–(50) for a quartet set of zero-energy states to show that they satisfy \(Z_4\) clock symmetry relation.
3.2 Breathing square lattice

The quadrupole insulator has been proposed on the breathing square lattice [2]. We propose a model possessing $Z_4$ clock symmetry on the breathing square lattice. The bulk Hamiltonian is given by

$$H = \begin{pmatrix} 0 & f_x^* & 0 & i f_y^* \\ -f_x & 0 & i f_y & 0 \\ 0 & -i f_y & 0 & -f_x \\ -i f_y & 0 & f_x^* & 0 \end{pmatrix},$$  \hspace{1cm} (51)

where

$$f_x = t_a + t_b e^{ik_x}, \quad f_y = t_a + t_b e^{ik_y},$$  \hspace{1cm} (52)

with $t_a$ and $t_b$, which are shown in Fig. 1b4.

The Hamiltonian (51) has the $Z_4$ clock symmetry [56]

$$\tau H (\mathbf{k}) \tau^\dagger = -iH (\mathbf{Rk}),$$  \hspace{1cm} (53)

where $R$ rotates the momentum by 90 degrees as

$$R (k_x, 0) = (0, -k_y),$$  \hspace{1cm} (54)

$$R (0, k_y) = (k_x, 0),$$  \hspace{1cm} (55)

making the energy spectrum have $Z_4$ symmetry in the complex-energy plane as in Eq. (46).

3.3 Edge and corner states

The topological number is defined by (19), where $\psi_{\text{bulk}}$ is the eigen function of the Hamiltonian (51) for the bulk. We find the topological insulator phase for $|t_a/t_b| < 1$, where $Q = 1$ and the trivial insulator phase for $|t_a/t_b| > 1$, where $Q = 0$.

We calculate the energy spectrum in square geometry numerically. $C_{4\sigma}$ symmetry is manifested, as shown in Fig. 8a. It is because the square lattice respects the $Z_4$ clock symmetry. We also show the energy spectrum as a function of $t_a/t_b$ in Fig. 8b, where the emergence of the zero-energy states is manifested in the topological phase.

Consequently, the present model is a second-order topological insulator in the region $|t_a/t_b| < 1$, being characterized by the emergence of topological corner states.

It is possible to obtain numerically the wave functions of the four corner states. We have confirmed that they satisfy the relations (47)–(50).

3.4 Electric-circuit implementation

In the Hamiltonian (51), there are nonreciprocal hopping terms along the $x$ axis. Nonreciprocal hopping is constructed by a combination of operational amplifier and capacitors [51]

$$\left( \begin{array}{c} I_{ij} \\ I_{ji} \end{array} \right) = i\omega \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} V_i \\ V_j \end{pmatrix}. \hspace{1cm} (56)$$

It corresponds to the Hamiltonian

$$H = C \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \hspace{1cm} (57)$$

We add an inductor or a capacitor to cancel the diagonal term.

We construct an electric circuit, as shown in Fig. 9a, where the $x$ axis is constructed by Fig. 9c and the $y$ axis is constructed by Fig. 9b.

4 $Z_6$ clock-symmetric model

Similarly, we construct a $Z_6$ clock-symmetric model. The clock symmetry operators are represented by the shift operator [56–58]

$$\gamma_1 \equiv \tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \hspace{1cm} (58)$$

and the clock operator [56–58]

$$\gamma_2 \equiv \sigma = \text{diag.} \ (1, \eta, \eta^2, \eta^3, \eta^4, \eta^5). \hspace{1cm} (59)$$
Here, $\tau$ and $\sigma$ satisfy the $Z_6$ clock symmetry relations

$$\tau^6 = \sigma^6 = 1, \quad \tau \sigma = \eta \sigma \tau.$$ (60)

In the $Z_6$ clock-symmetric model, the energy spectrum is composed of sextets $E_n^{(0,1,2,3,4,5)}$, $n = 0, 1, 2, \ldots$

$$E_n^{(0,1,2,3,4,5)} = \varepsilon_n, \quad \eta \varepsilon_n, \quad \eta^2 \varepsilon_n, \quad \eta^3 \varepsilon_n, \quad \eta^4 \varepsilon_n, \quad \eta^5 \varepsilon_n.$$ (61)

The system is necessarily non-Hermitian, because the eigen energies are complex except for zero-energy states.

A Hermitian second-order topological insulator has been proposed on the breathing honeycomb lattice [69]. We generalize it to a non-Hermitian model with the $Z_6$ clock-symmetric model on the breathing honeycomb lattice by introducing complex hoppings. The bulk Hamiltonian is defined on the breathing honeycomb lattice and given by

$$H = \begin{pmatrix}
0 & \eta^2 t_b & 0 & 0 & 0 & 0 \\
\eta^5 t_b & 0 & \eta^3 t_b & 0 & 0 & 0 \\
0 & \eta^3 t e^{-ik_y} & t_b & 0 & 0 & 0 \\
0 & \eta^4 t e^{i k_y} & \eta^2 t e^{-i k_y} \sqrt{\eta^3 - k_y} & 0 & 0 & 0 \\
\eta^2 t_b & 0 & 0 & 0 & \eta^5 t_b & 0 \\
0 & \eta^3 t_b & 0 & 0 & 0 & 0
\end{pmatrix}. \quad (62)$$

We diagonalize this Hamiltonian for a hexagon shown in Fig. 1c2. $C_{6v}$ symmetry is manifested in the complex-energy plane as in Fig. 10a. Furthermore, we find six zero-energy topological corner states for $|t_a/t_b| < 1$, as shown in Fig. 10. We also find the trivial insulator phase for $t_a/t_b < -2$ and $t_a/t_b > 1$. Additionally, there is metallic phase for $-2 < t_a/t_b < -1$.

5 Conclusion

We have constructed a $Z_3$ clock-symmetric model on the breathing Kagome lattice, a $Z_4$ clock-symmetric model on the breathing square lattice, and a $Z_6$ clock symmetric model on the breathing honeycomb lattice by imposing the $Z_d$ clock symmetry ($d = 3, 4, 6$). These model exhaust all the possible realization of the $Z_d$ clock symmetric models, since there are only threefold, fourfold, and sixfold rotational symmetries that are compatible with the periodic lattices.

We have shown that the clock symmetry is implemented in electric circuits rather easily. On the other hand, it is very hard to realize clock symmetry in condensed matter physics. A merit of electric circuits is that we may realize novel topological phases which are unreachable in condensed matter.

Recently, it is pointed out that the corner states in the breathing Kagome lattice are fragile in the presence of a certain type of long-range hopping terms [70]. These terms are possible mathematically, but they are very artificial. Indeed, the topological corner states in the breathing Kagome lattice have been experimentally observed in many physical systems including phononic [15, 16], photonic [17, 18] systems, solid-state surfaces [19, 20], and electric circuits [21].

A comment is in order. The topological classification of non-Hermitian topological phases with reflection symmetry has been studied [71]. On the other hand, there is no reflection symmetry in the present system due to the $\omega$ and $\omega^2$ contributions in the hopping terms although the lattice structure itself has reflection symmetry. It is an interesting problem to make a classification table in the presence of clock symmetry as a future problem.

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Data Availability Statement The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request. This manuscript has no associated data or the data will not be deposited. [Authors’ comment: All numerical results have been presented as graphical results in the text.]

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