Unity of CP and T violation in neutrino oscillations

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New Journal of Physics 6 (2004) 130
Received 21 April 2004
Published 7 October 2004
Online at http://www.njp.org/
doi:10.1088/1367-2630/6/1/130

Abstract. In a previous work, a simultaneous $P-CP[P]$ and $P-T[P]$ bi-probability plot was proposed as a useful tool for a unified graphical description of CP and T violation in neutrino oscillations. The diamond-shaped structure of the plot is understood as a consequence of the approximate CP–CP and T–CP relations obeyed by the oscillation probabilities. In this paper, we take a step forward towards a deeper understanding of the unified graphical representation by showing that these two relations are identical in their content, suggesting a truly unifying view of CP and T violation in neutrino oscillations. We suspect that the unity reflects the underlying CPT theorem. We also present a calculation of corrections to the CP–CP and the T–CP relations to leading order in $\Delta m_{21}^2/\Delta m_{31}^2$ and $s_{13}^2$. 

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1. Introduction

Exploring leptonic CP and T violation is one of the most challenging endeavours in particle physics. Confirming (or refuting) unsuppressed CP violation analogous to that in the quark sector must shed light on a deeper understanding of the lepton–quark correspondence, a concept whose importance was recognized early in the 1960s [1]. We should note, however, that it is only after the KamLAND experiment [2], which confirmed the MSW large mixing-angle (LMA) solution [3, 4] of the solar neutrino problem, that we are able to talk about detecting CP or T violation in an experimentally realistic setting. An almost maximal mixing of $\theta_{23}$ discovered by the atmospheric neutrino observation by Super-Kamiokande [5], which broke new ground in the field of research, also greatly encourages attempts towards measuring the leptonic Kobayashi–Maskawa phase $\delta$.

Yet, we might have the last impasse to observing leptonic CP violation, a too-small value of $\theta_{13}$, which lives in the unique unexplored (1–3) sector of the MNS matrix [6]. Currently, it is bounded from above by a modest constraint $\sin^2 2\theta_{13} \leq 0.15 - 0.25$ obtained by the Chooz reactor experiment [7]. Towards removing the last impasse, two different methods for measuring $\theta_{13}$ are proposed and materialized into a number of concrete experimental programs. The first is the measurement of appearance probability $P(\nu_\mu \rightarrow \nu_e)$ in long-baseline (LBL) experiments using an accelerator neutrino beam, being and to be performed by ongoing [8, 9] and next-generation projects [10]–[12]. The second is the reactor measurement of $\theta_{13}$. It is a pure measurement of $\theta_{13}$ independent of other oscillation parameters, $\delta$ and $\theta_{23}$, and thus will play a role complementary to the LBL experiments [13]. This property is expected to help in resolving the parameter degeneracy [14]–[19] related to $\theta_{23}$ [13]. A spur of experimental projects that took place worldwide for the relatively new opportunity are now summarized in the White Paper Report [20].

If such challenges are blessed by nature we will be able to proceed to measure the leptonic CP or T violating phase $\delta$. The relatively large value of $\theta_{13}$ will allow us to measure it via $\nu_e$ and $\bar{\nu}_e$ appearance measurement using conventional superbeam experiments, an idea which may be traced back to [21]. Feasible experimental programs for such appearance measurement with upgraded beams as well as detectors are proposed. See e.g., [10, 22] for the JPARC-Hyper-Kamiokande project and [11] for NO$\nu$A. It is also proposed that a fast search for CP violation...
can be performed by combining neutrino mode operation of such experiments with high statistics reactor measurement of $\theta_{13}$ [23].

If $\theta_{13}$ is too small to be seen in the above experiments, an entirely new strategy is called for. We will probably need a more aggressive approach with ambitious beam technologies, neutrino factory [24] or beta beam [25] or even both. Here also, vigorous worldwide activities for developing beam and target technologies as well as studying physics capabilities are underway [26]–[28]. Intense neutrino beams from a muon storage ring and the clean background for wrong sign muon detection are expected to lead to an enormous sensitivity of $\theta_{13}$ up to $\sim 1^\circ$. Enriched by the golden ($\nu_e \rightarrow \nu_\mu$) [29] as well as the silver ($\nu_e \rightarrow \nu_\tau$) [30] channels, it will be able to resolve all the parameter degeneracies, as claimed in [31]. See [32] for a review of old and new ideas on how to measure leptonic CP violation.

How does T violation measurement fit into the scene? In our understanding, it will probably come later than CP violation measurement because the measurement is more difficult to carry out. In neutrino factories it requires electron charge identification which is highly non-trivial, if not impossible. The beta beam, if built, would give us an ideal apparatus because it can deliver a pure $\nu_e$ beam which comes from decaying radioactive nuclei. By combining with superbeam (or neutrino factory) measurement of $P(\nu_\mu \rightarrow \nu_e)$, it will provide us with a unique opportunity for exploring leptonic T violation.

Keeping in mind the scope of experimental realization of CP and T violation measurement in the future, we discuss in this paper a unified view of leptonic CP and T violation, one of the most fundamental problems in particle physics. We hope that our discussion is illuminating and contributes to a deeper understanding of the problem. In this paper we use, except for the appendix, the standard notation of the MNS matrix [33].

2. CP and T violation in neutrino oscillation

It has been known for a long time that CP and T conservation are intimately related to each other by the CPT theorem. For neutrino oscillation in vacuum the invariance leads to a relation between neutrino and antineutrino oscillation probabilities

$$P(\nu_\alpha \rightarrow \nu_\beta; \delta) = P(\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha; \delta).$$

(1)

Then, the question might be ‘does there exist an analogous relation in neutrino oscillation in matter?’ It was shown in [34] that indeed there exists such a relationship,

$$P(\nu_\alpha \rightarrow \nu_\beta; \Delta m_{31}^2, \Delta m_{21}^2, \delta, a) = P(\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha; \Delta m_{31}^2, \Delta m_{21}^2, \delta, -a),$$

(2)

which comes from the classical time reversal and the complex conjugate of the neutrino evolution equation assuming that the matter profile is symmetric about the mid-point between production and detection. Let us call (2) the CPT relation in matter. Here, $a = 2\sqrt{2}G_FB E$ is the fundamental quantity which is related to the neutrino’s index of refraction in matter [3] with $G_F$ being the Fermi constant, $E$ the neutrino energy, and $N_e(x)$ an electron number density on Earth. The mass-squared difference of neutrinos is defined as $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ where $m_i$ is the mass of the $i$th eigenstate.
There is an immediate consequence of the CPT relation in matter, equation (2). If we define

\[ \Delta P_{CPT} \equiv P(\nu_\alpha \rightarrow \nu_\beta; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) - P(\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a), \]  

then, \( \Delta P_{CPT} \) is an odd function of \( a \). The property may be used to formulate the method for detecting extrinsic CPT violation in neutrino oscillation due to the matter effect [35].

Do the CPT relation and various other relationships between oscillation probabilities give a unified picture of CP and T violation in neutrino oscillation in matter? In this paper we argue that the answer is indeed yes. Although our argument in this paper is based on the line of thought in [34], we believe that we have taken a step forward from the previous work.

3. Unified graphical representation of CP and T violation

Towards the goal of this paper, let us introduce a graphical representation of the characteristic features of neutrino oscillations relevant for leptonic CP violation [15]. For simplicity of notation let us define the symbols for CP and T conjugate probabilities, \( CP[P] \equiv P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) \) and \( T[P] \equiv P(\nu_\beta \rightarrow \nu_\alpha) \), for a given probability \( P(\nu_\alpha \rightarrow \nu_\beta) \). It is the CP trajectory diagram in the \( P–CP[P] \) bi-probability space, which can be extended to incorporate the \( P–T[P] \) bi-probability plot [34].

Given two observables \( P \) and \( CP[P] \), one can draw a dot in \( P–CP[P] \) space, and it becomes a closed ellipse when \( \delta \) is varied. In figure 1 the ellipses labelled \( V_\pm \) are the ones for vacuum oscillation probabilities where the subscripts \( \pm \) denote the sign of \( \Delta m^2_{31} \). When the matter effect is turned on they split into two ellipses labelled \( CP_\pm \) in the CP bi-probability plot and into \( T_\pm \) in the T bi-probability plot, both of which are simultaneously depicted in figure 1.

Let us first focus on the \( P–CP[P] \) bi-probability plot. We first note that the oscillation probability \( P(\nu_\alpha \rightarrow \nu_\beta) \) can be written on very general grounds (for \( \alpha = e, \beta = \mu, \tau \), or vice versa)\(^2\) as [38, 39]

\[ P(\nu_\alpha \rightarrow \nu_\beta) = A \cos \delta + B \sin \delta + C, \]  

where \( A, B, \) and \( C \) are functions of \( \Delta m^2_{31}, \Delta m^2_{21} \) and \( a \). (Of course, the previously obtained approximate formulae do have such a form; see, e.g., [29, 40, 41].) It is nothing but (4) that guarantees the elliptic nature of the trajectories. Then, one can show that the lengths of major and minor axes (the ‘polar’ and ‘radial’ thickness of the ellipses) represent the size of the \( \sin \delta \) and the \( \cos \delta \) terms, respectively, whereas the distance between two ellipses with positive and negative \( \Delta m^2_{31} \) displays the size of the matter effect [15]. Finally, the distance to the centre of the ellipse from the origin is essentially given by \( \sin^2 2\theta_{13} \). Notice that all the features of the bi-probability plot except for the distance between \( \Delta m^2_{31} = \pm \) ellipses are essentially determined by the vacuum parameters in the setting of \( E \) and \( L \) relevant for the superbeam experiments. Therefore, one can easily guess how it looks like in the other experimental settings. As indicated in figure 1, the CP violating and CP conserving effects of \( \delta \) are comparable in size with the matter effect even at such high energy and long baseline.

\(^2\) In the \( \nu_\mu – \nu_\tau \) channel, there exist \( \sin 2\delta \) and \( \cos 2\delta \) terms in the oscillation probability, and the bi-probability diagrams are no longer elliptic.
Figure 1. A simultaneous $P-T[P]$ and $P-CP[P]$ bi-probability plot with experimental parameters corresponding to the baseline distance and about twice the optimal energy corresponding to maximal enhancement of T violating effect [36]. Notice the difference between movement of the direction in $P-T[P]$ and $P-CP[P]$ plot; they are orthogonal to each other for reasons explained in the text. The figure is the same as figure 1 of [34]; apart from that we have changed the convention of $\delta$ to the standard one used in the text of the paper. The convention is employed by almost everybody who works in the field (see, e.g., [14, 18]) including all our previous works, [15, 19, 34, 37], but is different from that of [33]. Although the convention of the MNS matrix is the same, the latter takes a convention such that $U$ in the neutrino evolution equation (A.1) is replaced by $U^*$. We notice that in the $P-T[P]$ bi-probability plot, the matter effect splits the vacuum ellipses $V_{\pm}$ in quite a different way from the $P-CP[P]$ bi-probability plot. This is because the T violating measure $\Delta P_T$, which is given by

$$\Delta P_T \equiv P(\nu_\alpha \to \nu_\beta; \Delta m_{13}^2, \delta, a) - P(\nu_\beta \to \nu_\alpha; \Delta m_{13}^2, \delta, a) = 2B \sin \delta$$

for symmetric matter profiles, vanishes at $\delta = 0$. Therefore, the T (or CP) conserving point must remain on the diagonal line in the $P-T[P]$ bi-probability plot.

Equation (5) stems from the fact that the coefficients except for $B$ are symmetric under the interchange $\alpha \leftrightarrow \beta$. Therefore, if $\Delta P_T \neq 0$, then $\delta \neq 0$ even in matter. The matter effect cannot create a fake T violation for symmetric matter profiles [42]. For modifications which occur for asymmetric matter profiles, see e.g. [43]–[45].
Notice that the matter effect cannot create fake T violation; it does modify the coefficient $B$ in equation (5), whose feature is made transparent in [36]. Among other things, it was shown in [36] that the matter effect can enhance the T asymmetry up to a factor of 1.5. Other earlier references on T violation in neutrino oscillation include [46]–[49].

4. Diamond shaped structure of CP and T bi-probability plot

In view of figure 1 we notice a remarkable feature of the simultaneous $P–CP[P]$ and $P–T[P]$ diagrams; it has a square or diamond shape. The diamond-shaped structure of combined $P–CP[P]$ and $P–T[P]$ diagrams can be understood by the following two relations which are called the CP–CP and the T–CP relations in [34]. Their precise statements are:

CP–CP relation:

$$P(v_e \rightarrow v_\mu; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(\bar{v}_e \rightarrow \bar{v}_\mu; -\Delta m^2_{31}, -\Delta m^2_{21}, \delta, a)$$

$$\approx P(\bar{v}_e \rightarrow \bar{v}_\mu; -\Delta m^2_{31}, +\Delta m^2_{21}, \pi + \delta, a).$$  (6)

T–CP relation:

$$P(v_\mu \rightarrow v_e; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(\bar{v}_e \rightarrow \bar{v}_\mu; -\Delta m^2_{31}, -\Delta m^2_{21}, 2\pi - \delta, a)$$

$$\approx P(\bar{v}_e \rightarrow \bar{v}_\mu; -\Delta m^2_{31}, +\Delta m^2_{21}, \pi - \delta, a).$$  (7)

These relations are meant to be valid to leading order in $\Delta m^2_{21}/\Delta m^2_{31}$, i.e., to zeroth order in $\delta$-independent and to first order in $\delta$-dependent terms, respectively.

Roughly speaking, the CP–CP relation guarantees that the locations of the first and the third bases are approximately symmetric under reflection with respect to the diagonal line in $P–CP[P]$ space, whereas the T–CP relation guarantees that the ordinates of the $T_\pm$ ellipse are approximately the same as those of $CP_\pm$. Of course, one has to specify the values of the CP phase $\delta$ to make the relationship precise, and that is why the change in $\delta$ is involved between the RHS and the LHS of equations (6) and (7).

A rough sketch of the argument given in [34] is as follows. The first equality in (6) is obvious by noticing

$$P(v_e \rightarrow v_\mu; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(v_e \rightarrow v_\mu; -\Delta m^2_{31}, -\Delta m^2_{21}, -\delta, -a),$$  (8)

which follows from the fact that a complex conjugate of the neutrino evolution equation gives the same oscillation probability, and the simple relation (see e.g. [41])

$$P(\bar{v}_e \rightarrow \bar{v}_\mu; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(v_e \rightarrow v_\mu; \Delta m^2_{31}, \Delta m^2_{21}, -\delta, -a).$$  (9)

Then, the second approximate equality follows to leading order in $\Delta m^2_{21}/\Delta m^2_{31}$ after an appropriate shift of $\delta$ which takes care of the sign change in $\delta$-dependent terms.

For the T–CP relation, the first equality in (7) can be derived by using (8) in the CPT relation in matter (2). Then, the second approximate equality holds for small $\Delta m^2_{21}/\Delta m^2_{31}$ with the same adjustment of the phase $\delta$. 

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5. Equivalence between the CP–CP and the T–CP relations

We now point out that the CP–CP and the T–CP relations are equivalent to each other in their physics contents. Roughly speaking, the T–CP relation is a ‘T conjugate’ of the CP–CP relation. It reflects the relationships among various neutrino oscillation probabilities discussed in the previous section. Their equivalence again testifies to the unity of CP and T violation in neutrino oscillations.

We present the proof of the above statement through the computation of the corrections to the CP–CP and the T–CP relations. Let us define for clarity of notation the deviation from the CP–CP and the T–CP relations as

\[ \Delta_{PCP} \equiv P(\nu_e \rightarrow \nu_\mu; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) - P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu; -\Delta m^2_{31}, +\Delta m^2_{21}, \pi + \delta, a), \]  

\[ \Delta_{PT} \equiv P(\nu_\mu \rightarrow \nu_e; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) - P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu; -\Delta m^2_{31}, +\Delta m^2_{21}, \pi - \delta, a). \]

Because of the first equality in (6), \( \Delta_{PCP} \) can be written as

\[ \Delta_{PCP} = P(\nu_e \rightarrow \nu_\mu; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) - P(\nu_e \rightarrow \nu_\mu; +\Delta m^2_{31}, -\Delta m^2_{21}, \pi + \delta, a). \]  

On the other hand, by using the first equality in (6) and the relation

\[ P(\nu_\mu \rightarrow \nu_e; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(\nu_e \rightarrow \nu_\mu; \Delta m^2_{31}, \Delta m^2_{21}, -\delta, a), \]

which is valid for symmetric matter density profiles, \( \Delta_{PT} \) can be cast into the form

\[ \Delta_{PT} = P(\nu_e \rightarrow \nu_\mu; \Delta m^2_{31}, \Delta m^2_{21}, -\delta, a) - P(\nu_e \rightarrow \nu_\mu; +\Delta m^2_{31}, -\Delta m^2_{21}, \pi - \delta, a). \]  

Therefore, it holds that

\[ \Delta_{PT}(\delta) = \Delta_{PCP}(-\delta) \text{ (mod $2\pi$)}. \]

Namely, the relation (11) is the T-conjugate of (10). This completes the proof that the CP–CP and the T–CP relations are identical in their content.

In passing, we note the following: it was noted in [15] that there exists an approximate symmetry in the vacuum oscillation probability under the simultaneous transformations \( \Delta m^2_{31} \rightarrow -\Delta m^2_{31} \) and \( \delta \rightarrow \pi - \delta \) which explains the almost overlap of \( V_+ \) and \( V_- \) trajectories as in figure 1. A generalization of the approximate symmetry in the case with matter effect has been obtained [34],

\[ P(\nu_\alpha \rightarrow \nu_\beta; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) \approx P(\nu_\alpha \rightarrow \nu_\beta; -\Delta m^2_{31}, \Delta m^2_{21}, \pi - \delta, -a), \]

from which the CP–CP and the T–CP relations also follow. Clearly, the correction to the approximate symmetry is also related to \( \Delta P_{CP-CP} \). To show this, we define the difference \( \Delta P_{flip} \) between the LHS and RHS of (16),

\[ \Delta_{flip} \equiv P(\nu_\alpha \rightarrow \nu_\beta; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) - P(\nu_\alpha \rightarrow \nu_\beta; -\Delta m^2_{31}, \Delta m^2_{21}, \pi - \delta, -a). \]

Using the frequently used identity, one can show that

\[ \Delta_{flip} = P(\nu_\alpha \rightarrow \nu_\beta; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) - P(\nu_\alpha \rightarrow \nu_\beta; +\Delta m^2_{31}, -\Delta m^2_{21}, \pi + \delta, a). \]

Thus, \( \Delta_{flip} = \Delta_{CP-CP} \); they are identical.
6. Leading-order corrections to the CP–CP and the T–CP relations

We now compute the leading-order corrections to the CP–CP and the T–CP relations. During the course of the computation, we will give an explicit proof of these relations. We start from the Kimura–Takamura–Yokomakura (KTY) formula [38] of the oscillation probability, equation (4). We note that the coefficients \( A, B \) and \( C \) are functions of \( \Delta m^2_{21}, \Delta m^2_{31}, \) and the matter coefficient \( a \), but we here suppress dependences on the latter two quantities. We also note that \( A \) and \( B \) start from first order in \( \Delta m^2_{21} \), so that we can write \( A(x) = x\alpha(x) \) and \( B(x) = x\beta(x) \). Using the fact that \( \sin(\pi + \delta) = -\sin(\delta) \) and \( \cos(\pi + \delta) = -\cos(\delta) \), we obtain

\[
\Delta P_{CP-CP} \equiv P(\Delta m^2_{21}, \delta) - P(-\Delta m^2_{21}, \pi + \delta)
\]

\[
= [A(\Delta m^2_{21}) + A(-\Delta m^2_{21})] \cos \delta + [B(\Delta m^2_{21}) + B(-\Delta m^2_{21})] \sin \delta 
+ [C(\Delta m^2_{21}) - C(-\Delta m^2_{21})]
\]

\[
= \alpha(\Delta m^2_{21}) - \alpha(-\Delta m^2_{21}) \Delta m^2_{21} \cos \delta + [\beta(\Delta m^2_{21}) - \beta(-\Delta m^2_{21})] \Delta m^2_{21} \sin \delta 
+ [C(\Delta m^2_{21}) - C(-\Delta m^2_{21})].
\] (19)

Therefore, we have shown that the RHS of (19) is of order \( \epsilon \equiv \Delta m^2_{21}/\Delta m^2_{31} \) (C term), or \( \epsilon^2 \) (\( A \) and \( B \) terms). This is an explicit proof of the CP–CP relation, and hence also the T–CP relation.

We are now left with the computation of the first-order terms of \( \alpha, \beta \) and \( C \). The exact form of these coefficients are calculated in [38]. Therefore, it is straightforward to compute the RHS of (19). It reads

\[
\Delta P_{CP-CP} = + 16 \frac{a \Delta m^2_{21} \Delta m^2_{31}}{(\Delta m^2_{31} - a)^3} s^2_{13} c^2_{13} s^2_{12} \sin^2 \left[ \frac{(\Delta m^2_{31} - a)L}{4E} \right]
- 8 \left( \frac{\Delta m^2_{31}L}{4E} \right) \left( \frac{\Delta m^2_{31}}{\Delta m^2_{31} - a} \right)^2 s^2_{13} c^2_{13} s^2_{12} \sin^2 \left[ \frac{(\Delta m^2_{31} - a)L}{2E} \right]
- 16 J_r \left( \frac{\Delta m^2_{31}}{\Delta m^2_{31} - a} \right)^2 \left( \frac{\Delta m^2_{31}}{a} \right)^2 (s^2_{12} - c^2_{12}) + \left( \frac{\Delta m^2_{31}}{a} \right) (c^2_{12} - s^2_{12}) - s^2_{12} \right] 
\times \sin \left( \frac{aL}{4E} \right) \sin \left( \frac{\Delta m^2_{31} - a) L}{4E} \right) \cos \left( \frac{\Delta m^2_{31}L}{4E} \right)
+ 8 J_r \left( \frac{\Delta m^2_{31}L}{4E} \right) \left( \frac{\Delta m^2_{31}}{a} \right) \left( \frac{\Delta m^2_{31}}{\Delta m^2_{31} - a} \right) \left[ (s^2_{12} - c^2_{12}) \sin \left( \frac{\Delta m^2_{31}L}{2E} \right) 
+ c^2_{12} \sin \left( \frac{\Delta m^2_{31}L}{2E} \right) - s^2_{12} \sin \left( \frac{\Delta m^2_{31}L}{2E} \right) \right].
\] (20)

where \( J_r \equiv c_{12}s_{12}c_{23}s_{23}c^2_{13}c_{13} \). \( \Delta P_{T-CP} \) can be obtained by replacing \( \delta \) by \( 2\pi - \delta \) in \( \Delta P_{CP-CP} \), as dictated in (15). We have kept in the expression of the oscillation probability the terms up to order \( O(\epsilon^2 s^2_{13}) \) and \( O(s^4_{13}) \) in \( C \), and to \( O(\epsilon^2 s^2_{13}) \) and \( O(\epsilon^3 s^3_{13}) \) in \( A \) and \( B \). But the contributions from higher order terms in \( s_{13} \) cancel in (20).

\footnote{Note, however, that there is an error in the sign of the term denoted as \( A_k^{(1)} \) in equation (44) of the first reference in [38].}
The feature that the coefficients $A$ and $B$ start with first-order terms of $\Delta m^2_{21}$ played an important role in proving the CP–CP relation to leading order. It comes from the fact that they vanish in the two-flavour limit $\Delta m^2_{21} \to 0$ and that the probabilities allow Taylor expansion in terms of the variable. The former statement is proved in the appendix on very general grounds without assuming adiabaticity or constant matter density.

7. Concluding remarks

In this paper, we have presented a new unified view of the leptonic CP and T violation in neutrino oscillations. Based on the CPT relation in matter and other relations obeyed by the oscillation probabilities which are derived in [34] we were able to complete our understanding of the structure of a unified description of CP and T violation in terms of the bi-probability plot. Namely, the diamond-shaped structure of simultaneous $P–CP[P]$ and $P–T[P]$ bi-probability plot is now understood as a consequence of a unique relation, the CP–CP (or equivalently, the T–CP) relation. Based on this observation and relying on the KTY formula we have computed leading-order corrections to the CP–CP relation.

We have also briefly touched upon the basic features of the T violating measure which are in contrast with those of CP violation. They include vanishing T violating measure at vanishing CP phase $\delta$, and the enhancement of T-violating asymmetry by the matter effect up to a factor of 1.5. Though measurement of T violation should give us a cleaner way of detecting genuine CP violating effects, it is not easy to carry out experimentally. We must wait for the construction of an intense electron (anti-) neutrino beam either by beta beam [25] or in neutrino factories [24].

Acknowledgments

We thank Hiroshi Nunokawa and Stephen Parke for kindly redrawing the figure and for the useful comments. MB wishes to thank the Sweden–Japan Foundation for financing his visit to Tokyo Metropolitan University. This work was supported by the Grant-in-Aid for Scientific Research in Priority Areas No. 12047222, Japan Ministry of Education, Culture, Sports, Science, and Technology, and the Grant-in-Aid for Scientific Research, No. 16340078, Japan Society for the Promotion of Science.

Appendix. No $\delta$-dependence in the two-flavour limit

Although it should be the case on physics grounds, it is not entirely trivial to show that $\delta$-dependence disappears from the oscillation probabilities in the two-flavour limit $\Delta m^2_{21} \to 0$. We carry it out explicitly in this appendix. It is a slight modification of the method [50] that allows us to show that $\delta$-dependence disappears in the survival probability $P(\nu_e \to \nu_e)$.

We write down the evolution equation of three flavour neutrinos in matter which is valid to leading order in electroweak interaction:

$$i \frac{d}{dx} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \frac{1}{2E} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{pmatrix} U^+ + \begin{pmatrix} a(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}. \tag{A.1}$$

In this appendix we take a slightly different parametrization of the mixing matrix

$$U = e^{i \delta_2 \beta_2} \Gamma_3 e^{i \delta_3 \theta_{13}} e^{i \delta_1 \beta_1}, \tag{A.2}$$
where $\lambda_i$ are SU(3) Gell-Mann’s matrices and $\Gamma$ contains the CP-violating phase

$$\Gamma_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix}. \quad (A.3)$$

We then rewrite the evolution equation (A.1) in terms of the new basis defined by

$$\tilde{\nu}_\alpha = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\lambda_7 \theta_{23}} \end{array} \right] \tilde{\nu}_\alpha \equiv (T_t)_{\alpha\beta} \tilde{\nu}_\beta. \quad (A.4)$$

In the vanishing $\Delta m^2_{12}$ limit, it reads

$$i \frac{d}{dx} \begin{bmatrix} \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \end{bmatrix} = \frac{1}{2E} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta m^2_{31} \end{bmatrix} e^{-i\lambda_2 \theta_{13}} + a(x) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \end{bmatrix}. \quad (A.5)$$

Now we observe that the CP phase $\delta$ disappears from the equation. It is due to the specific way that the matter effect comes in; $a(x)$ only appears in the (1.1) element in the Hamiltonian matrix and therefore the matter matrix diag($a$, 0, 0) is invariant under rotation in 2–3 space by $e^{i\lambda_7 \theta_{23}}$. Then the rotation by the phase matrix $\Gamma$ does nothing. Notice that $\tilde{\nu}_\mu$ does not have time evolution due to (A.5).

It is clear from (A.5) that any transition amplitude computed in the $\tilde{\nu}_\alpha$ basis is independent of the CP violating phase. Of course, it does not immediately imply that the CP violating phase $\delta$ disappears in the physical transition amplitude $\langle \nu_\beta | \nu_\alpha \rangle$. The latter is related to the transition amplitude defined in the $\tilde{\nu}_\alpha$ basis as

$$\langle \nu_\beta | \nu_\alpha \rangle = T_{\alpha\gamma}^* T_{\beta\delta} \langle \tilde{\nu}_\gamma | \tilde{\nu}_\delta \rangle, \quad (A.6)$$

where $T$ is defined in (A.4) and its explicit form in our parametrization (A.2) of the mixing matrix reads

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta} \\ 0 & -s_{23} & c_{23} e^{i\delta} \end{bmatrix}. \quad (A.7)$$

One can show that the amplitude of $\nu_e \to \nu_\mu$ has a pure phase factor $e^{i\delta}$ and hence $P(\nu_e \to \nu_\mu)$ is independent of phase $\delta$;

$$\langle \nu_\mu(x) | \nu_e(0) \rangle = c_{23} \langle \tilde{\nu}_\mu(x) | \tilde{\nu}_e(0) \rangle + s_{23} e^{i\delta} \langle \tilde{\nu}_\tau(x) | \tilde{\nu}_e(0) \rangle. \quad (A.8)$$

The first term, however, vanishes because $\langle \tilde{\nu}_\mu(x) | \tilde{\nu}_e(0) \rangle = \langle \tilde{\nu}_\mu(0) | \tilde{\nu}_e(0) \rangle = 0$. (No evolution in $\tilde{\nu}_\mu$.) Notice that the same statement does apply to the $P(\nu_e \to \nu_\tau)$ and $P(\nu_\mu \to \nu_\tau)$ as well. One can show that the same conclusion holds for different choices of the phase matrix from that in (A.2).

Since absence or presence of T violation should not depend on the parametrization used, this completes the proof that the $\delta$ dependence disappears from all the oscillation probabilities in the limit $\Delta m^2_{12} \to 0$. 

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