Valuation of Survivorship Life Insurance with Stochastic Rates of Return and Dependent Mortality

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Abstract. Life insurance has been one of the many options for people with concerns about the uncertainty of their future, because it is designed to protect against the serious financial impact that results from an individual’s death. An important variation of the single life insurance is the survivorship life insurance which covers two or more lives. Under such contract, a death benefit is paid out only on the last death. Insurance valuation is a very important tool for many insurance companies to be able to know its financial status in order to meet its future obligations. The valuation of an insurance policy comprises of 2 main components, they are: rates of return and mortality assumption. Traditionally, actuaries assume a constant interest rates and an independent mortality assumption in valuing joint-life insurance for the sake of simplicity. However, there has been considerable interest in the actuarial literature in studying the use of stochastic interest rate models and dependent mortality assumptions for insurance valuations. In this study, the mathematical expressions to value survivorship life insurance with stochastic rates of return and dependent mortality are presented. The valuation is conducted by the means of calculation of the expected value of the prospective loss random variable, by assuming an AR(1) process and Frank’s Copula to model the rates of return and the dependent mortality of the lifetimes of the policy holders, respectively.

1. Introduction
In the world of insurance, valuation is an important tool for an insurance company to estimate a future liability or asset of the company at different points in time. Therefore, survivorship life insurance valuation is a necessity for the company issuing the policy. In this study, the valuation is done by calculating reserves the company needs to prepare to fulfill its future obligations. In Cunningham [1], reserve is expressed as the expected value of the loss random variable. In this study, we focus on the prospective loss random variable described in Chen et al. [2].

Suppose there is a portfolio consisting of \( m \) fully discrete survivorship life insurance policies that are still in force at current time \((t = 0)\). The \( i \)th policy, with policy term \( n_i \), is issued at time \( r_i \), to a married couple aged \((x_i : y_i)\). Suppose the policy is valued at time \( t \) in the future, \( 0 < t < n_i - r_i \), with the following conditions:

(A1) The death benefit, \( b_i \), is payable at the end of the year of the last death (second death) if the death of the last person occurs before the maturity time \( n_i - r_i \).

(A2) The pure endowment benefit, \( c_i \), is payable at maturity if at least one of the two of the policyholders is still alive at the maturity time \( n_i - r_i \).
(A3) The level premium, \( \pi_i \), payable by the policyholder at the beginning of each year as long as at least one of the policyholders is alive.

Unlike single-life policies, a survivorship policy that is in force could be in one of three states. At current valuation time, \( t = 0 \), at least one life insured in the \( i \)th contract (\( i = 1, 2, ..., m \)) has survived \( r_i \) years (\( r_i < n_i \)), and therefore its survivorship status, denoted as \( u_i \), becomes:

\[
u_i = \begin{cases} 
x_i + r_i : y_i + r_i, & \text{if both insureds are alive.} 
\ x_i + r_i, & \text{if only the husband is alive.} 
\ y_i + r_i, & \text{if only the wife is alive.}
\end{cases}
\]  

(1)

2. The Prospective Loss Random Variable

As described earlier, the \( i \)th policy in the portfolio with current status \( u_i \) was issued \( r_i \) years ago. Let \( tL_{r_i}(u_i) \) be the prospective loss random variable at time \( t \) for the \( i \)th contract with conditions (A1)-(A3), given that the policy is still in force at time \( t < n_i - r_i \). For \( t < n_i - r_i \), the prospective loss random variable for the \( i \)th policy, \( tL_{r_i}(u_i) \), is expressed as:

\[
tL_{r_i}(u_i) = \begin{cases} 
\ b_i v(t, t + K_{u_i+t}) - \pi_i \bar{a}(t, t + K_{u_i+t}), & 1 \leq K_{u_i+t} \leq n_i - r_i - t, 
\ c_i v(t, n_i - r_i) - \pi_i \bar{a}(t, n_i - r_i), & K_{u_i+t} > n_i - r_i - t.
\end{cases}
\]  

(2)

where \( K_{u_i+t} \) is the discrete time-to-death random variable for the last death of the \( i \)th policy with status \( u_i + t \), and then \( v(.) \) and \( \bar{a}(.) \) denote the discount factor and annuity factor for present value calculations. Equation (2) illustrates that in the occurrence of \( tL_{r_i}(u_i) \), there are two possible events. The first event, denoted by Event I, occurs when \( 1 \leq K_{u_i+t} \leq n_i - r_i - t \), and the second event, denoted by Event II, occurs when \( K_{u_i+t} > n_i - r_i - t \). Further explanations about these two events are described in figure 1.

![Timeline of loss of the ith policy](image)

Figure 1. Timeline of loss of the \( i \)th policy \( tL_{r_i}(u_i) \).

To simplify the notation, let \( f_{i,k_i}(t) \) be the random variable that represents the occurrence of Event I and \( s_i(t) \) be the random variable that represents the occurrence of Event II. \( f_{i,k_i}(t) \) and \( s_i(t) \) are defined as follows for \( 0 < t < n_i - r_i \):

\[
f_{i,k_i}(t) = b_i v(t, t + k_i) - \pi_i \bar{a}(t, t + k_i), \quad 1 \leq k_i \leq n_i - r_i - t, \\
s_i(t) = c_i v(t, n_i - r_i) - \pi_i \bar{a}(t, n_i - r_i), \quad k_i > n_i - r_i - t, \quad i = 1, 2, ..., m.
\]  

(3)  

(4)
Then, $tL_{r_i}(u_i)$ for $i = 1, 2, ..., m$, can be rewritten in the following form:

$$tL_{r_i}(u_i) = f_i k_i(t) 1_{\{K_{m_{i+t}}-k_i, 1 \leq k_i \leq n_i-r_i-t\}} + s_i(t) 1_{\{K_{m_{i+t}}-k_i, k_i > n_i-r_i-t\}}, \quad 0 < t < n_i - r_i. \quad (5)$$

where $1_{\{K_{m_{i+t}}\}}$ is an indicator variable. Therefore, it follows from equation (5) that

$$E[tL_{r_i}(u_i)] = \sum_{k_i=1}^{n_i-r_i-t} E[f_i k_i(t)] 1_{\{k_i, k_i \leq n_i-r_i-t\}} + E[s_i(t)] n_i-r_i-t \sum_{k_i=1}^{n_i-r_i-t}, \quad 0 < t < n_i - r_i. \quad (6)$$

where $E[f_i k_i(t)] = b_i E[v(t, t+k_i)] - \pi_i E[u(t, t+k_i)]$, $E[s_i(t)] = c_i E[v(t, n_i-r_i)] - \pi_i E[u(t, n_i-r_i)]$.

Now let $tL$ denote the prospective loss random variable of the portfolio evaluated at future time $t$, that is: $tL = \sum_{i=1}^{m} tL_{r_i}(u_i) \cdot 1_{\{t < n_i-r_i\}}$, which sums the random variables $tL_{r_i}(u_i)$ defined in equation (5) over all policies in force, for $i = 1, 2, ..., m$. Therefore it follows that $E[tL] = \sum_{i=1}^{m} E[tL_{r_i}(u_i)]$, which can be calculated based on equation (6).

### 3. AR(1) Process for the Rates of Return

In this section, we use the assumption in Chen et al. [2] to illustrate the valuation approach. Suppose the rate of return is constant throughout the year of each policy year, and let $\delta(k)$ denote the future rate of return earned during the $k$th year, that is $(k-1, k]$, for $k \in \mathbb{N}^+$. A stochastic process AR (1) for the sequence $\{\delta(k); k = 1, 2, \ldots\}$ is defined as follows:

$$\delta(k) = \delta + \phi(\delta(k-1) - \delta) + e_k, \quad k \in \mathbb{N}^+, \quad (7)$$

where $\delta(k)$ is the rate of return in year $k$, $\delta(k-1)$ is the rate of return observed in the previous year, $\{e_k; k \in \mathbb{N}^+\}$ is a sequence of i.i.d. random variables following a normal distribution with mean zero and finite variance $\sigma^2$, i.e. $e_k \sim N(0, \sigma^2)$, $\delta$ is the long-term mean force of the process, and $\phi$ is the autoregressive coefficient of the process, with $|\phi| < 1$.

The alternative form of equation (7) is as follows:

$$\delta(k) = \delta + \phi^k [\delta(0) - \delta] + \sum_{i=0}^{k-1} \phi^i e_{k-i}, \quad k \in \mathbb{N}^+, \quad (8)$$

so if it is known that $\delta(0) = \delta_0$, the expectation and variance of $\delta(k)$, given $\delta(0)$, are as follows:

$$E[\delta(k)] = \delta + \phi^k [\delta_0 - \delta], \quad Var[\delta(k)] = \frac{\sigma^2 (1 - \phi^{2k})}{1 - \phi^2}, \quad Cov[\delta(s), \delta(k)] = \frac{\sigma^2}{1 - \phi^2} \phi^{k-s} (1 - \phi^2),$$

where $s < k$. Furthermore, the accumulated function of the rate of return from time $t$ to time $j$ is expressed as follows:

$$I(t, j) = \begin{cases} \sum_{k=t+1}^{j} \delta(k), & t < j, \\ 0, & t = j. \end{cases}$$

By assuming an AR (1) process, Chen [2] proposes a proposition stating that if the rate of return in year $k$, $\delta(k)$, follows the AR (1) process, then according to $E[\delta(k)]$, $Var[\delta(k)]$, and $Cov[\delta(s), \delta(k)]$, the mean and variance of $I(t, j)$, given $\delta(0) = \delta_0$, are as follows:

$$E[I(t, j)] = (j - t)\delta + (\delta_0 - \delta) \frac{\phi(\delta^j - \delta^t)}{1 - \phi}, \quad (9)$$

...
Suppose $v(t, j)$ denotes the discount factor which discounts a payment made at time $j$ to time $t$, for $t < j$. $v(t, j)$ is defined as follows:

$$v(t, j) = e^{-I(t,j)}, \quad 0 \leq t \leq j, \quad t, j \in \mathbb{N}^+.$$  \hfill (11)

From equation (7), it is known that given $\delta(0)$, each $\delta(k)$, for $k \in \mathbb{N}^+$, is normally distributed, which implies that $v(t, j) = e^{-I(t,j)}$ is lognormally distributed. Therefore, the expectation of $v(t, j)$ based on the property of the lognormal distribution is as follows:

$$E[v(t, j)] = e^{-E[I(t,j)]+0.5\text{Var}[I(t,j)]}, \quad t \leq j, \quad t, j \in \mathbb{N}^+.$$  \hfill (12)

Suppose $\bar{a}(t, j)$ denotes the present value at time $t$ of an annuity that pays 1 at the beginning of each year starting from time $t$ to $j - 1$. Then from equation (12) it follows that:

$$E[\bar{a}(t, j)] = E\left[\sum_{k=t}^{j-1} v(t, k)\right] = \sum_{k=t}^{j-1} E[v(t, k)], \quad t < j, \quad t, j \in \mathbb{N}^+. \hfill (13)$$

$E[\bar{a}(t, j)]$ and $E[v(t, j)]$ is the annuity factor and the discount factor needed to calculate $E[f_{i,k}(t)]$ and $E[s_i(t)]$, which is then required to calculate $E\left[\sum_{i=1}^{t} L_{r_i}^{(u_i)}\right]$ in equation (6).

### 4. Frank’s Copula Model for Dependent Mortality

In this section, we describe the mortality assumption for illustration, a Copula model with the marginal distribution function of Gompertz, to model the random variable $K_{u+t}$. In this study, as in Frees et al. [3], we use Frank’s Copula by Frank (1979) as follows:

$$C(u, v) = \frac{1}{\alpha} \ln \left(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1}\right), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1,$$

where $U$ and $V$ are random variables with marginal cumulative distributions $u$ and $v$, $\alpha \neq 0$ is a parameter measuring the dependence between the two underlying random variables, and the resulting Frank’s Copula, $C(u, v)$, is the joint distribution for $U$ and $V$.

Suppose the lifetimes of the policyholders follows the Gompertz model. Suppose $X$ and $Y$ is Gompertz distributed with distribution functions $G_1(x)$ dan $G_2(y)$, that is:

$$G_1(x) = 1 - e^{-v_1/\tau_1(1-e^{x/\tau_1})}, \quad x > 0, \quad \text{and} \quad G_2(y) = 1 - e^{-v_2/\tau_2(1-e^{y/\tau_2})}, \quad y > 0, \hfill (14)$$

where parameters $v_i$ and $\tau_i, \ i = 1, 2,$ represents the mode and the scale parameter of the distribution. Under the assumption of Frank’s Copula, the joint distribution of $X$ and $Y$ for $x, y > 0$ is as follows:

$$G(x, y) = \frac{1}{\alpha} \ln \left(1 + \frac{(e^{\alpha G_1(x)} - 1)(e^{\alpha G_2(y)} - 1)}{e^{\alpha} - 1}\right). \hfill (15)$$

Thus, by using Frank’s Copula, $k_{p_{xy}}$, which is the probability of status $(xy)$ will survive up to time $k$ is expressed in the form of $G, G_1,$ and $G_2$ as follows:

$$k_{p_{xy}} = \frac{1 - G_1(x + k) - G_2(y + k) + G(x + k, y + k)}{1 - G_1(x) - G_2(y) + G(x, y)}, \quad k \in \mathbb{N}^+.$$  \hfill (16)
and $kq_{xy}$, which is the probability of status $(x, y)$ will fail or die at time $k$ is as follows:

$$
kq_{xy} = \frac{G(x + k, y + k) - G(x, y + k) - G(x + k, y) + G(x, y)}{1-G_1(x) - G_2(y) + G(x, y)}, \quad k \in \mathbb{N}^+.
$$

In Cunningham [1], it is known that $tp_x = S_X(x + t) = S_X(x) = 1 - F_X(x + t) = 1 - F_X(x)$, therefore we have

$$
kp_x = 1 - kq_x = 1 - \frac{1-G_1(x+k)}{1-G_1(x)}, \quad \text{and} \quad kp_y = 1 - kq_y = 1 - \frac{1-G_2(y+k)}{1-G_2(y)}.
$$

The probabilities $kp_{xy}$, $kq_{xy}$, $kp_x$, and $kp_y$ are useful for the calculations of $k_{i-1|q_{ui+t}}$ and $n_{i-r_i-tp_{ui+t}}$, which is required for the calculation of $\mathbb{E}\left[tL_{r_i}^{(ui)}\right]$ in equation (6).

5. The Calculation of the Expected Loss of the Portfolio $\mathbb{E}[L]$ 
In this section, we will show the calculation of an illustrative non-homogeneous portfolio that consists of 6 survivorship life insurance policies as described in table 1. Since the focus of this study is to conduct a valuation, then we refer the parameters for the Copula mortality model in Frees et al. [3], they are $v_1 = 85.82$, $\tau_1 = 9.98$, $v_2 = 89.40$, $\tau_2 = 8.12$, and $\alpha = -3.367$. The rate of return is modeled by the AR(1) process in equation (7), with the parameters $\delta = 0.06$, $\delta_0 = 0.04$, $\phi = 0.9$, and $\sigma = 0.01$. The calculation results are presented in table 2.

| Policy |
|--------|
| $i$    |
| $x_i$  |
| $y_i$  |
| $b_i$  |
| $c_i$  |
| $n_i$  |
| 1      | 35 | 30 | 3000 | 0 | 20 |
| 2      | 35 | 30 | 3000 | 3000 | 10 |
| 3      | 45 | 40 | 1000 | 0 | 20 |
| 4      | 45 | 40 | 1000 | 1000 | 30 |
| 5      | 55 | 50 | 2000 | 0 | 30 |
| 6      | 55 | 50 | 2000 | 2000 | 15 |

| Table 1. Description of the illustrative non-homogeneous portfolio. |

| Policy year elapsed $r_i$ | $\mathbb{E}\left[tL_{r_i}^{(ui)}\right]$ | Future valuation time $t$ | $\mathbb{E}\left[tL_0^{(ui)}\right]$ |
|---------------------------|---------------------------------|--------------------------|---------------------------------|
| 1                         | 360.9397                        | 1                        | 348.4374                        |
| 2                         | 739.5166                        | 2                        | 712.4657                        |
| 3                         | 1136.205                        | 3                        | 1093.988                        |
| 4                         | 1551.404                        | 4                        | 1494.927                        |
| 5                         | 1985.414                        | 5                        | 1917.216                        |
| 6                         | 2438.407                        | 6                        | 2362.797                        |
| 7                         | 2910.387                        | 7                        | 2833.575                        |
| 8                         | 3401.151                        | 8                        | 3331.627                        |

| Table 2. Calculation results for the non-homogeneous portfolio. |
The calculation results in table 2 will be explained. On the left side of the table, suppose an insurance company issues the non-homogeneous portfolio $r_i$ years ago, for $r_i = 1, 2, ..., 8$. At the present time, namely $t = 0$, the insurance company wants to know how much fund it needs to prepare to cover the loss that might occur from the portfolio. Therefore, the company needs to prepare a fund of 360,9397 for a portfolio issued 1 year ago, a fund of 739,5166 for a portfolio issued 2 years ago, and so on. Furthermore, on the right side of the table, suppose an insurance company issues the same non-homogeneous portfolio at the present time, i.e. $r_i = 0$, and wants to know how much fund it needs to prepare $t$ years from now, for $t = 1, 2, ..., 8$. Therefore, within 1 year from now, the company needs to set aside a fund of 348,4374, within 2 years from now, a fund of 712,4657, and so on.

In the end, information from this calculation can be useful for insurance companies to determine the amount of reserves they need to prepare at present time or in the future.

6. Conclusion

This study incorporates the valuation approach presented by Chen [4] and Chen et al. [2], which explored the valuation of a general (non-homogeneous) survivorship insurance portfolio in a stochastic interest rate environment and dependent mortality assumptions. Since it is still a new topic in its infancy, the idea of this study is to mainly show the formulas of the expected value of prospective loss random variable to value a survivorship life insurance with stochastic rates of return and dependent mortality assumption, by assuming an AR (1) process and Frank’s Copula to illustrate the valuation approach. The models used in this study are in no way the best models and the authors recommend further research regarding this subject.

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