Dark Radiation Dynamics on the Brane

Rui Neves* and Cenalo Vaz†
Área Departamental de Física/CENTRA, FCT, Universidade do Algarve
Campus de Gambelas, 8000-117 Faro, Portugal

Abstract

We investigate the dynamics of a spherically symmetric vacuum on a Randall and Sundrum 3-brane world. Under certain natural conditions, the effective Einstein equations on the brane form a closed system for spherically symmetric dark radiation. We determine exact dynamical and inhomogeneous solutions, which are shown to depend on the brane cosmological constant, on the dark radiation tidal charge and on its initial energy configuration. We identify the conditions defining these solutions as singular or as globally regular. Finally, we discuss the confinement of gravity to the vicinity of the brane and show that a phase transition to a regime where gravity is not bound to the brane may occur at short distances during the collapse of positive dark energy density on a realistic de Sitter brane.

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1 Introduction

It is possible to localize matter fields in a 3-brane world embedded in a higher dimensional space if gravity is allowed to propagate away from the brane in the extra dimensions (see for example [1] and the early related work in [2]). The extra dimensions were usually compactified to a finite volume but it was realized that they could also be infinite and non-compact as in the model of Randall and Sundrum (RS) who proposed gravity to be confined to the vicinity of the brane by the warp of a single extra dimension [3].

In the RS brane world scenario the observable universe is a 3-brane boundary of a non-compact $Z_2$ symmetric 5-dimensional Anti-de Sitter (AdS) space. The matter fields are restricted to the brane but gravity exists in the whole AdS bulk. The classical dynamics is defined by the 5-dimensional Einstein field equations with a
negative bulk cosmological constant and a Dirac delta source representing the brane. The original RS solution involves a non-factorizable bulk metric for which the light-cone into the fifth dimension is closed by an exponential warp factor

$$d\tilde{s}^2 = e^{-2\gamma|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2,$$

where $\eta_{\mu\nu}$ is the Minkowski metric in 4 dimensions, $\gamma = \sqrt{-\Lambda\tilde{\kappa}^2/6}$ with $\Lambda$ the negative bulk cosmological constant and $\tilde{\kappa}^2 = 8\pi/M_p^2$ with $M_p$ denoting the fundamental 5-dimensional Planck mass. The brane cosmological constant was assumed to be fine-tuned to zero and so $\Lambda = -\tilde{\kappa}^2\lambda^2/6$ or $\gamma = \tilde{\kappa}^2\lambda/6$ where $\lambda$ is the brane tension. Weak field perturbations around this static vacuum solution were also analysed to reveal a new way to solve the hierarchy problem and show that 4-dimensional Einstein gravity is effectively recovered on the brane at low energy scales if the AdS radius $1/2\gamma$ is small enough. In particular Newton’s potential $V_N$ generated by a mass $M$ receives warp corrections which to leading order give

$$V_N = \frac{G_N M}{r} \left(1 + \frac{1}{\gamma^2 r^2}\right),$$

where $G_N$ is Newton’s gravitational constant. Extensions of this scenario were rapidly developed to allow for example non-fine-tuned branes and thick branes. The Gauss-Codazzi formulation used in was discussed earlier in.

Friedmann-Robertson-Walker cosmologies, static black holes and stars, the Vaidya solution as well as Oppenheimer-Snyder gravitational collapse solutions were also discovered within the RS brane world scenario. However, it should be noted that the static black hole solutions discussed in are for a 4-dimensional bulk and the existence of black hole solutions in the bulk that reduce to the a static black hole localized on the brane remain unknown. Moreover, cosmological perturbations have started to be implemented. So far a consistent adjustment to the available data coming out of astrophysics, cosmology and also high energy particle collision experiments has been shown to hold. In addition, it has been shown that the AdS/CFT duality is compatible with the RS scenario.

Studies of the gravitational collapse of matter have been limited to static or homogeneous dynamical models. Notable examples are the 5-dimensional black string solution, the tidal Reissner-Nordström black hole, the Vaidya solution on the brane and the Oppenheimer-Snyder model. As a result it has become clear that the exterior vaccum of a collapsing distribution of matter on the brane cannot be static. This was shown to be a consequence of the presence of gravitational modes in the bulk, which act as a kind of intrinsically dynamical dark matter in the whole brane world. These gravitational degrees of freedom are generated by the existence of Weyl curvature in the bulk and are a source of inhomogeneities in matter clouds located on the brane.
The purpose of this paper is to analyse the spherically symmetric, inhomogeneous vacuum on the brane. This is a necessary step towards the understanding of a realistic collapse problem in the RS scenario. We take an effective 4-dimensional point of view following the Gauss-Codazzi covariant geometric approach [7, 10, 11] (see also Ref. [27] for a review and notation). In section 2 we introduce the Einstein vacuum field equations on the brane. Although these equations do not, in general, form a closed system, additional conditions may be imposed to form a closed system described by two free parameters, viz., the brane cosmological constant \( \Lambda \) and the dark radiation tidal charge \( Q \). In this section we also present a discussion on the localization of gravity near the brane. For small but positive \( \Lambda \) we show that while for \( Q < 0 \) the tidal acceleration in the off-brane direction is always negative, implying acceleration towards the brane, for \( Q > 0 \) the acceleration changes sign at a critical short distance. In section 3 we present \( \Lambda \) and \( Q \) dependent inhomogeneous dynamical solutions. We show that they take the LeMaître-Tolman-Bondi form and depend on the initial configuration of the dark radiation, which is parametrized by a single function, \( f = f(r) > -1 \). This is interpreted as the energy function. The marginally bound solution \( (f(r) = 0) \) is the static, zero mass tidal Reissner-Nordström black hole solution of [17]. In section 4 we analyse the dynamics of the non-marginally bound, inhomogeneous solutions and characterize precisely how they depend on \( \Lambda \), \( Q \) and \( f(r) \). We also identify the conditions under which the dark radiation dynamics leads to the formation of a singularity as the final outcome of collapse or to a globally regular evolution with a bounce developing inside the vacuum. We conclude in section 5.

## 2 Vacuum Field Equations on the Brane

In the RS brane world scenario the Einstein field equations in the bulk are [3, 27]

\[
\tilde{G}_{AB} = \tilde{\kappa}^2 \left[ -\tilde{\Lambda}\tilde{g}_{AB} + \delta(y) (-\lambda g_{AB} + T_{AB}) \right],
\]

(3)

The 4-dimensional 3-brane is located at \( y = 0 \) and is a fixed point of the \( Z_2 \) symmetry. Its induced metric is \( g_{AB} = \tilde{g}_{AB} - n_An_B \) where \( n_A \) is the spacelike unit normal to the brane. Matter confined to the brane is characterized by the energy-momentum tensor \( T_{AB} \) and satisfies \( T_{AB}n^B = 0 \). The 5-dimensional metric is of the RS form

\[
d\tilde{s}^2 = \tilde{g}_{AB}dx^Adx^B = dy^2 + g_{\mu\nu}dx^\mu dx^\nu,
\]

(4)

where \( g_{\mu\nu} \) should display a dependence on the fifth dimension which localizes gravity near the brane and closes the light-cone.

According to the effective geometric approach [7, 10, 11, 27] the induced Einstein field equations on the brane are obtained from Eq. (3), the Gauss-Codazzi equations and the Israel conditions with \( Z_2 \) symmetry. In the vaccum \( T_{AB} \) is set to zero, as a consequence of which the induced equations take the form

\[3\]
\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} - \mathcal{E}_{\mu\nu}, \]  

where

\[ \Lambda = \frac{\tilde{\kappa}^2}{2} \left( \tilde{\Lambda} + \frac{\tilde{\kappa}^2 \lambda^2}{6} \right) \]

is the brane cosmological constant and \( \mathcal{E}_{\mu\nu} \) is the limit on the brane of the projected 5-dimensional Weyl tensor,

\[ \mathcal{E}_{\mu\nu} = \lim_{y \to 0^\pm} \delta^A_{\mu} \delta^B_{\nu} \mathcal{E}_{AB} = \lim_{y \to 0^\pm} \delta^A_{\mu} \delta^B_{\nu} \tilde{C}_{ACBD} \tilde{n}^C \tilde{n}^D. \]

It is a symmetric and traceless tensor due to the Weyl symmetries and is constrained by the conservation equations

\[ \nabla_{\mu} \mathcal{E}_{\nu}^{\mu} = 0, \]

obtained from Eq. (3) as a result of the Bianchi identities.

The system of vacuum field equations on the brane defined by Eqs. (5) and (8) may naturally be interpreted as defining the interaction of 4-dimensional Einstein gravity with matter represented by the traceless energy-momentum tensor

\[ \kappa^2 T_{\mu\nu} = -\mathcal{E}_{\mu\nu}, \]

where \( \kappa^2 = 8\pi/M_p^2 \) with \( M_p = 1/\sqrt{G_N} \) the effective Planck mass on the brane and

\[ \kappa^2 = \frac{\lambda \tilde{\kappa}^4}{6}. \]

Using general algebraic symmetry properties it is possible to write \( \mathcal{E}_{\mu\nu} \) as

\[ \mathcal{E}_{\mu\nu} = -\left( \frac{4}{\kappa} \right)^4 \mathcal{U} \left( u_{\mu} u_{\nu} + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_\mu u_{\nu} + \mathcal{Q}_\nu u_{\mu}, \]

where \( u_\mu \) such that \( u^\mu u_\mu = -1 \) is the 4-velocity field and \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) is the tensor which projects orthogonally to \( u_\mu \). The forms \( \mathcal{U}, \mathcal{P}_{\mu\nu} \) and \( \mathcal{Q}_\mu \) represent the effects on the brane of the free gravitational field in the bulk. Thus, \( \mathcal{U} \) is the effective energy density,

\[ \mathcal{U} = -\left( \frac{\kappa}{\kappa} \right)^4 \mathcal{E}_{\mu\nu} u^\mu u^\nu, \]

\( \mathcal{P}_{\mu\nu} \) is the anisotropic stress,

\[ \mathcal{P}_{\mu\nu} = -\left( \frac{\kappa}{\kappa} \right)^4 \left[ \frac{1}{2} \left( h^\alpha_{\mu} h^\beta_{\nu} + h^\alpha_{\nu} h^\beta_{\mu} \right) - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta} \right] \mathcal{E}_{\alpha\beta}, \]
and $Q_\mu$ is the effective energy flux

$$Q_\mu = \left(\frac{\kappa}{\tilde{\kappa}}\right)^4 h_\mu^\alpha \mathcal{E}_{\alpha\beta} u^\beta.$$  \hspace{1cm} (14)

Even though $E_{\mu\nu}$ is trace free and has to satisfy the conservation equations, in general it cannot be fully determined on the brane. This is to be expected because observers confined to the brane cannot predict all bulk effects without a knowledge of the solution of the full 5-dimensional Einstein equations. As a consequence, the effective 4-dimensional geometric theory is not closed. However, under additional simplifying assumptions about the bulk degrees of freedom it is possible to close the system if $E_{\mu\nu}$ is constrained in such a way that it may be completely determined by its symmetries and by the conservation equations. This is what happens when $P_{\mu\nu} = 0$ \cite{25} or if $P_{\mu\nu} \neq 0$ when a spherically symmetric brane is static \cite{17}.

Let us now show that it is possible to take a non-static spherically symmetric brane with $P_{\mu\nu} \neq 0$ and still close the system of dynamical equations. Consider the general, spherically symmetric metric in comoving coordinates $(t, r, \theta, \phi)$,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^\sigma dt^2 + A^2 dr^2 + R^2 d\Omega^2,$$  \hspace{1cm} (15)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, $\sigma = \sigma(t, r)$, $A = A(t, r)$, $R = R(t, r)$ and $R$ is interpreted as the physical spacetime radius. If there is no net energy flux, then $Q_\mu = 0$. Again, if the stress is isotropic, $P_{\mu\nu}$ will have the general form

$$P_{\mu\nu} = \mathcal{P} \left( r_\mu r_\nu - \frac{1}{3} h_{\mu\nu} \right),$$  \hspace{1cm} (16)

where $\mathcal{P} = \mathcal{P}(t, r)$ and $r_\mu$ is the unit radial vector, given in the above metric by $r_\mu = (0, A, 0, 0)$. The projected Weyl tensor then takes the diagonal form

$$\mathcal{E}^\nu = \left(\frac{\kappa}{\tilde{\kappa}}\right)^4 \text{diag}(\rho, -p_r, -p_T, -p_T),$$  \hspace{1cm} (17)

where the energy density and pressures are respectively $\rho = \mathcal{U}$, $p_r = (1/3) (\mathcal{U} + 2\mathcal{P})$ and $p_T = (1/3) (\mathcal{U} - \mathcal{P})$. Substituting in the conservation Eq. \cite{8} we obtain the following expanded system \cite{30}

$$2\frac{\dot{A}}{A} (\rho + p_r) = -2\dot{\rho} - 4\frac{\dot{R}}{R} (\rho + p_r),$$

$$\sigma' (\rho + p_r) = -p_r' + 4\frac{R'}{R} (p_T - p_T),$$  \hspace{1cm} (18)

where the dot and the prime denote, respectively, derivatives with respect to $t$ and $r$. A synchronous solution is obtained by taking the equation of state $\rho + p_r = 0$, giving $\mathcal{P} = -2\mathcal{U}$. Then the conservation Eqs. \cite{18} separate and we obtain
\[ \dot{U} + 4 \frac{\dot{R}}{R} U = 0 = U' + 4 \frac{R'}{R} U. \]  \hspace{1cm} (19)

Consequently the effective energy density has the dark radiation form [13]

\[ U = \left( \frac{\kappa}{\tilde{\kappa}} \right)^4 \frac{Q}{R^4}, \]  \hspace{1cm} (20)

where the dark radiation tidal charge \( Q \) is constant. Thus for the gravitational dynamics of spherically symmetric but still inhomogeneous dark radiation \( E_{\mu\nu} \) is fully determined to be

\[ E_{\mu\nu} = -\frac{Q}{R^4} \left( u_\mu u_\nu - 2 r_\mu r_\nu + h_{\mu\nu} \right). \]  \hspace{1cm} (21)

Note that this projected Weyl tensor \( E_{\mu\nu} \) is a simple dynamical extension of the corresponding form associated with the static tidal Reissner-Nordström solution on the brane [17]. In spite of the existing pressures in the dark radiation setting we may safely take the synchronous comoving frame, for which \( \sigma = 0 \). Then the intricate general field equations [30] simplify and substituting Eq. (21) in Eq. (5) we obtain

\[ G_{\mu\nu} = -\Lambda g_{\mu\nu} + \frac{Q}{R^4} \left( u_\mu u_\nu - 2 r_\mu r_\nu + h_{\mu\nu} \right), \]  \hspace{1cm} (22)

a consistent and exactly solvable closed system for the two unknown functions, \( A(t, r) \) and \( R(t, r) \).

Clearly, the dark radiation dynamics depends on \( \Lambda \) and \( Q \) [21, 22]. An important point to note is that these parameters have a direct influence on the localization of gravity in the vicinity of the brane (see also Ref. [14]). To understand how consider the tidal acceleration away from the brane as measured by brane observers [25]. For the dark radiation vacuum we are considering such acceleration is [17]

\[ -\lim_{y \to 0^\pm} \tilde{R}_{ABCD} n^A \tilde{u}^B n^C \tilde{u}^D = \tilde{\kappa}^2 \tilde{\Lambda} + \frac{Q}{R^4}, \]  \hspace{1cm} (23)

where \( \tilde{u}_A \) is the extension off the brane of the 4-velocity field which satisfies \( \tilde{u}^A n_A = 0 \) and \( \tilde{u}^A \tilde{u}_A = -1 \). For the gravitational field to be localized near the brane the tidal acceleration must be negative. The condition for this to happen is

\[ \tilde{\Lambda} R^4 < -\frac{6Q}{\tilde{\kappa}^2}. \]  \hspace{1cm} (24)

Consequently, the localization of gravity for all \( R \) is only possible if \( \tilde{\Lambda} < 0 \) and \( Q \leq 0 \) or \( \tilde{\Lambda} = 0 \) and \( Q < 0 \). In terms of brane parameters this implies that \( \Lambda < \Lambda_c \) with \( \Lambda_c = \tilde{\kappa}^4 \Lambda^2 / 12 \) and \( Q \leq 0 \) or \( \Lambda = \Lambda_c \) and \( Q < 0 \). For \( \Lambda < \Lambda_c \) and \( Q > 0 \) the gravitational field will only remain localized near the brane if \( R > R_c \) where \( R_c^4 = 3Q / (\Lambda_c - \Lambda) \). On the other hand for \( \Lambda > \Lambda_c \) and \( Q < 0 \) confinement is
restricted to the epochs $R < R_c$. If $\Lambda \geq \Lambda_c$ and $Q \geq 0$ then gravity is always free to propagate far away from the brane.

Since the dark radiation term comes from the electric or Coulomb part of the 5-dimensional Weyl tensor \[7\] the geometry of the bulk may assign to $Q$ any real value. Note however that in our settings the weak, strong and dominant energy conditions \[31\] all independently imply $Q \geq 0$ because $\rho = \mathcal{U}$, $p_r = -\mathcal{U}$ and $p_T = \mathcal{U}$. This is consistent with other studies of dark radiation effects \[13\] but not with perturbation theory and the negative induced energy condition \[18\] which need $Q < 0 \ [7, 17, 18\]$. Current observations do not yet constrain the sign of $Q \ [17, 26, 27, 32\]$. So in what follows we will keep $Q$ as real parameter. The same does not hold for the cosmological constant $\Lambda$. Indeed, according to the present data $\Lambda \sim 10^{-84}$GeV$^2 \ [33\]$. On the other hand $\tilde{M}_p > 10^8$GeV and $M_p \sim 10^{19}$GeV imply by Eq. \[26\] $\lambda > 10^{8}$GeV$^4 \ [26\]$. Since using once again Eq. \[10\] we may write $\Lambda_c = \kappa^2 \lambda / 2$ then $\Lambda_c$ has a lower limit, $\Lambda_c > 10^{-29}$GeV$^2$. Hence according to observations $\Lambda$ must be positive and below the critical value $\Lambda_c$, $0 < \Lambda < \Lambda_c$. This implies a bulk with negative cosmological constant, $\Lambda < 0$. Note as well that the same conclusion holds if $M_p$ is in the TeV range as $\Lambda_c$ increases when $M_p$ decreases. Finally note that when $0 < \Lambda < \Lambda_c$ and $Q > 0$ then gravity is not confined near the brane for $R \leq R_c$. This is a non-perturbative result which is consistent with the energy conditions \[31\] for dark radiation on the brane. So far experiments have not been able to detect any confinement phase transition to a high energy regime where gravity is not bound to the brane. If such a transition does occur at $1/R_c \sim 1$TeV then for $\Lambda_c \sim 10^{-29}$GeV$^2$ we get $Q \sim 10^{-42}$GeV$^{-2}$, a value well below current astrophysical or weak gravity experimental constraints on $Q \ [17, 26, 27, 32\]$.

For completeness in what follows we will also discuss the de Sitter (dS) models with $\Lambda \geq \Lambda_c$, which correspond to a dS bulk, the brane AdS setting where $\Lambda < 0$ and the brane with $\Lambda = 0$.

3 Dynamical Inhomogeneous Solutions

To completely define the brane world dark radiation vacuum dynamics we need to solve Eq. \[22\]. The solutions determine the evolution of the brane world as a whole and so are cosmological in nature. In the proper frame $\sigma = 0$ and so the remaining components of the spherically symmetric metric given in Eq. \[13\] must satisfy the off-diagonal equation

\[ G_{tr} = \frac{2}{AR} \left( \dot{A} R' - \dot{R} A \right) = 0, \]  

which implies

\[ A = \frac{R'}{H}, \]  

7
where $H = H(r)$ is an arbitrary positive function of $r$. Then introducing Eq. (26) in the trace equation

$$- G_t^t + G_r^r + 2G_\theta^\theta = - \frac{2}{AR} \left( \ddot{A}R + 2\dot{R}A \right) = -2\Lambda + \frac{2Q}{R^4}$$

and integrating in $r$ leads to

$$\dot{R} = \frac{\Lambda}{3} R + \frac{Q}{R^3}.$$  (28)

Note that at this point we have chosen to set the arbitrary integration function (of time) to zero. We make this choice here because such a function would correspond to an initial dust mass. A further integration gives

$$\dot{R}^2 = \frac{\Lambda}{3} R^2 - \frac{Q}{R^2} + f,$$  (29)

where $f = f(r)$ is an arbitrary function of $r$. This function is naturally interpreted as the energy inside a shell labelled by $r$ in the dark radiation vacuum. Integrating Eq. (29) we obtain

$$\pm t + \psi = \int \frac{RdR}{\sqrt{\frac{\Lambda}{3} R^4 + f R^2 - Q}},$$

where $\psi = \psi(r)$ is another arbitrary function of $r$ and the signs $+$ or $-$ refer respectively to expansion or collapse.

As it stands the dark radiation is characterized by three arbitrary functions of $r$, namely, $H$, $f$ and $\psi$. However, we have the freedom to rescale the radial coordinate $r$. Then we may impose on the initial hypersurface $t = 0$ the condition

$$R(0, r) = r$$  (31)

and so prescribe $\psi$ to be given by

$$\psi = \int \frac{RdR}{\sqrt{\frac{\Lambda}{3} R^4 + f R^2 - Q}},$$

where the r.h.s is evaluated at $t = 0$. Note that condition (31) also defines the initial dark radiation effective density profile. Using the initial distribution of velocities we can also determine the energy function $f$. $H$ is then determined by the remaining Einstein equation,

$$G_r^r = -\Lambda - \frac{Q}{R^3},$$

as $H = \sqrt{1 + f}$ and consequently the metric takes the LeMaître-Tolman-Bondi form.
\[ ds^2 = -dt^2 + \frac{R'^2}{1 + f} dr^2 + R^2 d\Omega^2, \]  

(34)

where \( f > -1 \).

Note that for the marginally bound models corresponding to \( f = 0 \) this metric describes a static solution. Indeed performing a transformation from the Tolman-Bondi coordinates \((t, r)\) to the curvature coordinates \((T, R)\) such that

\[ T = t + \int dR \frac{R \sqrt{-Q + \frac{\Lambda}{3} R^4}}{\frac{2}{3} R^4 - R^2 - Q}, \]  

(35)

we find

\[ ds^2 = -\left(1 + \frac{Q}{R^2} - \frac{\Lambda}{3} R^2\right) dt^2 + \left(1 + \frac{Q}{R^2} - \frac{\Lambda}{3} R^2\right)^{-1} dR^2 + R^2 d\Omega^2, \]  

(36)

which is the inhomogeneous static exterior of a collapsing sphere of homogeneous dark radiation \[21\] (see also Ref. \[22\]). When \( \Lambda = 0 \) this corresponds to the zero mass limit of the tidal Reissner-Nordström black hole solution on the brane \[17\]. The single black hole horizon only exists for \( Q < 0 \) and is located at \( R_h = \sqrt{-Q} \). Consequently for \( Q > 0 \) the singularity at \( R_s = 0 \) is naked. For non-zero \( \Lambda \) the solution has horizons at \[21\]

\[ R^\pm_h = \sqrt{\frac{3}{2\Lambda}} \left(1 \pm \sqrt{1 + \frac{4Q\Lambda}{3}}\right). \]  

(37)

If \( \Lambda > 0 \) and \( Q < 0 \) then we have an inner horizon \( R^-_h \) and an outer horizon \( R^+_h \). The two horizons merge for \( Q = -3/(4\Lambda) \) and for \( Q < -3/(4\Lambda) \) the singularity at \( R_s = 0 \) becomes naked. For \( \Lambda > 0 \) and \( Q > 0 \) there is a single horizon at \( R^+_h \). If \( \Lambda < 0 \) and \( Q > 0 \) the singularity at \( R_s = 0 \) is again naked. For \( \Lambda < 0 \) and \( Q < 0 \) this no longer happens as a single horizon forms at \( R^-_h \).

In the absence of dark radiation we simply find the homogeneous dS or AdS spaces \[34\]. In particular for \( \Lambda > 0 \) the solution takes the Friedmann-Robertson-Walker steady state form where the spatial curvature \( k \) is zero,

\[ R = r \exp \left( \pm \sqrt{\frac{\Lambda}{3}} t \right). \]  

(38)

Then it represents (half) homogeneous dS space \[34\]. The epoch \( R = 0 \) is regular but fixed as it is impossible to leave it classically.

For \( \Lambda = 0 \) the marginally bound models allow us to calculate the leading order contribution to Newton’s potential. Following Ref. \[17\] and keeping the coordinates \( T \) and \( R \) we may introduce a mass \( M \) for the gravitational source to find the tidal Reissner-Nordström metric
Then Newton’s potential is given by

\[ V_N = \frac{G_N M}{R} - \frac{Q^2}{2R^2} + o\left(\frac{1}{R^3}\right). \]

Contrary to Eq. (2) this is a short distance perturbative expansion [17]. So just as in the static limit our dynamical solutions with non-zero \( f \) should be more adequate to the high energy regime as long as quantum gravity effects can be neglected. The potential shows that perturbatively it should be \( Q \leq 0 \) if anti-gravity effects are to be avoided at short distances (see also Ref. [18]). Consequently if a positive \( Q \) is to be allowed as is by the standard energy conditions [31] than it must be seen as an intrinsically non-perturbative effect.

Let us now consider \( f \neq 0 \). Depending on \( f \) and on \( \Lambda \) and \( Q \) we distinguish several possible dynamical inhomogeneous solutions.

### 3.1 dS dynamics: \( \Lambda > 0 \)

In this setting Eq. (30) can be cast in the form

\[ \pm t + \psi = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \int \frac{dY}{\sqrt{Y^2 - \beta}}, \]

where \( Y = R^2 + 3f/(2\Lambda) \) and \( \beta = \beta(r) \) is given by

\[ \beta = \frac{3}{4\Lambda} \left[ 3f^2 + Q \right]. \]

Direct evaluation of this integral determines \( R \) and shows that the solutions are organized by \( \beta \). For \( \beta > 0 \) we find

\[ R^2 + \frac{3f}{2\Lambda} = \sqrt{\beta} \cosh \left[ \pm 2 \sqrt{\frac{\Lambda}{3}} t + \cosh^{-1} \left( \frac{r^2 + \frac{3f}{\sqrt{\beta}}}{2\Lambda} \right) \right], \]

where for \( Q > 0 \) the energy function may span all its range, \( f > -1 \), but for \( Q < 0 \) it must further satisfy \( |f| > 2\sqrt{-Q\Lambda}/3 \). If \( \beta < 0 \) we obtain

\[ R^2 + \frac{3f}{2\Lambda} = \sqrt{-\beta} \sinh \left[ \pm 2 \sqrt{\frac{\Lambda}{3}} t + \sinh^{-1} \left( \frac{r^2 + \frac{3f}{\sqrt{-\beta}}}{2\Lambda} \right) \right], \]

where \( Q < 0 \) and \( |f| < 2\sqrt{-Q\Lambda}/3 \). For \( \beta = 0 \) we get
\[ \left| R^2 + \frac{3f}{2\Lambda} \right| = \left| r^2 + \frac{3f}{2\Lambda} \right| \exp \left( \pm 2\sqrt{\frac{\Lambda}{3} t} \right), \]  

(45)

where \( Q < 0 \) and \( f = \pm 2\sqrt{-Q\Lambda/3} \).

Clearly, these cosmological solutions are explicitly dependent on \( r \) in a way which cannot be evaded by any coordinate transformation. As such they imply an intrinsically inhomogeneous dark radiation effective energy density. Indeed, \( R \) is not a factorizable function, a fact preventing a reduction to the standard homogeneous dS or Robertson-Walker spaces [34].

3.2 AdS dynamics: \( \Lambda < 0 \)

In this scenario Eq. (30) can be written as

\[ \pm t + \psi = \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} \int \frac{dY}{\sqrt{\beta - Y^2}}. \]  

(46)

Now the only possibility is \( \beta > 0 \). Integrating we obtain the solutions

\[ R^2 + \frac{3f}{2\Lambda} = \sqrt{\beta} \sin \left[ \pm 2\sqrt{-\frac{\Lambda}{3} t} + \sin^{-1} \left( \frac{r^2 + \frac{3f}{2\Lambda}}{\sqrt{\beta}} \right) \right], \]  

(47)

where for \( Q < 0 \) the energy function can take any value in the range \( f > -1 \) but for \( Q > 0 \) it must also be in the interval \( |f| > 2\sqrt{-Q\Lambda/3} \).

3.3 Absence of cosmological constant: \( \Lambda = 0 \)

In the limit \( \Lambda = 0 \) Eq. (31) is written as

\[ \pm t + \psi = \int \frac{RdR}{\sqrt{fR^2 - Q}}. \]  

(48)

Then the solutions for \( f \neq 0 \) are given by

\[ R^2 = \frac{1}{f} \left[ Q + \left( \pm ft + \sqrt{fr^2 - Q} \right)^2 \right], \]  

(49)

where \( fr^2 - Q > 0 \). Note that if \( Q > 0 \) then \( f > 0 \).

3.4 Absence of dark radiation: \( Q = 0 \)

When dark radiation is not present in the vacuum we have \( Q = 0 \) and then the brane world dynamics depends on \( \Lambda \) and \( f \) as can be seen by going back to Eq. (29). Integrating it we obtain
\[ \pm t + \psi(r) = \int \frac{dR}{\sqrt{\frac{2}{3}R^2 + f}}. \]  

Let us first consider \( \Lambda > 0 \). The solutions are now organized by \( f \). For \( f > 0 \) we find

\[ R = \sqrt{\frac{3f}{\Lambda}} \sinh \left[ \pm \sqrt{\frac{\Lambda}{3}} t + \sinh^{-1} \left( \sqrt{\frac{\Lambda}{3f}} r \right) \right]. \tag{51} \]

If \(-1 < f < 0\) we have

\[ R = \sqrt{-\frac{3f}{\Lambda}} \cosh \left[ \pm \sqrt{\frac{\Lambda}{3}} t + \cosh^{-1} \left( \sqrt{-\frac{\Lambda}{3f}} r \right) \right]. \tag{52} \]

Note that it is only for \( f = -\Lambda r^2/3 \) that solution (52) takes the homogeneous dS form [34].

When \( \Lambda < 0 \) the relevant integral is only defined for \( f > 0 \) and the corresponding solution is

\[ R = \sqrt{-\frac{3f}{\Lambda}} \sin \left[ \pm \sqrt{-\frac{\Lambda}{3}} t + \sin^{-1} \left( \sqrt{-\frac{\Lambda}{3f}} r \right) \right]. \tag{53} \]

In this case the homogeneous AdS space solution [34] is again obtained with the choice \( f = -\Lambda r^2/3 \).

### 4 Physical Singularities and Regular Rebounces

Both shell focusing singularities \( (R = 0) \) and shell crossing singularities \( (R' = 0) \) are found depending on the dynamics prescribed by Eq. (29). Shell crossing singularities are generally not considered to be “real” singularities, so we will concentrate always on shell focusing singularities in the following.

To determine the inhomogeneous dynamical solutions corresponding to dark radiation gravitation on the brane we have implicitly assumed the validity of the following regularity condition

\[ V = V(R, r) = \frac{\Lambda}{3} R^4 + f R^2 - Q > 0. \tag{54} \]

If Eq. (54) is verified for all \( R \geq 0 \) then collapsing dark radiation shells will meet at the physical shell focusing singularity \( R = R_s = 0 \). This is what happens in the gravitational collapse of ordinary dust matter when \( \Lambda \) is set to zero [35]. However, if \( \Lambda \neq 0 \) the evolution curve of a collapsing dust cloud has special rebouncing points at \( R > 0 \) for which the dust potential is zero [36]. These points may exist both when a singularity forms at \( R_s = 0 \) and when it does not. In the latter case the solution is
globally regular and the gravitational evolution of the dust cloud involves a collapsing phase followed by continuous expansion after velocity reversal at the rebounce point.

Not surprisingly, an analogously rich structure of solutions can also be uncovered in the gravitational dynamics of vacuum dark radiation on the brane (see also the discussion in Ref. [21]). We start by writing Eq. (29) in the following way

\[ R^2 \dot{R}^2 = V. \] (55)

Starting from an initial (collapsing) state, a rebounce will occur for a given shell whenever \( \dot{R} = 0 \) before that shell becomes singular. This happens when \( V(R, r) = 0 \). Introducing \( Y \) and \( \beta \) we write the potential \( V \) as the following simple quadratic polynomial

\[ V = \frac{\Lambda}{3} (Y^2 - \beta) \] (56)

and it becomes clear that no more than two regular rebounce epochs can be found. Depending on \( f \) and on the values of \( \Lambda \) and \( Q \) none, one or two such epochs may be associated with globally regular solutions.

### 4.1 dS dynamics: \( \Lambda > 0 \)

The dS dynamics corresponding to \( \Lambda > 0 \) is characterized by a concave phase space curve because

\[ \frac{d^2V}{dY^2} = \frac{2\Lambda}{3} > 0. \] (57)

This implies the existence of a phase of continuous expansion towards infinity with ever increasing speed. Other phases are possible depending on the parameters.

For \( \beta < 0 \) we have \( Q < 0 \) and \( |f| < 2\sqrt{-Q\Lambda}/3 \). Clearly, \( V \) is positive for all \( R \geq 0 \). It satisfies \( V(0, r) = -Q \) and grows to infinity with \( R \) as \( \Lambda R^4 \) (see Fig. 1). The dark radiation shells may either expand continuously with ever increasing kinetic energy or collapse to a shell focusing singularity at \( R_s = 0 \) after a proper time \( t = t_s(r) \) given by

\[ t_s(r) = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \left[ \sinh^{-1} \left( \frac{r^2 + \frac{3f}{2\Lambda}}{\sqrt{-\beta}} \right) - \sinh^{-1} \left( \frac{3f}{2\Lambda\sqrt{-\beta}} \right) \right]. \] (58)

For \( \Lambda \leq \Lambda_c \) gravity is always confined near the brane during the cycle of evolution. However, if \( \Lambda > \Lambda_c \) then the gravitational field only stays confined for \( R < R_c \). As a consequence an expanding dark radiation vacuum goes through a phase transition at \( R = R_c \) where the gravitational field ceases to be confined to the brane. As we have seen this case and the large value \( \Lambda = \Lambda_c \) are currently ruled out by observations.

For \( \beta > 0 \), \( Q > 0 \) and \( f > -1 \) (see Fig. 2) we find globally regular solutions with a single rebounce epoch at \( R = R_s \) where
Figure 1: Plots of $V$ for $\Lambda > 0$, $\beta < 0$ and $Q < 0$. Non-zero values of $f$ belong to the interval $f > -1 \wedge |f| < 2\sqrt{-Q\Lambda/3}$ and correspond to shells of constant $r$. In the lefthand plot $\Lambda \leq \Lambda_c$ and gravity is always bound to the brane. In the righthand plot $\Lambda > \Lambda_c$ and the shaded region indicates where gravity is not confined near the brane.

\[ R^2_\ast = -\frac{3f}{2\Lambda} + \sqrt{\beta}. \quad (59) \]

This is the minimum possible radius a collapsing dark radiation shell can have. The collapsing shells reach this epoch at the proper time $t = t_\ast(r)$ where

\[ t_\ast(r) = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \cosh^{-1} \left( \frac{r^2 + 3f}{2\Lambda\sqrt{\beta}} \right). \quad (60) \]

At this point they reverse their motion and expand forever with ever increasing rate. The phase space of allowed dynamics defined by $V$ and $R$ is thus restricted to the region $R \geq R_\ast$. Below $R_\ast$ there is a forbidden region where the potential $V$ is negative. In particular $V(0, r) = -Q < 0$ which means that the singularity at $R_\ast = 0$ does not form and so the solutions are globally regular. Note that if gravity is to stay confined to the vicinity of the brane for $R > R_\ast$ then $\Lambda < \Lambda_c$ and $R_\ast > R_c$. If not then there is a phase transition epoch $R = R_c$ such that for $R \leq R_c$ the gravitational field is no longer confined to the brane. For $\Lambda \geq \Lambda_c$ gravity will always be free to propagate far away from the brane a situation that as we have seen is not allowed by current observations.

If $\beta > 0$ and $Q < 0$ we have seen that $f > -1$ has to further satisfy $|f| > 2\sqrt{-Q\Lambda/3}$. If $f > 2\sqrt{-Q\Lambda/3}$ there no rebounce points in the allowed dynamical region $R \geq 0$ and the evolution follows the case $\beta < 0$ illustrated in Fig. 1. The time to reach the singularity at $R_\ast = 0$ is now given by

\[ t_\ast(r) = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \left[ \cosh^{-1} \left( \frac{r^2 + 3f}{2\Lambda\sqrt{\beta}} \right) - \cosh^{-1} \left( \frac{3f}{2\Lambda\sqrt{\beta}} \right) \right]. \quad (61) \]
Figure 2: Plots of $V$ for $\Lambda > 0$, $\beta > 0$ and $Q > 0$. Non-zero values of $f$ belong to the interval $f > -1$ and correspond to shells of constant $r$. In the left-hand plot $\Lambda \geq \Lambda_c$ gravity is always free to move away from the brane. In the righthand plot $\Lambda < \Lambda_c$ and the shaded region indicates where gravity is not bound to the brane.

On the other hand for $-1 < f < -2\sqrt{-QA/3}$ (see Fig. 3) there are two rebounce epochs at $R = R_{s\pm}$ with

$$R_{s\pm}^2 = -\frac{3f}{2\Lambda} \pm \sqrt{\beta}. \quad (62)$$

Since $V(0, r) = -Q > 0$ a singularity also forms at $R_s = 0$. The region between the two rebounce points is forbidden because there $V$ is negative. The phase space of allowed dynamics is thus divided in two disconnected regions separated by the forbidden interval $R_{s-} < R < R_{s+}$. For $0 \leq R \leq R_{s-}$ the dark radiation shells may expand to a maximum radius $R = R_{s-}$ in the time $t_{s-} = t_s$ where

$$t_s(r) = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \cosh^{-1} \left( \frac{r^2 + \frac{3f}{2\Lambda}}{\sqrt{\beta}} \right). \quad (63)$$

At this rebounce epoch the shells start to fall to the singularity which is met after the proper time

$$t_s(r) = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \cosh^{-1} \left( \frac{3|f|}{\Lambda\sqrt{\beta}} \right). \quad (64)$$

If $R \geq R_{s+}$ then there is a collapsing phase to the minimum radius $R = R_{s+}$ taking the time $t_{s+} = t_s$ followed by reversal and subsequent accelerated continuous expansion. The singularity at $R_s = 0$ does not form and so the solutions are globally regular.

Once more note that for $\Lambda \leq \Lambda_c$ gravity is always confined near the brane but for $\Lambda > \Lambda_c$ there is a phase transition epoch at $R = R_c$ such that for $R > R_c$ gravity is free to propagate out of the brane. Only if $R_{s-} < R_c < R_{s+}$ this phase transition will not occur. For $R < R_{s-}$ gravity is confined near the brane and for $R > R_{s+}$ it will not.
Figure 3: Plots of $V$ for $\Lambda > 0$, $\beta > 0$ and $Q < 0$. Non-zero values of $f$ belong to the interval $-1 < f < -2\sqrt{-QA/3}$ and correspond to shells of constant $r$. In the lefthand plot $\Lambda \leq \Lambda_c$ and gravity is always confined near the brane. In the righthand plot $\Lambda > \Lambda_c$ and the shaded region indicates where gravity is not bound to the brane.

Figure 4: Plots of $V$ for $\Lambda > 0$, $\beta = 0$ and $Q < 0$. Here $f = -2\sqrt{-QA/3}$. In the lefthand plot $\Lambda \leq \Lambda_c$ and gravity is always confined near the brane. In the righthand plot $\Lambda > \Lambda_c$ and the shaded region indicates where gravity is not bound to the brane.

If $\beta = 0$ then again $Q < 0$. For $f = -2\sqrt{-QA/3}$ (see Fig. 4) the single candidate to be a rebounce point is $R = R_*$ with

$$R_* = \sqrt{-\frac{3Q}{\Lambda}}. \quad (65)$$

In this case $V(0, r) = -Q > 0$ and then a singularity also forms at $R_* = 0$. There is no forbidden region in phase space but the point at $R_*$ turns out to be a regular fixed point which divides two distinct dynamical regions. Indeed if a shell starts at $R = R_*$ then it will not move for all times. If initially $R < R_*$ then either the shell expands towards $R_*$ or it collapses to the singularity. The time to meet the singularity is finite,
Figure 5: Plots of $V$ for $\Lambda < 0$, $\beta > 0$ and $Q < 0$. Non-zero values of $f$ belong to the interval $f > -1$ and correspond to shells of constant $r$. Here gravity is always bound to the vicinity of the brane.

$$t_s(r) = \frac{3}{2\Lambda} \ln \left( \frac{\sqrt{-3Q}}{r^2 - \sqrt{-3Q/\Lambda}} \right),$$  \hspace{1cm} (66)$$

but the time to expand to $R_*$ is infinite. If initially $R > R_*$ then the collapsing dark radiation shells also take an infinite time to reach $R_*$. If $f = 2\sqrt{-Q\Lambda}/3$ there are no real rebounce epochs and the collapsing dark radiation simply falls to the singularity at $R_s = 0$. The collision proper time is

$$t_s(r) = -\frac{3}{2\Lambda} \ln \left( \frac{\sqrt{-3Q}}{r^2 + \sqrt{-3Q/\Lambda}} \right).$$  \hspace{1cm} (67)$$

Of course for $\Lambda \leq \Lambda_c$ then the gravitational field is always confined to the brane but for $\Lambda > \Lambda_c$ there is a phase transition epoch $R = R_c$ in either of the two dynamical regions such that for $R > R_c$ gravity may freely move far into the bulk.

### 4.2 AdS dynamics: $\Lambda < 0$

The AdS dynamics corresponding to $\Lambda < 0$ is characterized by a convex phase space curve because

$$\frac{d^2V}{dy^2} = \frac{2\Lambda}{3} < 0,$$  \hspace{1cm} (68)$$

which does not allow a phase of continuous expansion to infinity. Also in this scenario the only possibility is $\beta > 0$.

If $Q < 0$ then we may consider $f > -1$ (see Fig. 3). Just like the corresponding dS case there is a single rebounce point at $R = R_*$ with
Figure 6: Plots of $V$ for $\Lambda < 0$, $\beta > 0$ and $Q > 0$. Non-zero values of $f$ belong to the interval $f > 2\sqrt{-Q\Lambda/3}$ and correspond to shells of constant $r$. In the lefthand plot $R_c < R_{s-}$ and gravity is confined near the brane in $R_{s-} < R < R_{s+}$. In the righthand plot $R_{s-} < R_c < R_{s+}$ and the shaded region indicates where gravity is not bound to the brane.

$$R^2_s = \frac{-3f}{2\Lambda} + \sqrt{\beta}. \quad (69)$$

Due to the convexity of the phase space evolution curve this is the maximum radius which an expanding dark radiation shell can reach. The time for this to happen is

$$t_s(r) = \frac{1}{2} \sqrt{-3\Lambda} \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{f^2}{2\Lambda} \right) \right]. \quad (70)$$

Because $V(0,r) = -Q > 0$ a singularity also forms at $R_s = 0$. Consequently, the reversal at $R = R_s$ will be followed by a collapsing phase to the singularity in the time

$$t_s(r) = \frac{1}{2} \sqrt{-3\Lambda} \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{3f}{2\Lambda \sqrt{\beta}} \right) \right]. \quad (71)$$

This is in contrast with the dS scenario where there was no formation of a singularity at $R_s = 0$ and after reversal at the rebounce epoch the dark radiation shells expand forever. Also, because $\Lambda < \Lambda_c$ gravity remains confined near the brane for all times.

If $Q > 0$ then $f > 2\sqrt{-Q\Lambda/3}$ (see Fig. 6). We find two regular rebouncing epochs at $R = R_{s\pm}$ where

$$R^2_{s\pm} = \frac{-3f}{2\Lambda} \pm \sqrt{\beta}. \quad (72)$$

The allowed phase space region is precisely the interval between these two roots, $R_{s-} \leq R \leq R_{s+}$. Since the singularity at $R_s = 0$ is inside the forbidden region it does not form and so we are in the presence of globally regular solutions. In this case an initially expanding dark radiation shell will rebounce at $R = R_{s+}$ after the time
\[ t_{s+}(r) = \frac{1}{2} \sqrt{-\frac{3}{\Lambda}} \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{r^2 + \frac{3f}{\sqrt{\Lambda}}}{\sqrt{3}} \right) \right]. \] (73)

Then it reverses its motion to collapse until it reaches \( R = R_s^- \) in the time

\[ t_{s-}(r) = \frac{\pi}{2} \sqrt{-\frac{3}{\Lambda}}. \] (74)

It rebounces again to expand to \( R = R_{s+} \) in the same amount of time and subsequently repeats the cycle. Thus these are oscillating globally regular solutions. Again note the contrast with the dS scenario where no oscillating solutions were found. Since \( \Lambda < \Lambda_c \) there is only a gravity confinement phase transition epoch in the cycle if \( R_s^- < R_c < R_{s+} \).

### 4.3 Absence of cosmological constant: \( \Lambda = 0 \)

When \( \Lambda = 0 \) the potential is simply \( V = fr^2 - Q \). If \( Q < 0 \) and \( f > 0 \) then there are no rebounce epochs in the allowed dynamical range \( R \geq 0 \) and the singularity is at \( R_s = 0 \) where \( V(0, r) = -Q > 0 \). The collapsing dark radiation shells reach the singularity after the time

\[ t_s(r) = \frac{1}{f} \left( \sqrt{fr^2 - Q} - \sqrt{-Q} \right). \] (75)

In this case gravity never propagates out of the brane. If \( Q > 0 \) and \( f > 0 \) or \( Q < 0 \) and \(-1 < f < 0 \) there is a single rebounce root at \( R_s = \sqrt{Q/f} \). For \( Q > 0 \) and \( f > 0 \) the allowed dynamical region is \( R \geq R_s \) and so the singularity does not form at \( R_s = 0 \). A collapsing shell falls to a minimum radius in the time

\[ t_s(r) = \frac{\sqrt{fr^2 - Q}}{f} \] (76)

and then expands continuously to infinity. If \( R_s > R_c \) gravity is confined for all \( R > R_s \). Otherwise the dark radiation goes through a confinement phase transition at \( R = R_c \). For \( Q < 0 \) and \(-1 < f < 0 \) the dynamical range in phase space is \( 0 \leq R \leq R_s \). The shells may expand out to reach \( R_s \) at

\[ t_s(r) = -\frac{\sqrt{fr^2 - Q}}{f}. \] (77)

Then it reverses its motion to fall into the singular epoch \( R_s = 0 \) in the proper time

\[ t_s(r) = -\frac{\sqrt{-Q}}{f}. \] (78)

In this setting gravity remains confined in the allowed dynamical range, \( 0 \leq R \leq R_s \).
### 4.4 Absence of dark radiation: $Q = 0$

When the dark radiation degrees of freedom are not present $Q = 0$ and the potential to be considered is

$$V = V(R, r) = \frac{\Lambda}{3} R^2 + f.$$  \hfill (79)

For $\Lambda > 0$ we may distinguish three types of evolution. If $-1 < f < 0$ there is a rebounce epoch towards infinite continuous expansion at $R_* = \sqrt{-3f/\Lambda}$ and the allowed dynamical region is $R \geq R_*$. The time to collapse to minimum radius is

$$t_s(r) = \sqrt{\frac{3}{\Lambda}} \cos^{-1} \left( r \sqrt{-\frac{\Lambda}{3f}} \right).$$ \hfill (80)

If $f > 0$ there are no rebounce epochs for $R \geq 0$ and a singularity forms at $R_s = 0$ where the potential is $V(0, r) = f$. The time to meet the singularity is

$$t_s(r) = \sqrt{\frac{3}{\Lambda}} \sinh^{-1} \left( r \sqrt{\frac{\Lambda}{3f}} \right).$$ \hfill (81)

If $\Lambda < 0$ the only possibility is to take $f > 0$. Then we find a rebounce epoch at $R_* = \sqrt{-3f/\Lambda}$. After expansion to a maximum radius $R_*$ in the time

$$t_s(r) = \sqrt{-\frac{3}{\Lambda}} \left[ \frac{\pi}{2} - \sin^{-1} \left( r \sqrt{-\frac{\Lambda}{3f}} \right) \right],$$ \hfill (82)

the brane collapses to the singularity at $R_s = 0$ reaching it after the time

$$t_s(r) = \frac{\pi}{2} \sqrt{-\frac{3}{\Lambda}}.$$ \hfill (83)

For $\Lambda < \Lambda_c$ the gravitational field is always confined. Otherwise it never is.

### 5 Conclusions

We have analysed some aspects of the gravitational dynamics of inhomogeneous dark radiation on a RS brane. This is an important first step towards the understanding of a realistic collapse setting on a brane. Indeed, the behaviour of dark radiation is nothing else than the dynamics of the vacuum on the brane when the gravitational modes are excited only as energy density.

We have taken an effective 4-dimensional viewpoint to show that with certain simplifying, but natural assumptions, the Einstein field equations form a solvable, closed system. The solutions obtained were shown to depend on the brane cosmological constant $\Lambda$, the dark radiation tidal charge, $Q$, and on the energy function $f(r)$. 

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20
We have given a precise description of the dynamics of the solutions and characterized how this depends on $\Lambda$, $Q$ and $f(r)$. We have also presented the conditions defining the solutions as singular or as globally regular.

Finally we have discussed the confinement of gravity to the vicinity of the brane. We have seen that it depends on $\Lambda$, $Q$ and also on the brane tension $\lambda$. If $\Lambda \leq \Lambda_c$ where $\Lambda_c = \kappa^4 \lambda^2/12$ and $Q < 0$ then the gravitational field is always confined to the vicinity of the brane during its evolution. Alternatively, if $\Lambda < \Lambda_c$ and $Q > 0$ gravity is only bound to the brane for $R > R_c$ where $R_c^4 = 3Q/(\Lambda_c - \Lambda)$. For $\Lambda > \Lambda_c$ and $Q < 0$ confinement is restricted to the epochs $R < R_c$. We have noted that current observations do not allow $\Lambda < 0$ and $\Lambda \geq \Lambda_c$. Consequently, the gravitational degrees of freedom may eventually propagate away from the brane at high energies. Such a phase transition would occur at the epoch $R = R_c$ and should be detected in a change in Newton’s gravitational constant.

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