Quasi-equilibrium problems with generalized monotonicity

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Abstract

In this work, we propose a new existence result for quasi-equilibrium problems using generalized monotonicity in an infinite dimensional space. Also, we show that the notions of generalized monotonicity can be characterized in terms of solution sets of equilibrium problems and convex feasibility problems. Moreover, we show that the concepts of pseudomonotonicity and upper sign property are related under suitable assumptions.

Keywords: Equilibrium problem, Convex feasibility problem, Generalized monotonicity, Upper sign property

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1 Introduction

Given $C$ a subset of a Banach space $X$ and a bifunction $f : X \times X \rightarrow \mathbb{R}$, an equilibrium problem, see Blum and Oettli [6], is the problem of finding:

$$x_0 \in C \text{ such that } f(x_0, y) \geq 0, \text{ for all } y \in C.$$ (1)

Problem (1) has been extensively studied in recent years (see [5–7, 9, 13] and their reference therein). A recurrent theme in the analysis of the conditions for the existence of solutions of equilibrium problems is the connection between them and the solutions of the so-called convex feasibility problem which turns out to be convex under suitable conditions of $f$ and which corresponds to a sort of dual formulation of the following equilibrium problem

$$\text{find } x \in C \text{ such that } f(y, x) \leq 0, \text{ for all } y \in C.$$ (2)

It was proved in [3] that if $f$ has the upper sign property then every solution of (2) is a solution of (1), and moreover both solution sets coincide under pseudomonotonicity of $f$.

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The classical example of equilibrium problem is the variational inequality problem (see e.g. [4, 14]), which is defined as follows: a Stampacchia variational inequality problem is formulated as

\[ \text{find } x \in C \text{ such that there exists } x^* \in T(x), \]

\[ \langle x^*, y - x \rangle \geq 0, \text{ for all } y \in C, \] \hspace{1cm} (3)

where \( T : X \rightrightarrows X^* \) is a set-valued map, \( X^* \) is the dual space of \( X \) and \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( X \) and \( X^* \). So, if \( T \) has weak* compact values, and we define the representative bifunction \( f_T \) of \( T \) by

\[ f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \] \hspace{1cm} (4)

it follows that every solution of the equilibrium problem associated to \( f_T \) and \( C \) is a solution of the variational inequality problem associated to \( T \) and \( C \), and conversely. Now, the convex feasibility problem associated to \( f_T \) is equivalent to

\[ \text{find } x \in C \text{ such that } \langle y^*, y - x \rangle \geq 0, \text{ for all } y \in C, y^* \in T(y), \] \hspace{1cm} (5)

which is known as the Minty variational inequality problem.

It was shown in [14] that generalized monotonicity for a set valued map is characterized in terms of the solution sets of problems (3) and (5). In this work we extend some of them to bifunctions and give a new existence result to equilibrium problems without convexity of the constraint set.

We consider next the quasi-equilibrium problem, which is defined as follows

\[ \text{find } x \in K(x) \text{ such that } f(x, y) \geq 0, \text{ for all } y \in K(x), \] \hspace{1cm} (6)

where \( K : C \rightrightarrows C \) is a set-valued map. One of the reasons for studying quasi-equilibrium problems lies in the relationship between them and many problems, for instance, generalized Nash equilibrium problems, quasi-variational inequality problems, among others (see [8, 10]).

Recently, in [5] the authors showed an existence result for quasi-equilibrium problems using generalized monotonicity, but in finite dimensional spaces. We propose a similar result, but in an infinite dimensional setting.

The paper is organized as follows. First, in Section 2 we consider important notions and comments concerning the mainly used concepts. Section 3 is devoted to the characterization of generalized monotonicity for bifunctions. Additionally, we show that under suitable assumptions the pseudomonotonicity and the upper sign property are related. Finally, in Section 4 we give a new existence result for quasi-equilibrium problems and an application is considered.

## 2 Preliminaries

This section is devoted to recall the main notions of generalized monotonicity and upper sign property which will be used in the sequel and we give comments and some preliminary results.

All along the paper, \( X \) stands for a real Banach space, \( X^* \) its dual space and \( \langle \cdot, \cdot \rangle \) the duality pairing between \( X \) and \( X^* \). Given a set \( C \subseteq X \), \( \text{co}(C) \) is the convex hull of \( C \).

Let us now recall some classical definitions of generalized convexity. A real-valued function \( h : X \rightarrow \mathbb{R} \) is said to be
- **quasiconvex** if, for any $x, y \in X$ and $t \in [0, 1]$, we have 
  \[ h(tx + (1-t)y) \leq \max\{h(x), h(y)\}, \]

- **semistrictly quasiconvex** if it is quasiconvex and, for any $x, y \in X$ such that $h(x) \neq h(y)$ the following holds 
  \[ h(tx + (1-t)y) < \max\{h(x), h(y)\} \text{ for all } t \in [0, 1]. \]

A good reference for quasiconvex functions and in particular for quasiconvex optimization is [1].

2.1 Generalized monotonicity

We now recall some different definitions of generalized monotonicity (the ones we will be use from now on).

A set-valued map $T : X \Rightarrow X^*$ is said to be:

- **pseudomonotone** if for all $x, y \in X$ and $x^* \in T(x)$, $y^* \in T(y)$ it holds that 
  \[ \langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq 0, \]

- **quasimonotone** if for all $x, y \in X$ and $x^* \in T(x)$, $y^* \in T(y)$ it holds that 
  \[ \langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, y - x \rangle \geq 0, \]

- **properly quasimonotone** if for all $x_1, \ldots, x_n \in X$, and all $x \in \text{co}\{x_1, \ldots, x_n\}$ there exists $i$ such that 
  \[ \langle x_i^*, x - x_i \rangle \leq 0, \forall x_i^* \in T(x_i). \]

In a similar way, a bifunction $f : X \times X \rightarrow \mathbb{R}$ is said to be:

- **pseudomonotone** if for all $x, y \in X$ the following implication holds 
  \[ f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0; \]

- **quasimonotone** if for all $x, y \in X$ the following implication holds 
  \[ f(x, y) > 0 \Rightarrow f(y, x) \leq 0; \]

- **properly quasimonotone** if for all $x_1, \ldots, x_n \in X$, and all $x \in \text{co}\{x_1, \ldots, x_n\}$ there exists $i$ such that 
  \[ f(x_i, x) \leq 0 \]

In the case of set-valued maps, pseudomonotonicity implies proper quasimonotonicity which implies quasimonotonicity. However, for bifunctions, pseudomonotonicity implies proper quasimonotonicity provided the bifunction is quasiconvex with respect to its second argument, see [5, Proposition 1.1]. Moreover, no relationship exists between quasimonotonicity and proper quasimonotonicity of bifunctions, see the counter-examples in [5].

It is very well known that a set-valued map $T : X \Rightarrow X^*$, with nonempty and weak* compact values, satisfies some generalized monotonicity if and only if its bifunction $f_T$ defined like (4) does too. In a similar spirit we have the following.
Proposition 2.1. Let $T : X \rightrightarrows X^*$ be a set-valued map with compact values. If $-T$ is pseudomonotone then $-f_T$ is too.

Proof. Let $x, y \in X$ such that $-f_T(x, y) \geq 0$. Thus, for each $x^* \in T(x)$, it holds $\langle x^*, y - x \rangle \leq 0$ which in turn implies $\langle y^*, y - x \rangle \leq 0$ for all $y^* \in T(y)$, due to pseudomonotonicity of $-T$. Hence, $f_T(y, x) \geq 0$ or equivalently $-f_T(y, x) \leq 0$. □

The converse of the previous result does not hold in general as the following example shows.

Example 2.1. Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued map defined by

\[ T(x) = \{-1, 1\} \text{ for all } x \in \mathbb{R}. \]

Clearly, $-T$ is not pseudomonotone but $-f_T$ is pseudomonotone, because $f_T \geq 0$ and it only vanishes on the diagonal of $\mathbb{R} \times \mathbb{R}$.

We finish this subsection with the following proposition which gives sufficient conditions in order to obtain the proper quasimonotonicity of a bifunction.

Proposition 2.2. Let $f : X \times X \to \mathbb{R}$ be a bifunction such that it vanishes on the diagonal of $X \times X$. If $-f(\cdot, y)$ is quasiconvex for all $y \in X$ then $f$ is properly quasimonotone.

Proof. Given $x_1, \ldots, x_m \in X$ and $x \in \text{co}(\{x_1, \ldots, x_m\})$, by quasiconvexity of $-f(\cdot, x)$ we have $0 \leq \max_{i=1,\ldots,m} -f(x_i, x) = -\min_{i=1,\ldots,m} f(x_i, x)$. Therefore, the result follows. □

2.2 Upper sign property

We start this subsection recalling the classical notions of continuity for set-valued maps. Let $K : A \rightrightarrows B$ be a set-valued map with $A$ and $B$ two topological spaces. The map $K$ is called:

- **closed** when its graph is a closed subset of $A \times B$;
- **lower semicontinuous at** $x_0$ when for each open set $V$ such that $K(x_0) \cap V \neq \emptyset$ there exists a neighborhood $U$ of $x_0$ such that $K(x) \cap V \neq \emptyset$ for all $x \in U$;
- **upper semicontinuous at** $x_0$ when for any neighborhood $V$ of $K(x_0)$, there exists a neighborhood $U$ of $x_0$ such that $K(U) \subset V$.

The usual definition of a lower semicontinuous set-valued map using sequences is equivalent to the one given here using sets (see for instance Proposition 2.5.6 in [11]).

Let $C$ be a convex subset of $X$.

- A set-valued map $T : X \rightrightarrows X^*$ is said to be upper sign-continuous (12) on $C$ if for all $x \in C$ and any $v \in X$ the following implication holds:

\[ \left( \forall t \in [0, 1], \inf_{x^* \in T(x)} \langle x^*, v \rangle \geq 0 \right) \Rightarrow \sup_{x^* \in T(x)} \langle x^*, v \rangle \geq 0 \]

- A bifunction $f : X \times X \to \mathbb{R}$ is said to have the **upper sign property** (13) on $C$ if for all $x \in C$ and for every $y \in C$ the following implication holds:

\[ (f(x, t) \leq 0, \ \forall t \in [0, 1]) \Rightarrow f(x, y) \geq 0, \]
where \( x_t = (1 - t)x + ty \).

Upper sign-continuity is a very weak notion of continuity. For instance, any upper semicontinuous set-valued map is upper sign-continuous. Moreover, any positive function on \( \mathbb{R} \) is upper sign-continuous. This notion plays an important role for proving the existence of solutions of variational inequalities and quasi-variational inequalities, see [2, 4]. In a similar way, the upper sign property plays an important role in order to establish the existence of solutions of equilibrium problems and quasi-equilibrium problems, see [3, 7].

On the other hand, from definition of \( f_T \) we have that \( T \) is upper sign-continuous if and only if \( f_T \) has the upper sign property. Finally, note that every bifunction \( f \) which has the upper sign property on \( C \) is nonnegative on the diagonal of \( C \times C \).

### 3 Canonical relations

John, in [14], characterized the proper quasimonotonicity of set-valued maps by the nonemptiness of the solution set of Minty variational inequality problems associated to this set-valued map on compact sets. Bianchi and Pini established a similar result for bifunctions under lower semicontinuity and quasi-convexity, see [5, Theorem 2.1]. However, this is not true in general as the following examples, which were considered in [5], put in evidence. First, let us denote by \( \text{EP}(f, C) \) and \( \text{CFP}(f, C) \) the solution sets of problems (1) and (2) associated to \( f \) and \( C \), respectively.

**Example 3.1.** The bifunction \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) defined as follows

\[
 f(x, y) = \begin{cases} 
 1, & \text{if } y = 0, \ x > 0, \ or \ x = 0, \ y > 0 \\
 0, & \text{otherwise}
\end{cases}
\]

is properly quasimonotone, but \( \text{CFP}(f, [0, 1]) = \emptyset \).

**Example 3.2.** The bifunction \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) defined as follows

\[
 f(x, y) = \begin{cases} 
 1, & \text{if } x + y < 1, \ y > x \geq 0, \ or \ x + y > 1, \ y < x \leq 1 \\
 0, & \text{otherwise}
\end{cases}
\]

is neither properly quasimonotone nor quasi-convex with respect to its second variable, but \( \text{CFP}(f, C) \neq \emptyset \) for any convex compact subset \( C \) of \( \mathbb{R} \).

In a similar way to [14, Theorem 2 and Corollary of Theorem 1], the next result characterizes quasimonotonicity and pseudomonotonicity.

**Proposition 3.1.** Let \( f : X \times X \rightarrow \mathbb{R} \) be a bifunction. Then the following hold:

1. \( f \) is quasimonotone if and only if \( \text{CFP}(f, \{x, y\}) \neq \emptyset \) for all \( x, y \in X \).

2. \( f \) is pseudomonotone if and only if \( \text{EP}(f, C) \subset \text{CFP}(f, C) \) for every subset \( C \) of \( X \).

3. If \( -f \) is pseudomonotone then \( \text{CFP}(f, C) \subset \text{EP}(f, C) \) for every subset \( C \) of \( X \). The converse holds provided that \( f \) vanishes on the diagonal of \( X \times X \).

**Proof.**

1. It follows from the fact that \( f \) is not quasimonotone if and only if there exists \( x, y \in X \) such that \( f(x, y) > 0 \) and \( f(y, x) > 0 \), which is equivalent to \( \text{CFP}(f, \{x, y\}) = \emptyset \).
2. It is a straightforward adaptation from [14, Theorem 2].

3. Let \( x \in \text{CFP}(f, C) \), that means \( f(y, x) \leq 0 \) for all \( y \in C \). By pseudomonotonicity of \( -f \) we have \( f(x, y) \geq 0 \). Hence, \( x \in \text{EP}(f, C) \).

Conversely, let \( x, y \in X \) such that \( -f(x, y) \geq 0 \). We take \( C = \{x, y\} \) and since \( f(y, y) = 0 \) we have \( y \in \text{CFP}(f, C) \). Thus, \( f(y, x) \geq 0 \) or equivalently \( -f(y, x) \leq 0 \).

The following example says that in part 3 of Proposition 3.1 the reciprocal does not hold in general.

**Example 3.3.** The bifunction \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined as follows

\[
f(x, y) = \begin{cases} 
-1, & \text{if } (x, y) = (0, 1) \\
1, & \text{if } (x, y) = (0, 0) \\
0, & \text{otherwise}
\end{cases}
\]

satisfies that \( \text{CFP}(f, C) \subset \text{EP}(f, C) \) for every subset \( C \) of \( \mathbb{R} \). However, \( -f \) is not pseudomonotone.

Now we are ready to establish an existence result for equilibrium problems where the constraint set is not necessary convex.

**Proposition 3.2.** Let \( C \) be a compact subset of \( X \) and \( f : X \times X \to \mathbb{R} \) be a properly quasimonotone bifunction such that it is lower semicontinuous with respect to its second argument. If \( -f \) is pseudomonotone then the set \( \text{EP}(f, C) \) is nonempty.

**Proof.** By [3, Proposition 2.4] we have \( \text{CFP}(f, C) \neq \emptyset \). The result follows from part 3 of Proposition 3.1.

**Example 3.4.** Let \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be the bifunction defined as follows

\[
f(x, y) = \begin{cases} 
-x, & \text{if } y = 0, x > 0 \\
y, & \text{if } x = 0, y > 0 \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly \( -f \) is pseudomonotone and quasiconvex with respect to its first argument. Furthermore, \( f \) is lower semicontinuous with respect to its second variable. Since \( f \) vanishes on the diagonal of \( \mathbb{R} \times \mathbb{R} \) we have that \( f \) is properly quasimonotone, due to Proposition 2.2. Therefore, by Proposition 3.2 \( \text{EP}(f, \{0, 1/2, 1\}) \neq \emptyset \). Finally, note that we cannot apply [4, Theorem 5] in order to establish the nonemptiness of the solution set because \( f \) does not have the upper sign property nor the constraint set is convex.

As a direct consequence of Proposition 2.1 and the previous one we have an existence result for variational inequality problems.

**Corollary 3.1.** Let \( T : X \rightrightarrows X^* \) be a set-valued map with weak* compact values and \( C \) be a compact subset of \( X \). If \( T \) is properly quasimonotone and \( -T \) is pseudomonotone then problem (3) has at least one solution.

**Remark 1.** Notice that Corollary 3.1 can not be deduced from [4, Theorem 2.1] because the constraint set is not convex.
It was shown in [3, Proposition 3.1] that under upper the sign property the inclusion in part 3 of Proposition 3.1 holds. Example 3.4 shows that the pseudomonotonicity of $-f$ does not imply the upper sign property of $f$. The following example says that the converse statement is false too.

**Example 3.5.** The bifunction $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 
1, & x \in \mathbb{Q}, \ y \notin \mathbb{Q} \text{ or } x \notin \mathbb{Q}, \ y \in \mathbb{Q} \\
0, & x = y \\
-1, & \text{otherwise}
\end{cases}$$

clearly has the upper sign property, but $-f$ is not pseudomonotone.

Now, the two next propositions show that pseudomonotonicity and upper sign property are related under suitable assumptions.

**Proposition 3.3.** Let $C$ be a convex subset of $X$ and $f : X \times X \to \mathbb{R}$ be a bifunction such that one of the following assumptions hold:

1. $f(\cdot, y)$ is lower semicontinuous, for all $y \in C$;
2. $f(x, \cdot)$ is upper semicontinuous, for all $x \in C$;
3. $f$ vanishes on the diagonal of $C$ and $-f(\cdot, y)$ is semistrictly quasiconvex for all $y \in C$;
4. $f$ vanishes on the diagonal of $C$ and $f(x, \cdot)$ is semistrictly quasiconvex for all $x \in C$.

If $-f$ is pseudomonotone then $f$ has the upper sign property on $C$.

**Proof.** Let $x$ and $y$ be two elements of $C$ such that

$$f(x_t, x) \leq 0, \text{ for all } t \in ]0, 1[ \tag{7}$$

where $x_t = tx + (1-t)y$.

1. If $f(\cdot, y)$ is lower semicontinuous then $f(y, x) \leq 0$. Thus, the result follows from the pseudomonotonicity of $-f$.
2. Since $-f$ is pseudomonotone, condition (7) implies $f(x, x_t) \geq 0$ for any $t \in ]0, 1[$. By upper semicontinuity of $f(x, \cdot)$ we deduce that $f(x, y) \geq 0$.
3. If $f(x, y) < 0$ then $f(y, x) > 0$ which in turn implies $f(x_t, x) > 0$ for all $t \in ]0, 1[$, due to semistrict quasiconvexity of $-f(\cdot, x)$. However, this fact is a contradiction with (7). Hence, $f(x, y) \geq 0$.
4. It follows from part 3 of Proposition 3.1 and [3, Proposition 3.1].

As a direct consequence of part 4 of Proposition 3.3 and Proposition 2.1 we have the following corollary.

**Corollary 3.2.** Let $T : X \rightrightarrows X^*$ be a set-valued map with weak * compact values. If $-T$ is pseudomonotone then it is actually upper sign continuous.
Proposition 3.4. Let \( f : X \times X \rightarrow \mathbb{R} \) be a bifunction such that the following assumptions hold:

1. \( f \) vanishes on the diagonal of \( X \times X \) and
2. \( f(\cdot, y) \) is quasiconvex for all \( y \in X \).

If \( f \) has the upper sign property on \( X \) then \( -f \) is pseudomonotone.

Proof. Let \( x \) and \( y \) be two elements of \( X \) such that
\[
\begin{align*}
    f(x, y) &\leq 0 \quad \text{and} \quad f(y, x) < 0
\end{align*}
\]
By quasiconvexity we obtain \( f(x_t, y) \leq 0 \) for all \( t \in [0, 1] \), where \( x_t = tx + (1 - t)y \).
We now apply the upper sign property of \( f \) and deduce that \( f(y, x) \geq 0 \) which is a contradiction with (8).

Remark 2. A bifunction satisfying condition 3 in Proposition 3.3 is actually properly quasimonotone, due to Proposition 2.2.

On the other hand, Example 3.5 shows that the quasiconvexity assumption in Proposition 3.4 cannot be dropped.

4 Existence result

Given a bifunction \( f : X \times X \rightarrow \mathbb{R} \), a subset \( C \) of \( X \) and a set-valued map \( K : C \rightrightarrows C \), we define the set-valued map \( T : C \rightrightarrows C \) as follows
\[
    T(x) = \{ z \in K(x) : f(y, z) \leq 0 \text{ for all } y \in K(x) \}.
\]
In other words, for any \( x \in C \) we have \( T(x) = \text{CFP}(f, K(x)) \).

The following says that under closeness and lower semicontinuity of the set-valued map \( K \) and some regularity of the bifunction \( f \) the map \( T \) is closed.

Proposition 4.1. Let \( f : X \times X \rightarrow \mathbb{R} \) be a bifunction, \( C \) be a subset of \( X \) and \( K : C \rightrightarrows C \) be a set-valued map. If \( K \) is closed and lower semicontinuous with convex values; \( f \) is properly quasimonotone and vanishes on the diagonal of \( C \times C \); \( f(x, \cdot) \) is quasiconvex for all \( x \in C \); the set \( \{(x, y) \in C \times C : f(x, y) \leq 0\} \) is closed; then \( T \) is closed.

Proof. It is a straightforward adaptation of [10, Proposition 2.2].

Finally, we now present our existence result for quasi-equilibrium problems with some similar assumptions to [3, Theorem 4.5], but in the infinite dimensional setting.

Proposition 4.2. Let \( f : X \times X \rightarrow \mathbb{R} \) be a bifunction, \( C \) be a compact convex subset of \( X \) and \( K : C \rightrightarrows C \) be a set-valued map. If the following assumptions hold:

1. \( K \) is closed and lower semicontinuous with convex values;
2. \( f \) is properly quasimonotone and vanishes on the diagonal of \( C \times C \);
3. \( f(x, \cdot) \) is quasiconvex for all \( x \in C \);
4. the set \( \{(x, y) \in C \times C : f(x, y) \leq 0\} \) is closed;
5. \( -f \) is pseudomonotone;
then there exists at least a solution of problem \( (6) \) associated to \( f \) and \( K \).

**Proof.** By Propositions 3.2 and 4.1 and the assumptions considered we have that the set-valued map \( T \) is closed with closed, convex and nonempty values. Since \( C \) is compact, we obtain that \( T \) is upper semicontinuous. Thus, by Kakutani’s fixed point theorem there exists \( x \in C \) such that \( x \in T(x) \), that means \( x \in \text{CFP}(f, K(x)) \). The result follows from Proposition 3.1.

As a direct consequence of Proposition 4.2 we have the following corollary.

**Corollary 4.1 (Quasi-variational inequality).** Let \( T : X \rightrightarrows X^* \) be a set-valued map with weak \( * \) compact values, \( C \) be a compact convex subset of \( X \) and \( K : C \rightrightarrows C \) be a set-valued map. If the following assumptions hold:

1. \( K \) is closed and lower semicontinuous with convex values;
2. \( T \) is properly quasimonotone;
3. \( -T \) is pseudomonotone;
4. \( \{ (x, y) \in C \times C : \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0 \} \) is closed;

then the following set

\[
\{ x \in C : x \in K(x) \text{ and exists } x^* \in T(x) \text{ such that } \langle x^*, y - x \rangle \geq 0, \text{ for all } y \in K(x) \}\]  

(9)

is nonempty

**Proof.** It is enough to notice that \( f_T \) defined as (4) verifies all assumptions of Proposition 4.2 and that the solution set of the quasi-equilibrium problem associated to \( f_T \) and \( K \) coincides with the set (9).

Two remarks are needed about assumption 4 of our corollary.

**Remark 3.**

- Assumption 4 in Corollary 9 was considered in [10, Proposition 4.1] which established an existence result for quasi-variational inequalities without generalized monotonicity.

- An analogous result in [2] Proposition 3.5] was proved, where they do not require our assumption 4, instead they assume the following technical condition: for all \( x_\alpha \to x \) and all \( y_\alpha \to y \)

\[
\liminf_{x_\alpha \in T(x_\alpha)} \sup_{x^* \in T(x)} \langle x^*, y_\alpha - x_\alpha \rangle \leq 0 \Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0. \]  

(10)

This condition implies assumption 4 in Corollary 9 but it is stronger than it. Indeed, consider for instance the set-valued map \( T : \mathbb{R} \rightrightarrows \mathbb{R} \) defined by

\[
T(x) = \begin{cases} 
\{x\}, & x \neq 0 \\
\{1\}, & x = 0 
\end{cases}
\]

which satisfies assumption 4 in Corollary 9. However, for \( x_n = 1/n \) and \( y_n = 1 \) for all \( n \in \mathbb{N} \) we have

\[
\liminf \langle x_n, y_n - x_n \rangle = 0 \text{ and } \langle 1, 1 - 0 \rangle > 0.
\]

Thus, it fails (10).
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