Higher–order recurrence relations, Sobolev–type inner products and matrix factorizations

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Abstract
It is well known that Sobolev-type orthogonal polynomials with respect to measures supported on the real line satisfy higher-order recurrence relations and these can be expressed as a (2N+1)-banded symmetric semi-infinite matrix. In this paper we state the connection between these (2N+1)-banded matrices and the Jacobi matrices associated with the three-term recurrence relation satisfied by the standard sequence of orthonormal polynomials with respect to the 2-iterated Christoffel transformation of the measure.

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1 Introduction
Given a vector of measures \((\mu_0, \mu_1, \cdots, \mu_m)\) such that \(\mu_k\) is supported on a set \(E_k, \ k = 0, 1, \cdots, m,\) of the real line, let consider the Sobolev inner product

\[ \langle f, g \rangle_S = \sum_{k=0}^{m} \int_{E_k} f^{(k)}(x)g^{(k)}(x)d\mu_k(x). \]

Several examples of sequences of orthogonal polynomials with respect to the above inner products have been studied in the literature (see [23] as a recent survey).

1. When \(E_k, \ k = 0, 1, \cdots, m,\) are infinite subsets of the real line. (Continuous Sobolev)
2. When $E_0$ is an infinite subset of the real line and $E_k$, $k = 1, \ldots, m$, are finite subsets (Sobolev type).

3. When $E_m$ is an infinite subset of the real line and $E_k$, $k = 0, \ldots, m - 1$, are finite subsets.

In the above cases, the three term recurrence relation that every sequence of orthogonal polynomials with respect to a measure supported on an infinite subset of the real line does not hold. This is a direct consequence of the fact that the multiplication operator by $x$ is not symmetric with respect to any of the above mentioned situations.

In the Sobolev type case, you get a multiplication operator by a polynomial intimately related with the support of the discrete measures. In [17] an illustrative example when $d\mu_0 = x^\alpha e^{-x}dx$, $\alpha > -1$, $x \in [0, +\infty)$ and $d\mu_k(x) = M_k\delta(x)$, $M_k \geq 0$, $k = 1, 2, \ldots, m$, has been studied. In general, there exists a symmetric multiplication operator for a general Sobolev inner product if and only if the measures $\mu_1, \ldots, \mu_m$ are discrete (see [10]). On the other hand, in [8] the study of the general inner product such that the multiplication operator by a polynomial is a symmetric operator with respect to the inner product has been done. The representation of such inner products is given as well as the associated inner product. Assuming some extra conditions, you get a Sobolev-type inner product. Notice that there is an intimate relation about these facts and higher order recurrence relations that the sequences of orthonormal polynomials with respect to the above general inner products satisfy. A connection with matrix orthogonal polynomials has been stated in [9].

When you deal with the Sobolev type inner product, a lot of contributions have emphasized on the algebraic properties of the corresponding sequences of orthogonal polynomials in terms of the polynomials orthogonal with respect to the measure $\mu_0$. The case $m = 1$ has been studied in [1], where representation formulas for the new family as well as the study of he distribution of their zeros have been analyzed. The particular case of Laguerre Sobolev type orthogonal polynomials has been introduced and deeply analyzed in [19]. Outer ratio asymptotics when the measure belongs to the Nevai class and some extensions to a more general framework of Sobolev type inner products have been analyzed in [22] and [20]. For measures supported on unbounded intervals, asymptotic properties of Sobolev type orthogonal polynomials have been studied for Laguerre measures (see [14], [24]) and, in a more general framework, in [21].

The aim of our contribution is to analyze the higher order recurrence relation that a sequence of Sobolev type orthonormal polynomials satisfies when you consider $d\mu_0 = d\mu + M\delta(x-c)$ and $d\mu_1 = N\delta(x-c)$, where $M, N$ are nonnegative real numbers. In a first step, we obtain connection formulas between such Sobolev type orthonormal polynomials and the standard ones associated with the measures $d\mu$ and $(x-c)^2d\mu$, respectively. A matrix analysis of the five diagonal symmetric matrix associated with such a higher order recurrence relation is presented taking into account the QR factorization of the shifted symmetric Jacobi associated with the orthonormal polynomials with respect to the measure $d\mu$. The shifted Jacobi matrix associated with $(x-c)^2d\mu$ is $RQ$ (see [4], [16]). Our approach is quite different and it is based on the iteration of the Cholesky factorization of the symmetric Jacobi matrices associated with $d\mu$ and $(x-c)d\mu$, respectively (see [2], [12]).

These polynomial perturbations of measures are known in the literature as Christoffel perturbations (see [13] and [29]). They constitute examples of linear spectral transformations. The set of linear spectral transformations is generated by Christoffel and Geronimus transformations (see [29]). The connection with matrix analysis appears in [6] and [7] in terms of an inverse
problem for bilinear forms. On the other hand, Christoffel transformations of the above type are related to Gaussian rules as it is studied in [12]. For a more general framework about perturbations of bilinear forms and Hessenberg matrices as representations of a polynomial multiplication operator in terms of sequences of orthonormal polynomials associated with such bilinear forms, see [3].

The structure of the manuscript is as follows. Section 2 contains the basic background about polynomial sequences orthogonal with respect to a measure supported on an infinite set of the real line. We will call them standard orthogonal polynomial sequences. In Section 3 we present several connection formulas between the sequences of standard orthonormal polynomials associated with the measures \(d\mu\) and \((x-c)^2d\mu\) and the orthonormal polynomials with respect to a Sobolev-type inner product. We give alternative proofs to those presented in [15]. In Section 4, we deduce the coefficients of the three term recurrence relation for the orthonormal polynomials associated with the measure \((x-c)^2d\mu\). In Section 5 we study the five term recurrence relation that orthonormal polynomials with respect to the Sobolev-type inner product satisfy. Section 6 deals with the connection between the shifted Jacobi matrices associated with the measures \(d\mu\) and \((x-c)^2d\mu\) in terms of QR factorizations. In a next step, taking into account the Cholesky factorization of the symmetric five diagonal matrix associated with the multiplication operator \((x-c)^2\) in terms of the Sobolev-type orthonormal polynomials by commuting the factors we get the square of the shifted Jacobi matrix associated with the measure \((x-c)^2d\mu\). Finally, in Section 7 we show an illustrative example in the framework of Laguerre-Sobolev type inner products when \(c = -1\). Notice that in the literature, the authors have focused the interest in the case \(c = 0\) and the analysis of the corresponding differential operator such that the above polynomials are their eigenfunctions (see [18], [25] and [26]).

2 Preliminaries

Let \(\mu\) be a finite and positive Borel measure supported on an infinite subset \(E\) of the real line such that all the integrals
\[
\mu_n = \int_E x^n d\mu(x),
\]
exist for \(n = 0, 1, 2, \ldots\). \(\mu_n\) is said to be the moment of order \(n\) of the measure \(\mu\). The measure \(\mu\) is said to be absolutely continuous with respect to the Lebesgue measure if there exists a non-negative function \(\omega(x)\) such that \(d\mu(x) = \omega(x)dx\).

In the sequel, let \(P\) denote the linear space of polynomials in one real variable with real coefficients, and let \(\{P_n(x)\}_{n \geq 0}\) be the sequence of polynomials in \(P\) with leading coefficient equal to one (monic OPS, or MOPS in short), orthogonal with respect to the inner product \(\langle \cdot, \cdot \rangle_\mu : P \times P \rightarrow \mathbb{R}\) associated with \(\mu\)
\[
\langle f, g \rangle_\mu = \int_E f(x)g(x)d\mu(x). \tag{1}
\]
It induces the norm \(\|f\|_\mu^2 = \langle f, f \rangle_\mu\). Under these considerations, these polynomials satisfy the following three term recurrence relation
\[
xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 0, \tag{2}
\]
where for every \(n \geq 1\), \(\gamma_n\) is a positive real number and \(\beta_n, n \geq 0\) is a real number.
The $n$–th reproducing kernel for $\omega(x)$ is

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{||P_k||_\mu^2}, \quad n \geq 0. \tag{3}$$

Because of the Christoffel-Darboux formula, see [5], it may also be expressed as

$$K_n(x, y) = \frac{1}{||P_n||_\mu^2} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}, \quad n \geq 0. \tag{4}$$

The confluent formula becomes

$$K_n(x, x) = \sum_{k=0}^{n} \frac{[P_k(x)]^2}{||P_k||_\mu^2} = \frac{P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x)}{||P_n||_\mu^2}, \quad n \geq 0.$$ 

We introduce the following usual notation for the partial derivatives of the $n$-th reproducing kernel $K_n(x, y)$

$$\frac{\partial^{j+k}K_n(x, y)}{\partial x^j \partial y^k} = K_n^{(j,k)}(x, y), \quad 0 \leq j, k \leq n.$$ 

We will use the expression of the first $y$–derivative of (3) evaluated at $y = c$

$$K_n^{(0,1)}(x, c) = \frac{1}{||P_n||_\mu^2} \times \frac{P_{n+1}(x)P_n(c) - P_n(x)P_{n+1}(c)}{(x - c)^2} \quad \frac{P_{n+1}(x)P_n'(c) - P_n(x)P_{n+1}'(c)}{x - c} \tag{5}$$

and the following confluent formulas

$$K_n^{(0,1)}(c, c) = \frac{1}{||P_n||_\mu^2} \left[ \frac{P_n(c)P'_{n+1}(c) - P_{n+1}(c)P'_n(c)}{2} \right], \quad n \geq 0, \tag{6}$$

$$K_n^{(1,1)}(c, c) = \frac{1}{||P_n||_\mu^2} \times \left[ \frac{P_n(c)P''_{n+1}(c) - P_{n+1}(c)P''_n(c)}{6} + \frac{P'_n(c)P''_{n+1}(c) - P''_n(c)P'_{n+1}(c)}{2} \right], \quad n \geq 0,$$

whose proof can be found in [15, Sec. 2.1.2].

We will denote by $\{p_n(x)\}_{n \geq 0}$ the orthonormal polynomial sequence with respect to the measure $\mu$. Obviously,

$$p_n(x) = \frac{P_n(x)}{||P_n||_\mu} = r_n x^n + \text{lower degree terms}.$$ 

Notice that

$$r_n = \frac{1}{||P_n||_\mu'}.$$ 

Using orthonormal polynomials, the Christoffel-Darboux formula (4) reads

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x)p_k(y) = \frac{r_n}{r_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y} \tag{7}.$$
and its confluent form is

\[ K_n(x, x) = \sum_{k=0}^{n} [p_k(x)]^2 = \frac{r_n}{r_{n+1}} (p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)). \]

Next we define the Christoffel canonical transformation of a measure \( \mu \) (see \[2\], \[28\] and \[29\]). Let \( \mu \) be a positive Borel measure supported on \( E \subseteq \mathbb{R} \), and assume \( c \notin E \). Here and in the sequel, \( \{P_n^{[k]}(x)\}_{n \geq 0} \) will denote the MOPS with respect to the inner product

\[ \langle f, g \rangle [k] = \int_E f(x)g(x)d\mu^{[k]}, \quad d\mu^{[k]} = (x - c)^k d\mu, \quad k \geq 0, \quad c \notin E. \quad (8) \]

\( \{P_n^{[k]}(x)\}_{n \geq 0} \) is said to be the \( k \)-iterated Christoffel MOPS with respect to the above \textbf{standard} inner product. If \( k = 1 \) we have the \textbf{Christoffel canonical} perturbation of \( \mu \). It is well known that, in such a case, \( P_n(c) \neq 0 \), and (see \[3\] (7.3))

\[ P_n^{[1]}(x) = \frac{1}{(x - c)} \left[ P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)} P_n(x) \right] = \frac{\|P_n\|_{\mu}^2}{P_n(c)} K_n(x, c), \]

are the monic polynomials orthogonal with respect to the modified measure \( d\mu^{[1]} \). They are known in the literature as monic \textit{kernel polynomials}. If \( k > 1 \), then we have the \( k \)-iterated Christoffel transformation of \( d\mu \). In the sequel, we will denote

\[ ||P_n^{[k]}||_{[k]}^2 = \int_E [P_n^{[k]}(x)]^2 (x - c)^k d\mu \]

and \( x_n^{[k]}, r = 1, 2, ..., n, \) will denote the zeros of \( P_n^{[k]}(x) \) arranged in an increasing order. Since \( P_n^{[2]}(x) \) are the polynomials orthogonal with respect to \( \mu \) when \( k = 2 \) we have

\[ (x - c)^2 P_n^{[2]}(x) = \begin{vmatrix} P_{n+2}(x) & P_{n+1}(x) & P_n(x) \\ P_{n+2}(c) & P_{n+1}(c) & P_n(c) \\ P_{n+2}'(c) & P_{n+1}'(c) & P_n'(c) \end{vmatrix}, \]

i.e.,

\[ (x - c)^2 P_n^{[2]}(x) = P_{n+2}(x) - d_n P_{n+1}(x) + e_n P_n(x), \quad (9) \]

where

\[
\begin{align*}
d_n &= P_{n+2}(c)P_n'(c) - P_{n+2}'(c)P_n(c) \\
e_n &= P_{n+2}(c)P_{n+1}'(c) - P_{n+2}'(c)P_{n+1}(c) \\
&= \frac{\|P_{n+1}\|_{\mu}^2}{\|P_n\|_{\mu}^2} K_{n+1}(c, c) - \frac{r_n^2}{r_{n+1}^2} K_n(c, c) > 0.
\end{align*}
\]
Similar determinant formulas can be obtained for \( k > 2 \). For orthonormal polynomials the above expression reads

\[
(x - c)^2 p_n^{[2]}(x) = \frac{p_{n+2}(x)}{r_n^{[2]}} = \frac{p_n(x) + e_n p_{n+1}(x)}{r_n} - d_n \frac{p_{n+1}(x)}{r_n} + e_n \frac{p_n(x)}{r_n}
\]

or, equivalently.

\[
(x - c)^2 p_n^{[2]}(x) = \frac{r_n^{[2]}}{r_{n+2}} p_{n+2}(x) - d_n \frac{r_n^{[2]}}{r_{n+1}} p_{n+1}(x) + e_n \frac{r_n^{[2]}}{r_n} p_n(x).
\]

 Furthermore, from [27, Theorem 2.5] we conclude that

\[
||p_{n+1}^{[1]}(c)||_2 = ||p_{n+2}^{[1]}(c)||_2 \left( \frac{p_{n+2}(c)}{p_n(c)} \right) ||P_n^{[2]}(c)||^2.
\]

On the other hand, taking (9) into account

\[
e_n = \frac{\int_E (x - c)^2 p_n^{[2]}(x) P_n(x) d\mu}{\int_E P_n^{[2]}(x) d\mu} = \frac{||P_n^{[2]}||_2}{||P_n||_2} K_{n+1}(c,c)
\]

which implies that

\[
r_n^{[2]} = r_{n+1} \left( \frac{K_n(c,c)}{K_{n+1}(c,c)} \right)^{1/2}.
\]

Replacing in (11), the orthonormal version of the connection formula (9) reads

\[
(x - c)^2 p_n^{[2]}(x) = \left( \frac{K_n(c,c)}{K_{n+1}(c,c)} \right)^{1/2} \times \left( \frac{r_{n+1}}{r_{n+2}} p_{n+2}(x) - d_n p_{n+1}(x) + e_n \frac{r_{n+1}}{r_n} p_n(x) \right)
\]

In this contribution we will focus our attention on following inner product (Sobolev type inner product)

\[
(f, g)_S = \int_E f(x) g(x) d\mu + M f(c) g(c) + N f'(c) g'(c), \quad f, g \in \mathbb{P},
\]

where \( \mu \) is a positive Borel measure supported on \( E = [a, b] \subseteq \mathbb{R} \), \( c \notin E \), and \( M, N \geq 0 \). In general, \( E \) can be a bounded or unbounded interval of the real line. Let \( \{S_n^{M,N}(x)\}_{n \geq 0} \) denote the monic orthogonal polynomial sequence (MOPS in short) with respect to (13). These polynomials are known in the literature as Sobolev-type or discrete Sobolev orthogonal polynomials. It is worth to point out that many properties of the standard orthogonal polynomials are lost when an inner product as (13) is considered.
3 The 3TRR for the 2-iterated orthogonal polynomials

In order to obtain the corresponding symmetric Jacobi matrix, in this section will find the coefficients of the three term recurrence relation satisfied by the 2—iterated orthonormal polynomials \( \{p_{n}^{[2]}(x)\}_{n \geq 0} \). First, we deal with the monic orthogonal polynomials \( \{P_{n}^{[2]}(x)\}_{n \geq 0} \). Taking into account it is a standard sequence we will have

\[
x P_{n}^{[2]}(x) = P_{n+1}^{[2]}(x) + \kappa_{n} P_{n}^{[2]}(x) + \tau_{n} P_{n-1}^{[2]}(x), \quad n \geq 0,
\]

where

\[
\kappa_{n} = \frac{\langle x P_{n}^{[2]}(x), P_{n}^{[2]}(x) \rangle_{[2]} P_{n}^{[2]}(x), \quad \tau_{n} = \frac{\langle x P_{n}^{[2]}(x), P_{n-1}^{[2]}(x) \rangle_{[2]} P_{n-1}^{[2]}(x)}. \]

In order to obtain the explicit expression of the above coefficients, we first study the numerator in \( \kappa_{n} \). Taking into account (3) and (4) we have

\[
\langle x P_{n}^{[2]}(x), P_{n}^{[2]}(x) \rangle_{[2]} = \langle x P_{n}^{[2]}(x), (x - c)^2 P_{n}^{[2]}(x) \rangle
\]

\[
= \langle x P_{n}^{[2]}(x), P_{n+2}(x) \rangle - d_{n} \langle x P_{n}^{[2]}(x), P_{n+1}(x) \rangle + e_{n} \langle P_{n}^{[2]}(x), x P_{n}(x) \rangle
\]

\[
= -d_{n}||P_{n+1}||^{2}_{\mu} + e_{n} \langle P_{n}^{[2]}(x), x P_{n}(x) \rangle.
\]

Next, applying (2)

\[
\langle P_{n}^{[2]}(x), x P_{n}(x) \rangle = \langle P_{n}^{[2]}(x), P_{n+1}(x) \rangle + \beta_{n} \langle P_{n}^{[2]}(x), P_{n}(x) \rangle + \gamma_{n} \langle P_{n}^{[2]}(x), P_{n-1}(x) \rangle
\]

\[
= \beta_{n} ||P_{n}||^{2}_{\mu} + \gamma_{n} \langle P_{n}^{[2]}(x), P_{n-1}(x) \rangle.
\]

Taking into account (5)

\[
P_{n-1}(x) = \frac{1}{e_{n-1}} (x - c)^2 P_{n-1}^{[2]}(x) - \frac{1}{e_{n-1}} P_{n+1}(x) + \frac{d_{n}}{e_{n-1}} P_{n}(x)
\]

we obtain

\[
\langle P_{n}^{[2]}(x), P_{n-1}(x) \rangle = \langle P_{n}^{[2]}(x), \frac{1}{e_{n-1}} (x - c)^2 P_{n-1}^{[2]}(x) - \frac{1}{e_{n-1}} P_{n+1}(x) + \frac{d_{n}}{e_{n-1}} P_{n}(x) \rangle
\]

\[
= \frac{1}{e_{n-1}} \langle P_{n}^{[2]}(x), P_{n-1}^{[2]}(x) \rangle_{[2]} - \frac{1}{e_{n-1}} \langle P_{n}^{[2]}(x), P_{n+1}(x) \rangle
\]

\[
+ \frac{d_{n}}{e_{n-1}} \langle P_{n}^{[2]}(x), P_{n}(x) \rangle
\]

\[
= \frac{d_{n}}{e_{n-1}} \langle P_{n}^{[2]}(x), P_{n}(x) \rangle.
\]

Thus

\[
\langle x P_{n}^{[2]}(x), P_{n}^{[2]}(x) \rangle_{[2]} = \left( \beta_{n} + \gamma_{n} \frac{d_{n}}{e_{n-1}} \right) e_{n} ||P_{n}||^{2}_{\mu} - d_{n} ||P_{n+1}||^{2}_{\mu}.
\]

Next, we study the denominator in the expression of \( \kappa_{n} \). From (6) we have

\[
\langle P_{n}^{[2]}(x), P_{n}^{[2]}(x) \rangle_{[2]} = \langle P_{n}^{[2]}(x), (x - c)^2 P_{n}^{[2]}(x) \rangle
\]

\[
= \langle P_{n}^{[2]}(x), P_{n+2}(x) \rangle - d_{n} \langle P_{n}^{[2]}(x), P_{n+1}(x) \rangle
\]

\[
+ e_{n} \langle P_{n}^{[2]}(x), x P_{n}(x) \rangle
\]

\[
= e_{n} ||P_{n}||^{2}_{\mu}.
\]
Hence

\[ \kappa_n = \left( \beta_n + \gamma_n \frac{d_{n-1}}{e_{n-1}} \right) e_n \left( \frac{r_n}{r_{n-1}} \right)^2 - d_n \left( \frac{r_n}{r_{n+1}} \right)^2 \]

\[ = \left( \beta_n + \gamma_n \frac{d_{n-1}}{e_{n-1}} \right) e_n \left( \frac{r_n}{r_{n-1}} \right)^2 - d_n \left( \frac{r_n}{r_{n+1}} \right)^2, \]

\[ \tau_n = e_n \left( \frac{\|P_n\|_2^2}{\|P_{n-1}\|_2^2} \right) = \left( \frac{r_n}{r_{n-1}} \right)^2 e_n > 0, \]

where

\[ d_n = \frac{r_{n+1} p_{n+1}(c)}{r_{n+2} p_{n+1}(c)} + \frac{r_n}{r_{n+1} p_{n+1}(c)} K_{n+1}(c, c), \]

\[ e_n = \frac{\|P_{n+1}\|_2^2}{\|P_n\|_2^2} K_{n+1}(c, c) = \left( \frac{r_n}{r_{n+1}} \right)^2 K_{n+1}(c, c) > 0. \]

Hence, we have proved the following

**Proposition 1** The monic sequence \( \{P_n^{[2]}(x)\}_{n \geq 0} \) satisfies the three term recurrence relation

\[ x P_n^{[2]}(x) = P_{n+1}^{[2]}(x) + \kappa_n P_n^{[2]}(x) + \tau_n P_{n-1}^{[2]}(x), \quad n \geq 0, \]

with \( P_{-1}^{[2]}(x) = 0, \ P_0^{[2]}(x) = 1, \) and

\[ \kappa_n = \left( \beta_n + \gamma_n \frac{d_{n-1}}{e_{n-1}} \right) e_n \left( \frac{r_n}{r_{n-1}} \right)^2 - d_n \left( \frac{r_n}{r_{n+1}} \right)^2, \]

\[ \tau_n = \left( \frac{r_n}{r_{n+1}} \right)^2 K_{n+1}(c, c) > 0, \]

where, taking into account the explicit expressions for \( d_n \) and \( e_n \) given in (9), we also have

\[ d_n = \frac{r_{n+1} p_{n+1}(c)}{r_{n+2} p_{n+1}(c)} + \frac{r_n}{r_{n+1} p_{n+1}(c)} K_{n+1}(c, c), \]

\[ e_n = \left( \frac{r_n}{r_{n+1}} \right)^2 K_{n+1}(c, c) > 0. \]

Observe that the orthonormal version of the above proposition is

\[ p_{n+1}^{[2]}(x) = (x - \kappa_n) \frac{p_{n}^{[2]}(x)}{r_{n}^{[2]} / r_{n+1}^{[2]}} - \tau_n \frac{p_{n+1}^{[2]}(x)}{r_{n+1}^{[2]}} / r_{n+1}^{[2]}, \quad n \geq 0, \]

and, according to [13] Th. 1.29, p.12-13,

\[ p_{n+1}^{[2]}(x) = (x - \kappa_n) \frac{p_{n}^{[2]}(x) / \sqrt{r_{n+1}^{[2]}}}{r_{n+1}^{[2]}} - \tau_n \frac{p_{n+1}^{[2]}(x) / \sqrt{r_{n+1}^{[2]}}}{\sqrt{r_{n+1}^{[2]}}}, \quad n \geq 0, \]
Therefore

\[
\sqrt{r_{n+1}} = \frac{r_n^{[2]}}{r_{n+1}^{[2]}}, \quad \sqrt{r_{n+1}r_n} = \frac{r_n^{[2]}}{r_{n+1}^{[2]}}, \quad \sqrt{r_{n+1}r_n} = \frac{r_n^{[2]}}{r_{n+1}^{[2]}}.
\]

Therefore

\[
\tau_n = \left( \frac{r_{n-1}^{[2]}}{r_{n+1}^{[2]}} \right)^2 \frac{K_{n+1}(c, c)}{K_n(c, c)} = \left( \frac{r_{n-1}^{[2]}}{r_n^{[2]}} \right)^2.
\]

As a consequence,

\[
\frac{K_{n+1}(c, c)}{K_n(c, c)} = \left( \frac{r_{n-1}^{[2]}}{r_n^{[2]}} \right)^2 \left( \frac{r_{n+1}^{[2]}}{r_n^{[2]}} \right)^2 = \left( \frac{r_{n+1}}{r_n} \right)^2,
\]

\[
e_n = \left( \frac{r_n}{r_n^{[2]}} \right)^2 > 0, \quad (14)
\]

\[
d_n = \frac{p_{n+2}(c)}{r_{n+2}p_{n+1}(c)} + \left( \frac{r_n}{r_n^{[2]}} \right)^2 \frac{p_n(c)}{r_n p_n(c)}.
\]

Replacing in \( \kappa_n \) these alternative expressions for \( e_n \) and \( d_n \) we have

\[
\kappa_n = \left( \beta_n + \frac{\gamma_n d_{n-1}}{e_{n-1}} \right) e_n \left( \frac{r_n^{[2]}}{r_n} \right)^2 - d_n \left( \frac{r_n^{[2]}}{r_n} \right)^2,
\]

\[
\frac{d_{n-1}}{e_{n-1}} = \left( \frac{r_{n-1}^{[2]} p_{n+1}(c)}{r_n p_{n}(c)} + \left( \frac{r_{n-1}^{[2]} p_{n-1}(c)}{r_n p_{n}(c)} \right)^2 \right) = \left( \frac{r_{n-1}^{[2]} r_{n-1} p_{n+1}(c)}{r_{n-1} r_{n+1} p_{n}(c)} + \frac{r_n p_{n-1}(c)}{r_{n-1} p_{n}(c)} \right).
\]

Therefore

\[
\kappa_n = \beta_n + \gamma_n \left( \frac{r_{n-1}^{[2]} r_{n-1}^{[2]} p_{n+1}(c)}{r_{n-1} r_{n+1} p_{n}(c)} + \frac{r_n p_{n-1}(c)}{r_{n-1} p_{n}(c)} \right)
\]

\[
- \left( \frac{r_{n-1}^{[2]} r_n^{[2]} p_{n+2}(c)}{r_{n+1} r_{n+2} p_{n+1}(c)} + \frac{r_n p_{n}(c)}{r_{n+1} p_{n+1}(c)} \right).
\]

We have then proved the following

Corollary 1 The orthonormal polynomial sequence \( \{ p_{n}^{[2]}(x) \} \) satisfies the three term recurrence relation

\[
\sqrt{r_{n+1}} p_{n+1}^{[2]}(x) = (x - \kappa_n) p_{n}^{[2]}(x) - \sqrt{r_n} p_{n-1}^{[2]}(x), \quad n \geq 0,
\]
4 Connection formulas

As we have seen in previous sections, the connection formulas are the main tool to study the analytical properties of new families of OPS, in terms of other families of OPS with well-known analytical properties. Indeed, the problem of finding such expressions is called the connection problem, and it is of great importance in this context.

In this Section we present some results of [15], which will be useful later. We will give some alternative proofs of them. From now on, let us denote by \{s_{n}^{M,N}\}_{n \geq 0}, \{p_{n}\}_{n \geq 0} the sequences of polynomials orthonormal with respect to (13) and (1), respectively. We will write

\[ s_{n}^{M,N}(x) = t_{n} x^{n} + \text{lower degree terms}, \quad t_{n} > 0, \]

\[ p_{n}(x) = r_{n} x^{n} + \text{lower degree terms}, \quad r_{n} > 0, \]

\[ p_{n}^{[k]}(x) = r_{n}^{[k]} x^{n} + \text{lower degree terms}, \quad r_{n}^{[k]} > 0. \]

In the sequel the following notation will be useful. For every \( k \in \mathbb{N}_{0} \), let us define \( J_{[k]} \) as the semi-infinite symmetric Jacobi matrix associated with the measure \((x - c)^{k} d\mu\), verifying

\[ x \tilde{p}^{[k]} = J_{[k]} \tilde{p}^{[k]}, \]

where \( \tilde{p}^{[k]} \) stands for the semi-infinite column vector with orthonormal polynomial entries \( \tilde{p}^{[k]} = [p_{0}^{[k]}(x), p_{1}^{[k]}(x), p_{2}^{[k]}(x), \ldots]^{T} \), being \( \{p^{[k]}(x)\}_{n \geq 0} \) the orthonormal polynomial sequence with respect to the measure \((x - c)^{k} d\mu\) [8]. One has \( \tilde{p}^{[0]} = \tilde{p} = [p_{0}(x), p_{1}(x), p_{2}(x), \ldots]^{T} \) being \( \{p(x)\}_{n \geq 0} \) the orthonormal polynomial sequence with respect to the standard measure \( \mu \), and \( J_{[0]} = J \) is the corresponding Jacobi matrix.

Next, we will present an expansion of the monic polynomials \( S_{n}^{M,N}(x) \) in terms of polynomials \( P_{n}(x) \) orthogonal with respect to \( \mu \). When necessary, we refer the reader to [15, Th. 5.1] for alternative proofs to those presented here.

Lemma 1

\[ S_{n}^{M,N}(x) = P_{n}(x) - M S_{n}^{M,N}(c) K_{n-1}(x,c) - N [S_{n}^{M,N}]'(c) K_{n-1}^{(0,1)}(x,c) \quad (16) \]
where

\[
S_n^{M,N}(c) = \begin{vmatrix}
P_n(c) & NK_{n-1}^{(0,1)}(c, c) \\
[P_n]'(c) & 1 + NK_{n-1}^{(1,1)}(c, c)
\end{vmatrix},
\]

(17)

\[
[S_n^{M,N}]'(c) = \begin{vmatrix}
1 + MK_{n-1}(c, c) & NK_{n-1}^{(0,1)}(c, c) \\
MK_{n-1}^{(1,0)}(c, c) & 1 + NK_{n-1}^{(1,1)}(c, c)
\end{vmatrix}.
\]

(18)

**Proof.** We search for the expansion

\[
S_n^{M,N}(x) = P_n(x) + \sum_{j=0}^{n-1} q_{n,j} P_j(x),
\]

where

\[
q_{n,j} = \frac{\int_E S_n^{M,N}(x) P_j(x) d\mu}{\|P_j\|_2^2} = -\frac{M S_n^{M,N}(c) P_j(c)}{\|P_j\|_2^2} - \frac{N [S_n^{M,N}]'(c) P_j'(c)}{\|P_j\|_2^2}.
\]

From these coefficients (16 follows). Next, having its first derivative with respect to \(x\), and taking \(x = c\) we get

\[
P_n(c) = (1 + MK_{n-1}(c, c)) S_n^{M,N}(c) + NK_{n-1}^{(0,1)}(c, c) [S_n^{M,N}]'(c),
\]

\[
[P_n]'(c) = MK_{n-1}^{(0,1)}(c, c) S_n^{M,N}(c) + (1 + NK_{n-1}^{(1,1)}(c, c)) [S_n^{M,N}]'(c).
\]

Solving the above linear system for \(S_n^{M,N}(c)\) and \([S_n^{M,N}]'(c)\) we obtain (17) and (17).

This completes the proof. ■

From the above lemma, we can also express \(S_n^{M,N}(x)\) as follows

\[
S_n^{M,N}(x) = \begin{vmatrix}
P_n(x) & MK_{n-1}(x, c) & NK_{n-1}^{(0,1)}(x, c) \\
P_n(c) & 1 + MK_{n-1}(c, c) & NK_{n-1}^{(1,1)}(c, c) \\
[P_n]'(c) & MK_{n-1}^{(1,0)}(c, c) & 1 + NK_{n-1}^{(1,1)}(c, c)
\end{vmatrix}.
\]

In terms of the orthonormal polynomials (16) becomes

\[
s_n^{M,N}(x) = \frac{t_n}{r_n} p_n(x) - M s_n^{M,N}(c) K_{n-1}(x, c) - N [s_n^{M,N}]'(c) K_{n-1}^{(0,1)}(x, c).
\]

(19)

As a direct consequence of Lemma [1] we get the following result concerning the norm of the Sobolev type polynomials \(S_n^{M,N}\)
Lemma 2 For \( c \in \mathbb{R}_+ \) the norm of the monic Sobolev type polynomials \( S_{n}^{M,N} \), orthogonal with respect to \((13)\) is

\[
\frac{1}{t_n} = \|S_{n}^{M,N}\|_2^2 = \|P_n\|_2^2 + M S_{n}^{M,N}(c)P_n(c) + N [S_{n}^{M,N}]'(c)[P_n]'(c).
\]

Proof. From \((16)\) we have

\[
S_{n}^{M,N}(x) = P_n(x) - M S_{n}^{M,N}(c)K_{n-1}(x,c) - N [S_{n}^{M,N}]'(c)K_{n-1}^{(0,1)}(x,c)
\]

and according to \((13)\) we get

\[
\langle S_{n}^{M,N}(x), S_{n}^{M,N}(x) \rangle_S = \langle S_{n}^{M,N}(x), P_n(x) \rangle + M S_{n}^{M,N}(c)P_n(c) + N [S_{n}^{M,N}]'(c)[P_n]'(c).
\]

This completes the proof. ■

Next, we represent the Sobolev type orthogonal polynomials in terms of the polynomial kernels associated with the sequence of orthonormal polynomials \( \{p_n(x)\}_{n \geq 0} \) and its derivatives. Another proof of this result can be found in \([13\) Prop. 5.6, p. 115].

Lemma 3 The sequence of Sobolev type orthonormal polynomials \( \{s_{n}^{M,N}(x)\}_{n \geq 0} \) can be expressed as

\[
s_{n}^{M,N}(x) = \alpha_{n+1,n}p_{n+1}(x) + \alpha_{n,n}p_n(x) - M s_{n}^{M,N}(c)K_{n+1}(x,c) - N [s_{n}^{M,N}]'(c)K_{n+1}^{(0,1)}(x,c),
\]

where

\[
\alpha_{n+1,n} = M s_{n}^{M,N}(c)p_{n+1}(c) + N [s_{n}^{M,N}]'(c)[p_{n+1}]'(c),
\]

\[
\alpha_{n,n} = \frac{t_n}{r_n} + M s_{n}^{M,N}(c)p_{n}(c) + N [s_{n}^{M,N}]'(c)[p_{n}]'(c).
\]

Proof. From \((17)\)

\[
K_{n-1}(x,c) = K_{n+1}(x,c) - p_{n+1}(x)p_{n+1}(c) - p_n(x)p_n(c),
\]

\[
K_{n+1}^{(0,1)}(x,c) = K_{n+1}^{(0,1)}(x,c) - p_{n+1}(x)[p_{n+1}]'(c) - p_n(x)[p_n]'(c).
\]

Replacing in \((19)\) yields

\[
s_{n}^{M,N}(x) = \frac{t_n}{r_n}p_n(x) - M s_{n}^{M,N}(c) (K_{n+1}(x,c) - p_{n+1}(x)p_{n+1}(c) - p_n(x)p_n(c))
\]

\[
- N [s_{n}^{M,N}]'(c) \left( K_{n+1}^{(0,1)}(x,c) - p_{n+1}(x)[p_{n+1}]'(c) - p_n(x)[p_n]'(c) \right)
\]

\[
= \left[ M s_{n}^{M,N}(c)p_{n+1}(c) + N [s_{n}^{M,N}]'(c)[p_{n+1}]'(c) \right] p_{n+1}(x)
\]

\[
+ \left[ \frac{t_n}{r_n} + M s_{n}^{M,N}(c)p_{n}(c) + N [s_{n}^{M,N}]'(c)[p_{n}]'(c) \right] p_n(x)
\]

\[
- M s_{n}^{M,N}(c)K_{n+1}(x,c) - N [s_{n}^{M,N}]'(c)K_{n+1}^{(0,1)}(x,c).
\]

This completes the proof. ■

Next we expand the polynomials \( \{p_n(x)\}_{n \geq 0} \) in terms of the polynomials \( \{p_n^{[2]}(x)\}_{n \geq 0} \). This result is already addressed in \([15\) Prop. 5.7, p.116] as well as in \([11\) but we include here an alternative proof.
Lemma 4 The sequence of polynomials \( \{p_n(x)\}_{n \geq 0} \), orthonormal with respect to \( d\mu \), is expressed in terms of the 2-iterated orthonormal polynomials \( \{p_n^{[2]}(x)\}_{n \geq 0} \) as follows

\[
p_n(x) = \xi_{n,n} p_n^{[2]}(x) + \xi_{n-1,n} p_{n-1}^{[2]}(x) + \xi_{n-2,n} p_{n-2}^{[2]}(x),
\]

where

\[
\xi_{n,n} = \frac{r_n}{r_n^{[2]}} = \frac{r_n}{r_n+1} \left( \frac{K_{n+1}(c,c)}{K_n(c,c)} \right)^{1/2} = e_n^{1/2},
\]

\[
\xi_{n-1,n} = -d_{n-1} \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2},
\]

\[
\xi_{n-2,n} = \frac{r_{n-1}}{r_n} \left( \frac{K_{n-2}(c,c)}{K_{n-1}(c,c)} \right)^{1/2} = \frac{r_{n-1}}{r_n r_{n+1}} e_n^{1/2}.
\]

Proof. Taking into account (9), (12) and (11), for the first coefficient we immediately have

\[
\xi_{n,n} = \langle p_n(x), p_n^{[2]}(x) \rangle_{[2]} = \langle p_n(x), (x-c)^2 p_n^{[2]}(x) \rangle
\]

\[
= \frac{r_n}{r_n^{[2]}} = \frac{r_n}{r_n+1} \left( \frac{K_{n+1}(c,c)}{K_n(c,c)} \right)^{1/2} = e_n^{1/2}.
\]

For the second coefficient, from (11) we have

\[
\xi_{n-1,n} = \langle p_n(x), p_{n-1}^{[2]}(x) \rangle_{[2]} = \langle p_n(x), (x-c)^2 p_{n-1}^{[2]}(x) \rangle
\]

\[
= \langle p_n(x), -d_{n-1} \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2} p_n(x) \rangle = -d_{n-1} \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2}.
\]

Finally, for the last coefficient, we get

\[
\xi_{n-2,n} = \langle p_n(x), p_{n-2}^{[2]}(x) \rangle_{[2]} = \langle p_n(x), (x-c)^2 p_{n-2}^{[2]}(x) \rangle
\]

\[
= \frac{r_{n-2}}{r_n} = \frac{r_{n-1}}{r_n} \left( \frac{K_{n-2}(c,c)}{K_{n-1}(c,c)} \right)^{1/2} = \frac{r_{n-1}}{r_n r_{n+1}} e_n^{1/2}.
\]

This completes the proof. ■

Next, let us obtain a third representation for the Sobolev type OPS in terms of the polynomials orthonormal with respect to \( (x-c)^2 d\mu \). This expression will be very useful to find the connection of these polynomials with the matrix orthogonal polynomials, and we include the proof for the convenience of the reader.

Theorem 1 Let \( \{s_n^{M,N}(x)\}_{n \geq 0} \) be the sequence Sobolev-type polynomials orthonormal with respect to (13), and let \( \{p_n^{[2]}(x)\}_{n \geq 0} \) be the sequence of polynomials orthonormal with respect to the inner product (5) with \( k = 2 \). Then, the following expression holds

\[
s_n^{M,N}(x) = \gamma_{n,n} p_n^{[2]}(x) + \gamma_{n-1,n} p_{n-1}^{[2]}(x) + \gamma_{n-2,n} p_{n-2}^{[2]}(x),
\]

where

\[
\gamma_{n,n} = \frac{t_n}{r_n^{[2]}} = \frac{t_n}{r_n+1} \left( \frac{K_{n+1}(c,c)}{K_n(c,c)} \right)^{1/2},
\]

\[
\gamma_{n-1,n} = -d_{n-1} \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2},
\]

\[
\gamma_{n-2,n} = \frac{t_{n-1}}{r_n} \left( \frac{K_{n-2}(c,c)}{K_{n-1}(c,c)} \right)^{1/2} = \frac{t_{n-1}}{r_n r_{n+1}} e_n^{1/2}.
\]
\[ \gamma_{n-1,n} = - \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2} \]
\[ \times \left( \frac{d_n - t_n}{r_n} + e_n - \frac{r_n}{r_n-1} \right) \left[ M s^{M,N}_n(c)p_{n-1}(c) + N \left[ s^{M,N}_n(c)[p_n]'(c) \right] \right], \]
\[ \gamma_{n-2,n} = \frac{r_{n-1}}{t_n} \left( \frac{K_{n-2}(c,c)}{K_{n-1}(c,c)} \right)^{1/2}. \]

**Proof.** For \( \gamma_{n,n} \), matching the leading coefficients of \( s^{M,N}_n(x) \) and \( p_n^{[2]}(x) \), it is a straightforward consequence to see that
\[ \gamma_{n,n} = \langle s^{M,N}_n(x), p_n^{[2]}(x) \rangle_{[2]} = \frac{t_n}{r_n^{[2]}} \]
Next from \( \text{(12)} \)
\[ \gamma_{n,n} = \frac{t_n}{r_n^{[2]}} = \frac{t_n}{r_{n+1}} \left( \frac{K_{n+1}(c,c)}{K_n(c,c)} \right)^{1/2}. \]

For \( \gamma_{n-1,n} \) we need some extra work. From \( \text{(20)} \) we have
\[ \gamma_{n-1,n} = \langle s^{M,N}_n(x), p_{n-1}^{[2]}(x) \rangle_{[2]} = \int_E s^{M,N}_n(x)(x - c)^2 p_{n-1}^{[2]}(x) d\mu \]
\[ = \int_E s^{M,N}_n(x) \left[ -d_n - r_n^{[2]} p_n(x) + e_n - \frac{r_n^{[2]}}{r_n} p_{n-1}(x) \right] d\mu \]
\[ = -d_n \frac{r_n}{r_n} \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2} + e_n \frac{r_n}{r_n - 1} \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2} \int_E s^{M,N}_n(x) p_{n-1}(x) d\mu. \]
The last integral can be computed using \( \text{(19)} \)
\[ \int_E s^{M,N}_n(x) p_{n-1}(x) d\mu = \]
\[ \int_E \left( -M s^{M,N}_n(c) K_{n-1}(x,c) - N \left[ s^{M,N}_n(c)[p_n]'(c) \right] K_{n-1}(x,c) \right) p_{n-1}(x) d\mu \]
\[ = -M s^{M,N}_n(c) p_{n-1}(x) - N \left[ s^{M,N}_n(c)[p_n]'(c) \right]. \]
Thus
\[ \gamma_{n-1,n} = - \left( \frac{K_{n-1}(c,c)}{K_n(c,c)} \right)^{1/2} \]
\[ \times \left( \frac{d_n - t_n}{r_n} + e_n - \frac{r_n}{r_n - 1} \right) \left[ M s^{M,N}_n(c)p_{n-1}(c) + N \left[ s^{M,N}_n(c)[p_n]'(c) \right] \right]. \]

Finally, for the last coefficient we have
\[ \gamma_{n-2,n} = \langle s^{M,N}_n(x), p_{n-2}^{[2]}(x) \rangle_{[2]} = \langle s^{M,N}_n(x), (x - c)^2 p_{n-2}^{[2]}(x) \rangle_{S} \]
\[ = t_n r_n^{[2]} (S^{M,N}_n(x), (x - c)^2 p_{n-2}^{[2]}(x) \rangle_{S} \]
\[ = t_n r_n^{[2]} ||S^{M,N}_n||_S^2 = t_n^{r_n-2} = \frac{r_n}{t_n} \left( \frac{K_{n-2}(c,c)}{K_{n-1}(c,c)} \right)^{1/2}. \]

This completes the proof. \( \blacksquare \)
5 The five term recurrence relation

In this section, we will obtain the five term recurrence relation that the sequence of Sobolev-type orthonormal polynomials \( \{ s_n^{M,N}(x) \}_{n \geq 0} \) satisfies. We use orthonormal polynomials because all the matrices associated with the multiplication operators we are dealing with are symmetric. Later on, we will derive an interesting relation between the five diagonal matrix \( H \) associated with the multiplication operator by \((x-c)^2\) in terms of the orthonormal basis \( \{ s_n^{M,N}(x) \}_{n \geq 0} \), and the tridiagonal Jacobi matrix \( J^2 \) associated with the three term recurrence relation satisfied by the 2-iterated orthonormal polynomials \( \{ p_n^2(x) \}_{n \geq 0} \).

To do that, we will use the following remarkable fact

**Theorem 2** The multiplication operator by \((x-c)^2\) is a symmetric operator with respect to the discrete Sobolev inner product (13). In other words, for any \( p(x), q(x) \in \mathbb{P} \), it satisfies

\[
((x-c)^2 p(x), q(x))_S = (p(x), (x-c)^2 q(x))_S.
\]

**Proof.** The proof is a straightforward consequence of (13). \( \blacksquare \)

Next, we will obtain the coefficients of the aforementioned five term recurrence relation. Let consider the Fourier expansion of \((x-c)^2 s_n^{M,N}(x)\) in terms of \( \{ s_n^{M,N}(x) \}_{n \geq 0} \)

\[
(x-c)^2 s_n^{M,N}(x) = \sum_{k=0}^{n+2} \rho_{k,n} s_k^{M,N}(x),
\]

where

\[
\rho_{k,n} = \langle (x-c)^2 s_n^{M,N}(x), s_k^{M,N}(x) \rangle_S, \quad k = 0, \ldots, n+2.
\]

From (22),

\[
\rho_{k,n} = \langle s_n^{M,N}(x), (x-c)^2 s_k^{M,N}(x) \rangle_S, \quad k = 0, \ldots, n+2.
\]

Hence, \( \rho_{k,n} = 0 \) for \( k = 0, \ldots, n-3 \). Taking into account that

\[
((x-c)^2 s_n^{M,N}(x))[x=c] = [(x-c)^2 s_n^{M,N}(x)]'|x=c = 0,
\]

and using [24] Th. 1, p. 174] we get

\[
\langle (x-c)^2 s_n^{M,N}(x), s_k^{M,N}(x) \rangle_S = \langle s_n^{M,N}(x), s_k^{M,N}(x) \rangle_S^{[2]}.
\]

Notice that

\[
\langle (x-c)^2 s_n^{M,N}(x), s_k^{M,N}(x) \rangle_S = \langle (x-c)^2 s_n^{M,N}(x), s_k^{M,N}(x) \rangle_S.
\]

Next, using the connection formula (24) we have

\[
\rho_{n+2,n} = \langle (x-c)^2 s_n^{M,N}(x), s_{n+2}^{M,N}(x) \rangle_S = \langle s_n^{M,N}(x), s_{n+2}^{M,N}(x) \rangle_S^{[2]} = \gamma_{n,n+2} \frac{n}{n+2},
\]

\[
\rho_{n+1,n} = \langle (x-c)^2 s_n^{M,N}(x), s_{n+1}^{M,N}(x) \rangle_S = \langle s_n^{M,N}(x), s_{n+1}^{M,N}(x) \rangle_S^{[2]} = \gamma_{n,n+1} \frac{p_n^{[2]}(x)}{p_{n+1}^{[2]}(x)} + \gamma_{n-1,n+1} \frac{p_n^{[2]}(x)}{p_{n-1}^{[2]}(x)}
\]

\[
= \gamma_{n,n+1} + \gamma_{n-1,n} \gamma_{n-1,n+1},
\]
\[ \rho_{n,n} = ((x - c)^2 s_n^M(x), s_n^N(x))_S = \langle s_n^M(x), s_n^M(x) \rangle_2 \]
\[ = \gamma_{n,n}^2 \langle p_n^2(x), p_n^2(x) \rangle_2 + \gamma_{n-1,n}^2 \langle p_{n-1}^2(x), p_{n-1}^2(x) \rangle_2 + \gamma_{n-2,n}^2 \langle p_{n-2}^2(x), p_{n-2}^2(x) \rangle_2 \]
\[ = \gamma_{n,n}^2 + \gamma_{n-1,n}^2 + \gamma_{n-2,n}^2, \]

\[ \rho_{n-1,n} = ((x - c)^2 s_n^M(x), s_{n-1}^M(x))_S = \langle s_n^M(x), s_{n-1}^M(x) \rangle_2 \]
\[ = \gamma_{n-1,n} \gamma_{n-1,n-1} \langle p_{n-1}^2(x), p_{n-1}^2(x) \rangle_2 + \gamma_{n-2,n} \gamma_{n-2,n-1} \langle p_{n-2}^2(x), p_{n-2}^2(x) \rangle_2 \]
\[ = \gamma_{n-1,n} \gamma_{n-1,n-1} + \gamma_{n-2,n} \gamma_{n-2,n-1} = \rho_{n,n-1} \]

\[ \rho_{n-2,n} = \langle (x - c)^2 s_n^M(x), s_{n-2}^N(x) \rangle_S = \langle s_n^M(x), s_{n-2}^M(x) \rangle_2 \]
\[ = \gamma_{n-2,n} \gamma_{n-2,n-2} \langle p_{n-2}^2(x), p_{n-2}^2(x) \rangle_2 \]
\[ = \gamma_{n-2,n} \gamma_{n-2,n-2} = \frac{t_n^2 - t_n}{t_n}. \]

Introducing the following notation
\[ \rho_{n-2,n} = a_n, \quad \rho_{n-1,n} = b_n, \quad \rho_{n,n} = c_n, \]

(23) reads
\[ (x - c)^2 s_n^M(x) = \]
\[ a_n s_{n+2}^M(x) + b_{n+1} s_{n+1}^M(x) + c_n s_n^M(x) + b_n s_{n-1}^M(x) + a_n s_{n-2}^M(x), \quad n \geq 0, \tag{26} \]

where, by convention,
\[ s_{-2}^M(x) = s_{-1}^M(x) = 0. \]

6 A matrix approach

In this Section we will deduce an interesting relation between the five diagonal matrix \( H \) associated with the multiplication operator by \((x - c)^2\) associated with the orthonormal Sobolev type orthonormal polynomials and the Jacobi matrix \( J_{[2]} \) associated with the 2-iterated orthonormal polynomials \( \{ p_n^2(x) \}_{n \geq 0} \).

First, we deal with the matrix representation of (26)
\[ (x - c)^2 s^M,N = H s^M,N, \tag{27} \]

where \( H \) is the five diagonal semi-infinite symmetric matrix
\[ H = \begin{bmatrix}
    c_0 & b_1 & a_2 & 0 & \cdots \\
    b_1 & c_1 & b_2 & a_3 & \cdots \\
    a_2 & b_2 & c_3 & b_3 & \cdots \\
    0 & a_3 & b_3 & c_3 & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}, \tag{28} \]
and \( \bar{s}^{M,N} = [s_0^{M,N}(x), s_1^{M,N}(x), s_2^{M,N}(x), \ldots]^T \).

Next, from (21) we get
\[
\bar{s}^{M,N} = T \bar{p}^{[2]}
\]
where \( T \) is the lower triangular, semi-infinite, and nonsingular matrix with positive diagonal entries
\[
T = \begin{bmatrix}
\gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} & \gamma_{0,3} & \cdots \\
\gamma_{0,1} & \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \cdots \\
\gamma_{0,2} & \gamma_{1,2} & \gamma_{2,2} & \gamma_{2,3} & \cdots \\
\gamma_{0,3} & \gamma_{1,3} & \gamma_{2,3} & \gamma_{3,3} & \cdots \\
& & & & \ddots
\end{bmatrix}
\]
and \( \bar{p}^{[2]} = [p_0^{[2]}(x), p_1^{[2]}(x), p_2^{[2]}(x), \ldots]^T \). We will denote by \( J \) the Jacobi matrix associated with the orthonormal sequence \( \{p_n(x)\}_{n \geq 0} \), with respect to the measure \( d\mu \). As a consequence, we have
\[
x \bar{p} = J \bar{p}.
\]
Let \( J^{[2]} \) be the Jacobi matrix associated with the 2-iterated OPS \( \{p_n^{[2]}(x)\}_{n \geq 0} \). Notice that from
\[
x \bar{p}^{[2]} = J^{[2]} \bar{p}^{[2]},
\]
we get
\[
(x - c)^2 \bar{p}^{[2]} = (J^{[2]} - cI)^2 \bar{p}^{[2]}.
\]
Starting with \( (J - cI) \), and assuming \( c \) is located in the left hand side of \( \text{supp}(\mu) \), all their leading principal submatrices are positive definite, so we get the following Cholesky factorization
\[
J - cI = LL^T.
\]
Here \( L \) is a lower bidiagonal matrix with positive diagonal entries. From [3] we know
\[
J^{[1]} - cI = L^T L = L_1 L_1^T,
\]
where \( L_1 \) is a lower bidiagonal matrix with positive diagonal entries. Notice that if \( c \) is located in the right hand side of the support, then you must deal with the Cholesky factorization of the matrix \( cI - J \).

Next, we show that the five diagonal matrix \( H \) associated with (26) can be given in terms of the five diagonal matrix \( (J^{[2]} - cI)^2 \). Combining (27) with (29), we get
\[
T(x - c)^2 \bar{p}^{[2]} = HT \bar{p}^{[2]}.
\]
Substituting (31) into (34)
\[
T (J^{[2]} - cI)^2 \bar{p}^{[2]} = HT \bar{p}^{[2]}.
\]
Hence, we state the following

**Proposition 2** The semi-infinite five diagonal matrix \( H \), can be obtained from the matrix \( (J^{[2]} - cI)^2 \) as follows
\[
H = T (J^{[2]} - cI)^2 T^{-1}.
\]
Next, we repeat the above process commuting the order of factors in $L_1 L_1^\top$. Thus
\[ L_1^\top L_1 = J_{[2]} - c I. \] (35)
From (33) we have $L_1 = L^\top L L_1^{-1\top}$, and replacing this expression as above, it yields
\[ J_{[2]} - c I = L_1^\top \left( L^\top L L_1^{-1\top} \right) = (L_1^\top L')(LL_1^{-1\top}) = (LL_1)^\top (LL_1^{-1\top}) = RQ. \]
Notice that $R = (LL_1)^\top$ is an upper triangular matrix, with positive diagonal entries because $L$ and $L_1$ are lower bidiagonal matrices. Now for the matrix $Q = LL_1^{-1\top}$ we have
\[
QQ^\top = LL_1^{-1\top} (LL_1^{-1\top})^\top = LL_1^{-1\top} (L_1^{-1\top} L^\top) = L \left( L_1^{-1\top} L_1^{-1\top} \right) L^\top = L (L_1 L_1^\top)^{-1} L^\top.
\]
Next, from (33) $L_1 L_1^\top = L^\top L$. Thus
\[ QQ^\top = L (L^\top L)^{-1} L^\top = LL^{-1\top} L^\top = I, \]
as well as
\[ Q^\top Q = (L_1^{-1\top} L^\top) (LL_1^{-1\top}) = L_1^{-1} (L^\top L) L_1^{-1\top} = L_1^{-1} (L_1 L_1^\top) L_1^{-1\top} = I. \]
This means that $Q$ is an orthogonal matrix. Thus, we have proved the following

**Proposition 3** The positive definite matrix $J_{[2]} - c I$ can be factorised as follows
\[ J_{[2]} - c I = RQ, \] (36)
where $R$ is an upper triangular matrix, and $Q$ is an orthogonal matrix, i.e. $QQ^\top = Q^\top Q = I$.

Notice that the above result has been also proved in [12], but the fact that also $QQ^\top = I$ holds is not proved.

Taking into account the previous result, we come back to (32) to observe
\[ J - c I = LL^\top = L \left( L_1^{-1\top} L_1^\top \right) L^\top = \left( LL_1^{-1\top} \right) (L_1^\top L_1^\top) = (LL_1^{-1\top}) (LL_1^\top) = QR. \]
Thus, we can summarize the above as follows

**Proposition 4** Let $J$ be the symmetric Jacobi matrix such that
\[ x \bar{p} = J \bar{p}. \]
If $\bar{p} = [p_0(x), p_1(x), p_2(x), \ldots]^\top$ is the infinite vector associated with the orthonormal polynomial sequence with respect to $d\mu$ and we assume $p_n(c) \neq 0$ for $n \geq 1$, then the following factorization
\[ J - c I = QR \]
holds. Here $R$ is an upper triangular matrix, and $Q$ is an orthogonal matrix, i.e. $QQ^\top = Q^\top Q = I$. Under these conditions,
\[ RQ = J_{[2]} - c I, \]
where $J_{[2]}$ is the symmetric Jacobi matrix such that $x \bar{p}^{[2]} = J_{[2]} \bar{p}^{[2]}$, where $\bar{p}^{[2]}$ is the infinite vector associated with the orthonormal polynomial sequence with respect to $(x - c)^2 d\mu$. 


Observe that this is an alternative proof of Theorem 3.3 in [4].

Since \( J_2 \) is a symmetric matrix, from (36) and \( \mathcal{Q}\mathcal{Q}^\top = \mathbf{I} \), we easily observe
\[
(J_2 - c\mathbf{I})^2 = (J_2 - c\mathbf{I})(J_2 - c\mathbf{I})^\top = \mathcal{R}\mathcal{Q}\mathcal{Q}^\top \mathcal{R}^\top = \mathcal{R}\mathcal{R}^\top.
\]
Thus

**Proposition 5** The square of the positive definite symmetric matrix \( J_2 - c\mathbf{I} \) has the following factorization
\[
(J_2 - c\mathbf{I})^2 = \mathcal{R}\mathcal{R}^\top,
\]
where \( \mathcal{R} \) is an upper triangular matrix. Furthermore
\[
(J - c\mathbf{I})^2 = \mathcal{R}'\mathcal{R}.
\]

Next, we are ready to prove that there is a very close relation between the five diagonal semi-infinite symmetric matrix \( \mathcal{H} \) defined in (28), and the lower triangular, semi-infinite, nonsingular matrix \( \mathcal{T} \) defined in (30).

We will use the following notation. Let \( \mathcal{f} \) be any semi-infinite column vector with polynomial entries \( \mathcal{f} = [f_0(x), f_1(x), f_2(x), \ldots]^\top \). Then \( \langle \mathcal{f}, \mathcal{g} \rangle \) will represent the given inner product of \( \mathcal{f} \) and \( \mathcal{g} \) componentwise, that is, we get the following semi-infinite square matrix
\[
\begin{bmatrix}
\langle f_0, g_0 \rangle & \langle f_0, g_1 \rangle & \langle f_0, g_2 \rangle & \cdots \\
\langle f_1, g_0 \rangle & \langle f_1, g_1 \rangle & \langle f_1, g_2 \rangle & \cdots \\
\langle f_2, g_0 \rangle & \langle f_2, g_1 \rangle & \langle f_2, g_2 \rangle & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Next, let us recall (29), i.e. \( \mathcal{s}^{M,N} = \mathcal{T} \bar{\mathcal{p}}^{[2]} \). Let us consider the inner product
\[
\langle \mathcal{s}^{M,N}, \mathcal{s}^{M,N} \rangle_{[2]} = \langle \mathcal{T} \bar{\mathcal{p}}^{[2]}, \mathcal{T} \bar{\mathcal{p}}^{[2]} \rangle_{[2]} = \mathcal{T} \langle \bar{\mathcal{p}}^{[2]}, \bar{\mathcal{p}}^{[2]} \rangle_{[2]} \mathcal{T}^\top = \mathcal{T}\mathcal{T}^\top,
\]
where \( \langle \bar{\mathcal{p}}^{[2]}, \bar{\mathcal{p}}^{[2]} \rangle_{[2]} = \mathbf{I} \) because we deal with orthonormal polynomials. On the other hand, from (13) and (27), one has
\[
\langle \mathcal{s}^{M,N}, \mathcal{s}^{M,N} \rangle_{[2]} = \langle (x - c)^2 \mathcal{s}^{M,N}, \mathcal{s}^{M,N} \rangle_{[2]} = \langle \mathcal{H} \mathcal{s}^{M,N}, \mathcal{s}^{M,N} \rangle_{[2]} = \mathcal{H} \langle \mathcal{s}^{M,N}, \mathcal{s}^{M,N} \rangle_{[2]} = \mathcal{H},
\]
where again \( \langle \mathcal{s}^{M,N}, \mathcal{s}^{M,N} \rangle_{[2]} = \mathbf{I} \) since we deal with orthonormal polynomials. Thus, we have proved the following

**Proposition 6** The five diagonal semi-infinite symmetric matrix \( \mathcal{H} \) defined in (28), has the following Cholesky factorization
\[
\mathcal{H} = \mathcal{T}\mathcal{T}^\top,
\]
where \( \mathcal{T} \) is the lower triangular, semi-infinite matrix defined in (30).

Finally, from (29) and (27), we have
\[
(x - c)^2 \mathcal{s}^{M,N} = \mathcal{H} \mathcal{s}^{M,N} = (x - c)^2 \mathcal{T} \bar{\mathcal{p}}^{[2]} = \mathcal{T}(x - c)^2 \bar{\mathcal{p}}^{[2]}.
\]
According to (31), we get
\[ T(x - c)^2 \hat{p}^{[2]} = T(J[2] - cI)^2 \hat{p}^{[2]} . \]

Next, from (37) we obtain
\[ T(J[2] - cI)^2 \hat{p}^{[2]} = TRR^\top \hat{p}^{[2]} = H s^{M,N} = TT^\top \hat{p}^{[2]} . \]

Therefore,
\[ TRR^\top \hat{p}^{[2]} = TT^\top \hat{p}^{[2]} \]
and, as a consequence,
\[ RR^\top = T^\top T . \]

Proposition 7. For any positive Borel measure \( d\mu \) supported on \( E \subseteq \mathbb{R} \), if \( J \) is the corresponding semi-infinite symmetric Jacobi matrix and \( c \not\in E \), then for the 2-iterated perturbed measure \((x - c)^2 d\mu\) such that if \( J[2] \) is the corresponding semi-infinite symmetric Jacobi matrix we get
\[ (J[2] - cI)^2 = RR^\top = T^\top T . \]

This is the symmetric version of Theorem 5.3 in [6], where the authors use other kind of factorization based on monic orthogonal polynomials.

7 An example with Laguerre polynomials

In [14] and Section 5 the coefficients of (26) for the monic Laguerre Sobolev-type orthogonal polynomials have been deduced. In the sequel we illustrate the matrix approach presented in the previous section for the Laguerre case with a particular example. First, let us denote by \( \{\ell_n^\alpha(x)\}_{n \geq 0}, \{\ell_n^{\alpha,[2]}(x)\}_{n \geq 0}, \{s^M,N_n(x)\}_{n \geq 0} \) the sequences of orthonormal polynomials with respect to the inner products (1), (8) and (13), respectively, when \( d\mu(x) = x^\alpha e^{-x} dx, \alpha > -1 \), is the classical Laguerre weight function supported on \((0, +\infty)\).

In order to obtain compact expressions of the matrices, in this section we will particularize all of those presented in the previous section for the choice of the parameters \( \alpha = 0, c = -1, M = 1, \) and \( N = 1 \). In these conditions, using any symbolic algebra package as, for example, Wolfram Mathematica©, the explicit expressions of the sequences of orthogonal polynomials appearing in our study.

From section 5 we know

\[
H = \begin{bmatrix}
\frac{5}{2} & \frac{11}{2 \sqrt{2}} & \frac{1}{2} \sqrt{\frac{80}{7}} & \frac{1}{2} \sqrt{\frac{35705}{178}} & 4 \sqrt{\frac{26690173}{3177745}} \\
\frac{11}{2 \sqrt{2}} & \frac{129}{\sqrt{89}} & 5321 & 1503493 & 72140663342 \\
\frac{1}{2} \sqrt{\frac{80}{7}} & \frac{129}{\sqrt{89}} & \frac{1}{2} \sqrt{\frac{35705}{178}} & 4151289273 & 2375425397 \\
\frac{1}{2} \sqrt{\frac{35705}{178}} & \frac{1503493}{178 \sqrt{11410}} & 4 \sqrt{\frac{26690173}{3177745}} & \frac{72140663342}{35705} & 108116532681297 \\
4 \sqrt{\frac{26690173}{3177745}} & 72140663342 & 2375425397 & 952972626965 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]
On the other hand, from (29) we obtain
\[
T = \begin{bmatrix}
\frac{1}{2} \\
\sqrt{\frac{2}{5}} \\
\frac{1}{2} \\
\frac{1}{2} \sqrt{\frac{20}{5}} \\
\frac{1601}{2 \sqrt{20705}} \\
\frac{2911082 \sqrt{2}}{\sqrt{359628247685}} \\
\frac{1841622937}{63447785} \\
\frac{\sqrt{529026997485485}}{4046188065} \\
\vdots \\
\end{bmatrix}.
\]
(38)

Notice that if we multiply \(T\) by its transpose then one recovers \(H\) according to the statement of Proposition 6.

The tridiagonal symmetric Jacobi matrix associated with the standard orthonormal family \(\{\ell_n^{\alpha,2}(x)\}_{n \geq 0}\) reads
\[
J_{[2]} = \begin{bmatrix}
1 & \sqrt{\frac{69}{5}} \\
\sqrt{\frac{69}{5}} & \frac{1501}{43} \\
\frac{2 \sqrt{8885}}{69} & \frac{790003}{122613} \\
\frac{2 \sqrt{8885}}{69} & \frac{4 \sqrt{3975797}}{1777} \\
\frac{3 \sqrt{4975797}}{109136499} & \frac{4 \sqrt{177535457}}{72113} \\
\frac{4 \sqrt{177535457}}{72113} & \frac{302365554333}{3195035811691} \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\end{bmatrix},
\]
and from this expression it is straightforward to check Proposition 2. Next, from the symmetric Jacobi matrix
\[
J = \begin{bmatrix}
1 & 1 & 2 & 3 & 4 & \cdots \\
1 & 3 & 2 & 5 & 3 & \cdots \\
2 & 5 & 3 & 7 & 4 & \cdots \\
3 & 7 & 4 & 9 & \ddots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix},
\]
(39)
associated with \(\{\ell_n^{\alpha}(x)\}_{n \geq 0}\), we can implement the Cholesky factorization of \(J - cI = LL^T\) in such a way the lower bidiagonal matrix is
\[
L = \begin{bmatrix}
\sqrt{\frac{2}{7}} & \sqrt{\frac{7}{7}} & \sqrt{\frac{44}{7}} & \sqrt{\frac{209}{31}} & \sqrt{\frac{1546}{209}} & \cdots \\
\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{7}} & \frac{4}{\sqrt{31}} & \frac{5}{\sqrt{209}} & \frac{1546}{209} & \cdots \\
\frac{2}{\sqrt{7}} & \frac{4}{\sqrt{31}} & \frac{209}{31} & \frac{209}{31} & \frac{209}{31} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}.
\]
Following (33), we commute the order of \( L \) and its transpose to obtain \( L^T L = J_{[1]} - cI \), where

\[
J_{[1]} = \begin{bmatrix}
\frac{3}{7} & \frac{2\sqrt{7}}{21} & \frac{\sqrt{7}}{14} \\
\frac{2\sqrt{7}}{21} & \frac{4\sqrt{7}}{21} & \frac{3\sqrt{13}}{28} \\
\frac{\sqrt{7}}{14} & \frac{3\sqrt{13}}{28} & \frac{3\sqrt{13}}{34}
\end{bmatrix}
\]

The computation of a new Cholesky factorization of \( J_{[1]} - cI \) yields \( L_1 L_1^T \), where

\[
L_1 = \begin{bmatrix}
\sqrt{\frac{4}{7}} & \sqrt{\frac{5}{14}} & \sqrt{\frac{2\sqrt{7}}{21}} & \sqrt{\frac{3\sqrt{13}}{28}} & \frac{\sqrt{7}}{14} & \frac{3\sqrt{13}}{34} \\
\sqrt{\frac{5}{14}} & \sqrt{\frac{2\sqrt{7}}{21}} & \sqrt{\frac{3\sqrt{13}}{28}} & \frac{\sqrt{7}}{14} & \frac{3\sqrt{13}}{34} & \frac{\sqrt{7}}{14} \\
\sqrt{\frac{2\sqrt{7}}{21}} & \sqrt{\frac{3\sqrt{13}}{28}} & \frac{\sqrt{7}}{14} & \frac{3\sqrt{13}}{34} & \frac{\sqrt{7}}{14} & \frac{3\sqrt{13}}{34} \\
\end{bmatrix}
\]

Commuting the order of the matrices in the decomposition then we finally deduce the expression (35), i.e. \( L_1^T L_1 = J_{[2]} - cI \). With these last matrices in mind we find \( R \) and \( Q \) at Proposition (3) Thus

\[
Q = L L_1^{-T} = \begin{bmatrix}
\sqrt{\frac{5}{9}} & \sqrt{\frac{6}{9}} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{6}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} \\
\end{bmatrix}
\]

and

\[
R = (L L_1)^T = \begin{bmatrix}
\sqrt{\frac{5}{9}} & \sqrt{\frac{6}{9}} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\sqrt{\frac{6}{9}} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{\sqrt{5}}{9} & \frac{2\sqrt{5}}{9} & \frac{\sqrt{5}}{9} \\
\end{bmatrix}
\]

Observe that \( Q \) is a matrix whose rows are orthogonal vectors, and multiplying (40) above by
its transpose (in this order) we get

\[
QQ^T \approx \begin{bmatrix}
0.99657 & 0.0068687 & -0.27601 & 0.019461 & -0.029907 & \cdots \\
0.0068687 & 0.98626 & 0.55201 & -0.038923 & 0.059815 & \cdots \\
-0.27601 & 0.55201 & 0.95793 & -0.73549 & -0.10468 & \cdots \\
0.019461 & -0.038923 & -0.73549 & 0.88972 & 0.16948 & \cdots \\
-0.029907 & 0.059815 & -0.10468 & 0.16948 & 0.73956 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \approx I.
\]

Notice that we implement our algorithm with finite matrices. Notwithstanding the foregoing, multiplying the transpose of (40) by (40) we indeed have \(Q^TQ = I\).

Employing these matrices above it is easy to test numerically expressions

\[
H = T(J_2 - cI)^2 T^{-1},
\]

\[
J - cI = QR, \text{ and } J_2 - cI = RQ
\]

according to the statements of Propositions 2, 3, 4 respectively. It is also possible to check that using the numerical expression (38), and alternatively the expression (41), we recover

\[
(J_2 - cI)^2 =
\]

\[
\begin{bmatrix}
13 & \frac{118}{\sqrt{69}} & 2\sqrt{\frac{177}{345}} & \frac{946913}{122613} & \frac{2\sqrt{1081695}}{49871} & \frac{36}{128144801} \\
\frac{118}{\sqrt{69}} & \frac{227476}{69\sqrt{8885}} & \frac{84432374}{1777} & \frac{1636442385}{128144801} & \frac{628405520264}{72113\sqrt{7450856157}} & \frac{5757253044081}{302365554333} \\
\frac{2\sqrt{177}}{345} & \frac{946913}{122613} & \frac{84432374}{1777} & \frac{1636442385}{128144801} & \frac{628405520264}{72113\sqrt{7450856157}} & \frac{5757253044081}{302365554333} \\
\frac{2\sqrt{1081695}}{49871} & \frac{1636442385}{128144801} & \frac{628405520264}{72113\sqrt{7450856157}} & \frac{5757253044081}{302365554333} & \cdots & \cdots \\
\frac{36}{128144801} & \frac{628405520264}{72113\sqrt{7450856157}} & \frac{5757253044081}{302365554333} & \cdots & \cdots & \cdots
\end{bmatrix},
\]

according to Proposition 7.

Finally, Proposition 5 can be numerically tested from (39), (41) and (42).

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References

[1] M. Alfaro, F. Marcellán, M. L. Rezola, A. Ronveaux, On orthogonal polynomials of Sobolev type: algebraic properties and zeros. SIAM J. Math. Anal. 23 (3) (1992), 737–757.
REFERENCES

[2] M. I. Bueno, F. Marcellán, *Darboux transformation and perturbation of linear functionals*, Linear Algebra Appl. **384** (2004), 215–242.

[3] M. I. Bueno, F. Marcellán, *Polynomial perturbations of bilinear functionals and Hessenberg matrices*, Linear Algebra Appl. **414** (2006), 64–83.

[4] M. D. Buhmann, A. Iserles, *On orthogonal polynomials transformed by the QR algorithm*, J. Comput. Appl. Math. **43** (1-2) (1992), 117–134.

[5] T. S. Chihara, *An Introduction to Orthogonal Polynomials*. Mathematics and its Applications Series, Gordon and Breach, New York, 1978.

[6] M. Derevyagin, F. Marcellán, *A note on the Geronimus transformation and Sobolev orthogonal polynomials*, Numer. Algorithms **67** (2) (2014), 271–287.

[7] M. Derevyagin, J. C. García-Ardila, F. Marcellán, *Multiple Geronimus transformations*, Linear Algebra Appl. **454** (2014), 158–183.

[8] A. J. Durán, *A generalization of Favard’s theorem for polynomials satisfying a recurrence relation*. J. Approx. Theory **74** (1) (1993), 83–109.

[9] A. J. Durán, W. Van Assche, *Orthogonal matrix polynomials and higher-order recurrence relations*. Linear Algebra Appl. **219** (1995), 261–280.

[10] W. D. Evans, L. L. Littlejohn, F. Marcellán, C. Markett, A. Ronveaux, *On recurrence relations for Sobolev orthogonal polynomials*. SIAM J. Math. Anal. **26** (2) (1995), 446–467.

[11] J. C. García-Ardila, F. Marcellán, P. H. Villamal-Hernández, *Associated orthogonal polynomials of the first kind and Darboux transformations*. J. Math. Anal. Appl. **508** (2022) 125883, 26 pp.

[12] W. Gautschi, *The interplay between classical analysis and (Numerical) Linear Algebra- A tribute to Gene Golub*, ETNA. **13** (2002), 119–147.

[13] W. Gautschi, *Orthogonal polynomials: computation and approximation*. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2004.

[14] E. J. Huertas, F. Marcellán, M. F. Pérez-Valero, Y. Quintana, *Asymptotics for Laguerre-Sobolev type orthogonal polynomials modified within their oscillatory regime*, Appl. Math. Comput. **236** (2014), 260–272.

[15] E. J. Huertas, *Analytic properties of Krall-type and Sobolev-type orthogonal polynomials* (Doctoral Dissertation), Universidad Carlos III de Madrid, 2012.

[16] J. Kautský, G. H. Golub, *On the calculation of Jacobi matrices*. Linear Algebra Appl. **52/53** (1983), 439–455.

[17] R. Koekoek, *Generalizations of Laguerre polynomials*. J. Math. Anal. Appl. **153** (2) (1990), 576–590.

[18] J. Koekoek, R. Koekoek, H. Bavinck, *On differential equations for Sobolev-type Laguerre polynomials*. Trans. Amer. Math. Soc. **350** (1) (1998), 347–393.
[19] R. Koekoek, H. G. Meijer, *A generalization of Laguerre polynomials*. SIAM J. Math. Anal. **24** (3) (1993), 768–782.

[20] G. López, F. Marcellán, W. Van Assche, *Relative asymptotics for polynomials orthogonal with respect to a discrete Sobolev inner product*. Constr. Approx. **11** (1) (1995), 107–137.

[21] F. Marcellán, J. J. Moreno Balcázar, *Asymptotics and zeros of Sobolev orthogonal polynomials on unbounded supports*. Acta Appl. Math. **94** (2) (2006), 163–192.

[22] F. Marcellán, W. Van Assche, *Relative asymptotics for orthogonal polynomials with a Sobolev inner product*. J. Approx. Theory **72** (2) (1993), 193–209.

[23] F. Marcellán, Y. Xu, *On Sobolev orthogonal polynomials*. Expo. Math. **33** (3) (2015), 308–352.

[24] F. Marcellán, R. Xh. Zejnullahu, B. Xh. Fejzullahu, E. J. Huertas *On orthogonal polynomials with respect to certain discrete Sobolev inner product*, Pacific J. Math. **257** (1) (2012), 167–188.

[25] C. Markett, *On the differential equation for the Laguerre-Sobolev polynomials*. J. Approx. Theory **247** (2019), 48–67.

[26] C. Markett, *Symmetric differential operators for Sobolev orthogonal polynomials of Laguerre- and Jacobi-type*. Integral Transforms Spec. Funct. **32** (5-8) (2021), 568–587.

[27] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. Series, vol **23**, Amer. Math. Soc. Providence, RI, 1975.

[28] G. J. Yoon, *Darboux transforms and orthogonal polynomials*, Bull. Korean Math. Soc. **39** (2002), 359–376.

[29] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*. J. Comput. Appl. Math. **85** (1997), 67–83.