Critical exponents and the pseudo-\(\epsilon\) expansion

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(Dated: September 8, 2017)

Abstract

We present the pseudo-\(\epsilon\) expansions (\(\tau\)-series) for the critical exponents of a \(\lambda\phi^4\) three-dimensional \(O(n)\)-symmetric model obtained on the basis of six-loop renormalization-group expansions. Concrete numerical results are presented for physically interesting cases \(n = 1\), \(n = 2\), \(n = 3\) and \(n = 0\), as well as for \(4 \leq n \leq 32\) in order to clarify the general properties of the obtained series. The pseudo-\(\epsilon\)-expansions for the exponents \(\gamma\) \(\alpha\) have small and rapidly decreasing coefficients. So, even the direct summation of the \(\tau\)-series leads to fair estimates for critical exponents, while addressing Padé approximants enables one to get high-precision numerical results. In contrast, the coefficients of the pseudo-\(\epsilon\) expansion of the scaling correction exponent \(\omega\) do not exhibit any tendency to decrease at physical values of \(n\). But the corresponding series are sign-alternating, and to obtain reliable numerical estimates, it also suffices to use simple Padé approximants in this case. The pseudo-\(\epsilon\) expansion technique can therefore be regarded as a specific resummation method converting divergent renormalization-group series into expansions that are computationally convenient.

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I. INTRODUCTION

Although no exact solution of the problem of the phase transition in the three-dimensional $n$-vector model has yet been obtained, the critical exponents of systems described by this model (Heisenberg and uniaxial ferromagnets, Bose superfluids, etc.) have been calculated with a rather high accuracy. One of the most efficient methods, which allows obtaining precise quantitative results, is the field-theory renormalization group (RG) method. The calculations of five-, six-, and seven-loop RG expansions showed the way for calculating the critical exponents, the ratios of critical amplitudes, and other universal characteristics of the critical behavior of three-dimensional systems with an absolute error less than or equal to 0.002 to 0.003.

The field-theory RG technique is based on the renormalized perturbation theory, i.e., on a regular mathematical procedure that allows obtaining the critical exponents and other observables as power series in the effective dimensionless coupling constant (renormalized charge) $g$. The asymptotic value of $g$ is a nontrivial zero of the Gell-Mann-Low function $\beta(g)$, which can also be calculated by the perturbation theory methods. But the perturbation series, as is known, diverge, and the expansion parameter is not small for models with spaces of physical dimension ($D = 3, D = 2$), i.e., $g \sim 1 \div 2$. To obtain quantitative results, we must therefore use different methods for summing the divergent series (see, e.g., [10–14]). When using an alternative approach based on $(4 - \epsilon)$-dimensional models, we must use resummation procedures because the $\epsilon$ expansions have the same drawbacks in the physical limit $\epsilon \to 1$ as the expansions in the charge.

At the same time, there is a method for transforming the initial RG expansions into series with small coefficients rapidly decreasing in absolute value. We mean the pseudo-$\epsilon$ expansion method proposed by Nickel (see reference [19] in [6]). The main idea of the method is to replace the coefficient of the linear term in the expansion of the $\beta$ function with a fictitious small parameter $\tau$, to calculate the Wilson fixed-point coordinate as a power series in $\tau$, and to construct the $\tau$-expansions of the critical exponents and other universal quantities. The structure of the series obtained with this technique turns
out to be very convenient computationally, and reliable numerical results can hence be obtained by directly summing these series or by using simple Padé approximants \[15–21\]. The pseudo-\(\epsilon\) expansion method proved to be highly efficient even in the case of two-dimensional systems, where the known RG expansions are shorter and diverge faster than in the case of three-dimensional models \[22–25\].

Our goal here is to present pseudo-\(\epsilon\) expansions of critical exponents of a three-dimensional \(n\)-vector model, to analyze their structure for different values of \(n\), and to discuss the numerical results obtained in the framework of this approach. We mainly focus on the cases \(n = 1\), \(n = 2\), \(n = 3\), and \(n = 0\), which are physically the most interesting. We obtain numerical estimates by using Padé approximants and by summing the obtained pseudo-\(\epsilon\) expansions directly.

II. PSEUDO-\(\epsilon\) EXPANSIONS OF CRITICAL EXPO NENTS FOR ARBITRARY \(n\)

The critical thermodynamics of three-dimensional systems with an \(O(n)\)-symmetric order parameter are described by the Euclidean field theory with a \(\lambda \phi^4\) type interaction whose Hamiltonian has the form

\[
H = \int d^3x \left[ \frac{1}{2}(m_0^2\phi^2 + (\nabla \phi)^2) + \frac{\lambda}{24}(\phi^2)^2 \right],
\]

(1)

where the squared bare mass \(m_0\) is proportional to \(T - T_c^{(0)}\) and \(T_c^{(0)}\) is the temperature of the phase transition without fluctuations. The RG expansions of the \(\beta\) function and the critical exponents, obtained in the framework of the massive theory with a propagator, a vertex function, and a three-leg function \(\Gamma_{1,2}^R\) normalized at zero external momenta,

\[
\begin{align*}
G_R^{-1}(0,m,g_4) &= m^2, \\
\frac{\partial G_R^{-1}(0,m,g_4)}{\partial p^2} \bigg|_{p^2=0} &= 1, \\
\Gamma_R(0,0,0,m,g) &= m^2 g_4, \\
\Gamma_{1,2}^R(0,0,m,g_4) &= 1,
\end{align*}
\]

(2)

are currently known in the six-loop approximation \[2, 3, 5\]. To obtain the sought pseudo-\(\epsilon\) expansions (\(\tau\)-series), it suffices to substitute the \(\tau\)-series for the Wilson fixed point
coordinate in the RG expansions of the critical exponents and to reexpand them in power series in $\tau$. Here, we present calculation results for the critical exponents of the susceptibility and heat capacity because these exponents are measured in experiments most frequently and most precisely. The pseudo-$\epsilon$ expansions of other critical exponents can be obtained based on the $\tau$-series for the parameters $\gamma$ and $\alpha$ using the well-known scaling relations. Hence, we have

$$
\gamma = 1 + \frac{(n + 2)\tau}{2(n + 8)} + \frac{\tau^2(n + 2)}{108(n + 8)^3} \left(27n^2 + 490n + 1088\right) + \frac{\tau^3(n + 2)}{8(n + 8)^5} \left(n^4 + 37.936996n^3 + 428.99211n^2 + 1073.0522n - 199.53302\right) + \frac{\tau^4(n + 2)}{16(n + 8)^7} \left(n^6 + 58.533991n^5 + 1336.0645n^4 + 13115.226n^3 + 46827.023n^2 + 60693.508n + 45180.873\right) + \frac{\tau^5(n + 2)}{32(n + 8)^9} \left(n^8 + 79.88230n^7 + 2753.499n^6 + 49024.95n^5 + 433177.7n^4 + 1573332n^3 + 961413.1n^2 - 7398997n - 1.49542410^7\right) + \frac{\tau^6(n + 2)}{64(n + 8)^{11}} \left(n^{10} + 102.140n^9 + 4723.96n^8 + 124384n^7 + 1.9342410^6n^6 + 1.7110410^7n^5 + 8.1792510^7n^4 + 2.2773310^8n^3 + 5.1932810^8n^2 + 1.3212010^9n + 1.9645810^9\right),
$$

\[(3)\]

$$
\alpha = \frac{1}{2} - \frac{3\tau(n + 2)}{4(n + 8)} - \frac{\tau^2(n + 2)}{72(n + 8)^3} \left(27n^2 + 506n + 1216\right) - \frac{3\tau^3(n + 2)}{16(n + 8)^5} \left(n^4 + 38.628325n^3 + 456.42625n^2 + 1330.8024n + 460.64396\right) - \frac{3\tau^4(n + 2)}{32(n + 8)^7} \left(n^6 + 59.190931n^5 + 1377.3824n^4 + 14095.696n^3 + 56076.311n^2 + 96550.551n + 94371.957\right) - \frac{3\tau^5(n + 2)}{64(n + 8)^9} \left(n^8 + 80.43442n^7 + 2802.339n^6 + 50883.40n^5 + 468728.9n^4 + 1911709n^3 + 2598713n^2 - 3400834n - 1.08722710^7\right) - \frac{3\tau^6(n + 2)}{128(n + 8)^{11}} \left(n^{10} + 102.515n^9 + 4770.89n^8 + 126952n^7 + 2.0113410^6n^6 + 1.8413910^7n^5 + 9.4077210^7n^4 + 2.9311510^8n^3\right)
$$
In the analysis of the experimental data, it is also important to know the critical exponent of the scaling correction $\omega$, whose pseudo-$\epsilon$ expansion has the form

$$
\omega = \tau - \frac{4\tau^2}{27(n+8)^2} \left(41n + 190\right) + \frac{\tau^3}{(n+8)^4} \left(2.6978855n^3 + 57.675086n^2 + 594.43112n + 1609.6102\right)
- \frac{\tau^4}{(n+8)^6} \left(-0.46693767n^5 + 34.440983n^4 + 1119.6990n^3 + 11012.281n^2 + 54280.495n + 103646.63\right)
+ \frac{\tau^5}{(n+8)^8} \left(0.2049447n^7 + 20.18796n^6 + 680.4944n^5 + 17055.42n^4 + 254491.6n^3 + 1892609n^2 + 6986639n + 1.019704 \times 10^7\right)
- \frac{\tau^6}{(n+8)^{10}} \left(-0.117121n^9 - 12.1743n^8 + 78.8694n^7 + 21145.0n^6 + 570921n^5 + 7.69420 \times 10^6n^4 + 6.34720 \times 10^7n^3 + 3.19979 \times 10^8n^2 + 9.00151 \times 10^8n + 1.07362 \times 10^9\right).
$$

These $\tau$-series are used below to obtain the numerical values of critical exponents for some special values of $n$.

### III. STRUCTURE OF THE $\tau$-SERIES AND NUMERICAL RESULTS

We consider pseudo-$\epsilon$ expansions (3)-(5) for physically interesting values of $n$. The most important cases are $n = 1$, $n = 2$ and $n = 3$, which correspond to phase transitions in simple liquids and binary mixtures, in easy-axis, easy-plane, and Heisenberg ferromagnets, in superconductors with $s$-pairing, in Bose superfluids, and in many other systems. The corresponding $\tau$-series have the forms

$$
\gamma = 1 + \frac{\tau}{6} + 0.061156836\tau^2 + 0.008519079\tau^3 + 0.00655499\tau^4
- 0.00467841\tau^5 + 0.0061748\tau^6,
$$

$$
\alpha = \frac{1}{2} - \frac{\tau}{4} - 0.099965706\tau^2 - 0.02179070\tau^3 - 0.01543751\tau^4
$$
\[
\omega = \tau - 0.42249657\tau^2 + 0.34513249\tau^3 - 0.32006015\tau^4
+ 0.4494775\tau^5 - 0.678421\tau^6
\] (8)

for \( n = 1, \)
\[
\gamma = 1 + \frac{\tau}{5} + 0.080592593\tau^2 + 0.01991018\tau^3 + 0.01205280\tau^4
- 0.00057920\tau^5 + 0.0065646\tau^6,
\] (9)
\[
\alpha = \frac{1}{2} - \frac{3\tau}{10} - 0.129777778\tau^2 - 0.03954735\tau^3 - 0.02432025\tau^4
- 0.0032498\tau^5 - 0.012109\tau^6,
\] (10)
\[
\omega = \tau - 0.40296296\tau^2 + 0.30507558\tau^3 - 0.26575046\tau^4
+ 0.3407266\tau^5 - 0.480435\tau^6
\] (11)

for \( n = 2, \) and
\[
\gamma = 1 + \frac{5}{22}\tau + 0.0974274425\tau^2 + 0.03099116\tau^3 + 0.01805659\tau^4
+ 0.00418601\tau^5 + 0.0080091\tau^6,
\] (12)
\[
\alpha = \frac{1}{2} - \frac{15\tau}{44} - 0.155323900\tau^2 - 0.05637685\tau^3 - 0.03357917\tau^4
- 0.0105846\tau^5 - 0.014517\tau^6,
\] (13)
\[
\omega = \tau - 0.38322620\tau^2 + 0.27216872\tau^3 - 0.22494669\tau^4
+ 0.2641538\tau^5 - 0.352583\tau^6
\] (14)

for \( n = 3. \) It is physically interesting to consider the limit \( n \to 0, \) where model (1) describes the critical behavior of polymers (self-avoiding walk). In the case \( n = 0, \) the pseudo-\( \epsilon \) expansions of the critical exponents become
\[
\gamma = 1 + \frac{\tau}{8} + 0.039351852\tau^2 - 0.00152232\tau^3 + 0.00269299\tau^4
- 0.00696361\tau^5 + 0.0071471\tau^6,
\] (15)
\[ \alpha = \frac{1}{2} - \frac{3}{16}\tau - 0.065972222\tau^2 - 0.00527165\tau^3 - 0.0084375\tau^4 + 0.0075942\tau^5 - 0.011897\tau^6, \]  
\[ \omega = \tau - 0.43981481\tau^2 + 0.39297124\tau^3 - 0.39538053\tau^4 + 0.6077908\tau^5 - 0.999887\tau^6. \]

We note that the pseudo-\(\epsilon\) expansion method was previously used to determine numerical values of critical exponents in [6, 7], but no \(\tau\)-series (3)-(17) were given in those papers. The resummation procedure based on the BorelLeroy transformation and the conformal map technique were used there. Here, we determine the values of critical exponents without using the Borel summation, which is a canonical tool in the theory of critical phenomena.

It follows from formulas (6)-(17) that the \(\tau\)-series for the exponents \(\gamma\) and \(\alpha\) have small coefficients rapidly decreasing in absolute value. It is therefore natural to try to find the numerical values of these exponents using the simplest way, namely, using Padé approximants and directly summing pseudo-\(\epsilon\) expansions. It is necessary to keep in mind that the obtained series, despite their favorable structure, diverge in this case; the divergence, in particular, is manifested by the behavior of the higher-order terms, which tend to increase. On the other hand, because the initial RG expansions are asymptotic, there are reasons to believe that the pseudo-\(\epsilon\) expansions have the same property because their coefficients can be obtained from the coefficients of the RG series by finitely many algebraic operations with the coefficients of the same and lower orders [21, 26]. This implies that in the case of direct summation of some pseudo-\(\epsilon\) expansion, the most precise estimate can be obtained by calculating the partial sum of the series bounded by the term that is least in absolute value.

The results of calculating the critical exponents of the susceptibility and heat capacity using the Padé approximants [L/M] and the efficiency of this technique applied to pseudo-\(\epsilon\) expansions are illustrated in Tables 14, where the Padé triangles are given for \(\gamma\) and \(\alpha\) at \(n = 1\) (the Ising model) and \(n = 3\) (the Heisenberg model). The bottom rows of
these tables (RoC) show the character and the rate of convergence of the Padé estimates to the limit values. The k-th order Padé estimate is here the number given by the corresponding diagonal approximant or by the half-sum of the values given by approximants of the form \([M/M-1]\) and \([M-1/M]\) in the case without any diagonal approximant.

It can be seen that summing pseudo-\(\epsilon\) expansions of the critical exponents by the Padé method generates an iteration procedure that converges to the asymptotic values very rapidly. The following question arises: How close are these values to the most precise numerical estimates obtained by other methods? This question is answered in Tables 5 and 6, where we present the values of the exponents \(\gamma\) and \(\alpha\) for different \(n\) obtained by the method described above (\(\tau\), Padé) and their analogues obtained by resumming six-loop RG expansions by the PadéBorelLeroy (PBL) method and by the conform-Borel (CB) method, by processing the strong coupling (SC) expansions, by processing the five-loop \(\epsilon\) expansions (\(\epsilon\)-exp), and by lattice calculations (LC). These tables also contain the estimates of \(\gamma\) and \(\alpha\) obtained by optimally truncated direct summation of the pseudo-\(\epsilon\) expansions, i.e., of expansions truncated at the term that is least in absolute value (\(\tau\), OTDS). To clarify the general properties of pseudo-\(\epsilon\) expansions, we calculated the critical exponents not only for physical values of \(n\) but also for \(n > 3\). The corresponding results are also given in Tables 5 and 6.

It can be seen from these tables that the pseudo-\(\epsilon\) expansion method together with the Padé approximant technique gives values of critical exponents that are very close to their counterparts obtained by alternative field theory methods and by lattice calculations. Moreover, even the direct summation of pseudo-\(\epsilon\) expansions of \(\gamma\) and \(\alpha\) leads to numerical estimates that differ from the most reliable values by their spread order, i.e., by the expected error of theoretical determination of the exponents. The proximity between the numbers in the first two rows in Tables 5 and 6 and the precise values obtained by different methods confirms that the pseudo-\(\epsilon\) expansion technique is numerically highly efficient in such problems.

We further consider the critical exponent of the scaling correction \(\omega\). In this case, the computational situation is not so favorable as in the case of the exponents \(\gamma\) and \(\alpha\).
The coefficients of pseudo-$\epsilon$ expansions (8), (11), (14), and (17) are not small and do not exhibit any pronounced tendency to decrease. But the $\tau$-series are sign-alternating and have a regular structure, i.e., their coefficients are first monotonically decreasing (in absolute value) and then monotonically increasing as their number grows. This allows concluding that using Padé approximants can also give reliable numerical results in this case. Tables 7 and 8, where the Padé triangles are given for pseudo-$\epsilon$ expansions (8) and (14) and the summary Table 9 confirm this conclusion. We can see that the Padé estimates of the exponent $\omega$ agree well with the result of field theory and lattice computations for all values of $n$ up to $n = 32$. At the same time, the optimally truncated direct summation gives acceptable results in this case only for sufficiently large $n$.

**IV. SCALING AND NUMERICAL EFFICIENCY**

Resumming the pseudo-$\epsilon$ expansions by the Padé method, we can calculate the critical exponents of three-dimensional systems with an accuracy comparable to the accuracy of the most efficient lattice and field theory iteration schemes. It is interesting to verify whether the results thus obtained are intrinsically consistent. For this, we must use expressions (3) and (4) to determine the pseudo-$\epsilon$ expansions of the other critical exponents, sum the corresponding $\tau$-series in the Padé sense, and verify the extent to which the obtained numbers satisfy the scaling relations. In this case, it is necessary to keep in mind that the relation $\alpha = 2 - D\nu$ cannot be a source of the required information, because it is exactly satisfied in our case for trivial reasons. The formulas

$$\nu = \frac{\gamma}{2 - \eta}, \quad \nu = \frac{2\beta}{1 + \eta}, \quad \alpha + 2\beta + \gamma = 2,$$

are efficient in this case, and we use them as test formulas. Table 10 shows the results of substituting the critical exponent values derived from the corresponding $\tau$-series by Padé resummations in these formulas. We see that the pseudo-$\epsilon$ expansion technique reproduces the scaling relations with an accuracy at the level of 0.005 and higher. This confirms that the approach discussed here allows obtaining high-precision numerical estimates even by the simplest resummation methods.
What are the reasons for this very high numerical efficiency of the pseudo-\(\epsilon\) expansion method? We believe that the main distinguishing characteristics of this method ensuring its computational power are the following:

1. The coefficients of pseudo-\(\epsilon\) expansions are significantly less (in absolute value) and begin to increase much later than the coefficients of the corresponding RG series. This can be explained by the structure of the formulas relating the former to the latter. This structure ensures multiple subtractions (destructive interference) of the coefficients of the initial RG expansions when the coefficients of the \(\tau\)-series are calculated \[21\].

2. The expressions for the coefficients of the \(k\)th-order pseudo-\(\epsilon\) expansions contain not only the \(k\)-th order coefficients of the RG series but also the RG coefficients of lower orders down to the first order \[26\]. This means that the operations with pseudo-\(\epsilon\) expansions use more information contained in the RG series than in the case where the RG expansions themselves are processed.

3. In the pseudo-\(\epsilon\) expansion method, the physical value of the expansion parameter \(\tau\) is equal to unity. The field theory RG technique is based on the expansion in the Wilson fixed-point coordinate \(g\). We have \(g \approx 1.4\) for three-dimensional systems with \(0 \leq n \leq 3\) and \(g \approx 1.8\) for the two-dimensional Ising model, i. e., \(g\) is significantly greater than unity. This difference is very significant, especially in the case of high perturbation orders.

4. The pseudo-\(\epsilon\) expansion technique also has some advantages over the canonical WilsonFisher \(\epsilon\) expansion method. The point is that in the construction of the \(\epsilon\) expansions, the integrals corresponding to the Feynman diagrams are calculated in the \((4 - \epsilon)\)-dimensional space, while the coefficients of the pseudo-\(\epsilon\) expansions are expressed in terms of integrals calculated for models with spaces of physical dimensions.

The results obtained above and the arguments listed above allow concluding that the pseudo-\(\epsilon\) expansion method can be regarded as a distinctive resummation method that converts divergent RG series into expansions very convenient for obtaining reliable numerical results.

This research is supported by St. Petersburg State University (Grant No. 11.38.636.2013), the Russian Foundation for Basic Research (Grant No. 15-02-04687),
and the Dynasty Foundation.

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TABLE I: Padé triangle for the pseudo-\( \epsilon \) expansion of the critical exponent \( \gamma \) at \( n = 1 \). The approximant [3/1] has a pole close to 1, which makes the corresponding value of \( \gamma \) unreliable. The bottom row (RoC) illustrates the character of convergence of the Padé estimates to the asymptotic value. Hereafter, the \( k \)-th order Padé estimate is the number given by the corresponding diagonal approximant or the half-sum of the quantities given by approximants of the form [M/M-1] and [M-1/M] in the case when diagonal approximant is absent.

| \( M \) \( \setminus \) L | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|----------------|-----|-----|-----|-----|-----|-----|-----|
| 0              | 1   | 1.1667 | 1.2278 | 1.2363 | 1.2429 | 1.2382 | 1.2444 |
| 1              | 1.2000 | 1.2633 | 1.2377 | 1.2648 | 1.2402 | 1.2409 |
| 2              | 1.2501 | 1.2408 | 1.2434 | 1.2412 | 1.2411 |
| 3              | 1.2389 | 1.2430 | 1.2419 | 1.2411 |
| 4              | 1.2455 | 1.2415 | 1.2409 |
| 5              | 1.2358 | 1.2410 |
| 6              | 1.2475 |
| RoC            | 1   | 1.1833 | 1.2633 | 1.2393 | 1.2434 | 1.2416 | 1.2411 |

TABLE II: Padé triangle for the pseudo-\( \epsilon \) expansion of the critical exponent \( \gamma \) at \( n = 3 \). The approximant [2/4] has a pole close to 1, which makes the corresponding value of \( \gamma \) unreliable.

| \( M \) \( \setminus \) L | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|----------------|-----|-----|-----|-----|-----|-----|-----|
| 0              | 1   | 1.2273 | 1.3247 | 1.3557 | 1.3737 | 1.3779 | 1.3859 |
| 1              | 1.2941 | 1.3978 | 1.3701 | 1.3990 | 1.3792 | 1.3692 |
| 2              | 1.3756 | 1.3728 | 1.3809 | 1.3829 | 1.3866 |
| 3              | 1.3727 | 1.3751 | 1.3830 | 1.3796 |
| 4              | 1.3858 | 1.3820 | 1.3999 |
| 5              | 1.3805 | 1.3842 |
| 6              | 1.3925 |
| RoC            | 1   | 1.2607 | 1.3978 | 1.3715 | 1.3809 | 1.3829 | 1.3796 |
TABLE III: Padé triangle for the pseudo-ε expansion of the critical exponent α at n = 1.

| M \ L | 0          | 1        | 2       | 3       | 4       | 5       | 6       |
|-------|------------|----------|---------|---------|---------|---------|---------|
| 0     | 0.5        | 0.25     | 0.1500  | 0.1282  | 0.1128  | 0.1162  | 0.1051  |
| 1     | 0.3333     | 0.0834   | 0.1222  | 0.0753  | 0.1156  | 0.1136  |
| 2     | 0.2564     | 0.1254   | 0.1101  | 0.1119  | 0.1096  |
| 3     | 0.2157     | 0.0959   | 0.1120  | 0.1109  |
| 4     | 0.1890     | 0.1232   | 0.1097  |         |
| 5     | 0.1718     | 0.0777   |         |         |
| 6     | 0.1583     |          |         |         |         |
| RoC   | 0.5        | 0.2917   | 0.0834  | 0.1238  | 0.1101  | 0.1120  | 0.1109  |

TABLE IV: Padé triangle for the pseudo-ε expansion of the critical exponent α at n = 3. The number in the bottom row (RoC) marked by an asterisk is the estimate averaged over the approximants [4/2] and [2/4] because the approximant [3/3] has a pole close to 1, which makes the corresponding value of α unreliable. Hereafter, the results obtained by averaging over the exponent values given by two off-diagonal approximants are marked by asterisks.

| M \ L | 0          | 1        | 2       | 3       | 4       | 5       | 6       |
|-------|------------|----------|---------|---------|---------|---------|---------|
| 0     | 0.5        | 0.1591   | 0.0038  | −0.0526 | −0.0862 | −0.0968 | −0.1113 |
| 1     | 0.2973     | −0.1262  | −0.0847 | −0.1356 | −0.1016 | −0.0577 |
| 2     | 0.2035     | −0.0827  | −0.1029 | −0.1086 | −0.1157 |
| 3     | 0.1510     | −0.1262  | −0.1084 | −0.0942 |
| 4     | 0.1169     | −0.0991  | −0.1145 |
| 5     | 0.0934     | −0.1414  |         |         |
| 6     | 0.0759     |          |         |         |         |         |
| RoC   | 0.5        | 0.2282   | −0.1262 | −0.0837 | −0.1029 | −0.1085 | −0.1151* |
TABLE V: Critical exponent $\gamma$ for different values of $n$ obtained using Padé approximants ($\tau$, Padé) and the optimally truncated direct summation of pseudo-$\epsilon$ expansions (3) ($\tau$, OTDS). The numbers marked by asterisks are the estimates averaged over the approximants $[4/2]$ and $[2/4]$ because the approximant $[3/3]$ has poles close to 1 in these cases. For comparison, we present the values of $\gamma$ obtained by resumming six-loop RG expansions using the Padé-Borel-Leroy (PBL) method and the conform-Borel (CB) method and the results obtained by processing strong-coupling (SC) expansions, by processing five-loop $\epsilon$ expansions ($\epsilon$-exp), and by lattice calculations (LC). The bc note means that resumming an $\epsilon$ expansion is based on using exact values of the exponents known for two-dimensional models. The sc and bcc notes means that the results were obtained on simple cubic and face-centered cubic lattices.

| $n$ | 0    | 1    | 2    | 3    | 4    | 8    | 16   | 32   |
|-----|------|------|------|------|------|------|------|------|
| $\tau$, Padé | 1.1617 | 1.2411 | 1.3154 | 1.3796 | 1.4572* | 1.6456* | 1.8254 | 1.9138 |
| $\tau$, OTDS | 1.1628 | 1.2382 | 1.3120 | 1.3779 | 1.4358 | 1.6174 | 1.7769 | 1.8798 |
| PBL [3] | 1.161 | 1.241 | 1.316 | 1.39 | | | | |
| CB [6] | 1.1615 | 1.241 | 1.316 | 1.386 | | | | |
| PBL [5] | 1.160 | 1.239 | 1.315 | 1.386 | 1.449 | 1.637 | 1.807 | 1.908 |
| CB [7] | 1.1596 | 1.2396 | 1.3169 | 1.3895 | 1.456 | | | |
| SC [8] | 1.161 | 1.241 | 1.318 | 1.390 | 1.451 | 1.638 | 1.822 | 1.920 |
| CB [9] | 1.1615 | 1.2411 | 1.3172 | 1.3876 | | | | |
| $\epsilon$-exp [7] | 1.1575 | 1.2355 | 1.311 | 1.382 | 1.448 | | | |
| $\epsilon$-exp [7] | 1.1571 | 1.2380 | 1.317 | 1.392 | 1.460 | | | |
| LC [27] | 1.1594 | 1.2388 | 1.325 | 1.406 | | | | |
| LC [27] | 1.1582 | 1.2384 | 1.322 | 1.402 | | | | |
| LC | 1.1575 [28] | 1.2372 [29] | 1.3177 [30] | 1.3960 [31] | 1.68 [32] | | | |
| LC | 1.1573 [33] | 1.2373 [34] | 1.3178 [35] | | | | | |
TABLE VI: Critical exponent $\alpha$ for different values of $n$ obtained using Padé approximants ($\tau$, Padé) and the optimally truncated direct summation of pseudo-$\epsilon$ expansions (4) ($\tau$, OTDS). The numbers marked by asterisks were obtained by averaging the values given by the approximants [4/2] and [2/4]; the diagonal approximant [3/3] has poles close to 1 in these cases. For comparison, we present the values of $\alpha$ obtained by resumming six-loop RG expansions by the Padé-Borel-Leroy (PBL) method and the conform-Borel (CB) technique and the results obtained by processing five-loop $\epsilon$ expansions ($\epsilon$-exp) and by lattice calculations (LC). The bc, sc, and bcc notes are the same as in Table 5.

| $n$ | 0   | 1   | 2   | 3   | 4   | 8   | 16  | 32  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\tau$, Padé | 0.2355 | 0.1109 | −0.0023 | −0.1151* | −0.2105* | −0.4828* | −0.7181* | −0.8841 |
| $\tau$, OTDS | 0.2413 | 0.1162 | 0.0031 | −0.0968 | −0.2011 | −0.4549 | −0.6859 | −0.8319 |
| PBL [3] | 0.236 | 0.110 | −0.007 | −0.115 |
| CB [6] | 0.236 | 0.110 | −0.007 | −0.115 |
| PBL [5] | 0.231 | 0.107 | −0.010 | −0.117 | −0.213 | −0.489 | −0.732 | −0.875 |
| CB [7] | 0.235 | 0.109 | −0.011 | −0.122 | −0.223 |
| $\epsilon$-exp [7] | 0.2375 | 0.1130 | −0.0040 | −0.1135 | −0.211 |
| $\epsilon$-exp [7], bc | 0.2366 | 0.1085 | −0.013 | −0.124 | −0.226 |
| LC [36], sc | 0.24 | 0.103 | −0.014 | −0.11 | −0.22 |
| LC [36], bcc | 0.235 | 0.105 | −0.019 | −0.13 | −0.25 |
| LC | 0.2370 | 0.1096 | −0.0151 | −0.1336 |

[33] [34] [35] [31]
TABLE VII: Padé triangle for the pseudo-ε expansion of the critical exponent \( \omega \) for \( n = 1 \). The Padé approximants \([L/M]\) are calculated for the ratio \( \omega/\tau \), i.e., in the case where the trivial multiplier \( \tau = 1 \) is neglected.

| \( M \setminus L \) | 0  | 1   | 2    | 3    | 4    | 5    |
|-----------------|----|-----|------|------|------|------|
| 0               | 1  | 0.5775 | 0.9226 | 0.6026 | 1.0521 | 0.3736 |
| 1               | 0.7030 | 0.7675 | 0.7566 | 0.7895 | 0.7817 |
| 2               | 0.7963 | 0.7577 | 0.7652 | 0.7830 |
| 3               | 0.7355 | 0.7752 | 0.7906 |
| 4               | 0.8720 | 0.7856 |
| 5               | 0.6867 |
| RoC             | 1  | 0.6402 | 0.7675 | 0.7571 | 0.7652 | 0.7868 |

TABLE VIII: Padé triangle for the pseudo-ε expansion of the critical exponent \( \omega \) for \( n = 3 \). The Padé approximants \([L/M]\) are obtained for the ratio \( \omega/\tau \), i.e., in the case where the multiplier \( \tau \) is neglected.

| \( M \setminus L \) | 0  | 1    | 2    | 3    | 4    | 5    |
|-----------------|----|------|------|------|------|------|
| 0               | 1  | 0.6168 | 0.8889 | 0.6640 | 0.9281 | 0.5756 |
| 1               | 0.7229 | 0.7759 | 0.7658 | 0.7855 | 0.7771 |
| 2               | 0.7950 | 0.7669 | 0.7729 | 0.7793 |
| 3               | 0.7516 | 0.7777 | 0.7807 |
| 4               | 0.8234 | 0.7803 |
| 5               | 0.7281 |
| RoC             | 1  | 0.6699 | 0.7759 | 0.7664 | 0.7729 | 0.7800 |
TABLE IX: Critical exponent $\omega$ for different values of $n$ obtained by resumming expansion (5) by using Padé approximants ($\tau$, Padé) and by optimally truncated direct summation ($\tau$, OTDS). For comparison, we present the values of $\omega$ obtained by processing six-loop RG expansions by the Padé-Borel-Leroy (PBL) method and the conform-Borel (CB) technique and the results obtained by processing strong coupling (SC) expansions, by processing five-loop $\epsilon$ expansions ($\epsilon$-exp), and by lattice calculations (LC).

| $n$ | 0  | 1  | 2  | 3  | 4  | 8  | 16 | 32 |
|-----|----|----|----|----|----|----|----|----|
| $\tau$, Padé | 0.7947 | 0.7868 | 0.7812 | 0.7800 | 0.7825 | 0.8082 | 0.8607 | 0.9201 |
| $\tau$, OTDS | 0.9532 | 0.6026 | 0.6364 | 0.6640 | 0.6877 | 0.8557 | 0.8478 | 0.9181 |
| PBL [3] | 0.794 | 0.788 | 0.78 | 0.78 |
| CB [6] | 0.80 | 0.79 | 0.78 | 0.78 |
| PBL [37] | 0.781 | 0.780 | 0.780 | 0.783 | 0.808 | 0.861 | 0.919 |
| CB [7] | 0.812 | 0.799 | 0.789 | 0.782 | 0.774 |
| SC [8] | 0.810 | 0.805 | 0.800 | 0.797 | 0.795 | 0.810 | 0.862 | 0.924 |
| CB [9] | 0.790 | 0.782 | 0.778 | 0.778 |
| $\epsilon$-exp [7] | 0.828 | 0.814 | 0.802 | 0.794 | 0.795 |
| $\epsilon$-exp [38] | 0.82 | 0.81 | 0.80 | 0.79 |
| LC | 0.83 [34] | 0.785 [35] | 0.773 [39] | 0.765 [39] |

TABLE X: Accuracy of scaling relations (18) for the critical exponents obtained by resumming the corresponding pseudo-$\epsilon$ expansions using Padé approximants.

| $n$ | 0  | 1  | 2  | 3  | 4  | 8  | 16 | 32 |
|-----|----|----|----|----|----|----|----|----|
| $\frac{\gamma}{2-\eta} - \nu$ | 0.0009 | 0.0012 | 0.0017 | 0.0037 | 0.0014* | 0.0010* | −0.0006 | −0.0001 |
|  | 0.0019* | 0.0142 |
| $\frac{2\beta}{1+\eta} - \nu$ | −0.0021 | −0.0024 | −0.0033 | −0.0055 | −0.0009* | −0.0015* | 0.0040 | 0.0007 |
|  | −0.0006* | −0.0076 |
| $\beta + \frac{\alpha+\gamma}{2} - 1$ | −0.0002 | −0.0001 | 0.0000 | 0.0008 | 0.0047* | 0.0079* | 0.0014 | 0.0003 |
|  | 0.0042* | −0.0101 |