On the Twist 2 and Twist 3 Contributions to the Spin-dependent Electroweak Structure Functions

Johannes Blümlein\textsuperscript{a} and Nikolai Kochelev\textsuperscript{a,b}

\textsuperscript{a}DESY–Zeuthen
Platanenallee 6, D–15735 Zeuthen, Germany
\textsuperscript{b}Bogoliubov Laboratory of Theoretical Physics, JINR,
RU–141980 Dubna, Moscow Region, Russia

Abstract

The twist 2 and twist 3 contributions of the polarized deep-inelastic structure functions are calculated both for neutral and charged current interactions using the operator product expansion in lowest order in QCD. The relations between the different structure functions are determined. New integral relations are derived between the twist 2 contributions of the structure functions $g_3(x, Q^2)$ and $g_5(x, Q^2)$ and between combinations of the twist 3 contributions to the structure functions $g_2(x, Q^2)$ and $g_3(x, Q^2)$. The sum rules for polarized deep inelastic scattering are discussed in detail.
1 Introduction

The investigation of the nucleon structure by polarized deep-inelastic lepton scattering off polarized targets revealed a rich structure of phenomena during the last years [1]. So far only the case of polarized deep-inelastic photon scattering has been studied experimentally for lower values of $Q^2$. In the future possible polarized deep-inelastic scattering experiments at the RHIC collider and HERA may be considered in a much wider kinematic range [2] in which the contributions of also the weak currents are important. For this general case the scattering cross sections are determined by five different polarized structure functions, if lepton mass effects are disregarded.

The light–cone expansion [3] proofed to be one of the most powerful techniques to describe the behavior of deep-inelastic scattering structure functions in the Bjorken limit [4]. Whereas the case of unpolarized deep-inelastic scattering and polarized scattering with pure photon exchange are well understood [5] still differing results are reported for polarized deep-inelastic scattering including weak neutral and charged current interactions, see e.g. refs. [6]–[18].

In previous investigations different techniques were used to derive relations between the polarized structure functions. In refs. [6]–[9] the structure functions were calculated in the parton model. Some of the investigations deal with the case of longitudinal polarization only [10]. In other studies light-cone current algebra [11]–[13] and the operator product expansion were used [14]–[16]. Furthermore, the structure functions $g_1^{em}$ and $g_2^{em}$ were also calculated in the covariant parton model [13] in refs. [17, 18]. Still a thorough agreement between the different approaches has not been obtained.

Unlike the structure functions in deep-inelastic scattering off unpolarized targets, two of the structure functions of polarized nucleons, $g_2(x, Q^2)$ and $g_3(x, Q^2)$, contain besides the twist 2 terms also twist 3 contributions. The determination of these contributions to polarized structure functions is a great challenge to experiment. It requires a precise measurement of the deep-inelastic scattering structure functions for transversely polarized targets. Polarized deep-inelastic scattering allows thus to test predictions of QCD in a new domain. Furthermore, to unfold also the flavor structure of these contributions charged current polarized deep-inelastic scattering has to be studied which is experimentally extremely difficult.

It is the aim of the present paper to derive the relations for the complete set of the polarized structure functions including weak interactions, which are not associated with terms in the scattering cross section vanishing as $m_{lepton} \to 0$, accounting both for their twist 2 and twist 3 contributions. The calculation is performed applying the operator product expansion.

As it turns out, the twist 2 contributions for only two out of the five polarized structure functions, corresponding to the respective current combinations, are linearly independent. Therefore three linear operators have to exist which determine the twist 2 contributions of the remaining three structure functions over a basis of two in lowest order QCD. Two of them are given by the Wandzura–Wilczek [21] relation and a relation by Dicus [11]. A third new relation is derived.

The structure function $g_3(x, Q^2)$ contributes only in the exchange of a weak current. We derive also a new relation between combinations of its twist 3 contribution and corresponding contributions to the structure function $g_2(x, Q^2)$. New sum rules based on this relation can be

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1 For first experimental results see ref. [20].
2 This possibility has been considered already earlier in ref. [1].
obtained. A brief summary of our results for the twist 2 terms for neutral current deep-inelastic scattering has already appeared [22].

The paper is organized as follows. In section 2 basic notations are introduced and the Born cross sections for polarized deep-inelastic scattering are discussed. The structure of the forward Compton amplitude and the crossing relations for the case of neutral and charged electroweak currents are derived in section 3. In section 4 a detailed derivation of the operator product expansion is presented. Relations between the moments of the structure functions are derived in section 5. In section 6 a critical account is given on sum rules discussed in the literature, and section 7 contains the conclusions.

2 The Born Cross Sections

The differential Born cross section for polarized lepton–polarized nucleon scattering reads

$$\frac{d^3\sigma}{dx dy d\theta} = \frac{y\alpha^2}{Q^4} \sum_i \eta_i(Q^2) L^\mu_\nu W_i^\mu\nu.$$  (1)

Here the index $i$ denotes the different current combinations, i.e. $i = |\gamma|^2, |\gamma Z|, |Z|^2$ for the neutral, and $i = |W^-|^2$ or $|W^+|^2$ for the charged current interactions. $\theta$ is the azimuthal angle of the final-state lepton, $x = Q^2/(2P.q)$ and $y = P.q/k.P$ are the Bjorken variables, with $q = k - k'$ the four momentum transferred, $k$ and $k'$ the initial and final-state lepton, $P$ the proton 4-momenta, respectively, and $Q^2 = -q^2$. The factors $\eta_i(Q^2)$ denote the ratios of the corresponding propagator terms to the photon propagator squared,

$$\eta^{|\gamma|^2}(Q^2) = 1,$$
$$\eta^{|\gamma Z|^2}(Q^2) = \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha} \frac{Q^2}{Q^2 + M_Z^2},$$
$$\eta^{|Z|^2}(Q^2) = (\eta^{|\gamma Z|^2})^2(Q^2),$$
$$\eta^{|W^\pm|^2}(Q^2) = \frac{1}{2} \left( \frac{G_F M_W^2}{4\pi\alpha} \frac{Q^2}{Q^2 + M_W^2} \right)^2.$$  (2)

$\alpha$ denotes the fine structure constant, $G_F$ is the Fermi constant and $M_Z$ and $M_W$ are the $Z$ and $W$ boson masses.

The leptonic tensor has the following form:

$$L^i_{\mu\nu} = \sum_{\lambda' \lambda} \left[ \bar{u}(k', \lambda') \gamma_\mu (g_{1V}^{i1} + g_{1A}^{i1} \gamma_5) u(k, \lambda) \right]^* \bar{u}(k', \lambda') \gamma_\nu (g_{2V}^{i2} + g_{2A}^{i2} \gamma_5) u(k, \lambda).$$  (3)

$\lambda$ and $\lambda'$ denote the helicity of the initial and final-state lepton. The indices $i_1$ and $i_2$ refer to the currents forming the combinations $i$ in eq. (1). The vector and axial vector couplings are

$$g_V^\gamma = 1, \quad g_A^\gamma = 0,$$
$$g_V^Z = -\frac{1}{2} + 2\sin^2\theta_W, \quad g_A^Z = \frac{1}{2},$$
$$g_V^{W-} = 1, \quad g_A^{W-} = -1.$$  (4)
for negatively charged initial-state leptons and neutrinos. Here $\theta_W$ denotes the weak mixing angle. For positively charged leptons and antineutrinos the sign of the axial vector couplings is reversed.

The hadronic tensor is given by

$$ W^{i}_{\mu \nu} = \frac{1}{4\pi} \int d^4x e^{iqx} \langle PS \mid [J^i_{\mu}(x) \dagger, J^j_{\nu}(0)] \mid PS \rangle. \quad (5) $$

$S$ denotes the nucleon spin 4-vector with, $S \cdot P = 0$. In the following we normalize $S^2 = -M^2$, where $M$ is the nucleon mass. In the framework of the quark–parton model the currents $J^i_{\mu}$ read

$$ J^i_{\mu}(x) = \sum_{f,f'} \bar{q}'_{f'}(x) \gamma_{\mu} (g^V_{q_{f}f} + g^A_{q_{f}f} \gamma_5) q_{f}(x) U_{ff'}, \quad (6) $$

where $g^V_{q_{f}f}$, $g^A_{q_{f}f}$ are the electroweak couplings of the quark labeled by $f$. For charged current interactions $U_{ff'}$ denotes the Cabibbo-Kobayashi-Maskawa matrix and $g^V_{q} = 1$, $g^A_{q} = -1$, whereas for neutral current interactions $U_{ff'} = \delta_{ff'}$, $g^V_{q} = \delta_{ff}$, $g^A_{q} = 0$ for $\gamma$, and

$$ g^q_V = \frac{1}{2} - \frac{4}{3} \sin^2 \theta_W, \quad g^q_A = -\frac{1}{2}, \quad \text{for } q = u, c $$
$$ g^q_V = -\frac{1}{2} + \frac{2}{3} \sin^2 \theta_W, \quad g^q_A = \frac{1}{2}, \quad \text{for } q = d, s \quad (7) $$

for $Z$ boson exchange. For antiquarks the sign of $g^q_A$ reverses.

The hadronic tensor is constructed requiring Lorentz and time-reversal invariance, as well as current conservation. The general structure of the hadronic tensor is

$$ W_{\mu \nu} = (-g_{\mu \nu} + \frac{q_{\mu}q_{\nu}}{q^2}) F_1(x, Q^2) + \frac{\hat{P}_{\mu} \hat{P}_{\nu}}{P \cdot q} F_2(x, Q^2) - i \varepsilon_{\mu \nu \lambda \sigma} \frac{q^\lambda P^\sigma}{2P \cdot q} F_3(x, Q^2) $$
$$ + i \varepsilon_{\mu \nu \lambda \sigma} \frac{q^\lambda S^\sigma}{P \cdot q} g_1(x, Q^2) + i \varepsilon_{\mu \nu \lambda \sigma} \frac{q^\lambda (P \cdot q S^\sigma - S \cdot q P^\sigma)}{(P \cdot q)^2} g_2(x, Q^2) $$
$$ + \left[ \frac{\hat{P}_{\mu} \hat{S}_{\nu} + \hat{S}_{\mu} \hat{P}_{\nu}}{2} - S \cdot q \frac{\hat{P}_{\mu} \hat{P}_{\nu}}{(P \cdot q)^2} g_3(x, Q^2) \right] $$
$$ + S \cdot q \frac{\hat{P}_{\mu} \hat{P}_{\nu}}{(P \cdot q)^2} g_4(x, Q^2) + (-g_{\mu \nu} + \frac{q_{\mu}q_{\nu}}{q^2}) \frac{(S \cdot q)}{P \cdot q} g_5(x, Q^2), \quad (8) $$

with

$$ \hat{P}_{\mu} = P_{\mu} - \frac{P \cdot q}{q^2} q_{\mu}, \quad \hat{S}_{\mu} = S_{\mu} - \frac{S \cdot q}{q^2} q_{\mu}. \quad (9) $$

Here the current indices were suppressed. The hadronic tensor depends in general on three unpolarized structure functions $F_i$ and five polarized structure functions $g_i$. The notation for the structure functions $F_i$, $g_1$ and $g_2$ is widely unique in the literature, however, different

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Footnotes:

3 Hadronic tensors with a violation of current conservation have been discussed in the literature, see e.g. ref. [15, 23].
notations are used for the structure functions $g_3$, $g_4$ and $g_5$. For later comparisons, we list the main conventions used by other authors in Table 1.

From eqs. (1), (3) and (8) one obtains the differential scattering cross sections of a lepton with helicity $\lambda$ off a polarized nucleon. For convenience we will consider two projections of the nucleon spin vector, choosing the spin direction longitudinally and transversely to the nucleon momentum. In the nucleon rest frame one has

\begin{align}
S_L &= (0, 0, 0, M), \\
S_T &= M(0, \cos \alpha, \sin \alpha, 0).
\end{align}


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\end{align}


\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
our notation & [8] & [10] & [11] & [12] & [13, 23] & [16] & [17] & [24] \\
\hline
$g_1$ & $g_1$ & $g_1$ & $g_1$ & $F_3 + F_4$ & $2(F_1 + F_2)$ & $G_1 + G_2$ & $g_1$ & $\tilde{F}_1$ & $M \nu \text{Im } G_1 / \pi$
\hline
$g_2$ & $g_2$ & $g_2$ & $g_2$ & $-F_4$ & $-2F_2$ & $-G_2$ & $g_2$ & $\tilde{F}_2 / (2x)$ & $-\nu^2 \text{Im } G_2 / (M \pi)$
\hline
$g_3$ & $-g_3$ & $g_5$ & $(g_4 - g_5) / 2$ & $2F_8$ & $4F_4$ & $2G_3$ & $(A_2 - A_3) / 2$ & $-2\tilde{F}_3$ & \hline
$g_4$ & $g_4 - g_3$ & $g_4 + g_5$ & $g_4$ & $2F_8 + F_7$ & $2(F_6 + 2F_4)$ & $2G_4 + G_5$ & $a_2 + b_1 + b_2$ & $A_2$ & \hline
$g_5$ & $-g_5$ & $-g_3$ & $g_3$ & $F_7 / (2x) - F_6$ & $F_7 / (2x) - 2F_5$ & $-G_4$ & $a_1$ & $A_1$ & \hline
\end{tabular}
\caption{A comparison of different conventions to denote the polarized nucleon structure functions.}
\end{table}

The scattering cross section for longitudinal nucleon polarization reads

\begin{align}
\frac{d^2 \sigma(\lambda, \pm S_L)}{dx dy} &= 2\pi S \alpha^2 Q^4 \sum_i C_i q_i(Q^2) \left\{ y^2 2x F_1^i + 2 \left( 1 - y - \frac{xy M^2}{S} \right) F_2^i - 2\lambda y \left( 1 - \frac{y}{2} \right) x F_3^i \\
&\quad \pm \left[ -2\lambda y \left( 2 - y - \frac{2xy M^2}{S} \right) x g_1^i + 8\lambda \frac{y^2 x^2 M^2}{S} g_2^i + \frac{4x M^2}{S} \left( 1 - y - \frac{xy M^2}{S} \right) g_3^i \\
&\quad - 2 \left( 1 + \frac{2x M^2}{S} \right) \left( 1 - y - \frac{xy M^2}{S} \right) g_4 - 2xy^2 \left( 1 + \frac{2x M^2}{S} \right) g_5^i \right]\}. \tag{11}
\end{align}

Correspondingly, for transversely polarized nucleons one obtains
\[
\frac{d^3\sigma(\lambda, \pm S_T)}{dx dy d\phi} = \sum_i C_i \tilde{\eta}(Q^2) \left[ y^2 2xF^i_1 + 2 \left( 1 - y - \frac{xyM^2}{S} \right) F^i_2 - 2\lambda y \left( 1 - \frac{y}{2} \right) xF^i_3 \right]
\]

\[\pm 2\sqrt{\frac{M^2}{S}} \sqrt{xy \left( 1 - y - \frac{xyM^2}{S} \right)} \cos(\alpha - \phi) \left[ -2\lambda y x g^i_1 - 4\lambda g^i_2 \right]
\]

\[\frac{1}{y} \left( 2 - y - \frac{2xyM^2}{S} \right) g^i_3 + \frac{2}{y} \left( 1 - y - \frac{xyM^2}{S} \right) g_4 + y^2 2xg^i_5 \right] \right). \tag{12}
\]

Here \(C^\gamma = 1\), \(C^{\gamma Z} = g_V + \lambda g_A\), \(C^Z = (g_V + \lambda g_A)^2\), and \(C^{\gamma \pm} = (1 \pm \lambda)\). As well known, the contributions of the structure functions \(g^Z_2\) and \(g^Z_3\) are suppressed by a factor of \(M^2/S\) in the longitudinal spin asymmetries \(\Delta^L = d^2\sigma(\lambda, S_L) - d^2\sigma(\lambda, -S_L)\). However, they contribute at the same strength as the other structure functions to the transverse spin asymmetries \(\Delta^T = d^2\sigma(\lambda, S_T) - d^2\sigma(\lambda, -S_T)\).

### 3 The Forward Compton Amplitude

The forward Compton amplitude \(T_{\mu\nu}^i\) is related to the hadronic tensor by

\[W^i_{\mu\nu} = \frac{1}{2\pi} \text{Im} T^i_{\mu\nu}, \tag{13}\]

with

\[T^i_{\mu\nu} = i \int d^4x e^{iqx} \langle PS | (T_{\mu\nu}(x)J^i_{\nu}(0))|PS \rangle. \tag{14}\]

It can be represented in terms of the amplitudes \(T^i_k\) and \(A^i_k\) as

\[T^i_{\mu\nu} = (-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2})T^i_1(q^2, \nu) + \hat{P}_{\mu} \hat{T}_{\nu} T^i_2(q^2, \nu) - i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^3 P^\sigma}{2M^2} T^i_3(q^2, \nu)
\]

\[+ i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^3 S^\sigma}{M^2} A^i_1(q^2, \nu) + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^3(P \cdot qS^\sigma - S \cdot qP^\sigma)}{M^2} A^i_2(q^2, \nu)
\]

\[+ \left( \frac{\hat{P}_{\mu} \hat{S}_{\nu} + \hat{S}_{\mu} \hat{P}_{\nu}}{2} - S \cdot \frac{\hat{P}_{\mu} \hat{P}_{\nu}}{P \cdot q} \right) A^i_3(q^2, \nu) + (-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2}) \frac{S \cdot q}{M^2} A^i_4(q^2, \nu)
\]

\[+ S \cdot \frac{\hat{P}_{\mu} \hat{P}_{\nu}}{M^2} A^i_4(q^2, \nu) + \left( \frac{q_{\mu}q_{\nu}}{q^2} \right) \frac{S \cdot q}{M^2} A^i_5(q^2, \nu), \tag{15}\]

where \(\nu = P \cdot q\). The structure functions \(g_i(x, Q^2)\) and amplitudes \(A_i(q^2, \nu)\) are related by

\[g_{1,3,5}(x, Q^2) = \frac{1}{2\pi M^2} \text{Im} A_{1,3,5}(q^2, \nu), \]

\[g_{2,4}(x, Q^2) = \frac{1}{2\pi M^2} \text{Im} A_{2,4}(q^2, \nu). \tag{16}\]

Subsequently we will consider the polarized part of \(T^i_{\mu\nu}\) only.
For neutral current interactions the current operators obey
\[ J_{\mu}^{\gamma,Z} = J_{\mu}^{\gamma,Z}. \] (17)

Therefore, the crossing relation for the amplitude for \( q \to -q, P \to P \) reads
\[ T_{\mu\nu}^i(q^2, -\nu) = T_{\mu\nu}^i(q^2, \nu). \] (18)

The corresponding relations for the amplitudes \( A_{i}^{NC}(q^2, \nu) \) are
\[ A_{1,3}^{NC}(q^2, -\nu) = A_{1,3}^{NC}(q^2, \nu), \]
\[ A_{2,4,5}^{NC}(q^2, -\nu) = -A_{2,4,5}^{NC}(q^2, \nu). \] (19)

Furthermore, the amplitudes obey the following forward dispersion relations:
\[ A_{1,3}^{NC}(q^2, \nu) = \frac{2}{\pi} \int_{Q^2/2}^{\infty} d\nu' \frac{\nu'}{\nu'^2 - \nu^2} \text{Im} A_{1,3}^{NC}(q^2, \nu'), \]
\[ A_{2,4,5}^{NC}(q^2, \nu) = \frac{2}{\pi} \int_{Q^2/2}^{\infty} d\nu' \frac{\nu}{\nu'^2 - \nu^2} \text{Im} A_{2,4,5}^{NC}(q^2, \nu'). \] (20)

For the charged current interactions it is suitable to study the linear combination of amplitudes
\[ T_{\mu\nu}^{\pm}(q^2, \nu) = T_{\mu\nu}^{W-}(q^2, \nu) \pm T_{\mu\nu}^{W+}(q^2, \nu). \] (21)

Due to the transformation
\[ J_{\mu}^{W\pm\dagger} = J_{\mu}^{W\mp}, \] (22)
the following crossing relations hold:
\[ T^{\pm}(q^2, -\nu) = \pm T^{\pm}(q^2, \nu). \] (23)

Correspondingly, one obtains for the combination of the amplitudes
\[ A_{i}^{\pm} = A_{i}^{W-} \pm A_{i}^{W+} \] (24)
the relations
\[ A_{1,3}^{\pm}(q^2, -\nu) = \pm A_{1,3}^{\pm}(q^2, \nu), \]
\[ A_{2,4,5}^{\pm}(q^2, -\nu) = \mp A_{2,4,5}^{\pm}(q^2, \nu). \] (25)

The respective dispersion relations for \( A_{i}^{+}(q^2, \nu) \) and \( A_{i}^{-}(q^2, \nu) \) are:
\[ A_{1,3}^{+}(q^2, \nu) = \frac{2}{\pi} \int_{Q^2/2}^{\infty} d\nu' \frac{\nu'}{\nu'^2 - \nu^2} \text{Im} A_{1,3}^{+}(q^2, \nu'), \]
\[ A_{2,4,5}^{+}(q^2, \nu) = \frac{2}{\pi} \int_{Q^2/2}^{\infty} d\nu' \frac{\nu}{\nu'^2 - \nu^2} \text{Im} A_{2,4,5}^{+}(q^2, \nu'), \]
\[ A_{1,3}^{-}(q^2, \nu) = \frac{2}{\pi} \int_{Q^2/2}^{\infty} d\nu' \frac{\nu'}{\nu'(\nu'^2 - \nu^2)} \text{Im} \left[ \nu'A_{1,3}^{-}(q^2, \nu') \right], \]
\[ A_{2,4}^{-}(q^2, \nu) = \frac{2}{\pi} \int_{Q^2/2}^{\infty} d\nu' \frac{\nu}{\nu'^2(\nu'^2 - \nu^2)} \text{Im} \left[ \nu'^2 A_{2,4}^{-}(q^2, \nu') \right], \]
\[ A_{5}^{-}(q^2, \nu) = \frac{2}{\pi} \int_{Q^2/2}^{\infty} d\nu' \frac{\nu}{\nu'^2 - \nu^2} \text{Im} A_{5}^{-}(q^2, \nu'). \] (26)
For the case of charged current interactions we introduce the structure function combinations

\[ g_i^\pm(x, Q^2) = g_i^{W-}(x, Q^2) \pm g_i^{W+}(x, Q^2). \]  

The integral representations of the amplitudes \( A_i^{\text{NC}} \) and \( A_i^\pm \) can be finally expressed by the moments of the corresponding structure functions as

\[
A_{1,3}^{\text{NC}+}(q^2, \nu) = \frac{4M^2}{\nu} \sum \frac{1}{x^{n+1}} \int_0^1 dy y^n g_{1,3}^{\text{NC}+}(y, Q^2),
\]

\[
A_{2,4}^{\text{NC}+}(q^2, \nu) = \frac{4M^4}{\nu^2} \sum \frac{1}{x^{n+1}} \int_0^1 dy y^n g_{2,4}^{\text{NC}+}(y, Q^2),
\]

\[
A_5^{\text{NC}+}(q^2, \nu) = \frac{4M^2}{\nu} \sum \frac{1}{x^{n+1}} \int_0^1 dy y^n g_5^{\text{NC}+}(y, Q^2),
\]  

and

\[
A_{1,3}^-(q^2, \nu) = \frac{4M^2}{\nu} \sum \frac{1}{x^{n+1}} \int_0^1 dy y^n g_{1,3}^-(y, Q^2),
\]

\[
A_{2,4}^-(q^2, \nu) = \frac{4M^4}{\nu^2} \sum \frac{1}{x^{n+1}} \int_0^1 dy y^n g_{2,4}^-(y, Q^2),
\]

\[
A_5^-(q^2, \nu) = \frac{4M^2}{\nu} \sum \frac{1}{x^{n+1}} \int_0^1 dy y^n g_5^-(y, Q^2),
\]  

performing a Taylor expansion of eqs. (20, 26) and using eqs. (16, 27), where \( x = Q^2/(2\nu) \) and \( y = Q^2/(2\nu') \).

\section{4 Operator product expansion}

The operator product expansion is one of the most general formalisms to analyze the properties of the structure functions in deep-inelastic scattering. We apply it to the \( T \)-product of two electroweak currents,

\[
\hat{T}_{\mu\nu}^i = T(J_{\mu}^{i+}(x)J_{\nu}^{i2}(0)).
\]  

Near the light cone one obtains for neutral currents

\[
\hat{T}_{\mu\nu}^{\text{NC}} = \bar{q}(x)\gamma_\mu(g_{V1} + g_{A1}\gamma_5)S(-x)\gamma_\nu(g_{V2} + g_{A2}\gamma_5)P^+q(0)
+ \bar{q}(0)\gamma_\nu(g_{V2} + g_{A2}\gamma_5)S(-x)\gamma_\mu(g_{V1} + g_{A1}\gamma_5)P^+q(x),
\]  

and for the charged current combinations

\[
\hat{T}_{\mu\nu}^\pm = \hat{T}_{\mu\nu}^{W-} \pm \hat{T}_{\mu\nu}^{W+},
\]

\[
\hat{T}_{\mu\nu}^\pm = \bar{q}(x)\gamma_\mu(g_{V1} + g_{A1}\gamma_5)S(-x)\gamma_\nu(g_{V2} + g_{A2}\gamma_5)P^\pm q(0)
 \pm \bar{q}(0)\gamma_\nu(g_{V2} + g_{A2}\gamma_5)S(-x)\gamma_\mu(g_{V1} + g_{A1}\gamma_5)P^\pm q(x).
\]
Here we used the projectors
\[ P^+ = 1, \quad P^- = \tau_3, \] (34)
with
\[ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
and suppressed the flavor indices and the Cabibbo-Kobayashi-Maskawa matrix for brevity. If not stated otherwise, we will not distinguish the neutral current case from the charged current + combination subsequently. The quark and antiquark states in eqs. (31, 33) are understood as singlets for neutral current interactions and as doublets for charged current interactions. In eqs. (31) and (33)
\[ S(x) \approx \frac{2i x}{(2\pi)^2(x^2 - i0)^2} \] (35)
denotes the free quark propagator.

We rewrite the above relations for \( \hat{T}^+_{\mu\nu} \) and \( \hat{T}^-_{\mu\nu} \) using
\[ \gamma_\mu \not\gamma_\nu = x^\alpha[S_{\mu\alpha\beta}\gamma_\beta - i\varepsilon_{\mu\alpha\beta}\gamma_5], \]
\[ S_{\mu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta}. \] (36)
For the spin dependent part of \( \hat{T}^i_{\mu\nu} \) one obtains
\[ \hat{T}^+_{\mu\nu, spin} = \frac{2ix^\alpha}{(2\pi)^2(x^2 - i0)^2} \left\{ -i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu\alpha\beta}u^\beta_+ + (gV_1gA_1 + gA_2)S_{\mu\alpha\beta}u^\beta_- \right\}, \] (37)
\[ \hat{T}^-_{\mu\nu, spin} = \frac{2ix^\alpha}{(2\pi)^2(x^2 - i0)^2} \left\{ -i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu\alpha\beta}v^\beta_- + (gV_1gA_1 + gA_2)S_{\mu\alpha\beta}v^\beta_+ \right\}, \] (38)
with
\[ u^\beta_+ = \bar{q}(x)\gamma^\beta\gamma_5P^+q(0) \pm \bar{q}(0)\gamma^\beta\gamma_5P^+q(x), \]
\[ u^\beta_- = \bar{q}(x)\gamma^\beta\gamma_5P^-q(0) \pm \bar{q}(0)\gamma^\beta\gamma_5P^-q(x). \]
(39)
The operators in eq. (37) can be represented by the following Taylor series around \( x = 0 \)
\[ \bar{q}(x)\gamma^\beta\gamma_5P^\pm q(0) = \sum_n \frac{(-1)^n}{n!}x_{\mu_1}...x_{\mu_n}\bar{q}(0)\gamma^\beta\gamma_5D^{\mu_1}...D^{\mu_n}P^\pm q(0), \]
\[ \bar{q}(0)\gamma^\beta\gamma_5P^\pm q(x) = \sum_n \frac{(+1)^n}{n!}x_{\mu_1}...x_{\mu_n}\bar{q}(0)\gamma^\beta\gamma_5D^{\mu_1}...D^{\mu_n}P^\pm q(0), \] (41)
for which
\[ \hat{T}^+_{\mu\nu, spin} = \frac{4ix^\alpha}{(2\pi)^2(x^2 - i0)^2} \left\{ -i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu\alpha\beta}\rho^\beta_+ \
- (gV_1gA_2 + gA_1gV_2)S_{\mu\alpha\beta}\rho^\beta_- \right\}, \] (42)
\[
\begin{align*}
\hat{T}_{\mu \nu, \text{spin}}^{-} &= \frac{4ix^\alpha}{(2\pi)^2(x^2 - i0)^2} \left\{ -i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu \nu \beta} \rho^{-\beta} - (gV_1gA_2 + gA_1gV_2)S_{\mu \nu \beta} \rho^{+\beta} \right\} \\
&= -i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu \nu \beta} \rho^{-\beta} + (gV_1gA_2 + gA_1gV_2)S_{\mu \nu \beta} \rho^{+\beta} \\
&= \Theta^{\pm \beta\{\mu_1...\mu_n\}}
\end{align*}
\]

is obtained. Here we used the abbreviations

\[
\rho^{\pm \beta} = \sum_{n \text{ even/odd}} \frac{1}{n!} x_{\mu_1}...x_{\mu_n} \tilde{q}(0)^{\gamma_5} \gamma^\beta D^{\mu_1}...D^{\mu_n} P^\pm 0(q(0)),
\]

for which it is convenient to define (cf. e.g. [23])

\[
\rho^{\pm \beta} = \sum_{n \text{ even/odd}} \frac{1}{n!} i x_{\mu_1}...ix_{\mu_n} \Theta^{\pm \beta\{\mu_1...\mu_n\}}
\]

for later analysis. The Fourier transforms of eqs. (42, 43) read

\[
\hat{T}_{\mu \nu, \text{spin}}^{+} = i \int d^4x e^{ixq} \hat{T}_{\mu \nu, \text{spin}}^{+}
\]

\[
= \frac{1}{\pi^2} \int d^4x e^{ixq} \frac{x^\alpha}{(x^2 - i0)} \left\{ i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu \nu \beta} \rho^{+\beta} + (gV_1gA_2 + gA_1gV_2)S_{\mu \nu \beta} \rho^{-\beta} \right\}
\]

\[
= -i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu \nu \beta} \rho^{+\beta} + (gV_1gA_2 + gA_1gV_2)S_{\mu \nu \beta} \rho^{-\beta}
\]

\[
+ \sum_{n \text{ even}} q^{\mu_1}...q^{\mu_n} \left( \frac{2}{Q^2} \right)^{n} \Theta^{+\mu\{\nu_1...\nu_n\}} + \Theta^{-\mu\{\nu_1...\nu_n\}}
\]

\[
\text{and}
\]

\[
\hat{T}_{\mu \nu, \text{spin}}^{-} = i \int d^4x e^{ixq} \hat{T}_{\mu \nu, \text{spin}}^{-}
\]

\[
= \frac{1}{\pi^2} \int d^4x e^{ixq} \frac{x^\alpha}{(x^2 - i0)} \left\{ i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu \nu \beta} \rho^{-\beta} + (gV_1gA_2 + gA_1gV_2)S_{\mu \nu \beta} \rho^{+\beta} \right\}
\]

\[
= -i(gV_1gV_2 + gA_1gA_2)\varepsilon_{\mu \nu \beta} \rho^{-\beta} + (gV_1gA_2 + gA_1gV_2)S_{\mu \nu \beta} \rho^{+\beta}
\]

\[
+ \sum_{n \text{ odd}} q^{\mu_1}...q^{\mu_n} \left( \frac{2}{Q^2} \right)^{n} \Theta^{-\mu\{\nu_1...\nu_n\}} + \Theta^{+\mu\{\nu_1...\nu_n\}}
\]

\[
\text{and}
\]

The operators \( \Theta^{\pm \beta\{\mu_1...\mu_n\}} \) can be decomposed into a symmetric and a remainder part, \( \Theta_S \) and \( \Theta_R \), respectively

\[
\Theta^{\pm \beta\{\mu_1...\mu_n\}} = \Theta_S^{\pm \beta\{\mu_1...\mu_n\}} + \Theta_R^{\pm \beta\{\mu_1...\mu_n\}},
\]

\[
(47)
\]

\[
(43)
\]
where
\[
\Theta^{±β(μ₁…μₙ)}_S = \frac{1}{n + 1} \left[ \Theta^{±β(μ₁…μₙ)} + \Theta^{±μ₁(β…μₙ)} + … + \Theta^{±μₙ(μ₁…β)} \right],
\]
\[
\Theta^{±β(μ₁…μₙ)}_R = \frac{1}{n + 1} \left[ \Theta^{±β(μ₁…μₙ)} - \Theta^{±μ₁(β…μₙ)} + \Theta^{±μ₂(μ₁β…μₙ)} - \Theta^{±μₙ(μ₁β…μ₂)} + … \right].
\]

The nucleon matrix elements of these operators are
\[
\langle PS|\Theta^{±β(μ₁…μₙ)}_S|PS\rangle = \frac{a^+_n}{n + 1} \left[ S^β P^μ₁ P^μ₂ … P^μₙ + S^μ₁ P^β P^μ₂ … P^μₙ + … - \text{traces} \right],
\]
\[
\langle PS|\Theta^{±β(μ₁…μₙ)}_R|PS\rangle = \frac{d^+_n}{n + 1} \left[ \left( S^β P^μ₁ - S^μ₁ P^β \right) P^μ₂ … P^μₙ \\
+ \left( S^β P^μ₂ - S^μ₂ P^β \right) P^μ₁ P^μ₃ … P^μₙ \\
+ … + \left( S^β P^μₙ - S^μₙ P^β \right) P^μ₁ P^μ₃ … P^{μₙ-1} - \text{traces} \right].
\]

The matrix elements \(a^+_n\) and \(d^+_n\) are the expectation values of twist 2 and twist 3 operators, respectively. For the contractions related to the structure functions \(g_2^j\) one may use a simplified version of (51), cf. (44),
\[
\langle PS|\Theta^{±β(μ₁…μₙ)}_R|PS\rangle \approx \frac{d^+_n}{n + 1} \left[ \left( S^β P^μ₁ - S^μ₁ P^β \right) P^μ₂ … P^μₙ - \text{traces} \right].
\]

Note, however, that this is not possible for the contractions emerging in the case of the structure functions \(g_3^j\).

Finally we obtain for the expectation values of the forward Compton amplitude between nucleon states
\[
T_{μν,\text{spin}}^{+} = -i(g_{ν₂} + g_{A₁} g_{A₂}) \frac{ɛ_{μνβρ} q^α}{ν} \sum_{n \text{ even}} \frac{1}{x^{n+1}} \left[ \frac{a^+_n + nd^+_n}{n + 1} S^β + \frac{n(a^+_n - d^+_n)}{n + 1} \left( S^μ P^ν + P^μ S^ν \right) ν \right] S^β \\
+ \left( g_{ν₂} g_{A₂} + g_{A₁} g_{ν₂} \right) \left\{ -g_{μν} \frac{(S,q)}{ν} \sum_{n \text{ odd}} \frac{a^+_n}{x^{n+1}} \right\} \\
+ \frac{2}{ν} \sum_{n \text{ odd}} \frac{1}{x^n} \left[ \frac{2a^+_n + (n - 1)d^+_n}{n + 1} \left( S^μ P^ν + P^μ S^ν \right) ν \right] \left( S,q \right) \\
+ a^+_n \frac{P^μ P^ν}{ν} (S,q) \right\},
\]
\[
T_{μν,\text{spin}}^{-} = -i(g_{ν₂} + g_{A₁} g_{A₂}) \frac{ɛ_{μνβρ} q^α}{ν} \sum_{n \text{ odd}} \frac{1}{x^{n+1}} \left[ \frac{a^-_n + nd^-_n}{n + 1} S^β + \frac{n(a^-_n - d^-_n)}{n + 1} \left( S^μ P^ν + P^μ S^ν \right) ν \right] S^β \\
+ \left( g_{ν₂} g_{A₂} + g_{A₁} g_{ν₂} \right) \left\{ -g_{μν} \frac{(S,q)}{ν} \sum_{n \text{ even}} \frac{a^-_n}{x^{n+1}} \right\} \\
+ \frac{2}{ν} \sum_{n \text{ even}} \frac{1}{x^n} \left[ \frac{2a^-_n + (n - 1)d^-_n}{n + 1} \left( S^μ P^ν + P^μ S^ν \right) ν \right] \left( S,q \right) \\
+ a^-_n \frac{P^μ P^ν}{ν} (S,q) \right\}.
\]

Here we arranged the structure as in eq. (15) according to the contributions to the different amplitudes.
5 Relations between the Moments of Structure Functions

From eqs. (52) and (53) one derives the following representations for the amplitudes $A_{i}^{NC}(q^{2}, \nu)$ and $A_{i}^{\pm}(q^{2}, \nu)$:

\[
A_{1}^{NC+}(q^{2}, \nu) = \frac{M^{2}}{\nu} \sum_{n \text{ even}} \frac{((g_{1}^{q})^{2} + (g_{A}^{q})^{2})a_{n}^{+q}}{x^{n+1}}
\]
\[
A_{2}^{NC+}(q^{2}, \nu) = \frac{M^{4}}{\nu^{2}} \sum_{n \text{ even}} \frac{((g_{1}^{q})^{2} + (g_{A}^{q})^{2})n(d_{n}^{+q} - a_{n}^{+q})}{x^{n+1}(n+1)}
\]
\[
A_{3}^{NC+}(q^{2}, \nu) = \frac{M^{2}}{\nu} \sum_{n \text{ odd}} 4g_{1}^{q}g_{A}^{q}(2a_{n}^{+q} + (n-1)d_{n}^{+q})
\]
\[
A_{4}^{NC+}(q^{2}, \nu) = \frac{M^{4}}{\nu^{2}} \sum_{n \text{ odd}} 4g_{1}^{q}g_{A}^{q}a_{n}^{+q}
\]
\[
A_{5}^{NC+}(q^{2}, \nu) = \frac{M^{2}}{\nu} \sum_{n \text{ odd}} 2g_{1}^{q}g_{A}^{q}a_{n}^{+q}
\]

and

\[
A_{1}^{-}(q^{2}, \nu) = \frac{M^{2}}{\nu} \sum_{n \text{ odd}} \frac{((g_{1}^{q})^{2} + (g_{A}^{q})^{2})a_{n}^{-q}}{x^{n+1}}
\]
\[
A_{2}^{-}(q^{2}, \nu) = \frac{M^{4}}{\nu^{2}} \sum_{n \text{ odd}} \frac{((g_{1}^{q})^{2} + (g_{A}^{q})^{2})n(d_{n}^{-q} - a_{n}^{-q})}{x^{n+1}(n+1)}
\]
\[
A_{3}^{-}(q^{2}, \nu) = \frac{M^{2}}{\nu} \sum_{n \text{ even}} 4g_{1}^{q}g_{A}^{q}(2a_{n}^{-q} + (n-1)d_{n}^{-q})
\]
\[
A_{4}^{-}(q^{2}, \nu) = \frac{M^{4}}{\nu^{2}} \sum_{n \text{ even}} 4g_{1}^{q}g_{A}^{q}a_{n}^{-q}
\]
\[
A_{5}^{-}(q^{2}, \nu) = \frac{M^{2}}{\nu} \sum_{n \text{ even}} 2g_{1}^{q}g_{A}^{q}a_{n}^{-q}
\]

On the other hand, the representations eq. (28) and (29) are valid, from which the following relations between the operator matrix elements $a_{n}^{\pm,q}$ and $d_{n}^{\pm,q}$ and the moments of the structure functions $g_{i}^{NC,\pm}(x, Q^{2})$ are obtained:

\[
\int_{0}^{1} dx x^{n} g_{1}^{NC+}(x, Q^{2}) = \sum_{q} \frac{((g_{1}^{q})^{2} + (g_{A}^{q})^{2})a_{n}^{+q}}{4}, \quad n = 0, 2 \ldots
\]
\[
\int_{0}^{1} dx x^{n} g_{2}^{NC+}(x, Q^{2}) = \sum_{q} \frac{((g_{1}^{q})^{2} + (g_{A}^{q})^{2})n(d_{n}^{+q} - a_{n}^{+q})}{4(n+1)}, \quad n = 2, 4 \ldots
\]
\[
\int_{0}^{1} dx x^{n} g_{3}^{NC+}(x, Q^{2}) = \sum_{q} \frac{g_{1}^{q}g_{A}^{q}(2a_{n+1}^{+q} + nd_{n+1}^{+q})}{(n+2)}, \quad n = 0, 2 \ldots
\]
\[
\int_{0}^{1} dx x^{n} g_{4}^{NC+}(x, Q^{2}) = \sum_{q} \frac{g_{1}^{q}g_{A}^{q}a_{n+1}^{+q}}{2}, \quad n = 2, 4 \ldots
\]
\[
\int_{0}^{1} dx x^{n} g_{5}^{NC+}(x, Q^{2}) = \sum_{q} \frac{g_{1}^{q}g_{A}^{q}a_{n}^{+q}}{2}, \quad n = 1, 3 \ldots
\]

\[\text{Note that a misprint in eq. (17), 23, was corrected in eq. (64).}\]
and

\[
\int_0^1 dx x^n g_1^i (x, Q^2) = \sum_q \frac{((g_V^q)^2 + (g_A^q)^2)a_n^{-q}}{4}, \quad n = 1, 3 \ldots \tag{61}
\]

\[
\int_0^1 dx x^n g_2^i (x, Q^2) = \sum_q \frac{((g_V^q)^2 + (g_A^q)^2)n(d_n^{-q} - a_n^{-q})}{4(n + 1)}, \quad n = 1, 3 \ldots \tag{62}
\]

\[
\int_0^1 dx x^n g_3^i (x, Q^2) = \sum_q \frac{g_V^q g_A^q (2a_{n+1}^{-q} + nd_{n+1}^{-q})}{(n + 2)}, \quad n = 1, 3 \ldots \tag{63}
\]

\[
\int_0^1 dx x^n g_4^i (x, Q^2) = \sum_q g_V^q g_A^q a_{n+1}^{-q}, \quad n = 1, 3 \ldots \tag{64}
\]

\[
\int_0^1 dx x^n g_5^i (x, Q^2) = \sum_q \frac{g_V^q g_A^q a_{n+1}^{-q}}{2}, \quad n = 0, 2 \ldots \tag{65}
\]

Eqs. (58, 63) differ from corresponding results obtained in ref. [14] for charged current interactions. As evident from eqs. (56–65) only the structure functions \(g_2^i (x, Q^2)\) and \(g_3^i (x, Q^2)\) contain twist 3 contributions.

Let us first consider the twist 2 contributions to the different structure functions. From eqs. (56, 57, 61, 62)

\[
\int_0^1 dx x^n g_1^i (x, Q^2) = -\frac{n + 1}{n} \int_0^1 dx x^n g_2^i (x, Q^2), \tag{66}
\]

follows, where \(n = 2, 4\ldots\) for \(i = NC, +\) and \(n = 1, 3\ldots\) for \(i = -\). If an analytic continuation of the moment–index \(n\) to the complex plane is performed for eq. (66), one obtains the Wandzura–Wilczek relation [21]

\[
g_2^i (x, Q^2) = -g_1^i (x, Q^2) + \int_x^1 \frac{dy}{y} g_1^i (y, Q^2). \tag{67}
\]

Although it is formally consistent with the Burkhardt-Cottingham sum rule [24]

\[
\int_0^1 dx g_2^i (x, Q^2) = 0, \tag{68}
\]

the 0th moment of the structure functions \(g_2^i (x, Q^2)\) is not described by the local operator product expansion.

Eqs. (58, 60) and (63, 65) yield the following relations between the parity violating spin–dependent structure functions

\[
\int_0^1 dx x^n g_3^i (x, Q^2) = \frac{n + 2}{2} \int_0^1 dx x^n g_3^i (x, Q^2), \tag{69}
\]

with \(n = 2, 4\ldots\) for the \(i = NC, +\) exchange and \(n = 1, 3\ldots\) for the \(i = -\), and

\[
\int_0^1 dx x^n g_3^i (x, Q^2) = \frac{n + 1}{4} \int_0^1 dx x^{n-1} g_3^i (x, Q^2), \tag{70}
\]

with \(n = 1, 3\ldots\) for the \(i = NC, +\) and \(n = 2, 4\ldots\) for the \(i = -\). Analytic continuation leads to the new relations

\[
g_3^i (x, Q^2) = 2x \int_x^1 \frac{dy}{y^2} g_4^i (y, Q^2), \tag{71}
\]
and
\[ g_3^i(x, Q^2) = 4x \int_x^1 \frac{dy}{y} g_5^i(y, Q^2). \] (72)

From (58) and (60) a new sum rule
\[ \int_0^1 dx [g_3^{NC, +}(x, Q^2) - 2xg_5^{NC, +}(x, Q^2)] = 0, \] (73)
is derived, which is not sensitive to the twist 3 contributions.

From eqs. (59, 60, 64, 65) follows the relation
\[ \int_0^1 dx x^n g_4^n(x, Q^2) = 2 \int_0^1 dx x^{n+1} g_5^n(x, Q^2), \] (74)
where \( n = 2, 4 \ldots \) for \( i = NC, + \) and \( n = 1, 3 \ldots \) for \( i = - \). Eq. (74) yields the Dicus relation (75):
\[ g_4^i(x, Q^2) = 2xg_5^i(x, Q^2). \] (75)
The latter equation is a strict relation for the structure functions \( g_4 \) and \( g_5 \) in lowest order QCD, since both structure functions contain no twist 3 contributions.

The relations (57, 72), and (75) provide the linear maps of a basis formed by the twist 2 contributions to \( g_1^i(x, Q^2) \) and \( g_5^i(x, Q^2) \) to the complete set of the twist 2 terms of the polarized structure functions in lowest order QCD.

Let us finally consider the twist 3 parts of the structure functions \( g_2(x, Q^2) \) and \( g_3(x, Q^2) \). We define the twist 3 contribution to \( g_2(x, Q^2) \) and \( g_3(x, Q^2) \) by
\[ g_2^\text{III}(x, Q^2) = g_2(x, Q^2) + g_1(x, Q^2) - \int_x^1 \frac{dy}{y} g_1(y, Q^2), \] (76)
\[ g_3^\text{III}(x, Q^2) = g_3(x, Q^2) - 4x \int_x^1 \frac{dy}{y} g_5(y, Q^2). \] (77)
The right hand sides of the eqs. (74) describe the violation of the Wandzura–Wilczek relation and the twist 2 relation, eq. (72), in lowest order QCD, respectively, in the presence of twist 3 terms.

The twist 3 contribution to charged current and electromagnetic structure functions can be obtained using the relations
\[ \int_0^1 dx x^n (4g_5^-(x, Q^2) - \frac{n+1}{x} g_3^-(x, Q^2)) = \sum_q (n-1)d_n^q, \] (78a)
\[ \int_0^1 dx x^n (ng_1^+(x, Q^2) + (n+1)g_2^+(x, Q^2)) = \sum_q \frac{n\epsilon_q^2}{4} d_n^{q+}, \quad n = 2, 4 \ldots . \] (78b)
The definitions of the matrix elements \( d_n^{q\pm} \) imply
\[ \int_0^1 dx x^n \{4g_5 - \frac{n+1}{x} g_3\}^{\mu n-\nu p} = \frac{12(n-1)}{n} \int_0^1 dx x^n \{ng_1 + (n+1)g_2\}^{\nu-\gamma n}, \quad n = 2, 4 \ldots . \] (79)

The analytic continuation in \( n \) for this relation finally yields the new integral relation among only twist 3 contributions
\[ g_3^\text{III,}\nu n-\nu p(x, Q^2) = 12 \left[ xg_2^\text{III}(x, Q^2) - \int_x^1 dy g_2^\text{III}(y, Q^2) \right]^{\nu-\gamma n}. \] (80)

This relation corresponds to the Callan–Gross relation for unpolarized structure functions since the spin dependence enters the tensors of \( g_4 \) and \( g_5 \) in \( W^\mu_{\mu
u} \), eq. (8), in terms of the overall factor \( S_q \).
6 Sum Rules

A summary of sum rules derived by different techniques in the literature is given in Table 2. Here we discuss first the twist 2 contributions and consider deep-inelastic scattering off massless quarks only. Most of the sum rules concern relations between the different neutral and charged current structure functions. In some cases relations to the first moment of a combination of polarized parton densities as

\[
\begin{align*}
  g_A &= \int_0^1 dx \left[ \Delta u(x) + \Delta \bar{u}(x) - \Delta d(x) - \Delta \bar{d}(x) \right] \\
  g_A^* &= \int_0^1 dx \left[ \Delta u_V(x) - \Delta d_V(x) \right] \\
  g_A^8 &= \int_0^1 dx \left[ \Delta u(x) + \Delta \bar{u}(x) + \Delta d(x) + \Delta \bar{d}(x) - 2(\Delta s(x) + \Delta \bar{s}(x)) \right] \\
  g_A^{\circ 8} &= \int_0^1 dx \left[ \Delta u_V(x) + \Delta d_V(x) \right] 
\end{align*}
\]

are considered. In most of the studies three massless flavors were assumed. Other sum rules being based on SU(6) symmetry were discussed in refs. [8, 13].

Before we will consider specific sum rules, we summarize the basic relations obtained in the previous sections. We consider the longitudinal and transverse projection of the hadronic tensor, \( W_{\mu \nu}^\parallel \) and \( W_{\mu \nu}^\perp \), respectively. The different relations between the twist 2 contributions to the structure functions are illustrated schematically in Figure 1.
The structure functions $g_1(x)$, $g_4(x)$, and $g_5(x)$ contribute to the longitudinal projection, whereas the transverse projection contains $g_1(x) + g_2(x)$ and $g_3(x)$. As shown above, the twist 2 contributions to the structure functions $g_1(x)$ and $g_2(x)$ are $\propto \Delta q(x) + \Delta q(x)$, while $g_3(x)$, $g_4(x)$ and $g_5(x) \propto \Delta q(x) - \Delta q(x)$. Except for valence approximations, strict relations between the twist 2 contributions to the structure functions can only be obtained either for $g_1(x)$ and $g_2(x)$ or $g_3(x)$, $g_4(x)$ and $g_5(x)$. While the two non-singlet structure functions in $W_{\mu\nu}$ are related by a factor of two in the right hand side from the corresponding relations (29) and (30), Table 2, eq. (23) in Table 2 are consistent with the analytic continuations of the relations derived in section 5, but are not obtainable in the framework of the local operator product expansion.

A series of sum rules derived previously is verified in the context of the operator product expansion, see Table 2. However, some relations could not be confirmed. Among the latter are those which were derived using the collinear parton model [7], in which the structure functions

$\propto$ and $\propto$ and

$\propto$ and $\propto$ where the transverse projection contains $\propto$ and $\propto$ whereas the transverse projection contains $\propto$ and $\propto$ contained in $W_{\mu\nu}$ and $W_{\mu\nu}$, respectively.

Some other relations, as the Burkhardt–Cottingham sum rule, for different currents, and eq. (23) in Table 2 in more detail. They differ by a factor of two in the right hand side from the corresponding relations (29) and (30), Table 2,

$$24x[(g_1 + g_2)^{ep} - (g_1 + g_2)^{en}] = g_3^{en} - g_3^{ep},$$

$$24x[g_2^{ep} - g_2^{en}] = (g_3 - 2g_4)^{en} - (g_3 - 2g_4)^{ep}.$$ (84)

This difference has a subtle reason, which can be traced back to the ansatz for the polarization vector $n_\alpha$. We will use the covariant parton model to illustrate this aspect. As shown in ref. [22] the results obtained in this approach are completely equivalent to those found using the local operator product expansion for $m_q \to 0$ in lowest order QCD.

In refs. [11, 12, 9] (see also [3]) the relation

$$n_\alpha \propto S_\alpha$$ (86)

was used, with $S_\alpha$ nucleon spin vector. As a result, linear relations between longitudinal and transverse spin-dependent structure functions are implied, which violate the Wandzura-Wilczek relation, the new relation (71), and yield the extra factor of two in eqs. (84,85).

Let us compare the derivation of eq. (84) in refs. [11, 12, 9] with the calculation performed in the covariant parton model in ref. [22]. The spin–dependent part of the hadronic tensor has the following form

$$W_{\mu\nu}(q, P, S) = \sum_q \int d^4k \Delta f_q(q, k, S) w^{\mu\nu}_{\mu\nu}[k + q]^2 - m_q^2],$$ (87)

In the valence approximation a similar relation is obtained for $g_1(x) + g_2(x)$ and $g_3(x)$. 


where \( w_q^{\mu\nu} \) is quark tensor, and \( \Delta f_q(P,k,S) \) is a function, which is related to the polarized quark distributions.

In lowest order QCD the spin-dependent part of the quark tensor has the following form

\[
\Delta f_q^{\mu\nu}(P,k,S) = \frac{1}{4} Tr[(1 + \gamma_5 \frac{\not{P} - \not{k}}{m_q})(\not{k} + m_q) \gamma_\mu(g_{V_1} + g_{A_1} \gamma_5)(\not{k} + \not{q} + m_q) \gamma_\nu(g_{V_2} + g_{A_2} \gamma_5)]
\]

where \( n \) is the partonic spin vector.

While in the covariant parton model \([18]\) the (off–shell) parton spin vector reads

\[
\not{n} \approx \frac{m_q \not{p} \cdot \not{k}}{\sqrt{(\not{p} \cdot \not{k})^2 k^2 - M^2 k^4}} (k_\sigma - \frac{k^2}{p \cdot k} p_\sigma),
\]

the representation

\[
n_\sigma \approx \frac{m_q}{M} S_\sigma.
\]

One obtains

\[
W_{\mu\nu}^{spin} = \sum_q \left[ \frac{m_q}{2(Pq)M} \int d^2k_\perp dy \Delta f_q(y,k^2) \right] \left\{ i\varepsilon_{\mu\nu\alpha\beta} \left[ \frac{g_{A_1} g_{A_2}}{Pq} ((S.q)q_\alpha P_\beta - (P.q)q_\alpha S_\beta) + (g_{A_1} g_{A_2} + g_{V_1} g_{V_2}) q_\alpha S_\beta \right] + g_{V_1} g_{V_2} \left[ 2x P_\mu S_\nu - (S.q)g_{\mu\nu} \right] + g_{A_1} g_{V_2} \left[ 2x S_\mu P_\nu - (S.q)g_{\mu\nu} \right] \right\}.
\]

\( \Delta f_q(y,k^2) \) is independent from the nucleon spin, with \( k = x P + y q' + k_\perp \) and \( q' = q + x P \).

The projections on the structure function combinations for photon and charged current interactions are

\[
W_{\mu\nu}^{\gamma, spin} = i\varepsilon_{\mu\nu\alpha\beta} q_\alpha S_\beta \sum_q \left[ \frac{e^2 m_q}{2(Pq)M} \int d^2k_\perp dy \Delta f_q(y,k^2) \right] \]

and

\[
W_{\mu\nu}^{W^\pm, spin} = i\varepsilon_{\mu\nu\alpha\beta} \left[ \frac{S.q}{Pq} (q_\alpha P_\beta + q_\alpha S_\beta) - 2x P_\mu S_\nu + P_\nu S_\mu \right] \]

\[
+ g_{\mu\nu}(S.q) \sum_q \left[ \frac{m_q}{(P.q)M} \int d^2k_\perp dy \Delta f_q(y,k^2) \right].
\]

From (92) and (93)

\[
12x[(g_1 + g_2)^{ep} - (g_1 + g_2)^{en}] = g_3^{en} - g_3^{ep}
\]

follows, which differs from eq. (54).

In the covariant parton model, on the other hand, the distribution

\[
\Delta f(p \cdot k, k \cdot S, k^2) = - \left( \frac{n \cdot S}{M^2} \right) f_{\text{cov}}(p \cdot k, k^2),
\]

16
depends on the nucleon spin. As lined out in refs. [18] and [22] the structure functions $g_i(x)$ are represented by

\begin{align}
\tag{96}
g_1(x) &= \frac{a\pi x M^2}{8} \int_x^1 dy(2x - y)\tilde{h}(y), \\
g_2(x) &= \frac{a\pi x M^2}{8} \int_x^1 dy(2y - 3x)\tilde{h}(y), \\
g_1(x) + g_2(x) &= \frac{a\pi x M^2}{8} \int_x^1 dy(y - x)\tilde{h}(y), \\
g_3(x) &= \frac{b\pi x^2 M^2}{2} \int_x^1 dy(y - x)\tilde{h}(y), \\
g_4(x) &= \frac{b\pi x^2 M^2}{4} \int_x^1 dy(2x - y)\tilde{h}(y), \\
g_5(x) &= \frac{b\pi x M^2}{8} \int_x^1 dy(2x - y)\tilde{h}(y),
\end{align}

where $a = g_{A_1}g_{A_2} + g_{V_1}g_{V_2}$, $b = g_{V_1}g_{A_2} + g_{V_2}g_{A_1}$, $y = x + k_{\perp}^2/(xM^2)$ and

\[\tilde{h}(y) = \int dk^2 \Delta \tilde{j}^{\text{cov}}(y, k^2).\] 

From these relations eq. (84) is obtained. This example clearly demonstrates, how carefully calculations in partonic approaches have to be performed to obtain correct results for the polarized structure functions contributing to $W^{\perp}_{\mu\nu}$. 
Table 2: A comparison of different structure function relations (twist 2) derived in the literature with the results obtained in the local operator product expansion in section 5. The signs in the last column mark agreement or disagreement.

|   | sum rule                                                                                                                                                                                                                                                                                                                                 | ref.      | \( m_q = 0 \) |
|---|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|---------------|
| 1 | \( g_4 = 2xg_5 \)                                                                                                                                                                                                                                                                                                                      | [11, 10, 16, 23] | +             |
| 2 | \( 12x[(g_1 + g_2)^{ep} - (g_1 + g_2)^{en}] = g_3^{en} - g_3^{ep} \)                                                                                                                                                                                                           | [11, 12, 9]    | -             |
| 3 | \( 12x[g_2^{ep} - g_2^{en}] = (g_3 - 2g_4)^{en} - (g_3 - 2g_4)^{ep} \)                                                                                                                                                                                                          |           | -             |
| 4 | \( 12x(g_1^{ep} - g_1^{en}) = g_4^{en} - g_4^{ep} \)                                                                                                                                                                                                                             | [12]       | +             |
| 5 | \( 3 \int_0^1 dx(g_1^{ep} - g_1^{en}) - \int_0^1 dx(g_1^{ep} + g_1^{en}) = -\frac{1}{6}g_8^A \)                                                                                                                                                                         |           | -             |
| 6 | \( 6 \int_0^1 dx(g_2^{ep} - g_2^{en}) - \int_0^1 dx(g_1^{ep} - g_1^{en}) = -g_A^* \)                                                                                                                                                                                   |           | +             |
| 7 | \( 12 \int_0^1 dx(g_2^{ep} - g_2^{en}) - \int_0^1 \frac{dx}{x}(g_4^{ep} - g_4^{en}) = -2g_A \)                                                                                                                                                                       |           | +             |
| 8 | \( 12x[g_1^{ep} - g_1^{en}] = g_3^{en} - g_3^{ep} \)                                                                                                                                                                                                                           |           | -             |
| 9 | \( \int_0^1 dx[(g_1 + g_2)^{ep} - (g_1 + g_2)^{en}] = g_A^* \)                                                                                                                                                                                                                  | [17]       | +             |
| 10| \( \int_0^1 \frac{dx}{x}[g_3^{ep} - g_3^{en}] = -g_A^* \)                                                                                                                                                                                                                   |           | +             |
| 11| \( \int_0^1 dxg_2^2 = 0 \)                                                                                                                                                                                                                                               | [24]       |               |
|   | sum rule                                      | ref. | $m_q = 0$ |
|---|---------------------------------------------|-----|----------|
| 12 | $\int_0^1 dx (g_1 + 2g_2)^{\nu_p-\nu_p} = 0$ | 13  | +        |
| 13 | $\int_0^1 dx (g_3 - 2xg_5)^{\nu_p+\nu_p} = 0$ |     | +        |
| 14 | $\int_0^1 dx (g_4 - g_3)^{\nu_p+\nu_p} = 0$ |     | +        |
| 15 | $\int_0^1 dx (g_5^{\nu_p} - g_5^{\nu_n}) = g_A$ | 15  | +        |
| 16 | $\int_0^1 dx \left[ (g_4 - g_3)^{\nu_p} - (g_4 - g_3)^{\nu_n} \right]/x = 0$ |     | -        |
| 17 | $\int_0^1 dx \left( g_3^{\nu_p} - g_3^{\nu_n} \right)/x = 2g_A$ |     | -        |
| 18 | $g_4 - g_3 = 2xg_5$ | 15  | -        |
| 19 | $\int_0^1 dx x^n (n-7/n+1)g_4 + 2g_3 = 0$ | 16  | -        |
| 20 | $g_3 = 2xg_5$ | 15  | -        |
| 21 | $g_3 = g_4$ |     | -        |
| 22 | $g_2^\gamma = g_2^\gamma Z = 0$ |     | -        |
| 23 | $g_1^{W^\pm} = -2g_2^{W^\pm}$ |     | -        |
Recently, a sum rule for the valence part of the structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$

$$\int_0^1 dx (g_1(x) + 2g_2(x)) = 0 \quad (98)$$

was discussed in ref. [27] in the context of a ‘field-theoretic’ framework.

Since the valence parts $g_1^V(x)$ and $g_2^V(x)$ cannot be isolated for electromagnetic interactions from the complete structure functions, a formulation of eq. (98) with the help of the local operator product expansion is thus not straightforward. On the other hand, one may consider eqs. (61) and (62) from which

$$\int_0^1 dx [g_1^V(x) + 2g_2^V(x)] = 0 \quad (99)$$

This sum rule was found firstly in ref. [13] for a specific flavor combination.

| $m_q = 0$ | sum rule | ref. |
|-----------|----------|-----|
| $\int_0^1 dx (g_3 - g_4)^{(\nu + \bar{\nu}), \gamma, Z} = 0$ | 24 | 23 |
| $\int_0^1 dx g_2^{\nu + \bar{\nu}} = 0$ | 25 |  |
| $\int_0^1 dx (g_1 + 2g_2)^{W^+ - W^-} = 0$ | 26 |  + |
| $\int_0^1 dx (g_3 - 2xg_5)^{(\nu + \bar{\nu}), \gamma, Z} = 0$ | 27 |  + |
| $\int_0^1 dx \frac{g_{3p}^\nu - g_{3n}^\nu}{x} = 4g_A$ | 28 |  this paper + |
| $24x[(g_1 + g_2)^{ep} - (g_1 + g_2)^{en}] = g_3^{en} - g_3^{ep}$ | 29 |  + |
| $24x[g_{2p}^e - g_{2n}^e] = (g_3 - 2g_4)^{en} - (g_3 - 2g_4)^{ep}$ | 30 |  + |
| $\int_0^1 dx (g_1^{ep} + g_1^{en}) - \frac{2}{9} \int_0^1 dx (g_1^{ep} + g_1^{en}) = \frac{1}{18}g_A^8$ | 31 |  + |

\[7\] This sum rule was found firstly in ref. [3] for a specific flavor combination.
follows for the charged current case. It is easily seen that the left hand side of eq. (99) includes only valence quark contributions. We may even rewrite eq. (99) for individual quark flavors separately

\[ \int_0^1 dx x^n (g_1^{Vq}(x, Q^2) + 2g_2^{Vq}(x, Q^2)) = \frac{e_q^2 (n d_n^{Vq} - (n - 1) a_n^{Vq})}{4(n + 1)}, \quad n = 1, 3..., \quad (100) \]

where \( d_n^{Vq}, a_n^{Vq} \) are the valence parts of the corresponding matrix elements.

For the first moment of this equation one obtains

\[ \int_0^1 dx (g_1^{Vq}(x, Q^2) + 2g_2^{Vq}(x, Q^2)) = \frac{e_q^2}{8} d_1^{Vq}. \quad (101) \]

The right hand side of eq. (101) vanishes in the case of massless quarks, because

\[ \langle PS|\bar{q}(\gamma_\beta \gamma_5 D^\mu - \gamma_\mu \gamma_5 D^\beta)q|PS \rangle = d_1^{Vq}(S^\beta P^\mu - S^\mu P^\beta) = m_q \langle PS|\bar{q}i\gamma_5 \sigma^{\beta\mu} q|PS \rangle. \quad (102) \]

Therefore eq. (98) can be derived in the operator product expansion.

We mention that the local operator product expansion for the valence parts of the \( g_1 \) and \( g_2 \) contains only the odd moments of the structure functions, whereas the operator product expansion for the complete structure functions \( g_1(x, Q^2) \) and \( g_2(x, Q^2) \) (56), (57) concerns the even moments.

The matrix element in (102) is related to the first moment of the structure function \( h_1(x) \) [28]

\[ \langle PS|\bar{q}i\gamma_5 \sigma^{\beta\mu} q|PS \rangle = \frac{\delta q}{M} (P^\beta S^\mu - P^\mu S^\beta), \quad (103) \]

where

\[ \delta q = \int_0^1 dx (h_1^{q}(x) - \bar{h}_1^{q}(x)), \quad (104) \]

and \( h_1^{q}(x), \bar{h}_1^{q}(x) \) are the quark and antiquark transversity functions, respectively, which can be measured in the Drell-Yan process.

### 7 Conclusions

We have derived the twist 2 and twist 3 contributions to the polarized structure functions in lowest order QCD for the general case of both neutral and charged current electroweak interactions. The results were obtained using the local operator product expansion. The twist 2 parts of the five structure functions per current combination are not independent but related by three linear operators, the Wandzura–Wilczek relation, a relation by Dicus and a new integral relation. A new relation between twist 3 contributions to the structure function \( g_2(x, Q^2) \) and \( g_3(x, Q^2) \) was derived. It was shown, that a relation between the valence contributions of the structure functions \( g_1(x, Q^2) \) and \( g_2(x, Q^2) \), eq. (98), can be derived using the local operator product expansion.

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