WEAK NULL SINGULARITIES IN GENERAL RELATIVITY

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ABSTRACT. We construct a class of spacetimes (without symmetry assumptions) satisfying the vacuum Einstein equations with singular boundaries on two null hypersurfaces intersecting in the future on a 2-sphere. The metric of these spacetimes extends continuously beyond the singularities while the Christoffel symbols fail to be square integrable in a neighborhood of any point on the singular boundaries. The construction shows moreover that the singularities are stable in a suitable sense. These singularities are stronger than the impulsive gravitational spacetimes considered by Luk-Rodnianski and are conjecturally the generic type of singularity in the interior of black holes arising from gravitational collapse.

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1. Introduction

In this paper, we study the existence and stability of weak null singularities in general relativity without symmetry assumptions. More precisely, a weak null singularity is a singularity on a null hypersurface in a spacetime \((\mathcal{M}, g)\) solving the Einstein equations

\[ \text{Ric}_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \]

such that the Christoffel symbols blow up and are not square integrable. This can be interpreted as a “terminal” singularity of the spacetime as it cannot be made sense of as a weak solution\(^1\) to the Einstein equations along the singular boundary. While the singularity is sufficiently strong to be “terminal”, the class of singularities that we consider is at the same time sufficiently weak such that the metric in an appropriate coordinate system is continuous up to the boundary.

The study of weak null singularities began with the attempts to understand the (in)stability of the Cauchy horizon in the black hole interior of Reissner-Nordström spacetimes. Reissner-Nordström spacetimes are the unique two-parameter family of asymptotically flat (with two ends), spherically symmetric, static solutions to the Einstein-Maxwell equations. Their Penrose diagrams\(^2\) are given by Figure 1. As seen in the Penrose diagram, the Reissner-Nordström solution possesses a smooth Cauchy horizon \(\mathcal{C}H^+\) in the interior of the black hole such that the spacetime can be extended as a smooth solution non-uniquely to the Einstein-Maxwell system. This feature is also shared by the Kerr family of solutions to the vacuum Einstein equations, which\(^3\) can also be depicted by a Penrose diagram given by Figure 1. According to the strong cosmic censorship conjecture (see Section 1.1 below), the Reissner-Nordström and Kerr spacetimes are expected to be non-generic and the smooth Cauchy horizons are expected to be unstable.

In a seminal work, Dafermos \cite{7}, \cite{8} showed that for a spacetime solution to the spherically symmetric Einstein-Maxwell-real scalar field system, if an appropriate upper and lower bound for the scalar field is assumed on the event horizon, then in a neighborhood of timelike infinity, the black hole terminates in a weak null singularity. The necessary upper bound was shown to hold for non-extremal black hole spacetimes arising from asymptotically flat initial data by Dafermos-Rodnianski \cite{10}. In particular, this implies that near timelike infinity, the terminal boundary of the Cauchy development does not contain a spacelike portion.

In a more recent work \cite{9}, Dafermos showed that if, in addition to assuming the two black hole exterior regions settle to Reissner-Nordström with appropriate rates, the initial data are moreover globally close to that of Reissner-Nordström, then the maximal Cauchy development of the data possesses the same Penrose diagram as Reissner-Nordström. In particular, the spacetime terminates in a global bifurcate weak null singularity and the singular boundary does not contain any spacelike portion.

The works \cite{7}, \cite{8} was in part motivated by the physics literature on the instability of Cauchy horizons, weak null singularities and the strong cosmic censorship conjecture. It will be discussed below in Section 1.1.

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\(^1\)One can define a weak solution to the Einstein equations by requiring \( \int_{\mathcal{M}} \text{Ric}(X,Y) + \frac{1}{2} R g(X,Y) - T(X,Y) dVol = 0 \) in the weak sense for all compactly supported smooth vector fields \(X\) and \(Y\). After integration by parts, the minimal regularity required for the spacetime for this to be defined is that the Christoffel symbols are square-integrable. See the discussion in the introduction of \cite{5}.

\(^2\)for \(0 < Q < M\)

\(^3\)for \(0 < |a| < M\)
While the works of Dafermos [7], [8], [9] are restricted to the class of spherically symmetric spacetimes, they nonetheless suggest the genericity of weak null singularities in the black hole interior. In particular, they motivate the following conjecture for the vacuum Einstein equations

\[ \text{Ric}_{\mu\nu} = 0 : \] (1)

Conjecture 1. (1) Suppose the domain of outer communication of a solution \((M, g)\) to the vacuum Einstein equations (1) settles to the exterior of a subextremal Kerr solution (with \(0 < a < M\)) with a sufficiently fast polynomial rate, then the maximal Cauchy development of \((M, g)\) possesses a null boundary “emanating from timelike infinity \(i^+\)” through which the spacetime is extendible with a continuous metric. Moreover, this piece of null boundary is generically a weak null singularity with non-square-integrable Christoffel symbols.

(2) (Ori, see discussion in [9]) If, in addition, the data for \((M, g)\) is globally a small perturbation of two-ended Kerr initial data (with \(0 < a < M\)), then the maximal Cauchy development possesses a global bifurcate future null boundary \(\partial M\). Moreover, for generic such perturbations of Kerr, \(\partial M\) is a global bifurcate weak null singularity which intersects every futurely causally incomplete geodesic.

If Conjecture 1 is true, then in particular there exist local stable weak null singularities for the vacuum Einstein equations without symmetry assumptions. We show in this paper that there is in fact a large class of such singularities, parameterized by initial data. More specifically, we solve a characteristic initial value problem with singular initial data and construct a class of stable bifurcate weak null singularities.

To motivate the strength of the singularity considered in this paper, we first recall the strength of the spherically symmetric weak null singularities in a neighborhood of Reissner-Nordström studied in [8]. The instability of the Reissner-Nordström Cauchy horizon is in fact already suggested by a linear analysis (see [21], [4]). For a spherically symmetric solution to the linear wave equation which has a polynomially decaying (in the Eddington-Finkelstein coordinates) tail along the event horizon, there is a singularity in a \((C^0)\)-regular coordinate...
system near the Cauchy horizon of the strength

$$|\partial_u \phi| \sim (u_* - u)^{-1} \log \left( \frac{e}{u_* - u} \right)^{-p},$$

(2)

for some $p > 1$ as $u \to u_*$. In particular, along an outgoing null curve, $\partial_u \phi$ is integrable but not $L^q$-integrable for any $q > 1$. In the spacetimes constructed by Dafermos [7], [8], it was shown moreover that even in the nonlinear setting, $\partial_u \phi$ is also singular but remains integrable. A more precise analysis will show that in fact the spherically symmetric scalar field in the nonlinear setting of [8] also blows up at a rate given by (2).

Returning to the problem of constructing stable weak null singularities in vacuum, our construction is based on solving a characteristic initial value problem with singular data. We will in fact construct spacetimes not only with one weak null singularity, but instead contain two weak null singularities terminating at a bifurcate sphere. More precisely, the data on the initial characteristic hypersurface $H_0$ (resp. $H_0$) is determined by the traceless part of the null second fundamental form $\hat{\chi}$ (resp. $\hat{\chi}$). We consider singular initial data satisfying in particular

$$|\hat{\chi}| \sim (u_* - u)^{-1} \log \left( \frac{e}{u_* - u} \right)^{-p}, \quad \text{for some } p > 1,$$

and

$$|\hat{\chi}| \sim (u_* - u)^{-1} \log \left( \frac{e}{u_* - u} \right)^{-p}, \quad \text{for some } p > 1.$$

This singularity is consistent with the strength of the weak null singularities in (2).

The following is the main result of this paper:

**Theorem 1.** [Main theorem, first version] For a class of singular characteristic initial data with the singular profile as above (see precise requirements on the data in Section 1.3) and for $\epsilon$ sufficiently small, there exists a unique smooth spacetime $(\mathcal{M}, g)$ endowed with a double null foliation $(u, u)$ in $0 \leq u < u_*, 0 \leq u < u_*$, where $u_*, u_* \leq \epsilon$. Associated to $(\mathcal{M}, g)$, there exists a coordinate system $(u, u, \theta^1, \theta^2)$ such that the metric extends continuously to the boundary but the Christoffel symbols are not in $L^2$.

**Remark 1.** This class of stable local weak null singularities that we construct in particular provides the first construction of weak null singularities for the vacuum Einstein equations.$^5$

Theorem 1 allows singularities on both initial null hypersurface and is valid in the region where $u_*$ and $u_*$ are sufficiently small. In the context of the interior of black holes, this corresponds to the darker shaded region in Figure 2. The existence theorem clearly implies an existence result when the data are only singular on one of the initial null hypersurfaces. In that context, we can in fact combine the methods in this paper with that in [17] to show that the domain of existence can be extended so that only one of the characteristic length scales is required to be small. More precisely, we allow that data on $H_0$ such that

$$|\hat{\chi}| \sim (u_* - u)^{-1} \log \left( \frac{e}{u_* - u} \right)^{-p}, \quad \text{for some } p > 1,$$

on $0 \leq u < u_* \leq C$ and the data on $H_0$ are smooth on $0 \leq u \leq u_* \leq \epsilon$. Then for $\epsilon$ sufficiently small, the spacetime $(\mathcal{M}, g)$ remains smooth in $0 \leq u < u_*$, $0 \leq u < u_*$ (see for example the lightly shaded region in Figure 2). We will omit the details of the proof of this result.

$^5$We recall the Birkhoff’s Theorem which states that the only spherically symmetric vacuum spacetimes are the Minkowski and Schwarzschild solutions. Thus to construct stable examples of weak null singularities in vacuum, one necessarily works outside the class of spherically symmetric spacetimes.
Theorem 1, which proves the existence and stability of the conjecturally generic weak null singularities, can be viewed as a first step towards Conjecture 1. A next step is an analogue of [8] for the vacuum Einstein equations without symmetry assumptions, i.e., to solve the characteristic initial value problem inside the black hole with data prescribed on the event horizon that is approaching Kerr at appropriate rates. This requires an understanding of the formation of weak null singularities from smooth data on the event horizon. A full resolution of Conjecture 1.2, however, requires in addition an understanding of the decay rates of gravitational radiation along the event horizon for generic perturbations of Kerr spacetime. This latter problem is intimately tied to the problem of the nonlinear stability of Kerr spacetimes, which remains to be one of the most important and challenging open problems in mathematical general relativity. Nevertheless, significant progress has been made for the corresponding linear problem in the past decade. We refer the readers to the survey of Dafermos-Rodnianski [11] for more about this linear problem.

Figure 2. Domains of existence

The approach for the main theorem applies equally well to the Einstein-Maxwell-scalar field system without symmetry assumptions. Thus, we show that the weak null singularity of Dafermos [8], which arises from appropriately decaying data on the event horizon, is stable against non-spherically symmetric perturbations on the hypersurface Σ sufficiently far within the black hole region (see Figure 3).

1.1. Weak Null Singularities and Strong Cosmic Censorship Conjecture. The study of weak null singularities can be viewed in the larger context of Penrose’s celebrated strong cosmic censorship conjecture in general relativity. The conjecture states that for generic asymptotically flat initial data for “reasonable” Einstein-matter systems, the maximal Cauchy development is future inextendible as a suitably regular Lorentzian manifold. This would guarantee general relativity to be a deterministic theory.

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6This can be easily seen by decomposing the Maxwell field and the gradient of the scalar field in terms of the null frame below. The components in this decomposition obey equations that can be put in the same schematic form as in Section 2.4. Therefore, the Maxwell field and the scalar field and their derivatives satisfy estimates similar to those for the Ricci coefficients and curvature components.
As pointed out above, the Kerr and Reissner-Nordström families of solutions (of the Einstein vacuum and Einstein-Maxwell equations respectively) have maximal Cauchy developments that are extendible as larger smooth spacetimes unless the angular momentum or the charge vanishes. This is connected with the existence of a smooth Cauchy horizon in the black hole interior such that the spacetime can be extended beyond as a smooth solution. According to the strong cosmic censorship conjecture, this is expected to be non-generic.

On the other hand, the situation for the Schwarzschild spacetime is more preferable from the point of view of the deterministic nature of the theory. The maximal development of the Schwarzschild spacetime terminates with a spacelike singularity at which the Hawking mass and the curvature scalar invariants blow up. In particular, the spacetime cannot be extended in $C^2$.

The early motivation for the strong cosmic censorship conjecture besides the desirability of a deterministic theory is a linear heuristic argument by Penrose [21] suggesting that the Reissner-Nordström Cauchy horizon is unstable. This was also confirmed by the numerical work by Simpson and Penrose [25]. It is thus conjectured that a small global perturbation would lead to a singularity in the interior of the black hole in such a way that the maximal Cauchy development is future inextendible.

However, the nature of the singular boundary in the interior of black holes was not well-understood until the first study of weak null singularity carried out by Hiscock [13]. In an attempt to understand the instability of the Reissner-Nordström Cauchy horizon, he considered the Vaidya model allowing for a self-gravitating ingoing null dust. In this model, an explicit solution can be found and he showed that various components of the Christoffel symbols blow up. This, however, was called a whimper singularity as the Hawking mass and the curvature scalar invariants remain bounded.

In subsequent works, Poisson-Israel [23], [24] added an outgoing null dust to the model considered by Hiscock. While explicit solutions were not available, they were able to deduce

\footnote{In particular, it was believed that a perturbation of the Reissner-Nordström Cauchy horizon would lead to a Schwarzschild type singularity.}
that the second outgoing null dust would cause the Hawking mass to blow up at the null singularity. It was then thought of as a stronger singularity than that of Hiscock.

However, from the point of view of partial differential equations, it is more natural to view this singularity at the level of non-square-integrability of the Christoffel symbols, which is exactly the threshold such that the spacetime cannot be defined as a weak solution to the Einstein equations. From this perspective, the singularity of Poisson-Israel is as strong as that of Hiscock and both singularities can be viewed as terminal boundaries for the spacetimes in question.

While the Christoffel symbols blow up Cauchy horizon, one can also think that the Cauchy horizon is “stable” in the sense that no singularity arises before the “original Cauchy horizon”. In particular, there is no spacelike portion of the singular boundary in a neighborhood of timelike infinity. This is thus contrary to the case of the Schwarzschild spacetime. Heuristic arguments moreover suggest that these null singularities are stable [12]. This weak null singularity picture has been further explored and justified in many numerical works (see [1], [2], [3]).

As we described before, the aforementioned picture of the interior of black holes was finally established by Dafermos in the context of the spherically symmetric Einstein-Maxwell-scalar field system [7]. This is the main motivation for our present work in which we initiate the study of weak null singularities of similar strength in vacuum without any symmetry assumptions.

1.2. Comparison with Impulsive Gravitational Waves. As pointed out by Dafermos [9], the weak null singularities that we consider in this paper share many similarities with impulsive gravitational waves. The latter are vacuum spacetimes admitting null hypersurfaces which support delta function singularities in the Riemann curvature tensor. Explicit examples were first constructed by Penrose [22], Khan-Penrose [14] and Szekeres [26]. In these spacetimes, while the Christoffel symbols are not continuous, they remain bounded. Therefore, in contrast with the weak null singularities that we consider here, these impulsive gravitational waves are not terminal singularities. In fact, the solution to the vacuum Einstein equation extends beyond the singularity and is smooth except across the singular hypersurface. Nevertheless, both scenarios represent singularities propagating along null hypersurfaces and from a mathematical point of view, the proofs of the existence theory for these singularities share many common features.

In recent joint works with Rodnianski [18], [19], we initiated the rigorous mathematical study for general impulsive gravitational waves without symmetry assumptions. We constructed the impulsive gravitational waves via solving the characteristic initial problem such that the initial data admit curvature delta singularities supported on an embedded 2-sphere. One of the new ideas in the proof is the use of renormalized energy estimates for the curvature components, i.e., instead of controlling the spacetime curvature components in $L^2$, we subtract off an $L^\infty$ correction from some curvature components. This allowed us to derive a closed system of $L^2$ estimates which is completely independent of the singular curvature components.

In [19], when the interaction of impulsive gravitational waves was studied, we also extended the analysis to include a class of spacetimes such that when measured in the worst direction, the Christoffel symbols are merely in $L^2$. While these spacetimes are rough, we proved an existence and uniqueness theorem for them and showed that the spacetime can be extended beyond the singularities. In view of the fact the square-integrability of Christoffel symbols
is the minimal requirement for the definition of a weak solution to the Einstein equations, this result is in fact sharp.

By contrast, the weak null singularities considered in this paper are spacetimes such that the Christoffel symbols are not in $L^2$. Even though these are terminal singularities such that there cannot be an existence theory beyond the singularity, the theory developed in [18], [19] can be extended to control the spacetime up to the singularity. Moreover, our main theorem, which allows for two weak null singularities terminating at their intersection, can be viewed as an extension of the result in [19] on the interaction of two impulsive gravitational waves. In particular, the renormalized energy of [18], [19] plays an important role in the proof of our main theorem. However, even after renormalization, the renormalized curvature is still singular (i.e., not in $L^2$) and has to be dealt with using an additional weighted estimate.

1.3. Description of the Main Results. Our setup is the characteristic initial value problem with initial data given on two null hypersurfaces $H_0$ and $H_{0}$ intersecting at a 2-sphere $S_{0,0}$ (see Figure 4). We will follow the general notations in [15], [5], [16].

![Figure 4. The Basic Setup](image)

We introduce a null frame $\{e_1, e_2, e_3, e_4\}$ adapted to a double null foliation (see Section 2.1). Decompose the Riemann curvature tensor with respect to the null frame $\{e_1, e_2, e_3, e_4\}$:

\[
\alpha_{AB} = R(e_A, e_4, e_B, e_4), \quad \alpha_{AB} = R(e_A, e_3, e_B, e_3),
\]

\[
\beta_A = \frac{1}{2} R(e_A, e_4, e_3, e_4), \quad \beta_A = \frac{1}{2} R(e_A, e_3, e_3, e_4),
\]

\[
\rho = \frac{1}{4} R(e_4, e_3, e_4, e_3), \quad \sigma = \frac{1}{4} R(e_4, e_3, e_4, e_3)
\]

We define also the Gauss curvature of the 2-spheres associated to the double null foliation to be $K$. Define also the following Ricci coefficients with respect to the null frame:

\[
\chi_{AB} = g(D_A e_4, e_B), \quad \chi_{AB} = g(D_A e_3, e_B),
\]

\[
\eta_A = -\frac{1}{2} g(D_3 e_A, e_4), \quad \eta_A = -\frac{1}{2} g(D_4 e_A, e_3)
\]

\[
\omega = -\frac{1}{4} g(D_4 e_3, e_4), \quad \omega = -\frac{1}{4} g(D_3 e_4, e_3),
\]

\[
\zeta_A = \frac{1}{2} g(D_A e_4, e_3)
\]

Let $\hat{\chi}$ (resp. $\hat{\chi}$) be the traceless part of $\chi$ (resp. $\chi$).

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8In fact, we allow initial data to be in $L^p$ only for $p = 1$. 
The data on $H_0$ are given on $0 \leq u < u_*$ such that $\chi$ becomes singular as $u \to u_*$. Similarly, the data on $H_0^*$ is given on $0 \leq u < u_*$ such that $\chi$ becomes singular as $u \to u_*$. More precisely, let $f_1 : [0, u_*) \to \mathbb{R}$ be a function such that $f(x) \geq 0$ is decreasing and

$$\int_0^{u_*} \frac{1}{f_1(x)^2} dx < \infty.$$ (resp. let $f_2 : [0, u_*) \to \mathbb{R}$ be a function such that $f(x) \geq 0$ is decreasing and

$$\int_0^{u_*} \frac{1}{f_2(x)^2} dx < \infty.$$)

For example, $f_1$ can be taken to be $f_1(x) = (u_* - x)^{1/2} \log^{p}(\frac{e}{u_* - x})$ for $p > \frac{1}{2}$.

Our main theorem shows local existence for a class of singular initial data with

$$|\chi(0, u)| \lesssim f_1(u)^{-2}, \quad |\chi(u, 0)| \lesssim f_2(u)^{-2}.$$ We construct a (unique) solution $(M, g)$ to the vacuum Einstein equations in the region $u < u_*$, $u < u_*$, where $u_*$, $u_*$, $\epsilon$, and

$$\int_0^{2u_*} f_1(u)^{-2} du, \quad \int_0^{u_*} f_2(u)^{-2} du \leq \epsilon^2.$$ (3)

Here, $(u, u)$ is a double null foliation for $(M, g)$ and the metric $g$ takes the form

$$g = -2\Omega^2(du \otimes du + du \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du)$$

in the $(u, u, \theta^1, \theta^2)$ coordinate system (to be defined in Section 2.2). More precisely, we have

**Theorem 2.** Consider the characteristic initial value problem with data that are smooth on $H_0 \cap \{0 \leq u < u_*\}$ and $H_0^* \cap \{0 \leq u < u_*\}$ such that

- There exists an atlas such that in each coordinate chart with local coordinates $(\theta^1, \theta^2)$, the initial metric $\gamma_0$ on $S_{0,0}$ obeys

$$d \leq \det \gamma_0 \leq D,$$

and

$$\sum_{i \leq 6} |(\frac{\partial}{\partial \theta^i})^i \gamma_{BC}| \leq D.$$ $$

- The metric on $H_0$ and $H_0^*$ satisfies the gauge conditions

$$\Omega = 1, \quad b^A = 0.$$

- The Ricci coefficients on the initial hypersurface $H_0$ verify

$$\sum_{i \leq 5} \sup_u ||f_2^2(u) \nabla^i \chi||_{L^2(S_{0,u})} \leq D,$$

$$\sum_{i \leq 4} \sup_u ||\nabla^i \zeta||_{L^2(S_{0,u})} \leq D,$$

$$\sum_{i \leq 4} \sup_u ||\nabla^i tr \chi||_{L^2(S_{0,u})} \leq D.$$
• The Ricci coefficients on the initial hypersurface $H_0$ verify
\[
\sum_{i \leq 5} \sup_u \| f_1^2(u) \nabla^i \chi \|_{L^2(S_{u,0})} \leq D,
\]
\[
\sum_{i \leq 4} \sup_u \| \nabla^i \zeta \|_{L^2(S_{u,0})} \leq D,
\]
\[
\sum_{i \leq 4} \sup_u \| \nabla^i \text{tr} \chi \|_{L^2(S_{u,0})} \leq D.
\]

Then for $\varepsilon$ sufficiently small (depending only on $d$ and $D$) and $u_*$, $u_* \leq \varepsilon$, $\| f_2(u)^{-1} \|_{L^2_2}$, $\| f_1(u)^{-1} \|_{L^2_2} < \varepsilon$, there exists a unique spacetime solution $(M, g)$ to the vacuum Einstein equations \[1\] endowed with a double null foliation $(u, u)$ in $0 \leq u < u_*$ and $0 \leq u < u_*$. Moreover, the spacetime remains smooth in $0 \leq u < u_*$ and $0 \leq u < u_*$.

**Remark 2.** In the following, we will only prove a priori estimates for spacetimes arising from these initial data (see Theorem 5 below). The existence of spacetime and the propagation of regularity follows from standard arguments.

**Remark 3.** In order to simplify notations, we will omit the subscripts 1 and 2 in the weight functions $f_1$ and $f_2$. They can be inferred from whether $f$ is a function of $u$ or $u$.

**Remark 4.** In Section 4 we will construct a class of characteristic initial data which satisfies the assumptions of Theorem 2.

**Remark 5.** The assumptions of the main theorem imply the boundedness of weighted $L^2$ norms of the curvature components:
\[
\sum_{i \leq 3} \| f(u) \nabla^i \beta \|_{L^2_2(S_{0,u})} + \sum_{i \leq 4} \| f(u) \nabla^i (K, \hat{\sigma}) \|_{L^2_2(S_{0,u})} \leq \bar{D},
\]
and
\[
\sum_{i \leq 3} \| f(u) \nabla^i \beta \|_{L^2_2(S_{0,u})} + \sum_{i \leq 4} \| f(u) \nabla^i (K, \hat{\sigma}) \|_{L^2_2(S_{0,u})} \leq \bar{D},
\]
for some $\bar{D}$ depending only on $D$ and $d$, where $\sigma := \sigma + \frac{1}{2} \hat{\chi} \land \hat{\chi}$. These estimates for $\beta$, $\hat{\sigma}$ and $\beta$ follow immediately from the constraint equations (see \[9\] below) on the 2-spheres. The bound for $K$ follows after integrating the null Bianchi equations for $K$ on each of the initial null hypersurfaces \[9\].

While the weight $f$ in the spacetime norms allows the spacetime to be singular, the spacetime metric can be extended beyond the singular hypersurfaces $H_{u_*}$ and $\bar{H}_{u_*}$ continuously.

**Theorem 3.** The spacetime $(M, g)$ can be extended continuously up to and beyond the singular boundaries $\bar{H}_{u_*}$, $H_{u_*}$. Moreover, the induced metric and null second fundamental form on the interior of the limiting hypersurfaces $H_{u_*}$ and $H_{u_*}$ are regular. More precisely, for any fixed $U < u_*$, we have the following estimates on $H_{u_*}$ for the metric functions $\gamma$, $b$, $\Omega$
\[
\sum_{i \leq 4} \sup_{0 \leq u \leq U} \| (\frac{\partial}{\partial \theta})^i (\gamma, b, \Omega) \|_{L^2(S_{u,u_*})} \leq C_U,
\]
\[9\] Notice that it is precisely for the initial bound for $K$ that we require an extra derivative for $\chi$ on $H_0$ (and $\chi$ on $\bar{H}_0$) in the assumptions of the theorem. This is related to the intrinsic loss of derivatives for the characteristic initial value problem for second order hyperbolic systems (see \[20\]).
and the Ricci coefficients \( \dot{\chi}, \text{tr} \chi, \varpi, \eta, \eta \)
\[
\sum_{i \leq 3} \sum_{j \leq 1} \sup_{0 \leq u \leq U} \| \nabla^3 \nabla^i (\dot{\chi}, \text{tr} \chi, \varpi, \eta, \eta) \|_{L^2(S_u, u)} \leq C_U.
\]

Similar regularity statements hold on \( H_u \).

**Remark 6.** If we assume in addition that the higher angular derivatives of \( \chi \) is bounded in \( L^1_u L^\infty(S) \), then the metric and the second fundamental form also inherit higher regularity in the interior of \( H_u \). In particular, if all angular derivatives of \( \chi \) are bounded in \( L^1_u L^\infty(S) \), then the metric restricted to \( H_u \cap \{0 \leq u \leq U\} \) is smooth along the directions tangential to \( H_u \). Similar statements hold on \( H_u^\ast \). We will omit the details.

Moreover, we show that if initially the data are indeed singular, then \( H_u \) and \( H_u^\ast \) are terminal singularities of the spacetime in the following sense:

**Theorem 4.** If, in addition to the assumptions of Theorem 2, we also have the following for the initial data
\[
\int_0^{u^\ast} |\dot{\chi}(0, u)|^2 du = \infty
\]
along every null generator on \( H_0 \), then the Christoffel symbols in the coordinate system \((u, \theta^1, \theta^2)\) do not belong to \( L^2 \) in a neighborhood of any point on \( H_u \).

Similarly if the initial data
\[
\int_0^{u^\ast} |\dot{\chi}(u, 0)|^2 du = \infty
\]
along every null generator on \( H_0 \), then the Christoffel symbols in the coordinate system \((u, \theta^1, \theta^2)\) do not belong to \( L^2 \) in a neighborhood of any point on \( H_u \).

**Remark 7.** Theorem 4 guarantees that if we extend the spacetime metric continuously in the obvious differentiable structure given by the coordinate system \((u, \theta^1, \theta^2)\), the Christoffel symbols are non-square-integrable in the extension. In general, one can show that for every proper extension of the spacetime, the metric is not \( C^2 \). However, it is an open problem whether the spacetime admits any continuous extensions with square integrable Christoffel symbols.

### 1.4. Main Ideas of the Proof

All the known proofs of regularity for the Einstein equations without symmetry assumptions rely on \( L^2 \) estimates on the metric and its derivatives or the Riemann curvature tensor and its derivatives. Let us denote schematically by \( \Gamma \) a general Ricci coefficient and by \( \Psi \) a general curvature component decomposed with respect to a null frame adapted to the double null foliation. In the double null foliation gauge (see for example, [15], [5]), the standard approach to obtain a priori bounds is to couple the \( L^2 \) estimates for the curvature components
\[
\int_{H} \Psi^2 + \int_{H} |\Psi|^2 \leq \text{Data} + \int \int \Gamma \Psi \Psi
\]
with the estimates for the Ricci coefficients obtained using the transport equations
\[
\nabla_3 \Gamma = \Psi + \Gamma \Gamma,
\]
\[
\nabla_4 \Gamma = \Psi + \Gamma \Gamma.
\]
However, in the setting of two weak null singularities, none of the spacetime curvature components $\alpha, \beta, \rho, \sigma, \beta, \alpha$ are in $L^2$.

Nevertheless, while these curvature components are singular, the nature of their singularity is specific. More precisely, while $\rho$ and $\sigma$ are not in $L^2$, one can subtract off a correction that is quadratic in $\Gamma$ in such a way that the renormalized curvature components $\rho - \Gamma\Gamma, \sigma - \Gamma\Gamma$ belong to $L^2$. Moreover, this renormalization allows us to remove all appearances of $\rho$ and $\sigma$ in the estimates and so that we do not have to deal with the singularities of $\alpha$ and $\alpha$! On the other hand, $\beta$ and $\beta$ are singular only towards the singular boundary $H_{u_+}^*$ and $H_{u_*}$ respectively. We therefore introduce degenerate $L^2$ norms that incorporate these singularities. We will explain the renormalization and the degenerate estimates in more detail below.

1.4.1. Renormalized Energy Estimates. As described above, a main ingredient of the proof of the main theorem is the renormalized energy estimates introduced in \cite{18, 19} in the study of impulsive gravitational waves. This can be seen as follows. For the class of weak null singularities that we consider, while the $L^2$ derivative of the spacetime metric blows up, the metric restricted to the 2-sphere remains regular in the angular directions. Since the Gauss curvature $K$ is intrinsic to the 2-spheres, it remains bounded. Recall now the Gauss equation:

\[ K = -\rho + \frac{1}{2}\hat{\chi} \cdot \hat{\chi} - \frac{1}{4}\text{tr}\chi \text{tr}\chi. \]

Since at the weak null singularity $u = u_*$, tr$\chi$ and $\hat{\chi}$ blow up, we conclude that $\rho$ also blows up. In view of this, we estimate the Gauss curvature instead of the spacetime curvature component $\rho$.

Indeed, we can see that the Gauss curvature $K$ satisfies equations such that the right hand side contains terms that are less singular than the terms in the corresponding equation for $\rho$. More precisely, for the curvature component $\rho$, we have the Bianchi equation

\[ \nabla_4 \rho + \frac{3}{2}\text{tr}\chi \rho = \text{div} \beta - \frac{1}{2}\hat{\chi} \cdot \alpha + ..., \]

where $...$ denotes terms that are products of Ricci coefficients and curvature components and are integrable along null geodesics. Notice now that we have the null structure equations

\[ \nabla_4 \hat{\chi} = -\text{tr}\chi \hat{\chi} - 2\omega \hat{\chi} - \alpha + ..., \]

\[ \nabla_4 \hat{\chi} = -\frac{1}{2}\text{tr}\chi \hat{\chi} + 2\omega \hat{\chi} - \frac{1}{2}\text{tr}\chi \hat{\chi} + ..., \]

\[ \nabla_4 \text{tr}\chi = -\frac{1}{2}(\text{tr}\chi)^2 - |\hat{\chi}|^2 - 2\omega \text{tr}\chi, \]

and

\[ \nabla_4 \text{tr}\chi = -\frac{1}{2}\text{tr}\chi \text{tr}\chi + 2\omega \text{tr}\chi + 2\rho - \hat{\chi} \cdot \hat{\chi} + ... \]

The first two equations give

\[ \frac{1}{2} \nabla_4 (\hat{\chi} \cdot \hat{\chi}) = -\frac{3}{4}\text{tr}\chi \hat{\chi} \cdot \hat{\chi} - \frac{1}{4}\text{tr}\chi |\hat{\chi}|^2 - \frac{1}{2} \hat{\chi} \cdot \alpha, \]

and the last two equations give

\[ \frac{1}{4} \nabla_4 (\text{tr}\chi \text{tr}\chi) = -\frac{1}{4}\text{tr}\chi (\text{tr}\chi)^2 + \frac{1}{2}\text{tr}\chi \rho - \frac{1}{4}\text{tr}\chi \hat{\chi} \cdot \hat{\chi} - \frac{1}{4}\text{tr}\chi |\hat{\chi}|^2. \]
Substituting the above into the equation for $\rho$, we get that
\[ \nabla_4 K + \text{tr} \chi K = \text{div} \beta + ... \]
where the term containing $\alpha$ drops off and all the terms that are quadratic in $\text{tr} \chi, \dot{\chi}$ and $\omega$ cancel! The right hand side of the equation now does not contain any term that is non-integrable in the $u$ direction.\(^{10}\)

In a similar fashion, the equation for $\sigma$ has a term on the right hand side containing $\alpha$ and is not integrable in the $u$ direction:
\[ \nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma = -\text{div} \star \beta + \frac{1}{2} \dot{\chi} \cdot \alpha + ... \]
Instead, if we consider the renormalized curvature component
\[ \check{\sigma} := \sigma + \frac{1}{2} \dot{\chi} \wedge \dot{\chi}, \]
we see that it satisfies the equation
\[ \nabla_4 \check{\sigma} + \frac{3}{2} \text{tr} \chi \sigma = -\text{div} \star \beta + ... \]
where all the terms on the right hand side are integrable in the $u$ direction.

An additional consequence of the renormalization is that we have completely removed the appearances of the curvature component $\alpha$ in the equations. In fact, as in [18], [19], this allows us to derive a set of estimates for the renormalized curvature component without requiring any information on the curvature component $\alpha$.

Moreover, when considering the equations for $\nabla_3 K$ and $\nabla_3 \check{\sigma}$ for the renormalized curvature components, one sees that $\alpha$ does not appear and all the terms are integrable in the $u$ direction. Therefore, although $\alpha$ or $\alpha^\perp$ can be very singular near one of the singular boundaries, we do not need to derive any estimates for them!

1.4.2. Degenerate $L^2$ Estimates. Since the renormalization above deals with the singularity in the $\rho$ and $\sigma$ components and avoids any information on $\alpha$ and $\alpha^\perp$, it remains to derive appropriate $L^2$ estimates for $\beta$ and $\beta^\perp$.

The main observation is that while $\beta$ and $\beta^\perp$ are both singular and fail to be in $L^2$, their singularity can be captured quantitatively. Consider the curvature component $\beta$. Since $\text{tr} \chi$ and $\dot{\chi}$ blows up with the rate $f(u)^{-2}$, in view of the Codazzi equations, $\beta$ also blows up as $f(u)^{-2}$ and is only in $L_2^1$ but not in $L_p^p$ for any $p > 1$. Nevertheless, the assumptions on the initial data allow us to control $f(u) \beta$ in $L_2^2$. We will thus incorporate this blow up in the norms and will be able to still use an $L^2$ based estimate.

The energy estimates will be obtained directly from two sets of Bianchi equations instead of using the Bel-Robinson tensor. Notice that since the energy estimates for $K, \check{\sigma}$ are obtained either together with that for $\beta$ or that for $\beta^\perp$, even though $K$ and $\check{\sigma}$ are regular, their energy estimates degenerate. Therefore, at the highest level of derivatives, we have to content with the weaker $L^2$ estimates for these curvature components.

\(^{10}\) The can be compared with the renormalization introduced in [18] and [19], where we estimated $\check{\rho} = \rho - \frac{1}{2} \dot{\chi} \cdot \dot{\chi}$ instead of $\rho$. Whereas the renormalization using $\check{\rho}$ allows one to eliminate $\alpha$ in the estimates, it nonetheless introduces a term $\frac{1}{4} \text{tr} \chi |\dot{\chi}|^2$, which is not integrable in the $u$ direction in the setting of the present paper. Instead, by studying the equation for $K$, we see none of these terms which are quadratic in $\text{tr} \chi, \dot{\chi}$ or $\omega$! This fact can also be derived directly by considering the equations for $\nabla_4 K$ using the intrinsic definition of the Gauss curvature.
A potentially more serious challenge is that the introduction of the degenerate weights in $u$ and $\underline{u}$ would create terms that cannot be estimated by the energy estimates themselves. Nevertheless, it can be shown that these uncontrollable terms possess a good sign!

1.4.3. Estimates for the Ricci Coefficients. As indicated above, the Ricci coefficients enter as error terms in the energy estimates. Thus, to close all the estimates, we need to control the Ricci coefficients $\Gamma$ by using the transport equations which in turn have the curvature components in the source terms. Since the various Ricci coefficients have different singular behavior, we separate them according to the bounds that they obey. More precisely, denote by $\psi_H$ the components that behave like $f(u)^{-2}$ as $u \to u_*$; by $\psi_H$ the components that behave like $f(u)^{-2}$ as $u \to u_*$; and by $\psi$ the components that are bounded.

For the singular Ricci coefficients $\psi_H$, we have the following schematic transport equations:

$$\nabla_4 \psi_H = K + \nabla \psi + \psi \psi + \psi_H \psi_H.$$

The first three terms on the right hand side of this equation are bounded while the last term is singular. Nevertheless, the singularity of $\psi_H$ still allows it to be controlled in $L^1$ along the $e_3$ direction. Thus, this equation can be integrated to show that the initial (singular) bounds for $\psi_H$ can be propagated. It is important that the terms of the form $\psi_H \psi_H$ do not appear in the equations. A similar structure can also be seen in the equation for the other singular Ricci coefficients $\psi_H$, which takes the form

$$\nabla_3 \psi_H = K + \nabla \psi + \psi \psi + \psi_H \psi_H.$$

For the regular Ricci coefficients $\psi$, we have transport equations of the form

$$\nabla_4 \psi = \beta + \psi \psi_H, \quad \text{or} \quad \nabla_3 \psi = \beta + \psi \psi_H.$$

The bounds that we prove show that the right hand side is integrable and therefore $\psi$ remains bounded. For example, in the $\nabla_4$ equation, it is important that we do not have terms of the form $\psi_H \psi_H, \psi_H \psi_H$ and $\psi_H \psi_H$, which are not uniformly bounded after integrating along the $e_4$ direction.

1.4.4. Null Structure in the Energy Estimates. A priori, the degenerate $L^2$ estimates that we introduce may not be sufficient to control the error terms. Nevertheless, the vacuum Einstein equations possess a remarkable null structure which allows one to close the estimates using only the degenerate $L^2$ estimates.

For example, in the energy estimates for the singular component $\beta$, we have

$$\|f(u)\beta\|^2_{L^2(H)} \leq \text{Data} + \|f(u)(\beta \psi_H \beta + \beta \psi_H \beta + \beta \psi K)\|_{L^1_t L^1(x)}.$$

To estimate the first term, it suffices to note that $\psi_H$, while singular, can be shown to be small after integrating along the $u$ direction. Thus the first term can be controlled using Gronwall’s inequality. For the second term, since the singularity for $\beta$ has the same strength as that for $\psi_H$ (and similarly the singularity for $\beta$ has the same strength as that for $\psi_H$), the singularity in this term is similar to that in the first term and can also be bounded. The final term is less singular since $\psi$ and $K$ are both uniformly bounded. Notice that if other combinations of curvature terms and Ricci coefficients such as $\beta \psi_H \beta \psi$ or $\beta \psi_K$ appear in the error terms, the degenerate energy will not be strong enough to close the bounds!

\footnote{Although as pointed out before, the highest derivative estimates for $K$ in the energy norm suffer a loss as one approaches the singular boundaries, this term can nevertheless be controlled.}
In order to close all the estimates, we need to commute also with higher derivatives. As in [18], [19], we will only commute with angular covariant derivatives. These commutations will not introduce terms that are more singular. Moreover, the null structure of the estimates indicated above is also preserved under these commutations.

Similar to [18], [19], the renormalization introduces terms in the energy estimates such that the Ricci coefficients have the highest order derivatives and cannot be estimated via transport equations. These terms are controlled via elliptic estimates on the spheres. A form of null structure similar to that described above also makes an appearance in these elliptic estimates, allowing all the bounds to be closed.

1.5. Outline of the Paper. We end the introduction with an outline of the remainder of the paper. In Section 2 we introduce the basic setup of the paper, including the double null foliation, the coordinate system and the Einstein vacuum equations recast in terms of the geometric quantities associated to the double null foliation. In Section 4, we construct a class of characteristic initial data satisfying the assumptions of Theorem 2. In Section 3, we introduce the norms used in the paper and state a theorem on a priori estimates (Theorem 5) which imply our main existence theorem (Theorem 2). In Sections 5-8 we prove Theorem 5. In Section 5, we obtain the estimates for the metric components and derive functional inequalities useful in our setting. Then in Sections 6 and 7, we prove bounds for the Ricci coefficients assuming control of the curvature components. In Section 8, we close all the estimates by obtaining bounds for the curvature components. Finally, in Section 9, we discuss the nature of the singular boundary and prove Theorems 3 and 4.

2. Basic Setup

2.1. Double Null Foliation. For a spacetime in a neighborhood of $S_{0,0}$, we define a double null foliation as follows: Let $u$ and $\overline{u}$ be solutions to the eikonal equation

$$g^{\mu\nu} \partial_\mu u \partial_\nu u = 0, \quad g^{\mu\nu} \partial_\mu \overline{u} \partial_\nu \overline{u} = 0,$$

satisfying the initial conditions $u = 0$ on $H_0$ and $\overline{u} = 0$ on $\overline{H}_0$. Let

$$L^\mu = -2g^{\mu\nu} \partial_\nu u, \quad \overline{L}^\mu = -2g^{\mu\nu} \partial_\nu \overline{u}.$$

These are null and geodesic vector fields. Let

$$2\Omega^{-2} = -g(L', \overline{L}').$$

Define

$$e_3 = \Omega L', \quad e_4 = \overline{\Omega} L'$$

to be the normalized null pair such that

$$g(e_3, e_4) = -2$$

and

$$L = \Omega^2 L', \quad \overline{L} = \overline{\Omega}^2 L'$$

to be the so-called equivariant vector fields.

In this paper, we will consider spacetime solutions to the vacuum Einstein equations (1) in the gauge such that

$$\Omega = 1, \quad \text{on } H_0 \text{ and } \overline{H}_0.$$

The level sets of $u$ (resp. $\overline{u}$) are denoted by $H_u$ (resp. $\overline{H}_{\overline{u}}$). The eikonal equations imply $H_u$ and $\overline{H}_{\overline{u}}$ are null hypersurface. The intersections of the hypersurfaces $H_u$ and $\overline{H}_{\overline{u}}$ are
topologically 2-spheres, which we denote by \( S_{u,u} \). Note that the integral flows of \( L \) and \( L \) respect the foliation \( S_{u,u} \).

2.2. The Coordinate System. We define a coordinate system \((u, u, \theta^1, \theta^2)\) in a neighborhood of \( S_{0,0} \) as follows: On the sphere \( S_{0,0} \), we have an atlas such that in the local coordinate system \((\theta^1, \theta^2)\) in each coordinate chart, the metric \( \gamma \) is smooth, bounded and positive definite. Recall that in a neighborhood of \( S_{0,0} \), \( u \) and \( u \) are solutions to the eikonal equations:

\[
g^{\mu\nu}\partial_\mu u \partial_\nu u = 0, \quad g^{\mu\nu}\partial_\mu u \partial_\nu u = 0.
\]

We then require the coordinates on the initial hypersurfaces to satisfy

\[
\frac{\partial}{\partial u} \theta^A = 0 \text{ on } H_0, \quad \text{and } \frac{\partial}{\partial u} \theta^A = 0 \text{ on } H_0,
\]

i.e.,

\[
\mathcal{L}_L \theta^A = 0,
\]

where \( \mathcal{L}_L \) denotes the restriction of the Lie derivative to \( TS_{u,u} \). (See [5], Chapter 1) and \( L \) is defined as in the Section 2.1. Relative to the coordinate system \((u, u, \theta^1, \theta^2)\), the null pair \( e_3 \) and \( e_4 \) can be expressed as

\[
e_3 = \Omega^{-1} \left( \frac{\partial}{\partial u} + b^A \frac{\partial}{\partial \theta^A} \right), \quad e_4 = \Omega^{-1} \frac{\partial}{\partial u},
\]

for some \( b^A \) such that \( b^A = 0 \) on \( H_0 \), while the metric \( g \) takes the form

\[
g = -2\Omega^2(du \otimes du + du \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du).
\]

2.3. Equations. We will recast the Einstein equations as a system for Ricci coefficients and curvature components associated to a null frame \( e_3, e_4 \) defined above and an orthonormal frame \( e_1, e_2 \) tangent to the 2-spheres \( S_{u,u} \). Using the indices \( A, B \) to denote 1, 2, we define the Ricci coefficients relative to the null frame:

\[
\chi_{AB} = g(D_A e_4, e_B), \quad \chi_{AB} = g(D_A e_3, e_B),
\]

\[
\eta_A = -\frac{1}{2} g(D_3 e_A, e_4), \quad \eta_A = -\frac{1}{2} g(D_4 e_A, e_3)
\]

\[
\omega = -\frac{1}{4} g(D_4 e_3, e_4), \quad \omega = -\frac{1}{4} g(D_3 e_4, e_3),
\]

\[
\zeta_A = \frac{1}{2} g(D_A e_4, e_3)
\]

where \( D_A = D_{e(A)} \). We also introduce the null curvature components,

\[
\alpha_{AB} = R(e_A, e_4, e_B, e_4), \quad \alpha_{AB} = R(e_A, e_3, e_B, e_3),
\]

\[
\beta_A = \frac{1}{2} R(e_A, e_4, e_3, e_4), \quad \beta_A = \frac{1}{2} R(e_A, e_3, e_3, e_4),
\]

\[
\rho = \frac{1}{4} R(e_4, e_3, e_4, e_3), \quad \sigma = \frac{1}{4} R(e_4, e_3, e_4, e_3),
\]

Here \( R^* \) denotes the Hodge dual of \( R \). We denote by \( \nabla \) the induced covariant derivative operator on \( S_{u,u} \) and by \( \nabla_3, \nabla_4 \) the projections to \( S_{u,u} \) of the covariant derivatives \( D_3, D_4 \) (see precise definitions in [15]).
Observe that,
\[
\begin{align*}
\omega &= -\frac{1}{2} \nabla_4 (\log \Omega), \\
\eta_A &= \zeta_A + \nabla_A (\log \Omega), \\
\bar{\eta}_A &= -\zeta_A + \nabla_A (\log \Omega).
\end{align*}
\] (6)

Let \( \phi^{(1)} \cdot \phi^{(2)} \) denote an arbitrary contraction of the tensor product of \( \phi^{(1)} \) and \( \phi^{(2)} \) with respect to the metric \( \gamma \). We also define
\[
(\phi^{(1)} \otimes \phi^{(2)})_{AB} := \phi_A^{(1)} \phi_B^{(2)} + \phi_B^{(1)} \phi_A^{(2)} - \delta_{AB} (\phi^{(1)} \cdot \phi^{(2)})
\]
for one forms \( \phi_A^{(1)}, \phi_A^{(2)} \),
\[
(\phi^{(1)} \wedge \phi^{(2)})_{AB} := \epsilon^{ABCD} \phi_A^{(1)} \phi_B^{(2)}
\]
for symmetric two tensors \( \phi^{(1)}_{AB}, \phi^{(2)}_{AB} \),

where \( \epsilon \) is the volume form associated to the metric \( \gamma \). For symmetric tensors, the \( \text{div} \) and \( \text{curl} \) operators are defined by the formulas
\[
(\text{div} \, \phi)_{A_1 \ldots A_r} := \nabla^B \phi_{BA_1 \ldots A_r},
\]
\[
(\text{curl} \, \phi)_{A_1 \ldots A_r} := \epsilon^{BC} \nabla_B \phi_{CA_1 \ldots A_r}.
\]

Define also the trace to be
\[
(\text{tr} \, \phi)_{A_1 \ldots A_{r-1}} := (\gamma^{-1})^{BC} \phi_{BCA_1 \ldots A_{r-1}}.
\]

We separate the trace and traceless part of \( \chi \) and \( \hat{\chi} \). Let \( \hat{\chi} \) and \( \hat{\hat{\chi}} \) be the traceless parts of \( \chi \) and \( \hat{\chi} \) respectively. Then \( \chi \) and \( \hat{\chi} \) satisfy the following null structure equations:
\[
\begin{align*}
\nabla_4 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 &= -|\hat{\chi}|^2 - 2\omega \text{tr} \chi \\
\nabla_4 \hat{\chi} + \text{tr} \hat{\chi} &= -2\omega \hat{\chi} - \alpha \\
\nabla_3 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 &= -2\omega \text{tr} \chi - |\hat{\chi}|^2 \\
\nabla_3 \hat{\chi} + \text{tr} \hat{\chi} &= -2\omega \hat{\chi} - \alpha \\
\nabla_4 \text{tr} \hat{\chi} + \frac{1}{2} (\text{tr} \hat{\chi})^2 &= 2\omega \text{tr} \hat{\chi} + 2\rho - \hat{\chi} \cdot \hat{\hat{\chi}} + 2 \text{div} \eta + 2|\eta|^2 \\
\nabla_4 \hat{\hat{\chi}} + \frac{1}{2} (\text{tr} \hat{\chi})^2 &= \nabla \hat{\hat{\chi}} + 2\omega \hat{\chi} + \frac{1}{2} \text{tr} \hat{\chi} + \eta \hat{\hat{\eta}} \\
\nabla_3 \text{tr} \hat{\chi} + \frac{1}{2} (\text{tr} \hat{\chi})^2 &= 2\omega \text{tr} \hat{\chi} + 2\rho - \hat{\chi} \cdot \hat{\hat{\chi}} + 2 \text{div} \eta + 2|\eta|^2 \\
\nabla_3 \hat{\hat{\chi}} + \frac{1}{2} (\text{tr} \hat{\chi})^2 &= \nabla \hat{\hat{\chi}} + 2\omega \hat{\chi} + \frac{1}{2} \text{tr} \hat{\chi} + \eta \hat{\hat{\eta}}
\end{align*}
\] (7)

The other Ricci coefficients satisfy the following null structure equations:
\[
\begin{align*}
\nabla_4 \eta &= -\chi \cdot (\eta - \bar{\eta}) - \beta \\
\nabla_3 \bar{\eta} &= -\hat{\chi} \cdot (\eta - \bar{\eta}) + \bar{\beta} \\
\nabla_4 \omega &= 2\omega \omega + \frac{3}{4} |\eta - \bar{\eta}|^2 - \frac{1}{4} (\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8} |\eta + \bar{\eta}|^2 + \frac{1}{2} \rho \\
\nabla_3 \omega &= 2\omega \omega + \frac{3}{4} |\eta - \bar{\eta}|^2 + \frac{1}{4} (\eta - \bar{\eta}) \cdot (\eta + \bar{\eta}) - \frac{1}{8} |\eta + \bar{\eta}|^2 + \frac{1}{2} \rho
\end{align*}
\] (8)
The Ricci coefficients also satisfy the following constraint equations

\[
\begin{align*}
\text{div } \hat{\chi} &= \frac{1}{2} \nabla \text{tr} \chi - \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \text{tr} \chi) - \beta, \\
\text{div } \hat{\chi} &= \frac{1}{2} \nabla \text{tr} \chi + \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \text{tr} \chi) + \beta \\
\text{curl } \eta &= -\text{curl } \eta = \sigma + \frac{1}{2} \hat{\chi} \wedge \hat{\chi} \\
K &= -\rho + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \frac{1}{4} \text{tr} \chi \text{tr} \chi
\end{align*}
\]  

(9)

with \( K \) the Gauss curvature of the spheres \( S_{u,u} \). The null curvature components satisfy the following null Bianchi equations:

\[
\begin{align*}
\nabla_3 \alpha + \frac{1}{2} \text{tr} \chi \alpha &= \nabla \hat{\otimes} \beta + 4 \omega \alpha - 3(\hat{\chi} \rho + \hat{\chi} \sigma) + (\zeta + 4 \eta) \hat{\otimes} \beta, \\
\nabla_4 \beta + 2 \text{tr} \chi \beta &= \text{div } \alpha - 2 \omega \beta + \eta \alpha, \\
\nabla_3 \beta + \text{tr} \chi \beta &= \nabla \rho + 2 \omega \beta + \nabla \sigma + 2 \hat{\chi} \cdot \beta + 3(\eta \rho + \hat{\chi} \sigma), \\
\nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma &= -\text{div } \hat{\chi} \cdot \alpha - \zeta \cdot \beta - 2 \eta \cdot \beta, \\
\nabla_3 \sigma + \frac{3}{2} \text{tr} \chi \sigma &= -\text{div } \hat{\chi} \cdot \alpha - \zeta \cdot \beta - 2 \eta \cdot \beta, \\
\nabla_4 \rho + \frac{3}{2} \text{tr} \chi \rho &= \text{div } \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2 \eta \cdot \beta, \\
\nabla_3 \rho + \frac{3}{2} \text{tr} \chi \rho &= -\text{div } \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta - 2 \eta \cdot \beta, \\
\nabla_4 \beta + \text{tr} \chi \beta &= -\nabla \rho + \omega \cdot \nabla \sigma + 2 \omega \beta + 2 \hat{\chi} \cdot \beta - 3(\eta \rho + \hat{\chi} \sigma), \\
\nabla_3 \beta + 2 \text{tr} \chi \beta &= -\nabla \rho + 2 \omega \beta + \eta \cdot \alpha, \\
\nabla_4 \alpha + \frac{1}{2} \text{tr} \chi \alpha &= -\nabla \hat{\otimes} \beta + 4 \omega \alpha - 3(\hat{\chi} \rho + \hat{\chi} \sigma) + (\zeta - 4 \eta) \hat{\otimes} \beta
\end{align*}
\]  

(10)

where \( \ast \) denotes the Hodge dual on \( S_{u,u} \).

We now rewrite the Bianchi equations in terms of the Gauss curvature \( K \) of the spheres \( S_{u,u} \) and the renormalized curvature component \( \hat{\sigma} \) defined by

\[
\hat{\sigma} = \sigma + \frac{1}{2} \hat{\chi} \wedge \hat{\chi}.
\]
The Bianchi equations take the following form
\[
\nabla_3 \beta + \text{tr} \chi \beta = \nabla K + \nabla \sigma + 2 \omega \beta + 2 \hat{\chi}, \beta + 3 \eta K + \eta \sigma + \frac{1}{2} (\nabla (\hat{\chi} - \nabla (\hat{\chi} \wedge \hat{\chi})))
\]
\[+ 3 (\eta \hat{\chi} \cdot \hat{\chi} - \eta \hat{\chi} \wedge \hat{\chi}) - \frac{1}{4} (\nabla \text{tr} \chi \text{tr} \chi + \text{tr} \chi \nabla \text{tr} \chi) - \frac{3}{4} \eta \text{tr} \chi \text{tr} \chi,
\]
\[
\nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma = - \text{div} * \beta - \zeta * \beta - 2 \eta * \beta + \frac{1}{2} \hat{\chi} * (\nabla \hat{\eta}) + \frac{1}{2} \hat{\chi} * (\nabla \hat{\eta}),
\]
\[
\nabla_4 K + \text{tr} \chi K = \text{div} \beta + \zeta \cdot \beta - \frac{1}{2} \hat{\chi} \cdot \hat{\eta} - \frac{1}{2} \hat{\chi} \cdot (\nabla \hat{\eta}) + \frac{1}{2} \text{tr} \chi \text{div} \eta + \frac{1}{2} \text{tr} \chi | \eta |^2,
\]
\[
\nabla_5 \sigma + \frac{3}{2} \text{tr} \chi \sigma = - \text{div} * \beta - \zeta * \beta - 2 \eta * \beta + \frac{1}{2} \hat{\chi} * (\nabla \hat{\eta}) + \frac{1}{2} \hat{\chi} * (\nabla \hat{\eta}),
\]
\[
\nabla_5 K + \text{tr} \chi K = - \text{div} \beta + \zeta \cdot \beta + \frac{1}{2} \hat{\chi} \cdot \hat{\eta} - \frac{1}{2} \hat{\chi} \cdot (\nabla \hat{\eta}) + \frac{1}{2} \text{tr} \chi \text{div} \eta + \frac{1}{2} \text{tr} \chi | \eta |^2,
\]
\[
\nabla_4 \beta + \text{tr} \chi \beta = - \nabla K + \nabla \sigma + 2 \omega \beta + 2 \hat{\chi}, \beta - 3 \eta K + \eta \sigma - \frac{1}{2} (\nabla (\hat{\chi} - \nabla (\hat{\chi} \wedge \hat{\chi})))
\]
\[+ \frac{1}{4} (\nabla \text{tr} \chi \text{tr} \chi + \text{tr} \chi \nabla \text{tr} \chi) - 3 (\eta \hat{\chi} \cdot \hat{\chi} + \eta \hat{\chi} \wedge \hat{\chi}) + \frac{3}{4} \eta \text{tr} \chi \text{tr} \chi,
\]
\[
(11)
\]

Notice that we have obtained a system for the renormalized curvature components in which the curvature components $\alpha$ and $\alpha$ do not appear.\(^{12}\)

From now on, we will use capital Latin letters $A \in \{1, 2\}$ for indices on the spheres $S_{u, \omega}$ and Greek letters $\mu \in \{1, 2, 3, 4\}$ for indices in the whole spacetime.

2.4. Schematic Notation. We define a schematic notation for the Ricci coefficients according to the estimates that they obey. Let\(^{13}\)

\[
\psi \in \{ \eta, \omega \}, \quad \psi_H \in \{ \text{tr} \chi, \hat{\chi}, \omega \}, \quad \psi_H \in \{ \text{tr} \chi, \hat{\chi}, \omega \}.
\]

We will use this schematic notation only in the situations where the exact constant in front of the term in irrelevant to the argument. We will denote by $\psi \psi$ (or $\psi \psi_H$, etc) an arbitrary contraction with respect to the metric $\gamma$ and by $\nabla \psi$ an arbitrary angular covariant derivative. $\nabla^i \psi^{j}$ will be used to denote the sum of all terms which are products of $j$ factors, such that each factor takes the form $\nabla^k \psi$ and that the sum of all $i_k$’s is $i$, i.e.,

\[
\nabla^i \psi^j = \sum_{i_1 + i_2 + \ldots + i_j = i} \nabla^{i_1} \psi \nabla^{i_2} \psi \cdots \nabla^{i_j} \psi.
\]

We will use brackets to denote terms with one of the components in the brackets. For instance, the notation $\psi(\psi, \psi_H)$ denotes the sum of all terms of the form $\psi \psi$ or $\psi \psi_H$.

In this schematic notation, the Ricci coefficients $\psi_H$ satisfy

\[
\nabla_3 \psi_H = K + \nabla \psi + \psi \psi + \psi_H \psi_H.
\]

The Ricci coefficients $\psi_H$ similarly obey

\[
\nabla_4 \psi_H = K + \nabla \psi + \psi \psi + \psi_H \psi_H.
\]

\(^{12}\)Moreover, compared to the renormalization in [18], this system do not contain the terms $\text{tr} \chi | \hat{\chi} |^2$ and $\text{tr} \chi | \hat{\chi} |^2$ which would be uncontrollable in the context of this paper.

\(^{13}\)Notice that this definition is different form that in [18] since in the context of the present paper, $\text{tr} \chi$ and $\text{tr} \hat{\chi}$ verify different bounds compared to [18].
The Ricci coefficients $\psi$ obey either one of the following equations:

$$\nabla_3 \psi = \beta + \psi \psi_H$$

or

$$\nabla_4 \psi = \beta + \psi \psi_H.$$ 

We also rewrite the Bianchi equations in the schematic notation:

\[
\begin{align*}
\nabla_3 \beta - \nabla K - ^* \nabla \bar{\sigma} &= \sum_{i_1+i_2=1} \psi_H \psi^{i_1} \nabla^{i_2} \psi_H + \psi K + \sum_{i_1+i_2=1} \psi^{i_1} \psi \nabla^{i_2} \psi \\
\nabla_4 \bar{\sigma} + \text{div}^* \beta &= \psi_H \bar{\sigma} + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H, \\
\nabla_4 K - \text{div} \beta &= \psi_H K + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H, \\
\nabla_3 \bar{\sigma} + \text{div}^* \bar{\beta} &= \psi_H \bar{\sigma} + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H, \\
\nabla_3 K + \text{div} \bar{\beta} &= \psi_H K + \psi \sum_{i_1+i_2+i_3=1} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H, \\
\n\nabla_4 \beta + \nabla K - ^* \nabla \bar{\sigma} &= \sum_{i_1+i_2=1} \psi_H \psi^{i_1} \nabla^{i_2} \psi_H + \psi K + \sum_{i_1+i_2=1} \psi^{i_1} \psi \nabla^{i_2} \psi.
\end{align*}
\]

(12)

3. Norms

We define the following norms for the Ricci coefficients $\psi$:

$$O_{i,p}[\psi] := ||\nabla^i \psi||_{L^\infty L^2 L^p(S)}.$$ 

Define the following norms for the Ricci coefficients $\psi_H$:

$$O_{i,p}[\psi_H] := ||f(u) \nabla^i \psi_H||_{L^2 L^\infty L^p(S)}.$$ 

Similarly, we define the following norms for the Ricci coefficients $\psi_H$:

$$O_{i,p}[\psi_H] := ||f(u) \nabla^i \psi_H||_{L^2 L^\infty L^p(S)}.$$ 

We make two remarks concerning these norms:

**Remark 8.** While the norms for $\psi_H$ and $\psi_H$ are based on $L^2$ in $u$ and $u$ respectively, by virtue of the weights $f(u)$ and $f(u)$, they actually control the $L^1$ norms. More precisely, since $\int_0^u \frac{1}{f^2(w')} du' < \epsilon$ and $\int_0^u \frac{1}{f^2(w')} du' < \epsilon$, by Cauchy-Schwarz, we have

$$||\nabla^i \psi_H||_{L^1 L^\infty L^p(S)} \leq C \epsilon O_{i,p}[\psi_H],$$

and

$$||\nabla^i \psi_H||_{L^1 L^\infty L^p(S)} \leq C \epsilon O_{i,p}[\psi_H].$$

**Remark 9.** The norm $O_{i,p}[\psi_H]$ (resp. $O_{i,p}[\psi_H]$) allows us to first take $L^\infty$ along $u$ direction (resp. $u$ direction) before the $L^2$ norm in $u$ (resp. $u$) is taken. This is stronger than the norms such that the order is reversed, i.e., we have

$$||f(u) \nabla^i \psi_H||_{L^\infty L^2 L^p(S)} \leq C O_{i,p}[\psi_H],$$

and

$$||f(u) \nabla^i \psi_H||_{L^\infty L^2 L^p(S)} \leq C O_{i,p}[\psi_H].$$
In addition to the above norms, we need to define norms for the highest derivatives for the Ricci coefficients. Let

\[
\mathcal{O}_{4,2} := \|f(u)^2 \nabla^4 \text{tr} \chi\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u)^2 \nabla^4 \text{tr} \chi\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u) \nabla^4 \chi, \omega\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u) \nabla^4 \eta, \eta\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u) \nabla^4 \chi, \omega\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u) \nabla^4 \eta, \eta\|_{L_2^\infty L_2^2 \mathcal{L}^2}.
\]

**Remark 10.** Here, note that for the norms for \(\hat{\chi}, \omega, \eta, \eta, \hat{\chi} \) and \(\omega, L_2^\infty \) in \(u \) (or \(u \)) is taken after \(L_2^2 \) in \(u \) (or \(u \)). According to Remark 3, this is weaker than the \(O_{i,2} \) norms defined above.

We also define the curvature norms for the curvature components. Let

\[
\mathcal{R}_i := \|f(u) \nabla^i \beta\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u) \nabla^i (K, \sigma)\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u) \nabla^i (K, \sigma)\|_{L_2^\infty L_2^2 \mathcal{L}^2} + \|f(u) \nabla^i \beta\|_{L_2^\infty L_2^2 \mathcal{L}^2}.
\]

As a shorthand, we also let

\[
\mathcal{R} := \sum_{i \leq 3} \mathcal{R}_i.
\]

Finally, let \(O_0 \) and \(R_0 \) denote the corresponding norms for the initial data, i.e.

\[
O_0 := \sum_{i \leq 3} \left( \|\nabla^i \psi\|_{L_2^\infty L^p(S_{0,1})} + \|\nabla^i \psi\|_{L_2^\infty L^p(S_{1,0})} + \|f(u) \nabla^i \psi_H\|_{L_2^\infty L^p(S_{0,1})} + \|f(u) \nabla^i \psi_H\|_{L_2^\infty L^p(S_{1,0})} + \|f(u)^2 \nabla^4 \text{tr} \chi\|_{L_2^\infty L^2(S_{0,1})} + \|f(u)^2 \nabla^4 \text{tr} \chi\|_{L_2^\infty L^2(S_{1,0})} + \|\nabla^4 \text{tr} \chi\|_{L_2^\infty L^2(S_{0,1})} + \|\nabla^4 \text{tr} \chi\|_{L_2^\infty L^2(S_{1,0})} + \|f(u) \nabla^4 (\hat{\chi}, \omega)\|_{L_2^\infty L^2(S_{0,1})} + \|f(u) \nabla^4 (\hat{\chi}, \omega)\|_{L_2^\infty L^2(S_{1,0})} + \|f(u)^2 \nabla^4 (\hat{\chi}, \omega)\|_{L_2^\infty L^2(S_{0,1})} + \|f(u)^2 \nabla^4 (\hat{\chi}, \omega)\|_{L_2^\infty L^2(S_{1,0})} + \|f(u) \nabla^4 (\hat{\chi}, \omega)\|_{L_2^\infty L^2(S_{0,1})} + \|f(u) \nabla^4 (\hat{\chi}, \omega)\|_{L_2^\infty L^2(S_{1,0})} \right)
\]

and

\[
R_0 := \|f(u) \nabla^i \beta\|_{L_2^\infty L^2(S_{0,1})} + \|\nabla^i (K, \sigma)\|_{L_2^\infty L^2(S_{0,1})} + \|\nabla^i (K, \sigma)\|_{L_2^\infty L^2(S_{1,0})} + \|f(u) \nabla^i \beta\|_{L_2^\infty L^2(S_{0,1})} + \|f(u) \nabla^i \beta\|_{L_2^\infty L^2(S_{1,0})}.
\]

In order to prove Theorem 2, we will establish a priori estimates for the geometric quantities in the above norms:

**Theorem 5.** Assume that the initial data for the characteristic initial value problem satisfy the assumptions of Theorem 2 with \(\epsilon\) sufficiently small. Then there exists \(B\) depending only on \(D\) and \(d\) such that

\[
\sum_{i \leq 3} O_{i,2} + \mathcal{O}_{4,2} + R \leq B.
\]

In the remainder of the paper, we will focus on the proof of Theorem 5 (after constructing initial data sets in the next section). Standard methods show that Theorem 5 implies Theorem 2. We will omit the details and refer the readers to [5] for a proof that the a priori estimates imply the existence theorem.
4. Construction of Initial Data Set

In this section, we construct initial data sets satisfying the assumptions of Theorems \ref{thm:2} and \ref{thm:4}. In particular, we show that the constraint equations can be solved for $|\hat{\chi}(0, u)| \sim (f(u))^{-2}$ and $|\hat{\chi}(u, 0)| \sim (f(u))^{-2}$. Our approach in this section follows closely that of Christodoulou in Chapter 2 of \cite{5}.

Assume for simplicity that $S_{0,0}$ is a standard sphere of radius 1. Introduce the standard stereographic coordinates $(\theta^1, \theta^2)$ such that the standard metric $\gamma_{AB}^0$ on the sphere takes the form

$$\gamma_{AB}^0 = \frac{\delta_{AB}}{(1 + \frac{1}{4}|\theta|^2)^2}.$$

Clearly, it suffices to construct initial data on $H_0$. The construction on $H_0$ is similar. On $H_0$, let $\frac{\partial}{\partial u}$ be the null geodesic vector field and define the coordinate system $(u, \theta^1, \theta^2)$ on $H_0$ by requiring

$$\mathcal{L}_{\frac{\partial}{\partial u}} \theta^A = 0,$$

where, as before, $\mathcal{L}$ denotes the restriction of the Lie derivative to $TS_{u,u}$. We will construct a metric on $H_0$ in these coordinates taking the form

$$\gamma_{AB} = \frac{\Phi^2 m_{AB}}{(1 + \frac{1}{4}|\theta|^2)^2},$$

where $\det m_{AB} = 1$ and $\Phi |_{S_{0,0}} = 1$. We will denote

$$\hat{\gamma}_{AB} = \frac{m_{AB}}{(1 + \frac{1}{4}|\theta|^2)^2}.$$

In order to ensure that $m$ satisfies $\det m = 1$, we can express $m = \exp \Psi$, with $\Psi \in \hat{S}$, where $\hat{S}$ denotes the set of all matrices taking the form

$$\begin{pmatrix}
a & b \\
b & -a
\end{pmatrix}$$

Let

$$m = \exp \Psi$$

such that $\Psi \in \hat{S}$ and satisfying

$$\sum_{j \leq N} |(\frac{\partial}{\partial \theta^i})^j \Psi| \lesssim 1$$

$$\sum_{j \leq N} |(\frac{\partial}{\partial \theta^i})^j \frac{\partial}{\partial u} \Psi| \sim f(u)^{-2}. \quad (13)$$

Following \cite{5}, we have

$$\hat{\chi}_{AB} = \frac{1}{2} \Phi^2 \frac{\partial}{\partial u} \hat{\gamma}_{AB}, \quad (14)$$

\footnote{While we only write down one coordinate chart, it is implicit that we have two stereographic charts - the north pole chart and the south pole chart. In the following, when we derive the estimates for the geometric quantities, we only prove the bounds in a sufficiently large ball $B_{\rho}$ in each of these charts.}
and
\[
\text{tr}\chi = \frac{2}{\Phi} \frac{\partial \Phi}{\partial u}. \tag{15}
\]
We can also derive that
\[
|\hat{\chi}|_\gamma^2 = \frac{1}{4} (\hat{\gamma}^{-1})^{AC} (\hat{\gamma}^{-1})^{BD} \frac{\partial}{\partial u} \hat{\gamma}_{AB} \frac{\partial}{\partial u} \hat{\gamma}_{CD}.
\]
Thus by (13), we have
\[
|\hat{\chi}|_\gamma^2 \sim f(u)^{-4}. \tag{16}
\]
In particular, this implies that the requirement in Theorem 4 is satisfied if
\[
\int_0^{u^*} f(u)^4 du = \infty.
\]
By the equation
\[
\mathcal{L}_u \text{tr}\chi = -\frac{1}{2} (\text{tr}\chi)^2 - |\hat{\chi}|^2,
\]
\Phi can be solved from the ODE
\[
\frac{\partial^2 \Phi}{\partial u^2} + \frac{1}{8} ((\hat{\gamma}^{-1})^{AC} (\hat{\gamma}^{-1})^{BD}) \frac{\partial}{\partial u} \hat{\gamma}_{AB} \frac{\partial}{\partial u} \hat{\gamma}_{CD}) \Phi = 0. \tag{17}
\]
We prescribe \(\text{tr}\chi\) on \(S_{0,0}\) to obey the initial conditions
\[
\Phi \big|_{S_{0,0}} = 1,
\]
\[
\frac{\partial \Phi}{\partial u} \big|_{S_{0,0}} = \frac{1}{2} \text{tr}\chi \big|_{S_{0,0}} < 1.
\]
Finally, we prescribe \(\zeta\) on \(S_{0,0}\) such that
\[
\sum_{j \leq N-1} |(\frac{\partial}{\partial \theta^j}) \zeta |_\gamma^2 \lesssim 1. \tag{18}
\]
We check that this initial data obey all the estimates required by Theorem 2.

**Estimates for \(\nabla^i \chi\) and the metric**

To satisfy the upper bounds in Theorem 2 we need to show that
\[
\sum_{i \leq N} |\nabla_i \chi|_{(0,u)} \lesssim f(u)^{-2} \tag{19}
\]
We will show the estimates separately for \(\text{tr}\chi\) and \(\hat{\chi}\). By (16), (19) is satisfied by \(\hat{\chi}\). To derive this bound for \(\text{tr}\chi\), notice that by the ODE for \(\Phi\), the initial conditions, and the bound for \(|\hat{\chi}|^2\) in (16), we have
\[
\frac{1}{2} \leq \Phi \leq 1
\]
and
\[
\left|\frac{\partial \Phi}{\partial u}\right| \lesssim 1 + \int_0^u f(u')^{-4} du' \leq 1 + f(u)^{-2} \int_0^{u^*} f(u')^{-2} du' \leq 1 + \epsilon^2 f(u)^{-2}.
\]
By (15), we thus have
\[
|\text{tr}\chi| \lesssim f(u)^{-2}.
\]
We now move on to control the angular derivatives of \(\chi\). By (13),
\[
\sum_{j \leq N} \left|(\frac{\partial}{\partial \theta^j}) \frac{\partial}{\partial u} m_{AB}\right| \sim f(u)^{-2}.
\]
Using this bound and commuting the ODE (17) with $\frac{\partial}{\partial \theta^i}$, we also have that for up to $N$ coordinate angular derivatives $\frac{\partial}{\partial \theta^i}$,

$$
\sum_{j \leq N} \left| \left( \frac{\partial}{\partial \theta^i} \right)^j \Phi \right| \sim 1. \tag{20}
$$

This implies that the metric $\gamma$ obeys the bounds

$$
\sum_{j \leq N} \left| \left( \frac{\partial}{\partial \theta^i} \right)^j \gamma_{AB} \right| \sim 1, \quad \sum_{j \leq N} \left| \left( \frac{\partial}{\partial \theta^i} \right)^j (\gamma^{-1})^{AB} \right| \sim 1. \tag{21}
$$

Together with (13) and (14), (20) implies

$$
\sum_{j \leq N} \left| \left( \frac{\partial}{\partial \theta^i} \right)^j \hat{\chi} \right| \sim f(u)^{-2}. \tag{22}
$$

By (15) and (15), we also have

$$
\sum_{j \leq N} \left| \left( \frac{\partial}{\partial \theta^i} \right)^j \text{tr} \chi \right| \lesssim f(u)^{-2}. \tag{23}
$$

Finally, we notice that by (21), the angular covariant derivatives of $\text{tr} \chi$ and $\hat{\chi}$ are comparable to the angular coordinate derivatives of $\text{tr} \chi$ and $\hat{\chi}$. Therefore, (19) follows from (22) and (23).

**Estimates for $\nabla^i \zeta$**

On $H_0$, since $\Omega = 1$, $\eta = \zeta$. Thus combining the transport equation for $\zeta$ in (8) and the Codazzi equation for $\beta$ in (9) and rewriting in $\mathcal{L}$ (instead of $\nabla_4$), we have

$$
\mathcal{L} \frac{\partial}{\partial u} \zeta + \text{tr} \chi \zeta = \text{div} \chi - \nabla \text{tr} \chi.
$$

Recall from (18) that the initial data for $\zeta$ and its angular derivatives are bounded. Therefore, by the estimates for $\text{tr} \chi$ and $\hat{\chi}$ (and their angular derivatives) above, we have

$$
\sum_{j \leq N-1} \left| \left( \frac{\partial}{\partial \theta^i} \right)^j \zeta \right| \lesssim 1.
$$

The bounds for the metric and Christoffel symbols on the sphere imply

$$
\sum_{j \leq N-1} \left\| \nabla^j \zeta \right\|_{L^\infty_u L^\infty(S_0, u)} \lesssim 1
$$

as desired.

**Estimates for $\nabla^i \text{tr} \chi$**

Similar to $\zeta$, $\text{tr} \chi$ obeys a transport equations along the null generators of $H_0$. More precisely, (8) and the Gauss equation in (9) imply that

$$
\mathcal{L} \frac{\partial}{\partial u} \text{tr} \chi + \text{tr} \chi \text{tr} \chi = -2K - 2\text{div} \zeta + 2|\zeta|^2.
$$

Thus, the previous estimates imply

$$
\sum_{j \leq N-2} \left\| \nabla^j \text{tr} \chi \right\|_{L^\infty_u L^\infty(S_0, u)} \lesssim 1
$$
5. The Preliminary Estimates

All estimates in this section will be proved under the following bootstrap assumption:

\[ \sum_{i \leq 1} O_{i,\infty} + \sum_{i \leq 2} O_{i,4} + \sum_{i \leq 3} O_{i,2} \leq \Delta_1. \]  

(A1)

5.1. Estimates for Metric Components. We first show that we can control \( \Omega \) under the bootstrap assumption (A1):

**Proposition 1.** There exists \( \epsilon_0 = \epsilon_0(\Delta_1) \) such that for every \( \epsilon \leq \epsilon_0 \),

\[ \frac{1}{2} \leq \Omega \leq 2. \]

Moreover, \( \Omega \) is continuous up to \( u = u_* \) and \( \bar{u} = \bar{u}_* \).

**Proof.** Consider the equation

\[ \omega = -\frac{1}{2} \nabla_4 \log \Omega = \frac{1}{2} \Omega \nabla_4 \Omega^{-1} = \frac{1}{2} \frac{\partial}{\partial u} \Omega^{-1}. \]  

Fix \( \bar{u} \). Notice that both \( \omega \) and \( \Omega \) are scalars and therefore the \( L^\infty \) norm is independent of the metric. We can integrate equation (24) using the fact that \( \Omega^{-1} = 1 \) on \( H_0 \) to obtain

\[ ||\Omega^{-1} - 1||_{L^\infty(S_{u,\bar{u}})} \leq C \int_0^\bar{u} ||\omega||_{L^\infty(S_{u,\omega}')} du' \leq C ||f(\bar{u})^{-1}||_{L^2_{\bar{u}}} ||f(\bar{u})\omega||_{L^\infty H^1 L^2 L^\infty(S)} \leq C \Delta_1 \epsilon. \]

This implies both the upper and lower bounds for \( \Omega \) for sufficiently small \( \epsilon \). To show continuity, take a sequence of points \((u_n, \bar{u}_n, \theta^n_1, \theta^n_2)\) such \( u_n \to u_{\infty}, \bar{u}_n \to \bar{u}_{\infty}, \theta^n_1 \to \theta^1_0 \) and \( \theta^n_2 \to \theta^2_0 \). Then

\[ |\Omega^{-1}(u_n, \bar{u}_n, \theta^n_1, \theta^n_2) - \Omega^{-1}(u_m, \bar{u}_m, \theta^1_1, \theta^1_2)| \]

\[ \leq |\Omega^{-1}(u_n, \bar{u}_n, \theta^n_1, \theta^n_2) - \Omega^{-1}(u_n, \bar{u}_n, \theta^1_1, \theta^2_2)| + |\Omega^{-1}(u_n, \bar{u}_n, \theta^1_1, \theta^2_2) - \Omega^{-1}(u_n, \bar{u}_n, \theta^1_1, \theta^1_2)| \]

\[ + |\Omega^{-1}(u_n, \bar{u}_n, \theta^1_1, \theta^1_2) - \Omega^{-1}(u_m, \bar{u}_m, \theta^1_1, \theta^2_2)| \]

\[ \leq C ||\nabla \log \Omega||_{L^\infty(S_{u_n, \bar{u}_n})} \text{dist}(S_{u_n, \bar{u}_n}, (\theta_n, \theta_m)) + 2 \int_{u_n}^{2u_m} ||\omega||_{L^\infty(S_{u', \bar{u}'})} du' \]

\[ + 2 \int_{u_n}^{2u_m} ||\omega||_{L^\infty(S_{u', \bar{u}'})} du'. \]

Since by the bootstrap assumption (A1), \( \nabla \log \Omega = \frac{1}{2}(\eta + \bar{\eta}) \) is uniformly bounded, \( ||\omega||_{L^\infty(S_{u, \bar{u}})} \) is uniformly integrable in \( u \) for all \( \bar{u} \) and \( ||\omega||_{L^\infty(S_{u, \bar{u}})} \) is uniformly integrable in \( u \) for all \( \bar{u} \), the right hand side can be made arbitrarily small by taking \( n, m \geq N \) for \( N \) sufficiently large. The conclusion thus follows. \( \square \)

We then show that we can control \( \gamma \) under the bootstrap assumption (A1):

**Proposition 2.** There exists \( \epsilon_0 = \epsilon_0(\Delta_1) \) such that for \( \epsilon \leq \epsilon_0 \), in the \((u, \bar{u}, \theta^1, \theta^2)\) coordinate system, we have

\[ c \leq \det \gamma \leq C, \quad |\gamma_{AB}|, |(\gamma^{-1})^{AB}| \leq C. \]

Moreover, \( \gamma \) remains continuous up to \( u = u_* \) and \( \bar{u} = \bar{u}_* \).
Proof. The first variation formula states that
\[ \mathcal{L}_L^\gamma = 2\Omega \chi. \tag{25} \]
In coordinates, this means
\[ \frac{\partial}{\partial u} \gamma_{AB} = 2\Omega \chi_{AB}. \]
From this we derive that
\[ \frac{\partial}{\partial u} \log(\det \gamma) = \Omega \text{tr} \chi. \]
Define \( \gamma_0(u, u, \theta^1, \theta^2) = \gamma(u, 0, \theta^1, \theta^2) \). Then
\[ \left| \frac{\det \gamma}{\det(\gamma_0)} \right| \leq C \exp(\int_0^u |\text{tr}\chi| du') \leq C \exp(\|f(u)\|_{L^\infty(S) \to L^\infty(S_u)} \|\phi u\|_{L^\infty(S_u)}) \leq C \exp(C \Delta_1 \epsilon). \tag{26} \]
This implies that the \( \det \gamma \) is bounded above and below. Let \( \Lambda \) be the larger eigenvalue of \( \gamma \). Clearly,
\[ \Lambda \leq C \sup_{A,B=1,2} \gamma_{AB}, \tag{27} \]
and
\[ \sum_{A,B=1,2} |\chi_{AB}|^2 \leq C \Lambda \|\chi\|^2_{L^\infty(S_u)}. \]
Then
\[ |\gamma_{AB} - (\gamma_0)_{AB}| \leq C \int_0^u |\chi_{AB}| du' \leq C \epsilon A \Lambda (\int_0^u \|\chi\|^2_{L^\infty(S_u)} du')^{1/2} \leq C \Lambda \Delta_1 \epsilon^{1/2}. \]
Using the upper bound \( \gamma_0 \) [27], we thus obtain the upper bound for \( |\gamma_{AB}| \). The upper bound for \( \|(\gamma^{-1})^{AB}\| \) follows from the upper bound for \( |\gamma_{AB}| \) and the lower bound for \( \det \gamma \). Continuity of \( \gamma \) up to the boundary follows as in the proof of continuity for \( \Omega \) in Proposition 1. \( \Box \)

With the estimates on \( \gamma \), it follows that the \( L^p \) norms defined with respect to the metric and the \( L^p \) norms defined with respect to the coordinate system are equivalent.

**Proposition 3.** Given a covariant tensor \( \phi_{A_1...A_r} \) on \( S_{u,\bar{u}} \), we have
\[ \int_{S_{u,\bar{u}}} <\phi, \phi>^p \sim \sum_{i=1}^{r} \sum_{A_i=1,2} \int \int |\phi_{A_1...A_r}|^p \sqrt{\det \gamma} d\theta^1 d\theta^2. \]

We can also bound \( b \) under the bootstrap assumption, thus controlling the full spacetime metric:

**Proposition 4.** In the coordinates system \( (u, \bar{u}, \theta^1, \theta^2) \),
\[ |b^A| \leq C \Delta_1 \epsilon. \]
Moreover, \( b^A \) is continuous up to \( u = u_* \) and \( \bar{u} = \bar{u}_* \).

**Proof.** \( b^A \) satisfies the equation
\[ \frac{\partial b^A}{\partial u} = -4\Omega^2 \zeta^A. \tag{28} \]
This can be derived from
\[ [L, L] = \frac{\partial b^A}{\partial u} \frac{\partial}{\partial \theta^A}. \]
Now, integrating (28) and using Proposition 3 gives the bound on $b$. Continuity of $b$ up to the boundary follows as in the proof of Proposition 1.

5.2. Estimates for Transport Equations. In this subsection, we prove general propositions for obtaining bounds from the covariant null transport equations. Such estimates require the integrability of $\text{tr}\chi$ and $\text{tr}\chi'$, which is consistent with our bootstrap assumption $[A1]$. This will be used in the following sections to derive some estimates for the Ricci coefficients and the null curvature components from the null structure equations and the null Bianchi equations respectively. Below, we state two propositions which provide $L^p$ estimates for general quantities satisfying transport equations either in the $e_3$ or $e_4$ direction.

**Proposition 5.** There exists $\epsilon_0 = \epsilon_0(\Delta_1)$ such that for all $\epsilon \leq \epsilon_0$ and for every $2 \leq p < \infty$, we have

$$
\|\phi\|_{L^p(S_{u, \omega})} \leq C(\|\phi\|_{L^p(S_{u', \omega'})} + \int_{u'}^{u} \|\nabla_4 \phi\|_{L^p(S_{u, \omega})} du''),
$$

for any $S$-tensor $\phi$.

**Proof.** The following identity holds for any scalar $f$:

$$
\frac{d}{du} \int_{S_{u, \omega}} f = \int_{S_{u, \omega}} \left( \frac{df}{du} + \Omega \text{tr}\chi f \right) = \int_{S_{u, \omega}} \Omega (e_4(f) + \text{tr}\chi f).
$$

Similarly, we have

$$
\frac{d}{du} \int_{S_{u, \omega}} f = \int_{S_{u, \omega}} \Omega (e_3(f) + \text{tr}\chi f).
$$

Hence, taking $f = |\phi|^p_\gamma$, we have

$$
\|\phi\|_{L^p(S_{u, \omega})} = \|\phi\|_{L^p(S_{u', \omega'})} + \int_{u'}^{u} \int_{S_{u, \omega}} p |\phi|^{p-2} \Omega \left( <\phi, \nabla_4 \phi > + \frac{1}{p} \text{tr}\chi |\phi|_\gamma^2 \right) du''
$$

$$
\|\phi\|_{L^p(S_{u, \omega})} = \|\phi\|_{L^p(S_{u', \omega'})} + \int_{u'}^{u} \int_{S_{u', \omega'}} p |\phi|^{p-2} \Omega \left( <\phi, \nabla_3 \phi > + \frac{1}{p} \text{tr}\chi |\phi|_\gamma^2 \right) du''
$$

The bootstrap assumption $[A1]$ implies that $\text{tr}\chi$ and $\text{tr}\chi'$ are integrable (and in fact it also implies that $\|\text{tr}\chi\|_{L^1_w L^\infty(S)}$ and $\|\text{tr}\chi\|_{L^1_w L^\infty(S)}$ are small after choosing $\epsilon$ to be small depending on $\Delta_1$). Thus the proposition can be proved by using Cauchy-Schwarz inequality and Gronwall’s inequality, together with the bound for $\Omega$ given in Proposition 1.

We also have the following bounds for the $p = \infty$ case by integrating along the integral curves of $e_3$ and $e_4$:

**Proposition 6.** There exists $\epsilon_0 = \epsilon_0(\Delta_1)$ such that for all $\epsilon \leq \epsilon_0$, we have

$$
\|\phi\|_{L^\infty(S_{u, \omega})} \leq C \left( \|\phi\|_{L^\infty(S_{u', \omega'})} + \int_{u'}^{u} \|\nabla_4 \phi\|_{L^\infty(S_{u, \omega})} du'' \right)
$$

$$
\|\phi\|_{L^\infty(S_{u, \omega})} \leq C \left( \|\phi\|_{L^\infty(S_{u', \omega'})} + \int_{u'}^{u} \|\nabla_3 \phi\|_{L^\infty(S_{u', \omega'})} du'' \right)
$$

for any tensor $\phi$. 

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Proof. This follows simply from integrating along the integral curves of $L$ and $\mathcal{L}$, and the estimate on $\Omega$ in Proposition 1. \hfill \square

5.3. Sobolev Embedding. Using the upper and lower bounds of the volume form, Sobolev embedding theorems in our setting follows from standard Sobolev embedding theorems (see [18]):

Proposition 7. There exists $\epsilon_0 = \epsilon_0(\Delta_1)$ such that as long as $\epsilon \leq \epsilon_0$, we have

$$||\phi||_{L^4(S_{u,\Omega})} \leq C \sum_{i=0}^{1} ||\nabla^i \phi||_{L^2(S_{u,\Omega})}$$

for any tensor $\phi$.

We also have the following Sobolev embedding theorem for the $L^\infty$ norm:

Proposition 8. There exists $\epsilon_0 = \epsilon_0(\Delta_1)$ such that as long as $\epsilon \leq \epsilon_0$, we have

$$||\phi||_{L^\infty(S_{u,\Omega})} \leq C (||\phi||_{L^2(S_{u,\Omega})} + ||\nabla \phi||_{L^4(S_{u,\Omega})})$$

for any tensor $\phi$. As a consequence,

$$||\phi||_{L^\infty(S_{u,\Omega})} \leq C \sum_{i=0}^{2} ||\nabla^i \phi||_{L^2(S_{u,\Omega})}.$$  

5.4. Commutation Formulae. We have the following formula from [15]:

Proposition 9. The commutator $[\nabla_4, \nabla]$ acting on an $(0, r)$ tensor is given by

$$[\nabla_4, \nabla_3, \nabla] \phi_{A_1...A_r} = D_4 D_B \phi_{A_1...A_r} + (\nabla_B \log \Omega) \nabla_4 \phi_{A_1...A_r} - (\gamma^{-1})_{D}^{C} x_{D} B \nabla C \phi_{A_1...A_r}$$

$$\sum_{i=1}^{r} (\gamma^{-1})_{D}^{C} x_{D} B \nabla C \phi_{A_1...A_r}. $$

Similarly, the commutator $[\nabla_3, \nabla]$ is given by

$$[\nabla_3, \nabla_3, \nabla] \phi_{A_1...A_r} = D_3 D_B \phi_{A_1...A_r} + (\nabla_B \log \Omega) \nabla_3 \phi_{A_1...A_r} - (\gamma^{-1})_{D}^{C} x_{D} B \nabla C \phi_{A_1...A_r}$$

$$\sum_{i=1}^{r} (\gamma^{-1})_{D}^{C} x_{D} B \nabla C \phi_{A_1...A_r}. $$

Recall the schematic notation

$$\psi \in \{\eta, \eta\}, \quad \psi_B \in \{\text{tr} \chi, \hat{\chi}, \omega\}, \quad \psi_H \in \{\text{tr} \chi, \hat{\chi}, \omega\}.$$  

By induction and the schematic Codazzi equations

$$\beta = \nabla \chi + \psi \chi = \nabla \psi_H + \psi \psi_H,$$

$$\beta = \nabla \chi + \psi \chi = \nabla \psi_H + \psi \psi_H,$$

we get the following schematic formula for repeated commutations (see [18]):

Proposition 10. Suppose $\nabla_4 \phi = F_0$ for some tensors $\phi$ and $F_0$. Let $F_i$ be the tensor defined by $\nabla_4 \nabla^i \phi = F_i$. Then

$$F_i \sim \sum_{i_1 + i_2 + i_3 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} F_0 + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} \phi.$$
Similarly, suppose $\nabla_3 \phi = G_0$ for some tensors $\phi$ and $G_0$. Let $G_i$ be the tensor defined by $\nabla_3 \nabla^i \phi = G_i$. Then

$$G_i \sim \sum_{i_1+i_2+i_3 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} G_0 + \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \nabla^{i_4} \phi.$$  

5.5. General Elliptic Estimates for Hodge Systems. We recall the definition of the divergence and curl of a symmetric covariant tensor of rank $r$:

$$(\text{div } \phi)_{A_1...A_r} = \nabla^B \phi_{BA_1...A_r},$$

$$(\text{curl } \phi)_{A_1...A_r} = \epsilon^{BC} \nabla_B \phi_{CA_1...A_r},$$

where $\epsilon$ is the volume form associated to the metric $\gamma$. Recall also that the trace is defined to be

$$(\text{tr} \phi)_{A_1...A_{r-1}} = (\gamma^{-1})^{BC} \phi_{BCA_1...A_{r-1}}.$$  

The following elliptic estimate is standard (see for example [6] or [3]):

**Proposition 11.** Let $\phi$ be a symmetric $r$ covariant tensor on a 2-sphere $(S^2, \gamma)$ satisfying

$$(\text{div } \phi) = f, \quad (\text{curl } \phi) = g, \quad (\text{tr} \phi) = h.$$  

Suppose also that

$$\sum_{i \leq 2} \| \nabla^i K \|_{L^2(S)} < \infty.$$  

Then for $i \leq 4$, there exists a constant $C_E$ depending only on $\sum_{i \leq 2} \| \nabla^i K \|_{L^2(S)}$ such that

$$\| \nabla^i \phi \|_{L^2(S)} \leq C_E \left( \sum_{j=0}^{i-1} (\| \nabla^j f \|_{L^2(S)} + \| \nabla^j g \|_{L^2(S)} + \| \nabla^j h \|_{L^2(S)} + \| \phi \|_{L^2(S)}) \right).$$

For the special case that $\phi$ a symmetric traceless 2-tensor, we only need to know its divergence:

**Proposition 12.** Suppose $\phi$ is a symmetric traceless 2-tensor satisfying

$$(\text{div } \phi) = f.$$  

Suppose moreover that

$$\sum_{i \leq 2} \| \nabla^i K \|_{L^2(S)} < \infty.$$  

Then for $i \leq 4$, there exists a constant $C_E$ depending only on $\sum_{i \leq 2} \| \nabla^i K \|_{L^2(S)}$ such that

$$\| \nabla^i \phi \|_{L^2(S)} \leq C_E \left( \sum_{j=0}^{i-1} (\| \nabla^j f \|_{L^2(S)} + \| \phi \|_{L^2(S)}) \right).$$

**Proof.** This follows from Proposition 11 and the fact that

$$\text{curl } \phi = * f.$$  

□
6. Estimates for the Ricci Coefficients via Transport Equations

In this section, we prove $L^2$ estimates for the Ricci coefficients and their first, second and third derivatives. We will assume bounds for $R$ and $\mathcal{O}_{4,2}$ and show that for $\epsilon_0$ chosen to be sufficiently small, $\sum_{i \leq 3} \mathcal{O}_{i,2}$ is likewise bounded. In order to achieve this, we will improve the bootstrap assumption (A1).

Recall that we will use the following notation: $\psi \in \{\eta, \tilde{\eta}\}$, $\psi H \in \{\text{tr} \chi, \hat{\chi}, \omega\}$ and $\psi H' \in \{\text{tr} \chi, \hat{\chi}, \omega\}$.

We first show bounds for $\psi$.

**Proposition 13.** Assume $R < \infty$.

Then there exists $\epsilon_0 = \epsilon_0(\Delta_1, R)$ such that whenever $\epsilon \leq \epsilon_0$,

$$\sum_{i \leq 3} \mathcal{O}_{i,2}[\psi] \leq C(\mathcal{O}_0),$$

i.e., the bounds depend only on the initial data norm $\mathcal{O}_0$. In particular, $C(\mathcal{O}_0)$ is independent of $\Delta_1$.

**Proof.** We first estimate $\eta$, the estimates for $\eta$ is similar after we replace $u$ with $\tilde{u}$ and 3 with 4. Using the null structure equations, we have a schematic equation of the type

$$\nabla^4 \eta = \beta + \psi H' \psi.$$  \hspace{1cm} (30)

We also commute the null structure equations with angular derivatives to get

$$\nabla^4 \nabla^i \eta = \sum_{i_1 + i_2 + i_3 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \beta + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi H'.$$  \hspace{1cm} (30)

By Proposition 5, in order to estimate $|| \nabla^i \eta ||_{L^\infty L^2 L^2(S)}$, it suffices to estimate the initial data and the $|| \cdot ||_{L^\infty L^2 L^2(S)}$ norm of the right hand side (30). We note that our bootstrap assumption imply that the right hand side is bounded in a weighted $L^2_u$ norm, which by Cauchy-Schwarz controls the $L^1_u$ norm.

We first estimate the curvature term

$$\sum_{i_1 + i_2 + i_3 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \beta.$$  

For the terms such that at most 1 derivative falling on $\psi$, the bootstrap assumption [A1] allows us to control $\sum_{i \leq 1} || \nabla^i \psi ||_{L^\infty L^2 L^\infty(S)}$ by $\Delta_1$. We then need to control $\sum_{i \leq 3} \nabla^i \beta$ in $L^\infty_u L^2_u L^2(S)$. By Cauchy-Schwarz, since $f(\psi)^{-1}$ is smaller than $\epsilon$ in $L^2_u$, we can bound this by $\sum_{i \leq 3} \nabla^i \beta$ in
the weighted norms. More precisely, we have

\[ \left\| \sum_{i_1 \leq 1, i_2 \leq 3, i_3 \leq 3} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \beta \right\|_{L^\infty_u L^2_x L^2(S)} \]

\[ \leq C \left( \sum_{i_1 \leq 1, i_2 \leq 3} \left\| \nabla^{i_1} \psi \right\|_{L^\infty_u L^2_x L^2(S)}^2 \right)^{1/2} \left( \sum_{i_3 \leq 3} \left\| \nabla^{i_3} \beta \right\|_{L^\infty_u L^1_x L^2(S)} \right) \]

\[ + C \left( \sum_{i_1 \leq 1, i_2 \leq 3} \left\| \nabla^{i_1} \psi \right\|_{L^\infty_u L^2_x L^2(S)}^2 \right)^{1/2} \left( \sum_{i_3 \leq 3} \left\| \nabla^{i_3} \beta \right\|_{L^\infty_u L^1_x L^2(S)} \right) \left\| f(u)^{-1} \right\|_{L^\infty_u L^2_x L^\infty(S)} \]

\[ \leq C \epsilon (1 + \Delta_1)^3 R, \tag{31} \]

For the term where exactly 2 derivatives fall on \( \psi \) (notice that this is the highest number of derivatives that can fall on \( \psi \)), we control \( \nabla^3 \psi \) in \( L^\infty_u L^\infty_x L^2(S) \) by \( \Delta_1 \) (using (A1)). Thus we are left with \( \beta \) in \( L^\infty_u L^1_x L^\infty(S) \). By Sobolev embedding (Proposition 8), this can be bounded by \( \sum_{i \leq 3} \left\| \nabla^i \beta \right\|_{L^\infty_u L^2_x L^2(S)} \), which in turn can be controlled by \( R \) after applying Cauchy-Schwarz as in (31). More precisely,

\[ \left\| \nabla^2 \psi \beta \right\|_{L^\infty_u L^2_x L^2(S)} \]

\[ \leq C \left\| \nabla^2 \psi \right\|_{L^\infty_u L^2_x L^2(S)} \left\| \beta \right\|_{L^\infty_u L^1_x L^\infty(S)} \]

\[ \leq C \left\| \nabla^2 \psi \right\|_{L^\infty_u L^2_x L^2(S)} \left( \sum_{i \leq 2} \left\| \nabla^i \beta \right\|_{L^\infty_u L^1_x L^2(S)} \right) \]

\[ \leq C \left\| \nabla^2 \psi \right\|_{L^\infty_u L^2_x L^2(S)} \left( \sum_{i \leq 2} \left\| f(u) \nabla^i \beta \right\|_{L^\infty_u L^2_x L^\infty(S)} \right) \left\| f(u)^{-1} \right\|_{L^\infty_u L^2_x L^\infty(S)} \]

\[ \leq C \epsilon \Delta_1 R. \tag{32} \]

Combining (31) and (32), we have

\[ \left\| \sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \beta \right\|_{L^\infty_u L^2_x L^2(S)} \leq C \epsilon (1 + \Delta_1)^3 R. \]

We then estimate the nonlinear term. We separate the terms where more derivatives fall on \( \psi_H \) and those where more derivatives fall on \( \psi \):

\[ \left\| \sum_{i_1 \leq 1, i_2 \leq 4, i_3 \leq 3} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \psi \right\|_{L^\infty_u L^2_x L^2(S)} \]

\[ \leq C \left( \sum_{i_1 \leq 1, i_2 \leq 4} \left\| \nabla^{i_1} \psi \right\|_{L^\infty_u L^\infty_x L^2(S)} \left( \sum_{i_3 \leq 3} \left\| \nabla^{i_3} \psi \right\|_{L^\infty_u L^1_x L^2(S)} \right) \right) \]

\[ + C(1 + \left\| \psi \right\|_{L^\infty_u L^\infty_x L^2(S)}) \left( \sum_{i_1 \leq 3} \left\| \nabla^{i_1} \psi \right\|_{L^\infty_u L^\infty_x L^2(S)} \right) \left( \sum_{i_2 \leq 1} \left\| \nabla^{i_2} \psi \right\|_{L^\infty_u L^1_x L^\infty(S)} \right) \]

\[ \leq C \Delta_1 (1 + \Delta_1)^3 \sum_{i_3 \leq 3} \left\| \nabla^{i_3} \psi \right\|_{L^\infty_u L^1_x L^2(S)} + \sum_{i_2 \leq 1} \left\| \nabla^{i_2} \psi \right\|_{L^\infty_u L^1_x L^\infty(S)} \]

\[ \leq C \Delta_1 (1 + \Delta_1)^3 \left\| f(u)^{-1} \right\|_{L^\infty_u L^2_x L^\infty(S)} \]

\[ \times \left( \sum_{i_3 \leq 3} \left\| f(u) \nabla^{i_3} \psi \right\|_{L^\infty_u L^2_x L^2(S)} + \sum_{i_2 \leq 1} \left\| f(u) \nabla^{i_2} \psi \right\|_{L^\infty_u L^2_x L^\infty(S)} \right) \]

\[ \leq C \Delta_1^2 (1 + \Delta_1)^3 \epsilon. \]
Hence, by Proposition 5, we have
\[ \sum_{i \leq 3} ||\nabla^i \eta||_{L^\infty_u L^2(S)} \leq C(O_0) + C\epsilon(\Delta_1^3(1 + \Delta_1)^3 + R(1 + \Delta_1)^3) \leq C(O_0), \]
after choosing \( \epsilon \) to be sufficiently small. Similarly, we consider the equation for \( \nabla_3 \nabla^i \eta \) to get
\[ \sum_{i \leq 3} ||\nabla^i \eta||_{L^\infty_u L^2(S)} \leq C(O_0). \]

We now move to the terms that we denote by \( \psi_H \), i.e., \( \text{tr}\chi, \hat{\chi}, \omega \). All of them obey a \( \nabla_4 \) equation. Unlike the previous estimates for \( \psi_H \), the initial data for the quantities \( \psi_H \) are not in \( L^\infty_u \). We will therefore prove only a bound for \( \psi_H \) in the weighted norm \( ||f(u)||_{L^2_\omega L^\infty(S)}. \)

**Proposition 14.** Assume
\[ R < \infty, \quad \tilde{O}_{42} < \infty. \]
Then there exists \( \epsilon_0 = \epsilon_0(\Delta_1, R, \tilde{O}_{42}) \) such that whenever \( \epsilon \leq \epsilon_0, \)
\[ \sum_{i \leq 3} O_{i,2}[\psi_H] \leq C(O_0). \]
In particular, as before, this estimate is independent of \( \Delta_1. \)

**Proof.** According to the definition of the \( O_{i,2} \) norm, we need to control the weighted \( L^2_u L^\infty_u L^2(S) \) norm of \( \psi_H \). Using the null structure equations, for each \( \psi_H \in \{\text{tr}\chi, \hat{\chi}, \omega\} \), we have an equation of the type
\[ \nabla_4 \psi_H = K + \nabla \eta + \psi \psi + \psi_H \psi_H. \]
We also use the null structure equations commuted with angular derivatives:
\[ \nabla_4 \nabla^i \psi_H = \sum_{i_1 + i_2 + i_3 = i} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} (K + \nabla \eta) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi \]
\[ + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} \psi_H. \]
We estimate the curvature term using the curvature norm. Recall that the curvature norm for \( K \) along the \( H_u \) is weighted with \( f(u) \). Using the Sobolev embedding theorem in Propositions 7 and 8 we have
\[ || \sum_{i_1 + i_2 + i_3 \leq 3} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} K ||_{L^2_u L^2(S)} \]
\[ \leq C f(u)^{-1} \left( \sum_{i_1 \leq 1, i_2 \leq 3} ||\nabla^{i_1} \psi||_{L^\infty_u L^2(S)} ||f(u)\nabla^{i_3} K||_{L^2_u L^2(S)} \right) 
+ C f(u)^{-1} ||\nabla^2 \psi||_{L^2_u L^4(S)} ||f(u) K||_{L^2_u L^4(S)} \]
\[ \leq C f(u)^{-1} \left( \sum_{i_1 \leq 3, i_2 \leq 3} ||\nabla^{i_1} \psi||_{L^2_u L^2(S)} ||f(u)\nabla^{i_3} K||_{L^2_u L^2(S)} \right) 
\leq C \epsilon \frac{1}{2} f(u)^{-1} (1 + \Delta_1)^3 R. \]
The term linear in $\nabla^3 \eta$ can be estimated analogously but using the $\tilde{O}_{4,2}$ norms instead of the $\mathcal{R}$ norms:

$$
\| \sum_{i_1+i_2+i_3 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3+1} \eta \|_{L_\infty^2 L^2(S)} \\
\leq C \varepsilon \bigg( \sum_{i_1 \leq 1} \sum_{i_2 \leq 2} \| \nabla^{i_1} \psi \|_{L_\infty^2 L^2(S)} \bigg) + C \varepsilon \frac{3}{2} f(u)^{-1} \| f(u) \nabla^4 \eta \|_{L_\infty^2 L^2(S)} \\
\leq C \varepsilon (1 + \Delta_1)^4 + C \varepsilon \frac{3}{2} f(u)^{-1} \tilde{O}_{4,2} \tag{35}
$$

We now move to control the terms that are nonlinear in the Ricci coefficients. First, we estimate the terms without $\psi_H$ or $\psi_H^*$:

$$
\| \sum_{i_1+i_2+i_3+i_4 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \|_{L_\infty^2 L^2(S)} \\
\leq C \varepsilon \bigg( \sum_{i_1 \leq 1, i_2 \leq 3} \| \nabla^{i_1} \psi \|_{L_\infty^2 L^\infty(S)} \bigg) \bigg( \sum_{i_3 \leq 3} \| \nabla^{i_3} \psi_H \|_{L_\infty^2 L^2(S)} \bigg) \bigg( \sum_{i_4 \leq 3} \| \nabla^{i_4} \psi_H \|_{L_\infty^2 L^2(S)} \bigg) \\
\leq C \varepsilon (1 + \Delta_1)^5. \tag{36}
$$

We then control the term with both $\psi_H$ and $\psi_H^*$:

$$
\| \sum_{i_1+i_2+i_3+i_4 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \psi_H \|_{L_\infty^2 L^2(S)} \\
\leq C \bigg( \sum_{i_1 \leq 1, i_2 \leq 3} \| \nabla^{i_1} \psi \|_{L_\infty^2 L^\infty(S)} \bigg) \bigg( \sum_{i_3 \leq 3} \| \nabla^{i_3} \psi_H \|_{L_\infty^2 L^2(S)} \bigg) \bigg( \sum_{i_4 \leq 3} \| \nabla^{i_4} \psi_H \|_{L_\infty^2 L^2(S)} \bigg) \\
+ C \bigg( \sum_{i_1 \leq 1, i_2 \leq 3} \| \nabla^{i_1} \psi \|_{L_\infty^2 L^\infty(S)} \bigg) \bigg( \sum_{i_3 \leq 3} \| \nabla^{i_3} \psi_H \|_{L_\infty^2 L^2(S)} \bigg) \bigg( \sum_{i_4 \leq 3} \| \nabla^{i_4} \psi_H \|_{L_\infty^2 L^\infty(S)} \bigg) \\
+ C \| \nabla^2 \psi \|_{L_\infty^2 L^2(S)} \| \psi_H \|_{L_\infty^2 L^\infty(S)} \| \psi_H^* \|_{L_\infty^2 L^\infty(S)} \\
\leq C \varepsilon (1 + \Delta_1)^3 \bigg( \sum_{i_1 \leq 3} \| f(u) \nabla^{i_1} \psi_H \|_{L_\infty^2 L^2(S)} \bigg) \bigg( \sum_{i_2 \leq 3} \| \nabla^{i_2} \psi_H \|_{L_\infty^2 L^2(S)} \bigg) \\
\leq C \varepsilon \Delta_1 (1 + \Delta_1)^3 \bigg( \sum_{i_3 \leq 3} \| \nabla^{i_3} \psi_H \|_{L_\infty^2 L^2(S)} \bigg). \tag{37}
$$

Therefore, by equations (34), (35), (36) and (37), we have that for every fixed $u$,

$$
\sum_{i_3 \leq 3} \| \nabla^2 \psi_H \|_{L_\infty^2 L^\infty(S)} \\
\leq C O_0 + C \varepsilon \frac{1}{2} f(u)^{-1} (\mathcal{R} + \tilde{O}_{4,2}) + C \varepsilon (1 + \Delta_1)^5 + C \varepsilon \Delta_1 (1 + \Delta_1)^3 \bigg( \sum_{i_3 \leq 3} \| \nabla^{i_3} \psi_H \|_{L_\infty^2 L^2(S)} \bigg).$$
We now multiply this inequality by $f(u)$ and take the $L^2$ norm in $u$ to get
\[
\sum_{i \leq 3} \|f(u)\nabla^i \psi_H\|_{L^2_u L^\infty_w L^2(S)} \\
\leq C(O_0) + C\epsilon^2 \|f(u)(f(u)^{-1})\|_{L^2_u(R + \tilde{O}_{4,2})} + C\epsilon(1 + \Delta_1)^5 \\
+ C\epsilon \Delta_1(1 + \Delta_1)^3 \left(\sum_{i \leq 3} \|f(u)\nabla^i \psi_H\|_{L^2_u L^\infty_w L^2(S)}\right) \\
\leq C(O_0),
\]
for $\epsilon$ sufficiently small. \hfill \Box

Using instead the equation for $\nabla_3 \psi_H$, we obtain the following estimates in a completely analogous manner:

**Proposition 15.** Assume
\[
R < \infty, \quad \tilde{O}_{4,2} < \infty.
\]
Then there exists $\epsilon_0 = \epsilon_0(\Delta_1, R, \tilde{O}_{4,2})$ such that whenever $\epsilon \leq \epsilon_0$, 
\[
\sum_{i \leq 3} O_{i,2}[\psi_H] \leq C(O_0).
\]
In particular, this estimate is independent of $\Delta_1$.

By the Sobolev embedding theorems given by Propositions \[7\] and \[8\], we have thus closed our bootstrap assumption \(A1\) and proved the desired estimates for the Ricci coefficients and their first two angular derivatives. We summarize this in the following proposition.

**Proposition 16.** Assume
\[
R < \infty, \quad \tilde{O}_{4,2} < \infty.
\]
Then there exists $\epsilon_0 = \epsilon_0(O_0, R, \tilde{O}_{4,2})$ such that whenever $\epsilon \leq \epsilon_0$, 
\[
\sum_{i \leq 3} O_{i,2}[\psi, \psi_H, \psi_H] \leq C(O_0).
\]

### 7. Elliptic Estimates for Fourth Derivatives of the Ricci Coefficients

We now estimate the 4th derivative of the Ricci coefficients. We introduce the following bootstrap assumption:
\[
\tilde{O}_{4,2} \leq \Delta_2. \tag{A2}
\]
The estimates for the 4th derivative of the Ricci coefficients cannot be achieved only by the transport equations since there would be a loss in derivatives. We can however use the transport equation - Hodge system type estimates as in \[13\], \[3\], \[10\]. We will first derive estimates for some chosen combination of $\nabla^4(\psi, \psi_H, \psi_H) + (\beta, K, \sigma, \beta)$ by using transport equations. We will then show that the estimates for all the 4th derivatives of the Ricci coefficients can be proved via elliptic estimates.

In order to apply elliptic estimates using Proposition \[5.3\], we need to first control the Gauss curvature and its first and second derivatives in $L^2(S)$. 


Proposition 17. Assume
\[ R < \infty. \]
Then there exists \( \epsilon_0 = \epsilon_0(\Delta_2, R) \) such that whenever \( \epsilon \leq \epsilon_0 \),
\[ \sum_{i \leq 2} ||\nabla^i K||_{L_0^\infty L_2^\infty L^2(S)} \leq C(O_0, R_0). \]

Proof. \( K \) obeys the following Bianchi equation:
\[ \nabla_4 K = \nabla \beta + \psi_H K + \sum_{i_1 + i_2 + i_3 \leq 1} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H. \]
Commuting with angular derivatives, we have, for \( i \leq 2 \),
\[ \nabla_4 \nabla^i K = \sum_{i_1 + i_2 + i_3 \leq 2} \nabla^{i_1} \psi^{i_2} \nabla^{i_3 + 1} \beta + \sum_{i_1 + i_2 + i_3 + i_4 \leq 2} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} K + \sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} K. \]
By Proposition 5 in order to control \( \nabla^i K \) in \( L_0^\infty L_2^\infty L^2(S) \), we need to bound the right hand side in \( L_0^\infty L_2^1 L^2(S) \). We first control the term containing \( \beta \):
\[ \sum_{i_1 + i_2 + i_3 \leq 2} \nabla^{i_1} \psi^{i_2} \nabla^{i_3 + 1} \beta \leq C(\sum_{i_1 \leq 1, i_2 \leq 2} ||\nabla^{i_1} \psi||_{L_0^\infty L_2^\infty L^\infty(S)} ||f(u)||^{-1} ||\nabla^{i_2} K||_{L_0^\infty L_2^2 L^2(S)} \leq C(O_0) \epsilon R, \]
where we have used the estimates for \( \psi \) given by Proposition 16. The term containing \( K \) can be controlled by
\[ \sum_{i_1 + i_2 + i_3 + i_4 \leq 2} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} K \leq C(\sum_{i_1 \leq 1, i_2 \leq 2} ||\nabla^{i_1} \psi||_{L_0^\infty L_2^\infty L^\infty(S)} \int_0^u \sum_{i_3 + i_4 \leq 2} ||\nabla^{i_3} \psi_H \nabla^{i_4} K||_{L_2^\infty L^2(S_{u,u'})} du' \leq C(O_0) \int_0^u \sum_{i_1 \leq 2} ||\nabla^{i_1} \psi_H||_{L_2^\infty L^2(S_{u,u'})} (\sum_{i_2 \leq 2} ||\nabla^{i_2} K||_{L_2^\infty L^2(S_{u,u'})} du'). \]
The remaining term has been bounded in the previous section. By Proposition 16 and Proposition 16
\[ || \sum_{i_1 + i_2 + i_3 \leq 2} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H ||_{L_0^\infty L_2^1 L^2(S)} \leq C(O_0) \epsilon. \]
Therefore, by Proposition 5
\[ \sum_{i \leq 2} ||\nabla^i K||_{L_0^\infty L^2(S_{u,u'})} \leq C(O_0)(1 + \epsilon + \epsilon R + \int_0^u (\sum_{i_1 \leq 2} ||\nabla^{i_1} \psi_H||_{L_0^\infty L^2(S_{u,u'})} (\sum_{i_2 \leq 2} ||\nabla^{i_2} K||_{L_0^\infty L^2(S_{u,u'})} du'). \]
Gronwall’s inequality implies
\[
\sum_{i \leq 2} ||\nabla^i K||_{L^\infty_u L^2(S_u,u)} \leq C(O_0) \exp\left(\sum_{i \leq 2} ||\nabla^i \psi_h||_{L^2_u L^2(S)}\right) \leq C(O_0)
\]
since by Proposition 16, \[\sum_{i \leq 1} ||\nabla^i \psi_h||_{L^2_u L^2(S)} \leq C(O_0)\] for \(\epsilon\) sufficiently small. \(\Box\)

It is easy to see that since \(\bar{\sigma}\) satisfies a similar schematic Bianchi equation as \(K\), we also have the following estimates for \(\bar{\sigma}\) and its derivative:

**Proposition 18.** Assume \(\mathcal{R} < \infty\).

Then there exists \(\epsilon_0 = \epsilon_0(\Delta_2, \mathcal{R})\) such that whenever \(\epsilon \leq \epsilon_0\),
\[
\sum_{i \leq 2} ||\nabla^i \bar{\sigma}||_{L^\infty_u L^2(S)} \leq C(O_0).
\]

Using Proposition 17, we now control the fourth derivatives of the Ricci coefficients. We first bound \(\nabla^4 \text{tr} \chi\) using the transport equation.

**Proposition 19.** There exists \(\epsilon_0 = \epsilon_0(O_0, \Delta_2)\) such that whenever \(\epsilon \leq \epsilon_0\),
\[
||f(u)\nabla^4 \text{tr} \chi||_{L^2_u L^\infty(S_v,v)} \leq C(O_0).
\]

**Proof.** Consider the following equation:
\[
\nabla_4 \text{tr} \chi = \psi_H \psi_H,
\]
After commuting with angular derivatives, we have
\[
\nabla_4 \nabla^4 \text{tr} \chi = \sum_{i_1 + i_2 + i_3 + i_4 = 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} \psi_H.
\]
By Proposition 5 in order to control \(\nabla^4 \text{tr} \chi\) in \(L^2(S_u,u)\), we need to bound the right hand side in \(L^1_u L^2(S)\). Using the fact that \(f(u)\) is decreasing, this can be achieved using Sobolev embedding (Propositions 7 and 8) by
\[
||\sum_{i_1 + i_2 + i_3 + i_4 \leq 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} \psi_H||_{L^2_u L^2(S)} \leq C f(u)^{-2} \sum_{i_1 \leq 3, i_2 \leq 2} ||\nabla^{i_1} \psi||_{L^2_u L^2(S)} \left(\sum_{i_3 \leq 2} ||f(u)\nabla^{i_3} \psi_H||_{L^2_u L^2(S)}\right) \left(\sum_{i_4 \leq 4} ||f(u)\nabla^{i_4} \psi_H||_{L^2_u L^2(S)}\right) \leq C f(u)^{-2} \Delta_2.
\]
By Proposition 5 we have
\[
||\nabla^4 \text{tr} \chi||_{L^2(S_u,u)} \leq C(O_0) + C(O_0) f(u)^{-2} \Delta_2. \tag{38}
\]
Multiplying (38) by \(f(u)\) and taking first the \(L^\infty\) norm in \(u\) and then the \(L^2\) norm in \(u\), we have
\[
||f(u)\nabla^4 \text{tr} \chi||_{L^2_u L^\infty(S_v,v)} \leq C(O_0) + C(O_0) ||f(u)\|^{-1} \|L^2_u \Delta_2 \leq C(O_0) + C \epsilon \Delta_2,
\]
where we have used
\[ \| f(u)^{-1} \|_{L^2_u} \leq C \epsilon. \]
Thus, the conclusion follows by choosing \( \epsilon \) to be sufficiently small depending on \( \Delta_2 \).

Once we have the estimates for \( \nabla^4 \text{tr} \chi \), we can control \( \nabla^4 \hat{\chi} \) using elliptic estimates:

**Proposition 20.** Assume

\[ \mathcal{R} < \infty. \]

Then there exists \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, \Delta_2, \mathcal{R}) \) such that whenever \( \epsilon \leq \epsilon_0 \),
\[ \| f(u) \nabla^4 \hat{\chi} \|_{L^\infty_u L^2_u L^2(S)} \leq C(\mathcal{O}_0) + C \mathcal{R}. \]

**Proof.** We now use the Codazzi equation
\[ \text{div} \, \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \beta + \psi \nabla H \]
and apply elliptic estimates from Proposition 12 to get
\[
\| \nabla^4 \hat{\chi} \|_{L^2(S)} \leq C \left( \sum_{i \leq 4} \| \nabla^i \text{tr} \chi \|_{L^2(S)} + \sum_{i \leq 3} \| \nabla^i \beta \|_{L^2(S)} + \sum_{i_1 + i_2 \leq 3} \| \nabla^{i_1} \psi \nabla^{i_2} \psi \|_{L^2(S)} \right).
\]
(39)

Notice that we can apply elliptic estimates using Proposition 12 since we have estimates for the Gauss curvature from Proposition 17. Multiply (39) by \( f(u) \) and take \( L^\infty_u L^2_u \) norm to get
\[
\| f(u) \nabla^4 \hat{\chi} \|_{L^\infty_u L^2_u L^2(S)} \leq C \left( \sum_{i \leq 4} \| f(u) \nabla^i \text{tr} \chi \|_{L^\infty_u L^2_u L^2(S_{0,u})} + \sum_{i \leq 3} \| f(u) \nabla^i \beta \|_{L^\infty_u L^2_u L^2(S)} \right.
\]
\[
+ \sum_{i_1 + i_2 \leq 3} \| f(u) \nabla^{i_1} \psi \nabla^{i_2} \psi \|_{L^\infty_u L^2_u L^2(S)} + \sum_{i \leq 3} \| f(u) \nabla^i \hat{\chi} \|_{L^\infty_u L^2_u L^2(S)} \bigg) \leq C(\mathcal{O}_0) + C \mathcal{R} + C \sum_{i_1 + i_2 \leq 3} \| f(u) \nabla^{i_1} \psi \nabla^{i_2} \psi \|_{L^\infty_u L^2_u L^2(S)}.
\]

By Proposition 16 and Sobolev embedding theorem in Propositions 7 and 8, we have
\[
\sum_{i_1 + i_2 \leq 3} \| f(u) \nabla^{i_1} \psi \nabla^{i_2} \psi \|_{L^\infty_u L^2_u L^2(S)} \leq C \left( \sum_{i_1 \leq 3} \| \nabla^{i_1} \psi \|_{L^\infty_u L^\infty_u L^2(S)} \right) \sum_{i_2 \leq 3} \| f(u) \nabla^{i_2} \psi \|_{L^\infty_u L^2_u L^2(S)} \leq C(\mathcal{O}_0).
\]

Therefore,
\[ \| f(u) \nabla^4 \hat{\chi} \|_{L^\infty_u L^2_u L^2(S)} \leq C(\mathcal{O}_0) + C \mathcal{R}. \]

\[ \square \]

The \( \mathcal{O}_{4,2} \) estimates for \( \nabla^4 \text{tr} \chi \) and \( \nabla^4 \hat{\chi} \) follow identically as that for \( \nabla^4 \text{tr} \chi \) and \( \nabla^4 \hat{\chi} \):
Proposition 21. Assume \( \mathcal{R} < \infty \).

Then there exists \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, \Delta_2, \mathcal{R}) \) such that whenever \( \epsilon \leq \epsilon_0 \),
\[
\| f(u) \nabla^4 \text{tr} X \|_{L^\infty_u L^2_x L^2(S)} \leq C(\mathcal{O}_0),
\]
and
\[
\| f(u) \nabla^4 \hat{X} \|_{L^\infty_u L^2_x L^2(S)} \leq C(\mathcal{O}_0) + C\mathcal{R}.
\]

We then prove estimates for \( \nabla^4 \eta \). To do so, we first prove estimates for third derivatives of \( \mu = -\text{div} \eta - K \) and recover the control for \( \nabla^4 \eta \) via elliptic estimates.

Proposition 22. Assume \( \mathcal{R} < \infty \).

Then there exists \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, \Delta_2, \mathcal{R}) \) such that whenever \( \epsilon \leq \epsilon_0 \),
\[
\| f(u) \nabla^4 \eta \|_{L^\infty_u L^2_x L^2(S)} \leq C(\mathcal{O}_0)(\epsilon^{\frac{1}{2}} + \mathcal{R}),
\]
and
\[
\| f(u) \nabla^4 \hat{\eta} \|_{L^\infty_u L^2_x L^2(S)} \leq C(\mathcal{O}_0)(\epsilon^{\frac{1}{2}} + \mathcal{R}).
\]

Proof. Recall that \( \mu = -\text{div} \eta + K \).

\( \mu \) satisfies the following equation:
\[
\nabla^4 \mu = \psi_H(K, \sigma) + \sum_{i_1 + i_2 + i_3 = 1} \psi^{j_1} \nabla^{i_2} \psi^{i_3} \psi_H.
\]

After commuting with angular derivatives, we get
\[
\nabla^4 \nabla^3 \mu = \sum_{i_1 + i_2 + i_3 + i_4 = 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4}(K, \sigma) + \sum_{i_1 + i_2 + i_3 + i_4 = 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \psi_H.
\]

We now control each of the terms on the right hand side in \( L^1_u L^2_x L^2(S) \). The first term, which contains curvature components, can be estimated by
\[
\| \sum_{i_1 + i_2 + i_3 + i_4 = 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4}(K, \sigma) \|_{L^1_u L^2_x L^2(S)}
\]
\[
\leq C f(u)^{-1} f(u)^{-1} \left( \sum_{i_1 \leq 3} \sum_{i_2 \leq 2} \| \nabla^{i_1} \psi \|_{L^\infty_u L^2_x L^2(S)} \left( \sum_{i_3 \leq 3} \| f(u) \nabla^{i_3} \psi_H \|_{L^2_u L^2_x L^2(S)} \right) \right)
\]
\[
\times \left( \sum_{i_4 \leq 3} \| f(u) \nabla^{i_4}(K, \sigma) \|_{L^2_u L^2_x L^2(S)} \right)
\]
\[
\leq C(\mathcal{O}_0) f(u)^{-1} f(u)^{-1} \mathcal{R},
\]
\[\text{[15]}\]

It is important to note that the potentially harmful term \( \psi_H \psi_H \psi_H \) is absent in this equation. This required structure is the reason that we perform this renormalization instead of using \( \mu = -\text{div} \eta + \hat{\rho} + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \) as in [18, 19].
using the bounds obtained in Proposition 16. The second term can be controlled using Sobolev embedding in Propositions 7 and 8 by

\[ \| \sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \psi \|_{L^2_u L^2(S)} \]

\[ \leq C f(u)^{-1} f(u)^{-1} \left( \sum_{i_1 \leq 4, i_2 \leq 5} \| \nabla^{i_1} \psi \|_{L^2_u L^2(S)} \right) \left( \sum_{i_3 \leq 4} \| f \nabla^{i_3} \psi \|_{L^2_u L^2(S)} \right) \left( \sum_{i_4 \leq 4} \| f \nabla^{i_4} \psi \|_{L^2_u L^2(S)} \right) \]

\[ \leq C(\mathcal{O}_0) f(u)^{-1} f(u)^{-1} (1 + \Delta_2)^2 \]

using the estimates in Proposition 16. Therefore, by Proposition 5, we have

\[ \| \nabla^3 \mu \|_{L^2(S)} \leq C(\mathcal{O}_0)(1 + f(u)^{-1} f(u)^{-1} (\mathcal{R} + (1 + \Delta_2)^2)). \tag{40} \]

Recall that the \( L^2_u \) norm of \( f(u)^{-1} \) is bounded by \( \epsilon \). Thus, multiplying (40) by \( f(u) \) and taking the \( L^2 \) norm in \( u \), we get

\[ \| f(u) \nabla^3 \mu \|_{L^2_u L^2(S)} \leq C(\mathcal{O}_0)(\epsilon^{\frac{1}{2}} + \epsilon (\mathcal{R} + (1 + \Delta_2)^2)) \leq C(\mathcal{O}_0) \epsilon^{\frac{1}{2}}, \]

for \( \epsilon \) sufficiently small. Similarly, multiplying (40) by \( f(u) \) and taking the \( L^2 \) norm in \( u \), we get

\[ \| f(u) \nabla^3 \mu \|_{L^2_u L^2(S)} \leq C(\mathcal{O}_0) \epsilon^{\frac{1}{2}}. \]

We can obtain bounds for \( \nabla^4 \eta \) from the control of \( \nabla^3 \mu \) using elliptic estimates as follows. By the div-curl systems

\[ \text{div } \eta = -\mu + K, \quad \text{curl } \eta = \hat{\sigma}, \]

and the elliptic estimates given by Propositions 11 and 17, we have

\[ \| \nabla^4 \eta \|_{L^2(S)} \leq C \left( \sum_{i \leq 3} \| \nabla^i \mu \|_{L^2(S)} + \sum_{i \leq 3} \| \nabla^i (K, \hat{\sigma}) \|_{L^2(S)} + \sum_{i \leq 3} \| \nabla^i \eta \|_{L^2(S)} \right). \]

Therefore,

\[ \| f(u) \nabla^4 \eta \|_{L^2_u L^2(S)} \]

\[ \leq C \left( \sum_{i \leq 3} \| f(u) \nabla^i \mu \|_{L^2_u L^2(S)} + \sum_{i \leq 3} \| f(u) \nabla^i (K, \hat{\sigma}) \|_{L^2_u L^2(S)} + \sum_{i \leq 3} \| f(u) \nabla^i \eta \|_{L^2_u L^2(S)} \right) \]

\[ \leq C(\mathcal{O}_0)(\epsilon^{\frac{1}{2}} + \mathcal{R}). \]

Similarly,

\[ \| f(u) \nabla^4 \eta \|_{L^2_u L^2(S)} \leq C(\mathcal{O}_0)(\epsilon^{\frac{1}{2}} + \mathcal{R}). \]

A similar proof shows that the conclusion of Proposition 22 holds also for \( \nabla^3 \eta \).

**Proposition 23.** Assume

\[ \mathcal{R} < \infty. \]

Then there exists \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, \Delta_2, \mathcal{R}) \) such that whenever \( \epsilon \leq \epsilon_0 \),

\[ \| f(u) \nabla^4 \eta \|_{L^2_u L^2 S} \leq C(\mathcal{O}_0)(\epsilon^{\frac{3}{2}} + \mathcal{R}), \]

and

\[ \| f(u) \nabla^4 \eta \|_{L^2_u L^2 S} \leq C(\mathcal{O}_0)(\epsilon^{\frac{3}{2}} + \mathcal{R}). \]

We now move to the estimates for \( \nabla^4 \omega \):
Proposition 24. Assume

\[ R < \infty. \]

Then there exists \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, \Delta_2, R) \) such that whenever \( \epsilon \leq \epsilon_0 \),

\[ \| f(u) \nabla^4 \omega \|_{L^2_\omega L^2(S)} \leq C(\mathcal{O}_0)(1 + R). \]

Proof. Let \( \omega^\dagger \) be defined as the solution to

\[ \nabla_4 \omega^\dagger = \frac{1}{2} \sigma \]

with zero data and

\[ K := -\nabla \omega + \nabla^* \omega^\dagger - \frac{1}{2} \beta. \]

By the definition of \( \omega^\dagger \), it is easy to see that using Proposition 5,

\[ \sum_{i \leq 3} \| \nabla^i \omega^\dagger \|_{L^2_\omega L^2(S)} \leq C \epsilon R \leq C(\mathcal{O}_0). \]

In other words, \( \omega^\dagger \) satisfies the same estimates as \( \psi^H \). In the proof of this proposition, we will also use \( \psi^H \) to denote \( \omega^\dagger \) (in addition to \( \text{tr}_\chi \), \( \hat{\chi} \) and \( \omega \)).

\( K \) then obeys the following equation:

\[ \nabla_4 K = \psi(K, \sigma) + \sum_{i_1 + i_2 + i_3 = 1} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi + \sum_{i_1 + i_2 + i_3 = 1} \psi^{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi^H. \]

After commuting with angular derivatives, we get

\[ \nabla_4 \nabla^3 K = \sum_{i_1 + i_2 + i_3 + i_4 = 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (K, \sigma) + \sum_{i_1 + i_2 + i_3 + i_4 = 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi. \]

Therefore,

\[ \| \nabla^3 K \|_{L^2(S_{u, \Omega})} \leq C \| \nabla^3 K \|_{L^2(S_{u, 0})} + C \| \sum_{i_1 + i_2 + i_3 + i_4 = 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (K, \sigma) \|_{L^1_\omega L^2(S)} \]

\[ + C \| \sum_{i_1 + i_2 + i_3 + i_4 = 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi \|_{L^1_\omega L^2(S)} \]

\[ + C \| \sum_{i_1 + i_2 + i_3 + i_4 = 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi^H \|_{L^1_\omega L^2(S)}. \]

Multiplying by \( f(u) \) and taking the \( L^2 \) norm in \( u \), we get

\[ \| f(u) \nabla^3 K \|_{L^2_\omega L^2(S)} \]

\[ \leq C \| f(u) \nabla^3 K \|_{L^2_\omega L^2(S_{u, 0})} + C \| f(u) \sum_{i_1 + i_2 + i_3 + i_4 = 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (K, \sigma) \|_{L^1_\omega L^1_\omega L^2(S)} \]

\[ + C \| f(u) \sum_{i_1 + i_2 + i_3 + i_4 = 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi \|_{L^2_\omega L^1_\omega L^2(S)} \]

\[ + C \| f(u) \sum_{i_1 + i_2 + i_3 + i_4 = 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi^H \|_{L^2_\omega L^1_\omega L^2(S)}. \]
The first term is an initial data term and it is bounded by a constant depending only on \( O_0 \). We estimate each of the nonlinear terms. The second term can be controlled by

\[
\|f(u)\sum_{i_1+i_2+i_3+i_4=3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} (K, \tilde{\sigma})\|_{L^2_u L^1_u L^2(S)} \\
\leq C\epsilon \left( \sum_{i_1 \leq 3} \sum_{i_2 \leq 4} \|\nabla^{i_1} \psi\|_{L^2_u L^\infty_u L^2(S)} \right) \left( \sum_{i_3 \leq 3} \|f(u)\nabla^{i_3} (K, \tilde{\sigma})\|_{L^\infty_u L^2_u L^2(S)} \right) \\
\leq C(O_0)\epsilon \mathcal{R}.
\]

The third term can be bounded by

\[
\|f(u)\sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \|_{L^2_u L^1_u L^2(S)} \\
\leq C\epsilon \left( \sum_{i_1 \leq 3} \sum_{i_2 \leq 4} \|\nabla^{i_1} \psi\|_{L^2_u L^\infty_u L^2(S)} \right) \left( \sum_{i_3 \leq 3} \|\nabla^{i_3} \psi\|_{L^\infty_u L^2_u L^2(S)} \right) \left( \sum_{i_4 \leq 4} \|f(u)\nabla^{i_4} \|_{L^\infty_u L^2_u L^2(S)} \right) \\
\leq C(O_0)\epsilon \Delta_2.
\]

The fourth term can be estimated by

\[
\|f(u)\sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \psi_H \|_{L^2_u L^1_u L^2(S)} \\
\leq C\epsilon \left( \sum_{i_1 \leq 3} \sum_{i_2 \leq 4} \|\nabla^{i_1} \psi\|_{L^2_u L^\infty_u L^2(S)} \right) \left( \sum_{i_3 \leq 3} \|\nabla^{i_3} \psi_H\|_{L^1_u L^\infty_u L^2(S)} \right) \left( \sum_{i_4 \leq 4} \|f(u)\nabla^{i_4} \psi_H\|_{L^\infty_u L^2_u L^2(S)} \right) \\
+ C\|f(u)^{-1}\|_{L^2_u} \|f(u)\nabla^4 \psi_H\|_{L^\infty_u L^2_u L^2(S)} \|f(u)\psi_H\|_{L^2_u L^\infty_u L^2(S)} \\
\leq C(O_0)\epsilon \left( 1 + \Delta_2 \right).
\]

Therefore,

\[
\|\nabla^2_{\xi}\|_{L^2_u L^\infty_u L^2(S)} \leq C(O_0)(1 + \epsilon(1 + \Delta_2 + \mathcal{R})) \leq C(O_0),
\]  
(41)

after choosing \( \epsilon \) to be sufficiently small. Finally, we retrieve the estimates for \( \nabla^4 \omega \) and \( \nabla^4 \omega^\dagger \) from the bounds for \( \nabla^2_{\xi} \). To this end, consider the div-curl system

\[
\text{div} \ \nabla \omega = -\text{div} \ K - \frac{1}{2} \text{div} \ \beta, \\
\text{curl} \ \nabla \omega = 0, \\
\text{div} \ \nabla \omega^\dagger = -\text{curl} \ K - \frac{1}{2} \text{curl} \ \beta, \\
\text{curl} \ \nabla \omega^\dagger = 0.
\]

By elliptic estimates given by Propositions [11] and [17] we have

\[
\|\nabla^4(\omega, \omega^\dagger)\|_{L^2(S_{u, \bar{u}})} \\
\leq C \left( \sum_{i \leq 3} \|\nabla^i \kappa\|_{L^2(S_{u, \bar{u}})} + \sum_{i \leq 3} \|\nabla^i \beta\|_{L^2(S_{u, \bar{u}})} + \sum_{i \leq 3} \|\nabla^i (\omega, \omega^\dagger)\|_{L^2(S_{u, \bar{u}})} \right).
\]

Therefore, using Proposition [13] (41) and the curvature norm,

\[
\|\nabla^4(\omega, \omega^\dagger)\|_{L^\infty_u L^2_u L^2(S)} \leq C(O_0)(1 + \mathcal{R}).
\]

By switching \( \omega \) and \( \omega \) as well as \( e_3 \) and \( e_4 \), we also have the following estimates for \( \nabla^4 \omega \):

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]
Proposition 25. Assume

\[ R < \infty. \]

Then there exists \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, \Delta_2, R) \) such that whenever \( \epsilon \leq \epsilon_0, \)

\[ \| f(u) \nabla^4 \omega \|_{L^\infty L^2(S)} \leq C(\mathcal{O}_0)(1 + R). \]

We have thus controlled the fourth angular derivatives of all Ricci coefficients and have closed the bootstrap assumption (A2). We summarize in the following proposition:

Proposition 26. Assume

\[ R < \infty. \]

There exists \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, R) \) such that whenever \( \epsilon \leq \epsilon_0, \)

\[ \tilde{\mathcal{O}}_{4,2} \leq C(\mathcal{O}_0)(1 + R). \]

8. Estimates for Curvature

In this section, we derive and prove the energy estimates. To this end, we introduce the following bootstrap assumptions:

\[ \mathcal{R} \leq \Delta_3. \quad (A3) \]

In order to derive the energy estimates, we need the following integration by parts formula, which can be proved by direct computations:

Proposition 27. Let \( D_{u, \underline{u}} \) be defined as the spacetime region whose coordinates \((u', \underline{u}')\) satisfy \( 0 \leq u' \leq u \) and \( 0 \leq \underline{u}' \leq \underline{u} \). Suppose \( \phi_1 \) and \( \phi_2 \) are tensors of rank \( r \), then

\[
\int_{D_{u, \underline{u}}} \phi_1 \nabla_4 \phi_2 + \int_{D_{u, \underline{u}}} \phi_2 \nabla_4 \phi_1 = \int_{H_{u}(0, u)} \phi_1 \phi_2 - \int_{H_{\underline{u}}(0, \underline{u})} \phi_1 \phi_2 + \int_{D_{u, \underline{u}}} (2\omega - tr \chi) \phi_1 \phi_2,
\]

\[
\int_{D_{u, \underline{u}}} \phi_1 \nabla_3 \phi_2 + \int_{D_{u, \underline{u}}} \phi_2 \nabla_3 \phi_1 = \int_{H_{u}(0, u)} \phi_1 \phi_2 - \int_{H_{\underline{u}}(0, \underline{u})} \phi_1 \phi_2 + \int_{D_{u, \underline{u}}} (2\omega - tr \chi) \phi_1 \phi_2.
\]

Proposition 28. Suppose we have a tensor \(^{(1)}\phi \) of rank \( r \) and a tensor \(^{(2)}\phi \) of rank \( r - 1 \). Then

\[
\int_{D_{u, \underline{u}}}^{(1)}\phi A_1 A_2 ... A_r \nabla A_r^{(2)} \phi A_1 ... A_{r-1} + \int_{D_{u, \underline{u}}} \nabla A_r^{(1)} \phi A_1 A_2 ... A_r^{(2)} \phi A_1 ... A_{r-1} = - \int_{D_{u, \underline{u}}} (\eta + \bar{\eta})^{(1)} \phi^{(2)}\phi.
\]

With these we are now ready to derive energy estimates for \( K, \bar{\sigma} \) in \( L^2(H_u) \) and for \( \beta \) in \( L^2(H_{\underline{u}}) \). The most important observation is that the two uncontrollable terms have favorable signs.
Proposition 29. The following $L^2$ estimates for the curvature hold:

$$
\sum_{i \leq 3} (\| f(u) \nabla^i(K, \tilde{\sigma}) \|^2_{L^2(S)} + \| f(u) \nabla^i \beta \|^2_{L^2(S,0)}) \leq \sum_{i \leq 3} (\| f(u) \nabla^i(K, \tilde{\sigma}) \|^2_{L^2(S_0, \omega)} + \| f(u) \nabla^i \beta \|^2_{L^2(S,0)}) \leq \sum_{i \leq 3} (\| f(u) \nabla^i(K, \tilde{\sigma}) \|^2_{L^2(S_0, \omega)} + \| f(u) \nabla^i \beta \|^2_{L^2(S,0)})
$$

Proof. Consider the following schematic Bianchi equations:

$$
\nabla_3 \tilde{\sigma} + \text{div} \beta = \psi_H \tilde{\sigma} + \sum_{i_1 + i_2 + i_3 = 1} \psi_{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H.
$$

$$
\nabla_3 K + \text{div} \beta = \psi_H K + \sum_{i_1 + i_2 + i_3 = 1} \psi_{i_1} \nabla^{i_2} \psi \nabla^{i_3} \psi_H.
$$

$$
\nabla_4 \beta + \nabla K - \text{div} \sigma = \psi(K, \tilde{\sigma}) + \sum_{i_1 + i_2 + i_3 = 1} \psi_{i_1} \nabla^{i_2} \psi_H \nabla^{i_3} \psi_H.
$$

Commute the first equation with $i$ angular derivatives for $i \leq 3$, we get the equation for $\nabla_3 \nabla^i \tilde{\sigma}$:

$$
\nabla_3 \nabla^i \tilde{\sigma} + \text{div} \nabla^i \beta = \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \psi_H.
$$

Notice that in the above equation, there are terms arising from the commutator $[\nabla^i, \text{div }] \beta$, which can be expressed in terms of the Gauss curvature. After substituting also the Codazzi equations for $\beta$, we get that these terms have the form of the first term in the above expression. The equation for $\nabla_3 \nabla^i K$ has a similar structure:

$$
\nabla_3 \nabla^i K + \text{div} \nabla^i \beta = \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \psi_H.
$$
Finally, we have the following structure for $\nabla_4 \nabla^i \beta$:

$$
\nabla_4 \nabla^i \beta + \nabla \nabla^i K - \nabla \nabla^i \bar{\sigma} \\
= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (K, \bar{\sigma}) + \sum_{i_1+i_2+i_3=i-1} \psi^{i_1} \nabla^{i_2} K \nabla^{i_3} (K, \bar{\sigma}) \\
+ \sum_{i_1+i_2+i_3+i_4=i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} \psi_H.
$$

(44)

As a shorthand, we denote by $F_{i,1}$ the terms of the form

$$
F_{i,1} := \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} (K, \bar{\sigma}) + \sum_{i_1+i_2+i_3+i_4=i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi_H.
$$

and by $F_{i,2}$ the terms of the form

$$
F_{i,2} := \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (K, \bar{\sigma}) + \sum_{i_1+i_2+i_3=i-1} \psi^{i_1} \nabla^{i_2} K \nabla^{i_3} (K, \bar{\sigma}) \\
+ \sum_{i_1+i_2+i_3+i_4=i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} \psi_H.
$$

Integrating $(44)$ in the region $D_{u,\omega}$, applying Proposition $28$ and using equations $(42)$ and $(43)$ yield the following identity on the derivatives of the curvature:

$$
\int_{D_{u,\omega}} \frac{f(u)^2}{2} < \nabla^i \beta, \nabla_4 \nabla^i \beta >_\gamma \\
= \int_{D_{u,\omega}} f(u)^2 < \beta, -\nabla \nabla^i K + * \nabla \nabla^i \sigma >_\gamma + f(u)^2 < \nabla^i \beta, F_{i,2} >_\gamma \\
= \int_{D_{u,\omega}} f(u)^2 < \text{div} \nabla^i \beta, \nabla^i K >_\gamma + f(u)^2 < \text{div} \nabla^i \beta, \nabla^i \sigma >_\gamma + f(u)^2 < \nabla^i \beta, F_{i,2} >_\gamma \\
= \int_{D_{u,\omega}} -f(u)^2 < \nabla^3 \nabla^i K, \nabla^i K >_\gamma -f(u)^2 < \nabla^3 \nabla^i \sigma, \nabla^i \sigma >_\gamma \\
+ \int_{D_{u,\omega}} f(u)^2 < \nabla^i \beta, F_{i,2} >_\gamma + f(u)^2 < \nabla^i (K, \bar{\sigma}), F_{i,1} >_\gamma.
$$

Using Proposition $27$, since $\nabla_4 f(u) = 0$, we have

$$
\frac{1}{2} (\int_{H_\omega} f(u)^2 |\nabla^i \beta|^2 - \int_{H_0} f(u)^2 |\nabla^i \beta|^2) + \int_{D_{u,\omega}} f(u)^2 (\omega - \frac{1}{2} \text{tr} \chi) |\nabla^i \beta|^2.
$$

(46)
For the terms with $\nabla_3 \nabla^i K$ and $\nabla_3 \nabla^i \tilde{\sigma}$, we similarly apply Proposition 27 but noting that there is an extra contribution coming from $\nabla_3 f(u)$:

$$
\int_{D_{u,\frac{H^i}{2}}} f(u)^2 < \nabla^i K, \nabla_3 \nabla^i K > \gamma 
= - \int_{D_{u,\frac{H^i}{2}}} f(u) \nabla_3 f(u) |\nabla^i K|^2 + \frac{1}{2} \left( \int_{H_u} f(u)^2 |\nabla^i K|^2 - \int_{H_0} f(u)^2 |\nabla^i K|^2 \right) 
+ \int_{D_{u,\frac{H^i}{2}}} f(u)^2 (\omega - \frac{1}{2} \text{tr} \chi) |\nabla^i K|^2. 
\tag{47}
$$

Similarly,

$$
\int_{D_{u,\frac{H^i}{2}}} f(u)^2 < \nabla^i \tilde{\sigma}, \nabla_3 \nabla^i \tilde{\sigma} > \gamma 
= - \int_{D_{u,\frac{H^i}{2}}} f(u) \nabla_3 f(u) |\nabla^i \tilde{\sigma}|^2 + \frac{1}{2} \left( \int_{H_u} f(u)^2 |\nabla^i \tilde{\sigma}|^2 - \int_{H_0} f(u)^2 |\nabla^i \tilde{\sigma}|^2 \right) 
+ \int_{D_{u,\frac{H^i}{2}}} f(u)^2 (\omega - \frac{1}{2} \text{tr} \chi) |\nabla^i \tilde{\sigma}|^2. 
\tag{48}
$$

Combining (45)-(48), we thus have the identity

$$
\int_{H_u} f(u)^2 |\nabla^i \beta|^2 + \int_{H_u} f(u)^2 |\nabla^i K|^2 + \int_{H_u} f(u)^2 |\nabla^i \tilde{\sigma}|^2 
- 2 \int_{D_{u,\frac{H^i}{2}}} f(u) \nabla_3 f(u) |\nabla^i K|^2 
- 2 \int_{D_{u,\frac{H^i}{2}}} f(u) \nabla_3 f(u) |\nabla^i \tilde{\sigma}|^2 
= \int_{H_u} f(u)^2 |\nabla^i \beta|^2 + \int_{H_u} f(u)^2 |\nabla^i K|^2 + \int_{H_u} f(u)^2 |\nabla^i \tilde{\sigma}|^2 
- 2 \int_{D_{u,\frac{H^i}{2}}} f(u)^2 (\omega - \frac{1}{2} \text{tr} \chi) |\nabla^i \beta|^2 
- 2 \int_{D_{u,\frac{H^i}{2}}} f(u)^2 (\omega - \frac{1}{2} \text{tr} \chi) (|\nabla^i K|^2 + |\nabla^i \tilde{\sigma}|^2) 
+ \int_{D_{u,\frac{H^i}{2}}} f(u)^2 < \nabla^i \beta, F_{i,2} > \gamma 
+ \int_{D_{u,\frac{H^i}{2}}} f(u)^2 < \nabla^i (K, \tilde{\sigma}), F_{i,1} > \gamma . 
$$

The terms

$$
- 2 \int_{D_{u,\frac{H^i}{2}}} f(u) \nabla_3 f(u) |\nabla^i K|^2 
- 2 \int_{D_{u,\frac{H^i}{2}}} f(u) \nabla_3 f(u) |\nabla^i \tilde{\sigma}|^2 
$$

on the left hand side, which cannot be controlled by the curvature flux (i.e., the integrals of $\nabla^i$ of the curvature components along $H_u$ or $H_{u'}$), have a favorable sign! This is because

\footnote{In fact, if we do not drop this term, we can control the spacetime integral

$$
\int_{D_{u,\frac{H^i}{2}}} (-f(u) \nabla_3 f(u)) |\nabla^i (K, \tilde{\sigma})|^2 
$$

where the weight $(-f(u) \nabla_3 f(u))$ can be singular. For weights such as $f(u) = (u - u_*)^\alpha$ for $\alpha < \frac{1}{2}$ or $f(u) = (u - u_*)^{\frac{1}{2}} \log^\beta \left( \frac{1}{u - u_*} \right)$ for $\beta > \frac{1}{2}$, this bound is “logarithmically” stronger than simply taking the bound for $\int_{H_u} f(u)^2 |\nabla^i (K, \tilde{\sigma})|^2$ and integrating in $u$.}
the weight function $f$ satisfies $f(u) \nabla_3 f < 0$. Therefore, we get an inequality for every $i$:

$$
\int_{H_u} f(u)^2 |\nabla^i \beta|^2 + \int_{H_u} f(u)^2 |\nabla^i K|^2 + \int_{H_u} f(u)^2 |\nabla^i \dot{\sigma}|^2
\leq \int_{H_{u'}} f(u)^2 |\nabla^i \beta|^2 + \int_{H_{u'}} f(u)^2 |\nabla^i K|^2 + \int_{H_{u'}} f(u)^2 |\nabla^i \dot{\sigma}|^2
+ C \|f(u)^2(\omega - \frac{1}{2}\text{tr} \chi) \nabla^i \beta \nabla^i \beta\|_{L^1_1 L^2_1 L^1(S)} + C \|f(u)^2(\omega - \frac{1}{2}\text{tr} \chi) \nabla^i (K, \dot{\sigma}) \nabla^i (K, \dot{\sigma})\|_{L^1_1 L^2_1 L^1(S)}
+ C \|f(u)^2 \nabla^i \beta F_{i,2}\|_{L^1_1 L^2_1 L^1(S)} + C \|f(u)^2 \nabla^i (K, \dot{\sigma}) F_{i,1}\|_{L^1_1 L^2_1 L^1(S)}.
$$

We now add the above inequalities for $i \leq 3$. One can easily check that the terms

$$
\sum_{i \leq 3} \|f(u)^2(\omega - \frac{1}{2}\text{tr} \chi) \nabla^i (K, \dot{\sigma}) \nabla^i (K, \dot{\sigma})\|_{L^1_1 L^2_1 L^1(S)},
$$

$$
\sum_{i \leq 3} \|f(u)^2 \nabla^i \beta F_{i,2}\|_{L^1_1 L^2_1 L^1(S)}
$$

and

$$
\sum_{i \leq 3} \|f(u)^2 \nabla^i (K, \dot{\sigma}) F_{i,1}\|_{L^1_1 L^2_1 L^1(S)}
$$

have the form of one of the terms in the statement of the proposition. After applying the Codazzi equation

$$
\beta = \nabla \psi_H + \psi(\psi + \psi_H)
$$

to one of the $\beta$'s, we note that the term

$$
\sum_{i \leq 3} \|f(u)^2(\omega - \frac{1}{2}\text{tr} \chi) \nabla^i \beta \nabla^i \beta\|_{L^1_1 L^2_1 L^1(S)}
$$

is also one of the terms in the statement of the proposition. \qed

To close the energy estimates, we also need to control $\beta$ in $L^2(H)$ and $(K, \dot{\sigma})$ in $L^2(H)$. It is not difficult to see, by virtue of the structure of the Einstein equations, that Proposition 29 also holds when all the barred and unbarred quantities are exchanged. The proof is exactly analogous to that of Proposition 29.
Proposition 30. The following $L^2$ estimates for the curvature components hold:

$$\sum_{i \leq 3} (\| f(u) \nabla^i (K, \sigma) \|_{L^2_\infty L^2_2 L^2(S)}^2 + \| f(u) \nabla^i \beta \|_{L^2_\infty L^2_2 L^2(S)}^2) \leq \sum_{i \leq 3} (\| f(u) \nabla^i (K, \sigma) \|_{L^2_\infty L^2_2 L^2(S_u, u)} + \| f(u) \nabla^i \beta \|_{L^2_\infty L^2_2 L^2(S_u, u)})$$

$$+ \| f(u)^2 \sum_{i \leq 3} \nabla^i (K, \sigma) (\sum_{i_1 + i_2 + i_3 + i_4 \leq 4} \nabla^{i_1 i_2 i_3 i_4} \psi \nabla^i \psi_H) \|_{L^1_\infty L^1_2 L^1(S)}$$

$$+ \| f(u)^2 \sum_{i \leq 3} \nabla^i (K, \sigma) (\sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1 i_2 i_3 i_4} \psi \nabla^{i_4} (K, \sigma)) \|_{L^1_\infty L^1_2 L^1(S)}$$

$$+ \| f(u)^2 \sum_{i \leq 3} \nabla^i \beta (\sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1 i_2 i_3 i_4} \psi \nabla^{i_4} (K, \sigma)) \|_{L^1_\infty L^1_2 L^1(S)}$$

$$+ \| f(u)^2 \sum_{i \leq 3} \nabla^i \beta (\sum_{i_1 + i_2 + i_3 + i_4 \leq 2} \nabla^{i_1 i_2 i_3 i_4} \psi \nabla^{i_4} \psi_H) \|_{L^1_\infty L^1_2 L^1(S)}$$

We now show that we can control all the nonlinear error terms in the energy estimates. We show this for $K$ and $\sigma$ in $L^2(H_u)$ and $\beta$ in $L^2(H_u)$. The other case can be dealt with in a similar fashion.

Proposition 31. There exist $\epsilon_0 = \epsilon_0(\mathcal{O}_0, \mathcal{R}_0, \Delta_3)$ sufficiently small such that whenever $\epsilon \leq \epsilon_0$,

$$\sum_{i \leq 3} (\| f(u) \nabla^i (K, \sigma) \|_{L^2_\infty L^2_2 L^2(S)} + \| f(u) \nabla^i \beta \|_{L^2_\infty L^2_2 L^2(S)}) \leq C(\mathcal{O}_0, \mathcal{R}_0).$$

Proof. To prove the curvature estimates, we use Proposition 29. By assumptions of Theorem 2 (see Remark 5 after the theorem), the two terms corresponding to the initial data are bounded by a constant $C(\mathcal{R}_0)$ depending only on initial data. Therefore, we need to control the remaining five error terms in Proposition 29. We first look at the term

$$\| f(u)^2 \sum_{i \leq 3} \nabla^i (K, \sigma) (\sum_{i_1 + i_2 + i_3 + i_4 \leq 4} \nabla^{i_1 i_2 i_3 i_4} \psi \nabla^i \psi_H) \|_{L^1_\infty L^1_2 L^1(S)}.$$
The term
\[ \| f(u)^2 \sum_{i \leq 3} \nabla_i (K, \tilde{\sigma}) ( \sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^H \nabla^{i_4} (K, \tilde{\sigma})) \|_{L^1_u L^2_t L^1(S)} \leq C(\mathcal{O}_0) \Delta_3 (1 + \Delta_3) \epsilon \]
similarly as in the previous estimate since by Propositions 17 and 18, \( \nabla^i (K, \tilde{\sigma}) \) satisfies exactly the same estimates as \( \nabla^{i+1} \psi \). We then consider the third nonlinear term
\[ \| f(u)^2 \sum_{i \leq 3} \nabla_i (K, \tilde{\sigma}) ( \sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^i (K, \tilde{\sigma})) \|_{L^1_u L^2_t L^1(S)}. \]
Using Propositions 16 and 26 and the bootstrap assumptions (A3), we have
\[ \| f(u)^2 \sum_{i \leq 3} \nabla_i (K, \tilde{\sigma}) ( \sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^i (K, \tilde{\sigma})) \|_{L^1_u L^2_t L^1(S)} \leq C(\mathcal{O}_0) \Delta_3 (1 + \Delta_3) \epsilon. \]
The fourth nonlinear term can be estimated analogously as the third nonlinear term by
\[ \| f(u)^2 \sum_{i \leq 3} \nabla_i (K, \tilde{\sigma}) ( \sum_{i_1 + i_2 + i_3 + i_4 \leq 4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^i (K, \tilde{\sigma})) \|_{L^1_u L^2_t L^1(S)} \leq C(\mathcal{O}_0) \Delta_3 (1 + \Delta_3) \epsilon. \]
As before, this is because by Propositions 17 and 18, \( \nabla^i (K, \tilde{\sigma}) \) satisfies exactly the same estimates as \( \nabla^{i+1} \psi \). Thus it remains to control
\[ \| f(u)^2 \sum_{i \leq 3} \nabla_i \|_{L^1_u L^2_t L^1(S)}. \]
This term can be bounded as follows:
\[
\begin{align*}
& \| f(u)^2 \sum_{i \leq 3} \nabla_i \|_{L^1_u L^2_t L^1(S)} \\
& \leq C \left( \sum_{i \leq 3} \| f(u) \nabla_i \|_{L^\infty_u L^2_t L^2(S)} \right)^2 \left( \sum_{i_1 + i_2 + i_3 + i_4 \leq 4} \| \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^i (K, \tilde{\sigma}) \|_{L^\infty_u L^2_t L^2(S)} \right) \\
& \quad \times \left( \sum_{i_3 \leq 3} \| f(u) \nabla^{i_3} \psi^H \|_{L^\infty_u L^2_t L^2(S)} \right) \left( \sum_{i_4 \leq 4} \| f(u) \nabla^{i_4} \psi^i (K, \tilde{\sigma}) \|_{L^\infty_u L^2_t L^2(S)} \right) \| f(u) \|_{L^\infty_u L^2_t L^\infty(S)}^{-1} \| L^2_u \| \\
& \quad + C \left( \sum_{i \leq 2} \| f(u) \nabla_i \|_{L^\infty_u L^2_t L^2(S)} \right) \| f(u) \|_{L^\infty_u L^2_t L^\infty(S)}^{-1} \| L^2_u \| \\
& \leq C(\mathcal{O}_0) \Delta_3 (1 + \Delta_3) \epsilon.
\end{align*}
\]
Therefore, gathering all the above estimates, we have
\[ \sum_{i \leq 3} \| f(u) \nabla^i (K, \tilde{\sigma}) \|_{L^\infty_u L^2_t L^2(S)}^2 + \| f(u) \nabla^i \|_{L^\infty_u L^2_t L^2(S)}^2 \leq C(\mathcal{O}_0, R_0) + C(\mathcal{O}_0) \Delta_3 (1 + \Delta_3) \epsilon, \]
which implies the conclusion of the proposition after taking \( \epsilon \) to be sufficiently small. \qed

Notice that the schematic equations are symmetric under the change \( \nabla_3 \leftrightarrow \nabla_4, u \leftrightarrow u \) and \( \psi_H \leftrightarrow \psi^H \). Since the conditions for the initial data are also symmetric, we also have the following analogous energy estimates for \( \nabla^i \beta \) on \( H_u \) and \( \nabla^i (K, \tilde{\sigma}) \) on \( H_{\tilde{\sigma}} \).
Proposition 32. There exist \( \epsilon_0 = \epsilon_0(\mathcal{O}_0, \mathcal{R}_0, \Delta_3) \) sufficiently small such that whenever \( \epsilon \leq \epsilon_0 \),

\[
\sum_{i \leq 3}(\|f(u)\nabla^i \beta\|_{L^\infty L^2 L^2(S)} + \|f(u)\nabla^i (K, \tilde{\sigma})\|_{L^\infty L^2 L^2(S)}) \leq C(\mathcal{O}_0, \mathcal{R}_0).
\]

Propositions 31 and 32 together imply

Proposition 33. There exists \( \epsilon_0 = (\mathcal{O}_0, \mathcal{R}_0) \) such that whenever \( \epsilon \leq \epsilon_0 \),

\[
\mathcal{R} \leq C(\mathcal{O}_0, \mathcal{R}_0).
\]

Proof. Let

\[
\Delta_3 \gg C(\mathcal{O}_0, \mathcal{R}_0),
\]

where \( C(\mathcal{O}_0, \mathcal{R}_0) \) is taken to be the maximum of the bounds in Propositions 31 and 32. Hence, the choice of \( \Delta_3 \) depends only on \( \mathcal{O}_0 \) and \( \mathcal{R}_0 \). Thus, by Propositions 31 and 32, the bootstrap assumption (A3) can be improved by choosing \( \epsilon \) sufficiently small depending on \( \mathcal{O}_0 \) and \( \mathcal{R}_0 \). □

This concludes the proof of Theorem 5.

9. Nature of the Singular Boundary

As described by Theorems 3 and 4, we will also prove the regularity and singularity of the boundary \( H_u \) and \( H_{\tilde{u}} \). We first prove the regularity of the boundary asserted in Theorem 3.

Proof of Theorem 3. The fact that \((\mathcal{M}, g)\) can be extend continuous up to and beyond \( H_u \) and \( H_{\tilde{u}} \) simply follows from the continuity of the metric components \( \Omega, \gamma \) and \( b \) proved in Propositions 1-4. To obtain the higher regularity for \( \gamma \), we recall the equations (24), (25) and (28):

\[
\partial_u \Omega^{-1} = 2\omega, \\
\mathcal{L} L \gamma = 2\Omega \chi, \\
\partial_u b^A = -4\Omega^2 \zeta^A.
\]

Commuting these equations with \( (\frac{\partial}{\partial \theta})^i \) and using the bounds for the Ricci coefficients obtained in the proof of Theorem 3, we can conclude that

\[
\sum_{i \leq 4} \sup_{0 \leq u \leq U} \sup_{0 \leq u \leq u_\ast} \| (\frac{\partial}{\partial \theta})^i(\Omega, \gamma, b) \|_{L^2(S_{u, u_\ast})}.
\]

The boundedness of \( \psi \) and its angular derivatives

\[
\sum_{i \leq 3} \| \nabla^i \psi \|_{L^\infty L^2 L^2(S)} \leq C.
\]

Notice that by controlling \( \gamma \) and its coordinate angular derivatives \( (\frac{\partial}{\partial \theta})^i \gamma \), we can show also that \( \frac{\partial}{\partial \theta} \) and \( \nabla \) are comparable up to lower order terms, which allows us to apply the estimates for \( \nabla^i \text{tr} \chi, \nabla^i \tilde{\chi}, \nabla^i \eta \) and \( \nabla^i \tilde{\eta} \) to bounded the coordinate angular derivatives of the metric components.
are already proved in Theorem 5. To control $\psi_H$ and its angular derivatives on the singular boundary $H_{u_0}$, we first note that by the smoothness assumption on the interior of $H_0$, we have that for every fixed $U \in [0, u_*)$,

$$\sup_{0 \leq u \leq U} \|\nabla^i \psi_H\|_{L^2(S_{u,0})} \leq C_{i,U}$$

for some finite $C_{i,U}$ for every $i$. We revisit the proof of Proposition 14. Restricting to $[0, U]$, $f(u)^{-1}$ is bounded. Therefore, the estimates in (34), (35) and (36) are bounded uniformly in $u$. Finally, (37) can be replace by the estimate

$$\int_0^U \sum_{i_1 + i_2 + i_3 + i_4 \leq 3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_H \nabla^{i_4} \psi_H \|_{L^2(S_{u, u')}, du'} \leq C \int_0^U (\sum_{i_1 \leq 3} \|\nabla^{i_1} \psi_H\|_{L^2(S_{u, u'})}) (\sum_{i_2 \leq 3} \sup_{0 \leq u'' \leq u} \|\nabla^{i_2} \psi_H\|_{L^2(S_{u, u'')}}) du'.$$

Putting these bounds together, we have

$$\sum_{i \leq 3} \sup_{0 \leq u \leq U} \|\nabla^{i} \psi_H\|_{L^2(S_{u, u'})} \leq C + C \int_0^U (\sum_{i_1 \leq 3} \|\nabla^{i_1} \psi_H\|_{L^2(S_{u, u'})}) (\sum_{i_2 \leq 3} \sup_{0 \leq u'' \leq u} \|\nabla^{i_2} \psi_H\|_{L^2(S_{u, u'')}}) du',$$

which implies

$$\sum_{i \leq 3} \sup_{0 \leq u \leq U} \|\nabla^{i} \psi_H\|_{L^2(S_{u, u'})} \leq C$$

after applying Gronwall’s inequality.

To conclude the proof, it remains to control $\nabla_3 \nabla^i \psi$ and $\nabla_3 \nabla^i \psi_H$ for $i \leq 2$. Since $\eta$ obeys a $\nabla_3$ equation, $\nabla_3 \nabla^i \eta$ can be estimated by directly controlling the right hand side of the null structure equation (commuted with angular derivatives) and using the bounds in Theorem 5. To control the term $\nabla_3 \nabla^i \eta$, we consider the equation for $\nabla_4 \nabla_3 \nabla^i \eta$. To this end, we use the commutation formula

$$[\nabla_3, \nabla_4] f = -2 \omega \nabla_3 f + 2 \omega \nabla_4 f + 4 \zeta \cdot \nabla f$$

(as well as Proposition 10) to derive

$$\nabla_4 \nabla_3 \nabla^i \eta = \sum_{j_1 + j_2 + j_3 + j_4 = 1} \nabla^{i_1} \psi^{j_1} \nabla^{j_2} \psi_H \nabla^{j_3} \psi_H \nabla^{j_4} \psi_H$$

$$+ \sum_{j_1 + j_2 + j_3 + j_4 = 1} \nabla^{i_1} \psi^{j_1} \nabla_3 \nabla^{j_2} \psi_H \nabla^{j_3} \psi_H \nabla^{j_4} \psi_H$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 = i + 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi.$$

For $i \leq 2$, the bounds derived in the proof of Theorem 5 control every term on the right hand side in $\|\cdot\|_{L^2(S)}$ except for the term

$$\sum_{i_1 + i_2 + i_3 + i_4 = 3} \nabla^{i_1} \psi^{i_2} \nabla_3 \nabla^{i_3} \psi \nabla^{i_4} \psi_H.$$
Therefore
\[ \sum_{i \leq 2} \| \nabla_{3, i}^{3} \eta \|_{L^2(S_{u, g})} \leq C + C \int_{0}^{u} \sum_{i_1 + i_2 + i_3 + i_4 \leq 2} \| \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi_{H} \|_{L^2(S_{u, g})} du'. \]

Since we have
\[ \sum_{i_1 + i_2 + i_3 \leq 2} \| \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi_{H} \|_{L^2(S_{u, g})} \leq C, \]
we conclude by Gronwall’s inequality that
\[ \sum_{i \leq 2} \| \nabla_{3, i}^{3} \eta \|_{L^2(S_{u, g})} \leq C. \tag{(49)} \]

Finally, we control the terms \( \nabla_{3, i}^{3} \psi_{H} \). Commuting the null structure equations with \( \nabla_{3, i}^{3} \), we have
\[
\nabla_{4} \nabla_{3}^{3} \psi_{H} = \sum_{j_1+j_2+j_3+j_4=1} (\nabla^{j_1} \psi^{j_2} \nabla_{3, j_3}^{3} \psi_{H} \nabla_{3}^{3} \psi_{H} + \nabla^{j_1} \psi^{j_2} \nabla_{3}^{3} \psi_{H} \nabla_{3, j_3}^{3} \psi_{H} \nabla_{3}^{3} \psi_{H}) + \sum_{j_1+j_2+j_3=1} \nabla^{j_1} \psi^{j_2} \nabla_{3, j_3}^{3} \psi_{H} \nabla_{3}^{3} \psi_{H} + \sum_{j_1+j_2+j_3+j_4=1} \nabla^{j_1} \psi^{j_2} \nabla_{3}^{3} \psi_{H} \nabla_{3, j_3}^{3} \psi_{H} \nabla_{3}^{3} \psi_{H}.
\]

Estimating directly the right hand side of the null structure equations or the Bianchi equations, we can easily show that
\[ \sum_{i \leq 2} \| \nabla_{3, i}^{3} \psi_{H}, \nabla_{3}^{3} \psi_{H}, \nabla_{3}^{3} \psi_{H} \|_{L^2(S_{u, g})} \leq C. \]

Using also (49), we thus have
\[ \sum_{i \leq 2} \| \nabla_{3, i}^{3} \psi_{H} \|_{L^2(S_{u, g})} \leq C + C \int_{0}^{u} \sum_{i_1 + i_2 + i_3 + i_4 \leq 2} \| \nabla^{i_1} \psi^{i_2} \nabla_{3}^{3} \psi_{H} \nabla_{3}^{3} \psi_{H} \|_{L^2(S_{u, g})} du'. \]

Using Gronwall’s inequality, we get
\[ \sum_{i \leq 2} \| \nabla_{3, i}^{3} \psi_{H} \|_{L^2(S_{u, g})} \leq C. \]

In particular, the above estimates show that
\[ \sum_{i \leq 3-j, j \leq 1} \sup_{0 \leq u \leq U} \| \nabla_{3}^{3} \psi_{H} \|_{L^2(S_{u, g})} \leq C \]
on \( H_{u, g} \), as desired.

Finally, we move to the proof of Theorem 1. First, we prove

**Proposition 34.** Suppose, in addition to the assumptions in Theorem 2, \( \hat{\chi} \) initially obeys
\[ \int_{0}^{u} |\hat{\chi} |_{\gamma(u')}^2 du' = \infty, \]
along an outgoing null generator \( \gamma \) of \( H_0 \). Let \( \Phi_u(\gamma) \) be the image of \( \gamma \) under the 1-parameter family of diffeomorphism generated by \( L \). Then
\[
\int_0^{u_*} (\text{tr} \chi |_{\Phi_u(\gamma)}(u'))^2 + |\hat{\chi} |_{\Phi_u(\gamma)}(u')^2 du' = \infty,
\]
holds for every \( 0 \leq u < u_* \).

Similarly suppose, in addition to the assumptions in Theorem 5, \( \hat{\chi} \) initially obeys
\[
\int_0^{u_*} |\hat{\chi} |_{\gamma}(u')^2 du' = \infty,
\]
along an outgoing null generator \( \gamma \) of \( H_0 \). Let \( \Phi_{\hat{u}}(\gamma) \) be the image of \( \gamma \) under the 1-parameter family of diffeomorphism generated by \( L \). Then
\[
\int_0^{\hat{u}_*} (\text{tr} \hat{\chi} |_{\Phi_{\hat{u}}(\gamma)}(u'))^2 + |\hat{\chi} |_{\Phi_{\hat{u}}(\gamma)}(u')^2 du' = \infty,
\]
holds for every \( 0 \leq \hat{u} < \hat{u}_* \).

**Proof.** Fix \( U \in (0, u_*) \). Suppose
\[
\int_0^{\hat{u}_*} \hat{\chi} |_{\Phi_{\hat{u}}(\gamma)}(u')^2 du' < \infty. \tag{50}
\]
We want to show that
\[
\int_0^{u_*} \hat{\chi} |_{\Phi_u(\gamma)}(u')^2 du' = \infty.
\]
Using (50), define \( h : [0, u_*] \to \mathbb{R} \) by
\[
h(u) = |\text{tr} \chi |_{\Phi_u(\gamma)}(u)|
\]
such that
\[
\int_0^{u_*} h(u')^2 du' < \infty.
\]
Consider the following null structure equation for \( \text{tr} \chi \):
\[
\nabla_3 \text{tr} \chi + \text{tr} \chi \text{tr} \chi = 2\omega \text{tr} \chi - 2K + 2 \text{div} \eta + 2|\eta|^2
\]
Along the integral curve of \(-e_3\) emanating from \( \Phi_u(\gamma) \), we thus have
\[
\frac{d}{du}(e^\int_0^u (\text{tr} \chi - 2\omega)|_{\Phi_u(\gamma)}(w) dw) \text{tr} \chi |_{\Phi_u(\gamma)}(u) = e^\int_0^u (\text{tr} \chi - 2\omega)|_{\Phi_u(\gamma)}(w) dw (-2K + 2 \text{div} \eta + 2|\eta|^2).
\]
By the estimates derived in the proof of Theorem 5 \( K, \nabla \eta, \eta \) are bounded and \( \text{tr} \chi, \omega \) are in \( L^1_u L^\infty(S) \). Therefore,
\[
|\text{tr} \chi |_{\Phi_u(\gamma)}(u)| \leq Ch(u) \quad \text{for all } u. \tag{51}
\]
Consider the following null structure equation for \( \hat{\chi} \):
\[
\nabla_3 \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} = \nabla \hat{\omega} \eta + 2\omega \hat{\chi} - \frac{1}{2} \text{tr} \chi \hat{\chi} + \eta \hat{\omega} \eta.
\]
Contract this equation with \( \hat{\chi} \) to get
\[
\frac{1}{2} \nabla_3 |\hat{\chi}|^2 + \frac{1}{2} \text{tr} \chi |\hat{\chi}|^2 - 2\omega |\hat{\chi}|^2 = (\nabla \hat{\omega} \eta - \frac{1}{2} \text{tr} \chi \hat{\chi} + \eta \hat{\omega} \eta) \cdot \hat{\chi},
\]
which implies
\[
|\nabla_3 |\hat{\chi}| + \frac{1}{2} \text{tr} \chi |\hat{\chi}| - 2\omega |\hat{\chi}| \leq |\nabla \hat{\omega} \eta| + \frac{1}{2} \text{tr} \chi \hat{\chi} + |\eta \hat{\omega} \eta|.
\]
This implies that along the integral curve of \( e_3 \), we have
\[
\left| \frac{d}{du} \left( e_i^\nu (\frac{1}{2} \text{tr} \chi - 2\omega) |\Phi_u (\gamma) (u) \rangle du' \right) \right| |\Phi_u (\gamma) (u) \rangle |^2 \leq 2 e_i^\nu (\frac{1}{2} \text{tr} \chi - 2\omega) |\Phi_u (\gamma) (u) \rangle | du' \right| \leq C (1 + h(u)).
\]
Using again the fact that \( K, \nabla \eta, \eta \) are bounded and \( \text{tr} \chi, \omega \) are in \( L_u^1 L^\infty (S) \), as well as the estimate \( (51) \), we have
\[
\left| (e_i^\nu (\frac{1}{2} \text{tr} \chi - 2\omega) |\Phi_u (\gamma) (u) \rangle |^2 \leq C (1 + h(u)).
\]
Notice that \( e_i^\nu (\frac{1}{2} \text{tr} \chi - 2\omega) |\Phi_u (\gamma) (u) \rangle du' \) is bounded above and below uniformly in \( u \). Taking the \( L_u^2 \) norm implies that
\[
\int_0^u |\tilde{\chi} |^2 du' \geq c \int_0^u |\tilde{\chi} | du' - C \int_0^u h^2 (u') du' = \infty
\]
by the assumption of the proposition. The blow up for \( \chi \) can be proved in a similar manner.

This implies

**Proposition 35.** Suppose the assumptions of Theorem 7 hold. Then, in a neighborhood of any point on \( H_{u_\ast} \), \( |\tilde{\chi}|^2 \) is not integrable with respect to the spacetime volume form. Similarly, in a neighborhood of any point on \( H_{u_\ast} \), \( |\tilde{\chi}|^2 \) is not integrable with respect to the spacetime volume form.

**Proof.** The proposition is easier for \( |\tilde{\chi}|^2 \) near \( H_{u_\ast} \). This is because by the definition of the coordinate system in Section 2.2, the image of the initial outgoing null generator under the map \( \Phi_u \) defined in Proposition 34 has constant \( u, \theta^1 \) and \( \theta^2 \) values. Also, by Propositions 1 and 2, the spacetime volume element \( 2 \eta^2 \sqrt{\det \gamma} \) is bounded above and below. Therefore, for any neighborhood \( N \) of \( p = (u, u_\ast, \theta^1, \theta^2) \) in \( H_{u_\ast} \), we have
\[
\int_N (\text{tr} \chi)^2 + |\tilde{\chi}|^2 \geq c \int_{\theta^2 - \delta}^{\theta^2 + \delta} \int_{\theta^1 - \delta}^{\theta^1 + \delta} \int_{u - \delta}^{u + \delta} \int_{u - \delta}^{u + \delta} ((\text{tr} \chi)^2 + |\tilde{\chi}|^2)(u', u', (\theta^1)', (\theta^2)') du' du' d(\theta^1)' d(\theta^2)' = \infty,
\]
by Proposition 34.

To prove the corresponding statement for \( |\tilde{\chi}|^2 \) near \( H_{u_\ast} \), we first change to the coordinate system \( (u, \theta^1 (u, \theta), \theta^2 (u, \theta)) \) such that the image of the initial outgoing null generator under the map \( \Phi_u \) defined in Proposition 34 has constant \( u, \theta^1 \) and \( \theta^2 \) values. This coordinate system can be constructed by solving the ordinary differential equation
\[
\frac{d}{du} \tilde{\theta}^A (u, \theta) = b^A (u, \theta, \tilde{\theta}^1, \tilde{\theta}^2),
\]
with initial data
\[
\tilde{\theta}^A (0; u, \theta) = \theta^A.
\]
By (28), as well as the estimates for $\zeta$, $\Omega$ and their derivatives, $b^A$ is $C^1$ (in all of the variables), therefore
\[
\frac{\partial \tilde{\theta}^A}{\partial u}, \frac{\partial \tilde{\theta}^A}{\partial \theta^B} \leq C.
\]
In the new coordinate system, we apply the same argument as in the case for $|\hat{\chi}|^2$ near $H_u$, and have the estimate
\[
\int_N |\tilde{\chi}|^2 = \infty,
\]
for a neighborhood $N$ of any point $p \in H_u^*$, as desired.

Finally, this allows us to conclude that the Christoffel symbols do not belong to $L^2$:

**Proposition 36.** Suppose the assumptions of Theorem 4 hold. Then, the Christoffel symbols in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system are not in $L^2$ in a neighborhood of any point on $H_u^*$ or $H_{\underline{u}}$.

**Proof.** Recall that the metric in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinates takes the form
\[
g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - b^A du) \otimes (d\theta^B - b^B du).
\]
Note that
\[
g^{uu} = -\frac{1}{2}\Omega^{-2}, \quad g^{uA} = 0 \text{ for } \alpha \neq u.
\]
One computes that
\[
\Gamma^u_{AB} = -\frac{1}{2}g^{uu} \frac{\partial}{\partial u} g_{AB} = \frac{1}{4\Omega^2} \frac{\partial}{\partial \underline{u}} \gamma_{AB} = \frac{1}{2\Omega} \chi_{AB}.
\]
Since $\frac{1}{2} \leq \Omega \leq 2$ and $\gamma$ is uniformly bounded and positive definite, $\Gamma^u_{AB}$ is not in $L^2$ in a neighborhood of any point on the singular boundary $H_u^*$, in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system.

To show that the incoming hypersurface $H_u^*$ is singular, first notice that
\[
g^{uu} = -\frac{1}{2}\Omega^{-2}, \quad g^{uA} = -\frac{1}{2}\Omega^{-2} b^A, \quad g^{uu} = 0.
\]
We then compute
\[
\Gamma^u_{AB} = \frac{1}{2}g^{uu}(\frac{\partial}{\partial \theta^A} g_{Bu} + \frac{\partial}{\partial \theta^B} g_{Au} - \frac{\partial}{\partial u} g_{AB}) + \frac{1}{2}g^{uc}(\frac{\partial}{\partial \theta^B} g_{AC} + \frac{\partial}{\partial \theta^A} g_{BC} - \frac{\partial}{\partial \theta^C} g_{AB})
\]
\[
= \frac{1}{4\Omega^2}(\frac{\partial}{\partial u} \gamma_{AB} - \frac{\partial}{\partial \theta^B}(\gamma_{AC} b^C) - \frac{\partial}{\partial \theta^A}(\gamma_{BC} b^C) - b^C(\frac{\partial}{\partial \theta^B} \gamma_{AC} + \frac{\partial}{\partial \theta^A} \gamma_{BC} - \frac{\partial}{\partial \theta^C} \gamma_{AB}))
\]
\[
= \frac{1}{2\Omega} \chi_{AB} + \text{regular terms},
\]
where the regular terms denote metric components and their derivatives that are uniformly bounded by the estimates proved in the previous sections. By the same reasoning as in the case near $H_{\underline{u}}^*$, $\Gamma^u_{AB}$ is not in $L^2$ in a neighborhood of any point on the singular boundary $H_u^*$ in the $(u, \underline{u}, \theta^1, \theta^2)$ coordinate system.
This concludes the proof of Theorem 4.

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