A new method for the computation of eigenvalues

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Abstract

In this paper we are concerned to find the eigenvalues and eigenvectors of a real symmetric matrix by applying a new numerical method similar to Jacobi method. Our approach consists to use a new orthogonal matrix. The computation of the eigenvalues and eigenvectors by using this method appears easier if compared with Jacobi method in the sense of the functions used in the orthogonal matrix.

Key words: eigenvalues, eigenvectors, symmetric matrix, numerical method, Jacobi method.

AMS Subject Classification: 65F15, 65F10

1 Introduction

As we know, for a given matrix $A \in \mathbb{C}^{n \times m}$, the computation of its eigenvalues and eigenvectors is easy when its dimension is small but this computation will become difficult when the dimension of the matrix is big. For the matrices with big dimensions many researchers have contributed and gave different numerical methods to compute their eigenvalues and eigenvectors, as examples, we cite the QR method (see [5, 2]), the power method [6] and Sturm sequences method which can be found in the book of Quarteroni et al [7]; and for a real symmetric matrix usually we use the Jacobi method [3].

This paper deals with the computation of eigenvalues and eigenvectors of a real symmetric matrix $A \in \mathbb{R}^{n \times n}$, by changing the Givens matrix using on the Jacobi method by another orthogonal matrix, i.e., we will replace the matrix:

$$
G = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \cos(\theta) & -\sin(\theta) & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \sin(\theta) & \cos(\theta) & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}
$$
by the following matrix $H$:

$$
H = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{\sqrt{x + \delta}}{\sqrt{-x - \delta + 1}} & -\frac{\sqrt{-x - \delta + 1}}{\sqrt{x + \delta}} \\
0 & 0 & 1
\end{pmatrix}
$$

such that $\delta \in \mathbb{R}$ and $-\delta \leq x \leq 1 - \delta$.

The paper is organized as follows: in Section 2 we present our main results and in Section 3 we give a MATLAB program to this new method. In the remainder of this paper and without loss of generality we will choose $\delta = \frac{1}{2}$.

## 2 The computation of eigenvalues and eigenvectors

In this section, in order to find the eigenvalues and eigenvectors of a real symmetric matrix $A \in \mathbb{R}^{n \times n}$, we will repeat the procedures of Jacobi method by using the matrix $H$ introduced above and we shall give all the steps of the calculus.

Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix of $n$ dimensions with coefficients $(a_{ij})_{1 \leq i, j \leq n}$. The Jacobi method [7] is an iterative method consists to build a sequences of orthogonal matrices $A^{(k)}$ such that on the $k$-iteration we have

$$
B^{(k)} = H_{pq}^T A^{(k-1)} H_{pq}, \quad (A^{(0)} = A)
$$

where $a_{ij}^{(k)} = 0$ if $(i, j) = (p, q)$ and the matrix $B^{(k)}$ converge to the matrix of eigenvalues.

In a block $A^{(k-1)}$ of the matrix $A$, we find

$$
\begin{pmatrix}
    a_{pp}^{(k-1)} & a_{pq}^{(k-1)} \\
    a_{pq}^{(k-1)} & a_{qq}^{(k-1)}
\end{pmatrix}
= \begin{pmatrix}
    \sqrt{x + \frac{1}{2}} & \sqrt{-x + \frac{1}{2}} \\
    -\sqrt{-x + \frac{1}{2}} & \sqrt{x + \frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
    a_{pp}^{(k-1)} & a_{pq}^{(k-1)} \\
    a_{pq}^{(k-1)} & a_{qq}^{(k-1)}
\end{pmatrix}
\begin{pmatrix}
    \sqrt{x + \frac{1}{2}} & -\sqrt{-x + \frac{1}{2}} \\
    -\sqrt{-x + \frac{1}{2}} & \sqrt{x + \frac{1}{2}}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
    a_{pp}(x + \frac{1}{2}) + 2a_{pq}\sqrt{-x^2 + \frac{1}{4} - (x + \frac{1}{2})a_{qq}^*} & (-a_{pp} + a_{qq})\sqrt{-x^2 + \frac{1}{4} + 2a_{pq}x} \\
    (-a_{pp} + a_{qq})\sqrt{-x^2 + \frac{1}{4} + 2a_{pq}x} & a_{pp}(-x + \frac{1}{2}) + 2a_{pq}\sqrt{-x^2 + \frac{1}{4} + (x + \frac{1}{2})a_{qq}^*}
\end{pmatrix}
$$

where $1 \leq p < q \leq n$.
Now, when we solve the equation

\[(a_{qq} - a_{pp})\sqrt{-x^2 + \frac{1}{4} + 2apqx} = 0\]  \hspace{1cm} (1)

we find that

\[x_0 = \pm \frac{|a_{qq} - a_{pp}|}{2\sqrt{(a_{qq} - a_{pp})^2 + 4a_{pq}^2}}\]

i.e., if \(a_{pq} > 0\), we have

\[x_0 = \frac{a_{pp} - a_{qq}}{2\sqrt{(a_{qq} - a_{pp})^2 + 4a_{pq}^2}}\]

then by substitution in (*) and (**) we find

\[\lambda_* = \frac{a_{qq} + a_{pp}}{2} + \frac{(a_{pp} - a_{qq})^2 + 4a_{pq}^2}{2\sqrt{(a_{qq} - a_{pp})^2 + 4a_{pq}^2}}\]

\[\lambda_{**} = \frac{a_{qq} + a_{pp}}{2} - \frac{(a_{pp} - a_{qq})^2 - 4a_{pq}^2}{2\sqrt{(a_{qq} - a_{pp})^2 + 4a_{pq}^2}}\]

and if \(a_{pq} < 0\), the root of the equation is

\[x_0 = \frac{a_{qq} - a_{pp}}{2\sqrt{(a_{qq} - a_{pp})^2 + 4a_{pq}^2}}\]

by substitution \(x_0\) by its value in (*) and (**) we find

\[\lambda_* = \frac{a_{qq} + a_{pp}}{2} - \frac{(a_{pp} - a_{qq})^2 - 4a_{pq}^2}{2\sqrt{(a_{qq} - a_{pp})^2 + 4a_{pq}^2}}\]

\[\lambda_{**} = \frac{a_{qq} + a_{pp}}{2} + \frac{(a_{pp} - a_{qq})^2 + 4a_{pq}^2}{2\sqrt{(a_{qq} - a_{pp})^2 + 4a_{pq}^2}}\]

### 2.1 Existence and uniqueness of solution of the equation 1

In this section we show that the equation (1) has a unique solution \(x_0\). The idea of the proof consists to divide the interval \([-\frac{1}{2}, \frac{1}{2}]\) on two open intervals, \([-\frac{1}{2}, 0[\) and \(]0, \frac{1}{2}[^\).
\[ f(x) = (a_{qq} - a_{pp}) \sqrt{-x^2 + \frac{1}{4}} + 2a_{pq}x \]

with the derivative

\[ f'(x) = (a_{qq} - a_{pp}) \frac{-x}{\sqrt{-x^2 + \frac{1}{4}}} + 2a_{pq} \]

now, for \( a_{qq} - a_{pp} > 0 \), \( a_{pq} > 0 \) and for \( x \in \left[ -\frac{1}{2}, 0 \right] \), we find

\[ -a_{pq} < f(x) < \frac{a_{qq} - a_{pp}}{2} \]

(2)

also we can find when \( x \in \left] 0, \frac{1}{2} \right[ \) the following

\[ 0 < f(x) < \frac{a_{qq} - a_{pp}}{2} + a_{pq} \]

(3)

By (2) and according to the intermediate value theorem, \( f \) has at least one root in the interval \( \left[ -\frac{1}{2}, 0 \right[ \) and by using the fact that \( f' \) is strictly positive in this interval, we can say that the root is unique. Now the expression (3) shows that \( f \) does not have any roots in this interval, so from the above we deduce that \( f \) has a unique root in the interval \( \left[ -\frac{1}{2}, \frac{1}{2} \right] \).

Processing by the same manner we can find:

1. if \( a_{pq} > 0 \) and \( a_{qq} - a_{pp} < 0 \) we have respectively on the intervals \( \left] -\frac{1}{2}, 0 \right[ \) and \( \left] 0, \frac{1}{2} \right[ \)

\[ \frac{a_{qq} - a_{pp}}{2} - a_{pq} < f(x) < 0 \]

\[ \frac{a_{qq} - a_{pp}}{2} < f(x) < a_{pq} \]

then \( x_0 \in \left] 0, \frac{1}{2} \right[ \)

2. if \( a_{pq} < 0 \) and \( a_{qq} - a_{pp} > 0 \) we have respectively on the intervals \( \left] -\frac{1}{2}, 0 \right[ \) and \( \left] 0, \frac{1}{2} \right[ \)

\[ 0 < f(x) < \frac{a_{qq} - a_{pp}}{2} - a_{pq} \]

\[ a_{pq} < f(x) < \frac{a_{qq} - a_{pp}}{2} \]

then \( x_0 \in \left] 0, \frac{1}{2} \right[ \)

3. if \( a_{pq} < 0 \) and \( a_{qq} - a_{pp} < 0 \) we have respectively on the intervals \( \left] -\frac{1}{2}, 0 \right[ \) and \( \left] 0, \frac{1}{2} \right[ \)

\[ \frac{a_{qq} - a_{pp}}{2} < f(x) < -a_{pq} \]

\[ \frac{a_{qq} - a_{pp}}{2} + a_{pq} < f(x) < 0 \]

then \( x_0 \in \left] -\frac{1}{2}, 0 \right[ \)
3 MATLAB Program

Although this method is very similar to the Jacobi method, which is of course convergent, this does not prevent us to give it an associate program. In what follows we shall give the program of this new method. Our approach is based directly upon the program of cyclic Jacobi method given in [7] (Program 23-33, 35-37). A few changes were made since the functions of the orthogonal matrix were changed.

It is clear that the numerical estimations of the Jacobi method is still here unchanged. First of all, Let give the following quantity

\[
\Psi(A) = \left( \sum_{i,j=1 \atop i \neq j}^{n} a_{ij}^2 \right)^{1/2} = \left( \|A\|_F^2 - \sum_{i=1}^{n} a_{ii}^2 \right)^{1/2}
\]

such that \( \| \cdot \|_F \) is the Frobenius norm. And it is well known that in the k-iteration we have

\[
\Psi(A^{(k)}) \leq \Psi(A^{(k-1)}), \quad \text{for } k \geq 1
\]

Let also give the following estimation

\[
\Psi(A^{(k+n)}) \leq \frac{1}{\delta \sqrt{2}} \left( \Psi(A^{(k)}) \right)^2, \quad k = 1, 2, \ldots
\]

this last is obtained in the cyclic Jacobi method, where \( N = n(n - 1)/2 \) and \( \delta \), by hypothes, satisfies the following inequality

\[
|\lambda_i - \lambda_j| \geq \delta \quad \text{for } i \neq j
\]

Now, we give the MATLAB program with the changes required.

- Let start by the program that allows us to calculate the product \( H(i, k, x)M \)

```matlab
function [M]=pro1(M,irr1,irr2,i,k,j1,j2)
for j=j1:j2
    t1=M(i,j);
    t2=M(k,j);
    M(i,j)=irr1.*t1+irr2.*t2;
    M(k,j)=-irr2.*t1+irr1.*t2;
end
return
```

such that \( irr1 = \sqrt{x + 1/2} \) and \( irr2 = \sqrt{-x + 1/2} \)
• Secondly, we give the program of the product $MH(i, k, x)^T$

```matlab
function [M] = pro2(M, irr1, irr2, j1, j2, i, k)
for j = j1:j2
    t1 = M(j, i);
    t2 = M(j, k);
    M(j, i) = irr1 * t1 + irr2 * t2;
    M(j, k) = -irr2 * t1 + irr1 * t2;
end
return
```

• Now, we give the program which allows us to evaluate $\Psi(A)$ in the cyclic new method

```matlab
function [psi] = psinorm(A)
[n, m] = size(A);
if n ~= m, error('only for square matrix'); end
psi = 0;
for i = 1:n-1
    j = [i + 1 : n];
    psi = psi + sum(A(i, j).^2 + A(j, i).^2)
end
psi = sqrt(psi);
return
```

• Afterwards, the program which allows us to evaluate $irr1$ and $irr2$

```matlab
function [irr1, irr2] = symschur2(A, p, q)
if A(p, q) == 0
    irr1 = 1; irr2 = 0;
else
    if A(p, q) > 0
        z1 = (A(p, p) - A(q, q));
        z2 = ((A(q, q) - A(p, p)).^2) + (4.*(A(p, q)).^2);
        z3 = sqrt(z2);
        z4 = 2.*z3;
        x = z1 ./ z4;
    else
        v1 = (A(q, q) - A(p, p));
        v2 = ((A(q, q) - A(p, p)).^2) + (4.*(A(p, q)).^2);
```
v3 = sqrt(v2);
v4 = 2.*v3;
x = v1./v4;
end
irr1 = sqrt(x + (1/2));
irr2 = sqrt(-x + (1/2));
end
return

• Finally, here is the program of the new method

function [D, sweep, psi] = cycjacobi2(A, tol, nmax)
    [n, m] = size(A);
    if n ~= m, error('only for the square matrix'); end
    D = A;
    psi = norm(A, 'fro');
    epsi = tol*psi;
    psi = psinorm(D);
    sweep = 0;
    iter = 0;
    while psi > epsi and iter <= nmax
        iter = iter + 1;
        sweep = sweep + 1;
        for p = 1:n-1
            for q = p+1:n
                irr1, irr2 = symshur2(D, p, q);
                D = pro1(D, irr1, irr2, p, q, 1, n);
                D = pro2(D, irr1, irr2, 1, n, p, q);
            end
        end
        psi = psinorm(D);
    end
    return

such that tol is the tolerance and nmax is the maximum number of iterations.

4 Example

Let

\[
A = \begin{pmatrix}
1 & 0 & 2 \\
0 & 3 & 0 \\
2 & 0 & 4 \\
\end{pmatrix}
\]
and let $A^{(0)} = A$, then we have

$$A^{(1)} = H^T AH$$

such that

$$H = \begin{pmatrix}
\sqrt{x + \frac{1}{2}} & 0 & -\sqrt{-x + \frac{1}{2}} \\
0 & 1 & 0 \\
\sqrt{-x + \frac{1}{2}} & 0 & \sqrt{x + \frac{1}{2}} 
\end{pmatrix}$$

Using the expressions defined in page (3) we get

$$x_0 = \frac{1-4}{2\sqrt{(4-1)^2+4x^2}} = \frac{-3}{10}$$

$$\lambda_1 = \frac{5}{2} + \frac{(1-4)\cdot 2+4x^2}{2\sqrt{(4-1)^2+4x^2}} = 5$$

$$\lambda_2 = 3$$

$$\lambda_3 = \frac{5}{2} + \frac{-1\cdot 2-4\cdot x^2}{2\sqrt{(4-1)^2+4x^2}} = 0$$

And the corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 5\sqrt{5} \\
0 \\
0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\
3 \\
0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\
0 \\
0 \end{pmatrix}$$

5 Conclusion

In this paper, we gave another method for the computation of eigenvalues and eigenvectors of a real symmetric matrix, and we well noticed the relationship between the two orthogonal matrices, i.e., these matrices allows us to calculate the same eigenvalues of a real symmetric matrix but with two different values. Indeed, in the Jacobi method this value $\theta \in ]-\pi/4, \pi/4[$ and in this new method $x \in ]-\delta, 1-\delta[$ such that $\delta \in \mathbb{R}$, so we can deduce that there is a bijection between these two intervals.
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