Symmetric two-field oscillons

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Oscillons are spatially localized structures that appear in scalar field theories and exhibit extremely long life-times. We go beyond single-field analyses and study oscillons comprised of two interacting fields, each field having an identical potential with quadratic, quartic and sextic terms. We consider a quartic interaction term of either attractive or repulsive nature. We construct semi-analytical oscillon profiles for different values of the potential parameters and coupling strength using the two-timing small-amplitude formalism. We show that the interaction sign, attractive or repulsive, leads to different oscillon solutions, albeit with similar characteristics, like the emergence of “flat-top” shapes. In the case of attractive interactions, the oscillons can reach higher values of the energy density and smaller values of the width. For repulsive interactions we identify a threshold for the coupling strength, above which oscillons do not exist within the two-timing small-amplitude framework. We extend the Vakhitov-Kolokolov stability criterion, which has been used to study single-field oscillons, and show that the symmetry of the potential leads to similar equations as in the single-field case, albeit with modified terms. Finally we explore the basin of attraction of stable oscillon solutions and show that, depending on the initial perturbation size, unstable oscillons can either completely disperse or relax to the closest stable configuration.

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I. INTRODUCTION

Oscillons are part of a large family of soliton-like structures. The stability of these solutions can be traced back to non-linear terms in the theory, since a linear theory will in general force rapid dispersion of any localized wave-packet. In general, solitons can be classified according two characteristics. They are either completely static (ignoring propagation) or oscillate during their lifetime. Furthermore, their stability is either due to some conserved charge or to interactions between nonlinearities and dispersive effects.

Among such non-linear dynamics, oscillons [1-10], localized long-lived objects, have attracted significant attention in the early universe cosmology community. Oscillons can arise in scalar field models where the potential is quadratic near the minimum and becomes shallower than quadratic ("flattens out") at larger field values. Currently, inflationary models that are preferred by observations are models that contain plateau-type potentials [11-14], which satisfy the necessary condition needed to support oscillons. Furthermore, numerical simulations have shown that the
post-inflationary fragmentation of the inflaton in such models leads to copious production of oscillons [15].

Recently oscillons have seen renewed interest. Numerical simulations have revealed that oscillon formation after inflation is accompanied by the generation of significant amounts of gravitational waves [16–23]. Since gravitational waves can be one of the very few tools able to probe the end of inflation, the dynamics of oscillon formation, evolution and stability are becoming an essential part of early universe cosmology.

Our current understanding of fundamental physics suggests that multiple scalar fields are likely to be present at high energies. In spite of this, most of the work on oscillons has focused on single-field models, ignoring decay channels of oscillons to other fields and –more interesting– the dynamics that can lead to oscillons comprised of multiple fields. Numerical simulations have uncovered two-field oscillons arising after hybrid inflation [24] and in an $SU(2)$ gauged Higgsed model [25], for which a semi-analytical construction of the observed oscillons was given in Ref. [10], along with a detailed study of their parameter dependent stability. Finally, oscillons were found in an $SU(2) \times U(1)$ model [26, 27], inspired by the electroweak sector of the Standard Model. In the case of the $SU(2)$ Higgsed oscillon, a particular mass ratio between the Higgs and W fields was required for the oscillon to be stable. In particular, oscillons where the Higgs mass was almost twice the W mass were found to be stable over very long time-scales. This observation (which was examined in detail in Ref. [10]) can be taken to imply that rather special conditions need to exist for multi-field oscillons to arise.

The current work aims to provide a detailed look into the conditions for the existence and stability of multi-component oscillons, albeit in a simplified model. We organize our presentation as follows. In Section II we describe the two-field model that we consider. In Section III we use the small-amplitude two-timing analysis to construct small amplitude solutions for composite oscillons, comprised of the two fields oscillating in unison. Section IV contains the stability analysis of oscillons against long-wavelength perturbations, which is an extension of the Vakhitov-Kolokolov criterion in the case of two interacting fields with identical potential structure. We provide our conclusions and outlook on future work in Section V.

II. MODEL

We consider a model of two identical real scalar fields, each with a quadratic-quartic-sixtic potential and a two-to-two interaction term, which can be either attractive or repulsive. This form of the potential was first analyzed for single-field oscillons in Ref. [6], where a large sextic term was
introduced to stabilize three-dimensional oscillons in a symmetric quadratic-quartic potential. We ignore the expansion of the universe and restrict ourselves to an action consisting of the two scalar fields on a Minkowski background

\[ S = \int d^3x dt \left[ \sum_{I=1,2} \left( \frac{1}{2} \partial_\mu \phi_I \partial^\mu \phi_I + \frac{1}{2} m^2 (\phi_I)^2 - \frac{\lambda}{4} (\phi_I)^4 + \frac{g}{6} (\phi_I)^6 \right) - \frac{\Lambda}{2} (\phi^1)^2 (\phi^2)^2 \right]. \tag{1} \]

Before proceeding, it is worth noting both the restrictions that the action of Eq. (1) poses, as well as some theoretical motivation for considering it. We draw our example from the well-motivated area of \( \alpha \)-attractors, specifically the T-model potential. Since metrics on hyperbolic spaces can be formulated in many different forms, following Möbius transformations \cite{28}, we follow the metric definition used in Refs. \cite{29–31}

\[ ds^2 = d\chi^2 + e^{2b(\chi)} d\phi^2, \tag{2} \]

where \( b(\chi) = \log (\cosh(\beta \chi)) \). The two-field potential for the T-model in this field basis is

\[ V(\phi, \chi) = \alpha \mu^2 \left( \frac{\cosh(\beta \phi) \cosh(\beta \chi) - 1}{\cosh(\beta \phi) \cosh(\beta \chi) + 1} \right) (\cosh(\beta \chi))^{2/\beta^2}, \tag{3} \]

where \( \beta = \sqrt{2/3\alpha} \).

For small value of \( \alpha \), the potential around the origin is expanded as

\[ V(\phi, \chi) \approx \mu^2 \chi^2 - \frac{\mu^2}{54\alpha} \chi^4 + \frac{17\mu^2}{9720\alpha^2} \chi^6 + \mu^2 \phi^2 - \frac{\mu^2}{54\alpha} \phi^4 + \frac{17\mu^2}{9720\alpha^2} \phi^6 + \frac{\mu^2}{6} \phi^2 \chi^2 - \frac{\mu^2}{16\alpha^2} \phi^2 \chi^2 (\phi^2 + \chi^2) + \ldots \tag{4} \]

where we included terms up to sextic order in the fields and neglected higher order contributions in \( \alpha \) in each term. For that we assume that the parameter \( \alpha \), which is inversely related to the field-space curvature, is small. We see that this expansion has the characteristics that we consider in the action of Eq. (1): two scalar fields with identical potential parameters, a negative quartic term and a very large sextic term (for \( \alpha \ll 1 \)), as well as a quartic coupling. Ref. \cite{29} reported the formation of oscillons during two-field preheating after inflation on a T-model potential, but did not provide a detailed analysis of their properties. Of course, the true T-model expansion differs from the idealized action of Eq. (1), in that it contains sextic interaction terms and it provides less freedom for choosing the various potential coupling strengths (see Fig. 1). Furthermore, the analysis of the T-model must also take into account the non-canonical kinetic structure, leading to the derivative couplings

\[ L_{\text{kin}} \supset \frac{\lambda^2}{3\alpha} (\partial \phi)^2 + \frac{2\chi^4}{27\alpha^2} (\partial \phi)^2 + \ldots \tag{5} \]
Figure 1. The T-model potential for $\chi = 0$ (blue solid) and $\chi = 0$ (red dashed), along with the sextic Taylor expansion (green dotted). The brown dot-dashed curve shows the quadratic term, which is steeper than the total potential, allowing in principle for oscillon formation in both $\phi$ and $\chi$ fields. The vertical lines show the field value of $\phi$ at the end of inflation.

Oscillons in systems with non-canonical kinetic terms have been considered in the literature for single-field systems [8]. In the present work, we instead focus on the symmetric sextic potential of Eq. (1) and consider canonical kinetic terms for the two fields. An extension of the current analysis to realistic $\alpha$-attractor models, taking into account the intricacies of derivative couplings, can reveal interesting phenomenology for this family of well-motivated and observationally relevant models and is left for future work.

III. TWO TIMING ANALYSIS

In this section we construct approximate profiles of two-field oscillons and compare them to the single-field oscillons found in Ref. [6]. Eq. (1) contains several physical parameters, such as the mass of the two fields $m$, the coupling $\Lambda$ and self couplings $g$ and $\lambda$. It is convenient to work with dimensionless space-time variables and fields. This is done by the rescalings $x^\mu \rightarrow x^\mu = x^\mu m$, $\phi^I \rightarrow \tilde{\phi}^I = m^{-1/2} \phi^I$, $g \rightarrow \tilde{g} = (m/\lambda)^2 g$ and $\Lambda \rightarrow \tilde{\Lambda} = (1/\lambda) \Lambda$. The Lagrangian is then re-written as

$$L = \frac{\lambda}{m^4} \left( \sum_{I=1,2} \frac{1}{2} \partial_\mu \tilde{\phi}^I \partial^\mu \tilde{\phi}^I + \frac{1}{2} \left( \tilde{\phi}^I \right)^2 - \frac{1}{4} \left( \tilde{\phi}^I \right)^4 + \frac{\tilde{g}}{6} \left( \tilde{\phi}^I \right)^6 \right) - \frac{\tilde{\Lambda}}{2} \left( \tilde{\phi}^1 \right)^2 \left( \tilde{\phi}^2 \right)^2$$

(6)
For simplicity, in what follows we define \( \{ \tilde{\phi}_1, \tilde{\phi}_2 \} \equiv \{ \tilde{\phi}, \tilde{\chi} \} \) and drop all tildes. The Lagrangian of Eq. (6) leads to a system of two coupled equations of motion for \( \phi \) and \( \chi \). We are interested in spherically symmetric solutions \( \phi(\vec{x},t) \to \phi(r,t) \), leading to the equations of motion

\[
\partial_t^2 \phi - \left( \partial_r^2 + \frac{2}{r} \partial_r \right) \phi + \phi = \phi^3 - g\phi^5 + \Lambda \phi \chi^2, \\
\partial_t^2 \chi - \left( \partial_r^2 + \frac{2}{r} \partial_r \right) \chi + \chi = \chi^3 - g\chi^5 + \Lambda \chi \phi^2.
\]

(7)

The restriction of spherical symmetry will be imposed throughout this work. The attentive reader will immediately notice that these equations are symmetrical under exchange of the fields, meaning that sending \( \chi \to \phi \) in either of the equations gives back the other. This greatly simplifies the analytical search for oscillons which is more difficult (if at all possible) in general multi-component systems.

It is well established in the literature that oscillons live on long time- and large spatial scales \([32,34]\). This suggests a perturbative approach to extract oscillon solutions from the equations of motion, known as the small-amplitude two-timing analysis. The idea is that oscillons exhibit behaviour on two time scales, one capturing the natural frequency of the free field theory and one capturing the correction to this frequency characteristic for the non-linear potential. Furthermore, we attempt to capture oscillons which are slowly varying in space, hence are "broad". This behaviour is found by introducing new time and space variables

\[
\tau = \alpha \epsilon^2 t, \quad \rho = \epsilon r
\]

(8)

where \( \alpha \sim O(1) \) and \( \epsilon \ll 1 \). While we expect all spatial variation to occur on the scale of \( \rho \), the time variation occurs on two scales: \( t \) and \( \tau \). The (double) time derivatives in Eqs. (7) must now be interpreted as full time derivatives, leading to the equations

\[
\partial_t^2 \phi + 2\alpha \epsilon^2 \partial_r \partial_r \phi - \epsilon^2 \left( \partial_r^2 + \frac{2}{\rho} \partial_\rho \right) \phi + \phi = \phi^3 - g\phi^5 + \Lambda \phi \chi^2 + O(\epsilon^4), \\
\partial_t^2 \chi + 2\alpha \epsilon^2 \partial_r \partial_r \chi - \epsilon^2 \left( \partial_r^2 + \frac{2}{\rho} \partial_\rho \right) \chi + \chi = \chi^3 - g\chi^5 + \Lambda \chi \phi^2 + O(\epsilon^4).
\]

(9)

Lastly, we assume that the amplitude of the oscillon profile also scales with the expansion parameter \( \epsilon \), allowing us to compute the effects of the non-linearities order-by-order. We therefore consider solutions of the form

\[
\phi(x,t) = \sum_{n=1}^\infty \epsilon^n \phi_n(\rho,t,\tau), \\
\chi(x,t) = \sum_{n=1}^\infty \epsilon^n \chi_n(\rho,t,\tau).
\]

(10)
Inserting Eqs. (10) into Eqs. (9), the lowest order equations in $\epsilon$ are

$$\partial_t^2 \phi_1 + \phi_1 = 0,$$
$$\partial_t^2 \chi_1 + \chi_1 = 0,$$

which are identical to the harmonic oscillator equation and capture the main oscillatory behavior of the oscillon. The equations in the next non-trivial order in $\epsilon$ are

$$\partial_t^2 \phi_3 + \phi_3 = \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) \phi_1 - 2\alpha \partial_t \partial_\rho \phi_1 + \phi_1^3 - \kappa \phi_1^5 + \Lambda \phi_1 \chi_1^2,$$
$$\partial_t^2 \chi_3 + \chi_3 = \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) \chi_1 - 2\alpha \partial_t \partial_\rho \chi_1 + \chi_1^3 - \kappa \chi_1^5 + \Lambda \chi_1 \phi_1^2 .$$

Notice that we used the fact that $g$ is large and have therefore written $g = \kappa/\epsilon^2$, with $\kappa \sim O(1)$. The solutions to the harmonic oscillator Eqs. (11) are trivial. Therefore we can write $\phi_1 = \text{Re} \{ A(\rho, \tau) e^{-it} \}$ and $\chi_1 = \text{Re} \{ B(\rho, \tau) e^{-it} \}$; where $A(\rho, \tau)$ and $B(\rho, \tau)$ are complex functions. Inserting this solution into Eqs. (12) and eliminating secular terms gives us the envelope equations of the system of oscillons

$$2i\alpha \partial_\tau A + \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) A + \frac{3}{4} |A|^2 A + \frac{\Lambda}{2} |A|^2 B + \frac{\Lambda}{4} A^* B^2 - \frac{\kappa}{8} |A|^4 A = 0,$$
$$2i\alpha \partial_\tau B + \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) B + \frac{3}{4} |B|^2 B + \frac{\Lambda}{2} |B|^2 A + \frac{\Lambda}{4} B^* A^2 - \frac{\kappa}{8} |B|^4 B = 0.$$

These equations are of the Nonlinear Schrödinger type [35]. They control the behaviour of the oscillon on long time- and large spatial scales. To find solutions of these equations that are localized in space we insert the “oscillon ansatz”. Since by assumption the oscillon is just some localized structure oscillating in time we should look for solutions of the form, $A(\rho, \tau) = a(\rho) e^{ic_1 \tau}$ and $B(\rho, \tau) = b(\rho) e^{ic_2 \tau}$; where $a$ and $b$ are real functions determining the spatial character of the oscillon. Due to the symmetry of the potential we can also set $c_1 = c_2 = c$. Inserting this into Eqs. (13) we obtain the profile equations of the oscillons.

$$-\alpha a + \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) a + \frac{3}{4} a^3 + \frac{3\Lambda}{4} a^2 b - \frac{\kappa}{8} a^5 = 0,$$
$$-\alpha b + \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) b + \frac{3}{4} b^3 + \frac{3\Lambda}{4} b^2 a - \frac{\kappa}{8} b^5 = 0,$$

where we set $c = 1/2$. A few remarks about the parameter $c$ are now in order. $c$ is in essence a free parameter, as long as it is not too large. This is because it can always be absorbed into a redefinition of $\epsilon$. We do require it to be positive, since the oscillon should behave as $a, b \to 0$ for $\rho \to \infty$, since they are localized solutions. We can therefore set $c = 1/2$ without loss of generality.
An interesting property arising due to the symmetry of the system now becomes apparent. Namely, any localized solution $a(\rho)$ of the equation

$$-\alpha a + \left(\frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial}{\partial \rho}\right) a + \frac{3}{4} (1 + \Lambda)a^3 - \frac{5}{8} \kappa a^5 = 0 \quad (15)$$

directly solves the system of Eqs. (14) if we set $a = b$. We arrive at the same equation if we instead choose $a = -b$, meaning that the two fields oscillate out of phase. It is worth noting that in the one other case in the literature, where two-field oscillons were constructed using the two-timing analysis, Ref. [10], the phase was irrelevant, since the masses of the two fields were chosen to have a $2 : 1$ ratio, meaning that for half the period of the light field, they were in phase, and for the other half they were out of phase.

The difficulty of finding oscillon solutions in the coupled system of Eqs. (14) is therefore greatly reduced and can be related to the solutions for single-field oscillons in a quartic-sextic potential, which were studied in Ref. [9].

A. Oscillon profiles

In order to solve Eq. (15), we must first define the boundary conditions. At large distance from the origin, far away from the oscillon core, the non-linear terms become subdominant, hence Eq. (15) can be approximated as $\partial^2 a + (2/\rho) \partial a \approx \alpha a$, leading to $a(\rho) \propto e^{-\sqrt{\alpha} \rho}/\rho$. The boundary condition at $\rho \to 0$ is slightly more complicated. Eq. (15) can be re-written as

$$\frac{dE_\rho}{d\rho} = -\frac{2}{\rho} \left( \frac{\partial a}{\partial \rho} \right)^2 , \quad (16)$$

where

$$E_\rho = \frac{1}{2} \left( \frac{\partial a}{\partial \rho} \right)^2 - \frac{1}{2} \alpha a^2 + \frac{3}{16} (1 + \Lambda) a^4 - \frac{5}{48} \kappa a^6 \quad (17)$$

would be the conserved energy for a one-dimensional system with the same potential. The right hand side of Eq. (16) is non-positive, hence the energy of any solution $E_\rho$ will decrease as $\rho \to \infty$. Since the energy of a localized solution must be zero in the far-distance regime we conclude that $E_\rho \geq 0$ at $\rho \to 0$. It is best to think about this in terms of the trajectories of solutions in phase space $(a, \partial a)$. Since we’re interested in functions that are smooth at the origin, all trajectories start on the $\partial a = 0$ axis. Any trajectory will continuously intersect curves with $E_\rho = \text{const}$, and the constant value of these curves must decrease as $\rho \to \infty$. The $E_\rho = 0$ curve then defines the boundary between localized and non-localized solutions. Namely, a localized solution must
intersect this curve in the origin. The requirement that the oscillon is smooth at the origin, meaning \( \partial_\rho a |_{\rho=0} = 0 \), leads to \( \alpha < \alpha_c \equiv (27/160)(1 + \Lambda)^2 \), generalizing the constraint found in Ref. [6] for single-field oscillons\(^1\) or equivalently \( \Lambda = 0 \). For \( \alpha \to \alpha_c \) the oscillon becomes infinitely wide with an amplitude \( a(\rho = 0) \to 9/10(1 + \Lambda) \).

It turns out that only a countable set \( a_n(\rho) \) of these kind of trajectories can be drawn in phase space \([36]\). Here \( n \) is the amount of nodes of the solution. We conclude that there is exactly one zero-node solution of the profile equation, and thus one zero-mode oscillon for every choice of \( \epsilon \) and \( \alpha \). We do not pursue solutions with \( n \geq 1 \), since they will have a higher energy than their zero-node counterpart, and thus are expected to be unstable. We used both the shooting and relaxation methods in order to find the corresponding oscillon profiles and check our results. Fig. 2 shows some characteristic profiles for \( \Lambda = \pm 0.5 \).

![Oscillon profiles](image)

Figure 2. *Left:* Oscillon profiles as a function of the rescaled radius \( \rho \) for repulsive fields with \( \Lambda = -0.5 \) and \( \alpha = 0.005, 0.01, 0.03, 0.037 \) (blue, red, green and brown respectively). *Right:* Oscillon profiles for attractive fields with \( \Lambda = 0.5 \) and \( \alpha = 0.045, 0.09, 0.22, 0.33 \) (blue, red, green and brown respectively). We see the emergence of a flat-top shape in both cases. Furthermore, we compute that \( \alpha_c = 0.042 \) for \( \Lambda = -0.5 \) and \( \alpha_c = 0.38 \) for \( \Lambda = 0.5 \), leading to, \( a(\rho = 0) \approx 0.67 \) and \( a(\rho = 0) \approx 1.16 \) respectively, meaning that oscillons can acquire larger amplitudes in a potential with repulsive interactions.

Fig. 3 shows the width and height of the oscillons as a function for various interaction strengths between the two fields, attractive or repulsive, rescaled by the small-amplitude parameter \( \epsilon \). We see that in the non-interacting case of \( \Lambda \to 0 \), we recover the single-field results of Ref. [6]. Furthermore,

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\(^1\) We must note that our definition of the parameter \( \alpha \) is related to the square of the parameter \( \alpha \) used in Ref. [6].
fields which interact repulsively lead to a smaller range of oscillon amplitudes, ending in no oscillons for $\Lambda \to -1$, as is also evident by the form of $\alpha_c \sim (1 + \Lambda)^2$. On the contrary, attractive interactions allow for taller oscillons.

Figure 3. The heigh-width relation for the oscillon profile for $\Lambda = -0.5, -0.2, 0, 0.2, 0.5$ (blue, red, green, brown and black respectively). We see that fields which interact repulsively lead to a smaller range of oscillon amplitudes, leading to no oscillons for $\Lambda \to -1$. On the contrary, attractive interactions allow for taller, as well as wider oscillons. The colored dots show the point where the oscillons change their stability properties, which is explained in detail in Section IV.

Fig. 4 shows several characteristic values for the height and width of oscillons as a function of the coupling strength $\Lambda$. We see that the spread of oscillon heights, calculated as the difference between the minimum height for stable oscillons and the maximum height, grows with $\Lambda$, being almost double for $\Lambda = 0.5$ compared to $\Lambda = -0.5$. The spread of the width is infinite, since arbitrarily wide oscillons can exist in theory. For larger values of the interaction strength $\Lambda$, narrower oscillons exist. In particular, the minimum width for oscillons with $\Lambda = -0.5$ is almost triple the corresponding value for $\Lambda = 0.5$. 


Figure 4. The height and width of oscillons for the point of marginal stability (blue), the point corresponding to the minimum oscillon width (green) and the point corresponding to the maximum oscillon height (red). The black curve corresponds to the asymptotic oscillon height $\frac{9}{10}(1 + \Lambda)$.

IV. STABILITY ANALYSIS

It has been shown repeatedly in the literature that no true breather solutions like the oscillon can exist in nonlinear systems. In general there will always be (classical) outgoing radiation in the tails of the oscillon. Classically, this radiation is exponentially suppressed, which is why oscillons can be extremely long-lived. Quantum mechanical radiation might play a more important role in real physical systems [37], but is beyond the scope of our present work. Radiation will in general perturb the oscillon system and it is therefore necessary to assess the stability of oscillons with respect to small perturbations. In the model we are investigating, let us consider small fluctuations $\delta(x, t), \Delta(x, t) \ll O(\epsilon)$ added to the oscillon solutions as

$$\phi(x, t) = \phi_{osc}(x, t) + \delta(x, t),$$
$$\chi(x, t) = \chi_{osc}(x, t) + \Delta(x, t).$$

Plugging these into Eqs. (7), the equations of motion for the two fields, and keeping only terms linear in $\delta$ and $\Delta$ leads to

$$\partial_t^2 \delta + \delta - (\partial_r^2 + \frac{2}{r}\partial_r)\delta - 3\phi_{osc}^2\delta + 5g\phi_{osc}^4\delta - \Lambda\chi_{osc}^2\delta - 2\Lambda\phi_{osc}\chi_{osc}\Delta = 0,$$
$$\partial_t^2 \Delta + \Delta - (\partial_r^2 + \frac{2}{r}\partial_r)\Delta - 3\chi_{osc}^2\Delta + 5g\chi_{osc}^4\Delta - \Lambda\phi_{osc}^2\Delta - 2\Lambda\chi_{osc}\phi_{osc}\delta = 0.$$
Since the exact shape of the initial perturbation is in essence unpredictable, a full linear stability analysis would require solving Eq. (19) for arbitrary initial conditions. This requires solving the full Floquet matrix for the coupled nonlinear system. It turns out that, for this system, there is a useful simplification if we only consider perturbations that are about the same size as the oscillon itself. This is in essence an extension of the Vakhitov-Kolokolov criterion that has been used to assess stability of single-component oscillons [6].

A. Extension of the V-K criterion

In Section III we have already shown that this system supports oscillons of the form

\[ \phi_{osc}(x,t) = \epsilon \text{Re} \left\{ a(\epsilon x)e^{it(1-\frac{\epsilon^2}{2})} \right\} = a(\epsilon x) \cos \omega t, \]

\[ \chi_{osc}(x,t) = \epsilon \text{Re} \left\{ b(\epsilon x)e^{it(1-\frac{\epsilon^2}{2})} \right\} = b(\epsilon x) \cos \omega t, \]

where the frequency is \( \omega = 1 - \epsilon^2 \alpha/2 \). The symmetry of the system dictates that we have \( a(\rho) = \pm b(\rho) \), as we have discussed in Section III. This allows us to turn Eqs. (20) into a single-variable system in order to derive and extend the V-K criterion, similarly to what was done in Ref. [6] for single-field oscillons. For clarity, we will separately analyze the two cases of oscillons where the two fields oscillate either in phase \( (a = b) \) or out of phase \( (a = -b) \).

1. Two-field system oscillating in phase

For oscillons comprised of the two fields oscillating in phase, \( a = b \), the system of Eqs. (19) becomes

\[ \partial_t^2 \delta + \delta - (\partial_r^2 + \frac{2}{r} \partial_r) \delta - \epsilon^2 a^2 (3 + \Lambda) \cos(\omega t)^2 \delta + \epsilon^4 5ga^4 \cos(\omega t)^4 \delta - \epsilon^2 2\Lambda a^2 \cos(\omega t) \Delta = 0, \]

\[ \partial_t^2 \Delta + \Delta - (\partial_r^2 + \frac{2}{r} \partial_r) \Delta - \epsilon^2 a^2 (3 + \Lambda) \cos(\omega t)^2 \Delta + \epsilon^4 5ga^4 \cos(\omega t)^4 \Delta - \epsilon^2 2\Lambda a^2 \cos(\omega t) \delta = 0. \]  

While Eqs. (21) is rather complicated, the symmetries of the initial Lagrangian allow us to simplify the stability analysis, by introducing the new variable \( \xi(x,t) = \delta(x,t) + \Delta(x,t) \), which reduces Eqs. (21) to

\[ \partial_t^2 \xi + \xi - (\partial_r^2 + \frac{2}{r} \partial_r) \xi - \epsilon^2 3a^2 (1 + \Lambda) \cos(\omega t)^2 \xi + \epsilon^4 5ga^4 \cos(\omega t)^4 \xi = 0 \]  

In several oscillon studies, the frequency is taken to be of the form \( \omega = \sqrt{1 - c \cdot h^2} \), where \( h \) is the oscillon height and \( c \) is some \( O(1) \) constant. This matches our expression to lowest order in perturbation theory when \( h \propto \epsilon \ll 1 \), which is true in the context of the small amplitude analysis used to construct oscillon solutions.
This equation only depends on $\xi(x, t)$, greatly simplifying our calculations. Since we are interested in perturbations that are about the same size as the oscillon itself we perform the same change of variable as before $r \to \rho = \epsilon r$. Furthermore, we expect there perturbations to oscillate near the oscillon frequency. To capture the growth or decay of the perturbation we should also introduce a new “slow" time variable $\tau = \epsilon^2 t$. We also introduce a time-scale related to the corrected oscillation frequency of the oscillon $T = \omega t$. As before we expect behaviour on both time scales, so $\xi(x, t) \to \xi(\rho, T, \tau)$, and partial time-derivatives turn into full derivatives. Looking for solutions that can be written as a power series in $\epsilon \ll 1$

$$\xi(\rho, T, \tau) = \sum_{n=0}^{\infty} \epsilon^n \xi_n(\rho, T, \tau)$$  \hspace{1cm} (23)

allows for a perturbative analysis of Eq. (22) The zeroth and first order equations are respectively

$$\partial_T^2 \xi_0 + \xi_0 = 0$$  \hspace{1cm} (24)

and

$$\partial_T^2 \xi_1 + \xi_1 = -\left( \partial_T \partial_r - \alpha \partial^2_T - \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) - 3a^2(1 + \Lambda) \cos T^2 + 5ka^4 \cos T^4 \right) \xi_0 .$$  \hspace{1cm} (25)

The general solution of Eq. (24) has the form

$$\xi_0(\rho, T, \tau) = u(\rho, \tau) \cos T + v(\rho, \tau) \sin(T) ,$$  \hspace{1cm} (26)

where the functions $u(\rho, \tau)$ and $v(\rho, \tau)$ capture the potential growth of the perturbation on timescales of order $\tau$. The V-K criterion provides a simple way to distinguish cases where perturbation grow exponentially based on the form of the oscillon itself. Namely, inserting Eq. (26) into Eq. (25) and eliminating secular terms on the right-hand side leads to equations for $u$ and $v$

$$\partial_\tau u = H_1 v ,$$  \hspace{1cm} (27)

$$\partial_\tau v = -H_2 u ,$$  \hspace{1cm} (28)

where $H_1$ and $H_2$ are Hermitian, linear operators defined as

$$H_1 = \alpha - \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) - \frac{3}{4}(1 + \Lambda)a^2 - \frac{5}{8} \kappa a^4 = 0 \, ,$$  \hspace{1cm} (29)

$$H_2 = \alpha - \left( \partial^2_\rho + \frac{2}{\rho} \partial_\rho \right) - \frac{9}{4}(1 + \Lambda)a^2 + \frac{25}{8} \kappa a^4 = 0 \, .$$  \hspace{1cm} (30)

Separating variables as $u(\rho, \tau) \to u(\rho)e^{i\Omega \tau}$ and $v(\rho, \tau) \to v(\rho)e^{i\Omega \tau}$ the problem reduces to an eigenvalue problem

$$\Omega^2 u = -H_1 H_2 u .$$  \hspace{1cm} (31)
The question of whether the oscillon is stable to general long-wavelength perturbation is thus reduced to an eigenvalue problem. If the operator $-H_1H_2$ has at least one positive eigenvalue $\Omega^2 > 0$, perturbations can grow and the oscillon will in general be unstable. If not, perturbations simply oscillate, leading to an oscillon that is stable, within the limits of the perturbative expansion used to construct it. The problem can be solved using a similar procedure as Vakhitov and Kolokolov [38]. We do not present the entire proof here, as its intricacies will add little to the main goal of analyzing two-field oscillons. The criterion for stability states that $\max(\Omega^2) < 0$ if and only if $\frac{dN}{d\alpha} > 0$, where

$$N = \int a^2(\rho)d^3\rho.$$  \hspace{1cm} (32)

Before proceeding, we must make a final remark about the validity of this derivation. The criterion we presented above only states whether perturbation of the form $\xi(x,t) = \delta(x,t) + \Delta(x,t)$ will grow. In principle, the growth of the perturbation might only be present in $\delta$ or $\Delta$. However, since the system is completely symmetric under exchange of the fields, we can only conclude that, if $\xi(x,t)$ grows, both $\delta(x,t)$ and $\Delta(x,t)$ will grow exponentially. The criterion is therefore valid for the full two-field system and can be used as an indicator for the stability of two-field oscillons, at least ones that are comprised of interchangeable fields.

2. Two-field system oscillating out of phase

For oscillons comprised of the two fields oscillating out of phase, $a = -b$, the system of Eqs. (19) becomes

$$\partial_t^2 \delta + \delta - \left(\partial_r^2 + \frac{2}{r}\partial_r\right)\delta - \epsilon^2 a^2(3 + \Lambda)\cos(\omega t)^2\delta + \epsilon^4 5ga^4\cos(\omega t)^4\delta + \epsilon^2 2\Lambda a^2\cos(\omega t)^2\Delta = 0,$$

$$\partial_t^2 \Delta + \Delta - \left(\partial_r^2 + \frac{2}{r}\partial_r\right)\Delta - \epsilon^2 a^2(3 + \Lambda)\cos(\omega t)^2\Delta + \epsilon^4 5ga^4\cos(\omega t)^4\Delta + \epsilon^2 2\Lambda a^2\cos(\omega t)^2\delta = 0.$$  \hspace{1cm} (33)

Now, instead of looking for unstable modes in the combination $\xi = \delta + \Delta$, we introduce the corresponding variable $\psi \equiv \delta - \Delta$. By now subtracting the equation in (33) The corresponding equation for $\psi$ becomes

$$\partial_t^2 \psi + \psi - \left(\partial_r^2 + \frac{2}{r}\partial_r\right)\psi - \epsilon^2 3a^2(1 + \Lambda)\cos(\omega t)^2\psi + \epsilon^4 5ga^4\cos(\omega t)^4\psi = 0.$$  \hspace{1cm} (34)

We see that the equation of motion for $\psi$ in the case of $a = -b$ is identical to the one for $\xi$ for $a = b$. The rest of the derivation follows exactly the same steps as before so we won’t repeat them here.
Hence the stability of the oscillons will be independent of the phase (0 or \(\pi\)) between the two fields, resulting in exactly the same V-K criterion as before. Namely, there are unstable perturbations of the oscillon if and only if \(dN/d\alpha > 0\), where \(N\) is defined in Eq. (32).

In this section we have derived a nontrivial extension of the V-K criterion for assessing the stability of oscillons in this coupled system to long wave-length perturbations (perturbations about the same size as the oscillon). It is, to our knowledge, the first time that the criterion has been derived for multi-component oscillons.

Fig. 5 shows the most unstable modes computed by solving the eigenvalue problem of Eq. (31) using the variational method for \(\Lambda = -0.5\) and \(\Lambda = 0.5\), corresponding to repulsive and attractive interactions respectively. We choose some values of \(\alpha\) for each case, which give \(\Omega^2 > 0\) in Eq. (31) and thus lead to (radially) unstable oscillon solutions. We see that the spatial size of the unstable fluctuations is indeed similar to the width of the corresponding oscillons.

![Figure 5](image-url)

*Figure 5.* *Left:* The profiles of the most unstable modes of the corresponding oscillons with \(\Lambda = -0.5\) and \(\alpha = 0.00028, 0.00071, 0.0025, 0.0046\), (blue, red, green and brown respectively). These are calculated numerically using the variational principle. *Right:* The profiles of the most unstable modes of the corresponding oscillons with \(\Lambda = 0.5\) and \(\alpha = 0.0025, 0.0064, 0.0225, 0.04\) (blue, red, green and brown respectively). The profiles were rescaled such that \(\delta(\rho = 0) = 1\). We see that the size of the unstable modes are similar to the size of the oscillon profile, shown in Fig. 2.
Figure 6. Schematic representation of the Vakhitov-Kolokolov stability criterion for two-field oscillons. We plot the quantity $N$, defined in Eq. (32), as a function of the phase-shift parameter $\alpha$ for $\Lambda = -0.5, -0.2, 0, 0.2, 0.5$ (blue, red, green, brown and black respectively). The vertical dashed lines correspond to the value of $\alpha_c$ for each case, while the solid (dotted) branches correspond to stable (unstable) oscillons.

B. Oscillon Dynamics and Decay

Having constructed two-field oscillon solutions and having extended the V-K criterion to examine their stability, we move to numerically demonstrate our results and understand the relevant timescales of interest. As a benchmark case of unstable oscillons, we use profiles corresponding to $\Lambda = 0.5, \alpha = 0.0064$ and $\epsilon = 0.08$, which we perturb using the most unstable mode-functions computed in Section IV. However, while the perturbation shape is given, the perturbation amplitude is a free parameter. We define the perturbation amplitude as the relative size of the oscillon and perturbation mode-function at the origin

$$\delta_{\text{magn.}} = \left| \frac{\delta(r = 0, t = 0)}{\phi_{\text{osc}}(r = 0, t = 0)} \right|,$$

and consider three values $\delta_{\text{magn.}} = 1\%, 5\%, 10\%$.

Fig. 7 shows the time-evolution of the oscillon peak energy density and width as a function of
Figure 7. The evolution of the oscillon peak energy density (left) and width (right) as a function of slow time $\tau = \epsilon^2 t$ for $\Lambda = 0.5$, $\alpha = 0.0064$ and $\epsilon = 0.08$. We perturb the oscillon by the most unstable mode with amplitude $\delta_{magn.} = 1\%, 5\%, 10\%$ (blue, red and green respectively). The brown curve corresponds to perturbing only one of the two fields with $\delta_{magn.} = 10\%$ and initializing the other field by the “perfect” oscillon profile.

slow time $\tau = \epsilon^2 t$. First of all, it is evident that the slow time controls the relevant time-scales for the oscillon dynamics. This can be traced back to the evolution Eqs. (13) for the complex functions $A(\rho, \tau)$ and $B(\rho, \tau)$, which are of the non-linear Schrödinger type, and control the dynamics of the oscillon envelope, on top of the fast oscillations, which occur on a timescale of $O(m^{-1})$. We see that the magnitude of the initial perturbation seems to play a crucial role for the ultimate fate of the oscillon: small fluctuations $\delta_{magn.} \lesssim 5\%$ lead to a new semi-stable point for the width and height of the localized configuration. On the other hand, for $\delta_{magn.} = 10\%$ the initial localized wave-packet simply disperses. We must note here, that initializing the two fields out of phase $a = -b$ and using the corresponding perturbation derived using Eqs. (33) gives indistinguishable curves, and thus we do not plot them for clarity. This however confirms our analysis, that the phase of the two fields in the initial oscillon configuration, 0 or $\pi$, does not affect the subsequent dynamics.

Furthermore, one can ask whether having the exact form of the most unstable modes computed in the previous section is a necessary condition for oscillon decay. In single-field oscillons, one can see that any perturbation can be decomposed in a basis of eigen-functions of the operator $H_1 H_2$, meaning a generic perturbation will include a component along the most unstable mode. In two-field systems, it is interesting to see how an initial perturbation is transferred from one field to the
other. Finally, we perturbed only the $\phi$ field, leaving the $\chi$ field profile identical to the perfect oscillon solution. Again, the evolution of the oscillon was almost identical to the case where both fields were perturbed with an important difference. Perturbing one field with $\delta_{\text{magn.}} = 10\%$ and leaving the other field intact leads to a similar evolution as perturbing both fields with $\delta_{\text{magn.}} = 5\%$. Furthermore, while the initial height and width of the perturbed and unperturbed fields can differ by more than 10\%, the values at the end of our simulation are very close to unity. This indicates that due to the interaction of the two fields and the symmetry of the system, a small initial perturbation is quickly distributed evenly among the two fields. We thus see that it is not necessary to perturb the oscillon with the exact form of the perturbation given by the stability analysis of Section IV, since unstable oscillons will eventually decay if perturbed, albeit on slightly different time-scales and with different final state, depending on the details of the perturbation. On the other hand, we perturbed oscillons that are characterized as stable by the two-field version of the V-K criterion, seeing no decay over the total run of the simulation, lasting several hundred units of “slow time” $\tau = \epsilon^2 t$.

It is now important to understand this behavior. Fig. 8 shows the trajectories of several perturbed oscillons on the height-width curve for oscillons with $\Lambda = 0.5$. We see that unstable oscillons will surely move away from their initial configuration if perturbed. However their ultimate fate is not unique. It is known that localized solutions of the three-dimensional non-linear Scroedinger equation are unstable and undergo a collapse instability, meaning that their width becomes smaller and their amplitude becomes larger. For oscillons that are “mildly perturbed”, $\delta_{\text{magn.}} \lesssim 5\%$ this is exactly what we see at first: the unstable oscillons follow the height-width relation, on which their amplitude increases and their width decreases. This is the regime, where one can safely neglect the sextic term in the initial Lagrangian, equivalently the $|A|^4 A$ term in Eq. (13). Once the oscillon amplitude grows enough, so that the $|A|^4 A$ term becomes important, the oscillon is stabilized, which occurs near the minimum of the height-width curve. Because our simulation box does not allow for the energy to escape (it does not possess absorbing boundary conditions), the oscillon will never truly reach a “perfect” shape, and will instead stay in the vicinity of the first stable point on the height-width curve.

Oscillons that are perturbed by a large amount, are too far away from the height-width curve, and do not undergo this slow collapse instability, which would take them to a stable configuration. Using different perturbation types on the initial oscillon configuration amounts to examining the

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3 By choosing the initial profile of one field to be significantly suppressed compared to the other, we observed the complete dispersion of one field and the relaxation of the other towards a single-oscillon solution.

4 In this plot, the height and width curve is derived from the energy density, not the profile of the oscillons. It is thus related to the one shown in Fig. 8 but is not identical to it.
Figure 8. The height and width of oscillons with $\Lambda = 0.5$ and $\epsilon = 0.08$. The different colors correspond to different values of $\alpha$ and $\delta_{\text{magn.}}$, as follows: $\{\alpha, \delta_{\text{magn.}}\} = \{0.0064, 1\%\}$ blue, $\{0.0064, 10\%\}$ red, $\{0.0025, 1\%\}$ green, $\{0.0025, 10\%\}$ brown, $\{0.04, 1\%\}$ purple and $\{0.04, 10\%\}$ maroon. The black-dashed curve corresponds to the height-width curve for oscillons and the vertical back-dotted line shows the transition from unstable to stable oscillons. The colored dots correspond to the initial profiles. We see that in all cases $\delta_{\text{magn.}} = 1\%$ leads to the relaxation of the oscillon towards a stable configuration, while $\delta_{\text{magn.}} = 10\%$ leads to the complete dispersion of the oscillon (height reducing to zero and width increasing without bound).

basis of attraction for two-field oscillons, finding the approximate size of the initial perturbation below which the oscillon is allowed to “rearrange” itself into a stable configuration and above which the oscillon is completely destabilized. An important question remains, as to the oscillon configurations that arise naturally after inflation: it is not known if a universe that is governed by the Lagrangian of Eq. (1) during preheating will be dominated by single or multi-field oscillons. We are currently working on the corresponding lattice simulations required to address this point and will present the results in a subsequent publication.
V. SUMMARY AND DISCUSSION

Despite the oscillons’ ubiquity in non-linear scalar field theories and their assumed presence in the early Universe, very few multi-field oscillons have been found and studied. In the present work we explored a symmetric system comprised of two interacting scalar fields and showed how genuinely two-field oscillon solutions can be constructed.

We found two-field oscillon solutions in potentials with either an attractive or a repulsive interaction term. They are qualitatively similar to “flat-top” oscillons found in Ref. [6], with quantitative differences that depend on the sign and strength of the interaction term. The oscillons that emerge in the presence of a attractive non-linearity can be both taller (can reach higher field values) and also narrower (have smaller widths) than their single-field counterparts. In the case of repulsively interacting fields, the range of possible oscillon amplitudes shrinks, until the repulsive term is strong enough to completely forbid the existence of oscillons, at least in the context of the small amplitude two-timing framework.

The stability of non-linearity-supported localized structures can be assessed through the Vakhitov-Kolokolov (VK) stability criterion. We formally extended the VK criterion, in order to be used in two-field systems. We checked the VK criterion against numerical simulations (assuming spherical symmetry throughout the evolution), finding excellent agreement between semi-analytical and fully numerical results. Our current proof holds for symmetric potentials and the generalization to arbitrary multi-field oscillons is left for future work. Finally we explored the basin of attraction of stable oscillons, by perturbing unstable initial configurations. We found that, depending on the size of the initial perturbation, the unstable oscillons can either completely disperse, or relax to a stable oscillon configuration with a smaller width and larger height.

While we explored in detail the shapes and stability properties of two-field oscillons in a symmetric quadratic-quartic-sexstic system, lattice simulations are required to test their emergence after inflation. This analysis is currently under way and will be presented in a subsequent publication. Overall, our current work provides analytical tools for studying multi-component oscillons and opens up several avenues for future work, such as relaxing the symmetry structure of the potential and the assumption of spherical symmetry.

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