The Normalization of Perturbative String Amplitudes: Weyl Covariance and Zeta Function Regularization

Shyamoli Chaudhuri
214 North Allegheny Street
Bellefonte, PA 16823

Abstract

This is a self-contained pedagogical review of Polchinski’s 1986 analysis from first principles of the Polyakov path integral based on Hawking’s zeta function regularization technique for scale-invariant computations in two-dimensional quantum gravity, an approach that can be adapted to any of the perturbative string theories. In particular, we point out the physical significance of preserving both Weyl and global diffeomorphism invariance while taking the low energy field theory limit of scattering amplitudes in an open and closed string theory, giving a brief discussion of some physics applications. We review the path integral computation of the pointlike off-shell closed bosonic string propagator due to Cohen, Moore, Nelson, and Polchinski. The extension of their methodology to the case of the macroscopic loop propagator in an embedding flat spacetime geometry has been given by Chaudhuri, Chen, and Novak. We examine the macroscopic loop amplitude from the perspective of both the target spacetime massive type II supergravity theory, and the boundary state formalism of the worldsheet conformal field theory, clarifying the precise evidence it provides for a Dirichlet (-2)brane, an identification made by Chaudhuri. The appendices contain a comprehensive presentation of the covariant path integral technique for the one loop amplitudes of the supersymmetric, and unoriented, open and closed type I string theory, and in an external two-form background field of generic strength.

1Current Address: 1312 Oak Dr, Blacksburg, VA 24060. Email: shyamolic@yahoo.com
1 Introduction

The bread-and-butter tool of high-energy theorists, whether in particle physics or cosmology, is effective quantum field theory: a spacetime Lagrangian holding at some given mass scale $m_e$ with, in general, non-renormalizable corrections accompanied by inverse powers of $m_s$. Many computations in string theory also begin from the perspective of an effective field theory, namely, a ten-dimensional supergravity-Yang-Mills theory defined at the string scale, $m_s$. An important distinction from generic effective field theories is that a detailed worldsheet prescription exists, at least in principle, for the computation of the nonrenormalizable terms in the string theory spacetime Lagrangian, and that it holds to all orders in $m_s$. Note that, roughly speaking, it is conventional to identify $m_s = \alpha'^{-1/2}$, where $\alpha'$ is the inverse closed string tension, with the ten-dimensional Planck scale, $M_P$, as befitting a theory of quantum gravity. But it should be kept in mind that the effective four-dimensional string scale can be considerably lower in specific vacua of the theory, due to a dependence on either geometric moduli, or on the background fields and fluxes characterizing the spacetime geometry. This feature has become popular in recent phenomenological model building.

In this paper, we wish to explore a key feature that distinguishes perturbative string theory from generic effective field theories. This distinction was already highlighted in Polchinski’s 1986 observation [1] that the vacuum energy in string theory is unambiguously normalized, inclusive of numerical factors, in terms of just two parameters: the string scale, $m_s$, and the dimensionless string coupling, or dilaton vev, $g=e^{\phi_0}$. Since this is quite unlike the expectation in generic effective quantum field theories, the significance of this observation should be clearly appreciated prior to serious investigation of low energy cosmo-particle physics in String/M Theory. We should also note that this result has crucial consequences for the thermodynamics of the canonical ensemble of perturbative strings [1, 38]. The vacuum energy of perturbative strings in a flat spacetime geometry is the most fundamental dimensionful quantity we would think to compute in string theory. But it should be emphasized that even when the vacuum energy receives additional, tree-level corrections from possible Dbranes, or from a background two-form field strength, as in many braneworld models of the type I and type II string theories, it is nonetheless true that such mass contributions are unambiguously compatible in terms of the single mass scale $m_s$, plus a specified number of dimensionless parameters that include the string coupling, or dilaton vev. This second remarkable observation follows from Polchinski’s 1995 worldsheet computation of the Dbrane tension in type I string theory [2]. In the next three sections, and in appendices A thru C, we will provide a pedagogical review of some relevant technical details and background that are helpful in understanding the derivation of these two key results.

In section 5, we review an additional far-reaching insight, first noticed by us during the course of the work reported on in Ref. [36]. Our observation follows by adapting the arguments that led to the worldsheet computation of the Dbrane tension [2] to the opposite limit of the annulus graph, dominated by the lowest-lying open string modes in the worldvolume gauge theory. We will find that the remarkable normalizability property of the open string scattering amplitude survives in the low energy effective field theory limit: it is possible to determine all of the couplings in the worldvolume effective Lagrangian, inclusive of numerical factors, in terms of the single mass scale, $m_s$, plus a finite number of dimensionless scalar field expectation values, or moduli. Our observation exploits an insufficiently exploited property of the type I and type I’ open string theories, first noted by
us in \[36\]. Consider the one-loop scattering amplitudes. In order to extract the contribution from the lowest-lying open string modes dominating the short cylinder limit of some given scattering amplitude, there is no need to set the modulus of the annulus to some ad-hoc value, thereby explicitly breaking reparametrization invariance. Instead, we can first expose the asymptotics of the integrand by Taylor expansion, followed by explicit evaluation of the modular integral. The physical significance of having thus preserved all of the manifest symmetries of the worldsheet formalism, namely, exact Weyl and diffeomorphism invariance, is that we obtain an unambiguously normalized expression for each of the couplings in the worldvolume effective field theory (EFT). Each is derived directly from an unambiguously normalized perturbative string scattering amplitude \[36\]. This observation could have significant consequences for the derivation of perturbative gauge theory results from perturbative string scattering amplitude computations.

Finally, in section 6, and in appendix D, we review the path integral computation of the point-like off-shell closed bosonic string propagator due to Cohen, Moore, Nelson, and Polchinski \[9\]. The extension of their methodology to the case of macroscopic boundary loops in an embedding flat spacetime geometry has been given by Chaudhuri, Chen, and Novak \[35\]. We explain this modification of the familiar one-loop string amplitude from the perspective of the boundary state formalism of the worldsheet conformal field theory, pointing out that it gives concrete worldsheet evidence for a Dirichlet (-2)brane boundary state in string theory. The consequences for the D(-2)brane spectrum of String/M theory will be explored elsewhere. We should note that in the case of more complicated loop geometries, including the interesting possibility of corners \[35\], there exist a number of potential physics applications for macroscopic loop amplitudes both in cosmology, and in condensed matter physics. These relate to cosmic string production from the vacuum, cosmic string scattering, as well as the study of radiation from cusps and/or corners on a cosmic string. It would be remarkable to derive such phenomena directly from a fundamental, and fully renormalizable, quantum theory.

This paper contains a topical review of pedagogical material which we feel is, regretfully, unavailable in any of the standard string theory textbooks and reviews. It should be useful to both students, and to practitioners of effective field theory, who wish to better understand the full import of the worldsheet approach to string theory. It should also be helpful to string theorists unfamiliar with the strengths of the covariant path integral approach. We have tried to provide an equivalent discussion in the operator formalism whenever this is absent in the current literature, as in our discussion in section 6 of the D(-2)brane boundary state. More generally, a description of the operator formulation for the type IB and type I' string theories, and of boundary conformal field theory techniques, can be easily found in the standard sources in the literature.

2 The Normalization of String Amplitudes

Let us begin by reviewing the computation of the vacuum functional in quantum field theory. In order to enable a clearer comparison, we will work in the Feynman path integral formulation for a perturbative quantum field theory. The vacuum functional of quantum field theory is given by the sum over classical field configurations, with the natural requirement that the measure of the path integral preserve the continuous symmetries of the classical action. In the case of a non-abelian gauge theory coupled to a massless scalar field, for example, we define a gauge invariant measure,
factor out the volume of the gauge group by choosing a slice in field space that intersects each gauge orbit exactly once, and eliminate the overall infinity introduced by the choice of a redundant gauge-invariant measure by dividing by the volume of the gauge group:

\[ Z[A, \Phi] = \frac{1}{\text{Vol}[\text{gauge}]} \int [dA][d\Phi] e^{-S_{YM}[A]-S[\Phi,A]} = \int_{\text{orbit}} [d\hat{A}][d\Phi] e^{-S[\hat{A}]+S_{FP}-S[\Phi,\hat{A}]} \, , \tag{1} \]

Although this gives an elegant starting point for calculations in perturbative non-abelian gauge theory, the vacuum functional of the gauge theory is not normalizable because of an overall infinity introduced by the ultraviolet divergent vacuum energy density. Despite having removed the redundancy due to gauge invariance by implementing the Faddeev-Popov procedure, the right-hand-side of Eq. (1) only becomes well-defined when we introduce an appropriate ultraviolet regulator for the gauge theory. This property is exactly the same as in any other quantum field theory, whether renormalizable, or not. Thus, in order to obtain physically meaningful results which hold independent of the choice of ultraviolet regulator, we restrict ourselves in quantum field theory to computing normalized correlation functions defined as follows:

\[ \langle O_1[A, \Phi] \cdots O_n[A, \Phi] \rangle = \frac{1}{Z[A, \Phi]} \frac{1}{\text{Vol}[\text{gauge}]} \int [dA][d\Phi] O_1[A, \Phi] \cdots O_n[A, \Phi] e^{-S_{YM}[A]-S[\Phi,A]} \, . \tag{2} \]

The ambiguity introduced by the introduction of an ultraviolet regulator in both the numerator and denominator has been cancelled by taking the ratio. A suitable choice of regulator in this example would be dimensional regularization, which preserves the non-abelian gauge invariance. We assume the reader is well-acquainted with the remarkably successful computational scheme for perturbative Yang-Mills theory that follows from this prescription. However, as is well-known, the cosmological constant problem has not been addressed in this field theoretic analysis: the vacuum energy density is regulator-dependent, formally infinite, or at least of the same order as the overall mass scale of the quantum field theory.

The path integral for two-dimensional quantum gravity coupled to \( d \) massless scalar fields, namely, the worldsheet action for \( d \)-dimensional bosonic string theory, can be analysed exactly along these same lines [5, 1]. Two-dimensional quantum gravity is almost pure gauge, as one might expect from the fact that the Einstein action in two-dimensions is a topological invariant equal to the Euler number of the two-dimensional manifold:

\[ S[g] = \int_M d^2\xi \sqrt{-g} R + 2 \int_{\partial M} dsk = 4\pi \chi_M = 4\pi(2-2h-b-c) \, , \tag{3} \]

where \( h, b, \) and \( c, \) are, respectively, the number of handles, boundaries, and cross-caps on the two-dimensional Riemannian manifold. \( R \) and \( k \) are, respectively, the Ricci scalar curvature, and the geodesic curvature on the boundary. However, as was shown by Polyakov in 1983 [5] by a careful analysis of the Faddeev-Popov functional determinant, in any sub-critical spacetime dimension, \( d < 26, \) the Weyl mode, \( \phi, \) in the two-dimensional metric, \( g_{ab} = e^{-\phi} \hat{g}_{ab}, \) acquires non-trivial dynamics given by the Liouville field theory:

\[ Z[g, X] = \frac{1}{\text{Vol}[\text{Diff}_0]} \int [dg][dX] e^{-\lambda S[g]-S[X,g]} = \int_{\text{orbit}} [d\phi][dX] e^{-\lambda S[\phi]+(d-26)S_L[\phi]-S[X,\phi]} \, , \tag{4} \]

where \( \lambda \) is an arbitrary constant that fixes the loop expansion parameter for two-dimensional quantum gravity: \( e^{-\lambda H_{\mu\nu}}. \) In string theory, \( \lambda \) will be determined by the dimensionless closed string
coupling, $\lambda = e^{-\Phi_0}$. The notation $\text{Diff}_0$ denotes the group of connected diffeomorphisms of the world-sheet metric. Notice that the coefficient of the Liouville action vanishes in the critical spacetime dimension, $d=26$. Thus, in critical string theory, $\phi$ drops out of the classical action, and we must divide by the volume of the Weyl group in order to eliminate the redundancy in the measure due to Weyl invariance:

$$W_{\text{string}}[g, X] = \frac{1}{\text{Vol}[\text{Diff}_0 \times \text{Weyl}]} \int [dg][dX] e^{-\lambda S[g] - S[X, g]} = \int [d\tau]_{\chi M} e^{-\lambda S[g]} \int [dX] e^{-S[X, \hat{g}]} \equiv \int [d\tau]_{\chi M} e^{-S[g]} (\det \Delta)^{-(d-2)/2} .$$

(5)

The path integral simplifies to an ordinary finite-dimensional integral over the moduli of the world-sheet metric, parameters that describe the shape and topology of the Riemannian manifold. The first-principles derivation of this result is reviewed in the appendix following [1, 10, 34]. $\Delta$ denotes the Laplacian acting on two-dimensional scalars on a Riemann surface with Euler number $\chi_M$ and moduli, $\tau$. The measure for moduli, $[d\tau]$, can be unambiguously determined by the requirement of gauge invariance, as was shown in [1, 10, 34]. What remains is to obtain an expression for the functional determinant on the right-hand-side of the equation, in as explicit a form as is feasible. This requires specification of an ultraviolet regulator for the two-dimensional quantum field theory.

The beauty of Weyl-invariant two-dimensional quantum gravity is that there is only one choice of ultraviolet regulator that preserves all of the gauge invariances of the theory, namely, both connected diffeomorphisms and Weyl transformations of the metric. That regularization scheme is zeta-function regularization [4], as was pointed out by Polchinski in [1], and it gives, therefore, an unambiguously normalized expression for both the vacuum functional of string theory, as well as, in principle, all of the loop correlation functions at arbitrary order in the perturbation expansion [36]. In practice, other than on worldsurfaces of vanishing Euler number which contribute to the one-loop amplitudes of string theory, the eigenspectrum of the scalar Laplacian is insufficiently well-known to enable explicit calculation of the functional determinant [10, 34]. But the fact remains that, unlike what happens in quantum field theory, there is no ambiguity in the normalization of generic loop correlation functions introduced by an arbitrariness in the choice of regulator.

Let us recall the basic idea underlying the zeta function regularization of the functional determinant of a differential operator in a Euclidean quantum field theory [4]. Hawking begins by noting that the functional determinant of a second-order differential operator $\Delta$ on a generic $k$-dimensional manifold $M$ with a discrete eigenvalue spectrum, $\{\lambda_n\}$, and normalized eigenfunctions, $\{\Psi_n\}$, can be interpreted as the generalization of an ordinary Riemann zeta function. This generalized zeta function is, formally, given by the sum over the eigenvalues of $\Delta$:

$$\Delta \Psi_n = \lambda_n \Psi_n, \quad \int \Psi_m \Psi_n \sqrt{-g} \ d^k \xi = C_n \delta_{mn}, \quad \Phi = \sum_n a_n \Psi_n, \quad \int [d\Phi] = \prod_n \mu_n(C_n) \int da_n ,$$

(6)

where $\mu$ is a normalization constant. In the generic case of higher-dimensional quantum gravity there is little need to belabour the issue of what determines the renormalization of $\mu$, since summing over the eigenvalue spectrum will give a divergent result. This divergence needs to be regularized

---

2It should be emphasized that the path integral for two-dimensional Weyl-invariant quantum gravity computes the sum over connected vacuum graphs in string theory. We denote this quantity by the usual symbol $W$ [1].
[4]. Any such regularization will introduce a scheme-dependent ambiguity in the normalization. However, as we have emphasized above, this is not true in two-dimensional Weyl invariant quantum gravity.

Following [4], we can write the functional determinant in Eq. (5) in terms of a generalized zeta function as follows. We begin with:

\[
\int \prod_n \mu_n da_n e^{-\frac{1}{2} \lambda_n a_n^2} = \prod_n (2\pi)^{-1/2} \mu_n \lambda_n^{-1/2} = \left[ \det\left( \frac{1}{2} \mu - \pi^{-1} \Delta \right) \right]^{-1/2}.
\]

The infinite product can be rewritten as an infinite sum by taking the logarithm, giving a formal expression on the right-hand-side that takes the form of a generalized zeta function:

\[
\ln[\det(\frac{1}{2} \mu - \frac{1}{2} \pi^{-1} \Delta)]^{-1/2} = -\frac{1}{2} (2\pi)^{-1/2} \sum_n \mu_n \log \lambda_n = \lim_{s \to 0} \frac{d}{ds} \left[ (\frac{1}{2} (2\pi)^{-1/2})^{-s} \sum_n \mu_n \lambda_n^{-s} \right].
\]

The infinite product can be rewritten as an infinite sum by taking the logarithm, giving a formal expression on the right-hand-side that takes the form of a generalized zeta function:

\[
\ln[\det(\frac{1}{2} \mu - \frac{1}{2} \pi^{-1} \Delta)]^{-1/2} = -\frac{1}{2} (2\pi)^{-1/2} \sum_n \mu_n \log \lambda_n = \lim_{s \to 0} \frac{d}{ds} \left[ (\frac{1}{2} (2\pi)^{-1/2})^{-s} \sum_n \mu_n \lambda_n^{-s} \right].
\]

In the case of free embedding scalars, \(\mu_n(C_n)\) is independent of \(n\), as was shown in [1]. Thus, the normalization of the path integral for Weyl-invariant two-dimensional quantum gravity is uniquely determined by the form of the action, and by the gauge invariant measure for moduli. This was shown clearly in Polchinski's 1986 derivation of the measure for moduli in the one-loop bosonic string path integral [1], and in subsequent work on higher genus Riemann surfaces including those with boundaries and crosscaps in [10, 33, 34, 36]. The key point that remains is explicit evaluation of the formally divergent right-hand-side of this equation. Since the choice of worldsheet ultraviolet regulator is unique, \(\mu\) is unambiguously renormalized, and the renormalization is, therefore, scheme-independent. The details of such computations are reviewed in Appendix C using the contour integral prescription given in [4, 1, 36].

Before leaving this general discussion, we should emphasize that, although the quickest route to computing string correlation functions employs conformal field theory techniques in the operator formalism and, quicker still, operator product expansions, these results are not unambiguously normalized. What is unambiguous in operator formalism computations is the ratio of two different correlation functions, and often, that information suffices to describe all of the interesting physics. This is precisely as in a generic quantum field theory: we have not exploited the full power of the worldsheet formalism. However, when we are interested in the numerical value of the vacuum energy density and the cosmological constant per se, the physics is in the string vacuum functional itself. We have no alternative but to compute it from first principles, if possible, using the path integral formalism. That such a calculation is viable in a full-fledged ten-dimensional string theory as a consequence of its relationship to two-dimensional quantum gravity, is nothing short of a miracle. This is the significance of Polchinski's first principles analysis of the Weyl-invariant Polyakov path integral for critical string theory. As mentioned in the Introduction, the result for the vacuum functional derived in [1] is only the first step in deriving a number of remarkable properties of the

---

3 More generally, the orthonormality constants, \(C_n\), can depend on the background fields of string theory. The normalization, \(\mu_n(C_n)\), turns out to be independent of \(n\), as shown in [36], where the precise form of the background field dependence in an external two-form field has been derived.

4 Many authors, including myself [34], are guilty of performing the apparent sacrilege of introducing a non-Weyl invariant worldsheet regulator into an analysis of the string theory path integral. In doing so, the gauge invariance of the two-dimensional quantum gravity has been explicitly broken. It should be emphasized, however, that for the purposes of many calculations, such as those of asymptotic bounds on the high energy string mass spectrum
canonical ensemble of perturbative strings, explored further in [38]. We should emphasize that this 
approach can be adapted to any of the supersymmetric string theories, as illustrated in detail for 
the interesting type I and type I′ unoriented, open and closed string theories in appendices A thru 
C.

A final comment on string theory correlation functions. In quantum field theory we calculate 
what are called normalized correlation functions by taking the ratio of an $N$-point function divided 
by the vacuum functional, thus eliminating the ambiguity introduced by a choice of ultraviolet 
regulator as explained above. It should be emphasized that, besides the vacuum amplitude at 
arbitrary loop order, the generic loop correlation function in string theory is also unambiguously 
normalized without the need to take a ratio [36]. In section 4, we will see an indication of this in 
the worldsheet computation of the quantum of Dpbrane charge [2]. Since the quantum of Dpbrane 
charge ia obtained by simply taking the zero slope, or massless field theory, limit of the factorized 
one-loop graph of the type I′ string theory in the background of a pair of Dpbranes, it is required to be 
unambiguously normalized [2]. Thus, Dbrane charge quantization simply follows as a consequence 
of the normalizability of the vacuum amplitude in string theory [1].

3 A Little Supergravity Background: Dpbrane Solitons

It is helpful to begin our review of Polchinski’s 1995 result for the quantum of Dpbrane charge 
[1, 18, 2] by sketching the relevant insights from both the worldsheet and the low-energy effective 
field-theory pictures that motivated this calculation. The relationship of the worldsheet computation 
of the normalized one-loop vacuum amplitude of type I string theory to the quantum of Dpbrane 
charge will be described in the next section. The low-energy effective field theory limit of the type 
II closed string theories is a ten-dimensional $N=2$ supergravity theory without Yang-Mills gauge 
fields. The massless supergravity Lagrangian can be extended by the inclusion of kinetic and Chern-
Simons terms for antisymmetric tensor fields, $F_{p+2}$, corresponding to gauge potentials, $C_{p+1}$ [16, 33], 
where $p$ lies in the range, $-2 \leq p \leq 8$. This covers the full range of field strength tensors in ten 
spacetime dimensions, namely, scalar to ten-form. Such a gauge potential can couple to an extended 
object with a $p+1$ dimensional worldvolume, or $p$-brane, and it is natural to ask whether the type 
II supergravities have classical solutions describing dilaton-gravitational-antisymmetric-tensor field 
configurations that have the geometry of a $p$-brane? The answer is yes, and a large variety of 
such gravitational solitons have been discovered over the years. The $p$-brane solitons of the type 
II theories can be distinguished by whether they carry charge for a Ramond-Ramond sector, or 
Neveu-Schwarz-Neveu-Schwarz sector, antisymmetric tensor field strength [19]: the kinetic term in 
the Lagrangian for the corresponding field strength differs in the powers of $e^{-\phi}$ appearing in the 
pre-factor [21, 33].

The next key point to note is that one of the ten-dimensional supersymmetries is spontaneously 
broken by such a choice of vacuum in either the type IIA or type IIB theory. We now know that 
[10, 34], the results are independent of the choice of worldsheet regulator. This is not true of the computation 
of the normalization of the vacuum energy where the use of a gauge invariant regulator is crucial. Zeta-function 
regularization [4] provides the only correct answer. Note that in analyses where the worldsheet gauge invariance has 
been explicitly broken, the string theory path integral is primarily being invoked for its intuitive value rather than 
as a high-precision computational tool.
the corresponding ten-dimensional $N=1$ supergravities are the low-energy limits of the type I' and type IB string theories [33], an identification originally made by Witten [21] by direct comparison of the ten-dimensional effective Lagrangians and spectrum of low-lying masses. This also identified such vacua as BPS states of the type II string theory. A further step was the recognition that the p-brane solitons of the type II string theories could be characterized by how their tension scales with the string coupling [21]: $1/g$ for R-R sector solitons vs $1/g^2$ for NS-NS sector solitons. Shenker [28] made the important observation that R-R sector solitons would be responsible for $e^{-1/g}$ corrections to the standard closed string perturbation expansion in powers of $1/g^2$. This is unlike the NS-NS solitons which give nonperturbative corrections of the form $e^{-1/g^2}$, indistinguishable from those of an ordinary Yang-Mills gauge theory soliton.

All of these observations about the low-energy field theory limit of string theory fall in place with Polchinski's insight that Dirichlet-branes [18], suitably supersymmetrized, are the carriers of R-R charge in the type II string theories: in a vacuum that breaks half of the supersymmetries and carries non-trivial R-R charge, the spectrum of a type IIA or IIB closed string theory is extended by an open string sector with Dirichlet boundary conditions imposed on the worldsheet fields [2]. The key point is that the closed string coupling scales as the square of the open string coupling, and while the perturbation expansion of a pure closed string theory closes on itself, it might allow extension by an open string sector. Exceptions are the closed heterotic string theories where the chiral worldsheet current algebra prevents such an extension. However, this obstruction did not exist for the type II string theories.

The Dbrane vacua of the type II string theories could thus be said to represent the more generic class of type II vacua, having both an open and a closed string sector [30]. However, since only half of the supersymmetries of the type IIA or type IIB theories are preserved in such a vacuum, the Dbrane solutions could equivalently be viewed as the classical vacua of an $N=1$ ten-dimensional open and closed string theory. This string theory is, respectively, either the type I' or type IB string theory. As is well-known, type IIA and type IIB were related by a T-duality transformation. This is also true for type I' and type IB. The advantage of the latter viewpoint is that questions which appeared obscure in the Ramond-Neveu-Schwarz (RNS) worldsheet formalism of the type II closed string theory, such as the prescription for Ramond-Ramond vertex operators or the computation of the Ramond-Ramond sector partition function, can now be straightforwardly answered in the RNS formalism of the corresponding open and closed string theories [33].

The spacetime geometry of the type II vacuum carrying pbrane charge is that of a $p+1$-dimensional hypersurface embedded in ten-dimensional spacetime [2]. It is well-known that a Hodge-star duality links a $p$-brane with a $(d-4-p)$-brane in $d$ dimensions, and that the corresponding charges must satisfy a Dirac quantization condition [16]. This is a simple consequence of applying quantum mechanics to extended objects. Such objects were independently discovered as the natural extension of the pointlike magnetic monopoles of 4d gauge theory, by Savit, Orland, and Nepomechie [16]. This early work on lattice field theories exploits the well-known correspondence between the phase structure of two-dimensional nonlinear sigma models and four-dimensional gauge theories, and was followed by a more classical presentation of higher pforn gauge theories due to Teitelboim [16]. p-form generalized electric-magnetic duality generalizes the electric-magnetic duality of the abelian gauge field discovered by Dirac. We can think of the magnetic monopole as a nonperturbative configuration of the vector potential which couples to the electron, the fundamen-
tal charge carrier of the electromagnetic gauge field. Of course, a more careful analysis only finds monopoles as stable configurations in field theories of scalars coupled to Yang-Mills fields. It is the same for any pair of pbranes that satisfy a Poincare duality relation [33]:

\[ \nu_p \nu_{d-4-p} = 2\pi n, \quad n \in \mathbb{Z} \quad , \]

where \( \nu_{d-4-p} \) is the quantized flux of the \((p+2)\)-form field strength. Such pbrane solitons are only found in \(d\)-dimensional field theories with both scalars and antisymmetric tensor fields of rank \(p+2\), with \(d \geq p+2\). In the case of the Dirichlet p-brane solitons of the type I-I’ string, where we have a clear understanding of the low-energy effective field theory limit of an open and closed string theory, we can show that both gravity and Yang-Mills gauge fields exist in the worldvolume of the Dpbrane. This is because the worldvolume represents the hypersurface in ten-dimensional spacetime where open string end-points are permitted to lie, in addition to the closed strings which, of course, lie in all ten embedding spacetime dimensions. Thus, the scalar fields of the ten-dimensional field theory split into worldvolume scalars and bulk scalars, some of the latter representing the fluctuations of the Dpbrane in the ten-dimensional embedding spacetime. Thus, in the Dpbrane vacuum, the ten-dimensional Lagrangian obtained in the low-energy limit of the relevant type II string theory acquires a new term proportional to the Dpbrane worldvolume action. In the physically relevant Einstein frame metric, related to the string frame metric by a spacetime Weyl transformation:

\[ G_{\mu\nu} = e^{-4\Phi/(d-2)} G_{\text{string}}^{\mu\nu} \]

the worldvolume action takes the simple form [18, 2, 33]:

\[ S_p = \tau_p \int d^{p+1} X e^{(p-3)\Phi/4} \sqrt{-\det (G_{\mu\nu} + B_{\mu\nu} + 2\pi \alpha' F_{\mu\nu})} + \mu_p \int C_{p+1} \quad , \]

where \( \tau_p \) is the physical value of the Dpbrane tension. \( \mu_{6-p} \) is the quantum of magnetic D(6-p)brane charge, related to the quantized flux of the field strength, \( F_{p+2} \), obtained by integrating it over a sphere in \((p+2)\) dimensions:

\[ \int_{S_{p+2}} F_{p+2} = 2\kappa^2_{10} \mu_{6-p} \quad . \]

Thus, the flux quantum satisfying a Hodge duality relation of the form given in Eq. (9) is identified as, \( \nu_p \equiv 2\kappa^2_{10} \mu_p \). The flux quantum, \( \nu_p \), and the quantum of Dpbrane charge, \( \mu_p \), differ from the Dpbrane-tension, \( \tau_p \), in their lack of dependence on the closed string coupling. As a consequence, they can be calculated unambiguously in weakly-coupled perturbative string theory. Namely, we have the relations:

\[ \kappa^2 \tau^2_p \equiv g^2 \kappa^2_{10} \tau^2_p = \kappa^2_{10} \mu_p^2, \quad \kappa^2_{10} \equiv \frac{1}{2}(2\pi)^7 \alpha'^4 \quad . \]

Here, \( \kappa \) is the physical value of the ten-dimensional gravitational coupling, and the dimensionless closed string coupling, \( g = e^{\Phi_0} \), where \( \Phi_0 \) is the vacuum expectation value of the dilaton field.

4 Worldsheet Computation of Dpbrane Charge

Given the remarkably simple worldsheet ansatz for what are now called Dpbrane solitons of the type II string theory—pbrane solitons that carry charge under the \(p\)-form gauge potentials of the R-R sector, and whose tension, \( \tau_p \), scales as \(1/g\), it is natural to ask whether the quantization of Dpbrane charge can be inferred directly from a worldsheet calculation? In fact, the result turns out to be stronger than what one might have expected. Not only can Dpbrane charge,
\( \mu_p \), be determined unambiguously in terms of the fundamental string mass scale, \( m_s \), but, upon substituting Polchinski’s result for \( \nu_p \) in the left hand side of Eq. (9), we find that the integer, \( n \), on the right hand side equals unity when \( d=10 \):

\[
\nu_p^2 = 2\pi (4\pi^2 \alpha')^{3-p}, \quad \nu_p \nu_{d-p} = 2\pi \quad \text{(String Theory)} \quad .
\]

We emphasize that this string miracle has its origin in the beautiful property of the one-loop vacuum amplitude of string theory described earlier [1]: unlike the vacuum energy density of a quantum field theory, the vacuum energy density of an infra-red finite string theory is both ultraviolet finite, and normalizable. Let us see how this works in detail.

The tree-level Dpbrane tension is measured by the tree-level coupling of the massless dilaton-graviton field to the Dpbrane: in worldsheet language, this is the one-point function of the massless dilaton-graviton closed string vertex operator on the disk. Extracting the factor of \( q \) by using the relation \( \mu_p = g\tau_p \) gives the value of the quantum of Dpbrane charge, \( \mu_p \). It would be nice to determine this one-point function directly from a first-principles path integral computation. However, the normalization is tricky because of the dependence on the volume of the conformal Killing group of the disk [12, 33]. In addition, one requires the result in the supersymmetric type I string, and that computation has never been done in the path integral formalism. In practice, it is much easier to invoke the factorization of the annulus amplitude in type I string theory to infer the normalization of the massless closed string one-point function on the disk indirectly. This was the method used to compute the tree-level Dpbrane tension in [2].

Referring to Eq. (89) in appendix B, we extract the contribution from the annulus to the sum over connected worldsurfaces of vanishing Euler character, \( W_{\text{ann}} \), in the background of a pair of parallel and static Dpbranes in the type I string theory:

\[
W_{\text{ann}} = \prod_{\mu=0}^{p} L^\mu \int_{0}^{\infty} \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\left(p+1\right)/2} e^{-R^2 t/2\pi \alpha'}
\times \frac{1}{\eta(it)^8} \left[ \frac{(\Theta_{00}(0, it))^4}{\eta(it)} - \frac{(\Theta_{01}(0, it))^4}{\eta(it)} - \frac{(\Theta_{10}(0, it))^4}{\eta(it)} \right].
\]

The amplitude vanishes as a consequence of spacetime supersymmetry, as can be seen by application of the abstruse identity for the Jacobi theta functions. We will focus, therefore, on the contribution from massless spacetime bosons alone, namely, the leading contributions from the (00) and (01) sectors. The factorization limit corresponds to the long cylinder, \( t \rightarrow 0 \). We can expose the correct asymptotic behavior of the theta functions by expressing them in terms of the theta functions with modular transformed argument, \( t \rightarrow 1/t \):

\[
W_{\text{ann}} = \prod_{\mu=0}^{p} L^\mu \int_{0}^{\infty} \frac{dt}{2t} (8\pi^2 \alpha' t)^{-\left(p+1\right)/2} e^{-R^2 t/2\pi \alpha'}
\times \frac{1}{t^4 \eta(-1/it)^8} \left[ \frac{(\Theta_{00}(0, -1/it))^4}{\eta(-1/it)} - \frac{(\Theta_{01}(0, -1/it))^4}{\eta(-1/it)} - \frac{(\Theta_{10}(0, -1/it))^4}{\eta(-1/it)} \right].
\]

\(^5\)Polchinski’s 1995 paper uses the operator formalism, motivating the normalization of the amplitude rather than deriving it. Although pedagogical, this presentation leaves the origin of the lack of ambiguity in the quantum of Dpbrane charge obscure. We will instead use the result of a path integral derivation of the annulus amplitude following his 1986 paper [1, 34], reviewed in Appendix B.
Expanding the theta functions in powers of $e^{-2\pi/t}$, and keeping only the leading term in the expansion of the (00) and (01) sectors gives:

$$W_{1\text{-ann}} = \prod_{\mu=0}^{p} L^\mu \int_{0}^{\infty} \frac{dt}{2t} (8\pi^2 \alpha')^{-(p+1)/2} e^{-R^2 t/2 \pi \alpha'} 2^4 t^{(7-p)/2}$$

$$= - \prod_{\mu=0}^{p} L^\mu (8\pi^2 \alpha')^{-(p+1)/2} 2^4 (2\pi \alpha')^{(7-p)/2} \Gamma \left( \frac{7-p}{2} \right) |R|^{p-7} .$$

In the right-hand-side of this equality we can recognize the propagator for a massless scalar field in 9–p dimensions, namely, in the bulk spacetime:

$$V_{p+1}(8\pi^2 \alpha')^{-(p+1)/2} 2^4 (2\pi \alpha')^{(7-p)/2} \Gamma \left( \frac{7-p}{2} \right) |R|^{p-7} = 4\pi (4\pi^2 \alpha')^{3-p} V_{p+1} G_{9-p}(|R|) ,$$

Inverting the position space Green’s function will enable us to make contact with the field theory expression for the tree-level exchange of a graviton-dilaton between two Dpbranes, the result for which depends explicitly on the Dpbrane tension.

The tree-level field theory calculation proceeds as follows. We begin with the spacetime action in the Einstein frame metric given in Eqs. (36), switching off both the ten-form, and the Yang-Mills, backgrounds. The worldvolume action for the Dpbrane background takes the form given in Eq. (10). Let us expand about the flat space background metric in perturbation theory, $h_{\mu\nu} = \gamma_{\mu\nu} - \eta_{\mu\nu}$, keeping terms up to quadratic order in the field variations. We will perform this calculation in the covariant Lorentz gauge:

$$F_\nu = \partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_{\mu} h^\lambda_{\lambda} = 0 .$$

Thus, the gauge-fixed spacetime action takes the form:

$$S'[h, \Phi] = \frac{1}{2\kappa^2} \int d^{10} X \sqrt{-G} \left( R_G - \frac{4}{d-2} (\partial \Phi)^2 \right) - \frac{1}{4\kappa^2} \int d^{10} X \sqrt{-G} F_{\mu\nu} F^{\mu\nu} + \tau_p \int d^{p+1} X e^{-(p-3)\Phi/4} \sqrt{-\text{det} G_{\mu\nu}}$$

$$= -\frac{1}{8\kappa^2} \int d^{10} X \left( \partial_{\mu} h_{\nu\lambda} \partial^\mu h^{\nu\lambda} - \frac{1}{2} \partial_{\mu} h_{\nu\lambda} \partial^\mu h^{\nu\lambda} + \frac{16}{d-2} \partial_{\mu} \Phi \partial^\mu \Phi \right)$$

$$- \tau_p \int d^{p+1} X \left( \frac{p-3}{4} \Phi - \frac{1}{2} h_{\lambda}^{\lambda} \right) .$$

The Feynman graph of interest to us in the diagrammatic expansion of this field theory is the amplitude for the tree-level exchange of the massless graviton-dilaton multiplet between a pair of Dpbranes in $d=10$ spacetime dimensions. Notice that both background, and propagators, for the massless dilaton and graviton fields are decoupled when the action is written in the Einstein frame metric. Thus, we need only sum the corresponding tree-level Feynman graphs for each. We will only need the trace-dependent piece of the graviton propagator which mixes with the massless dilaton scalar exchange. Given the action, Eq. (19), we can write down the form of the free field propagators in the momentum space representation:

$$< \Phi \Phi > = - \frac{(d-2)\kappa^2}{4\kappa^2}$$

$$< h_{\mu\nu} h_{\sigma\rho} > = - \frac{2\kappa^2}{k^2} \left( \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\rho} \eta_{\nu\sigma} - \frac{2}{d-2} \eta_{\mu\rho} \eta_{\sigma\nu} \right) .$$

(20)
Thus, for massless dilaton-graviton exchange in ten spacetime dimensions between two Dpbranes, we have the tree-level Feynman amplitude:

$$A(k) = \frac{(p-3)}{4}\tau_p \cdot \frac{2\kappa^2}{k^2} \cdot \frac{(p-3)}{4}\tau_p + \frac{1}{2}\tau_p \cdot \frac{2\kappa^2}{k^2} \left(2(p+1) - \frac{1}{4}(p+1)^2\right) \cdot \frac{1}{2}\tau_p$$

(21)

Recall that the momentum space representation of the Green’s function, $1/k^2$, is the Fourier inverse transform of the following position space Green’s function:

$$G_d = 2^{-2}\pi^{-d/2}\Gamma(\frac{1}{2}d - 1)R^{2-d},$$

a relation that holds in any dimension $d$. Project $k$ on to a $(9-p)$-dimensional momentum vector orthogonal to the worldvolume of the Dpbranes, and take the Fourier transform of the result in Eq. (21) in $9-p$ space dimensions. Comparison with Eq. (16) gives the following result for $\tau_p$:

$$\nu_p^2 = 2\tau_p^2\kappa^2 = 2\pi(4\pi\alpha')^{3-p}.$$  (22)

Notice that the dependence on the dimensionless string coupling, $g$, is eliminated in the product of physical parameters, $\tau_p\kappa$, giving results for the flux quantum, $\nu_p$, and the quantum of Dpbrane charge, $\mu_p$, that are valid even outside of the realm of perturbation theory. This is as it should be, not surprisingly, since $\nu_p$ is a spacetime topological invariant which should always be computable, in principle, from spacetime anomaly inflow arguments [27]. Such computations, carried out explicitly for the top-form field strength in [27], have given independent confirmation of the correctness of Polchinski’s worldsheet interpretation of the carriers of R-R charge [2].

## 5 Normalization of Worldvolume EFT Couplings

Consider a background of the type I or type I’ string theory with given set of Dbranes. The spacetime geometry and background fields and fluxes on the worldvolume of the Dbranes, or intersections thereof, constitute a braneworld, with associated worldvolume effective field theory Lagrangian. Let us explain why it is possible to determine all of the couplings in such a worldvolume Lagrangian, inclusive of numerical factors, in terms of the single mass scale, $m_s$, plus a finite number of dimensionless scalar field expectation values, or moduli. Our observation exploits a property of the annulus graph of the open and closed string theory first noted by us in [36]: in order to extract the contribution from the lowest-lying open string modes dominating the short cylinder limit of a given scattering amplitude, there is no need to set the modulus of the annulus to some ad-hoc value, thereby explicitly breaking reparametrization invariance. Instead, we will first expose the asymptotics of the integrand by Taylor expansion, followed by an explicit evaluation of the modular integral. The physical significance of having thereby preserved all of the manifest symmetries of the worldsheet formalism, namely, exact Weyl and diffeomorphism invariance, is that we obtain an unambiguously normalized expression for each of the couplings in the worldvolume effective field theory.

We begin with some general statements on loop correlation functions in string theory. The full beauty of perturbative string theory becomes transparent upon detailed examination of the loop expansion. The world-sheet representation of string loop amplitudes implies the existence of a single graph at each order in loop perturbation theory. In precise analogy with the vacuum functional,
the scattering amplitudes of string theory can be equivalently interpreted as quantum correlators in a two-dimensional “gauge” theory, where the gauge symmetry in question is two-dimensional diffeomorphism and Weyl invariance [5]. The S-matrix describing the scattering of asymptotic string states is obtained by invoking two-dimensional conformal invariance to represent asymptotic on-shell states as operator insertions that are local in the two-dimensional sense, i.e., localized on the worldsheet. Off-shell string states are macroscopic boundaries of the worldsheet—either macroscopic closed loops or macroscopic line segments, localized instead in the embedding spacetime [9, 20, 35]. This correspondence between worldsheet and spacetime pictures has remarkable consequences. Consider a gauge invariant path integral expression for the generic Greens function at arbitrary order in the string loop expansion. Upon gauge fixing to conformal gauge, the path integral over metrics is restricted to the fiducial representative from each conformally inequivalent class of metrics. This is an ordinary integral over the finite number of moduli parameters of Riemann surfaces of fixed topology. Remarkably, in any critical string theory, both the measure of the path integral, the functional determinants, and vertex operator insertions, can be unambiguously computed as functions of the moduli at least in principle, while preserving the full Diffeomorphism × Weyl gauge invariance. The result is an unambiguously normalized and ultraviolet finite expression for both the Greens function and generic N-point interactions, with a well-defined, zero string tension, field theory limit.

The resulting expressions for the field theory Greens functions are free of ultraviolet regulator ambiguity. More importantly, in an infrared finite string theory, they are also free of ambiguity in the choice of renormalization scheme [1, 36]. Both properties are a consequence of having maintained manifest two dimensional general coordinate invariance in computing the full string theory Greens function prior to taking the field theory limit which we define as follows: we factorize on massless mode exchange in either open or closed string sectors, projecting also onto the massless on- or off-shell modes in any external states, and integrating out the worldsheet modulus dependence of the resulting expression. The result is a coupling in the field theory in which we can smoothly take the zero string tension limit. Thus, the expression for any renormalized string theory N-point Greens function including, in particular, the corresponding field theory limit, is independent of dependence on the string tension, \( m_s \sim \alpha'^{-1/2} \), which plays the role of an ultraviolet cutoff in spacetime.

As an illustration of our comments, let us quote two results from the work in Ref. [36], done in collaboration with Novak. This describes a first-principles covariant path integral derivation of the N-point one-loop scattering amplitudes of open and closed bosonic string theory in an external two-form background field of generic strength. This is a straightforward adaptation of the derivation of the N-point scattering amplitude of the closed bosonic string given in [1]. The main new subtleties in the zeta function regularization technique resolved in [36] have to do with the extension to both planar and nonplanar amplitudes, and as a consequence of the external field dependence in the expressions. The results provide confirmation of the basic insights into renormalization given by the path integral approach [5, 1], in the framework of an open and closed string theory with nontrivial backgrounds. Evaluating the Greens function for two open string tachyon vertex operators on the same boundary of the annulus in an external twoform background of generic strength [36]:

\[
G'_{ij}(\sigma, \tau) = -\frac{2\alpha'}{1 + B^2} \ln \left| \Theta_1 \left( \frac{i\sigma + i\frac{B}{2}}, \frac{i\tau + i\frac{B}{2}}{2} \right) \right|^2 + \frac{2\pi\alpha' t}{1 + B^2} \left[ (\sigma_{ij})^2 - \frac{2B\sigma_{ij}}{t} + \frac{2B^2}{3t^2} \right] + \frac{2\pi\alpha' B}{1 + B^2} |\sigma_{ij}| \tag{23}
\]

Note that the short distance divergence on the worldsheet can be extracted by subtractive renormal-
ization while preserving two-dimensional reparametrization invariance: we express the subtraction in terms of the geodesic distance between the two sources on the worldsheet [1, 10, 36]. The short distance limit gives:

\[
G'(\sigma_i, \sigma_i) = -\frac{2\alpha'}{1 + B^2} \ln |t\sigma_{ij}|^2 - \frac{2\alpha'}{1 + B^2} \ln |\eta(it)|^2 + \frac{4\pi\alpha'B^2}{3t(1 + B^2)}, \quad \sigma_i^2 = 0, 1.
\]

and we can define the renormalized Greens function:

\[
\lim_{\sigma_i \to \sigma_j} G'_{\mathcal{P}}(\sigma_i, \sigma_j) = -2\alpha' \ln d^2(|\sigma_{ij}|) + f(\sigma_i, \sigma_j),
\]

where \(d\) is the distance between the sources as measured on the world-she et, and the function \(f\) is finite in the limit \(\sigma_i = \sigma_j\). Notice that the leading short distance divergence has the same form as on the boundary of the disk [33], except that it is defined with respect to the fiducial metric on the annulus. From the viewpoint of the correlation function itself, the divergent terms should be understood as having been absorbed in a renormalization of the bare open string coupling [1, 36].

The precise expression for the one-loop, \(N\)-point, planar scattering amplitude can be found in [36]; we will extract only its low energy field theory limit by exposing the large \(t\) asymptotics of the short cylinder limit. A full discussion of both planar and nonplanar amplitudes, with external field dependence intact, can be found in [36]. For clarity, we consider the massless limit of the planar amplitude, with all \(N\) on-shell vertex operators on the Dpbrane, and with the background field set to zero:

\[
\mathcal{A}_p|_{\text{massless}} = ig^N \delta(\sum_{i=1}^N p_i) \prod_{i=1}^N \int d\sigma_i \int_0^\infty \frac{dt}{2t} (2t)^{-N-(p+1)/2} \times \prod_{i \neq j; \ i, j = 1}^N \left[ e^{-\pi t \sigma_{ij}^2} k_i \cdot k_j \left( 2k_i \cdot k_j e^{-\pi t \sigma_{ij}} \right) \right] + O(e^{-2\pi t})
\]

Notice that the integral over the cylinder modulus can be performed explicitly, giving a closed form expression for the \(N\)-point function in the worldvolume gauge theory: the Feynman graph is the circle with \(N\) external legs, and a single photon running around the loop. It should be emphasized that the corrections from massive open string states circulating in the loop do not change the basic form of this result, the single term given here being replaced by a series of the form:

\[
\sum_{n=0}^\infty F_n^{(\text{open})} \left( \text{Sinh} \left( \frac{1}{2} \pi t \sigma_{ij} \right) \right) e^{- (\sigma_{ij} + n) \pi t}
\]

where \(F_n^{(\text{open})}\) is a polynomial function of the \(\text{Sinh} \left( \frac{1}{2} \pi t \sigma_{ij} \right)\), \(i, j = 1, \cdots, N\).

A more realistic calculation requires extension to a stack of coincident Dpbranes, with specific nonabelian gauge group realized on them. The \(N\) open string tachyon vertex operator extensions would be replaced by \(N\) massless gluon vertex operators. Finally, this particular computation would be performed for the open string sector of a fully consistent, anomaly-free, type I or type I’ string theory ground state. None of these features brings in insurmountable difficulty: the requisite worldsheet technology is well-developed, and can be found in the standard sources. An interesting
observation can be made about bulk-boundary couplings by considering the opposite limit of the planar amplitude, namely, the long cylinder limit. Factorization of the generic $K$-point planar correlation function on the annulus will yield the normalized $(K, 1)$-point function on the disk, with $K$ open string, and one massless closed string, vertex operator insertions. The field theory limit of this computation gives the tree-level coupling of an arbitrary number of worldvolume gauge bosons to the bulk dilaton-graviton. Notice, in particular, that the worldvolume Yang-Mills gauge theory does not decouple from the bulk supergravity theory, even at lowest order in the Yang-Mills gauge coupling.

6 The Macroscopic Loop Amplitude

Let us return to the familiar computation of the contribution to the one-loop amplitude of bosonic open and closed string theory from world-surfaces with the topology of a cylinder, reviewed from first principles in the covariant path integral approach in appendix A.2. From the perspective of the closed string channel, this graph represents the tree-level propagation of a single closed string, exchanged between a spatially-separated pair of Dpbranes. A crucial observation is as follows: although the Dpbrane vacuum corresponds to a spontaneous breaking of translation invariance in the bulk $25-p$ dimensional space orthogonal to the pair of Dpbranes, notice that spacetime translational invariance is preserved within the $p+1$-dimensional worldvolume of each Dpbrane.

It is interesting to ask whether it is possible to modify this calculation such that all $26$ spacetime translation invariances are broken. We emphasize that we ask this question not only for the point-like boundary limit of the annulus graph, but for the annulus with macroscopic boundary loops. The former limit with pointlike boundaries corresponds to the tree-level exchange of a closed string between a pair of Dpbranes: their worldvolumes are spacetime points, and each boundary of the annulus is therefore mapped to a point in the embedding 26d spacetime. The latter case corresponds to a genuinely new worldsheet amplitude, and the corresponding analysis of the covariant string path integral brings in many new features first described in [9, 35], and reviewed in appendix D of this paper. Remarkably, we will find that this computation leads us to discover a new Dirichlet vacuum of open and closed bosonic string theory, namely, that of a Dirichlet (-2)brane. This identification was first made by Chaudhuri in [37], examining evidence from both the target spacetime and worldsheet perspectives of the type I’ string. A complementary interpretation from the perspective of the boundary state formalism appears in our recent work [41]; we also give a discussion of the relation to the (-1)form potential in the global symmetry algebra of the massive type IIA supergravity [40].

It is convenient to align the macroscopic loops, $C_i, C_f$, which we will choose to have the common length $L$, such that their distance of nearest separation, $R$, is parallel to a spatial coordinate, call it $X^{25}$. As in appendix A.2, the Polyakov action contributes a classical piece corresponding to the saddle-point of the quantum path integral: the saddle-point is determined by the minimum action worldsurface spanning the given loops $C_i, C_f$. The result for a generic classical solution of the Polyakov action was given by Cohen, Moore, Nelson, and Polchinski in Ref. [9]. For coaxial circular loops in a flat spacetime geometry, we have a result identical to that which holds for a spatially separated pair of generic Dpbranes in flat spacetime [35].

The details of the computation are reviewed in appendix D. The main difference from the
analysis of the annulus diagram of open and closed string theory is the implementation of boundary reparametrization invariance: although all 26 coordinates of the embedding worldvolume of the space-filling D25brane are chosen to satisfy the Dirichlet boundary condition, reparametrization invariance of the map from the boundaries of the annulus to a given pair of loops in spacetime requires that we allow nontrivial boundary reparametrizations of the worldsheet metric (einbein) on the boundary [9, 35]. The result is an additional contribution to the measure for moduli in the path integral, computible in terms of the functional determinant of the scalar Laplacian on the one-dimensional boundary of the worldsheet. Our result for the connected sum over worldsurfaces with the topology of an annulus with boundaries mapped onto spatially separated macroscopic loops, \( C_i, C_f \), of common length \( L \) takes the form [9, 35]:

\[
A = i \left[ L^{-1}(4\pi^2\alpha')^{1/2} \right] \int_0^\infty \frac{dt}{2t} \cdot (2t)^{1/2} \cdot \eta(it)^{-24} e^{-R^2 t/2\pi\alpha'} .
\]  

The only change in the measure for moduli is the additional factor of \((2t)^{1/2}\) contributed by the functional determinant of \( J \). The pre-factor in square brackets is of interest; recall that there is no spacetime volume dependence in this amplitude since we have broken translational invariance in all 26 directions of the embedding spacetime. If we were only interested in the point-like off-shell closed string propagator, as in [9], the result as derived is correct without any need for a pre-factor.\(^6\) However, we have required that the boundaries of the annulus are mapped to loops in the embedding spacetime of an, a priori, fixed length \( L \). Since a translation of the boundaries in the direction of spacetime parallel to the loops is equivalent to a boundary diffeomorphism, we must divide by the (dimensionless) factor: \( L(4\pi^2\alpha')^{-1/2} \). This accounts for the pre-factor present in our final result. Note that for more complicated loop geometries, including the possibility of loops with corners, the pre-factor in this expression will take a more complicated form.

As mentioned above, we suspect that this expression can be interpreted as computation of the one-loop vacuum amplitude in a distinct Dirichlet background of the familiar open and closed string theory in critical spacetime dimension. As a check, let us take the factorization limit of the amplitude, expressing the eta function in an expansion in powers of \( q = e^{-2\pi/t} \). The small \( t \) limit is dominated by the lowest-lying closed string modes and the result is:

\[
A = i \left[ L^{-1}(4\pi^2\alpha')^{1/2} \right] \int_0^\infty dt \cdot (2t)^{-1/2} \cdot t^{12} \cdot q^{-1} \left( 1 + 24q + O(q^2) \right) e^{-R^2 t/2\pi\alpha'}
\]

\[
\rightarrow iL^{-1} 24 \cdot 2^{-12} (4\pi^2\alpha')^{13} \pi^{-25/2} \Gamma \left( \frac{25}{2} \right) |R|^{-25} .
\]  

Repeating the steps reviewed in section 4 [2, 33], we infer the existence of a Dirichlet (-2)brane in open and closed bosonic string theory with tension with:

\[
\tau_{-2}^2 = \frac{\pi}{256\alpha'} (4\pi^2\alpha')^{13} .
\]  

How does one give physical meaning to a Dirichlet (-2)brane? In order to understand its origin in the spectrum of pbranes, it will be helpful to consider the meaning of a D(-2)brane from the perspective of both the boundary state formalism, and of the ten-dimensional supergravity dualities.

\(^6\)Comparing with the final expression for the off-shell point-like propagator given in Eq. (4.5) of [9], and setting \( t \rightarrow 2\lambda \) in order to match with the notation in [9], the reader should ignore an extraneous factor of \( \lambda^{-13} \), which should clearly be absent in an all-Dirichlet string amplitude.
Let us begin from the perspective of the boundary state formalism, originally developed by Callan, Lovelace, Nappi, and Yost in [14], assuming open strings with Neumann boundary conditions, and later adapted to the Dirichlet case by Green [20]. The extension to the boundary state of a generic Dpbrane in the superstring is due to Li [26]. The boundary state is simply the physical state in the Hilbert space that is annihilated by the boundary conditions on all two-dimensional fields, when interpreted as operator statements [14]. We remind the reader that the worldsheet metric of the classical Polyakov action is eliminated in favor of the Faddeev-Popov ghosts in the course of covariant quantization. Thus, in bosonic string theory, all we have left are the Dirichlet boundary conditions acting on the 26 two-dimensional free scalar fields, and corresponding conditions on the ghosts. In terms of the free bosonic oscillators, the Dinstanton boundary state therefore satisfies the constraint [20]:

\[(\alpha^\mu_n - \alpha^\mu_{-n})|B, y^\mu >= 0, \ [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{mn}\delta^{\mu\nu}, \ \forall \mu = 0, \ldots, 25 \]  

For the b, c ghosts, we have corresponding constraints:

\[(b_n - \tilde{b}_{-n})|B, y^\mu >= 0, \ (c_n + \tilde{c}_{-n})|B, y^\mu >= 0, \ [b_n, c_m] = \delta_{m+n} = [\tilde{b}_n, \tilde{c}_m] \]  

Since the boundary state is a physical state in the Hilbert space, |B> is annihilated by the BRST charge, the operator responsible for implementing two-dimensional reparametrization invariance. In particular, |B> is also required to be invariant under diffeomorphisms of the boundary: \(\sigma^2 \rightarrow f(\sigma^2)\), of the worldsheet annulus. This last constraint is trivially satisfied by the Dirichlet boundary states, \(-1 \leq p \leq 25\), since all two-dimensional fields vanish on the boundary. In the extreme case of the Dinstanton, where the worldvolume is simply a spacetime point, the map from the boundary of the annulus to the worldvolume of the Dinstanton is trivial. The result takes the form [20]:

\[|B> = \exp \left[ \sum_{n=1}^{\infty} (\alpha^\mu_{-n}\tilde{\alpha}^\mu_{n} - b_{-n}\tilde{c}_{-n} - \tilde{b}_{-n}c_{-n}) \right] |\Omega> \]  

Since we have assumed rigid Dirichlet boundaries, |\Omega> is the SL(2, C) invariant vacuum with zero (transverse) momentum orthogonal to the boundary [33].

The boundary conditions we have imposed in our calculation of the macroscopic loop amplitude do not permit a trivial resolution to this last constraint. Requiring that the boundaries of the annulus map into loops at fixed locations in the embedding spacetime implies that we have broken all 26 spacetime translations. Thus, it is helpful to begin with the Dinstanton boundary state described above. Next, we must insert an operator in the transfer matrix whose purpose is to smear the location of the Dirichlet end-point over the specified loop \(C_i\), and likewise for \(C_f\). It is convenient to identify the direction parallel to the pair of coaxial circular loops as one of the coordinates of spacetime, call it \(X^i\). It follows that the operator that must be inserted in the transfer matrix is none other than the momentum generator, \(\hat{P}_i\). Thus, we conclude that the boundary state that describes a Dirichlet(-2)brane is obtained by simply acting with the momentum generator on the boundary state of the Dinstanton.

Is it possible to make this identification more precise? We remind the reader that the normalization of a boundary state is, a priori, ambiguous, as already noted in [14, 20]. We should point out, however, that the normalization of the boundary state could be determined by matching to the
factorization limit of the covariant path integral computation: as shown in section 4, and above, the
one-loop amplitude factorizes on a closed string propagator connecting (normalized) one-point
functions on the disk, each localized on a Dpbrane.

The corresponding one-loop amplitudes in the operator formalism can be written as follows.
Identifying the modular parameter of the one-loop amplitude as a fictitious “time”, the integrand
can be interpreted as the transfer matrix for two-dimensional field theory data from an initial
boundary state, $|B_i>$, to a final state, $|B_f>$. The precise measure of the one-loop string amplitude
is difficult to motivate from first principles without the path integral derivation, but the integral over
t is easily justified by invoking two-dimensional conformal invariance. Thus, the sum over world-
surfaces with the topology of an annulus, and with boundaries terminating on a spatially separated
pair of Dinstantons, or D(-1)branes, can be equivalently interpreted in terms of the transfer matrix
formalism as follows [14, 26]:

$$A_{-1}(R) = \int_0^\infty dt <B,R|e^{-(L_0+\tilde{L}_0-1)t}|B,0 > e^{-R^2 t/2\pi \alpha'}$$,

(34)

where $L_0, \tilde{L}_0$, are the zero modes of the Virasoro generators, including both matter and ghost degrees
of freedom, the vacuum energy of the $SL(2,C)$ invariant vacuum is $-1$, and we have separated the
classical contribution to the Hamiltonian. Comparing with the expressions in appendix A.2, and
in section 4, the reader will note that the normalization of the Dinstanton boundary state and,
consequently, of the one-loop string amplitude is, apriori, unknown and ambiguous [14, 20]. The
corresponding expression for the channel amplitude in the case of the Dirichlet(-2)branes takes the form:

$$A_{-2}(R) = \int_0^\infty dt <B,R|\hat{P}^t e^{-(L_0+\tilde{L}_0-1)t} \hat{P}|B,0 > e^{-R^2 t/2\pi \alpha'}$$,

(35)

where $\hat{P}$ acts in the direction parallel to the coaxial circular loops. The path integral expression
derived in appendix D gives concrete meaning to the various terms in the operator representation.

Additional insight into the nature of the D(-2)brane is provided by supergravity considerations.
Recall that the ten-form field strength is the Hodge dual of a scalar field strength in ten space-
time dimensions. Indeed, Roman’s massive IIA supergravity theory is known to have a nine-form
potential which couples to the D8brane of the IIA string theory [17, 2]. Let us write down the
relevant equations for the coupling to a D8brane in the ten-form formulation of the massive type
IIA supergravity theory. In the Einstein frame metric, the bosonic part of the massive IIA action
takes the simple form [33]:

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-G} \left( R_G - \frac{4}{d-2} (\partial \Phi)^2 - \frac{1}{2} e^{5\Phi/2} M^2 \right) + \frac{1}{2\kappa_{10}^2} \int MF_{10},$$

(36)

where $M$ is an auxiliary field which will be eliminated by its equation of motion. $F_{10}$ is the non-
dynamical ten-form field strength, which can be dualized to a zero-form, or scalar, field strength, $*F_{10}$. This non-dynamical constant field generates a uniform vacuum energy density that permeates
the ten-dimensional spacetime, thus behaving like a cosmological constant: $F_{10} = dC_9$, varying with
respect to the gauge potential, $C_9$, gives $M = constant$, and varying with respect to $M$ gives, $F_{10} = Me^{5\Phi/2}V_{10}$, where $V_{10}$ is the volume of spacetime. Thus, we can identify the dualized scalar field strength: $*F_{10} = Me^{5\Phi/4}$. Notice that, since the objects appearing explicitly in the spacetime
Lagrangian are the field strengths, there is, of course, nothing puzzling about the existence of a
scalar field strength in the Ramond-Ramond sector of the type IIA string theory. The Dirichlet (-2)brane should be identified as the source for the associated topological charge.

7 Conclusions

A basic insight that follows from the observations summarized in this paper is that perturbative string theory is more simply understood as an exactly renormalizable two-dimensional field theory with precisely one independent Wilsonian renormalized parameter [36]. This insight is already apparent in the computation of the one-loop renormalization of the string coupling in closed bosonic string theory by Polchinski in [1]. We have added further clarifications, especially in the interesting case of the open and closed type I and type I′ string theories in generic twoform background [36, 38].

It should be noted that the exact renormalizability of perturbative string theory is obscure from the viewpoint of the spacetime effective Lagrangian: computation of the renormalized Greens functions for massless fields in an $\alpha'$ expansion, while keeping only a finite number of terms in the nonrenormalizable Wilsonian effective action, does not enable one to infer that the ultraviolet cutoff can, in fact, be removed. Thus, at any finite order in the $\alpha'$ expansion, string theory is indeed nonrenormalizable. Our assertion that perturbative string theory is exactly renormalizable relies crucially on knowledge of an equivalent, all-orders in $\alpha'$, worldsheet representation of the Greens function and loop correlation functions. It should be emphasized that such an analysis can only be carried out in explicit form in the flat spacetime background, known to be infra-red finite to all orders in string perturbation theory for an anomaly-free choice of nonabelian gauge group [38].

Perhaps the most important issues raised by this paper are the observations in section 5, where we have noted that the remarkable normalizability property of the perturbative string scattering amplitudes can survive in the low energy effective field theory limit: all of the couplings in the worldvolume effective Lagrangian, inclusive of numerical factors, can be determined in terms of the single mass scale, $m_s$, plus a finite number of dimensionless scalar field expectation values, or moduli [36]. As mentioned earlier, this could have significant consequences for the derivation of precision perturbative gauge theory results from perturbative string scattering amplitude computations. This is an area of considerable import for the continued development of high energy physics, both in precision particle physics and astrophysics, and for braneworld cosmologies. It appears to us very unfortunate that the full power of perturbative string theory computations has not been fully exploited in its most immediate application to low energy physics: namely, the direct derivation of scattering amplitudes in perturbative gauge theories.

The physical significance of the Dirichlet(-2)brane described in section 6, and the implications for Poincare duality in String/M theory, are discussed elsewhere [41]. As mentioned earlier, macroscopic loop amplitudes in string theory have the potential to provide many new applications to physics. The preliminary analysis of loop geometries given in [9, 35], including the interesting possibility of corners, is deserving of further development. There exist a number of potential applications relating to cosmic string production from the vacuum, cosmic string scattering, as well as the study of radiation from cusps and/or corners on a cosmic string. It would be remarkable to have a first principles formalism for deriving such phenomena directly from a fundamental, and fully renormalizable, quantum theory.
ACKNOWLEDGEMENTS

The notes that appear in Appendix C on the zeta function regularization of divergent sums typically appearing in one-loop string amplitudes were written up in the Spring ’03, during the course of an email collaboration with Kristján Kristjánsson and Larúsf Thorlacius. I thank them for helpful input on the presentation of the derivations. A concise presentation of this technique, but with more sophisticated applications, can be found in my paper with Eric Novak [36]. I thank Peter Orland for bringing the work in Ref [15] to my notice.

A Open and Closed Bosonic String Theory

Consider a pair of parallel Dpbranes separated by a distance $R$ in the direction $X^{p+1}$ with $p < 25$. There are four worldsheet diagrams that contribute to the one-loop amplitude in an unoriented open and closed string theory. Recall that worldsheets with Euler number, $\chi = 2 - 2h - b - c$, where $h$, $b$, and $c$, are, respectively, the number of handles, boundaries, and crosscaps on the worldsheet, contribute at $O(g^\chi)$ to the string perturbative expansion [33]. We will perform the sum over surfaces for each of these topologies in turn, starting with the closed bosonic string amplitude derived in [1].

A.1 One-loop Vacuum Amplitude: Torus

This is essentially a review of Polchinski’s 1986 derivation of the one-loop vacuum amplitude for closed bosonic string theory. Since closed strings cannot couple to a background magnetic field, the background field strength, and Dpbrane geometry, are of no relevance to the computation of this diagram. Note that the worldsheets of toroidal topology are embedded in all 26 spacetime dimensions.

The generic metric on a torus can be parameterized by two shape parameters, or worldsheet moduli, $\tau = \tau_1 + i\tau_2$, and it takes the form:

$$ds^2 = e^{\phi}|d\sigma^1 + \tau d\sigma^2|^2, \quad \sqrt{g} = e^{\phi}\tau_2, \quad -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \quad \tau_2 > 0, \quad |\tau| > 1,$$

(37)

with worldsheet coordinates, $\sigma^a$, $a=1,2$, scaled to unit length. The combination, $\sqrt{g}g^{ab}$, is both diffeomorphism and Weyl invariant, leading to a gauge-invariant norm for quadratic deformations of the scalar fields, $X$, as well as for traceless deformations of the metric field, $g_{ab}$. Note that an arbitrary reparameterization of the metric can be decomposed into trace-dependent and traceless components, the latter including the effect of a variation in the worldsheet moduli, $\tau_i$:

$$\delta g_{ab} = g_{ab}(\delta \phi - \nabla_c \delta \sigma^c) - (\nabla_a \delta \sigma_b + \nabla_b \delta \sigma_a - g_{ab} \nabla_c \delta \sigma^c) + \delta \tau_i \partial_i g_{ab}.$$

(38)

Sections in Polchinski’s text that may be helpful in reading this appendix are 3.2, 3.5, 5.1, 5.2, 7, 10.8, and 13.5. The presentation here is based on his 1986 paper [1], and the follow-up works [10, 34], rather than on the textbook. One reason is that we wish to highlight the relationship to path integrals in quantum gravity, eschewing the use of worldsheet holomorphicity and ghosts, although, as we have emphasized earlier, conformal field theory in the operator formalism has its uses. More importantly, the casual reader of Polchinski’s text will fail to notice that many results, such as Eqs. (7.2.3), (7.2.4), and all of secs. 7.4, 10.8, and 13.5, cannot be obtained in closed form without the zeta-regularized path integral derivation. The author has done a nice job of motivating the results but, in the interests of pedagogy, many significant derivations have been skipped.
Thus, the only obstruction to a fully Weyl invariant measure for the string path integral is the norm for the field $\phi$:

$$|\delta \phi|^2 = \int d^2 \tau \sqrt{g}(\delta \phi)^2 = \int d^2 \tau_2 e^\phi(\delta \phi)^2 .$$  \hspace{1cm} (39)

The natural choice of diffeomorphism invariant norm violates Weyl invariance explicitly. Fortunately, this obstruction is absent in the critical spacetime dimension since the worldsheet action turns out to be independent of $\phi$. The differential operator that maps worldsheet vectors, $\delta \sigma^a$, to symmetric traceless tensors, usually denoted $(P_1 \delta \sigma)_{ab}$, has two zero modes, or Killing vectors, on the torus. These are the constant diffeomorphisms: $\delta \sigma^a_0$. Following [5], we invoke the unique diffeomorphism invariant norm on the tangent space to the space of classical metric configurations at a given metric, $g_{ab}$. This norm is also Weyl invariant in the critical dimension [1]:

$$|\delta g_{ab}|^2 = \int d^2 \sigma \sqrt{g} \left( g^{ac}g^{bd} + Cg^{ab}g^{cd} \right) \delta g_{ab}\delta g_{cd}, \quad dg dX = J(\tau_1)(d\phi d\delta \sigma)'d^2 \tau dX ,$$  \hspace{1cm} (40)

where the prime denotes exclusion of the zero mode. The diffeomorphism and Weyl invariant measure for moduli in the string path integral is derived as shown in [1, 10]. Let us arbitrarily pick the normalization unity for the gaussian path integral of any field on the worldsheet. This property is assumed to hold for either set of field variables: $(\delta g, \delta X)$, or $((\delta \phi d\sigma^a)'e, d^2 \tau, \delta X')$:

$$1 = \int d\delta g_{ab} e^{-\frac{1}{2} |\delta g_{ab}|^2} = \int d\delta \phi e^{-\frac{1}{2} |\delta \phi|^2} = \int d\delta \sigma^a e^{-\frac{1}{2} |\delta \sigma^a|^2} = \int d\delta X e^{-\frac{1}{2} |\delta X|^2} .$$  \hspace{1cm} (41)

Notice that the arbitrary normalization will drop out in the change of variables, leaving an unambiguously normalized expression for the Jacobian, $J(\tau_1)$. In the critical spacetime dimension, $J$ does not depend on $\phi$, and is also independent of the unknown constant $C$ in Eq. (40) [1]. To see this, we begin by accounting for all of the ordinary gaussian integrals over constant parameters that contribute to the measure. For the two real worldsheet moduli, it is easy to check that:

$$1 = (2\pi)^{-1} \left( \int d^2 \sigma \sqrt{g} \right) \int d\tau_1 d\tau_2 e^{-\frac{1}{2} |\delta \tau_1|^2} \int d^2 \sigma \sqrt{g} ,$$  \hspace{1cm} (42)

which gives the normalization of the integral over moduli. Likewise, separating out the two real zero modes of the vector Laplacian:

$$d\phi d\delta \sigma^a = (d\phi d\delta \sigma^a)'d\delta \sigma^a_0 d\delta \sigma^1_0 ,$$  \hspace{1cm} (43)

the gaussian integral for the corresponding field variations is found to be normalized as follows:

$$1 = \int d\delta \sigma^a_0 d\delta \sigma^1_0 e^{-\delta \sigma^a_0 \delta \sigma^b_0} \int d^2 \sigma \sqrt{g}(\delta_a \phi \delta_b \phi + g_{ab}) \int (d\delta \phi d\delta \sigma^a)'e^{-\frac{1}{2} |\delta \phi|^2 - \frac{1}{2} |\delta \sigma^a|^2} \equiv 2\pi (\det Q_{ab})^{-1/2} \int (d\delta \phi d\delta \sigma^a)'e^{-\frac{1}{2} |\delta \phi|^2 - \frac{1}{2} |\delta \sigma^a|^2} .$$  \hspace{1cm} (44)

Finally, we should account for the constant modes of the scalar Laplacian. We can distinguish the $p + 1$ noncompact embedding coordinates parallel to the Dpbrane worldvolume, assumed to have a common box regularized volume, from the $25 - p$ compact bulk coordinates. Denoting the size of each embedding coordinate by $L^\mu$, $\mu = 0, \cdots, 25$, we have:

$$1 = \int d\delta X e^{-\frac{1}{2} |\delta X|^2} = \prod_{\mu = 0}^{25} \int d\delta \bar{x} e^{-\frac{1}{2} |\delta \bar{x}^\mu|^2} \int d^2 \sigma \sqrt{g} \int d\delta X e^{-\frac{1}{2} |\delta X'|^2}$$
\[ \int d\delta X' e^{-\frac{1}{2} \delta g' |\delta X|^2} = (2\pi)^{d/2} \left( \int d^2\sigma \sqrt{g} \right)^{-d/2} \int d\delta X' e^{-|\delta X'|^2/2}, \]

with \( d = 26 \). Substituting Eqs. (42), (44), and (45), in Eq. (41) gives an unambiguously normalized expression for the Jacobian \( J(\tau) \):

\[ 1 = \int d\delta g X e^{-\frac{1}{2} \delta g |\delta g|^2 - \frac{1}{2} |\delta X|^2} \]

\[ = (\text{det}Q)^{-1/2} (2\pi)^{-d/2} \left( \int d^2\sigma \sqrt{g} \right)^{1+d/2} (\text{det}'\mathcal{M})^{1/2} \int (d\delta X' e^{-\frac{1}{2} \delta g |\delta g|^2 - \frac{1}{2} |\delta X|^2}), \]

where \( \mathcal{M} \) arises from the change of variables, and is a self-adjoint differential operator on the worldsheet. Its explicit form is obtained by substitution of Eq. (38) in Eq. (40) [1, 10]. We have factored out the redundant integrations over the gauge parameters \( (\delta \phi, \delta \sigma^a)' \). Dividing by the volume of the gauge group to eliminate this redundancy gives the following simplified expression for the one-loop closed bosonic string vacuum functional in the critical spacetime dimension:

\[ W_{\text{tor}} = \frac{1}{\text{Vol}[\text{Diff}_0 \times \text{Weyl}]} \int_{\text{tor}} [dg][dX] e^{-S[X,g]} = \int [d\tau]_{\text{tor}} \int [dX] e^{-S[X,\delta]} \]

\[ = \prod_{\mu=0}^{25} L^n \int d^2\tau (\text{det}Q_{ab})^{-1/2} (2\pi)^{-d/2} \left( \int d^2\sigma \sqrt{g} \right)^{1+d/2} (\text{det}'\mathcal{M})^{1/2} \int d\delta X' e^{-S[X,\delta]}, \]

where \( d = 26 \). It remains to perform the integration over embeddings of the closed worldsheets with toroidal topology, namely, the 26 scalar fields, \( X'(\sigma) \). On the torus, the Laplacian on scalars acts as: \( \Delta_\tau = \tau_2^{-2} |\partial_2 + \tau_1|'^2 \). The orthonormal basis for the scalar field, defined with respect to the Weyl and diffeomorphism invariant measure [4, 1], is given by the complete set of eigenfunctions:

\[ X'(\sigma^a) = \sum_{n_2,n_1} \Psi_{n_2,n_1} (\sigma^a), \quad \text{with} \quad (\sigma^a) = \frac{1}{\sqrt{2}} e^{2\pi i (n_2\sigma^2 + n_1\sigma^1)} \]

and the discrete set of eigenvalues:

\[ \omega_{n_2,n_1} = 4\pi^2 (g^{ab} n_a n_b) = \frac{4\pi^2}{\tau_2} |n_2 - \tau n_1|^2 \]

where the subscripts take values in the range \(-\infty \leq n_2, n_1 \leq \infty\), \(-\infty \leq n_1 \leq \infty\). Following Hawking [4], we note that the measure in the tangent space to the space of embeddings is ultralocal, a point that has also been stressed by Polchinski [1, 10, 34]. Namely, the functional integral over embeddings, \( Z_X(\sigma^a) \), is the product of ordinary integrals defined at some base point, \( \sigma^a \), on the two-dimensional domain, followed by an integration of the location of the base point in the domain \( 0 \leq \sigma^a \leq 1 \). The normalization of the sum over embeddings, denoted \( \mu \) in [4], is determined unambiguously by the form of the classical action and the gauge invariant norm on the space of eigenfunctions [1]:

\[ \int d\delta X' e^{-\frac{1}{4\pi \alpha'} \int d^2\sigma X' \sqrt{g} \partial_a \partial_b X} \]

\[ = \mu \prod_{n_2=-\infty}^{\infty} \prod_{n_1=-\infty}^{\infty} \int d\alpha' \omega_{n_2,n_1}^{-d/2} \left( \sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \omega_{n_2,n_1} |a_{n_2,n_1}|^2 \right) \]

\[ = (2\pi \alpha')^{-d/2} \prod_{n_2=-\infty}^{\infty} \prod_{n_1=-\infty}^{\infty} (\omega_{n_2,n_1})^{-d/2} \equiv (2\pi \alpha')^{-d/2} (\text{det}'\Delta)^{-d/2}. \]
More significantly, the lack of ambiguity in this normalization is preserved even after the introduction of a regulator for the infinite products in Eq. (50). This is due to the fact that the choice of worldsheet ultraviolet regulator is uniquely determined by the gauge symmetries.

One last substitution relates the functional determinants of the vector and scalar Laplacians [1, 10]:

\[
(\det' \mathcal{M})^{1/2} = (\det' 2 \Delta^c_d)^{1/2} \left(\frac{2}{\tau_2^2}\right) = \frac{1}{2} \det' \Delta = \frac{1}{2} \prod_{n_2=-\infty}^{\infty} \prod_{n_1=-\infty}^{\infty} \omega_{n_2,n_1} .
\]  

(51)

Notice that the dependence of \( \mathcal{M} \) on the unknown constant, \( C \), appearing in Eq. (40) drops out in the critical spacetime dimension since the \( \phi \) field decouples [1]. Substituting Eqs. (50) and (51) in the expression for the vacuum functional gives the simplified expression:

\[
W_{\text{tor}} = \prod_{\mu=0}^{25} L^\mu \int_F \frac{d^2\tau}{2\tau_2^2} \tau_2^{-3+2+1+d/2} (2\pi)^{-d/2} (2\pi\alpha')^{-d/2} \prod_{n_2=-\infty}^{\infty} \prod_{n_1=-\infty}^{\infty} (\omega_{n_2,n_1})^{-12} .
\]  

(52)

The infinite product in Eq. (50) can be zeta-function regulated using a Sommerfeld-Watson integral transform following [1]. We review this derivation in Appendix C. The result is [1]:

\[
\det' \Delta = \tau_2^2 e^{-\pi \tau_2/3} \prod_{n=1}^{\infty} |1 - e^{2\pi i n \tau}|^4 = \tau_2^2 |\eta(\tau)|^4 .
\]  

(53)

Substituting in Eq. (52) gives Polchinski’s result for the sum over connected worldsheets with the topology of a torus [1]:

\[
W_{\text{tor}} = \prod_{\mu=0}^{25} L^\mu \int_F \frac{d^2\tau}{2\tau_2^2} (4\pi^2 \alpha' \tau_2)^{-13} |\eta(\tau)|^{-48} .
\]  

(54)

It is instructive to compare the zeta-function regulated expressions in [1] with previous results for the one-loop closed string amplitudes obtained in the operator formalism. Friedan’s calculation of the closed string vacuum amplitude [6] misses its numerical coefficient. Shapiro’s result for the N-point closed string tachyon scattering amplitude at one loop order [7] misses the numerical relation between the one-loop renormalized closed string coupling and the fundamental string mass scale. Notice that the appropriate Jacobi theta functions appear in either worldsheet formalism simply as a consequence of modular invariance. This symmetry is responsible for the finiteness of one-loop string amplitudes, and it holds independent of their normalization.

### A.2 One-loop Vacuum Amplitude: Boundaries and Crosscaps

There is only one orientable open Riemann surface of Euler number zero, the annulus, with two boundaries. The corresponding nonorientable surfaces of vanishing Euler number are obtained by plugging, respectively, one or both holes of the annulus with a crosscap. The Mobius strip has a single hole, and the Klein bottle has none. Thus, the fiducial metric on each nonorientable surface can be chosen identical to that on the annulus, and the derivation for the gauge-invariant measure for moduli is unchanged [34]. The nonorientable surfaces share the same fundamental domain as the annulus, but the eigenspectrum is appropriately modified by application of the orientation reversal.
projection, $\Omega$. We therefore begin with a detailed discussion of the annulus. The Mobius strip and Klein bottle will be obtained as simple modifications of this result.

One further clarification is required to distinguish results in the presence, or absence, of Dirichlet $p$-branes. In the absence of $Dp$-branes, the boundary of the worldsheet can lie in all 26 dimensions, and we impose Neumann boundary conditions on all $d=26$ scalars. This gives the traditional open and closed string theory, whose supersymmetric generalization is the type IIB string theory. $T$-dualizing $25-p$ embedding coordinates gives the open and closed string theory in the background geometry of a pair of $Dp$-branes\cite{18}. Its supersymmetric generalization is the type I$'$ string theory, when $p$ is even, and the type IIB string theory in generic $Dp$-brane background, when $p$ is odd\cite{2}. The $Dp$-branes define the hypersurfaces bounding the compact bulk spacetime, which is $(25-p)$-dimensional. Since the bulk spacetime has edges, these $25-p$ embedding coordinates are Dirichlet worldsheet scalars. It is conventional to align the $Dp$-branes so that the distance of nearest separation, $R$, corresponds to one of the Dirichlet coordinates, call it $X_{25}$.

In the presence of a pair of $Dp$-branes, the classical worldsheet action contributes a background term given by the Polyakov action for a string of length $R$ stretched between the $Dp$-branes\cite{2,33,34}:

$$S_{\text{cl}}[G,g] = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} G^{25,25}(X) \partial_a X^{25} \partial_b X^{25} = \frac{1}{2\pi\alpha'} R^2 t \quad ,$$

where the second equality holds in the critical dimension on open world-surfaces of vanishing Euler number. The background dependence, $e^{-S_{\text{cl}}[G,g]}$, appears in the sum over connected open Riemann surfaces of any topology, orientable or nonorientable. Notice that the background action is determined both by the fiducial worldsheet metric, and by the bulk spacetime metric, $G_{\mu\nu}[X]$. Notice also that the boundary of an open worldsheet is now required to lie within the worldvolume of the $Dp$-brane, although the worldsheet itself is embedded in all 26 spacetime dimensions.

The metric on the generic annulus can be parameterized by a single real worldsheet modulus, $t$, and it takes the form:

$$ds^2 = e^\phi ((d\sigma^1)^2 + 4t^2(d\sigma^2)^2), \quad \sqrt{g} = e^{\phi/2} 2t, \quad 0 \leq t \leq \infty \quad ,$$

with worldsheet coordinates, $\sigma^a$, $a=1,2$, parameterizing a square domain of unit length. $2t$ is the physical length of either boundary of the annulus. The differential operator mapping worldsheet vectors, $\delta\sigma^a$, to symmetric traceless tensors, usually denoted ($P_1\delta\sigma)_{ab}$, has only one zero mode on the annulus. This is the constant diffeomorphism in the direction tangential to the boundary: $\delta\sigma_0^2$. Likewise, the analysis of the zero modes of the scalar Laplacian must take into account the $Dp$-brane geometry: the $p+1$ noncompact embedding coordinates satisfying Neumann boundary conditions are treated exactly as in the case of the torus. The $25-p$ Dirichlet coordinates lack a zero mode. Thus, the analog of Eq. (45) reads:

$$1 = \int d\delta X e^{-\frac{1}{2}|\delta X|^2} = \prod_{\mu=0}^{p} \int d\delta \bar{x} e^{-\frac{1}{2}(\delta \bar{x}^\mu)^2} \int d^2\sigma \sqrt{g} \int d\delta X' e^{-\frac{1}{2}|\delta X'|^2}$$

$$= (2\pi)^{(p+1)/2} \left( \int d^2\sigma \sqrt{g} \right)^{-(p+1)/2} \int d\delta X' e^{-|\delta X'|^2/2} \quad .$$

The analysis of the diffeomorphism and Weyl invariant measure for moduli follows precisely as for
the torus, differing only in the final result for the Jacobian \[34\]. The analog of Eq. (46) is given by:

\[
1 = \int \! dg dX e^{-\frac{1}{2} |\delta g|^2 - \frac{1}{2} |\delta X|^2} - \frac{1}{2} |\delta g|^2 - \frac{1}{2} |\delta X|^2
\]

where \((\det Q_{22}) = 2t\) in the critical dimension, cancelling the factor of \(2t\) arising from the normalization of the integral over the single real modulus. As shown in \[34\], \(M\) takes the form:

\[
(\det M')^{1/2} = (\det 2 \Delta_d^{1/2}) \frac{1}{2t} \prod_{n=\infty}^{\infty} \prod_{n=\infty}^{\infty} \omega_{n_2, n_1} .
\]

The Laplacian acting on free scalars on an annulus with boundary length \(2t\) takes the form, \(\Delta = (2t)^{-2} \partial_2^2 + \partial_1^2\), with eigenspectrum:

\[
\omega_{n_2, n_1} = \frac{\pi}{t^2} (n_2^2 + n_1^2 t^2), \quad \Psi_{n_2, n_1} = \frac{1}{\sqrt{2t}} e^{2\pi i n_2 \sigma^2} \sin(\pi n_1 \sigma^1) ,
\]

where the subscripts take values in the range \(-\infty \leq n_2 \leq \infty\), and \(n_1 \geq 0\) for a Neumann scalar, or \(n_1 \geq 1\) for a Dirichlet scalar.

In the case of a background electromagnetic field, \(F_{p-1,p}\), it is convenient to complexify the corresponding pair of scalars: \(Z = X^p + i X^{p-1}\) \[13, 29, 31\]. They satisfy the following twisted boundary conditions:

\[
\partial_1 \text{Re} Z = \text{Im} Z = 0 \quad \sigma^1 = 0 \quad \partial_1 \text{Re} e^{-i \phi} Z = \text{Im} e^{-i \phi} Z = 0 \quad \sigma^1 = 1
\]

Expanding in a complete set of orthonormal eigenfunctions gives:

\[
Z = \sum_{n_2, n_1} z_{n_2, n_1} \Psi_{n_2, n_1} = \frac{1}{\sqrt{2t}} e^{2\pi i n_2 \sigma^2} \sin(\pi (n_1 + \alpha) \sigma^1) ,
\]

where \(\pi \alpha = \phi\), and \(\pi - \phi\), respectively \[31\], and where the subscripts take values in the range \(-\infty \leq n_2 \leq \infty\), \(n_1 \geq 0\). The twisted complex scalar has a discrete eigenvalue spectrum on the annulus given by:

\[
\omega_{n_2, n_1}(\alpha) = \frac{\pi}{t^2} (n_2^2 + (n_1 + \alpha)^2 t^2) .
\]

Thus, the connected sum over worldsurfaces with the topology of an annulus embedded in the spacetime geometry of a pair of parallel Dpbranes separated by a distance \(R\) in the direction \(X^9\), and in the absence of a magnetic field, takes the form \[34\]:

\[
W_{\text{ann}} = \prod_{\mu=0}^{p} \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-(p+1)/2} \eta(it)^{-24} e^{-R^2 t/2\pi \alpha'} .
\]
In the presence of a worldvolume electromagnetic field, \( F_{p-1,p} \), the scalars \( X^{p-1}, X^{p} \), are complexified. Substituting the result for the eigenspectrum of the twisted complex scalar gives:

\[
W_{\text{ann}}(\alpha) = \prod_{\mu=0}^{p-2} L^\mu \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha^\prime t)^{-(p-1)/2} \eta(it)^{-22} e^{-Rt/2\pi \alpha^\prime} \frac{e^{\pi \sigma^2 \eta(it)}}{i\Theta_{11}(it\alpha, it)} ,
\]

with \( \alpha = i\phi/\pi \) and \( q = e^{-2\pi t} \).

The corresponding expressions for the Mobius strip and Klein bottle are easily derived by making the appropriate orientation projection on the eigenspectrum on the annulus. For the Mobius strip, and in the presence of the magnetic field, we have:

\[
W_{\text{mob}} = \prod_{\mu=0}^p L^\mu \int_0^\infty \frac{dt}{2t} (16\pi^2 \alpha^\prime t)^{-(p-1)/2} \eta(2it)^{-22} e^{-Rt/\pi \alpha^\prime} \prod_{n_1=0}^{\infty} \prod_{n_2=-\infty}^{\infty} (\det \Delta_{\text{mob}})^{-1} .
\]

The extra factor of two in the contribution from zero modes, and in the classical action of the stretched string, is easily understood as follows. Recall that a cylinder is a strip of boundary length \( 2t \) and height 1. The Mobius strip is the strip twisted by \( \Omega \), orientation reversal, and sewn back upon itself. Thus, the corresponding eigenfunction spectrum is the same as that on a cylinder with boundary length \( 4t \), height 2, and twisted boundary conditions on the scalar parameterized by an angle \( \alpha = \pi/2 \). Note that the extra factor of two cancels out in the integration measure since it has the scale invariant form \( dt/t \). The spectrum of the Laplacian takes the form:

\[
\Psi_{n_2n_1} = \frac{1}{\sqrt{2t}} e^{4\pi i(n_2+\frac{1}{2})\sigma^2} \sin 2\pi (n_1 + \frac{1}{2}) \sigma^1
\]

\[
\Psi_{n_2n_1} = \frac{1}{\sqrt{2t}} e^{4\pi i n_2 \sigma^2} \sin 2\pi (n_1) \sigma^1 ,
\]

where we have separated the \( \pi/2 \)-twisted (odd), and untwisted (even), eigenfunction sectors on the equivalent annulus. To obtain the corresponding results in the presence of the magnetic field, simply introduce the parameter \( \alpha \) in these expressions, where \( \alpha \) takes the values, \( \phi/\pi \), or \( 1-\phi/\pi \):

\[
\Psi_{n_2n_1}(\alpha) = \frac{1}{\sqrt{2t}} e^{4\pi i(n_2 + \frac{1}{2})\sigma^2} \sin 2\pi (n_1 + \frac{1}{2} + \frac{1}{2}\alpha) \sigma^1
\]

\[
\Psi_{n_2n_1}(\alpha) = \frac{1}{\sqrt{2t}} e^{4\pi i n_2 \sigma^2} \sin 2\pi (n_1 + \frac{1}{2}\alpha) \sigma^1 ,
\]

The corresponding eigenvalues are:

\[
\omega_{n_2n_1}^{\text{odd}}(\alpha) = \frac{4\pi^2}{t^2} (n_2 + \frac{1}{2})^2 + 4\pi^2 \left( n_1 + \frac{1}{2} + \frac{1}{2}\alpha \right)^2
\]

\[
\omega_{n_2n_1}^{\text{even}}(\alpha) = \frac{4\pi^2}{t^2} n_2^2 + 4\pi^2 \left( n_1 + \frac{1}{2}\alpha \right)^2 ,
\]

where \(-\infty \leq n_2 \leq \infty \), and \( 0 \leq n_1 \leq \infty \).

Thus, the functional determinant of the complex scalar Laplacian on the Mobius strip in the presence of the background magnetic field, is given by the infinite product over both sets of eigenvalues: gives:

\[
\det \Delta_{\text{mob}} = \prod_{n_2=-\infty}^{\infty} \prod_{n_1=0}^{\infty} \left[ \frac{4\pi^2}{t^2} n_2^2 + 4\pi^2 \left( n_1 + \frac{1}{2}\alpha \right)^2 \right] \left[ \frac{4\pi^2}{t^2} (n_2 + \frac{1}{2})^2 + 4\pi^2 \left( n_1 + \frac{1}{2} + \frac{1}{2}\alpha \right)^2 \right] .
\]
Regulating the divergent products by the method of zeta-function regularization, as shown in Appendix C.2, gives the result:

\[
\det \Delta_{\text{mob}} = q^{\frac{1}{2}(\alpha^2 + \frac{1}{6} - \alpha)} (1 - q^\alpha)^{-1} \prod_{n_1 = 1}^{\infty} \left[ (1 - q^{2n_1 - \alpha})(1 - q^{2n_1 + \alpha}) \right] \\
\prod_{n_1 = 0}^{\infty} \left[ (1 + q^{2n_1 - 1 - \alpha})(1 + q^{2n_1 - 1 + \alpha}) \right].
\]  

(72)

This can be expressed in terms of the Jacobi theta functions as follows:

\[
(\det \Delta_{\text{mob}})^{-1} = -ie^{2\pi \alpha^2} \left\{ \frac{\eta(2it)}{\Theta_{11}(2it\alpha, 2it)} \frac{\eta(2it)}{\Theta_{00}(2it\alpha, 2it)} \right\},
\]

(73)

where \(\alpha\) denotes \(\phi/\pi\).

Combining with the functional determinants for the free Dirichlet and Neumann scalars, our final expression for the sum over connected worldsurfaces with the topology of a Mobius strip is:

\[
W_{\text{mob}} = \prod_{\mu = 0}^{p} \int_0^\infty \frac{dt}{2t} (16\pi^2 \alpha^t)^{-(p+1)/2} e^{-R^2/2\pi \alpha^t} \left[ \eta(2it) \Theta_{00}(0, 2it) \right]^{-12}.
\]

(74)

In the presence of the background magnetic field, the result takes the form:

\[
W_{\text{mob}}(\alpha) = \prod_{\mu = 0}^{p-2} \int_0^\infty \frac{dt}{2t} (16\pi^2 \alpha^t)^{-(p-1)/2} e^{-R^2/2\pi \alpha^t} \\
\times \left[ \eta(2it) \Theta_{00}(0, 2it) \right]^{-10} e^{2\pi \alpha t} \left\{ \frac{\eta(2it)}{\Theta_{11}(2it\alpha, 2it)} \frac{\eta(2it)}{\Theta_{00}(2it\alpha, 2it)} \right\}.
\]

(75)

The amplitude for the Klein bottle follows from similar considerations. The Klein bottle is a strip twisted by \(\Omega\), orientation reversal, and with both ends sewn back upon themselves. Note that the Klein bottle is a closed worldsurface. Thus, the corresponding eigenfunction spectrum is the same as that on a cylinder with boundary length \(4t\), height 2, but with periodicity imposed on both edges. The periodicity condition implies that we include both left- and right-moving annulus modes, and with equal weight. For convenience, we present the results directly in the presence of the background magnetic field:

\[
\Psi_{n_2n_1}(\alpha) = \frac{1}{\sqrt{2t}} e^{4\pi in_2 \sigma^2 \sin 2\pi (n_1 + \frac{1}{2} \alpha) \sigma^1},
\]

(76)

and the eigenvalues are given by:

\[
\omega_{n_2n_1} = \frac{4\pi^2}{t^2} n_2^2 + 4\pi^2 (n_1 + \frac{1}{2} \alpha)^2,
\]

(77)

with the usual range for \(n_1\) and \(n_2\). The functional determinant takes the form:

\[
\det \Delta_{kb} = \prod_{n_2 = -\infty}^{\infty} \prod_{n_1 = 0}^{\infty} \left[ 4\pi^2 \frac{n_2^2}{t^2} + 4\pi^2 (n_1 + \frac{1}{2} \alpha)^2 \right].
\]

(78)
Following zeta-function regularization of the divergent eigenvalue sums, the result for a complex twisted scalar takes the form:

\[
\left(\det \Delta_{kb}\right)^{-1} = \left(q^{\alpha^2+\frac{1}{2}-\alpha}(1-q^{2\alpha})^{-1}\prod_{n_1=1}^{\infty} \left(1-q^{2n_1-\alpha}(1-q^{2n_1+\alpha})\right)^{-1}\right)
= e^{2\pi\alpha^2} \left[\frac{\eta(2it)}{\Theta_{11}(2it\alpha, 2it)}\right].
\] (79)

Combining with the contributions from the free Dirichlet and Neumann scalars, we obtain:

\[
W_{kb} = \prod_{\mu=0}^{p} L^{\mu} \int_{0}^{\infty} \frac{dt}{2t} (16\pi^2 \alpha'^t)^{-\frac{(p+1)}{2}} \eta(2it)^{-24} e^{-R^2t/\pi\alpha'}.
\] (80)

In the presence of the background magnetic field, the corresponding result is:

\[
W_{kb}(\alpha) = \prod_{\mu=0}^{p-2} L^{\mu} \int_{0}^{\infty} \frac{dt}{2t} (16\pi^2 \alpha'^t)^{-\frac{(p-1)}{2}} \eta(2it)^{-22} e^{-R^2t/\pi\alpha'} e^{\pi\alpha^2} \left[\frac{\eta(2it)}{\Theta_{11}(2it\alpha, 2it)}\right].
\] (81)

### B Type I-I’ String Theory in an Electromagnetic Background

The derivation of the gauge invariant measure for moduli given in Appendix A can be easily extended to the case of the supersymmetric unoriented open and closed string theories, type I and type I’ string theory [10, 33]. We begin with the contribution from worldsheets with the topology of an annulus in the presence of a background electromagnetic field, and in the background spacetime geometry of a pair of Dpbranes separated by a distance \( R \). The first principles derivation of the result for the annulus was derived in [35]. We will also derive the results for the sum over unoriented world-surfaces with the topology of a Mobius strip, and a Klein bottle, in the discussion that follows below. Beginning with the annulus:

\[
W_{ann-I}(\alpha) = \prod_{\mu=0}^{p-2} \int_{0}^{\infty} \frac{dt}{2t} (8\pi^2 \alpha'^t)^{-\frac{p-1}{2}} \eta(it)^{-6} e^{-R^2t/2\pi\alpha'} \frac{e^{\pi\alpha^2} \eta(it)}{i\Theta_{11}(it\alpha, it)}
\times \prod_{n_2=0}^{\infty} \prod_{n_1=-\infty}^{\infty} \left(\det_{ann-I} \Delta_{n_2+\frac{1}{2}, n_1+\frac{1}{2}+\alpha}\right)^1 \left(\det_{ann-I} \Delta_{n_2+\frac{1}{2}, n_1+\frac{1}{2}}\right)^3,
\] (82)

where we have included the contribution from worldsheet bosonic fields derived in the previous section.

Let us understand the eigenvalue spectrum of the worldsheet fermions in more detail. Recall that the functional determinant of the two-dimensional Dirac operator acting on a pair of Majorana Weyl fermions satisfying twisted boundary conditions is equivalent, by Bose-Fermi equivalence, to the functional determinant of the scalar Laplacian raised to the inverse power. This provides the correct statistics. In addition, we have the constraint of world-sheet supersymmetry. This requires
that the four complexified Weyl fermions satisfy identical boundary conditions in each sector of the theory in the $\sigma^4$ direction. For a complex Weyl fermion satisfying the boundary condition:

\[
\psi(1, \sigma^2) = -e^{\pi i a} \psi(0, \sigma^2) \\
\psi(\sigma^1, 1) = -e^{\pi i b} \psi(\sigma^1, 0)
\]

the Bose-Fermi equivalent scalar eigenspace takes the form:

\[
\Psi_{n_2 + \frac{1}{2}, n_1 + \frac{1}{2} + (1 + a)} = \frac{1}{\sqrt{2t}} e^{2\pi i (n_2 + \frac{1}{2}(1 + b)) \sigma^2} \cos \pi(n_1 + \frac{1}{2}(1 + a))
\]

where we sum over $-\infty \leq n_2 \leq \infty$, $n_1 \geq 0$. Notice that the unrotated oscillators are, respectively, half-integer or integer moded as expected for the scalar equivalent of antiperiodic or periodic worldsheet fermions. Finally, we must sum over periodic and antiperiodic sectors, namely, with $a, b$ equal to 0, 1. As reviewed in the appendix, weighting the $(a, b)$ sector of the path integral by the factor $e^{\pi i ab}$ gives the following result for the fermionic partition function [33]:

\[
Z_b^a(\alpha, q) = \frac{1}{q^{2\pi^2 - 4\pi} e^{\pi i ab}} \prod_{m=1}^{\infty} \left( 1 + e^{\pi i b q^{m - \frac{1}{2}(1 + a)}} (1 + e^{-\pi i b q^{m - \frac{1}{2}(1 - a)}}) \right)
\]

\[
= \frac{1}{e^{\pi i a^2 \eta(it)}} \Theta_{ab}(\alpha it, it)
\]

We have included a possible rotation by $\alpha$ or $1-\alpha$ as in the previous section. This applies for the Weyl fermion partnering the twisted complex worldsheet scalar. Substituting in the path integral, and summing over $a, b=0, 1$, for all fermions, and over $\alpha$ and $1-\alpha$ for the Weyl fermion partnering the twisted complex scalar, gives the result:

\[
W_{\text{ann}} = \prod_{\mu=0}^{p-2} \int_0^\infty dt \left( 8\pi^2 \alpha' t \right)^{-\frac{1}{2}} e^{-R^2 t / 2 \pi \alpha'} \times \left\{ \eta(it)^{-6} \left[ \frac{e^{2\pi i \alpha \eta(it)}}{\Theta_{11}(it\alpha, it)} \right] \right\}
\]

\[
\times \left[ \frac{\Theta_{00}(it\alpha, it)}{e^{\pi i a^2 \eta(it)}} \left( \frac{\Theta_{00}(0, it)}{\eta(it)} \right)^3 - \frac{\Theta_{01}(it\alpha, it)}{e^{\pi i a^2 \eta(it)}} \left( \frac{\Theta_{01}(0, it)}{\eta(it)} \right)^3 - \frac{\Theta_{10}(it\alpha, it)}{e^{\pi i a^2 \eta(it)}} \left( \frac{\Theta_{10}(0, it)}{\eta(it)} \right)^3 \right]
\]

where we have used the fact that $\Theta_{11}(0, it)$ equals zero. The factor within curly brackets is the contribution from world-sheet bosons; that within square brackets in the second line of the expression is the contribution from world-sheet fermions.

Likewise, we can write down the corresponding results for the sum over worldsurfaces with the topology of Mobius strip or Klein bottle by invoking Bose-Fermi equivalence, and by using the appropriate twisted cylinder eigenspaces. The detailed derivation is outlined in appendix C. For the Mobius strip we have the result:

\[
W_{\text{mob}} = \prod_{\mu=0}^{p-2} \int_0^\infty dt \left( 16\pi^2 \alpha' t \right)^{-\frac{1}{2}} e^{-R^2 t / \pi \alpha'} \times \left\{ \eta(2it)^{-6} \left[ \frac{e^{2\pi i \alpha \eta(2it)}}{\Theta_{11}(2it\alpha, 2it)} \right] \right\}
\]

\[
\times \left[ \left[ \frac{\Theta_{01}(2it\alpha, 2it)}{\eta(2it)} \right]^3 \left( \frac{\Theta_{00}(2it\alpha, 2it)}{\eta(2it)} \right)^3 \right] \times \left[ \left( \frac{\Theta_{00}(0, 2it)}{\eta(2it)} \right)^3 - \left( \frac{\Theta_{10}(0, 2it)}{\eta(2it)} \right)^3 \right] \right]
\]

\[.87\]
We have used the fact that $\Theta_{11}(0,2it)$ vanishes. The factor within curly brackets in the first line of this expression is the contribution from the eight transverse bosonic modes, in the presence of the electromagnetic background. The second, and third, lines of the expression are the contributions from the worldsheet fermions. Notice that the Ramond-Ramond, and Neveu-Schwarz-Neveu-Schwarz, sectors give identical contributions of opposite sign.

The corresponding result for the Klein bottle is:

$$W_{kb-1} = \prod_{\mu=0}^{p-2} L^\mu \int_0^\infty \frac{dt}{2t} (16\pi^2\alpha'^t)^{(p-1)/2} e^{-R^2t/\pi\alpha'} \times \left\{ [\eta(2it)]^{-6} e^\frac{2\pi\alpha'^2}{\Theta_{11}(it\alpha,2it)} \right\}$$

$$\times \left[ \left( \frac{\Theta_{00}(0,2it)}{\eta(it)} \right)^3 \frac{\Theta_{00}(2it\alpha,2it)}{e^{2\pi\alpha'^2\eta(2it)}} - \left( \frac{\Theta_{10}(0,2it)}{\eta(it)} \right)^3 \Theta_{10}(2it\alpha,2it) e^{2\pi\alpha'^2\eta(2it)} \right]$$

$$- \left( \frac{\Theta_{01}(0,2it)}{\eta(it)} \right)^3 \Theta_{01}(2it\alpha,2it) e^{2\pi\alpha'^2\eta(2it)} \right].$$

(88)

For simplicity, setting $\alpha=0$, let us write down the result for the sum over connected one-loop vacuum graphs in the Dpbrane background geometry, but without an electromagnetic field. The sum over worldsheets with the topology of a torus decouples from the sum over worldsheets with boundary and/or crosscap, since it is insensitive to the Dpbranes. It also vanishes as a consequence of the unbroken spacetime supersymmetry. The sum over unoriented connected worldsheets with vanishing Euler character in type I(I') string theory in the background of a pair of parallel and static Dpbrane stacks, each with $N$ coincident Dpbranes, takes the form:

$$W_I = \prod_{\mu=0}^p L^\mu \int_0^\infty \frac{dt}{2t} (8\pi^2\alpha'^t)^{-(p+1)/2} e^{-R^2t/2\pi\alpha'}$$

$$\times \left\{ \frac{N^2}{\eta(it)^8} \left[ \left( \frac{\Theta_{00}(0, it)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{01}(0, it)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{10}(0, it)}{\eta(it)} \right)^4 \right] \right\}$$

$$- 2^6 N \frac{\eta(it)^4}{\Theta_{00}(it)} \left[ \left( \frac{\Theta_{01}(0, it)}{\eta(it)} \right)^4 \left( \frac{\Theta_{10}(0, it)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{10}(0, it)}{\eta(it)} \right)^4 \left( \frac{\Theta_{01}(0, it)}{\eta(it)} \right)^4 \right]$$

$$+ 2^{10} \frac{\eta(it)^8}{\eta(it)^8} \left[ \left( \frac{\Theta_{00}(0, it)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{01}(0, it)}{\eta(it)} \right)^4 - \left( \frac{\Theta_{10}(0, it)}{\eta(it)} \right)^4 \right] \right\},$$

(89)

where we have used the fact that $t_{MS} = 2t_{ann}$, $t_{KB} = 2t_{ann}$. $\Theta_{11}(0, it)$ vanishes as a consequence of the zero mode in the Ramond-Ramond sector for worldsheet fermions [33]. $W_I$ vanishes as a consequence of target spacetime supersymmetry, as can be seen by use of the abstruse identity relating the Jacobi theta functions in the zero external field annulus, and Klein bottle, amplitudes [33]; notice that the Mobius strip gives a vanishing contribution even in the presence of the external field. The generalization of Eq. (89) in the presence of an external electromagnetic field is obtained by combining the expressions in Eqs. (86), (87), and (88).

It is important to understand that in the type I vacuum with $N=32$ Dpbranes, the vanishing of the one-loop vacuum amplitude in zero external field can also be understood as a consequence
As has been stressed by Polchinski [15, 2, 3, 33], the number of $D_p$-branes is fixed to be 32 by the requirement of Ramond-Ramond sector tadpole cancellation: $N - 2^5 = 0$, ensuring the absence of a propagating unphysical Ramond-Ramond state. The latter fact was first noted in [15]. In order to see this, note that the factor within curly brackets in lines 2–4 of the expression in Eq. (89) is the partition function of the worldsheet superconformal field theory (SCFT) on the strip of length $2t$, with appropriate orientation projection on the spectrum in the case of the Mobius strip and Klein bottle contributions, as explained above. Expanding in powers of $e^{-2\pi t}$ yields the so-called $q$-expansion of the partition function of the boundary SCFT. The leading terms of order $e^{\pi t}$ vanish, a precise cancellation indicating the absence of target spacetime tachyons in the open string mass spectrum.

The next order, $q^0$, counts the number of massless target spacetime bosons and target spacetime fermions, contributing with opposite signs to the one-loop string vacuum functional. If we collect the contributions at this order from target spacetime bosons alone, namely, the $O(q^0)$ term in the NS-NS sector of the annulus, Mobius strip, and Klein bottle amplitudes, we find a precise cancellation because the overall numerical coefficient vanishes when $N=2^5$ [15, 3, 33]:

$$W_{I(\text{NS-NS})} = \frac{1}{2}(N - 2^5)^2 \left[ \prod_{\mu=0}^{p+1} L^\mu \right] (8\pi^2\alpha')^{-(p+1)/2} \int_0^\infty dt t^{-(p+1)/2} e^{-R^2 t/2\pi\alpha'} \left[ 16 + O(e^{\pi t}) \right] , \quad (90)$$

and where $W_{I(\text{R-R})}$ is an expression that takes identical form, but with opposite overall sign [15, 3, 33]. Finally, we should note that the contribution from the annulus amplitude to $W_1$ from the NS-NS sector alone, namely, the $O(q^0)$ term in the NS-NS sector of the annulus, Mobius strip, and Klein bottle amplitudes, carries the significant information about the charge of the Dirichlet-pbrane [2]. This is explained at length in section 2.3.

C Zeta-function Regularization of Infinite Sums

The regularization of a divergent sum over the discrete eigenvalue spectrum of a self-adjoint differential operator by the zeta function method can always be carried out in closed form when the eigenvalues are known explicitly [4]. This is the case for all of the infinite sums encountered in one-loop string amplitudes [1, 33, 34]. We illustrate the basic method with a review of Polchinski’s calculation of the zeta-regularized functional determinant of the scalar Laplacian on the torus [1]:

$$\ln \det' \Delta_{\text{tor}} = \lim_{m \to 0} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \ln \left[ \frac{4\pi^2}{\tau^2} (n_2 - \tau n_1)(n_2 - \bar{\tau} n_1) + m^2 \right] - \ln \left[ \frac{4\pi^2 m^2}{\tau^2} \right] . \quad (91)$$

The $n_1=n_2=0$ term has been included in the infinite sum by introducing an infrared regulator mass, $m$, for the zero mode. We will take the limit $m \to 0$ at the end of the calculation. Following Hawking [4], we begin by expressing the first term in Eq. (91) in the equivalent form:

$$S_{\text{tor}} = - \lim_{s,m \to 0} \frac{d}{ds} \left\{ \left( \frac{4\pi^2}{\tau^2} \right)^{-s} \sum_{n_2,n_1=-\infty}^{\infty} \left[ (n_2 - \tau n_1)(n_2 - \bar{\tau} n_1) + m^2 \right]^{-s} \right\} . \quad (92)$$

---

8This is the basic observation used by us in recent works [38, 39] in arriving at an expression for the free energy of the canonical ensemble of type I strings: target spacetime supersymmetry is broken by the introduction of thermal phases in the finite temperature vacuum. But the one-loop string free energy nevertheless vanishes as a consequence of R-R sector tadpole cancellation alone.
Notice that the infinite sums are manifestly convergent for \( \Re s > 1 \). The required \( s \to 0 \) limit can be obtained by analytic continuation in the variable \( s \). The analogous step for the second term in Eq. (91) yields the relation:

\[
+ \lim_{m \to 0} \lim_{s \to 0} \frac{d}{ds} \left( \frac{4\pi^2 m^2}{\tau_2^2} \right)^{-s} = 2 \log \tau_2 - \lim_{m \to 0} 2 \log (2\pi m) .
\]

(93)

The finite term in this expression contributes the overall factor of \( \tau_2^2 \) to the result given in Eq. (53).

The infinite summation over \( n_2 \) is carried out using a Sommerfeld-Watson transform as in [1]. We invoke the Residue Theorem in giving the following contour integral representation of the infinite sum as follows:

\[
\sum_{n=-\infty}^{\infty} \left[ \frac{n^2 + x^2}{2\pi} \right]^{-s} = \oint_{C_{\pm}} \frac{dz}{2\pi i} \frac{\pi \cot(\pi z)}{z^2 + x^2} .
\]

(94)

where \( C_n \) is a small circle enclosing the pole at \( z=n \) in the counterclockwise sense. The contours may be deformed without encountering any new singularities into the pair of straight line contours, \( C_{\pm} \), where the line \( C_{\pm} \) runs from \( +i\epsilon \) to \( -i\epsilon \), respectively, in the upper, or lower, half-plane. This ensures that the contours are required to avoid the branch cuts which run, respectively, from \( +i\epsilon \) to \( +i\infty \), and from \( -i\epsilon \) to \( -i\infty \). Let us evaluate these integrals as before. The constant pieces from the square brackets in Eq. (96) combine to give:

\[
\sum_{n_2=-\infty}^{\infty} \left( \frac{n_2 - n_1 \tau_1}{2} + x \right)^{-s} = \oint_{C_{\pm}} dz \frac{e^{i\pi z}}{e^{i\pi z} - e^{-i\pi z} - \frac{1}{2}} \left( (z - n_1 \tau_1)^2 + x^2 \right)^{-s}
\]

\[
+ \oint_{C_{\pm}} dz \frac{e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z} + \frac{1}{2}} \left( (z - n_1 \tau_1)^2 + x^2 \right)^{-s} .
\]

(96)

Note that the contours are required to avoid the branch cuts which run, respectively, from \( +ix \) to \( +i\infty \), and from \( -ix \) to \( -i\infty \). Let us evaluate these integrals as before. The constant pieces from the square brackets in Eq. (96) combine to give:

\[
I_1(s, x) = \frac{1}{2} \left( \int_{C_{\pm}} dz - \int_{C_{\pm}} dz \right) \left( (z - n_1 \tau_1)^2 + x^2 \right)^{-s} = x^{-2s+1} \int_{-\infty}^{\infty} du (1 + u^2)^{-s} .
\]

(97)

The integral simply yields the beta function, \( B\left(\frac{1}{2}, s - \frac{1}{2}\right) \) (see Eq. (97)). Taking the \( s \)-derivative followed by the \( s=0 \) limit gives,

\[
\lim_{s \to 0} \frac{d}{ds} x^{-2s+1} B\left(\frac{1}{2}, s - \frac{1}{2}\right) = \lim_{s \to 0} \frac{d}{ds} x^{-2s+1} \frac{\sin(\pi s)}{\sqrt{\pi}} \Gamma(1-s) \Gamma(s - \frac{1}{2}) = -2\pi x .
\]

(98)
Substituting for $x^2 = n_1^2 \tau_1^2 + m^2$, and taking the $m=0$ limit, gives
\[ -4\pi \tau_2 \sum_{n_1=1}^{\infty} n_1 - 2\pi \lim_{m \to 0} m = -4\pi \tau_2 \zeta(-1) = 2\pi \tau_2 B_2(0) = \frac{1}{3} \pi \tau_2 , \tag{99} \]
where we have used $B_2(q) = q^2 - q + \frac{1}{6}$.

Next, we tackle the non-constant pieces from the square brackets in Eq. (96). It is helpful to take the $s$-derivative and the $s \to 0$ limit prior to performing the contour integral. We begin with the contour integral in the upper half-plane:
\[ I_2(x, s) = \int_{C^+} dz \frac{e^{i\pi z}}{e^{i\pi z} - e^{-i\pi z}} ( (z - n_1 \tau_1)^2 + x^2 )^{-s} \]
\[ = -2\sin(\pi s) \int_x^\infty dy \frac{e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} ( (y + in_1 \tau_1)^2 - x^2 )^{-s} . \tag{100} \]
Taking the derivative with respect to $s$, and setting $s=0$, gives:
\[ \frac{d}{ds} I_2(x, s)|_{s=0} = -2\pi \int_x^\infty dy \frac{e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} \]
\[ = -2\log \left| 1 - e^{-2\pi(x+in_1 \tau_1)} \right| . \tag{101} \]
The $C^-$ integral gives an identical contribution since,
\[ I_3(x, s) = -\int_{C^-} dz \frac{e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} ( (z - n_1 \tau_1)^2 + x^2 )^{-s} \]
\[ = -2\sin(\pi s) \int_x^\infty dy \frac{e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} ( (y + in_1 \tau_1)^2 - x^2 )^{-s} . \tag{102} \]
Upon taking the $s$-derivative, and setting $s=0$, we get the same result as in Eq. (100):
\[ \frac{d}{ds} I_3(x, s)|_{s=0} = -2\pi \int_x^\infty dz \frac{e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} \]
\[ = -2\log \left| 1 - e^{-2\pi(x+in_1 \tau_1)} \right| . \tag{103} \]
Combining all of the contributions to $S_{\text{tor}}$ gives the following result in the $m \to 0$ limit:
\[ S_{\text{tor}} = -\frac{\pi \tau_2}{3} + 4 \sum_{n_1=1}^{\infty} \log |1 - e^{2\pi in_1 \tau}| - \lim_{m \to 0} \left[ \pi m + 2 \log(2\pi m) - 2\log(1 - e^{-2\pi m}) \right] , \tag{104} \]
Notice that the divergent terms in the $m \to 0$ limit cancel as is seen by Taylor expanding the logarithm in the last term.

Combining with the result from Eq. (93) gives Polchinski’s result for the functional determinant of the scalar Laplacian on the torus:
\[ \prod_{n_2=-\infty}^{\infty} \prod_{n_1=-\infty}^{\infty} \omega_{n_1 n_2} = \tau_2 e^{-\pi \tau_2 / 3} \prod_{n_1=1}^{\infty} \left| 1 - q^{n_1} \right|^4 , \tag{105} \]
where $q = e^{2\pi \tau}$. 

32
C.1 Annulus: Twisted Complex Scalar Eigenspectrum

The eigenspectrum of the scalar Laplacian on a surface with boundary includes a dependence on an electromagnetic background, reflected as a twist in the boundary conditions satisfied by the scalar. As an illustration, let us work out the functional determinant of the scalar Laplacian for worldsheets with the topology of an annulus. The case of the Mobius strip and Klein bottle are simple extensions which do not introduce any significant new feature into the nature of the infinite summation. Since the required sums only differ in the choice of “twist”, the results can be straightforwardly written down given the result for the annulus with generic twist $\alpha$.

Begin with the eigenspectrum on the annulus. In the case of the free Neumann scalars, we must introduce an infrared regulator mass for the zero mode, as shown in the case of the torus. The functional determinant of the Laplacian can be written in the form [34]:

$$\ln \det' \Delta = \lim_{m \to 0} \sum_{n_1=0}^{\infty} \sum_{n_2=-\infty}^{\infty} \log \left[ \frac{\pi^2}{t^2} (n_2^2 + n_1^2 t^2 + m^2) \right] - \log \left( \frac{\pi^2 m^2}{t^2} \right) .$$  \hspace{1cm} (106)

The first term in Eq. (106) is a special case of the infinite sum with generic twist, $\alpha$, and the zeta-regulated result can be obtained by setting $\alpha=0$ in the generic calculation which will be derived below. The second term in Eq. (106) yields the result:

$$+ \lim_{m \to 0} \lim_{s \to 0} \frac{d}{ds} \left[ 4 \pi^2 m^2 \right]^{-s} = 2 \log 2t - \lim_{m \to 0} -2 \log (2\pi m) .$$  \hspace{1cm} (107)

This contributes the correct power of $2t$ to the measure of the path integral for a free Neumann scalar [34]. For the Dirichlet scalar, we must remember to drop the $n_1=0$ modes from the double sum above since the sine eigenfunction vanishes for all values of $\sigma^1$, not only at the boundary. Thus, the $n_1$ summation begins from $n_1=1$.

Now consider the case of the twisted Neumann scalar. There is no need to introduce an infrared regulator in the presence of a magnetic field since there are no zero modes in the eigenvalue spectrum. Thus, the analysis of the infinite eigensum is similar to that for a Dirichlet scalar, other than the incorporation of twist. We begin with:

$$S_{\text{ann}} = - \lim_{s \to 0} \frac{d}{ds} \left[ \frac{\pi^2}{t^2} \right]^{-s} \sum_{n_2=-\infty}^{\infty} \sum_{n_1=1}^{\infty} \left[ n_2^2 + (n_1 + \alpha)^2 t^2 \right]^{-s} ,$$  \hspace{1cm} (108)

and identical statements can be made about its convergence properties as in the previous subsection. The $n_2$ summation in $S_{\text{ann}}$ is carried out using a contour integral representation identical to that in Eq. (96) except that $x=(n_1 + \alpha)t$. The $n_1$ summation following the analog of Eq. (96) can be recognized as the Riemann zeta function with two arguments,

$$\sum_{n_1=0}^{\infty} (n_1 + \alpha)^{-2s+1} t^{-2s+1} \equiv \zeta(2s-1,\alpha) t^{-2s+1} .$$  \hspace{1cm} (109)

Taking the $s$-derivative followed by the $s=0$ limit gives,

$$\lim_{s \to 0} \frac{d}{ds} \zeta(2s-1,\alpha) t^{-2s+1} B \left( \frac{1}{2}, s - \frac{1}{2} \right) = \lim_{s \to 0} \zeta(2s-1,\alpha) t^{-2s+1} \frac{\sin(\pi s)}{\sqrt{\pi}} \Gamma(1-s) \Gamma(s - \frac{1}{2}) = -2\pi t \zeta(-1,\alpha) .$$  \hspace{1cm} (110)
Substituting the relation \( \zeta(-n,q) = -B_{n+2}^\prime(q)/(n+1)(n+2) \), and combining the contributions for \( q=\alpha \), and \( 1-\alpha \), gives:

\[
\frac{\pi t}{3} [B_3^\prime(1-\alpha) + B_3^\prime(\alpha)] = 2\pi t \left[ \alpha^2 + \frac{1}{6} - \alpha \right],
\]

where we have used \( B_n^\prime(q) = nB_{n-1}(q) \), and \( B_2(q) = q^2 - q + \frac{1}{6} \).

Next, we tackle the non-constant pieces from the square brackets in the analog of Eq. (96). It is helpful to take the \( s \)-derivative and the \( s \to 0 \) limit prior to performing the contour integral. We begin with the contour integral in the upper half-plane:

\[
I_2(x, s) = \int_{C_+} dz e^{i\pi z} \left( z^2 + x^2 \right)^{-s} = -2\sin(\pi s) \int_x^\infty dy e^{\pi y} - e^{-\pi y} \left( y^2 - x^2 \right)^{-s}.
\]

Taking the derivative with respect to \( s \), and setting \( s=0 \), gives:

\[
\frac{d}{ds} I_2(x, s) |_{s=0} = -2\pi \int_x^\infty dy \frac{e^{-\pi y}}{e^{\pi y} - e^{-\pi y}} = -\log \left( 1 + e^{-2\pi x} \right).
\]

The \( C_- \) integral gives an identical contribution as before. Combining the contributions to \( S_{\text{ann}} \) from terms with \( \alpha = \phi/\pi \), and \( 1-\alpha \), respectively, gives the following result in the \( m \to 0 \) limit:

\[
S_{\text{ann}} = 2\pi t (\alpha^2 + \frac{1}{6} - \alpha) - 2 \sum_{n_1=0}^\infty \log \left( (1 + e^{-2\pi(n_1+\alpha)t})(1 + e^{-2\pi(n_1+1-\alpha)t}) \right).
\]

The result for the functional determinant of the Laplacian acting on a twisted complex scalar takes the form:

\[
\left[ \prod_{n_2=0}^{\infty} \prod_{n_1=0}^{\infty} \omega_{n_2,n_1} \right]^{-1} = q^{\frac{1}{2}(\alpha^2 + \frac{1}{6} - \alpha)} (1 - q^{n_1})^{-1} \prod_{n_1=1}^{\infty} \left( (1 - q^{n_1 - \alpha})(1 - q^{n_1 + \alpha}) \right)^{-1},
\]

where \( q = e^{-2\pi t} \). The result can be expressed in terms of the Jacobi theta function as follows:

\[
\frac{q^{\frac{1}{2}(\alpha^2 + \frac{1}{6} - \alpha)}}{-2i\sin(\pi t \alpha/2)} \prod_{n_1=1}^{\infty} \left[ (1 - q^{n_1 - \alpha})(1 - q^{n_1 + \alpha}) \right]^{-1} = -i \frac{e^{\pi t \alpha^2} \eta(it)}{\Theta_{11}(it \alpha, it)},
\]

with \( \alpha = \phi/\pi \).

Setting \( \alpha = 0 \) in this expression, and combining with the result in Eq. (107), gives the functional determinant of the Laplacian acting on a free Neumann scalar:

\[
\left( \prod_{n_2=-\infty}^{\infty} \prod_{n_1=0}^{\infty} \omega_{n_1,n_2} \right)^{-1/2} = \left( \frac{1}{2t} \right) \frac{1}{q^{\frac{1}{2t}}} \prod_{n_1=1}^{\infty} (1 - q^{n_1})^{-1} = \frac{1}{2t} \left[ \eta(it) \right]^{-1},
\]

where \( q = e^{-2\pi t} \). The expression for the Dirichlet determinant is identical except for the absence of the overall factor of \( 1/2t \).
C.2 Mobius Strip: Twisted Complex Fermion Eigenspectrum

As a final illustration, we work out the contribution from worldsheet fermions to the Mobius strip amplitude for type I string theory in a background magnetic field. As explained in the text, we must identify the scalar eigenspace inferred by application of Bose-Fermi equivalence. Under the action of $\Omega$, the eigenspace in the $(a, b)$ fermionic sector of the theory for the corresponding complex scalar takes the form:

$$
\Psi_{n_2n_1} = \frac{1}{\sqrt{2t}} e^{4\pi i (n_2 + \frac{1}{2} + \frac{1}{2} (1 + b)) \sigma^2} \sin 2\pi (n_1 + \frac{1}{2} + \frac{1}{2} (1 \pm a) + \frac{1}{2} \alpha) \sigma^1
$$

where we will set $\alpha$ equal to $\phi/\pi$, and $1 - \phi/\pi$. Note that eigenfunctions of odd (even) mass level are weighted differently in the trace. The corresponding eigenvalues are:

$$
\begin{align*}
\omega_{n_2n_1}^{\text{odd}} &= \frac{4\pi^2}{t^2} (n_2 + \frac{1}{2} + \frac{1}{2} (1 + b))^2 + 4\pi^2 \left(n_1 + \frac{1}{2} + \frac{1}{2} (1 \pm a) + \frac{1}{2} \alpha\right)^2 \\
\omega_{n_2n_1}^{\text{even}} &= \frac{4\pi^2}{t^2} (n_2 + \frac{1}{2} (1 + b))^2 + 4\pi^2 \left(n_1 + \frac{1}{2} (1 \pm a) + \frac{1}{2} \alpha\right)^2
\end{align*}
$$

where $-\infty \leq n_2 \leq \infty$, and $0 \leq n_1 \leq \infty$. We will compute the product over both sets of eigenvalues and then take the square root of the result. This gives:

$$
\det \Delta_{\text{mob}}^{(a, b)} = \prod_{n_2 = -\infty}^{\infty} \prod_{n_1 = 0}^{\infty} \left[ \frac{4\pi^2}{t^2} (n_2 + \frac{1}{2} (1 + b))^2 + 4\pi^2 \left(n_1 + \frac{1}{2} (1 \pm a) + \frac{1}{2} \alpha\right)^2 \right]^{1/2}
= \left[ \frac{4\pi^2}{t^2} (n_2 + \frac{1}{2} (1 + b))^2 + 4\pi^2 \left(n_1 + \frac{1}{2} (1 \pm a) + \frac{1}{2} \alpha\right)^2 \right]^{1/2}.
$$

Denoting $\phi/\pi$ by $\alpha$, this gives the results:

$$
\begin{align*}
\det \Delta_{\text{mob}}^{(0, 0)} &= +ie^{-2\pi t a^2} \left[ \eta(2it) \Theta_{11}(2ita, 2it) \right]^{-1/2} \left[ \eta(2it) \Theta_{00}(2ita, 2it) \right]^{-1/2} \\
\det \Delta_{\text{mob}}^{(0, 1)} &= -e^{-2\pi t a^2} \left[ \eta(2it) \Theta_{10}(2ita, 2it) \right]^{-1/2} \left[ \eta(2it) \Theta_{01}(2ita, 2it) \right]^{-1/2} \\
\det \Delta_{\text{mob}}^{(1, 0)} &= +e^{-2\pi t a^2} \left[ \eta(2it) \Theta_{01}(2ita, 2it) \right]^{-1/2} \left[ \eta(2it) \Theta_{10}(2ita, 2it) \right]^{-1/2} \\
\det \Delta_{\text{mob}}^{(1, 1)} &= -ie^{-2\pi t a^2} \left[ \eta(2it) \Theta_{11}(2ita, 2it) \right]^{-1/2} \left[ \eta(2it) \Theta_{00}(2ita, 2it) \right]^{-1/2}.
\end{align*}
$$

Thus, allowing for the phase in front of each contribution, the contribution to the Mobius strip amplitude from four complex worldsheet fermions, one of which is twisted, takes the form:

$$
\begin{align*}
\det \Delta_{\text{mob}}^{F} &= e^{-2\pi t a^2} \left[ \eta(2it) \Theta_{01}(2ita, 2it) \eta(2it) \Theta_{10}(2ita, 2it) \right]^{-1} \left[ \eta(2it) \Theta_{01}(2ita, 2it) \eta(2it) \Theta_{10}(2ita, 2it) \right]^{-3} \\
&- e^{-2\pi t a^2} \left[ \eta(2it) \Theta_{10}(2ita, 2it) \eta(2it) \Theta_{01}(2ita, 2it) \right]^{-1} \left[ \eta(2it) \Theta_{10}(2ita, 2it) \eta(2it) \Theta_{01}(2ita, 2it) \right]^{-3}.
\end{align*}
$$
where $\alpha$ denotes $\phi/\pi$.

## D Off-shell Propagator of a Closed String

Let us return to our review in appendix A.2 of the contribution to the one-loop amplitude of bosonic open and closed string theory from world-surfaces with the topology of a cylinder. From the perspective of the closed string channel, this graph represents the tree-level propagation of a single closed string, exchange between a spatially-separated pair of D$p$branes. A crucial observation is as follows: although the D$p$brane vacuum corresponds to a spontaneous breaking of translation invariance in the bulk $25-p$ dimensional space orthogonal to the pair of D$p$branes, notice that spacetime translational invariance is preserved within the $p+1$-dimensional worldvolume of each D$p$brane.

It is interesting to ask whether it is possible to modify this calculation such that all $26$ spacetime translation invariances are broken. We emphasize that we ask this question not only for the point-like boundary limit of the annulus graph, but for the annulus with *macroscopic* boundary loops. The former limit with pointlike boundaries corresponds to the tree-level exchange of a closed string between a pair of Dinstantons: their worldvolumes are spacetime points, and each boundary of the annulus is therefore mapped to a point in the embedding $26$d spacetime. The latter case corresponds to a genuinely new worldsheet amplitude, and the corresponding analysis of the covariant string path integral brings in many new features, first described in [9, 35].

It is convenient to align the macroscopic loops, $C_i, C_f$, which we will choose to have the common length $L$, such that their distance of nearest separation, $R$, is parallel to a spatial coordinate, call it $X^{25}$. As in appendix A.2, the Polyakov action contributes a classical piece corresponding to the saddle-point of the quantum path integral: the saddle-point is determined by the minimum action worldsurface spanning the given loops $C_i, C_f$. The result for a generic classical solution of the Polyakov action was given in [9]. For coaxial circular loops in a flat spacetime geometry, we have a result identical to that which holds for a spatially separated pair of generic D$p$branes in flat spacetime, namely:

$$S_{cl}[G,g] = \frac{1}{4\pi \alpha'} \int d^2 \sigma \sqrt{g} g^{ab} G^{25,25}(X) \partial_a X^{25} \partial_b X^{25} = \frac{1}{2\pi \alpha'} R^2 t . \quad (123)$$

Notice, in particular, that there is no $L$ dependence in the saddle-point action as a consequence of the Dirichlet boundary condition on all $26$ scalars. The metric on the annulus is parameterized as before by a single real worldsheet modulus, $t$, and it takes the form:

$$ds^2 = e^\phi ((d\sigma^1)^2 + 4t^2 (d\sigma^2)^2), \quad \sqrt{g} = e^{\phi} 2t, \quad 0 \leq t \leq \infty \quad , \quad (124)$$

with worldsheet coordinates, $\sigma^a$, $a=1,2$, parameterizing a square domain of unit length. $2t$ is the physical length of either boundary of the annulus, as measured in the two-dimensional field theory. As in the case of the Dinstanton, we will evaluate the determinant of the scalar Laplacian for all $26$ embedding coordinates with the Dirichlet boundary condition. In addition, we should note that there is no contribution from coordinate zero modes, since all of the $X^\mu$ are Dirichlet. Thus, the
usual box-regularized spacetime volume dependence originating in the Neumann sector is absent, precisely as in the vacuum of a pair of D-instantons. The analog of Eq. (57) reads:

\[ 1 = \int d\delta X e^{-\frac{1}{2} |\delta X|^2} = \int d\delta X' e^{-|\delta X'|^2/2} \]  

The crucial difference in the path integral computation when the boundaries of the annulus are mapped to macroscopic loops in embedding spacetime has to do with the implementation of boundary reparametrization invariance: we must include in the path integral a sum over all possible maps of the worldsheet boundary to the loops \( C_i, C_f \) [9]. Notice that the analysis of reparametrization invariance in the bulk of the worldsheet is unaltered. As a consequence, the conditions for Weyl invariance, and for the crucial decoupling of the Liouville mode, are unchanged.

The path integral computation we are about to perform simply yields the one-loop amplitude of the open and closed bosonic string theory in a distinct vacuum. We will understand the nature of the new boundary state characterizing this vacuum in a moment.

Let us proceed with the analysis of the measure following the steps in appendix A.2. The differential operator mapping worldsheet vectors, \( \delta \sigma^n \), to symmetric traceless tensors, usually denoted \( (P_1 \delta \sigma)_{ab} \), has only one zero mode on the annulus. This is the constant diffeomorphism in the direction tangential to the boundary: \( \delta \sigma^2 \). The analysis of the diffeomorphism and Weyl invariant measure for moduli follows precisely as for the annulus [34]. The only difference is an additional contribution from the vector Laplacian, accounting for diffeomorphisms of the metric which are nontrivial on the boundary [9]. The analog of Eq. (58) now takes the form:

\[ 1 = \int de \int dgdXe^{-\frac{1}{2} |\delta g|^2 - \frac{1}{2} |\delta X|^2 - \frac{1}{2} |\delta e|^2} \]

\[ = (\det Q_{22})^{-1/2} \int d^2 \sigma \sqrt{g} (\det' J)^{1/2} (\det' M)^{1/2} \int (d\phi d\delta \sigma') dt dX' e^{-\frac{1}{2} |\delta g|^2 - \frac{1}{2} |\delta X|^2 - \frac{1}{2} |\delta e|^2}, \]  

(126)

where \( (\det Q_{22}) = 2t \) in the critical dimension, cancelling the factor of \( 2t \) arising from the normalization of the integral over the single real modulus. As shown in [34], the functional determinant of the vector Laplacian acting in the worldsheet bulk takes the form:

\[ (\det' M)^{1/2} = (\det' 2 \Delta_d^c)^{1/2} \left( \frac{1}{2t} \right) = \frac{1}{2} (2t)^{-1} \det' \Delta = \frac{1}{2} (2t)^{-1} \prod_{n_2=-\infty}^{\infty} \prod_{n_1=-\infty}^{\infty} \omega_{n_2,n_1}, \]  

(127)

and the infinite product is computed precisely as in appendix A.2. The functional determinant of the operator \( J \) can likewise be expressed in terms of the functional determinant of the Laplacian acting on free scalars on the one-dimensional boundary, parametrized here by \( \sigma^2 \) [9]. Thus, for boundary length \( 2t \), we have \( \Delta_b = (2t)^{-2} \partial_\sigma^2 \), with eigenspectrum:

\[ \omega_{n_2} = \frac{\pi^2}{L^2} n_2^2, \quad \Psi_{n_2} = \frac{1}{\sqrt{2t}} e^{2\pi i n_2 \sigma^2}, \]  

(128)

where the subscripts take values in the range \(-\infty \leq n_2 \leq \infty\).

Thus, the connected sum over worldsheets with the topology of an annulus with boundaries mapped onto spatially separated macroscopic loops, \( C_i, C_f \), of common length \( L \) takes the form
\[ A_{i,f} = \left[ L^{-1} (4\pi^2 \alpha')^{1/2} \right] \int_0^\infty \frac{dt}{2t} \cdot (2t)^{1/2} \cdot \eta(it)^{-24} e^{-2t^2/2\pi \alpha'} . \] 

The only change in the measure for moduli is the additional factor of \((2t)^{1/2}\) contributed by the functional determinant of \(\mathcal{J}\). The pre-factor in square brackets is of interest; recall that there is no spacetime volume dependence in this amplitude since we have broken translational invariance in all 26 directions of the embedding spacetime. If we were only interested in the point-like off-shell closed string propagator, as in [9], the result as derived is correct without any need for a pre-factor.\(^9\) However, we have \textit{required} that the boundaries of the annulus are mapped to loops in the embedding spacetime of an, a priori, fixed length \(L\). Since a translation of the boundaries in the direction of spacetime parallel to the loops is equivalent to a boundary diffeomorphism, we must divide by the (dimensionless) factor: \(L(4\pi^2 \alpha')^{-1/2}\). This accounts for the pre-factor present in our final result. Note that for more complicated loop geometries, including the possibility of loops with corners, the pre-factor in this expression will take a more complicated form.

References

[1] J. Polchinski, \textit{Evaluation of One-loop String Path Integral}, Comm. Math. Phys. \textbf{104} (1986) 37.

[2] J. Polchinski, \textit{Dirichlet-Branes and Ramond-Ramond Charges}, Phys. Rev. Lett. \textbf{75} (1995) 4724.

[3] J. Polchinski, S. Chaudhuri, and C. Johnson, \textit{Notes on Dbranes}, hep-th/9602052, electronic review paper.

[4] S. W. Hawking, \textit{Quantum Gravity and Path Integrals}, Phys. Rev. \textbf{D18} (1978) 1747; \textit{The Path Integral Approach to Quantum Gravity}, contribution to General Relativity: An Einstein Centenary Survey, (Cambridge) 1979; \textit{Zeta Function Regularization of Path Integrals}, Comm. Math. Phys. \textbf{55} (1977) 133.

[5] A. M. Polyakov, \textit{Quantum Geometry of Bosonic Strings}, Phys. Lett. \textbf{B103} (1982) 207.

[6] D. Friedan, \textit{Introduction to Polyakov’s String Theory}, 1982 Les Houches Summer School Lectures, 1982:0839 (QCD175:G7:1982).

[7] J. A. Shapiro, \textit{Loop Graph in the Dual Tube Model}, Phys. Rev. \textbf{D5} (1972) 1945. Reviewed in J. A. Schwarz, Phys. Rep. \textbf{89} (1982) 223.

[8] O. Alvarez, \textit{Theory of Strings with Boundaries: Fluctuations, Topology and Quantum Geometry}, Nucl. Phys. \textbf{B216} (1983) 125.

[9] A. Cohen, G. Moore, P. Nelson, and J. Polchinski, \textit{An Off-shell Propagator for String Theory}, Nucl. Phys. \textbf{B267} (1986) 143.

\(^9\)Comparing with the final expression for the off-shell point-like propagator given in Eq. (4.5) of [9], and letting \(t\to2\lambda\) in order to match with the notation in [9], the reader should ignore an extraneous factor of \(\lambda^{-13}\), which should clearly be absent in an all-Dirichlet string amplitude.
[10] E. D'Hoker and D. H. Phong, *Geometry of String Perturbation Theory*, Rev. Mod. Phys. **60** (1988) 917.

[11] M. Green and J. Schwarz, *Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory*, Phys. Lett. **B149** (1984) 117; *Infinity Cancellations in SO(32) Superstring Theory*, **B151** (1985) 21.

[12] M. Douglas and B. Grinstein, *Dilaton Tadpole for the Open String*, Phys. Lett. **B183** (1987) 552, Erratum-ibid:187B:442, 1987. J. Dai and J. Polchinski, *The Decay of Macroscopic Fundamental Strings*, Phys. Lett. **B220** (1989) 387. J. Polchinski, *The Phase of the Sum over Spheres*, Phys. Lett. **B219** (1989) 251.

[13] E. Fradkin and A. Tseytlin, Phys. Lett. **B163** (1985) 123. A. Abouesaoood, C. Callan, C. Nappi, and S. Yost, Nucl. Phys. **B280** [FS18] (1987) 599.

[14] C. Callan, C. Lovelace, C. Nappi, and S. Yost, Nucl. Phys. **B308** (1988) 221.

[15] J. Polchinski and Y. Cai, Nucl. Phys. **B296** (1988) 91. Also, N. Ohta, Phys. Rev. Lett. **59** (1987) 176, where the absence of regulator-related ambiguities in SO(32) type I string amplitudes was first clarified.

[16] See R. Savit, *Duality in Field Theory and Statistical Systems*, Rev. Mod. Phys. **52** (1980) 452, for early discussions, and citations, on flux quantization and duality for extended objects in lattice field theories. P. Orland, *Instantons and Disorder in Antisymmetric Tensor Gauge Fields*, Nucl. Phys. **B205**[FS5] (1982) 107. R. Nepomechie, *Magnetic Monopoles from Antisymmetric Tensor Gauge Fields*, Phys. Rev. **D31** (1985) 1921. C. Teitelboim, *Monopoles of Higher Rank*, Phys. Lett. **B167** (1986) 63.

[17] L. Romans, *Massive N=2A Supergravity in Ten Dimensions*, Phys. Lett. **B169** (1986) 374.

[18] J. Dai, R. Leigh, and J. Polchinski, *New Connections between Old String Theories*, Mod. Phys. Lett. **A4** (1989) 2073.

[19] G. Gibbons and K. Maeda, *Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields*, Nucl. Phys. **B298** (1988) 741. A. Dabholkar, G. Gibbons, J. Harvey, and F. Ruiz-Ruiz, *Superstrings and Solitons*, Nucl. Phys. **B340** (1990) 33. G. Horowitz and A. Strominger, *Black Strings and Branes*, Nucl. Phys. **B360** (1991) 197. M Duff and H. Lu, *The Self-dual type IIB superthreebrane*, Phys. Lett. **B273** (1991) 409; *Elementary Fivebrane Solutions of D=10 Supergravity*, Nucl. Phys. **B354** (1991) 141; *Black and Super pbranes in Various Dimensions*, Nucl. Phys. **B416** (1994) 301. P. Townsend, *p-brane Democracy*, hep-th/9507048.

[20] M. Green, *Point-like States for Type IIB Superstrings*, Phys. Lett. **B320** (1994) 435.

[21] E. Witten, *String Theory Dynamics in Various Dimensions*, Nucl. Phys. **B443** (1995) 85.

[22] S. Chaudhuri, G. Hockney, and J. Lykken, *Maximally Supersymmetric Theories in Four Dimensions*, Phys. Rev. Lett. (1995). S. Chaudhuri and J. Polchinski, *Moduli Space of CHL Strings*, Phys. Rev. **D52** (1995) 7168. S. Chaudhuri and D. Lowe, *Type IIA-Heterotic Duals*
with Maximal Supersymmetry, Nucl. Phys. B459 (1996) 113; Monstrous String Duality, Nucl. Phys. B469 (1996) 21.

[23] J. Polchinski and A. Strominger, New Vacua for Type II String Theory, Phys. Lett. B388 (1996).

[24] J. Polchinski and E. Witten, Evidence for Heterotic-Type I String Duality, Nucl. Phys. B460 (1996) 525.

[25] P. Horava and E. Witten, Heterotic and Type I String Dynamics from Eleven Dimensions, Nucl. Phys. B460 (1996) 506.

[26] M. Li, Boundary States of Dbranes and DyStrings, Nucl. Phys. B460 (1996) 351.

[27] M. Green, J. Harvey, and G. Moore, I-brane Inflow and Anomalous Couplings on Dbranes, Class. Quant. Grav. 14 (1997) 47.

[28] S. Shenker, Another Length Scale for String Theory?, hep-th/9509132.

[29] C. P. Bachas and M. Porratti, Phys. Lett. B296 (1992) 77. C. P. Bachas, Dbrane Dynamics, Phys. Lett. B374 (1996) 37.

[30] E. Gimon and J. Polchinski, Consistency Conditions for Orientifolds and D-Manifolds, Phys. Rev. D54 (1996) 1667.

[31] M. Berkooz, M. Douglas, and R. Leigh, Branes Intersecting at Angles, Nucl. Phys. B480 (1996) 265.

[32] C. Bachas and M. Green, On the Anomalous Creation of Branes, hep-th/9705074. G. Ferretti, and I. Klebanov, Creation of Fundamental Strings by Crossing Dbranes, Phys. Rev. Lett. 79 (1997) 1984. O. Bergmann, M. Gaberdiel, and G. Lifschytz, Branes, Orientifolds, and the Creation of Elementary Strings, Nucl. Phys. B509 (1998) 194; String Creation and Heterotic-Type I Duality, Nucl. Phys. B524 (1998) 524. The distinction between string creation in brane-antibrane and brane-brane crossings is discussed in T. Kitao, N. Ohta, and J.-Ge Zhao, Phys. Lett. B428 (1998) 68.

[33] J. Polchinski, String Theory, Volumes I & II (Cambridge) 1998.

[34] S. Chaudhuri, Path Integral Evaluation of Dbrane Amplitudes, Phys. Rev. D60 (1999) 106007; Ultraviolet Limit of Open String Theory, JHEP 9908:003 (1999).

[35] S. Chaudhuri, Y. Chen, and E. Novak, Pair Correlation Function of Wilson Loops, Phys. Rev. D62 (2000) 026004. S. Chaudhuri and E. Novak, Supersymmetric Pair Correlation Function of Wilson Loops, Phys. Rev. D62 (2000) 046002.

[36] S. Chaudhuri and E. Novak, Effective String Tension and Renormalizability: String Theory in a Noncommutative Space, JHEP 0008:027 (2000).

[37] S. Chaudhuri, Confinement and the Short Type I Flux Tube, Nucl. Phys. B591 (2000) 243; hep-th/0007056.
[38] S. Chaudhuri, *Deconfinement and the Hagedorn Transition in String Theory*, Phys. Rev. Lett. 86 (2001) 1943; Err. 87 (2001) 199901; *Finite Temperature Bosonic Closed Strings: Thermal Duality and the Kosterlitz-Thouless Transition*, Phys. Rev. D65 (2002) 066008.

[39] S. Chaudhuri, *Decompactification and the g-theorem*, hep-th/0408206; *Thermal Duality Transformations and the Canonical Ensemble: The Deconfining Long String Phase Transition*, hep-th/0409301; *Microcanonical Ensemble of Type I Strings*, hep-th/0502141.

[40] I. Schnakenburg and P. West, *Massive IIA Supergravity as a Nonlinear Realization*, Phys. Lett. 540 (2002) 137; hep-th/0107181.

[41] S. Chaudhuri, *Electric–Magnetic Duality and the Brane Spectrum of M Theory*, hep-th/0409033.