HARDY-LITTLEWOOD MAXIMAL OPERATOR ON THE
ASSOCIATE SPACE OF A BANACH FUNCTION SPACE

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Abstract. Let $E(X, d, \mu)$ be a Banach function space over a space of homogeneous type $(X, d, \mu)$. We show that if the Hardy-Littlewood maximal operator $M$ is bounded on the space $E(X, d, \mu)$, then its boundedness on the associate space $E'(X, d, \mu)$ is equivalent to a certain condition $A_{\infty}$. This result extends a theorem by Andrei Lerner from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

1. Introduction.

We begin with the definition of a space of homogeneous type (see, e.g., [4]). Given a set $X$ and a function $d : X \times X \to [0, \infty)$, one says that $(X, d)$ is a quasi-metric space if the following axioms hold:

(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) for all $x, y, z \in X$ and some constant $\kappa \geq 1$,
\[
\kappa d(x, y) \leq \kappa d(x, y) + d(y, z). \tag{1.1}
\]

For $x \in X$ and $r > 0$, consider the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ centered at $x$ of radius $r$. Given a quasi-metric space $(X, d)$ and a positive measure $\mu$ that is defined on the $\sigma$-algebra generated by quasi-metric balls, one says that $(X, d, \mu)$ is a space of homogeneous type if there exists a constant $C_\mu \geq 1$ such that for any $x \in X$ and any $r > 0$,
\[
\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)). \tag{1.2}
\]

To avoid trivial measures, we will always assume that $0 < \mu(B) < \infty$ for every ball $B$. Consequently, $\mu$ is a $\sigma$-finite measure.

Given a complex-valued function $f \in L^1_{\text{loc}}(X, d, \mu)$, we define its Hardy-Littlewood maximal function $Mf$ by
\[
(Mf)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(x)| \, d\mu(x), \quad x \in X,
\]
where the supremum is taken over all balls $B \subset X$ containing $x \in X$. The Hardy-Littlewood maximal operator $M$ is a sublinear operator acting by the rule $f \mapsto Mf$.

The aim of this paper is the studying the relations between the boundedness of the

2010 Mathematics Subject Classification. Primary 43A85, Secondary 46E30.

Key words and phrases. Hardy-Littlewood maximal operator, Banach function space, associate space, space of homogeneous type, dyadic cubes.

This work was partially supported by the Fundação para a Ciência e Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).
operator $M$ on a so-called Banach function space $E(X, d, \mu)$ and on its associate space $E'(X, d, \mu)$ in the setting of general spaces of homogeneous type $(X, d, \mu)$.

Let us recall the definition of a Banach function space (see, e.g., [3, Chap. 1, Definition 1.1]). Let $L^0(X, d, \mu)$ denote the set of all complex-valued measurable functions on $X$ and let $L^0_+(X, d, \mu)$ be the set of all non-negative measurable functions on $X$. The characteristic function of a set $E \subset X$ is denoted by $\chi_E$. A mapping $\rho : L^0_+(X, d, \mu) \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+(X, d, \mu)$ with $n \in \mathbb{N}$, for all constants $a \geq 0$, and for all measurable subsets $E$ of $X$, the following properties hold:

\begin{align*}
(A1) & \quad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \quad \rho(af) = a\rho(f), \quad \rho(f + g) \leq \rho(f) + \rho(g), \\
(A2) & \quad 0 \leq g \leq f \text{ a.e.} \Rightarrow \rho(g) \leq \rho(f) \text{ (the lattice property)}, \\
(A3) & \quad 0 \leq f_n \uparrow f \text{ a.e.} \Rightarrow \rho(f_n) \uparrow \rho(f) \text{ (the Fatou property)}, \\
(A4) & \quad \mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty, \\
(A5) & \quad \int_E f(x) \, d\mu(x) \leq C_E \rho(f)
\end{align*}

with a constant $C_E \in (0, \infty)$ that may depend on $E$ and $\rho$, but is independent of $f$. When functions differing only on a set of measure zero are identified, the set $E(X, d, \mu)$ of all functions $f \in L^0_+(X, d, \mu)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in E(X, d, \mu)$, the norm of $f$ is defined by

$$\|f\|_E := \rho(|f|).$$

The set $E(X, d, \mu)$ under the natural linear space operations and under this norm becomes a Banach space (see [3, Chap. 1, Theorems 1.4 and 1.6]). If $\rho$ is a Banach function norm, its associate norm $\rho'$ is defined on $L^0_+(X, d, \mu)$ by

$$\rho'(g) := \sup \left\{ \int_X f(x)g(x) \, d\mu(x) : f \in L^0_+(X, d, \mu), \, \rho(f) \leq 1 \right\}.$$ 

It is a Banach function norm itself [3, Chap. 1, Theorem 2.2]. The Banach function space $E'(X, d, \mu)$ determined by the Banach function norm $\rho'$ is called the associate space (Köthe dual) of $E(X, d, \mu)$.

Hytönen and Kaaremaa [10], developing further ideas of Christ [4], show that a space of homogeneous type $(X, d, \mu)$ can be equipped with a finite system of adjacent dyadic grids $\{D^t : t = 1, \ldots, K\}$, each of which consists of sets $Q$, called dyadic cubes, that resemble properties of usual dyadic cubes in $\mathbb{R}^n$. We postpone precise formulations of these results until Section 2.

Given a dyadic grid $D \in \bigcup_{t=1}^K D^t$, a sparse family $S \subset D$ is a collection of dyadic cubes $Q \in D$ for which there exists a collection of sets $\{E(Q)\}_{Q \in S}$ such that the sets $E(Q)$ are pairwise disjoint, $E(Q) \subset Q$, and

$$\mu(Q) \leq 2\mu(E(Q)).$$

Definition 1 (The condition $A_\infty$). Following [11], we say that a Banach function space $E(X, d, \mu)$ over a space of homogeneous type $(X, d, \mu)$ satisfies the condition $A_\infty$ if there exist constants $C, \gamma > 0$ such that for every dyadic grid $D \in \bigcup_{t=1}^K D^t$, every finite sparse family $S \subset D$, every collection of non-negative numbers $\{\alpha_Q\}_{Q \in S}$, and every collection of pairwise disjoint measurable sets $\{G_Q\}_{Q \in S}$
such that $G_Q \subset Q$, one has

$$\left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_E \leq C \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^\gamma \left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_E.$$  \hspace{1cm} (1.3)

The main aim of the present paper is to provide a self-contained proof of the following generalization of [11, Theorem 3.1] from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

**Theorem 2** (Main result). Let $E(X, d, \mu)$ be a Banach function space over a space of homogeneous type $(X, d, \mu)$ and let $E'(X, d, \mu)$ be its associate space.

(a) If the Hardy-Littlewood maximal operator $M$ is bounded on the space $E'(X, d, \mu)$, then the space $E(X, d, \mu)$ satisfies the condition $A_\infty$.

(b) If the Hardy-Littlewood maximal operator $M$ is bounded on the space $E(X, d, \mu)$ and the space $E(X, d, \mu)$ satisfies the condition $A_\infty$, then the operator $M$ is bounded on the space $E'(X, d, \mu)$.

The paper is organized as follows. In Section 2, we formulate results of Hytönen and Kairema [10] on a construction of a system of adjacent dyadic grids $\{D'_t : t = 1, \ldots, K\}$ on the underlying space of homogeneous type $(X, d, \mu)$. In Section 3, we prove that if a weight $w$ belongs to the dyadic class $A_D^1$ with $D \in \bigcup_{t=1}^K D'_t$, then it satisfies a reverse Hölder inequality. Section 4 contains a proof of a version of the Calderón-Zygmund decomposition of spaces of homogeneous type. Armed with these auxiliary results, following ideas of Lerner [11, Theorem 3.1], we give a self-contained proof of Theorem 2 in Sections 5 and 6.

We conjecture that reflexive variable Lebesgue spaces (see, e.g., [6, 8, 9, 11] for definitions) over spaces of homogeneous type satisfy the condition $A_\infty$. If this conjecture is true, then in view of Theorem 2, we can affirm that the Hardy-Littlewood maximal operator is bounded on a reflexive variable Lebesgue space over a space of homogeneous type if and only if it is bounded on its associate space. Note that in the Euclidean setting of $\mathbb{R}^n$, this result was proved by Diening [9, Theorem 8.1] (see also [11, Theorem 1.1]). We are going to embark on the proof of the above conjecture in a forthcoming paper.

## 2. DYADIC DECOMPOSITION OF SPACES OF HOMOGENEOUS TYPE.

Let $(X, d, \mu)$ be a space of homogeneous type. The doubling property of $\mu$ implies the following geometric doubling property of the quasi-metric $d$: any ball $B(x, r)$ can be covered by at most $N := N(C, \rho)$ balls of radius $r/2$. It is not difficult to show that $N \leq C_{\mu}^{6+3 \log_2 N}$. An important tool for our proofs is the concepts of an adjacent system of dyadic grids $D_t$, $t \in \{1, \ldots, K\}$, on a space of homogeneous type $(X, d, \mu)$. Christ [4, Theorem 11] (see also [5, Chap. VI, Theorem 14]) constructed a system of sets on $(X, d, \mu)$, which satisfy many of the properties of a system of dyadic cubes on the Euclidean space. His construction was further refined by Hytönen and Kairema [10, Theorem 2.2]. We will use the version from [2, Theorem 4.1].

**Theorem 3.** Let $(X, d, \mu)$ be a space of homogeneous type with the constant $\rho \geq 1$ in inequality (1.1) and the geometric doubling constant $N$. Suppose the parameter $\delta \in (0, 1)$ satisfies $96\rho^2\delta \leq 1$. Then there exist an integer number $K = K(\rho, N, \delta)$,
a countable set of points \( \{ z_{\alpha}^{k,t} : \alpha \in A_k \} \) with \( k \in \mathbb{Z} \) and \( t \in \{1, \ldots, K \} \), and a finite number of dyadic grids

\[ D^t := \{ Q_{\alpha}^{k,t} : k \in \mathbb{Z}, \alpha \in A_k \}, \]

such that the following properties are fulfilled:

(a) for every \( t \in \{1, \ldots, K \} \) and \( k \in \mathbb{Z} \) one has

(i) \( X = \bigcup_{\alpha \in A_k} Q_{\alpha}^{k,t} \) (disjoint union);

(ii) if \( Q, P \in D^t \), then \( Q \cap P \in \{ \emptyset, Q, P \} \);

(iii) if \( Q_{\alpha}^{k,t} \in D^t \), then

\[ B(z_{\alpha}^{k,t}, c_1 \delta^k) \subset Q_{\alpha}^{k,t} \subset B(z_{\alpha}^{k,t}, C_1 \delta^k), \]  

where \( c_1 = (12 \lambda^2)^{-1} \) and \( C_1 := 4 \lambda^2 \);

(b) for every \( t \in \{1, \ldots, K \} \) and every \( k \in \mathbb{Z} \), if \( Q_{\alpha}^{k,t} \in D^t \), then there exists at least one \( Q_{\beta}^{k+1,t} \in D^t \), which is called a child of \( Q_{\alpha}^{k,t} \), such that \( Q_{\beta}^{k+1,t} \subset Q_{\alpha}^{k,t} \), and there exists exactly one \( Q_{\gamma}^{k-1,t} \in D^t \), which is called the parent of \( Q_{\alpha}^{k,t} \), such that \( Q_{\gamma}^{k-1,t} \subset Q_{\alpha}^{k,t} \);

(c) for every ball \( B = B(x,r) \), there exists

\[ Q_B \in \bigcup_{t=1}^{K} D^t \]

such that \( B \subset Q_B \) and \( Q_B = Q_{\alpha}^{k-1,t} \) for some indices \( \alpha \in A_k \) and \( t \in \{1, \ldots, K \} \), where \( k \) is the unique integer such that \( \delta^{k+1} < r \leq \delta^k \).

The collections \( D^t \), \( t \in \{1, \ldots, K \} \), are called dyadic grids on \( X \). The sets \( Q_{\alpha}^{k,t} \in D^t \) are referred to as dyadic cubes with center \( z_{\alpha}^{k,t} \) and sidelength \( \delta^k \), see \((2.1)\). The sidelength of a cube \( Q \in D^t \) will be denoted by \( \ell(Q) \). We should emphasize that these sets are not cubes in the standard sense even if the underlying space is \( \mathbb{R}^n \). Parts (a) and (b) of the above theorem describe dyadic grids \( D^t \), with \( t \in \{1, \ldots, K \} \), individually. In particular, \((2.1)\) permits a comparison between a dyadic cube and quasi-metric balls. Part (c) guarantees the existence of a finite family of dyadic grids such that an arbitrary quasi-metric ball is contained in a dyadic cube in one of these grids. Such a finite family of dyadic grids is referred to as an adjacent system of dyadic grids.

Let \( D = \bigcup_{t=1}^{K} D^t \) be a fixed dyadic grid. One can define the dyadic maximal function \( M^D f \) of a function \( f \in L^1_{\text{loc}}(X,d,\mu) \) by

\[ (M^D f)(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_{Q} |f(x)| \, d\mu(x), \quad x \in X, \]

where the supremum is taken over all dyadic cubes \( Q \in D \) containing \( x \).

The following important result is proved by Hytönen and Kairema [10] Proposition 7.9].

**Theorem 4.** Let \((X,d,\mu)\) be a space of homogeneous type and let \( D \bigcup_{t=1}^{K} D^t \) be the adjacent system of dyadic grids given by Theorem \[3\]. There exist a constant \( C_{HK}(X) \geq 1 \) depending only \((X,d,\mu)\) such that for every \( f \in L^1_{\text{loc}}(X,d,\mu) \) and
A measurable non-negative locally integrable function $w$ on $X$ is said to be a weight. Given a weight $w$ and a measurable set $E \subset X$, denote
$$w(E) := \int_E w(x) \, d\mu(x).$$

Fix a dyadic grid $\mathcal{D} \in \bigcup_{k=1}^{K} \mathcal{D}^k$. A weight $w : X \to [0, \infty]$ is said to belong to the dyadic class $A^D_\infty$ if there exists a constant $c > 0$ such that for a.e. $x \in X$,
$$(M^D w)(x) \leq cw(x).$$

The smallest constant $c$ in this inequality is denoted by $[w]_{A^D_\infty}$.

Following [2, Definition 4.4], a generalized dyadic parent (gdp) of a cube $Q$ is any cube $Q^*$ such that $\ell(Q^*) = \frac{1}{2}\ell(Q)$ and for every $Q' \in \mathcal{D}$ such that $Q' \cap Q \neq \emptyset$ and $\ell(Q') = \ell(Q)$, one has $Q' \subset Q^*$. According to [2, Lemma 4.5], every cube $Q \in \mathcal{D}$ possesses at least one gdp.

For every $x \in X$ and $Q \in \mathcal{D}$, put
$$Q_Q := \{Q' \in \mathcal{D} : Q' \cap Q \neq \emptyset, \ell(Q') \leq \ell(Q)\}, \quad Q^*_Q := \{Q' \in Q_Q : x \in Q'\}.$$

It follows immediately that if $Q' \in Q_Q$, then $Q' \subset Q^*$. For every $Q \in \mathcal{D}$, the localized dyadic maximal operator $M_Q$ is defined by
$$(M_Qf)(x) = \begin{cases} \sup_{Q' \in Q_Q^*} \frac{1}{\mu(Q')} \int_{Q'} |f(y)| \, d\mu(y) & \text{if } Q^*_Q \neq \emptyset, \\ 0 & \text{if } Q^*_Q = \emptyset. \end{cases}$$

Following [2, Definition 4.7], one says that a weight $w : X \to [0, \infty]$ belongs to the dyadic class $A^D_\infty$ if
$$[w]_{A^D_\infty} := \sup_{Q \in \mathcal{D}} \inf_{Q^* \in Q_Q^*} \frac{1}{w(Q^*)} \int_X (M_Qw)(x) \, d\mu(x) < \infty.$$

Let $C_D \geq 1$ be a constant such that for all cubes $Q \in \mathcal{D}$ and $Q' \in Q_Q$ satisfying $\ell(Q) = \ell(Q')$, one has
$$\mu(Q^*) \leq C_D \mu(Q'). \quad (3.1)$$

**Lemma 5.** If $w \in A^D_{\infty}$, then $w \in A^D_{\infty}$ and $[w]_{A^D_{\infty}} \leq [w]_{A^D_{\infty}}$.

**Proof.** Fix a cube $Q \in \mathcal{D}$ and one of its gdp’s $Q^*$. It follows immediately from the definition of $Q_Q$ that if $Q' \in Q_Q$, then $Q' \subset Q^*$. Take any $x \in X$. If $Q^*_Q \neq \emptyset$, then there exists $Q' \in Q^*_Q$ such that $x \in Q' \subset Q^*$. Therefore, if $x \notin Q^*$, then
$$(M_Qw)(x) = (M_Qw)(x) \chi_{Q^*}(x) \leq (M^D w)(x) \chi_{Q^*}(x) \leq [w]_{A^D} w(x) \chi_{Q^*}(x),$$

and if $x \in Q^*$, then
$$(M_Qw)(x) \chi_{Q^*}(x) = (M^D w)(x) \chi_{Q^*}(x) \leq \frac{1}{w(Q^*)} \int_{Q^*} (M^D w)(x) \, d\mu(x) \leq [w]_{A^D} w(x) \chi_{Q^*}(x).$$

Hence, $[w]_{A^D_{\infty}} \leq [w]_{A^D_{\infty}}$.

Moreover, if $w \in A^D_{\infty}$, then $w \in A^D_{\infty}$ and $[w]_{A^D_{\infty}} \leq [w]_{A^D_{\infty}}$. Thus,
$$(M_Qw)(x) \leq [w]_{A^D} w(x) \chi_{Q^*}(x),$$

and if $x \in Q^*$, then
$$(M_Qw)(x) = (M_Qw)(x) \chi_{Q^*}(x) \leq (M^D w)(x) \chi_{Q^*}(x) \leq [w]_{A^D} w(x) \chi_{Q^*}(x).$$

Hence, $[w]_{A^D_{\infty}} \leq [w]_{A^D_{\infty}}$. 

Thus, for a.e. $x \in Q$,
whence

\[
[w]_{A^\infty_D} = \sup_{Q \in D} \inf_{Q^*} \frac{1}{w(Q^*)} \int_X (M_Q w)(x) \, d\mu(x)
\leq [w]_{A^\infty} \sup_{Q \in D} \inf_{Q^*} \frac{1}{w(Q^*)} \int_{Q^*} w(x) \, d\mu(x) = [w]_{A^\infty},
\]

which completes the proof. \(\square\)

The following result is an easy consequence of the weak reverse Hölder inequality for weights in \(A^\infty_D\) obtained recently by Anderson, Hytönen, and Tapiola [2, Theorem 5.4].

**Lemma 6.** Let \(K\) be the constant from Theorem 3 and \(C_D\) be the constant defined in (3.1). If \(w \in A^1_D\), then for every \(\eta\) satisfying

\[
0 < \eta \leq \frac{1}{2C^2_D K [w]_{A^1_D}} \tag{3.2}
\]

and every \(Q \in D\), one has

\[
\left( \frac{1}{2\mu(Q)} \int_Q w^{1+\eta}(x) \, d\mu(x) \right)^{\frac{1+\eta}{\eta}} \leq C_D [w]_{A^1_D} \frac{1}{\mu(Q)} \int_Q w(x) \, d\mu(x). \tag{3.3}
\]

**Proof.** We know from Lemma 5 that \(w \in A^\infty_D\) and \([w]_{A^\infty_D} \leq [w]_{A^\infty}.\) Then, by [2, Theorem 5.4], for every \(\eta\) satisfying (3.2), one has

\[
\left( \frac{1}{2\mu(Q)} \int_Q w^{1+\eta}(x) \, d\mu(x) \right)^{\frac{1+\eta}{\eta}} \leq C_D \frac{1}{\mu(Q^*)} \int_{Q^*} w(x) \, d\mu(x). \tag{3.4}
\]

Since \(w \in A^1_D\), for a.e. \(x \in Q \subset Q^*\), one has

\[
\frac{1}{\mu(Q^*)} \int_{Q^*} w(y) \, d\mu(y) \leq (M^D w)(x) \leq [w]_{A^1_D} w(x).
\]

Integrating this inequality over \(Q\), we obtain

\[
\frac{\mu(Q)}{\mu(Q^*)} \int_{Q^*} w(y) \, d\mu(y) \leq [w]_{A^1_D} \int_Q w(x) \, d\mu(x). \tag{3.5}
\]

Combining inequalities (3.4) and (3.5), we immediately arrive at inequality (3.3). \(\square\)

The main result of this section is the following reverse Hölder inequality.

**Theorem 7.** Let \(K\) be the constant from Theorem 3 and \(C_D\) be the constant defined in (3.1). If \(w \in A^1_D\), then for every \(\eta\) satisfying (3.2), every cube \(Q \in D\), and every measurable subset \(E \subset Q\), one has

\[
\frac{w(E)}{w(Q)} \leq 2^{1+\eta} C_D [w]_{A^1_D} \left( \frac{\mu(E)}{\mu(Q)} \right)^{\frac{\eta}{1+\eta}}. \tag{3.6}
\]
Proof. By Hölder’s inequality and reverse Hölder’s inequality \[ \|w(E)\|_{L^{p+\eta}(\mu)} \leq \left( \int_{Q} w^{1+\eta}(x) \, d\mu(x) \right)^{1\over 1+\eta} \left( \mu(E) \right)^{\eta\over 1+\eta} \]

\begin{align*}
\leq & \left( \int_{Q} w^{1+\eta}(x) \, d\mu(x) \right)^{1\over 1+\eta} \left( \mu(E) \right)^{\eta\over 1+\eta} \\
& \leq 2^{1\over 1+\eta} C_{\mathcal{D}}[w]_{A_{\mathcal{F}}} \left( \frac{\mu(Q)}{\mu(Q)} \right)^{1\over 1+\eta} w(Q) \left( \frac{\mu(E)}{\mu(Q)} \right)^{1\over 1+\eta},
\end{align*}

which immediately implies (3.6). \(\square\)

4. Calderón-Zygmund decomposition.

We start this section with the following important observation.

Lemma 8. Suppose \((X,d,\mu)\) is a space of homogeneous type with the constants \(\kappa \geq 1\) in inequality (1.1) and \(C_{\mu} \geq 1\) in inequality (1.2). Let \((X,d,\mu)\) be equipped with an adjacent system of dyadic grids \(\{\mathcal{D}^t, t = 1, \ldots, K\}\) and let \(\delta \in (0,1)\) be chosen as in Theorem 3. Then there is an \(\epsilon = \epsilon(\kappa, C_{\mu}, \delta) \in (0,1)\) such that for every \(t \in \{1, \ldots, K\}\) and all \(Q, P \in \mathcal{D}^t\), if \(Q\) is a child of \(P\), then

\[ \mu(Q) \geq \epsilon \mu(P). \]

This result is certainly known. For the construction of Christ, we refer to [5] Chap. VI, Theorem 14, where it is stated without proof (see also [4] Theorem 11, where it is implicit). In [11] Theorem 2.1 and [8] Theorem 2.5 it is stated without proof and attributed to Hytönen and Kairema [10], although it is only implicit in the latter paper. For the convenience of the readers, we provide its proof.

Proof. Let \(Q = Q_{\alpha}^{k,t}\) be a child of \(P = Q_{\beta}^{k,t}\) for some \(t \in \{1, \ldots, K\}\), \(k \in \mathbb{Z}\), and \(\alpha \in A_{k}, \beta \in A_{k+1}\). It follows from Theorem 3(a), part (iii), that \(P \subset B(z_{\alpha}^{k,t}, 4\kappa^{2}\delta^{k})\) and \(B(z_{\beta}^{k+1,t}, (12\kappa^{4})^{-1}\delta^{k+1}) \subset Q\), whence

\[ \mu(P) \leq \mu(B(z_{\alpha}^{k,t}, 4\kappa^{2}\delta^{k})) \quad \mu(B(z_{\beta}^{k+1,t}, (12\kappa^{4})^{-1}\delta^{k+1})) \leq \mu(Q). \]

(4.2)

It follows from [10] Lemma 2.10 with \(C_{0} = 2\kappa\) (cf. [10] Lemma 4.10) that if \(Q_{\beta}^{k+1,t}\) is a child of \(Q_{\alpha}^{k,t}\), then

\[ d(z_{\alpha}^{k,t}, z_{\beta}^{k+1,t}) < 2\kappa \delta^{k}. \]

(4.3)

If \(x \in B(z_{\alpha}^{k,t}, 4\kappa^{2}\delta^{k})\), then

\[ d(x, z_{\alpha}^{k,t}) < 4\kappa^{2}\delta^{k}. \]

(4.4)

Combining (4.1) with (4.3)–(4.4), we get

\[ d(x, z_{\beta}^{k+1,t}) \leq \kappa (d(x, z_{\alpha}^{k,t}) + d(z_{\beta}^{k+1,t}, z_{\alpha}^{k,t})) < \kappa (4\kappa^{2}\delta^{k} + 2\kappa \delta^{k}) = \kappa^{2}(4\kappa + 2)\delta^{k}, \]

whence \(x \in B(z_{\beta}^{k+1,t}, \kappa^{2}(4\kappa + 2)\delta^{k})\). Therefore

\[ B(z_{\alpha}^{k,t}, 4\kappa^{2}\delta^{k}) \subset B(z_{\beta}^{k+1,t}, \kappa^{2}(4\kappa + 2)\delta^{k}). \]
This inclusion immediately implies that
\[ \mu(B(z^k, t, \kappa^2 \delta^k)) \leq \mu(B(z^{k+1}, t, 2^k(4\kappa^2 + 2)^k)). \] (4.5)

Let \( s \) be the smallest natural number satisfying
\[ \log_2(12\kappa^6(4\kappa^2 + 2)\delta^{-1}) \leq s. \]

Then \( \kappa^2(4\kappa^2 + 2)\delta^k \leq 2^s(12\kappa^4)^{-1}\delta^{k+1} \) and, therefore,
\[ \mu(B(z^{k+1, t}, 2^s(4\kappa^2 + 2)^k)) \leq \mu(B(z^{k+1, t}, 2^s(12\kappa^4)^{-1}\delta^{k+1})). \] (4.6)

Applying inequality (1.2) \( s \) times, one gets
\[ \mu(B(z^{k+1, t}, 2^s(12\kappa^4)^{-1}\delta^{k+1})) \leq C_s^\mu \mu(B(z^{k+1, t}, (12\kappa^4)^{-1}\delta^{k+1})). \] (4.7)

Combining inequalities (4.2) with (4.5)–(4.7), we arrive at
\[ \mu(P) \leq \mu(B(z^{k+1, t}, \kappa^2(4\kappa^2 + 2)\delta^k)) \leq C_s^\mu \mu(B(z^{k+1, t}, (12\kappa^4)^{-1}\delta^{k+1})) \leq C_s^\mu \mu(Q), \]
which implies inequality (4.1) with \( \varepsilon = C_s^\mu \). \( \square \)

Once Lemma 8 is available, one can prove the following version of the Calderón-Zygmund decomposition for spaces of homogeneous type.

**Theorem 9.** Let \((X, d, \mu)\) be a space of homogeneous type and \(D \in \bigcup_{t=1}^K D_t\) be a dyadic grid. Suppose that \( \varepsilon \in (0, 1) \) is the same as in Lemma 8 and \( f \in L^1(X, d, \mu) \).

(a) If
\[ \lambda > \begin{cases} \frac{1}{\mu(X)} \int_X |f(x)| \, d\mu(x) & \text{if } \mu(X) < \infty, \\ 0 & \text{if } \mu(X) = \infty, \end{cases} \]
and the set
\[ \Omega_\lambda := \{ x \in X : (M^D f)(x) > \lambda \} \]
is nonempty, then there exists a collection \( \{Q_j\} \subset D \) that is pairwise disjoint, maximal with respect to inclusion, and such that
\[ \Omega_\lambda = \bigcup_j Q_j. \] (4.8)

Moreover, for every \( j \),
\[ \lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)| \, d\mu(x) \leq \frac{\lambda}{\varepsilon}. \] (4.9)

(b) Let \( a > 2/\varepsilon \) and, for \( k \in \mathbb{Z} \) satisfying
\[ a^k > \begin{cases} 0 & \text{if } \mu(X) = \infty, \\ \frac{1}{\mu(X)} \int_X |f(x)| \, d\mu(x) & \text{if } \mu(X) < \infty, \end{cases} \] (4.10)
let
\[ \Omega_k := \{ x \in X : (M^D f)(x) > a^k \}. \] (4.11)
If $\Omega_k \neq \emptyset$, then there exists a collection $\{Q^k_j\}_{j \in J_k}$ (as in part (a)) such that it is pairwise disjoint, maximal with respect to inclusion, and

$$\Omega_k = \bigcup_{j \in J_k} Q^k_j. \quad (4.12)$$

The collection of cubes

$$S = \{Q^k_j : \Omega_k \neq \emptyset, j \in J_k\}$$

is sparse, and for all $j$ and $k$, the sets

$$E(Q^k_j) := Q^k_j \setminus \Omega_{k+1}$$

satisfy

$$\mu(Q^k_j) \leq 2\mu(E(Q^k_j)). \quad (4.13)$$

Proof. The proof is analogous to the proof of [7, Proposition A.1]. For the convenience of the reader, we provide the proof in the case of $\mu(X) = \infty$. For $\mu(X) < \infty$, the proof is similar.

(a) Let $\Lambda_\lambda$ be the family of dyadic cubes $Q \in D$ such that

$$\lambda < \frac{1}{\mu(Q)} \int_Q |f(x)| \, d\mu(x). \quad (4.14)$$

Notice that $\Lambda_\lambda$ is nonempty because $\Omega_\lambda \neq \emptyset$. For each $Q \in \Lambda_\lambda$ there exists a maximal cube $Q' \in \Lambda_\lambda$ with $Q \subset Q'$, since

$$0 \leq \frac{1}{\mu(Q)} \int_Q |f(x)| \, d\mu(x) \leq \frac{1}{\mu(Q)} \int_X |f(x)| \, d\mu(x) \to 0 \quad \text{as} \quad \mu(Q) \to \infty.$$

Let $\{Q_j\}_k \subset \Lambda_\lambda$ denote the family of such maximal cubes. By the maximality, the cubes in $\{Q_j\}_k$ are pairwise disjoint. If $\tilde{Q}_j$ is the dyadic parent of $Q_j$, then $Q_j \subset \tilde{Q}_j$ and $\tilde{Q}_j$ does not belong to $\{Q_j\}_k$ in view of the maximality of the cubes in $\{Q_j\}_k$. Hence, taking into account Lemma 8, we see that

$$\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)| \, d\mu(x) \leq \frac{1}{\varepsilon \mu(Q_j)} \int_{\tilde{Q}_j} |f(x)| \, d\mu(x) \leq \frac{\lambda}{\varepsilon},$$

which completes the proof of (4.9).

If $x \in \Omega_\lambda$, then it follows from the definition of $M^D f$ that there exists a cube $Q \in D$ such that $x \in Q$ and (4.14) is fulfilled. Hence $Q \subset Q_j$ for some $j$. Therefore, $\Omega_\lambda \subset \bigcup_j Q_j$.

Conversely, since

$$\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)| \, d\mu(x),$$

if $x \in Q_j$, then $(M^D f)(x) > \lambda$. This means that $x \in \Omega_\lambda$. Therefore, $\bigcup_j Q_j \subset \Omega_\lambda$. Thus (4.8) holds. Part (a) is proved.

(b) Equality (4.12) follows from part (a). Since $\Omega_{k+1} \subset \Omega_k$ and for each fixed $k$, the cubes $Q^k_j$ are pairwise disjoint, it is clear that the sets $E(Q^k_j)$ are pairwise disjoint for all $j$ and $k$. If $Q^k_j \cap Q^{k+1}_i \neq \emptyset$, then by the maximality of the cubes in
and the hypothesis \( a > 2/\varepsilon \), we have \( Q_{i}^{k+1} \subseteq Q_{j}^{k} \). In view of part (a),

\[
\mu(Q_{j}^{k} \cap \Omega_{k+1}) = \sum_{\{i: Q_{i}^{k+1} \subseteq Q_{j}^{k}\}} \mu(Q_{i}^{k+1}) \leq \sum_{\{i: Q_{i}^{k+1} \subseteq Q_{j}^{k}\}} \frac{1}{a^{k+1}} \int_{Q_{i}^{k+1}} |f(x)| \, d\mu(x) \leq \frac{1}{a^{k+1}} \int_{Q_{j}^{k}} |f(x)| \, d\mu(x) \leq \frac{1}{a^{k+1}} \cdot \frac{a^{k} \mu(Q_{j}^{k})}{\varepsilon} = \frac{\mu(Q_{j}^{k})}{a \varepsilon}.
\]

Then

\[
\mu(E(Q_{j}^{k})) = \mu(Q_{j}^{k} \setminus \Omega_{k+1}) = \mu(Q_{j}^{k}) - \mu(Q_{j}^{k} \cap \Omega_{k+1}) \geq \left( 1 - \frac{1}{a \varepsilon} \right) \mu(Q_{j}^{k}) \geq \left( 1 - \frac{1}{2} \right) \mu(Q_{j}^{k}),
\]

whence \( \mu(Q_{j}^{k}) \leq 2 \mu(E(Q_{j}^{k})) \) for all \( j \) and \( k \), which completes the proof of (4.15). \( \square \)

**Corollary 10.** Let \((X, d, \mu)\) be a space of homogeneous type and \(D \in \bigcup_{t=1}^{K} \mathcal{D}_t\) be a dyadic grid on \(X\). Suppose \( \varepsilon \in (0, 1) \) is the same as in Lemma 8 and \( a > 2/\varepsilon \). For a non-negative function \( f \in L^{1}(X, d, \mu) \) and \( k \in \mathbb{Z} \) satisfying (4.10), let the sets \( \Omega_{k} \) be given by (4.11). For all \( \ell \in \mathbb{Z}_{+} \) and all \( j \) such that \( Q_{j}^{k} \subset \Omega_{k} \),

\[
\mu(Q_{j}^{k} \cap \Omega_{k+\ell}) \leq \frac{\mu(Q_{j}^{k})}{a^{\ell} \varepsilon}.
\]
(depending on \( f \)) such that for all \( x \in X \),
\[
(M^D f)(x) \leq a \sum_{Q \in S} \left( \frac{1}{\mu(Q)} \int_Q f(y) \, d\mu(y) \right) \chi_{E(Q)}(x).
\]

**Proof.** The proof is, actually, contained in the proof of [1, Theorem 3.1, p. 30]. We reproduce it here for completeness.

Let \( K \) denote the set of all \( k \in \mathbb{Z} \) satisfying (4.10). Then
\[
X = \bigcup_{k \in K} \Omega_k \setminus \Omega_{k+1}.
\]

Let \( S \) be the sparse family given by Theorem 9(b). For \( k \in K \) and a given \( x \in \Omega_k \setminus \Omega_{k+1} \), there exists a cube \( Q^k_j \in S \) such that \( x \in Q^k_j \setminus \Omega_{k+1} \) and
\[
(M^D f)(x) \leq a^{k+1} \mu(Q^k_j) \int_{Q^k_j} f(y) \, d\mu(y).
\]

Taking into account that by Theorem 9(b),
\[
\Omega_k \setminus \Omega_{k+1} = \bigcup_{j \in J_k} Q^k_j \setminus \Omega_{k+1} = \bigcup_{j \in J_k} E(Q^k_j),
\]
we obtain from (4.16)–(4.18) for all \( x \in X \),
\[
(M^D f)(x) = \sum_{k \in K} (M^D f)(x) \chi_{\Omega_k \setminus \Omega_{k+1}}(x)
\]
\[
\leq \sum_{k \in K} \sum_{j \in J_k} \left( \frac{a}{\mu(Q^k_j)} \int_{Q^k_j} f(y) \, d\mu(y) \right) \chi_{E(Q^k_j)}(x)
\]
\[
= a \sum_{Q \in S} \left( \frac{1}{\mu(Q)} \int_Q f(y) \, d\mu(y) \right) \chi_{E(Q)}(x),
\]
which completes the proof. \( \square \)

5. **Proof of part (a) of Theorem 2**

The scheme of the proof is borrowed from the proof of the necessity portion of [11, Theorem 3.1].

For a bounded sublinear operator on a Banach function space \( \mathcal{E}'(X, d, \mu) \), let \( \|T\|_{\mathcal{B}(\mathcal{E}')} \) denote its norm.

Fix \( \mathcal{D} \in \bigcup_{t=1}^K \mathcal{D}' \). It follows from the boundedness of the Hardy-Littlewood maximal operator \( M \) on \( \mathcal{E}'(X, d, \mu) \) in view of Theorem 4 and the lattice property (axiom (A2) in the definition of a Banach function space) that the dyadic maximal operator \( M^D \) is bounded on the space \( \mathcal{E}'(X, d, \mu) \) and
\[
\|M^D\|_{\mathcal{B}(\mathcal{E}')} \leq C_{HK}(X) \|M\|_{\mathcal{B}(\mathcal{E}')}. \tag{5.1}
\]

Let \( g \in L^0(X, d, \mu) \) with \( \|g\|_{\mathcal{E}'} \leq 1 \). Using an idea of Rubio de Francia [12] (see also [11, Section 2.1]), put
\[
(Rg)(x) := \sum_{k=0}^{\infty} \left( \frac{(M^D)^k g(x)}{2^k \|M^D\|_{\mathcal{B}(\mathcal{E}')}} \right)^k, \quad x \in X.
\]
where \((M^k)\) denotes the \(k\)-th iteration of \(M\) and \((M^0)g := g\). Then
\[
\|Rg\|_{E'} \leq 2\quad (5.2)
\]
and
\[g(x) \leq (Rg)(x) \text{ for a.e. } x \in X.\quad (5.3)
\]
Since \(M\) is sublinear, we have
\[
(M^kRg)(x) \leq \sum_{k=0}^{\infty} \frac{((M^k)^{g+1}g)(x)}{(2\|M\|_{B(E')}\)^{k+1}}
\]
\[
= 2\|M\|_{B(E')} \sum_{k=0}^{\infty} \frac{((M^k)^{g}g)(x)}{(2\|M\|_{B(E')}\)^{k}}
\]
\[
\leq 2\|M\|_{B(E')} \sum_{k=0}^{\infty} \frac{((M^k)^{g}g)(x)}{(2\|M\|_{B(E')}\)^{k}}
\]
\[
= 2\|M\|_{B(E')} (Rg)(x),
\]
whence \(Rg \in A^P\) with
\[
[Rg]_{A^P} \leq 2\|M\|_{B(E')}.
\]
Let the constants \(C_{D'} \geq 1\) be defined for each \(t \in \{1, \ldots, K\}\) by (5.1). Take \(\eta\) and \(\gamma\) such that
\[0 < \eta \leq \left(\frac{\max_{1 \leq t \leq K} C_{D'}^2}{K C_{H_K}(X)\|M\|_{B(E')}}\right)^{-1}, \quad \gamma = \frac{\eta}{1 + \eta}.
\]
Inequalities (5.4) and (5.1) imply that
\[
[Rg]_{A^P} \leq 2C_{H_K}(X)\|M\|_{B(E')}.
\]
whence \(\eta\) satisfies (5.2). Since \(Rg \in A^P\), it follows from Theorem 7 and inequality (5.5) that, for every finite sparse family \(S \subset D\), every collection of non-negative numbers \(\{\alpha_Q\}_{Q \in S}\), every collection of pairwise disjoint measurable subsets \(G_Q \subset Q\), and
\[
\int_{G_Q} (Rg)(x) d\mu(x) \leq 2^{1-\gamma}C_{D'} [Rg]_{A^P} \left(\frac{\mu(G_Q)}{\mu(Q)}\right)^{\gamma} \int_{Q} (Rg)(x) d\mu(x)
\]
\[
\leq \frac{C}{2} \left(\frac{\mu(G_Q)}{\mu(Q)}\right)^{\gamma} \int_{Q} (Rg)(x) d\mu(x), \quad (5.6)
\]
where
\[C := 2^{3-\gamma} \left(\frac{\max_{1 \leq t \leq K} C_{D'}}{K C_{H_K}(X)\|M\|_{B(E')}}\right).
\]
Taking into account inequalities (5.3), (5.6), Hölder’s inequality for Banach function spaces (see [3, Chap. 1, Theorem 2.4]), and inequality (5.2), we deduce that, for every finite sparse family \(S \subset D\), every collection of non-negative numbers \(\{\alpha_Q\}_{Q \in S}\), every collection of pairwise disjoint measurable subsets \(G_Q \subset Q\), and
every $g \in L^0_+ (X, d, \mu)$ satisfying $\|g\|_{\mathcal{E}} \leq 1$, one has
\[
\int_X \left( \sum_{Q \in S} \alpha_Q \chi_{G_Q}(x) \right) g(x) \, d\mu(x) \\
\leq \sum_{Q \in S} \alpha_Q \int_{G_Q} g(x) \, d\mu(x) \\
\leq \sum_{Q \in S} \alpha_Q \int_{G_Q} (Rg)(x) \, d\mu(x) \\
\leq \frac{C}{2} \sum_{Q \in S} \alpha_Q \left( \frac{\mu(Q)}{\mu(G_Q)} \right)^{\gamma} \int_Q (Rg)(x) \, d\mu(x) \\
\leq \frac{C}{2} \left( \max_{Q \in S} \frac{\mu(Q)}{\mu(G_Q)} \right)^{\gamma} \int_X \left( \sum_{Q \in S} \alpha_Q \chi_{G_Q}(x) \right) (Rg)(x) \, d\mu(x) \\
\leq \frac{C}{2} \left( \max_{Q \in S} \frac{\mu(Q)}{\mu(G_Q)} \right)^{\gamma} \left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_{\mathcal{E}} \|Rg\|_{\mathcal{E}},
\]

Then, in view of the Lorentz-Luxemburg theorem (see [3, Chap. 1, Theorem 2.7]),
\[
\left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_{\mathcal{E}} = \left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_{\mathcal{E}'} \\
= \sup \left\{ \int_X \left( \sum_{Q \in S} \alpha_Q \chi_{G_Q}(x) \right) g(x) \, d\mu(x) : g \in L^0_+ (X, d, \mu), \|g\|_{\mathcal{E}'} \leq 1 \right\} \\
\leq C \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{\gamma} \left\| \sum_{Q \in S} \alpha_Q \chi_{Q} \right\|_{\mathcal{E}},
\]

that is, the space $\mathcal{E}(X, d, \mu)$ satisfies the condition $\mathcal{A}_\infty$, which completes the proof of part (a) of Theorem [2]. \hfill \Box

6. PROOF OF PART (B) OF THEOREM [2]

We follow the proof of the sufficiency portion of the proof of [11, Theorem 3.1]. Let $\varepsilon \in (0, 1)$ be the same as in Lemma [8]. Take $a > 2/\varepsilon$. Assume that $f \in L^1_+ (X, d, \mu) \cap \mathcal{E}'(X, d, \mu)$ is a nonnegative function and fix any dyadic grid $\mathcal{D} = \bigcup_{k=1}^K \mathcal{D}'$. By Lemma [11] there exists a sparse family $S \subset \mathcal{D}$ (not necessarily finite) such that for all $x \in X$,
\[
(M^p f)(x) \leq a \sum_{Q \in S} \left( \frac{1}{\mu(Q)} \int_Q f(y) \, d\mu(y) \right) \chi_{E(Q)}(x). \quad (6.1)
\]

For every subfamily $S' \subset S$, put
\[
(A_{S'} f)(x) = \sum_{Q \in S'} \left( \frac{1}{\mu(Q)} \int_Q f(y) \, d\mu(y) \right) \chi_{E(Q)}(x).
\]
Let \( \{S_t\}_{t \in \mathbb{N}} \) be a sequence of subfamilies of \( S \) such that each subfamily \( S_t \) is finite, \( S_t \subset S_n \) if \( t < n \), and \( A_S f \uparrow A_S f \) a.e. on \( X \) as \( t \to \infty \). By the Fatou property (axiom (A3) in the definition of a Banach function space),

\[
\lim_{t \to \infty} \|A_S f\|_{E'} = \|A_S f\|_{E'}. \tag{6.2}
\]

By the Fubini theorem, for every \( g \in \mathcal{E}(X, d, \mu) \) and every \( t \in \mathbb{N} \), one has

\[
\int_X (A_S f)(x)g(x) \, d\mu(x)
= \int_X \int_X \sum_{Q \in S_t} \frac{1}{\mu(Q)} f(y) \chi_Q(y) \chi_{\mathcal{E}(Q)}(x) g(x) \, d\mu(y) \, d\mu(x)
= \int_X \int_X \sum_{Q \in S_t} \frac{1}{\mu(Q)} f(x) \chi_Q(x) \chi_{\mathcal{E}(Q)}(y) g(y) \, d\mu(y) \, d\mu(x)
= \int_X \sum_{Q \in S_t} \left( \frac{1}{\mu(Q)} \int_{\mathcal{E}(Q)} g(y) \, d\mu(y) \right) \chi_Q(x) f(x) \, d\mu(x)
= \int_X f(x)(A_{S_t}^* g)(x) \, d\mu(x), \tag{6.3}
\]

where

\[
(A_{S_t}^* g)(x) := \sum_{Q \in S_t} \left( \frac{1}{\mu(Q)} \int_{\mathcal{E}(Q)} g(y) \, d\mu(y) \right) \chi_Q(x), \quad x \in X.
\]

It follows from [6, §2.1] and Hölder’s inequality for Banach function spaces (see [3, Chap. 1, Theorem 2.4]) that

\[
\left| \int_X (A_S f)(x)g(x) \, d\mu(x) \right| \leq \|A_{S_t}^* g\|_{E} \|f\|_{E'}. \tag{6.4}
\]

Let \( C, \gamma > 0 \) be as in Definition [4]. Since \( a > 2/\varepsilon > 2 \), there exists \( \nu \in \mathbb{N} \) such that

\[
C \varepsilon^{-\gamma} \sum_{\ell=\nu}^{\infty} a^{-\ell\gamma} \leq \frac{1}{2}. \tag{6.5}
\]

For \( Q \in S_t \), let

\[
\alpha_Q := \frac{1}{\mu(Q)} \int_{\mathcal{E}(Q)} |g(x)| \, d\mu(x).
\]

Then for all \( x \in X \),

\[
|(A_{S_t}^* g)(x)| \leq \sum_{Q \in S_t} \alpha_Q \chi_Q(x)
= \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k}(x)
= \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k \setminus \Omega_{k,\nu}}(x) + \sum_{\{j,k: Q_j^k \in S_t\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap \Omega_{k,\nu}}(x)
=: \Sigma_1(x) + \Sigma_2(x), \tag{6.6}
\]

where the sets \( \Omega_k \) are defined by [4, (11)] for all \( k \in \mathbb{Z} \) satisfying [4, (10)].
Let \( K \) be the set of all those \( k \in \mathbb{Z} \) that satisfy (4.10). It is easy to see that for \( k \in K \) and \( \nu \in \mathbb{N} \),
\[
\Omega_k \setminus \Omega_{k+\nu} \subset \bigcup_{i=0}^{\nu-1} \Omega_{k+i} \setminus \Omega_{k+i+1}.
\]
(6.7)

It is also easy to see that if \( k \in K \) and \( x \in Q_j^k \), then
\[
\alpha_{Q_j^k} \leq \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} |g(x)| \, d\mu(x) \leq (M^P g)(x).
\]
(6.8)

Combining (6.7) and (6.8), we get for \( x \in X \),
\[
\Sigma_1(x) = \sum_{\{j,k:Q_j^k \in S_i\}} \alpha_{Q_j^k} \chi_{Q_j^k \setminus \Omega_{k+\nu}}(x)
\leq (M^P g)(x) \sum_{k \in K} \sum_{\nu=0}^{\nu-1} \chi_{\Omega_k \setminus \Omega_{k+i+1}}(x)
\leq (M^P g)(x) \sum_{i=0}^{\nu-1} \sum_{k \in K} \chi_{\Omega_k \setminus \Omega_{k+i+1}}(x)
= \nu (M^P g)(x).
\]
(6.9)

On the other hand, for \( x \in X \), we have
\[
\Sigma_2(x) = \sum_{\{j,k:Q_j^k \in S_i\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap \Omega_{k+\nu}}(x)
\leq \sum_{\{j,k:Q_j^k \in S_i\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap (\Omega_{k+i} \setminus \Omega_{k+i+1})}(x)
= \sum_{\ell=\nu}^{\infty} \sum_{\{j,k:Q_j^k \in S_i\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap (\Omega_{k+i} \setminus \Omega_{k+i+1})}(x).
\]
(6.10)

Since \( S_i \) is a finite sparse family, applying inequality (1.3) of Definition 1 we obtain for all \( \ell \geq \nu \),
\[
\left\| \sum_{\{j,k:Q_j^k \in S_i\}} \alpha_{Q_j^k} \chi_{Q_j^k \cap (\Omega_{k+i} \setminus \Omega_{k+i+1})} \right\|_{\infty} \leq C \left( \max_{\{j,k:Q_j^k \in S_i\}} \frac{\mu(Q_j^k \cap (\Omega_{k+i} \setminus \Omega_{k+i+1}))}{\mu(Q_j^k)} \right)^\gamma \|A_i g\|_{\mathcal{E}}.
\]
(6.11)

By Corollary 10 we get for all \( \ell \geq \nu \),
\[
\max_{\{j,k:Q_j^k \in S_i\}} \frac{\mu(Q_j^k \cap (\Omega_{k+i+1} \setminus \Omega_{k+i+1}))}{\mu(Q_j^k)} \leq \max_{\{j,k:Q_j^k \in S_i\}} \frac{\mu(Q_j^k \cap \Omega_{k+i})}{\mu(Q_j^k)} \leq \max_{\{j,k:Q_j^k \in S_i\}} \frac{\mu(Q_j^k)}{a_i^\ell \epsilon \mu(Q_j^k)} = \frac{1}{a_i^\ell \epsilon}.
\]
(6.12)
It follows from (6.10)–(6.12) and (6.5) that
\[
\|\Sigma_2\| \leq \sum_{\ell=0}^{\infty} \left( \sum_{j,k:Q_j^* \in S_{\ell}} 2^{\alpha j} |Q_j^* \cap (\Omega_{k+\ell} \setminus \Omega_{k+\ell+1})| \right) \leq C \sum_{\ell=0}^{\infty} \left( \frac{1}{2^{d\ell}} \right)^\gamma \|A_{S_{\ell}}^* g\| \leq \frac{1}{2} \|A_{S_{\ell}}^* g\| \varepsilon. \tag{6.13}
\]

Combining inequalities (6.10), (6.11), and (6.13), we arrive at
\[
\|A_{S_{\ell}}^* g\| \leq \nu \|M^D g\| \varepsilon + \frac{1}{2} \|A_{S_{\ell}}^* g\| \varepsilon.
\]
It follows from this inequality, Theorem 4 and the boundedness of the Hardy-Littlewood maximal operator on \(E(X,d,\mu)\) that for all finite sparse families \(S_i \subset S\) and all \(g \in E(X,d,\mu)\), one has
\[
\|A_{S_i}^* g\| \leq 2\nu \|M^D g\| \varepsilon \leq 2\nu C_{HK}(X) \|Mg\| \varepsilon \leq 2\nu C_{HK}(X) \|M\|_{B(E)} \|g\| \varepsilon. \tag{6.14}
\]
Combining (6.4) and (6.14) with [3, Chap. 1, Lemma 2.8], we see that
\[
\|A_{S_i} f\|_{E'} = \sup \left\{ \left| \int_X (A_{S_i} f)(x) g(x) \, d\mu(x) \right| : g \in E(X,d,\mu), \|g\| \varepsilon \leq 1 \right\}
\leq 2\nu C_{HK}(X) \|M\|_{B(E)} \|f\|_{E'}
\]
for all \(t \in \mathbb{N}\). Passing in this inequality to the limit as \(t \to \infty\) and taking into account (6.2), we get
\[
\|A_{S_i} f\|_{E'} \leq 2\nu C_{HK}(X) \|M\|_{B(E)} \|f\|_{E'}.
\]
It follows from this inequality and inequality (6.1) that
\[
\|M^D f\|_{E'} \leq 2\nu C_{HK}(X) \|M\|_{B(E)} \|f\|_{E'} \tag{6.15}
\]
for every dyadic grid \(D \in \bigcup_{i=1}^K D^i\). In turn, inequality (6.15) and Theorem 4 imply that
\[
\|M f\|_{E'} \leq 2\nu K C_{HK}^2(X) \|M\|_{B(E)} \|f\|_{E'} \tag{6.16}
\]
for every nonnegative function \(f \in L^1(X,d,\mu) \cap E'(X,d,\mu)\).

Now let \(f \in E'(X,d,\mu)\) be an arbitrary complex-valued function. Since \(X\) is \(\sigma\)-finite, there are measurable sets \(\{A_n\}_{n \in \mathbb{N}}\) such that \(\mu(A_n) < \infty\) for all \(n \in \mathbb{N}\), \(A_i \subset A_j\) for \(i < j\), and \(\bigcup_{n \in \mathbb{N}} A_n = X\). Let \(f_n = |f| \chi_{A_n}\) for \(n \in \mathbb{N}\). Then \(f_n \in L^1(X,d,\mu) \cap E'(X,d,\mu)\) for all \(n \in \mathbb{N}\) in view of axiom (A5) in the definition of a Banach function space. By (6.16), for all \(n \in \mathbb{N}\),
\[
\|M f_n\|_{E'} \leq 2\nu K C_{HK}^2(X) \|M\|_{B(E)} \|f_n\|_{E'}. \tag{6.17}
\]
Since \(f_n \uparrow |f|\) a.e., we have \(M f_n \uparrow Mf\) a.e. (cf. [6, Lemma 2.2]). Passing to the limit in inequality (6.17), we conclude from the Fatou property that inequality (6.16) holds for all \(f \in E'(X,d,\mu)\). Thus, the Hardy-Littlewood maximal operator \(M\) is bounded on the space \(E'(X,d,\mu)\) whenever it is bounded on the space \(E(X,d,\mu)\) and the latter space satisfies the condition \(A_{\infty}\). \(\square\)
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