Factorization of Correlation Functions and the Replica Limit of the Toda Lattice Equation

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Exact microscopic spectral correlation functions are derived by means of the replica limit of the Toda lattice equation. We consider both Hermitian and non-Hermitian theories in the Wigner-Dyson universality class (class A) and in the chiral universality class (class A\text{III}). In the Hermitian case we rederive two-point correlation functions for class A and class A\text{III} as well as several one-point correlation functions in class A\text{III}. In the non-Hermitian case the average spectral density of non-Hermitian complex random matrices in the weak non-Hermiticity limit is obtained directly from the replica limit of the Toda lattice equation. In the case of class A, this result describes the spectral density of a disordered system in a constant imaginary vector potential (the Hatano-Nelson model) which is known from earlier work. New results are obtained for the average spectral density in the weak non-Hermiticity limit of a quenched chiral random matrix model at nonzero chemical potential. These results apply to the ergodic or $\epsilon$ domain of the quenched QCD partition function at nonzero chemical potential. Our results have been checked against numerical results obtained from a large ensemble of random matrices. The spectral density obtained is different from the result derived by Akemann for a closely related model, which is given by the leading order asymptotic expansion of our result. In all cases, the replica limit of the Toda lattice equation explains the factorization of spectral one- and two-point functions into a product of a bosonic (noncompact integral) and a fermionic (compact integral) partition function. We conclude that the fermionic partition functions, the bosonic partition functions and the supersymmetric partition function are all part of a single integrable hierarchy. This is the reason that it is possible to obtain the supersymmetric partition function, and its derivatives, from the replica limit of the Toda lattice equation.
I. INTRODUCTION

A replicated system is one in which certain fields appear as exact copies of each other. For a given number, say \( n \), of replica fields the system may or may not have a physical realization, but it is nevertheless useful to study the system for any integer values of \( n \). Suppose, as is the case in this paper, that we wish to calculate the average resolvent of some operator \( D \),

\[
G(x) = \left< \frac{\text{Tr} \left\{ \frac{1}{x + D} \right\}}{n} \right>,
\]

and, as is also the case, that it is simpler to evaluate \( \langle \text{det}^n(D + x) \rangle \) for integer values of \( n \) than it is to evaluate the average of the resolvent. In that case the replica trick allows us to obtain the resolvent from the replicated system. The fermionic (bosonic) replica trick consists in evaluating \( \langle \text{det}^n(D + x) \rangle \) for all positive (negative) integers, \( n \), and then taking the limit \( n \to 0 \) as follows\[1\]

\[
G(x) = \lim_{n \to 0} \partial_x \left< \frac{\text{det}^n(D + x)}{n} \right> = \partial_x \left< \log \text{det}(D + x) \right>. \tag{2}
\]

The tricky part arises because the use of the replica trick assumes that an analytic continuation from positive or negative integers is uniquely defined. It has long been known\[2–12\] that this is the case for perturbative calculations such as for the expansion of the resolvent in an asymptotic series in \( 1/x \). Problems are known to occur when one attempts to derive exact nonperturbative results for the quenched average resolvent\[4,7\]. This is one of the reasons that alternative methods were introduced to evaluate quenched averages. Most notably, the supersymmetric method\[15,16\], where the resolvent is obtained from the generating function

\[
G(x) = \partial_f \left< \frac{\text{det}(D + x + J)}{\text{det}(D + x)} \right>_{J=0}, \tag{3}
\]

has been very successful. Among other alternatives to the replica trick, we only mention the Keldysh method\[17\] by means of which some nonperturbative results have been derived.

The purpose of this paper is to show that nonperturbative results can be obtained by means of the replica trick as well.

A determinant raised to an integer power inside a statistical average appears in a wide range of theories. In disordered condensed matter systems and random matrix theory, one is usually interested in quenched disorder where such a determinant enters in the generating function of Greens functions as discussed in our example of the resolvent. In Quantum Chromodynamics Dynamics (QCD), on the other hand, the physical theory is an average over gauge field configurations of a fermionic determinant raised to the number of flavors. In lattice gauge simulations, because of computational reasons, one often is forced to work with quenched averages (i.e. ignoring the fermion determinant) which are not realized in nature. Both in disordered systems and in QCD the low-energy effective theory of the quenched system is either a supersymmetric nonlinear sigma model or a nonlinear sigma model for replicated flavors. For perturbative calculations, both theories are completely equivalent\[4,8,11–14\]. To prefer one above the other is just a matter of convenience. On the other hand, nonperturbative results have only been derived by means of the supersymmetric method\[15,16,18–22\].

The replica trick has met substantial critique\[4,7\] in nonperturbative applications. For instance, it was shown that bosonic and fermionic replica limits lead to different results for the two point function of the Gaussian unitary ensemble and that neither limits lead to the correct result\[4\]. The original critique\[4\] motivated an insightful and challenging debate\[5–10,23–25\] the details of which have been accounted for in the introduction of\[26\]. The debate lead to the generally accepted belief that, although the replica trick correctly reproduces the asymptotic expansion of the microscopic spectral correlation functions, it can not be trusted for nonperturbative calculations except in a few cases, when the series can be resummed\[24\] or simply terminates\[7,9\]. However, generally this is not the case, and the asymptotic series leaves much to be desired. For example, since an asymptotic series does not determine the cut structure of an analytic function, it is not possible to obtain the density of eigenvalues from the discontinuity of the resolvent across its cut.

Over the past few years intimate connections have been established between random matrix theory, integrable systems, the theory of \( \tau \)-functions, Painlevé equations and unitary matrix integrals\[27–31,35,8,36,32–34\]. Kanzieper\[37\] exploited this connection to obtain exact analytical results by means of the replica limit of Painlevé equations for the generating function of spectral correlation functions. He noted that the replica index only appears as a coefficient in the associated Painlevé equation. To leading order in the replica index this equation is solvable and,
with appropriate boundary conditions, exact analytical results for spectral one- and two-point functions functions could be derived.

The ground breaking work of Kanzieper was followed up by the present authors [26] by showing that the replica limit can be taken in the Toda lattice equation rather than in the Painlevé equation. This method has two advantages. First, it automatically reveals the factorization of correlation functions or derivatives of correlation functions in terms of a product of a fermionic and a bosonic partition function. Second, in the examples we have considered, either no differential equation had to be solved or the desired result followed from the solution of a simple first order (inhomogeneous) differential equation. The factorization property, which had not been appreciated before, explains precisely why the original formulation of the replica trick is bound to be tricky: Fermionic partition functions are given by compact integrals which are analytic functions of the argument for all integer values of the replica index $n$. It is therefore a mystery how the replica limit, $n \to 0$, can be a nonanalytic function. This point of the original critique [7] is dealt with automatically in the replica limit of the Toda lattice equation which involves both fermionic and bosonic partition functions. The bosonic partition function is determined by a noncompact integral and carries the nonanalytic information.

The Toda lattice is well-known from the theory of exactly solvable systems [38,39]. It consists of masses interacting in one dimension through an exponential potential [38]. The Hamilton equations of motion are known as the Toda lattice equation and its solutions give the position of the masses as a function of time. The system is integrable and one particular solution can be written as a unitary matrix integral which, among others, is the QCD partition function in the $\epsilon$ regime. Similar connections exist for all of the classical random matrix ensembles (see for example [40]). Remarkably, the replica trick also enters in integrable systems related to matrix models for supersymmetric Yang-Mills theories [41], but it is not known if there is a relation with our approach.

In [26] we derived the two-point function of the Gaussian unitary ensemble and the quenched and partially quenched one-point functions of the chiral Gaussian unitary ensemble by means of the replica limit of the Toda lattice equation. The purpose of the present paper is to show that such strikingly simple results apply more generally. We will consider three different physical systems: the QCD partition function, the Hatano-Nelson model [42] and QCD at nonzero chemical potential. The Hatano-Nelson model is a disordered system in an imaginary vector potential that enters in the same way as a baryon chemical potential in the Euclidean QCD partition function. In both systems, the symmetries of the generating function are broken spontaneously. The corresponding Goldstone bosons interact according to a nonlinear sigma model that is completely determined by the pattern of spontaneous and explicit symmetry breaking [43]. In QCD, this is the well-known chiral Lagrangian which can be extended to include a nonzero chemical potential [44]. For the Hatano-Nelson model a similar effective Lagrangian can be derived [15]. Its static limit was obtained earlier [45] as the weak non-Hermiticity limit of a complex non-Hermitian random matrix model.

In both theories we will study the nonlinear sigma model in the ergodic domain which, in QCD, is also known as the $\epsilon$-regime, where the contribution from the zero-momentum modes (or constant fields) dominate the partition function. In this domain, the kinetic term of the $\sigma$ model can be ignored. The corresponding energy scale, which is known as the Thouless energy, is the quark mass for which the Compton wave length of the Goldstone bosons is of the order of the size of the box. In this regime, all theories with a given symmetry breaking pattern are universally described by the same partition function. Perhaps the simplest theories in this universality class are random matrix theories. In random matrix theories one seeks to describe average spectral properties on the scale of the average eigenvalue spacing in the limit of large matrices. It has been shown that spectral correlation functions in this microscopic limit are universal in the sense that they only depend on the symmetries of the random matrix model and not on its detailed form (see for example [46,47]). We now understand that [21,20] this universality is a direct consequence of the uniqueness of the static part of the effective Lagrangian. We will exploit the correspondence between random matrix theory and the static part of the effective Lagrangian to derive the bosonic partition function for QCD at nonzero chemical potential and for the Hatano-Nelson model. In both cases, the integration contour of the bosonic fields in the $\sigma$-model follows naturally from the Ingham-Siegel integral of the second kind [25].

The two-point functions of the unitary ensemble, the chiral unitary ensemble and the spectral density of the Hatano-Nelson model [45,15] and the chiral unitary ensemble at nonzero chemical potential all share one remarkable property. They factorize into a product of a fermionic and a bosonic partition function. We will show that this is a natural consequence of the Toda lattice equation. Indeed, the corresponding generating functions can be written as a determinant of derivatives. Therefore, they are $\tau$-functions of a Toda lattice hierarchy and satisfy a Toda lattice equation. Our result for the chiral unitary ensemble at nonzero chemical potential is a genuinely new result which is in complete agreement with results obtained from a numerical matrix diagonalization. The leading term of an asymptotic expansion of our result and the result for a closely related model for non-Hermitian spectra with a chiral symmetry [48] turn out to be the same.

The organization of this paper is as follows. As a warm up we start in sections II A and II B with simple examples that have already been discussed in [26]. The two-point function of the chiral unitary ensemble is discussed in
section II C. Turning to the complex spectra of non-Hermitian operators we consider the case of a disordered system in a nonzero imaginary vector potential in section III A and QCD at nonzero chemical potential in section III B. Concluding remarks are made in section IV. Our notation as well as additional technical details are explained in several appendices (A through D).

II. TODA LATTICE EQUATION FOR HERMITIAN THEORIES

In this section we start with three simple examples of the replica limit of the Toda lattice equation, i.e. the two-point function of the Unitary Ensemble (UE), the resolvent of the chiral Unitary Ensemble (chUE) in the quenched case for topological charge \( \nu \), and in the case of \( N_f \) massless flavors in the sector of zero topological charge. The first two cases were already discussed in [26]. In the second part of this section we discuss the calculation of the two-point function of the chUE. In this case the Toda lattice equation is obtained from a new representation of the finite volume partition function which might also be useful in other applications.

A. The Two-Point Function of the Unitary Ensemble

This section discusses the simplest example of the replica limit of the Toda lattice equation [26], namely the connected two-point function of the Gaussian Unitary Ensemble (GUE). Its generating function for \( n \) fermionic replicas is given by

\[
\int dH \det^n (H + E_1) \det^n (H + E_2) e^{-\frac{N}{2} \text{Tr} H^1 H}.
\]  

Following the usual reduction, this partition function can be written as a nonlinear \( \sigma \)-model. In the large \( N \)-limit, at fixed \( r = N(E_2 - E_1) \) near the center of the spectrum, the eigenvalue representation of this \( \sigma \)-model further simplifies (see eq. (1.2) of [4])

\[
Z_n(ir) = n! \int_{-1}^{1} \prod_{k=1}^{n} d\lambda_k e^{-ir\lambda_k} \Delta^2(\lambda).
\]  

Here, \( \Delta(\lambda) \) is the Vandermonde determinant \( \Delta(\lambda) = \prod_{k<l}(\lambda_k - \lambda_l) \). The connected two-point correlation function is then obtained simply as the replica limit

\[
G(r) = -\lim_{n \to 0} \frac{1}{n^2} \frac{\partial^2}{\partial r^2} \log Z_n(ir).
\]  

The generating function (5) can be rewritten as

\[
Z_n(r) = (nl)^2 \det[\partial_{r}^{i+j} Z_1(r)]_{i,j=0,\ldots,n-1}.
\]  

In a similar way the large \( N \) limit of the bosonic partition can be reduced to

\[
Z_{-n}(ir) = \frac{1}{n!(n-1)!^2} \int_{1}^{\infty} \prod_{k=1}^{n} d\lambda_k e^{ir\lambda_k} \Delta^2(\lambda).
\]  

where the sign of the exponent is determined by the sign of the imaginary part of \( r \). The \( r \) dependence can be scaled outside the integral resulting in

\[
Z_{-n}(r) = \frac{\kappa^n}{r^{n^2}} \prod_{k=1}^{n-2} (k!)^2.
\]  

This expression can be rewritten as

\[
Z_{-n}(r) = \frac{1}{[(n-1)!]^2} \det[\partial_{r}^{i+j} Z_{-1}(r)]_{i,j=0,\ldots,n-1}.
\]  

Therefore, both the fermionic and the bosonic partition function satisfy the Toda lattice equation [23,26].
\[-Z_n^2(\text{ir})\partial_r^2 \log Z_n(\text{ir}) = \frac{n^2}{(n+1)^2} Z_{n+1}(\text{ir}) Z_{n-1}(\text{ir}).\]

In the replica limit, the l.h.s. of this equation is \(n^2\) times the connected two-point function whereas the r.h.s. correctly gives

\[G(r) = Z_1(\text{ir}) Z_{-1}(\text{ir}) = \frac{2 \sin r \, e^{ir}}{r}.\]

We thus find that the factorization of the two-point correlation function into a bosonic and a fermionic partition function is not accidental but rather a consequence of the relation between random matrix theories and integrable hierarchies. The two-point function of the GUE can also be obtained from a solution of the Painlevé V equation [23]. Another feature of this result emphasized in [7] is that the asymptotic series in \(1/r\) terminates. This explains [7] that in this case the exact result could be obtained [5] by including a subleading saddle point manifold in the standard replica approach.

B. The Partially Quenched Resolvent in QCD

In this section we discuss the replica limit of the Toda lattice equation for the generating function of the partially quenched resolvent in chUE. As a simple extension of the results presented in [26] we obtain the microscopic limit of the partially quenched resolvent for an arbitrary number of flavors.

The partition function of the chiral Gaussian Unitary Ensemble (chGUE) with \(n\) fermionic replica flavors with mass \(m\) and \(N_f\) additional fermionic flavors with masses \(m_1, \ldots, m_{N_f}\) is defined by

\[Z_n^{(\nu)}(m, \{m_i\}) = \langle \det^n (D + m) \prod_{k=1}^{N_f} \det(D + m_k) \rangle.\]

The Dirac matrix is given by

\[D \equiv \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix},\]

where \(W\) is a rectangular \(l \times (l+\nu)\) matrix so that \(D\) has exactly \(\nu\) zero eigenvalues. The average is over the probability distribution of the matrix elements of \(W\) with the weight

\[P(W) = e^{-\frac{\nu}{N} \text{Tr}(WW^\dagger)}.\]

with \(N = 2l + \nu\). We will consider the microscopic limit where the thermodynamic limit is taken for fixed \(x \equiv mN\) and \(x_k \equiv m_kN\). In this case the partition function reduces to the unitary matrix integral [49]

\[Z_n^{(\nu)}(x, \{x_i\}) = \frac{1}{(2\pi)^{n(n+1)/2}} \int_{U \in U(n+N_f)} \, dU \, \det^\nu U e^{\frac{\nu}{2} \text{Tr}[M^\dagger U + MU^\dagger]},\]

where \(M = \text{diag}(x, \ldots, x, x_1, \ldots, x_{N_f})\) is the mass matrix. The normalization factor has been included for later convenience. The partially quenched resolvent, which we will derive below, is given by

\[G(x, \{x_i\}) = \lim_{n \to 0} \frac{1}{n} \partial_x \log Z_n^{(\nu)}(x, \{x_i\}).\]

The unitary matrix integral (16) is the partition function of QCD [49–51] in the \(\epsilon\)-regime where it coincides with the microscopic limit of the chGUE partition function (13). Partition functions for different numbers of flavors satisfy a recursive relation known as the Toda lattice equation [30]

\[\sum_{k=0}^{N_f} x_0 \partial_{x_0} x_k \partial_{x_k} \log Z_n^{(\nu)}(x_0, \{x_k\}) \bigg|_{x_0 = x} = nx^2 \frac{Z_n^{(\nu)}(x, \{x_k\})}{[Z_n^{(\nu)}(x, \{x_k\})]^2}.\]

For degenerate masses \(x_k\) the sum over the derivatives is only over different masses.
The bosonic chGUE partition function is defined as in (13) but with negative integer values of $n$. Its axial symmetry is not $U(n + N_f)$ but rather $GL(n + N_f)/U(n + N_f)$ [20,21,9]. Therefore, the Goldstone manifold for the bosonic partially quenched partition function is $GL(n + N_f)/U(n + N_f)$ rather than $U(n + N_f)$, i.e. the coset of positive definite matrices. In this way the integration manifold remains Riemannian [20] so that all integrals are convergent for positive masses. As bosonic partition function we thus obtain [21,9]

$$Z_n^{(\nu)}(x, \{x_i\}) = \frac{1}{C_n} \int_{Q \in SL(n)/SU(n)} dQ \operatorname{det}^\nu Q e^{-\frac{1}{2} \operatorname{Tr} [M^\dagger Q + MQ^{-1}]}.$$  

(19)

If we parameterize $Q$ as

$$Q = U \operatorname{diag}(e^{s_k}) U^\dagger, \quad U^\dagger U = 1$$

(20)

the invariant measure is given by

$$dQ = \prod_{k < l} (e^{s_k} - e^{s_l})(e^{-s_k} - e^{-s_l}) \prod_k ds_k dU.$$  

(21)

Using the Ingham-Siegel integral [25] this result can be derived directly from the bosonic random matrix partition function [52]. For zero topological charge this is discussed in [25] and the simplest nontrivial example is given in Appendix C. The explicit form of the partition function with any number of fermions and bosons in the topological sector $\nu$ conjectured in [26] was proven rigorously in [53] based on a general result in [54].

In [26], the $n \to 0$ limit of the Toda lattice equation (18) was solved for $N_f = 0$ and $N_f = 1$. Before we consider the case with $N_f$ massless flavors and zero topological charge, we first review the quenched limit ($N_f = 0$) of (18) in the sector of topological charge $\nu$. In this case, (18) simplifies to

$$[x \partial_x]^2 \log Z_n^{(\nu)}(x) = n x^2 \frac{Z_{n+1}^{(\nu)}(x) Z_{n-1}^{(\nu)}(x)}{[Z_n^{(\nu)}(x)]^2}.$$  

(22)

Collecting terms of $O(n)$ in the replica limit of this Toda lattice equation we find

$$\partial_x [xG(x)] = xZ_1^{(\nu)}(x)Z_{-1}^{(\nu)}(x).$$  

(23)

Again, we observe a factorization property of the resolvent. For topological charge $\nu$ we have that *

$$Z_1^{(\nu)}(x) = I_{\nu}(x) \quad \text{and} \quad Z_{-1}^{(\nu)}(x) = 2K_{\nu}(x).$$  

(24)

The general solution of the inhomogeneous differential equation (23) is given by

$$xG(x) = a + x^2 (K_{\nu}(x) I_{\nu}(x) + K_{\nu-1}(x) I_{\nu+1}(x)),$$

(25)

where the first term is a solution of the homogeneous differential equation. The integration constant is fixed by the small $x$ expansion of the resolvent which in the sector of topological charge $\nu$ is given by

$$G(x) \sim \frac{\nu}{x},$$  

(26)

so that $a = \nu$. This ends the derivation of the quenched resolvent in the topological sector $\nu$. The result is in exact agreement with the known result [55].

As a final introductory example, let us now consider the case of $N_f$ massless flavors and zero topological charge. The general Toda lattice equation (18) reduces to

$$\lim_{n \to 0} \frac{1}{n} \left( (x \partial_x + y \partial_y) x \partial_x \log Z_{n,N_f}^{(0)}(x, y) \right)_{y=0} = x^2 \frac{Z_{1,N_f}^{(0)}(x,0) Z_{-1,N_f}^{(0)}(x,0)}{[Z_{0,N_f}^{(0)}(x,0)]^2}.$$  

(27)

*The normalization factor of the bosonic partition function is chosen such that the large $x$ limit of both sides of (23) is the same. For large $x$ we have that $G(x) = 1$ whereas the large $x$ behavior of the r.h.s. of (23) is obtained from the leading order asymptotic expansion of the Bessel functions.
The partition function $Z_{1,N_f}^{(0)}(x,0)$ can be obtained by using the flavor-topology duality relation [56] and is given by

$$Z_{1,N_f}^{(0)}(x,0) = x^{-N_f} Z_{1,0}^{(ν=N_f)}(x) = x^{-N_f} I_{N_f}(x). \tag{28}$$

The partition function $Z_{-1,N_f}^{(0)}(x,0)$ has $N_f$ massless fermionic flavors and one bosonic flavor with mass $x$. Using a bosonic version of the flavor-topology duality relation we find

$$Z_{-1,N_f}^{(0)}(x,0) = x^{N_f} Z_{-1,0}^{(ν=N_f)}(x) = 2x^{N_f} K_{N_f}(x), \tag{29}$$

where the normalization factors are consistent with the asymptotic behavior of (27). The $y\partial_y$-derivative of $\log Z_{0,N_f}^{(0)}(x,y)$ vanishes for $y = 0$ so that (18) simplifies to

$$\partial_x [xG(x)] = 2xK_{N_f}(x)I_{N_f}(x). \tag{30}$$

As in the quenched case for topological charge $ν$, the solution is given by

$$xG(x) = a + x^2[K_{N_f}(x)I_{N_f}(x) + K_{N_f-1}(x)I_{N_f+1}(x)],$$

Because we are in the sector of zero topological charge, the resolvent approaches a constant for small $x$ so that the integration constant $a = 0$. This result agrees with the result obtained by integration the microscopic spectral density [55].

C. The Two-Point Function in QCD

In this section we consider the microscopic limit of the chUE partition function for $m$ flavors with mass $x$ and $n$ flavors with mass $y$. In the sector of topological charge $ν$ we will denote this partition function by $Z_{m,n}^{(ν)}(x,y)$. The microscopic spectral two-point function will be derived from the replica limit of the Toda lattice equation corresponding to this partition function. In the first subsection we will show that the partition function can be written as a determinant of derivatives. In the second subsection we shall use this form to show that $Z_{m,n}^{(ν)}(x,y)$ satisfies a Toda lattice equation and therefore is a $τ$-function of an integrable hierarchy. The replica limit of the Toda lattice equation then automatically gives us the exact analytical result for the two-point function that was obtained earlier [57] by means of the supersymmetric method.

1. The Generating Function for the Two-Point Function is a $τ$–Function

The finite volume partition function for $m$ fermionic flavors with mass $x$ and $n$ (with $n \geq m$) fermionic flavors with mass $y$ is given by [49–51]

$$Z_{m,n}^{(ν)}(x,y) \equiv \frac{1}{T_{m,n}} \int_{U(n+m)} dU \det^ν(U) \exp \frac{1}{2} \text{Tr} \left( x^1_m y^1_n U + \frac{1}{2} \text{Tr} \left( x^1_m y^1_n \right) U^⊥ \right). \tag{31}$$

This partition function can be derived from a random matrix theory with the symmetries of the QCD partition function [49] and is also known as the QCD partition function in the $ε$ regime. This integral and its generalizations have been well studied in the literature [58–62], but the known analytical expressions do not directly show that $Z_{m,n}^{(ν)}(x,y)$ is a $τ$-function. In order to write it in such a form we evaluate the integral over $U$ by decomposing $U(n+m)$ as $U(m) \times U(n) \times [U(n+m)/(U(n) \times U(m))]$, and evaluating the integral over the coset first. The normalization factor is chosen to be

$$T_{m,n} = (2π)^{(n+m)(n+m+1)/2}. \tag{32}$$

Our calculations simplify if we use the parameterization

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Lambda \begin{pmatrix} v_1^⊥ \\ v_2^⊥ \end{pmatrix}, \tag{33}$$
where $\Lambda$ is the block diagonal $(m+n) \times (m+n)$ matrix given by

$$
\Lambda = \begin{pmatrix}
\sqrt{1-\mu^2} & \mu \\
-\mu & \sqrt{1-\mu^2} \\
0 & 0 \\
0 & -1
\end{pmatrix}.
$$

(34)

The diagonal matrix $\mu = \text{diag}(\mu_1, \cdots, \mu_m)$, and we will use the notation that $\lambda_k = \sqrt{1-\mu_k^2}$ with $\lambda_k \in [0,1]$. The unit matrix $I$ in the lower right hand corner is of size $n-m$. Furthermore, $u_1$ and $v_1$ are unitary $m \times m$ matrices, $u_2 \in U(n)$ and $v_2 \in U(n)/(U^m(1) \times U(n-m))$. One easily verifies that the total number of parameters on both sides of (33) is the same. The Jacobian of this transformation is derived in Appendix B and is given by

$$
J = \prod_{1 \leq k < \ell \leq m} (\lambda_k^2 - \lambda_\ell^2)^2 \prod_{k=1}^m (2\lambda_k)^{2(n-m)}.
$$

(35)

The finite volume partition function can thus be rewritten as (the $v_k$ variables have been eliminated from the exponent by the transformation $u_k \to v_k u_k v_k^{-1}$)

$$
Z^{(v)}_{m,n}(x,y) = \frac{1}{m!T_{m,n}} \int_{U(m)} dv_1 \int_{U(n)/[U^m(1) \times U(n-m)]} dv_2 \int_{U(m)} du_1 \int_{U(n)} du_2 \int_0^1 \prod_{k=1}^m d\lambda_k \times J(\{\lambda_k\}) \text{det}'(u_1 u_2) \exp\left[\frac{1}{2} x^2 \text{Tr}\Lambda_{11}(u_1 u_1^\dagger) + \frac{1}{2} y \text{Tr}\Lambda_{22}(u_2 u_2^\dagger)\right].
$$

(36)

Here we have introduced the notation

$$
\Lambda_{11} \equiv \text{diag}(\lambda_1, \cdots, \lambda_m),
$$

(37)

and

$$
\Lambda_{22} \equiv \text{diag}(-\lambda_1, \cdots, -\lambda_m, -1, \cdots, -1).
$$

(38)

The integrals over $v_1$ and $v_2$ give an overall constant equal to the volume of the integration manifold. The integrals over $u_1$ and $u_2$ are well known. They are given by [59–62]

$$
\frac{1}{\text{vol}(U(m))} \int_{U(m)} du_1 \text{det}'(u_1) \exp\left[\frac{1}{2} x \Lambda_{11} u_1 + \frac{1}{2} x \Lambda_{11} u_1^\dagger\right] = C_m \frac{\text{det}[\phi_l(x \lambda_k)]_{l=0, \cdots, m-1}}{\Delta(\{x \lambda_k^2\})},
$$

(39)

where

$$
\phi_l(x_k) \equiv (x_k \partial_{x_k})^l I_\nu(x_k),
$$

(40)

and

$$
C_m \equiv 2^{m(m-1)/2} \prod_{k=1}^{m-1} k!.
$$

(41)

The volume of the unitary group is given by (see eg. [63])

$$
\text{vol}(U(n)) \equiv \int_{U(n)} dU = \frac{(2\pi)^{n(n+1)/2}}{\prod_{k=1}^{n-1} k!}.
$$

(42)

A similar formula can be derived for the $u_2$-integral. The only complication is that $n-m$ diagonal matrix elements of $y \Lambda_{22}$ are degenerate. To apply the formula (39) we add $\epsilon_k$ to the degenerate diagonal matrix elements. The $\epsilon_k \to 0$ limit can then be obtained conveniently by writing the Taylor expansion of $\phi_l(-y + \epsilon_k)$ to order $n-m$ in $\epsilon_k$ as a product of a matrix containing the powers of $\epsilon_k$ and a derivative matrix. This results in
\[
\frac{1}{\text{vol}(U(n))} \int_{U(n)} du_2 \det^{\nu}(u_2) \exp[\text{Tr}(\frac{1}{2} y \Lambda_{22} u_2 + \frac{1}{2} y \Lambda_{22} u_2^\dagger)]
\]

\[
= \frac{C_n}{C_{n-m}} \frac{y^{-(n-m)(n-m-1)/2}}{\prod_{k=1}^{m} (\lambda_k^2 y^2 - y^2)^{(n-m)} \prod_{1 \leq k < l \leq m} (\lambda_k^2 y^2 - \lambda_l^2 y^2)} \det \begin{vmatrix}
\phi_0(-y \lambda_1) & \cdots & \phi_{n-1}(-y \lambda_1) \\
\vdots & \ddots & \vdots \\
\phi_0(-y \lambda_m) & \cdots & \phi_{n-1}(-y \lambda_m) \\
\phi_0(-y) & \cdots & \phi_{n-1}(-y) \\
\vdots & \ddots & \vdots \\
\phi_0^{(n-m-1)}(-y) & \cdots & \phi_{n-1}^{(n-m-1)}(-y)
\end{vmatrix}, \tag{43}
\]

where we have used the notation \( \phi_k^{(l)}(y) \equiv \partial_y^l \phi_k(y) \). All factors in the Jacobian (35) except \( \prod (2 \lambda_k) \) are canceled by the corresponding factors from the \( u_1 \) and \( u_2 \) integrations. Multiplying the two determinants the partition function can be written as

\[
Z_{m,n}^{(\nu)}(x, y) = \frac{2^{(m+n)(m+n-1)/2} y^{(n-m)(n-m-1)/2}}{C_n C_m \lambda^m(m-1) y^{n-1}} \det \begin{vmatrix}
\int_0^1 d\lambda \lambda \phi_0(\lambda x) \phi_0(-\lambda y) & \cdots & \int_0^1 d\lambda \lambda \phi_0(\lambda x) \phi_{n-1}(-\lambda y) \\
\vdots & \ddots & \vdots \\
\int_0^1 d\lambda \lambda \phi_{m-1}(\lambda x) \phi_0(-\lambda y) & \cdots & \int_0^1 d\lambda \lambda \phi_{m-1}(\lambda x) \phi_{n-1}(-\lambda y) \\
\phi_0(-y) & \cdots & \phi_{n-1}(-y) \\
\vdots & \ddots & \vdots \\
\phi_0^{(n-m-1)}(-y) & \cdots & \phi_{n-1}^{(n-m-1)}(-y)
\end{vmatrix}. \tag{44}
\]

By the addition of rows, the derivatives in the last \( m \) rows can be rewritten in terms of the derivative operators

\[
\delta_x \equiv x \frac{d}{dx}.
\]

This results in the final form for the partition function of the chUE

\[
Z_{m,n}^{(\nu)}(x, y) = \frac{2^{(m+n)(m+n-1)/2}}{C_n C_m x^m(m-1) y^{n-1}} \det \begin{vmatrix}
\int_0^1 d\lambda \lambda \mu_\nu(\lambda x) I_\nu(-\lambda y) & \cdots & \int_0^1 d\lambda \lambda \mu_\nu(\lambda x) I_\nu(-\lambda y) \\
\vdots & \ddots & \vdots \\
\delta_x^{m-1} \int_0^1 d\lambda \lambda \mu_\nu(\lambda x) I_\nu(-\lambda y) & \cdots & \int_0^1 d\lambda \lambda \mu_\nu(\lambda x) I_\nu(-\lambda y) \\
\delta_x \delta_y \int_0^1 d\lambda \lambda \mu_\nu(\lambda x) I_\nu(-\lambda y) & \cdots & \int_0^1 d\lambda \lambda \mu_\nu(\lambda x) I_\nu(-\lambda y) \\
\delta_y^{n-m-1} I_\nu(-y) & \cdots & \delta_y^{n-m-2} I_\nu(-y)
\end{vmatrix}. \tag{46}
\]

If we denote the \( n \times n \) matrix that enters in this partition function by \( A^{(m,n)} \) it can be rewritten as

\[
Z_{m,n}^{(\nu)}(x, y) = \frac{2^{(m+n)(m+n-1)/2}}{C_n C_m x^m(m-1) y^{n-1}} \det A^{(m,n)}(x, y).
\]

Of course, \( A^{(m,n)} \) is a function of \( \nu \) even though we have not explicitly indicated it. Before we proceed to show that this partition function satisfies a Toda lattice equation, let us note that: \( Z_{1,1}^{(\nu)}(x, y) = 2 \int d\lambda \lambda \mu_\nu(\lambda x) I_\nu(-\lambda y) \). Hence, for \( n = m \) we obtain the compact expression

\[
Z_{n,n}^{(\nu)}(x, y) = \frac{2^{n(2n-1)}}{C_n x^n(y)^n} \det [\delta_x^i \delta_y^j Z_{1,1}^{(\nu)}(x, y)]_{i,j=0,\ldots,n-1}. \tag{48}
\]

Another special case is \( m = 0 \), where the partition function simplifies to

\[
Z_{m=0,n}^{(\nu)}(x, y) = \frac{2^{n(n-1)/2}}{C_n y^n} \det \begin{vmatrix}
I_\nu(-y) & \cdots & \delta_x^{n-1} I_\nu(-y) \\
\vdots & \ddots & \vdots \\
\delta_x^{n-1} I_\nu(-y) & \cdots & \delta_y^{2n-2} I_\nu(-y)
\end{vmatrix}. \tag{49}
\]

This expression can be easily rewritten in terms of one of the familiar forms for the finite volume partition function.
2. The Toda Lattice Equation

From the structure of $Z_{m,n}(x,y)$ in (46) it automatically follows that $Z_{m,n}^{(v)}(x,y)$ satisfies a Toda lattice equation. Below, this will be shown by a slight extension of an argument given in [40].

For the determinant of a matrix $A$ we have the Sylvester identity [64]

$$C_{ij}C_{pq} - C_{iq}C_{pj} = \det(A)C_{ij,pq}$$

where $C_{ij}$ is the cofactor of matrix element $ij$

$$C_{ij} = \frac{\partial \det(A)}{\partial A_{ij}}$$

and $C_{ij,pq}$ is the double cofactor of matrix elements $ij$ and $pq$

$$C_{ij,pq} = \frac{\partial^2 \det(A)}{\partial A_{ij} \partial A_{pq}}.$$  

We apply this relation to the matrix elements $A_{m-1,n-1}$, $A_{m-1,n}$, $A_{m,n-1}$ and $A_{m,n}$ of $A^{(m,n)}$ entering in eq. (47). This results in

$$C_{m-1,n-1}C_{mn} - C_{m-1,n}C_{mn-1} = \det(A^{(m,n)})C_{m-1,n-1,mn}.$$  

Using the explicit form of the derivatives in the determinant of (46) the cofactors can be expressed as derivatives

$$C_{mn} = \det A^{(m-1,n-1)},$$

$$C_{m-1,n} = -\delta_x \det A^{(m-1,n-1)},$$

$$C_{m-1,n-1} = \delta_x \delta_y \det A^{(m-1,n-1)},$$

$$C_{mn-1} = -\delta_y \det A^{(m-1,n-1)},$$

$$C_{m-1,n-1,mn} = \det A^{(m-2,n-2)}.$$  

This results in the recursion relation

$$[\delta_x \delta_y \det A^{(m-1,n-1)}] \det A^{(m-1,n-1)} - [\delta_x \det A^{(m-1,n-1)}][\delta_y \det A^{(m-1,n-1)}] = \det A^{(m,n)} \det A^{(m-2,n-2)}.$$  

Raising the indices of the recursion relation by one this relation can be rewritten as

$$\delta_x \delta_y \log \det(A^{(m,n)}) = \frac{\det A^{(m+1,n+1)} \det A^{(m-1,n-1)}}{\det^2 A^{(m,n)}}.$$  

The Toda lattice equation for the generating functional of the two-point function of the chUE follows immediately by applying this identity to (47). It is given by

$$\delta_x \delta_y \log Z_{m,n}^{(v)}(x,y) = 4nmx^2y^2 \frac{Z_{m+1,n+1}^{(v)}(x,y)Z_{m-1,n-1}^{(v)}(x,y)}{[Z_{m,n}^{(v)}(x,y)]^2}.$$  

3. The Replica Limit of the Toda Lattice Equation

The microscopic disconnected scalar susceptibility for the chUE is given by

$$\chi^{(v)}(x,y) \equiv \lim_{m \to 0, n \to 0} \frac{1}{nm} \frac{d}{dx} \frac{d}{dy} \log Z_{m,n}^{(v)}(x,y).$$  

Using the Toda lattice equation above in this definition we find
\[ \chi^{(\nu)}(x, y) = 4xyZ_{1,1}^{(\nu)}(x, y)Z_{-1,-1}^{(\nu)}(x, y), \]  

(59)

where, explicitly,

\[ Z_{1,1}^{(\nu)}(x, y) = \frac{1}{y^2 - x^2}(xI_{\nu+1}(x)I_{\nu}(y) - yI_{\nu+1}(y)I_{\nu}(x)), \]  

(60)

and

\[ Z_{-1,-1}^{(\nu)}(x, y) = \frac{1}{y^2 - x^2}(xK_{\nu+1}(x)K_{\nu}(y) - yK_{\nu+1}(y)K_{\nu}(x)). \]  

(61)

The two point spectral correlation function is given by the discontinuity of the disconnected scalar susceptibility across the imaginary axis. The result is in exact agreement with the analytical result for the two-point function found in [57]. Again, the replica limit of the Toda lattice equation explains that the spectral correlation function comes from the product of a fermionic and a bosonic partition function.

### III. TODA LATTICE EQUATION FOR NON-HERMITIAN THEORIES

In this section, we derive, in the limit of weak non-Hermiticity, the spectral density for non-Hermitian random matrix theories from the replica limit of the Toda lattice equation. We will consider the weak non-Hermiticity limit of two systems: 1) A disordered system in an imaginary vector potential (the Hatano-Nelson model [42]) which is in the universality class of the unitary ensemble. We will derive the known spectral density of this system [45,65] from the replica limit of the Toda lattice equation. 2) QCD at non zero baryon chemical potential which belongs to the universality class of the chiral unitary ensemble. We will derive the quenched microscopic spectral density of the non-Hermitian Dirac operator. This is a new result that we have checked against a high statistics numerical simulation. For a general discussion of non-Hermitian random matrix theories, we refer to a recent review by Fyodorov and Sommers [66].

#### A. Disordered System at Nonzero Imaginary Vector Potential

The spectral density of the Hamiltonian for a disordered system in an imaginary vector potential was derived in [45,65] using the supersymmetric method. In [45] this result was obtained starting from an ensemble of complex random matrices, whereas in [65] the starting point was a nonlinear \( \sigma \)-model which is also applicable beyond the ergodic domain. This nonlinear \( \sigma \) model is applicable to the Hatano-Nelson model [42] and belongs to the universality class of the (almost Hermitian) UE. In this section we analyze its partition function for \( n \) fermionic flavors. We find that it satisfies a Toda lattice equation enabling us to derive the known result [45,65] for the spectral density by means of the replica trick. The replicated bosonic partition function that enters in the replica limit of the Toda lattice equation is evaluated using the Ingham-Siegel integral [25].

1. **The fermionic partition function**

   The static limit of the partition function for \( n \) fermionic flavors is given by [65] (see also the discussion in [67])

\[ Z_n(y; a) = \frac{1}{\text{vol}(U(n))} \int dQ e^{-\frac{a^2}{2N} \text{Tr}[Q, \Sigma_3]^2 + \frac{yN}{2} \text{Tr} \Sigma_3 Q}, \]  

(62)

where the integral is over \( U(2n)/[U(n) \times U(n)] \). The matrix \( \Sigma_3 \) is defined by

\[ \Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  

(63)

and \( 1 \) is the \( n \times n \) unit matrix. The parameter \( a \) determines the strength of the imaginary vector potential, while \( y \) is the mass of the fermionic flavors (i.e. the imaginary part of the argument of the resolvent). In order to evaluate the integral we choose the parameterization [4,6]
\[ Q = T^\dagger \Sigma_3 T, \]  
with
\[ T = V^\dagger \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} V, \]  
and
\[ V = \begin{pmatrix} v_1^\dagger & 0 \\ 0 & v_2^\dagger \end{pmatrix}. \]

Here, \( v_1 \in U(n) \) and \( v_2 \in U(n)/U^n(1) \) and \( \theta = \text{diag}(\theta_1, \cdots, \theta_n) \). The integration measure is given by the product of the matrix elements of the off-diagonal blocks of
\[ \delta T \equiv V dT T^{-1} V^\dagger. \]

The calculation of the Jacobian of the transformation from the variables \{\( \delta T_{12}, \delta T_{21} \)\} to the variables \{\( dV V^\dagger, \delta \cos \theta \)\} is elementary and is given by
\[ J = \prod_k 2 \cos \theta_k \prod_{k<l} (\cos^2 \theta_k - \cos^2 \theta_l)^2. \]

The angular integrations factorize from the partition function and result in the volume of the integration manifold
\[ V_n = \text{vol}^2(U(n))/\text{vol}^n(U(1)). \]

In terms of the variables
\[ \lambda_k = \cos^2 \theta_k - \sin^2 \theta_k, \]
the partition function can thus be written as
\[ Z_n(y; a) = \frac{V_n}{2^{n(n-1)-nN!\text{vol}(U(n))}} \int_{-1}^1 \prod_k d\lambda_k \prod_{k<l} (\lambda_k - \lambda_l)^2 e^{-2a^2 N \sum_k (\lambda_k^2 - 1) + 2yN \sum_k \lambda_k}. \]

This partition function can be written as a determinant of derivatives
\[ Z_n(y; a) = \frac{V_n}{2^{2n(n-1)} N^n(n-1)!\text{vol}(U(n))^2} \det[\partial_y^{p+q} Z_1(y; a)]_{p,q=0,\ldots,n-1}. \]

where
\[ Z_1(y; a) = 2 \int_{-1}^1 d\lambda e^{-2a^2 N (\lambda^2 - 1) + 2yN \lambda}. \]

The partition function (62) therefore satisfies the Toda lattice equation (cf. the discussion in section II C 2)
\[ \frac{1}{N^2} \partial_y^2 \log Z_n(y; a) = \frac{8n}{\pi} \frac{Z_{n+1}(y; a)Z_{n-1}(y; a)}{Z_n^2(y; a)}. \]

2. The Spectral Density

The spectral density is given by \( z \equiv x + iy \)
\[ \rho(x, y; a) = \lim_{n \to 0} \frac{1}{n} \frac{1}{\pi} \frac{d}{dz} \frac{d}{dz^*} \log Z_n(z, z^*; a). \]

Taking the replica limit of the Toda lattice equation we thus find
\[ \rho(x, y; a) = \frac{8N^2}{\pi^2} Z_1(y; a) Z_{-1}(y; a) \]  

(76)

where (cf. (73))

\[ Z_1(y; a) = 4e^{2a^2N} \int_0^1 \cosh(2yNt) \exp\left(-2a^2Nt^2\right) dt \]  

(77)

and, as we will show below,

\[ Z_{-1}(y; a) = \frac{C_{-1}}{a} \sqrt{\frac{\pi}{2N}} \exp\left(-a^2N\right) \exp\left(-Ny^2/2a^2\right) \]  

(78)

The spectral density is \( x \)-independent. This implies that if we change the non-Hermiticity parameter \( a \) there is only spectral flow perpendicular to the \( y \)-axis. Therefore, the normalization \( C_{-1} \) of \( Z_{-1} \) has to be chosen such that the integral

\[ \frac{8N^2}{\pi^2} \int_{-\infty}^{\infty} dy Z_1(y; a) Z_{-1}(y; a) = \frac{32NC_{-1}}{\pi} e^{Na^2} \]  

(79)

is independent of \( a \). Normalizing the integrated spectral density to \( N \) per unit length we find

\[ C_{-1} = \frac{\pi}{32} e^{-Na^2}. \]  

(80)

Combining the previous five equations, we find the spectral density

\[ \rho(x, y; a) = \frac{N\sqrt{N}}{\sqrt{2\pi}a} \exp\left(-\frac{Ny^2}{2a^2}\right) \int_0^1 \cosh(2yNt) \exp\left(-2a^2Nt^2\right) dt, \]  

(81)

in complete agreement with earlier work [45,65]. Not only do we obtain the exact analytical result by means of the replica limit, but we have also shown why the spectral density factorizes into a bosonic and a fermionic partition function.

Before we derive \( Z_{-1} \) let us comment on the \( a \)-dependence of the normalization integral (79). The partition function \( Z_1 \) was derived from the effective partition function (62). The commutator term arises because of the requirement of local gauge invariance [43,44,65,68] of the effective Lagrangian with the kinetic term included. The non-Hermitian random matrix theory in [45] does not have this property. Starting from this random matrix model [45], which will be introduced in the next section, the exponential prefactor in (77) would have been \( \exp(Na^2) \) instead of \( \exp(2Na^2) \) and the normalization integral would have been automatically \( a \)-independent.

### 3. Calculation of \( Z_{-1} \) using the Ingham-Siegel integral

In this subsection we calculate \( Z_{-1} \) for a non-Hermitian random matrix model using the Ingham-Siegel integral as proposed in [25]. The saddle point manifold is obtained automatically as a result of this integral. This should be contrasted with the standard approach which uses the Hubbard-Stratonovitch transformation and where it is necessary to deform the integration contour in order to be able to interchange integrations [69].

The partition function \( Z_{-1} \) is defined by

\[ Z_{-1}(z, z^*; a) = \lim_{\epsilon \to 0, N \to \infty} C_{\epsilon} \left\langle \det^{-1} \begin{pmatrix} z^* + H - A & \epsilon \\ z + H + A & \epsilon \end{pmatrix} \right\rangle, \]  

(82)

where the average \( \langle \cdot \cdot \cdot \rangle \) is over the probability distribution

\[ P(H, A) = e^{-\frac{N}{2} \mathrm{Tr} H^\dagger H + \frac{N}{2} \mathrm{Tr} A^\dagger A}, \]  

(83)

and the \( N \times N \) matrices \( H \) and \( A \) are Hermitian, \( H^\dagger = H \), and anti-Hermitian, \( A^\dagger = -A \), respectively. The normalization constant will be chosen such that the limit \( \epsilon \to 0 \) is finite. In order to write the inverse determinant as a convergent bosonic integral
\[
\det^{-1} \left( z^* + H - A \right) = \int d\phi e^{i\epsilon \phi_k^* \phi_k + i\epsilon \phi_k^* \phi_k + i\epsilon (z_\delta + H_{\delta} + A_{\delta}) \phi_k^* + i\epsilon (z_\bar{\delta} + H_{\bar{\delta}} - A_{\bar{\delta}} \phi_k^*},
\]
the imaginary part of \(\epsilon\) has to be positive. We collect the bosonic fields into the positive definite \(2 \times 2\) matrix

\[
\bar{Q}_{ij} \equiv \sum_{k=1}^N (\phi_k^*)^k \phi_k^j.
\]
If we introduce a mass matrix by \((\sigma_i\) are the Pauli matrices\)

\[
\zeta = x\sigma_1 - y\sigma_2
\]
the partition function can be written as

\[
\mathcal{Z}^{-1}(z, z^*; a) = \lim_{\epsilon \to 0, N \to \infty} C_\epsilon \int d\phi_1 dQ dF e^{i\epsilon (\zeta^T + \epsilon) \bar{Q} + \epsilon \Theta(Q) + \epsilon \Theta(Q)} - \frac{\epsilon}{2} \Theta(Q)\sigma_1 \Theta(Q) - \frac{\epsilon}{2} \Theta(Q)\sigma_2 \Theta(Q).
\]
We calculate this integral in the weak non-Hermiticity limit and the microscopic limit, i.e. \(N \to \infty\) with \(\alpha^2 N\) and \(zN\) fixed. In this limit, the saddle point equation, given by

\[
\mathcal{Z}^{-1}(z, z^*; a) = \lim_{\epsilon \to 0, N \to \infty} C_\epsilon \int dQdF d\epsilon d\epsilon F e^{i\epsilon (\zeta^T + \epsilon) \bar{Q} + \epsilon \Theta(Q) + \epsilon \Theta(Q)} - \frac{\epsilon}{2} \Theta(Q)\sigma_1 \Theta(Q) - \frac{\epsilon}{2} \Theta(Q)\sigma_2 \Theta(Q).
\]
Under the assumption that the integrals can be interchanged we obtain

\[
\mathcal{Z}^{-1}(z, z^*; a) = \lim_{\epsilon \to 0, N \to \infty} C_\epsilon \int dQdF d\epsilon d\epsilon F e^{i\epsilon (\zeta^T + \epsilon) \bar{Q} + \epsilon \Theta(Q) + \epsilon \Theta(Q)} - \frac{\epsilon}{2} \Theta(Q)\sigma_1 \Theta(Q) - \frac{\epsilon}{2} \Theta(Q)\sigma_2 \Theta(Q).
\]
The integral over \(F\) is an Ingham-Siegel integral of the second kind. For \(\text{Im}(\epsilon) < 0\) it is given by [25]

\[
\int dF \epsilon^{-N}(\epsilon + \epsilon) e^{i\epsilon \Theta(Q)} = C_{N,p} \theta(Q) \epsilon^{-N}(Q) e^{-i\epsilon \Theta(Q)},
\]
where the integral is over \(p \times p\) Hermitian matrices, \(C_{N,p}\) is an irrelevant constant, and \(\theta(Q)\) denotes that \(Q\) is positive definite. After rescaling \(Q \to NQ\) we find (\(N > 2\))

\[
\mathcal{Z}^{-1}(z, z^*; a) = \lim_{\epsilon \to 0, N \to \infty} C_\epsilon \int dQ \epsilon \Theta(Q) \epsilon^{-N}(Q) e^{i\epsilon \Theta(Q)} = C_{N,p} \theta(Q) \epsilon^{-N}(Q) e^{-i\epsilon \Theta(Q)},
\]
We calculate this integral in the weak non-Hermiticity limit and the microscopic limit, i.e. \(N \to \infty\) with \(\alpha^2 N\) and \(zN\) fixed. In this limit, the saddle point equation, given by

\[
(\sigma_1 Q)^2 = 1,
\]
fixes two of the four degrees of freedom in \(Q\). In order to complete the evaluation of \(\mathcal{Z}^{-1}\), we will choose an explicit parametrization of \(Q\) and perform the integration over the two degrees of freedom not fixed by the saddle point equation exactly. Positive definite \(2 \times 2\) Hermitian matrices can be parameterized as

\[
Q = e^t \left( \begin{array}{cc} e^u \cosh s & i e^{i\phi} \sinh s \\ -i e^{-i\phi} \sinh s & e^{-u} \cosh s \end{array} \right).
\]
The Jacobian for the transformation from the \(Q\) variables to the variables of this parameterization is given by

\[
J = 4e^{4t} \cosh s \sinh s.
\]
The saddle point condition (93) is given by

\[
t = 0, \quad \phi = 0,
\]
whereas the variable $s$ and $u$ are not determined by the saddle point equations. They parameterize the saddle point manifold. The Gaussian fluctuations in the $t$ and $\phi$ variables about this saddle point manifold give rise to a factor $1/(2 \sinh s)$. As final result we thus obtain

$$Z_{-1}(z, z^*; a) = \lim_{\epsilon \to 0, N \to \infty} 2C_c \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} ds \cosh s e^{a^2 N (1 - 2 \cosh^2 s) - i2Ny \sinh s + 2iNc \cosh u \cosh s}.$$  

(97)

The integral over $u$ results in a contribution $\sim - \log \epsilon$. This leading order singularity in $\epsilon$ will be canceled by the normalization constant $C_c$ and the remaining prefactors, $2C_c$, will be denoted by $C_{-1}$. The integral over $\sinh s$ can then be performed by completing squares. This results in the Gaussian $y$ dependence (78)

$$Z_{-1}(y; a) = \frac{C_{-1}}{a} \sqrt{\frac{\pi}{2N}} \exp(-a^2 N) \exp \left(-\frac{Ny^2}{2a^2} \right).$$  

(98)

The Toda lattice equation suggests that the partition function with $n$ bosonic flavors can be expressed as a determinant of derivatives. This leads us to conjecture

$$Z_n(y; a) = C_{-n} \det[\partial^{p+q}_{y}] Z_{-1}(y; a)]_{p,q=0,...,n-1}$$  

(99)

$$\sim \exp\left[\frac{-nNy^2}{a^2} \right]$$  

(100)

where $Z_{-1}$ is given by (78). This conjecture will be discussed in a separate publication [52].

B. The Phase Quenched QCD Partition Function

In this section we derive the microscopic spectral density of the quenched Euclidean QCD Dirac operator at nonzero baryon chemical potential. The calculation is performed in the weak non-Hermiticity limit, where only terms to up to second order in the chemical potential contribute to the effective action. In this subsection we follow the usual convention that the massless Dirac operator at zero chemical potential is anti-Hermitian. Therefore “weak non-Hermiticity” is actually “weak non-anti-Hermiticity”, but we will refrain from using this name.

The generating function for the quenched spectral density at nonzero chemical potential is not the QCD partition function with $n$ flavors, but rather the QCD partition function with $n$ flavors and $n$ conjugate flavors $[71,72]$. Since the fermion determinants of the flavors and the conjugate flavors are each others complex conjugate this partition function is also known as the phase quenched QCD partition function. For $n$ flavors with mass $z$ and $n$ flavors with mass $z^*$ it is given by (in the sector of zero topological charge)

$$Z_n(z, z^*; \mu) = \left\langle \det^n \left( \begin{array}{cc} z & id + \mu \\ id^I + \mu & z \end{array} \right) \det^n \left( \begin{array}{cc} z^* & -id + \mu \\ -id^I + \mu & z^* \end{array} \right) \right\rangle_{Y^2M}, \quad z = x + iy,$$  

(101)

where $id$ is the covariant derivative and the average over gauge field configurations is weighted by the Yang-Mills action. For positive $n$ the low-energy limit of this theory in the chirally broken phase is $[73]$

$$Z_n(z, z^*; \mu) = \frac{1}{(2\pi)^{n(2n+1)} \text{vol}(U(n))} \int_{U \subseteq U(2n)} dU e^{-\frac{\mu^2}{4} \text{Tr}(U^T U) + \frac{1}{2\Sigma} \text{Tr}(MU + M^T)}$$  

(102)

with

$$B = \Sigma_3 \quad \text{and} \quad M = \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}. $$  

(103)

$^1$We stress that the combination $(MU + M^T)$ enters the action and not the usual combination $(M^T U + MU)$). The reason is that the phase quenched QCD action (101) has $n$ flavors of mass $z$ and other $n$ flavors has mass $z^*$. The mass term that follows from chiral invariance is given by $\text{Tr}(M_{RL} U + M_{LR} U^T)$ with $M_{RL}$ the right-left handed mass matrix and $M_{LR}$ the left-right handed mass matrix. The usual mass term is obtained if $M_{RL} = M_{LR}^T$. We however have that $M_{RL} = M_{LR} = M$ with $M$ complex.
The volume of space-time is denoted by $V$. The resolvent is defined by

$$G(z, z^*; \mu) = \lim_{n \to \infty} \frac{1}{nV} \partial_z \log Z_n(z, z^*; \mu). \quad (104)$$

At nonzero $\mu$ the support of the spectrum of the Dirac operator is a two-dimensional domain in the complex plane with spectral density given by

$$\rho(z; \mu) = \frac{1}{\pi} \partial_z \cdot G(z, z^*; \mu). \quad (105)$$

In the thermodynamic limit at fixed $\mu$ and $m$, the effective partition function (102) can be analyzed in terms of mean field theory [68,73,74]. The result is [73] that the spectral density is constant inside the domain

$$|\text{Re}z| < \frac{2\mu^2 F^2}{\Sigma} \quad (106)$$

and zero outside this domain.

1. Toda Lattice Equation

Here we will analyze the effective partition function in the microscopic limit where both $\mu^2 V$ and $zV$ remain fixed in the thermodynamic limit. Our aim is to calculate the spectral density of the quenched theory. To do this we will show that $Z_n$ satisfies the Toda lattice equation. The proof follows directly from the proof for two point function at zero chemical potential: In the parameterization discussed in section II C the trace of the first term in the action is given by

$$\text{Tr}[U, B][U^\dagger, B] = 8\sum_{k=1}^{n} (\lambda_k^2 - 1). \quad (107)$$

Repeating the steps (43)-(46) with this factor included we find (we recall the notation $\delta_z \equiv z\partial_z$)

$$Z_n(z, z^*; \mu) = \frac{D_n}{(zz^*)^{n(n-1)}} \det \left[ \delta_z \delta_{z^*} Z_1(z, z^*; \mu) \right]_{k,l=0,1,\ldots,n-1}, \quad (108)$$

with

$$Z_1(z, z^*; \mu) = \frac{1}{\pi} e^{2VF^2\mu^2} \int_0^1 d\lambda \lambda e^{-2VF^2\mu^2\lambda^2} I_0(\lambda z \Sigma V) I_0(\lambda z^* \Sigma V) \quad (109)$$

and

$$D_n = \frac{2^{n(n+1)/2}}{\pi^{n(n-1)/2} \prod_{k=1}^{n-1} k!} \quad (110)$$

This results in the Toda lattice equation

$$\delta_z \delta_{z^*} \log Z_n(z, z^*; \mu) = \frac{\pi n}{2} (zz^*)^2 \frac{Z_{n+1}(z, z^*; \mu) Z_{n-1}(z, z^*; \mu)}{Z_n^2(z, z^*; \mu)} \quad (111)$$

The quenched spectral density is then given by

$$\rho(x, y; \mu) = \lim_{n \to \infty} \frac{1}{\pi n} \partial_z \partial_{z^*} \log Z_n(z, z^*; \mu)$$

$$= \lim_{n \to \infty} \frac{zz^*}{2} \frac{Z_{n+1}(z, z^*; \mu) Z_{n-1}(z, z^*; \mu)}{Z_n^2(z, z^*; \mu)}. \quad (112)$$

In our normalization $Z_0(z, z^*) = 1$ so that

$$\rho(x, y; \mu) = \frac{zz^*}{2} Z_1(z, z^*; \mu) Z_1(z, z^*; \mu). \quad (113)$$

Our next task is to calculate $Z_{-1}(z, z^*; \mu)$. To do this we derive the static low energy partition function from a chiral random matrix model corresponding to (101). The main problem is to choose the correct integration manifold. As is discussed in the next section, this can be done without guess work if we use the Ingham-Siegel integral as proposed in [25]. As an introduction to this method we have included the calculation of $Z_{-1}(z, z^*)$ for $\mu = 0$ in Appendix C.
2. The effective partition function $Z_{-1}$

In this section we derive the effective phase quenched partition function in the microscopic and weak non-Hermiticity limit. The integration manifold is determined by symmetries and convergence requirements. Usually, these conditions define a suitable integration contour for the $\sigma$-field. A method that avoids this somewhat ad hoc procedure was advocated in [46,25]. Below we will follow this approach in the calculation of the phase quenched partition function for one bosonic flavor.

The partition function for one bosonic flavor and one conjugate bosonic flavor is given by the integral

$$\int dW e^{-\frac{1}{2} \text{Tr} W W^\dagger} \text{det}^{-1} \left( \begin{array}{cc} z & iW + \mu \\ iW + \mu & z^* \end{array} \right) \text{det}^{-1} \left( \begin{array}{cc} z^* & -iW + \mu \\ -iW + \mu & z \end{array} \right).$$

(114)

If $z$ is inside the domain of eigenvalues, the partition function potentially diverges. Therefore, we have to regularize the determinants. A suitable regularization is one where the determinant can be written as a convergent bosonic integral. This procedure, which is also known as hermitization [75], amounts to rewriting the product of the two determinants as

$$\lim_{\epsilon \to 0} \text{det}^{-1} \left( \begin{array}{cc} z & iW + \mu \\ iW + \mu & z^* \end{array} \right) \text{det}^{-1} \left( \begin{array}{cc} z^* & -iW + \mu \\ -iW + \mu & z \end{array} \right).$$

(115)

After rearranging the rows and columns inside the determinant, the partition function (114) can be written as

$$Z_{-1} = \lim_{\epsilon \to 0, N \to \infty} C_e \int dW e^{-\frac{1}{2} \text{Tr} W W^\dagger} \int d\phi_k d\phi_k^* \exp[i \left( \begin{array}{cccc} \phi_1^T & 0 & 0 & 0 \\ 0 & \phi_2^T & 0 & 0 \\ 0 & 0 & \phi_3^T & 0 \\ 0 & 0 & 0 & \phi_4^T \end{array} \right) \left( \begin{array}{cccc} \epsilon & z & iW + \mu & 0 \\ z^* & \epsilon & 0 & -iW + \mu \\ -iW + \mu & 0 & \epsilon & z^* \\ 0 & -iW + \mu & -z & \epsilon \end{array} \right) \left( \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{array} \right)].$$

(116)

After averaging over $W$ the limit $\epsilon \to 0$ is potentially singular. Such singularity will be absorbed as a multiplicative renormalization constant into $C_e$. As in the case of the Hatano-Nelson model the overall prefactor of $Z_{-1}$ will be fixed by the normalization of the complex density. For this reason we do not keep explicitly track of the prefactor, it will be simply labeled by $C_e$.

The Gaussian integration over $W$ results in a 4-boson interaction term given by

$$\exp \left[ -\frac{2}{N} \text{Tr} \tilde{Q}_1 \tilde{Q}_2 \right],$$

(117)

where

$$\tilde{Q}_1 \equiv \left( \phi_1^* \phi_1 \phi_2^* \phi_2, \phi_2^* \phi_1 \phi_2^* \phi_2 \right), \quad \tilde{Q}_2 \equiv \left( \phi_3^* \phi_3 \phi_4^* \phi_4, \phi_4^* \phi_3 \phi_4^* \phi_4 \right),$$

(118)

and we have used the notation

$$\phi_k^* \phi_l = \sum_{i=1}^{N/2} \phi_k^* \phi_i^* \phi_i.$$

(119)

Instead of the usual Hubbard-Stratonovitch transformation, we linearize the 4-boson interaction term with the help of matrix $\delta$ functions. For Hermitian matrices $Q_i$ and $\bar{Q}_i$, the $\delta$-function can be represented as

$$\delta(Q_i - \bar{Q}_i) = \frac{1}{(2\pi)^4} \int dF e^{-\text{Tr} F(Q_i - \bar{Q}_i)},$$

(120)

where the integral is over Hermitian matrices $F$. This results in
\[
\exp\left[-\frac{2}{N} \text{Tr} \bar{Q}_1 Q_2 \right] = \frac{1}{(2\pi)^N} \int dQ_1 dQ_2 \int dF dG e^{\text{Tr}[-iF(Q_1-Q_1)-iG(Q_2-Q_2)-\frac{\mu}{2}Q_1 Q_2]}, \tag{121}
\]

The integral over the \( \phi_k \) is uniformly convergent in \( F \) and \( G \) which allows us to interchange the order of these integrals. We finally obtain the partition function

\[
Z_{-1} = \lim_{\epsilon \to 0, N \to \infty} C \int dQ_1 dQ_2 \int dF dG e^{\text{Tr}[-iFQ_1-iGQ_2-\frac{\mu}{2}Q_1 Q_2]} \det^{-\frac{N}{2}} \left( \zeta + F^T \frac{\mu \sigma_3}{\mu \sigma_3} - \frac{\mu \sigma_3}{\mu \sigma_3} + G^T \right), \tag{122}
\]

where we have used a block notation and

\[
\zeta = \begin{pmatrix} \epsilon & z \\ z^* & \epsilon \end{pmatrix}. \tag{123}
\]

The anti-symmetric matrix \( I \) is defined by

\[
I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{124}
\]

We further simplify this integral by changing integration variables according to \( F \to F - \zeta^T \) and \( G \to G + I \zeta^T I \) and \( Q_i \to NQ_i/2, i = 1, 2 \). This results in

\[
Z_{-1} = \lim_{\epsilon \to 0, N \to \infty} C \int dQ_1 dQ_2 \int dF dG e^{\text{Tr}[-iFQ_1-iGQ_2-i\frac{\mu}{2} \zeta(Q_1-Q_2)-\frac{\mu}{2}Q_1 Q_2]} \det^{-\frac{N}{2}} \left( \epsilon + F \frac{\mu \sigma_3}{\mu \sigma_3} - \frac{\mu \sigma_3}{\mu \sigma_3} + G \right), \tag{125}
\]

where, for reasons of convergence we also left infinitesimal increments inside the determinant. Contrary to the example in Appendix C, we were not able to perform the \( F \) and \( G \) integrations analytically. However, in the weak non-Hermiticity limit, where \( \mu^2 N \) remains fixed in the limit \( N \to \infty \), the determinant is given by

\[
\det^{-\frac{N}{2}} \left( \epsilon + F \frac{\mu \sigma_3}{\mu \sigma_3} - \frac{\mu \sigma_3}{\mu \sigma_3} + G \right) = \det^{-\frac{N}{2}} \left( \epsilon + F \right) \det^{-\frac{N}{2}} \left( \epsilon + G \right) \exp \left[ \frac{N \mu^2}{2} \text{Tr} \left( \frac{1}{\epsilon + F} \frac{1}{\epsilon + G} \right) \right] \left( 1 + O \left( \frac{1}{N} \right) \right). \tag{126}
\]

In the limit \( N \to \infty \), at fixed \( \mu^2 N \), the \( F \) and \( G \) variables in the \( \mu^2 N \) term can be replaced with the saddle point values of \( F \) and \( G \) at \( \mu = 0 \). They are given by

\[
\frac{1}{\epsilon + F} = iQ_1, \quad \frac{1}{\epsilon + G} = iQ_2. \tag{127}
\]

The remaining integrals over \( F \) and \( G \) are Ingham-Siegel integrals (see (91)). They can be performed exactly resulting in

\[
Z_{-1}(z, z^*; \mu) = \lim_{\epsilon \to 0, N \to \infty} C \int dQ_1 dQ_2 \theta(Q_1) \theta(Q_2) \det^{-\frac{N}{2}}(Q_1 Q_2) e^{\text{Tr}[i\frac{\mu}{2} \zeta^T(Q_1-Q_2)-\frac{\mu}{2}Q_1 Q_2-\frac{\mu}{2}Q_1 \sigma_3 Q_2 \sigma_3]}, \tag{128}
\]

where we remind the reader that \( \zeta \) contains the regulator mass \( \epsilon \). (Recall that \( \theta(Q) \) is a matrix step function which is equal to unity if \( Q \) is Hermitian and positive definite and equal to zero otherwise). In the limit \( N \to \infty \), the integrals over the massive modes can be performed by a saddle point approximation. The saddle point equations are given by

\[
Q_1^{-1} - Q_2 = 0, \quad Q_2^{-1} - Q_1 = 0. \tag{129}
\]

Both equations can be rewritten as

\[
Q_1 = Q_2^{-1}, \tag{130}
\]

and therefore only four of the modes, which we choose to be \( Q_2 \), can be integrated out by a saddle point approximation. The quadratic fluctuations give rise to a factor \( \pi^2 / \text{det}^2 Q_1 \). The integral over the remaining modes has to be performed exactly. We thus arrive at the partition function

\[
Z_{-1}(z, z^*; \mu) = \lim_{\epsilon \to 0} C \int \frac{dQ_1}{\text{det}^2 Q_1} \theta(Q_1) e^{\text{Tr}[i\frac{\mu}{2} \zeta^T(Q_1^{-1}-1)-\frac{\mu}{2}Q_1 \sigma_3 Q_2 \sigma_3 \sigma_3]}, \tag{131}
\]

Comparing this result with the fermionic effective partition function (102) we can make the following identification

\[
N \leftrightarrow V, \quad \Sigma \leftrightarrow 1, \quad \mu^2 \leftrightarrow \mu^2 F_\pi^2. \tag{132}
\]

We will evaluate this integral explicitly in section III B 4. Before doing so we now rederive this partition function based on the symmetries of the QCD partition function.
3. Symmetries of $Z_{-1}$

In this section we show that the effective partition function (131) is completely determined by the symmetries of the underlying microscopic partition function. This implies that any theory with the same pattern of global symmetries and spontaneous symmetry breaking is described by the same low-energy effective theory. The only memory of the microscopic theory are the two constants, $\Sigma$ and $F_\pi$, in (131). In particular, this means that the low-energy limit of QCD and the random matrix model discussed in the previous section are the same. The virtue of the random matrix approach is that the low-energy limit of the partition function can be derived directly from the microscopic theory.

For $\zeta = 0$ and $\mu = 0$ the random matrix partition function (116) is invariant under $Gl(2)/U(2) \times U(2)$. Explicitly, the axial $Gl(2)/U(2)$ invariance is realized by

$$
\left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 
\end{array} \right) \rightarrow \left( \begin{array}{cc}
U_A & 0 \\
0 & U_A^{-1}
\end{array} \right) \left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{array} \right) \quad \text{with} \quad U_A^T U_A^{-1} = 1.
$$

(133)

For reasons of convergence we can allow only symmetry transformations that do not alter the complex conjugation structure of the partition function. Therefore,

$$
U_A = e^H, \quad \text{with} \quad H^T = H.
$$

(134)

The vector $U(2)$ symmetry is realized by

$$
\left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{array} \right) \rightarrow \left( \begin{array}{cc}
U_V & 0 \\
0 & U_V^{-1}
\end{array} \right) \left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{array} \right) \quad \text{with} \quad U_V^T U_V = 1.
$$

(135)

Imposing these symmetries at nonzero mass and chemical potential requires that, in the block notation,

$$
\left( \begin{array}{cc}
\zeta_1 & \mu_1 \\
\mu_2 & \zeta_2
\end{array} \right)
$$

(136)

the mass and chemical potential are transformed as

$$
\zeta_1 \rightarrow U_A^{-1} \zeta_1 U_A^{-1}, \quad \mu_1 \rightarrow U_A^{-1} \mu_1 U_A,
$$

$$
\zeta_2 \rightarrow U_A \zeta_2 U_A^{-1}, \quad \mu_2 \rightarrow U_A \mu_2 U_A^{-1},
$$

(137)

under axial transformations and as

$$
\zeta_1 \rightarrow U_V \zeta_1 U_V^{-1}, \quad \mu_1 \rightarrow U_V \mu_1 U_V^{-1},
$$

$$
\zeta_2 \rightarrow U_V \zeta_2 U_V^{-1}, \quad \mu_2 \rightarrow U_V \mu_2 U_V^{-1}
$$

(138)

under vector transformations. The matrix $\tilde{Q}_1$ introduced in (118) transforms as

$$
\tilde{Q}_1 \rightarrow U_A^T \tilde{Q}_1 U_A^T, \quad \tilde{Q}_1 \rightarrow U_V^T \tilde{Q}_1 U_V^T,
$$

(139)

under axial and vector transformations, respectively. To first order in the mass matrix and second order in the chemical potential, we therefore can write down the following nontrivial invariants ($Q_1$ has the same transformation properties as $\tilde{Q}_1$)

$$
\text{Tr} Q_1 \zeta_1 T, \quad \text{Tr} Q_1^{-1} \zeta_2 T, \quad \text{Tr} Q_1 \mu_2^T Q_1^{-1} \mu_1^T.
$$

(140)

Using that

$$
\zeta_1 = \zeta, \quad \zeta_2 = -I \zeta I, \quad \mu_1 = \mu_2 = \mu \sigma_3,
$$

(141)

we find the nontrivial invariant terms

$$
\text{Tr} Q_1 \zeta_1 T, \quad -\text{Tr} Q_1^{-1} I \zeta_1 T I, \quad \mu^2 \text{Tr} Q_1 \sigma_3 Q_1^{-1} \sigma_3.
$$

(142)
The Jacobian of this transformation is easily found to be
\[ \text{Tr} Q_1 \zeta^T - \text{Tr} Q_1^{-1} \zeta^T I. \]  
(143)

The numerical prefactors of the invariant terms are not fixed by the global symmetries of the random matrix model but follow from matching to the microscopic theory.

At nonzero chemical potential the phase quenched QCD partition function with one pair of bosonic flavors has to be hermiticized as in the random matrix model (116) resulting in a Dirac operator with the structure given in (116) and a mass matrix as in (123). The partition function has the global axial and vector symmetries of (133) and (135) if the mass and chemical potential matrices are transformed according to (137) and (138) as well as the discrete symmetry \( \zeta \to -I\zeta I \). The Goldstone manifold is parameterized by the axial transformations and is therefore the coset of positive definite matrices [21,22] which transform according to (139). At zero chemical potential, the effective Lagrangian density to order \( p^2 \) respecting these symmetries is therefore given by
\[ L_{\nu} = \frac{F^2}{4} \text{Tr} \partial_\mu Q \partial_\mu Q^{-1} - i \Sigma \frac{1}{2} (\text{Tr} Q \zeta^T - \text{Tr} Q^{-1} \zeta^T I). \]  
(144)

The phase of the mass term is determined by the condition that the integral over \( Q \) is convergent for \( \epsilon > 0 \). For real quark masses, \( z = z^* = m \), the mass term is minimized by the saddle-point solution \( Q = -i\sigma \). The vacuum energy is therefore given by \( 2m\Sigma V \) which identifies \( \Sigma \) as the chiral condensate. As is the case in the supersymmetric formulation of partially quenched chiral perturbation theory [14], the pion decay constant in \( L_{\nu} \) is taken to be the same as in the case of fermionic quarks.

The chemical potential enters in the QCD partition function as an external vector field,
\[ V_\nu = \mu \sigma_3 \delta_{\nu 0}. \]  
(145)

Therefore, the vector symmetry (135) can be promoted to a local symmetry by transforming \( V_\nu \) as a nonabelian gauge field
\[ V_\nu \to U_\nu^{-1} \partial_\nu U_\nu + U_\nu^{-1} V_\nu U_\nu. \]  
(146)

The effective Lagrangian at nonzero chemical potential should respect this symmetry. This is achieved by replacing the derivative in the kinetic term by the covariant derivative
\[ \partial_\nu Q = \nabla_\nu Q = \partial_\nu Q + [V_\nu, Q] \]  
(147)

with \( V_\nu \) given by (145). The low energy effective Lagrangian density is therefore given by
\[ L_{\nu} = \frac{F^2}{4} \text{Tr} \nabla_\nu Q \nabla_\nu Q^{-1} - i \Sigma \frac{1}{2} (\text{Tr} Q \zeta^T - \text{Tr} Q^{-1} \zeta^T I). \]  
(148)

In a parameter domain where the fluctuations of the zero momentum modes are much larger than the fluctuations of the nonzero momentum modes the zero momentum part factorizes from the partition function [50] and is given by
\[ Z_{\nu}(z, z^*; \mu) = \lim_{\epsilon \to 0} C \int \frac{dQ}{\det^2 Q} \theta(Q) e^{\text{Tr}[i \zeta^T (Q - iQ^{-1})] - \frac{1}{2} p^2 \mu^2 |Q| |Q^{-1}| \sigma_3}, \]  
(149)

where \( dQ/\det^2 Q \theta(Q) \) is the integration measure on positive definite Hermitian matrices. Up to an overall constant, this result agrees with the effective random matrix partition function imposing the identification (132).

4. Evaluation of \( Z_{\nu} \)

We parameterize the positive definite matrix \( Q_1 \) in (131) as
\[ Q_1 = e^t \left( \begin{array}{cc} e^r \cosh s & e^{i\theta} \sinh s \\ e^{-i\theta} \sinh s & e^{-r} \cosh s \end{array} \right). \]  
(150)

The Jacobian of this transformation is easily found to be
The range of the integration variables is given by
\[ r \in (-\infty, \infty), \quad s \in (-\infty, \infty), \quad t \in (-\infty, \infty), \quad \theta \in (0, \pi) \]

Inserting this parameterization in (131) we find that the partition function is given by the following integral
\[
Z_{-1} = \lim_{\epsilon \to 0} Z_{-1} \left( \epsilon, x, y, \theta \right) = C_{-1} \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} ds \int_{0}^{\pi} d\theta \ cosh s \sinh s \left| e^{\frac{1}{2}(4x \sinh s \cos \theta - 4y \sinh s \sin \theta + 4r \cosh s \cosh t)} - \mu^2 N(1 + 2 \sinh^2 s) \right|.
\]

The integral over \( r \) results in the modified Bessel \( K_0(2Nt \cosh s \cosh t) \) with leading singularity given by \( \sim - \log \epsilon \). This factor is absorbed in the normalization of the partition function. Then the integral over \( \theta \) gives a Bessel function. Introducing \( u = \sinh s \) as new integration variable we obtain
\[
Z_{-1}(z, z^*; \mu) = C_{-1} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\theta J_0(2Nu(x^2 \cosh^2 t + y^2 \sinh^2 t)^{1/2}) e^{-\mu^2 N(1 + 2u^2)},
\]

where the finite prefactor has been labeled \( C_{-1} \). The integral over \( u \) is a known integral over a Bessel function. The final result for the phase quenched partition function with one bosonic flavor is thus given by
\[
Z_{-1}(z, z^*; \mu) = \frac{C_{-1} e^{-\frac{N\mu^2}{4}}}{4\mu^2 N} \int_{-\infty}^{\infty} dt \frac{N \nu^2 \nu^2 + 2 \int_{-\infty}^{\infty} du \frac{N(x^2 + y^2)}{4\mu^2} K_0(\frac{N(x^2 + y^2)}{4\mu^2}).}
\]

5. The Spectral Density in the Complex Plane

Reminding that, with the identification (132), the partition function for one fermionic flavor is given by
\[
Z_1(z, z^*; \mu) = \frac{1}{N} e^{2N\mu^2} \int_{0}^{1} d\lambda e^{-2N\mu^2 \lambda^2} I_0(\lambda z N) I_0(\lambda z^* N),
\]

the final result for the spectral density of the chGUE in the weak non-Hermitian limit obtained from the replica limit (112) of the Toda lattice equation (111) is given by \((z = x + iy)\)
\[
\rho(x, y; \mu) = \frac{z z^*}{2} Z_1(z, z^*; \mu) Z_{-1}(z, z^*; \mu)
\]

As in section (III A) the constant \( C_{-1} \) can be obtained from a spectral flow argument. In this case, the spectral flow with variation of \( \mu \) becomes perpendicular to the \( y \) axis only for \( N y^2 \gg \mu \). In this limit we can use the large argument asymptotic approximation of the Bessel functions,
\[
K_0(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad I_0(z) = \frac{1}{\sqrt{2\pi z}} (e^z + ie^{-z}), \quad I_0(z^*) = \frac{1}{\sqrt{2\pi z^*}} (e^{z^*} - ie^{-z^*}),
\]

resulting in the large \( y \) asymptotic limit of the spectral density (note that the oscillating terms in \((z - z^*)N \lambda \) are subleading after integration over \( \lambda \))
\[
\rho(x, y; \mu) = \frac{C_{-1} e^{N\mu^2}}{4(2\pi)^{3/2} N^{5/2} \mu} \int_{0}^{1} d\lambda e^{-2N\mu^2 \lambda^2} (e^{2\lambda N x} + e^{-2\lambda N x}).
\]

The integral over \( x \) of this expression can be calculated analytically,
\[
\int_{-\infty}^{\infty} \rho(x, N y^2 \gg \mu^2; \mu) dx = \frac{C_{-1} e^{N\mu^2}}{4\pi N^3}
\]
and should be independent of $\mu$. In the normalization of $N$ eigenvalues per unit length in the $y$ direction this results in

$$C_{-1} = 4\pi N^4 e^{-N\mu^2}. \quad (161)$$

The microscopic spectral density, obtained in the limit $N \to \infty$ with $\mu_s = \mu^2 N$ and $\eta = zN$ fixed, is thus given by

$$\rho_s(\eta, \eta^*; \mu_s) = \lim_{N \to \infty} \frac{1}{N^2} \Re(\eta) \Im(\eta) \frac{\mu_s}{\sqrt{N}} \exp(-\eta^2 + \eta^*^2) K_0\left(\frac{\eta \eta^*}{4\mu_s^2}\right) \int_0^1 d\lambda \lambda e^{-2\eta^2\lambda^2} I_0(\lambda \eta) I_0(\lambda \eta^*). \quad (162)$$

Again we note that $Z_{-1}$ was calculated starting from a chiral random matrix model, whereas $Z_1$ was obtained from an effective partition function with a commutator term that was a remnant of the local gauge invariance of the chiral Lagrangian. If we also would have derived $Z_1$ from the random matrix model, the exponential prefactor would have been $e^{N\mu^2}$ instead of $e^{2N\mu^2}$, and the normalization integral would have been automatically $\mu$-independent.

The leading order asymptotic expansion of the bosonic partition function in $\mu^2/N(x^2 + y^2)$ is given by

$$Z_{-1}(x, y; \mu) = \frac{C_{-1}}{2\mu N \sqrt{2N(x^2 + y^2)}} e^{-\frac{8\mu^2}{2\mu N}(1 + \mathcal{O}\left(\frac{\mu^2}{N(x^2 + y^2)}\right))}. \quad (163)$$

If we insert this result in the expression for the spectral density (157) we reproduce the result in [48] which was obtained by means of the complex orthogonal method [76] for a closely related partition function defined in an eigenvalue representation. The discrepancy between the phase quenched partition function (102) and the partition function in [48] is even more remarkable since it was shown [77] that, in versions of both models with the phase of the fermion determinant not quenched, the two models coincide in the weak non-Hermiticity limit.

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**FIG. 1.** The radial microscopic spectral density of the quenched QCD partition function at nonzero chemical potential. The histogram shows the result of a numerical simulation in which 2,000,000 random matrices of size 200 × 200 were diagonalized. The value of the chemical potential is given by $\mu^2 N = 0.04$ with $N = 200$. The full line is the exact analytical result and is in agreement with the data. Also shown is the result with the asymptotic limit of the bosonic partition function in the expression for the density (dotted curve). This curve fails to describe the data when $r/\mu$ is not in the asymptotic domain $r/\mu \gg 1$. The left figure is a blown-up version of the right figure for $r < 0.005$. 

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To convince the reader of the correctness of our result, (162), we have performed a numerical diagonalization of 2000000 matrices of the form
\[
\begin{pmatrix}
0 & iW + \mu \\
iW^\dagger + \mu & 0
\end{pmatrix}.
\]
(164)
The matrices where of size \(N = 200\) and drawn on the weight (15). In Fig. 1, we have plotted a histogram of the radial density,
\[
\rho(r; \mu) = \int_0^{2\pi} d\phi \rho(x = r \cos \phi, y = r \sin \phi; \mu)
\]
(165)
for \(\mu^2 N = 0.04\). In the same figure we plot our analytical result (solid curve) and the leading order asymptotic approximation of our result (dotted curve) which coincides with the result of [48]. Recently, Dirac spectra were calculated for quenched lattice QCD at nonzero chemical potential [78]. Although the lattice spectra were shown [78] to be in agreement with [48], the statistical accuracy is not sufficient to distinguish between the two analytical results. We are looking forward to simulations that will resolve this issue.

In the limit \(y^2 N \gg \mu^2\) and \(\mu^2 N \gg 1\) the eigenvalues are distributed homogeneously in a strip parallel to the \(y\)-axis as has been shown by a mean field argument [73]. In this limit the Bessel functions in (162) can be approximated by their asymptotic expansion (see (159)) resulting in
\[
\rho_s(\eta, \eta^*; \mu_s) = \frac{1}{2\mu_s \sqrt{2\pi}} \exp\left(-\frac{(\eta + \eta^*)^2}{8\mu_s^2}\right) \int_0^1 d\lambda e^{-2\mu_s^2 \lambda^2 (e^{\lambda (\eta + \eta^*)} + e^{-\lambda (\eta + \eta^*)})}.
\]
(166)
The saddle point of the \(\lambda\)-integral is located at
\[
\bar{\lambda} = \frac{\eta + \eta^*}{4\mu_s^2}.
\]
(167)
If this saddle point is outside the integration domain, the integral is exponentially suppressed. We thus find that in the limit \(\mu_s \gg 1\) and \(\text{Im}(\eta) \gg \mu_s^2\) the spectral density vanishes for
\[
\frac{|\eta + \eta^*|}{4\mu_s^2} = \frac{|x|}{2\mu^2} > 1.
\]
(168)
If the saddle point is inside the integration domain, a saddle point approximation gives
\[
\rho_s(\eta, \eta^*; \mu_s) = \frac{1}{4\mu_s^2} \text{ for } \frac{|\eta + \eta^*|}{4\mu_s^2} = \frac{|x|}{2\mu^2} < 1,
\]
(169)
in agreement with the mean field result (106) [73] provided that we make the identification (132). The total number of eigenvalues in a strip of unit width perpendicular to the \(y\)-axis is thus given by
\[
N^2 \rho_s(\eta, \eta^*; \mu_s)4\mu^2 = N,
\]
(170)
in agreement with the choice of our normalization constant in (161).

Finally, let us stress that we expect that it is possible to express \(Z_{-n}\) as a determinant of matrix with elements given by derivatives of \(Z_{-1}\). This leads us to conjecture the following form of \(Z_{-n}\)
\[
Z_{-n}(z, z^*; \mu) = \frac{C_{-n}}{(zz^*)^{n(n-1)}} \det [\delta_k^l \delta_z \delta_{z^*} Z_{-l}(z, z^*; \mu)]_{k,l=0,1,\ldots,n-1}.
\]
(171)
We hope to prove this conjecture in a future publication [52].

**IV. CONCLUSIONS AND OUTLOOK**

We have shown that the replica limit of the Toda lattice equation is a powerful tool to derive the spectral correlation function of both Hermitian and non-Hermitian random matrix theories. We have obtained one-point and two-point
functions of Hermitian random matrix ensembles in the class $A$ (Wigner-Dyson) and in class $A_{III}$ (chiral) and one-point functions for non-Hermitian random matrix theories also both in class $A$ and in class $A_{III}$. In the case of the chiral ensemble, the non-Hermiticity arose because of a chemical potential that was introduced in the random matrix model consistent with the covariance properties of the QCD partition function. Therefore, this result is particularly relevant for the analysis of Dirac spectra of quenched QCD at nonzero chemical potential. We emphasize that our analytical results for this case were not previously known. These results convincingly show that it is possible to derive nonperturbative results by means of the replica trick contradicting the lore that the method only can be trusted for the calculation of asymptotic series of analytical expressions. In addition, the replica limit of the Toda lattice equation explains the factorization of correlation functions in terms of a product of a fermionic and a bosonic partition function. Such a structure is present both in Hermitian and non-Hermitian random matrix theories.

Although the results for the correlation functions in the other cases we have considered were known previously several new insights have emerged. In particular, we have conjectured that the fermionic, the bosonic and the supersymmetric partition functions are $\tau$-functions of a single integrable hierarchy which are related by means of the Toda lattice equations. In fact, this is the basis for the success of the replica trick in this approach. We have shown the validity of this conjecture for the fermionic generating function of the chGUE two-point function and the fermionic generating functions of the non-Hermitian theories. In several other cases this property was known previously. In the bosonic case, only the generating function for the GUE two-point function was shown to satisfy the fermionic Toda lattice equation for an arbitrary number of replicas. Of course, in cases were the spectral correlation functions were known previously it is clear that the Toda lattice equation can be continued at least to the partition function with one bosonic flavor. The difficulty with bosonic partition functions is that the weight in the integral has poles due to the inverse determinant. For Hermitian random matrix theories the poles are on the real axis and can be easily avoided. In the case of non-Hermitian random matrix theories with eigenvalues scattered in the complex plane, the poles are located in the very domain where we wish to calculate the spectral density. These problems can be dealt with by hermiticizing the generating function and using the Ingham-Siegel integral to avoid ambiguities in the choice of the integration contours. Although we have only evaluated the bosonic partition function for one bosonic flavor, we do not expect major technical problems for an arbitrary number of flavors and we expect to find the result that we have conjectured based on the Toda lattice equation. This problem will be addressed in a future publication.

Recently, quenched lattice QCD Dirac spectra at nonzero chemical potential were compared to a non-Hermitian chiral random matrix model that was formulated in terms of eigenvalues so that the method of complex orthogonal polynomials could be applied to this model. The spectral density of this model differs from our result, but since the leading order asymptotic expansion of the two models is the same, the two results are close and, within the statistical accuracy, they are both in agreement with the existing lattice data. However, in a numerical simulation of a chiral random matrix model at nonzero chemical potential, we convincingly have shown that the model of Akemann disagrees with the data, whereas our result is right on the mark. Based on universality arguments one would expect that the two models would coincide in the weak non-Hermiticity limit, but apparently, the model of Akemann is in a different universality class.

Having shown that it is possible to obtain exact analytical results by means of the replica trick some comments on the critique of the replica trick set forward in [4,7] are in order. One of the points that were raised is that the fermionic partition function or the bosonic partition functions alone are not sufficient to reproduce an exact analytical result. This is particularly clear in the fermionic case where the resolvent is an analytic function of the mass for all positive integer values of the replica index, so that its discontinuity is zero even in the replica limit. A careful analysis [7] shows that one cannot expect to obtain the full analytical answer in the bosonic case either. The conclusion of [7] was that, in order to derive nonperturbative results, the proper procedure is to combine information from bosonic and fermionic replicas. This is exactly what is achieved by the replica limit of the Toda lattice equation. However, the Toda lattice equation itself does not justify an analytical continuation from integer values of the number of replicated flavors. Only in the case of the one point function of the unitary ensemble with finite size matrices is it known how to derive a Toda lattice equation for any real number of flavors.

It would be of great conceptual and practical interest to generalize the results obtained in this paper to the orthogonal and symplectic ensembles. The orthogonal chiral ensemble is relevant for QCD with two colors where lattice simulations at nonzero chemical potential already have been performed. In this case there is no sign problem and one can go beyond the quenched approximation. Several predictions for macroscopic properties [44,68,73,79] have been confirmed by means of lattice gauge simulations [80]. A comparison to the microscopic spectral correlation functions will offer an independent crosscheck of both numerical and analytical methods.

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In this appendix we comment on the notation used in this paper. In the main text we consider several the generating functions or partition functions. A partition function in the class of the Unitary Ensemble (abbreviated UE) will be denoted by \( \mathcal{Z} \) and a partition function in the class of the chiral Unitary Ensemble (abbreviated chUE) by \( \mathcal{Z}_u \). If the probability distribution of the matrix elements of matrices in these ensembles is Gaussian, they will be called the Gaussian Unitary Ensemble (GUE) and the chiral Gaussian Unitary Ensemble (chGUE), respectively. If the form of the partition function does not depend on the specific form of the probability distribution we often omit the Gaussian adjective. The subscripts are used in the same way for both ensembles. We denote the partition function for a partition function does not depend on the specific form of the probability distribution we often omit the Gaussian adjective. The subscripts are used in the same way for both ensembles.

**Appendix A. NOTATION**

In this appendix we comment on the notation used in this paper. In the main text we consider several the generating functions or partition functions. A partition function in the class of the Unitary Ensemble (abbreviated UE) will be denoted by \( \mathcal{Z} \) and a partition function in the class of the chiral Unitary Ensemble (abbreviated chUE) by \( \mathcal{Z}_u \). If the probability distribution of the matrix elements of matrices in these ensembles is Gaussian, they will be called the Gaussian Unitary Ensemble (GUE) and the chiral Gaussian Unitary Ensemble (chGUE), respectively. If the form of the partition function does not depend on the specific form of the probability distribution we often omit the Gaussian adjective. The subscripts are used in the same way for both ensembles. We denote the partition function for a partition function does not depend on the specific form of the probability distribution we often omit the Gaussian adjective. The subscripts are used in the same way for both ensembles.

**Appendix B. CALCULATION OF A JACOBIAN**

In this Appendix, we calculate the Jacobian of the transformation

\[
U = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \Lambda \begin{pmatrix} v_1^\dagger \\ v_2^\dagger \end{pmatrix},
\]

where \( U \) is an \((n + m) \times (n + m)\) unitary matrix and \( \Lambda \) is the block diagonal matrix given by

\[
\Lambda_{k,l} = \begin{cases} \sqrt{1 - \mu^2} & k = 1, \cdots, 2m, \\ \frac{\mu}{\sqrt{1 - \mu^2}} & k, l = 1, \cdots, 2m, \\ 0 & k > 2m \text{ or } l > 2m, \ k \neq l, \\ -1 & k > 2m. \end{cases}
\]

The diagonal matrix \( \mu = \text{diag}(\mu_1, \cdots, \mu_n) \), and we will use the notation that \( \lambda_k = \sqrt{1 - \mu_k^2} \) with \( \lambda_k \in [0, 1] \). Here, \( u_1 \) and \( v_1 \) are unitary \( m \times m \) matrices, \( u_2 \in U(n) \) and \( v_2 \in U(n)/(U^m(1) \times U(n-m)) \). One easily verifies that the total number of parameters on both sides of (172) is the same. For the calculation of the Jacobian of this transformation it is convenient to use the differentials

\[
\delta U = \begin{pmatrix} v_1^\dagger \\ v_2^\dagger \end{pmatrix} U^{-1} dU \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},
\]

\[
\delta u_1 = v_1^\dagger u_1^{-1} du_1 v_1,
\]

\[
\delta u_2 = v_1^\dagger u_2^{-1} du_2 v_2,
\]

\[
\delta v_1 = v_1^{-1} dv_1,
\]

\[
\delta v_2 = v_2^{-1} dv_2
\]

\[
\delta \lambda = d\lambda.
\]

The matrices \( \delta U, \delta u_k \) and \( \delta v_k \) are anti-Hermitian. To calculate the Jacobian we have to distinguish six different cases depending on whether \( \delta U \) is diagonal or either of its indices are smaller or larger than \( 2m \). We split \( \delta U \) into 4 blocks.
\[
\delta U = \begin{pmatrix}
\delta U_{11} & \delta U_{12} \\
\delta U_{21} & \delta U_{22}
\end{pmatrix}
\]

(175)
of size \(m \times m, m \times n, n \times m\) and \(n \times n\), respectively.

The Jacobian of the transformation from \(\{\delta U_{ij}^{kl}, \delta U_{ij}^{kl}, \delta U_{ij}^{kl}, \delta U_{ij}^{kl}\}\) to the variables \(\{\delta v_1^{kl}, \delta v_2^{kl}, \delta u_1^{kl}, \delta u_2^{kl}\}\) will be denoted by \(J_{kl}\). For \(1 \leq k \leq m\) and \(1 \leq l \leq m\) and \(k \neq l\) the Jacobian of transformation from \(\{\delta U_{11}^{kl}, \delta U_{12}^{kl}, \delta U_{21}^{kl}, \delta U_{22}^{kl}\}\) to \(\{\delta v_1^{kl}, \delta v_2^{kl}, \delta u_1^{kl}, \delta u_2^{kl}\}\) is given by the determinant

\[
J_{k \leq m,l \leq m,k \neq l} = \det \begin{pmatrix}
\lambda_k \lambda_l - 1 & \mu_k \mu_l & \lambda_k \lambda_l & \mu_k \mu_l \\
\lambda_k \mu_l & -\mu_k \lambda_l & \lambda_k \lambda_l & -\mu_k \lambda_l \\
\mu_k \lambda_l & -\lambda_k \mu_l & \mu_k \lambda_l & -\lambda_k \mu_l \\
\mu_k \mu_l & \lambda_k \mu_l & \mu_k \lambda_l & \lambda_k \lambda_l
\end{pmatrix}
= \lambda_l^2 - \lambda_k^2,
\]

(176)
where we have used that \(\mu_k^2 + \lambda_k^2 = 1\). The Jacobian of the transformation from \(\{\delta U_{12}^{kl}, \delta U_{22}^{kl}\}\) to the variables \(\{\delta v_2^{kl}, \delta u_2^{kl}\}\) for \(1 \leq k \leq m\) and \(m < l \leq n\) is given by the determinant

\[
J_{k \leq m,l > m} = \det \begin{pmatrix}
\mu_k \\
-\lambda_k - 1
\end{pmatrix}
= \mu_k.
\]

(177)
The Jacobian of the transformation from \(\{\delta U_{21}^{kl}, \delta U_{22}^{kl}\}\) to the variables \(\{\delta v_2^{kl}, \delta u_2^{kl}\}\) for \(k > m\) and \(1 \leq l \leq m\) is given by the determinant

\[
J_{k > m,l \leq m} = \det \begin{pmatrix}
\mu_l \\
-\lambda_l - 1
\end{pmatrix}
= \mu_l.
\]

(178)
The Jacobian of the transformation from \(\{\delta U_{22}^{kl}\}\) to the variables \(\{\delta u_2^{kl}\}\) for \(k > m\) and \(l > m\) is simply given by the determinant

\[
J_{k > m,l > m} = 1.
\]

(179)
Finally, we consider the Jacobian of the transformation of the diagonal matrix elements. For the transformation \(\{\delta U_{11}^{kk}, \delta U_{12}^{kk}, \delta U_{21}^{kk}, \delta U_{22}^{kk}\}\) to the variables \(\{\delta v_1^{kk}, \delta u_1^{kk}, \delta u_2^{kk}, \delta \lambda_k\}\) with \(k < m\) we find the Jacobian

\[
J_{k \leq m,k \leq m} = \det \begin{pmatrix}
-\mu_k^2 & \lambda_k^2 & \mu_k^2 & 0 \\
\lambda_k \mu_k & \mu_k \lambda_k & -\lambda_k \mu_k & -1/\mu_k \\
\mu_k \lambda_k & \lambda_k \mu_k & -\mu_k \lambda_k & 1/\mu_k \\
\mu_k^2 & \mu_k^2 & \lambda_k^2 & 0
\end{pmatrix}
= 2\lambda_k.
\]

(180)
For \(k > m\) the only nonzero derivatives are \(\delta U_{kk}^{22}/\delta u_2^{kk} = 1\) resulting into a Jacobian of unity.

Multiplying the Jacobians calculated in previous paragraph we find the Jacobian of the transformation (172):

\[
J = \prod_{1 \leq k < l \leq m} (\lambda_k^2 - \lambda_l^2)^2 \prod_{k=1}^{m} (2\lambda_k)^{2(n-m)}.
\]

(181)

**Appendix C. CALCULATION OF Z_{-1} FOR \mu = 0**

This Appendix illustrates the use of the Ingham-Siegel integral for the chiral unitary ensemble at \(\mu = 0\) which was already considered in [25]. In this case there is no need to hermiticize the Dirac operator, and the inverse determinants can be represented as
\[
\begin{align*}
\text{det}^{-1} \left( \begin{array}{cc} z & iW \\ iW^\dagger & z^* \end{array} \right) \text{det}^{-1} \left( \begin{array}{cc} z^* & -iW \\ -iW^\dagger & z^* \end{array} \right) \\
= \text{det}^{-1} \left( \begin{array}{cc} z & iW \\ iW^\dagger & z^* \end{array} \right) \text{det}^{-1} \left( \begin{array}{cc} z^* & iW \\ iW^\dagger & z^* \end{array} \right), \\
= C \int d\phi_k d\phi^*_k \exp \left[ -\left( \begin{array}{c} \phi^*_1 \\
\phi^*_2 \\
\phi^*_3 \\
\phi^*_4 \end{array} \right) \left( \begin{array}{c} z \\
iW \end{array} \right) \left( \begin{array}{c} \phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \end{array} \right) ^T \left( \begin{array}{c} z^* \\
iW^\dagger \end{array} \right) \left( \begin{array}{c} \phi^*_1 \\
\phi^*_2 \\
\phi^*_3 \\
\phi^*_4 \end{array} \right) \right]. \quad (182)
\end{align*}
\]

Other constants that enter in the calculation will also be absorbed in the irrelevant normalization constant \(C\). The integrals converge if \(\text{Re}(z) > 0\). The Gaussian average with probability distribution as in (114) results in (up to an overall constant)

\[
\exp \left[ -\frac{2}{N} \text{Tr} \hat{Q}_1 \hat{Q}_2 \right] \quad (183)
\]

with

\[
\hat{Q}_1 = \left( \begin{array}{cc} \phi^*_1 \cdot \phi_1 & \phi^*_1 \cdot \phi_3 \\
\phi^*_2 \cdot \phi_1 & \phi^*_2 \cdot \phi_3 \end{array} \right), \quad \hat{Q}_2 = \left( \begin{array}{cc} \phi^*_2 \cdot \phi_2 & \phi^*_2 \cdot \phi_4 \\
\phi^*_3 \cdot \phi_2 & \phi^*_3 \cdot \phi_4 \end{array} \right). \quad (184)
\]

Introducing \(\delta\)-functions as in (120) and performing the \(\phi_k\)-integrations we obtain

\[
Z_{-1} = C \int dQ_1 dQ_2 \int dF dG \text{e}^{\text{Tr}\left[iFQ_1 + iGQ_2 - \zeta Q_1 - \zeta Q_2 - \frac{N}{2} \hat{Q}_1 \hat{Q}_2\right]} \text{det}^{-N/2}(F) \text{det}^{-N/2}(G),
\]

where an infinitesimal imaginary increment is included in \(F\) and \(G\) and \(\zeta\) is now defined by

\[
\zeta = \left( \begin{array}{c} z \\
0 \\
0 \\
z^* \end{array} \right). \quad (186)
\]

The integrals over \(F\) and \(G\) can be calculated by means of the Ingham-Siegel integral (91). After also rescaling \(Q_i \rightarrow NQ_i/2\) we find

\[
Z_{-1} = C \int dQ_1 dQ_2 \theta(Q_1) \theta(Q_2) e^{\text{Tr}\left[-NQ_1 - NQ_2 - \frac{N}{2} \hat{Q}_1 \hat{Q}_2\right]} \text{det}^{\frac{N}{2}}(Q_1 Q_2). \quad (187)
\]

In the microscopic limit where \(\zeta N\) is fixed as \(N \rightarrow \infty\) the saddle point equations are given by

\[
Q_1 Q_2 = 1. \quad (188)
\]

We choose the \(Q_1\) variables to parameterize the saddle point manifold. The Gaussian fluctuations about \(Q_2 = Q_1^{-1}\) give rise to a factor \(1/\text{det}^2(Q_1)\). As final result we obtain

\[
Z_{-1} = C \int \frac{dQ_1}{\text{det}^2 Q_1} \theta(Q_1) e^{\text{Tr}\left[-NQ_1^{-1} + Q_1^{-1}\right]} \quad (189)
\]

We parameterize the positive Hermitian matrices as

\[
Q_1 = U \left( \begin{array}{cc} e^{s_1} & 0 \\
0 & e^{s_2} \end{array} \right) U^{-1}. \quad (190)
\]

The integration measure is given by

\[
\frac{dQ_1}{\text{det}^2 Q_1} = (e^{s_1} - e^{s_2})(e^{-s_1} - e^{-s_2}) ds_1 ds_2 dU. \quad (191)
\]

The integral over \(U\) can be calculated as an Itzykson-Zuber integral [81]. This results in

\[
Z_{-1} = C \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 (e^{s_1} - e^{s_2})(e^{-s_1} - e^{-s_2}) e^{-Nz \cosh s_1 - Nz^* \cosh s_2 - e^{-Nz} \cosh s_2 - e^{-Nz^*} \cosh s_1} \frac{N(z - z^*) (\cosh s_1 - \cosh s_2)}{N(\cosh s_1 + \cosh s_2)}. \quad (192)
\]

Writing \(z = x + iy\), the limit \(y \rightarrow 0\) of this partition function is given by

\[
Z_{-1} = C \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 (e^{s_1} - e^{s_2})(e^{-s_1} - e^{-s_2}) e^{-Nz(\cosh s_1 + \cosh s_2)} = 8C(K_0^2(Nx) - K_1^2(Nx)), \quad (193)
\]

which, up to an (arbitrary) normalization constant, agrees with the result obtained by means of different methods [9]. For \(z\) purely imaginary we did not succeed to further simplify the integral (192).
Appendix D. THE $\mu = 0$ LIMIT OF EQ. (131)

To convince the reader that the partition function with one bosonic flavor is correct we consider in this Appendix
the $\mu = 0$ limit of (131) which is given by

$$Z_{-1}(z, z^*; \mu) = C \int \frac{dQ_1}{\det^2 Q_1} \theta(Q_1) e^{\text{Tr}[i\frac{N}{2} \zeta^T (Q_1 - IQ_1^{-1})]}.$$  \hfill (194)

where

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}. \hfill (195)$$

As in the previous Appendix, we parametrize positive definite Hermitian matrices as

$$Q_1 = U \begin{pmatrix} e^{s_1} & 0 \\ 0 & e^{s_2} \end{pmatrix} U^{-1}. \hfill (196)$$

with integration measure given by

$$\frac{dQ_1}{\det^2 Q_1} = (e^{s_1} - e^{s_2})(e^{-s_1} - e^{-s_2})ds_1 ds_2 dU. \hfill (197)$$

To rewrite the integral over $U$ as an Itzykson-Zuber integral, we use the property of $2 \times 2$ matrices

$$IQ^{-1}I = -\frac{Q^T}{\det Q}, \hfill (198)$$

so that

$$\text{Tr} \zeta^T (Q_1 - IQ_1^{-1}) = \text{Tr} \zeta^T Q_1 + \frac{1}{\det Q_1} \text{Tr} \zeta Q_1$$

$$= \text{Tr}(\zeta^T + \frac{1}{\det Q_1} \zeta)Q_1. \hfill (199)$$

Evaluating the Itzykson-Zuber integral we find as final result

$$Z_{-1} = C \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 (e^{s_1} - e^{s_2})(e^{-s_1} - e^{-s_2}) \frac{e^{i \frac{N}{2}(\lambda_1 e^{s_1} + \lambda_2 e^{s_2})} - e^{i \frac{N}{2}(\lambda_1 e^{s_2} + \lambda_2 e^{s_1})}}{i(N/2)(\lambda_1 - \lambda_2)(e^{s_1} - e^{s_2})}. \hfill (200)$$

where $\lambda_1$ and $\lambda_2$ are the solutions of

$$\lambda^2 = (z + z^* e^{-s_1-s_2})(z^* + ze^{-s_1-s_2}). \hfill (201)$$

We did not succeed to show that the partition function (200) agrees with (192) for arbitrary complex values of $z$. The reason is that the integration contour has to be modified appropriately so that the integral becomes convergent. However, for $z$ purely real or purely imaginary, (192) can be recovered from (200). For $z = iy$ (with $y$ real) one easily shows that the integrands of the two partition functions are the same. This is not the case for $z = x$ (with $x$ real). In this case a convergent integral is obtained if we first shift the integration contours in (200) by

$$s_1 \to s_1 + \frac{\pi}{2} i, \quad s_2 \to s_2 - \frac{\pi}{2} i \hfill (202)$$
in the first term and by

$$s_1 \to s_1 - \frac{\pi}{2} i, \quad s_2 \to s_2 + \frac{\pi}{2} i \hfill (203)$$
in the second term. We then find

$$Z_{-1}(x) = C \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 (e^{s_1} + e^{s_2})(e^{-s_1} + e^{-s_2}) \frac{e^{-Nx(cosh s_1 + cosh s_2)} + e^{-Nx(cosh s_2 + cosh s_1)}}{-2Nx(cosh s_1 + cosh s_2)}. \hfill (204)$$

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and finally obtain
\[
\frac{\partial}{\partial x}[xZ_{-1}(x)] = 8(K_0^2(Nx) + K_1^2(Nx)).
\] (205)

The solution of this equation is given by
\[
Z_{-1}(x) = 8C(K_0^2(Nx) - K_1^2(Nx)),
\] (206)
in complete agreement with (193).
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