On two-point boundary value problems for
the Sturm-Liouville operator

Alexander Makin

Abstract

In this paper, we study spectral problems for the Sturm-Liouville operator with arbitrary complexvalued potential \( q(x) \) and two-point boundary conditions. All types of mentioned boundary conditions are considered. We investigate in detail the completeness property and the basis property of the root function system.

1. Introduction. The spectral theory of two-point differential operators was begun by Birkhoff in his two papers [1, 2] of 1908 where he introduced regular boundary conditions for the first time. It was continued by Tamarkin [3, 4] and Stone [5, 6]. Afterwards their investigations were developed in many directions. There is an enormous literature related to the spectral theory outlined above, and we refer to [7-18] and their extensive reference lists for this activity.

The present communication is a brief survey of results in the spectral theory of the Sturm-Liouville equation

\[ u'' - q(x)u + \lambda u = 0 \]  

with two-point boundary conditions

\[ B_i(u) = a_{i1}u'(0) + a_{i2}u'(%\pi) + a_{i3}u(0) + a_{i4}u(%\pi) = 0, \]  

where the \( B_i(u) (i = 1, 2) \) are linearly independent forms with arbitrary complex-valued coefficients and \( q(x) \) is an arbitrary complex-valued function of class \( L_1(0, \pi) \).

Our main focus is on the non-self-adjoint case, and, in particular, the case when boundary conditions are degenerate. We will study the completeness property and the basis property of the root function system.

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of operator (1), (2). The convergence of spectral expansions is investigated only in classical sense, i.e. the question about the summability of divergent series by a generalized method is not considered.

We consider the operator \( Lu = u'' - q(x)u \) as a linear operator on \( L_2(0, \pi) \) with the domain \( D(L) = \{ u \in L_2(0, \pi) | u(x), u'(x) \) are absolutely continuous on \([0, \pi], u'' - q(u)u \in L_2(0, \pi), B_i(u) = 0 \) \( (i = 1, 2) \} \).

By an eigenfunction of the operator \( L \) corresponding to an eigenvalue \( \lambda \in \mathbb{C} \) we mean any function \( 0^0 u(x) \in D(L) (0^0 u(x) \not\equiv 0) \) which satisfies the equation

\[ L \, 0^0 u + \lambda \, 0^0 u = 0 \]

almost everywhere on \([0, \pi]\).

By an associated function of the operator \( L \) of order \( p \) \((p = 1, 2, \ldots)\) corresponding to the same eigenvalue \( \lambda \) and the eigenfunction \( 0^0 u(x) \) we mean any function \( p^0 u(x) \in D(L) \) which satisfies the equation

\[ L \, p^0 u + \lambda \, p^0 u = p^{-1} u \]

almost everywhere on \([0, \pi]\). One can also say that an eigenfunction \( 0^0 u(x) \) is an associated function of zero order. The set of all eigen- and associated functions (or root functions) corresponding to the same eigenvalue \( \lambda \) together with the function \( u(x) \equiv 0 \) forms a root linear manifold. This manifold is called a root subspace if its dimension is finite.

Let the set of the eigenvalues of the operator \( L \) be countable and all root linear manifolds be root subspaces. Let us choose a basis in each root subspace. Any system \( \{ u_n(x) \} \) obtained as the union of chosen bases of all the root subspaces is called a system of eigen- and associated functions (or root function system) of the operator \( L \).

The main purpose of this lecture is to study the basis property of the root function system of the operator \( L \). Before starting our in-
vestigation we must verify completeness of the root function system in $L_2(0, \pi)$.

It is convenient to write conditions (2) in the matrix form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

and denote the matrix composed of the ith and jth columns of $A (1 \leq i < j \leq 4)$ by $A(ij)$; we set $A_{ij} = \text{det}A(ij)$.

Denote by $c(x, \mu), s(x, \mu) (\lambda = \mu^2)$ the fundamental system of solutions to equation (1) with the initial conditions $c(0, \mu) = s'(0, \mu) = 1, c'(0, \mu) = s(0, \mu) = 0$. The eigenvalues of problem (1), (2) are the roots of the characteristic determinant

$$\Delta(\mu) = \begin{vmatrix} B_1(c(x, \mu)) & B_1(s(x, \mu)) \\ B_2(c(x, \mu)) & B_2(s(x, \mu)) \end{vmatrix}.$$ 

Simple calculations show that

$$\Delta(\mu) = -A_{13} - A_{24} + A_{34}s(\pi, \mu) - A_{23}s'(\pi, \mu) - A_{14}c(\pi, \mu) - A_{12}c'(\pi, \mu).$$

It is easily seen that if $q(x) \equiv 0$ then the characteristic determinant $\Delta_0(\mu)$ of the corresponding problem (1), (2) has the form

$$\Delta_0(\mu) = -A_{13} - A_{24} + A_{34}\frac{\sin \pi \mu}{\mu} - (A_{23} + A_{14}) \cos \pi \mu + A_{12} \mu \sin \pi \mu.$$

Boundary conditions (2) are called nondegenerate if they satisfy one of the following relations:

1) $A_{12} \neq 0, \quad 2) A_{12} = 0, A_{14} + A_{23} \neq 0, \quad 3) A_{12} = 0, A_{14} + A_{23} = 0, A_{34} \neq 0.$

Evidently, boundary conditions (2) are nondegenerate iff $\Delta_0(\mu) \neq \text{const}.$

Notice, that for any nondegenerate boundary conditions an asymptotic representation for the characteristic determinant $\Delta(\mu)$ as $|\mu| \to \infty$ one can find in [10].
Theorem 1 ([10]). For any nondegenerate conditions the spectrum of problem (1), (2) consists of a countable set \( \{ \lambda_n \} \) of eigen-values with only one limit point \( \infty \), and the dimensions of the corresponding root subspaces are bounded by one constant. The system \( \{ u_n(x) \} \) of eigen- and associated functions is complete and minimal in \( L^2(0, \pi) \); hence, it has a biorthogonally dual system \( \{ v_n(x) \} \).

For convenience, we introduce numbers \( \mu_n \), where \( \mu_n \) is the square root of \( \lambda_n \) with nonnegative real part.

It is known that nondegenerate conditions can be divided into three classes:

1) strengthened regular conditions;
2) regular but not strengthened regular conditions;
3) irregular conditions.

The definitions are given in [8]. These three cases should be considered separately.

2. Strengthened regular conditions. Let boundary conditions (2) belong to class 1). According to [8], this is equivalent to the fulfillment one of the following conditions:

\[
A_{12} \neq 0; \quad A_{12} = 0, A_{14} + A_{23} \neq 0, A_{14} + A_{23} \neq \mp(A_{13} + A_{24}); \\
A_{12} = 0, A_{14} + A_{23} = 0, A_{13} + A_{24} = 0, A_{13} = A_{24}, A_{34} \neq 0.
\]

It is well known that, all but finitely many eigenvalues \( \lambda_n \) are simple (in other words, they are asymptotically simple), and the number of associated functions is finite. Moreover, the \( \lambda_n \) are separated in the sense that there exists a constant \( c_0 > 0 \) such that, for any sufficiently large different numbers \( \lambda_k \) and \( \lambda_m \), we have

\[
|\mu_k - \mu_m| \geq c_0. \tag{3}
\]

Theorem 2. The system of root functions \( \{ u_n(x) \} \) forms a Riesz basis in \( L^2(0, \pi) \).

This statement was proved in [21], [22] and [9, Chapter XIX].
Class 1) contains many types of boundary conditions, for example, the Dirichlet boundary conditions \( u(0) = u(\pi) = 0 \), the Neumann boundary conditions \( u'(0) = u'(\pi) = 0 \), the Dirichlet-Newmann boundary conditions \( u(0) = u'(\pi) = 0 \) and others.

3. Regular but not strengthened regular conditions. Let boundary conditions belong to class 2). According to [8], this is equivalent to the fulfillment of the conditions

\[
A_{12} = 0, \quad A_{14} + A_{23} \neq 0, \quad A_{14} + A_{23} = (-1)^{\theta+1}(A_{13} + A_{24}), \quad (4)
\]

where \( \theta = 0, 1 \). It is well known [10] that the eigenvalues of problem (1), (2) form two series:

\[
\lambda_0 = \mu_0^2, \quad \lambda_{n,j} = (2n + o(1))^2 \quad (5)
\]

(if \( \theta = 0 \)) and

\[
\lambda_{n,j} = (2n - 1 + o(1))^2 \quad (6)
\]

(if \( \theta = 1 \)). Here, in both cases, \( j = 1, 2 \) and \( n = 1, 2, \ldots \). We denote \( \mu_{n,j} = \sqrt{\lambda_{n,j}} = 2n - \theta + o(1) \). It follows from [8] that asymptotic formulas (6) and (7) can be refined. Specifically,

\[
\mu_{n,j} = 2n - \theta + O(n^{-1/2}).
\]

Obviously, \( |\mu_{n,1} - \mu_{n,2}| = O(n^{-1/2}) \); i.e. \( \mu_{n,1} \) and \( \mu_{n,2} \) become infinitely close to each other as \( n \to \infty \). If \( \mu_{n,1} = \mu_{n,2} \) for all \( n \), except, possibly, a finite set, then the spectrum of problem (1), (2) is called asymptotically multiple. If the set of multiple eigenvalues is finite, then the spectrum of problem (1), (2) is called asymptotically simple.

There exist numerous examples when the number of multiple eigenvalues is finite or infinite, and the total number of associated functions is finite or infinite also. We see that separation condition (3) never holds. Depending on the particular form of the boundary conditions and the potential \( q(x) \) the system of root functions may have or may not have the basis property [17], [22], [23], and even for fixed boundary conditions, this property may appear or disappear under arbitrary
small variations of the coefficient $q(x)$ in the corresponding metric [24].
Thus, the considered case is much more complicated than the previous one, so we will study it in detail.

For any problem (1), (2) let $Q$ denote the set of potentials $q(x)$ from the class $L_1(0, \pi)$ such that the system of root functions forms a Riesz basis in $L_2(0, \pi)$, $\bar{Q} = L_1(0, \pi) \setminus Q$.

To analyze this class of problems, it is reasonable [12] to divide conditions (2) satisfying (4) into three types:

I) $A_{14} = A_{23}$, $A_{34} = 0$;
II) $A_{14} = A_{23}$, $A_{34} \neq 0$;
III) $A_{14} \neq A_{23}$

The eigenvalue problem for operator (1) with boundary conditions of type I, II, or III, is called the problem of type I, II, or III, respectively.

At first we consider the problems of type I. It was shown in [12] that any boundary conditions of type I are equivalent to the boundary conditions specified by the matrix

$$A = \begin{pmatrix} 1 & (-1)^{\theta+1} & 0 & 0 \\ 0 & 0 & 1 & (-1)^{\theta+1} \end{pmatrix},$$

i.e., to periodic or antiperiodic boundary conditions. These boundary conditions are selfadjoint.

**Theorem 3 ([25]).** The sets $Q$ and $\bar{Q}$ are everywhere dense in $L_1(0, \pi)$.

Recently, (see [26-37] and their extensive reference lists) by a number of authors, a very nice theory of the problems of type I was built.

Let us consider the problems of type II. It was also established in [12] that any boundary conditions of type II are equivalent to the boundary conditions specified by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & a_{14} \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 1 & 1 & 0 & a_{14} \\ 0 & 0 & 1 & 1 \end{pmatrix},$$
where \( a_{14} \neq 0 \) in both cases. If \( a_{14} \) is a real number and \( q(x) \) is a real function, then the corresponding boundary value problem is selfadjoint.

**Theorem 4 ([38])**. If \( A_{14} = A_{23} \) and \( A_{34} \neq 0 \), then the system \( \{u_n(x)\} \) forms a Riesz basis in \( L_2(0, \pi) \), and the spectrum is asymptotically simple. Denote by \( \{v_n(x)\} \) the biorthogonally dual system. The key point in the proof of Theorem 4 is obtaining the estimate

\[
\max_{(x,\xi)\in[0,\pi]\times[0,\pi]} |u_n(x)v_n(\xi)| \leq C,
\]

which is valid for any number \( n \). It follows from (7) and [39] that the system \( \{u_n(x)\} \) forms a Riesz basis in \( L_2(0, \pi) \).

A comprehensive description of boundary conditions of type III was given in [12]. In particular, it is known that all of them are non-self-adjoint.

**Theorem 5 ([38])**. If \( A_{14} \neq A_{23} \), then the system of root functions \( \{u_n(x)\} \) of problem (1), (2) is a Riesz basis in \( L_2(0, \pi) \) if and only if the spectrum is asymptotically multiple.

Thus, we have established that for problems of type III the question about the basis property for the system of eigen- and associated functions is reduced to the question about asymptotic multiplicity of the spectrum. The presence of this property depends essentially on the particular form of the boundary conditions and the function \( q(x) \).

**Theorem 6 ([40, 41])**. If \( A_{14} \neq A_{23} \), then, for any function \( q(x) \in L_2(0, \pi) \) and any \( \varepsilon > 0 \), there exists a function \( \tilde{q}(x) \in L_2(0, \pi) \) such that \( \|q(x) - \tilde{q}(x)\|_{L_2(0, \pi)} < \varepsilon \) and problem (1), (2) with the potential \( \tilde{q}(x) \) has an asymptotically multiple spectrum.

For \( A_{14} = A_{23} \) and \( A_{34} = 0 \), a similar proposition was deduced in [42].

Theorems 3, 4, 6 and the results of [43] imply that the whole class of regular but not strengthened regular boundary conditions splits into two subclasses (a) and (b). Subclass (a) coincides with the second
type of boundary conditions and is characterized by the fact that the system of root functions of problem (1), (2) with boundary conditions from this subclass forms a Riesz basis in $L_2(0, \pi)$ for any potential $q(x) \in L_1(0, \pi)$; i.e. $Q = L_1(0, \pi), \bar{Q} = \emptyset$. We will see below that boundary conditions from the subclass (a) are the only boundary conditions (in addition to strengthened regular ones) that ensure the Riesz basis property of the system of root functions for any potential $q(x) \in L_1(0, \pi)$.

Subclass (b) contains the remaining regular but not strengthened regular boundary conditions. An entirely different situation takes place in this case. For any problem with boundary conditions from this subclass, the sets $Q$ and $\bar{Q}$ are dense everywhere in $L_1(0, \pi)$.

4. Irregular conditions. Let boundary conditions (2) belong to class 3). According to [8, 12], this is equivalent to the fulfillment one of the following conditions:

$$A_{12} = 0, \quad A_{14} + A_{23} = 0, \quad A_{13} + A_{24} = 0, \quad A_{13} \neq A_{24}, \quad A_{34} \neq 0;$$

$$A_{12} = 0, \quad A_{14} + A_{23} = 0, \quad A_{13} + A_{24} \neq 0, \quad A_{34} \neq 0.$$

According to [12], any boundary conditions of the considered class are equivalent to the boundary conditions determined by the matrix

$$A = \begin{pmatrix} 1 & \pm 1 & 0 & b_0 \\ 0 & 0 & 1 & \mp 1 \end{pmatrix}, \quad \text{where} \quad b_0 \neq 0,$$

or

$$A = \begin{pmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & -b_1 \end{pmatrix}, \quad \text{where} \quad b_1 \neq \pm 1, \quad b_0 \neq 0,$$

or

$$A = \begin{pmatrix} 0 & 1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad a_0 \neq 0.$$

In case 3), as well as in case 1), all but finitely many eigenvalues $\lambda_n$ are simple, the number of associated functions is finite, and separation
condition (3) holds. However, the system \( \{u_n(x)\} \) never forms even a usual basis in \( L_2(0, \pi) \), because \( ||u_n||_{L_2(0,\pi)}||v_n||_{L_2(0,\pi)} \rightarrow \infty \) as \( n \rightarrow \infty \). Here \( \{v_n(x)\} \) is the biorthogonally dual system. This case was investigated in [5], [6], [44].

5. Degenerate conditions. Let boundary conditions (2) be degenerate. According to [10, 12], this is equivalent to the fulfillment of the following conditions:

\[
A_{12} = 0, \quad A_{14} + A_{23} = 0, \quad A_{34} = 0.
\]

According to [12], any boundary conditions of the considered class are equivalent to the boundary conditions determined by the matrix

\[
A = \begin{pmatrix} 1 & d & 0 & 0 \\ 0 & 0 & 1 & -d \end{pmatrix}, \quad \text{or} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

If in the first case \( d = 0 \) then for any potential \( q(x) \) we have the initial value problem (the Cauchy problem) which has no eigenvalues. The same situation takes place in the second case.

Further we will consider the first case if \( d \neq 0 \). Then the boundary conditions can be written in more visual form

\[
u'(0) + du'(\pi) = 0, \quad u(0) - du(\pi) = 0. \quad (8)\]

5.1. Spectrum.

By \( PW_\sigma \) we denote the class of entire functions \( f(z) \) of exponential type \( \leq \sigma \) such that \( ||f(z)||_{L_2(R)} < \infty \), and by \( PW_\sigma^- \) we denote the set of odd functions in \( PW_\sigma \).

By performing simple manipulations, we obtain the relation

\[
\Delta(\mu) = \frac{d^2 - 1}{d} + c(\pi, \mu) - s'(\pi, \mu) = \frac{d^2 - 1}{d} + \int_0^\pi K(t)\sin\frac{\mu t}{\mu} dt = \frac{d^2 - 1}{d} + \frac{f(\mu)}{\mu},
\]

where \( K(t) \in L_1(0, \pi) \). If \( q(x) \in L_2(0, \pi) \) then \( K(t) \in L_2(0, \pi) \) and \( f(\mu) \in PW_\pi^- \). Notice, that simple calculations show that if \( d = \pm 1 \),
and \( q(x) \equiv 0 \) then any \( \lambda \in \mathbb{C} \) is an eigenvalue of infinite multiplicity. This abnormal example constructed by Stone illustrates the difficulty of investigation of problems with boundary conditions of the considered class.

It is well known that the characteristic determinant \( \Delta(\mu) \) of problem (1), (8) is an entire function of the parameter \( \mu \), consequently, for the operator (1), (8) we have only the following possibilities:

1) the spectrum is absent;
2) the spectrum is a finite nonempty set;
3) the spectrum is a countable set without finite limit points;
4) the spectrum fills the entire complex plane.

One can prove [45] that case 2) is impossible. It is known that \( c(\pi, \mu) - s'(\pi, \mu) \equiv 0 \) if and only if the function \( Q(x) = q(x) - q(\pi - x) = 0 \) almost everywhere on the segment \([0, \pi]\). Evidently, \( \lim_{\mu \to \infty} (c(\pi, \mu) - s'(\pi, \mu)) = 0 \). Hence it follows that if \( c(\pi, \mu) - s'(\pi, \mu) \equiv C' \) then \( C' = 0 \). We see that case 1) takes place if and only if condition (9) holds and \( d \neq \pm 1 \), and case 4) takes place if and only if condition (9) holds and \( d = \pm 1 \). If condition (9) does not hold we have case 3).

5.2. Completeness.

Completeness of the root function system of problem (1), (8) was investigated in [46-47]. The main result of the mentioned papers is:

**Theorem 7 ([47]).** If \( q(x) \in C^k[0, \pi] \) for some \( k = 0, 1, \ldots \) and \( q^{(k)}(0) \neq (-1)^k q^{(k)}(\pi) \), then the system of root functions is complete in the space \( L_p(0, \pi) \) if \( 1 \leq p < \infty \).

It follows from [47] that depending on the potential \( q(x) \) the system of root functions may have or may not have the completeness property, moreover, this property may appear or disappear under arbitrary small variations of the coefficient \( q(x) \) in the corresponding metric even for fixed boundary conditions.
If the conditions $d^2 \neq 1$ and $q(x) \in C[0, \pi]$ hold necessary and sufficient conditions of the completeness of root function system of problem (1), (8) were found in [48].

**Theorem 8 ([49]).** If for a number $\rho > 0$

$$\lim_{h \to 0} \frac{\int_{\pi-h}^{\pi} Q(x)dx}{h^\rho} = \nu,$$

and $\nu \neq 0$, then the root function system of problem (1), (8) is complete in the space $L_p(0, \pi)$ if $1 \leq p < \infty$.

Since for a wide class of potentials $q(x)$ the root function system of problem (1), (8) is complete in $L_2(0, \pi)$ one can set a question whether the mentioned system forms a basis.

**5.3. Basis property.**

Recently, it was proved in [50] that the root function system never forms an unconditional basis in $L_2(0, \pi)$ if multiplicities of the eigenvalues are uniformly bounded by some constant. Moreover, under the condition mentioned above it was established there that if the eigen- and associated function system of general ordinary differential operator with two-point boundary conditions forms an unconditional basis then the boundary conditions are regular. Article [50] was published in 2006. At that time it was unknown whether there exists a potential $q(x)$ providing unbounded growth of multiplicities of the eigenvalues.

**5.4. Inverse problem.**

However, in 2010 in [45] an example of a potential $q(x)$ for which the characteristic determinant has the roots of arbitrary high multiplicity was constructed. Hence, the corresponding root function system $\{u_n(x)\}$ contains associated functions of arbitrary high order. It means, that paper [50] does not give the definitive solution of basis property problem. Below we will show a method to construct a potential $q(x)$ providing unbounded growth of multiplicities of eigenvalues.

**Theorem 9 ([51]).** Suppose that a function $v(\mu)$ can be repre-
presented in the form

\[ v(\mu) = \gamma + \frac{f(\mu)}{\mu}, \tag{10} \]

where \( \gamma \) is some complex number, the function \( f(\mu) \in PW_\pi^- \) satisfies the condition

\[ \int_{-\infty}^{\infty} |\mu^m f(\mu)|^2 d\mu < \infty, \]

where \( m \) is a nonnegative integer number. Then there exists a function \( q(x) \in W_2^m(0, \pi) \) such that the characteristic determinant \( \Delta(\mu) \) of problem (1), (8), where either \( d = (\gamma + \sqrt{\gamma^2 + 4})/2 \) or \( d = (\gamma - \sqrt{\gamma^2 + 4})/2 \) and with the potential \( q(x) \) is identically equal to the function \( v(\mu) \).

Therefore, Theorem 9 reduces the problem on the structure of the spectrum of problem (1), (8) with degenerate boundary conditions to the problem on the expansion of a function of the form (10) into a canonical product.

5.5. Nontrivial examples.

Example 1. Let us define a sequence \( \{a_k\} (k = 1, 2, \ldots) \) in this way: \( a_1 = 1, a_2 = 3, a_3 = 5, a_{k+1} = a_k + 2p, \) if \( 2^p < k < 2^{p+1} (p = 1, 2, \ldots) \), and \( a_{k+1} = a_k + (a_k - a_{k-1}) + 2, \) if \( k = 2^p (p = 2, 3, \ldots) \). Set

\[ F_1(\mu) = \prod_{k=1}^{\infty} \left( 1 - \frac{\mu^2}{a_k^2} \right)^{a_{k+1}-a_k-\delta_k}, \]

where \( \delta_k = 0, \) if \( k \neq 2^p, \) and \( \delta_k = 1, \) if \( k = 2^p, (p = 2, 3, \ldots) \).

Theorem 10 ([52]). For any real \( x \) the following inequality holds

\[ |F_1(x)| \leq C_3(|x| + 1)^{M_1}, \]

where \( M_1 \) is a sufficiently large number.

Denote

\[ f_1(\mu) = \mu \prod_{k=M_1+1}^{\infty} \left( 1 - \frac{\mu^2}{a_k^2} \right)^{a_{k+1}-a_k-\delta_k}. \]
It is easily shown that $f_1(\mu) \in PW_{\pi}^-$. This, together with Theorem 9 implies that there exists a potential $q_1(x) \in L_2(0, \pi)$, such that for the characteristic determinant $\Delta_1(\mu)$ of problem

$$u'' - q_1(x)u + \lambda u = 0, \quad u'(0) + du'(\pi) = 0, \quad u(0) - du(\pi) = 0 \quad (11)$$

$(d = \pm 1)$ we have the equality

$$\Delta_1(\mu) = f_1(\mu)/\mu.$$

It follows from the definition of sequence $\{a_k\}$ that multiplicities of zeros $a_k$ of constructed above function $f_1(\mu)$ monotonically not decrease and tend to infinity as $k \to \infty$. Therefore, the eigenvalues $\lambda_n = \mu_n^2$ of problem (11) have the desired property: their multiplicities $m(\lambda_n)$ tend to infinity and the corresponding root function system contains associated functions of arbitrary high order, i.e. the dimensions of root subspaces infinitely grow. Moreover, the following inequality takes place

$$c_1 \ln \mu_n \leq m(\lambda_n) \leq c_2 \ln \mu_n.$$

**Theorem 11 ([52]).** The root function system $\{u_n(x)\}$ of problem (11) is complete in $L_2(0, \pi)$.

**Example 2.** Denote $\tilde{a}_k = a_k - \alpha_k + i\beta_k$, $(k = 1, 2, \ldots)$ where $\alpha_k = a_k - \sqrt{a_k^2 - \beta_k^2}$, $\beta_k = (a_k - a_{k-1})/10$ $(a_0 = 0)$. Denote $h_k = a_{k+1} - a_k - \delta_k$,

$$F_2(\mu) = \prod_{k=1}^{\infty} \left( 1 - \frac{\mu^2}{a^2_k} \right)^{[h_k/2]} \left( 1 - \frac{\mu^2}{a^2_k} \right)^{[h_k/2]} \left( 1 - \frac{\mu^2}{a^2_k} \right)^{h_k-2[h_k/2]}.$$

**Theorem 12 ([53]).** For any real $x$ the following inequality holds

$$|F_2(x)| \leq C_3(|x| + 1)^{M_2},$$

where $M_2$ is a sufficiently large number.

Denote

$$f_2(\mu) = \mu \prod_{k=M_2+1}^{\infty} \left( 1 - \frac{\mu^2}{a^2_k} \right)^{[h_k/2]} \left( 1 - \frac{\mu^2}{a^2_k} \right)^{[h_k/2]} \left( 1 - \frac{\mu^2}{a^2_k} \right)^{h_k-2[h_k/2]}.$$
It is easily shown that $f_2(\mu) \in PW^{-\pi}$. This, together with Theorem 9 implies that there exists a potential $q_2(x) \in L_2(0, \pi)$, such that for the characteristic determinant $\Delta_2(\mu)$ of problem

$$u'' - q_2(x)u + \lambda u = 0, \quad u'(0) + du'(\pi) = 0, \quad u(0) - du(\pi) = 0 \quad (d = \pm 1)$$

we have the equality

$$\Delta_2(\mu) = f_2(\mu)/\mu.$$ 

It follows from the definition of sequence $\{\tilde{a}_k\}$ that multiplicities of zeros $\tilde{a}_k, \bar{a}_k$ of constructed above function $f_2(\mu)$ monotonically not decrease and tend to infinity as $k \to \infty$. Therefore, the eigenvalues $\tilde{\lambda}_n = \hat{\mu}_n^2$ of problem (12) have two properties: their multiplicities $m(\tilde{\lambda}_n)$ tend to infinity, hence, the corresponding root function system contains associated functions of arbitrary high order, and $|Im\hat{\mu}_n| \to \infty$ as $n \to \infty$. Moreover, the following two inequalities hold

$$c_1 \ln |\hat{\mu}_n| \leq m(\lambda_n) \leq c_2 \ln |\hat{\mu}_n|, \quad c_1 |Im\hat{\mu}_n| \leq m(\tilde{\lambda}_n) \leq c_2 \ln |\hat{\mu}_n|.$$ 

**Theorem 13** ([53]). The root function system $\{u_n(x)\}$ of problem (12) is complete in $L_2(0, \pi)$.

For any problem (1), (8) let $\Omega$ denote the set of potentials $q(x)$ from the class $L_1(0, 1)$ such that the system of root functions contains associated functions of arbitrary high order, $\bar{\Omega} = L_1(0, \pi) \setminus \Omega$.

**Theorem 14** ([54]). The sets $\Omega$ and $\bar{\Omega}$ are everywhere dense in $L_1(0, \pi)$.

Since for a wide class of potentials $q(x)$ the root function system of problem (1), (8) is complete in $L_2(0, \pi)$ one can set a question whether the mentioned system forms a basis.

5.6. **Again on the basis property.**

Let $\lambda_n = \mu_n^2 (Re\mu_n \geq 0, n = 1, 2, \ldots)$ be the eigenvalues of problem (1), (8) numbered neglecting their multiplicities in nondecreasing order of absolute value. By $m(\lambda_n)$ we denote the multiplicity of an eigenvalue $\lambda_n$. 

14
Theorem 15 ([55-57]). If
\[
\lim_{n \to \infty} \frac{m(\lambda_n)}{\sqrt{|\mu_n|}} = 0,
\]
then the system of eigenfunctions and associated functions of problem (1), (8) is not a basis in \(L_2(0, \pi)\).

Clearly, since Theorem 15 contains supplementary condition (13), it does not give the definitive solution of the basis property problem. If this condition does not hold then the mentioned problem has not been solved.

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E-mail: alexmakin@yandex.ru