Deformed covariance in spherically symmetric vacuum models of loop quantum gravity:
Consistency in Euclidean and self-dual gravity

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Abstract

Different versions of consistent canonical realizations of hypersurface deformations of spherically symmetric space-times have been derived in models of loop quantum gravity, modifying the classical dynamics and sometimes also the structure of space-time. Based on a canonical version of effective field theory, this paper provides a unified treatment, showing that modified space-time structures are generic in this setting. The special case of Euclidean gravity demonstrates agreement also with existing operator calculations.

1 Introduction

Several independent studies have shown that holonomy and inverse-triad corrections from loop quantum gravity (LQG) modify hypersurface-deformation brackets for spherically symmetric gravity and related midisuperspace models \cite{1,10}, thereby realizing a deformation of general covariance \cite{11,13}. These modifications are closely related \cite{14} to anomaly-free models of perturbative cosmological inhomogeneity constructed within the same framework \cite{15,19}, suggesting that modified space-time structures may be a generic consequence of quantum-geometry effects in loop quantum gravity. In \cite{20} (see also \cite{21}), however, it has been shown that such modifications may be avoided if one uses self-dual connections and a densitized lapse function, as in \cite{22,24}, instead of real variables \cite{25}. These models, valid for self-dual Lorentzian gravity with Barbero–Immirzi parameter $\gamma = \pm i$ or Euclidean

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gravity with Barbero–Immirzi parameter $\gamma = \pm 1$, are rather special because the Hamiltonian constraint simplifies considerably compared with general $\gamma$. It is therefore of interest to compare the structures encountered in various models in order to determine whether undeformed space-time structures could be realized more broadly.

Such a comparison is not obvious, for instance because the modifications considered in [20] are different from those found in anomaly-free models using real variables. In particular, those modifications cannot be implemented in an anomaly-free manner for arbitrary choices of the Barbero–Immirzi parameter: We will show that the classical form of the constraint brackets can be retained only with a specific class of holonomy modifications for $\gamma = \pm i$ (self-dual Lorentzian gravity) or $\gamma = \pm 1$ (a special version of Euclidean gravity). More general treatments of the self-dual or Euclidean case, implemented in close analogy with the real connection formulation, lead to either anomalies or deformations of the space-time structure. This result then allows us to draw conclusions about properties of the Hamiltonian constraint required for certain types of modifications to be consistent.

At a technical level, an analysis of the Hamiltonian constraint and its Poisson brackets indicates a formal relationship between modifications of space-time structures and the appearance of spatial derivatives of the densitized triads (canonically conjugate to the connection). Spatial derivatives of the triad generically appear in the Hamiltonian constraints of gravitational theories because they are required for curvature components. But for $\gamma^2 = \pm 1$, and only in this case, they are completely absorbed in the connection components through the spin connection which, in combination with extrinsic-curvature components, forms the Ashtekar connection in the self-dual case [22], or the Ashtekar–Barbero connection in the real case [25].

This structural statement allows us to draw a first conclusion about the genericness of modified space-time structures. Using standard arguments from effective field theory (generalized here to a canonical setting), modified brackets should be considered generic, unless one can show that the full quantum theory has a symmetry that protects the derivative structure of terms in the Hamiltonian constraint as encountered for self-dual variables, or more generally for $\gamma^2 = \pm 1$. No such symmetry is known. Although it has been shown that the real Ashtekar–Barbero connection, unlike the self-dual one, cannot be identified with the pull-back of a space-time connection, this result is of an “aesthetic nature” [26] and does not characterize the case of $\gamma^2 = \pm 1$ via a physical symmetry that could restrict possible quantum corrections. Moreover, applying this result in the present context would amount to pre-supposing the classical space-time structure in a model of quantum gravity. In canonical quantum gravity, the structure of space-time is determined intrinsically, based on the observation that space-time symmetries of a gravitational theory are gauge transformations, generated in Hamiltonian form by the constraints that are to be quantized in order to define canonical quantum gravity. Poisson brackets of these constraints, or commutators of their operator versions, then encode the structure of space-time. An analysis of possible consistent modifications of these brackets, such that they remain closed but possibly with non-classical structure functions, show whether the symmetries remain unviolated after quantization. As we will see, such modifications with intact (but possibly deformed) symmetry exist for any value of $\gamma$. Therefore, no value of $\gamma$ is distinguished by
the presence of a symmetry.

In this work, we will mainly focus on an interpretation of the constraints as representing Euclidean gravity. We will then be exempt from having to consider a possible role of reality conditions, the implementation of which remains poorly understood in a quantum theory of self-dual variables. However, as the constraints are formally identical in Euclidean gravity with $\gamma = \pm 1$ and self-dual Lorentzian gravity, our results can formally be used also in the latter case.

2 Unsolved Gauss constraint

The model considered in [20], following [23], consists of three canonical pairs of fields — $A_i(x)$ and $E^i(x)$ for $i = 1, 2, 3$ depending on the radial coordinate $x$ of a spherically symmetric manifold — subject to three constraints. Two of the constraints function as generators of hypersurface deformations in space-time and therefore encode the structure of space-time. The third one, a Gauss constraint, implements an internal symmetry of SO(2)-rotations of two of the canonical pairs.

While the form of the Gauss constraint and the spatial generator of hypersurface deformations (the diffeomorphism constraint) is strictly determined by the canonical structure together with the corresponding Lie algebras of infinitesimal rotations and 1-dimensional diffeomorphisms, respectively, there is much freedom in specifying the normal generator of hypersurface deformations, or the Hamiltonian constraint, even if the physical dynamics is fixed. The version used in [20, 23] is rather special in that it is quadratic in the canonical fields and does not contain spatial derivatives of $E^i$ (while first-order spatial derivatives of $A_i$ do appear). In the first part of this section we will strengthen the result of [20] by showing that the consistent deformation found in this paper is unique within a family of models that preserve the quadratic nature and derivative structure of the Hamiltonian constraint. In the second part of this section, however, we will show that this rigidity is not stable within a larger class of models that determine the same classical dynamics but do not respect the restricted derivative structure (parameterized by the so-called Barbero–Immirzi parameter $\gamma$ [25, 27]). The following sections will then place our discussion in a setting of effective field theory, and highlight the role played by the Gauss constraint.

2.1 Regaining the quadratic Hamiltonian constraint

In order to derive our rigidity result, we start from the condition that the Poisson brackets of constraints be closed and see what kind of restrictions it imposes on the form of constraints. The specific procedure follows the classical (and classic) result [28] that the full Hamiltonian constraint, up to second order in derivatives, can be regained uniquely from the classical hypersurface-deformation brackets, as specified in [29]. This procedure has already been applied to spherically symmetric models in [11], but only for modifications of the dependence of the Hamiltonian constraint on the triad variables $E^i$. Our calculations here differ from [11] in that we use connection variables $A_i$, and take into account potential
modifications of the dependence on these variables.

As already indicated, we assume for now that the Hamiltonian constraint is quadratic in the canonical fields without spatial derivatives of the triad $E^i$. This version of the constraint is realized in spherically symmetric gravity if one uses self-dual connection variables $[22]$ in Lorentzian signature, or real Barbero-type variables $[25]$ in Euclidean signature such that the Barbero–Immirzi parameter is equal to $\gamma = \pm 1$. (One should also smear the Hamiltonian constraint with a lapse function of density weight minus one to guarantee the quadratic nature.) This parameter is therefore fixed and does not appear in the remainder of this subsection. Working with

\[ \{ A_1(x), E^1(y) \} = 2G\delta(x, y) \]  

and

\[ \{ A_2(x), E^2(y) \} = G\delta(x, y), \quad \{ A_3(x), E^3(y) \} = G\delta(x, y) \]  

while all other brackets of basic variables vanish. (Note the missing factor of 2 in the last two brackets, compared with (1), which is a consequence of the fact that $(A_2, E^2)$ and $(A_3, E^3)$ encode the same degree of freedom after the Gauss constraint is implemented.)

\[ \{ A_1(x), E^1(y) \} = 2\{ A_{2/3}(x), E^{2/3}(y) \} = 2\delta(x, y). \]  

This canonical structure completely determines the Gauss constraint

\[ G[\Lambda] = \frac{1}{2G} \int \! dx \Lambda \left( (E^1)' - 2E^2 A_3 + 2E^3 A_2 \right) \]  

and the diffeomorphism constraint

\[ D[M] = \frac{1}{2G} \int \! dx M \left( 2A'_3 E^3 + 2A'_2 E^2 - A_1(E^1)' \right) \]  

but not the Hamiltonian constraint. Sometimes, it is convenient to combine the diffeomorphism constraint $D[M]$ and the Gauss constraint $G[\Lambda]$ to form the vector constraint

\[ V[M] = D[M] + G[A_1 M] = \frac{1}{G} \int \! dx M \left( (A'_3 + A_1 A_2) E^3 + (A'_2 - A_1 A_3) E^2 \right). \]  

We will now use these constraints and attempt to derive the most general form of the Hamiltonian constraint, purely quadratic in the canonical fields and with up to first derivatives of $A_i$ but no derivatives of $E^i$, such that all constraints have closed Poisson brackets. With this assumption, we can write the local (unsmeread) constraint as

\[ \mathcal{H} = H^{110} E^1 E^2 + H^{101} E^1 E^3 + H^{011} E^2 E^3 + H^{001}(E^1)^2 + H^{020}(E^2)^2 + H^{002}(E^3)^2, \]  

where we use the convention that $H[N] = (2G)^{-1} \int \! dx N(x) \mathcal{H}$, $H^{ijk}$ may be functions of $A_1, A_2, A_3$ and their spatial derivatives up to first order.
2.1.1 Diffeomorphism constraint

We first consider the bracket of the Hamiltonian and diffeomorphism constraints, writing it in local form as

\[
\{H(x), D(y)\} = G \int dz \left( 2 \frac{\delta H(x)}{\delta A_1(z)} \frac{\delta D(y)}{\delta E^1(z)} - 2 \frac{\delta H(x)}{\delta E^1(z)} \frac{\delta D(y)}{\delta A_1(z)} + \frac{\delta H(x)}{\delta A_2(z)} \frac{\delta D(y)}{\delta E^2(z)} - \frac{\delta H(x)}{\delta E^2(z)} \frac{\delta D(y)}{\delta A_2(z)} + \frac{\delta H(x)}{\delta A_3(z)} \frac{\delta D(y)}{\delta E^3(z)} - \frac{\delta H(x)}{\delta E^3(z)} \frac{\delta D(y)}{\delta A_3(z)} \right) \tag{8}
\]

where \( D[M] = (2G)^{-1} \int dx M(x) D(x) \). If this bracket is to correspond to classical hypersurface deformations, it should be equal to

\[
\{H(x), D(y)\} = 2G (H'(x) \delta(x,y) + 2H(x) \delta'(x,y)) , \tag{9}
\]

using the convention that a prime on a delta function always indicates a derivative with respect to the first argument. Therefore,

\[
\delta'(x,y) = -\delta'(y,x) . \tag{10}
\]

If the bracket is of the given form, the smeared constraints have the bracket

\[
\{H[N], D[M]\} = \frac{1}{4G^2} \int dxdy \int dzn M(y) \{H(x), D(y)\} = \frac{1}{2G} \int dxdy N(x) M(y) \left( (\partial_x H(x)) \delta(x,y) - 2H(x) \partial_y \delta(x,y) \right) = -H[N'M'] + 2H[NM'] = -H[M'N'] = -H[MN'] \tag{11}
\]

as required if \( N \) has density weight minus one for the purpose of having a quadratic Hamiltonian constraint.

We proceed by evaluating the Poisson bracket. Considering the assumed dependence
therefore, must be zero, so that we already know that \( H \) of \( \mathcal{H} \) on the canonical variables, we have

\[
\{\mathcal{H}(x), \mathcal{D}(y)\} = 2G \int dz \left( \left( \frac{\partial \mathcal{H}(x)}{\partial A_1(z)} \delta(x, z) + \frac{\partial \mathcal{H}(x)}{A'_1(z)} \delta'(x, z) \right) (-A_1(y)\delta'(y, z)) \right.
\]

\[
- \left. \left( \frac{\partial \mathcal{H}(x)}{\partial E_1(z)} \delta(x, z) \left( -(E_1)'(y)\delta(y, z) \right) \right) \right.
\]

\[
+ \left( \frac{\partial \mathcal{H}(x)}{\partial A_2(z)} \delta(x, z) + \frac{\partial \mathcal{H}(x)}{A'_2(z)} \delta'(x, z) \right) A'_2(y)\delta(y, z)
\]

\[
- \left. \left( \frac{\partial \mathcal{H}(x)}{\partial E_2(z)} \delta(x, z) E^2(y)\delta'(y, z) \right) \right.
\]

\[
+ \left( \frac{\partial \mathcal{H}(x)}{\partial A_3(z)} \delta(x, z) + \frac{\partial \mathcal{H}(x)}{A'_3(z)} \delta'(x, z) \right) A'_3(y)\delta(y, z)
\]

\[
- \left. \left( \frac{\partial \mathcal{H}(x)}{\partial E_3(z)} \delta(x, z) E^3(y)\delta'(y, z) \right) \right.
\]

\[
= 2G \left( \frac{\partial \mathcal{H}(x)}{\partial A_2(x)} A'_2(x) + \frac{\partial \mathcal{H}(x)}{\partial A_3(x)} A'_3(x) + \frac{\partial \mathcal{H}(x)}{\partial E_1(x)}(E_1)'(x) \right) \delta(x, y)
\]

\[
- \left( \frac{\partial \mathcal{H}(x)}{\partial A_1(x)} A_1(y) + \frac{\partial \mathcal{H}(x)}{\partial E_2(x)} E^2(y) + \frac{\partial \mathcal{H}(x)}{\partial E_3(x)} E^3(y) \right)
\]

\[
+ \left( \frac{\partial \mathcal{H}(x)}{\partial A_2(x)} A_2(y) + \frac{\partial \mathcal{H}(x)}{\partial A_3(x)} A_3(x) \right) \delta'(y, x)
\]

\[
- \int dz \frac{\partial \mathcal{H}(x)}{\partial A'_1(z)} A_1(y)\delta'(x, z)\delta'(y, z)
\]

where we used (10).

The last term has a product of two derivatives of delta functions, which does not occur in (9). Integrating by parts can remove one of the derivatives, but it also gives a second-order derivative of a delta function which does not appear either in (9). The term, therefore, must be zero, so that we already know that \( \mathcal{H} \) cannot depend on \( A_1' \). In order to bring the remaining terms to a form close to (9), we use the identity

\[
A(x)B(y)\delta'(y, x) = A(x)\partial_y(B(y)\delta(y, x)) - A(x)B'(y)\delta(x, y)
\]

\[
= A(x)\partial_y(B(x)\delta(y, x)) - A(x)B'(x)\delta(x, y)
\]

\[
= A(x)B(x)\delta'(y, x) - A(x)B'(x)\delta(x, y)
\]

(13)
and write
\[
\{ \mathcal{H}(x), \mathcal{D}(y) \} = 2G \left( \frac{\partial \mathcal{H}(x)}{\partial A_1(x)} A_1'(x) + \frac{\partial \mathcal{H}(x)}{\partial A_2(x)} A_2'(x) + \frac{\partial \mathcal{H}(x)}{\partial A_3(x)} A_3'(x) \right. \\
+ \frac{\partial \mathcal{H}(x)}{\partial A_2'(x)} A_2'(x) + \frac{\partial \mathcal{H}(x)}{\partial A_3'(x)} A_3'(x) \\
+ \frac{\partial \mathcal{H}(x)}{\partial E^1(x)} (E^1)'(x) + \frac{\partial \mathcal{H}(x)}{\partial E^2(x)} (E^2)'(x) + \frac{\partial \mathcal{H}(x)}{\partial E^3(x)} (E^3)'(x) \delta(x, y) \left. \right) \\
+ 2G \left( \frac{\partial \mathcal{H}(x)}{\partial A_1(x)} A_1(x) + \frac{\partial \mathcal{H}(x)}{\partial A_2(x)} A_2(x) + \frac{\partial \mathcal{H}(x)}{\partial A_3(x)} A_3(x) \\
+ \frac{\partial \mathcal{H}(x)}{\partial E^2(x)} E^2(x) + \frac{\partial \mathcal{H}(x)}{\partial E^3(x)} E^3(x) \right) \delta'(x, y). \tag{14}
\]

Since \( \mathcal{H} \) does not depend on \( A_1' \), the first parenthesis (multiplied by a delta function) is equal to \( \mathcal{H}' \) without any further restriction on the dependence on other canonical variables. In order to evaluate the second parenthesis, which according to (12) should equal \( 4G \mathcal{H} \), we use the quadratic form (7) and obtain the condition
\[
\frac{\partial \mathcal{H}(x)}{\partial A_1(x)} A_1(x) + \frac{\partial \mathcal{H}(x)}{\partial A_2(x)} A_2(x) + \frac{\partial \mathcal{H}(x)}{\partial A_3(x)} A_3(x) \\
+ H^{110} E^1 E^2 + H^{101} E^1 E^3 + 2H^{011} E^2 E^3 + 2H^{020} (E^2)^2 + 2H^{002} (E^3)^2 \\
= 2 \left( H^{110} E^1 E^2 + H^{101} E^1 E^3 + H^{011} E^2 E^3 + H^{020} (E^2)^2 + H^{002} (E^3)^2 \right) \tag{15}
\]
or
\[
\frac{\partial \mathcal{H}(x)}{\partial A_1(x)} A_1(x) + \frac{\partial \mathcal{H}(x)}{\partial A_2(x)} A_2(x) + \frac{\partial \mathcal{H}(x)}{\partial A_3(x)} A_3(x) = H^{110} E^1 E^2 + H^{101} E^1 E^3 + 2H^{200} (E^1)^2
\]
after some cancellations. Comparing coefficients of \( E^i E^j \) in this equation, we obtain
\[
\frac{\partial H^{110}}{\partial A_1} A_1 + \frac{\partial H^{110}}{\partial A_2} A_2 + \frac{\partial H^{110}}{\partial A_3} A_3' = H^{110} \tag{16}
\]
\[
\frac{\partial H^{101}}{\partial A_1} A_1 + \frac{\partial H^{101}}{\partial A_2} A_2 + \frac{\partial H^{101}}{\partial A_3} A_3' = H^{101} \tag{17}
\]
\[
\frac{\partial H^{011}}{\partial A_1} A_1 + \frac{\partial H^{011}}{\partial A_2} A_2 + \frac{\partial H^{011}}{\partial A_3} A_3' = 0 \tag{18}
\]
\[
\frac{\partial H^{200}}{\partial A_1} A_1 + \frac{\partial H^{200}}{\partial A_2} A_2 + \frac{\partial H^{200}}{\partial A_3} A_3' = 2H^{200} \tag{19}
\]
\[
\frac{\partial H^{020}}{\partial A_1} A_1 + \frac{\partial H^{020}}{\partial A_2} A_2 + \frac{\partial H^{020}}{\partial A_3} A_3' = 0 \tag{20}
\]
\[
\frac{\partial H^{002}}{\partial A_1} A_1 + \frac{\partial H^{002}}{\partial A_2} A_2 + \frac{\partial H^{002}}{\partial A_3} A_3' = 0. \tag{21}
\]
If we assume polynomial dependence of $\mathcal{H}$ on the connection variables, we can conclude that the coefficients $H_1^{10}$ and $H_1^{11}$ must be linear in $A_1$, $A_2'$ and $A_3'$, while $H_2^{00}$ must be quadratic in these variables. The coefficients $H_0^{11}$, $H_0^{02}$ and $H_0^{00}$ cannot depend on $A_1$, $A_2'$ or $A_3'$.

2.1.2 Bracket of Hamiltonian constraints

The Poisson bracket of two Hamiltonian constraints can be computed in a similar way. Classically, we expect

$$\{\mathcal{H}(x), \mathcal{H}(y)\} = 2G\left(E^1(x)^2\mathcal{V}(x)\delta'(y, x) - E^1(y)^2\mathcal{V}(y)\delta'(x, y)\right)$$

(22)

with the local vector constraint $\mathcal{V}(x)$ such that $\mathcal{V}[M] = (2G)^{-1}\int dx M(x)\mathcal{V}(x)$. If the space-time structure is deformed, the bracket is multiplied by a non-constant function $\beta$ which, for a comparison with [20], we assume to depend only on the $E_i$. (This function should approach $\beta = 1$ in some classical limit, usually for small $A_i$.) After using (7) and comparing coefficients of $E_i E_j$, we obtain the equations

$$2\left(-2\frac{\partial H_{110}}{\partial A'_1} H_{200} - \frac{\partial H_{200}}{\partial A'_1} H_{110}\right) - \frac{\partial H_{110}}{\partial A'_2} H_{110} - 2 \frac{\partial H_{200}}{\partial A'_2} H_{020} - \frac{\partial H_{110}}{\partial A'_3} H_{101} - \frac{\partial H_{200}}{\partial A'_3} H_{011}\right) = 4\beta(A'_2 - A_1 A_3)$$

(23)

$$2\left(-2\frac{\partial H_{101}}{\partial A'_1} H_{200} - \frac{\partial H_{200}}{\partial A'_1} H_{101}\right) - \frac{\partial H_{101}}{\partial A'_2} H_{110} - 2 \frac{\partial H_{200}}{\partial A'_2} H_{020} - \frac{\partial H_{101}}{\partial A'_3} H_{101} - \frac{\partial H_{200}}{\partial A'_3} H_{011}\right) = 4\beta(A'_3 + A_1 A_2)\right)\) (24)

which are sensitive to the modification function $\beta$, as well as several $\beta$-independent equations:

$$4 \frac{\partial H_{200}}{\partial A'_1} H_{200} + \frac{\partial H_{200}}{\partial A'_2} H_{110} + \frac{\partial H_{200}}{\partial A'_3} H_{101} = 0$$

(25)

$$2\left(\frac{\partial H_{110}}{\partial A'_1} H_{110} + 2 \frac{\partial H_{020}}{\partial A'_1} H_{200}\right) + 2 \frac{\partial H_{110}}{\partial A'_2} H_{020} + \frac{\partial H_{020}}{\partial A'_2} H_{110} + \frac{\partial H_{110}}{\partial A'_3} H_{011} + \frac{\partial H_{020}}{\partial A'_3} H_{101} = 0$$

(26)

$$2\left(\frac{\partial H_{101}}{\partial A'_1} H_{101} + 2 \frac{\partial H_{002}}{\partial A'_1} H_{200}\right) + \frac{\partial H_{101}}{\partial A'_2} H_{002} + \frac{\partial H_{002}}{\partial A'_2} H_{101} + 2 \frac{\partial H_{101}}{\partial A'_3} H_{011} + \frac{\partial H_{002}}{\partial A'_3} H_{101} = 0$$

(27)

$$2\left(\frac{\partial H_{011}}{\partial A'_1} H_{200} + \frac{\partial H_{011}}{\partial A'_2} H_{110} + \frac{\partial H_{011}}{\partial A'_3} H_{101}\right) + \frac{\partial H_{011}}{\partial A'_2} H_{101} + 2 \frac{\partial H_{011}}{\partial A'_2} H_{020} + \frac{\partial H_{011}}{\partial A'_3} H_{020} + 2 \frac{\partial H_{011}}{\partial A'_3} H_{011} = 0$$

(28)
Four additional equations,

\[
2 \frac{\partial H^{020}}{\partial A_1'} H^{110} + 2 \frac{\partial H^{020}}{\partial A_2'} H^{110} + \frac{\partial H^{020}}{\partial A_3'} H^{011} = 0 \quad (30)
\]
\[
2 \frac{\partial H^{002}}{\partial A_1'} H^{110} + \frac{\partial H^{002}}{\partial A_2'} H^{011} + 2 \frac{\partial H^{002}}{\partial A_3'} H^{002} = 0 \quad (31)
\]
\[
2 \left( \frac{\partial H^{011}}{\partial A_1'} H^{110} + \frac{\partial H^{020}}{\partial A_1'} H^{110} \right) + \frac{\partial H^{011}}{\partial A_2'} H^{011} + 2 \frac{\partial H^{011}}{\partial A_3'} H^{011} + 2 \frac{\partial H^{002}}{\partial A_3'} H^{002} + \frac{\partial H^{002}}{\partial A_3'} H^{002} = 0 \quad (32)
\]
\[
2 \left( \frac{\partial H^{011}}{\partial A_1'} H^{110} + \frac{\partial H^{002}}{\partial A_1'} H^{110} \right) + \frac{\partial H^{011}}{\partial A_2'} H^{011} + 2 \frac{\partial H^{011}}{\partial A_3'} H^{011} + 2 \frac{\partial H^{002}}{\partial A_3'} H^{002} + \frac{\partial H^{002}}{\partial A_3'} H^{002} = 0 \quad (33)
\]

are identically satisfied, given that \( H^{011}, H^{020} \) and \( H^{002} \) cannot depend on \( A_1' \). Because \( \mathcal{H} \) cannot depend on \( A_1' \), we may simplify the set of equations to

\[
- \frac{\partial H^{110}}{\partial A_2'} H^{110} - 2 \frac{\partial H^{200}}{\partial A_2'} H^{020} - \frac{\partial H^{110}}{\partial A_3'} H^{110} - \frac{\partial H^{200}}{\partial A_3'} H^{011} = 4 \beta (A_2' - A_1 A_3) \quad (34)
\]
\[
- \frac{\partial H^{110}}{\partial A_2'} H^{110} - 2 \frac{\partial H^{200}}{\partial A_2'} H^{011} - \frac{\partial H^{110}}{\partial A_3'} H^{110} - \frac{\partial H^{200}}{\partial A_3'} H^{020} = 4 \beta (A_3' + A_1 A_2) \quad (35)
\]
\[
\frac{\partial H^{200}}{\partial A_2'} H^{110} + \frac{\partial H^{200}}{\partial A_3'} H^{011} = 0 \quad (36)
\]
\[
2 \frac{\partial H^{110}}{\partial A_2'} H^{020} + \frac{\partial H^{110}}{\partial A_3'} H^{011} = 0 \quad (37)
\]
\[
2 \frac{\partial H^{110}}{\partial A_2'} H^{011} + 2 \frac{\partial H^{110}}{\partial A_3'} H^{002} = 0 \quad (38)
\]
\[
2 \frac{\partial H^{110}}{\partial A_2'} H^{011} + \frac{\partial H^{110}}{\partial A_3'} H^{011} + 2 \frac{\partial H^{110}}{\partial A_3'} H^{002} = 0 \quad (39)
\]

### 2.1.3 Gauss constraint

The Gauss constraint further restricts the combinations of basic variables which can appear in the Hamiltonian constraint. The gauge-invariant combinations that contribute to the classical constraint are \( E^1, (E^2)^2 + (E^3)^2, A_2 E^2 + A_3 E^3, A_2^2 + A_3^2 \) and \( A_1 (A_2 E^2 + A_3 E^3) - (A_2^2 E^3 - A_3^2 E^2) \). (The identity (13) is useful for seeing that the last combination has a vanishing Poisson bracket with the unsmeared Gauss constraint.) These expressions show that \( A_1, A_2 \) and \( A_3 \) can appear in gauge-invariant form only in combination with \( E^2 \) and \( E^3 \). It is therefore impossible to fulfill the condition that \( H^{200} \) be quadratic in \( A_1, A_2 \) and \( A_3 \) because \( H^{200} \) is defined as the \( E \)-independent coefficient of \( (E^1)^2 \) in the Hamiltonian constraint. For Hamiltonian constraints quadratic in \( E^i \), we have \( H^{200} = 0 \).
To determine the on-shell behavior of the Hamiltonian constraint, it restricts possible modifications of the dynamics. For instance, the extrinsic-curvature component transforms generated by the Gauss constraint, the modified term $\beta^{-1/2}(H_{cl}^{020} + H_{cl}^{002})$ is an arbitrary (positive) function of $A_2^2 + A_3^2$, which is equivalent to the modification found in [20] and therefore strengthens their result.

If we relax the condition that the Hamiltonian constraint not depend on spatial derivatives of the densitized triad, additional gauge invariant combinations are possible. For instance, the extrinsic-curvature component

$$K_1 = A_1 - \frac{(E^2)' E^3 - E^2 (E^3)'}{(E^2)^2 + (E^3)^2}$$

is gauge invariant. Moreover, if spatial derivatives of the densitized triad are allowed, the Gauss constraint can be used to rewrite the Hamiltonian constraint without changing the on-shell behavior. For instance, the identity

$$A_1(A_2 E^2 + A_3 E^3) + 2 E^2 A_3' - 2 E^3 A_2' = (E^1)'' + A_2(A_1 E^2 + 2(E^3)') + A_3(A_1 E_3^3 - 2(E^2)') - G'$$

eliminates spatial derivatives of $A_2$ and $A_3$ from the Hamiltonian constraint, in favor of a second-order spatial derivative of $E^1$. This new form is much closer to the expression of the Hamiltonian constraint in extrinsic-curvature variables [30], and may allow different modified brackets than the quadratic version (7) even if one works with the reduced Ashtekar connections $A_1$.

The possibility of rewriting the Hamiltonian constraint by using the Gauss constraint explains why different formulations of the same classical theory may give rise to different modified brackets: The Gauss constraint depends on $A_2$ and $A_3$, and therefore, depending on how it is used in writing the Hamiltonian constraint, it restricts possible modifications. In extrinsic-curvature variables, this ambiguity does not appear because the Gauss constraint is solved explicitly.
From the perspective of effective field theory, applied here to the classical structure of up to second-order derivatives, restricting the dependence of the Hamiltonian constraint on spatial derivatives of $E^i$ leads to non-generic models. The classical constraint is quadratic in $A_i$, which, according to the field equations implied by the theory, amounts to terms with up to two derivatives. Any term that is consistent with the symmetries of the theory (generated by the constraints) and has up to two derivatives (temporal or spatial) should then be allowed for a generic model. Such theories should include terms with up to second-order spatial derivatives of $E^i$, in addition to the quadratic terms in $A_i$ which contribute two time derivatives. (A higher-derivative theory beyond second order would be obtained by including quantum back-reaction effects, which is not the purpose of this paper.)

2.2 Arbitrary Barbero–Immirzi parameter

We will now show that the preceding rigidity result is not stable within a class of models in which spatial derivatives of the densitized triad are allowed to appear. A suitable set of constraints that describes the same classical physics as, depending on the signature, Euclidean or self-dual gravity is obtained by letting the Barbero–Immirzi parameter vary, instead of fixing it to a specific value such that $\gamma^2 = \pm 1$. The modification found in \cite{20} is therefore not generic. To this end, we will now switch to a general setting of spherically symmetric gravity in which the Barbero–Immirzi parameter and other numerical factors (as well as the gravitational constant $G$) are included.

Spherically symmetric gravity can be formulated as a Hamiltonian theory with phase space given by the canonical pairs, subject to three constraints. This setting has been formulated in \cite{23} for self-dual variables and in \cite{30} for real variables. In order to avoid having to impose reality conditions, we follow the latter notation, in which the canonical pairs $(A_1, E^1)$, $(A_2, E^2)$ and $(A_3, E^3)$ are such that

$$\{A_1(x), E^1(y)\} = 2\gamma G\delta(x, y) \tag{44}$$

and

$$\{A_2(x), E^2(y)\} = \gamma G\delta(x, y) , \quad \{A_3(x), E^3(y)\} = \gamma G\delta(x, y) \tag{45}$$

(a version of (1) and (2) for arbitrary real $\gamma$). They are subject to the Gauss constraint

$$G[\Lambda] = \frac{1}{2\gamma G} \int dx \Lambda ((E^1)' + 2A_2E^3 - 2A_3E^2) \tag{46}$$

smeared with a multiplier $\Lambda$, the diffeomorphism constraint

$$D[N^x] = \frac{1}{2\gamma G} \int dx N^x \left( -A_1(E^1)' + 2A_3' E^3 + 2A_2' E^2 \right) \tag{47}$$
smeared with the shift vector \( N^x \), and the Hamiltonian constraint
\[
H[N] = \frac{1}{2G} \int dx \, \overset{\sim}{N} (2A_1 E^1 (A_2 E^2 + A_3 E^3) \\
+ (A_2^2 + A_3^2 - 1) ((E^2)^2 + (E^3)^2) + 2E^1 (E^2 A_3' - E^3 A_2') \\
+ (\epsilon - \gamma^2) (2K_1 E^1 (K_2 E^2 + K_3 E^3) + ((K_2)^2 + (K_3)^2)((E^2)^2 + (E^3)^2))) \quad (48)
\]
smeared with the lapse function \( \overset{\sim}{N} \) of density weight \(-1\). The non-polynomial relationship between the extrinsic-curvature components \( K_1, K_2 \) and \( K_3 \) with the basic variables is given below.

In all three constraints, the prime represents a derivative with respect to the radial coordinate \( x \). Moreover, \( \gamma \) in (48) is the Barbero–Immirzi parameter \([25,27]\) and \( \epsilon = \pm 1 \) the space-time signature, such that \( \epsilon = 1 \) in the Euclidean case and \( \epsilon = -1 \) in the Lorentzian case. As usual, it is convenient to split the Hamiltonian constraint into the Euclidean part
\[
H^E[N] = \frac{1}{2G} \int dx \, \overset{\sim}{N} (2A_1 E^1 (A_2 E^2 + A_3 E^3) \\
+ (A_2^2 + A_3^2 - 1) ((E^2)^2 + (E^3)^2) + 2E^1 (E^2 A_3' - E^3 A_2')) \quad (49)
\]
and the “Lorentzian” contribution
\[
H^L[N] = -\frac{\gamma^2 - \epsilon}{2G} \int dx \, \overset{\sim}{N} (2K_1 E^1 (K_2 E^2 + K_3 E^3) + ((K_2)^2 + (K_3)^2)((E^2)^2 + (E^3)^2)) \quad (50)
\]
Thus, \( H[N] = H^E[N] \) for \( \gamma = \pm 1 \) in Euclidean signature (\( \epsilon = 1 \)), while the “Lorentzian” contribution (a slight misnomer) also contributes in Euclidean signature if \( \gamma \neq \pm 1 \). (The Lorentzian contribution is always required in Lorentzian signature if one works with real \( \gamma \) such that the Poisson brackets are real.) The canonical variables \( A_1, E^2 \) and \( E^3 \) have density weight one.

The geometrical meaning of the phase-space variables is determined as follows: The fields \( E^1, E^2 \) and \( E^3 \), as the components of a spherically symmetric densitized triad, describe a spatial metric \( q_{ab} \) according to the line element
\[
ds^2 = g_{ab} dx^a dx^b = \frac{(E^2)^2 + (E^3)^2}{|E^1|} dx^2 + |E^1|(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (51)
\]
The densitized triad also determines a spin connection such that it is constant with respect to the resulting covariant derivative. The components of this spin connection are functions of the densitized triad and its first spatial derivatives:
\[
\Gamma_1 = \frac{E^3(E^2)' - E^2(E^3)'}{(E^2)^2 + (E^3)^2}, \quad \Gamma_2 = -\frac{1}{2} \frac{(E^1)'E^3}{(E^2)^2 + (E^3)^2}, \quad \Gamma_3 = \frac{1}{2} \frac{(E^1)'E^2}{(E^2)^2 + (E^3)^2}. \quad (52)
\]
The densitized triad is canonically conjugate to components of extrinsic curvature, \(K_i\), \(i = 1, 2, 3\). Since the \(\Gamma_i\) depend only on \(E^i\), one can add them to \(\Gamma_i\) without changing the latter’s canonical relationships with \(E^i\). In this way, the canonical connection components \(A_i = \Gamma_i + \gamma K_i\) are obtained, using the Barbero–Immirzi parameter \(\gamma\).

The constrained system is first class, with brackets of the constraints \(D[N]\) and \(H[N]\) according to Dirac’s hypersurface deformations \cite{[20]} (taking into account the density weight of \(N\) in the Hamiltonian constraint used here). In particular, the bracket \(\{H[N]\}, H[M]\} should be proportional to the diffeomorphism constraint, up to possible contributions from the Gauss constraint. We display the relevant derivations in a more general setting, following the observation \cite{[20]} that, for \(\gamma^2 = \epsilon\), the constraint brackets remain closed in the presence of a “magnetic-field” modification, replacing \(B_1 := A_2^2 + A_3^2 - 1\) in the Euclidean part of the Hamiltonian constraint with an arbitrary function \(f(A_2^2 + A_3^2 - 1)\). Our aim is to determine whether this modification can be carried over to the Lorentzian contribution.

We begin with the bracket of two modified Euclidean parts, \(\{H^E[N], H^E[M]\}\). Thanks to antisymmetry of the bracket in \(N\) and \(M\), we need consider only those brackets of terms that lead to derivatives of delta functions. There are two such contributions,

\[
\{2A_1(x)E^2(x)(A_2(x)E^2(x) + A_3(x)E^3(x)), 2E^1(y)(E^2(y)A_3(y) - E^3(y)A_2(y))\} = (\cdots)\delta(x, y) - 4\gamma GA_1(x)E^1(x)E^1(y) (A_3(x)E^2(y) - A_2(x)E^3(y)) \partial_y \delta(x, y)
\]

and

\[
\{2E^1(x)(E^2(x)A_3(x) - E^3(x)A_2(x)), 2E^1(y)(E^2(y)A_3(y) - E^3(y)A_2(y))\} = (\cdots)\delta(x, y) - 4\gamma GE^1(x)E^1(y) \left( (E^2(x)A_2(y) + E^3(x)A_3(y)) \partial_x \delta(x, y) - (E^2(y)A_2(x) + E^3(y)A_3(x)) \partial_y \delta(x, y) \right).
\]

With these two ingredients, we obtain

\[
\{H^E[N], H^E[M]\} = \frac{\gamma}{G} \int dx \left( N'M - NM' \right) (E^1)^2 \left( A_1(E_2E_3 - A_3E_2) + E_2A_2 + E_3A_3' \right)
\]

and

\[
\{H^E[N], H^E[M]\} = \gamma^2 V ((E^1)^2 (N'M - M'N))
\]

where

\[
V[\Lambda] = \frac{1}{\gamma G} \int dx \Lambda \left( A_1(E_2E_3 - A_3E_2) + A_3' E_3 + A_2' E_2 \right)
\]

is the vector constraint constraint \cite{[4]}, \(V[\Lambda] = D[\Lambda] + G[A_1\Lambda]\), related to the diffeomorphism constraint \(D\) through a contribution from the Gauss constraint \cite{[46]}.

Using \(\sqrt{\text{det} q} = \sqrt{|E^1|((E^2)^2 + (E^3)^2)}\) from \cite{[51]}, we can write the smearing function in (55) as

\[
(E^1)^2 \left( N'M - M'N \right) = \frac{|E^1|}{(E^2)^2 + (E^3)^2} (N'M - M'N)
\]

where \(N = \sqrt{|E^1|((E^2)^2 + (E^3)^2)}N\) and \(M = \sqrt{|E^1|((E^2)^2 + (E^3)^2)}M\) are lapse functions without density weight. The coefficient \(|E^1|/(E^2)^2 + (E^3)^2\) in (57) is, according to (51),
the radial component of the inverse spatial metric, in agreement with the classical form of hypersurface-deformation brackets. The system is therefore anomaly-free for any modification \( f \) in \([48]\) without any modification of the constraint brackets and the space-time structure — provided the Lorentzian part does not contribute to the Hamiltonian constraint, that is in Euclidean gravity with \( \gamma = \pm 1 \) or in Lorentzian gravity with \( \gamma = \pm i \). This is consistent with the results reported in \([20]\).

It is easy to see that any function \( f(A_2^2 + A_3^2 - 1) \) can be used in the modified Euclidean part because this term does not produce derivatives of delta functions in the Poisson bracket of two Euclidean constraints. Moreover, because \( A_2 \) and \( A_3 \) are scalars without density weight, any such term has the correct Poisson bracket with the diffeomorphism constraint. However, if \( \gamma^2 \neq \epsilon \), the cross-term \( \{H^E[N], H^L[M]\} \) in the Poisson bracket of two Hamiltonian constraints does receive a contribution from \( f(A_2^2 + A_3^2 - 1) \) in \( H^E[N] \) because \( H^L[M] \), written in the canonical variables \( A_i \) and \( E^i \), contains spatial derivatives of \( E^3 \) through \( \Gamma_i \). An explicit calculation is therefore required to check whether the bracket can still be closed for \( f(A_2^2 + A_3^2 - 1) \neq A_2^2 + A_3^2 - 1 \).

We first compute The Poisson brackets of each individual term in \( H^E[N] \) with the full \( H^L[M] \): We obtain

\[
\frac{1}{G} \{ \int dx N(x) A_1(x) E^1(x)(A_2(x) E^2(x) + A_3(x) E^3(x)), H^L[M] \} = \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx dy N(x) M(y) \{ (-2A_1(x) E^1(x) E^1(x) E^1(y)(A_2(y) E^2(y) + A_3(y) E^3(y))\{A_2(x) E^2(x) + A_3(x) E^3(x), \Gamma_1(y) \}) + E^1(y) E^2(x) + A_3(x) E^3(x)) (E^2(y)^2 + E^3(y)^2) \times \{ A_1(x), 2A_2(y)\Gamma_2(y) + A_3(y)\Gamma_3(y) + \Gamma_2(y)^2 + \Gamma_3(y)^2 \} \}
\]

\[
= \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx dy N(x) M(y) \left( -2A_1(x) E^1(x) E^1(y)(A_2(y) E^2(y) + A_3(y) E^3(y)) \right) \frac{E^2(x) E^3(y) - E^2(y) E^3(x)}{E^2(y)^2 + E^3(y)^2} + 2 E^1(x) (E^2(y)^2 + E^3(y)^2)(A_2(x) E^2(x) + A_3(x) E^3(x)) \times \frac{A_2(y) E^3(y) - A_3(y) E^2(y) - E^3(y)\Gamma_2(y) + E^2(y)\Gamma_3(y)}{E^2(y)^2 + E^3(y)^2} \right) \partial_\gamma \delta(x, y)
\]

\[
= -\frac{\gamma^2}{2\gamma G} \int dx N(x) M'(x) E^1(A_2 E^2 + A_3 E^3) \left( (E^1)^2 + 2A_2 E^3 - 2A_3 E^2 \right)
\]

\[
= -(\gamma^2 - \epsilon) G[N M' E^1(A_2 E^2 + A_3 E^3)]
\]

Up to terms that cancel out when inserted in the antisymmetric \( \{H^E[N], H^L[M]\} + \{H^L[N], H^E[M]\} \).

In the detailed calculations, we have used the explicit expressions for the \( \Gamma_i \), from which we also obtain the useful identity

\[
\gamma(K_2 E^2 + K_3 E^3) = A_2 E^2 + A_3 E^3
\]
because $\Gamma_2 E^2 + \Gamma_3 E^3$ is identically zero.

The second term,

$$\frac{1}{2G} \left\{ \int dx N(x) f(A_2(x)^2 + A_3(x)^2 - 1)(E^2(x)^2 + E^3(x)^2), H^1[\sim] \right\}$$

$$= \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx dy N(x) M(y) \left( \cdots \delta(x, y) \right.$$

$$\left. - 2 \hat{f}(x)(E^2(x)^2 + E^3(x)^2) E^1(y)(A_2(x)E^2(y) + A_3(y)E^3(y)) \{ A_2(x)^2 + A_3(x)^2, \Gamma_1(y) \} \right)$$

$$= \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx dy N(x) M(y) \left( \cdots \delta(x, y) \right.$$

$$\left. - 2 \hat{f}(x)(E^2(x)^2 + E^3(x)^2) E^1(y)(A_2(x)E^2(y) + A_3(y)E^3(y)) \right.$$

$$\times \frac{2A_2(x)E^3(y) - A_3(x)E^2(y)}{E^2(y)^2 + E^3(y)^2} \partial_y \delta(x, y)$$

$$= 2(\gamma^2 - \epsilon) G[N M' \hat{f} E^1(A_2 E^2 + A_3 E^3)] - \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx N M' \hat{f} E^1(E^1)'(A_2 E^2 + A_3 E^3), \quad (60)$$

does not vanish on the constraint surface. Therefore, the function $f$, whose derivative by its argument we have denoted by $\hat{f}$, is now relevant for closed brackets. In particular, the last contribution containing $(E^1)'$ must be canceled by a corresponding term in the remaining bracket.

In this last bracket,

$$B := \frac{1}{G} \left\{ \int dx N(x) E^1(x)(E^2(x)A_3(x) - E^3(x)A_2(x)), H^1[\sim] \right\}$$

$$= \frac{\gamma^2 - \epsilon}{2\gamma G^2} \int dx dy N(x) M(y) \left( \cdots \delta(x, y) \right.$$

$$\left. + 2E^1(x)E^1(y)(A_2(y)E^2(y) + A_3(y)E^3(y)) \{ E^2(x) A_3(x)' - E^3(x) A_2(x)', \Gamma_1(y) \} \right.$$

$$\left. + 2E^1(x)E^1(y)(A_1(y) - \Gamma_1(y)) \{ E^2(x) A_3(x)' - E^3(x) A_2(x)', A_2(y)E^2(y) + A_3(y)E^3(y) \} \right.$$

$$\left. - 2E^1(x)(E^2(y)^2 + E^3(y)^2) \{ (A_2(y) - \Gamma_2(y)) \{ E^2(x) A_3(x)' - E^3(x) A_2(x)', \Gamma_2(y) \} \right.$$

$$\left. + (A_3(y) - \Gamma_3(y)) \{ E^2(x) A_3(x)' - E^3(x) A_2(x)', \Gamma_3(y) \} \} \right)$$

$$= \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx dy N(x) M(y) \left( \cdots \delta(x, y) \right.$$

$$\left. - 2E^1(x)E^1(y)(A_2(y)E^2(y) + A_3(y)E^3(y)) \frac{E^2(x)E^2(y) + E^3(x)E^3(y)'}{E^2(y)^2 + E^3(y)^2} \partial_x \delta(x, y) \right.$$

$$\left. + 2E^1(x)E^1(y)(A_2(y)E^2(y) + A_3(y)E^3(y)) \frac{E^2(x)E^3(y) + E^3(x)E^2(y)}{E^2(y)^2 + E^3(y)^2} \partial_x \partial_y \delta(x, y) \right.$$

$$\left. + 2(A_1(y) - \Gamma_1(y)) E^1(x)E^1(y)(E^2(x)A_3(y) - E^3(x)A_2(y)) \partial_x \delta(x, y) \right.$$

$$\left. + E^1(x)E^1(y) \left( (A_2(y) - \Gamma_2(y)) E^2(x) + (A_3(y) - \Gamma_3(y)) E^3(y) \right) \partial_x \delta(x, y) \right), \quad (61)$$

we have a contribution from a second-order derivative of the delta function. Integrating by parts once in this term and taking into account its contributions to $NM'$ and $N'M$,
respectively, (noting that terms with $N'M'$ cancel out in the final antisymmetric bracket) we write

$$B = \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx dy N(x) M(y) \left( (\cdots) \delta(x, y) \right. $$

$$\left. -2 \frac{E^1(x) E^1(y)}{E^2(y)^2 + E^3(y)^2} \left( E^2(x) E^2(y) + E^3(x) E^3(y) \right) \right. $$

$$\left. + (E^3(y) E^2(y)' - E^2(y) E^3(y)')(E^2(x) A_3(y) - E^3(x) A_2(y)) \right) \partial_x \delta(x, y) $$

$$+ E^1(x) E^1(y) (2 A_1(y) (E^2(x) A_3(y) - E^3(x) A_2(y)) $$

$$+ E^1(x) E^1(y)' (A_2(y) E^2(x) + A_3(y) E^3(x)) \right) \partial_x \delta(x, y) $$

$$- 2E^1(x) E^1(y) \left( A_2(y) E^2(y)' + A_3(y) E^3(y)' \right) \partial_x \delta(x, y) $$

$$\left. - 2(A_2(y) E^2(y) + A_3(y) E^3(y)) \frac{E^2(x) E^2(y)' + E^3(x) E^3(y)'}{E^2(y)^2 + E^3(y)^2} \right) \partial_x \delta(x, y) $$

$$= \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx dy N(x) M(y) \left( (\cdots) \delta(x, y) \right. $$

$$\left. + 2E^1(x) E^1(y) \right) $$

$$\times \left( A_1(y) (E^2(x) A_3(y) - E^3(x) A_2(y)) - (A_2(y)' E^2(y) + A_3(y)' E^3(y)) \right) \partial_x \delta(x, y) $$

$$\left. = (\gamma^2 - \epsilon) \left( D[(E^1)^2 N'M] + G[A_1(E^1)^2 N'M] \right) \right) $$

$$\left. - \frac{\gamma^2 - \epsilon}{2\gamma G} \int dx N'M E^1(E^1)' (A_2 E^2 + A_3 E^3). \right) \quad (62)$$

This result provides the diffeomorphism constraint as well as a term which cancels the previous non-constraint contribution in (60), but only if $\tilde{f} = 1$. Therefore, if the Lorentzian contribution is included, no modification of the classical $A_2^2 + A_3^2 - 1$ is allowed. The final bracket now equals

$$\{H[N], H[M]\} = \{H^E[N], H^E[M]\} + \{H^L[N], H^L[M]\} - \{H^E[M], H^L[N]\} $$

$$= \gamma^2 D[(E^1)^2 (N'M - N'M')] + \gamma^2 G[A_1(E^1)^2 (N'M - N'M')] $$

$$- (\gamma^2 - \epsilon) G[E^1 (A_2 E^2 + A_3 E^3) (1 - 2\tilde{f})(N'M - N'M')] $$

$$- (\gamma^2 - \epsilon) \left( D[(E^1)^2 (N'M - N'M')] + G[A_1(E^1)^2 (N'M - N'M')] \right) $$

$$\approx \epsilon \left( D[(E^1)^2 (N'M - N'M')] + G[A_1(E^1)^2 (N'M - N'M')] \right) $$

$$+ (\gamma^2 - \epsilon) G[E^1 (A_2 E^2 + A_3 E^3) (N'M - N'M')] $$

$$\approx -\epsilon D[(E^1)^2 (N'M - N'M')], \quad (63)$$

using $\tilde{f} = 1$ in the last step because the bracket would not be closed otherwise. (Note that $\{H^L[N], H^L[M]\} = 0$, which can most easily be seen if one uses the canonical variables $K_i$ and $E^i$, of which no spatial derivatives appear in the Lorentzian contribution.)
3 Connection variables in a canonical effective field theory

We have seen a crucial difference between gravitational theories governed by the Euclidean Hamiltonian constraint $H^E$ and the full $H^E + H^L$, respectively. Formally, the reason is the difference in derivative structures implied by the spin-connection terms in $H^L$: While $H^E$ contains derivatives only of the spatial connection, $H^L$ also contributes spatial derivatives of the triad. As a consequence, the two versions allow different modifications while maintaining closed brackets.

Derivative structures are best dealt with in a setting of effective field theory, in which one formulates generic theories by selecting the basic fields and the maximum order of derivatives to which they contribute, as well as relevant symmetries. For our purposes, we need an adaptation of the usual arguments to a canonical formulation, in which some derivatives may not be explicit because they appear only if some of the canonical equations are used, mainly in the relationship between momenta and “velocities.”

In order to determine the correct derivative orders in a canonical theory, we must first choose which of the basic fields should play the role of configuration variables and therefore are considered free of time derivatives. We are looking for a canonical theory of triads, which will correspond to a space-time metric or triad theory, and therefore choose as our basic fields a densitized spatial triad with momenta. The latter may be given in terms of a connection or extrinsic curvature. The derivative order depends on the quantum effects we wish to include. For now, we will analyze the classical setting and therefore consider up to second-order derivatives of the fields. Symmetries are implemented by the requirement that the constraint brackets be closed, and in the classical case amount to hypersurface-deformation brackets.

3.1 Basic strategy

In our explicit calculations of generic terms, we again follow the conventions of section 2.2 and set $\gamma = 1$ for simplicity. For our effective Hamiltonian, we choose to allow up to second-order in derivatives of densitized triads. Since the conjugate momenta are of the form $A \sim \partial E$, using the equations of motion for $\dot{E}$, we have the following general form of the Hamiltonian constraint $H[\tilde{N}] = (2G)^{-1} \int dx N(x) H(x)$ with

$$H = \alpha^i (E^j, \partial E^j) A_i + \beta^{ij}(E^k) A_{ij} + \gamma^i(E) \partial A_i + Q(E, \partial E, \partial^2 E) ,$$

(64)

where we have introduced the notation $\partial \equiv \partial / \partial x$, $A_{ij...k} = A_i A_j \cdots A_k$ and $E^{ij...k} = E^i E^j \cdots E^k$. We can already observe some preliminary restrictions on the coefficients $\alpha^i(E, \partial E)$ and $Q(E, \partial E, \partial E \partial E, \partial^2 E)$. Both coefficients are initially allowed to depend on $\partial E^i$ and $\partial^2 E^i$. But since we only allow up to second-order derivatives in the Hamiltonian constraint, the dependence cannot be arbitrary. Specifically, we have

$$\begin{align*}
\alpha^i &= \bar{\alpha}^i(E) + a_j^i(E) \partial E^j \\
Q &= \bar{Q}(E) + a_i(E) \partial E^i + b_{ij}(E) \partial E^i \partial E^j + c_i(E) \partial^2 E^i.
\end{align*}$$
We want the Hamiltonian density \( \mathcal{H} \) to respect the classical symmetries,

\[
\begin{align*}
\{ \mathcal{H}(x), \mathcal{G}(y) \} &= 0 \\
\{ \mathcal{H}(x), \mathcal{D}(y) \} &= 2G(\partial \mathcal{H}(x) \delta_{xy} + 2 \mathcal{H}(x) \delta'_{xy}) \\
\{ \mathcal{H}(x), \mathcal{H}(y) \} &\approx -2G(\partial (E^{11} \mathcal{D}(x)) \delta_{xy} + 2E^{11} \mathcal{D}(x) \delta'_{xy}),
\end{align*}
\]

where \( G[A] = (2G)^{-1} \int dx \Lambda(x) \mathcal{G}(x) \) and \( D[N] = (2G)^{-1} \int dx N(x) \mathcal{D}(x) \) are the diffeomorphism and Gauss constraints, respectively. We have introduced the shorthand notation \( \delta'_{xy} := \partial \delta(x-y) \), and \( \approx \) means “equal” when setting \( \mathcal{G} = 0 \) in the final step of calculation. These symmetries will impose restrictions on the coefficients \( \alpha_i, \beta^{ij}, \gamma^i, Q \) in (64), telling us what a generic Hamiltonian constraint looks like.

### 3.2 Brackets

The first bracket, \( \{ \mathcal{H}, \mathcal{G} \} \), represents the restriction to gauge-invariant terms for any allowed \( \mathcal{H} \). Inserting (64), we have

\[
\{ \mathcal{H}(x), \mathcal{G}(y) \} = 2G \int \! dz [(\alpha^1 + 2\beta^{1j} A_j) \delta_{xz} + \gamma^1 \delta'_{xz}](x) \delta'_{yz} \\
+ [(\alpha^2 + 2\beta^{2j} A_j) \delta_{xz} + \gamma^2 \delta'_{xz}](x)(-A_3(y) \delta_{yz}) \\
- [(\delta_{xz} \partial_2 + \delta'_{xz} \partial_2)(\alpha^2) A_i + (\delta_{xz} \partial_2 + \delta'_{xz} \partial_2 + \delta''_{xz} \partial_2)(\gamma^i A_i)](x) E^3(y) \delta_{yz} \\
+ \delta_{xz} \partial_2 \beta^{ij} A_{ij} + \delta_{xz} \partial_2 \gamma^i \partial A_i](x) E^3(y) \delta_{yz} \\
+ [(\alpha^3 + 2\beta^{3j} A_j) \delta_{xz} + \gamma^3 \delta'_{xz}][x A_2(y) \delta_{yz}) \\
- [(\delta_{xz} \partial_3 + \delta'_{xz} \partial_3)(\alpha^3) A_i + (\delta_{xz} \partial_3 + \delta'_{xz} \partial_3 + \delta''_{xz} \partial_3)(\gamma^i A_i)](x) E^3(y) \delta_{yz}) \\
+ \delta_{xz} \partial_3 \beta^{ij} A_{ij} + \delta_{xz} \partial_3 \gamma^i \partial A_i](x) E^3(y) \delta_{yz}) \\
= 0,
\]

where we have introduced further shorthand notation \( \partial_i := \partial/\partial E^i \) and \( \partial' := \partial/\partial (\partial_x E^i) \).

To make the right-hand side of the equation vanish, we need several cancellations. We can do this by first making all functions depend on \( x \) using delta functions and integrating over \( z \). Then we group terms with the same dependence on \( A_i \) and derivatives of \( \delta_{xy} \) together and demand that each grouping vanish by itself. (Different order of derivatives on \( \delta \) may be dependent, for instance in \( \delta'_{yz} A(x) = A(y) \delta'_{yz} + \partial_y A(y) \delta_{yz} \). Therefore, some \( \delta' \) can produce terms that group with a \( \delta \).) This procedure produces several dozens of partial differential equations which we will list later along with those from the \( \{ \mathcal{H}, \mathcal{D} \} \) bracket.
Inserting our form of $\mathcal{H}$ into the $\mathcal{H}$-$\mathcal{D}$ bracket, we obtain

$$\{\mathcal{H}(x), \mathcal{D}(y)\} = 2G \int dz [\delta_{x}^{\prime}(\alpha^1 + 2\beta^{ij}A_j) + \gamma^{i}\delta_{xz}^{y}](-A_1(y)\delta_{yz})$$

$$-[(\delta_{xz}\partial_t + \delta_{xz}\partial_{t'})^i A_i + \delta_{xz}\partial_1\beta^{ij}A_{ij}]$$

$$+\delta_{xz}\partial_t\gamma^i\partial A_i + (\delta_{xz}\partial_1 + \delta_{xz}\partial_{y'} + \delta_{xz}\partial_{y''})(Q)](x)(-\partial E^1(y)\delta_{yz})$$

$$+\delta_{xx}^{zz}(\alpha^2 + 2\beta^{2j}A_j + \gamma^2\delta_{x}^{y})(\partial A_2(y)\delta_{yz})$$

$$-[(\delta_{x}^{z}\partial_2 + \delta_{xz}\partial_{2'})^i A_i + \delta_{x}^{z}\partial_2\beta^{ij}A_{ij}]$$

$$+\delta_{x}^{z}\partial_2\gamma^i\partial A_i + (\delta_{x}^{z}\partial_2 + \delta_{x}^{z}\partial_{2'} + \delta_{x}^{z}\partial_{2''})(Q)](x)(E^2(y)\delta_{yz})$$

$$+\delta_{x}^{z}(\alpha^3 + 2\beta^{3j}A_j + \gamma^3\delta_{x}^{y})(\partial A_3(y)\delta_{yz})$$

$$-[(\delta_{x}^{z}\partial_3 + \delta_{x}^{z}\partial_{3'})^i A_i + \delta_{x}^{z}\partial_3\beta^{ij}A_{ij}]$$

$$+\delta_{x}^{z}\partial_3\gamma^i\partial A_i + (\delta_{x}^{z}\partial_3 + \delta_{x}^{z}\partial_{3'} + \delta_{x}^{z}\partial_{3''})(Q)](x)(E^3(y)\delta_{yz})$$

$$= 2G(\partial_x\mathcal{H}(x)\delta_{xy} + 2\mathcal{H}(x)\delta_{xy}').$$

Similarly to how we dealt with the condition of gauge invariance, we first integrate over $z$ to make all functions depend on $x$, then match term by term with the right-hand side, expanded in $A_i$ and derivatives of $\delta_{xy}$. Again, we obtain a few dozen partial differential equations.

We next list the partial differential equations that the coefficients of terms in $\mathcal{H}$ have to obey. These equations will completely determine the dependence on $E^2$ and $E^3$, leaving free functions of $E^1$ which the $\mathcal{H}$-$\mathcal{H}$ bracket will further restrict. These conditions then determine possible modifications of the classical $\mathcal{H}_d$. In the following equations, we use the differential operators $D := E^2\partial_2 + E^3\partial_3$ and $C := E^2\partial_3 - E^3\partial_2$.

### 3.2.1 The $\mathcal{H}$-$\mathcal{G}$ bracket

For $\beta^{ij}$ and $\gamma^i$ we have

$$\begin{cases}
\hat{\mathcal{C}}\beta^{11} = 0 \\
\hat{\mathcal{C}}\beta^{12} = -\beta^{13} \\
\hat{\mathcal{C}}\beta^{13} = \beta^{12} \\
\hat{\mathcal{C}}\beta^{22} = -2\beta^{23} \\
\hat{\mathcal{C}}\beta^{23} = 2\beta^{23} \\
\hat{\mathcal{C}}\beta^{33} = \beta^{22} - \beta^{33} \\
\hat{\mathcal{C}}\gamma^1 = 0 \\
\hat{\mathcal{C}}\gamma^2 = -\gamma^3 \\
\hat{\mathcal{C}}\gamma^3 = \gamma^2
\end{cases}$$

(66)

For $\alpha^i$ we have

$$\begin{cases}
\hat{\mathcal{C}}\alpha^1 = 0 \\
\hat{\mathcal{C}}\alpha^2 = -\alpha^3 \\
\hat{\mathcal{C}}\alpha^3 = \alpha^2 \\
\hat{\mathcal{C}}\alpha^1 = 0 \\
\hat{\mathcal{C}}\alpha^2 = -\alpha^1 \\
\hat{\mathcal{C}}\alpha^3 = \alpha_1 \\
\hat{\mathcal{C}}\alpha^2 = -\alpha^4 \\
\hat{\mathcal{C}}\alpha^3 = -\alpha_4 \\
\hat{\mathcal{C}}\alpha^2 = -\alpha^2 - \alpha^3 \\
\hat{\mathcal{C}}\alpha^3 = \alpha^2 - \alpha_3 \\
\hat{\mathcal{C}}\alpha^3 = \alpha^2 + \alpha_3
\end{cases}$$

(67)

For $Q$ we have

$$\hat{\mathcal{C}}Q = 0$$

(68)
\[
\begin{align*}
\hat{C}a_1 &= 0 \\
\hat{C}b_{11} &= 0 \\
\hat{C}b_{22} &= -2b_{32} \\
\hat{C}c_1 &= 0 \\
\hat{C}a_2 &= -a_3 \\
\hat{C}b_{12} &= -b_{13} \\
\hat{C}b_{33} &= 2b_{32} \\
\hat{C}c_2 &= -c_3 \\
\hat{C}a_3 &= a_2 \\
\hat{C}b_{13} &= b_{12} \\
\hat{C}b_{23} &= b_{22} - b_{33} \\
\hat{C}c_3 &= c_2
\end{align*}
\] (69)

The remaining equations mix different coefficients:
\[
\begin{align*}
E^2a_3 - E^3a_2 &= \bar{\alpha}^1 \\
(-\alpha^1_j + 2E^2b_{3j} - 2E^3b_{2j})\partial E^j &= -2(\partial E^2c_3 - \partial E^3c_2) \\
E^2c_3 - E^3c_2 &= \gamma^1
\end{align*}
\] (70)

3.2.2 The \(\mathcal{H}\)-\(\mathcal{D}\) bracket

For \(\beta^{ij}\) and \(\gamma^i\) we have
\[
\begin{align*}
\hat{D}\beta^{11} &= 0 \\
\hat{D}\beta^{12} &= \beta^{12} \\
\hat{D}\beta^{13} &= \beta^{13} \\
\hat{D}\gamma^1 &= 0 \\
\hat{D}\gamma^2 &= \gamma^2 \\
\hat{D}\gamma^3 &= \gamma^3
\end{align*}
\] (71)

For \(\alpha^i\) we have
\[
\begin{align*}
\hat{D}\alpha^1 &= \bar{\alpha}^1 \\
\hat{D}\alpha^2 &= 2\bar{\alpha}^2 \\
\hat{D}\alpha^3 &= 2\bar{\alpha}^3 \\
\hat{D}\alpha_1 &= 0 \\
\hat{D}\alpha_2 &= -\alpha_2 \\
\hat{D}\alpha_3 &= -\alpha_3 \\
\hat{D}\alpha_1 &= 0 \\
\hat{D}\alpha_2 &= 0 \\
\hat{D}\alpha_3 &= 0 \\
\end{align*}
\] (72)

For \(Q\) we have
\[
\begin{align*}
\hat{D}\bar{Q} &= 2\bar{Q} \\
E^2c_2 + E^3c_3 &= 0 \\
E^2a_2 + E^3a_3 &= 0 \\
\end{align*}
\] (73)

For \(Q\) we have
\[
\begin{align*}
\hat{D}c_1 &= 0 \\
\hat{D}c_2 &= -c_2 \\
\hat{D}c_3 &= -c_3 \\
\hat{D}a_1 &= a_1 \\
\hat{D}b_{11} &= 0 \\
\hat{D}b_{22} &= -2b_{22} \\
\hat{D}b_{12} &= -b_{12} \\
\hat{D}b_{33} &= -b_{33} \\
\hat{D}c_{13} &= 0 \\
\hat{D}c_{23} &= -b_{23}
\end{align*}
\] (74)

One equation mixes different coefficients:
\[
E^2\alpha^1_2 + E^3\alpha^1_3 = -\gamma^1.
\] (75)

3.2.3 The \(\mathcal{H}\)-\(\mathcal{H}\) bracket

Matching term by term for \(\mathcal{H}\)-\(\mathcal{H}\) is quite tedious, mainly because the classical bracket \(\{\mathcal{H},\mathcal{H}\}\) is fully determined only after setting \(G = 0\). For example, if there is a term \(f(\alpha, \beta, \gamma, Q)\partial E^1\) on the left-hand side of \(\{\mathcal{H}(x), \mathcal{H}(y)\} \approx -2G(E^{11}\partial x\mathcal{D}(x)\delta_{xy} + 2E^{11}\mathcal{D}(x)\delta_{xy}')\) which is not on the right hand side, do we demand \(f(\alpha, \beta, \gamma, Q) = 0\) or do we demand
\[ f(\alpha, \beta, \gamma, Q) \propto G \text{ or } \partial G, \text{ or does } f(\alpha, \beta, \gamma, Q) \partial E^1 \text{ combine with possible } f(\alpha, \beta, \alpha, Q)(-E^2 A_3 + E^3 E_2) \text{ terms to become something proportional to } G? \] There are about 10^2 terms on the left-hand side of the Hamiltonian bracket, each of which has several possibilities of respecting the symmetry (in the form of second-order polynomial equations of \( \alpha, \beta, \gamma, Q \)). It is therefore necessary to check whether these \( (10^2)^n, n \sim 10^9 \) possibilities are consistent with one another, rendering our current strategy impractical. Luckily, we can use an alternative strategy to find a subset of the most generic Hamiltonian by adding “semi-symmetric Gaussian” terms to the classical Hamiltonian constraint.

### 3.3 Real vs. self-dual variables

We define a **semi-symmetric** term to be any term in a generic Hamiltonian constraint that is allowed by the \( \{H, D\} \) and \( \{H, G\} \) brackets. These terms are solutions to our previous partial differential equations (66)-(75). We define a **Gaussian** term to be any term that is a polynomial of \( G \) and \( \partial G \), with coefficients denoted collectively as \( C(E) \), which may depend on densitized triads and its derivatives. Namely, for a semi-symmetric Gaussian term \( g(x) := g[G(x), \partial G(x), C(E(x))] \) we demand

\[
\begin{align*}
\{g(x), G(y)\} &= 0 \quad (76) \\
\{g(x), D(y)\} &= 2G(\partial g(x)\delta_{xy} + 2g(x)\delta'_{xy}),
\end{align*}
\]

Any semi-symmetric Gaussian term, \( g[G, \partial G, C(E)] \), that we add to the classical Hamiltonian constraint \( \mathcal{H}_{cl} \) is guaranteed to respect all our symmetries as shown below.

Suppose we add one semi-symmetric Gaussian term \( g[G, \partial G, C(E)] \) to the classical Hamiltonian constraint \( \mathcal{H}_{cl} \)

\[
\mathcal{H}[N] = \frac{1}{2G} \int d^3x \mathcal{N}(x)(\mathcal{H}_{cl} + g).
\]

Since \( \mathcal{H}_{cl} \) respects all symmetries by definition and \( g \) is built out of semi-symmetric Gaussian terms,

\[
\{\mathcal{H}[N], \mathcal{G}[M]\} = 0 \quad (78)
\]

is trivial. Similarly, the \( H-D \) bracket is satisfied:

\[
\begin{align*}
\{\mathcal{H}[N], \mathcal{D}[M]\} &= \frac{1}{4G^2} \int d^3x d^3y \mathcal{N}(x) M(y) \{\mathcal{H}_{cl}, D\} + \{g, D\} \\
&= \frac{1}{2G} \int d^3x d^3y \mathcal{N}(x) M(y)(\partial_x \mathcal{H}_{cl}(x)\delta_{xy} + 2\mathcal{H}_{cl}(x)\delta'_{xy} + \partial_x g(x)\delta_{xy} + 2g(x)\delta'_{xy}) \\
&= \frac{1}{2G} \int d^3x d^3y \mathcal{N}(x) M(y)(\partial_x \mathcal{H}(x)\delta_{xy} + 2\mathcal{H}(x)\delta'_{xy}) = -\mathcal{H}[MN - M'N] \quad (79)
\end{align*}
\]

because \( g \) is built out of semi-symmetric Gaussian terms. The \( \mathcal{H}[N]-\mathcal{H}[M] \) bracket then has additional terms compared with the classical case, given by \( \{\mathcal{H}_{cl}, g\} \) and \( \{g, g\} \). Both
terms are of the form \( \{ f, g \} \) with some semi-symmetric \( f \), and share the property that \( \int dx dy N(x) M(y) \{ f(x), g(y) \} \) vanishes when \( \mathcal{G} = 0 \): In

\[
\int dx dy N(x) M(y) \{ f(x), g \mathcal{G}(y), \partial^n \mathcal{G}(y), C(E) \}
= \int dx dy N(x) M(y) \left( \{ f(x), \mathcal{G}(y) \} \frac{\partial g}{\partial \mathcal{G}}(y) + \{ f(x), \partial^n \mathcal{G}(y) \} \frac{\partial g}{\partial (\partial^n \mathcal{G})(y)}(y) \right)
+ \{(x), C(E)\} \frac{\partial g}{\partial C(E)}
= \int dx dy N(x) M(y) \left( \{ f(x), \mathcal{G}(y) \} \frac{\partial g}{\partial \mathcal{G}}(y) + \{ f(x), C(E) \} \frac{\partial g}{\partial C(E)} \right)
+ \int dx dy N(x) (-\partial_y)^n \left( M(y) \frac{\partial g}{\partial (\partial^n \mathcal{G})(y)} \right) \{ f(x), \mathcal{G}(y) \}
\]

\( (80) \)

the first and last term vanish because \( f \) is semi-symmetric, while \( \partial g/\partial C(E) \approx 0 \) because \( C(E) \), by definition, represents coefficients in \( g \) of the Gauss constraint or its spatial derivatives.

With this result, we confirm that

\[
\{ H[\bar{N}], H[\bar{M}] \} = \frac{1}{4G^2} \int dx dy N(x) M(y) \{ \mathcal{H}_{cl}(x), \mathcal{H}_{cl}(y) \}
+ \{ g[\mathcal{G}(x), \partial^n \mathcal{G}(x), C(E)], g[\mathcal{G}(y), \partial^n \mathcal{G}(y), C(E)] \}
+ \{ \mathcal{H}_{cl}(x), g[\mathcal{G}(y), \partial^n \mathcal{G}(y), C(E)] \}
+ \{ g[\mathcal{G}(x), \partial^n \mathcal{G}(x), C(E)], \mathcal{H}_{cl}(y) \}
\approx \frac{1}{4G^2} \int dx dy N(x) M(y) \{ \mathcal{H}_{cl}(x), \mathcal{H}_{cl}(y) \}
\]

\( (81) \)

obeys the classical brackets for any semi-symmetric \( g \). Thus, semi-symmetric Gaussian terms indeed preserve all symmetries.

When written in real variables, the classical Hamiltonian constraint contains a term with second-order derivative of \( E^1 \sim E^x \), given by \( 2\partial E^x = \partial (\partial E^x / (E^x)) E^x \). But when using self-dual variables, there are no second-order derivative of triads. As already mentioned, this discrepancy is caused by the fact that \( \mathcal{G} \approx 0 \) is already solved in the real variable case. Indeed, using semi-symmetric terms (see appendix [A]) for constructing modifications we have the following allowed terms when using self-dual variables

\[
\mathcal{H}_2(A, E) = \mathcal{H}_{cl}(A, E) + c_1(E^1) \left( \partial \mathcal{G} - \frac{1}{2} \frac{\partial((E^x)^2)}{(E^x)^2} \mathcal{G} \right) + \partial E^1[b_{11}(E^1) \partial E^1 + \tilde{C}_{\alpha_1}(E^1)(E^3 A_2 - E^2 A_3)],
\]

\( (82) \)

where \( \partial \mathcal{G} \sim \partial^2 E^1 \) provides the second-order derivative. Note that the second semi-symmetric term (proportional to \( \partial E^1 \)) becomes a semi-symmetric Gaussian term if we pick \( b_{11} = \frac{1}{2} \tilde{C}_{\alpha_1} \).
Substituting $A_i = \gamma K_i + \Gamma_i$, $c_1 = E^1$, $b_{11} = \frac{1}{2} \tilde{C}_a \gamma^a_1 = 1/2$ in the classical Hamiltonian constraint and de-densitizing, we obtain

$$\mathcal{H}_2(K, E) = |E|^{|-1/2}\left(K_\varphi^2 E^\varphi + 2K_\varphi K_x E_x - \left(1 - \frac{\partial E^x}{2E^\varphi} \right)^2 E^\varphi + \frac{E^x \partial^2 E^x}{E^\varphi} - \frac{E^x \partial E^x \partial E^\varphi}{(E^\varphi)^2}\right),$$

where we used the Gauss constraint in real variables. This result matches the standard classical Hamiltonian constraint in real variables. Thus, including semi-symmetric Gaussian terms in the quadratic constraint, it is equivalent to the classical one written in real variables.

Revisiting the setting of the previous section, it follows that a further restriction of our $\mathcal{H}$ to be only quadratic in densitized triads implies that all allowed modifications to the classical $\mathcal{H}_{cl}$ are in the form of semi-symmetric Gaussian terms:

$$\mathcal{H}_{quad} = C_1(\partial A_3 E^{21} - \partial A_2 E^{31} + A_{12} E^{12} + A_{13} E^{13}) + C_2\left(A_{22} + A_{33} + \frac{C_2}{C_4}\right)(E^{22} + E^{33})$$

$$+ C_4 \partial E^1 \mathcal{G} + C_5(A_2 E^2 + A_3 E^3) \mathcal{G}.$$  

The first two terms are present in $\mathcal{H}_{cl}$ while the last two are new semi-symmetric Gaussian terms and all $C_i$ are constants. However, the complexity of the general equations makes it difficult to show that all possible modifications to the Hamiltonian constraint up to second order in derivatives can be constructed from semi-symmetric Gaussian terms.

### 4. Eliminating the Gauss constraint

Our analysis of gravitational theories in a setting of effective field theory has highlighted the role of the Gauss constraint, which implies that the hypersurface-deformation generators are not uniquely defined. Since the Gauss constraint contains a spatial derivative, and spatial derivatives of this constraint can also be added to the hypersurface-deformation generators, the derivative structure and therefore the possibility of modifications is ambiguous as long as the Gauss constraint remains unsolved. We will therefore now solve the Gauss constraint explicitly and analyze the resulting hypersurface-deformation generators and their brackets.

#### 4.1. Gauge-invariant variables

We begin with the classical constraint

$$H[N] = \int dx \frac{N}{\sqrt{E^1((E^2)^2 + (E^3)^2)}} \left(2E^1(E^2A_3^1 - E^3A_2^1)ight. + 2A_1 E^1(A_2 E^2 + A_3 E^3) + (A_2^2 + A_3^2 - 1)((E^2)^2 + (E^3)^2)$$

$$+ (\epsilon - \gamma^2)(2K_1 E^1(K_2 E^2 + K_3 E^3) + (K_2^2 + K_3^2)((E^2)^2 + (E^3)^2)),$$
in which the lapse function no longer has a density weight. The next few transformations closely follow the derivations given in [30], but are presented here in a different form using vector notation.

The pairs \((E^2, E^3)\) and \((A_2, A_3)\) (as well as \((K_2, K_3)\)) transform under the defining representation of \(\text{SO}(2)\) with respect to the Gauss constraint. It will be convenient to arrange them in 3-vectors, such that

\[
\vec{E} = E^2 \vec{e}_2 + E^3 \vec{e}_3, \quad \vec{A} = A_2 \vec{e}_2 + A_3 \vec{e}_3, \quad \vec{K} = K_2 \vec{e}_2 + K_3 \vec{e}_3
\]

with standard basis vectors \(\vec{e}_i\). Obvious invariant variables are therefore

\[
E^\varphi = |\vec{E}| = \sqrt{(E^2)^2 + (E^3)^2}, \quad A^\varphi = |\vec{A}| = \sqrt{A_2^2 + A_3^2}, \quad K^\varphi = |\vec{K}| = \sqrt{K_2^2 + K_3^2}.
\]

Moreover, we obtain another invariant \(\alpha\) from the angle between \(\vec{E}\) and \(\vec{A}\),

\[
\cos \alpha = \frac{\vec{E} \cdot \vec{A}}{E^\varphi A^\varphi}.
\]

While \(E^1\) and \(K_1\) are also invariant, \(A_1\) has a non-trivial transformation. A final gauge-invariant expression can be written as \(A_1 + \beta',\) where

\[
\cos \beta = \frac{\vec{e}_2 \cdot \vec{A}}{A^\varphi}.
\]

Using our definitions of \(\alpha\) and \(\beta\), we can write the unit vectors

\[
\vec{e}_A = \frac{\vec{A}}{A^\varphi} = \vec{e}_2 \cos(\beta) + \vec{e}_3 \sin(\beta) \quad (90)
\]

\[
\vec{e}_E = \frac{\vec{E}}{E^\varphi} = \vec{e}_2 \cos(\alpha + \beta) + \vec{e}_3 \sin(\alpha + \beta). \quad (91)
\]

From the last relation one can derive the spin-connection component \(\Gamma_1 = -(\alpha + \beta)' [30]\). Therefore, \(\gamma^{-1}(A_1 + \alpha' + \beta') = K_1\) is nothing but an extrinsic-curvature component. Since \(\alpha\) and \(K_1\) are gauge invariant, \(A_1 + \beta'\) must be gauge invariant, as claimed above.

Moreover, computing the extrinsic curvature and spin connection for a spherically symmetric triad [30] shows that the angular part \(\vec{K}\) points in the same internal direction as the triad,

\[
\vec{e}_K = \vec{e}_E, \quad (92)
\]

while the angular part of the spin connection, \(\vec{\Gamma}\), is orthogonal,

\[
\vec{e}_\Gamma = -\vec{e}_1 \times \vec{e}_E, \quad (93)
\]

with coefficient

\[
\Gamma^\varphi = -\frac{(E^1)'}{2E^\varphi}; \quad (94)
\]

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see (52). Therefore,
\[ A_\varphi^2 = |\vec{A}|^2 = |\Gamma_\varphi \vec{e}_\Gamma + \gamma K_\varphi \vec{e}_K|^2 = \Gamma_\varphi^2 + \gamma^2 K_\varphi^2. \] (95)

The term in (85) containing spatial derivatives of the connection can now be written as
\[ E^2 A_3' - E^3 A_2' = \vec{e}_1 \cdot (\vec{E} \times \vec{A}) = E^\varphi \vec{e}_1 (\vec{e}_E \times (A_\varphi \vec{e}_A))' = E^\varphi (-A_\varphi' \sin(\alpha) + A_\varphi \beta' \cos(\alpha)). \] (96)

We then express connection terms through spin connection and extrinsic curvature, using
\[ A_\varphi \sin(\alpha) = A_\varphi \vec{e}_A \cdot \vec{e}_\Gamma = \Gamma_\varphi \] (97)
and
\[ A_\varphi \cos(\alpha) = A_\varphi \vec{e}_A \cdot \vec{e}_K = \gamma K_\varphi. \] (98)

Therefore,
\[ E^2 A_3' - E^3 A_2' = E^\varphi (-A_\varphi \sin(\alpha) + A_\varphi (\alpha' + \beta') \cos(\alpha)) = E^\varphi (-\Gamma_\varphi' + \gamma K_\varphi (\alpha' + \beta')). \] (99)

The angles in the last term can be combined with a similar contribution from the second term in (85), which adds \( A_1 \) to \( \alpha' + \beta' \). (In (85), \( A_1 \) is multiplied with \( A_2 E^2 + A_3 E^3 = \vec{A} \cdot \vec{E} = \gamma K_\varphi E^\varphi \), which does not depend on \( \Gamma_\varphi \) because \( \vec{e}_\Gamma \cdot \vec{e}_E = 0 \).) Since \( \alpha' + \beta' = -\Gamma_1 \) and \( A_1 - \Gamma_1 = \gamma K_1 \), we have
\[ E^2 A_3' - E^3 A_2' + A_1 (A_2 E^2 + A_3 E^3) = E^\varphi (-\Gamma_\varphi' + \gamma^2 K_\varphi K_1). \] (100)

Thus, by using variables invariant under transformations generated by the Gauss constraint, we have been led to an expression in which all spatial derivatives of the connection have been replaced by spatial derivatives of the triad (through \( \Gamma_\varphi \)).

Again in [30], the Poisson brackets
\[ \{ K_\varphi(x), E^\varphi(y) \} = G \delta(x, y), \quad \{ K_1(x), E^1(y) \} = 2G \delta(x, y) \] (101)
for the new gauge-invariant variables have been derived. If we express the diffeomorphism and Hamiltonian constraints in these variables, we restrict the previous theory to the solution space of the Gauss constraint. We obtain
\[ D[N^\varphi] = \frac{1}{2G} \int dx N^\varphi \left( 2E^\varphi K_\varphi' - K_1(E^1)' \right) \] (102)
and
\[ H[N] = \frac{1}{2G} \int dx \frac{N}{\sqrt{E^1}} \left( K_\varphi^2 E^\varphi (\epsilon - \gamma^2) + 2\epsilon K_\varphi K_1 E^1 + (\Gamma_\varphi^2 + \gamma^2 K_\varphi^2 - 1) E^\varphi - 2E^1 \Gamma_\varphi' \right). \] (103)
4.2 Modified constraint with classical brackets

In the Hamiltonian constraint, the two terms with \( \gamma^2 K^2 \) cancel out, showing that, for \( \epsilon = -1 \), we obtain the Hamiltonian constraint as considered in [30]. Our calculation here extends this result to Euclidean signature, \( \epsilon = 1 \). Since all \( \gamma \)-dependent terms drop out of the final expression, it is no longer clear why \( \gamma^2 = \epsilon \) should lead to different options for modified constraints. Nevertheless, the previous distinction between \( \gamma^2 = \epsilon \) and \( \gamma^2 \neq \epsilon \) can still be realized if we do not cancel the \( \gamma \)-dependent terms in (103) before we try to modify the constraint. In particular, the previous modification, using an arbitrary function of \( f(A^2_2 + A^2_3 - 1) \), can still be implemented in the invariant version if we recognize the combination \( \Gamma^2 + \gamma^2 K^2 - 1 \) as the correct substitute of \( A^2_2 + A^2_3 - 1 = A^2_\varphi - 1 \). We therefore consider the modified constraint

\[
H[N] = \frac{1}{2G} \int dx \frac{N}{\sqrt{E^1}} \left( K^2 \phi E^\phi (\epsilon - \gamma^2) + 2\epsilon K^2 K^1 E^1 + f(\Gamma^2 + \gamma^2 K^2 - 1)E^\varphi - 2E^1 \Gamma' \right).
\]

Given the form of this new constraint, it is not obvious that it can lead to closed brackets for \( f \) not the identity because, compared with our previous derivation, we now have up to second-order spatial derivatives of the triad (through \( \Gamma_\varphi \)) instead of first-order derivatives of its momenta.

Thanks to antisymmetry of the Poisson bracket, the only terms that give non-zero contributions to \( B_{NM} := \{H[N], H[M]\} \) are combinations of a term from \( H[N] \) depending on one of the \( K_i \) and a term from \( H[M] \) depending on a (first or second order) spatial derivative of one of the \( E_i \), or vice versa. Therefore,

\[
B_{NM} = \frac{1}{4G^2} \int dxdy \frac{N(x)M(y)}{\sqrt{E^1(x)E^1(y)}} \left( (\epsilon - \gamma^2) \{K^2_\varphi(x), (E^\varphi)'\} \frac{E^1(y)E^1(y)'E^\varphi(x)}{(E^\varphi(y))^2} \right.
\]
\[
- 2\epsilon \{K_\varphi(x), E^\varphi(y)\}' K^1(x) \frac{E^1(x)E^1(y)E^1(y)'}{(E^\varphi(y))^2}
\]
\[
- 2\epsilon K_\varphi(x) \{K_1(x), E^1(y)\}' \frac{E^1(x)E^1(y)E^\varphi(y)'}{(E^\varphi(y))^2}
\]
\[
- \{f, E^\varphi(y)\}' \frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi(y))^2} + 2\epsilon K_\varphi(x) \{K_1(x), f\} E^1(x)E^\varphi(y)
\]
\[
+ 2\epsilon K_\varphi(x) \frac{E^1(y)}{E^\varphi(y)} \{K_1(x), E^1(y)''\} E^1(x) \right) - (N \leftrightarrow M). \quad (105)
\]
Integrating by parts, we obtain
\[
B_{NM} = \frac{1}{4G} \int d\gamma\gamma M' \left( (2\epsilon - \gamma^2)K^{(1)} E^\varphi + 2\epsilon E^1 (E^\varphi)^2 K_1 (E^\gamma)' + 4\epsilon K_\varphi (E^\varphi)' \frac{E^1}{(E^\varphi)^2} 
- 4\epsilon E^1 (E^\varphi)' E^\varphi K_\varphi' - 4\epsilon K_\varphi \frac{E^1 (E^\varphi)'}{(E^\varphi)^2} + \frac{\partial f}{\partial K_\varphi} (E^\varphi)' - 4\epsilon K_\varphi E^\varphi \frac{\partial f}{\partial (E^\gamma)'} \right) - (N \leftrightarrow M)
\]
\[
\begin{align*}
&= -\frac{\epsilon}{2G} \int d\gamma\gamma M' \gamma (N M' - N'M) (2E^\varphi K_\varphi' - K_1 (E^\gamma)') \\
&\quad + \frac{1}{4G} \int d\gamma\gamma M' \gamma (2\epsilon - \gamma^2)K_\varphi \frac{(E^1)'}{E^\varphi} + \frac{\partial f}{\partial K_\varphi} (E^\varphi)' - 4\epsilon K_\varphi E^\varphi \frac{\partial f}{\partial (E^\gamma)'} \\
&= -\epsilon D \left[ \frac{E^1 (E^\varphi)^2}{(E^\varphi)^2} (N M' - N'M) \right] \\
&\quad + \frac{1}{4G} \int d\gamma\gamma M' \gamma (2\epsilon - \gamma^2)K_\varphi \frac{(E^1)'}{E^\varphi} + \frac{\partial f}{\partial K_\varphi} (E^\varphi)' - 4\epsilon K_\varphi E^\varphi \frac{\partial f}{\partial (E^\gamma)'} \right).
\end{align*}
\]
For a closed bracket, therefore,
\[
2(\epsilon - \gamma^2)K_\varphi \frac{(E^1)'}{E^\varphi} + \frac{\partial f}{\partial K_\varphi} (E^\varphi)' - 4\epsilon K_\varphi E^\varphi \frac{\partial f}{\partial (E^\gamma)'} = 0. \tag{107}
\]
Since \( f \) depends on \( K_\varphi \) and \( (E^1)' \) only through \( \frac{1}{2}(E^1)^2/(E^\varphi)^2 + \gamma^2 K_\varphi^2 - 1 \), the chain rule implies that
\[
\frac{\partial f}{\partial K_\varphi} = 2\gamma^2 K_\varphi \dot{f} \quad \text{and} \quad \frac{\partial f}{\partial (E^\gamma)'} = \frac{1}{2(E^\varphi)^2} (E^1)' \dot{f}, \tag{108}
\]
and (107) is equivalent to
\[
2(\epsilon - \gamma^2)K_\varphi \frac{(E^1)'}{E^\varphi} \left( 1 - \dot{f} \right) = 0. \tag{109}
\]
If \( \gamma^2 = \epsilon \), the equation holds identically for any \( f \). If \( \gamma^2 \neq \epsilon \), however, \( \dot{f} = 1 \), and only the classical case is allowed. The modification found in [20] can therefore be found also in gauge-invariant variables, in which case the Hamiltonian constraint contains second-order derivatives of the triad, with the same restriction that it is allowed only for a specific value of \( \gamma \).

4.3 Modified brackets

A generic modification which does not require a specific value of \( \gamma \) can be obtained for the theories considered here, as has been known for some time for real variables [15]. Since the Hamiltonian constraint in real variables has the same form as the general spherically symmetric constraint in gauge-invariant variables, the same modification can be transferred also to self-dual type variables \( (\gamma^2 = \epsilon) \) provided we implement it at the gauge-invariant level. At the level of variables that are not gauge invariant, this new modification (compared
with [20]) is possible provided we use the Gauss constraint to reintroduce second-order derivatives of triads in the Hamiltonian constraint.

Starting with (103), the new modification is derived in a way very similar to the case of real variables, found in [1]. Nevertheless, we reproduce the calculation of brackets here for the sake of completeness. We modify (103) to

$$H[N] = \frac{1}{2G} \int dx N(x)(E^1)^{-1/2} \left( \epsilon f_1(K_\varphi)E^\varphi + 2\epsilon f_2(K_\varphi)E^1K_1 \right.$$  

$$+ \left( \frac{(E^1)^2}{4(E^\varphi)^2} - 1 \right) E^\varphi + \frac{E^1(E^1)''}{E^\varphi} - \frac{E^1(E^1)'(E^\varphi)'}{(E^\varphi)^2} \right)$$  

(110)

with two functions, $f_1$ and $f_2$, that will be restricted further by the condition of having closed brackets. We first interpret this modification based on arguments within canonical effective field theory. We are now allowing for a non-quadratic dependence of the Hamiltonian constraint on $K_\varphi$. If $K_\varphi$ is still a first-order time derivative, a non-quadratic dependence would be non-generic unless we also allow for higher-order spatial derivatives of the densitized triad, which we do not do in (110).

However, modifying the Hamiltonian constraint in this form also modifies the equations of motion that classically imply the first-order nature of $K_\varphi$. An analysis of these modified equations of motion should then be performed in order to reveal the derivative order of the Hamiltonian constraint. Schematically, we obtain the modified derivative dependence of $K_\varphi$ from the equation of motion

$$\dot{E}^1 = 2N\sqrt{E^1}f_2(K_\varphi) + N^1(E^1)'$$  

(111)

$$\dot{E}^\varphi = N\sqrt{E^1}K_1 \frac{df_2(K_\varphi)}{dK_\varphi} + \frac{NE^\varphi}{2\sqrt{E^1}} \frac{df_1(K_\varphi)}{dK_\varphi} + (N^1E^\varphi)'$$  

(112)

provided we can invert the function $f_2$. This can explicitly be done only in examples, which we restrict here to the common case of $f_1(K_\varphi) = \sin^2(K_\varphi)$, which implies $f_2(K_\varphi) = \sin(K_\varphi)\cos(K_\varphi)$ or $f_2(K_\varphi)^2 = f_1(K_\varphi)(1 - f_1(K_\varphi))$. The latter equation can be solved for

$$f_1(K_\varphi) = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4f_2(K_\varphi)^2} \right) = f_2(K_\varphi)^2 + f_2(K_\varphi)^4 + \cdots.$$  

(113)

According to (111), $f_2(K_\varphi)$ is strictly of first order in derivatives, but $f_1(K_\varphi)$ is not polynomial in $f_2(K_\varphi)$, and therefore a derivative expansion of $f_1(K_\varphi)$ does not terminate. Similarly,

$$\frac{df_2(K_\varphi)}{dK_\varphi} = \cos(2K_\varphi) = 1 - 2f_1(K_\varphi) = \sqrt{1 - f_2(K_\varphi)^2}$$  

(114)

has a derivative expansion that does not terminate. Therefore, $K_1$ has a non-terminating derivative expansion because $K_1\sqrt{1 - f_2(K_\varphi)^2}$ must be of first order according to (111).

We conclude that the constraint (110) contains a derivative expansion in both space and time derivatives, which can consistently be truncated at any finite derivative order. The resulting effective theory is therefore meaningful, but it may not be the most general
one because the derivative expansion results only from the $K$-dependent terms in (110), while we have not included higher-derivative corrections of the $E$-dependent terms. The mismatch does not violate (deformed) covariance because the constraint brackets still close. However, unless the symmetries implied by the closed constraints select only this specific derivative structure, the modified theory is not generic. (It resembles Born–Infeld type theories.) Since no other consistent modifications are known as of now, it remains unclear whether the apparently non-generic model is selected by symmetries.

In order to confirm that the constraint brackets can be closed, we compute

\[
\{H[N], H[M]\} = \frac{1}{4G^2} \int \frac{dxdy}{E^1(x)E^1(y)} \left( -\epsilon \frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi)^2(y)} \{f_1(K_\varphi(x)), E^\varphi(y)\} 
-2\epsilon \frac{E^1(x)c_{\varphi}(y)c_{\varphi}(y)K_1(x)}{(E^\varphi)^2(y)} \{f_2(K_\varphi(x)), E^\varphi(y)\}
+\epsilon \frac{f_2(K_\varphi(x))E^1(x)}{2E^\varphi(y)} \{K_1(x), (E^1(y))'\}
+2\epsilon f_2(K_\varphi(x))E^1(x) \frac{E^1(y)E^\varphi(y)'}{E^\varphi(y)^2} \{K_1(x), E^1(y)''\}
-2\epsilon f_2(K_\varphi(x))E^1(x) \frac{E^1(y)E^\varphi(y)'}{E^\varphi(y)^2} \{K_1(x), E^1(y)'\} \right) - (N \leftrightarrow M),
\]

writing only terms that produce non-zero contributions. All terms are multiplied with $\epsilon$, and therefore the possibility of modifications does not depend on the space-time signature.

The first two lines contain Poisson brackets of $f_1(K_\varphi)$ and $f_2(K_\varphi)$ and therefore lead to derivatives of the modification functions:

\[
\frac{1}{G} \frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi)^2(y)} \{f_1(K_\varphi(x)), E^\varphi(y)\}' = \frac{E^\varphi(x)E^1(y)E^1(y)'}{(E^\varphi)^2(y)} \frac{d f_1(K_\varphi)}{dK_\varphi} \partial_y \delta(x, y)
\]

and

\[
\frac{2}{G} \frac{E^1(x)E^1(y)E^1(y)'K_1(x)}{(E^\varphi)^2(y)} \{f_2(K_\varphi(x)), E^\varphi(y)'\} = \frac{2}{G} \frac{E^1(x)E^1(y)E^1(y)'K_1(x)}{(E^\varphi)^2(y)} \frac{d f_2(K_\varphi)}{dK_\varphi} \partial_y \delta(x, y).
\]

Another derivative of $f_2(K_\varphi)$ results from the second-order derivative of the delta function obtained after evaluating $\{K_1, (E^1)'\}$ in the fourth line of (115). This contribution follows from

\[
2 f_2(K_\varphi(x)) \frac{E^1(x)E^1(y)}{E^\varphi(y)} \{K_1(x), E^1(y)''\} = 4 f_2(K_\varphi(x)) \frac{E^1(x)E^1(y)}{E^\varphi(y)} \partial_y^2 \delta(x, y).
\]

Upon integrating by parts twice in the resulting expression in (115), we initially produce a term with $N(x)M(y)''$ times a delta function without derivatives. Integrating over $y$, the delta function is eliminated and we can integrate by parts once again to obtain a term with
\(N'M'\) (which cancels out in the antisymmetric bracket) and a term with \(NM'\) times the derivative of the entire coefficient in (118):

\[
-4 \left( f_2(K_\varphi) \left( \frac{(E^1)^2}{E^\varphi} \right) \right)' = -4 \left( \frac{d f_2}{dK_\varphi} K_\varphi \left( \frac{(E^1)^2}{E^\varphi} \right) + f_2(K_\varphi) \left( 2 \frac{E^1}{E^\varphi} \left( \frac{(E^1)'}{E^\varphi} - \frac{(E^1)^2}{(E^\varphi)^2} \right) \right) \right).
\]

(119)

The last term (containing \((E^\varphi)')\) cancels out with the fifth line of (115), while only half the second term cancels out with the third line of (115), for any \(f_2\). In order for the remaining terms to be proportional to the diffeomorphism constraint, only expressions proportional to \(K_1\) or \(K'_\varphi\) can remain. Therefore, the other half of the second term in (119) must cancel out with (116), which requires

\[
f_2(K_\varphi) = \frac{1}{2} \frac{d f_1(K_\varphi)}{dK_\varphi}.
\]

(120)

Only two terms are then left, (117) and the first contribution in (119). They are both proportional to \(d f_2(K_\varphi)/dK_\varphi\) and combine to form the diffeomorphism constraint:

\[
\{H[N], H[M]\} = -\frac{\epsilon}{2G} \int dx N'M' \frac{E^1}{(E^\varphi)^2} \frac{d f_2}{dK_\varphi} (2E^\varphi K'_\varphi - K_1(E^1)') - (N \leftrightarrow M)
\]

\[
= -\epsilon D \left[ \frac{d f_2(K_\varphi)}{dK_\varphi} \frac{E^1}{(E^\varphi)^2} (NM' - N'M') \right].
\]

(121)

This modification, following [1,5], differs from the modification of [20] in that it modifies not only the constraints but also their brackets (while the latter remain closed). It therefore implies a new, non-classical space-time structure [12,13]. This modification is consistent for all \(\gamma\) and is therefore generic. From this perspective, the modification of [20], which preserves the brackets, requires \(\gamma^2 = \epsilon\) and is not generic; it does not provide a way to avoid non-classical space-time structures without fine-tuning. Our derivations have shown that the different outcomes of [20] versus [15] are not a consequence of working with self-dual connections (used in [20]) or real variables (used in [15]). The crucial difference is that modified constraints with unmodified brackets, as in [20], can be obtained only for specific \(\gamma\), while modifications of constraints as well as brackets exist for all \(\gamma\).

5 Conclusion

We have shown that deformations of the classical space-time structure appear generically in spherically symmetric models of loop quantum gravity. For self-dual variables or Euclidean gravity with \(\gamma = \pm 1\), we have derived the most general form of the quadratic Hamiltonian constraint free of triad derivatives, such that a system with unmodified closed brackets is obtained. This rigidity result, just like the setting of [20] which it generalizes, relies on the absence of derivative terms of the triad. However, from the point of view of an effective field theory, this result is not generic because it depends on a restriction of derivative terms even
within the classical structure of second-order derivatives. Moreover, this rigidity result can be obtained only for specific values of the Barbero–Immirzi parameter $\gamma$.

The results of [20] have sometimes been interpreted as saying that deformations arising in the hypersurface-deformation brackets, obtained originally using holonomy modifications in real-valued variables, might be avoided in the self-dual case. Self-dual variables represent a specific choice for the Immirzi parameter, and therefore do not lead to generic results. These variables (or the values of $\gamma$ they correspond to) are not distinguished intrinsically by symmetries because constraint brackets, which define the symmetries of a canonical theory, can be closed for any $\gamma$.

Moreover, we have shown that the possibility of modifications, even within a self-dual setting, formally depends on the derivative structure which can be changed by adding multiples of the Gauss constraint or its spatial derivatives to the Hamiltonian constraint. This ambiguity can be eliminated by solving the Gauss constraint explicitly, following [30], in which case the same derivative structure is obtained in self-dual type variables and in real variables, which agrees with the form originally used in an analysis of modified brackets [15]. We therefore conclude that modified brackets and non-classical space-time structures are generic in any spherically symmetric model with holonomy modifications, even for self-dual variables. We also pointed out that currently known modifications may not be generic from the point of view of canonical effective theory introduced here: After translating momenta into time derivatives, different derivative orders appear in the terms of a modified Hamiltonian constraint. This observation suggests that there is room for further explorations of possibly new models. A likely candidate for a generic extension is the inclusion of canonical quantum back-reaction effects [31–33], which in an action formulation provide higher-curvature terms with generic higher derivatives. However, quantum back-reaction on its own does not modify the hypersurface-deformation brackets of constraints [34] and is therefore unlikely to change our conclusions about modified space-time structures.

Euclidean and self-dual type variables are special also in an analysis of cosmological perturbations [35,36], in which case non-generic modifications of constraint brackets have been observed as well. Our results present useful indications for operator calculations [37–43] which have demonstrated the possibility of off-shell closure of commutators of constraint operators, mainly in the Euclidean case. So far, these investigations have not yet given rise to indications that the commutators of constraint operators may be modified, in contrast to effective derivations as well as the operator constructions in [31–33]. (However, it is not always clear how to read off modifications of structure functions in the operator setting, which should be some function of a spatial metric or densitized triad and therefore requires a suitable notion of states of a semiclassical geometry which does not yet exist in the operator formulation.) Our results show that the Euclidean setting is, in fact, inconclusive as regards modifications of structure functions because it is a non-generic case that allows closed brackets with and without modifications. Current effective and operator treatments are therefore consistent with one another. For a complete picture of space-time structures in loop quantum gravity, it will be important to extend off-shell operator calculations to the full Lorentzian constraint.
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A Restrictions on coefficients of semi-symmetric Gaussian terms

We list the solutions to partial differential equations resulting from the $\mathcal{H}$-$\mathcal{G}$ and $\mathcal{H}$-$\mathcal{D}$ brackets. These will give us the so called semi-symmetric Gaussian terms. Denoting $(E^\varphi)^2 = E^{22} + E^{33}$, for $\beta^{ij}$ we have

$$\begin{cases}
\beta^{11} = \beta^{11}(E^1) \\
\beta^{12} = E^3 \tilde{C}_\beta(E^1) + E^2 \tilde{C}_\beta(E^1) \\
\beta^{13} = E^3 \tilde{C}_\beta(E^1) - E^2 \tilde{C}_\beta(E^1)
\end{cases}$$

$$\begin{cases}
\beta^{22} = 1/2[-8 \tilde{C}_{\beta 23}(E^1) E^{23} + (C_{\Sigma}(E^1) + \tilde{C}_{\beta 23}(E^1)) E^{22} + (C_{\Sigma}(E^1) - \tilde{C}_{\beta 23}(E^1)) E^{33}] \\
\beta^{33} = 1/2[8 \tilde{C}_{\beta 23}(E^1) E^{23} + (C_{\Sigma}(E^1) + \tilde{C}_{\beta 23}(E^1)) E^{33} + (C_{\Sigma}(E^1) - \tilde{C}_{\beta 23}(E^1)) E^{22}] \\
\beta^{23} = \tilde{C}_{\beta 23}(E^1) E^{23} + 2(E^{22} - E^{33}) \tilde{C}_{\beta 23}(E^1)
\end{cases}$$

For $\gamma^i$ we have

$$\begin{cases}
\gamma^1 = \gamma^1(E^1) \\
\gamma^2 = E^3 \tilde{C}_\gamma(E^1) + E^2 \tilde{C}_\gamma(E^1) \\
\gamma^3 = E^3 \tilde{C}_\gamma(E^1) - E^2 \tilde{C}_\gamma(E^1)
\end{cases}$$

For $\alpha^i$ we have

$$\begin{cases}
\alpha^1 = C_{\alpha 1}(E^1) E^\varphi \\
\alpha^2 = (\tilde{C}_{\alpha}(E^1) E^3 + C_{\alpha}(E^1) E^2) E^\varphi \\
\alpha^3 = (-\tilde{C}_{\alpha}(E^1) E^2 + C_{\alpha}(E^1) E^3) E^\varphi
\end{cases}$$
\[
\begin{align*}
\alpha_1 &= \alpha_1^1(E^1)
\alpha_2 &= E^3\tilde{C}_{\alpha_1^2}(E^1) + E^2\tilde{C}_{\alpha_2^2}(E^1)
\alpha_3 &= E^3\tilde{C}_{\alpha_1^2}(E^1) - E^2\tilde{C}_{\alpha_2^2}(E^1)
\alpha_1 &= (E^3\tilde{C}_{\alpha_1^2}(E^1) + E^3\tilde{C}_{\alpha_2^2}(E^1))\frac{1}{(E^\varphi)^2}
\alpha_2 &= (E^2\tilde{C}_{\alpha_1^2}(E^1) + E^3\tilde{C}_{\alpha_2^2}(E^1))\frac{1}{(E^\varphi)^2}
\alpha_3 &= (-E^2\tilde{C}_{\alpha_1^2}(E^1) + E^3\tilde{C}_{\alpha_2^2}(E^1))\frac{1}{(E^\varphi)^2}
\end{align*}
\]

For \( Q \) we have

\[
\begin{align*}
\tilde{Q} &= (E^\varphi)^2C_\varphi(E^1)
&= E^\varphi C_{a_1}(E^1)
&= E^\varphi C_{a_2}(E^1)
&= E^\varphi C_{a_3}(E^1)
\end{align*}
\]

\[
\begin{align*}
\alpha_1 &= c_1(E^1)
\alpha_2 &= \frac{E^3}{(E^\varphi)^2}C_k(E^1)
\alpha_3 &= \frac{E^2}{(E^\varphi)^2}C_k(E^1)
\end{align*}
\]

\[
\begin{align*}
b_{11} &= b_{11}(E^1)
b_{12} &= b_{12}(E^1)
b_{13} &= b_{13}(E^1)
b_{22} &= b_{22}(E^1)
b_{23} &= b_{23}(E^1)
b_{33} &= b_{33}(E^1)
\end{align*}
\]

We also have mixing conditions

\[
\begin{align*}
C_k(E^1) &= -\gamma_1(E^1) = \tilde{C}_{\alpha_1^2}(E^1)
C_{a_2}(E^1) &= -C_{\alpha_1}(E^1)
C_b(E^1) &= -\frac{1}{2}\alpha_1^1(E^1)
C_{b_2}(E^1) &= -\frac{1}{2}\tilde{C}_{\alpha_1^2}(E^1)
\end{align*}
\]

\[
\begin{align*}
\tilde{C}_{\alpha_1^2}(E^1) &= -2\beta_1^1(E^1)
\tilde{C}_{\alpha_2^2}(E^1) &= 2\tilde{C}_\beta(E^1) - \tilde{C}_\gamma(E^1)
\tilde{C}_{\alpha_3^2}(E^1) &= 2\tilde{C}_\beta(E^1) + \tilde{C}_\gamma(E^1)
\end{align*}
\]

**B Some useful identities**

In calculating the \{H[N(x)], H[M(x)]\} bracket, we can often make use of antisymmetry and integration by parts to simplify our calculations. Suppose we only have one canonical pair, then typically we have

\[
H[N(x)] \sim \int dx N(x) [\cdots + f(E(x), K(x))n(x) + \cdots] \tag{122}
\]
where \( n(x) \) is a function of phase-space variables depending on \( x \). Plugging this form of Hamiltonian into the Poisson bracket we obtain non-trivial term

\[
\{H[N(x)], H[M(x)]\} \ni \int dxdy\{N(x)M(y)[n(x)\{f(E(x), K(x)), \partial_y E(y)\}m(y)]
- (N \leftrightarrow M)\}
\]

(123)

Denote \( f'(x) \equiv \partial f(E(x), K(x))/\partial K(x) \) and \( K^{(n)}_{NM} \) for the above integral term (including the \((N \leftrightarrow M)\)), then for \( n = 1 \) we have

\[
K^{(1)}_{NM} = - \int dx[M'(x)N(x) - N'(x)M(x)]n(x)m(x)f'(x)
\]

(124)

For \( n=2 \) we have

\[
K^{(2)}_{NM} = \int dx[M'(x)N(x) - N'(x)M(x)][n(x)f(x)m'(x) - m(x)(n(x)f(x))']
\]

(125)

References

[1] J. D. Reyes, *Spherically Symmetric Loop Quantum Gravity: Connections to 2-Dimensional Models and Applications to Gravitational Collapse*, PhD thesis, The Pennsylvania State University, 2009

[2] M. Bojowald, J. D. Reyes, and R. Tibrewala, Non-marginal LTB-like models with inverse triad corrections from loop quantum gravity, *Phys. Rev. D* 80 (2009) 084002, [arXiv:0906.4767]

[3] A. Kreienbuehl, V. Husain, and S. S. Seahra, Modified general relativity as a model for quantum gravitational collapse, *Class. Quantum Grav.* 29 (2012) 095008, [arXiv:1011.2381]

[4] A. Kreienbuehl, V. Husain, and S. S. Seahra, Model for gravitational collapse in effective quantum gravity, [arXiv:1109.3158]

[5] M. Bojowald, G. M. Paily, and J. D. Reyes, Discreteness corrections and higher spatial derivatives in effective canonical quantum gravity, *Phys. Rev. D* 90 (2014) 025025, [arXiv:1402.5130]

[6] S. Brahma, Spherically symmetric canonical quantum gravity, *Phys. Rev. D* 91 (2015) 124003, [arXiv:1411.3661]

[7] M. Bojowald, S. Brahma, and J. D. Reyes, Covariance in models of loop quantum gravity: Spherical symmetry, *Phys. Rev. D* 92 (2015) 045043, [arXiv:1507.00329]
[8] M. Bojowald and S. Brahma, Covariance in models of loop quantum gravity: Gowdy systems, Phys. Rev. D 92 (2015) 065002, [arXiv:1507.00679]

[9] M. Bojowald and S. Brahma, Signature change in loop quantum gravity: Two-dimensional midisuperspace models and dilaton gravity, Phys. Rev. D 95 (2017) 124014, [arXiv:1610.08840]

[10] M. Bojowald and S. Brahma, Signature change in 2-dimensional black-hole models of loop quantum gravity, Phys. Rev. D 98 (2018) 026012, [arXiv:1610.08850]

[11] M. Bojowald and G. M. Paily, Deformed General Relativity and Effective Actions from Loop Quantum Gravity, Phys. Rev. D 86 (2012) 104018, [arXiv:1112.1899]

[12] M. Bojowald, S. Brahma, U. Büyükkıçman, and F. D’Ambrosio, Hypersurface-deformation algebroids and effective space-time models, Phys. Rev. D 94 (2016) 104032, [arXiv:1610.08355]

[13] M. Bojowald, S. Brahma, and D.-H. Yeom, Effective line elements and black-hole models in canonical (loop) quantum gravity, Phys. Rev. D 98 (2018) 046015, [arXiv:1803.01119]

[14] A. Barrau, M. Bojowald, G. Calcagni, J. Grain, and M. Kagan, Anomaly-free cosmological perturbations in effective canonical quantum gravity, JCAP 05 (2015) 051, [arXiv:1404.1018]

[15] M. Bojowald, G. Hossain, M. Kagan, and S. Shankaranarayanan, Anomaly freedom in perturbative loop quantum gravity, Phys. Rev. D 78 (2008) 063547, [arXiv:0806.3929]

[16] M. Bojowald, G. Hossain, M. Kagan, and S. Shankaranarayanan, Gauge invariant cosmological perturbation equations with corrections from loop quantum gravity, Phys. Rev. D 79 (2009) 043505, [arXiv:0811.1572]

[17] T. Cailleteau, J. Mielczarek, A. Barrau, and J. Grain, Anomaly-free scalar perturbations with holonomy corrections in loop quantum cosmology, Class. Quant. Grav. 29 (2012) 095010, [arXiv:1111.3535]

[18] T. Cailleteau, A. Barrau, J. Grain, and F. Vidotto, Consistency of holonomy-corrected scalar, vector and tensor perturbations in Loop Quantum Cosmology, Phys. Rev. D 86 (2012) 087301, [arXiv:1206.6736]

[19] T. Cailleteau, L. Linsefors, and A. Barrau, Anomaly-free perturbations with inverse-volume and holonomy corrections in Loop Quantum Cosmology, Class. Quantum Grav. 31 (2014) 125011, [arXiv:1307.5238]

[20] J. Ben Achour, S. Brahma, and A. Marciano, Spherically symmetric sector of self dual Ashtekar gravity coupled to matter: Anomaly-free algebra of constraints with holonomy corrections, Phys. Rev. D 96 (2017) 026002, [arXiv:1608.07314]
[21] J. Ben Achour and S. Brahma, Covariance in self dual inhomogeneous models of effective quantum geometry: Spherical symmetry and Gowdy systems, *Phys. Rev. D* 97 (2018) 126003, [arXiv:1712.03677](https://arxiv.org/abs/1712.03677)

[22] A. Ashtekar, New Hamiltonian Formulation of General Relativity, *Phys. Rev. D* 36 (1987) 1587–1602

[23] T. Thiemann and H. A. Kastrup, Canonical Quantization of Spherically Symmetric Gravity in Ashtekar’s Self-Dual Representation, *Nucl. Phys. B* 399 (1993) 211–258, [gr-qc/9310012](https://arxiv.org/abs/gr-qc/9310012)

[24] H. A. Kastrup and T. Thiemann, Spherically Symmetric Gravity as a Completely Integrable System, *Nucl. Phys. B* 425 (1994) 665–686, [gr-qc/9401032](https://arxiv.org/abs/gr-qc/9401032)

[25] J. F. Barbero G., Real Ashtekar Variables for Lorentzian Signature Space-Times, *Phys. Rev. D* 51 (1995) 5507–5510, [gr-qc/9410014](https://arxiv.org/abs/gr-qc/9410014)

[26] J. Samuel, Is Barbero’s Hamiltonian formulation a Gauge Theory of Lorentzian Gravity?, *Class. Quant. Grav.* 17 (2000) L141–L148, [gr-qc/0005095](https://arxiv.org/abs/gr-qc/0005095)

[27] G. Immirzi, Real and Complex Connections for Canonical Gravity, *Class. Quantum Grav.* 14 (1997) L177–L181

[28] S. A. Hojman, K. Kuchař, and C. Teitelboim, Geometrodynamics Regained, *Ann. Phys. (New York)* 96 (1976) 88–135

[29] P. A. M. Dirac, The theory of gravitation in Hamiltonian form, *Proc. Roy. Soc. A* 246 (1958) 333–343

[30] M. Bojowald and R. Swiderski, Spherically Symmetric Quantum Geometry: Hamiltonian Constraint, *Class. Quantum Grav.* 23 (2006) 2129–2154, [gr-qc/0511108](https://arxiv.org/abs/gr-qc/0511108)

[31] M. Bojowald and A. Skirzewski, Effective Equations of Motion for Quantum Systems, *Rev. Math. Phys.* 18 (2006) 713–745, [math-ph/0511043](https://arxiv.org/abs/math-ph/0511043)

[32] M. Bojowald and A. Skirzewski, Quantum Gravity and Higher Curvature Actions, *Int. J. Geom. Meth. Mod. Phys.* 4 (2007) 25–52, [hep-th/0606232](https://arxiv.org/abs/hep-th/0606232)

[33] M. Bojowald, S. Brahma and E. Nelson, Higher time derivatives in effective equations of canonical quantum systems, *Phys. Rev. D* 86 (2012) 105004, [arXiv:1208.1242](https://arxiv.org/abs/1208.1242)

[34] M. Bojowald and S. Brahma, Effective constraint algebras with structure functions, *J. Phys. A: Math. Theor.* 49 (2016) 125301, [arXiv:1407.4444](https://arxiv.org/abs/1407.4444)

[35] J. Ben Achour, S. Brahma, J. Grain, and A. Marciano, A new look at scalar perturbations in loop quantum cosmology: (un)deformed algebra approach using self dual variables (2016), [arXiv:1610.07467](https://arxiv.org/abs/1610.07467)
[36] J.-P. Wu, M. Bojowald, and Y. Ma, Anomaly freedom in perturbative models of Euclidean loop quantum gravity, *Phys. Rev. D* 98 (2018) 106009, [arXiv:1809.04465](https://arxiv.org/abs/1809.04465)

[37] A. Henderson, A. Laddha, and C. Tomlin, Constraint algebra in LQG reloaded: Toy model of a U(1)$^3$ Gauge Theory I, *Phys. Rev. D* 88 (2013) 044028, [arXiv:1204.0211](https://arxiv.org/abs/1204.0211)

[38] A. Henderson, A. Laddha, and C. Tomlin, Constraint algebra in LQG reloaded: Toy model of an Abelian gauge theory – II Spatial Diffeomorphisms, *Phys. Rev. D* 88 (2013) 044029, [arXiv:1210.3960](https://arxiv.org/abs/1210.3960)

[39] M. Varadarajan, Towards an Anomaly-Free Quantum Dynamics for a Weak Coupling Limit of Euclidean Gravity: Diffeomorphism Covariance, *Phys. Rev. D* 87 (2013) 044040, [arXiv:1210.6877](https://arxiv.org/abs/1210.6877)

[40] C. Tomlin and M. Varadarajan, Towards an Anomaly-Free Quantum Dynamics for a Weak Coupling Limit of Euclidean Gravity, *Phys. Rev. D* 87 (2013) 044039, [arXiv:1210.6869](https://arxiv.org/abs/1210.6869)

[41] M. Varadarajan, The constraint algebra in Smolins’ $G \to 0$ limit of 4d Euclidean Gravity, *Phys. Rev. D* 97 (2018) 106007, [arXiv:1802.07033](https://arxiv.org/abs/1802.07033)

[42] A. Laddha and M. Varadarajan, The Diffeomorphism Constraint Operator in Loop Quantum Gravity, *Class. Quant. Grav.* 28 (2011) 195010, [arXiv:1105.0636](https://arxiv.org/abs/1105.0636)

[43] A. Laddha, Hamiltonian constraint in Euclidean LQG revisited: First hints of off-shell Closure (2014), [arXiv:1401.0931](https://arxiv.org/abs/1401.0931)

[44] A. Perez and D. Pranzetti, On the regularization of the constraints algebra of Quantum Gravity in 2+1 dimensions with non-vanishing cosmological constant, *Class. Quantum Grav.* 27 (2010) 145009, [arXiv:1001.3292](https://arxiv.org/abs/1001.3292)