Modeling of dynamic branching of a longitudinal shear crack at an inclusion

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Abstract. Mathematical modeling of branching of the leading edge of a longitudinal shear crack on a porous inclusion is proposed. The leading edge of the crack is approximated by the \( \delta \)-neighborhood of the plastic deformation of the material with the front of longitudinal or shear waves, at which the longitudinal or transverse velocities break when passing through the leading edge of the crack. Expressions are obtained and graphs of the relative intensity of the leading edges of the reflected cracks of longitudinal and transverse shear and the conditions for the generation of reflected cracks depending on the angle of intersection of the crack with inclusion and physical parameters of the material are constructed.

1. Introduction

By a crack, we mean the spatial surface \( S \) of a bounded curve \( L \) propagating in space at a speed \( C_1 \) of longitudinal (or \( C_2 \) transverse) elastic waves [1, 3, 6]. The leading edge \( L \) is approximated by the \( \delta \)-neighborhood of the elasto-viscoplastic behavior of the material [1, 2, 4-8]. The criterion for the motion of the crack tip (the leading edge of the crack \( L \)) is to achieve the ultimate stress state in the neighborhood of the vertex of the peak satisfying the Mises plasticity condition for the longitudinal or transverse shear crack, or to achieve the ultimate value of the tensile stress for the separation crack.

It was shown in [6, 8] that the velocity jump at the leading edge \( L \) of the longitudinal shear crack occurs at the front of the longitudinal waves \( \Sigma_1 \), and the velocity jump at the edge \( L \) of the separation or transverse shear crack occurs at the front \( \Sigma_2 \) of shear waves. Thus [1, 6-9], the leading edges of the longitudinal \( \Sigma_1 \) or transverse \( \Sigma_2 \) waves of plastic loading can bear the leading edges \( L \) of the cracks. The propagation of the cracks themselves, as orthogonal trajectories of the leading edges \( L \) on the wave surfaces \( \Sigma_1 \) or \( \Sigma_2 \), reduces to the analysis of the stress state behind the fronts of the waves of plastic loading.

2. Materials and methods

2.1. Statement of the problem

Within the framework of the above proposals, the problem of branching a longitudinal shear crack on a porous inclusion is considered as the problem of reflecting a longitudinal wave of plastic loading from the free surface of the pore (figure 1). In figure 1 shows the geometry of the incidence of the...
crack $T_r_1$ of the longitudinal shear, $T_r_2$ of the transverse shear, as well as the geometry of the incident $\Sigma_1$ and reflected $\Sigma'_1, \Sigma'_2$ waves of plastic loading in the $\delta$-neighborhood of the plastic deformation of the material at the leading edge $L$ of the crack.

![Figure 1](image1.png)

**Figure 1.** Image a) S-porous inclusion; $\Sigma_1$ is the tangent to the front of the $\delta$-neighborhood of the plastic behavior of the material at the leading edge of the incident longitudinal shear crack; $\Sigma'_1, \Sigma'_2$ – reflected fronts of longitudinal and shear waves; $\phi$ is the angle of incidence of the crack trace $T_r_1$ relative to the normal $\vec{n}$ to the surface $S$; c) traces of: incident and reflected cracks of the longitudinal shear $T_r_1, T_r'_1$; reflected transverse shear crack $- T_r_2$.

The kinematics of material deformation in the vicinity of the leading edges of cracks of longitudinal and transverse shears is presented in figure 2, 3.

![Figure 2](image2.png)

**Figure 2.** Image of the surface $\Sigma_1$ of the wave longitudinal to $\Sigma_1$ the discontinuity of velocity and stress, the surface $S$ of the longitudinal shear crack with a leading edge $L$ lying on the front $\Sigma_1$, and the behavior of the material near the tip of the longitudinal shear crack.

Permissible discontinuous solutions with jumps in the normal velocity component $[v_n] = \omega_n$ at the front $\Sigma_1$ moving with speed $c_1$ (figure 2) and the tangent velocity component $[v_\tau] = \omega_\tau$ at the front $\Sigma_2$ moving with speed $c_2$ (figure 3) take place in the vicinity of the front edge of the crack.
Figure 3. Image of the surface $\Sigma_2$ of the wave of the discontinuity of velocity and stresses tangent to $\Sigma_2$ and $L$, and, the surface $S$ with the leading edge $L$ lying on the front $\Sigma_2$ and the behavior of the material near the top of the transverse shear crack.

2.2. A mathematical model of the leading edge of a crack

Select $\delta$ - the vicinity of the front edge $L$ of the crack in the form of a cylinder $S_\delta$ with a curved axis. The surface $\Sigma_\delta$ is the surface of a discontinuity generated by the initial perturbation during the initiation of the crack edge. The curve $L_\delta$ on $\Sigma_\delta$ will be considered a harbinger of the crack edge.

Let us consider the trace of the leading edge of the crack at the front $\Sigma_\delta$ of the wave precursor, which is represented by the spatial curve $L_\delta$ belonging to the surface $\Sigma_\delta$ (figure 4).

Figure 4. Schematic representation of a precursor $\Sigma_\delta$ of a crack and a trace $L_\delta$ of a crack on a precursor. The signs $\wedge$ and $\vee$ are used to denote the values of the functions beyond at the points above (sign $\wedge$) or below (sign $\vee$) of the crack trace $L_\delta$ on for the case of separation crack $L$, transverse or longitudinal shear.

The correspondence of the kinematics of the material behavior near the crack tip in displacements and the kinematics of the material behavior in the vicinity of the longitudinal and shear wave fronts at the displacement velocities leads to the conclusion:

1. The leading edge of the precursor of a spatial crack $L$ propagates as an orthogonal trajectory of the wave front $\Sigma_\delta$ with its own line $L_\delta$ with discontinuities of velocities and stresses when crossing across the line itself.

1.1. The leading edge of the longitudinal shear crack propagates with the speed $c_1$ of the longitudinal strain waves in the elastic body (figure 2, 4).

1.2. The leading edge of a transverse shear crack propagates with shear wave velocity $c_2$ (figure 3).

2. As the intensity of the leading edges of the cracks, it is convenient to choose a jump in speed when passing through the leading edge $L_\delta$ of the crack $S$.

2.1. For a longitudinal shear crack on $\Sigma_1$ (figure 2)
\[ \omega_{nl} = \left( \hat{v}^\prime - \hat{v} \right)_n = \left[ v^\prime_n \right]_n \]  

(1)

2.2. For a transverse shear crack on \( \Sigma_2 \) (figure 3)

\[ \omega_{Ll} = \left( \hat{v}^\prime - \hat{v} \right)_L = \left[ v^\prime_L \right]_L \]  

(2)

2.3. For the separation gap on \( \Sigma_2 \) (figure 4)

\[ \omega_{Ll} = \left( \hat{v}^\prime - \hat{v} \right)_L = \left[ v^\prime_L \right]_L . \]  

(3)

Thus, the type of crack generates the form of a precursor \( \Sigma_1 \) or \( \Sigma_2 \), and the very existence of the leading edge is determined by the intensity of the leading edge – the excess of the dynamic second invariant of the stress tensor deviator over the static plasticity limit: \( \sigma_{ij}' \sigma_{ij}' \geq K^2 \).

The intensity of the crack \( \omega_{Ll} \) \((1 - 3)\) is determined by the intensity of the corresponding waves of strong discontinuity \( \Sigma_1 \) and \( \Sigma_2 \).

In the vicinity of the precursor \( L \), cracks do occur

\[ \hat{\nu} = \nu^\prime - [\hat{\nu}] ; \quad \tilde{\nu} = \nu^\prime - [\tilde{\nu}] ; \quad \omega_L = \hat{\nu} - \tilde{\nu} = -[\hat{\nu}] + [\tilde{\nu}] . \]  

(4)

We assume that the existence of a crack is determined by the presence of plastic deformation on at least one of the sides \( L \) of the leading edge, and we consider the very problem of analyzing the intensity of the edges of the cracks as the problem of analyzing the plastic stress state behind the fronts of strong discontinuity waves.

3. Reflection of the crack tip of the longitudinal shear from the porous inclusion

The leading edges of the cracks lying on the wave fronts are orthogonal trajectories of these fronts.

By the intensity of the leading edge of the crack, we mean the difference in the longitudinal velocities (1) taken above and below the plane of the crack. The values of the velocities behind the wave fronts themselves satisfy the basic laws of the model of elasto-viscoplastic deformation of the material and the laws of reflection of wave fronts from the boundary.

From the condition of the simultaneous existence at the point of three waves \( \Sigma_1, \Sigma_1' \) and \( \Sigma_2' \), the angles of reflection \( \phi \) and \( \psi \) are related by Snell's law.

\[ \frac{c_1}{\sin \phi} = \frac{c_1}{\sin \phi'} = \frac{c_2}{\sin \psi} . \quad \sin \psi' = \frac{c_1}{c_2} \sin \phi, \quad \phi' = \phi \]  

(5)

The compatibility conditions for stresses and velocities at the front of a strong \( \Sigma_1 \) wave have the form:

\[ -c_1 \left[ \sigma_{ij} \right] = \left( \lambda \delta_{ij} + 2\mu n_i n_j \right) \omega_n , \]  

(6)

where \( \omega_n \) is the jump of the normal velocity component at the wave front and \( \omega_n = [V]n_i \).

Conditions (5, 6) reflect the fact that the front \( \Sigma_2 \) of the spatial fracture precursor can be a longitudinal strain wave for a longitudinal shear crack (figure 2), propagating with speed \( c_1 \left( \rho c_1^2 = \lambda + 2\mu \right) \) and a shear strain wave for separation or transverse shear cracks (figure 3), propagating with speed \( c_2 \left( \rho c_2^2 = \mu \right) \).
Stresses jumps are expressed through speed jumps [6, 9]

\[ \left[ \sigma_{ij} \right] = \frac{1}{c} \lambda \left[ v_n \right] \delta_{ij} - \frac{1}{c} \mu \left[ \left( v_x \right) n_j + \left( v_y \right) n_i \right] \]  

(7)

Jumps of velocities and stresses on reflected waves and are related by the relations

\[ -c_1 \left[ \sigma_{ij} \right]_1 = \left( \lambda \delta_{ij} + 2 \mu n_i n_j \right) \omega_n, \quad \left[ V_j \right]_1 = \omega_n \cdot n_i \]  

(8)

\[ -c_2 \left[ \sigma_{ij} \right]_2 = \mu \left( \rho_i^{(2)} n_j^{(2)} + \rho_j^{(2)} n_i^{(2)} \right) \omega_2, \quad \left[ V_j \right]_2 = \omega_2 \cdot n_i^{(2)} \]  

(9)

We supplement the system of equations (8, 9) with the boundary conditions of absence \( \sigma^{(3)} \cdot N_j = 0 \) stresses at the crack reflection boundary.

From (7, 8, 9) we obtain

\[ -\left[ \sigma_{ij} \right] = \frac{1}{c_1} \left( \lambda \delta_{ij} + 2 \mu n_i n_j \right) \omega_n, \]  

(10)

\[ -\left[ \sigma_{ij} \right]_2 = \frac{1}{c_1} \left( \lambda \delta_{ij} + 2 \mu n_i n_j \right) \omega_n, \quad -\left[ \sigma_{ij} \right]_2 = \frac{1}{c_2} \mu \left( \rho_i^{(2)} n_j^{(2)} + \rho_j^{(2)} n_i^{(2)} \right) \omega_2 \]  

(11)

Excluding stress jumps from (11) and satisfying the boundary conditions (10) in zone (3), we obtain

\[ \left( \lambda \delta_{ij} + 2 \mu n_i n_j \right) \omega_n N_j + \left( \lambda \delta_{ij} + 2 \mu n_i n_j \right) \omega_n N_j + \mu \frac{c_1}{c_2} \left( \rho_i^{(2)} n_j^{(2)} + \rho_j^{(2)} n_i^{(2)} \right) \omega_2 N_j = 0 \]  

(12)

Projecting equality (7) onto the normal and tangent to S we get:

\[ \left( \lambda + \mu \sin 2 \varphi \right) \omega_n + \left( \lambda - \mu \sin 2 \varphi \right) \omega_n + \mu \frac{c_1}{c_2} \cos 2 \varphi' \cdot \omega_2 = 0 \]  

(13)

\[ \mu \sin 2 \varphi \cdot \omega_n - \mu \sin 2 \varphi' \cdot \omega_n + \mu \frac{c_1}{c_2} \cos 2 \varphi' \cdot \omega_2 = 0 \]  

(14)

We express the intensity and reflected waves through the intensity of the incident wave \( \omega_n \).

\[ \omega_n = \frac{1 + 2 \frac{c_2^2}{c_1^2}}{1 - 2 \frac{c_2^2}{c_1^2}} \cdot \omega_n \]  

(15)
Figure 5. A graphical representation of the reflection coefficient $\omega_{1n}/\omega_n$ of a longitudinal shear wave from a stress-free boundary depending on the physical properties of the material $t = \frac{c_2^2}{c_1^2}$.

On the limiting longitudinal plastic wave $\omega_n = \frac{\sqrt{3}K}{2\mu}$ and, introducing the notation $\omega_n = \frac{\sqrt{3}K}{2\mu}$, we obtain the expression for $\omega_n$ through $\tilde{c} = \frac{c_2^2}{c_1^2}$

$$\omega_n^2 = \frac{K^2}{\rho^2\tilde{c}c_1^2\left(\frac{10}{3}\tilde{c} - 1\right)}.$$  \hfill (16)

Figure 6. A graphical representation of the dependence of the relative velocity $\omega_n$ at the front $\Sigma_1$ of a longitudinal plastic incident wave on the physical properties of the material $t = \frac{c_2^2}{c_1^2}$.

The intensity of the reflected shear wave $\omega_2$ is conveniently represented in the form (17) and the graph is presented in figure 7.
\[
\omega_2 = -\frac{-3c_1^3 \sin 2\varphi}{c_1^2 (c_1^2 - 1) \left( \frac{1}{2} - \frac{c_2^2}{c_1^2} \sin^2 \varphi \right)} \cdot \omega_n
\]  

(17)

**Figure 7.** A graphical representation of the shear wave reflection coefficient \(\omega_2/\omega_n\) as a function of the angle of incidence and material properties \(t = \frac{c_2^2}{c_1^2}\).

4. The stress state behind the fronts of the precursors of reflected cracks of longitudinal and transverse shears.

The fronts of the reflected longitudinal \(\Sigma_1\) and \(\Sigma_2\) shear waves are the precursors of the cracks of the longitudinal and transverse shears, which exist as long as the stress state remains in the plastic zone.

We calculate the intensity of the maximum tangential stresses \(I_2\) and evaluate the possibility of plastic deformation depending on the angle \(\varphi\) of the longitudinal wave \(\Sigma_1\) and the wave propagation velocity \(\varepsilon\) in zones (2) and (3).

For the stress state \(\sigma_{ij}^{(2)}\) and the second invariant \(I_{(2)}^{(2)}\) of the tensor \(\sigma_{ij}^{(2)}\) are determined

\[
I_{(2)}^{(2)} = \frac{6K^2}{\varepsilon(10\varepsilon - 3)} \left( 2(1 - 2\varepsilon)(3 - 2\varepsilon) + \frac{1}{3} (3 - 4\varepsilon)^2 - 4\varepsilon^2 \cdot \cos 2\varphi \right)
\]  

(18)
Figure 8. A graphical representation of the magnitude of the second invariant of the deviator of the stress tensor behind the reflected longitudinal wave in zone (2).

In zone (3):

\[
I_2^2 = \frac{6\mu}{c_2^2-\omega_1^2} \left( \frac{3\sqrt{\tilde{c}} \sin 2\varphi}{(\tilde{c} - 1)\left(\frac{1}{2} - \tilde{c}\sin^2\varphi\right)} \cdot \frac{K}{\rho_c\sqrt{\tilde{c}}\sqrt{\frac{10}{3} \tilde{c} - 1}} \right)^2.
\]  

(19)

Figure 9. A graphical representation of the magnitude of the second invariant in zone (3) behind the reflected shear wave.

The intensity graph of shear stresses in zone (3) reflects the fact of an increase in their intensity with increasing angle $\phi$ and magnitude $t = \frac{c_2^2}{c_1^2}$.
3. Conclusion

1. An analysis of the kinematics $\omega_n$ behavior at the front of the incident crack, which is necessary to achieve a plastic state at the front edge of the crack, shows that with an increase in the velocity $c_2$, the shear waves $\omega_n$ decrease (16).

2. The intensity of the stress state $I'_2(2)$ at the front of the reflected crack $Tr'_1$ of the longitudinal shear increases with increasing angle $(\pi/2-\varphi)$ of the incident crack with the pore surface (18).

3. The intensity of the stress state $I'_2(3)$ at the leading edge of the reflected crack $Tr'_2$ of the longitudinal shear increases with increasing $(\pi/2-\varphi)$ angle of the incident crack with the pore surface.

4. The branching of a longitudinal shear crack is determined by the intensity $\omega$ of the feed crack, the condition for achieving a plastic state at the leading edge of the reflected cracks, and the angle of incidence $(\pi/2-\varphi)$ of the crack relative to the pore surface.

References

[1] Kachanov L M 1974 Fundamentals of fracture mechanics (Moscow: Nauka) (in Russian) p 312
[2] Ievlev V M 2008 Thin films of inorganic materials: mechanisms of growth and structure: textbook (Voronezh CPI of Voronezh state University) (in Russian) p 172
[3] Parton V Z and Boriskovsky V G 1988 Dynamics of brittle fracture (Moscow: Nauka) (in Russian) p 240
[4] Kukudzhanov V N 1967 Propagation of elastoplastic waves in a rod taking into account the influence of the deformation rate (Computer Center of the USSR Academy of Sciences. M: VTs AN SSSR) p 48
[5] Ivlev D D 2005 Theory of limit state and ideal plasticity: selected works (Voronezh State University) (in Russian) p 357
[6] Verveiko N D 1997 Radiation theory of elastic-viscoplastic waves and shock waves (Voronezh state University Voronezh) (in Russian) p 204
[7] Verveiko N D, Shashkin A I and Krupenko E S 2017 The origin and motion of crack tips for the fronts elastoviscoplastic waves (Voronezh state University Voronezh: Quarta) (in Russian) p 124
[8] Verveiko N D, Shashkin A I and Krupenko E S 2018 Mathematical modeling of the propagations of precursors of the front edges of crack as spatial curves on the fronts of waves of a strong discontinuity of rates and stresses J. Phys.: Conf. Ser. 1203 012010 URL: iopscience.iop.org/article/10.1088/1742-6596/1203/1/012010
[9] Thomas T 1964 Plastic flow and destruction in solids (Moscow: Mir) (in Russian) p 308