Abstract—We are interested in investigating the security of source encryption with a symmetric key under side-channel attacks. In this paper, we propose a general framework of source encryption with a symmetric key under the side-channel attacks, which applies to any source encryption with a symmetric key and any kind of side-channel attacks targeting the secret key. We also propose a new security criterion for strong secrecy under side-channel attacks, which is a natural extension of mutual information, i.e., the maximum conditional mutual information between the plaintext and the ciphertext given the adversarial key leakage, where the maximum is taken over all possible plaintext distribution. Under this new criterion, we successfully formulate the rate region, which serves as both necessary and sufficient conditions to have secure transmission even under side-channel attacks. Furthermore, we also prove another theoretical result on our new security criterion, which might be interesting in its own right: in the case of the discrete memoryless source, no perfect secrecy under side-channel attacks in the standard security criterion, i.e., the ordinary mutual information, is achievable without achieving perfect secrecy in this new security criterion, although our new security criterion is more strict than the standard security criterion.

I. INTRODUCTION

As more cryptographic devices are deployed in open physical spaces, a new security challenge has arisen in the form of attackers which launch side-channel attacks, where an attacker does not only collect the encrypted data sent to the public communication channel, but also the physical information related to the private data which are leaked by the devices in the form such as power consumption, electromagnetic radiation, running time, etc. Therefore, we consider that designing an encryption scheme that is guaranteed to be secure even under side-channel attacks is very important.

In this paper, we propose a general framework for analyzing any source encryption with a symmetric key under any kind of side-channel attacks from which the adversary obtains some leaked information on the secret key. Although Santoso and Oohama investigated a similar problem in [11], their work is limited only to a single specific encryption scheme, i.e., one-time-pad encryption. In contrast, our framework here covers any encryption scheme. Then, we propose a new security criterion for secrecy which is defined as the maximum of all conditional mutual information between the ciphertext and plaintext given the adversarial key leakage. The maximum is taken over all probability distributions of plaintexts. Note that a security criterion is basically a metric for representing the total amount of leakage of private information, i.e., the loss of secrecy. Thus, our new security criterion is more strict than the standard security criterion, i.e., ordinary mutual information. Nevertheless, we show that we can construct a concrete encryption scheme with reliable decoding and secrecy under side-channel attacks based on the new security criterion.

We also prove that the perfect secrecy in the new security criterion is the necessary condition for the perfect secrecy in the standard security criterion in the case of the discrete memoryless source. In [2], Csiszár and Narayan introduced another security criterion in the form of summation of mutual information and some other terms. However, we believe that our security criterion is much more natural as an extension of mutual information compared to theirs.

The most important result in this paper is that we prove the strong converse, i.e., the necessary condition under the new security criterion to have an encryption scheme with both reliable decoding and secrecy under side-channel attacks. The key for deriving the lower bound is our main lemma which can be seen as an extension of Birkhoff-von Neumann theorem [12]. In a nutshell, our main lemma states that in any symmetric key encryption scheme where the plaintexts and the secret keys are independent, regardless of the information obtained via side-channel attacks on the secret key, the adversary can see the encryption process as a stochastic matrix. Based on this, we prove further that as long as the plaintexts are uniformly distributed, the adversary will see the ciphertexts as uniformly random, regardless of the information on the secret key it gets from side-channel attacks.

II. PROBLEM FORMULATION

A. Preliminaries

In this subsection, we show the basic notations and related consensus used in this paper.

Random Source of Information and Key: Let $X$ be a random variable from a finite set $\mathcal{X}$. Let $\{X_t\}_{t=1}^\infty$ be a stationary discrete memoryless source (DMS) such that for each $t = 1, 2, \ldots$, $X_t$ takes values in finite set $\mathcal{X}$ and obeys the same distribution as that of $X$ denoted by $p_X = \{p_X(x)\}_{x \in \mathcal{X}}$. The stationary DMS $\{X_t\}_{t=1}^\infty$ is specified with $p_X$. Also, let $K$ be a random variable taken from the same finite set $\mathcal{X}$ representing the key used for encryption. Similarly, let $\{K_t\}_{t=1}^\infty$ be a stationary DMS such that for each $t = 1, 2, \ldots$, $K_t$ takes values in the
finite set $\mathcal{X}$ and obeys the same distribution as that of $K$ denoted by $p_K = \{p_K(k)\}_{k \in \mathcal{X}}$. The stationary DMS $\{K_t\}_{t=1}^\infty$ is specified with $p_K$.

**Random Variables and Sequences:** We write the sequence of random variables with length $n$ from the information source as follows: $X^n := X_1X_2 \cdots X_n$. Similarly, the strings with length $n$ of $\mathcal{X}$ are written as $x^n := x_1x_2 \cdots x_n \in \mathcal{X}^n$. For $x^n \in \mathcal{X}^n$, $p_{X^n}(x^n)$ stands for the probability of the occurrence of $x^n$. When the information source is memoryless specified with $p_X$, we have the following equation holds:

$$p_{X^n}(x^n) = \prod_{i=1}^{n} p_X(x_i).$$

In this case we write $p_{X^n}(x^n)$ as $p_X^n(x^n)$. Similar notations are used for other random variables and sequences.

**Consensus and Notations:** Without loss of generality, throughout this paper, we assume that $\mathcal{X}$ is a finite field. The notation $\oplus$ is used to denote the field addition operation, while the notation $\ominus$ is used to denote the field subtraction operation, i.e., $a \ominus b = a \oplus (-b)$ for any elements $a, b \in \mathcal{X}$. Throughout this paper all logarithms are taken to the base natural.

### B. Basic System Description

In this subsection we explain the basic system setting and basic adversarial model we consider in this paper. First, let the information source and the key be generated independently by different parties $S_{\text{gen}}$ and $K_{\text{gen}}$, respectively. We further assume that the source is generated by $S_{\text{gen}}$ and independent of the key.

**Source coding without encryption:** The random source $X^n$ from $S_{\text{gen}}$ be sent to node $E$. Further settings of the system are described as follows. Those are also shown in Fig. 1.

1) **Encoding Process:** At the node $E$, the encoder function $\phi^{(n)} : \mathcal{X}^n \to \mathcal{X}^m$ observes $X^n$ to generate $\hat{X}^m = \phi^{(n)}(X^n)$. Without loss of generality we may assume that $\phi^{(n)}$ is surjective.

2) **Transmission:** Next, the encoded source $\hat{X}^m$ is sent to the destination $D$ through a noiseless channel.

3) **Decoding Process:** In $D$, the decoder function observes $\hat{X}^m$ to output $\hat{X}^n$, using the one-to-one mapping $\psi^{(n)}$ defined by $\psi^{(n)} : \mathcal{X}^m \to \mathcal{X}^n$. Here we set

$$\hat{X}^n := \psi^{(n)}(\hat{X}^m) = \psi^{(n)}\left(\phi^{(n)}(X^n)\right).$$

More concretely, the decoder outputs the unique pair $\hat{X}^n$ from $(\phi^{(n)})^{-1}(\hat{X}^m)$ in a proper manner.

For the above $\left(\phi^{(n)}, \psi^{(n)}\right)$, define the set $D^{(n)}$ of correct decoding by $D^{(n)} := \{x^n \in \mathcal{X}^n : \phi^{(n)}(\phi^{(n)}(x^n)) = x^n\}$. On $|D^{(n)}|$, we have the following property.

### Property 1:

**Under the conditions 1)-3) that we assume in the distributed source coding without encryption we have $|D^{(n)}| = |\mathcal{X}^m|$.**

**Proof of Property 1** is given in Appendix A.

**Source coding with encryption:** The source $X^n$ from $S_{\text{gen}}$ is sent to the node $L$. The random key $K^n$ from $K_{\text{gen}}$, is sent to $L$. Further settings of our system are described as follows. Those are also shown in Fig. 2.

1) **Source Processing:** At the node $E$, $X^n$ is encrypted with the key $K^n$ using the encryption function $\Phi^{(n)} : \mathcal{X}^n \times \mathcal{X} \to \mathcal{X}^m$. The ciphertext $C^n$ of $X^n$ is given by $C^n = \Phi^{(n)}(K^n, X^n)$. On the encryption function $\Phi^{(n)}$, we use the following notation:

$$\Phi^{(n)}(K^n, X^n) = \Phi^{(n)}_K(X^n) = \Phi^{(n)}_K(K^n).$$

2) **Transmission:** Next, the ciphertext $C^n$ is sent to the destination $D$ through the public communication channel. Meanwhile, the key $K^n$ is sent to $D$ through the private communication channel.

3) **Sink Node Processing:** In $D$, we decrypt the ciphertext $C^n$ from $C^n$ using the key $K^n$ through the corresponding decryption procedure $\Psi^{(n)}$ defined by $\Psi^{(n)} : \mathcal{X}^n \times \mathcal{X} \to \mathcal{X}^m$. Here we set $\hat{X}^n := \Psi^{(n)}(K^n, C^n)$. More concretely, the decoder outputs the unique $\hat{X}^n$ from $(\Phi^{(n)}_K)^{-1}(C^n)$ in a proper manner. On the decryption function $\Psi^{(n)}$, we use the following notation:

$$\Psi^{(n)}(K^n, C^n) = \Psi^{(n)}_K(C^n) = \Psi^{(n)}_K(K^n).$$

Fix any $K^n = k^n \in \mathcal{X}^n$. For this $K^n$ and for $(\Phi^{(n)}, \Psi^{(n)})$, we define the set $D^{(n)}_k$ of correct decoding by

$$D^{(n)}_k := \{x^n \in \mathcal{X}^n : \Psi^{(n)}_k(\Phi^{(n)}_k(x^n)) = x^n\}.$$

We require that the cryptosystem $(\Phi^{(n)}, \Psi^{(n)})$ must satisfy the following condition.

**Condition:** For each distributed source coding system $(\Phi^{(n)}, \Psi^{(n)})$, there exists a source coding system $(\phi^{(n)}, \psi^{(n)})$ such that for any $k^n \in \mathcal{X}^n$ and for any $k^n \in \mathcal{X}^n$,

$$\Psi^{(n)}_k(\Phi^{(n)}_k(x^n)) = \psi^{(n)}(\phi^{(n)}(x^n)).$$

The above condition implies that $D^{(n)} = D^{(n)}_k$, $\forall k^n \in \mathcal{X}^n$. We have the following properties on $D^{(n)}$.

**Property 2:**
extension of the Birkhoff-von Neumann theorem \[3\].

This lemma is shown below.

Proof of Property 2 is given in Appendix D.

Side-Channel Attacks by Eavesdropper Adversary: An adversary \(A\) eavesdrops the public communication channel in the system. The adversary \(A\) also uses a side information obtained by side-channel attacks. Let \(Z\) be a finite set and let \(W: X \rightarrow Z\) be a noisy channel. Let \(Z\) be a channel output from \(W\) for the input random variable \(K\). We consider the discrete memoryless channel (DMC) with \(W\). Let \(Z^n \in Z^n\) be a random variable obtained as the channel output by connecting \(K^n, X^n\) to the input of channel. We write a conditional distribution on \(Z^n\) given \(K^n\) as

\[ W^n = \{ W^n(z^n|K^n) \}_{(k^n,z^n) \in X^n \times Z^n} \]

On the output above \(Z^n\) of \(W^n\) for the input \(K^n\), we assume the followings.

- The three random variables \(X, K, Z\) satisfy \(X \perp (K, Z)\), which implies that \(X^n \perp (K^n, Z^n)\).
- \(W\) is given in the system and the adversary \(A\) cannot control \(W\).
- By side-channel attacks, the adversary \(A\) can access \(Z^n\).

We next formulate side information the adversary \(A\) obtains by side-channel attacks. For each \(n = 1, 2, \ldots\), let \(\varphi^n: Z^n \rightarrow M_A^n\) be an encoder function. Set \(\varphi_A^n = \{ \varphi^n( Z^n ) \} \in M_A^n\). We assume that \(\| \varphi_A^n \| = |M_A^n|\) must satisfy the rate constraint \(\| \varphi_A^n \| \leq e^nR_A\).

C. Security Criterion and Problem Set Up

In this subsection, we propose a new security criterion. We first state a lemma having a close connection with the new security criterion. This lemma is shown below.

**Lemma 1:** \(\forall (c^n, a) \in X^n \times M_A^n\), we have

\[ \sum_{x^n \in D(n)} P_{C^n|x^n}(c^n | a, x^n) = 1. \]

Lemma 1 can easily be proved by Property 2. The detail is found in Appendix D. This lemma can be regarded as an extension of the Birkhoff-von Neumann theorem [1].

In the following arguments all logarithms are taken to the base natural. The adversary \(A\) tries to estimate \(X^n \in X^n\) from \((C^n, M_A^n)\). Note that since \(X^n \perp (K^n, Z^n)\), we have \(X^n \perp (K^n, M_A^n)\). The mutual information (MI) between \(X^n\) and \((C^n, M_A^n)\) denoted by

\[ \Delta_M(n) := I(C^n, M_A^n; X^n) = I(C^n, X^n | M_A^n) \]

indicates a leakage of information on \(X^n\) from \((C^n, M_A^n)\).

In this sense it seems to be quite natural to adopt the mutual information \(\Delta_M(n)\) as a security criterion. On the other hand, directly using \(\Delta_M(n)\) as a security criterion of the cryptosystem has some problem that this value depends on the statistical property of \(X^n\). In this paper we propose a new security criterion, which is based on \(\Delta_M(n)\) but overcomes the above problem.

**Definition 1:** Let \(\bar{x}^n\) be an arbitrary random variable taking values in \(X^n\). Set \(\bar{C}^n = \Phi(n)(K^n, \bar{X}^n)\). The maximum mutual information criterion denoted by \(\Delta_{\text{max-MI}}(n)\) is as follows.

\[ \Delta_{\text{max-MI}}(n) := \max_{p_{\bar{C}^n} \in P(X^n)} I(\bar{C}^n, \bar{X}^n | M_A^n). \]

By definition it is obvious that \(\Delta_{\text{MI}}(n) \leq \Delta_{\text{max-MI}}(n)\). We have the following proposition on \(\Delta_{\text{max-MI}}(n)\).

**Proposition 1:**

- a) If \(\Delta_{\text{MI}}(n) = I(C^n; X^n | M_A^n) = 0\), then, we have \(\Delta_{\text{max-MI}}(n) = 0\). This implies that \(\Delta_{\text{max-MI}}(n)\) is valid as a measure of information leakage.
- b) We have the following.

\[ \Delta_{\text{max-MI}}(n) \geq \log |X^n| - H(K^n | M_A^n). \]

Proof of Proposition 1 is given in Appendix D.

**Remark 1:** The part a) in the above proposition is quite essential. If we have a security criterion \(\Delta(n)\) not satisfying this condition, it may happen that \(\Delta_{\text{MI}}(n) = I(C^n; X^n | M_A^n) = 0\), but \(\Delta(n) > 0\). Such \(\Delta(n)\) is invalid for the security criterion.

**Remark 2:** The property stated in the part b) is a key important property of \(\Delta_{\text{max-MI}}(n)\) which plays an important role in establishing the strong converse theorem. Lemma 1 is a key result for the proof of the part b).

**Defining Reliability and Security:** The decoding process is successful if \(X^n = \hat{X}^n\) holds. Hence the decoding error probability is given by

\[ Pr[\psi^n(K^n, \Phi^n(K^n)(X^n)) \neq X^n] \]

but overcomes the above security criterion.

Since the above quantity depends only on \((\Phi(n), \varphi_A^n)\) and \(\rho_K^n\), we write the error probability \(p_e\) of decoding as

\[ p_e = p_e(\psi^n, \varphi_A^n) \rho_K^n := Pr[X^n \notin D(n)]. \]

Since \(\Delta_{\text{max-MI}}(n)\) depends only on \((\Phi(n), \varphi_A^n)\) and \(\rho_K^n\), we write this quantity as \(\Delta_{\text{max-MI}}(n) = \Delta_{\text{max-MI}}(\Phi(n), \varphi_A^n, \rho_K^n)\). Define

\[ \Delta_{\text{max-MI}}(\Phi(n), \varphi_A^n, \rho_K^n) := \max_{\varphi_A^n} \left\{ \Delta_{\text{max-MI}}(\Phi(n), \varphi_A^n, \rho_K^n) : \| \varphi_A^n \| \leq e^nR_A \right\}. \]
Definition 2: We fix some positive constant $\epsilon_0$. For a fixed pair $(\epsilon, \delta)$ in $[0, \epsilon_0] \times (0, 1)$, a quantity $R$ is $(\epsilon, \delta)$-admissible under $R_D > 0$ for the system if $\exists \{ (\Phi^{(n)}, \Psi^{(n)}) \}_{n \geq 1}$ such that $\forall \gamma > 0$, $\exists n_0(n_0(\gamma)) \in \mathbb{N}$, $\forall n \geq n_0$, 
\[
\frac{1}{m} \log |X|^m = \frac{m}{n} \log |X| \in [R - \gamma, R + \gamma],
\]
$P_e(\Phi^{(n)}, \Psi^{(n)}|x)^n \leq \delta$, and $\Delta(n)(\Phi^{(n)}, R_A^n|x^{n_0}) \leq \epsilon$.

Definition 3: (Reliable and Secure Rate Region) Let $\mathcal{R}_{sys}(\epsilon, \delta|p_{X^n}, p_{K^n})$ denote the set of all $(R_A, R)$ such that $R$ is $(\epsilon, \delta)$-admissible under $R_A$. Furthermore, set $\mathcal{R}_{sys}(p_{X^n}, p_{K^n}) := \bigcap_{(\epsilon, \delta) \in (0, \epsilon_0) \times (0, 1)} \mathcal{R}_{sys}(\epsilon, \delta|p_{X^n}, p_{K^n})$.

We call $\mathcal{R}(p_{X^n}, p_{K^n})$ the reliable and secure rate region.

III. DIRECT CODING THEOREM

In this section we derive an explicit inner bound of $\mathcal{R}_{sys}(\epsilon, \delta|p_{X^n}, p_{K^n})$. To derive this result we use our previous result [1]. A condition for reliable transmission is an immediate consequence from the direct coding theorem for single discrete memoryless sources. We derive a sufficient condition for secure transmission under the security criterion measured by $\Delta^{(n)}_{\text{max-MI}}$.

A. Coding Scheme, Reliability and Security Analysis

Our coding scheme is illustrated in Fig. 4. In this coding scheme we assume the following rate constraint:

\[(1/n) \log |X|^m = (m/n) \log |X| \in [R - (1/n), R].\]  

In the coding scheme in Fig. 4 we first provide a universal code construction of $\{ (\Phi^{(n)}, \Psi^{(n)}) \}_{n \geq 1}$ deriving an exponential upper bound on $P_e(\Phi^{(n)}, \Psi^{(n)}|x)^n$. We next state a concrete construction of $\varphi^{(n)}$ in Fig. 4. Based on $(\Phi^{(n)}, \Psi^{(n)})$ and $\varphi^{(n)}$, we construct $(\Phi^{(n)}, \Psi^{(n)})$ in Fig. 4. We further provide some preliminary observation on an upper bound of $\Delta^{(n)}_{\text{max-MI}}(\Phi^{(n)}, \Psi^{(n)}|p_{X^n}, p_{K^n})$.

Universal Code Construction of $\{ (\Phi^{(n)}, \Psi^{(n)}) \}_{n \geq 1}$: Let $X$ be an arbitrary random variable over $\mathcal{X}$ and has a probability distribution $p_X$. Let $\mathcal{P}(X)$ denote the set of all probability distributions on $\mathcal{X}$. Fix $\gamma > 0$, arbitrary. For $R_0 \geq 0$ and $p_{X^n} \in \mathcal{P}(X)$, we define the following function:

\[E_\gamma(R|p_{X^n}) := \min_{p_{X^n} \in \mathcal{P}(X)} D(\mu_{\gamma}\|p_{X^n}).\]

Set $\delta_n := (1/n) \{ |X| \log (n + 1) + 1 \}$. Note that $\delta_n \to 0$ as $n \to \infty$. Let $n_0(n_0(\gamma))$ be the minimum integer such that we have $\delta_n \leq \gamma$ for $n \geq n_0(\gamma)$. Then we have the following proposition.

Proposition 2: $\forall \gamma > 0$, $\exists \{ (\Phi^{(n)}, \Psi^{(n)}) \}_{n \geq 1}$ satisfying [1], such that $p_{X^n}$ with $R > H(X)$ and $\forall n \geq n_0(\gamma)$,

\[p_e(\Phi^{(n)}, \Psi^{(n)}|x)^n \leq (m/n)\gamma - nE_\gamma(R|p_{X^n}).\]  

Proposition 3 is a well known result on the universal coding for discrete memoryless sources. We omit the proof, which is found in [2].
B. Several Definitions

In this subsection we define sets related to inner bounds of $\mathcal{R}_{\text{sh}}(P_X, p_{KZ})$. We further define functions related to upper bounds of $\Delta_A^{(n)}(\Phi^{(n)}, R_A[p_{KZ}])$, which hold for $(\Phi^{(n)}, \Psi^{(n)})$ proposed in the previous subsection. Let $U$ be an auxiliary random variable taking values in a finite set $\mathcal{U}$. We assume that the joint distribution of $(U, Z, K)$ is

$$p_{UZK}(u, z, k) = p_U(u)p_Z(z|u)p_{K|Z}(k|z).$$

The above condition is equivalent to $U \leftrightarrow Z \leftrightarrow K$. Define the set of probability distributions $p = p_{UZK}$ by

$$\mathcal{P}(p_{UZK}) := \{p_{UZK} : |U| \leq |Z| + 1, U \leftrightarrow Z \leftrightarrow K\}.$$

Let $\mathbb{R}^2_+ := \{R_A \geq 0, R \geq 0\}$. Let $\mathcal{R}_{\text{AKW}}(p_{UZK})$ be the subset of $\mathbb{R}^2_+$ such that for some $U$ with $p_{UZK} \in \mathcal{P}(p_{UZK})$,

$$R_A \geq I(Z; U), R \geq H(K|U).$$

The region $\mathcal{R}_{\text{AKW}}(p_{UZK})$ is equal to the rate region for the one helper source coding problem posed and investigated by Ahlswede and Körner [5] and Wyner [6]. The subscript “AKW” in $\mathcal{R}_{\text{AKW}}(p_{UZK})$ is derived from their names. We can easily show that the region $\mathcal{R}_{\text{AKW}}(p_{UZK})$ satisfies the following property.

**Property 3:**

a) The region $\mathcal{R}_{\text{AKW}}(p_{UZK})$ is a closed convex subset of $\mathbb{R}^2_+ := \{R_A \geq 0, R \geq 0\}$.

b) The point $(0, H(K))$ always belongs to $\mathcal{R}_{\text{AKW}}(p_{UZK})$.

Furthermore, for any $p_{KZ}$,

$$\mathcal{R}_{\text{AKW}}(p_{KZ}) \subseteq \{(R_A, R) : R_A + R \geq H(K)\} \cap \mathbb{R}^2_+.$$

We next explain that the region $\mathcal{R}_{\text{AKW}}(p_{KZ})$ can be expressed with a family of supporting hyperplanes. To describe this result, we define a set of probability distributions on $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}$ by

$$\mathcal{P}_{\text{sh}}(p_{KZ}) := \{p = p_{XZY} : |U| \leq |Z|, U \leftrightarrow Z \leftrightarrow K\}.$$

For $\mu \in [0, 1]$, define

$$R^{(\mu)}(p_{KZ}) := \min_{p \in \mathcal{P}_{\text{sh}}(p_{KZ})} \{\mu I(Z; U) + \mu H(K|U)\},$$

where $\mu = 1 - \mu$. Furthermore, define

$$\mathcal{R}_{\text{AKW}, \text{sh}}(p_{KZ}) := \bigcap_{\mu \in [0, 1]} \{(R_A, R) : \mu R_A + \mu R \geq R^{(\mu)}(p_{KZ})\}.$$

Then, we have the following property.

**Property 4:**

a) The bound $|U| \leq |X|$ is sufficient to describe $R^{(\mu)}(p_{KZ})$.

b) For any $p_{KZ}$, we have

$$\mathcal{R}_{\text{AKW}, \text{sh}}(p_{KZ}) = \mathcal{R}_{\text{AKW}}(p_{KZ}).$$

(5) Proof this property is found in Oohama [7]. We next define a function related to an exponential upper bound of $\Delta_A^{(n)}(\Phi^{(n)}, \Psi^{(n)} | p_{KZ})$. Set

$$\mathcal{Q}(p_{KZ}) := \{q = q_{KZU} : |U| \leq |Z|, U \leftrightarrow Z \leftrightarrow K, p_{KZ} = q_{KZ}\}.$$ 

For $(\mu, \alpha) \in [0, 1]^2$ and for $q = q_{KZU} \in \mathcal{Q}(p_{KZ})$, define

$$\omega_{q_{KZ}}^{(\mu, \alpha)}(z, k|u) := -\log E_q \left[ \exp \left\{ -\omega_{q_{KZ}}^{(\mu, \alpha)}(Z, K|U) \right\} \right],$$

$$\Omega^{(\mu, \alpha)}(q_{KZ}) := \min_{p \in \mathcal{Q}(p_{KZ})} \Omega^{(\mu, \alpha)}(p_{KZ}).$$

Furthermore, define

$$F(R_A, R|p_{KZ}) := \sup_{(\mu, \alpha) \in [0, 1]^2} \left[ \Omega^{(\mu, \alpha)}(p_{KZ}) - \alpha (\mu R_A + \mu R) \right].$$

We finally define a function serving as a lower bound of $F(R_A, R|p_{KZ})$. For $\lambda \geq 0$ and for $p_{UZK} \in \mathcal{P}_{\text{sh}}(p_{KZ})$, define

$$\tilde{\omega}_{q_{KZ}}^{(\mu, \alpha)}(z, k|u) := \mu \log \frac{p_{Z|U}(z|u)}{p_Z(z)} + \log \frac{1}{p_{KZ}|U(K|U)},$$

$$\tilde{\Omega}_{(\mu, \lambda)}(p) := -\log E_p \left[ \exp \left\{ -\omega_{q_{KZ}}^{(\mu, \alpha)}(Z, K|U) \right\} \right],$$

$$\tilde{\Omega}_{(\mu, \lambda)}(p_{KZ}) := \min_{p \in \mathcal{P}_{\text{sh}}(p_{KZ})} \tilde{\Omega}_{(\mu, \lambda)}(p_{KZ}).$$

Furthermore, define

$$\tilde{F}(R_A, R|p_{KZ}) := \sup_{\lambda, \mu \in [0, 1]} \left[ \tilde{\Omega}_{(\mu, \lambda)}(p_{KZ}) - \lambda (\mu R_A + \mu R) \right].$$

We can show that the above functions satisfy the following property.

**Property 5:**

a) The cardinality bound $|U| \leq |X|$ in $\mathcal{Q}(p_{KZ})$ is sufficient to describe the quantity $\Omega_{(\mu, \lambda)}(p_{KZ})$. Furthermore, the cardinality bound $|U| \leq |X|$ in $\mathcal{P}_{\text{sh}}(p_{KZ})$ is sufficient to describe the quantity $\tilde{\Omega}_{(\mu, \lambda)}(p_{KZ})$.

b) For any $R_A, R \geq 0$, we have

$$F(R_A, R|p_{KZ}) \geq \tilde{F}(R_A, R|p_{KZ}).$$

c) When $(R_A + \tau, R + \tau) \notin \mathcal{P}(p_{KZ})$ for $\tau > 0$, there exist $\lambda_0 > 0$ and $\mu_0 \in [0, 1]$ such that

$$\tilde{F}(R_A, R|p_{KZ}) > \frac{\tau}{2} \cdot \frac{\lambda_0}{2 + \lambda_0(5 - \mu_0)}.$$

Proofs the parts a) and b) of this property is found in Oohama [7]. Proof of the part c) is given in Appendix [ ].
C. Sufficient Condition for Secure Transmission

In this subsection we find a sufficient condition for secure transmission, deriving an inner bound of \( R_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ}) \). We first derive an explicit upper bound of \( \Delta_{\text{max}}^{(n)}(\Phi^{(n)}, \varphi_A^{(n)}|p_X^{(n)}) \). By Lemma 2, it suffices to derive an upper bound of \( m \log |X| - H(\bar{K}^n|M_A^{(n)}) \). According to Santoso and Oohama, we have the following result.

Proposition 3: \( \exists \{\varphi^{(n)}\}_{n \geq 1} \) such that \( \forall \varphi^{(n)}_A \) satisfying \( ||\varphi_A^{(n)}|| \leq e^{\alpha R_A} \),

\[
m \log |X| - H(\bar{K}^n|M_A^{(n)}) \leq 5nRe^{-nF(R_A; R; p_{KZ})}.
\]

We have two remarks on this proposition.

Remark 3: Proposition 3 has a close connection with the exponential strong converse theorem for the one helper source coding problem established by Ahlswede et al. in [2].

Remark 4: A result similar to Proposition 3 is obtained by Watanabe and Oohama [3]. Let \( V^m \) be the uniform random vector over \( \Lambda^m \) and let \( \Delta^{(n)}(p_{K-M_A^{(n)}}, p_{M_A^{(n)}|X} \times p_{V^m}) \in [0, 1] \) be the normalized variational variational distance between \( p_{K-M_A^{(n)}}^{(n)} \) and \( p_{M_A^{(n)}|X}^{(n)} \times p_{V^m}^{(n)} \). Here \( \bar{K}^n = f^{(n)}(K^n) \) is an image of the map \( f^{(n)}: \Lambda^n \rightarrow \Lambda^n \). Watanabe and Oohama proved that there exists \( \forall \varphi^{(n)}_A \) such that \( \gamma \in [0, 1] \), \( \exists n_0 \neq 0, \gamma \geq n_0, \varphi^{(n)}_A \) satisfying \( ||\varphi^{(n)}_A|| \leq e^{\eta R_A}, \Delta^{(n)}(p_{K-M_A^{(n)}}, p_{M_A^{(n)}|X} \times p_{V^m}) \leq \gamma \). To obtain this result they use the strong converse theorem for the one helper source coding problem established by Ahlswede et al. in [2].

Combining Propositions 2, 3 and Lemma 2 we have the following result.

Theorem 1: \( \forall \gamma > 0, \forall R_A, \forall R > 0, \exists n_0 \neq 0, p_{KZ} \) with \( (R_A, R) \in R_{\text{AKW}}(p_{KZ}) \), \( \exists \{\varphi^{(n)}_A, \Psi^{(n)}_A\}_{n \geq 1} \) satisfying

\[
\log |X|^m = \left(\frac{m}{n}\right) \log |X| \in [R - (1/n), R]
\]

such that \( \forall p_X \) with \( R > H(X) \),

\[
\psi_0(\varphi^{(n)}_A, \varphi^{(n)}_A|p_X^{(n)}) \leq (n + 1)|X^m|e^{-nE_\psi(\psi^{(n)}_A|p_X^{(n)})},
\]

\[
\Delta^{(n)}(\Phi^{(n)}, R_A|p_X^{(n)}) \leq 5ne^{-nF(R_A; R; p_{KZ})}.
\]

Set \( \mathcal{R}(p_X, p_{KZ}) := \{R \geq H(X)\} \cup \{\varphi A(\bar{K}^n, \Phi^{(n)}, \varphi^{(n)}_A|p_X^{(n)})\}. \)

The functions \( E_\psi(R|p_X) \) and \( F(R_A; R, p_{KZ}) \) take positive values if and only if \( (R_A, R) \) belongs to the inner point of \( \mathcal{R}(p_X, p_{KZ}) \). By Theorem 1, under \( (R_A, R) \in \mathcal{R}(p_X, p_{KZ}) \), we have the following:

- On the reliability, \( \psi_0(\varphi^{(n)}_A, \varphi^{(n)}_A|p_X^{(n)}) \) goes to zero exponentially as \( n \) tends to infinity, and its exponent is lower bounded by the function \( E_\psi(E_\psi(p_X^{(n)})) \).

- On the security, \( \Delta^{(n)}(\Phi^{(n)}, R_A|p_X^{(n)}) \) goes to zero exponentially as \( n \) tends to infinity, and its exponent is lower bounded by the function \( F(R_A, R, p_{KZ}) \).

The code that attains the exponent functions \( E_\psi(R|p_X) \) is the universal code that depends only on \( R \) not on the value of the distribution \( p_X \).

From Theorem 1 we have the following corollary.

Corollary 1:

\[
\mathcal{R}(p_X, p_{KZ}) \subseteq \mathcal{R}_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ}) \subseteq \mathcal{R}_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ})
\]

A typical shape of the region \( \mathcal{R}(p_X, p_{KZ}) \) is shown in Fig. 5.

IV. STRONG CONVERSE THEOREM

We first derive one simple outer bound for source coding. By the strong converse coding theorem for source coding we have that if \( R < H(X) \) then \( \forall \gamma > 0, \forall \gamma > 0, \forall \{\phi^{(n)}, \psi^{(n)}\}_{n \geq 1}, \exists n_0 = n_0(\gamma) \in \mathbb{N}, \forall n \geq n_0, \varphi^{(n)}_A \) satisfying

\[
\frac{m}{n} \log |X|^m \leq R + \gamma, p_{\phi^{(n)}, \psi^{(n)}|p_X^{(n)}} \geq 1 - \gamma.
\]

Hence we have the following theorem.

Theorem 2: For each \( (\varepsilon, \delta) \in [0, \varepsilon_0] \times (0, 1) \), we have

\[
\mathcal{R}_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ}) \subseteq \{R \geq H(X)\}.
\]

We next prove that for some \( \varepsilon_0 > 0 \), the set \( \mathcal{R}(\varepsilon, \delta; p_X, p_{KZ}) \) serves as an outer bound of \( \mathcal{R}_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ}) \) for \( (\varepsilon, \delta) \in [0, \varepsilon_0] \times (0, 1) \). From the definition of the region \( \mathcal{R}_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ}) \) and Proposition 1, part b), we immediately obtain the following proposition.

Proposition 4:

If \( (R_A, R) \in \mathcal{R}_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ}) \), then we have that \( \forall \gamma > 0, \exists n_0(\gamma) \in \mathbb{N}, \forall n \geq n_0(\gamma), \forall \varphi_A \in \{\varphi^{(n)}_A\}_{n = 1}^\infty \),

\[
R_A \geq \frac{1}{n} \mathcal{R}(p_X, p_{KZ}) + \gamma + \frac{\varepsilon_0}{n},
\]

From this proposition we have the following theorem.

Theorem 3: For each \( (\varepsilon, \delta) \in [0, \varepsilon_0] \times (0, 1) \), we have

\[
\mathcal{R}_{\text{sys}}(\varepsilon, \delta; p_X, p_{KZ}) \subseteq \{R \geq H(X)\}.
\]

Proof of Theorem 3 is given in Appendix 3. This theorem can be proved by Proposition 4 and a method used in the direct part of the one helper helper source coding problem 5, 6.

Combining Corollary 1, Theorems 2 and 3, we obtain the following:
Theorem 4: For each \((\varepsilon, \delta) \in (0, \varepsilon_0] \times (0, 1)\), we have
\[ R(p_X, p_{PKZ}) = \{ R \geq H(X) \} \cap \{ R_{\text{AKW}}(p_{PKZ}) \} = R_{\text{sys}}(p_X, p_{PKZ}) = \mathcal{R}_{\text{sys}}(\varepsilon, \delta| p_X, p_{PKZ}). \]

APPENDIX

A. Proof of Property \[1\]
In this appendix we prove the property on the decoding set \( \mathcal{D}(n) \) stated in Property \[1\].

**Proof of Property \[1\]**
We have the following:
\[ \mathcal{D}(n) \ni x^n = \sum_{m=1}^{n} a_m x^m : x^m \in (\phi(n))(\mathcal{X}^n) \]
\[ \exists k_n(x^n) = \sum_{m=1}^{n} a_m x^m \in \mathcal{X}^n. \]

Step (a) follows from that every pair \( x^m \in (\phi(n))(\mathcal{X}^n) \) uniquely determines \( x^n \in \mathcal{D}(n) \). Step (b) follows from that \( \phi(n) \) are surjective. Since \( \psi(n) \) is a one-to-one mapping and \( x^n \neq y^n \), we have \( |\mathcal{D}(n)| = |\mathcal{X}^n| \).

B. Proof of Property \[2\]
We first prove the part a) and next prove the part b).

**Proof of Property \[2\] part a):** Under \( x^n, y^n \in \mathcal{D}(n) \) and \( x^n \neq y^n \), we assume that
\[ \Phi_{k_n}(x^n) = \Phi_{k_n}(y^n). \]

Then we have the following
\[ x^n = (\phi(n))(\phi(n))(k_n) \]
\[ = (\psi(n))(\Phi_{k_n}(x^n)) \]
\[ = (\psi(n))(\Phi_{k_n}(y^n)) \]
\[ = (\psi(n))(\phi(n))(y^n) \]
\[ = y^n. \]

Steps (a) and (e) follow from the definition of \( \mathcal{D}(n) \). Step (c) follows from (10). Steps (b) and (d) follow from the relationship between \( (\phi(n), \psi(n)) \) and \( (\Phi_{k_n}, \Phi_{k_n}) \). The equality \[11\] contradicts the first assumption. Hence we must have Property \[2\] part a).

**Proof of Property \[2\] part b):** We assume that \( \exists k_n \) and \( \exists x^n \) such that \( \forall x^n \in \mathcal{D}(n), \Phi_{k_n}(x^n) \neq c^n \). Set
\[ B := \{ \Phi_{k_n}(x^n) : x^n \in \mathcal{D}(n) \}. \]

Then by the above assumption we have
\[ B \subseteq \mathcal{X}^n \setminus \{ c^n \}. \]

On the other hand we have
\[ \Psi_{k_n}(B) = \{ \psi(n)(\Phi_{k_n}(x^n)) : x^n \in \mathcal{D}(n) \} \]
\[ = \{ \psi(n)(\phi(n))(x^n) : x^n \in \mathcal{D}(n) \} = \mathcal{D}(n), \]

which together with that \( \Psi_{k_n} : \mathcal{X}^n \rightarrow \mathcal{X}^n \) is a one-to-one mapping yields that
\[ |B| = |\Psi_{k_n}(B)| = |\mathcal{D}(n)| = |\mathcal{X}^n|. \]

The above equality contradicts (12). Hence we must have that \( \forall k_n, \forall x^n, \exists x^n \in \mathcal{D}(n) \) such that \( \Phi_{k_n}(x^n) = c^n \).

C. Proof of Lemma \[7\]
In this appendix we prove Lemma \[7\]. For \( x^n \in \mathcal{X}^n \), we set
\[ A_x(c^n) := \{ k_n : \Phi_{k_n}(c^n) = c^n \}. \]

**Proof of Lemma \[7\]** Property \[2\] part a) implies that
\[ A_x(c^n) \cap A_y(c^n) = \emptyset \text{ for } x^n \neq y^n \in \mathcal{D}(n). \]

Furthermore, Property \[2\] part b) implies that
\[ \bigcup_{x^n \in \mathcal{D}(n)} A_x(c^n) = \mathcal{X}^n. \]

From (15), for each \( (c^n, a) \in \mathcal{X}^n \times M^A_n \), we have the following chain of equalities:
\[ \sum_{x^n \in \mathcal{D}(n)} p_{C^n|M^A_n,X_n}(c^n|a, x^n) \]
\[ \overset{(a)}{=} \Pr \left\{ K^n \in \bigcup_{x^n \in \mathcal{D}(n)} A_x(c^n) \left| M^A_n = a \right. \right\} \]
\[ \overset{(b)}{=} 1. \]

Step (a) follows from (13). Step (b) follows from (14).

D. Proof of Proposition \[7\]
In this appendix we prove Proposition \[7\]. We first give some preliminary results for the proof. By the definition of \( A_x(c^n) \), we have that for each \( (c^n, a, x^n) \in \mathcal{X}^n \times M^A_n \times \mathcal{X}^n \)
\[ p_{C^n|M^A_n,X_n}(c^n|a, x^n) \]
\[ = \Pr \left\{ K^n \in A_x(c^n) \left| M^A_n = a \right. \right\} = \Pr \left\{ K^n \in A_x(c^n) \right\} \]
\[ = \Pr \left\{ K^n \in \bigcup_{x^n \in \mathcal{D}(n)} A_x(c^n) \right\} = 1. \]

Step (a) follows from (15). Step (b) follows from (15).

We first prove the part a). Using the quantities
\[ \Gamma_{K^n|M^A_n,x^n}(c^n|a, x^n) \]
\[ \in \mathcal{X}^n \times \mathcal{X}^n \times M^A_n. \]
components $p_{C^m|M^k}(c^m|a)$ of the stochastic matrix $p_{C^m|M^k}$ can be computed as
\[ p_{C^m|M^k}(c^m|a) = \sum_{x^n} p_X(x^n) \Gamma_{K^m,M^k}(x^n,c^m|a). \]

Set
\[ \Gamma_{K^m,M^k}(x^n,c^m|a) = \sum_{x^n} p_X(x^n) \Gamma_{K^m,M^k}(x^n,c^m|a) = p_{C^m|M^k}(c^m|a). \]

Furthermore, set
\[ K^m,M^k(c^m|a) := \left\{ \Gamma_{K^m,M^k}(x^n,c^m|a) \right\}_{(c^m,a) \in \mathcal{X}^m \times \mathcal{M}^k}. \]

Using $\Gamma_{K^m,M^k}(x^n,c^m|a) \in \mathcal{X}^m$ and $\Gamma_{K^m,M^k}(x^n,c^m|a)$, we compute $\Delta_{\text{MI}}^k$ to obtain
\[ \Delta_{\text{MI}}^k = I(C^m;\mathcal{X}^n|M^k) = \sum_{x^n \in \mathcal{X}^n} p_X(x^n) \Gamma_{K^m,M^k}(x^n) \times D \left( \Gamma_{K^m,M^k}(x^n) \right| \Gamma_{K^m,M^k}(x^n) \left| p_{M^k}(x^n) \right). \] (16)

We note that since $X^n$ is from the discrete memoryless source specified with $p_X$, we have that
\[ p_X(x^n) = \prod_{t=1}^n p_X(x_t) > 0, \forall x^n \in \mathcal{X}^n. \] (17)

Now we suppose that $\Delta_{\text{MI}}^k = 0$. Then from (16) and (17), we have
\[ \Gamma_{K^m,M^k}(x^n,c^m|a) = \Gamma_{K^m,M^k}(x^n,c^m|a) \in \mathcal{X}^m \times \mathcal{M}^k. \] (18)

Let $\overline{p}_{T^k}$ be the optimal random variable, the distribution $\overline{p}_{T^k}$ of which attains the maximum in the definition of $\Delta_{\text{MI}}^k$. Set $\overline{p}_{T^k} = \Phi_{K^k}(\overline{X}^n)$. By definition we have $\Delta_{\text{MI}}^k = I(C^m;\mathcal{X}^n|M^k)$. Using (18), we compute
\[ \Gamma_{K^m,M^k}(c^m|a) = \sum_{x^n \in \mathcal{X}^n} p_{T^k}(x^n) \Gamma_{K^m,M^k}(x^n,c^m|a) = \Gamma_{K^m,M^k}(x^n,c^m|a). \]

Hence we have
\[ \Gamma_{K^m,M^k}(x^n,c^m|a) = \Gamma_{K^m,M^k}(x^n,c^m|a), \forall x^n \in \mathcal{X}^n. \] (19)

From (19), we have
\[ \Delta_{\text{MI}}^k = I(C^m;\mathcal{X}^n|M^k) = \sum_{x^n \in \mathcal{X}^n} p_{T^k}(x^n) \Gamma_{K^m,M^k}(x^n,c^m|a) \times D \left( \Gamma_{K^m=M^k}(x^n) \left| \Gamma_{K^m=M^k}(x^n) \right| p_{M^k}(x^n) \right) = 0. \]

We next prove the part b). Let $\overline{X}^n$ be a uniformly distributed random variable over $\mathcal{D}^k$. Set $\overline{C}^m := \Phi_{K^m}(\overline{X}^n)$. The three random variables $\overline{X}^n$, $\overline{C}^m$, and $M^k$ are shown in Fig. 6 $\overline{C}^m$ is the uniformly distributed random variable over $\mathcal{M}^k$ and independent of $M^k$. In fact for each $(c^m, a) \in \mathcal{X}^m \times \mathcal{M}^k$, we have the following chain of equalities:
\[ p_{C^m|M^k}(c^m|a) = \sum_{x^n \in \mathcal{D}^k} p_{C^m|M^k}(c^m|a) \times \frac{1}{|\mathcal{X}^m|}. \] (20)

Step (a) follows from Lemma 10. Since we have (20) for every $(c^m, a) \in \mathcal{X}^m \times \mathcal{M}^k$, we have that $\overline{C}^m$ is the uniformly distributed random variable over $\mathcal{M}^k$ and independent of $M^k$. We have the following chain of inequalities:
\[ \Delta_{\text{MI}}^k \geq I(C^m;\mathcal{X}^n|M^k) = H(C^m|M^k) - H(C^m|M^k, \overline{X}^n) \leq m \log |\mathcal{X}| - H(\overline{C}^m|M^k, \overline{X}^n) \geq m \log |\mathcal{X}| - H(K^m|M^k, \overline{X}^n) \geq m \log |\mathcal{X}| - H(K^m|M^k). \] (21)

Step (a) follows from Lemma 10 part c). Step (b) follows from the data processing inequality.

E. Proof of Property 5 (part c)

Proof of Property 5 (part c): By simple computation we have that for any $\mu \in [0, 1]$ and any $p \in \mathcal{P}_{AB}(\mu K Z)$, we have the following:
\[ \lim_{\lambda \to 0} \frac{\Omega(\mu, \lambda)(p)}{\lambda} = \frac{d}{d\lambda} \Omega(\mu, \lambda)(p) |_{\lambda=0} = \mu I(U; Z) + \gamma H(K|U). \] (22)

By the hyperplane expression $\mathcal{R}_{AKW,AB}(\mu K Z)$ of $\mathcal{R}_{AKW}(\mu K Z)$ stated Property 5 (part b) we have that when $(R_A, \tau, R) \notin \mathcal{R}_{AKW}(\mu K Z)$, we have
\[ \mu_0 R_A + \frac{\gamma R}{\mu_0} < R(\mu_0)(\mu K Z) - \tau \] (23)
for some $\mu_0 \in [0, 1]$. We fix $p \in \mathcal{P}_{AB}(\mu K Z)$ arbitrary. By (21), there exists $\lambda_0 > 0$ such that
\[ \frac{\Omega(\mu_0, \lambda_0)(q)}{\lambda_0} \geq \mu_0 I(Z; U) + \frac{\gamma R}{\mu_0} H(K|U) - \tau \]

\[ \geq R(\mu_0)(\mu K Z) - \tau \geq \mu_0 R_A + \frac{\gamma R}{\mu_0} + \tau. \] (24)
Step (a) follows from (22). Since (23) holds for any $q \in \mathcal{P}_{sh}(p_{KZ})$, we have

$$\Omega^{(\mu_0, \lambda_0)}(p_{KZ}) \geq \lambda_0 \left[ \mu_0 R_A + \frac{p}{2} \right].$$

(24)

Then, we have the following chain of inequalities:

$$F(R_A, R|p_{KZ}) \geq \frac{\Omega^{(\mu_0, \lambda_0)}(p_{KZ}) - \lambda_0 (\mu_0 R_A + \frac{p}{2})}{1 + \lambda_0 (5 - \mu_0)}$$

(a) $\frac{\tau \lambda_0}{2} \lambda_0 (5 - \mu_0)$.

Step (a) follows from (24).

F. Proof of Theorem 3

Proof of Theorem 3 We assume that $(R_A, R) \in \mathcal{R}_{Sys}(\varepsilon, \delta|p_{X}, p_{KZ})$. Then by Proposition 4 we have that $\forall \gamma > 0, \exists \eta_0(\gamma), \forall n \geq \eta_0(\gamma)$, and $\forall \varphi_A = \{\varphi_A^{(n)}\}_{n=1}^{\infty}$,

$$R_A \geq \frac{1}{n} I(Z^n; \varphi_A^{(n)}(Z^n))$$

\begin{align*}
R & \leq \frac{1}{n} H(K^n|\varphi_A^{(n)}(Z^n)) + \gamma + \varepsilon/n
\end{align*}

(25)

Here we choose any $U$ such that $p = p_{UZK} \in \mathcal{P}(p_{KZ})$. Then by a method used in the proof of direct coding theorem for one helper source coding problem [5, 6], we can show that $\exists \varphi_A^{(n)}: Z^n \rightarrow M_A$ such that

$$\begin{aligned}
\frac{1}{n} I(Z^n; \varphi_A^{(n)}(Z^n)) & \geq I(Z; U) - \nu_{1,n}, \\
\frac{1}{n} H(K^n|\varphi_A^{(n)}(Z^n)) & \leq H(K|U) + \nu_{2,n},
\end{aligned}$$

(26)

where $\{\nu_{i,n}\}_{n=1}^{\infty}, i = 1, 2$ are some suitable sequences such that $\nu_i \rightarrow 0, n \rightarrow \infty$ for $i = 1, 2$. From (25) and (26), we have

$$\begin{aligned}
R_A & \geq I(Z; U) - \nu_{1,n}, \\
R & \leq H(K|U) + \nu_{2,n} + \gamma + \varepsilon/n
\end{aligned}$$

(27)

Letting $n \rightarrow \infty$ in (27), we have

$$R_A \geq I(Z; U), \ R \leq H(K|U) + \gamma.$$  

(28)

Since $\gamma > 0$ is arbitrary in (28), we obtain that $\forall U$ with $p_{UZK} \in \mathcal{P}(p_{KZ})$,

$$R_A \geq I(Z; U), \ R \leq H(K|U).$$  

(29)

Define

$$R_{min}(R_A) := \min_{U: p_{UZK} \in \mathcal{P}(p_{KZ}), \ I(Z; U) \leq R_A} H(K|U).$$

Then we have that

$$\forall U \text{ with } p_{UZK} \in \mathcal{P}(p_{KZ}), \text{ we have }$$

$$\iff \forall U \text{ with } p_{UZK} \in \mathcal{P}(p_{KZ}) \text{ and } I(Z; U) \leq R_A,$$

we have $R \leq H(K| U)$

$$\iff R \leq R_{min}(R_A) \iff (R_A, R) \in cl[\mathcal{R}_{AKW}(p_{KZ})],$$

completing the proof.

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