Model of stress-strain state of wooden rod under eccentric compression and transverse load

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Abstract. The analytical model of the stress-strain state of a wooden compressed-bent rod at short-term loading by forces with unequal eccentricities and at presence of transverse loading is offered. In the calculation model the real diagrams of work of wood on compression and stretching are laid. The approximation of the curved compression chart takes into account the elastic-plastic properties of the wood. The method allows to solve the problem of determination of the limiting bearing capacity both by the criterion of buckling and by criteria of durability. The theoretical results are confirmed by experimental data and can be used for practical calculations of compressed-bent elements.

1. Introduction
The share of wooden structures in the total volume of building constructions in Russia remains very insignificant. But at the same time there are some предпосылки for the development of wooden building. This is facilitated by the availability of the constantly renewable resource base, modern production technologies and some general tendencies in the building market connected with the decrease in the production cost. The current regulations in the design of wooden structures were introduced in the 1930s and have not altered significantly since that time. The computational models no longer reflect the real work and bearing capacity of structures as well as being non-economical.

Modern software systems allow to compile any engineering design based on the computational models which take into account the peculiarities of the construction material. One of the major ways to improve the methods of wooden structures calculating is accounting of the non-linear mechanical properties of wood.

In this paper, an analytical model of the stress-strain state is developed and the problems of stability of eccentrically compressed wooden rods are solved under the application of compressive forces with unequal terminal eccentricities and transverse loads. The proposed method of calculation allows more accurately than in the known solutions, to simulate the work of compressible-bending elements with virtually any, actually encountered in practice, the conditions for fixing the supporting ends and loads.

Presented in this paper, the research results are aimed at improving the methods of calculating compressible-bending wooden elements of building structures.

2. Problem statement and basic assumptions
The general theoretical approaches of analytical modeling of the stress-strain state of an eccentrically compressed rod are presented in [1] and [2].
A hingedly supported wooden rod, compressed by a longitudinal force $P$, applied with unequal eccentricities $e_1$ and $e_2$ and bendable by a transverse load $q$ (figure 1) is considered. The stress-strain state of the most loaded cross section of the rod is investigated.

Figure 1. The design-scheme of the compressible-bending rod taking into account the lateral load.

To simplify the solution, a number of general assumptions are made. The curvature of the element is determined by the approximate expression:

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}. \quad (1)$$

It is assumed that the curved axis of the rod is a half-wave sinusoid:

$$y = f \sin \frac{\pi x}{l}, \quad (2)$$

where $l$ – estimated rod length; $f$ – deflection (maximum displacement) when $x=l/2$.

The stress distribution over the cross section is determined by the distribution of strains in the cross section and the dependence $\sigma-\epsilon$. It is assumed that deformations are distributed according to a linear law and the axis of zero deformations coincides with the axis of zero stresses. The dependence $\sigma-\epsilon$ for any fiber corresponds to the experimental $\sigma-\epsilon$ diagram obtained from the results of tensile and compression wood tests.

To approximate the work of wood under compression, we use a cubic parabola of the form:
\[ \sigma = A_1 \varepsilon - A_2 \varepsilon^3. \]  

(3)

The first term of this expression corresponds to Hooke’s law and the coefficient \( A_1 \) is taken to be equal to the modulus of elasticity of wood under compression along the fibers. When \( \varepsilon = 0, \frac{d\sigma}{d\varepsilon} = A_1 = E \). The second term provides the \( \sigma - \varepsilon \) dependences for the decrease in the rate of increase of stresses with respect to the rate of increase in deformations and the transition beyond the tensile strength to a falling section. When \( \varepsilon = \varepsilon_{\text{um}}, \frac{d\sigma}{d\varepsilon} = 0 \).

The relationship between stresses and strains in the stretched zone of the section corresponds to Hooke's law:

\[ \sigma_r = E_r \varepsilon. \]  

(4)

In the process of deformation of the rod in the cross section there are compressed and stretched zones. The distribution of stresses and strains over the cross section of the rod is shown in figure 2.

**Figure 2.** The design scheme of stresses and strains in the cross section of the rod.

Using an approximate expression for the curvature and the representation of the curved axis of the rod by a sinusoid, one can write:

\[ \frac{1}{\rho} = \frac{\varepsilon_2 - \varepsilon_1}{h} = f \frac{\pi^2}{12}. \]  

(5)

From here we find the deflection of the middle section:

\[ f = \frac{l^2}{\pi^2 n} (\varepsilon_2 - \varepsilon_1). \]  

(6)

3. The determination of bearing capacity

The equilibrium equations are written as follows:

\[ P_{\text{in}} = P; \]  

(7)

\[ M_{\text{in}} = Pf - Q_2 \left( x_0 + L - \frac{l}{2} \right) - \frac{q}{2} \left( x_0 + L - \frac{l}{2} \right)^2, \]

where \( q \) – a lateral distributed load;

\[ Q_2 = \frac{p(\varepsilon_2 - \varepsilon_1) - \frac{ql^2}{2}}{l}. \]

For convenience, the following ratio is introduced:

\[ q = \frac{nP}{L}. \]  

(8)

where \( n \) is lateral load intensity.

Finally, in the equilibrium equation (7) we have:

\[ M_{\text{in}} = Pf - P \left( x_0 + L - \frac{l}{2} \right) \left( \frac{\varepsilon_2 - \varepsilon_1}{L} - \frac{n}{2} + \frac{n}{2L} (x_0 + L - \frac{l}{2}) \right). \]  

(9)
The main vector \( P_n \) and the main moment \( M_n \) are diagrams of normal stress:

\[
P_{nh} = b \left( \int_{h}^{\epsilon x_{1}} \sigma_c(x) dx - \int_{0}^{\epsilon x_{2}-\epsilon_{1}} \sigma_p(x) dx \right);
\]

\[
M_{nh} = b \left( \int_{h}^{\epsilon x_{1}} \sigma_c(x)(x - \frac{h}{2}) dx + \int_{0}^{\epsilon x_{2}-\epsilon_{1}} \sigma_p(x)(\frac{h}{2} - x) dx \right).
\]

Or, taking into account (3) and (4):

\[
P_{nh} = b \left( \int_{0}^{\epsilon x_{2}}(A_{1}\epsilon - A_{2}\epsilon^{3}) \frac{h}{\epsilon x_{2} - \epsilon_{1}} d\epsilon - \int_{\epsilon x_{1}}^{0} E_p \epsilon \frac{h}{\epsilon x_{2} - \epsilon_{1}} d\epsilon \right);
\]

\[
M_{nh} = b \left( \int_{0}^{\epsilon x_{2}}(A_{1}\epsilon - A_{2}\epsilon^{3}) \left( \frac{\epsilon - \epsilon_{1}h}{\epsilon x_{2} - \epsilon_{1}} - \frac{h}{2} \right) d\epsilon + \int_{\epsilon x_{1}}^{0} E_p \epsilon \left( \frac{h}{2} - (\epsilon - \epsilon_{1}h) \right) \frac{h}{\epsilon x_{2} - \epsilon_{1}} d\epsilon \right).
\]

After integration and mathematical transformations, we finally have:

\[
P_{nh} = \frac{bh}{\epsilon x_{2} - \epsilon_{1}} \left( 0.5A_{1}\epsilon_{2}^{2} - 0.25A_{2}\epsilon_{2}^{4} - 0.5E_p \epsilon_{2}^{2} \right);
\]

\[
M_{nh} = \frac{bh^{2}}{(\epsilon x_{2} - \epsilon_{1})^{2}} \left( \frac{A_{1}}{12}(\epsilon_{3}^{2} - 3\epsilon_{2}^{2}\epsilon_{1}) - \frac{A_{2}}{48}(3\epsilon_{2}^{2} - 5\epsilon_{2}^{2}\epsilon_{1}) - \frac{E_p}{12}\epsilon_{1}(\epsilon_{3}^{2} - 3\epsilon_{2}^{2}\epsilon_{2}) \right).
\]

The compressive force applied to the ends of the rod with unequal eccentricities \( \epsilon_{1} \) and \( \epsilon_{2} \) (picture 1):

\[
e_{1} = f \sin \frac{\pi x_{0}}{l}; \quad e_{2} = f \sin \frac{\pi (x_{0}+l)}{l}.
\]

From conditions (13) we have the equations of communication:

\[
\Phi_{1}(\epsilon_{1}, \epsilon_{2}, l, x_{0}) = 0;
\]

\[
\Phi_{2}(\epsilon_{1}, \epsilon_{2}, l, x_{0}) = 0.
\]

Let the load on the rod increase in time according to a linear law:

\[
P = P_{n} + \alpha t.
\]

where \( P_{n} \) - initial load value; \( \alpha \) - a linear load rise parameter; \( t \) - time.

Differentiating equilibrium equations (7) and equations (14), we obtain a system of differential equations of motion:

\[
\left\{ \begin{array}{l}
\frac{\partial P_{nh}}{\partial \epsilon_{1}} \dot{\epsilon}_{1} + \frac{\partial P_{nh}}{\partial \epsilon_{2}} \dot{\epsilon}_{2} = \alpha; \\
\frac{\partial M_{nh}}{\partial \epsilon_{1}} \dot{\epsilon}_{1} + \frac{\partial M_{nh}}{\partial \epsilon_{2}} \dot{\epsilon}_{2} = k_{1} \dot{l} + k_{2} \dot{x}_{0} = k_{3};
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\frac{\partial P_{nh}}{\partial \epsilon_{1}} \dot{\epsilon}_{1} + \frac{\partial P_{nh}}{\partial \epsilon_{2}} \dot{\epsilon}_{2} = \frac{\partial P_{nh}}{\partial \epsilon_{1}} \dot{\epsilon}_{1} + \frac{\partial P_{nh}}{\partial \epsilon_{2}} \dot{\epsilon}_{2} = 0; \\
\frac{\partial M_{nh}}{\partial \epsilon_{1}} \dot{\epsilon}_{1} + \frac{\partial M_{nh}}{\partial \epsilon_{2}} \dot{\epsilon}_{2} = \frac{\partial M_{nh}}{\partial \epsilon_{1}} \dot{\epsilon}_{1} + \frac{\partial M_{nh}}{\partial \epsilon_{2}} \dot{\epsilon}_{2} = 0;
\end{array} \right.
\]

where

\[
\frac{\partial P_{nh}}{\partial \epsilon_{1}} = bh \left( \frac{0.5A_{1}\epsilon_{2}^{2} - 0.25A_{2}\epsilon_{2}^{4} + E_{p}(0.5\epsilon_{1}^{2} - \epsilon_{2}^{2}))}{(\epsilon_{2} - \epsilon_{1})^{2}} \right);
\]

\[
\frac{\partial P_{nh}}{\partial \epsilon_{2}} = bh \left( A_{1}(0.5\epsilon_{2}^{2} - \epsilon_{2}^{4}) - A_{2}(0.75\epsilon_{2}^{4} - 3\epsilon_{2}^{2}\epsilon_{1}) + 0.5E_{p}\epsilon_{2}^{5} \right);
\]

\[
\frac{\partial M_{nh}}{\partial \epsilon_{1}} = bh \left( \frac{A_{1}(-\epsilon_{2}^{2} + 3\epsilon_{2}^{2}\epsilon_{1}) - A_{2}(6\epsilon_{2}^{4} - 5\epsilon_{2}^{4}\epsilon_{1}) - E_{p}(3\epsilon_{2}^{2}\epsilon_{2} - 6\epsilon_{2}^{2}\epsilon_{1})}{(\epsilon_{2} - \epsilon_{1})^{2}} \right);
\]

\[
\frac{\partial M_{nh}}{\partial \epsilon_{2}} = bh \left( \frac{A_{1}(3\epsilon_{2}^{3} - 3\epsilon_{2}^{4}\epsilon_{1} + 6\epsilon_{2}^{2}\epsilon_{1}) - A_{2}(15\epsilon_{2}^{4} - 15\epsilon_{2}^{4}\epsilon_{1} + 25\epsilon_{2}^{2}\epsilon_{1}) - E_{p}(3\epsilon_{2}^{3}\epsilon_{2} - 6\epsilon_{2}^{3}\epsilon_{1})}{(\epsilon_{2} - \epsilon_{1})^{3}} \right);
\]

\[
k_{1} = -p \left( \frac{2l}{\pi^{2}h}(\epsilon_{2} - \epsilon_{1}) + \frac{e_{2} - e_{1}}{2l} \right) - \frac{p}{2} \left( \frac{e_{2} - e_{1}}{l} \right) \left( x_{0} + L - \frac{l}{2} \right);
\]

\[
k_{2} = p \left( \frac{e_{2} - e_{1}}{l} \right) \left( x_{0} + L - \frac{l}{2} \right) \left( x_{0} + L - \frac{l}{2} \right);
\]

\[
k_{3} = \alpha \left( x_{0} + L - \frac{l}{2} \right) \left( \frac{e_{2} - e_{1}}{l} \right) \left( x_{0} + L - \frac{l}{2} \right);
\]

\[
\frac{\partial \Phi_{1}}{\partial \epsilon_{1}} = \frac{p}{\pi^{2}h} \sin \frac{\pi x_{0}}{l}.
\]
\[
\begin{align*}
\frac{\partial \phi_1}{\partial \varepsilon_1} &= \frac{i^2}{\pi^2} \sin \frac{\pi x_0}{l}, \\
\frac{\partial \phi_2}{\partial \varepsilon_2} &= \frac{i^2}{\pi^2} \sin \frac{\pi (x_0 + l)}{l}, \\
\frac{\partial \phi_1}{\partial l} &= \frac{\varepsilon_2 - \varepsilon_1}{h} \left( \frac{2i}{\pi^2} \sin \frac{\pi x_0}{l} - \frac{x_0}{\pi} \cos \frac{\pi x_0}{l} \right), \\
\frac{\partial \phi_2}{\partial l} &= \frac{\varepsilon_2 - \varepsilon_1}{h} \left( \frac{2i}{\pi^2} \sin \frac{\pi (x_0 + l)}{l} - \frac{x_0 + l}{\pi} \cos \frac{\pi (x_0 + l)}{l} \right), \\
\frac{\partial \phi_1}{\partial x_0} &= \frac{i^2}{\pi^2} \sin \frac{\pi x_0}{l} \cos \frac{\pi x_0}{l}, \\
\frac{\partial \phi_2}{\partial x_0} &= \frac{i^2}{\pi^2} \sin \frac{\pi (x_0 + l)}{l} \cos \frac{\pi (x_0 + l)}{l}.
\end{align*}
\]

Solving the system (16) is a Cauchy problem with initial conditions obtained from the joint solution of nonlinear equations (7) and (13) for \(\varepsilon_1, \varepsilon_2, l, x_0\). The numerical integration of differential equations can be done using the Runge-Kutta method.

System (16) is linear with respect to derivatives \(\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{l}, \dot{x}_0\). In a matrix form, it looks:

\[
A \vec{X} = \vec{B},
\]

where \(A\) is a coefficient matrix; \(\vec{X}\) a column matrix (vector) of unknowns; \(\vec{B}\) a free column matrix.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}, \quad
\vec{X} = \begin{bmatrix}
\dot{\varepsilon}_1 \\
\dot{\varepsilon}_2 \\
\dot{l} \\
\dot{x}_0
\end{bmatrix}, \quad
\vec{B} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}.
\]

The roots of the system are expressed by Kramer’s formulas:

\[
\dot{\varepsilon}_1 = \frac{\Delta \varepsilon_1}{\Delta}, \quad \dot{\varepsilon}_2 = \frac{\Delta \varepsilon_2}{\Delta}, \quad \dot{l} = \frac{\Delta l}{\Delta}, \quad \dot{x}_0 = \frac{\Delta x_0}{\Delta},
\]

where \(\Delta, \Delta \varepsilon_1, \Delta \varepsilon_2, \Delta l, \Delta x_0\) are system identifiers.

To find the critical state condition, we use the criterion proposed by R.S. Sanzharovsky [3]:

\[
\frac{\delta M}{\delta \varepsilon_1} = \frac{\delta M}{\delta \varepsilon_2} = \frac{\delta M}{\delta l} = \frac{\delta M}{\delta x_0} = 0.
\]

The condition for stability loss is the determinant of the system (16) being equal to zero and composed of coefficients with variations.

The solution of the system (16) allows the evaluation of the stress-strain state of the wooden rod during any short-term loading.

4. Conclusion

The proposed analytical model provides information on the stress distribution over the cross section and on the deflection in the process of loading a wooden rod. Introduction to the calculation of the relevant criteria (limit deflection, ultimate strength) allows us to perform checks on both the first and the second group of limit states. In addition, for practical use, the buckling criterion is introduced into the design model. The reliability of the theoretical results of the proposed calculation method was confirmed by experimental studies [1, 2].

The developed technique is substantially closer to the actual work of the element than the normative method. By means of using a curvilinear stress-strain compression diagram, the physical non-linearity of the material is taken into account. Thus, the developed model of the stress-strain state of a wooden rod under off-center compression and lateral load can be proposed for introduction into computational practice of compressible-bending wooden elements.

References

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