Abstract

In Communications in Mathematical Physics, no. 199, (1998), we have considered the Heun operator \( H = a^* (a + a^*) a \) acting on Bargmann space where \( a \) and \( a^* \) are the standard Bose annihilation and creation operators satisfying the commutation relation \([a, a^*] = I\).

We have used the boundary conditions at infinity to give a description of all maximal dissipative extensions in Bargmann space of the minimal Heun’s operator \( H \). The characteristic functions of the dissipative extensions have been computed and some completeness theorems have been obtained for the system of generalized eigenvectors of this operator.

In this paper we study the deficiency numbers of the generalized Heun’s operator \( H^{p,m} = a^* (a^m + a^{*m}) a^p; (p, m = 1, 2, \ldots) \) acting on Bargmann space. In particular, here we find some conditions on the parameters \( p \) and \( m \) for that \( H^{p,m} \) to be completely indeterminate. It follows from these conditions that \( H^{p,m} \) is entire of the type minimal. And we show that \( H^{p,m} \) and \( H^{p,m} + H^{p,m*} \) (where \( H^{p,m*} \) is the adjoint of the \( H^{p,m} \)) are connected at the chaotic operators. We will give a description of all maximal dissipative extensions and all selfadjoint extensions of the minimal generalized Heun’s operator \( H^{p,m} \) acting on Bargmann space in separate paper.

Keywords: Weighted shift unbounded operators; Heun’operator; entire operators, chaotic operators; Bargmann space; Reggeon field theory.
1. Introduction and preliminaries results

Let \( \mathcal{B} \) be the Bargmann space \([3]\) defined as a subspace of the space \( O(\mathbb{C}) \) of holomorphic functions on \( \mathbb{C} \), given by

\[
\mathcal{B} = \{ \phi \in O(\mathbb{C}); <\phi,\phi> < \infty \} \tag{1.1}
\]

where the paring

\[
<\phi,\psi> = \int_{\mathbb{C}} \overline{\phi(z)} \psi(z) e^{-|z|^2} dx dy \quad \forall \, \phi, \psi \in O(\mathbb{C}) \tag{1.2}
\]

and \( dx dy \) is Lebesgue measure on \( \mathbb{C} \).

This space with \( ||\phi|| = \sqrt{<\phi,\phi>} \) is a Hilbert space and \( e_k(z) = \frac{z^k}{\sqrt{k!}}; k = 0, 1, \ldots \) is complete orthonormal basis of \( \mathcal{B} \).

In this representation, the standard Bose annihilation and creation operators are defined by

\[
\begin{align*}
\left\{ \begin{array}{l}
a\phi(z) = \phi'(z) \\
\text{with maximal domain} \\
D(a) = \{ \phi \in \mathcal{B} \text{ such that } a\phi \in \mathcal{B} \}
\end{array} \right. \tag{1.3}
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
a^*\phi(z) = z\phi(z) \\
\text{with maximal domain} \\
D(a^*) = \{ \phi \in \mathcal{B} \text{ such that } a^*\phi \in \mathcal{B} \}
\end{array} \right. \tag{1.4}
\end{align*}
\]

Accordingly, for the operator \( H^{p,m} = a^{*p}(a^m + a^{s*m})a^p; (p, m = 1, 2, \ldots) \) we have

\[
\begin{align*}
\left\{ \begin{array}{l}
H^{p,m}\phi(z) = z^p\phi^{(p+m)}(z) + z^{p+m}\phi^{(p)}(z) \\
\text{with maximal domain} \\
D(H^{p,m}_{\text{max}}) = \{ \phi \in \mathcal{B}; H^{p,m}\phi \in \mathcal{B} \}
\end{array} \right. \tag{1.5}
\end{align*}
\]

It follows from (1.3) and (1.4) that the action of the operator \( a^{*r}a^s \)

\((r \in \mathbb{N}, s \in \mathbb{N})\) on an element \( \phi \in \mathcal{B}; \phi(z) = \sum_{k=0}^{\infty} a_k e_k(z) \) is given by:

\[
a^{*r}a^s\phi(z) = \sum_{k=0}^{\infty} k(k-1)\ldots(k-s+1) a_k \overline{z}^{k+r-s} \sqrt{k!} \tag{1.6}
\]

\[
= \sum_{k=0}^{\infty} (k+s-r)(k+s-r-1)\ldots(k-r+1) a_{k+s-r} \overline{z}^k \sqrt{(k+s-r)!} \tag{1.7}
\]

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If $s \geq r$, we have $\sqrt{(k + s - r)!} = \sqrt{k!} \sqrt{(k + 1)!} \ldots \sqrt{(k + s - r)!}$ this implies that

$$a^* \ast a \phi(z) = \sum_{k=0}^{\infty} \{ \sqrt{(k + s - r)} \sqrt{(k + s - r - 1)} \ldots \sqrt{(k + 1)} \ldots (k - r + 1) a_{k + s - r} \} e_k(z)$$

Let $u_k = \sqrt{(k + s - r)} \sqrt{(k + s - r - 1)} \ldots \sqrt{(k + 1)} \ldots (k - r + 1)$ then we give the below obvious lemma that we will use in the following of this paper

**Lemma 1.1**

i) if $s \geq r$ then $u_k \sim k^{s+r}$

ii) Also, if $s \leq r$ then $u_k \sim k^{s+r}$

iii) For $r = p$ and $s = p + m$ then $u_k \sim k^{-\frac{2p+m}{2}}$

**Proof**

As $u_k \sim k^{s+r} k^r$ then $u_k \sim k^{s+r}$. In a similar manner, we obtain $u_k$ in ii) when $s \leq r$ and iii) is obvious particular case of i). The proof of Lemma 1.1 is complete.

Now the action of the operator $H_{p,m}$ on an element $\phi \in \mathbb{B}$ is given by:

$$H_{p,m} \phi(z) = \sum_{k=0}^{\infty} k(k - 1) \ldots (k - p - m + 1) a_k \frac{z^{k-m}}{\sqrt{k!}} +$$

$$\sum_{k=0}^{\infty} k(k - 1) \ldots (k - p + 1) a_k \frac{z^{k+m}}{\sqrt{k!}}$$

As $H_{p,m}$ is polynomial in $(a^*, a)$ of degree $2p + m$, we give the next lemma for study in separate paper an perturbation of $H_{p,m}$ by selfadjoint operators of the form $(a^*j a^j)$ with $j > p + \frac{m}{2}$, in particular we will show the non chaoticity of this perturbation and we will give a complete spectral analysis for the genre of these operators.

**Lemma 1.2**

For all $j > p + \frac{m}{2}$ the following statement holds:

$$\forall \ \epsilon > 0 \ \exists \ C_\epsilon > 0 \text{ such that}$$

$$\forall$$
∀ φ ∈ D(a^{2j}), |< H^{p,m} φ, φ >| ≤ ε || φ ||^2 + C_ε || φ ||^2.

Proof

Let φ ∈ D(a^{2j}) and u_k = \sqrt{(k + m)\sqrt{(k + m - 1)\ldots\sqrt{(k + 1)\ldots(k - p + 1)}}} then

\[ a^* a^{p+m} φ, φ > = \sum_{k=0}^{∞} u_k a_{k+m} \bar{a}_k \]

\[ \leq \sum_{k=0}^{∞} u_k \ | a_{k+m} \ | \bar{a}_k | \]

\[ \leq \frac{1}{2} \sum_{k=0}^{∞} u_k \ | a_{k+m} |^2 + \frac{1}{2} \sum_{k=0}^{∞} u_k \ | a_k |^2 \]

\[ \leq \frac{1}{2} \sum_{k=0}^{∞} (u_{k-m} + u_k) \ | a_k |^2 \]

By virtue of iii) of lemma 1.1, we have u_k \sim k^{2p+m} then there exist c_0 > 0 and c_1 > 0 such that u_k \leq c_0 + c_1 k^{2p+m}

Now, for j > p + \frac{m}{2} we apply the Young’s inequality to get

∀ δ > 0 \exists c_δ > 0; k^{2p+m} \leq δ k^j + c_δ

this implies that

\[ a^* a^{p+m} φ, φ > \leq c_1 δ \sum_{k=0}^{∞} k^j \ | a_k |^2 + (c_δ + c_0) \sum_{k=0}^{∞} | a_k |^2 \]

and

∀ ε > 0 \exists C_ε > 0 such that ∀ φ ∈ D(a^{2j}),

\[ |< a^* a^{p+m} φ, φ >| \leq ε |< a^* a^{p+m} φ, φ >| + C_ε || φ ||^2. \]

As < a^{p+m} a^p φ, φ >= < φ, a^{p+m} a^p φ > then we get

∀ ε > 0 \exists C_ε > 0 such that ∀ φ ∈ D(a^{2j}),

\[ |< H^{p,m} φ, φ >| \leq ε || a^p φ ||^2 + C_ε || φ ||^2. \]
The proof of the Lemma 1.2 is complete. \diamond

Now, the differential operations $a^*$ and $a$ act on the functions $e_k$ according to the formulas

$$a^*e_k = \sqrt{k+1}e_{k+1}, \quad ae_k = \sqrt{k}e_{k-1}; \quad e_{-1} = 0, k = 0, 1, \ldots \quad (1.6)$$

It follows from (1.6) that

$$H^{p,m}e_k = \begin{cases} \sqrt{k!(k+m)!} \frac{(k+p)!}{(k-p)!} e_{k+m} & \text{if } p \leq k < p + m \\ \frac{k!}{(k-p)!} e_{k-m} & \text{if } k \geq p + m \end{cases} \quad (1.7)$$

and

$$H^{p,m}e_k = \frac{k!(k-m)!}{(k-p-m)!} e_{k-m} + \frac{k!(k+m)!}{(k-p)!} e_{k+m} \quad (1.8)$$

Thus from (1.7), if we denote $\mathbb{B}_p = \{ \phi \in \mathbb{B}; \phi(0) = \phi'(0) = \ldots = \phi^{(p-1)}(0) = 0 \}$ then this space is generated by $\{e_p, e_{p+1}, \ldots\}$ and the matrix representation of the minimal operator $H^{p,m}$ in the basis $e_k; k = p, p + 1, \ldots$ is given by the symmetric Jacobi matrix $H$ which has only two nonzero diagonals. Namely, its numerical entries are the matrices $H_{ij}$ of order $m$ defined by:

$$H_{i,j} = H_{i,j} = O \quad \text{if } |i-j| > 1 \quad (i,j = 1, 2, \ldots)$$

where $O$ is the zero $m \times m$ matrix

and

$$H_{i+1,i} = H_{i,i+1} \quad \text{where } H_{i,i+1} \text{ is diagonal } m \times m \text{ matrix} \quad (1.9)$$

such that its numerical entries are

$$\beta_k^i = \frac{k!(k+m)!}{(k-p)!}; \quad (i-1)m + 1 \leq k \leq im$$

Let $A_i$ and $B_i = B_i^* \ (i = 1, 2, \ldots)$ be $m \times m$ matrices whose entries are complex numbers then the matrix (1.9) is a particular case of the infinite matrix whose general form
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\[ J = \begin{pmatrix} A_1 & B_1 & O & \cdots & \cdots & \cdots \\ B_1^* & A_2 & B_2 & \cdots & \cdots & \cdots \\ O & B_2^* & A_3 & B_3 & \cdots & \cdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \\ & & & & O & \cdots \end{pmatrix} \]  \quad (1.10)

where \( O \) is the zero \( m \times m \) matrix and the asterisk denotes the adjoint matrix.

Let \( l^2_m(\mathbb{N}) \) be the Hilbert space of infinite sequences \( \phi = (\phi_1, \phi_2, \ldots, \phi_i, \ldots) \) with the inner product \( \langle \phi, \psi \rangle = \sum_{i=1}^{\infty} \phi_i \overline{\psi_i} \)

where \( \phi_i = (\phi_{1i}, \phi_{2i}, \ldots, \phi_{mi}) \in \mathbb{C}^m \) and \( \phi_i \overline{\psi_i} = \sum_{j=1}^{m} \phi_{ij} \overline{\psi_{ij}} \)

The matrix \( J \) defines a symmetric operator \( \mathfrak{T} \) in \( l^2_m(\mathbb{N}) \) according to the formula

\[ (\mathfrak{T}\phi)_i = B_{i-1}\phi_{i-1} + A_i\phi_i + B_i\phi_{i+1}, \quad i = 1, 2, \ldots \]  \quad (1.11)

where \( \phi_0 = (\phi_{10}, \phi_{20}, \ldots, \phi_{m0}) = (0, 0, \ldots, 0) \)

then for our operator, we have:

\[ (H\phi)_i = H_{i,i-1}\phi_{i-1} + H_{i,i+1}\phi_{i+1}, \quad i = 1, 2, \ldots \]  \quad (1.12)

where \( \phi_0 = (\phi_{10}, \phi_{20}, \ldots, \phi_{m0}) = (0, 0, \ldots, 0) \)

**Remark 1.3**

1) The closure \( \mathfrak{T} \) with domain \( D(\mathfrak{T}) \) of the operator \( \mathfrak{T} \) is the minimal closed symmetric operator generated by the expression (1.11) and the boundary condition \( \phi_0 = 0 \)
2) According to Berezanskii, by Chap VII, [4], it is well known that the deficiency numbers \( n_+ \) and \( n_- \) of the operator \( \mathbb{T} \) satisfy the inequalities 
\[
0 \leq n_+ \leq m \quad \text{and} \quad 0 \leq n_- \leq m
\]
where \( n_+ \) is the dimension of \( \mathfrak{M}_z = (\mathbb{T} - zI)D(\mathbb{T}); Imz \neq 0 \) and \( n_- \) is the dimension of the eigensubspace \( \mathfrak{N}_z \) corresponding to the eigenvalue \( \tau \) of the operator \( \mathbb{T} \).

3) According to Krein [23] that the operator \( \mathbb{T} \) is said completely indeterminate if \( n_+ = n_- = m \) and to Kostyuchenko-Mirsoev [22] that the completely indeterminate case holds for the operator \( \mathbb{T} \) if and only if all solutions of the vector equation
\[
(\Sigma \phi)_i = z\phi_i \quad i = 1, 2, ....
\]
for \( z = 0 \) belongs to \( l^2_m(\mathbb{N}) \)

In 1949 Krein developed the theory of entire operators with arbitrary finite defect numbers, we refer to [4, 8, 9, 10, 12, 22, 23, 27] and the references therein which are closely connected with this theory. In section 2, we give some properties associated to this theory for the generalized Heun’s operator \( H^{p,m} = a^* a p^m(a^m + a^{*m})a^p \), in particular its completely indeterminacy in Bargmann space.

In section 3, we show that the operators \( \tilde{H} = a^* a^{p+m}U \) and \( \tilde{H} + \tilde{H}^* \); (\( p = 1, 2, ...., m = 1, 2, .... \)). are chaotic where \( U^*e_k = e_{k+1} \) and \( \tilde{H}^p, \ U^* \) are respectively the adjoint of \( \tilde{H} \) and of \( U \).

\section{2. On the completely indeterminacy of generalized Heun operator in Bargmann space}

In [22], Kostyuchenko and Mirzoev gave some tests for the complete indeterminacy of a Jacobi matrix \( \mathfrak{J} \) in terms of entries \( A_i \) and \( B_i \) of that matrix. In the following, we give two lemmas which permit us to show the complete indeterminacy of generalized Heun operator in Bargmann space.

For \( m = 1, 2, .... \), let \( \mathbb{C}^m \) be the euclidean \( m \)-dimensional space and \( B_i = H_{i,i+1} \) be the diagonal \( m \times m \) matrix such that its numerical entries are
\[
\beta^i_k = \frac{\sqrt{k!(k + m)!}}{(k - p)!}; \quad (i - 1)m + 1 \leq k \leq im(i = 1, 2, ....).
\]
By $||\cdot||$ we denote the spectral matrix norm, then we have:

1) $||B_i|| = \sqrt{(im)!(i+1)!/ (im-p)!} \sim i^{p+m/2}$

2) $||B^{-1}_i|| = \frac{1}{||B_i||}$

**Lemma 2.1**

Let $B_i = H_{i,i+1}, (i = 1, 2, \ldots)$ be the diagonal $m \times m$ matrix $(m = 1, 2, \ldots)$ such that its numerical entries are 

$$
\beta^i_k = \frac{k!(k+m)!}{(k-p)!}; \quad (i-1)m+1 \leq k \leq im(i = 1, 2, \ldots).
$$

then the following inequality holds

$$
||B_{i-1}|| || B_{i+1}|| \leq \frac{1}{||B^{-1}_i||^2}
$$

holds starting from some $i \geq m$

**Proof**

By using the lemma 1.1 or the behavior of Gamma function $\Gamma(x)$ as $\mathcal{R}ex \rightarrow +\infty$ given by Stirling’s formula $\Gamma(x) \sim \sqrt{2\pi e^{-x}x^{x-1/2}}$, we deduce that

$$
\beta^i_k \sim k^{p+m/2} \text{ as } k \rightarrow +\infty.
$$

As $||B_i|| = \beta^i_{im} = \sqrt{(im)!(m(i+1)!/(im-p)!} \sim (im)^{p+m/2} = (i)^{p+m/2}(m)^{p+m/2}$

then $||B_{i-1}|| \sim (i-1)^{p+m/2}(m)^{p+m/2}$, $||B_{i+1}|| \sim (i+1)^{p+m/2}(m)^{p+m/2}$ and

$$
||B_{i-1}|| || B_{i+1}|| \sim (i)^{2p+m}(1 - \frac{1}{i^2})^{p+m/2}(m)^{2p+m}
$$

Now as

$$
(1 - \frac{1}{i^2})^{p+m/2} \leq 1
$$

then

$$
||B_{i-1}|| || B_{i+1}|| \leq ||B_i||^2 \text{ and as } ||B^{-1}_i|| = \frac{1}{||B_i||} \text{ then (1.16) holds.}
$$
Lemma 2.2

Let \( B_i = H_{i,i+1}, (i = 1, 2,...) \) be the diagonal \( m \times m \) matrix , \( (m = 1, 2,...) \) such that its numerical entries are

\[
\beta^i_k = \sqrt{k!(k + m)!} \frac{1}{(k - p)!}; \quad (i - 1)m + 1 \leq k \leq im
\]

then if \( 2p + m > 2 \) the following inequality holds

\[
\sum_{i=1}^{+\infty} \frac{1}{\| B_i \|} < +\infty
\]

(1.17)

Proof

As \( \| B_i \| \sim i^{p+\frac{m}{2}} \) then if \( p + \frac{m}{2} > 1 \), the serie \( \sum_{i=1}^{+\infty} \frac{1}{i^{p+\frac{m}{2}}} < +\infty \), it follows that (1.17) holds.

\[\diamondsuit\]

Now, we prove the following theorem

Theorem 2.3

If \( p + \frac{m}{2} > 1 \) then the operator \( \mathbb{H} \) is completely indeterminate and its deficient numbers satisfy the conditions \( n_+ = n_- = m \).

Proof

By applying the results of Kostyuchenko and Mirsoev [22] to our operator then the completely indeterminate case holds for the operator \( \mathbb{H} \) if and only if all solutions of the vector equation

\[
B_{i-1} \phi_{i-1} + B_i \phi_{i+1} = \lambda \phi_i \quad (i = 1, 2, ....)
\]

for \( \lambda = 0 \) belongs to \( l^2_m(\mathbb{N}) \)

where \( B_i = H_{i,i+1} \) is the diagonal \( m \times m \) matrix such that its numerical entries are given by

\[
\beta^i_k = \sqrt{k!(k + m)!} \frac{1}{(k - p)!}; \quad (i - 1)m + 1 \leq k \leq im
\]

Now from (1.12) we consider the system

\[
B_{i-1} \phi_{i-1} + B_i \phi_{i+1} = 0 \quad (i = 1, 2, ....)
\]
where \( \phi_0 = (\phi_0^1, \phi_0^2, \ldots, \phi_0^m) = (0, 0, \ldots, 0) \)

As \( B_i^{-1}, \quad (i = 1, 2, \ldots) \) exist we deduce that the solutions of the above equation have the following explicit form

If \( i = 2j, \quad (j = 1, 2, \ldots) \) we have

\[
\phi_{2j} = 0 \quad \text{and} \quad \phi_{2j+1} = -(1)^j B_{2j}^{-1} B_{2j-1} \times B_{2j-2}^{-1} B_{2j-3} \ldots \times B_2^{-1} B_1 \phi_1
\]

This solution belongs to \( l_2(\mathbb{N}) \) if

\[
\sum_{j=1}^{+\infty} || B_{2j}^{-1} B_{2j-1} \times B_{2j-2}^{-1} B_{2j-3} \ldots \times B_2^{-1} B_1 ||^2 < +\infty
\]

and

if \( i = 2j - 1, \quad (j = 1, 2, \ldots) \) we have

\[
\phi_{2j-1} = 0 \quad \text{and} \quad \phi_{2j} = -(1)^j B_{2j-1}^{-1} B_{2j-2} \times B_{2j-3}^{-1} B_{2j-4} \ldots \times B_3^{-1} B_2 \phi_2
\]

This solution belongs to \( l_2(\mathbb{N}) \) if

\[
\sum_{j=1}^{+\infty} || B_{2j-1}^{-1} B_{2j-2} \times B_{2j-3}^{-1} B_{2j-4} \ldots \times B_3^{-1} B_2 ||^2 < +\infty
\]

Then the solution generated by the above solutions belongs to \( l_2(\mathbb{N}) \) if

\[
\sum_{j=1}^{+\infty} || B_{2j-1+\epsilon}^{-1} B_{2j-2+\epsilon} \times \ldots \times B_{3+\epsilon}^{-1} B_{2+\epsilon} B_{1+\epsilon}^{-1} B_\epsilon ||^2 < +\infty \quad (1.18)
\]

where \( \epsilon = 0 \) or \( \epsilon = 1 \) and \( B_0 = B_1^{-1} \)

Now as

\[
|| B_{2j-1+\epsilon}^{-1} B_{2j-2+\epsilon} \times \ldots \times B_{3+\epsilon}^{-1} B_{2+\epsilon} B_{1+\epsilon}^{-1} B_\epsilon ||^2 \leq \\quad \leq \quad \leq \quad \leq \quad \leq
\]

\[
|| B_{2j-1+\epsilon}^{-1}||^2 || B_{2j-2+\epsilon}||^2 \ldots \times || B_{3+\epsilon}^{-1}||^2 || B_{2+\epsilon}||^2 || B_{1+\epsilon}||^2 || B_\epsilon||^2
\]

then it follows from (1.16) of lemma 2.1 that

\[
|| B_{2j-1+\epsilon}^{-1}||^2 \times \ldots \times || B_{3+\epsilon}^{-1}||^2 || B_{1+\epsilon}^{-1}||^2 \leq \frac{1}{|| B_{2j-2+\epsilon}||^2 \times \ldots \times || B_{2+\epsilon}||^2 || B_{1+\epsilon}||^2 || B_{2j+\epsilon}||}
\]

and consequently the general term of the series (1.18) do not exceed
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\[ \sum_{j=1}^{+\infty} \frac{1}{\| B_{2j+\epsilon} \|} \]

and (1.17) of the lemma 2.2 ensures the convergence of the series. The proof of Theorem 2.3 is complete.

Let \( \hat{H} = a^p a^{p+m} U \) (\( p = 1, 2, \ldots, m = 1, 2, \ldots \)) and \( \hat{H}^* \) is its adjoint and \( U e_k = e_{k+m-1} \). In next section, we study the chaoticity of the operators \( \hat{H} \) and \( \hat{H}^* \) on Bargmann space in the sense of following Denavey’s definition [2], [14]:

**Definition 2.4**

A linear unbounded densely defined operator \((T, D(T))\) on a Banach space \( X \) is called chaotic if the following conditions are met:

1) \( T^n \) is closed for all positive integers \( n \).

2) there exists an element \( \phi \in D(T^\infty) = \cap_{n=1}^{\infty} D(T^n) \) whose orbit
   \( \text{Orb}(T, \phi) = \{ \phi, T\phi, T^2\phi, \ldots \} \) is dense in \( X \), i.e. \( T \) is said to be hyper-cyclic.

3) the set \( \{ \phi \in X; \exists j \in \mathbb{N} \text{ such that } T^j \phi = \phi \} \) of periodic points of operator \( T \) is dense in \( X \).

**Remark 2.5**

i) It is well known that linear operators in finite-dimensional linear spaces can’t be chaotic but the nonlinear operator may be. Only in infinite-dimensional linear spaces can linear operators have chaotic properties. These last properties are based on the phenomenon of hypercyclicity or the phenomenon of nonwandercity.

ii) The study of the phenomenon of hypercyclicity originates in the papers by Birkhoff [7] and Maclane [24] that show, respectively, that the operators of translation and differentiation, acting on the space of entire functions are hyper-cyclic.

iii) Ansari asserts in [1] that powers of a hyper-cyclic bounded operator are also hyper-cyclic

iv) For an unbounded operator, Salas exhibit in [25] an unbounded hyper-
cyclic operator whose square is not hyper-cyclic.

v) H.N. Salas found in [26] an example of bilateral weighted Shift $T$ such that both $T$ and $T^*$ are hypercyclic. The operator $T \oplus T^*$ is not even cyclic and therefore the direct sum of hypercyclic operators is not always hypercyclic.

vi) In Bargmann representation the annihilation operator $a$ is chaotic but $a + a^*$ is not chaotic where $a^*$ is its adjoint satisfying $[a, a^*] = I$.

vii) The result of Salas show that one must be careful in the formal manipulation of operators with restricted domains. For such operators it is often more convenient to work with vectors rather than with operators themselves.

3. On the chaoticity of generalized Heun operator in Bargmann space

We begin by recalling some sufficient conditions on hypercyclicity of unbounded operators given by the following (Bès-Chan-Seubert theorem)

**Theorem 3.1** (Bès-Chan-Seubert [6], p.258)

Let $X$ be a separable infinite dimensional Banach and let $T$ be a densely defined linear operator on $X$. Then $T$ is hypercyclic if

(i) $T^m$ is closed operator for all positive integers $m$.

(ii) There exist a dense subset $Y$ of the domain $D(T)$ of $T$ and a (possibly nonlinear and discontinuous) mapping $S : Y \to Y$ so that $TS = I|_Y$ ($I|_Y$ is identity on $Y$) and $T^m, S^n \to 0$ pointwise on $Y$ as $n \to \infty$.

**Lemma 3.2**

Let $\mathcal{B}$ be the Bargmann space and $a$ and $a^*$ are the annihilation and creation operators defined on $\mathcal{B}$ by $a\phi(z) = \phi'(z)$ and $a^*\phi(z) = z\phi(z)$ then

1) $H^{0,m} = a^m + a^{*m}; m = 1, 2$ and $a^{*p}a^p; p = 1, 2, \ldots$ are not chaotic operators.

2) $a$ and $H^{1,1} = a^*(a + a^*)a$ are chaotic operators.

**Proof**
1) In Bargmann representation we note that:

- The operators $a$ and $a^*$ have same domain and that the operators $a^m + a^{*m}$, $m = 1, 2$ are symmetric then they are self-adjoint and consequently they are not chaotic in Bargmann space. We can also use the Carleman criterion.

- The operators $a^* p a^p; p = 1, 2, \ldots$ are self-adjoint operators with compact resolvent then they are not chaotic in Bargmann space.

2) In [16], it showed that $a^* a^p a^{p+1}$ is chaotic for all $p \geq 0$ in particular the operator $a$ is chaotic on $\mathbb{B}$ and this result is generalized in [17] to $z^p D^{p+1}$ where $D$ is Gelfond-Leontiev operator of generalized differentiation [13] acting on generalized Fock-Bargmann space. Recently, for some weighted shift $M$ defined on $(\Gamma, \chi)$-Theta Fock-Bargmann spaces, we showed in [21] that the operators $M^p D^{p+1}$ are chaotic for all $p \geq 0$ where $D$ is adjoint of $M$.

It is showed in [11] that $H^{1,1} = a^*(a + a^*)a$ is chaotic on $\mathbb{B}$, this last operator play an essential role in Reggeon field theory (see [19] and [20]). Also, we can show that $H^{1,1} = a^*(a + a^*)a$ is chaotic on $\mathbb{B}$ by choosing $\gamma_n = \sqrt{n \log n}$ in the theorem [18] below on the chaoticity of the sum of chaotic shifts with their adjoint in Hilbert space.

**Theorem 3.3** [18]

Let a linear unbounded densely defined chaotic shift operator $(\mathbb{T}, D(\mathbb{T}))$ on a Hilbert space $E = \{\phi; \phi = \sum_{n=1}^{\infty} a_n e_n\}$ such that its adjoint is defined by:

$$\mathbb{T}^* e_n = \omega_n e_{n+1}$$  \hspace{1cm} (3.1)

where $\{e_n\}$ is an orthonormal basis of $E$ and $\omega_n$ is positive weight associated to $\mathbb{T}$

We assume that

(Assumption $Hyp_1$) $\sum_{n=1}^{\infty} \frac{1}{\omega_n} < \infty$ \hspace{1cm} (3.2)

(Assumption $Hyp_2$) $\omega_{n-1} \omega_{n+1} \leq \omega_n^2$ \hspace{1cm} (3.3)

(Assumption $Hyp_3$) there exist $\alpha > 0$, $\beta > 0$, $a > 0$, and a sequence $\gamma_n$ that:
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\begin{align}
(1) \quad & \frac{\omega_n \gamma_n}{\gamma_{n+1}} \geq n^{1+\alpha} \\
(2) \quad & \frac{\omega_{n-1} \gamma_{n+1}}{\omega_n \gamma_{n-1}} = 1 - \frac{a}{n} + O\left(\frac{1}{n^{1+\beta}}\right) \\
\text{and} \quad & \\
(3) \quad & \sum_{k=1}^{\infty} \frac{1}{\gamma_n^2} < \infty
\end{align}

Then for $\lambda \in \mathbb{C}$ the following recurrence sequence

\begin{align}
(\ast) \quad & \left\{ \\
& u_1(\lambda) = 1 \\
& u_2(\lambda) = \frac{\lambda}{\omega_1} \\
& \omega_{n-1}u_{n-1}(\lambda) + \omega_n u_{n+1}(\lambda) = \lambda u_n(\lambda)
\right.
\end{align}

(i) is solvable for all $\lambda \in \mathbb{C}$.

(ii) $\sum_{n=1}^{\infty} |u_n(\lambda)|^2 < \infty$ for all $\lambda \in \mathbb{C}$.

(iii) the spectrum of $T + T^*$ is the all complex plane $\mathbb{C}$.

(iv) $(T + T^*)^m$ is closed $\forall m \in \mathbb{N}$.

(v) $T + T^*$ is hypercyclic operator.

(vi) $T + T^*$ is chaotic operator.

Remark 3.4

i) In Bargmann representation, the operator $a$ give an example of linear unbounded densely defined chaotic shift operator $(T, D(T))$ on a Hilbert space $\mathcal{B}$ such that $T + T^*$ is not chaotic operator.

ii) In Bargmann representation, let $T$ be linear unbounded densely defined chaotic shift operator such that $n_+ = n_- \neq 0$ where $n_+$ and $n_-$ are the defect numbers of $T + T^*$, then the operator $T + T^*$ is it chaotic ?
Now we recall the operator $U$ defined by

$$U e_k = e_{k+m-1}$$

and its adjoint $U^* e_k = e_{k-m+1}; k \geq m$

Let $\hat{H} = a^* a^p + m$ and $\hat{H}^* = U^* a^s + m a^p$ then

$$\hat{H} e_k = \omega_{k-1}^{p,m} e_{k-1}$$

where

$$\omega_{k-1}^{p,m} = \sqrt{\frac{(k-1)!(k-1+m)!}{(k-1-p)!}}.$$

and

$$\hat{H}^* e_k = \omega_k^{p,m} e_{k+1}$$

In the following, we show that the operator $\hat{H}$ is chaotic and we apply the theorem 3.3 to prove that $\hat{H} + \hat{H}^*$ is also chaotic.

Theorem 3.5

Let $\mathbb{B}_p, p = 0, 1, \ldots$ be the subspace of Bargmann space generated by $e_k; k \geq p$ and the operator $\hat{H}$ with domain $D(\hat{H}) = \{ \phi \in \mathbb{B}_p; \hat{H} \phi \in \mathbb{B}_p \}$ defined by

$$\hat{H} e_k = \omega_{k-1}^{p,m} e_{k-1}$$

where

$$\omega_{k-1}^{p,m} = \sqrt{\frac{(k-1)!(k-1+m)!}{(k-1-p)!}}.$$

then

$\hat{H}$ is chaotic on $\mathbb{B}_p$.

Proof

To use the theorem of Bèes and al, we begin by observing that for $\phi(z) = \sum_{k=p}^{\infty} a_k e_k(z)$ such that $\sum_{k=p}^{\infty} |a_k|^2 < \infty$ we have the obvious properties

(i) $\hat{H}^l \phi(z) = \sum_{k=p}^{\infty} \left( \prod_{j=p}^{l+k-1} \omega_j^{p,m} \right) a_{k+l} e_k(z)$ of domain

$$D(\hat{H}^l) = \{ \phi = \sum_{k=p}^{\infty} a_k e_k; \sum_{k=p}^{\infty} |a_k|^2 < \infty \ \text{and} \ \sum_{k=p}^{\infty} \left( \prod_{j=p}^{l+k-1} \omega_j^{p,m} \right)^2 |a_{k+l}|^2 < +\infty \}.$$
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witch is dense in \( \mathbb{B}_p \) \( \forall \ l \in \mathbb{N} \)

(ii) \( \hat{\mathbb{H}}^l \) is closed \( \forall \ l \in \mathbb{N} \) and \( \hat{\mathbb{H}}^l e_k(z) = 0 \) \( \forall \ l > k \geq p \geq 0 \)

(iii) As \( \omega^{p,m}_n \to +\infty \) then the spectrum of \( \hat{\mathbb{H}} \) is all the complex plane.

In fact, let \( \phi_\lambda = \sum_{k=p}^{\infty} a_k e_k \) with \( a_k = \prod_{j=p}^{k-1} \frac{\lambda}{\omega^{p,m}_j} \) i.e

\[
\phi_\lambda = \sum_{k=p}^{+\infty} \prod_{j=p}^{k-1} \frac{\lambda}{\omega^{p,m}_j} e_k
\]

then as \( a_p = 0 \) we deduce that

\[
\hat{\mathbb{H}} \phi_\lambda = \lambda \phi_\lambda, \forall \ \lambda \in \mathbb{C}.
\] (3.8)

and as \( \sum_{k=p}^{\infty} \left( \prod_{j=p}^{k} \frac{\lambda}{\omega^{p,m}_j} \right)^2 < +\infty \) then \( \phi_\lambda \in D(\hat{\mathbb{H}}) \)

Now, take \( \mathbb{Y} \) the linear subspace generated by finite combinations of basis \( \{ e_k \}_{k=p}^{\infty} \), this subspace \( \mathbb{Y} \) is dense in \( \mathbb{B}_p \) and we define on it the operator \( S \) acting on \( \phi = \sum_{k=p}^{N} a_k e_k \) as following

\[
S \phi = \sum_{k=p}^{N+1} \frac{a_{k-1}}{\omega^{p,m}_{k-1}} e_k
\] (4.5)
then

\[
S^n e_k = \frac{1}{\prod_{j=k}^{n+k} \omega^{p,m}_j} e_{k+n}
\]

as \( \prod_{j=p}^{n} \omega^{p,m}_j \to +\infty \) as \( n \to +\infty \) we get

\[
S^n e_k \to 0 \text{ in } \mathbb{B}_p \text{ as } n \to +\infty
\] (3.9)

By noting that \( \hat{\mathbb{H}}^n e_k = 0 \) for \( n > k \) and any element of \( \mathbb{Y} \) can be annihilated by a finite power of \( \hat{\mathbb{H}} \) and \( \hat{\mathbb{H}} S = S \mathbb{Y} \) then the hypercyclicity of \( \hat{\mathbb{H}} \) follows from the theorem of Bès and al. recalled above.

We shall now show that \( \hat{\mathbb{H}} \) has a dense set of periodic points.

To see this, it suffices to show that for every element \( \phi \) in the dense sub-
space $\mathbb{Y}$ there is a periodic point $\psi$ arbitrarily close to it.

For $s \geq p$ and $N \geq s$ we put

$$\varphi_{s,N}(z) = e_s(z) + \sum_{k=s+1}^{\infty} \left[ \prod_{j=s}^{kN+s-1} \frac{1}{\omega_{j,m}} \right] e_{kN+s}(z)$$

Then we have the following obvious lemma

**Lemma 3.6**

(i) $\overline{H}^N_{kN-1} \prod_{j=0}^{kN-1} \frac{1}{\omega_{j,m}} e_{kN} = \prod_{j=0}^{(k-1)N-1} \frac{1}{\omega_{j,m}} e_{(k-1)N} \quad \forall \quad k \geq p$

(ii) $\overline{H}^N_{kN-s} \prod_{j=s}^{kN+s} \frac{1}{\omega_{j,m}} e_{kN+s} = \prod_{j=s}^{(k-1)N-s} \frac{1}{\omega_{j,m}} e_{(k-1)N+s}$ for $s \geq p, N \geq s$ and $k \geq p$

(iii) $\varphi_{s,N}$ is $N$-periodic point of $\overline{H}$.

(iv) $\varphi_{s,N} \in D(\overline{H}^N)$.

Now, Let

$$\phi(z) = \sum_{s=p}^{M} a_s e_s(z)$$

such that

$$|a_s \prod_{j=p}^{s-1} \omega_{j,m}^p| < 1; \ s = p, p + 1, \ldots, M$$

and we choose the periodic point for $\overline{H}$ as $\psi(z)$

$$\psi(z) = \sum_{s=p}^{M} a_s \varphi_{s,N}(z)$$

then there exists an $N \geq M$ such that

$$|| \phi - \psi || \leq \epsilon \quad \forall \quad \epsilon > 0.$$
Remark 3.7

We can also use the results of Bermudez et al [5] to prove the chaoticity of our operator \( \mathbb{H} \) is chaotic.

Theorem 3.8

Let \( \mathbb{B}_p \) the subspace of Bargmann space generated by \( e_k; k \geq p \) then

\( \mathbb{H} + \mathbb{H}^* \) is chaotic where \( \mathbb{H}^* \) is adjoint operator of \( \mathbb{H} \)

Proof

For \( \omega_{k}^{p,m} = \sqrt{k! \frac{(k+m)!}{(k-m)!}} \sim k^{p + \frac{m}{2}} \) the assumptions (3.2), (3.3) and (3.6) of theorem 3.3 hold. It remains to check the validity of assumptions (3.4) and (3.5)

i) The assumption (3.4) is valid. In fact, as \( \omega_{k}^{p,m} = \sqrt{k! \frac{(k+m)!}{(k-m)!}} \sim k^{p + \frac{m}{2}} \) then

\[ \frac{\omega_{k-1}^{p,m}}{\omega_{k}^{p,m}} = (1 - \frac{1}{k})^{p + \frac{m}{2}} = 1 - \frac{p + \frac{m}{2}}{k} + O\left(\frac{1}{k^2}\right) \]

ii) The assumption (3.5) is valid. In fact, if we choose \( \gamma_k = k^\frac{m}{2} Log(k^p) = pk^\frac{m}{2} Log(k) \) then

\[ \frac{\gamma_k}{\gamma_{k+1}} = 1 - \frac{1 + \frac{m}{2}}{k} + O\left(\frac{1}{k^2}\right) \]

and

for \( p + \frac{m}{2} > 2 \) we get

\[ \omega_k \frac{\gamma_k}{\gamma_{k+1}} \geq k^{1+\beta} \] where \( 1 + \beta = p + \frac{m}{2} - 1 \). \( \diamond \)
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