Componentwise perturbation analysis for the generalized Schur decomposition

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Abstract
By defining two important terms called basic perturbation vectors and obtaining their linear bounds, we obtain the linear componentwise perturbation bounds for unitary factors and upper triangular factors of the generalized Schur decomposition. The perturbation bounds for the diagonal elements of the upper triangular factors and the generalized invariant subspace are also derived. From the former, we present an upper bound and a condition number of the generalized eigenvalue. Furthermore, with numerical iterative method, the nonlinear componentwise perturbation bounds of the generalized Schur decomposition are also provided. Numerical examples are given to test the obtained bounds. Among them, we compare our upper bound and condition number of the generalized eigenvalue with their counterparts given in the literature. Numerical results show that they are very close to each other but our results don’t contain the information on the left and right generalized eigenvectors.

Keywords Generalized Schur decomposition · Linear componentwise perturbation bound · Nonlinear componentwise perturbation bound · Chordal metric · Condition number

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1 Introduction

As we know, for a complex square matrix $A \in \mathbb{C}^{n \times n}$, there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ and an upper triangular matrix $T \in \mathbb{C}^{n \times n}$ such that

$$A = UTU^H.\tag{1}$$

This decomposition is called the Schur decomposition of the matrix $A$, which is an important and effective tool in numerical linear algebra (e.g., [1–4]).

For the matrix pair $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$, the generalized Schur decomposition is

$$A = UTV^H, B = URV^H,$$

where $T = (t_{ij}) \in \mathbb{C}^{n \times n}$ and $R = (r_{ij}) \in \mathbb{C}^{n \times n}$ are upper triangular matrices, and $U$ and $V$ are unitary matrices. Throughout this paper, we assume that the matrix pair $(A, B)$ is regular, that is, there exists $\lambda \in \mathbb{C}$ satisfying $\det(A - \lambda B) \neq 0$. Then the pairs $(t_{ii}, r_{ii})$ or $\lambda_i = t_{ii}/r_{ii}$, where $t_{ii}$ and $r_{ii}$ are the diagonal elements of $T$ and $R$, respectively, are the generalized eigenvalues of the matrix pair $(A, B)$. We assume that the matrix pair $(A, B)$ has distinct eigenvalues throughout this paper. The generalized Schur decomposition also has many important applications in many fields (e.g., [1, 2, 5, 6]).

There are many works on the applications, computations, and perturbation analysis for the Schur and generalized Schur decompositions (e.g., [7–11]). Here, we mainly focus on the perturbation analysis. In 1994, Konstantinov et al. [10] presented the perturbation bounds of the Schur decomposition by using the operator splitting, Lyapunov majorants and fixed point theorem. Soon afterwards, based on these techniques, Sun [11] presented the perturbation bounds of the generalized Schur decomposition. Later, some authors considered the perturbation analysis for the periodic Schur decomposition [7] and the antitriangular Schur decomposition [8]. All the above perturbation analysis belong to the normwise perturbation analysis, because they are obtained by using some matrix norms. As we know, normwise perturbation bounds cannot measure the abnormal perturbation of individual elements. Motivated by this, recently, Petkov [9] presented the componentwise perturbation analysis of the Schur decomposition.

With the technique in [9], in this paper, we consider the componentwise perturbation analysis of the generalized Schur decomposition. One of the most important contribution is that we present an upper bound and a condition number of the generalized eigenvalue. Unlike the corresponding results in [2, 12, 13], they don’t include the information on the left and right generalized eigenvectors. Numerical results show that our upper bound and condition number are close to their counterparts given in [2, 12].

The rest of this paper is organized as follows. In Sect. 2, we introduce some preliminaries which include some notations, the perturbation equations about the generalized Schur decomposition and the definitions of the basic perturbation
vectors and their linear bounds. The linear perturbation bounds for $U$ and $V$, $T$ and $R$, diagonal elements of $T$ and $R$, and generalized invariant subspace are presented in Sects. 3, 4, 5, and 6, respectively. In Sect. 7, we provide the nonlinear perturbation bounds of the generalized Schur decomposition. Finally, some numerical results are shown to test the obtained bounds.

2 Preliminaries

We first introduce some notations used in this paper. For the matrices $A$ and $B$, $A \otimes B$ denotes their Kronecker product. Given a matrix $A \in \mathbb{C}^{n \times n}$, $A^T$, $A^H$, $|A|$, $\|A\|_2$ and $\|A\|_F$ denote its transpose, conjugate, conjugate transpose, absolute values, spectral norm and Frobenius norm, respectively. In addition, let $I_n$ be the $n \times n$ identity matrix, $A_i$ be the $i$th row of $A$, $A_{i:j}$ be the $j$th column of $A$ and $A_{i_1:j_1,i_2:j_2}$ be the part of $A$ consisting of rows from $i_1$ to $i_2$ and columns from $j_1$ to $j_2$.

For the matrix $A = [a_{11}, a_{12}, \ldots, a_{nn}] = (a_{ij}) \in \mathbb{C}^{n \times n}$, we denote the vector $\text{vec}(A)$ by $\text{vec}(A) = \left[ a_{11}^T, a_{12}^T, \ldots, a_{nn}^T \right]^T$, the vector of the last $i$ elements of $a_j$ by $a_j^{[i]}$ and the upper triangular part of $A$ by $\text{triu}(A)$. It is easy to check that

$$|\text{triu}(AB)| = \text{triu}(|AB|) \leq \text{triu}(|A| |B|),$$

where $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$. Meanwhile,

$$\text{vec}(A^T) = \Pi_n \text{vec}(A),$$

where $\Pi_n$ is the vec-permutation matrix. For simplicity of explicit expressions, we consider the notations in [14],

$$\text{slvec}(A) = \left[ a_1^{[n-1]}, a_2^{[n-2]}, \ldots, a_{n-1}^{[1]} \right] \in \mathbb{C}^{\nu},$$

$$M_{\text{slvec}} = \left[ \text{diag}(J_1, J_2, \ldots, J_{n-1}), 0_{\nu \times n} \right] \in \mathbb{C}^{\nu \times n^2}, J_i = \left[ 0_{(n-i) \times i}, I_{n-i} \right] \in \mathbb{C}^{(n-i) \times n},$$

where $\nu = \frac{n(n-1)}{2}$ and $0_{m \times n}$ is the $m \times n$ zero matrix. It is easy to show that

$$\text{slvec}(A) = M_{\text{slvec}} \text{vec}(A).$$

Assume that the perturbed matrix pair $(\tilde{A}, \tilde{B})$ has the following generalized Schur decomposition

$$\tilde{A} = A + \Delta A = \tilde{U}
\tilde{T}
\tilde{V}^H, \tilde{B} = B + \Delta B = \tilde{U}
\tilde{R}
\tilde{V}^H,$$

where
\[ \tilde{U} = U + \Delta U, \quad \tilde{V} = V + \Delta V, \]
\[ \tilde{T} = T + \Delta T, \quad \tilde{R} = R + \Delta R. \]

To analyze the perturbation bounds of the generalized Schur decomposition conveniently, we first define two important matrices, that is,
\[ \Delta W = U^H \Delta U, \quad \Delta K = V^H \Delta V. \] (6)

Further, we define the following two vectors
\[ \begin{cases} x = \text{slvec}(\Delta W), \\ y = \text{slvec}(\Delta K), \end{cases} \] (7)
and call them the basic perturbation vectors. Considering (4), (7) can be written as
\[ \begin{cases} x = M_{\text{slvec vec}}(\Delta W), \\ y = M_{\text{slvec}}(\Delta K). \end{cases} \]

In the following, we will obtain the linear bounds of \( x \) and \( y \). To this end, let
\[ U = [u_1, u_2, \ldots, u_n], \quad \Delta U = [\Delta u_1, \Delta u_2, \ldots, \Delta u_n], \]
\[ V = [v_1, v_2, \ldots, v_n], \quad \Delta V = [\Delta v_1, \Delta v_2, \ldots, \Delta v_n], \]
where \( u_j \) and \( v_j \) are the \( j \)th column vectors of \( U \) and \( V \), respectively. Thus, based on the relationship between \( U \) and \( \tilde{U} \), and the relationship between \( V \) and \( \tilde{V} \), we have
\[ \tilde{U} = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n], \quad \tilde{u}_j = u_j + \Delta u_j, \]
\[ \tilde{V} = [\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n], \quad \tilde{v}_j = v_j + \Delta v_j. \]

Since \( T, R, \tilde{T} \) and \( \tilde{R} \) are upper triangular matrices, considering (1) and (5), it follows that
\[ \tilde{u}_i^H (A + \Delta A) \tilde{v}_j = u_i^H A v_j = 0, \]
\[ \tilde{u}_i^H (B + \Delta B) \tilde{v}_j = u_i^H B v_j = 0, \]
where \( 1 \leq j < i \leq n \). Hence
\[ \begin{cases} u_i^H A \Delta v_j + \Delta u_i^H A v_j + \Delta u_i^H A \Delta v_j = -\tilde{u}_i^H \Delta A \tilde{v}_j, \\ u_i^H B \Delta v_j + \Delta u_i^H B v_j + \Delta u_i^H B \Delta v_j = -\tilde{u}_i^H \Delta B \tilde{v}_j. \end{cases} \] (8)

Note that the componentwise representation of (1) can be written by
\[ \begin{cases} u_i^H A = \sum_{k=1}^n t_{ik} v_k^H, \\ A v_j = \sum_{k=1}^n t_{kj} u_k, \end{cases} \]
\[ \begin{cases} u_i^H B = \sum_{k=1}^n r_{ik} v_k^H, \\ B v_j = \sum_{k=1}^n r_{kj} u_k, \end{cases} \] (9)
where \( 1 \leq j < i \leq n \). Thus by combining (8) with (9), we get
\[
\begin{align*}
\left\{ \begin{array}{l}
\sum_{k=1}^{n} t_{ik} v^H_k \Delta v_j + \sum_{k=1}^{j} t_{kj} \Delta u^H_i u_k + \Delta u^H_i A \Delta v_j = -\bar{u}^H_i \Delta A \bar{v}_j, \\
\sum_{k=1}^{n} r_{ik} v^H_k \Delta v_j + \sum_{k=1}^{j} r_{kj} \Delta u^H_i u_k + \Delta u^H_i B \Delta v_j = -\bar{u}^H_i \Delta B \bar{v}_j,
\end{array} \right. \\
\text{where } 1 \leq j < i \leq n. \text{ Because of the unitarity of } U \text{ and } \bar{U}, \text{ it follows that}
\end{align*}
\]

\[U^H \Delta U = -\Delta U^H U - \Delta U^H \Delta U,\]

whose componentwise representation can be written by

\[\Delta u^H_i u_j = -u^H_i \Delta u_j - \Delta u^H_i \Delta u_j.\] (11)

Substitute (11) into (10) getting the following equations

\[
\left\{ \begin{array}{l}
\sum_{k=1}^{n} t_{ik} v^H_k \Delta v_j - \sum_{k=1}^{j} t_{kj} u^H_i u_k - \sum_{k=1}^{j} t_{kj} \Delta u^H_i u_k + \Delta u^H_i A \Delta v_j = -\bar{u}^H_i \Delta A \bar{v}_j, \\
\sum_{k=1}^{n} r_{ik} v^H_k \Delta v_j - \sum_{k=1}^{j} r_{kj} u^H_i u_k - \sum_{k=1}^{j} r_{kj} \Delta u^H_i u_k + \Delta u^H_i B \Delta v_j = -\bar{u}^H_i \Delta B \bar{v}_j,
\end{array} \right. \\
\text{where } 1 \leq j < i \leq n. \text{ The } (12) \text{ is very important to find the linear bounds of the basic perturbation vectors } x \text{ and } y. \text{ To continue, we need to write it as a matrix-vector equation. Some notations are necessary.}
\]

\[
F = -\bar{U}^H \Delta A \bar{V}, f = \text{svec}(F) = M_{\text{svec}} \text{vec}(F),
\]

\[G = -\bar{U}^H \Delta B \bar{V}, g = \text{svec}(G) = M_{\text{svec}} \text{vec}(G),\]

\[\Delta^x_j = \sum_{k=1}^{j} t_{kj} \Delta u^H_i u_k - \Delta u^H_i A \Delta v_j,\]

\[\Delta^y_j = \sum_{k=1}^{j} r_{kj} \Delta u^H_i u_k - \Delta u^H_i B \Delta v_j,\]

where \(1 \leq j < i \leq n\) and \(l = i + (j - 1)n - \frac{(j+1)}{2}\). Then, (12) can be written as

\[
\left\{ \begin{array}{l}
-L_{T,X} x + L_{T,Y} y = f + \Delta^x, \\
-L_{R,X} x + L_{R,Y} y = g + \Delta^y,
\end{array} \right. \\
\text{where the vectors } \Delta^x \in \mathbb{C}^v \text{ and } \Delta^y \in \mathbb{C}^v \text{ consist of } \Delta^x_j \text{ and } \Delta^y_j, \text{ respectively, and}
\]

\[L_{T,X} = M_{\text{svec}}(T^T \otimes I_n) M_{\text{svec}}^T \in \mathbb{C}^{v \times v}, L_{T,Y} = M_{\text{svec}}(I_n \otimes T) M_{\text{svec}}^T \in \mathbb{C}^{v \times v},\]

\[L_{R,X} = M_{\text{svec}}(R^T \otimes I_n) M_{\text{svec}}^T \in \mathbb{C}^{v \times v}, L_{R,Y} = M_{\text{svec}}(I_n \otimes R) M_{\text{svec}}^T \in \mathbb{C}^{v \times v}.\]

Notice that \(L_{T,X}\) and \(L_{R,X}\) are block lower triangular matrices, and \(L_{T,Y}\) and \(L_{R,Y}\) are block diagonal matrices. For example, for \(n = 5\), the matrices \(L_{T,X}\) and \(L_{T,Y}\) have the following forms
Let

\[
L = \begin{bmatrix}
-L_{T,X} & L_{T,Y}
\end{bmatrix} = \begin{bmatrix}
-M_{\text{svec}} & 0 \\
0 & -M_{\text{svec}}
\end{bmatrix} \begin{bmatrix}
T^T \otimes I_n & I_n \otimes T \\
R^T \otimes I_n & I_n \otimes R
\end{bmatrix} \begin{bmatrix}
M_{\text{svec}}^T & 0 \\
0 & M_{\text{svec}}^T
\end{bmatrix} \in \mathbb{C}^{2\nu \times 2\nu}.
\]

(14)

Then (13) can be simplified as

\[
L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.
\]

(15)

Note that \( L \) is nonsingular [11]. Thus, neglecting the second order terms, we obtain the linear approximation of \( x \) and \( y \),

\[
\begin{bmatrix} x \\ y \end{bmatrix}_{\text{lin}} = L^{-1} \begin{bmatrix} f \\ g \end{bmatrix},
\]

(16)

which implies that the \( l \)th component of \( \begin{bmatrix} x \\ y \end{bmatrix}_{\text{lin}} \) can be written as

\[
\begin{bmatrix} x \\ y \end{bmatrix}_{\text{lin},l} = L_{l,:}^{-1} \begin{bmatrix} f \\ g \end{bmatrix},
\]

(17)

where \( l = 1, 2, \ldots, 2\nu \). Since
\[
\begin{cases}
\|f\|_2 \leq \|\Delta A\|_F, \\
\|g\|_2 \leq \|\Delta B\|_F.
\end{cases}
\]
we have the linear bound of the \(l\)th component
\[
\left| \begin{bmatrix} x \\ y \end{bmatrix} \right|_{\text{lin},l} \leq \left\| L_{k;:}^{-1} \right\|_2 \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F,
\]
and hence the linear bounds of the \(j\)th component of vectors \(x\) and \(y\)
\[
|x_{\text{lin},j}| \leq \left\| L_{j;:}^{-1} \right\|_2 \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F, \\
y_{\text{lin},j} \leq \left\| L_{j+1;:}^{-1} \right\|_2 \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F,
\]
where \(j = 1, 2, \ldots, v\).

The above discussions are summarized in the following theorem.

**Theorem 1** Let \((A, B)\) and \((\widetilde{A}, \widetilde{B})\) have the generalized Schur decompositions in (1) and (5), respectively. For \(x\) and \(y\) defined in (7), the linear bounds (19) hold.

In the following sections, we will use Theorem 1, i.e., the bounds (19), to investigate the componentwise perturbation bounds of the generalized Schur decomposition.

### 3 Linear perturbation bounds for \(U\) and \(V\)

We first rewrite the matrices \(\Delta W\) and \(\Delta K\) in (6) as the sum of matrices whose elements only contain the first or second order terms. To do this, note that \(\widetilde{U}\) and \(U\) are unitary matrices, then
\[
\Delta u_i^H u_j = -u_i^H u_j - u_i^H \Delta u_j
\]
or
\[
\bar{u}_j^H \Delta u_i = -u_i^H \Delta u_j - u_i^H u_j,
\]
where \(\bar{u}_j^H \Delta u_i\) with \(1 \leq i < j \leq n\) is the conjugate of the element \(x_i\), that is \(\bar{u}_j^H \Delta u_i = \bar{x}_j\) with \(l = j + (i - 1)n - \frac{(i+1)}{2}\). For the matrices \(\widetilde{V}\) and \(V\), similar results hold. Thus, \(\Delta W\) and \(\Delta K\) can be written as
\[
\begin{cases}
\Delta W = \Delta W_1 + \Delta W_2 - \Delta W_3, \\
\Delta K = \Delta K_1 + \Delta K_2 - \Delta K_3,
\end{cases}
\]
where
ΔW₁ = \[
\begin{bmatrix}
0 & -\bar{x}_1 & -\bar{x}_2 & \cdots & -\bar{x}_{n-1} \\
\bar{x}_1 & 0 & -\bar{x}_2 & \cdots & -\bar{x}_{2n-2} \\
\bar{x}_2 & \bar{x}_n & 0 & \cdots & -\bar{x}_{3n-6} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_{n-1} & x_{2n-3} & x_{3n-6} & \cdots & 0
\end{bmatrix} \in \mathbb{C}^{n \times n},
\]

ΔW₂ = diag(Δu₁, Δu₂, ..., Δuₙ) ∈ \mathbb{C}^{n \times n},
\tag{21}

ΔW₃ = \[
\begin{bmatrix}
0 & Δu₁^{H} \Delta u₂ & Δu₆^{H} Δu₃ & \cdots & Δu₆^{H} Δuₙ \\
0 & 0 & Δu₂^{H} Δu₃ & \cdots & Δu₆^{H} Δuₙ \\
0 & 0 & 0 & \cdots & Δu₆^{H} Δuₙ \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{C}^{n \times n},
\tag{22}
\]

and ΔK₁, ΔK₂, and ΔK₃ are the same as ΔW₁, ΔW₂, and ΔW₃, respectively, except that \(x_i\) is replaced by \(y_i\) and \(Δu_j\) is replaced by \(Δv_j\). Obviously, the factors ΔW₃ and ΔK₃ are comprised of the high order terms.

To obtain the desired bounds, we assume that the perturbed unitary matrices \(\tilde{U}\) and \(\tilde{V}\) are such that the imaginary parts of diagonal elements of ΔW₂ and ΔK₂ are zero. This assumption also appears in [9] and is common in perturbation analysis for Schur and generalized Schur decompositions (e.g., [10, 11]). Then it’s easy to show that \(u_i^{H} Δu_i = -\frac{Δu_i^{H} Δu_i}{2}\) and \(v_j^{H} Δv_j = -\frac{Δv_j^{H} Δv_j}{2}\). Hence the elements of ΔW₂ and ΔK₂ are also the high order terms. Thus, together with (19), we can obtain the linear bounds of ΔW and ΔK, i.e., |ΔW| and |ΔK|, which are given as follows:

\[
|\tilde{W}| = \[
\begin{bmatrix}
0 & |x_1| & |x_2| & \cdots & |x_{n-1}| \\
|x_1| & 0 & |x_2| & \cdots & |x_{2n-2}| \\
|x_2| & |x_n| & 0 & \cdots & |x_{3n-6}| \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
|x_{n-1}| & |x_{2n-3}| & |x_{3n-6}| & \cdots & 0
\end{bmatrix},
\tag{23}
\]

\[
|\tilde{K}| = \[
\begin{bmatrix}
0 & |y_1| & |y_2| & \cdots & |y_{n-1}| \\
|y_1| & 0 & |y_2| & \cdots & |y_{2n-2}| \\
|y_2| & |y_n| & 0 & \cdots & |y_{3n-6}| \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
|y_{n-1}| & |y_{2n-3}| & |y_{3n-6}| & \cdots & 0
\end{bmatrix}.
\tag{24}
\]

Since \(U\) and \(V\) are unitary matrices, the linear perturbation bounds of \(U\) and \(V\) follow as

\[
\begin{cases}
|ΔU| \leq |U||U^{H} ΔU| = |U||\tilde{W}|, \\
|ΔV| \leq |V||V^{H} ΔV| = |V||\tilde{K}|.
\end{cases}
\tag{25}
\]
In summary, we have the following theorem.

**Theorem 2** Let \((A, B)\) and \((\widetilde{A}, \widetilde{B})\) have the generalized Schur decompositions in (1) and (5), respectively. Then the linear perturbation bounds of \(U\) and \(V\) given in (25) hold.

### 4 Linear perturbation bounds for \(T\) and \(R\)

**Theorem 3** Let \((A, B)\) and \((\widetilde{A}, \widetilde{B})\) have the generalized Schur decompositions in (1) and (5), respectively. Then the linear perturbation bounds for \(T\) and \(R\) are

\[
|\Delta T| = \text{triu}\left( |U^H||A| + |\Delta A||V| |\Delta K| + |\Delta W^H||A| + |\Delta A||V| \right),
\]

(26)

and

\[
|\Delta R| = \text{triu}\left( |U^H||B| + |\Delta B||V| |\Delta K| + |\Delta W^H||B| + |\Delta B||V| \right),
\]

(27)

where \(|\Delta W|\) and \(|\Delta K|\) are given in (23) and (24), respectively.

**Proof** From (5), \(\Delta T\) can be written as

\[
\Delta T = \widetilde{T} - T = \widetilde{U}^H\widetilde{A}\widetilde{V} - U^HAV.
\]

Then

\[
\Delta T = U^H\Delta V + \Delta U^HAV + \Delta U^H\Delta V + U^H\Delta A\Delta V + U^H\Delta A\Delta V
\]

(28)

\[
+ \Delta U^H\Delta AV + \Delta U^H\Delta A\Delta V.
\]

Since \(\Delta T\) is an upper triangular matrix, applying triu to (28) implies

\[
\Delta T = \text{triu}(U^H\Delta V + \Delta U^HAV + \Delta U^H\Delta V + U^H\Delta AV + U^H\Delta A\Delta V + \Delta U^H\Delta AV + \Delta U^H\Delta A\Delta V).
\]

Combining the aforementioned equation with (2), we have
\[ |\Delta T| \leq \text{triu} \left( |U^H||A||\Delta V| + |\Delta U^H||A||\Delta V| + |\Delta U^H||A||\Delta V| + |U^H||\Delta A||V| \right. \\
\left. + |U^H||\Delta A||\Delta V| + |\Delta U^H||\Delta A||V| + |\Delta U^H||\Delta A||\Delta V| \right). \]

(29)

Considering (25) and (29), the perturbation bound of \(T\), i.e., the bound (26), holds. Similarly, we can obtain the perturbation bound of \(R\), i.e., the bound (27).

5 Linear perturbation bounds for diagonal elements of \(T\) and \(R\)

The method is similar to the one of obtaining the linear bounds of \(x\) and \(y\). That is, by using the componentwise representations of (1) and (5), we can construct matrix-vector equations and then find the desired bounds. Specifically, noting that the matrices \(T, R, \bar{T}\) and \(\bar{R}\) are upper triangular matrices, we have

\[
\begin{cases}
\Delta t_{ij} = \tilde{t}_{ij} - t_{ij} = u_i^H A \Delta v_j + \Delta u_i^H A \Delta v_j + \tilde{u}_i^H \Delta A v_j, \\
\Delta r_{ij} = \tilde{r}_{ij} - r_{ij} = u_i^H B \Delta v_j + \Delta u_i^H B \Delta v_j + \tilde{u}_i^H \Delta B v_j,
\end{cases}
\]

(30)

where \(1 \leq i \leq j \leq n\). Hence, when \(i = j\), (30) can be written as

\[
\begin{cases}
\Delta t_i = \sum_{k=1}^n t_{ik} \tilde{v}_k^H \Delta v_i - \sum_{k=1}^n t_{ik} u_i^H \Delta u_k - \sum_{k=1}^i t_{ik} \Delta u_k \Delta u_k + \Delta u_k A \Delta v_i + \tilde{u}_i^H \Delta A v_i, \\
\Delta r_i = \sum_{k=1}^n r_{ik} \tilde{v}_k^H \Delta v_i - \sum_{k=1}^n r_{ik} u_i^H \Delta u_k - \sum_{k=1}^i r_{ik} \Delta u_k \Delta u_k + \Delta u_k B \Delta v_i + \tilde{u}_i^H \Delta B v_i,
\end{cases}
\]

(31)

where \(i = 1, 2, \ldots, n\). Next, we write it as a matrix-vector equation. Some notations are necessary.

\[
f_1 = \begin{bmatrix}
\tilde{u}_1^H \Delta A \tilde{v}_1 \\
\tilde{u}_2^H \Delta A \tilde{v}_2 \\
\vdots \\
\tilde{u}_n^H \Delta A \tilde{v}_n
\end{bmatrix}, \quad s_1 = \begin{bmatrix}
\tilde{u}_1^H \Delta B \tilde{v}_1 \\
\tilde{u}_2^H \Delta B \tilde{v}_2 \\
\vdots \\
\tilde{u}_n^H \Delta B \tilde{v}_n
\end{bmatrix}, \quad \Delta \lambda' = \begin{bmatrix}
\Delta t_{11} \\
\Delta t_{22} \\
\vdots \\
\Delta t_{nn}
\end{bmatrix}, \quad \Delta \lambda' = \begin{bmatrix}
\Delta r_{11} \\
\Delta r_{22} \\
\vdots \\
\Delta r_{nn}
\end{bmatrix},
\]

(32)

\[
d_r = \begin{bmatrix}
t_{11} (v_1^H \Delta v_1 - u_1^H \Delta u_1) \\
t_{22} (v_2^H \Delta v_2 - u_2^H \Delta u_2) \\
\vdots \\
t_{nn} (v_n^H \Delta v_n - u_n^H \Delta u_n)
\end{bmatrix}, \quad d_r = \begin{bmatrix}
r_{11} (v_1^H \Delta v_1 - u_1^H \Delta u_1) \\
r_{22} (v_2^H \Delta v_2 - u_2^H \Delta u_2) \\
\vdots \\
r_{nn} (v_n^H \Delta v_n - u_n^H \Delta u_n)
\end{bmatrix},
\]

(33)

\[
\Delta dt = \begin{bmatrix}
\Delta t_{11} \\
\Delta t_{22} \\
\vdots \\
\Delta t_{nn}
\end{bmatrix}, \quad \Delta dr = \begin{bmatrix}
\Delta r_{11} \\
\Delta r_{22} \\
\vdots \\
\Delta r_{nn}
\end{bmatrix},
\]

where
Clearly, the vectors $\Delta \lambda^t$ and $\Delta \lambda^r$ consist of diagonal elements of the matrices $ΔT$ and $ΔR$, respectively, and $Δ^t, Δ^r, d_t$ and $d_r$ contain the second order terms. Thus, considering (31), we get

$$
\begin{align*}
\begin{bmatrix}
\Delta \lambda^t \\
\Delta \lambda^r
\end{bmatrix}
&= A_1 \begin{bmatrix}
x \\
y
\end{bmatrix}
+ f_1 + \begin{bmatrix}
\Delta^t \\
\Delta^r
\end{bmatrix}
+ \begin{bmatrix}
d_t \\
d_r
\end{bmatrix},
\end{align*}
$$

(34)

where

$$
A_1 = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
- t_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & - t_{13} & 0 & \cdots & - t_{23} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & - t_{1n} & 0 & 0 & \cdots & - t_{2n} & 0 & 0 & \cdots & - t_{3n} & \cdots & - t_{n-1,n}
\end{bmatrix},
$$

$$
A_2 = \begin{bmatrix}
t_{12} & t_{13} & t_{14} & \cdots & t_{1n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & t_{23} & t_{24} & t_{25} & \cdots & t_{2n} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & t_{34} & t_{35} & t_{36} & \cdots & t_{3n} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix},
$$

and the matrices $B_1$ and $B_2$ have the same forms as the matrices $A_1$ and $A_2$ except that the element $t_{ij}$ is replaced by $r_{ij}$. Setting

$$
E = \begin{bmatrix}
A_1 & A_2 \\
B_1 & B_2
\end{bmatrix},
Z = [EL^{-1}, I_{2n}],
$$

(35)

neglecting the second order term in (34), and using (16), we have

$$
\begin{bmatrix}
\Delta \lambda^t \\
\Delta \lambda^r
\end{bmatrix}
= E L^{-1} \begin{bmatrix}
f \\
g
\end{bmatrix}
+ \begin{bmatrix}
f_1 \\
g_1
\end{bmatrix}
= [EL^{-1}, I_{2n}]
\begin{bmatrix}
f \\
g
\end{bmatrix}.
$$

Since

$$
\left\| \begin{bmatrix}
f \\
g
\end{bmatrix} \right\|_2 \leq \left\| \begin{bmatrix}
\Delta A \\
\Delta B
\end{bmatrix} \right\|_F,
$$

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the linear componentwise perturbation bounds of $\Delta \lambda^t$ and $\Delta \lambda^r$ are obtained as follows

$$
\begin{bmatrix}
\Delta \lambda^t \\
\Delta \lambda^r
\end{bmatrix}_i \leq \|Z_i\|_2 \left\| \begin{bmatrix}
\Delta A \\
\Delta B
\end{bmatrix} \right\|_F,
$$

(36)

where $i = 1, 2, \ldots, 2n$ and $\begin{bmatrix}
\Delta \lambda^t \\
\Delta \lambda^r
\end{bmatrix}_i$ stands for the $i$th component of the vector $\begin{bmatrix}
\Delta \lambda^t \\
\Delta \lambda^r
\end{bmatrix}$.

In summary, we have the following theorem.

**Theorem 4** Let $(A, B)$ and $(\tilde{A}, \tilde{B})$ have the generalized Schur decompositions in (1) and (5), respectively. Then the linear perturbation bounds for the diagonal elements of $T$ and $R$ given in (36) hold.

Using the linear bounds of $\Delta \lambda^t$ and $\Delta \lambda^r$, we can consider the perturbation analysis of the generalized eigenvalue $\lambda$ of the regular matrix pair $(A, B)$, that is, $A\xi = \lambda B\xi$, where $\xi$ is a right eigenvector corresponding to the generalized eigenvalue $\lambda$. Recall that the eigenvalue $\lambda$ can also be represented as a pair of numbers $\langle \alpha, \beta \rangle$ such that $\beta A\xi = \alpha B\xi$. If $\beta \neq 0$, $\lambda = \alpha / \beta$, otherwise $\lambda = \infty$. As mentioned in Sect. 1, $\alpha$ and $\beta$ can be the diagonal elements of the matrices $T$ and $R$, respectively.

Stewart and Sun [2] defined the following chordal metric to measure the eigenvalue perturbation of a regular matrix pair,

$$
\chi(\langle \alpha, \beta \rangle, \langle \tilde{\alpha}, \tilde{\beta} \rangle) = \frac{|\alpha \tilde{\beta} - \beta \tilde{\alpha}|}{\sqrt{|\alpha|^2 + |\beta|^2} \sqrt{|\tilde{\alpha}|^2 + |\tilde{\beta}|^2}},
$$

(37)

where $\langle \tilde{\alpha}, \tilde{\beta} \rangle$ is the perturbed eigenvalue, and presented an upper bound

$$
\chi(\langle \alpha, \beta \rangle, \langle \tilde{\alpha}, \tilde{\beta} \rangle) \leq \frac{\|\xi\|_2 \|\eta\|_2}{\sqrt{|\alpha|^2 + |\beta|^2} \left\| \begin{bmatrix}
\Delta A \\
\Delta B
\end{bmatrix} \right\|_2},
$$

(38)

where $\eta$ is the left eigenvector corresponding to the eigenvalue $\langle \alpha, \beta \rangle$. Note that the chordal metric can also be used for the generalized singular value (e.g., [15, 16]).

On the basis of (37), we have

$$
\chi(\langle t_{ii}, r_{ii} \rangle, \langle \tilde{t}_{ii}, \tilde{r}_{ii} \rangle) = \frac{|t_{ii} \tilde{r}_{ii} - r_{ii} \tilde{t}_{ii}|}{\sqrt{|t_{ii}|^2 + |r_{ii}|^2} \sqrt{|\tilde{t}_{ii}|^2 + |\tilde{r}_{ii}|^2}}, \quad i = 1, 2, \ldots, n,
$$

which combined with Theorem 4, i.e.,
\[ |\tilde{r}_{ii} - t_{ii}| \leq \|Z_{i,:}\|_2 \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F, \quad |\tilde{r}_{ii} - r_{ii}| \leq \|Z_{i+n,:}\|_2 \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F, \]

implies a new bound for generalized eigenvalue with the chordal metric:

\[
\mathcal{A}(t_{ii}, r_{ii}, \langle \tilde{r}_{ii}, \tilde{r}_{ii} \rangle) \leq \frac{|t_{ii}|r_{i} + |r_{ii}|t_{i}}{\sqrt{|t_{ii}|^2 + |r_{ii}|^2} \sqrt{|t_{ii}|^2 + |r_{ii}|^2 - 2(|t_{ii}|t_{i} + |r_{ii}|r_{i})}},
\]

where

\[ t_{i} = \|Z_{i,:}\|_2 \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F, \quad r_{i} = \|Z_{i+n,:}\|_2 \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F. \]

This is because

\[
|t_{ii}\tilde{r}_{ii} - r_{ii}\tilde{r}_{ii}| = |t_{ii}(G_{ii} - r_{ii}) - r_{ii}(\tilde{G}_{ii} - t_{ii})| \leq |t_{ii}(G_{ii} - r_{ii})| + |r_{ii}(\tilde{G}_{ii} - t_{ii})|
\]

\[
\leq |t_{ii}| |\tilde{r}_{ii} - r_{ii}| + |r_{ii}| |\tilde{r}_{ii} - t_{ii}| \leq |t_{ii}|r_{i} + |r_{ii}|t_{i},
\]

and

\[
|\tilde{r}_{ii}|^2 + |\tilde{r}_{ii}|^2 = |\tilde{r}_{ii} - t_{ii} + t_{ii}|^2 + |\tilde{r}_{ii} - r_{ii} + r_{ii}|^2
\]

\[
\geq (|\tilde{r}_{ii} - t_{ii}| - |t_{ii}|)^2 + (|\tilde{r}_{ii} - r_{ii}| - |r_{ii}|)^2
\]

\[
= |\tilde{r}_{ii} - t_{ii}|^2 + |\tilde{r}_{ii} - r_{ii}|^2 + |t_{ii}|^2 + |r_{ii}|^2 - 2(|\tilde{r}_{ii} - t_{ii}|t_{ii} + |\tilde{r}_{ii} - r_{ii}|r_{ii})
\]

\[
\geq |t_{ii}|^2 + |r_{ii}|^2 - 2(|t_{ii}|t_{i} + |r_{ii}|r_{i}) > 0,
\]

where the penultimate inequality relies on \(|\tilde{r}_{ii} - t_{ii}| \leq t_{i}\) and \(|\tilde{r}_{ii} - r_{ii}| \leq r_{i}\), and the last inequality comes from the assumption that \(t_{i} \ll |t_{ii}|\) and \(r_{i} \ll |r_{ii}|\).

**Remark 1** Compared with the bound (39), our bound (39) doesn’t contain the information on the left and right generalized eigenvectors. A numerical example in Sect. 8 shows that our bound can be more accurate than (38) in most cases.

Now we consider the condition number of the generalized eigenvalue \(\lambda\) (e.g., [12, 13, 17]). A definition similar to the one in [13] is first given as follows

\[
\text{cond}(\lambda) := \lim_{\epsilon \to 0} \sup \left\{ \frac{|\Delta \lambda|}{\epsilon |\lambda|} : (A + \Delta A)(\xi + \Delta \xi) = (\lambda + \Delta \lambda) \right. \]

\[
(B + \Delta B)(\xi + \Delta \xi), \left. \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F \leq \epsilon \right\}.
\]

(40)
And, as done in [13], we assume that $\Delta \xi \to 0$ as $\epsilon \to 0$ in order to prevent $\text{cond}(\lambda) = \infty$. In the following, we will obtain an explicit expression of $\text{cond}(\lambda)$. First, note that a fact $\langle \alpha, \beta \rangle = (\eta^H A \xi, \eta^H B \xi)$, which was proved by Stewart and Sun [2, p.277]. Thus, we have

$$\tilde{\alpha} = \tilde{\eta}^H \tilde{A} \tilde{\xi} = (\eta^H + \Delta \eta^H)(A + \Delta A)(\xi + \Delta \xi)$$

$$= \alpha + \eta^H \Delta A \tilde{\xi} + O(\epsilon^2),$$

where $\tilde{\eta}$ and $\tilde{\xi}$ are the left and right generalized eigenvectors corresponding to the eigenvalue $\langle \tilde{\alpha}, \tilde{\beta} \rangle$ of the matrix pair $(\tilde{A}, \tilde{B})$, respectively. Similarly, we obtain

$$\tilde{\beta} = \beta + \eta^H \Delta B \tilde{\xi} + O(\epsilon^2).$$

Thus, combining the equation $(A + \Delta A)(\xi + \Delta \xi) = (\lambda + \Delta \lambda)(B + \Delta B)(\xi + \Delta \xi)$ with (41) and (42), implies

$$\Delta \lambda = \frac{\eta^H \Delta A \xi - \lambda \eta^H \Delta B \xi}{\eta^H B \xi} + O(\epsilon^2)$$

$$= \frac{\tilde{\alpha} - \alpha - \lambda(\tilde{\beta} - \beta)}{\beta} + O(\epsilon^2)$$

$$= \frac{\Delta \alpha}{\beta} - \frac{\alpha \Delta \beta}{\beta^2} + O(\epsilon^2),$$

where $\Delta \alpha = \tilde{\alpha} - \alpha$ and $\Delta \beta = \tilde{\beta} - \beta$ are the perturbations of $\alpha$ and $\beta$, respectively. Alternatively, we can get $\Delta \lambda$ more easily by Taylor expansion, that is,

$$\Delta \lambda = \lambda + \Delta \lambda - \lambda = \frac{\alpha + \Delta \alpha}{\beta + \Delta \beta} - \frac{\alpha}{\beta} = \frac{\alpha}{\beta} (1 - \frac{\Delta \beta}{\beta}) - \frac{\alpha}{\beta} + O(\epsilon^2)$$

$$= \frac{\Delta \alpha}{\beta} - \frac{\alpha \Delta \beta}{\beta^2} + O(\epsilon^2).$$

Therefore, from (40), we have

$$\frac{|\Delta \lambda|}{|\lambda|} \leq \left| \frac{\Delta \alpha}{\alpha} \right| + \left| \frac{\Delta \beta}{\beta} \right| \leq \left( \left| \frac{\Delta \alpha}{\alpha} \right| + \left| \frac{\Delta \beta}{\beta} \right| \right) \frac{\epsilon}{\|\Delta A, \Delta B\|_F}. \quad (43)$$

Dividing (43) by $\epsilon$ and considering (36), the condition number of the $i$th eigenvalue $\lambda_i$ can be obtained:

$$\frac{|\Delta \lambda_i|}{\epsilon |\lambda_i|} \leq \left( \left| \frac{\Delta \alpha}{\alpha} \right| + \left| \frac{\Delta \beta}{\beta} \right| \right) \frac{1}{\|\Delta A, \Delta B\|_F}$$

$$\leq \frac{\|Z_{i_r}\|_2}{|t_{ii}|} + \frac{\|Z_{i_{t+n}}\|_2}{|r_{ii}|},$$

$$\quad (44)$$
where \( i = 1, 2, \ldots, n \) and \( Z \) is given in (35).

From the above discussions, we conclude the following theorem.

**Theorem 5** The condition number of the \( i \)th generalized eigenvalue \( \lambda_i \) of \( (A, B) \) can be given in (44).

**Remark 2** Although the derivation of the condition number (44) is done using the left and right generalized eigenvectors, the condition number itself doesn’t include the information on generalized eigenvectors. In addition, it should be pointed out that (44) is actually an upper bound for the true condition number.

**Remark 3** According to (38), Frayssé and Toumazou [12] derived a upper bound of \( \Delta \lambda \)

\[
|\Delta \lambda| \leq \sqrt{1 + \frac{\| \xi \|_2 \| \eta \|_2}{|\lambda|}} \left\| \left[ \Delta A, \Delta B \right] \right\|_2.
\]

Thus, by setting \( \| \left[ \Delta A, \Delta B \right] \|_2 \leq \epsilon \), in our notation, we can find a condition number from the results in [2, 12]

\[
\frac{|\Delta \lambda|}{\epsilon |\lambda|} \leq \sqrt{1 + \frac{|\lambda|^2}{|\lambda|} \frac{\| \xi \|_2 \| \eta \|_2}{|\eta^H B \xi|}}.
\]

(45)

Actually, (45) is also an upper bound for the true condition number. In addition, Higham [13] presented the following normwise condition number

\[
\frac{|\Delta \lambda|}{\epsilon |\lambda|} \leq \text{normcond}(\lambda) = \frac{\| \eta \|_2 \| \xi \|_2 (\| E \|_2 + |\lambda| \| F \|_2)}{|\lambda| |\eta^H B \xi|},
\]

(46)

by setting \( \| A \|_2 \leq \epsilon \| E \|_2, \| B \|_2 \leq \epsilon \| F \|_2 \), and the following componentwise condition number

\[
\frac{|\Delta \lambda|}{\epsilon |\lambda|} \leq \text{compcond}(\lambda) = \frac{\| \eta^H [E] \xi + |\lambda| \eta^H [F] \xi \|}{|\lambda| |\eta^H B \xi|},
\]

(47)

by setting \( |\Delta A| \leq \epsilon E, |\Delta B| \leq \epsilon F \). Note that the definitions of the above four condition numbers are different. Specifically, the definitions for the condition numbers (44)–(47) are (40),

\[
\lim_{\epsilon \to 0} \sup \left\{ \frac{|\Delta \lambda|}{\epsilon |\lambda|} : (A + \Delta A)(\xi + \Delta \xi) = (\lambda + \Delta \lambda)(B + \Delta B)(\xi + \Delta \xi), \left\| \left[ \Delta A, \Delta B \right] \right\|_2 \leq \epsilon \right\},
\]

\[
\lim_{\epsilon \to 0} \sup \left\{ \frac{|\Delta \lambda|}{\epsilon |\lambda|} : (A + \Delta A)(\xi + \Delta \xi) = (\lambda + \Delta \lambda)(B + \Delta B)(\xi + \Delta \xi), \| \Delta A \|_2 \leq \epsilon \| E \|_2, \| \Delta B \|_2 \leq \epsilon \| F \|_2 \right\},
\]
and
\[ \lim_{\varepsilon \to 0} \sup \left\{ \frac{|\Delta \lambda|}{|\lambda|} : (A + \Delta A)(\xi + \Delta \xi) = (\lambda + \Delta \lambda)(B + \Delta B)(\xi + \Delta \xi), |\Delta A| \leq \varepsilon E, |\Delta B| \leq \varepsilon F \right\}, \]
respectively. So, it is difficult to compare the condition numbers (44)–(47) in theory, and they are usually not equal either. In Sect. 8, we will compare them numerically.

### 6 Linear perturbation bounds for generalized invariant subspace

We first present the definition of the generalized invariant subspace.

**Definition 1** [15] Let \((A, B)\) be a regular matrix pair. If there are unitary matrices \(Y\) and \(X = [X_1, X_2]\), where \(X_1 \in \mathbb{C}^{n \times p}\), such that
\[
Y^HAX = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad Y^HBX = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},
\]
where \(A_{11}, B_{11} \in \mathbb{C}^{p \times p}\), we call the column subspace of \(X_1\), i.e., \(R(X_1)\), the \(p\)-dimensional generalized invariant subspace of \((A, B)\).

Combining Definition 1 with the generalized Schur decomposition (1.1), we find that the perturbation of \(p\)-dimensional generalized invariant subspace is the perturbation of the span of the first \(p\) columns of the matrix \(V\).

We denote by \(\mathcal{X}\) the \(p\)-dimensional unperturbed generalized invariant subspace of \((A, B)\), by \(\mathcal{X}'\) the \(p\)-dimensional perturbed generalized invariant subspace of \((A, B)\), and by \(V_X\) and \(\tilde{V}_X\) the orthonormal bases for \(\mathcal{X}\) and \(\mathcal{X}'\), respectively. From [18], the distance between \(\mathcal{X}\) and \(\mathcal{X}'\) is measured by
\[
\sin \left( \Theta_{\max} \left( \mathcal{X}, \mathcal{X}' \right) \right) = \| V_X^H \tilde{V}_X \|_2,
\]
where \(V_X^L\) is the unitary complement of \(V_X\). Since
\[
\tilde{V}_X = V_X + \Delta V_X, V_X^L V_X = 0,
\]
we have
\[
\sin \left( \Theta_{\max} \left( \mathcal{X}, \mathcal{X}' \right) \right) = \| V_X^H \Delta V_X \|_2. \quad (48)
\]
According to (48), it is easy to find that the perturbation of generalized invariant subspace is related to the strictly lower triangular part of the matrix \(\Delta K\) given in
(6), that is, the elements \( y_l = v_i^H \Delta v_j \) with \( l = i + (j - 1)n - \frac{j(j+1)}{2} \) and \( 1 \leq j < i \leq n \). Hence,

\[
\sin \left( \Theta_{\text{max}} \left( \mathcal{D}, \mathcal{D}^* \right) \right) = \| \Delta K_{p+1:n,1:p} \|_2. \tag{49}
\]

Then the maximum angle between the perturbed and unperturbed invariant subspace of dimension \( p \) is

\[
\Theta_{\text{max}} \left( \mathcal{D}, \mathcal{D}^* \right) = \arcsin \left( \| \Delta K_{p+1:n,1:p} \|_2 \right). \tag{50}
\]

Note that from (17), the linear approximation of \( y_j \) is

\[
y_{\text{lin},l} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad l = 1, 2, \ldots, v. \tag{51}
\]

Thus, letting the elements \( y_j \) of the matrix \( \Delta K \) be replaced by the linear approximation, we have the linear approximation of \( \Delta K \):

\[
\Delta K_{\text{lin}} = \begin{bmatrix}
\star & \star & \star & \cdots & \star \\
y_{\text{lin},1} & \star & \star & \cdots & \star \\
y_{\text{lin},2} & y_{\text{lin},n} & \star & \cdots & \star \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_{\text{lin},n-1} & y_{\text{lin},2n-3} & y_{\text{lin},3n-6} & \cdots & \star
\end{bmatrix} \in \mathbb{C}^{n \times n}.
\]

Here the stars \( \star \) are numbers we’re not interested in. Further, set

\[
\tilde{L} = \begin{bmatrix}
\star & \star & \star & \cdots & \star \\
L_{v+1}^{-1} & \star & \star & \cdots & \star \\
L_{v+2}^{-1} & L_{v+n}^{-1} & \star & \cdots & \star \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
L_{v+n-1}^{-1} & L_{v+2n-3}^{-1} & L_{v+3n-6}^{-1} & \cdots & \star
\end{bmatrix} \in \mathbb{C}^{n \times 2nv}, \tag{52}
\]

\[
D_p = \text{diag} \left( \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix}, \ldots, \begin{bmatrix} f \\ g \end{bmatrix} \right) \in \mathbb{C}^{2pv \times p}.
\]

Thus, considering (51), we have

\[
\Delta K_{\text{lin},(p+1:n,1:p)} = \tilde{L}_{p+1:n,1:2pv} D_p. \tag{53}
\]

Combining (50), (53), and the following result

\[
\| D_p \|_2 = \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_2 \leq \left\| \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix} \right\|_F,
\]

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we obtain the linear perturbation bound for the $p$-dimensional generalized invariant subspace:

$$\Theta_{\max}\left(\mathcal{X}, \tilde{\mathcal{X}}\right) \leq \arcsin\left(\frac{\|\tilde{L}_{p+1:n,1:2p}\|_2}{\|A\|_F}\right).$$

(54)

We summary the above discussions in the following theorem.

**Theorem 6** The linear perturbation bound for the $p$-dimensional generalized invariant subspace of the regular matrix pair $(A, B)$ given in (54) holds.

**Remark 4** As done in [9], we can regard the term

$$\text{cond}(\Theta) = \|\tilde{L}_{p+1:n,1:2p}\|_2$$

(55)

as a condition number of the $p$-dimensional generalized invariant subspace.

**Remark 5** Let $T$ and $R$ in (1) be partitioned as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},$$

where $T_{11}$ and $R_{11} \in \mathbb{C}^{p \times p}$. By defining a linear operator as

$$T(P, Q) = (QT_{11} - T_{22}P, QR_{11} - R_{22}P),$$

(56)

and writing its matrix as

$$H = \begin{bmatrix} -I_p \otimes T_{22} & T_{11} \otimes I_{n-p} \\ -I_p \otimes R_{22} & R_{11} \otimes I_{n-p} \end{bmatrix} \in \mathbb{C}^{2p(n-p) \times 2p(n-p)},$$

Sun [15] obtained the following bound

$$\|\sin(\Theta(\mathcal{X}, \tilde{\mathcal{X}}))\|_F \leq \|\tan(\Theta(\mathcal{X}, \tilde{\mathcal{X}}))\|_F \leq s \left\|\frac{\Delta A}{\Delta B}\right\|_F,$$

(57)

where $s = \|S_1\|_2$ with $S_1 = H^{-1}_{1:p(n-p),:} \in \mathbb{C}^{p(n-p) \times 2p(n-p)}$, and regarded $s$ as the condition number for the $p$-dimensional generalized invariant subspace of $(A, B)$. In addition, according to the linear operator (6.9), Stewart [19] defined the function dif as follows:

$$\text{dif}(T_{11}, R_{11}; T_{22}, R_{22}) = \inf_{\|PQ\|_F = 1} \|T(P, Q)\|_F,$$

where $\|P, Q\|_F = \max(\|P\|_F, \|Q\|_F)$. If the linear operator $T$ is invertible, then
From Sun [15, p.395], we know that $s \leq \text{dif}^{-1} = \|T^{-1}\|_2$. In Sect. 8, we will compare (55), $s$, and $\text{dif}^{-1}$ numerically.

7 Nonlinear perturbation bounds

In the previous discussions for componentwise perturbation bounds, all the second order terms are neglected. In this part, we try to use numerical method to calculate the perturbation bounds including the second order terms, i.e., the nonlinear perturbation bounds. The numerical method is an iterative method for deriving the solution $\tilde{x}$ of the equation $T\tilde{x} = g - \varphi(\tilde{x})$ (e.g., [19]). Specifically, the solution $\tilde{x}$ can be calculated by the following procedure

1. $\tilde{x}_0 = 0$
2. for $i = 0, 1, 2, \ldots$
   1) $r_i = g - \varphi(\tilde{x}_i) - T\tilde{x}_i$,
   2) $d_i = S^{-1}r_i$,
   3) $\tilde{x}_{i+1} = \tilde{x}_i + d_i$,

where $S$ is an approximation to $T$. More details can be found in [19]. Petkov [9] ever used this approach to derive the nonlinear perturbation bounds of the Schur decomposition. In the following, we apply it to first calculate the nonlinear bounds of $x$ and $y$ in (15) and of $\Delta W$ and $\Delta K$ in (6) and then the nonlinear perturbation bounds of the generalized Schur decomposition.

Let $\Delta W = [\Delta w_1, \Delta w_2, \ldots, \Delta w_n]$ and $\Delta K = [\Delta k_1, \Delta k_2, \ldots, \Delta k_n]$. Then considering (6), we have

$$\Delta w_j = U^H \Delta u_j, \Delta k_j = V^H \Delta v_j,$$

where $j = 1, 2, \ldots, n$, which together with the facts that $U$ and $V$ are unitary matrices yield

$$\Delta u_i^H \Delta u_j = \Delta w_i^H \Delta w_j, \Delta v_i^H \Delta v_j = \Delta k_i^H \Delta k_j,$$

where $i, j = 1, 2, \ldots, n$. Thus, combining (21), (22), (59) and the fact that $u_i^H \Delta u_i = -\frac{\Delta u_i^H \Delta u_i}{2}$ and $v_i^H \Delta v_i = -\frac{\Delta v_i^H \Delta v_i}{2}$, which are proved in Sect. 3, we have
From the corresponding columns of the above inequality, we have

\[
\Delta W_2 = \begin{bmatrix}
-\|\Delta w_1\|^2_2 & 0 & 0 & \cdots & 0 \\
0 & -\|\Delta w_2\|^2_2 & 0 & \cdots & 0 \\
0 & 0 & -\|\Delta w_3\|^2_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\|\Delta w_n\|^2_2 \\
\end{bmatrix}
\]

Similarly, the matrices \(\Delta K_2\) and \(\Delta K_3\) have the same forms as the matrices \(\Delta W_2\) and \(\Delta W_3\) except that \(\Delta w_j\) is replaced by \(\Delta k_i\). Thus, considering (20), we get

\[
|\Delta W| = [|\Delta w_1|, |\Delta w_2|, \ldots, |\Delta w_n|] \leq |\Delta W_1| + |\Delta W_2| + |\Delta W_3|
\]

\[
= \begin{bmatrix}
0 & |x_1| & |x_2| & \cdots & |x_{n-1}| \\
|x_1| & 0 & |x_n| & \cdots & |x_{2n-3}| \\
|x_2| & |x_n| & 0 & \cdots & |x_{3n-6}| \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
|x_{n-1}| & |x_{2n-3}| & |x_{3n-6}| & \cdots & 0 \\
\end{bmatrix} + \begin{bmatrix}
\|\Delta w_1\|^2_2 & 0 & 0 & \cdots & 0 \\
0 & \|\Delta w_2\|^2_2 & 0 & \cdots & 0 \\
0 & 0 & \|\Delta w_3\|^2_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \|\Delta w_n\|^2_2 \\
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & |\Delta w_1^H| |\Delta w_2| & |\Delta w_1^H| |\Delta w_3| & \cdots & |\Delta w_1^H| |\Delta w_n| \\
0 & 0 & |\Delta w_2^H| |\Delta w_3| & \cdots & |\Delta w_2^H| |\Delta w_n| \\
0 & 0 & 0 & \cdots & |\Delta w_3^H| |\Delta w_n| \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

From the corresponding columns of the above inequality, we have

\[
\begin{cases}
|\Delta w_1| = |\Delta W_{11}| + |\Delta W_{21}|, \\
|\Delta w_j| \leq |S_j^w|^{-1} (|\Delta W_{1j}| + |\Delta W_{2j}|),
\end{cases}
\]

\[
\begin{bmatrix}
|\Delta w_1^H| \\
|\Delta w_2^H| \\
\vdots \\
\Delta w_{j-1}^H \\
0 \\
\vdots \\
0 \\
\end{bmatrix} = \begin{bmatrix}
|\Delta w_1| \\
|\Delta w_2| \\
\vdots \\
|\Delta w_{j-1}| \\
0 \\
\vdots \\
e_n^T
\end{bmatrix} - \begin{bmatrix}
e_1^T \\
e_2^T \\
\vdots \\
e_{j-1}^T \\
e_j^T \\
\vdots \\
e_n^T
\end{bmatrix}
\]

(60)
where \( j = 2, 3, \ldots, n \) and \( \Delta W_{1j} \) and \( \Delta W_{2j} \) are the \( j \)th column of \( \Delta W_1 \) and \( \Delta W_2 \), respectively. Similarly, we can also obtain

\[
\begin{aligned}
|\Delta k_1| &= |\Delta K_{11}| + |\Delta K_{21}|, \\
|\Delta k_j| &\leq |S_j^k|^{-1}(|\Delta K_{1j}| + |\Delta K_{2j}|), \\
|S_j^k| &= I_n - \begin{bmatrix}
|\Delta k_{j-1}^H| \\
|\Delta k_j^H| \\
\vdots \\
0
\end{bmatrix}, \\
\begin{bmatrix}
\Delta k_1^H \\
\Delta k_2^H \\
\vdots \\
0
\end{bmatrix} &= \begin{bmatrix}
e_1^T - |\Delta k_1^H| \\
e_2^T - |\Delta k_2^H| \\
\vdots \\
e_n^T
\end{bmatrix},
\end{aligned}
\] (61)

where \( j = 2, 3, \ldots, n \) and \( \Delta K_{1j} \) and \( \Delta K_{2j} \) are the \( j \)th column of \( \Delta K_1 \) and \( \Delta K_2 \), respectively. Furthermore, from (59), the estimates of \( |\Delta x| \) and \( |\Delta y| \) can be written as

\[
\begin{aligned}
|\Delta x| &= \sum_{k=1}^j |t_{kj}| |\Delta w_i^H| |\Delta w_k| + |\Delta w_i^H| |U^H AV| |\Delta k_j|, \\
|\Delta y| &= \sum_{k=1}^j |r_{kj}| |\Delta w_i^H| |\Delta w_k| + |\Delta w_i^H| |U^H BV| |\Delta k_j|,
\end{aligned}
\] (62)

where \( 1 \leq j < i \leq n \) and \( l = i + (j - 1)n - \frac{j(j+1)}{2} \). Thus, putting (2.14), (60)–(62) together, we can use the aforementioned iterative approach to obtain the nonlinear bounds of \( x, y, \Delta W \) and \( \Delta K \), i.e., \( |x_n|, |y_n|, |\Delta W_n| \) and \( |\Delta K_n| \), respectively. The details are summarized in Algorithm 1.
Based on Algorithm 1, we can find the nonlinear perturbation bounds for generalized Schur decomposition. Specifically, considering (3.6), we have the following nonlinear perturbation bounds of $U$ and $V$:

\[
\begin{align*}
|\Delta U_{nl}| &= |U||\Delta W_{nl}|, \\
|\Delta V_{nl}| &= |V||\Delta K_{nl}|.
\end{align*}
\] (63)

Using (26) and (27), we obtain the nonlinear perturbation bounds of $T$ and $R$:

\[
\begin{align*}
|\Delta T_{nl}| &= \text{triu}(U^H([A]+|A|)V|\Delta K_{nl}| + |\Delta W_{nl}||U^H([A]+|A|)V| + \|\Delta W_{nl}[U^H([A]+|A|)V|\Delta K_{nl}| + |\Delta W_{nl}||U^H|[A]+|A|)]V), \\
|\Delta R_{nl}| &= \text{triu}(U^H([B]+|B|)V|\Delta K_{nl}| + |\Delta W_{nl}||U^H([B]+|B|)V| + \|\Delta W_{nl}[U^H([B]+|B|)V|\Delta K_{nl}| + |\Delta W_{nl}||U^H|[B]+|B|)]V).
\end{align*}
\] (64)

In addition, considering (50), we have the nonlinear perturbation bound for the $p$-dimensional generalized invariant subspace:

\[
\Theta_{\max, nl} = \arcsin(\|\Delta K_{nl}\|p+1:n:1:p). \tag{65}
\]

At last, considering (33), the bounds for the second order terms in the diagonal elements of $T$ and $R$ can be obtained.
\[
\begin{align*}
\Delta^d_{ij} &= \sum_{k=1}^j |t_{kj}| |\Delta W_{nl}|_{i,k}^H |\Delta W_{nl}|_{:,k} + |\Delta W_{nl}|_{i,k}^H |U^H AV||\Delta K_{nl}|_{i,j}, \\
\Delta^d_{ij} &= \sum_{k=1}^j |r_{kj}| |\Delta W_{nl}|_{i,k}^H |\Delta W_{nl}|_{:,k} + |\Delta W_{nl}|_{i,k}^H |U^H BV||\Delta K_{nl}|_{i,j},
\end{align*}
\]

where \( i = 1, 2, \ldots, n \). Then we can obtain the nonlinear perturbation bounds for the diagonal elements of \( T \) and \( R \).

In summary, we have the following theorem.

**Theorem 7** Let \((A, B)\) and \((\tilde{A}, \tilde{B})\) have the generalized Schur decompositions in (1) and (5), respectively. Then the nonlinear perturbation bounds for \( U \) and \( V \), \( T \) and \( R \), and the \( p \)-dimensional generalized invariant subspace are given in (63), (64) and (65) respectively, and the bounds for the second order terms in the diagonal elements of \( T \) and \( R \) are given in (66) and (67).

### 8 Numerical experiments

All computations are performed in MATLAB R2018b. We compute the generalized Schur decomposition by the MATLAB function `qz` and set all numerical results to be eight decimal digits unless otherwise specified.

In the specific experiments, we consider the matrix pair

\[
A = \begin{bmatrix}
-5.9 & -2.9 & -4.1 & -1.3 & -11 \\
5.1 & -19.5 & -6 & -5.1 & 0 \\
-5.9 & -1 & -2.5 & 1 & -10.1 \\
-16 & 4 & -20 & 0 & -2 \\
6.5 & -4.5 & -2.1 & -14.5 & -2.1
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
4.5 & 4 & 0.5 & 7.9 & -1.1 \\
3.9 & 5.5 & 4.9 & -5.5 & 1.4 \\
-5.9 & 0.1 & 7.4 & 2.9 & 1.1 \\
-4 & 5.4 & -4.1 & 0.5 & 5.9 \\
3.9 & -4.5 & 1.1 & 1.1 & 7.9
\end{bmatrix},
\]

whose eigenvalues are

\[
\lambda_1 = 2.71596442, \lambda_2 = 1.41815971,
\]

\[
\lambda_3 = -0.01627420, \lambda_4 = -1.24264120, \lambda_5 = -2.43201588,
\]

and let

\[
\sum_{k=1}^j \left| t_{kj} \right| \left( \left| \Delta W_{nl} \right|_{i,k}^H \right) \left| \Delta W_{nl} \right|_{:,k} + \left| \Delta W_{nl} \right|_{i,k}^H \left| U^H AV \right| \left| \Delta K_{nl} \right|_{i,j},
\]

\[
\sum_{k=1}^j \left| r_{kj} \right| \left( \left| \Delta W_{nl} \right|_{i,k}^H \right) \left| \Delta W_{nl} \right|_{:,k} + \left| \Delta W_{nl} \right|_{i,k}^H \left| U^H BV \right| \left| \Delta K_{nl} \right|_{i,j},
\]
8.1 On the linear bounds of $x$ and $y$

By (19), the linear bounds of $x$ and $y$ are given by

$$
\Delta A = 10^{-6} \times \begin{bmatrix}
3 & -2 & 1 & 4 & -1 \\
1 & 0 & -5 & -4 & 2 \\
-4 & 9 & -3 & 6 & 1 \\
-1 & 0 & 3 & 5 & 4 \\
7 & 3 & 2 & 8 & -2
\end{bmatrix},
\Delta B = 10^{-6} \times \begin{bmatrix}
1 & 2 & 3 & -2 & 6 \\
-2 & 7 & -4 & 3 & 0 \\
8 & -2 & 4 & 5 & 1 \\
-6 & 0 & -3 & 1 & 8 \\
-5 & -2 & 3 & -9 & 1
\end{bmatrix}.
$$

$$\Delta A = 10^{-6} \times \begin{bmatrix}
0.17419294 \\
0.40018945 \\
0.10935913 \\
0.12034032 \\
0.14269152
\end{bmatrix},
\Delta B = 10^{-6} \times \begin{bmatrix}
0.18192223 \\
0.65324092 \\
0.31267546 \\
0.21624433 \\
0.20683921
\end{bmatrix},
$$

which compared with the exact absolute value of $x$ and $y$

$$
|x_{\text{lin}}| = 10^{-5} \times \begin{bmatrix}
0.14269152 \\
0.12535076 \\
0.40248315 \\
0.21364869 \\
0.38402942
\end{bmatrix},
|y_{\text{lin}}| = 10^{-5} \times \begin{bmatrix}
0.20683921 \\
0.32197838 \\
0.65547555 \\
0.36200403 \\
0.24169720
\end{bmatrix},
$$

which compared with the exact absolute value of $x$ and $y$

$$
|x| = 10^{-5} \times \begin{bmatrix}
0.00074295 \\
0.12082488 \\
0.03316662 \\
0.02389607 \\
0.00734417
\end{bmatrix},
|y| = 10^{-5} \times \begin{bmatrix}
0.04106670 \\
0.19483908 \\
0.06753171 \\
0.04902623 \\
0.04312303
\end{bmatrix},
$$

implies that the linear bounds of $x$ and $y$ are mostly an order of magnitude larger than the corresponding exact values.
8.2 On the linear perturbation bounds of $U$ and $V$

Based on (25), the linear perturbation bounds of $U$ and $V$ are

$$|\Delta U| = 10^{-5} \times \begin{bmatrix}
0.38468475 & 0.27246825 & 0.29945131 & 0.26729935 & 0.55913832 \\
0.26374593 & 0.33476081 & 0.38962465 & 0.38790461 & 0.38347382 \\
0.38583882 & 0.27766067 & 0.29551131 & 0.25619419 & 0.53648226 \\
0.24587855 & 0.21005819 & 0.38563454 & 0.23109513 & 0.34735174 \\
0.32841755 & 0.44142030 & 0.35890629 & 0.46268755 & 0.27965613 \\
\end{bmatrix},$$

$$|\Delta V| = 10^{-5} \times \begin{bmatrix}
0.38424572 & 0.41249836 & 0.74084379 & 0.35938612 & 0.50724759 \\
0.36311177 & 0.54870742 & 0.64496767 & 0.50670545 & 0.59384236 \\
0.58005598 & 0.38020000 & 0.62000334 & 0.49848125 & 0.66018186 \\
0.65402225 & 0.71478018 & 0.49806233 & 0.51246907 & 0.36743149 \\
0.68978250 & 0.56860387 & 0.39610912 & 0.55848956 & 0.56426987 \\
\end{bmatrix},$$

While the exact perturbations of $U$ and $V$ are

$$|\Delta U| = 10^{-5} \times \begin{bmatrix}
0.03271947 & 0.01004977 & 0.00134424 & 0.02137930 & 0.03189687 \\
0.00412770 & 0.01651989 & 0.03445908 & 0.00637473 & 0.01312251 \\
0.08702525 & 0.01759884 & 0.02623930 & 0.01757783 & 0.01775241 \\
0.05253147 & 0.00445104 & 0.10172491 & 0.02620602 & 0.03710982 \\
0.06963978 & 0.02897990 & 0.05360180 & 0.03621912 & 0.00487242 \\
\end{bmatrix},$$

$$|\Delta V| = 10^{-5} \times \begin{bmatrix}
0.08740596 & 0.01411635 & 0.17705352 & 0.03493969 & 0.01064313 \\
0.05844298 & 0.03364732 & 0.07484455 & 0.00846659 & 0.02244902 \\
0.10897767 & 0.04577163 & 0.08868471 & 0.06560471 & 0.04119389 \\
0.11657303 & 0.02633445 & 0.00605251 & 0.02382701 & 0.05804865 \\
0.10045922 & 0.01604350 & 0.00772145 & 0.06331427 & 0.05092626 \\
\end{bmatrix}.$$

Comparing $|\Delta U|$ and $|\Delta V|$ with $|\Delta U|$ and $|\Delta V|$, respectively, we find that the elements of the former are mostly an order of magnitude larger than the latter.

8.3 On the linear perturbation bounds of $T$ and $R$

Using (26) and (27), the linear perturbation bounds of $T$ and $R$ are
And the exact perturbation of $T$ and $R$ are

$$|\hat{T}| = 10^{-3} \times \begin{bmatrix} 0.53691509 & 0.51512020 & 0.59550180 & 0.52017664 & 0.52944730 \\ 0 & 0.53438475 & 0.58334135 & 0.53135660 & 0.54212988 \\ 0 & 0 & 0.57265449 & 0.55323306 & 0.52551356 \\ 0 & 0 & 0 & 0.40707800 & 0.38956929 \\ 0 & 0 & 0 & 0 & 0.48359999 \end{bmatrix}.$$  

$$|\hat{R}| = 10^{-3} \times \begin{bmatrix} 0.32986610 & 0.31603168 & 0.33529134 & 0.31919222 & 0.31500156 \\ 0 & 0.34828626 & 0.36523305 & 0.33561782 & 0.33582967 \\ 0 & 0 & 0.37103687 & 0.34669668 & 0.34254328 \\ 0 & 0 & 0 & 0.27212787 & 0.26436216 \\ 0 & 0 & 0 & 0 & 0.33100323 \end{bmatrix}.$$  

Comparing $|\hat{T}|$ and $|\hat{R}|$ with $|\Delta T|$ and $|\Delta R|$, respectively, we can see that the elements of the former are mostly two orders of magnitude larger than the latter. Hence, the bounds for $\Delta T$ and $\Delta R$ are a little loose.

### 8.4 On the linear perturbation for diagonal elements of $T$ and $R$

Considering (36), the linear perturbation bounds of diagonal elements of $T$ and $R$ are

$$|\Delta T| = 10^{-4} \times \begin{bmatrix} 0.08915691 & 0.13322844 & 0.35968307 & 0.09510800 & 0.11156715 \\ 0 & 0.06238591 & 0.09816516 & 0.09494656 & 0.03412775 \\ 0 & 0 & 0.04598561 & 0.09234727 & 0.05254959 \\ 0 & 0 & 0 & 0.00708673 & 0.05780197 \\ 0 & 0 & 0 & 0 & 0.02090659 \end{bmatrix}.$$  

$$|\Delta R| = 10^{-4} \times \begin{bmatrix} 0.03110455 & 0.05821619 & 0.09853095 & 0.02211806 & 0.00590593 \\ 0 & 0.05622772 & 0.01862746 & 0.00221196 & 0.00571853 \\ 0 & 0 & 0.03168093 & 0.02843183 & 0.00131302 \\ 0 & 0 & 0 & 0.04257788 & 0.11412344 \\ 0 & 0 & 0 & 0 & 0.00949379 \end{bmatrix}.$$  

### Table 1 Comparison of different bounds for generalized eigenvalues

| Eigenvalue                  | $C$         | $C_1$         | $C_2$         |
|-----------------------------|-------------|---------------|---------------|
| ⟨26.8689, 9.8929⟩           | 0.0056e−06  | 0.1531e−05    | 0.1477e−05    |
| ⟨14.3158, 10.0946⟩          | 0.2991e−06  | 0.2545e−05    | 0.2522e−05    |
| ⟨−0.1630, 10.0143⟩         | 0.0777e−06  | 0.3618e−05    | 0.3586e−05    |
| ⟨−12.4136, 9.9897⟩         | 0.0358e−06  | 0.2296e−05    | 0.2610e−05    |
| ⟨−24.0902, 9.9054⟩         | 0.2907e−06  | 0.1853e−05    | 0.1501e−05    |
which compared with the exact absolute value of vectors $\Delta \lambda'$ and $\Delta \lambda''$

\[
|\Delta \lambda'| = 10^{-5} \times \begin{bmatrix} 0.89156905 \\ 0.62385906 \\ 0.07086728 \\ 0.20906590 \end{bmatrix},
|\Delta \lambda''| = 10^{-5} \times \begin{bmatrix} 0.31104545 \\ 0.56227722 \\ 0.42577879 \\ 0.09493786 \end{bmatrix}
\]

implies that the linear perturbation bounds of $T$ and $R$ are an order of magnitude larger than the corresponding exact perturbation bounds.

Combining with (37), (38) and (39), we denote the chordal metric by $C$, Stewart and Sun’s upper bound by $C_1$ and our upper bound by $C_2$, respectively. In Table 1, we provide the corresponding results. For simplicity, we only keep four decimal digits in Table 1, from which, it is easy to see that $C_2$ is closer to $C$ than $C_1$ in most cases. That is, our bound is usually a little tighter.

We denote our condition number by $\text{cond}_1$ (see (44)), Stewart and Sun’s condition number by $\text{cond}_2$ (see (45)), Higham’s normwise condition number by $\text{cond}_3$ (see (46)) and componentwise condition number by $\text{cond}_4$ (see (47)). In Table 2, we present the corresponding results for different eigenvalue $\Delta \lambda$. It should be pointed out that for Higham’s condition numbers, we let $E = 10^6|\Delta A|$ and $F = 10^6|\Delta B|$. From Table 2, we find that our condition number $\text{cond}_1$ is very close
to Stewart and Sun’s condition number $\text{cond}2$, and both of them are smaller than Higham’s condition numbers $\text{cond}3$ and $\text{cond}4$. It should be clarified here that we can’t guarantee the condition number $\text{cond}1$ is the smallest one. In fact, the main advantage of our condition number is that it can avoid computing the generalized eigenvectors.

8.5 On the linear perturbation bound of generalized invariant subspace

Denoting the linear perturbation bound of generalized invariant subspace (54) by $\Theta_{\text{max,lin}}$ and the exact bounds by $\Theta_{\text{max}}$, we obtain the values of $\text{cond}(\Theta)$ (see (55)), $s$ (see (57)), $\text{dif}^{-1}$ (see (58)), $\Theta_{\text{max,lin}}$ and $\Theta_{\text{max}}$ in Table 3 for different values of dimension $p$. The numerical results in this table show that our condition numbers $\text{cond}(\Theta)$ are close to the corresponding values $s$. Especially, when the dimension $p = 1$, $\text{cond}(\Theta) = s$. And, both $\text{cond}(\Theta)$ and $s$ are smaller than the quantity $\text{dif}^{-1}$. For the linear perturbation bounds of generalized invariant subspace, the ones of the first two dimensions are tighter than those of the last two.

8.6 On the nonlinear perturbation bounds

The nonlinear bounds for the basic perturbation vectors $x$ and $y$ and the nonlinear perturbation bounds for the matrices $U$, $V$, $T$ and $R$, the diagonal elements of $T$ and $R$, and the generalized invariant subspace are shown below.

| $|x_{lin}|$ | $|x_{lin,p}|$ | $|x_{nl}|$ | $|x_{nl,p}|$ |
|----------|----------|----------|----------|
| 0.00009686 | 0.00003954 | 0.00009791 | 0.00003972 |
| 0.00003153 | 0.00001956 | 0.00003213 | 0.00001968 |
| 0.00003657 | 0.00002073 | 0.00003708 | 0.00002085 |
| 0.00001545 | 0.00001382 | 0.00001549 | 0.00001386 |
| 0.00006733 | 0.00004203 | 0.00006966 | 0.00004258 |
| 0.00007867 | 0.00004553 | 0.00008057 | 0.00004601 |
| 0.00002619 | 0.00002343 | 0.00002631 | 0.00002350 |
| 0.00231591 | 0.00159306 | 0.00238816 | 0.00161609 |
| 0.00009954 | 0.00008903 | 0.00010140 | 0.00008985 |
| 0.00008418 | 0.00007530 | 0.00008656 | 0.00007634 |
We can easily see that the above nonlinear bounds are a little larger than their linear counterparts.
8.7 On the case $B = I_n$

We use the example from [9] with $B = I_5$ to compare our results with the corresponding ones for the Schur decomposition from [9]. Specifically, let

$$
A = \begin{bmatrix}
2.5 & -0.4 & 1 & -0.1 & -0.9 \\
2 & 1.1 & 0 & -0.1 & -1.9 \\
2.5 & 1.5 & -1 & -1 & -3 \\
1.5 & -1.4 & 2 & 1.9 & -0.9 \\
-0.5 & -1.4 & 2 & 0.9 & 2.1
\end{bmatrix}, \quad B = I_5,
$$

whose eigenvalues are

$$
\lambda_1 = 2 + i, \quad \lambda_2 = 2 - i, \quad \lambda_3 = 1.1, \quad \lambda_4 = 1, \quad \lambda_5 = 0.5,
$$

and let

$$
\Delta A = 10^{-6} \times \begin{bmatrix}
-3 & 1 & 7 & -4 & 1 \\
6 & 0 & 4 & 2 & 9 \\
-3 & 2 & 7 & 1 & -5 \\
8 & 6 & -9 & -3 & 4 \\
7 & 4 & -3 & 2 & 6
\end{bmatrix}, \quad \Delta B = 0_{5 \times 5}.
$$

The numerical results are reported in Tables 4, 5 and 6, where the results for the Schur decomposition are labeled by the subscript $P$. From these tables, we find that our bounds or condition numbers are worse than the corresponding ones for the Schur decomposition. Comfortingly, except for the bounds for invariant subspace, the orders of magnitude of our results and the corresponding ones for the Schur decomposition are the same. The main reason for the difference may be that the
matrix $L$ defined in (14) is different from the matrix $M$ given in [9]. For this example, they can be written in the following form:

$$
L = \begin{bmatrix}
-M_{s\text{vec}}(T^T \otimes I_5)M_{s\text{vec}}^T & M_{s\text{vec}}(I_5 \otimes T)M_{s\text{vec}}^T \\
-I_{10} & I_{10}
\end{bmatrix} \in \mathbb{C}^{20 \times 20},
$$

$$
M = M_{s\text{vec}}(I_5 \otimes T - T^T \otimes I_5)M_{s\text{vec}}^T \in \mathbb{C}^{10 \times 10}.
$$

As a result, their inverses are very different, which leads to the final bounds and condition numbers are also different. We find that this phenomenon also appears in [2, p.294].

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Declarations

Conflict of interest This study does not have any conflicts to disclose.

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