Ultradiscrete Plücker relation specialized for soliton solutions

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Abstract
We propose an ultradiscrete analog of the Plücker relation specialized for soliton solutions. It is expressed by an ultradiscrete permanent which is obtained by ultradiscretizing the permanent, that is, the signature-free determinant. Using this relation, we also show soliton solutions to the ultradiscrete Kadomtsev–Petviashvili equation and the ultradiscrete two-dimensional Toda lattice equation, respectively.

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1. Introduction

Soliton equations have been researched for several decades. There are many equations expressed by different levels of discreteness. Now we have continuous, semi-discrete, discrete and ultradiscrete soliton equations. The continuous soliton equation is expressed by a partial differential equation and the semi-discrete soliton equation by a system of ordinary or partial differential equations. The Kadomtsev–Petviashvili (KP) equation and the two-dimensional Toda lattice equation are continuous and semi-discrete, respectively, and they are fundamental for the soliton theory [1, 2]. These equations are transformed into bilinear forms, and their solutions are expressed by Wronski determinants.

In general, soliton solutions in the determinant form obey Plücker relations and the relations are transformed into the soliton equations replacing the operations on the determinants by the differential or difference operators [3, 4]. This structure enables us to view the hierarchy and the common structure of soliton equations. In fact, many soliton equations including the Korteweg–de Vries (KdV) equation, the Toda lattice equation and the sine-Gordon equation are obtained from the KP equation or the two-dimensional Toda lattice equation by the reduction of variables.

The discrete soliton equation is an equation whose independent variables are all discrete. It is also expressed by the bilinear form and its determinant solution satisfies the Plücker relation. In this case, the solution is expressed by the Casorati determinant.
The ultradiscrete soliton equation is an equation whose all dependent and independent variables can take integer values. It is derived from a discrete soliton equation by ultradiscretization [5], which is a limiting procedure of the dependent variable using a key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{a/\varepsilon} + e^{b/\varepsilon}) = \max(a, b).$$  \hspace{1cm} (1)

The ultradiscrete soliton equation also has soliton solutions [6, 7]. Some interesting properties of the equation are discovered recently. For instance, Nakamura discovered a soliton solution with a periodic phase for the ultradiscrete hungry Lotka–Volterra equation [8]. Nakata proposed the vertex operator for the ultradiscrete KdV (uKdV) equation or the non-autonomous ultradiscrete KP (uKP) equation and showed their solutions [9, 10].

Moreover, the authors and Hirota proposed the ultradiscrete analog of determinant solutions though the determinant cannot be ultradiscretized directly [11–13]. Instead of the determinant, they used an ultradiscrete permanent (UP) defined by

$$\max_{1 \leq i, j \leq N} a_{ij} = \max_{1 \leq i, j \leq N} \sum_{\pi} a_{\pi_i \pi_j},$$  \hspace{1cm} (2)

where $a_{ij}$ is an arbitrary $N \times N$ matrix and $\pi = \{\pi_1, \pi_2, \ldots, \pi_N\}$ is an arbitrary permutation of $1, 2, \ldots, N$. The soliton solutions in the UP form for the uKdV equation and the ultradiscrete Toda equation are shown in [11, 12]. There exist Bäcklund transformations of ultradiscrete soliton equations [13].

The $(i, j)$ element of these UP soliton solutions is generally expressed by $|y_i + j r_j|$, where $y_i$ and $r_j$ are arbitrary parameters, and $|x|$ denotes an absolute value of $x$. For example, the soliton solution to the uKdV equation is given by

$$f_n^{(i)} = \max \left[ \begin{array}{cccc} |s_1(n, i) + 2p_1| & |s_1(n, i) + 4p_1| & \cdots & |s_1(n, i) + 2Np_1| \\ \cdots & \cdots & \cdots & \cdots \\ |s_N(n, i) + 2p_N| & |s_N(n, i) + 4p_N| & \cdots & |s_N(n, i) + 2Np_N| \end{array} \right].$$  \hspace{1cm} (3)

where

$$s_j(n, i) = p_j n - q_j i + c_j \quad q_j = \frac{1}{2}(|p_j + 1| - |p_j - 1|).$$  \hspace{1cm} (4)

Though the expression of an ultradiscrete solution is analogous to that of the discrete solution, we have not established the ultradiscretized Plücker relation. Therefore, we have used the individual method to find the solution for every ultradiscrete soliton equation.

This is due to the differences of basic operations between the determinant and the UP. We show an example of such differences as follows. The determinant satisfies

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21},$$  \hspace{1cm} (5)

for any $a_{ij}$ ($1 \leq i, j \leq 2$). When we consider the UP corresponding to the left-hand side of (5), we have

$$\max \begin{vmatrix} a_{11} & \max(a_{11}, a_{12}) \\ a_{21} & \max(a_{21}, a_{22}) \end{vmatrix}.$$  \hspace{1cm} (6)

Then, using a property of the UP

$$\max[b_1 \cdots b_{j-1}, \max(b_j, b'_j), b_{j+1} \cdots b_N] = \max[\max[b_1 \cdots b_{j-1}, b_j, b_{j+1} \cdots b_N], \max[b_1 \cdots b_{j-1}, b'_j, b_{j+1} \cdots b_N]],$$  \hspace{1cm} (7)
where \( b_j \) and \( b'_j \) \((1 \leq j \leq N)\) are arbitrary \( N \)-dimensional vectors and \( \max(b_j, b'_j) \) denotes
\[
\max(b_j, b'_j) = \left( \begin{array}{c}
\max(b_1, b'_1) \\
\max(b_2, b'_2) \\
\vdots \\
\max(b_N, b'_N)
\end{array} \right). \tag{8}
\]
we can expand (6):
\[
\max\left[ \begin{array}{c}
a_{11} \\
\max(a_{11}, a_{12}) \\
\max(a_{21}, a_{22})
\end{array} \right] = \max\left( \max\left[ \begin{array}{c}
a_{11} \\
a_{21}
\end{array} \right], \max\left[ \begin{array}{c}
a_{11} \\
a_{21}
\end{array} \right] \right)
= \max(a_{11} + a_{21}, a_{11} + a_{22}, a_{12} + a_{21}). \tag{9}
\]
In contrast to the determinant case, the first argument on the right-hand side cannot be neglected. Hence (6) is not always equal to
\[
\max\left[ \begin{array}{c}
a_{11} \\
a_{21}
\end{array} \right]. \tag{10}
\]
and it means that the UP does not have the relation such as (5).

The above kinds of differences cause many troubles when we verify the solutions. For example, one of the simplest Plücker relations is
\[
|a_1 \ldots a_{N-1} b_1| \times |a_1 \ldots a_{N-2} b_2 b_3| = |a_1 \ldots a_{N-1} b_2| \times |a_1 \ldots a_{N-2} b_1 b_3|
+ |a_1 \ldots a_{N-1} b_3| \times |a_1 \ldots a_{N-2} b_1 b_2| = 0 \tag{11}
\]
for any \( N \)-dimensional column vectors \( a_j \) and \( b_j \). However, the similar identity does not exist for the UP case. Instead, Hirota showed that UPs satisfy the following identity\(^1\) [14]:
\[
\max(\max(a_1 \ldots a_{N-1} b_1) + \max(a_1 \ldots a_{N-2} b_2 b_3),
\max(a_1 \ldots a_{N-1} b_2) + \max(a_1 \ldots a_{N-2} b_1 b_3))
= \max(a_1 \ldots a_{N-1} b_1) + \max(a_1 \ldots a_{N-2} b_2 b_3),
\max(a_1 \ldots a_{N-1} b_2) + \max(a_1 \ldots a_{N-2} b_1 b_3))
= \max(a_1 \ldots a_{N-1} b_1) + \max(a_1 \ldots a_{N-2} b_1 b_3),
\max(a_1 \ldots a_{N-1} b_2) + \max(a_1 \ldots a_{N-2} b_2 b_3)). \tag{12}
\]
This identity is not useful for the verification on ultradiscrete solutions since the anti-symmetry does not hold as shown in (5) for determinants.

In this paper, we consider a general UP expression specialized for ultradiscrete soliton solutions. The \((i, j)\) element of the specialized UP is defined by \(|y_i + jr|\) where \( y_i \) and \( r \) are arbitrary constants. Imposing this condition, we give a relation which corresponds to (11) in section 2. We call this relation the conditional ultradiscrete Plücker relation. In sections 3 and 4, we present UP soliton solutions to the uKP equation and the ultradiscrete two-dimensional (u2D) Toda lattice equation, respectively, and show that these solutions are verified by means of the conditional uPlücker relation. Finally, we give the concluding remarks in section 5.

2. Conditional ultradiscrete Plücker relation

We give the following theorem in this section.

\(^1\) Hirota gives an identity of ultradiscrete analog of Pfaffian in [14], and it reduces to (12) with proper conditions. We give another proof in terms of the UP in appendix A.
Theorem 2.1. Let \( x_j \) be an \( N \)-dimensional vector defined by
\[
x_j = \begin{pmatrix}
|y_1 + jr_1| \\
|y_2 + jr_2| \\
\vdots \\
|y_N + jr_N|
\end{pmatrix}
(13)
\]

Then
\[
\max[x_1 \cdots \hat{x}_k \cdots x_{N+1}] + \max[x_1 \cdots \hat{x}_h \cdots x_{N+2}] = \max(\max[x_1 \cdots \hat{x}_k \cdots x_{N+1}] + \max[x_1 \cdots \hat{x}_h \cdots x_{N+2}],
\max[x_1 \cdots \hat{x}_h \cdots x_{N+1}] + \max[x_1 \cdots \hat{x}_k \cdots x_{N+2}] 
\]
holds. Here \( 1 \leq k_1 < k_2 < k_3 \leq N + 1 \) and the symbol \( \hat{x}_k \) means that \( x_k \) is omitted.

Let us call (14) the ‘conditional ultradiscrete Plücker (uPlücker) relation’. We note that (14) can be rewritten as
\[
\max[M x_k, x_{k+1}] + \max[M x_k, x_{N+2}] = \max(\max[M x_k, x_{k+1}] + \max[M x_k, x_{N+2}],
\max[M x_k, x_{k+1}] + \max[M x_k, x_{N+2}] 
\]
with an \( N \times (N - 2) \) matrix \( M \) defined by
\[
M \equiv [x_1 \cdots \hat{x}_k \cdots x_{N+1}].
\]

In order to prove theorem 2.1, we give several lemmas.

Lemma 2.1. If the inequality
\[
\max[x_1 \cdots \hat{x}_k \cdots x_{N+1}] + \max[x_1 \cdots \hat{x}_h \cdots x_{N+2}] \geq \max[x_1 \cdots \hat{x}_k \cdots x_{N+1}] + \max[x_1 \cdots \hat{x}_h \cdots x_{N+2}] 
\]
holds, then (14) holds.

Lemma 2.2. Relation (14) can be rewritten as
\[
\max[x_2 \cdots \hat{x}_k \cdots x_{N+2}] + \max[x_1 \cdots \hat{x}_h \cdots x_{N+2}] = \max(\max[x_2 \cdots \hat{x}_k \cdots x_{N+2}] + \max[x_1 \cdots \hat{x}_h \cdots x_{N+2}],
\max[x_2 \cdots \hat{x}_k \cdots x_{N+2}] + \max[x_1 \cdots \hat{x}_h \cdots x_{N+2}] 
\]
where \( 1 < k_1 < k_2 < k_3 \leq N + 2 \).

Lemma 2.3. If
\[
0 \leq |r_1| \leq |r_2| \leq \cdots \leq |r_{N-1}| \leq r_N, 
\]
then the \( N \)-th order UP can be reduced to the \((N - 1)\)-th order UP as
\[
\max[x_{j_1} x_{j_2} \cdots x_{j_N}] = \max(y_N + j_N r_N + \max[\hat{x}_{j_1} \hat{x}_{j_2} \cdots \hat{x}_{j_{N-1}}],
-y_N - j_N r_N + \max[\tilde{x}_{j_1} \tilde{x}_{j_2} \cdots \tilde{x}_{j_N}]),
\]
where \( j_1 < j_2 < \cdots < j_N \) and \( \tilde{x}_j \) denotes an \((N - 1)\)-dimensional vector
\[
\tilde{x}_j = \begin{pmatrix}
|y_1 + jr_1| \\
|y_2 + jr_2| \\
\vdots \\
|y_{N-1} + jr_{N-1}|
\end{pmatrix}.
\]
Lemma 2.1 is derived from (12). Lemma 2.2 is obtained since each \( x_j \) of (14) can be rewritten as \( x_{j+N3} \) with suitable transformations. For lemma 2.3, the UP is expressed by

\[
\max[x_{j_1} x_{j_2} \cdots x_{j_N}] = \max_{\rho = \pm 1, \pi} \sum_{1 \leq i \leq N} \rho_i (y_i + \pi r_i)
\]

\[
= \max_{\rho = \pm 1} \left( \sum_{1 \leq i \leq N} \rho_i y_i + \max_{\pi} \sum_{1 \leq i \leq N} \rho_i \pi r_i \right),
\]

(22)

where \((\pi_1, \pi_2, \ldots, \pi_N)\) denotes an arbitrary permutation of 1, 2, \ldots, \( N \). The maximum of (22) is given by \( \pi_N = j_N \) in the case of \( \rho_N = 1 \) and \( \pi_N = j_1 \) in the case of \( \rho_N = -1 \) [11]. Thus, we obtain lemma 2.3.

For lemma 2.1, theorem 2.1 is proved if we show (17). Then let us prove (17) with a mathematical induction. We first prove in the case of \( N = 2 \),

\[
\max[1 3] + \max[2 4] \geq \max[1 2] + \max[3 4].
\]

(23)

Here, we adopt a simple notation \( j \) for \( x_j \). The UP defined as \( \max[1 3] \) is expanded as

\[
\max \begin{bmatrix} |y_1 + r_1| & |y_1 + 3r_1| \\ |y_2 + r_2| & |y_2 + 3r_2| \end{bmatrix} = \max(y_2 + 3r_2 + |y_1 + r_1|, -y_2 - r_2 + |y_1 + 3r_1|)
\]

(24)

by lemma 2.3. Other UPS are also expanded and we obtain

\[
\max[1 3] + \max[2 4] = \max(y_2 + 3r_2 + |y_1 + r_1|, -y_2 - r_2 + |y_1 + 3r_1|)
\]

\[
+ \max(2y_2 + 7r_2 + |y_1 + 2r_1|, -2y_2 - 3r_2 + |y_1 + 3r_1| + |y_1 + 4r_1|, r_2 + |y_1 + r_1| + |y_1 + 4r_1|, 3r_2 + |y_1 + 3r_1| + |y_1 + 2r_1|).
\]

(25)

Especially, using the formula \( \max(x, y) - \max(z, w) \leq \max(x - z, y - w) \) for any real numbers \( x, y, z \) and \( w \), the inequality

\[
||y_1 + j_1 r_1| - |y_1 + j_2 r_1|| \leq |j_1 - j_2|r_1
\]

(26)

holds for any \( j_1, j_2 \), and (25) is reduced to

\[
\max(2y_2 + 7r_2 + |y_1 + 2r_1|, -2y_2 - 3r_2 + |y_1 + 3r_1| + |y_1 + 4r_1|, 3r_2 + |y_1 + 3r_1| + |y_1 + 2r_1|).
\]

(27)

Similarly, we get

\[
\max[1 2] + \max[3 4] = \max(2y_2 + 6r_2 + |y_1 + r_1| + |y_1 + 3r_1|, 2y_2 - 5r_2 + |y_1 + r_1| + |y_1 + 4r_1|, 2r_2 + |y_1 + r_1| + |y_1 + 3r_1|).
\]

(28)

Each argument in (27) is greater than that in (28). Therefore, (23) holds.

Then let us show the inequality

\[
\max[1 \cdots k_1 \cdots N + 2] + \max[1 \cdots k_1 \cdots k_2 \cdots N + 3] \geq \max[1 \cdots k_2 \cdots N + 2] + \max[1 \cdots k_3 \cdots N + 3]
\]

(29)

for \( 1 \leq k_1 < k_2 < k_3 \leq N + 2 \) under the assumptions (17) and

\[
0 \leq |r_1| \leq |r_2| \leq \cdots \leq |r_N|.
\]

(30)

We note that (30) can be assumed without loss of generality.
In the case of $1 < k_1 < k_2 < k_3 < N + 2$ and $r_{N+1} > |r_N|$, the UPs of the left-hand side in (29) are rewritten as

$$\max[1 \cdots \hat{k}_3 \cdots N + 2] = \max(y_{N+1} + (N + 2)r_{N+1} + \max[1 \cdots \hat{k}_2 \cdots N + 1],$$

$$- y_{N+1} - r_{N+1} + \max[2 \cdots \hat{k}_2 \cdots N + 2])$$

(31)

and

$$\max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3] = \max(y_{N+1} + (N + 3)r_{N+1}$$

$$+ \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2],$$

$$- y_{N+1} - r_{N+1} + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3]),$$

(32)

respectively, by lemma 2.3. Therefore, the sum of (31) and (32) is expressed by

$$\max[1 \cdots \hat{k}_2 \cdots N + 2] + \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3]$$

$$= \max(2y_{N+1} + (2N + 5)r_{N+1} + \max[1 \cdots \hat{k}_2 \cdots N + 1]$$

$$+ \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2], -2y_{N+1} - 2r_{N+1}$$

$$+ \max[2 \cdots \hat{k}_2 \cdots N + 2] + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3],$$

$$(N + 2)r_{N+1} + \max[2 \cdots \hat{k}_2 \cdots N + 2]$$

$$+ \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2],$$

$$(N + 1)r_{N+1} + \max[1 \cdots \hat{k}_2 \cdots N + 1]$$

$$+ \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3]).$$

(33)

Similarly, the right-hand side in (29) is expressed by

$$\max(2y_{N+1} + (2N + 5)r_{N+1} + \max[1 \cdots \hat{k}_1 \cdots \hat{k}_2 \cdots N + 2],$$

$$- 2y_{N+1} - 2r_{N+1} + \max[2 \cdots \hat{k}_3 \cdots N + 2] + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_2 \cdots N + 3],$$

$$(N + 2)r_{N+1} + \max[2 \cdots \hat{k}_3 \cdots N + 2] + \max[1 \cdots \hat{k}_1 \cdots \hat{k}_2 \cdots N + 2],$$

$$(N + 1)r_{N+1} + \max[1 \cdots \hat{k}_3 \cdots N + 1] + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_2 \cdots N + 3]).$$

(34)

The first and second arguments of (33) on the right-hand side are greater than those of (34) from the assumption. The third argument of (33) on the right-hand side is also greater than that of (34) from lemma 2.2. Moreover, the following lemma holds.

**Lemma 2.4.** Inequalities

$$r_{N+1} + \max[2 \cdots \hat{k}_2 \cdots N + 2] + \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2]$$

$$\geq \max[1 \cdots \hat{k}_2 \cdots N + 1] + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3]$$

(35)

and

$$r_{N+1} + \max[2 \cdots \hat{k}_3 \cdots N + 2] + \max[1 \cdots \hat{k}_1 \cdots \hat{k}_2 \cdots N + 2]$$

$$\geq \max[1 \cdots \hat{k}_3 \cdots N + 1] + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_2 \cdots N + 3]$$

(36)

hold for $1 < k_1 < k_2 < k_3 < N + 2$.

Lemma 2.4 is proved by a mathematical induction shown in appendix B. Thus, the fourth argument is smaller than the third one in (33) and (34), respectively. Therefore, (29) holds in the case of $1 < k_1 < k_2 < k_3 < N + 2$ and $r_{N+1} > |r_N|$. A similar procedure enables us to prove in the other cases. Hence, we obtain the conditional uPlücker relation.
3. The ultradiscrete KP equation and its UP solution

Let us consider the following tau function defined by the UP:

\[ \tau(l, m, n) = \max[\phi_i(l, m, n, s + j - 1)]_{1 \leq i, j \leq N}, \]

where \( s \) is an auxiliary variable and \( \phi_i(l, m, n, s) \) is defined by

\[ \phi_i(l, m, n, s) = \max(\eta_i(l, m, n, s), \eta'_i(l, m, n, s)) \]

with

\[ \eta_i(l, m, n, s) = p_i s + \max(0, p_i - a_1)l + \max(0, p_i - a_2)m + \max(0, p_i - a_3)n + c_i, \]
\[ \eta'_i(l, m, n, s) = -p_i s + \max(0, -p_i - a_1)l \]
\[ + \max(0, -p_i - a_2)m + \max(0, -p_i - a_3)n + c'_i. \]

Here \( a_1, a_2 \) and \( a_3 \) are the parameters satisfying \( a_1 > a_2 > a_3 \), and \( p_i, c_i \) and \( c'_i \) are arbitrary parameters. One can obtain the following relations:

\[ \phi_i(l + 1, m, n) = \max(\phi_i(l, m, n, s), \phi_i(l, m, n, s + 1) - a_1), \]
\[ \phi_i(l, m + 1, n) = \max(\phi_i(l, m, n, s), \phi_i(l, m, n, s + 1) - a_2), \]
\[ \phi_i(l, m, n + 1) = \max(\phi_i(l, m, n, s), \phi_i(l, m, n, s + 1) - a_3) \]

and

\[ \phi_i(l, m, n, s + j) + \phi_i(l, m, n, s + j) \leq \max(\phi_i(l, m, n, s + j - 1) + \phi_i(l, m, n, s + j + 1), \phi_i(l, m, n, s + j - 1) + \phi_i(l, m, n, s + j + 1)) \]

for \( 1 \leq i, i_1, i_2 \leq N \). We first rewrite the tau function with (40), (41), (42) and (43) in subsection 3.1. Second we give the relation shown by the conditional uPlücker relation in subsection 3.2. Finally, in subsection 3.3, we give the UP solution for the uKP equation.

3.1. Rewriting the tau function

Using (40), \( \tau(l + 1, m, n) \) is expanded as

\[ \tau(l + 1, m, n) = \max[\phi_i(l + 1, m, n, s + j - 1)]_{1 \leq i, j \leq N} \]
\[ = \max[\max(\phi_i(l, m, n, s + j - 1), \phi_i(l, m, n, s + j - 1) - a_1)]_{1 \leq i, j \leq N}. \]

In particular, using the simple notations,

\[ \begin{pmatrix} \phi_1(l, m, n, s + j) \\ \phi_2(l, m, n, s + j) \\ \vdots \\ \phi_N(l, m, n, s + j) \end{pmatrix} \equiv \begin{pmatrix} \phi_1(j) \\ \phi_2(j) \\ \vdots \\ \phi_N(j) \end{pmatrix} \equiv \phi(j), \]

(44) is expressed by

\[ \tau(l + 1, m, n) = \max[\max(\phi(j - 1), \phi(j) - a_1 \cdot 1)]_{1 \leq j \leq N}, \]

where \( 1 \) and \( \max(\phi(j - 1), \phi(j)) \) denote

\[ 1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \]

(47)
and (8), respectively. Furthermore, by applying a property of UP (7) to each column in (46),
\( \tau(l + 1, m, n) \) is expanded as the maximum of the following 2\(^N\) UPs:
\[
\max[\phi(0) \phi(1) \phi(2) \cdots \phi(N - 1)], \\
\max[\phi(1) - a_1 \cdot 1 \phi(1) \phi(2) \cdots \phi(N - 1)], \\
\max[\phi(0) \phi(2) - a_1 \cdot 1 \phi(2) \cdots \phi(N - 1)], \\
\cdots \\
\max[\phi(1) - a_1 \cdot 1 \phi(2) - a_1 \cdot 1 \phi(3) - a_1 \cdot 1 \cdots \phi(N) - a_1 \cdot 1].
\] (48)

Let us call a set of the above UPs \( S \). Moreover, using another property of the UP,
\[
\max[b_1 \cdots b_{j-1} b_j + c \cdot 1 b_{j+1} \cdots b_N] = \max[b_1 \cdots b_{j-1} b_j b_{j+1} \cdots b_N] + c,
\] (49)

where \( b_j (1 \leq j \leq N) \) is an arbitrary \( N \)-dimensional vector and \( c \) an arbitrary constant, we can divide \( S \) into \( N + 1 \) sets as
\[
S = \{S_0, S_1 - a_1, S_2 - 2a_1, \ldots, S_N - Na_1\}. 
\] (50)

For example, \( S_0 \) is expressed as
\[
S_0 = [\max\{0 \, 1 \, 2 \cdots N - 1\}] 
\] (51)

where \( j \) denotes \( \phi(j) \), and
\[
S_1 = [\max\{1 \, 1 \, 2 \cdots N - 1\}, \max\{0 \, 2 \, 2 \, 3 \cdots N - 1\}, \ldots, \max\{0 \, 1 \, 2 \cdots N - 2 \, N\}]. 
\] (52)

For these sets of UPs, we give the following lemma.

**Lemma 3.1.** The inequality
\[
\max[M \ j \ j] \leq \max[M \ j - 1 \ j + 1] 
\] (53)
holds for any \( j \), where \( M \) denotes an arbitrary \( N \times (N - 2) \) matrix.

**Lemma 3.1** is proved since each UP is expanded as
\[
\max[M \ j \ j] = \max_{1 \leq i_1, i_2 \leq N} \left( \max_{N - 1} [M \ j \ j]_{i_1 \, i_2} + \phi_1(j) + \phi_2(j) \right), \\
\max[M \ j - 1 \ j + 1] = \max_{1 \leq i_1, i_2 \leq N} \left( \max_{N - 1} [M \ j - 1 \ j + 1]_{i_1 \, i_2} + \phi_1(j - 1) + \phi_2(j + 1) \right),
\] (54)

where \( \max A_{i_1 \, i_2 \, N - 1, N} \) denotes the \( (N - 2) \)nd-order UP obtained by eliminating the \( i_1 \)th and \( i_2 \)th rows and the \((N - 1)\)st and \( N \)th columns from the \( N \times N \) matrix \( A \). Inequality (53) is derived from
\[
\max[M \ j \ j]_{i_1 \, i_2 \ N - 1, N} = \max[M \ j - 1 \ j + 1]_{i_1 \, i_2 \ N - 1, N},
\] (55)

and (43).

Therefore, \( \max S_1 \) is determined as \( \max\{0 \, 1 \, 2 \cdots N - 2 \, N\} \) since
\[
\max\{1 \, 1 \, 2 \cdots N - 1\} \leq \max\{0 \, 2 \, 2 \, 3 \cdots N - 1\} \\
\leq \cdots \\
\leq \max\{0 \, 1 \, 2 \cdots N - N - 1\} \\
\leq \max\{0 \, 1 \, 2 \cdots N - 2 \, N\}
\] (56)

holds. Similarly, other \( \max S_k \) \((0 \leq k_1 \leq N)\) are determined, and \( \tau(l + 1, m, n) \) is reduced to the maximum of \( (N + 1) \) UPs and we obtain the following lemma.

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Lemma 3.2. The tau function $\tau(l+1, m, n)$ is reduced to
\[ \tau(l+1, m, n) = \max_{0 \leq i \leq N} (\tau_c(N - k_1, N + 1 - k_1 a_1)), \] (57)
where $\tau_c(\alpha, \beta) (\alpha < \beta)$ is the UP defined by
\[ \tau_c(\alpha, \beta) = \max[0 \cdots \beta \cdots N + 1]. \] (58)

Furthermore, using (41) and (42), $\tau(l, m + 1, n + 1)$ is also reduced to the maximum of $(N+1)^2$ UPs as follows.

Lemma 3.3. The tau function $\tau(l, m + 1, n + 1)$ is reduced to
\[ \tau(l, m + 1, n + 1) = \max_{0 \leq k_2, k_3 \leq N} (\Psi(k_2, k_3) - k_2 a_2 - k_3 a_3), \] (59)
where $\Psi(k_2, k_3)$ is defined by
\[ \Psi(k_2, k_3) = \begin{cases} \max_{0 \leq i \leq N - k_1} (\tau_c(N - k_3 - i, N - k_2 + 1 + i)) & (k_3 \geq k_2, N - k_2) \\ \max_{0 \leq i \leq k_1} (\tau_c(N - k_2 + i, N + 1 - i)) & (N - k_2 \geq k_3 \geq k_2) \\ \max_{0 \leq i \leq N - k_2} (\tau_c(N - k_2 - i, N - k_3 + 1 + i)) & (k_2 \geq k_3 \geq N - k_2) \\ \max_{0 \leq i \leq k_1} (\tau_c(N - k_2 - k_3 + i, N + 1 - i)) & (k_2, N - k_2 \geq k_3) \end{cases} \] (60)
for $0 \leq k_2, k_3 \leq N$. Especially, (60) gives
\[ \Psi(k_2 - 1, k_3) = \max(\Psi(k_2 - 1, k_2 - 1), \tau_c(N - k_3 + 1, N - k_2 + 1)), \] (61)
\[ \Psi(k_2 - 1, k_3) = \Psi(k_2, k_3 - 1), \] (61)
\[ \max(\Psi(k_2 - 1, k_3), \tau_c(N - k_2 + 1, N - k_3 + 1)) = \Psi(k_2, k_3 - 1), \] (61)
for $0 \leq k_2, k_3 \leq N$.

The proof of lemma 3.3 is shown in appendix C. We can obtain the similar expressions for $\tau(l, m + 1, n)$, $\tau(l, m, n + 1)$, $\tau(l + 1, m, n + 1)$ and $\tau(l + 1, m + 1, n)$.

3.2. Identity for $\tau_c$

For the function $\tau_c$, the following identity holds:
\[ \tau_c(k_2, N + 1) + \tau_c(k_1, k_3) = \max(\tau_c(k_1, N + 1) + \tau_c(k_2, k_3), \tau_c(k_3, N + 1) + \tau_c(k_1, k_2)), \] (62)
where $0 < k_1 < k_2 < k_3 < N + 1$. It is proved as below. Equation (62) is rewritten as
\[ \max[\phi(0) \cdots \phi(2) \cdots \phi(N)] + \max[\phi(0) \cdots \phi(1) \cdots \phi(2) \cdots \phi(N + 1)] \]
\[ = \max(\max[\phi(0) \cdots \phi(3) \cdots \phi(N)] + \max[\phi(0) \cdots \phi(1) \cdots \phi(3) \cdots \phi(N + 1)]), \]
\[ \max[\phi(0) \cdots \phi(1) \cdots \phi(3) \cdots \phi(N + 1)]. \] (63)

Especially, let us recall the definition of $\phi(j)$:
\[ \phi(j) = \begin{pmatrix} \max(\eta_1 + j p_1, \eta'_1 - j p_1) \\ \max(\eta_2 + j p_2, \eta'_2 - j p_2) \\ \vdots \\ \max(\eta_N + j p_N, \eta'_N - j p_N) \end{pmatrix}, \] (64)
where $\eta_i$ and $\eta'_i$ denote $\eta_i(l, m, n, s)$ and $\eta'_i(l, m, n, s)$ for short. By adding $\sum_{i=1}^{N}(\eta_i - \eta'_i)/2$ to both sides in (63), it is reduced to the conditional uPlücker relation, and hence, proved.
3.3. Equations for the tau functions

Substituting the expression of tau functions into

\[
\max(\tau(l + 1, m, n) + \tau(l, m + 1, n + 1) - a_1 - a_2, \\
\tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_2 - a_3, \\
\tau(l, m, n + 1) + \tau(l + 1, m + 1, n) - a_1 - a_3)
\]

(65)

and

\[
\max(\tau(l + 1, m, n) + \tau(l, m + 1, n + 1) - a_1 - a_3, \\
\tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_1 - a_2, \\
\tau(l, m, n + 1) + \tau(l + 1, m + 1, n) - a_2 - a_3)
\]

(66)

respectively, we obtain

\[
\max_{0 \leq k_1, k_2, k_3 \leq N} (\tau_c(N - k_1, N + 1) + \Psi(k_2, k_3) - (k_1 + 1)a_1 - (k_2 + 1)a_2 - k_3a_3, \\
\tau_c(N - k_2, N + 1) + \Psi(k_1, k_3) - k_1a_1 - (k_2 + 1)a_2 - (k_3 + 1)a_3, \\
\tau_c(N - k_3, N + 1) + \Psi(k_1, k_2) - (k_1 + 1)a_1 - k_2a_2 - (k_3 + 1)a_3)
\]

(67)

and

\[
\max_{0 \leq k_1, k_2, k_3 \leq N} (\tau_c(N - k_1, N + 1) + \Psi(k_2, k_3) - (k_1 + 1)a_1 - k_2a_2 - (k_3 + 1)a_3, \\
\tau_c(N - k_2, N + 1) + \Psi(k_1, k_3) - (k_1 + 1)a_1 - (k_2 + 1)a_2 - k_3a_3, \\
\tau_c(N - k_3, N + 1) + \Psi(k_1, k_2) - k_1a_1 - (k_2 + 1)a_2 - (k_3 + 1)a_3).
\]

(68)

Let us show that (67) is equal to (68). For this purpose, we compare the arguments which have the same \(-k_1a_1 - k_2a_2 - k_3a_3\) in both.

In the case of \(k_1 = 0\), the argument in (67) is expressed as

\[
\tau_c(N - k_2, N + 1) + \Psi(0, k_3) - (k_2 + 1)a_2 - (k_3 + 1)a_3.
\]

(69)

On the other hand, the argument in (68) is expressed as

\[
\tau_c(N - k_3, N + 1) + \Psi(0, k_2) - (k_2 + 1)a_2 - (k_3 + 1)a_3.
\]

(70)

They are equivalent for (60). Similarly, if \(k_2 = 0\) or \(k_3 = 0\), then the arguments are equivalent.

Next, we consider the case \(k_1 = N + 1\). When \(k_2\) or \(k_3\) is also \(N + 1\), both are obviously equivalent. When \(1 \leq k_2, k_3 \leq N\), each argument is expressed as

\[
\max(\tau_c(0, N + 1) + \Psi(k_2, k_3) - (N + 1)a_1 - k_2a_2 - k_3a_3, \\
\tau_c(N - k_3 + 1, N + 1) + \Psi(N, k_2) - (N + 1)a_1 - k_2a_2 - k_3a_3),
\]

(71)

\[
\max(\tau_c(0, N + 1) + \Psi(k_2, k_3 - 1) - (N + 1)a_1 - k_2a_2 - k_3a_3, \\
\tau_c(N - k_2 + 1, N + 1) + \Psi(N, k_3) - (N + 1)a_1 - k_2a_2 - k_3a_3),
\]

(72)

respectively. It is trivial that they coincide when \(k_2 = k_3\). When \(k_2 > k_3\), (71) and (72) reduce to

\[
\max(\tau_c(0, N + 1) + \max(\Psi(k_2, k_3 - 1), \tau_c(N - k_3 + 1, N - k_2 + 1)), \\
\tau_c(N - k_3 + 1, N + 1) + \tau_c(0, N - k_2 + 1) - (N + 1)a_1 - k_2a_2 - k_3a_3),
\]

(73)

\[
\max(\tau_c(0, N + 1) + \Psi(k_2, k_3 - 1), \\
\tau_c(N - k_2 + 1, N + 1) + \tau_c(0, N - k_3 + 1) - (N + 1)a_1 - k_2a_2 - k_3a_3
\]

(74)
Lemma 3.4. The UP (37) defined by (38) and (39) satisfies the equation for (60) and (61). They also coincide since

\[
\max(\tau_c(0, N + 1) + \tau_c(N - k_3 + 1, N - k_2 + 1), \tau_c(N - k_3 + 1, N + 1) + \tau_c(0, N - k_2 + 1))
\]

\[
= \tau_c(N - k_2 + 1, N + 1) + \tau_c(0, N - k_3 + 1)
\]

(75)

holds for \(1 \leq k_3 < k_2 \leq N\) because of (62). It is also shown in the case \(k_2 < k_3\).

Finally, we consider the case \(1 \leq k_1, k_2, k_3 \leq N\). The arguments in (67) and (68) are expressed as

\[
\max(\tau_c(N - k_1 + 1, N + 1) + \Psi(k_2 - 1, k_3), \tau_c(N - k_2 + 1, N + 1) + \Psi(k_1, k_3 - 1), \tau_c(N - k_3 + 1, N + 1) + \Psi(k_1 - 1, k_3))
\]

\[
\tau_c(N - k_2 + 1, N + 1) + \Psi(k_1, k_3 - 1), \tau_c(N - k_3 + 1, N + 1) + \Psi(k_1 - 1, k_3))
\]

(76)

\[
\tau_c(N - k_2 + 1, N + 1) + \Psi(k_1, k_3 - 1), \tau_c(N - k_3 + 1, N + 1) + \Psi(k_1 - 1, k_3))
\]

(77)

It is clear that both correspond if \(k_i = k_j\) \((i, j = 1, 2, 3\) and \(i \neq j\)). Then, we assume \(k_1 > k_2 > k_3\) and have

\[
\max(\tau_c(N - k_1 + 1, N + 1) + \max(\Psi(k_2, k_3 - 1), \tau_c(N - k_2 + 1, N - k_3 + 1)), \tau_c(N - k_2 + 1, N + 1) + \max(\Psi(k_1, k_2 - 1), \tau_c(N - k_1 + 1, N - k_2 + 1))), \tau_c(N - k_2 + 1, N + 1) + \max(\Psi(k_1, k_3 - 1), \tau_c(N - k_1 + 1, N - k_3 + 1)), \tau_c(N - k_3 + 1, N + 1) + \Psi(k_1, k_2 - 1))
\]

(78)

\[
\tau_c(N - k_2 + 1, N + 1) + \max(\Psi(k_1, k_3 - 1), \tau_c(N - k_1 + 1, N - k_3 + 1)), \tau_c(N - k_3 + 1, N + 1) + \Psi(k_1, k_2 - 1))
\]

(79)

They coincide since

\[
\max(\tau_c(N - k_1 + 1, N + 1) + \tau_c(N - k_2 + 1, N - k_3 + 1), \tau_c(N - k_2 + 1, N + 1) + \tau_c(N - k_1 + 1, N - k_2 + 1))
\]

\[
= \tau_c(N - k_2 + 1, N + 1) + \tau_c(N - k_1 + 1, N - k_3 + 1)
\]

(80)

holds by (62).

Therefore, we obtain the following lemma.

Lemma 3.4. The UP (37) defined by (38) and (39) satisfies the equation

\[
\max(\tau(l + 1, m, n) + \tau(l, m + 1, n + 1) - a_1 - a_2, \tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_2 - a_3, \tau(l, m, n + 1) + \tau(l + 1, m + 1, n) - a_1 - a_3)
\]

\[
= \max(\tau(l + 1, m, n) + \tau(l, m + 1, n + 1) - a_1 - a_3, \tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_2 - a_3, \tau(l, m, n + 1) + \tau(l + 1, m + 1, n) - a_2 - a_3)
\]

(81)

In particular, it can be reduced to the uKP equation [10, 15]

\[
\tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_2 = \max(\tau(l + 1, m, n) + \tau(l, m + 1, n + 1) - a_1 - a_2, \tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_2 - a_3)
\]

(82)

since

\[
\tau(l + 1, m, n) + \tau(l, m + 1, n + 1) - a_1 - a_2 < \tau(l + 1, m, n) + \tau(l, m + 1, n + 1) - a_1 - a_3, \tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_2 - a_3 > \tau(l, m + 1, n) + \tau(l + 1, m, n + 1) - a_1 - a_2, \tau(l, m, n + 1) + \tau(l + 1, m + 1, n) - a_1 - a_3 < \tau(l, m, n + 1) + \tau(l + 1, m + 1, n) - a_2 - a_3
\]

(83)
hold for $a_1 > a_2 > a_3$. We therefore obtain theorem 3.1.

**Theorem 3.1.** The UP (37) defined by (38) and (39) satisfies the uKP equation (82).

4. The ultradiscrete 2D Toda lattice equation and its UP solution

In this section, we give the UP soliton solution to the u2D Toda lattice equation \([2]\]

\[
\tau(l, m - 1, n) + \tau(l + 1, m, n) = \max(\tau(l, m, n) + \tau(l + 1, m - 1, n),
\tau(l, m - 1, n + 1) + \tau(l + 1, m - 1, n) - \delta - \varepsilon),
\tag{84}
\]

where $\delta, \varepsilon > 0$. The procedure is similar to the previous section. We only show the points of the proof.

Considering the tau function defined by the UP

\[
\tau(l, m, n) = \max[\phi_i(l, m, n + j - 1)]_{1 \leq i, j \leq N},
\tag{85}
\]

where $\phi_i(l, m, n + j - 1)$ is defined by

\[
\phi_i(l, m, n) = \max(\eta_i(l, m, n), \eta'_i(l, m, n))
\tag{86}
\]

with

\[
\eta_i(l, m, n) = \max(0, r_i - \delta)l - \max(0, -r_i - \varepsilon)m + r_in + c_i,
\eta'_i(l, m, n) = \max(0, -r_i - \delta)l - \max(0, r_i - \varepsilon)m - r_in + c'_i.
\tag{87}
\]

Here, $r_i, c_i$ and $c'_i$ are arbitrary parameters. In particular, $\phi_i(l, m, n)$ satisfies

\[
\phi_i(l, m, n) = \max(\phi_i(l, m, n), \phi_i(l, m, n + 1) - \delta),
\tag{88}
\phi_i(l, m, n + 1) = \max(\phi_i(l, m, n), \phi_i(l, m, n - 1) - \varepsilon).
\]

Moreover, using the notation $\phi_i(l, m, n + j) \equiv \phi_i(j)$, we have

\[
\phi_i(j) + \phi_i(j) \leq \max(\phi_i(j - 1) + \phi_i(j + 1), \phi_i(j - 1) + \phi_i(j + 1)),
\tag{89}
\]

where $1 \leq i_1, i_2 \leq N$. The above relation gives the reduced expression of tau functions.

**Lemma 4.1.** The tau functions are reduced to

\[
\tau(l + 1, m, n) = \max_{0 \leq k_1 \leq N} (\tau_c(-1, N - k_1) - k_1\delta),
\tag{90}
\]

\[
\tau(l, m - 1, n) = \max_{0 \leq k_2 \leq N} (\tau_c(k_2 - 1, N) - k_2\varepsilon),
\tag{91}
\]

\[
\tau(l + 1, m, n - 1) = \max_{0 \leq k_1 \leq N} (\tau_c(N - k_1 - 1, N) - k_1\delta),
\tag{92}
\]

\[
\tau(l, m - 1, n + 1) = \max_{0 \leq k_2 \leq N} (\tau_c(-1, k_2) - k_2\varepsilon),
\tag{93}
\]

\[
\tau(l, m, n) = \tau_c(-1, N)
\tag{94}
\]

and

\[
\tau(l + 1, m - 1, n) = \max_{0 \leq k_1, k_2 \leq N} (\Psi(k_1, k_2) - k_1\delta - k_2\varepsilon),
\tag{95}
\]

where $\tau_c(\alpha, \beta) (\alpha < \beta)$ is defined by

\[
\tau_c(\alpha, \beta) = \max[-1 \ldots \hat{\alpha} \ldots \hat{\beta} \ldots N].
\tag{96}
\]
We use \( j \) for \((\phi_i(j))_{1 \leq i \leq N}\) and define \(\Psi(k_1, k_2)\) as follows:

\[
\Psi(k_1, k_2) = \begin{cases} 
\max_{0 \leq i \leq k_2} (\tau_i(k_2 - i - 1, N - k_1 + i)) & (k_1 \geq k_2 \text{ and } N - k_1 \geq k_2) \\
\max_{0 \leq i \leq k_1} (\tau_i(k_2 - i - 1, N - k_1 + i)) & (N - k_1 \geq k_2 \geq k_1) \\
\max_{0 \leq i \leq N - k_1} (\tau_i(i - 1, N - k_1 + k_2 - i)) & (k_1 \geq k_2 \geq N - k_1) \\
\max_{0 \leq i \leq N - k_2} (\tau_i(N - k_1 - i - 1, k_2 + i)) & (k_2 \geq N - k_1 \text{ and } k_2 \geq k_1)
\end{cases}
\]

(97)

for \(1 \leq k_1, k_2 \leq N\). In the case of \(1 \leq k_1, k_2 \leq N\),

\[
\Psi(k_1, k_2) = \max(\Psi(k_1 - 1, k_2 - 1), \tau_c(k_2 - 1, N - k_1)) \quad (k_2 - 1 < N - k_1)
\]

\[
\Psi(k_1 - 1, k_2 - 1) = \max(\Psi(k_1, k_2), \tau_c(N - k_1, k_2 - 1)) \quad (k_2 - 1 > N - k_1)
\]

(98)

\[
\Psi(k_1, k_2) = \Psi(k_1 - 1, k_2 - 1) \quad (k_2 - 1 = N - k_1)
\]

hold.

Moreover, we can obtain the following equation by the conditional upPlücker relation:

\[
\tau_c(k_1, N + 1) + \tau_c(0, k_2) = \max(\tau_c(k_2, N + 1) + \tau_c(0, k_1), \tau_c(0, N + 1) + \tau_c(k_1, k_2)),
\]

(99)

where \(1 \leq k_1 < k_2 < N + 1\). Then, comparing the arguments which have the same \(-k_1 \delta - k_2 \varepsilon\) in

\[
\max(\tau(l, m - 1, n) + \tau(l + 1, m, n), \tau(l, m, n) + \tau(l + 1, m - 1, n) - \delta - \varepsilon)
\]

(100)

and

\[
\max(\tau(l, m, n) + \tau(l + 1, m - 1, n), \tau(l, m - 1, n + 1) + \tau(l + 1, m, n - 1) - \delta - \varepsilon)
\]

(101)

with lemma 4.1 and (99), we get lemma 4.2.

**Lemma 4.2.** The UP (85) defined by (86) and (87) satisfies the equation

\[
\max(\tau(l, m - 1, n) + \tau(l + 1, m, n), \tau(l, m, n) + \tau(l + 1, m - 1, n) - \delta - \varepsilon)
\]

\[
= \max(\tau(l, m, n) + \tau(l + 1, m - 1, n), \tau(l, m - 1, n + 1) + \tau(l + 1, m, n - 1) - \delta - \varepsilon)
\]

(102)

Since (102) can be reduced to the u2D Toda lattice equation (84), we obtain the following theorem.

**Theorem 4.1.** The UP (85) defined by (86) and (87) satisfies the u2D Toda lattice equation (84).

5. Concluding remarks

In this paper, we consider the specialized UP and give the conditional upPlücker relation. Moreover, we show that it solves both the uKP and the u2D Toda lattice equation. Since the determinant solution of the continuous or discrete soliton equation is derived from the Plücker relation, the conditional upPlücker relation can be regarded as the ultradiscrete analog of the Plücker relation. However, Plücker relations used for continuous or discrete soliton equations are quite general formulae on determinants, but strong conditions are necessary for the entry of the UP in the upPlücker relation. In fact, we note that there exists a difference between determinant and UP solutions as below. The UP solution for the uKP equation (37) is defined...
by (38) and (39) and they derive (40), (41) and (42). On the other hand, the discrete KP equation
\[
a_1(a_2 - a_1)\tau(l + 1, m, n)\tau(l, m + 1, n + 1) + a_2(a_3 - a_1)\tau(l, m + 1, n)\tau(l + 1, m, n + 1) + a_3(a_1 - a_2)\tau(l, m, n + 1)\tau(l + 1, m + 1, n) = 0
\]
has the determinant solution
\[
\tau(l, m, n) = |\varphi_i(l, m, n, s + j - 1)|_{1 \leq i, j \leq N}
\]
with
\[
\varphi_i(l + 1, m, n, s) = \varphi_i(l, m, n, s) + a_1\varphi_i(l, m, n, s + 1),
\]
\[
\varphi_i(l, m + 1, n, s) = \varphi_i(l, m, n, s) + a_2\varphi_i(l, m, n, s + 1),
\]
\[
\varphi_i(l, m, n + 1, s) = \varphi_i(l, m, n, s) + a_3\varphi_i(l, m, n, s + 1).
\]
Equation (105) corresponds to (40), (41) and (42). Then it is expected that the UP solution with only (40), (41) and (42) also satisfies the uKP equation. However, it does not. In fact, for \(N = 2\), when we set the function \(\varphi_i(l, m, n, s)\) as
\[
\begin{align*}
\varphi_i(l, m, n, s) &= 10 & \varphi_i(l, m, n, s) &= 30 \\
\varphi_i(l, m, n, s + 1) &= 50 & \varphi_i(l, m, n, s + 1) &= 0 \\
\varphi_i(l, m, n, s + 2) &= 0 & \varphi_i(l, m, n, s + 2) &= 40 \\
\varphi_i(l, m, n, s + 3) &= 100 & \varphi_i(l, m, n, s + 3) &= 0
\end{align*}
\]
and \((a_1, a_2, a_3) = (30, 2, 1)\), they satisfy (40), (41), (42) and also (43). Nevertheless, the UP solution provided with the above functions does not satisfy the uKP equation. Thus, it means that the form \(|y + jr_i|\) is necessary for the UP solution. It is one of the future problems to clarify the difference between these structures.

**Appendix A. Identity of UPs**

We prove an identity of UPs (12). In this appendix, we use the simple notations of the \(N \times N\) matrices
\[
A_j = [a_1 \ldots a_{N-1} b_j] \quad (1 \leq j \leq 3),
\]
\[
A_{j,j'} = [a_1 \ldots a_{N-2} b_j b_{j'}] \quad (1 \leq j < j' \leq 3),
\]
and the \((N - 1) \times (N - 1)\) matrix obtained by eliminating the \(k\)th row and the \(l\)th column from \(A_j\) as \(A_{j,k,l}\). In the same way, the \((N - n) \times (N - n)\) matrix obtained by eliminating the \(k_1, k_2, \ldots, k_n\) rows and the \(l_1, l_2, \ldots, l_n\) columns from \(A_j\) is denoted by \(A_j|_{l_1 \ldots l_n \ldots l_n}^{k_1 \ldots k_n \ldots k_n}\). These notations give
\[
\begin{align*}
A_{11 \ldots 1} &= A_{22 \ldots 2} = A_{33 \ldots 3} = 0 \\
A_{12 \ldots 1} &= A_{21 \ldots 2} = A_{31 \ldots 3} = 0 \\
A_{13 \ldots 1} &= A_{23 \ldots 2} = A_{33 \ldots 3} = 0
\end{align*}
\]
for \(1 \leq l_1 < l_2 < \cdots < l_{n-1} \leq N - 1\) and
\[
\begin{align*}
A_{23 1 \ldots 1} &= A_{13 1 \ldots 1} = 0 \\
A_{23 2 \ldots 2} &= A_{13 2 \ldots 2} = 0 \\
A_{23 3 \ldots 3} &= A_{13 3 \ldots 3} = 0
\end{align*}
\]
for \(1 \leq l_1 < l_2 < \cdots < l_{n-1} \leq N - 2\).

We can expand \(A_1\) as
\[
\max A_1 = \max_{1 \leq l_1 \leq N} \left( \max_{1 \leq l_1 \leq N} (a_1 t_1 + b_{l_1}) \right).
\]
Here $b_{k,1}$ stands for the $k_i$th element of $b_1$. This expansion corresponds the cofactor expansion. Similarly, we can derive max $A_{23}$ by expanding with respect to the $k_i$th row
\[
\max A_{23} = \max \left( \max_{1 \leq k_i \leq N} \left( \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + a_{k_i,l_i} \right), \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,2}, \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,3} \right)
\]
(A.6)

for $1 \leq k_i \leq N$. The symbols $a_{k_i,l_i}, b_{k_i,2}, b_{k_i,3}$ mean the $k_i$th element of $a_{i}, b_2, b_3$, respectively. Thus, we have
\[
\max A_1 + \max A_{23} = \max_{1 \leq k_i \leq N} \left( \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + a_{k_i,l_i} \right),
\]
\[
\max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,1} + \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + a_{k_i,l_i},
\]
\[
\max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,1} + \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,2},
\]
\[
\max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,1} + \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,3} \right).
\]
(A.7)

On the other hand,
\[
\max A_2 + \max A_{13} = \max_{1 \leq k_i \leq N} \left( \max_{1 \leq l_i \leq N^2} (A_{13})_{l_i} + a_{k_i,l_i} \right),
\]
\[
\max_{1 \leq l_i \leq N^2} (A_{13})_{l_i} + b_{k_i,2} + \max_{1 \leq l_i \leq N^2} (A_{13})_{l_i} + b_{k_i,1},
\]
\[
\max_{1 \leq l_i \leq N^2} (A_{13})_{l_i} + b_{k_i,2} + \max_{1 \leq l_i \leq N^2} (A_{13})_{l_i} + b_{k_i,3} \right).
\]
(A.8)

Then using (A.2), the second argument of (A.7) is rewritten as
\[
\max A_{13} + b_{k_i,1} + \max A_{13} + b_{k_i,2}.
\]
(A.9)

Hence, the second argument of (A.7) is equal to that of (A.8); in other words,
\[
\max A_1 + b_{k_i,1} + \max A_{23} + b_{k_i,2} \leq \max A_2 + \max A_{13}.
\]
(A.10)

Similarly, it follows that the third argument of (A.7) is smaller than or equal to max $A_3 + max A_{12}$. Next, let us consider the first argument of (A.7):
\[
\max A_1 + \max \left( \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + a_{k_i,l_i} \right) + b_{k_i,1}.
\]
(A.11)

We can derive the first term by expanding with respect to the $l_i (\neq N)$th column:
\[
\max A_1 = \max_{1 \leq l_i \leq N^2} \left( \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + a_{k_i,l_i} \right),
\]
(A.12)

and the second term with respect to the $k_2$th row
\[
\max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + a_{k_i,l_i} = \max_{1 \leq l_i \leq N^2} \left( \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,2} \right) + \max_{1 \leq l_i \leq N^2} (A_{23})_{l_i} + b_{k_i,3} \right).
\]
(A.13)

Recursively, any argument of max $A_1 + \max A_{23}$ is expressed by either
\[
\max A_1 + \max \left( \sum_{1 \leq l_i \leq N^2} a_{k_i,l_i} + b_{k_i,1} + \max A_{23} + \sum_{1 \leq l_i \leq N^2} a_{k_i,l_i} + b_{k_i,2} \right),
\]
(A.14)

or
\[
\max A_1 + \max \left( \sum_{1 \leq l_i \leq N^2} a_{k_i,l_i} + b_{k_i,1} + \max A_{23} + \sum_{1 \leq l_i \leq N^2} a_{k_i,l_i} + b_{k_i,3} \right),
\]
(A.15)

Using (A.3) and (A.4), (A.14) is expressed by
\[
\max A_{23} + \max \left( \sum_{1 \leq l_i \leq N^2} a_{k_i,l_i} + b_{k_i,2} + \max A_{13} + \sum_{1 \leq l_i \leq N^2} a_{k_i,l_i} + b_{k_i,1} \right),
\]
(A.16)
and it is small than or equal to max $A_2 + \max A_{13}$. We can prove (A.15) is smaller than or equal to max $A_3 + \max A_{12}$ similarly. Therefore, we obtain
\[
\max A_1 + \max A_{23} \leq \max(\max A_2 + \max A_{13}, \max A_3 + \max A_{12})
\] (A.17)
since any argument of max $A_1 + \max A_{23}$ is smaller than or equal to either max $A_2 + \max A_{13}$ or max $A_3 + \max A_{12}$. Moreover,
\[
\max A_2 + \max A_{13} \leq \max(\max A_1 + \max A_{23}, \max A_3 + \max A_{12}),
\]
\[
\max A_3 + \max A_{12} \leq \max(\max A_1 + \max A_{23}, \max A_2 + \max A_{13})
\] (A.18)
also hold from the symmetry, and we get (12).

**Appendix B. Proofs of inequalities (35) and (36)**

We prove only (35) in this appendix since (36) is proved by the similar way. We note that the idea of the proof is given in [9]. Let us define $H_1^N$ by
\[
H_1^N \equiv \max[1 \cdots \hat{k}_2 \cdots \hat{k}_4 \cdots N] + \max[2 \cdots \hat{k}_2 \cdots \hat{k}_3 \cdots N + 2]
\]
\[
= \max[2 \cdots \hat{k}_2 \cdots N + 1] - \max[1 \cdots \hat{k}_2 \cdots \hat{k}_3 \cdots N + 1],
\]
(B.1)
where $1 < k_1 < k_2 < k_3 < N + 1$ and $N$ is a natural number satisfying $N \geq 4$. We use a mathematical induction to prove $H_1^N \leq r_N$. For $N = 4$, we can calculate
\[
\max[1 \cdots 2 \cdots 4] = \max[3 \cdots 5 \cdots 6] - \max[2 \cdots 4 \cdots 5] = \max[1 \cdots 3 \cdots 4] \leq r_4.
\] (B.2)
Let us suppose $H_1^N \leq r_N$ and prove $H_1^{N+1} \leq r_{N+1}$. Using lemma 2.3, we have
\[
\max[1 \cdots \hat{k}_2 \cdots N + 1] = \max(y_N + (N + 1)r_N + \max[1 \cdots \hat{k}_2 \cdots N],
\]
\[
= \max[2 \cdots \hat{k}_2 \cdots N + 1] - \max[1 \cdots \hat{k}_2 \cdots \hat{k}_3 \cdots N + 1],
\]
(B.3)
\[
\max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3] = \max(y_N + (N + 3)r_N + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2],
\]
\[
= \max[3 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3],
\]
(B.4)
\[
\max[2 \cdots \hat{k}_2 \cdots N + 2] = \max(y_N + (N + 2)r_N + \max[2 \cdots \hat{k}_2 \cdots N + 1],
\]
\[
= \max[3 \cdots \hat{k}_2 \cdots N + 2],
\]
(B.5)
\[
\max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2] = \max(y_N + (N + 2)r_N + \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 1],
\]
\[
= \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2].
\]
(B.6)
In the case of $k_1 = 2$, we define
\[
\max[3 \\hat{2} \cdots \\hat{k}_3 \cdots N + 3] = -r_N + \max[4 \cdots \hat{k}_3 \cdots N + 3].
\] (B.7)
Here, we have the inequalities
\[
\max[1 \cdots \hat{k}_2 \cdots N + 1] - \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2]
\]
\[
\leq \max(-r_N + \max[1 \cdots \hat{k}_2 \cdots N] - \max[1 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 1]
\]
\[
\max[2 \cdots \hat{k}_2 \cdots N + 1] - \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2])
\] (B.8)
and
\[ \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3] - \max[2 \cdots \hat{k}_2 \cdots N + 2] \leq \max(r_N + \max[2 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 2] - \max[2 \cdots \hat{k}_2 \cdots N + 1], \]

\[ \max[3 \cdots \hat{k}_1 \cdots \hat{k}_3 \cdots N + 3] - \max[3 \cdots \hat{k}_2 \cdots N + 2] \] (B.9)

from the formula \( \max(x, y) - \max(z, w) \leq \max(x - z, y - w) \) for any real numbers \( x, y, z \) and \( w \). Then, the sum of the above inequalities gives
\[ H_{N+1}^l \leq \max(H_N^l, r_N^l, -r_N^l + H_N^N + H_2^N, H_2^N) \leq r_N \] (B.10)

for assumption. Therefore, we obtain \( H_{N+1}^l \leq r_{N+1} \).

**Appendix C. Proofs of lemma 3.3**

In this appendix, we prove lemma 3.3. Relation (41) derives
\[ \tau(l, m + 1, n + 1) = \max[\phi(l, m + 1, n + 1, s + j - 1)]_{1 \leq l, j \leq N} \]
\[ = \max_{0 \leq k_2 \leq N} (\tau_c(N - k_2, N + 1|n + 1) - k_2a_2), \] (C.1)

where \( \tau_c(N - k_2, N + 1|n + 1) \) is the same as \( \tau_c(N - k_2, N + 1) \) except that the label \( n \) in \( \tau_c(N - k_2, N + 1) \) is replaced by \( n + 1 \). Furthermore, applying (42) to each column in \( \tau_c(N - k_2, N + 1|n + 1) \), we have
\[ \tau_c(N - k_2, N + 1|n + 1) = \max[\phi(j - 1), \phi(j) - a_3] \] \( \cup \leq j \leq N+1 \).

(C.2)

Let us consider the maximum of the UPs which have \(-k_3a_3\) in (C.2). In the case of \( k_3 \geq k_2, N - k_2 \), for example, it is expressed by
\[
\max \left( \begin{array}{l}
\max[0 1 \cdots N - k_3 - 1 \underbrace{N - k_3 + 1 \cdots N - k_2}_{k_2 - k_3} N - k_2 + 2 \cdots N N + 1], \\
\max[0 1 \cdots N - k_3 - 2 \underbrace{N - k_3 \cdots N - k_2}_{k_2 - k_3 + 1} N - k_2 + 1 \underbrace{N - k_2 + 3 \cdots N N + 1}_{k_2 - 1}], \\
\vdots, \\
\max[1 2 \cdots N - k_2 \underbrace{N - k_2 + 1 \cdots 2 N - k_2 - k_3}_{k_2 - N - k_2} \underbrace{2 N - k_2 - k_3 + 2 \cdots N N + 1}_{k_2 - (N - k_2)}] \\
\end{array} \right) 
\] (C.3)
due to (49) and (53). Then, the above is expressed by
\[ \max_{0 \leq l \leq N - k_3} (\tau_c(N - k_3 - i, N - k_2 + 1 + i)) \] (C.4)
and it is equal to \( \Psi(k_2, k_3) \) in the case of \( k_3 \geq k_2, N - k_2 \). We can derive (60) in other conditions by a similar procedure.

Relations (60) derive (61). For example, in the case \( k_2 < k_3 \),
\[
\Psi(k_2 - 1, k_3) = \begin{cases} 
\max_{0 \leq l \leq N - k_3} (\tau_c(N - k_3 - i, N - k_2 + 2 + i)) & (k_3 \geq N - k_2 + 1) \\
\max_{0 \leq l \leq k_2 - 1} (\tau_c(N - k_2 + 1 - k_3 - i, N + 1 - i)) & (N - k_2 + 1 \geq k_3) \\
\max_{1 \leq l \leq N - k_2 + 1} (\tau_c(N - k_3 - i + 1, N - k_2 + 1 + i)) & (k_3 \geq N - k_2 + 1) \\
\max_{0 \leq l \leq k_2 - 1} (\tau_c(N - k_2 + 1 - k_3 - i, N + 1 - i)) & (N - k_2 + 1 \geq k_3). 
\end{cases} 
\] (C.5)
On the other hand,

$$\Psi(k_2, k_3 - 1) = \begin{cases} 
\max_{0 \leq i \leq N - k_3 + 1} (\tau_i(N - k_3 + 1 - i, N - k_2 + 1 + i)) & (k_3 - 1 \geq N - k_2) \\
\max_{0 \leq i \leq k_2} (\tau_i(N - k_2 - k_3 + 1 + i, N + 1 - i)) & (N - k_2 \geq k_3 - 1).
\end{cases} \quad (C.6)$$

The other relations also hold for the symmetry. Therefore, we have completed the proofs. In addition, (97) and (98) are also given by the similar procedure.

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