KINETIC GAS DISKS SURROUNDING SCHWARZSCHILD BLACK HOLES

CARLOS GABARRETE, OLIVIER SARBAECH

Instituto de Física y Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Edificio C-3, Ciudad Universitaria, 58040 Morelia, Michoacán, México

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We describe stationary and axisymmetric gas configurations surrounding black holes. They consist of a collisionless relativistic kinetic gas of identical massive particles following bound orbits in a Schwarzschild exterior spacetime and are modeled by a one-particle distribution function which is the product of a function of the energy and a function of the orbital inclination associated with the particle’s trajectory. The morphology of the resulting configuration is analyzed.

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1. Introduction

In recent years, there has been interest in analyzing the properties of solutions to the Vlasov equation on a fixed, curved background spacetime. In particular, such an analysis has been performed for a Schwarzschild background with the aim of understanding the Bondi–Michel and Bondi–Hoyle–Littleton accretion models for a collisionless kinetic gas [1–3]. Also, kinetic analogues of the perfect fluid “Polish doughnuts” configurations are discussed in [4]. Similarly to their fluid counterparts, they describe stationary and axisymmetric disks around black holes, where the individual gas particles follow bound timelike geodesics in a Schwarzschild spacetime. In [4], these configurations are modeled by a one-particle distribution function (DF) depending only on the energy $E$, azimuthal $L_z$, and total angular momentum $L$ of the particles. Examples are given in which the DF is described by a generalized polytropic ansatz $[5, 6]$ depending only on $E$ and $L_z$. In this article, we provide additional examples where the DF is a function of $E$ and

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the inclination angle $i$ defined by $\cos i = L_z/L$. We analyze the behavior of the resulting particle density and compute the total number of particles of the gas cloud as a function of the free parameters in our ansatz.

2. The model

We work in the Schwarzschild exterior spacetime, written in the usual coordinates $(t, r, \vartheta, \varphi)$, with metric

$$g := -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad N(r) := 1 - \frac{2M}{r} > 0,$$

where $M > 0$ is the mass of the black hole. Since this spacetime is static and spherically symmetric, the particle’s rest mass $m$ is conserved along with $E$, $L$, and $L_z$. In terms of the orthonormal tetrad $e^0_\mu = N(r)^{-1/2}\partial_t$, $e^1_\mu = N(r)^{1/2}\partial_r$, $e^2_\mu = r^{-1}\partial_\vartheta$, $e^3_\mu = (r \sin \vartheta)^{-1}\partial_\varphi$, the four-momentum of the particles can be parametrized as $p = p^\mu e^\mu_\mu$ with (see [2, Eq. (58)])

$$
\begin{pmatrix}
  E \\
  \sqrt{N(r)} \\
  \epsilon_r \sqrt{E^2 - V_L(r)} \\
  \epsilon_\vartheta r \sqrt{L^2 - L_z^2} / \sin^2 \vartheta \\
  L_z \\
\end{pmatrix},
$$

where the signs $\epsilon_r = \pm 1$ and $\epsilon_\vartheta = \pm 1$ determine the direction of motion in the radial and polar directions, respectively, and $V_L(r) = N(r)(m^2 + L^2/r^2)$ is the effective potential for the radial motion.

A collisionless relativistic gas consisting of identical massive particles of mass $m$ trapped in $V_L$ is described by a DF which relaxes in time to a DF depending only on integrals of motion. This is due to phase mixing, see e.g. [7, 8] and references therein. Here, we assume, in addition, that the final configuration is axisymmetric, which implies that the DF has the form of

$$f(x, p) = F(E, L, L_z)$$

for some function $F$ which we shall specify shortly. The relevant spacetime observables are the particle current density vector field $J$ and the energy-momentum-stress tensor $T$ defined by

$$J_\mu(x) := \int_{P_x^+(m)} f(x, p)p_\mu d\text{vol}_x(p), \quad T_{\mu\nu}(x) := \int_{P_x^+(m)} f(x, p)p_\mu p_\nu d\text{vol}_x(p),$$

where $d\text{vol}_x(p) = dp^1 \wedge dp^2 \wedge dp^3 / p^0$ is the Lorentz-invariant volume form on the future mass hyperboloid $P_x^+(m)$ of mass $m$ at $x$, see [9] for details.

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1 We use units in which the speed of light and the gravitational constant are one.
For the following, we focus on the particular ansatz

$$F(E, L, L_z) := F_0(E) \cos^{2s}(i), \quad F_0(E) = \alpha \left(1 - \frac{E}{m}\right)^{k - \frac{3}{2^s}}, \quad (5)$$

where $\alpha > 0$, $k > 1/2$ are constants, $i$ is the inclination angle, and $s \geq 0$ is a parameter. The notation $f_+$ refers to the positive part of the quantity $f$, that is $f_+ = f$ if $f > 0$ and $f_+ = 0$ otherwise. Here, the function $F_0$ is the general relativistic generalization of the polytropic ansatz [10], while the parameter $s$ controls the concentration of the orbits near the equatorial plane $\vartheta = \pi/2$ (see Fig. 1).

![Fig. 1. Illustration of the effect of the parameter $s$ (left panel for small $s$, right panel for large $s$). As $s$ increases, orbits confined to planes lying close to the equatorial one become more populated, such that the configuration becomes a thin disk in the limit $s \to \infty$.](image)

For the following, we introduce the dimensionless quantities $\xi := r/M$, $\lambda := L/(Mm)$, $\varepsilon := E/m$, and $U_\lambda(\xi) := V_L(r)/m^2$, and parametrize the future mass hyperboloid $P^+_{x}(m)$ in terms of the quantities $(\varepsilon, \lambda, \chi)$, where the angle $\chi$ is defined by $(p^2, p^3) = \frac{m\lambda}{\xi}(\cos \chi, \sin \chi)$ which implies $\cos i = \sin \vartheta \sin \chi$. For bound orbits, these quantities are restricted to the following domain (see [11, Appendix A] and [4, Appendix A]):

$$\varepsilon_c(\xi) < \varepsilon < 1, \quad \lambda_c(\varepsilon) \leq \lambda \leq \lambda_{\max}(\varepsilon, \xi), \quad \text{and} \quad 0 \leq \chi \leq 2\pi, \quad (6)$$

where $\varepsilon_c(\xi)$ is the minimum energy at radius $\xi$, $\lambda_c(\varepsilon)$ is the critical value for the total angular momentum for which the maximum of the potential barrier in $U_\lambda(\xi)$ is exactly equal to $\varepsilon^2$, and $\lambda_{\max}(\varepsilon, \xi)$ is the maximum angular momentum permitted at the energy $\varepsilon$ and radius $\xi$. Note that the domain (6) is empty if $\xi < 4$, since for a Schwarzschild black hole, the minimum radius for bound orbits is $r = 4M$. 
For the ansatz (5), the fibre integrals in Eq. (4) yield

\[
J_{\hat{\mu}}(x) = \frac{m^2 \sin^{2s} \vartheta}{\xi^2} \sum_{r=\pm 1} \int_{\varepsilon_c(\xi)}^{1} \frac{\lambda_{\text{max}}(\epsilon, \xi)}{\lambda_c(\epsilon)} 2\pi \int_{0}^{\lambda_{\text{max}}(\epsilon, \xi)} p_{\hat{\mu}} F_0(E) \sin^{2s} \chi \frac{d\varepsilon \lambda d\lambda d\chi}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}},
\]

and similarly for \( T_{\hat{\mu}\hat{\nu}}(x) \). Using expressions (2) for the four-momentum, the non-vanishing orthonormal components of \( J_{\hat{\mu}} \) and \( T_{\hat{\mu}\hat{\nu}} \) are

\[
J^0 = 4\sqrt{\pi} \sin^{2s} \vartheta \frac{\Gamma(s + 1/2)}{N^3/2} m^3 \int_{\varepsilon_c(\xi)}^{1} d\varepsilon \varepsilon Y(\varepsilon, \xi)^{1/2} F_0(m\varepsilon),
\]

\[
T^{\hat{0}\hat{0}} = -4\sqrt{\pi} \sin^{2s} \vartheta \frac{\Gamma(s + 1/2)}{N^2} \frac{m^4}{\Gamma(s + 1)} \int_{\varepsilon_c(\xi)}^{1} d\varepsilon \varepsilon^2 Y(\varepsilon, \xi)^{1/2} F_0(m\varepsilon),
\]

\[
T^{\hat{1}\hat{1}} = \frac{4\sqrt{\pi}}{3} \frac{\sin^{2s} \vartheta}{N^2} \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} \frac{m^4}{\Gamma(s + 2)} \int_{\varepsilon_c(\xi)}^{1} d\varepsilon \varepsilon Y(\varepsilon, \xi)^{3/2} F_0(m\varepsilon),
\]

\[
T^{\hat{2}\hat{2}} = \frac{4\sqrt{\pi}}{3} \frac{\sin^{2s} \vartheta}{N^2} \frac{\Gamma(s + 1/2)}{\Gamma(s + 2)} \frac{m^4}{\Gamma(s + 2)} \int_{\varepsilon_c(\xi)}^{1} d\varepsilon \varepsilon Y(\varepsilon, \xi)^{1/2} Z(\varepsilon, \xi) F_0(m\varepsilon),
\]

\[
T^{\hat{3}\hat{3}} = (2s + 1)T^{\hat{2}\hat{2}},
\]

where we have introduced the shorthand notation

\[
Y(\varepsilon, \xi) := \varepsilon^2 - N(r) \left[ 1 + \frac{\lambda_c(\varepsilon)^2}{\xi^2} \right], \quad Z(\varepsilon, \xi) := \varepsilon^2 - N(r) \left[ 1 - \frac{\lambda_c(\varepsilon)^2}{2\xi^2} \right].
\]

The quantities (8)–(12) determine the relevant macroscopic observables, namely the particle density \( n = J^0 \), energy density \( E = -T^{\hat{0}\hat{0}} \), and the principal pressures \( P_1 = T^{\hat{1}\hat{1}} \), \( P_2 = T^{\hat{2}\hat{2}} \), and \( P_3 = (2s + 1)P_2 \). Note that all of these quantities have the dependency of \( \sin^{2s} \vartheta \) with respect to the polar angle \( \vartheta \). In the limit \( s = 0 \), the configurations describe a spherical shell of gas trapped in the region of \( \xi > 4 \), while for \( s = 1/2, 1, 3/2, \ldots, \) they are axisymmetric, the macroscopic variables being zero for \( \xi \leq 4 \) and along the axis \( \vartheta = 0, \pi \). In the next section, we analyze the morphology of these configurations as a function of the parameters \( k \) and \( s \) for a fixed total particle number.
3. Total particle number and behavior of the particle density

The (conserved) total particle number \( \mathcal{N} \) is defined as minus the flux integral of the current density vector field with respect to a Cauchy surface. This in turn can be rewritten as an integral over the six-dimensional phase space parametrized by \((x^i, p_i)\). To compute this integral, it is convenient to transform \((x^i, p_i)\) to action-angle variables \((Q^i, J_i)\). The integral over the angle variables \( Q_i \) yields a factor \((2\pi)^3\), while the integral over the action variables can be rewritten in terms of the conserved quantities \((E, L, L_z)\), taking into account that \( d^3J = T(E, L) dE dL dL_z / 2\pi \), where \( T(E, L) \) is the period function for the radial motion. For the Schwarzschild spacetime, this function can be expressed in terms of elliptic integrals and has the form of \( T(E, L) = 2M\varepsilon [H_2 - H_0] \) (see \[7, Appendix A\] and \[4\] for the explicit form of \( H_2 \) and \( H_0 \) in the Schwarzschild case). For ansatz (5), this yields the following expression for the total particle number:

\[
\mathcal{N} = \frac{16\pi^2}{2s + 1} (Mm)^3 \alpha \int_{\varepsilon_{\text{min}}}^1 d\varepsilon \varepsilon (1 - \varepsilon)^{k - \frac{3}{2}} \int_{\lambda_{c}(\varepsilon)}^{\lambda_{ub}(\varepsilon)} d\lambda \lambda (H_2 - H_0), \tag{14}
\]

where \( \varepsilon_{\text{min}} = \sqrt{8/9} \) and \( \lambda_{ub}(\varepsilon) \) is given in \[11, Appendix A\]. To compute this integral, it is convenient to re-parametrize the orbits in terms of their eccentricity \( e \) and “semi-latus rectum” \( P \), related to the turning points \((\xi_1, \xi_2)\) by \( \xi_1 = P/(1 + e) \) and \( \xi_2 = P/(1 - e) \), and to the conserved quantities \((\varepsilon, \lambda)\) according to \[4, 7, 12, 13\] \((\varepsilon^2, \lambda^2) = (P^{-1}[(P - 2)^2 - 4e^2], P^2)/(P - e^2 - 3)\). Here, \((P, e)\) are restricted to the domain \(0 < e < 1\) and \( P > 6 + 2e\). The resulting integral is then calculated numerically using Mathematica. The total mass is simply \( m\mathcal{N} \) and the total energy is given by the same expression as in Eq. (14) with an extra factor \( m\varepsilon \) inside the integral.

Fig. 2. Left panel: Dimensionless profile of the particle density in the equatorial plane for \( k = 3, 4, 5 \) and \( s = 1 \) in a logarithmic scale. Right panel: The same quantity multiplied with \( \xi^2 \) which shows that even though configurations with higher values of \( k \) have a larger maximum, they have a faster decay at infinity.
In Fig. 2, we show the dimensionless quantity $M^3n/N$ in the equatorial plane for several values of $k$ and $s = 1$. In Fig. 3, we show contour plots of the same quantity in the $xz$-plane for $k = 3$ and two different values of $s$.

Fig. 3. Contour plots for the particle density in the $xz$-plane for the configurations with $k = 3$ and $s = 1$ (left panel) and $k = 3$ and $s = 3$ (right panel). Here, $(x, z) = r(\sin \vartheta, \cos \vartheta)$, the black region represents the black hole interior and the dashed black circles the interior boundary of the disk. As it is visible from these plots, the configuration with higher $s$ yields a thinner disk.

4. Conclusions

We described a family of stationary and axisymmetric collisionless gas configurations which are trapped in the gravitational potential of a Schwarzschild black hole. This family depends on two parameters $s$ and $k$ which control the thickness of the disk and its radial density distribution. An alternative model is discussed in detail in [4]. We expect these configurations to serve as a first approximation for the description of low-luminosity disks surrounding black holes.

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