The Influence of Seed Selection on the Solvency II Ratio

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Abstract

This article contains the first published example of a real economic balance sheet where the Solvency II ratio substantially depends on the seed selected for the random number generator (RNG) used. The theoretical background and the main quality criteria for RNGs are explained in detail. To serve as a gauge for RNGs, a definition of true randomness is given. Quality tests that RNGs should pass in order to generate stable results when used in risk management under Solvency II are described.

Introduction. Most German insurance companies use the standard formula for risk capital calculation, implying in almost all cases the use of the simulation model Branchensimulationsmodell (BSM), provided by the German Insurance Association (GDV) in order to evaluate the Best Estimate Liabilities (BEL), Own Funds (OF) and the Solvency Capital Requirements (SCR) within the Solvency II framework. Some insurance companies, especially the European-wide capital stock companies, use an internal model. In this paper we consider exclusively the BSM. According to legislation (see [EIO15]), the methodology for calculating the BEL has to be accurate and robust. In particular for life insurance a very important component of the calculation of BEL is the Economic Scenario Generator (ESG). Here, Guidelines 55-59, [EIO15], explicitly define requirements for its quality. The reliability of a Monte Carlo simulation of the BSM (or any other internal model) does not only depend on the quality of the model from a classical actuarial point of view but also on how well the underlying ESG performs. In this article, we put our focus on a mathematically challenging aspect within the ESG, namely the random numbers used therein serving as driver of stochasticity. Guideline 59, [EIO15], explicitly demands the proper testing of random number
generators used in the ESG. The ESG provided by the GDV is implemented in Excel and is based on a combined linear congruential generator (LCG) of Wichmann and Hill (see [Wie82]) in order to generate the underlying random numbers.

**Example.** To demonstrate the impact of the random number generator, let us consider an example of a typical German life insurance company using the BSM with a strong focus on endowment business in their portfolio. From a regulatory point of view, it is required to have stable results depending only minimal on the random number generator of the ESG. However, our data shows a significant impact of one of these settings within the ESG, namely the seed selection, on relevant financial statement data in the Economic Balance Sheet and ultimately the Solvency II ratio. The endowment business is typical for the German market and the BSM was mainly developed for its valuation. Based on our data the Solvency II ratio for YE 2016 was in the range of typical life insurance companies in the German market. Since focus on the stability of the results, we look at the relative changes and differences in percentage points and not absolute figures. In Table 1 a short comparison of the relevant financial data of two seeds is given. These two seeds are two of in total 30 seeds analysed, and show typical differences and not the maximal as observed in the set.

| Relevant financial information | Delta Own Funds [%] | -8.2 |
|------------------------------|---------------------|------|
|                              | Delta SCR [%]       | 7.2  |
|                              | Delta Risk Margin [%]| 9.9  |

| Final Solvency II ratios     | Delta SII Ratio [%] | -14.2 |
|------------------------------|---------------------|-------|
|                              | Delta SII Ratio [pp] | -35.0 |

Table 1. Comparison of relevant financial data among two seeds.

As we can assume that the BSM model works properly, there are essentially two explanations why the results in the example differ by such a large amount: Either the rates of convergence of the Monte Carlo simulation distinguish or one of the seeds produces a *more random* sequence than the other.

**What does randomness mean?** The term “random” describes the absence of patterns and predictability. For example, it would be correct to say that if $x$ is chosen *at random* according to the uniform distribution from the unit interval $[0, 1]$, then (by the strong law of large numbers), it satisfies the following property:
P1. Each of the digits 0-9 shows up in $x$’s decimal expansion with frequency $\frac{1}{10}$; e.g.

$$\lim_{n \to \infty} \frac{\text{the number of 7’s in the first } n \text{ digits of } x}{n} = \frac{1}{10}.$$ 

P1 is one example of a randomness property but its satisfaction is certainly not sufficient to guarantee randomness; there are numbers, like 0.0123456789, that are intuitively nonrandom, but satisfy P1.

So instead of satisfying only P1, one might require it also satisfy some other (almost-sure) property, P2. But then an example of something intuitively nonrandom satisfying P1 and P2 can be exhibited.

It is then natural to attempt to define randomness by considering only $x$’s that satisfy all randomness properties. The problem with this approach is that there are too many randomness properties, since the property “not equal to $x$” holds almost-surely and hence is a randomness property. If a random number were required to satisfy that property for each $x$, there would then be nothing left to call random.

The theory of computation gives a natural way to restrict the kind of randomness property; only those properties that can be effectively checked by a computer. (For sufficiently complicated $x$’s, a computer cannot check “not equal to $x$”.)

Definition 1 abstracts the notion of a randomness property by exploiting outer regularity; any probability-zero event can be covered by open set of arbitrarily small probability. Any almost-sure property that one would want a truly random $x \in [0, 1]$ to satisfy is representable as a so-called Martin-Löf test.

**Definition 1** ([Nie09]). *A (Martin-Löf) test for randomness is a sequence $U_1, U_2, \ldots$ of open subsets of $[0, 1]$, with the following properties*

- $U_n = \bigcup_k (a_{n,k}, b_{n,k})$ for rational numbers $a_{n,k}, b_{n,k}$,
- $\Pr(U_n) \leq 2^{-n}$,
- there is an (abstract) computer program that, given inputs $n$ and $k$, outputs $a_{n,k}$ and $b_{n,k}$.

*A number $x \in [0, 1]$ passes the test if $x \notin \bigcap U_n$ and $x$ is called (Martin-Löf) random if it passes every such test.*

Furthermore, Definition 1 gives a definition that is inherently not practical; finitely many digits of a given $x \in [0, 1]$ do not determine its randomness. However, from a practicioner’s perspective only finite samples are of relevance. As John Von Neumann said, “Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin.” (See [VNWiki])
Random Number Generators. Before we can explain how to test practically, if a sequence at hand is random or not we have to better understand how a computer produces random numbers and what is meant by seed selection. At the core of every Monte Carlo simulation there is a sequence of random variables $X_1, X_2, ...$ which is

(i) uniformly distributed in the interval $[0, 1]$ and 

(ii) satisfies the property that the $X_i$ are mutually independent.

A random number generator is a mechanism producing such a sequence. The economic scenario generator is fed by these numbers. One of the most ubiquitous random number generators is Mersenne-Twister. Like most other commonly used random number generators it is based on congruence calculations. Since we only intend to familiarize the reader with the underlying ideas but not to treat the topic in its full generality (see e.g. [Gla03], Chapter 2, [Nie92], Chapter 7 and most importantly [PTVF07], Chapter 7 for more details), we explain here only the simple linear congruential generator introduced by Lehmer in [Leh51] which suffices for an overall understanding.

At first we choose a large integer $m$, called the modulus, an integer $a$, called the multiplier, with $0 < a < m$, a number $c$, called the increment, with $\gcd(c, m) = 1$ and a starting value $Y_0$ with $0 \leq Y_0 < m$. A sequence of numbers is obtained by the recurrence

$$Y_{n+1} = (aY_n + c) \mod m, \quad n \geq 0.$$ 

From that a sequence in $[0, 1]$ can be derived by taking $X_{n+1} = \frac{Y_{n+1}}{m}$. In fact, a sophisticated choice of the numbers $a, c, m, Y_0$ is essential for getting a good random sequence. This process is called seed selection. Note that the sequence $Y_N$ will repeat after applying the recurrence a certain number of times. The repeating cycle is called period. Of course, it is a desirable property that the period is as great as possible. There is a well-known theorem stating that the period is at most $m$ and it can also be described precisely when this is the case.

1Property (i) is not a restriction since there are many ways to transform random variables uniformly distributed on $[0, 1]$ to arbitrary distributed random variables, compare [Gla03], Chapter 2.2. These methods are applied by the economic scenario generators which are used by insurance companies.

2An example of a random number generator not relying on congruence is John von Neumann’s middle-square method which is for several reasons not a good method in practice.

3For certain seeds, like $m = 10$, $Y_0 = a = c = 7$, the resulting sequence $X_n$ is obviously not random, see [Kim08].
Theorem 1. (see [Knu98], Chapter 1, Theorem A) The linear congruential sequence defined by \( m, a, c \) and \( Y_0 \) has period length \( m \) if and only if

(i) \( c \) is relatively prime to \( m \),

(ii) \( b = a - 1 \) is a multiple of \( p \) for every prime \( p \) dividing \( m \),

(iii) \( b \) is a multiple of 4 if \( m \) is a multiple of 4.

Quality Criteria for Random Number Generators. Since the random number generators commonly used in industry are based on deterministic algorithms\(^4\) which are similar to the linear congruential method they inherit periodicity and do not produce independent random variables in the mathematical sense. Therefore, it is more appropriate to speak of pseudorandom numbers in this context. The main questions of the remainder of our article are now if and how we can distinguish pseudorandom numbers and random numbers. In other words: given one sequence of pseudorandom numbers and one sequence of random number can we decide statistically which one was made up by a computer? The less possible it is to detect that a sequence of numbers has been created by a computer the better the random number generator (or the random seed) is deemed.

In the following we assume that we have a finite sequence of numbers \( x_1, \ldots, x_N \) in \([0, 1]\) at hand. The present article does not aim to give an exhaustive list of all known tests but should rather be regarded as a selection of meaningful and functional tests the authors consider to be particularly intuitive or easy-to-implement in practice. For a more comprehensive overview we refer to [ES07] and [Knu98]. While reading this paragraph the reader should always keep in mind Definition 1 which makes clear that no single test or even (finite) battery of tests can guarantee that a random number generator is indeed a good mimic of randomness. Still these tests can improve our confidence on the random number generator under consideration.

Global Uniformity Test. A very simple starting point is to calculate the empirical mean and variance of the finite sequence \( x_1, \ldots, x_N \) and compare it to the values \( 1/2 \) and \( 1/12 \) expected theoretically. This can be done in a statistical precise way by applying Student’s t-test and Levene’s test.

\(^4\)Indeed, it is necessary that the algorithms be deterministic because of demands by the Solvency II delegated acts [EC15].
Moreover, the empirical distribution may in a next step be compared graphi-
cally to uniform distribution. Finally a goodness-of-fit test like Kolmogorov-
Smirnov test, Chi-squared test or Anderson-Darling test may be used to judge
statistically if the empirically observed and theoretically desired distribution
coincide globally. Note that only a very bad random number generator would
fail these basic tests.

**Permutation Test.** We choose an arbitrary integral number $2 \leq k \ll N - 1$ and consider the tuples $(x_n, x_{n+1}, \ldots, x_{n+k-1})$ for $n = 1, \ldots, N - k$. All the $k!$ relative orderings among the entries of a generic $k$-tuple should be equiprobable, i.e. have probability $\frac{1}{k!}$. By counting the empirical frequency of the orderings in the sample and applying a goodness-of-fit test we may judge if the null-hypothesis that the relative orderings are equidistributed has to be rejected or not.

**Serial Test.** A uniformly distributed sequence on $[0, 1]$ is equally dense everywhere. Even $l$-tuples of successive numbers should be uniformly and independently distributed on $[0, 1]^l$. The serial test analyses if this property is satisfied. For that purpose we at first partition each copy of $[0, 1]$ into $d$ equally long pieces for some integer $d \geq 2$. By this construction we obtain $k := d^l$ subcubes of volume $\frac{1}{k}$ and each subcube is therefore expected to contain $\lambda := \frac{N}{k}$ elements of the finite sequence. The serial test measures the discrepancy between the empiric counts $Y_j$ of the subcubes and $\lambda$. The corresponding test statistics is given by

$$X^2 := \sum_{j=0}^{r-1} \frac{(Y_j - \lambda)^2}{\lambda}$$

and it is approximately Chi-square distributed with $k - 1$ degrees of freedom. Hence, we may at the very end apply a Chi-square test to see if equidistribu-
tion holds statistically. The crucial point about the serial test is the choice of $k$. In order to have a (good) approximate Chi-square distribution it is often recommended to have $N/k \geq 5$. This gives an upper bound on $d$ because otherwise the number of categories would be too large and the test would not be exact enough (see [Knu98], 3.3.2).

**Birthday Spacings Test.** The birthday spacings test is a refinement of the serial test. Its name is inspired by the birthday “paradox” which states that the likelihood that some pair of 23 randomly chosen people has the same

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For instance, the relative ordering of the 4-tuple $(0.8, 0.1, 0.2, 0.05)$ is $(4, 2, 3, 1)$. 
As for the serial test we decompose \([0, 1]\) into \(k\) cells and consider \(n\) points of the sequence. Let \(I_1 \leq I_2 \leq \ldots \leq I_n\) be the cell numbers where these \(n\) points fall and let \(Y\) be the number of collisions. It is well-known that \(Y\) is approximately Poisson distributed with mean \(n^3/(4k)\) (see [ES07]). A Chi-squared test can then be applied to the \(N/n\) sample values of \(Y\).

\[\text{Figure 1. Pairs } (x_k, x_{k+1}) \text{ of pseudorandom numbers.}\]

**Spectral Test.** While all tests so far were driven by statistics we complete our list by a purely geometrical test. Figure 1 and Figure 2 were generated using a pseudorandom number sequence similar to the one implemented in Excel\[^7\]. They show the sets consisting of all pairs \((x_k, x_{k+1})\) respectively triples \((x_k, x_{k+1}, x_{k+2})\). While Figure 1 does not seem to be particularly remarkable at first sight, Figure 2 is astonishing since all points lie on a few planes.

\[^6\]In the language of the birthday paradox the length of the sub-sequence \(n\) corresponds to the number of birthdays and the number of cells \(k\) corresponds to the number of days in a year.

\[^7\]In fact, we used the linear congruential method with \(m = 2^{18}, a = 4649\) and \(c = 819\). This example has been chosen because the described effects become visible already in lower dimensions.
On the other hand random number generators are neither expected nor desired to produce any symmetric geometric structures. This is a drawback of all random number generators which rely on linear congruence as we will explain now. Let $1/\nu_2$ be the maximum distance between lines, taken over all families of parallel straight lines that cover all points $(x_k, x_{k+1})$. The greater $1/\nu_2$ is, the fewer lines suffice to cover all points. In other words, the greater $\nu_2$ is the less the random number generator produces symmetric structures. Hence, $\nu_2$ is the two-dimensional **accuracy** of the random number generator. The concept of accuracy $\nu_d$ can easily be carried over to higher dimensions $d$, e.g. in dimension 3 we replace lines by planes. Another interpretation of the numbers $\log \nu_d$ is that they measure how many digits of $d$-tuples may be regarded as being independent. The difference between pseudorandom numbers and random numbers truncated to multiples of some $1/\nu$ is that random number sequences have approximately the same accuracy in all dimensions. It has been suggested in the literature to accept a random number generator for application whenever $\nu_d \geq 2^{30/d}$ for $2 \leq d \leq 6$. Practical implementation of spectral test demands some lines of programing code but can still be achieved with reasonable effort (see [Knu98] for details). The number theory behind the spectral test is explained wonderfully in [Kon03].

**Conclusion.** This paper has revealed a significant impact of seed selection on the Solvency II ratio. Moreover, we explained how randomness can be defined in a mathematical rigorous sense and showed how statistical and geometrical tests can be used to distinguish good seeds of an random number generator from bad ones. We strongly suggest that these quality criteria for random number generators will be considered by insurance undertakings, regulators and external auditors as well in the future. Despite proving appro-
The Hull-White model is used in the ESG of the BSM.

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