Noncommutative Extended Waves and Soliton-like Configurations in N=2 String Theory

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ABSTRACT: The Seiberg-Witten limit of fermionic N=2 string theory with nonvanishing $B$-field is governed by noncommutative self-dual Yang-Mills theory (ncSDYM) in 2+2 dimensions. Conversely, the self-duality equations are contained in the equation of motion of N=2 string field theory in a $B$-field background. Therefore finding solutions to noncommutative self-dual Yang-Mills theory on $\mathbb{R}^{2,2}$ might help to improve our understanding of nonperturbative properties of string (field) theory. In this paper, we construct nonlinear soliton-like and multi-plane wave solutions of the ncSDYM equations corresponding to certain D-brane configurations by employing a solution generating technique, an extension of the so-called dressing approach. The underlying Lax pair is discussed in two different gauges, the unitary and the hermitean gauge. Several examples and applications for both situations are considered, including abelian solutions constructed from GMS-like projectors, noncommutative $U(2)$ soliton-like configurations and interacting plane waves. We display a correspondence to earlier work on string field theory and argue that the solutions found here can serve as a guideline in the search for nonperturbative solutions of nonpolynomial string field theory.
1 Introduction

The study of noncommutative field theory has become an important subject in modern theoretical physics. Even though the idea of noncommuting coordinates is a rather old one [1], research in this direction has been boosted only after the discovery that noncommutativity naturally emerges in string theory with a $B$-field background in a certain zero slope ($\alpha' \to 0$) limit [2, 3, 4]. Of special interest is the study of nonperturbative objects in its low energy field theory limits, like solitons or instantons, which may be interpreted as D-branes in the context of string theory (for a review see [5, 6, 7, 8]). The goal is to gain some insight into the nonperturbative sector of these theories.

The discovery of Ooguri and Vafa that open $N=2$ string theory at tree level can be identified with self-dual Yang-Mills theory [9] sparked new interest in the study of this area. That noncommutative self-dual Yang-Mills (ncSDYM) appears as the effective field theory describing the open critical $N=2$ string in 2+2 dimensions with nonvanishing $B$-field was shown later in [10, 11]. Furthermore, (commutative) self-dual Yang-Mills has been conjectured to be a universal integrable model ([12], see also [13] and references therein), meaning that all (or at least most) of the integrable equations in $d < 4$ can be obtained from the self-dual Yang-Mills equations. Therefore it is worthwhile to study the noncommutative generalization of this theory and, more specifically, plane wave and soliton-like solutions to the ncSDYM equations.

A lot of work concerning noncommutative solitons has been carried out during the past years (see e.g. [14] – [34]). In particular, noncommutative solitons and plane waves in an integrable $U(N)$ sigma model in 2+1 dimensions were discussed in [35] – [38]. In the present work we will show that all the cases studied in [35] – [38] can be obtained from ncSDYM theory by dimensional reduction, i.e., by demanding that the solutions do not depend on one of the time directions (see section 4.2). The self-duality equations will be regarded as the compatibility conditions of two linear equations (Lax pair). Solutions $\psi$ to residue equations of the latter can then be used to find solutions to the self-duality equations. By employing a solution generating technique, namely a noncommutative extension of the so-called dressing approach [39, 40, 41], we are able to compute the aforementioned auxiliary field $\psi$. Starting from a simple first order pole ansatz for $\psi$, one can easily construct higher order pole solutions corresponding to multi-soliton configurations. Recently it has been shown that a variant of the dressing method can be used to construct exact solutions of Berkovits’ string field theory [42, 43]. We will elucidate the relation between the field theoretic solutions and possible solutions in string field theory.

The paper is organized as follows: Section 2 contains a review of some results from $N=2$ string theory with nonvanishing $B$-field, motivating the program carried out in this paper from a string theory point of view and placing it in this context. In section 3 we introduce our notation and conventions as well as the Moyal-Weyl map as a useful tool for later computations. After that, the dressing approach will be discussed in section 4; a clarification of the relation between the field theory approach given herein and its string field theoretic variant introduced in [42] is added at the end of section 4. We present various calculations and examples of solutions in this framework in sections 5 and 6. The discussion of some mathematical preliminaries like twistor spaces and the moduli space of complex structures on $\mathbb{R}^{2,2}$ is relegated into appendix A. An example of an abelian pseudo-instanton which is somewhat detached from the rest of the paper will be discussed in appendix B.

2 Noncommutativity from string theory

$N=0$ and $N=1$ string theories. It is well known for $N=0$ and $N=1$ string theories that turning on a $B$-field in the presence of D-branes modifies the dynamics of open strings [4]. It alters the
ordinary Neumann boundary conditions along the brane, which results in a deformation of the space-time metric $G_{\mu\nu}$ seen by open strings. Another consequence is the emergence of space-time noncommutativity in the world-volume of the brane [3],

$$[X^\mu(\tau), X^\nu(\tau)] = i\theta^{\mu\nu}. \quad (2.1)$$

This noncommutativity pertains to the low energy field theory capturing the dynamics of open strings on the brane. In this discussion, $G^{\mu\nu}$ and $\theta^{\mu\nu}$ can be extracted from the closed string metric $g_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$ as the symmetric and antisymmetric part of

$$[(g + 2\pi\alpha' B)^{-1}]^{\mu\nu} = G^{\mu\nu} + \frac{1}{2\pi\alpha'} \theta^{\mu\nu}. \quad (2.2)$$

In the Seiberg-Witten limit [4]

$$\alpha' \to 0, \quad \text{keeping } G^{\mu\nu} \text{ and } \theta^{\mu\nu} \text{ fixed}, \quad (2.3)$$

open string theory reduces to noncommutative Yang-Mills.¹ The effective open string coupling $G_s$, which is related to the closed string coupling $g_s$ via $G_s = g_s[(\det G/\det(g + 2\pi\alpha' B))^{-1/2}]$, in this limit reduces to

$$G_s \xrightarrow{\alpha' \to 0} \frac{g_{YM}^2}{2\pi}. \quad (2.4)$$

Note that bulk effects (due to closed string modes) only decouple from the open string modes if we take the Seiberg-Witten limit.

N=2 string theory. In the case of (critical) N=2 fermionic string theory in 2+2 dimensions, an analysis of $B$-field effects was carried out in [10]. In the following, we shall briefly delineate the results of this paper. In critical N=2 string theory with nonvanishing Kähler two-form field $B = (B_{\mu\nu})$, the dynamics of fields on $N$ coincident space-time filling D-branes² in the Seiberg-Witten limit is governed by $U(N)$ ncSDYM in the Leznov gauge.³ As a nontrivial check, the authors of [10] showed the vanishing of the noncommutative field-theory four-point amplitude at tree level. This is in accordance with the expectation from N=2 string theory, which features trivial $n$-point tree-level scattering amplitudes for $n > 3$, due to a certain kinematical identity in 2+2 dimensions. In this context it is worthwhile to emphasize two points: The failure of the Moyal-Weyl commutator to close in $su(N)$ necessitates the enlargement of the gauge group from $SU(N)$ to $U(N)$ [45]. Furthermore, to obtain ncSDYM in the Yang gauge [46], which will mostly be used in this paper, one has to consider N=2 string theory restricted to the zero world-sheet instanton sector.

After this brief string theoretic overture, let us now turn to ncSDYM, whose nonperturbative solutions shall concern us for the rest of this paper.

¹Alternatively, one can keep $\alpha'$ and $g_{\mu\nu}$ fixed and take $B \to \infty$. This formulation will be useful for the string field theory discussion in section 4.4.

²Due to the absence of R-R forms in the closed string spectrum of N=2 string theory, D-branes are simply defined in parallel to bosonic string theory as submanifolds on which open strings can end.

³N=2 string theory with $N$ coincident D2-branes yields an integrable modified $U(N)$ sigma model on noncommutative $R^{2,1}$ [11] (generalizing the commutative case considered by Ward [41]).
3 Noncommutative self-dual Yang-Mills on $\mathbb{R}^{2,2}$

3.1 Notation and conventions

In this paper we will consider solutions to the self-duality equations for the noncommutative version of $U(N)$ Yang-Mills theory on the space $\mathbb{R}^{2,2}$. We choose coordinates $(x^\mu) = (x,y,\tilde{t},-\tilde{t})$ such that the metric will take the form $(\eta_{\mu\nu}) = \text{diag}(+1,+1,-1,-1)$.\(^4\)

**Coordinates.** The signature $(++--)$ allows for two different choices of isotropic (light-like) coordinates (see appendix A). The set of real isotropic coordinates (suitable for the discussion of the unitary gauge, see section 4.1) is

\[
\begin{align*}
  u &:= \frac{1}{2}(t + y), & v &:= \frac{1}{2}(t - y), \\
  \tilde{u} &:= \frac{1}{2}(\tilde{t} + x), & \tilde{v} &:= \frac{1}{2}(\tilde{t} - x),
\end{align*}
\]

(3.1a)

(3.1b)

giving rise to

\[
\begin{align*}
  \partial_u &= \partial_t + \partial_y, & \partial_v &= \partial_t - \partial_y, \\
  \partial_{\tilde{u}} &= \partial_{\tilde{t}} + \partial_x, & \partial_{\tilde{v}} &= \partial_{\tilde{t}} - \partial_x.
\end{align*}
\]

(3.2a)

(3.2b)

For the discussion of the hermitean gauge (section 4.3), the other choice of isotropic coordinates, namely complex ones,

\[
\begin{align*}
  z^1 &:= x + iy, & \bar{z}^1 &= x - iy, \\
  z^2 &:= \tilde{t} - it, & \bar{z}^2 &= \tilde{t} + it,
\end{align*}
\]

(3.3a)

(3.3b)

turns out to be useful. These yield the following partial derivatives

\[
\begin{align*}
  \partial_{z^1} &= \frac{1}{2}(\partial_x - i\partial_y), & \partial_{\bar{z}^1} &= \frac{1}{2}(\partial_x + i\partial_y), \\
  \partial_{z^2} &= \frac{1}{2}(\partial_{\tilde{t}} + i\partial_t), & \partial_{\bar{z}^2} &= \frac{1}{2}(\partial_{\tilde{t}} - i\partial_t).
\end{align*}
\]

(3.4a)

(3.4b)

**Star product.** The multiplication law used to multiply functions is the standard Moyal-Weyl star product given by

\[
(f \star g)(x) := e^{\frac{i}{2}(\theta_{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu})} f(x)g(y) \bigg|_{y=x}.
\]

(3.5)

The noncommutativity of the coordinates is encoded in the usual structure of the commutator\(^5\)

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu}.
\]

(3.6)

As a constant antisymmetric matrix, $\theta^{\mu\nu}$ is taken to be

\[
\theta^{\mu\nu} := \begin{pmatrix}
  0 & \theta^{12} & 0 & 0 \\
  \theta^{21} & 0 & 0 & 0 \\
  0 & 0 & 0 & \theta^{34} \\
  0 & 0 & \theta^{43} & 0
\end{pmatrix}.
\]

(3.7)

\(^4\)All conventions are chosen to match those of [35, 42]. The choice $x^4 = -t$ is motivated by the fact that the hyperplane $\tilde{t} = 0$ then has the same orientation as in earlier work on self-dual Yang-Mills theory dimensionally reduced to this hyperplane.

\(^5\)[x^\mu, x^\nu] := x^\mu \star x^\nu - x^\nu \star x^\mu.
where \( \theta^{12} = -\theta^{21} =: \theta \) and \( \theta^{34} = -\theta^{43} =: \tilde{\theta} \). Without loss of generality we assume \( \theta > 0 \), and choose \( \tilde{\theta} \geq 0 \), which, in the case \( \theta = \tilde{\theta} \), corresponds to self-dual \( \theta^{\mu\nu} \). Note that we are dealing with two time directions which mutually do not commute, but that the commutator of one temporal and one spatial coordinate still vanishes.

**Yang-Mills theory.** The action of noncommutative Yang-Mills theory on \( \mathbb{R}^{2,2} \) reads

\[
S_{\text{ncYM}} = -\frac{1}{2g_{\text{YM}}^2} \int \! d^4x \, \text{tr}_{u(N)} F_{\mu\nu} \star F^{\mu\nu}. \tag{3.8}
\]

Here, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). The self-duality equations in \( x^\mu \)-coordinates are given by

\[
F_{12} = F_{34}, \quad F_{13} = F_{24} \quad \text{and} \quad F_{14} = -F_{23}. \tag{3.9}
\]

Due to the Bianchi identities for \( F_{\mu\nu} \), each solution to (3.9) will also be a solution to the equations of motion of noncommutative Yang-Mills theory.

### 3.2 Moyal-Weyl map and operator formalism

The Moyal-Weyl map provides us with the possibility to switch between two equivalent noncommutative formalisms. The noncommutativity of the configuration space may be captured by deforming the multiplication law for functions (the Moyal-Weyl- or \( \star \)-product formalism), which in turn are defined over a commutative space. Equivalently, we may pass to the operator formalism, which often simplifies calculations considerably.

**Fock space.** In the operator formalism, the coordinates \( x^\mu \) become operator-valued, thus satisfying \( [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \). More specifically, the commutation relations among the coordinates \( (x,y,\tilde{t},-\tilde{t}) \) are:

\[
\begin{align*}
[\hat{x}, \hat{t}] &= [\hat{y}, \hat{t}] = [\hat{x}, \hat{\tilde{t}}] = [\hat{y}, \hat{\tilde{t}}] = 0, \tag{3.10a} \\
[\hat{x}, \hat{y}] &= i\tilde{\theta} \Rightarrow [\hat{z}^1, \hat{z}^1] = 2\theta, \tag{3.10b} \\
[\hat{t}, \hat{\tilde{t}}] &= i\tilde{\theta} \Rightarrow [\hat{z}^2, \hat{z}^2] = 2\tilde{\theta}. \tag{3.10c}
\end{align*}
\]

The last two lines lead us to construct creation and annihilation operators (for \( \theta, \tilde{\theta} > 0 \)):

\[
\begin{align*}
a_1 &:= \frac{1}{\sqrt{2\theta}} \hat{z}^1, \quad a_2 := \frac{1}{\sqrt{2\theta}} \hat{z}^2, \tag{3.11a} \\
a_1^\dagger &:= \frac{1}{\sqrt{2\theta}} \hat{\bar{z}}^1, \quad a_2^\dagger := \frac{1}{\sqrt{2\theta}} \hat{\bar{z}}^2. \tag{3.11b}
\end{align*}
\]

These operators act, as usual, in a Fock space \( \mathcal{H} \) constructed from the action of the two creation operators \( a_1^\dagger, a_2^\dagger \) on the vacuum \( |0,0\rangle \). We introduce an orthonormal basis for \( \mathcal{H} \), i.e., \( \{ |n_1, n_2\rangle; n_1, n_2 \in \mathbb{N}_0 \} \) subject to

\[
\begin{align*}
N_i |n_1, n_2\rangle := a_i^\dagger a_i |n_1, n_2\rangle &= n_i |n_1, n_2\rangle, \quad i \in \{1, 2\}, \\
a_1 |n_1, n_2\rangle &= \sqrt{n_1 |n_1 - 1, n_2\rangle}, \quad a_1^\dagger |n_1, n_2\rangle = \sqrt{n_1 + 1 |n_1 + 1, n_2\rangle}, \\
a_2 |n_1, n_2\rangle &= \sqrt{n_2 |n_1, n_2 - 1\rangle}, \quad a_2^\dagger |n_1, n_2\rangle = \sqrt{n_2 + 1 |n_1, n_2 + 1\rangle}.
\end{align*}
\]

In the case \( \tilde{\theta} = 0 \) we can only introduce \( a_1 \) and \( a_1^\dagger \); \( \mathcal{H} \) will then be a one-oscillator Fock space.

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*In the self-dual case, \( \theta^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \theta^{\rho\sigma} \), where \( \epsilon_{1234} := 1 \).*
Moyal-Weyl map. It can be shown that there exists a bijective map, which maps functions $f(z^i, \bar{z}^i)$ (also called Weyl symbols) to operators $\hat{f} := O_f(a_i, a_i^\dagger)$ (cf. e.g. [6, 21]):

$$f(z^i, \bar{z}^i) \mapsto O_f(a_i, a_i^\dagger) = -\int \frac{d^2k_1 d^2k_2}{(2\pi)^4} d^2z^1 d^2z^2$$

$$\times f(z^i, \bar{z}^i) e^{-i(k_1(\sqrt{2\theta}a_1 - \bar{z}^i) + k_1(\sqrt{2\theta}a_1^\dagger - z^i) + k_2(\sqrt{2\theta}a_2 - z^i) + k_2(\sqrt{2\theta}a_2^\dagger - \bar{z}^i))},$$

where $\int \frac{d^2k_1 d^2k_2}{(2\pi)^4} d^2z^1 d^2z^2 := \int \frac{dk_1 dk_2}{(2\pi)^2} dz^1 d\bar{z}^1 \int \frac{dk_1 dk_2}{(2\pi)^2} dz^2 d\bar{z}^2$. Note that this formula implies an ordering prescription, the so-called Weyl ordering. The inverse transformation is given by:

$$O_f(a_i, a_i^\dagger) \mapsto f(z^i, \bar{z}^i) = 4\pi^2 \theta \int \frac{d^2k_1 d^2k_2}{(2\pi)^4}$$

$$\times \text{Tr}_H \left[ O_f(a_i, a_i^\dagger) e^{i\{k_1(\sqrt{2\theta}a_1 - \bar{z}^i) + k_1(\sqrt{2\theta}a_1^\dagger - z^i) + k_2(\sqrt{2\theta}a_2 - z^i) + k_2(\sqrt{2\theta}a_2^\dagger - \bar{z}^i))}\right].$$

It is understood that, under the Moyal-Weyl map,

$$f * g \mapsto \hat{f} \hat{g}.\quad (3.14)$$

Also, an integral $\int d^4x$ over the configuration space becomes a trace $\text{Tr}_H$ over the Fock space $H$ (modulo pre-factors) and derivatives are mapped to commutators, e.g.,

$$\partial_x f \mapsto \frac{i}{\theta} [\hat{y}, \hat{f}], \quad \partial_{z^i} f \mapsto -\frac{1}{\sqrt{2\theta}} [a^\dagger_i, \hat{f}],$$

and analogously for the other possible combinations. From now on, we will work in the operator formalism; exceptions will be mentioned explicitly. In order to slenderize the notation, hats will be omitted everywhere.

4 Dressing approach

As explained in appendix A, exact solutions to the self-duality equations (3.9) can be constructed by means of an associated linear system. Solutions to this linear system will be obtained via the so-called dressing method. It was originally invented for commutative integrable models as a solution generating technique to construct solutions to the equations of motion (see, e.g. [39, 40, 41]). New solutions can be constructed from a simple vacuum seed solution by recursively applying a dressing transformation. It was shown in [35] that the dressing approach can easily be extended to noncommutative models. In the following we will apply such an extension of the dressing method to construct solutions for the Lax pairs related to the self-duality equations of ncYM on $\mathbb{R}^{2,2}$.

4.1 Unitary gauge

Lax pair. Let us start the discussion by considering the Lax pair given in terms of real isotropic coordinates [13]:

$$(\zeta \partial_{\bar{v}} + \partial_{v})\psi = -(\zeta A_{\bar{v}} + A_v)\psi, \quad (4.1a)$$

$$(\zeta \partial_{\bar{u}} - \partial_{u})\psi = -(\zeta A_{\bar{u}} - A_u)\psi, \quad (4.1b)$$
where $A = (A_\mu)$ is the antihermitean gauge potential for the self-dual field strength $F = (F_{\mu\nu})$ and $\psi \in GL(N, \mathbb{C})$.\footnote{For a detailed discussion concerning the appearance of the complexified gauge group, we refer to [47, 48].} As shown in appendix A, $\psi$ may be chosen to satisfy the following reality condition:

$$\psi(u, v, \tilde{u}, \tilde{v}, \zeta)[\psi(u, v, \tilde{u}, \tilde{v}, \bar{\zeta})]^\dagger = 1.$$  

(4.2)

The compatibility conditions for the linear equations (4.1) are given by the self-duality equations expressed in real isotropic coordinates:

$$F_{\nu\tilde{u}} = 0,$$

(4.3a)

$$F_{uv} + F_{\tilde{u}\tilde{v}} = 0,$$

(4.3b)

$$F_{v\tilde{v}} = 0.$$  

(4.3c)

If we require

$$\psi(u, v, \tilde{u}, \tilde{v}, \zeta \to 0) = g_1^{-1}(u, v, \tilde{u}, \tilde{v}) + O(\zeta)$$

(4.4)

for some $U(N)$ matrix $g_1$ and

$$A_u = g_1^{-1}\partial_u g_1,$$

(4.5a)

$$A_{\tilde{u}} = g_1^{-1}\partial_{\tilde{u}} g_1,$$

(4.5b)

then eqs. (4.1) in the limit $\zeta \to 0$ are identically satisfied [46]. Thus, solving (4.1) (without knowing the gauge fields explicitly, simply by exploiting the asymptotics of $\psi$) amounts to solving (4.3a).

In the limit $\zeta \to \infty$, we can read off from (4.1) that

$$A_{\tilde{v}} = g_2^{-1}\partial_{\tilde{v}} g_2,$$

(4.6a)

$$A_v = g_2^{-1}\partial_v g_2,$$

(4.6b)

where $g_2^{-1} := \psi(u, v, \tilde{u}, \tilde{v}, \zeta = \infty) \in U(N)$; clearly, (4.6) solves eq. (4.3c).

\textbf{Gauge fixing.} Note that we can choose a gauge in which $A_v$ and $A_{\tilde{v}}$ vanish: Consider the gauge transformation

$$\psi \mapsto \psi' := g_2 \psi.$$  

(4.7)

Its action on the gauge field yields

$$A_{\tilde{v}} \mapsto A_{\tilde{v}}' = g_2 A_{\tilde{v}} g_2^{-1} + g_2 \partial_{\tilde{v}} g_2^{-1} = 0,$$

(4.8a)

$$A_v \mapsto A_v' = g_2 A_v g_2^{-1} + g_2 \partial_v g_2^{-1} = 0;$$

(4.8b)

this is equivalent to $\psi'(u, v, \tilde{u}, \tilde{v}, \zeta = \infty) = 1$. For the remaining components we find

$$A'_{\tilde{v}} = \Omega^{-1} \partial_{\tilde{v}} \Omega,$$

(4.9a)

$$A'_v = \Omega^{-1} \partial_v \Omega,$$

(4.9b)

with $\Omega^{-1} := g_2 g_1^{-1} = \psi'(u, v, \tilde{u}, \tilde{v}, \zeta = 0)$ (Yang prepotential, cf. [49]). This gauge is called (unitary) Yang gauge.

The gauge-fixed linear equations read

$$\zeta \partial_{\tilde{v}} + \partial_u) \psi' = -A'_v \psi',$$

(4.10a)

$$\zeta \partial_v - \partial_{\tilde{u}}) \psi' = A'_u \psi'.$$

(4.10b)
Moreover, since $g_2 \in U(N)$, the reality condition (4.2) is “preserved” under (4.7):

$$\psi'(u, v, \bar{u}, \bar{v}, \zeta)[\psi'(u, v, \bar{u}, \bar{v}, \zeta)]^\dagger = g_2 g_2^\dagger = 1.$$  \hfill (4.11)

In the following we shall omit the primes on the gauge transformed quantities. Using the above expressions (4.9) for $A'_u$ and $A'_\bar{u}$, the remaining self-duality equation (4.3b) in this gauge takes the form

$$\partial_\nu (\Omega^{-1} \partial_\mu \Omega) + \partial_\mu (\Omega^{-1} \partial_\nu \Omega) = 0.$$ \hfill (4.12)

**Action functional.** Let us introduce an antisymmetric rank two tensor $\omega^{\mu\nu}$ with components $\omega^{yt} = -\omega^{ty} = -1$, $\omega^{xt} = -\omega^{tx} = -1$. Then $\omega_{\mu\nu}$ coincides with $f^{\mu\nu}$, the analogue to the ’t Hooft tensor in 2+2 dimensions introduced in [13]; it is anti-self-dual. One can interpret $\omega = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu$ as the Kähler form w.r.t. the complex structure $\tilde{J} = -\left(\begin{smallmatrix} 0 & \sigma^3 \\ \sigma^1 & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^{2,2}$ ($\sigma^3$ denotes the third Pauli matrix). In $x^\mu$-coordinates, we can rewrite eq. (4.12) as

$$(\eta^{\mu\nu} - \omega^{\mu\nu}) \partial_\mu (\Omega^{-1} \partial_\nu \Omega) = 0.$$ \hfill (4.13)

In contrast to the metric $\eta_{\mu\nu}$, the Kähler form is not invariant under SO(2,2) rotations; it therefore breaks the rotational invariance of the equation of motion even in the commutative case. A straightforward computation shows that this is the equation of motion for the Nair-Schiff type action [50, 51]

$$S = -\frac{1}{2} \int_{\mathbb{R}^{2,2}} d^4x \eta^{\mu\nu} \text{tr}(\partial_\mu \Omega^{-1} \partial_\nu \Omega) - \frac{1}{3} \int_{\mathbb{R}^{2,2} \times [0,1]} \omega \wedge \text{tr}(\tilde{A} \wedge \tilde{A} \wedge \tilde{A}).$$ \hfill (4.14)

Here the gauge potential $A = \Omega^{-1} d\Omega$ and the Kähler form $\omega$ have nonvanishing components only along $\mathbb{R}^{2,2}$; in the Wess-Zumino term, $\tilde{A} = \Omega^{-1} d\Omega$ is defined via a homotopy $\tilde{\Omega}$ from a fixed element $\Omega_1$ from the homotopy class of $\Omega$ to $\Omega$, i.e., $\tilde{\Omega}(0) = \Omega_1$, $\tilde{\Omega}(1) = \Omega$. Star products are implicit. Note that the variation w.r.t. $\tilde{\Omega}$ of the Wess-Zumino term is a total divergence. An “energy-momentum” tensor can be easily obtained from this action; however, we do not want to embark on a discussion whether it can serve to give a sensible definition of energy or momentum in 2+2 dimensions. As a simplification, we will sometimes nevertheless speak of soliton solutions if we can verify that the solutions have finite energy in 2+1 dimensional subspaces at asymptotic times.

**Dressing approach and ansatz.** Note that, due to (4.11), eq. (4.10) can be rewritten as

$$\psi(\zeta \partial_u + \partial_u)\psi^\dagger = A_u,$$ \hfill (4.15a)

$$\psi(\bar{\zeta} \partial_{\bar{u}} - \partial_{\bar{u}})\psi^\dagger = -A_{\bar{u}}.$$ \hfill (4.15b)

It is possible to solve the gauge-fixed linear equations (4.15) without knowing $A_{\bar{u}}$ and $A_u$ explicitly, simply by fixing the pole structure$^8$ of $\psi$ in such a way that the left hand sides of (4.15) are independent of $\zeta$. Inserting an ansatz for $\psi$, we obtain conditions on its residues which can be solved. From the solution $\psi$, we may determine $A_{\bar{u}}$ and $A_u$ via eqs. (4.4) and (4.5). Suppose we have constructed a seed solution $\psi_0$ by solving some appropriate (gauge-fixed) linear equations, in the present case eqs. (4.10). Then we can look for a new solution of the form

$$\psi_1 = \chi_1 \psi_0 \quad \text{with} \quad \chi_1 = 1 + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1} P_1,$$ \hfill (4.16)

$^8$A nontrivial $\psi(\zeta)$ cannot be holomorphic in $\zeta$, since $\zeta \in \mathbb{C}P^1$, which is compact.
where \( \mu_1 \in \mathbb{H}_- \) (lower half plane) is a complex constant and where \( P_1(u, v, \bar{u}, \bar{v}) \) is an \( N \times N \) matrix independent of \( \zeta \). It can be shown that \( \mu_1 \) may be interpreted as a modulus parametrizing the velocity of the lump solution (see e.g. [41, 35] for the 2+1 dimensional case).

Let us start from the vacuum seed solution \( \psi_0 = 1 \) (the corresponding gauge potential vanishes). The reality condition (4.11) for \( \psi_1 \) is satisfied if we choose \( P_1 \) to be a hermitean projector, i.e., \((P_1)^2 = P_1\) and \((P_1)^\dagger = P_1\). The transformation \( \psi_0 \mapsto \psi_1 \) is called dressing. An \( m \)-fold repetition of this procedure yields

\[
\psi_m = \prod_{p=1}^{m} \left( 1 + \frac{\mu_p - \bar{\mu}_p}{\zeta - \mu_p} P_p \right),
\]

(4.17)
corresponding to an \( m \)-soliton type configuration if all \( \mu_p \in \mathbb{H}_- \). For (4.17), the reality condition (4.11) is automatically satisfied if we choose the \( P_p \) to be hermitean (not necessarily orthogonal) projectors. We will see below that taking all \( \mu_p \) to be mutually different will lead us to interacting plane wave and non-interacting solitons, whereas second-order poles in (4.17) (i.e., \( \mu_i = \mu_j \) for some \( i \neq j \)) entail scattering in soliton-like configurations.

**First-order pole ansatz.** For now, let us restrict to an ansatz (4.17) containing only first-order poles in \( \zeta \), i.e., choose all \( \mu_p \) to be mutually different. Then, performing a decomposition into partial fractions, we can rewrite the multiplicative ansatz (4.17) in the additive form

\[
\psi_m = 1 + \sum_{p=1}^{m} \frac{R_p}{\zeta - \mu_p}.
\]

(4.18)
The \( N \times N \) matrices \( R_p(u, v, \bar{u}, \bar{v}) \) are constructed from multiplicative combinations of the \( P_p \); as in [35], we take the \( R_k \) to be of the form

\[
R_p = \sum_{l=1}^{m} T_l \Gamma_{lp} T_p^\dagger,
\]

(4.19)
where the \( T_l(u, v, \bar{u}, \bar{v}) \) are \( N \times r \) matrices and the \( \Gamma_{lp}(u, v, \bar{u}, \bar{v}) \) are \( r \times r \) matrices for some \( r \geq 1 \). The ansatz (4.18) has to satisfy the reality condition (4.11). Since the right hand side of the latter is independent of \( \zeta \), the poles on the left hand side must be removable. Therefore we should equate the corresponding residues at \( \zeta = \bar{\mu}_k \) and \( \zeta = \mu_k \) of the left hand side to zero.\(^9\) This yields

\[
\left( 1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \bar{\mu}_k} \right) \ T_k = 0.
\]

(4.20)
These algebraic conditions on \( T_k \) imply that the \( \Gamma_{lp} \) invert the matrices

\[
\tilde{\Gamma}_{pk} := \frac{T_{p}^\dagger T_k}{\mu_p - \bar{\mu}_k}, \quad \text{i.e.,} \quad \sum_{p=1}^{m} \Gamma_{lp} \tilde{\Gamma}_{pk} = \delta_{lk}.
\]

(4.21)
\(^9\)This is the simplest solution to the algebraic conditions on \( P_1 \) emerging from the reality condition (4.11).

\(^{10}\)In fact, the equation for \( \zeta = \bar{\mu}_k \) is the hermitean adjoint to the equation for \( \zeta = \mu_k \). In general, this will hold for any two residue equations if the points are related by complex conjugation (or, for \( \lambda \) from section 4.3, by the mapping (A.9)).
Furthermore, our ansatz should satisfy the gauge-fixed linear equations (4.15). Putting to zero the residues of the left hand sides of (4.15) at $\zeta = \mu_k$ and $\zeta = \bar{\mu}_k$, we learn that

\[
\begin{align*}
&\left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \mu_k}\right) (\bar{\mu}_k \partial_v + \partial_u) \bar{R}_k^1 = 0, \\
&\left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \mu_k}\right) (\bar{\mu}_k \partial_v - \partial_u) \bar{R}_k^2 = 0,
\end{align*}
\]

(4.22a, b)

Thus, we may define new isotropic coordinates (note that $\mu_k$ is complex) $w_k^1$ and $w_k^2$ in the kernel of the differential operators in (4.22):

\[
\begin{align*}
w_k^1 := \bar{\mu}_k^{-1} \bar{u} - u & \quad \text{and} \quad w_k^2 := \bar{\mu}_k^{-1} v + \bar{v} \\
\Rightarrow \bar{w}_k^1 = \mu_k^{-1} \bar{v} - u & \quad \text{and} \quad \bar{w}_k^2 = \mu_k^{-1} v + \bar{u}.
\end{align*}
\]

(4.23a, b)

The Lax operators can be written as antiholomorphic vector fields in terms of these new isotropic coordinates\(^{11}\)

\[
\begin{align*}
\bar{L}_k^1 := \bar{\mu}_k \partial_v + \partial_u = \mu_k^{-1} (\bar{\mu}_k - \mu_k) \frac{\partial}{\partial \bar{w}_k^1}, \\
\bar{L}_k^2 := \bar{\mu}_k \partial_v - \partial_u = \mu_k^{-1} (\bar{\mu}_k - \mu_k) \frac{\partial}{\partial \bar{w}_k^2}.
\end{align*}
\]

(4.24a, b)

As long as $T_k$ is in the kernel of $\bar{L}_k^1$ and $\bar{L}_k^2$, all functions $R_k$ from (4.19) automatically solve eqs. (4.22). Thus, special solutions to (4.22) are given by (4.19) with arbitrary differentiable functions $T_k(w_k^1, w_k^2)$, i.e., $\partial_{w_k^1} T_k = 0 = \partial_{w_k^2} T_k$ (for each $k = 1, \ldots, m$). By inserting such $T_k$ into

\[
\Omega^{-1} = \psi(u, v, \bar{u}, \bar{v}, \zeta) = 1 - \sum_{l,p=1}^{m} \frac{T_l \Gamma^l p T_p^\dagger}{\mu_p},
\]

(4.25)

explicit expressions for $A_u$, $A_{\bar{u}}$ can be derived from (4.9) and (4.21).

4.2 Dimensional reduction to 2+1 dimensions

**Dimensional reduction.** In order to establish the connection between the solutions obtained above and previous work carried out in 2+1 dimensions\(^{12}\) (see [35]–[38]) we have to perform a dimensional reduction. This can be done by imposing the condition that all fields are independent of one of the time coordinates in $\mathbb{R}^{2+1}$. As a consequence, we may put $\theta = 0$. To be precise, let us impose

\[
\partial_\theta T_k = 0.
\]

(4.26)

We switch to the complex isotropic coordinates introduced in (4.23). Using (3.2), we can reexpress $\partial / \partial t$ as

\[
\frac{\partial}{\partial t} = \frac{1}{2} \left\{ \bar{\mu}_k^{-1} \frac{\partial}{\partial \bar{w}_k^1} + \mu_k^{-1} \frac{\partial}{\partial w_k^1} + \frac{\partial}{\partial w_k^2} + \frac{\partial}{\partial \bar{w}_k^2} \right\}.
\]

(4.27)

\(^{11}\)In general, $\bar{L}^1(\zeta) := \zeta \partial_v + \partial_u$ and $\bar{L}^2(\zeta) := \zeta \partial_v - \partial_u$ correspond to the antiholomorphic vector fields introduced in appendix A (in the coordinates $u, v, \bar{u}, \bar{v}$). Defining coordinate functions $w^1(\zeta) := \zeta^{-1} \bar{v} - u$, $w^2(\zeta) := \zeta^{-1} v + \bar{u}$ in their kernel, we may write $\bar{L}_k^{1,2}(\zeta = \bar{\mu}_k)$ and $\bar{w}_k^{1,2}(\zeta = \bar{\mu}_k)$. Furthermore we have $\bar{L}_k^{1,2}(\zeta = \zeta^{-1}(\zeta - \bar{\zeta}) \partial_{\text{antihol}, 2}(\zeta)$. \(^{12}\)To recover the linear system of [35], we need to choose the unitary gauge for the linear equations, i.e., Lax pair (4.1) as discussed in section 4.1.
As derived in section 4.1, eqs. (4.22) are solved by matrices $T_k$ independent of $\bar{w}_k^1$ and $\bar{w}_k^2$; therefore (4.26) reads
\[
\left[\bar{\mu}_k^{-1} \frac{\partial}{\partial w_k^1} + \frac{\partial}{\partial w_k^2}\right] T_k(w_k^1, w_k^2) = 0, \tag{4.28}
\]
i.e., $T_k$ can only be a function of
\[
w_k := \nu_k(w_k^2 - \bar{\mu}_k w_k^1) = \nu_k \left(x + \frac{1}{2}(\bar{\mu}_k - \bar{\mu}_k^{-1})y + \frac{1}{2}(\bar{\mu}_k + \bar{\mu}_k^{-1})t\right), \tag{4.29}
\]
if it is independent of the second time direction. The normalization constant
\[
\nu_k := \left[\frac{4i}{\bar{\mu}_k - \bar{\mu}_k^{-1} + \mu_k^{-1}} \frac{\mu_k - \mu_k^{-1} - 2i}{\mu_k - \mu_k^{-1} + 2i}\right]^{1/2} \tag{4.30}
\]
has been introduced for later convenience. Note that the “co-moving” coordinates $w_k$ become static (i.e., independent of $t$) when choosing $\mu_k = -i$; they “degenerate” to the complex coordinates $\bar{z}_1^1$ from (3.3). Conversely, they can be obtained from the “static” coordinates $\bar{z}_1^1, \bar{z}_1^2$ by an inhomogeneous $SU(1, 1)$ transformation [35]:
\[
\begin{pmatrix} w_k \\ \bar{w}_k \end{pmatrix} = \begin{pmatrix} \cosh \tau_k & -e^{i\beta_k} \sinh \tau_k \\ -e^{-i\beta_k} \sinh \tau_k & \cosh \tau_k \end{pmatrix} \begin{pmatrix} \bar{z}_1^1 \\ \bar{z}_1^2 \end{pmatrix} - \sqrt{2\theta} \begin{pmatrix} \beta_k \\ \beta_k \end{pmatrix} t, \tag{4.31}
\]
where
\[
\beta_k = -\frac{1}{2}(2\theta)^{-1/2} \nu_k(\bar{\mu}_k + \bar{\mu}_k^{-1}), \tag{4.32a}
\]
and
\[
cosh \tau_k - e^{i\beta_k} \sinh \tau_k = \nu_k, \quad e^{i\beta_k} \tanh \tau_k = \frac{\bar{\mu}_k - \bar{\mu}_k^{-1} - 2i}{\mu_k - \mu_k^{-1} + 2i}. \tag{4.32b}
\]
Recall that a general solution $T_k$ in 2+2 dimensions is an arbitrary function of $w_k^1, w_k^2$. Hence, dimensional reduction to 2+1 dimensions can be accomplished for $T_k$ depending only on $w_k$: \[
\frac{\partial w_k^1}{\partial w_k^1} T_k = 0 = \frac{\partial w_k^2}{\partial w_k^2} T_k \quad \text{and} \quad \frac{\partial t}{\partial t} T_k = 0 \quad \Leftrightarrow \quad T_k = T_k(w_k). \tag{4.33}
\]
The Lax operators acting in this 2+1 dimensional subspace are given by:
\[
\begin{align*}
\bar{L}_k^1 &= \bar{\mu}_k \partial_x - \bar{\mu}_k \partial_x + \partial_x^{2+1} - \bar{\nu}_k(\bar{\mu}_k - \mu_k) \partial_{\bar{w}_k^1}, \\
\bar{L}_k^2 &= \bar{\mu}_k \partial_x - \partial_x^{2+1} \bar{\nu}_k \mu_k^{-1}(\bar{\mu}_k - \mu_k) \partial_{\bar{w}_k^2}.
\end{align*} \tag{4.34a}
\]
This exactly matches the results of [35].

Note that an alternative reduction can be done if $T_k$ is independent of $t$ (but depends on all other coordinates):
\[
\frac{\partial}{\partial t} = \frac{1}{2} \left\{ \frac{\partial}{\partial \bar{w}_k^1} + \frac{\partial}{\partial \bar{w}_k^2} - \bar{\mu}_k^{-1} \frac{\partial}{\partial \bar{w}_k^1} - \mu_k^{-1} \frac{\partial}{\partial \bar{w}_k^2}\right\}. \tag{4.35}
\]
Then, the condition $\partial_t T_k = 0$ and eqs. (4.22) are satisfied for $T_k = T_k(\bar{w}_k)$ with
\[
\bar{w}_k := \nu_k(w_k^1 + \bar{\mu}_k w_k^2) = \nu_k \left(-y + \frac{1}{2}(\bar{\mu}_k - \bar{\mu}_k^{-1})x + \frac{1}{2}(\bar{\mu}_k + \bar{\mu}_k^{-1})t\right). \tag{4.36}
\]
Note that a “boost” transformation analogous to (4.31) can be found for the coordinates $\tilde{w}_k$:

$$\left( \frac{\tilde{w}_k}{w_k} \right) = i \left( \begin{array}{cc} \cosh \tau_k & e^{i\theta_k} \sinh \tau_k \\ -e^{-i\theta_k} \sinh \tau_k & -\cosh \tau_k \end{array} \right) \left( \begin{array}{c} z_k^1 \\ \bar{z}_k^1 \end{array} \right) - \sqrt{2} \theta \left( \begin{array}{c} \beta_k \\ \beta_k^* \end{array} \right) \tilde{t}. \tag{4.37}$$

Such $T_k(w_k)$ or $T_k(\tilde{w}_k)$ lead to $\Omega$ which are given by (4.25) and do not depend on $\tilde{t}$ or $t$, respectively.

**Map to operator formalism.** If we translate the co-moving coordinates $w_k$ and $\tilde{w}_k$ into the operator formalism, this yields co-moving creation and annihilation operators:

$$\hat{\tilde{w}}^+_k = \hat{\tilde{w}}_k \Rightarrow [\hat{\tilde{w}}_k, \hat{\tilde{w}}^*_k] = 2\theta, \tag{4.38a}$$

$$\hat{w}^+_k = \hat{w}_k \Rightarrow [\hat{w}_k, \hat{w}^*_k] = 2\theta. \tag{4.38b}$$

Note that, in general, the commutators between $\hat{w}_k$ and $\hat{\tilde{w}}_k$ will not vanish. Therefore, derivatives with respect to $w_k$ and $\tilde{w}_k$ translate into commutators (cf. (3.15)) of the simple form

$$2\theta \partial w_k = -[\hat{w}_k, ], \quad 2\theta \partial \tilde{w}_k = [\hat{\tilde{w}}_k, ], \tag{4.39}$$

only when acting on functions of $\hat{w}_k$ and $\hat{\tilde{w}}_k$. Analogous statements hold for derivatives with respect to $\hat{w}_k$ and $\hat{\tilde{w}}_k$. In this framework, the transformations (4.31) and (4.37) may be interpreted as Bogoliubov transformations relating $z^1$ and $\tilde{z}^1$ to the operators in (4.38) [35].

**Energy.** In 2+1 dimensions it is possible to define the notion of energy in a straightforward manner and to show that it is conserved. Dimensional reduction of the Nair-Schiff type action (4.14) leads to the action for a modified noncommutative sigma model in 2+1 dimensions as presented in [35]. From this, an energy-momentum tensor can easily be derived:

$$T_{cd} = (\delta_c^a \delta_d^b - \frac{1}{2} \eta_{cd} \eta^{ab}) \text{tr}(\partial_a \Omega^{-1} \partial_b \Omega), \tag{4.40}$$

$a, b, c$ and $d$ running over $x, y, t$. For the proof that $T_{cd}$ is divergence-free we need to apply the equation of motion

$$(\eta^{ab} - \omega^{ab}) \partial_a (\Omega^{-1} \partial_b \Omega) = 0 \tag{4.41}$$

obtained by dimensional reduction (by imposing $\partial_t (\Omega^{-1} \partial_b \Omega) = 0$) from eq. (4.13). Using the explicit form of $\omega^{ab}$, it is obvious that one can rewrite eq. (4.41) as

$$(\eta^{ab} + V_c e^{cab}) \partial_a (\Omega^{-1} \partial_b \Omega) = 0, \tag{4.42}$$

where $(V_c) = (V_x, V_y, V_t) = (1, 0, 0)$ manifestly breaks Lorentz-invariance even in the commutative case. With this, one finds that $\int d^2x \partial^a T_{at}$ vanishes due to the chosen form of $\omega_{ab}$. For the energy density, one obtains

$$\mathcal{E} = T_{tt} = \frac{1}{2} \text{tr}[(\partial_t \Omega^\dagger) \partial_t \Omega + (\partial_x \Omega^\dagger) \partial_x \Omega + (\partial_y \Omega^\dagger) \partial_y \Omega]; \tag{4.43}$$

obviously $\partial_t \int d^2x \mathcal{E} = 0$.

**Nonabelian soliton in 2+1 dimensions.** As an illustrative example, consider a nonabelian one-soliton ($m = 1$) in 2+1 dimensions as described in [35]. Since $m = 1$, we may start from (4.17) with $P_1 \equiv P = (T^T T)^{-1} T^\dagger$, cf. (4.19) and (4.21). For definiteness, we take the soliton to be embedded into the $xyt$-plane, i.e., $T_1 \equiv T$ is a function of $w_1 \equiv w$ (cf. (4.29)). Such a function $T$ trivially solves (4.22).
Exemplarily, we briefly review a solution corresponding to a moving $U(2)$ soliton [35]. Using the inverse Moyal-Weyl map, we can deduce from the simplest ansatz $T = (\frac{1}{\mu})$ that

$$\Omega_\star = 1 - \frac{\bar{\mu} - \mu}{\mu} \left( \frac{2\theta}{\mu + \theta} \frac{\sqrt{2\theta + \theta^2}}{(\mu + \theta)^2} \right),$$

with the ordinary product between $w$ and $\overline{w}$, solves the self-duality equation. With the help of (4.43), the energy of this configuration can be shown to be $E = 8\pi \cosh \eta \sin \phi$ where $e^{\eta - i\phi} = \mu$.

A remark on the interpretation of solitons in terms of D-branes is in order: We start out from ncSDYM on a space-time filling D-brane. If a solution $\psi$ is independent of one coordinate, we are allowed to compactify and subsequently T-dualize this direction. This alters the Neumann boundary conditions for open strings living on the space-time filling branes to Dirichlet boundary conditions. In this case we therefore consider gauge theory on a D2-brane. Although there exists no Hodge self-duality condition in such a three-dimensional gauge theory, we will (in a slight abuse of language) still speak of solitonic solutions (implicitly referring to the four-dimensional gauge theory before T-dualization).

Since $\Omega_\star$ from (4.44) and the corresponding energy density are independent of $\tilde{t}$, a T-dualization in the $\tilde{t}$-direction leads to a gauge configuration on a pair of D2-branes. Taking into account that $\Omega$ depends only on two variables $w, \overline{w}$ in three dimensions, we conclude that it corresponds to a D0-brane moving in the world-volume of two D2-branes.

4.3 Hermitean gauge

Lax pair. Instead of using $\zeta$, the Riemann sphere $\mathbb{C}P^1$ may alternatively be parametrized by the variable

$$\lambda = \frac{\zeta - i}{\zeta + i}.$$ (4.45)

The map $\zeta \mapsto \lambda$ carries the lower half plane in $\zeta$ to the exterior of the unit disk $\{|\lambda| > 1\}$ in the $\lambda$-plane. In terms of $\lambda$ and the coordinates $z^1, z^2, \overline{z}^2$ on $\mathbb{R}^{2,2} \cong \mathbb{C}^{1,1}$, the Lax pair (4.1) becomes

$$\begin{align*}
(\partial_{z^1} - \lambda \partial_{\overline{z}^2})\psi &= -(A_{z^1} - \lambda A_{\overline{z}^2})\psi, \\
(\partial_{\overline{z}^2} - \lambda \partial_{z^1})\psi &= -(A_{\overline{z}^2} - \lambda A_{z^1})\psi,
\end{align*}$$

and its compatibility conditions are the self-duality equations

$$\begin{align*}
F_{z^1 z^2} &= 0, \\
F_{z^1 \overline{z}^2} &= 0, \\
F_{\overline{z}^1 z^2} &= 0.
\end{align*}$$

(4.46a) (4.46b) (4.46c)

Here, $\psi$ may be chosen to satisfy the reality condition

$$\begin{align*}
\psi(z^1, \overline{z}^1, z^2, \overline{z}^2, \lambda)[\psi(z^1, \overline{z}^1, z^2, \overline{z}^2, \lambda^{-1})]^\dagger &= 1.
\end{align*}$$

Equations (4.47a) and (4.47c) imply that there exist $g, \tilde{g} \in GL(N, \mathbb{C})$ such that:

$$\begin{align*}
A_{\overline{z}^1} &= g^{-1} \partial_{\overline{z}^1} g, & A_{z^2} &= g^{-1} \partial_{z^2} g, \\
A_{z^1} &= \tilde{g}^{-1} \partial_{z^1} \tilde{g}, & A_{\overline{z}^2} &= \tilde{g}^{-1} \partial_{\overline{z}^2} \tilde{g}.
\end{align*}$$

(4.49a) (4.49b)

13The conventions for $z^1, z^2$ are such that for $\mu_k' = \infty$ we obtain holomorphic functions $T$ as solutions of (4.61).
We read off that a possible choice for $g$ and $\tilde{g}$ is given by

$$
g := \left[\psi(z^i, \bar{z}^i, \lambda \to \infty)\right]^{-1}, \quad (4.50a)$$

$$
\tilde{g} := \left[\psi(z^i, \bar{z}^i, \lambda \to 0)\right]^{-1}. \quad (4.50b)
$$

Since we are using antihermitean generators for the gauge group $U(N)$, the $GL(N, \mathbb{C})$-valued fields $g, \tilde{g}$ have to be related:

$$
A_{zi}^i = -A_{zi}, \quad i = 1, 2 \quad \Rightarrow \quad \tilde{g} = (g^\dagger)^{-1}. \quad (4.51)
$$

**Gauge fixing.** As in section 4.1, we can perform a gauge transformation to set two components of the gauge potential to zero. Contrary to the (unitary) gauge choice there, in the following we set to zero those components which are *not* multiplied by the respective spectral parameter in eqs. (4.46). Explicitly,

$$
\psi' = \tilde{g}\psi, \quad (4.52a)
$$

$$
A'_{z1} = h^{-1}\partial_{z1}h, \quad A'_{z2} = h^{-1}\partial_{z2}h, \quad (4.52b)
$$

$$
A'_{\bar{z}1} = 0, \quad A'_{\bar{z}2} = 0, \quad (4.52c)
$$

where $h := gg^{-1} = gg^\dagger \in GL(N, \mathbb{C})$ is hermitean. This gauge is “asymmetric”, i.e., the gauge potential does not obey (4.51), but instead it satisfies $(A'_{z1})^\dagger = -hA'_{z2}h^{-1} - h\partial_{z1}h^{-1}$. After solving (4.47) we are free to gauge back to a “symmetric” gauge, where (4.51) is restored. This is ensured by the hermiticity of $h$, which is the remnant of (4.51) in the asymmetric gauge. We will from now on work in the asymmetric gauge and omit all primes on the gauge-transformed quantities.

Now, the gauge-fixed linear equations read

$$
(\partial_{z1} - \lambda\partial_{z2})\psi = \lambda A_{z2}\psi, \quad (4.53a)
$$

$$
(\partial_{z2} - \lambda\partial_{z1})\psi = \lambda A_{z1}\psi. \quad (4.53b)
$$

Due to (4.51), the reality condition (4.48) transforms into

$$
\psi(\lambda)[\psi(\bar{\lambda})]^\dagger = \tilde{g}g^{-1} = h^{-1}. \quad (4.54)
$$

In the asymmetric gauge the remaining self-duality equation (4.47b) reduces to

$$
\partial_{z1}(h^{-1}\partial_{z2}h) - \partial_{z2}(h^{-1}\partial_{z1}h) = 0. \quad (4.55)
$$

**First-order pole ansatz.** Since the reality condition (4.54) is different from the one in the unitary gauge, we are forced to employ a modified ansatz for $\psi(\lambda)$. The first-order pole ansatz for $\psi$ takes the form

$$
\psi_m(\lambda) = 1 + \sum_{p=1}^{m} \frac{\lambda\tilde{R}_p}{\lambda - \mu_p'}, \quad (4.56)
$$

where

$$
\tilde{R}_p := -\sum_{q=1}^{m} \mu_p' T_p \Gamma^{pq} T_q^\dagger. \quad (4.57)
$$

---

14 In this way, we facilitate a comparison with [42].

15 This is the reason why we call this gauge hermitean. It coincides with the hermitean gauge introduced in [46].

16 The parameters $\mu_p'$ are the images of $\mu_p$ under (4.45). However, the ansatz (4.56) is not simply the transform of (4.18).
The “inverse” matrix $\tilde{\Gamma} = (\tilde{\Gamma}_{pk})$, cf. (4.21), here reads:

$$\tilde{\Gamma}_{pk} = \mu'_p \frac{T'_{kp} T_k}{1 - \mu'_p \mu'_{pk}}, \quad (4.58)$$

The matrix-valued function $\psi_m$ should satisfy the linear equations (4.53) and is subject to a reality condition, eq. (4.54). Again, the requirement that the poles at $\lambda = \bar{\mu'}^{-1}_k$ and $\lambda = \mu'_k$ of (4.54) have to be removable yields\(^{17}\)

$$\left(1 - \sum_{p=1}^{m} \frac{\bar{R}_p}{\mu'_p \mu'_{pk} - 1}\right) T_k = 0, \quad (4.59)$$

which is solved by (4.57) with (4.58). Now we exploit the pole structure of the Lax pair which, using (4.54), may be rewritten as

$$\left[\left(\frac{1}{\lambda} \partial z_1 - \partial z_2\right) \psi\right] \psi^\dagger = A_{z1} h^{-1} \quad (4.60a)$$

$$\left[\left(\frac{1}{\lambda} \partial z_2 - \partial z_1\right) \psi\right] \psi^\dagger = A_{z2} h^{-1}. \quad (4.60b)$$

As before, the right hand sides do not feature poles in $\lambda$, therefore taking the residue at $\lambda = \bar{\mu'}^{-1}_k$ and $\lambda = \mu'_k$ leads to the conditions

$$\left(1 - \sum_{p=1}^{m} \frac{\bar{R}_p}{\mu'_p \mu'_{pk} - 1}\right) (\partial z_1 - \bar{\mu'}^{-1}_k \partial z_2) \tilde{R}_k = 0, \quad (4.61a)$$

$$\left(1 - \sum_{p=1}^{m} \frac{\bar{R}_p}{\mu'_p \mu'_{pk} - 1}\right) (\partial z_2 - \bar{\mu'}^{-1}_k \partial z_1) \tilde{R}_k = 0. \quad (4.61b)$$

If we define

$$\eta^1(\mu') := z_1 + \lambda z_2 \quad \Rightarrow \quad \bar{\eta}^1(\bar{\lambda}) = \bar{z}_1 + \bar{\lambda} z_2, \quad (4.62a)$$

$$\eta^2(\mu') := z_2 + \lambda z_1 \quad \Rightarrow \quad \bar{\eta}^2(\bar{\lambda}) = \bar{z}_2 + \bar{\lambda} z_1, \quad (4.62b)$$

and denote $\eta_k^i := \eta^i(\mu' = \bar{\mu'}^{-1}_k)$, $\bar{\eta}_k^i := \bar{\eta}^i(\lambda = \mu'^{-1}_k)$, the Lax operators can be written as antiholomorphic vector fields in these coordinates:

$$\bar{L}_k^1 = \partial z_1 - \bar{\mu'}^{-1}_k \partial z_2 = (1 - |\mu'_{k}|^{-2}) \frac{\partial}{\partial \eta_k^1}, \quad (4.63)$$

$$\bar{L}_k^2 = \partial z_2 - \bar{\mu'}^{-1}_k \partial z_1 = (1 - |\mu'_{k}|^{-2}) \frac{\partial}{\partial \eta_k^2}. \quad (4.64)$$

Functions $T_k = T_k(\eta_k^1, \eta_k^2)$ are in the kernel of $\bar{L}_k^1$ and $\bar{L}_k^2$; therefore $\tilde{R}_k$ constructed via (4.57) from such $T_k$ automatically satisfy eqs. (4.61). Due to (4.50) and (4.54), we can determine $A_{z1}$ and $A_{z2}$ in eq. (4.52) from

$$h^{-1} = 1 + \sum_{p=1}^{m} \tilde{R}_p. \quad (4.65)$$

\(^{17}\)Note that $h^{-1}$ is independent of $\lambda$.  

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### 4.4 Relation to string field theory

In this section, we want to clarify the relation to N=2 string field theory in the presence of a B-field. In the next paragraph, we will show that the zero-mode part of the string field theory equation of motion contains the field theory self-duality equation. After that, it will turn out that, in the Seiberg-Witten limit, an analogous discussion of the dressing approach [42] leads to Lax operators acting only on the oscillator (non-zero mode) part of a string field.\(^{18}\)

**Field theory content of string field theory.** Let us first briefly show that the gauge-fixed self-duality equation (4.55) is contained in the equation of motion of nonpolynomial string field theory for N=2 strings [52, 53, 54] (for finite \(\alpha'\)). Its equation of motion in the conventions of [42] reads

\[
\bar{G}^+ (e^{-\Phi} G^+ e^{\Phi}) = 0,
\]

(4.66)

where \(\Phi\) is a hermitean string field. The operators \(G^+\) and \(\bar{G}^+\) are constituents of a small N=4 superconformal algebra acting on string fields via contour integration, e.g.,

\[
(G^+ e^{\Phi})(w) = \oint \frac{dw'}{2\pi i} G^+(w') e^{\Phi}(w),
\]

(4.67)

where the integration contour runs around \(w\). Note that, in this context, \(w\) and \(w'\) are world-sheet coordinates. All string fields in (4.66) are multiplied by the Witten star product; this will be analyzed more deeply in the next subsection. If we denote the complex N=2 world-sheet bosons by \(Z^i\) and \(\bar{Z}^\bar{i}\) (\(i = 1, 2\)) and their NSR superpartners as \(\psi^{+i}\) and \(\psi^{-i}\), the superconformal generators in (4.66) can be realized as [55, 56]

\[
G^+ = \eta_{ij} \psi^{+i} \partial \bar{Z}^j \quad \text{and} \quad \bar{G}^+ = -\varepsilon_{ij} \psi^{+i} \partial Z^j.
\]

(4.68)

Here, \(\eta_{ij}\) denotes the (pseudo-)Kähler space-time metric with non-vanishing components \(\eta_{11} = -\eta_{22} = 1\); for the antisymmetric tensor we choose the convention that \(\varepsilon_{12} = -1\). Taking into account the bosonic operator product expansions

\[
Z^i(w, \bar{w}) \bar{Z}^\bar{j}(w', \bar{w}') \sim -2\alpha' \eta^{ij} \ln |w - w'|^2, \quad Z^i(w, \bar{w}) Z^j(w', \bar{w}') \sim 0, \quad \bar{Z}^i(w, \bar{w}) \bar{Z}^\bar{j}(w', \bar{w}') \sim 0,
\]

(4.69)

we see that due to (4.67) \(G^+\) and \(\bar{G}^+\) act as derivatives on string fields containing only world-sheet bosons. Concretely, the equations of motion (4.66) for such string fields can be written as

\[
\psi_0^{+1} \psi_0^{+2} \eta^{ij} \partial_{\bar{z}^j} (e^{-\Phi} \partial_z e^{\Phi}) + \ldots = 0.
\]

(4.70)

Here, the bosonic zero-modes \(z^i\) and \(\bar{z}^\bar{j}\) coincide with the space-time coordinates used in section 4.3; \(\psi_0^{+i}\) denote the zero-modes of \(\psi^{+i}\). The dots indicate the oscillator-dependent part of the equation of motion. The zero-mode part in (4.70) coincides with the remaining self-duality equation (4.55) in the Yang gauge (4.52) if we identify \(h(z^i, \bar{z}^\bar{j}) = e^{\Phi(z^i, \bar{z}^\bar{j})}\) (cf. (4.52b)).

In the same way, the linear equation given in [42] includes the field theory Lax pair (4.53). For \(A = e^{-\Phi} G^+ e^{\Phi}\), it can be written as

\[
0 = \{G^+ + \lambda \bar{G}^+ + \lambda A\} \Psi
\]

\[
= \frac{1}{2} \left\{ \psi_0^{+1} (\partial_{\bar{z}^2} - \lambda \partial_{\bar{z}^1} - \lambda e^{-\Phi} \partial_{\bar{z}^1} e^{\Phi}) + \psi_0^{+2} (\partial_{\bar{z}^1} - \lambda \partial_{\bar{z}^2} - \lambda e^{-\Phi} \partial_{\bar{z}^2} e^{\Phi}) + \ldots \right\} \Psi.
\]

\(^{18}\)Since no special form of the star product was demanded in [42], the discussion there is equally well valid in the case of nonvanishing B-field.
Because $\psi_0^{\pm 1}$ and $\psi_0^{\pm 2}$ are mutually independent, the zero-mode part coincides with (4.53).

**Star product in the Seiberg-Witten limit.** Now we will scrutinize the Seiberg-Witten limit of string field theory in a $B$-field background and argue that, in this limit, the above BRST-like operators $G^+$ and $\tilde{G}^+$ act only on the oscillator-part of a string field $e^{\Phi}$.\(^{19}\) To this aim, we switch to real coordinates $x^\mu$ according to (3.3). In covariant string field theory, strings are glued with Witten’s star product identifying the left half of the first string with the right half of the second string. This product is noncommutative even without a $B$-field background, but in order to make contact with the discussion of ncSDYM in this paper, we switch on a constant $B$-field.\(^{20}\) There are several ways to compute Witten’s star product; the one most suitable for our purposes is the use of an oscillator representation of the three-vertex $123\langle V_3 \rangle$ joining two string states $|A\rangle_1$ and $|B\rangle_2$ according to

$$
3\langle C \rangle = 123\langle V_3 || A \rangle_1 |B\rangle_2
$$

with

$$
123\langle V_3 \rangle = \left( \frac{3\sqrt{3}}{4} \right)^3 \delta(p^{(1)} + p^{(2)} + p^{(3)}) |0\rangle \otimes |0\rangle \otimes |0\rangle \exp(E_{\text{mat}}),
$$

$$
E_{\text{mat}} = \sum_{m,n=0}^{\infty} \frac{1}{2} \alpha^{(r)}(r)_n \alpha^{(s)}_m G_{\mu\nu} - \frac{i}{2} \theta_{\mu\nu} p^{(1)}(1) p^{(2)}(2) + \sum_{m,n=0}^{\infty} \frac{1}{2} \psi^{(r)}_n \psi^{(s)}_m \nu G_{\mu\nu},
$$

where $N_{nm}^{rs}$ and $V_{nm}^{rs}$ are the Neumann coefficients [59] for world-sheet bosons and fermions and $\alpha^{(r)}_n$ and $\psi^{(r)}_n$ denote the bosonic and fermionic oscillators of the $r$-th string, respectively. The open string metric $G_{\mu\nu}$ was introduced in eq. (2.2), and we write $\alpha_0^{\mu} = \sqrt{2}\alpha_p^{\mu}$. A summation over $r, s = 1, 2, 3$ and over $\mu, \nu = 1, \ldots, 4$ is implicit. This expression is valid for $N=2$ strings and is constructed analogous to [58, 60, 61]. Note that for $N=2$ nonpolynomial string field theory no world-sheet ghosts are needed.

We will now consider the properties of this vertex in the Seiberg-Witten limit $B \to \infty$ keeping fixed all other closed string parameters. For this purpose, we set $B = tB_0$ and take $t \to \infty$;\(^{21}\) then, the effective open string parameters scale as [58]

$$
G_{\mu\nu} \sim G_{0\mu\nu} t^2, \quad \theta^{\mu\nu} \sim \theta_0^{\mu\nu} t^{-1}.
$$

In checking the operator product expansions for the $N=4$ superconformal algebra, the relations

$$
\varepsilon_{ij} \eta^{j} \varepsilon_{ki} = \eta_{li}
$$

(4.74)

are needed. Since $\eta_{ij}$ in eq. (4.68) has to be replaced by $G_{ij}$ in the case of a nonvanishing $B$-field, the “(anti)holomorphic part of the volume element” $\varepsilon_{ij}$ is changed to $\varepsilon_{ij}$ with the same scaling behavior as $G_{ij}$ (cf. (4.74)).

For the commutation relations

$$
[\alpha^{\mu}_m, \alpha^{\nu}_n] = m \delta_{m+n,0} G^{\mu\nu},
$$

(4.75a)

$$
[x^{\mu}_m, x^{\nu}_n] = i \theta^{\mu\nu},
$$

(4.75b)

$$
[p^{\mu}_m, x^{\nu}_n] = -i G^{\mu\nu},
$$

(4.75c)

$$
\{\psi^{\mu}_m, \psi^{\nu}_n\} = \delta_{m+n,0} G^{\mu\nu}
$$

(4.75d)

\(^{19}\)This argument is along the lines of [57, 58].

\(^{20}\)For $N=2$ strings, the $B$-field must be a (pseudo-)Kähler two-form [10].

\(^{21}\)This limit is not to be confused with the large time limit in section 6.
to be invariant in the large $B$-field limit, we have to introduce rescaled oscillators

$$\tilde{\alpha}_m^\mu = t\alpha_m^\mu \quad \text{for } m \neq 0,$$

$$\tilde{p}^\mu = t^{3/2}p^\mu,$$

$$\tilde{x}_m^\mu = t^{1/2}x_m^\mu,$$

$$\tilde{\psi}_m^\mu = t\psi_m^\mu.$$  \hspace{1cm} (4.76a-d)

In terms of these modes, the matter part of the three-vertex (4.72c) takes the form

$$E_{\text{mat}} = \sum_{m,n=1}^{\infty} \frac{1}{2} \tilde{\alpha}_n^\mu \tilde{\alpha}_m^\mu \bar{\alpha}_m^\mu G_{0\mu\nu} - \frac{i}{2} \theta_0 p_0 \tilde{\alpha}_m^\mu \tilde{\alpha}_m^\mu \tilde{p}_\nu G_{0\mu\nu} + \frac{1}{\sqrt{t}} \sum_{n=1}^{\infty} \sqrt{\tilde{\alpha}_n^\mu (\tilde{N}_n^\mu + \tilde{N}_0^\mu)} \tilde{p}_\nu G_{0\mu\nu}$$

$$+ \frac{1}{t} \tilde{p}^{(r)\mu} \tilde{N}_n^\mu \bar{\psi}_m^\mu G_{0\mu\nu} + \sum_{m,n=0}^{\infty} \frac{1}{2} \tilde{\psi}_n^{(r)\mu} \bar{\psi}_m^\mu G_{0\mu\nu}.$$  \hspace{1cm} (4.77)

Note that, for $t \to \infty$, the terms coupling $\alpha$-oscillators and momenta $p$ vanish. Thus, the string star algebra $A$ factorizes into a zero-momentum part $A_0$ spanned by $\tilde{p}$, $\tilde{\alpha}$, and $\tilde{\psi}$-oscillators and a space-time part $A_1$ generated by $\tilde{x}_m^\mu$ [57]. The star product in $A_1$ “degenerates” to the usual Moyal-Weyl product with constant noncommutativity parameter $\theta_0$.

To read off the scaling behavior of the BRST-like operators $G^+$ and $\tilde{G}^+$, we switch back to complex coordinates (labeled by roman space-time indices) and exemplarily pick two typical terms from $G^+$:

$$\psi_0^+ i^\mu \tilde{p}^\mu G_{ij} = \frac{1}{\sqrt{t}} \tilde{\psi}_0^+ i^\mu \tilde{p}^\mu G_{0ij}$$  \hspace{1cm} (4.78)

and

$$\psi_1^+ i^\mu \tilde{\alpha}_-^\mu G_{ij} = \psi_1^+ i^\mu \tilde{\alpha}_-^\mu G_{0ij}.$$  \hspace{1cm} (4.79)

Eq. (4.78) is the only term in $G^+$ acting onto $A_1$; obviously it is suppressed for large $t$. Eq. (4.79) exemplifies a term in $\tilde{G}^+$ acting onto $A_0$; it is independent of $t$. This affirms the claim that, in the large $B$-field limit, $G^+$ and $\tilde{G}^+$ act only onto $A_0$.

As a consequence, all BRST-like operators in the equations in [42] in the Seiberg-Witten limit act only onto the oscillator algebra $A_1$. Thus, if we assume a factorized solution $\Phi = \Phi_0 \otimes \Phi_1$ with $\Phi_0 \in A_0$ and $\Phi_1 \in A_1$, the equation of motion can be restricted to $A_0$ if $\Phi_1$ is chosen to be a projector (i.e., $\Phi_1 \ast \Phi_1 = \Phi_1$):

$$0 = \tilde{G}^+(e^{-\Phi} G^+ e^{\Phi}) = \tilde{G}^+(e^{-\Phi_0} G^+ e^{\Phi_0}) \otimes \Phi_1.$$  \hspace{1cm} (4.80)

Nevertheless, for finite $B$, the string field theory equation of motion contains the ncSDYM equation of motion. Therefore, the solutions to be constructed in the following section can serve as a guide in the search for nonperturbative solutions to string field theory. Note that some proposals for string functionals $T$ were made in [42, 43]; indeed, these solutions were motivated by the above ideas.

5 Configurations without scattering

The aim of this section is to demonstrate the usability of the solution generating technique described in section 4 in two simple cases. In 2+2 dimensions, we will construct an abelian GMS-like solution of codimension four and a solution representing two nonabelian moving lumps without scattering. The description of configurations with scattering will be relegated to section 6. Although we do not check their physical properties like tension and fluctuation spectrum, we will refer to them as D-branes.
5.1 Abelian GMS-like solution

It is fairly easy to construct $U(1)$ solutions depending on all space-time coordinates (i.e., with codimension four) via the dressing approach in 2+2 dimensions (cf. [62] on euclidean instantons via dressing). To this aim, let us start from the discussion of the dressing approach in the hermitean gauge (section 4.3). For $m = 1$, we can omit all labels $k$; a comparison of (4.56)–(4.58) with the multiplicative ansatz shows that $\mathcal{R} = (|\mu'|^2 - 1)P$, where $P$ is a hermitean projector independent of $\lambda$. If we choose $\theta = \bar{\theta}$ and define harmonic oscillators\(^{22}\)

\[
    c_i := \frac{1}{\sqrt{2\theta(1 - |\mu'|^2)}} \eta^i \\
    c_i^\dagger := \frac{1}{\sqrt{2\theta(1 - |\mu'|^2)}} \bar{\eta}^i,
\]

then $[c_i, c_j^\dagger] = \delta_{ij}$; thus, we can easily invert their commutation relations and obtain

\[
    \sqrt{2\theta(1 - |\mu'|^2)} \partial_{\eta^i} = [c_i, .].
\]

With this, we may rewrite (4.61) as

\[
    (1 - P)c_1 P = 0, \quad (5.3a) \\
    (1 - P)c_2 P = 0. \quad (5.3b)
\]

Obviously, these equations can be solved by the projector $P = |0, 0\rangle \langle 0, 0|$ onto the new vacuum $|0, 0\rangle$ annihilated by $c_1$ and $c_2$.\(^{23}\) We may use the inverse Moyal-Weyl map (3.14) to transform it to the star formulation:

\[
    P_\star = \exp \left( - \frac{\eta^1 \bar{\eta}^1 + \eta^2 \bar{\eta}^2}{\theta(1 - |\mu'|^2)} \right) \quad (5.4)
\]

This is the analogue of the GMS-solution [14] in 2+2 dimensions; the projector $P_\star$ is an example for a projector $\Phi_1$ in (4.80). The gauge potential can be derived from (4.52) with $h^{-1} = 1 - (1 - |\mu'|^2)P_\star$. The computation of the value of the action for this solution turns out to be rather unwieldy.

5.2 $U(2)$ solitons without scattering

Let us now demonstrate how the additive ansatz (4.18) in the unitary gauge can be employed to construct a solution describing two moving lumps. A detailed description of the asymptotic space-time picture will be given at the end of this section.

Additive ansatz. We work in the star formulation and relax the condition that $\theta = \bar{\theta}$. The result of the first dressing step, corresponding to a soliton in 2+1 dimensions, has already been given in section 4.2. This lump moves w.r.t. $t$ in the $xy$-plane; its energy (which is well-defined in 2+1 dimensions) was computed in the same section. In the second dressing step, we want to add a soliton confined for large $t$ to the $xy\bar{t}$-plane. From the preceding discussion in section 4.1 it is clear that for $m = 2$, we can construct a solution to the self-duality equations using (cf. (4.18))

\[
    \psi_2 = 1 + \sum_{l,k=1}^{2} \frac{T_l T_k^\dagger}{\zeta - \mu_k}, \quad (5.5)
\]

\(^{22}\)Recall that $|\mu'| > 1$ since $\mu \in \mathbb{H}$. in section 4.1.
\(^{23}\)Since the $a$-oscillators (cf. (4.62) and (3.11)) and the $c$-oscillators are related by a unitary transformation $c_i = U a_i U^\dagger$, the (properly normalized) vacuum $|0, 0\rangle'$ can naturally be obtained as $|0, 0\rangle' = U|0, 0\rangle$. 19
with \( T_1 = \left( \frac{1}{w_1} \right) \) and \( T_2 = \left( \frac{1}{\bar{w}_2} \right) \). However, it is not obvious that this solution really represents two soliton-like objects, i.e., whether for large \( t \) the solution can be integrated over a \((\text{spatial})\) plane in the \( x\overline{y}t\)-subspace to give finite energy (and vice versa on a plane in the \( xyt\)-subspace at large \( \tilde{t} \)). To prove this, we compare the additive (first-order pole) and multiplicative ansätze for asymptotic times. Note that the two are only equivalent if the multiplicative ansatz features merely first-order poles in \( \mu \), that is, if \( \mu_1 \neq \mu_2 \).

**Multiplicative ansatz.** In the multiplicative ansatz, \( \psi_2 = \chi_2 \chi_1 \psi_0 \) may be constructed by two successive dressing steps from a seed solution \( \psi_0 = 1 \). As in eq. (4.17), we may write

\[
\psi_2 = \left( 1 + \frac{\mu_2 - \bar{\mu}_2}{\zeta - \mu_2} P_2 \right) \left( 1 + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1} P_1 \right),
\]

and this has to coincide with (5.5) for all times. Remember that for hermitean projectors \( P_k = \overline{T_k} (T_k^\dagger T_k)^{-1} \overline{T_k^\dagger} \) this ansatz guarantees the reality condition (4.11). The solution \( \psi_2 \) is subject to eqs. (4.15):

\[
\begin{align*}
\psi_2 (\zeta \partial_v + \partial_u) \psi_2^\dagger &= A_{2, u}, \quad (5.7a) \\
\psi_2 (\zeta \partial_v - \partial_u) \psi_2^\dagger &= -A_{2, \bar{u}}. \quad (5.7b)
\end{align*}
\]

The removability of the poles of the left hand sides at \( \zeta = \bar{\mu}_1 \) and \( \zeta = \mu_1 \) is assured if (for \( \mu_1 \neq \mu_2 \))

\[
(1 - P_1) \overline{L_1} P_1 = 0 \quad \text{and} \quad (1 - P_1) \overline{L_2} P_1 = 0,
\]

and this allows for a solution \( \overline{T}_1 = T_1 = \left( \frac{1}{w_1} \right) \). Using the inverse Moyal-Weyl map, we obtain for \( P_1 \) and its large-time limits

\[
P_{1*} = \left( \begin{array}{cc} 2 \omega & \sqrt{2 \omega_1 \overline{\omega}_2} \\ \sqrt{2 \omega_1 \overline{\omega}_2} & \omega_1 + \omega_2 \overline{\omega}_2 \end{array} \right) t \to \pm \infty \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) =: \Pi_{\pm \infty} \quad (5.9)
\]

with the ordinary product between \( w_1 \) and \( \overline{w}_1 \), as in (4.44).\(^{24}\) In contrast, \( P_2 \) will in general be a function \( P_2(t, \overline{w}_2, \overline{w}_2) \), i.e., \( \overline{T}_2 \neq T_2 \) may also depend on \( t \). Namely, the residue equation of (5.7) at \( \zeta = \bar{\mu}_2 \) and \( \zeta = \mu_2 \) yields:

\[
\begin{align*}
(1 - P_2) \left( 1 + \frac{\mu_1 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_1} P_1 \right) \overline{L_2} \left\{ \left( 1 + \frac{\mu_1 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_1} P_1 \right) P_2 \right\} &= 0, \quad (5.10a) \\
(1 - P_2) \left( 1 + \frac{\mu_1 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_1} P_1 \right) L_2 \left\{ \left( 1 + \frac{\mu_1 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_1} P_1 \right) P_2 \right\} &= 0. \quad (5.10b)
\end{align*}
\]

Due to the asymptotic constancy of \( P_1 \) for large \(|t|\), we can move the Lax operators next to \( P_2 \) in this limit, and a short calculation shows that this leads to

\[
(1 - P_2) \partial_{\overline{w}_2} P_2 = 0 \quad \text{for} \quad |t| \to \infty. \quad (5.11)
\]

Obviously, we have \( \overline{T}_2 = T_2 \) only asymptotically.

Thus, the energy of the second lump can be computed in the limit \(|t| \to \infty\) to give \( E_2 = 8\pi \cosh \eta_2 \sin \varphi_2 \) as in section 4.2. Analogously, the energy of the first lump in the limit \(|\tilde{t}| \to \infty\) equals \( E_1 = 8\pi \cosh \eta_1 \sin \varphi_1 \).

\(^{24}\)From the asymptotics, we can read off that the two lumps pass through each other without scattering.
For large and fixed $|t|$, the space-time interpretation of the above solution is as follows: Since $P_1$ is independent of $\tilde{t}$, the first soliton (at a fixed time $t$) has some definite position in the $xy$-plane and extends along the $\tilde{t}$-direction (see figure 1). Moreover, the world-volume of the second soliton in this snapshot corresponds to a tilted line (cf. eq. (5.11)). When $t$ varies in the asymptotic region, the first (vertical) line gets shifted in a direction determined by $\mu_1$, while the second line remains fixed. Generically, the two world-volumes intersect the $xy$-plane at different points. Since $\Omega$ depends on both $t$ and $\tilde{t}$, it is not possible to perform a T-dualization in one of the time directions. Thus, the solution has to be interpreted in terms of tilted D1-branes inside space-time filling D-branes.

Figure 1: Snapshot of the configuration discussed in section 5.2 for fixed large $|t|$. The support of the solution is concentrated around the solid lines.

### 6 Configurations with scattering

In this section we discuss two different setups entailing configurations with scattering, namely two $U(2)$ soliton-like objects and two noncommutative $U(2)$ plane waves, with world-volumes in 2+1 dimensional subspaces of $\mathbb{R}^{2,2}$. As will become clear in this section, the crucial difference between the two configurations lies in the fact that for these plane waves scattering occurs even if $\mu_1 \neq \mu_2$, whereas soliton-like objects only scatter nontrivially if $\mu_1 = \mu_2$. (In fact, we have already seen in the previous section that the solitonic lumps do not scatter for $\mu_1 \neq \mu_2$.)

#### 6.1 $U(2)$ solitons with scattering

The setup is as in section 5.2; one of the solitonic lumps is evolving with $t$ on the $xy$-plane. Since a first-order pole ansatz for the auxiliary field $\psi_2$ in eqs. (5.5) did not lead to scattering, we now scrutinize the multiplicative ansatz with $\mu_1 = \mu_2 = \mu$.

**First dressing step.** Starting from a seed solution $\psi_0 = 1$, we make the following ansatz for the first dressing step:

$$\psi_1 = 1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P_1.$$  \hfill (6.1)
This automatically fulfills the reality condition (4.11) as long as $P_1$ is a hermitean projector. The residue condition on the linear equations (4.15) leads to

$$(1 - P_1)\partial_{\psi}P_1 = 0,$$  \hspace{1cm} (6.2)

equation which states that, since $P_1$ varies in a $2 + 1$ dimensional subspace parametrized by $w$ as defined in (4.29).

Now we set out to find explicit expressions for the components of the gauge potential. First note that, since $P_1$ is chosen to be independent of $\bar{t}$, $A_{1,\bar{u}}$ effectively reduces to $A_{1,x}$. From eqs. (4.9) and using $\psi_2(\zeta = 0) = \Omega^{-1}$ or (4.25), we find

$$A_{1,u} = \bar{\rho}(1 - \rho P_1) \partial_u P_1,$$  \hspace{1cm} (6.3a)

$$A_{1,x} = \bar{\rho}(1 - \rho P_1) \partial_x P_1,$$  \hspace{1cm} (6.3b)

where $\rho = 1 - \bar{\mu}/\mu$ was introduced for convenience.

**Second dressing step.** The reality condition (4.11) for the new ansatz $\psi_2 = \chi_2 \psi_1$ will be satisfied if we choose $\chi_2$ to be of the same functional form as $\psi_1$, i.e.,

$$\psi_2 = \left(1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P_2\right) \left(1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P_1\right) \psi_1,$$  \hspace{1cm} (6.4)

with a hermitean projector $P_2 = T_2(T_2^\dagger T_2)^{-1}T_2^\dagger$ in general depending on all four coordinates. The corresponding gauge-fixed linear equations (4.15) are:

$$\psi_2(\zeta \partial_{\bar{u}} + \partial_u)\psi_2^\dagger = A_{2,u},$$  \hspace{1cm} (6.5a)

$$\psi_2(\zeta \partial_{\bar{v}} - \partial_\bar{u})\psi_2^\dagger = -A_{2,\bar{u}},$$  \hspace{1cm} (6.5b)

which is equivalent to

$$A_{2,u} = \chi_2 A_{1,u} \chi_2^\dagger + \chi_2(\zeta \partial_{\bar{u}} + \partial_u)\chi_2^\dagger,$$  \hspace{1cm} (6.6a)

$$-A_{2,\bar{u}} = \chi_2 A_{1,\bar{u}} \chi_2^\dagger - \chi_2(\zeta \partial_{\bar{u}} - \partial_\bar{u})\chi_2^\dagger.$$  \hspace{1cm} (6.6b)

Inserting $\chi_2 = 1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P_2$ and demanding that the right hand sides of eqs. (6.6a) and (6.6b) are free of poles for $\zeta \to \bar{\mu}$ and $\zeta \to \mu$ leads to

$$(1 - P_2) \left\{ \rho \partial_{\varpi\bar{u}} - A_{1,u} \right\} P_2 = 0,$$  \hspace{1cm} (6.7a)

$$(1 - P_2) \left\{ \rho \partial_{\varpiu} + A_{1,x} \right\} P_2 = 0.$$  \hspace{1cm} (6.7b)

Recall that we defined $\rho = 1 - \bar{\mu}/\mu$. In the following, we shall assume $P_2 = P_2(w, \varpi, \bar{t})$. By appropriately combining eqs. (6.7) and taking into account eqs. (6.3), we obtain the following equations for the projector $P_2$:

$$(1 - P_2) \left\{ \partial_{\varpi\bar{u}}P_2 - (\partial_{\varpiu}P_1)P_2 \right\} = 0,$$  \hspace{1cm} (6.8a)

$$(1 - P_2) \left\{ \partial_{\varpiu}P_2 + \nu \bar{\rho}(\partial_u P_1)P_2 \right\} = 0.$$  \hspace{1cm} (6.8b)

\footnote{Alternatively, we could parametrize $A_{1,u}$ and $A_{1,\bar{u}}$ in terms of the algebra-valued Leznov prepotential $\phi_1$ (cf. [49]):

$$A_{1,u} = \partial_v \phi_1, \quad A_{1,\bar{u}} = -\partial_v \phi_1,$$

where $\phi_1 = (\mu - \bar{\mu})P_1$ is defined by the asymptotic condition

$$\psi_1(u, v, \bar{u}, \bar{v}, \zeta \to \infty) = 1 + \zeta^{-1} \phi_1(u, v, \bar{u}, \bar{v}) + O(\zeta^{-2}).$$

This also leads to eqs. (6.8).}
In the derivation of the second equation we have also made use of the hermitean conjugate of eq. (6.2), that is
\[ P_1 \partial_w P_1 \equiv 0. \] (6.9)

In the operator formalism, all derivatives can be understood as commutators in the sense of (4.38). The projector identities
\[ (1 - P_2)P_2 \equiv 0 \quad \text{and} \quad (1 - P_2)T_2 \equiv 0 \] (6.10)
transform eqs. (6.8) into
\[ (1 - P_2) \{ wT_2 - [w, P_1] T_2 \} (T_2^\dagger T_2)^{-1} T_2^\dagger = 0, \] (6.11a)
\[ (1 - P_2) \{ \partial \tilde{t} T_2 - \eta' [\bar{w}, P_1] T_2 \} (T_2^\dagger T_2)^{-1} T_2^\dagger = 0, \] (6.11b)
where \( \eta' := \frac{\bar{w}}{\bar{w} + \bar{\mu} - \mu} \) and \( \nu = \nu_1 = \nu_2 \) from (4.30). Due to (6.10), a sufficient condition for a solution is given by
\[ wT_2 - [w, P_1] T_2 = T_2 S_1 \] (6.12a)
\[ \partial \tilde{t} T_2 - \eta' \bar{w} P_1 T_2 = T_2 S_2, \] (6.12b)
for some functions \( S_1(w, \bar{w}, \tilde{t}) \) and \( S_2(w, \bar{w}, \tilde{t}) \).

**Explicit solutions.** For the example of \( U(2) \) soliton-like configurations, we choose \( T_1 = \left( \begin{array}{c} 1 \\ w \end{array} \right) \), which is the simplest nontrivial \( U(2) \) ansatz compatible with eq. (6.2). In the operator formalism,
\[ P_1 = T_1 (T_1^\dagger T_1)^{-1} T_1^\dagger = \left( \begin{array}{cc} (1 + \bar{w} w)^{-1} & (1 + \bar{w} w)^{-1} \bar{w} \\ w (1 + \bar{w} w)^{-1} & w (1 + \bar{w} w)^{-1} \bar{w} \end{array} \right). \] (6.13)
Our task is now to determine a possible solution for \( T_2 \). We employ the ansatz
\[ T_2 = \left( \begin{array}{c} u_1(\tilde{t}, w, \bar{w}) \\ u_2(\tilde{t}, w, \bar{w}) \end{array} \right). \] (6.14)
Setting \( S_1 = w \) and inserting (6.14) into eq. (6.12a) yields
\[ [w, u_1] = [w, (1 + \bar{w} w)^{-1}] (u_1 + \bar{w} u_2) + 2\theta (1 + \bar{w} w)^{-1} u_2, \] (6.15a)
\[ [w, u_2] = w [w, (1 + \bar{w} w)^{-1}] (u_1 + \bar{w} u_2) + 2\theta w (1 + \bar{w} w)^{-1} u_2. \] (6.15b)
The last two equations immediately imply
\[ [w, w u_1 - u_2] = 0. \] (6.16)

**The case \( \tilde{\theta} = 0 \).** Evidently, if we restrict ourselves to \( [\tilde{t}, \tilde{t}] = i\tilde{\theta} = 0 \),
\[ u_2 = w u_1 - f(\tilde{t}, w) \] (6.17)
solves eq. (6.16) with an arbitrary function \( f \) (depending only on \( \tilde{t} \) and \( w \)) yet to be determined. Exploiting eqs. (6.15a) and (6.16), we find a solution
\[ u_1 = 1 + (1 + \bar{w} w)^{-1} \bar{w} f(\tilde{t}, w), \quad u_2 = w - (1 + \bar{w} w + 2\theta)^{-1} f(\tilde{t}, w). \] (6.18)
From (6.12b) we obtain in a similar fashion
\[ \partial_{\tilde{t}} u_1 = \eta' \left[ \left[ \overline{w}, (1 + \overline{w} w)^{-1} \right] (u_1 + \overline{w} u_2) \right] \] (6.19)
by setting \( S_2 = 0 \). Taking into account eq. (6.19) the explicit \( \tilde{t} \)-dependence of \( f(\tilde{t}, w) \) can be easily deduced:
\[ f = 2 \theta \eta' \left( \tilde{t} + h(w) \right), \] (6.20)
for some function \( h \) meromorphic in \( w \). Finally, substituting the results into (6.14) leads to
\[ T_2 = \left( \frac{1}{w} \right) + \left( \frac{-\overline{w}}{-1} \right) (1 + \overline{w} w + 2 \theta)^{-1} f(\tilde{t}, w). \] (6.21)
Translating this to the star formalism, we easily read off that \( T_1 = T_2 \) at the zero locus of \( f_*(\tilde{t}, w) \); moreover, if we restrict \( \mu \) to be purely imaginary, \( \mu = -ip, p \in (1, \infty) \), \( \Omega \) degenerates at these points to the identity. If we choose \( h_* = w * w = w^2 \), which corresponds to two moving soliton-like objects, this leads to right angle scattering [36]:
\[ f_* = 0 \Rightarrow w = \pm \sqrt{-\tilde{t}}. \] (6.22)
For the points in this locus, \( w \) is purely real for \( \tilde{t} < 0 \), and \( w \) is purely imaginary for \( \tilde{t} > 0 \). Since for the above choice of \( \mu \),
\[ \omega = \left( \frac{2}{p + p^{-1}} \right)^{1/2} \left( x + \frac{i}{2} (p + p^{-1}) y + \frac{i}{2} (p - p^{-1}) t \right), \] (6.23)
we see that e.g. for \( t = 0 \), the point where \( \Omega = 1 \) moves along the positive \( x \)-axis accelerating towards the origin for negative \( \tilde{t} \). For positive \( \tilde{t} \) it decelerates during its motion along the positive (or negative, depending on the sign in (6.22)) \( y \)-axis.

**The case \( \tilde{\theta} \neq 0 \).** If the two time directions do not mutually commute, i.e., \( \tilde{\theta} \neq 0 \), eq. (6.12b) can be written as
\[ \frac{1}{i \tilde{\theta}} [t, T_2] - \eta'[w, P_1] T_2 = T_2 S_2. \] (6.24)
Now, we can still solve eq. (6.16) by
\[ u_2 = w u_1 - g(\tilde{t}, \overline{w}, w). \] (6.25)
The difference to the case \( \tilde{\theta} = 0 \) is that now the vanishing of the commutator (6.16) can only be achieved by a nontrivial choice for \( g(\tilde{t}, \overline{w}, w) \), e.g.,
\[ g(\tilde{t}, \overline{w}, w) = \tilde{t} + \alpha \overline{w} + h(w), \] (6.26)
where \( \alpha := -\frac{1}{4} (\mu + \mu^{-1}) \tilde{\theta} \) and \( h(w) \) is again an arbitrary function meromorphic in \( w \). Let us restrict \( \mu \) again to be purely imaginary, \( \mu = -ip, p \in (1, \infty) \), then \( \alpha \in \mathbb{R}_+ \).

Apparently we also need \( \theta \neq 0 \); then, the contributions of \([w, \overline{w}]\) and \([w, \tilde{t}]\) add up to zero. If we use the inverse Moyal-Weyl map to translate to the star product and choose \( h_*(w) = w * w = w^2 \), we obtain
\[ g_*(\tilde{t}, \overline{w}, w) = \tilde{t} + \alpha \overline{w} + w^2. \] (6.27)
The subsequent calculation is analogous to the case \( \tilde{\theta} = 0 \). It turns out that \( P_1 \) and \( P_2 \) coincide and \( \Omega = 1 \) at the locus of \( g_*(\tilde{t}, \overline{w}, w) \), i.e., \( \tilde{t} + \alpha \overline{w} + w^2 = 0 \). If we split \( w \) into real and imaginary parts,
\[ w = a + ib, \] (6.28)
we can easily read off $a$ and $b$ from eq. (6.23), and the locus where $\Omega = 1$ is given by

$$-\tilde{t} = \alpha a + a^2 - b^2 + i(2a - \alpha)b. \quad (6.29)$$

Since $\tilde{t}$ is real, obviously either $b = 0$ or $a = \alpha/2$. We obtain

$$b = 0 \quad \implies \quad a = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \tilde{t}} \quad \text{for} \quad \tilde{t} \leq \frac{\alpha^2}{4}, \quad (6.30a)$$

$$a = \frac{\alpha}{2} \quad \implies \quad b = \pm \sqrt{\frac{3}{4}\alpha^2 + \tilde{t}} \quad \text{for} \quad \tilde{t} \geq -\frac{3}{4}\alpha^2. \quad (6.30b)$$

Figure 2: Motion of the “point of degeneracy” where $\Omega = 1$ in $\tilde{t}$ (bold lines). Its coordinates $a$ and $b$ are plotted for $\alpha = 2$. For $b(\tilde{t})$, exemplarily the upper branch was chosen.

This can be interpreted as follows: The “point of degeneracy” where $\Omega = 1$ moves along $a = -\alpha/2 + \sqrt{\alpha^2/4 - \tilde{t}}$ and $b = 0$ as $\tilde{t}$ grows until $\tilde{t} = -3\alpha^2/4$. Then, as $\tilde{t}$ grows larger, it moves along $a = \alpha/2$ and $b = \pm \sqrt{3\alpha^2/4 + \tilde{t}}$ (see figure 2). With the help of (6.23), it is easy to interpret this motion in the $xy$ plane (for fixed $t$). Therefore we have shown that it is possible to construct nontrivial configurations with scattering also for the case of noncommuting time directions. More complicated solutions in both cases may be constructed by making different choices for $h(w)$ or by choosing a more sophisticated ansatz for $T_1$ and $T_2$.

### 6.2 Colliding plane waves

Beside the soliton-like solutions (discussed above), there is another class of exact solutions to the self-duality equations (4.3), namely extended plane waves. For asymptotic times, each of them has
codimension two. In the commutative case, these were constructed and discussed in [63, 64, 65]. In the context of the $U(N)$ sigma model in 2+1 dimensions, this type of solution was first discussed by Leese [66]; the noncommutative generalization was given in [37]. In [21], plane waves were described in (noncommutative) D1-D3 systems. Here we want to show that one can construct noncommutative two-wave solutions in ncSDYM which entail nontrivial scattering even for $\mu_1 \neq \mu_2$.

**Additive ansatz.** We assume $\mu_1 \neq \mu_2$ henceforth and therefore make a single-pole ansatz for the auxiliary field $\psi$. In this section, exceptionally all products are understood to be star products (including the inverse and the exponential of coordinates). The calculation is largely parallel to the derivation in section 5.2 which gives us the opportunity to shorten the description here and to concentrate on the novel features.

We start from the additive ansatz (5.5), but now choose, inspired by [66, 37], the exponential ansätze

$$ T_1 = \left( \frac{1}{e^{b_1 w_1}} \right) \quad \text{and} \quad T_2 = \left( \frac{1}{e^{b_2 \bar{w}_2}} \right) $$

with $b_1 \in \mathbb{R}_{>0}, b_2 \in \mathbb{R}$. The discussion in section 4.1 guarantees that this will yield a solution to the self-duality equations. However, it is not obvious that this solution factorizes into two plane waves for asymptotic times; to prove this, we have to compare with the multiplicative ansatz again.

**Multiplicative ansatz.** The multiplicative ansatz takes the same form as eq. (5.6). It can be easily shown as in section 5.2 that $P_1 = \tilde{T}_1 (\tilde{T}_1^{-1})^{-1} \tilde{T}_1$ can consistently be constructed from $\tilde{T}_1 = T_1$, given in (6.31). Let us now scrutinize the $|t| \to \infty$ limits of $P_1$. For simplicity, we set $\mu_1 = ip$ strictly imaginary with $p > 1$. Therefore, $\beta_1$ in (4.31) is real and

$$ \beta_1 = -\frac{1}{2} \theta^{-1/2} (p - p^{-1})(p + p^{-1})^{-1/2} < 0. $$

(6.32)

If we consider the large $t$ limit, it turns out that $w_1$ is dominated by the term linear in $t$, namely:

$$ b_1 w_1 \simeq \pm b_1 \sqrt{2 \theta} |t|, \quad \text{for} \quad t \to \pm \infty. $$

(6.33)

Thus, $P_1$ in the large $t$ limit behaves as

$$ P_1 = \left( 
\begin{array}{cc}
(1 + e^{b_1 \bar{w}_1} e^{b_1 w_1})^{-1} & (1 + e^{b_1 \bar{w}_1} e^{b_1 w_1})^{-1} e^{b_1 \bar{w}_1} \\
 e^{b_1 w_1} (1 + e^{b_1 \bar{w}_1} e^{b_1 w_1})^{-1} & e^{b_1 w_1} (1 + e^{b_1 \bar{w}_1} e^{b_1 w_1})^{-1} e^{b_1 \bar{w}_1}
\end{array}
\right) \begin{cases}
\to \infty \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: \Pi_{+\infty}, \\
\to -\infty \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: \Pi_{-\infty}.
\end{cases} $$

(6.34a)

(6.34b)

In these limits, $P_1$ obviously becomes a constant projector. Again, the Lax operators in (5.10) can be moved next to $P_2$ in these limits to give (5.11). This concludes the proof that we may write $\tilde{T}_2 = T_2$ asymptotically.

In addition, we can conclude that this setup entails nontrivial scattering, again by analyzing $\Omega^\dagger = \psi_2(\zeta = 0)$ in the limits $t \to \pm \infty$:

$$ \Omega^\dagger \bigg|_{t \to +\infty} = \lim_{t \to +\infty} \psi_2(\zeta = 0) = (1 - \rho_2 P_2)(1 - \rho_1 \Pi_{+\infty}), $$

(6.35a)

$$ \Omega^\dagger \bigg|_{t \to -\infty} = \lim_{t \to -\infty} \psi_2(\zeta = 0) = (1 - \rho_2 P_2)(1 - \rho_1 \Pi_{-\infty}). $$

(6.35b)

For convenience, we have set $\rho_k = 1 - \mu_k/\mu_k$ for $k = 1, 2$. Clearly, $\Omega^\dagger \bigg|_{t \to +\infty}$ and $\Omega^\dagger \bigg|_{t \to -\infty}$ are different, which indicates nontrivial scattering behavior.
If we now additionally take $\tilde{t} \to \pm \infty$, we find that $P_2$ also becomes a constant projector,

$$\lim_{\tilde{t} \to \pm \infty} P_2 = \Pi_{\pm \infty}. \tag{6.36}$$

Therefore,

$$\Omega^\dagger |_{t, \tilde{t} \to -\infty} = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.37a}$$

and

$$\Omega^\dagger |_{t, \tilde{t} \to +\infty} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \tag{6.37b}$$

where $\gamma := \bar{\mu}_1 \bar{\mu}_2 \mu_1^{-1} \mu_2^{-1}$. Again, this result shows the existence of scattering in this two-wave configuration.

The above-described solutions represent 1+2 dimensional plane waves in the asymptotic domain, i.e., long before and after the interaction. This can be seen by analyzing the energy density in 2+1 dimensional subspaces, e.g., the energy density for a gauge field constructed from $P_1$ (at a fixed time $\tilde{t}$) turns out to depend only on one spatial direction [37]. The asymptotic space-time interpretation for this setup can be visualized by the following snapshot for fixed large $t$ (see figure 3). Since $P_1$ is independent of $\tilde{t}$, the corresponding wave extends along this direction. The above energy density argument explains its spatial extension. Observe that for this type of solutions, the moduli $\mu_1, \mu_2$ not only parametrize the velocities of the plane waves but also their respective parallel directions in the $xy$-plane (cf. eqs. (4.29) and (4.36) together with (6.31)). When $t$ varies in the asymptotic region, the world-volume of the first plane wave undergoes a parallel shift. Consider a space-like section (i.e., $t$ and $\tilde{t}$ fixed). Then, the intersection of the two plane waves with this $xy$-plane will consist of two lines which generically include some angle determined by the moduli $\mu_1$ and $\mu_2$. For later times $t$ or $\tilde{t}$, the lines corresponding to $P_1$ and $P_2$ have changed position in the $xy$-subspace but kept their directions.

7 Conclusions

In this note we have discussed exact solutions to the self-duality equations of noncommutative Yang-Mills theory on $\mathbb{R}^{2,2}$. To this aim, a Lax pair has been gauged in two inequivalent ways; appropriate ansätze for the auxiliary field $\psi$ have been discussed. From concrete solutions $\psi$ to the residue equations of the Lax pair explicit expressions for the gauge potentials have been constructed. We have shown that the Lax pair is included in the string field theoretic one; therefore, it seems plausible that this also applies to its solutions. Conversely, our field theoretic solutions could serve as a guideline to construct nonperturbative solutions of $\tilde{N}=2$ string field theory. It seems reasonable to expect that a similar program could be carried out for $\tilde{N}=1$ strings.

A GMS-like solution and solutions describing $U(2)$ solitons have been constructed. Moreover, it has been shown that dimensional reduction to 2+1 dimensions leads to results coinciding with those of [35]–[38]. Explicitly, the field theory description of D-brane scattering (for plane wave and soliton-like configurations) has been generalized to the 2+2 dimensional case. It would be interesting to trace this description to the string theory level, i.e., compute scattering in the given $B$-field background by closed string exchange. To corroborate the interpretation of our field theory solutions as lower-dimensional D-branes one could try to compute their Chern characters and examine the fluctuation spectrum around these solutions.
Figure 3: Snapshot for fixed large $|t|$ of one octant of the configuration discussed in section 6.2. For simplicity, the first plane wave was chosen to be static ($\mu_1 = -i$). This choice implies that the energy density for the first wave at fixed $\tilde{t}$ does not depend on $y$. The support of the solution is concentrated around the grey planes.

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A Self-duality, twistor space and holomorphicity

In this section, we explain the geometric setup underlying the method we use to solve the self-duality equations of Yang-Mills theory on $\mathbb{R}^{2,2}$. We mostly restrict ourselves to the commutative case, comments on the noncommutative generalization are added where appropriate.\textsuperscript{26} For our purposes, $U(N)$ Yang-Mills theory is formulated in terms of a $GL(N, \mathbb{C})$ principal bundle $P \cong \mathbb{R}^{2,2} \times GL(N, \mathbb{C})$ over the (pseudo-)Riemannian “space-time” manifold $\mathbb{R}^{2,2}$. This principal bundle should be endowed with an irreducible $GL(N, \mathbb{C})$ connection $A$ and its respective curvature $F$. We will impose a reality condition on $A$ below. The self-duality equations $F = *F$ are tackled with the help of a Lax pair, whose geometrical meaning will now be described.

A.1 Isotropic coordinates

We will see in section A.4 that the self-duality equations on $\mathbb{R}^{2,2}$ can be written in real coordinates $x^\mu$ as

\[
\overline{W_1} W_2 F_{\mu\nu} = 0
\]  

\textsuperscript{26}For a description of twistors in the noncommutative case, see [67, 68, 69].
for certain 4-vectors $\mathbf{W}_i$. To derive constraints on the $\mathbf{W}_i$, it turns out to be useful to switch to a spinor notation. Exploiting that $so(2, 2) \cong sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$, we can rewrite $\mathbf{W}_i$ as

$$\mathbf{W}_i = \mathbf{v}^\alpha \mathbf{\tau}_\alpha = \left( \frac{\mathbf{W}_1 + \mathbf{W}_2}{\mathbf{W}_1 + \mathbf{W}_2} \right) \left( \frac{\mathbf{W}_1 - \mathbf{W}_2}{\mathbf{W}_1 - \mathbf{W}_2} \right) \quad (A.2)$$

with the help of $SL(2, \mathbb{R})$-generators $\mathbf{\tau}_a$, $a = 1, 2, 3$ and $\mathbf{\tau}_4 = 1$. If we define as for the Pauli matrices $\mathbf{\tau}_{\beta\bar{\beta}} = \eta^{\mu\nu} \mathbf{\tau}_\mu \bar{\epsilon}_{\alpha\beta} \epsilon_{\alpha\beta}$ with $\epsilon_{12} = -1$, eq. (A.1) can be rewritten as

$$\mathbf{W}_1 \mathbf{W}_2 (F_{a\bar{b}} \bar{\epsilon}_{\alpha\beta} + F_{\alpha\beta}) = 0. \quad (A.3)$$

These are the self-duality equations $F_{a\bar{b}} = 0$ iff we choose

$$\mathbf{W}_1 = \xi^\alpha \pi^\alpha \quad \text{and} \quad \mathbf{W}_2 = \chi^\beta \pi^\beta, \quad (A.4)$$

with arbitrary commutative spinors $\xi^\alpha, \chi^\alpha$, and $\pi^\alpha$. That is, $\mathbf{W}_1$ and $\mathbf{W}_2$ have to span a null plane in $\mathbb{R}^{2,2}$.

On $\mathbb{R}^{2,2}$, there are two possibilities to satisfy (A.4), related to the existence of Majorana-Weyl spinors in 2+2 dimensions: One can choose complex or real spinors. Since the $\mathbf{\tau}$-matrices in (A.2) are real, this will lead to complex and real coordinates on $\mathbb{R}^{2,2}$.

### A.2 Complex coordinates

**Almost complex structures on $\mathbb{R}^{2,2}$.** To elucidate the meaning of the $\mathbf{W}_i$, it is necessary to introduce an *almost complex structure* on $\mathbb{R}^{2,2}$. An almost complex structure is a tensor field $J$ of type (1,1) such that $J_{\mu\nu} J_{\rho\lambda} = -\delta_{\mu\lambda}$. We shall consider translationally invariant (constant) and therefore integrable almost complex structures, i.e., complex structures. It is easy to see that complex structures on $\mathbb{R}^{2,2}$ are parametrized by the coset $SO(2, 2)/U(1, 1) \cong SO(2, 1)/SO(2)$.

Without loss of generality, we can restrict the discussion to almost complex structures compatible with the metric (so that the metric is hermitean). Then, (anti)holomorphic basis vectors are automatically null vectors.

One can realize [70] this coset space on $so(2,1)$ in the following way [13]: We start from a matrix representation of $so(2,1)$,

$$I_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (A.5)$$

---

27For a given almost complex structure $J$ one can choose coordinates $x^1, x^2, y^1, y^2$ such that in this basis, $J$ as an endomorphism of the tangent bundle maps $J(\frac{\partial}{\partial \sigma^x}) = \frac{\partial}{\partial \sigma^y}$ and $J(\frac{\partial}{\partial \sigma^y}) = -\frac{\partial}{\partial \sigma^x}$ for $k = 1, 2$. A linear combination $\partial_{s^k} - i J \partial_{s^k} =: \partial_{s^k}$ (as a section of the complexified tangent bundle to $\mathbb{R}^{2,2}$) obviously has eigenvalue 1, it only rotates homogeneously under rotations $M^s$ of the structure group $SO(2,2)$ if $J$ and $M$ commute. This singles out a subgroup $U(1,1)$ of $SO(2,2)$ which leaves the fixed complex structure invariant.
satisfying $I_a I_b = g_{ab} + f_{ab}^c I_c$ with structure constants $f_{12}^3 = -f_{23}^1 = -f_{31}^2 = 1$ and metric $(g_{ab}) = \text{diag}(1, 1, -1)$ on $so(2, 1)$. Then we can write a general complex structure in the form

$$J = -s^a I_a$$  \hspace{1cm} (A.6)

for $s^1, s^2, s^3 \in \mathbb{R}$. We easily read off

$$J^2 = g_{ab} s^a s^b = -1 \quad \iff \quad (s^1)^2 + (s^2)^2 - (s^3)^2 = -1. \hspace{1cm} (A.7)$$

Obviously, the $\{s^a\}$ parametrize a two-sheeted hyperboloid $H^2$. We can map the upper half $H^2_+$ of $H^2$ onto the interior of the unit disk in the $y$-plane by a stereographic projection

$$s^1 := \frac{2y^1}{1 - r^2}, \quad s^2 := \frac{2y^2}{1 - r^2}, \quad s^3 := \frac{1 + r^2}{1 - r^2}, \quad r^2 := (y^1)^2 + (y^2)^2. \hspace{1cm} (A.8)$$

Simultaneously, the lower half $H^2_-$ is mapped onto the exterior of the unit disk. If we define $\lambda := -(y^1 + iy^2)$, both regions are related by the map

$$\sigma: \lambda \mapsto 1/\lambda. \hspace{1cm} (A.9)$$

Note that for $|\lambda| \neq 1$, $\sigma$ has no fixed points. Recapitulating, we can state that the moduli space of complex structures on $\mathbb{R}^{2,2}$ is $\mathbb{C}P^1 \setminus S^1$ (the $S^1$ being given by $|\lambda| = 1$).

**Almost complex structure on $H^2$.** If we introduce the standard complex structure $\varepsilon$ on $H^2$,

$$\varepsilon^i_k \varepsilon^j_k = -\delta^i_j, \quad \varepsilon^2_1 = -\varepsilon^1_2 = 1, \quad \varepsilon^1_1 = \varepsilon^2_2 = 0 \hspace{1cm} (A.11)$$

for $i, j, k = 1, 2$, we can give explicit expressions for the (local) antiholomorphic vector field on $H^2_+ \subset \mathbb{C}P^1$:

$$W_3 = -\frac{1}{2} \left( \frac{\partial}{\partial y^1} + i \frac{\partial}{\partial y^2} \right) = \frac{\partial}{\partial \lambda}. \hspace{1cm} (A.12)$$

**Noncommutative description.** In the noncommutative framework, one has to incorporate some modifications to the above description. All functions now have to be multiplied by a deformed product; alternatively, the Moyal-Weyl map may be used to transform them into operators with the usual operator product. In this interpretation, the space-time manifold $\mathbb{R}^{2,2}$ has to be replaced by the Heisenberg algebra $\mathbb{R}^{2,2}_\theta$ generated by operators $\hat{x}^\mu$ subject to $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$. The (Lie algebra of) inner derivations of $\mathbb{R}^{2,2}_\theta$ corresponds to the (Lie algebra of) sections of the tangent
bundle $T\mathbb{R}^{2,2}$. If we denote by $\mathfrak{g}$ a four-dimensional representation of $SO(2,2)$, the Lie algebra of inner derivations can be understood as a free $\mathfrak{g}$-module. From these arguments it is clear that the construction of the moduli space of complex structures on $\mathbb{R}^{2,2}_g$ can be treated analogously to the commutative setup. Note that $H^2$ remains commutative.

Let us reconsider the noncommutative setup from a different point of view. To this aim, we assume without loss of generality that $\theta = \bar{\theta} > 0$. Then, just as $z^1$ and $z^2$ are mapped to annihilation operators (3.11a) under the Moyal-Weyl map, $\eta^1$ and $\eta^2$ are mapped to new annihilation operators

$$c_1 := (1 - \lambda \bar{\lambda})^{-1/2}(a_1 + \lambda a_1^\dagger) \quad \text{and} \quad c_2 := (1 - \lambda \bar{\lambda})^{-1/2}(a_2 + \lambda a_2^\dagger), \quad (A.13)$$

where $|\lambda| < 1$. The $SO(2,2)$ rotation of the commutative discussion above transforming old coordinates $z^i$ to new coordinates $\eta^i$ after transition to operators takes the form of a Bogoliubov transformation (A.13). In general, transformations $c_1 = Ua_1U^\dagger$, $c_2 = Ua_2U^\dagger$ yield equivalent representations of the Heisenberg algebra $\mathbb{R}^{2,2}_g$ if $U$ is unitary. One can easily show that this is the case for $|\lambda| \neq 1$ for (A.13). Obviously, the Bogoliubov transformations leaving a given representation invariant are parametrized by the maximal pseudo-unitary subgroup $U(1,1)$ of $SO(2,2)$ leading again to the same coset space as in the commutative case.

A.3 Real isotropic coordinates

Although $|\lambda| = 1$ according to the preceding discussion will not correspond to a complex structure on $\mathbb{R}^{2,2}$, the vector fields (A.10) in this case still span a null plane in $\mathbb{R}^{2,2}$ [71]. Using that now $\lambda = \bar{\lambda}^{-1}$, one readily sees that complex conjugation maps $\bar{W}_1$ to a multiple of $\bar{W}_2$ and vice versa, i.e., the isotropic plane is real. One is free to choose a real basis for this plane, which is most easily accomplished with the help of the map (4.45) sending the unit circle to the real axis in the $\zeta$-plane.

Real isotropic planes, being parametrized by $S^1 = \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \}$, supplement the moduli space of complex isotropic two-planes (or complex structures) to $\mathbb{C}P^1 \cong H^2 \cup S^1$ [49, 71]. So, $\mathbb{C}P^1$ can be considered as the moduli space of all null two-planes (or extended complex structures) in $\mathbb{R}^{2,2}$.

A.4 Extended twistor space for $\mathbb{R}^{2,2}$

Extended twistor space. In this section, Ward’s theorem [72] on a one-to-one correspondence between vector bundles $E$ with self-dual connections over euclidean $\mathbb{R}^4$ and holomorphic bundles $E'$ over the so-called twistor space is rephrased for the case of $\mathbb{R}^{2,2}$. The twistor space for $\mathbb{R}^{2,2}$ is the bundle $\mathbb{R}^{2,2} \times H^2 \rightarrow \mathbb{R}^{2,2}$ of all constant complex structures on $\mathbb{R}^{2,2}$. It can be endowed with the direct sum $\mathcal{J}$ of the complex structures $\mathcal{J}$ and $\mathcal{E}$. The vector fields (A.10) and (A.12) for $|\lambda| \neq 1$ are the $\mathcal{J}$-antiholomorphic vector fields on $\mathbb{R}^{2,2} \times H^2$ with respect to this complex structure. Admitting $|\lambda| = 1$ in (A.10) and (A.12), we can extend these vector fields naturally to $Z := \mathbb{R}^{2,2} \times \mathbb{C}P^1$.

Vector bundle over $Z$. Now, we can use the canonical projection $\pi : Z \rightarrow \mathbb{R}^{2,2}$ to lift the vector bundle $E := P \times_{\text{GL}(N,\mathbb{C})} \mathbb{C}^N$ to a bundle $\pi^* E$ over the extended twistor space $Z$. By construction, the connection on $\pi^* E$ is flat along the fibers $\mathbb{C}P^1$ of $Z$, so that the lifted connection $\pi^* A$ on $\pi^* E$ can be chosen to have only components along $\mathbb{R}^{2,2}$, $\pi^* A = A_\mu dx^\mu$. Thus, the lift takes the covariant derivative $D_\mu = \partial_\mu + A_\mu$ on $E$ to

$$\pi^* D = dx^\mu D_\mu + dy^i \frac{\partial}{\partial y^i}. \quad (A.14)$$

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on \( \pi^*E \). Now, the \( \mathcal{J} \)-antiholomorphic components of (A.14) are the \((0, 1)\) components of \( \pi^*D \) along the antiholomorphic vector fields \( \overline{W}_i \) on \( \mathcal{E} \):

\begin{align}
D^{(0,1)}_1 \equiv W_1^\mu D_\mu &= W_1 + \frac{1}{2} (A_1 + iA_2) - \frac{\lambda}{2} (A_3 - iA_4), \\
D^{(0,1)}_2 \equiv W_2^\mu D_\mu &= W_1 + \frac{1}{2} (A_3 + iA_4) - \frac{\lambda}{2} (A_1 - iA_2), \\
D^{(0,1)}_3 \equiv W^i_3 \partial_{y^i} &= \overline{W_3}.
\end{align}

(A.15)

**Holomorphic sections.** Local sections \( \varphi \) of the complex vector bundle \( \pi^*E \) are holomorphic if

\begin{align}
D^{(0,1)}_1 \varphi &= 0, \\
D^{(0,1)}_2 \varphi &= 0, \\
D^{(0,1)}_3 \varphi &= 0.
\end{align}

(A.16)

We can also view this as the local form of meromorphic sections of \( E' := \pi^*E \) in a given trivialization of the bundle. One can combine \( N \) such sections (as columns) into an \( N \times N \) matrix to obtain the matrix-valued function \( \psi \) used in (4.46). Using (A.15), a comparison with (4.46) shows that after solving (A.16c) these are exactly the linear equations (Lax pair) for self-dual Yang-Mills theory. In this framework, the self-duality equations (4.47) emerge as the condition that eqs. (A.16) are compatible, i.e., the \((0, 2)\) components of the curvature of \( E' \) vanish.

**A.5 Reality condition**

So far, we have been working with a complex vector bundle associated to a \( GL(N, \mathbb{C}) \)-principal bundle \( P \) to describe \( U(N) \) self-dual Yang-Mills theory. Therefore, we have to implement a reality condition on our gauge fields, i.e., impose the additional constraint \( A^\dagger_\mu = -A_\mu \).

Let us now scrutinize the action of hermitean conjugation on the linear equations (4.46). Eq. (4.46a) is equivalent to

\[(\partial_{\overline{z}^1} - \lambda \partial_{\overline{z}^2}) \psi^{-1}(\lambda) = \psi^{-1}(\lambda)(A_{\overline{z}^1} - \lambda A_{\overline{z}^2}),\]

(A.17)

where we suppress the additional dependence of \( \psi \) of the space-time coordinates \( z^i, \overline{z}^i \). Since we demand this to hold for all \( \lambda \), we can as well first apply \( \sigma \) from (A.9) and then take the hermitean conjugate,

\[ (\partial_{\overline{z}^2} - \lambda \partial_{\overline{z}^1}) \left[ \psi^{-1}(\overline{\lambda}^{-1}) \right]^\dagger = -(A_{\overline{z}^2} - \lambda A_{\overline{z}^1}) \left[ \psi^{-1}(\overline{\lambda}^{-1}) \right]^\dagger. \]

(A.18)

This coincides with (4.46b) if we choose \( \psi(\lambda) = [\psi(\overline{\lambda}^{-1})]^\dagger \), i.e., eq. (4.48). With these restrictions, the \( gl(N, \mathbb{C}) \)-curvature \( F \) naturally descends to a \( u(N) \)-valued curvature.

In the noncommutative case, the vector bundle \( E \) has to be replaced by a free module over \( \mathbb{R}_{\theta}^{2,2} \). Accordingly, \( D = d + A \) is chosen to be a connection on the module \( E \) [8]. It is understood that the above discussion can be applied analogously, taking into account that multiplication of \( A \) and \( \psi \) becomes noncommutative.
B Abelian pseudo-instantons

This appendix concludes our considerations with the discussion of a special class of abelian, i.e., $U(1)$ solutions with finite action (in contrast to their commutative counterparts).\textsuperscript{28} We work in the operator formalism. Let us introduce “shifted” operators acting on the two-oscillator Fock space $\mathcal{H}$:

$$X_\mu := A_\mu + i(\theta^{-1})_{\mu \nu} x_\nu,$$  \hspace{1cm} (B.1)

where $(\theta^{-1})_{\mu \sigma} \theta^{\sigma \nu} = \delta_\mu^{\nu}$. The operator-valued field strength $F_{\mu \nu}$ can be expressed in terms of the shifted operators $X_\mu$ as

$$F_{\mu \nu} = [X_\mu, X_\nu] - i(\theta^{-1})_{\mu \nu}.$$  \hspace{1cm} (B.2)

The incarnation of the ncYM equations in this context is

$$[X_\mu, [X_\mu, X_\nu]] = 0.$$  \hspace{1cm} (B.3)

They are, of course, automatically satisfied by $X_\mu$ subject to the ncSDYM equations (cf. [74])

$$[X_\mu, X_\nu] = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} [X^\rho, X^\sigma] + i(\theta_{\mu \nu} - \frac{1}{2} \epsilon_{\mu \nu \rho} \theta^{\rho \sigma}).$$  \hspace{1cm} (B.4)

Observe that the last term of the ncSDYM equations vanishes for self-dual $\theta^{\mu \nu}$, i.e., $\theta = \bar{\theta}$. Switching to complex coordinates\textsuperscript{29} and assuming self-dual $\theta^{\mu \nu}$ from now on, eqs. (B.4) become

$$[X_{\bar{z}_i}, X_{\bar{z}_j}] = [X_{\bar{z}_i}, X_{z_j}] = 0, \hspace{1cm} (B.5a)$$

$$[X_{\bar{z}_i}, X_{z_j}] - [X_{\bar{z}_j}, X_{z_i}] = 0, \hspace{1cm} (B.5b)$$

where $X_{\bar{z}_i} := A_{\bar{z}_i} + i(\theta^{-1})_{i j} \bar{z}^j$ and $X_{z_i} := A_{z_i} + i(\theta^{-1})_{i j} z^j$, $i \in \{1, 2\}$. It is easily checked that

$$X^0_{\bar{z}_1} = i(\theta^{-1})_{1 1} \bar{z}^1, \hspace{1cm} X^0_{z_1} = i(\theta^{-1})_{1 1} z^1, \hspace{1cm} (B.6a)$$

$$X^0_{\bar{z}_2} = i(\theta^{-1})_{2 2} \bar{z}^2, \hspace{1cm} X^0_{z_2} = i(\theta^{-1})_{2 2} z^2, \hspace{1cm} (B.6b)$$

i.e., $A_{\bar{z}_i} = A_{z_i} = 0$ yields a (trivial) solution of (B.5).

New solutions $X^1$ may be obtained by shift operator “dressing” of the solutions $X^0$, namely

$$X^1_{\bar{z}_1} = S X^0_{\bar{z}_1} S^\dagger, \hspace{1cm} X^1_{z_1} = S X^0_{z_1} S^\dagger, \hspace{1cm} (B.7a)$$

$$X^1_{\bar{z}_2} = S X^0_{\bar{z}_2} S^\dagger, \hspace{1cm} X^1_{z_2} = S X^0_{z_2} S^\dagger. \hspace{1cm} (B.7b)$$

In these expressions, $S$ and $S^\dagger$ are shift operators acting on the two-oscillator Fock space $\mathcal{H}$ according to

$$S^\dagger S = 1, \hspace{1cm} SS^\dagger = 1 - P_0, \hspace{1cm} P_0 S = S^\dagger P_0 = 0. \hspace{1cm} (B.8)$$

Apparently, the representation of $S$ on $\mathcal{H}$ is not unique (see, e.g., [19, 69] for various explicit forms of $S$ and $S^\dagger$). Here, $P_0$ denotes the projector onto the ground state $|0, 0\rangle$ of the Fock space $\mathcal{H}$:

$$P_0 = |0, 0\rangle \langle 0, 0|.$$  \hspace{1cm} (B.9)

The field strength for such configurations turns out to be of the form

$$F_{z_i \bar{z}_i} = [X^1_{\bar{z}_i}, X^1_{z_i}] - i(\theta^{-1})_{i i} = -i(\theta^{-1})_{i i} P_0 = -\frac{1}{2\theta} P_0, \hspace{1cm} i \in \{1, 2\}. \hspace{1cm} (B.10)$$

\textsuperscript{28}Noncommutative instantons in euclidean space were introduced in [73].

\textsuperscript{29}We denote $[z^i, \bar{z}^j] = i \theta^{ij}$ and $(\theta^{-1})_{i i} \theta^{ij} = \delta^i_j$. 

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This coincides with the solution first presented in [19] for the euclidean case, namely on $\mathbb{R}^4$. The action for this type of solution is known to be finite; this is also the case here:

$$S_1 = -\frac{1}{2g_Y^2}(2\pi \theta)^2 \text{Tr}_H F_{z^i z^j} F^{z^i z^j} = \frac{4\pi^2}{g_Y^2}. \quad \text{(B.11)}$$

In the context of D-branes, solutions of type (B.10) have been interpreted as a D-brane of codimension four sitting at the origin of a space-time filling D-brane [19]. This may be transferred to our case.

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