Mathematical analysis of the dynamic flow characteristic in a damping nozzle for a pressure transmitter

P. D. Howell
Mathematical Institute, 24–29 St Giles', Oxford OX1 3LB, UK
E-mail: howell@maths.ox.ac.uk

A. A. Lacey
School of Mathematical & Computer Sciences, Heriot–Watt University, Edinburgh EH14 4AS, UK
E-mail: a.a.lacey@hw.ac.uk

J. Vogler
Danfoss A/S, DK–6430 Nordborg, Denmark
E-mail: vogler@danfoss.com

A. Vonsild
Danfoss A/S, DK–6430 Nordborg, Denmark
E-mail: alv@danfoss.com

Abstract. We analyse the flow of a liquid along a pipe and through a small nozzle connecting the pipe to a pressure sensor. The aims are to predict the accuracy with which the pressure in the sensor reflects that in the pipe, and to determine whether or not cavitation occurs in the reservoir.

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1. Introduction

Danfoss manufacture sensors that dynamically measure the pressure in liquid flowing along a pipe. The pressure in the pipe is transmitted via a small tube to a reservoir of liquid, whose pressure is measured using a piezo-electric membrane. In pipe flows where the liquid is stopped suddenly, it is quite possible for the pressure to become low enough for the liquid to cavitate. However, it is believed that the membrane may be damaged if cavitation bubbles ever form in the reservoir. Danfoss have discovered that this can be prevented by inserting a so-called “pulse snubber”, which restricts the flow to a small hole between the pipe and the reservoir. It is not fully understood, however, how this protects the reservoir against cavitation, or how it affects
the accuracy with which the pressure in the reservoir reflects that in the pipe. Our aim in this report is to construct mathematical models that address these questions.

We begin in section 2 by presenting a model for the flow in the pipe, incorporating liquid and vapour phases with a simple cavitation law governing the transition between them. In section 3, the model is used to predict the pressure transient caused when a flowing column of liquid is impulsively brought to rest. The resulting flow through the pulse snubber is then analysed in section 4. This is used in section 5 to predict the pressure in the reservoir corresponding to a given pipe pressure. Unfortunately, the model is unable to explain how the pulse snubber prevents cavitation in the reservoir. We hypothesise that this is a consequence of our oversimplified cavitation law, and propose two possible improvements in section 6. Finally, we draw our conclusions in section 7.

2. Model for flow in the pipe

Our aim in this report is to understand how pressure pulses in a pipe are transmitted to an attached pressure sensor. Before attempting to do so, we first consider the question of how the pressure pulses arise in the first place. This provides the input for our models of the pressure sensor and the pulse snubby developed below. Although we will only consider pipes with uniform cross section, we develop a model in which the pipe radius $R$ and cross-sectional area $A = \pi R^2$ may both be functions of distance $x$ along the pipe. This will prove useful later, since the same governing equations may then be applied to the flow through the pulse snubby.

Since we wish to consider situations in which the liquid may cavitate, we suppose in general that the pipe contains a mixture of gas and liquid. Conservation of mass implies that the cross-sectionally averaged density $\bar{\rho}$ and velocity $\bar{u}$ satisfy

$$\frac{\partial}{\partial t} (A \rho) + \frac{\partial}{\partial x} (A \rho u) = 0. \quad (1)$$

Conservation of momentum for the mixture leads to the equation

$$\frac{\partial}{\partial t} (A \rho u) + \frac{\partial}{\partial x} (A \rho u^2) = -A \frac{\partial p}{\partial x} - D, \quad (2)$$

where $p$ is the average pressure and $D$ is the drag exerted on the fluid by the pipe wall. In (2), we have approximated the profile coefficient as unity,\(^1\) which is reasonable for turbulent flow \([1]\).

Our constitutive equation for the wall drag takes the general form

$$D = -C2\pi R\rho u|u|, \quad (3)$$

where $C$ is a drag coefficient. In general, $C$ may depend both on the flow regime (e.g. turbulent or laminar, single-phase or multiphase, annular or dispersed) and on the local Reynolds number,

$$R = \frac{\rho u R}{\mu}, \quad (4)$$

where $\mu$ is the viscosity of the mixture. For single-phase turbulent flow, though, $C$ typically depends only weakly on $R$: Blasius \([2]\), for example, gives the empirical relation

$$C \approx \frac{0.033}{R^{1/4}}. \quad (5)$$

\(^1\) that is, we have approximated the average of the velocity squared by the square of the average velocity.
For simplicity, we treat $C$ as a constant (of the order of 0.01), so (2) reduces to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{2C}{R} u|u|. \quad (6)$$

To close the model, we need an effective gas law, relating pressure and density, for the mixture. Let the vapour volume fraction be $\alpha$, so that $\alpha = 0$ in regions containing only liquid, $\alpha > 0$ where cavitation has occurred, and $\alpha = 1$ where there is only vapour. The density of the mixture is thus related to the liquid and vapour densities ($\rho_l$ and $\rho_v$ respectively) by

$$\rho = (1 - \alpha) \rho_l + \alpha \rho_v. \quad (7)$$

We assume that the vapour phase satisfies an ideal gas law, with gas constant $\Gamma$, while the liquid phase has constant compressibility $1/\beta$. The two densities are thus related to the pressure by

$$p = \rho_v \Gamma T = p_a + \beta \left( \frac{\rho_l}{\rho_0} - 1 \right), \quad (8)$$

where $\rho_0$ is the liquid density at atmospheric pressure $p_a$ and $T$ is the absolute temperature, assumed constant.

Equations (7) and (8) may be simplified by making the following observations.

(i) The vapour density $\rho_v$ is much less than the liquid density $\rho_l$.

(ii) The incompressibility $\beta$ is typically very large, so the liquid phase is effectively incompressible, with constant density $\rho_l \approx \rho_0$.

Thus the mixture density is well approximated by

$$\rho \approx \rho_0 (1 - \alpha), \quad (9)$$

and we need not solve for $\rho_v$.

The simplest cavitation model is that cavitation occurs whenever the pressure falls to the vapour pressure $p_v$, so that $p$ is bounded below by $p_v$. This approach is common in the modelling of cavities [3] and may be expressed as the complementarity condition

$$p \geq p_v, \quad \alpha \geq 0, \quad \alpha (p - p_v) = 0. \quad (10)$$

Equations (1), (6), (9) and (10) form a closed system for $\rho$, $u$, $\alpha$ and $p$.

3. Impulsively stopped flow

Now we apply the model developed in section 2 to a typical experimental setup, in which liquid travelling along a uniform pipe (i.e. with constant $A$ and $R$) is impulsively stopped by a valve. The situation is depicted schematically in figure 1. The liquid flows along the pipe at constant speed $U$ such that, at $t = 0$, a column of length $L$ has entered the pipe. The end is then closed instantaneously, but the liquid continues to flow, leaving behind a cavitated region. Our aim is to predict the pressure variations measured by a sensor placed a small distance downstream of the valve.

We denote the closed end by $x = 0$ and the boundary between the cavitated and non-cavitated regions by $x = X(t)$. The problems to be solved in the two regions are

(i) $x < X(t)$, $\alpha > 0$, $p = p_v$

$$- \frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial x} \left( (1 - \alpha) u \right) = 0, \quad (11)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{2C}{R} u|u|, \quad (12)$$

$$u = 0 \quad \text{at} \quad x = 0; \quad (13)$$
(ii) $x > X(t), \alpha = 0, p > p_v$

$$\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} - \frac{2C}{R} u |u|,$$  \hspace{1cm} (15)

$$u = U \quad \text{at} \quad t = 0.$$  \hspace{1cm} (16)

In $x < X(t)$, we integrate the hyperbolic equation (12) to obtain

$$u = \frac{R}{2Ct} \left(1 - e^{-2Ct/R}\right).$$  \hspace{1cm} (17)

Then, (11) implies that

$$\frac{d}{dt} \left[t(\alpha - 1)e^{-2Ct/R}\right] = 0$$  \hspace{1cm} (18)

along the characteristics $dx/dt = u$. The term in brackets is clearly zero when $t = 0$, and it follows that $\alpha$ must be equal to 1 for all positive $t$. The fluid in $x < X(t)$ is therefore entirely cavitated, with no liquid phase present at all. On the other hand, the vapour fraction $\alpha$ is zero in $x > X(t)$, so there must be a shock at $x = X(t)$. Conservation of mass and momentum across the shock leads to two Rankine–Hugoniot conditions, namely

$$\frac{dX}{dt} = \left[\frac{u(1 - \alpha)}{1 - \alpha}\right]^+ = \left[\frac{\rho_0(1 - \alpha)u^2 + p}{\rho_0(1 - \alpha)u}\right]^+.$$  \hspace{1cm} (19)

Now, in $x > X(t)$, (14) implies that $u$ is a function only of $t$ and, from (19) we deduce that

$$u(t) = \frac{dX}{dt}.$$  \hspace{1cm} (20)
Then (15) gives the pressure gradient as
\[ -\frac{1}{\rho_0} \frac{\partial p}{\partial x} = \frac{d^2 X}{dt^2} + \frac{2C}{R} \frac{dX}{dt} \left( \frac{dX}{dt} \right). \] (21)

The second Rankine-Hugoniot condition (19) implies that \( p \) is continuous across \( x = X(t) \), while \( p \) is equal to atmospheric pressure \( p_a \) at the free end \( x = X + L \):
\[ p = p_v \text{ at } x = X(t), \quad p = p_a \text{ at } x = X(t) + L. \] (22)

We assume here that the length \( L \) of the liquid column remains constant, which is consistent with our neglecting the mass of the cavitated gas. It follows that the pressure in the liquid is given by
\[ p = p_v + \frac{p_a - p_v}{L} (x - X), \] (23)
and (21) gives us a differential equation for \( X(t) \):
\[ \frac{d^2 X}{dt^2} + \frac{2C}{R} \frac{dX}{dt} \left( \frac{dX}{dt} \right) = -\frac{p_a - p_v}{\rho_0 L}. \] (24)

The solution of (24) with initial conditions
\[ X = 0, \quad \frac{dX}{dt} = U \quad \text{at } t = 0, \] (25)
is
\[ X = \frac{R}{2C} \log \left\{ \cos(\lambda t) + \frac{\rho_0 U L \lambda}{p_a - p_v} \sin(\lambda t) \right\}, \quad \text{where } \lambda = \sqrt{\frac{2C(p_a - p_v)}{\rho_0 RL}}. \] (26)

This solution is valid until \( \dot{X} \) reaches zero, which happens when \( t = t^* \), \( X = X^* \), where
\[ t^* = \frac{1}{\lambda} \tan^{-1} \left( \frac{\rho_0 U L \lambda}{p_a - p_v} \right), \quad X^* = \frac{R}{4C} \log \left( 1 + \frac{2\rho_0CLU^2}{R(p_a - p_v)} \right). \] (27)

For \( t > t^* \), \( \dot{X} \) is negative, and the solution of (24) with \( X = X^* \), \( \dot{X} = 0 \) at \( t = t^* \) is
\[ X = X^* - \frac{R}{2C} \log \left[ \cosh \left( \lambda(t - t^*) \right) \right] = \frac{R}{4C} \log \left\{ 1 + \frac{2\rho_0CLU^2}{R(p_a - p_v)} \right\}. \] (28)

Hence \( X \) reaches zero, so the liquid plug hits the valve again, when
\[ \cosh^2 \left( \lambda(t - t^*) \right) = 1 + \frac{2\rho_0CLU^2}{R(p_a - p_v)}. \] (29)

When this happens, the liquid is moving at speed \( u = -U^* \), where
\[ U^* = \frac{U}{\sqrt{1 + \frac{2\rho_0CLU^2}{R(p_a - p_v)}}}. \] (30)
Note that the liquid hits the valve at a speed less than $U$ because of the drag exerted during its journey up the pipe and back.

Now the liquid must bounce off the valve and set off up the pipe again. In the absence of any information about the elastic response of the valve, we make the simplest possible assumption that it is perfectly rigid. Then, to understand how the liquid manages to reverse direction, we have to include compressibility of the liquid, although this is assumed negligible elsewhere in our analysis. We can, however, still use the fact that the incompressibility $\beta$ is large compared to any typical pressure in the problem, so the equations for conservation of mass and momentum may be linearised to

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \quad \rho_0 \frac{\partial u}{\partial t} = -\frac{\beta}{\rho_0} \frac{\partial \rho}{\partial x}. \tag{31}$$

It follows that $u$, $\rho$ and $p$ all satisfy the linear wave equation, with wave-speed $c$ given by

$$c^2 = \frac{\beta}{\rho_0}. \tag{32}$$

Our linearisation of (31) is valid provided the flow is everywhere very subsonic, i.e. $|u| \ll c$.

The $(x, t)$-plane for the problem is sketched in figure 2. For simplicity we redefine the time origin such that, at $t = 0$, the water hits the valve at speed $u = -U^*$. Then its left-hand end $x = 0$ is reduced instantaneously to $u = 0$ and a pressure wave propagates into the water at speed $c$. Behind the wave, the water is stationary and the density is found by solving (31):

$$u = -U^* H(x - ct) \Rightarrow \rho = \rho_0 + \frac{\rho_0 U^*}{c} H(ct - x), \tag{33}$$

where $H$ is the Heaviside function. Hence the density and pressure behind the wave are given by

$$\rho^* = \rho_0 (1 + U^*/c), \quad p = p_a + \Delta p, \quad \Delta p = \frac{\beta U^*}{c}. \tag{34}$$

When the wave reaches the free boundary $x = L$, where $p$ is fixed at atmospheric pressure $p_a$, it bounces back towards $x = 0$. Thus the pressure and density are returned to their atmospheric
values, while $u$ ends up equal to $+U^*$. After a total time of $2L/c$, the whole liquid plug has reversed its direction of flow, and then sets off into $x > 0$. The whole process is then repeated, with the liquid now starting at speed $U^*$ instead of $U$.

We illustrate the typical behaviour of $X(t)$ in figure 3, using the following estimated parameter values:

$$R = 1 \text{ m}, \quad L = 10 \text{ m}, \quad U = 10 \text{ m s}^{-1},$$
$$\rho_0 = 10^3 \text{ kg m}^{-3}, \quad p_a - p_v = 10^5 \text{ Pa}, \quad \beta = 10^9 \text{ Pa}, \quad C = 0.01. \quad (35)$$

The plug sets off from $X = 0$ at speed $U$ and is decelerated both by the pressure difference $p_a - p_v$ applied across it and by the drag. It eventually turns around and returns to hit $x = 0$ at a speed slightly less than $U$. After a very short delay $2L/c = 0.02$ s, while the acoustic waves propagate through the liquid, it sets off again at the reduced speed and the whole process repeats as a series of gradually shortening excursions.

In figure 4 we show the corresponding pressure $p$ measured at a sensor placed a distance $x = 1 \text{ m}$ from the valve. Graph (b) shows how the pressure decreases as the liquid plug flows away from the valve, reaching the vapour pressure (effectively zero) when the cavitation bubble

**Figure 3.** Free position $X$(m) versus time $t$(s) for a liquid plug bouncing up and down a pipe; parameter values given in equation (35).

**Figure 4.** (a) Pressure $p$(Pa) at $x = 1$ m versus time $t$(s). (b) Close-up of low-pressure behaviour. Parameter values given in equation (35).
reaches the sensor. When the liquid returns and passes the sensor once more, the pressure starts to increase again. Then, when the liquid impacts with the valve, a very large pressure pulse $\Delta p$ occurs, as given in (34). The pressure peak is too large to be shown in graph (b) so its full magnitude is indicated in graph (a). Then the liquid once again propagates away from the valve and the process repeats. Notice that, after the first two oscillations, the cavitation bubble never reaches the sensor so the pressure is uniformly positive. This behaviour is qualitatively similar to that reported by Danfoss, although the pressure peaks seem to be more smeared out in practice. This may be due to imperfect rigidity of the valve, attenuation of the signal before it reaches the sensor, or some other damping in the system.

4. Flow through the pulse snubber

In section 3, we found a typical pressure profile to be expected in a pipe following an impulsive shutdown of the flow. In practice, this would be measured by a sensor similar to that shown schematically in figure 5. A reservoir of volume $V$ is connected to the pipe by a nozzle, and the pressure in the reservoir is measured by a piezoelectric membrane. The accuracy of the sensor relies on the pressure in the reservoir being close to that in the pipe. Furthermore, the piezoelectric membrane, although well able to withstand very high pressures, is thought to be susceptible to rupturing if the fluid in the reservoir cavitates. Our task now is, therefore, given a pressure profile in the pipe similar to that shown in figure 4, to predict the corresponding reservoir pressure.

The flow between the pipe and the reservoir is limited by the introduction of a “pulse snubber”. This blocks the nozzle except for a very small hole through which fluid can propagate, whose typical profile is illustrated in figure 6. As a first approximation, we model the flow through this hole using the same equations as those developed for flow in the pipe in section 2. This enables us to estimate roughly how the flux through the hole depends on the material parameters and the geometry, although CFD calculations might be used in future to obtain more detailed and accurate results.

Let $x$ and $t$ denote distance along the hole and time respectively. As before, the gas fraction is denoted by $\alpha$ and the fluid velocity and pressure by $u$ and $p$. Then, with the nozzle radius $R$ a given function of $x$ (similar to that illustrated in figure 6) and the density approximated by $\rho \approx \rho_0(1 - \alpha)$, the equations governing conservation of mass and momentum are

$$-R^2 \frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial x} \left( (1 - \alpha)R^2 u \right) = 0,$$

(36)
Figure 6. Schematic sketch of the cross-section through the hole in a pulse snubber.

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho_0(1-\alpha)} \frac{\partial p}{\partial x} - \frac{2C_R u}{\bar{u}}.
\]  

(37)

The problem is closed by our simple cavitation law

\[ p \geq p_v, \quad \alpha \geq 0, \quad \alpha(p-p_v) = 0. \]  

(38)

We assume that the flow through the hole is quasi-steady, so the time-derivatives in (36) and (37) may be neglected; this assumption will be justified later in section 5. Thus (36) implies that the mass flux \( q \) from the reservoir to the pipe is spatially uniform:

\[ \pi \rho_0 (1-\alpha) R^2 u = q(t). \]  

(39)

The momentum equation (37) takes two different forms depending on whether or not the liquid has cavitated:

\[ \alpha = 0, \quad p > p_v \quad \Rightarrow \quad \frac{\partial p}{\partial x} = \frac{2q}{\pi^2 \rho_0 R^5} \left( q \frac{dR}{dx} - C|q| \right), \]  

(40)

\[ \alpha > 0, \quad p = p_v \quad \Rightarrow \quad \frac{d\alpha}{dx} = \frac{2(1-\alpha)}{qR} \left( q \frac{dR}{dx} - C|q| \right). \]  

(41)

These are to be solved subject to specified reservoir pressure \( p_R(t) \) and pipe pressure \( p_P(t) \). In matching these with the pressure in the hole, we also include the dynamic term in the Bernoulli constant:

\[ p + \frac{1}{2} \rho_0 u^2 = p_R(t) \text{ at } x = 0, \quad p + \frac{1}{2} \rho_0 u^2 = p_P(t) \text{ at } x = L, \]  

(42)

where \( L \) is the length of the hole. Our aim is to determine the flux \( q \) in terms of the pressure difference \( \Delta p = p_R - p_p \).

We have to decide which of (40) and (41) to apply. The possibilities are: either the fluid cavitates nowhere in the pulse-snubber (\( \alpha = 0, \ p > p_v \)), or it cavitates everywhere (\( \alpha > 0, \ p = p_v \)), or cavitation occurs at some value of \( x \), where \( \alpha = 0 \) and \( p = p_v \).

Suppose first that \( q \) is positive, which will occur if \( \Delta p > 0 \). Then, since \( R \) is a decreasing function of \( x \), (40) and (41) imply that \( p \) and \( \alpha \) are both decreasing functions of \( x \). It follows that cavitation cannot occur anywhere within the pulse snubber. If it did, then we would have \( \alpha = 0 \) and \( p = p_v \) at (say) \( x = x^* \) and thus either \( \alpha \) or \( p - p_v \) would have to be negative in \( x > x^* \), neither of which is allowed. Moreover, if \( \alpha \) were positive throughout, both \( p_R \) and \( p_P \)
would have to be smaller than \( p_v \). Hence (40) applies throughout the pulse snubber, and the pressure difference is thus given by

\[
\Delta p = \frac{2Cq|q|}{\pi^2 \rho_0} \int_0^L dx \frac{dR}{R^5}. 
\] (43)

If there is no cavitation in the hole, then (43) still holds if \( q \) is negative. In this case, however, there is no guarantee that cavitation does not occur; the configuration shown schematically in figure 7 (where prime is used as shorthand for \( d/dx \)) is also possible. The pressure drop required to force fluid in through the narrow end reduces \( p \) to the vapour pressure at the point where the nozzle widens. The fluid is then cavitated throughout the wider region. If the length of the narrow section is denoted by \( l \), then the pressure drop in this case is related to the (negative) flux \( q \) by

\[
p_P - p_v = \frac{q^2}{2\pi^2 \rho_0 R(l)^5} \left( 4Cl + R(l) \right). 
\] (44)

This implies that equilibrium between \( p_R \) and \( pp \) may be attained very rapidly when \( q \) is negative, since \( p_P - p_v \) is usually very much larger than \( p_P - p_R \).

In summary, we have obtained a law relating the flux \( q \) through the pulse snubber to the pressure drop applied across it. Depending on the geometry of the hole, we may have to apply the two different relations (43) for \( q > 0 \) and (44) for \( q < 0 \). Henceforth, we assume for simplicity that this is not the case so that (43) may be applied for all values of \( q \).

Finally, we note the analogy between the analysis carried out here and the classical theory of compressible flow through a nozzle [5]. If our cavitation law (38) were replaced by a continuous functional relation between \( p \) and \( \alpha \), then (40) and (41) would become

\[
\left( 1 - \frac{c^2}{u^2} \right) \frac{d\alpha}{dx} = \frac{2(1 - \alpha)}{R} \left( \frac{dR}{dx} - C\text{sgn}(q) \right), 
\] (45)

where \( c \) is the speed of sound:

\[
c^2 = \frac{1}{\rho_0} \frac{dp}{d\alpha} 
\] (46)

The switch between \( p = p_v \) and \( \alpha = 0 \) is replaced in this regularised approach by a switch between supersonic \( (u > c) \) and subsonic \( (u < c) \) flow, which can only happen where \( dR/dx - C\text{sgn}(q) \) changes sign.
5. Model for the reservoir

In the previous section we derived a relation between the pressure difference $\Delta p = p_R - p_P$ and the flux $q$ through the pulse snubber. The pipe pressure $p_P$ is assumed to be a known function of time, but the reservoir pressure $p_R$ must be found by analysing the behaviour of the fluid in the reservoir. We assume that the fluid in the reservoir is well-mixed, with density $\rho(t)$. If the reservoir volume is $V$, then simple mass balance gives

$$\frac{d}{dt}(\rho V) = -q,$$  (47)

where $q$ is given by (43):

$$q = \pi \sqrt{\frac{\rho_0}{2CI}} \text{sgn}(\Delta p) \sqrt{|\Delta p|}.$$  (48)

The problem is closed by applying the equation of state between $p_R$ and $\rho$. If the fluid in the reservoir is uncavitated ($\alpha = 0$), then

$$p_R = p_a + \beta \left( \frac{\rho}{\rho_0} - 1 \right).$$  (49)

Combining (47)–(49), we obtain a single differential equation for $p_R(t)$:

$$\frac{dp_R}{dt} = -\frac{\pi \beta}{V \sqrt{2CI\rho_0}} \text{sgn}(p_R - p_P) \sqrt{|p_R - p_P|}. $$  (50)

Now we compare the left- and right-hand sides of (50) using the following estimates:

$$p_R \approx 10^6 \text{ Pa}, \quad \frac{dp_R}{dt} \approx 10^8 \text{ Pa s}^{-1}, \quad \beta \approx 10^9 \text{ Pa},$$

$$V \approx 1.5 \times 10^{-6} \text{ m}^3, \quad C \approx 0.01, \quad I \approx 10^{15} \text{ m}^{-4}, \quad \rho_0 \approx 10^3 \text{ kg m}^{-3}. $$  (51)

These imply that the right-hand side of (50) is of order $10^9$ Pa s$^{-1}$ and therefore dominates the left-hand side. It follows that $p_R$ and $p_P$ are approximately equal, and the first correction may be obtained by substituting $p_R = p_P$ into the left-hand side of (50):

$$p_R \sim p_P - \frac{2CI\rho_0V^2}{\pi^2\beta^2} \frac{dp_P}{dt} \left| \frac{dp_P}{dt} \right|. $$  (52)

The second term on the right-hand side of (52) indicates how the pressure in the reservoir lags behind that in the pipe. The fact that it is typically small compared with $p_P$ shows that the measured pressure $p_R$ is indeed close to the pipe pressure. However, the estimate (52) fails when the pipe pressure is constant, in particular when the fluid in the pipe has cavitated so that $p_P \equiv p_v$.

To examine the behaviour of $p_R$ when this happens, suppose that the pipe pressure takes the form

$$p_P = \begin{cases} p_v - \phi t, & t < 0 \\ p_v, & t \geq 0 \end{cases}$$  (53)

where $\phi$ represents the rate at which $p_P$ drops to $p_v$. For $t < 0$, the reservoir pressure is given approximately by

$$p_R \sim p_v - \phi t + \frac{2V^2CI\rho_0\phi^2}{\pi^2\beta^2}. $$  (54)
For $t > 0$, we have to solve
\[
\frac{dp_R}{dt} = -\frac{\pi \beta}{V \sqrt{2C I_0}} \sqrt{p_R - p_v},
\]
with
\[
p_R = p_v + \frac{2V^2 CI_0 \phi^2}{\pi^2 \beta^2} \quad \text{at } t = 0.
\]
The solution is
\[
p_R = p_v + \frac{\pi^2}{\beta^2} 8V^2 CI_0 \left( \frac{4V^2 CI_0 \phi}{\pi^2 \beta^2} - t \right)^2,
\]
so the reservoir pressure reaches the cavitation pressure after a finite time
\[
t_v = \frac{4V^2 CI_0 \phi}{\pi^2 \beta^2}.
\]

We can therefore determine whether or not the reservoir pressure has time to reach the cavitation pressure by comparing $t_v$ with the time for which $p_P$ remains at $p_v$. Using the estimates given in (51), we find that
\[
t_v \approx 10^{-6} \text{s},
\]
so that there will certainly be time for the reservoir pressure to reach $p_v$. This seems to imply that the liquid in the reservoir is likely to cavitate. To escape this conclusion, a more complicated cavitation model is required, so that the pressure falling to $p_v$ does not necessarily lead to cavitation. We briefly discuss some possible approaches in section 6.

Before doing so, we use the estimates obtained in this section to justify the quasi-steady approximation adopted in section 4. The flux implied by (48) and (52) is of order
\[
q \sim \rho_0 V \frac{dp_p}{dt}.
\]
The velocity is thus of order
\[
u \sim \frac{q}{\pi R^2 \rho_0} \sim \frac{V}{\pi R^2 \beta} \frac{dp_p}{dt}.
\]
The time derivatives in (36) and (37) may be neglected provided $L \ll ut$, where $t$ is a typical timescale. Thus we obtain the following condition for the quasi-steady approximation to apply:
\[
\frac{\pi R^2 L \beta}{V \Delta p_P} \ll 1,
\]
where $\Delta p_P$ is a typical variation in $p_P$. This has the interpretation that the volume of the nozzle must be much less than the volume of fluid displaced through it. The estimates given in (51) imply that the left-hand side of (62) is small but not tiny; maybe of order 0.2. Hence the quasi-steady approximation is acceptable as a first approximation but should be examined in future refinements of the model.

6. Alternative cavitation models

In this report we have restricted our attention to the simple cavitation model (10). This assumes that the fluid cavitates whenever the pressure drops to the vapour pressure $p_v$. Unfortunately, this assumption leads to the conclusion that the liquid in the reservoir is almost certain to cavitate under realistic pipe pressures, despite the presence of the pulse snubber, which is not observed in practice. Furthermore, it is well known that liquids can exist at pressures well below the vapour pressure; indeed negative pressures of minus several hundred atmospheres
have been measured under laboratory conditions [4]. Recent experiments carried out by Danfoss indicate that the fluid in the reservoir may remain in the liquid state although the pressure drops significantly below the vapour pressure. To explain this, a more complicated cavitation model is required, and we now describe briefly two possible approaches.

Cavitation is typically initiated via nucleation of tiny bubbles. A bubble of radius $a$ containing vapour at pressure $p_v$ will expand if the external pressure is less than $p_v - 2\gamma/a$, where $\gamma$ is the surface tension of the vapour-liquid interface. It thus appears that an infinite negative pressure is required to make a bubble grow from an initially zero radius. The generally-accepted resolution of this paradox is that bubbles are formed at pre-existing nucleation sites with small but nonzero radius $r$. This implies that the liquid pressure must fall to $p_v - 2\gamma/r$ before cavitation occurs.

To describe such behaviour, one might replace the degenerate $p$ versus $\alpha$ behaviour depicted in figure 8(a) with something like that shown in figure 8(b). Here, the pressure must overcome the negative “barrier” $p_v - 2\gamma/r$ before $\alpha$ can increase from zero. Notice, though, that the problem is ill-posed in the region where $dp/d\alpha$ is positive, since the speed of sound is imaginary. Some sort of regularisation (such as surface tension) would be required to avoid a catastrophic instability.

An alternative approach is to consider the kinetics of the transition from the liquid to vapour state. A simple evolution equation for this transition is of the form

$$\tau \frac{d\alpha}{dt} = f(\alpha) \left( \frac{p}{p_v} - 1 \right),$$

where $\tau$ is the associated timescale and $f(\alpha)$ has zeros at the equilibria of the system, namely $\alpha = 0$ and $\alpha = 1$. If $f(\alpha)$ has the form sketched in figure 9, then the system equilibrates to $\alpha = 0$ (uncavitated) when $p > p_v$ or $\alpha = 1$ (cavitated) when $p < p_v$. Such a model therefore allows the possibility that the liquid in the reservoir may resist cavitation even though the reservoir pressure drops to $p_v$.

### 7. Conclusions

In this report we have constructed models for the flow of liquid along a pipe, through the pulse snubber connecting the pipe to a reservoir and in the reservoir itself. We allowed for cavitation by including a vapour volume fraction $\alpha$ that varies between 0 for pure liquid to 1 for pure vapour. Our basic model consists of two partial differential equations, representing conservation of mass and momentum, for $\alpha$, the velocity $u$ and the pressure $p$. To close the system, a cavitation law relating $\alpha$ to $p$ is required, and we proposed the simple complementarity condition (10).

As a first application of our model, we considered the impulsively stopped flow of liquid along a pipe. The predicted damped oscillatory flow and corresponding pressure profile depicted
in figures 3 and 4 were found to show encouraging qualitative agreement with experimental measurements. We then used the same model to derive a relation between the pressure difference across the pulse snubber and the flux through it. We thus obtained the differential equation (50) for the reservoir pressure $p_R(t)$. This enabled us to predict the discrepancy between the reservoir and pipe pressures and to estimate whether or not the reservoir pressure is likely to fall to the vapour pressure.

There are many further avenues to be explored, including the following.

- The pressure peaks shown in figure 4 are significantly sharper than those reported in practice. Better predictions might be obtained by allowing for damped elastic response of the valve as the liquid plug hits it.
- We showed that it is impossible for the liquid to cavitate in the pulse snubby when the flow is from the reservoir to the pipe but, depending on the geometry, cavitation may occur when liquid flows from the pipe into the reservoir. This switch in behaviour warrants further study and may help to explain why the pulse snubby protects the reservoir from low pressures but not from high pressures.
- Our quasi-one-dimensional model for the flow through the pulse snubby cannot give more than order-of-magnitude estimates, since the geometry is far from slowly-varying. CFD calculations would enable more accurate predictions to be made, including time-dependent effects that we neglected.
- We proposed two possible improved cavitation laws in section 6, and further consideration of the underlying physics may yield more improvements. The effect of these on our predictions should be explored.
- In practice, the reservoir may be filled with air when it is first fitted. Our modelling approach should allow the time taken to expel all the air to be estimated.

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