A Coherent-State Approach to Two-dimensional Electron Magnetism

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Abstract

We study in this paper the possible occurrence of orbital magnetism for two-dimensional electrons confined by a harmonic potential in various regimes of temperature and magnetic field. Standard coherent state families are used for calculating symbols of various involved observables like thermodynamical potential, magnetic moment, or spatial distribution of current. Their expressions are given in a closed form and the resulting Berezin-Lieb inequalities provide a straightforward way to study magnetism in various limit regimes. In particular, we predict a paramagnetic behaviour in the thermodynamical limit as well as in the quasiclassical limit under a weak field. Eventually, we obtain an exact expression for the magnetic moment which yields a full description of the phase diagram of the magnetization.

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I Introduction

In a recent paper, Ishikawa and Fukuyama [1] describe the possible orbital magnetism for two-dimensional electrons confined by a harmonic potential in various regimes of temperature and magnetic field. They afford a quite large complement of information in regard to the previous paper [2] devoted to the same subject. According to the range of values assumed by the relative ratios between the three characteristic energy scales present in the model, namely the thermodynamical unit $k_B T$, the magnetic quantum
\(h\omega_c\), and the harmonic quantum \(h\omega_0\), they explain the existence of the different magnetic regimes. As a matter of fact, they distinguish between the “Mesoscopic Fluctuation” regime \((k_B T \lesssim h(\sqrt{\omega_c^2 + 4\omega_0^2} - \omega_c)/2)\), the “Landau Diamagnetism” regime \((k_B T \gtrsim h(\sqrt{\omega_c^2 + 4\omega_0^2} + \omega_c)/2)\), and the “de Haas-van Alphen” regime. Their studies rest upon the derivation of an approximate closed formula for the thermodynamical potential \(\Omega\) and from which they are able to get the magnetic moment. The isotropic two-dimensional harmonic potential offers us a complete analytical treatment of the problem whereas the orbital magnetism is expected not to depend strongly on the shape of the confinement. The crucial point in our derivation is that the Fermi-Dirac function is a fixed point of the Fourier transform. Exact series expansions ensue by simple application of the residue theorem as shown in the appendix. Our results for the thermodynamical potential, the orbital magnetic moment, and the spatial distribution of currents are easily tractable and analyzable in comparison with those given in Ref.[1].

One can show that the Hamiltonian under a perpendicular magnetic field can be decomposed into the sum of two harmonic oscillators with frequencies \(\omega_+\) and \(\omega_-\) respectively. This fact makes the present model completely integrable and also makes the use of the coherent states (CS)\(\text{[3]}\) possible. These coherent states are defined as tensor products of two standard one-dimensional coherent states and read as: \(\{ | z_d, z_g \rangle = | z_d \rangle \otimes | z_g \rangle, z_d, z_g \in \mathbb{C}\}\). This family of states provides us with a resolution of the unity and in this way we are allowed to use the so-called symbol calculus \(\text{à la Berezin-Lieb-Perelomov}\text{[3]}\). Consequently, the Berezin-Lieb inequalities give us the opportunity to explore the behavior of the magnetization of our model in extreme conditions like the thermodynamical limit or the quasi-classical limit.

In the next section, we shall give a short review about the physical model. The eigenvalues and eigenvectors of the Hamiltonian will be obtained by algebraic and analytical methods respectively. In particular, we shall explain the symmetries of the problem and their relevance in various approximations of the model. In Section III, the construction of the coherent states is presented and the main CS algebraic and functional properties are briefly recalled there. Section IV is devoted to the thermodynamical potential \(\Omega\). First we apply the Berezin-Lieb inequalities and study the magnetic moment in the different limit regimes. We then give the exact expression of \(\Omega\). The core of the paper lies in Section V, in which we establish exact expressions for the magnetic moment and the average number of electrons and discuss their behavior in different temperature and magnetic field regions. We show in Section VI how coherent symbol calculus yields a Fourier integral expression for the radial distribution of the current. This integral can be also given as an expansion series with the aid of the residue theorem. In the conclusion we shall give some remarks and comments on the possible extension of our approach to the other systems of physical interest.

\(\text{a)}\) See \(\text{[4]}\) and Lieb in \(\text{[5]}\). For a recent up-to-dated review on CS, specially on their group representation aspects we also refer to \(\text{[6]}\). Notice that here the operator calculus in Hilbert space is replaced by the functional calculus involving CS parameters.
II Hamiltonian and Symmetries

The Hamiltonian for two-dimensional spinless electrons confined by an isotropic harmonic potential and submitted to a constant perpendicular magnetic field is written as

$$\mathcal{H} = \frac{1}{2m}(\mathbf{P}^2 + \frac{e}{c}\mathbf{A})^2 + \frac{1}{2}m\omega_0^2\mathbf{R}^2,$$

(2.1)

where Coulomb interactions are neglected. This model is called the Fock-Darwin Hamiltonian in the literature [7]. The radius of the system, \(R_m\), is classically defined as

$$\frac{1}{2}m\omega_0^2R_m^2 = \mu,$$

(2.2)

where \(\mu\) is the chemical potential, and we shall give in Section IV a full quantum statistical mechanics interpretation of this relation. We shall work with the symmetric gauge \(\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{R}\). We first solve the eigenvalue and eigenvector problems by using the underlying Weyl-Heisenberg symmetries, and we shall next consider eigenvectors in another system of coordinates by following a more classical analytical method.

A. Solutions through Weyl-Heisenberg symmetries

The algebraic structure is easily displayed if we adopt the method of separation of Cartesian variables. It leads to the following form of the Hamiltonian:

$$\mathcal{H} = \left(\frac{P_x^2}{2m} + \frac{1}{8}m\omega^2X^2\right) + \left(\frac{P_y^2}{2m} + \frac{1}{8}m\omega^2Y^2\right) + \frac{\omega}{2}L_z \equiv \mathcal{H}_0 + \frac{\omega}{2}L_z,$$

(2.3)

where \(\omega_c = eB/mc\) is the cyclotron frequency, \(\omega = \sqrt{\omega_c^2 + 4\omega_0^2}\), and \(L_z = XP_y - YP_x\). Here, we clearly see the splitting of the Hamiltonian into two independent harmonic oscillator Hamiltonians plus the angular momentum operator. Instead of directly using the oscillator annihilation operators:

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{X}{l_0} + \frac{i\hbar}{\hbar}P_x\right), \quad a_y = \frac{1}{\sqrt{2}}\left(\frac{Y}{l_0} + \frac{i\hbar}{\hbar}P_y\right),$$

(2.4)

we work with two new ones, which are linear superposition of \(a_x\) and \(a_y\):

$$a_d = \frac{1}{\sqrt{2}}(a_x - ia_y), \quad a_g = \frac{1}{\sqrt{2}}(a_x + ia_y),$$

(2.5)

where \(l_0 = \sqrt{2\hbar/m\omega}\). Note that \(a_d\) and \(a_g\) are bosonic operators: \([a_d, a_d^\dagger] = I = [a_g, a_g^\dagger]\), and one has the useful identities:

$$X = \frac{l_0}{2}(a_d + a_d^\dagger + a_g + a_g^\dagger), \quad Y = \frac{l_0}{2i}(a_d - a_d^\dagger + a_g - a_g^\dagger),$$

$$P_x = \frac{\hbar}{2il_0}(a_d - a_d^\dagger + a_g - a_g^\dagger), \quad P_y = \frac{\hbar}{2il_0}(a_d + a_d^\dagger - a_g - a_g^\dagger).$$

(2.6)
The operators $\mathcal{H}_0$ and $L_z$ then can be simply expressed in terms of the number operators $N_d = a_d^\dagger a_d$ and $N_g = a_g^\dagger a_g$ as:

$$\mathcal{H}_0 = \frac{\hbar \omega}{2} (N_d + N_g + 1), \quad L_z = \hbar (N_d - N_g), \quad (2.7)$$

and so

$$\mathcal{H} = \frac{\hbar \omega}{2} (N_d + N_g + 1) + \frac{\hbar \omega_c}{2} (N_d - N_g) = \hbar (\omega_+ N_d + \omega_- N_g + \frac{\omega}{2}), \quad (2.8)$$

where $\omega_\pm = (\omega \pm \omega_c)/2$. Eigenvalues are trivially found from the expression:

$$\mathcal{H} | n_d, n_g \rangle = E_{n_d n_g} | n_d, n_g \rangle, \quad (2.9)$$

with $E_{n_d n_g} = \frac{\hbar}{2} (\omega_+ n_d + \omega_- n_g + \frac{\omega}{2})$ and where $n_d$ and $n_g$ are non-negative integers. The corresponding eigenvectors are tensor products of single Fock oscillator states:

$$| n_d, n_g \rangle = | n_d \rangle \otimes | n_g \rangle = \frac{1}{\sqrt{n_d! n_g!}} (a_d^\dagger)^{n_d} (a_g^\dagger)^{n_g} | 0, 0 \rangle. \quad (2.10)$$

### B. su(2) and su(1, 1) symmetries

Two distinct dynamical symmetries exist on the level of quadratic observables.

1. **su(2) symmetry**
   The first one is of the type su(2) and is put into evidence by introducing the operators
   $$S_+ = a_g^\dagger a_d, \quad S_- = a_d^\dagger a_g, \quad S_z = \frac{N_d - N_g}{2} = \frac{L_z}{2\hbar}. \quad (2.11)$$

   The commutation relations read as:
   $$[S_+, S_-] = 2S_z, \quad [S_z, S_\pm] = \pm S_\pm, \quad (2.12)$$

   and the invariant Casimir operator is given by
   $$C = \frac{1}{2} (S_+ S_- + S_- S_+) + S_z^2 = \left(\frac{N_d + N_g}{2}\right) \left(\frac{N_d + N_g}{2} + 1\right). \quad (2.13)$$

   Therefore, for a fixed value $\lambda = (n_d + n_g)/2$ of the operator $(N_d + N_g)/2 = \mathcal{H}_0/(\hbar \omega) - 1/2$, there exists a $(2\lambda + 1)$-dimensional UIR of su(2) in which the operator $S_z$ has its spectral values in the range $-\lambda \leq \varsigma = (n_d - n_g)/2 \leq \lambda$.

2. **su(1, 1) symmetry**
   The second one is of the type su(1, 1):
   $$T_+ = a_d^\dagger a_g^\dagger, \quad T_- = a_g a_d, \quad T_0 = \frac{1}{2} (N_d + N_g + 1) = \frac{\mathcal{H}_0}{\hbar \omega}, \quad (2.14)$$

   with
   $$[T_+, T_-] = -2T_0, \quad [T_0, T_\pm] = \pm T_\pm. \quad (2.15)$$
The Casimir operator then reads as:
\[ D = \frac{1}{2}(T_+ T_- + T_- T_+) - T_0^2 = -\frac{(N_d - N_g + 1)}{2} \left( \frac{N_d - N_g - 1}{2} \right) = -\frac{1}{4} \left( \frac{L_z^2}{\hbar} - 1 \right). \]

When \( n_d \geq n_g \), for a fixed value \( \eta = (n_d - n_g + 1)/2 \geq 1/2 \) of the operator \( (N_d - N_g + 1)/2 \), there exists a UIR of \( \mathfrak{su}(1, 1) \) in the discrete series, in which the operator \( T_0 \) has its spectral values in the infinite range \( \eta, \eta + 1, \eta + 2, \cdots \). Alternatively, when \( n_d \leq n_g \), for a fixed value \( \vartheta = (-n_d + n_g + 1)/2 \geq 1/2 \) of the operator \( (-N_d + N_g + 1)/2 \), there also exists a UIR of \( \mathfrak{su}(1, 1) \) in which the spectral value of the operator \( T_0 \) runs in the infinite range \( \vartheta, \vartheta + 1, \vartheta + 2, \cdots \).

### C. Solutions through analytical derivation

The analytical solutions are obtained by separation of polar coordinates in the stationary Schrödinger equation
\[ \mathcal{H}\Psi(r, \theta) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial_r^2}{r^2} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right) - \frac{i}{2} \omega_c \partial_\theta + \frac{m}{8} \omega^2 r^2 \right] \Psi(r, \theta) = E \Psi(r, \theta). \]

We consider \( \Psi(r, \theta) \) as an eigenfunction diagonal with respect to the conserved angular momentum and put \( \Psi(r, \theta) = R(r) e^{i\alpha \theta} \). The function \( R(r) \) is determined in terms of Laguerre polynomials. Explicitly we have
\[ \Psi(r, \theta) = \Psi_{n, \alpha}(r, \theta) = (-1)^n \times \frac{1}{\sqrt{\pi L_0}} \sqrt{\frac{n!}{(n + |\alpha|)!}} \exp \left[ -\frac{r^2}{2L_0} \right] \frac{L_n^{(|\alpha|)}(r)}{L_n^{(|\alpha|)}(L_0)} r^{|\alpha|} e^{i\alpha \theta}, \]

where \( n = 0, 1, 2, \cdots \) is the principal quantum number and \( \alpha = 0, \pm 1, \pm 2, \cdots \) the angular moment quantum number. The eigenenergies are obtained to be:
\[ E_{n\alpha} = \hbar \omega (n + |\alpha| + 1/2) + \frac{\hbar \omega_c}{2} \alpha, \]

Therefore, \( n \) and \( \alpha \) are related to \( n_d \) and \( n_g \) by the following relations:
\[ n_d = n + \frac{1}{2} (|\alpha| + \alpha), \quad n_g = n + \frac{1}{2} (|\alpha| - \alpha). \]

We shall denote the eigenfunction indifferently by \( \Psi_{n, \alpha}(r, \theta) = \langle r, \theta \mid n, \alpha \rangle = \langle r, \theta \mid n_d, n_g \rangle \).

### D. Filling the shells with fermions

Each of the above two symmetries affords a way of ordering the pairs \( (n_d, n_g) \). However, they do not provide any hint for ordering the energy eigenvalues \( E_{n_d n_g} \) with the exception of two limiting cases: weak field one and strong field one.

#### 1. Weak field case

Suppose the average number of electrons \( \langle N_e \rangle \) obeys \( \omega_c \langle N_e \rangle \ll \omega_0 \). In terms of the \( \mathfrak{su}(2) \) symmetry, the energy eigenvalues read as:
\[ E_{n_d n_g} = \hbar \omega \lambda + \hbar \omega_c \varsigma + \frac{1}{2} \hbar \omega, \]
and can be approximated by
\[ E_{n,d,n_g} \approx \hbar \omega_0 (2\lambda + 1) \equiv E_\lambda. \] (2.21)

Therefore, in the weak field limit, the \( \text{su}(2) \) symmetry becomes a true symmetry of the Hamiltonian, which explains the degeneracy of order \( 2\lambda + 1 \) for the level \( E_\lambda \).

Note that there are \( (\lambda_0 + 1)(2\lambda_0 + 1) \) (spinless) electrons which fill the shells up to \( \lambda_0 \).

2. Strong field case

In the limit of strong magnetic field \( \omega_c \gg \omega_0 \), we have \( E_{n,d,n_g} \approx \hbar \omega_c (n_d + \frac{1}{2}) \). Therefore, for a given value of \( n_d \), we have an infinite degeneracy labelled by \( n_g \) or by \( \alpha = n_d - n_g \leq n_d \). The quantum number \( n_d \) corresponds to the Landau level index (as well as \( n \) for negative \( \alpha \)). One can reinterpret it in terms of \( \text{su}(1,1) \) symmetry by noting that, for a given value of \( \alpha \leq 0 \), the energy eigenstates are ladder states for the discrete series representation labelled by \( \vartheta = -\alpha/2 + 1/2 \).

3. Generic intermediate case

In the uncommensurate intermediate case, which means \( \omega_+ / \omega_- \notin \mathbb{Q} \) and no approximation is relevant, we are faced to the problem of ordering the relatively dense (but not uniformly discrete) set of eigenenergies:
\[ E_{n_d,n_g} \equiv E_{n_d,n_g} - \frac{\omega_+}{2\omega_-} = \frac{\omega_+}{\omega_-} n_d + n_g. \] (2.22)

In the commensurate case, \( \omega_+ / \omega_- = p/q \in \mathbb{Q} \), degeneracy is possible:
\[ E_{n_d,n_g} = E_{n'_d,n'_g} \iff \frac{p}{q} = -\frac{n_g - n'_g}{n_d - n'_d}. \] (2.23)

III Standard Coherent States

A. Definitions and Properties

The fact that the eigenvectors issued from the algebraic method are just tensor products of Fock harmonic oscillator eigenstates allows one to construct the corresponding coherent states in a standard way:
\[ |z_d, z_g\rangle \equiv |z_d\rangle \otimes |z_g\rangle = \exp \left[ -\frac{1}{2}(|z_d|^2 + |z_g|^2) \right] \sum_{n_d,n_g} \frac{z_d^{n_d} z_g^{n_g}}{\sqrt{n_d! \sqrt{n_g!}}} |n_d, n_g\rangle, \]
\[ = \exp \left[ -\frac{1}{2}(|z_d|^2 + |z_g|^2) \right] e^{z_d a_d^\dagger + z_g a_g^\dagger} |0, 0\rangle. \] (3.1)

These (normalized) coherent states obey some of the usual CS properties:

P1 (Eigenvector property)
\[ a_d |z_d, z_g\rangle = z_d |z_d, z_g\rangle, \quad a_g |z_d, z_g\rangle = z_g |z_d, z_g\rangle. \]
P₂ (Minimal uncertainty relations)
\[
\Delta X \Delta P_x = \frac{\hbar}{2}, \quad \Delta Y \Delta P_y = \frac{\hbar}{2},
\]
where \(\Delta X \equiv [\langle z_d, z_g \mid X^2 \mid z_d, z_g \rangle - (\langle z_d, z_g \mid X \mid z_d, z_g \rangle)^2]^{1/2}\), etc.

P₃ (Temporal stability)
\[
e^{-i\hat{H}t/\hbar} | z_d, z_g \rangle = e^{-i\omega t/2} | z_d, e^{-i(\omega + \omega_c)t/2} z_d, e^{-i(\omega - \omega_c)t/2} z_g \rangle.
\]

P₄ (Action identity)
\[
\hat{\mathcal{H}}(z_d, z_g) \equiv \langle z_d, z_g \mid \hat{\mathcal{H}} \mid z_d, z_g \rangle = \hbar[\omega_+ |z_d|^2 + \omega_- |z_g|^2 + \frac{\omega}{2}], \quad (3.2)
\]
The function \(\hat{\mathcal{H}}(z_d, z_g)\) has been called lower (resp. contravariant) symbol of the operator \(\hat{\mathcal{H}}\) by Lieb [5] (resp. by Berezin [4]). It will plays an important role in the present context.

P₅ (Overlapping)
\[
\langle z_d', z_g' \mid z_d, z_g \rangle = e^{i\overline{\mathcal{A}}(z_d, z_g)} e^{-i(z_d - z_d')/2 + i(z_g - z_g')^2}/2.
\]

P₆ (Resolution of the unity)
\[
I = \frac{1}{\pi^2} \int_{\mathbb{C}^2} d^2z_d d^2z_g | z_d, z_g \rangle \langle z_d, z_g \mid d^2z_d d^2z_g.
\]
The last property is also crucial in our context. For any observable \(A\) with suitable operator properties (traceclass, ...) there exists a unique upper (or covariant) symbol \(\hat{A}(z_d, z_g)\) defined by
\[
A = \frac{1}{\pi^2} \int_{\mathbb{C}^2} \hat{A}(z_d, z_g) | z_d, z_g \rangle \langle z_d, z_g \mid d^2z_d d^2z_g, \quad (3.3)
\]
For instance, upper symbols for number operators are given by
\[
\hat{N}_d(z_d, z_g) = |z_d|^2 - 1, \quad \hat{N}_g(z_d, z_g) = |z_g|^2 - 1, \quad (3.4)
\]
and hence the upper symbol for our Hamiltonian (2.8)
\[
\hat{\mathcal{H}}(z_d, z_g) = \hbar[\omega_+ |z_d|^2 + \omega_- |z_g|^2 - \frac{\omega}{2}], \quad (3.5)
\]
Finally, we should mention the useful trace identity for a traceclass observable \(A\):
\[
\text{Tr} A = \frac{1}{\pi^2} \int_{\mathbb{C}^2} \hat{A}(z_d, z_g) d^2z_d d^2z_g = \frac{1}{\pi^2} \int_{\mathbb{C}^2} \hat{A}(z_d, z_g) d^2z_d d^2z_g, \quad (3.6)
\]
where \(\hat{A}(z_d, z_g) \equiv \langle z_d, z_g \mid A \mid z_d, z_g \rangle\).
B. Fock-Bargman space

The coherent state function \(\langle r, \theta \mid z_d, z_g \rangle\) is, up to an exponential factor, the integral kernel for the isometry mapping the Hilbertian span \(L^2(\mathbb{R}^2)\) of the set of eigenfunctions \(2.18\) onto the so-called Fock-Bargman space, \(\text{i.e.}\) the Hilbert space

\[
\mathcal{FB} \equiv L^2_{\text{entire}}(\mathbb{C}^2, \frac{1}{\pi^2} e^{-|z_d|^2 + |z_g|^2} d^2 z_d d^2 z_g)
\]  

(3.7)

of entire two complex variable functions \(f(z_d, z_g)\) that are square integrable with respect to the measure \(\pi^{-2} e^{-|z_d|^2 + |z_g|^2} d^2 z_d d^2 z_g\). This unitary mapping \(L^2(\mathbb{R}^2) \ni \Psi(r, \theta) \rightarrow \Phi(z_d, z_g) \in \mathcal{FB}\) and its reciprocal are explicitly given by

\[
\Psi(r, \theta) = \frac{1}{\pi^2} \int_{\mathbb{C}^2} \mathcal{K}(r, \theta, \bar{z}_d, \bar{z}_g) \Phi(z_d, z_g) e^{-(|z_d|^2 + |z_g|^2)} d^2 z_d d^2 z_g
\]  

(3.8)

\[
\Phi(z_d, z_g) = \int_{\mathbb{R}^2} \mathcal{K}(r, -\theta, z_d, z_g) \Psi(r, \theta) r dr d\theta.
\]  

(3.9)

The kernel is given by the following generating function:

\[
\mathcal{K}(r, \theta, z_d, z_g) = e^{(|z_d|^2 + |z_g|^2)/2} \langle r, \theta \mid z_d, z_g \rangle = \sum_{n_d, n_g} \frac{z_d^{n_d}}{\sqrt{n_d!}} \frac{z_g^{n_g}}{\sqrt{n_g!}} \Psi_{n, \alpha}(r, \theta)
\]

\[
= \frac{1}{\sqrt{\pi l_0}} e^{r^2/(2l_0^2)} e^{-(z_d - e^{-i\theta} r/l_0)(z_g - e^{i\theta} r/l_0)}.
\]  

(3.10)

Note that normalized eigenkets \(2.18\) are mapped to the corresponding normalized eigenstates

\[
\frac{z_d^{n_d}}{\sqrt{n_d!}} \frac{z_g^{n_g}}{\sqrt{n_g!}} = e^{(|z_d|^2 + |z_g|^2)/2} \langle \bar{z}_d, \bar{z}_g \mid n_d, n_g \rangle
\]  

(3.11)

in the \(\mathcal{FB}\) representation. Also note that in this representation the operators \(a_d\) and \(a_g^\dagger\) have the simple form:

\[
a_d = \frac{\partial}{\partial z_d}, \quad a_g^\dagger = \text{(multiplication by)} \ z_d, \quad a_g = \frac{\partial}{\partial z_g}, \quad a_g^\dagger = z_g.
\]  

(3.12)

IV Thermodynamical potential

Let us now enter the core of the physical question we addressed in the introduction. We assume that the total number \(\langle N_e \rangle\) of electrons is large enough for making no appreciable difference between a grand canonical ensemble and a canonical one. Then the magnetic moment \(M\) is given by

\[
M = -\left(\frac{\partial \Omega}{\partial H}\right)_\mu,
\]  

(4.1)

where \(\Omega\) is the thermodynamical potential,

\[
\Omega = -\frac{1}{\beta} \text{Tr} \log (1 + e^{-\beta(H - \mu)}),
\]  

(4.2)
with $\beta = 1/(k_B T)$. The average number of electrons is given by:

$$
\langle N_e \rangle = \sum_{n_d=0}^{\infty} \sum_{n_g=0}^{\infty} f(E_{n_d n_g}) = \text{Tr} f(\mathcal{H}) = -\partial_n \Omega, \tag{4.3}
$$

where $f(E) = 1/(1 + e^{\beta(E - \mu)})$ is the Fermi distribution function. The magnetic moment can be yielded from eqs. (4.1) and (4.2) and reads as:

$$
\frac{M}{\mu_B} = -\frac{2}{\omega} \text{Tr} \frac{(N_d + 1/2)\omega_+ - (N_g + 1/2)\omega_-}{(1 + e^{\beta(H - \mu)})}, \tag{4.4}
$$

where $\kappa_{\pm} = \exp (\beta(\mu \pm \hbar \omega/2)) = \kappa_{\pm}(H, T)$, and $\mu_B = \hbar e/(2mc)$ is the Bohr magneton. Despite its concise appearance, the computation of the double series (4.4) is not easily tractable on a numerical level. Hence we will give the preliminary estimates before presenting much more exploitable exact expressions.

### A. Berezin-Lieb inequalities for the thermodynamical potential

First we note that $\log (1 + e^{-\beta(H - \mu)})$ is a convex function of the positive Hamiltonian $\mathcal{H}$, and so we can apply the Berezin-Lieb inequalities to examine the quasi-classical behaviour of the thermodynamical potential. These inequalities say that, for any convex function $g(\hat{A})$ of the observable $\hat{A}$, we have

$$
\frac{1}{\pi^2} \int_{\mathbb{C}^2} g(\hat{A}) \, d^2 z_d \, d^2 z_g \leq \text{Tr} g(\hat{A}) \leq \frac{1}{\pi^2} \int_{\mathbb{C}^2} g(\hat{A}) \, d^2 z_d \, d^2 z_g. \tag{4.5}
$$

Applying them to the (concave) thermodynamical potential leads to the inequalities:

$$
-\frac{1}{\beta \pi^2} \int_{\mathbb{C}^2} \log (1 + e^{-\beta(\mathcal{H} - \mu)}) \, d^2 z_d \, d^2 z_g \leq \Omega \leq -\frac{1}{\beta \pi^2} \int_{\mathbb{C}^2} \log (1 + e^{-\beta(\mathcal{H} - \mu)}) \, d^2 z_d \, d^2 z_g. \tag{4.6}
$$

Let us insert eqs. (3.2) and (3.3) and perform the angular integrations. It leads to the inequalities:

$$
-\frac{1}{\beta} \int_{0}^{\infty} du_d \int_{0}^{\infty} du_g \log (1 + e^{-\beta(h(\omega_+ u_d + \omega_- u_g - 2) - \mu)}) \leq \Omega,
$$

$$
\Omega \leq -\frac{1}{\beta} \int_{0}^{\infty} du_d \int_{0}^{\infty} du_g \log (1 + e^{-\beta(h(\omega_+ u_d + \omega_- u_g + 2) - \mu)}), \tag{4.7}
$$

where $u_d = |z_d|^2$ and $u_g = |z_g|^2$. Changing the integration variables, say $u = \beta h(\omega_+ u_d + \omega_- u_g)$, $v = h(\omega_+ u_d + \omega_- u_g)$, performing an integration by part, and introducing the control parameters $\kappa_{\pm}$ defined in (4.4), we easily reduce the above inequalities to the following ones:

$$
\phi(\kappa_+) \leq \Omega \leq \phi(\kappa_-). \tag{4.8}
$$
where the function φ is given by:

\[
\phi(\kappa) = -\frac{\kappa}{2\beta(\beta\hbar\omega_0)^2} \int_0^\infty \frac{u^2 e^{-u}}{1 + \kappa e^{-u}} \, du
\]

\[
= \begin{cases} 
\frac{1}{\beta(\beta\hbar\omega_0)^2} F_3(-\kappa) & \text{for } \kappa \leq 1, \\
\frac{1}{\beta(\beta\hbar\omega_0)^2} \left[ -\frac{\pi^2 \log \kappa}{6} + F_3(-\kappa^{-1}) \right] & \text{for } \kappa > 1.
\end{cases}
\]

(4.9)

We have introduced here the function \( F_s \), of the Riemann-Fermi-Dirac type, defined as:

\[
F_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s}.
\]

(4.10)

In the high temperature region \(|\mu \pm \hbar\omega/2| \ll k_B T\) we have \( \kappa_\pm \approx 1 \). From (4.8) we see that the thermodynamical potential is approximately equal to:

\[
\Omega \approx k_B T \left( \frac{k_B T}{\hbar\omega_0} \right)^2 F_3(-1) \approx -0.901543 k_B T \left( \frac{k_B T}{\hbar\omega_0} \right)^2
\]

We now consider the more realistic case: \( \mu \gg \hbar\omega/2 \) and \( \mu \gg k_B T \). Let us split the function \( \phi \) into three parts:

\[
\phi(\kappa_\pm) = A \mp \Delta + S_\pm,
\]

(4.11)

with

\[
A = -\frac{\mu}{2} \left[ \frac{1}{3} \left( \frac{\mu}{\hbar\omega_0} \right)^2 + \frac{1}{4} \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar\omega_0} \right)^2 \right],
\]

\[
\frac{\Delta}{2} = \frac{\hbar\omega}{2} \left[ \frac{1}{2} \left( \frac{\mu}{\hbar\omega_0} \right)^2 + \frac{1}{24} \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{6} \left( \frac{k_B T}{\hbar\omega_0} \right)^2 \right],
\]

\[
S_\pm = k_B T \left( \frac{k_B T}{\hbar\omega_0} \right)^2 F_3(-\exp[-\beta(\mu \pm \hbar\omega/2)]).
\]

(4.12)

Then we see that \( \Omega \) lies in the interval \([A + S_+ - \Delta/2, A + S_- + \Delta/2]\). Replacing \( S_\pm \) by the approximate expression

\[
S_0 = k_B T \left( \frac{k_B T}{\hbar\omega_0} \right)^2 F_3(-e^{-\beta\mu}),
\]

(4.13)

and observing that the ratio

\[
\frac{\Delta}{|A + S_0|} = \frac{\hbar\omega}{\mu} \left[ \frac{3 + \pi^2 \left( \frac{k_B T}{\mu} \right)^2 + \frac{\hbar\omega}{\mu}^2}{1 + \frac{\pi^2 \left( \frac{k_B T}{\mu} \right)^2 + \frac{3}{4} \left( \frac{\hbar\omega}{\mu} \right)^2 - \left( \frac{k_B T}{\mu} \right)^2 F_3(-e^{-\beta\mu})}{F_3(-e^{-\beta\mu})} \right]
\]

(4.14)

tends to zero, we see that the thermodynamical potential can be estimated as:

\[
\Omega \approx A + S_0
\]

\[
= -\frac{\mu}{2} \left[ \frac{1}{3} \left( \frac{\mu}{\hbar\omega_0} \right)^2 + \frac{1}{4} \left( \frac{\omega}{\omega_0} \right)^2 + \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar\omega_0} \right)^2 \right] + k_B T \left( \frac{k_B T}{\hbar\omega_0} \right)^2 F_3(-e^{-\beta\mu})(4.15)
\]
A similar asymptotic behaviour holds for the thermodynamical limit \( \langle N_e \rangle \to \infty \). Indeed, in this quasiclassical regime, the average number of electrons is calculated to be:

\[
\langle N_e \rangle \approx -\partial_\mu (A + S_0) \\
= \left( \frac{\mu}{\hbar \omega_0} \right)^2 \left[ \frac{1}{2} + \frac{1}{8} \left( \frac{\hbar \omega}{\mu} \right)^2 + \frac{\pi^2}{6} \left( \frac{k_B T}{\mu} \right)^2 \left( \frac{\omega_0}{\hbar} \frac{\hbar \omega}{\mu} \right)^2 \cdot \frac{F_2(-e^{-\mu})}{\hbar \omega} \right] \\
\approx \frac{1}{2} \left( \frac{\mu}{\hbar \omega_0} \right)^2 \text{ for } \mu \gg k_B T \text{ and } \mu \gg \hbar / 2.
\]

(4.16)

The magnetic moment hence reads as:

\[
M = \chi_p H, \quad \text{with } \chi_p = \mu \left( \frac{\mu}{\hbar \omega_0} \right)^2.
\]

(4.17)

Hence, we can assert that the system shows an orbital paramagnetism in this limit. At this point, we refer to a recent work by Combescure and Robert [8] in which precise informations are given for the magnetisation of electron gas constrained by general confinement potentials.

B. Exact expressions for the thermodynamical potential

We now determine the thermodynamical potential in a precise way by applying the formulas (7.5) and (7.7) in the appendix. The function \( \Theta(k) \) defined by (7.6) takes the following closed form:

\[
\Theta(k) = \text{Tr}(e^{-\frac{ik}{2} H}) = e^{-\frac{ik+1}{2} \hbar \omega} \frac{1}{1 - e^{-(ik+1)2\hbar \omega+}} - \frac{1}{1 - e^{-(ik+1)2\hbar \omega-}}.
\]

(4.18)

The Fourier integral representation for the thermodynamical potential hence reads as:

\[
\Omega = -\frac{1}{\beta} \int_{-\infty}^{+\infty} e^{-\frac{ik+1}{2} \hbar \omega} \frac{1}{2 \cosh \frac{\pi}{2} k} \left( \frac{1}{ik+1} \right) \left( \frac{1}{1 - e^{-(ik+1)2\hbar \omega+}} \right) \left( \frac{1}{1 - e^{-(ik+1)2\hbar \omega-}} \right) dk
\]

(4.19)

As indicated in the formula (7.9), this Fourier integral is given as a series by using the residue theorem. One can easily see that the numbers \((2m+1)i, m \in \mathbb{Z}\) are simple poles of \( \text{sech} \frac{\pi}{2} k \), \( i \) is a double pole of \( \Theta(k) \), and \( i + 4\pi m / (\hbar \omega+) \), \( i + 4\pi m / (\hbar \omega-) \), \( m \in \mathbb{Z}^* \) are simple or double poles of \( \Theta(k) \) according to whether \( \omega_+ \) and \( \omega_- \) are uncommensurable or not (see Fig.1). In order to fulfill the requirements of the Jordan Lemma, one has to consider the following two cases: \( \mu \leq \hbar / 2 \) and \( \mu \geq \hbar / 2 \). In the first case we take an integration path lying in the lower half-plane and involving only the simple poles \((2m+1)i, m < 0\). It leads to the result:

\[
\Omega = \frac{1}{4\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{\beta \mu m} \frac{\hbar \omega_+}{\sinh \left( \frac{\hbar \omega_+}{2} m \right)} \sinh \left( \frac{\hbar \omega_-}{2} m \right),
\]

(4.20)

which corresponds to the case \( \kappa \leq 1 \) in Eq.(4.9). In the second case, an integration path in the upper half-plane is chosen. It encircles all the other poles: \((2m+1)i, m \geq 0,\)
\[ i + 4\pi m/(\beta \hbar \omega_+), \quad i + 4\pi m/(\beta \hbar \omega_-), \quad m \in \mathbb{Z}^*, \] as shown in Fig.1. We present the result in a manner which will render apparent the various regimes:

\[
\Omega = (\Omega_L + \Omega_{01}) + \Omega_{02} + \Omega_{\text{osc}}
\]

\[
= 2\pi i \left( a_{-1}(i) - \sum_{m \geq 1} a_{-1}(2m + 1)i \right) + \sum_{m \neq 0} (a_{-1}(i + \frac{4\pi}{\beta \hbar \omega_{\pm}}m_{\pm})).
\]

The first term is at the origin of the Landau diamagnetism:

\[
\Omega_L = \frac{\mu}{24} \left( \frac{\omega_c}{\omega_0} \right)^2 = -\frac{1}{2}\chi_L H^2,
\]

where \(\chi_L = -\frac{1}{3}\mu \left( \frac{\mu_B}{\hbar \omega_0} \right)^2 = -\frac{1}{3}D_0 \mu_B^2\) is the Landau diamagnetic susceptibility. The coefficient \(D_0 = \mu/(\hbar \omega_0)^2\) can be interpreted as the density of states at Fermi energy. Notice that the value of \(\chi_L\) is equal to one third of the “quasiclassical” one \(\chi_p\) obtained in (4.17), a feature which is reminiscent of which we encounter in the studies of free 3D electron gas. The second term, which gives no contribution to the magnetization, is written as:

\[
\Omega_{01} = -\frac{\mu}{6} \left[ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \pi^2 \left( \frac{k_B T}{\hbar \omega_0} \right)^2 - \frac{1}{2} \right].
\]

The third term is given as:

\[
\Omega_{02} = \frac{1}{4\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \exp \left( -\frac{\mu}{k_B T} m \right) \sinh \left( \frac{\omega_+}{\omega_-} \frac{m}{2} \right) \sinh \left( \frac{2k_B T}{\hbar \omega_-} \pi^2 m \right).
\]

It becomes negligible at low temperature regime \(k_B T \ll \mu\). The sum of \(\Omega_L\) and \(\Omega_{01}\) is analogue to the term \(A\) in Eq.(4.11) and \(\Omega_{02}\) corresponds to \(S_{\pm}\). The last term is responsible for the oscillatory behaviour. We need to distinguish between irrational values of \(\omega_+/\omega_-\) and rational ones:

- **Case** \(\omega_+/\omega_- \notin \mathbb{Q}\),

\[
\Omega_{\text{osc}} = \frac{1}{2\beta} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin \left( \frac{2\mu}{\hbar \omega_-} \frac{m}{\pi} \right) \sinh \left( \frac{\omega_+}{\omega_-} \frac{m}{2} \right) \sinh \left( \frac{2k_B T}{\hbar \omega_-} \pi^2 m \right)
\]

\[
+ \sin \left( \frac{2\mu}{\hbar \omega_+} \frac{m}{\pi} \right) \sinh \left( \frac{\omega_+}{\omega_-} \frac{m}{2} \right) \sinh \left( \frac{2k_B T}{\hbar \omega_+} \pi^2 m \right) \equiv \Omega_{\text{osc}}^-.\]

- **Case** \(\omega_+/\omega_- = p/q \in \mathbb{Q}\), \(\gcd(p,q) = 1\), \(\omega_+/p = \omega_-/q = 2l/(\hbar \beta) \in \mathbb{R}\),

\[
\Omega_{\text{osc}} = \frac{1}{2\beta} \sum_{m=1, m \neq 0 \mod q}^{\infty} \frac{(-1)^m}{m} \sin \left( \frac{2\mu}{\hbar \omega_-} \frac{m}{\pi} \right) \sin \left( \frac{\omega_+}{\omega_-} \frac{m}{2} \right) \sinh \left( \frac{2k_B T}{\hbar \omega_-} \pi^2 m \right)
\]

\[
+ \sum_{m=1, m \neq 0 \mod p}^{\infty} \frac{(-1)^m}{m} \sin \left( \frac{2\mu}{\hbar \omega_+} \frac{m}{\pi} \right) \sinh \left( \frac{\omega_+}{\omega_-} \frac{m}{2} \right) \sinh \left( \frac{2k_B T}{\hbar \omega_+} \pi^2 m \right)
\]

\[
+ \frac{1}{lpq} \sum_{k=1}^{\infty} \frac{(-1)^{(p+q)k}}{k \sinh \left( \frac{\pi^2 k}{l} \right)} \left[ \frac{\mu}{k_B T} \cos \left( \frac{\mu \pi k}{k_B T} \right) - \left( \pi \coth \left( \frac{\pi^2 k}{l} \right) + \frac{l}{\pi k} \right) \sin \left( \frac{\mu \pi k}{k_B T} \right) \right].
\]
C. The thermodynamical potential via symbol calculus

Let us define the thermodynamical potential operator as

\[ \mathcal{O} = -\frac{1}{\beta} \log (1 + \exp [-\beta(\hat{H} - \mu)]) \]

We have:

\[ \mathcal{O} = \pi^2 \int_C C^2 \hat{\mathcal{O}}(z_d, z_g) | z_d, z_g \rangle \langle z_d, z_g | d^2 z_d d^2 z_g, \]

\[ \hat{\mathcal{O}}(z_d, z_g) = \langle z_d, z_g | \mathcal{O} | z_d, z_g \rangle, \quad (4.27) \]

We then perform the angular integrations and take its trace. We get an integral representation of the thermodynamical potential \( \Omega = \text{Tr} \mathcal{O} \):

\[ \Omega = -\frac{1}{\beta} \int_0^\infty \int_0^\infty d u_d d u_g \hat{\mathcal{O}}(u_d, u_g) = -\frac{1}{\beta} \int_0^\infty \int_0^\infty d u_d d u_g \hat{\mathcal{O}}(u_d, u_g). \quad (4.28) \]

The problem turns out to evaluate the upper and lower symbols. The lower one is shown here with the integral representation:

\[ \hat{\mathcal{O}}(u_d, u_g) = -e^{-(u_d+u_g)} 2\beta \int_{-\infty}^{+\infty} e^{-(ik+1)\frac{\bar{\hbar}\omega_+}{2}} \exp \left( u_d e^{-(ik+1)\frac{\bar{\hbar}\omega_+}{2}} \right) \exp \left( u_g e^{-(ik+1)\frac{\bar{\hbar}\omega_-}{2}} \right) dk \]

\( \equiv \langle N_e \rangle_L + \langle N_e \rangle_{01} + \langle N_e \rangle_{02} + \langle N_e \rangle_{osc} + \langle N_e \rangle^+ \quad (5.1) \)

V Average number of electrons and magnetic moment

In this section, we will exploit the formulas (4.20)-(4.26) to obtain the exact expressions of the average number of electrons and the magnetization. We will restrict ourselves to the more realistic case: \( \mu \leq \bar{\hbar}\omega/2 \).

The average number of electrons is easily derived by taking the derivative of \(-\Omega\) with respect to \( \mu \). It is found to be:

\[ \langle N_e \rangle = -\frac{1}{24} \left( \frac{\omega_c}{\omega_0} \right)^2 + \frac{1}{2} \left[ \left( \frac{\mu}{\hbar \omega_0} \right)^2 + \pi^2 \left( \frac{k_B T}{\hbar \omega_0} \right)^2 - \frac{1}{6} \right] \]

\[ + \frac{1}{4} \sum_{m=1}^{\infty} (-1)^m \frac{e^{-\beta \mu m}}{\sinh (\frac{\beta}{2} \hbar \omega_+ m) \sinh (\frac{\beta}{2} \hbar \omega_- m)} \]

\[ - \pi \sum_{m=1}^{\infty} (-1)^m \left[ \frac{k_B T}{\hbar \omega_-} \sin (\frac{\omega_-}{\omega_+} \pi m) \sinh (\frac{2k_B T}{\hbar \omega_-} \pi m) \right] + \frac{k_B T}{\hbar \omega_+} \sin (\frac{\omega_+}{\omega_-} \pi m) \sinh (\frac{2k_B T}{\hbar \omega_+} \pi m) \]

\[ \equiv \langle N_e \rangle_L + \langle N_e \rangle_{01} + \langle N_e \rangle_{02} + \langle N_e \rangle_{osc} + \langle N_e \rangle^+. \quad (5.1) \]

The magnetic moment is decomposed into four parts and is expressed in Bohr magneton units:

\[ M = \chi_L H - 2\mu_B \left( \frac{\partial \Omega_{02}}{\partial \hbar \omega_+} \right)_\mu - 2\mu_B \left( \frac{\partial \Omega_{osc}}{\partial \hbar \omega_+} \right)_\mu \]

\[ \equiv 2\mu_B (\mathcal{M}_L + \mathcal{M}_0 + \mathcal{M}_{osc}^- + \mathcal{M}_{osc}^+), \quad (5.2) \]
where

\[ M_L = \frac{-\mu}{12\hbar\omega_0} \left( \frac{\omega_c}{\omega_0} \right) \equiv \frac{1}{2\mu_B} \chi_L H, \]  

\[ M_0 = \frac{1}{8\omega} \sum_{m=1}^{\infty} (-1)^m e^{-\beta \mu m} \frac{\omega_+ \coth (\beta \hbar \omega_+ m/2) - \omega_- \coth (\beta \hbar \omega_- m/2)}{\sinh (\beta \hbar \omega_+ m/2) \sinh (\beta \hbar \omega_- m/2)}, \]  

and, for the irrational case \( \omega_+/\omega_- \not\in \mathbb{Q} \),

\[ M_{osc}^- = -\frac{k_B T}{\hbar \omega_0} \sum_{m=1}^{\infty} \frac{(-1)^m \sin (2\pi m \mu/(\hbar \omega_-))}{\sin (\pi m \omega_+ / \omega_-) \sinh (2\pi^2 m k_B T/(\hbar \omega_-))} \times \]

\[ \left[ \frac{\pi \mu}{\hbar \omega_-} \cot \left( 2\pi m \frac{\mu}{\omega_-} \right) - \frac{\pi \omega_+}{\omega_-} \cot \left( \pi m \frac{\omega_+}{\omega_-} \right) - \frac{\pi^2 k_B T}{\hbar \omega_-} \coth \left( 2\pi^2 m \frac{k_B T}{\hbar \omega_-} \right) \right], \]

\[ M_{osc}^+ = \frac{k_B T}{\hbar \omega_+} \sum_{m=1}^{\infty} \frac{(-1)^m \sin (2\pi m \mu/(\hbar \omega_+))}{\sin (\pi m \omega_+ / \omega_+) \sinh (2\pi^2 m k_B T/(\hbar \omega_+))} \times \]

\[ \left[ \frac{\pi \mu}{\hbar \omega_+} \cot \left( 2\pi m \frac{\mu}{\omega_+} \right) - \frac{\pi \omega_-}{\omega_+} \cot \left( \pi m \frac{\omega_-}{\omega_+} \right) - \frac{\pi^2 k_B T}{\hbar \omega_+} \coth \left( 2\pi^2 m \frac{k_B T}{\hbar \omega_+} \right) \right]. \]

We will not give the expressions of \( M_{osc}^\pm \) in the rational case because the magnetization is a continuous function of \( \omega_c \) and its behavior can be fully understood from the irrational one.

**Discussion**

The temperature scale is compared to the two natural modes \( \omega_\pm \) of the system and draws three possible intrinsic regimes: high temperature regime \( k_B T > \hbar \omega_+ \), low temperature regime \( k_B T < \hbar \omega_- \), and intermediate temperature regime \( \hbar \omega_- < k_B T < \hbar \omega_+ \). Remember that we work in the large electron number region: \( \mu > \hbar \omega /2 \).

1. **High temperature regime**: \( k_B T > \hbar \omega_+ > \hbar \omega_- \).

   This inequality implies the following constraint on the field:

   \[ \frac{\omega_c}{\omega_0} < \frac{k_B T}{\hbar \omega_0} - \frac{\hbar \omega_0}{k_B T} \approx \frac{k_B T}{\hbar \omega_0}. \]  

   We can see that \( M_0 \) is the dominant term for the magnetic moment in regard to \( M_{osc} \) because of the smallness of arguments of the sinh (in the denominator) and coth (in the numerator) functions. Hence, \( M \approx 2 \mu_B (M_L + M_0) \), which shows mainly Landau diamagnetism. Similarly, we infer from (5.1) that \( \langle N_e \rangle \approx \langle N_e \rangle_L + \langle N_e \rangle_{01} + \langle N_e \rangle_{02} \).

2. **Low temperature regime**: \( k_B T < \hbar \omega_- \).

   The magnetic field is restricted by the inequality:

   \[ \frac{\omega_c}{\omega_0} < \frac{\hbar \omega_0}{k_B T} - \frac{k_B T}{\hbar \omega_0} \approx \frac{\hbar \omega_0}{k_B T}. \]  

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Now the $M_0$ term becomes excessively small due to the rapidly decreasing exponential factor and the large arguments of the hyperbolic functions present in the expression. The magnetization is hence approximately determined by the three terms: $M_L$, $M^+_{osc}$, and $M^-_{osc}$ and exhibits oscillating behavior.

(a) **Strong fields** $\omega_c \gg \omega_0$.

We make the following approximations:

$$\omega_+ \approx \omega_c \left(1 + \frac{(\omega_0/\omega_c)^2}{2}\right), \quad \omega_- \approx \frac{\omega_0^2}{\omega_c}, \quad \frac{\omega_+}{\omega_-} \approx \left(\frac{\omega_c}{\omega_0}\right)^2.$$ (5.9)

Assume also that $\omega_c \leq 2\omega_0 \sqrt{\mu/\hbar} - 1$ to ensure the validity of $\mu \geq \hbar \omega/2$. One can see that the denominator of $M^+_{osc}$ contains the product of sinh and two sinh’s with small argument and, hence, $M \approx 2\mu_B (M_L + M^+_{osc})$. In particular, after replacing $\omega_+$ by $\omega_c$ in the sinus argument of the numerator of $M^+_{osc}$, we find that the magnetization is periodic with respect to the inverse of the magnetic field, a characteristic fact of the “de Haas-van Alphen” regime. The similar behavior holds for the electron number: $\langle N_e \rangle \approx \langle N_e \rangle_L + \frac{\mu}{\hbar \omega_0^2}/2 + \langle N_e \rangle^+_{osc}$.

(b) **Weak fields** $\omega_c \ll \omega_0$.

In this case the two characteristic frequencies and their ratio are approximated to

$$\omega_\pm \approx \omega_0 \left[1 \pm \frac{\omega_0}{2\omega_0} + \frac{1}{8} \left(\frac{\omega_c}{\omega_0}\right)^2\right], \quad \frac{\omega_+}{\omega_-} \approx \frac{\omega_c}{\omega_0} + \frac{1}{2} \left(\frac{\omega_c}{\omega_0}\right)^2.$$ (5.10)

Now $M^-_{osc}$ and $M^+_{osc}$ have the same order of contribution due to the presence in their denominators of

$$(\sinh (2\pi^2 mk_B T/\hbar \omega_\pm))^2 \approx (\sinh (2\pi^2 mk_B T/\hbar \omega_0))^2$$ (5.11)

and of the strongly oscillating functions

$$(\sin (\pi m \omega_\pm/\omega_\pm))^2 \approx (\sin (\pi m \omega_c/\omega_0))^2.$$ (5.12)

Consequently, the system is considered as lying in the mesoscopic phase. Similar conclusions can be reached for the behavior of the average electron number.

3. **Intermediate temperatures**: $\hbar \omega_- < k_B T < \hbar \omega_+$.

These inequalities imply the constraint

$$\frac{\omega_c}{\omega_0} > \left| \frac{\hbar \omega_0 - k_B T}{k_B T} \right|.$$ (5.13)

The weak field case occurs only when $k_B T$ approaches $\hbar \omega_0$. In this case, one may think that the oscillatory terms $M^-_{osc}$ and $M^+_{osc}$ give their contribution to $M$ as what we have seen in the previous subsection. But, in fact, the hyperbolic sinus functions of the denominators have large arguments:

$$\sinh (2\pi^2 mk_B T/\hbar \omega_\pm) \approx \sinh(19.74 m) \approx 0.5 \exp(19.74 m),$$
and so overcomes the algebraic contribution of the sinus functions. The system hence goes to the Landau diamagnetic regime. On the other hand, for a strong field, we return to the approximation $M \approx 2\mu_B (M_L + M_{osc}^+)$ and the system shows the de Haas-van Alphen effect.

We have repeated in Fig.2 the phase diagram of magnetization proposed by Ishikawa and Fukuyama. Let us however mention that the two intrinsic frequencies $\omega_{\pm}$ do not represent the exact borders between the different magnetic phases. It should be taken in a qualitative sense only. In order to justify this, we choose the chemical potential equal to $100.0\hbar\omega_0$ and make temperature vary through the different magnetic field regimes. Fig.3(a) illustrates the weak field regime. One can see that at low temperature $k_B T = 0.001\hbar\omega_0$ the magnetization experiences strong fluctuations, called mesoscopic fluctuations. As $T$ increases ($k_B T = 0.1\hbar\omega_0$), the strength of these fluctuations decreases. As the temperature is getting higher (for example for $k_B T = 0.5\hbar\omega_0$), the fluctuations disappear and we reach the Landau diamagnetism. In Fig.3(b) the magnetic field lies between $1.9\omega_0 mc/e$ and $3.1\omega_0 mc/e$. When $k_B T = 0.01\hbar\omega_0$, the system shows large fluctuations. One can see that, if the magnetic filed increases, the fluctuations diminish and the cycloid-like curve appears (de Haas-van Alphen oscillations). At higher temperature, e.g. $k_B T = 0.1\hbar\omega_0$, one can see more clearly the phase change from the mesoscopic fluctuations to the de Haas-van Alphen oscillations. If the temperature continues increasing (for example, $k_B T = 0.5\hbar\omega_0$, $k_B T = 1.0\hbar\omega_0$), the system will go from the Landau diamagnetism to the de Haas-van Alphen oscillations. Fig.3(c) shows the de Haas-van Alphen phases. The position of the peak can be predicted from the simple formula:

$$\frac{\omega_c}{\omega_0} = \frac{2\mu}{n\hbar\omega_0} - \frac{n\hbar\omega_0}{2\mu},$$

where $n$ is some odd positive integer.

In Fig.3(d) we see that, at the extreme low temperature ($k_B T = 0.001\hbar\omega_0$), the curve shows some “width”, which comes from the limit of the picture resolution and which corresponds, in fact, to small fluctuations. This gives us an example of de Haas-van Alphen oscillations mixed with mesoscopic fluctuations.

VI Current Distribution

We now turn our attention to the spatial density of current. It can be obtained by the following formula:

$$\mathbf{J}(\mathbf{r}) = \Re \left\langle \hat{\psi}^\dagger(\mathbf{R}) \left( -\frac{e}{m} \right) \left( \mathbf{P} + \frac{e}{c} \mathbf{A}(\mathbf{R}) \right) \hat{\psi}(\mathbf{R}) \right\rangle,$$

where $\langle \cdot \rangle = \text{Tr}(f(\mathcal{H})(\cdot))$ is the thermal average, and $\hat{\psi}(\mathbf{R})$ the field operator:

$$\hat{\psi}(\mathbf{R}) \mid n, \alpha \rangle = \Psi_{n,\alpha}(\mathbf{r}).$$

Due to the symmetry of the system, the current distribution is purely orthoradial,

$$\mathbf{J}(\mathbf{r}) = J_\theta(r)e_\theta,$$
and its component is given by the series:

\[ J_\theta(r) = -ev_0 \sum_{n,\alpha} \left( \frac{\xi}{r} + \frac{\omega_c}{2\omega_0 \xi} \right) \Psi_{n,\alpha}(r, \theta) \Psi_{n,\alpha}(r, \theta) f(E_{n\alpha}), \]  

(6.3)

where \( \xi = \sqrt{\hbar/(m\omega_0)} \) is the characteristic length and \( v_0 = \omega_0 \xi \) the characteristic speed in the harmonic potential. The current distribution induces a local magnetic moment,

\[ \mathbf{M}(r) = \frac{1}{2c} \mathbf{r} \times \mathbf{J} = \frac{1}{2c} r J_\theta(r) \mathbf{e}_z \equiv M_z(r) \mathbf{e}_z, \]  

(6.4)

which in turn gives rise to the total magnetic moment,

\[ M = \int M_z(r) dS = \frac{\pi}{c} \int_0^\infty J_\theta(r) r^2 dr. \]  

(6.5)

By applying Eq. (3.8) to the expression (6.3) of \( J_\theta(r) \) we get

\[ J_\theta(r) = (-e)v_0 \sum_{n_d,n_g} \frac{f(\hbar(\omega_+ n_d + \omega_+ n_g + \frac{\omega}{2})(n_d - n_g)}{\left( \frac{r}{\xi} + \frac{\omega_c}{2\omega_0 \xi} \right)} \times \]

\[ \frac{1}{\pi^4} \int \int \int \int \mathbb{C}^2 e^{-|z_d|^2 + |z_g|^2 + |z_d|^2 + |z_g|^2} \mathcal{K}(z_d', z_g'; r, \theta) \mathcal{K}(r, \theta; z_d, z_g) \times \]

\[ \frac{(z_d' \bar{z}_d)^{n_d}}{n_d!} \frac{(z_g' \bar{z}_g)^{n_g}}{n_g!} d^2 z_d' d^2 z_g' d^2 z_d d^2 z_g. \]  

(6.6)

The above equation is rotational invariant and one can hence put \( \theta = 0 \). Let us now represent the Fermi-Dirac function by the Fourier integral (7.2) and perform the discrete summation. The four integrals on the complex plane can be easily performed by using well-known gaussian integrals of the type

\[ \int_{-\infty}^{\infty} e^{-t^2 + tZ} dt = \sqrt{\pi} \exp \left( \frac{1}{4} Z^2 \right). \]

We finally obtain the following expression:

\[ J_\theta(r) = \frac{-ev_0 r}{16\pi l_0^2 \xi} \int_{-\infty}^{\infty} \frac{dk}{\cosh \left( \frac{\pi}{2} k \right) \sinh^2 \left( (ik + 1) \beta \frac{\hbar \omega}{4} \right)} \times \]

\[ \left[ \frac{\omega_c}{\omega_0} \sinh \left( (ik + 1) \beta \frac{\hbar \omega}{4} \right) - \frac{\omega}{\omega_0} \sinh \left( (ik + 1) \beta \frac{\hbar \omega}{4} \right) \right] \times \]

\[ \exp \left[ -2r^2 \sinh \left( (ik + 1) \beta \frac{\hbar \omega}{4} \right) \sinh \left( (ik + 1) \beta \frac{\hbar \omega}{4} \right) \right]. \]  

(6.7)

Three types of singular points appear in the integrand:

1. Pole \( k = i \).

   This pole is triple since it appears once in \( \cosh \frac{\pi}{2} k \), and twice in \( \sinh \left( (ik + 1) \beta \frac{\hbar \omega}{4} \right)^2 \).

2. Poles \( k = (2m + 1)i \), \( m \in \mathbb{Z}^* \).

   They are the other (simple) poles of \( \cosh \frac{\pi}{2} k \).
3. Singular points $k = i + 4\pi m/(\beta \hbar \omega)$, $m \in \mathbb{Z}^*$. These points are essential singularities since they appear in the exponential. The current distribution is hence separated into three components responsible for the Landau diamagnetism, weakly diamagnetism, and oscillation behavior:

$$
J_\theta(r) = (J_\theta(r))_L + (J_\theta(r))_0 + (J_\theta(r))_{osc}
$$

$$
= 2\pi i \left( a_{-1}(i) + \sum_{m \geq 1} a_{-1}(2m + 1)i \right) + \sum_{m \neq 0} (a_{-1}(i + \frac{4\pi}{\beta \hbar \omega} m)).
$$

In a forthcoming paper, we shall go deeper into the analysis of such an expression for the current distribution.

VII Conclusion

In this paper, we have established exact formulas for the thermodynamical potential of a two-dimensional gas of spinless electrons, confined in an isotropic harmonic potential, and submitted to a constant perpendicular magnetic field. From this it has become possible to study the magnetic moment and other thermodynamical quantities at different regimes of temperature and field. This exhaustive study was made possible thanks to the specific simplicity of the isotropic harmonic potential. Of course, there exist other situations in which we could get similar exact expressions: anisotropic harmonic potential, harmonic groove, ..., as indicated by [4]. In a next future, we shall deal with less tractable but still integrable models (see for instance [4], like infinite cylinder potential or quantum rings, cases in which the expressing of the trace (7.6) in a closed form is quite not expected. We shall also intend to make use of other families of coherent states, possibly more adapted to these new situations, in order to use an efficient symbol calculus. For instance, for the infinite rectangular well potential, we are thinking about using tensor products of infinite well coherent states introduced and analysed in a recent paper [4]. Finally, we shall also explore the behaviour, at different temperature and field regimes, of other thermodynamical quantities of current experimental interest, like the heat capacity [4].

Appendix: Fermi-Dirac trace formulas

It is well known that, like the Gaussian function, the function sech$x = 1/\cosh x$ is a fixed point for the Fourier transform in the Schwartz space:

$$
\frac{1}{\cosh \sqrt{\frac{2}{\pi}} x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} \frac{dy}{\cosh \sqrt{\frac{2}{\pi}} y}.
$$

Hence, given an Hamiltonian $\mathcal{H}$, we can write for the corresponding Fermi operator:

$$
f(\mathcal{H}) \equiv \frac{1}{1 + e^{\beta(\mathcal{H} - \mu)}} = \int_{-\infty}^{+\infty} \frac{e^{-(ik+1)\frac{\beta}{2}(\mathcal{H} - \mu)}}{4 \cosh \frac{\beta}{2} k} dk.
$$
Similarly, we can write for the thermodynamical potential operator:

\[- \frac{1}{\beta} \log (1 + e^{-\beta(H-\mu)}) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} e^{-(ik+1)\frac{\beta}{2}H} \Theta(k) \, dk. \quad (7.3)\]

Therefore, the average number of fermions and the thermodynamical potential can be written (at least formally) as follows:

\[\langle N \rangle = \text{Tr} f(H) = \int_{-\infty}^{+\infty} e^{(ik+1)\frac{\beta}{2}\Theta(k)} \, dk, \quad (7.4)\]

\[\Omega = \text{Tr}(-\frac{1}{\beta} \log (1 + e^{-\beta(H-\mu)})) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} e^{(ik+1)\frac{\beta}{2}\Theta(k)} \Theta(k) \, dk, \quad (7.5)\]

where \(\Theta\) designates the function

\[\Theta(k) = \text{Tr}(e^{-(ik+1)\frac{\beta}{2}H}). \quad (7.6)\]

Observe that \((2m+1)i, \ m \in \mathbb{Z}\) are (simple) poles for the function \(1/\cosh \frac{\beta}{2}k\) and \(i\) is a pole for the functions \(\Theta(k)\) and \(1/(ik+1)\). These Fourier integrals can be evaluated by using residue theorems if the integrand functions \(\Phi_1(k) = \Theta(k)/\cosh \frac{\beta}{2}k\) and \(\Phi_2(k) = \Theta(k)/((ik+1) \cosh \frac{\beta}{2}k)\) satisfy the Jordan Lemma, that is, \(\Phi_1(Re^{i\theta}) \leq g(R), \ \Phi_2(Re^{i\theta}) \leq h(R)\), for all \(\theta \in [0, \pi]\), and \(g(R)\) and \(h(R)\) vanish as \(R \to \infty\). The quantities \(\langle N \rangle\) and \(\Omega\) are then formally given by

\[2\pi i \left[ a_{-1}(i) + \sum_{m=1}^{\infty} a_{-1}((2m+1)i) + \sum_{\nu} a_{-1}(k_{\nu}) \right], \quad (7.7)\]

where \(a_{-1}(\cdot)\) denotes the residue of the involved integrand at pole \((\cdot)\), and the \(k_{\nu}\)'s are the poles (with the exclusion of the pole \(i\)) of \(\Theta(k)\) in the complex \(k\)-plane.

We now introduce the spectral resolution of the (bounded below) self-adjoint operator \(H\):

\[\varphi(H) = \int_{-\infty}^{+\infty} \varphi(\lambda) \, E(d\lambda), \quad (7.8)\]

where \(\varphi\) is a complex-valued function and \(E_\lambda = \int_{-\infty}^{\lambda} E(d\lambda)\) is the resolution of the identity for the Hamiltonian \(H\). Define the density of states \(\nu(\lambda)\) as \(\text{Tr}E(d\lambda)/d\lambda\). The trace formula ensues:

\[\text{Tr} \varphi(H) = \int_{-\infty}^{+\infty} \varphi(\lambda) \, \nu(\lambda) \, d\lambda. \quad (7.9)\]

Let us now introduce the weighted density of states \(w(\lambda) = e^{-\frac{\beta}{2}\lambda} \nu(\lambda)\) and its Fourier transform

\[\hat{w}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ik\lambda} w(\lambda) \, d\lambda. \quad (7.10)\]

Then, from (7.4), (7.3) and (7.4), we can represent \(\langle N \rangle\) and \(\Omega\) as follows:

\[\langle N \rangle = \sqrt{2\pi} \int_{-\infty}^{+\infty} e^{(ik+1)\frac{\beta}{2}\hat{w}(\frac{\beta}{2}k)} \, dk = \frac{\pi}{\beta} e^{\frac{\beta}{2}\hat{Z}_1(-\mu)}, \quad (7.11)\]

\[\Omega = -\frac{\sqrt{2\pi}}{\beta} \int_{-\infty}^{+\infty} \frac{e^{(ik+1)\frac{\beta}{2}\hat{w}(\frac{\beta}{2}k)}}{(2 \cosh \frac{\beta}{2}k)(ik+1)} \, dk = -\frac{2\pi}{\beta^2} e^{\frac{\beta}{2}\hat{Z}_2(-\mu)}, \quad (7.12)\]
where we have introduced the weighted functions $Z_1 = \text{sech}(\pi k/\beta) \hat{w}(k)$ and $Z_2 = \text{sech}(\pi k/\beta)(i2k/\beta + 1)^{-1} \hat{w}(k)$.

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Figure caption

Fig.1 Poles of the Fourier representation of the thermodynamical potential $\Omega$. The poles lying on the imaginary axis are simple except for the point $i$ which is of order 4. The poles lying on the line $k = i$ in the complex $k$-plane ($i$ is excluded) may be simple or double depending on whether $\omega_+$ and $\omega_-$ are uncommensurable or not.

Fig.2 Phase diagram of the magnetization. In high temperature and low magnetic field region, the system shows Landau diamagnetism; in low temperature and low magnetic field region, mesoscopic fluctuations appear; and in strong magnetic field region, the system experiences the de Haas-van Alphen oscillation phase. The two curves, $k_B T = \hbar \omega_+$ and $k_B T = \hbar \omega_-$, give a qualitative indication about the phase borders.

Fig.3 Magnetization curves versus the magnetic field at different temperatures. We represent the cyclotron frequency in $\omega_0$ units and the magnetization in $2\mu_B$ units. The chemical potential $\mu$ is set up to $100.0\hbar \omega_0$.

(a) $\omega_c$ is less than $0.8\omega_0$. Temperatures are chosen to be $0.001\hbar \omega_0$, $0.1\hbar \omega_0$, and $0.5\hbar \omega_0$, respectively.

(b) $\omega_c$ is between $1.9\omega_0$ and $3.1\omega_0$. Temperatures are chosen to be $0.01\hbar \omega_0$, $0.1\hbar \omega_0$, $0.5\hbar \omega_0$, and $1.0\hbar \omega_0$, respectively.

(c) $\omega_c$ is between $4.0\omega_0$ and $15.0\omega_0$. Temperatures are chosen to be $0.1\hbar \omega_0$, $0.5\hbar \omega_0$, $1.0\hbar \omega_0$, and $5.0\hbar \omega_0$, respectively.

(d) $\omega_c$ is greater than $15.0\omega_0$. Temperatures are chosen to be $0.001\hbar \omega_0$, $0.5\hbar \omega_0$, $1.0\hbar \omega_0$, and $5.0\hbar \omega_0$, respectively.
Figure 1: FIG.1
Landau diamagnetism

de Haas-van Alphen oscillation

Mesoscopic fluctuation

$k_B T = \hbar \omega_+$

$k_B T = \hbar \omega_-$
