The Hubble Slow Roll Expansion for Multi Field Inflation

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We examine the dynamics of inflation driven by multiple, interacting scalar fields and derive a multi field version of the Hubble slow roll expansion. We show that the properties of this expansion naturally generalize those of the single field case. We present an analogous hierarchy of slow roll parameters, and derive the system of “flow” equations that describes their evolution, and show that when this system is truncated at finite order, it can always be solved exactly. Lastly, we express the scalar and tensor perturbation spectra in terms of the slow roll parameters.

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I. INTRODUCTION

Scalar field driven inflation solves the traditional problems (horizon, flatness, structure) of early universe cosmology. Fortuitously, the need for an almost scale invariant perturbation spectrum strongly suggests that inflation is well described by the slow roll approximation: the parameters that describe the departures from scale invariance are also those that must be small if the slow roll approximation is to hold. In practice, we normally consider only the two lowest order slow roll parameters. However, from a theoretical perspective we recognize this is a truncated expansion and we are implicitly assuming that there are no large high-order derivatives in the dynamical system describing inflation.

For a single field, the general slow roll expansion was written down by Liddle, Parsons and Barrow. The energy density and its proxy the Hubble parameter, $H$, is a function of the slowly rolling field, which serves as the independent variable of the dynamical system. The slow roll expansion is expressed as a hierarchy of differential equations, involving progressively higher order derivatives of $H(\phi)$. In many practical applications, this hierarchy is truncated at some finite order, and the properties of the system have been carefully studied in recent years. In particular, Kinney showed that this system possesses a set of attractors. Easther and Kinney built on this observation, using the slow roll hierarchy to generate random models of inflation, and this technique was later employed in the analysis of the first year of WMAP data and more recently in. However, the randomly generated models are not associated with a measure so one cannot quantify which region of parameter space is favored. Subsequently, Liddle pointed out that the truncated slow roll hierarchy can always be solved analytically. Leach and Liddle recently used this solution to implement and test different reconstruction algorithms, and their results illustrate how the lack of a measure makes it very difficult to draw quantitative conclusions about the parameter space of slow roll inflation.

The developments above all refer to models of inflation driven by a single scalar field. However, multi field models are common in the literature and have a number of attractive features. Firstly, there is no theoretical guarantee that any inflationary phase can be described in terms of a single evolving field: the primarily motivation for doing so is simply an application of Occam’s Razor. Moreover, many phenomenologically promising models of inflation (e.g.) require two or more interacting fields. Lastly, recent developments in string theory have motivated the study of multi field models in the context of the string landscape.

The purpose of this paper is to generalize the Hubble slow roll formalism to models with multiple fields. Section recapts the slow roll regime for the single field case and Section gives the multi field version, which is the main result of this paper. We see that almost all features of the single field case carry over to the multi field case. In particular, the exact solution found by Liddle for the truncated single field slow roll hierarchy generalizes to the multi field case, and we write down an exact algebraic expression for the potential which corresponds to the truncated

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1 An analagous representation outside the context of inflation was introduced by Starobinsky.
slow roll hierarchy at arbitrary order. Finally we express the scalar and tensor perturbation spectra in terms of the generalized slow roll parameters.

II. SINGLE FIELD FORMALISM

A. Background Evolution

In scalar field driven inflationary models, the inflaton field is governed by the following equation of motion,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{1}$$

The spacetime background obeys the Friedman equations

$$H^2 = \frac{8\pi G}{3} \left[ V(\phi) + \frac{1}{2}\dot{\phi}^2 \right] \tag{2}$$

and

$$\frac{\dot{a}}{a} = \frac{8\pi G}{3} \left[ V(\phi) - \dot{\phi}^2 \right], \tag{3}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Equations (2) and (3) can be combined to produce the Hamilton-Jacobi equation

$$H^2(\phi) \left[ 1 - \frac{1}{3} \epsilon(\phi) \right] = \frac{8\pi G}{3} V(\phi), \tag{4}$$

where the slow roll parameter $\epsilon$ is

$$\epsilon = \frac{1}{4\pi G} \left( \frac{1}{H} \frac{dH}{d\phi} \right)^2. \tag{5}$$

Note one frequently sees the potential slow roll expansion which can be obtained directly from the functional form of the potential

$$\epsilon_V = \frac{1}{16\pi G} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \tag{6}$$

$$\eta_V = \frac{1}{8\pi G} \frac{V''(\phi)}{V(\phi)}. \tag{7}$$

We can understand the physical significance of $\epsilon$ by rewriting (3) as

$$\left( \frac{\ddot{a}}{a} \right) = H^2 \left[ 1 - \epsilon \right], \tag{8}$$

and we immediately see that $\epsilon < 1$ corresponds to $\ddot{a} > 0$, the necessary condition for inflation.

B. Hierarchy of Parameters

The duration of inflation is most frequently parameterized by $N$, the number of e-foldings by which the scale factor grows during the inflationary phase, so $a \equiv e^N$. As a matter of definition, we obtain $N$ by integrating $H$,

$$N = \int_t^t H dt. \tag{9}$$

As written above, $N$ measures the number of e-folds before the end of inflation, and increases as $t$ decreases. We can write $N$ as a function of $\phi$ via

$$N = \int_{\phi_e}^{\phi} \frac{H}{\sqrt{\epsilon(\phi)}} d\phi = 2\sqrt{\pi G} \int_{\phi_e}^{\phi} \frac{d\phi}{\sqrt{\epsilon(\phi)}}. \tag{10}$$
We now write a differential equation for $\epsilon(N)$,

$$\frac{d\epsilon}{dN} = \epsilon(\sigma + 2\epsilon),$$

(11)

where

$$\sigma = \frac{1}{\pi \sqrt{G}} \left[ \frac{1}{2} \left( \frac{\partial^2 \phi}{\partial N^2} \right) - \left( \frac{\partial \phi}{\partial N} \right)^2 \right].$$

(12)

Equation (12) is the first flow equation and $\sigma$ depends on the second derivative of $H$. Defining additional parameters

$$n\lambda = \left( \frac{1}{4\pi G} \right)^n \frac{(\partial \phi H)^{n-1}}{H^n} \partial_{\phi}^{n+1} H,$$

(13)

we create an infinite system of coupled first order differential equations:\n
$$\frac{d\sigma}{dN} = -\epsilon(5\sigma + 12\epsilon) + 2(2\lambda)$$

(14)

$$\frac{dn\lambda}{dN} = n\lambda \left[ \frac{1}{2} (n-1)\sigma + (n-2)\epsilon \right] + n^{+1}\lambda.$$  

(15)

Looking at (13) we see that the derivative of $n\lambda$ depends on four (or fewer) quantities: $n\lambda$, $n^{+1}\lambda$, $\sigma$, and $\epsilon$. These flow equations can be truncated to a system of finite order if for some $n_i$, $n\lambda = 0$, for all $N$. Once we have specified the values of the $n\lambda$ at some point $N$ we can then calculated the full inflationary evolution as a function of $N$ and then write down an expression for $V(\phi)$.

### III. MULTIPLE FIELDS

#### A. Background Evolution

We now want to extend the Hubble slow roll expansion to an inflationary model driven by $M$ scalar fields, $\phi_i$. The Friedman equations become

$$H^2 = \frac{8\pi G}{3} \left[ V(\phi) + \sum_{i=0}^{M} \frac{1}{2} \dot{\phi}_i \right]^2$$

(16)

$$\ddot{a} = \frac{8\pi G}{3} \left[ V(\phi) - \sum_{i=0}^{M} \dot{\phi}_i \right]^2.$$  

(17)

The single equation of motion (11) turns into $M$ equations

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \frac{\partial V(\phi)}{\partial \phi_i} = 0,$$

(18)

where an $\phi$ is now being used as a shorthand for all $M \phi_i$. To produce multi field analogs of (5) and (8) we follow the method of (21). Firstly, the time derivative of (16) is

$$2H \frac{dH}{dt} = \frac{8\pi G}{3} \sum_{i=1}^{M} \frac{d\phi_i}{dt} \left( \frac{d^2 \phi_i}{dt^2} + \frac{\partial V(\phi)}{\partial \phi_i} \right),$$

(19)

Note that $\sigma = 2(\lambda) - 4\epsilon$
which, when combined with the corresponding equation of motion [18], produces

\[ 2 \frac{dH}{dt} = -8\pi G \sum_{i=1}^{M} \left( \frac{d\phi_i}{dt} \right)^2. \]  

(20)

In general a time derivative can be changed into a derivative with respect to the fields by

\[ \frac{d}{dt} = \sum_{j=1}^{M} \frac{d\phi_j}{dt} \frac{\partial}{\partial \phi_j}, \]  

(21)

and so

\[ 2 \sum_{j=1}^{M} \frac{\partial H}{\partial \phi_j} \frac{d\phi_j}{dt} = -8\pi G \sum_{i=1}^{M} \left( \frac{d\phi_i}{dt} \right)^2. \]  

(22)

Since the fields are independent, corresponding terms can be set equal,

\[ \dot{\phi}_j = -\frac{1}{4\pi G} \frac{\partial H}{\partial \phi_j}. \]  

(23)

This defines the unique relationship between the temporal derivative of a given field and the derivative of the Hubble parameter with respect to that field.

We use this result to rewrite the equation of motion as [20, 21]

\[ H^2 = \frac{1}{12\pi G} \sum_{i=1}^{M} \left( \frac{\partial H}{\partial \phi_i} \right)^2 + \frac{8\pi G}{3} V. \]  

(24)

or

\[ H^2 \left[ 1 - \frac{1}{H^2} \frac{1}{12\pi G} \sum_{i=1}^{M} \left( \frac{\partial H}{\partial \phi_i} \right)^2 \right] = \frac{8\pi G}{3} V. \]  

(25)

We define a new set of parameters \( \epsilon_i \), such that

\[ \epsilon_i = \sqrt{\frac{1}{4\pi G H} \frac{\partial H}{\partial \phi_i}}. \]  

(26)

In practice we will frequently look at the sum of squares, which can be written as a dot product:

\[ \epsilon = \sum_{i=1}^{M} \epsilon_i \epsilon_i = \sum_{i=1}^{M} \frac{1}{4\pi G H^2} \left( \frac{\partial H}{\partial \phi_i} \right)^2. \]  

(27)

In principle, we should denote this quantity \( \epsilon^2 \) but in practice we use \( \epsilon \) to emphasize the connection with the single field slow roll expansion, which is recovered by setting \( M = 1 \). Substituting (27) into (25) we see

\[ H^2(\phi) \left[ 1 - \frac{1}{3} \epsilon \right] = \frac{8\pi G}{3} V(\phi), \]  

(28)

which is identical in form to the single field analog [14], although \( \phi \) and \( \epsilon \) are now understood to have their multi-component forms. Similarly, we can rewrite (17) as

\[ \frac{\ddot{a}}{a} = H^2 \left[ 1 - \epsilon \right], \]  

(29)

confirming that \( \epsilon < 1 \iff \ddot{a} > 0 \) for the multi field case.
B. The Multi Field Slow Roll Hierarchy

Now that we have a well-defined parameter $\epsilon$, we can build our slow roll hierarchy. We start with

$$\frac{d\epsilon_i}{dN} = \sqrt{\frac{1}{4\pi G}} \left[ \frac{d}{dN} \left( \frac{\partial H}{\partial \phi_i} \right) \frac{1}{H} - \frac{dH}{dN} \frac{\partial H}{\partial \phi_i} \frac{1}{H^2} \right]$$

(30)

or in terms of $\frac{d}{dN} = \sum \frac{d\phi_i}{dN} \frac{\partial}{\partial \phi_i}$ and $\frac{dH}{dN} = \epsilon H$,

$$\frac{d\epsilon_i}{dN} = \sqrt{\frac{1}{4\pi G}} \left[ \sum_{j=1}^{M} \frac{d\phi_j}{dN} \frac{\partial}{\partial \phi_j} \left( \frac{\partial H}{\partial \phi_i} \right) \frac{1}{H} - \epsilon \frac{\partial H}{\partial \phi_i} \frac{1}{H} \right] .$$

(31)

We define our first evolution equation:

$$\frac{d\epsilon_i}{dN} = \left[ \sum_{j=1}^{M} \epsilon_j \lambda_{ij} - \epsilon_i \epsilon \right] ,$$

(32)

where $\lambda_{ij}$ a higher order parameter defined by

$$\lambda_{ij} = \left( \frac{1}{4\pi G} \right) \frac{1}{H} \frac{\partial^2 H}{\partial \phi_i \partial \phi_j} .$$

(33)

Additional parameters are a generalization of (33):

$$m \lambda_{\alpha_0 \ldots \alpha_m} = \left( \frac{1}{4\pi G} \right) \frac{1}{H^m} \left[ \prod_{i=2}^{m} \left( \frac{\partial H}{\partial \phi_i} \right) \right] \frac{\partial^{m+1} H}{\partial \phi_{\alpha_0} \ldots \partial \phi_{\alpha_m}} .$$

(34)

Note that the product in (34) begins at 2 and not 0; the first two indices are not included in this product.

Using these higher order parameters, we can then calculate a system of coupled first order differential equations,

$$\frac{d^m \lambda_{\alpha_0 \ldots \alpha_m}}{dN} = \sum_{i=1}^{M} \frac{d+1}{m} \lambda_{\alpha_0 \ldots \alpha_m} i + m \lambda_{\alpha_0 \ldots \alpha_m} \left( \sum_{i=1}^{M} \sum_{k=2}^{m} \frac{\lambda_{\alpha_k}}{\epsilon_i} \epsilon_i - m \epsilon \right)$$

(35)

where we’ve used the relationship $d\phi_i/dN$ by looking at

$$N = \int_t^{\epsilon} H dt = \int_{\phi_i}^{\phi_i(c)} \frac{H}{\dot{\phi_i}} d\phi_i = \sqrt{4\pi G} \int_{\phi_i(c)}^{\phi_i} \frac{d\phi_i}{\epsilon_i} ,$$

(36)

where the third equality is justified by the definition of $\epsilon_i$ (26) and (29).

We now explore the properties of this hierarchy. First, when $M = 1$ we have a single field and the lower indices become redundant; we drop them by

$$m \lambda_{i \ldots i} = m \lambda .$$

(37)

It is then trivial to check that we then recover the single field flow equations.

These parameters exhibit several useful symmetries. Looking at (34) we see that the $\lambda$ are symmetric in the first two indices and separately symmetric in every other index. We can further reduce the number of independent parameters by recognizing that interchange of one the first two indices, $\alpha_i$, with one of the other indices, $\alpha_j$, is equivalent to multiplying by a factor of $\epsilon_{\alpha_i}/\epsilon_{\alpha_j}$. This is easily verified by a straightforward calculation. Consequently, we find that any two $m \lambda$ can be obtained in terms of one another (and the $\epsilon_i$) if they possess the same set of numerical subscripts as each other. At root, this corresponds to an implicit assumption that $H$ is a well behaved function of the $\phi_i$ and that all mixed derivatives commute. The final ambiguity that arises in the multi field case is that we are always free to rotate or translate the fields in any way, provided that the kinetic terms retain their canonical form. This will necessarily redefine the $\lambda$’s, although the overall physics remains unchanged. This issue does not arise in the single field case, since there is only one field and it cannot be transformed in a nontrivial way without changing the form of the kinetic term.
C. Alternative Parameters

In developing a previous formalism for perturbations in multiple field slow roll inflation, Nibbelink and van Trent introduce the following parameters \[23, 24\]:

\[
\tilde{\epsilon}(\phi) = -\frac{\dot{H}}{H^2},
\]

\[
\tilde{\eta}^{(n)}(\phi) = \frac{\phi^{(n)}}{H^{n-1} \phi},
\]

where we use the ordinary time convention of \[23\] and the notation of \[24\]. We can quickly show that the lowest order parameter used in their expansion is the same as our \(\epsilon\):

\[
\tilde{\epsilon} = \frac{-\dot{H}}{H^2} = \sum_{i=1}^{M} \frac{1}{H} \frac{\partial H}{\partial \phi_i} \frac{\partial \phi_i}{\partial \phi} = \sum_{i=1}^{M} \frac{\epsilon_i}{\sqrt{4\pi G}} \frac{1}{H} \frac{\partial H}{\partial \phi_i} = \sum_{i=1}^{M} \epsilon_i^2 = \epsilon
\]

where our substitutions come from (36). The similarity here allows both treatments to reduce the Friedman equation to its convenient form \(28\).

Our analyses diverge at this point. Nibbelink and van Trent choose to differentiate with respect to a temporal parameter (either \(t\) \[23\] or a more general temporal measure \(\tau\) \[24\]), whereas with reconstruction in mind we choose to define the parameters with respect to derivatives of the field. Since they both refer to the same physical system, the expressions obtained by Nibbelink and van Trent must be equivalent to ours. However, they differ in their relative tractability and in the clear connection between our formalism and the single field slow roll expansion.

IV. EVOLUTION OF PARAMETERS AND RECONSTRUCTION

The infinite hierarchy of parameters introduced in Section III B provides a complete description of the evolution of \(\epsilon = \epsilon_i \epsilon_i\). We now follow the development of the single field analysis and truncate the hierarchy at finite order. As before, we see that if all the \(m^\lambda\) vanish for some \(m\), then all the higher order parameters are also zero. To make this precise, assume there is some \(\kappa\) such that for all \(m > \kappa\)

\[
\left. m+1^\lambda_{\alpha_0...\alpha_{m+1}} \right|_{\phi_i=0} = 0.
\]

The derivatives of these parameters vanish, thanks to \(35\). What remains is a closed system of first order differential equations.

We can now reconstruct the inflationary potential, analogously to the procedure outlined in \[7\]. We begin by inverting \(28\)

\[
V = H^2 \left( 3 \frac{3}{8\pi G} \right) \left[ 1 - \frac{1}{3} \epsilon \right].
\]

This requires knowledge of \(H(N), \epsilon(N),\) and \(\phi(N)\). We can trivially calculate \(\epsilon\) from the flow equations, while to track the evolution of the field one must look only at \(23\)

\[
\frac{d\phi_i}{dN} = -\sqrt{\frac{1}{4\pi G} \epsilon_i}.
\]

Using identities introduced previously, the differential equation for the Hubble parameter is

\[
\frac{dH}{dN} = \sum_{i=1}^{M} \frac{d\phi_i}{dN} \frac{\partial H}{\partial \phi_i} = H \sum_{i=1}^{M} \epsilon_i^2 = H \epsilon
\]

where our final unknown is the initial value of this parameter. For this, we use the same normalization as in \[7\]; the amplitude of density fluctuations is on the order of \(10^{-5}\) so

\[
\frac{\delta \rho}{\rho} = \frac{H}{2\pi} \sqrt{\frac{G}{\epsilon}}.
\]

(45)
From this it might appear that we have reconstructed the full potential. However, in order to put this to use we would need to measure a large number of $\lambda^m$ with considerable precision, so one cannot use this to reconstruct the inflationary dynamics in practice.

In [6] and [7] the single field flow equations were evolved numerically to recover the full inflationary dynamics after the slow-roll parameters were specified at a specific initial time. Subsequently, Liddle pointed out that the flow equations can be solved analytically to produce $H(\phi)$ and $V(\phi)$. This treatment yields the inflationary potential in terms of the $\lambda^m$, as evaluated at some initial time. We now show that a similar truncation works in the multi field case. Assume that $\lambda^m = 0$ for all choices of $\alpha_i$. Via (44) and (45), this corresponds to

$$\frac{\partial^m H}{\partial \phi_{a_0} \ldots \partial \phi_{a_m}} = 0.$$  (46)

The flow equations guarantee that the above equation holds for all values of $\phi_i$. This means that a Taylor expansion of $H$ around the origin of field space will contain only a finite number of terms, and provide an exact solution to the truncated flow equations. We use this knowledge to write $H$ as

$$H = \sum_{i,j,k\ldots} A_{ijk\ldots} \phi_i^j \phi_j^k \ldots.$$  (47)

In (47) the sum runs from $i + j + k + \ldots = 0$ to $i + j + k + \ldots = m$ so that the highest order terms in the expansion have total order $m$.

Given an expression for $H$, we can find the coefficients $A_{ijk\ldots}$ by taking appropriate partial derivatives. For instance, for any number of fields $H$ contains a constant term $A_{000\ldots}$ which can be found by evaluating $H$ at $\phi_i = 0$

$$H|_{\phi_i=0} = A_{000\ldots} = H_0.$$  (48)

The value of $H_0$ was, of course determined by (46).

The higher order co-efficients may be best understood via a specific example. For argument's sake, consider the three field model:

$$2A_{121} = \frac{\partial^3 H}{\partial \phi_1 \partial \phi_2 \partial \phi_3}.$$  (49)

where the prefactor arises from the multiplicity in the partial derivatives. As we know

$$3\lambda_{123} = \left(\frac{1}{4\pi G}\right)^2 \frac{1}{H^2} \frac{\partial H}{\partial \phi_2} \frac{\partial H}{\partial \phi_3} \frac{\partial^3 H}{\partial \phi_1 \partial \phi_2 \partial \phi_3}.$$  (50)

or when we evaluate at the origin, using (49),

$$A_{121} = \frac{(4\pi G)^2}{2} \frac{\lambda_{123}}{\epsilon}.$$  (51)

From these coefficients, we can solve for the explicit form of $\epsilon(\phi_i)$ by taking appropriate partial derivatives of $H(\phi_i)$:

$$\epsilon(\phi_i) = \left(\frac{1}{4\pi G}\right) \frac{1}{H^2} \sum_{k=1}^{M} \left(\frac{\partial H}{\partial \phi_k}\right)^2.$$  (52)

The full analytic form of the potential is a complicated combination of these expansion coefficients (inverting (46))

$$V(\phi) = \frac{3}{8\pi G} H^2(\phi) \left[1 - \frac{1}{3} \epsilon(\phi)\right]$$  (53)

$$= \frac{3}{8\pi G} \left(\sum_{i,j,k\ldots\phi_i^j \phi_j^k \ldots} A_{ijk\ldots} \frac{\phi_i^j \phi_j^k \ldots}{\phi_M^z}\right)^2 \left[1 - \frac{1}{12\pi G} \sum_{k=1}^{M} \left(\frac{\partial}{\partial \phi_k} \left(\frac{\phi_i^j \phi_j^k \ldots}{\phi_M^z}\right)\right)\right]^2.$$  (54)
Equation (55) provides an algorithm for finding an infinite number of exact inflationary solutions. This is simply the multi field generalization of a process that has long been applied to the single field case – there is of course no guarantee that the resulting solutions will correspond to well-motivated potentials. Moreover, as noted previously we can transform the fields in any way that leaves the form of the kinetic terms unchanged. In this case, the explicit form of $H(\phi)$ and $V(\phi)$ will change (since these functions are not invariant under these transformations) although the physics is obviously not affected. Consequently, if one wanted to implement a multi field reconstruction algorithm, one would need to take this ambiguity into account.

V. COSMOLOGICAL PERTURBATIONS

We now turn our attention to the power spectrum of scalar perturbations

$$P_R = \left(\frac{H}{2\pi}\right)^2 \sum_{i=1}^{M} \left(\frac{\partial N}{\partial \phi_i}\right)^2. \quad (56)$$

Consistent with we assume that $N$ is the number of $e$-foldings from a reference point after complete reheating. A full discussion of isocurvature perturbations is beyond the scope of the current paper, and we do not include them here. For a clear treatment discussion of the generation of isocurvature perturbations in multifield inflation see Bassett, Tsujikawa and Wands’ recent review.

This problem has been solved for a very general set of scalar fields which couple via their kinetic terms as well as through their potential. However, since we have assumed canonical kinetic terms from the outset, we can find considerable simplifications to the general expressions – in technical terms we are assuming that the metric on the scalar field space is flat, or that $h_{ab} = \delta_{ab}$ in equation (45) of . The spectral index $n_R$ is given by

$$n_R - 1 = \frac{d\ln P_R}{d\ln k}. \quad (57)$$

or, again by,

$$n_R - 1 = 2 \frac{\dot{H}}{H^2} - 2 \frac{dN}{d\phi} \left( 8\pi G \delta_{\phi_a\phi_b} \frac{\partial^2 V}{\partial \phi_a \partial \phi_b} \right) \frac{1}{8\pi G V}, \quad (58)$$

where we take advantage of summation convention. To evaluate this for our specific models, we use two important substitutions:

$$2 \frac{\dot{H}}{H^2} = \frac{2}{H^2} \left( \frac{\dot{a}}{a} - H^2 \right) \quad (59)$$

and

$$\frac{\partial^2 V}{\partial \phi_b \partial \phi_a} = 3H^2 [\epsilon_a \epsilon_b + \lambda_{ab}] - H^2 \sum_{j=1}^{M} \left[ \lambda_{bj} \lambda_{aj} + 2 \lambda_{abj} \right]. \quad (60)$$

Equation (58) becommes

$$n_R - 1 = -(2 + 4M) \epsilon + \frac{\epsilon}{3} \sum_{a,b=1}^{M} \left( 2 \left[ 1 + \frac{\lambda_{ab}}{\epsilon_a \epsilon_b} \right] - \frac{1}{3} \sum_{j=1}^{M} \left[ \frac{\lambda_{bj} \lambda_{aj} + 2 \lambda_{abj}}{\epsilon_a \epsilon_b} \right] \right). \quad (62)$$

In the single field limit, $m = 1$, we see that, to first order,

$$n_R - 1 = -4 \epsilon + 2 \lambda = \sigma \quad (63)$$

which agrees with the approximation used in .
The second observational parameter of interest is the ratio of the amplitudes of the tensor to scalar perturbations, \( r = T/S \). For this we use the combined results: the power spectrum for gravitational waves can be expressed as

\[
P_g = \left( \frac{H}{2\pi} \right)^2 \tag{64}
\]

and the spectral index for gravitational waves is given by

\[
n_g = 2 \frac{H}{H^2} = -2\epsilon, \tag{65}
\]

where the second equality translates the result into the slow roll parameters introduced in this paper.

The spectra obey a generalized consistency condition

\[
\frac{P_g}{P_R} \leq |n_g| \tag{66}
\]

which generalizes the consistency condition for inflation driven by a single scalar field. This ratio is proportional to the scalar:tensor ratio, \( r \).

### VI. DISCUSSION

In this paper, we have developed the Hubble slow roll formalism for multi field models of inflation. The properties of the multi field expansion are all easily understood as generalizations of their single field counterparts. In particular, we show that there is a hierarchy of differential (“flow”) equations that describe the evolution of the slow roll parameters. If only a finite number of terms in the hierarchy are non-zero at a given instant then the flow equations ensure that this truncation is preserved as the universe evolves.

In principle, one could generate a large set of multi field models and attempt to constrain the slow roll parameters with observational data, as was done for the single field case via Monte Carlo reconstruction. We briefly examined this possibility, but it appears that the larger number of parameters at each order in the multi field slow roll expansion makes it unlikely we can put meaningful constraints on the multi field parameter space for any realistic dataset. Moreover, while we have assumed that our fields have minimal kinetic terms, we can still rotate the fields into one another, preserving the form of the kinetic terms but modifying the functional form of the potential, and any reconstruction algorithm would have to take this ambiguity into account.

While Monte Carlo reconstruction may not be practical in the multi field setting, Liddle, Parsons and Barrow’s systematic treatment of single field inflation in terms of the Hubble slow roll expansion has found a large range of applications in thorough studies of the dynamical system that underlies these models. We consequently expect that this multi field generalization is a theoretical tool with a number of useful applications. We have shown there that there is a strong correspondence between the two systems and that many features of the single field case generalize naturally to their multi field counterpart. In particular, the truncated multi field flow equations can be solved exactly, yielding an expression for the potential analogous to that obtained from Liddle’s exact solution to the single field hierarchy. If one wished to do so, this solution to the flow equations could be used to generate any number of exact multi field inflationary solutions. Moreover, we can write down expressions for the perturbation spectrum using the multi field slow roll parameters, and in future work we plan to use this formalism to examine the perturbation spectrum produced when the inflaton does not follow a single, well-defined path in the multi dimensional inflationary potential.

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