Clustering Categorical Time Series into Unknown Number of Clusters: A Perfect Simulation based Approach

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Abstract

Pamminger and Früwirth-Schnatter (2010) considered a Bayesian approach to model-based clustering of categorical time series assuming a fixed number of clusters. But the popular methods for selecting the number of clusters, for example, the Bayes Information Criterion (BIC), turned out to have severe problems in the categorical time series context.

In this paper, we circumvent the difficulties of choosing the number of clusters by adopting the Bayesian semiparametric mixture model approach introduced by Bhattacharya (2008), who assume that the number of clusters is a random quantity, but is bounded above by a (possibly large) number of clusters. We adopt the perfect simulation approach of Mukhopadhyay and Bhattacharya (2012) for posterior simulation for completely solving the problems of convergence of the underlying Markov chain Monte Carlo (MCMC) approach.

Importantly, within our main perfect simulation algorithm, there arose the necessity to simulate perfectly from the joint distribution of a set of continuous random variables with log-concave full conditional densities. We propose and develop a novel and efficient perfect simulation methodology for joint distributions with log-concave full conditionals. This perfect sampling methodology is of independent interest as well since in a very large and important class of Bayesian applications the full conditionals turn out to be log-concave.

We will consider application of our model and methodology to the Austrian wage mobility data, also analysed by Pamminger and Früwirth-Schnatter (2010), and adopting the methods developed in Mukhopadhyay et al. (2011), Mukhopadhyay et al. (2012), will obtain the posterior modes

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of clusterings and also the desired highest posterior distribution credible regions of the posterior distribution of clusterings. With these summaries of the posterior distribution of clustering we will detail the consequences of ignoring uncertainty in the number of clusters in the approach of Pamminger and Frühwirth-Schnatter (2010).

**Keywords:** Bounding chains; Categorical time series; Dirichlet process; Gibbs sampling; Mixtures; Optimization; Perfect Sampling

## 1 Introduction

We consider the problem of clustering a panel of categorical time series \( y_i; i = 1, \ldots, N \) into several classes (components), assuming that the number of classes is unknown. The known number of components situation has been recently handled by Pamminger and Frühwirth-Schnatter (2010), who consider a Bayesian mixture model based approach with a fixed number of components. However, the authors reported serious difficulties in reliably determining the appropriate number of components using the traditional approaches like Bayes Information Criterion (BIC).

We completely avoid the difficulties of the fixed components approach by adopting the assuming that the number of components is unknown, but is bounded above by a number specified by experts, the upper bound signifying that the number of possible clusters of the time series can not exceed the specified upper limit. Such a model has been proposed by Bhattacharya (2008); see also Mukhopadhyay et al. (2012) and Mukhopadhyay et al. (2011). We develop a perfect simulation method for sampling exactly from the underlying posterior distribution. Perfect simulation for mixtures with unknown number of components has been developed by Mukhopadhyay and Bhattacharya (2012), but in the time series context there are some additional complications, to be explained in due course.

Indeed, these additional difficulties led us to develop a general perfect simulation methodology in the case of joint distributions with log-concave full conditional distributions, which is of independent interest.

With our new developments related to perfect sampling, we then proceed to analyze the Austrian wage mobility data, obtaining the modes of the posterior distribution of clustering as well as the desired highest posterior distribution credible regions, using the methods detailed in Mukhopadhyay et al. (2011) and Mukhopadhyay et al. (2012). In particular, we demonstrate that ignoring uncertainty in the number of clusters in the approach of Pamminger and Frühwirth-Schnatter (2010) seriously affects inference.

The rest of our paper is structured as follows. In Section 2, adopting the mixture model of Bhattacharya
we model the time series as mixtures of unknown number of components, and in Section 3 provide the full conditional distributions to be used for perfect simulation, along with the need for perfect simulation from joint distributions of continuous parameters with log-concave full conditionals in our problem. Some more details are presented in the supplement, the sections of which we refer to by using the prefix “S-”. In Section 4 we introduce our perfect simulation idea in the case of continuous joint distributions having log-concave full conditionals. Using this development, and adopting the perfect simulation idea for mixtures with unknown number of components proposed by Mukhopadhyay and Bhattacharya (2012) we then present the relevant perfect simulation methodology for our categorical time series problem.

2 Mixtures of categorical time series with unknown number of components

In this work we confine ourselves to mixtures of categorical time series with Markov chain clustering, which has also been the main aspect of study in Pamminger and Frühwirth-Schnatter (2010), albeit the latter consider only fixed number of components. In what follows we shall borrow some notation already described in Pamminger and Frühwirth-Schnatter (2010).

We consider the mixture model of the following form: for $i = 1, \ldots, N$,

$$f(y_i \mid \Theta) = \frac{1}{M} \sum_{h=1}^{M} \prod_{r=1}^{T_i} f(y_{ir} \mid y_{i,r-1}, \theta_h) = \frac{1}{M} \sum_{h=1}^{M} \prod_{s=1}^{K} \prod_{t=1}^{K} \theta_{h,st}^{N_{i,st}}, \quad (1)$$

where $N_{i,st} = \# \{ y_{ir} = t, y_{i,r-1} = s \}$ is the number of transitions from state $s$ to state $t$ observed in time series $i$, and, for each $h \in \{1, \ldots, M\}$, $\theta_h = (\theta_{h,st})$; $s, t = 1, \ldots, K$, is the transition matrix of the underlying Markov chain model consisting of $K$ states. The latter satisfies $\sum_{t=1}^{K} \theta_{h,st} = 1 \quad \forall h, s$. In (1), $M$ is the maximum number of components, specified, perhaps, by some expert; however, Mukhopadhyay and Bhattacharya (2013) show how $M$ can be obtained objectively and optimally from a Bayesian asymptotics perspective.

We next consider the following Dirichlet process (DP) prior for $\Theta$: for $h = 1, \ldots, M$,

$$\theta_h \overset{iid}{\sim} G \quad (2)$$

$$G \sim DP(\alpha G_0) \quad (3)$$

\[3\]
Under $G_0$, for $h = 1, \ldots, M$ and $s = 1, \ldots, K$,
\[
(\theta_{h,s1}, \ldots, \theta_{h,sK}) \sim Dirichlet(\gamma_{s1}, \ldots, \gamma_{sK})
\]
(4)

In addition, we assume that
\[
\gamma_{st} \sim Gamma(a_{st}, b_{st}); \quad s = 1, \ldots, K; \quad t = 1, \ldots, K,
\]
(5)

where $Gamma(a, b)$ denotes the gamma distribution of the form $\frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx)$, having mean $a/b$ and variance $a/b^2$. We remark that Pamminger and Frühwirth-Schnatter (2010) assumed a discrete prior distribution on $\{\gamma_{st}; s, t = 1, \ldots, K\}$, namely, the negative multinomial distribution. However, continuous priors, such as ours, are perhaps more appropriate and more natural.

The Dirichlet process prior assumption entails that mixture model (1) reduces to the following form:
\[
f(y_i \mid \Theta_M) = p \sum_{\ell=1}^p \pi_{\ell} T_i \prod_{r=1} f(y_{ir} \mid y_{i,r-1}, \phi_{\ell}) = p \sum_{\ell=1}^p \prod_{s=1}^K \prod_{t=1}^K \phi_{N_{i,st}}^{N_{i,st}},
\]
(6)

where $\phi_{\ell}$ denotes the $\ell$-th distinct component among $\Theta_M = \{\theta_1, \ldots, \theta_M\}$, and $\pi_{\ell} = M_{\ell}/M$, with $M_{\ell} = \#\{h : \theta_h = \phi_{\ell}\}$. In (6), $p$ ($1 \leq p \leq M$) denotes the random number of distinct mixture components.

### 3 Full conditional distributions

Let $Y = \{y_1, \ldots, y_n\}$ denote the data set. We define the set of allocation variables $Z = \{z_1, \ldots, z_n\}$, where $z_i = j$ if $y_i$ arises from the $j$-th component. Letting $\Phi = \{\phi_1, \ldots, \phi_k\}$ denote the distinct components in $\Theta_M$, the element $c_j$ of the configuration vector $C = \{c_1, \ldots, c_M\}$ is defined as $c_j = \ell$ if and only if $\theta_j = \phi_{\ell}; j = 1, \ldots, M, \ell = 1, \ldots, k$. Thus, $(Z, \Theta_M)$ is reparameterized to $(Z, C, k, \Phi)$, $k$ denoting the number of distinct components in $\Theta_M$.

#### 3.1 Full conditionals of $\{z_1, \ldots, z_n\}$

For $i = 1, \ldots, n$, let $Z_{-i} = \{z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\}$, and let $C$ consist of $k$ distinct components. Then, denoting the set $\{\gamma_{st}; s, t = 1, \ldots, K\}$ by $\gamma$, the full conditional distribution of $z_i$ is given by
\[
[z_i = r \mid Y, Z_{-i}, C, \Phi, \gamma, k] \propto \prod_{s=1}^K \prod_{t=1}^K \theta_{r_{i,st}}^{N_{i,st}}
\]
(7)
3.2 Full conditionals of \( \{c_1, \ldots, c_M\} \)

To obtain the full conditional of \( c_r; r = 1, \ldots, M \), first let \( k_r \) denote the number of distinct values in \( \Theta_{-rM} = \{\theta_1, \ldots, \theta_{r-1}, \theta_{r+1}, \ldots, \theta_M\} \), and let \( \phi_{(r^*)}^{(1)} \); \( \ell = 1, \ldots, k_r \) denote the distinct values. Also suppose that \( \phi_{(r^*)}^{(1)} \) occurs \( M_{tr} \) times. Then the conditional distribution of \( c_r \) is given by

\[
[c_r = \ell \mid Y, Z, C_{-r}, \Phi, \gamma, k_r] = \begin{cases} 
\kappa_{q_{tr}} & \text{if } \ell = 1, \ldots, k_r \\
\kappa_{q_{0r}} & \text{if } \ell = k_r + 1
\end{cases}
\]  

where

\[
q_{tr} \propto M_{tr} \times \prod_{s=1}^{K} \prod_{t=1}^{K} \phi_{t,s}^{(1)} N_i,s_t^{(1)}
\]

and

\[
q_{0r} \propto \propto \prod_{s=1}^{K} \frac{\Gamma\left(\sum_{t=1}^{K} \gamma_{st}\right)}{\prod_{t=1}^{K} \Gamma\left(\gamma_{st}\right)} \times \prod_{s=1}^{K} \frac{\prod_{t=1}^{K} \Gamma\left(\sum_{i:j = r} N_i,s_t + \gamma_{st}\right)}{\prod_{t=1}^{K} \Gamma\left(\sum_{i:j = r} N_i,s_t + \sum_{t=1}^{K} \gamma_{st}\right)}
\]

3.3 Full conditionals of \( \{\phi_{\ell}; \ell = 1, \ldots, k\} \)

Assuming that there are \( k \) distinct components in \( C \), the full conditional distribution of \( \phi_{\ell}; \ell = 1, \ldots, k \), is given by

\[
[\phi_{\ell} \mid Y, Z, C, \Phi_{-\ell}, \gamma, k] = \prod_{s=1}^{K} \prod_{t=1}^{K} \phi_{t,s}^{(1)} \frac{\sum_{i:j = r} N_i,s_t + \gamma_{st} - 1}{\prod_{t=1}^{K} \Gamma\left(\sum_{i:j = r} N_i,s_t + \sum_{t=1}^{K} \gamma_{st}\right)}
\]

which are conditionally independent of \( \Phi_{-\ell} \).

The conditional mean and variance of \( \phi_{\ell,s^*t^*} \) are given, respectively, by

\[
\zeta_{\ell,s^*t^*} = E[\phi_{\ell,s^*t^*} \mid Y, Z, C, \gamma, k] = \frac{\gamma_{s^*t^*} + \sum_{i:j = r} N_i,s^*t^*}{\sum_{t=1}^{K} \left(\gamma_{s^*t} + \sum_{i:j = r} N_i,s^*t\right)},
\]

and

\[
\varphi_{\ell,s^*t^*} = Var[\phi_{\ell,s^*t^*} \mid Y, Z, C, \gamma, k] = \frac{\left(\gamma_{s^*t^*} + \sum_{i:j = r} N_i,s^*t^*\right)}{\left\{\sum_{t=1}^{K} \left(\gamma_{s^*t} + \sum_{i:j = r} N_i,s^*t\right)\right\}^2} \left\{1 + \sum_{t=1}^{K} \frac{\left(\gamma_{s^*t} + \sum_{i:j = r} N_i,s^*t\right)}{\left\{\sum_{t=1}^{K} \left(\gamma_{s^*t} + \sum_{i:j = r} N_i,s^*t\right)\right\}}\right\}.
\]
It follows that
\[
\frac{\varphi_{\ell,s^*t^*}}{\zeta_{\ell,s^*t^*} (1 - \zeta_{\ell,s^*t^*})} = \frac{1}{1 + \sum_{t=1}^{K} (\gamma_{s^*t} + \sum_{i:z_i = j; c_j = \ell} N_{i,s^*t})}.
\] (14)

Accordingly, as in Pamminger and Frühwirth-Schnatter (2010), but somewhat differently, we can interpret \(\Sigma_{\ell,s^*} = \sum_{t=1}^{K} (\gamma_{s^*t} + \sum_{i:z_i = j; c_j = \ell} N_{i,s^*t})\) as a *conditional* measure of heterogeneity in the corresponding rows of \(\varphi_{\ell}\) of the \(\ell\)-th cluster. Small values of \(\Sigma_{\ell,s^*}\) implies high degree of variability of the individual transition probabilities \(\varphi_{\ell,s^*t}\) and large deviations of \(\varphi_{\ell,s^*t} = (\varphi_{\ell,s^*1}, \ldots, \varphi_{\ell,s^*K})\) from the group mean \(\zeta_{\ell,s^*} = (\zeta_{\ell,s^*1}, \ldots, \zeta_{\ell,s^*K})\). Large values of \(\Sigma_{\ell,s^*}\) indicate small variability in the \(s^*\)-th row, implying that the individual transition probabilities \(\varphi_{\ell,s^*t}\) are nearly the same as as the group means \(\zeta_{\ell,s^*t}\).

Interestingly, for the purpose of perfect simulation, the full or marginal conditional distributions of \(\varphi_{\ell,s^*t}\), given below, will be shown to be more important than those of \(\varphi_{\ell}\), even though the latter is just the standard Dirichlet distribution and straightforward to simulate from.

### 3.3.1 Full and marginal conditionals of \(\varphi_{\ell,s^*t}\)

The full conditional of \(\varphi_{\ell,s^*t}\) is given by
\[
[\varphi_{\ell,s^*t^*} \mid Y, Z, C, \Phi_{-\ell,-s^*,-t^*}, \gamma, k] 
\propto \varphi_{\ell,s^*t}^{\sum_{i:z_i = j} \sum_{j:z_j = \ell} N_{i,s^*t} + \gamma_{s^*t^*} - 1} 
\times \left(1 - \sum_{t=1}^{K} \varphi_{\ell,s^*t}\right)^{\sum_{i:z_i = j} \sum_{j:z_j = \ell} N_{i,s^*K} + \gamma_{s^*K} - 1},
\] (15)

In the above, \(\Phi_{-\ell,-s^*,-t^*}\) denotes \(\Phi\) without \(\varphi_{\ell,s^*t^*}\).

The marginal conditional of \(\varphi_{\ell,s^*t}\) is given by
\[
[\varphi_{\ell,s^*t^*} \mid Y, Z, C, \Phi_{-\ell,-s^*,-t^*}, \gamma, k] 
\propto \varphi_{\ell,s^*t}^{\sum_{i:z_i = j} \sum_{j:z_j = \ell} N_{i,s^*t} + \gamma_{s^*t^*} - 1} 
\times \left(1 - \varphi_{\ell,s^*t}\right)^{\sum_{t\neq t^*} \varphi_{\ell,s^*t}} \left(\sum_{i:z_i = j; c_j = \ell} N_{i,s^*t} + \gamma_{s^*t}\right)^{-1},
\] (16)

which is a *Beta* distribution with parameters \(\sum_{i:z_i = j; c_j = \ell} N_{i,s^*t} + \gamma_{s^*t}\) and \(\sum_{t\neq t^*} \left(\sum_{i:z_i = j; c_j = \ell} N_{i,s^*t} + \gamma_{s^*t}\right)\).
3.4 Full conditionals of \( \{\gamma_{st}; s, t = 1, \ldots, K\} \)

Assuming that \( C \) consists of \( k \) distinct components, the full conditional distribution of \( \gamma_{s^*t^*} \), for \( s^* = 1, \ldots, K \), and \( t^* = 1, \ldots, K \), is given by

\[
[\gamma_{\ell,s^*t^*} \mid Y, Z, C, \Phi, \gamma_{-s^*,-t^*}, k] \propto \left( \prod_{\ell=1}^{k} \phi_{\ell,s^*t^*}^{-1} \right) \times \left( \frac{\Gamma \left( \sum_{t=1}^{K} \gamma_{s^*t} \right)}{\Gamma (\gamma_{s^*t^*})} \right)^k \times \gamma_{s^*t^*}^{a_{s^*t^*}-1} \exp \left( -b_{s^*t^*} \gamma_{s^*t^*} \right) (17)
\]

In the above, \( \gamma_{-s^*,-t^*} \) denotes all elements of the \( \gamma \)-parameters except \( \gamma_{s^*t^*} \).

3.5 Relabeling \( C \)

Simulation of \( C \) by successively simulating from the full conditional distributions (8) incurs a labeling problem. For instance, it is possible that all \( c_j \) are equal even though each of them corresponds to a distinct \( \theta_j \). For an example, suppose that \( \Phi \) consists of \( M \) distinct elements, and \( c_j = M \forall j \). Then although there are actually \( M \) distinct components, one ends up obtaining just one distinct component. For perfect sampling Mukhopadhyay and Bhattacharya (2012) created a labeling method which relabels \( C \) such that the relabeled version, denoted by \( S = (s_1, \ldots, s_M)' \), coalesces if \( C \) coalesces. To construct \( S \) we first simulate \( c_j \) from (8); if \( c_j \in \{1, \ldots, k_j\} \), then we set \( \theta_j = \phi_{c_j} \) and if \( c_j = k_j + 1 \), we draw \( \theta_j = \phi_c \sim G_j \). The elements of \( S \) are obtained from the following definition of \( s_j \): \( s_j = \ell \) if and only if \( \theta_j = \phi_{\ell} \). Note that \( s_1 = 1 \) and \( 1 \leq s_j \leq s_{j-1} + 1 \). Mukhopadhyay and Bhattacharya (2012) proved that coalescence of \( C \) implies the coalescence of \( S \), irrespective of the value of \( \Phi \).

3.6 Full conditionals using \( S \)

With the introduction of \( S \) it is now required to modify some of the full conditionals of the unknown random variables, in addition to introduction of the full conditional distribution of \( S \). The form of the full conditional \( [z_i \mid Y, S, k, \Phi, \gamma] \) remains the same as (7), but \( \Theta_M \) involved in the right hand side of (7) is now obtained from \( S \) and \( \Phi \). The modified full conditional of \( c_j \), which we denote by \( [c_j \mid Y, Z, S_{-j}, k_j, \Phi] \), now depends upon \( S_{-j} \), rather than \( C_{-j} \), the notation being clear from the context. The form of this full conditional remains the same as (8) but now the distinct components \( \phi_j^{s^*} ; \ell = 1, \ldots, k_j \) are associated with the corresponding components of \( S \) rather than \( C \). The form of the modified full
conditional distribution of $\phi_\ell$, which we now denote by $[\phi_\ell | Y, Z, S, k]$, remains the same as (11), only $C$ must be replaced with $S$. Also $k$ and $k_j$ are now assumed to be associated with $S$. The conditional posterior $[S | Y, C, \Phi, \gamma, k]$ gives point mass to $S^*$, where $S^* = \{s^*_1, \ldots, s^*_M\}$ is the relabeling obtained from $C$ and $\Theta_M$ following the method described in Section 3.5.

For the construction of bounds, the individual full conditionals $[s_j | Y, S_{-j}, C, \Phi, \gamma, k]$, giving full mass to $s^*_j$, will be considered due to convenience of dealing with distribution functions of one variable. It follows that once $Z$ and $C$ coalesce, $S$ and $\Phi$ must also coalesce. In the next section we describe how to construct efficient bounding chains for $Z$, $C$ and $S$. Bounding chains for $S$ are not strictly necessary as it is possible to optimize the bounds for $Z$ and $C$ with respect to $S$, but the efficiency of the other bounding chains is improved, leading to an improved perfect sampling algorithm, if we also construct bounding chains for $S$.

### 3.7 Need for perfect simulation of $(\Phi, \gamma)$ given the rest

The perfect sampling methodology for mixtures of unknown number of components developed in Mukhopadhyay and (2012) can be envisaged for simulating exactly from the posterior in this categorical time series problem. Their method requires simulation of the discrete parameters $(Z, C, S)$ only and not the continuous parameters $\Phi$ and $\gamma$ until coalescence of the discrete parameters. Simulation of the continuous parameters is necessary only after the discrete parameters have coalesced. In our example, however, simulation of $\Phi$ and $\gamma$ given $Z$ and $C$, even after coalescence of the latter, is not straightforward. This is because there does not seem to exist any method of directly simulating from the (joint) full conditional of $(\Phi, \gamma)$ and so it is required to simulate from the component-wise full conditionals of $\phi_\ell$ given $\gamma$, and from the (non-standard) component-wise full conditionals of $\gamma_{st}$, given $Z$, $S$, and $\gamma_{-s,-t}$ and $\Phi$, and although the initial values of $Z$ and $S$ are the coalesced values of the respective bounding chains, the initial values of $\gamma$ for simulating $\phi_\ell$ or the initial values of $\Phi$ and $\gamma_{-s,-t}$ for simulating $\gamma_{st}$, are not available. The non-availability of starting values is due to the fact that before coalescence of $(Z, C, S)$, $\Phi$ and $\gamma$ are not simulated at all.

The above problem calls for the need for perfect simulation of $\Phi$ and $\gamma$ given $(Z, S)$, using the full conditionals of $\Phi_\ell$ and $\gamma_{st}$, given by (11) and (17), respectively. Thus, our main perfect simulation methodology must proceed via incorporation of another perfect sampling method involving the full conditionals of $\gamma_{st}$. But Gibbs sampling-based perfect simulation in the case of continuous pa-
rameters is not developed in the literature. In this paper, we propose and develop a novel and general perfect simulation methodology using full conditional distributions of continuous parameters. All we require is that the full conditionals are log-concave. We then specialize our general methodology to the problem of perfectly simulating $\Phi$ and $\gamma$ given $Z, S$ within the perfect sampling methodology of Mukhopadhyay and Bhattacharya (2012). However, as mentioned already, for perfect sampling, we shall need to utilize the full conditional of $\phi_{\ell, st}$, given by (15) rather than that of $\phi_{\ell}$, given by (11). Indeed, it is easy to see that the full conditionals of $\phi_{\ell, st}$ and $\gamma_{\ell, st}$ satisfy
\[
\frac{d^2}{d\phi_{\ell, st}^2} \left[ \phi_{\ell, st} \mid Y, Z, S, \Phi_{-\ell, -s, -t}, \gamma, k \right] < 0, \quad (18)
\]
\[
\frac{d^2}{d\gamma_{st}^2} \left[ \gamma_{st} \mid Y, Z, S, \Phi, \gamma_{-s, -t}, k \right] < 0, \quad (19)
\]
provided that $\sum_{i: z_i = j} \sum_{j: c_j = \ell} N_{i, st} + \gamma_{st} > 1$ and $a_{st} > 1$. The proof of (18) follows by simple differentiation, while the proof of (19) also requires the formula (see Bowman and Shenton (1988)):
\[
\frac{d^2}{dx^2} \log [\Gamma(x)] = \frac{1}{x} + \frac{1}{2x^2} + \frac{2\pi}{x} \int_0^\infty \frac{y \sqrt{t}}{(x^2 + t)(y - 1)^2} dt, \quad (20)
\]
where $y = \exp(2\pi \sqrt{t})$. Using the above formula the proof of (19) follows in similar lines as the proof of Proposition 2 of He and Sun (1998). Note that although it is possible to integrate out $\Phi$ thanks to conjugacy and obtain the marginalized full conditionals of $(Z, C, \gamma)$ (see Section S-6 of the supplement), it can be easily verified that the resulting expression for the full conditional of $\gamma_{st}$ need not admit log-concavity; see Section S-6.4 of the supplement for details. This lack of log-concavity makes it difficult to generate perfect samples from the full conditional of $\gamma$.

4 Perfect simulation in posteriors with log-concave full conditionals

Before introducing our perfect simulation idea in Gibbs sampling for continuous, log-concave, full conditionals, we first provide a brief description of adaptive rejection sampling (ARS) following Gilks and Wild (1992).
4.1 Overview of ARS

Assuming that it is required to sample from a log-concave density \( g(\cdot) \) with \( g(\cdot) \) continuous and differentiable everywhere on a set \( D \), let us suppose that \( h(x) = \log g(x) \) and \( h'(x) \), the first differential of \( h(\cdot) \) has been evaluated at \( m \) abscissae in \( D \): \( x_1 \leq x_2 \leq \cdots \leq x_m \). For \( j = 1, \ldots, m - 1 \), define

\[
v_j = \frac{h(x_{j+1}) - h(x_j) - x_{j+1}h'(x_{j+1}) + x_jh'(x_j)}{h'(x_j) - h'(x_{j+1})}
\]

(21)

For \( x \in [v_{j-1}, v_j] \); \( j = 1, \ldots, m \), define

\[
u_m(x) = h(x_j) + (x - x_j)h'(x_j),
\]

(22)

Here \( v_0 \) is the lower bound of \( D \) (or \(-\infty \) if \( D \) is not bounded below) and \( v_m \) is the upper bound of \( D \) (or \( \infty \) if \( D \) is not bounded above). Also define

\[
s_m(x) = \frac{\exp\{u_m(x)\}}{\int_D \exp\{u_m(x')\}dx'}
\]

(23)

Also define, for \( x \in [x_j, x_{j+1}] \); \( j = 1, \ldots, m - 1 \),

\[
\ell_m(x) = \frac{(x_{j+1} - x)h(x_j) + (x - x_j)h(x_{j+1})}{x_{j+1} - x_j},
\]

(24)

and for \( x < x_1 \) or \( x > x_m \), \( \ell_m(x) = -\infty \). Thus, for all \( x \in D \), we have, due to concavity of \( h(\cdot) \),

\[
\ell_m(x) \leq h(x) \leq u_m(x)
\]

(25)

To sample using ARS, draw \( x^* \sim s_m \) and \( w \sim \text{Uniform}(0, 1) \) independently and accept \( x^* \) if \( w \leq \exp\{\ell_m(x^*) - u_m(x^*)\} \). Else accept \( x^* \) if \( w \leq \exp\{h(x^*) - u_m(x^*)\} \). If the sampling is to be continued then the accepted values may be included in the set of abscissae (the latter to be re-arranged in ascending order) to adaptively make the bounds (25) finer and finer; this enhances efficiency as the sampling progresses.

For our purpose of perfect sampling using the log-concave full conditionals we shall need to represent the Gibbs transition kernel in a special form using the lower bound of the form given in (25), while using ARS in conjunction for sampling. We introduce our perfect sampling idea in the next section.

4.2 Construction of perfect simulation methodology in posteriors with log-concave full conditionals

For the sake of generality, we consider full conditionals of the form \( \pi_i(\xi_i) = \pi(\xi_i \mid \xi_{-i}); i = 1, \ldots, p \), where it is necessary to simulate perfectly from the joint distribution of \( \xi = \{\xi_1, \ldots, \xi_p\} \); here \( \xi_{-i} = \{\xi_{\ldots, i, \ldots} \mid \xi_i = \)
\( \xi \setminus \xi_i \). We assume that each \( \pi_i(\xi_i) \) is log-concave. It then follows from (25) that

\[
\pi_i(\xi) \geq \exp\{\ell_{m,\xi-i}(\xi)\},
\]

where \( \ell_{m,\xi-i}(\cdot) \) may depend upon \( \xi_{-i} \). Taking the infimum over \( \xi_{-i} \) yields

\[
\pi_i(\xi) \geq \exp\{\ell_{m,\xi-i}(\xi)\} \geq \inf_{\xi_{-i}} \exp\{\ell_{m,\xi-i}(\xi)\} = \exp\{\ell_{m,i}(\xi)\},
\]

where \( \exp\{\ell_{m,i}(\xi)\} = \inf_{\xi_{-i}} \exp\{\ell_{m,\xi-i}(\xi)\} \) is independent of \( \xi_{-i} \). However, the right hand side of (27) need not be a density in that it need not integrate to one. Firstly, finiteness of the integral can be ensured at least if \( \xi \) is restricted to a compact set. That restriction of the support of the parameters to some judiciously constructed compact set is not unrealistic is discussed in detail in Mukhopadhyay and Bhattacharya (2012). Let \( D_i \) denote a compact interval to which \( \xi_i \) is restricted. Let \( \epsilon_i = \int_{D_i} \exp\{\ell_{m,i}(\xi)\} \), and let \( g_{m,i}(\xi) = \epsilon_i^{-1} \exp\{\ell_{m,i}(\xi)\} \) denote the density corresponding to \( \exp\{\ell_{m,i}(\xi)\} \). Then, we have, for each \( i = 1, \ldots, p \),

\[
\pi_i(\xi) \geq \epsilon_i g_{m,i}(\xi),
\]

which implies that

\[
\prod_{i=1}^{p} \pi_i(\xi_i) \geq \prod_{i=1}^{p} \epsilon_i g_{m,i}(\xi_i) = \epsilon g_m(\xi),
\]

where \( \epsilon = \prod_{i=1}^{p} \epsilon_i \) and \( g_m(\xi) = \prod_{i=1}^{p} g_{m,i}(\xi_i) \). That \( 0 < \epsilon \leq 1 \) is clear since for each \( i = 1, \ldots, p \), \( 0 < \epsilon_i \leq 1 \), the latter following by integrating both sides of (28) over the support \( D_i \).

This then implies that the Gibbs transition kernel, given by

\[
P(\xi^{(t+1)} | \xi^{(t)}) = \prod_{i=1}^{p} \pi_i(\xi_i^{(t+1)} | \xi_i^{(t)}, \ldots, \xi_{i-1}^{(t+1)}, \xi_{i+1}^{(t+1)}, \ldots, \xi_p^{(t)}),
\]

can be represented as

\[
P(\xi^{(t+1)} | \xi^{(t)}) = \epsilon g_m(\xi^{(t+1)}) + (1 - \epsilon) R_m(\xi^{(t+1)} | \xi^{(t)}),
\]

where

\[
R_m(\xi^{(t+1)} | \xi^{(t)}) = \frac{P(\xi^{(t+1)} | \xi^{(t)}) - \epsilon g_m(\xi^{(t+1)})}{1 - \epsilon}
\]

is the residual density.

Hence, in order to simulate the Gibbs chain \( P(\xi^{(t+1)} | \xi^{(t)}) \) one can first draw \( \delta^{(t+1)} \sim \text{Bernoulli}(\epsilon) \); if \( \delta^{(t+1)} = 1 \), then \( \xi^{(t+1)} \) is drawn from \( g_m(\cdot) \), and if \( \delta^{(t+1)} = 0 \), \( \xi^{(t+1)} \sim R_m(\cdot | \xi^{(t)}) \). Thus, if \( \delta^{(t+1)} = 1 \),
then $\xi^{(t+1)}$ is drawn from $g_m(\cdot)$ which does not depend upon the previous iteration $\xi^{(t)}$. We shall exploit this fact for our perfect sampling algorithm. Indeed, we constructed the mixture representation just to achieve this independence of $\xi^{(t)}$ which happens with positive probability $\epsilon$. This implies that once $\delta^{(t)} = 1$ for some $t < 0$ in the associated coupling from the past algorithm (CFTP), all possible chains initialised at all possible values of the state-space, will coalesce!

But since drawing directly from $R_m(\cdot \mid \xi^{(t)})$ (necessary when $\delta^{(t)} = 0$) is not straightforward, we consider a rejection sampling method which we now describe. Note that

$$R_m(\xi^{(t+1)} \mid \xi^{(t)}) = \frac{P(\xi^{(t+1)} \mid \xi^{(t)}) - \epsilon g_m(\xi^{(t+1)})}{1 - \epsilon}$$  \hspace{1cm} (33)

$$\leq \frac{P(\xi^{(t+1)} \mid \xi^{(t)})}{1 - \epsilon}$$  \hspace{1cm} (34)

Hence, we consider the rejection sampling scheme provided in Algorithm 4.1.

**Algorithm 4.1**  

Rejection sampling from $R_m(\cdot \mid \xi^{(t)})$

1. Draw $\xi \sim P(\cdot \mid \xi^{(t)})$ by sampling from the full conditionals, and independently draw $U \sim \text{Uniform}(0,1)$.

2. Accept $\xi$ as a realization from $R_m(\cdot \mid \xi^{(t)})$ if 

$$U < (1 - \epsilon) \frac{R_m(\xi \mid \xi^{(t)})}{P(\xi \mid \xi^{(t)})}.$$  

Note that sampling from $P(\cdot \mid \xi^{(t)})$ may require ARS from the full conditionals. To avoid ARS one may further upper bound $P(\xi \mid \xi^{(t)})$ using the upper bounds available for log-concave densities as follows.

$$P(\xi \mid \xi^{(t)}) \leq \eta f_m(\xi),$$  \hspace{1cm} (35)

where $\eta = \prod_{i=1}^{p} \eta_i$, $\eta_i = \int_{D_i} \exp\{u_{m,i}(\xi)\} d\xi$, $u_{m,i}(\xi) = \sup_{\xi_{-i}} u_{m,\xi_{-i}}(\xi)$, and

$$f_m(\xi) = \prod_{i=1}^{p} f_{m,i}(\xi_i),$$  \hspace{1cm} (36)

with $f_{m,i}(\xi) = \eta_i^{-1} \exp\{u_{m,i}(\xi)\}$. Since we also have the lower bound $P(\xi \mid \xi^{(t)}) \geq \epsilon g_m(\xi)$, the following rejection sampling method given by Algorithm (4.2) can be employed to sample from $P(\cdot \mid \xi^{(t)})$. 

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Algorithm 4.2 Rejection sampling from $P(\cdot \mid \xi^{(t)})$

1. Draw $\xi \sim f_m(\cdot)$, and independently draw $U \sim \text{Uniform}(0,1)$.

2. Accept $\xi$ as a realization from $P(\cdot \mid \xi^{(t)})$ if
   $$U < \frac{\epsilon g_m(\xi)}{\eta f_m(\xi)}.$$

3. Else accept $\xi$ as a realization from $P(\cdot \mid \xi^{(t)})$ if
   $$U < \frac{P(\xi \mid \xi^{(t)})}{\eta f_m(\xi)}.$$

It is important to remark that whenever simulation from $P(\cdot \mid \xi^{(t)})$ is straightforward, that is, whenever the full conditionals $\pi_i(\xi_i)$ are of standard forms, rejection sampling or ARS will not be used for sampling from the Gibbs kernel.

Our mixture Gibbs kernel (31) resembles that associated with the “multigamma coupler” of Murdoch and Green (1998), but the latter is a representation of one-dimensional cases only. Moreover, such mixture representation is very rarely achievable in reality for densities that are not log-concave. Perfect simulation of high-dimensional variables, using the one-dimensional multigamma coupler for each univariate full conditional density, is possible in principle, but is likely to be extremely inefficient, particularly as the dimension of the random variable tends to be large. As is evident from our construction, we completely bypass such difficulties by representing the Gibbs kernel of the entire high-dimensional random variable $\xi$ as a mixture of two (high-dimensional) densities, obtained using properties of log-concavity of the full conditionals. We have also shown how to sample from the two high-dimensional densities. In particular, we have provided an explicit rejection sampling method for simulating from the residual density of the mixture representation, whatever the dimensionality. We remark that explicit methods of simulating from the residual density has not been provided in Murdoch and Green (1998) or Green and Murdoch (1999). Although Mykland et al. (1995) proposed a trick to completely avoid simulation from the residual density in the context of regenerative simulation, such trick is not applicable in perfect simulation.

For perfect simulation we exploit the following idea first presented in Murdoch and Green (1998). Note that there is a fixed probability $\epsilon$ that at any given time $T = t$, $\xi$ will be drawn from $g_m(\cdot)$. Hence, $T$ follows a geometric distribution given by $P(T = t) = \epsilon(1-\epsilon)^t$; $t = 0, 1, 2, \ldots$. As a result, it is possible
to simulate $T$ from the geometric distribution and then draw $\xi^{(T)} \sim g_m(\cdot)$. Then the chain only need to be carried forward in time till time $t = 0$, using $\xi^{(t+1)} = \psi(\xi^{(t)}; U^{(t+1)})$, where $\psi(\xi^{(t)}; U^{(t+1)})$ is the deterministic function corresponding to the simulation of $\xi^{(t+1)}$ from $R_m(\cdot \mid \xi^{(t)})$ using the set of appropriate random numbers $U^{(t+1)}$; the sequence $\{U^{(t)}; t = 0, -1, -2, \ldots \}$ being assumed to be available before beginning the perfect sampling simulation. The resulting draw $\xi^{(0)}$ sampled at time $t = 0$ is a perfect sample from $\pi(\xi)$. For subsequent reference we present this in an algorithmic way in Algorithm 4.3.

**Algorithm 4.3** Perfect simulation from $\pi(\xi)$

1. Draw $T \sim \text{Geometric}(\epsilon)$.
2. Draw $\xi^{(T)} \sim g_m(\cdot)$.
3. Carry the chain forward till time $t = 0$ using the deterministic functional relationship $\xi^{(t+1)} = \psi(\xi^{(t)}; U^{(t+1)})$ and the available sequence $\{U^{(t)}; t = 0, 1, 2, \ldots \}$.
4. Report $\xi^{(0)}$ as a perfect sample from $\pi(\xi)$.

The above perfect sampling algorithm is to be embedded in the perfect sampling algorithm for mixture simulation in the context of categorical time series. This we do in the next section.

## 5 Perfect simulation for mixtures of categorical time series with unknown number of components

We first note that coalescence of $(Z, C, S)$ (equivalently, coalescence of $(Z, C)$ since coalescence of $C$ implies coalescence of $S$) implies coalescence of $(\Phi, \gamma)$. We exploit the bounding chains construction approach of Mukhopadhyay and Bhattacharya (2012) for facilitating coalescence. The idea is to obtain stochastic lower and upper bounds for the discrete parts of the Gibbs sampler, namely for $(Z, C, S)$, by maximizing and minimizing their respective distribution functions with respect to the continuous parameters, simulating only from the lower and the upper bounding chains thus created, and noting their coalescence. Remarkably, there is no need to simulate the continuous parameters $\Phi$ and $\gamma$ before coalescence of $(Z, C, S)$. Simulation of $(\Phi, \gamma)$, conditional on $(Z, S)$, is necessary only after the coalescence
of the latter. However, as already discussed in Section 3.7, methods for directly simulating \((\Phi, \gamma)\) given \((Z, S)\) are not available, and we must resort to the perfect simulation method introduced in Section 4 using the available full conditionals which are, thankfully, log-concave.

We now proceed to construction of appropriate bounding chains for the discrete parameters \((Z, C, S)\).

### 5.1 Bounding chains

#### 5.1.1 Bounds for \(Z\)

Let \(F_{z_i}(\cdot \mid Y, S, k, \Theta_M)\) denote the distribution function of the full conditional of \(z_i\), and let \(F_{c_j}(\cdot \mid Y, S_{-j}, k_j, \Phi)\) and \(F_{s_j}(\cdot \mid Y, S_{-j}, C, \Theta_M)\) stand for those of \(c_j\) and \(s_j\), respectively. In addition, when required, we shall assume that \(\{\gamma_{st}\}\) have compact supports not containing zero. This assumption entails multiplication of a constant to the prior to take care of the truncation, but clearly this constant does not destroy the log-concavity of the full conditional of \(\gamma_{st}\). On the other hand, truncation of \(\phi_{\ell, st}\) would involve a factor that depends upon \(\gamma_{st}\), which might affect log-concavity of \(\gamma_{st}\). However, we did not find truncation of \(\phi_{\ell, st}\) to be necessary in our simulations.

Letting \(\bar{S}\) denote the set consisting of only those \(s_j\) that have coalesced, and let \(S^- = S \setminus \bar{S}\) consist of the remaining \(s_j\). Then

\[
F_{z_i}^L(\cdot \mid Y, \bar{S}) = \inf_{S^-, k} F_{z_i}(\cdot \mid Y, \bar{S}, S^-, k, \Phi) \tag{37}
\]

\[
F_{z_i}^U(\cdot \mid Y, \bar{S}) = \sup_{S^-, k} F_{z_i}(\cdot \mid Y, \bar{S}, S^-, k, \Phi) \tag{38}
\]

Fixing \(\bar{S}\) helps reduce the gap between (37) and (38). As in Mukhopadhyay and Bhattacharya (2012), we calculate the infimum and the supremum above by simulated annealing. For further details, see Mukhopadhyay and Bhattacharya (2012).

#### 5.1.2 Bounds for \(C\)

Let \(\bar{Z}\) denote the set of coalesced \(z_i\), and let \(Z^- = Z \setminus \bar{Z}\) consist of those \(z_j\) that did not yet coalesce. Then

\[
F_{c_j}^L(\cdot \mid Y, \bar{S}, \bar{Z}) = \inf_{S^-, k_j, Z^-} F_{c_j}(\cdot \mid Y, \bar{S}, S^-, k_j, \bar{Z}, Z^-, \Phi) \tag{39}
\]

\[
F_{c_j}^U(\cdot \mid Y, \bar{S}, Z) = \sup_{S^-, k_j, Z^-} F_{c_j}(\cdot \mid Y, \bar{S}, S^-, k_j, Z, Z^-, \Phi) \tag{40}
\]
As noted in Mukhopadhyay and Bhattacharya (2012), the supremum corresponds to \( k_j = 1 \) and the infimum corresponds to \( k_j = M - 1 \). For details on optimization using simulated annealing, see Mukhopadhyay and Bhattacharya (2012).

### 5.1.3 Bounds for \( S \)

Letting \( \bar{C} \) and \( C^- = C \setminus \bar{C} \) denote the sets of coalesced and the non-coalesced \( c_j \), the lower and the upper bounds for the distribution function of \( s_j \) are

\[
F_{s_j}^L (\cdot | Y, \bar{C}) = \inf_{C^- \Phi} F_{s_j} (\cdot | Y, \bar{C}, C^-, \Phi) \\
F_{s_j}^U (\cdot | Y, C) = \sup_{C^- \Phi} F_{s_j} (\cdot | Y, C, C^-, \Phi)
\]

(41) (42)

Optimization in this case requires careful attention; see Mukhopadhyay and Bhattacharya (2012) for details.

**Algorithm 5.1 CFTP for mixtures with unknown number of components**

(1) For \( j = 1 \ldots \), until coalescence of \((Z, C)\), repeat steps (2) and (3) below.

(2) Define \( S_j = \{-2^j + 1, \ldots, -2^j - 1\} \) for \( j \geq 2 \), and let \( S_1 = \{-1, 0\} \). For each \( m \in S_j \), generate random numbers \( R_{Z,m}, R_{C,m}, R_{S,m}, R_{\Theta_M,m}, \) and \( R_{\gamma,m} \), meant for simulating \( Z, C, S, \Theta_M, \) and \( \gamma \) respectively. Note that for each \( m \), \( R_{\Theta,m} \) and \( R_{\gamma,m} \) are random numbers corresponding to the perfect simulation algorithm given by Algorithm 4.3; \((\Phi, \gamma)\) in this problem corresponds to the random vector \( \xi \) in that algorithm. We need to generate and fix these random numbers even though we won’t actually simulate \((\Phi, \gamma)\) before coalescence of \((Z, C)\).

Once generated, treat the random numbers as fixed thereafter for all iterations. Since step \(-2^j\) is the initializing step, no random number generation is required at this step.

(3) For \( t = -2^j + 1, \ldots, -1, 0 \), implement steps (3) (i), (3) (ii) and (3) (iii):

(i) For \( i = 1, \ldots, n \),
(a) For \( \ell = 1, \ldots, M \), calculate \( F_{z_i}(\ell \mid Y, \bar{S}) \) and \( F_{\bar{z}_i}(\ell \mid Y, \bar{S}) \) using the simulated annealing techniques detailed in Mukhopadhyay and Bhattacharya (2012).

(b) Determine \( z_{it}^L = F_{z_i}^{-}(R_{zi,t} \mid Y, \bar{S}) \) and \( z_{it}^U = F_{\bar{z}_i}^{-}(R_{zi,t} \mid Y, \bar{S}) \).

(ii) For \( i = 1, \ldots, M \),

(a) For \( \ell = 1, \ldots, k_i + 1 \), calculate \( F_{c_i}^{L}(\ell \mid Y, \bar{S}, \bar{Z}) \) and \( F_{c_i}^{U}(\ell \mid Y, \bar{S}, \bar{Z}) \), using the simulated annealing techniques of Mukhopadhyay and Bhattacharya (2012). The supremum corresponds to \( k_i = \# \bar{S} \backslash \{s_i\} \), when \( S^- \) contains a single distinct element, and the infimum corresponds to the case where \( k_i = \# (\bar{S} \cup S^-) \backslash \{s_i\} \), when all elements of \( S^- \) are distinct, and so the set \( S^- \) will be set manually to have a single distinct element or all distinct elements.

(b) Set \( c_{it}^L = F_{c_i}^{U}^{-}(R_{ci,t} \mid Y, \bar{S}, \bar{Z}) \) and \( c_{it}^L = F_{c_i}^{U}^{-}(R_{ci,t} \mid Y, \bar{S}, \bar{Z}) \).

(iii) For \( i = 1, \ldots, M \),

(a) For \( \ell = 1, \ldots, M \), calculate \( F_{s_i}^{L}(\ell \mid Y, C) \) and \( F_{s_i}^{U}(\ell \mid Y, C) \), using simulated annealing techniques detailed in Mukhopadhyay and Bhattacharya (2012).

(b) Since, for some \( \ell^* \in \{1, \ldots, M\} \), \( F_{s_i}^{L}(\ell \mid Y, C) = 0 \) for \( \ell < \ell^* \) and 1 for \( \ell \geq \ell^* \), it follows that \( s_{it}^L = \ell^* \). Similarly, \( s_{it}^U \) can be determined.

(4) If, for some \( t^* < 0 \), \( z_{it}^L = z_{it}^U \) \( \forall i = 1, \ldots, n \), and \( c_{it}^L = c_{it}^U \) \( \forall i = 1, \ldots, M \), then run the following Gibbs sampling steps from \( t = t^* \) to \( t = 0 \):

(a) Let \( Z^* = \{z_1^*, \ldots, z_n^*\} \) and \( C^* = \{c_1^*, \ldots, c_M^*\} \) denote the coalesced values of \( Z \) and \( C \) respectively, at time \( t^* \). Given \((Z^*, C^*)\), arbitrarily choose any value of \( \Theta_M \) which is compatible with \( C^* \) (one way to ensure compatibility is to choose any \( \Theta_M \) having \( M \) distinct elements) then obtain \( S^* \) from \([S \mid Y, C, \Theta_M]\) using the algorithm given in Section 3.5.
(b) Finally, generate \((\Phi, \gamma)\) using the perfect simulation algorithm described in Algorithm 4.3, using the random numbers already generated. This yields the coalesced value \((Z^*, C^*, S^*, \Phi^*, \gamma^*)\) at time \(t = t^*\).

(b) Using the random numbers already generated, carry forward the above Gibbs sampling chain started at \(t = t^*\) till \(t = 0\), simulating, in order, from the full conditionals of the individual components of \((Z, C, S)\), provided in Sections 3.1, 3.2, and 3.6, and by perfectly simulating \((\Phi, \gamma)\) using Algorithm 4.3. Note that, once \((\Phi^*, \gamma^*)\) are generated by perfect sampling at time \(t = t^*\), further perfect sampling of \((\Phi, \gamma)\) for time \(t > t^*\) does not seem necessary since now Gibbs sampling can be employed. But somewhat ironically, we are forced to continue perfect sampling since changing the simulation method in the midway is not legitimate.

(c) Then, the output of the Gibbs sampler obtained at \(t = 0\), which we denote by \((Z_0, C_0, S_0, \Phi_0, \gamma_0)\), is a perfect sample from the true target posterior distribution.

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**Supplementary Material**

**S-6 Marginalized full conditional distributions**

**S-6.1 Marginalized full conditionals of \(\{z_1, \ldots, z_n\}\)**

For \(i^* = 1, \ldots, n\), let \(Z_{i^*} = \{z_1, \ldots, z_{i^*-1}, z_{i^*+1}, \ldots, z_n\}\), and let \(C\) consist of \(k\) distinct components. Then, denoting the set \(\{\gamma_{s,t}; s, t = 1, \ldots, K\}\) by \(\gamma\), the full conditional distribution of \(z_{i^*}\) is given by

\[
[z_{i^*} = r \mid Z_{-i^*}, C, \gamma, k] \propto \prod_{t=1}^k \prod_{s=1}^K \frac{\prod_{t=1}^K \Gamma \left( \sum_{j: c_j \leq \ell} N_{i^*, st} + \gamma_{s,t} \right) \Gamma \left( \sum_{i: z_i = j} \sum_{j: c_j = \ell} N_{i, st} + \gamma_{s,t} \right)}{\Gamma \left( \sum_{i: z_i = j} \sum_{j: c_j = \ell} N_{i, st} + \sum_{t=1}^K \gamma_{s,t} \right)}
\]

(43)

In the right hand side of (7), \(z_{i^*}\) must be replaced with \(r\).
### S-6.2 Marginalized full conditionals of \( \{c_1, \ldots, c_M\} \)

To obtain the full conditional of \( c_r; r = 1, \ldots, M \), first let \( k_r \) denote the number of distinct values in \( \Theta_{-r,M} = \{\theta_1, \ldots, \theta_{r-1}, \theta_{r+1}, \ldots, \theta_M\} \), and let \( \phi^r_\ell; \ell = 1, \ldots, k_r \) denote the distinct values. Also suppose that \( \phi^r_\ell \) occurs \( M_{\ell r} \) times. Then the conditional distribution of \( c_r \) is given by

\[
[c_r = \ell \mid Y, Z, C_{-r}, \gamma, k_r] = \begin{cases} 
\kappa q_{\ell r} & \text{if } \ell = 1, \ldots, k_r \\
\kappa q_{0r} & \text{if } \ell = k_r + 1 
\end{cases} \tag{44}
\]

where

\[
q_{\ell r} \propto M_{\ell r} \times \prod_{s=1}^{k_r} \frac{\Gamma \left( \sum_{t=1}^{K} \sum_{i:z_i=r,j:|j|\neq r} N_{i,s,t} + \sum_{t=1}^{K} \gamma_{s,t} \right)}{\prod_{t=1}^{K} \Gamma \left( \sum_{i:z_i=r,j:|j|\neq r} N_{i,s,t} + \gamma_{s,t} \right)} \tag{45}
\]

and

\[
q_{0r} \propto \alpha \times \prod_{s=1}^{k_r} \frac{\Gamma \left( \sum_{t=1}^{K} \gamma_{s,t} \right)}{\prod_{t=1}^{K} \Gamma \left( \gamma_{s,t} \right)} \times \prod_{s=1}^{K} \frac{\Gamma \left( \sum_{t=1}^{K} N_{i,s,t} + \sum_{t=1}^{K} \gamma_{s,t} \right)}{\Gamma \left( \sum_{t=1}^{K} N_{i,s,t} + \sum_{t=1}^{K} \gamma_{s,t} \right)} \tag{46}
\]

### S-6.3 Marginalized full conditionals of \( \{\gamma_{s,t}; s, t = 1, \ldots, K\} \)

Assuming that \( C \) consists of \( k \) distinct components, the full conditional distribution of \( \gamma_{s^*t^*} \), for \( s^* = 1, \ldots, K \), and \( t^* = 1, \ldots, K \), is given by

\[
[\gamma_{\ell,t^*s^*} \mid Z, C, \gamma_{-s^*,-t^*}, k] \propto \left\{ \prod_{t=1}^{K} \frac{\Gamma \left( \sum_{i:z_i=\ell,j:|j|\neq \ell} N_{i,s^*,t^*} + \gamma_{s^*,t^*} \right)}{\Gamma \left( \sum_{i:z_i=\ell,j:|j|\neq \ell} N_{i,s^*,t^*} + \sum_{t=1}^{K} \gamma_{s^*,t^*} \right)} \right\} \times \left\{ \frac{\Gamma \left( \sum_{t=1}^{K} \gamma_{s^*,t^*} \right)}{\Gamma \left( \gamma_{s^*,t^*} \right)} \right\}^{k} \tag{47}
\]

\[
\times \gamma_{s^*,t^*}^{a_{s^*,t^*}-1} \exp \left( -b_{jk} \gamma_{s^*,t^*} \right) \\
= \prod_{\ell=1}^{k} \left\{ \frac{\Gamma \left( \sum_{i:z_i=\ell,j:|j|\neq \ell} N_{i,s^*,t^*} + \gamma_{s^*,t^*} \right)}{\Gamma \left( \sum_{i:z_i=\ell,j:|j|\neq \ell} N_{i,s^*,t^*} + \sum_{t=1}^{K} \gamma_{s^*,t^*} \right)} \right\} \times \left\{ \frac{\Gamma \left( \sum_{t=1}^{K} \gamma_{s^*,t^*} \right)}{\Gamma \left( \gamma_{s^*,t^*} \right)} \right\} \tag{47}
\]

\[
\times \gamma_{s^*,t^*}^{a_{s^*,t^*}-1} \exp \left( -b_{jk} \gamma_{s^*,t^*} \right)
\]
S-6.4 Discussion on log-concavity of the full conditionals of \( \{ \gamma_{st}; s, t = 1, \ldots, K \} \)

Note that each factor in the product is of the form

\[
h(\gamma_{st}^\ast) = \frac{\Gamma(\gamma_{st}^\ast + a_t^\ast)}{\Gamma(\gamma_{st}^\ast)} \times \frac{\Gamma(\gamma_{st}^\ast + y_{t,t}^\ast)}{\Gamma(\gamma_{st}^\ast + a_t^\ast + b_{t,t}^\ast)},
\]

(48)

\[a_t^\ast = \sum_{t=1, t \neq t^\ast}^K \gamma_{st}^\ast, \quad y_{t,t}^\ast = \sum_{i: z_i = j} \sum_{j: c_j = t} N_{i,s+t} \quad \forall t = 1, \ldots, K, \quad \text{and} \quad b_{t,t}^\ast = \sum_{t=1, t \neq t^\ast}^K y_{t,t}^\ast.\]

Clearly, all the terms are non-negative, with \( y_{t,t}^\ast \) and \( b_{t,t}^\ast \) being integers. As a result, \( h(\gamma_{st}^\ast) \) admits the following simple form:

\[h(\gamma_{st}^\ast) = \prod_{t=1}^{y_{t,t}^\ast} (\gamma_{st}^\ast + a_t^\ast + y_{t,t}^\ast - i) \prod_{t=1}^{y_{t,t}^\ast + b_{t,t}^\ast} (\gamma_{st}^\ast + a_t^\ast + b_{t,t}^\ast - i).\]

(49)

Thus,

\[
\begin{align*}
\frac{d^2 \log h(\gamma_{st}^\ast)}{d\gamma_{st}^\ast} &= -\sum_{i=1}^{y_{t,t}^\ast} \left\{ \frac{1}{(\gamma_{st}^\ast + y_{t,t}^\ast - i)^2} - \frac{1}{(\gamma_{st}^\ast + a_t^\ast + y_{t,t}^\ast - i)^2} \right\} \\
&\quad + \sum_{i=y_{t,t}^\ast + 1}^{y_{t,t}^\ast + b_{t,t}^\ast} \frac{1}{(\gamma_{st}^\ast + a_t^\ast + y_{t,t}^\ast - i)^2}.
\end{align*}
\]

(50)

Unless \( b_{t,t}^\ast = 0 \) (that is, \( y_{t,t}^\ast = 0 \forall t \neq t^\ast \)), (50) need not be negative for all \( \gamma_{st}^\ast \) and \( \gamma_{st,=t^\ast}. \)

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