INTERNAL RAPID STABILIZATION OF A 1-D LINEAR TRANSPORT EQUATION WITH A SCALAR FEEDBACK

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ABSTRACT. We use a variant the backstepping method to study the stabilization of a 1-D linear transport equation on the interval $(0, L)$, by controlling the scalar amplitude of a piecewise regular function of the space variable in the source term. We prove that if the system is controllable in a periodic Sobolev space of order greater than 1, then the system can be stabilized exponentially in that space and, for any given decay rate, we give an explicit feedback law that achieves that decay rate. The variant of the backstepping method used here relies mainly on the spectral properties of the linear transport equation, and leads to some original technical developments that differ substantially from previous applications.

1. Introduction. We study the linear 1-D transport equation

$$\begin{cases}
\alpha_t + \alpha_x + \mu \alpha = u(t) \phi(x), & x \in [0, L], \\
\alpha(t, 0) = \alpha(t, L), & \forall t \geq 0,
\end{cases} \tag{1}$$

where $\phi$ is a given real-valued function that will have to satisfy certain conditions, and at time $t$, $y(t, \cdot)$ is the state and $u(t)$ is the control.

This linear transport equation is the simplest example of a linear hyperbolic system with a distributed scalar input and proportional boundary conditions. Such systems appear naturally in physical problems. For example, as is mentioned in [37], a linear wave equation which can be rewritten as a $2 \times 2$ first order hyperbolic system, the problem of a vibrating damped string, or the plucking of a string, can be modeled thus. In a different field altogether, chemical tubular reactors, in particular plug flow reactors (see [32, 35]), are modeled by hyperbolic systems with a distributed scalar input (the temperature of the reactor jacket), albeit with a boundary input instead of proportional boundary conditions.

Finally, an important example is the water tank system, introduced in [19]. It models a 1-D tank containing an inviscid, incompressible, irrotational fluid, in the
approximation that its acceleration is small compared with the gravitational constant, and that the height of the liquid is small compared with the length of the tank. In this setting, the motion of the fluid can be modeled by the Saint-Venant equations on the interval $[0, L]$ with impermeable boundary conditions, and the control is the force applied to the tank itself, which takes the form of a distributed scalar input. Actually in Riemann coordinates the boundary conditions of this system take the form

$$\begin{aligned}
&u_1(t, 0) + u_2(t, 0) = 0 \\
&u_1(t, L) + u_2(t, L) = 0,
\end{aligned}$$

which are equivalent to periodic boundary conditions if one considers the invertible mapping

$$(u_1, u_2) \mapsto \chi_{(0, L)}u_1 + \chi_{(L, 2L)}u_2(2L - \cdot).$$

In summary, our work on the linear transport equation is a first step towards more general and applicable systems. Moreover, as we will elaborate further on, the simplicity of the equation also makes for a nice illustration of a variant of backstepping method, with new technical developments.

1.1. Notations and definitions. We note $\ell^2$ the space of complex-valued square summable series $\ell^2(\mathbb{Z})$. To simplify the notations, we will note $L^2$ the space $L^2(0, L)$ of complex-valued $L^2$ functions, with its hermitian product

$$\langle f, g \rangle = \int_0^L f(x)\overline{g(x)}dx, \quad \forall f, g \in L^2,$$

and the associated norm

$$\|f\| = \sqrt{\langle f, f \rangle}.$$  

We also use the following notation

$$e_n(x) = \frac{1}{\sqrt{L}}e^{2\pi inx}, \quad \forall n \in \mathbb{Z},$$

for the usual Hilbert basis for $L^2$. For a function $f \in L^2$, we will note $(f_n) \in \ell^2$ its coefficients in this basis:

$$f = \sum_{n \in \mathbb{Z}} f_n e_n.$$

Note that with this notation, we have

$$\bar{f} = \sum_{n \in \mathbb{Z}} \overline{f_n} e_n,$$

so that, in particular, $f$ is real-valued if and only if:

$$f_{-n} = \overline{f_n}, \quad \forall n \in \mathbb{Z}.$$  

Functions of $L^2$ can also be seen as $L$-periodic functions on $\mathbb{R}$, by the usual $L$-periodic continuation: in this article, for any $f \in L^2$ we will also note $f$ its $L$-periodic continuation on $\mathbb{R}$.

We will use the following definition of the convolution product on $L$-periodic functions:

$$f \star g = \sum_{n \in \mathbb{Z}} f_n g_n e_n = \int_0^L f(s)g(x-s)ds \in \mathcal{L}^2, \quad \forall f, g \in \mathcal{L}^2,$$

where $g(x-s)$ should be understood as the value taken in $x-s$ by the $L$-periodic continuation of $g$. 

Let us now note $E$ the space of finite linear combinations of the $(e_n)_{n \in \mathbb{Z}}$. Then, any complex-valued sequence $(f_n)_{n \in \mathbb{Z}}$ defines an element $f$ of the dual $E'$:

$$\langle \sum_{n \in I} c_n e_n, f \rangle = \sum_{n \in I} c_n f_n.$$ 

In particular, if $(f_n) \in l^2$, the linear form thus defined can be extended from $E$ to $L^2$. The resulting linear form on $L^2$ is then represented by the function

$$f = \sum_{n \in \mathbb{Z}} f_n e_n \in L^2.$$ 

Accordingly, for $f \in E'$, the corresponding $(f_n)$ will be called its “Fourier coefficients”. On this space of linear forms, derivation can be defined by duality:

$$f' = \left( \frac{2i\pi n}{L} f_n \right)_{n \in \mathbb{Z}}, \quad \forall f \in E'.$$ 

We also define the following spaces:

**Definition 1.1.** Let $m \in \mathbb{N}$. We note $H^m$ the usual Sobolev spaces on the interval $(0, L)$, equipped with the Hermitian product

$$\langle f, g \rangle_m = \int_0^L \partial_x^m f \overline{\partial_x^m g} + f \overline{g}, \quad \forall f, g \in H^m,$$

and the associated norm $\| \cdot \|_m$.

For $m \geq 1$ we also define $H^m_{(pw)}$ the space of piecewise $H^m$ functions, that is, $f \in H^m_{(pw)}$ if there exists a finite number $d$ of points $0 < \sigma_1 < \cdots < \sigma_d < L$ such that, noting $\sigma_0 := 0$ and $\sigma_{d+1} := L$, $f$ is $H^m$ on every $[\sigma_j, \sigma_{j+1}]$ for $0 \leq j \leq d$. This space can be equipped with the norm

$$\| f \|_{m, pw} := \sum_{j=0}^d \| f |_{[\sigma_j, \sigma_{j+1}]} \|_{H^m(\sigma_j, \sigma_{j+1})}.$$ 

For $m \in \mathbb{N}$, we also define the periodic Sobolev space $H^m_{per}$ as the subspace of $L^2$ functions $f = \sum_{n \in \mathbb{Z}} f_n e_n$ such that

$$\sum_{n \in \mathbb{Z}} \left( 1 + \left| \frac{2i\pi n}{L} \right|^{2m} \right) |f_n|^2 < \infty.$$ 

$H^m_{per}$ is a Hilbert space, equipped with the Hermitian product

$$\langle f, g \rangle_m = \sum_{n \in \mathbb{Z}} \left( 1 + \left| \frac{2i\pi n}{L} \right|^{2m} \right) f_n \overline{g_n}, \quad \forall f, g \in H^m_{per},$$

and the associated norm $\| \cdot \|_m$, as well as the Hilbert basis

$$\left( \frac{e_n}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}}} \right).$$

Note that $H^m_{per}$ is a closed subspace of $H^m$, with the same scalar product and norm, thanks to the Parseval identity. Moreover,

$$H^m_{per} = \left\{ f \in H^m, \quad f^{(k)}(0) = f^{(k)}(L), \quad \forall k \in \{0, \cdots, m-1\} \right\}, \quad \forall m \in \mathbb{N}.$$
Indeed, for \( f = \sum_{n \in \mathbb{Z}} f_ne_n \in H^m \),
\[
f_n = \frac{(-1)^{m-1}}{(2\pi n/L)^{m-1}} \langle f, \partial_x^{m-1} e_n \rangle \quad \text{(by definition of } e_n)\]
\[
= \sum_{k=1}^{m} \frac{(-1)^{m-k}}{(2\pi n/L)^k} \langle \partial_x^{k-1} f(L) - \partial_x^{k-1} f(0) \rangle + \frac{1}{(2\pi n/L)^{m-1}} \langle \partial_x^m f, e_n \rangle, \quad \forall n \in \mathbb{Z}^*,
\]
by integration by parts \( m \) times. Then, for (10) to be satisfied, it is necessary and sufficient that
\[
f^{(k)}(0) = f^{(k)}(L), \quad \forall k \in \{0, \cdots, m-1\}. \quad (12)
\]
Finally, for any Banach spaces \( E,F \), we will denote by \( \|\cdot\|_E \) and \( \|\cdot\|_{E,F} \) the operator norms of \( L(E) \) and \( L(E,F) \) respectively.

### 1.2. Main result.

Let us first give a characterization of the controllability of (1) in periodic Sobolev spaces.

**Lemma 1.2.** Let \( m \in \mathbb{N}^* \). Let \( \varphi \in H^{m-1}_{\text{per}} \) and \( u \in L^2(0,T) \). Then, if there exists \( C > 0 \) such that
\[
|\varphi_n| \leq \frac{C}{\sqrt{1 + \left|\frac{2\pi n}{L}\right|^{2m}}}, \quad \forall n \in \mathbb{N}, \quad (13)
\]
the equation (1) is well-posed in \( H^m_{\text{per}} \). Moreover, if there exist \( c,C > 0 \) such that
\[
\frac{c}{\sqrt{1 + \left|\frac{2\pi n}{L}\right|^{2m}}} \leq |\varphi_n| \leq \frac{C}{\sqrt{1 + \left|\frac{2\pi n}{L}\right|^{2m}}}, \quad \forall n \in \mathbb{Z}, \quad (14)
\]
then (1) is controllable in time \( T \geq L \) with \( L^2(0,T) \) controls.

This is obtained using the moments method, and we refer to [38]*Equation (2.19) and pages 199-200 for more details. The controllability of system (1), in turn, will allow us to use a form of backstepping method to stabilize it.

To stabilize (1), we will be considering linear feedbacks of the form
\[
\langle \alpha(t), F \rangle = \sum_{n \in \mathbb{Z}} \overline{F_n} \alpha_n(t) = \int_0^L \bar{F}(s) \alpha(s) ds
\]
where \( F \in \mathcal{E}' \) and \( (F_n) \in \mathbb{C}^\mathbb{Z} \) are its Fourier coefficients, and \( F \) is real-valued, that is,
\[
F_{-n} = \overline{F_n}, \quad \forall n \in \mathbb{Z}.
\]
In fact, the integral notation will appear as purely formal, as the \( (F_n) \) will have a prescribed growth, so that \( F \notin L^2 \). The associated closed-loop system now writes
\[
\begin{cases}
\alpha_t + \alpha_x + \mu \alpha = \langle \alpha(t), F \rangle \varphi(x), & x \in [0,L], \\
\alpha(t,0) = \alpha(t,L), & \forall t \geq 0.
\end{cases} \quad (15)
\]
This is a linear transport equation, which we seek to stabilize with an internal, scalar feedback, given by a real-valued feedback law. This article aims at proving the following class of stabilization results:
Theorem 1.3 (Rapid stabilization in Sobolev norms). Let \( m \in \mathbb{N}^\ast \). Let \( \varphi \in H^m_{\text{pw}} \cap H^m_{\text{per}} \) satisfying (14) with optimal constants \( c, C > 0 \), that is,

\[
  c := \sup \left\{ \left| \varphi_n \right| \sqrt{1 + \frac{2i\pi}{L}}^m \right\}, \quad C := \inf \left\{ \left| \varphi_n \right| \sqrt{1 + \frac{2i\pi}{L}}^m \right\}.
\]

Then, for every \( \lambda > 0 \), for all \( \alpha^0 \in H^m_{\text{per}} \) the closed-loop system (15) with initial condition \( \alpha^0 \) and with the stationary feedback law \( F \in \mathcal{E}' \) given by

\[
  \langle e_n, F \rangle := -\frac{1 - e^{\lambda L}}{1 + e^{-\lambda L}} \frac{2}{L \varphi_n}, \quad \forall n \in \mathbb{Z},
\]

has a unique solution \( \alpha(t) \) which satisfies the estimate

\[
  \| \alpha(t) \|_m \leq \left( \frac{C}{c} \right)^2 e^{(\mu + \lambda) L} e^{-\lambda t} \| \alpha^0 \|_m, \quad \forall t \geq 0.
\]

Remark 1. The more general linear transport equation

\[
  \begin{cases}
    u_t + u_x + a(x)u = u(t)\tilde{\varphi}(x), & x \in [0, L], \\
    y(t, 0) = y(t, L), & \forall t \geq 0,
  \end{cases}
\]

where \( a \in C([0, L], \mathbb{R}) \), can actually be transformed into an equation of the form (1) by the change of variable

\[
  \alpha(t, x) := e^{\int_0^t a(s) ds} \mu x y(x, t),
\]

where \( \mu = \int_0^L a(s) ds \), and with

\[
  \varphi(x) := e^{\int_0^x a(s) ds - \mu x} \tilde{\varphi}(x).
\]

Thus our stabilization results hold for the wider class of linear transport equations (18).

Note that the estimate (17) is explicit, as it only depends on \( c, C \) (and thus, \( \varphi \)), \( \mu \) and \( \lambda \). Though it is not necessarily sharp for a given controller \( \varphi \) and the corresponding feedback law \( F \), it is the “least worse” a priori estimate one can get, in a sense that we will elaborate on an example in Subsection 3.3.

Remark 2. Note that the control operator

\[
  B : u \in \mathbb{C} \mapsto u\varphi
\]

is not a bounded operator in the state space \( H^m_{\text{per}} \), as \( \varphi \notin H^m_{\text{per}} \). Otherwise, rapid stabilizability would be a consequence of controllability, by [18]Corollary 1, p.13 (see also [49] item (i), Theorem 3.3, p.227).

Rather, Theorem 1.3 extends the implication “controllability \( \implies \) stabilizability” to a case where the control operator is unbounded, as is done in [36, 43].

On the other hand, the additional regularity \( \varphi \in H^m_{\text{pw}} \) gives us the following equality, first using the fact that \( \varphi \in H^m_{\text{per}} \), then by integration by parts on each
interval $[\sigma_j, \sigma_{j+1}]$, using the fact that $\partial_x^{m-1} \varphi \in H^1_{(pw)}$:

$$\varphi_n = \frac{(-1)^{m-1}}{(2\pi L)^{m-1}} \langle \varphi, \partial_x^{m-1} e_n \rangle \quad \text{(by property of the } e_n)$$

$$= \frac{1}{(2\pi L)^{m-1}} \langle \partial_x^{m-1} \varphi, e_n \rangle \quad \text{($\partial_x$ is anti-hermitian on } H^1_{(pw)}) \quad (20)$$

$$= -\frac{\tau_n^\varphi}{(2\pi n)^m} + \frac{1}{(2\pi n)^m} \sum_{j=0}^d \langle \chi_{[\sigma_j, \sigma_{j+1}]} \partial_x^m \varphi, e_n \rangle, \quad \forall n \in \mathbb{Z}^*,$$

by integration by parts on each interval $[\sigma_j, \sigma_{j+1}]$, where

$$\tau_n^\varphi := \frac{1}{\sqrt{L}} \left( \partial_x^{m-1} \varphi(L) - \partial_x^{m-1} \varphi(0) \right. \right.$$  

$$\left. + \sum_{j=1}^d e^{-2\pi n \sigma_j} (\partial_x^{m-1} \varphi(-\sigma_j) - \partial_x^{m-1} \varphi(\sigma_j^+)) \right), \quad \forall n \in \mathbb{Z}. \quad (21)$$

Note that, because $\varphi \in H^m_{(pw)},$

$$\sum_{j=0}^d \chi_{[\sigma_j, \sigma_{j+1}]} \partial_x^m \varphi \in L^2,$$

which implies

$$\left( \sum_{j=0}^d \langle \chi_{[\sigma_j, \sigma_{j+1}]} \partial_x^m \varphi, e_n \rangle \right) \in \ell^2. \quad (22)$$

In particular

$$\left( \frac{2i\pi}{L} n \right)^m \varphi_n = -\tau_n^\varphi + o(1), \quad (23)$$

so thanks to condition (14), there exist $C_1, C_2 > 0$ such that

$$C_1 \leq |\tau_n^\varphi| \leq C_2, \quad \forall n \in \mathbb{Z},$$

so that the $\tau_n^\varphi$ are the eigenvalues of a diagonal isomorphism of any periodic Sobolev space into itself:

$$\tau_n^\varphi e_n = \tau_n^\varphi e_n, \quad \forall n \in \mathbb{Z}. \quad (25)$$

Finally, (24) implies that $\tau_n^\varphi \neq 0$. This together with (20) and (22) implies that $\varphi \notin H^m_{per}$. This gives a clear picture of the regularity of $\varphi$: $\varphi \in (H_{per}^{m-1} \cap H^m_{(pw)}) \setminus H^m_{per}$.

Moreover, it is clear from the definition of its coefficients that $\tau^\varphi$ is a sum of translations:

$$\tau^\varphi f = \frac{1}{\sqrt{L}} \left( \partial_x^{m-1} \varphi(L) - \partial_x^{m-1} \varphi(0) \right) f$$

$$+ \sum_{j=1}^d (\partial_x^{m-1} \varphi(-\sigma_j) - \partial_x^{m-1} \varphi(\sigma_j^+)) f(\cdot - \sigma_j), \quad \forall f \in L^2, \quad (26)$$

where $f$ in the right-hand side is understood as its periodic continuation.
1.3. Related results. To investigate the stabilization of infinite-dimensional systems, there are four main types of approaches.

The first type of approach relies on abstract methods, such as the Gramian approach and the Riccati equations (see for example [44, 43, 23]). In these works, rapid stabilization was achieved thanks to a generalization of the well-known Gramian method in finite dimension (see [33, 23]). However, the feedback laws that are provided involve the solution to an algebraic Riccati equation, and the inversion of an infinite-dimensional Gramian operator, which makes them difficult to compute in practice.

The second approach relies on Lyapunov functions. Many results on the boundary stabilization of first-order hyperbolic systems, linear and nonlinear, have been obtained using this approach: see for example the book [2], and the recent results in [21, 22]. However, this approach can be limited, as it is sometimes impossible to obtain an arbitrary decay rate using Lyapunov functions (see [15] Remark 12.9, page 318 for a finite dimensional example).

The third approach is related to pole-shifting results in finite dimension. Indeed, it is well-known that if a linear finite-dimensional system is controllable, then its poles can be arbitrarily reassigned (shifted) with an appropriate linear feedback law (see [15] Chapter 10, p.275). There have been some generalizations of this powerful property to infinite-dimensional systems, notably hyperbolic systems. Let us cite [38], in which the author uses a sort of canonical form to prove a pole-shifting result for a class of hyperbolic systems with a distributed scalar control. In this paper, the feedback laws under consideration are bounded and pole-shifting property is not as strong as in finite dimension. This is actually inevitable, as was proved in [41], in a very general setting: bounded feedback laws can only achieve weak pole-shifting, which is not sufficient for exponential stabilization. However, if one allows for unbounded feedback laws, it is possible to obtain stronger pole-shifting, and in particular exponential stabilization in some cases. This is extensively studied in [36], in which the author gives a formula for a feedback law that achieves the desired pole placement. However, this formula requires to know a cardinal function for which the poles coincide with the initial spectrum, which might be difficult in practice.

The fourth approach, which we will be using in this article, is the backstepping method. This name originally refers to a way of designing feedbacks for finite-dimensional stabilizable systems with an added chain of integrators (see [15] Chapter 12, p.334, [40, 27], and [8] or [30] for some applications to partial differential equations). Another way of applying this approach to partial differential equations was then developed in [3] and [1]: when applied to the discretization of the heat equation, the backstepping approach yielded a change of coordinates which was equivalent to a Volterra transform of the second kind. Backstepping then took yet another successful form, consisting in mapping the system to stable target system, using a Volterra transformation of the second kind (see [36] for a comprehensive introduction to the method):

\[ f \mapsto \left( x \mapsto f(x) - \int_0^x k(x, y)f(y)dy \right) \]

This was used to prove a host of results on the boundary stabilization of partial differential equations: let us cite for example [24] and [39] for the wave equation,
for the Korteweg-de Vries equation, Chapter 7 for an application to first-order hyperbolic systems, and also [17], which combines the backstepping method with Lyapunov functions to prove finite-time stabilization in $H^2$ for a quasilinear $2 \times 2$ hyperbolic system.

The backstepping method has the advantage of providing explicit feedback laws, which makes it a powerful tool to prove other results, such as null-controllability or small-time stabilization (stabilization in an arbitrarily small time). This is done in [14], where the authors give an explicit control to bring a heat equation to 0, then a time-varying, periodic feedback to stabilize the equation in small time. In [47], the author obtains the same kind of results for the Korteweg-de Vries equation.

In some cases, the method was used to obtain stabilization with an internal feedback. This was done in [42] and [45] for parabolic equations, and [48] for first-order hyperbolic equations. The strategy in these works is to first apply a Volterra transformation as usual, which still leaves an unstable source term in the target, and then apply a second invertible transformation to reach a stable target system. Let us note that in the latter reference, the authors study a linear transport equation and get finite-time stabilization. However, their controller takes a different form than ours, and several hypotheses are made on the space component of the controller so that a Volterra transform can be successfully applied to the system. This is in contrast with the method in this article, where the assumption we make on the controller corresponds to the exact null-controllability of the system.

In this paper, we use another application of the backstepping method, which uses another type of linear transformations, namely, Fredholm transformations:

$$f \mapsto \left( x \mapsto \int_0^L k(x, y)f(y)dy \right).$$

These are more general than Volterra transformations, but they require more work: indeed, Volterra transformations are always invertible, which is not the case for Fredholm transformations, and the invertibility can prove difficult to study. Even though it is sometimes more involved and technical, the use of a Fredholm transformation proves more effective for certain types of control: for example, in [13] for the Korteweg-de Vries equation and [12] for a Kuramoto-Sivashinsky equation, the position of the control makes it more appropriate to use a Fredholm transformation. Other boundary stabilization results using a Fredholm transformation can be found in [10] for integro-differential hyperbolic systems, and in [11] for general hyperbolic balance laws.

Fredholm transformations have also been used in [9], where the authors prove the rapid stabilization of the Schrödinger equation with an internal feedback. Their method of proof relies on the assumption that the system is controllable, as is the case in [10] and the references therein. This is a remarkable aspect of the evolution of backstepping methods for PDEs. Indeed, the original form of the finite-dimensional backstepping method, and the backstepping method with Volterra transformations of the second kind, could be applied to uncontrollable systems. Hence, a controllability assumption makes for potentially powerful additional information, for example when one considers the more general Fredholm transformations instead of Volterra transformations of the second kind. It is interesting to note that the role played by controllability is also a feature of the pole-shifting approach and the Gramian method, although in this setting it leads to an expression involving the inverse of
the Gramian operator, or, in the case of [36], a cardinal function. In a way it closes a loop that started with the seminal idea of transforming systems into others, or building equivalence classes of systems, in [6]. In accordance with the pole-shifting mindset, the developments of [9] depart from the work in previous references by relying on fine spectral properties of the Schrödinger equation.

In this article, we apply the same method of proof, but due to the hyperbolic nature of the equation, it takes quite a different (and, to our knowledge, original) course than in [9]. We will see that it involves fine asymptotic analysis and a weaker type of convergence of Fourier series. As such, the simple linear transport equation provides but an illustration of these developments, which shows that the general strategy of this variant of backstepping can be adapted successfully to very contrasting spectral behaviours. The predominance of spectral analysis suggests that the ideas in this article can be applied to more general hyperbolic equations or systems with a scalar distributed input.

The general outline of the proof, similar to that of [9], can be understood from a simple finite-dimensional example which we present in the next section.

1.4. The backstepping method revisited: a finite-dimensional example.

Let us now give a finite-dimensional example to illustrate the role controllability can play in the backstepping method for PDEs. We refer here to [16, 9] for alternate presentations of this example. Let us point out that this example has nothing to do with the original finite-dimensional backstepping method. It is much more closely related to pole-shifting and the notion of canonical form ([6]).

Consider the finite-dimensional control system
\[ \dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, \ A \in \mathbb{C}^{n \times n}, \ B \in \mathbb{C}^{n \times 1}. \tag{27} \]
Assume that \((A, B)\) is controllable. Suppose that \(x(t)\) is a solution of system (27) with \(u(t) = Kx(t)\). Now, in the spirit of PDE backstepping, let us try to invertibly transform the resulting closed-loop system into another controllable system, namely
\[ \dot{x} = \hat{A}x, \tag{28} \]
which can be exponentially stable if \(\hat{A}\) is well chosen.

Such a transformation \(T\) would map the closed-loop system to
\[ (Tx) = T\dot{x} = T(A + BK)x. \]
In order for \(Tx\) to be a solution of (28), we would need
\[ T(A + BK) = \hat{A}T. \tag{29} \]
One can see quite clearly that this matrix equation is not well-posed, in that if it has a solution, it has an infinity of solutions. Moreover, the variables \(T\) and \(K\) are not separated because of the \(TBK\) term, and as a result the equation is nonlinear. Hence, we can add the following constraint to equation (29), to separate the variables, make the equation linear in \((T, K)\), and get a uniqueness property:
\[ TB = B. \tag{30} \]
We will refer to this equality as the \(TB = B\) equation. Injecting it into (29), we get the following equations:
\[ TA + BK = \hat{A}T, \]
\[ TB = B, \tag{31} \]
which are closely related to the notion of \(F\)-equivalence introduced in [6].
Now for this set of equations, one can prove the following theorem, using the Brunovsky normal form (or canonical form):

**Theorem 1.4.** If $(A, B)$ and $(\tilde{A}, B)$ are controllable, then there exists a unique pair $(T, K)$ satisfying conditions (31).

This shows that controllability can be very useful when one wants to transform systems into other systems. In the finite-dimensional case, using the canonical form is the most efficient way of writing it. However, in order to gain some insight on the infinite-dimensional case, there is a different proof, relying on the spectral properties of $A$ and $\tilde{A}$, which can be found in [9]. The idea is that the controllability of $A$ allows to build a basis for the space state, in which $T$ can then be constructed. Indeed, suppose $A$ is diagonalizable with eigenvectors $(e_n, \lambda_n)_{1 \leq n \leq N}$, and suppose that $A$ and $\tilde{A}$ have no mutual eigenvalues. Then, let us project (31) on $e_n$:

$$\lambda_n Te_n + (K e_n) B = \tilde{A} Te_n,$$

from which we get the following relationship

$$Te_n = (K e_n)(\tilde{A} - \lambda_n I)^{-1} B, \quad \forall n \in \{1, \cdots, N\}.$$  \hfill (33)

Then, using the Kalman rank condition on the pair $(\tilde{A}, B)$, one can prove that the $f_n := ((\tilde{A} - \lambda_n I)^{-1} B)$ form a basis of $\mathbb{R}^N$.

Knowing this, write

$$B = \sum_{n=1}^{N} b_n e_n,$$

$$B = \sum_{n=1}^{N} \tilde{b}_n f_n,$$

and $TB$ is written naturally in this basis:

$$TB = \sum_{n=1}^{N} (K e_n) b_n f_n,$$

so that the second equation of (31) becomes

$$\sum_{n=1}^{N} (K e_n) b_n f_n = \sum_{n=1}^{N} \tilde{b}_n f_n.$$ \hfill (36)

Using the Kalman rank condition on $(A, B)$, one can prove that $b_n \neq 0$ so that the $(K e_n)$ are uniquely determined. The only thing that remains to prove is the invertibility of $T$, as the $(K e_n)$ could be 0. In the end the invertibility is proven thanks to the Fattorini-Hautus test on the pair $(\tilde{A}, B)$, and the uniqueness is given by the $TB = B$ condition.

**Remark 3.** In this finite-dimensional example, one can see a relationship between the Gramian method and the variant of the backstepping method used in this article. Indeed, define the Gramian matrix

$$C^\omega := \int_0^\infty e^{-2\omega t} e^{-tA} BB^* e^{-tA^*} dt,$$

with $\omega > \max (|\lambda| \in \sigma(A) \cup \sigma(A^*))$ where $\sigma$ denotes the spectrum of a matrix. It is the solution of the Lyapunov equation (see [4] and the references therein, and also
on the related Algebraic Riccati Equation):
\[
C_\omega^\infty (A + \omega I)^* + (A + \omega I)C_\omega^\infty = BB^*.
\]

Now, injecting the feedback law \( K := -B^* (C_\omega^\infty)^{-1} \) given by the Gramian method, this equation becomes:
\[
C_\omega^\infty (A + \omega I)^* + (A + \omega I)C_\omega^\infty = -BKC_\omega^\infty,
\]
which becomes, after multiplication by \((C_\omega^\infty)^{-1}\) on the left and on the right,
\[
(C_\omega^\infty)^{-1}(A + BK) = (-A^* - 2\omega I)(C_\omega^\infty)^{-1},
\]
which is of the form (29), with \( \tilde{A} = -A^* - 2\omega I \) and \( T = (C_\omega^\infty)^{-1} \). The fundamental difference then comes from the fact that in the Gramian method, this backstepping-type equation is coupled with the definition of the feedback law
\[
B^* (C_\omega^\infty)^{-1} = -K,
\]
whereas in the backstepping method, the backstepping-type equation (29) is coupled with the \( TB = B \) condition, which can be recast as
\[
B^* T^* = B^*.
\]
The former leads to a Lyapunov-type analysis of stability, whereas the latter can be used when the operator \( A \) has a basis of eigenvectors (or generalized eigenvectors) to give explicit coefficients in that basis for the feedback law.

\textit{Another way to look at this is that (29), rewritten as}
\[
\tilde{A}T - TA = TBK,
\]
\textit{has a linear left hand side with some symmetry which the right hand side does not have. In order to give some more symmetry, one can either symmetrize the right hand side by}
\[
K = -B^* T^*,
\]
\textit{which yields the Lyapunov equation}
\[
\tilde{A}T - TA = TBK, = -TBB^* T^*
\]
\textit{and corresponds to the Gramian method; or one can remove} \( T \) \textit{from the right hand side in order to have a non homogeneous linear equation in} \( T \)
\[
\tilde{A}T - TA = BK,
\]
\textit{which corresponds to the variant of the backstepping method used in this article, as illustrated by the finite-dimensional example above.}

1.5. Structure of the article. The structure of this article follows the outline of the proof given above: in Section 2, we look for candidates for the backstepping transformation in the form of general integral transformations. Formal calculations (and a formal \( TB = B \) condition) lead to a PDE analogous to (32) which we solve, which is analogous to the derivation of (33). Using the properties of Riesz bases and the controllability assumption, we prove that such candidates are indeed invertible, under some conditions on the feedback coefficients \((F_n)\). For consistency, we then determine the feedback law \((F_n)\) such that the corresponding transformation indeed satisfies a weak form of the \( TB = B \) condition. Then, in Section 3, we check that the corresponding transformation indeed satisfies an operator equality analogous to (31), making it a valid backstepping transformation. We check the well-posedness of the closed-loop system for the feedback law obtained in Section 2, which allows
us to prove the stability result. Finally, Section 4 gives a few remarks on the result, as well as further questions on this stabilization problem.

2. Definition and properties of the transformation. Let $\lambda' > 0$ be such that $\lambda' - \mu > 0$, and $m \geq 1$.

As announced above, we follow the outline of Section 1.4 the equivalent of (27) will be (1), and the corresponding closed-loop system will be (15), thus

$$ A = -\partial_x, \quad B : u \in \mathbb{R} \mapsto u \varphi, \quad K : \alpha \mapsto \langle \alpha, F \rangle, $$

and the equivalent of (28) will be the following target system:

$$ \begin{cases} z_t + z_x + \lambda' z = 0, & x \in (0, L), \\ z(t, 0) = z(t, L), & t \geq 0. \end{cases} $$

so that, following the notations of Section 1.4,

$$ \tilde{A} = A - \lambda'I. $$

Then it is well-known that, taking $\alpha^0 \in L^2$, the solution to (41) with initial condition $\alpha^0$ writes

$$ z(t, x) = e^{-\lambda't} \alpha^0(x - t), \quad \forall (t, x) \in \mathbb{R}^+ \times (0, L), $$

where we have extended $\alpha_0$ to the whole real line $\mathbb{R}$ by $L$-periodic continuation as noted in the introduction. Hence,

**Proposition 2.1.** For all $s \geq 0$, the system (41) is exponentially stable for $\| \cdot \|_s$, for initial conditions in $H^s_{\text{per}}$.

This stable system will be the target of the backstepping method.

2.1. Formal computations and kernel equations. Let $\varphi \in H^m \cap H^{m-1}_{\text{per}}$ be a real-valued function satisfying (14). As mentioned in the introduction, we follow the outline of the PDE backstepping method: we want to build transformations $T$ in the form of general kernel operators:

$$ T : f \mapsto \left( x \mapsto \int_0^L k(x, y)f(y)dy \right), $$

that transform system (1) into the stable target system (41).

To have an idea of what this kernel looks like, we can do the following formal computation for some the general kernel operator $T$ given above: first the boundary condition

$$ \left( \int_0^L k(0, y)\alpha(y)dy \right) = \left( \int_0^L k(L, y)\alpha(y)dy \right), $$

which is satisfied if and only if

$$ k(0, \cdot) = k(L, \cdot) \text{ a.e.}, \quad (42) $$
then the equation of the target system, for \( x \in [0, L] \):

\[
0 = \left( \int_0^L k(x, y) \alpha(y) dy \right)_t + \left( \int_0^L k(x, y) \alpha(y) dy \right) + \lambda' \left( \int_0^L k(x, y) \alpha(y) dy \right)
+ \left( \int_0^L k(x, y) \alpha_1(y) dy \right) + \lambda' \left( \int_0^L k(x, y) \alpha(y) dy \right)
+ \left( \int_0^L k(x, y)(-\alpha_x(y) - \mu_0(y) + \langle \alpha, F \rangle \varphi(y)) dy \right)
+ \left( \int_0^L (k_x(x, y) + \lambda' k(x, y)) \alpha(y) dy \right)
= \left( \int_0^L k_y(x, y) \alpha(y) dy \right) - (k(x, L) \alpha(L) - k(x, 0) \alpha(0))
+ \left( \int_0^L k(x, y) \langle \alpha, F \rangle \varphi(y) dy \right) + \left( \int_0^L (k_x(x, y) + (\lambda' - \mu) k(x, y)) \alpha(y) dy \right)
= \left( \int_0^L (k_y(x, y) + k_x(x, y) + (\lambda' - \mu) k(x, y)) \alpha(y) dy \right)
+ \left( \int_0^L k(x, y) \left( \int_0^L F(s) \alpha(s) ds \right) \varphi(y) dy \right) - (k(x, L) \alpha(L) - k(x, 0) \alpha(0))
= \left( \int_0^L (k_y(x, y) + k_x(x, y) + (\lambda' - \mu) k(x, y)) \alpha(y) dy \right)
+ \left( \int_0^L F(s) \left( \int_0^L k(x, y) \varphi(y) dy \right) \alpha(s) ds \right) - (k(x, L) \alpha(L) - k(x, 0) \alpha(0)).
\]

Now, suppose we have the formal \( TB = B \) condition, following the notations of (40):

\[
\int_0^L k(x, y) \varphi(y) dy = \varphi(x), \quad \forall x \in [0, L].
\]

Then, we get, noting \( \lambda := \lambda' - \mu > 0 \),

\[
\left( \int_0^L \left( k_y(x, y) + k_x(x, y) + \lambda k(x, y) + \varphi(x) \bar{F}(y) \right) \alpha(y) dy \right)
- (k(x, L) \alpha(L) - k(x, 0) \alpha(0)) = 0.
\]

Hence we have formally derived the following kernel equation:

\[
\begin{cases}
    k_x + k_y + \lambda k = -\varphi(x) \bar{F}(y), \\
    k(0, y) = k(L, y), \\
    k(x, 0) = k(x, L),
\end{cases}
\]

(43)

together with the \( TB = B \) condition

\[
\langle k(x, \cdot), \varphi(\cdot) \rangle = \varphi(x), \quad \forall x \in [0, L].
\]

(44)
To study the solution to the kernel equation, we project it along the variable $y$. Let us write heuristically

$$k(x, y) = \sum_{n \in \mathbb{Z}} k_n(x) e_n(y),$$

so that

$$\int_0^L k(x, y) \alpha(y) dy = \sum_{n \in \mathbb{Z}} \alpha_n k_{-n}(x).$$

Projecting the kernel equations (43), we get

$$k_n' + \lambda_n k_n = -F_{-n} \varphi,$$

where

$$\lambda_n = \lambda + \frac{2i\pi}{L} n.$$ (46)

Note that

$$\frac{2i\pi p}{L} \frac{1}{\lambda_{n+p}} + \frac{1}{\lambda_{n+p}} = 1, \quad \forall n, p \in \mathbb{Z},$$ (47)

Now consider the $L^2$ function given by

$$\Lambda_n^\lambda(x) = \frac{\sqrt{L}}{1 - e^{-\lambda x}} e^{-\lambda_n x}, \quad \forall n \in \mathbb{Z}, \quad \forall x \in [0, L].$$ (48)

Then, for all $m \geq 0$, $\Lambda_n^\lambda \in H^m$, and we have

$$\langle \Lambda_n^\lambda, e_p \rangle = \frac{1}{\sqrt{L}} \int_0^L \frac{\sqrt{L}}{1 - e^{-\lambda x}} e^{-\lambda_n x} e^{-2i\pi p x} dx = \frac{1}{1 - e^{-\lambda x}} \int_0^L e^{-\lambda_{n+p} x} dx = \frac{1}{\lambda_{n+p}}, \quad \forall n, p \in \mathbb{Z},$$ (49)

so that, using (47) and (9),

$$(\Lambda_n^\lambda)' + \lambda_n \Lambda_n^\lambda = \sum_{p \in \mathbb{Z}} e_p \text{ in } \mathcal{E}'. $$ (50)

Remark 4. In $\mathcal{E}'$, $\sum_{p \in \mathbb{Z}} e_p$ is the equivalent of the Dirac comb, or the “Dirac distribution” on the space of functions on $[0, L]$. So, in a sense, $\Lambda_n^\lambda$ is the elementary solution of (45).

2.2. Construction of Riesz bases for Sobolev spaces. Now we will use the above heuristic considerations to rigorously build the transformation $T$. Let us define, in analogy with the elementary solution method,

$$k_{n, \lambda} = -F_{-n} \Lambda_n^\lambda \ast \varphi \in H^m_{\text{per}}, \quad \forall n \in \mathbb{Z},$$ (51)

where $F = (F_n)_{n \in \mathbb{Z}} \in \mathcal{E}'$ will be characterized below. The regularity comes from the definition of the convolution product, (14), (49) and (46), and one can check, using (47), that $k_{n, \lambda}$ is a solution of (45).

The next step to build an invertible transformation is to find conditions under which $(k_{n, \lambda})$ is some sort of basis. More precisely we use the notion of Riesz basis (see [7]*Chapter 4)

Definition 2.2. A Riesz basis in a Hilbert space $H$ is the image of an orthonormal basis of $H$ by a bounded isomorphism.
Proposition 2.3. Let $H$ be a Hilbert space. A family of vectors $(f_k)_{k \in \mathbb{Z}} \in H$ is a Riesz basis if and only if it is complete (i.e., $\text{Span}(f_k) = H$) and there exist constants $C_1, C_2 > 0$ such that, for any scalar sequence $(a_k)$ with a finite number of non-zero elements,

$$C_1 \sum |a_k|^2 \leq \left\| \sum a_k f_k \right\|^2_H \leq C_2 \sum |a_k|^2.$$  \hfill (52)

Let us now introduce the following growth condition:

Definition 2.4. Let $s \geq 0$, $(u_n) \in C^\mathbb{Z}$ (or $u \in \mathcal{L}^\prime$). We say that $(u_n)$ (or $u$) has $s$-growth if

$$c \sqrt{1 + \left| 2i\pi n L \right|^2} \leq |u_n| \leq C \sqrt{1 + \left| 2i\pi n L \right|^2}, \quad \forall n \in \mathbb{Z},$$  \hfill (53)

for some $c, C > 0$. The optimal constants for these inequalities are called growth constants.

We can now establish the following Riesz basis property for the $(k_{n,\lambda})$:

Proposition 2.5. Let $m \in \mathbb{N}$. If $(F_n)$ has $m$-growth, then the family of functions

$$(k_{n,\lambda}^m) := \left( \frac{k_{n,\lambda}}{\sqrt{1 + \left| 2i\pi n L \right|^2m}} \right)$$

is a Riesz basis for $H^m_{\text{per}}$.

Proof. We use the characterization of Riesz bases given in Proposition 2.3. First, let us prove the completeness of $(k_{n,\lambda})$. Let $f \in H^m_{\text{per}}$ be such that

$$(f, k_{n,\lambda}^m)_m = 0, \quad \forall n \in \mathbb{Z}. \hfill (54)$$

For all $n \in \mathbb{Z}$ we get, by (53),

$$F_n \neq 0, \quad \forall n \in \mathbb{Z}.$$  \hfill (53)

Then, using (49), (54) becomes

$$0 = (\Lambda^\lambda_n \ast \varphi, f)_m$$

$$= \sum_{p \in \mathbb{Z}} \frac{f_p \varphi_p}{\lambda_n + p} \left( 1 + \left| \frac{2i\pi p L}{L} \right|^{2m} \right)$$

$$= \left( \Lambda^\lambda_n, \sum_{p \in \mathbb{Z}} \left( 1 + \left| \frac{2i\pi p L}{L} \right|^{2m} \right) f_p \overline{\varphi_p} e_p \right)_m,$$

as, thanks to (14), and using the fact that $f \in H^m_{\text{per}},$

$$\sum_{p \in \mathbb{Z}} \left( 1 + \left| \frac{2i\pi p L}{L} \right|^{2m} \right) f_p \overline{\varphi_p} e_p \in \mathcal{L}^2.$$  \hfill (53)

Now, $(\Lambda^\lambda_n)$ is a complete family of $\mathcal{L}^2$, so that

$$f_p \varphi_p = 0, \quad \forall p \in \mathbb{Z}.$$  \hfill (14)

Recalling condition (14), this yields

$$f_p = 0, \quad \forall p \in \mathbb{Z},$$

which proves the completeness of $(k_{n,\lambda}^s)$.
Now let \( I \subset \mathbb{Z} \) be a finite set, and \((a_n) \in \mathbb{C}^I\). Then,

\[
\left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 = \left\| \sum_{n \in I} -a_n \frac{F_{-n}}{\sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \Lambda_{n,\lambda}^* \varphi \right\|_m^2 = \left\| \sum_{n \in I} a_n \frac{F_{-n}}{\sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \sum_{p \in \mathbb{Z}} \frac{\varphi_p}{\lambda_{n+p}} e_p \right\|_m^2 = \left\| \sum_{p \in \mathbb{Z}} \varphi_p \left( \sum_{n \in I} \frac{a_n F_{-n}}{\lambda_{n+p} \sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \right) e_p \right\|_m^2 = \sum_{p \in \mathbb{Z}} \left( 1 + \left| \frac{2\pi p}{L} \right|^{2m} \right) |\varphi_p|^2 \left\| \sum_{n \in I} \frac{a_n F_{-n}}{\lambda_{n+p} \sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \right\|_m^2.
\]

Now, using condition (14), we have

\[
c^2 \sum_{n \in I} \left| \sum_{p \in \mathbb{Z}} a_n \frac{F_{-n}}{\lambda_{n+p} \sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \right|^2 \leq \left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 \leq C^2 \sum_{p \in \mathbb{Z}} \left| \sum_{n \in I} a_n \frac{F_{-n}}{\lambda_{n+p} \sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \right|^2.
\]

This last inequality can be rewritten

\[
c^2 \left\| \sum_{n \in I} \frac{a_n F_{-n}}{\sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \lambda_{n,\lambda}^* \right\|_m^2 \leq \left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 \leq C^2 \left\| \sum_{n \in I} \frac{a_n F_{-n}}{\sqrt{1 + \left| \frac{2\pi n}{L} \right|^2 s}} \lambda_{n,\lambda}^* \right\|_m^2,
\]

as

\[
\Lambda_{n,\lambda} = \sum_{p \in \mathbb{Z}} \frac{1}{\lambda_{n+p}} e_p.
\]

We now use the fact that \((\Lambda_{n,\lambda})\) is a Riesz basis of \(L^2\): indeed, from (6) and (48), it is the image of the Hilbert basis \((e_n)\) by the bounded isomorphism

\[
\Lambda_{\lambda} : f \in L^2 \mapsto \Lambda_{\lambda} \sqrt{L} f.
\]

The norms of \(\Lambda_{\lambda}\) and its inverse are rather straightforward to compute using the maximizing sequences of piecewise constant functions

\[
\chi_{[0,1/n]} \text{ for } \Lambda_{\lambda}, \quad \chi_{[L,L-1/n]} \text{ for } \left(\Lambda_{\lambda}\right)^{-1}.
\]

We have

\[
\|\Lambda_{\lambda}\|_{L^2} = \frac{L}{1 - e^{-\lambda L}},
\]

\[
\|\left(\Lambda_{\lambda}\right)^{-1}\|_{L^2} = \frac{1 - e^{-\lambda L}}{L} e^{\lambda L},
\]

\[
\|\Lambda_{\lambda}\|_{L^2} = \frac{L}{1 - e^{-\lambda L}},
\]

\[
\|\left(\Lambda_{\lambda}\right)^{-1}\|_{L^2} = \frac{1 - e^{-\lambda L}}{L} e^{\lambda L},
\]
so that
\[
\frac{1}{\|\Lambda^{-1}\|_2^2} \sum_{n \in I} \left| \frac{a_n F_{-n}}{\sqrt{1 + \frac{2|\pi n|}{L} 2^s}} \right|^2 \leq \left\| \Lambda^{\lambda} \right\|_2^2 \sum_{n \in I} \left| \frac{a_n F_{-n}}{\sqrt{1 + \frac{2|\pi n|}{L} 2^s}} \right|^2.
\]
and we finally get, using the fact that \((F_n)\) has \(s\)-growth,
\[
c^2 C^2 \frac{1}{\|\Lambda^{-1}\|_2^2} \sum_{n \in I} |a_n|^2 \leq \left\| \sum_{n \in I} a_n k_{n,\lambda}^s \right\|_m^2 \leq C^2 C^2 \|\Lambda^{\lambda}\|_2^2 \sum_{n \in I} |a_n|^2.
\]
where \(C_1, C_2 > 0\) are the growth constants of \((F_n)\), so that the constants in the inequalities above are optimal. Hence, using again (52), \((k_{n,\lambda})\) is a Riesz basis of \(H^m_{\text{per}}\).

We now have a class of invertible transformations, under some conditions on \(F\):

**Corollary 2.6.** Let \(m \in \mathbb{N}^*\), and \(F\) such that \((F_n)\) has \(m\)-growth, with growth constants \(C_1, C_2 > 0\). Define
\[
T^{\lambda} \alpha := \sum_{n \in \mathbb{Z}} \sqrt{1 + \frac{2|\pi n|}{L} 2^m} a_n k_{n,\lambda}^{m} = \sum_{n \in \mathbb{Z}} a_n k_{n,\lambda} \in H^m_{\text{per}}, \quad \forall \alpha \in H^m_{\text{per}}, \quad (57)
\]
where \(\alpha = \sum_{n \in \mathbb{Z}} a_n e_n\). Then, \(T^{\lambda} : H^m_{\text{per}} \rightarrow H^m_{\text{per}}\) is an isomorphism. Moreover,
\[
\|T^{\lambda}\|_{H^m_{\text{per}}} \leq \frac{CC_2 L}{1 - e^{-\lambda L}},
\]
\[
\|(T^{\lambda})^{-1}\|_{H^m_{\text{per}}} \leq \frac{1 - e^{-\lambda L}}{cc_1 L} e^{\lambda L}.
\]
(58)

**Proof.** The invertibility of \(T^{\lambda}\) is clear thanks to the Riesz basis property of \((k_{n,\lambda})\), and (58) comes from the above calculations. \(\Box\)

Finally, let us note that for \(F\) with \(m\)-growth, the associated isomorphism \(T^{\lambda}\) also writes, using (51) and (55),
\[
T^{\lambda} \alpha = -\varphi \ast \left( \Lambda^{\lambda}(\alpha \ast \widetilde{F}) \right)
\]
where \(\widetilde{F} \in \mathcal{E}'\) is defined by:
\[
\langle e_n, \widetilde{F} \rangle = F_{-n}, \quad n \in \mathbb{Z}.
\]

2.3. **Definition of the feedback law.** In order to further determine the feedback law, and define our final candidate for the backstepping transformation, the idea is now to invoke to the \(TB = B\) condition, as we have used it in the formal computations of Section 2.1, in the equation (44). However, in this case, \(\varphi \notin H^m_{\text{per}}\), and so it is not clear whether \(T^{\lambda} \varphi\) is well-defined.
We can nonetheless obtain a $TB = B$ condition in some weak sense: indeed, let us set
\[ \phi^{(N)} := \sum_{n=-N}^{N} \varphi_n e_n \in H^m_{\text{per}}, \quad \forall N \in \mathbb{N}. \]
Then,
\[ \phi^{(N)} \xrightarrow{N \to \infty} \phi \]
and
\[ T^\lambda \phi^{(N)} = \sum_{n=-N}^{N} -\varphi_n F_n \lambda_n \ast \varphi \]
\[ = \sum_{n=-N}^{N} \sum_{p \in \mathbb{Z}} -\varphi_n F_n \lambda_\lambda_n \ast \varphi_p \]
\[ = \sum_{p \in \mathbb{Z}} \varphi_p \left( \sum_{n=-N}^{N} -\varphi_n F_n \lambda_\lambda_n \ast \varphi \right) e_p. \]

Now, notice that one can apply the Dirichlet convergence theorem for Fourier series (see for example [20]*Theorem 3.5.4, p.219) to $\Lambda_{\lambda}^p, p \in \mathbb{Z}$ at 0, and recalling (49):
\[ \sum_{n=-N}^{N} \frac{1}{\lambda_{-n+p}} = \sum_{n=-N}^{N} \frac{1}{\lambda_{n+p}} \]
\[ = \sqrt{L} \sum_{n=-N}^{N} (\Lambda_{\lambda}^p, e_n) e_n(0) \]
\[ \xrightarrow{N \to \infty} \sqrt{L} \Lambda_{\lambda}^p(0) + \Lambda_{\lambda}^p(L) = \frac{L}{2} \left( 1 - e^{-\lambda L} \right) \left( 1 + e^{-\lambda L} \right). \]

Let us note
\[ K(\lambda) := \frac{2}{L} \frac{1 - e^{-\lambda L}}{1 + e^{-\lambda L}} > 0, \]
and set
\[ F^\lambda_n := \frac{K(\lambda)}{\varphi_n}, \quad \forall n \in \mathbb{Z}. \]

This defines a feedback law $F^\lambda \in \mathcal{E}'$ which is real-valued (in the sense given in page 2), as $\varphi$ is real-valued, and which has $m$-growth thanks to condition (14). Then the $k_{n,\lambda}$ defined by (51) define an isomorphism $T^\lambda : H^m_{\text{per}} \to H^m_{\text{per}}$ by Corollary 2.6:
\[ T^\lambda \alpha = \sum_{n \in \mathbb{Z}} \alpha_n k_{-n,\lambda}, \quad \forall \alpha \in H^m_{\text{per}}. \]

and
\[ \|T^\lambda\|_{H^m_{\text{per}}} \leq \frac{C K(\lambda) \sqrt{L}}{c(1 - e^{-\lambda L})}, \]
\[ \|(T^\lambda)^{-1}\|_{H^m_{\text{per}}} \leq \frac{C(1 - e^{-\lambda L})}{cK(\lambda) \sqrt{L}} e^{\lambda L}. \]

Moreover, from (59) and (60) we have the following expression for the inverse:
\[ (T^\lambda)^{-1} \alpha = \frac{1}{K(\lambda)^2} \varphi \ast \left( \frac{1}{\Lambda_{\lambda}} (\alpha \ast \tilde{F}_\lambda) \right), \quad \forall \alpha \in H^m_{\text{per}}. \]
Finally, note that
\[
\langle T^\lambda \varphi^{(N)}, e_p \rangle = \varphi_p K(\lambda) \sum_{n=-N}^{N} \frac{1}{\lambda - n + p} \xrightarrow{N \to \infty} \langle \varphi, e_p \rangle, \quad \forall p \in \mathbb{Z},
\] (64)
which corresponds to the \( TB = B \) condition in some weak sense.

We now have a feedback and an associated invertible transformation. We will use properties of both to prove our stabilization result.

2.4. Regularity of the feedback law. In order to study the well-posedness of the closed-loop system corresponding to (60), we need some information on the regularity of \( F^\lambda \).

Let us first begin by a general lemma for linear forms with coefficients that have \( m \)-growth:

**Lemma 2.7.** Let \( m \geq 0 \), and \( G \in \mathcal{E}' \) with \( m \)-growth.

Then, for all \( s > 1/2 \), \( G \) can be extended to a linear form on \( H_{\text{per}}^{m+s} \), continuous for \( \| \cdot \|_{m+s} \), but not for \( \| \cdot \|_{m+s} \), for \( -m \leq \sigma < 1/2 \).

In particular, the feedback law \( F^\lambda \in \mathcal{E}' \) defined by (60) can be extended to a linear form on \( H_{\text{per}}^{m+1} \) which is continuous for \( \| \cdot \|_{m+1} \) but not for \( \| \cdot \|_m \).

**Proof.** Let \( s > 1/2 \), and let \( \alpha \in H_{\text{per}}^{m+s} \). Using the growth conditions (53), we can do the following computations for \( \alpha \in H_{\text{per}}^{m+s} \):

\[
\sum_{n \in \mathbb{Z}} |G_n| \|\alpha_n\| \leq C \sum_{n \in \mathbb{Z}} \sqrt{1 + \frac{2|2\pi n|^{2m}}{L}} |\alpha_n| \\
\leq C' \sum_{n \in \mathbb{Z}} \frac{1}{1 + |n|^s} \sqrt{1 + \frac{2|2\pi n|^{2m+2s}}{L}} |\alpha_n| \\
\leq C' \left( \sum_{n \in \mathbb{Z}} \frac{1}{(1 + |n|^s)^2} \right) \|\alpha\|_{m+s}
\]

where \( C, C' > 0 \) are constants that do not depend on \( \alpha \), and where the last inequality is obtained using the Cauchy-Schwarz inequality, and the series in the first factor converges because \( s > 1/2 \). Thus

\[
\langle \alpha, G \rangle := \sum_{n \in \mathbb{Z}} G_n \alpha_n, \quad \forall \alpha \in H_{\text{per}}^{m+s},
\]
defines a continuous linear form on \( H_{\text{per}}^{m+s} \).

On the other hand, let \( -m \leq \sigma < 1/2 \), and consider, for \( N \geq 1 \),

\[
\gamma^{(N)} := \sum_{|n| \geq N} \frac{1}{G_n (1 + |n|^{1+s})} e_n \in H_{\text{per}}^{m+s}.
\]

We have

\[
\|\gamma^{(N)}\|_{m+s}^2 = \sum_{|n| \geq N} \left( 1 + \frac{2|2\pi n|^{2m+2s}}{L} \right) \frac{1}{|G_n|^2} \frac{1}{(1 + |n|^{1+s})^2} \leq C \sum_{|n| \geq N} \frac{1}{1 + |n|^{2s+2-2\sigma}}
\]
for some constant $C > 0$. Then,

$$\left| \langle \gamma(N), G \rangle \right| = \sum_{|n| \geq N} \frac{1}{1 + |n|^{1+s}} \geq c \sum_{|n| \geq N} \frac{|n|^{1+s-2\sigma}}{1 + |n|^{2+2s-2\sigma}} \geq c N^{1+s-2\sigma} \sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}} \| \gamma(N) \|_{m+\sigma}$$

(65)

for some constants $c, c' > 0$. Now, we know that there exists $c'', > 0$ such that

$$c'' N^{1+2s-2\sigma} \leq \sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}},$$

so that

$$\frac{c''}{N^{1+2s-2\sigma}} \leq \sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}},$$

$$\left| \langle \gamma(N), G \rangle \right| \| \gamma(N) \|_{m+\sigma} = c' N^{1+s-2\sigma} \sum_{|n| \geq N} \frac{1}{1 + |n|^{2+2s-2\sigma}} \geq c' c'' N^{\frac{1}{2}-\sigma} \to \infty \quad \text{as } N \to \infty.$$

Together with (65), this proves that $G$ is not continuous for $\| \cdot \|_{m+\sigma}$. \hfill $\square$

Let us now give a more precise description of the domain of definition and regularity of $F^\lambda$. Recalling the identity (20), and the fact that $\varphi$ is real-valued together with (7), we can derive the following identity for $F^\lambda_n$ from (60):

$$F^\lambda_n = (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left( \frac{2i\pi n}{L} \right)^m \sum_{j=0}^d \langle \chi[\sigma_j, \sigma_{j+1}] \varphi, e_n \rangle$$

$$+ (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left( \frac{2i\pi n}{L} \right)^m \sum_{j=0}^d \langle \chi[\sigma_j, \sigma_{j+1}] \varphi, e_n \rangle, \quad \forall n \in \mathbb{Z}^*,

(66)

so that, using (24) and (22),

$$\left( \left( \frac{2i\pi n}{L} \right)^m \left( F^\lambda_n - (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left( \frac{2i\pi n}{L} \right)^m \right) \right)_{n \in \mathbb{Z}^*} \in \ell^2.$$

(67)

Let us then note

$$h^\lambda_n := (-1)^m \frac{K(\lambda)}{\tau_{-n}^\varphi} \left( \frac{2i\pi n}{L} \right)^m, \quad \forall n \in \mathbb{Z},

(68)$$

and $h^\lambda$ the associated linear form in $E'$. We now prove that $F^\lambda$ is the sum of a regular part $F^\lambda$ and a singular part corresponding to $h^\lambda$:
Proposition 2.8. The linear form \( h^\lambda \) can be extended to the following linear form on \( \tau^\varphi(H^{m+1}_{\text{pw}}) \), continuous for \( \| \cdot \|_{m+1,\text{pw}} \):

\[
\langle \alpha, h^\lambda \rangle = \sqrt{L} \frac{K(\lambda)}{2} \left( \partial_x^m ((\tau^\varphi)^{-1} \alpha)(0) + \partial_x^m ((\tau^\varphi)^{-1} \alpha)(L) \right), \quad \forall \alpha \in \tau^\varphi(H^{m+1}_{\text{pw}}).
\]

Moreover, \( F^\lambda := F^{\lambda} - h^\lambda \) is continuous for \( \| \cdot \|_m \), so that \( F^\lambda \) is defined on \( \tau^\varphi(H^{m+1}_{\text{pw}}) \cap H^m_{\text{per}}, \) and is continuous for \( \| \cdot \|_{m+1,\text{pw}} \), but not for \( \| \cdot \|_m \).

Proof. It is clear, by definition of \( H^m_{\text{per}} \), and using (67), that for \( \alpha \in H^m_{\text{per}} \), the expression:

\[
\langle \alpha, F^\lambda - h^\lambda \rangle = \sum_{n \in \mathbb{Z}} \alpha_n (\overline{F^\lambda_n} - \overline{h^\lambda_n}) = K(\lambda)\alpha_0 + \sum_{n \neq 0} \left( \frac{2i\pi n}{L} \right)^m \alpha_n \frac{1}{\tau^\varphi} \sqrt{L}
\]

(70)
defines a continuous linear form on \( H^m_{\text{per}} \).

On the other hand, let \( \alpha \in \tau^\varphi(H^{m+2}_{\text{pw}}) \), then

\[
\sum_{n=-N}^{N} \alpha_n \overline{h^\lambda_n} = \sqrt{L}K(\lambda) \sum_{n=-N}^{N} \left( \frac{2i\pi n}{L} \right)^m \alpha_n \frac{1}{\tau^\varphi} \sqrt{L}
\]

and we can use the Dirichlet convergence theorem (see [20]) on \( \partial_x^m ((\tau^\varphi)^{-1} \alpha) \) ∈ \( H^1_{\text{pw}} \) at 0, so that

\[
\sum_{n=-N}^{N} \alpha_n \overline{h^\lambda_n} = \sqrt{L}K(\lambda) \sum_{n=-N}^{N} \left( \frac{2i\pi n}{L} \right)^m \alpha_n \frac{1}{\tau^\varphi} \sqrt{L}
\]

(70)

\[
= \sqrt{L}K(\lambda) \sum_{n=-N}^{N} \langle \partial_x^m ((\tau^\varphi)^{-1} \alpha), e_n(0) \rangle \to \sqrt{L}K(\lambda) \left( \partial_x^m ((\tau^\varphi)^{-1} \alpha)(0) + \partial_x^m ((\tau^\varphi)^{-1} \alpha)(L) \right).
\]

Now, \( \tau^\varphi : H^{m+2}_{\text{pw}} \to \tau^\varphi(H^{m+1}_{\text{pw}}) \) is continuous, as it is a sum of translations by (26). We also know that \( H^{m+2}_{\text{pw}} \) is dense in \( H^{m+1}_{\text{pw}} \) for \( \| \cdot \|_{m+1,\text{pw}} \). Hence, \( \tau^\varphi(H^{m+2}_{\text{pw}}) \) is dense in \( \tau^\varphi(H^{m+1}_{\text{pw}}) \) for \( \| \cdot \|_{m+1,\text{pw}} \).

Moreover, using the Sobolev inequality for \( H^1 \) and \( L^\infty \) (see example [5]Chapter 8, Theorem 8.8), we have, for \( \alpha \in \tau^\varphi(H^{m+2}_{\text{pw}}) \),

\[
|\langle \alpha, h^\lambda \rangle| \leq \sqrt{L}K(\lambda) \left( \| \partial_x^m ((\tau^\varphi)^{-1} \alpha)(0) \|_{1,\text{pw}} + \| \partial_x^m ((\tau^\varphi)^{-1} \alpha)(L) \|_{1,\text{pw}} \right)
\]

(71)
\[
\leq \sqrt{L}K(\lambda) \| \partial_x^m ((\tau^\varphi)^{-1} \alpha) \|_{1,\text{pw}} \quad \text{(Sobolev inequality)}
\]

\[
\leq C\sqrt{L}K(\lambda) \| \alpha \|_{m+1,\text{pw}} \quad \text{(} \tau^\varphi \text{ is an isomorphism)}
\]

Hence \( h^\lambda \) is continuous for \( \| \cdot \|_{m+1,\text{pw}} \), so that we can extend it from \( \tau^\varphi(H^{m+2}_{\text{pw}}) \) to \( \tau^\varphi(H^{m+1}_{\text{pw}}) \), by density. We also get that \( h^\lambda \) is not continuous for \( \| \cdot \|_m \), as \( \alpha \in H^m \to \partial_x^m \alpha(0) \) and \( \alpha \in H^m \to \partial_x^m \alpha(L) \) are not continuous for \( \| \cdot \|_m \).

Thus, \( F^\lambda := F^\lambda + h^\lambda \) is defined on \( \tau^\varphi(H^{m+1}_{\text{pw}}) \cap H^m_{\text{per}} \), is continuous for \( \| \cdot \|_{m+1} \) but not for \( \| \cdot \|_m \).

外
3. Well-posedness and stability of the closed-loop system. Let \( m \geq 1, \varphi \in H_{(pu)}^m \cap H_{\per}^{m-1} \) satisfying growth condition (14). Let the feedback law \( F^\lambda \) be defined by (60).

3.1. Operator equality. Now that we have completely defined the feedback \( F^\lambda \) and the transformation \( T^\lambda \), let us check that we have indeed built a backstepping transformation. As in the finite dimensional example of subsection 1.4, this corresponds to the formal operator equality

\[
T(A + BK) = (A - \lambda I)T.
\]

Let us define the following domain:

\[
D_m := \{ \alpha \in \tau^\varphi (H_{(pu)}^{m+1}) \cap H_{\per}^m, \quad -\alpha_x - \mu \alpha + \langle \alpha, F^\lambda \rangle \varphi \in H_{\per}^m \}.
\]  

(72)

Notice that, as \( \varphi \in H_{(pu)}^m \), we have \( D_m \subset H_{(pu)}^{m+1} \). Let us first check the following property:

**Proposition 3.1.** For \( m \geq 1 \), \( D_m \) is dense in \( H_{\per}^m \) for \( \| \cdot \|_m \).

**Proof.** From (25) we know that \( \tau^\varphi (H_{\per}^{m+1}) = H_{\per}^{m+1} \). Moreover \( H_{\per}^{m+1} \subset H_{(pu)}^{m+1} \), so \( H_{\per}^{m+1} \subset \tau^\varphi (H_{(pu)}^{m+1}) \). Hence

\[
K_m := \{ \alpha \in H_{\per}^{m+1}, \langle \alpha, F^\lambda \rangle = 0 \} \subset D_m.
\]

Now, by Lemma 2.7, as \( F^\lambda \) has \( m \)-growth, \( K_m \) is dense in \( H_{\per}^{m+1} \) for \( \| \cdot \|_m \), as the kernel of the linear form \( F^\lambda \) which is not continuous for \( \| \cdot \|_m \). As \( H_{\per}^{m+1} \) is dense in \( H_{\per}^m \), then \( D_m \) is dense in \( H_{\per}^m \) for \( \| \cdot \|_m \). \( \square \)

Now, on this dense domain, let us establish the operator equality:

**Proposition 3.2.** We have:

\[
T^\lambda (-\partial_x - \mu I + \langle \cdot, F^\lambda \rangle \varphi) \alpha = (-\partial_x - \lambda I)T^\lambda \alpha \quad \forall \alpha \in D_m.
\]  

(73)

**Proof.** First let us rewrite (73) in terms of \( \lambda = \lambda' + \mu \):

\[
T^\lambda (-\partial_x + \langle \cdot, F^\lambda \rangle \varphi) \alpha = (-\partial_x - \lambda I)T^\lambda \alpha \quad \forall \alpha \in D_m.
\]

Let \( \alpha \in D_m \). By definition of the domain \( D_m \) and by construction of \( T^\lambda : H_{\per}^m \to H_{\per}^m \) in Corollary 2.6, the left-hand side of (73) is a function of \( H_{\per}^m \subset \mathcal{E}' \) and the right-hand side of (73) is a function of \( H_{\per}^{m-1} \subset \mathcal{E}' \). To prove that these functions are equal, it is thus sufficient to prove their equality in \( \mathcal{E}' \). Let us then write each term of the equality against \( e_n \) for \( n \in \mathbb{Z} \). One has

\[
\langle (-\partial_x - \lambda I)T^\lambda \alpha, e_n \rangle = \left\langle T^\lambda \alpha, \frac{2i \pi n}{L} e_n \right\rangle - \lambda \langle T^\lambda \alpha, e_n \rangle = -\lambda_n \langle T^\lambda \alpha, e_n \rangle. \quad \text{(recalling (46))}
\]

Let us now prove that

\[
\langle T^\lambda (-\partial_x \alpha + \langle \alpha, F^\lambda \rangle \varphi), e_n \rangle = -\lambda_n \langle T^\lambda \alpha, e_n \rangle, \quad \forall n \in \mathbb{Z}.
\]  

(74)
As we only have $\partial_x \alpha \in H_{per}^{m-1}$, $T^\lambda \partial_x \alpha$ is not defined a priori. In order to allow for more computations, let us define

$$\alpha^{(N)} := \sum_{n=-N}^{N} \alpha_n e_n, \quad \forall N \in \mathbb{N},$$

$$\phi^{(N)} := \sum_{n=-N}^{N} \phi_n e_n, \quad \forall N \in \mathbb{N}.$$ 

We then have, by linearity of the partial Fourier sum, and because $-\partial_x \alpha + \langle \alpha, F^\lambda \rangle \phi \in H_{per}^m$, by definition of $D_m$,

$$-(\partial_x \alpha)^{(N)} + \langle \alpha, F^\lambda \rangle \phi^{(N)} = (-\partial_x \alpha + \langle \alpha, F^\lambda \rangle \phi^{(N)}) \xrightarrow{H_{per}^m \to N \to \infty} -\partial_x \alpha + \langle \alpha, F^\lambda \rangle \phi,$$

so that in particular,

$$\langle T^\lambda (-\partial_x \alpha)^{(N)} + \langle \alpha, F^\lambda \rangle \phi^{(N)}, e_n \rangle \xrightarrow{N \to \infty} \langle T^\lambda (-\partial_x \alpha + \langle \alpha, F^\lambda \rangle \phi), e_n \rangle.$$  

(75)

Let $N \in \mathbb{N}$. We can write, using (57), for $n \in \mathbb{Z}$,

$$\langle T^\lambda (-\partial_x \alpha)^{(N)} + \langle \alpha, F^\lambda \rangle \phi^{(N)}, e_n \rangle = -(\partial_x \alpha)^{(N)}, e_n \rangle + \langle \alpha, F^\lambda \rangle \langle T^\lambda \phi^{(N)}, e_n \rangle$$

$$= -\left\langle \sum_{p=-N}^{N} \frac{2i\pi p}{L} \partial_x k_{-p, \lambda} + \lambda k_{-p, \lambda} + F_p \phi, e_n \right\rangle + \langle \alpha, F^\lambda \rangle \langle T^\lambda \phi^{(N)}, e_n \rangle.$$ 

Now, using (45) and (46), we get

$$\frac{2i\pi p}{L} k_{-p, \lambda} = \partial_x k_{-p, \lambda} + \lambda k_{-p, \lambda} + F_p \phi,$$

so that

$$-T^\lambda (-\partial_x \alpha)^{(N)} = \sum_{p=-N}^{N} \alpha_p \left( \partial_x k_{-p, \lambda} + \lambda k_{-p, \lambda} + F_p \phi \right)$$

$$= \partial_x \left( \sum_{p=-N}^{N} \alpha_p k_{-p, \lambda} \right) + \lambda \sum_{p=-N}^{N} \alpha_p k_{-p, \lambda} + \langle \alpha^{(N)}, F^\lambda \rangle \phi$$

$$= -\partial_x T^\lambda \left( \alpha^{(N)} \right) - \lambda T^\lambda \left( \alpha^{(N)} \right) + \langle \alpha^{(N)}, F^\lambda \rangle \phi.$$ 

Hence

$$-\langle T^\lambda (-\partial_x \alpha)^{(N)}, e_n \rangle = -\left\langle \partial_x T^\lambda \alpha^{(N)}, e_n \right\rangle - \lambda \left\langle T^\lambda \alpha^{(N)}, e_n \right\rangle - \langle \alpha^{(N)}, F^\lambda \rangle \phi_n,$$

and finally, by integration by parts in the first term, (46), and adding $\langle \alpha, F^\lambda \rangle T^\lambda \phi^{(N)}$ to both sides of the previous equality, we get:

$$\langle T^\lambda (-\partial_x \alpha)^{(N)} + \langle \alpha, F^\lambda \rangle \phi^{(N)}, e_n \rangle$$

$$= -\lambda \left\langle T^\lambda \alpha^{(N)}, e_n \right\rangle + \langle \alpha - \alpha^{(N)}, F^\lambda \rangle \phi_n + \langle \alpha, F^\lambda \rangle \left( \langle T^\lambda \phi^{(N)} - \phi, e_n \rangle \right).$$

(76)
To deal with the third term of the right-hand side of this equality, recall that we have chosen a feedback law so that the weak $TB = B$ condition (64) holds. Thus,

$$\langle T^\lambda \varphi^{(N)} - \varphi, e_n \rangle \xrightarrow{N \to \infty} 0. \quad (77)$$

To deal with the second term, recall that $F^\lambda$ is the sum of a regular part $F^\lambda$ and a singular part $h^\lambda$:

$$\langle \alpha - \alpha^{(N)}, F^\lambda \rangle = \langle \alpha - \alpha^{(N)}, F^\lambda \rangle + \langle \alpha - \alpha^{(N)}, h^\lambda \rangle.$$  

Now, by definition of $\alpha^{(N)}$ and continuity of $F^\lambda$ for $\| \cdot \|_m$,

$$\langle \alpha - \alpha^{(N)}, F^\lambda \rangle \xrightarrow{N \to \infty} 0. \quad (78)$$

On the other hand, for all $N \in \mathbb{N}$, from (68) we get

$$\langle \alpha^{(N)}, h^\lambda \rangle = K(\lambda) \sum_{n=-N}^{N} \frac{\alpha_n}{\tau_n} \left( \frac{2i\pi n}{L} \right)^m$$

$$= \frac{K(\lambda)}{2} \sum_{n=-N}^{N} \left( \frac{\alpha_n}{\tau_n} + (-1)^m \frac{\alpha_{-n}}{\tau_{-n}} \right) \left( \frac{2i\pi n}{L} \right)^m.$$

(79)

where

$$\tilde{\tau}^\varphi f = \sum_{n \in \mathbb{Z}} \left( \frac{f_n}{\tau_n} + (-1)^{m-1} \frac{f_{-n}}{\tau_{-n}} \right) e_n,$$  

defines a bounded operator $\tilde{\tau}^\varphi : H^m_{\text{per}} \to H^m_{\text{per}}$ thanks to (24).

**Remark 5.** When $\varphi \in H^m$, $\tilde{\tau}^\varphi$ is simply $(1/\sqrt{L})(\partial_x^{m-1} \varphi(L) - \partial_x^{m-1} \varphi(0))I$, $F^\lambda$ is defined on $H^{m+1}_{(\text{pw})} \cap H^m_{\text{per}}$, and $\tilde{\tau}^\varphi \alpha$ is simply, up to a constant factor, the symmetrisation $\alpha + (-1)^{m-1}(\alpha(L- \cdot))$, which is $H^m_{\text{per}}$ if $\alpha \in H^m \cap H^m_{\text{per}}$.

In general, $\tilde{\tau}^\varphi$ is an operator that computes the “smooth part” of a $H^m_{(\text{pw})}$ function.

Notice that, by definition of $\tilde{\tau}^\varphi$ and $D_m$,

$$\tilde{\tau}^\varphi \left( -\partial_x \alpha - \mu \alpha + \langle \alpha, F^\lambda \rangle \right) \in H^m_{\text{per}}. \quad (81)$$

Moreover, using (20) and (24), we have for $n \in \mathbb{Z}^*$:

$$\frac{\varphi_n}{\tau_n} + (-1)^{m-1} \frac{\varphi_{-n}}{\tau_{-n}} = \frac{\varphi_n}{\tau_n} + (-1)^{m-1} \frac{\varphi_{-n}}{\tau_{-n}} \quad \frac{r_n}{2\pi/L}$$

$$= -1 \frac{(-1)^{m-1}(-1)^m}{2\pi/L} + \frac{r_n}{2\pi/L} = \frac{r_n}{2\pi/L}^{m},$$

where $r_n \in L^2$ thanks to (24). Hence, $\tilde{\tau}^\varphi \varphi \in H^m_{\text{per}}$. This, together with (81), yields

$$\tilde{\tau}^\varphi \partial_x \alpha \in H^m_{\text{per}}.$$  

Then, by definition of $H^m_{\text{per}}$,

$$\tilde{\tau}^\varphi (\partial_x \alpha)^{(N)} \xrightarrow{N \to \infty} \tilde{\tau}^\varphi \partial_x \alpha,$$

as $\tilde{\tau}^\varphi (\partial_x \alpha)^{(N)}$ is the partial sum of $\tilde{\tau}^\varphi \partial_x \alpha \in H^m_{\text{per}}$.  

Hence, by continuity of \( \alpha \mapsto \partial_x^{-1} \alpha(0) \) for \( \| \cdot \|_m \), (79) implies that
\[
\left\langle \alpha - \alpha^{(N)} , h \right\rangle \xrightarrow{N \to \infty} 0. \tag{82}
\]

Finally, (76), (77), (78), (82), and the continuity of \( T^\lambda \) yield
\[
( T^\lambda (-(\partial_x \alpha)^{(N)} + \langle \cdot , F^\lambda \rangle \varphi^{(N)}), e_n ) \xrightarrow{N \to \infty} -\lambda_n \langle T^\lambda \alpha, e_n \rangle.
\]
This, put together with (75), gives (74) by uniqueness of the limit, which in turn proves (73).

3.2. Well-posedness of the closed-loop system. The operator equality we have established in the previous section means that \( T^\lambda \) transforms, if they exist, solutions of the closed-loop system with a well-chosen feedback into solutions of the target system. Let us now check that the closed-loop system in question is indeed well-posed in some sense.

**Proposition 3.3.** The operator \( A + BK := -\partial_x - \mu \alpha + \langle \cdot , F^\lambda \rangle \varphi \) defined on \( D_m \) is a dense restriction of the infinitesimal generator of a \( C^0 \)-semigroup on \( H^m_{per} \).

**Proof.** We know from Lemma 3.1 that \( A + BK \) is densely defined on \( D_m \subset H^m_{per} \).

Now, define the following semigroup on \( H^m_{per} \):
\[
S_\lambda(t)\alpha := e^{-\lambda t} \alpha(-t), \quad \forall \alpha \in H^m_{per}, \quad t \geq 0, \tag{83}
\]
where \( \alpha \) on the right hand side is understood as its \( L \)-periodic continuation. This corresponds to the target system (41). Its infinitesimal generator is given by
\[
D^\lambda := H^m_{per} \setminus \partial_x - \lambda I. \tag{84}
\]

Now, define a second semigroup on \( H^m_{per} \):
\[
S(t)\alpha := (T^\lambda)^{-1} S_\lambda(t) T^\lambda \alpha, \quad \forall \alpha \in H^m_{per}, \quad t \geq 0. \tag{85}
\]
The infinitesimal generator of \( S(t) \) is given by the limit (when it exists) of
\[
\frac{S(t)\alpha - \alpha}{t} = (T^\lambda)^{-1} S_\lambda(t) T^\lambda \alpha - T^\lambda \alpha, \tag{86}
\]
on its domain which we note \( D^{F^\lambda}_m \). By (84) and (86), \( D^{F^\lambda}_m = (T^\lambda)^{-1}(D^\lambda) = (T^\lambda)^{-1}(H^m_{per} \setminus \partial_x) \), and the infinitesimal generator itself is given by
\[
\frac{S(t)\alpha - \alpha}{t} \xrightarrow{H^m_{per} \setminus \partial_x \to 0^+} (T^\lambda)^{-1}(-\partial_x - \lambda I) T^\lambda \alpha, \quad \forall \alpha \in (T^\lambda)^{-1}(H^m_{per} \setminus \partial_x). \tag{87}
\]

Now, by the operator equality (73), for \( \alpha \in D_m \),
\[
T^\lambda \alpha \in H^m_{per} \setminus \partial_x, \tag{88}
\]
i.e.
\[
\alpha \in D^{F^\lambda}_m, \tag{89}
\]
so that
\[
D_m \subset D^{F^\lambda}_m. \tag{90}
\]

Hence, putting (87) and (73) together,
\[
\frac{S(t)\alpha - \alpha}{t} \xrightarrow{H^m_{per} \setminus \partial_x \to 0^+} (T^\lambda)^{-1}(-\partial_x - \lambda I) T^\lambda \alpha = (-\partial_x - \mu I + \langle \cdot , F^\lambda \rangle \varphi)\alpha = (A + BK)\alpha, \quad \forall \alpha \in D_m. \tag{91}
\]
Hence, on the dense subset \( D_m \subset D_{\mu}^n \) (as \( D_m \) is dense in \( H_{\text{per}}^m \)), the infinitesimal generator of \( S(t) \) is given by \( A + BK \), which proves the proposition.

\[ \]

3.3. Stability of the closed-loop system. We can now prove Theorem 1.3.

Let \( S(t) \) the semigroup defined by (85), \( \alpha \in H_{\text{per}}^m \).

By definition of \( S(t) \), and using (62), we then get, for \( t \geq 0 \),

\[
\|S(t)\alpha\|_m \leq \|(T^\lambda)^{-1}\|\|S\lambda(t)\|T^\lambda\alpha\|_m \\
\leq \|(T^\lambda)^{-1}\|\|e^{-\lambda t}\|T^\lambda\alpha\|_m \\
\leq \|(T^\lambda)^{-1}\|\|T^\lambda\|e^{-\lambda t}\|\|\alpha\|_m \\
\leq \left(\frac{e}{\mu}\right)^2 \|T^\lambda\|e^{-\lambda t}\|\alpha\|_m,
\]

which proves the exponential stability of the semigroup \( S(t) \).

Now consider the particular case where \( C = c > 0 \), and \( \mu = 0 \) (i.e. \( \lambda' = \lambda \)) to simplify notations, together with:

\[
\phi_n := \frac{C}{\sqrt{1 + \left|\frac{2\pi n}{L}\right|^{2m}}}, \quad \forall n \in \mathbb{Z}, \tag{92}
\]

so that, applying (60), we get

\[
F_n^\lambda = -\frac{K(\lambda)}{C} \sqrt{1 + \left|\frac{2\pi n}{L}\right|^{2m}}. \tag{93}
\]

Then,

\[
\|\alpha \ast \phi\|_m = C\|\alpha\|, \quad \forall \alpha \in L^2, \quad \|\alpha \ast F^\lambda\| = \frac{K(\lambda)}{C} \|\alpha\|_m, \quad \forall \alpha \in H_{\text{per}}^m. \tag{94}
\]

Now let \( \varepsilon > 0 \). Keeping in mind that \( (\chi_{[0,1/n]})_{n>0} \) and \( (\chi_{[L-1/n,L]})_{n>0} \) are maximizing sequences for \( \Lambda^\lambda \) and \( (\Lambda^\lambda)^{-1} \) respectively, we get for \( t_n := L - 1/n \):

\[
S(t_n)(\chi_{[0,1/n]} \ast \phi) = (T^\lambda)^{-1}S\lambda(t_n)T^\lambda(\chi_{[0,1/n]} \ast \phi) \\
= (T^\lambda)^{-1}S\lambda(t_n) \left( \phi \ast \left( \Lambda^\lambda \left( \chi_{[0,1/n]} \ast \phi \right) \ast \tilde{F}^\lambda \right) \right) \quad \text{(by (59))} \\
= -K(\lambda)(T^\lambda)^{-1}S\lambda(t_n) \left( \phi \ast \left( \Lambda^\lambda \left( \chi_{[0,1/n]} \right) \right) \right) \quad \text{(by (60))} \\
= -K(\lambda) e^{-\lambda \sqrt{L}} \frac{1}{e^{-\lambda L}} (T^\lambda)^{-1} \phi \ast \left( \chi_{[L-1/n,L]} e^{-\lambda (-t_n)} \right) \quad \text{(by (83))} \\
= -e^{-\lambda \sqrt{L}} e^\lambda (L-1/n) \chi_{[L-1/n,L]} \ast \phi \quad \text{(by (63))} \\
= -e^{-\lambda \sqrt{L}} e^\lambda (L-1/n) \chi_{[0,1/n]} \ast \phi(-t_n), \quad \forall n > 0,
\]

with \( L \)-periodic continuation in the last equality, so that

\[
\|S(t_n)(\chi_{[0,1/n]} \ast \phi)\|_m = e^{-\lambda \sqrt{L}} e^\lambda (L-1/n) \|\chi_{[0,1/n]} \ast \phi\|_m, \quad \forall n > 0. \tag{95}
\]

Then, there exists \( n > 0 \) such that

\[
\|S(t_n)(\chi_{[0,1/n]} \ast \phi)\|_m > e^{-\lambda \sqrt{L}} (e^\lambda L - \varepsilon) \|\chi_{[0,1/n]} \ast \phi\|_m. \tag{96}
\]

This shows that estimate (17) can be critical in some cases.
3.4. Application. Let $m = 1$, $\lambda > 0$, $\mu = 0$. Define

$$\varphi(x) = L - x, \quad \forall x \in (0, L),$$

so that $\varphi \in H^1$ but is not periodic, and satisfies (14), with

$$\varphi_n = -\frac{iL^\frac{3}{2}}{2\pi n}, \quad \forall n \in \mathbb{Z}^*,$$

$$\varphi_0 = \frac{L^\frac{3}{2}}{2}.$$

Then, from the above definition of $\varphi$, we have

$$\tau \varphi = -\sqrt{L}I,$$

and from (60), (66) and (69), we have

$$\langle \alpha, F^\lambda \rangle = -\frac{2K(\lambda)}{L^2} \alpha_0 - K(\lambda) \frac{\alpha_x(0) + \alpha_x(L)}{2}$$

$$= -\frac{2K(\lambda)}{L^2} \int_0^L \alpha - K(\lambda) \frac{\alpha_x(0) + \alpha_x(L)}{2}, \quad \forall \alpha \in H^2_{(pw)} \cap H^1_{per}.$$  

Note here that in this particular case, the feedback has a simple expression which does not rely on the Fourier decomposition of $\alpha$.

Then, using (99),

$$D_1 = \left\{ \alpha \in H^2_{(pw)} \cap H^1_{per}, \quad K(\lambda) \left( \frac{2}{L^2} \alpha_0 + \frac{\alpha_x(0) + \alpha_x(L)}{2} \right) = \frac{\alpha_x(L) - \alpha_x(0)}{L} \right\},$$

so that

$$\left\{ \begin{array}{l} \alpha_t + \alpha_x = \left( -\frac{2K(\lambda)}{L^2} \alpha_0 - K(\lambda) \frac{\alpha_x(0) + \alpha_x(L)}{2} \right) \varphi(x), \quad x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \quad \forall t \geq 0, \end{array} \right.$$  

has a unique solution for initial conditions in $D_1$.

Note that $\varphi \in H^1$, so that

$$-\alpha_x + \langle \alpha, F \rangle \varphi \in H^1_{per} \implies \alpha_x \in H^1, \forall \alpha \in L^2,$$

which implies

$$D_1 \subset H^2.$$  

The backstepping transformation can be written as:

$$T^\lambda \alpha = \frac{\sqrt{L}}{1 - e^{-\lambda L}} \left( e^{-\lambda x} \left( -\frac{K(\lambda)}{\sqrt{L}} \alpha_x - \frac{2K(\lambda)}{L^2} \alpha_0 \right) \right) * \varphi, \quad \forall \alpha \in H^1_{per}.$$
Let $\alpha(t) \in D_1$ be the solution of the closed loop system (101) with initial condition $\alpha^0 \in D_1$, and let us note $z(t) := T^0 \alpha(t)$, then
\[
\begin{align*}
z_t &= \sqrt{L} \left( e^{-\lambda x} \left( -\frac{K(\lambda)}{\sqrt{L}} \alpha_{xt} - \frac{2K(\lambda)}{L^2} \alpha_t \right) \right) * \varphi, \\
&= \sqrt{L} \left( e^{-\lambda x} \left( -\frac{K(\lambda)}{\sqrt{L}} \alpha_{xx} + \langle \alpha, F^\lambda \rangle \varphi_x - \frac{2K(\lambda)}{L^2} \alpha_t \right) \right) * \varphi, \\
z_x &= \sqrt{L} \left( e^{-\lambda x} \left( -\frac{K(\lambda)}{\sqrt{L}} \alpha_{xx} - \langle \alpha, F^\lambda \rangle \varphi_x \right) - \frac{2K(\lambda)}{L^2} \alpha_t \right) * \varphi, \\
z_t + z_x + \lambda z &= \sqrt{L} \left( e^{-\lambda x} \left( \frac{K(\lambda)}{\sqrt{L}} \alpha_{xx} - \langle \alpha, F^\lambda \rangle \varphi_x \right) - \frac{2K(\lambda)}{L^2} \alpha_t \right) * \varphi.
\end{align*}
\]
By projecting the closed loop system on $e^0$, we get
\[
\alpha'_0 = \langle \alpha, F^\lambda \rangle \varphi_0 = \langle \alpha, F^\lambda \rangle \frac{L^2}{2},
\]
so that
\[
z_t + z_x + \lambda z = 0.
\]

4. Further remarks and questions.

4.1. Controllability and the $TB = B$ condition. In the introduction we have mentioned that the growth constraint on the Fourier coefficients of $\varphi$ actually corresponds to the exact null controllability condition in some Sobolev space for the control system (1). As we have mentioned in the finite dimensional example, the controllability condition is essential to solve the operator equation: in our case, formal computations lead to a family of functions that turns out to be a Riesz basis precisely thanks to that rate of growth. Moreover, that rate of growth is essential for the compatibility of the $TB = B$ condition and the invertibility of the backstepping transformation. Indeed, as the transformation is constructed formally using a formal $TB = B$ condition, that same $TB = B$ condition fixes the value of the coefficients of $F^\lambda$, giving them the right rate of growth for $T^\lambda$ to be an isomorphism.

In that spirit, it would be interesting to investigate whether a backstepping approach is still valid if the conditions on $\varphi$ are weakened. For example, if we suppose approximate controllability in $T \geq L$ instead of exact controllability, which corresponds to the condition that (see [15])
\[
(t \mapsto B^* \alpha(t + \cdot)) = 0 \text{ in } L^2(0, L) \implies \alpha = 0, \quad \forall \alpha \in L^2, \quad (105)
\]
which is equivalent to the Fattorini-Hautus test
\[
\varphi_n \neq 0, \quad \forall n \in \mathbb{Z},
\]
then $F^\lambda$ can still be defined using a weak $TB = B$ condition. However, it seems delicate to prove, in the same direct way as we have done, that $T^\lambda$ is an isomorphism, as we only get the completeness of the corresponding $(k_{n, \lambda})$, but not the Riesz basis property. A related question concerns the use of a characterization of controllability in our proof: in [10] the controllability property is used without resorting to an explicit characterization such as (14). If the same could be done in our case, it
would point to interesting possible generalizations, for example to systems with an operator that is not diagonalizable.

Finally, it should be noted that, while in [9] the $TB = B$ condition is well-defined, in our case, it only holds in a rather weak sense. This is probably because of a lack of regularization, indeed in [9] the backstepping transformation has nice properties, as it can be decomposed in Fredholm form, i.e. as the sum of an isomorphism and a compact operator. Accordingly, the Riesz basis in that case is quadratically close to the orthonormal basis given by the eigenvectors of the Laplacian operator. That is not the case for our backstepping transformation, as it is closely linked to the operator $\Lambda^\lambda$, which does not have any nice spectral properties.

Nonetheless, it appears that thanks to some information on the regularity of $F^\lambda$, a weak sense is sufficient and allows us to prove the operator equality by convergence.

4.2. Regularity of the feedback law. As we have pointed out in Section 2.4, if $\varphi$ is such that system (1) is controllable in $H^m_{per}$, then the feedback law $F^\lambda$ defined by (60) is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$. This was actually to be expected, as we have mentioned in the introduction that bounded feedback laws can only achieve “compact” perturbations of the spectrum, which is not enough to get exponential stabilization. More precisely, it would be possible to get exponential stabilization only with very singular controllers. With a distributed control such as ours, it is necessary to consider unbounded feedback laws.

Moreover, the application in Section 3.4 shows that even though the feedback is not continuous, and is given by its Fourier coefficients, in practice it can be expressed quite simply for some controllers.

4.3. Null-controllability and finite-time stabilization. As we have mentioned in the introduction, one of the advantages of the backstepping method is that it can provide an explicit expression for feedbacks, thus allowing the construction of explicit controls for null controllability, as well as time-varying feedbacks that stabilize the system in finite time $T > 0$.

The general strategy (as is done in [14], [46]) is to divide the interval $[0, T]$ in smaller intervals $[t_n, t_{n+1}]$, the length of which tends to 0, and on which one applies feedbacks to get exponential stabilization with decay rates $\lambda_n$, with $\lambda_n \to \infty$. Then, for well-chosen $t_n, \lambda_n$, the trajectory thus obtained reaches 0 in time $T$. Though this provides an explicit control to steer the system to 0, the norm of the operators applied successively to obtain the control tends to infinity. As such, it does not provide a reasonably regular feedback. However, the previous construction of the control can be used, with some adequate modifications (see [14] and [47]) to design a time-varying, periodic feedback, with some regularity in the state variable, which stabilizes the system in finite time.

Let us first note that, due to the hyperbolic nature of the system, there is a minimal control time, and thus small-time stabilization cannot be expected. Moreover, even for $T > L$, the estimates we have established on the backstepping transforms prevent us from applying the strategy we have described above: indeed, for any sequences $(t_n) \to T, \lambda_n \to \infty$, we have

$$\|\alpha(t)\|_m \leq \prod_{k=0}^{n} \left(\frac{C}{c}\right)^{2n} e^{\kappa \mu L} \exp \left(\sum_{k=0}^{n} -\lambda_k (t_{k+1} - t_k - L)\right) \|\alpha_0\|_m, \quad \forall t \in [t_n, t_{n+1}],$$
where \( c, C \) are the decay constants in (14). Moreover, as \( t_{k+1} - t_k \to 0 \), we have

\[
\exp \left( \sum_{k=0}^{n} -\lambda_k (t_{k+1} - t_k - L) \right) \xrightarrow{n \to \infty} \infty.
\]

Another approach could be to draw from [11] and apply a second transformation to design a more efficient feedback law. It would also be interesting to adapt the strategy in [48], inspired from [42], to our setting.

4.4. Nonlinear systems. Finally, another prospect, having obtained explicit feedbacks that stabilize the linear system, is to investigate the stabilization of nonlinear transport equations. This has been done in [12], where the authors show that the feedback law obtained for the linear Korteweg-de Vries equation also stabilizes the nonlinear equation. However, as in [9], the feedback law we have obtained is not continuous in the norm for which the system is stabilized. This would require some nonlinear modifications to the feedback law in order to stabilize the nonlinear system.

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