ISOMETRIC DISKS ARE HOLOMORPHIC

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Abstract. This paper shows that every totally-geodesic isometry from the unit disk to a finite-dimensional Teichmüller space for the intrinsic Kobayashi metric is either holomorphic or anti-holomorphic; in particular, it is a Teichmüller disk. Additionally, a similar result is proved for a large class of disk-rigid domains, which includes strictly convex bounded domains, as well as Teichmüller spaces.

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1. Introduction

Let \( \mathbb{D} \) denote the unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) equipped with its Poincaré metric \( |dz|/(1 - |z|^2) \) and let \( \mathcal{T}_{g,n} \) denote a finite-dimensional Teichmüller space equipped with its intrinsic Kobayashi metric, which is the largest (Finsler) metric such that every holomorphic map \( f : \mathbb{D} \to \mathcal{T}_{g,n} \) is non-expanding: \( ||df|| \leq 1 \).

An important feature of the Kobayashi metric of \( \mathcal{T}_{g,n} \) is that every holomorphic map \( f : \mathbb{D} \to \mathcal{T}_{g,n} \), for which \( df \) is an isometry on tangent spaces, is totally geodesic: it sends real geodesics to real geodesics preserving their length. Moreover, there are such isometries through every point in every direction, known as Teichmüller disks. \( \text{[Kra]} \)

Holomorphic rigidity for Teichmüller spaces. Our main result in this paper is the following:

**Theorem 1.1.** Let \( \mathcal{T}_{g,n} \) be a finite-dimensional Teichmüller space equipped with its intrinsic Kobayashi metric. Every totally geodesic isometry \( f : \mathbb{D} \hookrightarrow \mathcal{T}_{g,n} \) is either holomorphic or anti-holomorphic; in particular, it is a Teichmüller disk.

Remark. Theorem \( \text{[Kra]} \) settles a long standing problem in Teichmüller theory \( \text{[FM]} \).

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\( \text{[FM]} \)

See problem 5.3 in \([\text{FM}]\).
The proof is geometric and rests on the idea of complexification; see §3. Informally, the theorem shows that the intrinsic Kobayashi metric of $\mathcal{T}_{g,n}$ determines its natural structure as a complex manifold.

As a corollary, we obtain the following general result about Teichmüller spaces.

**Corollary 1.2.** Let $\mathcal{T}_{g,n}$, $\mathcal{T}_{h,m}$ be two finite-dimensional Teichmüller spaces equipped with their intrinsic Kobayashi metric. Every totally geodesic isometry $f : \mathcal{T}_{g,n} \hookrightarrow \mathcal{T}_{h,m}$ is either holomorphic or anti-holomorphic.

We note that there are, indeed, many holomorphic isometries $f : \mathcal{T}_{g,n} \hookrightarrow \mathcal{T}_{h,m}$ between Teichmüller spaces for their Kobayashi metric. [Kra]

**Holomorphic rigidity for convex domains.** In addition to Theorem 1.1, we prove a similar result for a large class of disk-rigid domains, which include strictly convex bounded domains, as well as Teichmüller spaces. We discuss the general statement in §4 as a special case, we obtain:

**Theorem 1.3.** Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two strictly convex bounded domains equipped with their intrinsic Kobayashi metric. Every totally geodesic isometry $f : \mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ is either holomorphic or anti-holomorphic.

This result need not be true for general convex domains. For example, the diagonal map $\delta(z) = (z, z)$ is a totally-real embedding $\delta : \mathbb{C}^1 \hookrightarrow \mathbb{C}^1 \times \mathbb{C}^1$, which is a totally geodesic isometry for the Kobayashi metric. In particular, the result is not true for bounded symmetric domains with rank two or more.

**Notes and References.**

For an introduction to Teichmüller spaces and the Kobayashi metric on complex manifolds, we refer to [GL], [Hub] and [Ko], respectively.

H. L. Royden proved that the Kobayashi metric of $\mathcal{T}_{g,n}$ coincides with its classical Teichmüller metric. [Roy]. When $\dim \mathcal{T}_{g,n} = 1$, we can identify $\mathcal{T}_{g,n}$ equipped with its Kobayashi-Teichmüller metric with the unit disk $\mathbb{C}^1$ equipped with its Poincaré metric. In particular, the first instance of Theorem 1.1 is implicit in the natural isomorphism $\text{Aut}(\mathbb{C}^1) \cong \text{Isom}^+(\mathbb{C}^1)$ between the group of holomorphic automorphisms of the unit disk and the group of orientation-preserving isometries of its Poincaré metric.

There is a natural action of $\text{SL}_2(\mathbb{R})$ on the sphere bundle of unit-area quadratic differentials $Q_1\mathcal{T}_{g,n}$ over $\mathcal{T}_{g,n}$, so that every orbit projects to a holomorphic totally geodesic isometry $\mathbb{C}^\mathbb{H}^1 \cong \text{SO}_2(\mathbb{R}) \setminus \text{SL}_2(\mathbb{R}) \hookrightarrow \mathcal{T}_{g,n}$, which is known as a Teichmüller disk. It is a classical result that every holomorphic isometry $\mathbb{C}^\mathbb{H}^1 \hookrightarrow \mathcal{T}_{g,n}$ into a finite-dimensional Teichmüller space is a Teichmüller disk. However, neither this result, nor Theorem 1.1 remain true for infinite-dimensional Teichmüller spaces.

A complex analytic proof that totally geodesic disks are holomorphic for strictly convex domains with $C^3$-smooth boundary appears in [HH]. Theorem 1.3 gives an optimal result for maps between convex domains. We also note that Teichmüller spaces $\mathcal{T}_{g,n} \subset \mathbb{C}^{3g-3+n}$ cannot be realised as convex domains.


The Kobayashi metric.  Let $B \subset \mathbb{C}^N$ be a bounded domain. The intrinsic Kobayashi metric of $B$ is the largest complex Finsler metric such that every holomorphic map $f : \mathbb{C}^1 \to B$ is non-expanding: $||df||_B \leq 1$. It determines both a family of norms $|| \cdot ||_B$ on tangent spaces and a distance function $d_B(\cdot, \cdot)$ on pairs of points.

By Schwarz’s lemma, every holomorphic map $f : \mathbb{C}^1 \to \mathbb{C}^1$ is non-expanding. The Kobayashi metric provides a natural generalisation - it has the fundamental property that every holomorphic map between complex domains is non-expanding. In particular, a holomorphic automorphism is always an isometry and the Kobayashi metric of a complex domain depends only on its structure as a complex manifold.

Examples.

1. $\mathbb{C}^1 \cong \{ z \in \mathbb{C} : |z| < 1 \}$ with its Poincaré metric $|dz|/(1 - |z|^2)$ coincides with the Kobayashi metric, by Schwarz’s lemma. More generally, the Kobayashi metric on the unit ball $\mathbb{C}^N \cong \{ (z_i)_{i=1}^N \in \mathbb{C}^N : \sum_{i=1}^N |z_i|^2 < 1 \}$ coincides with its complete invariant (Kaehler) metric of constant holomorphic curvature $-4$.

2. The Kobayashi metric of the bi-disk $\mathbb{C}^1 \times \mathbb{C}^1$ is the maximum metric of the two factors. It is a complex Finsler metric, which is not Hermitian. The distance function is given by $d_{\mathbb{C}^1 \times \mathbb{C}^1}((z_1, z_2), (w_1, w_2)) = \max\{ d_{\mathbb{C}^1}(z_1, w_1), d_{\mathbb{C}^1}(z_2, w_2) \}$ for all points $(z_1, z_2), (w_1, w_2) \in \mathbb{C}^1 \times \mathbb{C}^1$.

3. The Kobayashi metric of $T_{g,n}$ coincides with the classical Teichmüller metric, which endows $T_{g,n}$ with the structure of a complete geodesic metric space. We discuss this example in more detail below.

Complex geodesics. A holomorphic (or anti-holomorphic) map $\gamma : \mathbb{C}^1 \to B$ is locally distance preserving for the Kobayashi metric if and only if it is a totally geodesic isometry: $\gamma$ sends real geodesics to real geodesics preserving their length. We call such a map a complex geodesic. We note that in this case, for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the map given by $\gamma(t) = \gamma(e^{i\theta} \tanh(t))$, for $t \in \mathbb{R}$, defines a complete, unit-speed, real geodesic line in $B$. When it is clear from the context, we will often identify real and complex geodesics with their image in $B$.

Teichmüller space. Let $\Sigma_{g,n}$ be a connected, oriented surface of genus $g$ and $n$ punctures and $T_{g,n}$ denote the Teichmüller space of Riemann surfaces marked by $\Sigma_{g,n}$. A point in $T_{g,n}$ is specified by an equivalence class of orientation preserving homeomorphisms $\phi : \Sigma_{g,n} \to X$, where $X$ is a Riemann surface of finite type.

Teichmüller space $T_{g,n}$ is the orbifold universal cover of the moduli space of Riemann surfaces $M_{g,n}$ and is naturally a complex manifold with dimension $3g - 3 + n$. It is known that Teichmüller space can be realized as a contractible bounded domain of holomorphy $T_{g,n} \subset \mathbb{C}^{3g-3+n}$ by the Bers embeddings.  

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2 Two marked Riemann surfaces $\phi : \Sigma_{g,n} \to X$, $\psi : \Sigma_{g,n} \to Y$ are equivalent if $\psi \circ \phi^{-1} : X \to Y$ is isotopic to a holomorphic bijection.
Teichmüller metric. For each \( X \in T_{g,n} \), we let \( Q(X) \) denote the space of holomorphic quadratic differentials \( q = q(z)(dz)^2 \) on \( X \) with finite total mass: \( ||q||_1 = \int_X |q(z)| |dz|^2 < +\infty \), which means that \( q \) has at worse simple poles at the punctures of \( X \). The tangent and cotangent spaces to the Teichmüller space at \( X \in T_{g,n} \) are described in terms of the natural pairing \( (q, \mu) \mapsto \int_X q \mu \) between the space \( Q(X) \) and the space \( M(X) \) of \( L^\infty \)-measurable Beltrami differentials on \( X \); in particular, the tangent \( T_X T_{g,n} \) and cotangent \( T^*_X T_{g,n} \) spaces are naturally isomorphic to \( M(X)/Q(X) \perp Q(X) \), respectively.

The Teichmüller-Kobayashi metric on \( T_{g,n} \) is given by norm duality on the tangent space \( T_X T_{g,n} \) from the norm \( ||q||_1 = \int_X |q| \) on the cotangent space \( Q(X) \) at \( X \). The corresponding distance function is given by the formula \( d_{T_{g,n}}(X, Y) = \inf_\phi \frac{1}{2} \log K(\phi) \) and measures the minimal dilatation \( K(\phi) \) of a quasiconformal map \( \phi : X \to Y \) respecting their markings.

The Teichmüller metric is complete and coincides with the Kobayashi metric of \( T_{g,n} \) as a complex manifold. In particular, it has the remarkable property that every holomorphic map \( f : \mathbb{C}^1 \to T_{g,n} \) is non-expanding: \( ||df||_{T_{g,n}} \leq 1 \).

Holomorphic disks. We summarise below the main results about holomorphic disks in Teichmüller space which we shall employ in the proof of Theorem 1.1.

Complex geodesics in Teichmüller space are abundant: there is one through every point in \( T_{g,n} \) in every complex direction, classically known as Teichmüller disks.

Every complex geodesic \( \gamma_C : \mathbb{C}^1 \hookrightarrow T_{g,n} \) gives rise to a unit-speed real geodesic \( \gamma : \mathbb{R} \hookrightarrow T_{g,n} \) by \( \gamma(t) = \gamma_C(\tanh(t)) \), for \( t \in \mathbb{R} \). Conversely, every unit-speed real geodesic \( \gamma : \mathbb{R} \hookrightarrow T_{g,n} \) extends uniquely to a complex geodesic \( \gamma_C : \Delta \cong \mathbb{C}^1 \hookrightarrow T_{g,n} \) such that \( \gamma(t) = \gamma_C(\tanh(t)) \), for \( t \in \mathbb{R} \).

The following result characterises the holomorphic disks in Teichmüller space which are complex geodesics for the Kobayashi metric. See [EKK], for a simple proof based on Slodkowski’s theorem [Sl].

**Theorem 2.1.** Let \( f : \Delta \cong \mathbb{C}^1 \to T_{g,n} \) be a holomorphic map with \( ||f'(0)||_{T_{g,n}} = 1 \), then \( f \) is a totally geodesic isometry for the Kobayashi metric. In particular, it is a Teichmüller disk.

The following important result shows that there are no non-trivial holomorphic families of essentially proper holomorphic disks in Teichmüller space. It is a consequence of Sullivan’s rigidity theorem [Sul]; see [Tan] for a proof and [Mc], [Sh] for further applications and related ideas.

**Theorem 2.2.** Let \( \{f_t\}_{t \in \Delta} \) be a holomorphic family of holomorphic maps \( f_t : \Delta \cong \mathbb{C}^1 \to T_{g,n} \). If \( \partial \Delta \setminus B_{f_0} \) has positive (Lebesgue) measure, where \( B_{f_0} \) denotes the set of bounded rays of \( f_0 \), i.e. \( B_{f_0} = \{ e^{i\theta} \in \partial \Delta : \sup_{t \in [0,1]} d_{T_{g,n}}(f_0(t), f_0(te^{i\theta})) < +\infty \} \), then the family is trivial: \( f_t = f_0 \) for all \( t \in \Delta \).

There are other bounded domains that satisfy the same properties about holomorphic disks as above. We will discuss this class of disk-rigid domains and formulate a generalisation of Theorem 2.1 in §4.
3. Holomorphic rigidity for Teichmüller spaces

In this section we prove:

**Theorem 3.1.** Every totally geodesic isometry \( f : \mathbb{C}H^1 \hookrightarrow T_{g,n} \) for the Kobayashi metric is either holomorphic or anti-holomorphic. In particular, it is a Teichmüller disk.

The proof of the theorem uses the idea of *complexification* and leverages the following two facts. Firstly, a complete real geodesic in \( T_{g,n} \) is contained in a unique holomorphic Teichmüller disk; and secondly, a holomorphic family \( \{ f_t \}_{t \in \Delta} \) of *essentially proper* holomorphic maps \( f_t : \mathbb{C}H^1 \to T_{g,n} \) is trivial: \( f_t = f_0 \) for \( t \in \Delta \) (Sullivan’s rigidity theorem, see [Tan] for a precise statement and proof).

**Outline of the proof.** Let \( \gamma \subset \mathbb{C}H^1 \) be a complete real geodesic and denote by \( \gamma_C \subset \mathbb{C}H^1 \times \mathbb{C}H^1 \) its maximal holomorphic extension to the bi-disk. We note that \( \gamma_C \cong \mathbb{C}H^1 \) and we define \( F|_{\gamma_C} \) to be the unique holomorphic extension of \( f|_{\gamma_C} \), which is a Teichmüller disk.

Applying this construction to all (real) geodesics in \( \mathbb{C}H^1 \), we will deduce that \( f : \mathbb{C}H^1 \to T_{g,n} \) extends to a holomorphic map \( F : \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \to T_{g,n} \) such that \( f(z) = F(z, \overline{z}) \) for \( z \in \Delta \cong \mathbb{C}H^1 \). Using that \( f \) is totally geodesic, we will show that \( F \) is essentially proper and hence, by Sullivan’s rigidity theorem, we will conclude that either \( F(z, w) = F(z, \overline{z}) \) or \( F(z, w) = F(w, w) \), for all \( (z, w) \in \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \). □

\[
\begin{array}{ccc}
\mathbb{C}H^1 \times \overline{\mathbb{C}H^1} & \xrightarrow{F} & T_{g,n} \\
\downarrow{\delta} & & \downarrow{f} \\
\mathbb{C}H^1 & \xrightarrow{\gamma} & \mathbb{C}H^1 \end{array}
\]

We start with some preliminary constructions.

**The totally real diagonal.** Let \( \overline{\mathbb{C}H^1} \) be the complex hyperbolic line with its conjugate complex structure. The identity map is a canonical anti-holomorphic isomorphism \( \mathbb{C}H^1 \cong \overline{\mathbb{C}H^1} \) and its graph is a totally real embedding \( \delta : \mathbb{C}H^1 \hookrightarrow \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \), given by \( \delta(z) = (z, \overline{z}) \) for \( z \in \Delta \cong \mathbb{C}H^1 \). We call \( \delta(\mathbb{C}H^1) \) the *totally real diagonal*.

**Geodesics and graphs of reflections.** Let \( \mathcal{G} \) denote the set of all real, unoriented, complete geodesics \( \gamma \subset \mathbb{C}H^1 \). In order to describe their maximal holomorphic extensions \( \gamma_C \subset \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \), such that \( \gamma_C \cap \delta(\mathbb{C}H^1) = \delta(\gamma) \), it is convenient to parametrize \( \mathcal{G} \) in terms of the set \( \mathcal{R} \) of hyperbolic reflections of \( \mathbb{C}H^1 \) - or equivalently, the set of anti-holomorphic involutions of \( \mathbb{C}H^1 \). The map that associates a reflection \( r \in \mathcal{R} \) with the set \( \gamma = \text{Fix}(r) \subset \mathbb{C}H^1 \) of its fixed points gives a bijection between \( \mathcal{R} \) and \( \mathcal{G} \).

Let \( r \in \mathcal{R} \) and denote its graph by \( \Gamma_r \subset \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \); there is a natural holomorphic isomorphism \( \mathbb{C}H^1 \cong \Gamma_r \), given by \( z \mapsto (z, r(z)) \) for \( z \in \Delta \cong \mathbb{C}H^1 \). We note that \( \Gamma_r \) is
the maximal holomorphic extension $\gamma_C$ of the geodesic $\gamma = \text{Fix}(r)$ to the bi-disk and it is uniquely determined by the property $\gamma_C \cap \delta(CH^1) = \delta(\gamma)$.

The foliation by graphs of reflections. The union of the graphs of reflections $\bigcup_{r \in \mathcal{R}} \Gamma_r$ gives rise to a (singular) foliation of $CH^1 \times CH^1$ with holomorphic leaves $\Gamma_r$ parametrized by the set $\mathcal{R}$. We have $\Gamma_r \cap \delta(CH^1) = \delta(\text{Fix}(r))$ for all $r \in \mathcal{R}$, and

\[(3.1) \quad \Gamma_r \cap \Gamma_s = \delta(\text{Fix}(r) \cap \text{Fix}(s))\]

which is either empty or a single point for all $r, s \in \mathcal{R}$ with $r \neq s$. In particular, the foliation is smooth in the complement of the totally real diagonal $\delta(CH^1)$.

We emphasize that the following simple observation plays a key role in the proof of the theorem. For all $r \in \mathcal{R}$:

\[(3.2) \quad (z, w) \in \Gamma_r \iff (w, z) \in \Gamma_r\]

Geodesics and the Klein model. The Klein model gives a real-analytic identification $CH^1 \cong \mathbb{R}H^2 \subset \mathbb{R}^2$ with an open disk in $\mathbb{R}^2$. It has the nice property that the hyperbolic geodesics are affine straight lines intersecting the disk. [$\mathbb{R}a\mathbb{L}$]

Remark. The holomorphic foliation by graphs of reflections defines a canonical complex structure in a neighborhood of the zero section of the tangent bundle of $\mathbb{R}H^2$.

The description of geodesics in the Klein model is convenient in the light of the following theorem of S. Bernstein.

**Theorem 3.2.** ([AhRo]; S. Bernstein) Let $M$ be a complex manifold, $f : [0,1]^2 \to M$ a map from the square $[0,1]^2 \subset \mathbb{R}^2$ into $M$ and $E \subset \mathbb{C}$ an ellipse with foci at 0, 1. If there are holomorphic maps $F_\ell : E \to M$ such that $F_\ell|_{[0,1]} = f|_\ell$, for all vertical and horizontal slices $\ell \cong [0,1]$ of $[0,1]^2$, then $f$ has a unique holomorphic extension in a neighborhood of $[0,1]^2$ in $C^2$.

We use this to prove:

**Lemma 3.3.** Every totally geodesic isometry $f : CH^1 \hookrightarrow T_{g,n}$ admits a unique holomorphic extension in a neighborhood of the totally real diagonal $\delta(CH^1) \subset CH^1 \times CH^1$.

**Proof of 3.3.** Using the fact that analyticity is a local property and the description of geodesics in the Klein model of $\mathbb{R}H^2$, we can assume - without loss of generality - that the map $f$ is defined in a neighborhood of the unit square $[0,1]^2$ in $\mathbb{R}^2$ and has the property that its restriction on every horizontal and vertical line segment $\ell \cong [0,1]$ is a real-analytic parametrization of a Teichmüller geodesic segment. Moreover, we can also assume that the lengths of all these segments, measured in the Teichmüller metric, are uniformly bounded from above and from below away from zero.

Since every segment of a Teichmüller geodesic extends to a (holomorphic) Teichmüller disk in $T_{g,n}$, there exists an ellipse $E \subset \mathbb{C}$ with foci at 0, 1 such that the restrictions $f|_\ell$ extend to holomorphic maps $F_\ell : E \to T_{g,n}$ for all horizontal and vertical line segments $\ell \cong [0,1]$ of $[0,1]^2$. Hence, the proof of the lemma follows from Theorem 3.2. \(\square\)

**Remark.** See [Shiff], for a strongest result regarding separate analyticity.
Proof of Theorem 3.1.

Let \( f : \mathbb{C}H^1 \hookrightarrow T_{g,n} \) be a totally geodesic isometry. Applying Lemma 3.3, we deduce that \( f \) has a unique holomorphic extension in a neighborhood of the totally real diagonal \( \delta(\mathbb{C}H^1) \subset \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \). We will show that \( f \) extends to a holomorphic map from \( \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \) to \( T_{g,n} \).

We start by defining a new map \( F : \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \to T_{g,n} \), satisfying:
1. \( F(z, z) = f(z) \) for all \( z \in \Delta \cong \mathbb{C}H^1 \).
2. \( F|_{\Gamma_r} \) is the unique holomorphic extension of \( f|_{\text{Fix}(r)} \) for all \( r \in \mathcal{R} \).

Let \( r \in \mathcal{R} \) be a reflection. There is a unique (holomorphic) Teichmüller disk \( \phi_r : \mathbb{C}H^1 \hookrightarrow T_{g,n} \) such that the intersection \( \phi_r(\mathbb{C}H^1) \cap f(\mathbb{C}H^1) \subset T_{g,n} \) contains the Teichmüller geodesic \( f(\text{Fix}(r)) \) and \( \phi_r(z) = f(z) \) for all \( z \in \text{Fix}(r) \).

We define \( F \) by \( F(z, r(z)) = \phi_r(z) \) for \( z \in \mathbb{C}H^1 \) and \( r \in \mathcal{R} \); equation (3.1) shows that \( F \) is well-defined and satisfies conditions (1) and (2) above.

We claim that \( F : \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \to T_{g,n} \) is the unique holomorphic extension of \( f : \mathbb{C}H^1 \hookrightarrow T_{g,n} \) such that \( F(z, z) = f(z) \) for \( z \in \mathbb{C}H^1 \).

Proof of claim. We note that the restriction of \( F \) on the totally real diagonal \( \delta(\mathbb{C}H^1) \) agrees with \( f \) and that there is a unique germ of holomorphic maps near \( \delta(\mathbb{C}H^1) \) whose restriction on \( \delta(\mathbb{C}H^1) \) coincides with \( f \). Let us fix an element of this germ \( \tilde{F} \) defined on a neighborhood \( U \subset \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \) of \( \delta(\mathbb{C}H^1) \). For every \( r \in \mathcal{R} \), the restrictions of \( F \) and \( \tilde{F} \) on the intersection \( U_r = U \cap \Gamma_r \), are holomorphic and equal along the real-analytic arc \( U_r \cap \delta(\mathbb{C}H^1) \subset U_r \); hence they are equal on \( U_r \).

Since \( \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} = \bigcup_{r \in \mathcal{R}} \Gamma_r \), we conclude that \( F|_U = \tilde{F} \) and, in particular, \( F \) is holomorphic near the totally real diagonal \( \delta(\mathbb{C}H^1) \). Since, in addition to that, \( F \) is holomorphic along all the leaves \( \Gamma_r \) of the foliation, we deduce \(^3\) that it is holomorphic at all points of \( \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \).

In order to finish the proof of the theorem, we use the key observation (3.2); which we recall as follows: the points \((z, w)\) and \((w, z)\) are always contained in the same leaf \( \Gamma_r \) of the foliation for all \( z, w \in \Delta \cong \mathbb{C}H^1 \). Using the fact that the restriction of \( F \) on every leaf \( \Gamma_r \) is a Teichmüller disk, we conclude that \( d_{T_{g,n}}(F(z, w), F(w, z)) = d_{\mathbb{C}H^1}(z, w) \).

Let \( \theta \in \mathbb{R}/2\pi\mathbb{Z} \), it follows that at least one of \( F(\rho e^{i\theta}, 0) \) and \( F(0, \rho e^{i\theta}) \) diverges in Teichmüller space as \( \rho \to 1 \). In particular, there is a subset \( I \subset \mathbb{R}/2\pi\mathbb{Z} \) with positive measure such that either \( F(\rho e^{i\theta}, 0) \) or \( F(0, \rho e^{i\theta}) \) diverges as \( \rho \to 1 \) for all \( \theta \in I \).

We assume first that the former of the two is true. Using that \( F : \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \to T_{g,n} \) is holomorphic, we deduce from [Tan] (Sullivan’s rigidity theorem) that the family \( \{F(z, w)\}_{w \in \Delta} \) of holomorphic maps \( F(\cdot, w) : \Delta \cong \mathbb{C}H^1 \to T_{g,n} \) for \( w \in \Delta \cong \mathbb{C}H^1 \) is trivial. Therefore, \( F(z, 0) = F(z, z) = f(z) \) for all \( z \in \Delta \) and, in particular, \( f \) is holomorphic. If we assume that the latter of the two is true we similarly deduce that \( F(0, z) = F(z, z) = f(z) \) for all \( z \in \Delta \) and, in particular, \( f \) is anti-holomorphic. \( \Box \)

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\(^3\)For a simple proof of this claim using the power series expansion of \( F \) at \((0, 0) \in \mathbb{C}H^1 \times \overline{\mathbb{C}H^1} \), see [Hor] Lemma 2.2.11.
4. The class of disk-rigid domains

In this section we formulate a general theorem that applies to a large class of bounded domains, which we apply to deduce Corollary 1.2 and Theorem 1.3.

Let $B \subset \mathbb{C}^N$ be a bounded domain and $f : \Delta \to B$ a holomorphic map. We call the map $f$ essentially proper if $\partial \Delta \setminus B_f$ has positive (Lebesgue) measure, where $B_f$ denotes the set of bounded rays, i.e., $B_f = \{ e^{i\theta} \in \partial \Delta : \sup_{t \in [0,1]} d_B(f(0), f(te^{i\theta})) < +\infty \}$.

**Definition 4.1.** A bounded domain $B \subset \mathbb{C}^N$ is disk-rigid, if it satisfies:

1. every unit-speed geodesic $\gamma : \mathbb{R} \hookrightarrow B$, for the Kobayashi metric, extends to a complex geodesic $\gamma_C : \Delta \sim CH_1 \hookrightarrow B$ such that $\gamma(t) = \gamma_C(\tanh(t))$, for $t \in \mathbb{R}$,
2. every holomorphic family $\{f_t\}_{t \in \Delta}$ of holomorphic maps $f_t : \Delta \sim CH_1 \to B$, with $f_0$ an essentially proper map, is trivial i.e., $f_t = f_0$ for all $t \in \Delta$.

**Examples.**

1. Teichmüller spaces $T_{g,n}$ of finite dimension are disk-rigid. See §2, Theorems 2.1, 2.2.
2. The bi-disk $CH_1 \times \overline{CH_1}$ is a convex domain that is not disk-rigid. A bounded symmetric domain $B \subset \mathbb{C}^N$ is disk-rigid if and only if it has rank one: $B \sim CHN$.
3. All strictly convex bounded domains $B \subset \mathbb{C}^N$ are disk-rigid. We recall that a domain $B \subset \mathbb{C}^N$ is strictly convex if $\{ t \cdot P + (1 - t) \cdot Q : t \in (0,1) \} \subset B$ for every pair of distinct points $P \neq Q$ in the closure $\overline{B} \subset \mathbb{C}^N$. See [NPZ]

The proof of Theorem 3.1 in §3 used only those features of $T_{g,n}$ captured in the definition of a disk-rigid domain. In particular, the following result follows as well.

**Theorem 4.2.** Let $B \subset \mathbb{C}^N$ be a disk-rigid domain. Every totally geodesic isometry $f : CH_1 \hookrightarrow B$ for the Kobayashi metric is either holomorphic or anti-holomorphic.

We also have the following generalisation, which implies Corollary 1.2 and Theorem 1.3. The proof follows from Theorem 4.2 and Weyl’s regularity lemma.

**Theorem 4.3.** Let $B_1$, $B_2$ be two complete disk-rigid domains for the Kobayashi metric. Every totally geodesic isometry $f : B_1 \hookrightarrow B_2$ is either holomorphic or anti-holomorphic.

**Proof:** In a sufficiently small neighborhood of a point, the Kobayashi metric is bi-Lipschitz to a Hermitian metric. It follows that a totally geodesic isometry $f : B_1 \hookrightarrow B_2$ is locally Lipschitz and hence it is differentiable at almost all points of $B_1$, by Rademacher’s theorem (see Theorem 3.1.6 in [Fed]).

Let $p \in B_1$ such that the (real) linear map $df_p : T_pB_1 \to T_pB_2$ exists. Using Theorem 4.2 we conclude that $f$ sends complex geodesics in $B_1$ through $p$ to complex geodesics in $B_2$ through $f(p)$ and, in particular, the linear map $df_p$ sends complex lines in $T_pB_1$ to complex lines in $T_pB_2$. We conclude that $df_p$ is either a complex linear map or complex anti-linear map.
The assumption that the Kobayashi metric of $B_1$ and $B_2$ is complete implies that there is a complex geodesic between any pair of distinct points in $B_1$ and $B_2$. Hence, $df_p$ is either complex linear for almost every $p \in B_1$ or complex anti-linear for almost every $p \in B_1$. In particular, up to conjugation, $f$ is holomorphic as a distribution and the theorem follows from Weyl’s regularity lemma. [Kran]

\[\square\]

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