THE REDUCIBLE DOUBLE CONFLUENT HEUN EQUATION AND A GENERAL SYMMETRIC UNFOLDING OF THE ORIGIN

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Abstract. The reducible double confluent Heun equation (DCHE) is the only DCHE whose general symmetric unfolding leads to a Fuchsian equation. Contrary to the general Heun equation the unfolded Fuchsian equation has 5 singular points: \(x_L = -\sqrt{\varepsilon}, x_R = \sqrt{\varepsilon}, x_{LL} = -1/\sqrt{\varepsilon}, x_{RR} = 1/\sqrt{\varepsilon}\) and \(x_\infty = \infty\). We prove that the monodromy matrix around the regular resonant singularity at the origin is realizable as a limit of the product of the monodromy matrices around resonant singularities \(x_L\) and \(x_R\) when \(\sqrt{\varepsilon} \to 0\) while the Stokes matrix at the irregular singularity at the origin is a limit of the part of the monodromy matrix around the resonant singularity \(x_L\). This geometrical difference between the unfolding of the two different kinds of singularities at the origin is attended with an analytic difference between the coefficients in the logarithmic terms including in the solution of the unfolded equation. While the coefficients related to the unfolding of a regular singularity have infinite limits when \(\sqrt{\varepsilon} \to 0\) this one related to the unfolding of an irregular singularity has a finite limit. We also show that the reducible DCHE possesses a holomorphic solution in the whole \(\mathbb{C}^*\) if and only if the parameters of the equation are connected by a Bessel function of first kind and order depending on the non-zero characteristic exponent at the origin.

Key words: Reducible double confluent Heun equation, Unfolding, Stokes phenomenon, Irregular singularity, Monodromy matrices, Regular singularity, Limit

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1. Introduction

The double confluent Heun equation (DCHE) is a second order linear ordinary differential equation having two irregular singular points of Poincaré rank 1 over \(\mathbb{C}P^1\). If we fix them at \(x = 0\) and \(x = \infty\) the standard form of DCHE writes

\[
 \left. \begin{aligned}
 w'' + \left[ \frac{\alpha}{x} + \frac{\beta}{x^2} + \gamma \right] w' + \frac{\delta x - q}{x^2} w = 0 ,
 \end{aligned} \right. 
\]

where \(\alpha, \beta, \gamma, \delta\) and \(q\) are arbitrary complex parameters. The DCHE belongs to the list of confluent Heun’s equations. They were introduced and firstly studied by Decarreau et al. in 1978 \[6, 7\]. All of them are obtained by different confluence procedures from the general Heun equation (GHE)

\[
 w'' + \left[ \frac{\alpha}{x-1} + \frac{\beta}{x-a} + \gamma \right] w' + \frac{\delta \epsilon x - q}{x(x-1)(x-a)} w = 0 ,
\]
which is a second order Fuchsian equation with 4 singular points. The DCHE is obtained by a coalescence of the regular singularities \( x = a, x = \infty \) and \( x = 0, x = 1 \) of the GHE. The first confluent procedure leads to the irregular singularity at \( x = \infty \) while the second leads to the irregular singularity \( x = 0 \) of the DCHE (see [25]). The double confluent Heun equation finds many applications in superconductivity [4], statistical mechanics [11], gravity [23].

In this paper we apply a reverse procedure that is different from an anti-confluent procedure. We start with the double confluent Heun equation. By introducing a small complex parameter \( \varepsilon \) we unfold the equation (1.1) to the second order equation

\[
\begin{align*}
\quad w'' + & \left[ \alpha \left( \frac{1}{x - \sqrt{\varepsilon}} + \frac{1}{x + \sqrt{\varepsilon}} \right) + \frac{\beta}{2 \sqrt{\varepsilon}} \left( \frac{1}{x - \sqrt{\varepsilon}} - \frac{1}{x + \sqrt{\varepsilon}} \right) \right] w' + \\
- & \frac{\gamma}{2 \sqrt{\varepsilon}} \left( \frac{1}{\sqrt{\varepsilon} - x} + \frac{1}{\sqrt{\varepsilon} + x} \right) w = 0,
\end{align*}
\]

We call such an unfolding a general symmetric unfolding. Contrary to the anti-confluent procedure the general symmetric unfolding of the DCHE does not lead to a Fuchsian equation in general. The unfolded equation has 5 singular points. The points \( x = \sqrt{\varepsilon}, x = -\sqrt{\varepsilon}, x = 1/\sqrt{\varepsilon} \) and \( x = -\sqrt{\varepsilon} \) are regular singularities. When \( \delta \) and \( q \) are together different from zero the point \( x = \infty \) is an irregular singularity for the unfolded equation. It becomes a regular singularity if and only if \( \delta = q = 0 \), i.e. when the DCHE is a reducible equation. This fact is the main motivation for giving our attention to the unfolding of the reducible DCHE

\[
w'' + \left[ \frac{\alpha}{x} - \frac{\beta}{x^2} - \gamma \right] w' = 0,
\]

which is obtained from the equation (1.1) with \( \delta = q = 0 \) after the transformation \( x \rightarrow -x \). Without loss of generality (after a rotation of \( x \)) throughout this paper we assume that \( \beta \) in (1.2) is a real non-negative parameter. The corresponding unfolded Fuchsian equation writes

\[
\begin{align*}
\quad w'' + & \left[ \frac{\alpha}{2} \left( \frac{1}{x - \sqrt{\varepsilon}} + \frac{1}{x + \sqrt{\varepsilon}} \right) - \frac{\beta}{2 \sqrt{\varepsilon}} \left( \frac{1}{x - \sqrt{\varepsilon}} - \frac{1}{x + \sqrt{\varepsilon}} \right) \right] w' + \\
- & \frac{\gamma}{2 \sqrt{\varepsilon}} \left( \frac{1}{\sqrt{\varepsilon} - x} + \frac{1}{\sqrt{\varepsilon} + x} \right) w = 0,
\end{align*}
\]

We denote the singular points of the unfolded equation by \( x_L = -\sqrt{\varepsilon}, x_R = \sqrt{\varepsilon}, x_{LL} = -1/\sqrt{\varepsilon}, x_{RR} = 1/\sqrt{\varepsilon} \) and \( x_\infty = \infty \). Obviously the singular points \( x_L \) and \( x_R \) are obtained by the unfolding of \( x = 0 \) while \( x_{LL} \) and \( x_{RR} \) are obtained by the unfolding of \( x = \infty \) of (1.2). It is expected that \( x_\infty \) is also a result of the unfolding of \( x = \infty \). In this paper comparing the analytic invariants of both equations we confirm this conjecture. More precisely, we will show that the analytic invariants of the DCHE around the origin are realizable as a limit when \( \sqrt{\varepsilon} \rightarrow 0 \) of the analytic invariants of the unfolded equation only around resonant singularities \( x_L \) and \( x_R \). This phenomenon implies that the monodromy around \( x_{LL}, x_{RR} \) and \( x_\infty \) is responsible for the unfolding of the analytic invariants around the singularity \( x = \infty \) of the DCHE. The study of the nature of the unfolding of \( x = \infty \) is left to another project. Similar kind of problems related to the unfolding and confluence
of singularities of the differential equations have been studied in the works of Bolibrukh \cite{2}, Glutsyuk \cite{3, 4, 10}, Hurtubise, Lambert and Rousseau \cite{12, 15, 16, 17}, Klimeš \cite{13, 14}, Ramis \cite{19}, Stoyanova \cite{26, 27}, Zhang Z. In the works of Buchstaber and Glutsyuk \cite{3}, El-Jaick and Figueiredo \cite{5}, Roseau \cite{22}, Tertychniy \cite{28} have been studied solutions space and Stokes phenomenon of the families of double confluent Heun equations.

The kind of singularity at the origin depends on the parameter $\beta$. When $\beta = 0$ the origin is a regular singular point and the DCHE \eqref{eq:1.2} degenerates into a Bessel type of equation. We introduce the notion of unfolded monodromy (see Definition 3.9) as an analog of the unfolded Stokes matrix introduced by Lambert and Rousseau in \cite{15}. The unfolded monodromy measures geometrically the transformation of the monodromy around the regular singularity at the origin after a general symmetric unfolding. The reducibility allows us to prove in Section 3.3, Theorem 3.10 that when $\beta = 0$ the monodromy around the origin of the equation \eqref{eq:1.2} is realizable as a limit when $\sqrt{\varepsilon} \to 0$ of the unfolded monodromy which depends analytically on $\sqrt{\varepsilon}$. The main result in Section 3 states that the monodromy matrix around the resonant singularity at the origin is realizable as a limit of product of the local monodromy matrices of the unfolded equation around resonant singular points $x_L$ and $x_R$ when $\sqrt{\varepsilon} \to 0$ (see Proposition 3.13). In Section 3.2, Lemma 3.5 we demonstrate by a direct computation that the coefficients in the logarithmic terms of the solution of the unfolded equation have limits when $\sqrt{\varepsilon} \to 0$ and both of the limits are equal to $\infty$ whose sign depends on the parameter $\alpha$. It turns out that the sum of these coefficients has a finite limit when $\sqrt{\varepsilon} \to 0$ which is equal to the monodromy around the origin of the solution of the DCHE (see Corollary 3.14). Lemma 3.5 together with Lemma 4.7 in Section 4.1 fix the main difference between the unfolding of a regular singularity and an irregular singularity. In Lemma 4.7 we show explicitly that when the origin is an irregular singularity the coefficient in the logarithmic term of the solution of the unfolded equation has a finite limit when $\sqrt{\varepsilon} \to 0$. Moreover, this limit multiplied by $2\pi i$ is equal to the corresponding Stokes multiplier. In \cite{27} we have shown by a direct computation that when $\alpha = 2, \beta \neq 0$ the Stokes matrices at $x = 0$ and $x = \infty$ of the reducible double confluent Heun equation \eqref{eq:1.2} are realizable as a limit of the part of the monodromy matrices around a resonant singularity of the general reducible Heun equation \eqref{eq:1.3}. In Section 4.1 based on the recent works of Lambert, Rousseau, Hurtubise and Klimeš \cite{12, 14, 15, 16} we extend the result in \cite{27} to an arbitrary reducible DCHE \eqref{eq:1.2} without studying this equation. In fact this theoretical result allows us to derive the Stokes multiplier at the origin from the unfolded equation. In Section 4.2 we build explicit fundamental matrix solution at the origin with respect to which the Stokes multiplier is equal to that one obtained in Section 4.1. It turns out that the reducible DCHE \eqref{eq:1.2} admits a solution which is holomorphic in whole $\mathbb{C}^*$ if and only if the parameters $\alpha, \beta$ and $\gamma$ satisfy either the relation

$$\sum_{k=0}^{\infty} \frac{(-1)^k \beta^k \gamma^k}{k! \Gamma(2 - \alpha + k)} = 0, \quad \alpha \notin \mathbb{N} , \tag{1.4}$$

or

$$\sum_{k=0}^{\infty} \frac{(-1)^k \beta^k \gamma^k}{k! \Gamma(\alpha + k)} = 0, \quad \alpha \in \mathbb{N} , \tag{1.5}$$
where $\Gamma(z)$ is the Euler Gamma function. The relations (1.4) and (1.5) associate the parameters $\alpha, \beta$ and $\gamma$ with the Bessel function

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n+\alpha+1)}$$

of the first kind of order $\alpha$.

This paper is organized as follows. In Section 2 we introduce the fundamental matrix solutions with respect to which we will compare the analytic invariants of both equations. We also determine the conditions on the parameters under which the solution of the unfolded equation can contain logarithmic terms near the singular points $x_L$ and $x_R$. In Section 3 we study the unfolding of the regular singularity at the origin and the corresponding monodromy. The main result of Section 3 is Proposition 3.13 which states that when both of the singular points $x_L$ and $x_R$ are resonant singularity the monodromy matrix around the origin of the DCHE is realizable as a limit of the product $M_R(\varepsilon) M_L(\varepsilon)$ of the monodromy matrices around $x_R$ and $x_L$ when $\sqrt{\varepsilon} \to 0$. In Section 4 we deal with the unfolding of the irregular singularity at the origin and the corresponding Stokes phenomenon. The main result of Section 4.1 is Theorem 4.8 which states that the Stokes matrix $St_\pi$ at the origin of the DCHE is realizable as a limit of the part of the monodromy matrix around resonant singularity $x_L$ of the unfolded equation when $\sqrt{\varepsilon} \to 0$. The main result of Section 4.2 is Theorem 4.14 which provides an actual fundamental matrix solution at the origin of the DCHE. The paper contains also an Appendix where we confirm Corollary 3.14 by a direct computation for lower values of the parameter $\alpha$.

Since this paper appears as an extension of [27] we use without any effort some definitions and facts from [27].

2. Global solutions and logarithms, singular direction

**Theorem 2.1.** The equation (1.2) possesses a fundamental set of solutions $\{w_1(x,0), w_2(x,0)\}$ of the form

$$(2.6) \quad w_1(x,0) = 1, \quad w_2(x,0) = \int_{\Gamma(x,0)} z^{-\alpha} e^{-\beta z} e^{\gamma z} \, dz.$$  

The path of integration $\Gamma(x,0)$ is taken in such a way that the function $w_2(x,0)$ is a solution of equation (1.2).

We have a similar result for the equation (1.3).

**Theorem 2.2.** The equation (1.3) possesses a fundamental set of solution $\{w_1(x,\varepsilon), w_2(x,\varepsilon)\}$ of the form

$$(2.7) \quad w_1(x,\varepsilon) = 1, \quad w_2(x,\varepsilon) = \int_{\Gamma(x,\varepsilon)} (z - \sqrt{\varepsilon})^{\frac{\alpha}{2}\varepsilon - \frac{\alpha}{2}} \left( z + \sqrt{\varepsilon} \right)^{-\frac{\alpha}{2}\varepsilon - \frac{\alpha}{2}} \left( \frac{1 + \frac{1}{\sqrt{\varepsilon}}} {1 + \frac{z}{\sqrt{\varepsilon}}} \right) \frac{\gamma z}{\varepsilon} \, dz,$$

which depends analytically on $\sqrt{\varepsilon}$. The path of integration $\Gamma(x,\varepsilon)$ such that $\Gamma(x,\varepsilon) \to \Gamma(x,0)$ when $\sqrt{\varepsilon} \to 0$ is a path with the same base point $x$ as the path $\Gamma(x,0)$ from Theorem 2.1 and taken in such a way that the function $w_2(x,\varepsilon)$ is a solution of the equation (1.3).

The paths $\Gamma(x,0)$ and $\Gamma(x,\varepsilon)$ will be determined more precisely below.

As a direct consequence of Theorem 2.1 and Theorem 2.2 we construct fundamental matrices of equations (1.2) and (1.3).
Corollary 2.3. The equations (1.2) and (1.3) possess a fundamental matrix solution \( \Phi(x, \cdot) \) in the form
\[
\Phi(x, \cdot) = \begin{pmatrix} 1 & w_2(x, \cdot) \\ 0 & w'_2(x, \cdot) \end{pmatrix}, \quad \cdot = \{0, \varepsilon\},
\]
where \( w_2(x, \cdot), \cdot = \{0, \varepsilon\} \) is defined by Theorem 2.1 and Theorem 2.2 respectively.

Let us determine when the solution \( w_2(x, \varepsilon) \) of the unfolded equation can contain logarithmic terms near the singular points \( x_j, j = L, R \). Recall that from the local theory of the Fuchsian singularity such a singular point is called a resonant singularity. When \( \beta = 0 \) the points \( x_L \) and \( x_R \) are together either non-resonant or resonant singularities for the unfolded equation. In particular, they both are resonant singularities if and only if \( \alpha \in 2\mathbb{N} \). In the next section we consider the equations (1.2) and (1.3) under the restriction
\[
\beta = 0, \quad \alpha \in 2\mathbb{N}.
\]

Note that under the restriction (2.9) the origin is a resonant regular singularity too. Using the rotation \( x \to x e^{i\delta} \) where \( \delta = \arg(\sqrt{\varepsilon}) \) we always can fix \( \sqrt{\varepsilon} \) to be a real and positive. Due to this property when \( \beta = 0 \) we choose the path \( \Gamma(x, 0) \) in (2.6) to be a path from 1 to \( x \) approaching 1 in the direction \( \mathbb{R}^+ \). The path \( \Gamma(x, \varepsilon) \) is a path taken in the same direction \( \mathbb{R}^+ \) from 1 + \( \sqrt{\varepsilon} \) to the same base point \( x \).

When \( \beta > 0 \) we choose the path \( \Gamma(x, 0) \) in (2.6) to be a path from 0 to \( x \) approaching 0 in the direction \( \mathbb{R}^+ \). Then the corresponding unfolded path \( \Gamma(x, \varepsilon) \) is a path taken in the same direction \( \mathbb{R}^+ \) from \( \sqrt{\varepsilon} \) to the same base point \( x \). This choice of the path \( \Gamma(x, \varepsilon) \) implies that \( \varepsilon \) is a real positive parameter of unfolding and that \( x_L \) will be the resonant singularity. In particular in Section 4 we consider the unfolded equation under the restriction
\[
\beta = 0, \quad \alpha \in 2\mathbb{N}.
\]

We denote by \( \Phi_0(x, 0) \) and \( \Phi_0(x, \varepsilon) \) the fundamental matrix solutions from (2.8) corresponding to the so chosen paths \( \Gamma(x, 0) \) and \( \Gamma(x, \varepsilon) \).

From Definition 6.15 in [27] it follows that \( \theta = \arg(0 - \beta) = \arg(-\beta) = \pi \) is the only possible singular direction at the origin of the DCHE.

3. The unfolding of the monodromy around the origin

In this section we deal with the equations (1.2) and (1.3) when the parameters \( \alpha \) and \( \beta \) satisfy the condition (2.9).

3.1. The monodromy around the origin of the DCHE. Since the origin is a regular point for the DCHE its unfolding causes an unfolding of the monodromy around it. To compute this monodromy we rewrite the fundamental matrix \( \Phi_0(x, 0) \) in an appropriate form. Directly from (2.6) and (2.8) we have

**Theorem 3.1.** Assume that the condition (2.9) holds. Then the fundamental matrix solution \( \Phi_0(x, 0) \) of the equation (1.2) is represented in a neighborhood of the origin as
\[
\Phi_0(x, 0) = \exp(Gx) \ H(x) \ x^\Lambda \ x^J,
\]
where
\[
G = \text{diag}(0, \gamma), \quad \Lambda = \text{diag}(0, -\alpha).
\]
The matrix $H(x)$ is defined as

$$H(x) = \begin{pmatrix} 1 & x \varphi(x) \\ 0 & 1 \end{pmatrix},$$

where $\varphi(x)$ is a holomorphic function in a neighborhood of the origin. The matrix $J$ is given by

$$J = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix},$$

where

$$\lambda = \frac{\gamma^{\alpha-1}}{\alpha - 1!}. \tag{3.12}$$

The monodromy of the fundamental matrix solution $\Phi_0(x,0)$ around the origin is described by the local monodromy matrix $M_0 \in GL_2(\mathbb{C})$

$$M_0 = e^{2\pi i \Lambda} e^{2\pi i J} = e^{2\pi i J} = \begin{pmatrix} 1 & 2\pi i \lambda \\ 0 & 1 \end{pmatrix}, \tag{3.13}$$

where $\lambda$ is introduced by (3.12).

3.2. The monodromy around $x_L$ and $x_R$ of the unfolded equation. In this section we compute the local monodromy matrices of the equation (1.3) under the restriction (2.9).

In the next theorem we describe the local behavior of the fundamental matrix $\Phi_0(x,\varepsilon)$ near the singular points $x_R$ and $x_L$ when both of them are resonant singularities.

**Theorem 3.2.** Assume that the condition (2.9) holds. Then the fundamental matrix solution $\Phi_0(x,\varepsilon)$ of the unfolded equation depends analytically on $\sqrt{\varepsilon}$ and it is represented in a neighborhood of the origin which contains only the singular points $x_L$ and $x_R$ as

$$\Phi_0(x,\varepsilon) = G(x,\varepsilon) H(x,\varepsilon) (x - x_L)^{2\Lambda} (x - x_R)^{2\Lambda} J_L(\varepsilon) J_R(\varepsilon),$$

where

$$G(x,\varepsilon) = (x - x_{LL})^{-\frac{x_{LL}}{2}} G (x_{RR} - x)^{-\frac{x_{RR}}{2}} G. \tag{3.14}$$

The matrix $H(x,\varepsilon)$ is a holomorphic matrix-function at the both singular points $x_L$ and $x_R$ such that $H(x_k,\varepsilon) = I_2, k = L, R$. The matrices $G$ and $\Lambda$ are introduced in Theorem 3.1. The matrix $J_k(\varepsilon), k = L, R$ is given by

$$J_k(\varepsilon) = \begin{pmatrix} 0 & q_k \\ 0 & 0 \end{pmatrix}, \tag{3.15}$$

where the number $q_k$ is defined as

$$q_k = \text{Res} \left( w_2'(x,\varepsilon), x = x_k \right), \quad k = L, R.$$

**Proof.** Let us present the fundamental matrix $\Phi_0(x,\varepsilon)$ in the form

$$\Phi_0(x,\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & w_2'(x,\varepsilon) \end{pmatrix} \begin{pmatrix} 1 & \int_1^{x_{1+\sqrt{\varepsilon}}} w_2'(z,\varepsilon) \, dz \\ 0 & \frac{1}{1} \end{pmatrix} = G(x,\varepsilon) [(x - x_L)(x - x_R)]^{2\Lambda} \begin{pmatrix} 1 & \int_1^{x_{1+\sqrt{\varepsilon}}} w_2'(z,\varepsilon) \, dz \\ 0 & \frac{1}{1} \end{pmatrix},$$

where

$$G(x,\varepsilon) = (x - x_{LL})^{-\frac{x_{LL}}{2}} G (x_{RR} - x)^{-\frac{x_{RR}}{2}} G.$$
Consider the function \( w_2(x, \varepsilon) \). Since when \( \alpha \in 2\mathbb{N} \) the function \( \frac{1}{[z-\sqrt{\varepsilon}]^{\alpha/2}} \) is a rational function it can be split into a finite sum in \( \alpha \) number simpler ratios \( \frac{c_j}{(z-\sqrt{\varepsilon})^{d_j}} \) and \( \frac{d_j}{(z+\sqrt{\varepsilon})^{d_j}} \), \( 1 \leq j \leq \alpha/2 \) where the coefficients \( c_j, d_j \) are uniquely determined. Then the function \( w_2(x, \varepsilon) \) can be written as

\[
\int_{1+\sqrt{\varepsilon}}^{x} w_2'(z, \varepsilon) \, dz = \int_{1+\sqrt{\varepsilon}}^{x} \frac{P(z, \varepsilon)}{(z-\sqrt{\varepsilon})^{\frac{\alpha}{2}}} \left( \frac{\frac{1}{\sqrt{\varepsilon}} - z}{\frac{1}{\sqrt{\varepsilon}} + z} \right)^{\frac{\gamma}{2}} \, dz 
+ \int_{1+\sqrt{\varepsilon}}^{x} \frac{Q(z, \varepsilon)}{(z + \sqrt{\varepsilon})^{\frac{\alpha}{2}}} \left( \frac{\frac{1}{\sqrt{\varepsilon}} - z}{\frac{1}{\sqrt{\varepsilon}} + z} \right)^{\frac{\gamma}{2}} \, dz
\]

\[
= q_R \log(x-x_R) + q_L \log(x-x_L) + (x-x_R)^{-\frac{\alpha}{2}+1} h(x-x_R) + (x-x_L)^{-\frac{\alpha}{2}+1} g(x-x_L).
\]

Here \( P(z, \varepsilon) \) and \( Q(z, \varepsilon) \) are polynomials of degree at most \( \alpha/2 - 1 \). The functions \( h(x-x_R) \) and \( g(x-x_L) \) are holomorphic functions at the both singular points \( x_R \) and \( x_L \) since the function \( \left( \frac{\frac{1}{\sqrt{\varepsilon}} - z}{\frac{1}{\sqrt{\varepsilon}} + z} \right)^{\frac{\gamma}{2}} \) is a holomorphic function at the both singular points \( x_j, j = L, R \).

Then we can present \( \Phi_0(x, \varepsilon) \) as

\[
G(x, \varepsilon) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \cdot \frac{(x-x_R)(x-x_L)^{\frac{\alpha}{2}} h(x-x_R) + (x-x_L)(x-x_R)^{\frac{\alpha}{2}} g(x-x_L)}{1}
\times \left( \begin{array}{c}
1 \\
0
\end{array} \right) q_L \log(x-x_L) + q_R \log(x-x_R)
\]

\[
G(x, \varepsilon) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \cdot (x-x_R)(x-x_L)^{\frac{\alpha}{2}} h(x-x_R) + (x-x_L)(x-x_R)^{\frac{\alpha}{2}} g(x-x_L)
\times \left( \begin{array}{c}
1 \\
0
\end{array} \right) [(x-x_L)(x-x_R)]^{\frac{\gamma}{2}} \Lambda \left( \begin{array}{c}
1 \\
0
\end{array} \right) q_L \log(x-x_L) + q_R \log(x-x_R)
\]

\[
G(x, \varepsilon) \left( \begin{array}{c}
1 \\
0
\end{array} \right) \cdot [(x-x_L)(x-x_R)]^{\frac{\gamma}{2}} \Lambda (x-x_L)^{J_L(\varepsilon)} (x-x_R)^{J_R(\varepsilon)} = \Phi_0(x, \varepsilon).\]

This ends the proof. □

Consider the DCHE (1.2) and its fundamental matrix solution at the origin in the punctured disk \( D_R \) around the origin with a finite small radius \( R \)

\[
D_R := \{ x \in \mathbb{C} \mid 0 < |x| < R \}.
\]

The radius \( R \) is so chosen that the points \( x_L \) and \( x_R \) belong to \( D_R \) while the points \( x_{LL} \) and \( x_{RR} \) do not belong to \( D_R \). Let \( x_0 \in D_R \setminus \mathbb{R} \). Let \( \gamma_L \) and \( \gamma_R \) be two closed loops, starting and ending at the point \( x_0 \). The loop \( \gamma_L \) (resp. \( \gamma_R \)) encircles only the point \( x_L \) (resp. \( x_R \)) in the positive sense as it is shown in Figure 1. Thanks to Theorem 3.2 we can fix explicitly the corresponding local monodromy matrices corresponding to the loops \( \gamma_L \) and \( \gamma_R \).

**Theorem 3.3.** The local monodromy matrices \( M_k(\varepsilon), k = R, L \) of the perturbed equation with respect to the fundamental matrix \( \Phi_0(x, \varepsilon) \), introduced by Theorem 3.2 are given by

\[
M_k(\varepsilon) = e^{2\pi i J_k(\varepsilon)}, \quad k = L, R.
\]
Proposition 3.4. Assume that the condition \( q \) holds. Then the numbers \( q_R \) and \( q_L \) given by

\[
q_R = \frac{1}{(\alpha-2)!} \sum_{k=0}^{\alpha-2} \binom{\alpha-2}{k} (-1)^{\alpha-2-k} \frac{\Gamma(\alpha-1-k)}{\Gamma(\frac{\alpha}{2})} (2\sqrt{\varepsilon})^{-\alpha+1+k} A_R,
\]

where

\[
A_R = \left( \frac{\sqrt{\varepsilon}}{1+\varepsilon} \right)^k \left( \frac{1+\varepsilon}{1-\varepsilon} \right)^{\frac{\gamma}{2\sqrt{\varepsilon}}} \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma(\frac{\gamma}{2\sqrt{\varepsilon}}+s) \Gamma(\frac{\gamma}{2\sqrt{\varepsilon}}+1)}{\Gamma(\frac{\gamma}{2\sqrt{\varepsilon}}-k+s+1)} \left( 1+\varepsilon \right)^s.
\]

Similarly, the number \( q_L \) is given by

\[
q_L = \frac{1}{(\alpha-2)!} \sum_{k=0}^{\alpha-2} \binom{\alpha-2}{k} (-1)^{\alpha-2-k} \frac{\Gamma(\alpha-1-k)}{\Gamma(\frac{\alpha}{2})} (-2\sqrt{\varepsilon})^{-\alpha+1+k} A_L,
\]

where

\[
A_L = \left( \frac{\sqrt{\varepsilon}}{1-\varepsilon} \right)^k \left( \frac{1-\varepsilon}{1+\varepsilon} \right)^{\frac{\gamma}{2\sqrt{\varepsilon}}} \sum_{s=0}^{k} \binom{k}{s} \frac{\Gamma(\frac{\gamma}{2\sqrt{\varepsilon}}+s) \Gamma(\frac{\gamma}{2\sqrt{\varepsilon}}+1)}{\Gamma(\frac{\gamma}{2\sqrt{\varepsilon}}-k+s+1)} \left( 1-\varepsilon \right)^s.
\]

It turns out the numbers \( q_R \) and \( q_L \) have a limit when \( \sqrt{\varepsilon} \to 0 \).

Lemma 3.5. Assume that the condition \( q \) holds. Then for each fixed \( \alpha \) the numbers \( q_R \) and \( q_L \) computed by Proposition 3.4 satisfy the limits

\[
\lim_{\sqrt{\varepsilon} \to 0} q_R = -(-1)\frac{\alpha}{2} \infty, \quad \lim_{\sqrt{\varepsilon} \to 0} q_L = (-1)\frac{\alpha}{2} \infty
\]

when \( \sqrt{\varepsilon} \to 0 \).
Proof. Applying the limit
\[
\lim_{|z| \to \infty} \frac{\Gamma(z + a)}{\Gamma(z) z^a} = 1
\]
we find that
\[
q_R \to - \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{\alpha}{2}} \left( \frac{1}{2} \right)^{\alpha - 1} \sum_{k=0}^{\alpha - \frac{1}{2}} \frac{(-1)^k \Gamma(\alpha - 1 - k)}{k! (\alpha - k)!} \left( \frac{2\sqrt{\varepsilon} \gamma}{1 - \varepsilon^2} \right)^k,
\]
\[
q_L \to \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{\frac{\alpha}{2}} \left( \frac{1}{2} \right)^{\alpha - 1} \sum_{k=0}^{\alpha - \frac{1}{2}} \frac{\Gamma(\alpha - 1 - k)}{k! (\alpha - k)!} \left( \frac{2\sqrt{\varepsilon} \gamma}{1 - \varepsilon^2} \right)^k
\]
when $\sqrt{\varepsilon} \to 0$. Now the statement follows from the observation that $q_R$ and $q_L$ are expressed as finite sums and from the limits
\[
\lim_{\sqrt{\varepsilon} \to 0} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{\alpha}{2}} = 1, \quad \lim_{\sqrt{\varepsilon} \to 0} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{\frac{\alpha}{2}} = 1.
\]
\[\square\]

From Lemma 3.5 it follows that the sign of the limit of the number $q_k, k = L, R$ depends on the parameter $\alpha$ but we always have that
\[
\lim_{\sqrt{\varepsilon} \to 0} (q_R + q_L) = \infty - \infty.
\]

**Remark 3.6.** The result of Lemma 3.5 is the identification mark of the unfolding of a resonant regular point. Recall that in our previous works [26, 27] all the coefficients place before the logarithmic terms in the solution of the unfolded equation have a finite limit when $\sqrt{\varepsilon} \to 0$. But in all previous cases these logarithmic terms measure how the Stokes matrices of the initial equation are transformed to the monodromy matrices of the unfolded equation. This time the logarithmic terms in the solution of the unfolded equation correspond to an unfolding of the monodromy matrix of the initial equation to two monodromy matrices of the unfolded equation.

### 3.3. The unfolded monodromy around the origin

In this section we connect by a radial limit $\sqrt{\varepsilon} \to 0$ the monodromy matrices $M_j(\varepsilon), j = R, L$ of the unfolded equation with the monodromy matrix $M_0$ of the DCHE. The following proposition is a key for our study.

**Proposition 3.7.** When $\sqrt{\varepsilon} \to 0$ the fundamental set of solutions $\{w_1(x, \varepsilon), w_2(x, \varepsilon)\}$ of the unfolded equation fixed by Theorem 2.2 converges uniformly on compact sets of $D_R$ to the fundamental set of solutions $\{w_1(x, 0), w_2(x, 0)\}$ of the DCHE fixed by Theorem 2.1.

Thanks to Proposition 3.7 we have the following property of the fundamental matrices $\Phi_0(x, 0)$ and $\Phi(x, \varepsilon)$.

**Corollary 3.8.** The fundamental matrix $\Phi_0(x, \varepsilon)$ of the unfolded equation given by Theorem 3.2 converges uniformly on compact sets of $D_R$ to the fundamental matrix $\Phi_0(x, 0)$ of the DCHE given by Theorem 3.1 when $\sqrt{\varepsilon} \to 0$.

Let $\gamma \in D_R$ be a closed loop starting and ending at the same point $x_0 \in D_R \setminus \mathbb{R}$ as in Section 3.2, encircling the origin and the points $x_R$ and $x_L$ and oriented counter-clockwise as in Figure 1. The fundamental matrix solution $\Phi_0(x, 0)$ of the DCHE introduced in Theorem 3.1 is a holomorphic multi-valued function on $D_R$. The analytic continuation of
Φ₀(\(x, 0\)) along \(\gamma\) leads to a new fundamental matrix solution \([Φ₀(x, 0)]_{\gamma}\) of the DCHE. The connection between these two fundamental matrix solutions is given by the monodromy matrix \(M₀ = e^{2\pi i \Lambda} e^{2\pi i J} = e^{2\pi i J}\) from (3.13)

\[ [Φ₀(x, 0)]_{\gamma} = Φ₀(x, 0) M₀. \]

Let \(D_R(\varepsilon)\) be a domain in \(C \setminus \{x_L, x_R\}\) such that \(D_R(\varepsilon)\) tends to the disk \(D_R\) when \(\sqrt{\varepsilon} \to 0\). Let \(\gamma(\varepsilon) \in D_R(\varepsilon)\) be a closed loop starting and ending with the same point \(x₀\) and such that \(\gamma(\varepsilon) = γ_L \circ γ_R\) where the loops \(γ_L\) and \(γ_R\) are defined in Section 3.2 and in Figure 1. Then when \(\sqrt{\varepsilon} \to 0\) the loop \(γ(\varepsilon)\) tends to a closed loop that belongs to the homotopy class \([γ]\) of the loop. Analytic continuation of the fundamental matrix \(Φ₀(x, \varepsilon)\) along \(γ(\varepsilon)\) yields a new fundamental matrix \([Φ₀(x, \varepsilon)]_{γ(\varepsilon)}\). The connection between these two fundamental matrices is measured geometrically by an invertible constant matrix \(M₀(\varepsilon)\)

\[ [Φ₀(x, \varepsilon)]_{γ(\varepsilon)} = Φ₀(x, \varepsilon) M₀(\varepsilon). \]  

**Definition 3.9.** We call the invertible matrix \(M₀(\varepsilon)\) defined by (3.20) the unfolded monodromy matrix around the origin.

The reducibility ensures the connection by a limit \(\sqrt{\varepsilon} \to 0\) between the fundamental matrix solutions \(Φ₀(x, 0)\) and \(Φ₀(x, \varepsilon)\), which ensures such a connection between the monodromy around the origin and the unfolded monodromy

**Theorem 3.10.** The unfolded monodromy matrix \(M₀(\varepsilon)\) around the origin depends analytically on \(\sqrt{\varepsilon}\) and converges when \(\sqrt{\varepsilon} \to 0\) to the monodromy matrix around the origin \(M₀\) defined by (3.13).

**Proof.** Since the fundamental matrix \(Φ₀(x, \varepsilon)\) converges uniformly on the compact sets of \(D_R\) to the fundamental matrix \(Φ₀(x, 0)\), so does the fundamental matrix \([Φ₀(x, \varepsilon)]_{γ(\varepsilon)}\) to the fundamental matrix \([Φ(x, 0)]_{γ}\). Then the matrix \(M₀(\varepsilon)\) must converge to the monodromy matrix \(M₀\) when \(\sqrt{\varepsilon} \to 0\). \(\square\)
When the origin is a resonant singularity we find that the unfolded monodromy matrix $M_0(\varepsilon)$ is expressed in terms of the monodromy matrices $M_L(\varepsilon)$ and $M_R(\varepsilon)$.

**Theorem 3.11.** Assume that the condition \((2.9)\) holds. Let $M_j(\varepsilon), j = L,R$ and $M_0(\varepsilon)$ be the monodromy matrices and the unfolded monodromy matrix of the unfolded equation with respect to the fundamental matrix solution $\Phi_0(x,\varepsilon)$ Then they satisfy the following relation

\[ M_0(\varepsilon) = M_R(\varepsilon) M_L(\varepsilon) = M_L(\varepsilon) M_R(\varepsilon). \]  

**Proof.** The connection $M_0(\varepsilon) = M_R(\varepsilon) M_L(\varepsilon)$ follows from the definition $\gamma_L \circ \gamma_R = \gamma(\varepsilon)$, where $\gamma_L$ and $\gamma_R$ are the loops from section 3.2. The equality $M_R(\varepsilon) M_L(\varepsilon) = M_L(\varepsilon) M_R(\varepsilon)$ follows from the fact that under the condition \((2.9)\) the matrices $M_L(\varepsilon)$ and $M_R(\varepsilon)$ commute. □

As an immediate consequence we have

**Corollary 3.12.** Assume that the condition \((2.9)\) holds. Then the unfolded monodromy matrix $M_0(\varepsilon)$ and the matrices $e^{2\pi i J_j(\varepsilon)}, j = L,R$ satisfy the following relation

\[ M_0(\varepsilon) = e^{2\pi i J_L(\varepsilon)} e^{2\pi i J_R(\varepsilon)} = e^{2\pi i J_R(\varepsilon)} e^{2\pi i J_L(\varepsilon)}. \]  

**Proof.** The statement follows immediately from \((3.21)\) and \((3.16)\). □

Combining Corollary 3.12 and Theorem 3.10 we have that

**Proposition 3.13.** Assume that the condition \((2.9)\) holds. Then the matrices $J_k(\varepsilon), k = L,R$ of the unfolded equation and the monodromy matrix $M_0(\varepsilon)$ of the DCHE are connected by the limit

\[ e^{2\pi i J_L(\varepsilon)} e^{2\pi i J_R(\varepsilon)} = e^{2\pi i J_R(\varepsilon)} e^{2\pi i J_L(\varepsilon)} = M_0 \quad \text{when} \quad \sqrt{\varepsilon} \to 0. \]

Thanks to Proposition 3.13 we find the limit of $q_R + q_L$ when $\sqrt{\varepsilon} \to 0$.

**Corollary 3.14.** Assume that the condition \((2.9)\) holds. Then

\[ \lim_{\sqrt{\varepsilon} \to 0} q_R + q_L = \lambda, \]

where $\lambda$ is given by \((3.12)\).

In the Appendix we demonstrate by a direct computation that the limit \((3.22)\) is valid for lower values of the parameter $\alpha$.

4. Unfolding of the Stokes matrix at the origin

Throughout this section we assume that $\beta \neq 0$ and therefore the origin is an irregular singularity for the DCHE. In \([27]\) we have shown by a direct computation, that when $\alpha = 2$ the Stokes matrix at the origin of the DCHE \((1.2)\) can be obtained by a limit of this part of the monodromy matrices around resonant singular points that governs the existence of the logarithmic term in the solution of the unfolded equation. In the section 4.1 we show that this result remains valid for every reducible DCHE \((1.2)\). In fact the realization of the Stokes matrix as a limit of the part of the monodromy matrix of the unfolded equation is an effect of the recent theoretical result of Hurtubise, Klimeš, Lambert and Rousseau \([12, 14, 15, 16]\). Using the obtained connection between the analytic invariants of both equations we provide the Stokes matrix at the origin without studying in details the DCHE. Instead we deal with the unfolded equation and its monodromy matrix around
a resonant singularity $x_L$. In the section 4.2 we build explicitly an actual fundamental
matrix solution of the DCHE (1.2) at the origin with respect to which the Stokes matrix
has the form obtained in the first part.

4.1. The Stokes matrix at the origin as a limit of the monodromy matrix around
$x_L$. Following Lambert and Rousseau [15, 16] we consider both equations in the ramified
domain \( \{ x \in \mathbb{C} : -\kappa < \arg(x) < \kappa \} \) where \( 0 < \kappa < \frac{\pi}{2} \). We cover this domain by two open
sectors

\[
\Omega_1 = \Omega_1(\rho, \kappa) = \left\{ x = re^{i\delta} : 0 < r < \rho, -\kappa - \pi < \delta < \kappa \right\},
\]

\[
\Omega_2 = \Omega_2(\rho, \kappa) = \left\{ x = re^{i\delta} : 0 < r < \rho, -\kappa < \delta < \kappa + \pi \right\}.
\]

The radius \( \rho \) is so chosen that \( x_{LL}, x_{RR} \notin \Omega_1 \cup \Omega_2 \) while \( x_L, x_R \in \Omega_1 \cup \Omega_2 \). Denote by \( \Omega_R \) and \( \Omega_L \) the connected components of the intersection \( \Omega_1 \cap \Omega_2 \), as \( x_R \in \Omega_R, x_L \in \Omega_L \). We have a proposition similar to Proposition 3.7.

**Proposition 4.1.** When \( \sqrt{\varepsilon} \to 0 \) the fundamental set of solutions \( \{ w_1(x, \varepsilon), w_2(x, \varepsilon) \} \) of
the unfolded equation fixed by Theorem 2.2 converges uniformly on compact sets of
\( \Omega_R \cup \Omega_L \) to the fundamental set of solutions \( \{ w_1(x, 0), w_2(x, 0) \} \) of the DCHE fixed by Theorem 2.1.

Consider the DCHE (1.2) over \( \Omega_1 \cup \Omega_2 \). From the sectorial normalization theorem of Sibuya [24] and the theorem of Hukuhara-Turrittin[29] it follows that the actual fundamental matrix solution \( \Phi_0(x, 0) \) of the DCHE can be represented as

\[
\Phi_j(x, 0) = \exp(Gx) H_j(x) \left[ x^\Lambda \exp\left( \frac{-B}{x} \right) \right]_j
\]
on the sectors \( \Omega_j, j = 1, 2 \), respectively. Here

\[
G = \text{diag}(0, \gamma), \quad \Lambda = \text{diag}(0, -\alpha), \quad B = \text{diag}(0, \beta)
\]
and \( [x^\Lambda \exp(-B/x)]_j \) is the branch of the matrix \( x^\Lambda \exp(-B/x) \) on \( \Omega_j, j = L, R \), respectively. The matrices \( H_j(x) \) are holomorphic matrix functions on \( \Omega_j \), respectively, as
both of them are asymptotic in the Gevrey 1 sense to the same formal matrix \( \hat{H}(x) \) on \( \Omega_j, j = L, R \). On the sector \( \Omega_R \) the fundamental matrix solutions \( \Phi_1(x, 0) \) and \( \Phi_2(x, 0) \) coincide. On the sector \( \Omega_L \) the jump of the solution \( \Phi_2(x, 0) \) to the solution \( \Phi_1(x, 0) \) is measured geometrically by the Stokes matrix \( St_\pi \), corresponding to the singular direction \( \theta = \pi \)

\[
\Phi_2(x, 0) = \Phi_1(x, 0) St_\pi,
\]

where

\[
St_\pi = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.
\]

Consider now the unfolded equation. The next theorem describes the behavior of the fundamental matrix solution \( \Phi_0(x, \varepsilon) \) near the singular points when \( x_L \) is a resonant singularity.

**Theorem 4.2.** Assume that the condition \( (2.10) \) holds. Then the fundamental matrix solution \( \Phi_0(x, \varepsilon) \) of the unfolded equation depends analytically on \( \sqrt{\varepsilon} \) and it is represented in a neighborhood of the resonant singularity \( x_L \) which does not contain the point \( x_R \) as

\[
\Phi_0(x, \varepsilon) = (I_L(\varepsilon) + O(x-x_L)) (x-x_L)^{\frac{1}{2} \Lambda^+ + \frac{1}{2} \varepsilon} B (x-x_L) T_L(\varepsilon)
\]

and in a neighborhood of the non-resonant singularity \( x_R \) which does not contain the point \( x_L \) as

\[
\Phi_0(x, \varepsilon) = (I_R(\varepsilon) + O(x-x_R)) (x-x_R)^{\frac{1}{2} \Lambda^+ + \frac{1}{2} \varepsilon} B.
\]

The matrices \( I_j(\varepsilon) + O(x-x_j), j = L, R \) are holomorphic matrix functions there. The matrices \( \Lambda \) and \( B \) are given by (4.23). The matrix \( T_L(\varepsilon) \) is defined as

\[
T_L(\varepsilon) = \begin{pmatrix} 0 & d_L \\ 0 & 0 \end{pmatrix},
\]

where

\[
d_L = \text{Res}(w'_2(x, \varepsilon), x = x_L).
\]

**Proof.** The proof is similar to the proof of Proposition 4.7 in [26].

With respect to the matrix solution \( \Phi_0(x, \varepsilon) \) from Theorem 4.2 the local monodromy matrices \( M_j(\varepsilon) \) around the singular point \( x_j, j = L, R \) have the form

\[
M_L(\varepsilon) = e^{2\pi i (\frac{1}{2} \Lambda^+ + \frac{1}{2} \varepsilon) B} e^{2\pi i T_L(\varepsilon)} = e^{2\pi T_L(\varepsilon)}, \quad M_R(\varepsilon) = e^{2\pi i (\frac{1}{2} \Lambda^+ + \frac{1}{2} \varepsilon) B}.
\]

Let \( \Omega_1(\varepsilon) \) and \( \Omega_2(\varepsilon) \) be the sectors obtained from the sectors \( \Omega_1 \) and \( \Omega_2 \) by making a cut between the points \( x_L \) and \( x_R \) through the real axis (see Figure 2). The origin belongs to this cut. When \( \sqrt{\varepsilon} \to 0 \) the sectors \( \Omega_j(\varepsilon) \) tend to the sectors \( \Omega_j, j = L, R \), respectively. Consider the unfolded equation over \( \Omega_1(\varepsilon) \cup \Omega_2(\varepsilon) \). The fundamental matrix solution \( \Phi_0(x, \varepsilon) \) writes also as

\[
\Phi_0(x, \varepsilon) = G(x, \varepsilon) H(x, \varepsilon) (x-x_L)^{\frac{1}{2} \Lambda^+ + \frac{1}{2} \varepsilon} B (x-x_R)^{\frac{1}{2} \Lambda^+ + \frac{1}{2} \varepsilon} B,
\]

where

\[
G(x, \varepsilon) = (x-x_{LL})^{-\frac{x_{LL}}{2} G} (x_{RR}-x)^{-\frac{x_{RR}}{2} G}
\]

and

\[
H(x, \varepsilon) = \begin{pmatrix} 1 & (x-x_L)^{-\frac{x}{2} + \frac{a}{2}} (x-x_R)^{\frac{b}{2} + \frac{a}{2}} w_2(x, \varepsilon) \\ 0 & 1 \end{pmatrix}.
\]

The next proposition that follows immediately from Proposition 4.1 is a key for the extension of the results in [27].
Corollary 4.3. The fundamental matrix solution $\Phi_0(x, \varepsilon)$ from (4.26) converges uniformly on compact sets of $\Omega_R \cup \Omega_L$ to the actual fundamental matrix solution $\Phi_0(x, 0)$ at the origin of the DCHE when $\sqrt{\varepsilon} \to 0$.

As in [27] Corollary 4.3 allows us to identify the so called unfolded Stokes matrix $St_L(\varepsilon)$ with the matrix $e^{2\pi i T_L}$ when the point $x_L$ is a resonant singularity.

Proposition 4.4. (Proposition 6.1 in [27]) Let $M_L(\varepsilon)$ and $St_L(\varepsilon)$ be the monodromy matrices and the unfolded Stokes matrix of the unfolded equation. Then when the condition (2.10) holds they satisfy the relations

$$M_L(\varepsilon) = St_L(\varepsilon) e^{\pi i (\lambda + \frac{1}{x_R} B)}$$

on the sector $\Omega_1(\varepsilon)$, and

$$M_L(\varepsilon) = e^{\pi i (\lambda + \frac{1}{x_R} B)} St_L(\varepsilon)$$

on the sector $\Omega_2(\varepsilon)$.

Corollary 4.5. (Corollary 6.2 in [27]) Assume that the condition (2.10) holds. Then

$$St_L(\varepsilon) = e^{2\pi i T_L(\varepsilon)}.$$

Let us compute the limit of the matrix $T_L(\varepsilon)$ when $\sqrt{\varepsilon} \to 0$. The next proposition and lemma gives us the number $d_L$ and its limit when $\sqrt{\varepsilon} \to 0$.

Proposition 4.6. Assume that the condition (2.10) holds. Then

$$d_L = \frac{1}{\left(\frac{\beta}{2\sqrt{\varepsilon}} + \frac{\alpha}{2} - 1\right)!} \sum_{k=0}^{\frac{\beta}{2\sqrt{\varepsilon}} + \frac{\alpha}{2} - 1} \left(\frac{\beta}{2\sqrt{\varepsilon}} + \frac{\alpha}{2} - 1\right) \Gamma\left(\frac{\beta}{2\sqrt{\varepsilon}} - \frac{\alpha}{2} + 1\right) (-2\sqrt{\varepsilon})^{-\alpha + 1 + k} \Gamma(2 + k - \alpha) A_L,$$
where $A_L$ is given by \[ (3.17) \].

**Lemma 4.7.** Assume that the condition \[ (2.10) \] holds. Then
\[
\lim_{\sqrt{\varepsilon} \to 0} d_L = ( -\beta )^{1-\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\beta^k \gamma^k}{k! \Gamma(2+k-\alpha)}.
\]

**Proof.** Applying the limit \[ (3.18) \] for $z = \frac{\gamma}{2\sqrt{\varepsilon}}$ we find that $A \to \gamma^k$ when $\sqrt{\varepsilon} \to 0$. Again applying the limit \[ (3.18) \] for $z = \frac{\beta}{2\sqrt{\varepsilon}}$ and using the limit \[ (3.19) \] we obtain the limit of $d_L$. \qed

In \[ 16 \] Lambert and Rousseau prove that the unfolded Stokes matrix $St_L(\varepsilon)$ depends analytically on $\sqrt{\varepsilon}$ and tends to the Stokes matrix $St_\pi$ when $\sqrt{\varepsilon} \to 0$. Then

**Theorem 4.8.** Assume that the condition \[ (2.10) \] holds. Then the Stokes matrix $St_\pi$ at the origin of the DCHE and the matrix $e^{2\pi i T_L(\varepsilon)}$ of the unfolded equation are connected as
\[
e^{2\pi i T_L(\varepsilon)} \to St_\pi
\]
when $\sqrt{\varepsilon} \to 0$.

In a consequence of Theorem 4.8 we have

**Corollary 4.9.** Assume that $\beta > 0$. Then there exists an actual fundamental matrix solution at the origin of the DCHE with respect to which the corresponding Stokes matrix $St_\pi$ is given by
\[
St_\pi = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},
\]
where
\[
\mu = 2\pi i (-\beta)^{1-\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\beta^k \gamma^k}{k! \Gamma(2+k-\alpha)}.
\]

### 4.2. Actual fundamental matrix solution at the origin of the DCHE

In this paragraph we present explicitly the actual fundamental matrix solution $\Phi_0(x,0)$ at the origin of the DCHE with respect to which the Stokes matrix $St_\pi$ has the form fixed by Theorem 4.8. When $\alpha \in \mathbb{Z}$ we apply the Borel-Laplace summation in order to build this actual solution (see \[ 18 \] \[ 21 \] for details). When $\alpha \notin \mathbb{Z}$ we directly express the solution $w_2(x,0)$ in terms of Laplace integrals without using the summability theory. The second approach is more general and also can be used when $\alpha \in \mathbb{Z}$. But the first approach allows us to distinguish special solutions of DCHE that are holomorphic in whole $\mathbb{C}^*$ even when $\alpha \notin \mathbb{Z}$.

We start by building a formal fundamental matrix solution at the origin.

**Theorem 4.10.** Assume that $\beta > 0$. Then the DCHE \[ (1.2) \] possesses an unique formal fundamental matrix solution $\hat{\Phi}_0(x,0)$ at the origin in the form
\[
\hat{\Phi}_0(x,0) = \exp(Gx) \hat{H}(x) x^{\Lambda} \exp \left( -\frac{B}{x} \right),
\]
where the matrices $G, \Lambda$ and $B$ are given by \[ (4.23) \]. The matrix $\hat{H}(x)$ is defined as follows:

1. If $\alpha \notin \mathbb{Z}$ then
\[
\hat{H}(x) = \begin{pmatrix} 1 & \frac{x \varphi(x)}{\beta} \\ 0 & 1 \end{pmatrix}
\]
\[ \hat{\varphi}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n S_n \Gamma(2 - \alpha + n)}{\beta^n} x^{n+1}. \]

Here \( S_n \) is the \( n \)-th partial sum of the absolutely convergent number series
\[ S = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k \gamma^k}{k! \Gamma(2 - \alpha + k)}. \]

In particular,
(a) If \( S = 0 \) then the power series \( \hat{\varphi}(x) \) is convergent.
(b) If \( S \neq 0 \) then the power series \( \hat{\varphi}(x) \) is divergent.

(2) If \( \alpha \in \mathbb{N} \) then
\[ \hat{H}(x) = \begin{pmatrix} 1 & x^\alpha \gamma^{\alpha-1} \hat{\varphi}(x) \beta x \left( \frac{1}{x} \right) + x^\alpha P \left( \frac{1}{x} \right) \\ 0 & 1 \end{pmatrix} \]

where
\[ \hat{\psi}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n W_n n!}{\beta^n} x^{n+1} \]
and \( P \left( \frac{1}{x} \right) \) is a polynomial in \( \frac{1}{x} \) of degree \( \alpha - 2 \) for \( \alpha \geq 2 \) and \( P \equiv 0 \) for \( \alpha = 1 \).
Here \( W_n \) is the \( n \)-th partial sum of the absolutely convergent number series
\[ W = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k \gamma^k}{k! \Gamma(\alpha + k)}. \]

In particular,
(a) If \( W = 0 \) then the power series \( \hat{\psi}(x) \) is convergent.
(b) If \( W \neq 0 \) then the power series \( \hat{\psi}(x) \) is divergent.

(3) If \( \alpha \in \mathbb{Z}_{\leq 0} \) then
\[ \hat{H}(x) = \begin{pmatrix} 1 & x^\alpha \gamma^{\alpha-1} \hat{\varphi}(x) \beta x \left( \frac{1}{x} \right) \\ 0 & 1 \end{pmatrix} \]

where
\[ \hat{\phi}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n Q_n \Gamma(2 - \alpha + n)}{\beta^{1-\alpha+n}} x^{n+1}. \]

Here \( Q_n \) is the \( n \)-th partial sum of the absolutely convergent number series
\[ Q = \sum_{k=0}^{\infty} \frac{(-1)^k \gamma^k \beta^k \gamma^{1-\alpha+k}}{k! \Gamma(2 - \alpha + k)}. \]

In particular,
(a) If \( Q = 0 \) then the power series \( \hat{\phi}(x) \) is convergent.
(b) If \( Q \neq 0 \) then the power series \( \hat{\phi}(x) \) is divergent.

Proof. The proof is similar to the proof of Proposition 4.2 in [27]. We only note that when \( \alpha \notin \mathbb{Z} \) we reduce the solution \( w_2(x,0) \) from (2.6) to the integral \( \int_0^x e^{\beta z} \frac{1}{z^\alpha} dz \) while when \( \alpha \in \mathbb{Z} \) to the integral \( \int_0^x e^{\beta z} \frac{1}{z^\alpha} dz \). \( \square \)
The application of summability theory to the differential equations ensures that the
divergent power series \( \hat{\varphi}(x) \), \( \hat{\psi}(x) \) and \( \hat{\phi}(x) \) are 1-summable in any direction \( \theta \) except for
the singular direction \( \theta = \pi \).

**Lemma 4.11.** 1. Assume that \( W \neq 0 \). Then for any direction \( \theta \neq \pi \) the function

\[
\psi_\theta(x) = \beta \int_0^{+\infty} e^{i\theta} \frac{v(\xi) e^{-\xi \frac{\xi}{\xi + \beta}}}{\xi + \beta} d\xi
\]

defines the 1-sum of the power series \( \hat{\psi}(x) \) from (4.29) in such a direction. Here

\[
v(\xi) = \sum_{n=0}^{\infty} \frac{\gamma^n \xi^n}{n! \Gamma(\alpha + n)}.
\]

2. Assume that \( Q \neq 0 \). Then for every direction \( \theta \neq \pi \) the function

\[
\phi_\theta(x) = \frac{\beta}{x^{1-\alpha}} \int_0^{+\infty} \frac{\xi^{1-\alpha} q(\xi) e^{-\xi \frac{\xi}{\xi + \beta}}}{\xi + \beta} d\xi
\]

defines the 1-sum of the power series \( \hat{\phi}(x) \) from (4.30) is such a direction. Here

\[
q(\xi) = \sum_{k=0}^{\infty} \frac{\gamma^k \xi^k}{k! \Gamma(2 - \alpha + k)}.
\]

The functions \( \psi_\theta(x) \) and \( \phi_\theta(x) \) are holomorphic functions in the open disc

\[
D_\theta(\gamma) = \left\{ x \in \mathbb{C}^* \mid \Re \left( \frac{e^{i\theta}}{x} \right) > |\gamma| \right\}.
\]

**Proof.** We will prove the second statement of the Lemma. The first is proved in a similar way.

Let \( \alpha \in \mathbb{Z}_{\leq 0} \). The formal Borel transform of order 1 of the series \( \hat{\phi}(x) \) from (4.30) yields
the convergent power series near the origin \( \xi = 0 \) of the Borel \( \xi \)-plane

\[
\phi(\xi) = (\mathcal{B}_1 \hat{\phi})(\xi) = \sum_{n=0}^{\infty} (-1)^n Q_n \frac{(1 - \alpha + n)! \xi^n}{\beta^{1-\alpha+n} n!}.
\]

The series \( \phi(\xi) \) can be regarded as the \( 1 - \alpha \)-th derivative of the series

\[
w(\xi) = \sum_{n=0}^{\infty} (-1)^n Q_n \frac{\xi^{1-\alpha+n}}{\beta^{1-\alpha+n}}.
\]

Then from Lemma 6.12 in [27] it follows that the Laplace transform of order 1 of the function \( \phi(\xi) \) is expressed by the Laplace transform of the function \( w(\xi) \)

\[
(L_1 \phi)(x) = \frac{1}{x^{1-\alpha}} (L_1 w)(x)
\]

since \( \frac{d^k w}{d\xi^k} = 0 \) for \( 0 \leq k \leq -\alpha \). It turns out that the power series \( w(\xi) \) is the Maclaurin
series of a well known function. More precisely,

\[
w(\xi) = (-\beta)^{1-\alpha} \left( \frac{1}{1 + \frac{\xi}{\beta}} - \sum_{k=0}^{-\alpha} (-1)^k \frac{\xi^k}{\beta^k} \right) q(\xi),
\]
where \( q(\xi) \) is the analytic function in \( \mathbb{C} \) defined by the power series

\[
q(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k! \Gamma(2 - \alpha + k)}.
\]

Since

\[
\sum_{k=0}^{-\alpha} (-1)^k \frac{\xi^k}{\beta^k} = \frac{1 - (-1)^{1-\alpha} \frac{\xi^{1-\alpha}}{\beta^{1-\alpha}}}{1 + \frac{\xi}{\beta}}
\]

then

\[
(\mathcal{L}_1 \phi)(x) = \frac{(-\beta)^{1-\alpha}}{x^{1-\alpha}} \left( \mathcal{L}_1 \frac{q(\xi)}{1 + \frac{\xi}{\beta}} \right)(x) - \frac{(-\beta)^{1-\alpha}}{x^{1-\alpha}} \left( \mathcal{L}_1 \frac{q(\xi)}{1 + \frac{\xi}{\beta}} \right)(x) + \frac{1}{x^{1-\alpha}} \left( \mathcal{L}_1 \frac{\xi^{1-\alpha} q(\xi)}{1 + \frac{\xi}{\beta}} \right)(x)
\]

\[
= \frac{\beta}{x^{1-\alpha}} \int_0^{+\infty} \frac{\xi^{1-\alpha} q(\xi) e^{-\xi \beta}}{\xi + \beta} d\xi.
\]

Thus the function

\[
\phi_\theta(x) = \frac{\beta}{x^{1-\alpha}} \int_0^{+\infty} \frac{\xi^{1-\alpha} q(\xi) e^{-\xi \beta}}{\xi + \beta} d\xi
\]

gives the 1-sum of the power series \( \hat{\phi}(x) \) in any direction \( \theta \neq \pi \). Since

\[
\left| \frac{q(\xi)}{\xi + \beta} \right| \leq A \sum_{k=0}^{\infty} \frac{|\gamma|^k |\xi|^k}{k!} = A e^{|\gamma| |\xi|}
\]

for an appropriate constant \( A > 0 \) the integral \( \phi_\theta(x) \) exists when \( \alpha \in \mathbb{Z}_{\leq 0} \) and defines a holomorphic function in the open disc \( \text{Re} \left( \frac{e^{i\theta}}{x} \right) > |\gamma| \).

This ends the proof. \( \square \)

**Remark 4.12.** Unfortunately till now we can not derive the 1-sum of the power series \( \hat{\phi}(x) \) from (4.28) in an explicit way. The formal Borel transform of this series

\[
(\hat{B}_1 \hat{\phi})(\xi) = \sum_{n=0}^{\infty} (-1)^n S_n \frac{\Gamma(2 - \alpha + n)}{\beta^n} \frac{\xi^n}{n!}
\]

is a convergent power series for \( |\xi| < \beta \). But we can not specify explicitly the function whose Maclaurin series is \( (\hat{B}_1 \hat{\phi})(\xi) \). For this reason we use a slightly different approach to build an actual solution of the DCHE when \( \alpha \notin \mathbb{Z} \).

**Remark 4.13.** Let \( I = (-\pi, \pi) \subset \mathbb{R} \). When we move the direction \( \theta \in I \) the holomorphic functions \( \psi_\theta(x) \) (resp. \( \phi_\theta(x) \)) glue together analytically and define a holomorphic function \( \tilde{\psi}(x) \) (resp. \( \tilde{\phi}(x) \)) on a sector \( \tilde{D} \) with opening \( > \pi \)

\[
(4.32) \quad \tilde{D} = \bigcup_{\theta \in I} \tilde{D}_\theta(|\gamma|),
\]

where \( \tilde{D}_\theta(|\gamma|) \) is the lifting of \( D_\theta(|\gamma|) \) on the Riemann surface of the natural logarithm. On \( \tilde{D} \) the function \( \tilde{\psi}(x) \) (resp. \( \tilde{\phi}(x) \)) is asymptotic to the power series \( \hat{\psi}(x) \) (resp. \( \hat{\phi}(x) \)) in Gevrey 1 sense and defines the 1-sum of this series there. The restriction of \( \tilde{\psi}(x) \) (resp. \( \tilde{\phi}(x) \)) on \( \mathbb{C}^* \) is a multivalued function. In every direction \( \theta \neq \pi \) the function \( \tilde{\psi}(x) \) (resp. \( \tilde{\phi}(x) \)) has only one value than coincides with the function \( \psi_\theta(x) \) from Lemma 4.11 (1) (resp. \( \phi_\theta(x) \) from Lemma 4.11 (2)). Near the singular direction \( \theta = \pi \) the function \( \psi(x) \)
Theorem 4.14. Assume that $\beta > 0$. Then

1. Assume that one of the following conditions holds: $(\alpha \in \mathbb{N}, W = 0)$, $(\alpha \in \mathbb{Z}_{\leq 0}, Q = 0)$ or $(\alpha \notin \mathbb{Z}, S = 0)$. Then the DCHE \((1.2)\) possesses an unique actual fundamental matrix solution $\Phi_0(x,0)$ at the origin in the form

$$\Phi_0(x,0) = \exp(Gx) H(x) x^{\Lambda} \exp\left(-\frac{B}{x}\right),$$

where the matrices $G, \Lambda$ and $B$ are given by \((4.23)\) and $H(x)$ is a holomorphic matrix function in whole $\mathbb{C}$. More precisely, when $\alpha \notin \mathbb{Z}$ and $S = 0$ the matrix $H(x)$ coincides with the matrix $\tilde{H}(x)$ from Theorem \(4.10(1)\). When $\alpha \in \mathbb{N}$ and $W = 0$ the matrix $H(x)$ coincides with the matrix $\tilde{H}(x)$ from Theorem \(4.10(2)\). When $\alpha \in \mathbb{Z}_{\leq 0}$ and $Q = 0$ the matrix $H(x)$ coincides with the matrix $\tilde{H}(x)$ from Theorem \(4.10(3)\).

2. Assume that $Q \neq 0$, $W \neq 0$, $S \neq 0$. Then the DCHE \((1.2)\) possesses an unique actual matrix solution $\Phi_0(x,0)$ at the origin in the form \((4.33)\) which is a holomorphic matrix function on the sector $\mathcal{D}$ from \((4.32)\) whose opening is $> \pi$. The restriction of $\Phi_0(x,0)$ on $\mathbb{C}^*$ is a multivalued function. For any direction $\theta \neq \pi$ this solution has only one value $\Phi_0^\theta(x,0)$ in the form

$$\Phi_0^\theta(x,0) = \exp(Gx) H_\theta(x) x^{\Lambda} \exp\left(-\frac{B}{x}\right),$$

where the matrices $G, \Lambda$ and $B$ are given by \((4.23)\). The matrix $H_\theta(x)$ is defined as follows:

(a) If $\alpha \notin \mathbb{Z}$ then

$$H_\theta(x) = \begin{pmatrix} 1 & \frac{x \varphi_\theta(x)}{\beta} \\ 0 & 1 \end{pmatrix},$$

where

$$\varphi_\theta(x) = \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \left( \int_0^{+\infty} e^{i \theta} \left( \frac{e^{-\frac{\xi}{x}}}{(1 + \frac{\xi}{x})^{2-\alpha+k}} \right) \, d\xi \right).$$

(b) If $\alpha \in \mathbb{N}$ then

$$H_\theta(x) = \begin{pmatrix} 1 & \frac{x \psi_\theta(x)}{\beta} + x^\alpha P\left(\frac{1}{2}\right) \\ 0 & 1 \end{pmatrix}$$

where $\psi_\theta(x)$ is defined by Lemma \(4.11(1)\) and extended by Remark \(4.13\).

(c) If $\alpha \in \mathbb{Z}_{\leq 0}$ then

$$H_\theta(x) = \begin{pmatrix} 1 & \frac{x \phi_\theta(x)}{\beta} \\ 0 & 1 \end{pmatrix}$$

where $\phi_\theta(x)$ is defined by Lemma \(4.11(2)\) and extended by Remark \(4.13\).

For the singular direction $\theta = \pi$ the DCHE \((1.2)\) possesses two actual fundamental matrix solution at the origin

$$\Phi_0^{\pi+}(x,0) = \Phi_0^{\pi+}(x,0),$$

$$\Phi_0^{\pi-}(x,0) = \Phi_0^{\pi-}(x,0),$$

(resp. $\hat{o}(x)$) has two different values: $\psi_\pi^+(x) = \psi_{\pi+\epsilon}(x)$ (resp. $\phi_\pi^+(x) = \phi_{\pi+\epsilon}(x)$) and $\psi_\pi^-(x) = \psi_{\pi-\epsilon}(x)$ (resp. $\phi_\pi^-(x) = \phi_{\pi-\epsilon}(x)$) for a small number $\epsilon > 0$. Now we can present an actual fundamental matrix solution at the origin.
where $\epsilon > 0$ is a small number and the matrices $\Phi_0^{\pi \pm \epsilon}$ are given by (4.34).

Proof. The proof of items (1), (2.b) and (2.c) follows directly from the theorem of Hukuhara-Turrittin-Martinet-Ramis [?, [21]. We give the proof of item (2.a).

Let $\alpha \notin \mathbb{Z}$. The solution $w_2(x, 0)$ from (2.6) becomes

$$w_2(x, 0) = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} \left( \int_0^x e^{\frac{-\beta}{2} - \frac{\alpha}{k}} \, dz \right) = e^{-\frac{\beta}{2}} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} \left( \int_0^x e^{\frac{-\beta}{2} + \frac{\beta}{2}} \, dz \right).$$

By setting

$$\frac{-\beta}{z} + \frac{\beta}{x} = -\frac{\xi}{x},$$

we transform the solution $w_2(x, 0)$ into

$$w_2(x, 0) = \frac{e^{-\frac{\beta}{x}} x^{1-\alpha}}{\beta} \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \left( \int_0^{+\infty} \frac{e^{-\frac{\xi}{x}}}{(1 + \frac{\xi}{\beta})^{2-\alpha+k}} \, d\xi \right).$$

Now we will show that this infinite sum defines a holomorphic function on the open disc $D_\theta(|\gamma|)$ defined by (4.32) for $\theta \in (-\pi, \pi)$. Since $1/\cos \theta \geq 0$ and $1/1 + \frac{\xi}{z} \leq 1/|\sin \theta|$ for $\cos \theta < 0$ we find that when $Re(\alpha) \leq 2$

$$\frac{1}{|1 + \frac{\xi}{\beta}|^{k-\alpha+2}} \leq \begin{cases} A & \text{for } \cos \theta \geq 0, \\ A & \text{for } \cos \theta < 0 \end{cases}$$

where $A = e^{-(Im \alpha) \arg(1 + \frac{\xi}{\beta})} > 0$. Thus in this case each integral can be analytically continued along any ray $\theta \neq \pi$ from 0 to $+\infty e^{i\theta}$ and defines a holomorphic function in the open disc $Re\left(\frac{e^{i\theta}}{x}\right) > 0$ whose opening is $< \pi$. Let $x \in D_\theta(|\gamma|)$ and let $\kappa = \arg(x)$. Note that from $x \in D_\theta(|\gamma|)$ it follows that $|x| < \frac{\cos(\theta - \kappa)}{|\gamma|}$. Then we find that when $\cos \theta \geq 0$ and $Re(\alpha) \leq 2$

$$\left| \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \left( \int_0^{+\infty} e^{i\theta} e^{\frac{-\xi}{x}} \frac{e^{-\frac{\xi}{x}}}{(1 + \frac{\xi}{\beta})^{2-\alpha+k}} \, d\xi \right) \right| \leq A \sum_{k=0}^{\infty} \frac{|\gamma|^k |x|^k}{k!} < A \sum_{k=0}^{\infty} \frac{\cos^k(\theta - \kappa)}{k!} < \infty,$$

where $c = Re\left(\frac{e^{i\theta}}{x}\right) > |\gamma| > 0$. Similarly, when $\cos \theta < 0$ and $Re(\alpha) \leq 2$ we have that

$$\left| \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \left( \int_0^{+\infty} e^{i\theta} e^{\frac{-\xi}{x}} \frac{e^{-\frac{\xi}{x}}}{(1 + \frac{\xi}{\beta})^{2-\alpha+k}} \, d\xi \right) \right| \leq \frac{A}{c |\sin \theta|^2 |Re(\alpha)|^2} \sum_{k=0}^{\infty} \frac{|\gamma|^k |x|^k}{k! |\sin \theta|^k} < \frac{A}{c |\sin \theta|^2 |Re(\alpha)|^2} \sum_{k=0}^{\infty} \frac{\cos^k(\theta - \kappa)}{k! |\sin \theta|^k} < \infty.$$

Thus from the Wiierstrass’s theorem it follows that when $Re(\alpha) \leq 2$ the functional series

$$\sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \left( \int_0^{+\infty} e^{i\theta} e^{\frac{-\xi}{x}} \frac{e^{-\frac{\xi}{x}}}{(1 + \frac{\xi}{\beta})^{2-\alpha+k}} \, d\xi \right) = \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \varphi^\theta_k(x)$$

converges uniformly on the compact sets of the open disc $D_\theta(|\gamma|)$. Since for all $k \geq 0$ the functions $\frac{\gamma^k x^k}{k!} \varphi^\theta_k(x)$ are holomorphic functions on $D_\theta(|\gamma|)$ so the sum.
Assume now that there exists \( k_1 \in \mathbb{N}_0 \) such that \( k + 2 - \text{Re}(\alpha) < 0 \) for \( 0 \leq k \leq k_1 \) while \( k_1 + 1 + 2 - \text{Re}(\alpha) \geq 0 \). Since \(|(1 + \frac{\xi}{\beta})^{\alpha-k-2}| \leq A (1 + \frac{\xi}{\beta})^{\text{Re}(\alpha)-k-2} \) where \( A \) is above we find that for \( 0 \leq k \leq k_1 \)

\[
\left| \int_0^{+\infty e^{\theta}} \frac{e^{-\xi}}{(1 + \frac{\xi}{\beta})^{2-\alpha+k}} \, d\xi \right| \leq A 2^{\text{Re}(\alpha)-k-2} \int_0^{\beta e^{\theta}} e^{-|\xi|c} \, d\xi
\]

\[+ \quad A \left( \frac{2}{\beta} \right)^{\text{Re}(\alpha)-k-2} \int_0^{+\infty e^{\theta}} |\xi|^{\text{Re}(\alpha)-k-2} e^{-c|\xi|} \, d\xi
\]

\[\leq \frac{2^{\text{Re}(\alpha)-k-2} A}{c} (1 - e^{-\beta c}) + \frac{A}{e^{\text{Re}(\alpha)-k-1}} \left( \frac{2}{\beta} \right)^{\text{Re}(\alpha)-k-2} \Gamma(\text{Re}(\alpha) - k - 1).\]

Therefore for \( 0 \leq k \leq k_1 \) we also can continue analytically each integral along any ray \( \theta \neq \pi \) and define holomorphic functions \( \varphi^\theta_k(x) \) in the open disc \( \text{Re} \left( \frac{\xi^\theta}{\beta} \right) > 0 \) whose opening is \( < \pi \). Then for \( x \in \mathcal{D}_\theta(|\gamma|) \) we have

\[
\left| \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \left( \int_0^{+\infty e^{\theta}} \frac{e^{-\xi}}{(1 + \frac{\xi}{\beta})^{2-\alpha+k}} \, d\xi \right) \right| \leq K \sum_{k=0}^{k_1} \frac{|\gamma|^k |x|^k}{2^k k!}
\]

\[+ \quad \frac{A}{e^{\text{Re}(\alpha)-1}} \left( \frac{2}{\beta} \right)^{\text{Re}(\alpha)-2} \sum_{k=0}^{k_1} \frac{|\gamma|^k c^k \beta^k \Gamma(\text{Re}(\alpha) - k - 1)}{2^k k!} x^k + |F_{k+1}(x)|
\]

\[< K \sum_{k=0}^{k_1} \frac{\cos^k(\theta - \kappa)}{2^k k!} + \frac{A}{e^{\text{Re}(\alpha)-1}} \left( \frac{2}{\beta} \right)^{\text{Re}(\alpha)-2} \sum_{k=0}^{k_1} \frac{\cos^k(\theta - \kappa)}{2^k k!}
\]

where \( K = \frac{2^{\text{Re}(\alpha)-2} A}{c} (1 - e^{-\beta c}) \). For the last addend we have that

\[|F_{k+1}(x)| \leq \frac{A}{c} \sum_{k=1+1}^{\infty} \frac{|\gamma|^k |x|^k}{k!} < \frac{A}{c} \sum_{k=1+1}^{\infty} \frac{\cos^k(\theta - \kappa)}{k!} < \infty
\]

when \( \cos \theta \geq 0 \) and

\[|F_{k+1}(x)| \leq \frac{A}{c} \sum_{k=1+1}^{\infty} \frac{|\gamma|^k |x|^k}{k!} \frac{1}{\sin \theta^{2-\text{Re}(\alpha)}} \sum_{k=1+1}^{\infty} \frac{\cos^k(\theta - \kappa)}{k!} \frac{1}{\sin \theta^{2-\text{Re}(\alpha)}} < \infty
\]

when \( \cos \theta < 0 \).

As a result the functional series (4.36) defines a holomorphic function on the disc \( \mathcal{D}_\theta(|\gamma|) \) whose opening is \( < \pi \). We denote this function by \( \varphi_\theta(x) \). When we move \( \theta \in (-\pi, \pi) \) the holomorphic functions \( \varphi_\theta(x) \) glue together analytically and define a holomorphic function \( \tilde{\varphi}(x) \) on an open sector \( \mathcal{D} \) from (4.32) whose opening \( > \pi \). The restriction of \( \tilde{\varphi}(x) \) on \( \mathbb{C}^* \) is a multivalued function. For every direction \( \theta \neq \pi \) it has only one value. Near the singular direction \( \theta = \pi \) it has two different values: \( \varphi^\pi_\theta(x) = \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \varphi_k^{\theta+\epsilon}(x) \) and \( \varphi^\pi_\theta(x) = \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \varphi_k^{\theta-\epsilon}(x) \) where \( \epsilon > 0 \) is a small number.

This ends the proof.
Remark 4.15. It seems that the function $\varphi_0(x)$ from (4.35) plays the part of 1 sum of the formal series

$$\hat{\phi}(x) = \sum_{k=0}^{\infty} \frac{\gamma^k x^k}{k!} \varphi_k(x),$$

where

$$\varphi_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2 - \alpha + k)_n}{\beta^n} x^{n+1}.$$

Here $(a)_n$ is the Pochhammer symbol.

We are going to discuss in our next work the summation of such a series $\hat{\phi}(x)$.

Theorem 4.16. Assume that $\beta > 0$. Then with respect to the actual fundamental matrix solution $\Phi_0(x,0)$ defined by Theorem 4.14 the DCHE (1.2) has a Stokes matrix $\text{St}_\pi$ at the origin in the form

$$\text{St}_\pi = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},$$

where $\mu$ is introduced by (4.27).

Proof. Let $\epsilon > 0$ be a small number and let $\theta = \pi$. Let $\Phi^+_0(x,0) = \Phi_0^{\pi+\epsilon}(x,0)$ and $\Phi^-_0(x,0) = \Phi_0^{\pi-\epsilon}(x,0)$ be the actual fundamental matrix solutions at the origin of the DCHE built by Theorem 4.16. To find the Stokes matrix $\text{St}_\pi$ we have to compare the solutions $\Phi^-_0(x,0)$ and $\Phi^+_0(x,0)$

$$\Phi^-_0(x,0) = \Phi^+_0(x,0) \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

When $\alpha \in \mathbb{N}$ we find that

$$\mu = \frac{\gamma^{\alpha-1} e^{-\frac{\beta}{\gamma}}}{\beta} \left[ \psi^-_\pi(x) - \psi^+_\pi(x) \right] = 2\pi i \gamma^{\alpha-1} e^{-\frac{\beta}{\gamma}} \text{Res} \left( \frac{v(\xi) e^{-\frac{\xi}{\gamma}}}{\xi + \beta}; \xi = -\beta \right)$$

$$= 2\pi i \gamma^{\alpha-1} v(-\beta) = 2\pi i \gamma^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-1)^n \beta^n \gamma^n}{n! \Gamma(\alpha + n)}.$$

Similarly, when $\alpha \in \mathbb{Z}_{\leq 0}$ we find that

$$\mu = \frac{x^{1-\alpha} e^{-\frac{\beta}{x}}}{\beta} \left[ \phi^-_\pi(x) - \phi^+_\pi(x) \right] = 2\pi i e^{-\frac{\beta}{x}} \text{Res} \left( \frac{\xi^{1-\alpha} q(\xi) e^{-\frac{\xi}{x}}}{\xi + \beta}; \xi = -\beta \right)$$

$$= 2\pi i (-\beta)^{1-\alpha} q(-\beta) = 2\pi i (-\beta)^{1-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n \beta^n \gamma^n}{n! \Gamma(2 - \alpha + n)}.$$

Now we will show that the so found multipliers coincide. Indeed, let $\alpha \in \mathbb{N}$. Then the multiplier $\mu$ corresponding to $\alpha \in \mathbb{Z}_{\leq 0}$ becomes

$$\mu = 2\pi i (-\beta)^{1-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n \beta^n \gamma^n}{n! (1 + n - \alpha)}$$

since $1/\Gamma(z) = 0$ for $z \in \mathbb{Z}_{\leq 0}$. Then

$$\mu = 2\pi i (-\beta)^{1-\alpha} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^p \gamma^p}{(p + \alpha - 1)! p!} = 2\pi i \gamma^{\alpha-1} \sum_{p=0}^{\infty} \frac{(-1)^p \beta^p \gamma^p}{p! \Gamma(p + \alpha)}$$

which is the multiplier $\mu$ corresponding to $\alpha \in \mathbb{N}$. 

Let now $\alpha \not\in \mathbb{Z}$ and let us compare the functions $\varphi_{k}^{\pi-\epsilon}(x)$ and $\varphi_{k}^{\pi+\epsilon}(x)$. We have
\[
\varphi_{k}^{\pi-\epsilon}(x) - \varphi_{k}^{\pi+\epsilon}(x) = (\beta)^{2-\alpha+k} \int_{-\beta}^{-\infty} (\beta + \xi)^{\alpha-k-2} e^{-\frac{\xi}{2}} d\xi,
\]
where $\gamma = (\pi - \epsilon) - (\pi + \epsilon)$. Without changing the integral we can deform the path $\gamma$ into a Henkel type path going along the negative real axis from $-\infty$ to $-\beta$, encircling $-\beta$ in the positive sense and backing to $-\infty$. Then
\[
\varphi_{k}^{\pi-\epsilon}(x) - \varphi_{k}^{\pi+\epsilon}(x) = (\beta)^{2-\alpha+k} \left(1 - e^{-2\pi i(\alpha-k-2)}\right) \int_{-\beta}^{-\infty} (\beta + \xi)^{\alpha-k-2} e^{-\frac{\xi}{2}} d\xi.
\]
where we have used the Euler’s reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$ for $z \not\in \mathbb{Z}$. Then for the multiplier $\mu$ we find
\[
\mu = x^{1-\alpha} e^{-\frac{\beta}{\pi}} \left[\varphi_{k}^{\pi-\epsilon}(x) - \varphi_{k}^{\pi+\epsilon}(x)\right]
= x^{1-\alpha} e^{-\frac{\beta}{\pi}} \sum_{k=0}^{\infty} \gamma^{k} x^{k} \left[\varphi_{k}^{\pi-\epsilon}(x) - \varphi_{k}^{\pi+\epsilon}(x)\right]
= 2\pi i (-\beta)^{1-\alpha} \sum_{k=0}^{\infty} \frac{(-1)^{k} \gamma^{k} \beta^{k}}{k! \Gamma(2 - \alpha + k)}.
\]
This ends the proof.

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**Appendix**

In this paragraph we will show by a direct computation that for lower values of $\alpha$
\[
\lim_{\sqrt{\pi} \to 0} (q_{R} + q_{L}) = \lambda
\]
where
\[
\lambda = \gamma^{\alpha-1} \frac{\pi}{(\alpha - 1)!}.
\]
Let $\alpha = 2$. Then the multipliers $q_R$ and $q_L$ become

$$q_R = \frac{1}{2\sqrt{\varepsilon}} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}} \quad q_L = -\frac{1}{2\sqrt{\varepsilon}} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}}.$$  

Note that in this case $q_R \to +\infty$ and $q_L \to -\infty$ when $\sqrt{\varepsilon} \to 0$. Next the functions

$\left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}}$ and $\left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}}$

are expressed as power series in $\sqrt{\varepsilon}$ as follows

\[
\left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}} = \sum_{k=0}^{\infty} \left( -2\gamma \sum_{p=0}^{\infty} \left( \frac{(\sqrt{\varepsilon})^{4p+1}}{4p+2} \right) \right) \frac{1}{k!} = 1 - 2\gamma \left( \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}^5}{6} + \cdots \right)
\]

(4.38)

$\left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}} = \sum_{k=0}^{\infty} \left( 2\gamma \sum_{p=0}^{\infty} \left( \frac{(\sqrt{\varepsilon})^{4p+1}}{4p+2} \right) \right) \frac{1}{k!} = 1 + 2\gamma \left( \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}^5}{6} + \cdots \right)
\]

+ $\frac{4\gamma^2}{2!} \left( \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}^5}{6} + \cdots \right)^2 - \frac{8\gamma^3}{3!} \left( \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}^5}{6} + \cdots \right)^3 + \cdots$

Then we find that when $\alpha = 2$

$$\lim_{\sqrt{\varepsilon} \to 0} (q_R + q_L) = \lim_{\sqrt{\varepsilon} \to 0} \frac{2\gamma \sqrt{\varepsilon} + O(\varepsilon)}{2\sqrt{\varepsilon}} = \gamma,$$

which coincides with (4.37).

Let $\alpha = 4$. Then the multipliers $q_R$ and $q_L$ become

$$q_R = -\left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}} \frac{1}{(2\sqrt{\varepsilon})^3} \left[ \Gamma(3) - \Gamma(2) \frac{2\sqrt{\varepsilon}}{1 - \varepsilon^2} \right],$$

$$q_L = \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}} \frac{1}{(2\sqrt{\varepsilon})^3} \left[ \Gamma(3) + \Gamma(2) \frac{2\sqrt{\varepsilon}}{1 - \varepsilon^2} \right].$$

Note that in this case $\lim_{\sqrt{\varepsilon} \to 0} q_R = -\infty$ and $\lim_{\sqrt{\varepsilon} \to 0} q_L = +\infty$. Again using the expressions (4.38) we find that

$$\lim_{\sqrt{\varepsilon} \to 0} (q_R + q_L) = \frac{4}{3} \lim_{\sqrt{\varepsilon} \to 0} \frac{(\sqrt{\varepsilon})^3 \gamma^3 + O(\varepsilon^3)}{(2\sqrt{\varepsilon})^3} = \frac{\gamma^3}{3!},$$

which coincides with (4.37).

Let $\alpha = 6$. Then the multipliers $q_R$ and $q_L$ become

$$q_R = \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}} \frac{1}{2! (2\sqrt{\varepsilon})^5} \left[ \Gamma(5) - \Gamma(4) \frac{2\sqrt{\varepsilon}}{1 - \varepsilon^2} + \Gamma(3) \frac{2\sqrt{\varepsilon}}{1 - \varepsilon^2} \right],$$

$$q_L = -\left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^{\frac{2}{\sqrt{\varepsilon}}} \frac{1}{2! (2\sqrt{\varepsilon})^5} \left[ \Gamma(5) + \Gamma(4) \frac{2\sqrt{\varepsilon}}{1 - \varepsilon^2} + \Gamma(3) \frac{2\sqrt{\varepsilon}}{1 - \varepsilon^2} \right].$$

Then $\lim_{\sqrt{\varepsilon} \to 0} q_R = +\infty$ while $\lim_{\sqrt{\varepsilon} \to 0} q_L = -\infty$. For the sum $q_R + q_L$ we have

$$\lim_{\sqrt{\varepsilon} \to 0} (q_R + q_L) = \frac{4}{15} \lim_{\sqrt{\varepsilon} \to 0} \frac{(\sqrt{\varepsilon})^5 \gamma^5 + O(\varepsilon^3)}{(2\sqrt{\varepsilon})^5} = \frac{\gamma^5}{5!},$$

which coincides with (4.37).
REFERENCES

[1] H. Bateman and A. Erdélyi, *Higher transcendental functions*, vol. 1, McGraw - Hill (New York, 1953).
[2] A. A. Bolibrukh, *On isomonodromic confluence of Fuchsian singularites*, Tr. Mat. Inst. Steklova, vol. 221 (1998), pp. 127-142.
[3] V. Buchstaber and A. Glytsyk, *On determinants of modified Bessel functions and entire solutions of double confluent Heun equations*, Nonlinearity, vol. 29, no. 12 (2016).
[4] V. Buchstaber, S. Tertychnyi, *Holomorphic solution of the double confluent Heun equation associated with the RSJ model of the Josephson junction*, Theoret. and Math. Phys., 182:3 (2015), pp. 329-355.
[5] L. El-Jaick and B. Figueiredo, *Solutions for confluent and double-confluent Heun equation*, J. Math. Phys., 49 (2008).
[6] A. Decarreau, M.-Cl. Dumont-Lepage, P. Maroni, A. Robert and A. Ronveaux, *Formes canoniques des équations confluentes de l’équation de Heun*, Ann. Soc. Sci. Bruxelles Sér. I-II T. 92 (1978), pp. 53-78.
[7] A. Decarreau, P. Maroni and A. Robert, *Sur les équations confluentes de l’équation de Heun*, Ann. Soc. Sci. Bruxelles Sér. III T. 92 (1978), pp. 151-189.
[8] A. Glutsyuk, *Stokes operators via limit monodromy of generic perturbation*, Journal of Dynamical and Control Systems, vol.5 (1999), no.1, pp. 101-135.
[9] A. Glutsyuk, *On the monodromy group of confluenting linear equation*, Moscow Math. J., 5 (2005), no.1, pp. 67-90.
[10] A. Glutsyuk, *Resonant confluence of singular points and Stokes phenomenon*, J. Dyn. Control Syst., 10 (2004), pp. 253-302.
[11] T. Grava and G. Mazzuca, *Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, Circular $\beta$-ensemble and double confluent Heun equation*, Commun. Math. Phys. (2023)
[12] J. Hurtubise, C. Lambert, C. Rousseau, *Complete system of analytic invariants for unfolded differential linear systems with an irregular singularity of Poincaré rank k*, Moscow Math. J. 14 (2013), pp.309-338.
[13] M. Klimeš, *Confdue of singularities of nonlinear differential equations via Borel - Laplace transformations*, J Dynam Control Syst., 22 (2016), 285-324.
[14] M. Klimeš, *Stokes phenomenon and confluence in non-autonomous Hamiltonian systems*, 17 (2018), 665-708.
[15] C. Lambert and C. Rousseau, *Complete system of analytic invariants for unfolded differential linear systems with an irregular singularity of Poincaré rank 1*, Moscow Math. J., vol. 13 (2013), no. 3, pp. 529-550, 553-554.
[16] C. Lambert and C. Rousseau, *The Stokes phenomenon in the confluence of the hypergeometric equation using Riccati equation*, J. Differential Equation, 244 (2008), no.10, pp. 2641-2664.
[17] M. Loday-Richaud, *Divergent series, summability and resurgence II. Simple and multiple summability*, Lecture notes in math., 2154 (2016), Spinger, Berlin
[18] J.-P. Ramis, *Confluence et résurgence*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., 36 (1989), no.3, pp. 703-716
[19] J.-P. Ramis, *Séries divergentes et théories asymptotiques*, Bull. Soc. Math. France 121 (1993) (suppl. ) (Panoramas et Synthèses)
[20] J.-P. Ramis, *Gevrey asymptotics and applications to holomorphic ordinary differential equations*, Differential Equations and Asymptotic Theory in Mathematical Physics (Series in Analysis vol 2 (2004)) ed. C. Hua and R. Wong (Singapore: World Scientific ) pp. 44-99
[21] A. Roseau, *On the solutions of double confluent Heun equations*, Aequat. Math., 60 (2000), pp. 116-136.
[22] A. Salatich, S. Slavyanov, *Antiquantization of the double confluent Heun equation. The Thuekolsky equation*, Russian J. of Nonlinear Dynamics, vol. 15, no. 1 (2019), pp. 79-85.
[23] Y. Sibuya, *Linear differential equations in the complex domain : problems of analytic continuation*, Translations of Mathematical Monographs, 82 (1990), RI: American Mathematical Society, Providence
[25] S. Slavyanov and W. Lay, *Special functions: a unified theory based on singularities*, Oxford: Oxford University Press, 2000.

[26] Ts. Stoyanova, *Zero level perturbation of a certain third-order linear solvable ODE with an irregular singularity at the origin of Poincaré rank 1*, J. Dyn. Control Syst., 24 (2018), No.4, pp. 511-539.

[27] Ts. Stoyanova, *Stokes matrices of a reducible double confluent Heun equation via monodromy matrices of a reducible general Heun equation with symmetric finite singularities*, J. Dyn. Control Syst., 28 (2022), No.1, pp. 207-245.

[28] S. Tertychniy, *Solution space monodromy of a special double confluent Heun equation and its application*, Theoretical and Mathematical Physics, 201 (2019), pp. 1426-1441.

[29] W. Wasow, *Asymptotic expansions for ordinary differential equations*, (1965) (New York:Dover).

[30] C. Zhang, *Confluence et phénomène de Stokes*, J. Math. Sci. Univ. Tokyo, 3 (1996), pp. 91-107.