Non-ergodic extended states in the SYK model

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We analytically study spectral correlations and many body wave functions of an SYK-model deformed by a one body contribution to the Hamiltonian. Our main result is the identification of a wide range of intermediate coupling strengths where the spectral statistics is of Wigner-Dyson type, while wave functions are non-uniformly distributed over Fock space and show fractal behavior. The structure of the theory suggests that such manifestations of non-ergodic extendedness may be a prevalent phenomenon in strongly interacting chaotic quantum systems.

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Introduction:—In recent years, classifications of many body quantum systems as either ‘ergodic’, or ‘many body localized’ (MBL) have become mainstream. This reflects the discovery of a growing number of systems supporting MBL phases [1][11] and naturally extends the distinction between single particle ergodic and Anderson localized systems to many body quantum disorder. However, recently, we are seeing mounting evidence [12][16] that the above dichotomy may be too coarse to capture the complexity of chaotic many body systems. Specifically, Ref. [12] suggests the existence of phases with non-ergodic extended states, i.e. quantum states different from localized in that they have unbounded support, and different from ergodic in that their amplitudes are not uniformly distributed. One reason why this phenomenon is noticed only now is that standard tools in diagnosing chaos — spectral statistics applied to systems of small size of $O(10^4)$ physical sites — tend to be too coarse to resolve the spatial structure of quantum states in Fock space. Indeed, the above indications are indirect in that they are based on numerical and analytic work on disordered graphs with high coordination numbers, artificial systems believed to share key characteristics with genuine random Fock spaces.

In this paper, we present a first principle analytic description of non-ergodic extended states in a deformed version of the SYK model [17][18]. The standard SYK model is a system of $2N \gg 1$ Majorana fermions, $[\chi_i, \chi_j]_+ = 2\delta_{ij}$, governed by the interaction Hamiltonian

$$\hat{H}_0 = \frac{1}{4!} \sum_{i,j,k,l=1}^{2N} J_{ijkl} \hat{\chi}_i \hat{\chi}_j \hat{\chi}_k \hat{\chi}_l, \tag{1}$$

where the coupling constants are drawn from a Gaussian distribution, $\langle |J_{ijkl}|^2 \rangle = 6J^2/(2N)^3$, and the constant $J$ defines the effective band width of the system as $\gamma = J/(2N)^{1/2}$ [19]. The model [1] is known to be in an ergodic phase with eigenfunctions uniformly distributed in Fock space [19][20]. To make the situation more interesting, we generalize the Hamiltonian to $\hat{H} = \hat{H}_0 + \hat{H}_V$, where

$$\hat{H}_V = \gamma \sum_n v_n |n\rangle \langle n| \tag{2}$$

is a sum over projectors onto the occupation number eigenstates $|n\rangle = |n_1, n_2, \ldots, n_N\rangle$, $n_i = 0, 1$, of a system of complex fermions $c_i = \chi_{2i-1} + i\chi_{2i}$, $i = 1, \ldots, N$ defined from the Majorana operators. The coefficients $v_n$ can be chosen to represent any operator diagonal in the occupation number basis, $\{|n\rangle\}$, pertaining to a a fixed one-body basis. For example, any one-body operator, $\hat{H}_0 = \frac{1}{2} \sum_{i,j} J_{ij} \hat{\chi}_i \hat{\chi}_j$, can be diagonalized in the fermion representation and described in this way. However, for our discussion below it will be sufficient to consider perturbations of maximal entropy and draw the coefficients $v_n$ from a box distribution of width $\Delta$ symmetric around zero. In this way $\Delta$ sets the effective strength of the perturbation in units of the SYK bandwidth, and in the limit of asymptotically large $\Delta$ enforces Fock space localization in states $n$ with energies $v_n$. The Hamiltonian $\hat{H}_0$ perturbs this ‘Poisson limit’ via transitions $|n\rangle \to |m\rangle$ between states nearby in Fock space. (Containing only four fermion operators, $\hat{H}_0$ can change the occupation of a state $|n\rangle$ by at most four, and it preserves the number parity.) It does so via only an algebraically small number $\sim N^4 \sim \ln(D)$ of independent matrix elements, and thus defines an operator with strong statistical correlation. However, we will see that $\hat{H}_0$ is very efficient in introducing many body chaos, as evidenced by the onset of Wigner-Dyson (WD) spectral statistics, including for values $\Delta \gg 1$ where the diagonal still dominates. Our main objective is to explore the profile of the many body wave functions in this setting.

Field-theory:— We approach this problem in the mindset of first quantization, where the many body Hamiltonian $\hat{H}$ is considered as a sparse matrix acting in a huge Fock-space. The advantage gained from this un-
usual perspective of a many body Hamiltonian is that powerful concepts of single particle quantum disorder become applicable. Specifically, the occupation number basis \{ |n⟩ \} plays a role analogous to the position basis of a fictitious quantum state and \( \hat{H}_0 \) and \( \hat{H}_V \) act as ‘hopping’ and ‘on-site potential’ Hamiltonians, respectively. Within the first quantized approach, information on both, spectral statistics and many body wave functions is contained in the matrix elements of the resolvent, \( G_{n \rightarrow n'}^\pm (\epsilon) = \langle n| (\epsilon \pm i \eta - \hat{H})^{-1} |n'\rangle \). Following procedures standard in the theory of disordered electronic systems [21], products of such matrix elements can be computed via replica field theoretical methods [2]. Referring to [23] for details, the approach starts from a representation of \( G_{n n'}^\pm \) as a Gaussian integral over a vector of Grassmann variables. This integral is averaged over the distribution of couplings \( J_{ijkl} \), which results in a functional action quartic in integration variables. The latter is decoupled via a \( 2D R \times 2D R \) dimensional matrix field \( Y = \{Y_{n n', r r'}\} \), where \( s, s' = \pm \) distinguishes between advanced and retarded degrees of freedom and \( r = 1, \ldots, R \) is the replica index. Physical observables are then obtained as expectation values of this matrix field, in the limit of zero replicas, \( R \to 0 \). For example, Fock space transition probabilities \( \langle n \rangle \to \langle n' \rangle \) assume the form, \( \langle G_{n n'}^+ G_{n n'}^- \rangle \sim \lim_{R \to 0} R^{-2} \langle Y_{n n', r r'}^+ Y_{n n', r r'}^- \rangle Y \), where \( \langle \ldots \rangle_Y \equiv \int dY \exp (-S[Y]) \), and the action of the matrix functional integral will be defined momentarily.

Mappings to matrix functional integrals of this architecture can be formulated for any model, and by themselves remain formal and non-tractable in general. However, what makes the SYK system special is the existence of a powerful representation of the integration variables which is key to far reaching analytic progress. To understand this point, consider \( Y = \{Y_{n n'}\} \) as a matrix in Fock space. Arbitrary elements \( A \) of the \( D^2 = 2^N \) dimensional space of such matrices can be expanded as [24]

\[
A = \sum_\mu a_\mu X_\mu, \quad a_\mu = \frac{1}{D} \text{Tr}(AX_\mu^\dagger), \tag{3}
\]

where the sum runs over the \( 2^N = D^2 \) basis elements of a Clifford algebra, i.e. products of Majorana operators \( X_\mu = \chi^{\mu_1}_1 \chi^{\mu_2}_2 \cdots \chi^{\mu_N}_N \) labeled by the index \( \mu = (\mu_1, \ldots, \mu_N) \), \( \mu_j = 0, 1 \). Notice the structural similarity of Eq. (3) with a Fourier transform. In fact, the Clifford representation \( A_{n n'} \to a_\mu \) plays a role very similar to that of the momentum representation in single particle problems. Much as the diffusion modes in a disordered medium assume a particularly simple form when expressed in momentum space, expansions in the Clifford basis provide an efficient formulation of the present problem [24].

Focusing on the band center for simplicity, the action of the \( Y \)-functional describing the correlation of Green functions of slightly different energy \( \epsilon = \pm \omega/2 \) reads [24]

\[
S[Y] = -\frac{D}{2} \text{Tr}(Y Y^\dagger) + \text{Tr} \ln \left( (i \eta + \frac{\omega}{2}) \sigma_3^{\text{ta}} - \hat{H}_V + i \gamma Y \right), \tag{4}
\]

where \( \text{Tr} \equiv \text{tr}_F \text{tr} \) contains a trace, \( \text{tr} \), over the internal \( 2R \)-dimensional representation space of the theory, and one, \( \text{tr}_F \), over the even fermion parity sector of Fock space. (The Hamiltonian conserves parity, and we focus on the even sector for definiteness.). The fermion determinant, \( \text{Tr} \ln (\ldots) \) results from integration over the original Gaussian integration variables and couples the theory to the \( Y \)-variables where the information on statistics of the randomness now sits in the Gaussian weight. This weight is most conveniently expressed in the \( \mu \)-representation in which it assumes the form \( \text{Tr}(Y Y^\dagger) = \sum_\mu \text{tr}(g_\mu^T) \Pi_\mu |_\mu \), with \( \Pi_\mu |_\mu \) is a numerical factor depending only on the number of Majoranas \( |_\mu \equiv \sum_\ell \mu_\ell \) appearing in the product \( X_\mu \). For small arguments \( k \ll N \), \( \Pi_k = 1 + 4k/N + O(k/N)^2 \), while \( \Pi_k \sim N^2 \) for generic \( k = O(N) \) [23].

**Mean field approach:**—Much as the \( G \Sigma \)-theory of the SYK model [17], the \( Y \)-theory with action [4] defines an exact reformulation of the problem. However, while the \( G \Sigma \)-formalism is tailored to the description of short times, polynomial in \( N \), the present theory works best in the opposite limit of long times, exponential in \( N \). To see how this comes about, consider the limit \( \omega = 0 \) and subject the action \( S[Y] \) to a variational procedure. Variation in the components of the \( 2R \)-dimensional matrices \( y_\mu \) then leads to the equation

\[
\delta_{y_\mu} S[Y] = - D \Pi_\mu |_{\mu} y_\mu + i \gamma \text{tr}_F \left( \frac{1}{i \eta \sigma_3^{\text{ta}} - \hat{H}_V + i \gamma Y} X_\mu \right). \tag{5}
\]

Considering the unperturbed SYK model first, \( \Delta = 0 \), we note that \( \text{tr}_F(X_\mu) = \delta_{\mu_0 D} \) and \( \Pi_\mu_0 = 1 \), to find that the equation is solved by Fock-space isotropic configuration \( y_0 \sigma_3^{\text{ta}} \otimes 1_R \). Here, the sign structure is fixed by the causality parameter, \( i \eta \), as usual [21] in this class of models. Next, observe that for infinitesimal \( \eta \) any unitarily transformed configuration \( y_0 Q \equiv y_0 (T \sigma_3^{a T})^{-1} \), \( T \in U(2R) \) solves the equation, too. (We here consider values \( N = 2, 6 \mod 8 \) where the SYK model is in the unitary symmetry class [19]. In other cases, the rotation matrices must be replaced by elements of the orthogonal or symplectic group, respectively.) Substituting these configurations back into the action and expanding to first order in the symmetry breaking frequency offset, \( \omega \), one obtains the effective action \( S[Q] \equiv i \frac{\pi}{\omega} \text{tr}(Q \sigma_3^{a T}) \), where \( \delta = \pi \gamma/D \) is the many body level spacing. This is the action of the zero-dimensional \( \sigma \)-model [21], and it is known that the integration over the fluctuations, \( T \), reproduces all results of random matrix statistics. In particular it
implies that the many body wave functions are ergodic and uniformly distributed over Fock space.

The above construction shows that the RMT limit of the SYK model emerges at the simplest mean field level of the present formalism. However, it also indicates how to obtain information beyond RMT: first, one needs to explore the role of fluctuations \( y_\mu \), \( \mu \neq 0 \) different from the ergodic mode \( y_0 \). This was done in Ref. [23] where it was shown that a treatment on the Gaussian level is controlled and leads to results which agree quantitatively [19], or at least qualitatively [20] with numerical observation. We here go down a different avenue to study how the presence of the perturbation [2] affects the picture. This defines a situation with two preferential bases, the occupation number basis preferred by \( \hat{H}_V \), and the Clifford basis tailored to the description of the many body correlations. The competition of these two is comparable to that of the position and the momentum basis in single particle localization. However, the more intricate structure of Fock space implies more complex behavior and in particular the existence of regimes with extended yet non-uniformly distributed (‘non-ergodic’) many body states. The key to understanding this phenomenon lies in a careful analysis of the full mean field equation (6) and its fluctuations:

Among the \( D^2 \) elements of the Clifford basis there is a subset of cardinality \( D \) which is diagonal in the occupation number basis preferred by \( \hat{H}_V \). With \( \chi_{2i-1}\chi_{2i} = i(2n_i - 1) \), they assume the form \( X_\lambda \equiv \Pi_{i=1}^{N}(1 - 2n_i)_{\lambda_i}^\lambda \) where \( \lambda = (\lambda_1, ..., \lambda_N) \) with \( \lambda_i = 0, 1 \). Being diagonal on both, the kernel \( \Pi \) and \( \hat{H}_V \), we expect these operators to play a distinguished role in the solution of the generalized equation. Substituting the ansatz \( Y = \sum_{\lambda} y_\lambda X_\lambda \otimes \sigma^a_{\lambda} \otimes 1_R \) into Eq. (5) we indeed obtain a closed set of coupled equations for the scalar coefficients \( y_\lambda \),

\[
y_\lambda = \frac{1}{D!{\Pi}_{|\lambda|}} \sum_{n=1}^{D} \frac{s_{\lambda,n}}{i\nu_n + y_0 + \sum_{\lambda \neq 0} y_\lambda s_{\lambda,n}} , \tag{6}\]

where \( s_{\lambda,n} \equiv \langle n | X_\lambda | n \rangle = \pm 1 \) is a sign factor. What complicates the solution of this equation is the simultaneous appearance of all components \( y_\lambda \) in the denominator. At the same time, the presence of the alternating sign factors, \( s_{\lambda,n} \), indicates that the contribution of this term might be small, at least for some range of \( \Delta \)'s. We thus start with an approximation neglecting this term and will verify its validity self consistently (and numerically) below. Second, we assume that in equations containing sums over large numbers \( D \gg 1 \) of states \( n \) it is legitimate to treat the coefficients \( \nu_n \) as independently box-distributed random coefficients. In this way, functions \( F(\{\nu_n\}) \) become random variables whose averages, \( \bar{F} \) and variances \( \text{var}(F) \) are obtained by the independent average over all \( \nu_n \). Under this assumption, a straightforward computation [23] shows that the coefficients \( y_\lambda \) are determined by the equations

\[
\bar{y}_0 = \frac{2}{\Delta} \arctan \left( \frac{\Delta}{2y_0} \right), \quad \bar{y}_{\lambda \neq 0} = 0, \quad \text{var}(y_\lambda) = \frac{\pi}{4D!{\Pi}_{|\lambda|}^2 y_0^2} \bar{F} \left( \frac{\Delta}{2y_0} \right), \tag{7}\]

where \( \bar{F}(x) \) is a function monotonously increasing from \( 0 \) for \( x = 0 \) to \( \sim 1/x \) for \( x \gg 1 \) [20]. The first of Eqs. (7) shows how the solution \( y_0 = 1 \) of the unperturbed SYK model crosses over to a limiting behavior \( y_0 \sim \pi/\Delta \) for large values \( \Delta \gg 10 \) exceeding the SYK band width. The other equations contain information on the statistics of the solution, and on its limitations. Referring for details to [23], we note the existence of two energy scales at marking qualitative change of behavior: beyond a scale \( \Delta_i \sim N^2 \) the self consistent assumption of irrelevancy of inhomogeneous contributions to the stationary solution breaks down. At a yet much larger scale, \( \Delta_P \sim D^{1/2} \), the Gaussian controlling the fluctuation of individual modes, \( y_\lambda \), become of \( \mathcal{O}(1) \) and the mean field approach as such breaks down. We associate this latter scale with the threshold to Poissonian behavior where states are localized at individual occupation number sites, \( |n\rangle \). However, in the present context the more interesting observation is that there exists a wide regime of coupling strengths where the mean field is still Fock-space homogeneous, \( y_\lambda \sim y_0 \delta_{\lambda,0} \), although the two-body term already overpowers the SYK Hamiltonian. As we discuss in the following, this results in a coexistence of WD spectral correlations with non-trivial structure of the many body wave functions.

Results:—As in the case \( \Delta = 0 \) above, we parameterize fluctuations around the stationary point as \( y_0 Q \) with \( Q = (\sigma^a_{\lambda} \otimes 1_R)T^{-1} \), and in this way obtain the effective action

\[
S[Q] = \text{Tr} \ln \left( i\sigma^a_{\lambda} - \hat{H}_V + i\gamma y_0 Q \right) , \tag{8}\]

This action is otherwise known to describe the Rosenzweig-Porter model [27] — a \( D \)-dimensional Gaussian random matrix ensemble perturbed by a fixed diagonal contribution, \( \hat{H}_V \). While in that case the Hilbert space isotropic saddle point comes out of the box, our above analysis yields the same structure for not too strong perturbation, \( \Delta < \Delta_i \), in the more structured context of the present problem. The observation that the two models share the same soft mode action is one of the most important results of our analysis: Since we see no evidence for a qualitative effect of residual massive fluctuations, this finding implies that for moderately strong diagonals the SYK model and the Rosenzweig porter model are in the same universality class. The extraction of observables from the action [3] has been discussed earlier [21, 27] and we refer to the original references for the technical details. Specifically, the average local density of
states \( \langle v_0(n) \rangle \equiv -\frac{1}{2} \text{Im}(G_{nm}^+ n) \) close to the band center is given by \( \langle v_0(n) \rangle = (1/\pi \gamma) \gamma y_0/(v_n^2 + y_0^2) \). Summation over \( n \) yields the average density of states as \( \langle v_0 \rangle = y_0 D/\pi \gamma \). For \( \Delta \gtrsim 1 \) exceeding the SYK bandwidth, the mean field amplitude goes down as \( y_0 \sim \Delta^{-1} \), indicating that the spectral width of the model is now set by the distribution of the diagonal elements, \( v_n \), and spectral weight in the center of the SYK-band is reduced as \( \sim 1/\Delta \). However, the level–level correlation function at the band center, \( K(\omega) = \langle v(\omega/2)v(\omega/2) \rangle/\langle v_0 \rangle^2 - 1 \), remains of Wigner-Dyson form, including for strong perturbations, \( \Delta_1 < \Delta < \Delta_p \). This reflects the fact that the Goldstone mode generating RMT fluctuations is robust and continues to operate even when the mean field texture itself changes at couplings \( \Delta \gtrsim \Delta_1 \). For a numerical confirmation of this statement, cf. the inset of Fig. 1 showing the relative entropy (Kullback-Leibler divergence) between the spectral distribution of an \( N = 13 \) system and the Wigner-Dyson (dashed) and Poisson (solid) distribution on a logarithmic scale. As expected, the change of statistics takes place only at large coupling, \( \Delta \sim \Delta_p \).

We next turn to the statistics of many body eigenfunctions, as characterized by the moments \( I_q = \sum_{n=1}^D \langle |\psi(n)|^q \rangle \). After the integration over the Goldstone mode, \( I_q = \sum_n \langle (v_0(n))/\langle v_0 \rangle \rangle^q \Gamma(q + 1) \), the moments are expressed in terms of mean field quantities, averaged over the SYK disorder, but still containing the realization specific diagonal elements, \( v_n \). Once more relying on the assumption of self averaging in sums of this architecture, we average the individual \( v_n \) over their box distribution to obtain

\[
I_q = -(-2)^q q D^{1-q} \frac{1}{2} \frac{1}{\Delta (2y_0)} \arctan (\Delta/2y_0). \tag{9}
\]

For \( \Delta \ll 1 \) smaller than the SYK bandwidth, this asymptotes to the random matrix result \( I(q) = q! (D/2)^{1-q} \), showing that the wave functions are uniformly distributed in this limit. In the opposite case, \( \Delta \gg 1 \), \( y_0 = \pi/\Delta \) and the moments

\[
I_q = (2\pi^2)^{1-q} q (2q - 3)! \Delta^{2(q-1)} D^{1-q}, \tag{10}
\]

show power law scaling in \( \Delta \). Finally, for very strong \( \Delta \sim N^\alpha \) the wave functions show a weak form of fractality \( I_q \sim [D/(\log D)]^{2\alpha} \). In this regime, the wave functions effectively occupy only a fraction of the accessible Hilbert space, which means that they are extended, yet non-ergodic (while the spectral statistics continues to be RMT like.) Fig. 1 shows that the analytic prediction is in excellent agreement with the direct numerical calculation of the wave function moments. For the relatively small values of \( N = 10^4 \) accessible to numerics, the two scales \( \Delta_1 \) and \( \Delta_p \) are not clearly separable, and it is difficult to tell whether \( \Delta_p \) or the more conservative \( \Delta_1 \) marks the breakdown of the mean field analysis. In the former case, large values \( \Delta \sim D^\gamma, \gamma < 1/2 \), would define a regime of strong fractality, \( I_q \sim D^{(1-2\gamma)/(1-q)} \), however, the existence of this scaling remains hypothetical.

**Summary and discussion:**— The model considered in this paper defines the perhaps simplest many body system showing a competition between Fock space localization and ergodicity. We are seeing unambiguous evidence that the passage between the two limits is not governed by a single many body localization transition but contains a parametrically extended intermediate phase characterized by a coexistence of Wigner-Dyson spectral statistics and non-trivial extension of wave functions over Fock space. Methodologically, this phenomenon emerged as the result of a competition: the ‘hopping’ in Fock space generated by the SYK two-body interaction stabilized a uniform mean field against the ‘localizing’ tendency the Fock-space diagonal operator, \( \hat{H}_v \). This mean field uniformity is the principal mechanism behind the emergence of a soft replica rotation mode and in the consequence of Wigner-Dyson statistics. At the same time, the continued presence of the diagonal operator in the effective action of the theory distorted the many body wave functions away from a uniform ergodic profile. This mechanism appears to be of rather general nature and makes one suspect that non-ergodic wave function statistics in coexistence with RMT spectral correlations could be a much more frequent phenomenon than previously thought.

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[28] We here estimate $\Delta_p \simeq 142$ from $D_{\bar{y}}^2/2 \simeq 1$ with $D = 2^{12}$ and $y_0 = \pi/\Delta_p$, and $\Delta_i \simeq 124$ from $\bar{y}_0 \simeq \sigma_d$ with variance $\sigma_d \simeq 0.02 \times 1.15$. Here 0.02 is the variance for $N = 12$ at $\Delta = 10$ (see Fig. 1 in Ref. [23]) and the factor 1.15 accounts for the increase of $\sigma_d$ with $\Delta$. 
Non-ergodic extended states in the SYK model: supplementary material

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In this supplemental material we give details on the derivation of the replica field theory and the analysis of the mean field equations.

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REPLICA FORMALISM

We here provide details on the derivation of the replica theory defined by Eq. (4). For our purposes, it will be sufficient to focus on the diagonal elements, \( G^{a}_{nn}(\epsilon) \), \( s = \pm \), of the resolvent operators defined in the main text. These matrix elements can be obtained as Gaussian integrals,

\[
G^{a}_{nn}(\epsilon) = \partial_{h} \lim_{R \to 0} \frac{1}{R} Z_{J}(h)|_{h=0},
\]

where \( P_{n} = |nangle \langle n| \) is a projection matrix, and \( \psi = \{ \psi_{r,n} \} \) is an \( DR \) component vector of Grassmann variables, specified by the Fock-space index \( n \), a replica index \( r = 1, \ldots, R \), and the causal index \( s = \pm \). The subscript \( J \) indicates that \( Z_{J} \) is defined with reference to a fixed set of couplings \( J = \{ J_{ijkl} \} \). The generalization to a generating functional for products of retarded and advanced Green’s functions reads

\[
Z_{J}(h) = \int D(\tilde{\psi},\psi) e^{-\tilde{\psi}(\tilde{h}+h \hat{P}_{n} + i\eta \sigma^{a}_{n} - \hat{H}_{V})\psi},
\]

where \( \tilde{\psi} = (\tilde{\psi}_{+}, \tilde{\psi}_{-}) \) is a \( 2D \) component vector, the Pauli matrix \( \sigma^{a}_{n} \) operates in the two dimensional causal \( \pm \) subspace, and \( \tilde{h} = \text{diag}(h^{+},h^{-}) \) are diagonal energy and source matrices in this space.

Effective action:—The average over couplings \( J \) generates a term quartic in the Grassmann fields,

\[
Z(h) = \int D(\psi,\bar{\psi}) e^{-\bar{\psi} \hat{\Omega} \psi - \frac{1}{2} \sum_{a} (\bar{\psi} X_{a} \psi)(\bar{\psi} X_{a} \psi)},
\]

where \( \bar{\langle Z_{J}(h) \rangle} \), \( \hat{\Omega} \equiv \hat{e} + \hat{h} + i\eta \sigma^{a}_{n} - \hat{H}_{V} \) sums all non-stochastic contributions to the action, and the \( \sum_{a} \) is a sum over all ordered index quadruples \( a_{1} < a_{2} < a_{3} < a_{4} \). We reorganize the quartic term as \( (\bar{\psi} X_{a} \psi)(\bar{\psi} X_{a} \psi) = -\text{Tr}(\bar{\psi} X_{a} \psi \psi X_{a} \psi) \) and Hubbard-Stratonovich decouple it via a set of \( 2DR \)-dimensional matrix fields \( A_{a} = \{ A_{a}^{T,s,s'} \} \sim \{ \psi_{n}^{s} \psi_{n'}^{s'} \} \) carrying a non-trivial representation under continuous transforma-tions in replica and causal space [2]:

\[
Z(h) = \int DA e^{-\frac{1}{2} \sum_{a} \text{Tr}(X_{a} A_{a})^{2} + \text{Tr} \ln(\hat{O} + i\eta N \sum_{a} A_{a})} \]

where \( n \equiv (2N)^{4} \), \( c_{N} \equiv 4\ln(2N)^{4} \), \( \gamma = \frac{1}{4}(2N)^{1/2} \), and the Gaussian integral over \( \psi \) has been carried out. (While \( c_{N} = 1 + O(N^{-1}) \) we keep the \( 1/N \)-corrections to improve the comparison to numerical results obtained for systems of small size, \( N = O(10^{3}) \).) We now expand the matrices \( A_{a} \) in the Clifford basis as \( A_{a} = \sum_{\mu} a_{a,\mu} X_{\mu} \) where \( a_{a,\mu} \) are \( 2R \)-dimensional matrices in the internal indices. Defining a sign factor \( s_{\mu} = \pm 1 \) distinguishing between hermitian and anti-hermitian basis elements, \( X_{\mu} \equiv X_{\mu}^{\dagger} s_{\mu} \), and another one \( s_{\mu,a} \) by \( X_{\mu} X_{a} \equiv X_{a} X_{\mu} s_{\mu,a} \) we use that \( \text{tr}_{F}(X_{\mu} X_{a} X_{\nu} X_{\rho}) = D s_{\mu} s_{\mu,a} \delta_{\mu,\nu} \) to obtain the Gaussian weight for the \( a \)-modes as \( \text{Tr}(X_{a} A_{a})^{2} = D s_{\mu} s_{\mu,a} \text{tr}(a_{a,\mu} a_{a,\mu})^{2} \). We next observe that the \( \text{Tr ln} \) couples only to the linear combination \( y \equiv \frac{1}{n} \sum_{a} a_{a} \). This motivates a variable change \( a_{a,\mu} \rightarrow y_{\mu} + a_{a,\mu} \), where \( \sum_{a} a_{a,\mu} = 0 \). Enforcing the constraint via Lagrange multipliers, it is straightforward to carry out the Gaussian integral over \( a_{a,\mu} \) (see Ref. [1] for a few more details) and to arrive at

\[
Z(h) = \int Dy e^{-\frac{1}{2} \sum_{\mu} \Pi_{\mu,\nu} \text{tr}(y_{\mu} y_{\nu}) - \text{Tr} \ln(\hat{O} + i\eta N \sum_{\mu} y_{\mu} X_{\mu})},
\]

where \( \Pi_{\mu}^{-1} \equiv \frac{c_{N}}{n} \sum_{a} s_{\mu,a} \). These numerical factors can be computed by straightforward combinatorics [1] and the relevant asymptotic behavior for arguments of small and generic order \( |\mu| \), respectively, is given in the main text. The action of the functional integral defines Eq.(4) of the main text, where \( Y = \sum_{\mu} y_{\mu} X_{\mu} \), the energy arguments are chosen as \( \epsilon^{\pm} = \pm(\frac{c_{N}}{2} + i\eta) \), the presence of sources is suppressed, \( \hat{h} = 0 \), we approximated \( c_{N} = 1 \), and the weight is implicitly defined through \( \text{Tr}(Y \Pi Y) \equiv \sum_{\mu} \Pi_{\mu,\nu} \text{tr}(y_{\mu} y_{\nu}) \).
DETAILS ON MEAN FIELD EQUATION

To derive Eqs. (7) (all equation numbers refer to the main text throughout) we first notice that for \( \lambda = 0 \), the sign \( \sigma_{0,n} = \langle n|X_0|n\rangle = \langle n|n\rangle = 1 \) is uniformly positive. With \( \Pi_0 = 1 \), the equation then assumes the form \( \bar{y}_0 = \frac{1}{D} \sum_{n} \frac{\sigma_{0,n} y_n}{\sqrt{2} + y_n^2} \), where we neglected the real parts \( \text{Re}(y_\lambda) \) which will at most lead to a physically meaningless shift of the energy denominator. Application of the averaging paradigm then leads to \( \bar{y}_0 = \frac{D}{\Delta} \int_{-\Delta/2}^{\Delta/2} \frac{y_0}{\sqrt{2} + y^2_0} \), and doing the integral we obtain the result stated in Eq. (7). The variance, \( \text{var}(y_0) \), is obtained in the same way by first squaring and then averaging. Turning to \( \lambda \neq 0 \), the thing to notice is that in this case there are equally many positive and negative signs \( s_{\lambda,n} \). The equation may then be organized as \( y_\lambda = \frac{1}{D\Pi_{|\lambda|}} \left( \sum_n - \sum_n \right) \frac{\sigma_{0,n} y_n}{\sqrt{2} + y_n^2} \), where the two sums run over \( n \)-states with positive and negative sign \( s_{\lambda,n} = \pm 1 \). Applying the averaging procedure we obtain \( \bar{y}_\lambda = 0 \), on account of the competing signs. However, the variances, \( \text{var}(y_\lambda) \), remain finite, as indicated in Eq. (7).

Statistics of mean field equation:—According to Eq. (7) there are no Fock space inhomogeneous contributions to the mean field solution, at least on average, \( \bar{y}_\lambda \neq 0 \). However, these components fluctuate and, if their added contribution in the denominator in Eq. (6) outweighs \( y_0 \), our self-consistency assumptions break down. Application of a central limit comparison, \( \sqrt{D} \text{rms}(y_\lambda) \sim \bar{y}_\lambda \), where the rms value is obtained for typical values, \( \Pi_{|\lambda|} \sim N^2 \), leads to the estimate of a threshold energy, \( \Delta_1 \sim N^2 \), see also below.

We have checked the quality of the self consistent solution by numerically solving equation (6). The numerical analysis is best implemented in an occupation number representation, \( Y = \sum_{n=1}^{D} d_n P_n \sigma_v^{2D} \otimes 1_R \). The linear transform between the bases elements \( P_n \) and \( X_\lambda \equiv O_{\lambda n} P_n \) amounts to a binary variant of a Fourier transform, \( O_{\lambda n} = e^{i\pi \sum_{i=1}^{N} \lambda_i n_i} \). The mean field equation then assumes the form

\[
(i v_n + d_n) \sum_{m=1}^{D} A_{nm} d_m = 1,
\]

where \( A_{nm} = \frac{1}{D} \sum_{\lambda=1}^{D} O_{n \lambda}^* O_{\lambda m} \Pi_{\lambda} O_{\lambda n} \). Typical solutions in the even-parity sector are shown in Fig. 2 for fixed size \( N = 14 \) and different values of \( \Delta \). Average numerical solutions \( d \equiv \frac{1}{D} \sum_{n=1}^{D} d_n = \bar{y}_0 \), follow well the analytical prediction indicated by the solid line. The variance of numerical solutions is compared in the right inset of Fig. 2 to the analytically calculated variance, \( \sigma_d^2 = \sum_{|\lambda| = 0}^{N} (\sigma_{|\lambda|}^2) \) for system sizes \( N = 8 - 15 \). The agreement improves with increasing \( N \), and from the analytical estimate we expect that \( \sigma_d \sim N^{-2} \) for large \( N \gg 1 \). The left inset of Fig. 2 compares the histogram of numerical solutions for fixed disorder configuration \( \Delta = 10 \) to the Gaussian distribution.

![Figure 2: Comparison of numerical and analytical solutions of the mean field equation. Eq. (7) predicts that \{\( d_n \)\} are Gaussian distributed around the mean value \( \bar{y}_0 \) with variance \( \sigma_d \) decreasing with \( N \) (see text). The main figure shows solutions for \( N = 14 \) and varying disorder strength \( \Delta \). The average solution follows well the analytical prediction (indicated by the solid line), and variance increases with \( \Delta \) as predicted by \( \mathcal{F} \) in the main text (shaded regions are analytically calculated values for \( 1 - 3\sigma \)). Left inset: Histogram of solutions for \( \Delta = 10 \) and \( N = 14 \); the analytically predicted Gaussian distribution is indicated by the solid line. Right inset: Standard deviation \( \sigma_d \) of numerical solutions for fixed \( \Delta = 10 \) as a function of system size \( N = 8 - 15 \) (here averaged over a small ensemble of \( \mathcal{O}(10) \) samples), the analytical prediction is indicated by the solid line.](attachment:image.png)

[1] A. Altland, D. Bagrets, Nucl. Phys. B 930, 45 (2018).
[2] F. Wegner, Z. Phys. B 35, 207 (1979); A. Altland and B. D. Simons, Condensed Matter Field Theory, 2nd ed. (Cambridge University Press, Cambridge, UK, 2010).