Learning in Stackelberg Games with Non-myopic Agents

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Abstract

We study Stackelberg games where a principal repeatedly interacts with a long-lived, non-myopic agent, without knowing the agent’s payoff function. Although learning in Stackelberg games is well-understood when the agent is myopic, non-myopic agents pose additional complications. In particular, non-myopic agents may strategically select actions that are inferior in the present to mislead the principal’s learning algorithm and obtain better outcomes in the future.

We provide a general framework that reduces learning in presence of non-myopic agents to robust bandit optimization in the presence of myopic agents. Through the design and analysis of minimally reactive bandit algorithms, our reduction trades off the statistical efficiency of the principal’s learning algorithm against its effectiveness in inducing near-best-responses. We apply this framework to Stackelberg security games (SSGs), pricing with unknown demand curve, strategic classification, and general finite Stackelberg games. In each setting, we characterize the type and impact of misspecifications present in near-best-responses and develop a learning algorithm robust to such misspecifications.

Along the way, we improve the query complexity of learning in SSGs with $n$ targets from the state-of-the-art $O(n^3)$ to a near-optimal $O(n)$ by uncovering a fundamental structural property of such games. This result is of independent interest beyond learning with non-myopic agents.

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1 Introduction

Stackelberg games are a canonical model for strategic principal-agent interactions. Consider a defense system that distributes its security resources across high-risk targets prior to attacks being executed; or a tax policymaker who sets rules on when audits are triggered prior to seeing filed tax reports; or a seller who chooses a price prior to knowing a customer’s proclivity to buy. In each of these scenarios, a principal first selects an action \( x \in X \) and then an agent reacts with an action \( y \in Y \), where \( X \) and \( Y \) are the principal’s and agent’s action spaces, respectively. In the examples above, agent actions correspond to which target to attack, how much tax to pay to evade an audit, and how much to purchase, respectively. Typically, the principal wants an \( x \) that maximizes their payoff when the agent plays a best response \( y = \text{br}(x) \); such a pair \((x, y)\) is a Stackelberg equilibrium. By committing to a strategy, the principal can guarantee they achieve a higher payoff than in the fixed point equilibrium of the corresponding simultaneous-play game. However, finding such a strategy requires knowledge of the agent’s payoff function.

When faced with unknown agent payoffs, the principal can attempt to learn a best response via repeated interactions with the agent. If a (naive) agent is unaware that such learning occurs and always plays a best response, the principal can use classical online learning approaches to optimize their own payoff in the stage game. Learning from myopic agents has been extensively studied in multiple Stackelberg games, including security games [LCM09; BHP14; PSTZ19], demand learning [KL03; BZ09], and strategic classification [DRS+18; CLP20].

However, long-lived agents will generally not volunteer information that can be used against them in the future. This is especially true in online environments where a learner seeks to exploit recently learned patterns of behavior as soon as possible, thus the agent can see a tangible advantage for deviating from its instantaneous best response and leading the learner astray. This trade-off between the (statistical) efficiency of learning algorithms and the perverse long-term incentives they may create brings us to the main questions of this work:

What design principles lead to efficient learning in Stackelberg games with non-myopic agents?

How can insights from learning with myopic agents be applied to non-myopic agents?

Non-myopic long-lived agents are typically modeled as receiving \( \gamma \)-discounted utility in the future; the discount factor \( \gamma \) can be interpreted as capturing the agent’s uncertainty on whether they will participate in future rounds or the present bias they experience. This modeling choice works well for algorithms that are slow to implement lessons learned from each individual round of feedback. These algorithms incentivize the agents to \( \varepsilon \)-approximately best respond each round, by making it unappealing for the agent to sacrifice payoff more than \( \varepsilon \) in the present for the effect their actions will have only far into the future.

A key technical challenge for learning in Stackelberg games in the presence of non-myopic agents is designing robust learning algorithms that can learn from inexact best-responses. Indeed, a similar approach is behind the working of a long line of research on handling non-myopic agents, e.g. in auction design as we discuss in Section 1.2. However, high-dimensional Stackelberg games pose a unique hurdle because the set of actions that can be rationalized by an \( \varepsilon \)-approximate best responding agent can be complex. This is made worse by possible discontinuities in the principal’s payoff function, the agent frequently failing to best respond, and the difficulty of identifying well-behaved optimization subproblems in large Stackelberg games.

Furthermore, the statistical efficiency of the principal’s learning algorithm must be traded off against its effectiveness to lead an agent to \( \varepsilon \)-best respond. The more reactive an algorithm, i.e., the faster it is in implementing lessons learned from individual rounds of feedback, the more robust it has to be in order to handle deviations from an agent’s best response. Therefore, another
technical challenge is to devise principled approaches for designing minimally reactive (or optimally “slowed-down”) robust learning algorithms that support and encourage \( \epsilon \)-best responding behavior.

1.1 Our contribution

We seek algorithms that let the principal achieve vanishing regret with respect to the Stackelberg equilibrium, in the presence of non-myopic agents. We provide a framework to address both of the aforementioned challenges that result from this reduction, namely designing robust online algorithms that learn effectively from erroneous best response queries in high dimensions, and identifying mechanisms for designing statistically efficient minimally reactive learning algorithms. Our framework, presented in Section 2, reduces learning in the presence of non-myopic agents to robust bandit optimization. We envision principal-agent interactions as taking place over an information channel regulated by a third party acting as an information screen. For example, the third party can delay the revelation of the agent’s actions to the principal by \( D \) rounds, which, combined with the agent’s \( \gamma \)-discounting future utility, induces the agent to \( \epsilon \)-approximately best-respond for \( \epsilon = \gamma^D/(1 - \gamma) \). The third party can also release the principal’s actions in delayed batches. This information screen lets us discuss the tradeoff between the learning algorithm’s statistical efficiency and agent’s incentives to deviate. The more transparent the screen, the more reason an agent has to deviate from their best response. However, the less transparent the screen, the less information is available to the learner.

Optimal query complexity for security games and extension to non-myopic agents. Our first application is to Stackelberg security games (SSGs), a canonical Stackelberg game that models strategic interactions between an attacker (agent) and a defender (principal). Here, the principal wishes to fractionally allocate defensive resources across \( n \) targets, and the agent aims to attack while evading the principal’s defense. Existing approaches solve \( n \) separate convex optimization subproblems, one per target \( y \) over the set of \( x \) with \( y = \text{br}(x) \), using feedback from the agent to learn the region each action \( x \) belongs to. However, \( \epsilon \)-approximate best responses \( \text{br}_{\epsilon}(x) \) can corrupt this feedback adversarially anywhere near the boundaries of these high-dimensional regions.

In Section 3, seeking an analytically tractable algorithm for this corrupted feedback setting, we uncover a clean structure that characterizes the principal’s optimal solution against best-responding agents in a single-shot game. We show that all \( n \) regions and sub-problems share a unique optimal solution when considering a conservative allocation of the principal’s resources. This leads to a single optimization problem, which we solve with a variant of the cutting plane method. The resulting algorithm—CLINCH—solves the myopic learning problem with near-optimal \( \tilde{O}(n) \) query complexity, improving upon the state-of-the-art \( O(n^3) \) dependence on the number of targets \([PSTZ19]\). The simplicity of our new algorithm lets us extend it to \( \epsilon \)-approximately best responding agents. The uniqueness of this optimal solution allows us to approach it from any direction in the principal’s strategy space while tolerating small perturbations. Implementing delays via a careful batching procedure, we provide a regret bound of \( \tilde{O}(n \log T + T_\gamma) \) against non-myopic agents, where \( T_\gamma = \frac{1}{1 - \gamma} \) is the discounted horizon.

Reduction from non-myopic learning to robust optimization. The approach undertaken in SSGs suggests a general recipe to deal with non-myopic agents. First, we need to identify the type of error present in agent’s \( \epsilon \)-approximate best response, i.e., the difference between \( \text{br}_{\epsilon}(x) \) and \( \text{br}(x) \) and its impact on the principal’s utility. In the case of SSGs, when \( x \) is close to the boundaries of any subproblem, \( \text{br}_{\epsilon}(x) \) can be an adversarial perturbation of \( \text{br}(x) \) and thus adversarially perturbs the principal’s utility. Second, having identified this error type, we need to design an optimization
Table 1: For each primary learning environment, we list the error type to which we must be robust, the main robust learning algorithm employed, and the non-myopic regret bound achieved with known discount factor $\gamma$. Here, $n$ is the number of targets in SSGs, $d$ is the dimension of the feature space in strategic classification, and $m$ is the number of principal actions for finite Stackelberg games.

| Environment                | Error Type          | Robust Learning Algorithm | Non-myopic Regret               |
|----------------------------|---------------------|---------------------------|----------------------------------|
| SSGs                       | bounded-region      | CLINCH [this work]        | $\tilde{O}(n(\log T + T\gamma))$ |
| demand learning            | pointwise           | SuccElimination [EMM06]  | $\tilde{O}(\sqrt{T} + T\gamma)$ |
| strategic classification   | pointwise           | GDwoG [FKM05]             | $\tilde{O}(T^{1/4} \sqrt{d} T^{3/4})$ |
| finite Stackelberg games   | bounded-region      | convex optimization w/ membership oracle [LSV18] | $\tilde{O}(T\gamma \log^3(T) \sqrt{m} + T\gamma m^3 \log^2 T)$ |

**Design of minimally reactive learning algorithms.** The aforementioned multi-copy approach is a simple, non-reactive mechanism that conforms to the feedback-delays information screen and leads to regret bounds with multiplicative dependence on $T\gamma$. To design more efficient algorithms for learning with delays, we investigate this information screen in more detail. In particular, we show that the design of batched bandit algorithms is equivalent to algorithm design for the feedback-delay screen. The batched paradigm facilitates the design of policies that utilize non-reactive action schedules to extract more information per round, leading to a better exploration-exploitation tradeoff. This lets us improve the multiplicative dependence on $T\gamma$ to additive in several settings. Table 1 presents these regret guarantees alongside their corresponding error types and robust algorithms.

Although our main algorithms assume that the principal knows the agent discount factor $\gamma$ and selects the delay $D$ accordingly, we also extend our results to unknown $\gamma$ for the SSG and demand learning settings. The relevant property shared by our algorithms for these settings is that they maintain a running uncertainty set of environments consistent with observed feedback. Adapting an approach originally developed for adversarial corruptions [LMP18], we run multiple copies of our algorithms with geometrically increasing guesses for the appropriate delay $D$, where information is shared between versions by intersecting their corresponding uncertainty sets. The resulting guarantees depend multiplicatively on $T\gamma$ but do not require knowledge of this parameter.
1.2 Related work

Learning in the presence of non-myopic agents has been well-studied in the context of auctions 
[ARS13; MM14; LHW18; ACK+19]. There, batching and delays are often used to limit the extent of 
strategic manipulation from bidders and non-myopia is frequently modeled via $\gamma$-discounted utility 
maximizing agents. Initial work focused on posted prices [ARS13; MM14; Dru17], while later work 
included multi-bidder auctions with reserve prices [LHW18; ACK+19], formal guarantees for incentive 
compatibility [KN14], and more nuanced, contextual valuations [GJM21; Dru20]. In Section 4, 
we provide direct comparisons to the posted-price setting, where we improve the state-of-the-art 
dependence on $T^\gamma$ for both fixed and stochastic buyer valuations. Compared to higher-dimensional 
settings like SSGs, analysis of approximate best response behavior is more simplistic in auctions, 
where the agents can be viewed as having slightly perturbed one-dimensional values. Differential 
privacy has also been employed as a tool for filtering information flow in various mechanism design 
settings [MT07; NST12; KPRU14; LHW18; ACK+19].

Our results in Section 4 relate to two lines of work in the multi-armed bandit literature. First, 
learning in multi-armed bandits with delayed feedback has been studied under various assumptions, 
initially with stochastic arm-independent delays [JGS13], later with arm-dependent delays [GVCV20; 
LSKM21], and also with the added difficulty of aggregated feedback [PASG18]. Our analysis in 
Section 4 extends that of Lancewicki, Segal, Koren, and Mansour [LSKM21] to account for adversarial 
perturbations. Second, multi-armed bandits with adversarial corruptions have been studied as a 
notion of robustness to corrupted feedback [LMP18; GKT19; ZS21]. Unlike our setting where 
rewards can be perturbed slightly in every round, this line of work assumes that the feedback can be 
completely adversarial in some of the rounds and purely stochastic otherwise.

SSGs [CS06; Tam11] have been well-studied in recent literature, with regret and query complexity 
bounds derived for both online and offline learning [BHP14; BBHP15; PSTZ19; XTJ16]. We 
replace the state-of-the-art here for regret and query complexity (see Section 3.5). Several works in 
security games have explored the aspect robustness, modeling noisy best responses via behavioral 
assumptions [HFN+16; PJM+12]. The resulting algorithms, however, are not robust to the adversarial 
perturbations needed by our framework.

2 Framework

We consider learning in general Stackelberg environments, in which a principal (the “leader”) aims to 
learn an optimal strategy while interacting repeatedly with a non-myopic agent (the “follower”). In 
this section, we first describe the basic model for principal-agent interaction and then introduce our 
general approach for learning in non-myopic principal-agent settings.

2.1 Model

A Stackelberg game is a tuple $(X, Y, u, v)$ of principal action set $X$, agent action set $Y$, principal 
payoff function $u: X \times Y \rightarrow [0, 1]$, and agent payoff function $v: X \times Y \rightarrow [0, 1]$. The principal leads 
with an action $x \in X$, observed by the agent and the agent follows with an action $y \in Y$, observed 
by the principal. Finally, the principal and the agent receive payoffs $u(x, y)$ and $v(x, y)$, respectively.

We consider repeated Stackelberg games, in which the same principal and agent play a sequence of 
Stackelberg games $(X, Y, u, v_i)_{1 \leq t \leq T}$ over $T$ rounds, with both participants observing the outcome of 
each game before proceeding to the next round. Notice that the agent’s payoff function $v$ may depend 
on the round $t$, possibly drawn from some distribution over possible payoff functions. Furthermore, 
we assume that the agent knows both the principal payoff function $u$ and the distribution over each
future agent payoff function \( v_t \), while the principal knows only \( u \). When considering the principal learning in this context, we also assume that the agent knows the principal’s learning algorithm and can thus compute its forward-looking utility (as we discuss further below).

**Discounting.** A common assumption in repeated games is that the agent discounts future payoffs; indeed, we assume that our agent acts with a discount factor of \( \gamma \), for some \( 0 < \gamma < 1 \). Formally, suppose a sequence \( ((x_1, y_1), (x_2, y_2), \ldots, (x_T, y_T)) \) of actions is played over the \( T \) rounds. Then, the principal’s total utility is \( \sum_{t=1}^{T} u(x_t, y_t) \) and the agent’s \( \gamma \)-discounted utility is \( \sum_{t=1}^{T} \gamma^t v_t(x_t, y_t) \). We make the behavioral assumption that the agent acts to maximize their \( \gamma \)-discounted utility and, consequently, may trade off present utility for future (discounted) payoffs. A canonical motivation for this assumption is that the agent leaves the game with probability \( \gamma \) at each round and is replaced by another agent from the same population.

**(Approximate) best responses.** To bound the loss in present utility compared to the (myopic) best response, we consider \( \varepsilon \)-approximate best responses. Considering approximate best responses lets us move beyond myopic agents who always maximize present-round utility, as typically studied in Stackelberg games, to non-myopic agents whose actions take future payoffs into account.

Generally, we consider \( \varepsilon \)-approximate best responses. Considering approximate best responses

\[
\text{Formally, define } \text{BR}(x) := \{ y \in Y : v(x, y) = \max_{y' \in Y} v(x, y') \} \text{ to be the agent’s best response set and } \text{BR}^\varepsilon(x) := \{ y \in Y : v(x, y) \geq \max_{y' \in Y} v(x, y') - \varepsilon \} \text{ to be their } \varepsilon \text{-approximate best response set to } x \in X. \text{ When the agent payoff functions } v_t \text{ vary with the round number } t, \text{ we write } \text{BR}_t(x) \text{ and } \text{BR}^\varepsilon_t(x) \text{ to denote the agent’s (}\varepsilon\text{-approximate) best response sets with respect to } v_t. \]

**Histories, policies, and regret.** A history \( H \) is an element of \( \mathcal{H} := \bigcup_{t \geq 0} (X \times Y)^t \) representing actions played in previous rounds. Let \( H_{t-1} \) denote the history \( ((x_s, y_s))^{t-1}_{s=1} \) of actions played before round \( t \). A principal policy \( \mathcal{A} : \mathcal{H} \rightarrow X \) is a (possibly random) function that takes a history \( H_{t-1} \) and outputs an action \( x_t = \mathcal{A}(H_{t-1}) \). An agent policy \( \mathcal{B} : \mathcal{H} \times X \rightarrow Y \) is a (possibly random) function that takes a history \( H_{t-1} \) and a principal action \( x_t \) and outputs an action \( y_t = \mathcal{B}(H_{t-1}, x_t) \).

The principal commits to a policy \( \mathcal{A} \) before the start of the repeated game. The agent, given \( \mathcal{A} \), then chooses a policy \( \mathcal{B} \). To measure the performance of \( \mathcal{A} \) against \( \mathcal{B} \), we use Stackelberg (or strategic) regret

\[
R_{\mathcal{A}, \mathcal{B}}(T) := \max_{x \in X} \left( \mathbb{E} \left[ \sum_{t=1}^{T} (\max_{y \in \text{BR}_t(x)} u(x, y) - u(x_t, y_t)) \right] \right),
\]

where the expectation is taken over the random history \( H_T \) induced by these policies. This regret compares the principal’s realized payoff to that obtained against a best-responding agent. When the optimal choice of \( y \in \text{BR}_t(x) \) is not unique, we consider the choice of \( y \) that corresponds to an agent tie-breaking in favor of the principal, as this yields the highest standard against which one can compete.

When agent payoffs \( v_t \) are stochastic and drawn i.i.d., the regret benchmarks \( \mathcal{A} \) against optimal Stackelberg equilibrium play in the stage game and decomposes into \( T \max_{x \in X} \mathbb{E}[u(x, \text{br}(x))] - \mathbb{E}[\sum_{t=1}^{T} u(x_t, y_t)] \), where \( \text{br}(x) \in \arg \max_{y \in \text{BR}_t(x)} u(x, y) \) again breaks ties in favor of the principal.

Generally, we consider \( \mathcal{B} \) belonging to a class of agent policies \( \mathcal{B} \) (potentially depending on \( \mathcal{A} \)) and minimize the worst-case Stackelberg regret \( R_{\mathcal{A}, \mathcal{B}}(T) := \sup_{\mathcal{B} \in \mathcal{B}} R_{\mathcal{A}, \mathcal{B}}(T) \). In our non-myopic setting, \( \mathcal{B} = \mathcal{B}_\gamma(\mathcal{A}) \) is the family of policies which are rational for a \( \gamma \)-discounting agent given \( \mathcal{A} \). Since our framework will relate non-myopic agents to approximately best-responding agents, we also consider the class \( \mathcal{B}^\varepsilon \) of policies \( \mathcal{B} \) with \( \mathcal{B}(H_{t-1}, x_t) \in \text{BR}^\varepsilon_t(x_t) \) for all \( t \), where \( \mathcal{B}^0 \) corresponds to the traditional myopic setting. Define \( R_{\mathcal{A}, \gamma}(T) := R_{\mathcal{A}, \mathcal{B}_\gamma(\mathcal{A})}(T) \) and \( R_{\mathcal{A}, \varepsilon}(T) := R_{\mathcal{A}, \mathcal{B}^\varepsilon}(T) \), respectively.
2.2 Reduction to robust and minimally reactive learning

As noted above, a major challenge in our learning setting is that agents may play actions that are far from best responses in any given round to obtain higher discounted future utility. At a high level, this is remedied by choosing a principal policy that is minimally reactive to agent feedback.

Concretely, a simple technique to decrease the influence that individual agent actions have on the principal policy (and thus the agent’s incentive to manipulate their action in the present round) is to delay the principal’s response to agent actions. Formally, we say that a principal policy $\mathcal{A}$ is $D$-delayed if each action $x_t = \mathcal{A}(H_{t-1})$ relies only on the prefix $H_{t-D}$ of $H_{t-1}$, i.e., $\mathcal{A}(H_{t-1}) = \mathcal{A}'(H_{t-D})$ for some $\mathcal{A}'$. With sufficient delay, the agent will have little incentive to manipulate their action and will play an approximate best response; this can be thought of as a (possibly adversarial) perturbation of the actual best response. Previous work has explored such an idea in the context of auctions with non-myopic agents (see Section 1.2); in contrast, we focus on distilling design principles that apply to general principal-agent settings. Towards this goal, we present a black-box reduction from learning with non-myopic agents to the better-understood problem of bandit learning from adversarially perturbed inputs.

**Proposition 2.1.** Suppose $0 < \gamma < 1$ and set $D = [T_\gamma \log(T_\gamma/\varepsilon)]$, where $T_\gamma = \frac{1}{\gamma \log 2}$ is the agent’s discounted time horizon. Then, if principal policy $\mathcal{A}$ is $D$-delayed, we have $R_{\mathcal{A}}(T, \gamma) \leq R_{\mathcal{A}}(T)$.

Our proof in Appendix A.1 observes that the total discounted utility for rounds after time $t$ is at most $\frac{1}{\gamma \log 2} \cdot \gamma^t$. While Proposition 2.1 simplifies the principal’s learning problem, it still leaves us with two new—but more tractable—challenges: (i) designing an adversarially robust bandit algorithm, and (ii) implementing such an algorithm with delayed feedback. For (i), we translate the guarantee $y_t \in \text{BR}^\varepsilon(x_t)$ to a more standard error type in a context-specific way, generally showing that $y_t \in \text{BR}(x'_t)$ for some $x'_t$ near $x_t$ and potentially bounding the deviation of $y_t$ from $\text{BR}(x_t)$.

**Design principles for minimally reactive learning.** For (ii), we note that any bandit algorithm $\mathcal{A}$ can be simply converted to a $D$-delayed algorithm with up to a multiplicative in $D$ overhead in regret, by interleaving $D$ copies of $\mathcal{A}$ and (somewhat wastefully) running them in parallel; this was first observed by Weinberger and Ordentlich [WO02]. However, this approach is far from optimal in most of our applications; often, we are able to collect less wasteful feedback using non-reactive but more diverse and variable query schedules that allow us to incur less regret while maintaining the same delay. To design these non-reactive schedules, we relate designing efficient delayed algorithms to designing batched algorithms, in which the principal makes queries and receives feedback in batches of size $B$. Formally, we say a principal policy is $B$-batched if each action $x_t = \mathcal{A}(H_{t-1})$ relies only on the prefix $H_{B[T_{t-1}]/B}$. By definition, any $D$-delayed policy is also $D$-batched, but there is also a useful reduction in the opposite direction.

**Proposition 2.2.** Any $B$-batched principal policy $\mathcal{A}$ can be converted into a $B$-delayed policy $\mathcal{A}'$ such that $R_{\mathcal{A}, \mathcal{B}}(T) \leq 2R_{\mathcal{A}, \mathcal{B}}(T)$ for any class of agent policies $\mathcal{B}$.

Our proof in Appendix A.2 runs two copies of $\mathcal{A}$ in parallel, alternating between batches, and even applies to a wider class of abstract bandit learning problems (though the relevant case for our setting is $\mathcal{B} = \mathcal{B}^\varepsilon$). Designing and analyzing batched algorithms is often simpler than doing so for their delayed counterparts and many of our algorithms can be naturally interpreted through this lens. We remark that batched policies can often be converted into delayed policies more directly than the conversion of Proposition 2.2, so the algorithms we present do not rely on this reduction. That said, batching still serves as a useful design principle for our final policies.
3 Stackelberg security games

We turn to Stackelberg security games (SSGs) as a primary application of learning in non-myopic principal-agent settings. These games model strategic interaction between a principal (the "defender") and an agent (the "attacker"). The principal—who has limited resources—aims to protect a set of targets, while the agent aims to attack advantageous targets left unprotected by the principal. The principal first commits to a strategy, i.e., a probabilistic assignment of defensive resources to targets, and the agent then chooses a target to attack based on the principal’s strategy.

For $n$-target games, we provide a robust search algorithm CLINCH that approximates an optimal strategy for the principal using $O(n)$ queries to a near best-responding agent (see Theorem 3.9). This is nearly optimal and improves upon the state-of-the-art of $O(n^3)$ queries for search with exact best responses and bounded bit precision [PSTZ19]. To achieve this, we identify and leverage new structural properties of SSG equilibria to cast the principal’s learning problem as quasi-convex optimization with a separation oracle. For the non-myopic setting, we then give an efficient batching procedure that translates our query complexity bound to a policy BATCHEDCLINCH with regret $O(n(\log T + T_\gamma))$ against $\gamma$-discounting agents (see Theorem 3.13). Finally, when the discount factor is unknown, we give a $\gamma$-agnostic policy MULTIITHEREDEDCLINCH with regret $O(n \log T(\log T + T_\gamma))$.

3.1 Model and preliminaries

A Stackelberg security game (SSG) is a Stackelberg game $(X, Y, u, v)$ where the principal commits to a defense in the strategy space $X \subseteq [0, 1]^n$ and the agent attacks a target from the set $Y = \{1, 2, \ldots, n\}$. A defense $x \in X$ corresponds to target $y \in Y$ being defended with probability $x_y$. We assume that $X$ is closed, convex, and downward closed (i.e., if $x \in X$ and $x' \in [0, 1]^n$ is such that $x'_y \leq x_y$ for all $y$, then $x' \in X$ as well). Finally, we assume payoffs depend only on the target attacked and the extent to which it was defended. Specifically, for each $y \in Y$, $u(x, y) = u^y(x_y)$ and $v(x, y) = v^y(x_y)$, where $u^y : [0, 1] \to [0, 1]$ and $v^y : [0, 1] \to [0, 1]$ are, respectively, strictly increasing and strictly decreasing continuous functions in $x_y$. We note that our model generalizes the standard setup where $X$ is the space of marginal coverage probabilities achievable by a randomized allocation of defensive resources to certain schedules under the "subsets of schedules are schedules" (SSAS) assumption [KYK+14], used by previous works on learning in SSGs [LCM09; BHP14; PSTZ19]. Throughout, we write $\text{Unif}(S)$ for the uniform distribution over a set $S \subseteq X$, $e_v \in [0, 1]^n$ for the standard basis vector corresponding to $y \in Y$, $0_0$ and $1_n$ for the all zeros and ones vectors. We write $x \preceq x'$ for $x, x' \in \mathbb{R}^n$ if $x_y \leq x'_y$ for all $y \in Y$.

Remark 3.1. One important case of interest is where the principal can defend only one target at a time (but is allowed to mix over which target to defend). Mathematically, this corresponds to the setting where $X$ is the downward closure of the probability simplex $\Delta_n := \{x : \|x\|_1 = 1 \land x \geq 0_0\}$, i.e., $X = \Delta_n^\perp := \{x : \|x\|_1 \leq 1 \land x \geq 0_0\}$. We use the specialization of our framework to the simplex in Appendix B.3 to facilitate the exposition of the main ideas of our approach.

We consider learning a fixed SSG over $T$ rounds. During the $t$-th round, the principal commits to a defense $x^{(t)} \in X$ (observed by the agent), the agent attacks a target $y_t \in Y$, and the players receive payoffs $u(x^{(t)}, y_t)$ and $v(x^{(t)}, y_t)$, respectively, with the agent payoff function $v$ unknown to the principal. Recall that Stackelberg regret is given by $T \max_{x \in X} u(x, \text{BR}(x)) - E \left[ \sum_{t=1}^T u(x^{(t)}, y_t) \right]$. When the agent is myopic and $y_t \in \text{BR}(x^{(t)})$ for all $t$, this task can be reframed as learning an optimal strategy for the principal using queries to a best response oracle that returns an arbitrary representative from $\text{BR}(x)$ when given $x \in X$. For non-myopic agents, we consider learning with an approximate best oracle returning an element of $\text{BR}^\varepsilon(x)$, for some small $\varepsilon > 0$. 
**Regularity assumptions.** Additional structural assumptions are standard, and in fact necessary, for learning in security games. The conditions we use map to the bit precision and non-degeneracy assumptions in previous work \[\text{LCM09, BHP14, PSTZ19}.\]

First, we require a slope bound \(C \geq 1\) such that

\[
\frac{1}{C} \leq \frac{v^y(s) - v^y(t)}{t - s} \leq C \quad \text{and} \quad 0 < \frac{u^y(t) - u^y(s)}{t - s} \leq C
\]

for all \(0 \leq s < t \leq 1\) and \(y \in \mathcal{Y}\). At a high level, this slope assumption bounds how quickly the defender and attacker utilities improve and degrade, respectively, with one extra unit of protection on an attacked target. When utility functions are linear, the upper bounds must be satisfied with \(C = 1\) to ensure payoffs in \([0, 1]\).\(^1\)

To state the second assumption, define for each target \(y \in \mathcal{Y}\) the best response region \(K_y \subseteq \mathcal{X}\) as the set of principal strategies for which \(y\) is a best response, i.e., \(K_y := \{x \in \mathcal{X} : y \in \text{BR}(x)\}\). We require that each non-empty best response region \(K_y\) has minimum width \(W > 0\) along its target’s dimension, i.e., for all \(y \in \mathcal{Y}\) with \(K_y \neq \emptyset\),

\[
\max_{x \in K_y} x_y - \min_{x \in K_y} x_y = \max_{y \in K_y} x_y \geq W,
\]

where the equality uses that \(\min_{x \in K_y} x_y = 0\) by downward closure of \(\mathcal{X}\) and monotonicity of \(v^y\). This assumption implies that any target that a rational attacker can be made to attack under some defense \(x\) is a best response to a sufficiently substantial set of defenses.

**Remark 3.2.** The state-of-the-art algorithm for learning SSGs \[\text{PSTZ19}\] imposes that utilities are linear with non-zero coefficients specified by \(L\) bits; this implies a slope bound of \(C = 2^L\). Moreover, \[\text{PSTZ19}\] requires that each non-empty region has minimum volume \(2^{-nL}\); this implies a width bound of \(W = 2^{-nL}\), as a region contained in \([0, 1]^n\) has volume bounded by its width along any dimension.

### 3.2 Structural properties of SSG equilibria

To characterize equilibrium structure in SSGs underlying the analysis of CLINCH, we introduce the notion of *conservative* strategies, where the principal wastes no defensive resources on targets not attacked by a best-responding agent. This property was originally defined for optimal strategies \[\text{BHP14}:\] our generalization enables a helpful decoupling in our analysis.

**Definition 3.3.** A strategy \(x \in \mathcal{X}\) is called *conservative* if \(x_y > 0\) only for \(y \in \mathcal{Y}\) such that \(y \in \text{BR}(x)\).

Figure 1 compares agent utility profiles for a wasteful versus conservative principal strategy. The red crosses indicate wasteful coverage that must be eliminated to conserve resources while maintaining the same best response payoff \(w = v(x, \text{br}(x))\) for the agent.

We now show each conservative strategy is uniquely determined by its best response payoff \(w\), with the coordinates of these strategies moving monotonically in \(w\).

**Lemma 3.4.** Suppose \(x, x' \in \mathcal{X}\) are conservative, and define \(w := v(x, \text{br}(x))\) and \(w' := v(x', \text{br}(x'))\). If \(w = w'\), then \(x = x'\). Otherwise, if \(w < w'\), we have \(\text{BR}(x') \subseteq \text{BR}(x)\), and \(x_y \geq x'_y\) for all \(y \in \mathcal{Y}\) with equality only if \(x_y = x'_y = 0\).

\(^1\)When each \(v_y\) is non-linear but continuously differentiable with derivative bounded away from 0, compactness of \([0, 1]\) implies that these inequalities hold for sufficiently large \(C\).
Proof. We first show that knowing $w$ lets us uniquely recover $x$. We solve for each $x_y$ given $w$, splitting our analysis into three cases: (i) If $v^y(0) > w$, as with targets 1 and 3 of the conservative strategy in Figure 1, then we must have $x_y > 0$ to ensure $v^y(x_y) \leq w$. Since $x$ is conservative, it must hold that $y \in BR(x)$ and thus $v^y(x_y) = w$. Since $v^y(x)$ is strictly decreasing, $w$ uniquely determines $x_y$. (ii) If $v^y(0) < w$, as with target 2 of the conservative strategy in Figure 1, then $v^y(x_y) \leq v^y(0) < w$. Hence $y \notin BR(x)$, and we must have $x_y = 0$ as $x$ is conservative. (iii) If $v^y(0) = w$, then either $x_y = 0$ or $x_y > 0$. But the latter implies $y \notin BR(x)$, which contradicts conservativeness. Hence we must have $x_y = 0$ here as well.

Next, suppose that $w < w'$ and fix any $y \in BR(x')$. Then $v^y(x'_y) = w' > w \geq v^y(x_y)$, and so monotonicity implies $x'_y < x_y$. Hence, $x_y > 0$ and so $y \in BR(x)$ by conservativeness. Now, for any $y$ with $x'_y > 0$, $y \in BR(x')$ and so we have $x_y > x'_y$. For $y$ with $x_y = 0$, we trivially have $x_y \geq x'_y$. \hfill$\square$

Using Lemma 3.4, we establish that the unique conservative $x^*$ maximizing the principal’s payoff also minimizes the agent’s payoff. We leverage this property for efficient optimization: while the principal’s payoff $u(x, br(x))$ is difficult to directly optimize, as it may be discontinuous in $x$, the equivalent objective $v(x, br(x)) = \max_{y \in Y} v^y(x_y)$ is quasi-convex in $x$ (and convex when the $v^y$ are linear).

Proposition 3.5. There exists a unique conservative strategy $x^* \in \arg\max_{x \in X} u(x, br(x))$. Moreover, this principal strategy $x^*$ is also the unique conservative strategy in $\arg\min_{x \in X} v(x, br(x))$.

Proof. We first show that there exists a conservative strategy $x^* \in \arg\max_{x \in X} u(x, br(x))$. Indeed, let $x$ be any strategy in $\arg\max_{x \in X} u(x, br(x))$. If $x$ is not conservative, we give a procedure to achieve this property while keeping $x \in \arg\max_{x \in X} u(x, br(x))$. For this, suppose there exists some $y \in Y$ such that $x_y > 0$ but $y \notin BR(x)$. By the downward closure property of $X$, we can reduce $x_y$ until either $x_y = 0$ or $y$ becomes a best response. Note that this adjustment does not decrease $u(x, br(x))$, since we leave all the other coordinates unchanged and only expand $BR(x)$. While $x$ is not conservative, we iterate this procedure. Since either $x_y$ gets set to 0 or $y$ gets added to $br(x)$, this procedure will terminate after $n$ iterations. Thus, there exists a conservative strategy $x^* \in \arg\max_{x \in X} u(x, br(x))$.

Next, we show that this $x^*$ is unique. Suppose for the sake of contradiction that there exists another conservative strategy $x' \neq x^*$ in $\arg\max_{x \in X} u(x, br(x))$. Define $w^* := v(x^*, br(x^*))$ and $w' := v(x', br(x'))$. By Lemma 3.4, we must have $w^* \neq w'$. Assume without loss of generality...
that $w^* < w'$. We claim $x'$ cannot be optimal for the principal. Indeed, Lemma 3.4 gives that $\text{BR}(x') \subseteq \text{BR}(x^*)$, and so

$$u(x', \text{BR}(x')) = \max_{y \in \text{BR}(x')} u^v(y) \leq \max_{y \in \text{BR}(x^*)} u^v(y) = u(x^*, \text{BR}(x^*)).$$

Let $y' = \text{BR}(x')$ and $y^* = \text{BR}(x^*)$ be the first and second maximizers in the above inequality. Note that if $x^*_y = 0$, then the best response region $K_y$ has width $\max_{x \in K_y} x_y = x^*_y = 0$, which is forbidden by our regularity assumption. Therefore, $x^*_y > 0$ and, by Lemma 3.4, we have that $x^*_y < x^*_y$ and $y' \in \text{BR}(x^*)$. In this case, monotonicity of the utilities and optimality of $y^*$ within $\text{BR}(x^*)$ imply that $u(x^*, y^*) \geq u(x^*, y') > u(x', y')$. This shows that $x'$ is not an optimal strategy for the principal.

Having shown that $x^*$ is well-defined, we now prove that it also minimizes $v(x, \text{BR}(x))$. We first show that there exists a conservative strategy $x'$ minimizing $v(x, \text{BR}(x)) = \max_y v(x, y)$ over $X$. As before, we start with any minimizer $x$ of $v(x, \text{BR}(x))$. Then, if $x$ is not conservative at some $y$, we can decrease $x_y$ until either $x_y = 0$ or $y$ is a best response. By minimality, we know that $v(x', \text{BR}(x')) \leq w^*$. If this inequality were strict, then the argument above would contradict $x^* \in \arg\max_{x \in X} u(x, \text{BR}(x))$. Hence we have equality and Lemma 3.4 implies that $x' = x^*$.

**Remark 3.6.** Theorem 3.8 of [KYK+14] identifies that all optimal strategies for the principal minimize the utility of a best-responding agent. Moreover, assuming a homogeneity condition on $X$ satisfied in the simplex setting (but not for general SSGs), Theorem 3.10 of [KYK+14] implies that Proposition 3.3 holds without requiring conservativeness. To the best of our knowledge, however, these properties have not been previously exploited for learning SSGs.

### 3.3 CLINCH: a (nearly) optimal robust search algorithm

Our algorithm CLINCH (Algorithm 1) estimates the unique conservative optimizer $x^*$ guaranteed by Proposition 3.5, even with inexact best response feedback. Its main loop searches for an approximate minimizer of the agent’s best response utility $v(x, \text{BR}(x))$, while the post-processing routine CONSERVE MASS (Algorithm 2) ensures that this strategy is nearly conservative. More precisely, CLINCH maintains an active search region $S$, determined by entry-wise lower and upper bounds $\underline{x}_y, \overline{x}_y$ which may be initialized using prior knowledge of $x^*$. Upon querying the centroid $\bar{x}$ of $S$ and receiving feedback $y \in Y$, we deduce that $x^*_y \geq x_y - \varepsilon$. By updating $\underline{x}_y$ accordingly, we “clinch” this progress and either shrink $S$ significantly or remove $y$ from the active target set $R$ in the next round, in which case $S$ is flattened along this dimension. The termination condition at Step 2 ensures that the agent’s utility in best response to the final query is sufficiently small. After this, CONSERVE MASS performs a binary search for each target $y$ that approximates the procedure described in the proof of Proposition 3.5, reducing $x_y$ until it nears the threshold where $y$ becomes a best response.

---

2We give our proof assuming that we can exactly compute the centroid $\mathbb{E}_{w \sim \text{Unif}(S)}[w]$ of each search region $S$. Handling an approximate centroid is standard (see, e.g., [BV04]), and we omit the details. (Moreover, sample complexity is not affected since we have full knowledge of the set $S$ at each iteration.)
We now give formal guarantees for each component of CLINCH.

**Lemma 3.7 (Minimize).** Fix $\delta \in (0, 1]$ and $\underline{x}, \overline{x} \in \mathbb{R}^n$ with $\underline{x} \leq x^* \leq \overline{x}$. Then, after $O(n \log \frac{C\alpha n}{\delta^2})$ queries to a $\frac{\delta^2}{33C^2 n}$-approximate best response oracle, CLINCH calls CONSERVE MASS with an $x \in \mathcal{X}$ such that $v(x, \text{br}(x)) \leq v(x^*, \text{br}(x^*)) + \frac{\delta^2}{33C^2 n}$, where $\alpha = \max_{y \in \mathcal{Y}} y \cdot x_y - x_y$.

**Lemma 3.8 (Conserve).** Fix $\lambda \in (0, 1]$, $x \in \mathcal{X}$, and $x \in \mathbb{R}^n$ with $\underline{x} \leq x$. Then CONSERVE MASS returns $\hat{x} \in \mathcal{X}$ with $\hat{x}_y > x_y$ only for $y \in \text{BR}^{3C\lambda}(\hat{x})$ and such that $v(\hat{x}, \text{br}(\hat{x})) \leq v(x, \text{br}(x)) + 2C\lambda$, while making $O(n \log \frac{\alpha^2}{\delta})$ queries to a $\frac{\alpha^2}{C}$-approximate best response oracle, where $\alpha = \max_{y \in \mathcal{Y}} y \cdot x_y - x_y$.

To prove Lemma 3.7 in Appendix B.2, we show that the volume of $S$ decreases by a constant factor in each round unless a target is removed from $\mathcal{R}$, in which case $S$ loses a dimension but still has lower-dimensional volume not too much larger than before. The volume decrease claim follows by an approximate version of Grünbaum’s inequality [Grü60]; we show that a half-space nearly passing through the centroid of $S$ will split the region into roughly balanced halves. For Lemma 3.8, the $n$ binary searches ensure that each coordinate $\hat{x}_y$ is within $O(\lambda)$ of the threshold $\sup\{p \in [\underline{x}_y, x_y] : y \in \text{BR}(x_1, \ldots, x_{y-1}, p, x_{y+1}, \ldots, x_n)\}$. In Appendix B.3, we use this to show that each target with substantial coverage under $\hat{x}$ is an approximate best response, with the agent’s utility in best response to $\hat{x}$ nearly matching that of $x$.

Equipped with these results, we prove that CLINCH finds a $\delta$-approximation for $x^*$ in $O(n \log \frac{1}{\delta})$ queries, and strengthen the guarantee when the provided bounding box is small.

**Theorem 3.9.** Fix $0 < \delta \leq 1$ and $\underline{x}, \overline{x} \in \mathbb{R}^n$ such that $\underline{x} \leq x^* \leq \overline{x}$. Then CLINCH returns an $\hat{x} \in \mathcal{X}$ such that $||\hat{x} - x^*||_\infty \leq \delta$ using $O(n \log \frac{C\alpha n}{\delta^2})$ queries to a $\frac{\delta^2}{33C^2 n}$-approximate best response oracle, where $\alpha = \max_{y \in \mathcal{Y}} y \cdot x_y - x_y$. In particular, fixing $\underline{x} = 0_n$ and $\overline{x} = 1_n$ gives query complexity $O(n \log \frac{Cn}{\delta^2})$.

**Proof.** Combining Lemma 3.7 and Lemma 3.8 (with $\lambda = \frac{\delta}{16C^2}$), we find that the strategy $\hat{x} \in \mathcal{X}$ returned by CLINCH satisfies $v(\hat{x}, \text{br}(\hat{x})) \leq v(x^*, \text{br}(x^*)) + \delta/C$, and $\hat{x}_y > x_y$ only for $y \in \text{BR}^{3C\lambda}(\hat{x})$, using $O(n \log \frac{C\alpha n}{\delta^2})$ queries. First, the approximate minimization guarantee requires that $\hat{x}_y \geq x^*_y - \delta$ for all $y \in \mathcal{Y}$. Indeed, if $y \not\in \text{BR}(x^*)$, then $x^*_y = 0$ and the claim holds trivially. Otherwise, we have

$$v^y(\hat{x}_y) \leq v(\hat{x}, \text{br}(\hat{x})) \leq v(x^*, \text{br}(x^*)) + \delta/C = v^y(x^*_y) + \delta/C.$$

Monotonicity of $v^y$ and the lower slope bound of $1/C$ then imply that $\hat{x}_y \geq x^*_y - \delta$. Next, we show that approximate conservativeness implies $\hat{x}_y \leq x^*_y + \delta$ for all $y \in \mathcal{Y}$. Indeed, if $y \in \text{BR}^{3C\lambda}(\hat{x})$,

$$v^y(\hat{x}_y) \geq v(\hat{x}, \text{br}(\hat{x})) - 3C\lambda \geq v(x^*, \text{br}(x^*)) - 3C\lambda \geq v^y(x^*_y) - 3C\lambda.$$
and so monotonicity and our slope bound imply that \( \tilde{x}_y \leq x^*_y + 3C^2\lambda < x^*_y + \delta \). Otherwise, we must have \( \tilde{x}_y = x^*_y \leq x^*_y \). All together, we have \( \|\tilde{x} - x^*\|_\infty \leq \delta \), as desired. \( \square\)

**Remark 3.10.** Note that no algorithm can approximate \( x^* \) up to \( \ell_\infty \) precision \( \delta \) in fewer than \( \tilde{\Omega}(n \log \frac{1}{\delta}) \) queries. Indeed, if \( X = [0, 1]^n \), then any \( x^* \in X \cap \delta \mathbb{Z}_{\geq 0} \) with \( x^*_1 = 1 \) can be attained via payoffs \( v^i(x_1) = 1 - x_1/2 \) and \( v^i(x_i) = (1 + x^*_i - x_i)/2 \) for \( i > 1 \). Clearly, \( \Theta(n \log \frac{1}{\delta}) \) bits are needed to specify \( x^* \), but each query provides \( O(\log n) \) bits. Among all optimal strategies, the search for \( x^* \) is well-motivated because this optimizer is not wasteful of the principal’s resources and remains optimal for any choice of the principal’s utilities.

**Remark 3.11.** The minimization stage of CLINCH is connected to cutting-plane methods from optimization. Indeed, feedback \( y \in \text{BR}_k(x) \) for a query \( x \) implies \( x^*_y \geq x_y - Ce \), so the cut \( \{z \in \mathbb{R}^n : z_y = x_y\} \) nearly separates \( x \) from \( x^* \). Treating the agent as a noisy separation oracle, CLINCH mirrors the query selection of center-of-gravity methods [Lev65; New65]. Unlike this classic setting, we lack an evaluation oracle for the objective \( v(x, \text{br}(x)) \) and must implement more modern adjustments. Specifically, CLINCH relates to the ProjectedVolume algorithm for multidimensional binary search [LLV18], with coordinate locking at Step 4 serving as an analog of their cylindrification procedure and preventing a quadratic dependence on \( n \). While Krishnamurthy, Lykouris, Podimata, and Schapire [KLPS21] give a robust variant of ProjectedVolume that handles an unknown bounded number of adversarial corruptions, this notion of data contamination is orthogonal to that of our work.

### 3.4 Batching and our extension to non-myopic agents

We now return to the repeated game with non-myopic agents. First, we observe that the search guarantee of CLINCH translates to a single-round regret bound against approximately best-responding agents, after a small perturbation. Specifically, given any sufficiently precise estimate \( \hat{x} \) for \( x^* \), we prove in Appendix B.4 that the principal can identify the true best response set \( \text{BR}(x^*) \) of \( x^* \) and should slightly reduce the weight placed on a target \( \hat{y} \) in this set which maximizes \( u(\hat{x}, \hat{y}) \).

**Lemma 3.12.** Fix \( 0 < \lambda \leq 1 \) and suppose \( \hat{x} \in X \) with \( \|\hat{x} - x^*\|_\infty \leq \frac{W\lambda}{2\epsilon^2} \). Then the perturbed strategy \( \tilde{x} = \text{PERTURB}(\hat{x}, \lambda) := \hat{x} - \frac{W\lambda}{2} e_y \), where \( y \in \arg\max_{y : \hat{x}_y > W/2} u(\hat{x}, y) \), belongs to \( X \) and satisfies \( u(\tilde{x}, y) \geq u(x^*, \text{br}(x^*)) - \lambda \) whenever \( y \in \text{BR}_k(\tilde{x}) \) for \( \epsilon = \frac{W\lambda}{200C\gamma n} \).

Now, consider the variant of CLINCH that finds such a strategy and commits to it for the remaining rounds, incurring regret \( O(n \log \frac{C\gamma n}{W\lambda}) + \lambda T \) against \( \epsilon \)-approximately best-responding agents, where \( \epsilon = \frac{W\lambda}{240C\gamma n} \). Fixing \( \lambda = nT_{\gamma}/T \) and enforcing delay \( D = T_{\gamma} \log \frac{T_{\gamma}}{\epsilon} \) by naively repeating each query \( D \) times (only incorporating feedback from the first query in each batch), Proposition 2.1 gives regret \( O(Dn \log \frac{C\gamma n}{W\lambda} + T_{\gamma}n) = O(n T_{\gamma} \log^2 \frac{C\gamma n}{W\lambda}) \) against \( \gamma \)-discounting agents. To improve this, we introduce the policy **BatchedClinch** (Algorithm 3), which maintains a running estimate \( \hat{x} \) of the current best strategy. For each epoch \( \phi = 1, 2, \ldots \), this algorithm runs CLINCH with accuracy \( \lambda = 2^{-\phi} \) in batches of size \( O(T_{\gamma} \log T_{\gamma} n) \), initialized with \( x \) and \( \bar{x} \) set to bounds implied by the previous search. We update CLINCH during the first round of each batch, playing \( \hat{x} \) for the others, and then set \( \tilde{x} \) to the estimate returned by CLINCH (after perturbation). When all epochs have completed, the final estimate \( \bar{x} \) is played for the remaining rounds. For ease of presentation, we identify entry-wise lower and upper bounds with their corresponding axis-aligned bounding box.
\textbf{Algorithm 3 BatchedClinch}

1: \(\tilde{x} \leftarrow (0, \ldots, 0) \in \mathbb{R}^n, B \leftarrow [0, 1]^n\)
2: \textbf{for} epoch \(\phi = 1, \ldots, [\log_2 T]\) \textbf{do}
3: \quad Initialize CLINCH with accuracy \(\delta = \frac{W^4}{6C^2}\) for \(\lambda = 2^{-\phi}\) and axis-aligned bounding box \(B\)
4: \quad \textbf{while} CLINCH has not terminated \textbf{do}
5: \quad \quad Simulate query/response for CLINCH with next \(x(t), y_t\) pair \(\triangleright\) Explore
6: \quad \quad Play strategy \(\tilde{x}\) for next \(\left[ T_\gamma \log \frac{2^{200T_\gamma C^3 n}}{W}\right]\) rounds \(\triangleright\) Exploit
7: \quad \quad \tilde{x} \leftarrow \text{PERTURB}(\tilde{x}, \lambda)\) for \(\tilde{x}\) returned by CLINCH
8: \quad \quad \(B \leftarrow \{x \in \mathbb{R}^d : ||x - \tilde{x}||_\infty \leq \delta\}\) \(\triangleright\) Update bounding box
9: \quad Play strategy \(\tilde{x}\) for remaining rounds \(\triangleright\) Exploit

\textbf{Theorem 3.13.} BatchedClinch incurs regret at most \(O\left(n \log \frac{Cn}{W} \log T + nT_\gamma \log^2 \frac{CnT_\gamma}{W}\right)\) against \(\gamma\)-discounting agents.

\textbf{Proof.} During and after epoch \(\phi\), when \(\delta = \frac{W^4}{6C^2}\) for \(\lambda = 2^{-\phi}\), the principal’s feedback delay incentivizes the agent to \(\frac{W}{200\sqrt{T_\gamma}}\)-approximately best respond, by Proposition 2.1. Moreover, the entry-wise lower and upper bounds are trivially valid at the start and remain valid due to the \(\ell_\infty\) guarantee of Theorem 3.9. By the same theorem, epoch \(\phi = 1\) terminates after \(O(n \log \frac{Cn}{W})\) batches of size \(O(T_\gamma \log \frac{Cn}{W})\) and epoch \(\phi > 1\) terminates after \(O(n \log Cn)\) batches of size \(O(T_\gamma \log (T_\gamma Cn^2)/W))\); moreover, the resulting strategy \(\tilde{x}\) incurs regret at most \(2^{-t}\) when played in a future round (by the initial observation and Lemma 3.12). We bound total regret obtained during exploration (Step 5) by

\[
O\left(n \log \frac{Cn}{W} + \log T \cdot n \log Cn\right) = O\left(n \log \frac{Cn}{W} \log T\right).
\]

Similarly, the regret from exploitation rounds (Step 6) is at most

\[
O\left(nT_\gamma \sum_{i=1}^{[\log_2 T]} \frac{i2^{-i} \log Cn}{W} \log \frac{T_\gamma Cn}{W}\right) = O\left(nT_\gamma \log \frac{Cn}{W} \log \frac{CnT_\gamma}{W}\right),
\]

where the equality uses that \(\sum_{i=1}^{m} i2^{-i} = O(1)\). Finally, Step 9 contributes \(O(1)\) regret. \(\square\)

\textbf{Unknown Discount Factor.} BatchedClinch requires the principal to know the discount factor \(\gamma\). When \(\gamma\) is unknown, we adapt the multi-layer approach of Lykouris, Mirrokni, and Paes Leme \cite{LMP18}, running \(\log T\) copies of CLINCH in parallel threads, where thread \(r\) experiences delay \(2^r\) between queries. Each thread’s delay corresponds to a guess \(\hat{\gamma}\) for \(\gamma\), with search guarantees only holding if \(\hat{\gamma} \geq \gamma\) and sufficiently accurate best response feedback is induced. As each thread performs a search to accuracy \(O(1/T)\), it maintains a shrinking bounding box around \(\tilde{x}\). When search completes, it performs exploitation by perturbing a strategy in the intersection of the current boxes for threads \(\geq q\), for the smallest \(q\) such that this intersection is non-empty. By design, the resulting box contains \(\tilde{x}\) and is contained by the box of thread \(r^*\) corresponding to the correct guess for \(\gamma\). In total, we show that the resulting \(\gamma\)-agnostic policy MultiThreadedClinch (Algorithm 4) achieves regret \(\tilde{O}(n \log T(T_\gamma + \log T))\).
While the Algorithm 4
Theorem 3.14. MultiThreadedClinch incurs regret $O(n \log CnW \log^2 T + nT_\gamma \log CnW \log CnT_\gamma T )$ against $\gamma$-discounting agents.

Proof. By design, thread $r$ runs on rounds $2^{r-1}(2k - 1)$ for $k = 1, 2, \ldots$ and hence experiences delay $2^r$. Write $r^* = \left\lfloor \log_2 \left( T_\gamma \log \frac{400C^3nT_\gamma T}{W} \right) \right\rfloor$ for the index of the first thread whose delay induces $\frac{W}{400C^3nT_\gamma T}$-approximate best responses, using Proposition 2.1 (we can assume that $r^*$ is a valid thread; otherwise the regret bound holds trivially). By Theorem 3.9 feedback for each thread $r \ge r^*$ is sufficiently accurate so that the bounding box $B^{(r)}$ always contains $x^*$. Consequently, the uncertainty set $B$ selected at Step 10 is always a subset of $B^{(r^*)}$ which contains $x^*$.

To bound regret, we note that all threads terminate exploration after $O(n \log CnW \log T)$ updates, using the same bound as in the proof Theorem 3.13 this gives a total exploration cost of $O(n \log CnW \log^2 T)$. For exploitation, we consider the separate runs of $\mathcal{A}^{(r^*)}$; by Theorem 3.9 the first run takes $O(n \log CnW)$ queries, and the remaining $O(\log T)$ runs take $O(n \log Cn)$ queries each. While the $i$th run is in progress, corresponding to $\delta = \frac{nW}{6C^3nT_\gamma T}$, Lemma 3.12 implies that no exploit round from any thread can incur regret more than $2^{1-i}$, since $x^* \in B \subseteq B^{(r^*)}$ holds at all times. Consequently, we bound exploitation regret by

$$O \left( 2^{r^*} \left[ n \log \frac{CnW}{W} + \frac{\log T}{\log(Cn)} \right] \right) = O \left( nT_\gamma \log \frac{CnW}{W} \log \frac{CnT_\gamma T}{W} \right). \quad \Box$$

### 3.5 Comparison to prior work on Stackelberg security games

Finally, we compare CLINCH to previous work for solving SSGs with best-responding agents.

**The classical approach: multiple LPs.** First, we recall the standard method for the full-information problem, which requires each agent utility function $v^\gamma$ to be linear. Here, each best
response region \( K_y \) is convex as the intersection of \( X \) with \( n \) half-spaces, and we may rewrite the Stackelberg benchmark as a set of \( n \) optimization problems, with the \( y \)-th optimizing \( u_y \) over \( K_y \):

\[
\max_{x \in X} u(x, br(x)) = \max_{y \in Y} \max_{x \in X} u^y(x) = \max_{y \in Y, x \in K_y} u^y(x).
\]

These inner problems can be solved efficiently, since each \( K_y \) is convex and each objective \( u^y \) is quasi-convex by monotonicity. This approach was originally developed by Conitzer and Sandholm [CS06] for the case when \( X \) is a polytope and each \( u^y \) is linear, where it is termed “multiple LPs”.

When \( v \) is unknown, but the agent is fully myopic, previous works [LCM09, BHP14, PSTZ19] use exact best-response feedback to learn each \( K_y \) and apply multiple LPs. Most recently, Peng, Shen, Tang, and Zuo [PSTZ19] provide an algorithm that finds an optimal strategy using \( O(n^3L) \) best-response queries when agent utilities are linear with coefficients specified by \( L \) bits each.

**Our improvements.** In contrast, CLINCH applies in more general environments and provides stronger query complexity guarantees. Even for the full information problem, our method only requires monotonicity of agent utilities and works when these are non-linear, as depicted in Figure 2. On the other hand, linearity was needed to induce convex best response regions that were crucial for previous methods. For the learning problem, our results do not require the stronger \( L \)-bit precision assumption and extend seamlessly to the approximate best response regime needed to handle non-myopic agents. When specializing to the setting of prior work, we show in Appendix B.5 that CLINCH finds an exact optimizer using \( O(n^2L) \) best-response queries, improving upon the state-of-the-art \( O(n^3L) \) complexity. For \( \delta \)-approximate search with \( \delta^{-1} = \text{poly}(n) \), our query complexity of \( \tilde{O}(nL) \) improves quadratically over prior guarantees (for exact search).

We attribute our improved bounds to two main factors. First, previous works [LCM09, BHP14, PSTZ19] fix a target \( y \) and use best responses primarily as a membership oracle for the best response polytope \( K_y \) (since \( y \in BR(x) \) if and only if \( x \in K_y \)). In contrast, we use this feedback to simulate a separation oracle, incorporating more information per query to obtain faster convergence. Since the objective \( v(x, br(x)) \) is quasi-convex, as the maximum of monotonic functions in each coordinate, cutting plane methods can be adapted to obtain \( O(n \log \frac{1}{\delta}) \) search complexity. Second, existing methods solve an LP for each \( K_y \), while our structural results imply that \( x^\star \) belongs to all non-empty best response regions. (Indeed, if \( y \not\in BR(x^\star) \), then \( x^\star \) is always a best response.) Hence, it suffices to solve \( \max_{x \in K_y} x \) for any non-empty \( K_y \) and then eliminate wasteful defensive resources, giving an immediate \( n \)-factor improvement.

### 4 Pricing with an unknown demand curve

As a second application of our framework, we consider a demand learning problem [KL03], in which a price-setting principal seeks to maximize revenue from selling a good to a returning buyer with demand curve induced by an unknown value distribution. Each round, the principal posts a price and the buyer decides whether to purchase based on their realized value. This problem was among the first examined with non-myopic agents [ARS13, MMI14], as it arises naturally in settings like online advertising where strategic buyers may try to trick the seller into providing low prices.

When the buyer’s value is fixed, our learning task mirrors binary search, and we adapt the batching method of BATCHEDCLINCH to obtain a policy BATCHEDBINARYSEARCH with regret \( \tilde{O}(\log T + T_y) \), improving upon the state-of-the-art when \( T_y = \Omega(\log T) \). For stochastic values, the problem reduces naturally to an instance of stochastic multi-armed bandits—typically solved by adaptive exploration rather than explore-then-commit—and so a different approach is required.
Fortunately, a classic policy SUCCESSIVEELIMINATION extends seamlessly to the delayed feedback setting and exhibits natural robustness to bounded adversarial perturbations. This policy achieves regret $\mathcal{O}(\sqrt{T} + T_\gamma)$, with only an additive overhead in $T_\gamma$ compared to the standard $\mathcal{O}(\sqrt{T})$ bound.

4.1 Model and preliminaries

A posted-price single-buyer auction is a Stackelberg game where the principal (“seller”) sets the price $p \in [0, 1]$ of a single good and the agent (“buyer”) decides whether to buy ($a = 1$) or not ($a = 0$) at the posted price. The buyer has value $v \in [0, 1]$ for the good and receives payoff $a(v - p)$, while the seller receives $pa$. In the stochastic setting, where $v$ is sampled from a distribution $\mathcal{D}$, the seller’s expected revenue for posting price $p$ is $f(p) = pd(p)$, where $d(p) = \Pr_{v \sim \mathcal{D}}(v \geq p)$ is called the buyer’s demand curve.

We consider the repeated game where values $v_1, \ldots, v_T$ of a returning buyer are sampled i.i.d. from $\mathcal{D}$, unknown to the seller, where $\mathcal{D}$ either (i) is supported on a single value or (ii) satisfies mild regularity assumptions described below. Denoting the game’s history by $\{(p_t, a_t)\}_{t=1}^T$, the seller seeks to maximize revenue $\sum_{t=1}^T p_t a_t$, while the buyer maximizes discounted profit $\sum_{t=1}^T v_t a_t(v_t - p_t)$. Stackelberg regret for the seller is $T \max_p pd(p) - \mathbb{E} \left[ \sum_{t=1}^T p_t a_t \right]$, since a myopic agent buys the good when their value exceeds the posted price. As before, we write $\text{BR}_T^\varepsilon(p) = \{a \in \{0,1\} : a(v_t - p) \geq \max\{v_t - p, 0\} - \varepsilon\}$ for the $\varepsilon$-approximate best response set at time $t$.

Connection to multi-armed bandits. In case (ii), Kleinberg and Leighton [KL03] reduce this task to a stochastic multi-armed bandits problem, a setting which we recall briefly (see Slivkins [Sli19] for a textbook treatment). In this model, a principal interacts with a set of $K$ “arms” over $T$ rounds. The arms are indexed by $i \in [K]$, and each arm has an associated reward distribution $\mathcal{D}_i$ supported on $[0, 1]$. Write $\mu_i = \mathbb{E}_{r \sim \mathcal{D}_i}(r)$ and $\Delta_i = \max_{v} \mu_v - \mu_i$. During the $t$-th round, the principal must pull an arm $i_t \in [K]$; after doing so, they observe a reward $r_t \sim \mathcal{D}_i$. The performance of a policy is benchmarked against the best arm in expectation, with regret defined as $\mathbb{E} \left[ \sum_{t=1}^T \Delta_{i_t} \right]$. To view pricing through this lens, Kleinberg and Leighton [KL03] discretize the space of possible prices into the set $\{\frac{i}{K} : 1 \leq i \leq K\}$. Each of these $K$ possible prices can then be thought of as a bandit arm that the seller can pull, where pulling the $i$-th arm corresponds to posting a price of $\frac{i}{K}$, and the two notions of regret coincide up to a small difference in benchmarks due to discretization.

Regularity assumptions. For case (ii), we assume that the demand curve $d(p) = \Pr_{v \sim \mathcal{D}}(v \geq p)$ is $L$-Lipschitz, which is standard for this setting (see, e.g., [ARS13]). Moreover, following Kleinberg and Leighton [KL03], we assume that $f(p)$ achieves its maximum for a unique $p^* \in (0,1)$ with $f''(p^*) < 0$; this implies that there exist constants $C_1, C_2$ with $C_1 (p^* - p)^2 \leq f(p^*) - f(p) \leq C_2 (p^* - p)^2$. Under this assumption, [KL03] show the following for the $K$-armed bandits problem described above.

Lemma 4.1 ([KL03, Corollary 3.13]. The discretization error $f(p^*) - \max_i f(i/K)$ is at most $\frac{C_2}{K^2}$.

Lemma 4.2 ([KL03, within Theorem 3.14]. The sum of inverse gaps $\sum_{\Delta_i > 0} \frac{1}{\Delta_i}$ is at most $\frac{6K^2}{C_1}$.

4.2 Fixed value

Returning to case (i) where $\mathcal{D}$ is concentrated on an unknown $v \in [0, 1]$, feedback $\mathbb{I}\{v \geq p_t\}$ from a myopic agent is sufficient to perform binary search, implying an explore-then-commit $O(\log T)$ regret bound. Looking closely, we can reinterpret the problem as a security game over the simplex $\Delta_2$, with two targets representing the buyer’s purchase choices; this motivates us to implement delays via the batching approach of BATCHEDCLINCH, lengthening batches as search progresses and committing to
prices which incur low regret during non-exploration rounds. For clarity, we translate and simplify this approach to a policy BatchedBinarySearch for the present setting.

**Algorithm 5 BatchedBinarySearch**

1: \( \ell \leftarrow 0, u \leftarrow 1, v \leftarrow 0 \)
2: while \( u - \ell > 1/T \) do
3:    Set \( \varepsilon = (u - \ell)/4 \) and post price \( p = (\ell + u)/2 \)
4:    if agent buys good then \( \ell \leftarrow p - \varepsilon \) else \( u \leftarrow p + \varepsilon \)
5:    Post price \( \hat{v} \) for next \( T_{\gamma} \log \frac{T_{\gamma}}{\varepsilon} \) rounds
6:    \( \hat{v} \leftarrow \max\{\ell - \varepsilon, 0\} \)
7:    Post price \( \hat{v} \) for remaining rounds

**Theorem 4.3.** BatchedBinarySearch incurs regret \( O(\log T + T_\gamma \log T_\gamma) \) against \( \gamma \)-discounting agents with a fixed value.

**Proof.** First, we note that \( v \) always lies in the interval \([\ell, u]\), and that this interval shrinks by a factor of \( 3/4 \) between iterations. Indeed, when this interval has width \( 4\varepsilon \), our feedback delay ensures that the buyer’s decision is a best response for some perturbed value \( v' \) with \( |v' - v| \leq \varepsilon \) (by Proposition 2.1), and so the updates are sound. Consequently, we always have \( \hat{v} < v \) (unless \( v = 0 \), in which case any policy suffices), and so the buyer purchases the good at Steps 5 and 7 since there is no incentive to deviate from best response during these rounds. Hence, we incur at most \( 4 \cdot \frac{4}{3} \varepsilon \) regret for each round of Step 5 and at most regret 1 after search concludes, giving a total bound of

\[
\sum_{i=1}^{\left[ \frac{\log_{4/3} T}{T} \right]} \left( 1 + 4 \cdot 0.75^{i-1} T_{\gamma} \log (T_{\gamma} 1.4^i) \right) = O(\log T + T_\gamma \log T_\gamma). \]

**Remark 4.4 (Comparison to prior work).** Noting that over-pricing is costlier than under-pricing for the seller, Kleinberg and Leighton [KL03] beat binary search with a policy attaining regret \( O(\log \log T) \) for the myopic setting. With non-myopic agents, a line of work [ARS13; MM14; Dru17] has brought regret down to \( O(T_\gamma \log T_{\gamma} \log T_{\gamma}) \) via a delayed search policy of Drutsa [Dru17], compared to a lower bound of \( \Omega(\log \log T + T_\gamma) \) implied by [KL03] and [ARS13]. When \( T_\gamma = O(1) \), this regret is optimal; however, for \( T_\gamma = \Omega(\log T) \), our bound of \( O(T_\gamma \log T_\gamma) \) is a \( \log \log T \) improvement.

### 4.3 Stochastic values

To address case (ii), we first consider stochastic bandits with perturbed and delayed feedback. Formally, we say that (potentially adversarially adaptive) feedback \( r_1, \ldots, r_T \) is \( \delta \)-perturbed from that of the original bandits problem if, conditioned on any arm sequence \( i_1, \ldots, i_T \) with positive probability, there exist independent random intervals \( \{[\ell_i, u_i]\}_{i=1}^{T} \) such that each \( r_i \) lies in \( [\ell_i, u_i] \) almost surely and that \( \mu_i - \delta \leq \mathbb{E}[\ell_i] \leq \mathbb{E}[u_i] \leq \mu_i + \delta \). (Note that this definition of perturbations via couplings strictly generalizes the setting where each reward is shifted by \( \pm \delta \) prior to observation.) Recall that classic algorithms for standard stochastic bandits like UCB ([ACF02]) and SUCCESSIVEELIMINATION ([EMM06]) achieve regret \( O\left( \sum_{\Delta_i > 0} \frac{\log T}{\Delta_i} \right) \). Here, we apply a simple variant SUCCESSIVEELIMINATIONDELAYED (Algorithm 6) of SUCCESSIVEELIMINATION, first analyzed by Lancewicki, Segal, Koren, and Mansour [LSKM21] for a broader class of delays and without perturbations. Each phase of this policy pulls all arms, updates confidence intervals for reward means based on \( D \)-delayed feedback, and removes arms which are suboptimal assuming that confidence intervals are valid. Our regret bound incurs overhead \( \delta T \) from perturbations and \( D \log K \) from delays.
Algorithm 6 SuccessiveEliminationDelayed
\textbf{Input:} arm count $K$, delay $D$, error bound $\delta$
\begin{algorithmic}[1]
1: $S \leftarrow \{1, \ldots, K\}$; $t \leftarrow 1$
2: \textbf{while} $t < T$ \textbf{do}
3: \hspace{0.5em} Pull each arm $i \in S$ and observe feedback
4: \hspace{0.5em} $t \leftarrow t + |S|$
5: \hspace{0.5em} UpdateConfidenceBounds($S, t - D, \delta$)
6: $S \leftarrow \{i \in S : \text{UCB}_i \geq \text{LCB}_i \text{ for all } j \in S\}$
\end{algorithmic}

Algorithm 7 UpdateConfidenceBounds
\textbf{Input:} remaining arms $S$, time $t$, error bound $\delta$
\textbf{Output:} bounds LCB$_i$ and UCB$_i$ for $i \in S$
\begin{algorithmic}[1]
1: \textbf{for} arm $i \in S$ \textbf{do}
2: \hspace{0.5em} $n \leftarrow \max\{\sum_{r=1}^{t} 1\{i_r = i\}, 1\}$
3: \hspace{0.5em} $\mu_i \leftarrow \frac{1}{n} \sum_{r=1}^{t} 1\{i_r = i\} r_t$
4: \hspace{0.5em} LCB$_i \leftarrow \bar{\mu} - \sqrt{2 \log(T)/n - \delta}$
5: \hspace{0.5em} UCB$_i \leftarrow \bar{\mu} + \sqrt{2 \log(T)/n + \delta}$
\end{algorithmic}

Lemma 4.5. For $K$-armed stochastic bandits with $\delta$-perturbed rewards and $D$-delayed feedback, SuccessiveEliminationDelayed achieves regret $O\left(\sum_{i=0}^{T} \frac{\log T}{\Delta_i} + D \log K\right)$.

Proof sketch. In [LSKM21], a $O\left(\sum_{i=0}^{T} \frac{\log T}{\Delta_i} + D \log K\right)$ regret bound is given for this policy with unperturbed feedback. They observe that if $m$ arms remain after an iteration where an arm would have been eliminated without delays, then this arm is pulled at most $O(D/m)$ extra times before elimination (since we round robin over remaining arms). Summing over all arms, the delay overhead is at most $D \frac{K}{m} = O(D \log K)$. We prove in Appendix C.1 that at most $\delta T$ additional regret is incurred due to the potential $\delta$ inaccuracy of the confidence bounds. \hfill \Box

To apply this policy to pricing with $\varepsilon$-approximately best-responding agents, we use the following lemma, whose proof in Appendix C.2 is an immediate consequence of a standard error bound ($a \in \text{BR}_t^e(p)$ implies $a \in \text{BR}_t(p')$ with $|p' - p_t| \leq \varepsilon$) and $L$-Lipschitzness of the demand curve $d$.

Lemma 4.6. Let $\ell = \mathbb{I}\{v_t > p - \varepsilon\}$ and $u = \mathbb{I}\{v_t \geq p + \varepsilon\}$ for some round $t$ and $\varepsilon \geq 0$. If $a \in \text{BR}_t^e(p)$, then $\ell \leq a \leq u$, and, if $v_t \sim D$, then $f(p) - L \varepsilon \leq p \mathbb{E}[\ell] \leq p \mathbb{E}[u] \leq f(p) + L \varepsilon$.

Finally, we apply Proposition 2.1 to obtain a $\tilde{O}(\sqrt{T} + T\gamma)$ regret guarantee for demand learning.

Theorem 4.7. Running SuccessiveEliminationDelayed on the Section 4.1 bandits problem with $K = T^{1/4}$, $D = T\gamma \log(LT\gamma T)$, and $\delta = L\varepsilon$ incurs regret $O\left((C_2 + C_1^{-1})\sqrt{T \log T} + T\gamma \log^2(LT\gamma T)\right)$ for demand learning against $\gamma$-discounting agents with stochastic values.

Proof. By Lemma 4.6 we see that the bandits problem is $L\varepsilon$-perturbed from the myopic setting if the agent is $\varepsilon$-approximately best responding. Combining Lemmas 4.1 and 4.3 we then find that SuccessiveEliminationDelayed with $K$ arms, delay $D$, and error bound $\delta = L\varepsilon$ achieves regret

$$O\left(\sum_{i=1}^{K} \frac{\log T}{\Delta_i} + L\varepsilon T + D \log K + \frac{C_2 T}{K^2}\right)$$

for demand learning with $\varepsilon$-approximately best-responding agents. Taking $D = T\gamma \log \frac{T}{\varepsilon}$, Proposition 2.1 gives the same bound for $\gamma$-discounting agents. Controlling the first term with Lemma 4.2 and fixing $\varepsilon = (LT)^{-1}$, $K = (T/\log(T))^{1/4}$, we bound total regret by

$$O\left(\sqrt{T \log T / C_1 + T\gamma \log(LT\gamma T)} \log T + C_2 \sqrt{T \log T}\right) = O\left((C_2 + C_1^{-1})\sqrt{T \log T} + T\gamma \log^2(LT\gamma T)\right).$$ \hfill \Box

Remark 4.8 (Comparison to Prior Work). Kleinberg and Leighton [KL03] address this setting with fully myopic agents, where they obtain regret $\tilde{O}(\sqrt{T})$. Amin, Rostamizadeh, and Syed [ARS13] examine the non-myopic setting but assume a finite price set without bounding discretization error. Mohri and Muñoz Medina [MM14] establish a regret bound of $\tilde{O}(\sqrt{T} + T^{1/4})$ using a variant of UCB for a weaker class of $\varepsilon$-strategic agents. In comparison, our bound fully decomposes $T\gamma$ from polynomial dependence on $T$. 

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Remark 4.9 (Adversarial Values). [KL03] also consider values chosen by an oblivious adversary, for which they apply a learning algorithm for adversarial bandits. Although non-myopic learning is well-defined in this setting, motivation for a returning buyer to maximize discounted profit with respect to such values is lacking. Moreover, while there is a $D$-delayed adversarial bandits algorithm with regret $O((K+D) T \log K)$ [CGM19], it is unclear how to control the overhead from perturbations.

Remark 4.10 (Unknown Discount Factor). For unknown $\gamma$, we can again apply the approach of [LMP18] as in Section 3.4 to obtain matching regret $\tilde{O}(\sqrt{\gamma T} + T \gamma)$ for stochastic values, since the overhead is logarithmic in $T$. Here, each of the log $T$ threads performs arm elimination at Step 6 according to an intersection of confidence bounds over multiple threads which mirrors Step 10 of MultiThreadedClinch. For fixed values, we inherit the $O(\log (T \gamma + \log T))$ regret bound for two-target security games.

5 Strategic classification

Next, we address the strategic classification environment of Dong et al. [DRS+18], where a learner collects data from agents that may manipulate their features to obtain a desired classification outcome. In the myopic setting, [DRS+18] reduce this problem to bandit convex optimization and obtain regret $O(\sqrt{d T^{3/4}})$ via gradient descent without a gradient [FKM05]. Here, we show that this algorithm is inherently robust to the perturbations arising from approximate best responses and apply a simple delay procedure to achieve non-myopic regret $\tilde{O}(T^{1/4} \sqrt{d T^{3/4}})$ for $d$-dimensional feature vectors, compared to $O(\sqrt{dT^{3/4}})$ for the original myopic setting.

5.1 Model and preliminaries

In a single round of classification, the agent is described by a tuple $a = (x, y, d)$, where $x \in \mathbb{R}^d$ is their original feature vector, $y \in \{-1, 1\}$ is their label, and $d : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a distance function describing the cost $d(x, \hat{x})$ of changing their feature vector from $x$ to $\hat{x}$. During the round,

- the principal commits to a linear classifier parameterized by $\theta \in \Theta \subset \mathbb{R}^d$;
- the agent responds with a manipulated feature vector $\hat{x} \in \mathbb{R}^d$;
- the label $y$ is revealed to the principal.

Following [DRS+18], agent payoff is given by $v_a(\theta, \hat{x}) = \langle \theta, \hat{x} \rangle - d(\hat{x}, x)$ when $y = -1$, and, when $y = 1$, the agent is assumed to be non-strategic with $v_a(\theta, \hat{x}) = -\infty$ for $\hat{x} \neq x$. Hence, we write $\text{BR}(\theta) = \arg \max_{\theta'} v(\theta, \theta')$ when $y = -1$ and $\text{BR}(\theta) = \{x\}$ otherwise. The principal’s payoff is given by $-\ell(\theta, \hat{x}, y)$, where $\ell$ is either logistic loss $\ell_\text{log}(\theta, \hat{x}, y) = \log(1 + e^{-y(\theta, \hat{x})})$, corresponding to logistic regression, or hinge loss $\ell_h(\theta, \hat{x}, y) = \max\{0, 1 - y(\theta, \hat{x})\}$, for a support vector machine.

We consider a repeated game determined by the sequence of types $\{a_t = (x_t, y_t, d_t)\}_{t=1}^T$ for a returning agent. Define $\text{BR}_t$ and $\text{BR}^c_t$ in the usual way and let $\text{br}_t(\theta)$ denote a representative from $\text{BR}_t(\theta)$, breaking ties in favor of the principal. Denoting the game’s history by $\{(\theta_t, \hat{x}_t)\}_{t=1}^T$, the agent seeks to maximize their expected $\gamma$-discounted utility $\mathbb{E} \left[ \sum_{t=1}^T \gamma^t v_a(\theta_t, \hat{x}_t) \right]$, while the principal seeks to minimize Stackelberg regret $\mathbb{E} \left[ \sum_{t=1}^T \ell(\theta_t, \hat{x}_t, y_t) \right] - \min_{\theta \in \Theta} \mathbb{E} \left[ \sum_{t=1}^T \ell(\theta, \text{br}_t(\theta), y_t) \right]$.

Remark 5.1. To justify the single agent assumption, consider an example where the agent is a tutoring school preparing students for exams and seeks to efficiently allocate its resources to assist unprepared individuals. Alternatively, we can relax this condition so long as all participating agents are $\gamma$-discounting (since the proof of Proposition 2.1 does not require persistence of a single agent).
Regularities. We mirror the requirements of Dong et al. \([\text{DRS}^+18]\), assuming that
(i) \(\Theta\) is convex, closed, and contains the unit \(\ell_2\)-ball \(\mathcal{B}\), (ii) each feature vector \(x_t\) and classifier
\(\theta \in \Theta\) lie within \(\mathbb{R}\mathcal{B}\), and (iii) each distance function is of the form \(d_t(\hat{x}, x) = f_t(\hat{x} - x)\), where \(f_t\) is
\(\alpha\)-strongly convex and positive homogeneous of degree 2. (Strong, rather than standard, convexity is
an extra assumption used for non-myopic learning.) These imply the following, which we prove in
Appendix D.1.

**Lemma 5.2.** At any strategic round \(t\), the agent’s payoff is bounded from above by \(R^2(1 + 1/\alpha)\) and
the best response \(\text{br}_t(\theta_t)\) is unique with payoff at least \(-R^2\).

**Lemma 5.3.** Each map \(\theta \mapsto \ell(\theta, \text{br}_t(\theta), y_t)\) is convex, \((R + 2R/\alpha)\)-Lipschitz, and bounded in absolute
value by \(1 + R^2 + R^2/\alpha\). Moreover, each map \(\hat{x} \mapsto \ell((\theta, \hat{x}, y_t)\) is \(R\)-Lipschitz.

**Bandit convex optimization.** By Lemma 5.3, each loss function \(\ell_t(\theta) = \ell(\theta, \text{br}_t(\theta), y_t)\) in the
myopic setting is convex, Lipschitz, and bounded. When \(y_t = 1\), feedback \(\hat{x}_t\) is sufficient to determine
the gradient \(\nabla \ell_t(\theta_t)\), suggesting regret minimization via online convex optimization (OCO). Although
this is not the case when \(y_t = -1\), since the agent’s manipulation costs encoded by \(d_t\) are hidden, the
regime of OCO with one-point function evaluations, or bandit convex optimization, is well-studied.
Dong et al. [DRS^+18] employ the classic “gradient descent without a gradient” procedure GDwoG of
Flaxman, Kalai, and McMahan [FKM05] (Algorithm 8) to obtain regret \(O(\sqrt{dT^{3/4}})\) against myopic
agents, computing unbiased gradient estimates from stochastic function evaluations.

**Algorithm 8** Online Gradient Descent without a gradient (GDwoG) [FKM05]

**Input:** domain \(S \subset \mathbb{R}^d\) with \(\mathcal{B} \subseteq S \subseteq \mathbb{R}\mathcal{B}\), sequence of \(L\)-Lipshitz convex functions \(c_1, \ldots, c_T : S \to [-C, C]\)

1: \(\delta \leftarrow \sqrt{\frac{RC}{4L^2C^2}} T^{-1/4}\), \(\eta \leftarrow \frac{R}{C\sqrt{T}}\), \(v_1 \leftarrow (0, \ldots, 0) \in \mathbb{R}^d\)
2: for round \(t = 1, \ldots, T\) do
3: Sample unit vector \(s_t \in S^{d-1}\) uniformly at random
4: \(u_t \leftarrow v_t + \delta s_t\)
5: \(v_{t+1} \leftarrow \Pi_{(1-\delta)S}(v_t - \eta c_t(u_t)s_t)\) \(\checkmark\) \(\Pi_K(w)\) is the Euclidean projection of \(w\) onto \(K\)

**Lemma 5.4** (FKM05, Theorem 2). If functions \(c_1, \ldots, c_T : S \to [-C, C]\) are convex and \(L\)-Lipschitz, and \(S \subseteq \mathbb{R}^d\) is convex with \(\mathcal{B} \subseteq S \subseteq \mathbb{R}\mathcal{B}\), then the queries \(u_1, \ldots, u_T\) of GDwoG satisfy

\[
\mathbb{E} \left[ \sum_{t=1}^{T} c_t(u_t) \right] - \min_{u \in S} \sum_{t=1}^{T} c_t(u) \leq 6T^{3/4} \sqrt{RdC(L + C)} + 5C(Rd)^2.
\]  

**5.2 Learning with non-myopic agents**

We now extend this approach to robust learning, due to an intrinsic robustness of gradient descent
without a gradient. First, we prove the relevant error bound.

**Lemma 5.5.** If the agent chooses \(\hat{x}_t \in \text{BR}^\varepsilon(\theta_t)\), then \(|\ell(\theta_t, \hat{x}_t, y_t) - \ell(\theta_t, \text{br}_t(\theta_t), y_t)| \leq R\sqrt{2\varepsilon/\alpha}\).

**Proof.** By the strong convexity assumption, we have \(\|\hat{x}_t - \text{br}_t(\theta_t)\|_2 \leq \sqrt{2\varepsilon/\alpha}\), and so the result
follows from the Lipschitz bound \(R\) from Lemma 5.3 \(\square\)

We next provide an appropriate robust learning guarantee for GDwoG.

**Lemma 5.6.** Under the setting of Lemma 5.4, GDwoG achieves the same regret up to an additive
factor of \(\lambda RdT/\delta\) if each \(c_t(u_t)\) is substituted with an adversarial perturbation \(\tilde{c}_t(u_t) \in [c_t(u_t) \pm \lambda]\).
**Proof sketch.** The proof of Lemma 5.4 for the unperturbed setting goes through the smoothed functions $\tilde{c}_t(u) = \mathbb{E}[c_t(u + \delta s_t)]$. The key observations are that each $\tilde{c}_t$ is convex and Lipschitz with $|\tilde{c}_t(u) - c_t(u)| \leq L\delta$ and, crucially, $\mathbb{E}[\frac{d}{d\tilde{c}_t(u_t)}] = \nabla \tilde{c}_t(v_t)$. With adversarial perturbations, we have

$$
\| \mathbb{E}[\frac{d}{d\tilde{c}_t(u_t)}] - \nabla \tilde{c}_t(v_t) \|_2 = \| \mathbb{E}[\frac{d}{d\tilde{c}_t(u_t)}] - \nabla \tilde{c}_t(v_t) \|_2 \leq \frac{d\delta}{\delta},
$$

and we show that this bias translates to the claimed additive regret overhead in Appendix D.2. □

Finally, by running several copies of this procedure in parallel to achieve the needed feedback delay, we obtain a non-myopic regret bound of $\tilde{O}(T^{1/4}dT^{3/4})$ for large $T$.

**Algorithm 9** Non-myopic learning algorithm for strategic classification

1: $\varepsilon \leftarrow \alpha(R^2dT^{2.5})^{-1}$, $D \leftarrow \lceil T \log(R^2(1 + 1/\alpha)T^\gamma/\varepsilon) \rceil$
2: Initialize independent copies $\mathcal{A}_1, \ldots, \mathcal{A}_D$ of Algorithm 8 with $S = \Theta$, $C = 1 + R^2 + R^2/\alpha$, and $L = R + 2R/\alpha$
3: for round $t = 1, \ldots, T$ do
4: Write $t = D(k - 1) + (r - 1)$ for $k, r \in \mathbb{Z}_{>0}$
5: Simulate query $u_k$ and perturbed feedback $\tilde{e}_k(u_k)$ for $\mathcal{A}_r$ using $\theta_t$ and $\ell(\theta_t, \tilde{v}_t, y_t)$

**Theorem 5.7.** Algorithm 9 achieves regret $\tilde{O}(T^{1/4}dT^{3/4} + d^2)$ for strategic classification against $\gamma$-discounting agents when $R, \alpha^{-1} = \text{polylog}(T, d)$.

**Proof sketch.** Since Stackelberg regret is subadditive over disjoint sequences of rounds, we obtain regret $\text{DR}^{R\varepsilon}_{\mathcal{A}_t}([T/D])$ against $\varepsilon$-approximate best-responding agents. Combining Lemmas 5.3 and 5.6 and substituting our choices of constants, in Appendix D.3, we bound the regret of any single copy by $R^{\varepsilon}_{\mathcal{A}_t}(T) = O\left(R^{5/2}\tilde{\alpha}^{-1}\sqrt{dT}^{3/4} + R^4\tilde{\alpha}^{-2}d^2\right)$, where $\tilde{\alpha} = \min\{\alpha, 1\}$. By Proposition 2.1 and Lemma 5.2, our feedback delay induces $\varepsilon$-approximate best responses, so we obtain a final regret bound of $O\left(R^{5/2}\tilde{\alpha}^{-1}T^{1/4}\sqrt{dT}^{3/4} \log^{1/4}(TR\alpha) + R^4\tilde{\alpha}^{-2}d^2\right)$. □

**Remark 5.8.** As noted in [D'S18], GGDWG can be replaced by more modern methods with regret $\tilde{O}(\text{poly}(d)\sqrt{T})$ [BEL21], at the cost of substantial complexity and worse scaling with $d$. While beyond the scope of this paper, robustifying such algorithms would imply improved non-myopic regret bounds.

6 Finite Stackelberg games

Finally, we treat the general case of finite Stackelberg games. Although this category encompasses security games (at least those with linear utilities induced by the standard combinatorial setup), the techniques of Section 3 do not extend clearly to this setting. Hence, we aim here to provide more restricted guarantees for a wider class of games. When the principal has $m$ actions available, we apply methods from robust convex optimization with membership queries to achieve regret $\tilde{O}(T\gamma(V^{-1}\sqrt{m} + T\gamma m^3))$ against $\gamma$-discounting agents, where $V$ is the volume of a ball contained within a certain best response region.

6.1 Model and preliminaries

Let $(X_0, Y, u_0, v_0)$ be a base Stackelberg game where $X_0 = \{1, \ldots, m+1\} = [m+1]$ and $Y = [n]$ are finite sets of pure strategies for the principal and agent, respectively, and $u_0, v_0$ are arbitrary. We consider the mixed strategy game $(X, Y, u, v)$ where the principal commits to a distribution $x \in X = \Delta_{m+1}$, the agent responds with an action $y \in Y$, and expected payoffs are given by $u(x, y) = \mathbb{E}_{i \sim X}[u_0(i, y)]$. 21
and \( v(x, y) = \mathbb{E}_{i \sim x} [v_0(i, y)] \). We define best response functions \( \text{BR}, \text{BR}^\varepsilon, \text{br} \) as well as Stackelberg regret for the corresponding repeated game in the usual way. Finally, we write \( B(A, r) \subset \mathbb{R}^m \) for the Minkowski sum of a set or point \( A \) in \( \mathbb{R}^m \) with the \( L_2 \) ball of radius \( r \), with \( B(r) = B(0_m, r) \), and, for a set \( A \subset \mathbb{R}^m \), write \( B(A, -r) = \{ x \in A : B(x, r) \subset A \} \). For convenience, we identify \( \mathcal{X} \) with its isometric embedding into the ball \( B(\sqrt{2}) \subset \mathbb{R}^m \) (this poses no difficulties as our analysis is coordinate-free).

**Multiple LPs.** We apply the “multiple LPs” paradigm of Conitzer and Sandholm [CS06], previously outlined in Section 3.3 for SSGs. Writing \( K_y := \{ x \in \mathcal{X} : y \in \text{BR}(x) \} = \{ x \in \mathcal{X} : v(x, y) \geq v(x, y') \forall y' \in \mathcal{Y} \} \) for the best response polytope associated with action \( y \in \mathcal{Y} \), the principal’s utility in Stackelberg equilibrium is given by \( \max_{x \in \mathcal{X}} u(x, \text{br}(x)) = \max_{y \in \mathcal{Y}} \max_{x \in K_y} u(x, y) \). For fixed \( y \), the objective \( u(x, y) \) is linear in \( x \), so previous works find an optimal strategy by solving the \( n \) outer LPs, originally for known \( v \) [CS06] and later for unknown \( v \) via queries to a best response oracle [LCM09].

**Approximate best-response regions and comparing sets.** To analyze inexact feedback, we define the \( \varepsilon \)-approximate best response polytope for \( y \in \mathcal{Y} \) and \( \varepsilon \in \mathbb{R} \) by \( K_\varepsilon := \{ x \in \mathcal{X} : y \in \text{BR}^\varepsilon(x) \} \). Observe that \( K_\varepsilon \subseteq K_\varepsilon' \) for \( \varepsilon \leq \varepsilon' \). Against an \( \varepsilon \)-approximate best-responding agent, the principal always receives feedback \( y \in \text{BR}^\varepsilon(x) \) for a query \( x \), and they can ensure that a desired target \( y' \) is attacked by querying \( x \in K_{\varepsilon'} \) for \( \varepsilon' > \varepsilon \). To compare such regions, we consider the \( L_2 \)-Hausdorff distance defined between sets \( A, B \subset \mathbb{R}^d \) by \( d_H(A, B) := \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \} \), where \( d(z, S) := \inf_{x \in S} \| z - x \|_2 \) is the distance between a point and set.

**Convex optimization with membership queries.** One classic algorithm for SSGs [BHP14] treats best response feedback \( I \{ y \in \text{BR}(x) \} \) as a membership oracle for the best response region \( K_y \) and performs linear optimization with membership queries via simulated annealing [KV06]. For our problem, we employ an algorithm LSV of Lee, Sidford, and Vempala [LSV18] for robust convex optimization with membership feedback. Given access to evaluations for a convex function \( f \) and approximate membership queries for a convex set \( K \subset \mathbb{R}^m \) with well-centered initial point, LSV approximately optimizes \( f \) over \( K \) using \( \tilde{O}(m^2 \log \frac{1}{\varepsilon}) \) queries.

**Regularity assumptions.** To learn with non-myopic agents, we assume that there exist optimal \( x^* = \arg\max_{x \in \mathcal{X}} u(x, \text{br}(x)) \) and \( y^* = \arg\max_{y \in \text{BR}(x^*)} u(x, y) \) such that (i) \( K_{y^*} \) contains an \( L_2 \) ball of radius \( 2 \varepsilon > 0 \), and (ii) for all \( \varepsilon \in \mathbb{R} \) with \( K_{y^*} \neq \emptyset \), we have \( d_H(K_{y^*}, K_y) \leq L |\varepsilon| \) for some \( L > 0 \). Condition (i) guarantees that the best response polytope of interest is substantial enough to be found via sampling, and (ii) ensures that the polytope is stable under approximate best responses, with \( L \) describing a “condition number” that has been studied for general polytopes [Li93]. As given, these requirements are rather mild, with the first assumed even in the fully myopic setting [BHP14]. On the other hand, useful bounds on the \( r \) and \( L \) may be difficult to obtain, though we describe a strategy for controlling \( L \) via standard polytope perturbation bounds in Appendix E.1.

### 6.2 Our extension to non-myopic agents

We consider learning with \( \varepsilon \)-approximate best responses, taking an approach based on convex optimization with membership queries. Fixing \( y \in \mathcal{Y} \), checking whether the response for a query \( x \in \mathcal{X} \) equals \( y \) now acts as approximate membership feedback for \( K_y \) (with formal accuracy guarantees available when \( y = y^* \) by conditioning). Unfortunately, if we employ LSV with this oracle directly and obtain a strategy \( \hat{x} \) too close to the boundary of \( K_y \), we have no way of knowing that the agent will still respond with \( y \) if \( \hat{x} \) is replayed. To remedy this, we apply a sampling-based approach to simulate a sufficiently conservative membership oracle for \( K_{y^*} \).
Equipped with such oracles, we provide a robust learning policy \textit{NoisyStack}, formally defined in Appendix E.3. We begin by randomly sampling a large set of points so that one is well-centered in \( K_y \) with high probability. Next, for each point, we run LSV with our conservative membership oracle to maximize \( u(\cdot, y) \) over \( K_y \). Then, we query the approximate maximizer which appears optimal for the remaining rounds. If unexpected feedback is ever received, can safely remove the current strategy from consideration and repeat. This argument is formalized in Appendix E.3.

**Theorem 6.1.** \textit{NoisyStack} admits regret \( O(V^{-1}\sqrt{m}\log^2(T)+m^3\log^O(1) \frac{LTm}{r}) \) against \( \varepsilon \)-approximately best-responding agents with \( \varepsilon = (\frac{2r}{m})^{O(1)} \), where \( V_r \) is the volume of the ball of radius \( r \) in \( \mathbb{R}^m \). Thus, there exists a principal policy with regret \( \tilde{O}(T\gamma(V^{-1}\sqrt{m}+T\gamma m^3)\log^O(1) L) \) against \( \gamma \)-discounting agents.

**Remark 6.2.** We note that our bound scales linearly with the inverse volume \( V^{-1} \). Although the algorithms of Blum, Haghtalab, and Procaccia [BHP14] and Peng, Shen, Tang, and Zuo [PSTZ19] exhibit reduced \( \log(V^{-1}) \) dependence, the former work is specific to security games and the latter appears to rely on exact feedback (and is only efficient under additional structural assumptions).

### 7 Conclusion

In this work, we developed a framework for learning in Stackelberg games with non-myopic agents, reducing the original learning problem to that of designing minimally reactive and robust bandit algorithms. For each of our application settings—SSGs, demand learning, strategic classification, and finite Stackelberg games, we then identified a robust and delayed learning policy that obtained low Stackelberg regret against \( \gamma \)-discounting agents. Our work opens up several interesting avenues of future research, three of which we now highlight.

Our first question regards the optimality and generality of our reduction framework. While we primarily considered the feedback-delay information screen to inspire new algorithm design principles, other forms of information screens may be also effective for learning in the presence of non-myopic agents. Examples of such screens may include using random delays, delays that involve non-monotone release of information, or differentially private information screens. Are these information screens inherently different in the power they provide algorithms? In particular, are there settings for which algorithms that conform to one of these channels outperform those conforming to others? Are there universally optimal information screens that achieve optimal regret for a wide range of discount factors across all principal-agent games?

Second, we ask whether a regret overhead of \( T\gamma \log T \) can be avoided when the discount factor \( \gamma \) is unknown. While our algorithms achieve an additive dependence on \( T\gamma \) for SSGs and demand learning, our extensions to their agnostic analogues have a multiplicative dependence arising from our multi-threading approach (see, e.g., Theorem 3.14). Is this gap necessary to derive \( \gamma \)-agnostic algorithms for these settings, or can one achieve an additive dependence with different methods?

Our last question concerns non-myopic agent learning beyond repeated game settings. Specifically, many repeated interactions with agents do not fit the framework of a repeated game, e.g., our demand learning setting when the principal (the seller) has a limited inventory of items that they may distribute over the \( T \) rounds [BZ09]. Such learning settings can be modeled as bandits with knapsacks [BKS18], in which exploration time is not the only limited resource. More broadly, learning in the context of state and repeated interaction is of interest in reinforcement learning. In such settings—in which state is carried over between interactions, what principles apply to the design of effective learning algorithm?
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We assume histories, and assume that any subsequence of a feasible history is also feasible.\[\gamma\]

The principal’s policy is \[B^\text{BR}\setminus \gamma\text{-discounting agent satisfies}B(H_{t-1}, x_t) \in BR^\gamma(x_t)\text{ for any pair } (H_{t-1}, x_t)\text{ that occurs with positive probability, where } \varepsilon = \frac{1}{1-\gamma}\gamma^D.\] If \[B(H_{t-1}, x_t) \notin BR^\gamma(x_t)\text{ for a pair } (H_{t-1}, x_t)\text{ that occurs with positive probability, we can construct a modified agent policy } B’\text{ with strictly higher expected payoff. Define } B’\text{ so that } B’(H_{t-1}, x_t) \in BR(x_t)\text{ and } B’(H’, x’’) = B(H’, x’)\text{ for all other pairs } (H’, x’’).\]

Conditioned on history \[H_{t-1},\text{ a } D\text{-delayed policy } A\text{ plays the same sequence of actions } x_{t+1}, \ldots, x_{t+D-1}\text{ under both } B\text{ and } B’.\] Therefore, conditioned on playing history \[H_{t-1}\text{ and observing action } x_t,\text{ the agent loses at most } \sum_{j=0}^{D-1} \gamma^j = \gamma^{D+1}/(1-\gamma)\text{ in discounted future payoff by switching to } B’\text{ because the principal’s policy is } D\text{-delayed. Moreover, the agent gains more than } \gamma^D\varepsilon = \gamma^D\varepsilon/(1-\gamma)\text{ payoff at time } t.\text{ Thus, switching from } B\text{ to } B’\text{ yields a strictly positive gain in expectation for the agent.} \]

### A.2 A batch-delay equivalence (proof of Proposition 2.2)

Before proving the result, we define a general framework for bandit problems that generalizes the Stackelberg setting. We consider bandit problems over \[T\text{ rounds. An } abstract bandit problem \text{ is defined by a tuple } (X, Y, r), \text{ where } X\text{ is the principal’s action set, } Y\text{ describes possible unknown states, and } r\text{ is a regret function } r: \mathcal{H} \to \mathbb{R}\text{ mapping the set } \mathcal{H} := \bigcup_{t \geq 0} (X \times Y)^t\text{ of histories to regret values. We assume } r\text{ is } subadditive: if a history } H \in \mathcal{H}\text{ is partitioned into two complementary subsequences } H’, H” \in \mathcal{H}, \text{ then } r(H) \leq r(H’) + r(H’’).\]

We further distinguish a subset \[\mathcal{H}^\mathcal{F}\subseteq \mathcal{H}\text{ of feasible histories, and assume that any subsequence of a feasible history is also feasible.}\]

The principal’s subadditivity is satisfied by common notions of regret: in stochastic settings, regret is simply the sum of regrets over individual rounds; in adversarial settings, regret is subadditive.

In our setting, feasible histories correspond to restrictions on the agent’s behavior, e.g., the agent plays an \(\varepsilon\)-approximate best response at each round.

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[^1]: Subadditivity is satisfied by common notions of regret: in stochastic settings, regret is simply the sum of regrets over individual rounds; in adversarial settings, regret is subadditive.

[^2]: In our setting, feasible histories correspond to restrictions on the agent’s behavior, e.g., the agent plays an \(\varepsilon\)-approximate best response at each round.
policy is a map \( \mathcal{A}: \mathcal{H} \to \mathcal{X} \). During the \( t \)-th round, the principal plays action \( x_t = \mathcal{A}(H_{t-1}) \) (where \( H_{t-1} \) is the history up to the start of round \( t \)) and observes \( y_t \) (which may be chosen randomly and adaptively based on \( x_t \)). We say that \( \mathcal{A} \) satisfies the regret bound \( R_\mathcal{A}(T) \) if, for each history \( H \in \mathcal{H}^* \) of length \( T \) such that \( x_t = \mathcal{A}(H_{t-1}) \), then \( r(H) \leq R_\mathcal{A}(T) \).

Given any abstract bandit problem \((\mathcal{X}, \mathcal{Y}, r)\), we define learning with delayed feedback and batched queries as follows. As before, \( \mathcal{A} \) is \( D\)-delayed if \( \mathcal{A}(H_{t-1}) \) depends only on the prefix \( H_{t-D} \), and \( \mathcal{A} \) is \( B\)-batched if \( \mathcal{A}(H_{t-1}) \) depends only on the prefix \( H_{B(t-1)/B} \).

To cast our principal-agent learning setting as an abstract bandit problem, we let \( \mathcal{X} \) be the set of agent actions, \( \mathcal{Y} \) be the set of agent actions, and regret be Stackelberg regret. Note that Stackelberg regret \((\Pi)\) is subadditive because \( \max_x (f(x) + g(x)) \leq \max_x (f(x)) + \max_x (g(x)) \). Finally, we take the set of feasible histories to be those where the agent policy belongs to a class \( \mathcal{B} \).

We now present a proof of the batch-delay equivalence (Proposition 2.2), which relies on the following lemma. It states that a 1-delayed policy \( \mathcal{A} \) can be converted into a 2-delayed policy by instantiating two independent copies of \( \mathcal{A} \) and following them on alternating rounds.

Lemma A.1. Let \( \mathcal{A} \) be a policy with regret bound \( R_\mathcal{A} \). Consider the policy \( \mathcal{A}' \) that instantiates two independent copies \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) of \( \mathcal{A} \), and on round \( t \) plays \( x_t = \mathcal{A}_i \left( (x_{t-1}, y_{t-1}) \right) \mod 2 \), where \( t \equiv r \mod 2 \). Then \( \mathcal{A}' \) is 2-delayed and satisfies a regret bound of \( R_\mathcal{A}(T) \leq 2R_\mathcal{A}(T) \).

Proof. To bound regret, note that \( \mathcal{A}_0 \) is run on the history \( ((x_{t'}, y_{t'}))_{t' \leq t} \mod 2 \), which is by definition feasible. Thus, it incurs a total of \( R_\mathcal{A}(T/2) \) regret on this subsequence of the actual history. Likewise, \( \mathcal{A}_1 \) incurs at most \( R_\mathcal{A}(T/2) \) regret as well. Therefore, by the subadditivity axiom, the total regret is at most \( 2R_\mathcal{A}(T/2) \leq 2R_\mathcal{A}(T) \), since regret is monotonic in the delay length. The lemma now follows, since by definition, \( \mathcal{A}' \) is 2-delayed. \( \square \)

Proof of Proposition 2.2. The first claim follows from the definition of a batched algorithm, since \( t - D \leq D \lfloor (t - 1)/D \rfloor \). The second claim follows from an application of Lemma A.1 to the “B-batched” \((\mathcal{X}^B, \mathcal{Y}^B, r^B)\) bandit problem, where \( \mathcal{X}^B \) and \( \mathcal{Y}^B \) be are the \( B \)-fold products of \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, and \( r^B \) is given by evaluating \( r \) on the history given by the concatenations of actions \( ((x_1, \ldots, x_B), (y_1, \ldots, y_B)) \). That is, in this new bandit problem, the principal simply chooses \( B \)-tuples of actions and receives feedback on these \( B \)-tuples at once, with regret measured according to the original \((\mathcal{X}, \mathcal{Y}, r)\) bandit problem. This new problem is by definition equivalent to our \( B \)-batched bandit problem defined above. By Lemma A.1, an algorithm \( \mathcal{A} \) for this equivalent problem can be converted into an algorithm \( \mathcal{A}' \) for the 2-delayed version of this problem achieving \( 2R_\mathcal{A}(T) \) regret. Forgetting about the batch structure, we see that this algorithm \( \mathcal{A}' \) is \( B \)-delayed, since any batch starting at time \( t = kB + 1 \) depends on the history \( H_{<k(B-1)+1} \). Hence the second claim is proven. \( \square \)

B Supplementary material for for Section 3

B.1 Simplify design & analysis of Clinch when \( \mathcal{X} = \Delta_n \)

When \( \Delta_n^L := \{ x : \|x\|_1 \leq 1 \land x_y \geq 0 \forall y \} \), we may as well restrict to \( \mathcal{X} = \Delta_n \). Indeed, since each agent utility function \( v^y \) is continuous and strictly decreasing, we can increase coverage probabilities of any \( x \in \mathcal{X} \setminus \Delta_n \) to obtain \( x' \in \Delta_n \), with \( \text{BR}(x') = \text{BR}(x) \) and \( v(x', br(x')) < v(x', br(x')) \). This argument also implies that the optimal stable strategy \( x^* \) guaranteed by Proposition 3.5 belongs to \( \Delta_n \). From now on, we thus fix \( \mathcal{X} = \Delta_n \).

For this setting, we present a simplified (simplexified?) algorithm Clinch.Simplex that achieves the same query complexity as Theorem 3.9 but admits a simpler analysis. Similarly to Clinch, Clinch.Simplex maintains an (approximate) entry-wise lower bound \( \gamma \) for \( x^* \) initialized to the 0
vector. This time, however, we envision the remaining mass $1 - \|\tilde{x}\|_1$ as a potential that is decreased with each step. To ensure a significant reduction, we query $x$ which distributes this remaining mass evenly across the coordinates of $\tilde{x}$ and update $\tilde{x}_y \leftarrow x_y$ for the attacked target $y \in Y$. Finally, we normalize $\tilde{x}$ so that it lies on the simplex and apply the same perturbation used by CLINCH.

**Proposition B.1.** Fix $0 < \lambda \leq 1$. Then CLINCH.SIMPLEX returns a $\lambda$-approximate equilibrium strategy using $O(n \log \frac{C}{W\lambda})$ queries to an $\varepsilon$-approximate best response oracle with $\varepsilon \leq \frac{W\lambda}{12C^2n}$.

**Algorithm 10** CLINCH.SIMPLEX: an robust algorithm for learning SSGs when $X = \Delta_n$

| Input: | target accuracy $\lambda \in (0, 1]$, best response oracle $\text{ORACLE}$ with $\text{ORACLE}(x) \in BR^c(x)$ |
| Output: | $\lambda$-approximate equilibrium strategy |

1. $\tilde{x} \leftarrow (0, 0, \ldots, 0) \in \mathbb{R}^n$, $\delta \leftarrow \frac{W\lambda}{6C^2}$
2. for $i = 1, 2, \ldots, \lfloor n \ln \frac{1}{\delta} \rfloor$ do
3. $x \leftarrow \tilde{x} + (1, 1, \ldots, 1) \cdot \frac{1}{n} (1 - \|\tilde{x}\|_1)$
4. $y \leftarrow \text{ORACLE}(x)$
5. $\tilde{x}_y \leftarrow x_y$
6. $\hat{x} \leftarrow \tilde{x}/\|\tilde{x}\|_1$
7. $\hat{y} \leftarrow \max_{y \in Y} \arg \max_{x, y} u(\hat{x}, y)$
8. return $\hat{x} - \frac{W\lambda}{2} e_y$

**Proof.** First, we note that $\tilde{x}_y \leq x^*_y + Ce$ for each $y \in Y$, by the same argument applied in the proof of Theorem 3.9. Next, we analyze convergence. Notice that the quantity $1 - \|\tilde{x}\|_1$ decreases by a factor of $1 - \frac{1}{n}$ after each iteration. Since $1 - \|\tilde{x}\|_1$ is initially 1, after $\lfloor n \ln \frac{1}{\delta} \rfloor$ iterations, it holds that

$$1 - \|\tilde{x}\|_1 \leq \left(1 - \frac{1}{n}\right)^{\lfloor n \ln \frac{1}{\delta} \rfloor} \leq \frac{\delta}{4},$$

We may thus conclude, for $\hat{x} := \tilde{x}/\|\tilde{x}\|_1$, that

$$\|\hat{x} - x^*\|_\infty \leq \|\hat{x} - x^*\|_1 \leq \left(\|\tilde{x}\|_1^{-1} - 1\right) \|\tilde{x}\|_1 + \|\tilde{x} - x^*\|_1 \leq 1 - \|\tilde{x}\|_1 + \|\tilde{x} - x^*\|_1,$$  

by the triangle inequality. By the entry-wise lower bound property of $\tilde{x}$, we further note that

$$\|\tilde{x} - x^*\|_1 \leq 1 - \|\tilde{x}\|_1 + Cn \varepsilon \leq 1 - \|\tilde{x}\|_1 + \frac{\delta}{2},$$

and so $\|\hat{x} - x^*\|_\infty \leq \delta$. Finally, Lemma 3.12 gives that the returned point is a $\lambda$-approximate Stackelberg equilibrium strategy, as desired. \qed

**B.2 Minimizing agent best response utility (proof of Lemma 3.7)**

To show that CLINCH makes continual progress, we require an approximate version of Grünbaum’s inequality [Gri60]. First, we state the classic result.

**Lemma B.2** (Theorem 2 in [Gri60]). If $K$ is a non-empty compact convex set in $\mathbb{R}^d$, then for any halfspace $H$ containing its centroid $x_0 = \mathbb{E}_{x \sim \text{Unif}(K)}[x]$, we have $\text{vol}_d(H \cap K) \geq \frac{1}{e} \text{vol}_d(K)$.

For our approximate case, we use the following, which implies that each update to $\tilde{x}$ will sufficiently shrink the volume of the active search region $S$ (so long as no target is removed from $\mathcal{R}$).
Lemma B.3. Let $K \subseteq [0, 1]^d$ be convex and downward closed with centroid $x = \mathbb{E}_{z \sim \text{Unif}(K)}[z]$, and write $\alpha = \sup_{z \in S} z_1$. Then for $\beta \geq 0$, we have $\text{vol}_d(\{z \in K : z_1 \geq 1-x_1 - \beta\}) < (1 + e\beta/\alpha)^d (1-1/e) \text{vol}_d(K)$.

Proof of Lemma [B.3]. For convenience, assume that $K$ is closed; this does not affect the volumes. Similarly replace $\beta$ with $\min\{\beta, x_1\}$ so that $x_1 - \beta \geq 0$. Proving the result with this update implies the original result.

Now define the halfspace $H := \{z \in \mathbb{R}^d : z_1 \geq 1-x_1 - \beta\}$. First, we observe that the related halfspace $H_0 = \{z \in \mathbb{R}^d : z_1 \geq x_1\}$ satisfies $\frac{1}{e} \text{vol}_d(K) \leq \text{vol}_d(H_0 \cap K) < (1 - \frac{1}{e}) \text{vol}_d(K)$ by Grünbaum’s inequality (Lemma B.2). By downward closure, we have $\alpha e_1 \in K$ and deduce that $\text{vol}_d(H_0 \cap K) \leq (1 - \frac{1}{e}) \text{vol}_d(K)$. This requires that $x_1 \leq (1 - \frac{1}{e}) \alpha$ to avoid violating the first inequality.

Next, consider the $(d-1)$-dimensional intersection of $K$ with the hyperplane defining $H_0$, denoted by $L_0 := \{z \in K : z_1 = x_1\}$ for $H_0$. Convexity requires that $H_0 \cap K$ contain the convex hull of $L_0$ and $\alpha e_1$, a cone we denote by $A$ with $\text{vol}_d(A) = \frac{\alpha - x_1}{d-1} \text{vol}_{d-1}(L_0)$. Moreover, every point in $H \cap K \setminus H_0$ is outside of $A$ and connected to $\alpha e_1$ by a line segment contained in $H$ and passing through $L_0$. Thus, $H \cap K \setminus H_0$ is disjoint from the cone $A$ but contained by the cone $B$ obtained by intersecting $H$ with the union of all rays emitted from $\alpha e_1$ and passing through $K_0$. Similarly to $A$, we compute the volume of $B$ to be $\frac{\alpha - x_1 + \beta}{d-1} \frac{\alpha - x_1 + \beta}{\alpha - x_1} \text{vol}_{d-1}(L_0)$. Consequently, we have

$$\text{vol}_d(H \cap K \setminus H_0) \leq \text{vol}_d(B) - \text{vol}_d(A)$$

$$= \left[\frac{\alpha - x_1 + \beta}{\alpha - x_1}\right]^{d-1} \frac{\alpha - x_1}{d} \text{vol}_{d-1}(L_0)$$

$$\leq \left[\frac{\alpha - x_1 + \beta}{\alpha - x_1}\right]^{d-1} \text{vol}_d(H_0 \cap K)$$

$$= \left[1 + \frac{\beta}{\alpha - x_1}\right]^{d-1} \text{vol}_d(H_0 \cap K)$$

Finally, we can bound

$$\text{vol}_d(H \cap K) = \text{vol}_d(H_0 \cap K) + \text{vol}_d(H \cap K \setminus H_0)$$

$$\leq \left[1 + \frac{e\beta}{\alpha}\right]^d \text{vol}_d(H_0 \cap K)$$

$$< \left[1 + \frac{e\beta}{\alpha}\right]^d \left(1 - \frac{1}{e}\right) \text{vol}_d(K),$$

as desired. \hfill \Box

We now prove the guarantee for the primary stage of CLINCH.

Proof of Lemma [3.7]. First, we observe that $x$ always approximately lower bounds $x^*$ in each entry. Note that whenever $x'$ gets updated, we set $x_y = x_y - C\varepsilon$ for some $x$ such that $y \in \text{BR}^\varepsilon(x)$. By monotonicity of $v^y$ and our slope bound, this implies $x^*_y \geq x_y - C\varepsilon = x^*_y$, as desired.

Next, we will show that the termination condition at Step 2 is satisfied after at most $O(n \log \frac{n}{\delta})$ rounds, recalling that $0 < \delta \leq 1$ is our desired accuracy. To start, we establish a bit of notation to keep track of variables between iterations. For each round $i = 1, 2, \ldots$ before termination, we write

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\( \mathcal{R}_i \) for the remaining targets and \( S_i \) for the active search region after Step 5. \( x^{(i)} \) for the queried point at Step 6, \( y_i \) for the oracle response at Step 7, and \( \hat{x}^{(i)} \) for the value of \( x \) after Step 8. Set \( n_i = \dim(S_i) \), defined as the minimum dimension of the subspace spanned by \( S_i \). Finally, write \( \lambda = \frac{\delta}{12C^2\epsilon n} \) for the threshold used to flatten \( S \). Now, we fix ourselves at some round \( i \) and consider two cases.

**Case 1:** \( \mathcal{R}_{i+1} = \mathcal{R}_i \). In this case, no targets are removed from \( \mathcal{R}_i \) and \( n_{i+1} = n_i \). Since we selected \( x^{(i)} \) as the centroid of \( S_i \), we can apply Lemma 13.3. Indeed, \( S_i \) is convex, and its translation \( K = S_i - \hat{x}^{(i-1)} \) is downward closed. Moreover, \( \sup_{z \in K} z y_i = \sup_{z \in S_i} z y_i - x^{(i-1)} \geq \lambda \) (otherwise, the target \( y_i \) would have been removed from \( \mathcal{R}_i \) to obtain \( \mathcal{R}_{i+1} \)). Consequently, we have

\[
\vol_{n_{i+1}}(S_{i+1}) = \vol_{n_i}(S_{i+1}) = \vol_{n_i}\left(\left\{ z \in S_i : z y_i \geq x^{(i)} y_i - C \epsilon \right\}\right) \\
\leq \left(1 + \epsilon \cdot \frac{C \epsilon}{\lambda}\right)^n \left(1 - \frac{1}{e}\right) \vol_d(S_i) \\
\leq \epsilon^{1/3} \left(1 - \frac{1}{e}\right) \vol_d(S_i) < \frac{9}{10} \vol_d(S_i),
\]

where the penultimate inequality uses that \( \epsilon \leq \frac{\lambda}{3Cen} = \frac{\delta}{12C^2\epsilon n} \).

**Case 2:** \( \mathcal{R}_{i+1} \subset \mathcal{R}_i \). In this case, \( n_{i+1} - n_i > 0 \) targets are removed from \( \mathcal{R}_i \) in the next step. Writing \( K_i = \{ x' \in S_i : x' = \hat{x}^{(i-1)} + e \} \forall z \notin \mathcal{R}_{i+1} \} \) for the region which enforces the locked coordinates for the next step — but not the updated lower envelope — we (loosely) bound

\[
\vol_{n_{i+1}}(S_{i+1}) \leq \vol_{n_{i+1}}(K_i) \leq \left( \frac{n}{\lambda} \right)^{n_{i+1} - n_i} \vol_{n_i}(S_i).
\]

The first inequality uses that \( S_{i+1} \subseteq K_i \). For the second, convexity requires that \( S_i \) contains the convex hull of \( K_i \) and the points \( \{ \hat{x}^{(i-1)} + e z : z \in \mathcal{R}_i \setminus \mathcal{R}_{i+1} \} \), which has volume loosely bounded from below by \( (\lambda/n)^{n_{i+1} - n_i} \vol_{n_{i+1}}(K_i) \).

Combining these cases inductively, we deduce that

\[
\vol_{n_i}(S_i) < \left( \frac{n}{\lambda} \right)^n \left( \frac{\epsilon}{10} \right)^{i-n} \alpha^n.
\]  

On the other hand, once \( \vol_{n_i}(S_i) < \lambda^n/n! \), every coordinate must have slack less than \( \lambda \), and the termination condition at Step 2 will be satisfied. Consequently, we compute that the outer loop must terminate after at most \( 15n \log \frac{2\alpha n}{\lambda} \leq 15n \log \frac{8C^2\alpha}{\delta \alpha} \) iterations. At this point, we have \( y \in BR^e(x) \) for \( x \in X \) with \( x \geq x \) and either \( x + e \notin X \) or \( x \geq x + \lambda > \overline{x}_y \). In the former case, downward closure of \( X \) implies \( x^*_\lambda \leq \overline{x}_y + \lambda \leq x + \lambda \), and the same relations hold for the latter, since \( x^*_\lambda \leq \overline{x}_y \) at this point from the input guarantee. Hence, we obtain

\[
v(x^*, \br(x^*)) \geq v^3(x^*_\lambda) \geq v^3(\overline{x}_y + \lambda) \\
\geq v^3(\overline{x}_y) - CA \\
\geq v(x, \br(x)) - 2CA \\
= v(x, \br(x)) - \frac{\delta}{2C}.
\]

\[\square\]
B.3 Mass conservation (proof of Lemma 3.8)

Proof of Lemma 3.8. Fixing \( y \in \mathcal{Y} \), define the thresholds

\[
\begin{align*}
    r_y &= \sup \left\{ p \in [\xi_y, x_y] : \text{BR}^{1/C}(x + [p - x_y]e_y) = \{y\} \right\}, \\
    s_y &= \sup \left\{ p \in [\xi_y, x_y] : \text{BR}(x + [p - x_y]e_y) = \{y\} \right\}, \\
    t_y &= \sup \left\{ p \in [\xi_y, x_y] : y \in \text{BR}^{1/C}(x + [p - x_y]e_y) \right\},
\end{align*}
\]

where we define each to be \( \xi_y \) if the corresponding set is empty. (Note that the set for \( s_y \) can contain at most one point by strict monotonicity of \( v^y \).) By monotonicity of \( v^y \) and our slope bound, we have \( s_y - \lambda \leq r_y \leq s_y \leq t_y \leq s_y + \lambda \). By our choice of binary search, either \( \hat{x}_y > x_y - \lambda \), or \( \hat{x}_y > m - \lambda \) at some iteration for which \( \text{ORACLE}(x - [x_y - m]e_y) \neq y \). In the latter case, monotonicity of \( v^y \) and the slope bound require that \( \hat{x}_y \geq r_y \), while, for the former, we have \( \hat{x}_y > r_y - \lambda \). Similarly, either \( \hat{x}_y = \xi_y \), or \( \hat{x}_y < m \) for a search iteration during which \( \text{ORACLE}(x - [x_y - m]e_y) = y \). In the latter case, monotonicity of \( v^y \) and the slope bound require that \( \hat{x}_y < t_y \), while, for the former, we have \( \hat{x}_y \leq t_y \). Combining, we have that \( \hat{x}_y \in (s_y - 2\lambda, s_y + \lambda) \).

By definition of \( s_y \), we must have \( v^y(s_y) \leq v(x, \text{br}(x)) \) (with equality unless \( s_y = \xi_y \)). Hence, \( v^y(\hat{x}_y) < v^y(s_y) + 2C\lambda \leq v(x, \text{br}(x)) + 2C\lambda \). Since this holds for all \( y \in \mathcal{Y} \), we have \( v(\hat{x}, \text{br}(\hat{x})) \leq v(x, \text{br}(x)) + 2C\lambda \), proving the second part of the claim. Now, if \( \hat{x}_y > \xi_y \), we must have \( t_y > \hat{x}_y > \xi_y \), and so \( y \in \text{BR}^{1/C}(x + [\hat{x}_y - x_y]e_y) \). The previous result then implies that \( y \in \text{BR}^{1/C + 2C\lambda}(x + [\hat{x}_y - x_y]e_y) \). We bound \( \lambda/C + 2C\delta \leq 3C\lambda \) for conciseness, and note that each binary search use \( O(\log \frac{d \lambda}{\Delta}) \) queries. \( \square \)

B.4 Perturbing estimates of \( x^* \) (proof of Lemma 3.12)

Proof of Lemma 3.12. The error bound implies that \( \hat{x}_y > W/2 \) only if \( x^*_y > 0 \). On the other hand, if \( x^*_y > 0 \), then our regularity width assumption requires that \( x^*_y \geq W \), and so \( \hat{x}_y \geq W - \frac{W\lambda}{6C^2} > W/2 \). As noted in the proof of Proposition 3.5, there are no \( y \in \text{BR}(x^*) \) with \( x^*_y = 0 \), and so \( \text{BR}(x^*) = \{y \in \mathcal{Y} : x^*_y > 0 \} = \{y \in \mathcal{Y} : \hat{x}_y > W/2 \} \).

Now fix \( \hat{y} \) as defined in Step 8, and consider the returned strategy \( \hat{x} := \hat{x} - \frac{W\lambda}{6C^2}e_{\hat{y}} \). We know that \( \hat{x} \in X \) since \( \hat{x}_y > W/2 \) and \( X \) is downward closed. We claim that \( \text{BR}(\hat{x}) = \{\hat{y}\} \). Indeed, we have \( \hat{x}_y < x^*_y - \left(\frac{W\lambda}{2} - \frac{W\lambda}{6C^2}\right) \leq x^*_y - W\lambda/3 \), and so our lower slope bound requires that

\[
v^y(\hat{x}_y) > v^y(x^*_y) + \frac{W\lambda}{3C} = v(x^*, \text{br}(x^*)) + \frac{W\lambda}{3C}.
\]

For \( y \neq \hat{y} \), we have \( \hat{x}_y > x^*_y - \frac{W\lambda}{6C^2} \), and so our upper slope bound requires that

\[
v^y(\hat{x}_y) < v^y(x^*_y) + \frac{W\lambda}{6C^2} \leq v(x^*, \text{br}(x^*)) + \frac{W\lambda}{6C} \leq v(x^*, \text{br}(x^*)) + \frac{W\lambda}{3C} - \varepsilon.
\]

Consequently, we have \( \text{BR}^x(\hat{x}) = \{\hat{y}\} \). Finally, we compute

\[
u(\hat{x}, \text{br}(\hat{x})) = u^y(\hat{x}_y)
\geq u^y(x^*_y) - C|x^*_y - \hat{x}_y|
\geq u(x^*, \text{br}(x^*)) - \frac{W\lambda}{6C}
> u(x^*, \text{br}(x^*)) - \lambda,
\]

verifying that \( \hat{x} \) is indeed a \( \lambda \)-approximate Stackelberg equilibrium strategy for the principal. \( \square \)
B.5 Exact search with bounded bit precision

We now analyze Clinch imposing the additional regularity assumptions of [PSTZ19].

Assumption B.4. Agent utilities are linear with rational coefficients whose denominators are at most \(2^L\), and that each non-empty best response region has volume at least \(2^{-nL}\). Moreover, \(X\) is a polytope represented as the intersection of a finite set of half-spaces, each of the form \(\{x \in [0,1]^n : x^T a \leq b\}\) where \(b \in \mathbb{R}\) and each entry of \(a \in \mathbb{R}^n\) are rational with numerators and denominators at most \(2^L\).

By the discussion of our regularity assumptions in Section 3.1, it suffices to take \(C = 2^L\). With these settings, Theorem 3.9 states that Clinch terminates in \(O(nL + n \log \frac{1}{\delta})\) oracle queries, and the returned strategy \(\hat{x} \in X\) satisfies \(\|x - \hat{x}\|_\infty < \delta\). Next, we bound the bit complexity of \(x^*\).

Lemma B.5. If agent utilities are linear with rational coefficients whose denominators are at most \(2^L\), then the entries of \(x^*\) are rational with denominators at most \(2^{5Ln}\).

Proof. Fix any \(y \in \text{BR}(x^*)\), and write \(v^y(t) = \bar{d}_y - c_y t\), where \(c_y, d_y \in (0,1]\) are rational with denominators at most \(2^L\). As noted in the proof of Proposition 3.5, we must have \(x^*_y > 0\). Writing \(w^* = v(x^*, \text{br}(x^*)) = v^y(x^*_y)\), we can solve for \(x^*_y = (d_y - w^*)/c_y\). For \(y \notin \text{BR}(x^*)\), we have \(x^*_y = 0\).

Now, since \(x^*\) minimizes the agent’s best response utility, we know that \(w^*\) is as small as possible so that the \(x^*\) as determined above lies in \(X\). In other words, \(x^*\) must lie on a face of \(X\), represented as \(\{x \in \mathbb{R}^n : a^T x = b\}\). Thus, we have \(\sum_{y \in \text{BR}(x^*)} a_y (d_y - w^*)/c_y = b\) and can compute

\[
w^* = \frac{\sum_{y \in \text{BR}(x^*)} a_y d_y/c_y - b}{\sum_{y \in \text{BR}(x^*)} a_y/c_y}.
\]

Our bit precision assumptions imply that \(w^* \in (0,1]\) is rational with denominator at most \(2^{5Ln+L}\), and so each \(x^*_y \in [0,1]\) must also be rational with denominator at most \(2^{5Ln+3L}\).  

Hence, rounding appropriately, we have the following.

Proposition B.6. Under Assumption B.4, running Clinch with \(\delta = \frac{1}{2^{5Ln}}\) and rounding each entry of the result to the nearest multiple of \(2^{-8Ln}\) gives \(x^*\) using \(O(n^2L)\) best response queries.

C Supplementary material for Section 4

C.1 Stochastic bandits with delays and perturbations (proof of Lemma 4.5)

Without loss of generality, we assume that each random interval \([\ell_i, u_i]\) is always contained within \([0,1]\). For analysis, it will be convenient to define empirical counts, means, and confidence bounds for all arms \(i\) and rounds \(t\) as

\[
n_i(t) = \max \left\{ \sum_{\tau=1}^t \mathbb{1}\{i_\tau = i\}, 1 \right\}, \quad \hat{\mu}_i(t) = \frac{1}{n_i(t)} \sum_{\tau=1}^t \mathbb{1}\{i_\tau = i\} r_\tau
\]

\[
\text{LCB}_i(t) = \hat{\mu}_i(t) - \sqrt{2 \log(T)/n_i(t) - \delta}, \quad \text{UCB}_i(t) = \hat{\mu}_i(t) + \sqrt{2 \log(T)/n_i(t) + \delta}
\]

To start, we show that the confidence intervals are valid with high probability.

---

5A more careful analysis can eliminate dependence on \(W\) — yielding query complexity \(O(nL + n \log \frac{1}{\delta})\) — by avoiding the final perturbation step of Clinch. However this improvement will not impact our final result.
Lemma C.1. With probability $1 - \frac{2}{T}$, we have $\text{LCB}_i(t) \leq \mu_i(t) \leq \text{UCB}_i(t)$ for each arm $i$ and round $t$.

Proof. If arm $i$ has not been pulled by time $t$, the confidence bound $[\text{LCB}_i(t), \text{UCB}_i(t)]$ is trivially valid. Otherwise, conditioning on any arm pulls $i_1, \ldots, i_r$ and considering the intervals $[t_i, u_t]$ guaranteed by the perturbation bound, Hoeffding’s inequality implies that the corresponding empirical mean $\hat{\mu}_i(t)$ satisfies

$$\hat{\mu}_i(t) = \frac{1}{n_i(t)} \sum_{t_i \leq t} r_t \leq \frac{1}{n_i(t)} \sum_{t_i \leq t} u_t \leq \mu_i + \delta + \sqrt{\frac{2 \log T}{n_i(t)}}$$

with probability at least $1 - \frac{1}{T^2}$. Likewise, we have $\hat{\mu}_i \geq \mu_i - \delta - \sqrt{2 \log(T)/n_i(t)}$ with the same probability. Taking a union bound gives $\mu_i \in [\text{LCB}_i(t), \text{UCB}_i(t)]$ with probability at least $1 - \frac{2}{T^2}$. Since one confidence interval is modified per round, a union bound over rounds gives the lemma. \qed

Next, we note an arm $i$ can only contribute $O(\delta)$ regret in a given round if $\Delta_i = O(\delta)$. Hence, we call an arm acceptable if $\Delta_i < 8\delta$ and unacceptable otherwise. Conditioned on the “clean” event above, we show that the number of unacceptable arm pulls is bounded, extending the analysis from Theorem 2 of \cite{LSKM21} to the perturbed setting.

Lemma C.2. Conditioned on the event from Lemma C.1, no unacceptable arm $i$ is pulled more than $128 \log T/\Delta_i^2 + D/m + 2$ times, where $m$ is the number of remaining arms when it is pulled last.

Proof. To simplify analysis, we split the history of the algorithm into epochs, where epoch $\ell = 1, 2, \ldots$ denotes the $\ell$-th iteration of SUCCESSIVEELIMINATIONDELAYED’s main while loop. With this convention, after $\ell$ epochs, the remaining arms in $S$ have been pulled exactly $\ell$ times.

Now fix any unacceptable arm $i$, and consider the first epoch $\ell$ such that running UPDATECONFIDENCEBONDS with the full (non-delayed) history through epoch $\ell$ would eliminate arm $i$ after the corresponding update to $S$. Then $i$ must be truly eliminated after $D + m$ additional rounds have passed, where $m$ is the number of arms remaining when $i$ is pulled for the last time. During these extra rounds, $i$ is pulled at most $D/m + 1$ times due to the round-robin nature of arm pulls; this is the overhead from delayed feedback.

Now if $\ell \leq 1$, we are done. Otherwise, fix $S$ as the set of arms remaining after the final round $t$ of epoch $\ell - 1$, and let $\hat{S} = \{j \in S : \text{UCB}_j(t) \geq \text{LCB}_k(t) \text{ for all } k \in S\}$ denote the hypothetical update to $S$ based on non-delayed data. By the minimality of $\ell$, we know that $i \in \hat{S}$, and so

$$\hat{\mu}_i(t) \geq \max_{j \in \hat{S}} \hat{\mu}_j(t) - 2\sqrt{\frac{2 \log T}{\ell - 1}} - 2\delta \geq \max_j \mu_j - 3\sqrt{\frac{2 \log T}{\ell - 1}} - 3\delta,$$

where the second inequality follows by conditioning (noting in particular that the optimal arm is not eliminated). On the other hand, we have

$$\hat{\mu}_i(t) \leq \mu_i + \sqrt{\frac{2 \log T}{\ell - 1}} + \delta = \max_j \mu_j - \Delta_i + \sqrt{\frac{2 \log T}{\ell - 1}} + \delta.$$

Combining, we find that

$$\ell \leq \frac{32 \log T}{(\Delta_i - 4\delta)^2} + 1 \leq \frac{128 \log T}{\Delta_i^2} + 1.$$

Adding this upper bound to the overhead from delays gives the lemma. \qed

Now we are equipped to prove the main result.
Proof of Lemma 4.5. By Lemma C.2 we control regret by

\[ \sum_{\Delta_i \geq 8\delta} n_i(t) \Delta_i + 8\delta T \leq 128 \sum_{\Delta_i > 0} \left( \frac{\log T}{\Delta_i} + \frac{D}{m_i} + 1 \right) + 8\delta T, \]

where \( m_i \) is the number of remaining arms when arm \( i \) is pulled last. Bounding \( \sum_i \frac{1}{m_i} \leq \sum_{i=1}^K \frac{1}{\delta_i} \leq \log(K) + 2 \), we obtain a final bound of \( 128 \sum_{\Delta_i > 0} \frac{\log(3T)}{\Delta_i} + 128D \log(K) + 8\delta T \).

\[ \square \]

C.2 Perturbation bound for stochastic values (proof of Lemma 4.6)

Proof of Lemma 4.6. If \( a = 1 \), then \( v_t \geq p - \epsilon \) and \( a = 1 \{ v_t \geq p - \epsilon \} = u \), while, if \( a = 0 \), then \( v_t \leq p + \epsilon \) and \( a = 1 \{ v_t \leq p + \epsilon \} = f \). Moreover, we have

\[ p \mathbb{E}[u] = p \Pr(v_t \geq p - \epsilon) \leq pd(p) + pL \epsilon \leq f(p) + L \epsilon, \]

using the definitions of \( d \) and \( f \), the Lipschitz property of \( d \), and that \( p \in [0, 1] \). Likewise, we bound

\[ p \mathbb{E}[f] = p \Pr(v_t > p + \epsilon) = p \Pr(v_t \geq p + \epsilon) \geq f(p) - L \epsilon. \]

\[ \square \]

D Supplementary material for Section 5

D.1 Implications of regularity assumptions (proofs of Lemmas 5.2 and 5.3)

Proof of Lemma 5.2. First, by the \( \alpha \)-strong convexity assumption on \( f_t, v_{a_t} \) is \( \alpha \)-strongly concave in \( \theta \), and so the best response \( br_t(\theta_t) \) is unique. In the proof of Theorem 2 on page 8 of \( \text{DRS}^{+18} \), the authors show that \( v_{a_t}(\theta_t, br_t(\theta_t)) \leq \langle \theta_t, br_t(\theta_t) \rangle = \langle x_t, \theta_t \rangle + 2f^*_t(\theta_t) \), where \( f^*_t \) is the convex conjugate of \( f_t \). In the proof of Claim 2 on page 20, they further show that \( f^*_t(\theta_t) = \sup_{v \in S^{d-1}} \frac{\langle \theta, v \rangle^2}{4f_t(v)} \). The numerator of this objective is bounded from above by \( R^2 \), while the denominator is bounded from below by \( 2\alpha \), since \( \|v\|_2 = 1 \) and \( f_t \) is \( \alpha \)-strongly convex with minimum of 0 the origin (due to homogeneity). Thus, \( f^*_t(\theta_t) \leq \frac{R^2}{2\alpha} \), and so \( v_{a_t}(\theta_t, \hat{x}_t) \leq v_{a_t}(\theta_t, br_t(\theta_t)) \leq R^2(1 + \frac{1}{2\alpha}) \). Finally, by playing \( \hat{x} = x \), the agent obtains payoff \( \langle \theta_t, x_t \rangle \geq -R^2 \).

\[ \square \]

Proof of Lemma 5.3. Convexity of \( \ell_t(\theta) = \ell(\theta, br_t(\theta), y_t) \) is implied by Theorem 2 of \( \text{DRS}^{+18} \). For Lipschitzness of \( \ell_t \), we turn to the proof of Theorem 7 on page 17 of \( \text{DRS}^{+18} \). The discussion there implies that \( \ell_t(\theta) \) is Lipschitz with constant \( \|x_t\|_2 \leq R \) plus twice the Lipschitz constant of \( f^*_t \). To compute this, we bound

\[ \left| \sup_{v \in S^{d-1}} \frac{\langle \theta, v \rangle^2}{4f_t(v)} - \sup_{v \in S^{d-1}} \frac{\langle \theta', v \rangle^2}{4f_t(v)} \right| \leq \sup_{v \in S^{d-1}} \left| \frac{\langle \theta, v \rangle^2}{4f_t(v)} - \frac{\langle \theta', v \rangle^2}{4f_t(v)} \right| \leq \frac{2R\|\theta - \theta'\|_2}{2\alpha}, \]

using the same lower bound on \( f_t(v) \) as in the proof of Lemma 5.2. Combining gives a Lipschitz constant of \( R + 2R/\alpha \) for \( \ell_t \). Finally, discussion on page 18 of \( \text{DRS}^{+18} \) implies that \( |\ell_t(\theta)| \leq 1 + \|\theta, x_t\| + 2f^*_t(\theta) \), which we bound by \( 1 + R^2(1 + \frac{1}{2\alpha}) \) as in the proof of Lemma 5.2. Finally, the same discussion on page 17 implies that the map \( \hat{x} \mapsto \ell(\theta, \hat{x}, y_t) \) has Lipschitz constant bounded by that of the map \( \hat{x} \mapsto \langle \theta, \hat{x} \rangle \), which is \( \|\theta\|_2 \leq R \).

\[ \square \]
D.2 Robust bandit convex optimization (proof of Lemma 5.6)

Proof of Lemma 5.6. Compared to the proof of Theorem 2 in [FKM05] for the unperturbed setting, we need to apply their Lemma 2 when gradient estimates \( g_t \) have bias \( b = \frac{d}{\delta} \) rather than \( b = 0 \). Switching to their notation for the lemma, if we have \( \mathbb{E}[g_t|x_t] = \nabla c_i(x_t) + \xi_t \) with \( \|\xi_t\|_2 \leq b \), then the final chain of inequalities in their proof of Lemma 2 still holds, up to an added term of \( \sum_{t=1}^{n}(\xi_t, x_t - x_*) \leq nbR \). Switching back to our notation and substituting our value for \( b \), this gives a regret overhead of \( O(\frac{4RTd}{\delta}) \).

D.3 Main result for strategic classification (proof of Theorem 5.7)

Proof of Theorem 5.7. Since Stackelberg regret is subadditive over disjoint sequences of rounds, we obtain regret \( DR_{A_1}^\epsilon([T/D]) \) against \( \epsilon \)-approximate best-responding agents. Combining Lemmas 5.3, 5.5 and 5.6 and substituting our choices of constants, we bound the regret of any single copy by

\[
R_{A_1}^\epsilon(T) \leq \mathbb{E} \left[ \sum_{t=1}^{T} \ell(\theta_t, \hat{x}_t, y_t) - \min_{\theta \in \Theta} \sum_{t=1}^{T} \ell(\theta, \hat{b}_t(\theta), y_t) \right] \\
\leq \mathbb{E} \left[ \sum_{t=1}^{T} \ell(\theta_t, \hat{b}_t(x_t), y_t) - \min_{\theta \in \Theta} \sum_{t=1}^{T} \ell(\theta, \hat{b}_t(\theta), y_t) \right] + TR\sqrt{2\epsilon/\alpha} \\
\leq 6T^{3/4} \sqrt{RDc(L+C)} + 5C(Rd)^2 + TR\sqrt{\frac{2\epsilon}{\alpha}}(Rd/\delta + 1) \\
= O \left( R^{5/2} \hat{\alpha}^{-1} \sqrt{dT}^{3/4} + R^4 \hat{\alpha}^{-2}d^2 \right),
\]

where \( \hat{\alpha} = \min\{\alpha, 1\} \). By Proposition 2.1 and Lemma 5.2 our feedback delay induces \( \epsilon \)-approximate best responses, so we obtain a final regret bound of \( O \left( R^{5/2} \hat{\alpha}^{-1}T^{1/4} \sqrt{dT}^{3/4} \log^{1/4}(TRd/\alpha) + R^4 \hat{\alpha}^{-2}d^2 \right) \).

E Supplementary material for Section 6

E.1 Polytope perturbation bounds

For the moment, we fix \( y = n \) and write \( K = K_y, K^\epsilon = K^\epsilon_y \) for conciseness. Moreover, we use \( m \) to denote the exact count of principal actions (before \( m + 1 \)) to simplify presentation. Now, consider the matrix \( W \in \mathbb{R}^{(n-1) \times m} \) with \( W(j,i) \) set to the marginal utility of the agent for playing \( j \) over \( n \) when the principal selects pure strategy \( i \), i.e.,

\[
W(j,i) := v_0(i,j) - v_0(i,n).
\]

This matrix was selected so that we can concretely write

\[
K^\epsilon = \{ x \in \mathbb{R}^m : Wx \leq \epsilon 1_{n-1}, 1_m x \geq 0_m, 1_m^T x = 1 \}.
\]

We now label the subsets of these linear constraints which are active (tight) at a unique point in \( \mathbb{R}^m \), defining

\[
\mathcal{M} := \left\{ (R,S) : R \subseteq [n-1], S \subseteq [m], \begin{bmatrix} W_R & (1_m)_S \\ 1_m^T & \end{bmatrix} \right\} \text{ is invertible with } |R| + |S| + 1 = m \right\},
\]

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where set subscripts for matrices denote row restriction. For each \( \varepsilon \geq 0 \), a pair \( I = (R, S) \in \mathcal{M} \) corresponds to a unique point \( x^{I, \varepsilon} \) defined by

\[
x^{I, \varepsilon} = \begin{bmatrix} W_R \\ (I_m)_S \\ 1_m^T \\ 0_m \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon 1_{n-1} \\ 0_m \end{bmatrix},
\]

(sometimes called a basic solution in mathematical programming). We can now state a useful bound, which follows from the argument surrounding the more general Equation 1.6 in Li \cite{Li93}.

**Lemma E.1** (Polytope perturbation bound). For all \( \varepsilon \geq 0 \), we have

\[
d_H(K, K^{\varepsilon}) \leq \max_{I \in \mathcal{M}} \| x^{I, 0} - x^{I, \varepsilon} \|_2 = \varepsilon \cdot \alpha(W),
\]

for a constant \( \alpha(W) \) independent of \( \varepsilon \). Consequently, for each \( x \in K^{\varepsilon} \), there exists \( x' \in K \) such that \( \| x - x' \|_2 \leq \varepsilon \alpha(W) \).

### E.2 Algorithm definitions

First, we define the conservative membership oracles.

**Algorithm 11** Conservative membership oracle adjustment

**Input:** action \( y \in \mathcal{Y} \), query \( x \in \mathcal{X} \), \( \varepsilon \)-approximate best response oracle \textsc{Oracle} 

**Output:** response of conservative membership oracle \textsc{Mem}\( \varepsilon \) for \( K_y \) given query \( x \)

1: for \( i = 1 \) to \( \lceil 2\sqrt{m} \log T \rceil \) do 
2: \( w_i \leftarrow z + 2L \varepsilon \sqrt{d} \cdot S_i \), where \( S_i \) is sampled uniformly at random from unit sphere \( S^{m-1} \)
3: if \textsc{Oracle}(\( w_i \)) \( \neq y \) then Return \text{False}
4: Return \text{True}

Now, we can state our algorithm for robust learning.

**Algorithm 12** NOISYSTACK: Robust learning for finite Stackelberg games

**Input:** Access to \( T \) queries from \( \varepsilon \)-approximate best response oracle \textsc{Oracle}

1: \( \mathcal{Y}_0 \leftarrow \emptyset \), \( V \leftarrow \) volume of ball with radius \( r \) in \( \mathbb{R}^m \)
2: for \( i = 1, \ldots, \lceil V^{-1} \log(T) \rceil \) do
3: Sample \( x \) uniformly at random from \( \mathcal{X} \), \( y \leftarrow \textsc{Oracle}(x) \)
4: if \( y \notin \mathcal{Y}_0 \) and \textsc{Mem}\( \varepsilon \)(\( x \)) = \text{True} \ then \( \mathcal{Y}_0 \leftarrow \mathcal{Y}_0 \cup \{y\}, x^{(y)} \leftarrow x \)
5: for \( y \in \mathcal{Y}_0 \) do 
6: Run LSV with \( x_0 = x^{(y)} \), \( \delta = \frac{1}{L \sqrt{m T}} \), \( f(\cdot) = -u(\cdot, y) \), oracle \textsc{Mem}\( \varepsilon \), radii \( R = \sqrt{2} \) and \( r \), halting after \( O(m^2 \log^{O(1)} \frac{1}{\varepsilon^2}) \) queries
7: if search completed then \( x^{(y)} \leftarrow \) result else \( \mathcal{Y} \leftarrow \mathcal{Y} \setminus \{y\} \)
8: while queries remain do 
9: \( y \leftarrow \textsc{Oracle}(\hat{x}^{(y)}) \) for \( \hat{y} \in \arg \max_{y' \in \mathcal{Y}_0} u(\hat{x}^{(y')}, y') \)
10: if \textsc{Oracle}(\( y \)) \( \neq \hat{y} \) and \( |\mathcal{Y}_0| > 1 \) then \( \mathcal{Y}_0 \leftarrow \mathcal{Y}_0 \setminus \{\hat{y}\} \)

### E.3 Main result for finite Stackelberg games (proof of Theorem 6.1)

First, we first provide formal guarantees for an existing approach to robust convex optimization with membership queries. Here, we say that \textsc{Mem} is a \( \Lambda \)-approximate membership oracle for a set \( S \subset \mathbb{R}^m \) if \textsc{Mem} returns \text{True} given query \( x \in \mathbb{R}^m \) if \( x \in B(S, -\lambda) \) and \text{False} if \( x \notin B(S, \lambda) \).
Lemma E.2 (Theorem 1 of [LSV18]). Fix $\delta \in (0, 1)$, $x_0 \in \mathbb{R}^m$, and $r, R > 0$. Let $f : B(R) \to [0, 1]$ be a convex function given by an evaluation oracle, and let $B(x_0, r) \subseteq K \subseteq B(R)$ be a convex set given by a \((\frac{\delta}{m\epsilon R})^{O(1)}\)-approximate membership oracle. There exists an algorithm LSV that computes $z \in B(K, \delta)$ such that $f(z) \leq \min_{x \in K} f(x) + \delta$ with probability $1 - \delta$, using $O(m^2 \log^{O(1)} \frac{mR}{\delta \epsilon})$ oracle queries.

Next, we state formal guarantees for the conservative membership oracle defined in Algorithm 11.

Lemma E.3. The oracle $\text{Mem}_x^\epsilon$, simulated by Algorithm 11 is a $L \sqrt{m}$-approximate membership oracle for the set $B(K_y^\ast, -L \sqrt{m} + 1)$ with failure probability at most $T^{-2}$ per query.

Proof. First, we show that feedback $1\{\text{Oracle}(x) = y^\ast\}$ simulates the response of a $L \epsilon$-approximate membership oracle for $K_y^\ast$. Indeed, if $\text{Oracle}(x) = y^\ast$, then $x \in K_y^\ast$ and $d(x, K_y^\ast) \leq d_H(K_y^\ast, K_y^\ast) \leq L \epsilon$. Otherwise, if $\text{Oracle}(x) \neq y^\ast$, then $x \notin K_y^{\epsilon'}$ for any $\epsilon' > \epsilon$. Since $K_y^{\epsilon'} \subset K_y^\ast$, we have $d(x, \mathbb{R}^{m-1} \setminus K_y^\ast) \leq d(K_y^{\epsilon'}, K_y^\ast) \leq L \epsilon'$ for all $\epsilon' > \epsilon$, implying that $d(x, \mathbb{R}^{m-1} \setminus K_y^\ast) \leq L \epsilon$.

Now, suppose that $x$ lies within $\ell_2$ distance $\epsilon L$ of the boundary of $K_y^\ast$, i.e., it is not robustly within this set. By conditioning, it must then lie within distance $2 \epsilon L$ of the boundary of $K_y^{\epsilon'}$. Consider any hyperplane separating $x$ from the closest point of the boundary. By standard bounds on the volume of a spherical cap (see, e.g., Lemma 9 of [FS02]), the probability that any $w_i$ lands on the other side of this hyperplane is at least $(1 - 1/d)(1-d^{3/2})d^{-1/2} \geq \frac{1}{2}d^{-1/2}$. Consequently, we have that each $w_i$ lies outside of $K_y^{\epsilon'}$ at least this often, and hence Mem$_y^\ast(x)$ returns FALSE with probability at least $1 - \exp(-\sqrt{m}2 \log(T)/(4\sqrt{m})) \geq 1 - T^{-2}$. On the other hand, if $B(x, L \epsilon \sqrt{2m} + 1) \subsetneq K_y^\ast$, then Oracle$(w_i)$ will always be True; hence, Mem$_y^\ast(x)$ will be TRUE as well. Combining, we see that Mem$_y^\ast$ acts as a $L \sqrt{m}$-approximate oracle for the set $B(K_y^\ast, -L \epsilon (\sqrt{m} + 1))$, as desired. □

Finally, we prove our main result.

Proof of Theorem 6.1. First, we note by Lemma E.3 and a union bound, we can assume that the membership oracle guarantee holds with certainty, incurring at most $O(1)$ regret otherwise. Next, we analyze the sampling loop at Step 2. Let $S$ denote the ball of radius $2\epsilon$ contained in $K_y^\ast$. The sample count is taken sufficiently high such that a sampled point will lie inside $B(S, -\epsilon) \subseteq B(K_y^\ast, -\epsilon)$ probability at least $1 - \frac{1}{T}$. Since $r \geq 3L \sqrt{m}$ (here and throughout, we assume an appropriate selection of hidden constants), this margin is sufficiently large such so that $j^\ast$ will be added to $\mathcal{Y}$ at Step 2.

In the loop at Step 5, Lemma E.2 implies that the search starting at the initial point provided for $y = y^\ast$ will terminate in $O(m^2 \log^{O(1)} \frac{Lm}{\epsilon r})$ queries with success probability $1 - O((T \sqrt{m})^{-1})$, given that $\epsilon = (\frac{\delta}{m \sqrt{\epsilon}})^{O(1)}$.

Conditioning on success without loss of generality, Lemma E.2 and Algorithm 2 imply that the returned $\hat{x}(y^\ast)$ lies within $B(K_y^\ast, -\epsilon L)$, which by conditioning is contained in $K_y^{\epsilon'}$. Hence the reward obtained by playing $\hat{x}(y^\ast)$ is $u(\hat{x}(y^\ast), y^\ast)$, which Lemma E.2 implies is within $O(1/T)$ of the global maximum $u(x^\ast, y^\ast)$. Indeed, although our membership oracle is for the set $B(K_y^\ast, -L \epsilon (\sqrt{m} + 1))$ rather than $K_y^\ast$, the objective is $O(1)$-Lipschitz; hence, we obtain added error at most $O(\sqrt{m}L/(TL \sqrt{m})) = O(1/T)$ due to feedback inaccuracy, on top of the initial $O(1/(LT \sqrt{m}))$ guarantee.

The payoff obtained over each remaining round in Step 6 is at least this high, leading to negligible regret unless the agent deviates from our expectations, in which case we throw out the responsible candidate strategy. We can remove at most $m$ candidate strategies before arriving on that of $j^\ast$, and so our regret for the remaining rounds is at most $m + O(1)$. The number of rounds used for the initial queries is at most $O(V^{-1/2} \log^2(T) \sqrt{m} + m^3 \log^{O(1)} \frac{Lm}{r})$, matching a bound for regret against $\epsilon$-approximate best responding agents. Finally, the parallel copies approach applied in the proof of Algorithm 9 gives a non-myopic regret bound of $O(T \log(T/\epsilon)[V^{-1/2} \log^2(T) \sqrt{m} + m^3 T \log^{O(1)} \frac{Lm}{r}])$ against $\gamma$-discounting agents. Plugging in our choice of $\epsilon = (\frac{T}{Tm})^{\Theta(1)}$ gives the stated bound. □