Abstract

We study risk-sensitive reinforcement learning in episodic Markov decision processes with unknown transition kernels, where the goal is to optimize the total reward under the risk measure of exponential utility. We propose two provably efficient model-free algorithms, Risk-Sensitive Value Iteration (RSVI) and Risk-Sensitive Q-learning (RSQ). These algorithms implement a form of risk-sensitive optimism in the face of uncertainty, which adapts to both risk-seeking and risk-averse modes of exploration. We prove that RSVI attains an $\tilde{O}(\lambda(|\beta|H^2) \cdot \sqrt{H^3S^2AT})$ regret, while RSQ attains an $\tilde{O}(\lambda(|\beta|H^2) \cdot \sqrt{H^4SAT})$ regret, where $\lambda(u) = (e^{3u} - 1)/u$ for $u > 0$. In the above, $\beta$ is the risk parameter of the exponential utility function, $S$ the number of states, $A$ the number of actions, $T$ the total number of timesteps, and $H$ the episode length. On the flip side, we establish a regret lower bound showing that the exponential dependence on $|\beta|$ and $H$ is unavoidable for any algorithm with an $\tilde{O}(\sqrt{T})$ regret (even when the risk objective is on the same scale as the original reward), thus certifying the near-optimality of the proposed algorithms. Our results demonstrate that incorporating risk awareness into reinforcement learning necessitates an exponential cost in $|\beta|$ and $H$, which quantifies the fundamental tradeoff between risk sensitivity (related to aleatoric uncertainty) and sample efficiency (related to epistemic uncertainty). To the best of our knowledge, this is the first regret analysis of risk-sensitive reinforcement learning with the exponential utility.
1 Introduction

Risk-sensitive reinforcement learning (RL) concerns learning to act in a dynamic environment while taking into account risks that arise during the learning process. Effective management of risks in RL is critical to many real-world applications such as autonomous driving [32], real-time strategy games [56], financial investment [44], etc. In neuroscience, risk-sensitive RL has been applied to model human behaviors in decision making [46, 52].

In this paper, we consider risk-sensitive RL with the exponential utility [34] under episodic Markov decision processes (MDPs) with unknown transition kernels. Informally, the agent aims to maximize a risk-sensitive objective function of the form

\[
V = \frac{1}{\beta} \log \left\{ \mathbb{E} e^{\beta R} \right\},
\]

where \( R \) is the total reward the agent receives, and \( \beta \neq 0 \) is a real-valued parameter that controls risk preference of the agent; see Equation (2) for a formal definition of \( V \). The objective \( V \) admits the Taylor expansion \( V = \mathbb{E}[R] + \frac{\beta^2}{2} \text{Var}(R) + O(\beta^2) \). It can be seen that for \( \beta > 0 \) the agent is risk-seeking (favoring high uncertainty in \( R \)), for \( \beta < 0 \) the agent is risk-averse (favoring low uncertainty in \( R \)), and a larger \( |\beta| \) implies higher risk-sensitivity. When \( \beta \to 0 \), the agent tends to be risk-neutral and the objective reduces to the expected reward objective \( V = \mathbb{E}[R] \) standard in RL. Therefore, the risk-sensitive objective in (1) covers the entire spectrum of risk sensitivity by varying \( \beta \). In addition, the formulation (1) is closely related to RL with constraints. For example, a negative risk parameter \( \beta \) controls the tail of a risk distribution so as to mitigate the chance of receiving a total reward \( R \) that is excessively low. We refer to [42, Section 2.1] for an in-depth discussion of this connection.

The challenge of risk-sensitive RL lies both in the non-linearity of the objective function and in designing a risk-aware exploration mechanism. In particular, as we elaborate in Section 2.2, the non-linear objective function (1) induces a non-linear Bellman equation. Classical RL algorithms are inappropriate in this setting, as their design crucially relies on the linearity of Bellman equations. On the other hand, effective exploration has been well known to be crucial to RL algorithm design, yet it is not clear how to design an algorithm that efficiently explores uncertain environments while at the same time adapting to the risk-sensitive objective (1) of agents with different risk parameter \( \beta \).

To address these difficulties, we propose two model-free algorithms, Risk-Sensitive Value Iteration (RSVI) and Risk-Sensitive Q-learning (RSQ). Specifically, RSVI is a batch algorithm and RSQ
is an online algorithm; both families of batch and online algorithms see broad applications in practice. We demonstrate in Section 3 that our proposed algorithms implement a form of risk-sensitive optimism for exploration. Importantly, the exact implementation of optimism depends on both the magnitude and the sign of the risk parameter, and therefore applies to both risk-seeking and risk-averse modes of learning. Letting $\lambda(u) = (e^{3u} - 1)/u$ for $u > 0$, we prove that RSVI attains an $\tilde{O}(\lambda(|\beta| H^2) \cdot \sqrt{H^3 S^2 A T})$ regret, and RSQ achieves an $\tilde{O}(\lambda(|\beta| H^2) \cdot \sqrt{H^4 S A T})$ regret. Here, $S$ and $A$ are the numbers of states and actions, respectively, $T$ is the total number of timesteps, and $H$ is the length of each episode. These regret bounds interpolate across different regimes of risk sensitivity and subsume existing results under the risk-neutral setting. Compared with risk-neutral RL (corresponding to $\beta \to 0$), our general regret bounds feature an exponential dependency on $|\beta|$ and $H$, even though the risk-sensitive objective (1) is on the same scale as the total reward; see Figure 1 for a plot of the exponential factor $\lambda(|\beta| H^2)$. Complementarily, we prove a lower bound showing that such an exponential dependency is inevitable for any algorithm and thus certifies the near-optimality of the proposed algorithms. To the best of our knowledge, our work provides the first regret analysis of risk-sensitive RL with the exponential utility.

Our upper and lower bounds demonstrate the fundamental tradeoff between risk sensitivity and sample efficiency in RL.² Broadly speaking, risk sensitivity is associated with aleatoric uncertainty, which originates from the inherent randomness of state transition, actions and rewards, whereas sample efficiency is associated with epistemic uncertainty, which arises from imperfect knowledge of the environment/system and can be reduced by more exploration [24, 20]. These two notions of uncertainty are usually decoupled in the regret analysis of risk-neutral RL—in particular, using the expected reward as the objective effectively suppresses the aleatoric uncertainty. In risk-sensitive RL, we establish that there is a fundamental connection and tradeoff between these two forms of uncertainty: the risk-seeking and risk-averse regimes both incur an exponential cost in $|\beta|$ and $H$ on the regret, whereas the regret is polynomial in $H$ in the risk-neutral regime.

Our contributions. The contributions of our work can be summarized as follows:

- We consider the problem of risk-sensitive RL with the exponential utility. We propose two provably efficient model-free algorithms, namely RSVI and RSQ, that implement risk-sensitive optimism in the face of uncertainty;

²By standard arguments, regret can be translated into sample complexity bounds and vice versa; see [38].
• We provide regret analysis for both algorithms over the entire spectrum of risk parameter $\beta$. As $\beta \to 0$, we show that our results recover the existing regret bounds in the risk-neutral setting;

• We provide a lower bound result that certifies the near-optimality of our upper bounds and reveals a fundamental tradeoff between risk sensitivity and sample complexity.

**Related work.** RL with risk-sensitive utility functions have been studied in several work. The work [45] proposes TD(0) and Q-learning-style algorithms that transform temporal differences instead of cumulative rewards, and proves their convergence. Risk-sensitive RL with a general family of utility functions is studied in [52], which also proposes a Q-learning algorithm with convergence guarantees. The work of [28] studies a risk-sensitive policy gradient algorithm, though with no theoretical guarantees. We remark that while substantial work has been devoted to designing risk-sensitive RL algorithms and proving their convergence, the issues of exploration, sample efficiency and regret bounds have rarely been studied. Our work narrows this gap in the literature by studying regret bounds of model-free algorithms for risk-sensitive RL.

The exponential utility has also been been investigated in the more classical setting of MDPs. Following the seminal work of [34], this line of work includes [7, 9, 10, 11, 14, 21, 25, 29, 30, 33, 43, 48, 51, 58]. Note that these papers impose more restrictive assumptions and study different types of results than ours. Specifically, they assume known transition kernels or access to simulators, and they do not conduct finite-time or finite-sample analysis. Another related direction to ours is RL with risk/safety constraints studied by [1, 2, 16, 19, 17, 18, 26, 27, 49, 54, 59, 61], and readers are also referred to [31] for an excellent survey on this topic. Compared to our work, that line of work focuses on constrained RL problems with different risk criteria. Other related problems include...
risk-sensitive games \([5, 6, 8, 15, 35, 37, 40, 57]\), and risk-sensitive bandits \([13, 23, 22, 42, 50, 53, 55, 60, 62]\). Bandit problems are special cases of the RL problem that we investigate, with both the number of states and episode length being equal to one. As such, both our settings and results are more general than those obtained in bandit problems.

**Notations.** For a positive integer \(n\), let \([n] := \{1, 2, \ldots, n\}\). For two non-negative sequences \(\{a_i\}\) and \(\{b_i\}\), we write \(a_i \lesssim b_i\) if there exists a universal constant \(C > 0\) such that \(a_i \leq Cb_i\) for all \(i\). We write \(a_i \asymp b_i\) if \(a_i \lesssim b_i\) and \(b_i \lesssim a_i\). We use \(\tilde{O}(\cdot)\) to denote \(O(\cdot)\) while hiding logarithmic factors.

### 2 Problem setup

#### 2.1 Episodic MDPs and risk-sensitive objective

We consider the setting of episodic MDPs, denoted by \(\text{MDP}(S, A, H, P, R)\), where \(S\) is the set of possible states, \(A\) is the set of possible actions, \(H\) is the length of each episode, and \(P = \{P_h\}_{h \in [H]}\) and \(R = \{r_h\}_{h \in [H]}\) are the sets of state transition kernels and reward functions, respectively. In particular, for each \(h \in [H]\), \(P_h(\cdot | s, a)\) is the distribution of the next state if action \(a\) is taken in state \(s\) at step \(h\). We assume that \(S\) and \(A\) are finite discrete spaces, and let \(S = |S|\) and \(A = |A|\) denote their cardinalities. We assume that the agent does not have access to \(\{P_h\}\) and that each \(r_h : S \times A \rightarrow [0, 1]\) is a deterministic function.

An agent interacts with an episodic MDP as follows. At the beginning of each episode, an initial state \(s_1\) is chosen arbitrarily by the environment. In each step \(h \in [H]\), the agent observes a state \(s_h \in S\), chooses an action \(a_h \in A\), and receives a reward \(r_h(s_h, a_h)\). The MDP then transitions into a new state \(s_{h+1} \sim P_h(\cdot | s_h, a_h)\). We use the convention that the episode terminates when a state \(s_{H+1}\) at step \(H + 1\) is reached, at which the agent does not take an action and receives no reward.

A policy \(\pi = \{\pi_h\}_{h \in [H]}\) of an agent is a sequence of functions \(\pi_h : S \rightarrow A\), where \(\pi_h(s)\) is the action that the agent takes in state \(s\) at step \(h\) of an episode. For each \(h \in [H]\), we define the value function \(V^\pi_h : S \rightarrow \mathbb{R}\) of a policy \(\pi\) as the expected value of cumulative rewards the agent receives under a risk measure of exponential utility by executing policy \(\pi\) starting from an arbitrary state at step \(h\). Specifically, we have

\[
V^\pi_h(s) := \frac{1}{\beta} \log \left\{ \mathbb{E} \left[ \exp \left( \beta \sum_{h=1}^{H} r_h(s_h, \pi_h(s_h)) \right) \mid s_h = s \right] \right\},
\]  

(2)
for each \((h, s) ∈ [H] × S\). Here \(β \neq 0\) is the risk parameter of the exponential utility: \(β > 0\) corresponds to a risk-seeking value function, \(β < 0\) corresponds to a risk-averse value function, and as \(β → 0\) the agent tends to be risk-neutral and we recover the classical value function \(V_h^π(s) = E[\sum_{h=1}^{H} r_h(s_h, π_h(s_h)) \mid s_h = s]\) in RL. The goal of the agent is to find a policy \(π\) such that \(V_1^π(s)\) is maximized for all state \(s ∈ S\). Note the logarithm and rescaling by \(1/β\) in the above definition, which puts the objective \(V_1^π(s)\) on the same scale as the total reward; this scaling property is made formal in Lemma 1 below.

### 2.2 Bellman equations and regret

We further define the action-value function \(Q_h^π : S × A \rightarrow \mathbb{R}\), which gives the expected value of the risk measured by the exponential utility when the agent starts from an arbitrary state-action pair at step \(h\) and follows policy \(π\) afterwards; that is,

\[
Q_h^π(s, a) := \frac{1}{β} \log \left\{ \frac{\exp(β \cdot r_h(s, a)) \cdot E \left[ \exp \left( β \sum_{h'=h+1}^{H} r_{h'}(s_{h'}, a_{h'}) \right) \mid s_h = s, a_h = a \right]}{\exp(β \cdot r_h(s, a))} \right\},
\]

for all \((h, s, a) ∈ [H] × S × A\). The Bellman equation associated with policy \(π\) is given by

\[
Q_h^π(s, a) = r_h(s, a) + \frac{1}{β} \log \left\{ \frac{\exp(β \cdot r_h(s, a)) \cdot E \left[ \exp \left( β \cdot V_{h+1}^π(s') \right) \right]}{\exp(β \cdot r_h(s, a))} \right\},
\]

(3)

which holds for all \((s, a) ∈ S × A\).

Under some mild regularity conditions, there always exists an optimal policy \(π^∗\) which gives the optimal value \(V_h^∗(s) = \sup_π V_h^π(s)\) for all \((h, s) ∈ [H] × S\) [7]. The Bellman optimality equation is given by

\[
Q_h^∗(s, a) = r_h(s, a) + \frac{1}{β} \log \left\{ \frac{\exp(β \cdot r_h(s, a)) \cdot E \left[ \exp \left( β \cdot V_{h+1}^∗(s') \right) \right]}{\exp(β \cdot r_h(s, a))} \right\},
\]

(4)

This equation implies that the optimal policy \(π^∗\) is the greedy policy with respect to the optimal action-value function \(\{Q_h^∗\}_{h ∈ [H]}\). Hence, to find the optimal policy \(π^∗\), it suffices to estimate the optimal action-value function. We note that both Bellman equations (3) and (4) are non-linear in the value and action-value functions due to non-linearity of the exponential utility. This is in contrast with their linear risk-neutral counterparts.

Under the episodic MDP setting, the agent aims to learn the optimal policy by interacting with the environment throughout a set of episodes. For each \(k ≥ 1\), let us denote by \(s_k^i\) the initial state
chosen by the environment and \(\pi^k\) the policy chosen simultaneously by the agent at the beginning of episode \(k\). The difference in values between \(V_{1}^{\pi^k}(s_{1}^k)\) and \(V_1^*(s_1^k)\) measures the expected regret or the sub-optimality of the agent in episode \(k\). After \(K\) episodes, the total regret for the agent is

\[
\text{Regret}(K) := \sum_{k \in [K]} \left[ V_{1}^{\pi^k}(s_{1}^k) - V_1^*(s_1^k) \right].
\]

We record the following simple worst-case upper bounds on the value functions and regret.

**Lemma 1.** For any \((h, s, a) \in S \times A \times [H]\), policy \(\pi\) and risk parameter \(\beta \neq 0\), we have

\[
0 \leq V_{h}^\pi(s) \leq H \quad \text{and} \quad 0 \leq Q_{h}^\pi(s, a) \leq H.
\]

Consequently, for each \(K \geq 1\), all policy sequences \(\pi^1, \ldots, \pi^K\) and any \(\beta \neq 0\), we have

\[
0 \leq \text{Regret}(K) \leq KH.
\]

**Proof.** Recall the assumption that the reward functions \(\{r_h\}\) are bounded in \([0, 1]\). The lower bounds are immediate by definition. For the upper bound, we have \(V_h^\pi(s) \leq \frac{1}{\beta} \log \{\mathbb{E} [\exp(\beta H)]\} = H\). Upper bounds for \(Q_h^\pi\) and the regret follow similarly. \(\square\)

While straightforward, the above lemma highlights an important point: the risk and regret are on the same scale as the reward. In particular, the upper bounds above are independent of \(\beta\) and linear in the horizon length \(H\)—the same as in the standard MDP setting—because the log and exp functions in the definition of the objective function (2) cancel with each other in the worst case. Therefore, the exponential dependence of the regret on \(|\beta|\) and \(H\), which we establish below in Section 4, is not merely a consequence of scaling but rather is inherent in the risk-sensitive setting.

### 3 Algorithms

The non-linearity of the Bellman equations, discussed in Section 2.2, creates challenges in algorithmic design. In particular, standard model-free algorithms such as least-squares value iteration (LSVI) and Q-learning are no longer appropriate since they specialize to the risk-neutral setting with linear Bellman equations. In this section, we present risk-sensitive LSVI and Q-learning algorithms that adapt to both the non-linear Bellman equations and any valid risk parameter \(\beta\).
3.1 Risk-Sensitive Value Iteration

We first present Risk-Sensitive Value Iteration (RSVI) in Algorithm 1. Algorithm 1 is inspired by LSVI-UCB of [39], which is in turn motivated by the idea of LSVI [12, 47] and the classical value-iteration algorithm. Like LSVI-UCB, Algorithm 1 applies the Upper Confidence Bound (UCB) by incorporating a bonus term to value estimates of state-action pairs, which therefore implements the principle of Optimism in the Face of Uncertainty (OFU) [36].

Mechanism of Algorithm 1. The algorithm mainly consists of the value estimation step (Line 6–13) and the policy execution step (Line 14–18). In Line 7, the algorithm computes the intermediate value $w_h$ by a least-squares update

$$w_h \leftarrow \argmin_{w \in \mathbb{R}^{SA}} \sum_{\tau \in [k-1]} \left[ e^{\beta [r_h(s_{\tau h}^T, a_{\tau h}) + V_{h+1}(s_{\tau h}^{h+1})]} - w^T \phi(s_{\tau h}^T, a_{\tau h}) \right]^2. \tag{8}$$

Here, $\{(s_{\tau h}^T, a_{\tau h}^T, s_{\tau h}^{h+1})\}_{\tau \in [k-1]}$ are accessed from the dataset $D_h$ for each $h \in [H]$, and $\phi(\cdot, \cdot)$ denotes the canonical basis in $\mathbb{R}^{SA}$. Line 7 can be efficiently implemented by computing sample means of $e^{\beta [r_h(s, a) + V_{h+1}(s')]}$ over those state-action pairs that the algorithm has visited. Therefore, it can also be interpreted as estimating the sample means of exponentiated $Q$-values under visitation measures induced by the transition kernels $\{P_{h}\}$. This is a typical feature of the family of batch algorithms, to which Algorithm 1 belongs. Then, in Line 10, the algorithm uses the intermediate value $w_h$ to compute the estimate $Q_h$, by adding/subtracting bonus $b_h$ and thresholding the sum/difference at $e^{\beta (H-h+1)}$, depending on the sign of $\beta$. It is not hard to see that the logarithmic-exponential transformation in Line 10 conforms and adapts to the non-linearity in Bellman equations (3) and (4). In addition, the thresholding operator ensures that the estimated action-value function $Q_h$ of step $h$ stays in the range $[0, H-h+1]$ and so does the estimated value function $V_h$ in Line 11. This is to enforce the estimates $Q_h$ and $V_h$ to be on the same scale as the optimal $Q_h^*$ and $V_h^*$.

Besides the logarithmic-exponential transformation, another distinctive feature of Algorithm 1 is the way the bonus term $b_h > 0$ is incorporated in Line 10. At first sight, it might appear counter-intuitive to subtract $b_h$ from $w_h$ when $\beta < 0$. We demonstrate next that subtracting bonus when $\beta < 0$ in fact implements the idea of OFU in a risk-sensitive fashion.

Risk-Sensitive Upper Confidence Bound. For the purpose of illustration, let us consider a “promising” state $s^+ \in S$ at step $h$ that allows us to transition to states $\{s'\}$ in the next step with high values $\{V_{h+1}(s')\}$ regardless of actions taken. This means that the intermediate value $w_h(s^+, \cdot) \propto
Algorithm 1 RSVI

**Input:** number of episodes $K \in \mathbb{Z}_{>0}$, confidence level $\delta \in (0,1]$, and risk parameter $\beta \neq 0$

1: $Q_h(s,a) \leftarrow H-h+1$ and $N_h(s,a) \leftarrow 0$ for all $(h,s,a) \in [H] \times \mathcal{S} \times \mathcal{A}$
2: $Q_{H+1}(s,a) \leftarrow 0$ for all $(s,a) \in \mathcal{S} \times \mathcal{A}$
3: Initialize datasets $\{D_h\}$ as empty

4: for episode $k = 1, \ldots, K$ do

5: $V_{H+1}(s) \leftarrow 0$ for each $s \in \mathcal{S}$

6: for step $h = H, \ldots, 1$ do

7: Update $w_h$ via Equation (8)

8: for $(s,a) \in \mathcal{S} \times \mathcal{A}$ such that $N_h(s,a) \geq 1$ do

9: $b_h(s,a) \leftarrow c_\gamma \left| e^{\beta H} - 1 \right| \sqrt{\frac{2\log(2SAT/\delta)}{N_h(s,a)}}$ for some universal constant $c_\gamma > 0$

10: $Q_h(s,a) \leftarrow \begin{cases} \frac{1}{\beta} \log \left( \min \{ e^{\beta(H-h+1)}, w_h(s,a) + b_h(s,a) \} \right), & \text{if } \beta > 0; \\ \frac{1}{\beta} \log \left( \max \{ e^{\beta(H-h+1)}, w_h(s,a) - b_h(s,a) \} \right), & \text{if } \beta < 0 \end{cases}$

11: $V_h(s) \leftarrow \max_{a' \in \mathcal{A}} Q_h(s,a')$

12: end for

13: end for

14: for step $h = 1, \ldots, H$ do

15: Take action $a_h \leftarrow \arg\max_{a \in \mathcal{A}} Q_h(s_h,a)$ and observe $r_h(s_h,a_h)$ and $s_{h+1}$

16: $N_h(s_h,a_h) \leftarrow N_h(s_h,a_h) + 1$

17: Insert $(s_h, a_h, s_{h+1})$ into $D_h$

18: end for

19: end for

$\sum_{s'} e^{\beta V_{H+1}(s')}$ tends to be small, given that $\beta < 0$ and $\{V_{H+1}(s')\}$ are large. By subtracting a positive $b_h$ from $w_h$, we obtain an even smaller quantity $w_h(s^+,\cdot) - b_h(s^+,\cdot)$. We can then deduce that $Q_h(s^+,\cdot) \approx \frac{1}{\beta} \log[w_h(s^+,\cdot) - b_h(s^+,\cdot)]$ is larger compared to $\frac{1}{\beta} \log[w_h(s^+,\cdot)]$ which does not incorporate bonus, since the logarithmic function is monotonic and again $\beta < 0$ (we ignore thresholding for the moment). Therefore, subtracting bonus serves as a UCB for $\beta < 0$. Since the exact form of the UCB depends on both the magnitude and sign of $\beta$ (as shown in Lines 9 and 10), we name it Risk-Sensitive Upper Confidence Bound (RS-UCB) and this results in what we call Risk-Sensitive Optimism in the Face of Uncertainty (RS-OFU).
Algorithm 2 RSQ

Input: number of episodes $K \in \mathbb{Z}_{>0}$, confidence level $\delta \in (0, 1]$, learning rates $\{\alpha_t\}$ and risk parameter $\beta \neq 0$

1: $Q_h(s, a), V_h(s, a) \leftarrow H - h + 1$ and $N_h(s, a) \leftarrow 0$ for all $(h, s, a) \in [H] \times S \times A$

2: $Q_{H+1}(s, a), V_{H+1}(s, a) \leftarrow 0$ for all $(s, a) \in S \times A$

3: for episode $k = 1, \ldots, K$ do

4: Receive the initial state $s_1$

5: for step $h = 1, \ldots, H$ do

6: Take action $a_h \leftarrow \text{argmax}_{a' \in A} Q_h(s_h, a')$, and observe $r_h(s_h, a_h)$ and $s_{h+1}$

7: $t = N_h(s_h, a_h) \leftarrow N_h(s_h, a_h) + 1$

8: $b_t \leftarrow c \left| e^{\beta H} - 1 \right| \sqrt{\frac{H \log(SAT/\delta)}{t}}$ for some sufficiently large universal constant $c > 0$

9: $w_h(s_h, a_h) \leftarrow (1 - \alpha_t)e^{\beta Q_h(s_h, a_h)} + \alpha_te^{\beta[r_h(s_h, a_h) + V_{k+1}(s_{k+1})]}$

10: $Q_h(s_h, a_h) \leftarrow \begin{cases} \frac{1}{\beta} \log \left[ \min \left\{e^{\beta(H-h+1)}, w_h(s_h, a_h) + \alpha_t b_t \right\} \right], & \text{if } \beta > 0; \\ \frac{1}{\beta} \log \left[ \max \left\{e^{H-h+1}, w_h(s_h, a_h) - \alpha_t b_t \right\} \right], & \text{if } \beta < 0 \end{cases}$

11: $V_h(s_h) \leftarrow \max_{a' \in A} Q_h(s_h, a')$

12: end for

13: end for

3.2 Risk-Sensitive Q-learning

Although Algorithm 1 is model-free, it requires storage of historical data $\{D_h\}$ and computation over them (Line 7). A more efficient class of algorithms is Q-learning algorithms, which update Q values in an online fashion as each state-action pair is encountered. We therefore propose Risk-Sensitive Q-learning (RSQ) and formally describe it in Algorithm 2.

Mechanism of Algorithm 2. Algorithm 2 is based on Q-learning with UCB studied in the work of [38] and we use the same learning rates therein

$$\alpha_t := \frac{H + 1}{H + t} \quad (9)$$

for every integer $t \geq 1$. Similar to Algorithm 1, Algorithm 2 consists of the policy execution step (Line 6) and value estimation step (Lines 9–11). Line 9 updates the intermediate value $w_h$ in an online fashion, in contrast with the batch update in Line 7 of Algorithm 1, and Algorithm 2 can thus be seen as an online algorithm. Line 10 then applies the same logarithmic-exponential transform to the intermediate value and bonus as in Algorithm 1. Note the similar way we use
the bonus term $b_t$ in estimating Q-values in Line 10 of Algorithm 2 as in Line 10 of Algorithm 1. Algorithm 2 therefore also implements RS-UCB and follows the principle of RS-OFU.

**Comparisons of Algorithms 1 and 2.** It is interesting to compare the bonuses used in Algorithms 1 and 2. The bonuses in both algorithms depend on the risk parameter $\beta$ through a common factor $|e^{\beta H} - 1|$. A careful analysis (see our proofs in appendices) on the bonuses and the value estimation steps reveals that the effective bonuses added to the estimated value function is proportional to $|e^{\beta H} - 1|$. This means that the more risk-seeking/averse an agent is (or the larger $|\beta|$ is), the larger bonus it needs to compensate for its uncertainty over the environment. Such risk sensitivity of the bonus is also reflected in the regret bounds; see Theorems 1 and 2 below. Also, it is not hard to see that both algorithms have polynomial time and space complexities in $S$, $A$, $K$ and $H$. Moreover, thanks to its online update procedure, Algorithm 2 is more efficient than Algorithms 1 in both time and space complexities, since it does not require storing historical data (in particular, $\{D_h\}$ of Algorithm 1) nor computing statistics based on them for value estimation.

## 4 Main results

In this section, we first present regret bounds for Algorithms 1 and 2, and then we complement the results with a lower bound on regret that any algorithm has to incur.

### 4.1 Regret upper bounds

The following theorem gives an upper bound for regret incurred by Algorithm 1. Let $T := KH$ be the total number of timesteps for which an algorithm is run, and recall the function $\lambda(u) := (e^{3u} - 1)/u$.

**Theorem 1.** For any $\delta \in (0,1]$, with probability at least $1 - \delta$, the regret of Algorithm 1 is bounded by

$$\text{Regret}(K) \leq \lambda(|\beta|H^2) \cdot \sqrt{H^3S^2AT\log^2(2SAT/\delta)}.$$ 

The proof is given in Appendix C. We see that the result of Theorem 1 adapts to both risk-seeking ($\beta > 0$) and risk-averse ($\beta < 0$) settings through a common factor of $\lambda(|\beta|H^2)$.

As $\beta \to 0$, the setting of risk-sensitive RL tends to that of standard and risk-neutral RL, and we have an immediate corollary to Theorem 1 as a precise characterization.
Corollary 1. Under the setting of Theorem 1 and when $\beta \to 0$, with probability at least $1 - \delta$, the regret of Algorithm 1 is bounded by

$$\text{Regret}(K) \lesssim \sqrt{H^3S^2AT\log^2(2SAT/\delta)}.$$ 

Proof. The result follows from Theorem 1 and the fact that $\lim_{\beta \to 0} \lambda(|\beta|H^2) = 3$. 

The result in Corollary 1 recovers the regret bound of [4, Theorem 2] under the standard RL setting and is nearly optimal compared to the minimax rates presented in [3, Theorems 1 and 2]. Corollary 1 also reveals that Theorem 1 interpolates between the risk-sensitive and risk-neutral settings.

Next, we give a regret upper bound for Algorithm 2 in the following theorem.

Theorem 2. For any $\delta \in (0, 1]$, with probability at least $1 - \delta$ and when $T$ is sufficiently large, the regret of Algorithm 2 is bounded by

$$\text{Regret}(K) \lesssim \lambda(|\beta|H^2) \cdot \sqrt{H^4SAT\log(SAT/\delta)}.$$ 

The proof is given in Appendix E. Similarly to Theorem 1, Theorem 2 also covers both risk-seeking and risk-averse settings via the same factor $\lambda(|\beta|H^2)$, which gives the risk-neutral bound when $\beta \to 0$ as shown in the following.

Corollary 2. Under the setting of Theorem 2 and when $\beta \to 0$, with probability at least $1 - \delta$, the regret of Algorithm 2 is bounded by

$$\text{Regret}(K) \lesssim \sqrt{H^4SAT\log(SAT/\delta)}.$$ 

The proof follows the same reasoning as in that of Corollary 1. According to Corollary 2, the regret upper bound for Algorithm 2 matches the nearly optimal result in [38, Theorem 2] under the risk-neutral setting. As such, Theorems 1 and 2 strictly generalizes the existing nearly optimal regret bounds (up to polynomial factors).

The crux of the proofs of both Theorems 1 and 2 lies in a local linearization argument for the non-linear Bellman equations and non-linear updates of the algorithms, in which action-value and value functions are related by a logarithmic-exponential transformation. Although logarithmic and exponential functions are not Lipschitz globally, we show that they are locally Lipschitz in the domain of our interest, and their combined local Lipschitz factors turn out to be the exponential factors in the theorems. Once the Bellman equations and algorithm estimates are linearized,
we can apply standard techniques in RL to obtain the final regret. It is noteworthy that, as suggested by [38], the regret bounds in Theorems 1 and 2 can automatically be translated into sample complexity bounds in the probably approximately correct (PAC) setting, which did not previously exist even given access to a simulator.

In the risk-sensitive setting where $\beta$ is bounded away from 0, our regret bounds of Theorems 1 and 2 depend exponentially in the horizon length $H$ and the risk sensitivity $|\beta|$. In what follows, we argue that such exponential dependence is unavoidable.

4.2 Regret lower bound

We now present a fundamental lower bound on the regret, which complements the upper bounds in Theorems 1 and 2.

**Theorem 3.** For sufficiently large $K$ and $H$, the regret of any algorithm obeys

$$E[\text{Regret}(K)] \gtrsim e^{|\beta|H/2} - 1 \sqrt{T \log T}.$$ 

The proof is given in Appendix F. In the proof, we construct a bandit model that can be seen as a special case of our episodic fixed-horizon MDP problem, and then we show that any bandit algorithm has to incur an expected regret, in terms of the logarithmic-exponential objective, that grows as predicted in Theorem 3.

Theorem 3 shows that the exponential dependence on the $|\beta|$ and $H$ in Theorems 1 and 2 is essentially indispensable. In addition, it features a sub-linear dependence on $T$ through the $O(\sqrt{T})$ factor. In view of Theorem 3, therefore, both Theorems 1 and 2 are nearly optimal in their dependence on $\beta$, $H$ and $T$. One should contrast Theorem 3 with Lemma 1, which shows that the worst-case regret is linear in $H$ and $T$. Such a linear regret can be attained by any trivial algorithm that does not learn at all. In sharp contrast, in order to achieve the optimal $\sqrt{T}$ scaling (which by standard arguments implies a finite sample-complexity bound), an algorithm must incur a regret that is exponential in $H$. Therefore, our results show a (perhaps surprising) tradeoff between risk sensitivity and sample efficiency.

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Appendices

A Preliminaries

We set some notations and shorthands before the proofs. For both Algorithms 1 and 2, we let \( s_k^h, a_k^h, w_k^h, Q_k^h \) and \( V_k^h \) denote the values of \( s_h, a_h, w_h, Q_h \) and \( V_h \) in episode \( k \), and we denote by \( N_k^h \) the value of \( N_h \) at the end of episode \( k - 1 \). For Algorithm 1, we let \( D_k^h \) be the value of \( D_h \) at the end of episode \( k - 1 \). Next, we introduce a simple yet powerful result.

**Fact 1.** Consider \( x, y, b \in \mathbb{R} \) such that \( x \geq y \).

(a) if \( y \geq g \) for some \( g > 0 \), then \( \log(x) - \log(y) \leq \frac{1}{g}(x - y) \);

(b) Assume further that \( y \geq 0 \). If \( b \geq 0 \) and \( x \leq u \) for some \( u > 0 \), then \( e^{bx} - e^{by} \leq be^b(x - y) \); if \( b < 0 \), then \( e^{by} - e^{bx} \leq (-b)(x - y) \).

**Proof.** The results follow from Lipschitz continuity of the functions \( x \mapsto \log(x) \) and \( x \mapsto e^{bx} \).

We record a simple fact about exponential factors.

**Fact 2.** Define \( \lambda_0 := \frac{e^{\beta H - 1}}{|\beta|} \) and \( \lambda_2 := e^{\beta |H^2 + H|} \). Then we have \( \lambda_0 \lambda_2 H \leq \frac{e^{3\beta H^2 - 1}}{|\beta|} \).

B Proof warmup for Theorem 1

First, we set some notations and definitions. Define \( d := SA, i := \log(2dT/\delta) \) for a given \( \delta \in (0, 1] \), and \( I \) to be the \( d \times d \) identity matrix. To streamline some parts of the proof, we define \( \phi(s, a) \) to be a vector in \( \mathbb{R}^d \) whose \((s, a)\)-th entry is equal to one and other entries equal to zero (so \( \phi(s, a) \) is a canonical basis of \( \mathbb{R}^{SA} \)). Also let \( \Lambda_k^h \) be a diagonal matrix in \( \mathbb{R}^{d \times d} \) with each \((s, a)\)-th diagonal entry equal to \( \max\{N_k^{h-1}(s, a), 1\} \). It can be seen that \( \Lambda_k^h \) is positive definite. We adopt the shorthands \( \phi^\top_h := \phi(s_h^\top, a_h^\top) \) and \( r^\top_h := r_h(s_h^\top, a_h^\top) \) for \((\tau, h) \in [K] \times [H]\).

From now on, we fix a tuple \((k, h) \in [K] \times [H]\) and then fix \((s, a) \in S \times A \) such that \( N_h^{k-1}(s, a) \geq 1 \). We also fix a policy \( \pi \). We set

\[
w_h^\pi = e^{\beta Q_h^\pi (\cdot)}.
\]

It can be verified that by the definition of \( \phi(s, a) \), we have

\[
Q_h^\pi (s, a) = \frac{1}{\beta} \log \left( e^{\beta Q_h^\pi (s, a)} \right).
\]
Define
\[
\Lambda = \text{our goal is to derive lower and upper bounds for } G
\]
in words, 
\[
q \text{ and } \Lambda
\]
replaces the quantity 
\[
\log \beta
\]
and
\[
\log \log k
\]
for 
\[
\{ s, w, a \}, q = 1
\]
and
\[
\pi
\]
observe that
\[
Q \log \beta
\]
and
\[
\phi
\]
\[
\min \{ e^{\beta (H-h+1)}, q_1^+ \}, \text{ if } \beta > 0,
\]
\[
\max \{ e^{\beta (H-h+1)}, q_1^- \}, \text{ if } \beta < 0.
\]
By the definition of \( \Lambda_h^k \) and \( \phi_h^k \), observe that
\[
\bar{w}_h^k (s, a) = \langle \phi(s, a), w_h^k \rangle = \left< \phi(s, a), \left( \Lambda_h^k \right)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau \left[ e^{\beta \mathcal{Q}_h^\tau (s_h^\tau, a_h^\tau)} \right] \right>,
\]
Define
\[
G_0 := (Q_h^k - Q_h^\tau) (s, a) = \frac{1}{\beta} \log \{ q_1 \} - \frac{1}{\beta} \log \{ \langle \phi(s, a), w_h^\tau \rangle \},
\]
and our goal is to derive lower and upper bounds for \( G_0 \). From Equation (14), we have
\[
G_0 = \frac{1}{\beta} \log \{ q_1 \} - \frac{1}{\beta} \log \left< \phi(s, a), (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau \left[ e^{\beta \mathcal{Q}_h^\tau (s_h^\tau, a_h^\tau)} \right] \right>
\]
\[
= \frac{1}{\beta} \log \{ q_1 \} - \frac{1}{\beta} \log \left< \phi(s, a), (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau \left[ \mathcal{E}_{s' \sim \mathcal{P}_h} (\cdot | s_h^\tau, a_h^\tau) e^{\beta (r_h^\tau + V_{h+1}^\tau (s'))} \right] \right>
\]
\[
= \frac{1}{\beta} \log \{ q_1 \} - \frac{1}{\beta} \log \{ q_3 \}.
\]
The first step above holds by Equation (12), and the second step follows from Equation (3). In order to control \( G_0 \), we define an intermediate quantity
\[
q_2 := \left< \phi(s, a), (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^\tau \left[ \mathcal{E}_{s' \sim \mathcal{P}_h} (\cdot | s_h^\tau, a_h^\tau) e^{\beta (r_h^\tau + V_{h+1}^\tau (s'))} \right] \right>;
\]
in words, \( q_2 \) replaces the quantity \( V_{h+1}^\tau \) in \( q_3 \) by \( V_{h+1}^k \). It can be seen that
\[
G_0 = G_1 + G_2,
\]
where
\[
G_1 := \frac{1}{\beta} \log \{q_1\} - \frac{1}{\beta} \log \{q_2\},
\]
\[
G_2 := \frac{1}{\beta} \log \{q_2\} - \frac{1}{\beta} \log \{q_3\}.
\] (16)

Note that \(G_0, G_1\) and \(G_2\) are all well-defined, according to the following result.

**Lemma 2.** We have \(q_i \in [\min\{1, e^{\beta(H-h+1)}\}, \max\{1, e^{\beta(H-h+1)}\}]\) for \(i \in [3]\).

**Proof.** We prove the result by focusing on \(q_1\). By the definitions of \(\Lambda_h^k\) and \(\phi\), the \((s,a)\)-th entry of the vector \((\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_{\tau}^T \cdot u_{\tau}^T\) equals \(\frac{1}{N_h^{k-1}(s,a)} \sum_{\tau \in [k-1]} u_{\tau}^T \cdot I\{\left(\left(s_{\tau}, a_{\tau}\right) = (s,a)\right)\}\) for any sequence \(\{u_{\tau}^T\}_{\tau \in [k-1]}\). Then, the result follows from the fact that \(e^{\beta[r_{\tau + V^\tau_h(s')}]} \in [\min\{1, e^{\beta(H-h)}\}, \max\{1, e^{\beta(H-h)}\}]\) for \((\tau, s') \in [K] \times S\) and the definition of \(q_1\).

Therefore, we have the following equivalent form of Equation (14):
\[
(Q^k_h - Q^\tau_h)(s,a) = G_1 + G_2.
\] (17)

Thanks to the identity (17), our goal is now to control \(G_1\) and \(G_2\), which is done in the following lemma.

**Lemma 3.** For all \((k,h,s,a) \in [K] \times [H] \times S \times A\) that satisfies \(N_{h}^{k-1}(s,a) \geq 1\), there exist universal constants \(c_1, c_\gamma > 0\) (where \(c_\gamma\) is used in Line 9 of Algorithm 1) such that
\[
0 \leq G_1 \leq c_1 \cdot \frac{e^{\beta|H|} - 1}{|B|} \cdot d \sqrt{k} \sqrt{\phi(s,a)^T (\Lambda_h^k)^{-1} \phi(s,a)}
\]
with probability at least \(1 - \delta/2\). Furthermore, if \(V_{h}^{k}(s') \geq V_{h+1}^{\tau}(s')\) for all \(s' \in S\), then we have
\[
0 \leq G_2 \leq e^{\beta|H|} \cdot E_{s'' \sim P_h(\cdot | s, a)}[V_{h+1}^{k}(s') - V_{h+1}^{\tau}(s')].
\]

**Proof.** Case \(\beta > 0\). To control \(G_1\), we note that \(N_{h}^{k-1}(s,a) = \phi(s,a)^T (\Lambda_h^k)^{-1} \phi(s,a)\) and by Equation (13) we can compute
\[
\left|q_1^+ - q_2 - b^k_h(s,a)\right| = \left|\langle \phi(s,a), (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_{\tau}^T \left[ e^{\beta[r_{\tau + V^\tau_h(s')}]} - E_{s'' \sim P_h(\cdot | s_{\tau}, a_{\tau})} e^{\beta[r_{\tau + V^\tau_h(s')}] - E_{s'' \sim P_h(\cdot | s, a)} e^{\beta[r_h(s,a) + V^k_{h+1}(s')]}} \right]\right|
\]
\[
\leq \frac{1}{N_{h}^{k-1}(s,a)} \sum_{(s,a,s'') \in D_h^{k-1}} \left| e^{\beta[r_h(s,a) + V^k_{h+1}(s')]} - E_{s'' \sim P_h(\cdot | s, a)} e^{\beta[r_h(s,a) + V^k_{h+1}(s')]}} \right|
\]
\[
\leq \frac{1}{N_h^{k-1}(s,a)} \sum_{t \in [N_h^{k-1}(s,a)]} c' \left| e^{\beta H} - 1 \right| \sqrt{\frac{S_t}{t}}
\]

\[
\leq \frac{1}{N_h^{k-1}(s,a)} \int_{t \in [0,N_h^{k-1}(s,a)]} c' \left| e^{\beta H} - 1 \right| \sqrt{\frac{S_t}{t}} dt
\]

\[
= \frac{1}{N_h^{k-1}(s,a)} \cdot c \left| e^{\beta H} - 1 \right| \sqrt{S_t \cdot N_h^{k-1}(s,a)}
\]

\[
= c \left| e^{\beta H} - 1 \right| \sqrt{S_t} \cdot \sqrt{\phi(s,a)^\top (\Lambda_h^k)^{-1} \phi(s,a)},
\]

where the fourth step holds by Lemma 6, and the last step holds by the definition of $\Lambda_h^k$; in the above, $c' > 0$ is a universal constant and $c = 2c'$. If we choose $c_p = c$ in the definition of $b_h^k(s,a)$ in Line 9 of Algorithm 1, we have

\[
0 \leq q_1^+ - q_2 \leq 2c \cdot \left| e^{\beta H} - 1 \right| \sqrt{S_t} \cdot \sqrt{\phi(s,a)^\top (\Lambda_h^k)^{-1} \phi(s,a)}.
\]

Therefore, we have $q_1 \geq q_2$, and thus $G_1 \geq 0$, by the first inequality above, the definition of $q_1$ and Lemma 2 (in particular, $q_2 \leq e^{\beta(H-h+1)}$). By Lemma 2 and Fact 1(a) (with $g = 1$, $x = q_1$ and $y = q_2$), we have

\[
G_1 \leq \frac{1}{\beta} (q_1 - q_2) \leq \frac{1}{\beta} (q_1^+ - q_2),
\]

which together with the second inequality displayed above implies the desired upper bound on $G_1$.

Now we control the term $G_2$. For $\beta > 0$, it is not hard to see that the assumption $V_{h+1}^k(s') \geq V_{h+1}^{\pi}(s')$ for all $s' \in S$ implies that $q_2 \geq q_3$ and therefore $G_2 \geq 0$. We also have

\[
G_2 \leq \frac{1}{\beta} (q_2 - q_3)
\]

\[
\leq e^{\beta H} \left( \phi(s,a)^\top (\Lambda_h^k)^{-1} \sum_{\tau \in [k-1]} \phi_h^{\tau} \left[ E_{s' \sim P_h(\cdot|s,a)}[V^{k}_{h+1}(s') - V^{\pi}_{h+1}(s')] \right] \right)
\]

\[
= e^{\beta H} E_{s' \sim P_h(\cdot|s,a)}[V^{k}_{h+1}(s') - V^{\pi}_{h+1}(s')],
\]

where the first step holds by Fact 1(a) (with $g = 1$, $x = q_2$, and $y = q_3$) and the fact that $q_2 \geq q_3 \geq 1$ (with the last inequality suggested by Lemma 2), and the second step holds by Fact 1(b) (with $b = \beta$, $x = r_h^T + V_{h+1}^{k}(s)$, and $y = r_h^T + V_{h+1}^{\pi}(s)$) and $H \geq r_h^T + V_{h+1}^{k}(s) \geq r_h^T + V_{h+1}^{\pi}(s) \geq 0$.

**Case** $\beta < 0$. Similar to the case of $\beta > 0$, we have

\[
\left| q_1^+ - q_2 + b_h^k(s,a) \right|
\]

\[
\leq c \cdot \left| e^{\beta H} - 1 \right| \sqrt{S_t} \cdot \sqrt{\phi(s,a)^\top (\Lambda_h^k)^{-1} \phi(s,a)}.
\]
If we choose $c_r = c$ in the definition of $b_h^k(s, a)$ in Line 9 of Algorithm 1, the above equation implies
\[
0 \leq q_2 - q_1^+ \leq 2c \cdot |e^{\beta H} - 1| \sqrt{S_t} \cdot \sqrt{\phi(s, a)^\top (\Lambda_h^k)^{-1} \phi(s, a)}.
\]

Therefore, we have $q_1 \leq q_2$, and thus $G_1 \geq 0$, by the first inequality displayed above, the definition of $q_1$ and Lemma 2 (in particular, $q_2 \geq e^{\beta(H - h + 1)}$). By Lemma 2 and Fact 1(a) (with $g = e^{\beta H}, x = q_2$ and $y = q_1$), we further have
\[
G_1 = \frac{1}{(-\beta)} \left( \log \{q_2\} - \log \{q_1\} \right)
\leq \frac{e^{-\beta H}}{|\beta|} (q_2 - q_1)
\leq \frac{e^{-\beta H}}{|\beta|} (q_2 - q_1^+),
\]
which together with the second inequality displayed above and the fact that $|e^{\beta H} - 1| = 1 - e^{\beta H}$ implies the desired upper bound on $G_1$.

Next we control $G_2$. The assumption $V_{h+1}^k(s') \geq V_{h+1}^\pi(s')$ for all $s' \in S$ implies that $q_2 \leq q_3$ and therefore $G_2 \geq 0$. We also have
\[
G_2 = \frac{1}{(-\beta)} \left( \log \{q_3\} - \log \{q_2\} \right)
\leq \frac{e^{-\beta H}}{|\beta|} (q_3 - q_2)
\leq e^{\beta H} \left( e^{-\beta H} \sum_{s' \in S} \phi_h^\top \left[ \mathbb{E}_{s' \sim P_h(. | s_h, a_h)} [V_{h+1}^k(s') - V_{h+1}^\pi(s')] \right] \right)
\leq e^{\beta H} \mathbb{E}_{s' \sim P_h(. | s_h, a_h)} [V_{h+1}^k(s') - V_{h+1}^\pi(s')],
\]
where the second step holds by Fact 1(a) (with $g = e^{\beta H}, x = q_3, y = q_2$) and the fact that $q_3 \geq q_2 \geq e^{\beta H}$ (with the last inequality suggested by Lemma 2), and the third step holds by Fact 1(b) (with $b = \beta, x = r_h^T + V_{h+1}^k(s), y = r_h^T + V_{h+1}^\pi(s)$) and $r_h^T + V_{h+1}^k(s) \geq r_h^T + V_{h+1}^\pi(s) \geq 0$.

The proof is hence completed. \hfill \Box

The next lemma establishes the dominance of $Q_h^k$ over $Q_h^\pi$.\hfill \Box

**Lemma 4.** On the event of Lemma 3, we have $Q_h^k(s, a) \geq Q_h^\pi(s, a)$ for all $(k, h, s, a) \in [K] \times [H] \times S \times A$.

**Proof.** For the purpose of the proof, we set $Q_{h+1}^\pi(s, a) = Q_{h+1}^*(s, a) = 0$ for all $(s, a) \in S \times A$. We fix a tuple $(k, s, a) \in [K] \times S \times A$ and use strong induction on $h$. The base case for $h = H + 1$
Proof. The result follows from Lemma 4 and Equation (18).

Proof. The proof follows the same reasoning as [4, Lemma 12].

We also assume that \( s, a \) satisfies \( N_h^{k-1}(s, a) \geq 1 \), since otherwise \( Q^k_h(s, a) = H - h + 1 \geq Q^\pi_h(s, a) \) and we are done. This assumption and Equation (18) together imply \( G_2 \geq 0 \) by Lemma 3. We also have \( G_1 \geq 0 \) on the event of Lemma 3. Therefore, it follows that \( (Q^k_h - Q^\pi_h)(s, a) \geq 0 \) by Equation (17). The induction is completed and so is the proof. \( \square \)

Lemma 4 leads to an immediate and important corollary.

**Lemma 5.** For any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta / 2 \), we have \( V^i_h(s) \geq V^\pi_h(s) \) for all \( (k, h, s) \in [K] \times [H] \times S \).

**Proof.** The result follows from Lemma 4 and Equation (18). \( \square \)

### B.1 Supporting lemmas

We first present a concentration result.

**Lemma 6.** Define

\[
\tilde{V}_{h+1} := \left\{ \tilde{V}_h : S \rightarrow \mathbb{R} \mid \forall s \in S, \tilde{V}_h(s) \in [\min\{e^{\beta(H-h)}, 1\}, \max\{e^{\beta(H-h)}, 1\}] \right\}.
\]

There exists a universal constant \( c > 0 \) such that with probability \( 1 - \delta \), we have

\[
\left| e^{\beta \mathbb{E}_{s' \sim P_h(|s|_h \leq a_{h+1})} r_h(s_{h+1}) + \mathbb{V}(s_{h+1})} - \mathbb{E}_{s' \sim P_h(|s|_h \leq a_{h+1})} e^{\beta (r_h(s_{h+1}) + \mathbb{V}(s'))} \right| \leq c \left| e^{\beta H} - 1 \right| \sqrt{\frac{S_i}{N^k_h(s, a)}}
\]

for all \( (k, h, s, a) \in [K] \times [H] \times S \times A \) and all \( \mathbb{V} \in \tilde{V}_{h+1} \).

**Proof.** The proof follows the same reasoning as [4, Lemma 12]. \( \square \)

The next few lemmas help control \( \sum_{k \in [K]} (\phi^k_h)^\top (\Lambda^k_h)^{-1} \phi^k_h \).

**Lemma 7** ([39, Lemma D.2]). Let \( \{\phi_t\}_{t \geq 0} \) be a bounded sequence in \( \mathbb{R}^d \) satisfying \( \sup_{t \geq 0} \|\phi_t\| \leq 1 \). Let \( \Lambda_0 \in \mathbb{R}^{d \times d} \) be a positive definite matrix with \( \lambda_{\min}(\Lambda_0) \geq 1 \). For any \( t \geq 0 \), we define \( \Lambda_t := \Lambda_0 + \sum_{i \in [t]} \phi_i^\top \phi_i \). Then, we have

\[
\log \left[ \frac{\det(\Lambda_t)}{\det(\Lambda_0)} \right] \leq \sum_{i \in [t]} \phi_i^\top \Lambda_{i-1}^{-1} \phi_i \leq 2 \log \left[ \frac{\det(\Lambda_t)}{\det(\Lambda_0)} \right].
\]
Lemma 8. Recall the definitions of $\phi^k_h$ and $\Lambda^k_h$. For any $h \in [H]$, we have
\[
\sum_{k \in [K]} (\phi^k_h)\top (\Lambda^k_h)^{-1} \phi^k_h \leq 2d_t,
\]
where $t = \log(2d_T/\delta)$

Proof. Define $\Gamma^k_h := \lambda I + \sum_{\tau \in [k-1]} \phi^\tau_h (\phi^\tau_h)\top$ with $\lambda = 1$. It is not hard to see that by the definition of $\Lambda^k_h$ we have $\Lambda^k_h \preceq \Gamma^k_h$ for $h \in [H]$. Since $\lambda_{\min}(\Gamma^k_h) \geq 1$ and $\|\phi^k_h\| \leq 1$ for all $(k, h) \in [K] \times [H]$, by Lemma 7 we have for any $h \in [H]$ that
\[
\sum_{k \in [K]} (\phi^k_h)\top (\Lambda^k_h)^{-1} \phi^k_h \leq \sum_{k \in [K]} (\phi^k_h)\top (\Gamma^k_h)^{-1} \phi^k_h \leq 2 \log \left[ \frac{\det(\Gamma^k_{h+1})}{\det(\Gamma^k_h)} \right].
\]
Furthermore, note that $\|\Gamma^k_{h+1}\| = \|\lambda I + \sum_{\tau \in [k]} \phi^\tau_h (\phi^\tau_h)\top\| \leq \lambda + k$. This implies
\[
\sum_{k \in [K]} (\phi^k_h)\top (\Lambda^k_h)^{-1} \phi^k_h \leq 2d \log \left[ \frac{\lambda + k}{\lambda} \right] \leq 2d_t,
\]
as desired. \hfill \square

C Proof of Theorem 1

Define $\delta^k_h := V^k_h(s^k_h) - V^\tau_h(s^k_h)$, and $\zeta^k_{h+1} := \mathbb{E}_{s' \sim P_h(\cdot|s_h^k,a_h^k)}[V^k_{h+1}(s') - V^\tau_{h+1}(s')] - \delta^k_{h+1}$. For any $(k, h) \in [K] \times [H]$, we have
\[
\delta^k_h = (Q^k_h - Q^\tau_h) (s^k_h, a^k_h)
\leq c_1 \cdot e^{\beta|H| - 1/|\beta|} \cdot \sqrt{S_i} \sqrt{\phi(s^k_h, a^k_h)\top (\Lambda^k_h)^{-1} \phi(s^k_h, a^k_h)}
+ e^{\beta|H|} \cdot \mathbb{E}_{s' \sim P_h(\cdot|s_h^k,a_h^k)}[V^k_{h+1}(s') - V^\tau_{h+1}(s')]
= c_1 \cdot e^{\beta|H| - 1/|\beta|} \cdot \sqrt{S_i} \sqrt{\phi(s^k_h, a^k_h)\top (\Lambda^k_h)^{-1} \phi(s^k_h, a^k_h)}
+ e^{\beta|H|} (\delta^k_{h+1} + \zeta^k_{h+1}). \tag{19}
\]

In the above equation, the first step holds by the construction of Algorithm 1 and the definition of $V^\tau_h$ in Equation (3); the second step is a consequence of combining Equation (17) as well as Lemmas 3 and 5; the last step follows from the definitions of $\delta^k_h$ and $\zeta^k_{h+1}$.

Noting that $V^k_{h+1}(s) = V^\tau_{h+1}(s) = 0$ and the fact that $\delta^k_{h+1} + \zeta^k_{h+1} \geq 0$ implied by Lemma 5, we can continue by expanding the recursion in Equation (19) and get
\[
\delta^k_1 \leq \sum_{h \in [H]} e^{(\beta|H|)h} \zeta^k_{h+1}
\]
where

\[
\text{Regret}(K) = \sum_{k \in [K]} \left[ (V^*_1 - V^1_k)(s^k) \right]
\]

\[
\leq \sum_{k \in [K]} \delta_k^k
\]

\[
\leq e^{\lvert \beta \rvert^2} \sum_{k \in [K]} \sum_{h \in [H]} \xi_{k+1}^k
\]

\[
+ c_1 \cdot \frac{e^{\lvert \beta \rvert^2} - 1}{\lvert \beta \rvert} \cdot e^{\lvert \beta \rvert^2} \cdot \sqrt{\delta} \sum_{k \in [K]} \sum_{h \in [H]} \sqrt{\phi(s_h^k, a_h^k) \Lambda_h^k (s_h^k, a_h^k)}.
\]  

Therefore, we have

\[
\text{Regret}(K) \leq e^{\lvert \beta \rvert^2} \cdot 2H \cdot \sqrt{T} \cdot \log(2/\delta) \leq 2H \sqrt{T}i,
\]

where the second step holds by Lemma 5 with \( \pi \) therein set to the optimal policy, and in the last step we applied Equation (20) along with the Cauchy-Schwarz inequality.

We proceed to control the two terms in Equation (21). Since the construction of \( V^k_h \) is independent of the new observation \( s^k_h \) in episode \( k \), we have that \( \{ \xi_{k+1}^k \} \) is a martingale difference sequence satisfying \( \lvert \xi_{k+1}^k \rvert \leq 2H \) for all \( (k, h) \in [K] \times [H] \). By the Azuma-Hoeffding inequality, we have for any \( t > 0 \),

\[
\Pr \left( \sum_{k \in [K]} \sum_{h \in [H]} \xi_{k+1}^k \geq t \right) \leq \exp \left( -\frac{t^2}{2T \cdot H^2} \right).
\]

Hence, with probability \( 1 - \delta/2 \), there holds

\[
\sum_{k \in [K]} \sum_{h \in [H]} \xi_{k+1}^k \leq \sqrt{2TH^2 \cdot \log(2/\delta)} \leq 2H \sqrt{T}i,
\]

where \( i = \log(2dT/\delta) \). For the second term in Equation (21), we apply Lemma 8 and the Cauchy-Schwarz inequality to obtain

\[
\sum_{k \in [K]} \sum_{h \in [H]} \sqrt{\phi(s_h^k, a_h^k) \Lambda_h^k (s_h^k, a_h^k)}
\]

\[
\leq \sum_{h \in [H]} \sqrt{K} \sum_{k \in [H]} \phi(s_h^k, a_h^k) \Lambda_h^k (s_h^k, a_h^k)
\]

\[
\leq H \sqrt{2dKt}.
\]

Plugging Equations (22) and (23) back to Equation (21) yields

\[
\text{Regret}(K) \leq e^{\lvert \beta \rvert^2} \cdot 2H \sqrt{T}i + c_1 \cdot \frac{e^{\lvert \beta \rvert^2} - 1}{\lvert \beta \rvert} \cdot e^{\lvert \beta \rvert^2} \cdot H \sqrt{2dSKt^2}
\]

\[
\leq (c_1 + 2) \cdot \frac{e^{\lvert \beta \rvert^2} - 1}{\lvert \beta \rvert} \cdot e^{\lvert \beta \rvert^2} \cdot \sqrt{2dHSTi^2},
\]

\[
\leq (c_1 + 2) \cdot \frac{e^{\lvert \beta \rvert^2} - 1}{\lvert \beta \rvert} \cdot e^{\lvert \beta \rvert^2} \cdot \sqrt{2dHSTi^2}.
\]
where the last step holds since \( \frac{e^{\beta H} - 1}{|\beta|} \geq H \). The proof is completed in view of Fact 2 and the identity \( d = SA \).

DProof warmup for Theorem 2

Recall the learning rates \( \{\alpha_t\} \) defined in Equation (9). Define the quantities

\[
\alpha^0_t := \prod_{j=1}^{t} (1 - \alpha_j), \quad \alpha^i_t := \alpha_i \prod_{j=i+1}^{t} (1 - \alpha_j)
\]

for integers \( i, t \geq 1 \). By convention, we set \( \alpha^0_0 = 1 \) and \( \sum_{i \in [t]} \alpha^i_t = 0 \) if \( t = 0 \), and \( \alpha^i_t = \alpha_i \) if \( t < i + 1 \).

Define the shorthand

\[
\iota := \log(\text{SAT}/\delta)
\]

for \( \delta \in (0, 1] \).

The following fact describes some key properties of the learning rates \( \{\alpha_t\} \).

Fact 3. The following properties hold for \( \alpha^i_t \).

(a) \( \frac{1}{\sqrt{t}} \leq \sum_{i \in [t]} \alpha^i_t \leq \frac{2}{\sqrt{t}} \) for every integer \( t \geq 1 \).

(b) \( \max_{i \in [t]} \alpha^i_t \leq \frac{2H}{t} \) and \( \sum_{i \in [t]} (\alpha^i_t)^2 \leq \frac{2H}{t} \) for every integer \( t \geq 1 \).

(c) \( \sum_{i=1}^{\infty} \alpha^i_i = 1 + \frac{1}{H} \) for every integer \( i \geq 1 \).

(d) \( \sum_{i \in [t]} \alpha^i_t = 1 \) and \( \alpha^0_t = 0 \) for every integer \( t \geq 1 \), and \( \sum_{i \in [t]} \alpha^i_t = 0 \) and \( \alpha^0_t = 1 \) for \( t = 0 \).

Proof. The first three facts can be found in [38, Lemma 4.1], and the last one follows from direct calculation in view of Equation (24).

We also present a lemma that controls the deviation of the exponentiated value function from its expectation.

Lemma 9. There exists a universal constant \( c > 0 \) such that for any \( (k, h, s, a) \in [K] \times [H] \times \mathcal{S} \times \mathcal{A} \) and \( k_1, \ldots, k_t < k \) with \( t = N^k_h(s, a) \), we have

\[
\left| \frac{1}{|\beta|} \sum_{i \in [t]} \alpha^i_t \left[ e^{\beta[r_h(s,a) + V^*_{h+1}(s_{h+1})]} - E_{s' \sim P_h(\cdot|s,a)} e^{\beta(r_h(s,a) + V^*_{h+1}(s'))} \right] \right| \\
\leq c \frac{|e^{\beta H} - 1|}{|\beta|} \sqrt{\frac{H_i}{t}}.
\]

with probability at least \( 1 - \delta \), and

\[
\frac{1}{|\beta|} \sum_{i \in [t]} \alpha^i_t b_i \in \left[ c \frac{|e^{\beta H} - 1|}{|\beta|} \sqrt{\frac{H_i}{t}}, 2c \frac{|e^{\beta H} - 1|}{|\beta|} \sqrt{\frac{H_i}{t}} \right].
\]
Proof. For any \((k, h, s, a) \in [K] \times [H] \times S \times A\), define

\[
\psi(i, k, h, s, a) := e^{\beta [r_h(s,a)+V^{*}_{h+1}(s^i)]} - E_{s' \sim P_h(\cdot|s,a)} e^{\beta [r_h(s,a)+V^{*}_{h+1}(s')]}.
\]

Let us fix a tuple \((k, h, s, a) \in [K] \times [H] \times S \times A\). It can be seen that \(\{\mathbb{I}(k_i \leq K) \cdot \psi(i, k, h, s, a)\}_{i \in [\tau]}\) for \(\tau \in [K]\) is a martingale difference sequence. By the Azuma-Hoeffding inequality and a union bound over \(\tau \in [K]\), we have with probability at least \(1 - \delta/(HSA)\), for all \(\tau \in [K]\),

\[
\left| \sum_{i \in [\tau]} a^i_\tau \cdot \mathbb{I}(k_i \leq K) \cdot \psi(i, k, h, s, a) \right| \leq c \left| e^{\beta H} - 1 \right| \frac{2}{\sqrt{\tau}} \sum_{i \in [\tau]} (a^i_\tau)^2 \leq c \left| e^{\beta H} - 1 \right| \sqrt{\frac{Ht}{t}}
\]

where \(c > 0\) is some universal constant, the first step holds since \(r_h(s,a)+V^{*}_{h+1}(s') \in [0, H]\) for \(s' \in S\), and the last step follows from Fact 3(b). Since the above equation holds for all \(\tau \in [K]\), it also holds for \(\tau = t = N_h^k(s,a) \leq K\). Note that \(\mathbb{I}(k_i \leq K) = 1\) for all \(i \in [N_h^k(s,a)]\). Therefore, applying another union bound over \((h, s, a) \in [H] \times S \times A\), we have that the following holds for all \((k, h, s, a) \in [K] \times [H] \times S \times A\) and with probability at least \(1 - \delta\):

\[
\left| \sum_{i \in [t]} a^i_\tau \cdot \psi(i, k, h, s, a) \right| \leq c \left| e^{\beta H} - 1 \right| \sqrt{\frac{Ht}{t}},
\]

where \(t = N_h^k(s,a)\). Using the fact that \(r_h + V^{*}_{h+1} \in [0, H]\), we have

\[
\left| \frac{1}{\beta} \sum_{i \in [t]} a^i_\tau \left[ E_{s' \sim P_h(\cdot|s,a)} e^{\beta [r_h(s,a)+V^{*}_{h+1}(s^i)]} - E_{s' \sim P_h(\cdot|s,a)} e^{\beta [r_h(s,a)+V^{*}_{h+1}(s')]} \right] \right| = \left| \frac{1}{\beta} \sum_{i \in [t]} a^i_\tau \cdot \psi(i, k, h, s, a) \right| \leq c \left| e^{\beta H} - 1 \right| \sqrt{\frac{Ht}{t}}.
\]

To prove the result for \(\sum_{i \in [t]} a^i_\tau b_i\), we recall the definition of \(\{b_i\}\) in Line 8 of Algorithm 2 and compute

\[
\frac{1}{\beta} \sum_{i \in [t]} a^i_\tau b_i = c \left| e^{\beta H} - 1 \right| \frac{\sqrt{Ht}}{\beta} \sum_{i \in [t]} a^i_\tau \sqrt{\frac{Ht}{t}} \leq \left[ c \left| e^{\beta H} - 1 \right| \sqrt{\frac{Ht}{t}} \right] \left( 2c \left| e^{\beta H} - 1 \right| \sqrt{\frac{Ht}{t}} \right) \leq \left[ c \left| e^{\beta H} - 1 \right| \sqrt{\frac{Ht}{t}} \right] \left( 2c \left| e^{\beta H} - 1 \right| \sqrt{\frac{Ht}{t}} \right)
\]

where the last step holds by Fact 3(a). \(\square\)
We fix a tuple \((k,h,s,a)\in[K]\times[H]\times\mathcal{S}\times\mathcal{A}\) with \(k_i\leq k\) being the episode in which \((s,a)\) is taken the \(i\)-th time at step \(h\). Let us define

\[
q_1^+ := \begin{cases} 
\alpha_0^1 e^{\beta (H-h+1)} + \sum_{i\in[t]} \alpha_i^1 \left[ e^{\beta [r_h(s,a)+v_{h+1}^k(s_h^{k_i})] + b_i} \right], & \text{if } \beta > 0, \\
\alpha_0^1 e^{\beta (H-h+1)} + \sum_{i\in[t]} \alpha_i^1 \left[ e^{\beta [r_h(s,a)+v_{h+1}^k(s_h^{k_i})] - b_i} \right], & \text{if } \beta < 0,
\end{cases}
\]

and

\[
q_2^+ := \begin{cases} 
\alpha_0^1 e^{\beta (H-h+1)} + \sum_{i\in[t]} \alpha_i^1 \left[ e^{\beta [r_h(s,a)+v_{h+1}^k(s_h^{k_i})] + b_i} \right], & \text{if } \beta > 0, \\
\alpha_0^1 e^{\beta (H-h+1)} + \sum_{i\in[t]} \alpha_i^1 \left[ e^{\beta [r_h(s,a)+v_{h+1}^k(s_h^{k_i})] - b_i} \right], & \text{if } \beta < 0,
\end{cases}
\]

\[
q_2 := \begin{cases} 
\min\{e^{\beta (H-h+1)}, q_1^+\}, & \text{if } \beta > 0, \\
\max\{e^{\beta (H-h+1)}, q_1^+\}, & \text{if } \beta < 0,
\end{cases}
\]

\[
q_2' := \alpha_0^1 e^{\beta (H-h+1)} + \sum_{i\in[t]} \alpha_i^1 \left[ e^{\beta [r_h(s,a)+v_{h+1}^k(s_h^{k_i})]} \right],
\]

and

\[
q_3 := \alpha_0^1 e^{\beta Q_h^*(s,a)} + \sum_{i\in[t]} \alpha_i^1 \left[ \mathbb{E}[e^{\beta [r_h(s,a)+v_{h+1}^k(s_h^{k_i})]}] \right].
\]

We have a simple fact on \(q_2\) and \(q_2'\).

**Fact 4.** If \(\beta > 0\), we have \(q_2' \leq q_2\); if \(\beta < 0\), we have \(q_2' \geq q_2\).

**Proof.** We focus on the case of \(\beta > 0\). Note that \(r_h(s,a) + V_{h+1}^* (s_h^{k_i}) \in [0, H-h+1]\), which implies \(e^{\beta [r_h(s,a)+v_{h+1}^k(s_h^{k_i})]} \leq e^{\beta (H-h+1)}\). We also have \(\alpha_0^1, \sum_{i\in[t]} \alpha_i^1 \in \{0, 1\}\) with \(\alpha_0^1 + \sum_{i\in[t]} \alpha_i^1 = 1\) by Fact 3(d). These together imply that \(q_2' \leq e^{\beta H}\) and \(q_2' - q_2^+ = -\sum_{i\in[t]} \alpha_i^1 b_i \leq 0\) by definition of \(b_i\) in Line 8 of Algorithm 2. Therefore, \(q_2' \leq \min\{e^{\beta (H-h+1)}, q_2^+\} = q_2\). The case of \(\beta < 0\) can be proved in a similar way and thus omitted. \(\square\)

Next, we establish a representation of the performance difference \((Q_h^k - Q_h^*) (s,a)\) using the quantities \(q_1\) and \(q_3\).

**Lemma 10.** For any \((k,h,s,a)\in[K]\times[H]\times\mathcal{S}\times\mathcal{A}\), let \(t = N_h^k(s,a)\) and suppose \((s,a)\) was previously taken at step \(h\) of episodes \(k_1, \ldots, k_t < k\). We have

\[
(Q_h^k - Q_h^*)(s,a) = \frac{1}{\beta} \log\{q_1\} - \frac{1}{\beta} \log\{q_3\}.
\]
Proof. The Bellman optimality equation (4) implies
\[ e^\beta Q_h^i(s,a) = e^\beta r_h(s,a) \left[ \mathbf{E}_{s' \sim P_h(\cdot|s,a)} e^{\beta V_{h+1}^*(s')} \right]. \]
By Fact 3(d), we have
\[ e^\beta Q_h^i(s,a) = \alpha_i^0 e^\beta Q_h^i(s,a) + \sum_{i \in [t]} \alpha_i^0 e^\beta r_h(s,a) \left[ \mathbf{E}_{s' \sim P_h(\cdot|s,a)} e^{\beta V_{h+1}^*(s')} \right] = q_3 \]
for each integer \( t \geq 0 \), and therefore
\[ Q_h^* = \frac{1}{\beta} \log \{ q_3 \}. \] (26)
We finish the proof by combining Equation (26) and the fact that \( Q_h^*(s,a) = \frac{1}{\beta} \log \{ q_1 \} \), which follows from Line 10 of Algorithm 2.

We define the quantities
\[ G_1 := \frac{1}{\beta} \log \{ q_1 \} - \frac{1}{\beta} \log \{ q_2 \}, \]
\[ G_2 := \frac{1}{\beta} \log \{ q_2 \} - \frac{1}{\beta} \log \{ q_3 \}, \] (27)
It is not hard to see that \((Q_h^k - Q_h^*) (s,a) = G_1 + G_2\) by Lemma 10. The next lemma establishes upper and lower bounds for \((Q_h^k - Q_h^*) (s,a)\).

Lemma 11. For all \((k,h,s,a) \in [K] \times [H] \times S \times A\) such that \( t = N_h^k(s,a) \geq 1 \), let
\[ \gamma_t := 2 \sum_{i \in [t]} \alpha_i^1 b_i \cdot \begin{cases} \frac{1}{|\mathcal{P}|}, & \text{if } \beta > 0, \\ \frac{e^{-\beta H}}{|\mathcal{P}|}, & \text{if } \beta < 0, \end{cases} \]
and with probability at least \( 1 - \delta \) we have
\[ 0 \leq (Q_h^k - Q_h^*) (s,a) \leq \alpha_i^0 H e^{\beta |H|} + \sum_{i \in [t]} \alpha_i^0 e^{\beta |H|} \left[ V_{h+1}^{k_i} (s_{h+1}^{k_i}) - V_{h+1}^* (s_{h+1}^{k_i}) \right] + 2 \gamma_t, \]
where \( k_1, \ldots, k_t < k \) are the episodes in which \((s,a)\) was taken at step \( h \), and \( \gamma_t \leq \frac{4c(|\mathcal{P}| - 1)}{|\mathcal{P}|} \sqrt{H_t / t} \).

Proof. We prove the lower bound for \((Q_h^k - Q_h^*) (s,a)\) and then use it to prove the upper bound.

**Lower bound for \( Q^k - Q^* \).**

For the purpose of the proof, we set \( Q_{H+1}^h(s,a) = Q_{H+1}^*(s,a) = 0 \) for all \((k,s,a) \in [K] \times S \times A\).
We fix a \((s,a) \in S \times A\) and use strong induction on \( k \) and \( h \). Without loss of generality, we assume that there exists a \((k,h)\) such that \((s,a) = (s_h^k, a_h^k)\) (that is, \((s,a)\) has been taken at some point in
Algorithm 2), since otherwise $Q^k_h(s,a) = H - h + 1 \geq Q^*_h(s,a)$ for all $(k,h) \in [K] \times [H]$ and we are done. The base case for $k = 1$ and $h = H + 1$ is satisfied since $(Q^k_{h+1} - Q^*_h)(s,a) = 0$ for $k' \in [K]$ by definition. We fix a $(k,h) \in [K] \times [H]$ and assume that $0 \leq (Q^k_{h+1} - Q^*_h)(s,a)$ for each $k_1, \ldots, k_t < k$ (here $t = N_h^k(s,a)$). Then we have for $i \in [t]$ that

$$V^k_{h+1}(s) = \max_{a' \in A} Q^k_{h+1}(s,a') \geq \max_{a' \in A} Q^*_k(s,a') = V^*_h(s).$$

Recall the quantities $G_1$ and $G_2$ defined in Equation (27). The above equation implies $G_1 \geq 0$. We also have $G_2 \geq 0$ by the fact $Q^*_h(s,a) \leq H$ and on the event of Lemma 9. Therefore, it follows that $(Q^k_h - Q^*_h)(s,a) = G_1 + G_2 \geq 0$. The induction is completed and we have proved that $0 \leq (Q^k_{h+1} - Q^*_h)(s,a)$ for all $(k,h,s,a) \in [K] \times [H] \times S \times A$.

Upper bound for $Q^k - Q^*$.

Let us fix a $(k,h,s,a) \in [K] \times [H] \times S \times A$. Since $0 \leq (Q^k_h - Q^*_h)(s,a)$, we have for $i \in [t]$ that

$$V^k_{h+1}(s) = \max_{a' \in A} Q^k_{h+1}(s,a') \geq \max_{a' \in A} Q^*_h(s,a') = V^*_h(s).$$

Case $\beta > 0$. We have

$$G_1 = \frac{1}{\beta} \log\{q_1\} - \frac{1}{\beta} \log\{q_2\}$$

$$\leq \frac{1}{\beta} (q_1 - q_2)$$

$$\leq \frac{1}{\beta} (q_1^+ - q_2^-)$$

$$\leq \frac{1}{\beta} \sum_{i \in [t]} a_i \left[ e^{\beta [r_h(s,a) + V^k_{h+1}(s_{h+1})]} - e^{\beta [r_h(s,a) + V^*_h(s_{h+1})]} \right] + \frac{1}{\beta} \sum_{i \in [t]} a_i b_i$$

$$\leq e^{\beta H} \sum_{i \in [t]} a_i \left[ (V^k_{h+1} - V^*_h)(s_{h+1}) \right] + \gamma_t,$$

where the second step holds by Fact 1(a) with $g = 1$ and the fact that $V^k_{h+1}(s) \geq V^*_h(s)$ and by noticing that $a_i^0, \sum_{i \in [t]} a_i^i \in \{0,1\}$ with $a_i^0 + \sum_{i \in [t]} a_i^i = 1$ by Fact 3(d) (so that $q_1 \geq q_2$), the third step holds since by definition $q_1^+ \geq q_1$ and by Fact 4 $q_2^- \leq q_2$, and the last step holds by Fact 1(b) and the fact that $H \geq r_h(s,a) + V^k_{h+1}(s) \geq r_h(s,a) + V^*_h(s) \geq 0$. For $G_2$, we have

$$G_2 = \frac{1}{\beta} \log\{q_2\} - \frac{1}{\beta} \log\{q_3\}$$

$$\leq \frac{1}{\beta} (q_2 - q_3)$$

$$\leq \frac{1}{\beta} (q_2^+ - q_3^-).$$
\[
\begin{align*}
&= \frac{a_0}{\beta} \left[ e^{\beta H} - e^{\beta Q_h^*(s,a)} \right] + \frac{1}{\beta} \sum_{i \in [t]} a_i b_i \\
&+ \frac{1}{\beta} \sum_{i \in [t]} a_i \left[ e^{\beta (r_h(s,a) + V_{h+1}^i(s_{h+1}^i))} - \mathbb{E}_{s' \sim P_h (\cdot | s,a)} e^{\beta [r_h(s,a) + V_{h+1}^i(s')]} \right] \\
&\leq a_0^H H e^{\beta H} + \gamma_t, 
\end{align*}
\]

In the above, the second step holds by Fact 1(a) with \( g = 1 \) and

\[
\sum_{i \in [t]} a_i^* b_i \geq \left| \sum_{i \in [t]} a_i^* \left[ e^{\beta (r_h(s,a) + V_{h+1}^i(s_{h+1}^i))} - \mathbb{E}_{s' \sim P_h (\cdot | s,a)} e^{\beta [r_h(s,a) + V_{h+1}^i(s')]} \right] \right| 
\]

on the event of Lemma 9 (so that \( q_2 \geq q_3 \)); the third step holds by Fact 4; the last step holds by Fact 1(b) and \( Q_h^*(s,a) \in [0, H] \) and on the event of Lemma 9.

Case \( \beta < 0 \). We have

\[
G_1 = \frac{1}{(-\beta)} \log \{ q_2 \} - \frac{1}{(-\beta)} \log \{ q_1 \} \\
= e^{-\beta H} (q_2 - q_1) \\
\leq e^{-\beta H} (q_2 - q_1^+) \\
= e^{-\beta H} \sum_{i \in [t]} a_i^* \left[ e^{\beta (r_h(s,a) + V_{h+1}^i(s_{h+1}^i))} - e^{\beta (r_h(s,a) + V_{h+1}^i(s_{h+1}^i))} \right] + e^{-\beta H} \sum_{i \in [t]} a_i^* b_i \\
\leq e^{H \beta} \sum_{i \in [t]} a_i^* \left[ (V_{h+1}^{k_i} - V_{h+1}^*) (s_{h+1}^{k_i}) \right] + \gamma_t, 
\]

where the second step holds by Fact 1(a) with \( g = e^{\beta H} \) and the fact that \( V_{h+1}^{k_i} (s) \geq V_{h+1}^* (s) \) (so that \( q_2 \geq q_1 \)), the third step holds since \( q_2^+ \geq q_2 \) by Fact 4 and \( q_1^+ \leq q_1 \) by definition, and the last step holds by Fact 1(b) and the fact that \( H \geq r_h(s,a) + V_{h+1}^{k_i} (s) \geq r_h(s,a) + V_{h+1}^* (s) \geq 0 \). For \( G_2 \), we have

\[
G_2 = \frac{1}{(-\beta)} \log \{ q_3 \} - \frac{1}{(-\beta)} \log \{ q_2 \} \\
= e^{-\beta H} (q_3 - q_2) \\
\leq e^{-\beta H} (q_3 - q_3^+) \\
= e^{-\beta H} a_0^H \left[ e^{\beta Q_h^*(s,a)} - e^{\beta H} \right] + e^{-\beta H} \sum_{i \in [t]} a_i^* b_i \\
+ e^{-\beta H} \sum_{i \in [t]} a_i^* \left[ \mathbb{E}_{s' \sim P_h (\cdot | s,a)} e^{\beta [r_h(s,a) + V_{h+1}^i(s')]} - e^{\beta [r_h(s,a) + V_{h+1}^i(s_{h+1}^i) + V_{h+1}^i(s_{h+1}^i) + V_{h+1}^i(s_{h+1}^i)]} \right]
\]
\[ \leq e^{-\beta H} a_i^0 [H - Q_h^*(s, a)] + 2e^{-\beta H} \left( -\beta \right) \sum_{i \in [H]} a_i^l b_i \]
\[ \leq a_i^0 H e^{\beta |H|} + \gamma_l. \]

where the second step holds by Fact 1(a) given \( q_3 \geq q_2 \), the second to the last step holds by Fact 1(b), the fact that \( Q_h^*(s, a) \leq H \) and on the event of Lemma 9, and the last step holds by the definition of \( \gamma_l \).

Combining the bounds of \( G_1 \) and \( G_2 \) with the identity \( (Q_k^h - Q_h^*) (s, a) = G_1 + G_2 \) yields the upper bound for \( (Q_k^h - Q_h^*) (s, a) \). The proof is completed in view of Lemma 9 and the definition of \( \gamma_l \) that imply
\[ \gamma_l \leq \frac{4c(e^{\beta |H|} - 1)}{\sqrt{\frac{H_i}{t}}}. \]

\[ \square \]

### E Proof of Theorem 2

We first introduce some notations. Let \( G \) be a discrete space. Define the shorthand
\[ \text{lse}_\beta (P, f) := \frac{1}{\beta} \log \{ \mathbb{E}_{x \sim P} [\exp (\beta \cdot f(x))] \}, \tag{28} \]
for a probability distribution \( P \) supported on \( G \) and function \( f : G \rightarrow \mathbb{R} \). We record a useful lemma that shows \( \text{lse}_\beta (\cdot, \cdot) \) is Lipschitz continuous in the second argument.

**Lemma 12.** Let \( G \) be a discrete space and \( \bar{f} \geq 0 \) be a non-negative number. Let the functions \( f, f' : \mathbb{R}^d \mapsto [0, \bar{f}] \) be such that \( f(x) \geq f'(x) \) for all \( x \in \mathbb{R}^d \). Also let \( P \) be a probability distribution supported on \( G \). We have
\[ \text{lse}_\beta (P, f) - \text{lse}_\beta (P, f') \leq e^{\beta |f'} \cdot \mathbb{E}_{x \sim P} [f(x) - f'(x)]. \]

The proof is given in Appendix E.1.

Define \( \hat{P}_k^h (s' | s, a) \) to be the delta function centered at \( s_{h+1}^k \) for all \( (k, h, s, a) \in [K] \times [H] \times S \times A \), and this means \( \mathbb{E}_{s' \sim \hat{P}_k^h (\cdot | s, a)} [f(s')] = f(s_{h+1}^k) \) for any function \( f : S \rightarrow \mathbb{R} \). Also define
\[ \phi_h^k := (V_h^k - V_{\pi_h}^*(s_h^k)) \quad \text{and} \quad \phi_h^k := (V_h^k - V_h^*) (s_h^k). \]

Also define
\[ \xi_h := [(P_h - \hat{P}_h^k) (V_h^* - V_{\pi_h}^*) (s_h^k)] (a_h^k). \]
Note that for each \((k, h) \in [K] \times [H]\), we have
\[
\delta^k_h = (Q^k_h - Q^*_h)(s^k_h, a^k_h) \\
= (Q^k_h - Q^*_h)(s^k_h, a^k_h) + (Q^*_h - Q^*_{h+1})(s^k_h, a^k_h) \\
\leq \alpha^0_i H e^{\beta|H} + \sum_{i \in [t]} \alpha^i e^{\beta|H} \phi^k_{h+1} + 2\gamma_t \\
+ \|s\| \big(P_h(s^k_h, a^k_h, V^*_h) - s\big) \\
\leq \alpha^0_i H e^{\beta|H} + \sum_{i \in [t]} \alpha^i e^{\beta|H} \phi^k_{h+1} + 2\gamma_t + e^{\beta|H} \big(P_h(V^*_h) - s^k_{h+1}\big)(s^k_h, a^k_h) \\
= \alpha^0_i H e^{\beta|H} + \sum_{i \in [t]} \alpha^i e^{\beta|H} \phi^k_{h+1} + 2\gamma_t + e^{\beta|H} \big(\phi^k_{h+1} - \phi^k_{h+1} + \xi^k_{h+1}\big),
\]
where the third step holds by Lemma 11 and the Bellman equations (3) and (4), the fourth step holds by Lemma 12 and the fact that \(0 \leq V^*_h(s) \leq V^*_h(s) \leq H\) for all \(s \in S\), and the last step follows by definition that \(\delta^k_{h+1} - \phi^k_{h+1} = (V^*_h - V^*_{h+1})(s^k_h, a^k_h) = e^{\beta|H} \big(P_h(V^*_h) - s^k_{h+1}\big)(s^k_h, a^k_h)\) and the definition of \(\xi^k_{h+1}\).

We now compute \(\sum_{k \in [K]} \delta^k_h\) for a fixed \(h \in [H]\). Denote by \(n^k_h := N^k_h(s^k_h, a^k_h)\) and we have
\[
\sum_{k \in [K]} \alpha^0_{n^k_h} H e^{\beta|H} = H e^{\beta|H} \sum_{k \in [K]} \|\{n^k_h = 0\}\| \leq H e^{\beta|H} SA.
\]

Then we turn to control the second term in Equation (29) summed over \(k \in [K]\), that is,
\[
\sum_{k \in [K]} \sum_{i \in [t]} \alpha^i e^{\beta|H} \phi^k_{h+1} = e^{\beta|H} \sum_{k \in [K]} \sum_{i \in [n^k_h]} \alpha^i_{n^k_h} \phi^k_{h+1},
\]
where \(k_i(s^k_h, a^k_h)\) denotes the episode in which \((s^k_h, a^k_h)\) was taken at step \(h\) for the \(i\)-th time. We re-group the above summation in a different way. For every \(k' \in [K]\), the term \(\phi^k_{h+1}\) appears in the summand with \(k > k'\) if and only if \((s^k_h, a^k_h) = (s^{k'}_h, a^{k'}_h)\). The first time it appears we have \(n^k_h = n^k_{h+1} + 1\), the second time it appears we have \(n^k_h = n^k_{h+1} + 2\), and etc. Therefore,
\[
e^{\beta|H} \sum_{k \in [K]} \sum_{i \in [n^k_h]} \alpha^i_{n^k_h} \phi^k_{h+1} \leq e^{\beta|H} \sum_{k' \in [K]} \phi^k_{h+1} \sum_{i \geq n^k_{h+1}} \alpha^i_{n^k_{h+1}} \leq e^{\beta|H} \left(1 + \frac{1}{H}\right) \sum_{k' \in [K]} \phi^k_{h+1},
\]
where the last step follows Fact 3(c). Collecting the above results and plugging them into Equation (29), we have
\[
\sum_{k \in [K]} \delta^k_h \leq H e^{\beta|H} SA + e^{\beta|H} \left(1 + \frac{1}{H}\right) \sum_{k \in [K]} \phi^k_{h+1} + e^{\beta|H} \sum_{k \in [K]} (\delta^k_{h+1} - \phi^k_{h+1}) + \sum_{k \in [K]} (2\gamma n^k_h + e^{\beta|H} \xi^k_{h+1})
\]
35
\[ \leq H e^{\beta|H|S} + e^{\beta|H|} \left( 1 + \frac{1}{H} \right) \sum_{k \in [K]} \delta^k_{h+1} + \sum_{k \in [K]} (2 \gamma n_k^h + e^{\beta|H|} \xi^{k}_{h+1}), \]  

(30)

where the last step holds since \( \delta^k_{h+1} \geq \phi^k_{h+1} \) (due to the fact that \( V^*_h(s) \geq V^*_{h+1}(s) \) for all \( x \in S \)). Since it holds that

\[ \left[ e^{\beta|H|} \left( 1 + \frac{1}{H} \right) \right]^H \leq e^{\beta|H|^2}, \]

we can expand the quantity \( \sum_{k \in [K]} \delta^k \) recursively in the form of Equation (30), apply Holder’s inequality and use the fact that \( \delta^k_{H+1} = 0 \) to get

\[ \sum_{k \in [K]} \delta^k \leq e^{\beta|H|^2 + 1} \left[ H^2 e^{\beta|H|S} + \sum_{h \in [H]} \sum_{k \in [K]} (2 \gamma n_k^h + e^{\beta|H|} \xi^{k}_{h+1}) \right]. \]  

(31)

By the pigeonhole principle, for any \( h \in [H] \) we have

\[ \sum_{k \in [K]} \gamma n_k^h \leq \frac{e^{\beta|H|} - 1}{H^2} \sum_{k \in [K]} \sqrt{Hti} \]

\[ = \frac{e^{\beta|H|} - 1}{|\beta|} \sum_{(s,a) \in S \times A} \sum_{n \in [N^K_h(s,a)]} \sqrt{Hti} \]

\[ \leq \frac{e^{\beta|H|} - 1}{|\beta|} \sqrt{HSAKt} \]

\[ = \frac{e^{\beta|H|} - 1}{|\beta|} \sqrt{SATl}, \]  

(32)

where the third step holds since \( \sum_{(s,a) \in S \times A} N^K_h(s,a) = K \) and the RHS of the second step is maximized when \( N^K_h(s,a) = K/(SA) \) for all \( (s,a) \in S \times A \). Finally, the Azuma-Hoeffding inequality implies that with probability at least \( 1 - \delta \), we have

\[ \left| \sum_{h \in [H]} \sum_{k \in [K]} \delta^k_{h+1} \right| \leq H \sqrt{Ti}. \]  

(33)

Putting together Equations (32) and (33) and plugging them into (31), we have

\[ \sum_{k \in [K]} \delta^k \leq e^{\beta|H|^2 + H} \cdot H^2 SA \]

\[ + e^{\beta|H|^2} \cdot \frac{e^{\beta|H|} - 1}{|\beta|} \sqrt{H^2 SATl} \]

\[ + e^{\beta|H|^2 + H} \cdot H \sqrt{Ti}, \]

\[ \leq e^{\beta|H|^2 + H} \cdot H^2 SA \]

\[ + e^{\beta|H|^2 + H} \cdot \frac{e^{\beta|H|} - 1}{|\beta|} \sqrt{H^2 SATl} \]

The proof is completed in view of Fact 2 and when \( T \) is sufficiently large.
E.1 Proof of Lemma 12

We have the following two cases.

Case $\beta > 0$. We have

\[
\text{lse}_\beta(P, f) - \text{lse}_\beta(P, f') \leq \frac{1}{\beta} \mathbb{E}_{x \sim P} \left[ e^{\beta f(x)} - e^{\beta f'(x)} \right] \\
\leq \frac{1}{\beta} \mathbb{E}_{x \sim P} \left[ \beta e^{\bar{f}(x)} (f(x) - f'(x)) \right] \\
= e^{\bar{f}(x)} \cdot \mathbb{E}_{x \sim P} [f(x) - f'(x)],
\]

where the first step holds by Fact 1(a) with $g = 1$ and the fact that $e^{\beta f(x)} \geq e^{\beta f'(x)} \geq 1$, and the second holds by Fact 1(b) with $u = \bar{f}$ and the fact that $f(x) \geq f'(x)$.

Case $\beta < 0$. We have

\[
\text{lse}_\beta(P, f) - \text{lse}_\beta(P, f') = - \left[ \text{lse}_\beta(P, f') - \text{lse}_\beta(P, f) \right] \\
\leq \exp(-\bar{f}) \cdot \mathbb{E}_{x \sim P} \left[ \exp(\beta f'(x)) - \exp(\beta f(x)) \right] \\
\leq \exp(-\bar{f}) \cdot \mathbb{E}_{x \sim P} \left[ (\beta)(f(x) - f'(x)) \right] \\
= \exp(-\bar{f}) \cdot \mathbb{E}_{x \sim P} [f(x) - f'(x)],
\]

where the second step holds by Fact 1(a) with $g = e^{\bar{f}}$ given that $x \in [e^{\bar{f}}, 1]$, and the third step holds by Fact 1(b) and the fact $1 \geq e^{\beta f'(x)} \geq e^{\beta f(x)} > 0$.

F Proof of Theorem 3

For each $\rho \in [0, 1]$, let Ber($\rho$) denote the Bernoulli distribution with parameter $\rho$. Before diving into the proof, let us record two important results.

Lemma 13. Let $p, p' \in (0, 1)$ and $p > p'$. Define $D := D_{KL}(\text{Ber}(p')||\text{Ber}(p))$ to be the KL divergence between Ber($p'$) and Ber($p$). For any policy $\pi$ and a positive integer $K$, let $K_0 := K_0(K, \pi)$ be the number of times that the sub-optimal arm is pulled in the $K$-round two-arm bandit problem (with Ber($p'$) and Ber($p$) being the two arms) when executing policy $\pi$. When $K$ is sufficiently large, we have

\[
\mathbb{E}K_0 \gtrsim \log \frac{K}{D}.
\]

Proof. This is an intermediate result in the proof of [41, Theorem 16.2].

Lemma 14. Let $p, p' \in (0, 1)$ be such that $p > p'$. We have $D_{KL}(\text{Ber}(p')||\text{Ber}(p)) \leq \frac{(p-p')^2}{p(1-p)}$.

The proof is provided in Appendix F.3. We consider two cases: $\beta > 0$ and $\beta < 0$. 37
F.1 Case $\beta > 0$

Consider a two-arm bandit problem with $K$ rounds, where the reward for pulling arm $i \in \{1, 2\}$ is given by the scaled Ber($p_i$) random variable

$$X_i = \begin{cases} 
H & \text{w.p. } p_i, \\
0 & \text{w.p. } 1 - p_i,
\end{cases}$$

where $H \geq 1$ specifies the range of the reward, and the parameters $p_1 > p_2$ are to be specified later. Let $\Delta := p_1 - p_2 > 0$.

By Lemma 14, we have

$$D_{KL}(X_2 \| X_1) \geq \frac{\Delta^2}{p_1(1 - p_1)}. \quad (34)$$

It then follows from Lemma 13 that

$$\mathbb{E}K_0 \gtrsim \frac{\log K \cdot p_1(1 - p_1)}{\Delta^2} \quad (35)$$

Let us choose

$$\Delta = C \sqrt{\frac{\log K \cdot p_1(1 - p_1)}{K}}$$

for an universal constant $C > 0$. Note that under this choice we have $K \geq \mathbb{E}K_0$ as should be expected. Now, we set $p_2 = e^{-\beta H}$. Since $p_1(1 - p_1) \leq \frac{1}{4}$, we have $\Delta \lesssim \sqrt{\frac{\log K}{K}}$. By choosing $K$ and $H$ large enough, we can ensure $\Delta \leq e^{-\beta H}$ and $p_1 = p_2 + \Delta \leq \frac{3}{4}$.

Define $X_i^k$ to be the outcome of arm $X_i$ (if pulled) in round $k$, and $Y^k$ to be the outcome of the arm actually pulled in round $k$. Then, conditional on $K_0$, we have

$$\text{Regret}(K) = \frac{1}{\beta} \log \left[ \mathbb{E} \exp \left( \beta \sum_{k=1}^{K} X_1^k \right) \right] - \frac{1}{\beta} \log \left[ \mathbb{E} \exp \left( \beta \sum_{k=1}^{K} Y^k \right) \right]$$

$$\geq \frac{1}{\beta} \log \left[ \prod_{k=1}^{K} \mathbb{E} \exp \left( \beta X_1^k \right) \right] - \frac{1}{\beta} \log \left[ \prod_{k=1}^{K} \mathbb{E} \exp \left( \beta Y^k \right) \right]$$

$$\geq \frac{K}{\beta} \log \left[ \mathbb{E} \exp (\beta X_1) \right] - \frac{K}{\beta} \log \left[ \mathbb{E} \exp (\beta X_2) \right]$$

$$\geq \frac{K_0}{\beta} \log \left[ \mathbb{E} \exp (\beta X_1) \right] - \frac{K_0}{\beta} \log \left[ \mathbb{E} \exp (\beta X_2) \right], \quad (36)$$

where step $(i)$ holds because of the independence among $\{X_1^k\}$ and independence among $\{Y^k\}$.

Taking expectation over $K_0$ on both sides of Equation (36), we have

$$\mathbb{E}[\text{Regret}(K)] \geq \frac{\mathbb{E}K_0}{\beta} \left( \log \mathbb{E} e^{\beta X_1} - \log \mathbb{E} e^{\beta X_2} \right)$$
\[
E_0 \frac{\beta}{\log \left( \frac{p_1 e^{\beta H} + (1-p_1)}{p_2 e^{\beta H} + (1-p_2)} \right)}
= E_0 \frac{\beta}{\log \left( 1 + \frac{\Delta (e^{\beta H} - 1)}{p_2 e^{\beta H} + (1-p_2)} \right)}
\]
\[\geq (i) \frac{E_0}{\beta} \log \left( 1 + \frac{\Delta (e^{\beta H} - 1)}{1+1} \right)\]
\[\geq (ii) \frac{E_0}{\beta} \cdot \frac{1}{4} \Delta (e^{\beta H} - 1)\]
\[\geq (iii) \frac{1}{\beta} \cdot \log K \cdot p_1 (1-p_1) \cdot (e^{\beta H} - 1)\]
\[\geq (iv) \frac{1}{\beta} \cdot \sqrt{K \log K} \cdot (e^{\beta H/2} - 1)\]
\[\geq (v) \frac{1}{\beta} \cdot \sqrt{T \log T} \cdot (e^{\beta H/2} - 1),\]

where step (i) holds since \(p_2 = e^{-\beta H}\), step (ii) holds since \(\Delta \leq e^{-\beta H}\) and \(\log(1+x) \geq \frac{x}{2}\) for \(x \in [0,1]\), step (iii) holds by Equation (35), step (iv) holds since \(e^{-\beta H} = p_2 \leq p_1 \leq \frac{3}{4}\) by construction, and the last step holds since \(\frac{1}{\beta} (e^{\beta H/2} - 1) \gtrsim H\) implied by Fact 5 below.

**Fact 5.** For any \(G > 0\), the function

\[f_G(x) = \frac{e^{Gx} - 1}{x} , \quad x > 0\]

is increasing and satisfies \(\lim_{x \to 0} f_G(x) = G\).

Finally, note that the aforementioned \(K\)-round two-arm bandit model is a special case of an \(K\)-episode \((H+2)\)-horizon MDP with the per-step reward in \([0,1]\), illustrated in Figure 2. The MDP is equipped with \(A = \{a_1, a_2\}, S = \{s_1, s_2, s_3\}\), where state \(s_1\) is the initial state, and states \(s_2\) and \(s_3\) are absorbing regardless of actions taken. The states satisfy that \(r_h(s_2, a) = 1, r_h(s_1, a) = r_h(s_3, a) = 0\) for all \(h \in [H+2]\) and \(a \in A\). At the initial state \(s_1\), we may choose to take action \(a_1\) or \(a_2\). If \(a_1\) is taken at state \(s_1\), then we transition to \(s_2\) with probability \(p_1\) and to \(s_3\) with probability \(1 - p_1\). If \(a_2\) is taken at state \(s_1\), then we transition to \(s_2\) with probability \(p_2\) and to \(s_3\) with probability \(1 - p_2\).

### F.2 Case \(\beta < 0\)

The proof of the case \(\beta < 0\) is similar to that of the case \(\beta > 0\). For \(\beta < 0\), consider a 2-arm bandit model with \(K\) rounds, where the reward for pulling arm \(i \in \{1, 2\}\) is given by the scaled
\[
X_i = \begin{cases} 
0 & \text{w.p. } p_i, \\
H & \text{w.p. } 1 - p_i.
\end{cases}
\]

Let \( p_2 = e^{\beta H} > p_1 \) and \( \Delta := p_1 - p_2 < 0 \). Note that Equations (34) and (35) remain valid (by invoking Lemmas 14 and 13 with \( p = 1 - p_1 \) and \( p' = 1 - p_2 \)). Therefore, we choose

\[
\Delta = -C \sqrt{\frac{\log K \cdot p_1(1 - p_1)}{K}}
\]

for some universal constant \( C > 0 \). Since \( p_1(1 - p_1) \leq \frac{1}{4} \), we have \( \Delta \gtrsim -\sqrt{\frac{\log K}{K}} \). By choosing \( H \) large enough, we have \( 1 - p_1 \geq 1 - p_2 = 1 - e^{\beta H} \geq \frac{1}{4} \). And by choosing \( K \) large enough, we can ensure \( \Delta \geq -\frac{1}{2} e^{\beta H} \) so that \( p_1 = p_2 + \Delta \geq \frac{1}{2} e^{\beta H} \).

Taking the expectation over \( K_0 \) on both sides of Equation (36), we have

\[
\mathbb{E}[\text{Regret}(K)] = \frac{\mathbb{E}K_0}{\beta} \left( \log \mathbb{E}e^{\beta X_1} - \log \mathbb{E}e^{\beta X_2} \right)
\]

\[
= \frac{\mathbb{E}K_0}{\beta} \log \left( \frac{(1 - p_1)e^{\beta H} + p_1}{(1 - p_2)e^{\beta H} + p_2} \right) 
\]

\[
= \frac{\mathbb{E}K_0}{\beta} \log \left( 1 + \frac{\Delta(1 - e^{\beta H})}{(1 - p_2)e^{\beta H} + p_2} \right).
\]

\[
\geq \frac{\mathbb{E}K_0}{\beta} \cdot \frac{\Delta(1 - e^{\beta H})}{(1 - p_2)e^{\beta H} + p_2}
\]

\[
\geq \frac{\mathbb{E}K_0}{\beta} \cdot \frac{\Delta(1 - e^{\beta H})}{2e^{\beta H}}
\]

\[
\geq \frac{1}{(-\beta)} \cdot \frac{\log K \cdot p_1(1 - p_1)}{(-\Delta)} \cdot (e^{-\beta H} - 1)
\]
\[ \frac{1}{(-\beta)} \cdot \sqrt{K \log K \cdot p_1 (1 - p_1) \cdot (e^{-\beta H} - 1)} \]

\[ \approx \frac{1}{(-\beta)} \cdot \sqrt{K \log K \cdot (e^{-\beta H/2} - 1)} \]

In the above, step (i) holds since \( \beta < 0 \) and \( \log(1 + x) \leq x \) for all \( x > -1 \); step (ii) holds since \( p_2 = e^{\beta H} \) and \( \Delta, \beta < 0 \); step (iii) holds by Equation (35); step (iv) holds since \( p_1 \geq \frac{1}{2} e^{\beta H} \) and \( 1 - p_1 \geq \frac{1}{4} \) by construction; and the last step holds since \( \frac{1}{(-\beta)} (e^{-\beta H/2} - 1) \gtrsim H \) implied by Fact 5.

It is not hard to see that the two-arm bandit model discussed above is also a special case of an \( K \)-episode \((H + 2)\)-horizon MDP with the per-step reward in \([0, 1]\), similar to the case \( \beta > 0 \).

**F.3 Proof of Lemma 14**

Recall that \( \Delta := p - p' \). The KL divergence can be upper bounded as follows:

\[
D_{KL}(\text{Ber}(p')||\text{Ber}(p)) = p' \log \left( \frac{p'}{p} \right) + (1 - p') \log \left( \frac{1 - p'}{1 - p} \right)
\]

\[
= p' \log \left( 1 + \frac{p' - p}{p} \right) + (1 - p') \log \left( 1 + \frac{p - p'}{1 - p} \right)
\]

\[
\leq p' \cdot \frac{p' - p}{p} + (1 - p') \cdot \frac{p - p'}{1 - p}
\]

\[
= (\Delta - p) \cdot \frac{\Delta}{p} + (1 - p) \cdot \frac{\Delta}{1 - p}
\]

\[
= \frac{\Delta^2}{p} + \frac{\Delta^2}{1 - p}
\]

\[
= \frac{\Delta^2}{p(1 - p)}
\]

where step (i) holds since \( \log(1 + x) \leq x \) for all \( x > -1 \). The proof is completed.