Research Article

On $F$-Algebras $M^p$ $(1 < p < \infty)$ of Holomorphic Functions

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Received 31 August 2013; Accepted 4 November 2013; Published 28 January 2014

Academic Editors: A. Ibeas and G.-Q. Xu

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We consider the classes $M^p$ $(1 < p < \infty)$ of holomorphic functions on the open unit disk $D$ in the complex plane. These classes are in fact generalizations of the class $M$ introduced by Kim (1986). The space $M^p$ equipped with the topology given by the metric $\rho_p$ defined by

$$
\rho_p(f, g) = \left( \int_0^{2\pi} \log^+(1 + M(f-g)(\theta))(d\theta/2\pi) \right)^{1/p},
$$

with $f, g \in M^p$ and $Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|$, becomes an $F$-space. By a result of Stoll (1977), the Privalov space $N^p$ $(1 < p < \infty)$ with the topology given by the Stoll metric $d_p$ is an $F$-algebra. By using these two facts, we prove that the spaces $M^p$ and $N^p$ coincide and have the same topological structure. Consequently, we describe a general form of continuous linear functionals on $M^p$ (with respect to the metric $\rho_p$). Furthermore, we give a characterization of bounded subsets of the spaces $M^p$. Moreover, we give the examples of bounded subsets of $M^p$ that are not relatively compact.

1. Introduction and Preliminaries

Let $D$ denote the open unit disk in the complex plane and let $T$ denote the boundary of $D$. Let $L^q(T)$ $(0 < q \leq \infty)$ be the familiar Lebesgue spaces on the unit circle $T$.

Following Kim ([1, 2]), the class $M$ consists of all holomorphic functions $f$ on $D$ for which

$$
\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} < \infty,
$$

where $\log^+ |a| = \max\{\log a, 0\}$ and

$$
Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|,
$$

is the maximal radial function of $f$. The Privalov class $N^p$ $(1 < p < \infty)$ consists of all holomorphic functions $f$ on $D$ for which

$$
\sup_{0 < r < 1} \int_0^{2\pi} \left( \log^+ |f(re^{i\theta})| \right)^p \frac{d\theta}{2\pi} < +\infty.
$$

These classes were firstly considered by Privalov in [3, page 93], where $N^p$ is denoted as $A_q$.

Notice that for $p = 1$, the condition (3) defines the Nevanlinna class $N$ of holomorphic functions in $D$. Recall that the Smirnov class $N^+$ is the set of all functions $f$ holomorphic on $D$ such that

$$
\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < +\infty,
$$

where $f^*$ is the boundary function of $f$ on $T$; that is,

$$
f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})
$$

is the radial limit of $f$, which exists for almost every $e^{i\theta}$. We denote by $H^q$ $(0 < q \leq \infty)$ the classical Hardy space on $D$. It is known (see [4, 5]) that

$$
N^r \subset N^p \quad (r > p), \quad \bigcup_{q > 0} H^q \subset \bigcap_{p > 1} N^p,
$$

where the above containment relations are proper.

The study of the spaces $N^p$ $(1 < p < \infty)$ was continued in 1977 by Stoll [6] (with the notation $(\log^+ H^q)^\alpha$ in [6]). Further,
the topological and functional properties of these spaces were studied in [4, 5, 7–14]; typically, the notation varied and these spaces are called the Privalov spaces in [12–15].

It is well known [16, page 26] that a function \( f \in N^+ \) if and only if \( f = 1F \), where \( I \) is an inner function on \( D \) and \( F \) is an outer function given by

\[
F(z) = \exp \left( \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |F^*(e^{i\theta})| \frac{dt}{2\pi} \right),
\]

where \( \log |F^*| \in L^1(T) \).

Privalov [3, page 98] showed that \( f \in N^+ \) if and only if \( f = 1F \), where \( I \) is an inner function on \( D \) and \( F \) is an outer function as given above with \( \log^* |f^*| \in L^p(T) \).

Stoll [6, Theorem 4.2] showed that the space \( N^p \) (with the notation \( (\log^* H)^p \) in [6]) with the topology given by the metric \( d_p \) defined by

\[
d_p(f, g) = \left( \int_0^{2\pi} \left( \frac{\log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|)}{2\pi} \right)^p d\theta \right)^{1/p},
\]

becomes an \( F \)-algebra. Recall that the function \( d_1 = d \) defined on the Smirnov class \( N^* \) by (8) with \( p = 1 \) induces the metric topology on \( N^* \). Yanagihara [17] showed that, under this topology, \( N^* \) is an \( F \)-space.

Furthermore, in connection with the spaces \( N^p \) (\( 1 < p < \infty \)), Stoll [6] (also see [7] and [12, Section 3]) also studied the spaces \( F^q \) (\( 0 < q < \infty \)) (with the notation \( F^q \) in [6]), consisting of those functions \( f \) holomorphic on \( D \) for which

\[
\lim_{r \to 1} (1 - r)^{1/q} \log^* M_{\infty}(r, f) = 0,
\]

where

\[
M_{\infty}(r, f) = \max_{|z| = r} |f(z)|.
\]

Stoll [6, Theorem 3.2] proved that the space \( F^q \) with the topology given by the family of seminorms \( \| \cdot \|_{q,c} \) defined for \( f \in F^q \) as

\[
\|f\|_{q,c} = \sum_{n=0}^{\infty} \left| \widehat{f}(n) \right| e^{-c\sqrt{2\pi}n^q} < \infty,
\]

for each \( c > 0 \), where \( \widehat{f}(n) \) is the \( n \)th Taylor coefficient of \( f \), becomes a countably normed Fréchet algebra. By a result of Eoff [7, Theorem 4.2], \( F^q \) is the Fréchet envelope of \( N^q \), and hence \( F^q \) and \( N^q \) have the same topological duals.

Here, as always in the sequel, we will need some of Stoll’s results concerning the spaces \( F^q \) only with \( 1 < q < \infty \), and hence we will assume that \( q = p > 1 \) is any fixed number.

The study of the class \( M \) has been extensively investigated by Kim in [1, 2], Gavrilov and Zaharyan [18], and Nawrocky [19]. Kim [2, Theorems 3.1 and 6.1] showed that the space \( M \) with the topology given by the metric \( \rho \) defined by

\[
\rho(f, g) = \int_0^{2\pi} \log(1 + M(f - g)(\theta)) \frac{d\theta}{2\pi}, \quad f, g \in M
\]

becomes an \( F \)-algebra. Furthermore, Kim [2, Theorems 5.2 and 5.3] gave an incomplete characterization of multipliers of \( M \) into \( H^\infty \). Consequently, the topological dual of \( M \) is not exactly determined in [2], but, as an application, it was proved in [2, Theorem 5.4] (also cf. [19, Corollary 4]) that \( M \) is not locally convex space. Furthermore, the space \( M \) is not locally bounded ([2, Theorem 4.5] and [19, Corollary 5]).

Although the class \( M \) is essentially smaller than the class \( N^+ \), Nawrocky [19] showed that the class \( M \) and the Smirnov class \( N^+ \) have the same corresponding locally convex structure which was already established by Yanagihara for the Smirnov class in [17, 20]. More precisely, it was proved in [19, Theorem 1] that the Fréchet envelope of the class \( M \) can be identified with the space \( F^+ \) of holomorphic functions on the open unit disk \( D \) such that

\[
\|f\| := \sum_{n=0}^{\infty} |\widehat{f}(n)| e^{-c\sqrt{n}} < \infty,
\]

for each \( c > 0 \), where \( \widehat{f}(n) \) is the \( n \)th Taylor coefficient of \( f \). Notice that \( F^+ \) coincides with the space \( F^1 \) defined above. It was shown in [17, 21] that \( F^+ \) is actually the containing Fréchet space for \( N^+ \). Moreover, Nawrocky [19, Theorem 1] characterized the set of all continuous linear functionals on \( M \) which by a result of Yanagihara [17] coincides with those on the Smirnov class \( N^+ \).

Motivated by the mentioned investigations of the classes \( M \) and \( N^+ \), and the fact that the classes \( N^p \) (\( 1 < p < \infty \)) are generalizations of the Smirnov class \( N^+ \), in Section 2, we consider the classes \( M^p \) (\( 1 < p < \infty \)) as generalizations of the class \( M \). Accordingly, the class \( M^p \) (\( 1 < p < \infty \)) consists of all holomorphic functions \( f \) on \( D \) for which

\[
\int_0^{2\pi} \log^+(Mf(\theta)) \frac{d\theta}{2\pi} < \infty.
\]

Obviously,

\[
\bigcup_{p=1}^{\infty} M^p \subset M.
\]

Following [2], by analogy with the space \( M \), the space \( M^p \) is equipped with the topology induced by the metric \( \rho_p \) defined as

\[
\rho_p(f, g) = \int_0^{2\pi} \log^+(1 + M(f - g)(\theta)) \frac{d\theta}{2\pi} \]^{1/p},
\]

with \( f, g \in M^p \).

In Section 2, we give the integral limit criterion for a function \( f \) holomorphic on the disk \( D \) to belong to the class \( M^p \) (Lemma 3). Furthermore, we prove that the space \( M^p \) is closed under integration (Theorem 4).

In Section 3 we study and compare the uniform convergence on compact subsets of \( D \) and the convergences induced by the metrics \( \rho_p \) and \( d_p \) in the space \( M^p \), respectively. It is proved (Theorem 11) that \( M^p = N^p \) for each \( p > 1 \).
It is proved in Section 4 that the space of all polynomials on $C$ is a dense subset of $M_p$ (Theorem 13). Hence, $M_p$ is a separable metric space. We show that the space $M_p$ with the topology given by the metric $\rho_p$ becomes an $F$-space (Theorem 15). As an application, we prove that the metric spaces $(M_p, \rho_p)$ and $(N_p, d_p)$ have the same topological structure (Theorem 16). Consequently, we obtain a characterization of continuous linear functionals on $M_p$ (Theorem 17). Notice that Theorem 17 with $p = 1$ characterizes the set of all continuous linear functionals on the space $M$, which is in fact the Nawrocky result [19, Theorem 1] mentioned above.

In Section 5 we obtain a characterization of bounded subsets of the spaces $M^p(= N_p^p)$ (Theorem 19). It is also given another necessary condition for a subset of $M^p (N^p)$ to be bounded (Theorem 22). Finally, we give the examples of bounded subsets of $M_p$ that are not relatively compact (Theorem 24).

### 2. The Classes $M^p$ ($1 < p < \infty$)

Recall that, for a fixed $1 < p < \infty$, the class $M^p$ consists of all holomorphic functions $f$ on $D$ for which

$$\int_0^{2\pi} (\log^+ Mf(\theta))^p d\theta < \infty. \quad (17)$$

Combining the inequalities $\log(|a| + 1) \leq \log^+ |a| + \log 2$ and $(|b| + |c|)^p \leq 2^{p-1}(|b|^p + |c|^p)$, we obtain $\log^+ (|a| + 1) \leq 2^{p-1}(\log^+ |a|)^p + (\log 2)^p$ ($a, b, c \in \mathbb{C}$). The last inequality implies the fact that the condition (17) is equivalent to

$$\|f\|_p := \left( \int_0^{2\pi} (\log^+ (1 + Mf(\theta)))^p d\theta \right)^{\frac{1}{p}} < \infty. \quad (18)$$

**Lemma 1.** The function $\| \cdot \|_p$ defined on $M^p$ by (18) satisfies the following conditions:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad \forall f, g \in M^p, \quad (19)$$

$$\|fg\|_p \leq \|f\|_p \|g\|_p \quad \forall f, g \in M^p. \quad (20)$$

Hence, $M^p$ is an algebra with respect to the pointwise addition and multiplication of functions.

**Proof.** Combining the inequality

$$\log (1 + Mf(\theta)) \leq \log (1 + Mf(\theta)) + \log (1 + Mg(\theta)), \quad (21)$$

with $f, g \in M^p$ with Minkowski's integral inequality (with the power $p$), we immediately obtain (19). Similarly, combining the inequality

$$\log (1 + (fg)(\theta)) \leq \log (1 + (f)(\theta)) + \log (1 + (g)(\theta)), \quad (22)$$

with $f, g \in M^p$ with Minkowski's integral inequality (with the exponent $p$), we obtain (20).

**Theorem 2.** The function $\rho_p$ defined on $M^p$ as

$$\rho_p (f, g) = \|f - g\|_p$$

is a translation invariant metric on $M^p$. Further, the space $M^p$ is a complete metric space with respect to the metric $\rho_p$.

**Proof.** If we suppose that $\rho_p (f, g) = 0$, for some $f, g \in M^p$, then by (23) it follows that $M(f - g)(\theta) = 0$ for almost every $\theta \in [0, 2\pi]$. Hence, $f^\ast (e^{i\theta}) = g^\ast (e^{i\theta})$ for almost every $e^{i\theta} \in \mathbb{T}$, and, by Riesz uniqueness theorem, we infer that $f(\rho z) = g(\rho z)$ for all $\rho \in D$. As, by (19), the triangle inequality is satisfied, it follows that $\rho_p$ is a metric on $M^p$. Finally, by the obvious inequality

$$\rho_p (f + h, g + h) = \rho_p (f, g), \quad f, g, h \in M^p, \quad (24)$$

we see that $\rho_p$ is a translation invariant metric. This concludes the proof.

For simplicity, here as always in the sequel, we shall write $M^p$ instead of the metric space $(M^p, \rho_p)$. For a function $f$ holomorphic in $D$ and for any fixed $0 < \rho < 1$, denote by $f_{\rho}$ the function defined on $D$ as $f_{\rho}(z) = f(\rho z)$, $z \in D$. Furthermore, for a given holomorphic function $f$ on $D$, let

$$Mf_{\rho}(\theta) = \sup_{0 < r < \rho} \left| f_{\rho}(e^{i\theta}) \right| = \sup_{0 < r < \rho} \left| f_{\rho}(e^{i\theta}) \right|, \quad 0 \leq \rho < 1. \quad (25)$$

**Lemma 3.** A function $f$ holomorphic on the unit disk $D$ belongs to the class $M^p$ if and only if it satisfies

$$\lim_{\rho \to 1} \int_0^{2\pi} \left( \log^+ Mf_{\rho}(\theta) \right)^p d\theta = \int_0^{2\pi} \left( \log^+ Mf(\theta) \right)^p d\theta < \infty. \quad (26)$$

**Proof.** The condition (26) implies that $f \in M^p$. Conversely, assume that $f \in M^p$. Then

$$Mf_{\rho}(\theta) \to Mf(\theta) \quad \text{as} \quad \rho \to 1 \quad (27)$$

for almost every $\theta \in [0, 2\pi]$.

Since, by the assumption, $f \in M^p$; that is, $\int_0^{2\pi} (\log^+ Mf(\theta))^p d\theta < \infty$, using (27) and applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\rho \to 1} \int_0^{2\pi} \left( \log^+ Mf_{\rho}(\theta) \right)^p d\theta = \int_0^{2\pi} \left( \log^+ Mf(\theta) \right)^p d\theta < \infty. \quad (28)$$

which completes the proof.
Theorem 4. The space $M^p$ is closed under integration.

Proof. For a given function $f \in M^p$, define

$$F(z) = \int_0^z f(z) \, dz = \int_0^z f(te^{i\theta}) e^{i\theta} \, dt. \quad (29)$$

It follows that $|F(re^{i\theta})| \leq Mf(\theta)$, and thus $MF(\theta) \leq Mf(\theta)$ for almost every $\theta \in [0, 2\pi]$. Therefore $F \in M^p$, as desired. \qed

3. Convergences in the Space $M^p$

Theorem 5. For each function $f \in M^p$, $f_\rho \to f$ in $M^p$ as $\rho \to 1-$.

Proof. Assume that $f \in M^p$. Since $f \in N^p$, by Fatou's theorem, the radial limit $f^\ast(e^{i\theta}) = \lim_{\rho \to 1-} f(re^{i\theta})$ exists for almost every $\theta \in [0, 2\pi]$. Hence, for such a fixed $\theta$, the function $t \to f(te^{i\theta})$ is continuous on $[0, 1]$, and thus it is uniformly continuous on $[0, 1]$. Therefore, for such a $\theta$, we have

$$M\left(f - f_\rho \right)(\theta) \to 0 \quad \text{as } \rho \to 1-. \quad (30)$$

By the inequality

$$\log(1 + M\left(f - f_\rho \right)(\theta)) \leq \log(1 + Mf(\theta)) + \log(1 + Mf_p(\theta)) \quad (31)$$

$$\leq 2 \log(1 + Mf(\theta)), \quad \text{as } \theta \to 0. \quad (32)$$

From this and (30), by the Lebesgue dominated convergence theorem, we obtain

$$\int_0^{2\pi} \left(\log(1 + M\left(f - f_\rho \right)(\theta))\right)^p \frac{d\theta}{2\pi} \to 0, \quad (33)$$

as $\rho \to 1-$.

That is, $f_\rho \to f$ in $M^p$ as $\rho \to 1-$.

For the proof of completeness of the metric space $(M^p, \rho_p)$ we will need the following lemmas.

Lemma 6. If $\{f_n\}$ is a Cauchy sequence in $M^p$, then $(f_n)_\rho \to f_n$ in $M^p$ as $\rho \to 1-$, where this convergence is uniform with respect to $n \in \mathbb{N}$.

Proof. Suppose that $\{f_n\}$ is an arbitrary Cauchy sequence in $M^p$. Then for a given $\epsilon > 0$ there is a $k \in \mathbb{N}$ such that

$$\rho_p\left(f_n, f_m\right) < \frac{\epsilon}{3} \quad \forall n, m \geq k. \quad (34)$$

So by the triangle inequality, for each $n \geq k$, we have

$$\rho_p\left(f_n, f_k\right) \leq \rho_p\left(f_n, f_m\right) + \rho_p\left(f_m, f_k\right) \quad (35)$$

By Theorem 5, there exists $0 < \rho_0 < 1$ sufficiently near to 1, for which

$$\rho_p\left(f_l, f_k\right) < \frac{\epsilon}{3} \quad \text{for each } \rho_0 < \rho < 1, \quad (36)$$

Hence, by (35), we immediately obtain

$$\rho_p\left(f_n, f_k\right) < \epsilon \quad \text{for each } \rho_0 < \rho < 1, \text{ for each } n \in \mathbb{N}. \quad (37)$$

This completes proof of Lemma 6. \qed

Lemma 7. For any $p > 1$, $M^p \subseteq N^p$ and

$$d_p\left(f, g\right) \leq \rho_p\left(f, g\right) \quad \text{for each } f, g \in M^p. \quad (38)$$

where $d_p$ is the metric of $N^p$ defined by (8).

Proof. The inclusion $M^p \subseteq N^p$ is obvious, and (38) follows by the definition of the metrics $d_p$ and $\rho_p$. \qed

Lemma 8. The convergence with respect to the metric $d_p$ of the space $N^p$ is stronger than the metric of uniform convergence on compact subsets of the disk $D$.

Proof. The assertion immediately follows from the inequality on [5, page 898], which implies that, for any function $f \in N^p$ and $0 \leq r < 1$, we have

$$\max_{|z|=r} |f(z)| \leq \exp\left(\left(1 + \frac{r}{1 - r}\right)^{1/p} d_p(f, 0)\right). \quad (39)$$

\qed

Lemma 9. If $\{f_n\}$ is a Cauchy sequence in the space $M^p$, then $\{f_n\}$ converges uniformly on compact subsets of $D$ to some holomorphic function $f$ on $D$.

Proof. From the inequality (38) of Lemma 7, it follows that $\{f_n\}$ is a Cauchy sequence in $N^p$. Therefore, there exists $f \in N^p$ such that $f_n \to f$ in $N^p$, and so, by Lemma 8, $f_n \to f$ uniformly on compact subsets of $D$. \qed

The following result is a maximal theorem of Hardy and Littlewood.
Lemma 10 (see [16, page 11]). Let $1 < p \leq +\infty$ and let $\varphi$ be a function in the Lebesgue space $L^p(T)$. Let

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \varphi(t) \, dt, \quad 0 \leq r < 1$$

be the Poisson integral of the function $\varphi$. Define

$$U(\theta) = \sup_{0 \leq r < 1} |u(r,\theta)|, \quad \theta \in [0, 2\pi].$$

Then $U \in L^p(T)$ and there is a constant $A_p$ depending only on $p$ such that

$$\|U\|_{L^p} \leq A_p \|\varphi\|_{L^p},$$

where $\cdot \| \cdot _{L^p}$ is the usual norm of the space $L^p(T)$.

We are now ready to state the following result.

Theorem 11. $M^p = N^p$ for each $p > 1$; that is, the spaces $M^p$ and $N^p$ coincide.

Proof. By Lemma 7, $M^p \subseteq N^p$ for each $p > 1$. For the proof of the converse of this inclusion, assume that $f \in N^p$. We will show that $f \in M^p$. As noticed in Section 1, $f$ can be factorized as

$$f(z) = I(z) F(z), \quad z \in \mathbb{D},$$

where $I(z)$ is the inner function and $F(z)$ is an outer function for the class $N^p$; that is,

$$F(z) = \omega \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |f^*(e^{i\theta})| \, d\theta \right),$$

where $\omega$ is a constant of unit modulus. Furthermore, $|f(z)| \leq 1$, for each $z \in \mathbb{D}$, the previous factorization and the fact that $F \in M^p$ immediately imply that $f \in M^p$. Since

$$\Re \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2}, \quad z = re^{i\theta},$$

from (44), we immediately obtain

$$\log|F(re^{i\theta})| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log |f^*(e^{i\theta})| \, dt, \quad 0 \leq r < 1,$$

whence it follows that, for $0 \leq r < 1$,

$$\log^+ |F(re^{i\theta})| = \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log |f^*(e^{i\theta})| \, dt \right)^+$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log^+ |f^*(e^{i\theta})| \, dt.$$

The above inequality yields

$$\log^+ MF(\theta) \leq \sup_{0 \leq r < 1} \left( \log^+ |F(re^{i\theta})| \right)$$

$$\leq \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log^+ |f^*(e^{i\theta})| \, dt \right).$$

From the above inequality and the fact that $\log^+ |f^*| \in L^p(T)$, we conclude by Lemma 10 that $\log^+ MF(\theta) \in L^p(T)$. This means that $F \in M^p$ and therefore $f \in M^p$. Thus $N^p \subseteq M^p$, and therefore $M^p = N^p$. This completes the proof.

Corollary 12. Let $f \in M^p$. Then

$$\int_0^{2\pi} (\log^+ Mf(\theta))^p \, d\theta \leq C_p \int_0^{2\pi} (\log^+ |f^*(e^{i\theta})|)^p \, d\theta,$$

where $C_p$ is a nonnegative constant depending only on $p$.

Proof. Let $F$ be the outer factor in the canonical factorization of $f \in M^p$. From the proof of Theorem 11, we see that for the functions $U(\theta) = \log^+ M(\theta)$ and $\varphi(\theta) = \log^+ |f^*(e^{i\theta})|$ the inequality (42) can be applied from Lemma 10. The obtained inequality is in fact (49) with $F$ instead of $f$. Since $|f(z)| \leq |F(z)|$, for each $z \in \mathbb{D}$, it follows that $Mf(\theta) \leq MF(\theta)$ at almost every $\theta \in [0, 2\pi]$; thus (49) is obviously satisfied.

4. $M^p$ as an $F$-Algebra

Theorem 13. The space of all polynomials over $\mathbb{C}$ is a dense subset of $M^p$. Hence, $M^p$ is a separable metric space.

Proof. Suppose that $f \in M^p$. Since, for a fixed $0 \leq \rho < 1$, $f_\rho$ is a holomorphic function on the closed unit disk $\overline{\mathbb{D}} : |z| \leq 1$, by Runge's theorem, $f_\rho$ can be uniformly approximated by polynomials on $\overline{\mathbb{D}}$. This together with the fact that, by Theorem 5, $f_\rho \to f$ in $M^p$ as $\rho \to 1$ − yields that the space of all polynomials over $\mathbb{C}$ is a dense subset of $M^p$. Therefore, the set of all polynomials whose coefficients have rational real parts and rational imaginary parts becomes a countable dense subset of $M^p$. This concludes the proof.

Theorem 14. $M^p$ is a complete metric space.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $M^p$. Then since $N^p$ is complete, there is a $f \in N^p$ such that $f_n \to f$ in $N^p$. Since, by Theorem 11, $M^p = N^p$, it follows that $f \in M^p$, and thus it remains to show that $f_n \to f$ in $M^p$. By Theorem 5 and Lemma 6, there exist $0 < r < 1$ and $n_1 \in \mathbb{N}$ such that

$$\rho_p(f, f_n) < \frac{\epsilon}{3},$$

$$\rho_p(f_n, f) < \frac{\epsilon}{3}$$

for each $n \geq n_1$. (50)
Since, by Lemma 9, a sequence \( \{f_n\} \) converges uniformly on each closed disk \( |z| \leq \rho < 1 \) to some function \( f \), it follows that there exists \( n_2 \in \mathbb{N} \) such that
\[
\rho_p ((f_n)_p, f) < \frac{\varepsilon}{3} \quad \text{for each } n \geq n_2. \tag{51}
\]
Taking \( n_0 = \max\{n_1, n_2\} \), by (50) and (51), the triangle inequality implies that
\[
\rho_p (f_n, f) < \varepsilon \quad \forall n \geq n_0. \tag{52}
\]
This shows that \( f_n \to f \) in \( M^p \), which completes the proof. \( \Box \)

**Theorem 15.** \( M^p \) with the topology given by the metric \( \rho_p \) defined by (23) becomes an \( F \)-space.

**Proof.** By [22, page 51], it suffices to show the following properties:

(i) \( \rho_p \) is an additive-invariant metric,

(ii) for any fixed \( f \in M^p \), \( c \mapsto cf \) is a continuous map from \( \mathbb{C} \) into \( M^p \),

(iii) for any fixed \( c \in \mathbb{C} \), \( f \mapsto cf \) is a continuous map from \( M^p \) into \( M^p \), and

(iv) \( M^p \) is a complete metric space.

The assertion (i) follows from Theorem 2.

By the Lebesgue dominated convergence theorem, we have
\[
\rho_p (cf, 0) = \left( \int_0^{2\pi} \log^p (1 + |c| Mf (\theta)) \frac{d\theta}{2\pi} \right)^{1/p} \to 0
\]
as \( c \to 0 \).

Let \( k \in \mathbb{N} \) such that \( |c| \leq k \). Then the triangle inequality yields
\[
\rho_p (cf, 0) \leq \rho_p (kf, 0) \leq k \rho_p (f, 0), \tag{54}
\]
whence we see that \( f \mapsto cf \) is a continuous map from \( M^p \) into \( M^p \).

The assertion (iv) is in fact the assertion of Theorem 14. This concludes the proof. \( \Box \)

We are now ready to prove that the (metric) spaces \( (M^p, \rho_p) \) and \( (N^p, d_p) \) have the same topological structure.

**Theorem 16.** For each \( p > 1 \), the classes \( M^p \) and \( N^p \) coincide, and the metric spaces \( (M^p, \rho_p) \) and \( (N^p, d_p) \) have the same topological structure.

**Proof.** Consider the identity map \( j : M^p \to N^p \). Then, by the inequality (38) of Lemma 7, \( j \) is continuous. Since, by Theorem 11, \( M^p = N^p \), \( j \) maps \( M^p \) onto \( N^p \). Since \( M^p \) and \( N^p \) are both \( F \)-spaces, it follows, by the open mapping theorem [23, Corollary 2.12 (b)], that the inverse map \( j^{-1} \) of \( j \) is continuous. Hence, \( j \) is a homeomorphism, and so the metrics \( d_p \) and \( \rho_p \) induce the same topology on \( N^p \) and \( M^p \), respectively.

As an application of Theorem 16 and using the characterization of topological dual of the space \( F^p \) (which is by [7, Theorem 4.2] the Fréchet envelope of \( N^p \)) given by Stoll [6, Theorem 3.3] (cf. also [12, Theorem 3.5] and [13, Theorem 2]), we immediately get the following result.

**Theorem 17.** If \( \gamma \) is a continuous linear functional on \( M^p \), then there exists a sequence \( \{\gamma_n\}_n \) of complex numbers with \( \gamma_n = O(\exp(-cn^{1/(p+1)})) \), for some \( c > 0 \), such that
\[
\gamma (f) = \sum_{n=0}^{\infty} a_n \gamma_n, \tag{55}
\]
where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in M^p \), with convergence being absolute. Conversely, if \( \{\gamma_n\}_n \) is a sequence of complex numbers for which
\[
\gamma_n = O(\exp(-cn^{1/(p+1)})) \tag{56}
\]
then (55) defines a continuous linear functional on \( M^p \).

**Corollary 18.** \( M^p \) is an \( F \)-algebra.

**Proof.** By Theorem 15, \( M^p \) becomes an \( F \)-space. As \( N^p \) is an \( F \)-algebra, by Theorem 16, the multiplication is also continuous on \( M^p \). Hence, \( M^p \) is an \( F \)-algebra. \( \Box \)

**5. Bounded Subsets of \( M^p \)**

It is proved in Section 4 (Theorem 16) that the spaces \( M^p \) and \( N^p \) coincide and have the same topological structure. Since \( N^p \) and \( M^p \) are not Banach spaces, it is of interest to obtain a characterization of bounded subsets of these spaces in terms of both metrics \( d_p \) and \( \rho_p \).

Recall that, for a function \( f \in N^p \), its boundary function \( f^* \) is defined as the radial limit \( f^*(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta}) \) which exists for almost every \( e^{i\theta} \in T \).

The following result gives a characterization of bounded subsets of \( N^p (= M^p) \). Recall that the assertion (i)\( \Rightarrow \) (iii) is analogous to Theorem 1 in [21] that describes bounded subsets of \( N^r \).

**Theorem 19.** For given set \( L \subset M^p \), the following conditions are equivalent:

(i) \( L \) is a bounded subset of \( M^p \);

(ii) for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\int_E (\log^+ Mf (\theta))^p \frac{d\theta}{2\pi} < \varepsilon \quad \forall f \in L, \tag{57}
\]
for every measurable set \( E \subset \mathbb{T} \) with the Lebesgue measure \( |E| \leq \delta \);

(iii) for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\int_E (\log^+ |f^* (e^{i\theta})|^p) \frac{d\theta}{2\pi} < \varepsilon \quad \forall f \in L, \tag{58}
\]
for each measurable set \( E \subset \mathbb{T} \) with the Lebesgue measure \( |E| \leq \delta \).
Proof. (ii)⇒(iii). It follows that from the obvious inequality $|f^*(e^{i\theta})| \leq Mf(\theta)$, $f \in M^p$, for almost every $\theta \in [0,2\pi]$.

(iii)⇒(i). Let

$$V = \{g \in N^p : d_p(g,0) < \eta\}$$

be an arbitrary neighbor of zero in $N^p$. Choose sufficiently small $\epsilon > 0$ such that

$$\log^p (1 + \epsilon) + 2^{-p-1} \log^p 2\delta + 2^{-p-1}\epsilon < \eta^p.$$  

Now it follows that there exists $\delta, 0 < \delta < \epsilon$, such that (iii) holds. Choose an $n \in \mathbb{N}$ for which $1/n < \delta$. Set

$$E_k = \left\{e^{i\theta} : \theta \in \left[\frac{2(k-1)\pi}{n}, \frac{2k\pi}{n}\right]\right\}, \quad k = 1,2,\ldots,n.$$  

Then $|E_k| = 1/n < \delta$, and thus by (iii) we have

$$\int_0^{2\pi} \left(\log^p |f^*(e^{i\theta})|\right)^p \frac{d\theta}{2\pi} =: n \sum_{k=1}^{n} \int_{E_k} < ne, \quad \forall f \in L.$$  

By (62) and Chebyshev’s inequality, we conclude that for every function $f \in N^p$ there exists a measurable set $E_f \subset \mathbb{T}$ depending on $f$ such that

$$\left|\mathbb{T} \setminus E_f\right| < \delta, \quad \left(\log^p |f^*(e^{i\theta})|\right)^p \leq \frac{ne}{\delta} \quad \text{on } E_f.$$  

From (63), we obtain

$$\left|f^*(e^{i\theta})\right| \leq \exp\left(\frac{ne}{\delta}\right)^{1/p} = K(\delta) = K \quad \text{on } E_f.$$  

Choose $\alpha$ such that $0 < \alpha < \epsilon/\delta$. Then using the inequality

$$\log^p (1 + |a|) \leq 2^{-p-1} \left((\log^+ |a|)^p + \log^p 2\right),$$  

(60) and (iii), for every $f \in L$, we obtain

$$\left(d_p(\alpha f,0)\right)^p = \int_0^{2\pi} \log^p (1 + |\alpha f^*(e^{i\theta})|) \frac{d\theta}{2\pi} \leq \int_{E_f} \log^p (1 + \epsilon) \frac{d\theta}{2\pi} + 2^{-p-1} \left(\int_{E_f} \log^p e \frac{d\theta}{2\pi} + \int_{E_f} \left(\log^+ |f^*(e^{i\theta})|\right)^p \frac{d\theta}{2\pi}\right)$$

$$\leq \log^p (1 + \epsilon) + 2^{-p-1} \log^p 2\delta + 2^{-p-1}\epsilon < \eta^p.$$  

Therefore, $d_p(\alpha f,0) < \eta$, from which it follows that $\alpha L \subset V$. Hence, $L$ is a bounded subset of $N^p$.

(i)⇒(ii). Assume that $L$ is a bounded subset of $M^p$. Then for any given $\eta > 0$ there is an $\alpha_0 = \alpha_0(\eta), 0 < \alpha_0 < 1$, such that

$$\left(\frac{\eta}{\alpha_0}\right)^p = \int_0^{2\pi} \log\left(1 + |\alpha Mf(\theta)|\right) \frac{d\theta}{2\pi} < \eta^p$$

for each $f \in L$ and $|\alpha| \leq \alpha_0$. It follows that

$$\int_0^{2\pi} \left(\log^p |\alpha Mf(\theta)|\right)^p \frac{d\theta}{2\pi} < \eta^p$$

for each $f \in L, |\alpha| \leq \alpha_0$.

Since

$$\log^p Mf(\theta) \leq \log^p \alpha Mf(\theta) + \log^p \frac{1}{\alpha_0},$$

we obtain

$$\left(\log^p Mf(\theta)\right)^p \leq 2^{-p-1} \left((\log^+ \alpha_0 Mf(\theta))^p + \left(\log^p \frac{1}{\alpha_0}\right)^p\right).$$

For given $\epsilon > 0$, choose $\eta > 0$ satisfying

$$\eta < \frac{\epsilon^{1/p}}{2},$$

and $\alpha_0 = \alpha_0(\eta)$ satisfying (67) and so also satisfying (68). Next, take $\delta > 0$ such that

$$\delta \log^p \frac{1}{\alpha_0} < \frac{\epsilon}{2p^p}.$$  

Then for each set $E \subset \mathbb{T}$ with $|E| < \delta$, by (68)–(72), for every $f \in L$, we obtain

$$\int_E \left(\log^p Mf(\theta)\right)^p \frac{d\theta}{2\pi} \leq 2^{-p-1} \left(\int_E \left(\log^+ \alpha_0 Mf(\theta)\right)^p \frac{d\theta}{2\pi} + \int_E \log^p \frac{1}{\alpha_0} \frac{d\theta}{2\pi}\right)$$

$$\leq 2^{-p-1}\eta^p + 2^{-p-1} |E| \log^p \frac{1}{\alpha_0} \leq \epsilon.$$  

Therefore, the condition (ii) of the theorem is satisfied, which concludes the proof.

Remark 20. Note that the condition (ii) from Theorem 19 in fact means that the family $\{\log^p Mf(\theta)\} : f \in L$ is uniformly integrable on $\mathbb{T}$. The same assertion is also valid for the condition (iii). On the other hand, from the proof of Theorem 19, we see that (ii) implies that the family $\{\log^p Mf(\theta)\} : f \in L$ forms a bounded subset of the space $L^1(\mathbb{T})$. That is, there holds

$$\limsup_{f \in L} \int_0^{2\pi} \left(\log^p Mf(\theta)\right)^p \frac{d\theta}{2\pi} < +\infty.$$  


Similarly, it follows from (iii) that the family \( \{ \log^+ |f(e^{i\theta})|^p : f \in L \} \) is bounded in \( L^1(\mathbb{T}) \).

**Corollary 21.** If \( L \) is a subset of \( M^p \) for which the family
\[
\left\{ \left( \log^+ |f(e^{i\theta})| \right)^p : f \in L \right\}
\]
is uniformly integrable, then the family
\[
\left\{ \left( \log^+ |f(e^{i\theta})| \right)^p : f \in L, 0 \leq r < 1 \right\}
\]
is also uniformly integrable.

**Proof.** The condition of Corollary 21 and (iii)⇒(ii) of Theorem 19 immediately yield that the family \( \{ \log^+ |f(e^{i\theta})|^p : f \in L \} \) is uniformly integrable on the circle \( \mathbb{T} \). This fact and the obvious inequality \( |f(e^{i\theta})| \leq Mf(\theta), f \in M^p, 0 \leq r < 1 \), for almost every \( \theta \in [0, 2\pi] \), imply that the family \( \{ \log^+ |f(e^{i\theta})|^p : f \in L, 0 \leq r < 1 \} \) is uniformly integrable.

The following result gives a necessary condition for a subset of \( M^p (= N^p) \) to be bounded.

**Theorem 22.** Let \( L \) be a subset of \( M^p \). If \( L \) is bounded in \( M^p \), then
\[
M_{\infty}(r, f) \leq K \exp \left( \frac{\omega(r)}{(1 - r)^{1/p}} \right) \text{ for each } f \in L,
\]
where \( M_{\infty}(r, f) = \max_{0 \leq \theta < 2\pi} |f(e^{i\theta})|, K \) is a positive constant, and \( \omega(r), 0 \leq r < 1 \), is a positive continuous function that does not depend on \( f \in L \) and for which \( \omega(r) \downarrow 0 \) as \( r \to 1 \).

**Proof.** By the inequqlity (5.4) from the proof of Theorem 5.2 in [4], for all \( f \in N^p \), we have
\[
\left( \log^+ |f(e^{i\theta})| \right)^p \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos (\theta - t) + r^2} \left( \log^+ |f^*(e^{i\theta})| \right)^p dt.
\]

as, by the assumption, \( L \) is a bounded subset of \( N^p \), by Theorem 19 (iii), for all \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \), such that
\[
\int_0^{2\pi} \left( \log^+ |f^*(e^{i\theta})| \right)^p d\theta < \frac{\varepsilon}{2} \forall f \in L
\]
and for every measurable set \( E \subset \mathbb{T} \) with the Lebesgue measure \( |E| < \delta \).

Further, from the proof of (iii)⇒(i) of Theorem 19, we see that for each \( f \in N^p \) there is a measurable set \( E_f \subset \mathbb{T} \) depending on \( f \) for which
\[
|\mathbb{T} \setminus E_f| < \delta, \quad \left( \log^+ |f^*(e^{i\theta})| \right)^p \leq \frac{\varepsilon}{\delta}
\]
for almost every \( e^{i\theta} \in E_f \). From (78)–(80), we obtain
\[
\left( \log^+ |f(re^{i\theta})| \right)^p = \int_{E_f} + \int_{E_f} \leq \frac{\varepsilon}{\delta} + \frac{1}{1 - r} \frac{\varepsilon}{2},
\]
whence it follows that
\[
(1 - r) \left( \log^+ M_{\infty}(r, f) \right)^p \leq \frac{(1 - r) \varepsilon}{\delta} + \frac{\varepsilon}{2}.
\]

Choose a sequence \( \{ \varepsilon_k \} \) of positive numbers such that \( \varepsilon_k \downarrow 0 \). For each \( k \in \mathbb{N} \), let \( r_k > 0 \) be a number such that
\[
(1 - r_k) \frac{\varepsilon_k}{\delta_k} + \frac{\varepsilon_k}{2} < \varepsilon_k,
\]
where \( \varepsilon_k = \delta(\varepsilon_k) \) and
\[
r_{k-1} < r_k < 1, \quad r_k \uparrow 1 \text{ as } k \to \infty.
\]

Put
\[
\omega_1(r) = \varepsilon_k \text{ for } r_k \leq r < r_{k+1}, \quad k = 1, 2, \ldots.
\]

From (82), (83), and (85) we obtain
\[
\left( \log^+ M_{\infty}(r, f) \right)^p \leq \frac{\omega_1(r)}{1 - r} \forall 0 \leq r < 1.
\]

Since
\[
\omega_1(r) \to 0 \text{ as } r \to 1,
\]
we conclude that there exists a continuous function \( \omega_2(r) \) satisfying
\[
\omega_1(r) \leq \omega_2(r), \quad \omega_2(r) \downarrow 0 \text{ as } r \to 1.
\]

Therefore,
\[
\left( \log^+ M_{\infty}(r, f) \right)^p \leq \frac{\omega_2(r)}{1 - r} \text{ for each } 0 \leq r < 1,
\]
whence by setting
\[
\omega(r) = \left( \omega_2(r) \right)^{1/p} \text{ for each } 0 \leq r < 1,
\]
we obtain
\[
M_{\infty}(r, f) \leq \exp \left( \frac{\omega(r)}{(1 - r)^{1/p}} \right) \forall f \in L.
\]

This concludes the proof. \( \square \)

**Remark 23.** The condition of Theorem 22 is not a sufficient condition for a set \( L \subset M^p \) to be bounded. To show this, define
\[
f_n(z) = a_n z^n, \quad a_n = \exp \left( \lambda_n r^{1/(p+1)} \right),
\]
where
\[ \lambda_n = n^{-1/2(p+1)}. \] (93)
Then as in the proof of Lemma 1 in [21] it is easy to verify that the set \( L = \{ f_n \} \subset M^p \) satisfies the condition of Theorem 22. Since
\[ \log |f_n^* (e^{i\theta})| = n^{1/2(p+1)}, \] (94)
we see that \( L \) is not bounded in \( M^p \).

**Theorem 24.** There exist bounded subsets of \( M^p \) that are not relatively compact.

**Proof.** Define a sequence \( \{ h_n \} \) of functions on [0, 2\( \pi \)] as
\[ h_n(t) = 1 + \sin (nt), \quad t \in [0, 2\pi], \] (95)
and set
\[ f_n(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} + z h_n(t) \, dt \right) \]
\[ = \exp (1 - iz^n), \quad z \in \mathbb{D}. \] (96)
Obviously, \( \{ f_n \} \subset N^p \) and for each measurable set \( E \subset \mathbb{T} \) we have
\[ \int_0^{2\pi} h_n(t) \, dt = 2\pi, \]
\[ 0 \leq \int_E h_n(t) \, dt \leq 2 |E|, \] (97)
where \( |E| \) denotes the Lebesgue measure of \( E \). From this and Theorem 19, we see that the set \( L = \{ f_n \} \) is bounded in \( N^p \).

Now suppose that \( E \) is relatively compact. This means that there exists a subsequence \( \{ f_{nk} \} \) of \( \{ f_n \} \) and a function \( f \in N^p \) such that
\[ d_p (f_{nk}, f) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \] (98)
and thus
\[ f_{nk} (z) \rightarrow f(z), \] (99)
uniformly on each closed disk \( |z| \leq r < 1 \).

Therefore, by (96), it follows that \( f(z) \equiv e \) on \( \mathbb{D} \). On the other hand, from (98), it follows that
\[ f_{nk} (e^{i\theta}) \rightarrow f*(e^{i\theta}) \quad \text{in measure on} \quad \mathbb{T}. \] (100)
Therefore,
\[ \log |f_{nk}^* (e^{i\theta})| = 1 + \sin (nk\theta) \rightarrow \log |f^* (e^{i\theta})| = 1 \]
in measure on \( \mathbb{T}. \) (101)
This contradiction shows that \( L \) is not relatively compact in \( N^p \).

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**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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