Short fans and the 5/6 bound for line graphs

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March 9, 2022

Abstract

In 2011, the second author conjectured that every line graph $G$ satisfies $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G) + 8}{6}\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by $k$ parallel edges, and taking the line graph. In this paper we prove the conjecture. We also develop more general techniques and results that will likely be of independent interest, due to their use in attacking the Goldberg–Seymour conjecture.

1 Overview

By graph we mean multigraph without loops. Our notation follows Diestel [3]1. In [8], the second author showed that $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G) + 10}{8}\}$ for every line graph $G$. In the same paper, he conjectured that $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G) + 8}{6}\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by $k$ parallel edges, and taking the line graph. In this paper we prove the latter inequality. Along the way, we develop more general techniques and results that will likely be of independent interest. The main result of this paper is the following theorem.

Theorem 20 (5/6-Theorem). If $Q$ is a line graph, then

$$\chi(Q) \leq \max\{\omega(Q), \frac{5\Delta(Q) + 8}{6}\}.$$ 

For every graph $G$, we have $\chi'(G) \geq \left\lceil \frac{\|E\|}{\|V\|} \right\rceil$, since in any proper edge-coloring each color class has size at most $\left\lfloor \frac{|\Gamma|}{2} \right\rfloor$. Likewise, the same bound holds for any subgraph $H$. Thus

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1In particular, $|G|$ denotes $|V(G)|$ and $\|G\|$ denotes $|E(G)|$.
\[ \chi'(G) \geq \max_{H \subseteq G} \left\lceil \frac{|H|}{\Delta(H)} \right\rceil \text{ (where the max is over all subgraphs } H \text{ with at least two vertices)}.

For convenience, we let \[ W(G) := \max_{H \subseteq G} \left\lceil \frac{|H|}{\Delta(H)} \right\rceil \]. Goldberg \[ 5, 6 \] and Seymour \[ 10, 11 \] each conjectured that this lower bound holds with equality, whenever \( \chi'(G) > \Delta(G) + 1 \).

**Goldberg–Seymour Conjecture.** Every graph \( G \) satisfies

\[ \chi'(G) \leq \max \{ W(G), \Delta(G) + 1 \}. \]

The Goldberg–Seymour Conjecture is the major open problem in the area of edge-coloring multigraphs. Most of our work goes toward proving the following intermediate result, in Section 4. This theorem is a weakened version of both the Goldberg–Seymour Conjecture and our main result, the \( \frac{5}{6} \)-Theorem.

**Theorem 11** (Weak \( \frac{5}{6} \)-Theorem). If \( Q \) the line graph of a graph \( G \), then

\[ \chi(Q) \leq \max \left\{ W(G), \Delta(G) + 1, \frac{5\Delta(Q) + 8}{6} \right\}. \]

Finally, in Section 5 we show that the Weak \( \frac{5}{6} \)-Theorem does indeed imply the \( \frac{5}{6} \)-Theorem. To conclude, in Section 6 we prove strengthenings of Reed’s Conjecture for line graphs that follow from the general lemmas we prove earlier in the paper.

## 2 Tashkinov Trees

A graph \( G \) is **elementary** if \( \chi'(G) = W(G) \). Let \([k]\) denote \{1, \ldots, k\}. For a path or cycle \( Q \), let \( \ell(Q) \) denote the length of \( Q \). A graph \( G \) is **critical** if \( \chi'(G - e) < \chi'(G) \) for all \( e \in E(G) \).

For a graph \( G \) and a partial \( k \)-edge-coloring \( \varphi \), for each vertex \( v \in V(G) \), let \( \varphi(v) \) denote the set of colors used in \( \varphi \) on edges incident to \( v \). Let \( \overline{\varphi}(v) = [k] \setminus \varphi(v) \). A color \( c \) is **seen** by a vertex \( v \) if \( c \in \varphi(v) \) and \( c \) is **missing** by \( v \) if \( c \in \overline{\varphi}(v) \). Given a partial \( k \)-edge-coloring \( \varphi \), a set \( W \subseteq V(G) \) is **elementary** with respect to \( \varphi \) (henceforth, \( w.r.t. \) \( \varphi \)) if each color in \([k]\) is missing at no more than one vertex of \( W \). More formally, \( \overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset \) for all distinct \( u, v \in W \). A **defective color** for a set \( X \subseteq V(G) \) (\( w.r.t. \) \( \varphi \)) is a color used on more than one edge from \( X \) to \( V(G) \setminus X \). A set \( X \) is **strongly closed** \( w.r.t. \) \( \varphi \) if \( X \) has no defective color. Elementary and strongly closed sets are of particular interest because of the following theorem, proved implicitly by Andersen \[ 11 \] and Goldberg \[ 6 \]; see also \[ 12 \] Theorem 1.4.

**Theorem 1.** Let \( G \) be a graph with \( \chi'(G) = k + 1 \) for some integer \( k \geq \Delta(G) \). If \( G \) is critical, then \( G \) is elementary if and only if there exists \( uv \in E(G) \), a \( k \)-edge-coloring \( \varphi \) of \( G - uv \), and a set \( X \) with \( u, v \in X \) such that \( X \) is both elementary and strongly closed \( w.r.t. \) \( \varphi \).
A Tashkinov tree w.r.t. $\varphi$ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_{t-1}, e_t, v_t$ such that all $v_i$ are distinct, $e_i = v_jv_i$ and $\varphi(e_i) \in \varphi(v_i)$ for some $j$ and $\ell$ with $0 \leq j < i$ and $0 \leq \ell < i$. A Vizing fan (or simply fan) is a Tashkinov tree that induces a star. Tashkinov trees are of interest because of the following lemma.

**Tashkinov’s Lemma.** Let $G$ be a graph with $\chi'(G) = k + 1$, for some integer $k \geq \Delta(G) + 1$ and choose $e \in E(G)$ such that $\chi'(G - e) < \chi'(G)$. Let $\varphi$ be a $k$-edge-coloring of $G - e$. If $T$ is a Tashkinov tree w.r.t. $\varphi$ and $e$, then $V(T)$ is elementary w.r.t. $\varphi$.

In view of Theorem 1 and Tashkinov’s Lemma, to prove that a graph $G$ is elementary, it suffices to find an edge $e$, a $k$-edge-coloring $\varphi$ of $G - e$, and a Tashkinov tree $T$ containing $e$ such that $V(T)$ is strongly closed. This motivates our next two lemmas. But first, we need a few more definitions.

Let $t(G)$ be the maximum number of vertices in a Tashkinov tree over all $e \in E(G)$ and all $k$-edge-colorings $\varphi$ of $G - e$. Let $\mathcal{T}(G)$ be the set of all triples $(T, e, \varphi)$ such that $e \in E(G)$, $\varphi$ is a $k$-edge-coloring of $G - e$, and $T$ is a Tashkinov tree with respect to $e$ and $\varphi$ with $|T| = t(G)$. Notice that, by definition, we have $\mathcal{T}(G) \neq \emptyset$. For a $k$-edge-coloring $\varphi$ of $G - e$, a maximal Tashkinov tree starting with $e$ may not be unique. However, if $T_1$ and $T_2$ are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$. Let $G$ be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let $\varphi$ be a $k$-edge-coloring of $G - e_0$ for some $e_0 \in E(G)$. For $v \in V(G)$ and colors $\alpha, \beta$, let $P_v(\alpha, \beta)$ be the maximal connected subgraph of $G$ that contains $v$ and is induced by edges with color $\alpha$ or $\beta$. So $P_v(\alpha, \beta)$ is a path or a cycle. For a $k$-edge-coloring $\varphi$ of $G - v_0v_1$, we often let $P = P_{v_1}(\alpha, \beta)$ for some $\alpha \in \varphi(v_0)$ and $\beta \in \varphi(v_1)$. Clearly $P$ must end at $v_0$ (or we can swap colors $\alpha$ and $\beta$ on $P$ and color $v_0v_1$ with $\alpha$), so let $v_1, \ldots, v_r, v_0$ denote the vertices of $P$ in order. To rotate the $\alpha, \beta$ coloring on $P \cup \{v_0v_1\}$ by one, we uncolor $v_1v_2$ and use its color on $v_0v_1$. To rotate the $\alpha, \beta$ coloring on $P \cup \{v_0v_1\}$ by $j$, we rotate the $\alpha, \beta$ coloring by one $j$ times in succession. (When we do not specify $j$, we allow $j$ to take any value from 1 to $r$.)

**Lemma 2.** Let $G$ be a non-elementary critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. For every $v_0v_1 \in E(G)$, a $k$-edge-coloring $\varphi$ of $G - v_0v_1$, $\alpha \in \varphi(v_0)$, and $\beta \in \varphi(v_1)$, we have $|P_{v_1}(\alpha, \beta)| < t(G)$.

**Proof.** Suppose the lemma is false and choose $v_0v_1 \in E(G)$, a $k$-edge-coloring $\varphi$ of $G - v_0v_1$, $\alpha \in \varphi(v_0)$, and $\beta \in \varphi(v_1)$, such that $|P_{v_1}(\alpha, \beta)| \geq t(G)$. Let $P = P_{v_1}(\alpha, \beta)$; see Figure 1. Let $(T, v_0v_1, \varphi)$ be a Tashkinov tree that begins with edges $v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$. Now $V(T) = V(P)$ since $t(G) \geq |T| \geq |P| \geq t(G)$. By hypothesis $G$ is non-elementary, so Theorem 1 implies that $V(T)$ is not strongly closed; thus, $T$ has a defective color $\delta$ with respect to $\varphi$. Choose $\tau \in \varphi(v_2)$. Let $Q = P_{v_2}(\tau, \delta)$. Since $T$ is maximal, $\delta$ is not missing at any vertex of $T$, and since $V(T)$ is elementary, $\tau$ is not missing at any vertex of $T$ other than $v_2$. As a result, $Q$ ends outside $V(T)$. Now $Q$ could leave $V(T)$ and re-enter it repeatedly, but $Q$ ends outside $V(T)$, so there is a last vertex $w \in V(Q) \cap V(T)$; say $Q$ ends at $z \in V(G) \setminus V(T)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at $w$. Since $\tau \in \varphi(v_2)$ and $\pi \in \varphi(w)$ and $|T| = t(G)$,
no edge colored $\tau$ or $\pi$ leaves $V(T)$. So we can swap $\tau$ and $\pi$ on every edge in $G - V(T)$ without changing the fact that $T$ is a Tashkinov tree with $|T| = t(G)$. After swapping $\tau$ and $\pi$, we swap $\delta$ and $\pi$ on the subpath of $Q$ from $w$ to $z$. Since $\pi$ is missing at $w$, the $\delta - \pi$ path starting at $z$ must end at $w$. Now $\delta$ is missing at $w$, but $\delta$ was defective in $\varphi$, so some other edge $e$ colored $\delta$ still leaves $V(T)$. Adding $e$ gets a larger Tashkinov tree, which is a contradiction.

3 Short vertices and long vertices

A vertex $v \in V(G)$ is short if every Vizing fan rooted at $v$ (taken over all $k$-colorings of $G - e$, over all edges $e$ incident to $v$) has at most 3 vertices, including $v$. Otherwise, $v$ is long. Let $\nu(T)$ be the number of long vertices in a Tashkinov tree $T$.

Now we can outline our proof of the $\frac{5}{6}$-Conjecture. We will show in Section 4 (and at the end of Section 4) that the $\frac{5}{6}$-Conjecture is implied by the Goldberg–Seymour Conjecture. More precisely, if $Q$ is the line graph of graph $G$ and $\chi(Q) = \chi'(G) \leq \max \{W(G), \Delta(G) + 1\}$, then also $\chi(Q) \leq \max \{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$. So here it suffices to prove the bound $\chi'(G) \leq \max \{W(G), \Delta(G) + 1, \frac{5\Delta(Q)+8}{6}\}$. We consider cases based on $\nu(T)$, for some Tashkinov tree $T \in \tilde{T}(G)$.

In the present section, we show that if $G$ has a maximum Tashkinov tree $T$ that contains no long vertices, i.e., $\nu(T) = 0$, then $G$ is elementary. In fact, Lemma 7 implies that the same is true when $\nu(T) = 1$. In the proof of Theorem 11 we show that if $G$ is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then every long vertex $v$ has $d(v) < \frac{3}{4} \Delta(G)$. This implies that $\nu(T) < 4$, since otherwise the number of colors missing at vertices of $T$ is more than $4(k - \frac{3}{4} \Delta(G)) > k$, which contradicts that $V(T)$ is elementary. So it remains to consider the case $\nu(T) \in \{2, 3\}$.
In Section 4, we introduce \( k \)-thin graphs; these are essentially graphs for which \( \mu(G) \) is not too large. Using a lemma from [8], we show that every minimal counterexample to the \( \frac{5}{6} \)-Conjecture must be \( k \)-thin. We then extend the ideas of the present section to handle the case when \( \nu(T) \in \{2, 3\} \). Much like when \( \nu(T) \geq 4 \), we show that \( T \) has too many colors missing at its vertices to be elementary.

Short vertices were introduced in [2], where they were motivated by a version of the following lemma in the context of proving a strengthening of Reed’s Conjecture for line graphs.

**Parallel Edge Lemma.** Let \( G \) be a critical graph with \( \chi'(G) = k + 1 \) for some integer \( k \geq \Delta(G) + 1 \). Let \( \varphi \) be a \( k \)-edge-coloring of \( G - v_0v_1 \). Choose \( \alpha \in \varphi(v_0) \) and \( \beta \in \varphi(v_1) \). Let \( P = v_1v_2 \cdots v_r \) be an \( \alpha, \beta \) path with edges \( e_i = v_iv_{i+1} \) for all \( i \in [r-1] \). If \( v_i \) is short for all odd \( i \), then for each \( \tau \in \varphi(v_0) \) and for all odd \( i \in [r-1] \) there are edges \( f_i = v_iv_{i+1} \) such that \( \varphi(f_i) = \tau \).

**Proof.** Suppose not and choose a counterexample minimizing \( r \). By minimality of \( r \), we have \( \varphi(v_{r-1}v_r) = \alpha \) and we have \( f_i = v_iv_{i+1} \) for all odd \( i \in [r-2] \) such that \( \varphi(f_i) = \tau \); see Figure 2. Swap \( \alpha \) and \( \beta \) on \( e_i \) for all \( i \in [r-3] \) and then color \( v_0v_1 \) (call this edge \( e_0 \)) with \( \alpha \) and uncolor \( e_{r-2} \). Let \( \varphi' \) be the resulting coloring. Since \( k \geq \Delta(G) + 1 \), some color other than \( \alpha \) is missing at \( v_{r-2} \); let \( \gamma \) be such a color. Now \( v_{r-1} \) is short since \( r - 1 \) is odd (since \( P \) starts and ends with \( \alpha \)), so there is an edge \( e = v_{r-1}v_r \) with \( \varphi'(e) = \gamma \). Swap \( \tau \) and \( \alpha \) on \( e_i \) for all \( i \) with \( 0 \leq i \leq r - 3 \) to get a new coloring \( \varphi^* \). Now \( \gamma \) and \( \tau \) are both missing at \( v_{r-2} \) in \( \varphi^* \). Since \( v_{r-1} \) is short, the fan with \( v_{r-2}, v_{r-1}, v_r \) and \( e \) implies that there is an edge \( f_{r-1} = v_{r-1}v_r \) with \( \varphi^*(f_{r-1}) = \tau \). But we have never recolored \( f_{r-1} \), so \( \varphi(f_{r-1}) = \tau \), which is a contradiction. \( \square \)

**Lemma 3.** Let \( G \) be a non-elementary critical graph with \( \chi'(G) = k + 1 \) for some integer \( k \geq \Delta(G) + 1 \). Choose \( (T, v_0v_1, \varphi) \in \mathcal{T}(G) \) for some \( v_0v_1 \in E(G) \). Choose \( \alpha \in \varphi(v_0) \) and \( \beta \in \varphi(v_1) \) and let \( P = P_{v_1}(\alpha, \beta) \). Now \( P \) contains a long vertex. In particular, \( \nu(T) \geq 1 \).
Proof. Suppose every vertex of $P$ is short. Applying the Parallel Edge Lemma to $P$ shows that for every $\tau \in \varphi(v_0)$, there is an edge in $T$ colored $\tau$ incident to every $v \in V(P - v_0)$. The same is also true of every other color missing at some vertex of $P$; to see this, we rotate the $\alpha, \beta$ coloring of $P \cup \{v_0v_1\}$ and repeat the same argument. Hence $V(P) = V(T)$, which contradicts Lemma 2.

**Theorem 4.** If $G$ is a critical graph in which every vertex is short, then

$$\chi'(G) \leq \max \{W(G), \Delta(G) + 1\}.$$  

Proof. Suppose not and let $G$ be a counterexample. Let $k = \chi'(G) - 1$, and note that $k \geq \Delta(G) + 1$. Since $T(G) \neq \emptyset$, by applying Lemma 3 we conclude that $G$ is elementary. Hence $\chi'(G) = W(G)$, which is a contradiction.  

For a path $Q$, recall that $\ell(Q)$ denotes the length of $Q$. For $x, y \in V(Q)$, let $xQy$ denote the subpath of $Q$ with end vertices $x$ and $y$, and let $d_Q(x, y) = \ell(xQy)$, i.e., the distance from $x$ to $y$ along $Q$.

**Lemma 5.** Let $G$ be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let $\varphi$ be a $k$-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \varphi(v_0)$ and $\beta \in \varphi(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If $\tau \in \varphi(x)$ for some $x \in V(C)$ and there is a $\tau$-colored edge from $y \in V(C)$ to $w \in V(G) \setminus V(C)$, then $C$ has a subpath $Q$ with long endpoints $z_1, z_2$ such that $x \in V(Q)$, $y \notin V(Q - z_1 - z_2)$ and the distance from $x$ to $z_i$ along $Q$ is odd for each $i \in [2]$. Moreover, for each $i \in [2]$, there are no $\tau$-colored edges between $z_i$ and its neighbors along $C$.

Proof. Let $G, \alpha, \beta, \tau, x,$ and $y$ be as in the statement of the lemma. Choose $z_1$ (resp. $z_2$) to be the first vertex at an odd distance from $x$ along $C$ in the clockwise (resp. counterclockwise) direction with no incident $\tau$-colored edge parallel to some edge of $C$. Let $Q$ be the subpath of $C$ with endpoints $z_1$ and $z_2$ that contains $x$. By the choice of $z_1$ each vertex $w$ between $x$ and $z_1$ with $d_Q(x, w)$ odd has a $\tau$-colored edge parallel to some edge of $C$. The presence of these edges implies the same for each $w$ for which $d_Q(x, w)$ is even. By the proof of the Parallel Edge Lemma, $z_1$ must be long, since otherwise it would have an incident $\tau$-colored edge parallel to some edge of $C$. The same argument applies to $z_2$.

4 Thin graphs

Let $G$ be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. For vertices $x \in V(G)$ and $S \subseteq V(G) \setminus \{x\}$, we say that $x$ is **$S$-short** if every Vizing fan $F$ rooted at $x$ with $(\{x\} \cup S) \subseteq V(F)$, has $|F| \leq 3$ (with respect to any $k$-edge-coloring of $G - xy$). Otherwise, $x$ is **$S$-long**. For brevity, when $S = \{y\}$, we may write $y$-short instead of $\{y\}$-short. It is worth noting that in the Parallel Edge Lemma we can weaken the hypothesis that $v_i$ is short for all odd $i$ to require only that $v_i$ is $v_{i-1}$-short for all odd $i$, since this is what we use in the proof.
A graph \( G \) is \( k \)-thin if \( \mu(G) < 2k - d(x) - d(y) \) for all distinct long \( x, y \in V(G) \). In the proof of Theorem 11, we will show that every minimum counterexample to the \( \frac{5}{6} \)-Conjecture must be \( k \)-thin.

**Lemma 6.** Let \( G \) be a \( k \)-thin, critical graph with \( \chi'(G) = k + 1 \) for some integer \( k \geq \Delta(G) + 1 \). Let \( \varphi \) be a \( k \)-edge-coloring of \( G - v_0v_1 \). Choose \( \alpha \in \varphi(v_0) \) and \( \beta \in \varphi(v_1) \) and let \( C = P_{v_1}(\alpha, \beta) + v_0v_1 \). Let \( Q \) be a subpath of \( C \) with long end vertices. If all internal vertices of \( Q \) are short and \( 2 \leq \ell(Q) \leq \ell(C) - 2 \), then \( \ell(Q) \) is even.

**Proof.** Suppose to the contrary that we have a subpath \( Q \) of \( C \) with end vertices long, all internal vertices short, \( 2 \leq \ell(Q) \leq \ell(C) - 2 \), and \( \ell(Q) \) odd. Let \( x \) and \( y \) be the end vertices of \( Q \). Say \( C = v_1v_2 \cdots v_rv_1 \). By rotating the \( \alpha, \beta \) coloring of \( C \), we may assume that \( x = v_0 \) and \( y = v_a \), where \( a \geq 3 \) is odd.

We now apply the **Parallel Edge Lemma** twice, to show that \( \mu(v_1v_2) \geq 2k - d(v_0) - d(v_a) \), which contradicts that \( G \) is \( k \)-thin. More specifically, assume that the edges \( v_0v_1, v_1v_2, \ldots \) go clockwise around \( C \). We apply the **Parallel Edge Lemma** once going clockwise starting from \( v_0 \) and once going counterclockwise starting from \( v_a \). The first application implies that every color in \( \varphi(v_0) \) appears on some edge parallel to \( v_1v_2 \); the second implies the same for every color in \( \varphi(v_a) \). Since \( |\varphi(v_i)| = k - d(v_i) \) for each \( i \in \{0, a\} \) and \( \varphi(v_0) \cap \varphi(v_a) = \emptyset \), the conclusion follows.

**Lemma 7.** Let \( G \) be a \( k \)-thin, critical graph with \( \chi'(G) = k + 1 \) for some integer \( k \geq \Delta(G) + 1 \). Let \( \varphi \) be a \( k \)-edge-coloring of \( G - v_0v_1 \). Suppose \( \alpha \in \varphi(v_0) \) and \( \beta \in \varphi(v_1) \) and let \( C = P_{v_1}(\alpha, \beta) + v_0v_1 \). If \( \ell(C) \geq 5 \) and \( C \) contains exactly 3 long vertices, then \( C = xyAzBx \) where \( A \) and \( B \) are paths of even length and \( x, y, z \) are all long. Moreover, \( x \) is \( y \)-long and \( y \) is \( x \)-long.

**Proof.** Let \( G \) be a graph satisfying the hypotheses, and let \( x, y, z \) be the three long vertices. The three subpaths of \( C \) with endpoints \( x, y, z \) either (i) all have odd length or (ii) include two paths of even length and one of odd length. If we are in (i), then the longest of
these three subpaths violates Lemma 6 so we are in (ii), and also the path of odd length is simply an edge. This proves the first statement. For the second statement, assume to the contrary that $x$ is $y$-short. By rotating the $\alpha, \beta$ coloring, we can assume that $y = v_0$ and $x = v_1$. As in the previous lemma, we use the Parallel Edge Lemma (and the comment in the first paragraph of Section 4) to conclude that $\mu(v_1v_2) \geq 2k - d(v_0) - d(z)$. As above, this contradicts that $G$ is $k$-thin; this contradiction proves the second statement.

\begin{lemma}
Let $G$ be a non-elementary, $k$-thin, critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. If $\alpha \in \varphi(v_0)$ and $\beta \in \varphi(v_1)$, then $P_{v_1}(\alpha, \beta) + v_0v_1$ contains consecutive long vertices.
\end{lemma}

\begin{proof}
Let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. By Lemma 2 there is $x \in V(C)$ and $\tau \in \varphi(x)$ such that there is a $\tau$-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. Lemma 5 implies that $C$ has a subpath $Q$ with $x \in V(Q)$ and long endpoints $z_1, z_2$ such that the distance from $x$ to $z_i$ along $Q$ is odd for each $i \in [2]$. Let $Q'$ be the subpath of $C$ with endpoints $z_1$ and $z_2$ that does not contain $x$. Since $C$ is an odd cycle, $\ell(Q')$ is odd. Let $Q^*$ be a minimum length subpath of $Q'$ with long ends. Now $\ell(Q^*) = 1$ by Lemma 6 as desired.
\end{proof}

\begin{lemma}
Let $G$ be a non-elementary, $k$-thin, critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. If $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ and $v(T) \leq 3$, then $T$ contains long vertices $z_1, z_2, z_3$ such that either

1. $z_1$ is $\{z_2, z_3\}$-long and $z_2$ is $z_1$-long; or

2. $z_i$ is $z_j$-long and $z_j$ is $z_i$-long for each $(i, j) \in \{(1, 2), (2, 3)\}$.
\end{lemma}

\begin{proof}
Choose $\alpha \in \varphi(v_0)$ and $\beta \in \varphi(v_1)$ so that $P_{v_1}(\alpha, \beta)$ contains as many long vertices as possible and, subject to that, $P_{v_1}(\alpha, \beta)$ is as long as possible. Let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. By Lemma 2 there is $x \in V(C)$ and $\tau \in \varphi(x)$ such that there is a $\tau$-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. By Lemma 8 $C$ has at least 2 long vertices.

First suppose that $C$ contains only 2 long vertices, $z_1$ and $z_2$; see the left side of Figure 4. By Lemma 8, $z_1$ and $z_2$ are consecutive on $C$. Lemma 5 implies that $C$ has a subpath $Q$ with endpoints $z_1, z_2$ such that $x \in V(Q)$ and $y \notin V(Q - z_1 - z_2)$ and for each $i \in [2]$ there are no $\tau$-colored edges between $z_i$ and its neighbors on $C$. By rotating the $\alpha, \beta$ coloring of $C$, we can assume that $x = v_0$ and $\alpha, \tau \in \varphi(v_0)$ and $\beta \in \varphi(v_1)$. Note that $P_{v_1}(\tau, \beta)$ must end at $v_0$ (since otherwise we can recolor the Kempe chain and color $v_0v_1$ with $\tau$). Let $C' = P_{v_1}(\tau, \beta) + v_0v_1$. Note that $C'$ must include $v_1Qz_1$ and also $v_0Qz_2$ (the $\beta$-colored edges are present by definition and the $\tau$-colored edges are present by the the Parallel Edge Lemma). Thus, $z_1, z_2 \in V(C')$. Since $z_1$ and $z_2$ are not consecutive on $C'$ and $C'$ contains no other long vertices by the maximality condition on $C$, Lemma 8 gives a contradiction.

So instead assume that $C$ contains exactly 3 long vertices. Now we prove that $\ell(C) \geq 5$. Suppose, to the contrary, that $C = v_0v_1v_2$ and each vertex is a long vertex. By Lemma 2 some color that is missing at a vertex of $C$ is used on an edge leaving $C$. By symmetry, assume that color $\tau$ is missing at $v_0$ and is used on an edge $v_1y$, where $y \notin V(C)$. Uncolor
\[ v_2v_0 \text{ and color } v_0v_1 \text{ with } \beta. \text{ Now the } \beta, \tau \text{ path starting at } v_0 \text{ ends at } v_2, \text{ has length at least } 4 \text{ and contains } v_0, v_1, v_2. \text{ So the union of } v_0v_2 \text{ with this path gives a longer } C, \text{ a contradiction.}

By Lemma 4, \( C = z_1z_2Az_3Bz_1 \) where \( A \) and \( B \) are paths of even length. Also, \( z_1 \) is \( z_2 \)-long and \( z_2 \) is \( z_1 \)-long. By Lemma 5, \( C \) has a subpath \( Q \) with endpoints \( z_1, z_3 \) and with \( x \in V(Q) \) and \( y \notin V(Q - z_1 - z_3) \) such that there are no \( \tau \)-colored edges between \( z_1 \) and its neighbors along \( C \) for each \( i \in \{ 1, 3 \} \) (it could happen that \( z_3 \) has a \( \tau \)-colored edge parallel to an edge of \( C \), so the endpoints of \( Q \) are \( z_1, z_2 \), but now we get a contradiction as in the previous case, by letting \( C' = P_{v_1}(\tau, \beta) + v_0v_1 \). By rotating the \( \alpha, \beta \) coloring of \( C \), we may assume that \( x = v_0 \). Again, let \( C' = P_{v_1}(\tau, \beta) + v_0v_1 \). We know that \( C' \) contains \( z_1 \) and \( z_3 \) and that \( z_1 \) and \( z_2 \) are not consecutive on \( C' \). Note also that all long vertices in \( V(C') \) must be among \( z_1, z_2, z_3 \), since otherwise \( \nu(T) \geq 4 \), contradicting our hypothesis. So by Lemma 8, either \( z_1 \) and \( z_3 \) are consecutive on \( C' \) or \( z_2 \) and \( z_3 \) are consecutive on \( C' \).

Suppose that \( z_2 \) and \( z_3 \) are consecutive on \( C' \), and thus connected by a \( \tau \)-colored edge. Now applying Lemma 7 shows that \( z_2 \) is \( z_3 \)-long and \( z_3 \) is \( z_2 \)-long, so we satisfy (2) in the conclusion of the lemma (by swapping the names of \( z_1 \) and \( z_2 \)).

So instead \( z_1 \) and \( z_3 \) must be consecutive on \( C' \), and thus connected by a \( \tau \)-colored edge. If \( z_1 = v_1 \), then we have a fan with an \( \alpha \)-colored edge from \( z_1 \) to \( z_2 \) and a \( \tau \)-colored edge from \( z_1 \) to \( z_3 \), so \( z_1 \) is \( \{ z_2, z_3 \} \)-long.

Now assume that \( z_1 \neq v_1 \); see the right side of Figure 4. Let \( z'_1 \) be the predecessor of \( z_1 \) on the path from \( v_0 \) (through \( v_1 \)) to \( z_1 \). We can shift the coloring so that \( z'_1z_1 \) is uncolored and \( z_1z_2 \) is colored \( \alpha \) (as in the proof of the Parallel Edge Lemma). In fact, we can shift either the \( \alpha, \beta \) edges or the \( \tau, \beta \) edges. This gives the options that either \( \alpha \in \mathcal{T}(z'_1) \) or \( \tau \in \mathcal{T}(z'_1) \), whichever we prefer. Suppose we shift the \( \tau, \beta \) edges. Now choose \( \gamma \in \mathcal{T}(z'_1) - \alpha - \tau \). Consider the \( \gamma \)-colored edge \( e \) incident to \( z_1 \). If \( e \) goes to \( z_2 \), then we \( z_1 \) is \( \{ z_2, z_3 \} \)-long, by colors \( \gamma \) and \( \tau \); so we satisfy (1) in the conclusion of the lemma. If instead \( e \) goes to \( z_3 \), then instead of shifting the \( \tau, \beta \) edges we shift the \( \alpha, \beta \) edges; note that this recoloring preserves the fact that \( \gamma \) is missing at \( z'_1 \). Now again \( z_1 \) is \( \{ z_2, z_3 \} \)-long, this time by colors \( \alpha \) and \( \gamma \); so we again satisfy (1) in the conclusion of the lemma.

Finally, assume that the \( \gamma \)-colored edge incident to \( z_1 \) goes to some vertex other than \( z_2 \) and \( z_3 \). Now let \( C'' = P_{z_1}(\gamma, \beta) + z_1z'_1 \). Since \( V(C'') \subseteq V(T) \), Lemmas 8 and 7 imply that \( z_2 \) and \( z_3 \) are adjacent on \( C'' \) and furthermore \( z_2 \) is \( z_3 \)-long and \( z_3 \) is \( z_2 \)-long; thus, we satisfy (2) in the conclusion of the lemma.

We need the following result from [S], which we use to handle the case when \( G \) is not \( k \)-thin.

**Theorem 10** ([S]). If \( Q \) is the line graph of a graph \( G \) and \( Q \) is vertex critical, then

\[
\chi(Q) \leq \max \left\{ \omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2} \right\}.
\]

Now we prove the main result of this section, the Weak \( \frac{5}{6} \)-Theorem. It encapsulates most of what we will need from the first four sections when we prove our main result, the \( \frac{5}{6} \)-Theorem, in Section 5.
Theorem 11 (Weak $\frac{5}{6}$-Theorem). If $Q$ is the line graph of $G$, then

$$
\chi(Q) \leq \max \left\{ W(G), \Delta(G) + 1, \frac{5\Delta(Q) + 8}{6} \right\}.
$$

Proof. Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Let $k = \max \left\{ W(G), \Delta(G) + 1, \left\lfloor \frac{5\Delta(Q) + 8}{6} \right\rfloor \right\}$. Say $Q = L(G)$ for a graph $G$. The minimality of $Q$ implies that $G$ is critical and $\chi(Q) = k + 1$, for some $k \geq \Delta(G) + 1$.

The heart of the proof is Claim 1, which roughly says that if $x$ is long, then $d(x) < \frac{4}{5} \Delta(G)$. Moreover, we can improve this bound further if $x$ is the root of a long fan $F$ such that either (i) $F$ has length more than 3 or (ii) some of the other vertices in $F$ have degree less than $\Delta(G)$. The claims thereafter are all essentially applications of Claim 1.

Claim 1. Let $F$ be a fan rooted at $x$ with respect to a $k$-edge-coloring of $G - xy$. If $S \subseteq V(F) - x$ and $|S| \geq 3$, then

$$
d(x) \leq \frac{1}{5 |S| - 11} \left( 2 |S| - 12 + \sum_{v \in S} d(v) \right).
$$

In particular, if $|S| = 3$, then $d(x) \leq \frac{1}{4} \left( -6 + \sum_{v \in S} d(v) \right)$.

Proof: Since $F$ is elementary, we have

$$
2 + k - d(x) + \sum_{v \in S} k - d(v) \leq k,
$$

so

$$
2 + |S| k \leq d(x) + \sum_{v \in S} d(v).
$$
Using $k \geq \frac{5}{6}(\Delta(Q) + 1) - \frac{1}{3} \geq \frac{5}{6}(d(x) + d(v) - \mu(xv)) - \frac{1}{3}$ for each $v \in S$, we get

$$2 + \sum_{v \in S} \left( \frac{5}{6}(d(x) + d(v) - \mu(xv)) - \frac{1}{3} \right) \leq d(x) + \sum_{v \in S} d(v),$$

so multiplying by 6 and rearranging terms gives

$$12 + (5|S| - 6) d(x) - 2|S| \leq \sum_{v \in S} 5\mu(xv) + \sum_{v \in S} d(v).$$

Now $\sum_{v \in S} \mu(xv) \leq d(x)$, so this implies

$$12 + (5|S| - 11) d(x) - 2|S| \leq \sum_{v \in S} d(v).$$

Solving for $d(x)$ gives

$$d(x) \leq \frac{1}{5|S| - 11} \left( 2|S| - 12 + \sum_{v \in S} d(v) \right),$$

and when $|S| = 3$, we get $d(x) \leq \frac{1}{4} \left( -6 + \sum_{v \in S} d(v) \right)$. □

**Claim 2.** If $x \in V(G)$ is long, then $d(x) \leq \frac{3}{4}\Delta(G) - 1$.

**Proof:** This is immediate from Claim 1, since $d(v) \leq \Delta(G)$ for all $v \in S$. □

**Claim 3.** If $x_1x_2 \in E(G)$ such that $x_1$ is $x_2$-long and $x_2$ is $x_1$-long, then

$$d(x_i) \leq \frac{2}{3}\Delta(G) - 2 \text{ for all } i \in [2].$$

**Proof:** By Claim 1, for each $i \in [2]$,

$$d(x_i) \leq \frac{1}{4} \left( -6 + \sum_{v \in S} d(v) \right) \leq \frac{1}{4} \left( -6 + d(x_{3-i}) + 2\Delta(G) \right),$$

Substituting the bound on $d(x_{3-i})$ into that on $d(x_i)$ and simplifying gives for each $i \in [2]$,

$$d(x_i) \leq -2 + \frac{2}{3}\Delta(G).$$

□

**Claim 4.** If $x_1x_2, x_1x_3 \in E(G)$ such that $x_1$ is $\{x_2, x_3\}$-long, $x_2$ is $x_1$-long and $x_3$ is long, then

$$d(x_1) \leq -\frac{8}{5} + \frac{3}{5}\Delta(G),$$

Claim 4 follows immediately from Claims 1, 2, and 3.
\[ d(x_2) \leq -\frac{7}{5} + \frac{13}{20} \Delta(G). \]

**Proof:** By Claim 1, we have
\[
d(x_1) \leq \frac{1}{4} \left( -6 + \sum_{v \in S} d(v) \right) \leq \frac{1}{4} \left( -6 + d(x_2) + d(x_3) + \Delta(G) \right),
\]
\[
d(x_2) \leq \frac{1}{4} \left( -6 + \sum_{v \in S} d(v) \right) \leq \frac{1}{4} \left( -6 + d(x_1) + 2\Delta(G) \right).
\]

By the same calculation as in Claim 3, these together imply
\[
d(x_1) \leq -\frac{2}{5} \Delta(G) + \frac{4}{15} d(x_3).
\]

Since \( x_3 \) is long, using Claim 2, we get
\[
d(x_1) \leq -\frac{34}{15} + \frac{3}{5} \Delta(G),
\]
and hence
\[
d(x_2) \leq -\frac{61}{15} + \frac{13}{20} \Delta(G).
\]

**Claim 5.** The theorem is true.

**Proof:** Let \((T, v_0v_1, \varphi) \in \mathcal{T}(G)\). By Lemma \[\text{Claim 3}\], one of the following holds:

1. \( G \) is elementary; or
2. \( G \) is not \( k \)-thin; or
3. \( \nu(T) = 3 \) and \( V(T) \) contains vertices \( x_1, x_2, x_3 \) such that \( x_1 \) is \( x_2 \)-long, \( x_2 \) is \( x_1 \)-long, \( x_2 \) is \( x_3 \)-long, and \( x_3 \) is \( x_2 \)-long; or
4. \( \nu(T) = 3 \) and \( V(T) \) contains vertices \( x_1, x_2, x_3 \) such that \( x_1 \) is \( \{x_2, x_3\} \)-long, \( x_2 \) is \( x_1 \)-long, and \( x_3 \) is long; or
5. \( V(T) \) contains four long vertices \( x_1, x_2, x_3, x_4 \).

If (1) holds, then \( \chi(Q) = \mathcal{W}(G) \), which contradicts our choice of \( Q \) as a counterexample. If (2) holds, then Claim 2 implies that \( \mu(G) \geq 2k - \frac{3}{2} \Delta(G) + 2 \). Now Theorem \[\text{Claim 4}\] gives
\[
k + 1 \leq \Delta(Q) + 1 - \frac{2k - \frac{3}{2} \Delta(G) + 2}{2}
= \Delta(Q) + 1 - k + \frac{3}{4} \Delta(G) - 1,
\]
so
\[ 2(k + 1) \leq \Delta(Q) + 1 + \frac{3}{4}\Delta(G). \]
Substituting \( \Delta(G) \leq k \) and solving for \( k \) gives
\[ k \leq \frac{4}{5}\Delta(Q) - \frac{4}{5} < \frac{5}{6}\Delta(Q) + \frac{1}{2} \leq k, \]
which is a contradiction.

Suppose (3) holds. Now
\[ 2 + \sum_{i \in [3]} k - d(x_i) \leq k, \]
so Claim 3 implies
\[ 3 \left( \frac{2}{3}\Delta(G) - 2 \right) \geq 2k + 2, \]
which is a contradiction, since \( \Delta(G) \leq k \).

Suppose (4) holds. Now
\[ 2 + \sum_{i \in [3]} k - d(x_i) \leq k, \]
so Claims 2 and 4 give
\[ \left( \frac{3}{5} + \frac{13}{20} + \frac{3}{4} \right) \Delta(G) - \left( \frac{34}{15} + \frac{16}{15} + 1 \right) \geq 2k + 2, \]
which is
\[ 2\Delta(G) - \frac{13}{3} \geq 2k + 2, \]
again a contradiction, since \( \Delta(G) \leq k \).

So (5) must hold. But now
\[ 2 + \sum_{i \in [4]} k - d(x_i) \leq k, \]
so using Claim 2 gives
\[ 4 \left( \frac{3}{4}\Delta(G) - 1 \right) \geq 3k + 2, \]
a contradiction since \( \Delta(G) \leq k \).

This finishes the final case of Claim 5, which proves the theorem.

In the previous theorem, we showed that \( \chi(Q) \leq \max \left\{ \mathcal{W}(Q), \Delta(G) + 1, \frac{5\Delta(Q) + 8}{6} \right\} \). Now we show that if the maximum is attained by the second argument (and \( Q \) is vertex critical), then \( Q \) satisfies the \( \frac{5}{6} \)-Conjecture. We use the following lemma, which is implicit in \cite{8}; see the proof of Lemma 9 therein.
Lemma 12. If $Q$ is the line graph of a graph $G$ and $Q$ is vertex critical, then
\[ \chi(Q) \leq \max \{ \Delta(G), \Delta(Q) + 1 + 2\mu(G) - \Delta(G) \}. \]

Corollary 13. If $Q$ is the line graph of a critical graph $G$ and $\chi(Q) \leq \Delta(G) + 1$, then
\[ \chi(Q) \leq \max \left\{ \omega(Q), \frac{5\Delta(Q) + 8}{6} \right\}. \]

Proof. Let $k + 1 = \chi(Q) \leq \Delta(G) + 1$. Suppose $\chi(Q) > \omega(Q)$. Now Lemma 12 gives
\[ k + 1 = \chi(Q) \leq \Delta(Q) + 1 + 2\mu(G) - k, \]
so solving for $\mu(G)$ gives
\[ \mu(G) \geq k - \frac{\Delta(Q)}{2}. \]
Applying Theorem 11 gives
\[ k + 1 = \chi(Q) \leq \Delta(Q) + 1 - \frac{k - \frac{\Delta(Q)}{2} - 1}{2}, \]
and solving for $k + 1$ yields
\[ \chi(Q) = k + 1 \leq \frac{5}{6}\Delta(Q) + \frac{4}{3} = \frac{5\Delta(Q) + 8}{6}. \]

Since $\omega(Q) \leq \max \{ \Delta(G), W(G) \}$, Theorem 11 and Corollary 13 together imply the following.

Corollary 14. If $Q$ is the line graph of a graph $G$, then
\[ \chi(Q) \leq \max \left\{ \Delta(G), W(G), \frac{5\Delta(Q) + 8}{6} \right\}. \]

Proof. Let $Q$ be the line graph of a graph $G$. We assume that $G$ is critical. If not, then choose $\hat{G} \subseteq G$ such that $\hat{G}$ is critical and $\chi'(\hat{G}) = \chi'(G)$. Let $\hat{Q} := L(\hat{G})$. Now $\chi(Q) = \chi(\hat{Q}) \leq \max \{ \Delta(\hat{Q}), W(\hat{Q}), \frac{5\Delta(Q) + 8}{6} \} \leq \max \{ \Delta(Q), W(Q), \frac{5\Delta(Q) + 8}{6} \}$, as desired.

If $\chi(Q) > \Delta(G) + 1$, then Theorem 11 implies that $\chi(Q) \leq \max \{ W(G), \frac{5\Delta(Q) + 8}{6} \}$. Otherwise, $\chi(Q) \leq \Delta(G) + 1$; since $G$ is critical, Corollary 13 implies $\chi(Q) \leq \{ \omega(Q), \frac{5\Delta(Q) + 8}{6} \}$. Since $\omega(G) \leq \max \{ \Delta(G), W(G) \}$, the result follows.
5 The \(\frac{5}{6}\)-Conjecture

In this section, we prove our main result, Theorem 20, that \(\chi(Q) \leq \max\{\omega(Q), \frac{5\Delta(Q) + 8}{6}\}\), when \(Q\) is the line graph of a graph \(G\). Throughout, we may assume that \(Q\) is a minimal counterexample, so \(Q\) is vertex critical. Observe that if \(\chi(Q) \leq \Delta(G) + 1\), then the result follows immediately from Corollary 13. Thus, we may assume that \(\chi(Q) > \Delta(G) + 1\). Now, in view of Corollary 14, it suffices to show that \(\chi(Q) = \chi'(G) \leq W(G)\) implies \(\chi(Q) \leq \max\{\omega(Q), \frac{5\Delta(Q) + 8}{6}\}\).

Our approach is to show that if \(Q\) is a minimal counterexample and \(Q\) is the line graph of \(G\), then \(|N(x)| = 2\) for nearly every vertex \(x \in V(G)\). This implies that the simple graph underlying \(G\) is very close to a cycle. By Corollary 14, we can assume that \(G\) is elementary, i.e., \(\chi(Q) = \chi'(G) = W(G)\). Thus \(\chi(Q) = \left\lceil \frac{2|E(G)|}{|V(G)| - 1} \right\rceil \geq \Delta(G) + 2\). So, to show that \(|N(x)| = 2\) for nearly every \(x \in V(G)\), it suffices to show that when \(|N(x)| \geq 3\), we have \(d(x) \leq \frac{4}{5}\Delta(G)\). This is an immediate consequence of Lemma 18 which is the key technical result that we prove in this section.

It is helpful to note that in the present section the only results we use from previous sections are Corollary 13 and Corollary 14 (as well as Theorem 10 which is proved in [8]). In Lemmas 16 and 18 we prove bounds on \(|N(x)|\) for each \(x \in V(G)\). Ultimately, these lemmas yield Corollary 19 which says that \(|N(x)| \geq 3\) for at most one vertex \(x \in V(G)\). This corollary plays a key role in the proof of our main result, Theorem 20.

For reference, we record the following proposition, which is Proposition 1.3 in [12].

**Proposition 15.** Let \(G\) be a graph that is critical and elementary, with \(\chi'(G) = k + 1\) for some integer \(k \geq \Delta(G)\). Now \(||G||\) is odd and \(k = \frac{2(||G|| - 1)}{|G| - 1}\).

**Lemma 16.** Let \(G\) be a critical, elementary graph with \(\chi'(G) = k + 1\) where \(k \geq \Delta(G) + 1\). Put \(Q := L(G)\). If \(k = \epsilon(\Delta(Q) + 1) + \beta\), then for all \(x \in V(G)\),

\[
|N(x)| = \frac{\epsilon(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3 (|G| - 1))}{(1 - \epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},
\]

where

\[
S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),
\]

\[
S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),
\]

\[
S_3 := k - (\Delta(G) + 1).
\]

**Proof.** The details of the proof are somewhat tedious, however, the idea is simple. We write down a system of equations and solve for \(|N(x)|\). The main insight needed is knowing which quantities to consider. We use \(\epsilon, \beta, S_1, S_2, S_3,\) and \(\sum_{v \in N(x)} d_G(v)\).
Since $G$ is critical and elementary, Proposition 15 implies that $|G|$ is odd and
\[ k = \frac{2(|G| - 1)}{|G| - 1}. \]  
(1)

Choose $x \in V(G)$ and put $M := |N(x)|$ and
\[ P := \sum_{v \in N(x)} d_G(v). \]

Grouping edges by whether or not they are incident to any neighbor of $x$ gives
\[ 2(|G| - 1) = \Delta(G)(|G| - M) - S_2 + P. \]  
(2)

By (1), and by the definition of $S_3$,
\[ \frac{2(|G| - 1)}{|G| - 1} = k = \Delta(G) + 1 + S_3. \]

Now clearing the denominator and using (2) to substitute, we get
\[ P = (|G| - 1)(\Delta(G) + 1 + S_3) - \Delta(G)(|G| - M) + S_2. \]

After regrouping terms, this is
\[ P = \Delta(G)(M - 1) + |G| - 1 + S_2 + S_3(|G| - 1). \]  
(3)

Using $k = \epsilon(\Delta(Q) + 1) + \beta$, and the definition of $S_1$, we get
\[ kM = \beta M + \epsilon S_1 + \epsilon \sum_{v \in N(x)} d_G(x) + d_G(v) - \mu(xv). \]

Since $\sum_{v \in N(x)} \mu(xv) = d_G(x)$, we have
\[ kM = \beta M + \epsilon S_1 + \epsilon d_G(x)(M - 1) + \epsilon P. \]  
(4)

Substituting (3) into (4) and solving for $M$ gives
\[ M = \frac{\epsilon (|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3(|G| - 1))}{(1 - \epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3}, \]
as desired.

Using $\epsilon = \frac{5}{6}$ in Lemma 16, we get the following.
Lemma 17. Let $G$ be a critical, elementary graph with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$. Put $Q := L(G)$. If $k = \frac{5}{6} (\Delta(Q) + 1) + \beta$, then for all $x \in V(G)$,

$$|N(x)| = \frac{5(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3(|G| - 1))}{\Delta(G) - 5d_G(x) + 6(1 - \beta + S_3)},$$

where

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G) + 1).$$

Lemma 18. Let $G$ be a critical, elementary graph with $\chi'(G) = k + 1$, where $k \geq \Delta(G) + 1$. Put $Q := L(G)$. If $k = \frac{5}{6} (\Delta(Q) + 1) + \beta$, where $\beta \geq -\frac{1}{3}$, then for all $x \in V(G)$ with $|N(x)| \geq 3$,

$$d_G(x) \leq \frac{3}{5} \Delta(G) - \frac{1}{|N(x)| - 2} \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

If additionally, $|N(x)| \leq \frac{5}{8} |G|$, then

$$d_G(x) \leq \frac{|N(x)|}{5(|N(x)| - 2)} \Delta(G) - \frac{1}{|N(x)| - 2} \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

Proof. We may assume that $|G| \geq 5$, since $|G|$ is odd by Proposition 15 and if $|G| = 3$, then $|N(x)| \leq 2$ for all $x \in V(G)$, so there is nothing to prove.

Choose $x \in V(G)$ with $|N(x)| \geq 3$. Let $S_4 = |N(x)| - 2$, and note that $S_4 \geq 1$. Applying Lemma 17 and simplifying using $S_1 \geq 0$ and $\beta \geq -\frac{1}{3}$ gives

$$(5 + 5S_4)d_G(x) \leq (7 + S_4) \Delta(G) - 5|G| + 21 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5S_2. \quad (5)$$

Put

$$t := \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

Now $S_2 = t + 2 + \Delta(G) - d_G(x)$. Using this in (5), we get

$$5S_4d_G(x) \leq (2 + S_4) \Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5t. \quad (6)$$

When $|N(x)| \leq \frac{5}{8} |G|$, i.e., $S_4 \leq \frac{5}{8} |G| - 2$, we have

$$-5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 \leq 0,$$

so dividing (6) through by $5S_4$ gives the desired bound.
So instead assume $S_4 > \frac{5}{8} |G| - 2$. Rearranging (6) gives

$$5S_4d_G(x) \leq 3S_4\Delta(G) - (2S_4 - 2)\Delta(G) - 5|G| + 11 + S_3^2(-5|G| + 17 + 6S_4) + 8S_4 - 5t$$

(7)

Now $S_4 = |N(x)| - 2 = |G| - 3 + S_5$, where

$$S_5 := S_4 + 3 - |G| \leq 0.$$

Thus

$$-5|G| + 15 + 5S_4 + S_3^2(-5|G| + 15 + 5S_4) - 5S_5(1 + S_3) = 0.$$

Subtracting this equality from (7) gives

$$5S_4d_G(x) \leq 3S_4\Delta(G) - (2S_4 - 2)\Delta(G) - 4 + 2S_3 + (S_3 + 3)S_4 + 5S_5(1 + S_3) - 5t$$

(8)

If $-(2S_4 - 2)\Delta(G) - 4 + 2S_3 + (S_3 + 3)S_4 + 5S_5(1 + S_3) \leq 0$, then we have the desired bound

$$d_G(x) \leq \frac{3}{5}\Delta(G) - \frac{1}{|N(x)| - 2} \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

So assume instead that

$$-4 + 2S_3 + (S_3 + 3)S_4 - (2S_4 - 2)\Delta(G) + 5S_5(1 + S_3) > 0,$$

which we rewrite as

$$(2 + S_4)S_3 + 3S_4 > (2S_4 - 2)\Delta(G) + 4 - 5S_5(S_3 + 1).$$

(9)

By Shannon’s theorem $k + 1 \leq \frac{3}{2}\Delta(G)$, so $S_3 \leq \frac{\Delta(G)}{2} - 2$. After plugging in for $S_3$ on the left side and solving for $S_4$, we get

$$S_5 + |G| - 3 = S_4 < \frac{6\Delta(G) - 16 + 10S_5(S_3 + 1)}{3\Delta(G) - 2} = 2 + \frac{10S_5(S_3 + 1) - 12}{3\Delta(G) - 2},$$

so

$$|G| < 5 + \frac{(10S_3 - 3\Delta(G) + 12)S_5 - 12}{3\Delta(G) - 2}.$$

Since $S_5 \leq 0$, this implies $|G| \leq 3$, unless $10S_3 - 3\Delta(G) + 12 < 0$. So $S_3 \leq \frac{3}{10}\Delta(G) - \frac{6}{5}$. Since $S_5 \leq 0$, (9) implies

$$(2 + S_4)S_3 + 3S_4 > (2S_4 - 2)\Delta(G) + 4$$

Substituting $S_3 < \frac{3}{10}\Delta(G) - \frac{6}{5}$ gives

$$|G| - 3 \leq S_4 < \frac{26\Delta(G) - 64}{17\Delta(G) - 18} < 2,$$

which contradicts that $|G| \geq 5$. \qed
Corollary 19. Let $G$ be a critical, elementary graph with $\chi'(G) = k+1$, where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6} (\Delta(Q) + 1) + \beta$, where $\beta \geq -\frac{1}{3}$, then there is at most one $x \in V(G)$ with $|N(x)| \geq 3$.

Proof. Since $G$ is critical and elementary, Proposition 15 implies that $|G|$ is odd and
\[
2(\|G\| - 1) = k \geq \Delta(G) + 1,
\]
so
\[
2 \|G\| \geq \Delta(G) |G| + |G| - \Delta(G) + 1.
\]
In particular,
\[
\sum_{v \in V(G)} \Delta(G) - d_G(v) \leq \Delta(G) - 1 - |G|.
\]
(10)

By Lemma 18 every $x \in V(G)$ with $|N(x)| \geq 3$ has $d_G(x) \leq \frac{2}{3} \Delta(G)$, so there are at most two such $x$ since $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} > 1$. Suppose there are $x_1, x_2$ with $|N(x_1)| \geq |N(x_2)| \geq 3$.

Choose $z \in V(G)$ with $d_G(z) = \Delta(G)$. By Lemma 18 $|N(z)| = 2$, so $\mu(G) \geq \frac{1}{2} \Delta(G)$. By Theorem 10 $\mu(G) < \frac{1}{3} (\Delta(Q) + 1)$. We conclude
\[
\Delta(G) < \frac{2}{3} (\Delta(Q) + 1).
\]
(11)

First, suppose $x_1$ and $x_2$ are adjacent. Since $Q$ is vertex-critical,
\[
k \leq d_Q(x_1x_2) = d_G(x_1) + d_G(x_2) - \mu(x_1x_2) - 1 \leq \frac{6}{5} \Delta(G) - \mu(x_1x_2) - 1,
\]
So, $\Delta(G) > \frac{5}{6} k$. By (11),
\[
\frac{5}{6} k < \Delta(G) < \frac{2}{3} (\Delta(Q) + 1),
\]
and hence $k < \frac{4}{3} (\Delta(Q) + 1)$, a contradiction.

So instead assume $x_1$ and $x_2$ are non-adjacent. Suppose also that $|N(x_2)| \leq \frac{5}{3} |G|$. If $|N(x_2)| \geq 4$, then $d_G(x_2) \leq \frac{2}{5} \Delta(G)$. Now (10) and Lemma 18 give
\[
\Delta(G) - 1 - |G| \geq \sum_{v \in V(G)} \Delta(G) - d_G(v) \geq (\Delta(G) - d_G(x_1)) + (\Delta(G) - d_G(x_2)) \geq \left( \frac{2}{5} + \frac{3}{5} \right) \Delta(G) = \Delta(G),
\]
19
a contradiction.

So assume that $|N(x_2)| = 3$. Since $x_1$ and $x_2$ are non-adjacent, Lemma 18 gives

$$d_G(x_2) \leq \frac{3}{5} \Delta(G) - (\Delta(G) - d_G(x_1)) \leq \frac{1}{5} \Delta(G).$$

Similar to above, we get a contradiction since $\Delta(G) > \left( \frac{4}{5} + \frac{2}{5} \right) \Delta(G) > \Delta(G)$.

So assume that $|N(x_i)| > \frac{5}{8}|G|$ for each $i \in [2]$. In particular, there is $y \in N(x_1) \cap N(x_2)$. Since $|N(y)| = 2$, by symmetry we may assume $\mu(x_1y) \geq \frac{1}{2}d_G(y)$. Hence, using

$$k \leq d_Q(x_1y)$$

$$= d_G(x_1) + d_G(y) - \mu(x_1y) < d_G(x_1) + \frac{1}{2}d_G(y)$$

$$\leq \frac{3}{5} \Delta(G) + \frac{1}{2} \Delta(G) = \frac{11}{10} \Delta(G)$$

$$< \frac{11}{15} (\Delta(Q) + 1),$$

where the final inequality holds by (11). This contradiction completes the proof. \qed

Now we prove the Main Theorem.

**Theorem 20** ($\frac{5}{6}$-Theorem). If $Q$ is a line graph, then

$$\chi(Q) \leq \max \left\{ \omega(Q), \frac{5\Delta(Q) + 8}{6} \right\}.$$ 

**Proof.** It is convenient to note that, since $\chi(Q)$ is an integer, $\chi(Q) \leq \frac{5\Delta(Q) + 8}{6}$ if and only if $\chi(Q) \leq \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil$. For the present proof, it is simpler to work with the latter formulation.

Suppose the theorem is false and choose a counterexample $Q$ minimizing $|Q|$. Now $Q = L(G)$ for a critical graph $G$. Say $\chi(Q) = \chi'(G) = k + 1$. So $k = \max \left\{ \omega(Q), \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil \right\}$ by the minimality of $|Q|$. By Corollary 14,

$$k + 1 \leq \max \left\{ W(G), \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil \right\},$$

so

$$W(G) = \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil + 1 = \chi(Q).$$

Therefore $G$ is elementary and $k = \frac{5}{6} (\Delta(Q) + 1) + \beta$ for some $\beta \geq -\frac{1}{6}$. By Corollary 13, $k \geq \delta(G) + 1$. Let $H$ be the underlying simple graph of $G$. We may apply Corollary 19 to conclude that there is at most one $x \in V(G)$ with $d_H(x) \geq 3$. Since $G$ is critical, $\delta(H) \geq 2$ and $H$ has no cut vertices. Hence $H$ is a cycle.
Choose $t$ such that $|V(H)| = 2t + 1$. Let $x_1, \ldots, x_{2t+1}$ denote the multiplicities of the edges in $G$, and let $X = \sum_{i=1}^{2t+1} x_i$. Since $G$ is elementary, we have $\chi'(G) = \lceil \frac{X}{t} \rceil$. Let $Q = L(G)$ and let $v_i$ be a vertex of $Q$ corresponding to an edge of $G$ counted by $x_i$. Now $d_Q(v_i) = x_{i-1} + x_i + x_{i+1} - 1$. It suffices to show that there exists $j \in [2t+1]$ such that $\frac{X}{t} \leq \frac{5d_Q(v_j)+3}{6}$. We will prove the stronger statement that $\frac{X}{t} \leq \frac{5d+3}{6}$, where $d = \frac{1}{2t+1} \sum_{i=1}^{2t+1} d_Q(v_i)$. Since $\frac{5d+3}{6} = \frac{5X}{2(2t+1)} - \frac{1}{3}$, it suffices to have $\frac{5X}{2(2t+1)} - \frac{1}{3} \geq \frac{X}{t}$. Simplifying (for $t \geq 3$) gives $X \geq \frac{1}{3}(4t + 10 + \frac{20}{t-2})$. Since $X \geq 2t + 1$, this always holds when $t \geq 6$. When $t = 5$, it suffices to have $X \geq 13$. When $t = 4$, it suffices to have $X \geq 12$, and when $t = 3$, it suffices to have $X \geq 14$. Suppose $t = 5$ and $X \leq 12$. Now $\chi'(G) \leq \lceil \frac{12}{5} \rceil = 3 \leq \left\lceil \frac{5(2)+3}{6} \right\rceil$.

Suppose instead that $t = 4$ and $X \leq 11$. Now $\chi'(G) \leq \lceil \frac{11}{4} \rceil = 3 \leq \left\lceil \frac{5(2)+3}{6} \right\rceil$. Suppose instead that $t = 3$ and $X \leq 13$. If $X \leq 9$, then again $\chi'(G) \leq \lceil \frac{9}{3} \rceil = 3 \leq \left\lceil \frac{5(2)+3}{6} \right\rceil$. Assume that $X \geq 10$, which implies that $\Delta(G) \geq 4$. First suppose that $X \leq 12$. Now $\chi'(G) \leq \lceil \frac{12}{3} \rceil = 4 \leq \left\lceil \frac{5(4)+3}{6} \right\rceil$. So instead assume that $X = 13$, which implies that $\Delta(G) \geq 5$. Now $\chi'(G) \leq \lceil \frac{13}{3} \rceil = 5 \leq \left\lceil \frac{5(5)+3}{6} \right\rceil$, as desired. Finally, we consider $t = 2$. Now $\frac{5d+3}{6} = \frac{X}{2} - \frac{1}{3}$ and $\frac{X}{t} = \frac{X}{2}$, so always $\frac{5d+3}{6} < \frac{X}{t}$. However, $\left\lceil \frac{5d+3}{6} \right\rceil = \left\lceil \frac{X}{2} - \frac{1}{3} \right\rceil = \left\lceil \frac{X}{2} \right\rceil$, for all integers $X$, which completes the proof.

We suspect that our Main Theorem can be extended to the larger class of quasi-line graphs (those for which the neighborhood of each vertex is covered by two cliques).

**Conjecture 21.** If $Q$ is a quasi-line graph, then

$$
\chi(Q) \leq \max \left\{ \omega(Q), \frac{5\Delta(Q) + 8}{6} \right\}.
$$

## 6 Strengthenings of Reed’s Conjecture

In this section, we show how the results of Section 3 imply Reed’s Conjecture, as well as Local and Superlocal strengthenings of Reed’s Conjecture, for the class of line graphs.

Let $G$ be a graph. The **claw-degree** of $x \in V(G)$ is

$$
d_{\text{claw}}(x) := \max_{S \subseteq N(x), |S| = 3} \frac{1}{4} \left( d(x) + \sum_{v \in S} d(v) \right),
$$

where $d_{\text{claw}}(x) := 0$ when $|N(x)| \leq 2$. The **claw-degree** of $G$ is

$$
d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).
$$
Theorem 22. If $G$ is a graph, then

$$
\chi'(G) \leq \max \left\{ \mathcal{W}(G), \Delta(G) + 1, \left\lfloor \frac{4}{3} d_{\text{claw}}(G) \right\rfloor \right\}.
$$

Proof. Suppose not and choose a counterexample $G$ with the fewest edges; note that $G$ is critical. Let $k = \chi'(G) - 1$, so $k \geq \left\lfloor \frac{4}{3} d_{\text{claw}}(G) \right\rfloor$. By Theorem 4, $G$ has a long vertex $x$. Choose $xy_1 \in E(G)$ and a $k$-edge-coloring $\varphi$ of $G - xy_1$ such that $\varphi$ has a fan $F$ of length 3 rooted at $x$ with leaves $y_1, y_2, y_3$. Since no two vertices of $F$ miss a common color,

$$2 + k - d(x) + \sum_{i \in [3]} d(y_i) \leq k,$$

and hence

$$\frac{3k + 2}{4} \leq \frac{1}{4} \left( d(x) + \sum_{i \in [3]} d(y_i) \right) \leq d_{\text{claw}}(x).$$

This gives the contradiction

$$\left\lfloor \frac{4}{3} d_{\text{claw}}(G) \right\rfloor \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}. \quad \square$$

Reed conjectured that $\chi(G) \leq \left\lfloor \frac{\omega(G) + \Delta + 1}{2} \right\rfloor$ for every graph $G$. This is the average of a trivial lower bound $\omega(G)$ and a trivial upper bound $\Delta(G) + 1$. King conjectured the stronger bound $\chi(G) \leq \max_{v \in V(G)} \left\lfloor \frac{\omega(v) + d(v) + 1}{2} \right\rfloor$, where $\omega(v)$ is the size of the largest clique containing $v$; this bound is now known to hold for many classes of graphs, including line graphs. Here we show that for line graphs this bound is an easy consequence of our more general lemmas from Section 3. A thickened cycle is a multigraph that has a cycle as its underlying simple graph.

Theorem 23. If $G$ is a critical graph that is not a thickened cycle, then

$$\chi'(G) \leq \max \left\{ \Delta(G) + 1, \left\lfloor \frac{4}{3} d_{\text{claw}}(G) \right\rfloor \right\}.$$

Proof. The proof is in some ways similar to those of Lemmas 16 and 18. It consists largely of straightforward, albeit tedious, algebraic manipulations.

Suppose the theorem is false and let $G$ be a counterexample. By Theorem 22, $G$ is elementary. Since $G$ is critical and elementary, Proposition 15 implies that $|G|$ is odd and

$$k = \frac{2(|G| - 1)}{|G| - 1}.$$  

(12)

Let $x \in V(G)$ with $|N(x)| \geq 3$. Put $M := |N(x)|$,

$$P := \sum_{v \in N(x)} d_G(v),$$

and hence

$$\frac{3k + 2}{4} \leq \frac{1}{4} \left( d(x) + \sum_{i \in [3]} d(y_i) \right) \leq d_{\text{claw}}(x).$$

This gives the contradiction

$$\left\lfloor \frac{4}{3} d_{\text{claw}}(G) \right\rfloor \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}. \quad \square$$
\[ S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v), \]

\[ S_3 := k - (\Delta(G) + 1). \]

Now

\[ 2(\|G\| - 1) = \Delta(G)(|G| - M) - S_2 + P. \] (13)

Since

\[ \frac{2(\|G\| - 1)}{|G| - 1} = k = \Delta(G) + 1 + S_3, \]

using (13), we get

\[ P = (|G| - 1)(\Delta(G) + 1 + S_3) - \Delta(G)(|G| - M) + S_2, \]

which we rewrite as

\[ P = \Delta(G)(M - 1) + |G| - 1 + S_2 + S_3(|G| - 1). \] (14)

Let \( N(x) = \{v_0, v_1, \ldots, v_{M-1}\} \) and put

\[ R := \sum_{i=0}^{M-1} \frac{1}{3} (d_G(x) + d_G(v_i) + d_G(v_{i+1}) + d_G(v_{i+2})), \]

where indices are taken modulo \( M \). Since \( k \geq \frac{4}{3} d_{\text{claw}}(G) \), there is \( S_4 \geq 0 \) such that

\[ Mk - S_4 = R = \frac{M}{3} d_G(x) + P. \]

Now substituting (14) we get

\[ Mk = \frac{M}{3} d_G(x) + \Delta(G)(M - 1) + |G| - 1 + S_2 + S_3(|G| - 1) + S_4. \]

Since \( M \geq 3 \) by our choice of \( x \), solving for \( M \) gives (for some \( S_5 \geq 0 \))

\[ 3 + S_5 = M = \frac{S_2 + (S_3 + 1)(|G| - 1) + S_4 - \Delta(G)}{S_3 + 1 - \frac{1}{3} d_G(x)}. \]

Hence

\[ (3 + S_5)(S_3 + 1) - \left(1 + \frac{S_5}{3}\right) d_G(x) = S_2 + (S_3 + 1)(|G| - 1) + S_4 - \Delta(G). \]

Rearranging terms gives

\[ \left(1 + \frac{S_5}{3}\right) d_G(x) = \Delta(G) - (S_2 - 2) + (4 + S_5 - |G|) S_3 + (2 + S_5 - |G|) - S_4. \]
Suppose $4 + S_5 - |G| \leq 0$. Now

$$\left( 1 + \frac{S_5}{3} \right) d_G(x) \leq \Delta(G) - (S_2 - 2) - 2.$$  

By definition, $S_2 \geq 2 + \Delta(G) - d_G(x)$, so we have

$$\left( 1 + \frac{S_5}{3} \right) d_G(x) \leq d_G(x) - 2,$$

a contradiction since $S_5 \geq 0$. So, we must have $4 + S_5 - |G| > 0$, that is,

$$|G| \leq S_5 + 3 = |N(x)| \leq |G| - 1,$$

a contradiction.

For a graph $Q$ and $r \in \mathbb{N} \cup \{\infty\}$, put

$$C_r(Q) := \{ X \subseteq V(Q) : X \text{ is a maximal clique with } |X| < r \text{ or } X \text{ is a clique with } |X| = r \}.$$  

Put

$$\gamma_r(Q) := \max_{X \in C_r(Q)} \frac{1}{|X|} \sum_{v \in X} d(v) + \omega(v) + 1.$$

In terms of $\gamma_r$, the local version of Reed’s conjecture for line graphs, proved by Chudnovsky et al. [2], and the superlocal version, proved by Edwards and King [4], are the following two theorems.

**Theorem 24** (Chudnovsky et al.). If $Q$ is a line graph, then $\chi(Q) \leq \lceil \gamma_1(Q) \rceil$.

**Theorem 25** (Edwards and King). If $Q$ is a line graph, then $\chi(Q) \leq \lceil \gamma_2(Q) \rceil$.

Note that if $a, b \in \mathbb{N} \cup \{\infty\}$ with $a \leq b$, then $\gamma_a(Q) \geq \gamma_b(Q)$, so Theorem 25 implies Theorem 24. Edwards and King [4] conjectured that $\gamma_\infty(Q)$ is an upper bound on the fractional chromatic number for all graphs $Q$.

**Conjecture 26** (Edwards and King). If $Q$ is any graph, then $\chi_f(Q) \leq \gamma_\infty(Q)$.

A graph parameter $f$ is **monotone** if $f(G) \geq f(H)$ whenever $H$ is an induced subgraph of $G$. Unfortunately, $\gamma_\infty$ is not monotone and hence not very induction-friendly.

**Lemma 27.** $\gamma_1$ and $\gamma_2$ are monotone, but $\gamma_r$ is not monotone for all $r \geq 3$.

**Proof.** The first statement is clear. For the second statement, let $Q$ be the graph in Figure 3. For $r \geq 3$, we have $\gamma_r(Q - x) = \frac{11}{2} > \frac{16}{3} = \gamma_r(Q)$.

As a possible approach to Conjecture 26, we define the following variation on $\gamma_\infty(G)$, which should be more induction-friendly. For $r \in \mathbb{N} \cup \{\infty\}$, put

$$\tilde{\gamma}_r(Q) := \max_{H \subseteq Q} \gamma_r(H).$$
Conjecture 28. If $Q$ is any graph, then $\chi_f(Q) \leq \tilde{\gamma}_\infty(Q)$.

Conjecture 29. If $Q$ is a line graph, then $\chi(Q) \leq \lceil \tilde{\gamma}_\infty(Q) \rceil$.

Theorem 30. Conjecture 29 follows from Conjecture 28 and the Goldberg-Seymour Conjecture.

Proof. Assume that Conjecture 28 and the Goldberg–Seymour Conjecture are true and Conjecture 29 is false. Let $Q$ be a minimal counterexample, and say $Q = L(G)$. Since Conjecture 28 is true, $\chi_f(Q) \leq \tilde{\gamma}_\infty(Q)$. Since the Goldberg–Seymour Conjecture is true, $\chi(Q) \leq \max\{\Delta(G) + 1, \lceil \chi_f(Q) \rceil \} \leq \max\{\Delta(G) + 1, \lceil \tilde{\gamma}_\infty(Q) \rceil \}$. Since $Q$ is a counterexample $\lceil \tilde{\gamma}_\infty(Q) \rceil < \chi(Q) \leq \Delta(G) + 1 \leq \omega(Q) + 1$. This implies that $\lceil \tilde{\gamma}_\infty(Q) \rceil \leq \omega(Q)$. However, also $\tilde{\gamma}_\infty(Q) \geq \omega(Q)$, by taking $X$ in the definition of $\tilde{\gamma}_\infty$ to be a maximum clique. Further, since $Q$ is connected by minimality, the inequality is strict if $X \subseteq Q$. So $Q$ is a clique. But this yields the contradiction $\omega(Q) = \lceil \tilde{\gamma}_\infty(Q) \rceil < \chi(Q)$. \qed

We prove the next bound in the sequence begun by Theorems 24 and 25.

Theorem 31. If $Q$ is a line graph, then $\chi(Q) \leq \lceil \tilde{\gamma}_3(Q) \rceil$.

Before proving Theorem 31, we prove the following lemma, which aids in the proof of Theorem 31.

Lemma 32. Let $Q = L(G)$ where $G$ is a critical graph. If $G$ is not a thickened cycle, then $\chi(Q) \leq \lceil \gamma_3(Q) \rceil$.

Proof. Suppose $G$ is not a thickened cycle. For $uv \in E(G)$, put

$$f(uv) := \max\{d_G(u) + \frac{1}{2}(d_G(v) - \mu(uv)), d_G(v) + \frac{1}{2}(d_G(u) - \mu(uv))\}.$$
For $uv \in E(G)$, we have

$$f(uv) = \frac{d_G(u) + d_G(v) - \mu(uv) + \max \{d_G(u), d_G(v)\}}{2}$$

$$\leq \frac{d_Q(uv) + \omega(uv) + 1}{2}.$$ 

Since $G$ is critical, we have $|N(v)| \geq 2$ for all $v \in V(G)$. Since $G$ is not a thickened cycle, we may choose $x \in V(G)$ and $S \subseteq N(x)$ with $|S| = 3$ such that $x$ and $S$ achieve maximality in the definition of $d_{\text{claw}}(G)$. Say $S = \{v_1, v_2, v_3\}$. Then

$$\left\lceil \frac{4}{3}d_{\text{claw}}(G) \right\rceil = \left\lceil \left(\frac{4}{3}\right)\left(\frac{1}{4}\right) \left(d_G(x) + \sum_{i \in [3]} d_G(v_i)\right) \right\rceil$$

$$\leq \left\lceil \frac{1}{3} \sum_{i \in [3]} d_G(v_i) + \frac{1}{2}(d_G(x) - \mu(xv_i)) \right\rceil$$

$$\leq \left\lceil \frac{1}{3} \sum_{i \in [3]} f(xv_i) \right\rceil$$

$$\leq \left\lceil \frac{1}{3} \sum_{i \in [3]} d_Q(xv_i) + \omega(xv_i) + 1 \right\rceil$$

$$\leq \left\lceil \gamma_3(Q) \right\rceil.$$

By Theorem 23, we have

$$\chi(Q) \leq \max \{\Delta(G) + 1, \left\lceil \gamma_3(Q) \right\rceil\}.$$ 

Let $M \subseteq V(Q)$ be a maximum clique in $Q$. Since $G$ is not a thickened cycle, $|M| \geq 3$, so we can choose $X \subseteq M$ with $|X| = 3$ maximizing

$$\frac{1}{3} \sum_{v \in X} d(v) + \omega(v) + 1.$$ 

We have

$$\gamma_3(Q) \geq \frac{1}{3} \sum_{v \in X} d(v) + \omega(v) + 1$$

$$\geq \frac{1}{|M|} \sum_{v \in M} d(v) + \omega(v) + 1$$

$$\geq \omega(Q) + \sum_{v \in M} \frac{d(v) + 1 - \omega(v)}{2}.$$
If $V(Q) = M$, then $[\gamma_3(Q)] = \omega(Q) = \Delta(Q) = \chi(Q)$, as desired. Otherwise, some $v \in M$ has $d(v) \geq \omega(Q)$ and hence $[\gamma_3(Q)] \geq \omega(Q) + 1 \geq \Delta(Q) + 1$. Using (15), this gives $\chi(Q) \leq [\gamma_3(Q)]$.

**Proof of Theorem 34.** Suppose Theorem 31 is false, and choose a counterexample $Q$ minimizing $|Q|$. Say $Q = L(G)$. The minimality of $|Q|$ and monotonicity of $\bar{\gamma}$ imply that $G$ is critical. So Lemma 32 gives a contradiction unless $|Q| = \chi(G)$. Thus, Theorem 4 implies that $G$ is elementary, i.e., $\chi'(G) = \mathcal{W}(G)$. In fact, since $G$ is critical, $\chi'(G) = \lceil \frac{2|Q|}{|G|-1} \rceil$.

Since $G$ is elementary, $|G|$ is odd, so say $|G| = 2t + 1$. Denote the vertices of $G$ by \{v_1, \ldots, v_{2t+1}\}, and let $x_i = \mu(v_iv_{i+1})$ for each $i \in [2t + 1]$. Let $X = \sum_{i=1}^{2t+1} x_i$. First suppose there exists $j$ such that $x_j = 1$. So there exists $e \in E(G)$ such that $G - e$ is bipartite. Thus, $\chi'(G - e) = \Delta(G - e) \leq \Delta(G)$, so $\chi'(G) \leq \Delta(G) + 1$. Since $[\gamma_3(Q)] \geq \Delta(G) + 1$, as shown above, the theorem holds.

Now assume instead that $x_i \geq 2$ for all $i \in [2t + 1]$. For each $i$, let $u_i^1$ and $u_i^2$ be vertices of $Q$ corresponding to edges counted by $x_i$. Note that $\frac{d(u_i^1 \cup u_i^2)}{2} \geq \frac{x_{i-1} + 2x_i + 2x_{i+1}}{2}$, for each $i \in [2t + 1]$ and each $j \in [2]$. Now averaging over $\{u_i^1, u_i^{1+1}, u_i^{2+1}\}$ gives $\bar{\gamma}_3(Q) \geq \frac{1}{5} (x_{i-1} + 4x_i + 6x_{i+1} + 4x_{i+2})$. When we average over all $i \in [2t + 1]$, we get $\bar{\gamma}_3(Q) \geq \frac{5X}{4t+2}$. Since $G$ is elementary, we know $\chi'(G) = \lceil \frac{X}{t} \rceil$. Now the theorem holds since $\frac{1}{t} \leq \frac{5}{4t+2}$, whenever $t \geq 2$.

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