AN EQUIVALENCE BETWEEN VANISHING CYCLES AND MICROLOCALIZATION

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ABSTRACT. We define the functors of specialization and microlocalization along a quasi-smooth closed immersion using the theory of quasi-smooth blow-ups developed in [KR19].

Our first main result describes the entire microlocalization functor along the inclusion of a derived zero fiber as the vanishing cycles with respect to a certain function. The proof is based on the observation that the entire Fourier-Sato transform may be written as a vanishing cycles. An equivalent result in a slightly different setting was recently obtained by Tasuki Kinjo in [K21b] using a different method of proof.

Our second main result is an equivalence between the canonical twisted vanishing cycles sheaf introduced by Brav, Bussi, Dupont, Joyce, Szendrői, and Schürmann on the $-1$-shifted cotangent bundle of a quasi-smooth scheme and the quasi-smooth microlocalization of the perverse constant sheaf.

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1. Introduction

In the classical theory of $\mathcal{D}$-modules and microlocal sheaf theory, the functor of vanishing cycles and the functors of microlocalization and Fourier transform are intimately related in well-known ways. For example, if $f : X \to \mathbb{C}$ is a holomorphic function on a complex manifold $X$ and the zero locus of $f$ is non-singular, $\varphi_f$ is entirely determined by the microlocalization along $f^{-1}(0)$ ([KS90, Proposition 8.6.3]). On the other hand, the stalk of the Fourier transform at a generic covector is vanishing cycles with respect to the functional determined by that covector ([V83a]).

Yet, these results stop short of providing a full description of the microlocalization and Fourier-Sato transform functors as a vanishing cycles—and vice versa. In this paper, we present such a description in the setting of constructible sheaves on the complex analytic spaces associated to derived zero fibers and, more generally, quasi-smooth derived schemes.

1.1. Main results. In order to formulate the main results, we must use derived algebraic geometry in a non-trivial way. We begin by recalling the notion of quasi-smooth closed immersion.

The notion of a quasi-smooth closed immersion is the extension of the notion of a regular embedding to the derived setting. Such morphisms are Zariski locally modeled on the derived zero locus of a map from the target of the closed immersion to affine space of some finite rank. They are highly structured, having, for example, a well-behaved notion of normal bundle.

In this paper, we define the notions of specialization and microlocalization along a quasi-smooth, closed immersion $\mathcal{Z} \hookrightarrow X$. Our definition relies on the machinery of blow-ups along quasi-smooth, closed immersions developed by Adeel Khan and David Rydh in [KR19]. In loc. cit. the authors introduce the deformation to the normal bundle along $\mathcal{Z} \hookrightarrow X$ in a way completely analogous to the formation of the deformation to the normal cone along an arbitrary closed immersion in classical algebraic geometry. Likewise, in a way completely analogous to
the formation of the functor of Verdier specialization, we introduce the functor, $\text{Sp}_{Z/X} : \text{Shv}(X) \to \text{Shv}(\mathcal{N}_{Z/X})$, of quasi-smooth specialization. The functor of quasi-smooth microlocalization, $\mu_{Z/X} : \text{Shv}(X) \to \text{Shv}(\mathcal{N}_{Z/X})$, is then defined by composition with the Fourier-Sato transform, just as in the definition of the classical microlocalization functor of [KS90].

Once some basic properties of these new functors have been established, we are able to prove the main results of this paper: a pair of equivalences between quasi-smooth microlocalization and vanishing cycles—in both local and global forms.

1.1.1. Local version of equivalence. The following is the local version of the equivalence between microlocalization and vanishing cycles.

**Theorem 1.1** (= Theorem 5.1). Suppose that $X$ is a smooth scheme, and $f : X \to V$ is a regular function whose target is an $n$-dimensional complex vector space. Let $Z(f) \hookrightarrow X$ denote the derived zero locus of $f$, and $f : X \times V^\vee \to \mathbb{A}^1$ denote the map $(x, \lambda) \mapsto \langle \lambda, f(x) \rangle$. Then there exists a natural isomorphism of functors,

$$\varphi_f (\text{pr}_1^∗ -) \simeq \mu_{Z(f)/X} (-).$$

**Remark 1.2.** Theorem 1.1 is, with minimal work, implied by [K21b, Theorem 4.1], which was published while this paper was in preparation. We present an original, alternative method of proof. We briefly describe the difference.

The proof in [K21b] uses a clever argument to reduce the computation of the Fourier-Sato transform (applied to vanishing cycles) to the computation of the Fourier-Sato transform on a rank 1 bundle, where an explicit expression can be obtained in terms of $*!/!$ pushforward/pullbacks applied to $\mathcal{F}$. Further sheaf theoretic analysis is then used to show that the resultant sheaf is equivalent to the (pushforward of) Verdier specialization.

Our proof begins with the observation that Theorem 1.1 is easily verified in the case when $\mathcal{F}$ is monodromic because the full Fourier-Sato transform can be described as a certain vanishing cycles functor. The crux of our proof is deducing the non-monodromic case from this by proving that the vanishing cycles functor “factors” through specialization. Our conceptual emphasis, therefore, is on the theorem being a consequence of a certain property of vanishing cycles. There is also an attendant microlocal philosophy behind our proof, which we describe in Remark 1.9 below.

**Discussion 1.3.** As explained earlier, Theorem 1.1 (and [K21b] Theorem 4.1) describes the entire microlocalization functor as a kind of vanishing cycles. We believe that this equivalence between microlocalization and vanishing cycles is a fundamental one that should be true in any sheaf theory context where there are formal notions of nearby cycles, vanishing cycles, and Fourier transform—such as $\mathcal{D}$-modules, $\ell$-adic sheaves, mixed Hodge modules, etc.—though we in this paper (and Kinjo in [K21b]) only show it for $\mathbb{C}$-constructible sheaves.

1.1.2. Global version of equivalence. In [BBJ19], it is shown that $-1$-shifted symplectic schemes are locally modeled on the derived critical locus of a regular function whose target is $\mathbb{A}^1$. In a subsequent paper [BBDJSS15], Brav, Bussi, Dupont, Joyce, Szendrői, and Schürmann associate to any $-1$-shifted symplectic scheme, $(X, \omega_X)$, a canonical (up to unique isomorphism determined by choice of square root of the determinant line bundle, $\det(L_X)|_X$) perverse sheaf on the underlying
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classical scheme, \(X\), denoted by \(\varphi_{X,\omega_X}\), and characterized more-or-less by the following property: if \(X\) is locally modeled on the derived critical locus of \(f\), \(\varphi_{X,\omega_X}\) is locally modeled on the the perverse vanishing cycles sheaf with respect to \(f\)—up to a twist by \(\mathbb{Z}/2\mathbb{Z}\)-bundle.

The \(-1\)-shifted cotangent bundle of a derived scheme \(Z\) has a canonical \(-1\)-shifted symplectic structure. When \(Z\) is a quasi-smooth derived scheme, there is a canonical choice of square root for \(\det(L_{T[-1]Z})_{\mid T[-1]Z}\), so there is a canonical perverse sheaf, \(\varphi_{T[-1]Z}\), which we can think of as a “globalized” perverse vanishing cycles sheaf. The following theorem now serves as a global version of the equivalence between microlocalization and vanishing cycles.

**Theorem 1.4** (**= Theorem 6.1**). Given any closed immersion \(Z \hookrightarrow X\) of a quasi-smooth derived scheme \(Z\) into a smooth scheme \(X\), there exists an isomorphism, \(\varphi_{T[-1]Z} \cong \mu_{Z/X} [\text{codim.vir}(Z, X)](\mathcal{E}_X[\dim X])\), of perverse sheaves on \(T[-1]Z \subset N^\vee_{Z/X}\), where \(e\) is any well-behaved commutative ring.

**Remark 1.5.** Theorem 1.4 offers a new means of constructing \(\varphi_{T[-1]Z}\), via quasi-smooth microlocalization. The original construction found in [BBDJSS15] and the construction offered here are mutually complementary. On the one hand, it is readily apparent from the construction of \(\varphi_{T[-1]Z}\) in [BBDJSS15] that the sheaf so-defined is intrinsic to \(Z\), but the construction requires meticulous book-keeping of data. On the other hand, \(\mu_{Z/X}\) is defined in a friendlier way—using the derived version of a classical tool like Verdier specialization—but it is not at all apparent that it is intrinsic to \(Z\). At the cost of knowing our definition is intrinsic, we gain the benefit of a more transparent definition for the twisted vanishing cycles sheaf in the special case of a \(-1\)-shifted cotangent bundle.

**Remark 1.6.** A forthcoming paper of Adeel Khan, [AAK], independently develops the notion of derived microlocalization in the étale setting along arbitrary morphisms (i.e. not necessarily a quasi-smooth closed immersion).

**1.2. Methods.** In this section, we provide outlines of the proofs of Theorem 1.1 and Theorem 1.4, which are technical and somewhat involved.

**1.2.1. Proof of the local equivalence.** The proof of Theorem 1.1 begins by observing that if,

- (i) \(X := Z \times V\),
- (ii) \(f := \text{pr}_2 : X \to V\),
- (iii) and \(F\) is monodromic with respect to the scaling action on the fibers of \(X\), then \(\varphi_{\text{pr}_2^*}(\text{pr}_1^*F)\) is (almost tautologically) equivalent to \(\mu_{Z/X} F\). It is easily shown using the standard sequence on cotangent complexes that the normal bundle to the derived zero locus, \(Z(f)\), of an arbitrary regular function \(f : X \to V\), is trivial. If \(\pi : N_{Z(f)/X} \to \mathbb{A}^n\) denotes the bundle map, our observation therefore implies that, for arbitrary \(F \in \text{Shv}(X)\),

\[\varphi_{\pi}(\text{pr}_1^*\text{Sp}_{Z(f)/X} F) \cong \mu_{Z(f)/N_{Z(f)/X}}(\text{Sp}_{Z(f)/X} F),\]

\(\text{e.g. of finite homological dimension. The reader who wishes to will lose nothing by taking}\)
\(\epsilon := \mathbb{C}\) throughout this entire paper.
using the fact that quasi-smooth specialization is idempotent. This equivalence suggests our proof strategy of showing that the left-hand side factors through quasi-smooth specialization, in the sense that

\begin{equation}
\varphi_\#(pr_1^* Sp_{Z(f)/X} \mathcal{F}) \simeq \varphi_f(pr_1^* \mathcal{F}).
\end{equation}

Through a series of base change isomorphisms, it suffices to show (1.7) for the case when $Z$ is smooth, $X = Z \times V$, and $f = pr_2 : X \to V$. For ease of exposition, we denote $m := \tilde{pr}_2$. The proof proceeds in this special case in the following steps:

(i) We define a comparison map,

$$\psi_m(pr_1^* \mathcal{F})|_{Z \times V} \xrightarrow{\psi} \psi_m(pr_1^* Sp_{Z/X} \mathcal{F})|_{Z \times V}.$$  

(ii) We show that this comparison map is an isomorphism.

(iii) We use the defining fiber sequence for vanishing cycles to obtain an equivalence, (1.7).

We briefly elaborate on each of these steps.

Discussion 1.8. We begin the construction the comparison map in step (i) by obtaining convenient expressions for sections of the left- and right-hand sides of the comparison map.

The convenient expression for the sections of the left-hand side, $\psi_m(pr_1^* \mathcal{F})|_{Z \times V}$, over a convex, conic open subset of $Z \times V$ is as the filtered colimit of sections of $\mathcal{F}$ ranging over a certain family of open subsets in $X$. Clearly, sections of the right-hand side are also given by a filtered colimit of the same shape, but whose terms are now sections of $Sp_{Z/X} \mathcal{F}$ rather than $\mathcal{F}$ itself. A foundational result in classical microlocal sheaf theory on sections of the specialization functor further allows us to write each of these terms as a colimit of sections of $\mathcal{F}$. Substituting these colimits for each of the terms, we obtain the convenient expression for sections of the right-hand side of the comparison map over a convex, conic open subset of $Z \times V$ as the colimit of sections of $\mathcal{F}$ over a larger family of open subsets of $X$.

We note that convex, conic open subsets of $X$ form a basis for its conic topology. Using the aforementioned expressions for sections, we show that the basis sheaves obtained from the restriction of $\psi_m(pr_1^* \mathcal{F})|_{Z \times V}$ and $\psi_m(pr_1^* Sp_{Z/X} \mathcal{F})|_{Z \times V}$ to convex, conic open subsets are actually the left Kan extensions of two functors $Q \circ \alpha$ and $Q$, respectively, which we define in Section 5.5. As such, we obtain a map between these Kan extensions by standard abstract nonsense, at which point we have obtained a map between the induced basis sheaves mentioned in the previous sentence. A simple argument then shows that in our present set-up the category of basis sheaves is equivalent to the category of sheaves on the full site, yielding the desired comparison map, $\psi$.

In step (ii), we use the hypercompleteness of $\text{Shv}(Z \times V)$ to argue that $\psi$ is an isomorphism by showing that the map it induces on stalks is an isomorphism at each point. For a given point $(z, \lambda) \in Z \times V$, the stalks, $\psi_m(pr_1^* \mathcal{F})|_{(z, \lambda)}$ and $\psi_m(pr_1^* Sp_{Z/X} \mathcal{F})|_{(z, \lambda)}$ are each computed by filtered colimits of sections of $\mathcal{F}$ over certain opens in $X$. We identify a common cofinal subsequence for these colimits, compatible with the map $\psi$, thereby proving $\psi$ is an isomorphism.

Step (iii) is self-explanatory.

Remark 1.9. Lurking in the background of our proof is the following aspirational microlocal picture: $\varphi_f(pr_1^* \mathcal{F})$ and $\varphi_m(pr_1^* Sp_{Z(f)/X} \mathcal{F})$ should be thought of as
fibers of a sheaf on the deformation to the normal bundle, $D_{Z(f)/X}$, which we view as a family of sheaves over $\mathbb{C}$. This family should be non-characteristic along the $\mathbb{C}$-codirections in $D_{Z(f)/X}$ coming from the canonical map $D_{Z(f)/X} \rightarrow \mathbb{A}^1$, so locally constant along these codirections by non-characteristic propagation, meaning the fibers of this family over any two points of $\mathbb{C}$ are isomorphic.

The difficulty in making this picture rigorous is obtaining good estimates for the singular support of the sheaf on $D_{Z(f)/X}$, which proved troublesome in even the simplest cases.

1.2.2. **Proof of the global equivalence.** A closed immersion $Z \hookrightarrow X$ of a quasi-smooth derived scheme into a smooth scheme is always quasi-smooth. As such, $Z$ is modeled, Zariski locally on $X$, and the derived zero locus of a regular function $f : X \rightarrow \mathbb{A}^n$. The rank $n$ is locally constant on $Z$, and called the **virtual codimension** of the closed immersion $Z \hookrightarrow X$; it is the rank of the normal bundle, $\mathcal{N}_{Z/X}$. Let $U \subset X$ denote an open subset in which $Z$ has the above expression.

The local description of $Z$ as the derived zero locus $Z(f)$ gives a corresponding local description of $T^*[−1]Z$ as the derived critical locus of the function $\tilde{f}$, appropriating the notation used above. The canonical $-1$-shifted symplectic structure on $T^*[−1]Z(f)$ as a shifted cotangent bundle, and the $-1$-shifted symplectic structure coming from its expression as a derived critical locus coincide, so we obtain that $\varphi_{T^*[−1]Z} \simeq \varphi_{T^*[−1]Z(f)} \simeq \varphi_{\varphi^{[n]}([\dim U])}$. Application of smooth base change obtains the equivalence, $(\mu_{Z/X}^{\ast}F)|_{T^*[−1]Z(f)} \simeq \mu_{Z(f)/U}(F)|_{U}$.

Taking an open cover, $\{U_\alpha \subset X\}$, of $Z \subset X$ by such open sets, we obtain a corresponding open cover, $\{T^*[−1]Z(f_\alpha)\}$, of $T^*[−1]Z$ such that on $T^*[−1]Z(f_\alpha)$ there is an isomorphism,

\[(\alpha) \quad \varphi_{T^*[−1]Z} \simeq \mu_{Z/X}^{\ast}[n](\varphi_{\varphi^{[n]}([\dim U_\alpha])}|_{T^*[−1]Z(f_\alpha)}),\]

obtained by Theorem [14]. The central feature of the proof of Theorem [1.4] is showing that the various isomorphisms obtained on the elements of our open cover, as $\alpha$ ranges over all index values, glue to an isomorphism $\varphi_{T^*[−1]Z} \simeq \mu_{Z/X}^{\ast}[n](\varphi_{\varphi^{[n]}([\dim X])})$. The isomorphism obtained on $T^*[−1]Z(f_\alpha)$ of our open cover depends, very obviously, on the function, $f_\alpha$. At first glance, it is not obvious that, on their overlaps, the various functions in $\{f_\alpha\}$ are compatible with each other in a sense that implies that the isomorphisms, $(\alpha)$, glue. We demonstrate this by proving a lemma which allows us to compare any two overlapping Kuranishi charts of the same finite rank—a formal way of packaging the data that expresses $Z$, locally, as the derived critical locus $Z(f_\alpha)$—by finding, around any point in their overlap, a third Kuranishi chart of the same rank that openly embeds into each of the two. Roughly speaking, this means that in a sufficiently small neighborhood around any point, we can verify that the functions, $f_\alpha$ and $f_\beta$, are the same up to a “change in coordinates.” This clearly suffices to show that the various isomorphisms, $(\alpha)$, glue.

1.3. **Motivation from Donaldson-Thomas theory.** We give some naive motivation for our results coming from DT theory. Theorem [1.3] gives an alternative construction of the canonical twisted vanishing cycles sheaf for a class of $-1$-shifted symplectic schemes that feature prominently in categorical and motivic Donaldson-Thomas theory.

For example, let $\tilde{S}$ be a local surface (i.e. the total space of the canonical line bundle on a smooth quasi-projective surface $S$). Denote by $M_S$ and $\mathcal{M}_S$ the derived
moduli stacks of coherent sheaves with proper supports on \( X \) and \( S \), respectively. Then \( \tilde{M}_S \) carries a canonical \(-1\)-shifted symplectic structure, and \( M_S \) is quasi-smooth. In fact, \([K21a, \text{Theorem 5.1}]\) shows that \( \tilde{M}_S \simeq T^*[-1]M_S \) as \(-1\)-shifted symplectic stacks with the latter’s canonical shifted symplectic structure. When, for example, the underlying coarse moduli scheme \( \pi_0(M_S) \) is quasi-projective, \( \pi_0(M_S) \) admits an embedding into a smooth scheme, and Theorem 1.4 gives a microlocal interpretation of the categorification of the classical Donaldson-Thomas invariants of the Calabi-Yau 3-fold, \( \tilde{S}^2 \).

Similarly, we might take the category of representations of a quiver with super-potential. This furnishes a CY3 category whose moduli stack of objects can locally be written as the derived critical locus of a holomorphic function \( f : U \to \mathbb{C} \), for smooth \( U \). Simple examples of such stacks can sometimes globally be expressed as the critical locus of such a function, in which case, Theorem 1.4 (in fact, even Theorem 1.1) gives an alternative construction for its canonical twisted vanishing cycles sheaf.

1.4. Motivation from microlocal chains. Throughout this section, \( e := \mathbb{C} \).

Consider the now-familiar set-up of a derived Cartesian square:

\[
\begin{array}{ccc}
Z(f) & \xrightarrow{\iota} & X \\
\downarrow & & \downarrow f \\
\ast & \xrightarrow{0} & V,
\end{array}
\]

where \( X \) is smooth of pure dimension \( m \), \( V \) is an \( n \)-dimensional \( \mathbb{C} \)-vector space viewed as a scheme, and \( Z(f) \) is the derived zero fiber of \( f \). Let \( Z(f) \) denote the underlying classical scheme of \( Z(f) \)—equivalently, the classical zero fiber of \( f \). In a graduate course taught during the Fall 2013 quarter at UC Berkeley, David Nadler introduced a variant of the following definition.

**Definition 1.10** (c.f. \([Y13]\)). Let \( \Lambda \subset V^\vee \) be a closed conic subset. The microlocal homology of \( f \) with support \( \Lambda \) is defined as,

\[
M^f_\Lambda := \Gamma_\Lambda(V^\vee; \mu_{0/V}(f, \underline{\text{sec}}_X[m])),
\]

where \( \Gamma_\Lambda(V^\vee; -) \) is the global sections of the “sections with support in \( \Lambda \)” functor, using the terminology of \([KS90]\)—alternatively, the functor \( \pi_! \) (\( \Lambda \to V^\vee \)).

**Remark 1.11.** In Definition 1.10, one can take \( \mu_{0/V} \) to be the classical microlocalization or the quasi-smooth microlocalization—they coincide here.

The microlocal homology is meant to interpolate between the singular cohomology and Borel-Moore homology of \( Z(f) \) (with the complex analytic topology) as \( \Lambda \) ranges through the poset of conic subsets of \( V^\vee \). Indeed, supposing that \( X \) is oriented, a calculation using the standard properties of \( \mu_{0/V} \) yields the equivalences,

\[
\begin{align*}
M^f_0 & \simeq \Gamma(Z(f); \underline{\text{sec}}_{Z(f)}[m-n]) \\
M^f_m & \simeq \Gamma(Z(f); \omega_{Z(f)}),
\end{align*}
\]

meaning that global sections of the microlocal homology recover, up to a shift, singular cochains on \( Z(f) \) when \( \Lambda = 0 \) and Borel-Moore chains on \( Z(f) \) when \( \Lambda = V^\vee \). A natural line of inquiry is the following.

\[\text{c.f. [BBDJSS15, Remark 6.14]}\]
Question 1.14. What is a nice description of the intermediate complexes given by taking $0 \subseteq \Lambda \subseteq V$?

1.4.1. Geometric chains. On an oriented smooth manifold, singular chains and singular cochains are, in a sense, the same type of mathematical object; Poincaré duality furnishes an isomorphism between homology and cohomology groups.

Though $Z(f)$ may in general have singularities, it is a tame space in the sense that $X$ and $V$ can always be Whitney stratified in such a way that $f$ is a map of stratified spaces and $Z(f)$ is a union of strata. On such a space, the complex of singular chains and singular cochains are also not totally different species of mathematical objects: both may be represented by geometric chains and geometric cochains, which satisfy a kind of Poincaré duality. Both geometric chains and cochains are, roughly speaking, formal linear combinations of (suitable) subsets of $Z(f)$, together with some additional data (e.g. (co)orientation); the precise definition may be found in [G81]. A geometric $k$-chain is one whose formal terms are $k$-dimensional subsets of $Z(f)$, and a geometric $k$-cochain is one whose formal terms are $k$-co-dimensional subsets of $Z(f)$. Like in the case of a smooth manifold, there is a means of taking cocycles to cycles by taking the intersection of the “support” of a cocycle (the union of the subsets making up the formal sum) with the support of a cycle—giving the cap product in singular homology.

The animating idea behind the “geometric cycle” description of homology and cohomology—to view chains and cochains as concrete subsets of our space (with some extra data and decoration)—proved enormously fruitful to Mark Goresky and Robert MacPherson in their development of intersection homology. In their search for a robust homology theory for singular spaces, Goresky and MacPherson decided to look at the chain theories obtained by considering only some allowable geometric chains. If we work with $Y \subset X$, the closure of a single stratum inside the Whitney stratified manifold, $X$, allowable geometric chains are those whose intersections with the various strata of $Y$ have dimensions dictated by a chosen perversity function.

There are two extreme perversity functions, the zero perversity, given by the zero function, and the so-called “top” perversity function, given by the assignment,

$$Y_\alpha \mapsto \text{codim}Y_\alpha - 2,$$

for any stratum $Y_\alpha \subset Y$, and where, if $Y$ is of pure real dimension $n$, $\text{codim}Y_\alpha := n - \dim Y_\alpha$. By [GM80 §4.3], the chain theory determined by using allowable geometric chains for the top perversity is singular homology, and the chain theory determined by allowable chains for the zero perversity is singular cohomology. Thus, intersection homology interpolates between singular homology and cohomology, giving different geometric homology theories as the perversity function varies.

The central lesson of the Goresky-MacPherson construction of intersection homology is that considering collections of chains with fixed geometric properties leads to interesting theories of homology.

1.4.2. Currents. Returning to the microlocal homology, let us rewrite the Borel-Moore homology and the singular cohomology of $Z(f)$ using resolutions by currents on $X$. If $\mathcal{P}'_X$ denotes the sheaf of currents on $X$ (i.e. distributional differential

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3 An integer-valued function on the collection of singular strata of $Y$. See [M90 §1.1].
By the standard adjunction triangle, we obtain

\[ C[Z(f)] \simeq i^* D^* \]

(1.15)

and

\[ \omega[Z(f)][-m] \simeq i^* D^*[Z(f)] \]

(1.16)

where \( D^*[Z(f)] \) denotes the sheaf of currents on \( X \) supported on \( Z(f) \subset X \). Since \( X \) is smooth, \( \Omega^* \) is equivalent to \( D^*[f] \) so (1.15) follows from the algebraic de Rham theorem. On the other hand, (1.16) follows from the following slightly more general lemma.

**Lemma 1.17.** Suppose that \( i : Z \hookrightarrow X \) is the inclusion of a closed subset into a smooth, oriented manifold \( X \) of dimension \( m \). Then,

\[ \omega[Z][-m] \simeq i^* D^*[Z]. \]

**Proof.** Let \( j : U \rightarrow X \) denote the inclusion of the open subset complementary to \( Z \). By the standard adjunction triangle, we obtain \( i_* i^! \simeq i_* i^!(\omega_X[-m]) \simeq i_* \omega_Z[-m] \)

as the fiber of the map, \( \mathcal{C}_X \rightarrow j_* \mathcal{C}_U \). Since \( U \) and \( X \) are smooth, \( \mathcal{C}_X \) and \( \mathcal{C}_U \) have standard resolutions by the sheaves of currents, \( D^* \) and \( D^* \), respectively. The sheaf of \( p \)-currents, \( D^p \), is fine for each \( p \), so the push-forward \( j_* \mathcal{C}_U \) has the obvious resolution given by the pushforward, \( j_* D^* \). Now the map, \( \mathcal{C}_X \rightarrow j_* \mathcal{C}_U \), induces the obvious map on resolutions \( D^* \rightarrow j_* D^* \) given by domain restriction of distributions, and its fiber is clearly the sheaf of currents supported on \( Z \). \( \square \)

Using such resolutions, we may express the microlocal homology at \( \Lambda = \{0, V^\vee \} \) using the pullback of certain currents on \( X \):

\[ M_0^I \simeq \Gamma(Z(f); i^* D^*)[m-n] \]

(1.18)

\[ M_I^I \simeq \Gamma(Z(f); i^* D^*[Z(f)][m]). \]

(1.19)

While we have taken the perspective that currents are merely differential forms with distributional coefficients, in most standard accounts of currents (e.g. \cite{Fed69}) they are defined as the continuous dual to the topological vector space of compactly supported, smooth differential forms. The latter perspective suggests a connection between currents and geometric chains\(^4\) that can be made precise. For example, integration against a piece-wise smooth, oriented \( p \)-chain in \( X \) gives a \( p \)-current\(^5\).

In particular, any non-degenerate \( p \)-simplex determined by a general choice of \( p \) points in \( X \) gives a \( p \)-current. In fact, geometric measure theorists have defined the theories of “integral flat” homology and “integral rectifiable” homology based on special classes of currents called “integral flat chains” and “integral currents,” respectively\(^6\).

In light of this discussion, the philosophy behind the intersection homology of Goresky-MacPherson suggests that we address Question \([1.14]\) by fixing certain conditions indexed by \( \Lambda \) and by considering only those currents which satisfy these

\[^4\] The theory of currents has been developed mostly in the context of geometric measure theory, where they play the role of manifolds in differential geometry. For details of the theory, the reader is referred to [Fed69], the standard reference on the subject.

\[^5\] See pg. 385 of [GH78] for a proof of the so-called “smoothing of cohomology.”

\[^6\] Indeed, the author learned from [DP07] that currents were studied by Georges de Rham in order to “show the connections between differential forms and singular chains” ([DP07] pg. 168).

\[^7\] Beware notation conventions in the literature for currents.

\[^8\] The interesting article, [DP07], compares these homology theories with the theories of Čech and singular homology.
conditions. Inspired by this line of thought, the author’s graduate thesis advisor, David Ben-Zvi, made the following rough conjecture.

**Conjecture 1.20** (Ben-Zvi c. 2016). There should exist a resolution of $M^f_\Lambda$ by the pullback of currents on $X$ with prescribed regularities (e.g. wave front sets contained in a prescribed conic, subset of the cotangent bundle) dictated in some way by $\Lambda$.

Of course, we can view currents as differential forms with distributional coefficients as we did initially. From this perspective, Conjecture 1.20 states that there should be a $\Lambda$-indexed family of “distributional de Rham theorems” for the microlocal homology that interpolates between the usual de Rham theorem when $\Lambda = 0$ and the one proven in Lemma 1.17.

### 1.4.3. Singular support of chains

On a smooth manifold, the cochain theories obtained by using currents of any specified regularity—i.e. whether we consider all currents, only currents with Lipschitz regular coefficients, only currents with smooth coefficients, etc.—are equivalent. While we can model the de Rham complex with currents, the regularity (i.e. wave front sets) of individual currents is not meaningful since every current can be represented by a smooth form in its cohomology class. To the extent that currents can be viewed as chains, this is saying that any chain on a smooth manifold always has a smooth chain representative in homology. This suggests that, for a singular space $Z \subset X$ sitting inside of an ambient smooth $X$, when we model homology classes of chains on $Z$ using currents on $X$, there should be an obstruction to representing these currents by smooth forms which model the same homology class of $Z$. Furthermore, this obstruction should behave like a theory of singular support, assigning to each chain a conic subset of some vector space or vector bundle such that an assignment of the zero section implies smoothness of (a representative of) the chain.

When $Z := Z(f)$, the microlocal homology provides a systematic way of producing sheaves of currents with various regularity conditions (assuming some kind of positive resolution to Conjecture 1.20). At one extreme, $\Lambda = 0$ produces only currents with smooth coefficients. At the other extreme, $\Lambda = V^\vee$ produces all currents. The intermediate values for $\Lambda$ produce classes of currents somewhere in between these two, and as $\Lambda$ becomes larger, so too does the collection of currents it classifies. This suggests that $\Lambda \subset V^\vee$ controls the behavior of chains in a manner similar to the theory of singular support we seek. A working definition of the singular support of a current, $T$, might now be: the minimal closed, conic subset, $\Lambda$, such that $T \in M^f_\Lambda$. This train of thought leads to another conjecture of David Ben-Zvi, which reformulates some of the ideas behind Conjecture 1.20.

**Conjecture 1.21** (Ben-Zvi c. 2020). There is a good notion of singular support for chains on $Z(f)$ which measures the failure of a chain to be represented by “smooth chains” on $X$.

### 1.4.4. Main motivation

Clearly, in the remarks and discussion leading up to Conjecture 1.21, the central object has been, not $f$, but rather the closed immersion $i : Z(f) \hookrightarrow X$. Yet, the failure of a chain in $Z(f)$ to be represented by smooth chains in $X$ is ultimately the failure of $Z(f)$ to be smooth. Indeed, if a chain in $Z(f)$ avoids the singular locus, it is always representable by a smooth chain in $X$. 
Using elementary homological algebra and the standard sequence on cotangent complexes, we obtain the following closed immersion of schemes over \(Z(f)\),

\[
T^\ast[-1]Z(f) \hookrightarrow Z(f) \times V^\vee.
\]

Ideally, the singular support of a chain on \(Z(f)\) would live inside of a vector bundle over \(Z(f)\) itself, so rather than consider \(\Lambda \subset V^\vee\) we instead consider \(Z(f) \times \Lambda \subset Z(f) \times V^\vee\). The following conjecture now represents an important step toward showing a good theory of singular support of chains.

**Conjecture 1.23.** Let \(\tilde{\Lambda} := T^\ast[-1]Z(f) \cap (Z(f) \times \Lambda)\), for \(\Lambda \subset V^\vee\). Given \(\Lambda, \Lambda' \subset V^\vee\) such that \(\tilde{\Lambda} = \tilde{\Lambda}'\),

\[
M^f_\Lambda \simeq M^f_{\Lambda'}.
\]

Roughly speaking, this conjecture states that the microlocal homology should only depend on the part of \(\Lambda\) that intersects \(T^\ast[-1]Z(f)\) (the “scheme of singularities” of \(Z(f)\) in the language of [AG15]) when made into a trivial family over \(Z(f)\), viewed as a subfamily of the conormal bundle to \(Z(f)\). It suggests that the microlocal homology is actually something intrinsic to the quasi-smooth derived scheme \(Z(f)\) rather than to the map \(f\) itself.

### 1.4.5. Proving Conjecture 1.23

Theorem 1.4 allows us to make the following definition, and answer Conjecture 1.23 in the affirmative in the case when \(f\) is proper.

**Definition 1.24.** Given a quasi-smooth derived scheme \(Z\) which admits a closed immersion into a smooth scheme \(X\), the microlocal homology of \(Z\) with support in \(\Lambda\) is defined to be the sheaf,

\[
\mathcal{M}^Z_\Lambda := \Gamma_\Lambda \varphi_T T^\ast[-1]Z,
\]

where \(\Lambda \subset T^\ast[-1]Z\) is any closed conic subset.

When \(Z = Z(f) \hookrightarrow X\) is the derived zero locus of a proper map \(f : X \rightarrow V\), the following theorem justifies our terminology by showing that the global sections of the microlocal homology of \(Z(f)\) recovers the microlocal homology of \(f\).

**Theorem 1.26.** Let \(\Lambda \subset V^\vee\) be a closed conic subset, and suppose \(f : X \rightarrow V\) is a proper map from a smooth scheme \(X\). Then there exists an equivalence,

\[
\Gamma(T^\ast[-1]Z(f); \mathcal{M}^{Z(f)}_\Lambda) \simeq M^f_\Lambda[\operatorname{codim\,vir} Z(f)].
\]

**Proof.** Since the inclusion \(i : Z(f) \hookrightarrow X\) is quasi-smooth, it has a well-defined normal bundle, as described in Section 3. Furthermore, since the diagram,

\[
\begin{array}{ccc}
  Z(f) & \longrightarrow & X \\
  f \downarrow & & \downarrow f \\
  \ast & \longrightarrow & V^\vee
\end{array}
\]
is derived Cartesian, \(f\) induces a map of bundles over \(Z(f)\), which we call \(T_f := N_{Z(f)/X} \rightarrow f^\ast N_0/V^\vee \simeq Z(f) \times V\) — upon the trivialization \(N_{Z(f)/X} \simeq Z(f) \times V\), \(T_f\) becomes the identity. Composition of \(T_f\) with the projection, \(\operatorname{pr} : Z(f) \times V \rightarrow V\), gives the map of normal bundles, \(N_f : N_{Z(f)/X} \rightarrow N_0/V\), described in Section 3.
Let $T_f$ and $\pi$ denote the maps on dual bundles induced by $T_f$ and $pr$, respectively. Note the Cartesian diagram,

$$
\begin{array}{ccc}
Z(f) \times \Lambda & \xrightarrow{\iota} & Z(f) \times V^\vee \\
\downarrow & & \downarrow \\
\Lambda & \xrightarrow{i} & V^\vee.
\end{array}
$$

(1.27)

The following chain of equivalences proves the theorem.

(1.28) \quad \quad pt_* \mathcal{M}_A^{Z(f)} [-\text{codim.vir } Z(f)] \simeq pt_* i^* \mu_{Z(f)/X}(\mathcal{E}_X)

(1.29) \quad \quad \simeq pt_* \pi'_* i^! \mu_{Z(f)/X}(\mathcal{E}_X)

(1.30) \quad \quad \simeq pt_* i^! \pi_* \mu_{Z(f)/X}(\mathcal{E}_X)

(1.31) \quad \quad \simeq pt_* i^! \pi_* T_f^! \mu_{Z(f)/X}(\mathcal{E}_X)

(1.32) \quad \quad \simeq pt_* i^! \pi_* T_f^! \mathcal{F}(\text{Sp}_Z(f)/X(\mathcal{E}_X))

(1.33) \quad \quad \simeq pt_* i^! \mathcal{F}(pr_* T_f, \text{Sp}_Z(f)/X(\mathcal{E}_X))

(1.34) \quad \quad \simeq pt_* i^! \mathcal{F}(N_f, \text{Sp}_Z(f)/X(\mathcal{E}_X))

(1.35) \quad \quad \simeq pt_* i^! \mu_{Z(f)/X}(f_* \mathcal{E}_X)

(1.36) \quad \quad =: M_f^A.

Theorem 6.1 implies the first equivalence, (1.28); base change along diagram (1.27) gives (1.30); (1.31) follows from $T_f^!$ being the identity; and (1.35) follows from Lemma 4.4. \hfill \Box

**Corollary 1.37.** Conjecture 1.23 is true in the case when $f$ is proper.

1.5. **Structure of the paper.** The content of this paper is organized as follows.

In Section 2, we give an account of sheaf theory in the modern setting. Care is taken to justify the use of classical results in the theory of sheaves on locally compact spaces. The reader is reminded of various elements of the sheaf theory needed for the rest of the paper. The theory of constructible sheaves in the $\infty$-categorical setting is well-known to experts, but the author could not find a systematic account of it elsewhere in the literature.

In Section 3, we recall from [KR19] the construction of deformation to the normal bundle along a quasi-smooth closed immersion. We prove several properties about the construction in preparation for next section on the functors of quasi-smooth specialization and microlocalization.

In Section 4, we define the functors of quasi-smooth specialization and microlocalization. We establish some basic properties expected of the quasi-smooth specialization, such as base change and a lemma on restriction to the zero section. The classical notions of Verdier specialization and the real analytic specialization of [KS90] are also discussed and compared.

In Section 5, we state and prove one of the main results of this paper, Theorem 5.1.

Finally, in Section 6, we state and prove the other main result of this paper, Theorem 6.1 after recalling the basic concepts involved in its statement.
1.6. **Acknowledgments.** I am indebted to my graduate advisors, David Ben-Zvi and Sam Raskin; this paper would not exist without them. David first suggested I pursue the subject of the microlocal homology in my research as a graduate student. Meanwhile, Sam made the initial suggestion to look for a notion of microlocalization intrinsic to a quasi-smooth scheme, and later conjectured Theorem 6.1. Both of them have been extremely generous to me with their time and energy over the years, and I am truly grateful for them.

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1.7. **Notation and conventions.**

1.7.1. **Base ring and sheaf coefficients.** For the remainder of this paper, \( k := \mathbb{C} \) and \( e \) is as states in Theorem 1.4. This choice of notation was made to suggest that results analogous to those we prove should be true for sheaf theories other than constructible sheaves with \( e \) coefficients—such as \( \ell \)-adic sheaves, \( \mathcal{D} \)-modules, mixed Hodge modules, etc.—on schemes over a base other than \( (k :=) \mathbb{C} \) (see Discussion 1.3 above).

1.7.2. **\( \infty \)-categories.** We freely use the language of \( \infty \)-categories throughout the paper, and take Joyal’s quasi-categories (i.e. weak Kan complexes) as our particular model. Unless otherwise specified, our notation and conventions for the theory of \( \infty \)-categories and related subjects are taken to be those appearing in the canonical references, [HTT] and [HA].

1.7.3. **Derived algebraic geometry.** The theory of derived algebraic geometry is indispensable for the formulation of the results of this paper. There are three choices for a model of affine derived schemes over a commutative ring \( R \): (the opposite category of) simplicial commutative rings, edgas, and \( \mathbb{E}_\infty \)-algebras. If \( R \) contains the field of rational numbers, then all three choices give rise to the same theory of derived algebraic geometry.\(^9\) We will always work over the base ring \( \mathbb{C} \) in this paper, so our choice of local model is immaterial. Nonetheless, in order to access the results of [SAG] more directly, we take (the opposite category of) \( \mathbb{E}_\infty \)-algebras (i.e. \( \mathbb{E}_\infty \)-algebra objects in spectra) as our model for affine derived schemes. In light of the above commentary, we use the terms “spectral scheme” and “derived scheme” interchangeably.

1.7.4. **Miscellanea.** The following is a list of miscellaneous notation and conventions used in this paper, but which have no natural home in the main body of the text.

- Boldface letters such as \( X, Y, Z \) always denote derived schemes or stacks; likewise, boldface letters such as \( f, g, h \) always denote morphisms of derived schemes or stacks. The corresponding plain letters, \( X, Y, Z \) and \( f, g, h \), denote the underlying classical schemes and morphisms of underlying classical

---

\(^9\)See the introduction of [SAG] ch. 25] for a detailed comparison of the three choices.
schemes, respectively, of their boldface counterparts, when the meaning is clear from context.

- We denote by \( pt_X : X \to * \) the terminal map to a point from the topological space \( X \). We omit the subscript when \( X \) is clear from context.

- For \( * = \{ > 0, \geq 0, < 0, \leq 0 \} \), \( \mathbb{A}_1^* \) denotes the subset of \( \{ z \in \mathbb{C} | \Re(z) * \} \in \mathbb{C} \), consisting of complex numbers whose real parts are *.

- For \( * = \{ > 0, \geq 0, < 0, \leq 0 \} \), \( \ell_* \) denotes the subset \( \mathbb{R}_+ \subset \mathbb{C} = \mathbb{A}_1^1 \).

- \( \mathbb{R}^+ \) denotes the multiplicative group of strictly positive real numbers.

- If \( \mathcal{C} \) is a small, ordinary category, it can be viewed as an \( \infty \)-category by taking the nerve, \( N(\mathcal{C}) \), which gives a simplicial set. By abuse of notation, we often use \( \mathcal{C} \) to refer to the simplicial set, \( N(\mathcal{C}) \).

2. Sheaf theory

2.1. Sheaves valued in \( \text{Mod}_* \). The main reference for sheaves of spectra is [SAG]. Further details or justifications for unproven technical assertions in this subsection can be found there.

2.1.1. Suppose that \( X \) is a complex analytic space, locally of finite type. By abuse of notation, \( X \) will denote both the complex analytic space and its underlying topological space. Denote by \( \mathcal{U}(X) \) the partially ordered collection of open sets of \( X \), viewed as a category.

2.1.2. The category of sheaves. Let \( \text{Mod}_* \) be the \( \infty \)-category of \( e \)-module objects in the category of spectra and consider the category of \( \text{Mod}_* \)-valued presheaves on \( X \),

\[ \text{PrShv}(X; e) := \text{Fun}(\mathcal{U}(X)^{\text{op}}, \text{Mod}_*) \].

**Definition 2.1.** The category of \( e \)-sheaves on \( X \), denoted by \( \text{Shv}(X; e) \), is defined as the full subcategory of \( \text{Fun}(\mathcal{U}(X)^{\text{op}}, \text{Mod}_*) \) spanned by those functors \( \mathcal{F} \) which satisfy the following descent condition for open covers:

\((*)\) Let \( U \in \mathcal{U}(X) \) and let \( \{ U_\alpha \to U \} \) be a covering of \( U \), then the canonical map,

\[ \mathcal{F}(U) \to \lim_{\leftarrow V} \mathcal{F}(V) \]

is an equivalence in \( \text{Mod}_* \), where \( V \) ranges over all opens contained in \( U \) whose inclusion factors through some \( U_\alpha \).

**Remark 2.2.** The above definition makes sense for any topological space, not just those underlying a complex analytic space.

We may also define more generally the category of sheaves on \( X \) valued in an arbitrary \( \infty \)-category \( \mathcal{C} \), denoted \( \text{Shv}(X; \mathcal{C}) \), using the same descent condition (c.f. [SAG] Definition 1.1.2.1)). An important example for us will be the category of spectral sheaves on \( X \), \( \text{Shv}(X; \mathcal{S}) \). The stalk at a point and the support of a sheaf have the same definitions as in the setting of ordinary categories.

**Definition 2.3.** Suppose that \( \mathcal{F} \in \text{Shv}(X; \mathcal{C}) \), where \( \mathcal{C} \) is a stable \( \infty \)-category. The stalk at the point \( x \in X \), denoted \( \mathcal{F}_x \), is defined to be

\[ \mathcal{F}_x := \lim_{\{ U \in \mathcal{U}(X) | x \in U \}} \mathcal{F}(U) \].

\[^{10}\text{See} \ [\text{HA} \ §3.3] \text{for a general discussion of modules over an algebra object.} \]
The support of $\mathcal{F}$, denoted $\text{supp}\mathcal{F}$, is defined to be

$$\text{supp}\mathcal{F} := \{ x \in X | \mathcal{F}_x \neq 0 \}$$

The stalk and support of a section $s \in \mathcal{F}(U)$ are defined similarly.

**Remark 2.4.** The category $\text{Shv}(X; e)$ has an alternative description as the category of $e$–module objects in spectral sheaves, where $e$ is the constant sheaf on $X^{an}$ with stalk $e$, viewed as a discrete $E_\infty$–ring.

2.1.3. $t$-structures. If $\mathcal{C}$ is given as the stabilization of a presentable $\infty$-category $\mathcal{C}'$, there is a standard $t$-structure on $\mathcal{C}$ defined by the property that $\mathcal{C}_{<0}$ is the fiber of the “infinite loop space” functor $\mathcal{C} \simeq \text{Sp}(\mathcal{C}') \to \mathcal{C}'$. The category of $\mathcal{C}$-valued sheaves on $X$ is itself the stabilization of $\text{Shv}(X; \mathcal{C}')$ (c.f. [SAG, Remark 1.3.2.2]), so $\text{Shv}(X; \mathcal{C})$ inherits a standard $t$-structure.

[SAG, Proposition 1.3.2.7] states the resultant $t$-structure on spectral sheaves on $X$ (or more generally, on any $\infty$-topos), is compatible with filtered colimits (i.e. a filtered colimit of coconnective objects is coconnective) and right-complete. The $n$th homotopy sheaf of an object $\mathcal{F} \in \text{Shv}(X; \text{Sp})$, $\pi_n(\mathcal{F})$, may be identified ( [SAG, Example 1.3.2.4]) with the sheaf of abelian groups given by sheafifying the presheaf

$$U \mapsto \pi_n(\mathcal{F}(U)).$$

Because $e$ is connective as a sheaf of spectral rings, the category of $e$–sheaves, $\text{Shv}(X; e)$, admits a $t$-structure defined by letting $(\text{Shv}(X; e))_{\leq 0}$ be the inverse image of $(\text{Shv}(X; \text{Sp}))_{\leq 0} \subset \text{Shv}(X; \text{Sp})$ under the forgetful functor $\text{Shv}(X; e) \to \text{Shv}(X; \text{Sp})$. Like the $t$-structure on $\text{Shv}(X; \text{Sp})$, it is right complete and compatible with filtered colimits by [SAG, Proposition 2.1.1.1]. The heart of this $t$-structure can be identified with the abelian category of $e$–modules in sheaves of sets ([SAG, Remark 2.1.2.1]), meaning the homotopy sheaves of an object $\mathcal{F} \in \text{Shv}(X; e)$ have the additional structure of $e$–modules.

2.2. Hyperdescent. The classical results of sheaves on locally compact spaces rely on the property that a map in the derived category (of complexes of sheaves) is an equivalence if and only if the induced map on stalks is an equivalence at each point; and if and only if it induces isomorphisms on cohomology sheaves. While the latter property essentially defines the derived category, on an arbitrary $\infty$-topos, it is not always the case that the category of sheaves valued in an arbitrary stable $\infty$-category enjoys these properties.

2.2.1. By reducing to the case of sheaves of spaces on $X$, we show that

**Lemma 2.5.** In the category $\text{Shv}(X; e)$, a map is an equivalence:

(i) iff it is $\infty$-connective (i.e. induces isomorphisms on all homotopy sheaves).

(ii) iff the induced map on stalks is an equivalence at each point in $X$.

Both properties are consequences of the $\infty$-topos of sheaves of spaces on $X$ being hypercomplete.\[11\]

2.2.2. For a reasonable class of topological spaces—those which are suitably ‘finite’ in some sense—$\text{Shv}(-; \mathcal{S})$ is automatically hypercomplete. Fortunately, among them are finite dimensional complex analytic spaces locally of finite type, as we now show in the next two lemmas.

\[11\]See [HTT §6.5] for a thorough discussion of hypercompleteness.
Lemma 2.6. Any closed subspace $Z \subset S$ of a topological space $S$ has covering dimension less than or equal to that of $S$.

Proof. Take any open cover $\{U_i\}$ of the subspace $Z$. Each $U_i = U'_i \cap Z$ for some open $U'_i \subset S$. Complete $\{U'_i\}$ to an open cover of $S$ by adding the complement $S \setminus Z$ to the collection. Let $n$ be the covering dimension of $S$, and take a refinement, $\{V'_i\}$, of the completed cover of $S$ such that the $(n+2)$–fold intersection of pairwise disjoint members of $\{V'_i\}$ are empty. Now take collection $\{V'_i \cap Z\}$. This is clearly an open cover of $Z$, and $(n+2)$–fold intersections of pairwise disjoint members of $\{V'_i \cap Z\}$ are clearly empty. Finally, it is a refinement of $\{U_i\}$: take any nonempty element of the cover, $V'_i \cap Z$. Necessarily, $V'_i \subset U'_j$ for some $j$, since otherwise it would be contained in $S \setminus Z$, and $V'_i \cap Z = \emptyset$. Thus, $V'_i \cap Z \subset U_j$. \hfill $\square$

Lemma 2.7. Suppose $X$ is a finite dimensional complex analytic space, locally of finite type. Then the $\infty$-topos $\text{Shv}(X; S)$ is hypercomplete.

Proof. By [HTT] Corollary 7.2.1.12, it suffices to show that $\text{Shv}(X; S)$ is locally of homotopy dimension $\leq n$ for some integer $n$. We recall that this means that there exists a collection of objects $\{F_n\}$ in $\text{Shv}(X; S)$ which generate $\text{Shv}(X; S)$ under colimits, such that $\text{Shv}(X; S)/F_n$ is of homotopy dimension $\leq n$.

An obvious collection of objects which generates $\text{Shv}(X; S)$ is the collection of delta sheaves, $\{\underline{s}_U\}$, where $U$ ranges over $\mathcal{U}(X)$. Observe that $\text{Shv}(X; S)/\underline{s}_U \simeq \text{Shv}(U; S)$ via the adjoint pair $(j_U, j_U^*)$ induced by the inclusion $j_U$, so it suffices to show that $\text{Shv}(U; S)$ is of homotopy dimension $\leq n$ for any open subset $U$.

In fact, it suffices to show that $\text{Shv}(U; S)$ is of homotopy dimension $\leq n$ for some chosen open cover $\{U_i\}$ of $X$ because if $U \subset U'$, the homotopy dimension of $\text{Shv}(U'; S)$ is less than or equal to that of $\text{Shv}(U'; S)$, and $\text{Shv}(U'; S)$ is $n$-connective, $\mathcal{F}' := (U \hookrightarrow U')^* \mathcal{F}$ is $n$-connective since $(U \hookrightarrow U')$ is left exact. If $\mathcal{F}'$ admits a global section $s|_{U'} \rightarrow \mathcal{F}'$, then $s|_{U} = (U \hookrightarrow U')^* s|_{U'} \rightarrow (U \hookrightarrow U')^* \mathcal{F}' \simeq \mathcal{F}$ is a global section of $\mathcal{F}$.

Note that $X$ locally takes the form $\mathcal{O}_{\mathbb{C}^n}/(f_1, \ldots, f_m)$ for some natural numbers $n \leq \dim X$ and $m$. Choose an open cover $\{U_i\}$ of $X$ on each member of which $X$ has such a presentation. Each open subset $U_i$ of our cover, embeds into $\mathbb{C}^n_i$ as the intersection of an open subset $V \subset \mathbb{C}^n_i$, and the closed analytic subset given by

$$\{f_1 = \cdots = f_m = 0\}.$$  

By [HTT] Theorem 7.2.3.6, $\text{Shv}(U_i; S)$ is of homotopy dimension $\leq \dim X$ if $U_i$ has covering dimension $\leq \dim X$, but this follows immediately from ??, since any open subset of $\mathbb{C}^n_i$ is a smooth manifold of dimension $n < \dim X < \infty$, and the covering dimension of smooth manifolds is their usual dimension. \hfill $\square$

2.2.3. We now prove Lemma 2.5(i) and (ii). In each case, we only prove the “if” direction since the “only if” direction is obvious.

Proof of Lemma 2.5. Suppose $\mathcal{F} \xrightarrow{\pi_i} \mathcal{F}' \in \text{Shv}(X; e)$ were an $\infty$-connective morphism, i.e. $\pi_i \mathcal{F} \simeq \pi_i \mathcal{F}'$ for every integer $i$. Since the forget functor $\text{oblv} : \text{Shv}(X; e) \to \text{Shv}(X; \text{Sp})$ is conservative and preserves small colimits ([SAG] Proposition 2.1.0.3), $\text{oblv}(\alpha)$ is also $\infty$-connective; and it suffices to show that $\text{oblv}(\alpha)$ is an equivalence. By abuse of notation we refer to $\text{oblv}(\mathcal{F})$ as $\mathcal{F}$. Now the induced map $\Omega^\infty \mathcal{F} \to \Omega^\infty \mathcal{F}'$ is $\infty$-connective for each integer $n$ by design, so it is an
equivalence by hypercompletion using Lemma 2.7. Combining [SAG, Proposition 1.3.3.3] and [SAG, Remark 1.1.3.3], we obtain (i).

Now suppose $\alpha: F \to F' \in \text{Shv}(X; e)$ were a morphism such that the induced maps of stalks, $F_x \to F'_x$, were equivalences for all $x \in X$. Recall the definition of the stalk at $x$ as the filtered colimit of sections over open sets containing $x$:

$$F_x := \lim_{x \in U} F(U).$$

Because the forgetful functor to spectral sheaves is conservative and preserves small colimits, $\text{oblv}(\alpha)$ also induces isomorphisms on stalks, and it suffices to prove that that $\text{oblv}(\alpha)$ is an equivalence. As remarked earlier, the $t$-structure on spectra is compatible with filtered colimits. This is equivalent via the fiber sequence (for $S \in \text{Sp}$),

$$\tau_{\geq 1} S \to S \to \tau_{\leq 0} S,$$

to the truncation functor $\tau_{\leq 0}$ preserving filtered colimits. The functor $\tau_{\geq 0}$ is a left adjoint, so it preserves all colimits; therefore, $\pi_0 S := \tau_{\geq 0} \tau_{\leq 0} S$ preserves filtered colimits, as well as $\pi_i$ for all integers $i$.

In particular, we obtain that

$$\lim_{x \in U} \pi_i(F(U)) \xrightarrow{\cong} \pi_i(\lim_{x \in U} F(U)) =: \pi_i(F_x).$$

The left-hand side of this map is, by definition, the stalk of the presheaf whose sheafification is $\pi_i F$. Sheafification preserves stalks as a left adjoint, so the left-hand side is also $(\pi_\alpha F)_x$.

Thus, we obtain that the maps

$$\pi_i F \xrightarrow{\pi_i(\alpha)} \pi_i F'$$

induce isomorphisms on stalks, for all integers $i$. Since each $\pi_i F$ is a sheaf of sets on a topological space, $\pi_\alpha$ is an isomorphism for each $i$. By property (i), proven above, $\text{oblv}(\alpha)$ is an equivalence, and we have proven (ii).

2.2.4. Later in this paper, we will need to consider sheaves on closed subspaces of manifolds, $Z \subset M$. The following lemma is an immediate consequence of Lemma 2.6 and implies that the conclusions of Lemma 2.5 hold for $\text{Shv}(Z; e)$.

**Lemma 2.8.** Suppose that $Z$ is a closed subspace of $\mathbb{R}^n$ for some value of $n$. Then the $\infty$-topos $\text{Shv}(Z; S)$ is hypercomplete.

2.3. **Bounded variations.** We may also consider sheaves valued in the category of $e$–modules objects in one of the bounded variations of spectra—eventually connective, eventually coconnective, or bounded spectra: $\text{Mod}_e^+$, $\text{Mod}_e^-$, or $\text{Mod}_e^b$. We denote the resultant categories by $\text{Shv}^+(X; e)$, $\text{Shv}^-(X; e)$, and $\text{Shv}^b(X; e)$, respectively. The results of the previous sections all hold mutatis mutandis for these variants. The most important category for us will be $\text{Shv}^b(X; e)$, which we call the **bounded derived $\infty$-category of $e$-sheaves** on $X$. This terminology is justified on the ground that the homotopy category of $\text{Shv}^b(X; e)$ (with its standard $t$-structure) is the usual bounded derived category $D^b(X; e)$, where $* = +, -, b$.

**Remark 2.9.** Suppose that $X$ is a classical scheme, locally of finite type over $k$. Associated to $X$ is the topological space $X^{an}$, defined by endowing the set of closed points of $X$ with the complex analytic topology. One may further define
the analytification of $X$ by introducing a particular sheaf of rings $\mathcal{O}_{X^{an}}$ on $X^{an}$ in such a way that the ringed space $(X^{an}, \mathcal{O}_{X^{an}})$ is a complex analytic space, though we will not need details of this construction other than to note that the resulting complex analytic space is also locally of finite type.\textsuperscript{12} In this case, we denote the corresponding categories of $e$-sheaves on $X^{an}$ by $\text{Shv}^e(X; e)$, as well as refer to them as categories of sheaves $X$ rather than on $X^{an}$. If $X$ is finite dimensional, so is its associated complex analytic space. In that case, Lemma 2.5 holds for $\text{Shv}^e(X; e)$.

**Definition 2.10.** If $X$ is a derived $k$-scheme whose underlying classical scheme $X$ is locally of finite type, we define

$$\text{Shv}(X; e) := \text{Shv}(X; e)$$

2.4. **Six functor formalism.** In the sheaf theory context we have just defined, there is the usual Grothendieck six functor formalism.

2.4.1. Let $f : X \to Y$ be a map of locally compact topological spaces (not necessarily complex analytic spaces). Then there exist the following functors,

1. $f_* : \text{Shv}(X; e) \to \text{Shv}(Y; e)$
2. $f_! : \text{Shv}(X; e) \to \text{Shv}(Y; e)$
3. $f^* : \text{Shv}(Y; e) \to \text{Shv}(X; e)$
4. $f^! : \text{Shv}(Y; e) \to \text{Shv}(X; e)$

There are, additionally, two functors

5. $\boxtimes : \text{Shv}(X; e) \times \text{Shv}(Y; e) \to \text{Shv}(X \times Y; e)$
6. $\text{Hom}(-, -) : \text{Shv}(X; e) \times \text{Shv}(X; e) \to \text{Shv}(X; e)$,

which round out the list of the six functors.

**Notation 2.11.** If $S \hookrightarrow X$ is the inclusion of a subspace $S$ and $\mathcal{F} \in \text{Shv}(X; e)$, we sometimes use “$\mathcal{F}_S^*$” or “$\mathcal{F}_S^!$” to denote the $*$-pullback $(S \hookrightarrow X)^* \mathcal{F}$, and “$\mathcal{F}_S^!$” to denote the $!$-pullback $(S \hookrightarrow X)^! \mathcal{F}$.

2.4.2. $f^*$ and $f_*$. We briefly describe the existence of functors $\boxtimes$ and $\boxminus$. The push-forward $f_*$ of presheaves is defined by precomposition with the functor of $\infty$-sites, $\mathcal{U}(Y) \to \mathcal{U}(X)$, induced by the map $f$. It admits a left adjoint, given by left Kan extension along $f$, which we denote by $f^!$. By [PYY16, Lemma 2.13], $f_*$ preserves sheaves (like one would expect from the classical definition), so we define,

$$f_* := f_* \circ \iota_X$$
$$f^* := L_X f_*^! \circ \iota_Y,$$

where $\iota_X$ and $\iota_Y$ denotes the inclusions of sheaves into presheaves and where $L_X$ is sheafification. By [PYY16, Lemma 2.14],

$$f^* : \text{Shv}(Y; e) \rightleftharpoons \text{Shv}(X; e) : f_*$$

are also adjoints. See [PYY16, §2.4] for more details about the construction and properties of these functors.

\textsuperscript{12} Details of the complex analytic topology and the analytification of $X$ may be found in e.g. [N07].
2.4.3. $f_!$ and $f^!$. We briefly sketch the construction of functors $f_!$ and $f^!$. For an individual sheaf $\mathcal{F} \in \text{Shv}(X; e)$, $f_! \mathcal{F}$ is defined as the subsheaf of $f_* \mathcal{F}$ given by the formula,

$$f_! \mathcal{F}(U) := \{ s \in \mathcal{F}(f^{-1}(U)) \mid \text{supp } s \to Y \text{ is proper} \}.$$

Given a map $\mathcal{F} \to \mathcal{F}' \in \text{Shv}(X; e)$, we obtain the map $f_\ast \mathcal{F} \to f_\ast \mathcal{F}'$ by functoriality; it is easy to see that the image of the restriction of this map to the subsheaf $f_! \mathcal{F}$ is the subsheaf $f_! \mathcal{F}'$. From the functoriality of $f_\ast$, we therefore obtain the $\infty$-functor $f_!$. The functor $f_!$ is defined as the right adjoint to $f^!$. Its existence can be shown either by imitating the construction in [KS90, §3.1] in an $\infty$-setting or by inferring its existence from the Adjoint Functor Theorem [HA, Corollary 5.5.2.9] after showing $f_\ast$ preserves small colimits.

2.4.4. Tensor product. Using the first four functors, we may also define a tensor product of sheaves using the functor $- \otimes -$. It may be defined using the diagonal map $X \to X \times X$ in one of two inequivalent ways.

**Definition 2.12.** The $\ast$-tensor product is defined as,

$$- \ast - := \Delta^\ast (- \boxtimes -).$$

The $!$-tensor product is defined as,

$$- ! - := \Delta^! (- \boxtimes -).$$

Each of these functors endows $\text{Shv}(X; e)$ with a symmetric monoidal structure, which are generally not equivalent to each other. The practical difference between the two tensor products is that $f^\ast$ is symmetric monoidal for $\otimes$, while $f_!$ is symmetric monoidal for $\boxtimes$. The $\ast$-tensor product coincides with the canonical tensor product of module spectra (c.f. [SAG, §2.1]). For simplicity, we use the undecorated symbol, $\otimes$, to refer to the $\ast$-tensor product, and refer to it simply as the “tensor product of sheaves.”

**Remark 2.13.** While the exterior product, $- \boxtimes -$, is the more primordial functor, the tensor product is usually taken as one of the Grothendieck six functors instead.

2.4.5. The functor $\text{Hom}$ should not be confused with the space of maps in the $\infty$-category $\text{Shv}(X; e)$, which is denoted by $\text{Hom}$. In general, $\text{Hom}(-, -)$ is always the space of maps between two objects in a given category while $\text{Hom}(-, -)$ is always a sheaf.

The relation between the two for the category $\text{Shv}(X, e)$ is as follows: global sections of $\text{Hom}(\mathcal{F}, \mathcal{G})$ is $\text{Hom}(\mathcal{F}, \mathcal{G})$. $\text{Hom}$ is the internal hom functor for $\text{Shv}(X; e)$.

2.4.6. Adjunction triangles. Given a complementary pair of open and closed subsets $Z \hookrightarrow X \overset{i}{\leftarrow} U$, there exists the following two canonical fiber sequences, which we call the “adjunction triangles”:

$$i ji^! \mathcal{F} \to \mathcal{F} \to j_\ast j^! \mathcal{F}$$

$$j_\ast j^! \mathcal{F} \to \mathcal{F} \to i_! i^* \mathcal{F}.$$
2.5. **Accessing classical results in the ∞-categorical setting.** The canonical reference for the theory of sheaves on manifolds (or more generally, locally compact spaces) is the tome [KS90]. It and other standard references (e.g. [Dim04]) on the subject were written during a time when the theory of ∞-categories and sheaves valued therein were either not established or not adopted in widespread use. Rather, these references work in the (left, right, or bounded) derived category of complexes of sheaves.

Since then, the treatment of ∞-categories in the literature has matured immensely. Much recent work in the subject of microlocal sheaf theory and symplectic geometry takes place in the setting of sheaves valued, not just in certain stable ∞-categories, but in the category of ∞-categories itself. The toolbox of (microlocal) sheaf theory, however, continues to be accessed by referring to the work of Kashiwara, Schapira. Usually these references are accompanied by a short note assuring the reader that the same definitions, theory, and results hold in the ∞-categorical setting. This is one such note.

Section 2.2 established the ability to check equivalences in the category Shv(\(X; e\)) by checking on stalks or homotopy sheaves. Section 2.4 established the existence of the six functors in our setting. We make the meta-claim that the results of [KS90] from this ability and these six functors. As such, in the rest of this paper, we make reference to results in [KS90] and other classical sources with clear conscience and without further comment, whenever the corresponding ∞-categorical result is obvious.

2.6. **Constructible sheaves.** In this subsection we define the category of constructible sheaves, and recall its relevant properties.

**Definition 2.16.** Let \(M\) be an arbitrary object of Mod\(_e^b\), viewed as a sheaf on the point. Then the constant sheaf on \(X\) with value \(M\), denoted by \(M_X\), is defined as \((X_{\text{an}} \to * )^* M\).

**Definition 2.17.** A sheaf \(\mathcal{F} \in \text{Shv}(X; e)\) is locally constant if there exists a module \(M \in \text{Mod}_e^b\) and an open cover of \(X_{\text{an}}, \{U_\alpha\}\), such that \(\mathcal{F}|_{U_\alpha} \simeq M_{X_\alpha}\).

**Definition 2.18.** A sheaf \(\mathcal{F} \in \text{Shv}(X; e)\) is constructible if there exists a locally finite stratification of \(X\) by locally closed complex analytic subsets \(X_\alpha\) having the property that, for each index \(\alpha\), there exists a perfect \(e\)-module \(M_\alpha\) such that \(\mathcal{F}|_{X_\alpha}\) is locally constant with value \(M_\alpha\).

**Warning 2.19.** When \(X\) is a \(k\)-scheme locally of finite type, it is typical to use a narrower definition of constructible sheaf, considering only stratifications which are algebraic, i.e. in which the strata are algebraic subsets, rather than arbitrary analytic subsets. We do not adopt this convention here; we instead refer to the latter as “algebraically constructible” sheaves. We denote the category of such by Shv\(^{alg}\)(\(X\)).

**Definition 2.20.** We use Shv(\(X\)) to denote the full subcategory of Shv\(^h\)(\(X; e\)) spanned by constructible objects and call it the bounded derived category of constructible sheaves on \(X\).

**Remark 2.21.** Shv(\(X\)) is also a stable ∞-category.

\[^{13}\text{c.f. [Dim04] Remark 4.1.2(i)}\]
2.6.1. **Constructibility and the six functors.** The functors
\[- \otimes - , \mathcal{H}om(-, -)\]
preserve constructibility. Consider a map of complex analytic spaces, \(g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\). In general, the functors,
\([g^*, g^!]\)
each preserve constructibility. If, additionally, \(g\) is proper on \(\text{supp}(\mathcal{F})\), then the sheaves
\([f_* \mathcal{F}, f! \mathcal{F}]\)
are also constructible. In particular if \(g\) is a proper map, all six functors preserve \(\text{Shv}(X)\). The reference for these statements is \cite[Theorem 4.1.5]{Dim04}.

2.6.2. **Verdier duality.** Consider the terminal map from a scheme, \(\text{pt} : X \to *\).

**Definition 2.22.** The **dualizing sheaf** on \(X\) is defined as:
\[\omega_X := \text{pt}!e\]

The dualizing sheaf is used to define the following important functor.

**Definition 2.23.** The functor of **Verdier duality**, \(D: \text{Shv}(X; e) \to \text{Shv}(X; e)^{\text{op}}\), is defined by the formula,
\[DF := \mathcal{H}om(\mathcal{F}, \omega_X)\]

Below are the most immediate and relevant properties of Verdier duality.

(i) Verdier duality has the property that
\[D \circ D \simeq \text{id}_{\text{Shv}(X; e)}\]  
It is therefore a sort of anti-involution.

(ii) It follows immediately from the definition that Verdier duality also reverses shifts:
\[D(\mathcal{F}[n]) \simeq (D\mathcal{F})[-n],\]
for all \(\mathcal{F} \in \text{Shv}(X; e)\) and \(n \in \mathbb{Z}\).

(iii) \cite[Theorem 4.1.16]{Dim04} Verdier duality preserves constructibility, i.e. \(D: \text{Shv}(X) \to \text{Shv}(X)^{\text{op}}\).

(iv) \cite[Corollary 4.1.17]{Dim04} When \(f : X \to Y\) is map of schemes and \(\mathcal{F} \in \text{Shv}^{\text{alg}}(X)\) is an algebraically constructible sheaf,
\[f_!(D\mathcal{F}) \simeq D(f_* \mathcal{F}).\]

If \(f : X \to Y\) is a map of complex analytic spaces and \(\mathcal{F} \in \text{Shv}(Y)\),
\[f^*(D\mathcal{F}) \simeq D(f^! \mathcal{F}).\]

2.6.3. **Perverse sheaves.** Recall that a \(t\)-structure on a stable \(\infty\)-category, \(\mathcal{C}\), is a \(t\)-structure on its homotopy category, \(h\mathcal{C}\). \(h\text{Shv}(X)\) is the usual (i.e. 1–categorical) derived category of constructible sheaves defined, for example, in \cite[§4.1]{Dim04}, which we denote by \(D^b_c(X; e)\). Thus, we define the perverse \(t\)-structure on \(\text{Shv}(X)\) to be the usual perverse \(t\)-structure on \(D^b_c(X; e)\) for the middle perversity.

**Definition 2.24.** The category of **perverse sheaves** on \(X\), denoted by \(\text{Perv}(X)\), is the heart of the perverse \(t\)-structure on \(\text{Shv}(X)\).
2.7. Nearby and vanishing cycles. The six-functor formalism allows us to define the functors of nearby and vanishing cycles on Shv(\(X\)).

We choose the exponential map, \(\exp : \hat{\mathbb{G}}_m \to \mathbb{G}_m\), as a model for the universal cover of the punctured complex plane. Let \(X\) be a complex analytic space and \(f : X \to \mathbb{A}^1\) be a complex analytic function. We have the following diagram:

\[
\begin{array}{cccc}
X_0 & \xleftarrow{i} & X & \xleftarrow{j} & X^* & \xleftarrow{\tilde{\pi}} & \hat{X}^* \\
\downarrow{j} & & \downarrow{f} & & \downarrow{\tilde{\pi}} & & \downarrow{\tilde{f}} \\
0 & \xrightarrow{c} & \mathbb{A}^1 & \xleftarrow{\exp} & \mathbb{G}_m & \xrightleftharpoons{\sim} & \hat{\mathbb{G}}_m
\end{array}
\]

**Definition 2.26.** The nearby cycles functor with respect to \(f\), denoted by \(\psi_f : \text{Shv}(X; e) \to \text{Shv}(X_0; e)\), is defined by the formula,

\[
\psi_f F := i^* (j \circ \tilde{\pi})_* (j \circ \tilde{\pi})^* F.
\]

There is an obvious natural map,

\[
i^* F \xrightarrow{\sim} \psi_f F,
\]

induced by the unit of the \((j \circ \tilde{\pi})_* \dashv (j \circ \tilde{\pi})^*\) adjunction.

**Definition 2.28.** The vanishing cycles functor with respect to \(f\), denoted by \(\varphi_f : \text{Shv}(X; e) \to \text{Shv}(X_0; e)\), is defined to be the fiber of the natural map \(c\),

\[
\varphi_f F \to i^* F \to \psi_f F.
\]

**Remark 2.30.** Our definition of the vanishing cycles functor agrees with the one defined in \([\text{KS90}, \S 8.6]\), and differs from the ones in \([\text{Dim04}, \S 4.2]\) and \([\text{M16}, \S 3]\) by a shift of \([-1]\).

2.7.1. We record a few important properties of the nearby and vanishing cycles functors.

(i) \([\text{Dim04}, \text{pg. 103}]\) \(\psi_f\) and \(\psi_f\) preserve constructibility. In the case when \(X\) is a scheme, and \(f\) is an algebraic function, \(\psi_f\) and \(\psi_f\) preserve algebraic constructibility.

(ii) \([\text{Dim04}, \text{Theorem 5.2.21}]\) \(\psi_f[-1]\) and \(\varphi_f\) are \(t\)-exact with respect to the perverse \(t\)-structure.

(iii) \([\text{Dim04}, \text{Proposition 4.2.10}]\) \(\psi_f[-1]\) and \(\varphi_f\) commute with Verdier duality.

**Remark 2.31.** It is conventional to call the perverse sheaf obtained by applying \(\varphi_f\) to the perverse constant sheaf “the vanishing cycles along \(f\),” and denote it simply by \(\varphi_f(e[\dim X])\).

2.7.2. Monodromy automorphism. The nearby cycles functor comes equipped with a monodromy action of \(\mathbb{Z}\), natural in \(F\) and generated by the monodromy operator

\[
\psi_f F \xrightarrow{T} \psi_f F,
\]

which is induced by the action of the fundamental group of \(\mathbb{G}_m\) on the cover \(\hat{X}^*\). If \(i^* F\) is viewed as having a trivial monodromy action by \(\mathbb{Z}\), \(\varphi_f F\) also obtains a monodromy action as the fiber of the (trivially \(\mathbb{Z}\)-equivariant) map \(i^* F \xrightarrow{\sim} \psi_f F\) in the category of constructible sheaves valued in \(e[t, t^{-1}]\).
2.7.3. Nearby cycles at a fixed angle. In practice, there are equivalent expressions for $\psi f$ and $\varphi f$ called nearby and vanishing cycles at fixed angles which are helpful for computation. In order to state them, define:

$$H^\geq_\theta := f^{-1}(e^{i \theta} \mathbb{A}^1_{\geq 0})$$
$$H^\gt_\theta := f^{-1}(e^{i \theta} \mathbb{A}^1_{> 0})$$
$$\ell^\geq_\theta := f^{-1}(e^{i \theta} \mathbb{A}_1 \geq 0)$$
$$\ell^\gt_\theta := f^{-1}(e^{i \theta} \mathbb{A}_1 > 0),$$

and denote the inclusions into $X$ of each by:

$$I_\theta : H^\geq_\theta \hookrightarrow X,$$
$$J_\theta : H^\gt_\theta \hookrightarrow X,$$
$$i_\theta : \ell^\geq_\theta \hookrightarrow X,$$
$$j_\theta : \ell^\gt_\theta \hookrightarrow X.$$

The following equivalences hold:

(2.32) \[ \psi_f \mathcal{F} \simeq \psi_{\theta}^f \mathcal{F} \]

(2.33) \[ := (J_{\theta + \pi, \ast} J_{\theta + \pi, \ast}^\ast \mathcal{F})_{|f^{-1}(0)} \]

(2.34) \[ \simeq (j_{\theta, \ast} j_{\theta, \ast}^\ast \mathcal{F})_{|f^{-1}(0)} \]

(2.35) \[ \varphi_f \mathcal{F} \simeq \varphi_{\theta}^f \mathcal{F} \]

(2.36) \[ := (i_{\theta + \pi, \ast} \mathcal{F})_{|f^{-1}(0)} \]

(2.37) \[ \simeq (i_{\theta, \ast} \mathcal{F})_{|f^{-1}(0)} \]

Remark 2.38. One can show that there are natural isomorphisms

$$T_\theta^\psi : \psi_{\theta}^0 \mathcal{F} \xrightarrow{\sim} \psi_{\theta}^f \mathcal{F}$$
$$T_{2\pi}^\psi : \psi_{\theta}^0 \mathcal{F} \xrightarrow{\sim} \psi_{\theta}^f \mathcal{F}$$

which correspond to rotation counterclockwise over the origin in $\mathbb{A}^1$. Then $T_{2\pi}^\psi$ and $T_\theta^\psi$ correspond to the natural monodromy automorphisms on $\psi_f$ and $\varphi_f$ described in Section 2.7.2. See [K21b, §2.3] for the explicit construction of these isomorphisms (as well as proofs of many of the assertions in this section).

Remark 2.39. In this paper we only utilize expressions (2.33), (2.34), and (2.36) for the values $\theta = 0, \pi$.

2.8. Monodromic sheaves. We begin this section by recalling the definition of a monodromic sheaf, as first coined by Verdier in [V83b, Def. 3.1].

Let $X$ be a $k$-scheme of finite type, and $C$ a cone over $X$. Let $\theta : \mathbb{G}_m \times C \to C$ denote the scaling action on $C$, and for $p \in C$, let $W_p : \mathbb{G}_m \to C = \theta \circ (p \mapsto C)$ denote the orbit of $p$ under this action.

Definition 2.40. Let $\mathcal{F} \in \text{Shv}(C; e)$ be a sheaf on $C$. $\mathcal{F}$ is called monodromic if $W_p^\ast \mathcal{F}$ is locally constant for all values of $p$.

Notation 2.41. We denote the $\infty$-category of monodromic sheaves on $C$ by $\text{Shv}_{\mathbb{G}_m}(C; e)$.

---

\[14\] [M16] §3
In order to enumerate the properties of monodromic sheaves relevant to this work, we define the notions of equivariant maps and equivariant sheaves.

**Definition 2.42.** Suppose that $X$ and $Y$ are schemes with left $G$–actions, where $G$ is a group scheme. A map $f : X \rightarrow Y$ is **equivariant** if the following diagram commutes:

$$
\begin{array}{ccc}
G \times X & \xrightarrow{id \times f} & G \times Y \\
\downarrow{\text{act}_X} & & \downarrow{\text{act}_Y} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

**Definition 2.43.** Suppose that $X$ is a scheme with $G$–action, act : $G \times X \rightarrow X$. A **$G$-equivariant sheaf** is a sheaf on the stack quotient $X/G$ (i.e. an object $\mathcal{F} \in \text{Shv}(X/G; e)$). Alternatively, the category of $G$-equivariant sheaves on $X$ is the limit of the usual cosimplicial diagram,

$$
\text{Shv}(X; e) \xrightarrow{\pi^*} \text{Shv}(X \times_{X/G} X; e) \xrightarrow{\pi^*} \cdots,
$$

which we denote by $\text{Shv}^e_G(X; e)$.

We draw attention to the following well-known property of monodromic sheaves.

**Property 2.44.** Denote by $0_X : X \hookrightarrow C$ the inclusion of $X$ as the base of $C$, and by $\pi : C \rightarrow X$ the structure map of $C$ as a scheme over $X$. If $\mathcal{F} \in \text{Shv}_{G_m}(C; e)$, then

$$
\pi_! \mathcal{F} \simeq 0^*_X \mathcal{F}
$$

$$
\pi^* \mathcal{F} \simeq 0^!_X \mathcal{F}
$$

2.9. **Conic sheaves.** If we had taken $k = \mathbb{R}$, and left $C$ be any locally compact topological space with a $G_m = \mathbb{R}^+$–action, the above definition of monodromic sheaf coincides with what [KS90, §3.7] calls **conic sheaves**. Monodromic sheaves as we’ve defined them are obviously conic in this sense, and herein we will refer to such sheaves as either conic or monodromic according to whichever property we consider in the moment.

2.9.1. We denote the category of conic sheaves by $\text{Shv}_{\mathbb{R}^+}(-)$.

2.9.2. Conic sheaves on a real, finite dimensional vector space can be seen as sheaves on the much coarser topology consisting of open conic subsets.

**Definition 2.45.** Suppose $V$ is a real, finite dimensional vector space considered with the Euclidean topology. The collection of open conic subsets of $V$ is a topology which we call the **conic** topology on $V$. We denote the corresponding topological space by $V_{\text{con}}$.

There is an obvious continuous map $\varphi_{\text{con}} : V \rightarrow V_{\text{con}}$. Denote by $U_{\mathbb{R}^+}(V)$ the collection of conic open subsets of $V$, considered as elements of the topology on $V$.

**Proposition 2.46.** The functor $\varphi^*_{\text{con}} : \text{Shv}(V_{\text{con}}) \rightarrow \text{Shv}(V)$ induces an equivalence of categories between $\text{Shv}(V_{\text{con}})$ and $\text{Shv}_{\mathbb{R}^+}(V)$.
Proof. It suffices to show that the unit and counit of the \( \varphi_{con}^* \dashv \varphi_{con*} \) adjunction are equivalences. Suppose that \( F \in \text{Shv}_\mathbb{R}(V) \), and let \( x \in V \) be an arbitrary point. Then

\[
(\varphi_{con}^* \varphi_{con*} F)_x \to F_x
\]

\[
\lim_{\{x \in U \subseteq V_{con}\}} \Gamma(U; \varphi_{con*} F) \to F_x
\]

\[
\lim_{\{x \in U \mid U \in U_{\mathbb{R}}(V)\}} \Gamma(U; F) \cong F_x,
\]

where the last map is an equivalence because \( F \) is conic. Checking that the unit is an equivalence is done similarly. \( \square \)

2.9.3. The terminology and results of this section for conic sheaves on real f.d. vector spaces extend in an obvious way to sheaves on f.d. real vector bundles which are conic with respect to the scaling action on fibers. Proposition 2.46 will be used in section §3 to specify a map of conic sheaves by specifying maps of sections over each element of a basis for the conic topology on a vector bundle.

Remark 2.47. For sake of clarity later in the paper, we define a convex subset of a real vector bundle to be a subset whose intersection with any fiber is convex.

2.10. Fourier-Sato transform. Let \( X \) be a complex analytic space, and \( E \to X \) a complex vector bundle over \( X \). Denote by \( E^\vee \to X \) the (complex) dual vector bundle. Define the closed, subanalytic subset \( P \subset E \oplus E^\vee \) by

\[
P = \{(v, w) \in E \oplus E^\vee \mid \text{Re } w(v) \geq 0\}.
\]

Consider the following diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{i_P} & E \oplus E^\vee \\
\downarrow{pr_2} & & \downarrow{pr_1} \\
E^\vee & \leftarrow & E.
\end{array}
\]

Definition 2.48. ([V83a] §5, [KS90] §3.7) The Fourier-Sato transform is the functor \( \mathcal{F} : \text{Shv}_{\mathbb{G}_m}(E) \to \text{Shv}_{\mathbb{G}_m}(E^\vee) \) defined by the formula,

\[
\mathcal{F}(F) := \text{pr}_2 \circ i_P ! \circ \text{pr}_1^* F
\]

\[
\cong \text{pr}_2 \circ \Gamma_P (\text{pr}_1^* F)
\]

2.10.1. The above definition contains the hidden claim that \( \mathcal{F}(F) \) is monodromic, but this is easily seen from the formula as follows.

Proof that \( \mathcal{F}(F) \) is monodromic. Denote by \( \overline{E}^\vee \), the vector bundle whose fibers are the complex conjugates of the fibers of \( E^\vee \). Note that \( E^\vee \) and \( E^\vee \) have the same underlying topological space, \( |E^\vee| \). The orbit of \( p \in |E^\vee| \) under the complex conjugate action of \( \mathbb{G}_m \) is the same as under the original action of \( \mathbb{G}_m \). Thus, \( \text{Shv}_{\mathbb{G}_m}(E^\vee) \) and \( \text{Shv}_{\mathbb{G}_m}(\overline{E}^\vee) \) are identical subcategories of \( \text{Shv}(|E^\vee|) \).
Consider the bundle $E \oplus \overline{E}^\vee$ with the diagonal $\mathbb{G}_m$-action (i.e. its action as a vector bundle over $X$). The subset $P$ is invariant under this action, meaning we obtain the Cartesian square,

$$
P \times \mathbb{G}_m \xrightarrow{\text{id} \times \text{id}_m} P \xleftarrow{i_P \times \text{id}_m} E \oplus \overline{E}^\vee \times \mathbb{G}_m \xrightarrow{\text{id} \times i_P} E \oplus \overline{E}^\vee,
$$
demonstrating that $i_P$ is $\mathbb{G}_m$-equivariant. At the same time, $\text{pr}_1$ and $\text{pr}_2$ are clearly equivariant as maps of vector bundles, so we obtain that $\mathcal{F}(\mathcal{F}) := \text{pr}_2^* i_P^* i_P \text{pr}_1^* \mathcal{F}$ is monodromic. \hfill \Box

2.10.2. Naming conventions. The definition of $\mathcal{F}$ first appears in [V83a, §5] under the name “geometric Fourier transform”—as opposed to the Fourier transform of $\mathcal{D}$-modules, which Verdier calls the “analytic Fourier transform.” In keeping with modern convention (e.g. [K21b, §2.4]), however, we refer the functor $\mathcal{F}$ as the Fourier-Sato transform, a name which first appears in [KS90, §3.7].

2.10.3. There is an alternate definition of the Fourier-Sato transform using the subset

$$
P' := \{(v, w) \in E \oplus \overline{E}^\vee | \text{Re } w(v) \leq 0\},
$$
as well as an inverse functor for the Fourier-Sato transform exist, but we will not need these in this paper. A wonderful, detailed account of these functors and their various compatibilities is found in [K21b, §2.4], from which we are borrowing much of the notation in this section.

2.10.4. We will need the following two properties of the Fourier-Sato transform.

\begin{lemma}
Suppose that $f : X' \to X$ is a map of complex analytic spaces, and let $E \to X$ be a complex vector bundle. Denoting $E' := E \times_X X'$, we obtain the induced maps $f_E : E' \to E$ and $f_{E'} : E'^\vee \to E^\vee$. The following natural transformation is an equivalence:

$$
f_{E'}^* \mathcal{F}_{E'} \simeq \mathcal{F}_f f_E^*.
$$

\end{lemma}
\begin{proof}
\textit{c.f. [KS90, Proposition 3.7.13]} \hfill \Box
\end{proof}

\begin{lemma}
Suppose that $E \to X$ is a rank $n$ (i.e. $\dim_C = n$) complex vector bundle. Then,

$$
\mathcal{F}[n] : \text{Perv}(E) \to \text{Perv}(E^\vee).
$$

\end{lemma}
\begin{proof}
\textit{c.f. [KS90, Proposition 10.3.18]} \hfill \Box
\end{proof}

\footnote{The functor which appears in [KS90, §3.7] is technically different from that defined above, being defined for conic sheaves on real vector bundles, but its usage later in that work indicates that the authors were aware of the obvious generalization to complex vector bundles.}
3. Deformation to the normal bundle

Recall the functor of specialization introduced by Verdier in [V83b §8]. Letting $X$ be a separated scheme of finite type over $\text{Spec } \mathbb{C}$ and $Z$ a closed subscheme, the Verdier specialization along $Z$ of sheaves on $X$ is a functor

$$\text{Sp}_{Z/X} : \text{Shv}(X) \to \text{Shv}_{\mathbb{G}_m}(C_{Z/X})$$

where $C_{Z/X}$ is the normal cone of $Z$ inside $X$.

It is defined using the deformation to the normal cone of $Z$ inside $X$, a scheme defined as $D_{Z/X} := \text{Bl}_{Z} \times \{0\} (X \times \mathbb{A}^1) \setminus \text{Bl}_{Z}(X)$. The scheme $D_{Z/X}$ fits into the following commutative diagram,

$$
\begin{array}{ccccccc}
C_{Z/X} & \xrightarrow{i_{C_{Z/X}}} & D_{Z/X} & \xleftarrow{j} & X \times \mathbb{G}_m & \xrightarrow{\text{pr}_1} & X \\
\downarrow & & \downarrow{\iota} & & \downarrow & & \\
\{0\} & \xrightarrow{\iota} & \mathbb{A}^1 & \leftarrow & \mathbb{G}_m & & \\
\end{array}
$$

Definition 3.2. The functor of Verdier specialization is defined by the formula:

$$\text{Sp}_{Z/X}(F) := \psi_{\iota,j}\text{pr}_1^*(F).$$

A full account of the properties of the specialization functor is given in its original source [V83b §8].

In Section 4 we define a functor of quasi-smooth specialization generalizing the above construction. It is defined in the same manner as Verdier’s specialization functor, only replacing the deformation to the normal cone with an appropriate notion of deformation to the normal bundle of $Z \hookrightarrow X$. The appropriate such notion for us is that appearing in [KR19]. In this section, we recall the construction of deformation to the normal bundle presented in [KR19], and prove some general properties of it needed for a good notion of specialization.

3.1. Quasi-smooth morphisms.

Definition 3.3. A map $f : X \to Y$ of derived schemes is quasi-smooth if the relative cotangent complex $L_f$ is perfect of Tor-amplitude $[-1,0]$. $L_f$ and $L_X/Y$ will be used interchangeably when the map $f$ is clear from context.

As a special case of the above definition, we say that a derived scheme $X$ is quasi-smooth if $L_{X/k}$ is perfect of Tor-amplitude $[-1,0]$.

Lemma 3.4. Assume $X$ is smooth, and suppose $i : Z \hookrightarrow X$ is a closed immersion of derived schemes. Then $i$ is quasi-smooth iff $Z$ is quasi-smooth.

Proof. Since $X$ is smooth, $i^*L_X$ is locally free. $L_Z$ is perfect by assumption. Consider the standard sequence on cotangent complexes, $i^*L_X \xrightarrow{i} L_Z \to L_i$. If any two of the three terms in a distinguished triangle in $\text{QCoh}(Z)$ are perfect, the third is as well [Stacks Lemma 066R], so if either of $L_i$ or $L_Z$ is perfect, the other is as well. Meanwhile, $i^*L_X$ is flat, so has Tor amplitude $[0]$. By [Stacks Lemma 0655], this implies that if either $L_i$ or $L_Z$ has Tor-amplitude in $[-1,0]$, the other does as well. □

The following lemma gives a local model for quasi-smooth closed immersions.
Lemma 3.5. Suppose that \( i : Z \hookrightarrow X \) is a quasi-smooth closed immersion of derived schemes. Then Zariski-locally on \( X \), \( i \) fits into a Cartesian diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
* & \xrightarrow{0} & \mathbb{A}^n
\end{array}
\]

Moreover, \( \mathbb{L}_i[-1] \) is then locally free of finite rank. Denote this sheaf by \( N_{Z/X} \).

Proof. c.f. [KR19, Proposition 2.3.8], which in fact shows the converse is also true. □

Definition 3.6. ([KR19, §2.3.11]) The \textit{virtual codimension} of \( i \), defined Zariski-locally on \( Z \), is the rank of the \( \mathcal{O}_Z \)-module \( N_{Z/X} \). We denote it by \( \text{codim.vir}(Z, X) \), or \( \text{codim.vir} Z \) when there is no ambiguity.

Definition 3.7. The \textit{normal bundle} to \( Z \) inside \( X \) is the scheme,

\[
N_{Z/X} := \text{Spec}_Z(\text{Sym}_{\mathcal{O}_Z}(N_{Z/X})).
\]

Similarly, the \textit{conormal bundle} to \( Z \) inside \( X \), denoted \( N^*_Z/X \), is the total space of the sheaf dual to \( N_{Z/X} \).

3.1.1. Obviously the rank of \( N_{Z/X} \) is \( \text{codim.vir} Z \).

3.2. Definition of deformation to the normal bundle. The deformation to the normal bundle described in [KR19] is obtained by removing a divisor from a blow-up, just as in the classical construction—only the precise notions of “blow-up” and “divisor” differ. The bulk of [KR19] is spent developing the theory of blow-ups along quasi-smooth closed immersions of derived stacks. Roughly, quasi-smooth blow-ups are defined by a universal property that generalizes the classical one characterizing the blow-up of \( X \) along \( Z \) as the terminal object in schemes over \( X \) whose fiber over \( Z \) is a Cartier divisor. The exact definition is given in [KR19, §4].

Once the authors of loc. cit. obtain a good theory of quasi-smooth blow-ups, they define the deformation to the normal bundle of \( Z \hookrightarrow X \), which we denote \( D_{Z/X} \), as the open substack of \( \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1) \) given by removing the immersed copy of \( \text{Bl}_Z X \) inside it. We recall the theorem of [KR19], establishing the existence of deformation to the normal bundle.

Theorem 3.8 ([KR19, Theorem 4.1.13]). Let \( i : Z \hookrightarrow X \) be a quasi-smooth closed immersion of derived stacks with normal bundle \( N_{Z/X} = \text{Spec}_Z(\text{Sym}_{\mathcal{O}_Z}(N_{Z/X})) \). Then there exists a canonical factorization of \( i \times \text{id} \), the deformation to the normal bundle:

\[
i \times \text{id} : Z \times \mathbb{A}^1 \xrightarrow{j} D_{Z/X} \xrightarrow{\pi} X \times \mathbb{A}^1
\]

satisfying the following properties

(i) The factorization is stable under arbitrary derived base change along \( X \).
(ii) \( j \) is a quasi-smooth closed immersion.
(iii) \( \pi \) is quasi-smooth and quasi-projective.
(iv) Restricting to \( \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} \) we obtain

\[
i \times \text{id} : Z \times \mathbb{G}_m \xrightarrow{j_{\text{gen}}} X \times \mathbb{G}_m \xrightarrow{\pi_{\text{gen}}} X \times \mathbb{G}_m
\]

where \( j_{\text{gen}} = i \times \text{id} \) and \( \pi_{\text{gen}} = \text{id} \).
(v) Restricting to \(\{0\}\) we obtain

\[ i : Z \xrightarrow{j_0} N_{Z/X} \xrightarrow{\pi_0} X \]

where \(j_0\) is the zero-section and \(\pi_0\) the composition of the projection \(N_{Z/X} \rightarrow Z\) and \(i : Z \hookrightarrow X\).

### 3.2.1. Immediate Consequence of Theorem 3.8

An immediate consequence of Theorem 3.8 is the existence of the following commutative diagram of derived stacks, almost identical to diagram (3.1) above:

\[
\begin{array}{ccc}
N_{Z/X} & \xmapsto{s} & \tilde{D}_{Z/X} \\
\downarrow & & \downarrow e \\
\{0\} & \xmapsto{j_0} & \mathbb{A}^1 \\
\end{array}
\]

\[ \xleftarrow{i} \]

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi_0} & X \\
\downarrow & & \downarrow \\
\mathbb{G}_m & \xrightarrow{\text{pr}_1} & X \\
\end{array}
\]

3.2.2. If \(i : Z \hookrightarrow X\) is a quasi-smooth closed immersion of derived schemes, \(D_{Z/X}\) is (representable by) a derived scheme by [KR19, Theorem 4.1.5(i)].

### 3.3. Formal properties of deformation to the normal bundle.

3.3.1. Familiarity with the notation, terminology, and concepts of [KR19] is required to understand this subsection.

3.3.2. The following lemma says, roughly speaking, that a map of derived schemes induces a map of their blow-ups along compatible quasi-smooth, closed immersions.

**Lemma 3.10.** Suppose that \(f : X' \rightarrow X\) is a map of derived schemes, and that \(Z \hookrightarrow X, Z' \hookrightarrow X'\) are quasi-smooth closed immersions such that \(Z' \subset X' \times_X Z\); \(Z' \simeq X' \times_X Z\); and the canonical morphism \(f_\ast N_{Z/X} \rightarrow N_{Z'/X'}\) is surjective on \(\pi_0\). Then there exists a natural map \(\text{Bl}_f : \text{Bl}_{Z'}X' \rightarrow \text{Bl}_ZX\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{P}_Z(N_{Z'/X'}) & \xmapsto{f_{|Z'}\circ\pi'_X} & \text{Bl}_{Z'}X' \\
\downarrow & & \downarrow \text{Bl}_f \\
\mathbb{P}_Z(N_{Z/X}) & \xmapsto{\pi_X} & \text{Bl}_ZX' \\
\downarrow & & \downarrow f_{\ast}\pi_{X'} \\
Z & \xmapsto{\tau} & X. \\
\end{array}
\]

If, in fact, \(Z' \simeq X' \times_X Z\), then we furthermore have a commutative diagram:

\[
\begin{array}{ccc}
\text{Bl}_ZX' & \xmapsto{X' \setminus Z'} & X' \setminus Z' \\
\downarrow \text{Bl}_f & & \downarrow f_{|X' \setminus Z'} \\
\text{Bl}_ZX & \xmapsto{X \setminus Z} & X \setminus Z \\
\downarrow \pi_X & & \downarrow \pi_X \\
X & \xmapsto{\pi_X} & X \setminus Z. \\
\end{array}
\]

Moreover, all the squares in diagrams (3.11) and (3.12) are Cartesian on underlying classical schemes.
Proof. Given a derived scheme \( S \in \text{DSch}_{/X'} \), the mapping space \( \text{Hom}_{\text{DSch}_{/X'}}(S, \text{Bl}_Z X') \) is the space of virtual Cartier divisors on \( S \) lying over \((X', Z')\). Recall that a point in this space consists of a commutative square,

\[
\begin{array}{ccc}
D & \xrightarrow{i_D} & S \\
\downarrow{g'} & & \downarrow{h'} \\
Z' & \xrightarrow{i'} & X',
\end{array}
\]

satisfying the following properties:

(a) \( i_D : D \hookrightarrow S \) exhibits \( D \) as a virtual Cartier divisor on \( S \) (i.e. a quasi-smooth closed immersion whose normal bundle has rank 1).

(b) The underlying square of classical schemes is Cartesian.

(c) The canonical base change morphism

\[
g'^*N_{Z'/X'} \rightarrow N_{D/S}
\]

is surjective on \( \pi_0 \).

Denote the space of such by \( \text{VDiv}(S/(X', Z')) \).

Given a point \( i_D \in \text{VDiv}(S/(X', Z')) \), we show that it naturally obtains a virtual Cartier divisor on \( S \) living over \((Z, X)\). Indeed, by hypothesis, \( i_D \) fits into the commutative diagram,

\[
\begin{array}{ccc}
D & \xrightarrow{i_D} & S \\
\downarrow{g'} & & \downarrow{h'} \\
Z' & \xrightarrow{i'} & X' \\
\downarrow{f_z} & & \downarrow{f} \\
Z & \xrightarrow{i} & X.
\end{array}
\]

Denote the composition of the outer left and right vertical arrows by \( g \) and \( h \) respectively. The claim is that the outer commutative square of this diagram satisfies the conditions of a virtual Cartier divisor on \( S \) over \((X, Z)\).

Indeed, \( i \) was taken to be a virtual Cartier divisor on \( S \), so condition (a) is satisfied. Condition (b) is satisfied because of the hypothesis that \( Z' \simeq Z \times_X X' \).

Condition (c) is satisfied because the base change morphism \( g'^*N_{Z'/X'} \rightarrow N_{D/S} \) factors as

\[
g'^*f_z^*N_{Z/X} \rightarrow g'^*N_{Z'/X'} \rightarrow N_{D/S}
\]

where the first morphism is induced by \( f_z^*N_{Z/X} \rightarrow N_{Z'/X'} \), which is surjective by assumption. Surjectivity on \( \pi_0 \) then follows from that of \( g'^*N_{Z'/X'} \rightarrow N_{D/S} \) and the fact that \( g'^* \) preserves surjectivity on \( \pi_0 \) (this is just the classical statement that pull-back of quasi-coherent sheaves is right exact).

The assignment \( \text{VDiv}(S/(X', Z')) \rightarrow \text{VDiv}(S/(X, Z)) \) described above is obviously functorial in \( S \), so by Yoneda we obtain a map \( \text{Bl}_Z X' \rightarrow \text{Bl}_Z X \times_X X' \in \text{DSch}_{/X} \). Composition with the projection onto \( \text{Bl}_Z X \) yields the desired map \( \text{Bl}_f \) of derived schemes.

A map \( S \rightarrow S' \in \text{DSch}_{/X} \) induces a map on virtual Cartier divisors over \((X, Z)\) by precomposition with points in \( \text{Hom}_{\text{DSch}_{/X}}(S', \text{Bl}_Z X) \). Explicitly, the image of an element \( i_{D'} \in \text{VDiv}(S'/(X, Z)) \) under precomposition by \( S \rightarrow S' \) is given by
an element \( i_D \in \text{VDiv}(S/(X, Z)) \) whose diagram

\[
\begin{array}{ccc}
D & \xrightarrow{i_D} & S \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}
\]

factors as

\[
\begin{array}{ccc}
D & \xrightarrow{i_D} & S \\
\downarrow & & \downarrow \\
D' & \xrightarrow{i_D'} & S' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}
\]

Recall that the closed immersion \( \mathbb{P}_Z(N_Z/X) \hookrightarrow \text{Bl}_Z X \) is the virtual Cartier divisor lying over \((X, Z)\) given by the identity \( \text{Bl}_Z X \xrightarrow{\text{id}} \text{Bl}_Z X \) (cf. proof of [KR19 Theorem 4.1.4(iv)]). Note that the map \( \text{Bl}_f : \text{Bl}_Z X' \to \text{Bl}_Z X \) factors trivially through the identity:

\[
\text{Bl}_Z X' \to \text{Bl}_Z X \xrightarrow{\text{id}} \text{Bl}_Z X.
\]

As such, we see that the virtual Cartier divisor given by \( \text{Bl}_f, D_{\text{Bl}_f} \to \text{Bl}_Z X' \), factors as

\[
\begin{array}{ccc}
D_{\text{Bl}_f} & \hookrightarrow & \text{Bl}_Z X' \\
\downarrow & & \downarrow \\
\mathbb{P}_Z(N_Z/X) & \hookrightarrow & \text{Bl}_Z X \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}
\]

It remains to be shown that \( D_{\text{Bl}_f} \hookrightarrow \text{Bl}_Z X' \) is the divisor \( \mathbb{P}_Z(N_{Z'/X'} \hookrightarrow \text{Bl}_Z X' \). This follows easily by noting that, as illustrated here,

\[
\text{Hom}_{\text{Sch}/X}(\text{Bl}_Z X', \text{Bl}_Z X') \xrightarrow{\text{Bl}_f} \text{Hom}_{\text{Sch}/X}(\text{Bl}_Z X', \text{Bl}_Z X) \xrightarrow{\pi_0 \sim} \text{id} \xrightarrow{\text{Bl}_f}
\]

the divisor \( \text{Bl}_f \) is the image of the divisor \( \text{id}_{\text{Bl}_Z X'} \) under the map which views elements of \( \text{VDiv}(\text{Bl}_Z X'/\text{(X', Z')}) \) as elements of \( \text{VDiv}(\text{Bl}_Z X'/\text{(X, Z)}) \).

Let \( \pi : \mathbb{P}_Z(N_{Z'/X'}) \to \text{Z} \) and \( \tau' : \mathbb{P}_Z(N_{Z'/X'}) \to \text{Z}' \) denote the structure maps of \( \mathbb{P}_Z(N_{Z'/X'}) \) as a divisor lying over \((X, Z)\) and \((X', Z')\), respectively. Unraveling the definitions used in the argument above, observe that the natural map \( \mathbb{P}_Z(N_{Z'/X'}) \to \mathbb{P}_Z(N_{Z/X}) \), which we label by "\("", is classified by the data of the canonical, surjective (on \( \pi_0 \)) morphism

\[
\pi^*N_{Z/X} \to N_{\mathbb{P}_Z(N_{Z'/X'})/\text{Bl}_Z X'}
\]

associated to \( \mathbb{P}_Z(N_{Z'/X'}) \hookrightarrow \text{Bl}_Z X' \) as a virtual divisor over \((X, Z)\). This morphism in turn factors as

\[
(\pi^* \sim) \tau^* f_{Z'}^* N_Z/X \to \tau'^* N_{Z'/X'} \to N_{\mathbb{P}_Z(N_{Z'/X'})/\text{Bl}_Z X'}.
\]
This factorization then classifies a commutative triangle in $\text{DSch}_X$,

$$
\begin{array}{ccc}
\mathbb{P}^1(\mathcal{N}_{Z/X}) & \xrightarrow{\text{id}} & \mathbb{P}^1(\mathcal{N}_{Z/X}) \\
\downarrow & & \downarrow \\
\mathbb{P}^1(\mathcal{N}_{Z/X}) & \xrightarrow{\mathcal{N}_f} & \mathbb{P}^1(\mathcal{N}_{Z/X})
\end{array}
$$

by the moduli-theoretic description of $\mathbb{P}^1(\mathcal{N}_{Z/X})$. Thus $\mathcal{N}_f = \mathbb{P}^1(\mathcal{N}_{Z/X})$. This yields diagram \((3.11)\).

To see the existence of diagram Eq. \((3.12)\) observe that the map $\mathcal{N}_f : \text{Bl}_Z X' \setminus \mathbb{P}^1(\mathcal{N}_{Z'/X'}) \rightarrow \text{Bl}_Z X' \setminus \mathbb{P}^1(\mathcal{N}_{Z/X})$ classifies, in part, the following commutative triangle,

$$
\mathbb{P}^1(\mathcal{N}_{Z/X}) \xrightarrow{\mathcal{N}_f} \mathbb{P}^1(\mathcal{N}_{Z/X}) \xrightarrow{\mathcal{N}_f} \mathbb{P}^1(\mathcal{N}_{Z/X}) \\
\mathbb{P}^1(\mathcal{N}_{Z'/X'}) \xrightarrow{\mathcal{N}_f} \mathbb{P}^1(\mathcal{N}_{Z'/X'}) \xrightarrow{\mathcal{N}_f} \mathbb{P}^1(\mathcal{N}_{Z'/X'})
$$

Identifying $\text{Bl}_Z X' \setminus \mathbb{P}^1(\mathcal{N}_{Z'/X'})$ with $X' \setminus Z'$ and $\text{Bl}_Z X \setminus \mathbb{P}^1(\mathcal{N}_{Z/X})$ with $X \setminus Z$ using \([\text{KR19} \text{ Theorem 4.1.5(v)}]\), we obtain that $\mathcal{N}_f = \mathbb{P}^1(\mathcal{N}_{Z'/X'})$ as desired.

The claim that all underlying squares of classical schemes in diagrams \((3.11)\) and \((3.12)\) are Cartesian follows trivially from the definitions in \([\text{KR19}]\) and our assumptions.

**Remark 3.13.** Note that, in contrast with the theory of classical blow-ups, unless $Z \hookrightarrow X$ itself has virtual codimension 1, the square,

$$
\begin{array}{ccc}
\mathbb{P}^1(\mathcal{N}_{Z/X}) & \longrightarrow & \text{Bl}_Z X \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X,
\end{array}
$$

will never be Cartesian as a diagram of derived schemes, so in general the squares in diagram \((3.11)\) are only Cartesian on the level of underlying classical schemes.

An immediate corollary of Lemma \(3.10\) is the following.

**Corollary 3.14.** Suppose that $f : X' \rightarrow X$ is a map of derived schemes, and that $Z \hookrightarrow X$, $Z' \hookrightarrow X'$ are quasi-smooth closed immersions such that $Z' \simeq X' \times_X Z$. Then there exists a natural map $D_f : D_{Z'/X'} \rightarrow D_{Z/X}$ making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{N}_{Z'/X'} & \xrightarrow{s'} & D_{Z'/X'} \\
\downarrow & & \downarrow \\
\mathcal{N}_{Z/X} & \xrightarrow{s} & D_{Z/X}
\end{array}
\xrightarrow{\mathbb{P}^1} \begin{array}{ccc}
X' & \xrightarrow{\mathbb{P}^1} & X' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mathbb{P}^1} & X
\end{array}
\xrightarrow{\mathbb{P}^1} \begin{array}{ccc}
\mathbb{A}^1 & \xrightarrow{\mathbb{P}^1} & \mathbb{A}^1 \\
\downarrow & & \downarrow \\
\mathbb{G}_m & \xrightarrow{\mathbb{P}^1} & \mathbb{G}_m.
\end{array}
$$

**Proof.** Using Lemma \(3.10\) and \([\text{KR19} \text{ Theorem 4.1.5(iii)}]\) for the inclusions $X = X \times \{0\} \hookrightarrow X \times \mathbb{A}^1$, $X' = X' \times \{0\} \hookrightarrow X' \times \mathbb{A}^1$ we obtain the following map of
blow-ups,

\[
\begin{array}{ccc}
\text{Bl}_{Z^i \times \{0\}}(X^i \times \{0\}) & \xrightarrow{\text{Bl}_{f^i}} & \text{Bl}_{Z^i \times \{0\}}(X^i \times \mathbb{A}^1) \\
\downarrow_{\text{Bl}_{f^i}} & & \downarrow_{\text{Bl}_{f^i \times \text{id}_{\mathbb{A}^1}}} \\
\text{Bl}_{Z \times \{0\}}(X \times \{0\}) & \xrightarrow{\text{Bl}_i} & \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1),
\end{array}
\]

whose commutativity follows easily from unwinding the definition of each map.

Roughly speaking, each map in the above diagram is obtained by viewing virtual Cartier divisors lying over one pair as Cartier divisors lying over another pair. Clearly, the two ways specified in the above diagram of viewing a virtual Cartier divisor on $S$ lying over $(X', Z')$ as one lying over $(X \times \mathbb{A}^1, Z \times \{0\})$ are compatible.

**Claim 3.17.** The diagram $\text{3.16}$ is Cartesian.

**Proof.** Indeed, suppose we are given a commutative diagram,

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & \text{Bl}_{Z^i \times \{0\}}(X^i \times \mathbb{A}^1) \\
\downarrow_{\beta} & & \downarrow_{\text{Bl}_i} \\
\text{Bl}_{Z \times \{0\}}(X \times \{0\}) & \xrightarrow{\text{Bl}_i} & \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1).
\end{array}
\]

This classifies the data of an element $i_{D_Z} : D_Z \to S \in \text{VDiv}(S/(X, Z))$, an element $i_{D_{Z^i \times \{0\}}} : D_{Z^i \times \{0\}} \to S \in \text{VDiv}(S/(X \times \mathbb{A}^1, Z \times \{0\}))$, and an equivalence $D_Z \simeq D_{Z^i \times \{0\}}$ as divisors lying over $(X \times \mathbb{A}^1, Z \times \{0\})$. Alternatively, using [KR19, Remark 4.1.3], an equivalence of divisors lying over $(X \times \mathbb{A}^1, Z \times \{0\})$ is an equivalence $D_Z \simeq D_{Z^i \times \{0\}}$ of schemes over $S_{Z \times \{0\}} := S \times_{X \times \mathbb{A}^1} Z \times \{0\}$. More precisely we have the following commutative diagram,

\[
\begin{array}{ccc}
D := D_Z \simeq D_{Z^i \times \{0\}} & \longrightarrow & S_{Z^i \times \{0\}} \\
\downarrow & & \downarrow \\
S_Z & \longrightarrow & S_{Z \times \{0\}}
\end{array}
\]

which is classified by a map $D \to S_Z \times_{S_{Z \times \{0\}}} S_{Z^i \times \{0\}}$. It is now a routine exercise to check that $S_Z \times_{S_{Z \times \{0\}}} S_{Z^i \times \{0\}} \simeq S_{Z^i}$, yielding a map $D \to S_{Z^i}$ which classifies a virtual Cartier divisor on $S$ over $(X' \times \{0\}, Z' \times \{0\})$. Thus, we obtain a map $S \to \text{Bl}_{Z^i \times \{0\}}(X^i \times \{0\})$, verifying the universal property for products. \(\square\)

The claim having been shown, we see that the restriction of $\text{Bl}_{f \times \text{id}_{\mathbb{A}^1}}$ to $D_{Z^i} X^i := \text{Bl}_{Z^i \times \{0\}}(X^i \times \mathbb{A}^1) \setminus \text{Bl}_{Z \times \{0\}}(X^i \times \{0\})$ induces a map

\[
D_f : D_Z X^i \to D_Z X.
\]

The lemma now follows from Lemma [3.10] and [KR19, Theorem 4.1.13], which describes the complements of $\text{Bl}_{Z^i} X^i$ and $\text{Bl}_{Z} X$ inside each of the terms in diagrams (3.11) and (3.12). \(\square\)

---

\[16\] The product $S \times_{X \times \mathbb{A}^1} Z \times \{0\}$ is defined unambiguously because $\text{Bl}_i \circ \beta$ is canonically homotopic to $\text{Bl}_f \circ \alpha$ as specified by the commutative diagram.
3.3.3. Consistent with our notation conventions, denote by \( D_{Z/X} \) the underlying classical scheme of \( D_{Z/X} \), and by \( N_{Z/X} \) the underlying classical scheme of \( N_{Z/X} \). The following commutative diagram of schemes is obtained as the classical truncation of diagram (3.9):

\[
\begin{array}{ccc}
N_{Z/X} & \xrightarrow{s} & D_{Z/X} & \xrightarrow{j_{Z/X}} X \\
\downarrow & & \downarrow & \downarrow \text{pr}_1 \\
\{0\} & \xrightarrow{s} & \mathbb{A}^1 & \xleftarrow{\mathbb{G}_m} X
\end{array}
\]

(3.18)

and the following diagram of schemes is obtained as the classical truncation of diagram (3.15):

\[
\begin{array}{ccc}
N'_{Z/X} & \xrightarrow{s'} & D_{Z/X} & \xrightarrow{j'_{Z/X}} X' \\
\downarrow & & \downarrow & \downarrow \text{pr}_1' \\
N_{Z/X} & \xrightarrow{s} & D_{Z/X} & \xrightarrow{j_{Z/X}} X \\
\downarrow & & \downarrow & \downarrow \text{pr}_2 \\
\{0\} & \xrightarrow{s} & \mathbb{A}^1 & \xleftarrow{\mathbb{G}_m} X
\end{array}
\]

(3.19)

Remark 3.20. We draw attention to the fact that all the squares in diagrams (3.19) and (3.18) are Cartesian.

3.3.4. \( \mathbb{G}_m \)-action. The classical construction of deformation to the normal cone has a natural \( \mathbb{G}_m \)-action induced by the grading on the Rees algebra. While there is no obvious grading on the structure sheaf of \( D_{Z/X} \) (indeed, we have not even considered any explicit model for the blow-up as locally-ringed space), there is a natural \( \mathbb{G}_m \)-action obtained by application of Lemma 3.10.

Let \( m : X \times \mathbb{A}^1 \times \mathbb{G}_m \to X \times \mathbb{A}^1 \) be the obvious \( \mathbb{G}_m \)-action map given by scaling the second factor. The following square is Cartesian:

\[
\begin{array}{ccc}
Z \times \{0\} \times \mathbb{G}_m & \xrightarrow{j} & X \times \mathbb{A}^1 \times \mathbb{G}_m \\
\downarrow \text{pr}_{12} & & \downarrow m \\
Z \times \{0\} & \xleftarrow{j} & X \times \mathbb{A}^1
\end{array}
\]

We therefore obtain the map of blow-ups,

\[
\text{Bl}_m : \text{Bl}_{Z \times \{0\} \times \mathbb{G}_m}(X \times \mathbb{A}^1 \times \mathbb{G}_m) \to \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1).
\]

Identifying \( \text{Bl}_{Z \times \{0\} \times \mathbb{G}_m}(X \times \mathbb{A}^1 \times \mathbb{G}_m) \) and \( \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1 \times \mathbb{G}_m) \) via \( \text{Kr19} \) Theorem 4.1.5(ii)), \( \text{Bl}_m \) gives a \( \mathbb{G}_m \)-action on \( \text{Bl}_{Z \times \{0\}}(X \times \{0\}) \).

For reasons similar to those in the proof of Corollary 3.14, the following diagram is Cartesian:

\[
\begin{array}{ccc}
\text{Bl}_{Z \times \{0\}}(X \times \{0\}) \times \mathbb{G}_m & \xrightarrow{\text{Bl}_m} & \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1) \times \mathbb{G}_m \\
\downarrow \text{pr}_{12} & & \downarrow \text{Bl}_m \\
\text{Bl}_{Z \times \{0\}}(X \times \{0\}) & \xleftarrow{\text{Bl}_m} & \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1)
\end{array}
\]
Therefore, $Bl_m$ induces the map

$$act_{G_m} : D_{Z/X} \times G_m \to D_{Z/X},$$

endowing $D_{Z/X}$ with a $G_m$-action. We denote by $act : D_{Z/X} \times G_m \to D_{Z/X}$ the corresponding action on underlying classical schemes.

**Remark 3.21.** Lemma 3.10 ensures that the action on $X \times G_m \subset D_{Z/X}$ is given by expected scaling on the second factor, while the action on $N_{Z/X}$ is given the canonical scaling of fibers, so $act_{G_m}$ is indeed a reasonable notion of $G_m$-action on $D_{Z/X}$.

### 4. Quasi-smooth specialization and microlocalization

In this section, we generalize Verdier specialization by defining a functor of specialization along a quasi-smooth closed immersion $Z \hookrightarrow X$. We prove several properties of the construction which are analogous to those enjoyed by Verdier’s original functor. The functor we define is a generalization of Verdier’s in the sense that it coincides with Verdier specialization in the case when $Z$ and $X$ are classical, meaning the map $Z \hookrightarrow X$ is a regular embedding.

At the end of the section, we define quasi-smooth microlocalization as the Fourier-Sato transform of quasi-smooth specialization and briefly state one of its properties.

#### 4.1. Quasi-smooth specialization.

The following definition is very natural.

**Definition 4.1.** Suppose that $i : Z \hookrightarrow X$ is a quasi-smooth closed immersion of derived schemes. Then the specialization functor along $i$ is given by the formula:

$$\text{Sp}_{Z/X}(F) := \psi_t j_{\neq 0!} \mathcal{P}^* (F),$$

where $F \in \text{Shv}(X)$.

Like its classical counterpart, $\text{Sp}_{Z/X}(F)$ is monodromic, lying in $\text{Shv}_{G_m}(N_{Z/X})$, by virtue of the following well-known lemma.

**Lemma 4.2.** Suppose that $X$ is a scheme with a $G_m$-action, and that $f : X \to \mathbb{A}^1$ is $G_m$-equivariant. Then if $F \in \text{Shv}(X \setminus Z(f))$ is equivariant, $\psi_f(F)$ is monodromic with respect to the induced $G_m$-action on $Z(f)$.

The structure map $t : D_{Z/X} \to \mathbb{A}^1$ is obviously $G_m$-equivariant, and the choice of an equivalence $\mathcal{P}^* F \simeq F \otimes_{G_m} \mathcal{G}_{G_m}$ gives $\mathcal{P}^* F$ the structure of an equivariant sheaf. Therefore, $\text{Sp}_{Z/X}(F)$ is monodromic.

#### 4.2. Specialization and algebraic structure.

Deformation to the normal bundle, and by extension quasi-smooth specialization, is sensitive to the non-reduced, algebro-geometric information of the closed immersion $Z \hookrightarrow X$, even for regular embeddings of classical schemes.

While the category $\text{Shv}(Z) := \text{Shv}(Z)$ by definition only depends on the associated analytic space $Z^{an}$, which itself only depends on the underlying reduced, classical subscheme $Z^{red}$, performing deformation to the normal bundle along two closed immersions possessing the same underlying reduced subscheme can produce schemes with genuinely different reduced subschemes and analytifications thereof.
4.2.1. The dependence of quasi-smooth specialization on the nilpotent fuzz of $Z$ is readily exhibited by looking at the case when $Z \hookrightarrow X$ is given by the quotient of an discrete algebra by a regular ideal

$$A \to A/I$$

In this case, deformation to the normal bundle is given by the Rees algebra of $I$, $R(I)$, which we may explicitly compute in the following simple example.

**Example 4.3.** Consider the closed immersions given by the following maps of $k$-algebras:

$$k[x] \to k[x]/(x) \simeq k$$
$$k[x] \to k[x]/(x^2) =: k[\varepsilon].$$

The first is the usual embedding,

$$\{0\} \hookrightarrow \mathbb{A}^1,$$

and the second is an embedding of the dual numbers at zero,

$$\text{Spec}(k[\varepsilon]) \hookrightarrow \mathbb{A}^1.$$  
Both are regular closed embeddings, since neither $x$ nor $x^2$ is a zero divisor, and they share the same reduced subscheme; yet specialization along each of these two immersions yields different sheaves, as shown below.

4.2.2. We compute $\text{Sp}_{\{0\}/\mathbb{A}^1}(\varepsilon)$. The Rees algebra $R((x))$ is given by the $k$-algebra $k[x, t, h]/(th - x)$, and the canonical map from deformation to the normal cone to $\mathbb{A}^1$ is given by the obvious map $k[t] \to k[x, t, h]/(th - x)$. The analytification, $\text{Spec}(R((x)))^{an}$, is the affine variety $\{th - x = 0\} \subset \mathbb{C}^3$. The fibers of the map $p$ in the definition of the specialization above are subvarieties given by intersections of this variety with the hyperplanes $\{t = t_0\}$ for various values of $t_0$. The fiber at $t_0 \neq 0$ is a linear subspace defined by the equation $t_0 h = x$; the fiber at $t_0 = 0$ is the $h$-axis. One can solve a system of equations to find the region given by the intersection of a small ball in centered at a point $(0, 0, a)$ in the zero fiber of $p$ with a nearby fiber at $|t_0| \ll 1$ to find that the local Milnor fiber at any point is contractible. The stalk of nearby cycles is given by the cohomology of the local Milnor fiber, the stalk of $\text{Sp}_{\{0\}/\mathbb{A}^1}(\varepsilon)$ at any point is the one dimension vector space $e$.

4.2.3. For simplicity of notation, denote $D := \text{Spec}(k[\varepsilon])$. We compute $\text{Sp}_{D/\mathbb{A}^1}(\varepsilon)$. The Rees algebra $R((x^2))$ is given by the $k$-algebra $k[x, t, h]/(th - x^2)$, so the associated complex analytic space is a singular quadric surface with singularity at $(0, 0, 0)$. The fibers of the map $p$ are parabolas (over $\mathbb{C}$) defined at $t = t_0 \neq 0$ by the equation $h = x^2/t_0$; the fiber at $t_0 = 0$ (of the analytification) is again the $h$-axis. One can, as above, solve a system of equations to find the region given by the intersection of a small ball centered at a non-zero point $(0, 0, a)$ in the zero fiber of $p$ with a nearby fiber at $|t_0| \ll 1$ to find that the Milnor fiber at any non-zero point is now a disjoint union of two contractible subspaces. The intuition one should have (coming from the real picture) is that as we approach the zero fiber, the nearby parabolas become narrower and narrower–while still isomorphic to the complex line–and in the limit merge into a complex line over $t_0 = 0$. Any sufficiently small ball at a non-zero point on the $h$-axis will then intersect with a
nearby fiber in disjoint segments of each of the two “legs” of the parabola. Consequently, the stalk of $\text{Sp}_{D/\mathcal{A}}(e)$ at any such point is the two(!) dimensional vector space $e \oplus e$.

4.3. Properties of quasi-smooth specialization. The functor $\text{Sp}_{Z/X}$, which we call quasi-smooth specialization, possesses most of the properties enjoyed by its classical counterpart. We list and prove a few such properties.

4.3.1. Proper and smooth base change. The following two lemmas verify the base change properties expected of quasi-smooth specialization.

**Lemma 4.4.** Suppose that $f: X' \to X$ is a map of derived schemes, $Z \to X$ and $Z' \to X'$ are quasi-smooth closed immersions such that $Z' \simeq X' \times_X Z$. Then there exists a natural morphism

$$\epsilon_f: \text{Sp}_{Z/X}(f_*F) \to N_f(\text{Sp}_{Z'/X'}(F))$$

for $F \in \text{Shv}(X')$. Moreover, $\epsilon_f$ is an isomorphism when $f$ is proper.

**Proof.** Using diagram (3.19), we obtain the chain of base change morphisms:

$$\text{Sp}_{Z/X}(f_*F) := \psi_t(j\neq 0!pr_1^*F) \to \psi_t(j\neq 0!(f \times \text{id}_{\mathbb{A}^1})_*pr_1^*F) \to N_f(\text{Sp}_{Z'/X'}(F))$$

The composition of all these morphisms is, by definition, $\epsilon_f$. Since all the morphisms in the above chain are natural in $F$, $\epsilon_f$ is as well, proving the first part of our claim.

Clearly, all the arrows in the above chain of morphisms are isomorphisms when $f$ and $N_f$ are proper. Assume that $f$ is proper as a morphism of derived schemes. By [KR19, Theorem 4.1.5(ii)] or Lemma 3.10 we have the following Cartesian diagram:

$$\begin{array}{ccc}
\text{Bl}_{Z' \times \{0\}}(X' \times \mathbb{A}^1) & \xrightarrow{\text{Bl}_{f \times \text{id}_{\mathbb{A}^1}}(f \times \text{id}_{\mathbb{A}^1})} & \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1) \\
\pi & & \pi \\
X' \times \mathbb{A}^1 & \xrightarrow{f \times \text{id}_{\mathbb{A}^1}} & X \times \mathbb{A}^1.
\end{array}$$

Since $f \times \text{id}_{\mathbb{A}^1}$ is proper, $\text{Bl}_{f \times \text{id}_{\mathbb{A}^1}}$ is proper, as properness is preserved by base change. By the proof of Corollary 3.14 above, we also have the Cartesian diagram:

$$\begin{array}{ccc}
D_{Z'/X'} & \xrightarrow{f \times \text{id}_{\mathbb{A}^1}} & \text{Bl}_{Z' \times \{0\}}(X' \times \mathbb{A}^1) \\
D_f & & D_f \\
D_{Z/X} & \xrightarrow{\text{Bl}_{f \times \text{id}_{\mathbb{A}^1}}} & \text{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1),
\end{array}$$

so $D_f$ is proper as well.

---

17 This notion is defined and discussed at length in [SAG, ch. 5]. We use its results freely and without precise reference, but they may be all be found in there.
A proper map of derived schemes over $k (= \mathbb{C})$ is easily seen to induce a proper map on underlying classical schemes, so both $D_f$ and $f$ are proper. The classical notion of properness is also preserved under base change, so we deduce from diagram (3.19) that $N_f$ is indeed proper. □

**Lemma 4.5.** Suppose that $f : X' \to X$ is a map of derived schemes, $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ are quasi-smooth closed immersions such that $Z' \simeq X' \times_X Z$. Then there exists a natural morphism

$$\eta_f : N_f^*(\mathsf{Sp}_{Z/X}(\mathcal{F})) \to \mathsf{Sp}_{Z'/X'}(f^*\mathcal{F})$$

for $\mathcal{F} \in \mathsf{Shv}(X)$. Moreover, $\eta_f$ is an isomorphism on the locus where $N_f$ is smooth.

**Proof.** Using diagram (3.19), we obtain the chain of morphisms:

$$N_f^*\mathsf{Sp}_{Z/X}(\mathcal{F}) := N_f^*\psi_t(j_{\neq 0!}\text{pr}_1^*\mathcal{F})$$

$$\to \psi_t(D_f^*j_{\neq 0!}\text{pr}_1^*\mathcal{F})$$

$$\simeq \psi_t(j_{\neq 0!}(f \times \text{id}_{\mathbb{G}_m})^*\text{pr}_1^*\mathcal{F})$$

$$\simeq \psi_t(j_{\neq 0!}\text{pr}_1^*f^*\mathcal{F})$$

$$=: \mathsf{Sp}_{Z'/X'}(f^*\mathcal{F}).$$

The composition of all these morphisms is, by definition, $\eta_f$. Since all the morphisms in the above chain are natural in $\mathcal{F}$, $\eta_f$ is as well, proving the first part of our claim.

Clearly $\eta_f$ is an isomorphism at points in $N_{Z'/X'}$, where the base change morphism

$$N_f^*\psi_t(-) \to \psi_t(D_f^*(-))$$

is an isomorphism. This is true at points where the maps $N_f$ and $D_f$ are smooth, so to prove the second part of our claim it suffices to show that $D_f$ is smooth at those points where $N_f$ is smooth. Recall that a map of schemes is smooth at a point if it is finitely presented at that point and the stalk of the relative cotangent complex at that point is free.

We’ve shown earlier that $D_f$ is a morphism of finite presentation. Now choose $x \in N_{Z'/X'}$, where $N_f$ is smooth, and consider the stalk at $x$ of the fiber sequence:

$$(s'^*\mathbb{L}_{D_f})_x \to (\mathbb{L}_{D_f \circ s'})_x \to (\mathbb{L}_{s'})_x.$$

By assumption, $(\mathbb{L}_{N_f})_x$ is free. The fiber sequence,

$$N_f^*\mathbb{L}_s \to \mathbb{L}_{s \circ N_f} \to \mathbb{L}_{N_f},$$

exhibits an equivalence $\mathbb{L}_{N_f} \simeq \mathbb{L}_{s \circ N_f} = \mathbb{L}_{D_f \circ s'}$ since $s$ is unramified as a closed immersion. Thus $(\mathbb{L}_{D_f \circ s'})_x$ is free. As a closed immersion, $s'$ is also unramified, so $\mathbb{L}_{s'} \simeq 0$. Therefore, $(s'^*\mathbb{L}_{D_f})_x \simeq (\mathbb{L}_{D_f})_x$ is free as well. □

The following lemma verifies that quasi-smooth specialization commutes with Verdier duality.

**Lemma 4.6.** For any $\mathcal{F} \in \mathsf{Shv}(X)$,

$$\mathbb{D}(\mathsf{Sp}_{Z/X}(\mathcal{F})) \simeq \mathsf{Sp}_{Z/X}(\mathbb{D}\mathcal{F}).$$
Proof. The standard properties of Verdier duality and nearby cycles yield the following chain of isomorphisms:

\[ D(\psi_t(j_{\neq 0!}pr_1^*F)) \cong \psi_t(D(j_{\neq 0!}pr_1^*(\mathbb{D}F))[-1])[-1] \cong \psi_t(j_{\neq 0!}pr_1^*(\mathbb{D}F)[-1])[-1] \cong \psi_t(j_{\neq 0!}pr_1^*(\mathbb{D}F)) \cong \psi_t(j_{\neq 0!}pr_1^*(\mathbb{D}F)) =: \text{Sp}_{Z/X}(\mathbb{D}F). \]

\[ \square \]

4.3.2. Restriction to the base. The following lemma computing the restriction of the specialization to the zero section, is indispensable in the proof of Theorem 5.1.

**Lemma 4.7.** Suppose that \( X/k \) is a derived scheme locally of finite presentation (equivalently, locally of finite type). Let \( Z \hookrightarrow X \) be a quasi-smooth, closed immersion. For any \( F \in \text{Shv}(X) \),

\[ F|_Z \cong (\text{Sp}_{Z/X}F)|_Z \tag{4.8} \]

\[ F|_Z \cong \text{Sp}_{Z/X}(F)|_Z. \tag{4.9} \]

**Proof.** By Lemma 4.6 it suffices to verify just either of (4.8) or (4.9). We verify (4.8).

Let \( \widetilde{\pi} : \widetilde{D}^*Z/X \to D^*_Z/X \cong X \times \mathbb{G}_m \) denote the morphism indicated in diagram (2.25). Then,

\[ \text{Sp}_{Z/X}F := \psi_t(j_{\neq 0!}pr_1^*F) \cong s^*(\widetilde{\pi} \circ j_{\neq 0!})*p^*F, \]

where \( p \) is the morphism indicated in diagram (3.18). From this formula, we obtain the following comparison map, natural in \( F \),

\[ F|_Z \cong (s^*p^*F)|_Z \]

\[ \to (s^*(\widetilde{\pi} \circ j_{\neq 0!})*p^*F)|_Z \cong \text{Sp}_{Z/X}(F)|_Z. \]

where the first equivalence follows from the fact that the composition \( Z \to Df_{Z/X} \to X \) is the inclusion \( Z \hookrightarrow X \).

We show that this map is an isomorphism. Note that (4.8) is local on \( X^{an} \). That is, it suffices to show that \( \text{Sp}_{Z/X}F|_Z \cong F|_Z \) on an open cover of \( X^{an} \). Since \( Z \hookrightarrow X \) is a quasi-smooth closed immersion, by Lemma 3.5 it has the form

\[ \begin{array}{c}
Z \\
\downarrow \scriptstyle j \\
\{0\} \\
\downarrow \scriptstyle f \\
A^n
\end{array} \]

Zariski-locally on \( X \). Pick a Zariski open cover of \( X \) on each member of which \( Z \hookrightarrow X \) has the above form; then consider the pullback of this cover along the map \( X \to X \). Zariski open covers are in particular covers in the analytic topology on...
X, so we may assume without loss of generality that \( Z \hookrightarrow X \) is the zero locus of a finite set of regular functions on \( X, f = (f_1, \ldots, f_n) \).

The derived scheme \( X \) embeds as a closed subscheme of \( X \times \mathbb{A}^n \) via the graph map \( \Gamma_f := \text{id}_X \times f : X \to X \times \mathbb{A}^n \). This map is, in particular, proper, since \( X \times \mathbb{A}^n \) is locally Noetherian. Therefore,
\[
\text{Sp}(X \times \{0\}/(X \times \mathbb{A}^n)(\Gamma_f \ast \mathcal{F}) \simeq N_{\Gamma_f} \text{Sp}_Z(X)(\mathcal{F}).
\]

Restricting to \( X \times \{0\} \), we obtain
\[
(4.10) \quad \text{Sp}(X \times \{0\}/(X \times \mathbb{A}^n)(\Gamma_f \ast \mathcal{F})|_{(X \times \{0\})} \simeq (Z \hookrightarrow X)_\ast (\text{Sp}_Z(X)(\mathcal{F})|_Z).
\]

**Claim 4.11.** There is an equivalence of functors,
\[
\text{Sp}(X \times \{0\}/(X \times \mathbb{A}^n) \simeq \text{Sp}(X \times \{0\}/(X \times \mathbb{A}^n)),
\]
where the left-hand side is quasi-smooth specialization and the right-hand side is Verdier specialization.

**Proof.** By Theorem 3.8 (i), deformation to the normal bundle is stable under arbitrary base change, meaning
\[
D(X \times \{0\}/(X \times \mathbb{A}^n) \simeq D_0/\mathbb{A}^n \times (X \times \mathbb{A}^n) \simeq D_0/\mathbb{A}^n \times X.
\]
From this, we see clearly that
\[
D(X \times \{0\}/(X \times \mathbb{A}^n) \simeq D_0/\mathbb{A}^n \times (X \times \mathbb{A}^n),
\]
where the left-hand side is the underlying classical scheme of the derived deformation to the normal bundle and the right-hand side is the ordinary deformation to the normal cone along a closed subscheme. This suffices to prove the claim, since both quasi-smooth specialization and Verdier specialization are defined by the same formula. \(\square\)

The claim in hand, we have
\[
\text{Sp}(X \times \{0\}/(X \times \mathbb{A}^n)(\Gamma_f \ast \mathcal{F})|_{(X \times \{0\})} \xrightarrow{\simeq} (\Gamma_f \ast \mathcal{F})|_{(X \times \{0\})}
\]
by \[V83b\] §8, Property (SP5)]. The base change isomorphism yields
\[
(\Gamma_f \ast \mathcal{F})|_{(X \times \{0\})} \simeq (Z \hookrightarrow X)_\ast (\mathcal{F}|_Z),
\]
which, combined with \[(4.10),\] gives the result, since \((Z \hookrightarrow X)_\ast (Z \hookrightarrow X)_\ast \simeq \text{id}_{\text{Shv}(Z)}\). \(\square\)

4.3.3. Specialization preserves perversity. Finally, we show that quasi-smooth specialization preserves perversity.

**Lemma 4.12.** Suppose that \( \mathcal{F} \in \text{Perv}(X) \). Then \( \text{Sp}_Z(X)(\mathcal{F}) \in \text{Perv}(N_Z/X) \).

**Proof.** Recall that, with respect to the perverse t-structure,
(i) \( f \neq 0 \) is right t-exact \[\text{Dim04} \] Theorem 5.2.4(iii)].
(ii) \( f \neq 0 \) is left t-exact \[\text{Dim04} \] Theorem 5.2.4(v)].
(iii) \( pr_1^\ast[1] \) is t-exact \[\text{BBDJSS15} \] Theorem 2.6(d)].

Items (i) and (iii) imply:
\[
\psi_t(j \neq 0 \text{pr}_1^\ast[1](\mathcal{F})) = \psi_t(j \neq 0 \text{pr}_1^\ast[1](\mathcal{F}))[−1]
\]
\[
\simeq \psi_t(j \neq 0 \text{pr}_1^\ast \mathcal{F})
\]
\[
=: \text{Sp}_Z(X)(\mathcal{F}) \in \text{Shv}(N_Z/X)^{\leq 0}.
\]
On the other hand, \( \psi_{t}(j_{\neq 0}^{*}\text{pr}_{1}^{*}\mathcal{F}) \simeq \psi_{t}(j_{\neq 0}^{*}\text{pr}_{1}^{*}\mathcal{F}) \), and the latter lives in \( \text{Shv}(N_{Z/X})^{\geq 0} \) by items (ii) and (iii). We conclude that \( \text{Sp}_{Z/X}(\mathcal{F}) \in \text{Perv}(N_{Z/X}) \).

\[ \square \]

4.4. Comparing classical notions of specialization. Earlier in this section we recalled the notion of Verdier specialization, a functor which takes a sheaf on the analytic topology of a scheme \( X \) and produces a sheaf on the normal cone along a designated subscheme \( Z \).

4.4.1. When \( X \) and \( Z \) are smooth, their analytifications have the structure of smooth manifolds, and there is an alternative notion of specialization available to us as described in [KS90, §4.1 and §4.2]. In Section 5 we use a result from [KS90] §4 on the functor of specialization. In order to access these results, we sketch an argument that, in the case when \( Z \subset X \) is given as the zero locus of a section of a vector bundle on \( X \), the notions from [KS90] and [V83b] agree.

Discussion 4.13. Suppose that \( Z \subset X \) is given as the zero locus of the section \( s : X \to E \) of a vector bundle \( E \to X \). By [F98, Remark 5.1.1], \( D_{Z/X} \) is then described by the following construction attributed to MacPherson.

Consider the map,

\[
X \times \mathbb{A}^1_{\neq 0} \xrightarrow{\psi} E \times \mathbb{A}^1_{\neq 0}
\]

\[
(x, t) \quad \begin{array}{c}
\mapsto \\
(\frac{s(x)}{t}, t).
\end{array}
\]

Then \( D_{0Z/VZ} \) is given by the closure of the image of \( S \) inside of \( E \times \mathbb{A}^1 \). Intuitively: \( D_{0Z/VZ} \) is given by adding the “limit point” at 0 to the smooth \( \mathbb{A}^1_{\neq 0} \)-family of \( X \)-s whose slice at a point \( t \) is given by the embedded copy of \( X \) inside \( E \) given by \( s \), scaled by \( \frac{1}{t} \).

The diligent reader may now check that the following constructions are identical:

1. The restriction of the above construction to those values \( t \in \mathbb{R} \subset \mathbb{C} \) — i.e. consider the closure of the image of \( s|_{VZ \times \mathbb{R} \times 0} \) inside of \( V_{Z} \times \mathbb{R} \times V \).
2. The construction of deformation to the normal bundle in [KS90, §4.1, pg. 186].

In an open subset of \( X \) over which \( E \) is trivialized, the correspondence goes as follows. In the notation of [KS90, §4.1]: \( X \leftrightarrow U_{i}, E \leftrightarrow U_{i} \times \mathbb{R}^{l}, s \leftrightarrow (\text{id}_{U_{i}}, (\mathbb{R}^{n} \to \mathbb{R}^{l} \times \{0\}) \circ \phi_{i}) \), and \( D_{Z/X} \leftrightarrow V_{i} \).

Now denote by \( t : D_{Z/X} \to \mathbb{A}^1 \) and \( pr_{\neq 0} : (X \times \mathbb{A}^1_{\neq 0} \simeq)D_{Z/X}^{*} \to X \) the restrictions of the obvious projections.

The diligent reader may again check that \( t^{-1}(l_{>0}) \) coincides with the fiber over \( \mathbb{R}^{+} \) of the object outlined in (1), as well as the subset in [KS90, §4.1] that the authors denote by \( \Omega \). Clearly, \( \text{Sp}_{Z/X} \mathcal{F} \simeq \psi_{t}(pr_{\neq 0}^{*}\mathcal{F}) \). At the same time, formula (2.34) for nearby cycles at fixed angle \( \theta = 0 \) yields:

\[
\psi_{t}(pr_{\neq 0}^{*}\mathcal{F}) \simeq (t^{-1}(0) \hookrightarrow D_{Z/X})^{*}(t^{-1}(l_{>0}) \hookrightarrow D_{Z/X}), (t^{-1}(l_{>0}) \hookrightarrow D_{Z/X})^{*}pr_{\neq 0}^{*}\mathcal{F},
\]

which coincides precisely with the formula for the specialization in [KS90, §4.2], taking into account the agreement of constructions (1) and (2) sketched above.

\[\text{It is folklore that these two notions always agree, but we only verify this agreement in the current setting.}\]
4.4.2. Specialization of monodromic sheaves. MacPherson’s description of deformation to the normal cone allows for a nice proof of the following lemma, which is well-known but hard to find in the literature.

**Lemma 4.14.** Suppose that $E \to X$ is a finite dimensional complex vector bundle on the derived scheme $X$. If $F \in \text{Shv}_{\mathbb{G}_m}(E)$, then

$$\text{Sp}_{X/E} F \simeq F,$$

$$\text{Sp}_{X/E} F \simeq F,$$

where $X$ (resp. $X$) is seen as the zero section of $E$ (resp. $E$).

**Proof.** The following equivalence of schemes is easily seen from the properties of quasi-smooth blow-up:

$$D_{X/E} \simeq D_{X/E}.$$  

As such, $\text{Sp}_{X/E} \simeq \text{Sp}_{X/E}$, so it suffices to prove that $\text{Sp}_{X/E} F \simeq F$. Without loss of generality, we may assume $X = *$ and $E$ is a vector space $V$.

MacPherson’s description of deformation to the normal cone, outlined in [F98, Remark 5.1.1], identifies $D_{0/V}$ with the closure of the image of the map,  

$$V \times \mathbb{A}^1_{\neq 0} \xrightarrow{S} V \times V \times \mathbb{A}^1,$$

$$\psi$$

$$(v, t) \mapsto (v, \frac{v}{t}, t),$$

and identifies the maps $t : D_{0/V} \to \mathbb{A}^1$ and $p : D_{0/V} \to V$ with the restrictions to $D_{0/V}$ of $\text{pr}_3 : V \times V \times \mathbb{A}^1 \to \mathbb{A}^1$ and $\text{pr}_1 : V \times V \times \mathbb{A}^1 \to V$, respectively. Let $U := D_{0/V} \setminus (0, 0) \times \mathbb{A}^1$. By smooth base change,  

$$(\text{Sp}_{0/V} F|_{V \setminus 0}) \simeq \psi_{t_U} (p_U^* F)|_{V \setminus 0},$$

where $t_U$ and $p_U$ are the restriction to $U$ of $t$ and $p$, respectively. We show that $\psi_{t_U} (p_U^* F)|_{V \setminus 0} \simeq F_{V \setminus 0}$.

Let $\pi : V \setminus 0 \to (V \setminus 0) / \mathbb{R}^+ = S(V)$ be the map to the spherization of $V$. The adjoint functors  

$$\pi^* : \text{Shv}(S(V) \rightleftarrows \text{Shv}_{\mathbb{R}^+}(V \setminus 0) : \pi_*$$

induce an equivalence of categories because the counit map is an isomorphism. While the map $S$ doesn’t extend to a continuous map on $V \times \mathbb{A}^1$, the induced map,  

$$S(V) \times \mathbb{A}^1_{\neq 0} \xrightarrow{\tilde{S}} S(V \times V) \times \mathbb{A}^1,$$

$$(\tilde{V}, t) \mapsto (\tilde{V}, \frac{v}{t}, t)$$

does extend to a continuous map on $S(V) \times \mathbb{A}^1$. This extension is a homeomorphism onto its image, which we can identify with $S(D_{0/V})$, the fiberwise spherization of $D_{0/V}$. In all, we obtain the following commutative diagram, whose maps are
induced from maps on $D_{0/V}$:

\[
\begin{array}{ccc}
S(V) & \xrightarrow{pr_1} & S(V) \\
\downarrow{\bar{\rho}} & & \downarrow{\bar{\iota}} \\
S(V) \times \mathbb{A}^1 & \xrightarrow{\tilde{S}} & S(D_{0/V}) \\
\downarrow{pr_2} & & \downarrow{\tilde{\pi}} \\
\mathbb{A}^1 & \xrightarrow{\tilde{\psi}} & \mathbb{A}^1
\end{array}
\]

Denote $\tilde{F} := \pi_*(F|_{V\setminus 0})$. From the above diagram:

\[
\tilde{F} \simeq \psi_{pr_2} (pr_1^* \tilde{F})
\]

\[
\simeq \psi_{|_S} (\tilde{S} \circ \tilde{\rho} \cdot \pi_* \tilde{F})
\]

\[
\simeq \tilde{S} |_{\tilde{p}^{-1}(0)} \psi_1 (p_* \pi_* \tilde{F})
\]

\[
\simeq \text{Sp}_{0/V} F
\]

Because $\pi_*$ induces an equivalence of categories, we obtain an equivalence, $\mathcal{F}|_{V\setminus 0} \simeq (\text{Sp}_{0/V} \mathcal{F})|_{V\setminus 0}$. Combining this equivalence with that obtained in Lemma 4.7, we use an adjunction triangle for the complementary pair of subspaces $0, V \setminus 0 \subset V$ to obtain the desired equivalence, $\mathcal{F} \simeq \text{Sp}_{0/V} \mathcal{F}$. □

The following is an immediate corollary of Lemma 4.14 which is also well-known but hard to find in the literature.

**Corollary 4.15.** Both Verdier and quasi-smooth specialization are idempotent in the following sense. Suppose $Z \hookrightarrow X$ an arbitrary closed immersion of ordinary schemes. Then, for $\mathcal{F} \in \text{Shv}(X)$,

\[
\text{Sp}_{Z/N} (\text{Sp}_{Z/X} \mathcal{F}) \simeq \text{Sp}_{Z/X} \mathcal{F},
\]

after making the identification $N_{Z/N} \simeq N_{Z/X}$. If $Z \hookrightarrow X$ is a quasi-smooth embedding of derived schemes, the corresponding statement is obtained by substituting $\mathcal{N}, Z$, and $X$ for $\mathcal{N}, Z$, and $X$.

4.4.3. Lastly, we record a result from [KS90] that will be used in Section 5, as well as a definition needed to state it.

**Definition 4.16.** ([KS90, Definition 4.1.1(i)]) Suppose that $p : D_{Z/X} \to X$ is the canonical map, and $p_{>0} : t^{-1}(A^*_X) =: D_{Z/X, >0} \to X$ is its restriction. Suppose given a subset $S \subset X$. Then we define the normal cone of $S$ along $Z$ as:

\[
C_Z(S) := p_{>0}^{-1}(S) \cap C_{Z/X}.
\]

**Theorem 4.17.** ([KS90, Theorem 4.2.3(ii)]) Suppose $M$ is a smooth submanifold of a smooth manifold $X$. Let $\mathcal{F} \in \text{Shv}(X)$ and $\mathcal{V}$ be an open conic subset of $N_{M/X}$. Then:

\[
\lim_{\mathcal{U}} \Gamma(\mathcal{U}; \mathcal{F}) \xrightarrow{\sim} \Gamma(\mathcal{V}; \text{Sp}_{M/X} \mathcal{F}),
\]

where $\mathcal{U}$ ranges through the family of open subsets of $X$ such that $C_M(X \setminus \mathcal{U}) \cap \mathcal{V} = \emptyset$, which we denote by $\mathfrak{F}_\mathcal{V}$.

4.5. Quasi-smooth microlocalization.
4.5.1. We make the following obvious definition.

**Definition 4.19.** Suppose that $Z \hookrightarrow X$ is a quasi-smooth closed immersion, and that $\mathcal{F} \in \text{Shv}(X)$. The microlocalization of $\mathcal{F}$ along $Z \hookrightarrow X$ is defined as

$$\mu_{Z/X} \mathcal{F} := \mathcal{F}(\text{Sp}_{Z/X} \mathcal{F})$$

Clearly, $\mu_{Z/X}$ is a functor from $\text{Shv}(X)$ to $\text{Shv}_{Gm}(X)$. We also call $\mu_{Z/X}$ the functor of quasi-smooth microlocalization.

4.5.2. Unlike quasi-smooth specialization, $\mu_{Z/X}$ preserves perversity only up to a shift. The following is an immediate corollary of Lemma 4.12 and Lemma 2.50.

**Lemma 4.20.** $\mu_{Z/X}[\text{codim.vir } Z]: \text{Perv}(X) \rightarrow \text{Perv}(N^\vee_{Z/X})$.

**Remark 4.21.** Just as quasi-smooth specialization enjoys most of the formal properties of its classical counterpart, quasi-smooth microlocalization enjoys many more properties than the one stated in Lemma 4.20 above. Listing and proving them is beyond the scope of this paper.

5. The local equivalence

5.1. **Stating the equivalence.** For the remainder of this section, we use the following set-up. Let $X$ be a smooth $k$-scheme, and let $V$ be a $k$-vector space of dimension $n$, viewed as a $k$-scheme. Given a map $f: X \rightarrow V$, we consider the derived zero locus,

$$Z(f) \hookrightarrow X$$

$$\downarrow \quad \downarrow f$$

$$\{0\} \hookrightarrow V.$$

The closed immersion $Z(f) \hookrightarrow X$ is obviously quasi-smooth, and by application of base change for cotangent complexes, we find that

$$N_{Z(f)/X} \simeq Z(f) \times V.$$

For such a map, $f$, we consider the map $\tilde{f}: X \times V^\vee \rightarrow V$ whose definition is indicated by the following diagram:

$$\begin{array}{ccc}
X \times V^\vee & \xrightarrow{f \times \text{id}_{V^\vee}} & V \times V^\vee \\
\downarrow \tilde{f} & & \downarrow \langle \cdot, \cdot \rangle \\
A^1 & \rightarrow &
\end{array}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between a vector space and its dual.

5.1.1. The conormal bundle $N^\vee_{Z(f)/X}$ embeds into $X \times V^\vee$ as the closed subscheme $Z(f) \times V^\vee$, using the identification $N^\vee_{Z(f)/X} \simeq Z(f) \times V^\vee$ given above. Note in particular that, under the inclusion $Z(f) \times V^\vee \hookrightarrow Z(\tilde{f})$, $N^\vee_{Z(f)/X} \subset Z(\tilde{f})$, the (underived) zero locus of $\tilde{f}$.

5.1.2. Denote by $p_1$ and $p_2$ the canonical projections from $X \times V$ onto its factors. In this section, we prove the following theorem.

**Theorem 5.1.** There exists a natural isomorphism of functors,

$$\varphi_{\tilde{f}}(p_1^*(-)) \simeq \mu_{Z(f)/X}(-).$$
5.1.3. Prima facie, Theorem 5.1 is ill-posed: for a given $F \in \text{Shv}(X)$, the left-hand side and right-hand side produce sheaves that live on different spaces. However, $\varphi_f(\text{pr}_1^*F)$ has support contained in $Z(f) \times V^\vee$ by [Dav17, Lemma A.4], so it can be viewed as a sheaf on such. At the same time, the identification discussed in Section 5.1.1 allows us to view $\mu_{Z(f)/X}(F)$ as an element of $\text{Shv}(Z(f) \times V^\vee)$, as well, so the two functors of Theorem 5.1 are indeed comparable.

5.2. The monodromic case. The following lemma is the basis for our strategy of proof.

Lemma 5.2. Suppose that $p : E \to S$ is a complex vector bundle of finite rank. If $F \in \text{Shv}_{\mathbb{G}_m}(E)$, then

$$\mathcal{F}(F) \simeq \varphi_m(\text{pr}_1^*F),$$

where $\text{pr}_1 : E \oplus E^\vee \to E$ is the obvious projection and $m : E \oplus E^\vee \to \mathbb{A}^1$ is the natural pairing map.

Proof. The key observation in this proof is the following identity:

$$P := \{(v, w) \in E \oplus E^\vee | \Re w(v) \geq 0\} =: \{\Re m \geq 0\}$$

Recalling the expression for vanishing cycles at the fixed angle $\theta = \pi$, (2.36), we obtain the following chain of isomorphisms:

$$\mathcal{F}(F) := \text{pr}_2_* \Gamma_P(\text{pr}_1^*F)$$
$$= \text{pr}_2_* \Gamma_{\{\Re m \geq 0\}}(\text{pr}_1^*F)$$
$$\simeq (0 \oplus E^\vee \hookrightarrow E \oplus E^\vee)^* \Gamma_{\{\Re m \geq 0\}}(\text{pr}_1^*F)$$
$$\simeq (0 \oplus E^\vee \hookrightarrow m^{-1}(0))^* (m^{-1}(0) \to E \oplus E^\vee)^* \Gamma_{\{\Re m \geq 0\}}(\text{pr}_1^*F)$$
$$\simeq (0 \oplus E^\vee \hookrightarrow m^{-1}(0))^* \varphi_m(\text{pr}_1^*F),$$

where the third equivalence holds by monodromicity with respect to the $\mathbb{G}_m$-action given by scaling on the first factor alone.

Now observe that $\varphi_m(\text{pr}_1^*F)$ is supported on the closed subset $0 \oplus E^\vee$ by [Dav17, Lemma A.4], so we may identify $\varphi_m(\text{pr}_1^*F)$ with its pullback $(0 \oplus E^\vee \hookrightarrow m^{-1}(0))^* \varphi_m(\text{pr}_1^*F)$, which completes the proof. \qed

5.2.1. Suppose that in the statement of Lemma 5.2 we took $E \to S$ to be the normal bundle $N_{Z(f)/X} \to Z(f)$ and $F$ to be the specialization $\text{Sp}_{Z(f)/X} F$. Then we obtain,

$$\mu_{Z(f)/X}(F) := \mathcal{F}(\text{Sp}_{Z(f)/X} F) \simeq \varphi_m(\text{pr}_1^* \text{Sp}_{Z(f)/X} F).$$

Therefore, it suffices to prove that the functor $\varphi_m(\text{pr}_1^-)$ factors through the specialization along $Z(f)$ in the sense of the following lemma.

Lemma 5.3. $\varphi_f(\text{pr}_1^*F) \simeq \varphi_m(\text{pr}_1^* \text{Sp}_{Z(f)/X} F)$.
5.2.2. *Reducing to the case of a product.* In order to prove Lemma 5.3, it suffices to take \( X := Z \times V \), for \( Z \) smooth, \( V \) a finite dimensional vector space over \( k \), and \( f := \text{pr}_V \). Demonstrating this is a matter of simple, repeated application of proper base change; we write it out for completeness.

Consider the following diagram,

\[
\begin{array}{ccc}
X & \xrightarrow{f} & V \\
\downarrow{\text{pr}_1} & & \uparrow{\text{pr}_V} \\
X \times V & \xrightarrow{\Gamma_f \times \text{id}_V} & X \times V \times V^\vee \\
\downarrow{\tilde{\text{pr}}_V} & & \uparrow{\text{pr}_1} \\
A^1 & \xrightarrow{\text{pr}_1} & A^1 \\
\end{array}
\]

(5.4)

where the meaning of the overloaded notation “\( \text{pr}_1 \)” is inferred from context in what follows.

Using the graph map \( \Gamma_f : X \to X \times V \), we apply proper base change for vanishing cycles to obtain:

\[
(Z(\tilde{f}) \to Z(\tilde{\text{pr}}_V))_{\ast} \varphi(\text{pr}_1^\ast F) \simeq \varphi(\tilde{\text{pr}}_V) ((\Gamma_f \times \text{id}_V^\vee)_{\ast} \text{pr}_1^\ast F) \\
\simeq \varphi(\tilde{\text{pr}}_V) (\text{pr}_1^\ast \Gamma_f^\ast F)
\]

One the other hand, since \( f = \text{pr}_V \circ \Gamma_f \), the square,

\[
\begin{array}{ccc}
Z(f) & \xrightarrow{j} & X \times \{0\} = Z(\text{pr}_V) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Gamma_f} & X \times V,
\end{array}
\]

is derived Cartesian. Note that \( N_{(X \times \{0\})/(X \times V)} \simeq X \times V \), and \( N_{\text{pr}_V} = \text{pr}_V \). Meanwhile, \( N_{Z(f)/X} \simeq Z(f) \times V \), and \( N_f : Z(f) \times V \to V \) is the projection. The following diagram is induced from diagram (5.4):

\[
\begin{array}{ccc}
A^1 & \xleftarrow{m} & \overline{N_{\text{pr}_V}} = \tilde{\text{pr}}_V \\
\downarrow{\mathcal{N}_f} & & \downarrow{\mathcal{N}_f} \\
N_{Z(f)/X} \times V^\vee & \xrightarrow{\mathcal{N}_f \times \text{id}_V^\vee} & N_{(X \times \{0\})/(X \times V)} \times V^\vee \\
\downarrow{\text{pr}_1} & & \downarrow{\text{pr}_1} \\
Z(f) \times V & \simeq N_{Z(f)/X} & N_{(X \times \{0\})/(X \times V)} \simeq X \times V,
\end{array}
\]

where \( m \) is as follows:

\[
\begin{array}{ccc}
N_{Z(f)/X} \times V^\vee & \xrightarrow{m} & A^1 \\
\psi & & \psi \\
(z, v, w) & \xrightarrow{w} & w(v).
\end{array}
\]
Proper base change for quasi-smooth specialization (Lemma 4.4) and vanishing cycles now yields:

\[
(Z(m) \to Z(\tilde{p}_V))^* \varphi_m(\text{pr}_V^* \text{Sp}_{Z(f)/X} \mathcal{F}) \simeq \varphi_{\tilde{p}_V}((N_{G_f} \times \text{id}_V)_* \text{pr}_V^* \text{Sp}_{Z(f)/X} \mathcal{F}) \\
\simeq \varphi_{\tilde{p}_V}((\text{pr}_V^* X)_{\text{Sp}_{(X \times \{0\})/\{0 \times V\}}(\Gamma_f \mathcal{F})).
\]

Hence, if the equivalence

\[
\varphi_{\tilde{p}_V}(\text{pr}_V^* \mathcal{G}) \simeq \varphi_{\tilde{p}_V}(\text{pr}_V^* \text{Sp}_{(X \times \{0\})/\{0 \times V\}}(\mathcal{G}))
\]

holds for all \( \mathcal{G} \in \text{Shv}(X \times V), \) the equivalence

\[
(Z(\tilde{f}) \to Z(\tilde{p}_V))_* \varphi_{\tilde{f}}(\text{pr}_V^* \mathcal{F}) \simeq (Z(m) \to Z(\tilde{p}_V))_* \varphi_m(\text{pr}_V^* \text{Sp}_{Z(f)/X} \mathcal{F}),
\]

also holds. The latter then implies Lemma 5.3 once we take into account that each side is supported on \( Z \times \{0\} \times V^\vee \subset X \times V \times V^\vee \).

5.2.3. For the remainder of this section, take \( Z \) smooth, \( X := Z \times V \), and \( f := \text{pr}_V \), per Section 5.2.2 above.

5.2.4. To ease the exposition, we denote \( Z \times V \) by \( V_Z \), viewing \( X \) as a scheme over \( Z \) with structure map given by \( \text{pr}_1 \). Thus, products are taken in the overcategory \( \text{Sch}/Z \), meaning \( V_Z \times_Z V_Z^\vee \) denotes \( Z \times V \times V^\vee = X \times V^\vee \), \( 0_Z \subset V_Z \) denotes \( Z \times \{0\} \subset Z \times V \), etc. By abuse of notation, we denote the analytifications of all these schemes by the same. From hereon out, we denote \( \tilde{p}_V \) by \( m \). \( \text{pr}_1 \) will always denote the projection \( V_Z \times_Z V_Z^\vee \to V_Z \).

5.3. Proof strategy for the non-monodromic case. Our proof of Lemma 5.3 proceeds indirectly by first proving the equivalence

\[
\psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z^\vee (= \{0 \times V^\vee\} \times Z)} \xrightarrow{\simeq} \psi_m(\text{pr}_1^* \text{Sp}_{0_Z/V_Z} \mathcal{F})|_{V_Z^\vee},
\]

and then using the standard fiber sequence relating nearby and vanishing cycles. In order to prove this latter equivalence, we define a natural comparison map, \( \psi : \psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z^\vee} \to \psi_m(\text{pr}_1^* \text{Sp}_{0_Z/V_Z} \mathcal{F})|_{V_Z^\vee} \), which we then show is an equivalence by checking on stalks. Since obtaining the comparison map is somewhat involved, we briefly outline its construction below.

5.3.1. Outline of comparison map construction. Observe that that \( \varphi_m(\text{pr}_1^* \mathcal{F}) \) is conic by Lemma 4.2, so \( \psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z^\vee} \) is conic via the canonical fiber sequence (2.29). Let \( \mathcal{B}_{\mathcal{R}+}(V_Z^\vee) \subset \mathcal{U}_{\mathcal{R}+}(V_Z^\vee) \) denote the partially ordered subset consisting of convex, conic open subsets. These are a basis for the conic topology on \( V_Z^\vee \).

We begin the construction of the comparison map, \( \psi \), by finding convenient expressions for the sections of \( \psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z^\vee} \) and \( \psi_m(\text{pr}_1^* \text{Sp}_{0_Z/V_Z} \mathcal{F})|_{V_Z^\vee} \) over each element of \( \mathcal{B}_{\mathcal{R}+}(V_Z^\vee) \). These convenient expressions are colimits of sections of \( \mathcal{F} \) over certain families of open subsets in \( V_Z \). From these descriptions, we argue that the restrictions of \( \psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z^\vee} \) and \( \psi_m(\text{pr}_1^* \text{Sp}_{0_Z/V_Z} \mathcal{F})|_{V_Z^\vee} \) to \( \mathcal{B}_{\mathcal{R}+}(V_Z^\vee) \) are actually left Kan extensions of the functors (to be defined), \( Q \circ \alpha \) and \( Q \), respectively. Standard abstract nonsense gives a map between these two Kan extensions, which are sheaves on \( \mathcal{B}_{\mathcal{R}+}(V_Z^\vee) \). We then verify that the category of conic sheaves on \( V_Z^\vee \) is equivalent to the category of sheaves on the basis \( \mathcal{B}_{\mathcal{R}+}(V_Z^\vee) \), thereby obtaining the desired comparison map.
Notation 5.5. To ease the exposition, we sometimes choose to denote \( \psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z} \) and \( \psi_m(\text{pr}_1^* \text{Sp}_{0Z} / V_Z \mathcal{F})|_{V_Z} \) by \( \psi_m \) and \( \psi_m^{\text{Sp}} \), respectively, for the remainder of this section.

5.4. Obtaining expressions for sections of \( \psi_m \) and \( \psi_m^{\text{Sp}} \).

5.4.1. Expression for sections of \( \psi_m \). Let \( j_{>0} : m^{-1}(A_{>0}) \to V_Z \times Z V_Z^\vee \) and \( i_{V_Z^\vee} : V_Z^\vee \to V_Z \times Z V_Z^\vee \) denote the obvious inclusions. Using the expression (2.33) for nearby cycles at the fixed angle \( \theta = \pi \), we obtain:

\[
\psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z^\vee} \simeq i_{V_Z^\vee}^* j_{>0}^* \text{pr}_1^* \mathcal{F}.
\]

Now choose a convex, conic open subset \( U \subset V_Z^\vee \) (c.f. Remark 2.47). We compute the following chain of isomorphisms:

\[
\Gamma(U; \psi_m(\text{pr}_1^* \mathcal{F})|_{V_Z^\vee}) \simeq \lim_{\to \infty} \Gamma(U; j_{>0}^* \text{pr}_1^* \mathcal{F}) \tag{5.6}
\]

\[
\simeq \lim_{\to 0} \Gamma(B,(0) \times Z \mathcal{V}; j_{>0}^* \text{pr}_1^* \mathcal{F}) \tag{5.7}
\]

\[
\simeq \lim_{\to 0} \Gamma(B,(0) \times Z \mathcal{V} \cap m^{-1}(A_{>0}); j_{>0}^* \text{pr}_1^* \mathcal{F}) \tag{5.8}
\]

where, in the second line, \( B,(0) \) is the subset \( Z \times B,(0) \), determined by fixing an isomorphism of \( V \) with \( \mathbb{R}^{\text{dim} \mathcal{V}} \).

Now consider the diagram,

\[
\begin{array}{ccc}
(B,(0) \times Z \mathcal{V} \cap m^{-1}(A_{>0})) & \xrightarrow{j_{*}^* \mathcal{V}} & V_Z \times Z V_Z^\vee \\
\downarrow \text{pr}_1 & & \downarrow \text{pr}_1 \\
\mathcal{U}_{e \mathcal{V}} & \xrightarrow{j_{*}^* \mathcal{V}} & V_Z
\end{array}
\]

In the notation of this diagram, we have:

\[
\Gamma(B,(0) \times Z \mathcal{V} \cap m^{-1}(A_{>0}); j_{>0}^* \text{pr}_1^* \mathcal{F}) \simeq \text{pt}^* j_{*}^* \text{pr}_1^* \mathcal{F} \tag{5.9}
\]

\[
\simeq \text{pt}^* p_{e, \mathcal{V}}^* p_{e, \mathcal{V}}^* j_{*}^* \mathcal{F} \tag{5.10}
\]

\[
\simeq \text{pt}^* j_{*}^* \mathcal{F} \tag{5.11}
\]

\[
=: \Gamma(\mathcal{U}_{e \mathcal{V}}; \mathcal{F}) \tag{5.12}
\]

We use the following lemma to prove that the arrow in the third line is an isomorphism.

Lemma 5.13. Suppose \( E \xrightarrow{p} X \) is a continuous map of topological spaces such that \( E \subset W \) where \( W \) is a topological finite dimensional, real vector bundle over \( X \). Suppose also that the fibers of \( p \) are nonempty, convex subsets of \( W \) and that there exist continuous local sections of \( p \) around each \( x \in X \). Then \( \text{id} \xrightarrow{\simeq} p_\ast p^* \).

Proof. The statement that \( \text{id} \to p_\ast p^* \) is an isomorphism can be checked on stalks, so without loss of generality, we assume \( p \) actually has a global section, \( X \xrightarrow{\phi} E \).
By [KS90, Corollary 2.7.7(ii)] it suffices to show that \( s \circ p \) is homotopic to \( \text{id}_E : E \to E \) over \( X \). That is, we exhibit a continuous map \( H : E \times [0,1] \to E \) such that 
\[
H(-, 1) = s \circ p, \ H(-, 0) = \text{id}_E, \quad \text{and} \quad p \circ \text{pr}_1 = p \circ H. 
\]
Such a map is given by
\[
H(e, t) := t \cdot s(p(e)) + (1 - t) \cdot e,
\]
where the scalar multiplication, \( \cdot \), is performed in \( W \). Since the fiber \( E_x \) is convex, for all \( x \in X \), \( H(t, e) \in E_{p(e)} \) for all \( t \in [0,1] \).

**Claim 5.14.** \( p_{e,V} \) satisfies the hypotheses of Lemma 5.13.

**Proof.** Fix a point \((z_0, v_0) \in \mathcal{U}_{e,V}\). The fiber \( p_{e,V}^{-1}(z_0, v_0) \) is the intersection
\[
\{(v_0, \lambda) \in V_Z \times_{z_0} V_Z' | \Re \lambda(v_0) > 0 \} \cap \langle \langle V, z_0 \rangle \rangle.
\]
It is nonempty by the definition of \( \mathcal{U}_{e,V} \). Note that \( \Re \lambda(z_0, v_0) : V_Z' \to \mathbb{R} \) is map
of real vector bundles over \( Z \), so \( \{(v_0, \lambda) \in V_Z \times_{z_0} V_Z' | \Re \lambda(v_0) > 0 \} \)
is the product of an open half-plane in \( V' \) with \( Z \); in particular, it is a convex, conic open subset of \( V_Z' \).
On the other hand, \( V \) is a convex, conic open subset of \( V_Z' \) by assumption.

By elementary convex geometry, therefore, their intersection is also a convex, conic open subset of \( \{(z_0, v_0)\} \times V' \).

Now, choosing an arbitrary \( \lambda_0 \in p_{e,V}^{-1}(z_0, v_0) \), it is clear from the continuity of \( m \) that \((z, v, \lambda_0) \in p_{e,V}^{-1}(z, v) \) for all \((z, v)\) in a sufficiently small neighborhood of \( z_0, v_0 \). This shows that \( p_{e,V} \) has local sections (i.e. \( s(z, v) = \lambda_0 \)), so the hypotheses of Lemma 5.13 are satisfied. \( \square \)

Thus, we obtain the equivalence,

\[
\text{(5.8) } \lim_{\epsilon > 0} \Gamma(\mathcal{U}_{e,V}; \mathcal{F}) \simeq \Gamma(\mathcal{U}_V; \mathcal{F}).
\]

5.4.2. Expression for sections of \( \psi_m^{sp} \). Observe that \( \mathcal{U}_{e,V} \) is itself open (as \( \text{pr}_1 \) is an open map) and, not quite conic, but rather the intersection of the conic open subset \( \mathcal{U}_V := \text{pr}_1(V \times V \cap m^{-1}(\mathbb{A}^1_{>0})) \) and the open ball \( B_\epsilon(0_Z) \). Thus, in light of the above discussion,

\[
\text{(5.15) } \Gamma(\mathcal{V}; \psi_m(\text{pr}_1^* \text{Sp}_{0_Z/V_Z} \mathcal{F})|_{V_Z'}) \simeq \lim_{\epsilon > 0} \Gamma(\mathcal{U}_{e,V}; \text{Sp}_{0_Z/V_Z} \mathcal{F})
\]

\[
\text{(5.16) } \lim_{\epsilon > 0} \Gamma(\mathcal{U}_V; \text{Sp}_{0_Z/V_Z} \mathcal{F}) \simeq \Gamma(\mathcal{U}_V; \mathcal{F})
\]

\[
\text{(5.17) } \Gamma(\mathcal{U}_V; \mathcal{F}) \simeq \Gamma(\mathcal{U}; \mathcal{F}).
\]

where we have used that the restriction of sections along \( \mathcal{U}_{e,V} \to \mathcal{U}_V \) is an equivalence because \( \text{Sp}_{0_Z/V_Z} \mathcal{F} \) is conic.

Finally, we take advantage of (4.18) to write:

\[
\Gamma(\mathcal{U}_V; \text{Sp}_{0_Z/V_Z} \mathcal{F}) \simeq \lim_{U \in \mathcal{U}_V} \Gamma(U; \mathcal{F}).
\]

**Remark 5.18.** Note that \( \{\mathcal{U}_V\}_{\epsilon > 0} \) is a subcollection of \( \mathcal{F}_{\mathcal{U}_e} \), so there is a natural map of sections over \( \mathcal{V} \), which we denote by \( \psi_\mathcal{V} \), coming from the composition of
the following chain of morphisms:
\[
\Gamma(V; \psi_m(pr_1^* F)|_{V^\vee}) \simeq \lim_{\epsilon > 0} \Gamma(U, V; F)
\]
\[
\to \lim_{U \in \tilde{U} \in V} \Gamma(U; F)
\]
\[
\simeq \Gamma(U, Sp_{0_{V^\vee} V^\vee} F)
\]
\[
\simeq \Gamma(V; \psi_m(pr_1^* Sp_{0_{V^\vee} V^\vee} F)|_{V^\vee}).
\]

\textbf{5.5. Obtaining the comparison map.}

\textbf{5.5.1. Left Kan extensions.} We briefly recall the notion of left Kan extension found in [HTT, §4.3]. First suppose that \(i : C \to \tilde{C}\) is a fully faithful embedding, and let \(F : C \to D\) be an arbitrary functor. In the definition below, for \(\tilde{x} \in \text{ob}(\tilde{C})\): \(\tilde{C}/\tilde{x} := C \times_{\tilde{C}} \tilde{C}/\tilde{x}\) and \((\tilde{C}/\tilde{x})^\triangleright := (\tilde{C})^\triangleright \times_{\tilde{C}} \tilde{C}/\tilde{x}\), where the final object in \((\tilde{C})^\triangleright\) is mapped to \(\tilde{x}\).

\textbf{Definition 5.19.} A left Kan extension of \(F\) along \(i\) is a functor, \(\tilde{F} : \tilde{C} \to D\), making the following diagram commute,

\[
\begin{array}{ccc}
C & \xrightarrow{i} & \tilde{C} \\
\downarrow \tilde{F} & & \downarrow \tilde{F} \\
F & \to & D,
\end{array}
\]

such that for every object \(C \in \tilde{C}\), the induced diagram,

\[
\begin{array}{ccc}
(C/\tilde{x})^\triangleright & \xrightarrow{i} & C/\tilde{x} \\
\downarrow & & \downarrow \\
F & \to & D,
\end{array}
\]

exhibits \(\tilde{F}(\tilde{x})\) as the colimit of \(F_{/\tilde{x}}\).

\textbf{Definition 5.20.} A left Kan extension of \(F\) along an arbitrary functor \(\delta : C \to \tilde{C}\) is defined to be a left Kan extension of \(F\) along the inclusion of \(C\) into the mapping cylinder of \(F\), \(C \times \{0\} \to (C \times [1]) \coprod_{C \times \{1\}} \tilde{C} := M_F\).

\textbf{Remark 5.21.} Left Kan extensions are unique up to a contractible space of choices. We denote the left Kan extension of \(F\) along \(\delta\) by \(\text{LKE}_\delta(F)\).

\textbf{Remark 5.22.} Lurie shows that Definition \textbf{5.20} coincides with Definition \textbf{5.19} in the case when \(\delta\) is a fully faithful embedding.

\textbf{Remark 5.23.} Note that Definition \textbf{5.19} says the left Kan extension of \(F\) along \(i\) is any functor whose on an object \(\tilde{x} \in \tilde{C}\) is the colimit of the diagram, \(C_{/\tilde{x}} \to D\). Similarly, the value of \(\text{LKE}_\delta(F)\) on an object \(\tilde{x} \in \tilde{C}\) is the colimit of the diagram \(C \times_{M_F} M_F_{/\tilde{x}} \to D\). But the natural map \(C \times_{M_F} M_F_{/\tilde{x}} \to C_{/\tilde{x}}\) is an equivalence since \(C \simeq M_F\), so, in fact,

\[
\text{LKE}_\delta(F)(\tilde{x}) = \lim_{x \in C_{/\tilde{x}}} F(x).
\]
5.5.2. Let $I := B_{\mathbb{R}^+}(V_2')^{op} \times (\mathbb{R}^+)^{op}$ be the product category, where $\mathbb{R}^+$, being a totally ordered set, is considered as a category. Similarly, let $J$ denote the full subcategory of $B_{\mathbb{R}^+}(V_2')^{op} \times \mathcal{U}(V_2)^{op}$ on pairs \{$(V, U)|V \in \text{ob}(B_{\mathbb{R}^+}(V_2')), U \in \text{ob}(\mathcal{U}_V)$\}. There are obvious functors, $F : I \to B_{\mathbb{R}^+}(V_2')^{op}$ and $G : J \to B_{\mathbb{R}^+}(V_2')^{op}$ given by the respective projection functors, as well as a functor $\alpha : I \to J$ defined by $(V, \epsilon) \mapsto (V, U_{\epsilon, V})$. Now define a functor $Q : J \to \text{Mod}_e^{\epsilon}$ by the assignment $(V, U) \mapsto \Gamma(U; F) \in \text{Mod}_e^{\epsilon}$; in other words, define $Q$ to be the composition of the projection $J \to \mathcal{U}(V_2)$ and the sheaf $F$. Altogether, we have the following commutative diagram,

\[
\begin{array}{ccc}
I & \xrightarrow{\alpha} & J \\
\downarrow F & & \downarrow G \\
B_{\mathbb{R}^+}(V_2')^{op} & & \text{Mod}_e^{\epsilon}
\end{array}
\]

(5.24)

5.5.3. The category $B_{\mathbb{R}^+}(V_2')$ has a natural Grothendieck topology that we use to consider “basis” sheaves, $\text{Shv}(B_{\mathbb{R}^+}(V_2'))$. There is a canonical restriction functor, $\theta : \text{Shv}(V^{co}_V) \to \text{Shv}(B_{\mathbb{R}^+}(V_2'))$, given by restriction along the inclusion $B_{\mathbb{R}^+}(V_2') \hookrightarrow \mathcal{U}(V^{co}_V)$.

**Lemma 5.25.** $\theta(\psi_m)$ and $\theta(\psi_m^{\text{Sp}})$ are the left Kan extensions of $Q \circ \alpha$ along $F$ and of $Q$ along $G$, respectively.

**Proof.** We begin with the first claim. It was shown in the previous section that $\psi_m(V)$ is the colimit of the following $(\mathbb{R}^+)^{op}$-shaped diagram,

\[
(\mathbb{R}^+)^{op} \longrightarrow \text{Mod}_e^{\epsilon} \\
\Psi \downarrow \Psi \\
\epsilon \mapsto \Gamma(U_{\epsilon, V}; F)
\]

This is a diagram with the same terms as the diagram, which we denote $D_V$, whose colimit computes $\text{LKE}_F(Q \circ \alpha)(V)$ but it is indexed by a different category. Nonetheless, there is an obvious cofinal map $r_V : (\mathbb{R}^+)^{op} \to I_V$ given by the assignment $\epsilon \mapsto (V, \epsilon)$. Indeed, given any $(V', \epsilon) \in I_V$, there is a unique morphism $(V', \epsilon) \to (V, \epsilon)$ coming from the fact that $V \xrightarrow{id} V$ is final in $(B_{\mathbb{R}^+}(V_2'))^{op}/V$. Thus, $\text{colim}(D_V \circ r_V) \simeq \text{colim}(D_V)$, and we have shown that $\psi_m$ is the left Kan extension, $\text{LKE}_F(Q \circ \alpha)$.

Likewise, in the previous section we established that $\psi_m^{\text{Sp}}(V)$ is the colimit of the $\mathcal{U}_V$-shaped diagram,

\[
\mathcal{U}_V \longrightarrow \text{Mod}_e^{\epsilon} \\
\Psi \downarrow \Psi \\
U \mapsto \Gamma(U; F)
\]

On the other hand, $\text{LKE}_G(Q)(V)$ is the colimit of the diagram $D_V' : J_V \to \text{Mod}_e^{\epsilon}$. In this case, as well, there is an obvious map $\mathcal{U}_V \to J_V$ given by $U \mapsto (V, U)$, which is cofinal for the same reason as above. Thus, $\psi_m^{\text{Sp}} \simeq \text{LKE}_G(Q)$, and the claim is proven. $\square$
From the commutativity of (5.24) we obtain a map of Kan extensions, \( \psi : \text{LKE}_F(Q \circ \alpha) \to \text{LKE}_G(Q) \). Unwinding the definitions, it is clear that the map \( \psi(\mathcal{V}) : \text{colim}(D_{\mathcal{V}}) \to \text{colim}(D_{\mathcal{V}'}) \) is necessarily unique morphism coming from the universal property of the first colimit. Upon identification of \( \text{LKE}_F(Q \circ \alpha) \) and \( \text{LKE}_G(Q) \) with \( \psi_m \) and \( \psi_{sp} \), respectively, then, we see that \( \psi(\mathcal{V}) \) is precisely the map \( \psi_{\mathcal{V}} \) constructed in Remark 5.18. We thus obtain a map of basis sheaves, \( \theta(\psi_m) \xrightarrow{\sim} \theta(\psi_{sp}) \).

5.5.4. **Extension of basis sheaves.** The following lemma allows us to lift \( \psi \) to a map of conic sheaves on \( V'_Z \).

**Lemma 5.26.** The canonical restriction map

\[ \theta : \text{Shv}_{\mathbb{R}^+}(V'_Z) \to \text{Shv}((V'_Z)) \]

is an equivalence of categories.

**Proof.** The following argument is based on the discussion in [HTT] right before Warning 7.1.1.4. Observe that the category \( B_{\mathbb{R}^+}(V'_Z) \) has all finite products since the intersection of finitely many convex, conic open subsets is again convex, conic, and open. Moreover, there exists a final object: \( V'_Z \). By [Stacks] Lemma 002O, \( B_{\mathbb{R}^+}(V'_Z) \) therefore admits all finite limits. Moreover, it is a small 0-category. The proof of [HA] Proposition 6.4.5.7 shows that the \( \infty \)-topos, \( \text{Shv}(B_{\mathbb{R}^+}(V'_Z); \mathcal{S}) \), is a 0-localic. As such, it is determined by the locale \( Z \) of subobjects of the final object \( \mathbf{z} \in \text{Shv}(B_{\mathbb{R}^+}(V'_Z); \mathcal{S}) \), which in turn is equivalent to the locale given by the partially ordered set \( B_{\mathbb{R}^+}(V'_Z) \) itself. For the same reasons, the topos \( \text{Shv}_{\mathbb{R}^+}(V'_Z; \mathcal{S}) \) is determined by the partially ordered set \( U_{\mathbb{R}^+}(V'_Z) \).

Now observe that \( B_{\mathbb{R}^+}(V'_Z) \) being a basis means precisely that the inclusion of partially ordered sets, \( B_{\mathbb{R}^+}(V'_Z) \hookrightarrow U_{\mathbb{R}^+}(V'_Z) \) induces an isomorphism of locales. Thus, the restriction map \( \theta_S : \text{Shv}((V'_Z); \mathcal{S}) \to \text{Shv}(B_{\mathbb{R}^+}(V'_Z); \mathcal{S}) \) of \( \infty \)-topoi is an isomorphism.

Now recall [SAG] Proposition 1.3.1.7, which states that, for a small \( \infty \)-category \( \mathcal{T} \) equipped with a Grothendieck topology and \( \mathcal{E} \) an arbitrary \( \infty \)-category admitting small limits,

\[ \text{Shv}_{\mathcal{E}}(\text{Shv}(\mathcal{T}; \mathcal{S})) \xrightarrow{\sim} \text{Shv}(\mathcal{T}; \mathcal{E}), \]

where \( \text{Shv}_{\mathcal{E}}(-) \) denotes the category of small limit-preserving functors \( (-)^{\text{op}} \to \mathcal{E} \).

Then, in light of the equivalence \( \theta_S \), the restriction map, \( \text{Shv}_{\mathbb{R}^+}(V'_Z; \mathcal{S}) \to \text{Shv}(B_{\mathbb{R}^+}(V'_Z); \mathcal{S}) \), is an equivalence. This suffices to prove the lemma, as \( \text{Shv}(-) \) is a full subcategory of \( \text{Shv}(\mathcal{T}; \mathcal{E}) \).

Thus, \( \psi \) lifts to a map of conic sheaves (which we also call \( \psi \) by abuse of notation),

\[ \psi_m(\text{pr}_1^*\mathcal{F})|_{V'_Z} \xrightarrow{\psi} \psi_m(\text{pr}_1^*\text{Sp}_{0Z/V'_Z}\mathcal{F})|_{V'_Z}, \]

such that for \( \mathcal{V} \in B_{\mathbb{R}^+}(V'_Z) \), \( \psi(\mathcal{V}) = \psi_{\mathcal{V}} \), as desired.

5.6. **Proof of Lemma 5.3.** We proceed to show that the comparison morphism, \( (5.27) \), is an isomorphism by computing the induced map of stalks. The map of stalks is shown to be an isomorphism at each point by finding a common cofinal subsequence in each of the colimits computing the stalks of each the sheaves, \( \psi_m(\text{pr}_1^*\mathcal{F})|_{V'_Z} \) and \( \psi_m(\text{pr}_1^*\text{Sp}_{0Z/V'_Z}\mathcal{F})|_{V'_Z} \).
5.6.1. **Proving the induced map on stalks is an isomorphism.** Choose a covector, 
\((z, \lambda) \in V^\vee_Z\). Clearly, the comparison map (5.27) induces the following map of 
stalks at \((z, \lambda)\):

\[
\lim_{\mathcal{V}} \left( \lim_{\epsilon > 0} \Gamma(U, \mathcal{V}; F) \right) \to \lim_{\mathcal{V}} \left( \lim_{U \in \mathcal{V}} \Gamma(U; F) \right),
\]

where \(\mathcal{V}\) ranges over the family of convex, conic open subsets of \(V^\vee\) containing 
\((z, \lambda)\).

Let \(\{\mathcal{W}_m\}_{m \in \mathbb{N}}\) be a countable neighborhood basis of \(z \in Z\). If \(\lambda \neq 0\), let \(\gamma_n\), 
\(n \in \mathbb{N}\), be a countable neighborhood basis of \(\lambda\) in the conic topology on \(V^\vee\) such 
that \(\gamma_{n'} \subset \gamma_n \) for all \(n' > n\). If \(\lambda = 0\), let \(\gamma_n = V^\vee\) for all \(n\).

Clearly, the subcollection \(\{\mathcal{W}_{m,n} := \mathcal{W}_m \times \gamma_n\}\) is cofinal for both the outer left- 
and right-hand colimits above. Furthermore,

\[
\lim_{k, m, n} \Gamma(U, \mathcal{W}_{m,n}; F) \overset{\simeq}{\to} \lim_{\mathcal{V}} \left( \lim_{\epsilon > 0} \Gamma(U, \mathcal{V}; F) \right),
\]

where \(\{U, \mathcal{W}_{m,n}\}_{k \in \mathbb{N}}\) is the subcollection of \(\{U, \mathcal{W}_{m,n}\}\) where \(\epsilon = \frac{1}{k}\).

We now show that the subsequence \(\{U, \mathcal{W}_{m,n}\} \subset \{\mathcal{F}_n U\}\) is cofinal for the 
right-hand side as well by demonstrating that any elements of the latter contains 
an element of the former. In order to ease the exposition, we assume without loss 
of generality that \(Z = *\) (so that we can drop the index \(m\)), and denote \(U_n\), and 
\(\mathcal{F}_n U\), by \(U_n\) and \(\mathcal{F}_n\), respectively.

First suppose that \(\lambda \neq 0\). Fix \(n\), and suppose that \(U \in \mathcal{F}_n\). Recall that this 
means \(C_0(V \setminus U) \cap U_n = \emptyset\), where \(U_n = \text{pr}_1(V \times \gamma_n \cap m^{-1}(A^1_{>0}))\).

**Claim 5.29.** There exists an \(\epsilon_n > 0\) such that \(B_{\epsilon_n}(0) \cap U_t \subset U\) for all \(t > n\).

**Proof.** Suppose that no such \(\epsilon_n\) existed. Then for each \(\epsilon > 0\), \((B_{\epsilon}(0) \cap U_m) \cap \partial U \neq \emptyset\).

To see this, note that \((B_{\epsilon}(0) \cap U) \cap U \neq \emptyset\), for otherwise,

\[
C_0(V \setminus U) \cap U_n \supset C_0(V \setminus U) \cap U_t = C_0(V \setminus U) \cap U_t \cap C_0(B_{\epsilon}(0) \cap U_t)
= C_0(V \setminus U) \cap C_0(B_{\epsilon}(0) \cap U_t)
= C_0(B_{\epsilon}(0) \setminus (B_{\epsilon}(0) \cap U_t)) \cap C_0(B_{\epsilon}(0) \cap U_t)
\]

\[
\supset T_0 V \cap p_{>0}^{-1}(B_{\epsilon}(0) \setminus (B_{\epsilon}(0) \cap U_t)) \cap p_{>0}^{-1}(B_{\epsilon}(0) \cap U_t)
= T_0 V \cap p_{>0}^{-1}(B_{\epsilon}(0) \setminus (B_{\epsilon}(0) \cap U_t)) \cap B_{\epsilon}(0) \cap U_t
= U_t
\neq \emptyset,
\]

contradicting our assumption that \(U \in \mathcal{F}_n\).

Now suppose that \((B_{\epsilon}(0) \cap U) \cap \partial U = \emptyset\). Observe that \(\partial((B_{\epsilon}(0) \cap U_t) \cap U) \neq \emptyset\) 
because \((B_{\epsilon}(0) \cap U_t) \cap U\) is a nonempty, open proper subset of \(V \simeq \mathbb{R}^\dim\).

Thus, given a point \(x \in \partial((B_{\epsilon}(0) \cap U_t) \cap U)\), there is a sequence \(x_i \in (B_{\epsilon}(0) \cap U_t) \cap U\) 
such that \(x_i \to x\). If \(x \notin \partial(B_{\epsilon}(0) \cap U_t) \cap \partial U\), then \(x \in (B_{\epsilon}(0) \cap U_t) \cap U\) necessarily 
since \(x\) belongs to the closures of both \((B_{\epsilon}(0) \cap U_t) \cap U\). Since
Theorem 6.1. Given any closed immersion \( B_r(0) \cap \mathcal{U}_t \neq \mathcal{U} \) by assumption, there exists a point in \((B_r(0) \cap \mathcal{U}_t) \cap \partial \mathcal{U}\); simply choose \( x \in (B_r(0) \cap \mathcal{U}_t) \cap \partial((B_r(0) \cap \mathcal{U}_t) \cap \mathcal{U}) \).

Now choose a point \( x_j \in (B_r(0) \cap \mathcal{U}_t) \cap \partial \mathcal{U}. \) Doing so for each value \( \epsilon = \frac{1}{j} \) yields a sequence in \( V \setminus \mathcal{U}, \) \( x_j \to 0. \) Normalize the terms of this sequence as \( x' \) converging to \( v \in \mathcal{U}. \) By \( \text{KS90} \) Proposition 4.1.2(ii)], \( v \in C_0(V \setminus \mathcal{U}), \) so \( C_0(V \setminus \mathcal{U}) \cap \mathcal{U}_n \supset C_0(V \setminus \mathcal{U}) \cap \mathcal{U}_t \neq \emptyset, \) a contradiction.

Choose \( \frac{1}{k} < \epsilon_n. \) Then \( \mathcal{U}_{n,k} \subset \mathcal{U}, \) which proves the equivalence of stalks at \( \lambda \neq 0. \)

Finally, suppose that \( \lambda = 0. \) A neighborhood basis for 0 in the conic topology was given by the single, nonempty conic open containing: \( \gamma \) via the canonical fiber sequences (2.29). We deduce that the induced map \( \phi \) is also an equivalence.

Choose a point \( x \in \mathcal{U}. \) Thus, the comparison map, (5.27), is an equivalence.

Note that for each value \( \epsilon \neq 0 \) the equivalence of stalks reduces to the claim that, \( \lim_{\epsilon \to 0} \Gamma(B_r(0); \mathcal{F}) \xrightarrow{\cong} \lim_{\epsilon \to 0} \Gamma(\mathcal{U}; \mathcal{F}). \)

But this is identical to the claim that \( \mathcal{F}_0 \xrightarrow{\cong} (\mathcal{S}_0^0/\mathcal{V})_0 \) which was shown earlier, in Lemma 4.7 (alternatively, in the proof of \( \text{KS90} \) Theorem 4.2.1(iv) provided there). Thus, the comparison map, (5.27), is an equivalence.

5.6.2. Finishing the proof of Lemma 5.3. Observe that, by smooth base change, the naive \( \ast \)-restriction of \( \text{pr}^*_1 \mathcal{F} \) and \( \text{pr}^*_1 \mathcal{S}_0^0/\mathcal{V}_x \) to the subspace \( V'_x \) coincide:

\[
(\text{pr}^*_1 \mathcal{F})|_{V'_x} \cong \mathcal{F}|_{V_x} \cong (\mathcal{S}_0^0/\mathcal{V}_x)|_{V_x} \cong (\text{pr}^*_1 \mathcal{S}_0^0/\mathcal{V}_x)|_{V'_x}.
\]

As such, we obtain a commutative square which extends to the map of fiber sequences,

\[
\begin{array}{ccc}
\varphi_m(\text{pr}^*_1 \mathcal{F}) & \longrightarrow & (\text{pr}^*_1 \mathcal{F})|_{V'_x} \\
\downarrow \varphi & & \downarrow \cong \\
\varphi_m(\text{pr}^*_1 \mathcal{S}_0^0/\mathcal{V}_x) & \longrightarrow & (\text{pr}^*_1 \mathcal{S}_0^0/\mathcal{V}_x)|_{V'_x}
\end{array}
\]

via the canonical fiber sequences (2.29). We deduce that the induced map \( \varphi \) is also an equivalence by the three lemma for stable \( \infty \)-categories. This completes the proof of Lemma 5.3.

6. The global equivalence

In this section we prove the main result of this paper, which is a globalized version of Theorem 5.1, proven in the previous section.

Theorem 6.1. Given any closed immersion \( Z \hookrightarrow X \) of a quasi-smooth derived scheme \( Z \) into a smooth scheme \( X, \) there exists an isomorphism,

\[
\varphi_{T^*[-1]Z} \simeq \mu_{Z/X}[\text{codim.vir}(Z, X)][\xi_{Z}(\dim X)],
\]

of perverse sheaves on \( T^*[-1]Z \subset N^\vee_{Z/X}. \)

The left-hand side of this equivalence is the perverse sheaf associated to a \(-1\)-shifted symplectic scheme, as defined in \( \text{BBDSS15}. \) We briefly review its construction, and state some relevant results.
6.1. d-critical loci and the canonical perverse sheaf.

6.1.1. We assume familiarity with the basic concepts and definitions of shifted symplectic geometry, as appears in e.g. [PTVV13].

6.1.2. d-critical loci. An algebraic d-critical locus, defined in [J15], is a classical model for the structure of a −1-shifted symplectic derived scheme. Its definition consists of a classical scheme equipped with extra data. We state the precise definition after introducing some notation.

Notation 6.2. The exposition we present here is largely derived from [K21a], which in turn is a summary of the results and ideas of [J15].

To any \( C \)-scheme \( X \), locally of finite type, Joyce associates a \( C \)-sheaf \( S_X \), whose sections, loosely speaking, parametrize the various ways of writing \( X \) as the critical locus of a regular function on a smooth scheme.

The sheaf \( S_X \) enjoys the following (more-or-less defining) property ([J15]): for an open \( R \subset X \) and any closed embedding \( i : R \hookrightarrow U \) into a smooth scheme \( U \), there is an exact sequence of sheaves,

\[
0 \to S_X|_R \xrightarrow{i_{R,U} \cdot \iota^{-1}} i^{-1}O_U \xrightarrow{d} i^{-1}\Omega_U \to 0,
\]

which identifies \( S_X \) as the kernel of the de Rham differential \( d \); here \( I_{R,U} \subset i^{-1}(O_U) \) is the ideal sheaf of \( i^{-1}O_U \) corresponding to the closed embedding \( i : R \hookrightarrow U \).

The compositions,

\[
S_X|_R \xrightarrow{i^{-1}O_U \cdot \iota^{-1}} i^{-1}O_U \xrightarrow{i^{-1}\Omega_U} O_R,
\]

for various \( R \subset X \) are compatible and glue to a map \( S_X \xrightarrow{\beta_X} O_X \). Joyce defines a subsheaf \( S^0_X \subset S_X \) as the kernel of the composition,

\[
S_X \xrightarrow{\beta_X} O_X \to O_{X^{red}},
\]

and shows that there is decomposition \( S_X = \bigoplus_X \otimes S^0_X \).

Definition 6.3. [J15, Definition 2.5] An algebraic d-critical locus is a pair \((X, s)\), where \( X \) is a \( C \)-scheme (locally of finite type), and \( s \in \Gamma(X; S_X^0) \) satisfying the condition that \( X \) may be covered by Zariski open sets \( R \subset X \) with a closed embedding \( i : R \hookrightarrow U \) into a smooth scheme \( U \), and a regular function \( f \) on \( U \) such that \( i(R) = \text{crit}(f) \) and \( f + I_{R,U}^2 = s|_R \). The tuple \( \mathcal{A} = (R, U, f, i) \) as above is called a critical chart for \( (X, s) \).

Remark 6.4. There is an appropriate notion of embedding of critical charts [J15, Definition 2.18] which allows us to compare two different, overlapping critical charts. Indeed, by [J15, Theorem 2.20], around each point in the overlap of critical charts \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), we may find a third critical chart into which \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) commonly embed.

Remark 6.5. Joyce also defines the notion of a complex analytic d-critical locus in a very similar fashion. For our purposes, the notion of algebraic d-critical locus suffices, but the results of this section are equally valid should we consider d-critical structures on the analytifications of the schemes in consideration, rather than the schemes themselves.
6.1.3. Oriented d-critical loci. Joyce proves the existence of a canonical line bundle on any algebraic d-critical locus.

**Theorem 6.6 ([J15, Theorem 2.28]).** Let \((X, s)\) be an algebraic d-critical locus. There is a natural line bundle \(K_{X,s}\) on \(X^{\text{red}}\) called the canonical line bundle characterized by the following properties:

(i) If \(\mathcal{R} = (R, U, f, i)\) is a critical chart of \((X, s)\), we have an isomorphism,

\[
\iota_{\mathcal{R}} : K_{X,s}|_{R^{\text{red}}} \cong (i^*K_U)^{\otimes 2}|_{R^{\text{red}}},
\]

where \(K_U\) is the usual canonical bundle on \(U\).

(ii) If \(\Phi : \mathcal{R}_1 \hookrightarrow \mathcal{R}_2\) is an embedding of critical charts, we have

\[
\iota_{\mathcal{R}_2}|_{R_1^{\text{red}}} = J_{\Phi} \circ \iota_{\mathcal{R}_1},
\]

where \(J_{\Phi}\) is a natural isomorphism \((i_1^*K_{U_1})^{\otimes 2}|_{R_1^{\text{red}}} \cong (i_2^*K_{U_2})^{\otimes 2}|_{R_2^{\text{red}}}\).

An orientation on a d-critical locus is then defined in terms of this canonical bundle.

**Definition 6.7.** An orientation on a d-critical locus \((X, s)\) is a choice of square root \(K_{X,s}^{1/2}\) for the canonical bundle on \(X^{\text{red}}\),

\[
o : (K_{X,s}^{1/2})^{\otimes 2} \cong K_{X,s}.
\]

A d-critical locus \((X, s)\) equipped with an orientation \(o\) is called an oriented d-critical locus, which we denote by the tuple \((X, s, o)\).

6.1.4. The canonical perverse sheaf on an oriented d-critical locus. To any oriented d-critical locus \((X, s, o)\), Joyce et al. introduce a canonical perverse sheaf, \(\varphi_{X,s,o}\), on \(X\) which is a kind of globalized vanishing cycles (c.f. Remark 2.31) sheaf; namely, if \((X, s)\) is locally modeled on the critical locus \(\text{crit}(f : U \to \mathbb{A}^1)\), \(\varphi_{X,s,o}\) is locally modeled on the vanishing cycles along \(f\), up to a twist by a \(\mathbb{Z}/2\mathbb{Z}\)-bundle determined by the orientation \(o\). The precise statement follows.

**Theorem 6.8 ([BBDJSS15, Theorem 6.9(i)])**. Let \((X, s, o)\) be an oriented d-critical locus, with orientation given by square root \(K_{X,s}^{1/2}\). Then there exists a perverse sheaf \(\varphi_{X,s,o} \in \text{Perv}(X)\) which is natural up to canonical isomorphism, such that if \(\mathcal{R} = (R, U, f, i)\) is a critical chart on \((X, s)\), there is a natural isomorphism,

\[
\omega_{\mathcal{R}} : \varphi_{X,s,o}|_R \to i^*(\varphi_f) \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{\mathcal{R}},
\]

where \(Q_{\mathcal{R}} \to R\) is the principal \(\mathbb{Z}/2\mathbb{Z}\)-bundle parametrizing local isomorphisms \(\alpha : K_{X,s}^{1/2} \to (i^*K_U)|_{R^{\text{red}}}\) such that \(\alpha \otimes \alpha = \iota_{\mathcal{R}} \circ o\).

6.1.5. The canonical perverse sheaf on \(T^*[-1]Z\). We remarked above that d-critical loci are classical models for \(-1\)-shifted symplectic schemes. This remark is made precise by the following theorem.

**Theorem 6.9 ([BBJ19, Theorem 6.6]).** Let \((X, \omega_X)\) be a \(-1\)-shifted symplectic derived scheme. Then the underlying classical scheme \(X\) carries a canonical d-critical structure \(s_X\) characterized by the following property: if \((R = \text{crit}(f), \omega_R)\) is a \(-1\)-shifted symplectic scheme, where \(f\) is a regular function on a smooth scheme \(U\), \(\omega_R\) is the usual \(-1\)-shifted symplectic form on a derived critical locus, and an
open inclusion \( i : R \hookrightarrow X \) such that \( i^* \omega_X \cong \omega_R \), the tuple \((R, U, f, i)\) gives a critical chart for \((X, s_X)\), where \( i : R \hookrightarrow U \) is the natural closed embedding.

In particular, the \(-1\)-shifted cotangent bundle, \( \mathcal{T}^*[-1] \mathcal{Z} \), of any derived scheme \( \mathcal{Z} \) carries a canonical \(-1\)-symplectic structure, and therefore the underlying classical scheme \( \mathcal{T}^*[-1] \mathcal{Z} \) carries a canonical \( d \)-critical structure, the section of which is denoted by \( s_{\mathcal{T}^*[-1] \mathcal{Z}} \). When \( \mathcal{Z} \) is additionally quasi-smooth, the \( d \)-critical locus \((\mathcal{T}^*[-1] \mathcal{Z}, s_{\mathcal{T}^*[-1] \mathcal{Z}})\) carries a canonical orientation\(^{23}\), denoted by \( o_{\mathcal{T}^*[-1] \mathcal{Z}} \).

**Definition 6.10.** Given a quasi-smooth derived scheme \( \mathcal{Z} \), we denote the canonical perverse sheaf, \( \varphi((\mathcal{T}^*[-1] \mathcal{Z}, s_{\mathcal{T}^*[-1] \mathcal{Z}}, o_{\mathcal{T}^*[-1] \mathcal{Z}})) \), by \( \varphi([-1] \mathcal{Z}) \).

In general, \(-1\)-shifted symplectic schemes are Zariski locally modeled on a derived critical locus\(^{24}\). On the other hand, quasi-smooth derived schemes are locally modeled on the derived zero locus of a section \( s \in \Gamma(U; \mathcal{E}) \) of a vector bundle \( \mathcal{E} \) on a smooth affine scheme \( U \). When the quasi-smooth scheme \( \mathcal{Z} \) is locally modeled on the zero locus, \( \mathcal{Z}(s) \), \( \mathcal{T}^*[-1] \mathcal{Z} \) is locally modeled on the derived critical locus \( \text{crit}(\tilde{s}) \), where \( \tilde{s} \) is the associated regular function on \( E' := \text{Spec}_U(\text{Sym}^*_U((\mathcal{E}'))) \). The following lemma from \[K21a\] establishes the compatibility between the \(-1\)-shifted symplectic structures on \( \mathcal{T}^*[-1] \mathcal{Z}(s) \) coming from its structure as a derived critical locus and from its being a shifted cotangent bundle.

**Lemma 6.11 (\[K21a\], Lemma 2.7).** Let \( U = \text{Spec} \, A \) be a smooth affine scheme admitting a global étale coordinate, \( \mathcal{E} \) be a trivial vector bundle on \( U \), and \( s \in \Gamma(U; \mathcal{E}) \) be a section. Denote by \( \tilde{s} \) the regular function on \( E' \) corresponding to \( s \). Then we have an equivalence of \(-1\)-shifted symplectic schemes,

\[
(\text{crit}(\tilde{s}), \omega_{\text{crit}(\tilde{s})}) \simeq (\mathcal{T}^*[-1] \mathcal{Z}(s), \omega_{\mathcal{T}^*[-1] \mathcal{Z}(s)}).
\]

### 6.1.6. Triviality of \( \mathcal{Q}_{\mathcal{Z}} \).

Later in this section, we expend a good deal of effort proving that the equivalences of Theorem 5.1 that are obtained for each member of a particular critical chart cover of the oriented \( d \)-critical locus, \((\mathcal{T}^*[-1] \mathcal{Z}, s_{\mathcal{T}^*[-1] \mathcal{Z}}, o_{\mathcal{T}^*[-1] \mathcal{Z}})\), glue. Fortunately, our task is made much simpler by the apparent triviality of \( \mathcal{Q}_{\mathcal{Z}} \) for critical charts of \((\mathcal{T}^*[-1] \mathcal{Z}, s_{\mathcal{T}^*[-1] \mathcal{Z}}, o_{\mathcal{T}^*[-1] \mathcal{Z}})\) obtained from expressing \( \mathcal{Z} \) locally as a derived zero locus \( \mathcal{Z}(s) \).

**Lemma 6.12 (\[K21a\], Lemma 2.19).** Let \( U, s \) be as in Lemma 6.11. Denote by \( o_{\mathcal{T}^*[-1] \mathcal{Z}(s)} \) the canonical orientation and \( \mathcal{Z}' = (\text{crit}(\tilde{s}), E', \tilde{s}, i) \) the critical chart induced by the equivalence of Lemma 6.11. Then \( \mathcal{Q}_{\mathcal{Z}'}^{o_{\mathcal{Z}'}} \) is a trivial \( \mathbb{Z}/2\mathbb{Z} \)-bundle.

As such, \( \omega_{\mathcal{Z}'} : \varphi(\mathcal{T}^*[-1] \mathcal{Z}(s)|_{\text{crit}(\tilde{s})}) \xrightarrow{i^*} i^*(\varphi_{\tilde{s}}) \).

### 6.2. Local models of quasi-smooth schemes.

In order to prove Theorem 6.1 we will some particular results on the local structure of quasi-smooth closed immersions, which we now establish.

---

\(^{22}\) Constructed in e.g. \[K21a\] Example 2.5 or \[BBJ19\] Example 5.8 and 5.15

\(^{23}\) c.f. \[K21a\] Example 2.15 or \[T21\] Lemma 3.3.3

\(^{24}\) \[BBJ19\] Theorem 5.18

\(^{25}\) \[K21a\] Proposition 2.3
6.2.1. The following construction is a very slight variant of one that appears in the proof of [HA, Theorem 8.4.3.18], and virtually identical—apart from presentation—to the one that appears in [BBJ19, Example 2.8] of what the authors of loc. cit. call “standard form cdgas.” We use the conventions and notation of [HA].

**Construction 6.13.** Let $B$ be a connective $E_\infty$-algebra locally of finite presentation over a connective $E_\infty$-ring $A$ such that $\pi_0(B)$ is finitely generated as a $\pi_0(A)$-algebra, and suppose that $L_{B/A}$ is perfect of Tor-amplitude $[0,1]$. Assume additionally that $\pi_0A$ is Noetherian, and choose generators $\{x_i\}_{i=1}^n$ for $\pi_0B$ over $\pi_0A$.

Choose $B(0)$ a smooth $A$-algebra (i.e. $\underline{L}_{B(0)/A}$ is projective) and $f : B(0) \to B$ for which the induced map $\pi_0f : \pi_0B(0) \to \pi_0B$ is surjective$^{20}$.

The map $f$ endows $B$ with the structure of a $B(0)$-module, and so we can in fact view $f$ as a map in $\text{CAlg}_{B(0)}$. Since $B(0)$ is smooth over $A$, $\pi_0B(0)$ is a finitely generated $\pi_0A$-algebra. Since $\pi_0A$ is Noetherian, $\pi_0B(0)$ is Noetherian. Thus $I := \ker \pi_0f$ is finitely generated. Choose generators $\{g_j\}_{j=1}^m \subset \ker \pi_0f$ for $I$.

From the long exact sequence associated to a fiber sequence, the image of $\pi_0\text{fib}(f) \to \pi_0B(0)$ is identified with the ideal $I$. By abuse of notation we use $\{g_j\}$ to denote both the elements in $\pi_0B(0)$ and a choice of their lifts to $\pi_0\text{fib}(f)$. Choose a finitely generated, projective $B(0)$-module $N$ of rank $m$ and a map $N \to \text{fib}(f)$ sending the generators of $N$ to the elements $\{g_j\}$ (for example, the free module on $m$ elements and the map generated by the choice of $\{g_j\} \subset \pi_0\text{fib}(f)$). We thereby obtain a map $g : N \to \text{fib}(f) \to B(0)$. Hence, $g$ classifies a diagram

$$
\begin{array}{ccc}
N & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
B(0) & \xrightarrow{f} & B
\end{array}
$$

in $\text{Mod}_{B(0)}$. Adjoint to this is a diagram in $\text{CAlg}_{B(0)}$,

$$(A) \quad \begin{array}{ccc}
\text{Sym}_{B(0)}(N) & \xrightarrow{0} & B(0) \\
\downarrow & & \downarrow \\
B(0) & \xrightarrow{f} & B
\end{array},$$

which classifies a map $h_B : B(1) := B(0) \otimes_{\text{Sym}_{B(0)}(N)} B(0) \to B$.

**Notation 6.14.** To prevent possible confusion, we sometimes denote $N$ in the preceding construction by $N_B$.

**Lemma 6.15.** Take $B$ as in Construction 6.13. There exists an affine open neighborhood $\text{Spec} C$ of each point $x \in \text{Spec} B$, such the map $h_C : C(1) \to C$ is an isomorphism of $E_\infty$-algebras over $A$.

---

$^{20}$Such a choice might be the free $A$-algebra generated on $n$ elements with $f$ given by the choose of generators $\{x_i\} \subset \pi_0B$, by which we mean the following. The “forgetful” functor $\Omega^\infty : \text{Mod}_A \to \text{Spec}$ admits a left adjoint denoted $\Sigma^\infty : \text{Spec} \to \text{Mod}_A$, and $M$ is $\Sigma^\infty$ applied to the disjoint union of $n$ points. The choice of elements $\{x_i\}_{i=1}^n$ in the presentation of $\pi_0(B)$ lifts to a map of $A$-modules $f : M \to B$ as follows. By adjunction, $\pi_0\text{Map}(M,B) = \pi_0\text{Map}(\coprod_i \Sigma^\infty(B)) = \pi_0(\text{Map}(\coprod_i \Omega^\infty(B))) = \pi_0\Omega^\infty(B)^n = (\pi_0\Omega^\infty(B))^n = (\pi_0B)^n$. That is, an $n$-tuple of elements in $\pi_0B$ determine a map $M \to B$. The forgetful functor $\text{Obv} : \text{CAlg}_A \to \text{Mod}_A$ has a left adjoint $\text{Sym}_A : \text{Mod}_A \to \text{CAlg}_A$. By abuse of notation, we denote the map in $\text{CAlg}_A$ adjoint to $f$ by $f : \text{Sym}_A(M) \to B$. 

Proof. First observe that the hypotheses of Construction \cite{6.13} are stable under localization, so the statement of the lemma makes sense.

Fix \( x \in \text{Spec } B \), and take \( \text{Spec } B \) as an ansatz for such a neighborhood. The proof proceeds by first trying to show \( h_B \) is an isomorphism, then localizing when necessary.

Observe that \( \pi_0 h_B : \pi_0 B(1) \to \pi_0 B \) is an isomorphism by construction. By \cite{HA Corollary 7.4.3.2}, which relates the connectivity of the cofiber of a map to the connectivity of its cotangent complex, it therefore suffices to show that the relative cotangent complex \( L_{B/B(1)} = L_{h_B} \) vanishes.

Consider the fiber sequence
\[
L_{B/A}[-1] \to L_{B/B(0)}[-1] \to L_{B(0)/A} \otimes_{B(0)} B
\]
where here and in what follows \( L_{B/B(0)} = L_f \). By assumption, \( L_{B/A}[-1] \) is perfect of Tor-amplitude \([-1, 0] \). Since \( B(0) \) is the free algebra on \( A \)-module \( M, L_{B(0)/A} \cong M \otimes B(0) \). Hence, \( L_{B(0)/A} \otimes_{B(0)} B \cong M \otimes B \), which is a projective module over \( B \).

By \cite{HA Proposition 7.2.4.23}, we deduce that \( L_{B/B(0)}[-1] \) is perfect of Tor-amplitude \( \leq 0 \). Since the cofiber of \( f : B(0) \to B \) is 1-connective, \( L_{B/B(0)}[-1] \) is connective (see \cite{HA Corollary 7.4.3.2}).

Hence is a flat \( B \)-module by \cite{HA Remark 7.4.2.22}. As a flat \( B \)-module, by \cite{HA Remark 7.4.2.22} \( L_{B/B(0)}[-1] \) is a finitely generated, projective \( B \)-module. If \( L_{B/B(0)}[-1] \) is free, then our choice of \( N \) in Construction \cite{6.13} yields an isomorphism \( L_{B/B(0)} \cong N[1] \otimes B \).

If \( L_{B/B(0)}[-1] \) is not free, choose a localization of \( \pi_0 B \) over which \( \pi_0 L_{B/B(0)}[-1] \) is a free module. This localization of \( \pi_0 B \) is given by a multiplicative subset \( S \subseteq \pi_0 B \), a commutative ring, and as such is a set of homogeneous elements of the graded ring \( \pi_0 B \) which satisfies both the left and the right Ore conditions. Denote by \( C \) the localization \( B[S^{-1}] \), and observe that it is trivially also locally of finite presentation over \( A \).

Repeat the above discussion mutatis mutandis with \( C \) in place of \( B \). We see that \( L_{C/C(0)}[-1] \) is a finitely generated, projective \( C \)-module such that \( \pi_0 L_{C/C(0)}[-1] \) is free. Note that \( \pi_0 C(0) \to \pi_0 C \) is surjective. A careful accounting of the details in the proof of \cite{HA Lemma 7.2.2.19} reveals that \( L_{C/C(0)}[-1] \) is in the essential image of the functor \( \text{hProj} C \to \text{hProj} C \). Thus, there exists a projective \( C(0) \)-module \( N \) such that \( N[1] \otimes_{C(0)} C \cong L_{C/C(0)} \).

Using \( N \) as the choice of projective module surjecting onto \( (g_1, \ldots, g_m) \) in Construction \cite{6.13} we obtain that the first morphism in the fiber sequence
\[
L_{B(1)/B(0)} \otimes_{B(1)} B \to L_{B/B(0)} \to L_{B/B(1)}
\]
is an equivalence, so \( L_{B/B(1)} \) vanishes as desired.

\[\square\]

Remark 6.16. Note that in the proof of Lemma \cite{6.15} \( C(0) \) may be taken to be \( B(0) \) localized with respect to the preimage of \( S \) under the surjection \( \pi_0 B(0) \to \pi_0 B \), which is also a multiplicative subset of homogeneous elements. In particular, this means we could have required \( N \) in the above proof to be free after further localizing \( C \) and \( C(0) \). This observation will be crucial to the proof of Lemma \cite{6.20}.
6.2.2. The previous lemma provided us with a particular means of constructing a local model for a quasi-smooth derived scheme over an arbitrary affine base. A related means of describing quasi-smooth schemes locally is given by a Kuranishi chart, which expresses the scheme locally as the derived zero locus of a section of a vector bundle over a smooth base. We recall the definition given in [K21a].

**Definition 6.17 ([K21a, Definition 2.2])**. For a quasi-smooth derived scheme $X$, a Kuranishi chart is a tuple $(Z, U, E, s, \iota)$ where $Z$ is an open subscheme of $X$, $U$ is a smooth scheme, $E$ is a vector bundle on $U$, $s$ is a section of $E$, and $\iota : Z(s) \to X$ is an open immersion whose image is $Z$. A Kuranishi chart is said to be minimal at $p = i(q) \in Z$ if the differential $(ds)_q : T_q U \to E_q$ is zero. A Kuranishi chart is called good if $U$ is affine and has global étale coordinates (i.e. regular functions $x_1, x_2, \ldots, x_n$ such that $dR x_1, dR x_2, \ldots, dR x_n$ form a basis of $\Omega^1_U$), and $E$ is a trivial vector bundle of a constant rank.

**Remark 6.18.** Any smooth scheme will have étale coordinates Zariski locally around each point (see e.g. [Stacks, Lemma 054L]), so any Kuranishi chart locally gives a good Kuranishi chart.

**Remark 6.19.** When $A$ is a classical, Noetherian ring the condition that $B$ is locally finitely presented over $A$ is equivalent to the condition that $B$ is locally of finite type over $A$ in the sense of [GR17]. In this case, Lemma 6.15 furnishes a proof of the standard fact (see e.g. [AG15, Proposition 2.1.10]) that a quasi-smooth, derived scheme $Z$ locally of finite type over $k$ has a Zariski-local presentation as the derived zero locus of a section $s : U \to E$ where $U$ is smooth and $E$ is a vector bundle over $U$.

Indeed, given a point $x \in Z$, we may let $U := \text{Spec } C(0) = \mathbb{A}^n_k$ and $s : \text{Spec } C(0) \to \text{Spec}(\text{Sym}_C(0) N)$ be induced by the vertical map in the pushout diagram, $\mathbb{A}$. Applying the functor $\text{Spec}$ to the pushout diagram, $\mathbb{A}$, we obtain the Kuranishi chart:

$$
\begin{array}{ccc}
\text{Spec } B & \longrightarrow & U \\
\downarrow & & \downarrow 0 \\
U & \underset{s}{\longrightarrow} & E
\end{array}
$$

In fact, by the Quillen-Suslin theorem, $E$ is a trivial vector bundle over $U = \mathbb{A}^n_k$, so this is a good Kuranishi chart.

Conversely, suppose that $Z' := \text{Spec } B \hookrightarrow Z$ is an open immersion. Then, in a straightforward way, a good Kuranishi chart $(Z', U, E, s, \iota)$ whose image is $\text{Spec } \pi_0 B$ furnishes data of the sort used algorithmically in Construction 6.13 to produce $B(1) \in C\text{Alg}_k$.

In the notation of that construction:

1. $\pi_0 B = \mathcal{O}_{Z'}$;
2. $B(0) = \mathcal{O}_U$;
3. $N$ is the sheaf of sections of $E$;
4. $s$ furnishes the vertical map $\text{Sym}_k(N) \to B(0)$ in diagram $\mathbb{A}$;
5. $B(1) = \mathcal{O}_{Z(s)}$; and
6. the open immersion $\iota$ induces the natural isomorphism $B(1) \to B$.

---

27 So named because the “coordinate” map, $U \to \mathbb{A}^n$, defined by such a choice of functions is clearly étale, and conversely any such étale map will give such coordinates.
The following lemma is the central tool in our proof of Theorem 6.1. It will allow us to compare the isomorphisms obtained from Theorem 5.1 using local expressions of \(Z\) as the derived zero locus of different functions on \(X\). It is a variant of [K21a, Proposition 2.3(ii)] which in turn is a rephrasing of [BBJ19, Theorem 4.2].

Lemma 6.20. Let \(Z \to U\) be a closed immersion of a quasi-smooth derived scheme \(Z\) into smooth \(U\). Suppose that \(U\) is the union of two opens, \(U_1\) and \(U_2\), such that there exist Kuranishi charts, \((Z_i, U_i, E_i, s_i, \iota_i)\), on \(Z\), where \(E_i\) is constant rank \(n\).

Denote \(Z_i = Z \times_U U_i\) for \(i = 1, 2\).

Then around each point \(x \in Z_1 \times_Z Z_2\) there exists a third good Kuranishi chart \((Z', U', E', s', \iota')\) such that \(x \in \iota'(Z(s'))\) and \(E'\) is also constant rank \(n\); open immersions \(\eta_i : U' \to U_i\); and isomorphisms \(\tau_i : E' \to \eta_i^* E_i\) with the following properties:

(i) \(\tau_i(s') = \eta_i^*(s_i)\)
(ii) The composition \(Z(s') \to Z(s_i) \to U\) is equivalent to \(\iota'\) where the first map is induced by \(\eta_i\) and \(\tau_i\), and the second map is \(\iota_i\).

Proof. \(Z_i \to Z\) are clearly affine open immersions of derived schemes, so \(Z_1 \times_Z Z_2\) is an open neighborhood of the chosen point \(x\) trivially contained in \(Z_1 \times_Z Z_2\). Choose an affine open subscheme \(\text{Spec} C \subset Z_1 \times_Z Z_2\) containing \(x\).

By Remark 6.19, the data of a Kuranishi chart and the data used in Construction 6.13 are equivalent, so the hypotheses of the lemma ensure that \(Z(s_i) \simeq \text{Spec} B_i(1)\) for derived rings \(B_i(1)\) constructed in the manner of Construction 6.13.

With this in mind, the preceding discussion yields the following commutative diagram of derived schemes:

\[
\begin{array}{ccc}
\text{Spec } C & \downarrow & \\
Z_1 \times_Z Z_2 & & \\
\bigwedge \text{Spec } B_1(1) & & \bigwedge \text{Spec } B_2(1) \\
U_1 \simeq \text{Spec } B_1(0) & \leftarrow & U_2 \simeq \text{Spec } B_2(0) \\
\bigwedge U & \downarrow & \\
Z_1 \times_Z Z_2 & \leftarrow & \\
\end{array}
\]

From the above diagram, we have a map, \(\text{Spec } C \to U_1 \times U_2\), which is composition of the open immersion \(\text{Spec } C \to Z_1 \times_Z Z_2\) followed by the closed immersion \(Z_1 \times_Z Z_2 \to U_1 \times_Z U_2\).

Localizing if necessary, we may assume that the above map factors as the composition of a closed immersion \(\text{Spec } C \to U' = \text{Spec } C(0)\) and an open affine immersion \(U' = \text{Spec } C(0) \to U_1 \times_U U_2\). As an affine open subscheme of \(U\), it is clear that \(U'\) is smooth.
It is clear from the set-up that Spec $C$ is quasi-smooth, and $C$ therefore satisfies the hypotheses of Construction 6.13. Using Lemma 6.15 and localizing $C(0)$ and $C$ if necessary (c.f. Remark 6.16), we may use $C(0)$ (as notation suggests) and the induced map $C(0) \to C$, to obtain an object $C(1) \in \text{CAlg}_k$ equivalent to $C$. Thus, we have a map $\text{Spec } C(1) \to \text{Spec } B_i(1)$ and a specified factorization,

$$\text{Spec } C(1) \to \text{Spec } C(0) \to \text{Spec } B_i(0),$$

for $i = 1, 2$.

Since Kuranishi charts are not intrinsic to a quasi-smooth scheme, we need a more explicit model for derived rings to prove the desired statement. This more explicit model is furnished by the model category of cdgas over $k$ with the standard model structure, $\text{cdga}_k$. Since $k$ contains $\mathbb{Q}$, the underlying $\infty$-category of $\text{cdga}_k$ is $\text{CAlg}_k$.

Since $C(0)$ and $N_C$ are a discrete $k$-algebra and discrete module over a discrete ring, respectively, they are objects of ordinary commutative algebra. View $C(0)$ and $\text{Sym}_{C(0)}^* N_C = \text{Sym}_{C(0)}^* C(0)$ as cdgas concentrated in degree 0. Then the pushout

$$\begin{array}{ccc}
\text{Sym}_{C(0)}^* N & \to & C(0) \\
\downarrow & & \downarrow \\
C(0) & \xrightarrow{f} & C(0) \otimes_{\text{Sym}_{C(0)}^* C(0)} C(0)
\end{array}$$

provides a cdga model for $C(1)$. We similarly obtain cdga models for $B_i(1)$.

Assuming without loss of generality that $N_{B_i}$ (c.f. Remark 6.16) is free, the model for $B_i(1)$ is also free over $B_i(0)$ as a graded commutative $k$-algebra, generated by elements in degree $-1$. As such, the structure morphism $f : B_i(0) \to B_i(1)$ is a cofibration in the model category structure, so $B_i(1)$ is a cofibrant object in $\text{cdga}_{B_i(0)}$. Meanwhile, the specified map $B_i(0) \to C(0)$ allows us to view the map $B_i(1) \to C(1) \in \text{CAlg}_k$ as a map in $\text{CAlg}_{B_i(0)}$. All objects in $\text{cdga}_{B_i(0)}$ are fibrant, so

$$\pi_0 \text{Hom}_{\text{CAlg}_{B_i(0)}}(B_i(1), C(1)) \simeq \text{Hom}_{\text{cdga}_{B_i(0)}}(B_i(1), C(1)),$$

by the general yoga of model categories and $\infty$-categorical localization. The upshot is that the map of derived $k$-algebras, $B_i(1) \to C(1)$, is induced by a strict map of cdgas, for $i = 1, 2$. In particular, since $(B_i(1))^0 = B_i(0)$ and $(C(1))^0 = C(0)$, we obtain a commutative diagram (not necessarily Cartesian),

$$\begin{array}{ccc}
B_i(1) & \to & C(1) \\
\uparrow & & \uparrow \\
B_i(0) & \to & C(0)
\end{array}$$

in the category $\text{cdga}_k$, where the vertical arrows are the inclusions of the degree 0 parts of the cochain complexes. Passing to $\text{CAlg}_k$, the underlying $\infty$-category of $\text{cdga}_k$, this obtains a homotopy commutative diagram (i.e. the only kind of commutative diagram in $\text{CAlg}_k$).

There is a natural base change morphism,

$$\mathbb{L}_{B_i(1)/B_i(0)} \otimes_{B_i(0)} C(0) \to \mathbb{L}_{C(1)/C(0)},$$
arising from the commutativity of this diagram, where we have viewed \( L_{B_i(1)/B_i(0)} \) as an object in \( \text{Mod}_{B_i(1)/B_i(0)} \) and \( L_{C_i(1)/C_i(0)} \) as an object in \( \text{Mod}_{C_i(0)} \). Recall from the proof of Lemma 6.15 that \( L_{B_i(1)/B_i(0)} \) and \( L_{C_i(1)/C_i(0)} \) are projective modules over \( B_i(0) \) and \( C_i(0) \), respectively, concentrated in homotopy degree 1. We labeled them \( N_{B_i(1)}[1] \) and \( N_{C_i(1)}[1] \), respectively. As such, the base change morphism yields an isomorphism,

\[
N_{B_i(1)} \otimes_{B_i(0)} C(0) \simeq N_{C_i(1)}.
\]

With this last isomorphism, we have unwittingly proven Lemma 6.20. Explicitly,

- \((Z', U', E', s', t') := (\text{Spec} \pi_0 C(1), \text{Spec} C(0), \text{Spec}_{C'}(\text{Sym}_C(0) N_{C(1)}), \text{Spec}(C(0) \to C(1)), \text{Spec}(1) \to Z_1 \times Z_2 \to Z);\)
- \(\eta_i := \text{Spec}(B_i(0) \to C(0))\) for \(i = 1, 2;\)
- \(\tau_i\) is induced by the isomorphism \(N_{B_i(1)} \otimes_{B_i(0)} C(0) \simeq N_{C_i(1)}\), which are the dual sheaf of sections of \(\eta_i^* E_i\) and \(E'\), respectively.

The desired properties and compatibilities of these objects and maps follow clearly from their construction above. Furthermore, by shrinking \(U'\) around \(x\) we may take \((Z', U', E', s', t')\) to be good. \(\square\)

### 6.3. Gluing local statements.

6.3.1. Continue to suppose that \(Z\) is quasi-smooth and \(i : Z \hookrightarrow X\) is a closed immersion into a smooth scheme \(X\). By Lemma 3.4 \(i\) is quasi-smooth. Choose a Zariski open cover of \(X\), \(\{U_\alpha \hookrightarrow X\}_{\alpha \in \mathcal{I}}\), such that on each open, \(U_\alpha \hookrightarrow X\), \(i\) has the form of a derived zero locus, \(Z(f_\alpha)\), per Lemma 3.5:

\[
\begin{array}{ccc}
Z \times_{U_\alpha} X & \to & U_\alpha \\
\downarrow & & \downarrow f_\alpha \\
^0 \to & & \mathbb{A}^n
\end{array}
\]

Note that the target of \(f_\alpha\) will have the same rank for any choice of index, \(\alpha\). Let \(\alpha_0\) denote a fixed index value. Observe that \(N_{Z(f_\alpha_0)/U_\alpha_0} \simeq O_{Z(f_\alpha_0)}\), so the normal bundle is trivial of rank \(n\). To \(f_\alpha_0\) we associate the corresponding regular function \(\tilde{f}_{\alpha_0}\) on \(U_{\alpha_0} \times \mathbb{A}^{n^\vee}\). This is the setting of Theorem 5.1 so we obtain the isomorphism,

\[
g_{\alpha_0} : \varphi_{f_\alpha_0} \cong (pr_1^* O_{U_{\alpha_0}}[\dim X + n]) \cong \mu_{Z(f_\alpha_0)}(\xi_{U_{\alpha_0}}[\dim X + n]).
\]

6.3.2. The following lemmas show that the left- and right-hand sides of \(g_{\alpha_0}\) are the restrictions to \(T^*[\cdot]Z \times_{Z} Z(f_{\alpha_0})\) of the perverse sheaves, \(\varphi_{T^*[\cdot]Z}\) and \(\mu_Z[n]\xi_{\mathbb{A}^n} [\dim X]\), respectively.

**Lemma 6.22.** Suppose that \(j : Z_1 \hookrightarrow Z_2\) is an open immersion of derived Artin stacks. Then \(T^*[\cdot]Z_2 \times_{Z_2} Z_1 \simeq T^*[\cdot]Z_1\)

**Proof.** Observe that \(L_j \simeq 0\), so the natural map \(O_{Z_1} \otimes O_{Z_2} L_{Z_1} = j^* L_{Z_2} \to L_{Z_1}\) is an isomorphism. The result follows after shifting, dualizing, and applying the total space construction. \(\square\)
Lemma 6.23. Suppose that \( j : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2 \) is an open immersion of derived Artin stacks. Let \( T_j : T^*[-1] \mathcal{Z}_1 \rightarrow T^*[-1] \mathcal{Z}_2 \) be the map of stacks induced by projection on the first factor of \( T^*[-1] \mathcal{Z}_1 \times_{\mathcal{Z}_1} \mathcal{Z}_2 \) via Lemma 6.22. Then

\[
\varphi_{T^*[-1] \mathcal{Z}_1} \cong T_j^* \varphi_{T^*[-1] \mathcal{Z}_2} \quad \text{and} \quad \mu_{\mathcal{Z}_1} \cong T_j^* \mu_{\mathcal{Z}_2}.
\]

Proof. By [BBDJSS15, Corollary 6.11], the perverse sheaf \( \varphi_{T^*[-1] \mathcal{Z}_1} \) is defined uniquely up to canonical isomorphism as the perverse sheaf possessing the characteristics (i) and (ii) of [BBDJSS15, Theorem 6.9]. Thus, to prove (6.24), it suffices to show that \( T^*_j \varphi_{T^*[-1] \mathcal{Z}_2} \) possesses these characteristics.

Observe that if \( \mathcal{X} \) is a critical chart for the canonical \( \mathcal{D} \)-critical locus associated to \((T^*[-1] \mathcal{Z}_1, \omega_{T^*[-1] \mathcal{Z}_1})\), it is also a critical chart for the \( \mathcal{D} \)-critical locus \( T^*[-1] \mathcal{Z}_2 \), viewing \( j_R : R \rightarrow T^*[-1] \mathcal{Z}_2 \) as an open subset of \( T^*[-1] \mathcal{Z}_2 \) via composition with the open embedding \( T_j^* \), denoted \( j_R^* : R \leftrightarrow T^*[-1] \mathcal{Z}_2 \). As such, \( \varphi_{T^*[-1] \mathcal{Z}_2} \mid_R \cong j_R^* \varphi_{T^*[-1] \mathcal{Z}_2} \). Both characteristics (i) and (ii) now follow from the corresponding characteristics of \( \varphi_{T^*[-1] \mathcal{Z}_2} \).

The isomorphism (6.25) is obtained as the composition of the smooth base change isomorphism along \( j \) for quasi-smooth specialization (Lemma 4.5) and the general base change isomorphism for the Fourier-Sato transform (Lemma 2.49).

6.3.3. Letting \( \alpha \) range over all index values, we obtain a collection of isomorphisms,

\[
\{ \varphi_{f^\alpha} (\text{pr}_{U_{\alpha \beta}^\alpha}^* \mathcal{E}_{U_{\alpha \beta}^\alpha} \mid \dim X + n)] \cong \mu_{\mathcal{Z}(\alpha \beta)} (\mathcal{E}_{U_{\alpha \beta}^\alpha} \mid \dim X + n)] \}_{\alpha \in \mathcal{T}},
\]

whose sources and targets glue to the perverse sheaves \( \varphi_{T^*[-1] \mathcal{Z}} \) and \( \mu_{\mathcal{Z}[\alpha \beta]} (\mathcal{E}_X \mid \dim X) \) by Lemma 6.22 and Lemma 6.23. We finally prove Theorem 6.1.

Proof of Theorem 6.1. Let all notation and terms be as above. Since \( \text{Perv}(T^*[-1] \mathcal{Z}) \) is a groupoid-valued stack in the \( \text{\acute{e}tale} \) topology of \( T^*[-1] \mathcal{Z} \), it suffices to show that the isomorphisms \( g_\alpha \) agree on overlaps \( \{U_{\alpha \beta} \} \) for any values of \( \alpha \) and \( \beta \). We establish some alternative notation to ease the exposition. Denote \( Z_\alpha := Z(\alpha \beta) \) and \( Z_{\alpha \beta} := Z_\alpha \times_{Z_\alpha} Z_\beta \). Let \( i_{\alpha \beta}^\alpha : Z_{\alpha \beta} \rightarrow Z_\alpha \rightarrow Z \) denote the obvious inclusion. By abuse of notation, we also use \( i_{\alpha \beta}^\alpha \) to denote the corresponding map \( T^*[-1] \mathcal{Z}_{\alpha \beta} \rightarrow T^*[-1] \mathcal{Z}_\alpha \rightarrow T^*[-1] \mathcal{Z} \). Now choose arbitrary \( \alpha \) and \( \beta \). We proceed to show that the following equality holds:

\[
(i_{\alpha \beta}^\alpha)^* g_\alpha = (i_{\alpha \beta}^\beta)^* g_\beta,
\]

where \( g_\alpha \) and \( g_\beta \) are seen as maps in \( \text{Perv}(T^*[-1] \mathcal{Z}) \) using extension by 0.

Note that the Zariski open \( U_\alpha \subset X \), and the pull-back diagram,

\[
\begin{array}{ccc}
Z(f_\alpha) & \xrightarrow{i} & U_\alpha \\
\downarrow & & \downarrow f_\alpha \\
* & \xrightarrow{0} & \mathbb{A}^n
\end{array}
\]

comprise a good Kuranishi chart for \( Z \), which we denote by \( \mathcal{X}_\alpha \). Choose an arbitrary point \( x \in Z_{\alpha \beta} \). By Lemma 6.20 there exists an third good Kuranishi chart \( \mathcal{X}_x \) around \( x \) which openly embeds into each of \( \mathcal{X}_\alpha \) and \( \mathcal{X}_\beta \). The collection of all such charts as \( x \) ranges over all points in \( Z_{\alpha \beta} \) forms a Zariski open cover of \( Z_{\alpha \beta} \). Denote by \( t_x' \) both the open immersion \( Z_{\alpha \beta} \rightarrow Z_{\alpha \beta} \) and the induced open immersion

\[28\text{[BBDJSS15, Theorem 2.7]}\]
\( T^*[\{-1\}]Z'_\alpha \hookrightarrow T^*[\{-1\}]Z_{\alpha\beta} \). Since the \{\( Z'_\alpha \)\} form an open cover of \( Z \), in order to show \([6.26]\) it suffices to show the equality

\[
(i'_x)^*(i''_{\alpha\beta})^*g_\alpha = (i'_x)^*(i''_{\alpha\beta})^*g_\beta,
\]

for arbitrary \( x \).

The equality, \( i''_{\alpha\beta} \circ i'_x = i' \), is merely a reformulation of Lemma \([6.20]\) property (ii). We obtain the induced equality on the level of \(-1\)-shifted cotangent bundles, denoting (again by abuse of notation) the induced map \( T^*[\{-1\}]Z'_\alpha \hookrightarrow T^*[\{-1\}]Z \) by \( \iota' \). As such,

\[
(i'_x)^*(i''_{\alpha\beta})^*g_\alpha = (i'_x)^*(i''_{\alpha\beta})^*g_\beta \Longleftrightarrow \iota'^*g_\alpha = \iota'^*g_\beta,
\]

and the latter equation is true since all of the constructions in section Section \([5]\) are self-evidently compatible with open restrictions. That is, the isomorphism, \( \varphi_{f|U} \rightarrow \mu_{Z(f)} \) in \([6.20]\), obtained in the proof of Theorem \([5.1]\) for a Zariski open, \( U \subset X \), is naturally the restriction of the isomorphism \( \varphi_f \rightarrow \mu_{Z(f)} \) along \( Z(f|_U) \times V' \cong (f|_U)(f \times V') \). Indeed, this follows from the following canonical isomorphism of sections: \( F^*_{\iota_U}(V) \cong F(U \cap V) \).

Since the choices of \( \alpha \) and \( \beta \) in \([6.26]\) were arbitrary, this concludes the proof. \( \square \)

6.3.4. Since the canonical sheaf \( \varphi_{T^*[\{-1\}]Z} \) is intrinsic to \( Z \), Theorem \([6.1]\) immediately implies the following corollary.

**Corollary 6.27.** Suppose \( Z \) is a quasi-smooth derived scheme and \( Z \hookrightarrow X \) is a closed immersion into a smooth scheme \( X \). Then \( \mu_{Z/X}[\text{codim.vir}(Z,X)](\xi_X[\dim X]) \in \text{Perv}(T^*[\{-1\}]Z) \), is independent of the closed immersion \( Z \hookrightarrow X \). In particular, if \( Z \hookrightarrow X' \) is another closed immersion, there is a canonical isomorphism,

\[(6.28) \quad \mu_{Z/X}[\text{codim.vir}(Z,X)](\xi_X[\dim X]) \cong \mu_{Z/X'}[\text{codim.vir}(Z,X')](\xi_{X'}[\dim X']). \]

In light of Corollary \([6.27]\), we make the following definition for any quasi-smooth derived scheme which admits a closed immersion into a smooth scheme.

**Definition 6.29.** The microlocalization along \( Z \) is defined to be:

\[
\mu_Z := \mu_{Z/X}[\text{codim.vir}(Z,X)](\xi_X[\dim X]),
\]

for any choice of closed immersion, \( Z \hookrightarrow X \), into a smooth scheme, \( X \).

**Remark 6.30.** It should be possible to modify the given proof of Theorem \([6.1]\) to prove a variant in which \( Z \) is not assumed to admit a closed immersion into a smooth scheme, \( X \). Using the derived microlocalization functor of \([AAK]\) one might obtain such a variant by replacing the right-hand side of the isomorphism in Theorem \([6.1]\) by \( \mu_{Z/A}(e) \), which is manifestly independent of any embedding \( Z \) might admit. In the case that \( Z \) happens to admit an embedding into a smooth scheme \( X \), \( \mu_{Z/A}(e) \) and \( \mu_Z \) would necessarily agree. Such a generalization of our present theorem is the subject of ongoing work in progress by Adeel Khan and Tasuki Kinjo.

\[29\text{c.f. Remark } 1.6\]
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