A Generalization of the Kodama State for Arbitrary Values of the Immirzi Parameter

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Abstract

The Kodama State for Lorentzian gravity presupposes a particular value for the Immirzi-parameter, namely $\beta = -i$. However, the derivation of black hole entropy in Loop Quantum Gravity suggests that the Immirzi parameter is a fixed value whose magnitude is on the order of unity but larger than one. Since the Kodama state has de-Sitter spacetime as its classical limit, to get the proper radiation temperature, the Kodama state should be extended to incorporate a more physical value for $\beta$. Thus, we present an extension of the Kodama state for arbitrary values of the Immirzi parameter, $\beta$, that reduces to the ordinary Chern-Simons state for the particular value $\beta = -i$. The state for real values of $\beta$ is free of several of the outstanding problems that cast doubts on the original Kodama state as a ground state for quantum general relativity. We show that for real values of $\beta$, the state is invariant under large gauge transformations, it is CPT invariant (but not CP invariant), and it is expected to be delta-function normalizable with respect to the kinematical inner product. To aid in the construction, we first present a general method for solving the Hamiltonian constraint for imaginary values of $\beta$ that allows one to use the simpler self-dual and anti-self-dual forms of the constraint as an intermediate step.

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1 Introduction

The Kodama state\cite{1, 2} is the only known exact solution to the constraints of quantum gravity which also has a well-defined classical limit, namely de-Sitter spacetime\cite{3}. It is often overlooked that the Kodama state presupposes a particular value for the Immirzi parameter, namely $\beta = -i$. However, it has been shown that requiring consistency with the entropy of isolated horizons from Loop Quantum Gravity with the entropy of Hawking radiation fixes the magnitude of the Immirzi parameter on the order of unity but larger than one\cite{4}. Furthermore, the Kodama state has de-Sitter spacetime as its classical limit, and de-Sitter spacetime is thermal due to the presence of an isolated cosmological horizon. An analysis of the horizon entropy precisely parallels the derivation of black hole entropy in Loop Quantum Gravity since in both cases the entropy comes from boundary fields on an isolated horizon. This again suggests that in order to arrive at the proper horizon temperature the magnitude of the Immirzi parameter must be larger than one. For these reasons alone it seems imperative to extend the Kodama state to arbitrary values of the Immirzi parameter.

In addition, the Kodama state is plagued with various difficulties which have cast doubts on its viability as a ground state of quantum general relativity: it is not invariant under large gauge transformations, it is not normalizable under the physical inner product, and it is not CPT invariant\cite{5}. All of these difficulties can be traced down to the simple fact that there is no $i$ in front of the Chern-Simons functional, which, in turn, can be traced back to requirement that the Immirzi parameter is pure imaginary. Thus, extending the state to real values of the parameter has the potential of resolving some, if not all of these issues. We show that for real values of $\beta$ when the familiar pre-quantization condition for Chern-Simons theory is satisfied, the extended state is invariant under large gauge transformations. In addition, the state violates CP but is CPT invariant, and, drawing from analogy with the Euclidean theory, it is expected to be delta-function normalizable\cite{6}.

To aid in the construction of the state we employ what seems to be a general method for finding solutions to the Hamiltonian constraint for arbitrary values of the Immirzi parameter which allows one to use the simpler self-dual and anti-self dual forms as an intermediate step. We present a general outline of the method below.

The equivalence of the equations of motion from the Einstein-Cartan action and its (anti)self-dual counterpart is in part due to the fact that the
left and right handed components of the action are non-interacting. Since they don’t interact, often one can treat them as independent variables despite one being the complex conjugate of the other. In some ways this is analogous to complex Klein-Gordon theory where \( \phi \) and \( \bar{\phi} \) are treated as functionally independent variables. However, there are important differences. In our case, the action is irreducibly complex and there is no analogue of a \( U(1) \) symmetry between the fields. The general idea is to first treat the conjugate pair \((\omega_L, \Sigma_L)\) and \((\omega_R, \Sigma_R)\) as functionally independent variables all the way up to the construction of the quantum constraints and even solutions thereof, only then enforcing the condition

\[
\omega_R^i = \omega_L^i. \tag{1}
\]

As we will see, this allows one to work with the manageable self-dual and anti-self-dual forms of the Hamiltonian constraint while searching for solutions. Once a solution has been found, enforcing an appropriate form the condition (1) is simply a matter of making a change of variables in the solution and the constraints. Written in terms of the new variables, barring potential unforseen pitfalls, the new solution is automatically a solution to the Hamiltonian constraint for the large class of imaginary values of the Immirzi parameter subject to \(|\beta| \geq 1\). Furthermore, for many solutions including the Kodama state, once the extended solution has been found for these imaginary values of \( \beta \), nothing seems to prevent one from simply replacing \( \beta \) with \( i\beta \), thereby extending the solution to real values of the parameter.

2 Imaginary Immirzi Parameter

Holst has shown\[7\] that the constraint equations for \((3 + 1)\) gravity in terms of the Barbero connection\[5\] for an arbitrary value of the Immirzi parameter follow from a modification of the Einstein-Cartan action of the form:

\[
S_H = S_{EC} + S'_H
\]

\[
= \frac{1}{4k} \int_M \left( \epsilon_{IJJK} e^I \wedge e^J \wedge \Omega^{KL} - \frac{2}{\beta} e^I \wedge e^J \wedge \Omega_{IJ} \right) \]

where \( k = 8\pi G \), \( \beta \) is the Immirzi parameter, and \( \Omega^I_{IJ} = d\omega^I_{J} + \omega^K_{I} \wedge \omega^K_{J} \) is the curvature of a metric compatible (but not torsion free) spin connection, \( \omega^I_{J} \). Metric compatibility implies the gauge group is either \( SO(3, 1) \) or, as we will assume in this section, its covering group \( SL(2, C) \).
For imaginary values of $\beta$, the Holst modification term has a simple interpretation that will be essential for the rest of the paper: it reflects a partial parity violation in the gravitational interaction. To see this, we first write the Holst modified action in a slightly more suggestive form. For convenience of notation, we will work in the Dirac representation of the Lie algebra. Thus, take the tetrad to be one-forms valued in the Dirac representation of the Clifford algebra $e \equiv e^I \gamma_I$ and the connection coefficients to be one-forms valued in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, $\omega \equiv \omega_{IJK} \gamma^I \gamma^J \gamma^K$. Using the identity $\text{Tr}(\gamma^5 \gamma^I \gamma^J \gamma^K \gamma^L) = 4i \epsilon^{IJKL}$, the action (2) can then be written

$$S_H = \frac{1}{4k} \int_M \text{Tr} \left[ -i \gamma^5 e \wedge e \wedge \Omega + \frac{1}{\beta} e \wedge e \wedge \Omega \right] = \frac{-2i}{4k} \int_M \text{Tr} \left[ \left( 1 + \frac{(i/\beta) \gamma^5}{2} \right) \gamma^5 e \wedge e \wedge \Omega \right].$$

(3)

The operator $\frac{1}{2} \left[1 + (i/\beta) \gamma^5\right]$ is a projection operator only for the particular values $\beta = -i$ and $\beta = i$ where it projects the curvature into its left and right handed components respectively, or, put another way, projects the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{sl}_{\pm}(2, \mathbb{C})$. We point out, however, for imaginary values of $\beta$ such that $|\beta| > 1$, the operator also has a natural interpretation as the sum of two weighted chiral projection operators. Motivated by partial parity violating interactions we introduce coupling constants $\alpha_L = \cos^2 \theta$ and $\alpha_R = \sin^2 \theta$ where $0 \leq \theta \leq \pi/2$. Then we have,

$$\frac{\alpha_L}{2} (1 - \gamma^5) + \frac{\alpha_R}{2} (1 + \gamma^5) = \frac{1}{2} \left( 1 - \cos 2\theta \gamma^5 \right).$$

(4)

This simple observation has powerful consequences. It has long been known that the Holst action reduces to the self-dual or anti-self-dual action for the particular values $\beta = \mp i$. We emphasize here, that this connection can be extended to a much larger class of Immirzi parameter. With this goal in mind, using the above construction, the natural generalization of the Einstein-Cartan action to allow for partial-parity violation is (inserting a factor of two for convenience):

$$S = 2 \left( \alpha_L S_{EC}[\omega_L] + \alpha_R S_{EC}[\omega_R] \right)$$

$$= -\frac{2i}{4k} \int_M \text{Tr} \left[ \alpha_L \gamma^5 e \wedge e \wedge \Omega_{(L)} + \alpha_R \gamma^5 e \wedge e \wedge \Omega_{(R)} \right]$$

(5)

$^1$Our sign conventions are $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$, $\epsilon^{0123} = -1$, and $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. 

4
\[
\frac{-2i}{4k} \int_M Tr \left[ \frac{1 - \cos 2\theta \gamma^5}{2} \right] \gamma^5 e \wedge e \wedge \Omega
\]
\[
= \frac{1}{4k} \int_M \left[ \epsilon_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} - 2i \cos 2\theta e^I \wedge e^J \wedge \Omega_{IJ} \right].
\]

Comparing this with the Holst-modified action we find that the Immirzi parameter for this theory is given by

\[
\beta = \frac{-i}{(\alpha_L - \alpha_R)} = \frac{-i}{\cos 2\theta},
\]

and for these particular values, the physical meaning of the parameter is clear: it is a numerical measure of the degree of chiral asymmetry in the gravitational interaction\(^2\). Thus, the action \(\Box\) is the bridge between the self-dual and anti-self-dual actions, and the Holst action for the large class of Immirzi parameter in the form of \(\Box\). For this reason, it is a natural starting point for the generalization of the Kodama state to arbitrary values of the Immirzi parameter.

## 3 Legendre Transforms

Before attacking the action \(\Box\) we take the time to briefly review the Legendre Transform of the Holst Action. Our goal here is to point out several subtleties that many sources ignore for the sake of simplicity but play an important role in the generalization of the Kodama state. Here it is easiest to work in a vector representation. Since we eventually want to construct the Kodama state, we also add in a cosmological constant term to the action:

\[
S = S_H + S_{CC}
\]
\[
= \frac{1}{4k} \int_M \left( \epsilon_{IJKL} e^I \wedge e^J \wedge \left( \Omega^{KL} - \frac{\Lambda}{6} e^K \wedge e^L \right) - \frac{2}{\beta} e^I \wedge e^J \wedge \Omega_{IJ} \right)
\]

\(^2\)As a side we note that chiral symmetry is violated in the standard model so one might impose consistency relations which demand that the measure of chiral asymmetry in the matter interactions of the standard model is compatible with the measure of chiral violation in the gravitational interaction (i.e. the Immirzi parameter). This could yield an alternative fixing of the Immirzi parameter so long as one accepts imaginary values for \(\beta\). It is also worthy of commentary that for these particular values, the Immirzi parameter plays the dual role as the measure of chiral asymmetry and the renormalization factor of Newton’s constant, \(G \rightarrow G' = G/\beta\), as seen from the derivation of the entropy of thermal radiation in the Kodama ground state\[3\]. This interplay is intriguing, but it is not well understood.
Per usual, we decompose spacetime into spacelike foliations $M = \mathbb{R} \times \Sigma$ and introduce a timelike vector field $\bar{t} = N\bar{n} + \bar{N}$ where $\bar{n}$ is the metric normal to the foliation and $\bar{N}$ is metrically parallel to the foliation. The canonical one form $dt$ associated with $\bar{t}$ defines the time coordinate $t$. For the remainder of this section and the next (with some obvious exceptions) all forms, duals, and lower case Latin indices will be defined in the foliations $\Sigma$.

We will fix the gauge by assuming that the vector $\bar{n}$ is constant when viewed as a vector in the vector bundle and has components $n^I = \epsilon^{I\mu}n_\mu = (1, 0, 0, 0)$. Thus, the extrinsic curvature is $K^i = (q^a_\mu q^i_\nu \mathcal{D}_\mu n^\nu)dx^a = \omega^i_0$, where $q^i_\mu$ and $q^a_\mu$ project components in $TM$ to the foliations $T\Sigma$. It is easiest to begin by decomposing the action into components parallel and perpendicular to $\bar{n}$ and later making the substitution $\bar{n} = \frac{\bar{t} - \bar{N}}{N}$. However you do it the decomposition of (2) can be written:

$$
\frac{-1}{k\beta} \int_{\mathbb{R} \times \Sigma} dt \wedge [E^i \wedge E^j \wedge \mathcal{L}_{\bar{t}}A_{ij} - E^i \wedge E^j \wedge \mathcal{L}_{\bar{N}}A_{ij}
- A_{ij}(N\bar{n})G^{ij} - K_{ij}(N\bar{n})\tau^{ij} - NH].
$$

(8)

where $A_{ij} = \omega_{ij} - \beta K_{ij}$ is the Barbero connection ($K_{ij} \equiv \epsilon_{ijk}K^k$). We see in the first term the familiar phase space consisting of the “position” variable $A_{ij}$ and its conjugate momentum $P^{kl} = \frac{1}{k\beta}E^k \wedge E^l$ whose Poisson brackets become operator commutators in the quantum theory:

$$
[A_{ij}|_P, E^k \wedge E^l|_Q] = -ik\beta \delta^k_i \delta^l_j \tilde{\delta}(P, Q)
$$

(9)

where $\tilde{\delta}(P, Q)$ is the delta distribution valued 3-form satisfying:

$$
\int_{P \in \Sigma} f(P)\tilde{\delta}(P, Q) = f(Q).
$$

(10)

Here, the variables $N, \bar{N}, A_{ij}(N\bar{n}) = NA_{ij\mu}n^\mu$, and $K_{ij}(N\bar{n}) = N\epsilon_{ijk}K^k_\mu n^\mu$ are all Lagrange multipliers whose variation yields the constraints below:

**Diffeomorphism Constraint:**

$$
\mathcal{L}_V A_{ij} \wedge E^i \wedge E^j \approx 0
$$

(11)

for any smooth vector field $V$ lying completely in $\Sigma$. This constraint comes from varying the shift $\bar{N}$. The form of the constraint differs trivially from the form usually presented in the literature $E^i \wedge E^j \wedge F_{ij}(\bar{N}) = E^i \wedge E^j \wedge (\mathcal{L}_{\bar{N}}A_{ij} - D_A(A_{ij}(\bar{n})))$ because we have removed the Gauss part of the constraint. This...
is accomplished by taking $A_{ij}(N\tilde{n})$ as a Lagrange multiplier as opposed to $A_{ij}(\tilde{t})$.

**Torsion Constraint:**

$$-2\beta K^i_{[m} \wedge E^m \wedge E^j] \approx 0.$$  \hfill (12)

This constraint comes from varying $K^{ij}(N\tilde{n})$. One can show that it is classically equivalent to the vanishing of the normal component of the torsion of the connection $\omega$ on the spacelike hypersurfaces: $n_i D_\omega e^I = n_i T^I \approx 0$.

**Gauge Constraint:**

$$D_A(E^i \wedge E^j) + 2\beta K^i_{[m} \wedge E^m \wedge E^j] = D_\omega(E^i \wedge E^j) \approx 0.$$  \hfill (13)

This constraint comes from varying the action with respect to $A_{ij}(N\tilde{n})$. It is the usual Gauss constraint with an extra piece which is identical to the torsion constraint. It is classically equivalent to the vanishing of the three dimensional components of the torsion of the connection $\omega$: $D_\omega E^i = T^i \approx 0$.

For imaginary values of $\beta$, the Gauge and the Torsion constraints can be combined simply by adding them to yield the ordinary form of the Gauss constraint:

$$D_A(E^i \wedge E^j) \approx 0.$$  \hfill (14)

This changes nothing since the real and imaginary parts must vanish separately. For real values of $\beta$ one does not have this luxury and one or the other constraint must be solved prior to quantization. Typically one solves the gauge constraint by taking $\omega = \omega[E]$ to be torsion free and then replacing the Torsion and Gauge constraints by the single Gauss constraint (14).

**Hamiltonian Constraint:**

$$\epsilon_{ijk} E^i \wedge \left( \Omega_A^{jk} + \frac{1}{\beta}(1 + \beta^2)D_\omega K^{jk} - (1 + \beta^2)K^j_{[m} \wedge K^{mk} - \frac{\Lambda}{3} E^j \wedge E^k \right) \approx 0.$$  \hfill (15)

This comes from varying the lapse $N$. It has the usual form of the Hamiltonian constraint with the addition of the term proportional to $\epsilon_{ijk} E^i \wedge D_\omega K^{jk}$. This is usually ignored because it vanishes identically when the both the Torsion and Gauge constraints are applied. Keeping the extra term seems to complicate the constraint unnecessarily, but as we will see it actually simplifies our situation considerably because it facilitates the connection between this Hamiltonian constraint and that of the action [15].
4 Constraints from partial-parity violating action

We begin by constructing the quantum constraints assuming \( \omega_L \) and \( \omega_R \) are functionally independent. Using a bit of self-dualology, we can rewrite the action (5) with a cosmological constant:

\[
\frac{i \alpha_L}{k} \int_M \Sigma_{(L)IJ} \wedge \left( \Omega^{IJ}_{(L)} - \frac{\Lambda}{6} \Sigma^{IJ}_{(L)} \right) - \frac{i \alpha_R}{k} \int_M \Sigma_{(R)IJ} \wedge \left( \Omega^{IJ}_{(R)} - \frac{\Lambda}{6} \Sigma^{IJ}_{(R)} \right),
\]

where

\[
\Sigma^{IJ}_{(L/R)} = \frac{1}{2} \left( e^I \wedge e^J \mp \frac{i}{2} \epsilon^{IKL} e^K \wedge e^L \right).
\]

It is clear then that the action is essentially the difference of two functionally independent actions, one self-dual and one anti-self-dual, each of whose constraints are well known. As we will see, this makes the constraints much easier to solve. We now construct the Legendre transform of the above action. The technique is the same as before. Although technically we do not have to fix the gauge in the self-dual formalism, we do so as a matter of convenience to compare the resulting constraints to those above. The result is that the Legendre transformed action takes the form:

\[
S = -\frac{1}{k} \int_{R \times \Sigma} dt \wedge (i \alpha_L \Sigma_{(L)ij} \wedge \mathcal{L}_{\bar{L}i} \omega_{(L)}^{ij} - i \alpha_R \Sigma_{(R)ij} \wedge \mathcal{L}_{\bar{R}i} \omega_{(R)}^{ij})
\]

\[
- i \alpha_L \Sigma_{(L)ij} \wedge \mathcal{L}_N \omega_{(L)}^{ij} + i \alpha_R \Sigma_{(R)ij} \wedge \mathcal{L}_N \omega_{(R)}^{ij}
\]

\[
- i \alpha_L \omega_{(L)ij} (N \bar{n}) D_{(L)} \Sigma_{(L)}^{ij} + i \alpha_R \omega_{(R)ij} (N \bar{n}) D_{(R)} \Sigma_{(R)}^{ij}
\]

\[
- NH).
\]

The phase space now consists of the “position” variables \( \omega_{(L)ij} \) and \( \omega_{(R)ij} \) and their conjugate momenta \( P_{(L)}^{kl} = -\frac{i \alpha_L}{k} \Sigma_{(L)}^{kl} \), \( P_{(R)}^{kl} = \frac{i \alpha_R}{k} \Sigma_{(R)}^{kl} \) which yield the quantum commutators:\footnote{We apologize beforehand for being rather loose about the quantization. The primary intent of this paper is to construct the generalized Kodama state and outline a method for solving the quantum constraints by means of the Kodama state as an example. A more rigorous treatment of the quantization is certainly in order.}

\[
\left[ \omega_{(L)ij} | P, \Sigma_{(L)}^{kl} | Q \right] = -\frac{k}{\alpha_L} \delta_{[i}^{[k} \delta_{j]}^{l]} \delta(P, Q)
\]
\[
\begin{align*}
[\omega^{(L)}_{ij} | P, \Sigma^{kl}_{(R)} | Q] &= 0 \\
[\omega^{(R)}_{ij} | P, \Sigma^{kl}_{(R)} | Q] &= \frac{k}{\alpha_R} \delta^k_{ij} \delta^l_j \delta(P, Q) \\
[\omega^{(R)}_{ij} | P, \Sigma^{kl}_{(L)} | Q] &= 0
\end{align*}
\] (18)

These operators act on states in the expanded Hilbert space which we take at the kinematical level to be \( L^2[\Omega_{(L)} \otimes \Omega_{(R)}] \) with respect to some presently undefined inner product. Here \( \Omega_{(L/R)} \) is the space of generalized connections of \( \omega_{(L/R)} \).

Now, \( N, \bar{N}, \omega^{(L)}_{ij}(N\bar{n}) \), and \( \omega^{(R)}_{ij}(N\bar{n}) \) are to be treated as independent Lagrange multipliers whose variation yields the constraints:

\[
\begin{align*}
D_{(L)} \Sigma_{(L)}^{ij} &\approx 0 \\
D_{(R)} \Sigma_{(R)}^{ij} &\approx 0 \\
\alpha_L \mathcal{L} \bar{\mathcal{V}} \omega^{(L)}_{ij} \wedge \Sigma_{(L)}^{ij} - \alpha_R \mathcal{L} \bar{\mathcal{V}} \omega^{(R)}_{ij} \wedge \Sigma_{(R)}^{ij} &\approx 0 \\
\alpha_L \ast \Sigma_{(L)jk} \wedge (\Omega^{jk}_{(L)} - \frac{\Lambda}{3} \Sigma_{(L)}^{jk}) + \alpha_R \ast \Sigma_{(R)jk} \wedge (\Omega^{jk}_{(R)} - \frac{\Lambda}{3} \Sigma_{(R)}^{jk}) &\approx 0
\end{align*}
\] (20)

These constraints should be viewed as quantum operators (with the given choice of operator ordering) acting on the Hilbert space.

In all of the above we have treated \( \omega^{ij}_{(L)} \) and \( \omega^{ij}_{(R)} \) and their momenta as functionally independent variables. We now need to impose the condition that \( \omega^{ij}_{(R)} = \omega^{ij}_{(L)} \). Classically this would simply entail a reduction of the phase space. However, we have already quantized the theory, so the reduction is a reduction on the Hilbert space itself. This is implemented by imposing operator constraints on the configuration variables and momentum operators and rewriting all the states in terms of the new variables. Since the classical phase space for an arbitrary value of the Immirzi parameter consists of the position-momentum pair \((A_{ij}, -k\beta E^m \wedge E^n)\) we expect that in the connection representation the “position” operator will be the multiplicative operator \( A^{ij} \) and the momentum operator will be \( ik\beta \frac{4}{\delta A_{ij}} \) for the particular value \( \beta = -i/\cos 2\theta \). These operators will act on states in the kinematical Hilbert space given by \( L^2[\mathcal{A}] \) where \( \mathcal{A} \) is the space of generalized connections of \( A \).
To this end, we define the “real” and “imaginary” parts of $\omega_{(L)}$:

$$\omega_{ij} \equiv \text{Re}(\omega_{ij}^{(L)}) = \frac{1}{2}(\omega_{ij}^{(L)} + \omega_{ij}^{(R)})$$

$$K_{ij} \equiv \text{Im}(\omega_{ij}^{(L)}) = \frac{1}{2i}(\omega_{ij}^{(L)} - \omega_{ij}^{(R)})$$

(21)

and we define the new connection

$$A_{ij} \equiv \omega_{ij} + \frac{i}{\cos 2\theta} K_{ij} = \frac{\alpha_L \omega_{ij}^{(L)} - \alpha_R \omega_{ij}^{(R)}}{\alpha_L - \alpha_R}.$$ 

(22)

In addition, we define a surface form, which as we will see is proportional to the conjugate momentum of $A$:

$$\Sigma_{ij} \equiv \alpha_L \Sigma_{ij}^{(L)} + \alpha_R \Sigma_{ij}^{(R)} = E^i \wedge E^j.$$ 

(23)

With these definitions, from the commutation relations (19) one deduces commutation relations for the new variables:

$$[\omega_{ij}|_P, \Sigma_{kl}|_Q] = 0$$

$$[K_{ij}|_P, \Sigma_{kl}|_Q] = ik\delta^k_i \delta^l_j \delta(P,Q)$$

$$[A_{ij}|_P, \Sigma_{kl}|_Q] = \frac{-k}{\cos 2\theta} \delta^k_i \delta^l_j \delta(P,Q)$$

(24)

These operators now act on the reduced Hilbert space. We note the that vanishing of the commutator $[\omega, \Sigma]$ follows from the commutation relations (19) and does not require that $\omega$ is torsion free, which is effectively imposed by the Gauss constraint in the complex theory.

If we now make these substitutions in the constraints (20), after a bit of manipulation, the full set of constraints reduces to the set (11)-(15) for the class of Immirzi parameters given by $\beta = -i/\cos 2\theta$. This, incidentally, is the reason why we took the time to work through the constraints explicitly: the extra term in the Hamiltonian makes its appearance in this redefinition.

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4 The terms real and imaginary are only heuristic labels here since these are truly operators on the Hilbert space. Perhaps the terms Hermitian and anti-Hermitian would be more appropriate, but we have not yet defined the inner product or the Hermitian adjoint.

5 The last equality only holds because of our choice of gauge fixing. Without this gauge fixing there would be extra terms.
of variables. Thus, keeping this term as opposed to eliminating it by appeals to the Gauss constraint makes more transparent the connection between the simple Hamiltonian constraint of (5) and that of the Holst action. This suggests the following procedure for solving the constraint equations: first work with the simpler form of the constraints (20) in order to find solutions. Then, once a solution has been found, impose the condition $\omega_{ij}^R = \overline{\omega}_{ij}^L$ by rewriting the solution and the constraints in terms of the new variable $A$, whose real and imaginary parts, $\omega$ and $K$, might separate in the final form of the solution. The new solution should be a solution to the full set of constraints for $\beta = -i/\cos 2\theta$. In the next section, we demonstrate this explicitly for the generalization of the Kodama state.

5 The Generalized Kodama State

The quantum constraints (20) immediately admit the Kodama-like solution

$$\Psi[\omega(L), \omega(R), \alpha_L, \alpha_R] = \exp \left[ \frac{3}{2k\Lambda} \left( \alpha_L \int_\Sigma Y_{CS}[\omega(L)] - \alpha_R \int_\Sigma Y_{CS}[\omega(R)] \right) \right]$$

(25)

where

$$\int Y_{CS}[\omega] = \int Tr \left( \omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) = \int \left( \omega^i_k \wedge d\omega^k_i + \frac{2}{3} \omega^i_m \wedge \omega^m_n \wedge \omega^n_i \right).$$

We note that for the particular value $\alpha_L = 1$ this reduces to the original Kodama state. We also note that a strikingly similar state with $\alpha_L = \alpha_R$ and an overall factor of $i$ was found in the context of quantum supergravity in [9].

Our task now is to rewrite $\Psi[\omega(L), \omega(R), \alpha_L, \alpha_R]$ as an explicit function of $A$, $K$, and $\beta$ and check that it is in fact a solution to the constraints (11)-(15) for $\beta = -i/(\alpha_L - \alpha_R)$. Using the shift identity,

$$\int Y_{CS}[\omega + \kappa] = \int \left( Y_{CS}[\omega] + Tr(\kappa \wedge D\omega + \frac{2}{3} \kappa \wedge \kappa \wedge \kappa) \right)$$

(26)

one can show after a bit of algebra that the new state is,

$$\Psi[\omega(L), \omega(R), \alpha_L, \alpha_R] \Rightarrow \Psi[A, K, \beta]$$

$$= \exp \left[ \frac{-3i}{2k\Lambda \beta} \int_\Sigma \left( Y_{CS}[A] - (1 + \beta^2) Tr \left( K \wedge D\omega K - \frac{2}{3} \beta K \wedge K \wedge K \right) \right) \right].$$
To check that this is a solution to the Hamiltonian constraint \((15)\) for \(\beta = -i/\cos 2\theta\), we write the surface operator in the connection representation

\[ E_i \wedge E_j = i k \beta \frac{\delta}{\delta A^{ij}} \equiv ik \beta [abc] \, dx^a \wedge dx^b \frac{\delta}{\delta A^{ij}c} \]  

(28)

and the extrinsic curvature

\[ K^{ij} = -\frac{A^{ij} - \omega^{ij}}{\beta}. \]  

(29)

Some simple arithmetic shows that \(\psi[A, K, \beta]\) is in fact in the kernel of the operator

\[ \Omega^j_k + \frac{1}{\beta}(1 + \beta^2)D_\omega K^{jk} - (1 + \beta^2)K^k_m \wedge K^{mk} - \frac{\Lambda}{3}E^j \wedge E^k \]  

(30)

and therefore satisfies the Hamiltonian constraint for an appropriate choice of operator ordering. It should be clear that the state is invariant under (small) gauge transformations and infinitesimal diffeomorphisms so we won’t check that it satisfies the diffeomorphism and and Gauss constraint explicitly.

\section{Extension to real values of the Immirzi Parameter}

Although our derivation of the generalized Kodama state technically only holds for \(\beta = -i/\cos 2\theta\), the reader may have already noticed that the state is in fact in the kernel of the Hamiltonian for any finite value of \(\beta\), real or complex\(^7\). However, the state has very different functional properties when the Immirzi parameter is real. These new properties may resolve some of the outstanding issues associated with the Kodama state as a valid ground state for general relativity. However, we begin by briefly discussing in general how to extend the procedure used in this paper to real values of the Immirzi parameter.

\(^6\)Here we are using MTW’s notation where \([abc]\) is simply the pure alternating symbol to distinguish from the densitized alternating symbol \(\epsilon_{abc} \equiv \sqrt{|g|}[abc]\). In this expression, the density is implicitly contained in the operator \(\delta/\delta A^{ij}c\), which is proportional to the densitized triad operator.

\(^7\)The only subtlety is that for real values of \(\beta\) the Gauge constraint must be solved so \(\omega\) is torsion free and is explicitly a function of derivatives of the triad.
Once one has obtained a solution to the simpler self-dual form of the Hamiltonian constraint of (20) and rewritten the solution in terms of $A, K,$ and $\beta = -i/(\alpha_L - \alpha_R)$, one can then check whether the solution is still valid for arbitrary values of $\beta$. Since the Hamiltonian constraint takes the same form for any value of $\beta$, the solution $\psi[A, K, \beta]$ will still be a solution to the Hamiltonian constraint so long as $\psi[A, K, \beta]$ is functionally well-behaved after the substitution $\beta \to -i\beta$. Thus, we have reduced the problem to a problem analogous to the following: given a complex valued function $f(x, y)$ of two real variables $x$ and $y$, is the function still well-behaved when its domain is extended to the complex plane of $y$? We are not guaranteed that our solution will still be a solution when we make the substitution $\beta \to i\beta$, but so long as the function is not pathological upon a complex extension of $\beta$, the solution will still hold. However, there are other requirements for a physical solution which are dramatically affected by such an extension, for example, normalizability and CPT invariance. As a simple, but relevant example, the two functions $f(k, x) = e^{kx}$ and $f(K, x) = e^{Kx}$ satisfy equations of the same form (think of this as the Hamiltonian constraint): $\partial_x f(k, x) - kf(k, x) = 0$ and $\partial_x f(K, x) - Kf(K, x) = 0$. However, one is delta-function normalizable and CPT invariant, while the other is not. Thus, substituting $\beta \to i\beta$ could make a physical state unphysical or vice-versa. Indeed, this is true for the generalization of the Kodama state to real values of $\beta$.

We note that, for real values, because of the $i$ in front of the Chern-Simons functional, the state is invariant under large gauge transformations so long as $\kappa = \frac{3}{4G\Lambda} = \frac{a_H}{43a_0}$ is an integer where $a_H = 12\pi/\Lambda$ is the area of the cosmological horizon of de-Sitter spacetime and $a_0$ is the area of a sphere with a radius of the Planck length. This is the familiar pre-quantization condition of Chern-Simons theory. This is related to the analogous pre-quantization condition encountered in the analysis of the surface degrees of freedom on an isolated horizon, which also carries a Chern-Simons field. In the context of isolated horizons, the condition is related to the integer number of punctures on the horizon and the quantized deficit angles of the connection around these punctures. We expect the same interpretation in our context if the state is to reproduce some variant of de-Sitter spacetime with a cosmological horizon.

In addition, the state is CPT invariant for real values of $\beta$. Here we follow Soo’s definitions of the action of discrete symmetries. Since a graviton is its own anti-particle, $C$ acts trivially on the state. Under parity
\[ K^i = (q^\mu_i q^\mu_I n^I) dx^a \to -K^i. \] Since \( \epsilon_{jk}^i \to \epsilon_{jk}^i \) (this is consistent with \( P \) actively inverting the volume form) we have \( K^i_j \to -K^i_j. \) But under time reversal we also have \( K^i_j \to -K^i_j \) since, in our gauge, \( n^I = t^I/N \to -n^I. \) Thus under \( PT, K^i_j \to K^i_j. \) The connection coefficient \( \omega \) is unaffected by \( P \) and \( T. \) Now, \( T \) is anti-unitary so it also acts on the state by complex conjugation introducing an overall minus sign to the phase. However, the whole the Chern Simons functional is parity odd, and the terms involving the extrinsic curvature have the same overall behavior under parity (ignoring the effect of \( P \) on \( K^i_j \) which is canceled by the action of \( T. \) Thus, in total, the state violates \( CP, \) but it is \( CPT \) invariant.

Finally, we notice that for real values of \( \beta \) the state is pure phase. Thus, drawing from analogy with Kodama state in the Euclidean theory, we expect that the state is delta-function normalizable with respect to the kinematical inner-product\(^6\).

## 7 Concluding Remarks

We have shown that the Kodama state can be naturally extended to arbitrary values of the Immirzi parameter. The result reduces to the original Kodama state when the Immirzi parameter is \( \beta = -i. \) Along the way we have employed what appears to be a general method of solving the Hamiltonian constraint for arbitrary values of the Immirzi parameter which allows one to use the simpler self-dual and anti-self dual forms of the constraint as an intermediate step. This method capitalizes on the interpretation of a large class of the Immirzi parameter as the measure of chiral asymmetry in the gravitational interaction. Much work is left to be done. First, it would be valuable to know exactly what conditions must be satisfied for our method of solution to work. Furthermore, a rigorous extension of the method to spin network states is in order. Second, an exploration of the consequences of \( \beta = -i/\cos 2\theta \) as the measure of partial parity violation in gravity may yield interesting results. As alluded to in a footnote, this could lead to an alternative fixing of the Immirzi parameter. Finally, much work is left to be

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\(^8\)This is also seen more readily by noting that for \( \beta = -i, \omega^{ij}_{(L)} = \omega^{ij} + iK^{ij} \) is the pullback of the left-handed four dimensional spin connection, \( \omega^{ij}_{(L)}. \) Since under parity left-handed spinors go to right-handed spinors, the left-handed spin connection and its pullback must also go their right-handed counterparts. Thus, we deduce under parity \( K^{ij} \to -K^{ij}. \)
done on the generalized Kodama state itself. One needs to show that it has a well-defined classical limit which reproduces something like de-Sitter spacetime. The question of normalizability needs to be dealt with rigorously. One may also question what effect the generalization has on the loop transform of the Kodama state which we know has a simple and elegant connection with knot invariants[11]. This is currently under investigation.

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References

[1] H. Kodama Prog. Theor. Phys. 80 (1988) 1024.

[2] H. Kodama Phys. Rev. D 42 (1990) 2548.

[3] L. Smolin, “Quantum gravity with a positive cosmological constant,” arXiv:hep-th/0209079

[4] A. Ashtekar, J. C. Baez, and K. Krasnov, “Quantum geometry of isolated horizons and black hole entropy,” Adv. Theor. Math. Phys. 4 (2000) 1–94, arXiv:gr-qc/0005126

[5] E. Witten, “A note on the chern-simons and kodama wavefunctions,” arXiv:gr-qc/0306083

[6] L. Friedel and L. Smolin Class. Quant. Grav. 21 (2004) 3831–3844, arXiv:hep-th/0310224

[7] S. Holst, “Babero’s hamiltonian derived from a generalized hilbert-palatini action,” Phys.Rev. D 53 (1996) 5966–5969, arXiv:gr-qc/9511026

[8] J. F. Barbero, “Real ashtekar variables for lorentzian signature space-times,” Phys. Rev. D 51 no. 10, 5507–5510, arXiv:gr-qc/9410014
[9] Y. Ling and L. Smolin, “A holographic formulation of quantum supergravity,” *Phys. Rev. D* 63 (2001) arXiv:hep-th/0009018.

[10] C. Soo, “Self-dual variables, positive semi-definite action, and discrete transformations in four-dimensional quantum gravity,” *Phys. Rev. D* 52 (1995) 3484–3493, arXiv:gr-qc/9504042.

[11] E. Witten, “Quantum field theory and the jones polynomial,” *Commun. Math. Phys.* 121 (1989) 351–399.