A NON-AUTONOMOUS BIFURCATION PROBLEM FOR A NON-LOCAL SCALAR ONE-DIMENSIONAL PARABOLIC EQUATION

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Abstract. In this paper we study the asymptotic behaviour of solutions for a non-local non-autonomous scalar quasilinear parabolic problem in one space dimension. Our aim is to give a fairly complete description of the forward asymptotic behaviour of solutions for models with Kirchhoff type diffusion. In the autonomous case we use the gradient structure, symmetry properties and comparison results to obtain a sequence of bifurcations of equilibria, analogous to what is seen in the local diffusivity case. We provide conditions so that the autonomous problem admits at most one positive equilibrium and analyse the existence of sign changing equilibria. Also using symmetry and the comparison results (developed here) we construct what is called non-autonomous equilibria to describe part of the asymptotics of the associated non-autonomous non-local parabolic problem.

1. Introduction and setting of the problem. In this work we consider the following initial boundary value problem

\[\begin{align*}
  &u_t - a(||u_x||^2)u_{xx} = \lambda u - \beta(t)u^3, \quad x \in (0, \pi), \quad t > s, \\
  &u(0, t) = u(\pi, t) = 0, \quad t \geq s, \\
  &u(\cdot, s) = u_0(\cdot) \in H^1_0(0, \pi),
\end{align*}\]

(1.1)

where \(||u_x||^2 = \int_0^\pi |u_x(x)|^2 dx\) (usual norm of the Hilbert space \(H^1_0(0, \pi)\)), \(\lambda \in (0, \infty)\) is a parameter, \(a : \mathbb{R}^+ \to [1, 2]\) is a locally Lipschitz function, \(0 < b_1 < b_2\) and \(\beta : \mathbb{R}^+ \to [b_1, b_2]\) is a globally Lipschitz function.

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The study of the inner structure of attractors for scalar semilinear parabolic problems with local diffusivity (e.g. (1.1) with $a \equiv 1$) is considerably well understood and many very interesting results are available in the literature (see, for example, [10] and references therein). The description of the inner structure for non-local models is much less exploited. Our aim is to provide some techniques to unravel the dynamics of such non-local models in both autonomous and non-autonomous frameworks.

Our problem has its origin in the developments that are concerned with the so-called Chafee-Infante equation. The Chafee-Infante problem, which appeared in the literature for the first time in 1974 (see [12, 13]), can be represented by the following initial boundary value problem,

\begin{equation}
\begin{aligned}
    &u_t - u_{xx} = \lambda u - bu^3, \quad x \in (0, \pi), \ t > s, \\
    &u(0, t) = u(\pi, t) = 0, \quad t \geq s, \\
    &u(\cdot, s) = u_0(\cdot) \in H_0^1(0, \pi).
\end{aligned}
\end{equation}

(1.2)

This is the infinite dimensional model for which the asymptotics is best understood. We know that it is gradient, has a finite number of equilibria and, along a connection between two equilibria, the stable and unstable manifolds intersect transversally (see [18, 2]). Moreover, we also know precisely the diagram of connections between equilibria (see [17]) and that this diagram is stable under autonomous and non-autonomous perturbations (see [18, 2, 6]). In addition, when we replace $b$ in (1.2) by a time dependent function which is not close to a constant, there has been interesting developments ensuring that the asymptotics still resembles that of (1.2) with $b$ constant (see [11, 8]).

The introduction of a non-local diffusion changes everything. Although the gradient structure and some symmetry of solutions are still present, as we will see in the sequel, most of the techniques used in the constant diffusion case fail. For example, the phase plane analysis used to construct the sequence of bifurcations of equilibria does not hold anymore as well as some important tools, such as comparison and the Lap-Number, which will not hold automatically.

Let us comment, in more detail, which important properties remain true for the non-local diffusion case and the ones which do not.

Let $C(H_0^1(0, \pi))$ be the space of continuous functions from $H_0^1(0, \pi)$ to itself. It is well known that the problem (1.2) is globally well-posed. Denote by $\{T(t) : t \geq 0\}$ its solution operator, that is, if $\mathbb{R}^+ \ni t \mapsto u(t, u_0) \in H_0^1(0, \pi)$ is the global solution of (1.2), we write $T(t)u_0 = u(t, u_0)$. The family $\{T(t) : t \geq 0\} \subset C(H_0^1(0, \pi))$ is a gradient semigroup with Lyapunov function $V : H_0^1(0, \pi) \to \mathbb{R}$ given by

\[V(u) = \frac{1}{2} \int_0^\pi \left(u_x^2(x) - \lambda u^2(x) + \frac{b}{2} u^4(x)\right)dx\]

for all $u \in H_0^1(0, \pi)$, that is,

i) $V$ is continuous;

ii) $[0, \infty) \ni t \mapsto V(T(t)u) \in \mathbb{R}$ is non-increasing;

iii) $V(T(t)u) = V(u)$ for all $t \geq 0$ implies that $u$ is an equilibrium of (1.2).

Also, this semigroup has a global attractor $\mathcal{A}$ which, due to the gradient structure, is given by $\mathcal{A} = W^u(\mathcal{E})$, where $\mathcal{E}$ is the set of equilibria and $W^u(\mathcal{E})$ its unstable set. In addition, if the number of equilibria is finite, that is, $\mathcal{E} = \{e_1, \cdots, e_n\}$, then

\[\mathcal{A} = \cup_{i=1}^n W^u(e_i). \]
We observe that equilibria have a fundamental role in the description of the dynamics of gradient semigroups.

To ensure that the number of equilibria for (1.2) is finite, in [12], the authors employ a phase plane analysis, constructing a “time map” which gives information about the existence and non-existence of equilibria.

As the parameter $\lambda > 0$ varies, we have the following sequence of bifurcations:

**Theorem 1.1 ([12]).** If $N^2 < \lambda \leq (N + 1)^2$, for some $0 < N \in \mathbb{N}$, then there are $2N + 1$ equilibria $\phi_{j,b}^+ = \phi_{0,b} = 0$.

i) $\phi_{j,b}^+$ vanishes exactly at $\{k\pi : 0 \leq k \leq j\}$, $1 \leq j \leq N$;

ii) $\phi_{j,b}^- = -\phi_{j,b}^+$, $\phi_{j,b}^+$ does not vanish for $x \in (0, \frac{\pi}{j})$, $1 \leq j \leq N$;

iii) $\phi_j(x) = \phi_j(\frac{\pi}{j} - x)$, $x \in (0, \frac{\pi}{j})$ and $\phi_j(x) = -\phi_j(x - \frac{\pi}{j})$, $x > \frac{\pi}{j}$, $1 \leq j \leq N$;

iv) There is no other equilibrium of (1.2).

Consider now the autonomous non-local problem

$$
\begin{aligned}
\begin{cases}
    u_t = a(||u||^2)u_{xx} + \lambda u - bu^3, & x \in (0, \pi), \quad t > 0, \\
    u(0,t) = u(\pi,t) = 0, & t \geq 0, \\
    u(0) = u_0 \in H_0^1(0,\pi).
\end{cases}
\end{aligned}
$$

for some constant $b > 0$. We will see that (1.4) is globally well-posed. If $\{T_{a,b}(t) : t \geq 0\}$ denotes its solution operator for (1.4), the family $\{T_{a,b}(t) : t \geq 0\} \subset C(H_0^1(0,\pi))$ is a gradient semigroup with Lyapunov function $V : H_0^1(0,\pi) \to \mathbb{R}$ given by

$$
V(u) = \frac{1}{2} \int_0^\pi ||u||^2 a(s)ds + \int_0^\pi \left( -\frac{\lambda}{2} u^2(x) + \frac{b}{4} u^4(x) \right) dx.
$$

We will see that $\{T_{a,b}(t) : t \geq 0\}$ has a global attractor $A_n$ and that it has a finite number of equilibria $E = \{e_1, \ldots, e_n\}$, consequently

$$
A_n = \bigcup_{i=1}^n W^n(e_i^a).
$$

As the parameter $\lambda > 0$ varies, we have the following sequence of bifurcations for the equilibria of (1.4).

**Theorem 1.2.** If $a(0)N^2 < \lambda \leq a(0)(N + 1)^2$, then there are at least $2N + 1$ equilibria of the equation (1.4), each of them having the properties i) – iv), indicated in Theorem 1.1, for some j. If, in addition, the function $a$ is non-decreasing, then there are exactly $2N + 1$ equilibria of the equation (1.4) satisfying the properties indicated in Theorem 1.1.

The phase plane analysis that led to the results of Theorem 1.1 do not apply to the non-local diffusion case or to the non-autonomous case. To overcome this difficulty, we have pursued a different approach, based on comparison results (which do not hold automatically, but we were able to prove), to construct subspaces of $H_0^1(0,\pi)$ which are positively invariant where we use the properties of the gradient semigroup.

A similar approach, to that described in the above paragraph, has been used to study the inner structure of pullback attractors and uniform attractors (see [8, 11]) for a non-autonomous version of (1.2). In order to describe the results for the non-autonomous problem (1.1) we will need to introduce some terminology.
We will see that (1.1) is globally well-posed and, if \([s, \infty) \supseteq t \mapsto u(t, s, u_0) \in H^1_0(0, \pi)\) denotes the global solution of (1.1), we write \(S_{a, \beta}(t, s)u_0 = u(t, s, u_0)\) for \(t \geq s\). If \(\mathcal{P} = \{(t, s) \in \mathbb{R} \times \mathbb{R} : t \geq s\}\) we define

**Definition 1.3.** An evolution process \(\{S_{a, \beta}(t, s) : (t, s) \in \mathcal{P}\} \subset C(H^1_0(0, \pi))\) is a family of maps that satisfies the following conditions:

i) \(S_{a, \beta}(t, t) = I_{H^1_0(0, \pi)}\), for all \(t \in \mathbb{R}\), here \(I_{H^1_0(0, \pi)}\) denotes the identity in \(H^1_0(0, \pi)\);

ii) \(S_{a, \beta}(t, s)S_{a, \beta}(s, \tau) = S_{a, \beta}(t, \tau)\), for all \(t, s, \tau \in \mathbb{R}\) with \(t \geq s \geq \tau\);

iii) \(\mathcal{P} \times H^1_0(0, \pi) \ni (t, s, x) \mapsto S_{a, \beta}(t, s)x \in H^1_0(0, \pi)\) is a continuous map.

When \(\beta(t) = b\) for all \(t \in \mathbb{R}\), \(S_{a, b}(t, s) = S_{a, b}(t - s, 0)\), for all \((t, s) \in \mathcal{P}\). In this case, the process will be called an autonomous process or a semigroup and \(S_{a, b}(t, 0) = T_{a, b}(t)\), for all \(t \geq 0\), with \(\{T_{a, b}(t) : t \geq 0\}\) defined by (1.4).

A global solution of the process \(\{S_{a, \beta}(t, s) : (t, s) \in \mathcal{P}\}\) is a function \(\xi : \mathbb{R} \to H^1_0(0, \pi)\) such that

\[ S_{a, \beta}(t, s)\xi(s) = \xi(t), \quad \forall (t, s) \in \mathcal{P}. \]

If the set \(\{\xi(t) : t \in \mathbb{R}\}\) is bounded in \(H^1_0(0, \pi)\), \(\{\xi(t) : t \in \mathbb{R}\}\) is called a bounded solution. Next we define the solutions of (1.1) that will have the role that equilibria have in the description of the dynamics (see (1.5)).

**Definition 1.4.** A global solution \(\xi : \mathbb{R} \to H^1_0(0, \pi)\) of \(\{S_{a, \beta}(t, s) : (t, s) \in \mathcal{P}\}\) is a non-autonomous equilibrium if the zeros of \(\xi(t)\) are the same for all \(t \in \mathbb{R}\) and \(\xi\) is non-degenerate as \(t \to \pm \infty\), that is, we can find \(\phi \in H^1_0(0, \pi)\) such that

\[ |\xi(t)(x)| \geq \phi(x) > 0, \quad \text{for all } t \in \mathbb{R} \quad \text{and for all } x \in (0, \pi) \quad \text{such that } \xi(t)(x) \neq 0. \]

A notion of non-autonomous equilibria also appears in [16, page 3], associated with random dynamical systems. The notion we have used here is related to the global solutions that play the role of equilibria in the characterization (1.5), see [10, Definition 3.9] for the abstract theoretical situation and [8, 11] for (1.1) with \(a \equiv 1\).

In order to further describe the results we will need to introduce the notions of pullback and uniform attractor for evolution processes.

**Definition 1.5.** A family \(\{A(t) : t \in \mathbb{R}\} \subset H^1_0(0, \pi)\) is the pullback attractor of \(\{S_{a, \beta}(t, s) : (t, s) \in \mathcal{P}\}\) if

i) \(A(t)\) is compact, for each \(t \in \mathbb{R}\);

ii) \(S_{a, \beta}(t, s)A(s) = A(t)\), for all \(t \geq s\);

iii) The family \(\{A(t) : t \in \mathbb{R}\}\) pullback-attracts bounded sets of \(H^1_0(0, \pi)\), that is, for each bounded \(B \subset H^1_0(0, \pi)\), we have that

\[ \sup_{b \in B} \inf_{a \in A(t)} ||S_{a, \beta}(t, s)b - a||_{H^1_0(0, \pi)} \longrightarrow 0 \quad \text{as } s \to -\infty; \]

iv) \(\{A(t) : t \in \mathbb{R}\}\) is the minimal closed family that satisfies the condition iii).

If we assume \(\bigcup_{t \in \mathbb{R}} A(s)\) is bounded in \(H^1_0(0, \pi)\), then we have the following characterization for the pullback attractor:

\[ A(t) = \{\xi(t) : \xi : \mathbb{R} \to H^1_0(0, \pi) \text{ is a bounded global solution of } \{S_{a, \beta}(t, s)\}\}. \quad (1.6) \]

**Definition 1.6.** A set \(A\) is the uniform attractor of \(\{S_{a, \beta}(t, s) : (t, s) \in \mathcal{P}\}\) if it is a compact subset of \(H^1_0(0, \pi)\) with the property that \(\sup_{t \in \mathbb{R}} \sup_{b \in B} \inf_{a \in A} ||S_{a, \beta}(t + \tau, \tau)b - a||_{H^1_0(0, \pi)} \to 0\) for any \(B \subset H^1_0(0, \pi)\) bounded.

For more details about evolution processes and their attractors, see [10].

Concerning the local version of (1.1), namely, when \(a \equiv 1\), in [11] the authors prove that if \(N^2 < \lambda \leq (N + 1)^2\) there are \(2N + 1\) non-autonomous equilibria
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including the zero equilibrium), showing that the same sequence of bifurcation observed in the autonomous case, for (1.2), is also present in the non-autonomous case. These non-autonomous equilibria arise in the quest to find global bounded solutions that would play the role that the equilibria have, in the description of attractors for autonomous gradient systems (as in (1.3)).

In [8], it is shown that the non-autonomous equilibria play such a role. They are the key factor in the construction of the 'lifted-invariant isolated invariant sets' to which the solutions converge and the rest of the uniform attractor correspond to ‘connections’ between two of these, see [8] for more details.

A natural further development is asking what happens with the structure of autonomous and non-autonomous attractors when the diffusion is a non-local Kirchhoff-type term $a$ as in (1.1). We consider our equation (1.1) in the particular case when $\beta$ is time independent and when $\beta$ is time dependent. That simple natural extension leads to the need of completely different techniques. An important aspect, that deserves to be pointed out, is that the operator $a \left( \|u_x\|^2 \right) u_{xx}$ is non-linear and non-local, leading to several interesting and nontrivial mathematical issues about monotonicity (see [7, 5, 9]), comparison principles (c.f. [9, 1]), symmetry, (odd) extension and hyperbolicity among others.

To obtain the same sequence of bifurcations when $\beta$ is time dependent, the authors in [11] use the existence of a positive non-degenerate global solution (inspired in the work of [21]) and construct invariant regions in $H^1_0(0, \pi)$. That technique can also be used to obtain, in a different way, the sequence of bifurcations of equilibria of (1.2). Its application requires that the first eigenvalue of $-u_{xx} + \lambda u$ with Dirichlet boundary conditions to be positive, that comparison results are available and the use of the gradient structure of the problem. Since (1.1) has a non-linear principal part, this analysis will require several adjustments and new ideas.

In Section 2, we develop the comparison results that will be essential to deal with the non-autonomous problem. In Section 3, we prove our central result, Theorem 3.2, constructing bounded positively invariant regions in $H^1_0(0, \pi)$. In Section 4, we sharpen the bifurcation result, using a new technique, obtaining that the sequence of bifurcations happens, in the autonomous case, at $a(0)N^2$, under mild additional conditions on the diffusivity function $a$ and without using comparison. Finally, in Section 5, we present some open questions for further investigation as well as the difficulties involved in each of these questions.

2. Comparison results. First consider the initial boundary value problem

\[
\begin{align*}
& w_{\tau} = w_{xx} + \frac{1}{a(\|w_x\|^2)} \left( \lambda w - \beta \left( s + \int_s^\tau \frac{1}{a(\|w_x(\cdot, \theta)\|^2)} d\theta \right) w^3 \right), \quad x \in (0, \pi), \quad \tau > s, \\
& w(0, \tau) = w(\pi, \tau) = 0, \quad \tau \geq s, \\
& w(\cdot, s) = u_0(\cdot) \in H^1_0(0, \pi)
\end{align*}
\]

(2.1)

Problem (2.1) is locally well-posed and the solutions are jointly continuous with respect to time and initial conditions. We observe that the results of [10, Section 6.7] can be applied with the same proofs, even though the nonlinearity in (2.1) has a non-local in time and in space characteristic.

Now, making the change of variables $t = s + \int_s^\tau a(\|w_x(\cdot, \theta)\|^2)^{-1} d\theta$ we have that $w(x, t) = w(x, \tau)$ is the unique solution of (1.1).
By using the comparison result that we develop next, we can also prove that the solutions of (2.1) are globally defined. Therefore, those of (1.1) are also globally defined. Hence, we can define the solution operators in the following way: if \( w(t, s, u_0) \) is the solution of (2.1) we write \( U(t, s)u_0 = w(t, s, u_0) \). That defines an evolution process \( \{U(t, s) : t \geq s\} \subset C(H^1_0(0, \pi)) \) associated to (2.1).

Consider the auxiliary initial boundary value problems

\[
\begin{cases}
  z_t = z_{xx} + \lambda z - \frac{b_1}{2} z^3, & x \in (0, \pi), \ t > 0 \\
  z(0, t) = z(\pi, t) = 0, & t \geq 0, \\
  z(\cdot, 0) = z_0(\cdot) \in H^1_0(0, \pi),
\end{cases}
\]

(2.2)

and

\[
\begin{cases}
  v_t = v_{xx} + \frac{\lambda}{2} v - b_2 v^3, & x \in (0, \pi), \ t > 0 \\
  v(0, t) = v(\pi, t) = 0, & t \geq 0, \\
  v(\cdot, 0) = v_0(\cdot) \in H^1_0(0, \pi).
\end{cases}
\]

(2.3)

It is well known that (2.2) and (2.3) are globally well-posed and if \( \mathbb{R}^+ \ni t \mapsto z(t, z_0) \in H^1_0(0, \pi) \) and \( \mathbb{R}^+ \ni t \mapsto v(t, v_0) \in H^1_0(0, \pi) \) denote the solutions of (2.2) and (2.3), we define the semigroups \( \{T_1(t) : t \geq 0\} \) and \( \{T_2(t) : t \geq 0\} \) by

\[
T_1(t)z_0 = z(t, z_0) \quad \text{and} \quad T_2(t)v_0 = v(t, v_0), \ t \geq 0, \text{ see [10]}. 
\]

For \( X = H^1_0(0, \pi) \) or \( X = L^2(0, \pi) \) we introduce the following partial ordering

\[
u \gg v \in X \Leftrightarrow u(x) \geq v(x) \text{ a. e. for } x \in (0, \pi).
\]

Denote by \( \mathcal{C} = \{u \in X : u \geq 0\} \), the associated positive cone and by \( B_X(0, r) \) the open ball of radius \( r \) around 0 in \( X \).

This section will be dedicated to the proof of the following result:

**Theorem 2.1.** With the above notation, if \( u_0 \leq u_1 \leq u_2 \) in \( H^1_0(0, \pi) \), then

\[
T_2(t - s)u_0 \leq U(t, s)u_1 \leq T_1(t - s)u_2, \ \forall (t, s) \in \mathcal{P}. \tag{2.4}
\]

To prove that the result of Theorem 2.1 holds, we first write (2.1) (2.2) and (2.3) in the abstract form

\[
\dot{u} + Au = f(t, u) \tag{2.5}
\]

where

1. \( A : D(A) \subset L^2(0, \pi) \rightarrow L^2(0, \pi) \) is the linear operator defined by \( D(A) = H^2(0, \pi) \cap H_0^1(0, \pi) \) and \( Au = -u_{xx}, \ u \in D(A) \). It is clear that \( (0, \infty) \subset \rho(A) \) (resolvent of \( A \)) and that for all element \( u_0 \in L^2(0, \pi) \) with \( u_0 \geq 0 \) we have

\[
(\lambda - A)^{-1}u_0 \geq 0, \quad \forall \lambda > 0.
\]

We express this fact by saying that \( A \) has positive resolvent.

2. The nonlinearity \( f : \mathbb{R} \times H^1_0(0, \pi) \rightarrow L^2(0, \pi) \) satisfies: For each \( r > 0 \) there exists \( \gamma(r) > 0 \) such that, for all \( t \in [t_0, t_1] \) and \( u \in \mathcal{C} \cap B_{H^1_0(0, \pi)}(0, r) \), \( \gamma u + f(t, u) \) is positive.

Denote by \( u_f(t, s, u_0) \) the solution at time \( t \) of (2.5), satisfying \( u_f(s, s, u_0) = u_0 \). The following theorem provides the comparison result that we are seeking.

**Theorem 2.2.** [10, Theorem 6.4] If \( A \) is as above and \( f, g, \) and \( h \) are functions that satisfies (2), then we have the following

i. If for every \( r > 0 \) there is a constant \( \gamma = \gamma(r) > 0 \) such that \( f(t, \cdot) + \gamma I \) is increasing in \( B_{H^1_0(0, \pi)}(0, r) \), for all \( t \in [s, t_1] \) and \( u_0, u_1 \in H^1_0(0, \pi) \) with \( u_0 \geq u_1 \), then \( u_f(t, s, u_0) \geq u_f(t, s, u_1) \) as long as both solutions exist.
(ii) If $f(t,\cdot) \geq g(t,\cdot)$ for all $t \in \mathbb{R}$ and $u_0 \in H^1_0(0, \pi)$ then $u_f(t,s,u_0) \geq u_g(t,s,u_0)$ as long as both solutions exist.

(iii) If $f, g$ are such that for every $r > 0$ there exist a constant $\gamma = \gamma(r) > 0$ and an increasing function $h(t,\cdot)$ such that, for every $t \in [s,t_1]$

$$f(t,\cdot) + \gamma I \geq h(t,\cdot) \geq g(t,\cdot) + \gamma I$$

in $B_{H^1_0(0,\pi)}(0,r)$ and $u_0, u_1 \in H^1_0(0, \pi)$ with $u_0 \geq u_1$, then $u_f(t,s,u_0) \geq u_g(t,s,u_1)$ as long as both exist.

Proof of Theorem (2.1). Observe that, given $R > 0$ there exists $\gamma(R) > 0$ such that for $t \in \mathbb{R}$, $\nu \in \mathbb{R}^+$ and $|u| \leq R$

$$0 \leq \gamma u + \frac{\lambda}{2} u - b_2 u^3 \leq \gamma u + \frac{\lambda u - \beta(t) u^3}{a(\nu^2)} \leq \gamma u + \lambda u - \frac{b_1}{2} u^3 \quad (2.6)$$

with $\gamma u + \frac{\lambda u}{2} - b_2 u^3$ and $\gamma u + \lambda u - \frac{b_1}{2} u^3$ being increasing functions in the variable $u$ in the interval $[-R,R]$.

Now let us compare solutions of (2.1), (2.2) and (2.3). To that end, we define

$$g_1(t,u)(x)$$

$$= \frac{\lambda u(x) - \beta(s + \int_{1}^{3} a(||w_x(\cdot, \theta)||^2)^{-1} d\theta) u^3(x)}{a(||w_x||^2)} h_1(t,u)(x)$$

$$= f_1(t,u)(x) = \gamma u(x) - \frac{b_1}{2} u^3(x) \quad (2.7)$$

$$f_2(t,u)(x)$$

$$= \frac{\lambda u(x) - \beta(s + \int_{1}^{3} a(||w_x(\cdot, \theta)||^2)^{-1} d\theta) u^3(x)}{a(||w_x||^2)} g_2(t,u)(x)$$

$$= h_2(t,u)(x) = \frac{\lambda}{2} u(x) - b_2 u^3(x)$$

Noticing that $H^1_0(0, \pi)$ is embedded in $L^\infty(0, \pi)$ and using (2.6), Theorem 2.2, item iii) can be applied twice to obtain the result of Theorem 2.1.

3. The non-autonomous equation. In this section we will use the comparison result of Theorem 2.1 to construct the non-autonomous equilibria of (1.1).

The process $\{U(t,s): t \geq s\}$ defined by (2.1) admits a pullback attractor. The pullback uniform bounded dissipativity property follows exactly as in [4] using the comparison result of Theorem 2.1.

Now, note that, by Theorem 1.1, if $\lambda > 2$, we can find a positive equilibrium $\phi^+_{1,b_1}$ of (2.2) and a positive equilibrium $\phi^+_{1,b_2}$ of (2.3). Using Theorem 2.1 and the fact that $\{T_1(t): t \geq 0\}$ is gradient, we have

$$\phi^+_{1,b_2} = T_2(t) \phi^+_{1,b_2} \leq T_1(t) \phi^+_{1,b_2} \xrightarrow{t \to +\infty} \psi,$$

for some positive equilibrium $\psi$ of (2.2).

By the uniqueness of the positive equilibrium of (2.2), we conclude that $\psi = \phi^+_{1,b_1}$ and, consequently, $\phi^+_{1,b_2} \leq \phi^+_{1,b_1}$.

Define the set

$$X^+_1 = \left\{ u \in H^1_0(0,\pi) : \phi^+_{1,b_2}(x) \leq u(x) \leq \phi^+_{1,b_1}(x), \quad u(x) = u(\pi - x) \quad \text{in} \quad (0,\pi) \right\}.$$
3.1. Construction of a positive non-autonomous equilibrium. We will use the denomination “positive solution” for a global solution \( \xi(t) \in \mathcal{C} \) for all \( t \in \mathbb{R} \). If there exists a \( \phi \in \mathcal{C} \cap \{ \psi \in C^1(0, \pi) : \psi'(0) \cdot \psi'(\pi) < 0 \} \) and \( t_0 \in \mathbb{R} \) such that \( \phi \leq \xi(t) \) for all \( t \leq t_0 \) (for all \( t \geq t_0 \)) then \( \xi \) will be called non-degenerate as \( t \to -\infty \) (as \( t \to +\infty \)).

Note that, a positive global solution \( \xi \) of \( U(\cdot, \cdot) \) which is non-degenerate as \( t \to \pm \infty \) is non-autonomous equilibrium (see Definition 1.4).

To construct a positive non-autonomous equilibrium, we will prove that \( X^+_1 \) is positively invariant, which means \( U(t, s)X^+_1 \subset X^+_1 \), for all \((t, s) \in \mathcal{P}\).

Given \( u_0 \in X^+_1 \), for \( x \in (0, \pi) \) we have

\[
\phi^+_{1,b_1}(x) \leq T_2(t-s)u_0 \leq U(t,s)u_0 \leq T_1(t-s)u_0 \leq \phi^+_{1,b_1}(x)
\]

where we used the comparison result of Theorem 2.1 and that \( T_i(t-s)\phi^+_{1,b_1} = \phi^+_{1,b_1} \), for all \((t, s) \in \mathcal{P}\), \( i = 1, 2 \).

Since \( u_0(x) = u_0(\pi-x) \) for \( x \in (0, \pi) \), if \( u(t, s, u_0)(x) := U(t, s)u_0(x) \), then both \( u(t, s, u_0)(\cdot) \) and \( u(t, s, u_0)(\pi-\cdot) \) are \( H^1_0(0, \pi) \) with \( t \geq s \) are solutions to the problem (2.1). By uniqueness of solutions we conclude that \( u(t, s, u_0)(x) = u(t, u_0)(\pi-x) \), for all \( x \in (0, \pi) \).

**Theorem 3.1.** Suppose \( \lambda > 2 \). Then the process \( \{U(t, s) : (t, s) \in \mathcal{P}\} \) restricted to \( X^+_1 \) admits a pullback attractor. In particular, there exists a non-autonomous equilibrium in \( \mathcal{C} \).

**Proof.** The positive invariance follows from the reasoning that preceded the theorem. The fact that \( \{U(t, s) : (t, s) \in \mathcal{P}\} \) has a pullback attractor in \( H^1_0(0, \pi) \) ensures that it also has a pullback attractor when restricted to \( X^+_1 \).

Now, any global solution in the pullback attractor of \( \{U(t, s) : (t, s) \in \mathcal{P}\} \) restricted to \( X^+_1 \) is a non-autonomous equilibrium. \( \square \)

![Figure 1](image.png)

**Figure 1.** Region bounded by the positive equilibria \( \phi^+_{1,b_1} \) and \( \phi^+_{1,b_2} \).

3.2. Construction of other non-autonomous equilibria. Next, we assume that \( \lambda > 2N^2 \), for some positive integer \( N \). From Theorem 1.1, there is an equilibrium \( \phi^+_{j,b_i} \) with \( j+1 \) zeros in \([0, \pi]) \), \( 1 \leq j \leq N \), for the semigroup \( \{T_i(t) : t \geq 0\} \), \( i = 1, 2 \).

Now, if \( 1 < j \leq N \), we consider the set \( X^+_j = Y^+_j \cap Z_j \), where

\[
Y^+_j = \left\{ u \in H^1_0(0, \pi) : \min \left( \phi^+_{j,b_1}(x), \phi^+_{j,b_2}(x) \right) \leq u(x) \leq \max \left( \phi^+_{j,b_1}(x), \phi^+_{j,b_2}(x) \right) \right\}
\]

and

\[
Z_j = \left\{ u \in H^1_0(0, \pi) : u(x) = u \left( \frac{\pi}{j} - x \right), 0 < x < \frac{\pi}{j} \text{ and } u(x) = -u \left( \frac{\pi}{j} - x \right), x > \frac{\pi}{j} \right\}.
\]

Let us prove that these sets are positively invariant.

We will start with \( j = 2 \).
Consider $u_0 \in X^+_2$ then we know $u_0 \in Z_2$ which means that $u_0(x) = -u_0(\pi - x)$, for $x \in [0, \pi]$. And by the uniqueness of solution, we have $u(t, s, u_0)(x) = -u(t, s, u_0)(\pi - x)$, for all $x \in [0, \pi]$ and $t \geq s$. With this we proved that $u \in Z_2$. In particular, $u(t, s, u_0)(\frac{\pi}{2}) = 0$ for $t \geq s$.

Now we can use comparison restricted to the subintervals $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \pi]$:

$$
\begin{cases}
0 \leq T_2(t-s)u_0 \leq U(t, s)u_0 \leq T_1(t, s)u_0 & \text{in } [0, \frac{\pi}{2}]

T_1(t-s)u_0 \leq U(t, s)u_0 \leq T_2(t, s)u_0 \leq 0 & \text{in } [\frac{\pi}{2}, \pi]
\end{cases}
$$

Since $\phi^+_{2, b_1}$ is an equilibrium of $\{T_i(t) : t \geq 0\}$, $i = 1, 2$ and $0 \leq \phi^+_{2, b_2} \leq u_0 \leq \phi^+_{2, b_1}$ in $[0, \frac{\pi}{2}]$ and $\phi^+_{2, b_1} \leq u_0 \leq \phi^+_{2, b_2} \leq 0$ in $[\frac{\pi}{2}, \pi]$, we can write

$$
\begin{cases}
0 \leq \phi^+_{2, b_2} \leq U(t, s)u_0 \leq \phi^+_{2, b_1} & \text{in } [0, \frac{\pi}{2}]

\phi^+_{2, b_1} \leq U(t, s)u_0 \leq \phi^+_{2, b_2} \leq 0 & \text{in } [\frac{\pi}{2}, \pi].
\end{cases}
$$

Therefore, $X^+_2$ is positively invariant.

Before proving the case $X^+_4$, we will prove the positive invariance for $X^+_3$. Just observe that if $u_0 \in X^+_3$ then $u_0(x) = -u_0(\pi - x)$, for all $x \in (0, \pi)$ and, in particular, $u(t, s, \frac{\pi}{2}) = 0$. Now, we can analyse the following problem

$$
\begin{align*}
& u_t = u_{xx} + \frac{\lambda u - \beta(s + \frac{1}{a(\|u_x\|^2)} \int_0^t \frac{1}{a(\|u_x\|^2)} d\theta)u^3}{a(\|u_x\|^2)} , \quad x \in \left(0, \frac{\pi}{2}\right) , \quad t > s \\
& u(0, t) = u\left(\frac{\pi}{2}, t\right) = 0, \quad t \geq s \\
& u(\cdot, s) = u_0(\cdot) \in H^1_0\left(0, \frac{\pi}{2}\right)
\end{align*}
$$

(3.1)

where $\|u_x\|^2 = \int_0^\pi u_x^2(s)ds$. Moreover, using the uniqueness of solution for (3.1), we conclude that

$$
u(t, s, u_0)(x) = -u(t, s, u_0)\left(\frac{\pi}{2} - x\right), \quad \text{for } x \in \left[0, \frac{\pi}{2}\right]
$$

and

$$
u(t, s, u_0)(x) = -u(t, s, u_0)(\pi - x) \quad \text{for } x \in [0, \pi].
$$

In particular, $u(t, s, u_0, \frac{\pi}{2}) = 0$ for all $t \geq s$ and $u(t, s, u_0, \frac{3\pi}{4}) = -u(t, s, u_0, \frac{\pi}{4}) = 0$.

With this, we can prove that $u$ lies in $Z_4$. Now, we can use comparison to prove that $X^+_3$ is positively invariant.
To prove the invariance of $X^+_3$, we define the following set

$$W^+_3 = \left\{ u_0 \in H^1_0 \left(0, \frac{4\pi}{3}\right): u_0(x) = -u_0 \left(\frac{4\pi}{3} - x\right) \text{ in } \left[0, \frac{4\pi}{3}\right] \text{ and } u_0 |_{[0,\pi]} \in X^+_3 \right\}$$

and consider the problem (2.1) in the interval $[0, \frac{4\pi}{3}]$ and with $a(\cdot) = a(\frac{4\pi}{3} \cdot)$.

We have that $u(t, s, u_0, \frac{4\pi}{3}) = 0$ and we can use the same idea as in $X^+_4$ to prove that $u(t, s, u_0, \frac{4\pi}{3}) = 0$. The comparison in $[0, \pi]$ follows similarly to the previous cases.

Therefore $X^+_j$ is invariant under the action of $\{U(t, s): t \geq s\}$, since it is a restriction of $W^+_4$ to the interval $[0, \pi]$.

For the other cases, $u_0 \in X^+_j$, just observe that the invariance of $Z_j$ can be obtained using the reasoning applied in the previous cases and then we conclude that $u(t, s, u_0)(\frac{4\pi}{3}) = 0$, $k = 0, \ldots, j$, for all $t \geq s$.

Now, for all $(t, s) \in \mathcal{P}$, we have the following comparison:

$$0 \leq (T_2(t - s)u_0)(x) \leq (U(t, s)u_0)(x) \leq (T_1(t - s)u_0)(x), \quad x \in \left[0, \frac{\pi}{j}\right]$$

and

$$(U(t, s)u_0)(x) = -(U(t, s)u_0)(x - \frac{\pi}{j})$$

and

$$(T_1(t - s)u_0)(x) = -(T_1(t - s)u_0)(x - \frac{\pi}{j}), \quad x > \frac{\pi}{j}, i = 1, 2.$$

With this, we conclude that $U(t, s)u_0 \in X^+_j$ for all $(t, s) \in \mathcal{P}$. Therefore, $X^+_j$ is positively invariant under the action of $\{U(t, s): (t, s) \in \mathcal{P}\}$.

**Theorem 3.2.** Suppose that, for some $0 < N \in \mathbb{N}$, we have $2N^2 < \lambda \leq 2(N + 1)^2$. Then, for $j = 1, \ldots, N$, the process $\{U(t, s): (t, s) \in \mathcal{P}\}$ restricted to $X^+_j$ admits a pullback attractor.

In particular, for each $j = 1, \ldots, N$, there exists a non-autonomous equilibrium $\xi^+_j$ that has $j + 1$ zeros in $[0, \pi]$.

**Remark 1.** Note that if $\lambda > 2N^2$, for each $1 \leq j \leq N$, there exists an equilibrium $\phi_{j,b_1}^-$ of $\{T_1(t): t \geq 0\}$, $i = 1, 2$, with $j + 1$ zeros in $[0, \pi]$. Then, we can define the set $X^-_j = Y^-_j \cap Z_j$, where

$$Y^-_j = \left\{ u \in H^1_0(0, \pi): \min \left(\phi_{j,b_1}^-(x), \phi_{j,b_2}^-(x)\right) \leq u(x) \leq \max \left(\phi_{j,b_1}^-(x), \phi_{j,b_2}^-(x)\right) \right\}.$$

We can also prove that $U(t, s)X^-_j \subset X^-_j$, for all $(t, s) \in \mathcal{P}$.

Observe that the whole construction was carried out for solutions of (2.1). Recall that the change of variables only affects $t$, hence we have also constructed a set of bounded non-autonomous equilibria of (1.1). We can summarize the result in the following

**Theorem 3.3.** Suppose that $\lambda > 2N^2$, for $0 < N \in \mathbb{N}$. The problem (1.1) has at least $2N$ non-autonomous equilibria.

**Remark 2** (The autonomous problem). From the previous section, by taking $b_1 = b_2 = b$ in (2.2) and (2.3), we have the comparison (2.4) with $U(t, s)(t, s)$ replaced by $T_{a,b}(t - s)$.

Suppose that $\lambda > 2$. For $u_0 \in X^+_1$, there is a sequence $t_n \to \infty$ such that

$$\phi^+_{1,b_2} \leq T_{a,b}(t_n)u_0 \to \psi$$
where $\psi$ is an equilibrium of the equation (1.4) and we have used that $\{T_{a,b}(t) : t \geq 0\}$ is a gradient semigroup. Also, by construction, $\psi \in X^+_1$.

This proves that, if $\lambda > 2$, there exists a positive equilibrium of (1.4). We could use the same arguments for $\{T_{a,b}(t) : t \geq 0\}$ to guarantee that if $\lambda > 2N^2$, for some $0 < N \in \mathbb{N}$, we can construct $2N$ non-zero equilibria of (1.4), $\{\phi^+_j : 1 \leq j \leq N\}$, with the properties (ii), (iii) and (iv) of Theorem 1.1.

In fact we will see, in Section 4, that such positive symmetric equilibrium can be constructed for $\lambda > a(0)$. If we require the function $a$ to be non-decreasing, then we can find exactly one positive symmetric equilibrium for $\lambda > a(0)$. We will also show that the bifurcations of equilibria happen at $\lambda = a(0)N^2$, $0 \neq N \in \mathbb{N}$, using the existence of a positive solution and comparison results.

4. An auxiliary elliptic problem. In this section we derive a different approach to obtain the equilibria of the problem (1.4) by proving the existence of a positive, nontrivial and symmetric equilibria. That is achieved by minimizing the energy functional of the associated elliptic problem

$$
\begin{cases}
-a \left(\|u_x\|_2^2\right) u_{xx} = \lambda u - bu^3, x \in \left(0, \frac{\pi}{j}\right), \\
u(0) = u \left(\frac{\pi}{j}\right) = 0
\end{cases}
$$

(4.1)

where $0 \neq j \in \mathbb{N}$ and $\|u_x\|_j^2 = \int_0^{\pi} |u_x(x)|^2 dx$ is the usual norm in $H^1_0(0, \pi)$. The parameters $\lambda > 0$ and $b > 0$ are positive numbers and the function $a : \mathbb{R} \to \mathbb{R}$ is bounded away from zero, that is, there exists $\sigma > 0$ such that

$$
0 < \sigma \leq a(r), \quad \text{for all } r \in \mathbb{R}. \quad (4.2)
$$

We say that $u \in H^1_0(0, \pi)$ is a weak solution to the problem (4.1) if

$$
a \left(\|u_x\|_j^2\right) \int_0^{\pi} u_x(x)v_x(x) dx + \int_0^{\pi} \left(-\lambda u(x) + bu^3(x)\right)v(x) dx = 0, \quad (4.3)
$$

for all $v \in H^1_0(0, \pi)$.

A solution of (4.3) can be found as a critical point of the energy functional $E : H^1_0(0, \pi) \to \mathbb{R}$ defined by

$$
E(u) = \frac{1}{2} \int_0^{\pi} a(s)|u_x|^2 ds + \int_0^{\pi} \left(-\frac{\lambda}{2} u^2(x) + \frac{b}{4} u^4(x)\right) dx.
$$

In fact, for all $u, v \in H^1_0(0, \pi)$,

$$
\langle E'(u), v \rangle = a \left(\|u_x\|_j^2\right) \int_0^{\pi} u_x(x)v_x(x) dx + \int_0^{\pi} \left(-\lambda u(x) + bu^3(x)\right)v(x) dx.
$$

Note that, $-\frac{3}{2}s^2 + \frac{b}{4}s^4 \geq -\frac{\lambda^2}{4b} \pi^2$ for all $s \in \mathbb{R}$. It follows that $E(u) \geq \frac{\sigma}{2} \|u_x\|_j^2 - \frac{\lambda^2 \pi^2}{4b\pi}$ and that $E$ is coercive. Also, $E$ is easily seen to be weakly lower semicontinuous in $H^1_0(0, \pi)$. Thus,

Recall that

$$
L : H^2 \left(0, \frac{\pi}{j}\right) \cap H^1_0 \left(0, \frac{\pi}{j}\right) \subset L^2 \left(0, \frac{\pi}{j}\right) \rightarrow L^2 \left(0, \frac{\pi}{j}\right),
$$

$$
Lu = u_{xx}, \quad u \in H^2 \left(0, \frac{\pi}{j}\right) \cap H^1_0 \left(0, \frac{\pi}{j}\right)
$$
Theorem 1.2 of [22]. Consequently, there exists a minimum $u \in \Omega$.

Lemma 4.1. Assume only that the function $R^+ \ni t \mapsto a(t) \in [\sigma, \infty)$ is continuous and satisfies (4.2). If $\lambda > a(0)j^2$, $0 \neq j \in \mathbb{N}$, there exists a nontrivial positive weak solution of (4.1).

Proof. Since we are interested in positive weak solutions to the problem (4.1), we restrict the domain of $E$ to

$$\mathcal{M} = \left\{ v \in H^1_0 \left( 0, \frac{\pi}{j} \right) : 0 \leq v(x) \leq \sqrt{\frac{\lambda}{b}}, \ x \in \left( 0, \frac{\pi}{j} \right) \right\}.$$ 

Clearly $\mathcal{M}$ is weakly closed, that is $\mathcal{M}$ and $E$ satisfies all the conditions of Theorem 1.2 of [22]. Consequently, there exists a minimum $u \in \mathcal{M}$ of $E$ in $\mathcal{M}$. To show that $u$ is a weak solution to (4.1), we consider $\varphi \in C^\infty_c \left( 0, \frac{\pi}{j} \right)$ and $\epsilon > 0$. Let $v_\epsilon = u + \epsilon \varphi - \varphi^\epsilon + \varphi \in \mathcal{M}$, where

$$\varphi^\epsilon = \max \left\{ 0, u + \epsilon \varphi - \sqrt{\frac{\lambda}{b}} \right\} \geq 0 \quad \text{and} \quad \varphi = \max \{ 0, -(u + \epsilon \varphi) \} \geq 0,$$

we note that $\varphi^\epsilon, \varphi \in H^1_0 \left( 0, \frac{\pi}{j} \right) \cap L^\infty \left( 0, \frac{\pi}{j} \right)$.

Now we have the following estimates

$$\langle E'(u), \varphi \rangle = a(\| u \|_2^2) \int_0^{\frac{\pi}{j}} u_x \varphi'_x dx + \int_0^{\frac{\pi}{j}} (-\lambda u + bu^3) \varphi dx$$

$$= a(\| u \|_2^2) \int_{\Omega^c} u_x (u_x + \epsilon \varphi_x) + \int_{\Omega^c} (-\lambda u + bu^3) (u + \epsilon \varphi - \sqrt{\frac{\lambda}{b}}) dx$$

$$\geq a(\| u \|_2^2) \int_{\Omega^c} \epsilon \varphi_x dx + \int_{\Omega^c} (-\lambda u + bu^3) (u + \epsilon \varphi - \sqrt{\frac{\lambda}{b}}) dx$$

$$\geq a(\| u \|_2^2) \int_{\Omega^c} \epsilon \varphi_x dx + \int_{\Omega^c} (-\lambda u + bu^3) \epsilon \varphi dx$$

$$\geq -a(\| u \|_2^2) |\Omega^c| L\| \epsilon \varphi \|_{L^\infty(0, \frac{\pi}{j})} - \frac{2\lambda}{3} \left( \frac{\lambda}{3b} \right)^\frac{3}{2} |\Omega^c| \epsilon \varphi \|_{L^\infty(0, \frac{\pi}{j})}$$

$$\geq \left[ -a(\| u \|_2^2) \| \varphi \|_{L^\infty(0, \frac{\pi}{j})} - \frac{2\lambda}{3} \left( \frac{\lambda}{3b} \right)^\frac{3}{2} \| \varphi \|_{L^\infty(0, \frac{\pi}{j})} \right] \epsilon \|\Omega^c\|$$

where $\Omega^c := \{ x \in \left( 0, \frac{\pi}{j} \right) : u(x) + \epsilon \varphi(x) \geq \sqrt{\frac{\lambda}{b}} > u(x) \}$, that satisfies $|\Omega^c| \to 0$ as $\epsilon \to 0$.

Similarly

$$\langle E'(u), \varphi \rangle = a(\| u \|_2^2) \int_0^{\frac{\pi}{j}} u_x (\varphi_x)_x dx + \int_0^{\frac{\pi}{j}} (-\lambda u + bu^3) \varphi_x dx$$

$$= -a(\| u \|_2^2) \int_{\Omega_x} u_x (u_x + \epsilon \varphi_x) dx + \int_{\Omega_x} (-\lambda u + bu^3) (u + \epsilon \varphi) dx$$

$$\leq -a(\| u \|_2^2) \int_{\Omega_x} \epsilon \varphi_x dx + \int_{\Omega_x} (-\lambda u + bu^3) \epsilon \varphi dx$$

$$\leq \epsilon |\Omega_x| \left( a(\| u \|_2^2) \| \varphi \|_{L^\infty(0, \frac{\pi}{j})} + \frac{2\lambda}{3} \left( \frac{\lambda}{3b} \right)^\frac{3}{2} \| \varphi \|_{L^\infty(0, \frac{\pi}{j})} \right)$$
where $\Omega := \{ x \in (0, \frac{\pi}{J}) : u(x) + \epsilon \varphi(x) \leq 0 < u(x) \}$ and $|\Omega| \to 0$ as $\epsilon \to 0$.

Since $u, v \in \mathcal{M}$ we have that, for $h \in [0, 1]$, $u+h(v)-u = hv + (1-h)u \in \mathcal{M}$ and, consequently, $E(u) \leq E(u+h(v))$. Therefore, we have that

$$\langle E'(u), v - u \rangle = \lim_{h \to 0^+} \frac{E(u+h(v)-u) - E(u)}{h} \geq 0,$$

and, knowing that $v - u = \epsilon \varphi - \varphi^c + \varphi_c$, it follows that

$$\langle E'(u), \varphi \rangle \geq \langle E'(u), \varphi^c \rangle = C_1|\Omega| - C_2|\Omega^c|.$$

Making $\epsilon \to 0$ we obtain $\langle E'(u), \varphi \rangle \geq 0$, for each $\varphi \in C_0^\infty(0, \frac{\pi}{J})$. Hence $\langle E'(u), t\varphi \rangle \geq 0$, for each $t \in \mathbb{R}$ and $\varphi \in C_0^\infty(0, \frac{\pi}{J})$. Taking $t = \pm 1$ and recalling that $C_0^\infty(0, \frac{\pi}{J}) = H_0^1(0, \frac{\pi}{J})$ we obtain $E'(u) = 0$.

Finally, we will show that $u$ is nontrivial. For that, we will show that the minimum of energy is negative which guarantees that $u$ could not be zero. Let $\psi$ be a positive eigenfunction associated to the first eigenvalue $j^2$ of the operator $u_{xx}$ on $H_0^1(0, \frac{\pi}{J})$. Then $\psi$ is the solution to the following eigenvalue problem

$$\begin{cases}
-\psi_{xx} = j^2 \psi, & x \in (0, \frac{\pi}{J}), \\
\psi(0) = \psi\left(\frac{\pi}{J}\right) = 0.
\end{cases}$$

Since that $a(0)j^2 < \lambda$, from continuity of the function $a$, we have $a(t)j^2 < \lambda$ for each $t \in [0, j^2\delta^2 \int_0^\frac{\pi}{J} (\psi(x))^2 dx]$ for some $\delta > 0$ small enough. Note that $\delta \psi \in \mathcal{M}$, then for some $c_d \in [0, j^2\delta^2 \int_0^\frac{\pi}{J} (\psi(x))^2 dx]$,

$$E(\delta \psi) = \int_0^{\frac{\pi}{J}} a(s)ds + \frac{b}{4} \left(\delta \psi(x)\right)^4 - \frac{\lambda}{2} \left(\delta \psi(x)\right)^2 dx = \frac{\lambda}{2} \left(\delta \psi(x)\right)^4 + \frac{b}{2} \delta^2 \int_0^\frac{\pi}{J} \psi^4(x)dx < 0. \quad \square$$

**Lemma 4.2.** If $\mathbb{R}^+ \ni t \mapsto a(t) \in [\sigma, \infty)$ is continuous and non-decreasing and satisfies (4.2), then there exists a unique nontrivial positive solution of (4.1). Furthermore, this solution satisfies $u(x) = u(x_0 - x), x \in (0, \frac{\pi}{J})$.

**Proof.** The last statement follows from the uniqueness since, if $u(x)$ is a solution of (4.1), then so is $u(x_0 - x)$. Assume, by contradiction, that $u$ and $v$ are two distinct nontrivial non-negative solutions of (4.1). By regularity we have $u, v \in C^1(0, \frac{\pi}{J})$ and by the maximum principle we have $u, v > 0$ in $(0, \frac{\pi}{J})$ (see [20]), so that $\frac{u^2}{\int_0^\frac{\pi}{J} u^2 dx}, \frac{v^2}{\int_0^\frac{\pi}{J} v^2 dx} \in H_0^1(0, \frac{\pi}{J})$. Thus,

$$0 \leq \left( a\|u_x\|^2 - a(\|v_x\|^2)\right) \left(\|u_x\|^2 - \|v_x\|^2\right)$$

$$+ a(\|v_x\|^2) \int_0^{\frac{\pi}{J}} \left(u_x - \frac{u}{v} v_x\right)^2 + a(\|u_x\|^2) \int_0^{\frac{\pi}{J}} \left(v_x - \frac{v}{u} u_x\right)^2$$

$$= a(\|u_x\|^2)\|u_x\|^2 - a(\|u_x\|^2) \int_0^{\frac{\pi}{J}} u_x \left(\frac{u^2}{\int_0^\frac{\pi}{J} u^2 dx}\right) + a(\|v_x\|^2)\|v_x\|^2 - a(\|v_x\|^2) \int_0^{\frac{\pi}{J}} v_x \left(\frac{v^2}{\int_0^\frac{\pi}{J} v^2 dx}\right)$$




Obtain the following symmetry: \( \phi \) for \( x \in [0, \pi] \) and for \( \phi \) observe that \( \phi \) positive symmetric weak solution to the problem (4.4), that we will denote \( \phi^+ \).

Now, for \( a > 0 \), let us first note that, for \( \lambda < a \), the bifurcations occur at \( \lambda = N^2 \), with \( N \) being a positive integer. In this case we also obtain that, for \( a(0)N^2 < \lambda \leq a(0)(N + 1)^2 \) there are exactly \( 2N + 1 \) equilibria.

Notice that finding equilibria of the equation (1.4) corresponds to finding the solutions for the elliptic problem

\[
\begin{align*}
\nu(\|u_x\|^2)u_{xx} + \lambda u - bu^3 &= 0, \quad x \in (0, \pi) \\
u(0) &= u(\pi) = 0.
\end{align*}
\]

4.1. Equilibria of the autonomous problem. In this section we assume that \( a \) is non-decreasing and satisfies (4.2). Depending on the value of parameter \( \lambda > 0 \), we can construct equilibria of the problem (1.4) that change sign by an induction argument. Proving that, at least for this monotone case, the bifurcations occur at \( \lambda = N^2 \), with \( N \) being a positive integer. In this case we also obtain that, for \( a(0)N^2 < \lambda \leq a(0)(N + 1)^2 \) there are exactly \( 2N + 1 \) equilibria.

Notice that finding equilibria of the equation (1.4) corresponds to finding the solutions for the elliptic problem

\[
\begin{align*}
\nu(\|u_x\|^2)u_{xx} + \lambda u - bu^3 &= 0, \quad x \in (0, \pi) \\
\nu(0) &= u(\pi) = 0.
\end{align*}
\]

Let us first note that, for \( \lambda < a(0) \), 0 is the only solution to the problem (4.4). Now, for \( a(0) < \lambda \), Lemma 4.1 and Lemma 4.2, ensure that there exists a unique positive symmetric weak solution to the problem (4.4), that we will denote \( \phi^+_{1,b} \).

Observe that \( \phi^-_{1,b} = -\phi^+_{1,b} \) is also an equilibrium of (4.4).

For \( a(0)2^2 < \lambda \), we have the equilibria 0, \( \phi^+_{1,b} \) and \( \phi^-_{1,b} \) and, we can also construct a pair of equilibria that change sign once. For that, we are going to restrict ourselves to the following problem in \( [0, \frac{\pi}{2}] \):

\[
\begin{align*}
\left\{
\begin{array}{ll}
\nu(2\|u_x\|^2)u_{xx} + \lambda u - bu^3 &= 0, \quad x \in (0, \frac{\pi}{2}) \\
\nu(0) &= u\left(\frac{\pi}{2}\right) = 0
\end{array}
\right.
\end{align*}
\]

Since, in this case, we have that \( \lambda > a(0)2^2 \), the problem (4.5) has a positive solution \( \phi^+_{2,b} \) with \( \phi^+_{2,b}(0) = \phi^+_{2,b}\left(\frac{\pi}{2}\right) = 0 \) and that satisfies \( u(x) = u\left(\frac{\pi}{2} - x\right) \) for all \( x \in (0, \frac{\pi}{2}) \). Define

\[
\phi^+_{2,b}(x) = \begin{cases} 
\phi^+_{2,b}(x), & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\
-\phi^+_{2,b}(\pi - x), & \text{if } x \in \left[0, \frac{\pi}{2}\right].
\end{cases}
\]

Notice that \( \phi^+_{2,b} \) is a solution to (4.4), hence an equilibrium of (1.4). We also obtain the following symmetry: \( \phi^+_{2,b}(\pi - x) = -\phi^+_{2,b}(x) \), for all \( x \in (0, \pi) \). In fact, for \( x \in [0, \frac{\pi}{2}] \), we have

\[
\phi^+_{2,b}(\pi - x) = -\phi^+_{2,b}(\pi - (\pi - x)) = -\phi^+_{2,b}(\pi - x) = -\phi^+_{2,b}(x)
\]

and for \( x \in \left[\frac{\pi}{2}, \pi\right] \),

\[
\phi^+_{2,b}(\pi - x) = \phi^+_{2,b}(\pi - x) = -(-\phi^+_{2,b}(\pi - x)) = -\phi^+_{2,b}(x).
\]

Observe that \( \phi^+_{2} \) lies inside

\[
Y^+_2 = \left\{ u \in H^1_0(0, \pi) : u(x) = -u(\pi - x) \text{ in } [0, \pi] \text{ and } u(x) = u\left(\frac{\pi}{2} - x\right) \text{ in } \left[0, \frac{\pi}{2}\right] \right\}.
\]

A recursion argument shows that \( a(0)2^2 \) is a bifurcation point of the parameter \( \lambda > 0 \). Consequently, when \( a \) is non-decreasing, these are the only equilibria of (1.4). These results are summarized in Theorem 1.2.
5. **Conclusion.** We constructed non-autonomous equilibria of (1.1) depending on the parameter $\lambda > 0$. This was an interesting step, and it was the first one, to try to describe the structure of the attractor. There are still several interesting open questions that remain. One of them is that: Is there any other non-autonomous equilibrium besides the ones we have already constructed? We expect the answer to be no.

We still want to understand where the bifurcations happen. We believe that, for general $a$, they always happen at $\lambda = a(0)N^2$. One can also investigate the equilibria that we have constructed to study their stability and hyperbolicity and try to describe the connections among them.

For the autonomous Chafee-Infante (1.2), we know that all non-zero equilibria are hyperbolic and that the positive one is stable. We know exactly the diagram of connections between equilibria, namely: 0 is connected to all other equilibria and there is a connection between two equilibria if they have different number of zeros (connections go from equilibria with more zeroes to equilibria with less zeroes). The proof of this result requires the lap-number property and the fact the operator is a Sturm-Liouville type operator.

In the case (1.1) with $a \equiv 1$, the authors in [8] proved results concerning connections between the non-autonomous equilibria. That follows from a careful use of the lap-number property. Hyperbolicity, however, is still an open question. Hyperbolicity is always a challenging subject in the non-autonomous case.

In (1.1) with $a \equiv 1$, they also showed that the uniform attractor can be described using the non-autonomous equilibria of problems involving limit of translations of the function $\beta(\cdot)$ (functions in the global attractor of the driving semigroup). In [14] the authors prove that the $\omega-$limit sets of solutions consist of symmetric functions. To prove that they extend the Dirichlet problem to a periodic problem in an interval twice as big and show, using the lap-number, that the $\omega-$limit is a union of “sets of symmetric solutions”.

We believe that, if one can prove the lap-number-like properties for solutions, one will be able to give a full characterization of the uniform attractor.

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