Dimension of CPT posets

Atrayee Majumder · Rogers Mathew · Deepak Rajendraprasad

Abstract A collection of linear orders on $X$, say $L$, is said to realize a partially ordered set (or poset) $P = (X, \preceq)$ if, for any two distinct $x, y \in X$, $x \preceq y$ if and only if $x \prec L y, \forall L \in L$. We call $L$ a realizer of $P$. The dimension of $P$, denoted by $\dim(P)$, is the minimum cardinality of a realizer of $P$.

A containment model $M_P$ of a poset $P = (X, \preceq)$ maps every $x \in X$ to a set $M_x$ such that, for every distinct $x, y \in X$, $x \preceq y$ if and only if $M_x \subseteq M_y$. We shall be using the collection $(M_x)_{x \in X}$ to identify the containment model $M_P$. A poset $P = (X, \preceq)$ is a Containment order of Paths in a Tree (CPT poset), if it admits a containment model $M_P = (P_x)_{x \in X}$ where every $P_x$ is a path of a tree $T$, which is called the host tree of the model.

We show that if a poset $P$ admits a CPT model in a host tree $T$ of maximum degree $\Delta$ and radius $r$, then $\dim(P) \leq \lg \lg \Delta + (\frac{1}{2} + o(1)) \lg \lg \Delta + \lg r + \frac{1}{2} \lg \lg r + \frac{1}{2} \lg \pi + 3$. This bound is asymptotically tight up to an additive factor of $\min(\frac{1}{2} \lg \lg \Delta, \frac{1}{2} \lg \lg r)$. Further, let $P(1, 2; n)$ be the poset consisting of all the 1-element and 2-element subsets of $[n]$ under ‘containment’ relation and let $\dim(1, 2; n)$ denote its dimension. The proof of our main theorem gives a simple algorithm to construct a realizer for $P(1, 2; n)$ whose cardinality is only an additive factor of at most $\frac{3}{2}$ away from the optimum.

Keywords Poset dimension, Order dimension, 3-suitable family of permutations, Containment order of paths in a tree.

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1 Introduction

Dimension of a poset

A partially ordered set or poset \( P = (X, \preceq) \) is a tuple, where \( X \) represents a set, and \( \preceq \) is a binary relation on the elements of \( X \) that is reflexive, anti-symmetric and transitive. For any \( x, y \in X, x \) is said to be comparable with \( y \) if either \( x \preceq y \) or \( y \preceq x \). Otherwise, we say \( x \) and \( y \) are incomparable. A linear order is a partial order in which every two elements are comparable with each other. If a partial order \( P = (X, \preceq) \) and a linear order \( L = (X, \prec) \) are both defined on the same set \( X \), and if every ordered pair in \( P \) are also present in \( L \), then \( L \) is called a linear extension of \( P \). A collection of linear orders, say \( \mathcal{L} = \{L_1, L_2, \ldots, L_s\} \) with each \( L_k \) defined on \( X \), is said to realize a poset \( P = (X, \preceq) \) if, for any two distinct elements \( x_i, x_j \in X \), \( x_i \preceq x_j \in P \) if and only if \( x_i \prec_{L_k} x_j \), \( \forall L_k \in L \). We call \( \mathcal{L} \) a realizer for \( P \). The dimension of a poset \( P \), denoted by \( \dim(P) \), is defined as the minimum cardinality of a realizer for \( P \). The concept of poset dimension was introduced by Dushnik and Miller in [3] and has been extensively studied since then (see [14]). Let \( x, y \in X \) such that \( x \) and \( y \) are incomparable in \( P \). We say the ordered incomparable pair \((x, y)\) is critical if

1. \( \forall u \in X \setminus \{x\}, u \preceq x \implies u \preceq y \), and
2. \( \forall v \in X \setminus \{y\}, y \preceq v \implies x \preceq v \).

Critical pairs were introduced by Rabinovitch and Rival [11]. The following theorem is from their paper.

**Theorem 1 ([11])** A family \( \mathcal{L} \) of linear extensions of a poset \( P = (X, \preceq) \) is a realizer of \( P \) if and only if, for every critical pair \((x, y)\) in \( X \times X \), there is an \( L \in \mathcal{L} \) with \( y \prec_L x \).

Containment model for representing a poset

A containment model \( M_P \) of a poset \( P = (X, \preceq) \) maps every \( x \in X \) to a set \( M_x \) such that, for every distinct \( x, y \in X \), \( x \preceq y \) if and only if \( M_x \subseteq M_y \). We shall be using the collection \((M_x)_{x \in X}\) to identify the containment model \( M_P \). The reader may note that, for any poset \( P = (X, \preceq) \), \( M_x = \{y : y \preceq x\}, \forall x \in X \), is a valid containment model of \( P \). In [4,5,8,14,16] researchers have tried imposing geometric restrictions to the sets \( M_x \) to obtain geometric containment models. To cite a few: Containment models in which \( M_x \) is an interval on the \( x \)-axis [3,4], or every \( M_x \) is a \( d \)-box in the \( d \)-Euclidean space [6,8,16], or every \( M_x \) is a \( d \)-sphere in the \( d \)-Euclidean space [10].

Dimension of posets that admit a containment model

It was shown by Dushnik and Miller in [3] that \( \dim(P) \leq 2 \) if the poset \( P \) admits an interval containment model. Golumbic [6] and Golumbic and Scheinerman [8] generalized this further, showing that \( P \) is a containment poset of axis-parallel \( d \)-dimensional boxes in \( d \)-dimensional Euclidean space if and only if \( \dim(P) \leq 2d \). In [13] Sidney et al. stated that all posets of dimension 2 admit a containment model named circle order where elements of the partial order are mapped to circles in the Euclidean plane. Santoro and Urrutia showed in [12] that every poset of dimension 3 can be represented using a containment model where every element of the poset is mapped to an equilateral triangle in the Euclidean plane. They also showed that \( \dim(P) \leq n \) when the poset \( P \) admits a containment model where the elements of \( P \) are represented...
by regular $n$-gons all having the same orientation in the Euclidean plane. Trotter and Moore in \cite{15} studied the dimension of a poset that admits a containment model where every element of the poset is mapped to a subgraph of a given host graph. They proved the following interesting theorem.

**Theorem 2** \cite{15} If $G$ is a nontrivial connected graph with $n$ non-cut vertices, then the dimension of a poset $X(G)$ formed by the induced connected subgraphs of $G$ ordered by inclusion is $n$.

In this paper, we focus on *Containment order of Paths in a Tree* (CPT), which was first introduced by Corneil and Golumbic \cite{1}, and studied further by Alcón et al. in \cite{2}, and Golumbic and Limouzy \cite{7}. Below we define a CPT poset as outlined in \cite{2}.

**Definition 1** A poset $P = (X, \preceq)$ is a Containment order of Paths in a Tree (CPT poset), if there exists a tree $T$ such that $P$ admits a containment model $M_P = (P_x)_{x \in X}$ where every $P_x$ is a path of the tree $T$. $T$ will be called the host tree of the model.

The following theorem stated in \cite{2} follows from **Theorem 2**.

**Theorem 3** \cite{2} If a poset $P$ admits a CPT model in a host tree $T$ with $k$ leaves then $\dim(P) \leq k$.

The following observation follows directly from the definition of critical pairs.

**Observation 1** Let $P = (X, \preceq)$ be a poset that admits a containment model $M_P = (P_x)_{x \in X}$ in a host tree $T$, where each $P_x$ is a path in $T$. Then the critical pairs in $P$ have the form $(x, y)$ where $P_x$ is a singleton vertex in $T$, say $v$, that is not in $P_y$, and there is no path extending $P_y$ that does not contain $v (= P_x)$.

1.1 Notations and definitions

Unless mentioned explicitly, all logarithms used in the paper are to the base 2. Given any $n \in \mathbb{N}$, we shall use $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

**Definition 2** Let $P$ be a CPT poset that admits a containment model in a host tree $T$. Let $D$ be a planar drawing of $T$. A *linear extension* $L$ of $P$ corresponding to a tree traversal $\lambda$ of $D$ is calculated according to the following rules: Retain all the poset relations. Let $\text{last}_i(\lambda)$ denote the vertex of a path $P_i$ in $T$ which was listed after every other vertex of $P_i$ was listed in $\lambda$. Let $P_i$ and $P_j$, $i < j$, be two paths representing elements $x_i$ and $x_j$, respectively, of $P$. Suppose $\text{last}_i(\lambda) \neq \text{last}_j(\lambda)$. Then, $x_i \prec_L x_j$ if and only if $\text{last}_i(\lambda)$ was listed before $\text{last}_j(\lambda)$ in the tree traversal $\lambda$. Consider the case when $\text{last}_i(\lambda) = \text{last}_j(\lambda)$. Then, $x_i \prec_L x_j$ if and only if $|P_i| < |P_j|$.

It is easy to observe that the linear order $L$ thus constructed is unique and is a linear extension of $P$.

2 A 3-suitable family of permutations and $\dim(1, 2; n)$

**Definition 3** Let $S = \{R_1, R_2, \ldots, R_k\}$, where each $R_i$ is a permutation (or linear order or simple order) of $[n]$. We say $S$ is a 3-suitable family of permutations of $[n]$ if for every 3-subset of $[n]$, say $\{a_1, a_2, a_3\}$, and any distinguished element of the set, say $a_3$, there is some permutation $R_i \in S$ such that $a_j \prec_{R_i} a_3$ for every $1 \leq j < 3$, that is $a_3$ succeeds all the other elements of the 3-subset under $R_i$. Let $\alpha(n)$ denote the cardinality of a smallest 3-suitable family of permutations of $[n]$.

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1 Corneil, D. and Golumbic, M. C., Unpublished, but cited in \cite{8}, \cite{6}
Let \( \mathcal{P}(1, 2; n) \) denote the poset representing 1-element and 2-element subsets of an \( n \) element set ordered by the subset containment relation. Let \( \dim(1, 2; n) \) denote the dimension of the poset \( \mathcal{P}(1, 2; n) \). Given a realizer \( R \) of the poset \( \mathcal{P}(1, 2; n) \), it is easy to construct a 3-suitable family of permutations of \([n]\) of the same size from \( R \). In a similar way, given a 3-suitable family of permutations of \([n]\), one can construct from it a realizer of \( \mathcal{P}(1, 2; n) \) of the same cardinality. Thus, \( \dim(1, 2; n) = \alpha(n) \). In this section we discuss some of the bounds on \( \dim(1, 2; n) \). In [9], Hoşten and Morris proved the following theorem.

**Theorem 4 (Theorem 1.1 in [9])** Let \( n \geq 3 \). Then, \( \dim(1, 2; n) \) is the smallest integer \( t \) for which there are \( n \) antichains in the subset lattice of \([t - 1]\) that do not contain \([t - 1]\) or two sets whose union is \([t - 1]\).

The determination of the number of monotone Boolean functions on \( n \) variables or the number of antichains in the subset lattice of an \( n \) element set ordered by subset containment relation. Kleitman and Markovsky in [10] gave the following result regarding this problem.

**Theorem 5 ([10])** The size of the free distributive lattice, i.e. \( \Psi(n) \), on \( n \) generators (which is the number of monotone Boolean functions on \( n \) variables or the number of antichains in the subset lattice of \([n]\)), satisfies the following condition.

\[
\Psi(n) \leq 2^{(1 + O(\log n / n))(1 \vee n / 2)}
\]

Combining Theorem 4 and Theorem 5, Agnarsson, Felsner, and Trotter has given the following result in [1].

**Theorem 6 ([1])** For any \( n \geq 3 \), \( \dim(1, 2; n) \leq \lg \lg n + (1/2 + o(1)) \lg \lg \lg n \).

Following is a more technical result due to Trotter which combines Theorem 4 and Theorem 5.

**Theorem 7** For every \( \epsilon > 0 \), there is an integer \( n_0 \) so that if \( n > n_0 \) and

\[
s = \lg \lg n + \frac{1}{2} \cdot \lg \lg \lg n + \frac{1}{2} \cdot \lg \pi + \frac{1}{2},\text{ then }
\]

\[
s - \epsilon < \dim(1, 2; n) < s + 1 + \epsilon.
\]

### 3 The dimension of a CPT poset

Given a tree \( T \), let \( \mathcal{P}(T) \) denote the containment poset of all paths in \( T \) and let \( \dim_p(T) \) denote \( \dim(\mathcal{P}(T)) \).

An antichain \( A \) is called intersecting, if no pair of sets in \( A \) is disjoint. Let \( \beta(t) \) denote the smallest natural number \( \beta \) such that the subset lattice of \([\beta]\) contains an intersecting antichain of size \( t \). It suffices to ensure that \( \left( \frac{1}{(\beta+1/2)} \right) \geq t \). Hence \( \beta(t) \leq \left\lfloor \lg t + \frac{1}{2} \lg \lg t + \frac{1}{2} \lg \pi + 1 \right\rfloor \).

**Theorem 8** Let \( T \) be a rooted tree of height \( h \) in which every internal vertex has exactly \( k \) children (a perfect \( k \)-ary tree in Computer Science terms). Then \( \dim_p(T) \leq \alpha(k) + \beta(h) + 1 \).

**Proof** Let \( \alpha = \alpha(k) \) and \( \beta = \beta(h) \). Fix a planar drawing \( D \) of \( T \) with every child being below its parent. For each internal node of \( T \), identify its \( k \) children with \([k]\) in the left to right order they appear in \( D \). Let \( \psi = \{\sigma_1, \ldots, \sigma_\alpha\} \) be a 3-suitable family of linear orders of \([k]\). For each \( i \in [\alpha] \), let \( D_i \) be the drawing of \( T \) obtained by reordering the \( k \) children of every internal node in \( D \) according to \( \sigma_i \). Let \( \phi = \{A_0, \ldots, A_{2h-1}\} \) be an antichain in the subset lattice of \([\beta]\). For each \( i \in [\beta] \), let \( D_{\beta+i} \) be the drawing obtained from \( D_1 \) by reversing the following condition.

\[
\Psi(n) \leq 2^{(1 + O(\log n / n))(1 \vee n / 2)}
\]
the order of the children of every node \( v \) where \( j \in S(v) \). For each \( i \in [\alpha + \beta] \), let \( L_i \) be the linear extension of \( \mathcal{P}(T) \) corresponding (Definition 2) to the preorder traversal of the drawing \( D_i \). Finally let \( D_0 \) be the linear extension of \( \mathcal{P}(T) \) corresponding to any order of \( V(T) \) in which every parent appears only after all its children (a bottom to top traversal). We argue that \( L = \{L_0, \ldots, L_{\alpha+\beta}\} \) is a realizer of \( \mathcal{P}(T) \) and thus the theorem.

From Theorem 1 we know that, in order to show that \( L \) is a realizer for a poset \( P \), it is enough to show that \( L \) is a collection of linear extensions of \( \mathcal{P} \) such that for every critical pair \((x, y)\) of \( \mathcal{P} \), there is an \( L \in \mathcal{L} \) satisfying \( y <_L x \). Consider a critical pair \((x, y)\) in \( \mathcal{P}(T) \). It follows from Observation 1 and the fact that \( \mathcal{P}(T) \) contains every path in \( T \), that \( x \) is a vertex of \( T \) (call it \( p \)) and \( y \) is a path in \( T \) in which one end (call it \( q_1 \)) is a leaf of \( T \) and the other (call it \( q_2 \)) is either a leaf of \( T \) or a degree-2 neighbour of \( p \). We will call the least common ancestor of \( q_1 \) and \( q_2 \) as \( q \). For every path in a tree, the last vertex to be traversed in a preorder traversal will always be an end-vertex of the path. This is because every node in a path is an ancestor to at least one end-vertex of the path. Hence if \( p \) succeeds both \( q_1 \) and \( q_2 \) in some preorder traversal, then we have \( y <_L x \), in the corresponding linear extension.

First let us consider the case when \( q_2 \) is a neighbour of \( p \). If \( p \) is the parent of \( q_2 \), then \( p \) succeeds both \( q_1 \) and \( q_2 \) in \( L_0 \). If \( p \) is a child of \( q_2 \) and then \( p \) succeeds both \( q_1 \) and \( q_2 \) either in \( L_1 \) or in \( L_{\alpha+j} \) for every \( j \in S(q) \).

Now consider the case when \( q_2 \) (along with \( q_1 \)) is also a leaf of \( T \). With relabeling if necessary, we can assume that \( q_2 \) is to the right of \( q_1 \) in \( D_1 \) (i.e., \( q_1 < q_2 \) in \( L_1 \)). If \( p \) is an ancestor of \( q \), then \( p \) succeeds both \( q_1 \) and \( q_2 \) in \( L_0 \). If \( p \) is neither an ancestor nor a descendent of \( q \), then \( p \) succeeds both \( q_1 \) and \( q_2 \) either in \( L_1 \) or in \( L_{\alpha+j} \) for every \( j \in S(r) \), where \( r \) is the least common ancestor of \( p \) and \( q \). A careful observation will reveal that all the cases considered so far could have been handled with just three linear extensions - \( L_0 \), \( L_1 \) and a linear extension corresponding to the right to left preorder traversal of \( D_1 \). The only remaining case of \( p \) being a descendent of \( q \) is the most demanding and we are forced to split it further into the following three subcases.

Subcase 1. Let \( q \) be the only ancestor of \( p \) in the path from \( q_1 \) to \( q_2 \). Let \( q_1' \), \( q_2' \) and \( p' \) be, respectively, the ancestors of \( q_1 \), \( q_2 \) and \( p \) which are children of \( q \). Let \( D_i \), \( i \in [\alpha] \) be a drawing in which \( p' \) succeeds both \( q_1' \) and \( q_2' \). Such a drawing will exist since \( \psi \) is 3-suitable for \([k]\). One can verify that \( p \) succeeds both \( q_1 \) and \( q_2 \) in \( L_i \).

Subcase 2. Let the least common ancestor \( p_2 \) of \( p \) and \( q_2 \) be a proper descendent of \( q \). If \( p \) is to the right of \( q_2 \) in \( D_1 \), then \( p \) succeeds both \( q_1 \) and \( q_2 \) in \( L_1 \). Otherwise \( p \) succeeds both \( q_1 \) and \( q_2 \) in \( L_{\alpha+j} \), for every \( j \in S(p_2) \) \( \setminus S(q) \), which is non-empty since \( \phi \) is an antichain.

Subcase 3. Let the least common ancestor \( p_1 \) of \( p \) and \( q_1 \) be a proper descendent of \( q \). If \( p \) is to the right of \( q_2 \) in \( D_1 \), then \( p \) succeeds both \( q_1 \) and \( q_2 \) in \( L_{\alpha+j} \), for every \( j \in S(q) \) \( \setminus S(p_1) \), which is non-empty since \( \phi \) is an antichain. Otherwise \( p \) succeeds both \( q_1 \) and \( q_2 \) in \( L_{\alpha+j} \), for every \( j \in S(p_1) \cap S(q) \), which is non-empty since \( \phi \) is intersecting. \( \square \)

**Corollary 1** For the tree \( T \) in Theorem 1 \( \text{dim}_p(T) \leq \lg \lg k + (\frac{1}{2} + o(1)) \lg \lg \lg k + \lg h + \frac{1}{3} \lg \lg h + \frac{1}{3} \lg \pi + 3 \).

**Remark 1** The tree \( T \) in Theorem 1 has \( k^h \) leaves. The subposet of \( \mathcal{P}(T) \) induced on the singleton paths corresponding to every leaf of \( T \) and the maximal paths in \( T \) is isomorphic to \( \mathcal{P}(1, 2; k^h) \). Hence \( \text{dim}_p(T) \geq \text{dim}(1, 2; k^h) = \alpha(k^h) \). Using the bound given in Theorem 7 for \( \text{dim}(1, 2; n) \), we can say that the upper bound in Corollary 1 is tight up to an additive factor of \( \min \left( \frac{1}{2} \lg \lg \lg k, \frac{1}{2} \lg \lg h \right) + \frac{1}{2} \lg \pi + 4 \). Hence our bound in Corollary 1 is asymptotically tight when at least one of \( k \) or \( h \) is a constant.
Remark 2 For a perfect binary tree \( T \) (\( k = 2 \)) of height \( h \), we get \( \dim(1; 2; 2^h) \leq \dim_P(T) \leq \beta(h) + 2 \).

While considering only singleton paths corresponding to every leaf of \( T \) and the maximal paths in \( T \), the linear order \( L_0 \) defined in the proof of Theorem \( \ref{thm:dim_P} \) is not required as we do not encounter the case where \( p \) is an ancestor of \( q \) (or \( q_2 \)). This observation helps us in arriving at Remark \( \ref{rem:dim_P} \) below. Let \( \gamma(t) \) denote the smallest natural number \( \gamma \) such that the subset lattice of \( [\gamma] \) contains an intersecting antichain of size \( t \) that satisfies the following property: for every two sets in the antichain, their union is a proper subset of \( [\gamma] \).

Remark 3 For any natural number \( n \geq 3 \), \( \dim(1; 2; n) \leq \gamma(\lg n) \leq \beta(\lg n) + 1 \leq \lceil \lg \lg n + \frac{1}{2} \lg \lg\lg n + \frac{1}{2} \lg \pi + 1 \rceil + 1 \).

Observe that this bound for \( \dim(1; 2; n) \) is only an additive factor of at most \( \frac{1}{2} \) away from the upper bound given by Theorem \( \ref{thm:dim_P} \). Our proof yields a simple algorithm to construct a realizer for \( P(1; 2; n) \) (or a 3-suitable family of permutations of \([n] \)) which is near optimal in size.

Corollary 2 If a poset \( P = (X, \preceq) \) admits a CPT model in a host tree \( T \) of maximum degree \( \Delta \) and radius \( r \), then \( \dim(P) \leq \alpha(\Delta) + \beta(r) + 1 \).

Proof \( T \) is an induced subgraph of a perfect \( \Delta \)-ary tree \( T \) of height \( r \) and \( P \) is hence an induced subposet of \( P(T) \).

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