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The Killing Tensors on an $n$-dimensional Manifold with $SL(n, \mathbb{R})$-structure

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Abstract

In this paper we solve the problem of finding integrals of equations determining the Killing tensors on an $n$-dimensional differentiable manifold $M$ endowed with an equiaffine $SL(n, \mathbb{R})$-structure and discuss possible applications of obtained results in Riemannian geometry.

Key words: Differentiable manifold, $SL(n, \mathbb{R})$-structure, Killing tensors.

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1 Introduction

1.1. The “structural point of view” of affine differential geometry was introduced by K. Nomizu in 1982 in a lecture at Münster University with the title “What is Affine Differential Geometry?” (see [12]). In the opinion of K. Nomizu, the geometry of a manifold $M$ endowed with an equiaffine structure is called affine differential geometry.

In recent years, there has been a new wave of papers devoted to affine differential geometry. Today the number of publications (including monographs) on affine differential geometry reached a considerable level. The main part of these publications is devoted to geometry of hypersurfaces (see [15, 16] for the history and references).
1.2. In the present paper we solve the problem of finding integrals of equations determining the Killing tensors (see [8] for the definitions, properties and applications) on an \( n \)-dimensional differentiable manifold \( M \) endowed with an equiaffine structure. The paper is a direct continuation of [18]. The same notations are used here.

The first of two present theorems proved in our paper is an affine analog of the statement published in the paper [17], which appeared in the process of solving problems in General relativity.

2 Definitions and results

2.1. In order to clarify the approach to problem of finding integrals of equations determining the Killing tensors on an \( n \)-dimensional differentiable manifold \( M \) we shall start with a brief introduction to the subject which emphasizes the notion of an equiaffine \( SL(n, \mathbb{R}) \)-structure.

Let \( M \) be a connected differentiable manifold of dimension \( n \) (\( n > 2 \)), and let \( L(M) \) be the corresponding bundle of linear frames with structural group \( GL(n, \mathbb{R}) \). We define \( SL(n, \mathbb{R}) \)-structure on \( M \) as a principal \( SL(n, \mathbb{R}) \)-sub-bundle of \( L(M) \). It is well known that an \( SL(n, \mathbb{R}) \)-structure is simply a volume element on \( M \), i.e. an \( n \)-form \( \eta \) that does not vanishing anywhere (see [6, Chapter I, §2]).

We recall the famous problem of the existence of a uniquely determined linear connection \( \nabla \) reducible to \( G \) for each given \( G \)-structure on \( M \) (see [1, p. 213]). For example, if \( M \) is a manifold with a pseudo-Riemannian metric \( g \) of an arbitrary index \( k \), then the bundle \( L(M) \) admits a unique linear connection \( \nabla \) without torsion that is reducible to \( O(m, k) \)-structure. Such a connection is called the Levi-Civita connection. It is characterized by the following condition

\[
\nabla g = 0
\]

A linear connection \( \nabla \) having zero torsion and reducible to \( SL(n, \mathbb{R}) \) is said to be equiaffine and can be characterized by the following equivalent conditions (see [15, p. 99], [16, pp. 57–58]):

1. \( \nabla \eta = 0 \);
2. the Ricci tensor \( \text{Ric} \) of \( \nabla \) is symmetric; that means \( \text{Ric}(X, Y) = \text{Ric}(Y, X) \) for any vector fields \( X, Y \in C^\infty TM \).

An equiaffine \( SL(n, \mathbb{R}) \)-structure or an equiaffine structure on an \( n \)-dimensional differentiable manifold \( M \) is a pair \( (\eta, \nabla) \), where \( \nabla \) is a linear connection with zero torsion and \( \eta \) is a volume element which is parallel relative to \( \nabla \) (see [13, p. 43]).

The curvature tensor \( R \) of an equiaffine connection \( \nabla \) admits a point-wise \( SL(n, \mathbb{R}) \)-invariant decomposition of the form

\[
R = (n-1)^{-1}[\text{id}_M \otimes \text{Ric} \otimes \text{id}_M] + W
\]

where \( W \) is the Weyl projective curvature tensor (see [16, p. 73–74], [2, §40]). Then two classes of equiaffine structures can be distinguished in accordance
with this decomposition: the *Ricci-flat* equiaffine $SL(n, \mathbb{R})$-structures for which $\text{Ric} = 0$, and the *equiprojective* $SL(n, \mathbb{R})$-structures for which

$$R = (n - 1)^{-1}[\text{id}_M \otimes \text{Ric} - \text{Ric} \otimes \text{id}_M].$$

**Remark 1** Recall that a linear connection $\nabla$ with zero torsion is called *Ricci-flat* if the Ricci tensor $\text{Ric} = 0$ (see [9]). On the other hand, a connection $\nabla$ is called *equiprojective* if the Weyl projective curvature tensor $W = 0$ (see [15, §18]). In the literature equiprojective connections sometimes are called *projectively flat* (see, for example, [16, p. 73]).

An autodiffeomorphism of the manifold $M$ is an automorphism of $SL(n, \mathbb{R})$-structure if and only if it preserves the volume element $\eta$. Let $X$ be a vector field on $M$. The function $\text{div} X$ defined by the formula $(\text{div} X)\eta = L_X \eta$ where $L_X$ is the Lie differentiation in the direction of the vector field $X$ is called the divergence of $X$ with respect to the $n$-form $\eta$ (see [7, Appendix no. 6]). Obviously, $X$ is an infinitesimal automorphism of an $SL(n, \mathbb{R})$-structure if and only if $\text{div} X = 0$. Such a vector field $X$ is said to be *solenoidal*.

For an arbitrary vector field $X$ on $M$ with a linear connection $\nabla$ we can introduce the tensor field $A_X = L_X - \nabla_X$ regarded as a field of linear endomorphisms of the tangent bundle $TM$. If $M$ is an $n$-dimensional with an equiaffine $SL(n, \mathbb{R})$-structure then the formula $\text{trace} A_X = -\text{div} X$ can be verified directly (see [7, Appendix no. 6]).

We have the $SL(n, \mathbb{R})$-invariant decomposition

$$A_X = -n^{-1}(\text{div} X) \text{id}_M + \dot{A}_X$$

at every point $x \in M$.

Two classes of vector fields on $M$ endowed with an equiaffine $SL(n, \mathbb{R})$-structure can be distinguished in accordance with this decomposition: the *solenoidal vector fields* and the *concircular vector fields* for which, by definition (see [14, p. 322], [9]), we have $A_X = -n^{-1}(\text{div} X) \text{id}_M$.

The integrability conditions of the structure equation $A_X = -n^{-1}(\text{div} X) \text{id}_M$ of the concircular vector field $X$ is the Ricci’s identity

$$Y(\text{div} X)Z - Z(\text{div} X)Y = nR(Y, Z)X$$

for any vector fields $Y, Z \in C^\infty TM$ (see [2, §11]). This identity are equivalent to the condition $W(Y, Z)X = 0$ for any vector fields $Y, Z \in C^\infty TM$. It follows that an equiaffine $SL(n, \mathbb{R})$-structure on an $n$-dimensional manifold $M$ is equiprojective if and only if there exist $n$ linearly independent concircular vector fields $X_1, X_2, \ldots, X_p$ on $M$ (see also [24]). This statement is an affine analog of the well known fact for the Riemannian manifold $M$ of constant sectional curvature (see [3]).

**Remark 2** A pseudo-Riemannian manifold $(M, g)$ with a projectively flat Levi-Civita connection $\nabla$ is a manifold of constant section curvature (see [15, §18]). Therefore a manifold $M$ endowed with an equiprojective $SL(n, \mathbb{R})$-structure is an affine analog of a pseudo-Riemannian manifold of constant sectional curvature.
2.2. We consider an \( n \)-dimensional manifold \( M \) with an equiaffine \( SL(n, \mathbb{R}) \)-structure and denote by \( \Lambda^p M \) (\( 1 \leq p \leq n-1 \)) the \( p \)-th exterior power \( \Lambda^p (T^* M) \) of the cotangent bundle \( T^* M \) of \( M \). Hence \( C^\infty \Lambda^p M \), the space of all \( C^\infty \)-sections of \( \Lambda^p M \), is the space of skew-symmetric covariant tensor fields of degree \( p \) (\( 1 \leq p \leq n-1 \)).

Let \( \gamma : J \subset \mathbb{R} \to M \) be an arbitrary geodesic on \( M \) with affine parameter \( t \in J \). In this case, we have \( \nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0 \) for the tangent vector \( \frac{d\gamma}{dt} \) of \( \gamma \).

**Definition 1** (see [18]). A skew-symmetric tensor field \( \omega \in C^\infty \Lambda^p M \) (\( 1 \leq p \leq n-1 \)) on an \( n \)-dimensional manifold \( M \) with an equiaffine \( SL(n, \mathbb{R}) \)-structure is called Killing-Yano tensor of degree \( p \) if the tensor

\[
\frac{i_{\frac{d\gamma}{dt}} \omega}{\frac{d\gamma}{dt}} := \text{trace} \left( \frac{d\gamma}{dt} \otimes \omega \right)
\]

is parallel along an arbitrary geodesic \( \gamma \) on \( M \).

From this definition we conclude that

\[
\left( \nabla_{\frac{d\gamma}{dt}} \omega \right) \left( \frac{d\gamma}{dt}, X_2, \ldots, X_p \right) = 0
\]

for any vector fields \( X_2, \ldots, X_p \in C^\infty TM \). Since the geodesic \( \gamma \) may be chosen arbitrary, the above relation is possible if and only if \( \nabla \omega \in C^\infty \Lambda^{p+1} M \), which is equivalent to \( d\omega = (n+1) \nabla \omega \) for the exterior differential operator \( d : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M \).

Obviously, the set of Killing-Yano tensors of degree \( p \) (\( 1 \leq p \leq n-1 \)) constitutes an \( \mathbb{R} \)-module of tensor fields on \( M \), denoted by \( K^p(M, \mathbb{R}) \).

Let \( X_1, \ldots, X_p \) be \( p \) linearly independent concircular vector fields on \( M \) (\( 1 \leq p \leq n-1 \)). Then direct inspection shows that the tensor field \( \omega \) of degree \( n-p \) dual to the tensor field \( alt\{X_1 \otimes \cdots \otimes X_p\} \) relative to the \( n \)-form \( \eta \) is a Killing-Yano tensor (see also [18]). Therefore on any \( n \)-manifold \( M \) with equiprojective \( SL(n, \mathbb{R}) \)-structure, there exist at least \( n!p!(n-p)!^{-1} \) linearly independent Killing-Yano tensors (see [18]). Moreover the following theorem is true.

**Theorem 1** On an \( n \)-dimensional manifold \( M \) endowed with an equiprojective \( SL(n, \mathbb{R}) \)-structure \( (\eta, \nabla) \), there exist a local coordinate system \( x^1, \ldots, x^n \) in which an arbitrary Killing-Yano tensor \( \omega \) of degree \( p \) (\( 1 \leq p \leq n-1 \)) has the components

\[
\omega_{i_1 \ldots i_p} = \psi^{(p+1)} \left( A_{i_0 i_1 \ldots i_p} x^{i_0} + B_{i_1 \ldots i_p} \right) \quad (2.1)
\]

where \( A_{i_0 i_1 \ldots i_p} \) and \( B_{i_1 \ldots i_p} \) are arbitrary constants skew-symmetric w.r.t. all their indices and \( \psi = (n+1)^{-1} \ln(\eta) \).

From the theorem we conclude that the maximum of linearly independent the Killing–Yano tensors is by calculating the number \( K^n_p \) of independent \( A_{i_0 i_1 \ldots i_p} \) and \( B_{i_1 \ldots i_p} \) which exist after accounting for the symmetries on the indices. It follows that \( K^n_p = \frac{(n+1)!}{(p+1)!(n-p)!} \) is the maximum number linearly independent the Killing–Yano tensors.
Corollary 1  Let $M$ be an $n$-dimensional manifold endowed with an equiprojective $SL(n, \mathbb{R})$-structure then

$$\dim K^p(M, \mathbb{R}) = \frac{(n + 1)!}{(p + 1)!(n - p)!}.$$  

On our fixed manifold $M$ with an equiaffine $SL(n, \mathbb{R})$-structure, we denote by $S^p M$ the bundle of symmetric covariant tensor fields of degree $p$ on $M$. Hence $C^\infty S^p M$, the space of all $C^\infty$-sections of $S^p M$, is the space symmetric covariant tensor fields of degree $p$.

Definition 2 (see [18]). A symmetric tensor field $\varphi \in C^\infty S^p M$ on an $n$-dimensional manifold $M$ with an equiaffine $SL(n, \mathbb{R})$-structure is called Killing tensor of degree $p$ if

$$\varphi \left( \frac{d\gamma}{dt}, \ldots, \frac{d\gamma}{dt} \right) = \text{const.}$$

along an arbitrary geodesic $\gamma$ on $M$.

Let $\varphi \left( \frac{d\gamma}{dt}, \ldots, \frac{d\gamma}{dt} \right) = \text{const.}$ along an arbitrary geodesic $\gamma$ on $M$ and hence $\varphi$ is a Killing tensor. Then the above relation is possible if and only if

$$\delta^* \varphi := \sum_{cicl} \{ \nabla \varphi \} = 0$$

where for the local components $\nabla_{i_0} \varphi_{i_1 \ldots i_p}$ of $\nabla \varphi$ we define the sum

$$\sum_{cicl} \{ \nabla_{i_0} \varphi_{i_1 \ldots i_p} \}$$

as the sum of the terms obtained by a cyclic permutation of indices $i_0, i_1, \ldots, i_p$.

Obviously, the set of Killing tensors constitutes an $\mathbb{R}$-module of tensor fields on $M$, denoted by $T^p(M, \mathbb{R})$.

Let $M$ be an $n$-dimensional manifold endowed with an equiprojective $SL(n, \mathbb{R})$-structure $(\eta, \nabla)$, and $\omega_1, \ldots, \omega_p$ be $p$ linearly independent Killing-Yano tensors of degree 1 on $M$. Then direct inspection shows that the tensor field $\varphi := \text{sym}\{\omega_1 \otimes \cdots \otimes \omega_p\}$ is a Killing tensor of degree $p$. Therefore on any $n$-manifold $M$ with equiprojective $SL(n, \mathbb{R})$-structure, there exist at least $(n + p - 1)!p!(n - 1)!^{-1}$ linearly independent Killing tensors (see also [23]). Moreover the following theorem is true.

Theorem 2  On an $n$-dimensional manifold $M$ endowed with an equiprojective $SL(n, \mathbb{R})$-structure $(\eta, \nabla)$, there exist a local coordinate system $x^1, \ldots, x^n$ in which the components $\varphi_{i_1 \ldots i_p}$ of an arbitrary Killing tensor $\varphi$ of degree $p$ can be expressed in the form of an $p$th degree polynomial in the $x^i$’s

$$\varphi_{i_1 \ldots i_p} = e^{2p\psi} \sum_{q=0}^p A_{i_1 \ldots i_p j_1 \ldots j_q} x^{j_1} \cdots x^{j_q} \quad (2.2)$$
where the coefficients $A_{i_1...i_p,j_1...j_q}$ are constant and symmetric in the set of indices $i_1,...,i_p$ and the set of indices $j_1,...,j_q$. In addition to these properties the coefficients $A_{i_1...i_p,j_1...j_q}$ have the following symmetries

$$\sum_{cicl} \{A_{i_1...i_p,j_1...j_p-s}\}_{j_p-s+1} = 0 \quad (2.3)$$

for $s = 1,...,p-1$ and

$$\sum_{cicl} \{A_{i_1...i_p,j_1}\} = 0. \quad (2.4)$$

From the theorem we conclude that the maximum number of linearly independent the Killing tensors is obtained by calculating the number $T_{pn}$ of independent $A_{i_1...i_p,j_1...j_q}$ ($q = 0,1,...,n$) which exist after accounting for the symmetries on the indices the dependence relations (2.3) and (2.4). By [4] it follows that

$$T_{pn} = \frac{p(p+1)^2(p+2)^2 \ldots (m+p-1)^2(m+p)}{(p+1)!p!}$$

is the maximum number linearly independent the Killing–Yano tensors. Then we have the following proposition.

**Corollary 2** Let $M$ be an $n$-dimensional manifold endowed with an equiprojective $SL(n,\mathbb{R})$-structure then

$$\dim T_{pn}(M,\mathbb{R}) = \frac{p(p+1)^2(p+2)^2 \ldots (m+p-1)^2(m+p)}{p!(p+1)!}.$$

### 3 Proofs of theorems

**3.1.** We let $f: \bar{M} \to M$ denote the mapping of an $\bar{n}$-dimensional manifold $\bar{M}$ endowed with an equiaffine $SL(\bar{n},\mathbb{R})$-structure onto another an $n$-dimensional manifold $M$ endowed with an equiaffine $SL(n,\mathbb{R})$-structure, and let $f_*$ be the differential of this mapping. For any covariant tensor field $\omega$ on $M$, we can then define the covariant tensor field $f^*\omega$ on $\bar{M}$, where $f^*$ is the transformation transposed to the transformation $f_*$.

If $\dim \bar{M} = \dim M = n$ and $f: \bar{M} \to M$ is a projective diffeomorphism, i.e., a mapping that transforms an arbitrary geodesic in $\bar{M}$ into a geodesic in $M$, then we have the following lemma.

**Lemma 1** Let $f: \bar{M} \to M$ be a projective diffeomorphism of $n$-dimensional manifolds endowed with the equiaffine $SL(n,\mathbb{R})$-structures $(\bar{\eta}, \bar{\nabla})$ and $(\eta, \nabla)$ respectively. Then for an arbitrary Killing-Yano tensor $\omega$ of degree $p$ ($1 \leq p \leq n-1$) on the manifold $M$ the tensor field $\bar{\omega} = e^{-(p+1)\psi}(f^*\omega)$ with $\psi = (n+1)^{-1}\ln(\eta/\bar{\eta})$ will be the Killing-Yano tensor of degree $p$ on the manifold $\bar{M}$. 

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**Proof**  It is known that the diffeomorphism \( f : \bar{M} \to M \) can be realized following the principle of equality of the local coordinates \( \bar{x}^1 = x^1, \ldots, \bar{x}^n = x^n \) at the corresponding points \( \bar{x} \) and \( x = f(\bar{x}) \) of these manifolds. In this case, we have the equalities (see [15, §18], [9, 10, 26])

\[
\Gamma^k_{ij} = \Gamma^k_{ij} + \psi_i \delta_j^k + \psi_j \delta_i^k \tag{3.1}
\]

for the objects \( \Gamma^k_{ij} \) and \( \Gamma^k_{ij} \) of the a equiaffine connections \( \nabla \) and \( \bar{\nabla} \) in the coordinate system \( x^1, \ldots, x^n \) that is common w.r.t. the mapping \( f : \bar{M} \to M \), and for the gradient \( \psi_j = (n + 1)^{-1} \partial_j \ln[\eta/\bar{\eta}] \).

Equalities (3.1) imply that the mapping \( f^{-1} \), which in inverse to the projective diffeomorphism \( f : \bar{M} \to M \), is a projective mapping [10, p. 262].

We set \( \omega_{i_1 ... i_p} \) be the local components of a Killing-Yano tensor \( \omega \) of degree \( p \) (\( 1 \leq p \leq n - 1 \)) arbitrary defined on the manifold \( M \); by definition, these components satisfy the equations

\[
\nabla_{i_0} \omega_{i_1 ... i_p} + \nabla_{i_1} \omega_{i_0 ... i_p} = 0. \tag{3.2}
\]

From equalities (3.2) we find directly that the components

\[
\bar{\omega}_{i_1 ... i_p} = e^{-(p+1)\psi} \omega_{i_1 ... i_p} \tag{3.3}
\]

of the tensor field \( \bar{\omega} = e^{-(p+1)\psi} (f^* \omega) \) satisfy the equations

\[
\nabla_{i_0} \bar{\omega}_{i_1 ... i_p} + \nabla_{i_1} \bar{\omega}_{i_0 ... i_p} = 0. \tag{3.4}
\]

Hence, the tensor field \( \bar{\omega} \) is a Killing-Yano tensor of degree \( p \) (\( 1 \leq p \leq n - 1 \)) on the manifold \( \bar{M} \). \( \square \)

**3.2.** Let \( A^n \) be an \( n \)-dimensional affine space with a volume element given by the determinant \( \det(e_1, \ldots, e_n) = 1 \), where \( \{e_1, \ldots, e_n\} \) is the standard basis of the underlying vector space for \( A^n \). We denote by \( \nabla \) the standard linear connection in \( A^n \) relative to which the volume element “\( \det \)” is parallel (see [13], [16, p. 10]).

Let \( f : \bar{M} \to A^n \) be a projective diffeomorphism from a manifold \( \bar{M} \) endowed with equiaffine \( SL(n, \mathbb{R}) \)-structure onto an affine space \( A^n \) endowed with standard equiaffine \( SL(n, \mathbb{R}) \)-structure. It is well known that manifolds endowed with equivprojective \( SL(n, \mathbb{R}) \)-structures and only these manifolds are projectively diffeomorphic to an affine space \( A^n \) (see [15, §18], [9]) therefore in our case the \( SL(n, \mathbb{R}) \)-structure of the manifold \( \bar{M} \) must be an equivprojective structure.

If \( A^n \) is an affine space with the Cartesian system of coordinates \( \bar{x}_1, \ldots, \bar{x}^n \) then the components \( \bar{\omega}_{i_1 ... i_p} \) of the Killing-Yano tensor \( \bar{\omega} \) of degree \( p \) (\( 1 \leq p \leq n - 1 \)) in equation (3.4) must now satisfy

\[
\partial_j \bar{\omega}_{i_1 ... i_p} + \partial_i \bar{\omega}_{j_1 ... i_p} = 0 \tag{3.5}
\]

where \( \partial_j = \frac{\partial}{\partial x^j} \). From (3.5) we conclude the following equations

\[
\partial_k \partial_j \bar{\omega}_{i_1 ... i_p} + \partial_k \partial_i \bar{\omega}_{j_1 ... i_p} = 0; \tag{3.6}
\]
\[ \partial_j \partial_i \bar{\omega}_{ki_1...i_p} + \partial_j \partial_k \bar{\omega}_{i_1...i_p} = 0; \quad (3.7) \]
\[ \partial_i \partial_k \bar{\omega}_{j_1...i_p} + \partial_i \partial_j \bar{\omega}_{k_1...i_p} = 0. \quad (3.8) \]

From (3.6), (3.7), (3.8) we find
\[ \partial_k \partial_j \bar{\omega}_{i_1...i_p} = 0, \quad (3.9) \]

by using identities \( \frac{\partial^2 h}{\partial x^i \partial x^j} = \frac{\partial^2 h}{\partial x^j \partial x^i} \) which are carried out for an arbitrary smooth function \( h: \mathbb{A}^n \to \mathbb{R} \). The integrals of equations (3.9) take the form
\[ \bar{\omega}_{i_1...i_p} = A_{i_0i_1...i_p} \bar{x}^{i_0} + B_{i_1...i_p}, \quad (3.10) \]

for any skew-symmetric constants \( A_{i_0i_1...i_p} \) and \( B_{i_1...i_p} \) (see also [23, 19]). Taking the components (3.10) of the Killing-Yano tensor \( \bar{\omega} \) in \( \mathbb{A}^n \) and using Lemma 1, we can formulate Theorem 1.

### 3.3. Let \( \bar{M} \) be a manifold of dimension \( n \) endowed with the equiaffine \( SL(n, \mathbb{R}) \)-structure \( (\bar{\eta}, \bar{\nabla}) \) and \( M \) be a manifold of some dimension endowed with the equiaffine \( SL(n, \mathbb{R}) \)-structure \( (\eta, \nabla) \). Let there is given a projective diffeomorphism \( f: \bar{M} \to M \), then we have the following lemma.

**Lemma 2** Let \( f: \bar{M} \to M \) be a projective diffeomorphism of \( n \)-dimensional manifolds endowed with the equiaffine \( SL(n, \mathbb{R}) \)-structures \( (\bar{\eta}, \bar{\nabla}) \) and \( (\eta, \nabla) \) respectively. Then for an arbitrary Killing tensor \( \phi \) of degree \( p \) on the manifold \( \bar{M} \) the tensor field \( \bar{\phi} = e^{-2p\psi}(f^*\varphi) \) with \( \psi = (n+1)^{-1}\ln(\eta/\bar{\eta}) \) will be the Killing tensor of degree \( p \) on the manifold \( M \).

**Proof** We set \( \varphi_{i_1...i_p} \) to be components of the Killing tensor \( \varphi \) arbitrary defined on the manifold \( \bar{M} \); by definition, these components satisfy the following equations \( \sum_{cicl} \{ \nabla_i \varphi_{i_1...i_p} \} = 0 \). Then we find directly that the components \( \bar{\varphi}_{i_1...i_p} = e^{-2p\psi}\varphi_{i_1...i_p} \) of the tensor \( \bar{\phi} = e^{-2p\psi}\varphi \) satisfy the equations
\[ \sum_{cicl} \{ \bar{\nabla}_i \bar{\varphi}_{i_1...i_p} \} = e^{-2p\psi} \sum_{cicl} \{ \nabla_i \varphi_{i_1...i_p} \} = 0. \quad (3.11) \]

From (3.11) we conclude that the tensor field \( \bar{\phi} \) is a Killing tensor of degree \( p \) on the manifold \( \bar{M} \).

### 3.4. It follows from Nijenhuis (see [11]) that in an \( n \)-dimensional affine space \( \mathbb{A}^n \) the components \( \bar{\varphi}_{i_1...i_p} \) of the Killing tensor \( \bar{\phi} \) of degree \( p \) can be expressed in the form of an \( p^{th} \) degree polynomial in the \( \bar{x}^i \)'s
\[ \varphi_{i_1...i_p} = e^{-2p\psi} \sum_{q=0}^{p} A_{i_1...i_p,j_1...j_q} \bar{x}^{j_1} \ldots \bar{x}^{j_q}. \quad (3.12) \]

The coefficients \( A_{i_1...i_p,j_1...j_q} \) are constant and symmetric in the set of indices \( i_1, \ldots, i_p \) and the set of indices \( j_1, \ldots, j_q \). In addition to these properties the coefficients \( A_{i_1...i_p,j_1...j_q} \) have the following symmetries
\[ \sum_{cicl} \{ A_{i_1...i_p,j_1...j_{p-s}} \} J_{p-s+1} = 0 \]
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for $s = 1, \ldots, p - 1$ and

$$\sum_{cicd} \{A_{i1 \ldots i_p j_1}\} = 0.$$ 

Taking the components (3.12) of the Killing tensor $\bar{\varphi}$ in $A^n$ and using Lemma 2, we can formulate Theorem 2.

4 Applications to Riemannian geometry

4.1. Let $(M, g)$ be a pseudo-Riemannian manifold of dimensional $n$. Then from the present theorems 1 and 2 we conclude that an arbitrary Killing vector $\omega$ has the following local covariant components $\omega_i = e^{2\psi}(A_{ik} x^k + B_i)$ where $\psi = [2(n + 1)]^{-1} \ln |\det g|$, $A$'s and $B$'s are constants and $A_{ik} + A_{ki} = 0$ (see also [17]). It follows that the group of infinitesimal isometric transformations has $\frac{1}{2} n(n + 1)$ parameters (see also [2, §71]).

4.2. Following [25, 5], a skew-symmetric covariant tensor field $\vartheta$ of degree $p$ $(1 \leq p \leq n - 1)$ is called a conformal Killing tensor if $\vartheta \in \ker D$ for

$$D = \nabla - \frac{1}{p + 1} d - \frac{1}{n - p + 1} g \wedge d^*$$

where $d^*$ is the codifferential operator $d^*: C^\infty \Lambda^{p+1} M \to C^\infty \Lambda^p M$ and

$$(g \wedge d^* \vartheta)_{i_0 i_1 \ldots i_p} = \sum_{a=1}^p (-1)^{a+1} g_{i_0 i_a} (d^* \vartheta)_{i_1 \ldots \hat{i}_a \ldots i_p}.$$ 

Obviously, the set of conformal Killing tensors constitutes an vector space of tensor fields on $(M, g)$, denoted by $C^p(M, \mathbb{R})$ (see [21]). If a conformal Killing tensor $\vartheta$ belongs to $\ker d^*$, then it is a Killing-Yano tensor. On the other hand, if a conformal Killing tensor $\vartheta$ belongs to $\ker d$, it is called a closed conformal Killing tensor or a planar tensor (see [20, 21, 22]). We denote the vector space of these tensors by $P^p(M, \mathbb{R})$.

By [5] on an arbitrary $n$-dimensional pseudo-Riemannian manifold $(M, g)$ of constant nonzero sectional curvature $C$ ($C \neq 0$) the vector space $C^p(M, \mathbb{R})$ of conformal Killing tensors is decomposed uniquely in the form

$$C^p(M, \mathbb{R}) = K^p(M, \mathbb{R}) \oplus P^p(M, \mathbb{R}). \quad (4.1)$$

From (4.1) we conclude that any conformal Killing tensor $\vartheta$ of degree $p$ is decomposed uniquely in the form $\vartheta = \omega + \theta$ where $\omega$ is a Killing-Yano tensor of degree $p$ and $\theta$ is a closed conformal Killing tensor of degree $p$.

Following theorem 1, on an $n$-dimensional pseudo-Riemannian manifold $(M, g)$ of constant nonzero sectional curvature $C$ ($C \neq 0$) there is a local coordinate system $x^1, \ldots, x^n$ in which an arbitrary Killing-Yano tensor $\omega$ of degree $p$ $(2 \leq p \leq n - 1)$ has the components

$$\omega_{i_1 \ldots i_p} = e^{(p+1)\psi}(A_{i_0 i_1 \ldots i_p} x^{i_0} + B_{i_1 \ldots i_p}) \quad (4.2)$$
where $\psi = [2(n + 1)]^{-1} \ln |\det g|$, $\psi_k = \frac{\partial \psi}{\partial x^k}$ and $A_{i_0 i_1 \ldots i_p}$, $B_{i_1 \ldots i_p}$ are arbitrary skew-symmetric constants. On the other hand, by [19] on a pseudo-Riemannian manifold $(M, g)$ of constant nonzero curvature $C$ ($C \neq 0$) the components $\theta_{i_1 \ldots i_p}$ of a closed conformal Killing tensor $\theta$ of degree $p$ ($1 \leq p \leq n - 1$) can be found from the equations

$$\theta_{i_1 i_2 \ldots i_p} = -\frac{1}{pC} \nabla_{i_1} \omega_{i_2 \ldots i_p}$$

(4.3)

where $\nabla_{i_1} \omega_{i_2 \ldots i_p} = \partial_{i_1} \omega_{i_2 \ldots i_p} - \omega_{k \ldots i_p} \Gamma^k_{i_2 i_1} - \cdots - \omega_{i_2 \ldots k} \Gamma^k_{i_1 i_p}$ is the expression for the covariant derivative $\nabla \omega$ of the Killing-Yano tensor of degree $p - 1$. Moreover, by virtue of (3.1) on a pseudo-Riemannian manifold $(M, g)$ of constant curvature $C$ ($C \neq 0$) the Christoffel symbols $\Gamma^k_{ij}$ have the following form $\Gamma^k_{ij} = \psi_i \delta^k_j + \psi_j \delta^k_i$ (see also [17]). Therefore, we can deduce from (4.2) and (4.3) that

$$\theta_{i_1 \ldots i_p} = \frac{1}{C} e^{\psi} (\psi_{i_1} A_{[k|i_2 \ldots i_p]} x^k + \psi_{i_1} B_{i_2 \ldots i_p} + \frac{1}{p} A_{i_1 i_2 \ldots i_p}).$$

Consequently we have

**Theorem 3** On an $n$-dimensional pseudo-Riemannian manifold $(M, g)$ of constant nonzero sectional curvature $C$ ($C \neq 0$) there is a local coordinate system $x^1, \ldots, x^n$ in which an arbitrary conformal Killing tensor $\vartheta$ of degree $p$ ($2 \leq p \leq n - 1$) has the components

$$\vartheta_{i_1 \ldots i_p} = e^{(p+1)\psi} (A_{k_1 \ldots i_p} x^k + B_{i_1 \ldots i_p})$$

$$- \frac{1}{C} e^{\psi} \left( \psi_{i_1} C_{[k|i_2 \ldots i_p]} x^k + \psi_{i_1} D_{i_2 \ldots i_p} + \frac{1}{p} C_{i_1 i_2 \ldots i_p} \right)$$

where $\psi = [2(n + 1)]^{-1} \ln |\det g|$, $\psi_k = \frac{\partial \psi}{\partial x^k}$ and $A_{i_0 i_1 \ldots i_p}$, $B_{i_1 \ldots i_p}$, $C_{i_1 \ldots i_p}$ and $D_{i_1 \ldots i_p}$ are arbitrary skew-symmetric constants.

**Remark 3** For a conformal Killing vector field, see K. Yano and T. Nagano [27].

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