The anomaly equation can be derived from the ultraviolet properties of quantum field theory and should, therefore, not depend on infrared properties, such as the presence of a thermal heat bath. There is also an infrared explanation of anomalies which is related to fermionic zero modes. I show how the anomaly equation can be satisfied in a high temperature plasma in spite of the fact that all propagating fermionic excitations have a thermal mass.

1 Introduction

The anomaly of the conservation of baryon and lepton number currents is of obvious importance for baryogenesis. Though it can be derived purely from the UV behaviour of fermions in a background field it is reassuring to understand it also from a more intuitive IR point of view. In vacuum there is a close relation between the existence of zero modes and the anomaly, most conveniently formulated in terms of the index of the Dirac operator in a background field. A physical picture is that particles are pumped up from the Dirac sea and at the same time holes, representing antiparticles, move down into the Dirac sea. In vacuum this can only happen continuously if energy levels cross the Dirac surface. At finite temperature all propagating particles have a finite thermal mass of order $gT$ and there is no level crossing. It is therefore at first difficult to imagine how thermal particles can be created without overcoming the mass gap. We should keep in mind that an ordinary Dirac mass gap suppresses the creation of particles exponentially. If this were the case also for the effective thermal mass, which is not a local mass of Dirac type, the production of fermions from sphalarons could have been suppressed. To resolve the paradox it is absolutely essential to abandon the strict quasi-particle picture. The reason why the anomaly equation is satisfied even in presence of thermal masses, where the standard level crossing picture is not applicable since no levels ever cross the Dirac surface, is that the spectral weight $Z$ varies continuously from zero to one on each particle/hole branch (Fig. 1). Thus the topological argument of level crossing does not apply at finite temperature. I show explicitly, using the full spectral function in a background of electric and magnetic fields, how the produced chiral charge only depends on the ultraviolet properties of the spectral function. I do this in the context of the chiral anomaly in massless QED using the Hard Thermal Loop (HTL) effective action to describe the dynamics of the fermions at high temperature.
2 HTL spectral function in a magnetic field

The HTL effective action for QED can be written as:

\[
\mathcal{L}_{\text{HTL}} = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \frac{3}{4} \mathcal{M}^2 \mathcal{F}_{\mu\alpha} \frac{u^{\alpha} u^{\beta}}{(\partial \cdot u)^2} F_{\beta\mu}
\]

\[
+ \bar{\Psi} \left( \Pi - m \right) \Psi - \mathcal{M}^2 \mathcal{F}_{\gamma\mu} \left( \frac{u^\mu}{u \cdot \Pi} \right) \Psi ,
\]

where \( \Pi_{\mu} = i \partial_{\mu} - g A_{\mu} \) and the average \( \langle \cdot \rangle \) is defined by

\[
\langle f(u_0, \vec{u}) \rangle = \int \frac{d\Omega}{4\pi} f(u_0, \vec{u}) ,
\]

where \( u_0 = 1 \) and \( \vec{u} \) is a spatial unit vector. The thermal mass of the photon \( \mathcal{M}_\gamma^2 \) is given by \( g^2 T^2 / 9 \) and for the electron we have \( \mathcal{M}_e^2 = g^2 T^2 / 8 \). The equation of motion for \( \Psi \) that follows from Eq. (1) is

\[
\left[ \Pi - m - \mathcal{M}_e^2 \gamma_{\mu} \left( \frac{u^\mu}{u \cdot \Pi} \right) \right] \Psi = 0 .
\]
Equation (3) is a non-local and non-linear differential equation, which is, in general, very difficult to solve. What makes this equation much less tractable than the thermal Dirac equation, in the absence of an external electromagnetic field, is that the average over $\vec{u}$ is difficult to perform explicitly since $[\Pi_\mu, \Pi_\nu] = -igF_{\mu\nu} \neq 0$, i.e., not all components of $\Pi_\nu$ can be diagonalized simultaneously. We shall first only deal with an external magnetic field and fix it to be in the $z$-direction. The solutions to Eq. (3) in vacuum ($M_e = 0$) are given by the standard Landau levels. Since the spatial symmetries of the system are unchanged by the thermal heat bath, we expect the eigenfunctions to have the same spatial form as at zero temperature. In fact, after performing the $u$-integral in Eq. (3) the result can only be a function of the invariants $\Pi_\perp^2, p_0^2$ and $p_z^2$, and the $\gamma$-structure has to be proportional to $\gamma_\perp \Pi_\perp, \gamma_0 p_0$ and $\gamma_z p_z$. We shall therefore compute the matrix elements of Eq. (3) between the vacuum eigenstates $\Phi_\kappa$. To be specific we use the gauge $A_\mu = (0, 0, -Bx, 0)$. After computing the matrix elements of Eq. (3) we find indeed that they are diagonal in $\kappa$ for $u_0$ and $u_z$, and have a mixing with the first subdiagonals for $u_x$ and $u_y$. We define

$$\langle u_0, z, \pm \rangle = \left\langle \Phi_\kappa' \bigg| \frac{u_0 \pm i u_y}{u \cdot \Pi} \right| \Phi_\kappa \right\rangle = \left(2\pi\right)^3 \delta_{\kappa', \kappa} \langle u_0, z \rangle \kappa, \tag{4}$$

$$\langle u_x \pm i u_y, z, \pm \rangle = \left\langle \Phi_\kappa' \bigg| \frac{u_x \pm i u_y}{u \cdot \Pi} \right| \Phi_\kappa \right\rangle = \left(2\pi\right)^3 \delta_{\kappa', \kappa \mp 1} \langle u_x \pm \rangle \kappa, \tag{5}$$

and $\kappa \mp 1 = \{p_0, n \mp 1, p_y, p_z\}$, where $n$ labels the Landau levels.

In the equilibrium real-time finite temperature formalism the free propagator is a $2 \times 2$ matrix, but since we shall compute a one-point function to find the chiral charge we only need the 11-part:

$$iS^{0}_{F}(p) = iS^{0}_{F}(p_0) - f_F(p_0) \left( iS^{0}_{F}(p) - iS^{0}_{F}(p) \right), \tag{6}$$

where $f_F(p_0)$ is the thermal distribution function. The fermionic part of the HTL effective action is simply related to the inverse of the propagator by

$$\mathcal{L}_{\text{HTL}} = \mathcal{V}(x) S^{-1}(x, y) \Psi(y). \tag{7}$$

The Feynman propagator ($\Pi_0 = p_0 + i\epsilon p_0$) is the given by

$$iS^{0}_{F}(x, y) = \langle T | \Psi(x) \bar{\Psi}(y) \rangle = \left\langle x \left| \frac{i}{\Pi - m - M^*_\mu \gamma_\mu \langle \gamma_\mu \Pi \rangle} \right| y \right\rangle. \tag{8}$$

With the explicit expression of the inverse propagator in the Landau level basis it is straightforward to calculate the spectral function $\mathcal{A}$. In the lowest
Landau level, which is the only one we need to obtain the anomaly, we find for right-handed particles

\[
\begin{align*}
A_{\text{LLL}}^R(E, p_z) &= \text{tr} \left[ \frac{1}{2} (1 + \gamma_5) \gamma_0 A(E, p_z, n = 0) \right] \\
&= \frac{1}{2\pi i} \left( S_{\text{LLL}}^R(E - i\epsilon, p_z) - S_{\text{LLL}}^R(E + i\epsilon, p_z) \right), \\
S_{\text{LLL}}^R(E, p_z) &= \frac{1}{p_0 - p_z - M^2((u_0) - (u_z))}. 
\end{align*}
\]

(9)

3 The chiral anomaly

The classical action for massless fermions is invariant under chiral transformations, but the corresponding chiral current is not conserved on the quantum level due to the chiral anomaly. The divergence of the chiral current in 3+1 dimensions is given by

\[
\partial_\mu j_5^\mu = \partial_\mu \gamma_\mu \gamma_5 \Psi = \frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}. 
\]

(10)

Finite temperature effects do not break chirality and, as a classical action, the HTL effective action is still chirally invariant.

To verify Eq. (10) explicitly we shall compute the produced chiral charge

\[
\langle Q_5 \rangle = \int d^3x \langle \overline{\Psi}(x) \gamma_0 \gamma_5 \Psi(x) \rangle 
\]

in a constant magnetic field when a parallel electric field is applied. The chiral charge has to be defined using a gauge-invariant point splitting regularization in the spatial z-direction

\[
\langle Q_5 \rangle = \int dx dy dz dz' \exp \left[ -\frac{(z - z')^2}{2\gamma} \right] \sqrt{2\pi\gamma} \times \langle \overline{\Psi}(x, y, z, t) \gamma_0 \gamma_5 \Psi(x, y, z', t) \rangle \exp \left[ ie \int_{z'}^z A_z(z'', t) dz'' \right].
\]

(11)

The field expectation values can be related to the time-ordered Feynman Green’s function via

\[
\langle \overline{\Psi}(x) \gamma_0 \gamma_5 \Psi(y) \rangle = -i \text{tr} \left[ S_F(y, x)|_{x_a > y_b} \gamma_0 \gamma_5 \right].
\]

(12)

In last section we derived an explicit expression for the propagator in a background magnetic field, and it turns out to be rather easy to include a parallel electric field with arbitrary time dependence in the gauge

\[
A_\mu = (0, 0, 0, A_3(t))
\]

for the electric field. It only amounts to a phase shift of the eigenstates (for more details see Ref. [1]).
Just as in vacuum the higher Landau levels \((n \geq 1)\) do not contribute to the anomaly. It can also be shown\(^5\) that the purely thermal part of the propagator (the second part of the right-hand side of Eq.\((6)\)) does not give any contribution to the anomaly. We are left with possible contributions from the lowest Landau level. The relevant expectation value is (\(\kappa_0 = \{n = 0, p_y, p_z\}\)):

\[
\text{tr} S_F(\kappa_0)\gamma_0\gamma_5 = \int_{-\infty}^{\infty} dE \frac{A_{\text{LLL}}^L(E,p_z) - A_{\text{LLL}}^R(E,p_z)}{p_0 - E + i\epsilon p_0}, \quad (13)
\]

with \(A_{\text{LLL}}^L(E,p_z) = A_{\text{LLL}}^R(E,-p_z)\). The produced chiral charge reduces to

\[
\langle Q_5 \rangle_{\text{LLL}} = \frac{VeB}{4\pi^2} \int dp_z e^{-\frac{i}{2}(p_z - eA_z)^2} [Z(p_z) - Z(-p_z)] , \quad (14)
\]

where \(Z(p_z) = \int_{0}^{\infty} dE A_{\text{LLL}}^R(E,p_z)\). It is the spectral weight for the right-handed positive energy solution in the lowest Landau level. For very large \(|p_z|\) there are no collective excitations, such as holes, but only the standard particle solution. Therefore, \(Z(p_z \to -\infty) = 0\) and \(Z(p_z \to \infty) = 1\) (see Fig.\(^4\)), which implies

\[
\langle Q_5(t) \rangle_{\text{LLL}} = -\frac{VeB}{2\pi^2}eA_z = \int d^3x \int dt' \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} , \quad (15)
\]

in agreement with Eq.\((10)\).

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