Induced $W_\infty$ Gravity as a WZNW Model

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ABSTRACT

We derive the explicit form of the Wess-Zumino quantum effective action of chiral $W_\infty$-symmetric system of matter fields coupled to a general chiral $W_\infty$-gravity background. It is expressed as a geometric action on a coadjoint orbit of the deformed group of area-preserving diffeomorphisms on cylinder whose underlying Lie algebra is the centrally-extended algebra of symbols of differential operators on the circle. Also, we present a systematic derivation, in terms of symbols, of the “hidden” $SL(\infty; \mathbb{R})$ Kac-Moody currents and the associated $SL(\infty; \mathbb{R})$ Sugawara form of energy-momentum tensor component $T_{++}$ as a consequence of the $SL(\infty; \mathbb{R})$ stationary subgroup of the relevant $W_\infty$ coadjoint orbit.

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1. Introduction

The infinite-dimensional Lie algebra $W_\infty$ (and its generalizations $W_{1+\infty}$ etc.) \[1, 2, 3\] are nontrivial “large $N$” limits of the associative, but non-Lie, conformal $W_N$ algebras \[4\]. They arise in various problems of two-dimensional physics. The list of their principal applications includes self-dual gravity \[5\], first Hamiltonian structure of integrable KP hierarchy \[6\], string field actions in the collective field theory approach \[7\], conformal affine Toda theories \[8\]. One of the most remarkable manifestations of $W_\infty$-type algebras is the recent discovery of a subalgebra of their “classical” limit $w_\infty$ (the algebra of area-preserving diffeomorphisms) in $c = 1$ string theory as symmetry algebra of the special discrete states \[9\] or as the algebra of infinitesimal deformations of the ground ring \[10\]. Also, it is worth noting that similar algebras are found also in $D = 2$ quasitopological models, such as $D = 2$ Yang-Mills \[11\], where the metric dependence of the partition function degenerates into a dependence on the area only.

It is well known in the mathematical literature \[12\], that the family of possible deformations $W_\infty(q)$ of the initial “classical” $w_\infty$ depends on a single parameter $q$ and that, for each fixed value of $q$, $W_\infty(q)$ possesses an one-dimensional cohomology with values in $\mathbb{R}$. In particular, for $q = 1$ one finds that $W_\infty(1) \simeq D\tilde{O}\mathcal{P}(S^1)$ - the centrally extended algebra of differential operators on the circle, which was recently studied in refs.\[13\] The equivalence of $D\tilde{O}\mathcal{P}(S^1)$ to the original definition of $W_\infty(1) \[1, 3\]$ was explicitly demonstrated in \[14\].

In this letter we first derive a WZNW field-theory action $W_{D\tilde{O}\mathcal{P}(S^1)}[g]$ on a generic coadjoint orbit of the group $G = D\tilde{O}\mathcal{P}(S^1)$. The elements $g(\xi, x; t)$ of this group for fixed time $t$ are symbols of exponentiated differential operators on $S^1$ and in this sense $D\tilde{O}\mathcal{P}(S^1)$ is the formal Lie group corresponding to the Lie algebra $D\tilde{O}\mathcal{P}(S^1)$. As it was shown in \[15\], the Legendre transform $\Gamma[g] = -W[g^{-1}]$ of a group coadjoint orbit action $W[g]$ for a general infinite-dimensional group $G$ provides the exact solution for the quantum effective action of matter fields possessing an infinite-dimensional Noether symmetry group $G_0$ - the “classical” undeformed version of the group $G$. Thus, our WZNW action $W_{D\tilde{O}\mathcal{P}(S^1)}[g]$ is the explicit field-theoretic expression of the induced $W_\infty$-gravity effective action. In particular, we show that $W_{D\tilde{O}\mathcal{P}(S^1)}[g]$ reduces to the well-known Polyakov’s WZNW action of induced $D = 2$ gravity in the light-cone gauge \[16\] when restricting the WZNW field $g(\xi, x; t)$ to the Virasoro subgroup of $D\tilde{O}\mathcal{P}(S^1)$. Furthermore, the appearance of the “hidden” $SL(\infty; \mathbb{R})$ Kac-Moody symmetry and the associated $SL(\infty; \mathbb{R})$ Sugawara form of the $T_{++}$ component of the energy-momentum tensor are shown to be natural consequences of $SL(\infty; \mathbb{R})$ stationary subgroup the pertinent $D\tilde{O}\mathcal{P}(S^1)$ coadjoint orbit. Also, we present WZNW field-theoretic expressions in terms of $g(\xi, x; t)$ for the “hidden” currents and $T_{++}$.

2. Basic Ingredients

The object of primary interest is the infinite-dimensional Lie algebra $\mathcal{G} = D\tilde{O}\mathcal{P}(S^1)$ of symbols of differential operators \[\hat{\mathcal{G}} = \{ X \equiv \sum_k \xi_k X_k(x) \} \] on the circle $S^1$ with vanishing zero-order part $\hat{\mathcal{G}} = \{ X \equiv \sum_k \xi_k X_k(x) \}$. Let us recall \[17\] the correspondence between (pseudo)differential operators and symbols : $X(\xi, x) = \sum_k \xi_k X_k(x) \leftrightarrow \hat{X} = \sum_k X_k(x)(-i\partial_x)^k$. 

\[1\]
X(\xi, x) = \sum_{k \geq 1} \xi^k X_k(x) \right\}. For any pair X, Y \in \mathcal{G} = \mathcal{DOP}(S^1) the Lie commutator is given in terms of the associative (and non-commutative) symbol product denoted henceforth by a cirlce \circ:

\[ [X, Y] \equiv X \circ Y - Y \circ X \quad ; \quad X \circ Y \equiv X(\xi, x) \exp\left(\overrightarrow{\partial_\xi \partial_x} Y(\xi, x) \right) \tag{1} \]

In order to determine the dual space \( \mathcal{G}^* = \mathcal{DOP}^*(S^1) \), let us consider the space \( \Psi\mathcal{DO}(S^1) = \left\{ U \equiv U(\xi, x) = \sum_{k=1}^\infty \xi^{-k} U_k(x) \right\} \) of all purely pseudodifferential symbols \[17\] on \( S^1 \) and the following bilinear form on \( \Psi\mathcal{DO}(S^1) \otimes \mathcal{DOP}(S^1) \):

\[ \langle U | X \rangle \equiv \int dx \text{Res}_\xi U \circ X = \int dx \text{Res}_\xi \left( e^{-\partial_\xi \partial_x} U(\xi, x) \right) X(\xi, x) \tag{2} \]

The last equality in (2) is due to the vanishing of total derivatives w.r.t. the measure \( \int dx \text{Res}_\xi \), and \( \text{Res}_\xi U(\xi, x) = U_1(x) \). From (2) we conclude that any pseudodifferential symbol of the form \( U^{(0)} = e^{\partial_\xi \partial_x} \left( \frac{1}{\xi} u(x) \right) \) is “orthogonal” to any differential symbol \( X \in \mathcal{DOP}(S^1) \), i.e. \( \langle U^{(0)} | X \rangle = 0 \). Thus, the dual space \( \mathcal{G}^* = \mathcal{DOP}^*(S^1) \) can be defined as the factor space \( \Psi\mathcal{DO}(S^1) \setminus \left\{ e^{\partial_\xi \partial_x} \frac{1}{\xi} u(x) \right\} \) w.r.t. the “zero” pseudodifferential symbols. In particular, we shall adopt the definition:

\[ \mathcal{G}^* = \left\{ U_* ; U_*(\xi, x) = U(\xi, x) - e^{\partial_\xi \partial_x} \left( \frac{1}{\xi} \text{Res}_\xi U(\xi, x) \right) \right\} \quad \text{for } \forall U \in \Psi\mathcal{DO} \quad \tag{3} \]

Having the bilinear form (2) one can define the coadjoint action of \( \mathcal{G} \) on \( \mathcal{G}^* \) via:

\[ \langle \text{ad}^*(X) U | Y \rangle = - \langle U | [X, Y] \rangle \]
\[ (\text{ad}^*(X) U)(\xi, x) \equiv [X, U]_* \quad \tag{4} \]

Here and in what follows, the subscript (−) indicates taking the part of the symbol containing all negative powers in the \( \xi \)-expansion, whereas the subscript * indicates projecting of the symbol on the dual space (3). The Jacobi identity for the coadjoint action \( \text{ad}^*(\cdot) \) (4) is fulfilled due to the following important property:

\[ \left[ X, e^{\partial_\xi \partial_x} \left( \frac{1}{\xi} u(x) \right) \right] = e^{\partial_\xi \partial_x} \left( \frac{1}{\xi} \text{Res}_\xi [X, e^{\partial_\xi \partial_x} \frac{1}{\xi} u(x)] \right) \tag{5} \]

i.e., the coadjoint action of \( \mathcal{DOP}(S^1) \) on \( \Psi\mathcal{DO}(S^1) \) maps “zero” pseudodifferential symbols into “zero” ones.

The central extension in \( \tilde{\mathcal{G}} \equiv \mathcal{DOP}(S^1) = \mathcal{DOP}(S^1) \oplus \mathbb{R} \) is given by the two-cocycle \( \omega(X, Y) = -\frac{1}{4\pi} \langle \dot{s}(X)|Y \rangle \), where the cocycle operator \( \dot{s} : \mathcal{G} \longrightarrow \mathcal{G}^* \) explicitly reads \[13\] :

\[ \dot{s}(X) = [X, \ln \xi]_* \quad \tag{6} \]

Let us now consider the Lie group \( G = \text{DOP}(S^1) \) defined as exponentiation of the Lie algebra \( \mathcal{G} \) of symbols of differential operators on \( S^1 \):

\[ G = \left\{ g(\xi, x) = \text{Exp}X(\xi, x) \equiv \sum_{N=0}^\infty \frac{1}{N!} X(\xi, x) \circ X(\xi, x) \circ \cdots \circ X(\xi, x) \right\} \quad \tag{7} \]
and the group multiplication is just the symbol product \( g \circ h \). The adjoint and coadjoint action of \( G = DOP(S^1) \) on the Lie algebra \( DOP(S^1) \) and its dual space \( DOP^*(S^1) \), respectively, is given as:

\[
(Ad(g)X) = g \circ X \circ g^{-1} ; \quad (Ad^*(g)U) = (g \circ X \circ g^{-1})^*
\]  

The group property of (8) \( Ad^*(g \circ h) = Ad^*(g) Ad^*(h) \) easily follows from the “exponentiated” form of the identity (5).

After these preliminaries we are ready to introduce the two interrelated fundamental objects \( S[g] \) and \( Y[g] \) entering the construction of the geometric action on a coadjoint orbit of \( G \). To this end we shall follow the general formalism for geometric actions on coadjoint orbits of arbitrary infinite-dimensional groups with central extensions proposed in refs. [18, 19].

Namely, \( S[g] \) is a nontrivial \( \mathcal{G}^* \)-valued one-cocycle on the group \( G \) (also called finite “anomaly” or generalized Schwarzian), whose infinitesimal form is expressed through the Lie-algebra \( \mathcal{G} \) cocycle operator \( \hat{s}(\cdot) \) (infinitesimal “anomaly”):

\[
S[g \circ h] = S[g] + Ad^*(g)S[h] \quad ; \quad \left. \frac{d}{dt} S[\text{Exp}(tX)] \right|_{t=0} = \hat{s}(X)
\]  

The integrability condition for (11) implies that the one-form \( Y[g] \) satisfies the Maurer-Cartan equation and that it is a \( DOP(S^1) \)-valued group one-cocycle:

\[
dS[g] = -Ad^*(g) \hat{s}(Y[g^{-1}])
\]  

From (8) and (10)-(12) one easily finds:

\[
Y[g] = dg(\xi, x) \circ g^{-1}(\xi, x)
\]  

At this point it would be instructive to explicitate formulas (8), (11) and (12) when the elements of \( G = DOP(S^1) \) and \( \mathcal{G} = DOP(S^1) \) are restricted to the Virasoro subgroup (subalgebra, respectively):

\[
X(\xi, x) = \xi \omega(x) \leftarrow \omega(x) \partial_x \in \text{Vir}
\]

\[
g(\xi, x) = \text{Exp}(\xi \omega(x)) \leftrightarrow F(x) \equiv \exp(\omega(x)\partial_x) x \in \text{Diff}(S^1)
\]  

Substituting (14) into (8), (10) and (12), one obtains:

\[
Y[g] \bigg|_{g(\xi, x) = \text{Exp}(\xi \omega(x))} = \xi \frac{dF(x)}{\partial_x F(x)} \quad ; \quad \hat{s}(X) = [\xi \omega(x), \ln \xi]_* = -\frac{1}{6} \xi^{-2} \partial_x^3 \omega(x) + \cdots
\]

\[
S[g] \bigg|_{g(\xi, x) = \text{Exp}(\xi \omega(x))} = -\frac{1}{6} \xi^{-2} \left( \frac{\partial_x^3 F}{\partial_x F} - \frac{3}{2} \left( \frac{\partial_x^2 F}{\partial_x F} \right)^2 \right) + \cdots
\]
The dots in (13) indicate higher order terms $O(\xi^k), k \geq 3$, which do not contribute in bilinear forms with elements of $\mathcal{V}ir$ (14).

3. WZNW Action of $W_\infty$ Gravity

According to the general theory of group coadjoint orbits [20], a generic coadjoint orbit $O(U_0, c)$ of $G$ passing through a point $(U_0, c)$ in the extended dual space $\tilde{G}^* = G^* \oplus \mathbb{R}$:

$$O(U_0, c) \equiv \left\{ (U(g), c) \in \tilde{G}^* ; U(g) = Ad^*(g)U_0 + cS[g] \right\}$$

has a structure of a phase space of an (infinite-dimensional) Hamiltonian system. Its dynamics is governed by the following Lagrangian geometric action written solely in terms of the interrelated fundamental group and algebra cocycles $S[g], Y[g], \hat{s}(\cdot)$ (cf. eqs.(6),(9)-(13) ) [18, 19] :

$$W[g] = \int_L \langle U_0 \mid Y[g^{-1}] \rangle - c \int \left[ \langle S[g] \mid Y[g] \rangle - \frac{1}{2} d^{-1} \left( \langle \hat{s}(Y[g]) \mid Y[g] \rangle \right) \right]$$

(17)

The integral in (17) is over one-dimensional curve $L$ on the phase space $O(U_0, c)$ (16) with a “time-evolution” parameter $t$. Along the curve $L$ the exterior derivative becomes $d = dt \partial_t$. Also, $d^{-1}$ denotes the cohomological operator of Novikov [21] - the inverse of the exterior derivative, defining the customary multi-valued term present in any geometric action on a group coadjoint orbit.

In the present case of $G = DOP(S^1)$, the co-orbit action (17) takes the following explicit form, which (as discussed in section 1) is precisely the Wess-Zumino action for induced $W_\infty$-gravity (the explicit dependence of symbols on $(\xi, x; t)$ will in general be suppressed below) :

$$W[g] = - \int dt dx \text{Res}_\xi U_0 \circ g^{-1} \circ \partial_t g +$$

$$\frac{c}{4\pi} \int_L \int dx \text{Res}_\xi \left( \left[ \ln \xi , g \right] \circ g^{-1} \circ \partial_t g \circ g^{-1} - \frac{1}{2} d^{-1} \left( \left[ \ln \xi , g^{-1} \circ dg \right] \wedge \left( g^{-1} \circ dg \right) \right) \right)$$

(18)

The physical meaning of the first term on the r.h.s. of (18) is that of coupling of the chiral $W_\infty$ Wess-Zumino field $g = g(\xi, x; t)$ to a chiral $W_\infty$-gravity “background”. For simplicity, we shall consider henceforth the case $U_0 = 0$.

It is straightforward to obtain, upon substitution of eqs.(14)-(15), that the restriction of $g(\xi, x; t)$ to the Virasoro subgroup reduces the $W_\infty$ Wess-Zumino action (18) to the well-known Polyakov’s Wess-Zumino action of induced $D = 2$ gravity [10, 22].

The group cocycle properties (eqs.(9),(12)) of $S[g]$ (11) and $Y[g]$ (13) imply the following fundamental group composition law for the $W_\infty$ geometric action (18):

$$W[g \circ h] = W[g] + W[h] - \frac{c}{4\pi} \int dt dx \text{Res}_\xi \left( \left[ \ln \xi , h \right] \circ h^{-1} \circ g^{-1} \circ \partial_t g \right)$$

(19)

Eq.(19) is a particular case for $W_\infty$ of the group composition law for geometric actions on coadjoint orbits of arbitrary infinite-dimensional groups with central extensions [19]. It
generalizes the famous Polyakov-Wiegmann group composition law [23] for ordinary $D = 2$
WZNW models.

Using the general formalism for co-orbit actions in [18, 19] we find, that the basic Poisson
brackets for $S[g]$ (11) following from the action (18) read:

$$\{ S[g](\xi, x), S[g](\eta, y) \}_PB = \left[ S[g](\xi, x) + \ln \xi, \delta_{DOP}(\eta, \xi, x, \eta) \right]$$  

(20)

where $\delta_{DOP}(\cdot, \cdot) \in G^* \otimes G$ denotes the kernel of the $\delta$-function on the space of differential
operator symbols:

$$\delta_{DOP}(x, \xi, \eta, y) = e^{\partial_x \partial_{\xi}} \left( \sum_{k=1}^{\infty} \xi^{-(k+1)} \eta^k \delta(x - y) \right)$$  

(21)

Eq. (20) is a succinct expression of the Poisson-bracket realization of $W_\infty$, which becomes
manifest by rewriting (20) in the equivalent form:

$$\{ \langle S[g] | X \rangle, \langle S[g] | Y \rangle \}_PB = -\langle S[g] | [X, Y] \rangle + \langle \delta(X) | Y \rangle$$  

(22)

for arbitrary fixed $X, Y \in G = DOP(S^1)$. Alternatively, substituting into (20) (or (22)) the
$\xi$-expansion of the pseudodifferential symbol $S[g](\xi, x) = \sum_{r \geq 2} \xi^{-r} S_r(x)$, one recovers the
Poisson-bracket commutation relations for $W_\infty$ among the component fields $S_r(x)$ in the
basis of ref. [14] (which is a “rotation” of the more customary $W_\infty$ basis of refs. [2]).

In particular, for the component field $S_2(x) \equiv \frac{4\pi c}{3} T_{-1}(x)$ (the energy-momentum tensor component, cf. [15])
one gets from (20) the Poisson-bracket realization of the Virasoro algebra:

$$\{ S_2(x), S_2(y) \}_PB = -\frac{4\pi}{c} \left( 2S_2(x) \partial_x \delta(x - y) + \partial_x S_2(x) \delta(x - y) + \frac{1}{6} \partial^3_x \delta(x - y) \right)$$  

(23)

The higher component fields $S_r(x), r = 3, 4, \cdots$ turn out to be quasi-primary conformal
fields of spin $r$. The genuine primary fields $W_r(x) (r \geq 3)$ are obtained from $S_r(x)$ by adding derivatives of the lower spin fields $S_q(x) (2 \leq q \leq r - 1)$. For instance, for
$W_3(x) = S_3(x) - \frac{3}{2} \partial_x S_2(x)$, eq. (20) yields:

$$\{ S_2(x), W_3(y) \}_PB = -\frac{4\pi}{c} \left( 3W_3(x) \partial_x \delta(x - y) + 2\partial_x W_3(x) \delta(x - y) \right)$$  

(24)

4. Noether and “Hidden” Symmetries of $W_\infty$ Gravity

The general group composition law (13) contains the whole information about the symmetries of the $W_\infty$ geometric action (18). First, let us consider arbitrary infinitesimal left
group translation. The corresponding variation of the action (18) is straightforwardly obtained from (19):

$$\delta_\varepsilon W[g] \equiv W[(1 + \varepsilon) \circ g] - W[g] = \frac{c}{4\pi} \int dt \, dx \, \text{Res}_\xi \left\{ \left[ \ln \varepsilon, g \right] \circ g^{-1} \right\}_* \circ \partial_t \varepsilon$$  

(25)
From (25) one finds that (18) is invariant under \( t \)-independent left group translations and the associated Noether conserved current is the generalized “Schwarzian” \( S[g] \) whose components are the (quasi)primary conformal fields \( S_r(x; t) \) of spin \( r \).

Next, let us consider arbitrary right group translation. Now, from (19) the variation of the \( W_\infty \) action (18) is given by:

\[
\delta R \zeta W \equiv W[g \circ (\mathbb{I} + \zeta)] - W[g] = -\frac{c}{4\pi} \int dt \, dx \, \text{Res}_\xi \left( [\ln \xi, \zeta] \circ Y_t(g^{-1}) \right) = -\frac{c}{4\pi} \int dt \, dx \, \text{Res}_\xi \left( [\ln \xi, Y_t(g^{-1})] \circ \zeta \right)
\]

where \( Y_t(g^{-1}) \) denotes the Maurer-Cartan gauge field:

\[
Y_t(g^{-1}) = -\frac{1}{g - 1} \partial_t g
\]

Equality (27) implies the equations of motion:

\[
\hat{s} \left( Y_t(g^{-1}) \right) \bigg|_{\text{on-shell}} = 0
\]

As a matter of fact, the off-shell relation (11) exhibits the full equivalence between the Noether conservation law \( \partial_s S[g] = 0 \) (25) and the equations of motion (29).

On the other hand, equality (26) shows that the \( W_\infty \) geometric action (18) is gauge-invariant under arbitrary time-dependent infinitesimal right-group translations \( g(\xi, x; t) \rightarrow g(\xi, x; t) \circ (1 + \tilde{\zeta}(\xi, x; t)) \) which satisfy:

\[
\hat{s}(\tilde{\zeta}) \equiv -[\ln \xi, \tilde{\zeta}]_* = 0
\]

Equality (26) implies the equations of motion:

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\[
\hat{s}(\tilde{\zeta}) \equiv -[\ln \xi, \tilde{\zeta}]_* = 0
\]

For finite right group translations \( k = \text{Exp} \tilde{\zeta} \) the integrated form of (30) reads:

\[
S[k] \equiv -[\ln \xi, k \circ k^{-1}]_* = 0
\]

The solutions of eqs. (30) and (31) form a subalgebra in \( DOP(S^1) \), and a subgroup in \( DOP(S^1) \), respectively. From (16) one immediately concludes that the latter subgroup:

\[
G_{\text{stat}} = \left\{ k ; S[k] = 0 \right\}
\]

is precisely the stationary subgroup of the underlying coadjoint orbit \( \mathcal{O}_{(U_0 = 0, c)} \). The Lie algebra of (32):

\[
\mathcal{G}_{\text{stat}} = \left\{ \tilde{\zeta} ; \hat{s}(\tilde{\zeta}) \equiv -[\ln \xi, \tilde{\zeta}]_* = 0 \right\}
\]

is the maximal centerless (“anomaly-free”) subalgebra of \( \text{DO}(S^1) \), on which the cocycle (3) vanishes:

\[
\omega(\tilde{\zeta}_1, \tilde{\zeta}_2) = -\left\langle \hat{s}(\tilde{\zeta}_1) | \tilde{\zeta}_2 \right\rangle = 0 \quad \text{for any pair} \quad \tilde{\zeta}_1, \tilde{\zeta}_2 \in \mathcal{G}_{\text{stat}}.
\]

The restriction of eq. (29) to the Virasoro subgroup via (14)-(15) takes the well known from [16]

\[
\partial^3_x (\partial_t f \partial_x f) = 0, \quad \text{and} \quad f(x; t) \text{ is the inverse Virasoro group element : } f(F(x; t); t) = x.
\]
The Cartan subalgebra of \( \mathfrak{g}_{\text{stat}} \), can be written in the form:

\[
\zeta^{(l,m)}(\xi, x) = \sum_{q=1}^{l} \binom{l}{q} \frac{(l-1)!(l+q)!}{(q-1)!(2l)!} \frac{\xi^q x^{q+m}}{\Gamma(q+m+1)}
\]

where \( l = 1, 2, \ldots \) and \( m = -l, -l+1, \ldots, l-1, l \).

The basis \( \{ \zeta^{(l,m)} \} \) identifies the stationary subalgebra \( \mathfrak{g}_{\text{stat}} \) as the infinite-dimensional algebra \( \mathfrak{sl}(\infty; \mathbb{R}) \). Namely, \( \mathfrak{g}_{\text{stat}} \) decomposes (as a vector space) into a direct sum of irreducible representations \( \mathcal{Y}^{(l)}_{\text{sl}(2)} \) of its \( \mathfrak{sl}(2; \mathbb{R}) \) subalgebra with spin \( l \) and unit multiplicity: \( \mathfrak{g}_{\text{stat}} = \bigoplus_{l=1}^{\infty} \mathcal{Y}^{(l)}_{\text{sl}(2)} \). This \( \mathfrak{sl}(2; \mathbb{R}) \) subalgebra is generated by the symbols \( 2\zeta^{(1,1)} = \xi x^2, \zeta^{(1,0)} = \xi x \) and \( \zeta^{(1,-1)} = \xi \). The subspaces \( \mathcal{Y}^{(l)}_{\text{sl}(2)} \) are spanned by the symbols \( \{ \zeta^{(l,m)}; \ |m| \leq l \} \) with \( \zeta^{(l,l)} \) being the highest-weight vectors:

\[
[\xi x^2, \zeta^{(l,l)}] = 0; \quad [\xi x, \zeta^{(l,m)}] = m \zeta^{(l,m)}; \quad [\xi, \zeta^{(l,m)}] = \zeta^{(l,m-1)}
\]

The Cartan subalgebra of \( \mathfrak{g}(\infty; \mathbb{R}) \) is spanned by the subset \( \{ \zeta^{(l,0)}; l = 1, 2, \ldots \} \) of symbols \( \{ \zeta^{(l,0)} \} \).

The above representation of \( \mathfrak{sl}(\infty; \mathbb{R}) \) in terms of symbols \( \{ \zeta^{(l,m)} \} \) is analogous to the construction of \( \mathfrak{sl}(\infty; \mathbb{R}) \) as “wedge” subalgebra \( \mathfrak{W}(\mu) \) of \( \mathfrak{W}_\infty \) for \( \mu = 0 \) [2, 24], which in turn is isomorphic to the algebra \( \mathfrak{A}_\infty \) of Kac [23].

Now, accounting for (33)-(34), one can write down explicitly the solution to the equations of motion (29):

\[
Y_t(g^{-1}) \bigg|_{\text{on-shell}} = \sum_{l=1}^{\infty} \sum_{|m| \leq l} J^{(l,m)}(t) \zeta^{(l,m)}(\xi, x)
\]

with \( \zeta^{(l,m)} \) as in (34). The coefficients \( J^{(l,m)}(t) \) in (36) are arbitrary functions of \( t \) and represent the on-shell form of the currents of the “hidden” \( \mathfrak{g}_{\text{stat}} \equiv \mathfrak{sl}(\infty; \mathbb{R}) \) Kac-Moody symmetry of \( \mathfrak{W}[g] \) (28).

Indeed, upon right group translation with \( \zeta(a) = \sum a^{(l,m)}(x, t) \circ \zeta^{(l,m)}(\xi, x) \) with arbitrary coefficient functions (zero order symbols) \( a^{(l,m)}(x, t) \), one obtains from (26):

\[
\delta_{\zeta(a)} W[g] = -\frac{c}{4\pi} \int dt \, dx \sum_{l=1}^{\infty} \sum_{|m| \leq l} \partial_x a^{(l,m)}(x, t) J^{(l,m)}(x, t) \]

\[
\bar{J}^{(l,m)} \equiv \sum_{r=1}^{\infty} (-\partial_x)^{r-1} \left\{ (-1)^r l!(l+1)! \frac{x^m}{\Gamma(m)} \left( Y_t(g^{-1}) \right)_r \right\} - \frac{1}{r+1} \partial_x \left( \zeta^{(l,m)} \circ Y_t(g^{-1}) \right)_r \]

The subscripts \( r \) in (38) and below indicate taking the coefficient in front of \( \xi^r \) in the corresponding symbol.

The Noether theorem implies from (17) that \( \bar{J}^{(l,m)}(x, t) \) (38) are the relevant Noether currents corresponding to the symmetry of the \( \mathfrak{W}_\infty \) action (18) under arbitrary right group \( SL(\infty; \mathbb{R}) \) translations. Clearly, \( \bar{J}^{(l,m)}(x, t) \) are \( \mathfrak{sl}(\infty; \mathbb{R}) \)-valued and are conserved w.r.t. the “time-evolution” parameter \( x \equiv x^{-} : \)

\[
\partial_x \bar{J}^{(l,m)}(x, t) \bigg|_{\text{on-shell}} = 0
\]
Substituting the on-shell expression (36) into (38) we get:

$$\bar{J}^{(l,m)}(x, t) \bigg|_{\text{on-shell}} = \sum_{l=1}^{\infty} \sum_{|m| \leq l} K^{(l,m)(l',m')} J^{(l',m')}(t)$$

(40)

where $K^{(l,m)(l',m')}$ is a constant invariant symmetric $sl(\infty; \mathbb{R})$ tensor:

$$K^{(l,m)(l',m')} = \sum_{r=1}^{\infty} (-\partial_x)^{r-1} \left\{ \left( \zeta^{(l',m')} \right)_r \text{Res}_\xi \left[ \ln \xi, \zeta^{(l,m)} \right] - \frac{1}{r+1} \partial_x \left( \zeta^{(l,m)} \circ \zeta^{(l',m')} \right)_r \right\}$$

(41)

naturally representing the Killing metric of $sl(\infty; \mathbb{R})$.

The fact that the currents $J^{(l,m)}(t)$ in (36) generate a $sl(\infty; \mathbb{R})$ Kac-Moody algebra, can be shown most easily by considering infinitesimal right group translation $g \rightarrow g \circ (\mathbb{1} + \zeta_t)$ with $\zeta_t = \sum_{l,m} \varepsilon^{(l,m)}(t) \zeta^{(l,m)}(\xi, x) \in sl(\infty; \mathbb{R})$ on $Y_t(g^{-1}) \equiv -g^{-1} \circ \partial_t g$. Recall (cf. (37), (40)), that $J^{(l,m)}(t)$ are the corresponding Noether symmetry currents. From the cocycle property (12) one obtains:

$$\delta_{\zeta_t} Y_t(g^{-1}) \equiv Y_t \left( (\mathbb{1} - \zeta_t) \circ g^{-1} \right) - Y_t(g^{-1}) = -\partial_t \zeta_t + \left[ Y_t(g^{-1}), \zeta_t \right]$$

(42)

which upon substitution of (36) yields:

$$\delta_{\zeta_t} J^{(l,m)}(t) = -\partial_t \varepsilon^{(l,m)}(t) + f^{(l,m)(l',m')} J^{(l',m')}(t) \varepsilon^{(l',m')}(t)$$

(43)

Here $f^{(l,m)(l',m')}$ denote the the structure constants of $sl(\infty; \mathbb{R})$ in the basis $\zeta^{(l,m)}$ (i.e., $[\zeta^{(l,m)}, \zeta^{(l',m')}]) = f^{(l,m)(l',m')} \zeta^{(l',m')} \right)$.

Finally, let us also show that the canonical Noether energy-momentum tensor $T_{++}$ (the Noether current corresponding to the symmetry of the $W_\infty$ action (18) under arbitrary rescaling of $t \equiv x^+$) autohmatically has the (classical) Sugawara form in terms of the “hidden” $sl(\infty; \mathbb{R})$ Kac-Moody currents $J^{(l,m)}(t)$ (36). Indeed, the variation of (18) under a reparametrization $t \rightarrow t + \rho(t, x)$ reads:

$$\delta_{\rho} W[g] = -\frac{1}{4\pi} \int dt \, dx \, \partial_x \rho(t, x) \, T_{++}(t, x)$$

(44)

$$T_{++} \equiv \frac{1}{2e} \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} \partial_x^r \left\{ (Y_t(g^{-1}) \circ Y_t(g^{-1}))_r + \frac{1}{r!} \text{Res}_\xi \left( \partial_x^{r+1} Y_t(g^{-1}) \circ \left[ \ln \xi, Y_t(g^{-1}) \right] \right) \right\}$$

(45)

Substituting (36) into (43) and accounting for (37), one easily gets the $sl(\infty; \mathbb{R})$ Sugawara representation of the energy-momentum tensor (13):

$$T_{++}(t, x) \bigg|_{\text{on-shell}} = \frac{1}{2e} \sum_{(l,m),(l',m')} K^{(l,m)(l',m')} J^{(l,m)}(t) J^{(l',m')}(t)$$

(46)

where $K^{(l,m)(l',m')}$ is the $sl(\infty; \mathbb{R})$ Killing metric tensor (11).

In particular, substituting into (15) the restriction of $g(\xi, x; t)$ to the Virasoro subgroup via (14)-(15), we recover the well-known (classical) $sl(2; \mathbb{R})$ Sugawara form of $T_{++}$ in $D = 2$ induced gravity (26).
5. Conclusions and Outlook

According to the general discussion in [15], the Legendre transform $\Gamma[y] = -W[g^{-1}]$ of the induced $W_\infty$-gravity WZNW action (18) is the generating functional, when considered as a functional of $y \equiv Y_t(g^{-1})$, of the quantum correlation functions of generalized Schwarzians $S[g]$. Similarly, $W[J] \equiv -W_{DOP(S^1)}[g]$, when considered as a functional of $J \equiv -\frac{c}{4\pi}S[g]$, is the generating functional of all correlation functions of the currents $Y_t(g^{-1})$. These correlation functions can be straightforwardly obtained, recursively in $N$, from the functional differential equations (i.e., Ward identities):

$$\partial_t \frac{\delta \Gamma}{\delta y} + \left[ \frac{\delta \Gamma}{\delta y} - \frac{c}{4\pi} \ln \xi, y \right] = 0 \ ; \ \partial_t J + \left[ \frac{\delta W}{\delta J}, J - \frac{c}{4\pi} \ln \xi \right] = 0$$ (47)

An interesting problem is to derive the $W_\infty$ analogue of the Knizhnik-Zamolodchikov equations [27] for the correlation functions $\langle g(\xi_1, x_1; t_1) \cdots g(\xi_N, x_N; t_N) \rangle$. To this end we need the explicit form of the symbol $r((\xi, x); (\xi', x')) \in DOP(S^1) \otimes DOP(S^1)$ of the classical $r$-matrix of $W_\infty$. This issue will be dealt with in a forthcoming paper.

Another basic mathematical problem is the study of the complete classification of the coadjoint orbits of $DOP(S^1)$ and the classification of its highest weight irreducible representations.

Let us note that, in order to obtain the WZNW action of induced $W_{1+\infty}$ gravity along the lines of the present approach, one should start with the algebra of differential operator symbols containing a nontrivial zero order term in the $\xi$-expansion $X = X_0(x) + \sum_{k \geq 1} \xi^k X_k(x)$. In this case one can solve the “hidden” symmetry (i.e. the “anomaly” free subalgebra) equation $[\ln \xi, \xi] = 0$ and the result is the Borel subalgebra of $gl(\infty; \mathbb{R})$ spanned by the symbols $\zeta^{(p,q)} = \xi^p x^q$ with $p \geq q$. The $W_{1+\infty}$ WZNW action will have formally the same form as (18), however, now the meaning of the symbol $g^{-1}(\xi, x; t)$ of the inverse group element is obscure due to the nontrivial ($((\xi, x)$-dependent) zero order term in the $\xi$-expansion of $g(\xi, x; t)$.

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