Conjugacy Limits of the Cartan Subgroup in $SL_3(\mathbb{R})$

Arielle Leitner

Abstract

A limit group is the limit under a sequence of conjugations of the diagonal Cartan subgroup, $C \leq SL_3(\mathbb{R})$. We show $C$ has 5 possible limit groups, up to conjugacy. Each limit group is determined by an equivalence class of nonstandard triangle, and we give a criterion for a sequence of conjugates of $C$ to converge to each of the 5 limit groups.

1 Background and the Main Theorem

Let $G$ be a Lie group and $H$ a closed subgroup. A sequence of subgroups $H_n$ of $G$ converges to $H$ if the following two conditions are satisfied:

(a) For every $h \in H$ there is a sequence $h_n \in H_n$ converging to $h$
(b) For every sequence $h_n \in H_n$, if there is a subsequence which converges to $h$, then $h \in H$.

A subgroup $L \leq G$ is a conjugacy limit of a subgroup $H$ if there is a sequence of conjugating matrices $P_n$ such that $P_nHP_n^{-1}$ converges to $L$. Let $C$ be the Cartan subgroup of positive diagonal matrices in $SL_3(\mathbb{R})$. A conjugate of $C$ is the stabilizer of the vertices of a triangle in $\mathbb{R}P^2$. We show that a conjugacy limit of $C$ is the shadow of the stabilizer of a nonstandard triangle.

Theorem 1. 1. There are five distinct conjugacy classes of subgroups in $SL_3(\mathbb{R})$ isomorphic to $\mathbb{R}^2$:

$$
C = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}, \quad F = \begin{pmatrix} a & t & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{at} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 1 & s & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & s & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}
$$

where $a, b \in \mathbb{R}_{>0}$ and $s, t \in \mathbb{R}$.

2. A nonstandard triangle in $*\mathbb{R}P^2$ determines a limit group. Let $\theta$ be the largest infinitesimal angle, and $h$ the longest side. The groups are determined as follows:

| # infinitesimal angles | # infinitesimal sides | $\frac{h}{\theta}$ | nonstandard triangle |
|------------------------|-----------------------|---------------------|---------------------|
| 0                      | 0                     |                     | $C$                 |
| 1                      | 1                     | finite              | $F$                 |
| 2                      | 3                     | infinite            | $N_1$               |
| 0, 1, or 2             | 3                     | infinitesimal       | $N_2$               |
| 2                      | 0, 1, or 3            |                     | $N_3$               |

There is a bijection from conjugacy limits of $C$ to characteristic degenerate triangle configurations.

3. Every subgroup of $SL_3(\mathbb{R})$ isomorphic to $\mathbb{R}^2$ is a 1-parameter limit of $C$. There is a graph of limits, where an edge denotes that one group is the limit of another.
A configuration is a set with elements which are points and lines in $\mathbb{R}P^2$. A configuration, $T$, is a limit of a configuration, $S$, if there is a sequence of projective transformations, $P_n$, such that every element of $T$ is the limit under $P_n$ of an element of $S$, in the Hausdorff topology. We write $P_nS \to T$. A projective triangle consists of three points and three line segments connected in the usual way. A triangle configuration in $\mathbb{R}P^2$ consists of three lines in general position, and their three intersection points, obtained by extending the lines of a projective triangle in the natural way. A degenerate triangle configuration is the limit of images of a triangle configuration under a sequence of projective transformations. It has at most three points and three lines.

![Figure 1: The 5 characteristic degenerate triangle configurations](image)

We say a configuration, $C$, is maximal for a group, $G \leq SL_3(\mathbb{R})$, if $G$ acts on $\mathbb{R}P^2$ so that each element of $C$ is mapped to itself, and $G$ preserves no additional points or lines. A degenerate triangle configuration, $T$, is characteristic for $G$, if $G$ maps each point or line of $T$ to itself, and $G$ preserves no degenerate triangle configuration that has more points or lines than $T$. In configuration $N_3$, three points on the line are fixed, and any group which acts on $\mathbb{R}P^2$ and fixes three points on a projective line must fix every point on the line. In configuration $N_2$, three lines in the link of the vertex are preserved, and similarly, every line in the link is preserved. Under this interpretation, figure 1 shows the configuration which is maximal for each group.

In the next part of the paper, we find all 2 dimensional abelian subgroups of $SL_3(\mathbb{R})$, which will complete the proof of the first part of theorem 1. Next we give some background on hyperreal numbers, and explain the lower dimensional case, $SL_2(\mathbb{R})$. In proposition 18 we prove the second part of theorem 1 and show each characteristic degenerate triangle configuration is the shadow of a nonstandard triangle configuration. In section 4 we show each group is a 1-parameter limit, and in section 5 we prove the third part of theorem 1.

Most of the work in this paper follows Haettel, [7], which classifies the homogeneous space of diagonal Cartan subgroups of a group $G$ with the Chabauty topology. Haettel determines the compactification of the set of all closed connected abelian subalgebras of dimension the real rank of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$. This paper follows his work, but from the perspective of Lie groups, and introduces the geometric notions of characteristic degenerate triangles, nonstandard triangles, and a maximal configuration preserved by a group.
1.1 Classification of Abelian Subgroups of $SL_3(\mathbb{R})$ Isomorphic to $\mathbb{R}^2$

**Lemma 2.** Every limit of the $(n-1$ dimensional $)$diagonal Cartan subgroup in $SL_n(\mathbb{R})$ is $n-1$ dimensional.

**Proof.** Work in $GL_n(\mathbb{R})$, and consider the closure of the diagonal group in $M_n(\mathbb{R}) \cong \text{End}(\mathbb{R}^n)$, which is a vector space of 1 larger dimension, isomorphic to $\mathbb{R}^n$. Conjugates of $\mathbb{R}^n$ are isomorphic to $\mathbb{R}^n$, and limits of $\mathbb{R}^n$ are $\mathbb{R}^n$. □

**Remark 3.** We say two representations $\rho_1, \rho_2 \in \text{Hom}(\mathbb{R}^2, SL_3(\mathbb{R}))$ are equivalent if $\rho_1 = \tau \rho_2 \phi$, where $\phi \in \text{Aut}(\mathbb{R}^2)$ and $\tau \in \text{Inn}(SL_3(\mathbb{R}))$.

**Proof.** (1 of theorem 1.) Haettel gives eight 2-dimensional abelian subalgebras of $\mathfrak{s}(\mathbb{R})$ up to conjugacy in $B_0$, the subgroup of the Borel group with positive diagonal. (7, Corollary 5.2.)

$${\mathcal{C}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a - b \end{pmatrix}, \ i_\alpha = \begin{pmatrix} a & t & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix}, \ i_\beta = \begin{pmatrix} -2a & 0 & 0 \\ 0 & a & t \\ 0 & 0 & a \end{pmatrix}, \ i_{\alpha + \beta} = \begin{pmatrix} a & 0 & t \\ 0 & -2a & 0 \\ 0 & 0 & a \end{pmatrix},$$

$$i_{[0:0]} = \begin{pmatrix} 0 & s & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ i_{[0:1]} = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}, \ i_{[1:1]} = \begin{pmatrix} 0 & s & t \\ 0 & 0 & -s \\ 0 & 0 & 0 \end{pmatrix}.$$

Exponentiating Haettel’s 8 algebras and expanding to conjugacy in all of $SL_3(\mathbb{R})$, reduces to the five cases above, since $i_\alpha, i_\beta$ and $i_{\alpha + \beta}$ are conjugate, and $i_{[1:1]}$ is conjugate to $i_{[1:-1]}$. □

1.2 The Hyperreals, $^*\mathbb{R}$

We will work with the hyperreal numbers, $^*\mathbb{R}$. The hyperreal numbers are a non-Archimedian field, with $\mathbb{R} \subset ^*\mathbb{R}$. Elements of $^*\mathbb{R}$ are equivalence classes of sequences of real numbers. For a good introduction to the hyperreals, see [6].

**Definition 4.** 1. Fix $\mathcal{F}$ a non principal ultra filter on $\mathbb{N}$. Let $\alpha = (a_n)$ and $\beta = (b_n)$ be sequences of real numbers. Define an equivalence relation $\sim$ on sequences of real numbers by $(a_n) \sim (b_n)$ if $\{n \in \mathbb{N} : b_n = a_n\} \in \mathcal{F}$. If $\sim$ holds, we say that as hyperreal numbers $\alpha = \beta$.

2. A hyperreal number $\alpha$ is infinite if $|\alpha| > n$ for every $n \in \mathbb{N}$. A hyperreal number $\alpha$ is infinitesimal if $|\alpha| < \frac{1}{n}$ for every $n \in \mathbb{N}$. If $\alpha$ not infinite, $\alpha$ is finite, and denote the set of finite hyperreals by $\mathcal{F}$.

3. A hyperreal $\alpha \in ^*\mathbb{R}$ is appreciable if $\alpha$ is neither infinitesimal or infinite, i.e., $\alpha$ is a non-zero real number plus an infinitesimal.

4. Any hyperreal $\alpha = r + \beta$ where $r$ is appreciable and $\beta$ is infinite. The finite part of $\alpha$ is $\text{Fin}(\alpha) = r$.

5. Let $\alpha \in ^*\mathbb{R}$ be a finite hyperreal. The standard part or shadow of $\alpha$ is $\text{sh}(\alpha) \in \mathbb{R}$, where $\alpha - \text{sh}(\alpha)$ is infinitesimal. Note that $\text{sh} : \mathcal{F} \to \mathbb{R}$ is a ring homomorphism.

6. The galaxy of $\alpha \in ^*\mathbb{R}$ is $\mathcal{G}(\alpha) = \{x \in ^*\mathbb{R} : \alpha - x \in \mathcal{F}\}$.

7. The $\varepsilon$-galaxy of $\alpha$ in $^*\mathbb{R}$ is $\mathcal{G}_\varepsilon(\alpha) = \{x \in ^*\mathbb{R} : \alpha - x \in \varepsilon \cdot \mathcal{F}\}$.

One can check (or see [6]) that these are equivalence relations. We denote nonstandard objects in script $\mathcal{G}$ or $\mathcal{N}$, and denote their standardizations $G$ or $N$. We discuss $\mathbb{R}^n$ and $^*\mathbb{R}^n$ with the usual inner product.

**Definition 5.** 1. A projective basis for $\mathbb{R}P^n$ (or $^*\mathbb{R}P^n$) consists of $n + 1$ equivalence classes of vectors, such that any $n + 1$ vectors form a basis for the underlying vector space. We will write basis when we mean projective basis, and use the symbol $\mathcal{B}$ for both. The meaning should be clear from the context.
2. The usual basis for $^*\mathbb{R}^{n+1}$ (or $\mathbb{R}^{n+1}$) is $\{e_0, e_1, \ldots, e_n\}$. The usual basis for $^*\mathbb{R}^n$ (or $\mathbb{R}^n$) is $\{[e_0], [\ldots, [e_0 + e_1 + \ldots + e_n]\ldots]\}$.

3. The shadow map is $sh : ^*\mathbb{R}^n \to \mathbb{R}^n$ where $v \mapsto sh(\frac{v}{||v||})$, and we take the shadows of the coordinates. The shadow of a basis $B \subset ^*\mathbb{R}^n$ is $sh(B) = \{sh(v) | v \in B\}$.

4. A hyperreal projective basis $B$ is appreciable if $sh(B)$ is a projective basis for $\mathbb{R}^n$. A hyperreal projective transformation is finite if the image of every appreciable basis is an appreciable basis.

**Definition 6.** Given $\mathcal{G} \leq SL_n(^*\mathbb{R})$, the finite part, $\text{Fin}(\mathcal{G})$, is the subset of all elements that have finite entries. The subset of infinitesimal elements, $\mathcal{I}$, is the set of matrices that are the identity matrix plus a matrix with infinitesimal entries.

**Lemma 7.** $\text{Fin}(\mathcal{G})$ and $\mathcal{I}$ are subgroups of $\mathcal{G}$.

**Proof.** Let $A \in \text{Fin}(\mathcal{G}) \leq SL_n(^*\mathbb{R})$. Since $\det A = 1$, and the adjoint formula is finite, $A^{-1} \in \text{Fin}(\mathcal{G})$. Sums and products of finite hyperreals are finite, so $\text{Fin}(\mathcal{G})$ is closed under multiplication. Thus $\text{Fin}(\mathcal{G})$ is a group. The proof that $\mathcal{I}$ is a group is analogous.

**Definition 8.** Given $\mathcal{G} \leq SL_n(^*\mathbb{R})$ and $\mathcal{A} \in \text{Fin}(\mathcal{G})$, the shadow of $\mathcal{A}$, $sh(\mathcal{A})$, has entries that are the shadows of entries of $\mathcal{A}$. The standard part or shadow of $\mathcal{G}$ is $sh(\mathcal{G}) := \{sh(\mathcal{A}) | \mathcal{A} \in \text{Fin}(\mathcal{G})\}$.

**Lemma 9.** The $sh(\text{Fin}(\mathcal{G})) \cong \text{Fin}(\mathcal{G})/\mathcal{I}$.

**Proof.** Note $sh : \text{Fin}(\mathcal{G}) \to sh(\text{Fin}(\mathcal{G}))$ is a homomorphism, since the kernel, $\mathcal{I}$, is a normal subgroup. The homomorphism is surjective since $\mathbb{R} \to ^*\mathbb{R}$.

The inclusion $\otimes_{\mathbb{R}} : \mathbb{R} \hookrightarrow ^*\mathbb{R}$ is a faithful functor from the category of real vector spaces and invertible linear transformations, to the category of hyperreal vector spaces and invertible hyperreal linear transformations. In general, $\otimes_{\mathbb{R}} : \mathbb{R} \hookrightarrow ^*\mathbb{R}$ is not surjective, since there are infinite hyperreal transformations. Let $P(\otimes_{\mathbb{R}})$ be the induced functor on projective vector spaces. Then $P(\otimes_{\mathbb{R}})$ is surjective on objects, and since any hyperreal projective transformation differs from a transformation in the image of $P(\otimes_{\mathbb{R}})$ by an infinitesimal transformation, the image of the set of real projective transformations under $P(\otimes_{\mathbb{R}})$ is dense in the set of hyperreal projective transformations. Therefore $P(\otimes_{\mathbb{R}})$ is essentially surjective.

Instead of writing this discussion in terms of matrices, we could have considered nonstandard projective transformations over nonstandard vector spaces, in the context of appreciable bases. Let $B$ be an appreciable basis. If $B'$ is another appreciable basis, then $B'$ is the image of $B$ under an appreciable projective transformation, so the entries of $\text{Fin}(\mathcal{G})$ are finite in $B'$. Thus an appreciable projective transformation maps any appreciable basis to another appreciable basis.

A nonstandard triangle consists of three line segments in general position in $^*\mathbb{R}P^2$, which intersect in three distinct points in the usual way. Let $\mathcal{P}$ be a matrix of hyperreals, and $\mathcal{I}$ be the nonstandard triangle which is the image of a projective triangle under $\mathcal{P}$. A conjugate of $C$ by $\mathcal{P}$ is uniquely determined as the connected component of the identity in the stabilizer of the vertices of $\mathcal{I}$. Every nonstandard triangle determines a nonstandard triangle configuration, $\mathcal{I}$, by extending the lines in the obvious way.

**Lemma 10.** The shadow of a nonstandard triangle configuration is the degenerate triangle configuration that is the (Hausdorff) limit of images of a projective triangle configuration under a sequence of projective transformations.

**Proof.** Recall $sh(\mathcal{I})$ is the shadow of the image of a projective triangle configuration under a nonstandard projective transformation, which is an equivalence class of a sequence of real projective transformations.

**Theorem 11.** Every limit group of the diagonal Cartan subgroup in $SL_3(\mathbb{R})$ is given by $sh(\text{Fin}(\mathcal{G}))$, where $\mathcal{G}$ be a conjugate of the diagonal Cartan subgroup by a hyperreal matrix.
Theorem 12. The Cartan subgroup $\{a \in \mathbb{R} : a \neq 0\}$ of $SL_2(\mathbb{R})$ has one conjugacy limit: $\{(\frac{1}{0}, 1)\}$ as $n \to \infty$. The Cartan subgroup preserves a maximal configuration consisting of two points and the parabolic group preserves a maximal configuration consisting of a projective line with one fixed point.

Proof. $SL_2(\mathbb{R})$ has three conjugacy classes of elements: elliptic, parabolic, and hyperbolic. (See [3].) Elliptic subgroups are isomorphic to $S^1$, so any subgroup isomorphic to $\mathbb{R}$ must be hyperbolic or parabolic.

The diagonal Cartan subgroup preserves the maximal configuration of 2 fixed points an appreciable distance apart on a projective line. Conjugate by the sequence of projective transformations as $n \to \infty$:

$$\begin{pmatrix} 1 & \frac{1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{n} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a \frac{n(a-\frac{1}{a})} {n} \\ 0 \frac{n} {a} \end{pmatrix} \to \begin{pmatrix} \frac{1}{0} \end{pmatrix}.$$ 

Since we want the limit to be finite, i.e., we want $n(a-\frac{1}{a})$ to converge to some $t \in \mathbb{R}$, we need $a \to 1$. Since $a \in \mathbb{R}$ is arbitrary, the limit is a group where $t$ is any real number. Applying this sequence of projective transformations identifies the points in the limit, so that the limit configuration consists of one point on a projective line, which is the maximal configuration preserved by the parabolic group.

In $^*\mathbb{R}P^1$, conjugate by a hyperreal transformation to change to the infinitesimal basis $\{[1 : 0], [1 : \delta]\}$:

$$\begin{pmatrix} \frac{1}{0} \delta \\ 0 \frac{1}{\delta} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} \frac{1}{0} \delta \\ 0 \frac{1}{\delta} \end{pmatrix}^{-1} = \begin{pmatrix} a \frac{1}{\delta} \frac{(a-\frac{1}{a})}{\delta} \\ 0 \frac{1}{\delta} \frac{1}{\delta} \end{pmatrix},$$

the finite part of this group has $a$ is infinitesimally close to 1, so that the upper right entry is finite. In the shadow of the finite part, $a = 1$, so we have the parabolic group.

Let $p, q \in ^*\mathbb{R}P^1$ be distinct points, and define $\mathcal{A}(p, q) = \{\mathcal{A} \in SL_2(\mathbb{R}) | \mathcal{A}p = p \text{ and } \mathcal{A}q = q\}$. Recall $SH(\mathcal{A}(p, q)) = \text{Fin}(\mathcal{A}(p, q))/\mathcal{A}$, where $\mathcal{A}$ is the subgroup of infinitesimal transformations. If $sh(p) \neq sh(q)$, then $G(p, q)$ is conjugate to a group of parabolic projective transformation since $G$ acts on $\mathbb{R}P^1$ with two fixed points. Alternatively, if $sh(p) = sh(q)$, then $G(p, q)$ is conjugate to a group of parabolic projective transformations since $G$ acts on $\mathbb{R}P^1$ with one fixed point.

Corollary 13. Suppose $p, q \in ^*\mathbb{R}P^1$, and let $G(p, q) = sh(\text{Fin}(\mathcal{A}(p, q))) \leq SL_2(\mathbb{R})$. If the distance between $p$ and $q$ is appreciable, $G(p, q)$ is hyperbolic, otherwise, $G(p, q)$ is parabolic.

3 Non-Standard Triangles

So far, we have shown there are 5 limit groups in the first part of theorem 1 and that each group is determined by a characteristic degenerate triangle configuration in section 1.2. In this section, we establish the bijection between limit groups and equivalence classes of nonstandard triangles, which will prove the second part of theorem 1.
The table in theorem [1] defines a partition on the set of nonstandard triangles, which may be built from a nonstandard 1-simplex. Consider \( {}^*\mathbb{R}P^2 \) with the positive scalar curvature metric inherited from the sphere. This metric is not preserved by projective transformations. Three points, \( p, q, x \in {}^*\mathbb{R}P^2 \), determine a nonstandard triangle, \( \Delta(p, q, x) \). Measure the distance from each point to the nonstandard line containing the other points. Let \( \eta \) be the smallest such distance, and \( x \) the point from which the distance \( \eta \) is measured. Let \( p, q \) be the other points of the triangle, \( \mathcal{M} \) the \( {}^*\mathbb{R}P^1 \) containing \( p \) and \( q \), and \( \delta = \text{dist}(p, q) \). Let \( y \in \mathcal{M} \) be the closest point to \( x \). We may assume \( y \) is closer to \( p \) than \( q \), and \( p = [1 : 0] \). Let \( \varepsilon \) be the distance from \( p \) to \( y \), so \( y = [1 : \varepsilon] \).

The link of \( x \), \( \mathcal{L}(x) \cong {}^*\mathbb{R}P^1 \), is the set of all lines through \( x \).

Let \( \mathcal{G} := \mathcal{G}(p, q) \), and \( G = \text{sh}(\text{Fin}(\mathcal{G})) \). Let \( \mathcal{N} \leq \text{Fin}(\mathcal{G}) \) be the subgroup preserving \( \mathcal{G}(x) \cap \mathcal{M} \). The action of \( \mathcal{G} \) on \( \mathcal{M} \) is slowed down if \( \text{sh}(\mathcal{N}) \subset \text{sh}(\mathcal{G}) \). A projective transformation is finite if it is finite with respect to the usual basis. Let \( \mathcal{X} \) be a nonstandard projective subspace, and extend the usual basis of \( \mathcal{X} \) to the usual basis of \( \mathcal{G} \). We say \( \mathcal{G} \) acts finitely on \( \mathcal{X} \), if \( g|_{\mathcal{X}} \) is a finite transformation, for all \( g \in \mathcal{G} \).

Let \( x \in {}^*\mathbb{R}P^n \). Given a projective basis for \( {}^*\mathbb{R}P^n \), \( \{e_0 = x, e_1, \ldots, e_n\} \), a projective basis for \( \mathcal{L}(x) \cong {}^*\mathbb{R}P^{n-1} \) consists of lines \( \{\pi e_i : 1 \leq i \leq n\} \). A basis, \( \mathcal{B} \), for \( \mathcal{L}(x) \) is appreciable if \( \text{sh}(\mathcal{B}) \) is a basis for \( L(x) := \text{sh}(\mathcal{L}(x)) \), or equivalently if the angles between the projective lines in the basis for \( \mathcal{L}(x) \) are appreciable. Let \( \mathcal{L} \leq SL_2({}^*\mathbb{R}) \) act on \( \mathcal{M} \) as before. The action of \( \mathcal{G} \) on \( \mathcal{L}(x) \) is defined as follows. In projective space, every pair of lines intersect in a point, so every line in \( \mathcal{L}(x) \) intersects \( \mathcal{M} \) in a point, \( z \). Thus \( \mathcal{L}(x) = \{\pi z : z \in \mathcal{M}\} \), and the action of \( \text{Fin}(\mathcal{G}) \) on \( \mathcal{L}(x) \) is given as \( \pi z \mapsto Tg(z) \), for \( g \in \mathcal{G} \). As before, we say \( \mathcal{G} \) acts finitely on \( \mathcal{L}(x) \) if the image of an appreciable basis is an appreciable basis.

**Lemma 14.** \( \text{Fin}(\mathcal{G}) \) preserves \( \mathcal{G}(y) \) if and only if \( \text{Fin}(\mathcal{G}) \) acts finitely on \( \mathcal{L}(x) \).

**Proof.** Let \( \{v_1 = [1 : \varepsilon + \eta], v_2 = [1 : \varepsilon - \eta], v_3 = [1 : \varepsilon]\} \), and set \( \mathcal{B} = \{\pi v_i : i = 1, 2, 3\} \), a basis for \( \mathcal{L}(x) \). \( \mathcal{B} \) is appreciable since the lines form a 45° isosceles triangle with \( \mathcal{M} \). The distance a point \( z \) is moved in \( \mathcal{M} \) is \( |z - g.z| \) for \( g \in \mathcal{G} \). The distance a point is moved in \( \mathcal{L}(x) \) is \( \Delta(\pi z, x(g.z)) \approx \|z - g.z\|_\eta \). Then \( \text{Fin}(\mathcal{G}) \) acts finitely on \( \mathcal{L}(x) \) if and only if \( \text{Fin}(\mathcal{G}) \) acts finitely on \( \mathcal{B} \), if and only if \( \text{Fin}(\mathcal{G}) \) preserves \( \mathcal{G}(y) \).

**Lemma 15.** \( \text{Fin}(\mathcal{G}) \) moves a point in \( \mathcal{G}(y) \) a distance of order \( \varepsilon \delta \).

**Proof.** First, we find \( \mathcal{G}(p, q) \) in the standard basis:

\[
\begin{pmatrix}
1 & 1 & 0 \\
\frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\
0 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix}^{-1}
= \begin{pmatrix}
\frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\
0 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix}.
\]

In \( \text{Fin}(\mathcal{G}) \), we have \( \frac{1}{a}(a - \frac{1}{a}) = 2t \), a finite hyperreal. The action on \( y = [1 : \varepsilon] \) depends on \( \eta, \delta, \varepsilon \), and we want to find the subgroup of \( \text{Fin}(\mathcal{G}) \) that preserves \( \mathcal{G}(y) \). We have:

\[
\begin{pmatrix}
\frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\
0 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix} [1 : \varepsilon] = [a + \varepsilon(\frac{1}{a} - \frac{1}{a}) : \frac{\varepsilon}{a} | = [1 : \frac{\varepsilon}{a + \frac{\varepsilon}{a}}].
\]

We want to find the distance \( y \) is moved. Remember \( \frac{1}{a} - a = 2t \delta \).

\[
|\frac{\varepsilon}{a + \frac{\varepsilon}{a}}| = \varepsilon |\frac{1}{a} - a + 2t \varepsilon| = 2t \varepsilon |\frac{\delta + \varepsilon}{a + 2t \varepsilon}| \approx 2t \varepsilon |\delta + \varepsilon| \approx \varepsilon |\delta + \varepsilon| \approx \varepsilon \delta
\]

The first approximation holds since \( a \approx 1 \), and the last one since \( \delta >> \varepsilon \). Thus \( y \) is moved a distance of order \( \varepsilon \delta \).

**Corollary 16.** \( \text{Fin}(\mathcal{G}) \) acts finitely on \( \mathcal{L}(x) \) if and only if \( \frac{\varepsilon}{\eta} \) is finite. Moreover, the action of \( \text{Fin}(\mathcal{G}) \) on \( \mathcal{L}(x) \) is infinite if \( \frac{\varepsilon}{\eta} \) is infinite, and \( \text{sh}(\text{Fin}(\mathcal{G})) \) acts as the identity on \( L(x) \) if \( \frac{\varepsilon}{\eta} \) is infinitesimal.
Proof. By lemma 15, the distance \( y \in G_\eta(x) \) is moved in \( \mathcal{H} \) is of order \( \varepsilon \delta \), and so by the proof of lemma 14 \( \text{Fin}(\mathcal{I}) \) moves points in \( \mathcal{L}(x) \) a distance of order \( \varepsilon \delta \). There is a homomorphism:

\[
\theta : \text{Fin}(\mathcal{I}) \to \text{Fin}(\mathcal{J}), \quad \text{where } \theta(A) = \varepsilon \delta \eta A.
\]

The image of \( \theta \) is finite if and only if \( \varepsilon \delta \eta \) is finite. If \( \varepsilon \delta \eta \) is infinitesimal, then \( sh(\text{Im}(\theta)) \) is the identity. \( \square \)

Lemma 17. Let \( \mathcal{I} \leq SL_4(\ast \mathbb{R}) \) act on \( \mathcal{H} \) as above. If \( \text{Fin}(\mathcal{I}) \) acts finitely on \( \mathcal{L}(x) \), define \( \mathcal{G} \leq SL_3(\ast \mathbb{R}) \) to be the set of elements which fix \( x \), act appreciably on \( \ast \mathbb{R} P^2 \), and for every \( \hat{g} \in \mathcal{I} \), \( \hat{g}|_{\mathcal{H}} = g \) for some \( g \in \mathcal{I} \). Then \( \hat{G} := sh(\text{Fin}(\mathcal{I})) \) is a limit group, and the action of \( \hat{G} \) on \( H \) coincides with the action of \( G \) on \( H \).

Proof. We pick a projective basis \( \mathcal{B} = \{ e_1, e_2, e_3, e_1 + e_2 + e_3 \} \) for \( \ast \mathbb{R} P^2 \), and show the image of \( \mathcal{B} \) under \( \mathcal{G} \) is an appreciable basis. \( \mathcal{I} \) fixes \( e_1 \) in \( \mathcal{H} \). Let \( y \in \mathcal{H} \) be the point closest to \( e_1 \). Choose \( e_3 \in \mathcal{F} \) an appreciable distance from \( e_1 \). Then \( \alpha \mathcal{F}1 \) and \( \alpha \mathcal{F}3 = \alpha \mathcal{F}y \) are at an appreciable angle, since \( x \) is infinitesimally close to \( \mathcal{H} \). Pick \( e_2 \in \mathcal{H} \) an appreciable distance from \( e_1, e_3 \), and such that \( \mathcal{F}x, \mathcal{F}y, \mathcal{F}3 \) are at an appreciable angle. Again, \( \mathcal{F}x \) and \( \mathcal{H} \) are at an appreciable angle, since \( x \) is infinitesimally close to \( \mathcal{H} \).

For \( g \in \text{Fin}(\mathcal{I}) \), \( g(e_2) \) is an appreciable distance from \( g(e_1) = e_1 \). Let \( \hat{g} \in \mathcal{I} \), so that \( \hat{g}|_{\mathcal{H}} = g \). Since \( \hat{g}(e_3) \) lies on the line \( \mathcal{F}x \mathcal{F}y \), there is a 1 parameter hyperreal family of choice of image of \( e_3 \). Choose \( \hat{g}(e_3) \) to be an appreciable distance from \( \hat{g}(e_1) = e_1 \). The action of \( \text{Fin}(\mathcal{I}) \) on \( \mathcal{L}(x) \) is finite, so \( \mathcal{F}x \mathcal{F}e_1 \) and \( \mathcal{F}y \mathcal{F}e_2 \) are an appreciable distance apart, for all \( i \neq j \). Thus the image of an appreciable basis is an appreciable basis, and \( \{ p, q, x \} \) is the image of the usual basis under a nonstandard projective transformation. By theorem 11 \( sh(\text{Fin}(\mathcal{I})) \) is a limit group, and by construction \( \mathcal{G}|_{\mathcal{H}} \) is isomorphic to a subgroup of \( \mathcal{G} \). \( \square \)

Proposition 18. Given \( p, q, x \in \ast \mathbb{R} P^2 \) in general position as above, there are 5 groups up to conjugacy, \( \hat{G} := sh(\text{Fin}(\mathcal{I}(p, q, x))) \leq SL_3(\mathbb{R}) \), such that \( \hat{G}|_{\mathcal{H}} \) is isomorphic to a subgroup of \( G = sh(\text{Fin}(\mathcal{I}(p, q))) \leq SL_2(\mathbb{R}) \).

1. If \( \text{dist}(p, q) = \delta \) is appreciable, and
   (a) \( \eta \) is appreciable, then \( \hat{G} \) is conjugate to \( C \)
   (b) \( \eta \) is infinitesimal, and \( p \) or \( q \in G_\eta(x) \), \( ( \varepsilon \eta \) is appreciable), then \( \hat{G} \) is conjugate to \( F \)
Remark 20. Given a hyperreal conjugating matrix

\[ \epsilon \delta \]

Thus the ratio

Remark 19. Two lines, so \( \hat{G} \) acts conjugately to \( N_2 \).

2. If \( \text{dist}(p, q) = \delta \), an infinitesimal, and

(a) \( \frac{\epsilon}{\eta} \) is infinitesimal, \( \hat{G} \) is conjugate to \( N_2 \).

(b) \( \frac{\epsilon}{\eta} \) is appreciable, \( \hat{G} \) is conjugate to \( N_1 \).

(c) \( \frac{\epsilon}{\eta} \) is infinite, \( \hat{G} \) is conjugate to \( N_3 \).

Proof. Case 1a: The points are an appreciable distance apart, and each point is an appreciable distance from the hyperplane containing the other two points. Since \( \eta \) is appreciable, \( \mathcal{G}_\eta(x) \cap \mathcal{H} = \mathcal{H} \), so the action of \( \text{Fin}(\mathcal{H}) \) preserves \( \mathcal{G}_\eta(x) \), and \( G \) acts finitely on \( L(x) \). The action of \( \hat{G} \) on each of the three lines is hyperbolic, so \( \hat{G} \) has three distinct weights. Thus \( \hat{G} \) is conjugate to \( C \).

Case 1b: Suppose \( q \in \mathcal{G}_\eta(x) \). Two of the points in the nonstandard triangle are infinitesimally close, and the third point is an appreciable distance away from the line containing the other two. So, \( \text{sh}(\Delta(p, q, x)) \) has two distinct points, and two distinct lines. The action of \( \hat{G} \) on \( \text{sh}(\Delta(p, q, x)) \) is parabolic, since \( \Delta(p, q, x) \) has a unique fixed point. The action of \( \hat{G} \) on \( H \) is hyperbolic, since \( H \) has two distinct fixed points. Thus \( \hat{G} \) has two 0 dimensional invariant subspaces, and two 1 dimensional invariant subspaces, one with a parabolic action, and one with a hyperbolic action. So, \( \hat{G} \) is conjugate to \( F \).

Case 1c: If \( \eta \) is infinitesimal, and \( \mathcal{G}_\eta(x) \) contains neither \( p \) or \( q \), the action of \( \text{Fin}(\mathcal{H}) \) on \( \mathcal{L}(x) \) is infinite by corollary 19. The subgroup of \( \text{Fin}(\mathcal{H}) \) which acts finitely on \( \mathcal{L}(x) \) is infinitesimal, and its shadow in \( G \) is the identity. Thus \( \hat{G} \) is conjugate to \( N_3 \), since this is the only limit group which acts as the identity on a line.

Case 2a: The points \( p, q, x \) are infinitesimally close, so \( \text{sh}(\Delta(p, q, x)) \) has one point, the fixed point under the action of \( G \). Thus \( \hat{G} \) acts parabolically on any line it preserves. Since \( \frac{\epsilon}{\eta} \) is infinitesimal, \( \hat{G} \) acts as the identity on \( L(x) \) by corollary 19 and \( \hat{G} \) preserves at least two 1 dimensional invariant subspaces. There are two groups with a single fixed point, \( N_1 \) and \( N_2 \). But, \( N_1 \) preserves only one line, and \( \hat{G} \) preserves at least two lines, so \( \hat{G} \) is conjugate to \( N_2 \).

Case 2b: If \( \frac{\epsilon}{\eta} \) is appreciable, the action of \( \text{Fin}(\mathcal{H}) \) on \( \mathcal{L}(x) \) is finite by corollary 19 and \( G \) does not fix \( L(x) \). By lemma 17, the action of \( \hat{G} \) on \( H \) coincides with the nontrivial action of \( G \) on \( H \). The points \( p, q, x \) are infinitesimally close, so \( \text{sh}(\Delta(p, q, x)) \) has one point. \( \hat{G} \) acts parabolically on \( H \), since \( \hat{G} \) fixes a single point. Since \( \hat{G} \) does not fix \( L(x) \), \( \hat{G} \) has only one 1 dimensional invariant subspace and one 0 dimensional invariant subspace, so \( \hat{G} \) is conjugate to \( N_1 \).

Case 2c: If \( \frac{\epsilon}{\eta} \) is infinite, then \( \mathcal{H} \) acts infinitely on \( \mathcal{L}(x) \) by corollary 19. The subgroup of \( \text{Fin}(\mathcal{H}) \) which acts infinitely on \( \mathcal{L}(x) \) is infinitesimal, and its shadow in \( G \) is the identity. Thus \( \hat{G} \) is conjugate to \( N_3 \), since this is the only limit group which acts as the identity on a line.

\[ \text{Remark 19. Recall that for small angles, } \sin(\theta) \approx \theta, \text{ so } \frac{\epsilon}{\eta} \text{ is approximately the largest infinitesimal angle. Thus the ratio } \frac{\epsilon}{\eta} \text{ is the ratio of the longest side to the largest infinitesimal angle, as in theorem 7.} \]

\[ \text{Remark 20. Given a hyperreal conjugating matrix } \mathcal{P} = \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}, \]

the columns of \( \mathcal{P} \) are the coordinates of the vertices of a nonstandard triangle. The equivalence class of nonstandard triangle determines the limit group \( \hat{G} = \text{Sh}(\text{Fin}(\mathcal{P} \mathcal{P}^{-1})) \), or in \( SL_3(\mathbb{R}) \), the conjugacy limit of \( H \) under the sequence \( P_n \), with sequences in the strictly upper triangular portion, \( u_n, v_n, w_n \), converging to \( u, v, w \). By proposition 18, \( \hat{G} \) depends only on the relative orders of \( u \) and \( \frac{w}{v} \).
4 1-Parameter Limits

In this section, we discuss theorem [1] from the standard (real) perspective. We show first that the limit group preserves the limit configuration, and secondly that we may always take an upper triangular sequence of conjugating matrices.

**Proposition 21.** Let $G$ be the diagonal Cartan subgroup of $SL_3(\mathbb{R})$, and $S \subset \mathbb{R}P^2$, a projective triangle configuration. Let $P_t \in PSL_3(\mathbb{R})$ be a sequence of projective transformations, such that $S$ converges under $P_t$ to $S_\infty = \lim_{t \to \infty} P_t S$, a degenerate triangle configuration, and $G$ has conjugacy limit $G_\infty = \lim_{t \to \infty} P_t G P_t^{-1}$. Then $G_\infty$ preserves $S_\infty$.

*Proof.* Let $G_t = P_t G P_t^{-1}$, and assume $G_t$ preserves $S_t$ for all $t$. Suppose for contradiction that $G_\infty$ does not preserve $S_\infty$. Then there is some $x \in S_\infty$, and $g \in G_\infty$ such that $gx \notin S_\infty$, or $d(gx, S_\infty) > 0$. Take $g_t \in G_t$ such that $\lim_{t \to \infty} g_t = g$. Pick a point $y \in \mathbb{R}P^2$ so that $\lim_{t \to \infty} P_t y = x$, thus $y \in S_t$ for some $t$. But then $\lim_{t \to \infty} d(g_t y, S_t) = d(g x, S_\infty) > 0$, contrary to $d(g y, S_t) = 0$. □

**Lemma 22.** Suppose $H$ is a subgroup of $SL_3(\mathbb{R})$, and $P_n$ is a sequence of conjugating matrices such that $P_n H P_n^{-1}$ has conjugacy limit $L \leq SL_3(\mathbb{R})$. There is a sequence of upper triangular matrices, $P'_n$ such that $P'_n H P'_n^{-1}$ converges to a conjugate of $L$.

*Proof.* Recall the Iwasawa decomposition of a matrix, $P = KNA$, where $K$ is orthogonal, $N$ is unipotent, and $A$ is diagonal. Writing each $P_n$ in this way, we have $P_n H P_n^{-1} = K_n (N_n A_n H A_n^{-1} N^{-1}_n) K^{-1}_n$. The orthogonal group is compact, so every sequence has a convergent subsequence, and in particular, every sequence $K_n H' K_n^{-1}$ converges to $H'$. Thus we may assume $P_n = N_n A_n$, or $P_n$ is upper triangular. □

**Definition 23.** If each element of the sequence of conjugating matrices lies in a one parameter subgroup of $SL_n(\mathbb{R})$, then $G$ is a limit under a 1-parameter path of conjugacies.

**Proposition 24.** Each limit group is a limit under a 1-parameter path of conjugacies.

*Proof.* Each limit group is a limit under the 1 parameter path of conjugacies (as $n \to \infty$) given below:

$$
F : \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
N_1 : \begin{pmatrix} 1 & n & n^2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix},
N_2 : \begin{pmatrix} 1 & n & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
N_3 : \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.
$$

The degenerate triangle configuration characteristic for each group (figure [1]) is the limit of a projective triangle configuration under the path given for the limit group in proposition [24].

**Remark 25.** $N_3$ fixes every point on a line, and $N_2$ preserves every line through a fixed point. $N_2$ and $N_3$ are dual configurations, and duality is an automorphism of the digraph of limits.

5 Computing the Digraph of Limit Groups

In this section, we finish the proof of part 3 of Theorem [4] $F$ limits to each of the nilpotent groups by the paths of conjugating matrices:

$$
N_1 : \begin{pmatrix} n & 0 & n^2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix},
N_2 : \begin{pmatrix} 1 & n & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
N_3 : \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.
$$

To understand the limit of a characteristic degenerate triangle configurations, view the sequence of projective transformations as conjugation by a hyperreal matrix in $SL_3(\mathbb{R})$, which rearranges the vertices of a
nonstandard triangle in $\mathbb{R}P^2$, and then take the shadow.

The group $N_1$ limits to both $N_2$ and $N_3$, along the paths

$$N_3 : \begin{pmatrix} \frac{1}{n^2} & n & 0 \\ 0 & \frac{1}{n} & n^2 \\ 0 & 0 & \frac{1}{n^2} \end{pmatrix} \quad \text{and} \quad N_2 : \begin{pmatrix} n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n \end{pmatrix}.$$

**Proposition 26.** If $A, B \leq G = SL_n(\mathbb{R})$ and $A$ is a limit of $B$, then the dimension of the normalizer must increase: $\dim N_G(A) \leq \dim N_G(B)$, with equality if and only if $A$ and $B$ are conjugate.

**Proof.** See [4] proposition 3.2. \qed

Proposition 26 implies that if $A \to B$ the dimension of the orbits of $B$ is smaller than the dimension of the orbits of $A$. In the limit $N_1 \to N_2$, the action on the link of the fixed point is slowed down, so every line through the fixed point is preserved, which is the maximal configuration for $N_2$. In the limit $N_1 \to N_3$, the action on the preserved line is slowed down, so every point on the line is fixed, which is the maximal configuration for $N_3$.

We have given a path of limits for every arrow in the digraph. Now we show these are all possible arrows. Denote by $\text{Char}(h)$ the subset of $\mathbb{R}[x]$ consisting of characteristic polynomials of all elements in $h$.

**Proposition 27.** Suppose $H$ is a subgroup of $GL_n(\mathbb{R})$, and $L$ is a conjugacy limit of $H$. Then $\text{Char}(l) \subset \text{Char}(h)$, where $h,l \subset gl(n)$ denote the Lie algebras of $H$ and $L$ respectively.

**Proof.** See [4] proposition 3.4. \qed

Since $N_2$ and $N_3$ are both normalized by the Borel group, if either $N_2$ or $N_3$ limits to the other, proposition 26 implies $N_2$ and $N_3$ are conjugate. Adding any additional arrows to the graph would create a loop, and since we are working in dimension 3, proposition 27 implies the groups in the loop are conjugate or dual. But $N_2$ and $N_3$ are the only dual groups, and since none of the groups are conjugate, the graph has all possible arrows. This concludes the proof of theorem 1.

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