Duo Gamma Modules and Full Stability

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Abstract
In this work we study gamma modules which are implying full stability or implying by full stability. A gamma module $M$ is fully stable if $\theta(N) \subseteq N$ for each gamma submodule $N$ of $M$ and each $R_\Gamma$- homomorphism $\theta$ of $N$ into $M$. Many properties and characterizations of these classes of gamma modules are considered. We extend some results from the module to the gamma module theories.

Keywords: Gamma modules, fully stable gamma modules, uniserial gamma modules, $\Gamma$-Hopfian and $\Gamma$-coHopfian gamma modules.

مقياس كاما الأثنائية والاستقرارة التامة

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الخلاصه
في هذا العمل ندرس مقاسات من نمط كاما والتي تكون تامة الاستقرار ونقهل عن انها تامة الاستقرار $M$ لكل مقاس جزئي $N$ من $M$ إذا كان $\theta(N) \subseteq N$ لكل $\theta$ من $\theta$. العديد من الصفات والخصائص ممكن اعتبارها من مقاسات كاما. نحن وسعتوا بعض النتائج من المقاسات إلى نظرية المقاسات من نمط كاما.

1- Introduction:
In 1964, Nobusawa introduced the idea of gamma rings as a generalization of the idea of rings [1]. In 1966, Barnes summed up this idea and obtained entirely fundamental properties of gamma rings [2].

Let $R$ and $\Gamma$ be two additive abelian groups. $R$ is called a $\Gamma$-ring if there is a mapping $R \times \Gamma \times R \to R,(r,\alpha,\beta) \to r\alpha \beta$ such that the followings hold:

(i) $(r_1 + r_2) \alpha r_3 = r_1 \alpha r_3 + r_2 \alpha r_3$,
(ii) $r_1(\alpha + \beta)r_2 = r_1 \alpha r_2 + r_1\beta r_2$,
(iii) $r_1\alpha(r_2 + r_3) = r_1 \alpha r_2 + r_1\alpha r_3$ and
(iv) $(r_2\alpha r_2)\beta r_3 = r_1\alpha(r_2\beta r_3)$, for all $r_1, r_2, r_3 \in R, \alpha, \beta \in \Gamma$.

In 2010, Ameri and Sadeqi extended the idea of modules to gamma modules [3].

Let $R$ be a $\Gamma$-ring. An additive abelian group $M$ is called a left $R_\Gamma$-module, if there exists a mapping : $R \times \Gamma \times M \to M, r\alpha m$ denote the image of $(r,\alpha, m)$ such that the followings hold:

(i) $r\alpha(m_1 + m_2) = r\alpha m_1 + r\alpha m_2$,
(ii) $(r_1 + r_2)\alpha m = r_1 \alpha m + r_2\alpha m$,
(iii) $\alpha(m_1 + m_2) = r\alpha m_1 + r\alpha m_2$ and
(iv) $r\alpha_1 (r_2 \alpha_2 m) = (r_1 \alpha_1 r_2) \alpha_2 m$, for all $m, m_1, m_2 \in M, \alpha, \alpha_1, \alpha_2 \in \Gamma$ and $r, r_1, r_2, \in R$.

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An $R_{\Gamma}$–module $M$ is called unitary if there is $1 \in R$, $\alpha_0 \in \Gamma$ such that $1\alpha_0 m = m$ for all $m \in M$.

A previous article provided more details of gamma modules [3].

In 1973, Faith introduced the definition of duo modules. Let $M$ be an $R$–module, a submodule $N$ of $M$ is said to be fully invariant if $\theta(N) \subseteq N$ for each $R$–endomorphism of $M$ [4]. In the case that each submodule of $M$ is fully invariant, then $M$ is called duo.

In 1991, Abbas studied the relationship between the fully stable modules and the duo modules; an $R$–module $M$ is fully stable if for each submodule $N$ of $M$, $\theta(N) \subseteq N$ for each $R$–endomorphism $\theta$ from $N$ into $M$ [5].

In this paper, we consider the duo property in the category of gamma modules. A left $R_{\Gamma}$–module $M$ is called duo if $\theta(N) \subseteq N$ for each $R_{\Gamma}$–submodule $N$ of $M$ and $R_{\Gamma}$–endomorphism of $M$. For an arbitrary fixed $\alpha \in \Gamma$, a subset $A$ of $R$ and a subset $L$ of $M$, we define:

$$l^\alpha_{R}(L) = \{ r \in R \mid r\alpha L = o \} \text{ and } y^\alpha_{M}(A) = \{ m \in M \mid \alpha m = o \} .$$

We give many properties and characterizations of this class of gamma modules. A left $R_{\Gamma}$–module $M$ is a duo if and only if every $\alpha$–cyclic $R_{\Gamma}$–submodule $R\alpha x$ of $M$ is fully invariant where $x \in M$. We study the relationship between the duo and the multiplication gamma modules, while every fully stable gamma module is duo and the converse is true in principally quasi-injective gamma modules. We consider direct summand and sum of duo gamma modules. Finally, we consider some generalizations of full stability which are related to the duo property.

2. Basics of duo gamma modules

Let $M$ be an $R_{\Gamma}$–module. An $R_{\Gamma}$–submodule $N$ of $M$ is called fully invariant if $f(N) \subseteq N$ for each $R_{\Gamma}$–endomorphism $f$ of $M$. In case that each $R_{\Gamma}$–submodule of $M$ is fully invariant, then $M$ is called a duo. Clearly, $(0)$ and $M$ are fully invariant $R_{\Gamma}$–submodules, and hence, simple $R_{\Gamma}$–modules are duo. Let $M$ be an $R_{\Gamma}$–module, $\alpha \in \Gamma$ an arbitrary fixed element and $m \in M$. Then the set $Ram = \{ r\alpha m \mid r \in R \}$ is an $R_{\Gamma}$–submodule of $M$ and it is called an $\alpha$–cyclic. It is easy to see that an $R_{\Gamma}$–module $M$ is a duo if and only if every $\alpha$–cyclic $R_{\Gamma}$–submodule of $M$ is fully invariant, that is for each $x \in M$ and $R_{\Gamma}$–endomorphism $\theta$ of $M$, there exists $r \in R$ such that $\theta(x) = r\alpha x$.

In general, $R_{\Gamma}$–submodules of duo gamma modules may not be duo. However, every direct summand of duo gamma modules is a duo, for if $K$ is an $R_{\Gamma}$–submodule of a direct summand $N$ of an $R_{\Gamma}$–module $M$ and $\theta$ is an $R_{\Gamma}$–endomorphism of $N$, then $\theta$ can be extended in the usual way to an $R_{\Gamma}$–endomorphism $\bar{\theta}$ of $M$, $\bar{\theta}(K) = \bar{\theta}(K) \subseteq K$.

It is clear that any fully stable $R_{\Gamma}$–module is a duo, but the converse is not true generally. For example, the $Z_2$–module $Z$ is a duo, but not fully stable.

In the following, we consider conditions under which every gamma submodule of a duo module is a duo, as well as the homomorphic image, but first we introduce the following.

An $R_{\Gamma}$–module $M$ is said to be $\Gamma$–poorly injective, if each $R_{\Gamma}$–endomorphism of an $R_{\Gamma}$–submodule of $M$ can be extended to an $R_{\Gamma}$–endomorphism of $M$.

We call an $R_{\Gamma}$–module $M$ an $\Gamma$–quasi projective if, for any $R_{\Gamma}$–module $W$ and $R_{\Gamma}$–homomorphisms $f$, $g$: $M \rightarrow W$ with $f$ is surjective, there is an $R_{\Gamma}$–endomorphism $h$ of $M$ such that $g = fh$. Then we have the following.

Proposition (2.1): Let $M$ be a duo gamma module. Then:

i) If $M$ is $\Gamma$–poorly injective, then every gamma submodule of $M$ is a duo.

ii) If $M$ is $\Gamma$–quasi projective, then every $R_{\Gamma}$–homomorphic image of $M$ is a duo.

Proof (i): Let $K$ be an $R_{\Gamma}$–submodule of $M$, $\alpha$ an $R_{\Gamma}$–submodule of $K$, and $\theta$ an $R_{\Gamma}$–endomorphism of $K$. $\Gamma$–poor injectivity of $M$ implies that $\theta$ can be extended to an $R_{\Gamma}$–endomorphism $\bar{\theta}$ of $M$. Then $\bar{\theta}(N) = \bar{\theta}(N) \subseteq N$.

(ii): Let $K$ be an $R_{\Gamma}$–submodule of a duo $R_{\Gamma}$–module $M$, and $f$ be an $R_{\Gamma}$–endomorphism of $M/K$.

For each $R_{\Gamma}$–submodule $L/K$ of $M/K$ where $L$ is an $R_{\Gamma}$–submodule of $M$ containing $K$. $\Gamma$–quasi projectivity of $M$ implies that there is an $R_{\Gamma}$–endomorphism $g$ of $M$ such that $g(m + K) = f(m) + K$ for each $m$ in $M$. Duo property of $M$ implies that $g(L) \subseteq L$ and hence $f(L/K) \subseteq L/K$. This shows that $M/K$ is a duo.

An $R_{\Gamma}$–submodule $K$ is called countably $\alpha$–generated of an $R_{\Gamma}$–module $M$, where $\alpha$ is an arbitrary fixed element in $\Gamma$, if there is a countable subset $\{ K_i \mid i \in \mathbb{N} \}$ of $M$ such that $K = \sum_{i \in \mathbb{N}} R\alpha \eta_i$. Then we have the following result:
Proposition (2.2): Let $M$ be an $R_\Gamma$ -module in which every countably $\alpha$ -generated $R_\Gamma$ -submodule of $M$ is a duo. Then $M$ is a duo.

**Proof:** Suppose that $m$ is any element of $M$, $\alpha \in \Gamma$ and $f$ is an $R_\Gamma$ -endomorphism of $M$. Let $N = Rm + R\phi f(m) + R\phi f^2(m) + \ldots$. It is clear that $N$ is a countably $\alpha$ -generated $R_\Gamma$ -submodule of $M$ and $f \upharpoonright_{K'} K \to K$. So, $f(m) = r\alpha m$ for some $r \in R$. This implies that $M$ is a duo.

In the following, we show that there are a lot of gamma modules which are not duo. Let $S$ be a $\Gamma$ -ring. A nonempty subset $R$ of $S$ is called a $\Gamma$ -subring of $S$, if $R$ is itself a $\Gamma$ -ring.

**Proposition (2.3):** Let $R$ be a proper $\Gamma$ -subring of a $\Gamma$ -ring $S$. Then the $R_\Gamma$ -module $S$ is not a duo.

**Proof:** Let $t$ be any element of a $\Gamma$ -ring $S$ such that $t \notin R$. Then the mapping $f:S \to S$, defined by $f(a) = ta_{\alpha}a$ for all $a \in S$, is an $R_\Gamma$ -homomorphism. If $S$ is a duo, then $t = ta_{\alpha}a = f(1) \in R$, which is a contradiction.

In the following, we give some sources of duo gamma modules.

An $R_\Gamma$ -module $M$ is called a $\Gamma$ -multiplication if, for any $R_\Gamma$ -submodule $N$ of $M$, there exists a two-sided $\Gamma$ -ideal $I$ of $R$ such that $N = I\Gamma M$. It is easy to see that $N = [N: M] \Gamma M$ where $[N: M] = \{r \in R | r\Gamma M = 0\}$ [6].

**Proposition (2.4):** Every $\Gamma$ -multiplication $R_\Gamma$ -module is a duo.

**Proof:** If $N$ is an $R_\Gamma$ -submodule of an $\Gamma$ -multiplication $R_\Gamma$ -module $M$, then $N = I\Gamma M$ for some two-sided $\Gamma$ -ideal $I$ of $R$, and so for every $R_\Gamma$ -endomorphism $f$ of $M$, $f(N) = f(I\Gamma M) = I\Gamma f(M) \subseteq I\Gamma M = N$.

The converse of proposition (2.4) is not true generally.

Let $M$ be an $R_\Gamma$ -module and $\alpha$ be an arbitrary fixed element of $\Gamma$. A previous article [7] introduced the concept of the $\alpha$ -free gamma module. An $R_\Gamma$ -module $P$ is called an $\alpha$ -projective if $P$ is an $\alpha$ -direct summand of an $\alpha$ -free $R_\Gamma$ -module for an arbitrary fixed $\alpha \in \Gamma$. This is equivalent to saying that, for every $\alpha$ -generators set $\{x_i | i \in I\}$ of $P$, there exists a family $\{\phi_i | i \in I\}$ of $P^* = Hom_{R_\Gamma}(P, R)$, such that for each $x \in P$, $\phi_i(x) \neq 0$ for finitely many $i \in I$ and $x = \sum_{i \in I} \phi_i(x) x_i$.

With regard to these concepts, we have the following theorem.

**Theorem (2.5):** The followings are equivalent for an $\alpha$ -projective $R_\Gamma$ -module $M$ where $\alpha$ is an arbitrary fixed element in $\Gamma$.

1. $M$ is a duo.
2. $M$ is a $\Gamma$ -multiplication.
3. $M$ is a $\Gamma$ -ring.

**Proof:** Assume that $M$ is a duo and $N$ is an $R_\Gamma$ -submodule of $M$. $\alpha$ -projective of $M$ implies that, for every $\alpha$ -generators set $\{x_i | i \in I\}$ of $M$, there exists a family $\{\phi_i | i \in I\}$ of elements $\phi_i \in Hom(P, R)$, such that for every $m \in M$, $\phi_i(m) \neq 0$ for finitely many $i \in I$ and $m = \sum_{i \in I} \phi_i(m)x_i$. Let $A$ be the $\Gamma$ -ideal of $R$ in $A_\Gamma M$. Then $x = \sum_{i \in I} \phi_i(x) a x_i$, and hence $N \subseteq A_\Gamma M$.

For other inclusions, suppose that $x \in N$ and $m \in M$, define $\theta_a: R \to M$ by $\theta_a(r) = r \alpha m$ for all $r \in R$. Then $\theta_a \phi_i$ is an $R_\Gamma$ -endomorphism of $M$ and $\phi_i(x) a \alpha m = \theta_a(\phi_i(x)) \subseteq (\theta_a \phi_i)(R \alpha x) \subseteq R \alpha x \subseteq N$.

Since $M$ is a duo, then $A_\Gamma M \subseteq N$ and so $N = A_\Gamma M$.

A $\Gamma$ -ideal $I$ of a $\Gamma$ -ring $R$ is called a $\Gamma$ -idempotent if $I = I\Gamma I$. [8]. We call an $R_\Gamma$ -module $M$ as a $\Gamma$ -multiplication if for each $R_\Gamma$ -submodule $N$, there is a $\Gamma$ -idempotent $\Gamma$ -ideal $I$ of $R$, such that $N = I\Gamma M$. We define that a $\Gamma$ -ring $R$ is called regular if all its $\Gamma$ -ideals are $\Gamma$ -idempotent.

Then we have the following result:

**Corollary (2.6):** Let $M$ be an $\alpha$ -projective gamma module over a regular $\Gamma$ -ring $R$. Then the following statements are equivalent:

1. $M$ is a $\Gamma I \alpha$ -multiplication.
2. $M$ is fully stable.
3. $M$ is a duo.
4. $M$ is a $\Gamma$ -multiplication.

**Proof:** (1) $\Rightarrow$ (2) Let $N$ be an $R_\Gamma$ -submodule of $M$ and $\theta: N \to M$ an $R_\Gamma$ -homomorphism. By (1), there is a $\Gamma$ -idempotent $\Gamma$ -ideal $A$ of $R$ such that $N = A_\Gamma M$. Now, $\theta(N) = \theta(A_\Gamma M) = \theta(A_\Gamma A_\Gamma M) = A_\Gamma \theta(A_\Gamma M) = A_\Gamma \theta(N) \subseteq A_\Gamma M = N$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) follows from theorem (2.5).
A gamma module $M$ is called uniserial if, for all gamma submodules $K$ and $N$ of $M$, either $K \subseteq N$ or $N \subseteq K$ [9].

A $\Gamma -$ring $R$ is with a supper identity, if there is $1 \in R$ such that $ra1 = 1ar = r$ for all $r, a \in \Gamma$. And an $R_{\Gamma} -$module $M$ is supper unitary if there is $1 \in R$ such that $1am = m$ for all $m \in M$ and $a \in \Gamma$ [7].

**Proposition (2.7):** Let $R$ be a $\Gamma -$ring and $M$ a supper unitary $R_{\Gamma} -$module. If $M$ is a uniserial satisfying the a. c. c. on $\alpha -$cyclic $R_{\Gamma} -$submodules, then $M$ is a duo.

**Proof:** Let $m(\neq 0) \in M$ and $f$ an $R_{\Gamma} -$endomorphism of $M$. Suppose that $f(m) \not\in Ram$. Then $m \in Raff(m)$ and hence $m = raff(m)$ for some $r \in R$. It follows that $f^n(m) = f^n(raff(m)) = raff^{n+1}(m)$ for each positive integer $n$. Consider the a. c. c.:

$$Ram \subseteq Raff(m) \subseteq Raff^2(m) \subseteq \cdots$$

The hypothesis implies that there is a positive integer $n_0$ such that $Raff^t(m) = Raff^{t+1}(m)$, for all $t \geq n_0$ and there is $z \in R$ such that $f^{t+1}(m) = zaff^t(m) = f^t(zam)$. Hence $f(m) - zam \in \ker(f^t)$. If $Ram \subseteq \ker(f^t)$, then $f^t(m) = 0$ and hence $m = 0$ which is a contradiction. Thus $\ker(f^t) \not\subseteq Ram$ and hence $f(m) - zam \not\in Ram$. $f(m) \in Ram$ is a contradiction. Therefore $M$ is a duo.

It was previously proved [9] that a fully stable $R_{\Gamma} -$module $M$ satisfies for every pair of $R_{\Gamma} -$submodules $N_1, N_2$ of $M$ with $N_1 \cap N_2 = 0$. We have $\text{Hom}_{R_{\Gamma}}(N_1, N_2) = 0 = \text{Hom}_{R_{\Gamma}}(N_2, N_1)$, but the converse may not be true. However, the converse is true in case that $M$ is fully essential stable [9].

In the following Lemma we have the following:

**Lemma (2.8):** Let an $R_{\Gamma} -$module $M = N_1 \oplus N_2$ be a direct sum of $R_{\Gamma} -$submodules $N_1, N_2$. Then $N_1$ is a fully invariant if and only if $\text{Hom}_{R_{\Gamma}}(N_1, N_2) = 0$.

**Proof:** Denote $\rho_1$ (resp. $\rho_2$): $M \to N_1$ (resp. $N_2$) the canonical projection onto $N_1$ (resp. $N_2$) and $i_1$ (resp. $i_2$): $N_1$ (resp. $N_2$) $\to M$ denote the injection mapping of $N_1$ (resp. $N_2$). Suppose that $N_1$ is a fully invariant $R_{\Gamma} -$submodule of $M$ and $f: N_1 \to N_2$ is an $R_{\Gamma} -$homomorphism. Then $f' = i_2 \circ \rho_1$ is an $R_{\Gamma} -$endomorphism of , and hence $f(N_1) \subseteq N_1$, so that $f(N_1) \subseteq N_1 \cap N_2 = 0$. It follows that $f = 0$.

For any $R_{\Gamma} -$endomorphism $g$ of $M$, $g(N_1) \subseteq \rho_1 \circ i_2(N_1) + \rho_2 \circ i_1(N_1) = \rho_1 \circ g(N_1) \subseteq N_1$, because $\rho_2 \circ i_1 \in \text{Hom}_{R_{\Gamma}}(N_1, N_2) = 0$. It follows that $N_1$ is a fully invariant $R_{\Gamma} -$submodule $M$.

**Lemma (2.9):** Let an $R_{\Gamma} -$module $M = \oplus_{i \in I} M_i$ be a direct sum of $R_{\Gamma} -$submodules $M_i (i \in I)$ and $N$ be a fully invariant $R_{\Gamma} -$submodule of $M$. Then $N = \oplus_{i \in I}(N \cap M_i)$.

**Proof:** Suppose that $\rho_i: M \to M_i$ is the canonical projection for each $i \in I$, and that $j_i: M_i \to M$ is the injection, then $j_i \circ \rho_i: M \to M$, and hence $j_i \circ \rho_i(N) \subseteq N$ for each $j_i \in I$. It follows that $N \subseteq \oplus_{i \in I}j_i \circ \rho_i(N) \subseteq \oplus_{i \in I}(N \cap M_i) \subseteq N$ so that $N = \oplus_{i \in I}(N \cap M_i)$.

**Lemma (2.10):** Let an $R_{\Gamma} -$module $M = \oplus_{i \in I} M_i$ be a direct sum of $R_{\Gamma} -$submodules $M_i (i \in I)$, and it is supper unitary. Then the following statements are equivalent.

1. $R = l_{R_{\Gamma}}^2(m_i) + l_{R_{\Gamma}}^2(m_j)$ for all $m_i \in M_i, m_j \in M_j$ with $i \neq j \in I$.
2. $N = \oplus_{i \in I}(N \cap M_i)$ for every $\alpha -$cyclic $R_{\Gamma} -$submodules $N$ of $M$. Moreover, in this case $\text{Hom}(M_i, M_j) = 0$ for all distinct $i, j \in I$.

**Proof:** (1)$\Rightarrow$(2): Let $N$ be any $\alpha -$cyclic $R_{\Gamma} -$submodule of $M$, and $m \in N$. Then there exists a positive integer $n$, distinct elements $i_j \in I (1 \leq j \leq n)$, and elements $m_j \in M_{i_j}(1 \leq j \leq n)$, such that $m = m_1 + m_2 + \cdots + m_n$. For $n = 1$, then $m = m_1 \in N \cap M_{i_1}$, and hence $N = \oplus(N \cap M_{i_1})$ Suppose that $n \geq 2$. By the hypothesis, there exists elements $r, s \in R$, such that $1 = r + s, ram_1 = 0$ and $sam_m = 0$. Then:

- $sam = sam_1 + sam_2 + \cdots + sam_n = sam_1 + sam_2 + \cdots + sam_n$
- $sam_1 = sam_1 + sam_2 + \cdots + sam_{n-1} = 1am_1 - ram_1 + \cdots + sam_{n-1}$
- $sam = sam_1 + sam_2 + \cdots + sam_{n-1}$

Note that $sam_1 \in M_{i_1}(2 \leq j \leq n - 1)$ and $sam \in N$. By induction on $n, m_1 \in N \cap M_{i_1}$. Similarly $m_1 \in N \cap M_{i_1}(2 \leq j \leq n)$.
(2)⇒(1): Let \( i, j \) be distinct elements of \( I \), let \( x \in M_i \) and let \( y \in M_j \). If \( K = Ra(x + y) \), then \( K = \bigoplus_{i \in I} (K \cap M_i) \) and hence \( x + y \in (K \cap M_i) \). There exists \( a, b \in R \), such that \( x + y = aa(x + y) + ba(x + y) \), where \( aa(x + y) \in M_i \) and \( ba(x + y) \in M_j \). Then \( x = aa(x + y) \), so that:
\[
x = aax + aay \Rightarrow x - aax = aay \Rightarrow x(1 - aa1) = aay.
\]
So that \( x(1 - aa1) = 0 \) and \( aay = 0 \). Thus \( a1 \in l_{R_i}(y), (1 - aa1) \in l_{R_i}(x), 1 = (1 - aa1) + a1 \in l_{R_i}(x) + l_{R_i}(y) \).

Finally, let \( i, j \) be distinct elements of \( I \). Let \( f: M_i \to M_j \) be any \( R_i \)–homomorphism. Let \( n \in M_i \).
By (1), \( R = l_{R_i}(n) + l_{R_j}(f(n)) \) so that \( 1 = c + d \) for some \( c, d \in R \) in \( R \alpha \) in \( \Gamma \) with \( can = 0 \), \( daf(n) = 0 \). It follows that \( f(n) = caf(n) + daf(n) = f(caf) + f(daf) = 0 \), thus \( f = 0 \).

The following corollary follows from (2.10) and (2.9).

**Corollary (2.11):** Let a super unitary gamma module \( M = \bigoplus_{i \in I} M_i \) be a direct sum of gamma submodules \( M_i \) \( i \in I \), \( \alpha \in \Gamma \) be an arbitrary fixed element, and \( N \) be a fully invariant gamma submodule of \( M \). Then \( l_{R_i}(m_i) + l_{R_j}(m_j) = R \) for all \( m_i \in M_i, m_j \in M_j \) for all \( i \neq j \) in \( I \).

**Theorem (2.12):** Let an \( R_i \)–module \( M = \bigoplus_{i \in I} M_i \) be a direct sum of \( R_i \)–submodules \( M_i \) \( i \in I \). Then \( M \) is a duo \( R_i \)–module if and only if:
\[
\begin{align*}
(1) & \quad M_i \text{ is a duo gamma module for all } i \in I \text{ and} \\
(2) & \quad N = \bigoplus_{i \in I} (N \cap M_i) \text{ for every } R_i \text{–submodule } N \text{ of } M.
\end{align*}
\]

**Proof:** follows by Lemma (2.9).

\( \Leftarrow \) Suppose that \( M \) satisfies the above conditions. Let \( L \) be an \( R_i \)–submodule of \( M \) and \( f \) any \( R_i \)–endomorphism of \( M \). For \( i \in I \) let \( \rho_i: M \to M_i \) denotes the canonical projection and let \( i_i: M_i \to M \) denotes the injection. By (a), \( \rho_i \circ f \circ i_i \subseteq L \) for all \( i \in I \). Now (b) gives \( f(L) = \sum_{i \in I} f(L \cap M_i) \subseteq \sum_{i \in I} \rho_i \circ f \circ i_i \subseteq \sum_{i \in I} (L \cap M_i) \subseteq L \). Thus \( M \) is a duo.

**Corollary (2.13):** Let a super unitary gamma module \( M = \bigoplus_{i \in I} M_i \) be a direct sum of \( R_i \)–submodules \( M_i \) \( i \in I \). Then \( M \) is a duo gamma module if and only if \( M_i \bigoplus M_j \) is a duo gamma module for all \( i \neq j \) in \( I \).

**Proof:** The assumption that any direct summand of a duo gamma module is a duo proves the first direction.

Conversely, suppose that \( M_i \bigoplus M_j \) is a duo gamma module for all \( i \neq j \) in \( I \). Then \( M_i \) is a duo gamma module for all \( i \in I \). Furthermore, for all \( i \neq j \) in \( I \), \( R = l_{R_i}(m_i) + l_{R_j}(m_j) \) for all \( m_i \in M_i, m_j \in M_j \). By Lemma (2.9), Lemma (2.10) and Theorem (2.12), we get that \( M \) is a duo gamma module.

We introduce the following generalization of fully stable gamma modules.

An \( R_i \)–module \( M \) is called fully direct-summand stable (for short, fully ds-stable) if every direct summand of \( M \) is stable.

It is clear that a direct summand of a fully ds-stable is fully ds-stable.

**Theorem (2.14):** Let a gamma module \( M = \bigoplus_{i \in I} M_i \) be a direct sum of \( R_i \)–submodules \( M_i \) \( i \in I \). Then \( M \) is a fully ds-stable if and only if:
\[
\begin{align*}
(1) & \quad M_i \text{ is a fully ds-stable for all } i \in I, \\
(2) & \quad N = \bigoplus_{i \in I} (N \cap M_i) \text{ for every direct summand } N \text{ of } M.
\end{align*}
\]

**Proof:** Assume that \( M \) is a fully ds-stable \( R_i \)–module. Then, clearly, \( M_i \) is a fully ds-stable for all \( i \in I \) and hence we get (1). Lemma (2.9) gives (2).

Conversely, suppose that \( M \) satisfies the above conditions. Let \( L \) be a direct summand of \( M \) and \( g: L \to M \) an \( R_i \)–homomorphism. By (2), \( L = \bigoplus_{i \in I} (L \cap M_i) \) and from this we get \( g: \bigoplus_{i \in I} (L \cap M_i) \to \bigoplus_{i \in I} M_i \) for each \( i \) in \( I \). Let \( \rho_i: \bigoplus_{i \in I} M_i \to M_i \) denotes the canonical projection and let \( i_i: L \cap M_i \to L \) denotes the inclusion. Hence, \( \rho_i \circ g \circ i_i : L \cap M_i \to M_i \), by (1), \( \rho_i \circ g \circ i_i : L \cap M_i \subseteq L \cap M_i \) for all \( i \in I \). Now (2) gives \( g(L) = \sum_{i \in I} g(L \cap M_i) \subseteq \sum_{i \in I} \rho_i \circ g \circ i_i (L \cap M_i) \subseteq \sum_{i \in I} (L \cap M_i) \subseteq L \). Thus \( M \) is fully ds-stable.

An element \( r \neq 0 \) in \( R \) is called \( \Gamma \)--zero divisor if there exists \( \alpha \neq 0 \in \Gamma \) and \( s \neq 0 \in R \) such that \( ras = 0 \) [10]. Let \( M \) be an \( R_i \)–module, an element \( m \) in \( M \) is called a \( \Gamma \)--torsion if there is a non-zero divisor \( r \) in \( R \), and a non-zero element \( \alpha \in \Gamma \) such that \( r\alpha m = 0 \).

Denote the set of all \( \Gamma \)--torsion elements in \( M \) by \( T_{\Gamma}(M) \), if \( T_{\Gamma}(M) = M \) (resp. 0), then \( M \) is called \( \Gamma \)--torsion (resp. \( \Gamma \)--torsion free).

It is a matter of checking that \( T_{\Gamma}(M) \) is an \( R_i \)--submodule of \( M \).
In example, let $R$ be a $\Gamma -$ring, $M = \left\{ \left( \begin{array}{c} m_1 \\ m_2 \end{array} \right) ; m_1, m_2 \in R \right\}$ and $\Gamma = \left\{ \left( \begin{array}{c} \alpha \\ 0 \end{array} \right) ; \alpha \in R \right\}$, let $r = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \neq 0$, take $\alpha = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \neq 0$. Then $r \alpha \left( \begin{array}{c} m_1 \\ m_2 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ for any $\left( \begin{array}{c} m_1 \\ m_2 \end{array} \right) \in M$. So, $M$ is a $\Gamma -$torsion.

Lemma (2.15): Let a supper unitary $R_{\Gamma} -$module $M = M_1 \oplus M_2$ be a direct sum of a non-zero torsion free $R_{\Gamma} -$submodule $M_1$ and a non-zero $R_{\Gamma} -$submodule $M_2$. Then $M$ is not a duo gamma module.

Proof: Let $m_1$ and $m_2$ be non-zero elements of $M_1$ and $M_2$, respectively. Then $l_{R_{\Gamma}}(m_1) = 0$, hence $l_{R_{\Gamma}}(m_1) + l_{R_{\Gamma}}(m_2) = l_{R_{\Gamma}}(m_2) \neq R$. By Lemma (2.10) and Theorem (2.12), $M$ is not a duo gamma module.

Let $M$ be an $R_{\Gamma} -$module. An $R_{\Gamma} -$submodule $N$ of $M$ is called a $\Gamma -$essential if $N$ has a nontrivial intersection with every nonzero $R_{\Gamma} -$submodule of $M$ [10].

Dually, we say that an $R_{\Gamma} -$submodule $N$ of $M$ is small if $N + K$ is a proper $R_{\Gamma} -$submodule of $M$ for each proper $R_{\Gamma} -$submodule $K$ of $M$.

An $R_{\Gamma} -$module $M$ is called $\Gamma -$Hopfian (resp. generalized $\Gamma -$Hopfian) if every surjective $R_{\Gamma} -$endomorphism of $M$ is an isomorphism (resp. has a small kernel).

An $R_{\Gamma} -$module $M$ is called $\Gamma -$coHopfian (resp. weakly $\Gamma -$coHopfian) if every injective $R_{\Gamma} -$endomorphism of $M$ is an isomorphism (resp. has an $\Gamma -$essential image of $M$).

Proposition (2.16): Every fully stable gamma module is a $\Gamma -$coHopfian, and hence is a weakly $\Gamma -$coHopfian.

Proof: Let $M$ be a fully stable $R_{\Gamma} -$module and $f: M \rightarrow M$ be an $R_{\Gamma} -$monomorphism, then $M \cong f(M)$. Hence, we have $M = f(M)$ so that $f$ is an $R_{\Gamma} -$epimorphism. By Corollary (2.4) in a previous study [9], we have $M = f(M)$.

Proposition (2.17): Every duo gamma module is a $\Gamma -$Hopfian and a weakly $\Gamma -$coHopfian.

Proof: Let $f$ be any surjective $R_{\Gamma} -$endomorphism of $M$. Let $K \subseteq M$ such that $M = \text{ker}(f) + K$. Then $M = f(M) = f(\text{ker}(f) + K) = f(K) \subseteq K$. It follows that $\text{ker}(f)$ is a small $R_{\Gamma} -$submodule of $M$. Let $g$ be an injective $R_{\Gamma} -$endomorphism of $M$, let $N \subseteq M$ such that $N \cap g(M) = 0$. Since $N$ is fully invariant, we get $g(N) = 0$ and hence $N = 0$. It follows that $g(N)$ is an essential $R_{\Gamma} -$submodule of $M$.

Duo gamma modules are neither $\Gamma -$Hopfian nor $\Gamma -$coHopfian in general.

We have seen in a previous article [9] that $Z_{p^\infty}$ is a fully stable $Z_S-$module where $S$ is an arbitrary subring of $Z$. Let $s_0$ be an arbitrary fixed element in $S$. The mapping $f: Z_{p^\infty} \rightarrow Z_{p^\infty}$, defined by $f(x) = ps_0x$ for all $x$ in $Z_{p^\infty}$, is a surjective which is not an isomorphism, and hence $Z_{p^\infty}$ is a duo which is not a $\Gamma -$Hopfian. On the other hand, it is clear that $Z$ is a duo $Z_S-$module. We define $h: Z \rightarrow Z$ by $h(z) = 2s_0z$ for all $z \in Z$ is an injective which is not an isomorphism.

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