THREE CONCEPTS OF ROBUST EFFICIENCY FOR
UNCERTAIN MULTIOBJECTIVE OPTIMIZATION PROBLEMS
VIA SET ORDER RELATIONS

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Abstract. In this paper, we propose three concepts of robust efficiency for
uncertain multiobjective optimization problems by replacing set order rela-
tions with the minmax less order relation, the minmax certainly less order
relation and the minmax certainly nondominated order relation, respectively.
We make interpretations for these concepts and analyze the relations between
new concepts and the existent concepts of efficiency. Some examples are given
to illustrate main concepts and results.

1. Introduction. Optimization problems in most real world are affected by un-
certain data. A decision maker needs to address this important subject in order to
obtain some optimal solutions that remain feasible for an uncertain optimization
problem. Sometimes, it is very important to estimate the effects of the uncertainties
and so it is necessary to evaluate how sensitive an optimal solution is to pertur-
bations of the input data. Dealing with uncertain optimization problems yields
two basic approaches that are discussed in the literatures. In stochastic optimiza-
tion, the uncertain parameter is assumed to obey a probability distribution and the
objective is to find a feasible solution that optimizes the expected value of some ob-
jective or cost function. For an introduction to stochastic optimization, we refer to
Birge and Louveaux [3]. The other way is described by robust optimization which
is an active field of research. In the concept of robustness it is not assumed that all
data are known, but one allows different scenarios for the input various data and
looks for solutions that works well in every uncertain scenario. Robust optimization
has come to encompass several approaches to protect the decision maker who must
determine what it means for him or her to have a robust solution against parameter
ambiguity and stochastic uncertainty.

One of the first researchers to study robust optimization problems was Soyster
[23]. Minmax robustness called strict robustness was first mentioned in [23] and

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has been extensively investigated in [1]. In different situations robust solution sets were presented in [1, 15, 2, 25, 8, 9, 24, 19]. Recently, robust optimization has been becoming a focus of studying. In [16], it has shown that many different concepts of robustness and of stochastic programming were described by using a general nonlinear scalarization method, which leads to a unified concept to handle robust and stochastic optimization problems. Köbis [17] discussed the relation between an unconstrained multiobjective optimization problem and various well-known scalar robust optimization problems with a finite uncertainty set. Schöbel [22] investigated the trade-off between robustness and nominal quality and gave an approach called the generalized light robustness to compute non-dominated solutions. In all these papers, the uncertain optimization problem was replaced by a deterministic version, called the robust counterpart of the uncertain problem.

Kuroiwa and Lee firstly presented the concept of minmax robustness to multiobjective optimization in [21], who followed the approach of minimizing the vector of the worst case in each component. This definition has been extended in [4, 7], where the authors replaced the objective function by a set-valued objective function and provided some methods to compute the robust solutions of multiobjective optimization problems. Furthermore, a lot of different set order relations were proposed in [20, 14, 6, 13] and played an important role in set optimization. By replacing the set ordering with other set orderings, some new concepts of efficiency for uncertain multiobjective optimization problems were put forward in [10].

Based on the study of [4, 14, 10], in this paper, we propose three concepts called the minmax less ordered efficiency, the minmax certainly less ordered efficiency and the minmax certainly nondominated ordered efficiency for uncertain multiobjective optimization problems. We give interpretations for new concepts and some examples to illustrate them. We analyze the connections between new concepts and the existing concepts in [10] and use these concepts to the problem of tourist’s destination selection.

This paper is organized as follows: In Section 2, we introduce some preliminaries. In Section 3, we propose three concepts of efficiency for uncertain multiobjective optimization problems. In Section 4, we discuss the relationships between new concepts and the existent concepts of efficiency. An applied example is given to illustrate the usefulness of these concepts in Section 5. In the end, we give a short conclusion.

2. Preliminaries. Firstly, we introduce an uncertain multiobjective optimization problem. Given a set of feasible solutions \( X \subseteq \mathbb{R}^n \), the multiobjective optimization problem for an objective function \( f : X \rightarrow \mathbb{R}^k \) is given by

\[
\min_{x \in X} f(x).
\]

In order to minimize a vector valued function, we need to define the meaning of minimum on \( \mathbb{R}^k \). We use the relations \( \leq, \leq, < \) to compare vectors (see [5]). Let \( y_1, y_2 \in \mathbb{R}^k \), we define \( y_1 \leq y_2 \) if \( y_1 \) is smaller or equal to \( y_2 \) in every component, \( y_1 \leq y_2 \) if \( y_1 \) is smaller or equal to \( y_2 \) in every component and smaller in at least one component, and \( y_1 < y_2 \) if \( y_1 \) is smaller than \( y_2 \) in every component. Notice that this implies the equivalence of the relations \( \leq \) and \( < \) in \( \mathbb{R} \). Additionally, we define the sets \( \mathbb{R}_{\geq}^k, \mathbb{R}_{\geq}^k, \mathbb{R}_{>}^k \) as follows:

\[
\mathbb{R}_{[\geq / \geq / >]}^k := \{ x \in \mathbb{R}^k : x [\geq / \geq / >] 0 \}.
\]
Given a set of scenarios \( U \subseteq \mathbb{R}^m \) called uncertainty set, an uncertain optimization problem \( P(U) \) is given as the family \( P(\xi) (\xi \in U) \) of multiobjective optimization problems. Let \( f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^k \) and \( X \subseteq \mathbb{R}^n \), then \( P(\xi) \) is defined as the following

\[
\min_{x \in X} f(x, \xi).
\]

We use the notation

\[
f_U(x) := \{ f(x, \xi) : \xi \in U \} \subseteq \mathbb{R}^k.
\]

for the image of the uncertainty set \( U \) and all \( x \in X \) under \( f \).

Obviously for \( |U| = 1 \), \( P(U) \) reduces to a (deterministic) multiobjective optimization problem. We will use this special case to justify and compare our concepts and results.

We introduce the concept of robust efficiency (also called the upper set less ordered efficiency in [10]) for uncertain multiobjective optimization problems in [4, Definition 3.1]:

**Definition 2.1.** Given an uncertain multiobjective optimization problem \( P(U) \), we call a feasible solution \( \bar{x} \in X \)

(i) robust strictly efficient, if there is no \( x^0 \in X \setminus \{ \bar{x} \} \) such that \( f_U(x^0) \subseteq f_U(\bar{x}) - \mathbb{R}^k_\ge \);

(ii) robust efficient, if there is no \( x^0 \in X \setminus \{ \bar{x} \} \) such that \( f_U(x^0) \subseteq f_U(\bar{x}) - \mathbb{R}^k_\ge \);

(iii) robust weakly efficient, if there is no \( x^0 \in X \setminus \{ \bar{x} \} \) such that \( f_U(x^0) \subseteq f_U(\bar{x}) - \mathbb{R}^k_\ge \).

Or alternatively, a solution \( \bar{x} \in X \) is called upper set less ordered (strictly/weakly) efficient, if there is no \( x^0 \in X \setminus \{ \bar{x} \} \) such that \( f_U(x^0) \preceq_u f_U(\bar{x}) \) with respect to \( \mathbb{R}^k_{[\ge / \ge / >]} \), which is equivalent to

\[
\nexists x^0 \in X \setminus \{ \bar{x} \} : f_U(x^0) \subseteq f_U(\bar{x}) - \mathbb{R}^k_{[\ge / \ge / >]}.
\]

Let \( A \) be a nonempty subset of a partially ordered linear space \( Y := \mathbb{R}^k \) with a pointed cone \( \mathbb{R}^k_\ge \). We introduce the minimal element and the maximal element of the set \( A \) by using the definition given in [12, Definition 4.1], then the sets of minimal elements and maximal elements of \( A \) can be characterized as

\[
\min A := \{ \hat{a} \in A : A \cap (\hat{a} - \mathbb{R}^k_\ge) = \{ \hat{a} \} \},
\]

\[
\max A := \{ \check{a} \in A : A \cap (\check{a} + \mathbb{R}^k_\ge) = \{ \check{a} \} \}.
\]

We will use the following notations:

\[
\mathcal{P}(Y) := \{ A \subset Y : A \text{ is nonempty} \},
\]

\[
\mathcal{M} := \{ A \in \mathcal{P}(Y) : \min A \text{ and } \max A \text{ are nonempty} \},
\]

and for all \( x \in X \),

\[
\text{CMax}f_U(x) := \left( \sup_{\xi \in U} f_1(x, \xi), \ldots, \sup_{\xi \in U} f_k(x, \xi) \right)^T,
\]

\[
\text{CMin}f_U(x) := \left( \inf_{\xi \in U} f_1(x, \xi), \ldots, \inf_{\xi \in U} f_k(x, \xi) \right)^T,
\]

the latter is defined in [10, Lemma 31].
Now we recall the set less order relation, the set less ordered efficiency, the certainly less order relation and the certainly less ordered efficiency used in this paper:

**Definition 2.2.** [14] Let \( A, B \in \mathcal{P}(Y) \) be arbitrarily chosen sets. Then \( A \) is said to dominate \( B \) with respect to the set less order relation (we denote that by \( A \preceq_s B \)) and with respect to \( \mathbb{R}^k_{\geq / >} \) if \( A \subseteq B - \mathbb{R}^k_{\geq / >} \) and \( B \subseteq A + \mathbb{R}^k_{\geq / >} \).

**Definition 2.3.** [10] Given an uncertain multiobjective optimization problem \( P(U) \), a solution \( \bar{x} \in X \) is called set less ordered (strictly/weakly) efficient for \( P(U) \), if there is no \( x^0 \in X \setminus \{\bar{x}\} \) such that \( f_U(x^0) \preceq_s f_U(\bar{x}) \), which is equivalent to
\[
\not\exists x^0 \in X \setminus \{\bar{x}\} : f_U(x^0) \subseteq f_U(\bar{x}) - \mathbb{R}^k_{\geq / >} \quad \text{and} \quad f_U(\bar{x}) \subseteq f_U(x^0) + \mathbb{R}^k_{\geq / >}.
\]

**Definition 2.4.** [14] Let \( A, B \in \mathcal{P}(Y) \) be arbitrarily chosen sets. Then \( A \) is said to dominate \( B \) with respect to the certainly less order relation (we denote that by \( A \prec_c B \)) and with respect to \( \mathbb{R}^k_{\geq / >} \) if \((A = B) \) or \((A \neq B, B - A \subseteq \mathbb{R}^k_{\geq / >}) \).

**Definition 2.5.** [10] Given an uncertain multiobjective optimization problem \( P(U) \), a solution \( \bar{x} \in X \) to \( P(U) \) is called certainly less ordered (strictly/weakly) efficient, if there is no \( x^0 \in X \setminus \{\bar{x}\} \) such that
\[
\text{CMax}_U(x^0) \in \text{CMin}_U(\bar{x}) - \mathbb{R}^k_{\geq / >}.
\]

**Remark 1.** From Lemma 31 and Remark 32 of [10] we see that, for all \( \tilde{x}, \hat{x} \in X \),
\[
\text{CMax}_U(\tilde{x}) \in \text{CMin}_U(\hat{x}) - \mathbb{R}^k_{\geq / >} \quad \Rightarrow \quad f_U(\tilde{x}) \preceq_c f_U(\hat{x}) \quad \text{w.r.t} \quad \mathbb{R}^k_{\geq / >}.
\]

In addition, the lower set less ordered efficiency and the alternative set less ordered efficiency were defined in [10, Definitions 9 and 26], which are given in the following:

**Definition 2.6.** [10] Given an uncertain multiobjective optimization problem \( P(U) \), a solution \( \bar{x} \in X \) is called lower set less ordered (strictly/weakly) efficient, if there is no \( x^0 \in X \setminus \{\bar{x}\} \) such that \( f_U(x^0) \preceq_s f_U(\bar{x}) \) with respect to \( \mathbb{R}^k_{\geq / >} \), which is equivalent to
\[
\not\exists x^0 \in X \setminus \{\bar{x}\} : f_U(x^0) \subseteq f_U(\bar{x}) - \mathbb{R}^k_{\geq / >}.
\]

**Definition 2.7.** [10] Given an uncertain multiobjective optimization problem \( P(U) \), a solution \( \bar{x} \in X \) of \( P(U) \) is called alternative set less ordered (strictly/weakly) efficient, if there is no \( x^0 \in X \setminus \{\bar{x}\} \) such that \( f_U(x^0) \preceq_s f_U(\bar{x}) \) with respect to \( \mathbb{R}^k_{\geq / >} \), which is equivalent to
\[
\not\exists x^0 \in X \setminus \{\bar{x}\} : f_U(x^0) \subseteq f_U(\bar{x}) - \mathbb{R}^k_{\geq / >} \quad \text{or} \quad f_U(\bar{x}) \subseteq f_U(x^0) + \mathbb{R}^k_{\geq / >}.
\]

3. **New definitions of robust efficiency.** Ide and Köbis [10] derived new concepts of efficiency for uncertain multiobjective optimization problems by replacing the set ordering with others, such as lower set less ordered efficiency, set less ordered efficiency, alternative set less ordered efficiency and certainly less ordered efficiency. In addition, they gave a detail interpretation and discussed the relations between these concepts. Besides, Köbis [18] introduced the minmax less ordered robustness and the minmax certainly less ordered robustness which made the robustness concepts abundant. In [4] the concept of robust efficiency is closely connected to the upper set less ordering introduced by Kuroiwa [20]. Based on the study of [10, 4, 18]
and new order relations defined by Jahn and Ha in [14], we propose three definitions
of efficiency for uncertain multiobjective optimization. Compared with the existing
concepts, we replace the sets \( f_U(x) \) by their minimal and maximal elements, which
seems to be more clear to express the decision maker's preferences.

3.1. Minmax less ordered efficiency. The first proposed concept of efficiency
for uncertain multiobjective optimization problems is called minmax less ordered
efficiency which is useful and significant for decision making. Now we introduce the
minmax less order relation defined by Jahn and Ha [14, Definition 3.5].

**Definition 3.1.** Let \( \minmaxless \) order relation defined by Jahn and Ha [14, Definition 3.5].

Given an uncertain multiobjective optimization problem \( P(U) \), for all \( x \in X \) we define
\[
\begin{align*}
\min f_U(x) & := \{ f(x, \xi) : f_U(x) \cap (f(x, \xi) - \mathbb{R}^k_{\geq}) \subseteq \{ f(x, \xi) \} \}, \\
\max f_U(x) & := \{ f(x, \xi) : f_U(x) \cap (f(x, \xi) + \mathbb{R}^k_{\geq}) \subseteq \{ f(x, \xi) \} \}.
\end{align*}
\]

Considering the uncertain multiobjective optimization problem and combining
with Definition 2.2, we have the following set relation: for all \( x, \bar{x} \in X \),
\[
\begin{align*}
f_U(\bar{x}) & \preceq_m f_U(x) \text{ with respect to } \mathbb{R}^k_{[\geq/\geq/\geq]}, \\
\leftrightarrow & \min f_U(\bar{x}) \preceq_s \min f_U(x) \text{ and } \max f_U(\bar{x}) \preceq_s \max f_U(x) \\
\leftrightarrow & \min f_U(\bar{x}) \subseteq \min f_U(x) - \mathbb{R}^k_{[\geq/\geq/\geq]}, \ \min f_U(x) \subseteq \min f_U(\bar{x}) + \mathbb{R}^k_{[\geq/\geq/\geq]} \\
& \text{and } \max f_U(\bar{x}) \subseteq \max f_U(x) - \mathbb{R}^k_{[\geq/\geq/\geq]}, \ \max f_U(x) \subseteq \max f_U(\bar{x}) + \mathbb{R}^k_{[\geq/\geq/\geq]}.
\end{align*}
\]

Now we define the minmax less ordered efficient solutions for an uncertain multi-
objective optimization problem.

**Definition 3.2.** Given an uncertain multiobjective optimization problem \( P(U) \),
a solution \( \bar{x} \) to \( P(U) \) is called minmax less ordered (strictly/\-/weakly) efficient, if
there is no \( x^0 \in X \setminus \{ \bar{x} \} \) such that \( \min f_U(x^0) \preceq_s \min f_U(\bar{x}) \) and \( \max f_U(x^0) \preceq_s \max f_U(\bar{x}) \) with respect to \( \mathbb{R}^k_{[\geq/\geq/\geq]} \), which is equivalent to \( \not\exists x^0 \in X \setminus \{ \bar{x} \} \),
\[
\begin{align*}
\min f_U(x^0) & \subseteq \min f_U(\bar{x}) - \mathbb{R}^k_{[\geq/\geq/\geq]}, \ \min f_U(\bar{x}) \subseteq \min f_U(x^0) + \mathbb{R}^k_{[\geq/\geq/\geq]} \\
\max f_U(x^0) & \subseteq \max f_U(\bar{x}) - \mathbb{R}^k_{[\geq/\geq/\geq]}, \ \max f_U(\bar{x}) \subseteq \max f_U(x^0) + \mathbb{R}^k_{[\geq/\geq/\geq]}.
\end{align*}
\]

The concept of minmax less ordered efficiency is appealing for a decision maker
because its definition contains comparisons of minimal as well as maximal elements
of sets. In that manner it reflects optimism about the future as well as the risk averse
of the approaches containing maximal elements. Contrary to upper (lower) set less
ordered efficiency, the decision maker is now able to hedge against the minimal
(maximal) solutions of sets \( f_U(x) \) instead of the whole lower (upper) bound. This
enables a user to specify his/her wishes during the decision process even more. This
concept is less restrictive than set less ordered efficiency.

**Remark 2.** This concept, compared with [18, Definition 20], uses a specific cone
\( \mathbb{R}^k_{[\geq/\geq/\geq]} \) to define the minimal and maximal elements of \( f_U(x) \). Besides, the definition of
minmax less ordered (\)//weakly) efficient solution is also different. By this means, it
may be consistent with the existing concepts [10, 11] and convenient to investigate their relations.

An example is given to interpret the minmax less ordered efficiency.

**Example 1.** Consider Figure 1. An uncertain multiobjective optimization problem $P(U)$ with feasible set $X = \{x_1, \ldots, x_5\}$ and the five sets $f_U(x_1), \ldots, f_U(x_5)$ are described as polygons. In Figure 2 and Figure 3, the left pictures show the sets of their Min and Max. By adding $-\mathbb{R}^2_\leq$ and $\mathbb{R}^2_\leq$ to each of these sets, in the middle and right pictures we can see that $\nexists \ x^0 \in X \setminus \{x_1\}$ (respectively) such that $\min f_U(x^0) \preceq_s \min f_U(x_1)$ (respectively) and $\max f_U(x^0) \preceq_s \max f_U(x_1)$ (respectively), thus $x_1$ and $x_3$ are the minmax less ordered strictly efficient solutions.

![Figure 1. Sets $f_U(x_i)$ of objective values of $x_i$, $i = 1, \ldots, 5$](image)

We check if the minmax less ordered efficiency is consistent in the case $P(U)$ with $|U| = 1$. It is obvious that this result is a special case of [18, Lemma 12].

### 3.2. Minmax certainly less ordered efficiency

The next introduced concept is called minmax certainly less ordered efficiency which is helpful for us to have a better understanding of a decision maker’s preferences. Before the concept proposed, we firstly introduce the minmax certainly less order relation defined by Jahn and Ha [14, Definition 3.6].

**Definition 3.3.** Let $A, B \in \mathcal{M}$ be arbitrarily chosen sets. We say that $A$ dominates $B$ with respect to the minmax certainly less order relation (we denote this by $A \preceq_{mc} B$) with respect to $\mathbb{R}^k_{\geq/\geq/\geq}$ iff $(A = B)$ or $(A \neq B, \min A \preceq_c \min B$ and $\max A \preceq_c \max B)$.

Due to Definitions 2.4 and 3.3, we are able to define the minmax certainly less ordered efficient solutions.
Definition 3.4. Given an uncertain multiobjective optimization problem \( P(U) \), a solution \( \bar{x} \) to \( P(U) \) is called minmax certainly less ordered (strictly/weakly) efficient with respect to \( \mathbb{R}^k_{\geq/\geq/\geq} \), if there is no \( x^0 \in X \setminus \{ \bar{x} \} \) such that \( f_U(x^0) \preceq_{\mathrm{me}} f_U(\bar{x}) \), which is equivalent to

\[
\exists x^0 \in X \setminus \{ \bar{x} \}, \text{ s.t. } (f_U(x^0) = f_U(\bar{x})) \text{ or } (f_U(x^0) \neq f_U(\bar{x})), \\
\min_{\bar{x}} f_U(\bar{x}) - \min f_U(x^0) \subseteq \mathbb{R}^k_{\leq/\geq/\geq} \text{ and } \max f_U(\bar{x}) - \max f_U(x^0) \subseteq \mathbb{R}^k_{\leq/\geq/\geq}.
\]

The concept of minmax certainly less ordered efficiency based on the minmax certainly less order relation is less restrictive than the minmax less ordered efficiency and demands some solutions which strictly dominate others in both the minimal and maximal elements of \( f_U(x) \) (the best and worst scenarios). This concept would reflect a decision maker’s preferences precisely if he/she is optimistic or risk-averse about the future.

Remark 3. This concept, especially the (-/weakly) efficient solution of Definition 3.4, is different from [18, Definition 25] because of the exact definitions of minimal and maximal elements of \( f_U(x) \). Certainly, we could be able to discuss the relations between this concept and those ones in [10, 11].
Example 2. See Example 1, since \( \not\exists x^0 \in X \setminus \{x_1\} \) \((X \setminus \{x_2\}, X \setminus \{x_3\})\), respectively such that

\[
\begin{align*}
\min f_U(x_1) (\min f_U(x_2), \min f_U(x_3), \text{respectively}) - \min f_U(x^0) & \subseteq \mathbb{R}_k^k \\
\max f_U(x_1) (\max f_U(x_2), \max f_U(x_3), \text{respectively}) - \max f_U(x^0) & \subseteq \mathbb{R}_k^k
\end{align*}
\]

then \( x_1, x_2 \) and \( x_3 \) are minmax certainly less ordered strictly efficient.

Again, for \( P(U) \) with \( |U| = 1 \) reduced to a deterministic multiobjective optimization problem, the minmax certainly less ordered efficiency is a special case of [18, Lemma 19].

3.3. Minmax certainly nondominated ordered efficiency. The last introduced concept which is called minmax certainly nondominated ordered efficiency intends to filter out some bad solutions relatively. In order to propose this concept, we give the minmax certainly nondominated order relation defined by Jahn and Ha [14, Definition 3.8].

Definition 3.5. Let \( A, B \in \mathcal{M} \) be arbitrarily chosen sets. We say that \( A \) dominates \( B \) with respect to the minmax certainly nondominated order relation (we denote this by \( A \preceq_{mn} B \)) with respect to \( \mathbb{R}_k^k \) iff \( (A = B) \) or \( (A \neq B, \max A \leq, \min B) \).
Combined Definition 2.2 with Definition 3.5, we define the minmax certainly nondominated ordered efficient solutions.

Definition 3.6. Given an uncertain multiobjective optimization problem $P(U)$, a solution $\bar{x}$ is called minmax certainly nondominated ordered (strictly/weakly) efficient with respect to $\mathbb{R}^k_{\geq/\geq/\geq}$, if there is no $x^0 \in X \setminus \{\bar{x}\}$ such that $f_U(x^0) \preceq_{\min} f_U(\bar{x})$, which is equivalent to

\[ \not\exists x^0 \in X \setminus \{\bar{x}\}, \text{s.t.} \ (f_U(x^0) = f_U(\bar{x})) \text{ or } (f_U(x^0) \neq f_U(\bar{x})], \]

\[ \text{Max}_{f_U(x^0)} \subseteq \text{Min}_{f_U(\bar{x})} - \mathbb{R}^k_{\geq/\geq/\geq} \text{ and } \text{Min}_{f_U(\bar{x})} \subseteq \text{Max}_{f_U(x^0)} + \mathbb{R}^k_{\geq/\geq/\geq}. \]

The purpose of this concept is to sort out some solutions in the best scenarios which are dominated by others in the worst scenarios. If a design maker is risk-averse, those selected solutions are filtered out primarily. By this concept, it intuitively reflects some degree of a design maker’s risk-aversion. Additionally, this concept is less restrictive than the minmax less ordered efficiency and seems to be a completely new definition.

Example 3. See Example 1, since $\not\exists x^0 \in X \setminus \{x_1\}$ $(X \setminus \{x_2\}, X \setminus \{x_3\}$, respectively) such that

\[ \text{Max}_{f_U(x^0)} \subseteq \text{Min}_{f_U(x_1)} \ (\text{Min}_{f_U(x_2)}, \text{Min}_{f_U(x_3)}, \text{respectively}) - \mathbb{R}^k_{\geq} \text{ and } \]

\[ \text{Min}_{f_U(x_1)} \ (\text{Min}_{f_U(x_2)}, \text{Min}_{f_U(x_3)}, \text{respectively}) \subseteq \text{Max}_{f_U(x^0)} + \mathbb{R}^k_{\geq}. \]

then $x_1, x_2$ and $x_3$ are minmax certainly nondominated ordered strictly efficient.

Furthermore, we check the concept of minmax certainly nondominated ordered efficiency for consistency in the case that $P(U)$ is a deterministic multiobjective optimization problem:

Proposition 1. Given an uncertain multiobjective optimization problem $P(U)$ with $|U| = 1$, then a solution $\bar{x} \in X$ is the minmax certainly nondominated ordered (strictly/weakly) efficient iff it is (strictly/weakly) efficient.

4. Relationships between robust efficient concepts. In Section 3, we propose three definitions of efficiency for uncertain multiobjective optimization problems and give interpretations for them. At present, some of the existing concepts, i.e., [lower/upper/alternative/] set less ordered efficiency and certainly less ordered efficiency were introduced in [4, 10]. Before discussing the relations between new concepts and the existing concepts, we introduce the quasi domination property which relates a set and the sets of its minimal and maximal elements given in [14, Definition 3.7].

Definition 4.1. A set $A \in \mathcal{M}$ is said to have the quasi domination property iff $A \subseteq \max A - \mathbb{R}^k$ and $A \subseteq \min A + \mathbb{R}^k$ hold.

We assume that $f_U(x)$ has the quasi domination property for all $x \in X$. From Definition 4.1 we know that $\forall x \in X$,

\[ f_U(x) \subseteq \text{Max}_{f_U(x)} - \mathbb{R}^k \text{ and } f_U(x) \subseteq \text{Min}_{f_U(x)} + \mathbb{R}^k, \quad (1) \]

which will be used in the proofs of Theorems 4.2 and 4.5.

Firstly, we explore the relation between the set less ordered efficiency and the minmax less ordered efficiency. From [18, Theorem 33] we know that a set less ordered efficient solution must be minmax less ordered strictly efficient. The
following theorem implies that if a solution is set less ordered (weakly) efficient, then it is minmax less ordered (weakly) efficient.

**Theorem 4.2.** Given an uncertain multiobjective optimization problem \( P(U) \), if a solution \( \bar{x} \in X \) is set less ordered (weakly) efficient, then it is minmax less ordered (weakly) efficient.

**Proof.** It is analogous to the proof of Theorem 33 in [18] because of the equation \( \mathbb{R}^k_{\geq} + \mathbb{R}^k_{[\geq/>]} = \mathbb{R}^k_{[\geq/>]} \).

Unfortunately, the converse of Theorem 4.2 is, in general, not true. The following example illustrates this result.

**Example 4.** Consider Figure 4. An uncertain multiobjective optimization problem \( P(U) \) with feasible set \( X = \{x^1, x^2\} \) and two sets \( f_U(x^1) \), \( f_U(x^2) \) are depicted as rectangle and circle respectively. We can easily obtain \( f_U(x^1) \preceq_s f_U(x^2) \) with respect to \( \mathbb{R}^2_{\geq} \), thus \( x^1 \) is a set less ordered efficient solution. Since \( \text{Max} f_U(x^1) \nless_s \text{Max} f_U(x^2) \), then \( x^1, x^2 \) are minmax less ordered efficient. However, \( x^2 \) is not set less ordered efficient.

![Figure 4](image_url)

**Figure 4.** Sets \( f_U(x^i) \), \( \text{Max} f_U(x^i) \) and \( \text{Min} f_U(x^i) \) of objective values of \( x^i \), \( i = 1, 2 \).

Based on Theorems 39 and 40 in [18], a minmax less ordered strictly efficient solution is minmax certainly less ordered strictly efficient and a minmax certainly less ordered strictly efficient solution implies a certainly less ordered strictly efficient solution. Next we study the relations between their (weakly) efficient solutions.

**Theorem 4.3.** Given an uncertain multiobjective optimization problem \( P(U) \), if a solution \( \bar{x} \in X \) is minmax less ordered (weakly) efficient, then it is minmax certainly less ordered (weakly) efficient.

**Proof.** From the proof of [18, Theorem 39], we can obtain this conclusion. \( \square \)
Theorem 4.4. Given an uncertain multiobjective optimization problem $P(U)$, if a solution $\bar{x} \in X$ is minmax certainly less ordered (weakly) efficient, then it is certainly less ordered (weakly) efficient.

Proof. Combined Remark 1 and the proof of [18, Theorem 40], the conclusion is true. \qed

Generally, the converses of Theorems 4.3 and 4.4 are not true.

Example 5. See Example 1, since $\bar{x}$ is a solution $\bar{x} \in X \setminus \{x_1\}$ (or $X \setminus \{x_2\}, X \setminus \{x_3\}, X \setminus \{x_4\}$ respectively) such that

\begin{equation}
\text{CMax}_{f_U}(x^0) \subseteq \text{CMin}_{f_U}(x_i) \quad (i = 1, 2, 3, 4, \text{ respectively}) - \mathbb{R}_{\geq|\geq|}^2,
\end{equation}

then $x_1, x_2, x_3$ and $x_4$ are certainly less ordered efficient. From Examples 1 and 2 we know that $x_2$ is minmax certainly less ordered efficient but not minmax less ordered efficient, $x_4$ is certainly less ordered efficient rather than minmax certainly less ordered efficient. Therefore, Theorems 4.3 and 4.4 just provide sufficient conditions.

From [14, Proposition 3.10], we have known the relations between the minmax less order relation, the minmax certainly nondominated order relation and the certainly less order relation. We investigate the relationships among their concepts of efficiency as follows.

Theorem 4.5. Given an uncertain multiobjective optimization problem $P(U)$, if a solution $\bar{x} \in X$ is minmax less ordered (strictly/-/weakly) efficient, then it is minmax certainly less ordered (strictly/-/weakly) efficient.

Proof. If $\bar{x}$ is not minmax certainly nondominated ordered (strictly/-/weakly) efficient, there exists an $x^0 \in X \setminus \{\bar{x}\}$ such that

\begin{equation}
\text{Max}_{f_U}(x^0) \subseteq \text{Min}_{f_U}(\bar{x}) - \mathbb{R}_{\geq|\geq|}^k \quad \text{and} \quad \text{Min}_{f_U}(\bar{x}) \subseteq \text{Max}_{f_U}(x^0) + \mathbb{R}_{\geq|\geq|}^k.
\end{equation}

(2)

Since

\begin{equation}
\forall x \in X, \quad \text{Max}_{f_U}(x) \subseteq f_U(x) \quad \text{and} \quad \text{Min}_{f_U}(x) \subseteq f_U(x),
\end{equation}

combined with (1) and (2), we have

\begin{align*}
\text{Min}_{f_U}(x^0) & \subseteq f_U(x^0) \subseteq \text{Max}_{f_U}(x^0) - \mathbb{R}_{\geq|\geq|}^k
\subseteq \text{Min}_{f_U}(\bar{x}) - \mathbb{R}_{\geq|\geq|}^k - \mathbb{R}_{\geq|\geq|}^k \
\text{Min}_{f_U}(\bar{x}) & \subseteq \text{Max}_{f_U}(x^0) + \mathbb{R}_{\geq|\geq|}^k \subseteq f_U(x^0) + \mathbb{R}_{\geq|\geq|}^k
\subseteq \text{Min}_{f_U}(x^0) + \mathbb{R}_{\geq|\geq|}^k + \mathbb{R}_{\geq|\geq|}^k \subseteq \text{Min}_{f_U}(x^0) + \mathbb{R}_{\geq|\geq|}^k.
\end{align*}

This implies $\text{Min}_{f_U}(x^0) \leq_s \text{Min}_{f_U}(\bar{x})$. Analogously,

\begin{align*}
\text{Max}_{f_U}(x^0) & \subseteq \text{Min}_{f_U}(\bar{x}) - \mathbb{R}_{\geq|\geq|}^k - \mathbb{R}_{\geq|\geq|}^k \subseteq f_U(\bar{x}) - \mathbb{R}_{\geq|\geq|}^k \
\text{Max}_{f_U}(\bar{x}) & \subseteq f_U(x^0) + \mathbb{R}_{\geq|\geq|}^k + \mathbb{R}_{\geq|\geq|}^k \subseteq \text{Max}_{f_U}(x^0) + \mathbb{R}_{\geq|\geq|}^k \quad \text{and}
\end{align*}

\begin{align*}
\text{Max}_{f_U}(x^0) & \subseteq \text{Min}_{f_U}(\bar{x}) + \mathbb{R}_{\geq|\geq|}^k \subseteq f_U(\bar{x}) + \mathbb{R}_{\geq|\geq|}^k \
\text{Max}_{f_U}(\bar{x}) & \subseteq f_U(x^0) + \mathbb{R}_{\geq|\geq|}^k + \mathbb{R}_{\geq|\geq|}^k \subseteq \text{Max}_{f_U}(x^0) + \mathbb{R}_{\geq|\geq|}^k,
\end{align*}

which imply $\text{Max}_{f_U}(x^0) \leq_s \text{Max}_{f_U}(\bar{x})$. Therefore, $\bar{x}$ is not minmax less ordered (strictly/-/weakly) efficient, we arrive at a contradiction to the assumption. \qed
Then there exists an $x$.

Proof. Suppose that $\bar{x}$ is not certainly less ordered (strictly/\-weakly) efficient. Then there exists an $x^0 \in X \setminus \{\bar{x}\}$ such that
\[
\text{CMax}_U(x^0) \in \text{CMin}_U(\bar{x}) - \mathbb{R}^k_{\geq/\geq/}|\text{CMin}_U(\bar{x}) - \mathbb{R}^k_{\geq/\geq/}|
\]
which implies
\[
f_U(\bar{x}) - f_U(x^0) \subseteq \mathbb{R}^k_{\geq/\geq/}\]
Since $\text{Max}_U(x^0)$ is a subset of $f_U(x^0)$ and $\text{Min}_U(\bar{x})$ is a subset of $f_U(\bar{x})$, so for any
\[
f(\bar{x}, \xi) \in \text{Min}_U(\bar{x}) \quad \text{and} \quad f(x^0, \xi^*) \in \text{Max}_U(x^0),
\]
where $\bar{x}, \xi \in U$, we have $f(x^0, \xi^*) \geq \leq < f(\bar{x}, \xi)$. It is immediate that
\[
\text{Max}_U(x^0) \subseteq \text{Min}_U(\bar{x}) - \mathbb{R}^k_{\geq/\geq/}\quad \text{and} \quad \text{Min}_U(\bar{x}) \subseteq \text{Max}_U(x^0) + \mathbb{R}^k_{\geq/\geq/}
\]
in contradiction to the assumption of $\bar{x}$ being minmax certainly nondominated ordered (strictly/\-weakly) efficient for $P(U)$.

Normally the converses of Theorems 4.5 and 4.6 are also not true.

Example 6. See Example 1, combined with Examples 3 and 5, we know that $x_2$ is minmax certainly nondominated ordered strictly efficient but not minmax less ordered strictly efficient and $x_4$ is certainly less ordered strictly efficient instead of a minmax certainly nondominated ordered strictly efficient solution.

In the end, we discuss the relation between two concepts of the minmax certainly less ordered efficiency and the minmax certainly nondominated ordered efficiency. If a solution is minmax certainly less ordered (strictly/\-weakly) efficient, it is not necessarily minmax certainly nondominated ordered (strictly/\-weakly) efficient. Moreover, a minmax certainly nondominated ordered (strictly/\-weakly) solution may not be minmax certainly less ordered (strictly/\-weakly) efficient. Let us see the following two examples.

Example 7. Consider Figure 5. An uncertain multiobjective optimization problem $P(U)$ with feasible set $X = \{x_1, x_2\}$ and two sets $f_U(x_1), f_U(x_2)$ are depicted as circle and triangle respectively. We can easily obtain $f_U(x_1) \preceq_{\text{unc} f_U(x_2)$ with respect to $\mathbb{R}^2_{\geq/\geq/}$, thus $x_1$ is a minmax certainly nondominated ordered strictly efficient solution. Since $\text{Max}_U(x_1) - \text{Max}_U(x_2) \not\subseteq \mathbb{R}^2_{\geq/\geq/}$ and $\text{Max}_U(x_2) - \text{Max}_U(x_1) \not\subseteq \mathbb{R}^2_{\geq/\geq/}$, thus $x_1, x_2$ are minmax certainly less ordered strictly efficient. But $x_2$ is not minmax certainly nondominated ordered strictly efficient.

Example 8. Consider Figure 6. An uncertain multiobjective optimization problem $P(U)$ with feasible set $X = \{x_3, x_4\}$ and two sets $f_U(x_3), f_U(x_4)$ are depicted as circles. We can easily obtain $f_U(x_3) \preceq_{\text{unc} f_U(x_4)$ with respect to $\mathbb{R}^2_{\geq/\geq/}$, thus $x_3$ is minmax certainly less ordered strictly efficient. Since $\text{Max}_U(x_3) \not\subseteq \text{Min}_U(x_4) - \mathbb{R}^2_{\geq/\geq/}$ and $\text{Max}_U(x_4) \not\subseteq \text{Min}_U(x_3) - \mathbb{R}^2_{\geq/\geq/}$, then $x_3, x_4$ are minmax certainly nondominated ordered strictly efficient. Nevertheless, $x_4$ is not minmax certainly less ordered strictly efficient.
Combining Theorems 4.2, 4.3, 4.4, 4.5 and 4.6 with relations between the concepts in [10, 18], we supplement relationships between new concepts and the existing concepts in Figure 7, where “IK” stands for Ref. [10].
5. **An Application for Tourist’s destination selection.** With the rapid development of economy and the improvement of living level of people, more and more people go traveling on holidays. It is very significant for travellers to select suitable tourist destinations. In general, they value the most aspects that are the entertainment of tourist spots and the amount of tourist crowds as two objective functions. They want to maximize the former and minimize the latter. It is very difficult to make choices for them. For one thing they are not sure what the weather will be and how many tourists will visit the specific sight, for another with different weather conditions, the entertainment may be different and visiting a specific tourist spot may have less fun.

They evaluated the different tourist spots and edited the data in the following way: Assume there are four weather scenarios, each of them resulting in a different score on entertainment factor and tourist crowds for each sight. The score is estimated in grades from 1 to 20, 1 being perfect and 20 being very bad. We denote entertainment factor and tourist crowds as EF and TC respectively. Table 1 provides the results and $S_i, i \in I := \{1, \ldots, 10\}$ denote the tourist destinations respectively (A part of the data of Table 1 is from Table 1 in Ref. [10], the remaining data is estimated based on the fact.). Since this problem is multiobjective, they have to make trade-offs between the two objective functions. Moreover, since the problem is uncertain, they need to define what would be a suitable solution and make a decision through considering all four weather scenarios.

Now, we will apply these concepts of robustness to analyse this problem. Figure 8 shows the objective values of Table 1. From Figure 8, we can obtain that $S_i, i = 1, 4, 5, 6$ are minmax less ordered strictly efficient and $S_i, i \in I \setminus \{2, 3, 8, 9\}$ are

![Diagram of relationships between new concepts and the existent concepts of efficiency](image-url)
minimax certainly less ordered strictly efficient. See Figure 9, it is obvious that $S_i, i \in \mathcal{I}\{2,9\}$ are minimax certainly nondominated ordered strictly efficient and $S_i, i \in \mathcal{I}\{2\}$ are certainly less ordered strictly efficient.

![Figure 8. Objective values of Table 1](image)

We can obtain different solutions by using different robustness concepts, which not only reflects visitors’ preferences but also is more convenient and helpful for them to select appropriate tourist destinations. Furthermore, if they prefer to the entertainment of sights than the amount of tourist crowds, it is easier to make a choice, and vice versa. In addition, if EF and TC, weather scenarios and tourist destinations are substituted by cost and transportation speed, transportation means and warehouse locations respectively, these concepts can be applied to some special

| EF and TC | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ | $S_9$ | $S_{10}$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| Scenario 1 | (12,8) | (15,13) | (15,10) | (13,6) | (15,7) | (14,9) | (14,7) | (9,8) | (16,12) | (14,6) |
| Scenario 2 | (9,3) | (15,13) | (10,8) | (6,5) | (7,3) | (8,4) | (8,5) | (7,8) | (13,10) | (7,5) |
| Scenario 3 | (4,9) | (15,13) | (10,8) | (4,7) | (3,8) | (5,10) | (9,10) | (10,16) | (13,10) | (5,7) |
| Scenario 4 | (10,14) | (15,13) | (13,13) | (6,10) | (7,15) | (8,12) | (5,9) | (17,10) | (15,8) | (10,11) |
facility location problems. If we replace EF and TC, weather scenarios and tourist destinations by profit and risk, capital grades and investment proposals respectively, we can also use these concepts to portfolio problems.

6. Conclusions. In this paper, we introduced three concepts of robust efficiency which differ from the existent concepts in [10, 18] for uncertain multiobjective optimization problems. We made explanations for these concepts and discussed the relationships between new concepts and the existing concepts of efficiency. Some examples were also provided to illustrate main concepts and results. These concepts serve as a supplement to the existent concepts in [10] and make the concepts of efficiency for uncertain multiobjective optimization problems complete. In order to estimate the usefulness of robust efficient solutions, more applications of the presented concepts for real world problems are worth studying. Specifically, the robust solution and its properties for parametric generalized vector equilibrium problems [26] should be focused. Additionally, when we finish this work, we find that Ide and Schöbel [11] have presented and compared ten different concepts of robustness (including all known ones in Ide and Köbis [10]) for uncertain multi-objective optimization very recently. Combining with the concepts introduced herein, more relationships between these various concepts may be revealed in future work.

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THREE CONCEPTS OF ROBUST EFFICIENCY FOR UMOPS

REFERENCES

[1] A. Ben-Tal, L. El Ghaoui and A. Nemirovski, *Robust Optimization*, Princeton University Press, Princeton, 2009.

[2] A. Ben-Tal and A. Nemirovski, Robust solutions of linear programming problems contaminated with uncertain data, *Math. Program.*, 88 (2000), 411–424.

[3] J. R. Birge and F. V. Louveaux, *Introduction to Stochastic Programming*, Springer, New York, 1997.

[4] M. Ehrgott, J. Ide and A. Schöbel, Minmax robustness for multi-objective optimization problems, *European J. Oper. Res.*, 239 (2014), 17–31.

[5] M. Ehrgott, *Multicriteria Optimization*, Springer, New York, 2005.

[6] G. Eichfelder and J. Jahn, Vector optimization problems and their solution concepts, in *Recent Developments in Vector Optimization* (eds. Q.H. Ansari and J.C. Yao), Springer, Berlin, (2012), 1–27.

[7] J. Fliege and R. Werner, Robust multiobjective optimization & applications in portfolio optimization, *European J. Oper. Res.*, 234 (2014), 422–433.

[8] P. Gr. Georgiev, D. T. Luc and P. M. Pardalos, Robust aspects of solutions in deterministic multiple objective linear programming, *European J. Oper. Res.*, 229 (2013), 29–36.

[9] M. A. Goberna, V. Jeyakumar, G. Li and J. Vicente-Pérez, Robust solutions to multi-objective linear programs with uncertain data, *European J. Oper. Res.*, 242 (2015), 730–743.

[10] J. Ide and E. Köbis, Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations, *Math. Methods Oper. Res.*, 80 (2014), 99–127.

[11] J. Ide and A. Schöbel, Robustness for uncertain multi-objective optimization: A survey and analysis of different concepts, *OR Spectrum*, 38 (2016), 235–271.

[12] J. Jahn, *Vector Optimization Theory, Applications, and Extensions*, Springer, Berlin, 2004.

[13] J. Jahn, Vectorization in set optimization, *J. Optim. Theory Appl.*, 148 (2011), 209–236.

[14] J. Jahn and T. X. D. Ha, New order relations in set optimization, *J. Optim. Theory Appl.*, 148 (2011), 209–236.

[15] V. Jeyakumar, G. M. Lee and G. Li, Characterizing robust solution sets of convex programs under data uncertainty, *J. Optim. Theory Appl.*, 164 (2015), 407–435.

[16] K. Klamroth, E. Köbis, A. Schöbel and C. Tammer, A unified approach for different concepts of robustness and stochastic programming via non-linear scalarizing functionals, *Optimization*, 62 (2013), 649–671.

[17] E. Köbis, On robust optimization: Relations between scalar robust optimization and unconstrained multicriteria optimization, *J. Optim. Theory Appl.*, 167 (2015), 969–984.

[18] E. Köbis, *On Robust Optimization: A Unified Approach to Robustness Using a Nonlinear Scalarizing Functional and Relations to Set Optimization*, Ph.D. thesis, Martin-Luther-University in Halle-Wittenberg, 2014.

[19] L. S. Kong, C. J. Yu, K. L. Teo and C. H. Yang, Robust real-time optimization for blending operation of alumina production, *J. Ind. Manag. Optim.*, 13 (2017), 1149–1167.

[20] D. Kuroiwa, On set-valued optimization, *Nonlinear Anal.*, 47 (2001), 1395–1400.

[21] D. Kuroiwa and G. M. Lee, On robust multiobjective optimization, *Vietnam J. Math.*, 40 (2012), 305–317.

[22] A. Schöbel, Generalized light robustness and the trade-off between robustness and nominal quality, *Math. Methods Oper. Res.*, 80 (2014), 161–191.

[23] A. L. Soyster, Convex programming with set-inclusive constraints and applications to inexact linear programming, *Oper. Res.*, 21 (1973), 1154–1157.

[24] X. K. Sun, X. J. Long, H. Y. Fu and X. B. Li, Some characterizations of robust optimal solutions for uncertain fractional optimization and applications, *J. Ind. Manag. Optim.*, 13 (2017), 803–824.

[25] F. Wang, S. Y. Liu and Y. F. Chai, Robust counterparts and robust efficient solutions in vector optimization under uncertainty, *Oper. Res. Lett.*, 43 (2015), 293–298.

[26] X. Zuo, C. R. Chen and H. Z. Wei, Solution continuity of parametric generalized vector equilibrium problems with strictly pseudomonotone mappings, *J. Ind. Manag. Optim.*, 13 (2017), 475–486.

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