Eigenvalues of the truncated Helmholtz solution operator under strong trapping

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Abstract

For the Helmholtz equation posed in the exterior of a Dirichlet obstacle, we prove that if there exists a family of quasimodes (as is the case when the exterior of the obstacle has stable trapped rays), then there exist near-zero eigenvalues of the standard variational formulation of the exterior Dirichlet problem (recall that this formulation involves truncating the exterior domain and applying the exterior Dirichlet-to-Neumann map on the truncation boundary).

Our motivation for proving this result is that a) the finite-element method for computing approximations to solutions of the Helmholtz equation is based on the standard variational formulation, and b) the location of eigenvalues, and especially near-zero ones, plays a key role in understanding how iterative solvers such as the generalised minimum residual method (GMRES) behave when used to solve linear systems, in particular those arising from the finite-element method. The result proved in this paper is thus the first step towards rigorously understanding how GMRES behaves when applied to discretisations of high-frequency Helmholtz problems under strong trapping (the subject of the companion paper [MGSS21]).

Keywords. Helmholtz equation, trapping, quasimodes, eigenvalues, resonances, semiclassical analysis.

AMS subject classifications. 35J05, 35P15, 35B34, 35P25.

1 Introduction

1.1 Preliminary definitions

Let \( \Omega_- \subset \mathbb{R}^d, d \geq 2 \) be a bounded open set such that its open complement \( \Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-} \) is connected. Let \( \Gamma_D := \partial \Omega_- \), where the subscript \( D \) stands for “Dirichlet”. Let \( \Omega_1 \) be another bounded open set with connected open complement and such that \( \operatorname{conv}(\Omega_-) \Subset \Omega_1 \), where \( \operatorname{conv} \) denotes the convex hull and \( \Subset \) denotes compact containment. Let \( \Omega_{tr} := \Omega_1 \setminus \Omega_- \), and \( \Gamma_{tr} := \partial \Omega_1 \), where the subscript \( tr \) stands for “truncated”. We assume throughout that \( \Gamma_D \) and \( \Gamma_{tr} \) are both \( C^\infty \).

Let \( \gamma_D^\nu \) and \( \gamma_{tr}^\nu \) denote the Dirichlet traces on \( \Gamma_D \) and \( \Gamma_{tr} \) respectively, and let \( \gamma_D^\nu \) and \( \gamma_{tr}^\nu \) denote the respective Neumann traces, where the normal vector points out of \( \Omega_{tr} \) on both \( \Gamma_D \) and \( \Gamma_{tr} \).

Let

\[
H_0^1(\Omega_{tr}) := \{ v \in H^1(\Omega_{tr}) : \gamma_D^\nu v = 0 \}.
\]

Let \( \mathcal{D}(k) : H^{1/2}(\Gamma_{tr}) \rightarrow H^{-1/2}(\Gamma_{tr}) \) be the Dirichlet-to-Neumann map for the equation \( \Delta u + k^2 u = 0 \) posed in the exterior of \( \Omega_1 \) with the Sommerfeld radiation condition

\[
\frac{\partial u}{\partial r}(x) -iku(x) = o\left(\frac{1}{r^{(d-1)/2}}\right),
\]

as \( r := |x| \rightarrow \infty \), uniformly in \( \hat{x} := x/r \). We say that a function satisfying (1.1) is \( k\)-outgoing. When \( \Gamma_{tr} = \partial B_R \), for some \( R > 0 \), the definition of \( \mathcal{D}(k) \) in terms of Hankel functions and

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polar coordinates (when \( d = 2 \))/spherical polar coordinates (when \( d = 3 \)) is given in, e.g., [MS10, Equations 3.7 and 3.10].

**Definition 1.1** (Eigenvalues of the truncated exterior Dirichlet problem). We say \( \mu_\ell \) is an eigenvalue of the truncated exterior Dirichlet problem at frequency \( k_\ell > 0 \), with corresponding eigenfunction \( u_\ell \), if \( u_\ell \in H^1_{0,D}(\Omega_\alpha) \setminus \{0\} \) and \( \mu_\ell \in \mathbb{C} \) satisfies

\[
\langle \Delta + k_\ell^2 \rangle u_\ell = \mu_\ell u_\ell \quad \text{in} \quad \Omega_\alpha \quad \text{and} \quad \gamma^{1\nu}_{1\nu} u_\ell = D(k_\ell)(\gamma^{\nu\nu}_0 u_\ell).
\]

**Definition 1.2** (Quasimodes). A family of quasimodes of quality \( \epsilon(k) \) is a sequence \( \{(u_\ell,k_\ell)\}_{\ell=1}^\infty \subset H^2(\Omega_\alpha) \cap H^1_{0,D}(\Omega_\alpha) \times \mathbb{R} \) such that the frequencies \( k_\ell \to \infty \) as \( \ell \to \infty \) and there is a compact subset \( K \subset \Omega \) such that, for all \( \ell \), \( \text{supp } u_\ell \subset K \),

\[
\|\langle \Delta + k_\ell^2 \rangle u_\ell\|_{L^2(\Omega_\alpha)} \leq \epsilon(k_\ell) \quad \text{and} \quad \|u_\ell\|_{L^2(\Omega_\alpha)} = 1.
\]

**Remark 1.3.** By [Bur98, Theorem 2], we can assume that there exist \( S_1, S_2 > 0 \) such that \( \epsilon(k) \geq S_1 \exp(-S_2 k) \).

**Definition 1.4** (Quasimodes with multiplicity). Let \( \{u_\ell,k_\ell\}_{\ell=1}^\infty \) be a quasimode with quality \( \epsilon(k) \) and let \( \{(m_\ell,k_\ell^-,k_\ell^+)\}_{\ell=1}^\infty \subset \mathbb{N} \times \mathbb{R}^2 \) be such that \( k_\ell^- \to \infty \) and \( k_\ell^- \leq k_\ell^+ \). Define

\[
W_j := \{ \ell : k_\ell \in [k_\ell^-, k_\ell^+] \}.
\]

We say \( u_\ell \) has multiplicity \( m_j \) in the window \([k_\ell^-, k_\ell^+]\) if

\[
|W_j| = m_j, \quad |\langle u_\ell, u_{\ell'} \rangle_{L^2(\Omega_\alpha)}| \leq \epsilon(k_\ell^-) \quad \text{for} \quad \ell_1 \neq \ell_2, \quad \ell_1, \ell_2 \in W_j.
\]

We assume throughout that the quality, \( \epsilon(k) \), of a quasimode is a decreasing function of \( k \); this can always be arranged by replacing \( \epsilon(k) \) by \( \tilde{\epsilon}(k) := \sup_{k \geq k_0} \epsilon(k) \).

We use the notation that \( A = O(k^{-N}) \) as \( k \to \infty \) if, given \( N > 0 \), there exists \( C_N \) and \( k_0 \) such that \( |A| \leq C_N k^{-N} \) for all \( k \geq k_0 \), i.e. \( A \) decreases superalgebraically in \( k \).

### 1.2 The main results

**Theorem 1.5** (From quasimodes to eigenvalues). Let \( \alpha > 3(d+1)/2 \). Suppose there exists a family of quasimodes of quality \( \epsilon(k) \) with

\[
\epsilon(k) \ll k^{1-\alpha}.
\]

Then there exists \( k_0 > 0 \) (depending on \( \alpha \)) such that, if \( \ell \) is such that \( k_\ell \geq k_0 \), then there exists an eigenvalue of the truncated exterior Dirichlet problem at frequency \( k_\ell \) satisfying

\[
|\mu_\ell| \leq k_\ell^2 \epsilon(k_\ell).
\]

We now give three specific cases when the assumptions of Theorem 1.5 hold. The first two cases are via the quasimode constructions of [BCWG11, Theorem 2.8, Equations 2.20 and 2.21] and [CP02, Theorem 1] for obstacles whose exteriors support elliptic-trapped rays. The third case is via the “resonances to quasimodes” result of [Ste00, Theorem 1]; recall that the resonances of the exterior Dirichlet problem are the poles of the meromorphic continuation of the solution operator from \( \text{Im } k \geq 0 \) to \( \text{Im } k < 0 \); see, e.g., [DZ19, Theorem 4.4. and Definition 4.6].

**Lemma 1.6** (Specific cases when the assumptions of Theorem 1.5 hold).

(i) Let \( d = 2 \). Given \( a_1 > a_2 > 0 \), let

\[
E := \left\{ (x_1,x_2) : \left( \frac{x_1}{a_1} \right)^2 + \left( \frac{x_2}{a_2} \right)^2 < 1 \right\}.
\]

If \( \Gamma_D \) coincides with the boundary of \( E \) in the neighborhoods of the points \((0, \pm a_2)\), and if \( \Omega_+ \) contains the convex hull of these neighborhoods, then the assumptions of Theorem 1.5 hold with

\[
\epsilon(k) = \exp(-C_1 k).
\]
for some $C_1 > 0$ (independent of $k$).*

(ii) Suppose $d \geq 2$, $\Gamma_D \in C^\infty$, and $\Omega_+$ contains an elliptic-trapped ray such that (a) $\Gamma_D$ is analytic in a neighbourhood of the ray and (b) the ray satisfies the stability condition [CP02, (H1)]. If $q > 11/2$ when $d = 2$ and $q > 2d + 1$ when $d \geq 3$, then the assumptions of Theorem 1.5 hold with

$$\epsilon(k) = \exp(-C_2k^{1/q})$$

for some $C_2 > 0$ (independent of $k$).

(iii) Suppose there exists a sequence of resonances $\{\lambda_\ell\}_\ell$ of the exterior Dirichlet problem with

$$0 \leq -\text{Im} \lambda_\ell = O(|\lambda_\ell|^{-\infty}) \quad \text{and} \quad \text{Re} \lambda_\ell \to \infty \quad \text{as} \quad \ell \to \infty. \quad (1.3)$$

Then there exists a family of quasimodes of quality $\epsilon(k) = O(k^{-\infty})$ and thus the assumptions of Theorem 1.5 hold.

**Remark 1.7** (Resonances $\iff$ quasimodes $\iff$ eigenvalues). Part (iii) of Lemma 1.6 is the “resonances to quasimodes” result of [Ste00, Theorem 1]. The converse implication, i.e. that a family of quasimodes of quality $\epsilon(k) = O(k^{-\infty})$ implies a sequence of resonances satisfying (1.3), was proved in [TZ98], [Ste99] (following [SV95, SV96]), see also [DZ19, Theorem 7.6]. Therefore the “quasimodes to eigenvalues” result of Theorem 1.5 is equivalent to a “resonances to eigenvalues” result. In fact, in Appendix A we show that the existence of $O(k^{-\infty})$ eigenvalues implies the existence of quasimodes of quality $O(k^{-\infty})$. We therefore have that resonances $\iff$ quasimodes $\iff$ eigenvalues.

With $\{\mu_j(k)\}_j$ the set of eigenvalues, counting multiplicities, of the truncated exterior Dirichlet problem at frequency $k$ (with $\mu_j(k)$ depending continuously on $k$ for each $j$), let

$$\mathcal{E}(\varepsilon_1, \varepsilon_0, k_-, k_+) := \{j : \mu_j(k) \in (-2\varepsilon_1, 2\varepsilon_1) - i(0, 2\varepsilon_0) \text{ for some } k \in [k_-, k_+]\}; \quad (1.4)$$

$[\mathcal{E}]$ is therefore the counting function of the eigenvalues, $\mu_j(k)$, that pass through a rectangle next to zero in $\mu$ as $k$ varies in the interval $[k_-, k_+]$; see Figure 1.1.†

**Theorem 1.8** (From quasimodes to eigenvalues, with multiplicities). Let $k_j^-, k_j^+ \to \infty$ such that there is $C > 0$ satisfying $k_j^- \leq k_j^+ \leq Ck_j^-$. Suppose there exists a family of quasimodes of quality

*In [BCWG11, Theorem 2.8], $\Omega_+$ is assumed to contain the whole ellipse $E$. However, inspecting the proof, we see that the result remains unchanged if $E$ is replaced with the convex hull of the neighbourhoods of $(0, \pm a_2)$. Indeed, the idea of the proof is to consider a family of eigenfunctions of the ellipse localising around the periodic orbit $\{0, x_2 : |x_2| \leq a_2\}$.

†In Figure 1.1 we have drawn the paths of the eigenvalues as arbitrary curves. We see later in Figure 1.7 an example where the paths appear to be horizontal lines; this is consistent with the intuition that eigenvalues should be shifted resonances.
\[
\cos(\frac{3\pi}{10})
\]

\[
\cos(\frac{\pi}{10})
\]

\[x \cdot y \cdot 0.5 \cdot 0.6 \cdot 1.3 \cdot 1.0.\]

\[\text{Figure 1.2: The two obstacles } \Omega_- \text{ considered in the numerical experiments}
\]

\[
\epsilon(k) \ll k^{-\frac{(5d+3)}{2}} \text{ and multiplicity } m_j \text{ in the window } [k_j^-, k_j^+] \text{ (in the sense of Definition 1.4). If } \epsilon_0(k) \text{ is such that, for some } S > 0,
\]

\[
\epsilon_0(k) \leq S k^{-\frac{(d+1)}{2}} \text{ for all } k \text{ and } \epsilon_0(k) \gg k^{2d+1} \epsilon(k) \text{ as } k \to \infty,
\]

then there exists } k_0 > 0 \text{ such that if } k_j^- \geq k_0,
\]

\[
\left| E\left((k_j^-)^{(d+1)/2} \epsilon_0(k_j^-), \epsilon_0(k_j^-), k_j^-, k_j^+\right) \right| \geq m_j.
\]

Observe that if } k_j^+ = k_j^- \text{, then (up to algebraic powers of } k) \text{ Theorem 1.8 reduces to Theorem 1.5, except that now multiplicities are counted; therefore the “quasimodes to eigenvalues” result holds with multiplicities (just as the “quasimodes to resonances” result of [Ste99] includes multiplicities).

The ideas used in the proof of Theorems 1.5 and 1.8 are discussed in §1.5 below.

\textbf{Remark.} The reason why both the constant } \alpha \text{ in Theorem 1.5 and the exponent in the bound on the quality Theorem 1.8 depend on } d \text{ is because the right-hand side of the bound (1.15) below on the solution operator of the truncated problem depends on } d \text{, which in turn comes from the fact that the trace-class norm of compactly-supported pseudodifferential operators depends on } d.

\subsection*{1.3 Numerical experiments illustrating the main results}

\textbf{Description of the obstacles } \Omega_- \text{.} \text{ In this section, } \Omega_- \text{ is one of the two “horseshoe-shaped” 2-d domains shown in Figure 1.2. We define the small cavity as the region between the two elliptic arcs}

\[
(\cos(t), 0.5 \sin(t)), \quad t \in [-\phi_0, \phi_0] \quad \text{and} \quad (1.3 \cos(t), 0.6 \sin(t)), \quad t \in [-\phi_1, \phi_1]
\]

\[
\text{with } \phi_0 = 7\pi/10 \quad \text{and} \quad \phi_1 = \arccos\left(\frac{1}{1.3} \cos(\phi_0)\right);
\]

this corresponds to the interior of the solid lines in Figure 1.2. We define the large cavity as the region between the two arcs now with } \phi_0 = 9\pi/10. \text{ (Note that our small cavity is the same as the cavity considered in the numerical experiments in [BCWG+11, Section IV\:].) Recall that Theorems 1.5 and 1.8 require } \Gamma_D \text{ to be smooth, and thus these results do not strictly apply to the small and large cavities; however they do apply to smoothed versions of these.}
For both the small and large cavities, $\Gamma_D$ coincides with the boundary of the ellipse $E$ (1.2) with $a_1 = 1$ and $a_2 = 0.5$ in the neighbourhood of its minor axis. Part (i) of Lemma 1.6 (i.e., the results of [BCWG+11]) then implies that there exist quasimodes with exponentially-small quality. We choose these particular $\Omega^-$ because we can compute the frequencies $k_\ell$ in the quasimode. Indeed, the functions $u_\ell$ in the quasimode construction in [BCWG+11] are based on the family of eigenfunctions of the ellipse localising around the periodic orbit $\{ (0, x_2) : |x_2| \leq a_2 \}$; when the eigenfunctions are sufficiently localised, the eigenfunctions multiplied by a suitable cut-off function form a quasimode, with frequencies $k_\ell$ equal to the square roots of eigenvalues of the ellipse. By separation of variables, $k_\ell$ can be expressed as the solution of a multiparametric spectral problem involving Mathieu functions; see see [BCWG+11, Appendix A] and [MGSS21, Appendix E].

When giving specific values of $k_\ell$ below, we use the notation from [BCWG+11, Appendix A] and [MGSS21, Appendix E] that $k_{m,n}^e$ and $k_{m,n}^o$ are the frequencies associated with the eigenfunctions of the ellipse that are even/odd, respectively, in the angular variable, with $m$ zeros in the radial direction (other than at the centre or the boundary) and $n$ zeros in the angular variable in the interval $[0, \pi)$.

![Figure 1.3: The eigenvalues of the truncated exterior Dirichlet problem (Definition 1.1) near the origin when $\Gamma_D$ is equal to the small cavity. The eigenvalues are plotted at several frequencies, $k$, corresponding to eigenvalues of the ellipse. In each plot, the origin is marked with a black dot, and the eigenvalues are shown as green circles.](image)

![Figure 1.4: The eigenvalues of the truncated exterior Dirichlet problem (Definition 1.1) near the origin when $\Gamma_D$ is equal to the large cavity. The eigenvalues are plotted at several frequencies, $k$, corresponding to eigenvalues of the ellipse. In each plot, the origin is marked with a black dot, and the eigenvalues are shown as green circles.](image)

**Plots of the eigenvalues and eigenfunctions** Figures 1.3 and 1.4 plot the near-zero eigenvalues of the truncated exterior Dirichlet problem for the small and large cavity, respectively, at
Figure 1.4: The eigenvalues of the truncated exterior Dirichlet problem (Definition 1.1) near the origin when $\Gamma_D$ is equal to the large cavity. The eigenvalues are plotted at several frequencies, $k$, corresponding to eigenvalues of the ellipse. In each plot, the origin is marked with a black dot, and the eigenvalues are shown as green circles.

frequencies corresponding to eigenvalues of the ellipse. Figures 1.5 and 1.6 plot the corresponding eigenfunctions. In all these figures $\Gamma_{tr} = \partial B(0,2)$.

Figure 1.4 shows that the large cavity has an eigenvalue very close to zero at each of the four frequencies considered, qualitatively illustrating Theorem 1.5. In contrast, Figure 1.3 shows that the small cavity only has an eigenvalue very close to zero at the frequencies $k_{1,0}^e$ and $k_{3,0}^o$ (top right and bottom left in the figures) and not at $k_{0,3}^o$ and $k_{2,4}^e$ (top left and bottom right). The reason for this is clear from the plots of the eigenfunctions of the truncated exterior Dirichlet problem: looking at Figure 1.5, we see that at $k_{0,3}^o$ and $k_{2,4}^e$ the eigenfunctions are not well localised around the minor axis of the ellipse to be inside the small cavity – in the top left and bottom right of Figure 1.5 we see them “leaking out” of the small cavity. However, looking at Figure 1.6, we see that the corresponding eigenfunctions are localised sufficiently to be inside the large cavity, and thus generate an eigenvalue very close to zero. In these plots, the eigenfunctions are normalised so that their $L^2(\Omega_{tr})$ norm equals one.

Figure 1.7 plots the trajectories of the near-zero eigenvalues as functions of $k$ for both the small cavity (left plot) and large cavity (right plot) for $k \in (2.5, 12.5)$, with the spectra computed every 0.025. For Figure 1.7, $\Gamma_{tr} = \partial B(0,1.5)$; this change (compared to $\Gamma_{tr} = \partial B(0,2)$ for the earlier figures) is to reduce the cost of each eigenvalue solve, because each of the two plots in
Figure 1.5: Absolute value of the eigenfunction of the truncated exterior Dirichlet problem associated with the smallest eigenvalue the small cavity.

Figure 1.7 requires 400 such solves. Since we use the exact (up to discretisation error) Dirichlet-to-Neumann map on $\Gamma_{tr}$, we expect there to be no difference between choosing $\Gamma_{tr} = \partial B(0, 1.5)$ and $\Gamma_{tr} = \partial B(0, 2)$ (in particular Figures 1.3 and 1.4 are unchanged when $\Gamma_{tr}$ is changed from $\partial B(0, 2)$ to $\partial B(0, 1.5)$).

The eigenvalues that enter the red rectangle in Figure 1.7 are coloured green; these are members of $\mathcal{E}(0.2, 0.05, 2.5, 12.5)$, where $\mathcal{E}$ is defined by (1.4). Similar to the eigenvalues plots in Figures 1.3 and 1.4, Figure 1.7 shows that the large cavity has more near-zero eigenvalues for the range of $k$ considered than the small cavity. This is expected since a larger number of the eigenfunctions of the ellipse are localized in the large cavity than in the small cavity.

**How the eigenvalues and eigenfunctions were computed.** Definition 1.1 (of the eigenvalues of the truncated Dirichlet problem) implies that if $\mu_\ell$ is an eigenvalue at frequency $k_\ell$, and with corresponding eigenfunction $u_\ell$, then

$$a(u_\ell, v) = \mu_\ell (u_\ell, v)_{L^2(\Omega_{tr})} \quad \text{for all } v \in H^1_{0,D}(\Omega_{tr}),$$

where the sesquilinear form $a(\cdot, \cdot)$ is that appearing in the standard variational (i.e. weak) formulation of the Helmholtz exterior Dirichlet problem.

**Definition 1.9** (Variational formulation of Helmholtz exterior Dirichlet problem). Given $k > 0$, $\Omega_-$ as above, and $F \in (H^1_{0,D}(\Omega_{tr}))^*$, let $u \in H^1_{0,D}(\Omega_{tr})$ be the solution of the variational problem

$$\text{find } u \in H^1_{0,D}(\Omega_{tr}) \quad \text{such that } \quad a(u, v) = F(v) \quad \text{for all } v \in H^1_{0,D}(\Omega_{tr}),$$

where

$$a(u, v) := \int_{\Omega_{tr}} \left( \nabla u \cdot \nabla v - k^2 u v \right) - \left( \mathcal{D}(k)(\gamma_{tr}^N u), \gamma_{tr}^N v \right)_{\Gamma_{tr}},$$

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Figure 1.6: Absolute value of the eigenfunction of the truncated exterior Dirichlet problem associated with the smallest eigenvalue for the large cavity.

(a) $k_{0,3}^0 = 9.17017539835808$

(b) $k_{1,0}^1 = 9.977120156613617$

(c) $k_{5,0}^5 = 22.526496854613104$

(d) $k_{5,4}^5 = 22.6811692253925$

Figure 1.7: Paths of the eigenvalues for $k \in (2.5, 12.5)$ for the small cavity (left) and the large cavity (right). The eigenvalues that enter the red rectangle are coloured green.

where $\langle \cdot, \cdot \rangle_{\Gamma_{tr}}$ denotes the duality pairing on $\Gamma_{tr}$ that is linear in the first argument and antilinear in the second.

The figures above were created by solving the eigenvalue problem (1.5) using the finite-element method with continuous piecewise-linear elements (i.e. the polynomial degree, $p$, equals one) and meshwidth $h$, equal $(2\pi/30)k^{-3/2}$. The Dirichlet-to-Neumann map, $D(k)$, in $a(\cdot, \cdot)$ was computed using boundary integral equations – see Appendix B for details. The accuracy, uniform in frequency, of the finite-element applied the variational problem (1.6) with $p = 1$ and $hk^{3/2}$ sufficiently small.
has been known empirically for a long time, and was recently proved in [LSW19] for the case when
the Dirichlet-to-Neumann map is realised exactly.

Since computing the Dirichlet-to-Neumann map is relatively expensive, in practice one often
approximates it using a perfectly-matched layer (PML) or an absorbing boundary condition (such
as the impedance boundary condition). The plots of the eigenfunctions and near-zero eigenvalues
of the corresponding truncated exterior Dirichlet problems are very similar to those above; this
too is expected since the quasimode is supported in a neighbourhood of the obstacle.

1.4 Implications of the main results for numerical analysis of the Helmholtz
exterior Dirichlet problem

Theorems 1.5 and 1.8 are the first step towards rigorously understanding how iterative solvers such as
the generalised minimum residual method (GMRES) behave when applied to discretisations of
high-frequency Helmholtz problems under strong trapping (the subject of the companion paper
[MGSS21]). We now explain this in more detail.

As we saw in (1.5), the eigenvalues of truncated exterior Dirichlet problem (in sense of Definition
1.1) correspond to eigenvalues of sesquilinear form of standard variational formulation (Definition
1.9). The standard variational formulation is the basis of the finite-element method for computing
approximations to the solution of the variational problem (1.6). Indeed, the finite-element method
consists of choosing a piecewise-polynomial subspace of $H^1_0(D(\Omega))$ and solving the variational
problem (1.6) in this subspace.

A very popular way of solving the linear systems resulting from the finite-element method
applied to the Helmholtz scattering problems is via iterative solvers such as GMRES [SS86]; this
choice is made because the linear systems are (i) large and (ii) non-self-adjoint. Regarding (i): the
systems are large since the number of degrees of freedom must be $\gtrsim k^d$ to resolve the oscillations in
the solution, see, e.g., the literature review in [LSW19, §1.1]. Regarding (ii): non-self-adjointness
of the linear systems arises directly from the non-self-adjointness of the underlying Helmholtz
scattering problem; GMRES is applicable to such systems, unlike the conjugate gradient method.

There is currently large research interest in understanding how iterative methods behave when
applied to Helmholtz linear systems, and in designing good preconditioners for these linear systems;
see the literature reviews [Erl08, EG12, GZ19], [GSZ20, §1.3].

The location of eigenvalues, especially near-zero ones, is crucial in understanding the behaviour of iterative methods. In the Helmholtz context, eigenvalue analyses of iterative methods applied to nontrapping problems include, for finite-element discretisations, [EO99, EEO01, VGEV07, EG14, LXSdH20], and, for boundary-element discretisations,
[CH01, DDL13, CDLL17].

The paper [MGSS21] analyses GMRES applied to discretisations of Helmholtz problems with
strong trapping, using the “cluster plus outliers” GMRES convergence theory from [CIKM96] (with this idea arising in the context of the conjugate gradient method [Jen77] and used subsequently in, e.g., [ESW02]). The paper [MGSS21] obtains bounds on how the number of GMRES iterations
depends on the frequency, under various assumptions about the eigenvalues. In particular, Theorem
1.5 rigorously justifies [MGSS21, Observation O2(b)] for the standard variational formulation of the truncated exterior Dirichlet problem. We highlight that, although the results in [MGSS21] are
about unpreconditioned systems, they give insight into the design of preconditioners. Indeed, a
successful preconditioner for Helmholtz problems with strong trapping will need to specifically deal
with the near-zero eigenvalues created by trapping. Theorem 1.5 and 1.8 give information about
the location and multiplicities of these eigenvalues, and [MGSS21] shows how these locations and
multiplicities affect GMRES.

1.5 The ideas behind the proof of Theorem 1.5

Semiclassical notation. Instead of working with the parameter $k$ and being interested in
the large-$k$ limit, the semiclassical literature usually works with a parameter $h := k^{-1}$ and is interested
in the small-$h$ limit. So that we can easily recall results from this literature, we also work with
the small parameter $k^{-1}$, but to avoid a notational clash with the meshwidth of the FEM, we
let $h := k^{-1}$ (the notation $h$ comes from the fact that the semiclassical parameter is sometimes
related to Planck’s constant, which is written as $2\pi\hbar$; see, e.g., [Zwo12, §1.2]). Theorem 1.5 is then restated in semiclassical notation as Theorem 2.2 below.

**The solution operator of the truncated problem.** Let $R_{\Omega_{\tr}}(\lambda, z) : L^2(\Omega_{\tr}) \rightarrow L^2(\Omega_{\tr})$ be the solution operator for the truncated problem

$$
\begin{cases}
(-\hbar^2 \Delta - \lambda^2 - z)u = f & \text{in } \Omega_{\tr} \\
\gamma_{0}^D u = 0, \\
\gamma_{1}^I u = \mathcal{D}(\lambda/\hbar)\gamma_{0}^R u;
\end{cases}
$$

that is, $R_{\Omega_{\tr}}(\lambda, z)$ satisfies

$$
\begin{cases}
(-\hbar^2 \Delta - \lambda^2 - z)R_{\Omega_{\tr}}(\lambda, z)f = f & \text{in } \Omega_{\tr} \\
\gamma_{0}^D R_{\Omega_{\tr}}(\lambda, z)f = 0 \\
\gamma_{1}^I R_{\Omega_{\tr}}(\lambda, z)f = \mathcal{D}(\lambda/\hbar)\gamma_{0}^R R_{\Omega_{\tr}}(\lambda, z)f.
\end{cases}
$$

Note that, at this point, it is not clear that the problem (1.8) is well posed and that the family of operators $R_{\Omega_{\tr}}(\lambda, z)$ is well defined. We address this in Lemma 1.10 below.

We study $R_{\Omega_{\tr}}(\lambda, z)$ by relating it to the solution operator of a more-standard scattering problem. Namely, let $V \in L^{\infty}(\Omega_{+})$ with supp $V \subseteq \mathbb{R}^d$, and consider the problem

$$
\begin{cases}
(-\hbar^2 \Delta - \lambda^2 + V)u = f & \text{on } \Omega_{+}, \\
\gamma_{0}^D u = 0, \\
u \text{ is } \lambda/\hbar \text{ outgoing.}
\end{cases}
$$

By, e.g., [DZ19, Chapter 4], the inverse of (1.9) is a meromorphic family of operators (for $\lambda \in \mathbb{C}$ when $d$ is odd or $\lambda$ in the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when $d$ is even) $R_V(\lambda) : L^2_{\text{comp}}(\Omega_{+}) \rightarrow L^2_{\text{loc}}(\Omega_{+})$ with finite-rank poles satisfying

$$
\begin{cases}
(-\hbar^2 \Delta - \lambda^2 + V)R_V(\lambda)f = f & \text{in } \Omega \\
\gamma_{0}^D R_V(\lambda)f = 0, \\
R_V(\lambda)f \text{ is } \lambda/\hbar \text{ outgoing.}
\end{cases}
$$

Observe that, although both $R_{\Omega_{\tr}}(\lambda, z)$ and $R_V(\lambda)$ depend on $\hbar$, we omit this dependence in the notation to keep expressions compact.

The following two lemmas (proved in §2.2) relate $R_{\Omega_{\tr}}(\lambda, z)$ and $R_V(\lambda)$ and then characterise the eigenvalues of the truncated exterior Dirichlet problem as poles of $R_{\Omega_{\tr}}(\lambda, z)$ as a function of $z$.

We use three indicator functions: $1_{\Omega_{\tr}}$ denotes the function in $L^\infty(\Omega_{+})$ that is one on $\Omega_{\tr}$ and zero otherwise, $1_{\Omega_{\tr}}^{\text{ext}}$ denotes the restriction operator $L^2(\Omega_{+}) \rightarrow L^2(\Omega_{\tr})$, and $1_{\Omega_{\tr}}^{\text{ext}}$ denotes the extension-by-zero operator $L^2(\Omega_{\tr}) \rightarrow L^2(\Omega_{+})$.

**Lemma 1.10.** Define

$$R(\lambda, z) := R_V(\lambda) \quad \text{with } V(z) = -z1_{\Omega_{\tr}}. \quad (1.11)$$

Then

$$R_{\Omega_{\tr}}(\lambda, z) = 1_{\Omega_{\tr}}^{\text{ext}} R(\lambda, z) 1_{\Omega_{\tr}}^{\text{ext}}, \quad (1.12)$$

and thus $R_{\Omega_{\tr}}(\lambda, z)$ is a meromorphic family of operators in $\lambda$ for $\lambda \in \mathbb{C}$ when $d$ is odd and $\lambda$ in the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when $d$ is even.

**Lemma 1.11.** For $\lambda \in \mathbb{R} \setminus \{0\}$, $z \mapsto R(\lambda, z)$ is a meromorphic family of operators $L^2_{\text{comp}}(\Omega_{+}) \rightarrow L^2_{\text{loc}}(\Omega_{+})$ with finite rank poles.

**Corollary 1.12.** If $z_j$ is a pole of $z \mapsto R_{\Omega_{\tr}}(1, z)$, then $\mu_{\ell} := -\hbar^2 z_j$ is an eigenvalue of the truncated exterior Dirichlet problem (in the sense of Definition 1.1).

The key point is that we are interested in $R_{\Omega_{\tr}}(\lambda, z)$ as a meromorphic family in the variable $z$, in contrast to the more-familiar study of $R_V(\lambda)$ as a meromorphic family in the variable $\lambda$. 

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Recap of “from quasimodes to resonances”. Recall that resonances of $-h^2 \Delta + V$ are defined as poles of the meromorphic continuation of $R_V(w)$ into $\text{Im} \ w < 0$, see [DZ19, §4.2, §7.2]. The “quasimodes to resonances” argument of [TZ98] (following [SV95, SV96]; see also [DZ19, Theorem 7.6]) shows that existence of quasimodes (as in Definition 1.2) implies existence of resonances close to the real axis; the additional arguments in [Ste99] then prove the corresponding result with multiplicities.

These arguments use the semiclassical maximum principle (a consequence of the maximum principle of complex analysis, see Theorem 2.7 below) combined with the bounds

$$\| \chi R_V(\lambda) \chi \|_{L^2 \to L^2} \leq C \exp \left( C h^{-d} \log \delta^{-1} \right), \quad \lambda^2 \in \Omega \setminus \bigcup_{w \in \text{Res}(\-h^2 \Delta + V)} B(w, \delta),$$

for $\Omega \Subset \{ \text{Re} \ w > 0 \}$, and

$$\| R_V(\lambda) \|_{L^2 \to L^2} \leq \frac{1}{\text{Im}(\lambda^2)} \quad \text{for } \text{Im}(\lambda^2) > 0;$$

see [TZ98, Lemma 1], [TZ00, Proposition 4.3], [DZ19, Theorem 7.5].

From quasimodes to eigenvalues. Theorems 1.5 and 1.8 are proved using the same ideas as in the quasimodes to resonances arguments, except that now we work in the complex $z$-plane (with real $\lambda$) instead of the complex $\lambda$-plane. The analogue of the bounds (1.13) and (1.14) are given in the following lemma.

**Lemma 1.13** (Bounds on $R_{\Omega_\epsilon}(\lambda, z)$). Let $0 < a < b$ and let $z_j(h, \lambda)$ be the poles of $R_{\Omega_\epsilon}(\lambda, z)$ (as a meromorphic function of $z$). Then there exist $C_1, \varepsilon_1 > 0$ such that for all $0 < h < 1$, $\lambda^2 \in [a, b]$ and $\delta > 0$,

$$\| R_{\Omega_\epsilon}(\lambda, z) \|_{L^2(\{\lambda^2\}) \to L^2(\{\lambda^2\})} \leq \exp \left( C_1 h^{-d} \log \delta^{-1} \right) \quad \text{for } z \in B(0, \varepsilon_1 h) \setminus \bigcup_{j} B(z_j(h, \lambda), \delta). \quad (1.15)$$

Furthermore, there exists $C_2 > 0$ such that

$$\| R_{\Omega_\epsilon}(\lambda, z) \|_{L^2(\{\lambda^2\}) \to L^2(\{\lambda^2\})} \leq C_2 \frac{\langle z \rangle}{\text{Im} z} \quad \text{for } \text{Im} \ z > 0, \quad (1.16)$$

where $\langle z \rangle := (1 + |z|^2)^{1/2}$.

The bound (1.15) is proved by finding a parametrix for $-h^2 \Delta - \lambda^2 - z 1_{\Omega_\epsilon} \Delta$ (i.e. an approximation to $R_{\Omega_\epsilon}(\lambda, z)$) via a boundary complex absorbing potential. While parametrices based on complex absorption are often used in scattering theory (see, e.g., [DZ16, DG17] [DZ19, Theorem 7.4]), parametrices based on boundary complex absorption appear to be new in the literature. One of the main features of the argument below is that it relies on a comparison of the (in principle, trapping) billiard flow with the non-trapping free flow to obtain estimates on the parametrix. A similar argument should work for boundaries in any non-trapping background.

We also highlight that, while we consider the scattering by Dirichlet obstacles in this paper and therefore must use boundary complex absorption, smooth compactly-supported perturbations of $-\Delta$, e.g. metric perturbations or semiclassical Schrödinger operators, can be handled similarly. Indeed, for these problems, the parametrix based on boundary absorption could be replaced by one based on simpler complex absorbing potentials.

### 1.6 Outline of the rest of the paper

In §2 we prove Lemmas 1.10 and 1.11 and then collect preliminary results about the generalized bicharacteristic flow (§2.4), the geometry of trapping (§2.5), complex scaling (§2.6), and defect measures (§2.8). In §3 we find a parametrix for $R_{\Omega_\epsilon}(\lambda, z)$ via a boundary complex absorbing potential. In §4 we prove Lemma 1.13. In §5 we prove Theorems 1.5 and 1.8 using Lemma 1.13 and the semiclassical maximum principle.
2 Preliminary results

2.1 Restatement of Theorems 1.5 and 1.8 in semiclassical notation

Definition 2.1 (Quasimodes in \( h \) notation). A family of quasimodes of quality \( \varepsilon(h) \) is a sequence \( \{(u_{\ell}, h_{\ell})\}_{\ell=1}^{\infty} \subset H^2(\Omega_{tr}) \cap H^1_{0,D}(\Omega_{tr}) \times \mathbb{R} \) such that \( h_{\ell} \to 0 \) as \( \ell \to \infty \) and there is a compact subset \( K \subset \Omega_1 \) such that, for all \( \ell \), \( u_{\ell} \subset K \),

\[
\|(-h^2 \Delta - 1)u_{\ell}\|_{L^2(\Omega_{tr})} \leq \varepsilon(h_{\ell}) \quad \text{and} \quad \|u_{\ell}\|_{L^2(\Omega_{tr})} = 1.
\]

Let

\[
\varepsilon(h) := h^2 \varepsilon(h^{-1}). \tag{2.1}
\]

Remark 1.3 implies that we can assume that there exist \( \tilde{S}_1, \tilde{S}_2 > 0 \) such that

\[
\varepsilon(h) \geq \tilde{S}_1 \exp(-\tilde{S}_2/h). \tag{2.2}
\]

Theorem 1.5 is then equivalent to the following result in the sense that the following result holds if and only if Theorem 1.5 holds with \( \mu_{\ell} := h_{\ell}^{-1} \).

Theorem 2.2 (Analogue of Theorem 1.5 in \( h \) notation). Let \( \alpha > 3(d+1)/2 \). Suppose there exists a family of quasimodes in the sense of Definition 2.1 such that the quality \( \varepsilon(h) \) satisfies

\[
\varepsilon(h) \ll h^{1+\alpha}. \tag{2.3}
\]

Then there exists \( h_0 > 0 \) (depending on \( \alpha \)) such that, if \( \ell \) is such that \( h_{\ell} \leq h_0 \) then there exists \( z_{\ell} \in \mathbb{C} \) and \( 0 \neq u_{\ell} \in H^1_{0,D}(\Omega_{tr}) \) with

\[
(-h_{\ell}^2 \Delta - 1 + z_{\ell})u_{\ell} = 0 \quad \text{in} \quad \Omega_{tr}, \quad \gamma^j u_{\ell} = D(h_{\ell}^{-1})(\gamma^j u_{\ell}), \quad \text{and} \quad |z_{\ell}| \leq h_{\ell}^{-\alpha} \varepsilon(h_{\ell}). \tag{2.4}
\]

Definition 2.3 (Quasimodes with multiplicity in \( h \) notation). Let \( 0 \leq a(h) \leq b(h) < \infty \) be two functions of \( h \). A family of quasimodes of quality \( \varepsilon(h) \) and multiplicity \( m(h) \) in the window \([a(h), b(h)]\) is a sequence \( \{h_j\}_{j=1}^{\infty} \) such that \( h_j \to 0 \) as \( j \to \infty \) and for every \( j \) there exist \( \{(u_{j,\ell}, E_{j,\ell})\}_{\ell=1}^{m(h_j)} \subset H^2(\Omega_{tr}) \cap H^1_{0,D}(\Omega_{tr}) \times [a(h_j), b(h_j)] \) with

\[
\|(-h_j^2 \Delta - E_{j,\ell})u_{j,\ell}\|_{L^2(\Omega_{tr})} = \varepsilon(h_j), \quad \|u_{j,\ell}\|_{L^2(\Omega_{tr})} = 1, \quad |\{u_{j,\ell,1}, u_{j,\ell,2}\}_{L^2(\Omega_{tr})}| \leq h_j^{-2} \varepsilon(h_j) \quad \text{for} \quad \ell_1 \neq \ell_2,
\]

and \( \text{supp} \ u_{j,\ell} \subset K \) for all \( j \) and \( \ell \), where \( K \subset \Omega_1 \).

With \( \{z_p(h,\lambda)\}_p \) the set of poles of \( z \mapsto R_{\Omega_\nu}(\lambda, z) \) counting multiplicities (with \( z_p(h,\lambda) \) depending continuously on \( \lambda \) for each \( p \)), let

\[
Z(\varepsilon_1, \varepsilon_2, a, b ; h) := \left\{ p : z_p(h,\lambda) \in (-2\varepsilon_1, 2\varepsilon_2) - i(0, 2\varepsilon_0) \quad \text{for some} \quad \lambda^2 \in [a, b] \right\}; \tag{2.5}
\]

\(|Z|\) is therefore the counting function of the poles of \( z \mapsto R_{\Omega_\nu}(\lambda, z) \) that enter a rectangle next to zero in \( z \) as \( \lambda^2 \) varies from \( a \) to \( b \).

Theorem 2.4 (Analogue of Theorem 1.8 in \( h \) notation). Let \( 0 \leq a_0 \leq a(h) \leq b(h) < b_0 < \infty \) and suppose there exists a family of quasimodes with quality

\[
\varepsilon(h) \ll h^{(5d+3)/2}, \tag{2.6}
\]

and multiplicity m(h) in the window \([a(h), b(h)]\) (in the sense of Definition 2.3). If \( \varepsilon_0(h) \) is such that, for some \( \tilde{S} > 0 \),

\[
\varepsilon_0(h) \leq \tilde{S} h^{(d+1)/2} \quad \text{for all} \quad h, \quad \text{and} \quad \varepsilon_0(h) \gg h^{-2d-1} \varepsilon(h) \quad \text{as} \quad h \to 0, \tag{2.7}
\]

then there exists \( h_0 > 0 \) such that if \( h_j \leq h_0 \), then

\[
|Z(h_j^{-(d+1)/2} \varepsilon_0(h_j), \varepsilon_0(h_j), a(h_j), b(h_j) ; h_j)| \geq m(h_j).
\]
Proof of Theorem 1.8 from Theorem 2.4. We first show that if there exists a family of quasimodes \( u_j \) with multiplicity \( m \) in the window \([k_j^-, k_j^+]\) in \( k \) notation (i.e. in the sense of Definition 1.4), then there exists a family of quasimodes in \( h \) notation (in the sense of Definition 2.3).

Without loss of generality, each \( k \in [k_j^-, k_j^+] \) for some \( j \) (if necessary by adding a window with \( k_j^- = k_j^+ = k \)), i.e. given \( \ell \) in the index set of the quasimode, there exists \( j \) such that \( \ell \in W_j \). We now index the quasimode with the index \( j \) describing the windows \([k_j^-, k_j^+]\).

Let \( h_j := (k_j^-)^{-1}, \ m(h_j) := m_j, \ a(h_j) := 1, \ b(h_j) := \frac{(k_j^+)^2}{(k_j^-)^2}, \ \epsilon(h_j) := h_j^2(\epsilon(h_j)^{-1}), \) and \( E_{j, \ell} := \frac{(\epsilon(h_j)^2}{(k_j^-)^2} \) and \( u_{j, \ell} := u_{\ell} \) for \( \ell \in W_j \).

Then,
\[
\| (h_j^2 \Delta + E_{j, \ell}) u_\ell \|_{L^2(\Omega, \mu)} = (k_j^{-})^{-2} \| (\Delta + k_j^{2}) u_\ell \|_{L^2(\Omega, \mu)} = (k_j^{-})^{-2} \epsilon(k_\ell) \leq (k_j^-)^{-2} \epsilon(k_j^-) = \epsilon(h_j),
\]

where we have used that \( \epsilon(k) \) is a decreasing function of \( k \). Therefore, we have shown that there exists a family of quasimodes with multiplicity \( m(h) \) in the window \([a(h), b(h)]\) in \( h \) notation (i.e. in the sense of Definition 2.3).

The result of Theorem 1.8 then follows from the result of Theorem 2.4 since (a) if \( \lambda \in [a(h), b(h)] \) and \( \lambda/h = k \), then \( k \in [k_j^-, k_j^+] \), and (b) if
\[
z \in \mathcal{E} \left( h_j^{-(d+1)/2} \epsilon_0(h_j), \ a(h_j), \ b(h_j), \ h_j \right),
\]
then
\[
\mu := h_j^{-2} z \in \mathcal{E} \left( (k_j^-)^{(d+1)/2} \epsilon_0(k_j^-), \ k_j^-, \ k_j^+ \right).
\]

\[\square\]

2.2 Results about meromorphic continuation

Proof of Lemma 1.10. Once we show (1.12), the meromorphy of \( R_{\Omega, \gamma}(\lambda, z) \) in \( \lambda \) follows from the corresponding result for \( R_{\nu} (\lambda) \) [DZ19, Theorem 4.4].

We first show that the appropriate extension of a solution of (1.8) is a solution of (1.9) with \( V(z) = -z1_{\Omega_+} \). We then show that the appropriate restriction of the solution of (1.9) with \( V(z) = -z1_{\Omega_+} \) is a solution of (1.8).

Given \( f \in L^2(\Omega_+) \), suppose that \( u \) solves (1.8). Then, by the definition of the operator \( D \), there exists a \( \lambda/h \)-outgoing function \( v \in H^2_{\text{loc}}(\mathbb{R}^d \setminus \Omega_1) \) such that
\[
(-h^2 \Delta - \lambda^2) v = 0 \quad \text{on } \mathbb{R}^d \setminus \Omega_1, \quad \text{and} \quad \gamma^{tr}_{0} v = \gamma^{tr}_{1} u, \quad \gamma^{tr}_{1} v = \gamma^{tr}_{1} u.
\]
Therefore,
\[
\tilde{v} := 1^\text{ext}_{\Omega_+} u + 1^\text{ext}_{\mathbb{R}^d \setminus \Omega_1} v
\]
is in \( H^2_{\text{loc}}(\Omega_+) \) (since both its Dirichlet and Neumann traces match across \( \partial \Omega_1 \)) and
\[
(-h^2 \Delta - \lambda^2) \tilde{v} = z1_{\Omega_+} \tilde{v} + 1^\text{ext}_{\Omega_+} f \quad \text{on } \Omega_+.
\]
By the definition of \( R(\lambda, z) \) as the solution of (1.10) with \( V(z) = -z1_{\Omega_+} \),
\[
\tilde{v} = R(\lambda, z) 1^\text{ext}_{\Omega_+} f, \quad \text{which implies that} \quad u = 1^\text{res}_{\Omega_+} R(\lambda, z) 1^\text{ext}_{\Omega_+} f.
\]
Now suppose \( \tilde{f} \in L^2(\Omega_+) \). Then, by (1.11) and (1.9),
\[
\begin{align*}
(-h^2 \Delta - \lambda^2 - z1_{\Omega_+}) R(\lambda, z) \tilde{f} = \tilde{f} & \quad \text{in } \Omega, \\
R(\lambda, z) \tilde{f} = 0 & \quad \text{on } \Gamma_D, \\
R(\lambda, z) \tilde{f} & \text{is } \lambda/h\text{-outgoing.}
\end{align*}
\]
Therefore, if $\tilde{f} = 1^\text{ext}_{\Gamma_{tr}} f$ and $v := R(\lambda, z) \tilde{f}$, then $(-\hbar^2 \Delta - \lambda^2)R(\lambda, z)\tilde{f} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ and $v$ is $\lambda/\hbar$-outgoing. This last fact implies that
\[
\gamma_0^{tr}(1_{\Omega_{tr}}^{\text{res}} v) = \mathcal{D}(\lambda/\hbar)\gamma_0^{tr}(1_{\Omega_{tr}}^{\text{res}} v). \tag{2.9}
\]
Since $v = R(\lambda, z)\tilde{f} \in H^2_{\text{loc}}(\Omega_+)$, the Dirichlet and Neumann traces of $v$ across $\Gamma_{tr}$ do not have jumps, so that (2.9) implies that
\[
\gamma_0^{tr}(1_{\Omega_{tr}}^{\text{res}} v) = \mathcal{D}(\lambda/\hbar)\gamma_0^{tr}(1_{\Omega_{tr}}^{\text{res}} v). \tag{2.10}
\]
Then, by (2.8) and (2.10), $u := 1_{\Omega_{tr}}^{\text{res}} v$ solves (1.8) and the proof is complete. \hfill \Box

**Proof of Lemma 1.11.** Since
\[
(-\hbar^2 \Delta - \lambda^2 - z 1_{\Omega_0})R(\lambda, 0) = I - z 1_{\Omega_0} R(\lambda, 0),
\]
the definition of $R(\lambda, z)$ (1.11) implies that
\[
R(\lambda, z) = R(\lambda, 0) \left(I - z 1_{\Omega_0} R(\lambda, 0)\right)^{-1}. \tag{2.11}
\]
We now claim that, for any $\rho \in C^\infty(\overline{\Omega})$ with supp $\rho \subset \mathbb{R}^d$ and $\rho \equiv 1$ on $\Omega_{tr}$
\[
(I - z 1_{\Omega_0} R(\lambda, 0))^{-1} = (I - z 1_{\Omega_0} R(\lambda, 0)\rho)^{-1} \left(I + z 1_{\Omega_0} R(\lambda, 0)(1 - \rho)\right). \tag{2.12}
\]
Indeed,
\[
I - z 1_{\Omega_0} R(\lambda, 0) = \left(I - z 1_{\Omega_0} R(\lambda, 0)(1 - \rho)\right) \left(I - z 1_{\Omega_0} R(\lambda, 0)\rho\right)^{-1} \left(I - z 1_{\Omega_0} R(\lambda, 0)\right)^{-1}.
\]
and thus
\[
(I - z 1_{\Omega_0} R(\lambda, 0))^{-1} = (I - z 1_{\Omega_0} R(\lambda, 0)\rho)^{-1} \left(I - z 1_{\Omega_0} R(\lambda, 0)(1 - \rho)\right) \left(I - z 1_{\Omega_0} R(\lambda, 0)\rho\right)^{-1}.
\]
Observe that since $\rho R(\lambda, 0) : L^2(\Omega_+) \to L^2(\Omega_+)$ is compact, $1_{\Omega_0} R(\lambda, 0) \rho : L^2(\Omega_+) \to L^2(\Omega_+)$ is compact, and the analytic Fredholm theorem [DZ19, Theorem C.8] implies that
\[
z \mapsto (I - z 1_{\Omega_0} R(\lambda, 0)\rho)^{-1} \text{ is a meromorphic family of operators for } z \in \mathbb{C} \tag{2.14}
\]
with finite rank poles.

Now, since $(1 - \rho) 1_{\Omega_0} = 0$, for $|z|$ small enough,
\[
(1 - \rho)(I - z 1_{\Omega_0} R(\lambda, 0)\rho)^{-1} = (1 - \rho) \sum_{j=0}^{\infty} (z 1_{\Omega_0} R(\lambda, 0)\rho)^j = (1 - \rho). \tag{2.15}
\]
However, by (2.14) both the left- and right-hand sides of (2.15) are meromorphic for $z \in \mathbb{C}$. Therefore, (2.15) holds for all $z \in \mathbb{C}$ and hence
\[
(I - z 1_{\Omega_0} R(\lambda, 0)(1 - \rho))^{-1} = I + z 1_{\Omega_0} R(\lambda, 0)(1 - \rho). \tag{2.16}
\]
Using (2.15) and (2.16) in (2.13), we obtain (2.12). Therefore, for $\chi \equiv 1$ on $\Omega_{tr}$ and $\rho \equiv 1$ on supp $\chi$, (2.11), (2.12) and (2.15) imply that
\[
\chi R(\lambda, z)\chi = \chi R(\lambda, 0)\rho(I - z 1_{\Omega_0} R(\lambda, 0)\rho)^{-1}\chi.
\]
Using (2.14) again completes the proof. \hfill \Box
With \( z_0(h, \lambda) \) a pole of \( R_{\Omega_\nu}(\lambda, z) \), let
\[
\Pi_{z_0(h, \lambda)} := -\frac{1}{2\pi i} \int_{z_0(h, \lambda)} R_{\Omega_\nu}(\lambda, z) \, dz \quad \text{and} \quad m_R(z_0(h, \lambda)) := \text{rank} \Pi_{z_0(h, \lambda)},
\] (2.17)
where \( \int_{z_0(h, \lambda)} \) denotes integration over a circle containing \( z_0 \) and no other pole of \( R_{\Omega_\nu}(\lambda, z) \).

The following result then holds by, e.g., [DZ19, Theorem C.9].

**Lemma 2.5.** For \( \lambda \in \mathbb{R} \setminus \{0\} \), \( \Pi_{z_0(h, \lambda)} : L^2(\Omega_{\nu}) \to L^2(\Omega_{\nu}) \) is a bounded projection with finite rank.

The next result concerns the singular behaviour of \( R_{\Omega_\nu}(\lambda, z) \) near its poles in \( z \), and is analogous to (parts of) [DZ19, Theorem 4.7] concerning the singular behaviour of \( R_V(\lambda) \) near its poles in \( \lambda \).

**Lemma 2.6.** For \( \lambda \in \mathbb{R} \setminus \{0\} \), if \( z_0 = z_0(h, \lambda) \) and \( m_R(z_0) > 0 \), then there exists \( M_{z_0} > 0 \) such that
\[
R_{\Omega_\nu}(\lambda, z) = -\sum_{\ell=1}^{M_{z_0}} \Pi_{z_0} \frac{(-h^2 \Delta - \lambda^2 - z)^{\ell-1}}{(z - z_0)^\ell} + A(z, z_0, \lambda)
\]
where \( z \mapsto A(z, z_0, \lambda) \) is holomorphic near \( z_0 \).

**Proof.** By Lemma 1.11, for \( \lambda \in \mathbb{R} \setminus \{0\} \), \( z \mapsto R_{\Omega_\nu}(\lambda, z) \) is a meromorphic family of operators (in the sense of [DZ19, Definition C.7]) from \( L^2(\Omega_{\nu}) \to L^2(\Omega_{\nu}) \) and thus there exists \( M_{z_0} > 0 \), finite-rank operators \( A_\nu(\lambda) : L^2(\Omega_{\nu}) \to L^2(\Omega_{\nu}) \), \( \ell = 1, \ldots, M_{z_0} \), and a family of operators \( z \mapsto A(z, z_0, \lambda) \) from \( L^2(\Omega_{\nu}) \to L^2(\Omega_{\nu}) \), holomorphic near \( z_0 \), such that
\[
R_{\Omega_\nu}(\lambda, z) = \sum_{\ell=1}^{M_{z_0}} \frac{A_\nu(\lambda)}{(z - z_0)^\ell} + A(z, z_0, \lambda).
\]
By integrating around \( z_0 \) and using the residue theorem, we have \( A_1 = -\Pi_{z_0} \). Then, with \( \equiv \) denoting equality up to holomorphic operators,
\[
R_{\Omega_\nu}(\lambda, z)(-h^2 \Delta - \lambda^2 - z) \equiv \sum_{\ell=1}^{M_{z_0}} \frac{A_\nu(-h^2 \Delta - \lambda^2 - z_0)}{(z - z_0)^\ell} - \frac{A_\nu}{(z - z_0)^\ell},
\]
\[
= \sum_{\ell=1}^{M_{z_0}} \frac{A_\nu(-h^2 \Delta - \lambda^2 - z_0) - A_{\nu+1}}{(z - z_0)^\ell},
\]
where we define \( A_{M_{z_0}+1} := 0 \). Since \( R_{\Omega_\nu}(\lambda, z)(-h^2 \Delta - \lambda^2 - z) = I \) on \( H^2(\Omega_{\nu}) \cap H^0_0(\Omega_+) \), \( A_{\nu+1} = A_\nu(-h^2 \Delta - \lambda^2 - z) \), \( \ell = 1, \ldots, M_{z_0} \), and the result follows from density of \( H^2(\Omega_{\nu}) \cap H^0_0(\Omega_+) \) in \( L^2(\Omega_{\nu}) \).

\[
2.3 \quad \text{The semiclassical maximum principle}
\]

The following result is the semiclassical maximum principle of [TZ98, Lemma 2], [TZ00, Lemma 4.2] (see also [DZ19, Lemma 7.7]).

**Theorem 2.7 (Semiclassical maximum principle).** Let \( \mathcal{H} \) be an Hilbert space and \( z \mapsto Q(z, h) \in \mathcal{L}(\mathcal{H}) \) an holomorphic family of operators in a neighbourhood of
\[
\Omega(h) := (w - 2\beta(h), w + 2\beta(h)) + i(-\delta(h)h^{-L}, \delta(h)),
\] (2.18)
where
\[
0 < \delta(h) < 1, \quad \text{and} \quad \beta(h)^2 \geq C h^{-3L} \delta(h)^2
\] (2.19)
for some \( L > 0 \) and \( C > 0 \). Suppose that
\[
\|Q(z, h)\|_{\mathcal{H} \to \mathcal{H}} \leq \exp(C h^{-L}), \quad z \in \Omega,
\] (2.20)
\[ \|Q(z,h)\|_{H_{\to}H} \leq \frac{C}{\operatorname{Im} z}, \quad \operatorname{Im} z > 0, \quad z \in \Omega. \quad (2.21) \]

Then
\[ \|Q(z,h)\|_{H_{\to}H} \leq \frac{C}{\delta(h)} \exp(C + 1) \quad \text{for all } z \in [w - \beta(h), w + \beta(h)]. \quad (2.22) \]

References for proof. Let \( f, g \in H \) with \( \|f\|_H = \|g\|_H = 1 \), and let
\[ F(z,h) := \langle Q(z+w,h)g,f \rangle_{H}. \]

The result (2.22) follows from the “three-line theorem in a rectangle” (a consequence of the maximum principle) stated as [DZ19, Lemma D.1] applied to the holomorphic family \( \langle F(\cdot,h) \rangle_{0<h<1} \) with
\[ R = 2\beta(h), \quad \delta_+ = \delta(h), \quad \delta_- = \delta(h)h^{-L}, \]
\[ M = M_- = \exp(CH^{-L}), \quad M_+ = C \delta(h)^{-1}. \]

\[ \square \]

2.4 The generalized bicharacteristic flow

Recall that
\[ T_{\Omega^+}^+ \mathbb{R}^d := \{ (x, \xi) \in T^* \mathbb{R}^d, x \in \overline{\Omega^+} \} = \{ x \in \overline{\Omega^+}, \xi \in \mathbb{R}^d \} \]
and
\[ S_{\Omega^+}^+ \mathbb{R}^d := \{ (x, \xi) \in S^* \mathbb{R}^d, x \in \Omega^+ \} = \{ x \in \Omega^+, \xi \in \mathbb{R}^d \text{ with } |\xi| = 1 \}. \]

We write \( \varphi_t : S_{\Omega^+}^+ \mathbb{R}^d \rightarrow S_{\Omega^+}^+ \mathbb{R}^d \) for the generalized bicharacteristic flow associated to a symbol \( p \) (see e.g. [Hör85, §24.3]). Since the flow over the interior is generated by the Hamilton vector field \( H_p \), for any symbol \( b \in C^\infty_c(T_{\Omega^+}^+ \mathbb{R}^d) \),
\[ \partial_t (b \circ \varphi_t) = H_p b = \{ p, b \}, \]

(2.23)
where \( \{ \cdot, \cdot \} \) denotes the Poisson bracket; see [Zwo12, §2.4].

We primarily consider the case when \( p \) is the semiclassical principal symbol of the Helmholtz equation, namely \( p = |\xi|^2 - 1 \). By Hamilton’s equations, away from the boundary of \( \Omega^+ \), the corresponding flow satisfies \( \dot{x}_i = 2 \xi_i \) and \( \dot{\xi}_i = 0 \), and thus, for \( p = \langle x, \xi \rangle \) with \( x \) away from \( \Gamma_D \), \( \varphi_t(p) = x + 2t\xi \) for \( t \) sufficiently small; i.e., the flow has speed two.

We let \( \pi_R \) denote the projection operator onto the spatial variables; i.e.
\[ \pi_R : T_{\Omega^+}^+ \mathbb{R}^d \rightarrow \Omega^+, \quad \pi_R(\langle x, \xi \rangle) = x. \]

2.5 Geometry of trapping

Let \( \chi \in C^\infty_c(\Omega^+;[0,1]) \) with \( \operatorname{supp} \chi \subset \mathbb{R}^d \) and \( \chi \equiv 1 \) near \( \Omega^+ \) and define \( r : T_{\Omega^+}^+ \mathbb{R}^d \rightarrow \mathbb{R} \) by
\[ r(x,\xi) := (1 - \chi(x))|x| \]
so that there is \( c > 0 \) such that for \( r_0 > c \),
\[ \{ x : r > r_0 \} = \mathbb{R}^d \setminus B(0,r_0). \]

Moreover, note that \( \{ r \leq c \} \) is compact for every \( c \). Next, define the directly escaping sets,
\[ \mathcal{E}^+ := \{ (x,\xi) \in S^* \mathbb{R}^d \mid r(x,\xi) \geq r_0, \quad \pm(x,\xi)_{\mathbb{R}^d} \geq 0 \}, \]
\[ \mathcal{E}^- := \{ (x,\xi) \in S^* \mathbb{R}^d \mid r(x,\xi) \geq r_0, \quad \pm(x,\xi)_{\mathbb{R}^d} > 0 \}. \]
Then,
\[ \rho \in \mathcal{E}_\pm \text{ implies that } \varphi_{\pm t}(\rho) \in \mathcal{E}_\pm \text{ and } r(\varphi_{\pm t}(\rho)) \geq \sqrt{r(\rho) + 4t^2}, \text{ for all } t \geq 0. \] (2.24)
Therefore, \( r(\varphi_t(\rho)) \to \infty \) as \( t \to \pm \infty \) and hence \( \rho \in \mathcal{E}_\pm \) escapes forward/backward in time. This, in particular implies that
\[ r(\rho) \geq r_0, \quad r(\varphi_{t_0}(\rho)) \leq r(\rho) \text{ for some } t_0 > 0 \quad \Rightarrow \quad \pm \langle x(\rho), \xi(\rho) \rangle > 0. \] (2.25)

We now define the outgoing tail \( \Gamma_+ \subset S^*_R \mathbb{R}^d \), the incoming tail \( \Gamma_- \subset S^*_L \mathbb{R}^d \), and the trapped set, \( K \) by
\[ \Gamma_{\pm} := \{ q \in S^*_R \mathbb{R}^d \mid r(\varphi_t(q)) \neq \infty, t \to \mp \infty \}, \quad K := \Gamma_+ \cap \Gamma_-; \] (2.26)
i.e. the outgoing tail is the set of trajectories that do not escape as \( t \to -\infty \), the incoming tail is the set of trajectories that do not escape as \( t \to \infty \), and the trapped set is the set of trajectories that do not escape in either time direction.

We now recall some basic properties of \( \Gamma_{\pm} \) and \( K \), with these proved in a more general setting in [DZ19, §6.1].

**Lemma 2.8.**
(i) The sets \( \Gamma_{\pm}, K \) are closed in \( S^*_R \mathbb{R}^d \) and \( K \subset \{ r < r_0 \} \).
(ii) Suppose that \( \rho_n \in S^*_R \mathbb{R}^d \) with \( \rho_n \to \rho \) and there are \( t_n \to \infty \) such that \( \varphi_{\pm t_n}(\rho_n) \to \rho_{\infty} \).
Then \( \rho \in \Gamma_{\pm} \).

**Proof.** (i) We show that \( \Gamma_- \) is closed in \( S^*_R \mathbb{R}^d \). Suppose that \( \rho_0 \in S^*_R \mathbb{R}^d \setminus \Gamma_- \). Then \( r(\varphi_t(\rho_0)) \to \infty \) as \( t \to \infty \). In particular, there are \( 0 < t_1 < t_2 \) such that \( r(\varphi_{t_1}(\rho_0)) \geq r_0 \) and \( r(\varphi_{t_2}(\rho_0)) \leq r(\varphi_{t_2}(\rho_0)) \). So, applying (2.25) with \( \rho = \varphi_{t_2}(\rho_0) \), we have \( \varphi_{t_2}(\rho_0) \in \mathcal{E}_+^0 \). Since \( \mathcal{E}_+^0 \) is open and \( \varphi_{t_2} \) is continuous, we have \( \varphi_{t_2}(\rho) \in \mathcal{E}_+^0 \) for all \( \rho \) sufficiently close to \( \rho_0 \) and hence, by (2.24), \( \rho \notin \Gamma_- \). Therefore \( \Gamma_- \) is closed. By an identical argument \( \Gamma_+ \) and hence \( \Gamma_- \cap \Gamma_+ \) are closed.

Now we show that \( K \subset \{ r < r_0 \} \). Note that \( S^*_R \mathbb{R}^d \cap \{ r \geq r_0 \} \subset \mathcal{E}_+ \cup \mathcal{E}_- \). But, \( \mathcal{E}_+ \cap \mathcal{E}_- = \emptyset \) and \( \mathcal{E}_- \cap \Gamma_+ = \emptyset \) and hence \( S^*_R \mathbb{R}^d \cap \{ r \geq r_0 \} \cap \Gamma_+ \cap \Gamma_- = \emptyset \) as claimed.

(ii) We prove the result for \( t_n \to \infty \); the proof of the other case is similar. Seeking a contradiction, assume that \( \rho \notin \Gamma_- \). Then there exists \( T > 0 \) such that \( r(\varphi_T(\rho)) \in \mathcal{E}_+^0 \) and hence, since \( \varphi_T \) is continuous, \( \mathcal{E}_+^0 \) is open, for \( n \) large enough, \( \varphi_T(\rho_n) \in \mathcal{E}_+^0 \). But then, by (2.24) and (2.25) for \( t \geq T \),
\[ r(\varphi_T(\rho_n)) \geq \sqrt{r_0^2 + 4(T_n - T)^2} \to \infty \]
which contradicts the fact that \( r(\varphi_T(\rho_n)) \to \rho_{\infty} \).

\[ \square \]

### 2.6 Complex scaling

We now review the method of complex scaling following [DZ19, §4.5]. We first fix a small angle of scaling, \( \theta > 0 \), and the radius, \( r_1 > r_0 \), where the scaling starts; without loss of generality, we assume that \( \Omega_1 \subset \{ x : r \leq r_1 \} \). Let \( f_\theta \in C^\infty([0, \infty) \) satisfy
\[ f_\theta(r) \equiv 0, \quad -r \leq r_1; \quad f_\theta(r) = r \tan \theta, \quad r \geq 2r_1; \]
\[ f_\theta'(r) \geq 0, \quad r \geq 0; \quad \{ f_\theta(r) = 0 \} = \{ f_\theta(r) = 0 \} \].

Then, consider the totally real manifold (see [DZ19, Definition 4.28])
\[ \Gamma_\theta := \left\{ x + i f_\theta(|x|) \frac{x}{|x|} : x \in \mathbb{R}^d \right\} \subset \mathbb{C}^d \]
and note that we identify \( \Omega_- \) with its image on \( \Gamma_\theta \). We define the complex scaled operator \( P_\theta \) on \( \Omega \) by the Dirichlet realization of
\[ P_\theta := \left( \frac{1}{1 + i f_\theta(r)} h D_r \right)^2 - \frac{(d - 1)i}{(r + i f_\theta(r))(1 + i f_\theta(r))} h^2 D_r - \frac{h^2 \Delta_\theta}{(r + i f_\theta(r))^2}, \quad \{ r \geq r_0 \}. \]

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where $\Delta_\phi$ denotes the Laplacian on the round sphere $S^{d-1}$. Note that $P_\theta$ is a semiclassical differential operator of second order such that on $r \leq r_1$, $P_\theta = -\hbar^2 \Delta$ with principal symbol, $p_\theta$, satisfying $p_\theta(x, \xi) = |\xi|^2$ on $\{r \leq r_1\}$, and in polar coordinates $x = r\phi$,

$$p_\theta(r, \phi, \xi_r, \xi_\phi) = \frac{\xi_r^2}{(1 + i f_\theta(r))^2} + \frac{|\xi_\phi|^2}{(r + i f_\theta(r))^2}. \quad (2.27)$$

Now, by e.g. [DZ19, Theorems 4.36,4.38], for $\Im(e^{i\theta}) > 0$,

$$P_\theta - \lambda^2 : H^2(\Omega_+) \cap H^1_0(\Omega_+) \rightarrow L^2(\Omega_+)$$

is a Fredholm operator of index zero. \quad (2.28)

In particular, for $V \in L^\infty(\mathbb{R}^d)$, supp $V \subset \{r < r_1\}$, this implies that

$$P_\theta - \lambda^2 + V : H^2(\Omega_+) \cap H^1_0(\Omega_+) \rightarrow L^2(\Omega_+)$$

is a Fredholm operator of index zero. \quad (2.29)

Moreover, by [DZ19, Theorem 4.37], $(P_\theta - \lambda^2 + V)^{-1}$ has the same poles as $R_V(\lambda)$ and, for $\chi \in C^\infty(\{x : r \leq r_1\})$ with supp $\chi \subset \mathbb{R}^d$,

$$\chi(P_\theta - \lambda^2 + V)^{-1}\chi = \chi R_V(\lambda)\chi. \quad (2.30)$$

### 2.7 Semiclassical pseudodifferential operators

For simplicity of exposition, we begin by discussing semiclassical pseudodifferential operators on $\mathbb{R}^d$, and then outline below how to extend the results from $\mathbb{R}^d$ to a manifold $\Gamma$ (with these results then applied with $\Gamma = \Gamma_D$ or $\Gamma = \Gamma_+$).

A symbol is a function on $T^*\mathbb{R}^d := \mathbb{R}^d \times (\mathbb{R}^d)^*$ that is also allowed to depend on $\hbar$, and thus can be considered as an $\hbar$-dependent family of functions. A family $a = (a_h)_{0 < h \leq h_0}$, with $a_h \in C^\infty(T^*\mathbb{R}^d)$, is a symbol of order $m$, written as $a \in \Psi^m(\mathbb{R}^d)$, if for any multiindices $\alpha, \beta$,

$$|\partial^\alpha_x \partial^\beta_\xi a_h(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|} \quad \text{for all } (x, \xi) \in T^*\mathbb{R}^d \text{ and for all } 0 < h \leq h_0, \quad (2.31)$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ and $C_{\alpha, \beta}$ does not depend on $\hbar$; see [Zwo12, p. 207], [DZ19, §E.1.2].

For $a \in \Psi^m$, we define the semiclassical quantisation of $a$, denoted by $\Op_\hbar(a) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$, by

$$\Op_\hbar(a) \psi(x) := (2\pi \hbar)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp(i(x - y) \cdot \xi/h) a(x, \xi) \psi(y) dy d\xi; \quad (2.32)$$

[Zwo12, §4.1] [DZ19, §E.1 (in particular Page 543)]. The integral in (2.32) need not converge, and can be understood either as an oscillatory integral in the sense of [Zwo12, §3.6], [Hö83, §7.8], or as an iterated integral, with the $y$ integration performed first; see [DZ19, Page 543].

Conversely, if $A$ can be written in the form above, i.e. $A = \Op_\hbar(a)$ with $a \in \mathcal{S}^m$, we say that $A$ is a semiclassical pseudo-differential operator of order $m$ and we write $A \in \Psi^m_\hbar$. We use the notation $a \in \hbar^\ell \mathcal{S}^m$ if $\hbar^{-\ell} a \in \mathcal{S}^m$; similarly $A \in \hbar^\ell \Psi^m_\hbar$ if $\hbar^{-\ell} A \in \Psi^m_\hbar$. We write $\Psi^\infty_\hbar = \cap_m \Psi^{-m}_\hbar$.

Let the quotient space $\mathcal{S}^m / \hbar \mathcal{S}^{m-1}$ be defined by identifying elements of $\mathcal{S}^m$ that differ only by an element of $\hbar \mathcal{S}^{m-1}$. For any $m$, there is a linear, surjective map

$$\sigma^m_\hbar : \Psi^m_\hbar \rightarrow \mathcal{S}^m / \hbar \mathcal{S}^{m-1},$$

called the principal symbol map, such that, for $a \in \mathcal{S}^m$,

$$\sigma^m_\hbar(\Op_\hbar(a)) = a \mod \hbar \mathcal{S}^{m-1}; \quad (2.33)$$

see [Zwo12, Page 213], [DZ19, Proposition E.14] (observe that (2.33) implies that $\ker(\sigma^m_\hbar) = \hbar \Psi^{-m}_\hbar$). When applying the map $\sigma^m_\hbar$ to elements of $\Psi^m_\hbar$, we denote it by $\sigma_\hbar$ (i.e. we omit the $m$ dependence) and we use $\sigma_\hbar(A)$ to denote one of the representatives in $\mathcal{S}^m$ (with the results we use then independent of the choice of representative). Key properties of the principal symbol that we use below is that

$$\sigma_\hbar(AB) = \sigma_\hbar(A)\sigma_\hbar(B) \quad \text{and} \quad \hbar^{-1}\sigma_\hbar([\Op_\hbar(a), \Op_\hbar(b)]) = -i[a, b], \quad (2.34)$$

see [Zwo12, Page 213], [DZ19, Proposition E.14].
where (as in §2.4) \(\langle \cdot, \cdot \rangle\) denotes the Poisson bracket; see [DZ19, Proposition E.17] and [DZ19, Equation E.1.44], [Zwo12, Page 68].

While the definitions above are written for operators on \(\mathbb{R}^d\), semiclassical pseudodifferential operators and all of their properties above have analogues on compact manifolds (see e.g. [Zwo12, §14.2], [DZ19, §E.1.7]). Roughly speaking, the class of semiclassical pseudodifferential operators of order \(m\) on a compact manifold \(\Gamma\), \(\Psi^m_\hbar(\Gamma)\) are operators that, in any local coordinate chart, have kernels of the form (2.32) where the function \(a \in S^m\) modulo a remainder operator \(R\) that has the property

\[
\|R\|_{H^N_{\hbar} \to H^N_{\hbar}} \leq C N h^N. \tag{2.35}
\]

We say that an operator \(R\) satisfying (2.35) is \(O(h^\infty)\).

Semiclassical pseudodifferential operators on manifolds continue to have a natural principal symbol map

\[
\sigma_\hbar : \Psi^m_\hbar(\Gamma) \to \mathcal{S}_m(\Gamma)
\]

where now \(\mathcal{S}_m(\Gamma)\) is a class of functions on \(\Gamma\) which satisfy the estimates (2.31). Furthermore, (2.34) holds as before.

Finally, there is a noncanonical quantisation map \(\text{Op}_\hbar : \mathcal{S}_m(\Gamma) \to \Psi^m_\hbar(\Gamma)\) that satisfies

\[
\sigma_\hbar(\text{Op}_\hbar(a)) = a.
\]

and for all \(A \in \Psi^m_\hbar(\Gamma)\), there is \(a \in \mathcal{S}^m(\Gamma)\) such that

\[
A = \text{Op}_\hbar(a) + O(h^\infty)\Psi_{\hbar}^{-\infty}.
\]

### 2.8 Defect measures

We say that a sequence \(\{u_\hbar,n\}_{n=1}^{\infty}\) with \(\|u_\hbar,n\|_{L^2(\mathbb{R}^d)} \leq C\) for all \(n\) (with \(C\) independent of \(n\)) has defect measure \(\mu\) if for all \(a \in C_c^\infty(T^*\mathbb{R}^d)\),

\[
\langle \text{Op}_\hbar(a)u_\hbar,n, u_\hbar,n \rangle_{L^2(\mathbb{R}^d)} \to \int a \, d\mu,
\]

where \(\text{Op}_\hbar(a)\) is defined by (2.32). By, e.g., [Zwo12, Theorem 5.2], \(\mu\) is a positive Radon measure on \(T^*\mathbb{R}^d\). We say that \(u_\hbar,n\) and \(f_\hbar,n\) have joint defect measure \(\mu^j\) if

\[
\langle \text{Op}_\hbar(a)u_\hbar,n, f_\hbar,n \rangle_{L^2(\mathbb{R}^d)} \to \int a \, d\mu^j. \tag{2.36}
\]

We usually suppress the \(n\) in the notation and instead write that \(u_\hbar\) has defect measure \(\mu\) and \(u_\hbar\) and \(f_\hbar\) have joint defect measure \(\mu^j\).

**Lemma 2.9.** ([Zwo12, Theorem 5.3].) Let \(P \in \Psi^m_\hbar(\mathbb{R}^d)\) and suppose that \(u_\hbar\) has defect measure \(\mu\) and satisfies

\[
\|\text{Pu}_\hbar\|_{L^2(\mathbb{R}^d)} = o(1).
\]

Then, \(\text{supp} \mu \subset \{\sigma_\hbar(P) = 0\}\), where \(\sigma_\hbar(P)\) is the semiclassical principal symbol of \(P\).

The following lemma is the defect-measure analogue of the propagation of singularities result [DZ19, Theorem E.47].

**Lemma 2.10.** Let \(P \in \Psi^m_\hbar(\mathbb{R}^d)\) with \(\text{Im} \sigma_\hbar(P) \leq 0\). There exists \(C > 0\) such that the following holds: suppose that \(u_\hbar\) has defect measure \(\mu\) and satisfies

\[
\text{Pu}_\hbar = hf_\hbar,
\]

where \(\|f_\hbar\|_{L^2(\mathbb{R}^d)} \leq C_1\) and \(u_\hbar\) and \(f_\hbar\) have joint defect measure \(\mu^j\). Then, for all real valued \(a \in C_c^\infty(T^*\mathbb{R}^d)\),

\[
\mu(H_{\text{Re} \sigma_\hbar(P)}a^2) \geq -2 \text{Im} \mu^j(a^2) - C_2 \mu(\|\xi\|^{m-1}a^2).
\]

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Proof. Let $A = \text{Op}_h(a)$. Since $\sigma_h(A^*) = a$ (by [DZ19, Equation E.1.45]) and thus $\sigma_h(A^*A) = a^2$ (by [DZ19, Equation E.1.43]), by the definition of the joint measure (2.36),

$$2h^{-1}\text{Im}\langle A^*Au_h,Pu_h \rangle = 2\text{Im}\mu^l(a^2) + o(1), \tag{2.37}$$

and, by (2.34) and (2.23),

$$h^{-1}\text{Im}\langle [A^*A,\text{Re}P]u_h,u_h \rangle = \mu(H_{\text{Re}\sigma_h(P)}a^2).$$

Since $2\text{Im}z = \text{Im}(z - \pi)$ and $P = \text{Re}P + i\text{Im}P$ with $\text{Re}P$ and $\text{Im}P$ both self-adjoint,

$$-2h^{-1}\text{Im}\langle A^*Au_h,Pu_h \rangle,$$

$$= h^{-1}\text{Im}\left(\langle Pu_h,A^*Au_h \rangle - \langle A^*Au_h,Pu_h \rangle \right),$$

$$= h^{-1}\text{Im}\left(\langle (A^*A + \text{Re}P)u_h,u_h \rangle + i\langle (A^*A + \text{Im}P)u_h,u_h \rangle \right),$$

$$= h^{-1}\text{Im}\left(\langle (A^*A + \text{Re}P)u_h,u_h \rangle + 2h^{-1}\text{Re}\langle A^*A\text{Im}P\rangle u_h, u_h \rangle \right),$$

$$= \mu(H_{\text{Re}\sigma_h(P)}a^2) + o(1) + 2h^{-1}\text{Re}\langle \text{Im}P\rangle u_h, u_h \rangle + 2h^{-1}\text{Re}\langle A^*[A,\text{Im}P]\rangle u_h, u_h \rangle,$$

$$\leq \mu(H_{\text{Re}\sigma_h(P)}a^2) + o(1) + 2h^{-1}\text{Re}\langle A^*[A,\text{Im}P]\rangle u_h, u_h \rangle + C\|Au\|_{H(m - 1/2)}, \tag{2.38}$$

where the last line follows from the sharp Garding inequality (see, e.g., [DZ19, Proposition E.32]) and the fact that $\text{Im}\sigma_h(P) \leq 0$. By (2.34),

$$\text{Re}h^{-1}\sigma_h(A^*[A,\text{Im}P]) = \text{Re}(-ia\{a,\text{Im}\sigma_h(P)\}) = 0,$$

and therefore, since the kernel of $\sigma_h : \Psi_h^\infty \to S^{-\infty}/hS^{-\infty}$ is $h\Psi_h^\infty$, $h^{-1}\text{Re}A^*[A,\text{Im}P] \in h\Psi_h^\infty$ and, in particular,

$$\text{Re}\langle A^*[A,\text{Im}P]u_h,u_h \rangle = O(h^2). \tag{2.39}$$

The lemma follows from combining (2.38) with (2.39) and (2.37), and sending $h \to 0$. \hfill \Box

**Corollary 2.11.** Let $\Xi \geq 0$ and suppose the assumptions of Lemma 2.10 hold and, in addition, $\mu^l = 0$. Then, with $\varphi_t$ the bicharacteristic flow corresponding to the symbol $\text{Re}\sigma_h(P)$, for any $B \subset T^*\mathbb{R}^d \cap \{\rho \leq \Xi\}$,

$$\mu\langle \varphi_t(B) \rangle \leq e^{Ct(\Xi)^{-m-1}}\mu(B) \quad \text{for} \quad t \geq 0. \tag{2.40}$$

Corollary 2.11 shows that, under the assumptions of Lemma 2.10, we have information about the defect of the sets moving forward under the flow.

**Proof of Corollary 2.11.** Let $a \in C_c^\infty(T^*\mathbb{R}^d \cap \{\rho \leq \Xi\})$ By (2.23),

$$\partial_t \left(e^{Ct(\Xi)^{-m-1}}\int (a^2 \circ \varphi_t) \, d\mu \right) = \int \partial_t(a^2 \circ \varphi_t) + (C(\Xi)^{m-2}a^2) \circ \varphi_t \, d\mu,$$

$$\geq \int \partial_t(a^2 \circ \varphi_t) + (C(\Xi)^{m-2}a^2) \circ \varphi_t \, d\mu,$$

$$= \mu(H_{\text{Re}\sigma_h(P)}a^2 + C(\Xi)^{m-2}a^2) \geq 0,$$

and thus

$$e^{Ct(\Xi)^{-m-1}}\int a^2 \, d\mu \geq \int (a^2 \circ \varphi_{-t}) \, d\mu. \tag{2.41}$$

Let $1_B$ be the indicator function of $B \subset T^*\mathbb{R}^d \cap \{\rho \leq \Xi\}$. By approximating $1_B$ by squares of smooth symbols, compactly supported symbols (2.41) holds with $a^2 = 1_B$. Since $1_B \circ \varphi_{-t} = 1_{\varphi_{-t}(B)}$, the result (2.40) follows. More precisely, we first let $B$ open and $K_n \Subset B$ compact with $K_n \uparrow B$ and choose $a_n \in C_c^\infty(T^*\mathbb{R}^d \cap \{\rho \leq \Xi\})$ with $a_n \equiv 1$ on $K_n$ and supp $a_n \subset B$. The result for $B$ open follows by monotonicity of measure from below; the result for general $B$ follows by outer regularity of $\mu$. \hfill \Box
We now review some recent results from [GLS21] about defect measures when \( u_h \) satisfies the Helmholtz equation. Let \( f_h \in L^2_{\text{comp}}(\mathbb{R}^d) \) be such that \( \| f_h \|_{L^2(\mathbb{R}^d)} \leq C \).

We use Riemannian/Fermi normal coordinates \( (x_1, x') \) in which \( \Gamma_D \) is given by \( \{ x_1 = 0 \} \) and \( \Omega_+ \) is \( \{ x_1 > 0 \} \). The conormal and cotangent variables are given by \( (\xi_1, \xi') \). Recall the definition of the hyperbolic set
\[
\mathcal{H}_{\Gamma_D} := \{(x', \xi') \in T^* \Gamma_D : |\xi'|_g < 1 \} \subset T^* \Gamma_D,
\]
(where the metric \( g \) is that induced by \( \Gamma_D \) and the definition of the gliding set
\[
G := \left\{ x_1 = H_p x_1 = 0, \quad H^2_p x_1 < 0, \quad |\xi| = 1 \right\} \subset S^*_g \mathbb{R}^d.
\]

Let \( \mathcal{N} \in \Psi^m(\Gamma_D) \) and \( \mathcal{D} \in \Psi^{m+1}(\Gamma_D) \) have real-valued principal symbols satisfying
\[
|\sigma_h(\mathcal{N})|^{2} |\langle \xi' \rangle|^{2m} + |\sigma_h(\mathcal{D})|^{2} |\langle \xi' \rangle|^{2m-2} > c > 0 \quad \text{on} \quad T^* \Gamma_D,
\]
where \( B^* \Gamma_D := \{(x', \xi') : |\xi'|_g < 1 \} \) and \( |\xi'|_g \) denotes the norm of \( \xi' \) in the metric, \( g \), induced on \( \Gamma_D \) from \( \mathbb{R}^d \). Let \( u \in L^2_{\text{loc}}(\Omega_+) \) be a solution to
\[
-h^2 \Delta - 1) u_h = h f_h \quad \text{in} \quad \Omega_+, \quad (\mathcal{N} h D_v - D) u_h |_{\Gamma_D} = o(1).
\]

Later we restrict attention to specific \( \mathcal{N} \) and \( \mathcal{D} \), but we consider more-general operators here because we believe some of our intermediate results (specifically Lemma 3.3) are of independent interest; see [GMS21].

Suppose that \( 1^\text{ext}_{\Omega_+} u_h \) has defect measure \( \mu \) and \( 1^\text{ext}_{\Omega_+} f_h \) have joint defect measure \( \mu' \). On \( \Gamma_D \), let \( \nu_j \) be the joint measure associated with the Dirichlet and Neumann traces and \( \nu_n \) be the measure associated with the Neumann trace; see [GLS21, Theorem 2.3]. In what follows, we only use the fact that \( \delta \nu_n = \nu_j \), where \( \delta \nu_j = \sigma_h(\mathcal{N})/\sigma_h(\mathcal{D}) \) is bounded (see [GLS21, Lemma 2.14 and §4]).

With \( u \) as above, let \( \mu^\text{in/ou} \) be the positive measures on \( T^* \Gamma_D \), supported in the hyperbolic set \( \mathcal{H}_{\Gamma_D} \), and defined in [GLS21, Lemma 2.9][[Mil00, Proposition 1.7, Part (ii)].

In the following lemma, \( b^* T^* \Omega_+ \) denotes the b-cotangent bundle to \( \Omega_+ \) and \( \pi : T^* \Omega_+ \to b^* T^* \Omega_+ \) is defined in local coordinates by \( \pi(x_1, x', \xi_1, \xi') := (x_1, x', x_1 \xi_1, \xi') \) (for more details about \( b^* T^* \Omega_+ \), see, e.g., [Hör85, Section 18.3], [GSW20, Section 4B]).

**Lemma 2.12.** With \( u_h, \mu, \mu', \mu^\text{in}, \mu^\text{ou} \), and \( \delta \nu_j \) as above,
\[(i) \quad \text{supp} \ \mu \subset S^* \Omega_+\]
\[(ii) \quad \text{For all} \ \chi \in C^0_{\text{c}}(\mathbb{R}^d \setminus \Omega_-), \ \lim_{h \to 0} \| \chi u_h \|_{L^2}^2 = \mu(|\chi|^2)\]
\[(iii) \quad \text{For all} \ a \in C^0_{\text{c}}(T^* \Omega_+), \]
\[
\pi_* a \circ \chi - \pi_* a = \int_0^1 \left( -2 \text{Im} \pi_* a + \delta(x_1) \otimes (\mu^\text{in} - \mu^\text{ou}) + \frac{1}{2} \frac{\sigma_h(\mathcal{N})}{\sigma_h(\mathcal{D})} H^2_p x_1 \mu \right) (a \circ \varphi) \ ds
\]
\[
(\text{where the integral is understood as the integral of distributions acting on smooth functions})\]
\[(iv) \quad \text{On} \ \mathcal{H}_{\Gamma_D}, \ \mu^\text{ou} = \alpha \mu^\text{in}, \ \text{where} \]
\[
\alpha := \frac{1 - |\xi'|_g^2}{1 - |\xi'|_g^2} \frac{\sigma_h(\mathcal{N})}{\sigma_h(\mathcal{D})} \frac{\pi_* a}{\sigma_h(\mathcal{D})} (x', \xi') \]
\[
. \quad (2.43)
\]

**References for the proof.** Parts (i) and (ii) are proved in [GSW20, Lemma 4.2]. Part (iii) is proved in [GLS21, Theorem 2.15] (following [GSW20, Lemma 4.8]), and Part (iv) is proved in [GLS21, Lemmas 2.12 and 2.18] (following [Mil00, Proposition 1.10, Part (iii)]).
3 Parametrix for \((P_\theta - \lambda^2)\) via boundary complex absorption

We now find a parametrix for \((P_\theta - \lambda^2)\) using a complex absorbing potential on the boundary \(\Gamma_D\). We then obtain by perturbation a parametrix for \((P_\theta - \lambda^2 - z1_{\Omega_n})\) for \(z\) sufficiently small.

First, let

\[
P_\theta(\lambda) := \left( P_\theta - \frac{\lambda^2}{\gamma_D^0} \right) : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D).
\]

Then let \(E : H^{3/2}(\Gamma_D) \to H^2(\Omega_+)\) be an extension operator satisfying

\[
\gamma_D^0 E g = g, \quad g \in H^{\frac{3}{2}}(\partial \Omega).
\]

Simple calculation then implies that

\[
(\mathcal{P}_\theta(\lambda))^{-1} = \left( \mathcal{R}_\theta(\lambda), \ E - \mathcal{R}_\theta(\lambda)(P_\theta - \lambda^2)E \right),
\]

(3.1)

where \(\mathcal{R}_\theta(\lambda) := (P_\theta - \lambda^2)^{-1}\) is the inverse of \((2.28)\).

**Lemma 3.1.** The operator \(\mathcal{P}_\theta(\lambda)\) is Fredholm with index zero.

**Proof.** Recall that the map \((2.28)\) is Fredholm with index zero. First, note that if \(\mathcal{P}_\theta(\lambda)u = 0\), then \(u \in H_0^1(\Omega_+) \cap H^2(\Omega_+)\) and in particular, \(u \in \ker(P_\theta - \lambda^2)\). Therefore, since \(P_\theta - \lambda^2 : H_0^1(\Omega_+) \cap H^2(\Omega_+) \to L^2(\Omega_+)\) is Fredholm, \(\ker\mathcal{P}_\theta(\lambda)\) is finite dimensional. To see that the cokernel \(L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D)/\mathcal{P}_\theta(\lambda)H^2(\Omega_+)\) is finite dimensional, define the map

\[
\pi : L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D)/\mathcal{P}_\theta(\lambda)H^2(\Omega_+) \to L^2(\Omega_+)\big/(P_\theta - \lambda^2)(H_0^1(\Omega_+) \cap H^2(\Omega_+))
\]

\[
(f, g) + \mathcal{P}_\theta(\lambda)H^2(\Omega_+) \mapsto f - (P_\theta - \lambda^2)Eg + (P_\theta - \lambda^2)(H_0^1(\Omega_+) \cap H^2(\Omega_+)).
\]

First, observe that this map is well defined since if \((f_1, g_1) + \mathcal{P}_\theta(\lambda)H^2(\Omega_+) = (f_2, g_2) + \mathcal{P}_\theta(\lambda)H^2(\Omega_+)\) then there is \(u \in H^2(\Omega_+)\) such that

\[
(f_1 - f_2, g_1 - g_2) = ((P_\theta - \lambda^2)u, \gamma_D u).
\]

In particular,

\[
(f_1 - f_2, g_1 - g_2) = (P_\theta - \lambda^2)(u - E(g_1 - g_2)) \in (P_\theta - \lambda^2)(H_0^1(\Omega_+) \cap H^2(\Omega_+)),
\]

so \(\pi(f_1, g_1) = \pi(f_2, g_2)\).

Now, suppose that \(\pi(f, g) = 0\). Then, there is \(u \in H_0^1(\Omega_+) \cap H^2(\Omega_+)\) such that

\[
f - (P_\theta - \lambda^2)Eg = (P_\theta - \lambda^2)u.
\]

Therefore,

\[
(f, g) - \mathcal{P}_\theta(\lambda)Eg = (f - (P_\theta - \lambda^2)Eg, 0) = ((P_\theta - \lambda^2)u, 0) \in \mathcal{P}_\theta(\lambda)H^2(\Omega_+),
\]

and \(\pi\) is injective. For an injective operator, \(\dim(\text{domain}) \leq \dim(\text{range}) \leq \text{dim}(\text{codomain})\); therefore

\[
\dim \left( L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D)/\mathcal{P}_\theta(\lambda)H^2(\Omega_+) \right) \leq \dim \left( L^2(\Omega_+)\big/(P_\theta - \lambda^2)(H_0^1(\Omega_+) \cap H^2(\Omega_+)) \right) < \infty
\]

Since \(P_\theta - \lambda^2 : H_0^1(\Omega_+) \cap H^2(\Omega_+) \to L^2(\Omega_+)\) is Fredholm, \(\mathcal{P}_\theta(\lambda)\) is Fredholm. To see that \(\mathcal{P}_\theta(\lambda)\) has index zero, recall that the index is constant in \(\lambda\) by, e.g., [DZ19, Theorem C.5], and observe that the formula (3.1) implies that the inverse exists for some \(\lambda\).

We now define our complex absorbing operator. Let \(\psi \in C_0^\infty(\mathbb{R}; [0, 1])\) with \(\psi \equiv 1\) on \([-b, b]\) and \(\text{supp}\psi \subseteq [-2b, 2b]\). It will be convenient to have a specific notation for the Neumann trace with the standard derivative operator replaced by \(D := -ih\theta\). We therefore let \(\gamma_D^β := -ih^β\). Let

\[
\mathcal{P}_\theta,Q(\lambda) := \left( \frac{P_\theta - \lambda^2}{Q_\theta \gamma_D^β + \gamma_D^0} \right) : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D).
\]
where \( Q_b \in \Psi^\text{comp}_h(\Gamma_D) \) with symbol

\[
\sigma_h(Q_b) = -\psi(\xi')|_\gamma.
\] (3.2)

Note that

\[
P_{\theta,Q}(\lambda) = P_{\theta}(\lambda) + \left( \begin{array}{c} 0 \\ Q_b \gamma_{1,h}^D \end{array} \right)
\]

and hence \( P_{\theta,Q}(\lambda) \) is a compact perturbation of \( P_{\theta}(\lambda) \). Therefore, by Lemma 3.1, \( P_{\theta}(\lambda) \) is Fredholm with index zero.

**Lemma 3.2.** Let \( Q_b \) be as above and \( 0 < a < b \) and \( C_1 > 0 \). Then there exists \( C > 0 \) such that for all \( \lambda \in [a, b] + i[-C_1 h, C_1 h] \),

\[
\| \gamma_{1,h}^D u \|_{L^2(\Gamma_D)} + \| u \|_{H^2_h(\Omega_+)} \leq C h^{-1} \| (P_{\theta} - \lambda^2) u \|_{L^2(\Omega_+)} + C \left( \| Q_b \gamma_{1,h}^D + \gamma_0^D \| \| u \|_{H^3_0(\Gamma_D)} \right).
\] (3.3)

In particular, since \( P_{\theta,Q}(\lambda) \) is Fredholm with index zero,

\[
R_{\theta,Q}(\lambda) := (P_{\theta,Q}(\lambda))^{-1}
\]

exists and satisfies

\[
\| \gamma_{1,h}^D R_{\theta,Q}(\lambda)(f, g) \|_{L^2(\Gamma_D)} + \| R_{\theta,Q}(\lambda)(f, g) \|_{H^2_h(\Omega_+)} \leq C \left( h^{-1} \| f \|_{L^2(\Omega_+)} + \| g \|_{H^3_0(\Gamma_D)} \right).
\] (3.4)

Observe that the bound (3.4) has the same \( h \)-dependence as the standard non-trapping resolvent estimate.

Before proving Lemma 3.2 we show how a parametrix for the operator \( (P_{\theta} - \lambda^2 - z_1\Omega_\nu) \) can be expressed in terms of \( R_{\theta,Q}(\lambda) \). Let

\[
P_{\theta,Q}(\lambda, z) := \left( P_{\theta} - \lambda^2 - z_1\Omega_\nu, Q_b \gamma_{1,h}^D + \gamma_0^D \right) : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D).
\]

By Lemma 3.2, the bound (3.4), and inversion by Neumann series, for \( |z| \leq h/(2C) \) (where \( C \) is the constant from Lemma 3.2)

\[
R_{\theta,Q}(\lambda, z) := (P_{\theta,Q}(\lambda, z))^{-1}
\]

exists and satisfies

\[
\| \gamma_{1,h}^D R_{\theta,Q}(\lambda, z)(f, g) \|_{L^2(\Gamma_D)} + \| R_{\theta,Q}(\lambda, z)(f, g) \|_{H^2_h(\Omega_+)} \leq 2C \left( h^{-1} \| f \|_{L^2(\Omega_+)} + \| g \|_{H^3_0(\Gamma_D)} \right).
\] (3.5)

Next, let

\[
P_{\theta}(\lambda, z) := \left( P_{\theta} - \lambda^2 - z_1\Omega_\nu, \gamma_0^D \right) : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}_h(\Gamma_D).
\] (3.6)

If \( R_{\theta,Q}(\lambda, z) \) exists, then

\[
P_{\theta}(\lambda, z) = (I + K(\lambda, z)) P_{\theta,Q}(\lambda, z),
\]

where

\[
K(\lambda, z) := Q R_{\theta,Q}(\lambda, z) \quad \text{and} \quad Q := \left( \begin{array}{c} 0 \\ -Q_b \gamma_{1,h}^D \end{array} \right).
\] (3.7)

Since \( K(\lambda, z) : L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D) \) is compact, \( (I + K(\lambda, z))^{-1} \) is a meromorphic family of operators by [DZ19, Theorem C.8]. Therefore, for \( |z| \leq h/(2C) \),

\[
P_{\theta}(\lambda, z)^{-1} = R_{\theta,Q}(\lambda, z)(I + K(\lambda, z))^{-1}.
\] (3.8)

Let \( R_{\theta}(\lambda, z) \) be the inverse of the map (2.29) with \( V = -z_1\Omega_\nu \), i.e.

\[
R_{\theta}(\lambda, z) := (P_{\theta} - \lambda^2 - z_1\Omega_\nu)^{-1}.
\] (3.9)

Then, for \( |z| \leq h/(2C) \),

\[
R_{\theta}(\lambda, z) = P_{\theta}(\lambda, z)^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} = R_{\theta,Q}(\lambda, z)(I + K(\lambda, z))^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix},
\] (3.10)

which is the required parametrix.

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Proof of Lemma 3.2. Suppose that the estimate (3.3) fails with the left-hand side replaced by \( \|u\|_{L^2(\Omega_+)} \), then there are \( h_n \to 0 \), \( \lambda_n \in [a, b) + i[-C h, C h] \), and \((\tilde{u}_n, \tilde{f}_n, \tilde{g}_n) \in H^2(\Omega_+) \oplus L^2(\Omega_+) \oplus H^{3/2}_h(\Gamma_D) \) with
\[
\|\tilde{f}_n\|_{L^2(\Omega_+)} + \|\tilde{g}_n\|_{H^{3/2}_h(\Gamma_D)} = 1, \quad \|\tilde{u}_n\|_{L^2} = n,
\]
and with
\[
P_{\theta, Q}(\tilde{u}_n) = (h_n \tilde{f}_n, \tilde{g}_n).
\]
In particular, renormalizing \( u_n := \tilde{u}_n/n \), \( f_n := \tilde{f}_n/n \), and \( g_n := \tilde{g}_n/n \),
\[
\|f_n\|_{L^2(\Omega_+)} = h^{-1} \| (P_\theta - \lambda_n^2) u_n \|_{L^2(\Omega_+)} \leq \frac{1}{n}
\]
and
\[
\|g_n\|_{L^2(\Gamma_D)} = \| (Q_{\theta, D}^\gamma + \gamma_D) u_n \|_{L^2(\Gamma_D)} \leq \frac{1}{n}.
\]
Now, since \( 0 < a \leq \text{Re} \lambda_n \leq b \), we may rescale \( h_n \) to \( \tilde{h} := h_n/\text{Re} \lambda_n \) and hence replace \( \text{Re} \lambda_n \) by 1. Note that this rescaling does not cause any issues since \( b^{-1} h_n \leq \tilde{h}_n \leq a^{-1} h_n \). Extracting a subsequence, we can assume that \( 1_{\overline{\Omega}_n} u_n \) has defect measure \( \mu \) (see e.g. [Zwo12, Theorem 5.2]) and \( h_n^{-1} \text{Im} \lambda_n \to \text{Im} \beta_\infty \), and \( \text{Re} \lambda_n = 1 \). Since \( \|f_h\|_{L^2} \to 0 \), \( \mu^f = 0 \).

Let \( \chi, \chi_0 \in C_0^\infty(\mathbb{R}^d; [0, 1]) \) with \( \text{supp} \chi \subset \mathbb{R}^d \) and \( \chi, \chi_0 \equiv 1 \) in a neighborhood of \( \{ r \leq 2r_1 \} \) and \( \text{supp} \chi_0 \subset \{ \chi \equiv 1 \} \). We first show that
\[
\| (1 - \chi) u_n \|_{L^2(\Omega_+)} = O(h_n).
\]
To do this, observe that, by (2.27),
\[
|\sigma_k(P_\theta - \lambda_n^2)(x, \xi)| = \left| \frac{|\xi|^2}{(1 + i \tan \theta)^2} - 1 \right| \geq c(|\xi|^2 + 1), \quad r(x, \xi) \geq 2r_1.
\]
Therefore, by ellipticity, for \( W \) a neighborhood of \( \text{supp} \partial \chi \),
\[
\|u_n\|_{H^2_\theta(W)} \leq C \left( \| (P_\theta - \lambda_n^2) u_n \|_{L^2(\Omega_+)} + \| u_n \|_{L^2(\Omega_+)} \right).
\]
Now, by (3.12) and the definitions of \( \chi \) and \( \chi_0 \),
\[
\left| \sigma_k \left( \text{Op}_h ((1 + |\xi|^2)^{-1}) (1 - \chi_0)(P_\theta - \lambda_n^2)(1 - \chi_0) - i\chi_0 \right) \right| \geq c.
\]
Therefore, by [Zwo12, Theorem 4.29],
\[
\| (1 - \chi) u_n \|_{L^2(\Omega_+)} \leq C \| \text{Op}_h((1 + |\xi|^2)^{-1})(1 - \chi_0)(P_\theta - \lambda_n^2)(1 - \chi_0) - i\chi_0 \|_{L^2(\mathbb{R}^d)} \leq C \| \text{Op}_h((1 + |\xi|^2)^{-1})(1 - \chi_0)(P_\theta - \lambda_n^2)(1 - \chi) u_n \|_{L^2(\mathbb{R}^d)}.
\]
But,
\[
\| \text{Op}_h((1 + |\xi|^2)^{-1})(1 - \chi_0)(P_\theta - \lambda_n^2)(1 - \chi) u_n \|_{L^2(\mathbb{R}^d)} \leq C \| (1 - \chi) h_n f_n \|_{L^2(\Omega_+)} + \| (P_\theta, \chi) u_n \|_{H^{3/2}_h(\Omega_+)} \leq C \| (1 - \chi) h_n f_n \|_{L^2(\Omega_+)} + Ch_n \| u_n \|_{L^2(\Omega_+)} = O(h_n).
\]
where we have used that, by direct computation, \( \| (P_\theta, \chi) \|_{H^{3/2}_h(\Omega_+) \to H^{1/2}_h(\Omega_+)} \leq Ch \) in the second inequality; (3.11) then follows from combining (3.14) and (3.15).

We now show that \( \mu(T^\theta \mathbb{R}^d) = 1 \). First, observe that
\[
(P_\theta - \lambda_n^2) \chi u_n = [P_\theta, \chi] u_n + o(h_n)_{L^2}.
\]
Consequently, using (3.13) in (3.16) we find that
\[(P_0 - \lambda^2)\chi u_n = O(h_n)_{L^2}.
\]
Since \((P_0 - \lambda^2) = (-\hbar^2 \Delta - \lambda^2)\) on \(\text{supp } \chi\), we can now apply Lemma 2.12 (with \(u\) in that lemma replaced by \(\chi u_n\) here) to find that
\[\mu(\chi^2) = \lim_{h \to 0} \|\chi u_n\|^2_{L^2(\Gamma_+)} = \lim_{h \to 0} \|u_n\|^2_{L^2(\Omega_+)} = 1,
\]
where we have used (3.11) in the second equality. Moreover,
\[\mu(T^*\mathbb{R}^d) \leq \lim_{h \to 0} \|u_n\|^2_{L^2(\Omega_+)} = 1,
\]
so that in fact \(\mu(T^*\mathbb{R}^d) = 1\).

We now show that \(\mu = 0\) which is a contradiction. To do this, we start by observing that (3.11) implies that \(\mu(\{ \rho \geq 2r_1 \}) = 0\). In fact, by Lemma 2.9, \(\mu(\{\sigma_0(P_0) \neq 0\}) = 0\), and therefore, \(\text{supp } \mu \subset S^0_{G\nu} - \mathbb{R}^d \cap \{ \rho \leq 2r_1 \}\).

Now, Lemma 2.12, along with Lemma 2.10 together with the fact that \(\text{Im } \sigma_0(P_0) \leq 0\), allows us to propagate forward along the generalized bicharacteristic flow (in the sense of Corollary 2.11), but not backward. In particular, since \(\mu(\{ \rho \geq 2r_1 \}) = 0\), this implies that \(\text{supp } \mu \subset \Gamma_+\). Indeed, suppose that \(A \subset S^0_{G\nu} - \mathbb{R}^d \cap \{ \rho \leq 2r_1 \}\) is compact and \(A \cap \Gamma_+ \neq \emptyset\). Then, by the definition of \(\Gamma_+(2.26)\), for each \(\rho \in A\) there is \(t_\rho > 0\) such that \(r(\varphi_{-t_\rho}(\rho)) > \max(2r_1, r(\rho))\). Hence, by (2.24) for \(t \geq t_\rho\), \(r(\varphi_{-t}(\rho)) > 2r_1\), and by continuity of \(\varphi_{-t}\), there is a neighborhood \(U_\rho\) of \(\rho\) such that \(\varphi_{-t}(U_\rho) \subset \{r > 2r_1\}\) for \(t \geq t_\rho\). In particular, by compactness of \(A\), there is \(T > 0\) such that \(\varphi_{-T}(A) \subset \{r > 2r_1\}\).

Hence, by compactness of \(A\) in the \(\xi\) variable and (2.40), there is \(C > 0\) such that \(\mu(A) \leq \exp(CT)\mu(\varphi_{-T}(A)) = 0\). Now, by Lemma 2.8, \(\Gamma_+\) is closed and hence we may write \((\Gamma_+)^c = \bigcup_n A_n\) with \(A_n\) compact. In particular, \(\mu((\Gamma_+)^c) = 0\) by monotonicity from below.

Next, note that since \(\text{Im } \sigma_0(P_0 - \lambda^2) < 0\) on \(\{f_0 \neq 0\}\),
\[\text{supp } \mu \subset \{f_0 = 0\}\]
by Lemma 2.9. In particular, by the definition of \(f_0\),
\[\text{supp } \mu \subset \{r < 2r_1\}\]
To complete the proof, we need to show that in fact \(\mu(\Gamma_+) = 0\). This is where the boundary term \(Q_b\) is used.

We claim there are \(T, c > 0\) such that
\[\mu(\varphi_{-T}(A)) \geq c^\nu \mu(A)\tag{3.17}
\]
for all \(A\). Once this is done, we have that \(\mu = 0\). To see this, observe that if \(\mu(A) > 0\), then by induction \(\mu(\varphi_{-nT}(A)) \geq c^\nu \mu(A)\). Taking \(N > -\frac{(\log \mu(A))}{c}\), we have \(\mu(\varphi_{-NT}(A)) > 1\), which is a contradiction to \(\mu(T^*\mathbb{R}^d) = 1\).

We now prove (3.17). First, note that the statement is empty if \(\mu(A) = 0\). Therefore, we can assume that \(\mu(A) > 0\). Since \(\text{supp } \mu \subset \Gamma_+\), we assume that \(A \subset \Gamma_+\); since \(\Gamma_+\) is closed, we can assume that \(A\) is compact. Now, by (2.24), (2.25), and (2.26),
\[\Gamma_+ \cap \{r_0 \leq r \leq 2r_1\} \subset \bigcup_{t=0}^{\sqrt{(2r_1)^2 - r_0^2}} \varphi_t(\Gamma_+ \cap \{r \leq r_0\}).
\]
Therefore, increasing \(T\) by \(\sqrt{(2r_1)^2 - r_0^2}\), we may assume that \(A \subset \{r < r_0\} \cap \Gamma_+\).

Letting \(N = Q_b\) and \(D = -1\) and recalling (3.2), we see that \(N\) and \(D\) satisfy (2.42). Therefore, the proof of (3.17) is completed by the next lemma.

**Lemma 3.3.** Suppose that \(N\) and \(D\) are as in (2.42), \(\mu\) satisfies the conclusions of Parts (iii) and (iv) of Lemma 2.12 with \(\mu^0 = 0\), and \(A \subset \{r < r_0\} \cap \Gamma_+\). Then there are \(T, c > 0\) such that (3.17) holds.
Proof. We claim that there are $\varepsilon_1, T > 0$ such that for all $\rho \in \Gamma_+$ with $r(\rho) < r_0$,

$$
\int_0^T \left( -\frac{1}{2} \frac{\sigma_h(N)}{\sigma_h(D)} \| H_p x_1 \|^2_2 \varphi_{-t}(\rho) + |H_p x_1 (\varphi_{-t}(\rho))|^{-1} \delta(x_1(\varphi_{-t}(\rho))) \log \alpha (\pi_{\Gamma_+}(\varphi_{-t}(\rho))) \right) dt \leq -\varepsilon_1.
$$

(3.18)

where $\pi_{\Gamma_D} : S^*_\Gamma \mathbb{R}^d \rightarrow T^*\Gamma_D$ is the orthogonal projection and $\alpha$ is given by (2.43).

Once (3.18) is proved, we claim that Lemma 2.12 implies (3.17) with $(c,T) = (\varepsilon_1,T)$. Indeed, suppose that (3.18) holds and that $\mu(A) > 0, A \subset \Gamma_+ \cap \{ r < r_0 \}$ and $A$ is closed. Then, let $0 \leq a \in C_c^\infty(b^2 \mathbb{R}^d \setminus \Omega_-)$ with $a \equiv 1$ on $A$ and

$$
\bigcup_{t \in [0,-T]} \varphi_t(\text{supp } a) \subset \left\{ r < \frac{r_0 + r_1}{2} \right\}.
$$

(3.19)

Now, let $\chi \equiv 1$ on $\{ r \leq \frac{r_0 + r_1}{2} \}$ with supp $\chi \subset \{ r < r_1 \}$. Then,

$$
(-h^2 \Delta - 1) \chi u = [-h^2 \Delta, \chi] u + h f,
$$

\[
\chi u|_{\Gamma_D} = 0
\]

with $f = o(1)_{L^2}$ and hence by Lemma 2.12

$$
\pi_* \mu(\chi(a \circ \varphi_t)) - \pi_* \mu(\chi(a)) = \int_0^t \left( -4(\xi, \partial_t \chi) \mu + \delta(x_1) \otimes (\mu^\text{in} - \mu^\text{out}) + \frac{\sigma_h(N)}{2 \sigma_h(D)} H_p^2 x_1 \chi \right)(\chi(a \circ \varphi_s)) ds.
$$

But, by (3.19), $\chi^2 \equiv 1$ on supp $a \circ \varphi_t$ for $t \in [0,T]$. In particular, for $t \in [0,T]$

$$
\pi_* \mu(a \circ \varphi_t) - \pi_* \mu(a) = \int_0^t \left( \delta(x_1) \otimes (\mu^\text{in} - \mu^\text{out}) + \frac{\sigma_h(N)}{2 \sigma_h(D)} H_p^2 x_1 \chi \right)(a \circ \varphi_s) ds.
$$

Finally, since $A$ is closed we may approximate $1_A$ by smooth, compactly supported functions to obtain

$$
\pi_* \mu(\varphi_{-t}(A)) - \pi_* \mu(A) = \int_0^t \left( \delta(x_1) \otimes (\mu^\text{in} - \mu^\text{out}) + \frac{\sigma_h(N)}{2 \sigma_h(D)} H_p^2 x_1 \chi \right)(1_A \circ \varphi_s) ds.
$$

(3.20)

Now, to study (3.20), we first assume that $A$ and $t$ are such that for all $\rho \in A$ and $s \in [0,2t]$, $\varphi_{-s}(\rho)$ does not lie in the glancing region ($H_p x_1 = 0$) and each trajectory intersects $\Gamma_D$ exactly once and does so for $s \in (0,t)$. Shrinking the support of $a$ further if necessary, we can find $\Sigma \subset b^2 \mathbb{R}^d \setminus \Omega_-$ transverse to the vector field $H_p$ such that

$$
F : [-t,t] \times \Sigma \ni (s, \rho) \mapsto \varphi_{-s}(\rho) \in b^2 T^*_D \mathbb{R}^d
$$

are smooth coordinates and $\varphi_{-s}(A)$ is in the image of $F$ for all $s \in [0,t]$. Then, (3.20) reads

$$
\pi_* \mu(\varphi_{-t}(A)) - \pi_* \mu(A) = \int_0^t \left( \delta(x_1) \otimes (\mu^\text{in} - \mu^\text{out}) \right)(1_A \circ \varphi_s) dt' + \int_0^t \int_{-t}^t \int_{\Sigma} \left( |H_p x_1| |(s, \rho) \delta(s) \otimes (1_A(s - t', \rho)) \right) d(\mu^\text{in} - \mu^\text{out})(\rho) ds dt'.
$$

Now, arguing as in [GLS21, Lemma 2.16], we obtain that $\pi_* \mu = |H_p x_1| \mu^\text{out} 1_{s < 0} ds + |H_p x_1| \mu^\text{in} 1_{s > 0} ds$ and hence,

$$
\pi_* \mu(A) = \int_{-t}^t \int_{\Sigma} |H_p x_1| (0, \rho) 1_A(-t', \rho) d\mu^\text{out}(\rho) dt'.
$$

Therefore

$$
\pi_* \mu(\varphi_{-t}(A)) \geq \inf_{F^{-1}(A)} (\alpha(\rho))^{-1} \pi_* \mu(A)
$$

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In particular, shrinking $A$ where this last equality comes from evaluating the integral using the fact that $F$ is well-defined (since each trajectory intersects $\Gamma_D$ exactly once),

$$A \subset \{ \varphi_s(\{x_1 = H_p x_1 = 0\}) \setminus \{ H_p x_1 \neq 0, x_1 = 0 \} : s \in [0, t] \},$$

so that, in particular, trajectories from $A$ do not intersect the hyperbolic set. In this case, (3.20) implies that

$$\partial_s \pi_* \mu(\varphi_{-s}(A)) = \left(\frac{\sigma_h(N)}{2\sigma_h(D)}\right) H_p^2 x_1 1_G \pi_* \mu(\varphi_{-s}(A)), \quad (3.21)$$

In particular, shrinking $A$ if necessary, we may choose $\Sigma \subset \{ x_1 = H_p x_1 = 0 \}$ transverse to $H_p$ and work in coordinates

$$[0, t] \times \Sigma \ni (s, \rho) \mapsto \varphi_{-s}(\rho) \in \{ \varphi_{-s}(\{x_1 = H_p x_1 = 0\}) : s \in [0, t] \}.$$ 

In these coordinates, (3.21) implies that $\pi_* \mu$ is absolutely continuous with respect to $t$ in the sense that there is a family of measures, $t \mapsto \nu_t$ on $\Sigma$ such that $\nu_t(\Sigma) \in L^1$ and $\mu = \nu_t \, dt$. Moreover,

$$\int_B \nu_t(\rho) = \int_B \exp \left( \int \frac{\sigma_h(N)}{2\sigma_h(D)} H_p^2 x_1 1_G (\varphi_{-s}(\rho)) \, ds \right) \, d\rho(\rho).$$

In particular,

$$\pi_* \mu(\varphi_{-t}(A)) \geq \inf_A \exp \left( \int \frac{\sigma_h(N)}{2\sigma_h(D)} x_1 1_G (\varphi_{-s}(\rho)) \, ds \right) \pi_* \mu(A)$$

Putting everything together, we have for all $A$ and $0 \leq t \leq T$,

$$\pi_* \mu(\varphi_{-t}(A)) \geq \inf \left\{ \int \frac{\sigma_h(N)}{2\sigma_h(D)} x_1 1_G (\varphi_{-s}(\rho)) \, ds \right\} \pi_* \mu(A)$$

as claimed.

Therefore, it is enough to prove (3.18). Seeking a contradiction, we assume that for every $\varepsilon > 0$ and $T > 0$ there is $\rho \in \Gamma_+$ with $r(\rho) < r_0$ such that

$$\int_0^T \left( -\frac{\sigma_h(N)}{2\sigma_h(D)} H_p^2 x_1 1_G (\varphi_{-t}(\rho)) + |H_p x_1 (\varphi_{-t}(\rho))|^{-1} \delta(x_1 (\varphi_{-t}(\rho))) \log \alpha(\pi_{T,D}(\varphi_{-t}(\rho))) \right) \, dt \geq -\varepsilon_1.$$ 

(3.22)

Note that since both terms are non-positive (since $\alpha \leq 1$ and $\sigma_h(N) \sigma_h(D) > 0$), this implies that each term is $\geq -\varepsilon_1$.

Now, if $\varphi_{-t}(\rho) \in \mathcal{G}$ for $t \in [t_1, t_2]$, then, since the flow in $\mathcal{G}$ is given by the flow of the vector field

$$H_p^\mathcal{G} := H_p + \frac{H_p^2 x_1}{H_{\mathcal{G}}^2} H_{x_1}, \quad p = |\xi|^2 - 1,$$

(see [Hör85, Def. 24.3.6]), we obtain, using that $\sigma_h(N)/\sigma_h(D) > c > 0$ on $\mathcal{G}$ (since $\mathcal{G} \subset B^\Gamma_D$),

$$\varphi_{-t_2}(\rho) = \exp(-(t_2 - t_1) H_{|\xi|^2}(\rho)) + \mathcal{O}\left( \int_{t_1}^{t_2} H_p^2 x_1 (\varphi_{-t}(\rho)) \, dt \right)$$

$$= \exp(-(t_2 - t_1) H_{|\xi|^2}(\rho)) + \mathcal{O}\left( \int_{t_1}^{t_2} \frac{\sigma_h(N)}{2\sigma_h(D)} H_p^2 x_1 (\varphi_{-t}(\rho)) \, dt \right),$$

where both here and in the rest of this proof we write $a = b + \mathcal{O}(c)$ if $|a - b| \leq Cc$ for some $C > 0$ depending only on $\sigma_h(N)$ and $\sigma_h(D)$. On the other hand, if $\varphi_{-t}(\rho) \notin \mathcal{G}$ for $t \in [t_1, t_2]$, and has exactly one intersection with $\Gamma_D$, then

$$\varphi_{-t_2}(\rho) = \exp(-(t_2 - t_1) H_{|\xi|^2}(\varphi_{-t_1}(\rho))) + \mathcal{O}\left( \int_{t_2}^{t_1} \sqrt{1 - |\xi|^2} \, dt \right).$$

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where $|\xi'|_g$ is measured at the point of reflection. All together, since $\sigma_h(\mathcal{N})\sigma_h(D) > c > 0$ on $|\xi'|_g \leq 1$, and thus there is $c > 0$ such that
\[
\log \alpha = -4\sqrt{1 - |\xi'|_g^2 \sigma_h(\mathcal{N})/\sigma_h(D)} + \mathcal{O}(1 - |\xi'|_g^2) \
\leq -4\sqrt{1 - |\xi'|_g^2 \sigma_h(\mathcal{N})/\sigma_h(D)} + \mathcal{O}(1 - |\xi'|_g^2),
\]
we obtain from (3.22) that
\[
\varphi_{-T}(\rho) = \exp(-TH|\xi'|^2(\rho)) + \mathcal{O}(\varepsilon_1)
\]
Therefore, choosing $T \gg r_0$, and $\varepsilon_1$ small enough, we obtain
\[
\text{dist}(\pi_\mathbb{R}(\varphi_{-T}(\rho)), \pi_\mathbb{R}(\rho)) > 3r_0
\]
which is a contradiction to $\rho \in \Gamma_+ \cap \{r \leq r_0\}$.

We have therefore proved that
\[
\|u\|_{L^2(\Omega_\varepsilon)} \leq C h^{-1} \|(P_\theta - \lambda^2)u\|_{L^2(\Omega_\varepsilon)} + C \|(Q_b^D \gamma_1^D + \gamma_0^D)u\|_{H^2(D)},
\]
where here, and in the rest of the proof, $C$ denotes a constant, independent of $h$, $\lambda$, and $\varepsilon$, whose value may change from line to line. To complete the proof of Lemma 3.2, we now need to obtain a bound on the $H^2_\varepsilon$ norm of $u$, as opposed to just the $L^2$ norm in (3.23). By a standard elliptic parametrix construction, for $\chi_1 \in C^\infty(\Omega_\varepsilon)$ supported away from $\Gamma_D$, we have
\[
\|\chi_1 u\|_{H^2_\varepsilon(\Omega_\varepsilon)} \leq C \|\chi_1 u\|_{L^2(\Omega_\varepsilon)} + C \|\chi_0 u\|_{L^2(\Omega_\varepsilon)}
\leq C h^{-1} \|(P_\theta - \lambda^2)u\|_{L^2(\Omega_\varepsilon)} + C \|(Q_b^D \gamma_1^D + \gamma_0^D)u\|_{H^2(D)},
\]
by (3.23). Finally, using the trace estimate from [GLS21, Corollary 4.2] we have for $\chi_2 \in C^\infty(\{x : r \leq r_0\})$ with supp $\chi_2 \Subset \mathbb{R}^d$, $\|\gamma_1^D u\|_{L^2(\Gamma_D)} \leq C \|\chi_2 u\|_{L^2(\Omega_\varepsilon)} + \|(-h^2\Delta - 1)\chi_2 u\|_{L^2(\Omega_\varepsilon)}$.

Elliptic regularity for the Laplacian then implies that
\[
\|\chi_2 u\|_{H^2_\varepsilon(\Omega_\varepsilon)} \leq C \|\chi_2 u\|_{L^2(\Omega_\varepsilon)} + C \|(Q_b^D \gamma_1^D + \gamma_0^D)u\|_{H^2(D)}
\leq C h^{-1} \|(P_\theta - \lambda^2)u\|_{L^2} + C \|(Q_b^D \gamma_1^D + \gamma_0^D)u\|_{H^2(D)},
\]
where we have used (3.23). Combining the bounds on $\|\chi_1 u\|_{H^2_\varepsilon(\Omega_\varepsilon)}$, $\|\chi_2 u\|_{H^2_\varepsilon(\Omega_\varepsilon)}$, and $\|\gamma_1^D u\|_{L^2(\Gamma_D)}$, we obtain (3.4).

4 Proof of Lemma 1.13

With $R(\lambda, z)$ defined by (1.11), $R_\theta(\lambda, z)$ defined by (3.9), and $\chi \in C^\infty$ with supp $\chi \subset \{x : r \leq r_1\}$ and supp $\chi \Subset \mathbb{R}^d$, (2.30) implies that
\[
\chi R_\theta(\lambda, z) \chi = \chi R(\lambda, z) \chi.
\]
Recalling (1.12), we see that to prove the bounds (1.15), (1.16) it is sufficient to bound
\[
\|R_\theta(\lambda, z)\|_{L^2(\Omega_\varepsilon) \rightarrow L^2(\Omega_\varepsilon)}.
\]
We first focus on proving the bound for $\text{Im} \, z > 0$ (1.16). By the definitions of $P_\theta(\lambda, z)$ (3.6) and $R_\theta(\lambda, z)$ (3.9), the bound (1.16) follows if we can prove the following.

**Lemma 4.1.** There exists $C > 0$ such that if $\text{Re} \, \lambda > 0$, $\text{Im} \, \lambda = 0$, \[
\|P_\theta(\lambda, z)^{-1}\|_{L^2(\Omega_\varepsilon) \otimes H^{-1/2}(\Gamma_D) \rightarrow L^2(\Omega_\varepsilon)} \leq C(z)(\text{Im} \, z)^{-1} \quad \text{for} \; \text{Im} \, z > 0.
\]
Moreover, there exists $\varepsilon > 0$ small enough such that if $\text{Re} \, \lambda > 0$, $\text{Im} \, \lambda = 0$, \[
\|P_\theta(\lambda, z)^{-1}\|_{L^2(\Omega_\varepsilon) \otimes H^{-1/2}(\Gamma_D) \rightarrow H^2(\Omega_\varepsilon)} \leq C(\text{Im} \, z)^{-1} \quad \text{for} \; \text{Im} \, z > 0 \; \text{and} \; |z| \leq \varepsilon h.
\]
To prove Lemma 4.1, we need the following result about the sign of the Dirichlet-to-Neumann map.

**Lemma 4.2.** For \( \text{Re} \lambda > 0 \), and \( \text{Im} \lambda \geq 0 \) we have \( \text{Im} \mathcal{D}(\lambda/h) \geq 0 \).

**Proof.** Let \( G(\lambda) \) be the meromorphic continuation from \( \text{Im} \lambda > 0 \) of the solution operator satisfying

\[
(h^2 \Delta - \lambda^2)G(\lambda)g = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega}, \quad G(\lambda)g|_{\Gamma_\sigma} = g,
\]

and \( G \) is \( \lambda/h \)-outgoing; then \( \mathcal{D}(\lambda/h) = \gamma_1^\lambda G(\lambda) \). Note that for \( \text{Im} \lambda > 0 \), \( G(\lambda) : H^{1/2}(\Gamma_\sigma) \to H^1(\mathbb{R}^d \setminus \Omega_\sigma) \). Therefore, for \( \text{Re} \lambda > 0 \) and \( \text{Im} \lambda > 0 \), by integration by parts,

\[
0 = \langle (h^2 \Delta - \lambda^2)G(\lambda)g, G(\lambda)g \rangle_{\mathbb{R}^d \setminus \Omega_\sigma} = \|h\nabla G(\lambda)g\|_{L^2(\mathbb{R}^d \setminus \Omega_\sigma)}^2 - \lambda^2\|G(\lambda)g\|_{L^2(\mathbb{R}^d \setminus \Omega_\sigma)}^2 + h^2\langle \mathcal{D}(\lambda/h)g, g \rangle_{\Gamma_\sigma}.
\]

Therefore, taking imaginary parts

\[
2\text{Re} \lambda \text{Im} \|G(\lambda)g\|_{L^2(\mathbb{R}^d \setminus \Omega_\sigma)}^2 = h^2 \text{Im} \langle \mathcal{D}(\lambda/h)g, g \rangle_{\Gamma_\sigma} \geq 0
\]

and in particular, for \( \text{Re} \lambda > 0 \), \( \text{Im} \lambda > 0 \)

\[
0 \leq \text{Im} \langle \mathcal{D}(\lambda/h)g, g \rangle_{\Gamma_\sigma}
\]

Now, since the right hand side continues analytically from \( \text{Im} \lambda > 0 \) to \( \text{Im} \lambda = 0 \), we have

\[
\text{Im} \langle \mathcal{D}(\lambda/h)g, g \rangle_{\Gamma_\sigma} \geq 0
\]

for \( \text{Re} \lambda > 0 \) and \( \text{Im} \lambda = 0 \). \( \square \)

**Proof of Lemma 4.1.** Let \( u \in H^2_{\text{loc}}(\Omega_\sigma) \). Then, let \( v = u - E\gamma_D g \in H^2_{\text{loc}}(\Omega_\sigma) \cap H^1_{\text{loc}}(\Omega_\sigma) \). By integration by parts,

\[
-\text{Im} \langle (P_\theta - \lambda^2 - z1_{\Omega_\sigma})v, v \rangle_{\Omega_\sigma} = -\text{Im} \langle (h^2 \Delta - \lambda^2 - z1_{\Omega_\sigma})v, v \rangle_{\Omega_\sigma} = (\text{Im} z)\|v\|_{L^2(\Omega_\sigma)}^2 + h^2 \text{Im} \langle \mathcal{D}(\lambda/h)g, v \rangle_{\Gamma_\sigma} \geq (\text{Im} z)\|v\|_{L^2(\Omega_\sigma)}^2.
\]

Therefore, there exists \( C, C_1, C_2 > 0 \) such that for \( \text{Im} z > 0 \),

\[
\|u\|_{L^2(\Omega_\sigma)}^2 \leq \|v\|_{L^2(\Omega_\sigma)}^2 + \|E\gamma_D^D u\|_{L^2(\Omega_\sigma)}^2 \leq (\text{Im} z)^{-1}\|(-h^2 \Delta - \lambda^2 - z1_{\Omega_\sigma})v\|_{L^2(\Omega_\sigma)}^2 + C_1\|\gamma_0^D u\|_{H^{1/2}_h(\Gamma_\sigma)}^2 \leq (\text{Im} z)^{-1}\|(-h^2 \Delta - \lambda^2 - z1_{\Omega_\sigma})u\|_{L^2(\Omega_\sigma)}^2 + C_2(\text{Im} z)^{-1}\|E\gamma_D^D u\|_{H^2(\Omega_\sigma)}^2 + C_1\|\gamma_0^D u\|_{H^{1/2}_h(\Gamma_\sigma)}^2 \leq C(\text{Im} z)^{-1}\|P_\theta(\lambda, z)\|_{L^2(\Omega_\sigma) \cap H^{1/2}_h(\Gamma_\sigma)}^2
\]

by the definition of \( P_\theta(\lambda, z) \) (3.6). Having obtained the bound (4.2) on \( \|u\|_{L^2(\Omega_\sigma)}^2 \), we now prove the bound (4.3) on \( \|u\|_{H^2(\Omega_\sigma)}^2 \). Using, e.g., the trace estimate from [GLS21, Corollary 4.2] (in a similar way to the end of the proof of Lemma 3.2), we have

\[
\|\gamma_1^D u\|_{L^2(\Gamma_\sigma)} \leq Ch^{-1} \|(-h^2 \Delta - \lambda^2 - z1_{\Omega_\sigma})u\|_{L^2(\Omega_\sigma)} + C(\text{Im} z)^{-1}\|u\|_{L^2(\Omega_\sigma)}.
\]

Furthermore, by (3.5) there exists \( \epsilon > 0 \) small enough such that for \( \text{Im} z > 0 \) and \( |z| \leq \epsilon h \), \( (P_\theta, Q(\lambda, z)) \) exists, and then, by Lemma 3.2 and reducing \( \epsilon \) further if necessary,

\[
\|u\|_{H^2(\Omega_\sigma)}^2 \leq Ch^{-1} \|(P_\theta - \lambda^2 - z1_{\Omega_\sigma})u\|_{L^2(\Omega_\sigma)} + C \|(Q_\theta^D + 1)u\|_{H^{1/2}_h(\Gamma_\sigma)}^2 \leq C(\text{Im} z)^{-1}\|u\|_{L^2(\Omega_\sigma)}^2 + C \gamma_0^D u\|_{H^{1/2}_h(\Gamma_\sigma)}^2.
\]

By (3.2) and the Calderon-Vaillancourt theorem (see, e.g., [DZ19, Proposition E.24], [Zwo12, Theorem 13.13]), \( \|Q_\theta|_{L^2(\Gamma_\sigma)} \leq h^{1/2} \) \( \mathcal{D}(\lambda/h) \to H^1(\mathbb{R}^d \setminus \Omega_\sigma) \), and (4.2), we obtain

\[
\|u\|_{H^2(\Omega_\sigma)}^2 \leq C(h^{-1} + (z^2)(\text{Im} z)^{-1})\|(P_\theta - \lambda^2 - z1_{\Omega_\sigma})u\|_{L^2(\Omega_\sigma)} + C \gamma_0^D u\|_{H^{1/2}_h(\Gamma_\sigma)}^2,
\]

which implies (4.3); the proof is complete. \( \square \)
Having proved the bound (1.16), we now prove the bound (1.15). From (3.10),
\[
R_\theta(\lambda, z) = R_{\theta, Q}(\lambda, z)(I + K(\lambda, z))^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}
\]  
(4.5)
where $K(\lambda, z)$ is defined by (3.7). Since we have the bound (3.5) on $R_{\theta, Q}(\lambda, z)$, to bound $R_\theta(\lambda, z)$ we only need to bound $(I + K(\lambda, z))^{-1}$.

Let $H := L^2(\Omega) \oplus H^1_0(\Gamma_D)$. Recalling the definition of trace class operators (see [DZ19, Definition B.17]) and [DZ19, Equation B.4.7], since $R_{\theta, Q}(\lambda, z)$ exists for $|z| \leq \varepsilon h$, $K(\lambda, z)$ defined by (3.7) is trace class for $|z| \leq \varepsilon h$ with
\[
\|K(\lambda, z)\|_{\mathcal{L}(H; H)} \leq \|Q_b\|_{\mathcal{L}(L^2(\Gamma_D); H^3/2(\Gamma_D))} \|\gamma_{1, h}^D R_{\theta, Q}(\lambda, z)\|_{H \rightarrow L^2(\Gamma_D)},
\]
\[
\leq C \|\langle h D \rangle^{3/2} Q_b\|_{\mathcal{L}(L^2(\Gamma_D))} \|\gamma_{1, h}^D R_{\theta, Q}(\lambda, z)\|_{H \rightarrow L^2(\Gamma_D)}.
\]
Then, using similar reasoning to that in [DZ19, Page 343] to bound the norm of $\langle h D \rangle^{3/2} Q_b$ together with the bound (3.5) on $\gamma_{1, h}^D R_{\theta, Q}(\lambda, z)$, we have
\[
\|K(\lambda, z)\|_{\mathcal{L}(H; H)} \leq C h^{1-d} h^{-1} \leq C h^{-d}.
\]  
(4.6)
Furthermore, by [DZ19, Equation B.5.21] and [DZ19, Equation B.5.19],
\[
\|(I + K(\lambda, z))^{-1}\|_{H \rightarrow H} \leq \det (I + K(\lambda, z))^{-1} \det (I + \|K(\lambda, z)\|^{1/2} K(\lambda, z)),
\]
\[
\leq \det (I + K(\lambda, z))^{1/2} \exp \left(\|\|K(\lambda, z)\|_{\mathcal{L}(H)}^2\|_{\mathcal{L}(H)}\right),
\]
\[
\leq \det (I + K(\lambda, z))^{-1} \exp \left(\|\|K(\lambda, z)\|_{\mathcal{L}(H)}\right),
\]  
(4.7)
where we have used the definition of the trace class norm $\|\cdot\|_{\mathcal{L}}$ in terms of singular values (see [DZ19, Equation B.4.2]) to write
\[
\|\|K(\lambda, z)\|_{\mathcal{L}(H)}\|_{\mathcal{L}(H)} = \|K(\lambda, z)\|_{\mathcal{L}(H)}.
\]
Using (4.6) in (4.7), we find that
\[
\|(I + K(\lambda, z))^{-1}\|_{L^2 \rightarrow L^2} \leq \det (I + K(\lambda, z))^{-1} \exp(C h^{-d}) \quad \text{for } |z| \leq \varepsilon h.
\]  
(4.8)
To estimate $\det(I + K(\lambda, z))^{-1}$ we use the same idea used to prove the bound (1.13), namely the following complex-analysis result

**Lemma 4.3. ([DZ19, Equation D.1.13].)** Let $\Omega_0 \Subset \Omega_1 \Subset \mathbb{C}$, let $f$ be holomorphic in a neighbourhood of $\Omega_1$ with zeros $z_j$, $j = 1, 2, \ldots$, and let $z_0 \in \Omega_1$. There exists $C = C(\Omega_0, \Omega_1, z_0)$ such that for any $\delta > 0$ sufficiently small
\[
\log |f(z)| \geq -C \log (\delta^{-1}) \left( \max_{z \in \Omega_1} \log |f(z)| - \log |f(z_0)| \right) \quad \text{for } z \in \Omega_0 \setminus \bigcup_j B(z_j, \delta).
\]
Applying this result with $f(z) = \det(I + K(\lambda, z))$, we see that to get an upper bound on $\log \det(I + K(\lambda, z))^{-1}$ we only need a lower bound on $\det(I + K(\lambda, z))$ for some $|z_0| \leq \varepsilon h$ and an upper bound on $\det(I + K(\lambda, z))$ for all $|z| \leq \varepsilon h$.

To obtain the upper bound for all $|z| \leq \varepsilon h$, we again use [DZ19, Equation B.5.19] and (4.6) to obtain
\[
\|\|K(\lambda, z)\|_{\mathcal{L}(H)}\|_{\mathcal{L}(H)} \leq \exp(\|K(\lambda, z)\|_{\mathcal{L}(H)}^2) \leq \exp(C h^{-d}) \quad \text{for } |z| \leq \varepsilon h.
\]  
(4.9)
To obtain the lower bound for some $|z_0| \leq \varepsilon h$, we first observe that, from (3.8),
\[
(I + K(\lambda, z))^{-1} = P_{\theta, Q}(\lambda, z)P_{\theta}(\lambda, z)^{-1} = I - QP_{\theta}(\lambda, z)^{-1}.
\]
so that
\[
\|\|K(\lambda, z)\|_{\mathcal{L}(H)}\|_{\mathcal{L}(H)}^{-1} = \|\|I - QP_{\theta}(\lambda, z)^{-1}\|_{\mathcal{L}(H)}\|_{\mathcal{L}(H)}^{-1}.
\]
Since \( Q \mathcal{P}_\delta(\lambda, z) \) is trace class, we use [DZ19, Equation B.5.19], [DZ19, Equation B.4.7], (4.6), and (4.3) to obtain

\[
\log |\det (I + K(\lambda, z_0))|^{-1} \leq \|Q\|_{L^2(H^2(\Omega_+); \mathcal{H})} \|\mathcal{P}_\delta(\lambda, z_0)^{-1}\|_{\mathcal{H} \rightarrow H^2(\Omega_+)} \leq C h^{-d} \quad \text{for } z_0 = i \varepsilon h.
\]

(4.10)

Therefore, combining Lemma 4.3, (4.9), and (4.10), we have

\[
\log |\det (I + K(\lambda, z))|^{-1} \leq C h^{-d} \log \delta^{-1}, \quad z \in B(0, \varepsilon_1 h) \setminus \bigcup_{z_j} B(z_j, \delta)
\]

where \( z_j \) are the poles of \((I + K(\lambda, z))^{-1}\). Therefore, combining this last bound with (4.5), (4.8), and (3.5), we have

\[
\|R_0(\lambda, z)\|_{L^2(\Omega_+) \rightarrow L^2(\Omega_+)} \leq \exp \left( C h^{-d} \log \delta^{-1} \right) \quad \text{for } z \in B(0, \varepsilon_1 h) \setminus \bigcup_{z_j} B(z_j, \delta).
\]

where \( z_j \) are the poles of \( R(\lambda, z) \). The bound (1.15) and the fact that \( z_j \) are the poles of \( R_{\Omega_+}(\lambda, z) \) then follow from the relation (4.1) and Lemma 1.10.

5 Proofs of Theorems 2.2 and 2.4

5.1 Proof of Theorem 2.2

With Lemma 1.13 in hand, this proof is very similar to [DZ19, Proof of Theorem 7.6], except that now we work in the complex \( z \) plane as opposed to the complex \( \lambda \) plane. In addition, in this proof, the roles of \( \varepsilon_0 \) and \( \varepsilon \) are swapped compared to [DZ19, Proof of Theorem 7.6].

Let

\[
\varepsilon_0(h) := h^{-\alpha} \varepsilon(h),
\]

(5.1)

with \( \alpha > 3(d+1)/2 \) (we see later where this requirement comes from). The lower bound (2.2) then implies that, given \( h_0 \), there exists \( C' \) (depending on \( h_0 \) and \( \alpha \)) such that

\[
\log \left( \frac{2}{\varepsilon_0(h)} \right) \leq \frac{C'}{h} \quad \text{for all } 0 < h \leq h_0.
\]

(5.2)

Seeking a contradiction, we assume that when \( h = h_j \) there are no eigenvalues in \( B(0, \varepsilon_0(h_j)) \) (the exponential lower bound on \( \varepsilon_0(h) \) leading to (5.2) therefore limits how small this ball can be). Our goal is to show that this assumption implies that

\[
\|R(1, 0)\|_{L^2(\Omega_+) \rightarrow L^2(\Omega_+)} < \frac{1}{2} (\varepsilon(h_j))^{-1}.
\]

Indeed, since supp \( u_\ell \in \Omega_1 \),

\[
R(1, 0)(-h^2 \Delta - 1)u_\ell = u_\ell.
\]

(5.4)

Then, by taking the norm of (5.4) and using (5.3), we obtain that \( \|u_\ell\|_{L^2(\Omega_+)} = 1 \). We prove (5.3) by using Theorem 2.7 where \( \Omega(h) \) is a box (to be specified below) in \( B(0, \varepsilon_0(h)/2) \) with Lemma 1.13 providing the bounds (2.20) and (2.21).

We first use the bound (1.15) from Lemma 1.13. This bound is valid for \( z \in B(0, \varepsilon_1 h) \) and away from the poles. The definition of \( \varepsilon_0(h) \) (5.1) and the upper bound in (2.3) implies that \( B(0, \varepsilon_0(h)/2) \subset B(0, \varepsilon_1 h) \) for \( h \) sufficiently small. We then choose \( \delta \) in (1.15) to equal \( \varepsilon_0(h)/2 \) and use (5.2) so that, for all \( h_j \) sufficiently small,

\[
\|R(1, z)\|_{L^2(\Omega_+) \rightarrow L^2(\Omega_+)} \leq \exp \left( C_1 C' h_j^{(d+1)} \right) \quad \text{for all } z \in B(0, \varepsilon_0(h_j)/2),
\]

(5.5)

and thus for all \( z \in \Omega(h_j) \) (since \( \Omega(h_j) \subset B(0, \varepsilon_0(h_j)/2) \)). We now let

\[
Q(z, h) := R_{\Omega_+}(1, z), \quad L := d + 1, \quad \text{and } C := \max \{ C_1 C', C_2 c \},
\]

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and the result is that the bound (2.22) holds for all \( z \) which is satisfied if
\[
\hbar \text{ is sufficiently small since } \alpha > d + 1. \quad \text{The second requirement is}
\]
\[
\frac{1}{8} \hbar^{-2\alpha} \varepsilon(h)^2 \geq C\hbar^{-3(d+1)} M \varepsilon(h)^2;
\]
given \( M \), this inequality is satisfied when \( h \) is sufficiently small since \( \alpha > 3(d+1)/2 \).

Therefore, the assumptions of Theorem 2.7 are all satisfied at \( h = h_j \) (for \( h_j \) sufficiently small), and the result is that the bound (2.22) holds for all \( z \in [-\beta(h_j), \beta(h_j)] \), and thus, in particular, at \( z = 0 \). Therefore, for all \( h_j \) sufficiently small,
\[
\| R(1,0) \|_{L^2(\Omega_{tr}) \rightarrow L^2(\Omega_{tr})} \leq \frac{C}{M \varepsilon(h)} \exp(1 + C).
\]

We now choose
\[
M := 2C \exp(1 + C),
\]
and obtain (5.3), i.e. the desired contradiction to there being no eigenvalues in \( B(0, \varepsilon_0(h_j)) \).

### 5.2 Proof of Theorem 2.4

We first recall the following lemma proved in [Ste99, Lemma 4]; see also [Laz93, Lemma AI.20].

**Lemma 5.1.** Let \( f_1, \ldots, f_N \) be \( N \) vectors in a Hilbert space \( \mathcal{H} \) with
\[
|\langle f_i, f_j \rangle_{\mathcal{H}} - \delta_{ij} | \leq \varepsilon \quad \text{for all } i, j = 1, \ldots, N.
\]

If \( \varepsilon < N^{-1} \), then \( f_1, \ldots, f_N \) are linearly independent.

We use Lemma 5.1 both in the proof of Theorem 2.4 below, and in the proof of the following preparatory result.

**Lemma 5.2.** Let \( m(h_j) \) and \( \varepsilon(h) \) be as in Theorem 2.4 (so that, in particular \( \varepsilon(h) \ll h^{(5d+3)/2} \) as \( h \to 0 \)). Then there exists \( C > 0 \) (independent of \( h_j \)) such that
\[
m(h_j) \leq C h_j^{-d}.
\]

**Proof.** First observe that it is sufficient to prove the result for sufficiently small \( h_j \) (equivalently, sufficiently large \( j \)). Let \( P(h_j) = -h_j^2 \Delta \) with zero Dirichlet boundary conditions on \( \Gamma_D \) and \( \Gamma_{tr} \). \( P(h_j) \) is therefore self-adjoint with discrete spectrum and, since \( \text{supp } u_{j,\ell} \subset \mathcal{K} \subset \Omega \),
\[
\| (P(h_j) - E_{j,\ell}) u_{j,\ell} \|_{L^2(\Omega_{tr})} = \varepsilon(h_j) \quad \text{for all } j, \ell.
\]

Let \( \mu > c > 0 \), let \( \Pi(h_j) \) be the orthogonal projection on to the eigenspaces corresponding to all eigenvalues of \( P(h_j) \) in \([a_0 - \mu, b_0 + \mu]\), and let \( M(h_j) \) be the number of these eigenvalues (counting multiplicities). By the Weyl law (with no remainder term) on manifolds with boundary (see e.g. [Hör85, Theorem 17.5.3]),
\[
M(h_j) \leq C h_j^{-d}.
\]

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Furthermore, \( \text{rank } \Pi(h_j) \leq M(h_j) \) and thus to prove the result (5.6) it is sufficient to prove that \( m(h_j) \leq \text{rank } \Pi(h_j) \). To keep expressions compact, we now write \( P \) and \( \Pi \) instead of \( P(h_j) \) and \( \Pi(h_j) \).

Since \( \Pi \) commutes with \( (P - E_{j,\ell})^{-1} \), and \( (P - E_{j,\ell}) \) is invertible on \( (I - \Pi)L^2 \),

\[
(I - \Pi)u_{j,\ell} = (P - E_{j,\ell})^{-1} (I - \Pi) (P - E_{j,\ell}) u_{j,\ell}.
\]

(5.7)

Since \( P \) is self-adjoint, the spectral theorem (see, e.g., [DZ19, Theorem B.8]) implies that

\[
\| (P - E_{j,\ell})^{-1} (I - \Pi) \|_{L^2(\Omega_\nu) \rightarrow L^2(\Omega_\nu)} \leq \frac{1}{\mu}.
\]

(5.8)

Therefore, combining (5.7) and (5.8), we have

\[
\| (I - \Pi)u_{j,\ell} \|_{L^2(\Omega_\nu) \rightarrow L^2(\Omega_\nu)} \leq \frac{\varepsilon(h_j)}{\mu}
\]

(compare to [Laz93, Equation 32.2] and the first displayed equation in [Ste99, §3]). Then, for \( \ell_1, \ell_2 \in \{1, \ldots, m(h_j)\} \),

\[
\| (\Pi u_{j,\ell_1}, \Pi u_{j,\ell_2})_{L^2(\Omega_\nu)} - \delta_{\ell_1\ell_2} \|_{L^2(\Omega_\nu)} \leq \| (u_{j,\ell_1}, u_{j,\ell_2})_{L^2(\Omega_\nu)} - \delta_{\ell_1\ell_2} \|
\]

\[
+ \| (u_{j,\ell_1}, (I - \Pi) u_{j,\ell_2})_{L^2(\Omega_\nu)} \| + \| ((I - \Pi) u_{j,\ell_1}, \Pi u_{j,\ell_2})_{L^2(\Omega_\nu)} \|
\]

\[
\leq h_j^{-2} \varepsilon(h_j) + \frac{2}{\mu} \varepsilon(h_j),
\]

(5.9)

\[
\ll h_j^{(5d-1)/2} \quad \text{as } j \to \infty,
\]

where we have used that \( \| \Pi \|_{L^2(\Omega_\nu) \rightarrow L^2(\Omega_\nu)} \leq 1 \) since \( \Pi \) is orthogonal. By Lemma 5.1, any subset of \( \{\Pi u_{j,\ell}\}_{\ell=1}^{m(h_j)} \) with cardinality \( \ll h_j^{(5d-1)/2} \) is linearly independent. Seeking a contradiction assume that (5.6) does not hold, i.e. for all \( C > 0 \) there exists \( j \) such that \( m(h_j) > Ch_j^{-d} \). Choose a subset of \( \{\Pi u_{j,\ell}\}_{\ell=1}^{m(h_j)} \) with cardinality \( [Ch_j^{-d} + 1] \). By the above argument, this subset is linearly independent, and thus \( [Ch_j^{-d} + 1] \leq \text{rank } \Pi(h_j) = M(h_j) \leq Ch_j^{-d} \) which is the required contradiction. \( \square \)

Proof of Theorem 2.4. The proof is similar to that of the corresponding “quasimodes to resonances” result [Ste99, Theorem 1] (see also [DZ19, §7.7, Exercise 1]), except that we use the semiclassical maximum principle in the \( z \) plane (as in the proof of Theorem 2.2), and now we also work in an interval in \( \lambda \) (as opposed to at \( \lambda = 1 \) in the proof of Theorem 2.2). To keep the expressions compact, we write \( h \) instead of \( h_j \) and write functions of the index \( j \) as functions of \( h \); in particular, we drop the subscript \( j \) on \( h_j, E_j, \ell, \) and \( u_{j,\ell} \).

Let

\[
Z := \tilde{Z}(\varepsilon_1(h), \varepsilon_0(h), a(h), b(h); h),
\]

where \( \tilde{Z}(\varepsilon_1, \varepsilon_0, a, b; h) \) is defined by (2.5), \( \varepsilon_0(h) \) is as in the statement of the theorem, and \( \varepsilon_1(h) \ll h \) will be fixed later. We assume throughout that \( |Z| < \infty \), since otherwise the proof is trivial. Let \( \Pi(h) \) denote the orthogonal projection onto

\[
\bigcup_{\rho \in \tilde{Z}} \Pi_{z_{\rho}}(L^2(\Omega_\nu)),
\]

where \( \Pi_{z_{\rho}} \) is defined in (2.17). Let \( \tilde{Z}(\lambda) \) be the set of distinct values of \( z_{\rho}(h, \lambda) \) such that \( \rho \in \tilde{Z} \). (While \( \tilde{Z} \) is independent of \( \lambda \), \( \tilde{Z} \) depends on \( \lambda \) since the poles of \( z \to R_{\Omega_\nu}(z, \lambda) \) depend on \( \lambda \).)

Note that for \( z_\rho \neq z_q \), \( \text{rank } \Pi_{z_\rho} + \text{rank } \Pi_{z_q} = \text{rank } \Pi_{z_\rho} + \text{rank } \Pi_{z_q} \); therefore

\[
\text{rank } \Pi(h) = \sum_{z_\rho \in \tilde{Z}(\lambda)} \text{rank } \Pi_{z_\rho}(h, \lambda) = \sum_{z_\rho \in \tilde{Z}(\lambda)} m_R(z_\rho(h, \lambda)) = |Z|,
\]

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where \( m_R(z_0) \) is defined in (2.17). To prove the theorem, therefore, it is sufficient to show that 
\( m(h) \leq \text{rank} \Pi(h) \).

Seeking a contradiction, we assume that rank \( \Pi(h) < m(h) \). By Lemma 2.6, near \( z_p \), the singular part of \( R_{\Omega_{\epsilon}}(\lambda, z) \) is in the range of \( \Pi_{z_p}(h, \lambda) \), and therefore \( z \mapsto (I - \Pi(h))R_{\Omega_{\epsilon}}(\lambda, z) \) is holomorphic on 
\[
\Omega(h) := \{ -2\varepsilon_1(h), 2\varepsilon_1(h) \} - i[0, 2\varepsilon_0(h)]
\]
for all \( \lambda^2 \in [a(h), b(h)] \). Let \( \tilde{\Omega}(h) \subset \Omega(h) \) be defined by
\[
\tilde{\Omega}(h) := \{ -\varepsilon_1(h), \varepsilon_1(h) \} - i(0, \varepsilon_0(h)).
\]

Our goal is to apply the semiclassical maximum principle (Theorem 2.7) in subsets of \( \tilde{\Omega}(h) \) with \( Q(z, h) = (I - \Pi(h))R_{\Omega_{\epsilon}}(\lambda, z) \).

By Lemma 1.13, the fact that \( \max(\varepsilon_0, \varepsilon_1) \ll h \), and the fact that \( \Pi(h) \) is orthogonal (and so \( \|\Pi(h)\|_{L^2(\Omega_{\epsilon})} < 1 \)),
\[
\left\| (I - \Pi(h))R_{\Omega_{\epsilon}}(\lambda, z) \right\|_{L^2(\Omega_{\epsilon})} \leq \exp \left( C_1 h^{-d} \log \delta^{-1} \right) \quad \text{for} \quad z \in \tilde{\Omega}(h) \setminus \bigcup_m B(z_m(h, \lambda), \delta)
\]
and for \( \lambda^2 \in [a(h), b(h)] \), where the \( z_m(h, \lambda) \) are the poles of \( R_{\Omega_{\epsilon}}(\lambda, z) \) such that \( B(z_m(h, \lambda), \delta) \cap \tilde{\Omega}(h) \neq \emptyset \). If \( \delta > \min\{\varepsilon_0(h), \varepsilon_1(h)\} \), then these \( z_m(h, \lambda) \) might include poles that are not equal to \( z_0(h, \lambda) \) for some \( p \in \mathbb{Z} \), but we restrict \( \delta \) so that this is not the case. Indeed, we now choose \( \delta > 0 \) so that the bound in (5.10) holds for all \( z \in \tilde{\Omega}(h) \) and for all \( \lambda^2 \in [a(h), b(h)] \).

If \( \delta \) and \( z_m \) are such that \( B(z_m, \delta) \in \tilde{\Omega}(h) \), then the bound in (5.10) holds on \( \partial B(z_m, \delta) \), and then, since \( z \mapsto (I - \Pi(h))R_{\Omega_{\epsilon}}(\lambda, z) \) is holomorphic in \( \tilde{\Omega}(h) \), the maximum principle implies that the bound in (5.10) holds in \( B(z_m, \delta) \). We now restrict \( \delta \) so that there cannot be a connected union of \( B(z_m, \delta) \) that intersects both \( \tilde{\Omega}(h) \) and \( \partial \Omega(h) \). Once this is ruled out, the maximum principle and the fact that \( z \mapsto (I - \Pi(h))R_{\Omega_{\epsilon}}(\lambda, z) \) is holomorphic in \( \tilde{\Omega}(h) \) imply that the bound in (5.10) holds in \( \tilde{\Omega}(h) \). Since we have assumed that rank \( \Pi(h) < m(h) \), and \( m(h) \leq C h^{-d} \) by (5.6), there exist a maximum of \( C h^{-d} \) of balls of radius \( \delta \). In particular, the maximum distance between any two points in such a connected union is bounded by \( 2C \delta h^{-d} \) and hence, a connected union intersecting both \( \partial \Omega(h) \) and \( \tilde{\Omega}(h) \) is ruled out if
\[
2C \delta h^{-d} < \min \{ \varepsilon_0(h), \varepsilon_1(h) \}.
\]
We now assume that \( \varepsilon_0(h) \leq \varepsilon_1(h) \) and set
\[
\delta := \frac{\varepsilon_0(h) h^d}{4C},
\]
so that (5.11) holds. The lower bound on \( \varepsilon_0(h) \) in (2.7) and the lower bound on \( \varepsilon(h) \) (2.2) imply that, given \( h_0 \), there exists \( C' \) (depending on \( h_0 \)) such that
\[
\log \delta^{-1} \leq C' \frac{h}{h_0} \quad \text{for all} \quad 0 < h \leq h_0.
\]
Therefore, the end result is that, if \( h \) is sufficiently small,
\[
\left\| (I - \Pi(h))R_{\Omega_{\epsilon}}(\lambda, z) \right\|_{L^2(\Omega_{\epsilon})} \leq \exp \left( C h^{-d} \right) \quad \text{for} \quad z \in \tilde{\Omega}(h) \text{ and } \lambda^2 \in [a(h), b(h)],
\]
where \( C := \max\{C_1, C', cC_2\} \), where, as in the proof of Theorem 2.2, \( c = c(h_0) \) is chosen large enough such that \( (z) \leq c \left( \forall \right) z \in \tilde{\Omega}(h) \text{ and } h \leq h_0 \).

We apply the semiclassical maximum principle (Theorem 2.7) with
\[
w = 0, \quad \beta(h) = \varepsilon_1(h), \quad \delta(h) = h^{d+1} \varepsilon_0(h), \quad \text{and} \quad L = d + 1,
\]
and we now fix \( \varepsilon_1(h) \) as
\[
\varepsilon_1(h) := \frac{h^{(d+1)/2} \varepsilon_0(h)}{C}.
\]
observe that this definition of $\varepsilon(h)$ satisfies both the second requirement in (2.19) and our previous assumption that $\varepsilon_0(h) \leq \varepsilon_1(h)$. The result of Theorem 2.7 is that

$$\| (I - \Pi(h)) R_{\partial \Omega_c} (\lambda, z) \|_{L^2(\Omega_{\ell_c}) \to L^2(\Omega_{\ell_c})} \leq C \exp(C + 1) \frac{h^{-(d+1)}}{\varepsilon_0(h)}$$

(5.12)

for $z \in [-\varepsilon_1(h), \varepsilon_1(h)]$ and $\lambda^2 \in [a(h), b(h)]$.

The definitions of $E_\ell$ and $u_\ell$ imply that

$$(I - \Pi(h)) R_{\partial \Omega_c} (\sqrt{E_\ell}, 0) (-h^2 \Delta - E_\ell) u_\ell = (I - \Pi(h)) u_\ell,$$

for $\ell = 1, \ldots, m(h)$. Since $E_\ell \in [a(h), b(h)]$ for all $\ell$, the fact that the bound (5.12) holds for all $\lambda^2 \in [a(h), b(h)]$ implies that

$$\| (I - \Pi(h)) u_\ell \|_{L^2(\Omega_{\ell_c}) \to L^2(\Omega_{\ell_c})} \leq C \exp(C + 1) h^{-(d+1)} \frac{\varepsilon(h)}{\varepsilon_0(h)}$$

for $\ell = 1, \ldots, m(h)$. Therefore

$$\left| \langle (I - \Pi(h)) u_\ell^1, (I - \Pi(h)) u_\ell^2 \rangle_{L^2(\Omega_{\ell_c})} - \delta_{\ell_1 \ell_2} \right| \leq \varepsilon(h) + 2C \exp(C + 1) h^{-(d+1)} \frac{\varepsilon(h)}{\varepsilon_0(h)}$$

(compare to (5.9), but note that now the projection $\Pi$ is different). Using the inequality (2.6) and the second inequality in (2.7), we have

$$\left| \langle (I - \Pi(h)) u_\ell^1, (I - \Pi(h)) u_\ell^2 \rangle_{L^2(\Omega_{\ell_c})} - \delta_{\ell_1 \ell_2} \right| \ll h^d \quad \text{and thus} \quad \left| \langle (I - \Pi(h)) u_\ell^1, (I - \Pi(h)) u_\ell^2 \rangle_{L^2(\Omega_{\ell_c})} - \delta_{\ell_1 \ell_2} \right| \leq \frac{h^d}{C},$$

where $C$ is the constant in (5.6). By (5.6) and Lemma 5.1, $\{ (I - \Pi(h)) u_\ell \}_{\ell=1}^{m(h)}$ are linearly independent, and thus $\text{rank} \Pi(h) \geq m(h)$, which is the desired contradiction to the assumption that $\text{rank} \Pi(h) < m(h)$.

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**A From eigenvalues to quasimodes**

**Lemma A.1** (From eigenvalues to quasimodes in $h$ notation). Suppose that there exist $z = O(h^\infty)$ and $u$ satisfying (2.4) with $\|u\|_{L^2(\Omega_{\ell_c})} = 1$. Let $\chi \in C_\infty^\infty(\Omega_1)$ with $\chi \equiv 1$ in a neighborhood of $\pi_\Sigma(K)$. Then $\chi u$ is a quasimode (in the sense of Definition 2.1) of quality $\varepsilon(h) = O(h^\infty)$ satisfying

$$\|u - \chi u\|_{H^1_0(\Omega_{\ell_c})} = O(h^\infty).$$

**Proof.** The proof is similar to the proof of the “resonances to quasimodes” result of [Ste00, Theorem 1], except that we avoid using results about $\mathcal{D}$ for strictly convex obstacles that are used in [Ste00] and instead use a commutator argument.

First observe that

$$(-h^2 \Delta - 1 - z)u = 0 \quad \text{in } \Omega_{\ell_c},$$

so that

$$u = 1_{\Omega_{\ell_c}}^\text{res} R(1, 0) 1_{\Omega_{\ell_c}}^\text{ext} z u.$$
Therefore,  
\[ u = 1_{\Omega_{tr}}^{\text{res}} R_\theta(1, 0) 1_{\Omega_{tr}}^{\text{ext}} z u \]
by (2.30) and the definition of \( R_\theta(\lambda, z) \). Let  
\[ v = R_\theta(1, 0) 1_{\Omega_{tr}}^{\text{ext}} z u, \]  
(A.1)
and observe that \( v = u \) on \( \Omega_{tr} \).

We now claim that, since \( z = O(h^{\infty}) \) and \( \Omega_{tr} \subseteq \mathbb{R}^d \), \( \text{WF}_h(v) \subset \Gamma_+ \) (as defined by (2.26)). By the definition of the wavefront set \([\text{DZ19}, \text{Definition E.36}]\), this is equivalent to \( Av = O(h^{\infty}) \) for all \( A \) with \( \text{WF}_h(A) \subset (\Gamma^+)^c \). This then follows by noting that \( (P_\theta - 1) v = O(h^{\infty}) L_{\text{comp}}^\infty \)
and applying \([\text{DZ19, Theorem E.47}], [\text{Hoer85, Section 24.4}], [\text{Vaso08, Theorem 8.1}] \) \(^1\) (with, in the notation of \([\text{DZ19, Theorem E.47}], B_1 = I, B = P = P_\theta - 1 \) together with the facts that \( \sigma_0(\text{Im}(P_\theta - 1)) \leq 0 \) and that \( P_\theta - 1 \) is elliptic on \( \{ r \geq 2 r_1 \} \) (so that if \( x_0, \xi_0 \) in \( \text{WF}_h(A) \) then there exists \( T \geq 0 \) such that \( \varphi - t T(x_0, \xi_0) \in \text{dB}_h(P_\theta - 1) \)).

Now let \( \chi \in C_c^\infty(\Omega_1) \) with \( \chi \equiv 1 \) in a neighborhood of \( \pi_{\mathbb{R}}(K) \). We claim that \( \chi v = \chi u \) is a quasimode with quality \( \varepsilon(h) = O(h^{\infty}) \). To prove this, since
\[ \| u - \chi u \|_{H_2^2(\Omega_{tr})} = \| (1 - \chi) u \|_{H_2^2(\Omega_{tr})} = \| (1 - \chi) v \|_{H_2^2(\Omega_{tr} \setminus \{ \chi = 1 \})}, \]  
(A.2)
it is sufficient to prove that \( v = O(h^{\infty}) H_{2,\text{loc}}^\infty \) outside a compact set.

Our first step is to prove that, with \( r_0 < a < b < r_1 \), for \( h \) sufficiently small,
\[ \| v \|_{L^2(r > a)} \leq C h^{-1} \| (P_\theta - 1) v \|_{L^2(\Omega_+)} + C \| v \|_{L^2(a < r < b)}, \]  
(A.3)
where here, and in the rest of the proof, \( C \) denotes a constant, independent of \( h \) and \( z \), whose value may change from line to line. To prove (A.3), first observe that, since \( P_\theta - 1 \) is elliptic on \( r \geq 2 r_1 \), by \([\text{DZ19, Theorem E.33}] \) (more precisely its proof together with the calculus from \([\text{Zwo12, Chapter 4}])\),
\[ \| v \|_{L^2(r > 3 r_1)} \leq C \| (P_\theta - 1) v \|_{L^2(\Omega_+)} + C h N \| v \|_{L^2(r > 2 r_1)}, \]
and hence
\[ \| v \|_{L^2(r > 3 r_1)} \leq C \| (P_\theta - 1) v \|_{L^2(\Omega_+)} + C h N \| v \|_{L^2(2 r_1 < r < 4 r_1)}. \]  
(A.4)
Next, observe that there exists \( T > 0 \) such that for all \( \rho \in \Gamma_+ \cap \{ \frac{2 \pi h}{T} < r < 4 r_1 \} \), there exists \( 0 \leq t \leq T \) such that \( a < r(\varphi - t(\rho)) < b \). In particular, using \([\text{DZ19, Theorem E.47}] \) again, we have
\[ \| v \|_{L^2(\frac{2 \pi h}{T} < r < 4 r_1)} \leq C h^{-1} \| (P_\theta - 1) v \|_{L^2(\Omega_+)} + \| v \|_{L^2(0 < r < b)} + C h N \| v \|_{L^2(r > a)}. \]
Using this and (A.4) in
\[ \| v \|_{L^2(r > a)} \leq \| v \|_{L^2(0 < r < b)} + \| v \|_{L^2(\frac{2 \pi h}{T} < r < 4 r_1)} + \| v \|_{L^2(r > 3 r_1)}, \]
we obtain (A.3) for \( h \) sufficiently small.

The next part of the proof involves using a commutator argument to control (up to \( h^{\infty} \) errors) \( \| v \|_{L^2(0 < r < b)} \) by the norm on a slightly bigger region and with a gain of \( h \) (see (A.5) below). Let \( \psi \in C_c^\infty(-r_1, r_1) \) with \( \psi \equiv 1 \) on \( \{ |x| \leq r_0 \} \), \( \psi'(x) \leq 0 \), and \( \psi(x') < 0 \) on \( a \leq |x| \leq b \). Then,
\[
2 h^{-1} \text{Im} \left( \langle ( - h^2 \Delta - 1 ) v, ( - h^2 \Delta - 1 ) \psi(x) \rangle \right) \\
= - i h^{-1} \left( \langle ( - h^2 \Delta - 1 ) v, \psi'(x) \rangle \right) \left\| _{L^2(\Omega_+)} - \langle \psi(x) v, ( - h^2 \Delta - 1 ) \psi(x) \rangle \right\|_{L^2(\Omega_+)} \\
= i h^{-1} \left( \langle ( - h^2 \Delta - 1 ) \psi(x) v, \psi(x) \rangle \right) \left\| _{L^2(\Omega_+)} - \langle 2 \psi'(x) h D_r - i h (\Delta \psi(x)) \psi(x) \rangle \right\|_{L^2(\Omega_+)} .
\]  

\(^1\)Strictly speaking \([\text{DZ19, Theorem E.47}] \) is used away from the boundary and \([\text{Vaso08, Theorem 8.1}] \) is written for the time dependent problem, but the semiclassical version can be easily recovered by applying the time dependent results to \( e^{it \phi_t(x)} v \). It is then necessary to use the arguments in \([\text{Hoer85, Section 24.4}] \) to obtain the 'diffractive improvement' i.e. that singularities hitting a diffractive point follow only the flow of \( H_\nu \) rather than sticking to the boundary. A careful examination of \([\text{Hoer85, Lemma 24.4.7}] \) shows that the norm on the error term on \( (P_\theta - 1) v \) is correct.
By the definition of $\Gamma_+$ (2.26), $\sigma_h(\psi'(r)hD_x) = \psi'(r)\langle \xi, \frac{x}{\hbar} \rangle \leq -c < 0$ on $\Gamma_+ \cap \{a \leq r \leq b\}$. Therefore, since WF$_h(v) \subset \Gamma_+$, for $\psi_1 \in C_c^\infty(r_0 < r < r_1)$ with $\psi_1 \equiv 1$ in a neighbourhood of supp $\psi(r)$

$$2\hbar^{-1} \text{Im} \left( (\hbar^{-2}\Delta - 1)v, \psi(r)v \right)_{L^2(\Omega_s)} \leq -c\|\psi\|_{L^2(a<r<b)}^2 + C\|\psi_1\|_{L^2(\Omega_+ )}^2 + C\hbar h^N \|v\|_{L^2(\Omega_+) }^2,$$

by the microlocal Garding inequality [DZ19, Proposition E.34] (with $A = -\psi'(r)hD_x - c$ and $B$ supported in $\langle \xi, x/|x| \rangle < \epsilon$, i.e., away from $\Gamma^+$, and in $\{r_0 \leq r \leq r_1\}$). Therefore, by Young's inequality,

$$\|v\|_{L^2(a<r<b)}^2 \leq C h^{-N-2}\|(-\hbar^{-2}\Delta - 1)v\|_{L^2(r \leq r_1)}^2 + C\|\psi_1\|_{L^2(\Omega_+) }^2 + C\hbar h^N \|v\|_{L^2(\Omega_+) }^2 \quad (A.5)$$

We now use the propagation estimate again to control (up to $h^\infty$ errors) $\|\psi_1v\|_{L^2(\Omega_s)}^2$ by $\|v\|_{L^2(a<r<b)}^2$. Suppose that $\rho \in \mathfrak{r}^{-1}(\{\text{supp } \psi_1\}) \cap \Gamma_+$. Then, there exists $|t| \leq \sqrt{r_1^2 - r_0^2}$ such that $\varphi_\rho(t) \in \{a < r < b\}$. Therefore, by standard propagation estimates [DZ19, Theorem E.47], again using that $WF_h(v) \subset \Gamma_+$, we have

$$\|\psi_1v\|_{L^2(\Omega_s)}^2 \leq C h^{-1}\|(-\hbar^{-2}\Delta - 1)v\|_{L^2(r \leq r_1)}^2 + C\|v\|_{L^2(a<r<b)}^2 + C\hbar h^N \|v\|_{L^2(\Omega_+) }^2 \quad (A.6)$$

We next use the propagation estimate again to control $\|v\|_{L^2((\Omega_+ \setminus \chi=1))}$ by $\|v\|_{L^2(a<r<b)}$. To do this, we need that there exists $T > 0$ such that for all $\rho \in S^0_{\Omega_+ \setminus \chi=1} \Omega_+$ with $|\rho| \leq r_1$ there is $|t| \leq T$ with $a < r(\varphi_\rho(t)) < b$. Suppose not; then there exist $\rho_n \in S^0_{\Omega_+ \setminus \chi=1} \Omega_+$ with $|\rho(\rho_n)| \leq r_1$ and $T_n \to \infty$ such that

$$\bigcup_{|t| \leq T_n} \varphi_\rho(t) \cap \{a < r < b\} = \emptyset.$$ 

By (2.24), we have $\rho(\rho_n) \to r_0$ and also $\rho(\varphi_{\rho_n}(\rho_n)) \to r_0$. In particular, we may assume that $\rho_n \to \rho \in \{r \leq r_0\} \setminus K$ (since $\pi\mathfrak{r}(K) \subset \{\chi \equiv 1\}$) and $\varphi_{\rho_n}(\rho_n) \to \rho_\perp$. Then, by Lemma 2.8, $\rho \in \Gamma_+ \cap \Gamma_- = K$, which is a contradiction. Applying the propagation estimate (using the existence of the uniform time $T$), we have

$$\|v\|_{L^2((\varphi_{\rho_n}(\rho_n) = 1))}^2 \leq C h^{-1}\|(-\hbar^{-2}\Delta - 1)v\|_{L^2(r \leq r_1)}^2 + C\|v\|_{L^2(a<r<b)}^2 + C\hbar h^N \|v\|_{L^2(\Omega_+) }^2 \quad (A.7)$$

Finally, we control $\|v\|_{L^2(\Omega_+ \setminus \Omega_\chi)}$. For this, note that, $v = u_1 \chi_\Omega + v_1 \chi_\Omega$, and by (A.3) and (A.7) we have

$$\|v\|_{L^2(\Omega_+ \setminus \Omega_\chi)} \leq C h^{-1}\|(P_0 - 1)v\|_{L^2(\Omega_+) } + C\|v\|_{L^2(a<r<b)} + C\hbar h^N \|u\|_{L^2(\Omega_+)}. \quad (A.8)$$

Now, using (A.6) in (A.5), and then using the definition of $v$ (A.1) and that $v = u$ on $\Omega_{tr}$, we have

$$\|v\|_{L^2(a<r<b)}^2 \leq C h^{-N-2}\|(-\hbar^{-2}\Delta - 1)v\|_{L^2(r \leq r_1)}^2 + C\hbar h^N \|v\|_{L^2(\Omega_+) }^2.$$

Then, using (A.8),

$$\|v\|_{L^2(a<r<b)}^2 \leq C\hbar h^N \|u\|_{L^2(\Omega_\chi)}^2 + C\hbar h^N \|v\|_{L^2(a<r<b)}^2,$$

and, taking $\hbar$ small enough, we obtain

$$\|v\|_{L^2(a<r<b)} \leq C\hbar h^N \|u\|_{L^2(\Omega_\chi)} \leq C\hbar h^N,$$

since $\|u\|_{L^2(\Omega_\chi)} = 1$. Therefore, using (A.7), (A.8), the definition of $v$ (A.1), and the fact that $z = O(h^\infty)$, we have

$$\|\psi(\mathfrak{r})v\|_{L^2(\Omega_+ \setminus \chi=1)}^2 = O(h^\infty).$$

so that, since $WF_h(v) \subset S^*\mathbb{R}^d$ (which is fibre compact),

$$\|\psi(\mathfrak{r})v\|_{H^2(\Omega_+ \setminus \chi=1)} = O(h^\infty);$$

the result then follows from (A.2). \qed
B Details of how the eigenvalues/eigenfunctions were computed in §1.3

When discretising the sesquilinear form $a(\cdot, \cdot)$ defined by (1.7), we need to calculate the Dirichlet-to-Neumann map $D(k)$. Instead of approximating $D(k)$ using either a perfectly-matched layer (PML) or an absorbing boundary condition, we use boundary integral operators to find $D(k)$ “exactly” (i.e., up to the discretisation of these integral operators).

Recall that the single-layer potential on $\Gamma_{tr}$ is defined for $\varphi \in L^2(\Gamma)$ by

$$S_k \varphi(x) := \int_{\Gamma_{tr}} \Phi_k(x, y) \varphi(y) \, ds(y) \quad \text{for all } x \in \mathbb{R}^d \setminus \Gamma_{tr},$$

where, in 2-d, $\Phi_k(x, y) := i H_0^{(1)}(k|x-y|)/4$, where $H_0^{(1)}$ is the order zero Hankel function of the first kind. The single-layer and adjoint-double-layer operators are then defined, respectively, by $S_k := \gamma_0^T S_k$ and $D_k' := \gamma_0^T S_k - I/2$, where the traces are taken from inside $\Omega_{tr}$. With these definitions, for values of $k$ for which $S_k : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ is invertible,

$$D(k) = \left( -\frac{1}{2} I + D_k' \right) S_k^{-1}, \quad \text{(B.1)}$$

see, e.g., [CWGLS12, Page 136].

To avoid the operator product in (B.1), we introduce the auxiliary variable $\varphi_{\ell} = S_k^{-1}(\gamma_0^T u_{\ell}) \in H^{-1/2}(\Gamma_{tr})$. The eigenvalue problem (1.5) can therefore be rewritten as: find $u_{\ell} \in H^1_{0,D}(\Omega_{tr})$ and $\varphi_{\ell} \in H^{-1/2}(\Gamma_{tr})$ such that

$$\langle \nabla u_{\ell}, \nabla v \rangle_{L^2(\Omega_{tr})} - k^2 (u_{\ell}, v)_{L^2(\Omega_{tr})} - \left( \left( -\frac{1}{2} I + D_k' \right) \varphi_{\ell}, \gamma_0^T v \right)_{\Gamma_{tr}} = \mu_{\ell} (u_{\ell}, v)_{L^2(\Omega_{tr})},$$

and

$$\langle \gamma_0^T u_{\ell}, \psi \rangle_{\Gamma_{tr}} - \langle S_k \varphi_{\ell}, \psi \rangle_{\Gamma_{tr}} = 0, \quad \text{(B.2)}$$

for all $v \in H^1_{0,D}(\Omega_{tr})$ and $\psi \in H^{-1/2}(\Gamma_{tr})$. We note that this formulation is the transpose of the Johnson–Nédélec FEM-BEM coupling [JN80] applied to the eigenvalue problem (1.5); see, e.g., [GHS12, Equation 9].

We use continuous piecewise-linear basis functions to discretise (B.2), and obtain the following generalised eigenvalue problem

$$\tilde{A} u_{\ell} = \begin{pmatrix} A_k & 1/2 (M^T)^T & -D_k' \\ M^T & -S_k & \end{pmatrix} \begin{pmatrix} u_{\ell} \\ \mu_{\ell} \end{pmatrix} = \mu_{\ell} \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{\ell} \\ \mu_{\ell} \end{pmatrix} =: \mu_{\ell} B u_{\ell}, \quad \text{(B.3)}$$

where $M$ is the mass matrix on $\Omega_{tr}$, $S_k$ is a discretisation of the single-layer operator, and $A_k$ is the Galerkin matrix corresponding to the discretisation of $(\nabla u_{\ell}, \nabla v) - k^2 (u_{\ell}, v)$. The matrices $M^T$ and $D_k'$ are defined by

$$(M^T)_{i,j} = \langle \gamma_0^T v_j, \psi_i \rangle_{\Gamma_{tr}} \quad \text{and} \quad (D_k')_{j,i} = \langle D_k' \psi_i, \gamma_0^T v_j \rangle_{\Gamma_{tr}}, \quad \text{for } i = 1 \ldots M \text{ and } j = 1 \ldots N,$$

where $v_j$ and $\psi_i$ are, respectively, continuous piecewise-linear basis functions of the Galerkin discretisations $V_h(\Omega_{tr}) \subset H^1_{0,D}(\Omega_{tr})$ and $V_h(\Gamma_{tr}) \subset H^{-1/2}(\Gamma_{tr})$; the dimensions of these spaces are denoted $N$ and $M$ respectively.

To build the matrices in (B.3) and solve this problem, we use PETSc [BGMS97, BAA+19, BAA+20] and the eigensolver SLEPc [RCRT20, HRV05] via the software FreeFEM [Hec12]. Since we are interested in the eigenvalues near the origin, we use the shift-and-invert technique, i.e., we compute the largest eigenvalues of the problem $(\tilde{A})^{-1} B u_{\ell} = \nu_{\ell} u_{\ell}$, and then set $\mu_{\ell} = 1/\nu_{\ell}$. To obtain the action of $(\tilde{A})^{-1}$, we use SuperLU [LD03] to compute the LU factorisation of $\tilde{A}$.

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