A GEOMETRIC GENERALIZATION OF KAPLANSKY’S DIRECT FINITENESS CONJECTURE

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Abstract. Let $G$ be a group and let $k$ be a field. Kaplansky’s direct finiteness conjecture states that every one-sided unit of the group ring $k[G]$ must be a two-sided unit. In this paper, we establish a geometric direct finiteness theorem for endomorphisms of symbolic algebraic varieties. Whenever $G$ is a sofic group or more generally a surjunctive group, our result implies a generalization of Kaplansky’s direct finiteness conjecture for the near ring $R(k, G)$ which is $k[X_g : g \in G]$ as a group and which contains naturally $k[G]$ as the subring of homogeneous polynomials of degree one. We also prove that Kaplansky’s stable finiteness conjecture is a consequence of Gottschalk’s Surjunctivity conjecture.

1. Introduction

In [11], Kaplansky conjectured that for every group $G$ and for every field $k$, the group ring $k[G]$ is directly finite, i.e., for $a, b \in k[G]$ with $ab = 1$, one has $ba = 1$. The conjecture was settled by Kaplansky himself in [11] in characteristic zero but is still open in positive characteristic. However, in arbitrary characteristic, the conjecture is known for the wide class of sofic groups $G$ (see [7], [20]) while no examples of non sofic groups are known in the literature. Results in [6] proved that $k[G]$ is directly finite for every field $k$ and every sofic group $G$. More generally, the papers [6] (see also [1]), [2], and [12] show that $k[G]$ is directly finite for every sofic group $G$ when $k$ is respectively a division ring, an Artinian ring, a left Noetherian ring.

Our first goal is to establish an algebraic generalization of Kaplansky’s direct finiteness conjecture for the class of near rings $R(K, G)$ associated with a field $K$ and a sofic group $G$ that we describe briefly in the sequel.

Let $k$ be a field and let $G$ be a group. Following [15], we have a near ring $(R(k, G), +, \ast)$ where $(R(k, G), +) = (k[X_g : g \in G], +)$ as a group while the multiplication $\ast$ is induced naturally by the group $G$ as follows.

Let us consider the semi-ring:

\begin{equation}
N[G] := \{ f : G \to \mathbb{N} : f(g) \neq 0 \text{ for finitely many } g \in G \}.
\end{equation}
With each \( u \in \mathbb{N}[G] \), we associate a monomial \( X^u := \prod_{g \in G} X_g^{u(g)} \) with the convention \( X^0 = 1 \) so that

\[
(1.2) \quad R(K, G) = \left\{ \sum_{u \in \mathbb{N}[G]} \alpha(u)X^u : \alpha(u) \neq 0 \text{ for finitely many } u \in \mathbb{N}[G] \right\}.
\]

We now describe the natural actions of \( G \) on \( \mathbb{N}[G] \) and \( R(k, G) \). Let us fix \( g \in G \). For \( u \in \mathbb{N}[G] \), we define \( gu \in \mathbb{N}[G] \) by setting \( (gu)(h) := u(g^{-1}h) \) for every \( h \in G \). Then for \( \gamma = \sum_{w \in \mathbb{N}[G]} \gamma(w)X^w \in R(k, G) \), we set

\[
(1.3) \quad g\gamma := \sum_{w \in \mathbb{N}[G]} \gamma(w)X^{gw} \in R(k, G).
\]

For \( \alpha = \sum_{u \in \mathbb{N}[G]} \alpha(u)X^u \) and \( \beta = \sum_{v \in \mathbb{N}[G]} \beta(v)X^v \) elements of \( R(K, G) \), their multiplication \( \alpha \star \beta \) is given by:

\[
(1.4) \quad \alpha \star \beta := \sum_{u \in \mathbb{N}[G]} \alpha(u)X^u \star \beta := \sum_{u \in \mathbb{N}[G]} \alpha(u) \prod_{g \in G} (g\beta)^{u(g)}.
\]

Therefore, \( \alpha \star \beta = \sum_{u \in \mathbb{N}[G]} \alpha(u) \prod_{g \in G} \left( \sum_{v \in \mathbb{N}[G]} \beta(v)X^{gv} \right)^{u(g)} \).

**Example.** Let \( g, h, s, t \in G \) and \( \alpha = X_gX_h^2 + 1, \beta = X_s^2 - X_t^3 \). Then

\[
\alpha \star \beta = (X_gX_s - X_t^3)(X_hX_s - X_t^3)^2 + 1, \quad \beta \star \alpha = (X_{gs}X_{sh}^2 + 1)^2 - (X_{tg}X_{th}^2 + 1)^3.
\]

It follows from [15, Proposition 10.4] that \((R(k, G), +, \star)\) is a left near ring, i.e., \((R(k, G), \star)\) is a monoid with identity element \( X_{1_G} \), and we have \( (\alpha + \beta) \star \gamma = \alpha \star \gamma + \beta \star \gamma \) for all \( \alpha, \beta, \gamma \in R(k, G) \).

The first main result of the paper is the following generalization of Kaplansky’s direct finiteness conjecture for the near ring \( R(k, G) \) over sofic groups in arbitrary characteristic (see Theorem 5.1):

**Theorem A.** Let \( G \) be a sofic group and let \( k \) be a field. Suppose that \( \alpha, \beta \in R(k, G) \) verify \( \alpha \star \beta = X_{1_G} \). Then one has \( \beta \star \alpha = X_{1_G} \).

In particular, Theorem A generalizes [15] Theorem 10.12 where \( G \) is required to be a residually finite group.

On the other hand, by [15] Proposition 10.5, it is known that for every field \( k \) and for every group \( G \), the canonical map \( \Phi : k[G] \to R(k, G) \) determined by \( \sum_{g \in G} \alpha(g)g \mapsto \sum_{g \in G} \alpha(g)X_g \) is an embedding of near rings.

As a consequence, we deduce:

**Corollary 1.1.** Let \( G \) be a sofic group and let \( k \) be a field. Suppose that \( \alpha, \beta \in k[G] \) satisfy \( \alpha \beta = 1 \). Then one has \( \beta \alpha = 1 \).

Now fix a set \( A \) called the *alphabet*, and a group \( G \), the *universe*. A *configuration* \( c \in A^G \) is a map \( c : G \to A \). The Bernoulli shift \( G \times A^G \to A^G \) is defined by \((g, c) \mapsto gc\), where \((gc)(h) := c(g^{-1}h)\) for \( g, h \in G \) and \( c \in A^G \).
Introduced by von Neumann [14], a cellular automaton over the group \( G \) and the alphabet \( A \) is a map \( \tau: A^G \to A^G \) admitting a finite memory set \( M \subset G \) and a local defining map \( \mu: A^M \to A \) such that

\[
(\tau(c))(g) = \mu((g^{-1}c)|_M) \quad \text{for all } c \in A^G \text{ and } g \in G.
\]

The well-known Gottschalk’s conjecture [10] asserts that over any universe, every injective cellular automaton with finite alphabet is surjective. We call a group \( G \) surjunctive if it satisfies Gottschalk’s conjecture, i.e., for every finite alphabet \( A \), every injective cellular automaton \( \tau: A^G \to A^G \) must be surjective. Hence, it follows from Gromov-Weiss theorem ([7], [20]) that all sofic groups are surjunctive.

Generalizing the direct finiteness conjecture, Kaplansky’s stable finiteness conjecture for a field \( k \) and a group \( G \) states that the ring \( \text{Mat}_n(k[G]) \) of square matrices of size \( n \) with coefficients in \( k[G] \) is directly finite for every \( n \geq 1 \). We establish the following general result (Section 5.1):

**Theorem B.** Kaplansky’s stable finiteness conjecture holds for every surjunctive group \( G \) and every field \( k \). In particular, Gottschalk’s conjecture implies Kaplansky’s stable finiteness conjecture.

Let \( G \) be a group and let \( X \) be an algebraic variety over a field \( k \). Denote by \( A = X(k) \) the set of rational points of \( X \). Then the set \( CA_{\text{alg}}(G, X, k) \) of algebraic cellular automata consists of cellular automata \( \tau: A^G \to A^G \) which admit a memory \( M \subset G \) with local defining map \( \mu: A^M \to A \) induced by some \( k \)-morphism of algebraic varieties \( f: X^M \to X \), i.e., \( \mu = f|_{A^M} \), where \( X^M \) is the fibered product of copies of \( X \) indexed by \( M \).

The central geometric result of the paper is the following direct finiteness property of the class of algebraic cellular automata \( CA_{\text{alg}} \) (Section 4):

**Theorem C.** Let \( G \) be a surjunctive group. Let \( X \) be an algebraic variety over an algebraically closed field \( K \). Suppose that \( \tau, \sigma \in CA_{\text{alg}}(G, X, K) \) satisfy \( \sigma \circ \tau = \text{Id} \). Then one has \( \tau \circ \sigma = \text{Id} \).

By [15] Theorem 10.8, we have a canonical isomorphism of near rings \( R(k, G) \simeq CA_{\text{alg}}(G, \mathbb{A}^1, k) \) where \( \mathbb{A}^1 \) denotes the affine line and \( k \) is an infinite field. Consequently, we can deduce an extension (Theorem 5.1) of Theorem A directly from Theorem C where the group \( G \) is only required to be surjunctive since we can suppose without loss of generality that \( k \) is algebraically closed.

As another application of Theorem C, we obtain (see Section 5.1):

**Theorem D.** Let \( G \) be a surjunctive group. Let \( R \) be the endomorphism ring of a commutative algebraic group over an algebraically closed field. Then the group ring \( R[G] \) is stably finite.

Hence, we note that Theorem D is a generalization of the similar result [15] Corollary 1.3 where the group \( G \) is required to be sofic and \( R \) is required to be the endomorphism ring of a connected commutative algebraic group.
The paper is organized as follows. In Section 2, we recall basic facts in algebraic varieties and schemes of finite type. In Section 3, we formulate an extension of a lemma of Grothendieck to describe the set of closed points of relative fibered products of \( \mathbb{Z} \)-schemes of finite type. The proof of Theorem C is given in Section 4. Finally, in Section 5, we apply Theorem C to obtain the proofs of other main results and their extensions presented in the Introduction.

2. Algebraic varieties and morphisms of finite type

2.1. Algebraic varieties. Let \( X \) be an algebraic variety over a field \( k \), i.e., a reduced separated \( k \)-scheme of finite type. We denote by \( A := X(k) \) the set of \( k \)-points of \( X \). If the field \( k \) is algebraically closed, we can identify \( X \) with \( A \). The following auxiliary result states that morphisms of algebraic varieties are determined by their restrictions to the set of closed points.

**Lemma 2.1.** Let \( k \) be an algebraically closed field. Let \( f, g : X \to Y \) be \( k \)-scheme morphisms of \( k \)-algebraic varieties. Suppose that the restrictions \( f|_{X(k)}, g|_{X(k)} : X(k) \to Y(k) \) are equal. Then \( f \) and \( g \) are equal as morphisms of \( k \)-schemes.

**Proof.** See, e.g., [4, Lemma 7.2].

Recall also that an algebraic group is a group that is an algebraic variety with group operations given by algebraic morphisms (cf. [13]).

2.2. Models of morphisms of finite type. We shall need the following auxiliary lemma in algebraic geometry:

**Lemma 2.2.** Let \( n \in \mathbb{N} \) and let \( X, Y \) be algebraic varieties over a field \( k \). Let \( f_i : X^n \to Y, i \in I \), be finitely many morphisms of \( k \)-algebraic varieties. Then there exist a finitely generated \( \mathbb{Z} \)-algebra \( R \subset k \) and \( R \)-schemes of finite type \( X_R, Y_R \) and \( R \)-morphisms \( f_i : (X_R)^n \to Y_R \) of \( R \)-schemes with \( X = X_R \otimes_R k, Y = Y_R \otimes_R k \), and \( f_i = f_i \otimes_k k \) (base change to \( k \)). Moreover, if \( X = Y \), one can take \( X_R = Y_R \).

**Proof.** See, e.g., [9, Section 8.8], notably [9, Scholie 8.8.3], and [9, Proposition 8.9.1].

3. An extension of Grothendieck’s lemma

3.1. Jacobson schemes. We recall the notion of Jacobson schemes in [9]. Let \( U \) be a topological space. A subset of \( U \) which is the intersection of an open subset and a closed subset is said to be *locally closed*. A constructible subset of \( U \) is a finite union of locally closed subsets of \( U \). Observe that the complement of a constructible subset is also constructible.

A scheme is called *Jacobson scheme* if every nonempty constructible subset contains a closed point. For every field \( k \), it follows from [9, Proposition 10.4.2] and [9, Corollaire 10.4.6] that every \( \mathbb{Z} \)-scheme (resp. every \( k \)-scheme) of finite type is a Jacobson scheme.
For every scheme $X$, we denote by $\kappa(x)$ the residual field at a point $x \in X$. Then we have the following result:

**Lemma 3.1.** Let $X$ be a scheme and let $Y$ be a Jacobson scheme. Suppose that $f: X \to Y$ is a morphism of finite type. Then $X$ is Jacobson and for every closed point $x$ of $X$, the image $f(x)$ is a closed point of $Y$. Moreover, we have a canonical inclusion $\kappa(f(x)) \subseteq \kappa(x)$ of fields.

*Proof.* See [9, Lemma 10.4.11.1]. The last statement is a basic property in scheme theory. \qed

3.2. An extension of Grothendieck’s lemma. To describe the set of closed points of a $\mathbb{Z}$-scheme of finite type, we recall the following lemma due to Grothendieck:

**Lemma 3.2.** Let $X$ be a $\mathbb{Z}$-scheme of finite type. Then the set of closed points of $X$ is given by the following union of finite sets $\cup_{p,d} T_{p,d}$ over all prime numbers $p \in \mathbb{N}$ and all $d \in \mathbb{N}$ where

$$T_{p,d} = \{ x \in X : |\kappa(x)| = p^r, 1 \leq r \leq d \} \subseteq X \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

*Proof.* See [9, Lemma 10.4.11.1]. \qed

For each prime number $p$ and every $\mathbb{Z}$-scheme $V$, we denote by $V_p = V \otimes_{\mathbb{Z}} \mathbb{F}_p$ the fibre of $V$ above the prime $p$. Let $X$ be an $S$-scheme and let $M$ be a finite set. To keep track of the factor, we denote by $X^M_S$ the $M$-fold fibered product $X \times_S \cdots \times_S X$ of $S$-schemes indexed over the set $M$.

We deduce from Lemma 3.2 the following technical consequence that we shall need in the proof of Theorem C:

**Corollary 3.3.** Let $X$ be an $S$-scheme of finite type where $S$ is a $\mathbb{Z}$-scheme of finite type. Then for every finite set $M$, the set of closed points of $X^M_S$ is the union of finite sets $\cup_{s,p,d} T_{s,p,d}^M$ over all prime numbers $p \in \mathbb{N}$, all closed points $s \in S_p$, and all $d \in \mathbb{N}$ where

$$T_{s,p,d} = \{ x \in X_s : |\kappa(x)| = p^r, 1 \leq r \leq d \} \subseteq X_p,$$

where $X_s = X \otimes_S \kappa(s)$ is the fibre of $X$ above $s$.

*Proof.* First observe that $X$ and $X^M_S$ are $\mathbb{Z}$-schemes of finite type. Let us fix a prime number $p$ then we have

$$X^M_S \otimes \mathbb{F}_p = X^M_S \otimes_{\mathbb{Z}} \mathbb{F}_p = X_p \times_{S_p} \cdots \times_{S_p} X_p = (X_p)^M_{S_p}. \tag{3.1}$$

Let $f: X \to S$ be the structural morphism. Then we infer from Lemma 3.1 that every closed point of $(X_p)^M_{S_p}$ is given by an $M$-tuple $(x_g)_{g \in M}$ of closed points of $X_p$ such that $f(x_g) = s$ for all $g \in M$ for some closed point $s \in S_p$. The last condition is equivalent to the requirement that $x_g \in X_s$ for all $g \in M$ for some closed point $s \in S$.

On the other hand, Lemma 3.2 implies the set of closed points of $X_p$ is given by the union over $d \in \mathbb{N}$ of the finite sets $T_{p,d} = \{ x \in X_p : |\kappa(x)| = p^r, 1 \leq r \leq d \}$. From these descriptions, the conclusion of the corollary follows immediately. \qed
4. Proof of the main geometric result

We are now in position to prove the main result of the paper Theorem C, which is a geometric direct finiteness property of symbolic algebraic varieties.

Proof of Theorem C. Denote by $A = X(K)$ the set of $K$-points of $X$. In particular, we can identify $X$ with the set $A$. Let us fix $\tau, \sigma \in \text{CA}_{\text{alg}}(G, X, K)$ such that $\sigma \circ \tau = \text{Id}$. Let $\Gamma := \tau(\mathbb{A}^G) \subset \mathbb{A}^G$ be the image of $\tau$.

Now fix a finite subset $\Omega \subset G$. Choose a finite subset $M \subset G$ large enough such that $M$ is a common memory set of both $\tau$ and $\sigma$ and such that $1_G \in M$ and $M = M^{-1}$. Let $\mu: A^M \to A$ and $\eta: A^M \to A$ be respectively the algebraic local defining maps of $\tau$ and $\sigma$.

Up to enlarging $M$, we can clearly suppose that $\Omega \subset M$. For every subset $E \subset G$, we denote $\Gamma_E = \{x|_E: x \in \Gamma\} \subset A^E$. We will show that $\Gamma_M = A^M$.

Consider the algebraic $K$-morphism $\tau^+_{M}: A^{M^2} \to A^M$ of algebraic varieties given by $\tau^+(c)(g) = \mu((g^{-1}c)|_M)$ for every $c \in A^{M^2}$ and $g \in M$. Since $M$ is a memory set of $\tau$, it is clear that we have $\Gamma_M = \tau^+_{M}(A^{M^2})$.

Since $\sigma \circ \tau = \text{Id}$, it follows that for every $c \in A^{M^2}$, we have $\eta(\tau^+(c)) = c(1_G)$. Consequently, we infer from Lemma 2.1 that the composition morphism $\eta \circ \tau^+_M: A^{M^2} \to A$ is exactly the canonical projection morphism of $K$-schemes $\pi: A^{M^2} \to A^{\{1_G\}}$ induced by the inclusion $\{1_G\} \subset M^2$.

Consider the finitely generated $\mathbb{Z}$-sub-algebra $R \subset K$ given by Lemma 2.2 applied to the morphisms $\mu, \eta: A^M \to A$. Consequently, we can find a corresponding $R$-scheme of finite type $A_R$ and morphisms $\mu_R, \eta_R: A^M_R \to A_R$ of $R$-schemes of finite type such that $A = A_R \otimes_R K$, $\mu = \mu_R \otimes_R K$, and $\eta = \eta_R \otimes_R K$. Here, we denote by $A^M_R = A_R \times_R \cdots \times_R A_R$ the fibered product indexed by $M$ of the $R$-schemes $A_R$.

For each $g \in G$, we have the canonical isomorphisms $A^M_R \simeq A^M_R$ and $A_R(g) \simeq A^{\{1_G\}}_R$ induced respectively by the trivial set bijections $M \simeq gM$, $h \mapsto gh$, and $\{1_G\} \simeq \{g\}$. Then similarly as above, we obtain an $R$-morphism of finite type $\varphi: A^{M^2}_R \to A^M_R$ of $R$-schemes which is defined as a fibered product over $g \in M$ of the morphisms $\mu_{g,R}: A^M_{g,R} \to A^M_R$ induced by $\mu_R: A^M_R \to A_R$ via the canonical isomorphisms $A^M_R \simeq A^M_R$ and $A_R(g) \simeq A^{\{1_G\}}_R$. It follows immediately that $\tau^+_{M} = \varphi \otimes_R K$.

Since $R \subset K$ and the composition map $\eta \circ \tau^+_M: A^{M^2} \to A$ is exactly the canonical projection $A^{M^2} \to A^{\{1_G\}}$ as $K$-morphisms of $K$-schemes, we can suppose, up to adding a finite number of generators to $R$, that the model morphism $\eta_R \circ \varphi: A^{M^2}_R \to A_R$ is equal to the canonical projection $A^{M^2}_R \to A^{\{1_G\}}_R$ as $R$-morphisms of $R$-schemes (cf. [9], Scholie 8.8.3).

Claim. The morphism $\varphi$ is surjective.

Proof of the Claim. Indeed, let $\mathcal{P} \subset \mathbb{N}$ denote the set of prime numbers. For each $p \in \mathcal{P}$, let us define $A^p = A \otimes_{\mathbb{Z}} \mathbb{F}_p$ the fibre of $A$ above the prime $p$. 

Since $A_R$ is an $R$-scheme of finite type and since $R$ is a $\mathbb{Z}$-algebra of finite type, the scheme $A_R$ is also a $\mathbb{Z}$-scheme of finite type. We deduce that $A_R$ and thus $A^M_R$ are Jacobson schemes. Therefore, we infer from Corollary 3.3 that the set of closed points of $A^M_R$ is given by $\Delta = \bigcup_{p \in \mathcal{P}, s \in S_p, d \in \mathbb{N}} T^M_{p, s, d}$ where $S = \text{Spec } R$ and $s \in S_p$ denotes a closed point and

$$(4.1) \quad T^M_{p, s, d} = \{ x \in A_s : |\chi(x)| = p^r, 1 \leq r \leq d \}$$

is a finite subset of $A_s$ and thus of $A_p$.

Fix a prime $p \in \mathcal{P}$ and a closed point $s \in S_p$. Let $T_s = \bigcup_{d \in \mathbb{N}} T^M_{p, s, d}$. We obtain by restriction to the fibre $A_s \subset A$ above $s$ two cellular automata $\tau_s, \sigma_s : T^G_s \to T^G_s$ with $\tau_s = \tau|_{T^G_s}$ and $\sigma_s = \sigma|_{T^G_s}$. Note that the restrictions $\mu_s = \mu|_{T^M_s} : T^M_s \to T_s$ and $\eta_s = \eta|_{T^M_s} : T^M_s \to T_s$ are respectively well-defined local defining maps of $\tau_s$ and $\sigma_s$ by Lemma 3.1.

Since $\eta_R \circ \varphi : A^M_R \to A_R$ is the canonical projection $A^M_R \to A^1_R$ as $R$-morphisms of $R$-schemes, the restriction map $(\eta_R \circ \varphi)|_{T^M_s}$ is the same as the canonical projection $T^M_s \to T^1_s$. It follows that $\sigma_s \circ \tau_s = \text{Id}$. In particular, the cellular automaton $\tau_s$ is injective.

Now let $d \in \mathbb{N}$. Since $\mu_s(T^M_{p, s, d}) \subset T^M_{p, s, d}$ by Lemma 3.1, we obtain a well-defined restriction cellular automaton $\tau_{p, s, d} = \tau_s|_{T^G_{p, s, d}} : T^G_{p, s, d} \to T^G_{p, s, d}$ admitting $\mu_s|_{T^M_{p, s, d}} : T^M_{p, s, d} \to T^M_{p, s, d}$ as a local defining map. As $\tau_s$ is injective, $\tau_{p, s, d}$ is also an injective cellular automaton. Since the alphabet $T^M_{p, s, d}$ is finite by Corollary 3.3 and the group $G$ is surjective (see the Introduction), we can conclude that $\tau_{p, s, d}$ is surjective.

Consequently, we deduce that $T^M_{p, s, d} \subset \varphi(A^M_R)$ for all $p \in \mathcal{P}$ and $d \in \mathbb{N}$. Therefore, it follows that $\Delta = \bigcup_{p \in \mathcal{P}, s \in S_p, d \in \mathbb{N}} T^M_{p, s, d} \subset \varphi(A^M_R)$.

Observe on the one hand that $A^M_R$ and $A^M_{R^2}$ are Jacobson schemes and the image $\varphi(A^M_R)$ is a constructible subset of $A^M_R$ by Chevalley’s theorem ([8, Théorème 1.8.4]). On the other hand, Corollary 3.3 tells us that $\Delta$ is the set of all closed points of $A^M_R$.

Therefore, we must have $\varphi(A^M_R) = A^M_R$ by the definition of Jacobson schemes. The map $\varphi$ is thus surjective and the claim is proved.

Note that surjectivity is a stable property under base change (see, e.g., [19, Lemma 01S1]). Hence, it follows that $\tau^+_M = \varphi \otimes R K$ is also a surjective morphism. Therefore, we find that

$$\Gamma_M = \tau^+_M(A^M_R) = A^M_R.$$ 

Since $\Omega \subset M$ by our choice of $M$, we deduce that $\Gamma^M_{\Omega} = A^\Omega_{\Omega}$ for every finite subset $\Omega \subset G$. Therefore, $\tau(A^G) = \Gamma$ is dense in $A^G$ with respect to the prodiscrete topology. Since $\sigma \circ \tau = \text{Id}$ by hypothesis, $(\tau \circ \sigma)(x) = x$ for every $x \in \Gamma$. Thus, $(\tau \circ \sigma)(x) = x$ for all $x \in A^G$ as the prodiscrete topology on $A^G$ is Hausdorff and $\tau \circ \sigma$ is continuous since it is a cellular automaton (see [11 Proposition 3.3]). Hence, $\tau \circ \sigma = \text{Id}$ and the conclusion follows.
5. Applications

As applications of Theorem C, we present in this section the proofs of Theorem A, Theorem B, and Theorem D presented in the Introduction. We begin with the following generalization of Theorem A:

**Theorem 5.1.** Let $G$ be a surjunctive group and let $k$ be a field. Suppose that $\alpha, \beta \in R(k, G)$ verify $\alpha \star \beta = X_1G$. Then $\beta \star \alpha = X_1G$.

**Proof.** We can trivially embed $k$ into some algebraically closed field $K$. In particular, we have an inclusion $R(k, G) \subset R(K, G)$. Let $A_1$ denote the affine line then we infer from [15, Theorem 10.8] that there exists a canonical isomorphism of near rings $\Psi: R(K, G) \to CA_{alg}(G, A_1, K)$ where the multiplication on $CA_{alg}(G, A_1, K)$ is defined as the composition of maps. Therefore, we find that:

\[(5.1) \quad \Psi(\alpha) \circ \Psi(\beta) = \Psi(\alpha \star \beta) = \Psi(X_1G) = \text{Id}.\]

It follows from Theorem C that $\Psi(\beta) \circ \Psi(\alpha) = \text{Id}$. Hence, $\Psi(\beta \star \alpha) = \text{Id} = \Psi(X_1G)$ and we deduce that $\beta \star \alpha = X_1G$. The proof of the theorem is complete. □

The proof of Theorem B is very similar.

**Proof of Theorem B.** We have to show that every surjunctive group satisfies Kaplansky’s stable finiteness conjecture. Let $k$ be a field and let $G$ be a surjunctive group. Let $n \geq 1$ and let $\alpha, \beta \in \text{Mat}_n(k[G])$ be such that $\alpha \beta = 1$. We embed $k$ into an arbitrary algebraically closed field $K$. Then it follows trivially that $\alpha, \beta \in \text{Mat}_n(K[G])$ and $\alpha \beta = 1$. We infer from [3, Corollary 8.7.8] a ring isomorphism $\psi: \text{Mat}_n(K[G]) \to LCA(G, K^n)$ where $LCA(G, K^n)$ is the $K$-algebra of cellular automata $K^G \to K^G$ admitting $K$-linear local defining maps. The multiplication on $LCA(G, K^n)$ is given by the composition of maps. Therefore, $\psi(\alpha) \circ \psi(\beta) = \psi(\alpha \beta) = \psi(1) = \text{Id}$.

As $LCA(G, K^n) \subset CA_{alg}(G, A^n, K)$, where $A^n$ is the $n$-dimensional affine space, and as $G$ is surjunctive, Theorem C implies that $\psi(\beta) \circ \psi(\alpha) = \text{Id}$ and therefore $\psi(\beta \alpha) = \text{Id}$. It follows that $\beta \alpha = 1$ since $\psi$ is an isomorphism. The proof is thus complete. □

5.1. **Symbolic group varieties.** Let $G$ be a group and let $k$ be a field. Let $X$ be an algebraic group over $k$ and let $A = X(k)$. The set $CA_{alg}(G, X, k)$ of algebraic group cellular automata consists of cellular automata $\tau: A^G \to A^G$ which admit a memory set $M$ with local defining map $\mu: A^M \to A$ induced by some homomorphism of algebraic groups $f: X^M \to X$, i.e., $\mu = f|_{A^M}$. It is clear that we have an inclusion $CA_{alg}(G, X, k) \subset CA_{alg}(G, X, k)$. See also [15], [16], [17], or [15] for more details.

**Proof of Theorem D.** Let $Y$ be a commutative algebraic group over an algebraically closed field $K$. Let $R = \text{End}_K(Y)$ be the endomorphism ring (of $K$-homomorphisms of algebraic groups) of $Y$. Let $G$ be a surjunctive group. Let $n \in \mathbb{N}$ and let $X = Y^n$. Then $\text{Mat}_n(R) = \text{End}_K(X)$ by [15]...
Lemma 9.5]. By [15, Proposition 9.3], there exists a natural ring isomorphism $\text{End}_K(X)[G] \simeq CA_{\text{algr}}(G, X, K)$. We infer from Theorem C that $CA_{\text{algr}}(G, X, K)$ is directly finite as a subspace of $CA_{\text{alg}}(G, X, K)$. It follows that the ring $\text{Mat}_n(R)[G] = \text{End}_K(X)[G]$ is also directly finite. Finally, since we have an isomorphism $\text{Mat}_n(R[G]) \simeq \text{Mat}_n(R)[G]$ by [15, Lemma 9.4], the ring $\text{Mat}_n(R[G])$ is also directly finite and the conclusion follows. □

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