On $k$-ranks of Topological Spaces

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Abstract

In this paper, the concepts of $K$-subset systems and $k$-well-filtered spaces are introduced, which provide another uniform approach to d-spaces, s-well-filtered spaces (i.e., $U_S$-admissibility) and well-filtered spaces. We prove that the $k$-well-filtered reflection of any $T_0$ space exists. Meanwhile, we propose the definition of $k$-rank, which is an ordinal that measures how many steps from a $T_0$ space to a $k$-well-filtered space. Moreover, we derive that for any ordinal $\alpha$, there exists a $T_0$ space whose $k$-rank equals to $\alpha$. One immediate corollary is that for any ordinal $\alpha$, there exists a $T_0$ space whose $d$-rank (respectively, $wf$-rank) equals to $\alpha$.

Keywords: $k$-well-filtered space, $k$-Rudin set, $k$-well-filtered reflection, $k$-rank

1 Introduction

In non-Hausdorff topological spaces and domain theory, $d$-spaces and well-filtered spaces are two important classes of spaces. Let $\textbf{Top}_0$ be the category of all $T_0$ spaces, $\textbf{Top}_d$ the category of all $d$-spaces and $\textbf{Top}_w$ the category of all well-filtered spaces. It is well-known that $\textbf{Top}_d$ and $\textbf{Top}_w$ are reflective in $\textbf{Top}_0$, respectively. Different ways for constructing $d$-completions and well-filtered reflections of $T_0$ spaces were found in [3,10,12,17]. In [3], Ershov introduced one way to get $d$-completions of $T_0$ spaces using the equivalent classes of directed subsets, he called it $d$-rank which is an ordinal that measures how many steps from a $T_0$ space to a $d$-space. Inspired by his method, in [10], Liu, Li and Wu proposed one way to get well-filtered reflections of $T_0$ spaces using the equivalent classes of Rudin subsets, they called it $wf$-rank, which is an ordinal that measures how far a $T_0$ space is from being a well-filtered space.

In [16], based on irreducible subset systems, Xu provided a uniform approach to $d$-spaces, sober spaces and well-filtered spaces, and developed a general framework for dealing with all these spaces. In this paper, we will provide another uniform approach to $d$-spaces and well-filtered spaces and develop a general framework for dealing with all these spaces. Similar to the concept of irreducible subset systems in [16], we propose the concepts of $K$-subset systems and $k$-well-filtered spaces. For a $K$-subset system $Q_k : \textbf{Top}_0 \rightarrow \textbf{Set}$ and a $T_0$ space $X$, $X$ is called $k$-well-filtered if for any open set $U$ and a filtered family $\mathcal{K} \subseteq Q_k(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. The category of all $k$-well-filtered spaces with continuous

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mappings is denoted by $\mathbf{Top}_k$. It is not difficult to verify that $d$-spaces and well-filtered spaces are two special kinds of $k$-well-filtered spaces. Moreover, we find that $s$-well-filtered spaces (i.e., $U_0$-admissibility in [6]) is also a kind of $k$-well-filtered spaces which is different from $d$-spaces and well-filtered spaces. Just like directed subsets and Rudin subsets, the concept of $k$-Rudin sets will be introduced. Moreover, for a $K$-subset system $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$, we use the equivalent classes of $k$-Rudin sets to construct the $k$-well-filtered reflections of $T_0$ spaces. Meanwhile, we introduce the concept of $k$-rank, which is an ordinal that measures how far a $T_0$ space can become a $k$-well-filtered space. For a $T_0$ space $X$, we get that there exists an ordinal $\alpha$ such that the $k$-rank of $X$ is equal to $\alpha$.

In [3] and [8], for any ordinal $\alpha$, there exists a $T_0$ space whose $d$-rank (respectively, $wf$-rank) equals to $\alpha$. Consider a $T_0$ space whose $k$-rank equals to $\alpha$ may be more complex, because we know little about $Q_k(X)$. We have to find suitable conditions to characterize a class of $T_0$ spaces whose $k$-rank equals to $\alpha$. It turns out that finding these $T_0$ spaces is the hard part of our task, but how to prove the results is relatively simple.

Finally, we obtain that for any ordinal $\alpha$, there exists a $T_0$ space whose $k$-rank equals to $\alpha$.

### 2 Preliminaries

First, we briefly recall some standard definitions and notations to be used in this paper, for further details see [1], [4], [5] and [7].

Let $P$ be a poset and $A \subseteq P$. We denote $\uparrow A = \{ x \in P \mid x \geq a \text{ for some } a \in A \}$ and $\downarrow A = \{ x \in P \mid x \leq a \text{ for some } a \in A \}$. For every $a \in P$, we denote $\uparrow \{ a \} = \uparrow a = \{ x \in P \mid x \geq a \}$ and $\downarrow \{ a \} = \downarrow a = \{ x \in P \mid x \leq a \}$. A is called an upper set (resp., a lower set) if $A = \uparrow a$ (resp., $A = \downarrow a$). $A$ is called directed provided that it is nonempty and every finite subset of $A$ has an upper bound in $A$. The set of all directed sets of $P$ is denoted by $\mathcal{D}(P)$. Moreover, the set of all nonempty finite sets in $P$ is denoted by $P^{<\omega}$.

A poset $P$ is called a dcpo if every directed subset $D$ in $P$ has a supremum. A subset $U$ of $P$ is called Scott open if (1) $U = \uparrow U$ and (2) for any directed subset $D$ for which $\forall D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of $P$ form a topology, we call it the Scott topology on $P$ and denoted by $\sigma(P)$.

For a $T_0$ space $X$, let $\mathcal{O}(X)$ (resp., $\Gamma(X)$) be the set of all open subsets (resp., closed subsets) of $X$. For a subset $A$ of $X$, the closure of $A$ is denoted by $\overline{A}$ or $\bar{A}$. We use $\leq_X$ to represent the specialization order of $X$, that is, $x \leq_X y$ iff $x \in \overline{\{ y \}}$. A subset $B$ of $X$ is called saturated if $B$ equals the intersection of all open sets containing it (equivalently, $B$ is an upper set in the specialization order). Let $S(X) = \{ \{ x \} \mid x \in X \}$, $S_c(X) = \{ \{ x \} \mid x \in X \}$ and $\mathcal{D}(X) = \{ \overline{D} \mid D \in \mathcal{D}(X) \}$. A $T_0$ space $X$ is called a $d$-space (i.e., monotone convergence space) if $X$ (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ ([4]). The category of all $d$-spaces with continuous mappings is denoted by $\mathbf{Top}_d$.

For a $T_0$ space $X$, let $\mathcal{K}$ be a filtered family under the inclusion order in $Q(X)$, which is denoted by $\mathcal{K} \subseteq_{filt} Q(X)$, i.e., for any $K_1, K_2 \in Q(X)$, there exists $K_3 \in Q(X)$ such that $K_3 \subseteq K_1 \cap K_2$. $X$ is called well-filtered if for any open subset $U$ and any $\mathcal{K} \subseteq_{filt} Q(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. The category of all well-filtered spaces with continuous mappings is denoted by $\mathbf{Top}_w$ ([14]).

In what follows, $\mathbf{K}$ always refers to a full subcategory $\mathbf{Top}_0$ that contains $\mathbf{Sob}$, the full subcategory of sober spaces. The objects of $\mathbf{K}$ are called $\mathbf{K}$-spaces.

**Definition 2.1** [14] Let $X$ be a $T_0$ space. A $\mathbf{K}$-reflection of $X$ is a pair $(\hat{X}, \mu)$ comprising a $\mathbf{K}$-space $\hat{X}$ and a continuous mapping $\mu : X \rightarrow \hat{X}$ satisfying that for any continuous mapping $f : X \rightarrow Y$ to a $\mathbf{K}$-space, there exists a unique continuous mapping $f^* : \hat{X} \rightarrow Y$ such that $f^* \circ \mu = f$, that is, the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\mu} & \hat{X} \\
\downarrow{f} & & \downarrow{f^*} \\
Y & & 
\end{array}
$$
By a standard argument, K-reflections, if they exist, are unique up to homeomorphism. We shall use $X^k$ to denote the space of the K-reflection of X if it exists.

For $K = \text{Top}_w$, the K-reflection of X is called the well-filterification or well-filtered reflection of X, we denote it by $H_wf(X)$ if the well-filterification of X exists. For $K = \text{Top}_d$, the K-reflection of X is called the d-completion of X, we denote it by $H_d(X)$ if the d-completion of X exists.

**Definition 2.2** [13] Let $X = (X, \tau)$ be a topological space and $A \subseteq X$. A is called strongly compact in X if for each $U \in \tau$ with $A \subseteq U$, there is $F \in X^{<\omega}$ such that $A \subseteq \uparrow \tau F \subseteq U$.

**Proposition 2.3** [6] Every finite set is strongly compact, and every strongly compact set is compact.

**Proposition 2.4** [6] $A$ is strongly compact if and only if $\uparrow A$ is so.

We use $Q_s(X)$ to denote the set of all nonempty strongly compact saturated subsets of X. X is called s-well-filtered (i.e., $U_S$-admissibility in [6]) if it is $T_0$, and for any open subset $U$ and $K \subseteq_{filt} Q_s(X)$, $\bigcap K \subseteq U$ implies $K \subseteq U$ for some $K \in K$. The category of all s-well-filtered spaces with continuous mappings is denoted by $\text{Top}_{s-w}$.

3 k-well-filtered spaces

In this section, we provide a uniform approach to $d$-spaces and well-filtered spaces and develop a general framework for dealing with all these spaces.

**Definition 3.1** $Q_k : \text{Top}_0 \rightarrow \text{Set}$ is called a C-subset system if $S^u(X) \subseteq Q_k(X) \subseteq Q(X)$ for all $X \in \text{ob}(\text{Top}_0)$, where $S^u(X) = \{ \uparrow x \mid x \in X \}$.

**Definition 3.2** Let $Q_k : \text{Top}_0 \rightarrow \text{Set}$ be a C-subset system and $X$ a $T_0$ space. A nonempty subset A is said to have k-Rudin property, if there exists $K \subseteq_{filt} Q_k(X)$ such that A is a minimal closed set that intersects all members of K. We call such a set k-Rudin set or k-Rudin set. Let $K^R(X) = \{ A \subseteq X \mid A$ has k-Rudin property $\}$ and $K^R_c(X) = K^R(X) \cap \Gamma(X)$.

For $Q_k(X) = Q(X)$, a k-Rudin set of X is called a Rudin set (i.e., KF set) of X. The set of all Rudin sets of X is denoted by $\text{KF}(X)$. $RD(X) = \text{KF}(X) \cap \Gamma(X)$.

**Proposition 3.3** Let $Q_k : \text{Top}_0 \rightarrow \text{Set}$ be a C-subset system and X a $T_0$ space. Then $D(X) \subseteq K^R(X) \subseteq KF(X)$.

**Proof.** Clearly, $K^R(X) \subseteq \text{KF}(X)$. Now we prove that every directed subset $D$ of X is a k-Rudin set. Let $K = \{ \{ d \mid d \in D \} \}$. Then $K \subseteq Q_k(X)$ is filtered and $D$ interests all members of K. Assume that A is a closed subset in X and interests all members of K. This means that $A \cap \uparrow d \neq \emptyset$ for all $d \in D$. Since A is closed, it is a lower set, then $d \in A$ for all $d \in D$. Hence, $D \subseteq A$. So D is a minimal closed set that intersects all members of K.

**Definition 3.4** A C-subset system $Q_k : \text{Top}_0 \rightarrow \text{Set}$ is called a K-subset system provided that for any $T_0$ spaces $X, Y$ and any continuous mapping $f : X \rightarrow Y$, $f(A) \in K^R(Y)$ for all $A \in K^R(X)$.

**Definition 3.5** Let $Q_k : \text{Top}_0 \rightarrow \text{Set}$ be a K-subset system and X a $T_0$ space. X is called k-well-filtered if for any open set $U$ and $K \subseteq_{filt} Q_k(X)$, $\bigcap K \subseteq U$ implies $K \subseteq U$ for some $K \in K$. The category of all k-well-filtered spaces with continuous mappings is denoted by $\text{Top}_k$.

In the following, we give some special k-well-filtered spaces and their relations with $\text{Top}_d$, $\text{Top}_w$ and $\text{Top}_{s-w}$, respectively.

For a C-subset system $Q_k : \text{Top}_0 \rightarrow \text{Set}$ and a $T_0$ space X, here are some important examples of $Q_k(X)$:

1. $Q_k(X) = S^u(X)$ (i.e., $Q_k(X) = \{ \uparrow x \mid x \in X \}$).
2. $Q_k(X) = Q_f(X)$ (i.e., $Q_k(X) = \{ \uparrow F \mid \emptyset \neq F \in X^{<\omega} \}$).
(3) $Q_k(X) = Q_s(X)$ (i.e., $Q_k(X) = \{A \mid A$ is a nonempty strongly compact saturated subset in $X\}$).

(4) $Q_k(X) = Q(X)$ (i.e., $Q_k(X)$ is the set of all nonempty compact saturated subsets in $X$).

Let $Q_k : \mathbf{Top}_0 \to \mathbf{Set}$ be a C-subset system. For any $T_0$ space $X$, if $Q_k(X) = S^n(X)$ (i.e., $Q_k(X) = Q_f(X)$), then it follows directly from Definition 1, Example 1(1) and Theorem 1 in [9] that $X$ is $k$-well-filtered iff $X$ is a $d$-space. If $Q_k(X) = Q(X)$, it is trivial that $X$ is $k$-well-filtered iff $X$ is well-filtered. In the case $Q_k(X) = Q_s(X)$, $k$-well-filtered spaces are exactly $s$-well-filtered spaces.

From the above, for a $K$-subset system $Q_k : \mathbf{Top}_0 \to \mathbf{Set}$, it is not difficult to see that well-filtered spaces are $k$-well-filtered spaces and $k$-well-filtered spaces are $d$-spaces. That is

$$\mathbf{Top}_w \subseteq \mathbf{Top}_k \subseteq \mathbf{Top}_d.$$ 

In particular, well-filtered spaces are $s$-well-filtered spaces and $s$-well-filtered spaces are $d$-spaces. In Example 3.6 and Example 3.7 below, we will show that the converses are not true, respectively.

**Example 3.6** Consider set $N$ of natural numbers. Let $X = (N, \tau_{cof})$ be the space $N$ equipped with the co-finite topology (the empty set and the complements of finite subsets of $N$ are open sets). Then

(a) $\Gamma(X) = \{\emptyset, N\} \cup N^{<\omega}$, $X$ is $T_1$, hence it is a $d$-space.

(b) $X$ is $s$-well-filtered since a subset in $T_1$ spaces is strongly compact iff it is finite.

(c) $K(X) = 2^N \setminus \emptyset$.

(d) $\text{RD}(X) = \{N\} \cup \{\{n\} \mid n \in N\}$.

(e) $X$ is not well-filtered.

**Example 3.7** (Johnstone space) Recall the dcpo constructed by Johnstone in [5], which is defined as $\mathbb{J} = N \times (N \cup \{\infty\})$, with the order defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$ or $n = \infty$ and $k \leq m$. Let $X = (\mathbb{J}, \tau_o)$. Then

(a) $\mathbb{J}$ is a dcpo, thus $X$ is a $d$-space.

(b) $Q(X) = Q_s(X)$.

(c) $X$ is not well-filtered.

(d) $X$ is not $s$-well-filtered.

Using the equivalent classes of directed subsets, Ershov introduced one way to get $d$-completions of $T_0$ spaces in [3]. Inspired by his method, Liu, Li and Wu presented one way to get well-filtered reflections of $T_0$ spaces using the equivalent classes of $\text{KF}$-subsets in [10]. Now for a $K$-subset system $Q_k : \mathbf{Top}_0 \to \mathbf{Set}$, we use the equivalent classes of $k$-Rudin sets to construct the $k$-well-filtered reflections of $T_0$ spaces. Let $(X, \tau)$ be a $T_0$ space. Consider an equivalence relation $\sim$ on $K^R(X)$ which is defined as follows:

$$A_0 \sim A_1 \text{ if and only if } A_0 \cap U \neq \emptyset \text{ is equivalent to } A_1 \cap U \neq \emptyset \text{ for any } U \in \tau$$

where $A_0, A_1 \in K^R(X)$. Note that $A_0 \sim A_1$ if and only if $\text{cl}_X(A_0) = \text{cl}_X(A_1)$. Let

$$[A] = \{A' \in K^R(X) \mid A \sim A'\}, A \in K^R(X),$$

$$K(X) = \{[A] \mid A \in K^R(X)\},$$

$$U^* = \{[A] \mid A \cap U \neq \emptyset\}, U \in \tau,$$

$$\tau^* = \{U^* \mid U \in \tau\}.$$ 

Then $\tau^*$ is a topology. Moreover, $(K(X), \tau^*)$ is a $T_0$ space. For $Q_k(X) = S^n(X)$, we denote $K(X)$ by $D(X)$. For $Q_k(X) = Q(X)$, we denote $K(X)$ by $KF(X)$.

**Lemma 3.8** Let $Q_k : \mathbf{Top}_0 \to \mathbf{Set}$ be a $K$-subset system, $(X, \tau)$ a $T_0$ space and $\lambda : X \to K(X)$ the map defined by $\lambda(x) = [\{x\}]$. Then the map $\lambda$ is a homeomorphic embedding.
Lemma 3.9 Let \( Q_k : \text{Top}_0 \rightarrow \text{Set} \) be a \( K \)-subset system and \((X, \tau)\) a \( T_0 \) space. Then the following are equivalent:

1. \( X \) is a \( k \)-well-filtered space.
2. \( K(X) \cong X \) (under the map \( \lambda \)).

Let \( Q_k : \text{Top}_0 \rightarrow \text{Set} \) be a \( K \)-subset system and \((X, \tau)\) a \( T_0 \) space. Suppose that \( Y \) is a well-filtered space that has \( X \) as a subspace. Since \( Y \) is well-filtered, it is \( k \)-well-filtered. By Lemma 3.9, we have \( K(Y) \cong Y \). In general, we can consider \( K_\gamma(X) \) as a subspace of \( K_\beta(X) \) in the sense of embedding mappings for all ordinals \( \gamma \leq \beta \). The transfinite sequence of extensions is constructed as follows:

1. \( K_0(X) = X \),
2. \( K_{\beta+1}(X) = K(K_\beta(X)) \),
3. \( K_\beta(X) = \bigcup_{\gamma<\beta} K_\gamma(X) \) if \( \beta \) is a limit ordinal.

By [2] and [12], we have the following similar results.

Theorem 3.10 For a \( K \)-subset system \( Q_k : \text{Top}_0 \rightarrow \text{Set} \) and a \( T_0 \) space \((X, \tau)\), the \( k \)-well-filterification of \( X \) exists; i.e., there exists an ordinal \( \alpha \) such that \( H_k(X) = K_\alpha(X) \cong K_{\alpha+1}(X) \).

Proof. The proof is similar to the method of constructing the \( d \)-completion of \( T_0 \) spaces in [2] and the method of constructing the \( k \)-well-filterification of \( T_0 \) spaces in [12].

Definition 3.11 Let \( Q_k : \text{Top}_0 \rightarrow \text{Set} \) be a \( K \)-subset system and \((X, \tau)\) a \( T_0 \) space. The \( k \)-rank of \( X \) is the least ordinal \( \alpha \) such that \( K_\alpha(X) \cong K_{\alpha+1}(X) \). We denote the \( k \)-rank of a space \( X \) by \( \text{rank}_k(X) \).

Similarly, the \( d \)-rank of \( X \) is the least ordinal \( \alpha \) such that \( D_\alpha(X) \cong D_{\alpha+1}(X) \), it is denoted by \( \text{rank}_d(X) \) in [3]. The \( \text{uf-rank} \) of \( X \) is the least ordinal \( \alpha \) such that \( KF_\alpha(X) \cong KF_{\alpha+1}(X) \), it is denoted by \( \text{rank}_{\text{uf}}(X) \) in [8].

4. \( \alpha^k \)-special spaces

For a \( K \)-subset system \( Q_k : \text{Top}_0 \rightarrow \text{Set} \), in Theorem 3.10, there exists an ordinal \( \alpha \) such that \( \text{rank}_k(X) = \alpha \) for a \( T_0 \) space \( X \). Conversely, for any given ordinal \( \alpha \) it is natural to ask whether there exists a \( T_0 \) space \( X \) such that \( \text{rank}_k(X) = \alpha \). In this section, we prove that for any given ordinal \( \alpha \), there exists a \( T_0 \) space \( X \) such that \( \text{rank}_k(X) = \alpha \).

Definition 4.1 Let \( Q_k : \text{Top}_0 \rightarrow \text{Set} \) be a \( K \)-subset system and \((X, \tau)\) a \( T_0 \) space. For an ordinal \( \alpha \), \( X \) is called \( \alpha^k \)-special if the following conditions are satisfied:

1. \( \text{rank}_k(X) = \alpha \);
2. \( \alpha \) is the least ordinal for which \( K_\alpha(X) \) has a greatest element.

\( X \) is called \( \alpha^d \)-special (resp., \( \alpha^w \)-special), similarly, see [3] and [8], respectively.

Remark 4.2 If \( X \) is a \( \alpha^k \)-special space, then \( \alpha \) is not a limit ordinal.

Proof. In fact, let \((X, \tau)\) be a \( \alpha^k \)-special space. Suppose that \( \alpha \) is a limit ordinal. Then \( K_\alpha(X) = \bigcup_{\beta<\alpha} K_\beta(X) \). By Definition 4.1, \( K_\alpha(X) \) has a greatest element. Hence, there exists \( \beta < \alpha \) such that \( K_\beta(X) \) has a greatest element, which is a contradiction. So \( \alpha \) will not be a limit ordinal.

Lemma 4.3 For any nonlimit ordinal \( \alpha \), every \( \alpha^k \)-special space is irreducible.

Proof. The proof is similar to Lemma 3.3 in [3].
Recall the following construction in [3]. For topological spaces $X$ and $Y_x$, $x \in X$, let

$$Z = \bigcup_{x \in X} Y_x \times \{x\},$$

$$\tau = \{U \subseteq Z \mid (U)_x \in \tau(Y_x) \text{ for any } x \in X \text{ and } (U)_X \in \tau(X)\},$$

where $(U)_x = \{y \in Y_x \mid (y, x) \in U\}$ for any $x \in X$ and $(U)_X = \{x \in X \mid (U)_x \neq \emptyset\}$.

**Lemma 4.4 ([3])** Let $X$ be a $T_0$ space and $Y_x$ an irreducible $T_0$ space for every $x \in X$. Then

1. $\tau$ is a $T_0$ separable topology on $Z$.
2. The map $y \mapsto (y, x)$ determines a homeomorphic embedding of $Y_x$ in $Z$ for any $x \in X$.
3. If the space $X$ is irreducible, then the space $Z$ is also irreducible.

For any subset $A \subseteq Z$, put $\bar{X} = \{x \in X \mid Y_x \text{ has the greatest element } \top_x\}$ with the induced topology of $X$. Define

$$(A)_x = \{y \in Y_x \mid (y, x) \in A\} \text{ for any } x \in X,$$

$$(A)_X = \{x \in X \mid (A)_x \neq \emptyset\},$$

$$A_x = \{x \in \bar{X} \mid (\top_x, x) \in A\}.$$

And the space $(Z, \tau)$ is also denoted by $\sum X Y_x$.

**Lemma 4.5 ([3])** Let $X$ be a $T_0$ space, $Y_x$ an irreducible $T_0$ space for any $x \in X$, and $Z = \sum X Y_x$. For all $(y_0, x_0), (y_1, x_1) \in Z$, we have $(y_0, x_0) \leq (y_1, x_1)$ if and only if the following two alternatives hold:

1. $x_0 = x_1$ and $y_0 \leq_{Y_{x_0}} y_1$;
2. $x_0 <_X x_1$ and $y_1 = \top_{x_1}$ is the greatest element in $Y_{x_1}$.

**Lemma 4.6 ([3])** Let $X$ be a $T_0$ space, $Y_x$ an irreducible $T_0$ space for any $x \in X$, and $Z = \sum X Y_x$. Then an arbitrary set $S' \in D(Z)$ contains a cofinal subset $S \subseteq S'$ having one of the following forms:

1. $S = \{(y, x) \mid y \in (S)_x\}$ for some fixed $x \in X$ and $(S)_x \in D(Y_x)$;
2. $S = \{((\top_x, x) \mid x \in S_x\}$ for some $S_x \in D(\bar{X})$.

For any irreducible topological space $X$, put

$$Y^\top = \begin{cases} Y, & \text{if } Y \text{ has a greatest element}, \\ (Y \cup \{\top\}, \tau(Y)^\top), & \text{otherwise}, \end{cases}$$

where $\tau(Y)^\top = \{U \cup \{\top\} \mid \emptyset \neq U \in \tau(Y)\} \cup \{\emptyset\}$. It is easy to see that for any irreducible $T_0$ space $Y$, $Y^\top$ is also a $T_0$ space and has a greatest element. Let

$$X' = \{\{x\} \mid x \in X\} \cup D(\bar{X}) \subseteq D(X).$$

Then $X'$ with the induced topology is a subspace of $D(X)$ and the space $D(Y_x)$ is irreducible for any $x \in X$ from [3]. Moreover, for any $x' \in X'$, we define

$$Y_{x'}^\top = \begin{cases} D(Y_x), & \text{if } x' = \{x\} \text{ for some } x \in X, \\ \top, & \text{otherwise}, \end{cases}$$

where $\top = \{\{\top\}, \emptyset, \{\top\}\}$. Then we have the following theorem.
Theorem 4.7 ([3]) Let $X$ be a $T_0$ space, $Y_x$ an irreducible $T_0$ space for any $x \in X$, and $Z = \sum_X Y_x$. Then the spaces $D(Z)$ and $Z' = \sum_{X'} Y'_{X'}$, are homeomorphic.

For any ordinal $\alpha > 0$, consider the irreducible $T_0$ space

$$\mathbb{O}_\alpha = \{\downarrow \alpha \setminus \{\alpha\}, \emptyset \cup \{\uparrow \beta \mid \beta < \alpha \text{ is not a limit ordinal}\} \}.$$ 

Proposition 4.8 ([3]) Let $\alpha > 0$ be an ordinal.

(i) If $\alpha$ is a limit ordinal, then $H_d(\mathbb{O}_\alpha) = \mathbb{O}_{\alpha}^\top = D(\mathbb{O}_\alpha)$, i.e., the space $\mathbb{O}_\alpha$ is $1^d$-special.

(ii) If $\alpha$ is not a limit ordinal, then $H_d(\mathbb{O}_\alpha) = \mathbb{O}_\alpha$, i.e., the $d$-rank of $\mathbb{O}_\alpha$ is equal to 0.

(iii) If $\alpha$ is a limit ordinal, $\gamma$ is not a limit ordinal, a $T_0$ space $Y_\beta$ is $\gamma^d$-special for any $\beta < \alpha$ and the space $Z = \sum_{\mathbb{O}_\alpha} Y_\beta$, then the spaces $D_\delta(Z) \cong \sum_{\mathbb{O}_\alpha} D_\delta(Y_\beta)$ for any ordinal $\delta \leq \gamma$.

(iv) If $\alpha$ is a limit ordinal, $\gamma$ is not a limit ordinal and a $T_0$ space $Y_\beta$ is $\gamma^d$-special for any $\beta < \alpha$, then the space $Z = \sum_{\mathbb{O}_\alpha} Y_\beta$ is $(\gamma + 1)^d$-special.

(v) If $Y$ is an $\alpha(\alpha + 1)^d$-special for some ordinal $\alpha$, then $D_\beta(Y^\top) \cong D_\beta(Y)^\top$ for any $\beta \leq \alpha$ and $D_{\alpha + 1}(Y^\top) \cong D_{\alpha + 1}(Y) = H_d(Y)$.

(vi) If $\alpha$ is a limit ordinal, and a $T_0$ space $Y_\beta$ is $(\beta + 1)^d$-special for any $\beta < \alpha$, then the space $Z = \sum_{\mathbb{O}_\alpha} Y_\beta$ is $(\alpha + 1)^d$-special and the $d$-rank of a space $Z^\top$ is equal to $\alpha$.

Lemma 4.9 Let $N = (N, \tau_\alpha)$ denote the set $N$ of natural numbers endowed with the Scott topology. Then $N$ is $1^d$-special.

Proof. This directly follows from (i) of Proposition 4.8. 

Lemma 4.10 Let $\alpha > 0$ be an ordinal.

1. If $\alpha$ is not a limit ordinal and a $T_0$ space $X_n$ is $\alpha^d$-special for any $n \in N$, then the space $Z = \sum_X X_n$ is $(\alpha + 1)^d$-special.

2. If $\alpha$ is a limit ordinal and a $T_0$ space $X_n$ is $(\overline{\alpha} + n)^d$-special for any $n \in N$, then the space $Z = \sum_X X_n$ is $(\alpha + 1)^d$-special and the $d$-rank of the space $Z^\top$ is equal to $\alpha$, where $\overline{\alpha} = 0$ if $\alpha = \omega$, otherwise, $\overline{\alpha}$ denotes the largest limit ordinal less than $\alpha$.

Proof.

1. It follows directly from (iv) of Proposition 4.8.

2. First we prove that the spaces $D_\delta(Z)$ and $\sum N \delta_n$ are homeomorphic for every ordinal $\delta < \alpha$, where

$$W_\delta = \begin{cases} D_\delta(X_n), & \text{if } \delta < \overline{\alpha} + n, \\ H_d(X_n), & \text{if } \overline{\alpha} + n \leq \delta < \alpha. \end{cases}$$

We use induction on $\delta$. For $\delta = 0$, the statement follows from the definition of space $Z$.

Let $\delta$ be an ordinal such that $\delta + 1 < \alpha$, and suppose that $D_\delta(Z) \cong \sum N \delta_n$. Then the space $D_\delta(X_n)$ does not contain a greatest element for the arbitrary $n \in N$ such that $\delta < \overline{\alpha} + n$. Note that $\tilde{N} = \{n \in N \mid \overline{\alpha} + n \leq \delta\}$ is a finite subset in $N$, hence $\tilde{N}' = \{\{n\} \mid n \in N\} \cup D(\tilde{N}) \cong N$. By
Theorem 4.7, we have
\[ D_{\delta+1}(Z) = D(D_\delta(Z)) \cong D(\sum_N W^\delta_n) \cong \sum_N D(W^\delta_n) = \sum_N W^\delta_{n+1}. \]

Suppose now that \( \delta < \alpha \) is a limit ordinal and \( D_\beta(Z) \cong \sum_N W^\beta_n \) for any ordinal \( \beta < \delta \). By Theorem 4.7, we get
\[ D_\delta(Z) = \bigcup_{\beta<\delta} D_\beta(Z) \cong \bigcup_{\beta<\delta} \sum_N W^\beta_n \cong \sum_{\beta<\delta} \bigcup_N W^\beta_n = \sum_N W^\delta_n. \]

Thus by induction, we have \( D_\delta(Z) \cong \sum_N W^\delta_n \) for any ordinal \( \delta < \alpha \).

Therefore,
\[ D_\alpha(Z) = \bigcup_{\delta<\alpha} D_\delta(Z) \cong \bigcup_{\delta<\alpha} \sum_N W^\delta_n \cong \sum_{\delta<\alpha} \bigcup_N W^\delta_n = \sum_N W^\alpha_n \cong \sum_N H_d(X_n). \]

Moreover, for any \( n \in N \), the space \( H_d(X_n) \) has a greatest element, which implies that \( \bar{N} = N \). Hence \( N' \cong N^\top \). In the view of Lemma 4.5 and Theorem 4.7, we obtain
\[ D_{\alpha+1}(Z) = D(D_\alpha(Z)) \cong D(\sum_N H_d(X_n)) \cong \sum_{N^\top} X'_{n'} \cong (\sum_N H_d(X_n))^\top, \]
where
\[ X'_{n'} = \begin{cases} H_d(X_{n'}), & \text{if } n' \in N, \\ \top, & \text{if } n' = \top, \end{cases} \]
and
\[ D_{\alpha+2}(Z) = D(D_{\alpha+1}(Z)) \cong D(\sum_{N^\top} X'_{n'}) \cong \sum_{N^\top} D(X'_{n'}) \cong \sum_{N^\top} X'_{n'} \cong D_{\alpha+1}(Z). \]

Again by Lemma 4.5, the space \( D_\beta(Z) \) has not a greatest element for any ordinal \( \beta \leq \alpha \). Therefore, by virtue of Definition 4.1, the space \( Z \) is \( (\alpha + 1)^\alpha \)-special.

For \( Z^\top \), first, we claim that the spaces \( D_\delta(Z^\top) \cong \sum_{N^\top} W^\delta_{n'} \) for the arbitrary ordinal \( \delta \leq \alpha \), where
\[ W^\delta_{n'} = \begin{cases} D_\delta(X_{n'}), & \text{if } \delta < \overline{\alpha} + n' < \alpha, \\ H_d(X_{n'}), & \text{if } \overline{\alpha} + n' \leq \delta \leq \alpha, \\ \top, & \text{if } n' = \top, \end{cases} \]

By the part (v) of Proposition 4.8, we get
\[ D_\delta(Z^\top) \cong (D_\delta(Z))^\top \cong (\sum_{N^\top} W^\delta_{n'})^\top \cong \sum_{N^\top} W^\delta_{n'} \]
for every ordinal \( \delta \leq \alpha \). This implies that
\[ D_\alpha(Z^\top) \cong \sum_{N^\top} W^\alpha_{n'} \cong (\sum_N H_d(X_n))^\top \cong D_{\alpha+1}(Z), \]
which is a \( d \)-space by the above proof. Therefore, \( D_\alpha(Z^\top) \cong D_{\alpha+1}(Z^\top) \).
Next, we claim that $D_\delta(Z^\top)$ is not a $d$-space for any ordinal $\delta < \alpha$. Assume that there exists an ordinal $\delta < \alpha$ such that $D_\delta(Z^\top)$ is a $d$-space. Then by Lemma 3.9, $D_\delta(Z^\top) \cong D_\alpha(Z^\top)$. However, from the preceding discussion, we have that the spaces $D_\delta(Z^\top) \cong \sum_{N}^{\top} W_\alpha^{\delta}$ for the arbitrary ordinal $\delta < \alpha$. Note that there are at most finitely many $W_\alpha^{\delta}$'s are $d$-spaces. Furthermore, for $\delta < \alpha + n' < \alpha$, $W_\alpha^{\delta} = D_\delta(X_n')$ is not a $d$-space and $W_\alpha^{\delta} = H_\delta(X_n')$ is a $d$-space, which implies that $W_\alpha^{\delta}$ and $W_\alpha^{\delta}$ are not homeomorphic. Hence, $D_\delta(Z^\top)$ and $D_\alpha(Z^\top)$ are not homeomorphic, which is a contradiction. So the $d$-rank of the space $Z^\top$ is equal to $\alpha$.

For the $wf$-rank of a $T_0$ space, we have the following similar results.

**Lemma 4.11** [11] Let $X$ be a $T_0$ space, $Y_x$ an irreducible $T_0$ space for any $x \in X$, and $Z = \sum_{X} Y_x$. Then an arbitrary set $A' \in \text{KF}(Z)$ contains a subset $A \subseteq A'$ such that $A \sim A'$ having one of the following forms:

1. there exists an element $x \in X$ such that $A \subseteq Y_x \times \{x\}$ and $(A)_{x} \in \text{KF}(Y_x)$;

2. $A = \{(\top, x) \mid x \in A_x\}$ for some $A_x \in \text{KF}(\bar{X})$, where $\bar{X} = \{x \in X \mid Y_x \text{ has a greatest element}\}$.

**Theorem 4.12** Let $X$ be a $T_0$ space, $Y_x$ an irreducible $T_0$ space for any $x \in X$, and $Z = \sum_{X} Y_x$. Then the spaces $KF(Z)$ and $Z' = \sum_{X'} Y_x'$ are homeomorphic, where

$$X' = \{[\{x\}] \mid x \in X\} \cup KF(\bar{X}) \subseteq KF(X),$$

and for any $x' \in X'$

$$Y_{x'} = \begin{cases} KF(Y_x), & \text{if } x' = [\{x\}] \text{ for some } x \in X, \\ \top, & \text{otherwise, where } \top = \langle\{\top\}, \emptyset, \{\top\}\rangle. \end{cases}$$

**Lemma 4.13** $(N, \tau_\alpha)$ is $1^{wf}$-special.

**Lemma 4.14** Let $\alpha > 0$ be an ordinal.

1. If $\alpha$ is not a limit ordinal and a $T_0$ space $X_n$ is $\alpha^{wf}$-special for any $n \in N$, then the space $Z = \sum_{N} X_n$ is $(\alpha + 1)^{wf}$-special.

2. If $\alpha$ is a limit ordinal and a $T_0$ space $X_n$ is $(\alpha + n)^{wf}$-special for any $n \in N$, then the space $Z = \sum_{N} X_n$ is $(\alpha + 1)^{wf}$-special and the $wf$-rank of the space $Z^\top$ is equal to $\alpha$, where $\bar{\alpha} = 0$ if $\alpha = \omega$, otherwise, $\bar{\alpha}$ denotes the largest limit ordinal less than $\alpha$.

Next, let $Q_k$ be a $K$-subset system and $X$ a $T_0$ space. We denote

$$\overline{D}(X) = \{A \subseteq X \mid \text{there exists a directed subset } D \text{ in } X \text{ such that } \overline{A} = \overline{D}\}.$$

For the $k$-rank of $X$, we deduce the following results.

**Lemma 4.15** For $N = (N, \tau_\alpha)$, the following statements hold:

1. $N$ is $1^d$-special and $1^{wf}$-special.

2. $\overline{D}(N) = \text{KF}(N)$ and $D(N) = K(N) = KF(N)$.

3. $N$ is $1^k$-special.

**Lemma 4.16** If $\alpha$ is not a limit ordinal and a $T_0$ space $X_n$ satisfies the following conditions.
(1) $X_n$ is $\alpha^d$-special and $\alpha^wf$-special,

(2) $\overline{D}(K_m(X_n)) = K\overline{F}(K_m(X_n))$ and $D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n)$ for $0 \leq m < \alpha$,

for any $n \in N$, then the space $Z = \sum_{n} X_n$ satisfies:

(1) $Z$ is $(\alpha + 1)^d$-special and $(\alpha + 1)^wf$-special.

(2) $\overline{D}(K_m(Z)) = K\overline{F}(K_m(Z))$ and $D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z)$ for $0 \leq m < \alpha + 1$.

(3) $Z$ is $(\alpha + 1)^k$-special.

**Proof.** For (1), this directly follows from Lemma 4.10 (1) and Lemma 4.14 (1).

For (2), the proof is by induction on $m$.

Basic steps. For $m = 0$, obviously, $\overline{D}(Z) \subseteq K\overline{F}(Z)$. Conversely, let $A \in K\overline{F}(Z)$. From Lemma 4.11, there exists a subset $A' \subseteq A$ that $A' \sim A$ and $A'$ satisfies Type (i) in Lemma 4.11. This means that there exists $n \in N$ such that $A' \subseteq X_n \times \{n\}$ and $(A')_n \in K\overline{F}(X_n) = \overline{D}(X_n)$. By the definition of $\overline{D}(X_n)$, there is a directed subset $D \in X_n$ such that $\text{cl}_{X_n}(A'_n) = \text{cl}_{X_n}(D)$. We claim that $\text{cl}_Z(A') = \text{cl}_Z(D \times \{n\})$. For $(y, x) \in \text{cl}_Z(A')$, let $U$ be an open neighbourhood of $(y, x)$. Then we have that $U \cap A' \neq \emptyset$. This implies that $(U)_n \cap (A')_n \neq \emptyset$. Since $U$ is open in $Z$, we have $(U)_n \in \tau(X_n)$. By $\text{cl}_{X_n}(A'_n) = \text{cl}_{X_n}(D)$, we have $(U)_n \cap D \neq \emptyset$, that is $U \cap (D \times \{n\}) \neq \emptyset$. Hence $(y, x) \in \text{cl}_Z(D \times \{n\})$. The opposite direction is similar to prove. So $A \in \overline{D}(Z)$. That is $\overline{D}(Z) = K\overline{F}(Z)$, which implies that $\overline{D}(Z) = K(Z) = K\overline{F}(Z)$. Therefore, $D(Z) = K(Z) = K\overline{F}(Z)$.

Inductive steps. There are two cases to consider:

Case 1. Let $m$ be an ordinal such that $m + 1 < \alpha + 1$. Assume that $\overline{D}(K_m(Z)) = K\overline{F}(K_m(Z))$ and $D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z)$, by Lemma 4.8 (iii), we have

$$D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z) \cong \sum_{n} D_{m+1}(X_n).$$

To prove $\overline{D}(K_{m+1}(Z)) = K\overline{F}(K_{m+1}(Z))$, it is enough to show $\overline{D}(\sum_{n} D_{m+1}(X_n)) = K\overline{F}(\sum_{n} D_{m+1}(X_n))$. Clearly, $\overline{D}(\sum_{n} D_{m+1}(X_n)) \subseteq K\overline{F}(\sum_{n} D_{m+1}(X_n))$. Conversely, for any $A \in K\overline{F}(\sum_{n} D_{m+1}(X_n))$, by Lemma 4.11, there exists a subset $A' \subseteq A$ such that $A' \sim A$. Two options are possible:

Case 1.1. $A'$ is Type (i) in Lemma 4.11. This implies that there exists $n \in N$ such that $A' \subseteq D_{m+1}(X_n) \times \{n\}$ and $(A')_n \in K\overline{F}(D_{m+1}(X_n))$. By the condition (2) of $X_n$, we get

$$D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n) \text{ and } K\overline{F}(K_{m+1}(X_n)) = \overline{D}(K_{m+1}(X_n)).$$

Hence, $(A')_n \in K\overline{F}(D_{m+1}(X_n)) = \overline{D}(D_{m+1}(X_n))$. So $A' \in \overline{D}(\sum_{n} (D_{m+1}(X_n)))$. This implies that $A \in \overline{D}(\sum_{n} (D_{m+1}(X_n)))$.

Case 1.2. $A'$ is Type (ii) in Lemma 4.11. This means that there exists $A_* \in K\overline{F}(\tilde{N})$ such that $A' = \{(\bot, n) \mid n \in A_*\}$. Note that $m + 1 = \alpha$, then $\sum_{n} D_{m+1}(X_n) \cong \sum_{n} H_d(X_n) \cong \sum_{n} H_{wf}(X_n)$. So $\tilde{N} = N$. Hence, $K\overline{F}(\tilde{N}) = \overline{D}(\tilde{N})$, which implies that $A' \in \overline{D}(\sum_{n} D_{m+1}(X_n))$. Therefore, $A \in \overline{D}(\sum_{n} D_{m+1}(X_n))$.

In any case, we have $K\overline{F}(\sum_{n} D_{m+1}(X_n)) \subseteq \overline{D}(\sum_{n} D_{m+1}(X_n))$. So $\overline{D}(K_{m+1}(Z)) = K\overline{F}(K_{m+1}(Z))$. This implies that $D_{m+2}(Z) = K_{m+2}(Z) = KF_{m+2}(Z)$.

Case 2. Suppose that $m < \alpha + 1$ is a limit ordinal and the required statement holds for any $\delta < m$. Then

$$K_m(Z) = \bigcup_{\delta < m} K_\delta(Z) \cong \bigcup_{\delta < m} D_\delta(Z) \cong \bigcup_{\delta < m} \sum_{n} (D_\delta(X_n)) \cong \sum_{\delta < m} \bigcup_{n} D_\delta(X_n) \cong \sum_{\delta < m} D_\delta(X_n) \cong \sum_{\delta < m} D_m(X_n).$$
Now it is enough to show that \( \overline{KF}(\sum_N D_m(X_n)) = \overline{D}(\sum_N D_m(X_n)) \). Repeat the proof method of Case 1, we get that \( \overline{D}(K_m(Z)) = \overline{KF}(K_m(Z)) \). So \( \overline{D}(K_m(Z)) = \overline{K}(K_m(Z)) = \overline{KF}(K_m(Z)) \). Hence, \( D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z) \).

For (3), by (2), let \( m = \alpha \). We get

\[
D_{\alpha+1}(Z) = K_{\alpha+1}(Z) = KF_{\alpha+1}(Z).
\]

Since \( Z \) is \((\alpha + 1)^d\)-special and \((\alpha + 1)^{wf}\)-special, for any ordinal \( \delta < \alpha + 1 \), \( K_\delta(Z) \) is not a \( d \)-space and \( K_{\alpha+1}(Z) \) is well-filtered. Therefore, \( K_\delta(Z) \) is not \( k \)-well-filtered and \( K_{\alpha+1}(Z) \) is \( k \)-well-filtered. Then \( Z \) is \((\alpha + 1)^k\)-special.

**Lemma 4.17** If \( \alpha \) is a limit ordinal and a \( T_\delta \) space \( X_n \) satisfies the following conditions.

1. \( X_n \) is \((\overline{\alpha} + n)^d\)-special and \((\overline{\alpha} + n)^{wf}\)-special,
2. \( \overline{D}(K_m(X_n)) = \overline{KF}(K_m(X_n)) \) and \( D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n) \) for \( 0 \leq m < \overline{\alpha} + n \), for any \( n \in N \), then for the space \( Z = \sum_N X_n \), we have the following conclusions.

1. \( Z \) is \((\alpha + 1)^d\)-special and \((\alpha + 1)^{wf}\)-special.
2. \( \overline{D}(K_m(Z)) = \overline{KF}(K_m(Z)) \) and \( D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z) \) for \( 0 \leq m < \alpha + 1 \).
3. \( Z \) is \((\alpha + 1)^k\)-special.

Moreover, for the space \( Z^\top \), the following results hold.

1. \( \text{rank}_d(Z^\top) = \text{rank}_w(Z^\top) = \alpha \),
2. \( \overline{D}(K_m(Z^\top)) = \overline{KF}(K_m(Z^\top)) \) and \( D_{m+1}(Z^\top) = K_{m+1}(Z^\top) = KF_{m+1}(Z^\top) \) for \( 0 \leq m < \alpha \),
3. \( \text{rank}_k(Z^\top) = \alpha \).

**Proof.** First, we consider the space \( Z = \sum_N X_n \).

For (1), it follows directly from Lemma 4.10 (2) and Lemma 4.14 (2).

For (2), we proceed by induction. For \( m = 0 \), the statement follows from the proof of Lemma 4.16. Let \( m \) be an ordinal such that \( m + 1 < \alpha + 1 \). Assume that \( \overline{D}(K_m(Z)) = \overline{KF}(K_m(Z)) \) and \( D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z) \), by the proof of Lemma 4.10 (2), we derive

\[
D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z) \cong \sum_N W_n^{m+1}
\]

where

\[
W_n^{m+1} = \begin{cases} 
D_{m+1}(X_n), & \text{if } m + 1 < \overline{\alpha} + n, \\
H_d(X_n), & \text{if } \overline{\alpha} + n \leq m + 1 < \alpha + 1.
\end{cases}
\]

To prove \( \overline{D}(K_{m+1}(Z)) = \overline{KF}(K_{m+1}(Z)) \), it suffices to show that \( \overline{D}(\sum_N W_n^{m+1}) = \overline{KF}(\sum_N W_n^{m+1}) \). Clearly, \( \overline{D}(\sum_N W_n^{m+1}) \subseteq \overline{KF}(\sum_N W_n^{m+1}) \). For any \( A \in \overline{KF}(\sum_N W_n^{m+1}) \), by Lemma 4.11, there exists a subset \( A' \subseteq A \) such that \( A' \sim A \). There are two cases to consider:

Case 1. \( A' \) is Type (i) in Lemma 4.11. This implies that there exists \( n \in N \) such that \( A' \subseteq W_n^{m+1} \times \{n\} \) and \( (A')_n \in \overline{KF}(W_n^{m+1}) \). Again there are two cases to consider:

Case 1.1. If \( m + 1 < \overline{\alpha} + n \), then \( W_n^{m+1} = D_{m+1}(X_n) \). By the condition (2) of \( X_n \), we have

\[
D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n) \text{ and } \overline{KF}(K_{m+1}(X_n)) = \overline{D}(K_{m+1}(X_n)).
\]
Hence, \((A')_n \in \overline{KF}(D_{m+1}(X_n)) = \overline{D}(D_{m+1}(X_n))\). So \(A' \in \overline{D}(\bigcup_n W_{m+1}^n)\). This implies that \(A \in \bigcup_n \overline{D}(W_{m+1}^n)\).

Case 1.2. If \(\pi + n \leq m + 1 < \alpha + 1\), then \(W_{m+1}^n = H_{wf}(X_n) = H_d(X_n)\). Hence, \(\overline{KF}(W_{m+1}^n) = \overline{D}(W_{m+1}^n)\). It is straightforward to check that \(A \in \overline{D}(\bigcup_n W_{m+1}^n)\).

Case 2. \(A'\) is Type (ii) in Lemma 4.11. This means that there exists \(A_\alpha \in \overline{KF}(\overline{N})\) such that \(A_\alpha = \{ (\pi, n) \mid n \in A_\alpha \}\). Note that there are at most finitely many \(W_{m+1}^n\)s which have a greatest element; thus \(\overline{KF}(\overline{N}) = \overline{D}(\overline{N})\). This implies that \(A_\alpha \in \overline{D}(\bigcup_n W_{m+1}^n)\). Therefore, \(A \in \overline{D}(\bigcup_n W_{m+1}^n)\).

In any case, we have that \(\overline{KF}(\bigcup_n W_{m+1}^n) \subseteq \overline{D}(\bigcup_n W_{m+1}^n)\). So \(\overline{D}(K_{m+1}(Z)) = \overline{KF}(K_{m+1}(Z))\). This implies that \(D_{m+2}(Z) = K_{m+2}(Z) = KF_{m+2}(Z)\).

Suppose that \(m < \alpha + 1\) is a limit ordinal and that the required statement holds for any \(\delta < m\). Then

\[
K_m(Z) = \bigcup_{\delta < m} K_\delta(Z) = \bigcup_{\delta < m} D_\delta(Z) = \bigcup_{\delta < m} \sum_n W_\delta^n \approx \sum_n \bigcup_{\delta < m} W_\delta^n \approx \sum_n W_n
\]

where

\[
W_\delta^n = \begin{cases} 
D_\delta(X_n), & \text{if } \delta < \overline{\alpha} + n, \\
H_d(X_n), & \text{if } \overline{\alpha} + n \leq \delta < \alpha.
\end{cases}
\]

Now it is enough to show that \(\overline{KF}(\bigcup_n W_n^m) = \overline{D}(\bigcup_n W_n^m)\). Repeat the above proof method, we get that \(\overline{D}(K_m(Z)) = \overline{KF}(K_m(Z))\). Hence,

\[
\overline{D}(K_m(Z)) = \overline{K}(K_m(Z)) = \overline{KF}(K_m(Z)) \text{ and } D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z).
\]

For (3), let \(m = \alpha\). By (2), we deduce

\[
D_{\alpha+1}(Z) = K_{\alpha+1}(Z) = KF_{\alpha+1}(Z).
\]

Since \(Z\) is \((\alpha + 1)^d\)-special and \((\alpha + 1)^w\)-special, we have that for any ordinal \(\delta < \alpha + 1\), \(K_\delta(Z)\) is not a \(d\)-space and \(K_{\alpha+1}(Z)\) is well-filtered. Thus \(K_\delta(Z)\) is not \(k\)-well-filtered and \(K_{\alpha+1}(Z)\) is \(k\)-well-filtered. Then \(Z\) is \((\alpha + 1)^k\)-special.

Next, we analyze the space \(Z^\top\). For \(Z^\top\), the statement (1) also follows from Lemma 4.10 (2) and Lemma 4.14 (2).

For (2), The proof is by induction on \(m\).

Basic steps. For \(m = 0\), clearly, \(\overline{D}(Z^\top) \subseteq \overline{KF}(Z^\top)\). Conversely, let \(A \in \overline{KF}(Z^\top)\). From Lemma 4.11, we have that \(A \in \overline{KF}(Z)\) or \(A \sim \{ \top \}\). It is straightforward to check that \(A \in \overline{D}(Z^\top)\). Therefore,

\[
\overline{D}(Z^\top) = \overline{KF}(Z^\top) \text{ and } D(Z) = K(Z) = KF(Z).
\]

Inductive steps. There are two cases to consider:

Case 1. Let \(m\) be an ordinal such that \(m + 1 < \alpha\). Assume that

\[
\overline{D}(K_m(Z^\top)) = \overline{KF}(K_m(Z^\top)) \text{ and } D_{m+1}(Z^\top) = K_{m+1}(Z^\top) = KF_{m+1}(Z^\top).
\]
By the proof of Lemma 4.10 (2), we have that $D_{m+1}(Z^\uparrow) = K_{m+1}(Z^\uparrow) \cong \sum_{\mathbb{N}^+} W_{n'}^{m+1}$, where

$$W_{n'}^{m+1} = \begin{cases} 
D_{m+1}(X_{n'}), & \text{if } m + 1 < \alpha' + n' < \alpha, \\
H_d(X_{n'}), & \text{if } \alpha' + n' \leq m + 1 < \alpha, \\
\top, & \text{if } n' = \top.
\end{cases}$$

To prove $\overline{D}(K_{m+1}(Z^\uparrow)) = \overline{KF}(K_{m+1}(Z^\uparrow))$, it is sufficient to show $\overline{D}(\sum_{\mathbb{N}^+} W_{n'}^{m+1}) = \overline{KF}(\sum_{\mathbb{N}^+} W_{n'}^{m+1})$.

Clearly, $\overline{D}(\sum_{\mathbb{N}^+} W_{n'}^{m+1}) \subseteq \overline{KF}(\sum_{\mathbb{N}^+} W_{n'}^{m+1})$. For any $A \in \overline{KF}(\sum_{\mathbb{N}^+} W_{n'}^{m+1})$, again by Lemma 4.11, we have that $A \in \overline{D}(\sum_{\mathbb{N}^+} W_{n'}^{m+1})$. Therefore, $\overline{KF}(\sum_{\mathbb{N}^+} W_{n'}^{m+1}) \subseteq \overline{D}(\sum_{\mathbb{N}^+} W_{n'}^{m+1})$. So

$$\overline{D}(K_{m+1}(Z^\uparrow)) = \overline{KF}(K_{m+1}(Z^\uparrow)).$$

This implies

$$D_{m+2}(Z^\uparrow) = K_{m+2}(Z^\uparrow) = KF_{m+2}(Z^\uparrow).$$

**Case 2.** Suppose that $m < \alpha$ is a limit ordinal and that the required statement holds for any $\delta < m$. Then

$$K_m(Z^\uparrow) = \bigcup_{\delta < m} K_\delta(Z^\uparrow) \cong \bigcup_{\delta < m} D_\delta(Z^\uparrow) \cong \bigcup_{\delta < m} \sum_{\mathbb{N}^+} W_{n'}^\delta \cong \sum_{\mathbb{N}^+} \bigcup_{\delta < m} W_{n'}^\delta \cong \sum_{\mathbb{N}^+} W_{n'}^m.$$

Now it is enough to show $\overline{KF}(\sum_{\mathbb{N}^+} W_{n'}^m) = \overline{D}(\sum_{\mathbb{N}^+} W_{n'}^m)$. Repeat the proof method of Case 1, we get that $\overline{D}(K_m(Z^\uparrow)) = \overline{KF}(K_m(Z^\uparrow))$. So

$$\overline{D}(K_m(Z^\uparrow)) = \overline{KF}(K_m(Z^\uparrow)) = \overline{KF}(K_m(Z^\uparrow)).$$

Hence

$$D_{m+1}(Z^\uparrow) = K_{m+1}(Z^\uparrow) = KF_{m+1}(Z^\uparrow).$$

For (3), by (2), we get that

$$K_\alpha(Z^\uparrow) = \bigcup_{\delta < \alpha} K_\delta(Z^\uparrow) \cong \bigcup_{\delta < \alpha} KF_\delta(Z^\uparrow) \cong KF_\alpha(Z^\uparrow).$$

So $K_\alpha(Z^\uparrow)$ is well-filtered by (1) and hence $K_\alpha(Z^\uparrow)$ is $k$-well-filtered. Again by (1), we have that $K_\delta(Z^\uparrow)$ is not $k$-well-filtered for any ordinal $\delta < \alpha$. Therefore, $\text{rank}_k(Z^\uparrow) = \alpha.$

**Theorem 4.18** For any ordinal $\alpha$, there exists a $T_0$ space $X$ and the following statements hold.

1. $X$ is $(\alpha + 1)^{d}$-special and $(\alpha + 1)^{w_f}$-special.
2. $\overline{D}(K_m(X)) = \overline{KF}(K_m(X))$ and $D_{m+1}(X) = K_{m+1}(X) = KF_{m+1}(X)$ for $0 \leq m < \alpha + 1$.
3. $X$ is $(\alpha + 1)^{k}$-special.

**Proof.** As usual, we use induction on $\alpha$. Base cases. If $\alpha = 0$, by Lemma 4.15, $(\mathbb{N}, \tau_\sigma)$ satisfies the required statements.

Inductive steps. There are two cases to consider:

Case (1). Suppose that the statements of the theorem hold for any ordinal $\alpha$. From Lemma 4.16, we have that the statements of the theorem hold for the ordinal $\alpha + 1$.

Case (2). Let $\alpha$ be a limit ordinal. Assume for any ordinal $\beta < \alpha$, the statements of the theorem hold. Note that for every $n \in \mathbb{N}$, $\overline{\alpha} + n < \alpha$. So the statements of the theorem hold for $\overline{\alpha} + n$. By Lemma 4.17, it is straightforward to check that the statements of the theorem hold for $\alpha$. $\square$
Corollary 4.19 For any non-limit ordinal $\alpha$, there exists a $T_0$ space $X$ satisfying the following conditions.

1. $X$ is $\alpha^d$-special and $\alpha^{wf}$-special.
2. $\overline{D}(K_m(X)) = \overline{KF}(K_m(X))$ and $D_{m+1}(X) = K_{m+1}(X) = KF_{m+1}(X)$ for $0 \leq m < \alpha$.
3. $X$ is $\alpha^k$-special.

Proof. For any non-limit ordinal $\alpha$, there exists an ordinal $\beta$ such that $\alpha = \beta + 1$. By Theorem 4.18, for ordinal $\beta$, there exists a $T_0$ space $X$ which satisfies the required statements. $\square$

Theorem 4.20 For any ordinal $\alpha$, there exists an irreducible $T_0$ space $X$ such that the following statements hold.

1. $\operatorname{rank}_d(X) = \operatorname{rank}_{wf}(X) = \alpha$.
2. $\overline{D}(K_m(X)) = \overline{KF}(K_m(X))$ and $D_{m+1}(X) = K_{m+1}(X) = KF_{m+1}(X)$ for $0 \leq m < \alpha$.
3. $\operatorname{rank}_k(X) = \alpha$.

Proof. Let $\alpha$ be an ordinal. Now we consider two cases:

Case 1. If $\alpha$ is not a limit ordinal, the statements of the theorem follow directly from Corollary 4.19.

Case 2. If $\alpha$ is a limit ordinal, then $\beta + n$ is a non-limit ordinal for any $n \in \mathbb{N}$. Hence, the statements of the theorem follow from Lemma 4.17 and Corollary 4.19. $\square$

Corollary 4.21 For any ordinal $\alpha$, there exists an irreducible $T_0$ space $X$ whose $k$-rank is equal to $\alpha$.

Proof. For any ordinal $\alpha$, it is enough to show that the $k$-rank of the space $X$ required in Theorem 4.20 is $\alpha$. $\square$

We can directly obtain the following corollaries.

Corollary 4.22 (see also in [3]) For any ordinal $\alpha$, there exists an irreducible $T_0$ space $X$ whose $d$-rank is equal to $\alpha$.

Corollary 4.23 (see also in [8]) For any ordinal $\alpha$, there exists an irreducible $T_0$ space $X$ whose $wf$-rank is equal to $\alpha$.

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