The spectrum of an open vertex model based on the $U_q[SU(2)]$ algebra at roots of unity

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**Abstract**

We study the exact solution of an $N$-state vertex model based on the representation of the $U_q[SU(2)]$ algebra at roots of unity with diagonal open boundaries. We find that the respective reflection equation provides us one general class of diagonal $K$-matrices having one free-parameter. We determine the eigenvalues of the double-row transfer matrix and the respective Bethe ansatz equation within the algebraic Bethe ansatz framework. The structure of the Bethe ansatz equation combine a pseudomomenta function depending on a free-parameter with scattering phase-shifts that are fixed by the roots of unity and boundary variables.

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1 Introduction

A relevant class of two-dimensional integrable models are those based on the representations of a given quantum affine algebra $U_q[G]$ \[1\]. The quantum group framework permits us to generate solutions of the Yang-Baxter equation which can be seen as the Boltzmann weights of two-dimensional vertex models \[2\]. The standard paradigm is the spin-$s$ representation of the $U_q[SU(2)]$ algebra for generic values of $q$ \[3\] which leads us to an $N$-state extension of the six-vertex model \[4, 5\]. The corresponding one-dimensional spin chain turns out to be an integrable generalization of the Heisenberg model with arbitrary spin $s = (N - 1)/2$ \[6\].

The previous example by no means exhaust the possibility of constructing vertex models within the quantum $U_q[SU(2)]$ algebra. It still remains the representations for non-generic values of $q$ which are known to be quite different from the one with arbitrary $q$ \[7\]. In fact, a new family of $N$-state vertex model can be generated by exploring non-cyclic $U_q[SU(2)]$ representations when $q$ is a root of unity. The respective $R$-matrix depends on both an extra continuous variable besides the spectral parameter and on a discrete variable characterizing the possible roots of unity branches.

We remark that this $R$-matrix has been previously discussed in the literature in distinct contexts such as on the realm of new braid matrices with extra color variables \[8\], in the Baxterization of braids for the special cases of $N = 2, 3$ \[9\] and in the quantum group framework \[10, 11\].

The Bethe ansatz solution of such $N$-vertex model based on roots of unity has so far been restricted to the case of periodic boundary conditions \[12, 13\]. Results for fixed boundary conditions are concentrated on the calculations of the partition function with certain domain wall boundary conditions \[14\]. The purpose of this paper is to start the study of this class of vertex models with open integrable boundary terms. We shall here consider the simplest case of diagonal boundary conditions containing a free parameter. We find that the dependence of the Bethe equations on the continuous parameter characterizing the roots of unity representation appears only in the pseudomomenta function. By way of contrast the respective many-body phase-shifts are functions determined by the discrete roots of unity branches and the boundary parameters.

The outline of this paper is as follows. In section 2 we review the structure of the bulk $R$-matrix
and discuss properties that are useful to build up boundary $K$-matrices. In section 3 we discuss the most general diagonal solution of the reflection equations for both left and right boundaries. We also present the explicit expressions of the corresponding one-dimensional open spin chains for $N = 2$ and $N = 3$. In section 4 we adapt the algebraic Bethe ansatz construction of [15] in order to derive the eigenvalues of the double-row transfer matrix and the respective Bethe equations. Our conclusions are summarized in section 5. In Appendices A and B we present technical details concerning the form of projectors and $R$-matrix amplitudes.

2 The $R$-matrix properties

The corresponding $R$-matrix of the $N$-state vertex model based on the $U_q[SU(2)]$ algebra at roots of unity has been previously discussed by several authors [8, 9, 10, 11]. We shall follow here the recent presentation given by us in terms of the Baxterization approach [13]. For practical computations it is convenient to write the $R$-matrix introducing an auxiliary operator $\hat{R}(\lambda)$,

$$R_{12}(\lambda) = P_{12} \hat{R}_{12}(\lambda),$$

where $P_{12}$ denotes the $C^N \otimes C^N$ operator.

The matrix $\hat{R}(\lambda)$ can be expressed by means of the following linear combination of projectors,

$$\hat{R}(\lambda) = \rho(\lambda) \sum_{i=1}^{N} \prod_{j=1}^{N-1} \frac{\sinh\left[\frac{\pi k(j-1)}{N} + i\gamma + \lambda\right]}{\sinh\left[\frac{\pi k(j-1)}{N} + i\gamma - \lambda\right]} P_i(\gamma, k),$$

where $k$ is an integer coprime to $N$ and the overall normalization $\rho(\lambda)$ is chosen,

$$\rho(\lambda) = \prod_{j=1}^{N-1} \sinh[-\lambda + i\gamma + \frac{i\pi k(j-1)}{N}].$$

The $R$-matrix [13] is characterized by the continuous variable $\gamma$ and the discrete index $k$ parameterizing the non-cyclic $U_q[SU(2)]$ representation at the roots of unity $q = e^{2i\pi k/N}$. The projectors $P_i(\lambda, k)$ can be written in terms of the corresponding braid representation [8] whose expressions can be found in [13]. To make this paper self-consistent we have summarized the main formula for $P_i(\gamma, k)$ in Appendix A. In order to investigate integrable open boundary conditions it
is convenient to bring the bulk $R$-matrix to its most possible symmetrical form. As usual this is accomplished by performing a transformation in $R$-matrix that preserves the Yang-Baxter equation. We find that the suitable spectral parameter dependent transformation is,

$$
\bar{R}_{12}(\lambda) = V_1(\lambda)R_{12}(\lambda)V_1^{-1}(\lambda).
$$

(4)

where the gauge matrix $V(\lambda)$ is diagonal and it is given by

$$
V(\lambda) = \sum_{a=1}^{N} e^{\lambda(a-1)} e_{a,a}
$$

(5)

while $e_{a,b}$ denotes standard $N \times N$ Weyl matrices.

Let us now discuss the structure of the $R$-matrix $\bar{R}_{12}(\lambda)$ for few values of $N$. The simplest case $N = 2$ turns out to be directly related to a six-vertex model satisfying the free-fermion condition, namely

$$
\bar{R}_{12}(\lambda) = \begin{pmatrix}
\sinh[\nu \gamma + \lambda] & 0 & 0 & 0 \\
0 & \sinh[\lambda] & \sinh[\nu \gamma] & 0 \\
0 & \sinh[\nu \gamma] & \sinh[\lambda] & 0 \\
0 & 0 & 0 & \sinh[\nu \gamma - \lambda]
\end{pmatrix}
$$

(6)

The $R$-matrix for $N \geq 3$ gives origin to novel integrable vertex models. For the simplest case $N = 3$ one has a nineteen-vertex model and the structure of the corresponding $R$-matrix is,

$$
\bar{R}_{12}(\lambda) = \begin{pmatrix}
a_+(-\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_+(-\lambda) & 0 & c_+(-\lambda) & 0 & 0 & 0 & 0 \\
0 & 0 & f(\lambda) & 0 & d(\lambda) & 0 & e(\lambda) & 0 \\
0 & c_+(-\lambda) & 0 & b_+(-\lambda) & 0 & 0 & 0 & 0 \\
0 & 0 & d(\lambda) & 0 & g(\lambda) & 0 & d(\lambda) & 0 \\
0 & 0 & 0 & 0 & 0 & b_-(\lambda) & 0 & c_-(\lambda) \\
0 & 0 & e(\lambda) & 0 & d(\lambda) & 0 & f(\lambda) & 0 \\
0 & 0 & 0 & 0 & c_-(\lambda) & 0 & b_-(\lambda) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_-(\lambda)
\end{pmatrix}
$$

(7)
The respective Boltzmann weights are given by,

\[ a_\pm(\lambda) = \sinh[\gamma \pm \lambda] \sinh[\gamma + \frac{\pi k}{3} \pm \lambda] \]
\[ b_\pm(\lambda) = \sinh[\gamma + \frac{\pi k}{3} \pm \lambda], \quad b_\pm(\lambda) = \varepsilon_k \sinh[\gamma - \lambda] \]
\[ c_\pm(\lambda) = \sinh[\gamma] \sinh[\gamma + \frac{\pi k}{3} \pm \lambda], \quad c_\pm(\lambda) = \sinh[\gamma - \lambda] \sinh[\gamma + \frac{\pi k}{3}] \]
\[ d(\lambda) = \varepsilon_k^{3/2} \sinh[\lambda] \sqrt{\sinh[\gamma] \sinh[\gamma + \frac{\pi k}{3}]} \]
\[ e(\lambda) = \sinh[\lambda] \sinh[\gamma + \frac{\pi k}{3}], \quad g(\lambda) = \sinh[\lambda] \sinh[\gamma + \frac{\pi k}{3}] + \sinh[\lambda + \frac{2\pi k}{3}] \sinh[\lambda] \] (8)

where the phase \( \varepsilon_k = \exp[i\pi(k - 1)] \).

Considering Eqs. (1-5) and the expressions for the projectors (A.1-A.4) one is able to compute the \( R \)-matrix elements of \( \bar{R}_{12}(\lambda) \) on the Weyl basis,

\[ \bar{R}_{12}(\lambda) = \sum_{a,b,c,d=1}^{N} \bar{R}^{a,d}_{a,b}(\lambda)e_{a,c} \otimes e_{b,d} \] (9)

for rather high values of \( N \) with moderate computational effort. As further examples we have exhibited in Appendix B the non-null Boltzmann weights of the \( \bar{R}_{12}(\lambda) \) for \( N = 4 \) and \( N = 5 \).

We now turn our attention to describing the symmetry properties of \( \bar{R}_{12}(\lambda) \). Besides satisfying the unitarity property it is invariant by both temporal and parity symmetry, namely

\[ \bar{R}_{12}(\lambda)\bar{R}_{21}(-\lambda) = \rho(\lambda)\rho(-\lambda)I_N \otimes I_N \] (10)
\[ P_{12}\bar{R}_{12}(\lambda)P_{12} = \bar{R}_{12}(\lambda) \] (11)
\[ \bar{R}_{12}(\lambda)^{t_1t_2} = \bar{R}_{12}(\lambda) \] (12)

where \( I_N \) is the \( N \times N \) identity matrix and \( t_i \) denotes the transposition on the \( i \)-th space.

We also note that for arbitrary \( \gamma \) the matrix elements \( \bar{R}^{a,d}_{a,b}(\lambda) \) of \( \bar{R}_{12}(\lambda) \) are not self-conjugated under the charge invariance. As a consequence of that the \( \bar{R}_{12}(\lambda) \) does not satisfy standard crossing properties but only an analog of that which is given by,

\[ \bar{R}_{12}(\lambda)^{t_2}\bar{R}_{21}(-\lambda + \frac{2\pi k}{N})^{t_2} = \rho_1(\lambda)I_N \otimes I_N \] (13)

\[ ^1 \text{Recall that the charge symmetry relates the weights } \bar{R}^{a,d}_{a,b}(\lambda) \text{ and } \bar{R}^{N+1-c,N+1-d}_{N+1-a,N+1-b}(\lambda). \]
where \( \rho_1(\lambda) = \prod_{j=1}^{N-1} \sinh^2[\lambda + \frac{i\pi(k(j-1))}{N}] \).

In next section we shall use the properties \(10, 13\) to obtain the right \(K\)-matrix \(K^+(\lambda)\) from the left one \(K^-(\lambda)\) by means of an isomorphism.

### 3 The \(K\)-matrices and open spin chains

The integrability at the boundary is described in terms of two scattering matrices \(K^\pm(\lambda)\) [16]. The compatibility between bulk and boundary scattering lead us to an algebraic condition at one of the ends of an open chain, which reads [16]

\[
\bar{R}_{12}(\lambda - \mu)K_1^-(\lambda)\bar{R}_{21}(\lambda + \mu)K_2^-(\mu) = K_2^-(\mu)\bar{R}_{12}(\lambda + \mu)K_1^-(\lambda)\bar{R}_{21}(\lambda - \mu) \tag{14}
\]

where \(K_1^-(\lambda) = K^- (\lambda) \otimes I_N\) and \(K_2^-(\lambda) = I_N \otimes K^-(\lambda)\).

When the bulk \(R\)-matrix satisfies the properties \(10, 13\) one can follow a procedure devised in [17, 18] to obtain the matrix \(K^+(\lambda)\) at the opposite boundary. At this point we note that property \(13\) plays the role of the crossing symmetry [18] and the matrix \(K^+(\lambda)\) is then fixed by the following isomorphism,

\[
K^+(\lambda) = K^-(\lambda) \frac{i\pi k}{N} - \lambda)^t. \tag{15}
\]

As a consequence of that we are left with the task to solve a single reflection equation. Here we shall be searching only for diagonal solutions of the reflection equation (14), namely

\[
K^-(\lambda) = \sum_{a=1}^{N} K_a^-(\lambda)e_{a,a}. \tag{16}
\]

The next step is to substitute the ansatz (16) in the reflection equation (14) and look for the simplest relations constraining the unknown matrix elements \(K_a^-(\lambda)\). Among the many functional equations we find that there exists two of them that fix the ratios between the first two and the last two elements in a rather suitable way,

\[
\frac{K_2^-(\mu)}{K_1^-(\mu)} = \frac{\frac{K_1^-(\lambda)}{K_1^-(\lambda)} \bar{R}_{1,2}^{1,1}(\lambda - \mu)\bar{R}_{2,1}^{2,1}(\mu + \lambda) + \bar{R}_{2,1}^{2,1}(\lambda - \mu)\bar{R}_{1,2}^{1,1}(\mu + \lambda)}{\bar{R}_{2,1}^{1,2}(\lambda - \mu)\bar{R}_{2,1}^{2,1}(\mu + \lambda) + \frac{\frac{K_1^-(\lambda)}{K_1^-(\lambda)} \bar{R}_{2,1}^{2,1}(\lambda - \mu)\bar{R}_{1,2}^{1,1}(\mu + \lambda)}} \tag{17}
\]
and

$$\frac{K_{N-1}^{(-)}(\mu)}{K_{N-1}^{(-)}(\mu)} = \frac{K_{N-1}^{(-)}(\lambda)}{K_{N-1}^{(-)}(\lambda)} \frac{\bar{R}_{N-1,N}^{N-1,N}(\lambda + \mu) \bar{R}_{N-1,N}^{N-1,N}(\lambda - \mu) + \bar{R}_{N-1,N}^{N-1,N}(\lambda + \mu) \bar{R}_{N-1,N}^{N-1,N}(\lambda - \mu)}{\bar{R}_{N-1,N}^{N-1,N}(\lambda + \mu) \bar{R}_{N-1,N}^{N-1,N}(\lambda - \mu) + \bar{R}_{N-1,N}^{N-1,N}(\lambda + \mu) \bar{R}_{N-1,N}^{N-1,N}(\lambda - \mu)}.$$  (18)

Taking into account the explicit expressions for the Boltzmann weights we conclude that the right-hand sides of (17) and (18) are independent of variable \(\lambda\) only if the ratios \(\frac{K_{2}^{(-)}(\lambda)}{K_{1}^{(-)}(\lambda)}\) and \(\frac{K_{N}^{(-)}(\lambda)}{K_{N-1}^{(-)}(\lambda)}\) satisfy the following relations

$$\frac{K_{2}^{(-)}(\lambda)}{K_{1}^{(-)}(\lambda)} = \frac{\sinh[\beta_1 - \lambda]}{\sinh[\beta_1 + \lambda]}, \quad \frac{K_{N}^{(-)}(\lambda)}{K_{N-1}^{(-)}(\lambda)} = \frac{\sinh[\beta_{N-1} - \lambda]}{\sinh[\beta_{N-1} + \lambda]},$$  (19)

where \(\beta_1\) and \(\beta_{N-1}\) are free continuous parameters.

In the particular cases \(N = 2\) and \(N = 3\) the previous analysis already provides us a proposal for the \(K^{(-)}(\lambda)\) matrix. The strategy for \(N > 3\) is to substitute Eq. (19) into the reflection equation (14) and to search for further relations that are able to determine the remaining ratios among next-neighbor amplitudes. By performing this analysis up to \(N = 5\) we observed that some of the functional equations can be fulfilled once we extend the ansatz (19) to any ratio \(\frac{K_{a+1}^{(-)}(\lambda)}{K_{a}^{(-)}(\lambda)}\),

$$\frac{K_{a+1}^{(-)}(\lambda)}{K_{a}^{(-)}(\lambda)} = \frac{\sinh[\beta_a - \lambda]}{\sinh[\beta_a + \lambda]}, \quad a = 1, \ldots, N - 1.$$  (20)

Now by substituting this proposal back to the reflection equation and after systematic algebraic manipulations, we find that the parameters \(\beta_a\) have to satisfy the following recurrence relation

$$\beta_{a+1} - \beta_a = \frac{i\pi k}{N}, \quad a = 1, \ldots, N - 1.$$  (21)

Putting together all the above results we conclude that the most general solution for the amplitudes are,

$$K_{a}^{(-)}(\lambda) = \prod_{b=1}^{a-1} \sinh[\xi_- + \frac{\gamma}{2} - \frac{i\pi k}{N}(\frac{1}{2} - b) - \lambda] \prod_{b=a}^{N-1} \sinh[\xi_- + \frac{\gamma}{2} - \frac{i\pi k}{N}(\frac{1}{2} - b) + \lambda]$$  (22)

where for later convenience we choose \(\beta_1 = \xi_- + \frac{i\pi}{2} + \frac{ik}{2N}\) such that the variable \(\xi_-\) is a free continuous parameter. We also emphasize that we have checked that the K-matrix (22) satisfy the reflection equation (14) until \(N = 7\).
Considering the isomorphism (15) one can easily derive that the form of the respective elements of $K^{(\pm)}(\lambda)$ are given by

$$K^{(\pm)}(\lambda) = \prod_{b=a}^{a-1} \sinh[\xi_+ + i\gamma/2 - \frac{i\pi k}{N}(\frac{3}{2} - b) + \lambda] \prod_{b=a}^{N-1} \sinh[\xi_+ + i\gamma/2 + \frac{i\pi k}{N}(\frac{1}{2} + b) - \lambda]$$

(23)

where $\xi_+$ is yet another free continuous variable.

Having at hand the reflection matrices $K^{(\pm)}(\lambda)$ one can construct an integrable model with open boundaries following the double-row transfer matrix formulation proposed by Sklyanin [16],

$$t(\lambda) = Tr_A \left[ K_A^{(+)}(\lambda) T_A(\lambda) K_A^{(-)}(\lambda) T_A(-\lambda)^{-1} \right]$$

(24)

where $T_A(\lambda)$ is the monodromy matrix of the corresponding closed chain with $L$ sites,

$$T_A(\lambda) = \bar{R}_{AL}(\lambda) \bar{R}_{AL-1}(\lambda) \ldots \bar{R}_{A1}(\lambda).$$

(25)

To obtain the respective Hamiltonian with open boundaries one needs to expand the double-row transfer matrix $t(\lambda)$ in powers $\lambda$. The first derivative of $t(\lambda)$ is proportional to $Tr_A \left[ K_A^{(+)}(0) \right]$ which in our case is null for arbitrary values of $\gamma$ and $N$. In this situation we have to consider the expansion of $t(\lambda)$ up to the second order in the spectral parameter $\lambda$. Considering that $K_A^{(-)}(\lambda)$ has been normalized such that $K_A^{(-)}(0) = I_N$, the expression for the Hamiltonian is [19],

$$H = \sum_{j=1}^{L-1} H_{j,j+1} + \frac{\rho(0)}{2} \frac{d}{d\lambda} K_A^{(+)}(\lambda) \bigg|_{\lambda=0} + \frac{\xi}{\zeta} Tr_A \left[ \frac{d}{d\lambda} K_A^{(+)}(\lambda) \right]_{\lambda=0} H_{L,A}$$

$$+ \frac{1}{2} K_A^{(+)}(0) \frac{d^2}{d\lambda^2} R_{AL}(\lambda) \bigg|_{\lambda=0} P_{L,A} + \frac{1}{2\rho(0)} K_A^{(+)}(0) H_{L,A}^2,$$

(26)

where $H_{j,j+1}$ is the standard bulk Hamiltonian $H_{j,j+1} = P_{j,j+1} \frac{d}{d\lambda} \bar{R}_{jj+1}(\lambda) |_{\lambda=0}$ while $\zeta$ is a constant proportional to the following identity matrix

$$\zeta I_N = Tr_A \left[ \frac{d}{d\lambda} K_A^{(+)}(\lambda) \right]_{\lambda=0} + \frac{2}{\rho(0)} K_A^{(+)}(0) H_{L,A}$$

(27)

Let us now present the explicit expressions of the open Hamiltonians in the simplest cases $N = 2$ and $N = 3$. For $N = 2$ we have the XX chain in the presence of both bulk and boundary
magnetic fields,

\[ H = \sum_{i=1}^{L-1} \left( \frac{1}{2} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \frac{\cos[\gamma]}{2} \left( \sigma_i^z + \sigma_{i+1}^z \right) + \frac{\sin[\gamma]}{2} \cot \left[ -\xi - \frac{\gamma}{2} + \frac{\pi}{4} \right] \sigma_i^z \right) + \sin[\gamma] \cot \left[ \xi + \frac{\gamma}{2} + \frac{\pi}{4} \right] \sigma_L^z \]  

where \( \sigma_i^x, \sigma_i^y \) and \( \sigma_i^z \) are Pauli matrices.

On the other hand for \( N = 3 \) we find that the corresponding open Hamiltonian up to an additive constant is,

\[ H = \sqrt{3} \sum_{i=1}^{L-1} \sin[\gamma \varepsilon_k + \frac{\pi}{6}] \left( S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + \frac{3}{2} \left( S_i^z S_{i+1}^z + S_i^z S_{i+1}^z \right) \]  

such that \( S_i^x, S_i^y \) and \( S_i^z \) denotes the standard \( SU(2) \) spin-1 matrices.

In the next section we shall consider the diagonalization of the double-row transfer matrix associated to these models.

### 4 Bethe ansatz analysis

The purpose of this section is to present the eigenvalues \( \Lambda_n(\lambda) \) of the double-row transfer matrix,

\[ t(\lambda) |\phi_n\rangle = \Lambda_n(\lambda) |\phi_n\rangle . \]  

In the case of diagonal \( K \)-matrices the diagonalization problem \( \text{(30)} \) can be tackled within the algebraic Bethe ansatz framework. It turns out that the standard ferromagnetic highest vector
\( |\phi_0\rangle \) is an exact eigenvector of \( t(\lambda) \),

\[
|\phi_0\rangle = \prod_{i=1}^{L} \otimes |0\rangle_i,
\]

\[
|0\rangle_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_N
\]

playing the role of suitable reference state to start the Bethe ansatz analysis.

An algebraic procedure to generate other eigenvectors of \( t(\lambda) \) was first devised by Sklyanin for \( N = 2 \) [16]. The central object in this approach is the matrix elements of the double-row monodromy matrix,

\[
\bar{T}_A(\lambda) = T_A(\lambda) K_A^{(-)}(\lambda) T_A(\lambda)^{-1}
\]

which also satisfies the reflection equation (14),

\[
\bar{R}_{12}(\lambda - \mu) \bar{T}_A^1(\lambda) \bar{R}_{21}(\lambda + \mu) \bar{T}_A^2(\mu) = \bar{T}_A^2(\mu) \bar{R}_{12}(\lambda + \mu) \bar{T}_A^1(\lambda) \bar{R}_{21}(\lambda - \mu)
\]

The other eigenstates of \( t(\lambda) \) were then constructed by exploring a set of commutation relations derived from the quadratic algebra (33). This algebraic analysis has been extended to tackle three-state vertex models [20, 21] as well as the isotropic Heisenberg chain with arbitrary \( N \) up to the two-particle eigenstates [15]. In what follows we shall adapt the results of the latter work to include vertex models that are not invariant by charge symmetry. We shall not repeat the technical details discussed in [15] but only present the main relevant steps necessary to construct the eigenvalues \( \Lambda_n(\lambda) \). In order to describe that we shall first represent the double-monodromy matrix as

\[
\bar{T}_A(\lambda) = \begin{pmatrix} \bar{T}_{1,1}(\lambda) & \bar{T}_{1,2}(\lambda) & \ldots & \bar{T}_{1,N}(\lambda) \\ \bar{T}_{2,1}(\lambda) & \bar{T}_{2,2}(\lambda) & \ldots & \bar{T}_{2,N}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{T}_{N,1}(\lambda) & \bar{T}_{N,2}(\lambda) & \ldots & \bar{T}_{N,N}(\lambda) \end{pmatrix}
\]

(34)

Taking into account this representation, the diagonalization of the double-row transfer matrix \( t(\lambda) \) becomes equivalent to the problem

\[
\sum_{i=1}^{N} K_i^{(+)}(\lambda) \bar{T}_{i,i}(\lambda) |\phi_n\rangle = \Lambda_n(\lambda) |\phi_n\rangle
\]

(35)
The first step in the algebraic framework is to reformulate (35) in terms of a suitable combination of diagonal fields $\bar{T}_{i,i}(\lambda)$. It turns out that for arbitrary $N$ such linear combination is given by,

$$\bar{T}_{i,i}(\lambda) = \sum_{j=1}^{i} \left| \frac{M_{j,j}^{(+)}(2\lambda)}{M_{i,j}^{(+)}(2\lambda)} \right| \bar{T}_{j,j}(\lambda)$$

(36)

where the $j \times j$ matrix $M_{j,i}^{(+)}(\lambda)$ is build up from the $R$-matrix elements $R_{1,2}(\lambda)$ by the expression,

$$M_{j,i}^{(+)}(\lambda) = \begin{bmatrix}
    R_{1,1}^{1,1}(\lambda) & R_{1,1}^{1,2}(\lambda) & \cdots & R_{j-1,1}^{1,j-1}(\lambda) & R_{i,1}^{1,i}(\lambda) \\
    R_{1,2}^{2,1}(\lambda) & R_{2,2}^{2,2}(\lambda) & \cdots & R_{j-1,2}^{2,j-1}(\lambda) & R_{i,2}^{2,i}(\lambda) \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    R_{1,j}^{j,1}(\lambda) & R_{2,j}^{j,2}(\lambda) & \cdots & R_{j-1,j}^{j,j-1}(\lambda) & R_{i,j}^{j,i}(\lambda)
\end{bmatrix}_{j \times j}$$

(37)

Now by using Eqs (35), (36) the eigenvalue problem can be rewritten as,

$$\sum_{i=1}^{N} w_{i}^{(+)}(\lambda) \bar{T}_{i,i}(\lambda) |\phi_{n}\rangle = \Lambda_{n}(\lambda) |\phi_{n}\rangle$$

(38)

where functions $w_{i}^{(+)}(\lambda)$ are defined by

$$w_{i}^{(+)}(\lambda) = \sum_{j=i}^{N} \left| \frac{M_{j,i}^{(+)}(2\lambda)}{M_{i,j}^{(+)}(2\lambda)} \right| K_{j}^{(+)}(\lambda).$$

(39)

The basic property of the new diagonal fields $\bar{T}_{i,i}(\lambda)$ is that their action on the reference $|\phi_{0}\rangle$ are always proportional to functions having a single power in $L$. More precisely we find that,

$$\bar{T}_{i,i}(\lambda) |\phi_{0}\rangle = w_{i}^{(-)}(\lambda) \frac{[\bar{R}_{i,i}^{1,1}(\lambda)]^{2L}}{[\rho(\lambda)/\rho(-\lambda)]^{L}} |\phi_{0}\rangle$$

(40)

The functions $w_{i}^{(-)}(\lambda)$ depend now on the amplitudes $K_{i}^{(-)}(\lambda)$ by the expression,

$$w_{i}^{(-)}(\lambda) = K_{i}^{(-)}(\lambda) - \sum_{j=1}^{i-1} \left| \frac{M_{j-1,i}^{(-)}(2\lambda)}{M_{i-1,j}^{(-)}(2\lambda)} \right| K_{j}^{(-)}(\lambda)$$

(41)

where the second auxiliary $j \times j$ matrix $M_{j,i}^{(-)}(\lambda)$ is also build up from the $R$-matrix elements.
To this point we have gathered the basic informations to provide a closed expression for the eigenvalue \( \Lambda_0(\lambda) \) associated to the reference state \( |\phi_0\rangle \). In fact, by substituting Eq.(40) in Eq.(38) one finds,

\[
\Lambda_0(\lambda) = \sum_{i=1}^{N} \frac{w_i^+(\lambda)w_i^-(\lambda) [R_{i,1}^{i,1}(\lambda)]^{2L}}{[\rho(\lambda)\rho(-\lambda)]^L}.
\] (43)

To obtain the other eigenvalues of \( t(\lambda) \) we have to look for eigenstates generated by the action of the creation field \( \tilde{T}_{i,j}(\lambda) \) for \( i < j \) on the reference state \( |\phi_0\rangle \). This analysis has been already carried out in [15] up to the two-particle states. Recall that in integrable theories the results for the two-particle state are enough to propose an educated expression for the eigenvalues \( \Lambda_n(\lambda) \). By adapting the conclusion of [15] to our vertex model we find that the expression for the multi-particle eigenvalues are

\[
\Lambda_n(\lambda) = \sum_{i=1}^{N} \frac{[\tilde{R}_{i,1}^{i,1}(\lambda)]^{2L}}{[\rho(\lambda)\rho(-\lambda)]^L} w_i^+(\lambda)w_i^-(\lambda) \prod_{j=1}^{n} Q_i(\lambda, \lambda_j)
\] (44)

where \( \lambda_1, \ldots, \lambda_n \) are the variables parameterizing the \( n \)-particle state.

The functions \( Q_i(\lambda, \lambda_j) \) are determined in terms of the Boltzmann weights by the following
factorize in terms of three types of products of trigonometric functions, namely

\[
Q_i(\lambda, \mu) = \begin{cases}
\frac{R_{1,i}^1(\mu - \lambda) R_{1,i}^2(\lambda + \mu)}{R_{1,i}^2(\mu - \lambda) R_{1,i}^1(\lambda + \mu)}, & \text{for } i = 1 \\
\frac{R_{1,i+1}^1(\lambda - \mu) R_{2,i}^2(\lambda - \mu)}{R_{1,i+1}^2(\lambda - \mu) R_{2,i}^1(\lambda - \mu)} & \frac{R_{i-1,1}^1(\lambda + \mu) R_{i-1,1}^2(\lambda + \mu)}{R_{i-1,1}^2(\lambda + \mu) R_{i-1,1}^1(\lambda + \mu)}, & \text{for } 2 \leq i \leq N - 1 \\
\frac{R_{2,N}^{1,1}(\lambda - \mu)}{R_{N-1,1}^{1,1}(\lambda + \mu) R_{1,N}^{1,1}(\lambda + \mu)}, & \text{for } i = N
\end{cases}
\]

while the rapidities \( \lambda_j \) satisfy the following Bethe ansatz equations,

\[
\left[ \frac{R_{1,1}^1(\lambda_j)}{R_{2,1}^2(\lambda_j)} \right]^{2L} \frac{w_1^{(+)}(\lambda_j) w_1^{(-)}(\lambda_j)}{w_2^{(+)}(\lambda_j) w_2^{(-)}(\lambda_j)} = \frac{R_{1,1}^1(2\lambda_j) R_{2,2}^2(2\lambda_j) - R_{2,1}^1(2\lambda_j) R_{1,2}^2(2\lambda_j)}{[R_{2,1}^2(2\lambda_j)]^2} \prod_{i=1, i \neq j}^n Q_1(\lambda_j, \lambda_i) \prod_{i=1}^n Q_2(\lambda_j, \lambda_i) (46)
\]

for \( j = 1, \ldots, n. \)

We now have at hand the basic ingredients to exhibit explicit expressions for the eigenvalues \( \Lambda_n(\lambda) \). The first step is to simplify the expressions for \( w_1^{(\pm)}(\lambda), \ldots, w_N^{(\pm)}(\lambda) \) by using the \( K \)-matrices amplitudes as well as the \( R \)-matrix elements provided in section 2 and Appendix B. These simplifications require a considerable amount of algebraic work since we have to sum a number of distinct terms for each possible branch \( k \). Fortunately, an analysis up to \( N = 5 \) is enough to exhibit the uniform dependence of these functions on \( N \). We find that the final results factorize in terms of three types of products of trigonometric functions, namely

\[
w_n^{(-)}(\lambda) = \prod_{j=1}^{2N-1} \frac{\sinh[2\lambda + \frac{\nu k}{N}(j - 1)]}{\sinh[2\lambda + \nu(\lambda + \frac{\nu}{N}(a + j - 3))]} \prod_{j=1}^{2N-1} \sinh[\xi_\lambda - \lambda - \frac{\nu k}{N}(j - 3)] \\
\times \prod_{j=1}^{N-1} \sinh[\xi_\lambda + \lambda + \frac{\nu}{2} - \frac{\nu k}{N}(\frac{1}{2} - j)] (47)
\]
and

\[
\begin{align*}
\omega_a^{(+)}(\lambda) &= \prod_{j=a}^{N-1} \frac{\sinh[2\lambda + \frac{i\pi k}{N}(j-1)]}{\sinh[2\lambda + r\gamma + \frac{i\pi k}{N}(a+j-2)]} \prod_{j=1}^{a-1} \sinh[\xi_+ + \lambda + \frac{r\gamma}{2} - \frac{i\pi k}{N}(\frac{3}{2} - j)] \\
&\times \prod_{j=a}^{N-1} \sinh[\xi_+ - \lambda - \frac{r\gamma}{2} + \frac{i\pi k}{N}(N + \frac{1}{2} - j)]
\end{align*}
\] (48)

The next step is to carry out similar algebraic simplifications for functions \(Q_i(\lambda, \lambda_j)\). We notice that in order to make these polynomials as symmetrical as possible it is convenient to perform the shift \(\lambda_i \rightarrow \bar{\lambda}_i - i\frac{\gamma}{2}\). Considering this change of variables and after some manipulations we find that the double-row transfer matrix eigenvalue are,

\[
\Lambda_n(\lambda) = \sum_{a=1}^{N} \left[ \frac{\rho(-\lambda)}{\rho(\lambda)} \right]^L \prod_{j=1}^{a-1} \frac{\sinh[\lambda + \frac{i\pi k}{N}(j-1)]}{\sinh[\lambda + r\gamma + \frac{i\pi k}{N}(j-1)]]} \omega_a^{(+)}(\lambda) w_a^{(-)}(\lambda) \\
\times \prod_{i=1}^{n} \frac{\sinh[\lambda - \bar{\lambda}_i - i\frac{\gamma}{2}] \sinh[\lambda - \bar{\lambda}_i + i\frac{\gamma}{2} - \frac{i\pi k}{N}]}{\sinh[\lambda - \bar{\lambda}_i + i\frac{\gamma}{2} - (2 - a)\frac{i\pi k}{N}] \sinh[\lambda - \bar{\lambda}_i - \frac{i\pi k}{N}]} \\
\times \prod_{i=1}^{n} \frac{\sinh[\lambda + \bar{\lambda}_i - i\frac{\gamma}{2}] \sinh[\lambda + \bar{\lambda}_i + i\frac{\gamma}{2} - \frac{i\pi k}{N}]}{\sinh[\lambda + \bar{\lambda}_i + i\frac{\gamma}{2} - (2 - a)\frac{i\pi k}{N}] \sinh[\lambda + \bar{\lambda}_i - \frac{i\pi k}{N}]} 
\] (49)

The corresponding Bethe ansatz equations for the shifted variables \(\bar{\lambda}_i\) become,

\[
\left( \frac{\sinh[\bar{\lambda}_i + i\frac{\gamma}{2}]}{\sinh[\bar{\lambda}_i - i\frac{\gamma}{2}]} \right)^{2L} \frac{\sinh[\xi_+ + \bar{\lambda}_i + \frac{i\pi k}{N}]}{\sinh[\xi_+ - \bar{\lambda}_i + \frac{i\pi k}{N}]} \frac{\sinh[\xi_- - \bar{\lambda}_i - \frac{i\pi k}{N}]}{\sinh[\xi_- + \bar{\lambda}_i - \frac{i\pi k}{N}]} \\
= \prod_{j=1}^{n} \frac{\sinh[\bar{\lambda}_i - \bar{\lambda}_j - \frac{i\pi k}{N}]}{\sinh[\bar{\lambda}_i - \bar{\lambda}_j + \frac{i\pi k}{N}]} \frac{\sinh[\bar{\lambda}_i + \bar{\lambda}_j - \frac{i\pi k}{N}]}{\sinh[\bar{\lambda}_i + \bar{\lambda}_j + \frac{i\pi k}{N}]} 
\] (50)

We conclude with the following remark. Note that the roots of unity branches on the left-hand side of Eq. (50) can be absorbed by performing the change of variables \(\xi_- \rightarrow \xi_- - \frac{i\pi k}{2N}\) and \(\xi_+ \rightarrow \xi_+ + \frac{i\pi k}{2N}\). After this transformation, the dependence of the Bethe ansatz equations on the roots of unity remains restricted to the two-body scattering amplitudes.
5 Conclusion

The purpose of this paper was to solve the integrable vertex models based on the $U_q[SU(2)]$ algebra at roots of unity with open boundary conditions. We have solved the reflection equation and found one family of diagonal $K$-matrices having a free-parameter. For such diagonal boundary conditions we have been able to present the eigenvalues of the double-row transfer matrix and the corresponding Bethe ansatz equations. The next natural step would be to consider these vertex models with non-diagonal boundaries. In particular, to investigate if the functional relation approach developed for the open high spin $XXZ$ quantum chain [22, 24, 23, 25] can be applied to such roots of unity vertex models with non-diagonal $K$-matrices.

Appendix A: The projectors $P_j(\gamma, k)$

For completeness we shall here present the explicit expressions for the projectors $P_j(\gamma, k)$ given first by us in [13]. In the Weyl basis these projectors can be written as decomposition of the braid $S(\gamma, k)$,

$$P_j(\gamma, k) = \prod_{l=1}^{N} \frac{S(\gamma, k) - \xi_l}{\xi_j - \xi_l}$$  \hspace{1cm} (A.1)

where

$$\xi_j = (-1)^{j+1} e^{\frac{\pi k}{N} (j-2)(j-1) + 2\gamma(j-1)}.$$  \hspace{1cm} (A.2)

The corresponding matrix representation of the braid $S(\gamma, k)$ is

$$S(\gamma, k) = \sum_{a,b,c,d=1}^{N} S_{a,b,c,d}^{a,b}(\gamma, k) e_{b,d} \otimes e_{a,c}.$$  \hspace{1cm} (A.3)

where the amplitudes $S_{a,b}^{c,d}$ are given by,

$$S_{a,b,c,d}^{a,b}(\gamma, k) = \frac{\exp\left[\frac{2\pi i k}{N} (b-1)(d-1) + i\gamma(b + d - 2)\right]}{H\left(\frac{2\pi k}{N}, a - d\right)} \frac{H\left(\frac{2\pi k}{N}, a - 1\right)H\left(\frac{2\pi k}{N}, c - 1\right)}{\sqrt{H\left(\frac{2\pi k}{N}, d - 1\right)H\left(\frac{2\pi k}{N}, b - 1\right)}} \times \frac{\sqrt{H(2\gamma, a - 1)H(2\gamma, c - 1)}}{\sqrt{H(2\gamma, b - 1)H(2\gamma, d - 1)}} \delta_{a+b,c+d}.$$  \hspace{1cm} (A.4)
such that the auxiliary function \( H(\lambda, n) = \prod_{l=0}^{n-1} (1 - e^{i(\lambda + \frac{2\pi k}{N})}) \). 

Altogether the above expressions provide us the explicit matrix expressions for the projectors for arbitrary and \( k \) coprime with \( N \).

Appendix B: Boltzmann weights for \( N = 4 \) and \( N = 5 \)

In what follows we present the weights \( \tilde{R}_{a,b}^{c,d}(\lambda) \) of the \( R \)-matrix \( \tilde{R}_{12}(\lambda) \) \((B)\) for \( N = 4 \) and \( N = 5 \). We recall here that the only non-null amplitudes are those that satisfy the ice rule \( a + b = c + d \).

The non-trivial forty-four Boltzmann weights for \( N = 4 \) are,

\[
R_{1,1}^{1,1}(\lambda) = \sinh[\gamma + \lambda] \sinh[\gamma + \frac{i\pi k}{4} + \lambda] \sinh[\gamma + \frac{i\pi k}{2} + \lambda] \quad \text{(B.1)}
\]

\[
R_{1,2}^{1,2}(\lambda) = R_{2,1}^{2,1}(\lambda) = \sinh[\lambda] \sinh[\gamma + \frac{i\pi k}{4} + \lambda] \sinh[\gamma + \frac{i\pi k}{2} + \lambda] \quad \text{(B.2)}
\]

\[
R_{1,1}^{2,1}(\lambda) = R_{2,1}^{1,2}(\lambda) = \sinh[\gamma] \sinh[\gamma + \frac{i\pi k}{4} + \lambda] \sinh[\gamma + \frac{i\pi k}{2} + \lambda] \quad \text{(B.3)}
\]

\[
R_{1,2}^{1,3}(\lambda) = R_{3,1}^{3,1}(\lambda) = \sinh[\lambda] \sinh[\lambda + \frac{i\pi k}{4}] \sinh[\gamma + \frac{i\pi k}{2} + \lambda] \quad \text{(B.4)}
\]

\[
R_{1,2}^{2,2}(\lambda) = R_{2,2}^{1,3}(\lambda) = R_{2,1}^{3,1}(\lambda) = R_{3,1}^{2,2}(\lambda) = -\frac{2}{\varepsilon_k} \frac{1}{i} \sqrt{\sinh[\gamma] \sinh[\gamma + \frac{i\pi k}{4}]} \times \sinh[\gamma] \sinh[\gamma + \frac{i\pi k}{2} + \lambda] \quad \text{(B.5)}
\]

\[
R_{1,3}^{3,1}(\lambda) = R_{3,1}^{3,3}(\lambda) = \sinh[\gamma] \sinh[\gamma + \frac{i\pi k}{4}] \sinh[\gamma + \frac{i\pi k}{2} + \lambda] \quad \text{(B.6)}
\]

\[
R_{1,1}^{4,1}(\lambda) = R_{4,1}^{1,1}(\lambda) = \sinh[\lambda + \frac{i\pi k}{4}] \sinh[\lambda + \frac{i\pi k}{2}] \sinh[\lambda] \quad \text{(B.7)}
\]

\[
R_{1,4}^{2,3}(\lambda) = R_{2,3}^{1,4}(\lambda) = R_{3,2}^{1,4}(\lambda) = R_{4,1}^{3,2}(\lambda) = -\varepsilon_k \frac{1}{i} \sqrt{\sinh[\gamma] \sinh[\gamma + \frac{i\pi k}{2}] \sinh[\lambda] \sinh[\lambda + \frac{i\pi k}{4}]} \quad \text{(B.8)}
\]

\[
R_{1,4}^{3,2}(\lambda) = R_{3,2}^{1,4}(\lambda) = R_{2,3}^{4,1}(\lambda) = R_{4,1}^{2,3}(\lambda) = -\varepsilon_k \frac{1}{i} \sqrt{\sinh[\gamma] \sinh[\gamma + \frac{i\pi k}{2}] \sinh[\lambda] \sinh[\gamma + \frac{i\pi k}{4}]} \quad \text{(B.9)}
\]
\[R_{1,4}^4(\lambda) = R_{4,1}^1(\lambda) = \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{\nu \pi k}{4}] \sinh[\nu \gamma + \frac{\nu \pi k}{2}]\]  
(B.10)

\[R_{2,3}^2(\lambda) = R_{3,2}^3(\lambda) = \left[ \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{\nu \pi k}{4}] + \sinh[\lambda - \frac{\nu \pi k}{4}] \sinh[\lambda] \right] \sinh[\nu \gamma + \frac{\nu \pi k}{2} + \lambda] \]  
(B.11)

\[R_{2,3}^3(\lambda) = R_{3,2}^3(\lambda) = R_{4,5}^3(\lambda) = R_{3,3}^3(\lambda) = (2\varepsilon_k)^\frac{1}{2} \sinh[\lambda] \sqrt{\sinh[\nu \gamma + \frac{\nu \pi k}{4}] \sinh[\nu \gamma + \frac{\nu \pi k}{2}]} \times \sinh[\nu \gamma - \lambda] \sinh[\lambda] \]  
(B.12)

\[R_{2,4}^4(\lambda) = R_{4,2}^4(\lambda) = \sinh[\nu \gamma + \frac{\nu \pi k}{4}] \sinh[\nu \gamma + \frac{\nu \pi k}{2}] \sinh[\nu \gamma - \lambda] \]  
(B.13)

\[R_{3,3}^3(\lambda) = \sinh[\nu \gamma - \lambda] \left[ \sinh[\nu \gamma + \frac{\nu \pi k}{4}] \sinh[\nu \gamma + \frac{\nu \pi k}{2}] + \sinh[\lambda - \frac{\nu \pi k}{4}] \sinh[\lambda] \right] \]  
(B.14)

\[R_{3,4}^3(\lambda) = R_{4,5}^3(\lambda) = \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \frac{\nu \pi k}{4} - \lambda] \sinh[\lambda] \]  
(B.15)

\[R_{3,4}^3(\lambda) = R_{4,4}^3(\lambda) = \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \frac{\nu \pi k}{4} - \lambda] \sinh[\nu \gamma + \frac{\nu \pi k}{2}] \]  
(B.16)

where here we have \(k = 1\) or \(k = 3\).

For \(N = 5\) the eighty-five Boltzmann weights are,

\[R_{1,1}^1(\lambda) = \sinh[\nu \gamma + \lambda] \sinh[\nu \gamma + \frac{\nu \pi k}{5} + \lambda] \sinh[\nu \gamma + \frac{2\nu \pi k}{5} + \lambda] \sinh[\nu \gamma + \frac{3\nu \pi k}{5} + \lambda] \]  
(B.21)

\[R_{1,2}^2(\lambda) = R_{1,2}^2(\lambda) = \sinh[\nu \gamma + \frac{\nu \pi k}{5} + \lambda] \sinh[\nu \gamma + \frac{2\nu \pi k}{5} + \lambda] \sinh[\nu \gamma + \frac{3\nu \pi k}{5} + \lambda] \]  
(B.22)

\[R_{2,1}^2(\lambda) = R_{2,1}^2(\lambda) = \sinh[\nu \gamma + \frac{\nu \pi k}{5} + \lambda] \sinh[\nu \gamma + \frac{2\nu \pi k}{5} + \lambda] \sinh[\nu \gamma + \frac{3\nu \pi k}{5} + \lambda] \]  
(B.23)
\[ R_{1,3}^{1,3}(\lambda) = R_{3,1}^{3,1}(\lambda) = \sinh[\lambda] \sinh[\lambda + \frac{2\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \] (B.24)

\[ R_{1,3}^{2,2}(\lambda) = R_{2,2}^{3,1}(\lambda) = R_{3,1}^{2,2}(\lambda) = -\sqrt{2} \cosh[\frac{2\pi k}{5}] \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \]
\[ \times \sinh[\lambda] \sinh[\nu \gamma + \frac{2\pi k}{5} + \lambda] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \] (B.25)

\[ R_{1,3}^{3,1}(\lambda) = R_{3,1}^{3,1}(\lambda) = \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \] (B.26)

\[ R_{1,4}^{4,1}(\lambda) = R_{4,1}^{4,1}(\lambda) =\sinh[\lambda] \sinh[\lambda + \frac{2\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \] (B.27)

\[ R_{1,4}^{2,3}(\lambda) = R_{2,3}^{3,1}(\lambda) = R_{3,1}^{2,3}(\lambda) = -\sinh[\lambda] \sinh[\lambda + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \]
\[ \times \sqrt{2\varepsilon_k \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{2\pi k}{5}] \cosh[\frac{\nu \gamma}{5}] \cosh[\frac{3\pi k}{5} + \lambda] \] (B.28)

\[ R_{1,4}^{3,2}(\lambda) = R_{2,3}^{1,3}(\lambda) = R_{3,1}^{2,3}(\lambda) = -\sinh[\lambda] \sinh[\lambda + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \]
\[ \times \sqrt{2\varepsilon_k \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{2\pi k}{5}] \cosh[\frac{\nu \gamma}{5}] \cosh[\frac{3\pi k}{5} + \lambda] \] (B.29)

\[ R_{1,4}^{4,1}(\lambda) = R_{4,1}^{4,1}(\lambda) = \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \] (B.30)

\[ R_{1,5}^{5,1}(\lambda) = R_{5,1}^{5,1}(\lambda) = \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \] (B.31)

\[ R_{1,5}^{2,4}(\lambda) = R_{2,4}^{5,1}(\lambda) = R_{4,2}^{5,1}(\lambda) = -\sqrt{\varepsilon_k \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \cosh[\frac{3\pi k}{5}] \cosh[\frac{\nu \gamma}{5}] \cosh[\frac{2\pi k}{5}] \] (B.32)

\[ R_{1,5}^{3,3}(\lambda) = R_{3,3}^{1,5}(\lambda) = R_{3,1}^{3,3}(\lambda) = R_{5,1}^{5,1}(\lambda) = -2 \sinh[\lambda] \sinh[\lambda + \frac{3\pi k}{5}] \]
\[ \times \sqrt{\varepsilon_k \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \cosh[\frac{3\pi k}{5}] \cosh[\frac{\nu \gamma}{5}] \cosh[\frac{2\pi k}{5}] \] (B.34)

\[ R_{1,5}^{4,2}(\lambda) = R_{4,2}^{1,5}(\lambda) = R_{5,1}^{2,4}(\lambda) = R_{2,4}^{5,1}(\lambda) = -\sqrt{\varepsilon_k \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \cosh[\frac{3\pi k}{5}] \cosh[\frac{\nu \gamma}{5}] \cosh[\frac{2\pi k}{5}] \] (B.35)

\[ R_{1,5}^{5,1}(\lambda) = R_{5,1}^{1,5}(\lambda) = \sinh[\nu \gamma] \sinh[\nu \gamma + \frac{3\pi k}{5}] \sinh[\nu \gamma + \frac{3\pi k}{5} + \lambda] \] (B.36)
$$R_{2,2}^{2,2}(\lambda) = \left[ \sinh[\nu \gamma] \sinh[\nu \gamma \frac{i \pi k}{5}] + \sinh[\lambda \frac{i \pi k}{5}] \sinh[\lambda] \right] \sinh[\nu \gamma \frac{2 i \pi k}{5} + \lambda] \sinh[\nu \gamma \frac{3 i \pi k}{5} + \lambda]$$

(B.37)

$$R_{2,3}^{3,2}(\lambda) = R_{3,2}^{3,2}(\lambda) = \left[ 2 \sinh[\nu \gamma] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \cosh[\frac{i \pi k}{5}] + \sinh[-\frac{i \pi k}{5}] \sinh[\lambda] \right] \sinh[\nu \gamma + \frac{3 i \pi k}{5} + \lambda]$$

(B.38)

$$R_{2,3}^{2,2}(\lambda) = R_{2,3}^{2,2}(\lambda) = \left[ \sinh[\nu \gamma] \sinh[\nu \gamma \frac{2 i \pi k}{5}] + 2 \sinh[\lambda \frac{i \pi k}{5}] \cosh[\frac{i \pi k}{5}] \right] \sinh[\nu \gamma + \frac{3 i \pi k}{5} + \lambda]$$

(B.39)

$$R_{2,4}^{4,2}(\lambda) = R_{4,2}^{4,2}(\lambda) = \sinh[\lambda \frac{i \pi k}{5}] \sinh[\lambda \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] + 2 \epsilon_k \sinh[\nu \gamma]$$

(B.40)

$$R_{2,4}^{3,3}(\lambda) = R_{3,3}^{3,3}(\lambda) = R_{3,3}^{3,3}(\lambda) = 2 \sqrt{\epsilon_k \sinh[\nu \gamma \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \cosh[\frac{i \pi k}{5}]$$

$$\times \cos[\frac{i \pi k}{5}] \sinh[\lambda \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \cosh[\frac{i \pi k}{5}]$$

(B.41)

$$R_{2,4}^{2,4}(\lambda) = R_{4,2}^{2,4}(\lambda) = \sinh[\nu \gamma \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \cosh[\frac{i \pi k}{5}]$$

$$\times \cos[\frac{i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

(B.42)

$$R_{3,3}^{3,3}(\lambda) = \sinh[\nu \gamma \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

$$+ (\sinh[\lambda \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

$$\sinh[\nu \gamma \frac{5 i \pi k}{5}])$$

$$\times (\sinh[\nu \gamma \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

(B.43)

$$R_{2,5}^{5,2}(\lambda) = R_{5,2}^{5,2}(\lambda) = \epsilon_k \sinh[\nu \gamma \frac{i \pi k}{5}] \sinh[\lambda \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

(B.44)

$$R_{2,5}^{3,4}(\lambda) = R_{3,4}^{3,4}(\lambda) = R_{3,4}^{3,4}(\lambda) = 2 \epsilon_k \sinh[\nu \gamma \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

$$\times \cosh[\frac{i \pi k}{5}] \sqrt{2 \cosh[\frac{5 i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

(B.45)

$$R_{2,5}^{2,3}(\lambda) = R_{4,3}^{2,3}(\lambda) = R_{3,4}^{5,4}(\lambda) = 2 \sinh[\nu \gamma \frac{i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}]$$

$$\times \cosh[\frac{i \pi k}{5}] \sqrt{2 \cosh[\frac{5 i \pi k}{5}] \sinh[\nu \gamma \frac{2 i \pi k}{5}] \sinh[\nu \gamma \frac{3 i \pi k}{5}$$

(B.46)
\[ R_{2,5}^5(\lambda) = R_{5,2}^5(\lambda) = \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{\nu \pi k}{5}] \] (B.47)

\[ R_{3,4}^4(\lambda) = R_{4,3}^5(\lambda) = \varepsilon_k \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{\nu \pi k}{5}] \]
\times 2 \cosh[\nu \pi k] + \sinh[\nu \gamma - \nu \pi k] \sinh[\nu \gamma + \nu \pi k] \] (B.48)

\[ R_{3,4}^3(\lambda) = \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \nu \pi k] \]
\times 2 \cosh[\nu \pi k] \sinh[\nu \gamma - \nu \pi k] \sinh[\nu \gamma + \nu \pi k] \] (B.49)

\[ R_{3,5}^3(\lambda) = \varepsilon_k \sinh[\nu \gamma + \nu \pi k] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \nu \pi k] \]
\times \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \nu \pi k] \] (B.50)

\[ R_{4,4}^4(\lambda) = R_{5,3}^5(\lambda) = R_{4,4}^5(\lambda) = \sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \nu \pi k] \]
\times 2 \varepsilon_k \cosh[\nu \pi k] \sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{\nu \pi k}{5}] \] (B.51)

\[ R_{3,5}^5(\lambda) = \sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \nu \pi k] \]
\times \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \nu \pi k] \] (B.52)

\[ R_{4,4}^4(\lambda) = (\sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] + 4 \varepsilon_k \cosh[\nu \pi k]) \cosh[\nu \pi k] \]
\times \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \nu \pi k] \] (B.53)

\[ R_{4,4}^5(\lambda) = \varepsilon_k \sinh[\nu \gamma + \nu \pi k] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{\nu \pi k}{5}] \] (B.54)

\[ R_{5,5}^4(\lambda) = R_{5,4}^5(\lambda) = \sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \sinh[\nu \gamma - \lambda] \sinh[\nu \gamma + \nu \pi k] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{\nu \pi k}{5}] \] (B.55)

\[ R_{5,5}^5(\lambda) = \sinh[\nu \gamma + \nu \pi k] \sinh[\nu \gamma + \frac{2 \nu \pi k}{5}] \sinh[\nu \gamma + \frac{3 \nu \pi k}{5}] \] (B.56)

where the possible values of \( k = 1, 2, 3, 4 \).

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