A VERTEX ALGEBRAIC APPROACH TO THE WITTEN GENUS

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ABSTRACT. This work is divide in two cases. In the first case, we consider a spin manifold $M$ as the set of fixed points of an $S^1$-action on a spin manifold $X$, and in the second case we consider the spin manifold $M$ as the set of fixed points of an $S^1$-action on the loop space of $M$. For each case, we build on $M$ a vector bundle, a connection and a set of bundle endomorphisms resembling the theory of vertex algebras. These objects are used to build global operators on $M$ which define an analytical index in each case. In the first case, the analytical index is equal to the topological equivariant Atiyah Singer index, and in the second case the analytical index is equal to a topological expression where the Witten genus appears.

1. Introduction

Let $X$ be an oriented, compact, even dimensional, spin Riemannian manifold. On $X$ there is a bundle of complex spinors $\triangle(X)$, and a Dirac operator $\mathcal{D}_X$ acting on sections of this bundle. The orientation on $X$ defines a Riemannian volume form, and its image in the Clifford algebra splits the bundle of complex spinors $\triangle(X) = \triangle^+(X) \oplus \triangle^-(X)$, we call $\triangle^+(X)$ the positive spinor bundle and $\triangle^-(X)$ the negative spinor bundle. The index of the Dirac operator is calculated using the Atiyah Singer Index theorem

$$\text{Ind}(\mathcal{D}_X) = \dim \ker(\mathcal{D}_X|_{\triangle^+(X)}) - \dim \ker(\mathcal{D}_X|_{\triangle^-(X)}) = \int_M A(X),$$

where $A(M)$ is the topological $\hat{A}$ class, the notation $|E|$ means that the operator is restricted to the sections $\Gamma(E)$ for a vector bundle $E$. For an introduction to the Atiyah Singer index theorems see [20], [3]. Additionally, If $R \to M$ is a complex bundle equipped with a connection, then we have a twisted Dirac operator $\mathcal{D} : \Gamma(\triangle^+(X) \otimes R) \to \Gamma(\triangle^-(X) \otimes R)$ and the Atiyah Singer index theorem states

$$\text{Ind}(\mathcal{D} \otimes R) = \dim \ker(\mathcal{D}_X|_{\triangle^+(X) \otimes R}) - \dim \ker(\mathcal{D}_X|_{\triangle^-(X) \otimes R}) = \int_M A(M) \text{ch}(R).$$

Where $\text{ch}$ is the Chern character of the bundle $R$.

Now, let us assume that there is a $S^1$-action orientation preserving isometry acting on $X$ such that the action lifts to the spin bundle. Then this action preserves the splitting $\triangle(X) = \triangle^+(X) \oplus \triangle^-(X)$ and commutes with the Dirac operator. The equivariant index is defined by

$$\text{Ind}(\mathcal{D}_X, q) = Tr(q|\ker(\mathcal{D}_X|_{\triangle^+(X)})) - Tr(q|\ker(\mathcal{D}_X|_{\triangle^-(X)})),$$

where $q = e^{i\theta} \in \mathbb{C}^\times$ and the notation $q|E$ denotes the $S^1$ representation on the sections $\Gamma(E)$ of a vector bundle $E$. There is also a topological formula for the analytical equivariant index above,
in order to express this formula we consider the following definitions and properties. See [4] for details.

The set of fixed points of the $S^1$-action is a manifold that we denoted by $M$. The normal bundle $N'$ of $M$ in $X$ can be written as a direct sum $N' = \bigoplus_{r \in A} N_r'$ where each $N_r'$ has a complex structure and $A$ is given by finite positive integers, see theorem 2.2. We denote by $N'_r$ the underlying real bundle and by $N_r$ the complex bundle. On $N_r$ the circle action acts as $q^r$. Note that the fixed point manifold $M$ has a canonical orientation from the complex bundle $N'$ and the orientation on $X$. We define $S_t(E) := 1 + tE + t^2\text{Sym}^2(E) + \ldots$ as the formal power series with values in vector bundles on $M$. Then the equivariant index for circle actions is given by

$$\text{Ind}(\mathcal{D}, q) = q^{\frac{1}{2}c_1} \int_M A(M)ch(\sqrt{\det N} \otimes_{r \in A} S_{q^r}(N_r)),$$

where $c_1 := \sum_{r \in A} rd_r$ for $d_r := \dim \mathbb{C}N_r$. Note that $q$ is not branched because $\sum_{r \in A} rd_r = 0 \mod 2$, this follows from the lifting of the action to the spin bundle on $X$. The topological expression above makes sense even when the bundle $\det N$ does not have a square root. In this work we are interested in the case that $M$ is spin, then because $X$ is also a spin manifold we have that $0 = w_2(N') = c_1(N) \mod 2$ and this implies that the square root of $\det N$ exists. Finally, note that the Atiyah Hirzebruch theorem express that the index above is indeed zero. Even with this result the expression above can be easily generalized for more interesting cases. Our interest in (1.3) is that this expression can be seen as a finite dimensional analogous of the Witten genus, we are going to use this approach in this work.

For an introductory review to the elliptic genus and the Witten genus see [14]. Let $M$ be a compact, oriented manifold such that $\dim M = 4k$ the Witten genus is defined as follows

$$\phi_W(M) := \int_M 2^{2k} \prod_{i=1}^{2k} \exp(\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau)x_i^{2k}) \in \mathbb{Q}[[q]],$$

where $x_1, \ldots, x_{2k}$ are the Chern roots of the bundle $TM \otimes \mathbb{C}$, $q = e^{2\pi i \tau}$ and $G_{2k}(\tau)$ are Eisenstein series. The Witten genus $\phi_W(M)$ is a modular form of weight $2k$.

The Witten genus was defined in [28], in this work the expression below appeared considering the index of a Dirac operator on the loop space $LM$ in the same way as (1.3) is the equivariant index of a Dirac operator on $X$

$$\Phi(M) := \int_M A(M)ch(\bigotimes_{n=1}^{\infty} S_{q^n}(TM_C)) \prod_{n=1}^{\infty} (1 - q^n)^{\dim M} \in \mathbb{Q}[[q]],$$

where we used the notation $TM_C := TM \otimes \mathbb{C}$. It was proved in [30] that if the first Pontrjagin class is zero, $p_1(M) = 0$ in $H^4(M, \mathbb{R})$, then we have the identity $\Phi(M) = \phi_W(M)$. And if additionally $M$ is a spin manifold then $\Phi(M) \in \mathbb{Z}[[q]]$. The condition on the first Pontrjagin class was later refined in [23], where from a topological point of view the condition $\frac{1}{2}p_1(M, \mathbb{Z}) = 0$ in $H^4(M, \mathbb{Z})$ appeared considering extensions of spin loop groups.

In [28], the author was motivated by some physics construction in quantum field theory [1], [29], [10]. These physics works have motivated advances in different mathematical areas in order to provide explanations to the properties of the Witten genus and elliptic genus; some works in this direction are [7], [25], [9], [2] and [15], the following works in particular considered the theory of vertex algebras [5], [13], [21], [8], [24] [6] and [16].
In this work, we combine the geometrical constructions done in [25] with a language of operators motivated by vertex algebras. Now we express our main results.

Following [25], we consider first the case where $M$ is a fixed point spin manifold of an $S^1$-action in a finite dimensional spin manifold $X$, we call this case the finite dimensional case. In this case, we will define a set of objects: a $\mathbb{Z}_2$ graded vector bundle $V \cong V^+ \oplus V^- \rightarrow M$, a connection and a set of bundle endomorphisms resembling the theory of vertex algebras. These objects will be used to build a global operator $Q : \Gamma(V) \rightarrow \Gamma(V)$ and a bundle morphism $L_K : V \rightarrow V$ such that the analytical index defined below

$$\text{Ind}(Q, q) := \text{Tr}(q^{L_K}|_{\ker(Q|V^+)} - \text{Tr}(q^{L_K}|_{\ker(Q|V^-)}),$$

it is going to be proved, theorem 2.10, to satisfy the following identity

$$\text{Ind}(Q, q) = q^{\frac{1}{24}} \int_M A(M) \text{ch}(\sqrt{\det N} \otimes_{r \in A} S_q(N_r)).$$

In this work, the vector bundles, connections and bundle endomorphisms are defined on the fixed point manifold of the $S^1$-action; this is a main difference with [25] where the vector bundles, connections and bundle endomorphisms were defined on a neighborhood of the fixed point manifold. This approach in particular makes us to consider infinite dimensional vector bundles even for the finite dimensional case; this allows us to see the vertex algebraic language arising naturally.

And we consider a second case, where $M$ is a fixed point spin manifold of a $S^1$-action on the loop space $LM$, we call this case infinite dimensional case. In this case, analogous to the finite dimensional case, we will define a set of objects: a $\mathbb{Z}_2$ graded vector bundle $V_R \cong V^+_R \oplus V^-_R \rightarrow M$, a connection and a set of bundle endomorphisms resembling the theory of vertex algebras. These objects will be used to build a global operator $Q_R : \Gamma(V_R) \rightarrow \Gamma(V_R)$ and a bundle morphism $P : V_R \rightarrow V_R$ such that the index defined below

$$\text{Ind}(Q_R, q) := \text{Tr}(P|_{\ker(Q_R|V_R^+)}) - \text{Tr}(P|_{\ker(Q_R|V_R^-)}),$$

it is going to be proved, theorem 3.5, to satisfy the following identity

$$\text{Ind}(Q_R, q) = \frac{\Phi(M)}{\eta(q)^{dim \mathbb{M}}},$$

where $\eta(q)$ is the Dedekind eta function.

Before we explain the organization of this work, we mention what the expression resembling the theory of vertex algebras means mathematically. From a vertex algebraic point of view, the examples worked in this work motivate a definition of sheaves of vertex algebras, we define sheaves of vertex algebras formally\(^1\) in [26]. Additionally in [26], we use sheaves of vertex algebras to explain the following two problems in this work. The first problem comes from the definition of the operator $P : V_R \rightarrow V_R$, definition 3.3, where an extra term $-dim \mathbb{M}/24$ is introduced by definition. The second problem is that the condition on the first Pontrjagin class did not appear in the previous construction. We remark that although the term $-dim \mathbb{M}/24$ in the definition of $P$ allows us to obtain the previous expression $\frac{\Phi(M)}{\eta(q)^{dim \mathbb{M}}}$ and the condition $p_1(M) = 0$ in $H^4(M, \mathbb{R})$ makes this expression modular, we are interested in seeing these two conditions in the bundle and operator language.

We explain now the organization of this work.

\(^1\)These sheaves of vertex algebras are different to Chiral de Rham [22]
In the section 2, we consider the finite dimensional case. We assume that \( M \) is spin and connected, this simplifies the notation and we do not need more general assumptions in the following sections.

In the section 2.1, the following \( \mathbb{Z}_2 \) graded complex bundle is built

\[
V = \Delta(M) \otimes \sqrt{\det N} \otimes \bigwedge^A S(N_r) \otimes S(\bar{N}_r) \rightarrow M,
\]

where \( A \) is finite set of positive integers, \( \bigwedge^A(E) := 1+E+E^2(E)+.. \) and \( S(E) := 1+E+Sym^2(E)+.. \) as power series with values in vector bundles on \( M \). We will see later in the infinite dimensional case that we must change the finite set \( A \) by \( \mathbb{Z}_+ \), then the bundle above could be thought as a finite dimensional version of a sheaves of the vector space of states in a vertex algebras. In this section, it is also introduced an hermitian inner product \((\cdot,\cdot)\) on the bundle \( V \).

In the section 2.2, it is defined an hermitian connection \( D : \Gamma(V) \rightarrow \Gamma(V \otimes T^*M) \). And motivated geometrically by the derivation and multiplication of the complex and complex conjugate coordinates in the normal directions, it is first defined in 2.2.2 and later modified in the definition 2.2, a set of bundles morphisms \( \partial_r, \gamma : N_r \rightarrow End(V) \) and \( \partial_r, \gamma : \bar{N}_r \rightarrow End(V) \) for \( r \in A \). It is proved in the lemma 2.4 that these operators satisfy the following algebra

\[
[\partial_n, m] = (m,n)\delta_{r,-l}, \quad [\partial_n, \bar{m}] = (\bar{m},\bar{n})\delta_{r,-l},
\]

where \( n \in \Gamma(N_r), \ m \in \Gamma(N_l) \), and we used the notation \( N_{-l} = N_l \). Notice that we omitted the notation of the subscript \( r \in A \) to the bundles morphisms once an element in their respective domain is evaluated: \( \partial_r := \partial(n), n := \gamma(n) \) for \( n \in \Gamma(N_r) \) and \( \partial_n := \partial(n) \) and \( \bar{n} := \gamma(\bar{n}) \) for \( \bar{n} \in \Gamma(\bar{N}_r) \).

The definition of the bundles maps \( \partial_r : N_r \rightarrow End(V) \) and \( \partial_r : \bar{N}_r \rightarrow End(V) \) for \( r \in A \) satisfy that the sections of the trivial vector bundle \( \mathbb{C} \times M \), the space of functions \( C^\infty(M) \otimes \mathbb{C} \), are vacuum states for a set of Harmonic oscillator operators acting on the space of sections \( \Gamma(V) \). In particular, we have, definition 2.3, bundles maps \( a_r, b_{-r} : N_r \rightarrow End(V) \) and \( b_r, a_{-r} : \bar{N}_r \rightarrow End(V) \) for \( r \in A \) such that they satisfy, lemma 2.6, the following algebra

\[
[a_n, a_m] = (m,n)\delta_{r,-l}, \quad [b_n, b_m] = (\bar{m},\bar{n})\delta_{r,-l},
\]

where \( n \in \Gamma(N_r) \) and \( m \in \Gamma(N_l) \).

Finally, motivated geometrically by the wedge product and the interior product on the space of differential forms in the normal directions, it is defined in 2.3 a set of bundles morphisms \( \varphi_r : N_r \rightarrow End(V) \) and \( \varphi_r : \bar{N}_r \rightarrow End(V) \) for \( r \in A \). They satisfy, lemma 2.6, the following algebra

\[
[\varphi_m, \varphi_n] = (n,m)\delta_{r,-l},
\]

where \( n \in \Gamma(N_r) \) and \( m \in \Gamma(N_l) \). In this work we call operators indistinctly either the connection or the bundles morphism above. In the theorem 2.7, we express the adjoint operator for each one of the operators above regarding the hermitian inner product on \( V \).

In the section 2.3 are defined some global operators. Motivated by the Dirac operator on \( X \) it is defined a global operator \( \bar{D}_X \in End(\Gamma(V)) \), definition 2.5. By abuse of language we keep the same geometrical notation used previously to describe the equivariant index at (1.3). And motivated by the Killing vector field on \( X \), there are defined global operators \( K' \in End(V), dK' \in End(V) \) and \( D^X_K \in End(V) \), definition 2.6. Finally, the following global operators are defined

\[
Q := \bar{D}_X + iK' \in End(\Gamma(V)), \quad L_K := D^X_K + \frac{1}{4}dK' \in End(V),
\]
see definition 2.1, the motivation behind these definitions are also geometrical [27]. These operators are expressed locally on an open set \( U \subset M \) as follows

\[
Q|_U = \psi_0^i D_{e_i} + \sum_{r \in A} \sum_{k=1}^{\dim N_r} \psi^k_r \alpha^k_r + \psi^k_{-r} \alpha^k_r, \quad L_K|_U = \frac{1}{2} c_1 + \sum_{r \in A} \sum_{k=1}^{\dim N_r} \beta^{k}_{-r} \beta^{k}_r - \alpha^{k}_{-r} \alpha^{k}_r - r \psi^k_r \psi^k_{-r},
\]

where we used a basis \( \alpha^i_r, \alpha^i_{-r}, \beta^i_r, \beta^i_{-r} \) and \( \psi^i_r, \psi^i_{-r} \) for \( r \in A \) and \( i \in \{1, \ldots, \dim N_r\} \) on \( U \) of the operators \( a_r, b_{-r}, \varphi_r : N_r \to \text{End}(V) \) and \( b_r, a_{-r}, \varphi_r : \tilde{N}_r \to \text{End}(V) \).

In the section 2.4, it is defined the analytical index (1.5) where the splitting \( V = V^+ \oplus V^- \) comes from the \( \mathbb{Z}_2 \) gradation of the bundle, see (2.1) and (2.2). And it is proved the theorem 2.10 from where we obtain the equality (1.6).

In the section 3, we consider the infinite dimensional case. The normal bundle of \( M \) inside \( LM \) is defined by the bundle \( N := \bigoplus_{n \geq 0} N_n \), where \( N_n \cong TM \times \mathbb{C} \). This section is a generalization of the previous section and much of the definitions in the previous section are useful in this section.

In the section 3.1, it is built the vector bundle

\[
V_R := \Delta(M) \otimes_{n \in \mathbb{Z}_+} \wedge^n \mathcal{N}_n \otimes_{n \in \mathbb{Z}_+} \mathcal{S}(N_n) \otimes_{n \in \mathbb{Z}_+} \mathcal{S}(-N_n) \to M,
\]

where \( N_n \cong \tilde{N}_n \cong TM \otimes \mathbb{C} \). Notice that the isomorphism \( N_n \cong \tilde{N}_n \) implies that the determinant line bundle is a tensor products of line bundles and their dual bundles, therefore the determinant bundle is the trivial bundle in this case. In this section is defined an hermitian inner product on this bundle \( V_R \).

In the section 3.2, we introduce analogously operators to the operators defined in the section 2.2. In particular, it is defined an hermitian connection \( D : \Gamma(V_R) \to \Gamma(V_R \otimes T^*M) \). And we have bundle maps \( a_r, b_{-r}, \varphi_r : N_r \to \text{End}(V) \) and \( b_r, a_{-r}, \varphi_r : \tilde{N}_r \to \text{End}(V_R) \) for \( r \in \mathbb{Z}_+ \), definition 3.1, such that they satisfy the following algebra

\[
[a_n, a_m] = (m, n) \delta_{r,-l}, \quad [b_n, b_m] = (\bar{m}, \bar{n}) \delta_{r,-l}, \quad [\varphi_m, \varphi_n] = (n, m) \delta_{r,-l},
\]

where \( n \in \Gamma(N_r) \) and \( m \in \Gamma(N_l) \).

In the section 3.3, we must make some changes regarding the section 2.3. The problem comes from changing the finite sum, \( \sum_{r \in A} \) of composition of operators in the previous section to an infinite sum, \( \sum_{r \in \mathbb{Z}_+} \). The normal ordered is introduced in this section to solve this problem. The global operators \( Q_R \in \text{End}(\Gamma(V_R)) \) and \( P \in \text{End}(V_R) \) are defined as normal ordered compositions of operators, definitions 3.2 and 3.3. These operators locally on a open set \( U \subset M \) are given by

\[
Q_R|_U = \psi_0^i \nabla_{e_i} + \sum_{n \in \mathbb{Z}_+} \sum_{k=1}^{\dim M} \psi^k_n \alpha^k_n + \alpha^k_{-n} \psi^k_n, \quad P|_U = \sum_{n \in \mathbb{Z}_+} \sum_{k=1}^{\dim M} \beta^k_{-n} \beta^k_n - \alpha^k_{-n} \alpha^k_n - n \psi^k_{-n} \psi^k_n - \frac{\dim M}{24}.
\]

where we used a basis \( \alpha^i_r, \alpha^i_{-r}, \beta^i_r, \beta^i_{-r} \) and \( \psi^i_r, \psi^i_{-r} \) for \( r \in \mathbb{Z}_+ \) and \( i \in \{1, \ldots, \dim M\} \) on \( U \) of the operators \( a_r, b_{-r}, \varphi_r : N_r \to \text{End}(V) \) and \( b_r, a_{-r}, \varphi_r : \tilde{N}_r \to \text{End}(V_R) \). We notice that the term \(-\dim M/24\) on \( P|_U \) is introduced by definition.

In the section 3.4, it is defined the analytical index (1.7) where the splitting \( V_R = V_R^+ \oplus V_R^- \) comes from the \( \mathbb{Z}_2 \) gradation of the bundle, see (3.1) and (3.2). And it is proved the theorem 3.5 from where we obtain the equality (1.8).

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2. Localization, finite dimensional case

In this section, we assume that $M$ is a connected spin manifold. The theorems 2.1 and 2.5 in this section are not useful in the next section, readers interested in the infinite dimensional case can skip them.

2.1. Vector Space. Let $X$ be a spin manifold with an $S^1$-action by isometries such that the action lift to the spin bundle. Let $M$ be the set of fixed points, and $N'$ its normal bundle.

Theorem 2.1. The bundle $N'$ is a complex vector bundle, it means exists $J : N' \rightarrow N'$ s.t $J^2 = -Id$.

We use the notation $N'$ for the real bundle, $(N', J)$ for the complex bundle and $N$ is the complex bundle isomorphic to $(N', J)$

Proof. This theorem is proved in [19] and in [18] (theorem 5.3). Because we are going to use some properties of the complex structure, we give some details of the proof here. For $x \in M$ we consider $(\nabla K)_x : T_x X \rightarrow T_x X$ then by definition of $M$ we have that $T_x M = Ker (\nabla K)_x$ and $N'_x = Im (\nabla K)_x$. And from the the antisymmetry with respect to the metric for Killing vector fields of $(\nabla K)_x$ there is a basis such that $(\nabla K)_x$ is given by a matrix of the form

$$
\begin{pmatrix}
0 & \ldots & 0 & a_1 \\
& & 0 & a_1 \\
& & -a_1 & 0 \\
& & & \ldots & 0 & a_n \\
& & & & -a_n & 0
\end{pmatrix}, \quad a_i \in \mathbb{Z}_+
$$

Then we can consider the map $\nabla K : N' \rightarrow N'$ on $M$. This operator is parallel $\nabla(\nabla K) = 0$. Therefore the eigenvalues of $\nabla K$ remain constant on $M$, this splits the normal bundle $N' = \bigoplus_{\pm i r \in A} N'_r$ for $A = \{r_1, ..., r_k\} \subset \mathbb{Z}_+$, we use the notation $a_i$ for possibly equal values and $r_i$ for different. Let $J : N' \rightarrow N'$ be the endomorphism defined by $J|_{N'_r} := \frac{1}{r}(\nabla K)$, then $J^2 = -Id$.

The complex isomorphism between $(N', J)$ and $(N, i)$ is given by $u \rightarrow u - iJu$ for $u \in N'$. □

Now, using the theorem above we define the vector bundle

$$
V := \Delta(M) \otimes \sqrt{\det N} \otimes \wedge^*(\overline{N}) \otimes S(N) \otimes S(\overline{N}) \rightarrow M,
$$

where $S(E) := 1 + E + Sym^2(E) + ..$, $\wedge^*(E) := 1 + E + \wedge^2(E) + ..$ and $\Delta(M)$ is the spin bundle. A naive motivation behind the definition of this bundle is as follows. Expressions like (1.3) shows that there is a certain structure around the fixed point manifold $M$ that measures the index of the Dirac operator on $X$. We are going to show that the bundle $V$ gives us this index.
The bundle $V$ is the tensor product of the bundles $\Delta(M) \otimes \sqrt{\det N} \otimes \Lambda^* (\bar{N})$ and $S(N) \otimes S(\bar{N})$. The bundle $\Delta(M) \otimes \sqrt{\det N} \otimes \Lambda^* (\bar{N})$ comes from the bundle $\Delta(X)$ splitting the spinors on the tangent and normal directions of $M$. The bundle $S(N) \otimes S(\bar{N})$ is motivated from the space of functions on $X$, which depend on the normal coordinates. In [25] the construction of the bundle $\Delta(M) \otimes \sqrt{\det N} \otimes \Lambda^* (\bar{N})$ is revisited in detail. The bundle $S(N) \otimes S(\bar{N})$ appears naturally in the infinite dimensional case considering the creation and annihilation construction of a Verma module, in this work we introduce this bundle also to the finite dimensional case.

Observe that $\Delta(M)$ has a standard $\mathbb{Z}_2$ grading and $\Lambda^* \bar{N}$ is $\mathbb{Z}_2$ graded considering the parity $\Lambda^\text{even} \bar{N}$ and $\Lambda^\text{odd} \bar{N}$, therefore the $\mathbb{Z}_2$ graded tensor product on $\Delta(M) \otimes \Lambda^* \bar{N}$ induce a $\mathbb{Z}_2$ grading on the vector space $V$ as follows

\begin{align}
V^+ &= \left( \Delta^+(M) \otimes \Lambda^\text{even} \bar{N} \oplus \Delta^-(M) \otimes \Lambda^\text{odd} \bar{N} \right) \otimes \det(N) \otimes S(N) \otimes S(\bar{N}), \\
V^- &= \left( \Delta^-(M) \otimes \Lambda^\text{even} \bar{N} \oplus \Delta^+(M) \otimes \Lambda^\text{odd} \bar{N} \right) \otimes \det(N) \otimes S(N) \otimes S(\bar{N}).
\end{align}

From the previous theorem, we have

**Theorem 2.2.** The operator $\nabla K : N' \to N'$ splits the normal bundle as follows

$$N' \otimes \mathbb{C} = \bigoplus_{r \in A} N_r \oplus \bar{N}_r, \quad N_r := \{ v \in N' \otimes \mathbb{C} | \nabla v K = -i rv \},$$

where $A \subset \mathbb{Z}_+$ is the finite subset of eigenvalues. Therefore, we have the isomorphisms $N = \bigoplus_{r \in A} N_r$ and $\bar{N} = \bigoplus_{r \in A} \bar{N}_r$.

We express the bundle $V$ as follows

$$V = \Delta(M) \otimes \sqrt{\det N} \otimes r \in A \Lambda^* (\bar{N}_r) \otimes \bigoplus_{r \in A} S(N_r) \otimes S(\bar{N}_r) \to M.$$  

Observe that this decomposition respects the $\mathbb{Z}_2$ grading on $V$.

**2.1.1. Hermitian product.** The vector bundle $V$ is defined as a tensor product of other bundles, then we define the hermitian product on $V$ considering hermitian products for each one of these bundles.

First, $M$ is spin manifold and its spin bundle $\Delta(M)$ has an hermitian inner product, see [20].

Second, we have a real inner product on $N'$ coming from the Riemannian metric on $X$, from this inner product we get an hermitian inner product considering the complex structure.

**Theorem 2.3.** The complex structure on $N'$ is orthogonal, it means $J_p \in SO(2n, \mathbb{R})$ for every $p \in M$.

This theorem follows directly from the definition of $J$ in the theorem 2.1. A positive definite hermitian metric on the complex bundle $(N', J)$ is defined by $h(u, v) := \frac{1}{2} \langle g_X(u, v) - ig_X(Ju, v) \rangle$ for $u, v \in N'$, where we denoted as $g_X$ the metric induced from $X$ on the normal bundle $N'$. Additionally, we have an hermitian extension of the Riemannian metric to the bundle $N' \otimes \mathbb{C} = N \oplus \bar{N}$ defined by

$$\langle u \otimes a, v \otimes b \rangle := abg_X(u, v)$$

for $u, v \in N'$ and $a, b \in \mathbb{C}$. The bundles $N$ and $\bar{N}$ form an orthogonal decomposition for $(\cdot, \cdot)$. It follows from the definitions above that $(\cdot, \cdot)|_N = h_{(N', J)}$ and $(\cdot, \cdot)|_{\bar{N}} = \bar{h}_{(N', J)}$. In particular, for $\bar{n}, \bar{m} \in \bar{N}$ we have the identity $(\bar{n}, \bar{m}) = (m, n)$. 

Third, we use the previous hermitian inner product to induce an hermitian inner product on $T(N) := \sum_{l \geq 0} N^\otimes l$

$$(n_1 \otimes \ldots \otimes n_k, m_1 \otimes \ldots \otimes m_l)_{T(N)} = \delta_{kl}(n_1, m_1) \ldots (n_k, m_k),$$

where $n_1 \otimes \ldots \otimes n_k, m_1 \otimes \ldots \otimes m_l \in T(N)$. Therefore, we have an hermitian inner product on the subspaces $S(N)$ and $\wedge^r N$ given by

$$(n_1 \ldots n_k, m_1 \ldots m_l)_{S(N)} = \delta_{kl} \sum_{\sigma \in \delta_k} (n_{\sigma(1)}, m_1) \ldots (n_{\sigma(k)}, m_k),$$

$$(n_1 \wedge \ldots \wedge n_k, m_1 \wedge \ldots \wedge m_l)_{\wedge^r N} = \delta_{kl} \sum_{\sigma \in \delta_k} \epsilon(\sigma)(n_{\sigma(1)}, m_1) \ldots (n_{\sigma(k)}, m_k),$$

where $n_1 \ldots n_k, m_1 \ldots m_l \in S(N)$, $n_1 \wedge \ldots \wedge n_k, m_1 \wedge \ldots \wedge m_l \in \wedge^r N$, $\delta_n$ is the group of permutations and $\epsilon : \delta_n \to \{1, -1\}$ the sign homomorphism. Observe that $\det(N)$ is a subspace of $\wedge^* N$ and its hermitian inner product is defined by (2.6). The definitions of hermitian products for $S(\bar{N})$ and $\wedge^* \bar{N}$ follow similarly from the hermitian metric on $\bar{N}$.

Four, for $S(N) \otimes S(\bar{N})$ with do not use the tensor product of the hermitian inner products, instead we consider a different hermitian product motivated by the Harmonic oscillator. We are going to define bundles maps over $\text{End}(V)$, see definition 2.3, that satisfy the algebra of Harmonic oscillators, then the inner product defined below makes that these operators have the usual conjugate operators, lemma 2.7. In particular the definition below will be useful to prove the theorems 2.10 and 3.5.

The Hermitian inner product for $S(N_r) \otimes S(\bar{N}_r)$ is defined as follows: We have naturally an hermitian inner product restricted to subbundles $N_r$ from (2.3). Then for two arbitrary sections $a_1 \ldots a_k b_1 \ldots b_s$ and $c_1 \ldots c_t d_1 \ldots d_t$ in $S(N_r) \otimes S(\bar{N}_r)$, we relabel them as $n_1 \ldots n_k \bar{m}_{l+1} \ldots \bar{m}_{l+s}$ and $m_1 \ldots m_l \bar{n}_{k+1} \ldots \bar{n}_{k+t}$ respectively its hermitian inner product is given by

$$(n_1 \ldots n_k \bar{m}_{l+1} \ldots \bar{m}_{l+s}, m_1 \ldots m_l \bar{n}_{k+1} \ldots \bar{n}_{k+t}) := (n_1 \ldots n_k, m_1 \ldots m_l)$$

$$(n_1 \ldots n_k \bar{m}_{l+1} \ldots \bar{m}_{l+s}, m_1 \ldots m_l \bar{n}_{k+1} \ldots \bar{n}_{k+t}) := (n_1 \ldots n_k, m_1 \ldots m_l)$$

$$(n_1 \ldots n_k \bar{m}_{l+1} \ldots \bar{m}_{l+s}, m_1 \ldots m_l \bar{n}_{k+1} \ldots \bar{n}_{k+t}) := (n_1 \ldots n_k, m_1 \ldots m_l)$$

$$(n_1 \ldots n_k \bar{m}_{l+1} \ldots \bar{m}_{l+s}, m_1 \ldots m_l \bar{n}_{k+1} \ldots \bar{n}_{k+t}) = \frac{1}{k+l+s} \delta_{k+l+t} \sum_{\sigma \in \delta_{k+l+t}} (n_{\sigma(1)}, m_1) \ldots (n_{\sigma(k)}, m_l) (n_{\sigma(k+1)}, l_1) \ldots (n_{\sigma(k+t)}, l_s).$$

The restriction of this hermitian product to the subspaces $S(N_r)$ and $S(\bar{N}_r)$ is not the same hermitian product given in (2.5), the difference comes from a factor $\frac{1}{k}$ for a certain $k \in \mathbb{Z}_+$. We build the hermitian product on $S(N) \otimes S(\bar{N})$ from the tensor product of the hermitian bundles $S(N) \otimes S(\bar{N}) = \otimes_{r \in \mathbb{A}} S(N_r) \otimes S(\bar{N}_r)$.

Finally, the hermitian inner product on $V$ is given by the tensor product of the inner products on $\Delta(M)$, $\det N$, $\wedge \bar{N}$ and $S(N) \otimes S(\bar{N})$. Therefore $V$ is an hermitian bundle, we denote this hermitian product by $(.,.)$.

2.2. Operators. In this section, we define the operators acting on the bundle $V$. The operators in this section will be be used in the infinite dimensional case, the notation in this section is motivated by the infinite dimensional case.

2.2.1. Connections. The spin manifold $M$ has a Riemannian metric from the restriction of the metric on $X$. Therefore we have a connection $D^M : \Gamma(\Delta(M)) \to \Gamma(\Delta(M) \otimes T^*M)$ from the Levi-Civita connection on $M$. This connection is compatible with the hermitian metric on $\Delta(M)$.
Additionally, we have the Levi-Civita connection on $X$, we can use this connection to define on $M$ a connection $\nabla$ acting on the normal bundle $N'$. In general, evaluating normal sections on tangent vectors of $M$ we have that

$$\nabla : \Gamma(N') \to \Gamma(TX \otimes T^*M)$$

where $TX|_M = TM \otimes N'$.

**Theorem 2.4.** The fixed point manifold $M$ is totally geodesic, then $\nabla : \Gamma(N') \to \Gamma(N' \otimes T^*M)$. Additionally, the complex structure $J$ and $\nabla K$, defined on $M$, are parallel.

See the proof of the theorem 2.1 and the references there. The following theorem can be seen as a grading preservation of operators.

**Theorem 2.5.** The connection $\nabla$ induce an hermitian connection $D'$ s.t. $D' : \Gamma(N_r) \to \Gamma(N_r \otimes T^*M)$ and $\bar{D}' : \Gamma(\bar{N}_r) \to \Gamma(\bar{N}_r \otimes T^*M)$. Additionally, the connection $D'$ is compatible with the hermitian metric (2.4).

The proof of this theorem is a direct verification of the compatibility relation using the flatness and orthogonality of $J$ restricted to the sub bundles $N_r$ and $\bar{N}_r$.

These connections induce connections on the tensor products $T(N_r)$ and $T(\bar{N}_r)$ compatible with the hermitian product (2.4). Then we build naturally connections on the following bundles

$$D^s : \Gamma(S(N_r)) \to \Gamma(S(N_r) \otimes T^*M), \quad D^\wedge : \Gamma(\wedge^s(\bar{N}_r)) \to \Gamma(\wedge^s(\bar{N}_r) \otimes T^*M),$$

$$D^{\wedge} : \Gamma(\wedge^s(\bar{N}_r)) \to \Gamma(\wedge^s(\bar{N}_r) \otimes T^*M), \quad D^{\det} : \Gamma(\det N) \to \Gamma(\det N \otimes T^*M).$$

The compatibility of $D'$ implies the compatibility of the connections above with respect to their hermitian inner products, we prove the only nontrivial compatibility in the following theorem

**Theorem 2.6.** The connection $D^s \otimes D^\wedge : \Gamma(S(N) \otimes S(\bar{N})) \to \Gamma(S(N) \otimes S(\bar{N}) \otimes T^*M)$ is compatible with the hermitian metric (2.8).

**Proof.** If $k + t \neq l + s$ then the compatibility is trivial, now if $k + t = l + s$ we have from a direct calculation that

$$(D^s \otimes D^\wedge(n_1 \ldots n_k \bar{m}_{t+1} \ldots \bar{m}_{t+s}), m_1 \ldots m_l \bar{n}_{k+1} \ldots \bar{n}_{k+t}) + (n_1 \ldots n_k \bar{m}_{t+1} \ldots \bar{m}_{t+s}, D^s \otimes D^\wedge(m_1 \ldots m_l \bar{n}_{k+1} \ldots \bar{n}_{k+t}))
= (D^s(n_1 \ldots n_{k+t}), m_1 \ldots m_{l+s}) + (n_1 \ldots n_{k+t}, D^\wedge(m_1 \ldots m_{l+s}))$$

$$= d(n_1 \ldots n_{k+t}, m_1 \ldots m_{l+s}) = d(n_1 \ldots n_{k+t}, m_1 \ldots m_{l+s})$$

Therefore, we have the following connection on $V$

$$D_M := D^M \otimes D^{\det} \otimes D^{\wedge} \otimes D^s \otimes D^\wedge : \Gamma(V) \to \Gamma(V \otimes T^*M).$$

And this connection is compatible with the hermitian metric $(.,.)$ on $V$.

### 2.2.2 Sheaves of operators

Now, we define the operators on the fibers of the bundle $V \to M$.

**Definition 2.1.** The following $\mathbb{C}$-linear bundle maps on an arbitrary local open set $U \subset M$, define $\mathbb{C}$-linear sheaves of bundle morphisms on $M$, we express the local definitions on $U$ in the right.

$$\cdot_r := N_r \to \text{End}(S(N_r)), \quad n \cdot (n_1 \ldots n_i \ldots n_k) := nn_1 \ldots n_i \ldots n_k,$$

$$\wedge_r := N_r \to \text{End}(\wedge^r N_r), \quad n \wedge (n_1 \wedge \ldots \wedge n_i \ldots \wedge n_k) := n \wedge n_1 \wedge \ldots \wedge n_i \ldots \wedge n_k,$$
and considering the complex conjugate map \( \bar{N}_r \rightarrow N_r \),
\[
\partial'_r := \bar{N}_r \rightarrow \text{End}(S(N_r)), \quad \tilde{m} \mapsto \partial'_m(n_1...n_i...n_k) := \sum_{i=1}^{k} (n_i, m)n_1...\hat{n}_i...n_k,
\]
\[
i_r := \bar{N}_r \rightarrow \text{End}(\wedge^* N_r), \quad \tilde{m} \mapsto i_m(n_1 \wedge ...n_i \wedge ...n_k) := \sum_{i=1}^{k} (-1)^{i+1}(n_i, m)n_1 \wedge ...\hat{n}_i \wedge ...n_k,
\]
where \( n \in \Gamma(N_r)|_{\bar{U}}, \tilde{m} \in \Gamma(\bar{N}_r)|_{\bar{U}}, n_1...n_i...n_k \in \Gamma(S(N_r))|_{\bar{U}}, n_1 \wedge ...n_i \wedge ...n_k \in \Gamma(\wedge^* N_r)|_{\bar{U}} \) and \((,\cdot)\) is the Hermitian metric defined in (2.3). Using the transition functions for the bundles above it is easy to see that the maps are independent of the local chart \( U \), below we show this for the map \( \partial' \).

The bundle \( N_r \) is an hermitian bundle, then for an open set \( U \subset M \) we consider the transition functions \( g : U \cap U' \rightarrow U(d_r) \) and we have the following commutative diagram
\[
\begin{array}{ccc}
(n, n_1...n_i...n_k) \in \Gamma(\bar{N}) \times \Gamma(S(N_r))|_{U} & \xrightarrow{\partial'} & \sum_{i=1}^{k} (n_i, n)n_1...\hat{n}_i...n_k \in \Gamma(S(N_r))|_{U} \\
\downarrow{g} & & \downarrow{g} \\
(g\tilde{n}, (gn_1)\ldots(gn_k)) \in \Gamma(\bar{N}) \times \Gamma(S(N_r))|_{U} & \xrightarrow{\partial'} & \sum_{i=1}^{k} (gn_i, gn)(\hat{n}_1\ldots\hat{n}_k) \in \Gamma(S(N_r))|_{U}
\end{array}
\]

Notice that, as we already mentioned in the introduction, we omit the notation of the subscript \( r \in A \) to the bundles morphisms once an element in their respective domain is evaluated: \( n\wedge = \wedge_r(n) \), \( n = _r(n) \) for \( n \in \Gamma(N_r)|_{U} \) and \( \partial_n = \partial_r(\tilde{n}) \) and \( i_n = i_r(\tilde{n}) \) for \( \tilde{n} \in \Gamma(\bar{N}_r)|_{U} \).

In this work, we call extension of \( A \in \text{End}(S) \) to a vector space \( S \otimes R \), the map \( A \otimes Id_R \in \text{End}(S \otimes R) \). In particular, we consider the extensions of the bundles maps above to act on \( S(N) \otimes \wedge N \), we denote these operators as \( \cdot_r : N_r \rightarrow \text{End}(V), \wedge_r := N_r \rightarrow \text{End}(V), \partial'_r := \bar{N}_r \rightarrow \text{End}(V) \) and \( i_r := \bar{N}_r \rightarrow \text{End}(V) \). Now the \( \mathbb{Z}_2 \) grading of the vector bundle \( V \) induce a \( \mathbb{Z}_2 \) grading on these bundle morphisms; it is easy to see that the operators \( \cdot_r, \partial_r \) are even and the operators \( \wedge_r, i_r \) are odd.

**Lemma 2.1.** The operators \( \cdot_r : N_r \rightarrow \text{End}(V), \partial'_r : \bar{N}_r \rightarrow \text{End}(V), \wedge_r := N_r \rightarrow \text{End}(V) \) and \( i_r : \bar{N}_r \rightarrow \text{End}(V) \) satisfy an algebra considering the graded commutator, below we express the non zero relations on an open set \( U \subset M \)

\[
[\partial'_m, n] = (n, m), \quad [i_m, n \wedge] = (n, m),
\]

where \( n \in \Gamma(N)|_{U} \) and \( \tilde{m} \in \Gamma(\bar{N})|_{U} \).

The proof of this lemma follows directly from the definition 2.1. In the following lemma, we consider the extension of the operators \( \cdot_r \) and \( \partial'_r \) defined in 2.1 to the bundle \( S(N) \).

**Lemma 2.2.** The operators \( \cdot_r : N_r \rightarrow \text{End}(S(N)) \) and \( \partial'_r := \bar{N}_r \rightarrow \text{End}(S(N)) \) satisfy the following relations regarding the hermitian metric defined in (2.5) on an open set \( U \subset M \)

\[
(\partial'_m v, w)_{S(N)} = (v, mw)_{S(N)},
\]

where \( m \in \Gamma(N)|_{U} \) and \( v, w \in \Gamma(S(N))|_{U} \).

**Proof.** This follows from the identity

\[
(n_1...n_{k+1}, mm_1...m_k)_{S(N)} = \sum_{i=1}^{k+1} (n_i, m)(n_1...\hat{n}_i...n_{k+1}, m_1...m_k)_{S(N)}.
\]

And in the following lemma, we consider the extension of the operators \( \wedge_r \) and \( i_r \) defined in 2.1 to the bundle \( \wedge N \).
Lemma 2.3. The operators $\wedge_r : N_r \to \text{End}(\wedge N)$ and $i_r := \tilde{N}_r \to \text{End}(\wedge N)$ satisfy the following relations regarding the hermitian metric defined in (2.6) on an open set $U \subset M$

$$(i_m v, w)_{\wedge N} = (v, \wedge_m w)_{\wedge N},$$

where $m \in \Gamma(N_r)|_U$ and $v, w \in \Gamma(\wedge N)|_U$.

Proof. This follows from the identity

$$(n_1 \ldots n_{k+1}, m \wedge m_1 \wedge \ldots \wedge m_k)_{\wedge N} = \sum_{i=1}^{k+1} (-1)^{i+1} (n_i, m)(n_1 \wedge \ldots \wedge \hat{n}_i \wedge m_1 \wedge \ldots \wedge m_k)_{\wedge N}\tag{2.1}$$

Finally, we can define as in 2.1 analogous operators $\gamma_r : \tilde{N}_r \to \text{End}(S(\tilde{N}_r))$ and $\partial_r : N_r \to \text{End}(S(\tilde{N}_r))$. And analogously these operators satisfy the lemmas 2.1 and 2.2.

2.2.3. Harmonic oscillator. In this section, the operator $\partial'$ is modified; this is going to help us to construct harmonic oscillator operators acting on the bundle $V$, the motivation comes more specifically from the vacuum of the harmonic oscillator. Observe that the harmonic oscillator appeared naturally considering the geometric description in [27]. The operator $\partial'$ is analogous to the differential operator acting on a space of functions where the vacuum is given by the constant function 1. On the other hand, the vacuum of the harmonic oscillator is given by an exponential function $\exp(-\sum_{n \in \mathbb{Z}_+} \frac{1}{2} n z_n \tilde{z}_n)$, and this change can be seen as a modification on the operator $\partial'$.

Definition 2.2.

$$\partial_r := \partial'_r - \frac{r}{2} : \tilde{N}_r \to \text{End}(S(\tilde{N}_r)), \quad \partial_n = \partial'_n - \frac{r}{2} \tilde{n}$$

$$\tilde{\partial}_r := \tilde{\partial}'_r - \frac{r}{2} : N_r \to \text{End}(S(N_r)), \quad \tilde{\partial}_n = \tilde{\partial}'_n - \frac{r}{2} n$$

where $n \in \Gamma(N_r)|_U$ and $\tilde{n} \in \Gamma(\tilde{N}_r)|_U$.

Considering the extensions $\partial_r : \tilde{N}_r \to \text{End}(V)$ and $\tilde{\partial}_r : N_r \to \text{End}(V)$ and the direct sums $\partial : \tilde{N} \to \text{End}(V)$ and $\tilde{\partial} : N \to \text{End}(V)$, we obtain from the lemma 2.1 the following lemma

Lemma 2.4. The operators $\cdot : N \to \text{End}(V), \partial : \tilde{N} \to \text{End}(V), \cdot : \tilde{N} \to \text{End}(V)$ and $\tilde{\partial} : N \to \text{End}(V)$ satisfy an algebra considering the commutator, below we express the non zero relations on an open set $U \subset M$

$$[\partial_n, m] = (m, n), \quad [\tilde{\partial}_n, \tilde{m}] = (\tilde{m}, \tilde{n}),$$

where $m \in \Gamma(N)|_U$ and $n \in \Gamma(\tilde{N})|_U$.

Lemma 2.5. The operators $\cdot : N_r \to \text{End}(V), \partial_r : \tilde{N}_r \to \text{End}(V), \cdot : \tilde{N}_r \to \text{End}(V), \tilde{\partial}_r : N_r \to \text{End}(V), \wedge_r : N_r \to \text{End}(V)$ and $i_r := \tilde{N}_r \to \text{End}(V)$ satisfy the following relations regarding the hermitian metric of the bundle $V$

$$(\partial_n v, w) = (v, -\partial_n w), \quad (n v, w) = (v, \tilde{n} w), \quad (n \wedge v, w) = (v, i_n w),$$

where $n \in \Gamma(N_r)|_U$ and $v, w \in \Gamma(V)|_U$. 

Proof. The relation \((n \wedge v, w) = (v, i_n w)\) follows from the lemma 2.3. The relation \((nv, w) = (v, \bar{\eta} w)\) follows from definition of the hermitian inner product (2.8). And the relation \((\partial_a v, w) = (v, -\partial_{\bar{\eta}} w)\) follows from

\[
(\partial_n (n_1 \ldots n_k, m_1 \ldots m_{l+s}), m_1 \ldots m_{l+t}, \partial_{\bar{n}} (m_1 \ldots m_{l+t})) = (\partial_{\bar{n}} (n_1 \ldots n_{k+l}), m_1 \ldots m_{l+t}) - r(n_1 \ldots n_{k+l}, nm_1 \ldots m_{l+t}) = 0.
\]

The last expression is zero because in this case we have the identity

\[
(n_1 \ldots n_{k+l}, mm_1 \ldots m_k) = \frac{1}{r} \sum_{i=1}^{k+1} (n_i, m)(n_1 \ldots \hat{n_i} n_{k+l}, m_1 \ldots m_k).
\]

on the hermitian metric (2.8).

Finally, we define the following operators

**Definition 2.3.**

\[
\varphi_r = i_r : N_r \to \text{End}(\wedge N_r), \quad \varphi_{-r} = \wedge_r : \bar{N}_r \to \text{End}(\wedge \bar{N}_r),
\]

\[
a_r := i(\partial_r + \frac{r}{2}) : N_r \to \text{End}(S(N_r) \otimes S(\bar{N}_r)), \quad a_{-r} := i(\partial_{-r} - \frac{r}{2}) : \bar{N}_r \to \text{End}(S(N_r) \otimes S(\bar{N}_r)),
\]

\[
b_r := i(\partial_r + \frac{r}{2}) : \bar{N}_r \to \text{End}(S(N_r) \otimes S(\bar{N}_r)), \quad b_{-r} := i(\partial_{-r} - \frac{r}{2}) : N_r \to \text{End}(S(N_r) \otimes S(\bar{N}_r)).
\]

We extend the operators to act on \(V\) and we define the set \(-\mathcal{A} := \{-r| r \in \mathcal{A}\}\).

**Lemma 2.6.** The operators \(a_r : N_r \to \text{End}(V), b_r : N_r \to \text{End}(V)\) and \(\varphi_r : N_r \to \text{End}(V)\) for \(r \in \mathcal{A} \cup -\mathcal{A}\) satisfy an algebra considering the graded commutator, we express below the non zero relations on an open set \(U \subset M\)

\[
[a_n, a_m] = r(m, n)\delta_{r, -1}, \quad [b_n, b_m] = r(m, n)\delta_{r, -1}, \quad [\varphi_n, \varphi_m] = (m, n)\delta_{r, -1},
\]

where \(n \in \Gamma(N_r)|_U\) and \(m \in \Gamma(N_l)|_U\) for \(r, l \in \mathcal{A} \cup -\mathcal{A}\).

**Lemma 2.7.** The operators \(a_r : N_r \to \text{End}(V), b_r : N_r \to \text{End}(V)\) and \(\varphi_r : N_r \to \text{End}(V)\) for \(r \in \mathcal{A} \cup -\mathcal{A}\) satisfy the following relations regarding the hermitian metric of the bundle \(V\)

\[
(a_n v, w) = (v, a_n w), \quad (b_n v, w) = (v, b_n w), \quad (\varphi_n v, w) = (v, \varphi_n w),
\]

where \(n \in \Gamma(N_r)|_U\) and \(v, w \in \Gamma(V)|_U\).

Finally, we introduce the following notation. The complex Clifford bundle \(CL(TM_C)\) on \(M\) acts on the bundle \(\Delta(M)\). We denote as \(\varphi_0 : TM \to CL(TM_C) \to \text{End}(\Delta(M))\) the map composition, this maps unit tangent vectors to unitary operators on \(\Delta(M)\). We denote the extension of the Clifford action from \(\Delta(M)\) to \(V\) as follows

\[
(\varphi_0 : TM \to \text{End}(V),
\]

and by definition of the hermitian product on \(V\) we have

\[
(\varphi_0(x)v, w) = -(v, \varphi_0(x)w),
\]

where \(x \in \Gamma(TM_C)|_U\) and \(v, w \in \Gamma(V)|_U\).
2.3. **Global operators.** Now, we define the global operators. In order to make this section clear, we write after each definition the operator in a local basis on an open set $U \subset M$. The operators $\partial_r$, $\gamma_r$, $\wedge_r$ and $i_r$ form finite dimensional vector bundles. We express a basis for these operators below considering an orthonormal basis $\{z_r^k\}_{k=1, r \in \mathcal{A}}$ of $N = \oplus_{r \in \mathcal{A}} N_r$ on $U$.

$$z_r^k \in : N_r \rightarrow \text{End}(V), \quad \bar{z}_r^k \in : \bar{N}_r \rightarrow \text{End}(V)$$

$$\partial z_r^k \in \partial : \bar{N}_r \rightarrow \text{End}(V), \quad \bar{\partial} z_r^k \in \bar{\partial} : N_r \rightarrow \text{End}(V)$$

$$\psi_r^k \in i_r : N_r \rightarrow \text{End}(V), \quad \bar{\psi}_r^k \in \wedge r : \bar{N}_r \rightarrow \text{End}(V).$$

Additionally, we consider an orthonormal basis $\{e_i\}_{i=1}^{\dim M}$ of $TM$ on $U$, and $\varphi_0(e_i) = \psi_0^i \in \text{End}(V)$ on $U$.

**Definition 2.4.** The connection (2.9) and the map (2.10) define a twisted Dirac operator as the composition

$$(2.11) \quad \mathcal{D}_M : \Gamma(V) \xrightarrow{D_M} \Gamma(T^* M \otimes V) \rightarrow \Gamma(TM \otimes V) \xrightarrow{\varphi_0} \Gamma(V)$$

where we used the Riemannian metric on $M$ to identify $TM \simeq T^* M$.

Locally this operator ca be written as

$$\mathcal{D}_M|_U = \psi_0^i D_{e_i}$$

Motivate for the connection operator on $X$ we define the following operator\(^2\)

$$D_X := D_M + \partial + \bar{\partial} : \Gamma(V) \rightarrow \Gamma(V \otimes (T^* M \oplus N^* \oplus \bar{N}^*))$$

For language abuse we call this the connection $D_X$, and in the open set $U$ this operator can be written as

$$D_X|_U = D_{e_i} + \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} i \partial z_r^k + i \bar{\partial} z_r^k.$$

**Definition 2.5.** Motivate for the Dirac operator on $X$ we define

$$(2.12) \quad \mathcal{D}_X : \Gamma(V) \xrightarrow{D_X} \Gamma((T^* M \oplus \bar{N}_r^* \oplus \bar{N}^*_r) \otimes V) \rightarrow \Gamma((TM \oplus \bar{N}_r^* \oplus N_r) \otimes V) \xrightarrow{\varphi_0 \oplus \varphi_{-r} \oplus \bar{\varphi}_r} \Gamma(V)$$

where $TM \simeq T^* M$ and the hermitian metric (2.3) identify $N^*_r \simeq \bar{N}_r$ and $\bar{N}^*_r \simeq N_r$. For language abuse we call this Dirac operator $\mathcal{D}_X$.

Locally this operator is

$$\mathcal{D}_X|_U = \psi_0^i D_{e_i} + \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} \psi_r^k i \partial z_r^k + \bar{\psi}_r^k i \bar{\partial} z_r^k.$$

Additionally, from the $S^1$-action on $X$ we have a Killing vector field, and considering the metric on $X$ its dual on the cotangent space. These operators motivate the following definition

\(^2\)We are using that $\partial \in \text{Hom}_A(\bar{N}, \text{Hom}_A(V, V)) = \text{Hom}_A(V, V \otimes \bar{N}^*)$ for $A = C^\infty(M) \otimes \mathbb{C}$, analogously, the same identification is done for $\bar{\partial}$. 

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Definition 2.6. In the same way as we define the Dirac operators as a composition of maps we define the following bundle maps

\[(2.13) \quad K_r : V \xrightarrow{\frac{\partial}{\partial r}} (N_r^* \oplus \bar{N}_r^*) \otimes V \rightarrow (N_r \oplus N_r) \otimes V \xrightarrow{\mathcal{A}} V, \quad K := \sum_{r \in \mathcal{A}} K_r \in \text{End}(V),\]

where \(N^* \simeq \bar{N}\) and \(\bar{N}^* \simeq N\).

\[(2.14) \quad K'_r : V \xrightarrow{\frac{\partial}{\partial r} - \frac{\partial}{\partial \bar{r}}} (N_r^* \oplus \bar{N}_r^*) \otimes V \rightarrow (N_r \oplus N_r) \otimes V \xrightarrow{\mathcal{A}} V, \quad K' := \sum_{r \in \mathcal{A}} K'_r \in \text{End}(V).\]

And

\[(2.15) \quad K'_r : V \xrightarrow{\frac{\partial}{\partial r} - \frac{\partial}{\partial \bar{r}}} (N_r^* \oplus \bar{N}_r^*) \otimes V \rightarrow (N_r \oplus N_r) \otimes V \xrightarrow{\mathcal{A}} V \rightarrow V, \quad dK' := \sum_{r \in \mathcal{A}} dK'_r \in \text{End}(V),\]

The maps bundles are given respectively by

\[K|_U = \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} r(z_r^k \partial z_r^k - z_r^k \partial z_{\bar{r}}^k), \quad K'|_U = \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} r(z_r^k \psi^k_r - z_{\bar{r}}^k \psi^k_{\bar{r}}),\]

\[dK'|_U = \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} r(\psi^k_r \bar{\psi}^k_r - \psi^k_{\bar{r}} \bar{\psi}^k_{\bar{r}}) = c_1 - 2 \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} r \psi^k_r \bar{\psi}^k_{\bar{r}}.\]

Note that the value \(c_1 = \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} r\) appeared in the expression above.

The following operators can be expressed using the following basis of operators \(a_r, b_{-r} : N_r \to \text{End}(V)\) and \(b_r, a_{-r} : \bar{N}_r \to \text{End}(V)\) as follows

\[\alpha_r^k := i(\partial z_r^k + \frac{r}{2} z_r^k) : N_r \to \text{End}(V), \quad \alpha_{-r}^k := i(\partial \bar{z}_{-r}^k - \frac{r}{2} \bar{z}_{-r}^k) : \bar{N}_r \to \text{End}(V),\]

\[\beta_r^k := i(\partial z_r^k + \frac{r}{2} z_r^k) : \bar{N}_r \to \text{End}(V), \quad \beta_{-r}^k := i(\partial \bar{z}_{-r}^k - \frac{r}{2} \bar{z}_{-r}^k) : N_r \to \text{End}(V).\]

Now, we define the most important operators in this work.

Definition 2.1.

\[(2.16) \quad Q := \mathcal{D}_X + \frac{i}{2} K' \in \text{End}(\Gamma(V)), \quad L_K := D_K^X + \frac{1}{2} dK' \in \text{End}(V).\]

On the local open set \(U\) we have

\[Q|_U = \psi^i_0 D_{e_i} + \sum_{r \in \mathcal{A}} \sum_{k=1}^{\dim N_r} \psi^j_r \iota(\partial z_r^k + \frac{r}{2} z_r^k) + \psi^j_{\bar{r}} \iota(\partial \bar{z}_r^k - \frac{r}{2} \bar{z}_r^k)\]

\[= \psi^i_0 D_{e_i} + \sum_{r \in \mathcal{A}} \sum_{k=1}^{\dim N_r} \psi^j_r \alpha^k_r + \psi^j_{\bar{r}} \alpha^k_{-r}.\]

This operator is a finite dimensional version of the Dirac-Ramond operator in the Ramond \(N = 1\) vertex algebra. Also, we have easily the following identity

\[L_K|_U = \frac{1}{2} c_1 + \sum_{r \in \mathcal{A}} \sum_{k=1}^{d_r} \beta_{-r}^k \beta_r^k - \alpha_{-r}^k \alpha_r^k - r \psi^l_{-r} \psi^l_r.\]
The last expression is a finite dimensional version of the momentum operator, it is given by the subtraction of finite dimensional versions of Virasoro operators.

2.4. **The index and the supersymmetry.** The hermitian inner product for sections $\Gamma(V)$ is defined as follows

$$ <v, w> := \int_M (v(x), w(x))dM, \quad \forall v(x), w(x) \in \Gamma(V) $$

Where $dM$ is the volume form coming from the Riemannian metric on $M$. It is clear from the Lemma 2.7 that $\mathcal{D}_N := Q - \mathcal{D}_M$ is self adjoint, $\mathcal{D}_M$ is a twisted Dirac operator and its self adjointness follows from the divergence for vector fields.

**Theorem 2.7.** The operator $Q$ satisfy

$$ <Qv, w> = <v, Qw>, \quad \forall v(x), w(x) \in \Gamma(V) $$

For $V = V^+ \oplus V^-$, we define $Q^+ := Q|_{V^+}$ and $Q^- := Q|_{V^-}$, then $Q^+ : V_+ \to V_-$, $Q^- : V_- \to V_+$

and from the self adjointness of $Q$, we have $Q_+^\dagger = Q_-$. Now, we define an equivariant index motivated by (1.2), the equivariant index on $X$. The action of $S^1$ on the sections of the spinor bundle on $X$, is now replaced for the operator $q^{L_K}$ acting on sections of $V$ where $q = e^{\beta t} \in \mathbb{C}^\times$.

$$ \text{Ind}(Q, q) := \text{Tr}(q^{L_K}|_{\text{Ker}(Q_+^\dagger)}) - \text{Tr}(q^{L_K}|_{\text{Ker}(Q_-^\dagger)}) $$

This is well defined, we prove this finding that the index above is equal to (1.3), see theorem 2.10. The operator $q^{L_K}$ acts on the vector bundle $V$, in particular we have

$$ \text{Tr}(q^{L_K} S(N)) = q^{\frac{1}{2}c_1} \text{Tr}(q^{L_K} S(N)) = q^{\frac{1}{2}c_1} \otimes r \in A S_q^r (N_r), $$

$$ \text{Tr}(q^{L_K} S(N)) = q^{\frac{1}{2}c_1} \text{Tr}(q^{-L_K} S(N)) = q^{\frac{1}{2}c_1} \otimes r \in A S_q^{-r} (N_r) $$

$$ s \text{Tr}(q^{L_K} \wedge^*(\bar{N})) = q^{\frac{1}{2}c_1} s \text{Tr}(q^{L_{\psi}} \wedge^*(\bar{N})) = q^{\frac{1}{2}c_1} (\otimes r \in A \wedge^{even}_{q^{-r}} (N_r) - \otimes r \in A \wedge^{odd}_{q^{-r}} (N_r)) $$

In order to express the index above as a topological expression we consider first some theorems.

**Theorem 2.8.** The operators $Q$ and $L_K$ satisfy the following relations

- $[Q, L_K] = 0,$
- $Q^2 = \mathcal{D}_M^2 + L_\alpha + L_\psi$

**Proof.** It is enough to proof this on an open set on $M$, we denote $\{z^k_r\}_{r=1}^d$ of $N = \oplus r \in A N_r$ as a orthonormal basis on $U$. By definition of the connection, see theorem 2.5, we have that $D_t^i z_j = w^i_j z_k$ and $D_t^i z_j = \bar{w}^i_j \bar{z}_k$. The action of these operators extend to $\mathcal{V}$. In particular, we have the following algebra from the commutator of operators $[D_{e_i}, z^k] = w_{ij}^k z_k$, $[D_{e_i}, d z^k] = w_{ij}^k dz_k$, $[D_{e_i}, i \partial z^k] = -w_{ik}^j \partial z_k$ and $[D_{e_i}, \partial z^k] = -w_{ik}^j \partial z_k$. And considering their conjugates we have

$$ [D_{e_i}, \bar{z}^k] = \bar{w}_{ij}^k \bar{z}_k, \quad [D_{e_i}, \bar{d} z^k] = \bar{w}_{ij}^k \bar{d} z_k, \quad [D_{e_i}, i \partial \bar{z}^k] = -\bar{w}_{ik}^j i \partial \bar{z}_k \quad \text{and} \quad [D_{e_i}, \partial \bar{z}^k] = -\bar{w}_{ik}^j \partial \bar{z}_k. $$

Therefore, we have the following relations for the operators in the definition 2.3

$$ [D_{e_i}, \psi_r^j] = -\bar{w}_{ik}^j \psi_{r}^k, \quad [D_{e_i}, \psi_r^j] = \bar{w}_{ij}^k \psi_{r}^k, $$

$$ [D_{e_i}, \alpha_r^j] = -\bar{w}_{ik}^j \alpha_{r}^k, \quad [D_{e_i}, \alpha_r^j] = \bar{w}_{ij}^k \alpha_{r}^k. $$
\[ [D_{e_i}, \beta^j] = -w_{ik}^j\beta^k_r, \quad [D_{e_i}, \beta^{-j}_r] = w_{ij}^k\beta^k_r. \]

This implies the following commutative relation

\[ [\mathcal{D}_M, \mathcal{D}_N] = \psi_0^2 [D_{e_i}, \psi^j] \alpha^j_{-r} - \psi_r^j \psi^j_0 [D_{e_i}, \alpha^j_{-r}] = \psi_0^j (-w_{ik}^j \psi^k_r) \alpha^j_{-r} + \psi_0^j \psi^j_0 (w_{ij}^k \alpha^k_{-r}) = 0, \]

and

\[ \mathcal{D}_N^2 = [\mathcal{D}_N, \mathcal{D}_N] = 2 \sum_{r \in A} \sum_{k=1}^{\dim N_r} \alpha^k_{-r} \alpha^k_r + r\psi^k_r \psi^k_r + r - r = 2L_\alpha + 2L_\psi. \]

This proves the second relation.

Now, we proof the first relation

\[ [\mathcal{D}_N, L_\alpha + L_\psi] = [\psi_r \alpha_{-r}, \alpha_n \alpha_{-n} + n\psi_n \psi_{-n}] = -r\psi_r \alpha_{-r} + r\psi_r \alpha_{-r} = 0, \quad [\mathcal{D}_N, L_\beta] = 0 \]

\[ [\mathcal{D}_M, L_\alpha] = \psi_0^j (-w_{ik}^j \alpha^k_n) \alpha^j_{-n} + \psi_0^j \alpha^j_0 (w_{ij}^k \alpha^k_n) = 0 \]

\[ [\mathcal{D}_M, L_\psi] = \psi_0^j (-w_{ik}^j \psi^k_n) \psi^j_{-n} + \psi_0^j \psi^j_0 (w_{ij}^k \psi^k_n) = 0 \]

\[ [\mathcal{D}_M, L_\beta] = \psi_0^j (-w_{ik}^j \beta^k_n) \beta^j_{-n} + \psi_0^j \beta^j_0 (w_{ij}^k \beta^k_n) = 0 \]

Therefore \([Q, L_K] = 0\)

Now, we have the following theorem

**Theorem 2.9 (susy).** \(Ker(Q) \subset V' := \Delta(M) \otimes \sqrt{\det N} \otimes S(N) \subset V\)

**Proof.** Note that the eigenvalues of \(L_\alpha\) and \(L_\psi\) are strictly positive on \(\wedge^k N \otimes S(N)\). Now, for any element \(v \in \Gamma(V)\)

\[(Qv, Qv) = (Q^2v, v) = (Q_M v, Q_M v) + (L_\alpha + L_\psi v, v) \geq (L_\alpha + L_\psi v, v)\]

where we used the theorem 2.7 and 2.8.

Finally, we have the theorem that express the index in terms of a topology expression

**Theorem 2.10.**

\[ \text{Ind}(Q, q) = q^{\frac{\alpha}{2}} \int_M A(M)ch(\sqrt{\det N} \otimes \otimes_{r \in A} S_{q_r}^B(N_r)) \]

**Proof.** From the theorem 2.9 we have that \(Ker(Q) \subset V'\) and by definition of \(Q\) we have \(Q|_{V'} = D_M|_{V'}\), therefore

\[ \text{Ind}(Q, q) = \text{Ind}(D_M|_{V'}, q) = \text{Tr}(q^{L_K}|_{Ker(D_M|_{V'})}) - \text{Tr}(q^{L_K}|_{Ker(D_M|_{V'})}) \]

Considering the theorem 2.8, the eigenvalues of the operators commute then

\[ q^{L_K} (\Delta(M) \otimes \sqrt{\det N} \otimes S(N)) = q^{\frac{\alpha}{2}} \Delta (M) \otimes \sqrt{\det N} \otimes \otimes_{r \in A} S_{q_r}^B(N_r). \]

And

\[ \text{Ind}(D_M|_{V'}, q) = q^{\frac{\alpha}{2}} \text{Ind}(D_M \otimes (\Delta(M) \otimes \sqrt{\det N} \otimes \otimes_{r \in A} S_{q_r}^B(N_r))). \]

Then the theorem follows from the Atiyah Singer index theorem for twisted bundles (1.1).

### 3. Localization, Infinite Dimensional Case

In this section we consider the infinite dimensional case, where \(X = LM\).
3.1. Vector space. Let \( X = LM \) be the loop space of \( M \), on the loop space there is a natural \( S^1 \)-action s.t the set of fixed points is given by \( M \). We work on the normal bundle of the manifold \( M \) inside \( X \), this bundle is defined as follows

\[
\mathcal{N} := \bigoplus_{n>0} TM_C = \bigoplus_{n>0} N_n, \quad N_n = TM_C.
\]

Unlike the finite dimensional case the normal bundle is trivially complex. And motivated by the finite dimensional case, we consider the vector space

\[
V_R := \Delta(M) \otimes_{n \in \mathbb{Z}_+} \Lambda^* \bar{N}_n \otimes n \in \mathbb{Z}_+ S(N_n) \otimes n \in \mathbb{Z}_+ S(\bar{N}_n) \rightarrow M.
\]

In this case, the complex bundles are isomorphic \( N_n \approx \bar{N}_n \approx TM_C \). As in the finite dimensional case the bundle \( V \) is \( \mathbb{Z}_2 \)-graded, the graded comes from the graded tensor product of the bundles \( \Delta(M) \) and \( \otimes_{n \in \mathbb{Z}_+} \Lambda^* \bar{N}_n \),

(3.1) \[
V_R^+ = \left( \Delta^+(M) \otimes_{n \in \mathbb{Z}_+} \Lambda^{even} \bar{N}_n \oplus \Delta^-(M) \otimes_{n \in \mathbb{Z}_+} \Lambda^{odd} \bar{N}_n \right) \otimes_{n \in \mathbb{Z}_+} S(N_n) \otimes_{n \in \mathbb{Z}_+} S(\bar{N}_n),
\]

(3.2) \[
V_R^- = \left( \Delta^-(M) \otimes_{n \in \mathbb{Z}_+} \Lambda^{even} \bar{N}_n \oplus \Delta^+(M) \otimes_{n \in \mathbb{Z}_+} \Lambda^{odd} \bar{N}_n \right) \otimes_{n \in \mathbb{Z}_+} S(N_n) \otimes_{n \in \mathbb{Z}_+} S(\bar{N}_n).
\]

3.1.1. Hermitian Inner product. Now, we define an hermitian inner product on \( V_R \), we proceed as in the section 2.1.1.

The spin bundle \( \Delta(M) \) has an hermitian inner product, as we already mentioned in the previous section.

The vector spaces \( N_n = TM_C \) for all \( n \in \mathbb{Z}_+ \) has a natural hermitian metric described as follows

(3.3) \[
\langle av, bu \rangle = a \bar{b} g_M(u, v),
\]

where \( a, b \in \mathbb{C} \), \( u, v \in \Gamma(TM) \) and \( g_M \) is the Riemannian metric on \( M \). The hermitian metric on \( N_n \) is the one above \( (n, m)_{N_n} := (n, m) \) and the hermitian metric on \( \bar{N}_n \) is given by \( (n, m)_{\bar{N}_n} := (\bar{m}, \bar{n}) \).

The hermitian structure on \( \Lambda^* \bar{N}_n \) is defined by (2.6) and the hermitian structure on \( S(N_n) \otimes S(\bar{N}_n) \) is defined by (2.8).

Finally, the bundle \( V_R \) is a tensor product of hermitian bundles, therefore \( V_R \) is an hermitian bundle and we denoted the hermitian product by \( \langle, \rangle \).

3.2. Operators. In this section, we define the operators acting on the bundle \( V_R \).

3.2.1. Connection. Like in the finite dimensional case on the bundle \( \Gamma(\Delta(M)) \) there is a connection \( D_M : \Gamma(\Delta(M)) \rightarrow \Gamma(\Delta(M) \otimes T^*M) \) coming from the Levi Civita connection on \( M \).

Theorem 3.1. The connection \( \nabla \) induce an hermitian connection \( D' \) s.t. \( D' : \Gamma(N_n) \rightarrow \Gamma(N_n \otimes T^*M) \) and \( D' : \Gamma(\bar{N}_n) \rightarrow \Gamma(\bar{N}_n \otimes T^*M) \). Additionally, the connection \( D' \) is compatible with the hermitian metric defined in (3.3).

The theorem follows by definition of \( N_n \cong TM \otimes \mathbb{C} \) as a complexification of a real bundle. We can use this theorem to build a natural Hermitian connections on the bundles \( \Lambda^* \bar{N}_n \), and from the theorem 2.6, we have a natural connection on the bundle \( S(N_n) \otimes S(\bar{N}_n) \). Therefore, from the tensor product of these connections, we obtain a connection on \( V_R \)

\[
D_M : \Gamma(V_R) \rightarrow \Gamma(V_R \otimes T^*M)
\]

And this connection is compatible with the Hermitian metric on \( V_R \).
3.2.2. Sheaves of operators. All the definitions and lemmas in the section 2.2.2 and section 2.2.3 works on $V_R$. In particular, we restate the definition 2.3 considering the maps are extended to $V_R$, in this case the maps are defined on $N_n$ for arbitrary $n \in \mathbb{Z}_+$. 

Definition 3.1.

$$
\varphi_n = i_n : N_n \to \text{End}(V_R), \\
a_n := i(\partial + \frac{n}{2}) : N_n \to \text{End}(V_R), \\
b_n := i(\partial - \frac{n}{2}) : N_n \to \text{End}(V_R), \\
\varphi_n = \wedge_n : \tilde{N}_n \to \text{End}(V_R), \\
a_{-n} := i(\partial - \frac{n}{2}) : \tilde{N}_n \to \text{End}(V_R), \\
b_{-n} := i(\partial - \frac{n}{2}) : N_n \to \text{End}(V_R).
$$

The lemmas 2.6 and 2.7 are satisfied too. The Clifford algebra on $\Delta(M)$ is extended to $V_R$

$$
\varphi_0 : TM \to \text{End}(V_R).
$$

And we have the property

$$(\varphi_0(x)v, w) = -(v, \varphi_0(x)w),$$

where $x \in \Gamma(TM_C)|_U$ and $v, w \in \Gamma(V_R)|_U$.

3.3. Global operators. Now, we define the global operators. In order to make this section clear, we write after each definition the operator in a local basis. We consider an open set $U \subset M$. The operators $a_r, b_{-r}, \varphi_{-r} : N_r \to \text{End}(V_R)$ and $b_r, a_{-r}, \varphi_r \in \tilde{N}_n \to \text{End}(V_R)$ for $r \in \mathbb{Z}_+$ are finite dimensional vector bundles. We express a basis for these operators considering an orthonormal basis $\{z_n^{k_1} \}_{k=1}^{\dim M}$ of $N_n$ for any $n \in \mathbb{Z}_+$ on the open set $U$ as follows

$$
\alpha_n^{k} := i(\partial_z^{k} + \frac{r}{2}z_n^{k}) : N_n \to \text{End}(V_R), \\
\alpha_{-n}^{k} := i(\partial_{\bar{z}}^{k} - \frac{r}{2}z_n^{k}) : \tilde{N}_n \to \text{End}(V_R), \\
\beta_n^{k} := i(\partial_{\bar{z}}^{k} + \frac{r}{2}z_n^{k}) : \tilde{N}_n \to \text{End}(V_R), \\
\beta_{-n}^{k} := i(\partial_z^{k} - \frac{r}{2}z_n^{k}) : N_n \to \text{End}(V_R).
$$

Because in this case we work with an infinite set $\mathbb{Z}_+$, instead of $A$, some operators analogous to the operators in the section 2.3 are not well defined. In order to solve this problem, we need to consider the normal ordered product. Note that the sections of the vector bundle $V_R$ on a local open chart $U$ can be expressed as follows

$$(V_R)|_U \cong C[\psi_{-1}, \psi_{-2}, ...] \otimes C[\alpha_{-1}, \alpha_{-2}, ...] \otimes C[\beta_{-1}, \beta_{-2}, ...] \otimes C^\infty(U),$$

where $k_1, k_2, ...j_1, j_2, ...i_1, i_2 \in \{1, ..., \dim M\}$. Using this local form of the vector space, the normal order product is defined as a composition of operators writing the annihilation operators $\alpha_n^{k}, \beta_n^{k}$ and $\psi_n^{k} (n > 0)$ to the right of the creation operators $\alpha_{-n}^{k}, \beta_{-n}^{k}$ and $\psi_{-n}^{k}$. We use the symbol $\mathcal{O}$ to indicate the normal ordering. For example, for $n > 0$ and arbitrary $k, j \in \{1, ..., \dim M\}$ we have that

$$
\alpha_{-n}^{k}\alpha_{j}^{k} := \alpha_{j}^{k}\alpha_{-n}^{k}, \\
\beta_{-n}^{k}\beta_{j}^{k} := \beta_{j}^{k}\beta_{-n}^{k}, \\
\psi_{-n}^{k}\psi_{j}^{k} := \psi_{j}^{k}\psi_{-n}^{k}.
$$

Note that even when we have defined the normal ordered product locally, the normal ordered product is well defined for the operators $a_r, b_{-r}, \varphi_{-r} : N_r \to \text{End}(V_R)$ and $b_r, a_{-r}, \varphi_r \in \tilde{N}_n \to \text{End}(V_R)$ for $r \in \mathbb{Z}_+$. Motivated for the definition of the operator $Q$ in the finite dimensional case we define the following operator

$$
Q \psi := \psi \Delta(\alpha^{k_1} \alpha^{j_1} ... \alpha^{k_l} \alpha^{j_l} \beta^{k_1} \beta^{j_1} ... \beta^{k_l} \beta^{j_l} \psi), \\
Q \psi \in \text{End}(V_R).
$$
Definition 3.2. The Ramond Dirac operator $Q_R$ is defined as a normal ordered product composition of operators
\[
\Gamma(V_R) \xrightarrow{D+\alpha_n+a_{-n}} \Gamma((T^*M \oplus \mathbb{Z}_+ N_n^* \oplus \tilde{N}_n^*) \otimes V_R) \rightarrow \Gamma((TM \oplus \mathbb{Z}_+ \tilde{N}_n \oplus N_n) \otimes V_R) \xrightarrow{\varphi_n + \varphi_{-n} + \bar{\varphi}_n} \Gamma(V_R)
\]
where $TM \simeq T^*M$ and the hermitian metric (2.3) to identify $N_n^* \simeq \tilde{N}_n$ and $\tilde{N}_n^* \simeq N_n$.

In the local basis we have
\[
Q_R|_U = \psi_0 \nabla e_i + \sum_{n \in \mathbb{Z}_+} \psi_{-n}^k \alpha_n^k + \alpha_{-n}^k \psi_n^k.
\]
The normal ordered product is used in order to make the operator well defined.

Analogously, the operators $2L_\alpha$ and $2L_\psi$ are defined as the normal ordered product of the following composition of operators
\[
\begin{align*}
\Gamma(V) & \xrightarrow{a_n + a_{-n}} \Gamma((\oplus_{n \in \mathbb{Z}_+} N_n^* \oplus \tilde{N}_n^*) \otimes V) \rightarrow \Gamma((\oplus_{n \in \mathbb{Z}_+} \tilde{N}_n \oplus N_n) \otimes V) \xrightarrow{a_{-n} + a_n} \Gamma(V), \\
\Gamma(V) & \xrightarrow{\bar{\varphi}_n + \varphi_{-n}} \Gamma((\oplus_{n \in \mathbb{Z}_+} N_n^* \oplus \tilde{N}_n^*) \otimes V) \rightarrow \Gamma((\oplus_{n \in \mathbb{Z}_+} \tilde{N}_n \oplus N_n) \otimes V) \xrightarrow{n\varphi_{-n} + \bar{\varphi}_n} \Gamma(V),
\end{align*}
\]
where we use the hermitian metric (2.3) to identify $N_n^* \simeq \tilde{N}_n$ and $\tilde{N}_n^* \simeq N_n$.

Analogously for $L_\beta$.

Definition 3.3. The momentum operator is defined by
\[
P := L_\beta - L_\alpha - L_\psi - \frac{\dim M}{24} \in \text{End}(V_R).
\]

In the finite dimensional case the extra term is given by $c_1 = \sum_{r \in A} \sum_{k=1}^{d_r} r$, this term does not make sense in this case. Instead, we consider the central charge of the Neveu-Schwarz twisted module or Ramond fermions. In the local basis we have
\[
P|_U = \sum_{n \in \mathbb{Z}_+} \beta_{-n}^\alpha_n - \alpha_{-n}^\beta_n - n\psi_{-n}^\psi_n - \frac{\dim M}{24}.
\]

3.4. The index and the supersymmetry. The hermitian inner product for sections $\Gamma(V_R)$ is defined analogously like in the section 2.4
\[
<v, w> := \int_M (v(x), w(x)), \quad \forall v(x), w(x) \in \Gamma(V)
\]

And in the same way we have

Theorem 3.2. The operator $Q_R$ satisfy
\[
<v, w> = <Q_Rv, w>, \quad \forall v(x), w(x) \in \Gamma(V)
\]

\footnote{This assumption is usual in the physics literature since this value is the same that appear from the zeta regularization. From the math point of view, the central charge appears considering a change of coordinates to the torus, see [11] chapter 6.}
For $V_R = V_R^+ \oplus V_R^-$, we define $Q_+ := Q_R|_{V_R^+}$ and $Q_- := Q_R|_{V_R^-}$, then

$$Q_+: V_R^+ \to V_R^-, \quad Q_-: V_R^- \to V_R^+$$

and from the self adjointness of $Q_R$, we have $Q_+^\dagger = Q_-$. Considering the action of $q^{L_R}$ on sections of $V_R$ where $q$ is a formal variable, we define the index as follows

$$\text{Ind}(Q_R, q) := \text{Tra}(q^P|_{\text{Ker}(Q_+^\dagger)}) - \text{Tra}(q^P|_{\text{Ker}(Q_-^\dagger)}).$$

The proof of the following theorems can be translated *mutatis mutandis* from the theorems 2.8, 2.9 and 2.10.

**Theorem 3.3.** The operators satisfy the following relations

- $[Q_R, P] = 0$
- $Q_R^2 = Q^2_M + L_\alpha + L_\psi$

**Theorem 3.4 (susy).** $\text{Ker}(Q_R) \subset V_R' := \Delta(M) \otimes S(N) \subset V$

And, we have the theorem that express the index in terms of a topology expression

**Theorem 3.5.**

$$\text{Ind}(Q_R, q) = q^{-\frac{\dim M}{24}} \int_M A(M) \text{ch}(\bigotimes_{n=1}^\infty S_{q^n}(TM_C)) = \frac{\Phi(M)}{\eta(q)^{\dim M}}.$$
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