Derivation of stiffness matrix in constitutive modeling of magnetorheological elastomer

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Abstract. Magnetorheological elastomers (MREs) are a class of smart materials whose mechanical properties change instantly by the application of a magnetic field. Based on the specially orthotropic, transversely isotropic stress-strain relationships and effective permeability model, the stiffness matrix of constitutive equations for deformable chain-like MRE is considered. To valid the components of shear modulus in this stiffness matrix, the magnetic-structural simulations with finite element method (FEM) are presented. An acceptable agreement is illustrated between analytical equations and numerical simulations. For the specified magnetic field, sphere particle radius, distance between adjacent particles in chains and volume fractions of ferrous particles, this constitutive equation is effective to engineering application to estimate the elastic behaviour of chain-like MRE in an external magnetic field.

1. Introduction
Magnetorheological (MR) or magneto-sensitive (MS) elastomers are smart materials whose mechanical properties response rapidly with the external magnetic stimulus. These intelligent materials typically consist of micro-sized ferrous particles dispersed in an elastomer matrix. Subjected to uniaxial compression or shear, MREs have been utilized as a single non-linear variable stiffness spring in some applications such as adaptive tuned vibration absorbers, suspensions and automotive bushing [1, 2].

Curing without or with magnetic field, magnetorheological elastomers present isotropic or anisotropic property, respectively. For these two types of MREs, the compression and shear stiffness under uniform magnetic field have been measured by experiment [3-5]. In addition, several

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mathematical models for isotropic MRE are provided to investigate its constitutive relationship [6-9]. Most of the recent stress energy functions and constitutive equations of MRE were based on the classic work [10] in which the equations of motion for an isotropic non-polar continuum in an electromagnetic field are described by Maxwell’s equations and the mechanical and electro-magnetic-structural field for homogenous particle dispersed elastomers. However, it is demonstrated that the chain-like MREs have more obvious magnetic-field response effect. Hence, exploring the constitutive relationship of anisotropic MREs is more significant.

In this paper, based on the anisotropic stress-strain relationships and effective permeability model, the stiffness matrix for elastic deformable chain-like MRE is considered. To validate the constitutive equations of chain-like MRE, the magnetic-structural simulations by finite element method are presented.

2. Constitutive equations of chain-like MRE
For the anisotropic elastic material, the most linear general stress-strain relationships are given by equations of the form [11],

$$\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{ij}$$

(1)

where $\sigma_{ij}$ is the stress tensor having nine components, $C_{ijkl}$ is the stiffness tensor having eighty-one components, $\varepsilon_{ij}$ is the strain tensor having nine components.

As shown in any mechanics of material book, both stresses and strains are symmetric, so that there are only six independent stress components and six independent strain components. This means that the elastic constants must be symmetric, and that the number of nonzero elastic constants is reduced to twenty-one. Hence, the generalized Hooke’s law can now be simplified as,

$$\sigma_i = C_{ij} \cdot \varepsilon_j, \ i, j = 1,2 \ldots ,6$$

(2)

MRE is a typical material that employs before yield condition, and it exhibits elastic properties. Hence, the constitutive equations of MR elastomers can be expressed as equation (2). Further simplification is possible, as MRE has three mutually orthogonal planes of material properties symmetry (or orthotropic material). Considering a general 3D state of stress consisting of all possible normal and shear stresses for orthotropic materials, the resulting set of equations is given below,

$$\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E_1} & -\frac{V_{21}}{E_1} & -\frac{V_{31}}{E_1} & 0 & 0 & 0 \\
-\frac{V_{12}}{E_2} & \frac{1}{E_2} & -\frac{V_{32}}{E_2} & 0 & 0 & 0 \\
-\frac{V_{13}}{E_3} & -\frac{V_{23}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_{31} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_{12}
\end{bmatrix} \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix}$$

(3)
For the MR elastomers cured in an external magnet, the chain-like structure of particles is formed. It exhibits specially orthotropic, transversely isotropic material property, as shown in Figure 1.

![Figure 1](image.png)

**Figure 1.** The material plane of chain-like MREs.

For specially orthotropic, transversely isotropic material, the stiffness matrix have twelve nonzero coefficients with five independent constants. The subscripts 2 and 3 in equation (3) can be interchangeable,

\[
G_{13} = G_{12} \quad (4)
\]

\[
E_2 = E_3 \quad (5)
\]

\[
\nu_{21} = \nu_{31} \quad (6)
\]

\[
\nu_{23} = \nu_{32} \quad (7)
\]

In addition, the familiar relationship among the isotropic engineering constants is now valid for engineering constants associated with the 23 plane, so that [11]

\[
G_{23} = \frac{E_2}{2(1 + \nu_{32})} \quad (8)
\]

Hence, the equation (3) can be simplified as,

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
1/E_1 & -\nu_{21}/E_2 & -\nu_{21}/E_2 & 0 & 0 & 0 \\
-\nu_{12}/E_1 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0 \\
-\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_{12}
\end{bmatrix}
\begin{bmatrix}
\gamma_{12} \\
\gamma_{23} \\
\gamma_{31}
\end{bmatrix} \quad (9)
\]
An effective permeability model of MR elastomers is introduced to estimate the shear modulus in chain direction, $G_{23}$ [12]. This model is based on the Maxwell Garnett mixing rule [13], the field-induced shear modulus of chain direction can be expressed as [12],

$$ G = 6 \phi_p \cdot \mu_0 \cdot \mu_m \cdot H_0^2 \quad (10) $$

where $\phi_p$ is the volume fraction of particles, $\mu_0$ is the permeability of vacuum, $\mu_m$ is the permeability of matrix, $H_0$ is the magnetic field intensity. Before ferrous particle achieves magnetic saturation,

$$ H_0 = \frac{B}{\mu_{\text{eff}}} \quad (11) $$

where $B$ is the magnetic induction density, and $\mu_{\text{eff}}$ is the effective relative permeability of MREs,

$$ \mu_{\text{eff}} = \mu_{\text{eff}} \cdot \phi_c + \mu_m \cdot (1 - \phi_c) \quad (12) $$

where $\phi_c$ is the volume fraction of chain structure in MREs,

$$ \phi_c = \frac{3 \phi_p \cdot d}{4R} \quad (13) $$

where $d$ is the distance between adjacent particles in chain structure, $R$ is the radius of ferrous particle, and $\mu_{\text{eff}}$ is the effective permeability of chains, it can be predicted as [13],

$$ \mu_{\text{eff}} = \mu_m + 2 \cdot \phi \cdot \frac{\mu_p - \mu_m}{\mu_p + \mu_m - \phi (\mu_p - \mu_m)} \quad (14) $$

where $\phi$ is the volume fraction of particles in chain-structure,

$$ \phi = \frac{4R}{3d} \quad (15) $$

Based on equation (12) to equation (15),

$$ \mu_{\text{eff}} = \mu_m + 2 \cdot \phi_p \cdot \frac{\mu_p - \mu_m}{\mu_p + \mu_m - \frac{4R}{3d} (\mu_p - \mu_m)} \quad (16) $$

The shear modulus in chain structure, $G_{23}$, can be obtained,

$$ G_{23} = G_0 + G' \quad (17) $$
where $G_0$ is the shear modulus of the MRE under zero magnet,

$$G_0 = G_m \cdot \left( 1 + 2.5 \phi_p + 14.1 \phi_p^2 \right)$$  \hspace{1cm} (18)

where $G_m$ is the shear modulus of the elastomer matrix.

Through equation (9) to equation (13), the shear modulus in chain structure can be obtained,

$$G_{23} = G_m \cdot \left( 1 + 2.5 \phi_p + 14.1 \phi_p^2 \right) + 6 \phi_p \cdot \mu_n \cdot \frac{B^2}{\left( \mu_n + 2 \phi_p \cdot \mu_n + 4R \frac{\mu_p - \mu_n}{3d} \right)}$$  \hspace{1cm} (19)

According to equation (8), the Young’s modulus of chain structure, is

$$E_1 = 2 \cdot (1 + \nu_{12}) \cdot G_m \cdot \left( 1 + 2.5 \phi_p + 14.1 \phi_p^2 \right) + 6 \phi_p \cdot \mu_n \cdot \frac{B^2}{\left( \mu_n + 2 \phi_p \cdot \mu_n + 4R \frac{\mu_p - \mu_n}{3d} \right)^2}$$  \hspace{1cm} (20)

In the material plane 2 and 3 which are perpendicular to the chain direction, as shown in Fig. 1, the mutual magnetic force between ferrous particles are far less than that of adjacent particles in chain structure [14]. Hence, the shear modulus, $G_{12}$, is predicted as particle reinforced composites [15].

$$G_{12} = G_m \cdot \left( 1 + 2.5 \phi_p + 14.1 \phi_p^2 \right)$$  \hspace{1cm} (21)

Based on equation (8), the Young’s modulus of chain structure, $E_{21}$, is

$$E_2 = 2G_m \cdot \left( 1 + 2.5 \phi_p + 14.1 \phi_p^2 \right) \cdot (1 + \nu_{21})$$  \hspace{1cm} (22)

Through equation (16) to equation (19), each modulus and the stress-strain relationship of chain-like MREs, under different magnetic field, can be obtained. Based on the Passion’s ration, the Young’s modulus can be obtained through shear modulus. For the specified magnetic field, sphere particle radius, distance between adjacent particles in chains and volume fractions of ferrous particles, above-mentioned equations can predict the elastic stiffness matrix of MREs with aligned chains of particles, and then the stress-strain relationships can be obtained.

However, the accuracy of the shear modulus in both chain direction and vertical to chain direction is significant to this constitutive equation. Hence, it will be discussed below.

3. Numerical simulations in finite element method

The magnetic-structural simulation of chain-like MREs by finite element method is presented, in order to valid $G_{23}$ and $G_{12}$ in analytical method. The simulation flowchart of magnetic-structural method is illustrated in Figure. 2. In static simulation, ANSYS/EMAG is used to obtain the magnetic forces in an external magnetic field. Then the forces data are used as a boundary condition to simulate the
deformation of MRE specimen in ANSYS/STRUC. In this numerical simulation, the unit cell mode is used, and the periodic boundary conditions are applied. The cycle of simulation ends when the deformation of specimen keeps steadily. The parameters of unit cell model are shown in Table 1.

![Simulation Flowchart](image)

**Figure 2.** The simulation flowchart of coupling method.

| Part        | Young’s modulus (G Pa) | Shear modulus (M Pa) | Passion’s ratio | Density (kg m⁻³) | Size (mm) | Relative Permeability |
|-------------|------------------------|----------------------|-----------------|------------------|-----------|----------------------|
| Particle    | 210                    | 0.3                  |                 | 7800             | 4(Radius) | 1000                 |
| Elastomer   | 0.0098                 | 0.499                |                 | 2870             |           | 1                    |

In the 3D model, it is assumed that the volume fraction of micro-particles in chain-like MREs, $V_p$, is 27%. This volume fraction is predicted as the optimum one [15]. As [15] referred, the distance between the adjacent particles in chains, $d$, can be calculated,

$$d = 3.1 \times R$$  \hspace{1cm} (23)

Hence, $d$ is 8.4 mm. It is also assumed that the arrangement of particles in chain-like MREs is 3*3*3 rows, in total twenty-seven particles. The 3D unit cell model is shown in Figure 3. The Z direction is the chain direction.

![3D Model](image)

**Figure 3.** (a) The 3D MRE specimen and (b) one half of 3D model.
4. Results and discussion

The deformation of MREs in shear condition of both chain direction and vertical to chain direction is shown in Figure 4.

![Figure 4. (a)The shear deformation of MREs in chain direction, (b) the shear deformation in vertical to chain direction.](image)

Through equation (14), the relative effective permeability of MRE is 2.73. According to equation (19), the shear modulus of chain direction can be calculated, as Table 2 shows. The field dependence of shear modulus in chain direction in MREs is shown in Figure 5.

| Magnetic induction density (mT) | Shear Modulus of chain direction (M Pa) |
|--------------------------------|----------------------------------------|
|                                | Analytical method                      |
| 100                            | 0.262                                  |
| 200                            | 0.273                                  |
| 300                            | 0.283                                  |
| 400                            | 0.297                                  |
| 500                            | 0.315                                  |
From Table 2 and Figure 5, it can be seen that there is a same tendency of the shear modulus in chain direction of MREs between analytical equations and the finite element method. As the magnetic field intensity increases, the modulus increases correspondingly.

According to equation (21), the shear modulus in the material plane 2 and 3 which are perpendicular to the chain direction can be calculated, as Table 4 shows. The field dependence of shear modulus in material plane 2 and 3 is shown in Figure 6.

**Table 3.** Shear modulus in material plane 2 and 3 of MREs.

| Magnetic induction density (mT) | Shear Modulus in material plane 2 and 3 (MPa) |
|--------------------------------|----------------------------------------------|
|                                | Analytical method | FEM |
| 100                            | 0.26             | 0.162 |
| 200                            | 0.26             | 0.163 |
| 300                            | 0.26             | 0.171 |
| 400                            | 0.26             | 0.174 |

**Figure 5.** Chain-like shear modulus at different magnetic induction density.

**Figure 6.** Shear modulus in material 2 and 3 at different magnetic induction density.
From Table 3 and Figure 6, it can be seen that the shear modulus in material plane 2 and 3, in both analytical method and finite element method in low magnetic field intensity, is independent on the external magnetic field, nearly a constant. From Figure 5 and Figure 6, it can be seen that the results of shear modulus in two directions between the analytical method and numerical method have the same tendency.

5. Conclusions
In this paper, it applied the specially orthotropic, transversely isotropic stress-strain relationship and an effective permeability model to predict the stiffness matrix of chain-like MREs. To valid this model, the finite element method of magnetic-structural simulations is presented. An acceptable agreement is obtained between analytical equations and numerical simulations. For the specified magnetic field, sphere particle radius, distance between adjacent particles in chains and volume fractions of ferrous particles, the full system of constitutive equations for elastic deformable chain-like MRE is considered.

In the future, the analyses of constitutive equations at high magnetic field intensity should be studied. Also, the authors will intend to predict the Passion’s ratio of MREs to obtain more complete stress-strain equation related.

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