Polynomial stochastic games
via sum of squares optimization

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Abstract

Stochastic games are an important class of problems that generalize Markov decision processes to game theoretic scenarios. We consider finite state two-player zero-sum stochastic games over an infinite time horizon with discounted rewards. The players are assumed to have infinite strategy spaces and the payoffs are assumed to be polynomials. In this paper we restrict our attention to a special class of games for which the single-controller assumption holds. It is shown that minimax equilibria and optimal strategies for such games may be obtained via semidefinite programming.

I. INTRODUCTION

Markov decision processes (MDPs) are very widely used system modeling tools where a single agent attempts to make optimal decisions at each stage of a multi-stage process so as to optimize some reward or payoff [1]. Game theory is a system modeling paradigm that allows one to model problems where several (possibly adversarial) decision makers make individual decisions to optimize their own payoff [2]. In this paper we study stochastic games [3], a framework that combines the modeling power of MDPs and games. Stochastic games may be viewed as competitive MDPs where several decision makers make decisions at each stage to maximize their own reward. Each state of a stochastic game is a simple game, but the decisions made by the players affect not only their current payoff, but also the transition to the next state.

This research was funded in part by AFOSR MURI subawards 2003-07688-1 and 102-1080673.
Notions of solutions in games have been extensively studied, and are very well understood. The most popular notion of a solution in game theory is that of a Nash equilibrium. While these equilibria are hard to compute in general, in certain cases they may be computed efficiently. For games involving two players and finite action spaces, mixed strategy minimax equilibria always exist (see, e.g., [2]). These minimax saddle points correspond to the well-known notion of a Nash equilibrium. From a computational standpoint such games are considered tractable because Nash equilibria may be computed efficiently via linear programming. Stochastic games were introduced by Shapley [4] in 1953. In his paper, he showed that the notion of a minimax equilibrium may be extended to stochastic games with finite state spaces and strategy sets. He also proposed a value iteration-like algorithm to compute the equilibria. In 1981 Parthasarathy and Raghavan [5], [3] studied single controller games. Single controller games are games where the probabilities of transitions are controlled by the action of only one player. They showed that stochastic games satisfying this property could be solved efficiently via linear programming (thus proving that such problems with rational data could be computed in a finite number of steps).

While computational techniques for finite games are reasonably well understood, there has been some recent interest in the class of infinite games; see [6], [7] and the references therein. In this important class, players have access to an infinite number of pure strategies, and the players are allowed to randomize over these choices. In a recent paper [6], Parrilo describes a technique to solve two-player, zero-sum infinite games with polynomial payoffs via semidefinite programming. It is natural to wonder whether the techniques from finite stochastic games can be extended to infinite stochastic games (i.e. finite state stochastic games where players have access to infinitely many pure strategies). In particular, since finite, single-controller, zero-sum games can be solved via linear programming, can similar infinite stochastic games be solved via semidefinite programming? The answer is affirmative, and this paper focuses on establishing this result.

The main contribution of this paper is to provide a computationally efficient, finite dimensional characterization of the solution of single-controller polynomial stochastic games. For this, we extend the linear programming formulation that solves the finite action single-controller stochastic game (i.e., under assumption (SC) below), to an infinite dimensional optimization problem when the actions are uncountably infinite. We furthermore establish the following properties of this
infinite dimensional optimization problem:

1) Its optimal solutions correspond to minimax equilibria.

2) The problem can be solved efficiently by semidefinite programming.

Section II of this paper provides a formal description of the problem and introduces the basic notation used in the paper. We show that for two-player zero-sum polynomial stochastic games, equilibria exist and that the corresponding equilibrium value vector is unique. (This proof is essentially an adaptation of the original proof by Shapley in [4] for finite stochastic games). In Section III we also briefly review some elegant results about polynomial nonnegativity, moment sequences of nonnegative measures, and their connection to semidefinite programming. In Section III we briefly review the linear programming approach to finite stochastic games. Section IV states and proves the main result of this paper. In Section V we present an example of a two-player, two-state stochastic game, and compute the equilibria via semidefinite programming. Finally, in Section VI we state some natural extensions of this problem, conclusions, and directions of future research.

II. PROBLEM DESCRIPTION

A. Stochastic games

We consider the problem of solving two-player zero-sum stochastic games via mathematical programming. The game consists of finitely many states with two adversarial players that make simultaneous decisions. Each player receives a payoff that depends on the actions of both players and the state (i.e. each state can be thought of as a particular zero-sum game). The transitions between the states are random (as in a finite state Markov decision process), and the transition probabilities in general depend on the actions of the players and the current state. The process runs over an infinite horizon. Player 1 attempts to maximize his reward over the horizon (via a discounted accumulation of the rewards at each stage) while player 2 tries to minimize his payoff to player 1. If \((a_1^1, a_2^1, \ldots)\) and \((a_1^2, a_2^2, \ldots)\) are sequences of actions chosen by players 1 and 2 resulting in a sequence of states \((s_1, s_2, \ldots)\) respectively, then the reward of player 1 is given by:

\[
\sum_{k=1}^{\infty} \beta^k r(s_k, a_1^k, a_2^k).
\]

The game is completely defined via the specification of the following data:
Fig. 1. A two state stochastic game. The payoff functions associated to the states are denoted by \( r_1 \) and \( r_2 \). The edges are marked by the corresponding state transition probabilities.

1) The (finite) state space \( S = \{1, \ldots, S\} \).
2) The sets of actions for players 1 and 2 given by \( A_1 \) and \( A_2 \).
3) The payoff function, denoted by \( r(s, a_1, a_2) \), for a given set of state \( s \) and actions \( a_1 \) and \( a_2 \) (of players 1 and 2).
4) The probability transition matrix \( p(s'; s, a_1, a_2) \) which provides the conditional probability of transition from state \( s \) to \( s' \) given players’ actions.
5) The discount factor \( \beta \), where \( 0 \leq \beta < 1 \).

To fix ideas, consider the following example of a two-state stochastic game (i.e. \( S = \{1, 2\} \)). The action spaces of the two players are \( A_1 = A_2 = [0, 1] \). The payoff function in state 1 is \( r(1, a_1, a_2) = r_1(a_1, a_2) \) and the payoff function in state 2 is given by \( r(2, a_1, a_2) = r_2(a_1, a_2) \). Both are assumed to be polynomials in \( a_1 \) and \( a_2 \). The probability transition matrix is:

\[
P = \begin{bmatrix}
p_{11}(a_1, a_2) & p_{12}(a_1, a_2) \\
p_{21}(a_1, a_2) & p_{22}(a_1, a_2)
\end{bmatrix}.
\]

Every entry in this matrix is assumed to be a polynomial in \( a_1 \) and \( a_2 \). This stochastic game can be depicted graphically as shown in Fig. 1. We will return to a specific instance of this example in Section [V] where we explicitly solve for the equilibrium strategies of the two players.

Through most of this paper (except Section [II-C]) we make the following important assumption about the probability transition matrix:

**Assumption SC**

The probability transition to state \( s' \) conditioned upon the current state being \( s \) depends only on \( s, s' \), and the action \( a_1 \) of player 1 for every \( s \) and \( s' \). This probability is *independent of the action*.
of player 2. Thus, \( p(s'; s, a_1, a_2) = p(s'; s, a_1) \). This is known as the single-controller assumption.

In this paper we will mostly (except briefly, in Section III where finite strategy spaces are considered) be concerned with the case where the action spaces \( A_1 \) and \( A_2 \) of the two players are uncountably infinite sets. For the sake of simplicity we will often consider the case where \( A_1 = A_2 = [0, 1] \in \mathbb{R} \). The results easily generalize to the case where the strategy sets are finite unions of arbitrary intervals of the real line. For the sake of simplicity, we also assume that the action sets are the same for each state, though this assumption may be relaxed. We will denote by \( a_1 \) and \( a_2 \), the actual actions chosen by players 1 and 2 from their respective action spaces. The payoff function is assumed to be a polynomial in the variables \( a_1 \) and \( a_2 \) with real coefficients:

\[
    r(s, a_1, a_2) = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} r_{ij}(s) a_1^i a_2^j.
\]

Finally, we assume that the transition probability \( p(s'; s, a_1) \) is a polynomial in the action \( a_1 \).

The decision process runs over an infinite horizon, thus it is natural to restrict one’s attention to stationary strategies for each player, i.e. strategies that depend only on the state of the process and not on time. Moreover, since the process involves two adversarial decision makers, it is also natural to look for randomized strategies (or mixed strategies) rather than pure strategies so as to recover the notion of a minimax equilibrium. A mixed strategy for player 1 is a finite set of probability measures \( \mu = [\mu(1), \ldots, \mu(S)] \) supported on the action set \( A_1 \). Each probability measure corresponds to a randomized strategy for player 1 in some particular state, for example \( \mu(k) \) corresponds to the randomized strategy that player 1 would use when in state \( k \). Similarly, player 2’s strategy will be represented by \( \nu = [\nu(1), \ldots, \nu(S)] \). (A word on notation: Throughout the paper, indices in parentheses will be used to denote the state. Bold letters will be used indicate vectorization with respect to the state, i.e., collection of objects corresponding to different states into a vector with the \( i^{th} \) entry corresponding to state \( i \). The Greek letters \( \xi, \mu, \nu \) will be used to denote measures. Subscripts on these Greek letters will be used to denote moments of the measures. A bar over a greek letter indicates a (finite) moment sequence (the length of the sequence being clear from the context). For example \( \xi_j(i) \) denotes the \( j^{th} \) moment of the measure \( \xi \) corresponding to state \( i \), and \( \bar{\xi}(i) = [\xi_0(i), \ldots, \xi_n(i)] \).
A strategy $\mu$ leads to a probability transition matrix $P(\mu)$ such that $P_{ij}(\mu) = \int_{A_1} p(j; i, a_1) d\mu(i)$. Thus, once player 1 fixes a strategy $\mu$, the probability transition matrix is fixed, and can be obtained by integrating each entry in the matrix with respect to the measure $\mu$. (Since the entries are polynomials, upon integration, these entries depend affinely on the moments $\mu(i)$). Given strategies $\mu$ and $\nu$, the expected reward collected by player 1 in some stage $s$ is given by:

$$r(s, \mu(s), \nu(s)) = \int_{A_1} \int_{A_2} r(s, a_1, a_2) d\mu(s) d\nu(s).$$

The reward collected over the infinite horizon (for fixed strategies $\mu(s)$ and $\nu(s)$) starting at state $s$, $v_\beta(s, \mu(s), \nu(s))$, is given by the system of equations:

$$v_\beta(s, \mu(s), \nu(s)) = r(s, \mu(s), \nu(s)) + \beta \sum_{s' \in S} \left( \int_{A_1} p(s'; s, a_1) d\mu(s) \right) v_\beta(s', \mu(s'), \nu(s')) \quad \forall s.$$

Vectorizing $v_\beta(s, \mu(s), \nu(s))$, we obtain

$$v_\beta(\mu, \nu) = (I - \beta P(\mu))^{-1} r(\mu, \nu),$$

where $r(\mu, \nu) = [r(1, \mu(1), \nu(1)), \ldots, r(S, \mu(S), \nu(S))] \in \mathbb{R}^S$.

**B. Solution Concept**

We now briefly discuss the question: “What is a reasonable solution concept for stochastic games?” Recall that for zero-sum normal form games, a Nash equilibrium is a widely used notion of equilibrium in competitive scenarios. A Nash equilibrium in a two-player game is a pair of independent randomized strategies (say $\mu$ and $\nu$, one for each player) such that, given player 2 plays the $\nu$, player 1’s best response would be to play $\mu$ and vice-versa. It is an easy exercise that computation of Nash equilibria is equivalent to finding saddle points of the payoff-function. It is also well-known that Nash equilibria (or equivalently saddle points) correspond to the minimax notion of an equilibrium, i.e. points that satisfy the following equality:

$$\min_\mu \max_\nu v(\mu, \nu) = \max_\nu \min_\mu v(\mu, \nu).$$

While there may exist no pure strategies that satisfy this equality, it may be achieved by allowing randomization over the allowable strategies.

In his seminal paper [4], Shapley generalized the notion of Nash equilibria to stochastic games. He defined the notion of a “stationary equilibrium” to be a pair of randomized strategies (over
the action space) that depended only on the state of the game. (Of course, to be an equilibrium, these mixed strategies must also satisfy the no-deviation principle). For stochastic games, once one restricts attention to stationary equilibria, instead of having unique “values” (as in normal form games), one has a unique “value vector”. This vector is indexed by the state and the $i^{th}$ component is interpreted as the equilibrium value Player 1 can expect to receive (over the infinite discounted process) conditioned on the fact that the game starts in state $i$. Note that different states of the game may be favorable to different players. Since the actions affect both payoffs and state transitions, players must balance their strategies so that they receive good payoffs in a particular state along with favorable state transitions. The “no unilateral deviation” principle, saddle point inequality (interpreted row-wise, i.e., conditioned upon a particular state) and the equivalence of the minmax and maxmin over randomized strategies all extend to the stochastic game case, and when we restrict attention to games with just one state, we recover the classical notions of equilibrium.

**Definition 1:** A pair of vector of mixed strategies (indexed by the state) $\mu^0$ and $\nu^0$ which satisfy the saddle point property:

$$v_\beta(\mu, \nu^0) \leq v_\beta(\mu^0, \nu^0) \leq v_\beta(\mu^0, \nu)$$

for all (vectors of) mixed strategies $\mu, \nu$ are called *equilibrium strategies*. The corresponding vector $v_\beta(\mu^0, \nu^0)$ is called the *value vector* of the game.

One should note that $v_\beta(\mu, \nu)$ is a vector in $\mathbb{R}^S$ indexed by the initial state of the Markov process. Hence the above inequality is a vector inequality and is to be interpreted componentwise. More precisely, if $\mathcal{A}$ is the action space, let $\Delta(\mathcal{A})$ denote the space of probability measures supported on $\mathcal{A}$. Then the function $v_\beta$ is a function of the form:

$$v_\beta : \Pi_{i=1}^S \Delta(\mathcal{A}) \times \Pi_{i=1}^S \Delta(\mathcal{A}) \to \mathbb{R}^S,$$

and equilibrium strategies correspond to the saddle-points of this function. The mixed strategies of the players are indexed by the state (i.e. there is one probability measure per state per player). These probability measures (conditioned upon the state) are independent across states, and are also independent across the players.
C. Existence of Equilibria

In his original paper, Shapley [4] showed that stationary equilibria always exist (and that the corresponding value-vectors are unique) for two-player, zero-sum, finite state, finite action stochastic games. (Shapley considered games where at each state there was some probability of termination, where as in this paper we consider games over an infinite horizon with discounted rewards, as already mentioned. These two formulations are equivalent in the sense that starting from a discounted game one can construct a game with termination probabilities and vice-versa such that both have the same equilibrium value vectors.) In this subsection we address the existence and uniqueness issue, and prove that for two-player, zero-sum stochastic games over finite state spaces, infinite strategy spaces, and polynomial payoffs, stationary equilibria always exist, and that the value vectors are unique. Throughout the paper, we assume that the transition probabilities are polynomial functions of the actions of the players. It is important to note that the results of this subsection do not depend upon the single-controller assumption. As a by-product of this proof, we obtain a simple algorithm for computing equilibria for all such games. This algorithm is analogous to policy-iteration in dynamic programming, and consists of solving a sequence of simple (non-stochastic) games whose value-vectors converge to the true value vector.

Let \( p(x, y) \) be a polynomial, and \( A = [0, 1] \) be the strategy space of players 1 and 2. Let \( \text{val}(p(x, y)) \) be the value of the zero-sum polynomial game with the payoff function as \( p(x, y) \) and the strategy space \( A \). It can be shown that a mixed-strategy Nash equilibrium always exists for two-player zero-sum polynomial games [8], and they can be computed using semidefinite programming [6].

**Lemma 1:** Let \( p_1(x, y) \) and \( p_2(x, y) \) be given polynomials. Then

\[
|\text{val}(p_1(x, y)) - \text{val}(p_2(x, y))| \leq \max_{x, y \in [0, 1]} |p_1(x, y) - p_2(x, y)|.
\]

**Proof:** Let \( \mu_1, \nu_1 \) be the optimal strategies for the polynomial zero-sum game with payoff \( p_1(x, y) \) (so that \( E_{\mu_1, \nu_1} [p_1(x, y)] = \text{val}(p_1(x, y)) \)) and \( \mu_2, \nu_2 \) be the optimal strategies for the game with payoff \( p_2(x, y) \). If \( \text{val}(p_1) = \text{val}(p_2) \) the result is trivial, so without loss of generality, assume that \( \text{val}(p_1) > \text{val}(p_2) \). By the saddle point property,

\[
\int p_1(x, y) d\mu_1 d\nu_2 \geq \int p_1(x, y) d\mu_1 d\nu_1 \geq \int p_2(x, y) d\mu_2 d\nu_2 \geq \int p_2(x, y) d\mu_1 d\nu_2.
\]
Here the first inequality follows by considering $\nu_2$ to be a deviation of player 2 from his optimal strategy (i.e., $\nu_1$) for the game with payoff $p_1$, the second inequality follows by the preceding assumption, and the third inequality follows from a deviation argument for player 1 from his optimal strategy. Hence,

$$\left| \int p_1(x,y) d\mu_1 d\nu_1 - \int p_2(x,y) d\mu_2 d\nu_2 \right| \leq \left| \int (p_1(x,y) - p_2(x,y)) d\mu_1 d\nu_2 \right| \leq \max_{x,y \in [0,1]} |(p_1(x,y) - p_2(x,y))| \int d\mu_1 d\nu_2.$$  

Note that the quantity on the right is bounded because we are considering the maximum of a bounded continuous function on a compact set. Let $\alpha \in \mathbb{R}^S$. Given a polynomial game with payoff functions $r(s,a_1,a_2)$ and transition probabilities $p(t;s,a_1,a_2)$ (sometimes we will hide the state indices and write the entire matrix as $P(a_1,a_2)$), fix a state $s$ and define the polynomial $G^s(\alpha) = r(s,a_1,a_2) + \beta \sum_{t \in S} p(t;s,a_1,a_2)\alpha_t$. We will need to perform iterations using this vector $\alpha \in \mathbb{R}^S$. We call the iterates of these vectors $\alpha^k \in \mathbb{R}^S$ ($k$ is the iteration index), and denote $s^{th}$ component of this vector by $\alpha^k_s$. Pick the vector $\alpha^0 \in \mathbb{R}^S$ arbitrarily and define the recursion for the $s^{th}$ component at iteration $k$ by:

$$\alpha^k_s = \text{val}(G^s(\alpha^{k-1})), \quad k = 1,2,\ldots$$

Rephrasing the above in terms of operators, define $T_s$ to be the operator such that

$$T_s \alpha = \text{val}(G^s(\alpha)).$$

Let $T \alpha = [T_1 \alpha, \ldots T_S \alpha]^T$. Then the recursion simply consists of computing the terms $T^k(\alpha)$.

**Lemma 2:** The quantity

$$\lim_{k \to \infty} T^k(\alpha) = \phi$$

exists and is independent of $\alpha$. Moreover, $\phi$ is the unique fixed point solution to the equation:

$$\phi = T \phi.$$

**Proof:** For $\alpha \in \mathbb{R}^S$ define the norm $\|\alpha\| = \max_s |\alpha_s|$. Then,

$$\|T \gamma - T \alpha\| = \max_s |\text{val}(G^s(\gamma)) - \text{val}(G^s(\alpha))|$$

$$\leq \max_s \max_{a_1,a_2 \in [0,1]} |\beta \sum_t p(t;s,a_1,a_2)(\gamma_t - \alpha_t)| \quad \text{(using Lemma 1)}$$

$$\leq \max_s \max_{a_1,a_2 \in [0,1]} |\beta \sum_t p(t;s,a_1,a_2)| \max_t |(\gamma_t - \alpha_t)|$$

$$= \beta \|\gamma - \alpha\|. $$
Since the discount factor $\beta < 1$, we have a contraction, and by the contraction mapping principle, the iteration $T^k\alpha$ is convergent to the unique fixed point of the equation $T\phi = \phi$. \hfill \Box

Lemma 2 establishes that a fixed point solution to the iteration exists. We now show that the fixed point is in fact the value vector of the game. To show this, we show that if we compute the optimal strategies $\mu(s), \nu(s)$ to the game $G^s(\phi), s = 1, 2, \ldots, S$ then play according to these strategies achieves the value vector $\phi$. Since $\phi$ by definition satisfies the saddle point inequality (1), an equilibrium solution exists. To show that the value vector is unique, we show that any value vector satisfies the fixed point equation $Tv_\beta = v_\beta$. Since there is a unique fixed point by Lemma 2, the value vector must be unique.

**Theorem 1:** Let $\phi$ be the fixed point defined in Lemma 2. Then,

a. Let $\mu(s), \nu(s)$ denote the optimal measures to the polynomial game with payoff $G^s(\phi), s = \{1, \ldots, S\}$. Then $\mu = [\mu(1), \ldots, \mu(S)]^T, \nu = [\nu(1), \ldots, \nu(S)]^T$ are the optimal strategies for the stochastic game.

b. If $v_\beta(\mu, \nu)$ is a value vector for the game then $v_\beta$ satisfies $Tv_\beta = v_\beta$. Hence $v_\beta = \phi$ exists and is unique.

**Proof:** Let $\mu(s)$ and $\nu(s)$ be the optimal strategies for the game $G^s(\phi)$. Then by definition, the expected value of play under these strategies will be $\phi_s = T_s \phi = \ldots = T_s^k \phi$. Vectorizing this equation, we note that

$$\phi = T^k \phi = \mathbf{E}_{\mu,\nu} [r(a_1, a_2) + \beta P(a_1, a_2)r(a_1, a_2) + \cdots + \beta^{k-1} P^{k-1}(a_1, a_2)r(a_1, a_2) + \beta^k P^k(a_1, a_2)\phi].$$

Taking the limit as $k \to \infty$, we obtain that $\phi = \mathbf{E}_{\mu,\nu} [\sum_{k=0}^{\infty} \beta^k P^k(a_1, a_2)r(a_1, a_2)] = v_\beta(\mu, \nu)$.

Hence playing according to the stationary strategies $\mu(s), \nu(s), s = 1, \ldots, S$ achieves the value vector $\phi$. Suppose player 1 plays according to the strategy $\mu$, and suppose player 2 deviates from the prescribed stationary strategy $\nu$ to stationary strategy $\nu'$. Then, since $\mu, \nu$ are defined to be an equilibrium strategies for the game $G^s(\phi)$, we have the (vector) inequality for all $\nu'$:

$$\phi = \mathbf{E}_{\mu,\nu} [r(a_1, a_2) + \beta P(a_1, a_2)\phi] \leq \mathbf{E}_{\mu,\nu} [r(a_1, a_2) + \beta P(a_1, a_2)\phi] \leq \mathbf{E}_{\mu,\nu} [r(a_1, a_2) + \beta P(a_1, a_2)r(a_1, a_2) + \beta^2 P^2(a_1, a_2)\phi] \leq \mathbf{E}_{\mu,\nu} [r(a_1, a_2) + \beta P(a_1, a_2)r(a_1, a_2) + \cdots + \beta^k P^k(a_1, a_2)r(a_1, a_2) + \beta^k P^k(a_1, a_2)\phi].$$
In the first inequality a $\phi$ occurs on the right side. We substitute that inequality in the $\phi$ on the right side to obtain the second inequality and so on. Finally, we obtain the inequality:

$$\phi = E_{\mu, \nu} \left[ \sum_{k=0}^{\infty} \beta^k P^k(a_1, a_2) r(a_1, a_2) \right] \leq E_{\mu, \nu'} \left[ \sum_{k=0}^{\infty} \beta^k P^k(a_1, a_2) r(a_1, a_2) \right],$$

i.e. that $\phi = v_\beta(\mu, \nu) \leq v_\beta(\mu, \nu')$ for all $\nu'$. A similar argument for deviations $\mu'$ of player 1 shows that $v_\beta(\mu', \nu) \leq v_\beta(\mu, \nu) = \phi$. Hence $\mu(s), \nu(s)$ constructed as the strategies for the games $G^s(\phi)$ satisfy the saddle point inequality (1) component-wise. This establishes the existence of equilibria. For uniqueness, note that any strategies $\mu, \nu$ such that $v_\beta(\mu, \nu)$ satisfies the saddle point inequality (1), by definition we have $T v_\beta(\mu, \nu) = v_\beta(\mu, \nu)$. Since $T$ has a unique fixed point, the vector $v_\beta(\mu, \nu)$ must be unique.

It is interesting to note that the above proof also provides an algorithm to compute approximate equilibria. To compute each iterate $T_s(\alpha)$ one needs to solve a polynomial game in normal form (which can be done by solving a single semidefinite program), and by solving a sequence of such problems, one can compute $T^k(\alpha)$ which is provably close to the actual value-vector. However, the rate of convergence of this iteration is not very attractive. In the rest of this paper, we focus attention on single-controller games, for which equilibria can be computed by solving a single semidefinite program.

### D. SDP Characterization of Nonnegativity and Moments

Let $A$ be a closed interval on the real line. The set of univariate polynomials which are nonnegative on $A$ have an exact semidefinite description. The set of (finite) vectors in $\mathbb{R}^n$ which correspond to moment sequences of measures supported on $A$ also have an exact semidefinite description. We briefly review these notions here and introduce some related notation [6].

Let $\mathbb{R}[x]$ denote the set of univariate polynomials with real coefficients. Let $p(x) = \sum_{k=0}^{n} P_k x^k \in \mathbb{R}[x]$. We say that $p(x)$ is nonnegative on $A$ if $p(x) \geq 0$ for every $x \in A$. We denote the set of nonnegative polynomials of degree $n$ which are nonnegative on $A$ by $\mathcal{P}(A)$. (To avoid cumbersome notation, we exclude the degree information in the notation. Moreover the degree will usually be clear from the context.) The polynomial $p(x)$ is said to be a sum of squares if there exist polynomials $q_1(x), \ldots, q_k(x)$ such that $p(x) = \sum_{i=1}^{k} q_i(x)^2$. It is well known that a univariate polynomial is a sum of squares if and only if $p(x) \in \mathcal{P}(\mathbb{R})$. 

June 15, 2008  DRAFT
Let $\mu$ denote a measure supported on the set $A$. The $i^{th}$ moment of the measure $\mu$ is denoted by

$$\mu_i = \int_A x^i \, d\mu.$$ 

Let $\bar{\mu} = [\mu_0, \ldots, \mu_n]$ be a vector in $\mathbb{R}^{n+1}$. We say that $\bar{\mu}$ is a moment sequence of length $n + 1$ if it corresponds to the first $n + 1$ moments of some nonnegative measure $\mu$ supported on the set $A$. The moment space, denoted by $M(A)$ is the subset of $\mathbb{R}^{n+1}$ which corresponds to moments of nonnegative measures supported on the set $A$. We say that a nonnegative measure $\mu$ is a probability measure if its zeroth order moment satisfies $\mu_0 = 1$. The set of moment sequences of length $n + 1$ corresponding to probability measures is denoted by $M_p(A)$.

Let $S^n$ denote the set of $n \times n$ symmetric matrices and define the linear operator $\mathcal{H} : \mathbb{R}^{2n-1} \rightarrow S^n$ as:

$$\mathcal{H} : \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n-1} \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ a_2 & a_3 & \ldots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2n-1} & a_n & a_{n+1} & \ldots & a_{2n-1} \end{bmatrix}.$$ 

Thus $\mathcal{H}$ is simply the linear operator that takes a vector and constructs the associated Hankel matrix which is constant along the antidiagonals. We will also frequently use the adjoint of this operator, the linear map $\mathcal{H}^* : S^n \rightarrow \mathbb{R}^{2n-1}$:

$$\mathcal{H}^* : \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{12} & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1n} & m_{2n} & \ldots & m_{nn} \end{bmatrix} \mapsto \begin{bmatrix} m_{11} \\ 2m_{12} \\ m_{22} + 2m_{13} \\ \vdots \\ m_{nn} \end{bmatrix}.$$ 

This map flattens a matrix into a vector by adding all the entries along antidiagonals.

**Lemma 3:** Let $p(x) = \sum_{k=0}^{2n} p_k x^k$ be a polynomial. Let $\bar{p} = [p_0, \ldots, p_{2n}]^T$ be the vector of its coefficients. Then $p(x)$ is nonnegative (or SOS) if and only if there exists $S \in S^{n+1}$, $S \succeq 0$ such that:

$$\bar{p} = \mathcal{H}^*(S).$$
Proof: For univariate polynomials, nonnegativity is equivalent to SOS (see [9]). Let $[x]_n = [1, x, \ldots, x^n]^T$. We have for every $S \in S^{n+1}$,

$$p(x) = \bar{p}^T [x]_{2n} = \mathcal{H}^*(S) [x]_{2n} = [x]^T_n S [x]_n.$$  

Factoring $S \succeq 0$, we obtain a sum of squares decomposition. The converse is immediate. 

One can give a similar semidefinite characterization of polynomials that are nonnegative on an interval. Since in this paper we are typically considering the interval to be $[0, 1]$ we give an explicit semidefinite characterization of $\mathcal{P}([0, 1])$. We define the following matrices:

$$L_1 = \begin{bmatrix} I_{n \times n} \\ 0_{1 \times n} \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0_{1 \times n} \\ I_{n \times n} \end{bmatrix},$$

where $I_{n \times n}$ stands for the $n \times n$ identity matrix.

**Lemma 4:** The polynomial $p(x) = \sum_{k=0}^{2n} p_k x^k$ is nonnegative on $[0, 1]$ if and only if there exist matrices $Z \in S^{n+1}$ and $W \in S^n$, $Z \succeq 0, W \succeq 0$ such that

$$\begin{bmatrix} p_0 \\ \vdots \\ p_{2n} \end{bmatrix} = \mathcal{H}^*(Z + \frac{1}{2}(L_1 W L_2^T + L_2 W L_1^T) - L_2 W L_2^T).$$

**Proof:** The proof follows from the characterization of nonnegative polynomials on intervals.

It is well known that

$$p(x) \geq 0 \quad \forall x \in [0, 1] \iff p(x) = z(x) + x(1-x)w(x),$$

where $z(x)$ and $w(x)$ are sums of squares. A simple application of Lemma 3 yields the required condition. 

In this paper, we will also be using a very important classical result about the semidefinite representation of moment spaces [10], [11]. We give an explicit characterization of $\mathcal{M}([0, 1])$ and $\mathcal{M}_P([0, 1])$.

**Lemma 5:** The vector $\bar{\mu} = [\mu_0, \mu_1, \ldots, \mu_{2n}]^T$ is a valid set of moments for a nonnegative measure supported on $[0, 1]$ if and only if

$$\mathcal{H}(\bar{\mu}) \succeq 0$$

$$\frac{1}{2}(L_1^T \mathcal{H}(\bar{\mu}) L_2 + L_2^T \mathcal{H}(\bar{\mu}) L_1) - L_2^T \mathcal{H}(\bar{\mu}) L_2 \succeq 0.$$  

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Moreover, it is a moment sequence corresponding to a probability measure if and only if in addition to \((2)\) it satisfies \(\mu_0 = 1\).

**Proof:** The proof follows by dualizing Lemma 4. Alternatively, a direct proof may be found in [10].

For example, for \(2n = 2\) the sequence \([\mu_0, \mu_1, \mu_2]\) is a moment sequence corresponding to a measure supported on \([0, 1]\) if and only if the following inequalities are true:

\[
\begin{bmatrix}
\mu_0 & \mu_1 \\
\mu_1 & \mu_2
\end{bmatrix} \succeq 0
\]

\[
\mu_1 - \mu_2 \geq 0.
\]

### III. Finite Strategy Case

For the reader’s convenience and comparison purposes, we briefly review here the case where each player has only finitely many strategies at each state [3]. Again, for simplicity we assume that the set of pure strategies available to each player at each state is identical so that \(A_1 = A_2 = \{1, \ldots, m\}\). Under the finite strategy case, when assumption \(SC\) holds, a minimax solution may be computed via linear programming. We state the linear program in this section. In the next section, drawing motivation from this linear program, we write an infinite dimensional optimization problem for the case where each player has a choice from infinitely many pure strategies. The finite action game is completely defined via the specification of the following data:

1) The state space \(S = \{1, \ldots, S\}\).
2) The (finite) sets of actions for players 1 and 2 given by \(A_1 = A_2 = \{1, \ldots, m\}\).
3) The payoff function for a given state \(s\) (representable by a matrix indexed by the actions of each players) denoted by \(r(s, a_1, a_2)\).
4) The probability transition matrix \(p(s'; s, a_1)\) which provides the conditional probability of transition from state \(s\) to \(s'\) given player 1’s action \(a_1\).
5) The discount factor \(\beta\).

A *mixed* strategy for player 1 is a function \(f : S \times A_1 \to [0, 1]\) subject to the normalization constraint \(\sum_{a_1} f(s, a_1) = 1\) for each \(s \in S\) (so that \(f(s) = [f(s, 1), \ldots, f(s, m)]\) becomes a probability distribution over the strategy space \(A_1\)). Similarly the mixed strategy for player 2 in
a particular state \( s \) is given by \( g(s) = [g(s, 1), \ldots, g(s, m)] \). The collection of mixed strategies (indexed by the states) will be denoted by \( f = [f(1), \ldots, f(S)] \) (and \( g = [g(1), \ldots, g(S)] \) respectively). A strategy \( f \) leads to a probability matrix \( P(f) = \sum_{a_1 \in A_1} p(s'; s, a_1) f(s, a_1) \).

Again we consider a \( \beta \)-discounted process over an infinite horizon. Given strategies \( f \) and \( g \), the reward collected by player 1 in some stage \( s \) is given by:

\[
r(s, f(s), g(s)) = \sum_{a_1 \in A_1, a_2 \in A_2} r(s, a_1, a_2) f(s, a_1) g(s, a_2).\]

The reward collected over the infinite horizon starting at state \( s \), \( v_\beta(s, f(s), g(s)) \), is given by the system of equations:

\[
v_\beta(s, f(s), g(s)) = r(s, f(s), g(s)) + \\
\beta \sum_{s' \in S} \left( \sum_{a_1 \in A_1} p(s'; s, a_1) f(s, a_1) \right) v_\beta(s', f(s'), g(s')).
\]

Thus,

\[
v_\beta(f, g) = (I - \beta P(f))^{-1} r(f, g),
\]

where \( r(f, g) = [r(1, f(1), g(1)), \ldots, r(S, f(S), g(S))] \in \mathbb{R}^S \). The problem is to find equilibrium strategies \( f^0 \) and \( g^0 \) that satisfy the Nash equilibrium property:

\[
v_\beta(f, g^0) \leq v_\beta(f^0, g^0) \leq v_\beta(f^0, g)
\]

for all mixed strategies \( f, g \).

**Theorem 2 ([3]):** Consider the primal-dual pair of linear programs:

**Minimize**

\[
\sum_{s=1}^{S} v(s) \quad g(s, a_2), v(s)
\]

\[
v(s) \geq \sum_{a_2 \in A_2} r(s, a_1, a_2) g(s, a_2) + \\
\beta \sum_{s' = 1}^{S} p(s'; s, a_1) v(s') \quad \forall s \in \mathcal{S}, a_1 \in A_1
\]

\[
\sum_{a_2 \in A_2} g(s, a_2) = 1 \quad \forall s \in \mathcal{S}
\]

\[
g(s, a_2) \geq 0 \quad \forall s \in \mathcal{S}, a_2 \in A_2.
\]
and

$$\text{maximize } \sum_{s=1}^{S} z(s)$$

$$x(s, a_1), z(s)$$

$$\sum_{s=1}^{S} \sum_{a_1 \in A_1} [\delta(s, s') - \beta p(s', s, a_1)] x(s, a_1) = 1 \quad \forall s' \in S \tag{D}$$

$$z(s) \leq \sum_{a_1 \in A_1} x(s, a_1) r(s, a_1, a_2) \quad \forall s \in S, a_2 \in A_2,$$

$$x(s, a_1) \geq 0, \quad \forall s \in S, a_1 \in A_1.$$  

Let $p^*$ be the optimal value of $(P)$, and $d^*$ be the optimal value of $(D)$. Let $x^*(s, a_1)$ be the optimal values of the $x(s, a_1)$ variables obtained in $(D)$. Let

$$f^*(s, a_1) = \frac{x^*(s, a_1)}{\sum_{a_1} x^*(s, a_1)}$$

and $g^*(s, a_2)$ be the distribution obtained by the optimal solution of $(P)$. Then the following statements hold:

1) $p^* = d^*$.

2) Let $v^* = [v^*(1), \ldots, v^*(S)]$ be the optimal solution of $(P)$. Then $v^* = v_{\beta}(f^*, g^*)$.

3) $v_{\beta}(f^*, g^*)$ satisfies the saddle-point inequality (3).

Remark Note that statement 2 claims that the solution of the LP $(P)$ corresponds to the infinite horizon discounted reward obtained when players 1 and 2 play according to the distributions $f^*$ and $g^*$. Statement 3 claims that these distributions are in fact optimal for the two players in the Nash equilibrium sense.

Proof: See [3] pp. 93].

Remark Note that the primal problem $(P)$ has a natural interpretation in terms of security strategies. Feasible vectors $v$, and $g$ satisfy the first set of inequalities in $(P)$. The inequalities can be interpreted to mean that using strategy $g$ the payoff of player 2 will be at most $v$. 

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IV. INFINITE STRATEGY CASE

A. Problem Setup

In this section we consider the case where each player can choose from uncountably many different actions. In particular, each player can choose actions from the set \([0, 1]\). The number of states \(|S| = S\) is still finite. The payoff function \(r(s, a_1, a_2)\) is a polynomial in \(a_1\) and \(a_2\) for each \(s \in S\). The single controller case (Assumption SC) is studied. In this case, we assume that the probability of transition \(p(s'; s, a_1)\) is a polynomial in \(a_1\). Again we consider the two-player zero sum case where player 1 attempts to maximize his reward over the infinite horizon. We generalize the problem \((P)\) to this case. The variables \(f\) and \(g\) representing distributions over the finite sets \(A_1\) and \(A_2\) are replaced by measures \(\mu(s)\) and \(\nu(s)\). These measures represent mixed strategies over the uncountable action spaces. (We remind the reader that for each player there are \(S\) measures, each measure corresponding to a mixed strategy in a particular state. For example \(\mu(s)\) corresponds to the mixed strategy player 1 would adopt when the game is in state \(s\).)

B. Preliminary Results

We point out that the generalization of \((P)\) to this case is an optimization problem involving non-negativity of a system of univariate polynomials with coefficients that depend on the moments of these measures. The interpretation in terms of security strategies for player 2 holds. The following is the generalization of the linear program \((P)\) mentioned above:

\[
\begin{align*}
\text{minimize} \quad & \sum_{s=1}^{S} v(s) \\
\text{subject to} \quad & v(s) \geq \int_{a_2 \in A_2} r(s, a_1, a_2) d\nu(s) + \\
& \beta \sum_{s'=1}^{S} p(s'; s, a_1) v(s') \quad \text{for all } s \in S, a_1 \in A_1 \\
\text{and} \quad & \nu(s) \text{ is a measure supported on } A_2 \text{ for all } s \in S
\end{align*}
\]

Since \(\int r(s, a_1, a_2) d\nu(s) = q_{\nu}(s, a_1)\), a univariate polynomial in \(a_1\) for each \(s \in S\), for a fixed vector \(v(s)\), the constraints (a) are a system of polynomial inequalities. Note that the coefficients of \(q\) will depend on the measure \(\nu\) only via finitely many moments. More concretely, let \(r(s, a_1, a_2) = \sum_{i,j}^{n_s,m_s} r_{ij}(s) a_1^i a_2^j\) be the payoff polynomial. Then \(\int r(s, a_1, a_2) d\nu(s) = \)
$\sum_{i,j} r_{ij}(s)a_1^i \nu_j(s)$). Using this observation, this problem may be rewritten as the following problem.

$$\minimize \sum_{s=1}^{S} v(s), v(s)$$

$$(c) \quad v(s) - \sum_{i,j} r_{ij}(s)a_1^i \nu_j(s) - \beta \sum_{s'=1}^{S} p(s'; s, a_1)v(s') \in \mathcal{P}(A_1) \text{ for all } s \in S$$

$$(d) \quad \bar{v}(s) \in \mathcal{M}(A_2), \text{ and } \nu_0(s) = 1 \text{ for all } s \in S.$$

The constraints (c) give a system of polynomial inequalities in $a_1$, one inequality per state. Fix some state $s$. Let the degree of the inequality for that state by $d_s$. Let $[a_1]_{d_s} = [1, a_1, a_1^2, \ldots, a_1^{d_s}]$. The first term in constraint (c) can be rewritten in vector form as:

$$\sum_{i,j} r_{ij}(s)a_1^i \nu_j(s) = \bar{v}(s)^T R(s)^T [a_1]_{d_s},$$

where $R(s)$ is a matrix that contains the coefficients of the polynomial $r(s, a_1, a_2)$. Similar to the finite strategy case we define a vector by $v^* = [v^*(1), \ldots, v^*(S)]^T$ which will turn out to be the value vector of the stochastic game (which is indexed by the state). The second term in the constraint (c) which depends on the probability transition $p(s'; s, a_1)$ is also a polynomial in $a_1$ whose coefficients depend on the coefficients of $p(s'; s, a_1)$ and $v$. Specifically

$$\sum_{s'=1}^{S} p(s'; s, a_1)v(s') = v^T Q(s)^T [a_1]_{d_s},$$

for some matrix $Q(s)$ which contains the coefficients of $p(s'; s, a_1)$.

**Lemma 6**: Let $A_1 = A_2 = [0, 1]$. Let $E_s \in \mathbb{R}^{d_s \times S}$ be the matrix which has a 1 in the $(1, s)$
position. Then the semidefinite program \((SP)\) given by:

\[
\begin{align*}
\text{minimize} & \quad \sum_{s=1}^{S} v(s) \\
\quad & \quad \tilde{\nu}(s), v(s) \\
(e) & \quad \mathcal{H}^*(Z_s + \frac{1}{2}(L_1 W_s L_2^T + L_2 W_s L_1^T) - L_2 W_s L_2^T) \\
& \quad = E_s v - \beta Q(s)v - R(s)\tilde{\nu}(s) \quad \forall s \in \mathcal{S}
\end{align*}
\]

\[
(f) \quad \mathcal{H}(\tilde{\nu}(s)) \succeq 0 \quad \forall s \in \mathcal{S}
\]

\[
(g) \quad \frac{1}{2} \left( L_1^T \mathcal{H}(\tilde{\nu})(s)L_2 + L_2^T \mathcal{H}(\tilde{\nu})(s)L_1 \right) \\
& \quad - L_2^T \mathcal{H}(\tilde{\nu})(s)L_2 \succeq 0 \quad \forall s \in \mathcal{S}
\]

\[
(h) \quad e_1^T \tilde{\nu}(s) = 1 \quad \forall s \in \mathcal{S}
\]

\[
(i) \quad Z_s, W_s \succeq 0 \quad \forall s \in \mathcal{S}
\]

exactly solves the polynomial optimization problem \((P')\).

**Proof:** The polynomial in inequality (c) has the coefficient vector \(E_s v - \beta Q(s)v - R(s)\tilde{\nu}(s)\). The proof follows as a direct consequence of Lemma 4 concerning the semidefinite representation of polynomials nonnegative over \([0, 1]\), and Lemma 5 concerning the semidefinite representation of moment sequences of nonnegative measures supported on \([0, 1]\).
The dual of \((SP)\) is given by the following semidefinite program:

\[
\begin{align*}
\text{maximize} & \quad \sum_{s=1}^{S} \alpha(s) \\
\alpha(s), \bar{\xi}(s) \\
(j) & \quad \mathcal{H}'(A_s + \frac{1}{2}(L_1 B_s L_2^T + L_2 B_s L_1^T) - L_2 B_s L_2^T) = \\
& \quad R^T \bar{\xi}(s) - \alpha(s)e_1 \quad \forall s \in S \\
(k) & \quad \mathcal{H}(\bar{\xi}(s)) \succeq 0 \quad \forall s \in S \\
(l) & \quad \frac{1}{2} (L_1^T \mathcal{H}(\bar{\xi}(s)) L_1 + L_2^T \mathcal{H}(\bar{\xi}(s)) L_2) - \\
& \quad L_2^T \mathcal{H}(\bar{\xi}(s)) L_2 \succeq 0 \quad \forall s \in S \\
& \quad \sum_s (E_s - \beta Q(s))^T \bar{\xi}(s) = 1 \\
(m) & \quad A_s, B_s \succeq 0 \quad \forall s \in S.
\end{align*}
\]

**Lemma 7:** The dual SDP \((SD)\) is equivalent to the following polynomial optimization problem:

\[
\begin{align*}
\text{maximize} & \quad \sum_{s=1}^{S} \alpha(s) \\
\alpha(s), \bar{\xi}(s) \\
(n) & \quad \sum_{i,j} r_{ij}(s) \xi_i(s) a_j^s - \alpha(s) \geq 0 \quad \forall a_2 \in A_2, s \in S \\
(o) & \quad \bar{\xi}(s) \in \mathcal{M}(A_2) \quad \forall s \in S \\
(p) & \quad \sum_s \int_{A_1} (\delta(s, s') - \beta p(s', s, a_1)) d\xi(s) = 1 \quad \forall s' \in S.
\end{align*}
\]

**Proof:** This again follows as a consequence of Lemmas 4 and 5.

**Remark** Note that in the dual problem, the moment sequences do not necessarily correspond to probability measures. Hence, to convert them to probability measures, one needs to normalize the measure. Upon normalization, one obtains the optimal strategy for player 1.
Lemma 8: The polynomial optimization problems \((P')\) and \((D')\) are strong duals of each other.

Proof: We prove this by showing that the semidefinite program \((SP)\) satisfies Slater’s constraint qualification and that it is bounded from below. The result then follows from the strong duality of the equivalent semidefinite programs \((SP)\) and \((SD)\).

First pick \(\mu(s)\) and \(\nu(s)\) to be the uniform distribution on \([0, 1]\) for each state \(s \in S\). One can show [10] that the moment sequence of \(\mu\) is in the interior of the moment space of \([0, 1]\). As a consequence, constraints (f) and (g) are strictly positive definite. Using the strategies \(\mu\) and \(\nu\), evaluate the discounted value of this pair of strategies as:

\[
v_{\beta}(\mu, \nu) = [I - \beta P(\mu)]^{-1}r(\mu, \nu).
\]

Choose \(v > v_{\beta}\). The polynomial inequalities given by (c) are all strictly positive and thus constraints (i) are strictly positive definite. The equality constraints are trivially satisfied.

To prove that the problem is bounded below, we note that \(r(s, a_1, a_2)\) is a polynomial and that the strategy spaces for both players are bounded. Hence,

\[
\inf_{a_1 \in A_1, a_2 \in A_2} r(s, a_1, a_2)
\]

is finite and provides a trivial lower bound for \(v(s)\).

Lemma 9: Let \(\bar{\nu}^*(s)\) and \(\bar{\xi}^*(s)\) be optimal moment sequences for \((P')\) and \((D')\) respectively. Let \(\nu^*(s)\) and \(\xi^*(s)\) be the corresponding measures supported on \(A_1\) and \(A_2\) respectively. The following complementary slackness results hold for the optima of \((P')\) and \((D')\):

\[
v^*(s) \int_{A_1} d\xi^*(s) = \int_{A_2} \int_{A_1} r(s, a_1, a_2)d\xi^*(s)d\nu^*(s) + \beta \sum_{s'=s} v^*(s') \int_{A_1} p(s'; s, a_1)d\xi^*(s) \quad \forall s \in S
\]

\[
\alpha^*(s) \int_{A_2} d\nu^*(s) = \int_{A_2} \int_{A_1} r(s, a_1, a_2)d\xi^*(s)d\nu^*(s) \quad \forall s \in S
\]

Proof: The result follows from the strong duality of the equivalent semidefinite representations of the primal-dual pair \((P') - (D')\). The Lagrangian function for \((P')\) is given by:

\[
\mathcal{L}(\xi, \alpha) = \inf_{\nu, \nu'} \{ \sum_{s=1}^{S} v(s) - \int_{A_1} [v(s) - \int_{A_2} r(s, a_1, a_2)d\nu(s)] 
- \beta \sum_{s'} v^*(s') \int_{A_1} p(s'; s, a_1)d\xi(s) + \sum_{s} \alpha(s)(1 - \nu_0(s))\}.
\]

\(\mathcal{L}(\xi, \alpha)\) must satisfy weak duality, i.e. \(d^* \leq p^*\). At optimality \(p^* = \sum_s v^*(s)\) for some vector \(v^*\). However, strong duality holds, i.e. \(p^* = d^*\). This forces the first complementary slackness
relation. The second relation is obtained similarly by considering the Lagrangian of the dual problem.

We have shown that problem \((P')\) can be reduced to the semidefinite program \((SP)\), and is thus computationally tractable via convex optimization algorithms. We next show that the solution to problem \((P')\) is in fact the desired equilibrium solution.

C. Main Theorem

Let \(p^*\) be the optimal value of \((P')\), and \(d^*\) be the optimal value of \((D')\). Let \(\nu^*(s)\) and \(\xi^*(s)\) be the optimal measures recovered in \((P')\) and \((D')\). Let

\[
\mu^*(s) = \frac{\xi^*(s)}{\int_{A_1} d\xi^*(s)},
\]

so that \(\mu^*\) is a normalized version of \(\xi^*\) (i.e. \(\mu^*\) is a probability measure). Let \(v^*\) be the vector obtained as the optimal solution of \((P')\).

**Theorem 3:** The optimal solutions to the primal-dual pair \((P'), (D')\) satisfy the following:

1) \(p^* = d^*\).
2) \(v^* = v_\beta(\mu^*, \nu^*)\).
3) \(v_\beta(\mu^*, \nu^*)\) satisfies the saddle-point inequality:

\[
v_\beta(\mu, \nu) \leq v_\beta(\mu^*, \nu^*) \leq v_\beta(\mu^*, \nu)
\]

(6)

for all mixed strategies \(\mu, \nu\).

**Proof:**

1) Follows from the strong duality of the primal-dual pair \((P') - (D')\).
2) Using Lemma 9 equation (4) in normalized form (i.e. dividing throughout by \(\xi^*_0(s)\), which is the zeroth order moment of the measure \(\xi(s)\)) we obtain

\[
v^*(s) = \int_{A_2} \int_{A_1} r(s, a_1, a_2) d\mu^*(s) d\nu^*(s) + \beta \sum_{s'} v^*(s') \int_{A_1} p(s'; s, a_1) d\mu^*(s) \quad \forall s \in S.
\]

Upon simplification and vectorization of \(v^*(s)\) one obtains

\[
v^* = r(\mu^*, \nu^*) + \beta P(\mu^*)v^*.
\]

Using a Bellman equation argument or by simply iterating this equation (i.e. substituting repeatedly for \(v^*\)) it is easy to see that \(v^* = v_\beta(\mu^*, \nu^*)\).
3) Consider inequality (c) at its optimal value. We have for every state $s$:

$$v^*(s) \geq \int_{a_2 \in A_2} r(s, a_1, a_2) d\nu^*(s) + \beta \sum_{s'} p(s'; s, a_1) v^*(s').$$

Integrating with respect to some arbitrary probability measure $\mu(s)$ (with support on $A_1$), we get:

$$v^*(s) \geq \int_{a_2 \in A_2} \int_{a_1 \in A_1} r(s, a_1, a_2) d\mu(s) d\nu^*(s) + \beta \sum_{s'} \int_{a_1 \in A_1} p(s'; s, a_1) v^*(s') d\mu(s).$$

Thus,

$$v^*(s) \geq r(s, \mu(s), \nu^*(s)) + \beta \sum_{s'} \int_{a_1 \in A_1} p(s'; s, a_1) v^*(s') d\mu(s).$$

Iterating this equation, we obtain $v_{\beta}(\mu^*, \nu^*) = v^* \geq v_{\beta}(\mu, \nu^*)$ for every strategy $\mu$. This completes one side of the saddle point inequality.

Using the normalized version of equation (5), we get:

$$\frac{\alpha^*(s)}{\xi_0^*(s)} = \int_{a_2 \in A_2} \int_{a_1 \in A_1} r(s, a_1, a_2) d\mu(s) d\nu^*(s) = r(s, \mu^*(s), \nu^*(s)).$$

If we integrate inequality (n) in problem $(D')$ with respect to any arbitrary probability measure $\nu(s)$ with support on $A_2$, we obtain

$$\frac{\alpha^*(s)}{\xi_0^*(s)} \leq r(s, \mu^*(s), \nu(s)).$$

Thus $r(s, \mu^*(s), \nu^*(s)) \leq r(s, \mu^*(s), \nu(s))$ for every $s$. Multiplying throughout by $(I - \beta P(\mu^*))^{-1}$, we get $v_{\beta}(\mu^*, \nu^*) \leq v_{\beta}(\mu^*, \nu)$. This completes the other side of the saddle point inequality.

\[ \blacksquare \]

### D. Obtaining the measures

Solutions to the semidefinite programs $(SP)$ and $(SD)$ provide the moment sequences corresponding to optimal strategies. Additional computation is required to recover the actual measures. We briefly describe a classical procedure to recover the measures using linear algebra. For more details, the reader may refer to [11], [12].
Let $\bar{\mu} \in \mathbb{R}^{2n}$ be a given moment sequence. We wish to find a nonnegative measure $\mu$ supported on the real line with these moments. The resulting measure will be composed of finitely many atoms (i.e. a discrete measure) of the form $\sum w_i \delta(x - a_i)$ where 

$$\text{Prob}(x = a_i) = w_i \quad \forall i.$$ 

Construct the following linear system:

$$
\begin{bmatrix}
\mu_0 & \mu_1 & \ldots & \mu_{n-1} \\
\mu_1 & \mu_2 & \ldots & \mu_n \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1} & \mu_n & \ldots & \mu_{2n-2}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{n-1}
\end{bmatrix}
= -
\begin{bmatrix}
\mu_n \\
\mu_{n+1} \\
\vdots \\
\mu_{2n-1}
\end{bmatrix}.
$$

Note that the Hankel matrix that appears on the left hand side is a sub-matrix of $\mathcal{H}(\bar{\mu})$. We assume without loss of generality that the above matrix is strictly positive definite. (Suppose the above matrix is not full rank, construct a smaller $k \times k$ linear system of equations by eliminating the last $n - k$ rows and columns of the matrix so that the $k \times k$ submatrix is full rank, and therefore strictly positive definite.) By inverting this matrix we solve for $[c_0, \ldots, c_{n-1}]^T$. Let $x_i$ be the roots of the polynomial equation

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 = 0.$$ 

It can be shown that the $x_i$ are all real and distinct, and that they are the support points of the discrete measure. Once the supports are obtained, the weights $w_i$ may be obtained by solving the nonsingular Vandermonde system given by:

$$\sum_{i=1}^{n} w_i x_i^j = \mu_j \quad (0 \leq j \leq n-1).$$

V. Example

Consider the two player discounted stochastic game with $\beta = 0.5$, $S = \{1, 2\}$ with payoff function $r(1, a_1, a_2) = (a_1 - a_2)^2$ and $r(2, a_1, a_2) = -(a_1 - a_2)^2$. Let the probability transition matrix be given by:

$$P(a_1) = \begin{bmatrix}
a_1 & 1 - a_1 \\
1 - a_1^2 & a_1^2
\end{bmatrix}.$$
A two state stochastic game with transition probabilities dependent only on the action of Player 1. The payoffs associated to the states are indicated in the corresponding nodes. The edges are marked by the corresponding state transition probabilities.

Figure 2 graphically illustrates this stochastic game, consisting of two states (the nodes) with polynomial transition probabilities dependent on $a_1$ (as marked on the edges of the graph). Within the nodes, the payoffs associated to the corresponding states are indicated.

To understand this game, consider first the zero-sum (nonstochastic game) with payoff function $p(a_1, a_2) = (a_1 - a_2)^2$ over the strategy space $[0, 1]$. This game (called the “guessing game”) was studied by Parrilo in [6]. If Player 2 is able to guess the action of Player 1, he can simply imitate his action (i.e. set $a_2 = a_1$ and his payoff to player 1 would be zero (this is the minimum possible since $(a_1 - a_2)^2 \geq 0$). Player 1 would try to confuse player 2 as much as possible and thus randomize between the extreme actions $a_1 = 0$ and $a_1 = 1$ with a probability of $\frac{1}{2}$. Player 2’s best response would be to play $a_2 = \frac{1}{2}$ with probability 1.

In the game described in Fig. 2, in State 1 Player 1 plays the role of confuser and Player 2 plays the role of guesser. In state 2, the roles of the players are reversed, Player 1 is the guesser and Player 2 the confuser. However, the problem is complicated a bit by the fact that State 1 is advantageous to Player 1 so that at every stage he has incentive to play a strategy that gives him a good payoff as well as maximize the chances of transitioning to State 1.

The polynomial optimization problem that computes the minimax strategies and the equilib-
rium values is the following:

\[
\text{minimize } v(1) + v(2)
\]

\[
v(1) \geq \int (a_1 - a_2)^2 d\nu(1) + \beta(a_1 v(1) + (1 - a_1)v(2)) \quad \forall a_1 \in [0,1]
\]

\[
v(2) \geq -\int (a_1 - a_2)^2 d\nu(2) + \beta((1 - a_1^2)v(1) + a_1^2 v(2)) \quad \forall a_1 \in [0,1]
\]

\(\nu(1), \nu(2)\) probability measures supported on \([0,1]\).

This problem can be reformulated as follows:

\[
\text{minimize } v(1) + v(2)
\]

\[
v(1) \geq a_1^2 - 2a_1 \nu_1(1) + \nu_2(1) + \beta(a_1 v(1) + (1 - a_1)v(2)) \quad \forall a_1 \in [0,1]
\]

\[
v(2) \geq -a_1^2 + 2a_1 \nu_1(2) - \nu_2(2) + \beta((1 - a_1^2)v(1) + a_1^2 v(2)) \quad \forall a_1 \in [0,1]
\]

\([1, \nu_1(1), \nu_2(1)]^T, [1, \nu_1(2), \nu_2(2)]^T \in M([0,1])\).

Solving the SDP and its dual we obtain the following optimal cost-to-go and optimal moment sequences:

\[\mathbf{v}^* = [.298, -.158]^T\]

\[\bar{\mu}^*(1) = [1, .614, .614]^T \quad \bar{\mu}^*(2) = [1, .5, .25]^T\]

\[\bar{\nu}^*(1) = [1, .614, .377]^T \quad \bar{\nu}^*(2) = [1, .614, .614]^T.\]

The corresponding measures obtained as explained in subsection \[\text{IV-D}\] are supported at only
finitely many points, and are given by the following:

\[ \mu^*(1) = 0.386 \delta(a_1) + 0.614 \delta(a_1 - 1) \]
\[ \mu^*(2) = \delta(a_1 - 0.5) \]
\[ \nu^*(1) = \delta(a_2 - 0.614) \]
\[ \nu^*(2) = 0.386 \delta(a_2) + 0.614 \delta(a_2 - 1). \]

Consider, for example, play in State 1. If Player 1 were playing obliviously with respect to the state transitions, he would play actions \( a_1 = 0 \) and \( a_1 = 1 \) with one half probability each. However, to increase the probability of staying in State 1 he plays action 1 with a higher probability. Player 2 cannot affect the state transition probabilities directly, thus he must play a myopic best response. (A myopic best response is one that is a best response for the game in the current state). Note that in state 1, once Player 1’s strategy is fixed, the (only) best response for Player 2 is to play the action \( a_2 = 0.614 \) with probability 1. In state 2, player 1’s best strategy is to play \( a_1 = 0.5 \). Player 2 picks an action from his myopic best response set (in this case, all probability distributions that are supported on the points 0 and 1).

VI. Conclusions and Future Work

In this paper, we have presented a technique for solving two-player, zero-sum finite state stochastic games with infinite strategies and polynomial payoffs. We established the existence of equilibria for such games. As a by-product we got an algorithm that converged to unique value vector of the game (however this algorithm does not seem to have very attractive convergence rates). We focused mainly on the case where the single-controller assumption holds. We showed that the problem can be reduced to solving a system of univariate polynomial inequalities and moment constraints. We used techniques from the classical theory of moments and sum-of-squares to reduce the problem to a semidefinite programming problem. By solving a primal-dual pair of semidefinite programs, we obtained minimax equilibria and optimal strategies for the players.

It is known that finite-state, finite action, two-player zero-sum games which satisfy the or- derfield property [13], [5] may be solved via linear programming. The single-controller case, games with perfect information, switching controller stochastic games, separable reward-state
independent transition (SER-SIT) games and additive games all satisfy this property. We intend
to extend these cases to the infinite strategy case with polynomial payoffs. General finite action
stochastic games which do not satisfy the orderfield property still have an interesting math-
ematical structure, but efficient computational procedures are not available. Developing such
procedures present an interesting direction of future research.

Acknowledgement: The authors would like to thank Ilan Lobel and Prof. Munther Dahleh for
bringing to their attention the linear programming solution to single controller finite stochastic
games.

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June 15, 2008