An accurate and fast iterative scheme for estimating the ship rolling and capsizing in regular waves

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Abstract. A short number of steep breaking waves hitting a ship from the side may affect its dynamic behaviour and to determine extreme roll angles and, eventually, capsizing. This last phenomenon is a challenging task for naval engineers because it is responsible not only for a lot of material damages, but also for human lives. It is therefore not surprising that there have been published many relevant theoretical and experimental studies on heavy rolling and capsizing but the problem is not yet fully resolved. Generally, the ship rolling in beam regular seas could be described by a second-order non-linear differential equation with the roll angle as dependent variable. Non-linearity comes from the restoring and damping moments, which are usually represented by (odd) polynomials of roll angle or of its time derivative. Numerical integrators incorporated in modern software packages (e.g. ode45 in Matlab) encounter efficiency problems for the unbounded solutions associated with capsizing, the running time being extremely high. The alternative is to write a computer software, appropriate to the topic to be solved. In the paper, the roll equation was solved using a simple, fast and accurate iterative scheme based on Taylor expansion. Compared with Runge-Kutta 4th order numerical technique, the used scheme demonstrated not only an excellent agreement of the results, but also a significant reduction of the CPU time. The rapidity of the scheme allowed us to conduct a comprehensive investigation on fractal erosion of safe basins and to represent the normalized integrity curves and the amplitude-frequency response curves for different combinations of wave parameters. The restoring and damping coefficients corresponded to a vehicle ferry model, considered to be either intact or damaged, on the one hand, and with or without bilge keels, on the other.

1. Introduction
Rolling of a ship in waves is a swinging motion around its longitudinal axis. For small roll angles, the response of the ship can be described by a linear differential equation. Contrary, for severe environment conditions the amplitude of oscillations increases significantly, so the nonlinear effects come into play, especially because of damping, restoring and external loadings’ moments. Various nonlinear models for ship rolling have been proposed by researchers in the field [1-4].

By capsizing or keeling over of a ship we understand that situation when the ship is turned on its side or it is upside down. In the language of nonlinear dynamics, capsizing is a transition from a stable equilibrium point near the upright position to a stable equilibrium point near the upside-down position. This phenomenon is a challenging task for naval engineers because it is responsible not only for a lot of material damages, but also for human lives. Theoretical studies and experiments showed that the
common reasons for capsizing of ships are parametric roll resonance, broaching, water on deck, extreme and Fricke waves or loss of stability at a wave crest [5, 6].

Development of nonlinear dynamics in the 1980s and significant enhancement of computational capabilities opened new possibilities to study this dangerous phenomenon. Tools like safe basins and integrity curves, Lyapunov exponents, correlation dimension and system entropy are extremely used in an attempt to understand the mechanisms behind capsizing [7-10].

The ship rolling in beam regular seas is usually described by a second-order non-linear differential equation with the roll angle as dependent variable. Non-linearity comes from the restoring and damping moments, which are usually represented by (odd) polynomials of roll angle or of its time derivative. The sea action is incorporated in the study by a single frequency harmonic excitation [11, 12].

In the present study, we used a simple and accurate iterative scheme based on Taylor expansion to solve numerically a typical nonlinear roll equation, whose coefficients were determined for an existing ferry ship. The scheme’s flexibility and fastness made it easier to obtain extensive information about the safe basin’s fractal erosion and the amplitude-frequency response when the ship was considered intact/damaged and was/ was not equipped with bilge keels.

2. The nonlinear roll equation

The equation used throughout the paper for describing the rolling motion is written as

\[ \ddot{\theta} + d_1 \dot{\theta} + d_3 \dot{\theta}^3 + k_1 \theta + k_3 \dot{\theta}^3 + k_5 \theta^5 = m \cos \omega t \]

with \( d_1 \) and \( d_3 \) denoting the linear and cubic roll damping coefficients, \( k_1, k_3 \) and \( k_5 \) the coefficients of the roll restoring moment (written as a fifth order odd polynomial), \( m \) and \( \omega \) the forcing amplitude and, respectively, the wave frequency. Additionally, \( \theta \) represents the roll angle and a dot denotes the time differentiation.

For the estimation of ship rolling and capsizing, the coefficients \( d_1, i = 1, 3 \) and \( k_j, j = 1, 3, 5 \) were chosen for a real vehicle ferry ship [4]. Thus, if the ship was equipped with bilge keels, then \( d_1 = 0.0476542 \) \( s^{-1} \) and \( d_3 = 3.765 \) \( s \). The ship without bilge keels was characterized by \( d_1 = 0.01265913 \) \( s^{-1} \) and \( d_3 = 0.4954 \) \( s \).

The coefficients of the restoring moment polynomial were gained so they approximate the static stability curve in the best possible way. For the case of the intact ship it was found that \( k_1 = 0.69197033 \) \( s^{-2} \), \( k_3 = -0.53920393 \) \( s^{-2} \), \( k_5 = 0 \), while for the damaged ship \( k_1 = 0.65 \) \( s^{-2} \), \( k_3 = -1.2 \) \( s^{-2} \) and \( k_5 = 0.105 \) \( s^{-2} \). The initial conditions are \( \theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0 \).

3. The iterative scheme

Since there are no exact analytical solutions for equation (1), we will have to resort to a numerical technique. In [13], the authors proposed a simple and accurate iterative scheme based on Taylor theory for finding approximate solutions of nonlinear Duffing oscillator.

Adapted to our problem, the scheme reads as:

\[
\begin{align*}
\theta_{n+1} &= \theta_{n-1} + 2 \theta_{n-2} \Delta t + 2 \theta_{n-1} \Delta t^2,
\theta_{n+1} = \theta_{n-1} + 2 \theta_{n-2} \Delta t + 2 \theta_{n-1} \Delta t^2, \\
\theta_n &= (\theta_{n+1} - \theta_{n-1}) / 2 \Delta t, \\
\theta_{n+1} &= -d_1 \theta_n - d_3 \theta_n^3 - k_1 \theta_n - k_3 \theta_n^3 - k_5 \theta_n^5 + m \cos \omega (n - 1) \Delta t \]
\end{align*}
\]
The starting values are as follows:

$$\theta_1 = \theta(0), \quad \dot{\theta}_1 = \dot{\theta}(0), \quad \ddot{\theta}_1 = m - d_1 \theta(0) - d_3 \theta^3(0) - k_1 \theta(0) - k_3 \theta^3(0) - k_5 \theta^5(0) \quad (3)$$

The scheme is $O(\Delta t^2)$ of accuracy.

4. Numerical simulations

In this section, with the help of scheme (2), we carried out a thorough investigation on the safe basin’s fractal erosion and on amplitude-frequency response curve for different combinations $(\omega, m)$ of wave parameters, in the hypothesis that the ship is intact/damaged and equipped/not equipped with bilge keels. The computations were performed with the Matlab software package. Of course, there are high-performance numerical integrators in Matlab, such as `ode45`, but for what we are interested in, they proved to be slow and not very flexible. Thus, if equation (1) is solved one thousand times for the case of intact ship without bilge keels for $\omega = 0.9, m = 0.3, \Delta t = 0.01, t \in [0, 500]$, the running time was 7.65 s for a computer software based on scheme (2) (written by us) and 557.32 s for `ode45`.

Let us now analyse the accuracy of scheme (2). For an arbitrary $\omega$ and a small forcing $m$, the numerical solution of equation (1) is periodic. Figures 1(a) and 1(b) show an excellent agreement between the results provided by scheme (2) and the `ode45` results, not only for the transition period but also for the steady state. The used parameters were $\omega = 0.9, m = 0.3, \Delta t = 0.01$. An increasing in forcing $m$ will cause a strengthening of the roll amplitudes. For large steps of integration (e.g. $\Delta t = 0.01$), there will be some differences between the two solutions, but only for the transition stage. A decrease of the step $\Delta t$ will eliminate these differences, as proven in figures 1(c) and 1(d). Further increases of $m$ will result in an unbounded solution that is equivalent from physical point of view with ship capsizing.
Figure 1. Comparison between the numerical solutions of equation (1), provided by scheme (2) and ode45 in Matlab, for the intact ship without bilge keels. The stars stand for ode45 solution. (a) \( \omega = 0.9, m = 0.3, \Delta t = 0.01, t \in [0, 50] \); (b) \( \omega = 0.9, m = 0.3, \Delta t = 0.01, t \in [450, 500] \); (c) \( \omega = 0.9, m = 0.345, \Delta t = 0.01, t \in [0, 50] \); (d) \( \omega = 0.9, m = 0.345, \Delta t = 0.001, t \in [0, 50] \); (e) \( \omega = 0.9, m = 0.35, \Delta t = 0.01, t \in [0, 18] \); (f) \( \omega = 0.9, m = 0.35, \Delta t = 0.001, t \in [0, 20] \).

The size of the step \( \Delta t \) is now important only to keep the two solutions closer for a longer period of time, as illustrated in figures 1(e) and 1(f). This overlap of the solutions given by scheme (2) and ode45 was found for any tested combination of parameters, so that the following results were obtained with iterative technique (2) and \( \Delta t = 0.01 \).

Experimental studies and data from the reported capsizing conducted to the conclusion that the worst scenario and, in the same time, a more realistic representation of a sea state consists in a short sequence of no more ten steep waves hitting the vessel from a side and not in a long pulse of regular waves. As a consequence, the transient response is capital for our objective [14]. If capsize does not occur within 8 – 10 cycles of forcing than it is unlikely to appear later. As an example, one has figure 1(f), where the amplitude exceeds two radians in less than three periods \( T = 2\pi / \omega \). It is useful to observe that once the angle \( \theta \) has exceeded one radian, its increase becomes exponential.

The cumulative conditions of \( \theta > 2 \text{ rad} \) and \( t < 10T \) were the basis of the construction of figure 2, where the dark colour (green or black) corresponds to the pairs \((\omega, m)\) for which the solution of the equation (1) is bounded and the light one (yellow or brick) to the pairs in which the vessel is in a capsized state. It is observed that the presence of the damping devices greatly improves the ship stability and the damaged vessel is more susceptible to overturning.
Safe basin concept has been introduced in 1990s for the study of nonlinear ship rolling and capsize. The safe basin denotes the set of initial conditions in the phase plane \((\theta, \dot{\theta})\) which define the separatrix between capsize and non-capsize areas and illustrate the high sensitivity of capsize to initial conditions. In fact, for a given set of parameters \((\omega, m)\), the safe basin of attraction is formed by all initial conditions \((\theta(0), \dot{\theta}(0))\) that do not lead to capsize. The size, shape and locations of safe basins provide valuable information about the engineering integrity of the ship. The numerical simulation showed that an increasing of forcing amplitude \(m\) leads to a process of fractal erosion of the safe basin, finished with a sharp decrease of safe area. We will prove this in the following by examples.

The initial conditions were selected from a vast set having 40,401 = 201 x 201 elements, obtained by dividing the rectangle \([-1.2, 1.2] \times [-1.0, 1.0]\) in equally spaced segments. The running time was chosen equal to 10\(T\). Each point is tested against the escape criterion \((\theta > 2 rad)\), and a colored small rectangle around the initial condition is used to indicate the time after which this criterion was satisfied. It turns out that the white region will correspond to the safe basin.

**Case 1:** Intact ship without bilge keels

a) \(\omega = 0.6 \text{ rad/s} \quad (10T = 104.7 s)\): figure 2(a) tells us that this is one of the most dangerous wave frequencies for the ship, because the capsizing occurs for a minimum forcing amplitude. For \(m = 0\), the safe basin contains 66.47% from the tested initial conditions (see figure 3(a)). By increasing \(m\), the basin erodes slightly from the outside but the important loss occurs from the inside. Here, the start is given by a few thin whiskers that begin to invade into the safe basin. If \(m\) continues to grow, the whiskers increase in number and area and transform into thick fingers (see figure 3(b)). The safe basin of attraction diminishes and, at \(m \approx 0.18\), disappears completely (see figures 3(c) and 3(d)).

b) \(\omega = 0.4 \text{ rad/s} \quad (10T = 157 s)\): The fractal erosion experienced by the safe basin could follow other route, as indicated in figure 4. The basin contracts with the gradual increase of the forcing amplitude without presenting gaps in its “mass”.

For many wave frequencies, the two scenarios above occur simultaneously.
Figure 3. Safe basin’s fractal erosion for the intact ship without bilge keels if $\omega = 0.6$ (a) $m = 0$; (b) $m = 0.105$; (c) $m = 0.125$; (d) $m = 0.14$.

Figure 4. Safe basin’s fractal erosion for the intact ship without bilge keels if $\omega = 0.4$. (a) $m = 0.085$; (b) $m = 0.17$.

Case 2: Intact ship with bilge keels

It is the best situation for the ship, the safe basin having the maximum range. Again, the erosion of the safe basin can be done from the outside, from the inside or from both directions. The basin’s disappearance can occur very suddenly, as suggested in figure 5. Almost infinitesimal increases in forcing amplitude $m$ produce "catastrophic" effects for the ship safety.
Figure 5. Safe basin’s fractal erosion for the intact ship with bilge keels if $\omega = 0.4$:
(a) $m = 0$; (b) $m = 0.2584$; (c) $m = 0.25845$; (d) $m = 0.25855$.

An interesting situation is obtained for $\omega = 0.8$, when the erosion is done from the outside, gradually, by "snatching" some strips of initial conditions (see figure 6).

Case 3: Damaged ship without bilge keels

The vessel is in the most dangerous situation among those analysed in the paper. The safe basin represents only about one-third of that of the intact vessel and shrinks rapidly with increasing forcing amplitude. The erosion mechanisms are similar to those of the intact vessel, an example being shown in figure 7.
Figure 7. Safe basin’s fractal erosion for the damaged ship without bilge keels if $\omega = 0.6$. (a) $m = 0.375$; (b) $m = 0.055$.

**Case 4: Damaged ship with bilge keels**

The safe basin retains the shape corresponding to the intact vessel, but is much longer and thinner. For certain wave frequencies, its contraction begins from the inside in the form of a “dolphin head” that grows in size. The erosion continues in the form of strips parallel to the basin’s borders, which quickly covers the entire remaining available space. A typical example is reported in figure 8.

Figure 8. Safe basin’s fractal erosion for the damaged ship with bilge keels if $\omega = 0.8$ (a) $m = 0.45$; (b) $m = 0.454$.

Despite the large volume of calculations, the speed of iterative scheme (2) allowed the construction of each of the above panels in only 4-5 minutes. For comparison, it is worth mentioning that an identical operation based on *ode45* takes about 80 minutes on the same computer [7]. The integrity curves show the relative influence of wave excitation’s amplitude $m$ on capsize relative to vessel safety in the absence of incident waves. They could be generated by plotting the safe area, normalized to unity at $m = 0$. In order to have a common reporting base, this normalization was done relative to the case of the intact vessel with bilge keels. This situation corresponds to a maximum safe basin (81.45% of the tested initial conditions). Figure 9 presents the integrity curves associated to the intact/damaged ship without/with bilge keels for $\omega \in \{0.4, 0.6, 0.8\}$. They were obtained by interpolating 10 to 20 safe basins’ sizes for different forcing amplitudes $m$.

Our previous analysis on equation (1) aimed exclusively at separating the bounded solutions from the unlimited ones. In the second part of the paper we will focus on the size of the finite roll amplitudes. In this idea, the equation was integrated using the scheme (2) for $\omega \in [0.2, 1]$ and three forcing amplitudes $m$ in an attempt to build the amplitude – frequency response curves. Our findings
are displayed in figure 10. Features characteristic of Duffing-type equations, such as jumps, subharmonic resonances, growing of amplitude with the forcing, can be easily observed in the four panels. The missing points for certain wave frequencies and forcing amplitudes correspond to the capsized vessel.

Figure 9. Normalized integrity curves for intact/damaged ship without/with bilge keels.

These curves allow us to obtain the pairs \((\omega, m)\) such that some pre-specified operational limits are not exceeded. Thus, the boundaries between tolerable and not acceptable points in the parameter plane \((\omega, m)\) for three operational limits \((\theta_{\text{max}} = 10^\circ, \theta_{\text{max}} = 25^\circ, \theta_{\text{max}} = 40^\circ)\) are illustrated in figure 11. The need to use additional damping (in this case through the bilge keels) is above all doubt. These border curves are somewhat parallel to those that separate the capsizing and non-capsizing regions in figure 2.
5. Conclusions
In the present study, a simple iterative technique based on Taylor expansion was applied to a nonlinear roll equation for estimating the oscillation amplitudes and to predict the danger of ship capsizing. Comparing it with \texttt{ode45} in Matlab, a high quality numerical integrator, the scheme proved to be not only very precise but also much faster and more flexible. The coefficients of the equation associated with the damping and restoring moments corresponded to a real ferry boat, either in intact or damaged condition, equipped or not with bilge keels. The sea action was described by a single frequency harmonic excitation. The speediness of the scheme was a determining factor in revealing several mechanisms for eroding the ship’s safe basin, in the construction of the normalized integrity curves, amplitude-frequency response curves and of the boundary curves between acceptable and non-acceptable pairs of wave frequencies and forcing amplitudes for some given operational limits. The
usefulness of the bilge keels in preventing large roll amplitudes and ship capsizing, as well as the increased sensitivity of the damaged vessel, were well evidenced by the results obtained.

We conclude that the iterative scheme used in the paper can be considered in any detailed study on the behavior of nonlinear oscillators.

6. References

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