On the geometry of soft breaking terms and $N = 1$ superpotentials

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Abstract

We describe, in the context of $M$ theory on elliptically fibered Calabi Yau fourfolds, the change of variables that allow us to pass from the $U(1)$ invariant elliptic fibration to the one describing the uncompactified four dimensional limit, where the $U(1)$ symmetry is broken. These changes of variables are the analog to the ones used to derive from the Atiyah-Hitchin space the complex structure of the Seiberg-Witten solution for $N = 2$ pure supersymmetric Yang Mills. The connection between these changes of variables and the recently introduced rotation of branes is discussed.
1 Introductory remarks

One of the most interesting aspects of the celebrated Seiberg-Witten solution of $N = 2$ SSYM theories is the way this solution makes operative 't Hooft's confinement program based on the so called ”abelian projection” gauge \cite{3}. The beauty of the SW solution for the supersymmetric case is that it allow us to pass from the simple ”parametrization of ignorance”, summarized in the soft breaking term $\epsilon \text{tr} \phi^2$, to the confinement dynamics of $N = 1$ SSYM theory as described in terms of monopole condensation. In more concrete terms the solution of $N = 2$ SSYM constitutes an effective procedure to pass from the a priori ”blindness” - to concrete, $N = 1$ dynamics- soft breaking term, to a well defined $N = 1$ superpotential with the set of minima predicted by direct computation of $\text{Tr}(-1)^F$ \cite{4}, and gaugino condensates \cite{5, 6, 7, 8, 9}. How this marvelous thing is happening?. The answer is of course enclosed in the rich geometry of the $N = 2$ solution, more specifically in the types of singularities of the elliptic fibration defining the solution. The essential point consists in giving physical meaning to the SW curve itself. This can be done, in principle, in three different ways. One by considering field theory limits of string compactifications \cite{10, 11, 12}, another by working out brane configurations \cite{13}–\cite{30} and, finally, by considering the compactified theory in three dimensions \cite{31}.

The key for this last approach is of course based on the main result of reference \cite{31}, where it is proved that in terms of complex structure the moduli of the $N = 4$ three dimensional supersymmetric theory is the same of that of the $N = 2$ in four dimensions, the difference being determined by the geometry, namely the volume of the elliptic fiber becomes zero in the four dimensional limit (a typical F-theory limit). Most of the comments in this note will refer to issues that can be beautifully understood using brane configuration technology. However, we feel that a different point of view, based in the decompactification picture, can be of some value.

2 Kodaira singularities, soft breaking terms and hyperkähler structure

Let us consider a generic elliptic fibration \cite{32}:
\[ \phi : V \to \Delta, \]
with a set of points $a_i$ where the fiber becomes singular. Let assume $\Delta$ compact and of genus zero. The different singularities in $\Delta$ are characterized in terms of Dynkin diagrams and related monodromy matrices for the elliptic modulus. For the $A$ and $D$ cases we have
\[
A_{n-1} = \begin{pmatrix}
1 & n \\
0 & 1
\end{pmatrix},
\]
\[
D_n = \begin{pmatrix}
-1 & 4 - n \\
0 & -1
\end{pmatrix}.
\]
The singular fiber $C_i$ at the point $a_i$ is characterized by a set of irreducible components satisfying $\Theta_{ij}^2 = -2$, and with the intersection matrix the corresponding affine Dynkin diagram:

$$C_i = \sum_j \Theta_{ij}.$$  \hfill (3)

The singular fiber can be defined \cite{32} simply as the divisor of the holomorphic function $\tau(\phi)$, with $\tau$ a uniformization parameter on $\Delta$. What we will generically call a “geometric” soft breaking term would be defined exactly by this holomorphic function. Now we will consider the elliptic fibration defining the solution of the four dimensional $N = 2$ theory from the point of view of the elliptic fibration defining its $N = 4$ three dimensional compactification. The $N = 2$ soft breaking term, i.e., what is going to play the role of $N = 1$ superpotential, will appear as the ”rotation” in an appropriated sense of the ”geometrical” soft breaking term defined by the $N = 4$ elliptic fibration. In order to be a bit more explicit we need to consider in more detail the role played by the different $U(1)$ entering into the game. Let us recall that the basic dynamical fact underlying the instanton generation of superpotentials in three dimensional supersymmetric gauge theories \cite{33,31} is, on one side, the generation of effective fermionic vertices by three dimensional instantons and, on the other side, the existence of a non anomalous $U(1)$. The way these two facts combine is thanks to the role of the dual photon field as a Goldstone boson and to the form of the instanton contribution \cite{34} as $\exp - I + i \sigma$, with $\sigma$ the dual photon field and $I$ the standard instanton action in the Prasad-Sommerfeld limit. For $N = 4$ three dimensional theories the moduli space is an hyperkähler manifold, that for the case of $SU(2)$ coincides with the Atiyah-Hitchin space for the moduli of two static BPS monopoles \cite{35}. After distinguishing one particular complex structure, the hyperkähler moduli space becomes an elliptic fibration. The rotation group $SU(2)_R$ acting on the Atiyah-Hitchin space gets effectively reduced to a $U(1)$ once we distinguish a particular complex structure. This $U(1)$ is in fact not broken by instanton effects in three dimensions and therefore we should be able to see its action directly on the elliptic fiber. This was explicitely done in reference \cite{31} for the case of $SU(2)$. In fact from the Atiyah-Hitchin solution

$$y^2 = x^2 v + x$$  \hfill (4)

the $U(1)$ $C^*$ action on the curve is defined by:

$$
\begin{align*}
    y & \rightarrow \lambda y, \\
    x & \rightarrow \lambda^2 x, \\
    v & \rightarrow \lambda^{-2} v,
\end{align*}
$$  \hfill (5)

As we observe from equation (5), the $U(1)$ is acting on the $v$-plane. What we have called the ”geometric” soft breaking term can be read off from the curve defined in equation (4); namely, we get

$$v = -\frac{1}{x}.$$  \hfill (6)
Now we want to get from (6) the soft breaking term of the four dimensional $N = 2$ gauge theory. The solution for $SU(2)$ is defined by the following elliptic fibration:

$$y^2 = x^3 - x^2 u + x.$$  \hfill (7)

The first thing to be noticed is that the $U(1)$ is on the $u$-plane, effectively broken to a $Z_2$ subgroup. Moreover, as introduced in reference [31], in order to go from (4) to (7), we can use the following change of variables:

$$v = x - u$$  \hfill (8)

from which it is manifest the existence of only a $Z_2$ subgroup on the $u$ plane once we asign charges 2 and $-2$ to $v$ and $x$ as dictated by (3). The soft breaking term for the $N = 2$ theory, which is our candidate for the superpotential of the four dimensional $N = 1$ theory will be defined by combining the ”rotation” (8) in the $(x, v)$ plane with the ”geometric” soft breaking term given by equation (6). Of course we need to include some scale that is effectively characterizing the breaking of the $U(1)$ symmetry. This scale is $\Lambda_{N=1}$, and should vanish in the compactified $U(1)$ invariant case:

$$U = \Lambda_{N=1}^6 x + \frac{1}{x}.$$  \hfill (9)

This was the derivation of the superpotential in reference [31]. The crucial step is of course to pass from the complex structure with explicit $U(1)$ action, the hyperkähler Atiyah-Hitchin space defined in (4), to the elliptic fibration parametrized by $u$ where the $U(1)$ action is manifestly broken. The change of variables (8) is only allowed when we take out a point in the elliptic fiber. This is related to the fact that the volume of the elliptic fibration becomes infinity in the three dimensional limit. Notice also that the minima of (6), interpreted as a function of $x$, are at the point of infinity. This raise immediately the question on what can be the meaning of the change of variables (8) in the four dimensional limit. In the solution of the $N = 2$ theory the elliptic curve degenerates in two points (for the case of $SU(2)$) where the cycle of the torus vanishes, for that curve it is still possible to define the change of variables (8). In fact what we observe comparing (6) with (4) is that the minima at infinity of (6) become now two finite points, with two well defined ”critical values” precisely at the points in the $u$ plane where the curve degenerates.

3 $U(1)$ and M-theory instantons.

In reference [36] Witten has introduced M-theory instantons as six-cycles of aritmetic genus equal one in M-theory compactifications on Calabi-Yau fourfolds with $SU(4)$ holonomy. Let us consider a Calabi-Yau fourfold elliptically fibered. More specifically we will consider that on the base space $B$ we have a four dimensional locus $C$ on which the fiber
degenerates in some of the ADE Kodaira different types. Moreover and following reference [37] we assume that $C$ satisfies the condition $h_{1,0} = h_{2,0} = 0$. In order to mimic the effect of three dimensional instantons the first thing to do is to properly identify the non anomalous $U(1)$ symmetry for which the instanton defined by the six cycle produces the correct change of $U(1)$ charge. This was done in reference [36]. In fact the normal bundle to the six-cycle, in our case the normal to the locus $C$ in the base space $B$, is one complex dimensional, and the desired $U(1)$ can be identify with

$$z \to e^{i\theta} z. \quad (10)$$

By analogy with what we did in the $SU(2)$ case in the previous section we should now identify the coordinate $z$ on the normal bundle as the analog of the $v$ variable, as on both of them, there is a well defined $U(1)$ action \[1\]. Now we need the analog of equation (10); what we have called the "geometric" soft breaking term. The coordinate $z$ has the appropriate $U(1)$ charge to define a superpotential in the $N = 2$ three dimensional theory. Let us suppose that we are dealing with a singularity of type $A_{n-1}$, then following [37] we can associate with each irreducible component a six-cycle defined by just fibering the irreducible component on the locus $C$. These are six-cycles of arithmetic genus equal one. We will only consider the irreducible components that contribute to the Picard group, so for a singularity of type $A_{n-1}$ we have $n-1$ contributions. Each one can be represented by a variable $1/x$ with the appropriated $U(1)$ charge, hence the analog of equation (6) of the previous section should be

$$z = -\frac{n-1}{x} \quad (11)$$

Now the rule we have abstracted from the previous analysis of $N = 4$ three dimensional $SU(2)$ super Yang Mills would be -after interpreting the variable $z$ in parallel to the $v$ of the previous section - to define the soft breaking term of the $N = 2$ four dimensional theory i.e the superpotential of the four dimensional $N = 1$ theory. This can be done by the equivalent to the change of variables (8), which now should mix the $x$ and $z$ variables. In other words, we are rotating the elliptic fiber of the elliptically fibered Calabi-Yau fourfold to a new elliptic fibration variable, $u$, and the way to do it would be simply by, as was the case in the field theory analysis of the previous section, breaking the $U(1)$ symmetry this time to $Z_n$:

$$z = x^{n-1} - u, \quad (12)$$

which implies for $u$, using the $U(1)$ charges \[2\] for $v$ and $x$ a residual $Z_n$ symmetry. In fact, it follows from (12) that the residual symmetry is characterized by $u \to \lambda^2 u$ with $\lambda^{2n} = 1$. As before the superpotential will be now obtained combining (11) and (12) and including the scale of the $U(1)$ breaking,

$$U = \Lambda^{3n} \sum_{N=1}^{n-1} x^{n-1} + \frac{n-1}{x} \quad (13)$$

\[1\]We are effectively thinking of the elliptically fibered $z$-plane as the analog of the $N = 4$ moduli in three dimensions. The topological conditions we have imposed on $C$ seem to allow that picture.
The contribution (11) survives the decompactification limit since it is coming from vertical instantons [36]. This superpotential has the nice features we expect in $N = 1$ four dimensions. When we consider (12) as a superpotential and we look for its “critical values” in the $u$ plane it seems like if, by going from the $U(1)$ invariant picture in the $z$ plane to the one with broken $U(1)$ in the $u$ plane, we have effectively ”broken” the initial $A_{n-1}$ singularity into $n A_0$ singularities, separated by the scale of the $U(1)$ breaking i.e $\Lambda_{N=1}$. We would like to interpret this effective ”breaking” of the elliptic singularity, when we move to the $u$ plane as reflecting a type of confinement dynamics based on ”liquids” of fractional topological objects separated a distance of the order of the scale $\Lambda_{N=1}$ [38]. Here it is important to stress that fractional topological objects have nothing to do a priori with any fractionalization of the standard $\theta \rightarrow \theta + 2\pi$ symmetry transformation. In fact if one uses torons [39] as fractional topological objects to get the gaugino condensate, what one gets is $< \lambda \lambda > \sim \Lambda_{N=1}^{-2} e^{2\pi i l/n}$ with $l = 0, \ldots, n - 1$ representing an electric flux flow. By changing $\theta \rightarrow \theta + 2\pi$ we pass from one vacua to another as a direct consequence of Witten’s dyon effect [14]. In fact the electric flow changes as $l + \frac{\theta m}{2\pi}$. In reference [30] a beautiful proposal for QCD string has been made on the basis of a D-brane interpretation of domain walls. If we consider (13) in the same spirit as a Landau-Ginzburg superpotential the domain wall would be characterized by the difference between two consecutive critical values. Now, if we use the toron representation of the gaugino condensate in terms of ’t Hooft’s electric flux, the operator to transit from one vacua to another would be the Wilson loop, which would in principle allow the comparison of both quantities.

4 M-theory fivebranes.

The physical picture of instantons in [36] was that of fivebranes wrapping on the six cycles of the Calabi-Yau fourfold. Recently a description of four dimensional $N = 1$ theories has been done using branes configurations and M-theory [29, 30]. In that picture a rotation of branes [13] was introduced in order to break supersymmetry. We think that that type of rotations in the brane description of $N = 1$ four dimensional theories is equivalent to the change of variables (12) suggested by the three dimensional case as given by (8). Moreover the normal bundle is playing in this picture the analog role of the $(8, 9)$-plane of references [24, 30], and the conditions for rotating the curve are the analogous to those necessary for giving sense to the change of variables (8) and (12). The hope of this note is by relating the superpotentials of type (12), much in the spirit of reference [37], to M-theory instantons, to point out to the common dynamical mechanism of confinement and gaugino condensates. On the other hand it would maybe be interesting to work out in more detail the connection between brane manipulations, as for instance rotations, and the type of variable changes underlying the relation between the elliptic fibration for three dimensional theories and those of their uncompactified four dimensional limit, mostly taking into account that they share common complex structures.
Acknowledgments

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