\textbf{H-HARMONIC BERGMAN PROJECTION ON THE HYPERBOLIC BALL}

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\textbf{Abstract.} We determine precisely when the Bergman projection $P_\beta$ is bounded from Lebesgue spaces $L^p_\alpha$ to weighted Bergman spaces $B^p_\alpha$ of \( \mathcal{H} \)-harmonic functions on the hyperbolic ball, and verify a recent conjecture of M. Stoll. We obtain upper estimates for the reproducing kernel of the \( \mathcal{H} \)-harmonic Bergman space $B^2_\alpha$ and its partial derivatives. We also consider the projection from $L^\infty$ to the Bloch space $\mathcal{B}$ of \( \mathcal{H} \)-harmonic functions.

1. \textbf{Introduction}

We follow the notation and terminology of the book [PS] which we refer the reader for more details. For $n \geq 2$, let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product and $| \cdot |$ the corresponding norm in $\mathbb{R}^n$. Let $\mathbb{B} = \{ x \in \mathbb{R}^n : |x| < 1 \}$ be the unit ball and $S = \partial \mathbb{B}$ the unit sphere.

The hyperbolic ball is $\mathbb{B}$ equipped with the hyperbolic metric

$$ds = \frac{2}{1 - |x|^2} |dx|.$$  

For $f \in C^2(\mathbb{B})$, the Laplace-Beltrami operator associated with the hyperbolic metric (the invariant Laplacian) is given by (up to a constant factor)

$$\Delta_h f(x) = (1 - |x|^2)[(1 - |x|^2)\Delta f(x) + 2(n - 2)\langle x, \nabla f(x) \rangle],$$

where $\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$ and $\nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ are the Euclidean Laplacian and gradient. It is straightforward to show that $\Delta_h f(a) = \Delta(f \circ \varphi_a)(0)$, where

$$\varphi_a(x) = \frac{a|x - a|^2 + (1 - |a|^2)(a - x)}{|x, a|^2},$$

is a Möbius transformation mapping $\mathbb{B}$ to $\mathbb{B}$ and exchanging $a$ and $0$. Here $[x, a]$ is defined as

$$[x, a] := \sqrt{1 - 2(x, a) + |x|^2|a|^2} \quad (x, a \in \mathbb{B}).$$

A complex-valued function $f \in C^2(\mathbb{B})$ is called hyperbolic harmonic or \( \mathcal{H} \)-harmonic (sometimes also called \( \mathcal{M} \)-harmonic) on $\mathbb{B}$, if $\Delta_h f(x) = 0$ for all $x \in \mathbb{B}$. We denote the space of all $\mathcal{H}$-harmonic functions by $\mathcal{H}(\mathbb{B})$. In case $n = 2$, \( \mathcal{H} \)-harmonic functions coincide with Euclidean harmonic functions for which all the results mentioned below are well-known. Therefore, from now on we assume $n \geq 3$. 

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Let $\nu$ be the Lebesgue measure on $B$ normalized so that $\nu(B) = 1$. For $\alpha > -1$, define the weighted measures
$$d\nu_\alpha(x) = \frac{1}{V_\alpha(1 - |x|^2)} d\nu(x),$$
where the normalizing constant
$$V_\alpha = \frac{n}{2} B \left( \frac{n}{2}, \alpha + 1 \right)$$
so that $\nu_\alpha(B) = 1$. For $0 < p < \infty$ and $\alpha > -1$, the Lebesgue space $L^p_\alpha = L^p(B, d\nu_\alpha)$ consists of all measurable functions $f$ on $B$ such that
$$\|f\|_{p,\alpha} = \left( \int_B |f(x)|^p d\nu_\alpha(x) \right)^{1/p} < \infty.$$The subspace $B^p_\alpha$ consisting of $H$-harmonic functions is called the $H$-harmonic weighted Bergman space,
$$B^p_\alpha = \{ f \in H(B) : \|f\|_{p,\alpha} < \infty \}.$$
When $1 \leq p < \infty$, $B^p_\alpha$ is a Banach space on which point evaluation functionals $f \mapsto f(x)$ are bounded. In particular, $B^2_\alpha$ is a reproducing kernel Hilbert space and for each $x \in B$, there exists a unique $R_\alpha(x, \cdot) \in B^2_\alpha$ such that
$$f(x) = \int_B f(y) R_\alpha(x, y) d\nu_\alpha(y) \quad (f \in B^2_\alpha).$$The reproducing kernel $R_\alpha(\cdot, \cdot)$ is real-valued (so the conjugation in (2) can be removed), symmetric in its variables and is $H$-harmonic with respect to each variable.

For $\alpha > -1$, the Bergman projection operator $P_\alpha$ is defined by
$$P_\alpha f(x) = \int_B R_\alpha(x, y) f(y) d\nu_\alpha(y)$$for suitable $f$. It follows from (2) that $P_\alpha$ is the orthogonal projection of $L^2_\alpha$ onto $B^2_\alpha$.

The main purpose of this paper is to determine exactly when the operators $P_\beta : L^p_\alpha \to B^p_\alpha$ are bounded.

**Theorem 1.1.** Let $1 \leq p < \infty$ and $\alpha, \beta > -1$. Then $P_\beta : L^p_\alpha \to B^p_\alpha$ is bounded if and only if
$$\alpha + 1 < p(\beta + 1).$$In case (3) holds, $P_\beta$ restricted to $B^p_\alpha$ is identity.

This verifies a recent conjecture of M. Stoll (see [29, Section 6]). Actually, when the dimension $n$ is even, the sufficiency part of this theorem has already been proved in [24, Theorem IV.14]. The difference between even and odd dimensions and the reason why the problem is more difficult in odd dimensions will be explained below.

We remark that Theorem 1.1 is true for all $n \geq 3$ regardless of the parity of the dimension and shows that the condition (3) is both necessary and sufficient.

To prove Theorem 1.1 we need upper and lower estimates of the kernel $R_\alpha$. This is a nontrivial problem since there is no known closed formula for $R_\alpha$. However, it is known that it has the series expansion ([24, Corollary III.5], [29, Theorem 5.3])
$$R_\alpha(x, y) = \sum_{m=0}^{\infty} c_m(\alpha) S_m(|x|) S_m(|y|) Z_m(x, y),$$
where
\[ S_m(r) = \frac{F(m, 1 - \frac{n}{2}; m + \frac{n}{2}; r^2)}{F(m, 1 - \frac{n}{2}; m + \frac{n}{2}; 1)}, \]
with \( F = \frac{\pi}{2} F_1 \) is the Gauss hypergeometric function, \( Z_m \) is the zonal harmonic of degree \( m \) and the coefficients \( c_m(\alpha) \) are given by
\[ \frac{1}{c_m(\alpha)} = \frac{1}{V_\alpha} n \int_0^{1} r^{2m+n-1} (1 - r^2)^\alpha S_m^2(r) \, dr =: I_m. \]

We compare this with the Euclidean harmonic case. Harmonic weighted Bergman space \( b_\alpha^2 \) has the reproducing kernel (see [5, p. 164], [20, Proposition 3])
\[ R_\alpha(x, y) = \sum_{m=0}^{\infty} \gamma_m(\alpha) Z_m(x, y), \]
where
\[ \frac{1}{\gamma_m(\alpha)} = \frac{1}{V_\alpha} n \int_0^{1} r^{2m+n-1} (1 - r^2)^\alpha \, dr = \frac{\Gamma(\alpha + \frac{n}{2} + 1)\Gamma(m + \frac{n}{2})}{\Gamma(\frac{n}{2})\Gamma(m + \alpha + \frac{n}{2} + 1)} \frac{\Gamma(m + \alpha + n)}{\Gamma(m + n - 1)}. \]

Note that formulas may differ by a constant factor depending on whether the normalizing constant \( V_\alpha \) is used or not. Upper and lower estimates of the Euclidean Bergman kernel \( R_\alpha \) and its derivatives are by now well-established. If we compare (4) and (7), we see that in (4) there appears the extra factor \( S_m \). However it is not difficult to deal with this extra term and the main difficulty lies in the coefficient \( c_m(\alpha) \). Although the integral in (8) is easily evaluated, a closed formula for the hypergeometric integral in (6) is not known.

We remark that in even dimensions the hypergeometric function in (5) is just a polynomial since \( 1 - \frac{n}{2} \) is a negative integer and the hypergeometric series terminates. For this reason it is easier to work in even dimensions and when the dimension is also small the integral in (8) can be explicitly evaluated and a closed formula for \( c_m(\alpha) \) can be written. We will not use this fact and all our results and methods below will be independent of dimension and will equally work in both even and odd dimensions.

It is shown in [24, Theorem III.6] that
\[ c_m(\alpha) \sim m^{\alpha+1} \quad (m \to \infty). \]

We will show later that the following stronger estimate is true
\[ c_m(\alpha) = \frac{\Gamma(m + \alpha + n)}{\Gamma(m + n - 1)} \left( D_0 + O\left( \frac{1}{m} \right) \right) \quad (m \to \infty), \]
where the constant \( D_0 = D(\alpha, n) \) is given by
\[ D_0 = \frac{\Gamma^2(n-1)\Gamma(\alpha + n + 1)\Gamma(\frac{n}{2} + 1)\Gamma(\frac{n}{2} + n)}{\Gamma(\frac{n}{2})\Gamma(\alpha + \frac{n}{2} + 1)\Gamma(\alpha + 2n - 1)\Gamma(\frac{2n}{2} + 2)\Gamma(\frac{2n}{2} + 1)}. \]

However, neither of these are sufficient to estimate \( R_\alpha \). Our first aim is to show that \( 1/c_m(\alpha) = I_m \) is given by the following series.

**Theorem 1.2.** There exist constants \( A_k \) (\( k = 0, 1, 2, \ldots \)) depending only on \( \alpha \) and \( n \) such that for \( m \geq 1, \)
\[ I_m = \frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \sum_{k=0}^{\infty} \frac{A_k}{(m + \alpha + n)_k}. \]
The exact value of $A_k$ is given by

$$A_k = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(\alpha + n)\Gamma(\alpha + 2n - 1)\Gamma(\alpha + 1)k}{\Gamma^2(n - 1)\Gamma(\alpha + \frac{3n}{2})\Gamma^2(n - 1)\Gamma(\alpha + 3n^2)}$$

$$\times {}_3F_2\left[\frac{n}{2}, \alpha + n, 1 - \frac{n}{2}; \alpha + \frac{3n}{2} + 1 + k, \alpha + 3n^2; 1\right].$$

(12)

The formula for $A_k$ is complicated but these exact values will not be important for the size estimates of $R_\alpha$. What matters only is the form of the series in (11) which, in fact, is a convergent asymptotic series since $(m + \alpha + n)_k$ is a polynomial in $m$ of degree $k$. From this the asymptotic expansion of $c_m(\alpha)$ will easily follow (see Corollary 3.3 below) which will lead to the following upper estimate of $R_\alpha$.

**Theorem 1.3.** Let $\alpha > -1$. There exists a constant $C > 0$ depending only on $\alpha$ and $n$ such that for all $x, y \in \mathbb{B}$,

$$|R_\alpha(x, y)| \leq \frac{C}{|x, y|^{\alpha + n}}.$$  

Moreover, the exponent $(\alpha + n)$ is best possible.

Comparing this with the Euclidean case, we see that exactly same estimate holds for $R_\alpha$. This leads, naturally, to the fact that the condition (3) is same as the Euclidean case. As a matter of fact this condition is also same for Bergman spaces of holomorphic functions, see [10, Theorem 1.10].

By a common technique (applying Schur’s test) the upper estimate in Theorem 1.3 proves the “if” part of Theorem 1.1. However to show the “only if” part we need lower estimates of $R_\alpha$. To achieve this we first estimate first-order partial derivatives of $R_\alpha$.

**Theorem 1.4.** There exists a constant $C > 0$ depending only on $\alpha$ and $n$ such that for all $x, y \in \mathbb{B}$ and $i = 1, 2, \ldots, n$,

$$\frac{\partial}{\partial x_i} R_\alpha(x, y) \leq \frac{C}{|x, y|^{\alpha + n + 1}}.$$  

We remark that this theorem is true for all $n \geq 3$. On the other hand, presumably, higher-order derivatives of $R_\alpha$ will be dimension dependent and will behave differently in even and odd dimensions, see [9] and [12] for similar results in other contexts.

The required lower estimate will be obtained in Proposition 6.2 below which will lead to the following two-sided estimate of weighted integrals of $R_\alpha$ which is of independent interest.

**Theorem 1.5.** For $\alpha, \beta > -1$ and $0 < p < \infty$, set $c = p(\alpha + n) - (\beta + n)$. For all $x \in \mathbb{B}$,

$$\int_\mathbb{B} |R_\alpha(x, y)|^p (1 - |y|^2)^\beta \, dv(y) \sim \begin{cases} 
1, & \text{if } c > 0; \\
(1 - |x|^2)^c, & \text{if } c = 0; \\
1 + \log \frac{1}{1 - |x|^2}, & \text{if } c < 0.
\end{cases}$$

The implied constants depend only on $a, b, \alpha, n$ and are independent of $x$. 

The “only if” part of Theorem 1.1 will follow from this theorem. The projection theorem immediately leads to duality. The proof of the following corollary is similar to the proof of [10, Theorem 1.16] and is omitted.

**Corollary 1.6.** Let \( \alpha > -1 \) and \( 1 < p < \infty \). The dual of \( B^p_\alpha \) can be identified with \( B^q_\alpha \) under the pairing

\[
(f, g)_\alpha = \int_{B} f(x) \overline{g(x)} \, d\nu_\alpha(x),
\]

where \( 1/p + 1/q = 1 \).

Finally we consider the case \( p = \infty \). Denote the space of all essentially bounded functions on \( B \) by \( L^\infty = L^\infty(B) \). The gradient \( \nabla^h f \) of the hyperbolic metric (the invariant gradient) is given by

\[
(\nabla^h f)(a) = -\nabla (f \circ \varphi_a)(0) = (1 - |a|^2) \nabla f(a) \quad (a \in B),
\]

for \( f \in C^1(B) \). The \( \mathcal{H} \)-harmonic Bloch space \( \mathcal{H} \) consists of all \( f \in \mathcal{H}(B) \) such that

\[
\sup_{x \in B} |\nabla^h f(x)| = \sup_{x \in B} (1 - |x|^2) |\nabla f(x)| < \infty
\]

with norm

\[
\|f\|_\mathcal{H} = |f(0)| + \sup_{x \in B} |\nabla^h f(x)|.
\]

**Theorem 1.7.** For every \( \alpha > -1 \), \( P_\alpha : L^\infty \rightarrow \mathcal{H} \) is bounded.

This paper is organized as follows. In the Preliminaries section we recall some known facts about (generalized) hypergeometric functions and a few integral estimates that will be used later. In Section 3 we prove Theorem 1.2 and obtain asymptotic expansion of the coefficient \( c_m(\alpha) \). In Section 4 we obtain the upper estimate of \( \mathcal{R}_\alpha \). Our method will be general and will also give an upper estimate of the Hardy kernel improving [30, Theorem 3.2]. In Section 5 we estimate partial derivatives of \( \mathcal{R}_\alpha \) and prove Theorem 1.4, and in Section 6 we prove the two-sided estimate in Theorem 1.5. We prove the projection theorems, Theorems 1.1 and 1.7, in the final section.

## 2. Preliminaries

### 2.1. Notation

We use the letter \( C \) to denote positive constants whose exact value may differ at each occurrence. For two positive expressions \( X \) and \( Y \) we write \( X \lesssim Y \) to mean \( X \leq CY \) for some \( C > 0 \). The constant \( C \) may depend on the parameters \( \alpha, \beta, n \) etc. that are fixed beforehand, but will be independent of the variables \( x, y \in B \) etc. If both \( X \lesssim Y \) and \( Y \lesssim X \), we write \( X \sim Y \).

The Beta function is given by

\[
B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (\text{Re}\{a\} > 0, \text{Re}\{b\} > 0),
\]

where \( \Gamma \) is the Euler Gamma function.

For \( a \in \mathbb{C} \) and \( k \) non-negative integer, the rising factorial \( (a)_k \) is defined as \( (a)_0 := 1 \) and for \( k \geq 1 \), \( (a)_k := a(a+1) \cdots (a+k-1) \). If \( a \neq 0, -1, -2, \ldots \),

\[
(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.
\]
By Stirling’s formula, for \(a, b \in \mathbb{C}\) and \(k\) non-negative integer

\[
\frac{\Gamma(k + a)}{\Gamma(k + b)} \sim k^{a-b}, \quad \frac{(a)_k}{(b)_k} \sim k^{a-b}, \quad k \to \infty.
\]

For \(x, y \in \mathbb{R}^n\), the Ahlfors bracket \([x, y] = \sqrt{1 - 2(x, y) + |x|^2|y|^2}\). It is symmetric and if either of the variables is 0, then \([x, 0] = [0, y] = 1\). Otherwise

\[
[x, y] = \left|y|x - \frac{y}{|y|}\right| = \left|\frac{x}{|x|} - |x|y\right|.
\]

It is clear that

\[
[x, y] \geq 1 - |x||y|.
\]

For \(f \in L^1(\mathbb{S})\), integration in polar coordinates is (see [2, p. 6])

\[
\int_{\mathbb{S}} f \, d\nu(x) = n \int_{0}^{1} r^{n-1} \int_{\mathbb{S}} f(r\zeta) \, d\sigma(\zeta) \, dr,
\]

where \(\sigma\) is normalized surface-area measure on \(\mathbb{S}\).

2.2. Zonal Harmonics. We briefly review the basic properties of zonal harmonics that will be used later. For details we refer the reader to [2], Chapter 5]. Denote by \(H_m(\mathbb{R}^n)\), the linear space of all homogeneous harmonic polynomials of degree \(m\). It is finite dimensional with \(\dim H_m \sim m^{n-2}\) as \(m \to \infty\). By homogeneity, \(p_m \in H_m(\mathbb{R}^n)\) is determined by its restriction to \(\mathbb{S}\). This restriction is called a spherical harmonic and the space of all spherical harmonics of degree \(m\) is denoted \(H_m(\mathbb{S})\). Spherical harmonics of different degrees are orthogonal in \(L^2(\mathbb{S})\):

\[
\int_{\mathbb{S}} p_m(\zeta) \, p_k(\zeta) \, d\sigma(\zeta) = 0 \quad (m \neq k, \, p_m \in H_m(\mathbb{S}), \, p_k \in H_k(\mathbb{S})).
\]

Point evaluation functionals are bounded on \(H_m(\mathbb{S})\). So, for every \(\eta \in \mathbb{S}\) there exists \(Z_m(\cdot, \eta) \in H_m(\mathbb{S})\), called the zonal harmonic of degree \(m\) with pole \(\eta\), such that

\[
p_m(\eta) = \int_{\mathbb{S}} p_m(\zeta) Z_m(\zeta, \eta) \, d\sigma(\zeta) \quad (p_m \in H_m(\mathbb{S})).
\]

\(Z_m(\cdot, \cdot)\) is real-valued, symmetric, and homogenous of degree \(m\) with respect to each variable. When \(m = 0\), \(Z_0 = 1\) and when \(m \geq 1\),

\[
Z_m(\zeta, \zeta) = \dim H_m \sim m^{n-2} \quad \text{and} \quad |Z_m(\zeta, \eta)| \leq Z_m(\zeta, \zeta) \quad (\zeta, \eta \in \mathbb{S}).
\]

2.3. Hypergeometric Functions. Let \(a, b, c \in \mathbb{C}\) with \(c \neq 0, -1, -2, \ldots\). For \(|z| < 1\), the Gauss hypergeometric function \(F(a, b; c; z) = {}_2F_1(a, b; c; z)\) is defined by the series

\[
F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.
\]

The function \(F(a, b; c; z)\) can be analytically continued to \(\mathbb{C}\setminus[1, \infty)\) which we denote by the same symbol, see [16, §9.1].

Remark 2.1. If \(\text{Re}\{c - a - b\} > 0\), then by (15), the series in (22) converges absolutely and uniformly on the closed disk \(|z| \leq 1\) and at \(z = 1\) by [16] Eq. 9.3.4,

\[
F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}.
\]

For a proof of the following transformation formula, see [16] Eq. 9.5.3].
Lemma 2.2. Let \(a, b, c \in \mathbb{C}\) with \(c \neq 0, -1, -2, \ldots\). For \(z \in \mathbb{C}\setminus[1, \infty)\),
\[
F(a, b; c; z) = (1 - z)^{c-a}F(c-a, c-b; c; z).
\]

A proof of the following integral formula for \(F(a, b; c; z)\) can be found in [10] Eq. 9.1.4. The functions in the integrand take their principal values.

Lemma 2.3. Suppose \(\text{Re}\{c\} > \text{Re}\{a\} > 0\). Then for \(|z| < 1\),
\[
F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt.
\]
Moreover, the right-hand side gives the analytic continuation of \(F(a, b; c; z)\) to \(\mathbb{C}\setminus[1, \infty)\). Therefore the above formula is true for all \(z \in \mathbb{C}\setminus[1, \infty)\).

Remark 2.4. The integral above shows that if \(a, b, c\) are real and \(c > a > 0\), then \(F(a, b; c; z) > 0\) when \(-\infty < z < 1\). If, in addition \(b < 0\), then \(F(a, b; c; z)\) is decreasing on the interval \(-\infty < z < 1\).

The next identity appears in [7, Eq.(4), p.399]. For a proof, see [17, Lemma 2.1].

Lemma 2.5. Suppose \(\text{Re}\{c\} > 0, \text{Re}\{\rho\} > 0\) and \(\text{Re}\{c-a-b+\rho\} > 0\). Then
\[
\int_0^1 t^{c-1}(1-t)^{a-1}F(a, b; c; t) dt = \frac{\Gamma(c)\Gamma(\rho)\Gamma(c-a-b+\rho)}{\Gamma(c-a+\rho)\Gamma(c-b+\rho)}.
\]

2.4. Generalized Hypergeometric Function \(\mathbf{3}_F_2\). Let \(a, b, c, d, e \in \mathbb{C}\) with \(d\) and \(e\) are different than \(0, -1, -2, \ldots\). For \(|z| < 1\), the generalized hypergeometric series \(\mathbf{3}_F_2\left[\begin{array}{c}a, b, c \\ d, e \end{array}; z\right]\) is defined by
\[
\mathbf{3}_F_2\left[\begin{array}{c}a, b, c \\ d, e \end{array}; z\right] = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k k!} z^k.
\]
The function \(\mathbf{3}_F_2\) can be analytically continued to \(\mathbb{C}\setminus[1, \infty)\) which we denote by the same symbol.

Remark 2.6. If \(\text{Re}\{d + e - a - b - c\} > 0\), then by [15] the series in (24) converges absolutely and uniformly for \(|z| \leq 1\).

Remark 2.7. The \(\mathbf{3}_F_2\) is symmetric with respect to its upper variables \(a, b, c\) and they can be permuted. Same is true for the lower variables \(d, e\).

Similar to Lemma 2.3 there is an integral formula for \(\mathbf{3}_F_2\). It can be verified by expanding \((1 - zt_1t_2)^{-c}\) in series (see [10, Eq.(1.5)]).

Lemma 2.8. Suppose \(\text{Re}\{d\} > \text{Re}\{a\} > 0\) and \(\text{Re}\{\epsilon\} > \text{Re}\{b\} > 0\). For \(|z| < 1\),
\[
\mathbf{3}_F_2\left[\begin{array}{c}a, b, c \\ d, e \end{array}; z\right] = \frac{\Gamma(d)\Gamma(e)}{\Gamma(d-a)\Gamma(b)\Gamma(e-b)}
\]
\[
\times \int_0^1 \int_0^1 t_1^{a-1}(1-t_1)^{d-a-1}t_2^{b-1}(1-t_2)^{e-b-1}(1-zt_1t_2)^{-c} dt_1t_2.
\]
Moreover, the right-hand side gives the analytic continuation of \(\mathbf{3}_F_2\) to \(\mathbb{C}\setminus[1, \infty)\).

Lemma 2.9. Suppose \(a, b, c, d, e\) are all real with \(d, e \neq 0, -1, -2, \ldots\).

(i) If \(d > a > 0\) and \(e > b > 0\), then \(\mathbf{3}_F_2\left[\begin{array}{c}a, b, c \\ d, e \end{array}; z\right] > 0\) for \(-\infty < z < 1\).
(ii) If \( d > a > 0, \ e > b > 0 \) and \( c < 0 \), then \( \genfrac{[}{]}{}{3}{a, b, c}{d, e} \) is decreasing on the interval \((-\infty, 1)\).

(iii) If \( d > a > 0, \ e > b > 0 \) and \( c < 0 \), then \( 0 \leq \genfrac{[}{]}{}{3}{a, b, c}{d, e} \leq 1 \).

Proof. Parts (i) and (ii) follow from the integral in Lemma 2.8. For part (iii) note that by Remark 2.6, \( \genfrac{[}{]}{}{3}{a, b, c}{d, e} \) is well-defined and left continuous at \( z = 1 \).

Since at \( z = 0, \ genfrac{[}{]}{}{3}{a, b, c}{d, e} = 1 \), the result follows from parts (i) and (ii). \( \square \)

The next identity is taken from [7, Eq.(6), p. 399]. However, there is a typo in [7] where \((1 - z)^{-\sigma} \) is written as \((1 - z)^{\sigma} \). Therefore we give a proof.

Lemma 2.10. Suppose \( \Re\{c\} > 0, \ \Re\{\rho\} > 0 \) and \( \Re\{c - a - b + \rho\} > 0 \). Then for \( z \in \mathbb{C}\setminus[1, \infty) \),

\[
\int_0^1 t^{c-1}(1-t)^{\rho-1}(1-zt)^{-\sigma} F(a,b;c;t) \, dt = \frac{\Gamma(c)\Gamma(\rho)\Gamma(c-a-b+\rho)}{\Gamma(c-a+\rho)\Gamma(c-b+\rho)} (1-z)^{-\sigma} \times \genfrac{[}{]}{}{3}{\rho, \sigma, c-a-b+\rho}{c-a+\rho, c-b+\rho, z-1}.
\]

Proof. Call the integral on the left as \( I \). Assume first that \(|\frac{z}{z-1}| < 1 \). Because

\[
1-zt = 1-z+z(1-t) = (1-z)\left(1 - \frac{z}{z-1} (1-t)\right),
\]

we have

\[
(1-zt)^{-\sigma} = (1-z)^{-\sigma} \sum_{m=0}^{\infty} \frac{(\sigma)_k}{k!} \left(\frac{z}{z-1}\right)^k (1-t)^k,
\]

where the series uniformly converges for \( 0 \leq t \leq 1 \). Inserting this into the integral and changing the orders of the series and the integral we obtain

\[
I = (1-z)^{-\sigma} \sum_{m=0}^{\infty} \frac{(\sigma)_k}{k!} \left(\frac{z}{z-1}\right)^k \int_0^1 t^{c-1}(1-t)^{\rho+k-1} F(a,b;c;t) \, dt.
\]

Computing the integral on the right with Lemma 2.5 and applying (14) proves the result when \(|\frac{z}{z-1}| < 1 \). The general case follows from analytic continuation. \( \square \)

We will need two identities about the value of an \( \genfrac{[}{]}{}{3}{a}{d} \) at \( z = 1 \). The first one is due to Kummer (see [23, 16.4.11] or [11, Corollary 3.3.5, p. 142]).

Lemma 2.11. If \( \Re\{d + e - a - b - c\} > 0 \) and \( \Re\{e - a\} > 0 \), then

\[
\genfrac{[}{]}{}{3}{a, b, c}{d, e} = \frac{\Gamma(e)\Gamma(d + e - a - b - c)}{\Gamma(e - a)\Gamma(d + e - b - c)} \genfrac{[}{]}{}{3}{a, d - b, d - c}{d, d + e - b - c}.
\]

The next identity is called Dixon’s well-poised sum (see [23, 16.4.4] or [11, Theorem 3.4.1, p. 143]).

Lemma 2.12. If \( \Re\{a - 2b - 2c\} > -2 \), then

\[
\genfrac{[}{]}{}{3}{a, b, c}{a - b + 1, a - c + 1} = \frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma\left(\frac{a}{2}-b-c+1\right)}{\Gamma(a+1)\Gamma\left(\frac{a}{2}-b+1\right)\Gamma\left(\frac{a}{2}-c+1\right)\Gamma(a-b-c+1)}.
\]
2.5. The Factor $S_m(r)$. Let $p_m \in H_m(\mathbb{S})$. The solution of the Dirichlet problem for $p_m$ in the hyperbolic ball is given by (see [23] Section 6.1)

$$f(x) = S_m(|x|)p_m(x) \quad (x \in \mathbb{B})$$

That is, $f$ is $H$-harmonic on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$ and equals $p_m$ on $\mathbb{S}$. Here, the factor $S_m(r)$ is as given in [23]. It is a hypergeometric function normalized so that $S_m(1) = 1$. When $m = 0$, $S_0(r) = 1$. When $m \geq 1$ by [23],

$$F(m, 1 - \frac{n}{2}; m + \frac{n}{2}; 1) = \frac{\Gamma(m + \frac{n}{2})\Gamma(n - 1)}{\Gamma(\frac{n}{2})\Gamma(m + n - 1)} = \frac{B(m, n - 1)}{B(m, \frac{n}{2})}$$

so that

$$S_m(r) = \frac{B(m, \frac{n}{2})}{B(m, n - 1)} F(m, 1 - \frac{n}{2}; m + \frac{n}{2}; r^2).$$

The following estimate is proved in [30] Lemma 2.6. A different proof is given below.

**Lemma 2.13.** There exists a constant $C > 0$ depending only on $n$ such that for all $m \geq 1$ and $0 \leq r \leq 1$,

$$1 \leq S_m(r) \leq Cr^\frac{n}{2}.$$

**Proof.** By (25) and Lemma 2.3 the integral formula

$$S_m(r) = \frac{1}{B(m, n - 1)} \int_0^1 t^{m-1}(1 - t)^{\frac{n}{2}-1}(1 - r^2 t)^{\frac{n}{2}-1} dt$$

holds. As in Remark 2.4 it follows from the above integral that $S_m(r)$ is decreasing on the interval $0 \leq r \leq 1$. Thus $S_m(1) \leq S_m(r) \leq S_m(0)$. Putting $r = 0$ in (26) shows $S_m(0) = B(m, \frac{n}{2})/B(m, n - 1)$. Thus

$$1 \leq S_m(r) \leq \frac{B(m, \frac{n}{2})}{B(m, n - 1)} = \frac{\Gamma(\frac{n}{2})\Gamma(m + n - 1)}{\Gamma(\frac{n}{2})\Gamma(m + n - 1)},$$

and the result follows from (15). □

2.6. Integral Estimates. In the sequel we will need three integral estimates which have been repeatedly proved in various places and we don’t know the original sources. The first one is elementary.

**Lemma 2.14.** Let $b > -1$ and $c \in \mathbb{R}$. For $0 \leq r < 1$,

$$\int_0^1 \frac{(1 - t)^b dt}{(1 - rt)^{1+b+c}} \sim \begin{cases} \frac{1}{(1-r)^c}, & \text{if } c > 0; \\ \frac{1}{1 + \log \frac{1}{1-r}}, & \text{if } c = 0; \\ 1, & \text{if } c < 0. \end{cases}$$

The implied constants depend only on $b, c$ and are independent of $r$.

A proof of the next lemma can be found in, for example, [8] Lemma 6.1.

**Lemma 2.15.** Let $a, b > -1$ and $c \in \mathbb{R}$. For $x \in \mathbb{B}$ and $\eta \in \mathbb{S}$,

$$\int_0^1 \frac{t^a(1 - t)^b dt}{|tx - \eta|^{1+b+c}} \sim \begin{cases} \frac{1}{|x - \eta|^c}, & \text{if } c > 0; \\ \frac{1}{1 + \log \frac{1}{|x - \eta|}}, & \text{if } c = 0; \\ 1, & \text{if } c < 0. \end{cases}$$
The implied constants depend only on \( a, b, c, n \) and are independent of \( x \).

The following lemma is the real analog of [27, Proposition 1.4.10]. For a proof see, for example, [18, Proposition 2.2].

**Lemma 2.16.** Let \( b > -1 \) and \( c \in \mathbb{R} \). For \( x \in B \),

\[
\int_B \frac{1 - |y|^2b}{|x, y|^{n+b+c}} \, d\nu(y) \sim \begin{cases} 
1, & \text{if } c > 0; \\
(1 - |x|^2)^{c/2}, & \text{if } c = 0; \\
1 + \log \frac{1}{1 - |x|^2}, & \text{if } c < 0.
\end{cases}
\]

The implied constants depend only on \( b, c, n \) and are independent of \( x \).

3. **Asymptotic Expansion of the Coefficients of Reproducing Kernels**

The purpose of this section is to obtain an asymptotic expansion of the coefficient \( c_m(\alpha) \) of the reproducing kernel \( R_\alpha \). For this we first prove Theorem 1.2 and write \( I_m = 1/c_m(\alpha) \) as a series. By (6) and (1), for all \( m \geq 0 \),

\[
I_m = \frac{1}{B\left(\frac{m}{2}, \alpha + 1\right)} \int_0^1 r^{m+\frac{n}{2} - 1}(1-r)^\alpha S_m^2(\sqrt{r}) \, dr.
\]

If \( m = 0 \), then \( S_0(r) = 1 \) and so \( I_0 \) and \( c_0(\alpha) \) equal to 1. From now on, we assume \( m \geq 1 \).

**Proof of Theorem 1.2.** Inserting the formula of \( S_m(r) \) in (25) into (27) we obtain

\[
I_m = \frac{B^2(m, \frac{n}{2})}{B\left(\frac{m}{2}, \alpha + 1\right)B^2(m, n-1)} \int_0^1 t^{m+\frac{n}{2} - 1}(1-t)^{n-1} F(m, 1-\frac{n}{2}; m + \frac{n}{2}; r) \, dr.
\]

We do not know a simple closed formula for this integral and we look for an asymptotic expansion. However in the above integral there are too many terms that depend on \( m \) and our first aim is to manipulate this integral by using the formulas listed in Section 2 in such a way that the appearances of \( m \) will be reduced.

First, note that by Lemmas 2.2 and 2.3

\[
F(m, 1-\frac{n}{2}; m + \frac{n}{2}; r) = (1-r)^{n-1} F\left(\frac{n}{2}, m + n - 1; m + \frac{n}{2}; r\right)
= \frac{1}{B(m, \frac{n}{2})} \int_0^1 t^{m-1}(1-t)^{m-1}(1-rt)^{-(m+n-1)} \, dt.
\]

Inserting this into (28) for one of the \( F(m, 1-\frac{n}{2}; m + \frac{n}{2}; r) \) and changing the orders of the integrals (possible by Tonelli’s theorem since the integrand is non-negative by Remark 2.4), we obtain

\[
I_m = \frac{B(m, \frac{n}{2})}{B\left(\frac{m}{2}, \alpha + 1\right)B^2(m, n-1)} \int_0^1 t^{m-1}(1-t)^{m-1} \int_0^1 r^{m+\frac{n}{2} - 1}(1-r)^{\alpha+n-1}
\times (1-rt)^{-(m+n-1)} F(m, 1-\frac{n}{2}; m + \frac{n}{2}; r) \, dr \, dt.
\]
Computing the inner integral with Lemma 2.10 gives (with also Remark 2.7)
\[ I_m = \frac{B(m, \frac{n}{2})}{B(\frac{n}{2}, \alpha + 1)B^2(m, n - 1)} \cdot \frac{\Gamma(m + \frac{n}{2})\Gamma(\alpha + n)\Gamma(\alpha + 2n - 1)}{\Gamma(\alpha + \frac{3n}{2})\Gamma(m + \alpha + 2n - 1)} \]
\[ \times \int_0^1 t^{\frac{n}{2} - 1}(1 - t)^{-n} \; {}_3F_2 \left[ \begin{array}{c} m + n - 1, \alpha + n, \alpha + 2n - 1 \\ m + \alpha + 2n - 1, \alpha + \frac{3n}{2} \end{array} ; \frac{t}{t - 1} \right] dt. \]

Although in the integrand there is the term \((1 - t)^{-n}\), the above integral is convergent. This is because the \(_3F_2\) term is positive by Lemma 2.9 (i) and is dominated by \((1 - t)^{\min(m+n-1,\alpha+n)} \log^2\left(\frac{1}{t-1}\right)\) as \(t \to 1^-\) (see [3, p. 570]).

We next apply Lemma 2.8 and write the \(_3F_2\) in the above integral as a double integral. After the cancellations we obtain
\[ I_m = \frac{B(m, \frac{n}{2})}{B(\frac{n}{2}, \alpha + 1)B^2(m, n - 1)} \cdot \frac{\Gamma(m + \frac{n}{2})\Gamma(\alpha + 2n - 1)}{\Gamma(\alpha + \frac{3n}{2})\Gamma(m + \alpha + 2n - 1)} \int_0^1 t^{\frac{n}{2} - 1}(1 - t)^{-n} \]
\[ \times \int_0^1 \int_0^1 u^{m+n-2}(1 - u)^{\alpha+n-1}v^{\alpha+n-1}(1 - v)^{\frac{n}{2} - 1} \left(1 - uv \frac{t}{t - 1}\right)^{-(\alpha+2n-1)} dudv. \]

Observe that in the integrand there is only one term that depends on \(m\) and we can now obtain the asymptotic expansion of \(I_m\). Write
\[ 1 - uv \frac{t}{t - 1} = (1 - t)^{-1}(1 - t + uv) = (1 - t)^{-1}(1 - t(1 - uv)), \]
and change the orders of the integrals which is possible since every term in the integrand is non-negative. Replace also the Beta functions with Gamma functions as in (13) and simplify. This shows
\[ I_m = \frac{\Gamma(\alpha + \frac{n}{2} + 1)\Gamma(\alpha + 2n - 1)}{\Gamma(\frac{3n}{2})\Gamma^2(n - 1)\Gamma(\alpha + 1)\Gamma(\alpha + n)} \int_0^1 \int_0^1 u^{m+n-2}(1 - u)^{\alpha+n-1} \]
\[ \times v^{\alpha+n-1}(1 - v)^{\frac{n}{2} - 1} \int_0^1 t^{\frac{n}{2} - 1}(1 - t)^{\alpha+n-1} \left(1 - t(1 - uv)\right)^{-(\alpha+2n-1)} dt dudv. \]

The inner integral is a hypergeometric function by Lemma 2.3
\[ \int_0^1 t^{\frac{n}{2} - 1}(1 - t)^{\alpha+n-1} \left(1 - t(1 - uv)\right)^{-(\alpha+2n-1)} dt \]
\[ = \frac{\Gamma(\frac{n}{2})\Gamma(\alpha + n)}{\Gamma(\alpha + \frac{3n}{2})} F\left(\frac{n}{2}, \alpha + 2n - 1; \alpha + \frac{3n}{2}; 1 - uv\right) \]
\[ = \frac{\Gamma(\frac{n}{2})\Gamma(\alpha + n)}{\Gamma(\alpha + \frac{3n}{2})} (uv)^{-n+1} F\left(\alpha + n, 1 - \frac{n}{2}; \alpha + \frac{3n}{2}; 1 - uv\right), \]
where in the last equality we apply Lemma 2.2. The reason for this last step is to get the parameter \(1 - \frac{n}{2}\) in the hypergeometric function so that when \(n\) is even the sum will terminate. Thus
\[ I_m = \frac{\Gamma(\alpha + \frac{n}{2} + 1)\Gamma(\alpha + 2n - 1)}{\Gamma^2(n - 1)\Gamma(\alpha + 1)\Gamma(\alpha + \frac{3n}{2})} \frac{\Gamma(m + n - 1)}{\Gamma(m)} \]
\[ \times \int_0^1 \int_0^1 u^{m-1}(1 - u)^{\alpha+n-1} v^{\alpha}(1 - v)^{\frac{n}{2} - 1} F\left(\alpha + n, 1 - \frac{n}{2}; \alpha + \frac{3n}{2}; 1 - uv\right) dudv. \]
We next expand the above hypergeometric function in series. By Remark 2.4, this expansion converges uniformly for \(0 \leq u, v \leq 1\) and changing the orders of the
This is possible because except (1

\sum_{j=0}^{\infty} \frac{(\alpha + n)_j (1 - \frac{n}{2})_j}{(\alpha + \frac{n}{2})_j j!} \int_{0}^{1} \int_{0}^{1} u^{m-1} (1-u)^{\alpha+n-1} v^{\alpha} (1-v)^{\frac{n}{2}-1} (1-uv)^j \, du \, dv.

Since 1-uv = (1-v)+v(1-u), we have (1-uv)^j = \sum_{k=0}^{j} \binom{j}{k} v^k (1-u)^k (1-v)^{j-k}.

Inserting this into the above integral gives

\begin{align*}
I_m &= \frac{\Gamma(\alpha + \frac{n}{2} + 1) \Gamma(\alpha + 2n - 1)}{\Gamma^2(n-1) \Gamma(\alpha + 1) \Gamma(\alpha + \frac{n}{2})} \frac{\Gamma(m+n-1)}{\Gamma(m)} \sum_{j=0}^{\infty} \frac{(\alpha + n)_j (1 - \frac{n}{2})_j}{(\alpha + \frac{n}{2})_j j!} \\
&\quad \times \sum_{k=0}^{j} \binom{j}{k} \int_{0}^{1} \int_{0}^{1} u^{m-1} (1-u)^{\alpha+n-1+k} v^{\alpha+k} (1-v)^{\frac{n}{2}-1+j-k} \, du \, dv.
\end{align*}

We compute the integrals with \[13\] and after the cancellation obtain

\begin{align*}
I_m &= \frac{\Gamma(\alpha + \frac{n}{2} + 1) \Gamma(\alpha + 2n - 1)}{\Gamma^2(n-1) \Gamma(\alpha + 1) \Gamma(\alpha + \frac{n}{2})} \frac{\Gamma(m+n-1)}{\Gamma(m)} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(\alpha + n)_j (1 - \frac{n}{2})_j}{(\alpha + \frac{n}{2})_j j!} \sum_{k=0}^{j} \binom{j}{k} \frac{\Gamma(\alpha+n+k) \Gamma(\alpha+1+k) \Gamma(\frac{n}{2}+j-k)}{\Gamma(m+n+k) \Gamma(\alpha+\frac{n}{2}+1+j)}.
\end{align*}

We change the orders of the sums by using the identity (see \[26\] Lemma 10, p. 56)

\[\sum_{j=0}^{\infty} \sum_{k=0}^{j} f(j, k) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(j, k) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(j+k, k).\]

This is possible because except (1 - \frac{n}{2})_j, all the terms in the sum are positive. Also if \(n\) is even, then (1 - \frac{n}{2})_j = 0 when \(j \geq \frac{n}{2}\) and the sum is finite. If \(n\) is odd, then (1 - \frac{n}{2})_j has the same sign for every \(j \geq \frac{n-1}{2}\) and except for finitely many terms the summands are either all positive or all negative. Thus

\begin{align*}
I_m &= \frac{\Gamma(\alpha + \frac{n}{2} + 1) \Gamma(\alpha + 2n - 1)}{\Gamma^2(n-1) \Gamma(\alpha + 1) \Gamma(\alpha + \frac{n}{2})} \frac{\Gamma(m+n-1)}{\Gamma(m+n+k) k!} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(\alpha + n)_{j+k} (1 - \frac{n}{2})_{j+k}}{(\alpha + \frac{n}{2})_{j+k} j!} \frac{\Gamma(\frac{n}{2}+j)}{\Gamma(\alpha+\frac{n}{2}+1+j+k)}
\end{align*}

By \[13\], \(\Gamma(\alpha+n+k) = \Gamma(\alpha+n)\Gamma(\alpha+n)_k\) and a similar identity holds for the other Gamma functions in the summand. Writing these and doing the cancellations shows

\begin{align*}
I_m &= \frac{\Gamma(\alpha + 2n - 1) \Gamma(\alpha + \frac{n}{2}) \Gamma(m+n-1)}{\Gamma^2(n-1) \Gamma(\alpha + \frac{n}{2})} \frac{\Gamma(m+n-1)}{\Gamma(m+n) \Gamma(m+n+k) k!} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(\alpha + n)_{j+k} (1 - \frac{n}{2})_{j+k} \Gamma(\frac{n}{2})_j}{(\alpha + \frac{n}{2})_{j+k} \Gamma(\alpha+\frac{n}{2}+1+j+k) j!}
\end{align*}
We next use the elementary identity $(a)_{j+k} = (a)_j (a+k)_j$ and obtain
\[ I_m = \frac{\Gamma(\alpha + 2n - 1)\Gamma(\alpha + n)\Gamma\left(\frac{a}{2}\right)}{\Gamma^2(n-1)\Gamma(\alpha + \frac{3n}{2})}\frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \sum_{k=0}^{\infty} \frac{(\alpha + n)_k (\alpha + 1)_k (\alpha + n + 1)_k (1 - \frac{a}{2})_k}{(m + \alpha + n)_k (\alpha + \frac{n}{2} + 1)_k k!} \sum_{j=0}^{\infty} (\alpha + \frac{3n}{2} + k) (\alpha + \frac{a}{2} + 1 + k)_j j!.
\]

Observe that the inner sum is a \( \, _3F_2 \) evaluated at \( z = 1 \). We transform this sum by Lemma 2.11. We apply Lemma 2.11 for two reasons. First to obtain \( 1 - \frac{a}{2} \) as an upper parameter in \( \, _3F_2 \) so that the sum will terminate when \( n \) is even. Second there will be some cancellations and formula for \( I_m \) will simplify.

After the cancellations we obtain
\[ I_m = \frac{\Gamma(\alpha + 2n - 1)\Gamma(\alpha + n)\Gamma\left(\frac{a}{2}\right)}{\Gamma^2(n-1)\Gamma(\alpha + \frac{3n}{2})}\frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \times \sum_{k=0}^{\infty} \frac{(\alpha + 1)_k (\alpha + n)_k (1 - \frac{a}{2})_k}{(m + \alpha + n + 1)_k (\alpha + \frac{n}{2} + 1)_k k!} \, _3F_2 \left[ \frac{\alpha + n}{2}, \frac{\alpha + 1}{2} + 1 + k, \frac{\alpha + 3n}{2} + 1 \right].
\]

This shows that \( I_m \) can be written in the form (11) and finishes the proof. \( \square \)

We remark about the case \( n \) is even. In this case \( A_k = 0 \) for \( k \geq \frac{n}{2} \) because of the term \( (1 - \frac{a}{2})_k \), and therefore the sum in (11) is finite and is a rational function of \( m \). For \( 0 \leq k < \frac{n}{2} \), the coefficients \( A_k \) can also be computed in finite steps since the \( \, _3F_2 \) in (12) terminates.

The exact values of the constants \( A_k \) will not be important for us. Nevertheless let us find a closed formula for the dominating term \( A_0 \). When \( k = 0 \), we can compute the \( \, _3F_2 \) in (12) with Lemma 2.12 where we permute the upper parameters and take \( a = \alpha + n \).

\[ \, _3F_2 \left[ \alpha + n, \frac{\alpha + 1 + 1}{2}, 1 - \frac{a}{2} \right] = \frac{\Gamma(\alpha + n + 1)\Gamma(\alpha + \frac{3n}{2} + 1)\Gamma(\alpha + \frac{a + n}{2})}{\Gamma(\alpha + n + 1)\Gamma\left(\frac{a}{2}\right)\Gamma(\alpha + n + 1)\Gamma(\alpha + n + 1)}. \]

After cancellations we obtain
\[ A_0 = \frac{\Gamma(\alpha + \frac{a}{2})\Gamma(\alpha + \frac{a}{2} + 1)\Gamma(\alpha + 2n - 1)\Gamma(\alpha + \frac{a + n}{2})\Gamma(\alpha + \frac{a + n}{2} + 1)}{\Gamma^2(n-1)\Gamma(\alpha + n + 1)\Gamma(\alpha + n + 1)\Gamma(\alpha + n + 1)}. \]

To estimate the reproducing kernel \( R_n \), only finite part of the series in Theorem 1.2 will be sufficient. In the next corollary we estimate the remainder part.
Corollary 3.1. For $K = 1, 2, \ldots,$
\[ I_m = \frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \left( \sum_{k=0}^{K-1} \frac{A_k}{(m + \alpha + n)_k} + O\left(\frac{1}{m^K}\right) \right) \quad (m \to \infty), \]
where $A_k$ is as in (12). The implied constant of the $O$ term depends only on $\alpha, n, K.$

Proof. The remainder term in the series in (11) is
\[ R_K := \sum_{k=K}^{\infty} \frac{A_k}{(m + \alpha + n)_k} \]
\[
= C \sum_{k=K}^{\infty} \frac{(\alpha + 1)_k (\alpha + n)_k (1 - \frac{n}{k})_k}{(m + \alpha + n)_k (\alpha + \frac{n}{k} + 1)_k k!} F_2 \left[ \begin{array}{c} \frac{n}{2}, \alpha + n, 1 - \frac{n}{2} \\ \alpha + \frac{n}{2} + 1 + k, \alpha + \frac{3n}{2} \end{array} ; 1 \right],
\]
where $C$ is a constant that depends only on $\alpha$ and $n$. To estimate $R_K$, first note that by Lemma 2.9 (iii), for every $k \geq 1,$ we have
\[ 0 \leq F_2 \left[ \begin{array}{c} \frac{n}{2}, \alpha + n, 1 - \frac{n}{2} \\ \alpha + \frac{n}{2} + 1 + k, \alpha + \frac{3n}{2} \end{array} ; 1 \right] \leq 1. \]
Next when $n$ is odd, by (15), there exists a constant $C$ depending only on $\alpha$ and $n$ such that for every $k \geq 1,$
\[
\left| \frac{(\alpha + 1)_k (1 - \frac{n}{k})_k}{(\alpha + \frac{n}{k} + 1)_k k!} \right| \leq C \frac{1}{k^n}.
\]
The above estimate is certainly true when $n$ is even since left-hand side vanishes for $k \geq \frac{n}{2}$. Thus
\[
|R_K| \leq C \sum_{k=K}^{\infty} \frac{(\alpha + n)_k}{(m + \alpha + n)_k k^n} = C \frac{(\alpha + n)_K}{(m + \alpha + n)_K} \sum_{k=K}^{\infty} \frac{1}{(m + \alpha + n + K)_k K^n}
\leq C \frac{(\alpha + n)_K}{(m + \alpha + n)_K} \sum_{k=1}^{\infty} \frac{1}{k^n} = C \frac{(\alpha + n)_K}{(m + \alpha + n)_K},
\]
since $(\alpha + n + K)_j \leq (m + \alpha + n + K)_j$ and $\sum_{k=1}^{\infty} 1/k^n$ converges. Finally, because $(m + \alpha + n)_K \geq m^K,$ we conclude that there exists a constant $C$ depending only on $\alpha, n$ and $K$ but independent of $m$ such that $|R_K| \leq C/m^K.$ \[ \Box \]

Corollary 3.1 shows that the series (11) is a convergent asymptotic series
\[ I_m \approx \frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \sum_{k=0}^{\infty} \frac{A_k}{(m + \alpha + n)_k} \quad (m \to \infty). \]

Our main interest is in $c_m(\alpha) = 1/I_m$ and to invert $I_m$, we next write the above asymptotic expansion in powers of $1/m$. As a function of $m$, $(m + \alpha + n)_j$ is a polynomial of degree $j$ and for $j \geq 1,$ the rational function $1/(m + \alpha + n)_j$ has the Laurent series expansion
\[ \frac{1}{(m + \alpha + n)_j} = \sum_{k=j}^{\infty} \frac{C_k(j)}{m^k} \quad (m > \alpha + n + j - 1), \]
where $C_k(j)$ depends only on $\alpha$ and $n$. When $j = 0,$ $(m + \alpha + n)_0 = 1$ and we set $C_0(0) = 1$ and $C_k(0) = 0$ for $k \geq 1$. An explicit formula for the coefficients $C_k(j)$ can be written (the partial fraction expansion of $1/(m + \alpha + n)_j$ has a simple form)
but we will not do this since exact values of these coefficients are not important for our purposes. For \(k = 0, 1, 2, \ldots\), define

\[
B_k = \sum_{j=0}^{k} A_j C_k(j).
\]

In particular \(B_0 = A_0\) with \(A_0\) as in (29).

**Corollary 3.2.** There exist constants \(B_k\) \((k = 0, 1, 2, \ldots)\) depending only on \(\alpha\) and \(n\) such that for every \(K = 1, 2, \ldots\),

\[
I_m = \frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \left(\sum_{k=0}^{K-1} \frac{B_k}{m^k} + O\left(\frac{1}{m^K}\right)\right) \quad (m \to \infty).
\]

The constants \(B_k\) are given in (31). The implied constant of the \(O\) term depends only on \(\alpha, n, K\).

**Proof.** Fix \(K\). For \(j = 0, 1, 2, \ldots, K-1\), by the Laurent series in (30), we have

\[
\frac{1}{(m + \alpha + n)_j} = \sum_{k=j}^{K-1} \frac{C_k(j)}{m^k} + O\left(\frac{1}{m^K}\right) \quad (m \to \infty).
\]

Using Corollary 3.1 and changing the orders of the sums we obtain

\[
I_m = \frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \left(\sum_{j=0}^{K-1} A_j \sum_{k=j}^{K-1} \frac{C_k(j)}{m^k} + O\left(\frac{1}{m^K}\right)\right)
= \frac{\Gamma(m + n - 1)}{\Gamma(m + \alpha + n)} \left(\sum_{k=0}^{K-1} \frac{1}{m^k} \sum_{j=0}^{k} A_j C_k(j) + O\left(\frac{1}{m^K}\right)\right) \quad (m \to \infty),
\]

which is the desired result by (31). \(\square\)

Once we have the asymptotic expansion of \(I_m\) in powers of \(1/m\), it is straightforward to write the asymptotic expansion of \(c_m(\alpha) = 1/I_m\) (see, for example, [22, p. 20]). Let \(D_0 = 1/B_0\) (note that \(B_0 = A_0 \neq 0\)) and for \(k \geq 1\), define \(D_k\) recursively by the equation

\[
(32) \quad D_k B_0 + D_{k-1} B_1 + \cdots + D_0 B_k = 0.
\]

**Corollary 3.3.** There exist constants \(D_k\) \((k = 0, 1, 2, \ldots)\) with \(D_0 > 0\), depending only on \(\alpha\) and \(n\) such that for every \(K = 1, 2, \ldots\),

\[
c_m(\alpha) = \frac{\Gamma(m + \alpha + n)}{\Gamma(m + n - 1)} \left(\sum_{k=0}^{K-1} \frac{D_k}{m^k} + O\left(\frac{1}{m^K}\right)\right) \quad (m \to \infty).
\]

The constants \(D_k\) \((k \geq 1)\) are determined by (32). The implied constant of the \(O\) term depends only on \(\alpha, n, K\).

If we take \(K = 1\) in the above Corollary we obtain (10), since \(D_0 = 1/A_0\) with \(A_0\) given in (29).
4. Upper Estimates of Reproducing Kernels

For a multi-index \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i \) are non-negative integers, and for a smooth function \( f \) we write

\[
\partial^\lambda f(x) = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}},
\]

where \( |\lambda| = \lambda_1 + \cdots + \lambda_n \). The following estimate of the Euclidean Poisson kernel \( P(x, \eta) = \sum_{m=0}^{\infty} Z_m(x, \eta) \) is well-known (see, for example, [15]).

**Lemma 4.1.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a multi-index. There exists a constant \( C > 0 \) depending only on \( \lambda \) and \( n \) such that for all \( x \in \mathbb{B} \) and \( \eta \in \mathbb{S} \),

\[
|\partial^\lambda P(x, \eta)| \leq \frac{C}{|x - \eta|^{n-1+|\lambda|}}.
\]

Using Lemma 4.1 it is easy to estimate \( \sum_{m=1}^{\infty} m Z_m(x, \eta) = \frac{d}{dr} P(r x, \eta)|_{r=1} \) and applying the same argument repeatedly shows (for \( k = 1, 2, \ldots \))

\[
\sum_{m=1}^{\infty} m^k Z_m(x, \eta) \lesssim \frac{1}{|x - \eta|^{n-1+k}}.
\]

We need a more general version of the above estimate.

**Proposition 4.2.** Let \( a_i, b_i \geq 0 \) \( (i = 1, 2, \ldots, k) \) and for \( x \in \mathbb{B}, \eta \in \mathbb{S} \), let

\[
W(x, \eta) = \sum_{m=1}^{\infty} \frac{\Gamma(m + a_1)}{\Gamma(m + b_1)} \cdots \frac{\Gamma(m + a_k)}{\Gamma(m + b_k)} Z_m(x, \eta).
\]

For a multi-index \( \lambda \), set \( c = n - 1 + (a_1 - b_1) + \cdots + (a_k - b_k) + |\lambda| \). There exists a constant \( C > 0 \) depending only on \( a_i, b_i, n \) and \( \lambda \) such that for all \( x \in \mathbb{B} \) and \( \eta \in \mathbb{S} \),

\[|\partial^\lambda W(x, \eta)| \leq C \begin{cases} 
\frac{1}{|x - \eta|^c}, & \text{if } c > 0; \\
1 + \log \frac{1}{|x - \eta|}, & \text{if } c = 0; \\
1, & \text{if } c < 0.
\end{cases}\]

Observe that by [15], the coefficient \( \prod_{i=1}^{k} \Gamma(m + a_i) / \Gamma(m + b_i) \sim m^{\sum_{i=1}^{k} (a_i - b_i)} \) as \( m \to \infty \) and when \( |\lambda| = 0 \), \( c \) is \( (n-1) \) plus the exponent of \( m \). So the above theorem generalizes [33] by replacing \( m^k \) with more general coefficients that are asymptotic to \( m^k \). If there is differentiation, \( c \) increases by the order of the derivative. In the Euclidean case reproducing kernels \( R_\alpha \) of Bergman spaces (and also the generalized family of Bergman-Besov spaces with \( \alpha \in \mathbb{R} \) in [3]) are of the above form.

This proposition, in the form written above, is essentially proved in [8 Lemma 7.4] following the methods of [4 Section 3], [5 Chapter 7], [13 Section 2], [19 Section 3]) and [25 Section 2]. However, there is one detail that needs to be mentioned. In [8], the summation starts from \( m = 0 \) and therefore \( a_i, b_i \) are taken to be strictly positive. Since the case \( a_i \) or \( b_i \) equal to 0 will be important for us and will be repeatedly used later we give a proof of Proposition 4.2 below omitting the steps that are already written in detail in [8]. We begin with the following special case.
Lemma 4.3. For $b \geq 0$ and $l = 0, 1, 2, \ldots$, let
\[
W(x, \eta) = \sum_{m=1}^{\infty} \frac{\Gamma(m+b+l)}{\Gamma(m+b)} Z_m(x, \eta) \quad (x \in \mathbb{B}, \eta \in \mathbb{S}).
\]

For a multi-index $\lambda$, there exists a constant $C > 0$ such that for all $x \in \mathbb{B}$, $\zeta \in \mathbb{S}$,
\[
|\partial^\lambda W(x, \eta)| \leq \frac{C}{|x - \eta|^{n-1+l+|\lambda|}}.
\]

Proof. If $l = 0$, then $W(x, \eta) = P(x, \eta) - 1$ since $Z_0(x, \eta) = 1$, and the result follows from Lemma 4.1. If $l \geq 1$, consider first $\lambda = (0, \ldots, 0)$. By homogeneity of $Z_m$,
\[
r^{b+l-1} P(rx, \eta) = \sum_{m=0}^{\infty} r^{m+b+l-1} Z_m(x, \eta) = r^{b+l-1} + \sum_{m=1}^{\infty} r^{m+b+l-1} Z_m(x, \eta).
\]
Differentiating $l$ times with respect to $r$ and putting $r = 1$, we obtain
\[
\frac{\partial^l}{\partial r^l} (r^{b+l-1} P(rx, \eta)) \bigg|_{r=1} = (b)_l + W(x, \eta).
\]
Applying Leibniz and chain rules to the left and using Lemma 4.1 we obtain the desired result since highest order of differentiation applied to $P$ is $l$. For a general $\lambda$, we apply $\partial^\lambda$ to (34), change the orders of the derivatives and apply Lemma 4.1. See the proof of Lemma 7.2 in [8] for more details. \hfill \Box

Lemma 4.4. For $a, b \geq 0$, let
\[
W(x, \eta) = \sum_{m=1}^{\infty} \frac{\Gamma(m+a)}{\Gamma(m+b)} Z_m(x, \eta) \quad (x \in \mathbb{B}, \eta \in \mathbb{S}).
\]

For a multi-index $\lambda$, set $c = n - 1 + a - b + |\lambda|$. There exists a constant $C > 0$ such that for all $x \in \mathbb{B}$, $\zeta \in \mathbb{S}$,
\[
|\partial^\lambda W(x, \eta)| \leq C \begin{cases}
\frac{1}{|x - \eta|^c}, & \text{if } c > 0; \\
1 + \log \frac{1}{|x - \eta|}, & \text{if } c = 0; \\
1, & \text{if } c < 0.
\end{cases}
\]

Proof. We first consider the case $a > 0$. Pick a non-negative integer $l$ such that $b-a+l > 0$ and let
\[
g(x, \eta) = \sum_{m=1}^{\infty} \frac{\Gamma(m+b+l)}{\Gamma(m+b)} Z_m(x, \eta) \quad (x \in \mathbb{B}, \eta \in \mathbb{S}).
\]
We can write $W$ as an integral in the form
\[
W(x, \eta) = \frac{1}{\Gamma(b-a+l)} \int_{0}^{1} g(rx, \eta) r^{a-1} (1-r)^{b-a+l-1} \, dr.
\]
This can be verified by integrating the series expansion of $g(rx, \eta)$ which uniformly converges on $0 \leq r \leq 1$ for fixed $x \in \mathbb{B}$. Estimating $g$ with Lemma 4.3 we obtain
\[
|W(x, \eta)| \lesssim \int_{0}^{1} \frac{r^{a-1}(1-r)^{b-a+l-1}}{|x \eta|^{n-1+l}} \, dr.
\]
When \( \lambda = (0, \ldots, 0) \) the result follows from Lemma 2.15. In the case of a general multi-index \( \lambda \), apply \( \partial^\lambda \) to (37), push \( \partial^\lambda \) into the integral and apply chain rule. The result follows from applying first Lemma 4.3 and then Lemma 2.15.

We now consider the case \( a = 0 \) which we need to deal with separately because of the convergence problem in the integral in (37). If \( b = 0 \), then \( W(x, \eta) = P(x, \eta) - 1 \) and the lemma follows from Lemma 4.1. If \( b > 0 \), let (we take \( l = 0 \) in (36))

\[
g(x, \eta) = \sum_{m=1}^{\infty} Z_m(x, \eta) = P(x, \eta) - 1.
\]

Then \( g(rx, \eta) = O(r) \) as \( r \to 0^+ \), since for \( r \leq r_0 < 1 \), by (21) and \( |x| < 1 \),

\[
|g(rx, \eta)| = r \left| \sum_{m=1}^{\infty} r^{m-1} Z_m(x, \eta) \right| \leq Cr \sum_{m=1}^{\infty} r^{m-1} m^{n-2} = Cr
\]

As above

\[
W(x, \eta) = \frac{1}{\Gamma(b)} \int_0^1 g(rx, \eta) r^{-1} (1 - r)^{b-1} dr,
\]

where the integral absolutely converges. Writing \( \int_0^1 = \int_0^{r_0} + \int_{r_0}^1 \), the first integral is \( O(1) \), and the second integral is estimated by Lemmas 4.1 and 2.15 as above. If \(|\lambda| > 0\), by the chain rule, \( \partial^\lambda (g(rx, \eta)) = O(r) \) as \( r \to 0^+ \) and we argue similarly. \( \square \)

Proof of Proposition 4.2. We use induction on \( k \). If \( k = 1 \), the theorem follows from Lemma 4.4. We deduce the case \( k = 2 \), the general case is shown similarly. Let

\[
W(x, \eta) = \sum_{m=1}^{\infty} \frac{\Gamma(m + a_1) \Gamma(m + a_2)}{\Gamma(m + b_1) \Gamma(m + b_2)} Z_m(x, \eta),
\]

and

\[
g(x, \eta) = \sum_{m=1}^{\infty} \frac{\Gamma(m + a_1)}{\Gamma(m + b_1)} Z_m(x, \eta).
\]

Note that starting with the estimate \( \partial^\lambda P \), we estimated \( \partial^\lambda g \) in two steps with Lemmas 4.3 and 4.4. To estimate \( \partial^\lambda W \) we repeat the same arguments, the only difference is we replace the role of \( P \) by \( g \). As the first step choose a nonnegative integer \( l \) such that

\[
d := n - 1 + (a_1 - b_1) + l > 0 \quad \text{and} \quad b_2 - a_2 + l > 0,
\]

and let

\[
h(x, \eta) = \sum_{m=1}^{\infty} \frac{\Gamma(m + a_1) \Gamma(m + b_2 + l)}{\Gamma(m + b_1) \Gamma(m + b_2)} Z_m(x, \eta).
\]

We have (corresponding to (34))

\[
h(x, \eta) = \frac{\partial^l}{\partial r^l} (r^{b_2 + l - 1} g(rx, \eta)) \bigg|_{r=1}
\]

and retracing the proof of Lemma 4.3 with replacing references to Lemma 4.1 by Lemma 4.4 we see that

\[
|\partial^\lambda h(x, \eta)| \lesssim \frac{1}{|x - \eta|^{d + |\lambda|}}.
\]
Note that highest order of derivative applied to $g$ is $d + |\lambda|$ which is positive by the first assumption in (39). For the second step observe that (corresponding to (37))

$$W(x, \eta) = \frac{1}{\Gamma(b_2 - a_2 + l)} \int_0^1 h(rx, \eta) r^{a_2-1} (1-r)^{b_2-a_2+l-1} dr.$$ 

Retracing the proof of Lemma 4.4 with replacing references to Lemma 4.3 by the estimate (40) proves the case $k = 2$. In case $a_2 = 0$ we also use the fact that $h(rx, \eta) = O(r)$ as $r \to 0^+$ which can be shown as in (38) with using also (15). \hfill $\square$

In the next corollary we allow the second variable of $W$ to be in $\mathbb{R}$.

**Corollary 4.5.** Let $a_i, b_i \geq 0$ $(i = 1, 2, \ldots, k)$ and for $x \in \mathbb{B}$, $y \in \mathbb{R}$, let

$$W(x, y) = \sum_{m=1}^{\infty} \frac{\Gamma(m + a_1)}{\Gamma(m + b_1)} \cdots \frac{\Gamma(m + a_k)}{\Gamma(m + b_k)} \frac{Z_m(x, y)}{r^m}.$$ 

For a multi-index $\lambda$, set $c = n - 1 + (a_1 - b_1) + \cdots + (a_k - b_k) + |\lambda|$. There exists a constant $C > 0$ such that for all $x \in \mathbb{B}$, $y \in \mathbb{R}$,

$$|\partial^\lambda W(x, y)| \leq C \begin{cases} \left(\frac{1}{|x, y|}\right)^c, & \text{if } c > 0; \\ 1 + \log \left(\frac{1}{|x, y|}\right), & \text{if } c = 0; \\ 1, & \text{if } c < 0, \end{cases}$$

where differentiation is applied to the first variable.

**Proof.** If $y = |y, \eta|$ with $\eta \in \mathbb{S}$, then by homogeneity of $Z_m$ in both variables, we have $Z_m(x, y) = Z_m(|y|x, \eta)$. Therefore $W(x, y) = W(|y|x, \eta)$ and by the chain rule $\partial^\lambda W(x, y) = |y|^\lambda (\partial^\lambda W)(|y|x, \eta)$. Since $|y| \leq 1$, $\partial^\lambda W(x, y)$ is bounded from above by the same terms given in Proposition 4.2 only with $|x - \eta|$ replaced by $||y| x - \eta|$. Because $||y| x - \eta| = |x, y|$ by (18), the corollary follows. \hfill $\square$

We now insert the factors $S_m(|x|)$ and $S_m(|y|)$ into the series.

**Theorem 4.6.** Let $a_i, b_i \geq 0$ $(i = 1, 2, \ldots, k)$ and for $x \in \mathbb{B}$, $y \in \mathbb{R}$, let

$$h(x, y) = \sum_{m=1}^{\infty} \frac{\Gamma(m + a_1)}{\Gamma(m + b_1)} \cdots \frac{\Gamma(m + a_k)}{\Gamma(m + b_k)} S_m(|x|) S_m(|y|) Z_m(x, y).$$ 

Set $c = n - 1 + (a_1 - b_1) + \cdots + (a_k - b_k)$. There exists a constant $C > 0$ depending only on $a_i, b_i$ and $n$ such that for all $x \in \mathbb{B}$, $y \in \mathbb{R}$,

$$|h(x, y)| \leq C \begin{cases} \left(\frac{1}{|x, y|}\right)^c, & \text{if } c > 0; \\ 1 + \log \left(\frac{1}{|x, y|}\right), & \text{if } c = 0; \\ 1, & \text{if } c < 0. \end{cases}$$

**Proof.** For shortness, write $d_m = \prod_{i=1}^k \Gamma(m + a_i)/\Gamma(m + b_i)$. Inserting the integral formula (20) for both $S_m(|x|)$ and $S_m(|y|)$ and changing the orders of the integrals...
and the series we obtain
\[ h(x, y) = \int_0^1 \int_0^1 (1 - t)^{n-1} (1 - |x|^2 t) (1 - \tau)^{n-1} (1 - |y|^2 \tau)^{n-1} \times \sum_{m=1}^{\infty} \frac{d_m}{B^2(m, n-1)} t^{m-1} \tau^{m-1} Z_m(x, y) \, d\tau dt. \]

The interchange of the orders is possible since for fixed \( x \) and \( y \) the series uniformly converges for \( 0 \leq t, \tau \leq 1 \) by (15), homogeneity of \( Z_m \) and (21). When \( 0 \leq t \leq 1/2 \) or \( 0 \leq \tau \leq 1/2 \) above integrand is uniformly bounded for every \( x, y \) by the same reasoning. When \( 1/2 \leq t, \tau \leq 1 \), observe that \( t^{m-1} \tau^{m-1} Z_m(x, y) = Z_m(tx, \tau y)/\tau \). Therefore, if we let
\[ W(x, y) = \int_0^1 \int_0^1 (1 - t)(1 - |x|^2 t)(1 - \tau)(1 - |y|^2 \tau)^{n-1} \frac{1}{t \tau} |W(tx, \tau y)| \, d\tau dt. \]

We can get rid of the factor \( 1/(t \tau) \) and obtain
\[ |h(x, y)| \lesssim 1 + \int_0^1 \int_0^1 ((1 - t)(1 - |x|^2 t)(1 - \tau)(1 - |y|^2 \tau))^{n-1} \frac{1}{t \tau} |W(tx, \tau y)| \, d\tau dt. \]

We estimate \( W(x, y) \) with Corollary 4.5 in three cases depending on the sign of \( c + 2(n - 1) \), since the coefficient in the definition of \( W \) is \( d_m \Gamma^2(m + n - 1)/\Gamma^2(m) \).

Case (i). If \( c + 2(n - 1) > 0 \), then by Corollary 4.5
\[ |h(x, y)| \lesssim 1 + \int_0^1 \int_0^1 ((1 - t)(1 - |x|^2 t)(1 - \tau)(1 - |y|^2 \tau))^{n-1} \frac{1}{t \tau} \, d\tau dt. \]

Note that for \( 0 \leq t \leq 1 \),
\[ 1 - t \leq 1 - |x|^2 t \leq 1 - |x|^2 t^2 \leq 2(1 - |x| t) \leq 2(1 - |x| |y| \tau) \leq 2|tx, \tau y|, \]

where the last inequality follows from (17). Similarly,
\[ 1 - \tau \leq 1 - |y|^2 \tau \leq 2|tx, \tau y|. \]

Hence
\[ |h(x, y)| \lesssim 1 + \int_0^1 \int_0^1 \frac{1}{|tx, \tau y|^{c+2(n-1)}} \, d\tau dt = 1 + \int_0^1 \int_0^1 \frac{1}{|\tau t | |x - \eta|^{c+2}} \, d\tau dt, \]

where in the last equality we write \( y = |y| \eta, \eta \in S \) and use (19). We next estimate the inner integral with Lemma 2.15. This requires to consider three subcases.

If \( c > -1 \), applying Lemma 2.15 to the inner integral in (44) we obtain
\[ |h(x, y)| \lesssim 1 + \int_0^1 \frac{dt}{|t| |x - \eta|^{c+1}}. \]

Applying Lemma 2.15 one more time in the three cases \( c > 0, c = 0 \) and \( -1 < c < 0 \) (and noting that \( |y| |x - \eta| = |x, y| \) by (19) proves the theorem when \( c > -1 \).

If \( c = -1 \), applying Lemma 2.15 to the inner integral in (44) shows
\[ |h(x, y)| \lesssim 1 + \int_0^1 \left( 1 + \log \frac{1}{|t| |x - \eta|} \right) dt \leq 1 + \int_0^1 \left( 1 + \log \frac{1}{1 - t} \right) dt \lesssim 1, \]
since \(|t|y|x-\eta| \geq 1-t|y||x| \geq 1-t\), and last integral is finite.

If \(-2(n-1) < c < -1\), applying Lemma 2.15 to the inner integral in (44), we obtain

\[
|h(x, y)| \lesssim 1 + \int_0^1 dt \lesssim 1.
\]

Thus the theorem holds when \(c > -2(n-1)\).

Case (ii). If \(c + 2(n-1) = 0\), estimating \(W(tx, \tau y)\) in (41) with Corollary 4.5, we obtain

\[
|h(x, y)| \lesssim 1 + \int_0^1 \int_0^1 ((1-t)(1-|x|^2t)(1-\tau)(1-|y|^2\tau))^{-1} (1+\log \frac{1}{|tx, \tau y|}) d\tau dt.
\]

Since \(|tx, \tau y| \geq 1-\tau|tx||y| \geq 1-t\) by (17), and the other terms in the integrand are bounded from above by 1, we have

\[
|h(x, y)| \lesssim 1 + \int_0^1 \log \frac{1}{1-t} dt \lesssim 1.
\]

Case (iii). If \(c + 2(n-1) < 0\), then by (41) and Corollary 4.5,

\[
|h(x, y)| \lesssim 1 + \int_0^1 \int_0^1 ((1-t)(1-|x|^2t)(1-\tau)(1-|y|^2\tau))^{-1} d\tau dt \lesssim 1,
\]

since again the integrand is bounded by 1. This finishes the proof.

Using Theorem 4.6 we can find upper bounds for the reproducing kernels of both Bergman and Hardy spaces of \(\kappa\)-harmonic functions on \(B\). We begin with Hardy spaces. The reproducing kernel \(K(x, y)\) of the Hardy space \(H^2\) has the series expansion [30, Theorem 2.5]

\[
K(x, y) = \sum_{m=0}^{\infty} S_m(|x|)S_m(|y|) Z_m(x, y) \quad (x, y \in B).
\]

The following corollary of Theorem 4.6 improves the estimate of \(K(x, y)\) given in [30, Theorem 3.2]. We note that \(K(x, y) \geq 0\) by [30, Theorem 2.1].

**Corollary 4.7.** There exists a constant \(C > 0\) depending only on \(n\) such that for all \(x, y \in B\),

\[
K(x, y) \leq \frac{C}{|x, y|^{n-1}}.
\]

Moreover the exponent \((n-1)\) is non-improvable.

**Proof.** By Theorem 4.6 and the fact that \([x, y] \leq 2\),

\[
|K(x, y)| = \left| 1 + \sum_{m=1}^{\infty} S_m(|x|)S_m(|y|) Z_m(x, y) \right| \lesssim 1 + \frac{1}{|x, y|^{n-1}} \lesssim \frac{1}{|x, y|^{n-1}}.
\]

On the other hand, when \(y = x\),

\[
K(x, x) \gtrsim 1 + \sum_{m=1}^{\infty} m^{n-2}|x|^{2m} \sim \frac{1}{(1-|x|^2)^{n-1}} = \frac{1}{|x, x|^{n-1}},
\]

since \(S_m(r) \geq 1\) by Lemma 2.12 and \(Z_m(x, x) \sim |x|^{2m}m^{n-2}\) by (21). \(\square\)

We next deal with Bergman spaces and prove Theorem 1.3.
Proof of Theorem 1.3. Pick a positive integer $K$ such that $K > \alpha + 2n - 2$. By (11) and Corollary 3.3,

$$\mathcal{R}_\alpha(x, y) = 1 + \sum_{m=1}^{\infty} \frac{\Gamma(m + \alpha + n)}{\Gamma(m + n - 1)} \left( \sum_{k=0}^{K-1} \frac{D_k}{mk} + \delta_m(K) \right) S_m(|x|)S_m(|y|) Z_m(x, y),$$

where $\delta_m(K) = O(1/m^K)$, $m \to \infty$.

We first show that the last part with coefficient $\delta_m(K)$ is uniformly bounded for all $x, y \in \mathbb{B}$. Note that by Lemma 2.13, $S_m(r) \lesssim m^{\frac{n}{2} - 1}$, and by homogeneity and (21), $|Z_m(x, y)| \lesssim |x|^n |y|^m m^{n-2} \lesssim m^{n-2}$. Therefore using also (13),

$$\left| \sum_{m=1}^{\infty} \frac{\Gamma(m + \alpha + n)}{\Gamma(m + n - 1)} \frac{1}{m^k} S_m(|x|)S_m(|y|) Z_m(x, y) \right| \lesssim \sum_{m=1}^{\infty} \frac{1}{m^{K-(\alpha+2n-3)}} \lesssim 1,$

by the choice of $K$.

Next, for $k = 0, 1, \ldots, K - 1$, writing $1/m^k = \Gamma^k(m)/\Gamma^k(m + 1)$ and applying Theorem 1.6 with $c = n - 1 + (\alpha + 1) - k = \alpha + n - k$, we see that

$$\left| \sum_{m=1}^{\infty} \frac{\Gamma(m + \alpha + n)}{\Gamma(m + n - 1)} \frac{1}{m^k} S_m(|x|)S_m(|y|) Z_m(x, y) \right| \lesssim \begin{cases} \frac{1}{|x, y|^{\alpha+n-k}}, & \text{if } 0 \leq k < \alpha + n; \\ \log \frac{1}{|x, y|}, & \text{if } k = \alpha + n; \\ 1, & \text{if } k > \alpha + n. \end{cases}$$

Since $0 < |x, y| \leq 2$ the term with highest growth rate is $1/|x, y|^{\alpha+n}$ occurring when $k = 0$. We conclude that $|\mathcal{R}_\alpha(x, y)| \lesssim 1/|x, y|^{\alpha+n}$. The exponent $(\alpha + n)$ is non-improvable by Lemma 6.1 below. \hfill \square

5. First-Order Partial Derivatives of Reproducing Kernels

In this section we estimate first-order partial derivatives of $\mathcal{R}_\alpha$. We begin with the following lemma.

Lemma 5.1. There exists a constant $C > 0$ depending only on $n$ such that for all $x, y \in \mathbb{B}$ and $i = 1, 2, \ldots, n$,

$$\left| \frac{\partial}{\partial x_i} Z_m(x, y) \right| \leq C m^n |x|^{m-1} |y|^m.$$ 

It appears that this lemma can be improved by replacing $m^n$ with $m^{n-1}$, but for our purposes the above estimate will be sufficient.

Proof. Write $x = |x| \zeta$, $y = |y| \eta$ with $\zeta, \eta \in \mathbb{S}$. Comparing the expansions of $Z_m$ in [2 Theorem 5.38] and the Gegenbauer (ultraspherical) polynomials in [23 18.5.10] shows that (when $n \geq 3$)

$$Z_m(x, y) = \frac{n + 2m - 2}{n - 2} |x|^m |y|^m C^{(n/2-1)}_{m} \left( \left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle \right).$$

Differentiating and using $\frac{d}{dt} C^{(n/2-1)}_{m}(t) = (n - 2) C^{(n/2-2)}_{m-1}(t)$ (see [23 18.9.19]) shows

$$\frac{\partial}{\partial x_i} Z_m(x, y) = \frac{n + 2m - 2}{n - 2} |x|^{m-1} |y|^m \left( m \zeta_i C^{(n/2-1)}_{m}((\zeta, \eta)) + (n - 2)(\eta_i - \zeta_i) C^{(n/2)}_{m-1}((\zeta, \eta)) \right).$$
By [23] 18.14.4, for \( -1 \leq t \leq 1 \), \(|C_m^{(\lambda)}(t)| \leq (2\lambda)_m/m!\). Using this and (15) gives the desired result. \( \square \)

We state the next lemma as a preparation to the proof of Theorem 1.4.

**Lemma 5.2.** Let \( \alpha > -1 \) and \( c_m(\alpha) \) be as in (19). Let

\[
W(x, y) = \sum_{m=1}^{\infty} c_m(\alpha) \frac{\Gamma^2(m+n-1)}{\Gamma^2(m)} Z_m(x, y).
\]

There exists a constant \( C > 0 \) depending only \( \alpha \) and \( n \) such that for all \( x, y \in \mathbb{B} \),

(i) \( W(x, y) \leq \frac{C}{|x, y|^n+n+2(n-1)} \),

(ii) \( \left| \frac{\partial}{\partial x_i} W(x, y) \right| \leq \frac{C}{|x, y|^\alpha+n+2(n-1)+1} \quad (i = 1, 2, \ldots, n) \).

**Proof.** Since by (9), the coefficient \( c_m(\alpha)\Gamma^2(m+n-1)/\Gamma^2(m) \sim m^\alpha+2(n-1) \) as \( m \to \infty \), the lemma follows from Corollary 4.5. Nevertheless we provide the details.

Pick \( K \) with \( K > \alpha + 3n - 2 \). By Corollary 3.3

\[
W(x, y) = \sum_{m=1}^{\infty} \frac{\Gamma(m+\alpha+n)}{\Gamma(m+n-1)} \frac{\Gamma^2(m+n-1)}{\Gamma^2(m)} \left( \sum_{k=0}^{K-1} \frac{D_k}{m^k} + \delta_m(K) \right) Z_m(x, y),
\]

where \( \delta_m(K) = O(1/m^K) \). Corollary 4.5 implies that for the dominating term corresponding to \( k = 0 \),

\[
\sum_{m=1}^{\infty} \frac{\Gamma(m+\alpha+n)}{\Gamma(m+n-1)} \frac{\Gamma^2(m+n-1)}{\Gamma^2(m)} \frac{1}{m^k} Z_m(x, y) \lesssim \frac{1}{|x, y|^\alpha+n+2(n-1)}.
\]

and for \( k = 1, 2, \ldots, K - 1 \), a better estimate holds but since \( |x, y| \leq 2 \), (46) is true for these \( k \)’s also. For the part involving coefficient \( \delta_m(K) \), by (15) and (21),

\[
\sum_{m=1}^{\infty} \frac{\Gamma(m+\alpha+n)}{\Gamma(m+n-1)} \frac{\Gamma^2(m+n-1)}{\Gamma^2(m)} \frac{1}{m^k} Z_m(x, y) \lesssim \sum_{m=1}^{\infty} \frac{1}{m^{K-(\alpha+3n-3)}} \lesssim 1.
\]

To estimate the partial derivative, pick \( K > \alpha + 3n \) and differentiate (45). By Corollary 4.5 for all \( k = 0, 1, \ldots, K - 1 \),

\[
\left| \frac{\partial}{\partial x_i} \sum_{m=1}^{\infty} \frac{\Gamma(m+\alpha+n)}{\Gamma(m+n-1)} \frac{\Gamma^2(m+n-1)}{\Gamma^2(m)} \frac{1}{m^k} Z_m(x, y) \right| \lesssim \frac{1}{|x, y|^\alpha+n+2(n-1)+1}.
\]

For the part involving coefficient \( \delta_m(K) \), by Lemma 5.1 and (15),

\[
\sum_{m=1}^{\infty} \frac{\Gamma(m+\alpha+n)}{\Gamma(m+n-1)} \frac{\Gamma^2(m+n-1)}{\Gamma^2(m)} \frac{1}{m^k} \frac{\partial}{\partial x_i} Z_m(x, y) \lesssim \sum_{m=1}^{\infty} \frac{1}{m^{K-(\alpha+3n-3)}} \lesssim 1. \quad \square
\]

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Differentiating (4), we have

\[
\frac{\partial}{\partial x_i} R_\alpha(x, y) = \sum_{m=1}^{\infty} c_m(\alpha) \frac{\partial}{\partial x_i} S_m(|x|)S_m(|y|) Z_m(x, y)
\]

\[+ \sum_{m=1}^{\infty} c_m(\alpha)S_m(|x|)S_m(|y|) \frac{\partial}{\partial x_i} Z_m(x, y) =: T_1 + T_2. \]
The method we will use to estimate $T_1$ and $T_2$ is similar to the proof of Theorem 4.6 where every step is justified. Here we will be brief and will not repeat the same arguments unless there is an essential difference.

We first estimate $T_1$. Using the integral formula (20) for both $S_m(|x|)$ and $S_m(|y|)$ and differentiating $S_m(|x|)$ under the integral sign shows

$$T_1 = - (n - 2) x_i \int_0^1 \int_0^1 (1 - t)^{\frac{n}{2} - 1}(1 - |x|^2 t)^{\frac{n}{2} - 2}(1 - \tau)^{\frac{n}{2} - 1}(1 - |y|^2 \tau)^{\frac{n}{2} - 1}$$

$$\times \sum_{m=1}^{\infty} \frac{c_m(\alpha)}{B^2(m, n - 1)} t^m \tau^{m-1} Z_m(x, y) \, d\tau \, dt.$$

Observe that the power of $(1 - |x|^2 t)$ is $n/2 - 2$ which is negative when $n = 3$ and this case should be treated separately. Define $W(x, y)$ as in Lemma 5.2. Using that $t^m \tau^{m-1} Z_m(x, y) = Z_m(tx, \tau y) / \tau$ we obtain

$$|T_1| \leq C + C \int_0^1 \int_0^1 (1 - t)(1 - \tau)(1 - |y|^2 \tau)^{\frac{n}{2} - 1}(1 - |x|^2 t)^{\frac{n}{2} - 2} |W(tx, \tau y)| \, d\tau \, dt,$$

where we can get rid of the factor $1/\tau$ in the same way as done in the proof of Theorem 4.6. Applying Lemma 5.2 (i) we deduce

$$|T_1| \lesssim 1 + \int_0^1 \int_0^1 \frac{1}{|tx, \tau y|^\alpha+n+2(n-1)} \, d\tau \, dt \lesssim \frac{1}{|x, y|^\alpha+n+1},$$

where in the last inequality we use (10) and Lemma 2.15 twice. When $n = 3$, the power of $1 - |x|^2 t$ is negative and we repeat the same argument with the only change that we use the inequality $1 - |x|^2 t \geq 1 - t$. We obtain (47) again.

We now estimate $T_2$. Applying the integral formula (20) twice shows

$$T_2 = \int_0^1 \int_0^1 (1 - t)^{\frac{n}{2} - 1}(1 - |x|^2 t)^{\frac{n}{2} - 1}(1 - \tau)^{\frac{n}{2} - 1}(1 - |y|^2 \tau)^{\frac{n}{2} - 1}$$

$$\times \sum_{m=1}^{\infty} \frac{c_m(\alpha)}{B^2(m, n - 1)} t^m \tau^{m-1} \partial_{x_i} Z_m(x, y) \, d\tau \, dt.$$
and first using (12) and (13) and then (16) and Lemma 2.15 twice gives
\[ |T_2| \lesssim 1 + \int_0^1 \int_0^1 \frac{1}{(tx, ty)^{\alpha+n+3}} d\tau dt \lesssim \frac{1}{|x, y|^{\alpha+n+1}}. \]
With the estimate of \( T_1 \) in (17) and \( T_2 \) above, the proof is complete. \( \square \)

6. Weighted Integrals of Powers of Reproducing Kernels

In this section we prove the two-sided estimate given in Theorem 1.3. The upper estimate part immediately follows from Theorem 1.3 and the main problem is to obtain the lower estimate. For this we first estimate \( R_\alpha(x, y) \) from below when \( y \) is a positive multiple of \( x \).

**Lemma 6.1.** Let \( \alpha > -1 \). There exists a constant \( C > 0 \) depending only \( \alpha \) and \( n \) such that for all \( 0 \leq r, \rho < 1 \) and \( \zeta \in \mathbb{S} \),
\[ R_\alpha(r\zeta, \rho\zeta) \geq \frac{C}{(1 - \rho \rho)^{\alpha + n}}. \]

**Proof.** Note that every term in the sum
\[ R_\alpha(r\zeta, \rho\zeta) = 1 + \sum_{m=1}^{\infty} c_m(\alpha) S_m(r) S_m(\rho) Z_m(r\zeta, \rho\zeta) \]
is non-negative because \( S_m(r) \geq 1 \) by Lemma 2, (13) \( c_m(\alpha) > 0 \) by 13, and
\( Z_m(r\zeta, \rho\zeta) = (\rho)^m \dim H_m \geq 0 \) by 21. Since \( c_m(\alpha) \sim m^{\alpha+1} \) as \( m \to \infty \) by 13 and \( c_m(\alpha) \) is strictly positive, there exists a constant \( C \) such that \( c_m(\alpha) \geq C m^{\alpha+1} \) for all \( m \geq 1 \). Therefore using also 21, we have
\[ R_\alpha(r\zeta, \rho\zeta) \geq C \left( 1 + \sum_{m=1}^{\infty} m^{\alpha+n-1} (\rho)^m \right) \geq \frac{C}{(1 - \rho \rho)^{\alpha + n}}. \] \( \square \)

Lemma 6.1 is not sufficient to estimate the integral in Theorem 1.3 from below. Our aim is to extend Lemma 6.1 to a larger domain of \( y \).

For an orthogonal transformation \( U : \mathbb{R}^n \to \mathbb{R}^n \), by [2, Proposition 5.27 (c)], \( Z_m(Ux, Uy) = Z_m(x, y) \) and therefore \( R_\alpha(Ux, Uy) = R_\alpha(x, y) \). Since there exists such a \( U \) that \( Ux = (|x|, 0, \ldots, 0) \), for notational simplicity, we consider only \( x \) of the form \((r, 0, \ldots, 0)\). Let \( e_1 = (1, 0, \ldots, 0) \). For \( y \in \mathbb{R}^n \), we write \( y = (y_1, \hat{y}) \) with \( \hat{y} = (y_2, \ldots, y_n) \in \mathbb{R}^{n-1} \). For \( s > 0 \), define the nontangential approach region
\[ \Omega_s = \{ y = (y_1, \hat{y}) \in \mathbb{B} : 0 < y_1 < 1, |\hat{y}| < s(1 - y_1) \}. \]
The set \( \Omega_s \) is intersection of a cone with vertex at \( e_1, \mathbb{B} \), and right half-space.

**Proposition 6.2.** Let \( \alpha > -1 \). There exists an \( s < 1/2 \) and a constant \( C > 0 \) depending only \( \alpha, n, s \) such that for every \( x = re_1, 0 \leq r < 1 \) and \( y \in \Omega_s \),
\[ R_\alpha(x, y) \geq \frac{C}{(1 - ry_1)^{\alpha + n}}. \]

**Proof.** We follow [20, Proposition 5] and [8, Theorem 7.2]. Given \( y = (y_1, \hat{y}) \in \Omega_s \), with \( s < 1/2 \), let \( y_p = (y_1, 0) \) be the projection onto \( y_1 \)-axis. By Lemma 6.1 there exists a constant \( C_1 > 0 \) such that
\[ R_\alpha(x, y_p) \geq \frac{C_1}{(1 - ry_1)^{\alpha + n}}. \]
By the mean-value theorem of calculus

\[ \mathcal{R}_\alpha(x, y) \geq \mathcal{R}_\alpha(x, y_p) - \max_{z \in L(y_p, y)} |\nabla_z \mathcal{R}_\alpha(x, z)| |y - y_p|, \]

where \( L(y_p, y) \) is the line segment joining \( y_p \) and \( y \). By Theorem 1.3 there exists a constant \( C_2 > 0 \) such that (we also use the fact that \( \mathcal{R}_\alpha(\cdot, \cdot) \) is symmetric with respect to its two variables)

\[ |\nabla_z \mathcal{R}_\alpha(x, z)| \leq \frac{C_2}{|x, z|^{\alpha+n+1}} \leq \frac{C_2}{(1 - r|z|)^{\alpha+n+1}}, \]

where in the last inequality we use (17). Since \( s < 1/2 \) and \( L(y_p, y) \subset \Omega_s \), for \( z \in L(y_p, y) \) we have \( |z| \leq y_1 + s(1 - y_1) \leq (1 + y_1)/2 \). Therefore

\[ 1 - r|z| \geq 1 - r \cdot \frac{ry_1}{2} \geq \frac{1 - ry_1}{2}. \]

Combining above we deduce

\[ \mathcal{R}_\alpha(x, y) \geq \frac{C_1}{(1 - ry_1)^{\alpha+n}} - \frac{2^{\alpha+n+1}C_2|y - y_p|}{(1 - ry_1)^{\alpha+n+1}}. \]

Because \( |y - y_p| = |\hat{y}| < s(1-y_1) < s(1 - ry_1) \), we can find a sufficiently small \( s \) such that the lemma holds.

We are now ready to prove Theorem 1.5

Proof of Theorem 1.5. By Theorem 1.3

\[ J(x) := \int_{\Omega_s} |\mathcal{R}_\alpha(x, y)|^p (1 - |y|^{2p})^{-\beta} d\nu(y) \leq \int_{\Omega_s} (1 - |y|^{2p})^\beta d\nu(y) \]

and the upper estimate follows from Lemma 2.10.

For the lower estimate first note that since \( \mathcal{R}_\alpha(\cdot, \cdot), 1 - |y|^{2p} \) and \( d\nu(y) \) are all invariant under orthogonal transformations, without loss of generality, we can assume \( x = (r, 0, \ldots, 0) \), \( 0 \leq r < 1 \). Let \( s \) be as given in Proposition 6.2. Then

\[ J(x) \geq \int_{\Omega_s} |\mathcal{R}_\alpha(x, y)|^p (1 - |y|^{2p})^{-\beta} d\nu(y) \geq \int_{\Omega_s} (1 - |y|^{2p})^\beta d\nu. \]

For \( y = (y_1, \hat{y}) \in \Omega_s \), we have \( (1 - |y|^{2p}) \sim (1 - y_1) \). This is true since first, \( 1 - y_1 \geq 1 - |y| \geq (1 - |y|^{2p})/2 \). Second, \( |y| \leq y_1 + |\hat{y}| \leq y_1 + s(1-y_1) \) and so \( 1 - |y|^{2p} \geq 1 - |y| \geq (1-s)(1-y_1) \). Thus iterating the integral over \( \Omega_s \) we obtain

\[ J(x) \gtrsim \int_0^1 \int_{s(1-y_1)\mathbb{B}_{n-1}} \left( \frac{1 - y_1}{1 - ry_1} \right)^{\beta} d\nu_{n-1} dy_1 \sim \int_0^1 \left( \frac{1 - y_1}{1 - ry_1} \right)^{\beta+n-1} dy_1, \]

where \( \mathbb{B}_{n-1} \) is the \((n-1)\)-dimensional unit ball, \( \nu_{n-1} \) is the \((n-1)\)-dimensional volume measure and \( \nu_{n-1}(s(1-y_1)\mathbb{B}_{n-1}) \sim (1 - y_1)^{n-1} \). The lower estimate now follows from Lemma 2.14. \( \square \)
7. Bergman Projection

In this section we prove Theorems 1.1 and 1.7. With the estimates obtained earlier this will be straightforward and similar to the proof of [10, Theorem 1.9]. We provide the details for completeness.

We first recall that for every \( f \in \mathcal{H}(\mathbb{B}) \), there exists a unique sequence of homogeneous harmonic polynomials \( p_m \in H_m(\mathbb{R}^n) \) such that for all \( x \in \mathbb{B} \),

\[
(48) \quad f(x) = \sum_{m=0}^{\infty} S_m(|x|) p_m(x),
\]

where the above series converges absolutely and uniformly on compact subsets of \( \mathbb{B} \). A proof of this can be found in [11], [14], [21], [28, 6.3.1]. Using this we extend the reproducing property given in (2) to \( B_{1,\alpha} \).

**Lemma 7.1.** Let \( \alpha > -1 \). For all \( f \in B_{1,\alpha} \) and \( x \in \mathbb{B} \),

\[
f(x) = \int_{\mathbb{B}} R_\alpha(x, y) f(y) \, d\nu_\alpha(y).
\]

**Proof.** Fix \( x \in \mathbb{B} \) and let \( f \) has the expansion (48). We integrate in polar coordinates as in (18). The integrability condition is satisfied since for fixed \( x \in \mathbb{B} \), \( R_\alpha(x, y) \) is bounded by Theorem 1.3 and (17). Thus

\[
\int_{\mathbb{B}} R_\alpha(x, y) f(y) \, d\nu_\alpha(y) = \frac{1}{V_\alpha} \int_0^1 r^{n-1}(1-r^2)^\alpha \int_{\mathbb{S}} R_\alpha(x, r\zeta) f(r\zeta) \, d\sigma(\zeta) \, dr.
\]

The series expansions of \( R_\alpha(x, \cdot) \) in (4) and of \( f \) in (48) uniformly converge on \( r\mathbb{S} \). Writing these, changing the orders of the integral and the series, and then using the orthogonality in (19) and reproducing property of \( Z_m \) in (20) shows that

\[
\int_{\mathbb{B}} R_\alpha(x, y) f(y) \, d\nu_\alpha(y) = \frac{n}{V_\alpha} \int_0^1 r^{n-1}(1-r^2)^\alpha \sum_{m=0}^{\infty} c_m(\alpha) r^{2m} S_m(|x|) S_m^2(r) p_m(x) \, dr.
\]

We claim that the above series uniformly converges on \( 0 \leq r \leq 1 \). To see this pick \( \rho > 1 \) such that \( \rho x \in \mathbb{B} \). Then by the absolute convergence of the series in (48) and the fact that \( S_m \geq 1 \) by Lemma 2.13

\[
\sum_{m=0}^{\infty} |p_m(\rho x)| \leq \sum_{m=0}^{\infty} S_m(|x|) |p_m(\rho x)| < \infty,
\]

and so there exists \( C > 0 \) such that \( |p_m(\rho x)| \leq C \) for all \( m \). Thus by homogeneity

\[
|p_m(x)| = \frac{|p_m(\rho x)|}{\rho^m} \leq \frac{C}{\rho^m},
\]

which with (19) and Lemma 2.13 implies the uniform convergence. Changing the orders of the integral and the series gives the desired result. \( \square \)

**Lemma 7.2.** Let \( \alpha > -1 \) and \( \delta > -1 \). The integral

\[
\int_{\mathbb{B}} R_\alpha(x, y)(1-|y|^2)^\delta \, d\nu(y)
\]

is constant for all \( x \in \mathbb{B} \).
Proof. For fixed $x \in \mathbb{B}$, the integrability condition is satisfied again and we integrate in polar coordinates to obtain

$$\int_{\mathbb{B}} R_\alpha(x, y)(1 - |y|^2)^\delta \, d\nu(y) = n \int_0^1 r^{n-1}(1 - r^2)^\delta \int_{S} R_\alpha(x, r\zeta) \, d\sigma(\zeta) \, dr.$$ 

By the mean-value property of $\mathcal{H}$-harmonic functions, the inner integral over $S$ equals $R_\alpha(x, 0) = 1$. So the integral is $n \int_0^1 r^{n-1}(1 - r^2)^\delta \, dr = C$ for every $x \in \mathbb{B}$. □

Proof of Theorem A.1. Let $q$ be the conjugate exponent of $p$. Before starting the proof we mention that the reason we used the normalizing constant $V_\alpha$ is to normalize also the reproducing kernel $R_\alpha$, so that the series in (1) starts with 1 and we have $R_\alpha(x, 0) = R_\alpha(0, y) = 1$. This constant will show up in the formulas below, however it has no effect on the boundedness of $P_\beta$ and can be ignored.

We first show the sufficiency of the condition (3). We first note that if (3) holds and $f \in L^p_\alpha$, then $f \in L^1_\beta$. This is clear when $p = 1$, since in this case (3) implies $\beta > \alpha$. When $1 < p < \infty$, by Hölder’s inequality

$$\int_{\mathbb{B}} |f(y)|(1 - |y|^2)^\beta \, d\nu(y) \leq \left( \int_{\mathbb{B}} |f(y)|^p (1 - |y|^2)^{\alpha} \, d\nu(y) \right)^{\frac{1}{p}} \left( \int_{\mathbb{B}} (1 - |y|^2)^{\beta - \frac{\alpha}{p}} \, d\nu(y) \right)^{\frac{1}{q}}$$

and the last integral is finite since $q(\beta - \frac{\alpha}{p}) > -1$ by (3). So, $P_\beta f$ is a well-defined function on $\mathbb{B}$. Also, since $R_\alpha(\cdot, y)$ is $\mathcal{H}$-harmonic, $P_\beta f$ is $\mathcal{H}$-harmonic. Therefore, by Theorem 1.3 it suffices to show that the operator $Q_\beta$ defined by

$$Q_\beta f(x) := \int_{\mathbb{B}} \frac{1}{|x, y|^{\beta + \alpha}} f(y)(1 - |y|^2)^\beta \, d\nu(y)$$

is bounded from $L^p_\alpha$ to $L^p_\alpha$.

If $p = 1$, then by Fubini’s theorem and Lemma 2.16 (with symmetry of $[x, y]$),

$$\|Q_\beta f\|_{1, \alpha} = \int_{\mathbb{B}} |Q_\beta f(x)| \, d\nu(\alpha) = \frac{1}{\alpha} \int_{\mathbb{B}} |f(y)|(1 - |y|^2)^\beta \int_{\mathbb{B}} \frac{(1 - |x|^2)^\alpha}{|x, y|^{\beta + \alpha}} \, d\nu(x) \, d\nu(y)$$

$$\lesssim \int_{\mathbb{B}} |f(y)|(1 - |y|^2)^\beta \frac{1}{(1 - |y|^2)^{\beta - \alpha}} \, d\nu(y) = \|f\|_{1, \alpha},$$

since $\beta - \alpha > 0$ by (3).

If $1 < p < \infty$, we apply Schur’s test (see [10] Theorem 1.8)). According to this test, since

$$Q_\beta f(x) = V_\alpha \int_{\mathbb{B}} \frac{1 - |y|^2)^{\beta - \alpha}}{|x, y|^{\beta + \alpha}} f(y) \, d\nu(\alpha),$$

if we can find a positive function $h$ on $\mathbb{B}$ such that

$$\int_{\mathbb{B}} \frac{(1 - |y|^2)^\beta}{|x, y|^{\beta + \alpha}} h^q(y) \, d\nu(y) \lesssim h^q(x) \quad (x \in \mathbb{B})$$

and

$$\int_{\mathbb{B}} \frac{(1 - |x|^2)^\alpha}{|x, y|^{\beta + \alpha}} h^p(x) \, d\nu(x) \lesssim h^p(y) \quad (y \in \mathbb{B}),$$

we have

$$\|Q_\beta f\|_{p, \alpha} \leq C \|f\|_{p, \alpha}.$$
then \( Q_\beta : L_0^p \to L_\alpha^p \) is bounded. We take \( h(x) = (1 - |x|^2)^{-\frac{\alpha+1}{p}} \). Then (3) holds because by Lemma 2.16,

\[
\int_\mathbb{B} \frac{(1 - |y|^2)^\beta - \frac{\alpha+1}{p}}{[x, y]^{\beta+n}} \, d\nu(y) \lesssim \frac{1}{(1 - |x|^2)^{\frac{\alpha+1}{p}}} = h^p(x),
\]

since \( \beta - \frac{\alpha+1}{p} > -1 \) by (3), and \( \frac{\alpha+1}{q} > 0 \). To see (50) we again apply Lemma 2.16 to obtain

\[
(1 - |y|^2)^{\beta - \alpha} \int_\mathbb{B} \frac{(1 - |x|^2)^{\alpha - \frac{\alpha+1}{q}}}{[x, y]^{\beta+n}} \, d\nu(x) \lesssim \frac{(1 - |y|^2)^{\beta - \alpha}}{(1 - |y|^2)^{\beta - \alpha + \frac{\alpha+1}{q}}} = h^q(y),
\]

since \( \alpha - \frac{\alpha+1}{q} > -1 \), and \( \beta - \alpha + \frac{\alpha+1}{q} > 0 \) by (3). This proves the sufficiency of (3).

To show the necessity of (3), suppose \( P_\beta : L_0^p \to L_\alpha^p \) is bounded. Then the adjoint operator \( P_\beta^* : L_\alpha^q \to L_0^q \) is bounded (as usual we identify the dual of \( L_\alpha^q \) with \( L_\alpha^q \)), where \( P_\beta^* \) is given by

\[
P_\beta^* f(x) = V_\alpha^{-1} \frac{(1 - |x|^2)^{\beta - \alpha}}{V_\alpha} \int_\mathbb{B} R_\beta(x, y) f(y) \, d\nu(y).
\]

If \( 1 < p < \infty \), then take \( f = 1 \). Since \( f \in L_\alpha^q \), we must have \( P_\beta^* f \in L_\alpha^q \). But by Lemma 7.2 we have \( P_\beta^* f(x) = C (1 - |x|^2)^{\beta - \alpha} \) and this belongs to \( L_\alpha^q \) only if \( q(\beta - \alpha) + \alpha > -1 \) which is same as (3).

If \( p = 1 \), then \( P_\beta^* \) is bounded on \( L_\infty \) and we need to show that \( \beta > \alpha \). Taking \( f = 1 \) again, we have \( P_\beta^* f(x) = C (1 - |x|^2)^{\beta - \alpha} \) which belongs to \( L_\infty \) only if \( \beta \geq \alpha \). What remains is to show that \( \beta = \alpha \) can not be true. Assume now that \( \beta = \alpha \) and for \( x_0 \in \mathbb{B} \), define \( f_{x_0} \) on \( \mathbb{B} \) by

\[
f_{x_0}(y) = \begin{cases} \frac{|R_\beta(x_0, y)|}{R_\beta(x_0, y)}, & \text{if } R_\beta(x_0, y) \neq 0; \\ 1, & \text{if } R_\beta(x_0, y) = 0. \end{cases}
\]

Clearly \( \|f_{x_0}\| = 1 \), and by Theorem 1.5

\[
P_\beta^* f_{x_0}(x_0) = \int_\mathbb{B} |R_\beta(x_0, y)| \, d\nu(y) \sim 1 + \log \frac{1}{1 - |x_0|^2}.
\]

Because \( P_\beta^* f_{x_0} \) is continuous, this implies \( \|P_\beta^* f_{x_0}\| \geq 1 + \log \frac{1}{1 - |x_0|^2} \to \infty \) as \( |x_0| \to 1^- \). This contradicts to the boundedness of \( P_\beta^* \) on \( L_\infty \). Thus \( \beta = \alpha \) can not be true and the necessity of (3) is proved.

Finally, suppose (3) holds and \( f \in B_\alpha^p \). Then \( f \in B_\beta^1 \) as shown in the beginning of the proof, and it follows from Lemma 7.1 that \( P_\beta f = f \). \( \square \)

We finish by proving Theorem 1.7

**Proof of Theorem 1.7.** Let \( f \in L_\infty \). First, since \( R_\alpha(0, y) = 1 \),

\[
|P_\alpha f(0)| = \left| \int_\mathbb{B} R_\alpha(0, y) f(y) \, d\nu(y) \right| \leq \|f\|_\infty.
\]
Next, by Theorem [14] and Lemma [2,16] for \( i = 1, \ldots, n \),

\[
\left| \frac{\partial}{\partial x_i} P_\alpha f(x) \right| = \left| \int_\mathbb{B} \frac{\partial}{\partial x_i} R_\alpha(x, y) f(y) \, d\nu_\alpha(y) \right| \lesssim \|f\|_\infty \int_\mathbb{B} \frac{(1 - |y|^2)\alpha}{|x, y|^{n+\alpha+1}} \, dv(y)
\]

\[
\lesssim \|f\|_\infty \frac{1}{1 - |x|^2}.
\]

Hence \((1 - |x|^2)\left| \nabla P_\alpha f(x) \right| \lesssim \|f\|_\infty \) combining above we conclude that \( P_\alpha f \in \mathcal{B} \) with \( \|P_\alpha f\|_\mathcal{B} \lesssim \|f\|_\infty \). □

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