Heisenberg realization for $U_q(sl_n)$ on the flag manifold

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Abstract

We give the Heisenberg realization for the quantum algebra $U_q(sl_n)$, which is written by the $q$-difference operator on the flag manifold. We construct it from the action of $U_q(sl_n)$ on the $q$-symmetric algebra $A_q(Mat_n)$ by the Borel-Weil like approach. Our realization is applicable to the construction of the free field realization for the $U_q(\widehat{sl}_n)$ [AOS].

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1. Introduction

Recently, the quantum Knizhnik-Zamolodchikov equations \((q\text{-KZ eq.)}\) \cite{Sm, FR} have been analyzed \cite{M1, R}. This \(q\text{-KZ equations}\) are important both for physics and mathematics by the relationship with 2-dimensional integrable theories \cite{Sm, DFJMN}, quantum affine Lie algebras and elliptic \(R\)-matrices \cite{FR, DJO}.

To solve the classical \((q = 1)\) KZ equations, an important and powerful tools were the free field realization for the affine Lie algebra \(\widehat{G}\) \cite{W, FF} and the Heisenberg realization for the corresponding Lie algebra \(G\) which is written by the differential operator on the flag manifold \cite{SV, ATY, FM}. Even in the quantum case \((q \neq 1)\), for example for the algebra \(U_q(\widehat{sl}_2)\), the Heisenberg realization and the free field realization \cite{FJ, M2, ABG, Sh} are also important for the analysis of the \(q\text{-KZ equation}\) \cite{JMMN, KQS, M3}. We expect that this situation is the same for other quantum affine Lie algebras.

The aim of this paper is to construct the Heisenberg realization for the quantum algebra \(U_q(sl_n)\). In the forthcoming paper \cite{AOS}, the free field realization for the quantum affine algebra \(U_q(\widehat{sl}_n)\) will be constructed by using this Heisenberg realization.

2. Quantum algebra \(U_q(sl_n)\)

\section{First we fix some notations. The algebra \(U_q(sl_n)\) is generated by \(e_i, f_i\) and invertible \(k_i\) \((1 \leq i \leq n - 1)\) with relations

\begin{align*}
  k_i e_j k_i^{-1} &= q^{A_{ij}} e_j, \\
  k_i f_j k_i^{-1} &= q^{-A_{ij}} f_j, \\
  e_i f_j - f_j e_i &= \delta_{ij} \left( k_i - k_i^{-1} \right) / \left( q - q^{-1} \right),
\end{align*}

\begin{align*}
  1 - A_{ij} \sum_{m=0}^{1-A_{ij}} (-1)^m \left( \begin{array}{c} 1 - A_{ij} \\ m \end{array} \right) e_i^{1-A_{ij}-m} e_j^m &= 0, \\
  1 - A_{ij} \sum_{m=0}^{1-A_{ij}} (-1)^m \left( \begin{array}{c} 1 - A_{ij} \\ m \end{array} \right) f_i^{1-A_{ij}-m} f_j^m &= 0,
\end{align*}

where \(q \in \mathbb{C}\), \((A_{ij})_{1 \leq i, j \leq n-1}\) is the Cartan matrix such that \(A_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}\), \([n] = [n]!/[n-m]![m]!\) and \([n] = (q^n - q^{-n})/(q - q^{-1})\).

The algebra \(U_q(sl_n)\) is a Hopf algebra with the comultiplication \(\Delta\)

\begin{align*}
  \Delta(k_i) &= k_i \otimes k_i, \\
  \Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i, \\
  \Delta(f_i) &= f_i \otimes k_i^{-1} + 1 \otimes f_i,
\end{align*}
the antipode $S$ such that $S(k_i) = k_i^{-1}, S(e_i) = -k_i^{-1}e_i, S(f_i) = -f_ik_i$ and the co-unit $\epsilon$ such that $\epsilon(k_i) = 1, \epsilon(e_i) = 0, \epsilon(f_i) = 0$.

§ 2.2. Let $M_\lambda$ be the Verma module over $U_q(sl_n)$ generated by the highest weight vector $|\lambda\rangle$ such that $e_i|\lambda\rangle = 0, k_i|\lambda\rangle = q^{\lambda_i}|\lambda\rangle$ with $\lambda_i \in \mathbb{C}$. The dual module $M_\lambda^*$ is generated by $\langle \lambda|$, which satisfies $\langle \lambda|f_i = 0, \langle \lambda|k_i = q^{\lambda_i}\langle \lambda|$.

The quantum algebra $U_q(sl_n)$ is realized by the Heisenberg algebra $H_n$. We have

\textbf{Theorem I}. There exists the algebra homomorphism $\pi_\lambda : U_q(sl_n) \rightarrow H_n$ define as

\[
\pi_\lambda(k_i) = q^{i-1} \sum_{j=1}^{i-1} ((\vartheta_{ji} - \vartheta_{ji+1}) + (\lambda_i - 2\vartheta_{ji+1}) + \sum_{j=i+2}^n (\vartheta_{i+1j} - \vartheta_{ij})),
\]

\[
\pi_\lambda(e_i) = \sum_{k=1}^i q^{i-k} \sum_{j=1}^{k-1} (\vartheta_{ji} - \vartheta_{ji+1}) \frac{x_{ki}}{x_{k+1i}} [\vartheta_{k+1i}],
\]

\[
\pi_\lambda(f_i) = \sum_{k=1}^{i-1} \frac{x_{k+1i} [\vartheta_{k+1i}]}{x_{ki}} q^{-i-1} \sum_{j=k+1}^{i} ((\vartheta_{ji} - \vartheta_{ji+1}) - (\lambda_i - 2\vartheta_{ji+1}) - \sum_{j=i+2}^n (\vartheta_{i+1j} - \vartheta_{ij}))
\]

\[
+ x_{i+1}^i ((\lambda_i - \vartheta_{i+1i}) + \sum_{j=i+2}^n (\vartheta_{i+1j} - \vartheta_{ij}))
\]

\[
- \sum_{k=i+2}^n \frac{x_{ik}}{x_{i+1k}} [\vartheta_{i+1k}] q^{\lambda_i} + \sum_{j=k}^n (\vartheta_{i+1j} - \vartheta_{ij}),
\]

with $x_{ii} = 1$.

Here $[n]$ denotes the $q$ integer, so $\pi_\lambda(g)$’s are the $q$-difference operators. The proof will be given in the next section.

We also have the following dual generators†

† These dual generators relate to the screening currents of the free field realization for $U_q(sl_n)$ [AOS] which must important to the analysis of the $q$-KZ equation.

3. Heisenberg realization for $U_q(sl_n)$
Theorem II. There exists the algebra anti-homomorphism \( \tilde{\pi}_\lambda : U_q(\mathfrak{sl}_n) \to \mathcal{H}_n \), \( \tilde{\pi}_\lambda = \tilde{\sigma} \circ \pi_\lambda \circ \sigma \), with \( \sigma \) such that \( \sigma(k_i) = k_{n-i}, \sigma(e_i) = e_{n-i}, \sigma(f_i) = f_{n-i} \) and \( \tilde{\sigma} \) such that \( \tilde{\sigma}(x_{ij}) = x_{n+1-j \, n+1-i}, \tilde{\sigma}(\vartheta_{ij}) = -\vartheta_{n+1-j \, n+1-i}, \tilde{\sigma}(\lambda_i) = -\lambda_{n+1-i} \).

§ 3.2. Let \( \mathcal{F} = \mathbb{C}[x_{ij}]|0\rangle \) be the Fock module over Heisenberg algebra \( \mathcal{H}_n \) generated by the highest weight vector \( |0\rangle \) such that \( x_{ij}^{-1}|0\rangle = \vartheta_{ij}|0\rangle = 0 \). The dual module \( \mathcal{F}^* = \langle 0| \mathbb{C}[x_{ij}^{-1}] \rangle \) is generated by \( \langle 0| \) which satisfies \( \langle 0|x_{ij} = \langle 0|\vartheta_{ij} = 0 \). The bilinear form \( \mathcal{F}^* \otimes \mathcal{F} \to \mathbb{C} \) is uniquely defined by \( \langle 0|0 \rangle = 1 \) and \( \langle u|X|v \rangle = \langle u|(X|v) \rangle \) for any \( \langle u| \in \mathcal{F}^*, \langle v| \in \mathcal{F} \) and \( X \in \mathcal{H}_n \). For \( \langle 0|f(x_{ij}^{-1}) \in \mathcal{F}^* \) and \( g(x_{ij})|0\rangle \in \mathcal{F}, \langle 0|f(x_{ij}^{-1})g(x_{ij})|0\rangle \) is nothing but the constant part of \( f(x_{ij}^{-1})g(x_{ij}) \).

4. Construction of the Heisenberg realization for \( U_q(\mathfrak{sl}_n) \)

Next we prove above Theorems by a Borel-Weil like approach, which is based on the method in Ref. [N]. First we give some notations.

§ 4.1. The \( q \)-symmetric algebra \( \mathcal{A}_q(\text{Mat}_n) \) is generated by \( t_{ij} \) (1 \( \leq i, j \leq n \)) with relations

\[
\begin{align*}
t_{ik}t_{jk} &= qt_{jk}t_{ik}, & t_{il}t_{jk} &= t_{jk}t_{il}, \\
t_{ik}t_{il} &= qt_{il}t_{ik}, & t_{ik}t_{jl} - qt_{il}t_{jk} &= t_{jl}t_{ik} - q^{-1}t_{jk}t_{il},
\end{align*}
\]

for \( i < j \) and \( k < l \). Note that this algebra has the algebra automorphism \( \rho \) such that \( \rho(t_{ij}) = t_{n+1-j \, n+1-i} \), \( \rho(q) = q^{-1} \) and the algebra anti-automorphism \( \tilde{\rho} \) such that \( \tilde{\rho}(t_{ij}) = t_{n+1-j \, n+1-i} \), \( \tilde{\rho}(q) = q \).

The algebra \( \mathcal{A}_q(\text{Mat}_n) \) has the structure of a \( U_q(\mathfrak{sl}_n) \)-module. The action of \( U_q(\mathfrak{sl}_n) \) on \( \mathcal{A}_q(\text{Mat}_n) \) is

\[
k_m t_{ij} = t_{ij} q^{\delta_{m+1,j} - \delta_{m+1,i}}, \quad e_m t_{ij} = t_{i,j-1} \delta_{m+1,j}, \quad f_m t_{ij} = t_{i,j+1} \delta_{m,j},
\]

\[
g(uv) = \sum_a (g'_a u)(g''_a v), \quad g.1 = \epsilon(g)1,
\]

for all \( u, v \in \mathcal{A}_q(\text{Mat}_n) \) and for all \( g \in U_q(\mathfrak{sl}_n) \) with \( \Delta(g) = \sum_a g'_a \otimes g''_a \). Note that this action of \( g \in U_q(\mathfrak{sl}_n) \) can be written by the matrix \( g(g)_{ij} \) as \( g t_{ij} = \sum_k t_{jk} g(g)_{kj} \) with \( g(k_m) = q^{E_{mm} - E_{m+1 \, m+1}}, g(e_m) = E_{m+1,m}, g(f_m) = E_{m+1,m} \) and \( (E_{\alpha \beta})_{ij} = \delta_{\alpha i} \delta_{\beta j} \). These matrices are
noting but the vector representation for the \( U_q(sl_n) \). The action for the rows of matrix \( t_{ij} \) is given by the above automorphism \( \rho \) or \( \tilde{\rho} \).

§ 4.2. For the ordered set \( I = \{i_1 < \cdots < i_r\} \) and \( J = \{j_1 < \cdots < j_r\} \), let \( \xi^I_J \) be the quantum \( r \)-minor determinant with respect to rows \( I \) and columns \( J \) such that [TT, NYM]

\[
\xi^I_J = \sum_{\sigma \in S_r} (-q)^{l(\sigma)} t_{i_{\sigma(1)}j_1} \cdots t_{i_{\sigma(r)}j_r}.
\]

Here \( S_r \) is the permutation group of the set \( \{1, \cdots, r\} \) and \( l(\sigma) \) stands for the number of inversions involved in \( \sigma \); \( l(\sigma) = \sharp\{(i, j) : i < j, \sigma(i) > \sigma(j)\} \). From now on, \( \xi^I_J = 0 \) if \( I \) or \( J \) has same elements. Note that \( \xi^I_J \xi^{I'}_{J'} = \xi^I_J \xi^I_J \) if \( I' \subset I, J' \subset J \). We have

**Proposition.** With the lower triangular matrix \( B \), the Gauss decomposition of the matrix \( T = (t_{ij}) \) of the \( q \)-coordinates is given as

\[
t_{ij} = \sum_k B_{ik} X_{kj}, \quad B_{ij} = (\xi^1_{1: \cdots j-1})^{-1} \xi^1_{1: \cdots j-1} i, \quad X_{ij} = (\xi^1_{1: \cdots i})^{-1} \xi^1_{1: \cdots i-1 j},
\]

and \( B_{ij} = 0 \) for \( i < j \) and \( X_{ij} = 0 \) for \( i > j \). Here \( \{1 \cdots 0\} = \{\} \).

**Proof.** follows from

\[
t_{ij} = B_{i1} X_{1j} + (\xi^1_1)^{-1} \xi^1_{1j}, \quad (\xi^1_{1: \cdots r})^{-1} \xi^1_{1: \cdots r} j = B_{i r+1} X_{r+1 j} + (\xi^1_{1: \cdots r+1})^{-1} \xi^1_{1: \cdots r+1} j,
\]

which are obtained from the \( q \)-deformed Jacobi identity

\[
\xi^1_{1: \cdots r} \xi^1_{1: \cdots r+1} + 2 = \xi^1_{1: \cdots r+1} \xi^1_{1: \cdots r+2} - q \xi^1_{1: \cdots r} \xi^1_{1: \cdots r+2}.
\]

Q.E.D.

We regard \( X_{ij} (i < j) \) as a \( q \)-analogue of local coordinates of the flag manifold \( B \backslash GL_n \).

For \( i < i_1 \) and \( I = \{i_1 < \cdots < i_r\} \), we denote \( \eta^i_I = \xi^1_{1: i_1: \cdots i_r} \), then \( X_{ij} = (\eta^1_{i-1})^{-1 i} \eta^1_{j-1} \).

Since the principal minors \( \xi^1_{1: \cdots i} \)'s \( 1 \leq i \leq n \) commute with each other, one can consistently adjoin their inverse to the algebra \( \mathbb{C}[\xi^1_{1: \cdots} J] \).

§ 4.3. The quantum minor \( \eta^r_{ij} \)'s satisfy, for \( r < i < j < k < l \), the same relations as \( t_{ij} \)'s in (4.1) and Plücker relation ( Young symmetry ) [TT, NYM, N]

\[
\eta^r_{ij} \eta^r_{jk} - q \eta^r_{ij} \eta^r_{ik} + q^2 \eta^r_{ik} \eta^r_{jk} = 0,
\]

\[5\]
and the commutation relations
\[ η_i^r η_j^r = qη_j^r η_i^r, \quad η_i^r η_k^r = qη_k^r η_i^r, \quad η_i^r η_j^r = η_k^r η_i^r + η_i^r η_k^r, \]
\[ η_j^r η_k^r = qη_j^r η_k^r, \quad η_j^r η_k^r = q^2 η_k^r η_j^r. \]

The action of \( U_q(sl_n) \) on the quantum minor \( \eta_j^1 \) is
\[ k_m η_j^i = η_j^{i-1} q^δ_{m-j} δ_{m+1,j} + δ_{m,i}, \]
\[ e_m η_j^i = η_{j-1}^{i-1} δ_{m+1,j}, \quad f_m η_j^i = η_{j+1}^{i-1} δ_{m,j} + η_{i+1,j}^{i-1} δ_{m,i}. \]

Owing to the Plücker relation, \( η_j^{i+1} = η_{j+1}(η_j^{i-1})^{-1} η_j^{i-1} - qη_j^{i-1}(η_j^{i-1})^{-1} η_j^{i-1} \), the algebra \( \mathcal{A} = C[η_j^{i-1}, (η_j^{i-1})^{-1}]_{1≤i≤n, i≤j≤n} \) has the structure of a \( U_q(sl_n) \)-module.

§ 4.4. To relate the non-commutative algebra \( C[X_{ij}] \) with the commutative one \( C[x_{ij}] \), we fix the ordering of \( η_j^i \)'s. The algebra \( \mathcal{A} \) has the basis
\[ \{ (η_j^0)^{a_1n} \cdots (η_j^1)^{a_1i} (η_j^1)^{a_2n} \cdots (η_j^2)^{a_2i} \cdots (η_j^n)^{a_n-1n-n-1} | a_{ij} ∈ Z_{≥0}, i < j, a_{ii} ∈ Z \}, \]
which ordering we call normal ordering. We introduce the projection \( \circ : \mathcal{A} → \mathcal{A} \) such that
\[ \circ \text{ any ordered } \prod_{i≤j}^0 (η_j^{i-1})^{a_{ij}} \circ = \text{ normal ordered } \prod_{i≤j} (η_j^{i-1})^{a_{ij}}. \]

Let \( Z^a_λ = \circ \prod_i (η_j^{i-1})^{λ_i} \prod_{i<k} (X_{jk})^{a_{ik}} \circ \) with \( λ_i ∈ Z \) and \( a_{ij} ∈ Z_{≥0} \). If we denote \( Y^i = (η_j^{i-1})^{a_{i1}} \cdots (η_j^{i-1})^{a_{ii}} \) with \( a_{ii} = λ_i - \sum_{j=i+1} a_{ij} \), then \( Z^a_λ = Y^1 \cdots Y^{n-1} \). The algebra \( \mathcal{A} \) has the decomposition \( \mathcal{A} = \bigoplus_{λ_i ∈ Z} \mathcal{A}_λ \) such that \( \mathcal{A}_λ \) is the vector space spanned by the vectors \( \{ Z^a_λ | a_{ij} ∈ Z_{≥0}, i < j \} \). The algebra \( \mathcal{A}_λ \) also has the structure of a \( U_q(sl_n) \)-module, and we have

**Lemma.** The left action of \( U_q(sl_n) \) on \( \mathcal{A}_λ \) is as follows
\[ k_i Z^a_λ = Z^a_λ \sum_{j=1}^{i-1} (a_{ij} - a_{i+1}) + (λ_i - 2a_{i+1}) + \sum_{j=i+2}^n (a_{i+1,j} - a_{ij}), \]
\[ e_i Z^a_λ = \sum_{k=1}^i q \sum_{j=1}^{k-1} (a_{j+1} - a_{j+1}) [a_{k,i+1}]^o Z^a_λ (X_{ki+1})^{-1} X_{ki}^o, \]
\[ f_i Z^a_λ = \sum_{k=1}^{i-1} [a_{ki}]o X_{k,i+1} (X_{ki})^{-1} Z^a_λ o - \sum_{j=k+1}^{i-1} (a_{j+1} - a_{j+1}) - (λ_i - 2a_{i+1}) - \sum_{j=i+2}^n (a_{i+1,j} - a_{ij}) \]
\[ + [(λ_i - a_{i+1}) + \sum_{j=i+1}^n (a_{i+1,j} - a_{ij})] o X_{i,i+1} Z^a_λ o \]
\[ - \sum_{k=i+2}^n [a_{i+1,k}] o X_{ik} (X_{i+1,k})^{-1} Z^a_λ o q^{λ_i} + \sum_{j=k}^n (a_{i+1,j} - a_{ij}). \]
Proof. follows from
\[ k_m Y^i = Y^i q^{(a_{im} - a_{i,m+1})} \sum_{j=1}^{n-1} \delta_{mj} + \delta_{m,i-1} \sum_{j=i+1}^n a_{m+1,j}, \]
\[ e_m Y^i = [a_{im+1}] \circ Y^i (\eta_{m+1}^{-1})^{-1} \sum_{j=i+1}^n \delta_{m+1,j}, \]
\[ f_m Y^i = [a_{im}] \circ (\eta_{m+1}^{-1})^{-1} \sum_{j=i}^{n-1} \delta_{mj} + \delta_{m,i-1} \sum_{k=i+1}^n [a_{ik}] \circ (\eta_{k}^{-1})^{-1} Y^i q - \sum_{j=i+1}^{k-1} a_{ij} \]
\[ = [a_{im}] \circ (\eta_{m+1}^{-1})^{-1} \sum_{j=i}^{n-1} \delta_{mj} + \delta_{m,i-1} \sum_{k=i+1}^n [a_{ik}] \circ \eta_{k}^{-2} (\eta_{k-1}^{-1})^{-1} Y^i q - \sum_{j=k}^{m} a_{ij}, \]
here we use \( k_m (\eta_i^r)^a = (k_m \eta_i^r)^a, e_m (\eta_i^r)^a = [a](\eta_i^r)^{a-1}(e_m \eta_i^r)^a, f_m (\eta_i^r)^a = [a](f_m \eta_i^r)(\eta_i^r)^{a-1} \]
with \( a \in \mathbb{Z} \) and the identity \( \sum_k [a_k] q (\sum_{j<k} - \sum_{j>k}) a_j = [\sum_k a_k] \). The polynomials of \( q \) in \( e_i Z_\lambda^a \) and \( f_i Z_\lambda^a \) come from the Cartan parts of the comultiplication of \( e_i \) and \( f_i \) respectively.

Q.E.D.

§ 4.5. Proof of Theorem I.

We consider the commutative algebra \( \mathbb{C}[x_{ij}]_{1 \leq i < j \leq n} \) and define an isomorphism \( \pi_\lambda : \mathbb{A}_\lambda \rightarrow \mathbb{C}[x_{ij}] \) by \( \pi_\lambda(Z_\lambda^a) = z^a \), with \( z^a = \prod_{r < j} (x_{rj})^{a_{rj}} \). Applying this isomorphism \( \pi_\lambda \) to above Lemma, we obtain the \( q \)-difference operators on \( \mathbb{C}[x_{ij}] \) in Theorem I.

Q.E.D.

§ 4.6. Proof of Theorem II.

With the lower triangular matrix \( \tilde{B} \), the Gauss decomposition of inverse direction \( T = \tilde{X} \tilde{B} \) is obtained by the algebra anti-automorphism \( \tilde{\rho} \) in §4.1 from the Gauss decomposition \( T = BX \).

By the algebra automorphism \( \rho \) with some sign changing, we get the action of \( U_q(sl_n) \) on \( \mathbb{C}[\tilde{X}_{ij}] \) and the dual generators of Theorem II.

Q.E.D.

Conclusion and Discussion.

We constructed the Heisenberg realization for the \( U_q(sl_n) \) by the flag coordinate, which is applicable to the construction of the free field realization for the \( U_q(sl_n) \) [AOS]. In the Ref. [DJMM], they also gave the similar realization for the \( U_q(sl_n) \) but it seems that it can not be affinized.
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Appendix. The Jordan-Schwinger type realization and q-oscillator

§ A.1. If we consider only \( i = 1 \) of \( t_{ij} \in A_q(\text{Mat}_n) \), then we can obtain the \( n \) variables Jordan-Schwinger type realization for \( U_q(sl_n) \) \([H, Z]\). Let us denote \( t_j = t_{1j} \), the algebra \( A = C[t_i]_{1 \leq i \leq n} \) has the basis \( \{ t_{a_1} \cdots t_{a_n} | a_i \in \mathbb{Z}_{\geq 0} \} \), which ordering we call normal ordering, and has the structure of a \( U_q(sl_n) \)-module. By an isomorphism \( \pi : A \to C[x_i] \), \( \pi(t_{a_1} \cdots t_{a_n}) = x_{a_1} \cdots x_{a_n} \) and by the action of \( U_q(sl_n) \) on \( A \), we obtain

**Proposition.** There exists the algebra homomorphism \( \pi : U_q(sl_n) \to H_n \) define as

\[
\pi(k_i) = q^{\vartheta_i - \vartheta_{i+1}}, \quad \pi(e_i) = \frac{x_i}{x_{i+1}}[\vartheta_{i+1}], \quad \pi(f_i) = \frac{x_{i+1}}{x_i}[\vartheta_i].
\]

We introduce the projection \( \circ \ast \circ \) same as before. Denote \( X_i = t_i^{-1}t_i \) \((2 \leq i \leq n)\) and

\[
Z_{\lambda} = \circ \pi^A \prod_{i=2}^{n} X_i \circ = t_{a_1} \cdots t_{a_1} \text{ with } a_1 = \lambda - \sum_{i=2}^{n} a_i,
\]

then \( \pi : A[t_1^{-1}] = \oplus_{\lambda \in \mathbb{Z}} A_{\lambda} \) such that \( A_{\lambda} \) is the vector space spanned by the vectors \( \{ Z_{\lambda}^i | a_i \in \mathbb{Z}_{\geq 0}, i > 1 \} \). By an isomorphism \( \pi_{\lambda} : A_{\lambda} \to C[x_i] \), \( \pi_{\lambda}(t_{a_1} \cdots t_{a_1}) = x_{a_2} \cdots x_{a_n} \) and by the action of \( U_q(sl_n) \) on \( A_{\lambda} \), we obtain the \( n - 1 \) variables inhomogeneous realization for \( U_q(sl_n) \), which is the same as above Proposition with additional conditions \( x_1 = 1 \) and \( \vartheta_1 = \lambda - \sum_{i=2}^{n} \vartheta_i \). This realization corresponds with that in Theorem-I on \( C[x_{1j}] \) with \( \lambda_i = 0 \) for \( i \neq 1 \).

§ A.2. For the Heisenberg algebra \( \langle x, \vartheta \rangle \) with \( q^\vartheta x q^{-\vartheta} = qx \), if we denote

\[
a = x, \quad a^\dagger = \frac{1}{x}[\vartheta], \quad N = \vartheta,
\]

then \( \langle a, a^\dagger, N \rangle \) satisfies the q-oscillator algebra such that

\[
aa^\dagger = [N], \quad a^\dagger a = [N + 1],
\]

which is equivalent to \( a^\dagger a - q^{\pm 1}aa^\dagger = q^{\mp N} \). And they satisfy \( [N, a] = a, [N, a^\dagger] = -a^\dagger \).

So we can rewrite our Theorem by the q-oscillator algebra.
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