Asymptotics of Random Contractions

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Abstract: In this paper we discuss the asymptotic behaviour of random contractions $X = RS$, where $R$, with distribution function $F$, is a positive random variable independent of $S \in (0, 1)$. Random contractions appear naturally in insurance and finance. Our principal contribution is the derivation of the tail asymptotics of $X$ assuming that $F$ is in the max-domain of attraction of an extreme value distribution and the distribution function of $S$ satisfies a regular variation property. We apply our result to derive the asymptotics of the probability of ruin for a particular discrete-time risk model. Further we quantify in our asymptotic setting the effect of the random scaling on the Conditional Tail Expectations, risk aggregation, and derive the joint asymptotic distribution of linear combinations of random contractions.

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1 Introduction

Let $R, S$ be two independent random variables with $R > 0, S \in (0, 1)$ almost surely, and define $X = RS$. The random variable $X$ is a random contraction of $R$ via $S$. Random contractions or

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random scalings are common in insurance and finance applications. Typically $R$ models a random payment whereas $S$ is a random discount factor. Several authors have studied random contractions in quite different contexts. Some recent contributions dealing with distributional and asymptotic properties of random contractions are Kotz and Nadarajah (2000), Galambos and Simonelli (2004), Gomes et al. (2004), Maulik and Resnick (2004), Tang and Tsitsiashvili (2003, 2004), Jessen and Mikosch (2006), D’Auria and Resnick (2006, 2008), Tang (2006, 2008), Denisov and Zwart (2007), Pakes and Navarro (2007), Resnick (2007), Beutner and Kamps (2008), Charpentier and Segers (2007, 2009), Hashorva (2008, 2009, 2010), Hashorva and Pakes (2010), Liu and Tang (2010).

Our main goal in this paper is to investigate the tail asymptotics of random contractions when the tail asymptotics of $R$ is known. An important motivation for this investigation is the fact that in insurance and finance applications assumptions often are made on the tail behaviour of a random payment modeled by $R$. If $S$ represents the random discount factor applicable to the interval from the present to the payment time, then $RS$ is the present value of the later payment $R$. In cases where the distribution function $G$ of $S$ is unknown, it is of some interest to know how the tail behaviour of the random contraction $X = RS$ is determined by the corresponding asymptotic behaviours of the factors. One possible application is to approximating the Value at Risk in the presence of discounting, given information about the Value at Risk before discounting.

Without going into mathematical details, we mention briefly the main contributions of this paper:

a) Under the assumption that the distribution function $F$ of $R$ is in the max-domain of attraction of some univariate extreme value distribution we obtain the asymptotic behaviour of $P\{X > u\}$ as $u$ tends to the upper endpoint of $F$, provided that $\overline{G} = 1 - G$ satisfies a regular variation property (Theorem 3.1 below).

b) We determine corresponding results for the density function of $X$ assuming a regular variation property for the density function of $S$, and additional regularity properties in some cases.

c) We present four applications: c1) First we derive the asymptotics of the ruin probability for a particular discrete-time ruin problem, c2) then we discuss briefly the asymptotics of Conditional Tail Expectation in the random contraction framework, c3) and we obtain asymptotic expansions for the aggregation of two contractions, which lead to novel asymptotic characterisation of bivariate elliptical distributions, c4) finally we show the asymptotic independence of certain bivariate random contractions.

As mentioned above, we assume that a generic scaling factor $S$ with distribution function $G$ takes
values in (0, 1). In addition, we assume that \( G(1-y) \) is regularly varying at zero, the above mentioned regular variation property. However, the reader will easily appreciate that the scaling factors can be multiplied by a positive constant and hence can function as inflation or deflation factors. Since our results can easily be adjusted for this contingency, we will say no more about it beyond the closure property in Lemma 2.1.

It is interesting that under the setup of this paper the asymptotic tail behaviours of \( R \) and \( X \) are very similar. In particular, membership of a max-domain of attraction is insensitive to the distribution of bounded discount factors.

Our main results are presented in Section 3 followed by the applications in Sections 4. The proofs of all the results are relegated to Section 5.

## 2 Maximal Domains of Attraction

In this short section we present some details on max-domains of attraction. The distribution function \( F \) belongs to the max-domain of attraction of a univariate extreme value distribution function \( N \), written \( F \in \text{MDA}(N) \), if

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{F_n(a_n x + b_n) - N(x)}{F(x)} \right| = 0 \tag{2.1}
\]

holds for some constants \( a_n > 0, b_n \in \mathbb{R}, n \geq 1 \). See e.g., Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), De Haan and Ferreira (2006), or Resnick (2008) for more details on univariate max-domains of attraction. Only three choices for \( N \) are possible, namely the Fréchet distribution, the Gumbel distribution, or the Weibull distribution. We denote the corresponding distribution functions by \( \Phi_\gamma \), \( \Lambda \), and \( \Psi_\gamma \), respectively, where \( \gamma > 0 \) indexes members of the Fréchet and Weibull families.

The functional form of the Fréchet distribution function is \( \Phi_\gamma(x) = \exp(-x^{-\gamma}), x > 0 \). If \( F \in \text{MDA}(\Phi_\gamma) \), then (2.1) with \( N = \Phi_\gamma \) is equivalent to

\[
\lim_{u \to \infty} \frac{\overline{F}(xu)}{\overline{F}(u)} = x^{-\gamma}, \quad \forall x > 0. \tag{2.2}
\]

This means that the survival function \( \overline{F} = 1 - F \) is regularly varying at infinity with index \(-\gamma \) and further it has an infinite upper endpoint (denoted in the sequel by \( r_F \)).
The functional form of the standard Gumbel distribution function is \( \Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R} \), and (2.1) with \( N = \Lambda \) is equivalent to

\[
\lim_{u \uparrow r_F} \frac{F(u + x/w(u))}{F(u)} = \exp(-x), \quad \forall x \in \mathbb{R},
\]  

(2.3)

where \( w \) is a positive scaling function satisfying

\[
\lim_{u \uparrow r_F} uw(u) = \infty, \quad \text{and} \quad \lim_{u \uparrow r_F} w(u)(r_F - u) = \infty \quad \text{if} \ r_F < \infty.
\]  

(2.4)

Recall that the scaling function \( w \) can be defined asymptotically via the mean excess function (see e.g., Embrechts et al. (1997) or Resnick (2008)) by

\[
w(u) \sim \frac{1}{E\{R - u|R > u\}}, \quad u \uparrow r_F.
\]  

(2.5)

Throughout this paper he relation \( a(u) \sim b(u) \) means that the quotient of both sides tends to 1 according to the indicated limit procedure.

The functional form of the Weibull distribution function is \( \Psi_\gamma(x) = \exp(-|x|^\gamma), x \leq 0 \). If \( F \in \text{MDA}(\Psi_\gamma) \), then \( r_F \) is finite and (2.1) is equivalent to

\[
\lim_{u \to \infty} \frac{F(r_F - x/u)}{F(r_F - 1/u)} = x^\gamma, \quad \forall x > 0.
\]  

(2.6)

In some applications it is necessary to admit scaling factors \( S \in (0, c) \), where \( c \) is a positive constant. We always assume that \( c = 1 \). If \( F \in \text{MDA}(\Psi_\gamma) \) we assume too that \( r_F = 1 \). The next lemma explains why these conventions are not restrictive.

**Lemma 2.1.** Let \( W \) be a random variable whose distribution function \( F \) satisfies (2.1). If \( c, p \in (0, \infty) \), then \( cW^p \) has a distribution function in the same max-domain of attraction as \( W \).

We will need the following facts about subexponential distribution functions. See Embrechts et al. (1997, Appendix A3) for distribution functions supported in \([0, \infty)\), and Borovkov and Borovkov (2008, p. 13) for the general case. Let \( F^{2*} \) denote the convolution square of \( F \), i.e., the distribution function of \( R + R^* \), where \( R^* \) is an independent copy of \( R \). Assume that \( r_F = \infty \) in what follows.

In the case that \( F(0-) = 0 \), we say that \( F \) is subexponential, written \( F \in \mathcal{S}_+ \), if

\[
\lim_{u \to \infty} \frac{F^{2*}(u)}{F(u)} = 2.
\]  

(2.7)

In the case that \( F \) is two-sided, i.e., \( F(0-) > 0 \), define \( F_+(u) = 0 \) if \( u < 0 \) and \( F_+(u) = F(u) \) if \( u \geq 0 \). We say that \( F \) is subexponential if \( F_+ \in \mathcal{S}_+ \), and we write \( F \in \mathcal{S} \); see Borovkov and
Borovkov (2008, p. 14). These authors (p. 19) show that it is possible that a two-sided distribution function $F \not\in S$ satisfies (2.7), but imposing a condition described below will ensure that $F \in S$ is equivalent to (2.7).

Say that $F$ belongs to the class of long-tailed distribution functions, written $F \in \mathcal{L}$, if

$$
\lim_{u \to \infty} \frac{F(u+y)}{F(u)} = 1
$$

for all real $y$. The convergence here is locally uniform with respect to $y$. If $F(0-) = 0$, (2.7) implies that $F \in \mathcal{L}$. In the two-sided case, if $F \in \mathcal{L}$, then $\overline{F_{-\infty}}(u) \sim \overline{F_{+\infty}}(u)$ as $u \to \infty$ (Borovkov and Borovkov (2008, Theorem 1.2.4(vi)), and hence $F \in S$ is equivalent to (2.7). These concepts relate to attraction to the Gumbel distribution as follows.

Still assuming that $r_F = \infty$, assume too that $F \in \text{MDA}(\Lambda, w)$ with $\lim_{u \to \infty} w(u) = 0$. For real $y$ we can choose $u$ so large that $|y| \leq |x|/w(u)$ and hence conclude from (2.3) that $F \in \mathcal{L}$. Further conditions can be given to ensure that $F \in \mathcal{L} \cap S$. This holds if there is a positive constant $\lambda$ such that (see Mitra and Resnick (2008, Corollary 2.9))

$$
\lim_{u \to \infty} \frac{\left[F(\lambda/w(u))\right]^2}{F(u)} = 0. 
$$

(2.8)

Note that if $F(0-) = 0$, in view of Corollary 2.5 of Goldie and Resnick (1988), when $w$ is eventually non-increasing such that

$$
\lim_{u \to \infty} \frac{w(u)}{w(tu)} > 1
$$

holds for some constant $t > 1$, then $F \in S_+$.

We derive by our next result a self-contained proof of the Mitra-Resnick criterion (2.8).

**Lemma 2.2.** Let $F \in \text{MDA}(\Lambda, w)$ with $r_F = \infty$ and $\lim_{u \to \infty} w(u) = 0$ (hence $F \in \mathcal{L}$). Then $F \in S$ if and only if

$$
\lim_{u \to \infty} \frac{1}{F(u)} \int_{\lambda/w(u)}^{u-\lambda/w(u)} \overline{F}(u-y) dF(y) = 0 
$$

(2.10)

holds for some $\lambda > 0$.

By Lemma 2.2, the Mitra-Resnick criterion follows immediately, because the integral in (2.10) is bounded above by

$$
\overline{F}(\lambda/w(u)) \left[\overline{F}(\lambda/w(u)) - \overline{F}(u-\lambda/w(u))\right] \leq \left[\overline{F}(\lambda/w(u))\right]^2.
$$
Furthermore, Lemma 2.2 implies a generalization of Goldie’s sufficient condition asserting that if $F \in \mathcal{L}$ has a dominatedly varying right-hand tail, i.e., $\limsup_{u \to \infty} \frac{F(u/2)}{F(u)} < \infty$, then $F \in \mathcal{S}$ (see Embrechts et al. (1997, pp. 49, 52)). More precisely, suppose that $F \in \text{MDA}(\Lambda, w)$ with $r_F = \infty$ and $\lim_{u \to \infty} w(u) = 0$. Then by Lemma 2.2, $F \in \mathcal{S}$ if
\[
\lim_{u \to \infty} \frac{F(u/2)}{F(u)} \cdot \frac{F(\lambda/w(u))}{F(u/2)} = 0 \quad (2.11)
\]
for some $\lambda \in (0, \infty)$.

### 3 Principal Results

Let $R$ be a positive random variable with distribution function $F$, and let $S_1, \ldots, S_n$ be mutually independent, and independent of $R$. Denote by $G_i$ the distribution function of $S_i$, and assume it is supported on $(0, 1)$. In insurance and financial contexts $S_i$ represents the random discount factor over the interval $[i-1, i)$. Then
\[X_n = R \prod_{i=1}^{n} S_i\]
is the present value of the payment $R$ received at time $n$. Let $H_n$ denote the distribution function of the random product $X_n$.

If $F \in \text{MDA}(\Phi_\gamma)$ then, with no further conditions, it follows from Breiman’s lemma (Breiman (1965)) that
\[
\lim_{u \to \infty} \frac{P\{X_n > u\}}{P\{R > u\}} = E\left\{ \prod_{i=1}^{n} S_i^\gamma \right\} \quad (3.1)
\]
and, in particular, that $H_n \in \text{MDA}(\Phi_\gamma)$. See Jessen and Mikosch (2006), Denisov and Zwart (2007), and Resnick (2007) for details on Breiman’s lemma and for some of its generalisations. So we need to consider only the cases where $F$ is in the max-domain of attraction of the Gumbel or of the Weibull distribution. In these cases it turns out that the tail asymptotic behaviour of each $S_i$ is crucial. Our working assumption for the scaling random variables is that
\[
\lim_{u \to \infty} \frac{G_i(1-x/u)}{G_i(1-1/u)} = x^{\alpha_i}, \quad \forall x > 0 \quad (3.2)
\]
for some $\alpha_i \in [0, \infty)$. So if $\alpha_i > 0$, then $G_i \in \text{MDA}(\Psi_{\alpha_i})$. If $G_i$ possesses a positive density function $g_i$, and if $\alpha_i \in (0, \infty)$, then we will assume the von Mises condition
\[
\lim_{u \to \infty} \frac{g_i(1-x/u)}{g_i(1-1/u)} = x^{\alpha_i-1}, \quad \forall x > 0. \quad (3.3)
\]
The following is our first result, in which we denote by \( \Gamma(\cdot) \) the Euler gamma function.

**Theorem 3.1.** For \( i \leq n \) let \( S_i \) (with distribution function \( G_i \)) be mutually independent scaling random variables, and independent of \( R \) (with distribution function \( F \)). Assume that \( F(0^-) = 0 \) and \( r_F \in (0, \infty) \), and also that \( (3.2) \) holds for every \( i \leq n \) with \( \alpha_i \in [0, \infty) \).

a) If \( F \in \text{MDA}(\Lambda, w) \) with \( r_F \in (0, \infty] \), then

\[
P\{X_n > u\} \sim \prod_{i=1}^{n} \left[ \Gamma(\alpha_i + 1) \frac{1}{\Gamma(\gamma_i)} \right] F(u), \quad u \uparrow r_F. \tag{3.4}
\]

a1) If, in addition, \( r_F = \infty, \lim_{u \to \infty} w(u) = 0 \), and \( (2.10) \) holds, then \( H_n \in S_+ \).

b) If \( F \in \text{MDA}(\Psi, \gamma) \) with \( \gamma \in [0, \infty) \) (hence \( r_F = 1 \) by our convention), then

\[
P\{X_n > u\} \sim \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \sum_{i=1}^{n} \alpha_i + 1)} \prod_{i=1}^{n} \left[ \Gamma(\alpha_i + 1) \frac{1}{\Gamma(\gamma_i)} \right] F(u), \quad u \uparrow 1. \tag{3.5}
\]

In the light of Theorem 3.1 if \( F \) is in the Gumbel or the Weibull max-domain of attraction and each survival function \( G_i(1-u) \) is regularly varying at 0, then \( X_n \) has a distribution function in the Gumbel or the Weibull max-domain of attraction, respectively. See also the proof of Theorem 3.1 below. Part a1) of Theorem 3.1 shows that the subexponentiality is preserved under random scaling. Recent results on the subexponentiality of products are obtained in Tang (2006, 2008), Liu and Tang (2010).

The random contraction \( X_n \) possesses a density function \( h_n \) if one of the scaling random variables \( S_i \) has a density function \( g_i \), or if \( R \) has a density function \( f \). In the following theorem we derive asymptotic approximations of the density function \( h_n \). Part a) is a density version of Breiman’s lemma. The assumption under a1) below places conditions on one of the density functions \( g_i \) but not on \( F \) (beyond \( (2.2) \)). In particular, \( g_i \) can behave like a beta density near the origin, but it is bounded near unity. It seems that relaxing this condition requires restricting the slowly varying factor of \( F \). One possibility is to assume that this factor is normalized slowly varying, i.e., that the density function \( f \) exists and is regularly varying. We do this in a2), and then we need no conditions on \( G_i \). In parts b) and c) we assume for some \( i \) that \( g_i \) satisfies \( (3.3) \) with an additional technical condition for b).

**Theorem 3.2.** Let \( S_i, i \leq n, n \geq 1, \) and \( R \) (with distribution function \( F \)) be as in Theorem 3.1.

a) If \( F \) satisfies \( (2.2) \) with \( \gamma \in [0, \infty) \), then

\[
\lim_{u \to \infty} \frac{uh_n(u)}{P\{X_n > u\}} = \gamma \tag{3.6}
\]
holds under either of the following conditions:

a1) For some \( i \leq n \), the scaling factor \( S_i \) has a density function such that \( yg_i(y) \) is bounded in \((0, 1)\), or

a2) \( F \) has a density function \( f \) which is regularly varying at infinity with index \(- (\gamma + 1)\) for some \( \gamma \geq 0 \), and \( \int_0^1 y^{-\epsilon} dG_i(y) < \infty \) for some \( \epsilon > 0 \) and all \( i \leq n \) provided that \( \gamma = 0 \).

In addition, \( \lim_{u \to \infty} h_n(u)/f(u) \) exists and equals the limit in (3.1).

b) Assume that (3.2) holds for all \( i \leq n \) with \( \alpha_i \in (0, \infty) \), and \( F \in \text{MDA}(\Lambda, w) \) with \( r_F \in (0, \infty] \).

Suppose too for some \( i \leq n \) that \( S_i \) has a density function \( g_i \) satisfying (3.3) and that there exist \( c \in (0, 1) \) and \( p_i > 0 \) such that

\[
\sup_{0 < y \leq c} y^{p_i} g_i(y) < \infty.
\] (3.7)

Then

\[
\lim_{u \uparrow r_F} \frac{h_n(u)}{w(u) P\{X_n > u\}} = 1.
\] (3.8)

c) Suppose that the distribution function \( F \) satisfies (2.6) with \( \gamma \in [0, \infty) \) (hence, \( r_F = 1 \) by our convention) and that (3.2) holds for all \( i \leq n \) with \( \alpha_i \geq 0 \). If in addition (3.3) holds for some \( i \leq n \) with \( \alpha_i > 0 \), then

\[
\lim_{u \downarrow 0} \frac{uh_n(1 - u)}{P\{X_n > 1 - u\}} = \gamma + \sum_{i=1}^n \alpha_i.
\] (3.9)

Under the assumptions of part b) of Theorem 3.2 we obtain further

\[
\lim_{u \uparrow r_F} \frac{h_n(u + x/w(u))}{h_n(u)} = \exp(-x), \quad \forall x \in \mathbb{R}.
\] (3.10)

If \( F \in \text{MDA}(\Lambda, w) \) and it has a density function \( f \), then conditions exist under which

\[
\lim_{u \uparrow r_F} \frac{f(u + x/w(u))}{f(u)} = \exp(-x), \quad \forall x \in \mathbb{R};
\] (3.11)

see Resnick (2008). If (3.11) holds, then we can derive (3.10) under milder conditions than in Theorem 3.2(b). In essence, we assume below that all the \( g_i \) exist and satisfy a condition similar to intermediate regular variation.

**Theorem 3.3.** Let \( S_i, i \leq n, n \geq 1 \), be mutually independent scaling random variables, with positive density functions \( g_i \), and independent of the random variable \( R \geq 0 \) which has a distribution function \( F \) with an upper endpoint \( r_F \in (0, \infty] \). Suppose further that \( F \) possesses a positive density function
such that (3.11) holds. If for \( i \leq n, \ s \in (u, r_F) \), and any measurable function \( a : \mathbb{R}^2 \to [0, \infty) \) such that \( \lim_{u \uparrow r_F} a(u, s) = 1 \) we have
\[
\lim_{u \uparrow r_F} \frac{g_i((u/s)a(u, s))}{g_i(u/s)} = 1,
\]
then (3.10) is satisfied.

Note in passing that if \( r_F \) is finite and \( g_i(1 - u) \) is regularly varying at 0 with index \( \alpha_i - 1 \in (0, \infty) \), then (3.12) is satisfied.

We present next three illustrating examples.

**Example 1.** Let \( R \) be a random variable with distribution function \( F \in \text{MDA}(\Lambda, w) \) and upper endpoint \( r_F \in (0, \infty] \), and let \( S \) be a random variable with beta distribution with positive parameters \( \alpha, \beta \). Since
\[
P\{S > 1 - u\} \sim \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + 1)\Gamma(\beta)} u^\alpha, \quad u \downarrow 0,
\]
it follows from Theorem 3.1 that the distribution function \( H \) of \( RS \) satisfies
\[
\overline{H}(u) \sim \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} (uw(u))^{-\alpha} F(u), \quad u \uparrow r_F.
\]
If \( g \) is the positive density function of \( S \), then condition (3.7) holds, whence (3.8) implies that the density function \( h \) of \( H \) satisfies
\[
h(u) \sim w(u)\overline{H}(u), \quad u \uparrow r_F.
\]
Note in passing that condition (3.12) can be easily checked.

**Example 2.** Under the setup of the previous example suppose that
\[
P\{R > u\} \sim Ku^q \exp(-ru^\gamma), \quad K > 0, r > 0, \gamma > 0, q \in \mathbb{R}
\]
holds for \( u \to \infty \). It follows easily that \( F \in \text{MDA}(\Lambda, w) \) with \( w(u) = r_\gamma u^{\gamma - 1}, u > 0 \). Hence relation (3.13) implies
\[
\overline{H}(u) \sim Kr^\gamma u^{\alpha - 1} \exp(-ru^\gamma), \quad u \to \infty.
\]
Further, by (3.8), the density function \( h \) of \( RS \) satisfies
\[
h(u) \sim r_\gamma u^{\gamma - 1} P\{RS > u\}, \quad u \to \infty.
\]
Note in passing that condition (2.8) does not hold for any $\lambda \in (0, \infty)$. However, if $\gamma \in (0, 1)$, then $R$ and $RS$ have subexponential distributions.

**Example 3.** Let $R$ be a positive random variable with distribution function $F$ satisfying

$$F(u) \sim c_1 \exp(-c_2/(1-u)), \quad u \uparrow 1,$$

with $c_1, c_2$ two positive constants. Since for $w(u) = c_2/(1-u)^2$, $u \in (0, 1)$, and any $s \in \mathbb{R}$,

$$\frac{F(u + s/w(u))}{F(u)} \sim \exp(-c_2[1/(1-u+s/w(u)) - 1/(1-u)]) \to \exp(-s), \quad u \uparrow 1,$$

we have $F \in MDA(\Lambda, w)$. Let $S \in (0, 1)$ be a random variable independent of $R$ such that (3.3) holds. Applying Theorem 3.1 we obtain

$$H(u) \sim \Gamma(\alpha + 1)F(u)G\left(1 - \frac{(1-u)^2}{c_2u}\right), \quad u \uparrow 1.$$

### 4 Applications

#### 4.1 Ruin in the Presence of Risky Investments

Consider the following discrete-time insurance risk model. Within period $i$, the insurer’s net profit (total premium income less claim payment) is denoted by a real-valued random variable $Z_i$. The insurer positions him/herself in a discrete-time financial market consisting of a risk-free bond with a constant periodic interest rate $\delta_i > 0$ and a risky stock with a periodic stochastic return rate $\Delta_i$ taking values in $(-1, \infty)$. Suppose that, in the beginning of each period $i$, the insurer invests a fraction $\pi_i \in [0, 1)$ of his current wealth in the stock and keeps the remaining wealth in the bond. Denote by $U_i$ the insurer’s wealth at time $i$, with a deterministic initial value $U_0 = u \geq 0$. Then, $U_i$ evolves according to

$$U_i = [(1 - \pi_i)(1 + \delta_i) + \pi_i(1 + \Delta_i)]U_{i-1} + Z_i, \quad i = 1, 2, \ldots.$$

As usual, define the probability of ruin by time $n$ as

$$\psi(u; n) = P\left\{\min_{0 \leq i \leq n} U_i < 0 \mid U_0 = u\right\}, \quad n = 1, 2, \ldots.$$

Assume that $Z_1, Z_2, \ldots$ are independent and identically distributed random variables, that $\Delta_1, \Delta_2, \ldots$ are independent random variables, and that the two sequences $\{Z_1, Z_2, \ldots\}$ and $\{\Delta_1, \Delta_2, \ldots\}$
are mutually independent. Introduce
\[
\Upsilon_i = \frac{1}{1 + \Delta_i}, \quad R_i = -Z_i, \quad S_i = \frac{1}{(1 - \pi_i)(1 + \delta_i) + \pi_i (1 + \Delta_i)}, \quad i = 1, 2, \ldots \quad (4.1)
\]
The random variable \(\Upsilon_i\) is the random discount factor during period \(i\) of the risky asset and it takes values in \((0, \infty)\), the random variable \(R_i\) is the net loss during period \(i\), and the random variable \(S_i\) is the overall random discount factor during period \(i\) of the investment portfolio. Denote by \(F\) the common distribution of \(\{R_i, i = 1, 2, \ldots\}\).

According to Tang and Vernic (2010), if \(F \in \mathcal{S}\), then
\[
\psi(u; n) \sim \sum_{k=1}^{n} P\left\{ R_k \prod_{i=1}^{k} S_i > u \right\}, \quad u \to \infty. \quad (4.2)
\]
See related discussions in Tang and Tsitsiashvili (2003, 2004). The applicability of this formula requires explicit asymptotic expressions for the tail probabilities in \((4.2)\) and our main results clearly are crucial for this purpose.

Notice from \((4.1)\) that if \(P\{\Upsilon_i > u\}\) is regularly varying at infinity with index \(-\alpha_i\) for some \(\alpha_i \in [0, \infty)\), then \(P\{S_i > \hat{s}_i - 1/u\}\) is regularly varying at infinity with index \(-\alpha_i\), where \(\hat{s}_i = (1 - \pi_i)^{-1}(1 + \delta_i)^{-1} \in (0, \infty)\). Actually,
\[
P\left\{ S_i > \hat{s}_i - 1/u \right\} \sim P\left\{ \Upsilon_i > \pi_i \hat{s}_i^2 u \right\}, \quad u \to \infty. \quad (4.3)
\]
Hence, if \(F \in \mathcal{S} \cap \text{MDA}(\Lambda, w)\) and for each \(1 \leq i \leq n\), the tail probability \(P\{\Upsilon_i > u\}\) is regularly varying at infinity with index \(-\alpha_i\) for some \(\alpha_i \in [0, \infty)\), then applying Theorem 3.1(a) to relation \((4.2)\) we obtain, with \(u_k = u \prod_{i=1}^{k} 1/\hat{s}_i\),
\[
P\left\{ R_k \prod_{i=1}^{k} S_i > u \right\} = P\left\{ R_k \prod_{i=1}^{k} \frac{S_i}{\hat{s}_i} > u_k \right\}
\sim F(u_k) \prod_{i=1}^{k} \left[ \Gamma (\alpha_i + 1) P\left\{ \frac{S_i}{\hat{s}_i} > 1 - \frac{1}{u_kw(u_k)} \right\} \right]
\sim F(u_k) \prod_{i=1}^{k} \left[ \Gamma (\alpha_i + 1) \left( \frac{1}{\pi_i \hat{s}_i} \right)^{\alpha_i} P\{\Upsilon_i > u_kw(u_k)\} \right], \quad u \to \infty,
\]
where the last step is due to \((4.3)\). We summarize all this as follows.

**Theorem 4.1.** Consider the discrete-time risk model introduced above. If \(F \in \mathcal{S} \cap \text{MDA}(\Lambda, w)\) and for each \(1 \leq i \leq n\), the tail probability \(P\{\Upsilon_i > u\}\) is regularly varying at infinity with index \(-\alpha_i\) for
some $\alpha_i \in [0, \infty)$, then

$$
\psi(u; n) \sim \sum_{k=1}^{n} F(u_k) \prod_{i=1}^{k} \left[ \frac{\Gamma(\alpha_i + 1)}{(\pi_i \hat{s}_i)^{\alpha_i}} P\{ Y_i > u_k w(u_k) \} \right], \quad u \to \infty.
$$

Note in passing that if $F \in \text{MDA}(\Lambda, w)$ with $w$ such that $\lim_{u \to \infty} w(u) = 0$, then in order to show that $F \in \mathcal{S}$ we can utilise (2.10).

### 4.2 Asymptotics of Conditional Tail Expectation

Let $S_i, i \leq n, R$ (with distribution function $F$), and $X_n$ be as in Theorem 3.1 If $F \in \text{MDA}(\Lambda, w)$, then $w$ satisfies the self-neglecting property (see e.g., Reiss (1989) or Resnick (2008))

$$
\lim_{u \uparrow r_F} \frac{w(u + z/w(u))}{w(u)} = 1 \quad (4.4)
$$

uniformly with respect to $z$ in every compact set of $\mathbb{R}$. So if the conditions of Theorem 3.1(a) are satisfied, it follows from (3.2) and (3.5) that $H_n \in \text{MDA}(\Lambda, w)$. Consequently, we obtain the following asymptotic formula (recall (2.5))

$$
\lim_{u \uparrow r_F} \frac{\mathbb{E}\{X_n - u | X_n > u\}}{\mathbb{E}\{R - u | R > u\}} = 1. \quad (4.5)
$$

In several insurance and finance applications the mean excess function is a crucial quantity (see Embrechts et al. (1997), p. 294). The result in (4.5) shows that under the assumed conditions, the mean excess function is asymptotically invariant under random contractions.

The Conditional Tail Expectation (CTE) for $R$ with continuous distribution function is defined e.g., by

$$
\text{CTE}_R(u) := \mathbb{E}\{R | R > u\} = \mathbb{E}\{R - u | R > u\} + u, \quad u > 0.
$$

Then in view of (4.5),

$$
\lim_{u \uparrow r_F} \frac{\text{CTE}_R(u)}{u} = \lim_{u \uparrow r_F} \frac{\text{CTE}_X(u)}{u} = 1 + \lim_{u \uparrow r_F} \frac{1}{uw(u)} = 1.
$$

Consequently, if $\text{VaR}_p(X_n)$ denotes the Value at Risk (VaR) corresponding to the level $p \in (0, 1)$, i.e.,

$$
\text{VaR}_p(X_n) := \inf \{ x : P\{X_n \leq x\} \geq p \},
$$

then

$$
\text{CTE}_{X_n}(\text{VaR}_p(X_n)) \sim \text{VaR}_p(X_n), \quad p \uparrow 1.
$$
It is well-known that for continuous risks CTE is more conservative than VaR. The above asymptotics shows that in the Gumbel case CTE and VaR are asymptotically the same and that this relation is preserved under random scaling.

### 4.3 Linear Combinations of Random Contractions

In order to motivate the next applications we consider a bivariate scale mixture random vector \((U_1, U_2)\) with stochastic representation

\[
(U_1, U_2) \overset{d}{=} R(I_1 S, I_2 \sqrt{1 - S^2}), \tag{4.6}
\]

where \(R\), with distribution function \(F\), is almost surely positive, \(I_1, I_2\) assume values in \((-1, 1)\), and \(S\), with distribution function \(G\), is a scaling random variable taking values in \((0, 1)\). Furthermore, suppose that \(I_1, I_2, R, S\) are mutually independent. If \(S^2\) follows a beta distribution with parameters \(1/2, 1/2\) and \(P(I_1 = 1) = P(I_2 = 1) = 1/2\), then \((U_1, U_2)\) is a spherically distributed random vector, see Cambanis et al. (1981). (We shorten this by saying that \((U_1, U_2)\) is spherical). Hence by Lemma 6.1 of Berman (1983)

\[
c_1 U_1 + c_2 U_2 \overset{d}{=} \sqrt{c_1^2 + c_2^2} U_1, \quad \forall c_1, c_2 \in \mathbb{R}. \tag{4.7}
\]

If \(S\) follows a beta distribution with parameters \(a, b\), then \((U_1, U_2)\) is a generalised Dirichlet random vector (see Hashorva et al. (2007)), and \((4.7)\) does not hold in general.

Next, we derive the tail asymptotics of the aggregated risk

\[
U(\rho) = \rho U_1 + \sqrt{1 - \rho^2} U_2, \quad \rho \in (0, 1)
\]

for some general scaling random variable \(S\). If \(I_1 = I_2 = 1\), then \(U(\rho)\) is maximized with respect to \(S\) at \(S = \rho\). Hence we make the following assumption about the local form of \(G\) at \(\rho\):

\[
P\{|S - \rho| \leq t\} = L_\rho(t)^{\alpha_\rho}, \quad \alpha_\rho \in [0, \infty), \tag{4.8}
\]

for all \(t \in (0, \varepsilon), \varepsilon > 0\), where \(L_\rho\) is positive and slowly varying at 0, and \(L_\rho(0+) = 0\) if \(\alpha_\rho = 0\). Clearly, if \(G\) possesses a density function \(g\) continuous at \(\rho\), then \((4.8)\) holds for any \(\rho \in (0, 1)\) with \(\alpha_\rho = 1\) and \(L_\rho(t) = (2 + o(1))g(\rho)\) as \(t \downarrow 0\).
Lemma 4.2. Let \( I_1, I_2 \) be two random variables taking values \(-1, 1\) with \( q = P\{I_1 = 1, I_2 = 1\} \in (0, 1) \) and independent of a scaling random variable \( S \) with distribution function \( G \). For given \( \rho \in (0, 1) \) define a new random variable

\[
S(\rho) := \rho I_1 S + \sqrt{1 - \rho^2} I_2 \sqrt{1 - S^2}.
\]

If \( G \) satisfies \( (4.8) \) with \( L_\rho(0+) = 0 \) when \( \alpha_\rho = 0 \), then

\[
P\{S(\rho) > 1 - u\} \sim qL_\rho(\sqrt{u})(2u(1 - \rho^2))^{\alpha_\rho/2}, \quad u \downarrow 0. \quad (4.9)
\]

Note that if \( G \) is absolutely continuous with a positive density function \( g \) continuous at \( \rho \), then we have

\[
P\{S(\rho) > 1 - u\} \sim qg(\rho)\sqrt{8(1 - \rho^2)u}, \quad u \downarrow 0. \quad (4.10)
\]

In view of Theorem 3.1 and Lemma 4.2, we now study the tail behavior of \( U(\rho) \). First consider the Gumbel case i.e., \( F \in \text{MDA}(\Lambda, w) \) with \( r_F \in (0, \infty) \). Then with \( \eta(u) = uw(u) \) for \( u > 0 \) we have

\[
P\{U(\rho) > u\} \sim q\Gamma(\alpha_\rho/2 + 1)L_\rho(\eta(u))^{-1/2}\left(\frac{2(1 - \rho^2)}{\eta(u)}\right)^{\alpha_\rho/2} F(u), \quad u \uparrow r_F. \quad (4.11)
\]

Since \( \lim_{u \uparrow r_F} \eta(u) = \infty \) it follows that \( \lim_{u \uparrow r_F} P\{U(\rho) > u\}/F(u) = 0 \). If \( G \) possesses a density function \( g \) continuous at \( \rho \), then

\[
P\{U(\rho) > u\} \sim q\Gamma(1/2)g(\rho)\left(\frac{2(1 - \rho^2)}{\eta(u)}\right)^{1/2} F(u), \quad u \uparrow r_F. \quad (4.12)
\]

Now if \( (U_1, U_2) \) is spherical, then \( (1.7) \) (obviously!) implies the tail equivalence

\[
P\{U(\rho) > u\} \sim P\{U_1 > u\}, \quad u \uparrow r_F. \quad (4.13)
\]

The corresponding density function \( h_\rho \) of \( U(\rho) \) satisfies

\[
h_\rho(u) \sim h_1(u), \quad u \uparrow r_F. \quad (4.14)
\]

The following converse result characterises spherical random vectors.

Theorem 4.3. Suppose that \( I_1, I_2 \) are independent random variables assuming values \(-1, 1\) with probability \( 1/2 \), the distribution function \( F \) with \( F(0-) = 0 \) satisfies \( (2.1) \), and \( G \) possesses a continuous density function \( g \). Then \( (4.13) \) holds for all \( \rho \in (0, 1) \) if and only if \( (U_1, U_2) \) is spherical. Similarly, \( (4.14) \) holds for all \( \rho \in (0, 1) \) if and only if \( (U_1, U_2) \) is spherical.
Next consider the Weibull case. Assume that $F \in \text{MDA}(\Psi, \gamma)$ with $\gamma \in (0, \infty)$ (and $r_F = 1$). Applying Theorem 3.1 and Lemma 4.2 we obtain
\[
\mathbb{P}\{U(\rho) > 1-u\} \sim q \frac{\Gamma(\alpha/2 + 1/2) \Gamma(\gamma + 1)}{\Gamma(\alpha/2 + \gamma + 1)} L_\rho(u^{1/2})(2u(1-\rho^2))^{\alpha/2} F(1-u), \quad u \downarrow 0.
\]
If $G$ possesses a continuous density function $g$, then this simplifies to
\[
\mathbb{P}\{U(\rho) > 1-u\} \sim q \frac{\Gamma(1/2) \Gamma(\gamma + 1)}{\Gamma(3/2 + \gamma)} g(\rho)(2u(1-\rho^2))^{1/2} F(1-u), \quad u \downarrow 0.
\]

The following result gives the Weibull analogue of Theorem 4.3.

**Theorem 4.4.** Suppose that $I_1, I_2, g, G$ are as in Theorem 4.3. If $F \in \text{MDA}(\Psi, \gamma)$ with $\gamma \in (0, \infty)$ (and $r_F = 1$), then (4.13) holds for all $\rho \in (0, 1)$ if and only if $(U_1, U_2)$ is spherical.

We remark that if $F \in \text{MDA}(\Phi, \gamma)$ with $\gamma \in (0, \infty)$, then the tail behaviour of $U(\rho)$ follows from Breiman’s lemma. Indeed, under this assumption $U(\rho)$ has a distribution function in MDA(\Phi, \gamma).

### 4.4 Max-Domain of Attraction of Bivariate Samples

Suppose that $Q$ is the distribution function of a bivariate random vector $(X, Y)$. Extending (2.1), we say that $Q$ belongs to the max-domain of attraction of a bivariate max-stable distribution function $N$ if
\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| \frac{Q^n(a_n x + b_n, c_n y + d_n) - N(x, y)}{Q^n(a_n x, c_n y) - N(x, y)} \right| = 0 \quad (4.15)
\]
holds for some constants $a_n > 0, c_n > 0, b_n, d_n \in \mathbb{R}, n \geq 1$. This implies that each univariate marginal distribution of $Q$ is in the max-domain of attraction of the corresponding univariate marginal of $N$. Conversely, if further
\[
\lim_{n \to \infty} n \mathbb{P}\{X > b_n, Y > d_n\} = 0,
\]
then (4.15) holds with $N$ a product distribution function with univariate extreme value marginal distributions.

In this last application we discuss the max-domain of attraction for the distribution function of $(U_1, U(\rho))$, defined in the previous subsection, assuming that $F$ is in the max-domain of attraction of some univariate distribution. For any $\rho \in (0, 1)$ denote by $Q_\rho$ the distribution function of the bivariate random vector $(U_1, U(\rho))$, and let $Q_{i, \rho}, i = 1, 2$, denote the corresponding marginal distribution.
functions. We focus here on the cases where $F$ is in either the Gumbel or the Weibull max-domain of attraction. Assume that the distribution function $G$ of $S$ satisfies (4.8) with $\alpha_\rho \in [0, \infty)$, and $L_\rho$ positive, slowly varying at 0, and $L_\rho(0+) = 0$ if $\alpha_\rho = 0$.

First consider the Gumbel case. Assume that $F \in MDA(\Lambda, w)$. It follows from (4.11) that $Q_{i,\rho} \in MDA(\Lambda, w)$ for $i = 1, 2$. We show that, if $\rho \in [0, 1)$, then $Q_\rho$ is in the max-domain of attraction of a bivariate distribution function which is a product distribution. If $r_F$ is finite, then this follows from the evident fact that $U_1$ and $U(\rho)$ cannot simultaneously take their maximum values. If $r_F = \infty$, then set $b_i(n) := Q_{i,\rho}^{-1}(1 - 1/n), n > 1$ with $Q_{i,\rho}^{-1}$ the generalized inverse of the $i$-th marginal distribution of $Q_\rho$. It follows from Lemma 4.2 of Hashorva and Pakes (2010) that

$$
\lim_{n \to \infty} (b_2(n) - \rho b_1(n)) = \infty.
$$

(4.16)

Our claim follows if

$$
\lim_{n \to \infty} nP\{U_1 > b_1(n), U(\rho) > b_2(n)\} = \lim_{n \to \infty} P\{U(\rho) > b_2(n) | U_1 > b_1(n)\} = 0.
$$

(4.17)

In view of Theorem 2 of Hashorva (2009)

$$
\lim_{n \to \infty} P\left\{U(\rho) > \rho b_1(n)[1 + x/\sqrt{b_1(n)w(b_1(n))}] | U_1 > b_1(n) \right\} \in (0, \infty), \quad \forall x \in \mathbb{R}
$$

implying (4.17), and hence our claim. Note that condition (4.16) is satisfied for a distribution function $F$ with tail asymptotics given by (3.14).

Next consider the Weibull case. Assume that $F \in MDA(\Psi, \gamma)$ with $\gamma \in (0, \infty)$ and $r_F = 1$. In view of (4.11) it follows that $Q_{1,\rho} \in MDA(\Psi^\gamma, \alpha_{1/2})$ and $Q_{2,\rho} \in MDA(\Psi^{\gamma + \alpha_{1/2}})$. Since $r_F = 1$ it follows that $Q_\rho$ is in the max-domain of a bivariate distribution function which is a product distribution. By Lemma 2.1 this outcome can be formally generalized to the case where $r_F$ is an arbitrary positive constant.

5 Proofs

Proof of Lemma 2.1 In Lemma 5.2 of Hashorva and Pakes (2010) the proof of Lemma 2.1 is shown for $F \in MDA(\Lambda, w)$ and $c = 1$. The case $c \in (0, \infty)$ and $F$ is in the Fréchet or Weibull max-domain of attraction can be easily shown, therefore omitted here.
Proof of Lemma 2.2 First of all, if relation (2.10) holds for some $\lambda > 0$, then it holds for all $\lambda > 0$. This can be verified as follows. For all $0 < \lambda_1 < \lambda_2 < \infty$, by $F \in \text{MDA}(\Lambda, w)$ we have, as $u \to \infty$,
\[
\int_{u-\lambda_2/w(u)}^{u-\lambda_1/w(u)} F(u-y) dF(y) \leq F(\lambda_1/w(u)) [F(u-\lambda_2/w(u)) - F(u-\lambda_1/w(u))] = o \left( (e^{\lambda_2} - e^{\lambda_1}) F(u) \right),
\]
and similarly, $\int_{\lambda_1/w(u)}^{\lambda_2/w(u)} F(u-y) dF(y) = o \left( F(u) \right)$.

Next, we assume $F \in \mathcal{L} \cap \mathcal{S}$. Since $\lim_{u \to \infty} w(u) = 0$, it holds for every $\lambda > 0$, every $T > 0$, and all large $u > 0$ that
\[
\int_{\lambda/w(u)}^{u-\lambda/w(u)} F(u-y) dF(y) \leq F^{\ast 2}(u) - \int_{-\infty}^{T} F(u-y) dF(y) - \int_{u+T}^{\infty} F(u-y) dF(y).
\]
By $F \in \mathcal{L} \cap \mathcal{S}$, it holds that $F^{\ast 2}(u) \sim 2F(u)$, that
\[
\int_{-\infty}^{T} F(u-y) dF(y) \sim F(u)F(T), \quad (5.1)
\]
and that
\[
\overline{F}(-T) \leq \liminf_{u \to \infty} \frac{1}{F(u)} \int_{u+T}^{\infty} F(u-y) dF(y) \leq \limsup_{u \to \infty} \frac{1}{F(u)} \int_{u+T}^{\infty} F(u-y) dF(y) \leq 1. \quad (5.2)
\]
Then, relation (2.10) follows since $T$ can be arbitrarily large.

Finally, we assume that relation (2.10) holds for all $\lambda > 0$ and we prove $F \in \mathcal{S}$. This part of Lemma 2.2 is an easy consequence of Theorem 3.6 of Foss et al. (2009). Actually, for a distribution $F \in \text{MDA}(\Lambda, w)$ with $r_F = \infty$ and $\lim_{u \to \infty} w(u) = 0$, for any $\lambda_u > 0$ with $\lim_{u \to \infty} \lambda_u = 0$ the function $h(u) = \lambda_u/w(u)$ is $F$-insensitive in the sense of Foss et al. (2009), i.e.,
\[
\lim_{u \to \infty} \frac{F(u+h(u))}{F(u)} = 1.
\]
Hence by Theorem 3.6 of Foss et al. (2009), relation (2.10) implies $F \in \mathcal{S}$. Nevertheless, we give here another self-contained proof. Similarly as above, for every $\lambda > 0$ and $T > 0$ we write
\[
F^{\ast 2}(u) = \left( \int_{-\infty}^{T} + \int_{T}^{\lambda/w(u)} + \int_{\lambda/w(u)}^{u-\lambda/w(u)} + \int_{u-\lambda/w(u)}^{u+T} + \int_{u+T}^{\infty} \right) F(u-y) dF(y).
\]
Estimates of the first and last terms above have been given in (5.1) and (5.2), and an estimate of the third term is given by (2.10). For the second and the fourth terms, we have, as $u \to \infty$,
\[
\frac{1}{F(u)} \int_{T}^{\lambda/w(u)} F(u-y) dF(y) \leq \frac{F(u-\lambda/w(u))}{F(u)} (F(T) - F(\lambda/w(u))) \to e^\lambda \overline{F}(T),
\]
and similarly, $\int_{\lambda_1/w(u)}^{\lambda_2/w(u)} F(u-y) dF(y) = o \left( F(u) \right)$.
\[
\frac{1}{F(u)} \int_{u-\lambda/w(u)}^{u+T} F(u-y) dF(y) \leq \frac{F(u-\lambda/w(u)) - F(u+T)}{F(u)} \to e^\lambda - 1.
\]

By the arbitrariness of \(\lambda\) and \(T\), we easily conclude that \(F^\ast(u) \sim 2F(u)\).

**Proof of Theorem 3.1**  

a) The claim in (3.4) follows by Lemma A.5 of Tang and Tsitsiashvili (2004) and Theorem 3 of Hashorva (2009). We give here another direct proof. Let \( X = RS \), where the factors are independent and the distribution function \( G \) of \( S \) satisfies (3.2) with the subscript \( i \) omitted, i.e.,

\[
G(1-x) = x^\alpha L(1/x),
\]

with \( \alpha \geq 0 \) and \( L \) slowly varying at infinity. Clearly,

\[
P\{X > u\} = \int_u^{r_F} \overline{G}(u/y) dF(y).
\]

Substitute \( y = u + z/w(u) \) and define the random variable \( W_u \) by

\[
P\{W_u > z\} = \frac{F(u + z/w(u))}{F(u)}, \quad 0 \leq z < r(u),
\]

where \( r(u) = (r_F - u)w(u) < \infty \) if \( r_F < \infty \), and \( r(u) = \infty \) otherwise. Observe that (2.3) can be expressed as \( W_u \stackrel{d}{\to} W, u \uparrow r_F \), where \( W \) is a random variable following an exponential distribution with mean 1 and \( \stackrel{d}{\to} \) means convergence in distribution.

Next, (5.4) can be expressed as

\[
P\{X > u\} = F(u)E\{\overline{G}(1/(1 + W_u/\eta(u)))\}, \quad \eta(u) = uw(u).
\]

The survival function in this expectation is asymptotically proportional to \( W_u^\alpha \overline{G}(1 - 1/\eta(u)) \) almost surely as \( u \uparrow r_F \). We consider two cases.

i) If \( W_u \leq 1 \), then \( W_u/(W_u + \eta(u)) \leq 1/(1 + \eta(u)) \), and hence dominated convergence implies that

\[
\lim_{u \uparrow r_F} \frac{E\{\overline{G}(1 - W_u/(W_u + \eta(u))); W_u \leq 1\}}{\overline{G}(1 - 1/(1 + \eta(u)))} = E\{W^\alpha; W \leq 1\}.
\]

ii) If \( W_u \geq 1 \), then Potter’s bounds for slowly varying functions (Bingham et al. (1987, Theorem 1.5.6 (i))) imply that for chosen constants \( A > 1 \) and \( \delta > 0 \), there exists a number \( r > 0 \) such that, once \( \eta(u) > r \) (see (2.4)),

\[
\frac{L(1 + \eta(u)/W_u)}{L(1 + \eta(u))} \leq AW_u^\delta.
\]

Let \( M_n \) denote the maximum of \( n \) independent copies of \( R \). It follows from (2.1) that, for all positive \( k \), the \( k \)-th order moments of \( |M_n - b_n|/a_n \) converge to the \( k \)-th order moment of the
Gumbel distribution (Pickands (1968)). This can be utilised to show that \( \{W_u^\alpha + \delta; u < r_F\} \) is a uniformly integrable family. So it follows from (5.3) that

\[
\lim_{u \uparrow r_F} \frac{E\{G(1 - W_u/(W_u + \eta(u))); W_u \geq 1\}}{G(1 - 1/(1 + \eta(u)))} = E\{W^\alpha; W \geq 1\}.
\]

Since \( E\{W^\alpha\} = \Gamma(1 + \alpha) \), we conclude that, as \( u \uparrow r_F \),

\[
P\{X > u\} \sim \Gamma(1 + \alpha) \overline{G}\left(1 - \frac{1}{1 + \eta(u)}\right) F(u).
\]

The self-neglecting property of the scaling function (see (4.4)) implies in addition that \( H \in \text{MDA}(\Lambda, w) \). Hence (3.4) can be proved by induction.

a1) The proof follows directly from Lemma 2.2

b) Since \( r_F = 1 \), we express (5.4) as

\[
P\{X > 1 - u\} = \int_{1-u}^1 \overline{G}\left((1 - u)/y\right) dF(y) = \overline{F}(1 - u)E\left\{G\left(\frac{1 - u}{1 - uW_u}\right)\right\},
\]

where

\[
P\{W_u \leq z\} = \frac{\overline{F}(1 - uz)}{\overline{F}(1 - u)}, \quad 0 \leq z \leq 1.
\]

So as \( u \to 0 \), we have \( W_u \overset{d}{\to} W \), where \( P\{W \leq z\} = z^\gamma \) for \( 0 \leq z \leq 1 \). Since

\[
\frac{1 - u}{1 - uW_u} = 1 - \frac{u(1 - W_u)}{1 - uW_u},
\]

it follows from (5.3) that, almost surely,

\[
\overline{G}\left(\frac{1 - u}{1 - uW_u}\right) \sim (1 - W_u)^\alpha \overline{G}(1 - u), \quad u \to 0.
\]

Clearly, for any \( u \in (0, 1) \)

\[
\overline{G}\left(\frac{1 - u}{1 - uW_u}\right) \leq \overline{G}(1 - u),
\]

hence dominated convergence yields that

\[
\lim_{u \downarrow 0} \frac{1}{\overline{G}(1 - u)} E\left\{G\left(\frac{1 - u}{1 - uW_u}\right)\right\} = E\{(1 - W)^\alpha\} = \frac{\Gamma(\alpha + 1)\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)}.
\]

We prove (3.5) by induction as follows. Let

\[
C_n = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma_n + 1)} \prod_{i=1}^n \Gamma(\alpha_i + 1), \quad \gamma_n = \gamma + \sum_{i=1}^n \alpha_i,
\]

and assume that

\[
\overline{H}_{n-1}(1 - u) \sim C_{n-1} \prod_{i=1}^{n-1} \overline{G}_i(1 - u) \overline{F}(1 - u).
\]
Noting that the right-hand side is regularly varying at zero with index $\gamma_{n-1}$, the case $n = 1$ implies that

$$
\overline{H}_n(1-u) \sim C_{n-1} \mathbb{E}\left\{ (1 - W_{(n-1)})^{\alpha_n} \right\} \left[ \prod_{i=1}^{n} \overline{G}_i(1-u) \right] \overline{F}(1-u),
$$

where $P\{W_{(n-1)} \leq z\} = z^{\gamma_{n-1}}$. But

$$
C_{n-1} \mathbb{E}\left\{ (1 - W_{(n-1)})^{\alpha_n} \right\} = C_{n-1} \frac{\Gamma(\alpha_n + 1)\Gamma(\gamma_{n-1} + 1)}{\Gamma(\alpha_n + \gamma_{n-1} + 1)} = C_n,
$$

and the assertion follows. \(\square\)

**Proof of Theorem 3.2** In all cases it suffices to prove the case $n = 1$ because the general result follows from asymptotic estimates obtained from Theorem 3.1 applied to the product with $n - 1$ contraction factors. Thus, omitting subscripts, the density function of $X = RS$ is (see e.g., Lemma 2.1 of Pakes and Navarro (2007))

$$
h(u) = \int_{u}^{r_p} y^{-1} g(u/y) dF(y) = \int_{0}^{1} y^{-1} f(u/y) dG(y), \tag{5.5}
$$

where the first equality applies when $G$ has a density function, and the second equality applies when $F$ has a density function.

a1) Let $W_u$ be a random variable whose distribution function is

$$
P\{W_u \leq z\} = \frac{\overline{F}(u/z)}{\overline{F}(u)}, \quad z \in (0, 1].
$$

Clearly $W_u \overset{d}{\to} W$ as $u \to \infty$, where $P\{W \leq z\} = z^\gamma$ (so it is degenerate at 0 provided that $\gamma = 0$). Substituting $y = u/z$ in the first integral of (5.5) yields

$$
h(u) = \frac{\overline{F}(u)}{u} \mathbb{E}\{W_u g(W_u)\}.
$$

It follows from dominated convergence that the expectation converges to

$$
\lim_{u \to \infty} \mathbb{E}\{W_u g(W_u)\} = \mathbb{E}\{W g(W)\} = \gamma \int_{0}^{1} g(z) z^\gamma dz = \gamma \mathbb{E}\{S^\gamma\},
$$

and hence, from (3.1), that

$$
\lim_{u \to \infty} \frac{uh(u)}{H(u)} = \gamma. \tag{5.6}
$$

a2) The second integral form in (5.5) can be expressed as $h(u) = \mathbb{E}\{S^{-1}f(uS^{-1})\}$. Expressing the regular variation assumption as $f(u) = u^{-\gamma-1}L(u)$, where $L$ is slowly varying at infinity, we have

$$
\frac{h(u)}{f(u)} = \mathbb{E}\{S^\gamma[L(uS^{-1})/L(u)]\}.
$$
Choose \( \epsilon \in (0, \gamma) \) if \( \gamma > 0 \) or choose as in the assumptions if \( \gamma = 0 \). Since \( S \leq 1 \), it follows from Potter’s bounds that for chosen \( A > 1 \) there exists \( u' > 0 \) such that \( L(uS^{-1})/L(u) \leq S^{-\epsilon} \) for all \( u > u' \). Hence dominated convergence yields \( \lim_{u \to \infty} h(u)/f(u) = E\{S^\gamma\} \). The Karamata-Abelian theorem (Bingham et al. (1987, p. 26)) implies that (5.6) still holds.

b) As in the proof of Theorem 3.1 it suffices to consider the case \( n = 1 \), i.e., \( X = RS \), where the density function \( g \) of \( S \) has the form

\[
g(1 - 1/u) = \alpha u^{-\alpha + 1} L(u), \quad u > 1.
\]

Letting \( \eta(u) = uw(u) \), and recalling notation from the proof of Theorem 3.1(a), it follows from (5.5) that the density function of \( X \) satisfies

\[
h(u) \sim u^{-1} g(1 - 1/\eta(u)) \overline{F}(u) E\{(1 + W_u/\eta(u))^{-1} R_u\}, \quad u \uparrow r_F,
\]

where

\[
R_u = \frac{g(1 - W_u/(\eta(u) + W_u))}{g(1 - 1/(\eta(u) + 1))} = W_u^{-\alpha - 1} \left( \frac{1 + \eta(u)}{W_u + \eta(u)} \right)^{\alpha - 1} \frac{L(1 + \eta(u)/W_u)}{L(1 + \eta(u))}.
\]

Note that \((1 + W_u/\eta(u))^{-1} < 1 \) and that the second and third factors of \( R_u \) converge to unity almost surely.

Let \( l \) be a (large) positive constant and \( B_u = \{W_u \leq \eta(u)/l\} \). Note that \( P\{B_u\} \to 1 \) as \( u \uparrow r_F \). If \( \alpha \geq 1 \), then the second factor of \( R_u \) is bounded above by \((1 + 1/\eta(u))^{\alpha - 1} \to 1 \), and if \( 0 < \alpha < 1 \), then on \( B_u \) the second factor equals

\[
\left( \frac{W_u + \eta(u)}{1 + \eta(u)} \right)^{1-\alpha} \leq \left( \frac{(1 + l^{-1})\eta(u)}{1 + \eta(u)} \right)^{1-\alpha} \leq (1 + l^{-1})^{1-\alpha}.
\]

Next, choosing \( \delta \in (0, \max(\alpha, 1)) \) and \( A > 1 \), \( l \) can be made so large that on \( B_u \) and with \( u \) such that \( \eta(u) > l \), the third factor of \( R_u \) is dominated by Potter’s bound \( A \max(W_u^{\delta}, W_u^{-\delta}) \). Dominated convergence hence gives the conclusion

\[
\lim_{u \uparrow r_F} E\{(1 + W_u/\eta(u))^{-1} R_u; W_u \leq \eta(u)/l\} = E\{W^{\alpha - 1}\}.
\]

Assume that \( \eta(u) \geq l \) and consider outcomes on \( \overline{B}_u \), i.e., that \( l \leq \eta(u) < lW_u \). If \( \alpha \geq 1 \), then the second factor of \( R_u \) is bounded above by unity, and if \( 0 < \alpha < 1 \) then this factor is dominated by \( W_u^{1-\alpha} \). To deal with the third factor, observe that condition (5.7) is equivalent to the existence
of \( p > 0 \) such that \( L(1 + z) = O(z^{-p}) \) as \( z \to 0 \). Consequently, \( l \) can be chosen so large that the numerator in the third factor of \( R_u \) is \( O_p[(W_u/\eta(u))^p] \), as \( u \uparrow r_F \). It follows from these estimates that

\[
E\{R_u; B_u\} = O\left( \frac{E\{W_u^{p+\max(\alpha-1,0)}; B_u\}}{(\eta(u))^p L(\eta(u))} \right) \to 0, \quad u \uparrow r_F
\]
since the denominator tends to infinity.

Since \( E\{W^{\alpha-1}\} = \Gamma(\alpha) \), it follows that,

\[
h(u) \sim \Gamma(\alpha)u^{-1}g(1 - 1/\eta(u))\overline{F}(u).
\]

But \( g(1 - y) \sim (\alpha/y)\overline{G}(1 - y) \), so \((3.8)\) (with the subscripts omitted) follows. The general result follows from Theorem 3.1 after replacing \( F \) with the distribution function of \( R \prod_{j \neq i} S_j \).

c) The proof for the case \( n = 1 \) is similar to those above. Observe that if \( u, w \in (0,1) \), then

\[
(1 - uw)/(1 - w) > 1.
\]

Assuming \((5.7)\), then choosing \( \delta \in (0, \alpha) \), we have Potter’s bound

\[
\frac{L((1 - uw)/(u(1 - W_u)))}{L(1/u)} \leq A \max(1, (1 - uw)\delta(1 - W_u)^{-\delta}), \quad u > u',
\]

where \( W_u \) is as in the proof of Theorem 3.1(b). This leads to the asymptotic form

\[
h(1 - u) \sim g(1 - u)\overline{F}(1 - u)E\{(1 - W)^{\alpha-1}\}, \quad u \downarrow 0.
\]

The assertion for \( n = 1 \) then follows with no further assumptions about \( g \).

For the general case we may assume with no loss of generality that \((5.7)\) holds with \( i = n \), and then apply the single factor case with \( H_{n-1} \) replacing \( F \). It follows from Theorem 3.1(b) that

\[
h_n(1 - u) \sim g_n(1 - u)\overline{H}_{n-1}(1 - u)E\{(1 - W_{n-1})^{\alpha-1}\}.
\]

But \( g_n(1 - u) \sim (\alpha_n/u)\overline{G}_n(1 - u) \) and

\[
\alpha_nC_{n-1}E\{(1 - W_{n-1})^{\alpha_n-1}\} = C_{n-1}\frac{\alpha_n\Gamma(\alpha_n)\Gamma(\gamma_{n-1} + 1)}{\Gamma(\gamma_n)} = \gamma_nC_n.
\]

It follows that \( h_n(1 - u) \sim \gamma_n\overline{H}_n(1 - u)/u \), whence the general result \((3.9)\). \( \Box \)

**Proof of Theorem 3.3** It suffices to consider the case \( n = 1 \). We can, with no loss in generality, choose the scaling function to be differentiable and satisfy

\[
\overline{w}(s) := \frac{d}{ds} \frac{1}{w(s)} \to 0, \quad u \uparrow r_F.
\]
Replacing $u$ in (5.5) with $u + x/w(u)$, the substitution $y = s + x/w(s)$ yields

$$h(u + x/w(u)) = \int_0^y g \left( \frac{u + x/w(u)}{s + x/w(s)} \right) \frac{1 + w(s)}{s + x/w(s)} f(s + x/w(s)) \, ds.$$  

Write the argument of $g$ as $(u/s)a(u,s)$, where

$$a(u,s) = \frac{1 + x/(uw(u))}{1 + x/(sw(s))} \to 1, \quad u \uparrow r_F.$$  

Also, the middle factor in the above integrand is asymptotically proportional to $s^{-1}$. Hence (3.10) follows from (3.11) and (3.12); see Lemma 5.1 of Takahashi and Sibuya (1998).

\textbf{Proof of Lemma 4.2} Some algebra shows that $S(\rho) \leq 1$ and it is bounded away from unity unless $I_1 = I_2 = 1$. If this event occurs, then $S(\rho)$ is close to 1 if and only if $S$ is close to $\rho$. So on the event $\{I_1 = I_2 = 1\}$, algebra reveals that, for all small $u > 0$, $S(\rho) > 1 - u$ if and only if

$$(\rho - S)^2 + 2\rho Su < 2u - u^2,$$

and this condition is equivalent to

$$(\rho - S)^2 < (1 + o_p(1))2(1 - \rho^2)u.$$  

But

$$P\{(\rho - S)^2 \leq t\} \sim t^{\alpha_p/2}L_p(\sqrt{t}), \quad u \downarrow 0$$  

and the assertion follows.

\textbf{Proof of Theorem 4.3} The assumptions imply that (4.12) holds for all $\rho \in (0,1)$. The tail equivalence (4.13) implies that the factor $g(\rho)(1 - \rho^2)^{1/2}$ is constant with respect to $\rho \in (0,1)$. The density function of $S^2$ is $g(y^{1/2})/2y^{1/2}$, i.e., $S^2$ has the beta distribution with parameters $(1/2, 1/2)$. If (4.14) holds, then so does (4.13). Hence either condition implies that $(U_1, U_2)$ is spherical.

\textbf{Proof of Theorem 4.4} The proof is similar to that of Theorem 4.3.

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