Galois Theory of Hopf-Galois Extensions

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April 1, 2022

Abstract

We introduce Galois Theory for Hopf-Galois Extensions proving existence of a Galois connection between subalgebras of an $H$-comodule algebra and generalised quotients of the Hopf algebra $H$. Moreover, we show that these quotients $Q$ which define $Q$-Galois extension are closed elements of the Galois connection. We generalise an important results of Hopf-Galois Theory of M. Takeuchi and H.-J. Schneider by showing that there is a bijective correspondence between right ideals coideals and right coideal subalgebras of any finite dimensional Hopf algebra and we reformulate still open problem in the general (i.e. non finite dimensional) case. We apply this result to the theory of cleft extensions, where we characterise closed elements of the Galois connection as exactly these which are $Q$-Galois. Based on our Galois Theory for Hopf-Galois extensions the Chase–Sweedler Theorem follows immediately. We also describe the relation of our results to the work of F. van Oystaeyen, Y. Zhang and P. Schauenburg on biGalois extensions. We show that our approach improves their results.

We present a construction of a Galois connection between the complete lattice of subalgebras of an $H$-comodule algebra $A$ and the complete lattice of generalised quotients of a Hopf algebra $H$, i.e. quotients by coideals right ideals:

**Theorem** (Main Theorem). For an $H$-comodule algebra $A$ over a field $k$ there exists a Galois connection:

$$
\text{Sub}_{\text{alg}}(A/A^{co\, H}) \leftrightarrow \text{Quot}_{\text{gen}}(H)
$$

where $\text{Quot}_{\text{gen}}(H) = \{H/I : I \text{ - coideal right ideal of } H\}$.

Importance of this result is that the Galois connection always restricts to a bijection between so called closed elements, i.e. elements which belong to the image of a Galois connection. Thus on the left hand side, closed elements are

We would like to thank T. Brzeziński for his comments on our manuscript.
the extensions of the form: \( A/A^{coQ} \) for some \( Q \in \text{Quot}_{\text{gen}}(H) \). The idea is to describe closed elements of the right hand side or in other words to describe which quotients of \( H \) classifies the extensions of the form \( A/A^{coQ} \).

The above theorem can be easily extended to the case of \( C \)-comodule algebras where \( C \) is a coalgebra with a group-like element or to the case of corings. Then on the left hand side there is a lattice of submodules which possesses a coaction of a coring and on the right hand side is the lattice of coideals of the coring. In this setting there is also another Galois correspondence which was studied in the literature. In [2] it is proved the Jacobson-Bourbaki type correspondence for simple Artinian rings. It is a 1-1 correspondence between simple Artinian subrings of \( \text{End}_R(\Sigma) \) for an \( R \)-module \( \Sigma \) and an Artinian subring \( B \) of \( \text{End}_R(\Sigma) \) and coideals of the comatrix coring \( \Sigma^* \otimes_B \Sigma \) (definition in [2]):

\[
\text{Sub}_{\text{Artinian}}(\text{End}_R(\Sigma)/B) \cong \text{cold}(\Sigma^* \otimes_B \Sigma)
\]

This result is a generalisation of M. Sweedler result obtained in [18]: the predual version of the Jacobson-Bourbaki Theorem. In our context we would have to assume that \( R \) is a \( k \)-algebra over a field \( k \) and then we can get the following Galois correspondence:

\[
\text{cold}(\Sigma^* \otimes_B \Sigma) \Leftrightarrow \text{Sub}_R(\Sigma)
\]

We provide all the ingredients of the theory, namely: we prove that the inclusion relation on right ideals coideals defines a complete lattice structure on the set of (generalised) quotients of \( H \). We show that the set of subobjects, i.e. sub-Hopf algebras or even subalgebras right coideals\(^1\) of a Hopf algebra is a complete lattice. When the Hopf algebra is finite dimensional then all the lattices are algebraic and dually algebraic (Definition 1.11).

Our Main Theorem is a core of a Galois Theory and connects together interesting areas: classical Galois Theory for field extensions, Hopf-Galois Theory, which can be now formulated as a Galois Theory, and Lattice Theory. We prove that \( Q \), which are \( Q \)-Galois, are closed elements the Galois connection of the Main Theorem. Furthermore, we prove (Theorem 8.3) that in the case of the \( H \)-extension \( k \subseteq H \) and cleft extensions this is a complete characterisation of the closed elements of \( \text{Quot}_{\text{gen}}(H) \), i.e. \( Q \) is closed if and only if \( A/A^{coQ} \) is \( Q \)-Galois (Proposition 7.8 and Theorem 8.3). Closed elements of \( \text{Sub}_{\text{alg}}(A/B) \) are the \( H \)-subextensions, i.e. the extensions of the form \( A/A^{coQ} \) for some \( Q \in \text{Quot}_{\text{gen}}(H) \). Any Galois connection gives bijective correspondence between closed elements. Thus in our case the closed elements of \( \text{Quot}_{\text{gen}}(H) \) classifies \( H \)-subextensions.

This subject was already investigated in [20], [10] and [11] but lack of explicit formulas in terms of an algebra or a Hopf algebra led to difficulties to define the

\(^1\)Provided lattice theoretic results can be easily done in the cases of: subalgebras left coideals or other mixed sub- and quotient structures.
Galois connection for the generality that we are dealing with. To overcome these difficulties we use a new approach. We use Lattice Theory and an existence theorem for Galois connections (Theorem 3.6) which provides an explicit formula for a Galois connections between complete lattices in terms of the lattice structure.

Being based on our results we prove part of classical Galois Theory (injectivity of the map $\text{Gal}(E/F) \ni G \mapsto \text{Fix}(G) \in \text{Sub}(E/F)$, where $E/F$ is a finite Galois extension), the Chase–Sweedler Theorem (and its generalisation due to F. van Oystaeyen and Y. Zhang [20, Theorem 2.3]) and a generalisation of an interesting result in Hopf-Galois theory due to M. Takeuchi (Theorem 7.6).

We formulate finite dimensional Hopf-Galois Theory as an isomorphism of lattices:

**Theorem** (Finite Hopf-Galois Theory). Let $H$ be a finite dimensional Hopf algebra and $A/A^{coH}$ an $H$-Hopf-Galois extension. Then the Galois connection of the Main Theorem specialises to the isomorphism:

$$\text{Sub}_{H-ext}(A/A^{coH}) \simeq \text{Quot}_{\text{gen}}(H)$$

where on the left hand side is the lattice of all $H$-extensions, i.e. the extensions of the form $A/A^{coQ}$ for some $Q \in \text{Quot}_{\text{gen}}(H)$.

Thus the lattice $\text{Quot}_{\text{gen}}(H)$ classifies intermediate $H$-extensions of an $H$-Galois extension $A/A^{coH}$.

In commutative case Hopf-Galois theory has a geometrical meaning. If we consider an affine scheme $X = \text{Spec}(A)$ and an affine group scheme $G = \text{Spec}(H)$ represented by a commutative Hopf algebra $H$, then an action $\mu : X \times G \to X$ corresponds to a co-action $\mu^* : A \to A \otimes H$ making $A$ an $H$-comodule algebra. The action $\mu$ is free if and only if the map

$$X \times G \to X \times X \quad (x, g) \mapsto (x, xg)$$

is injective which is equivalent to

$$A \otimes A \to A \otimes H \quad a \otimes b \to ab_{(0)} \otimes b_{(1)}$$

being surjective. The affine quotient of $X$ by $G$ is defined to be $\text{Spec}(A^{coH})$, i.e. the equaliser:

$$A^{coH} \subseteq A \xrightarrow{\mu^*} A \otimes H$$

This corresponds to the cokernel in the category of affine schemes:

$$X \times G \xrightarrow{\mu} Y \quad p(x, g) = x$$
Provided $A$ is faithfully flat $A^{co\,H}$ module then the affine quotient $\text{Spec}(A^{co\,H})$ and the true quotient $X/G$ agrees.

We enclose a modern view on classical Galois Theory of finite field extension which is based on corings and we relate our results to this point of view. In [8] there is proved that a finite field extension $E/F$ is Galois if and only if the following map is an isomorphism of corings:

$$
can : E \otimes_F E \longrightarrow \text{Map}(\text{Gal}(E/F), E), \quad e_1 \otimes_F e_2 \longmapsto (g \mapsto e_1 g(e_2))
$$

The definition of the coring structure of $\text{Map}(\text{Gal}(E/F), E)$ was given by T. Maszczyk in [8, after Definition 2.4] and we include it in Example 5.3. Among other things we show that there is a Galois monomorphism form the lattice of quotients of the coring $\text{Map}(\text{Gal}(E/F), E)$ to the lattice of submonoids of the Galois group $\text{Gal}(E/F)$.

An interesting result which we obtain from our Main Theorem is the Chase–Sweedler Theorem which lives on the crossroads of Galois Theory for field extensions and Hopf-Galois Theory:

**Theorem (Chase–Sweedler).** Let $E/F$ be a Galois extension of fields which is an $H$-module coalgebra and an $H$-Hopf-Galois extension for a finite dimensional Hopf algebra $H$, i.e. the canonical map:

$$
can : E \otimes_F E \longrightarrow \text{Hom}(H, E), \quad e_1 \otimes_F e_2 \longmapsto (h \mapsto e_1 he_2)
$$

is bijective. Then the following map is injective:

$$
\text{Sub}_{\text{Hopf}}(H) \longrightarrow \text{Sub}_{\text{field}}(F \subseteq E) \quad H' \longmapsto \mathbb{E}^{H'} = \{ e \in E : \forall h \in H he = \epsilon(h)e \}
$$

Also the generalisation of Chase–Sweedler Theorem by F. van Oystaeyen and Y. Zhang [20, Theorem 2.3] to the case of a noncommutative $H$-module algebra $A$ such that $A$ is $H^*$-Hopf-Galois over a finite dimensional Hopf algebra $H$ follows from our theorem for finite Hopf-Galois extensions.

In the light of the Chase-Sweedler theorem the coring $\text{Map}(\text{Gal}(E/F), E)$ can be expressed using many Hopf algebras: take a finite field extension $K \subseteq F$ then there is a Hopf algebra $H$ over $K$ such that the coring $\text{Map}(\text{Gal}(E/F), E)$ is isomorphic to the coring $E \otimes_K H^* = \text{Hom}(H, E)$. The Hopf algebras are thus not canonical (there is the choice of $K$) while the coring is. This canonical property of the coring $\text{Map}(\text{Gal}(E/F), E)$ was first described by T. Maszczyk in [8].

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2This approach is a result of T. Maszczyk presented at the Marie Curie Action Transfer of Knowledge ‘Noncommutative Geometry and Quantum Groups’ lecture ‘Galois Structures’ by T. Maszczyk, G. Janelidze and T. Brzeziński [8].
We apply our Main Theorem to the $H$-extension $k \subseteq H$. In this case it was shown by M. Takeuchi that there is the following bijective correspondence:

\[
\begin{align*}
\{ K \subseteq H : & \text{ right coideal subalgebra } \notag \\
& H \text{ faithfully flat over } K \} & \psi & \Leftrightarrow & \left\{ H/I : & \text{ left ideal coideal } \notag \\
& H \text{ faithfully coflat over } H/I \right\}
\end{align*}
\]

\[
\psi(K) = H/HK^+, \quad \phi(H/I) := \co H/I H
\]

Additionally H.-J. Schneider proved that the above bijection restricts to normal/conormal elements:

\[
\begin{align*}
\{ K \subseteq H : & \text{ normal sub-Hopf algebra } \notag \\
& H \text{ faithfully flat over } K \} & \psi & \Leftrightarrow & \left\{ H/I : & \text{ normal Hopf ideal } \notag \\
& H \text{ faithfully coflat over } H/I \right\}
\end{align*}
\]

In this setting we show the following important result:

**Theorem.** Let $H$ be a Hopf algebra. Then $Q \in \Quot_{gen}(H)$ is closed element of the Galois connection (1) specified to the $H$-extension $H/k$ if and only if $H/H^Q$ is $Q$-Galois.

There is the question of S. Montgomery if the bijective correspondences (2) and (3) still survive if we don’t assume faithfully flat/coflat conditions. We positively answer this question in the case of finite dimensional Hopf algebras (Theorem 7.6). This was recently proved by S. Skryabin in [16, Corollary 6.5]. Our line of proof is different and we show an equivalent form of this statement in the general case:

**Proposition.** For a Hopf algebra $H$ over a field $k$ such that for every its generalised quotient $Q$ the extension $\co Q H \subseteq H$ is $Q$-Galois. Then there is a positive answer to the question of S. Montgomery, i.e. there is the bijective correspondence:

\[
\begin{align*}
\{ K \subseteq H : & \text{ right coideal subalgebra } \notag \\
& H \text{ faithfully flat over } K \} & \xrightarrow{\sim} & \left\{ H/I : & \text{ left ideal coideal } \notag \\
& H \text{ faithfully coflat over } H/I \right\}
\end{align*}
\]

if and only if $\co H/K^+H \subseteq K$ for every right coideal subalgebra $K$ of $H$.

At the end we are coming back to the origins of this subject. F. van Oystaeyen and Y. Zhang in [20] prove a noncommutative generalisation of the Chase–Sweedler theorem and they apply it to the theory of finite extensions of fields. In their paper for the first time appear a remarkable construction of an additional Hopf algebra making a commutative $H$-Hopf-Galois extension a biGalois extension. They construct a Hopf algebra $L(H, A)$ associated to a commutative faithfully flat $H$-Hopf-Galois extension $A/B$ ($H$ is assumed to be commutative). The extension $A/B$ becomes $L(H, A)$-$H$-bicomodule algebra and a biGalois extension. This additional structure Hopf algebra $L(H, A)$ classifies intermediate $H$-comodule subalgebras of $A/B$. Furthermore, when $A/B = \mathbb{E}/\mathbb{F}$ is a field extension then they prove the following Galois theory

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Theorem (F. van Oystaeyen and Y. Zhang [20, Theorem 4.7]). Let $k \subseteq \mathbb{F}$ be a field extension and let $H$ be a commutative and cocommutative $k$-Hopf algebra over $k$. Let $\mathbb{F} \subseteq \mathbb{E}$ be a field extension and an $H$-Hopf-Galois extension. Then there is one-to-one correspondence:

$$
\left\{ \text{Hopf ideals of } \mathbb{F} \otimes_k H \right\} \cong \left\{ \text{H-subcomodule subfields of } \mathbb{E} \right\} \tag{4}
$$

if $I$ and $\mathbb{M}$ corresponds to each other then $\mathbb{E}/\mathbb{M}$ is $\mathbb{F} \otimes_k (H/I)$-Hopf-Galois.

Moreover, there is the following bijection:

$$
\left\{ \text{Hopf subalgebras of } \mathbb{F} \otimes_k H \right\} \cong \left\{ \text{H-subcomodule subfields of } \mathbb{E} \right\} \tag{5}
$$

if $H'$ and $\mathbb{M}$ corresponds to each other then $\mathbb{E}/\mathbb{M}$ is $(\mathbb{F} \otimes_k H)/(\mathbb{F} \otimes_k H)H'^+$-Hopf-Galois

where $H'^+ := \ker \epsilon \cap H'$.

When $H$ is commutative and cocommutative then the Hopf algebra $L(H, A)$ is equal to $\mathbb{F} \otimes_k H$ ([20, Corollary 3.4]). The proof of the previous theorem is based on this fact, so that $L(H, A)$ is the Hopf algebras which plays the main role in their theory. In [10] P. Schauenburg generalised the construction of $L(H, A)$ to noncommutative extensions of rings $k \subseteq A$ over noncommutative (and noncommutative) Hopf algebras. In his work P. Schauenburg proves the following theorem which is a generalisation of the preceding result of F. van Oystaeyen and Y. Zhang.

Theorem (P. Schauenburg, [10, Theorem 6.4]). Let $k \subseteq A$ be a faithfully flat $H$-Hopf-Galois extension of a ring $k$. Then there is the following Galois connection:

$$
\left\{ \text{coideals left ideals of } L(H, A) \right\} \leftrightarrow \left\{ \text{H-subcomodule algebras of } A \right\} \tag{6}
$$

If $B \in \text{Sub}_{\text{alg}}^H(A)$ is an $H$-subcomodule algebra such that $A_B$ is a faithfully projective then it is a closed element of the Galois connection (6). The corresponding closed elements of the left hand side are those coideals left ideals which are $k$ direct summands of $L(H, A)$. Thus the above Galois connection gives an isomorphism of these objects. Furthermore, if $A$ is a skew field then the Galois connection (6) is an isomorphism. Another result of this type is given in [11, Theorem 3.6] where P. Schauenburg shows that the above Galois connections is a bijection on the set of (left, right) admissible objects (Definition 9.2). Where (right, left) admissibility is the (right, left) faithfully flat/coflat condition. In this paper we show a similar statement for the Galois connection (1). We show that the map $Q \mapsto A^{coQ}$ of the Galois connection (1) is injective on the subset of (right,
left) admissible objects provided $A/A^{coH}$ is a faithfully flat $H$-Hopf-Galois extension.

We shall remark, that the Galois correspondences between posets of generalised quotients of $L(H, A)$ is a special case of our Main Theorem.

## 1 Preliminaries

**Definition 1.1.** Partially ordered set, poset for short, is a set $P$ together with an order relation $\preceq$ which is reflexive, transitive and antisymmetric.

**Definition 1.2 (Galois connection).** Let $(P, \preceq)$ and $(Q, \leq)$ be two partially ordered sets. An antimonotonic morphisms of posets $\phi : P \rightarrow Q$ and $\psi : Q \rightarrow P$ establishes a Galois connection if

$$\forall p \in P \quad p \preceq \phi \circ \psi(p) \quad \text{and} \quad \forall q \in Q \quad q \leq \psi \circ \phi(q)$$

We refer to this property as the Galois property. An element of $P$ (or $Q$) will be called closed if it is invariant under $\psi \phi$ (or $\phi \psi$). Sets of closed elements will be denoted by $\overline{P}$ and $\overline{Q}$ respectively. A standard notation for a Galois connection is

$$P \xleftrightarrow{\phi, \psi} Q$$

Another name which appear in the literature for this notion is Galois correspondence.

**Proposition 1.3.** Let there be a Galois connection $P \xleftrightarrow{\phi, \psi} Q$. Then the following holds:

1. $\overline{P} = \psi(Q)$ and $\overline{Q} = \phi(P)$

2. The restrictions $\phi|_{\overline{P}}$ and $\psi|_{\overline{Q}}$ are inverse bijections of $\overline{P}$ and $\overline{Q}$.

3. Map $\phi$ is unique in the sense that there exists only one Galois connection of the form $(\tilde{\phi}, \psi)$ for some map $\tilde{\phi} : P \rightarrow Q$, i.e. $\tilde{\phi} = \phi$. A similar statement holds for $\psi$.

4. The map $\phi$ is mono (onto) if and only if the map $\psi$ is onto (mono).

5. If one of the two maps is an isomorphism then the second is its inverse.

A lattice is a poset in which there exists supremum and infimum of any two elementary subset or equivalently of any finite subset. A lattice can also be defined as an algebraic structure which has two binary operations: join (an abstract
supremum of two elements) denoted by $\lor$ and meet (an abstract infimum of two elements) denoted by $\land$ which satisfies the following equalities:

\[
\begin{align*}
    a \land a &= a & \quad \text{idempotent laws} \\
    a \lor b &= b \lor a & \quad \text{commutative laws} \\
    a \lor (b \lor c) &= (a \lor b) \lor c & \quad \text{associative laws} \\
    a \lor (a \land b) &= a & \quad \text{absorption laws}
\end{align*}
\]

for any element $a$ of a lattice $L$. There is a bijective correspondence between orders which have finite suprema and infima and lattice operations which is given as follows:

\[
\leq \quad \iff (\inf(-,-), \sup(-,-)), \quad (\land, \lor) \quad \iff (x \leq y \iff x \land y = x)
\]

The order given by join in a similar way is the same as this above. This follows from the absorption laws. From this bijective correspondence we get that the above equalities satisfied by meet and join are exactly the equalities which are satisfied by $\sup$ and $\inf$ in any poset with finite suprema and infima. It is some times more convenient to use the poset description of a lattice and some times the algebraic structure of a lattice. We refer to [5] for the Theory of Lattices.

**Definition 1.4.** A lattice $(L, \land, \lor)$ is complete if for every $B \subseteq L$ there exists $\sup B$ and $\inf B$.

**Note 1.5.** There exists arbitrary infima in a lattice $L$ if and only if there are arbitrary suprema.

**Note 1.6.** If a poset is closed under arbitrary infima (or suprema) then it is a complete lattice.

The above observations will be often used and they will simplify proofs. A lower (upper) semilattice is an algebraic structure with one binary operation which satisfies the equalities of meet (join) of a lattice: the idempotent law, the commutative and the associative laws. A sublattice of a lattice or a lower sub-semilattice of a lower semilattice is a subset closed under the given operations. Next we state an important lemma which will be extensively used in this work.

**Lemma 1.7.** Let $(M, \land_M, \lor_M)$ be a complete lattice and let there be two complete lattices which are upper sub-semilattices of $M$: $(K, \land_M, \lor_K)$ and $(L, \land_M, \lor_L)$. Let $K$ and $L$ have common the smallest element. Then the upper semilattice $(K \cap L, \land_M)$ is a complete lattice.

The lattice structure of $K \cap L$ such that $K \cap L$ is a lower (upper) sub-semilattice of $M$ is unique.
Proof. Let us define lattice operations on $K \cap L$ by the formulas:

\[
\begin{align*}
    a \lor_{K \cap L} b &:= a \lor M b \\
    a \land_{K \cap L} b &:= \lor_M \{ c \in K \cap L : c \leq a \text{ and } c \leq b \}
\end{align*}
\]

where $a, b \in K \cap L$. Join is well defined, because of our assumption that $K$ and $L$ are upper subsemilattices of the lattice $M$. Now let us prove that the above meet is well defined as well. Lattices $K$ and $L$ have the same smallest element, so the set $\{ c \in K \cap L : c \leq a \text{ and } c \leq b \}$, for any $a, b \in K \cap L$, is non-empty. The supremum $\lor_M \{ c \in K \cap L : c \leq a \text{ and } c \leq b \}$ exists and belongs to $K \cap L$, because both $K$ and $L$ are complete lattices. The axioms of lattice operations are trivially satisfied.

It remains to show that the lattice $(K \cap L, \land_{K \cap L}, \lor_{K \cap L})$ is complete. Let $B \subseteq K \cap L$. Then

\[
\sup_{K \cap L} B = \sup_M B \quad \text{and} \quad \inf_{K \cap L} B = \sup_M \{ x \in K \cap L : \forall b \in B \ x \leq b \}.
\]

These infimum and supremum exists, because $M$ is complete and it belongs to $K \cap L$ what comes from the property that $K$ and $L$ are upper sub-semilattices of $M$.

\begin{definition}
Let $L$ be a lattice and let $a, b \in L$. Then a subset $\{ x \in L : a \leq x \leq b \}$ of the lattice $L$ is called an interval and it will be denoted by $[b/a]$.
\end{definition}

\begin{definition}
An element $z$ of a lattice $L$ is called compact if for any subset $S \subseteq L$ such that $z \leq \lor S$ there exists a finite subset $S_f$ of $S$ with the property $z \leq \lor S_f$.
\end{definition}

\begin{example}
Let $V$ be a $k$-vector space. Then its subspace $W$ is a compact element of the lattice of subspaces $\text{Sub}_{\text{Vect}}(V)$ if and only if it is finite dimensional.
\end{example}

\begin{definition}
A lattice is algebraic if it is complete and every its element is a supremum of compact elements. A lattice is dually algebraic if its dual, i.e. the one with the dual order, is algebraic.
\end{definition}

\begin{example}
Let $A$ be a $k$-algebra then the lattices of ideals, left (right) ideals and subalgebras are algebraic.
\end{example}

It is a well known theorem of Universal Algebra that lattices of subalgebras and lattices of congruences (quotient structures) of any algebraic structure are algebraic. Thus all the lattices of sub-objects and quotient objects of classical algebraic structures like groups, semi-groups, rings, modules, etc. are algebraic. This is the origin of the name of this property.

\begin{lemma}
Let $L$ be an algebraic lattice such that every its element is compact. Then any its sublattice is algebraic.
\end{lemma}
Proof. First of all one has to observe that a sublattice of a complete lattice in which every element is compact is itself complete. For this, let $K$ be a sublattice of $L$ and let $B$ be a subset of $K$. Because $\sup B$ exists in $L$, we just have to show that it belongs to $K$. There exists a finite subset $B_f$ of $B$ such that $\sup B \geq \sup B_f$, because $\sup B$ is compact. So $\sup B = \sup B_f$. But $\sup B_f = b_1 \vee b_2 \vee \cdots \vee b_n$ where $B_f = \{b_1, b_2, \ldots, b_n\}$ thus by the assumption that $K$ is a sublattice of $L$ $\sup B \in K$. The lemma follows from the observation that if an element is compact in $L$ then it is also compact in a sublattice.

To the following argument we will refer later on:

Remark 1.14. Let $\text{Sub}_{\text{Vect}}(V)$ be the lattice of subspaces of a finite dimensional $k$-vector space $V$. By Example 1.10 every subspace of a finite dimensional vector space is a compact element of $\text{Sub}_{\text{Vect}}(V)$. From Lemma 1.13 we get that any sublattice of $\text{Sub}_{\text{Vect}}(V)$ is algebraic, but it is dually algebraic as well.

$$\text{Sub}_{\text{Vect}}(V) \cong \text{Sub}_{\text{Vect}}(V^*)$$

Thus a sublattice of $\text{Sub}_{\text{Vect}}(V)$ is anti-isomorphic to an algebraic sublattice of $\text{Sub}_{\text{Vect}}(V^*)$.

## 2 Lattices of substructures and quotient structures

**Proposition 2.1.** Let $(C, \Delta, \epsilon)$ be a coalgebra, $(B, m, u, \Delta, \epsilon)$ a bialgebra and $(H, m, u, \Delta, \epsilon, S)$ a Hopf algebra. All over a field $k$. Then subcoalgebras of $C - (\text{Sub}(C), \subseteq)$, subbialgebras of $B - (\text{Sub}_b(B), \subseteq)$ and sub-Hopf algebras of a Hopf algebra $H - (\text{Sub}_{\text{Hopf}}(H), \subseteq)$ are complete lattices which additionally are algebraic and dually algebraic when $C$, $B$, $H$ are finitely dimensional.

Proof. Let $A$ and $B$ be two subcoalgebras or subbialgebras or sub-Hopf algebras. Then $A \cap B$ is again such. By definition, supremum is the smallest element of all the subcoalgebras (bialgebras, Hopf algebras) that contain these two subcoalgebras (bialgebras, Hopf algebras), i.e.

$$\sup(A, B) = \bigcap\{D : A \cup B \subseteq D\}.$$ 

Where $D$ belongs to subobjects of $L$, $B$ or $H$. The infimum exists of any family of objects thus these posets are complete. By Remark 1.14, these lattices are algebraic and dually algebraic whenever $C$, $B$ and $H$ are finitely dimensional.

**Notation 2.2.** Let $C$ be a coalgebra, $B$ a bialgebra and $H$ an Hopf algebra. We use the following notation:

- $\text{cold}(C)$ - the set of coideals of $C$,
- $\text{cold}_l(C)$, $\text{cold}_r(C)$ - the sets of left, respectively right, coideals of $C$,
- $\text{cold}_b(B)$, $\text{cold}_{\text{Hopf}}(H)$ - the sets of coideals of the bialgebra $B$ and the Hopf algebra $H$. 

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Remark 2.3. The sets $\text{cold}(C)$, $\text{cold}_r(C)$, $\text{cold}_l(C)$, $\text{ld}_b(B)$ and $\text{ld}_{Hopf}(H)$ are posets under the inclusion relation.

Let us cite Propositions 1.4.5 and 1.4.6 [17, ]:

**Proposition 2.4.** Let $(C, \Delta, \epsilon)$ be a coalgebra over a field $k$, and let $C^* = \text{Hom}_k(C, k)$ be the dual space. $C^*$ has an algebra structure given by the convolution product:

$$(f \ast g)(a) := f(a(0))g(a(1))$$

for $f, g \in C^*$. The counit $\epsilon$ plays the role of identity. Let $V \subseteq C$. Then $V^\perp$ denotes the set of all linear functionals which are equal 0 on the subspace $V$. The following holds provided $C$ is finitely dimensional:

1. $V$ is a right (left) coideal if and only if $V^\perp$ is a right (left) ideal of the algebra $C^*$,
2. $V$ is a coideal if and only if $V^\perp$ is a subalgebra of $C^*$.

**Proposition 2.5.** Let $(C, \Delta, \epsilon)$ be a finite dimensional coalgebra then the posets

$$(\text{cold}(C), \subseteq), (\text{cold}_r(C), \subseteq), (\text{cold}_l(C), \subseteq)$$

are algebraic and dually algebraic lattices.

**Proof.** By Proposition 2.4, we have the following dual isomorphisms:

$$\text{Sub}_{alg}(C^*) \simeq \text{cold}(C), \quad I_r(C^*) \simeq \text{cold}_r(C), \quad I_l(C^*) \simeq \text{cold}_l(C).$$

where $\text{ld}_r(C^*)$ and $\text{ld}_l(C^*)$ are the algebraic lattices of right respectively left ideals of $C^*$. The lattices on the left hand sides are algebraic and thus those on the right hand side are dually algebraic. The theorem concludes by observing that the lattice $\text{Sub}_{Vect}(C)$ for a finite dimensional vector space $C$ has the property that every its element is compact, thus every its complete sublattice is algebraic. □

One can ‘cogenerate’ a coideal by a subset $Y$ of a coalgebra $C$. This is defined as a join of all the coideals contained in $Y$, i.e. it is the greatest coideal contained in $Y$. We use this notion to define a meet operation in the poset of coideals. This is dual to the case of algebras where the join is defined as ideal generated by the set-theoretic sum.

**Lemma 2.6.** Let $C$ be a coalgebra and $I_1, I_2$ two coideals. Then we have the following formulas for the meet and the join in the lattice of coideals:

$$I_1 \vee I_2 = I_1 + I_2$$

$$I_1 \wedge I_2 = +\{I \in \text{cold}(C) : I \subseteq I_1 \cap I_2\}$$
Proof. A lattice structure on the poset of coideals of a coalgebra $C$ is given by the dual isomorphism with the lattice of subalgebras of $C^*$. The meet (join) in the lattice of coideals of $C$ comes from the join (meet) in the lattice of subalgebras of $C^*$. We have the isomorphism:

$$\text{cold}(C) \ni I \overset{\phi}{\mapsto} I^\perp := \{ f \in C^* : f|_I = 0 \} \in \text{Sub}_{\text{alg}}(C^*)$$

Let $I_1$ and $I_2$ be two coideals. Then

$$I_1 \lor I_2 := \phi^{-1}(I_1^\perp \cap I_2^\perp) = \phi^{-1}(\{ f \in C^* : f|_{I_1 \lor I_2} = 0 \})$$

because $\phi^{-1}(I^\perp) = I$. Moreover,

$$I_1 \land I_2 = \sup \{ I \in \text{cold}(C) : I \subseteq I_1 \text{ and } I \subseteq I_2 \} = + \{ I \in \text{cold}(C) : I \subseteq I_1 \cap I_2 \},$$

where the first equality holds by the reason of completeness of the lattice of coideals.

Note 2.7. The above shows that there is a complete lattice structure on the poset of coideals (also right or left) even if $C$ is infinite dimensional and it also works over any base ring.

Proposition 2.8. Let $B$ be a bialgebra and $H$ a Hopf algebra then the posets

$$(\text{Id}_{\text{bi}}(B), \subseteq), (\text{Id}_{\text{Hopf}}(H), \subseteq)$$

are complete lattices which in finite dimensional case are algebraic and dually algebraic.

Proof. Let us take two biideals $I$ and $J$. Then $I + J$ is the join in the lattice of coideals, by $I \lor J$ we will denote their join in the lattice of ideals of the underlying algebra. We prove that $I \lor J = I + J$. First of all $I + J \subseteq I \lor J$ just because $I + J$ is a join in the lattice of vector subspaces and all objects here are supposed to be subspaces. So it is enough to prove that $I + J$ is an ideal. This is true, because $(I + J)A = IA + JA = I + J$ and $A(I + J) = AI + AJ = I + J$ and thus $I \lor_{\text{bi}} J := I + J = I \lor J$. From Lemma 1.7 it follows that biideals form a complete lattice: $\text{ld}_{\text{bi}}(B) = \text{ld}(B^A) \cap \text{cold}(B^C) \subseteq \text{Sub}_{\text{Vect}}(B)$ and all the three lattices (of ideals, coideals and vector subspaces which is algebraic one) have the same join and have common the smallest element.

It remains to consider the Hopf algebra case. We have the inclusion $\text{Id}_{\text{Hopf}}(H) \subseteq \text{ld}_{\text{bi}}(B)$. It is enough to show that if $I$ and $J$ are Hopf ideals then $I + J$ is a Hopf ideal. This is certainly true, because $S(I + J) = S(I) + S(J) \subseteq I + J$. Using Lemma 1.7 we get that Hopf ideals form a complete lattice.

In finite dimensional case these lattices are algebraic and dually algebraic by Remark 1.14.
The completeness of the lattices of coideals, biideals and Hopf ideals holds without the assumption that the dimension is finite (and works also in any abelian category not just in $\text{Vect}_k$) while the algebraic and dual algebraic properties we prove only in the finite dimensional case which is less general than we suspect: it is known that the lattices of ideals or subalgebras of an algebra are algebraic without any finiteness condition. Therefore, it is reasonable to suspect that coideals form dually algebraic lattice in any case (however one should bear in mind that in this setting we are mixing ‘costructures’ with ‘algebraic structures’ as all the objects are already defined in the category $\text{Vect}_k$).

**Definition 2.9.** Let $C$ be a coalgebra, $B$ a bialgebra and $H$ a Hopf algebra. Then the sets have a poset structure under the inclusion relation:

$$
\begin{align*}
\text{Quot}(C) &= \{ C/I \mid I \text{ is a coideal} \}, \\
\text{Quot}_l(C) &= \{ C/I \mid I \text{ is a left coideal} \}, \\
\text{Quot}_r(C) &= \{ C/I \mid I \text{ is a right coideal} \}, \\
\text{Quot}(B) &= \{ B/I \mid I \text{ is a biideal} \}, \\
\text{Quot}(H) &= \{ H/I \mid I \text{ is a Hopf ideal} \}.
\end{align*}
$$

**Remark 2.10.** The defined posets are complete lattices. In finite dimensional case they are algebraic and dually algebraic lattices as we show in Remark 1.14.

The notion presented below will play the most important role in this paper.

**Definition 2.11.** Fix a Hopf algebra $H$. We define $\text{Quot}_{\text{gen}}(H)$ as the set of all quotients of $H$ by coideal right ideal (quotient as a coalgebra and a right $H$-module).

$$\text{Quot}_{\text{gen}}(H) := \{ H/I : I \text{ coideal right ideal} \}$$

**Proposition 2.12.** The set $\text{Quot}_{\text{gen}}(H)$ is a poset under the relation:

$$Q_1 \succeq Q_2 \iff \text{there exists an epimorphism } Q_1 \xrightarrow{\pi} Q_2 \text{ such that the following diagram commutes}$$

$$\begin{array}{ccc}
H & \xrightarrow{\pi I_1} & Q_1 = H/I_1 \\
\pi I_2 & \downarrow & \downarrow \pi I_2 \\
& & Q_2 = H/I_2
\end{array}$$

where $\pi I_1$ and $\pi I_2$ are the canonical projections.

The proof is straightforward and it will be omitted.

**Theorem 2.13.** The poset $\text{Quot}_{\text{gen}}(H)$ is a complete lattice. When $H$ is finite dimensional then this lattice is algebraic and dually algebraic.

**Proof.** Let us take a look at the two lattices:

$$(\text{Id}_r(H), +, \cap) \quad (\text{cold}(H), +, \wedge)$$
They have the same join thus their intersection is an upper semilattice. Both lattices are complete thus there is well defined meet operation on:

\[ I_1 \land I_2 = + \{ I \in I_r(H) \cap \text{cold}(H) : I \leq I_1 \text{ and } I \leq I_2 \} \]

Furthermore, \((\text{Id}_r(H) \cap \text{cold}(H), +, \land)\) is a complete lattice what can be proved using Lemma 1.7: it is an intersection of two complete lattices which are upper sublattices of the lattice of all subspaces of \(H\) and they have common the smallest element. It is easy to see that the above formula gives a meet in this lattice. Let us note that both operations of a lattice are unique in the sense that for any meet (join) if there exists join (meet) with which the lower (upper) semilattice is a lattice then it is unique. In \(\text{Id}_r(H) \cap \text{cold}(H)\) there exists supremum of any family of elements thus the lattice \(\text{Quot}_{\text{gen}}(H)\) is complete. Using Remark 1.14, we get that this lattice is algebraic and dually algebraic whenever \(H\) is finite dimensional.

\[ \square \]

3 Galois connection in Hopf-Galois theory

In this section we formulate and prove our main theorem: existence of a Galois connection between lattice of subalgebras and generalised quotients of a Hopf algebra in the case of comodule algebras. In general we do not assume that \(H\) is finite dimensional, whenever we do not write it explicitly. Let us begin with definition of the lattices which will stand on one side of our Galois connection:

**Definition 3.1.** Let \(B \subseteq A\) be an extension of algebras then by \(\text{Sub}_{\text{alg}}(B \subseteq A)\) we denote the lattice of all subalgebras of \(A\) which contain \(B\).

The defined lattice is an interval in the algebraic lattice of subalgebras of \(A\), thus it is complete, and algebraic.

**Lemma 3.2.** The lattice \(\text{Sub}_{\text{alg}}(B \subseteq A)\) is an algebraic lattice.

**Proof.** We already know that it is complete so it remains to prove that every element is a supremum of a set of compact elements. We will use the fact that the lattice of subalgebras of a given algebra \(A\) is algebraic, i.e. for every subalgebra \(A_0 \in \text{Sub}_{\text{alg}}(B \subseteq A)\) there exists a set of compact elements, that means finitely generated subalgebras, \(I\) such that \(A_0 = \sup I \supseteq B\). Then \(\sup \{ I \cap \text{Sub}_{\text{alg}}(B \subseteq A) \} = A_0\). This ends the proof, because every compact element of \(\text{Sub}_{\text{alg}}(B \subseteq A)\) belonging to \(I\) is compact. \(\square\)

**Definition 3.3.** Let \(H\) be a Hopf algebra, and \(A\) an algebra. \(A\) is said to be a comodule algebra if it is an \(H\)-comodule, whose structure map is a map of algebras, i.e. there is a coassociative algebra map \(\delta : A \rightarrow A \otimes H\) compatible with the counit of \(H\):
Above, Δ stands for the comultiplication of H and ε is the counit of H. The set
\[ A^\co H := \{a \in A : \delta(a) = a \otimes 1\} \]
is a subalgebra of A and is called subalgebra of coinvariants.

**Definition 3.4** \((H\)-extension and \(Q\)-Galois extension). Let \( A \) be an \( H \)-comodule algebra. An intermediate extension of the form \( A^\co Q \subseteq A \) will be called an intermediate \( H \)-extension. The poset of all the intermediate \( H \)-extensions will be denoted as \( \text{Sub}_{H-\text{ext}}(B \subseteq A) \). An intermediate \( H \)-extension \( A^\co Q \subseteq A \) will be called \( Q \)-Galois if the canonical map:
\[ \text{can}_Q : A \otimes A^\co Q A \rightarrow A \otimes Q \]
is a bijection. The subposet of \( \text{Sub}_{H-\text{ext}}(B \subseteq A) \) consisting of all \( Q \)-Galois extensions will be denoted by \( \text{Sub}_{Q-\text{Galois}}(B \subseteq A) \).

If \( A \) and \( H \) are commutative the above defines affine torsor. \( A^\co H \) is then the affine quotient of \( A \) which agrees with the true quotient if \( A/A^\co H \) is faithfully flat. Moreover, in this case \( A/A^\co H \) is faithfully flat \( H \)-Galois extension if and only if \( \text{Spec}(A) \rightarrow \text{Spec}(A^\co H) \) is faithfully flat and the map
\[ \text{Spec}(A) \times \text{Spec}(H) \rightarrow \text{Spec}(A) \times_{\text{Spec}(A^\co H)} \text{Spec}(A), \quad (x, g) \mapsto (x, xg) \]
is an isomorphism of affine schemes. This correspond to the notion of a \( G \)-torsors [4], for an affine group scheme \( G \) represented by a commutative Hopf algebra \( H \). Next theorem is the most important result of this work which is a new result in the theory of comodule algebras over a Hopf algebra.

**Theorem 3.5.** Let \( H \) be a Hopf algebra over a field \( k \) and \( A \) be an \( H \)-comodule algebra then there exists a Galois connection between the complete lattices:
\[ \text{Quot}_{\text{gen}}(H) \xrightarrow{\phi} \text{Sub}_{\text{alg}}(A/A^\co H) \]
where \( \phi(Q) = A^\co Q \).

Moreover, the map \( \psi \) is unique what follows from general statement on Galois connections (Proposition 1.3 (3)). By the above existence theorem of Galois connection and Proposition 1.3 (2) we obtain that \( \phi(\text{Quot}_{\text{gen}}(H)) \) and \( \psi(\text{Sub}_{\text{alg}}(A^\co H \subseteq A)) \) are dually isomorphic posets.
The above statement can be proved in more general contexts: of coactions of coalgebras or even coactions of corings (the poset of generalised quotients of a Hopf algebra would be exchanged with the poset of quotient coalgebras or corings).

To prove the main theorem we will use the following general existence theorem for Galois connections which was already known.

**Theorem 3.6.** Suppose that $P, Q$ are posets and moreover $P$ is complete. Then antimonotonic function $\phi : P \rightarrow Q$ is part of a Galois connection if and only if it reflects all suprema into infima.

The above theorem has its more know categorical generalisation: the P.J. Freyd characterisation theorem which more or less states that a functor has a left (right) adjoint functor if and only if it preserves limits (colimits) ([7, Theorem 2, p.121]. One just have to keep in mind that the definition that we have for Galois connections would straightly generalise to contravariant functors rather than to covariant ones: $(\phi, \psi) : P \xleftrightarrow{\phi} Q$ is a Galois connection if an only if $(\phi, \psi) : P \xleftrightarrow{\psi} Q^{op}$ is a covariant adjunction with the usual category structure: the relation $p' \geq p$ is thought as a morphism from $p'$ to $p$.

**Proof.** Let us define a map $\psi : Q \rightarrow P$ by the following formula:

$$\psi(B) := \sup\{A \in P : \phi(A) \geq B\}$$  \hspace{1cm} (9)

This is definition works, because $P$ is complete. Clearly, this map reverses the order. Now we will show the two inequalities $\phi \psi \geq id$ and $\psi \phi \geq id$:

$$\phi \psi(B) = \inf\{\phi(A) : \phi(A) \geq B\} \geq B$$

$$\psi \phi(A) = \sup\{\tilde{A} : \phi(\tilde{A}) \geq \phi(A)\} \geq A$$

Hence $\phi$ and $\psi$ establishes a Galois connection. Now let $P \xleftrightarrow{\phi} Q$ be a Galois connection, we will show that $\phi$ reflects all suprema (then it follows that $\psi$ reflects all suprema that exists in $Q$). One has the two inequalities which hold whenever the left hand sides exist (we will show that in the case of the left inequality its left side always exists):

$$\inf_i \phi(A_i) \geq \phi\left(\sup_i A_i\right), \hspace{1cm} \psi\left(\inf_i B_i\right) \geq \sup_i \psi(B_i)$$  \hspace{1cm} (10)

where $B_i \in Q, A_i \in P$. Plugging $B_i = \phi(A_i)$ into the second one and using the Galois property (7) we get:

$$\psi\left(\inf_i \phi(A_i)\right) \geq \sup_i \psi\phi(A_i) \geq \sup_i A_i$$
Now applying $\phi$ and using the Galois property once again we obtain:

$$\inf_i \phi(A_i) \leq \phi \left( \inf_i \phi(A_i) \right) \leq \phi \left( \sup_i A_i \right)$$

together the first inequality of (10) it proves that $\phi$ reverses infinite suprema as we claimed. It remains to show that for any $A \subseteq P$ there exists $\inf \phi(A)$ in $Q$. The element $\phi(\sup A)$ is smaller than every element of $\phi(A)$, because $\phi$ reverses the order. Furthermore, there can not exist an element of $Q$ which is greater than $\phi(\sup A)$ and which is a lower bound of $\phi(A)$. For if $q \in Q$ is such then $\psi(q) = \sup A$ and thus $q \leq \phi \psi(q) = \phi(\sup A)$ so $q = \phi(\sup A)$. Line of reasoning proves that $\psi$ reverses existing in $Q$ infinite suprema.

For more on Galois connections we refer to [1], where among other results the above theorem is proved. Now we are ready to prove Theorem 3.5.

**Proof.** By Theorem 3.6, it is enough to show that the map $\phi : Q \mapsto A^{\text{co}Q}$ reverses suprema, i.e.

$$A^{\text{co}Q} \bigvee_{i \in I} Q_i = \bigcap_{i \in I} A^{\text{co}Q_i}$$

From the set of inequalities: $\bigvee_{i \in I} Q_i \geq Q_j$ ($j \in I$) it follows that

$$A^{\text{co}Q} \bigvee_{i \in I} Q_i \subseteq \bigcap_{i \in I} A^{\text{co}Q_i}$$

Fix an element $a \in \bigcap_{i \in I} A^{\text{co}Q_i}$. We let $I_i$ denote the coideal and right ideal such that $Q_i = H/I_i$. Then we can write the following

$$\forall_{i \in I} a \in A^{\text{co}Q_i} \iff \forall_{i \in I} \delta(a) - a \otimes 1 \in A \otimes I_i \iff \delta(a) - a \otimes 1 \in A \otimes \bigcap_{i \in I} I_i \iff a \in A^{\text{co}Q} \bigvee_{i \in I} Q_i$$

The equivalence in the middle is just because $\bigcap_{i \in I} A \otimes I_i = A \otimes \bigcap_{i \in I} I_i$ what we show below:

$$\bigcap_{i \in I} A \otimes I_i \ni \sum_{l=1}^n a_l \otimes b_l \iff \forall_{i \in I} \sum_{l=1}^n a_l \otimes b_l \in A \otimes I_i$$

$$\iff \forall_{i \in I} \forall_{l=1,...,n} b_l \in I_i$$

$$\iff \sum_{l=1}^n a_l \otimes b_l \in A \otimes \bigcap_{i \in I} I_i$$

$\square$
4 Closed elements of Galois connection for Hopf-Galois extensions

In this section we will show which elements of \( \text{Quot}_{\text{gen}}(H) \) are closed in the Galois connection (8). The importance of this theorem lies in the fact that closed elements of \( \text{Quot}_{\text{gen}}(H) \) classifies elements of \( \text{Sub}_{H - \text{ext}}(A/B) \). Our main result of this section states that \( Q \in \text{Quot}_{\text{gen}}(H) \) is closed whenever \( A/A^{\text{co}Q} \) is \( Q \)-Galois.

It follows that if \( H \) is finite dimensional then every generalised quotient is closed provided \( A/A^{\text{co}H} \) has surjective (thus injective) canonical map. As a corollary we obtain a bijective correspondence between \( \text{Sub}_{H - \text{ext}}(A/B) \) and \( \text{Quot}_{\text{gen}}(H) \) for finite dimensional Hopf algebras.

**Proposition 4.1.** ³ Let \( A \) be an \( H \)-comodule algebra (both \( A \) and \( H \) can be infinite dimensional) with surjective canonical map and let \( A \) be a \( Q_1 \)-Galois and a \( Q_2 \)-Galois extension where \( Q_1, Q_2 \in \text{Quot}_{\text{gen}}(H) \). Then:

\[
A^{\text{co}Q_1} = A^{\text{co}Q_2} \Rightarrow Q_1 = Q_2
\]

To prove this proposition we will need that

\[
\text{can}_Q : A \otimes_{A^{\text{co}Q}} A \rightarrow A \otimes Q
\]

is in some sense functorial.

**Lemma 4.2.** Let \( A \) be an \( H \)-comodule algebra and \( Q_1, Q_2 \in \text{Quot}_{\text{gen}}(H) \) be such that \( Q_1 \succeq Q_2 \). Then following diagram commutes:

\[
\begin{array}{ccc}
A \otimes_{A^{\text{co}Q_1}} A & \xrightarrow{\text{can}_{Q_1}} & A \otimes Q_1 \\
\downarrow p & & \downarrow \text{id} \otimes \pi \\
A \otimes_{A^{\text{co}Q_2}} A & \xrightarrow{\text{can}_{Q_2}} & A \otimes Q_2
\end{array}
\]  

(11)

where \( p \) and \( \pi \) exists by assumption.

**Proof.** Let \( \pi_i : H \rightarrow Q_i, \ i = 1, 2 \) be right \( H \) linear and colinear epimorphism. Then \( \pi : Q_1 \rightarrow Q_2 \) is such that \( \pi \circ \pi_1 = \pi_2 \) (exists by assumption, that \( Q_1 \succeq Q_2 \)), and there exists an epimorphism from \( A \otimes_{A^{\text{co}Q_1}} A \) onto \( A \otimes_{A^{\text{co}Q_2}} A \) sending \( a \otimes_{A^{\text{co}Q_1}} b \) to \( a \otimes_{A^{\text{co}Q_2}} b \), because \( A^{\text{co}Q_1} \subseteq A^{\text{co}Q_2} \). Now we will just compute the commutativity of the diagram (11):

\[
\text{can}_2 \circ p(a \otimes_{A^{\text{co}Q_1}} b) = \text{can}_2(a \otimes_{A^{\text{co}Q_2}} b) = ab(0) \otimes \pi_2(b(1))
\]

³We would like to thank P. Hajac for his insight which helped to prove this proposition.
and

\[(id \otimes \pi) \circ can_1(a \otimes_{A^{co}Q_1} b) = (id \otimes \pi)(ab_{(0)} \otimes \pi_1(b_{(1)})) = ab_{(0)} \otimes \pi_2 b_{(1)}\]

Now we can prove Proposition 4.1.

**Proof.** Let \(B = A^{co}Q_1 = A^{co}Q_2\) then we have the following commutative diagram:

\[
\begin{array}{ccc}
A \otimes B A & \xrightarrow{can} & A \otimes H \\
\downarrow & & \downarrow \text{id} \otimes \pi
\end{array}
\]

Commutativity of this diagram we proved in the previous lemma. Because \(can_1\) and \(can_2\) are isomorphisms we have the map \(can_1 \circ can_2^{-1} : A \otimes Q_2 \rightarrow A \otimes Q_1\). Let us see that \((can_1 \circ can_2^{-1}) \circ (id \otimes \pi_2) = (id \otimes \pi_1)\). Pre-composing these two arrows with \(can\) we get equal maps what follows from commutativity of the above diagram. Moreover, surjectivity of \(can\) yields the equality \((can_1 \circ can_2^{-1}) \circ (id \otimes \pi_2) = (id \otimes \pi_1)\). Now let us show that \(can_1 \circ can_2^{-1} = id \otimes \pi\) for some \(\pi : Q_1 \rightarrow Q_2\) such that \(\pi \circ \pi_2 = \pi_1\). Any simple tensor from domain of \(can_1 \circ can_2^{-1}\) is of the form \(a \otimes \pi_2(h)\) and its image under \(can_1 \circ can_2^{-1}\) is \(a \otimes \pi_1(h)\). It follows that \(\pi : \pi_2(h) \mapsto \pi_1(h)\) is a well defined map. By definition we get that \(\pi \circ \pi_2 = \pi_1\). From this equality it follows that \(\pi\) is colinear and right \(H\)-linear map of right \(H\)-modules coalgebras \((\pi_1\) and \(\pi_2\) are such):

\[
\begin{align*}
\pi(h\pi_2(h')) &= \pi \circ \pi_2(hh') = \pi_1(hh') = h\pi_1(h') = h\pi_2(h') \\
\delta_2(\pi_2(h)) &= \delta_2(\pi_1(h)) = \pi_1 \otimes \pi_1 \Delta(h) = \pi_1 \otimes \pi \otimes \pi_1 \circ \Delta(h) \\
&= \pi \otimes \pi \circ \delta_1(\pi_1(h))
\end{align*}
\]

\(\Delta\) denotes the coproduct of \(H\) and \(\delta_i\) is the coproduct of \(Q_i, i = 1, 2\). Thus we proved that \(Q_2 \succeq Q_1\). In the same way we prove that \(Q_1 \succeq Q_2\) (this time taking \(can_2 \circ can_1^{-1}\) instead of \(can_1 \circ can_2^{-1}\)); because \(\succeq\) is an order (antisymmetry) we get that \(Q_1 = Q_2\).

\[\square\]

**Proposition 4.3.** If \(A\) is an \(H\)-comodule algebra with epimorphic canonical map (both \(A\) and \(H\) can be infinite dimensional) then every \(Q \in \text{Quot}_{gen}(H)\) for which \(A^{co}Q \subseteq A\) is a \(Q\)-Galois extension is a closed element of the Galois connection (8).
Proof. Fix $A^{coQ}$ for some $Q \in \text{Quot}_{\text{gen}}(H)$. $\phi^{-1}(A^{coQ})$ is an upper-sublattice of $\text{Quot}_{\text{gen}}(H)$ (i.e. it is a subposet closed under finite suprema) which has the greatest element, namely $\tilde{Q} = \psi(A^{coQ})$. $\tilde{Q}$ is the only closed element belonging to $\phi^{-1}(A^{coQ})$. Both $Q \leq \psi(A^{coQ})$ and the observation that $A^{coQ} \subseteq A$ is a $Q$-Galois imply that $\tilde{Q}$ is also such. We have the commutative diagram:

\[
\begin{array}{ccc}
A \otimes_B A & \longrightarrow & A \otimes H \\
\downarrow & & \downarrow \\
A \otimes_{A^{co\tilde{Q}}} A & \longrightarrow & A \otimes \tilde{Q} \\
\downarrow & & \downarrow \\
A \otimes_{A^{coQ}} A & \longrightarrow & A \otimes Q
\end{array}
\]

From the lower commutative square we get that $\text{can}_{\tilde{Q}}$ is a monomorphism and from the upper commutative square one can deduce that $\text{can}_{\tilde{Q}}$ is onto. Unless $\tilde{Q} = Q$ we get a contradiction with Proposition 4.1. \qed

P. Schauenburg and H.-J. Schneider in [13, Corollary 3.3] give conditions under which an extension $^{coQ}A \subseteq A$ is a $Q$-Galois and thus when it is a closed element of the Galois connection 15. Let us cite it here:

**Proposition 4.4** (P. Schauenburg, H.-J. Schneider [13, Corollary 3.3]). Let $H$ be a Hopf algebra over a ring $k$ with bijective antipode and let $A$ be an $H$-comodule algebra with epimorphic canonical map. Let $Q \in \text{Quot}_{\text{gen}}(H)$, then it follows that in each of the case $A/A^{coQ}$ is $Q$-Galois and $A$ is a projective $A^{coQ}$-module:

1. $k$ is a field and $H$ is finite dimensional,
2. $H$ is finitely generated projective over $k$, coflat as a right $Q$-comodule, and the surjection $H \rightarrow Q$ splits as a left $Q$-comodule map,
3. $H$ has enough right integrals, is coflat as a right $Q$-comodule, and the surjection $H \rightarrow Q$ splits as a left $Q$-comodule map,
4. $k$ is a field, $H$ is co-Frobenius, and faithfully coflat both as a left and a right $Q$-comodule,
5. $H$ is $Q$-cleft and $Q$ is finitely generated projective,
6. $k$ is a field, $H$ has cocommutative coradical, and $Q$ is finite dimensional and of the form $Q = H/K^+H$ for a Hopf subalgebra $K$ of $H$. 

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Let us remark that due to Larson and M. Sweedler result [6] the assumption of bijectivity of the antipode follows in the case of finite dimensional Hopf algebras and thus is not needed in (1) above. The following theorem is a generalisation of [20, Theorem 4.7] which we included in the introduction.

**Theorem 4.5.** Let $H$ be a finite dimensional Hopf algebra and let $B \subseteq A$ be an $H$-Hopf-Galois extension. Then every intermediate $H$-extension is a $Q$-Galois and there is an anti-isomorphism of posets:

$$\text{Sub}_{H-\text{ext}}(B \subseteq A) \simeq \text{Quot}_{\text{gen}}(H)$$

**Proof.** It follows directly from Proposition 4.3 and Proposition 4.4 (1). □

**Remark 4.6.** It follows that intermediate $H$-extensions form a complete lattice. Moreover, the poset of Hopf-Galois subextensions is dually isomorphic to the lattice of Hopf algebra quotients of $H$:

$$\{ A^{\alpha H/I} \subseteq A : I \text{ Hopf ideal of } H \} \simeq \text{Quot}(H).$$

Later on we will see that this corollary implies part of the Galois Theorem for finite field extensions. The main result in finite Hopf-Galois theory – Theorem 4.5, shows that there is bijective correspondence with all $H$-subextensions. The map $\psi$ constructed as an adjunction of $Q \mapsto A^{\alpha Q}$ is not given by an explicit formula in terms of the theory of Hopf algebras. However, in some important cases: the finite Galois theory, the $k \subseteq H$ $H$-Hopf-Galois extension and cleft extensions we will see an explicit formula.

## 5 Comparison with classical Galois Theory

In this part we show, how the presented concept of Galois Theory includes classical Galois theory in Field Theory, and which part of classical theory can be covered by the developed theory.

We were taught by T. Maszczyk that one can expose classical Galois theory using corings. This point of view we will explain here as it is a canonical way what will emphasise. The material is essentially taken from lecture notes [8]. Our attempt to this exposition is the proof of lattice structures of posets of subcorings that will appear and the Galois epimorphisms in Propositions 5.6.

**Theorem 5.1** (T. Maszczyk [8, Corollary 2.3]). Let $F \subseteq E$ be finite field extension with the Galois group $G = \text{Gal}(E/F)$. Then $F \subseteq E$ is a Galois extension if and only if the map:

$$\text{can} : E \otimes_F E \longrightarrow \text{Map}(G, E)$$

$$e_1 \otimes_F e_2 \longmapsto (g \mapsto e_1 g(e_2))$$
is a bijection.

Proof. If \( \mathbb{E}/\mathbb{F} \) is a finite Galois extension then \( G \) is a finite group, so \( G = \{g_1, g_2, \ldots, g_n\} \). Defined map \( \text{can} \) is \( \mathbb{E} \)-linear. Let us show that it is injective and the dimension of domain and codomain are equal. Every element of \( \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \) can be written as a sum \( \sum_{i=1}^{n} e_i \otimes e_i \) where \( \{e_1, \ldots, e_n\} \) is a basis of \( \mathbb{E}/\mathbb{F} \), so that if such elements belong to kernel of \( \text{can} \) then

\[
\sum_{i=1}^{n} e_i g_j(e_i) = 0
\]

by the Dedekind theorem which says that all elements of \( G \) are linearly independent as endomorphisms of \( \mathbb{E} \) over \( \mathbb{F} \), \( g_j(e_i) \) are non singular thus all \( e_i \) are equal to 0. In this way we obtain that \( \text{can} \) is injective. Moreover,

\[
dim_{\mathbb{F}} \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} = [\mathbb{E} : \mathbb{F}]^2 = [\mathbb{E} : \mathbb{F}] |G| = \dim_{\mathbb{F}} \text{Map}(G, \mathbb{E})
\]

where the middle equality holds, because \( [\mathbb{E} : \mathbb{F}] = |G| \) when \( \mathbb{E}/\mathbb{F} \) is a finite Galois extension. Now let us show the converse. If \( \text{can} \) is bijective then \( \mathbb{F} = \mathbb{E}^G \) and thus \( \mathbb{E}/\mathbb{F} \) is Galois. We have:

\[
dim_{\mathbb{F}} \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} = \dim_{\mathbb{F}} \text{Map}(G, \mathbb{E})
\]

hence \( [\mathbb{E} : \mathbb{F}] = |G| \). We know that \( \mathbb{F} \subseteq \mathbb{E}^G \). Moreover, \( [\mathbb{E} : \mathbb{F}] = [\mathbb{E} : \mathbb{E}^G][\mathbb{E}^G : \mathbb{F}] \) so \( [\mathbb{E}^G, \mathbb{F}] = 1 \) and thus \( \mathbb{F} = \mathbb{E}^G \).

Furthermore, in [8, Proposition 2.5] it is shown that \( \text{can} \) is an isomorphism of corings.

Definition 5.2. A coring over ring \( k \) is a comonoid in the category of \( k \)-bimodules. In other words it is a bimodule \( L \) together with maps of bimodules: \( \Delta : L \to L \otimes_k L \) called coproduct and a counit \( \epsilon : L \to k \) which are subject to the equalities:

\[
\begin{array}{ccc}
L & \xrightarrow{\Delta} & L \otimes_k L \\
\Delta & \downarrow & \Delta \\
L \otimes_k L & \xrightarrow{id \otimes_k \Delta} & L \otimes_k L \\
\end{array}
\]

Morphism of \( k \)-corings is a morphism of comonoids in the category of \( k \)-bimodules.

Example 5.3. Here we give two important examples of corings.
(1) Let $E/F$ a be field extension then $E \otimes_F E$ is a coring with the comultiplication:

$$\Delta : E \otimes_F E \ni e' \otimes_F e \longmapsto (e \otimes_F 1_E) \otimes_E (1_E \otimes_F e') \in (E \otimes_F E) \otimes_E (E \otimes_F E)$$

and counit map $m : E \otimes_F E \longrightarrow E$ the multiplication map of $E$. This coring is called the canonical Sweedler coring or Sweedler coring for short.

(2) Let $G$ be a finite subgroup of $\text{Gal}(E/F)$ and let $E$ be a field. Then $\text{Map}(G, E)$ is a coring with $\Delta$ induced by the multiplication of $G$ $m : G \times G \longrightarrow G$.

$$\Delta : \text{Map}(G, E) \longrightarrow \text{Map}(G \times G, E) \simeq \text{Map}(G, E) \otimes_E \text{Map}(G, E)$$

where $\text{Map}(G, E) \otimes_E \text{Map}(G, E) \simeq \text{Map}(G \times G, E)$ is given by

$$\phi_1 \otimes_E \phi_2 \mapsto \left((g_1, g_2) \mapsto \phi_1(g_1)g_2(\phi_2(g_2))\right)$$

and the counit $\epsilon : \text{Map}(G, E) \longrightarrow E$ given by $\epsilon(\phi) = \phi(e_G)$, where $e_G$ denotes the identity of the group $G$.

The coring $\text{Map}(G, E)$ was introduced by T. Maszczyk and he observed [8, Proposition 2.5] that the canonical map $\text{can}$ of Theorem 5.1 is a morphism of corings.

Under the assumption that $F \subseteq E$ is a finite Galois extension with the Galois group $\text{Gal}(E/F) = G$ the coring $\text{Map}(G, E)$ can be realised by many Hopf algebras, i.e. there are many Hopf algebras $H$ such that $\text{Map}(G, E) \simeq E \otimes H$ as corings. For example $K[G]^*$ is such for any finite field extension $K \subseteq F$. Moreover, the canonical map of corings $E \otimes_F E \longrightarrow \text{Map}(G, E)$ is the canonical map of the $K[G]^*$-Hopf-Galois extension $F \subseteq E$ which structure map $\delta : E \longrightarrow E \otimes_K K[G]^*$ is given by

$$E \longrightarrow E \otimes_F E \xrightarrow{\text{can}} \text{Map}(G, E) \xrightarrow{\sim} E \otimes_K K[G]^* \xrightarrow{\epsilon} 1 \otimes ev_e$$

Under the assumption $G = \text{Gal}(E/F)$ $\delta$ is a $K$-linear algebra homomorphism. Thus we can realise the isomorphism of corings $\text{can}$ as a Hopf-Galois extension, but there is no canonical Hopf algebra: $K$ was any field such that $F$ is its finite extension. However, if $G$ is infinite then we cannot use the canonical coring $\text{Map}(G, E)$ but we can still use Hopf algebras.

**Definition 5.4.** A subcoring $L'$ of a $k$-coring $L$ is a $k$-coring $L'$ such that $L'$ is submodule of $L$ and the inclusion $L' \subseteq L$ is a morphism of $k$-corings.
If \( L \) is pure as a left and right \( k \)-module then a subcoring is a subbimodule such that \( \Delta|_L \) takes values in \( L' \otimes L' \subseteq L \otimes L \).

**Definition 5.5.** Let \( L \) be a \( k \)-coring. Then a coideal of \( L \) is a kernel of an epimorphism of \( k \)-corings.

If \( I \) is pure as left and right \( k \)-module then a coideal is a \( k \)-subbimodule \( I \) such that \( \Delta(I) \subseteq I \otimes_k L + L \otimes_k I \subseteq L \otimes L \) and \( I \subseteq \ker \epsilon \).

**Proposition 5.6.** Let \( G \) be a finite group. Then there exists a Galois epimorphism from the lattice of coideals of the coring \( \text{Map}(G, \mathbb{E}) \) to the lattice of submonoids of \( G \).

**Proof.** Let \( I \) be a coideal of \( \text{Map}(G, \mathbb{E}) \) then \( \bigcap_{f \in I} \ker f \), where \( \ker f \) denotes the set of elements of \( G \) which maps to 0, is a submonoid of \( G \). It contains identity of the group \( G \), because \( I \subseteq \ker \epsilon \), where \( \epsilon \) is evaluation at the identity of \( G \). \( \bigcap_{f \in I} \ker f \) is closed under multiplication: if \( g_1 \) and \( g_2 \) belongs to all of the kernels of elements of \( I \) then for any \( f \in I \) \( f \cdot (g_1, g_2) = \Delta(f)(g_1, g_2) = 0 \), because \( \Delta(f) \in I \otimes_k \text{Map}(G, \mathbb{E}) + \text{Map}(G, \mathbb{E}) \otimes_k I \). The map \( \theta : I \mapsto \bigcap_{f \in I} \ker f \) reverses the order. The second map of the Galois connection is given as follows. For any submonoid \( G_0 \) of \( G \), \( \text{Map}(G_0, \mathbb{E}) \) forms a coring (the identity element of \( G \) is needed to set the counit as evaluation on it). \( \text{Map}(G_0, \mathbb{E}) \) is a quotient coring of \( \text{Map}(G, \mathbb{E}) \) via the restriction map \( f \mapsto f|_{G_0} \). Thus we have a map \( \xi \) from submonoids of \( G \) to quotient corings of \( \text{Map}(G, \mathbb{E}) \) defined as \( \xi(G_0) = \ker(\text{Map}(G, \mathbb{E}) \twoheadrightarrow \text{Map}(G_0, \mathbb{E})) \) which reverses the order. Furthermore, one can easily verify that 

\[
\theta \xi = \text{id}_{\text{Sub}_{\text{mono}}(G)} \quad \text{and} \quad \xi \theta \geq \text{id}_{\text{coId}(\text{Map}(G, \mathbb{E}))}
\]

hence \( \theta \) is an epimorphism which is a part of the Galois connection \((\theta, \xi)\). \( \square \)

The lattice \( \text{Sub}_{\text{alg}}(\mathbb{E}/\mathbb{F}) \), where \( \mathbb{E} \) is thought as an \( \mathbb{F} \)-algebra (or \( \mathbb{K} \)-algebra for \( \mathbb{K} \subseteq \mathbb{F} \) a finite extension), consists of all intermediate fields of the field extension \( \mathbb{E}/\mathbb{F} \). We show that the lattice \( \text{Quot}_{\text{gen}}(\mathbb{F}[G]^*) \) is isomorphic to the lattice of submonoids of the Galois group \( G \). Whenever the Galois group is finite its submonoid is a subgroup. In this part we are dealing only with finite Galois extensions. In this case \( \mathbb{F}[G]^* \) is finite dimensional and thus [17, Proposition 1.4.6] can be applied:

**Proposition 5.7.** Let \( H \) be a finite dimensional Hopf algebra. Then \( I \) is its coideal if and only if \( I^\perp \) is a subalgebra of \( H^* \).

We use this characterisation to describe the lattice \( \text{Quot}_{\text{gen}}(\mathbb{F}[G]^*) \).

**Proposition 5.8.** Let \( H = \mathbb{F}[G]^* \) then

\[
\text{Quot}_{\text{gen}}(H) \simeq \text{Sub}_{\text{mono}}(G)
\]

where \( \text{Sub}_{\text{mono}}(G) \) denotes the algebraic lattice of submonoids\(^4\) of \( G \).

\(^4\)Monoid is a set closed under binary associative operation which necessarily has a unit.
Proof. Let us fix a right ideal coideal $I$ of $\mathbb{F}[G]^*$. Let us take a basis of $\mathbb{F}[G]^*$ consisting of $\delta_g$ given by $\delta_g(h) = \delta_{g,h}$. We show that any right ideal of $\mathbb{F}[G]^*$ must be generated by some subset of this basis. We have the equality $\delta_g \cdot \delta_h = \delta_{g,h} \delta_g$, where $\delta_{g,h}$ is the Kronecker symbol given by $\delta_{g,k} = \begin{cases} 1 & \text{iff } g = k \\ 0 & \text{otherwise} \end{cases}$. If we have a right ideal $I$ and its element $\sum_{k=1}^n \lambda_k \delta_{g_k}$ then 

$$
\left( \sum_{k=1}^n \lambda_k \delta_{g_k} \right) \cdot \delta_{g_i} = \lambda_i \delta_{g_i},
$$

and thus if $\lambda_i \neq 0$ then $\delta_{g_i} \in I$. A right ideal $I$ is a coideal if and only if the set $M_I := \{ g : \delta_g \notin I \}$ is a submonoid of $G$, i.e. it is closed under multiplication and contains the unit of $G$. This is because $I$ is a coideal of $\mathbb{F}[G]^*$ if and only if $I^\perp$ is a subalgebra of $\mathbb{F}[G]^{**} \simeq \mathbb{F}[G]$ and $M_I$ is a basis of $I^\perp$. A submonoid $M$ of $G$ defines a right ideal, coideal of $\mathbb{F}[G]^*$. The right ideal, coideal $I_M$ is spanned by all the $\delta_g$ for $g \notin M$, thus in fact we have a bijective correspondence between $\text{Quot}_{\text{gen}}(\mathbb{F}[G]^*)$ and $\text{Sub}_{\text{mono}}(G)$ as we stated. \hfill \square

Remark 5.9. The Hopf algebra $\mathbb{F}[G]^*$ is commutative, so every its right ideal is an ideal and thus:

$$
\text{Quot}_{\text{gen}}(\mathbb{F}[G]^*) = \text{Quot}(\mathbb{F}[G]^*)
$$

Remark 5.10. If $G$ is finite then every its submonoid is a subgroup, because every element has finite order and the inverse of an element $g$ is equal to $g^{\text{ord}(g)-1}$, where $\text{ord}(g)$ is the order of $g$. By this reason

$$
\text{Quot}_{\text{gen}}(\mathbb{F}[G]^*) = \text{Sub}(G)
$$

however if $G$ is infinite it is not true, for example take: $\mathbb{N} \subseteq \mathbb{Z}$.

Let $\text{Sub}_{\text{field}}(\mathbb{F} \subseteq \mathbb{E})$ denote the lattice of subfields of a field $\mathbb{E}$ containing a subfield $\mathbb{F}$.

Corollary 5.11. There is the following diagram of Galois connections in which the upper Galois connection is the classical one and the lower one is the Galois connection of our Main Theorem 3.5.

$$
\begin{array}{ccc}
\text{Sub}_{\text{field}}(\mathbb{E}/\mathbb{F}) & \xrightarrow{\text{Gal}} & \text{Sub}(\text{Gal}(\mathbb{E}/\mathbb{F})) \\
\downarrow & & \downarrow \psi \\
\text{Sub}_{\text{ring}}(\mathbb{E}/\mathbb{F}) & \xleftarrow{\text{Quot}_{\text{gen}}(\mathbb{F}[\text{Gal}(\mathbb{E}/\mathbb{F})]^*)} & \text{Sub}_{\text{mono}}(\text{Gal}(\mathbb{E}/\mathbb{F}))
\end{array}
$$

Commutativity of this diagram follows from the formula:

$$
\mathbb{E}^{\text{co} \mathbb{F}[G]^*} = \mathbb{E}^G,
$$

If $G$ is finite, then $\text{Sub}(\text{Gal}(\mathbb{E}/\mathbb{F})) = \text{Sub}_{\text{mono}}(\text{Gal}(\mathbb{E}/\mathbb{F}))$, furthermore $\phi$ is a monomorphism if.
It follows that \( \phi \) factorises through the embedding \( \text{Sub}_{\text{field}}(\mathbb{F} \subseteq \mathbb{E}) \subseteq \text{Sub}_{\text{ring}}(\mathbb{E}/\mathbb{F}) \), and thus the only closed elements of the Galois connection \( (\phi, \psi) \) in \( \text{Sub}_{\text{ring}}(\mathbb{E}/\mathbb{F}) \) are the one coming from closed elements of \( \text{Sub}_{\text{field}}(\mathbb{E}/\mathbb{F}) \). Normal subgroups of the Galois group corresponds to conormal quotients of \( H \), i.e. quotients by normal ideal, and also to normal subextensions.

### 6 Chase–Sweedler Theorem

As a direct consequence of our Main Theorem we get the Chase–Sweedler Theorem in the case of finite dimensional Hopf algebras.

**Definition 6.1.** Let \( H \) be a Hopf algebra over \( k \). We say that a \( k \)-algebra \( A \) is an \( H \)-module algebra if it is an \( H \)-module and the action \( h \cdot a \) satisfies the following identity:

\[
h(a_1a_2) = h(0)(a_1)h(1)(a_2)
\]

The set of invariants is defined as \( A^H := \{a \in A : \forall h \in H, ha = \epsilon(h)a\} \).

Let \( \mathbb{E}/\mathbb{F} \) be a field extension and \( \mathbb{E} \) be an \( H \)-module algebra. An algebra \( \mathbb{E}/\mathbb{F} = \mathbb{E}^H \) will be called Hopf-Galois extension if and only if the canonical map

\[
can : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \rightarrow \text{Hom}(H, \mathbb{E}), \quad e_1 \otimes e_2 \mapsto (h \mapsto e_1he_2)
\]

is a bijection.

**Theorem** (Chase–Sweedler). Let \( \mathbb{E}/\mathbb{F} \) be a Galois extension such that it is a Hopf-Galois extension under an action of a cocommutative Hopf algebra \( H \). Then there is the injective map:

\[
\text{Sub}_{\text{Hopf}}(H) \rightarrow \text{Sub}_{\text{field}}(\mathbb{F} \subseteq \mathbb{E})
\]

This theorem follows from Theorem 4.5 and the next proposition, assuming that \( H \) is finite dimensional:

**Proposition 6.2.** Let \( H \) be a finite dimensional Hopf algebra over a field \( k \). Then \( A \) is an \( H \)-comodule algebra under a coaction \( \delta : A \rightarrow A \otimes H \) if and only if \( A \) is an \( H^* \)-module algebra with the action: \( \phi \cdot a = a_{(0)}\phi(a_{(1)}) \). This correspondence establishes a bijection between \( H \)-comodule algebra structures and \( H^* \)-module algebra structures. Moreover, \( A^{coH} \subseteq A \) is an \( H \)-Hopf-Galois extension as a comodule algebra if and only if it is an Hopf-Galois extension as a \( H^* \)-module algebra, thus \( A^{coH} = A^{H^*} \).

The proof is straightforward. The inverse map which associates an \( H^* \)-comodule algebra structure to an \( H \)-module algebra structure is given by: \( a_{(0)} \otimes_k a_{(1)}(h) := h \cdot a \). Furthermore, a field extension \( \mathbb{F} \subseteq \mathbb{E} \) is an \( \mathbb{F}[G] \)-module algebra with the module structure induced be the action of \( G = \text{Gal}(\mathbb{E}/\mathbb{F}) \) on \( \mathbb{E} \). There is a commutative diagram of canonical maps:
A $\mathbb{F}[G]^*$-comodule algebra structure on $E$ can be defined by the $\mathbb{F}[G]$-module algebra structure. One can then observe that $E^{\comod \mathbb{F}[G]} = E^G$.

In [20, Theorem 2.3] there is proved an extension of Chase–Sweedler theorem for commutative algebras which are comodule algebras over finite dimensional Hopf algebra. Theorem 4.5 extends this result to non-commutative algebras.

7 Generalisation of H.-J. Schneider’s and M. Takeuchi’s results

In the case of an $H$-Hopf-Galois extension $k \subseteq H$ we show that Theorem 3.5 generalises M. Takeuchi’s and H.-J. Schneider’s results given in [19] and [15, Theorem 1.4]. They provide a bijection between some (normal) Hopf ideals and some (conormal) Hopf subalgebras of a given Hopf algebra. Below we cite these two theorems merged together. H.-J. Schneider added the normality/conormality condition to the M. Takeuchi original result. The definition of normal Hopf ideals and normal Hopf subalgebras can be found in [15, Definition 1.1].

**Theorem 7.1.** Let $H$ be a Hopf algebra. Then

\[
\left\{ K \subseteq H : K - \text{right coideal subalgebra} \right\} \left\{ H/I : I - \text{left ideal coideal} \right\} \\
\downarrow_{\psi} \delta \downarrow_{\phi} \\
\left\{ K \subseteq H : K - \text{normal sub-Hopf algebra} \right\} \left\{ H/I : I - \text{normal Hopf ideal} \right\}
\]

are inverse bijections. They restricts to normal/conormal elements:

\[
\left\{ K \subseteq H : K - \text{normal sub-Hopf algebra} \right\} \cong \left\{ H/I : I - \text{normal Hopf ideal} \right\}
\]

Also P. Schauenburg has a statement of this form. In [11, Theorem 3.10] he shows that for a $k$-flat Hopf algebra $H$ the above bijective correspondence restricts to (left, right) admissible objects (Definition 9.2) of right and left hand sides. We present the previous theorem in the same way as it was originally stated, however in this paper we work in a dual setting than H.-J. Schneider and M. Takeuchi:
right/left coideal subalgebras \(\Longleftrightarrow\) quotients left/right module coalgebras

All our results are true in both cases. We switch to the convention of H.-J. Schneider and M. Takeuchi.

**Definition 7.2.** We let \(\text{Sub}_{\text{gen}}(H)\) denote the poset of right coideals subalgebras of a Hopf algebra \(H\).

This poset has all infima and thus it has a unique structure of a complete lattice.

**Proposition 7.3** (M. Takeuchi [19, Proposition 1]). Let \(H\) be a Hopf algebra and \(I\) its left ideal coideal. Then \(\text{co}^H/I\ H\) is a right coideal subalgebra of \(H\). Let \(K\) be a right coideal subalgebra of \(H\). Then \(HK^+\) is a left ideal coideal of \(H\).

**Proof.** Fix \(I\) a left ideal coideal and \(h, k \in \text{co}^H/I\ H\). Let the canonical projection map \(H \to H/I\) be denoted by \(\pi\). Then

\[
\delta(hk) = \pi(h(1)k(1)) \otimes h(2)k(2)
\]

\[
= h(1)\pi(k(1)) \otimes h(2)k(2)
\]

\[
= h(1)\pi(1) \otimes h(1)k
\]

\[
= \pi(h(1)) \otimes h(2)k
\]

\[
= \pi(1) \otimes hk
\]

thus coinvariants form a subalgebra, the rest is straightforward. \(\square\)

**Example 7.4.** Let \(H\) be a Hopf algebra (possibly infinite dimensional) and let \(K\) be its right coideal subalgebra. Then \(\text{co}^H/HK^+\ H \subseteq H\) is \(H/HK^+\)-Galois. The inverse of

\[
\text{can} : H \otimes_{\text{co}^H/HK^+} H \to H/HK^+ \otimes H, \quad x \otimes y \mapsto \pi(1) \otimes x(1) \otimes x(2)y
\]

is given by: \(\text{can}^{-1}(\pi \otimes y) = x(1) \otimes S(x(2))y\), where \(\pi\) is the class of \(x\) in \(H/HK^+\). This map is well defined since \(K\) is a subcoalgebra: if \(x = hk\), where \(k \in K^+\) and \(h \in H\), then

\[
\text{can}^{-1}(hk \otimes y) = h(1)k(1) \otimes S(k(2))S(h(2))y
\]

\[
= h(1) \otimes k(1)S(k(2))S(h(2))y
\]

\[
= h(1) \otimes \epsilon(k)S(h(2))y = 0\quad \text{since } K \subseteq \text{co}^H/HK^+\ H
\]

where the tensor on the left hand side is over \(\text{co}^H/HK^+\ H\). Let us show that in fact \(K \subseteq \text{co}^H/HK^+\ H\):

\[
\delta(k) = \pi(1) \otimes k(2) = \pi(1) \otimes \epsilon(k(1)) \otimes k(2) = \pi \otimes 1
\]

The second equality holds since \(k(1) \in K\) and the two maps \(K \otimes \epsilon(k)\ k \to H/HK^+\ H\) have the same kernels and thus are equal. Moreover, \(K = \text{co}^H/HK^+\ H\) in the following two cases:
1. $H$ is finite dimensional, for the proof we refer to [3],

2. $H$ is left or right faithfully flat over $K$:

\[
\begin{array}{c}
K \subseteq H \\
i \\
\end{array}
\xrightarrow{\text{id}_H \otimes 1_H}
\xrightarrow{1_H \otimes \text{id}_H}
H \otimes K
\]

$H \otimes K$ is an equaliser. Furthermore, there is a morphism of exact sequences:

\[
\begin{array}{c}
K \\
i \\
\end{array}
\xrightarrow{\text{id} \otimes 1_H}
\xrightarrow{1_H \otimes \text{id}}
\xrightarrow{\text{can}}
H \otimes_K H
\]

The map \text{can} is an isomorphism and thus the inclusion $i : K \subseteq \co H/K^+H$ is onto.

3. Moreover, in these two cases it is known that if $K$ is a normal Hopf subalgebra then $HK^+ = K^+H$ is a normal Hopf ideal, and if $I$ is a normal Hopf ideal then $\co H/IH = H^{\co H/1}$ is a normal Hopf subalgebra what was originally proved by H.-J. Schneider in [15, Lemma 1.3] to show (13).

In the second case flatness is necessary as it shows the following example:

**Example 7.5.** Let $F_1$ be the free group with one generator $g$ and $M_1$ its free submonoid generated by $g$. Then the Hopf algebra $H = k[F_1]$ and its coideal (thus right ideal) subalgebra $K = k[M_1]$ are such that $HK^+ = HH^+$ and $k[F_1]$ is not flat over $k[M_1]$:

$k[Z_n] \otimes_{k[M_1]} k[F_1] = 0$, where $Z_n$ is the group of integers modulo $n$.

The two cases of the preceding theorem lead us to two new results which positively answers the question raised by S. Montgomery in her book [9]: is (12) a bijection without extra assumptions?

**Theorem 7.6.** Let $H$ be a Hopf algebra over a field $k$. Then $k \subseteq H$ is an $H$-Hopf-Galois extension and there exists a Galois connection:

\[
\begin{array}{c}
\{ K \subseteq H : K \text{-right coideal subalgebra} \} \\
\phi \\
\{ H/I : I \text{-left ideal coideal} \} \\
\end{array}
\xleftrightarrow{\psi}
\begin{array}{c}
\{ H/I : I \text{-left ideal coideal} \} \\
\phi \\
\{ K \subseteq H : K \text{-normal Hopf subalgebra} \} \\
\end{array}
=: \text{Sub}_{\text{gen}}(H)
=: \text{Quot}_{\text{gen}}(H)
\]

where $(\phi(Q) = \co Q H, \psi(K) = H/HK^+)$ is the Galois connection obtained in Theorem 3.5. Moreover, this Galois correspondence restricts to normal elements:

\[
\begin{array}{c}
\{ K \subseteq H : K \text{-normal Hopf subalgebra} \} \\
\phi \\
\{ H/I : I \text{-normal Hopf ideal} \} \\
\end{array}
\xleftrightarrow{\psi}
\begin{array}{c}
\{ H/I : I \text{-normal Hopf ideal} \} \\
\phi \\
\{ K \subseteq H \} \\
\end{array}
=: \text{Sub}_{\text{Hopf}}(H)
=: \text{Quot}_{\text{normal}}(H)
\]

\[\text{Here we changed one side of the Galois connection comparing to Theorem 3.5 but it doesn't make a difference.}\]
We claim that:

(1) \( K \in \text{Sub}_{\text{gen}}(H) \) such that \( H \) is faithfully flat over \( K \) is a closed element of the Galois connection (15).

(2) \( Q \in \text{Quot}_{\text{gen}}(H) \) such that \( H \) is faithfully coflat over \( Q \) is a closed element of the Galois connection (15).

(3) if \( k \) is a field and \( H \) is finite dimensional then \( \phi \) and \( \psi \) are inverse bijections.

Point (1) gives an alternative proof of the M. Takeuchi’s part of Theorem 7.1 observing that \( H \) is faithfully flat over a right coideal subalgebra \( K \) if and only if \( H \) is faithfully coflat over \( H/HK^+ \) (for Hopf algebra \( H \) over a field \( k \)). We refer to [13, Proposition 4.5] for the proof of this observation. Point (3) is the recent S. Skryabin result [16, Corollary 6.5]. However, our method of proof is different as it is based on the presented Galois theory. We will also see that the presented method will shed more light on the general case of S. Montgomery question.

Proof. In Proposition 7.3 we proved that both maps \( \phi \) and \( \psi \) are well defined. Equation (14) shows that \( K \subseteq \phi\psi(K) = \co H/HK^+H \) thus to show that \( (\phi, \psi) \) is a Galois connection it remains to prove that the following inequality holds: \( H/I \leq \psi\phi(H/I) = H/H(\co H/H)^+ \), i.e. \( I \supseteq H(\co H/H)^+ \) as \( I \) is a left ideal it is enough to show that \( I \supseteq (\co H/H)^+ \). Let \( x \in (\co H/H)^+ \) then

\[
\Delta(x) = x_{(1)} \otimes x_{(2)} = 1 \otimes x + \sum_k i_k \otimes x_k
\]

\[
\Delta(x) = x_{(1)} \otimes x_{(2)} = 1 \otimes x + \sum_k i_k \otimes x_k
\]

where \( i_k \in I, \; x_k \in H, \; k = 1, \ldots, n \), thus

\[
x = 1\epsilon(x) + \sum_k i_k\epsilon(x_k)
\]

\[
= \sum_k i_k\epsilon(x_k) \in I \quad \text{since } x \in \ker \epsilon
\]

This Galois connection is the same as (8) in Theorem 3.5, because of the uniqueness of Galois maps (Proposition 1.3 (3)). The minor difference is the codomain of \( \phi \): here it is \( \text{Sub}_{\text{gen}}(H) \) rather than \( \text{Sub}_{\text{alg}}(k \subseteq H) \) as in Theorem 3.5 according to the case \( A = H \). The map \( \phi \) restricts to normal elements as shown by H.-J. Schneider in [15]. In the case of finite dimensional Hopf algebra with bijective antipode every \( Q \)-extension: \( H^{\co Q} \subseteq H \) is \( Q \)-Galois (Proposition 4.4). From Proposition 4.3 we get that \( \phi \) is a monomorphism and so by the properties of Galois connections:

\[
\psi \circ \phi(Q) = H/H(\co Q H)^+ = Q
\]
Moreover, in any of the two cases: $H$ is finite dimensional or is faithfully flat over $K$, we have
\[ \phi \circ \psi (K) = \co H / H K^+ H = K \]
Thus in fact $K$ is closed element of the Galois connection. Point (2) follows from Theorem 7.1.

The question of S. Montgomery in [9]: exists there a bijection between normal elements without extra assumptions? has a positive answer in the case when $H$ is finite dimensional. Furthermore, the following proposition holds:

**Proposition 7.7.** For a Hopf algebra $H$ such that for every its generalised quotient $Q$ the extension $\co Q H \subseteq H$ is $Q$-Galois. Then there is a positive answer to the question of S. Montgomery:

\[ \{ K \subseteq H : K - \text{right coideal subalgebra} \} \xrightarrow{\sim} \{ H / I : I - \text{left ideal coideal} \} \]
if and only if $\co H / K H^+ H \subseteq K$ for every right coideal subalgebra $K$ of $H$.

**Proof.** The pair of maps:
\[ \begin{array}{ccc}
\text{Sub}_{\text{gen}} (H) & \xrightarrow{\phi} & \text{Quot}_{\text{gen}} (H) \\
\psi & \xrightarrow{\psi} & \text{Sub}_{\text{gen}} (H) \\
K & \xrightarrow{\phi} & H / H K^+ \\
\co Q H & \xleftarrow{\phi} & Q
\end{array} \]
is a Galois connection. As we proved in Proposition 4.1, under the given assumptions, $\phi$ is a monomorphism and thus by the Galois property $\psi \phi = id$. In the presence of a Galois correspondence, the equality $\phi \psi = id$ is equivalent to the inclusion $\co H / K H^+ H \subseteq K$.

In this setting we can prove inverse of Proposition 4.3, and thus obtain a full characterisation of closed elements of $\text{Quot}_{\text{gen}} (H)$.

**Proposition 7.8.** Let $H$ be a Hopf algebra. Then $Q \in \text{Quot}_{\text{gen}} (H)$ is a closed element of the Galois connection (15) if and only if $H / \co Q H$ is a Hopf-Galois extension.

**Proof.** It remains to prove that if $Q$ is a closed element then $\co Q H \subseteq H$ is an $H$-Hopf-Galois. If $Q$ is closed then $Q = H / H (\co Q H)^+$. One can show that for any $K \in \text{Sub}_{\text{gen}} (H)$ the following map is an isomorphism:
\[ H \otimes K \to H / H K^+ \otimes H, \quad h \otimes k \mapsto h_{(1)} \otimes h_{(2)} k \]
(16)

Its inverse is given by $H / H K^+ \otimes H \ni h_{(1)} \otimes k \mapsto h_{(1)} \otimes S (h_{(2)}) k \in H \otimes K H$ which is well defined because $\Delta (K) \subseteq K \otimes H$. Plugging $K = \co Q H$ to equation (16) we observe that this map is then nothing else than the canonical map of $Q$. Thus $H / \co Q H$ is a $Q$-Galois extension.\Box
S. Skryabin [16, Corollary 6.5] shows that the correspondence (15) is bijective for any finite dimensional Hopf algebra $H$, not assuming bijectivity of the antipode. Thus we get the following interesting corollary:

**Corollary 7.9.** Let $H$ be a finite Hopf algebra. Then every $Q \in \text{Quot}_{\text{gen}}(H)$ is $Q$-Galois.

## 8 Cleft extensions

Here we give definition of the normal basis property for $H$-extensions.

**Definition 8.1.** Let $B \subseteq A$ be an $H$-extension then it has the normal basis property if and only if $A \simeq B \otimes_k H$ as left $B$-modules and right $H$-comodules.

S. Montgomery in [9, Example 8.2.2] shows that a classical extension of fields has the classical normal basis property if and only if it has the above property. The crucial characterisation of cleft extension due to Takeuchi is as follows:

**Theorem 8.2.** Let $B \subseteq A$ be an $H$-extension. Then it is cleft if and only if it is a Hopf-Galois extension and it has the normal basis property.

We let $f$ denote the isomorphism $B \otimes H \simeq A$. The main theorem of this section Theorem 7.6 together and Proposition 7.8 yields the following result. Note that we do not need the assumption that the antipode of $H$ is bijective.

**Theorem 8.3.** Fix an $H$-cleft extension $A/B$. Then we have the Galois connection of Main Theorem:

$$\text{Sub}_{\text{alg}}(A/B) \leftrightarrow \text{Quot}_{\text{gen}}(H)$$

An element $Q$ of $\text{Quot}_{\text{gen}}(H)$ is closed if and only if the extension $A/A^{coQ}$ is $Q$-Galois. When $H$ is finite dimensional then there is a bijective correspondence:

$$\text{Sub}_{H-\text{ext}}(A) \simeq \text{Quot}_{\text{gen}}(H)$$

The normality condition can be added to both sides as it is done in Theorem 7.6. The above bijection is derived from the Galois correspondence (15).

## 9 BiGalois extensions

In [10] there is constructed a Hopf algebra $L(A, H)$ for a given $k \subseteq A$ $H$-Hopf-Galois extension with the property that $k \subseteq A$ is $(L(A, H), H)$-biGalois extension. This construction extends the one given by F. van Oystaeyen and Y. Zhang in [20] to the non-commutative case. The Hopf algebra $L(A, H)$ is unique up to isomorphism, and can be given explicitly by $L(A, H) = (A \otimes A)^{coH}$ under the codiagonal coaction of $H$ on $A \otimes A$. We only include here relevant parts of the whole Galois theory based on this additional Hopf algebra. For closer acknowledgement we refer the reader to the papers of P. Schauenburg [10], [11], [12] and also the work of F. van Oystaeyen and Y. Zhang [20].

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Definition 9.1. We call $A/k$ a $L$-$H$-biGalois extension if $A$ is left $L$ comodule algebra, $A/k$ is a left $L$-Hopf-Galois extension and $A$ is a right $H$-comodule algebra such that $A/k$ is a right $H$-Hopf-Galois extension. Moreover, $A$ is supposed to be $L$-$H$-bicomodule so that both coactions commute.

Let us define notions which plays an important role in a Galois connection between $\text{Quot}_{\text{gen}}(L)$ and $\text{Sub}_{\text{alg}}^H(A)$ – the complete lattice of $H$-subcomodule algebras of $A$.

Definition 9.2. Fix a coalgebra $C$ and its quotient coalgebra $C/I$ where $I$ is an coideal of $C$. We call $C/I$ right (left) admissible if it is $k$ flat (thus faithfully flat) and $C$ is right (left) faithfully coflat over $C/I$. We call a coideal $I$ of $C$ right (left) admissible if $C/I$ is. A quotient of a bialgebra or a Hopf algebra is admissible if it is admissible as quotient coalgebra.

A subalgebra $B$ of $A$ is right (left) admissible if $A$ is faithfully flat over $B$ as right (left) module. In both cases admissible will mean left and right admissible.

Here we quote [11, Proposition 3.2 and Theorem 3.6]:

Proposition 9.3. Let $A/k$ be a faithfully flat $L(H, A)$-$H$-biGalois extension of a ring $k$. Then there exists a Galois connection:

$$\text{Quot}_{\text{gen}}(L) \overset{\mathcal{F}}{\longrightarrow} \text{Sub}_{\text{alg}}^H(A)$$

such that $\mathcal{F}(L/I) = \co^{L/I} A$ and $I(B) = (A \otimes_B A)^{coH}$. If in addition the antipodes of $H$ and $L(H, A)$ are bijective then admissible objects are closed elements of the Galois connection $(\mathcal{F}, I)$. The bijection between closed objects restricts to the admissible objects.

It is shown in [10, Corollary 3.6] that the antipode of $L(A, H)$ is bijective if the antipode of $H$ is bijective and $A/k$ is faithfully flat.

Let us denote by $A^{op}$ the opposite algebra to an algebra $A$ (the underlying vector space of $A^{op}$ is the same as $A$ but the multiplication is precomposed with the flip of tensor factors subsequently denoted by $\tau$). An opposite bialgebra $B^{op}$ has opposite multiplication and comultiplication (i.e. $\Delta^{op} := \tau \circ \Delta, \mu^{op} = \mu \circ \tau$). The opposite bialgebra of a Hopf algebra is a Hopf algebra if and only if the antipode is bijective. Then $S^{op} = S^{-1}$.

Theorem 9.4. Let $H$ be a Hopf algebra over a field $k$ with bijective antipode and let $A$ be a faithfully flat $H$-Hopf-Galois extension of $A^{coH}$. Then the map of the Galois connection $(1) \phi : \text{Quot}_{\text{gen}}(H) \longrightarrow \text{Sub}_{\text{alg}}(A/A^{coH}) Q \mapsto A^{coQ}$ is injective on the set of (right, left) admissible quotients of $H$ and its image is in the set of (right, left) admissible subalgebras of $A$. 

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Proof. If $Q$ is left admissible ($H$ is faithfully coflat as left $Q$-comodule) then by [14, Remark 1.4 (2)] $A$ is a $Q$-Galois extension. Moreover, $A$ is faithfully flat as a left $A^{coQ}$-module and thus $\phi(Q) = A^{coQ}$ is left admissible. Now if $Q$ is right admissible then $Q^{op}$ is left admissible for $H^{op}$ ($A^{op}$ is a left $H^{op}$-Galois extension) and by the same reasoning as in [14, Theorem 1.4 (2)] we get that $A^{op}$ is a left faithfully flat $Q^{op}$-Galois extension. Then $A$ is a right faithfully flat $Q$-Galois extension. Thus $A^{coQ}$ is right admissible. By Proposition 4.1 the map $\phi$ is injective on the set of (right, left) admissible quotients of $H$.

\[\square\]

Corollary 9.5. Let $k \subseteq A$ be a $L(H, A)$-$H$-biGalois extension of a field $k$, where $H$ is a Hopf algebra with bijective antipode. The last two results show:

\[
\begin{align*}
\text{(left, right) admissible} & \quad \approx \quad \text{(left, right) admissible} \\
\text{quotients of } L(H, A) & \quad \approx \quad \text{H-comodule subalgebras of } A \\
\cap & \quad \cap \\
\text{(left, right) admissible} & \quad \approx \quad \text{(left, right) admissible} \\
\text{quotients of } H & \quad \approx \quad \text{subalgebras of } A
\end{align*}
\]

Moreover, admissible quotients of $L(H, A)$ and $H$ are in one-to-one correspondence [11, Corollary 3.13] so that admissible quotients of $L(H, A)$ and $H$ classify the same subextensions of $A$.

Acknowledgements

We would like to thank T. Brzeziński for reading our manuscript and giving his comments and advices.

References

[1] Roland Carl Backhouse, Roy L. Crole, and Jeremy Gibbons, editors. Algebraic and coalgebraic methods in the mathematics of program construction, volume 2297 of Lecture Notes in Computer Science. Springer-Verlag, Berlin, 2002. Lectures from the International Summer School and Workshop held at the University of Oxford, Oxford, April 10–14, 2000, Edited by Roland Backhouse, Roy Crole and Jeremy Gibbons.

[2] J. Cuadra and J. Gómez-Torrecillas. Galois corings and a Jacobson-Bourbaki type correspondence. J. Algebra, 308(1):178–198, 2007.

[3] Sorin Dăscălescu, Constantin Năstăsescu, and Şerban Raianu. Hopf algebras, volume 235 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc., New York, 2001. An introduction.
[4] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris, 1970. Avec un appendice it Corps de classes local par Michiel Hazewinkel.

[5] George Grätzer. *General lattice theory*. Birkhäuser Verlag, Basel, second edition, 1998. New appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung and R. Wille.

[6] Richard Gustavus Larson and Moss Eisenberg Sweedler. An associative orthogonal bilinear form for Hopf algebras. *Amer. J. Math.*, 91:75–94, 1969.

[7] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[8] Tomasz Maszczyk. Galois Structures. Lecture notes available at www.toknotes.mimuw.edu.pl, 2007.

[9] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[10] Peter Schauenburg. Hopf bi-Galois extensions. *Comm. Algebra*, 24(12):3797–3825, 1996.

[11] Peter Schauenburg. Galois correspondences for Hopf bi-Galois extensions. *J. Algebra*, 201(1):53–70, 1998.

[12] Peter Schauenburg. Hopf-Galois and bi-Galois extensions. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 469–515. Amer. Math. Soc., Providence, RI, 2004.

[13] Peter Schauenburg and Hans-Jürgen Schneider. On generalized Hopf Galois extensions. *J. Pure Appl. Algebra*, 202(1-3):168–194, 2005.

[14] Hans-Jürgen Schneider. Normal basis and transitivity of crossed products for Hopf algebras. *J. Algebra*, 152(2):289–312, 1992.

[15] Hans-Jürgen Schneider. Some remarks on exact sequences of quantum groups. *Comm. Algebra*, 21(9):3337–3357, 1993.

[16] Serge Skryabin. Projectivity and freeness over comodule algebras. *Trans. Amer. Math. Soc.*, 359(6):2597–2623 (electronic), 2007.

[17] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.
[18] Moss E. Sweedler. The predual theorem to the Jacobson-Bourbaki theorem. 
*Trans. Amer. Math. Soc.*, 213:391–406, 1975.

[19] Mitsuhiro Takeuchi. Relative Hopf modules — equivalences and freeness 
criteria. *J. Algebra*, 60(2):452–471, 1979.

[20] F. Van Oystaeyen and Y. Zhang. Galois-type correspondences for 
Hopf-Galois extensions. *K-Theory*, 8(3):257–269, 1994.