The weak Hilbert-Smith conjecture from a Borsuk-Ulam-type conjecture

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Abstract

We prove a number of results surrounding the Borsuk-Ulam-type conjecture of Baum, Dąbrowski and Hajac (BDH, for short), to the effect that given a free action of a compact group $G$ on a compact space $X$, there are no $G$-equivariant maps $X \ast G \to X$ (with $\ast$ denoting the topological join). In particular, we prove the BDH conjecture for locally trivial principal $G$–bundles. The proof relies on the non-existence of $G$-equivariant maps $G^{\ast(n+1)} \to G^{\ast n}$, which in turn is a slight strengthening of an unpublished result of M. Bestvina and R. Edwards. Moreover, we show that the BDH conjecture partially settles a conjecture of Ageev. In turn, the latter implies the weak version Hilbert-Smith conjecture stating that no infinite compact zero-dimensional group can act freely on a manifold such that the orbit space is finite-dimensional.

Key words: Hilbert-Smith conjecture, Ageev conjecture, Borsuk-Ulam theorem, $p$-adic integers, free action, Menger compactum, dimension

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Introduction

The fifth question on the Hilbert’s famous list of problems [15] concerned a characterization of Lie groups. In a modern and commonly accepted form [37, Theorem 2.7.1] the result reads as follows.

Theorem 0.1 [Hilbert’s fifth problem] Let $G$ be a topological group which is locally Euclidean. Then $G$ is isomorphic to a Lie group.

The above theorem was proved by Gleason [14, Theorem 3.1] and by Montgomery and Zippin [27, Theorem A] in 1952. The theorem and the mathematical tools developed in the course of proving it have many applications, including the celebrated Gromov theorem on groups of polynomial growth (see [37] for a survey of applications).

It is worth noting that in Hilbert’s lecture [15] groups were not treated abstractly, but rather as groups of transformations. Hence, one might argue that the following generalization of Theorem 0.1 in fact better captures the original intent behind Hilbert’s problem (see e.g. [37, Conjecture 2.7.2]).

Conjecture 0.2 [Hilbert-Smith conjecture] Let $G$ be a locally compact topological group acting continuously and effectively on a connected finite-dimensional topological manifold $M$. Then, $G$ is isomorphic to a Lie group.

It is known that the Hilbert-Smith conjecture can be reduced to studying actions of the groups that are very far from being Lie, namely the groups $\mathbb{Z}_p$ of $p$-adic integers for arbitrary primes $p$. This was proved by Lee [21, Theorem 3.1] using the Gleason-Yamabe theorem on the structure of locally compact groups [40, Theorem 5'], and some results of Newman [30]. We state the reduced conjecture following [37, Conjecture 2.7.3].
Conjecture 0.3 [Hilbert-Smith conjecture for p-adic actions] It is not possible for a p-adic group $\mathbb{Z}_p$ to act continuously and effectively on a connected finite-dimensional topological manifold $M$.

Let us recall some results that partially confirm the Hilbert-Smith conjecture. In [28] Montgomery and Zippin observed that Conjecture 0.2 is true for transitive actions on topological manifolds and for smooth actions on smooth manifolds. Repovš and Ščepin [34] proved it for Lipschitz actions. Then Martin announced the proof for quasiconformal actions on Riemannian manifolds in [23]. There is also a proof for Hölder actions given by Maleshich [22]. Recently, Pardon showed that Conjecture 0.2 is true for three-manifolds [32]. In the survey [12] Dranishnikov gives an account of various partial results and reduces a weaker version of the conjecture to two other problems. In its full generality, the Hilbert-Smith conjecture remains unsolved.

We base our approach to the Hilbert-Smith conjecture on classifying spaces for p-adic groups and the theory of Menger compacta [26, 7]. In [11] Dranishnikov shows that any zero-dimensional compact metric group (e.g. $\mathbb{Z}_p$) can act freely on universal Menger compacta. Such actions can be fairly exotic, e.g. for the $n$-dimensional Menger compactum $\mu^n$ the dimension of the orbit space $\mu^n/\mathbb{Z}_p$ can exceed $n$.

Nevertheless, there is a class of actions for which $\dim(\mu^n/\mathbb{Z}_p) = n$; these are all proven isomorphic (for given $G$, in the category of $G$-spaces) by Ageev in [2]. The results of the latter paper also suggest that the actions in question satisfy a universality condition to the effect that for any free action of $G$ on an at-most-$n$-dimensional compact space $Y$, every equivariant map $Y \to \mu^n$ is approximable by an equivariant embedding. This is referred to as ‘strong $G$-$n$ universality’ in the discussion preceding [2, Theorem B], and is an equivariant version of the celebrated Bestvina recognition criterion for Menger compacta ([7] and [2, Theorem A (b)]).

In [1] Ageev states a certain conjecture about Menger compacta which if true would imply a weaker version of the Hilbert-Smith conjecture for free actions on manifolds with finite-dimensional orbit spaces. The following statement appears as [1, Conjecture].

Conjecture 0.4 [Ageev] Suppose two Menger compacta $\mu^m$ and $\mu^n$ with $m > n$ are acted on freely by a non-trivial zero-dimensional compact metric group $G$. Then, there is no equivariant map $f : \mu^m \to \mu^n$.

These ideas were followed by Yang in [42], where a free Menger $\mathbb{Z}_p$-compactum is explicitly constructed.

In this paper we focus on the the generalized compact-Hausdorff-space Borsuk-Ulam conjecture proposed by Baum, Dąbrowski and Hajac [6, Conjecture 2.2]. In order to state it, recall first that for two spaces $X$ and $Y$, their topological join $X \ast Y$ is the space defined by

$$X \ast Y = X \times Y \times [0,1]/\sim,$$

where the equivalence relation identifies all $(x, y_0, 0), x \in X$ for fixed $y_0 \in Y$, and similarly identifies all $(x_0, y, 1), y \in Y$ for fixed $x_0 \in X$.

Conjecture 0.5 Let $X$ be a compact Hausdorff space equipped with a continuous free action of a non-trivial compact Hausdorff group $G$. Then, for the diagonal action of $G$ on the topological join $X \ast G$, there does not exist a $G$-equivariant continuous map $f : X \ast G \to X$.

Here we show in Section 2 that the positive answer to Conjecture 0.5 would prove Conjecture 0.4 as well. This result would then be sufficient to resolve affirmatively the finite-dimensional-orbit-space version of the Hilbert-Smith conjecture for free actions in Corollary 2.15.
The proposed reformulation of the problem is justified by recent work on and interest in Borsuk-Ulam-type theorems. Conjecture 0.5 is known to be true for actions of compact Hausdorff groups with torsion elements on compact Hausdorff spaces, which can be deduced from [25, Theorem 6.2.6] or [38, Corrolary 3.1], and was recently proved using different methods by Passer in [33, Theorem 2.8]. In Theorem 1.3 in Section 1, we obtain a new Borsuk-Ulam-type result, namely for actions of a compact Hausdorff group $G$ on a compact Hausdorff space $X$ such that $X \to X/G$ is a locally trivial bundle. We prove it by strengthening a theorem of M. Bestvina and R. Edwards (unpublished). Note that Theorem 1.3 includes previous Borsuk-Ulam-type results since free actions of finite groups automatically satisfy the locally trivial assumption. Furthermore, in [10, Section 3] Chirvasitu and Passer propose a possible approach to Conjecture 0.5 in its full generality.

1 Borsuk-Ulam-type theorem for locally trivial $G$-bundles

The aim of the present Section is to outline a proof of the following Borsuk-Ulam-type result:

**Theorem 1.1** Let $G$ be a non-trivial compact Hausdorff group, and for positive integers $n$ denote $E_n G := G^{*(n+1)}$. Then, there are no $G$-equivariant maps $E_{n+1} G \to E_n G$.

To the best of our knowledge, the above theorem was proved for zero-dimensional compact groups by M. Bestvina and R. Edwards. There is an immediate corollary:

**Corollary 1.2** Let $G$ be a compact Hausdorff group and for a $G$-space $X$ define

$$\text{ind}_G(X) := \min\{ n : \exists G\text{-map } X \to E_n G \}. \quad (1.1)$$

Then, $\text{ind}_G(E_n G) = n$.

The finiteness of $\text{ind}_G(X)$ is equivalent to the condition that $X \to X/G$ is a locally trivial principal $G$-bundle. Hence Theorem 1.1 is equivalent to the following (see Proposition 1.6 below):

**Theorem 1.3** Let $G$ be a non-trivial compact Hausdorff group and let $X$ be a compact Hausdorff $G$-space such that $X \to X/G$ is a locally trivial principal $G$-bundle. Then there is no $G$-equivariant map $X \ast G \to X$.

**Remark 1.4** The case when $G$ has non-trivial elements of finite order is known, e.g., see [38, 33]. We also note that Theorem 1.1 in the case when $G$ is a finite group can be found in [25, Theorem 6.2.5].

To prove that Theorem 1.1 and Theorem 1.3 are equivalent, we will need the following remark regarding the trivializability of the principal $G$-bundle

$$E_n G \to E_n G/G =: B_n G.$$ 

Note that the classifying space for $G$ is the direct limit of $B_n G$’s. We slightly abuse notation and denote the coordinates on $E_n G$ by the convex combinations

$$t_0 g_0 + \cdots + t_n g_n, \quad g_i \in G, \quad t_i \in [0,1], \quad \sum t_i = 1,$$

and the action of $G$ on it is diagonal. $B_n G$ thus consists of analogous convex combinations, considered up to simultaneous translation of the $g_i$ on the right:

$$t_0 g_0 + \cdots + t_n g_n = t_0 g_0 g + \cdots + t_n g_n g \text{ in } B_n G, \quad \forall g_i, g \in G.$$
Remark 1.5 The bundle $E_n G \to B_n G$ can be trivialized by the cover of $B_n G$ by the $n + 1$ open sets $U_i$, $0 \leq i \leq n$, consisting respectively of the classes of combinations

$$t_0 g_0 + \cdots + t_n g_n$$

for which $t_i \neq 0$. See e.g. the proof of [19, Theorem 4.11.2].

Proposition 1.6 Theorem 1.1 and Theorem 1.3 are equivalent.

Proof Remark 1.5 indicates that $E_n G$ is a locally trivial principal $G$-bundle, hence Theorem 1.3 implies Theorem 1.1. Conversely, if $X$ is a locally trivial principal $G$-bundle, then it has finite $G$-index. Suppose that $\text{ind}_G(X) = n$ and that there exists a $G$-map $X \ast G \to X$. We have the following chain of $G$-maps

$$E_{n+1} G \hookrightarrow X \ast G \ast \cdots \ast G \to \cdots \to X \ast G \ast G \to X \ast G \to X \to E_n G,$$

which contradicts Theorem 1.1.

Before going into the proof of Theorem 1.1, we need some preparation. First, recall the following notion of size for a topological space.

Definition 1.7 The Lusternik-Schnirelmann category of a topological space $X$, denoted $\text{LS}(X)$, is the minimal number $n$, for which there exists an open covering of $X$ consisting of sets $U_i$, $i = 0, 1, 2, \ldots, n$, such that, for every $i$, the embedding $U_i \to X$ is nullhomotopic.

The concept enters the present discussion via the following observation (whose routine proof we omit):

Proposition 1.8 The Lusternik-Schnirelmann category of the $n$-dimensional torus $\mathbb{T}^n$ is $n$. [24, §II.2]

One ingredient in the proof of Theorem 1.1 will be the existence of locally trivial principal $G$-bundles over $\mathbb{T}^n$ that are non-trivial over non-trivial loops. We will thus need to prove the existence of such bundles. By a non-trivial loop on a topological space $X$ we mean a non-nullhomotopic continuous map $\alpha : [0, 1] \to X$ such that $\alpha(0) = \alpha(1)$.

All of our topological spaces will be Hausdorff and locally path connected, to ensure a well-behaved theory of covering spaces (see e.g. [24, §II.2], where the notion is referred to as local arwise or pathwise connectivity). Let $X$ be a connected space, $G$ a compact group, and $\Gamma$ a discrete group acting on $X$ (from the left) freely and properly discontinuously, i.e. such that every point has a neighborhood $U$ with

$$\gamma U \cap U = \emptyset, \quad \forall \gamma \in \Gamma, \gamma \neq 1,$$

([24, p. 136]).

We also fix a morphism $\phi : \Gamma \to G$, and regard $\Gamma$ as acting on $G$ on the left by translations via $\phi$. Note that $\Gamma$ acts properly discontinuously on $X \times G$ as well: for any $U \subset X$ satisfying (1.2), so does $U \times G$. Moreover, the right action of $G$ on the right-most component of $X \times G$ commutes with the left $\Gamma$-action and hence descends to an action on

$$E := \Gamma \backslash (X \times G).$$

It is not difficult to check that we have
Lemma 1.9: In the setting above, $E$ is a locally trivial principal $G$-bundle over $Y := \Gamma \backslash X$.

**Proof** Let $y \in Y$, choose a point $x \in \pi^{-1}(y)$ where

$$\pi : X \to \Gamma \backslash X = Y$$

is the projection, and let $x \in U \subset X$ be an open subset satisfying (1.2). Then, $V = \pi(U)$ is an open neighborhood of $y$ and condition (1.2) ensures that the restriction of $E \to Y$ to $E|_V = V$ is isomorphic to $V \times G \to V$.

Now cover $Y$ with open sets $V_i$ of the form $\pi(U)$ for open $U \subset X$ satisfying (1.2). $\pi : X \to Y$ is a covering map and the opens $V_i$ are chosen so that the preimage $\pi^{-1}(V_i)$ is the disjoint union of open subsets $U_{i\ell} \subset X$ mapped onto $V_i$ homeomorphically by $\pi$. Identify each restriction $E|_{V_i}$ with the trivial principal $G$-bundle $U_{i\ell} \times G$ via $\pi$ for one of the $U_{i\ell}$'s:

$$h_i : U_{i\ell} \times G \cong V_i \times G \to E|_{V_i}.$$ 

This is an atlas for the bundle in the sense of [19], Chapter 5, Definition 2.1, and the choice of the atlas makes it clear that the resulting transition functions $g_{ij} : V_i \cap V_j \to G$ defined by

$$(h_i^{-1} \circ h_j)(y, g) = (y, g_{ij}g)$$

actually factor through $\phi : \Gamma \to G$. In other words, the structure group of the fiber bundle $E \to Y$ can be reduced from $G$ to the discrete group $\Gamma$ via $\phi$. The bundle thus admits a flat connection, and the corresponding monodromy morphism

$$\pi_1(Y) \to \Gamma \to G$$

is nothing but the composition of $\phi : \Gamma \to G$ with the canonical morphism $\pi_1(Y) \to \Gamma$ resulting from the quotient map

$$\pi : X \to \Gamma \backslash X \cong Y.$$ 

Before continuing, we need more preparation. Following [16, Chapter 8, Part 2] we denote by $G_a$ the normal subgroup of the compact group $G$ consisting of elements that are path-connected to the identity. Note that in general $G_a$ is not closed, so we will disregard the quotient topology on $G_a = G/G_a$, considering it simply as an abstract group.

**Definition 1.10** A principal $G$-bundle over $Y$ is incompressible if its restriction to every non-nullhomotopic loop is non-trivial.

**Lemma 1.11** In the setting above, suppose $X$ is simply connected and $\phi : \Gamma \to G \to \pi_0(G)$ is a group monomorphism. Then, the bundle $E \to Y$ constructed above is incompressible.

**Proof** By [20, Theorem 1.1 (ii)], the equivalence class under inner automorphisms of $G$ of the homotopy class of the monodromy map is an invariant of bundles. In particular, for trivial bundles the map is homotopic to the trivial morphism.

On the other hand, under the hypotheses of the lemma the monodromy

$$m : \pi_1(Y) \cong \Gamma \to G$$

is...
cannot be homotoped to $1 \in G$ on any non-trivial element $s$ of $\pi_1(Y)$ because $m(s)$ belongs to $G/G_a$. In conclusion, the above paragraph implies that the restriction of $E$ to the circle via a map

$$\mathbb{S}^1 \to Y$$

sending the generator of $\pi_1(\mathbb{S}^1)$ to $s \neq 1$ in $\pi_1(Y)$ is non-trivial. ■

**Proposition 1.12** If $G$ is a non-trivial torsion-free compact group then, for every positive integer $n$, $G$ contains a copy of $\mathbb{Z}^n$ which embeds into $\pi_0(G)$ via

$$\mathbb{Z}^n \to G \to \pi_0(G).$$

**Proof** We do this in stages.

**Step 1:** $G$ itself contains $\mathbb{Z}^n$. The torsion freeness of $G$ ensures that we have an embedding of the additive group $\mathbb{Z}_p$ of $p$-adic integers into $G$; it thus suffices to argue that $\mathbb{Z}^n$ embeds into $\mathbb{Z}_p$ for every $n$. This, however, is clear: the quotient field

$$\mathbb{Q}_p = \mathbb{Z}_p \left[ \frac{1}{p} \right]$$

of $\mathbb{Z}_p$ is a field of characteristic zero and continuum cardinality, and hence is a vector space of dimension $2^{\aleph_0}$ over $\mathbb{Q}$. If the $p$-adic rationals $v_1$ up to $v_n$ are linearly independent over $\mathbb{Q}$, then for sufficiently large $k$ the elements $p^kv_i$ belong to $\mathbb{Z}_p$ and generate a group isomorphic to $\mathbb{Z}^n$.

**Step 2:** $G = \hat{\mathbb{Q}}$. Since $G$ is abelian and $\mathbb{Z}^n$ is projective in the category of abelian groups, any embedding $\mathbb{Z}^n \to \pi_0(G)$ lifts to an embedding $\mathbb{Z}^n \to G$ as in the statement. In conclusion, it suffices to argue that $\pi_0(G)$ contains a copy of $\mathbb{Z}^n$.

By [16, Theorem 8.30 (iii)], we have the following isomorphism

$$\pi_0(\hat{\mathbb{Q}}) \cong \text{Ext}(\mathbb{Q}, \mathbb{Z}).$$

We know that $\text{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{R}$ from [39] and, as $\mathbb{R}$ is a $\mathbb{Q}$-vector space of dimension $2^{\aleph_0}$, using a similar argument as in Step 1, the result follows.

**Step 3:** the full result. As seen in Step 1, we are done if $G$ is totally disconnected. If it is not, then the connected component $G_0$ of the identity is a non-trivial connected torsion-free compact group. Restricting our attention to it, we can assume $G$ itself is connected to begin with.

By [16, Theorem 9.24 (ii)], being compact and connected, $G$ is isomorphic to the quotient group $(Z_0(G) \times \prod S_j)/D$, where $Z_0(G)$ is the connected component of the center of $G$, all the $S_j$’s are simple simply connected Lie groups and $D$ is a totally disconnected central subgroup of the domain of the surjection

$$Z_0(G) \times \prod S_j \to G. \tag{1.3}$$

Moreover, (1.3) identifies the $Z_0(G)$ factor with its copy inside $G$.

Since torsion elements are dense in compact connected Lie groups and $G$ is assumed torsion-free, the factors $S_j$ all map to $G$ trivially. It follows that in fact (1.3) factors through $Z_0(G) \to G$. In other words, $G$ must be abelian, and we can work with its discrete Pontryagin dual $\hat{G}$.

The connectedness of $G$ now implies that $\hat{G}$ is torsion-free, while the fact that $G$ itself is torsion-free means that $\hat{G}$ is divisible. The only divisible torsion-free abelian groups are the vector spaces over $\mathbb{Q}$, i.e. direct sums $\mathbb{Q}^{\oplus S}$ for some set $S$. This in turn implies that

$$G \cong \hat{\mathbb{Q}}^{\oplus S}.$$ 

In particular $G$ is abelian, and the same argument as in Step 2 reduces the problem to showing that $\mathbb{Z}^n$ embeds in $\pi_0(G)$. This, however, follows from Step 2, since $\pi_0(\hat{\mathbb{Q}})$ embeds into $\pi_0(G)$ via one of the factor embeddings $\hat{\mathbb{Q}} \subset G$. ■
Corollary 1.13 If \( G \neq \{1\} \) is torsion-free and \( n \) is a positive integer, then there are incompressible \( G \)-bundles on the \( n \)-dimensional torus \( \mathbb{T}^n \).

Proof This follows from Lemma 1.11 applied to \( \Gamma = \mathbb{Z}^n \equiv \pi_1(\mathbb{T}^n) \).

We are now ready for the

Proof (of Theorem 1.1) As noted in Remark 1.4, the case when \( G \) has torsion is settled in a stronger form. For this reason, throughout the proof we assume that \( G \) is torsion-free; then, Proposition 1.12 and Corollary 1.13 apply.

Let \( E \to \mathbb{T}^n \) be a bundle as in Corollary 1.13. The space \( E_n G \) has vanishing homotopy groups \( \pi_i, 1 \leq i < n \), and hence the bundle \( E_n G \to B_n G \) is \( n \)-universal in the sense of [36, §19.2]: for any locally trivial principal \( G \)-bundle \( X \to Y \) with \( \dim(Y) \leq n \) there exists a map of \( G \)-bundles

\[
\begin{array}{ccc}
X & \longrightarrow & E_n G \\
\downarrow & & \downarrow \\
Y & \longrightarrow & B_n G
\end{array}
\]

Now suppose there exists a \( G \)-map \( E_n G \to E_{n-1} G \) and consider the following commutative diagram.

\[
\begin{array}{ccc}
E & \longrightarrow & E_n G \\
\downarrow & & \downarrow \\
\mathbb{T}^n & \longrightarrow & B_n G \\
\downarrow & & \downarrow \\
& \longrightarrow & B_{n-1} G
\end{array}
\]

Recall that by construction, the bundle \( E \to \mathbb{T}^n \) is nontrivial over every nontrivial loop. We know from Lemma 1.5 that \( B_{n-1} G \) can be covered with \( n \) open trivializing sets. Pulling those sets back to \( \mathbb{T}^n \), we obtain a open trivializing cover consisting of sets \( U_i, i = 0, 1, 2, \ldots, n-1 \). No \( U_i \) can contain a nontrivial loop, and hence the maps \( U_i \to \mathbb{T}^n \) are nullhomotopic.

Indeed, the non-existence of non-trivial loops in the connected components \( U_i \) of \( U_i \) ensures that the image of \( \pi_1(U_i) \to \pi_1(\mathbb{T}^n) \) is trivial. This then implies that the map \( U_i \to \mathbb{T}^n \) factors through the contractible universal cover \( \mathbb{R}^n \to \mathbb{T}^n \). The conclusion follows.

We now have a contradiction: trivializing \( E \to \mathbb{T}^n \) with \( n \) open sets \( U_i \) whose embeddings into \( \mathbb{T}^n \) are nullhomotopic contradicts Proposition 1.8.

Remark 1.14 We would like to remark that the core idea of the proof of Theorem 1.1 is due to M. Bestvina and R. Edwards, while Lemma 1.11 and Proposition 1.12 contain new results that enabled us to generalize the theorem to arbitrary compact Hausdorff groups.

2 Relation to the Hilbert-Smith conjecture

2.1 Preliminaries

From now on, all of our topological spaces are at the very least second countable and metrizable (or equivalently, separable and metrizable). We record this formally:

Convention 2.1 Henceforth, unless specified otherwise, we only consider separable metrizable topological spaces.
Compacta are compact Hausdorff metrizable topological spaces. We denote by $\mu^n$ the universal $n$-dimensional Menger compactum characterized abstractly in a number of ways in [7].

Since we have to work extensively with quotients of spaces by group actions, it will be necessary for said quotients to be metrizable (and separable) as well. Since all of our actions are proper and all of our topological groups are locally compact and separable (because they act freely on locally compact separable metric spaces) [4, Theorem B and Corollary 1] confirm that indeed $X/G$ is metrizable and separable; we take this for granted repeatedly below.

We use [18, 13] as our sources of background for the dimension theory of separable metrizable spaces. We will not recall any of the precise definitions of dimension here (henceforth denoted by $\dim$), and refer instead e.g. to [18, Definition III 1] or [13, Definitions 1.1.1, 1.6.1, 1.6.7] for the various notions of dimension (small or large inductive as well as covering dimension) and [13, Theorem 1.7.7] for verification that they all coincide for separable metric spaces.

Suffice it to say the concept captures the usual intuition, and specializes to the standard notion of dimension for manifolds. The universality of $\mu^n$ referred to above consists in the fact that it is $n$-dimensional, and contains a homeomorphic copy of every $n$-dimensional compactum.

Following [1, Definition 1], we recall

**Definition 2.2** Let $n$ be a non-negative integer and $G$ a topological group acting freely on a space $X$. Then, $X$ equipped with the action in question is free $n$-universal (or simply universal for short if $n$ is understood) if for any free $G$-space $Y$ with

$$\dim(Y/G) \leq n$$

and any closed $G$-invariant subspace $Z \subseteq Y$ every equivariant map $Z \to X$ admits an extension to an equivariant map $Y \to X$.

In Section 2 below we will make crucial use of the following result (see [1, Theorem 1]).

**Theorem 2.3** A Menger compactum $\mu^n$ equipped with a free action by a zero-dimensional compact group is free $n$-universal in the sense of Definition 2.2.

Finally, we will on occasion refer to ANR spaces (for absolute neighborhood retract). These are spaces $X$ with the property that whenever embedded homeomorphically $\iota : X \to Y$ as a closed subspace of a topological space $Y$, there is an open neighborhood $U$ of $X$ in $Y$ that retracts onto $X$ in the sense that we have a diagram of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\iota} & Y \\
\downarrow{r} & & \downarrow{U} \\
U & \xrightarrow{r} & X
\end{array}
$$

for some $r : U \to X$ so that the $X$-based loop is the identity. Metrizable manifolds are automatically ANR, e.g. by [17, Theorem V.7.1] or the main result of [9].

### 2.2 The weak Hilbert-Smith conjecture

In [12, Conjecture 1.2] a version of the Hilbert-Smith conjecture was presented in its weak form, whereby free actions are substituted for effective ones, and an additional constraint is imposed on the orbit space. That is, one assumes that, for a free action of a compact group $G$ on a finite-dimensional topological manifold $M$, the orbit space $M/G$ is finite-dimensional. At first sight this is counterintuitive since it disagrees with the well-known equation

$$\dim(M/G) = \dim(M) - \dim(G),$$

2
which holds for example for free actions of compact Lie groups. However, in the case when $G = \mathbb{Z}_p$, the situation changes drastically. Although $p$-adic integers are zero-dimensional, Smith showed that the dimension of $M/\mathbb{Z}_p$ is not equal to $\dim(M)$ [35]. Next, combining the result of C.T. Yang [41] and Alexandroff’s theorem about the coincidence of the cohomological and covering dimension [3], one can deduce that for a free action of $\mathbb{Z}_p$ on an $n$-manifold we have that either $\dim(M/\mathbb{Z}_p) = n + 2$ or it is infinite. Therefore, proving the Hilbert-Smith conjecture for finite-dimensional orbit spaces would exclude the first possibility.

Along the lines of [37], we somewhat strengthen the statement of [12, Conjecture 1.2] by allowing our groups to be *locally* compact, so long as the action is well-behaved enough for the resulting orbit spaces to be locally compact Hausdorff spaces. Namely, we assume the actions to be *proper*. Let us list four equivalent definitions of when an action of a locally compact group $G$ on a locally compact Hausdorff space $X$ is proper:

- the map $G \times X \to X \times X$, $(g, x) \mapsto (xg, x)$ is a proper map, i.e. preimages of compact sets are compact,
- for any pair of points $x, y \in X$ there exist open neighborhoods $U$ of $x$ and $V$ of $y$ such that the set
  \[
  U.V := \{g \in G \mid Ug \cap V \neq \emptyset\}
  \]
  is relatively compact, i.e. its closure is compact,
- if $K$ is a compact subset of $X$, then the set $K.K$ is compact.
- if $K$ and $K'$ are compact subsets of $X$ then $K.K'$ is compact.

The equivalence of the above definitions follows from [8, Proposition 7 p. 104 and Proposition 7 p. 255] and [31, Theorem 1.2.9].

**Remark 2.4** Note that properness, like freeness, is a hereditary property, i.e. it descends to open or closed subgroups. Indeed, let $G$ be a locally compact Hausdorff topological group acting properly on a locally compact Hausdorff space $X$. This means that for any compact subset $K$ of $X$, the set $K.K$ is compact in $G$. Recall that an open subgroup of a topological group is automatically closed, so we only need to prove the claim for closed subgroups. Suppose that $H$ is a closed subgroup of $G$ and take any compact subset $K$ in $X$. We want to prove that the set

\[
\{h \in H : Kh \cap K \neq \emptyset\} = K.K \cap H
\]

is compact. This is clear as the above set is a closed subset of $K.K$, which is compact by properness of the action of $G$ on $X$.

We now state the weaker Hilbert-Smith conjecture:

**Conjecture 2.5** Let $G$ be a locally compact group acting freely and properly on a connected finite-dimensional topological manifold $M$ such that the orbit space $M/G$ is finite dimensional. Then $G$ is a Lie group.

An examination of the argument presented in [37, §2.7.2] will show that the reduction of the general Hilbert-Smith conjecture to its $p$-adic version (Conjecture 0.3) also goes through in the present setting of free actions and finite-dimensional orbit spaces:

**Conjecture 2.6** Let $p$ be a prime number and $M$ a connected finite-dimensional topological manifold. Then $\mathbb{Z}_p$ cannot act freely on $M$ so that $M/\mathbb{Z}_p$ is finite-dimensional.
2.3 Reduction to compact groups

We set about reducing Conjecture 2.5 to Conjecture 2.6 gradually, starting by transporting the discussion over to compact (as opposed to locally compact) groups:

**Theorem 2.7** If Conjecture 2.5 holds for compact groups then it does in general.

We need some preparation, in the form of a series of auxiliary results. First, given that the various reductions involve passing from actions of $G$ to the restricted actions of various subgroups of $G$ and the resulting conjectures stipulate the finite-dimensionality of orbit spaces, we will make repeated use of the following remark.

**Lemma 2.8** If Conjecture 2.5 holds for an open subgroup $H \leq G$ then it does for $G$ as well.

**Proof** Under hypothesis of Conjecture 2.5, $G$ acts freely and properly on a topological manifold $M$, hence, by Remark 2.4, $H$ is again acting freely and properly. If $H$ is a Lie group, then so is $G$, since $H$ is its open subgroup. The only observation needed here is that if $M/G$ is finite-dimensional then so is $M/H$. In fact, we will prove more:

**Claim:** Under the hypotheses, $M/H \to M/G$ is a local homeomorphism. Given this, it follows from the fact that the property of being of dimension $\leq n$ is a local property ([13, Definition 1.1.1]) that the two spaces in question have equal dimension; it thus remains to prove the claim.

We denote the various canonical projection maps as in the diagram

\[
\begin{array}{c}
\pi_G \\
\pi_H \\
\pi_{H,G}
\end{array}
\begin{array}{c}
M \\
M/H \\
M/G.
\end{array}
\]

Let $p_H \in M/H$ be an arbitrary point. We have to show that some open neighborhood $p_H \in U \subseteq M/H$ is mapped homeomorphically onto its image by $\pi_{H,G}$.

We propose to do this as follows:

(a) fix some $p \in \pi_H^{-1}(p_H)$;

(b) show that there is some open neighborhood $p \in V \subseteq M$

such that the set $V.V = \{ g \in G \mid Vg \cap V = \emptyset \}$

is contained in $H$;

(c) deduce from this containment that $VgH$ and $VH$ are disjoint if $gH \neq H$ are different cosets, and finally

(d) take $U \subseteq M/H$ to be $\pi_H(V)$.

Once (b) is in place (c) follows immediately: if $VgH \cap VH \neq \emptyset$ then

$vgh = v'$

for some choice of $v, v' \in V$, etc. We then have $gh \in VV \subseteq H$ by (b), and hence $g \in H$. 

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Next, we argue that the choice of $U$ proposed in (d) meets the requirements (i.e. is mapped homeomorphically onto $\pi_{H,G}(U) = \pi_G(V)$). To see this, note that

$$\pi^{-1}_{H,G}(\pi_{H,G}(U)) = \pi_H(VG) = \pi_H \left( VH \cup \bigcup_g VgH \right)$$  \hspace{1cm} (2.1)$$

where

- the union $\bigcup_g$ is over representatives of the cosets $gH \neq H$ and
- the union $\sqcup$ is disjoint by (c).

On the other hand, because the parenthetic sets on the right hand side of (2.1) are invariant under the action of $H$, that right hand side is

$$\pi_H(VH) \sqcup \pi_H \left( \bigcup_g VgH \right) = U \sqcup \pi_H \left( \bigcup_g VgH \right).$$

In other words, $\pi_{H,G}^{-1}(\pi_{H,G}(U))$ breaks up as a disjoint union between $U$ and another open set. This suffices to conclude that

$$\pi_{H,G}|_U : U \to \pi_{H,G}(U)$$

is an isomorphism.

It thus remains to find a neighborhood $V \ni p$ as in (b). In order to do this, note first that the properness of the action ensures the existence of a compact neighborhood $M \supseteq W \ni p$ such that $W.W \subset G$ is compact. Now consider the net $(V)$ of compact neighborhoods

$$p \in V \subseteq W$$

ordered by reverse inclusion (i.e. $V_1 \supseteq V_2$ if $V_1 \subseteq V_2$). Suppose that $V.V \not\subseteq H$ for any such $V$. This means that each of the sets $V.V \cap (G - H)$ is non-empty. Since these sets are contained in the compact set $W.W$, we can thus find a subnet $(V_\alpha)$ of $(V)$ and elements

$$g_\alpha \in V_\alpha.V_\alpha \cap (G - H)$$

converging to some $g \in G - H$. But then, because $(V)$ converges to $p \in M$ and hence so does $(V_\alpha)$, we have $pg = p$. Given that $g \in G - H$, this contradicts the freeness of the action and finishes the proof.

To reduce the problem to compact groups, we need another lemma.

**Lemma 2.9** Let $H$ be a locally compact group acting freely and properly on a locally compact separable metrizable space $X$ such that both $X$ and $X/H$ are finite-dimensional. We then have

$$\dim(X/N) < \infty$$

for every compact normal subgroup $N \leq H$.

**Proof** Since $N$ is normal, there is an action of $H$ on $X/N$ given by $[x]_{X/N} \cdot h := [xh]_{X/N}$. We begin with

**Claim: The action of $H$ on $X/N$ is proper.** To see this denote by $\pi$ the canonical surjection $X \to X/N$ and let

$$K, K' \subseteq X/N$$
be two compact subsets. We then have
\[ K.K' = (\pi^{-1}(K).\pi^{-1}(K'))N, \]
and the right hand side is compact because

- \( N \) is compact;
- this in turn implies that \( \pi \) is proper and hence the preimages \( \pi^{-1}(\cdot) \) on the right hand side are compact;
- finally, the properness of the original \( H \)-action on \( X \) shows that \( \pi^{-1}(K).\pi^{-1}(K') \) is compact.

This finishes the proof of the claim.

The orbits of the action of \( H \) on \( X/N \) are homeomorphic to \( H/N \) and the orbit space is homeomorphic to \( X/H \). Given the claim, [5, Theorem 3.10] (which requires properness) is applicable and we have
\[ \dim(X/N) \leq \dim(X/H) + \dim(H/N). \] (2.2)

Since \( H \) acts freely on a metrizable and finite-dimensional space, \( H \) is automatically metrizable and finite-dimensional. By [29, Corollary 2] \( H \to H/N \) is a locally trivial bundle and therefore \( \dim(H/N) < \infty \). It follows that both summands on the right hand side of (2.2) are finite, finishing the proof. ■

We are now ready for the

**Proof of Theorem 2.7** Assume that Conjecture 2.5 is true for compact groups. We follow the strategy pursued in [37, §2.7.2], using the Gleason-Yamabe theorem. Recall that the latter states that every locally compact group \( G \) has an open subgroup \( H \subseteq G \) which in turn has a normal compact subgroup \( N \) such that \( H/N \) is Lie.

Lemma 2.8 has already reduced the problem to its analogue for \( H \). Now, if \( N \) is Lie, then so is \( H \), as it is an extension of a Lie group \( H/N \) by Lie group \( N \). Again, we only need to prove the implication
\[ \dim(M/H) < \infty \Rightarrow \dim(M/N) < \infty. \] (2.3)

This, however, is nothing but Lemma 2.9 (\( N \) is a compact normal subgroup of \( H \)). ■

**2.4 Reduction to p-adic integers**

Finally, we can reduce Conjecture 2.5 to the \( p \)-adic case.

**Theorem 2.10** Conjecture 2.6 implies Conjecture 2.5.

We once more need some preparatory remarks. In the same spirit as Lemma 2.9 we have

**Lemma 2.11** Let \( X \) be a separable metrizable space acted upon freely by the compact group \( G \). Then, for every closed subgroup \( H \subseteq G \) we have
\[ \dim(X/H) \leq \dim(X/G) + \dim(G/H). \]
If either $X/G$ or $G/H$ is infinite-dimensional then there is nothing to prove, so we assume both are finite-dimensional.

Because $G$ is compact,

$$\pi_{H,G} : X/H \to X/G$$

is a closed, finite-dimensional map (in the sense that its fibers, homeomorphic to $G/H$, are finite-dimensional). Coupled with the observation made in the discussion following Convention 2.1 that all of our quotient spaces are metrizable, this ensures that [13, Theorem 4.39] applies and delivers the conclusion.

**Remark 2.12** The $H = \{1\}$ case of Lemma 2.11 is a particular instance of [5, Theorem 1.3].

Finally:

**Proof of Theorem 2.10** First, observe that we can reduce Conjecture 2.5 to compact groups by Theorem 2.7. We prove the contrapositive of Theorem 2.10, namely we assume that there is a free action of a compact non-Lie group $G$ on a connected finite-dimensional manifold $M$ with $M/G$ finite-dimensional, and we find an action of $\mathbb{Z}_p$ on $M$ as in Conjecture 2.6. By [21, Theorem 3.1], we know that $G$ contains $\mathbb{Z}_p$ and by Lemma 2.11 applied to $H = \mathbb{Z}_p$ we have

$$\dim(M/G) < \infty \Rightarrow \dim(M/\mathbb{Z}_p) < \infty.$$
Theorem 2.14 Let $m > n$ be non-negative integers, and let the zero-dimensional non-trivial compact group $G$ act freely on $\mu^n$ and $\mu^n$ so that $\dim(\mu^n/G) \leq m$. Suppose that there are no $G$-equivariant maps $\mu^n \ast G \to \mu^n$ for diagonal action on the join.

Then, there are no $G$-equivariant maps $\mu^m \to \mu^n$.

Notice that Theorem 2.14 assumes that Conjecture 0.5 is satisfied for a zero-dimensional compact metric group acting on a compact Hausdorff space of finite covering dimension. Before going into the proof, let us record the resolution of the weak Hilbert-Smith conjecture.

Corollary 2.15 Conjecture 0.5 implies Conjecture 2.6.

Proof Let $m > n$ be arbitrary positive integers.

We first apply Theorem 2.14 to actions of $\mathbb{Z}_p$ on $\mu^m$ and $\mu^n$ constructed as in [11], having the properties

$$\dim(\mu^m/\mathbb{Z}_p) = m, \dim(\mu^n/\mathbb{Z}_p) = n.$$ 

The non-existence of an equivariant map $\mu^m \to \mu^n$ resulting from this theorem then ensures that the hypotheses of Theorem 2.13 are satisfied for this specific choice of free actions, and hence $\dim(M/\mathbb{Z}_p) > n$. Since $n$ was arbitrary, this concludes the proof that the dimension of the orbit space cannot be finite.

Notation 2.16 In the context of a group $G$ acting on spaces $X$ and $Y$, we write $X \hat\times \Delta Y$ (with the group and actions being understood) to denote the Cartesian product $X \times Y$ equipped with the diagonal $G$-action.

Lemma 2.17 Let $X$ be a compact space equipped with a free action by a compact group $G$. Then, we have the inequality

$$\dim(X * G/G) \leq \max(\dim(X/G), \dim(X) + 1). \quad (2.4)$$

Proof The orbit space $X * G/G$ on the left hand side of (2.4) is the union of the two closed subspaces $X/G$ (at the endpoint $1 \in [0, 1]$; see the definition of joins in the introduction) and $\{*\} \cong G/G$ at $0 \in [0, 1]$, and the open subspace

$$X \times \Delta G \times J/G \cong (X \times \Delta G/G) \times J$$

where $J = (0, 1)$ and the $\Delta$ subscript is explained in Notation 2.16.

[18, Corollary 1 to Theorem III 2] implies the estimate

$$\dim(X * G/G) \leq \max(\dim(X/G), \dim((X \times \Delta G/G) \times J)).$$

Hence, for our purposes, it suffices to prove that we have

$$\dim((X \times \Delta G/G) \times J) \leq \dim(X) + 1. \quad (2.5)$$

First, notice that the subadditivity of dimension under taking Cartesian products ([18, Theorem III 4]) together with $\dim(J) = 1$ bounds the left hand side of (2.5) above by

$$\dim(X \times \Delta G/G) + 1.$$ 

Finally, the desired conclusion follows from the fact that

$$(x, g) \mapsto (xg, g)$$
implements an isomorphism of $G$-spaces between $X \times G$ with the right hand factor action and $X \times_{\Delta} G$. This, in turn, ensures

$$X \times_{\Delta} G/G \cong X,$$

and hence

$$\dim(X \times_{\Delta} G/G) = \dim(X).$$

This finishes the proof of the lemma.

As an immediate consequence of Lemma 2.17, we obtain

**Corollary 2.18** Let $m > n$ be non-negative integers. For a free action of a compact group $G$ on $\mu^n$ with $\dim(\mu^n/G) \leq m$ we have $\dim(\mu^n * G/G) \leq m$.  

**Proof of Theorem 2.14** The freeness of all actions in sight imply that all orbits are equivariantly homeomorphic to $G$ itself.

Now, our assumption that $\dim(\mu^n/G) \leq m$ shows via Corollary 2.18 below that we have

$$\dim(\mu^n * G/G) \leq m.$$  

(2.6)

The universality of the $G$-action on $\mu^m$ provided by Theorem 2.3 and (2.6) ensure that any equivariant homeomorphism from a $G$-orbit in $\mu^n * G$ onto a $G$-orbit in $\mu^m$ extends to an equivariant map

$$\mu^n * G \rightarrow \mu^m.$$  

Composition with an equivariant map $\mu^m \rightarrow \mu^n$ as in the statement of the theorem would then imply the existence of an equivariant map $\mu^n * G \rightarrow \mu^n$, contradicting the assumption.

**Remark 2.19** Theorems 2.13 and 2.14 are stated in such generality to emphasize their connection with Ageev’s original formulation [1]. However, working with the universal actions on Menger compacta $\mu^n$ such that $\dim(\mu^n/\mathbb{Z}_p) = n$ would still confirm Conjecture 2.6.

As mentioned in the introduction, the actions in question are proved unique up to isomorphism (for given $n$) in [2, Theorem B] and their existence is proven by [11, Theorem 1]. Remark 1 on p. 228 therein argues that the orbit space for the action on $\mu^n$ being constructed in the proof of the theorem is indeed $n$. 

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