Steinhaus conditions for convex polyhedra

Joël Rouyer

May 5, 2017

Abstract

On a convex surface $S$, the antipodal map $F$ associates to a point $p$ the set of farthest points from $p$, with respect to the intrinsic metric. $S$ is called a Steinhaus surface if $F$ is a single-valued involution. We prove that any convex polyhedron has an open and dense set of points $p$ admitting a unique antipode $F_p$, which in turn admits a unique antipode $F_{F_p}$, distinct from $p$. In particular, no convex polyhedron is Steinhaus.

Keywords: convex polyhedra, antipodal map.
MSC 2010: 52B10, 51A15, 53C45

1 Introduction

By a polyhedron, denoted by $P$, we mean the boundary of a compact and convex polyhedron in $\mathbb{R}^3$. $P$ is naturally endowed with its intrinsic metric: the distance between two points is the length of the shortest curve joining them. In this paper, we shall never consider the extrinsic distance. A segment is by definition a shortest path on the polyhedron between its endpoints. In general, it is not a line segment of $\mathbb{R}^3$, but becomes so if one unfolds the faces it crosses onto a same plane. An antipode of $p$ is a farthest point from $p$; the set of antipodes of $p$ is denoted by $F_p$. It is well-known that the mapping $F$ is upper semicontinuous.

When the context makes clear that $F_p$ is a singleton, we shall not distinguish between this singleton and its only element.

The study of antipodes on convex surfaces began with several questions of H. Steinhaus, reported in [3], most of them answered by Tudor Zamfirescu, see e.g. [11], [12], [13], [14]. However, one of those questions had remained open a little longer: does the fact that the antipodal map of a convex surface is a single-valued involution imply that the surface is a round sphere? As we shall see, the answer is negative. By definition, such a surface will be called a Steinhaus surface.

The first family of Steinhaus surfaces was discovered by C. Vălcu [9]. It consists of centrally symmetric surfaces of revolution, and includes the ellipsoids having two axes equal, and the third shorter than the two equal ones. (Note that if the third axis is longer than the two equal ones, the surface is no longer
Steinhaus [10]. Other examples were discovered afterward: cylinders of small height [4], and the boundaries of intersections of two solid balls, provided that the part of the surface of the smaller ball included in the bigger one does not exceed a hemisphere. This last example, as well as the mentioned family, were generalized to hypersurfaces in $\mathbb{R}^n$ [5].

Note that Steinhaus conditions are related to another one. Define the radius at a point $p$ as the distance between $p$ and its antipodes. It is known that, if some surface has a constant radius map, then it is a Steinhaus surface [10]. Moreover, all examples of Steinhaus surfaces hitherto discovered also satisfy the constant radius condition. So it is still open whether the two conditions are equivalent.

One can also notice that all known examples are surfaces of revolution, and in particular, are not polyhedral. The first attempt to find a polyhedral example was the investigation of the regular tetrahedron. An explicit computation of the antipodal map proved that it is not a Steinhaus surface [6]. Then we proved that no tetrahedra can be Steinhaus [7]. A few years later, we proved that no polyhedron can satisfy the radius condition, and that no centrally symmetric polyhedron can be Steinhaus [8]. Since then, the problem of the (non)existence of Steinhaus polyhedra was very natural. The aim of this paper to prove they do not exist.

In order to make this article self-contained, we give in Section 2 some preliminary, concerning the antipodes on a convex polyhedron. In Section 3 we prove the result. Then, in a last section, we discuss a few open questions related to this topic.

2 Preliminaries

The explicit computation of the antipodal map in the case of the regular tetrahedron [6] was generalized – as much as possible – to arbitrary convex polyhedra [7, 8]. We proved there the following theorem.

**Theorem 0.** Any convex polyhedron $P$ can be written as a disjoint union

$$P = \Gamma(P) \cup \bigcup_{i=1}^{N} Z_i$$

with the following properties.

1. The sets $Z_i$ are open, and the restricted maps $F|Z_i$ are singled-valued.

2. The set $\Gamma(P)$ is a finite union of algebraic arcs, of degree at most 10. It includes the edges of $P$.

3. The map $F|Z_i$ is a constant map if and only if its image is a singleton containing a vertex.
4. If $\text{Im} (F|Z_i) = \{v\}$, then there is exactly one segment between $v$ and any point of $Z_i$; otherwise, there are exactly three segments between $p \in Z_i$ and $F_p$.

From now on, the sets $Z_i$ defined in the above theorem will be referred to as zones.

Remark. Including a priori the edges in $\Gamma (P)$ is only a trick to simplify the proof and ensure that any zone is (isometric to) a plane domain. In general, nothing special happens while crossing an edge: if $Z$ and $Z'$ are two zones separated by an edge and one rotates the face of $Z$ onto the plane of $Z'$, then $F|Z$ and $F|Z'$ become rational prolongations of each other.

More generally, the notion of edges belongs to the extrinsic geometry and plays no role in a purely intrinsic problem.

The proof of Theorem $\Box$ is given in details in [7] and [8], we present next only its rough idea. The fourth point of the theorem has been stated for aim of completeness only. It will not be used here, so we shall omit its proof which involves extensive computation.

A first remark is that a segment cannot pass through any vertex, see for instance [2]. Secondly, note that a point $p$ on the polyhedron is joined to any of its antipodes $q$ by at least three segments, provided that $q$ is not a vertex. Suppose on the contrary that there are only two segments $\sigma_1$ and $\sigma_2$ between $p$ and $q$. They would determine two sectors at point $q$, one of them having measure at least $\pi$ (if there is only one segment, there is only one sector, measuring $2\pi$).

Consider a point $r$ on the bisector of this sector, tending to $q$. A segment $\sigma$ between $p$ and $r$ should tend to either $\sigma_1$ or $\sigma_2$, say $\sigma_1$. For $r$ close enough to $q$, the triangle composed of $\sigma_1$, $\sigma$, and the only segment between $r$ and $q$ contains no vertex, and so, is a (folded) Euclidean triangle. Moreover the angle at $q$ is obtuse or right; it follows that the length of $\sigma$ (which is also $d(p, r)$) is greater than the length of $\sigma_1$ (which is supposed to be the radius at $p$), and we get a contradiction.

Hence there exist three segments $\sigma_{-1}$, $\sigma_0$, $\sigma_1$ between $p$ and $q$. Let $\mathcal{F}'$ be the face of $p$ and $\mathcal{F}$ be the face of $q$; if one unfolds the union of the faces crossed by $\sigma_i$ ($i = -1, 0, 1$) onto the plane of $\mathcal{F}$, one obtains three images of the face $\mathcal{F}'$, say $\mathcal{F}_0$, $\mathcal{F}_{-1}$ and $\mathcal{F}_1$. One passes from $\mathcal{F}_0$ to $\mathcal{F}_i$ ($i = \pm 1$) by a planar affine direct isometry $f_i$. Since the segments $\sigma_i$ have the same length, $q = \text{cc} (p_0, f_{-1} (p_0), f_1 (p_0))$, where $p_0$ is the point of $\mathcal{F}_0$ corresponding to $p$ and $\text{cc} (\ldots)$ stands for the circumcenter (see Figure 1).

There is only a finite number of ways to unfold sequences of faces, inducing a finite number of pairs of isometries $f_{\pm 1}$, leading to a finite number of maps $\tau_k : p \mapsto \text{cc} (p_0, f_{-1} (p_0), f_1 (p_0))$. For each zone $Z$ the function $F|Z$ is either one $\tau_k$ or a constant map whose value is a vertex. Let $\delta_k (p)$ be the square of the distance between $p$ and $\tau_k (p)$. The algebraic arcs that compose $\Gamma (P)$ are parts of the locus of those points such that $\delta_k (p) = \delta_{k'} (p)$ for some indices $k$ and $k'$, such that $\delta_k (p) = d(p, v)^2$ for some index $k$ and some vertex $v$, or such that
Figure 1: Unfolding of $P$ in the proof of Theorem 0.

$d(p, v)^2 = d(p, w)^2$ for some vertices $v$ and $w$. A straightforward computation shows that they have degree at most 10. (Note that, in the only case that have been explicitly solved – the regular tetrahedron – the higher degree terms vanish and the actual degree drops to 4 [6]. So it is not entirely clear that 10 can be achieved.)

For further reasoning, it is necessary to notice that $f_1$ and $f_{-1}$ cannot be both translations (Lemma 1 in [8]). Moreover, if either $f_1$ or $f_{-1}$ is a translation then, by interchanging the roles of $F_0$ and $F_1$, we obtain that $f_{-1}$ and $f_1$ are both rotations, of the same angle. Hence, we can assume without loss of generality that $f_{\pm 1}$ are both rotations. Now, a straightforward computation proves the following lemma.

**Lemma 1.** Let $P$ be a convex polyhedron, let $Z$ be a zone of $P$, and assume that $F|Z$ is not constant. Endow the plane of $Z$ and the plane of $\text{Im}(F|Z)$ with orthonormal frames and let $(x, y)$ be the coordinates of $p \in Z$. The coordinates of $F_p$ are given by the formula:

$$F(x, y) = \frac{(X(x, y), Y(x, y))}{\varepsilon(x^2 + y^2) + L(x, y)},$$  

(1)

where $\varepsilon \in \{0, 1\}$, $L, X, Y \in \mathbb{R}[x, y]$, $\deg(X) \leq 2$, $\deg(Y) \leq 2$, $\deg(L) \leq 1$. Moreover, the zero set of the denominator is neither empty nor reduced to a single point.

Indeed, $\varepsilon = 0$ if and only if the function $f_{\pm 1}$ are rotations of the same angle. In this case, the zero set of the denominator of (1) is the line through the centers
of $f_1$ and $f_{-1}$. If the angles of the rotations are distinct (i.e., $\varepsilon = 1$), the zeroset of the denominator is the circle through the centers of $f_1$, $f_{-1}$, and $f_{-1} \circ f_1^{-1}$.

**Lemma 2.** With the notation of Lemma 1, in the case $\varepsilon = 0$, there exists orthonormal frames of the planes of $Z$ and $F_Z$ such that

\[
X(x, y) = 2xy \cos \theta + (x^2 - y^2 - 1) \sin \theta \\
Y(x, y) = (1 - \cos \theta) \left( x + y \cot \frac{\theta}{2} - 1 \right) \left( x + y \cot \frac{\theta}{2} + 1 \right) \\
L(x, y) = 2y.
\] (2)

**Proof.** Let $\theta$ be the angle common to both rotations. Choose the orthonormal direct frame in the plane of $\text{Im}(F|Z)$ in such a way that the coordinates of the center of $f_i$ are $(i, 0)$ $(i = \pm 1)$. In the plane of $Z$, choose the frame in which the coordinates of $p \in F$ equal the coordinates of $p_0 \in F_0$. Let $(x, y)$ be the coordinates of $p$; the coordinates of $f_i(p_0)$ are

\[
((x - i) \cos \theta - y \sin \theta + i, y \cos \theta (x - i) \sin \theta).
\]

The equation in coordinates $(\xi, \psi)$ of the mediator of the segment $p_0f_i(p_0)$ is

\[
\begin{align*}
(x (\cos \theta - 1) - y \sin \theta - i (\cos \theta - 1)) \xi \\
+ (y (\cos \theta - 1) + (x - i) \sin \theta) \psi \\
+ iy \sin \theta + (\cos \theta - 1) (1 - ix) = 0.
\end{align*}
\] (3)

In order to shorten the formulas, we put $u \overset{\text{def}}{=} \cos \theta - 1 = -2 \sin^2 \frac{\theta}{2}$ and $v \overset{\text{def}}{=} \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$. The half difference and half sum of the two equations given by (3) $(i = \pm 1)$ are respectively

\[
\begin{align*}
u \xi + v \psi &= -ux + vy \\
(ux - vy) \xi + (uy + vx) \psi &= -u.
\end{align*}
\]

It follows that

\[
\begin{align*}
\xi &= \frac{(-ux + vy) (uy + vx) + uw}{(uy + vx) u - (ux - vy) v} \\
&= \frac{(v^2 - u^2) xy + uv (y^2 - x^2 + 1)}{y (u^2 + v^2)} = \frac{X(x, y)}{2y}, \\
\psi &= \frac{(-ux + vy) (ux - vy) + u^2}{v (ux - vy) - u (uy + vx)} \\
&= \frac{-u^2 x^2 + 2uxy - v^2 y^2 + u^2}{2yu} \\
&= \frac{u (1 - x + \frac{v}{u} y) (1 + x - \frac{v}{u} y)}{2y} = \frac{Y(x, y)}{2y}.
\end{align*}
\]

$\square$

5
3 Main result

Now, we are in a position to prove the claimed result. The idea of proof is very simple: we just have to prove that the inverse of a function of the form $f$ cannot be of this form.

**Theorem.** Any convex polyhedron $P$ contains an open and dense set $D$ such that $F|D$ is single-valued, $F|F_D$ is single-valued, and $F_{F_D} \neq x$ for all $x \in D$. Consequently, no polyhedron satisfies, even locally, Steinhaus conditions.

**Proof.** It is sufficient to prove that any open set $U_0 \subset P$ contains an open subset $V$ such that $F|V$ and $F|F_V$ are single-valued, and such that $F_{F_V} \neq x$ for any $x \in V$. Then, the union of all those $V$ is a suitable $D$.

Let $U_0$ be a fixed open set; by virtue of Theorem [1] it contains a subset $U_1$ which is included in a zone. Since $F|U_1$ is continuous, there exists a smaller open set $U \subset U_1$ such that $F_U$ is wholly contained in a zone too. Since $F \circ F|U$ is single-valued and thereby continuous, the set $V$ of those points in $x \in U$ such that $F_{F_U} \neq x$ is open. If it is non-empty, the proof is over; suppose on the contrary that $F_{F_U} = \{p\}$ for all $p \in U$. Put $U' \triangleq F_U$. Let $\Pi$ be the plane of the face containing $U$ and $\Pi'$ be the plane of the face containing $U'$. By Lemma [1] $F|U$ and $F|U'$ are rational and admit natural continuations $G : \Pi \setminus C \to \Pi'$ and $G' : \Pi' \setminus C' \to \Pi$ respectively, where $C$ and $C'$ are either a circle or a line. By hypothesis, $G'|U' \circ G|U = \text{id}_U$; since $G$ and $G'$ are rational, $G' \circ G(p) = p$ wherever $G' \circ G$ is well defined.

Assume that $	ext{Im} G$ intersects $C'$. Let $a$ be a point of the boundary of $G^{-1}(C')$, and $p_n \in \Pi \setminus (C \cup G^{-1}(C'))$ be a point tending to $a$ when $n$ tends to infinity. Then $\|G' \circ G(p_n)\|$ tends to infinity, in contradiction with $G' \circ G(p_n) = p_n \to a$. Hence $\text{Im} G$ does not intersect $C'$, and similarly $\text{Im} G'$ does not intersect $C$.

Assume now that either $C$ or $C'$ is a circle. Since $G$ and $G'$ play symmetrical roles, we can assume that $C$ is a circle. Let $D$ be the interior of $C$ and $E$ its exterior; let $A, B$ be the two connected components of $\Pi \setminus C'$. Since $\|G'(x)\|$ tends to infinity when $x$ tends to some points of $C'$, $G'(A)$ (resp. $G'(B)$) cannot be included in $D$. It follows that $\text{Im} G' \subset E$. This contradict the fact that a point $x \in D$ equals $G' \circ G(x)$.

Now assume that $C$ and $C'$ are both straight lines. By Lemma [2] one can assume that the expression of $G$ in Euclidean coordinates is given by [1] and [2]. Let $p_1 \in \Pi$ be the point of coordinates $(\cos \theta, -\sin \theta)$, $p_2 \in \Pi$ be the point of coordinates $(-\cos \theta - 2, \sin \theta)$ and $q \in \Pi'$ the point of coordinates $(1, 0)$. A direct computation shows that

$$G(p_1) = G(p_2) = q.$$

Hence $p_1 = G'(q) = p_2$ and we get a contradiction. \qed
4 Further questions

In order to close the paper, we will mention a few open questions related to the above result. The first two were already suggested in the introduction.

**Open question i.** Do there exist Steinhaus surfaces with non-constant radius?

**Open question ii.** Do there exist Steinhaus surfaces without rotational symmetry?

Observe that the definition of Steinhaus surfaces combines two distinct conditions: the fact that $F$ is single-valued does not imply that it is an involution. The simplest known example is an ellipsoid of revolution whose axes measure respectively 1, 1, and $a \in (1, \sqrt{2})$. On such an ellipsoid, $F$ is a homeomorphism, but not an involution [10]. This naturally leads to the following question.

**Open question iii.** Does there exist a convex polyhedron such that $F$ is single-valued? Such that $F$ is a homeomorphism?

A certain trend nowadays is to consider the so-called *Alexandrov surfaces with curvature bounded below*, instead of the classical convex surfaces. Roughly speaking, the difference between these notions is twofold. On the one hand, the curvature bound is no longer necessarily zero, and on the other hand, the topology is no longer spherical. See [1] for details. In this context, it is natural to consider abstract polyhedra obtained by gluing several polygons along their boundaries, in such a way that (1) the gluing map is length preserving, (2) the resulting space is a (not necessarily spherical) topological surface, and (3) the singular curvature at each vertex is non-negative. It is easy to see that the theorem of this paper cannot be generalized to such abstract polyhedra, for rectangle flat tori are Steinhaus. As for projective planes, the main obstruction to be Steinhaus is purely topological.

**Proposition.** Any metric space homeomorphic to an even dimensional projective space admits at least one point with more than one antipodes.

**Proof.** Assume that $F$ is single-valued, and consequently continuous. Since, as a consequence of Lefschetz fixed point theorem, even dimensional projective spaces have the fixed point property, $F$ should have a fixed point, which is absurd.

Still in the context of Alexandrov surfaces, it is natural to consider abstract polyhedra whose faces are geodesic polygons of the unit sphere (*spherical polyhedra*), or of the hyperbolic plane (*hyperbolic polyhedra*).

**Open question iv.** Except from the sphere, is there a Steinhaus spherical polyhedron? A hyperbolic one? If yes, which are the admissible topologies?
Acknowledgement. The author was supported by the grant PN-II-ID-PCE-2011-3-0533 of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI.

Special thanks are due to the anonymous referee for his or her useful suggestions that strongly influenced the final form of this paper.

References

[1] Yu. Burago, M. Gromov, and G. Perel’man, A. D. Alexandrov spaces with curvature bounded below., Russ. Math. Surv. 47 (1992), no. 2, 1–58 (English. Russian original).

[2] Herbert Buseman, Convex surfaces, Dover, New York, 2008, originally published in 1958 by Interscience Publishers, Inc.

[3] Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy, Unsolved problems in geometry, Springer-Verlag, New York, 1991.

[4] Jin ichi Itoh and Costin Vîlcu, What do cylinders look like?, J. Geom. 95 (2009), no. 1-2, 41–48.

[5] Jin-Ichi Itoh, Joël Rouyer, and Costin Vîlcu, Antipodal convex hypersurfaces, Indag. Math., New Ser. 19 (2008), no. 3, 411–426 (English).

[6] Joël Rouyer, Antipodes sur le tétraèdre régulier, J. Geom. 77 (2003), no. 1-2, 152–170.

[7] , On antipodes on a convex polyhedron, Adv. Geom. 5 (2005), no. 4, 497–507.

[8] , On antipodes on a convex polyhedron (II), Adv. Geom. 10 (2010), no. 3, 403–417.

[9] Costin Vîlcu, On two conjectures of Steinhaus, Geom. Dedicata 79 (2000), no. 3, 267–275.

[10] Costin Vîlcu and Tudor Zamfirescu, Symmetry and the farthest point mapping on convex surfaces, Adv. Geom. 6 (2006), no. 3, 379–387.

[11] Tudor Zamfirescu, On some questions about convex surfaces, Math. Nach. 172 (1995), 312–324.

[12] , Points joined by three shortest paths on convex surfaces, Proc. Am. Math. Soc. 123 (1995), no. 11, 3513–3518.

[13] , Farthest points on convex surfaces, Math. Z. 226 (1997), no. 4, 623–630.

[14] , Extreme points of the distance function on a convex surface, Trans. Amer. Math. Soc. 350 (1998), no. 4, 1395–1406.
