A thermal noise model for a branched system of harmonic oscillators.

Paola Puppo
INFN Roma, P.le A. Moro 2, 00185, Roma, Italy.
E-mail: paola.puppo@roma1.infn.it

Abstract. We have calculated the thermal noise of a branched system of oscillators through the normal mode representation. This model describes well the mechanical behavior of the last stage suspension system like in the Virgo interferometer and is consistent with the predictions coming from the fluctuation-dissipation theorem. Moreover, the developed formalism can be useful to infer informations on the mechanical quantities of the uncoupled elements of the suspension and on the suspension thermal noise predictions for a third generation gravitational interferometer like the Einstein Telescope (ET).

1. Introduction
It is well understood that the pendulum thermal noise limits the sensitivity curve of a ground based interferometric g.w. antenna in the low frequency range and the prediction of its noise spectrum is important for a proper design of the detector. For this reason the use of low dissipative materials is a consolidated choice to improve the sensitivity of the enhanced and advanced interferometers [3, 4, 5]. Another challenging choice is to go at cryogenic temperatures pushing down to 1 Hz the sensitivity bandwidth of the third generation interferometers like ET [2]. The recent new calculation of the mirror thermal noise [1] for the Virgo-like detectors has shown that as soon as we are dealing with low losses mirror pendulum, the upper suspension losses starts to contribute via its recoil and must be included in the computation. This paper is meant to give an overview of the thermal noise for a Virgo-like suspension focusing on the normal-mode expansion treatment. A similar treatise was already performed for double systems by P. Rapagnani [8] for the study of the electromechanical transducers in the resonant g.w. antennas and by E. Majorana and Y. Ogawa [7] for a double suspension. The results are equivalent to the calculation performed using the Fluctuation-Dissipation theorem (FDT), however the developed formalism can be useful for the noise estimation of a cryogenic suspension at a thermally steady state but with the stages at different temperatures [10].

2. The branched model
A Virgo-like last stage suspension is a cascade of three pendula [6]. To the first pendulum (the marionette) the mirror and the recoil mass are hung as branches. The system is equivalent to a combination of three harmonic oscillators (see figure 1). This model can be used to calculate the thermal noise of the pendulum, and also of the vertical degrees of freedom by properly defining
Figure 1. A Virgo-like last stage suspension is a cascade of three pendula. To the first pendulum (the marionette, $M_1$) the mirror $M_2$ and the recoil mass $M_3$ are hung as branches.

the uncoupled frequencies as follows:

\[
\begin{align*}
\text{Pendulum modes } & \quad \omega_1^2 = \left(\omega_{g1}^2 + \omega_{w1}^2\right) / \mu_t; \quad \mu_t = M_1 / M_T \\
& \quad \omega_i^2 = \omega_{g1}^2 + \omega_{wi}^2; \quad i = 2, 3 \\
& \quad \omega_{gi} = \left(\frac{g}{L_i}\right)^{1/2} \omega_{wi} = \left(\frac{k_i^g}{M_i}\right)^{1/2} \\
& \quad k_{ei}^i = \frac{n_i f_i (\Lambda_i Y_i I_i)}{2 L_i^2} \quad ; \quad I_i = \frac{\pi}{4} r_i^2 
\end{align*}
\]

Vertical modes

\[
\begin{align*}
\omega_{vi}^2 &= (2 \pi 0.4)^2 / \mu_t \\
& = \left(\frac{\pi^2 Y_i}{M_i / 4}\right) ; \quad i = 2, 3
\end{align*}
\]

where $Y_i$, $r_{wi}$, $L_i$ are the Young modulus, the radius and the length of the suspension wires, $\Lambda_i$ is the wires tension, $M_T = M_1 + M_2 + M_3$ and $f_i, n_i$ the number of flexural points and the number of suspension wires for each mass.

The mechanical losses are defined by the mechanical quality factors $Q_i$ and $Q_{iv}$ of the oscillators describing the viscous dissipation mechanisms and by the structural loss angles which are the imaginary parts of the oscillators elastic constants $k_j = k_j'(1 + i \phi_j)$. Thus the overall loss angles are defined as follows:

\[
\begin{align*}
\Phi_1 (\omega) &= \left(\phi_1 (\omega) + \frac{\omega}{\omega_1 Q_1^{-1}}\right) / (\mu_t)^{1/2} \\
\Phi_i (\omega) &= \left(\phi_i (\omega) + \frac{\omega}{\omega_i Q_i^{-1}}\right); \quad i = 2, 3 \\
\Phi_{iv} (\omega) &= \left(\phi_{iv} (\omega) + \frac{\omega}{\omega_{iv} Q_{iv}^{-1}}\right); \quad i = 1, 2, 3
\end{align*}
\]

The equations of motion can be found starting from the Lagrangian and the dissipation functions:

\[
\begin{align*}
\mathcal{L} &= \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_2^2 + \frac{1}{2} M_3 \dot{x}_3^2 - \frac{1}{2} M_1 \omega_1^2 x_1^2 - \frac{1}{2} M_2 \omega_2^2 (x_1 - x_2)^2 - \frac{1}{2} M_3 \omega_3^2 (x_1 - x_3)^2 \\
E_d &= \frac{1}{2} M_1 \frac{\omega_1}{Q_1} \dot{x}_1^2 + \frac{1}{2} M_2 \frac{\omega_2}{Q_2} (\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2} M_3 \frac{\omega_3}{Q_3} (\dot{x}_1 - \dot{x}_3)^2
\end{align*}
\]
and, from the Lagrange equations \( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i} - \frac{\partial E_d}{\partial x_i} \) in the frequency space, they can be written in matricial form as:

\[
\hat{Z} \begin{pmatrix}
\hat{X}_1 \\
\hat{X}_2 \\
\hat{X}_3
\end{pmatrix} = \begin{pmatrix}
\mathcal{F}_{thI} \\
\mathcal{F}_{thII} \\
\mathcal{F}_{thIII}
\end{pmatrix}
\] (4)

where \( \hat{Z} \) is the mechanical impedance of the system. The stochastic generalized forces in the right side can be consistently related to the uncoupled stochastic terms as:

\[
f_{th_i} = f_{th_1} - f_{th_2} - f_{th_3}; f_{th_{II}} = f_{th_2}; f_{th_{IIII}} = f_{th_3}
\] (5)

where \( T_i \) are the temperatures and \( \tau_0 \) the decay times of each oscillator. Using the impedance matrix, the FDT leads to the thermal noise of the mirror in the case of homogeneous temperature \( T_1 = T_2 = T_3 = T \) [1]:

\[
X_{th2}(\omega)^2 = \frac{4k_b T}{\omega^2} \text{Re}\{(\hat{Z}(\omega))^{-1}\}_{22}
\] (6)

due to the fact that the whole mechanical system is at temperature \( T \), but it does not deal with the case of different temperature oscillators.

3. **Normal mode treatment**

In the following we will illustrate the main steps of the normal mode treatment for a triple oscillator like that one sketched in figure 1. The Lagrangian (3) can be expressed as a sum of independent quadratic terms obtained by diagonalizing the characteristic matrix of the motion equations. The eigenvectors of such a matrix are coordinates of modes \( (y_+, y_0, y_-) \) obtaining:

\[
\mathcal{L} = \frac{1}{2} m_- y_-^2 + \frac{1}{2} m_0 y_0^2 + \frac{1}{2} m_+ y_+^2 - \frac{1}{2} m_- \omega_-^2 y_-^2 - \frac{1}{2} m_0 \omega_0^2 y_0^2 - \frac{1}{2} m_+ \omega_+^2 y_+^2
\] (7)

while the dissipation function is:

\[
E_{dn} = \frac{1}{2} m_+ \omega_+ \frac{y_+^2}{Q_+} + \frac{1}{2} m_0 \omega_0 y_0^2 + \frac{1}{2} m_- \omega_- y_-^2 + \text{Cross}(m_i, \omega_i, Q_i, y_+, y_-, y_0 y_-)
\] (8)

The coupled quality factors can be found by imposing the equivalence between the dissipation function expressed in terms of the normal modes and the coupled system one, their full expression is:

\[
Q_- = \omega_- m_- \left[ \frac{\omega_1 M_1}{Q_1} + \frac{\omega_2 M_2}{Q_2} \left( \frac{\omega_-^2}{\omega^2 - \omega_-^2} \right)^2 + \frac{\omega_3 M_3}{Q_3} \left( \frac{\omega_-^2}{\omega^2 - \omega_+^2} \right)^2 \right]^{-1}
\]

\[
Q_+ = \omega_+ m_+ \left[ \frac{\omega_1 M_1}{Q_1} + \frac{\omega_2 M_2}{Q_2} \left( \frac{\omega_+^2}{\omega^2 - \omega_+^2} \right)^2 + \frac{\omega_3 M_3}{Q_3} \left( \frac{\omega_+^2}{\omega^2 - \omega_-^2} \right)^2 \right]^{-1}
\]

\[
Q_0 = \omega_0 m_0 \left[ \frac{\omega_1 M_1}{Q_1} \left( 1 - \frac{\omega_0^2}{\omega_-^2} \right)^2 + \frac{\omega_2 M_2}{Q_2} \left( \frac{\omega_0^2}{\omega_+^2} + \frac{\omega_0^2}{\omega_-^2} \left( \omega_+^2 + \omega_0^2 (1 + \mu_3) \omega_0^2 \right) \right) \right]^{-1}
\] (9)

the found formulas for the quality factors fully agrees with the calculations made in the paper [9]. We notice that in general, there is a cross-term \( \text{Cross}(m_i, \omega_i, Q_i, y_+, y_-, y_0 y_-) \) in the dissipative function which can be neglected for systems with high \( Q \) but is not null. For this
reason \((y_+, y_0, y_-)\) are called quasi-normal modes. The relationships between the physical and
the normal coordinates are related to the uncoupled frequencies and mass by means of the
equations:

\[
\begin{pmatrix}
Y_-
Y_0
Y_+
\end{pmatrix}
= \hat{\Lambda}^{-1}
\begin{pmatrix}
X_1
X_2
X_3
\end{pmatrix}
\]  

(10)

where the diagonalization matrix is:

\[
\hat{\Lambda} =
\begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\omega_{\lambda}^2 - \omega_0^2} & \frac{\omega_0^2 - \omega_+^2}{\omega_{\lambda}^2 - \omega_0^2} + \frac{\omega_{\lambda}^2 + \mu_{21} \omega_{\lambda}^2 - \omega_0^2}{\omega_{\lambda}^2 - \omega_0^2} \left(1 - \frac{\omega_0^2}{\omega_{\lambda}^2}\right) & \frac{1}{\omega_{\lambda}^2 - \omega_+^2} \\
\frac{\omega_0^2 - \omega_+^2}{\omega_{\lambda}^2 - \omega_0^2} & \frac{1}{\omega_{\lambda}^2 - \omega_+^2} & \frac{1}{\omega_{\lambda}^2 - \omega_0^2}
\end{pmatrix}
\]  

(11)

and \(\omega_-, \omega_0, \omega_+\) are the modal frequencies of the system. Within this formalism we can express
the mirror coordinate \(X_2\) in terms of the normal eigenvectors and find its thermal noise by
computing the power spectrum. Thus remembering that the stochastic thermal forces expressed
in the equation (5) are uncorrelated we can write:

\[
\langle X_{th2}(\omega)^2 \rangle = \langle |\lambda_{21}Y_-(\omega) + \lambda_{22}Y_0(\omega) + \lambda_{23}Y_+(\omega)|^2 \rangle
\]  

\[
\downarrow
\]  

\[
\langle X_{th2}(\omega)^2 \rangle = \langle f_{th1}(\omega)^2 |T_{n1}(\omega)|^2 + \langle f_{th2}(\omega)^2 |T_{n2}(\omega)|^2 + \langle f_{th3}(\omega)^2 |T_{n3}(\omega)|^2
\]  

(12)

where \(T_{ni}(\omega), i = 1, 2, 3\) are generalized transfer functions depending on the uncoupled
mechanical parameters of the pendulum. Their complete expression can be found on the
reference [1]. The relation (12) differs considerably by the naïve treatment of the thermal
noise in which the calculation is performed by a simple quadratic sum of the modal frequencies
thermal noise:

\[
\langle X_{th2}^{naive}(\omega)^2 \rangle = \lambda_{21}^2 |T_-(\omega)|^2 \langle F_{th-2}(\omega)^2 \rangle + \lambda_{22}^2 |T_0(\omega)|^2 \langle F_{th0}(\omega)^2 \rangle + \lambda_{23}^2 |T_+(\omega)|^2 \langle F_{th+2}(\omega)^2 \rangle
\]  

(13)

The main difference is the presence of a cross-term due to the correlation between the normal
forces that in the equation (13) are erroneously neglected. In figure 2 the general difference
between naïve and the modal curves is shown. We notice that the off-resonance zone at high
frequencies is quite different, and in some cases, the cross-term give rise to a helpful cancellation,
which reduces the thermal noise in the zone of the sensitivity. The result with the modal
treatment is fully equivalent to the FDT calculation shown in the equation (6) at homogenous
temperature and can be useful for the thermal noise computation of systems at a thermal steady
state but with different temperatures of the stages.

4. Numerical estimations
As an example we have computed the thermal noise of a Virgo-like last stage suspension with
a very low Q marionette stage. The result is sketched in figure 3 where it is compared with the
simple mirror pendulum. In presence of low dissipative mirror suspensions, the contributions
of the other last stage suspension elements cannot be neglected. In particular the marionetta
mechanical losses give a non negligible effect via its recoil, in the off-resonance high-freq. range.
The modal formalism agrees with the FDT approach [1].

The developed formalism can be useful to infer informations on the mechanical quality factor of
the uncoupled pendula using the formulas (9) and consequently helpful for their optimization.
Figure 2. Thermal noise prediction of for the Virgo Branched pendulum calculated with the Normal Mode method, compared with the naïve calculation.

Figure 3. The thermal noise spectrum of a Virgo-like last stage suspension with a very low Q marionette stage compared with the single mirror thermal noise. The light blue rectangle indicates the frequency zone at which the marionetta losses change the curve with respect to the single pendulum one. This difference can affect the sensitivity at low frequency and consequently lower the expected average sight for the BBH signals.

As an example we have calculated the quality factors of the uncoupled pendula using the measurements on the monolithic payload prototype installed in air, the results are shown in the Table 1. The fitted values can be used to predict the quality factors of the suspension coupled to the Superattenuator chain in Virgo and to have the right prediction of the overall sensitivity curve of the detector[11].
Table 1. From the measured quality factors of the mirror last stage suspension the uncoupled losses can be fitted using the formula (9)

| Coupled Measured Q's | Modal frequencies | Fitted uncoupled Q's from the fit |
|----------------------|-------------------|----------------------------------|
| $Q_-$                | $\nu_-$           | 0.421 Hz                         | 1000 $Q_{\text{mario}}$ |
| $Q_0$                | $\nu_0$           | 0.598 Hz                         | 200 $Q_0$ |
| $Q_+$                | $\nu_+$           | 0.842 Hz                         | 300 $Q_+$ |
| $Q_{\text{vert}}$   | $\nu_{\text{vert}}$ | 5.316 Hz                       | 900 $Q_{\text{mariov}}$ |
| $Q_{0\text{vert}}$  | $\nu_{0\text{vert}}$ | 8.746 Hz                       | 1300 $Q_{0\text{vert}}$ |
| $Q_{+\text{vert}}$  | $\nu_{+\text{vert}}$ | 19.95 Hz                        | 1500 $Q_{+\text{vert}}$ |

5. Conclusions
We have calculated the thermal noise of a branched system of oscillators through the normal mode representation. This model describes well the mechanical behavior of the last stage suspension system like in the Virgo interferometer and is consistent with the predictions coming by the Fluctuation-Dissipation Theorem. In the evaluation of a multimodal system thermal noise it is important to take into account the cross-correlation term among the various modes which is neglected in a naive calculation. In presence of low dissipative mirror suspensions, the contributions of the other suspension elements to thermal noise of the mirror cannot be neglected and a new thermal noise estimation must be done by including the viscous and structural dissipations of the marionette and recoil mass pendulum. The developed formalism can be useful to estimate the thermal noise of a system in a thermally steady state and with the stages at different temperatures. This study can be important for an evaluation of the thermal noise of the suspensions in a 3rd generation g.w. interferometers [12].

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