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Hedging crash risk in optimal portfolio selection

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\textbf{A B S T R A C T}

When almost all underlying assets suddenly lose a certain part of their nominal value in a market crash, the diversification effect of portfolios in a normal market condition no longer works. We integrate the crash risk into portfolio management and investigate performance measures, hedging and optimization of portfolio selection involving derivatives. A suitable convex conic programming framework based on parametric approximation method is proposed to make the problem a tractable one. Simulation analysis and empirical study are performed to test the proposed approach.

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1. Introduction

Modern investment theory originates from mean-variance analysis developed by Markowitz (1952), where the mean measures return performance and variance measures risk. Subsequently, a general return-risk analysis framework has been developed for financial study and practice. Risk measures that are different from variance have also been proposed one after another. For example, risk measure Value-at-Risk (VaR) (see Morgan, 1996; Duffie and Pan, 1997; Jorion, 2007) was used to capture downside risk, which has become a standard tool for risk management and supervision in the financial industry.

Under the return-risk framework, investment decisions optimize the return-risk profile of a portfolio, and are usually formulated as bi-criteria optimization models. An essential point of modern investment theory is the effect of diversification whereby the diversified portfolio can enhance the return-risk profile in the sense of Pareto optimality. A sufficiently diversified portfolio will eliminate the non-systematic risk and remain the systematic risk only. This can be further optimized for balance between return and risk to suit an investor’s preference.

Diversification, however, only works under normal market conditions where financial assets are only weakly connected and where some asset prices fall, some rise and still others remain unchanged. In extreme market conditions such as a crash, on the other hand, almost all asset prices suddenly fall, and the diversification no longer works as almost all of the asset returns are corrected perfectly. When a crash happens, traditional risk measures, including variance and VaR, are no longer effective, even though that is when risk measures are needed the most. The portfolio risk under a market crash is called crash risk. For simplicity, compared with the crash risk, we call the risk in normal market condition as normal risk in the sequel.

In this paper, we propose an approach for optimal portfolio choice, taking into account a crash, that combines the ideas of hedge and measure of crash risk. Our strategy and research roadmap are illustrated by Fig. 1. We propose a general nonlinear portfolio optimization model involving both, normal risk and crash risk, which can deal with risk in a more flexible way. The crash risk is hedged by derivatives and the model is formulated as a tractable one via parametric approximation. We then demonstrate that even if the return of derivative is usually asymmetric, the proposed model is reasonable when the portfolio is relatively diversified. We further derive an efficient convex programming approach to solve this general nonlinear portfolio problem.

A common belief in financial literature is that asset returns are distributed with heavy tails. Consequently, several tail risk measures have been proposed. VaR (see Jorion, 2007), Conditional VaR (CVaR) (see Rockafellar and Uryasev, 2000), Lower Partial Moments (LPM) (see Fishburn, 1977) are among the most popular ones. Time-varying tail risks defined as the average logarithmic shortfall with respect to a prescribed threshold are discussed by Kelly and Jiang (2014) and Fajia and A. Zambrano (2017). However, with the exception of the extreme returns associated with market crashes...
or booms, we find that the returns closely follow normal distributions. Therefore, estimating risk using traditional risk measures while falling to distinguish return data sets between normal situations and crash situations, we will overestimate risk in a normal market and underestimate it in a crash situation. Thus, measuring risk under normal versus extreme conditions separately is critical in portfolio decision-making.

Wilmott (2007) introduced a risk measuring system called CrashMetrics to deal with crash problems during a crash. It parallels RiskMetrics (Morgan, 1996) for market risk and CreditMetrics (Morgan, 1997) for credit risk in normal market conditions. Whereas these latter systems based on VaR work well in normal market conditions, CrashMetrics addresses risk management in extreme market conditions.

In CrashMetrics (Wilmott, 2007), the portfolio risk under crash conditions, i.e., crash risk, is defined as the worst-case realized return of the portfolio where uncertain asset returns are modeled as a suitable set containing all possible returns. This measure of crash risk makes sense since it is approximately the loss of the portfolio in the situation where all the assets fall more or less simultaneously.

Managing crash risk is an issue even more important than how to measure it. Since diversification does not work in a crash, hedging using options and/or other derivatives is a natural choice (see, e.g., Wilmott, 2007; Hull, 2009). Although much literature has investigated the hedging of portfolio risk using derivatives, there is relatively little research on portfolio optimization involving derivatives. Isakov and Morard (2001) and Liang et al. (2008) discuss the mean-variance portfolio selection using options. Alexander et al. (2006) investigate the Conditional Value-at-Risk (CVaR) model-based portfolio optimization problem considering only derivatives. Also based on CVaR risk measure, a stochastic portfolio model incorporating options is studied by Topaloglou et al. (2011), and the model is solved by its corresponding deterministic linear programming form with scenario generation techniques. Cui et al. (2013) propose a general hedged portfolio optimization approach based on risk measure calculated by the approximate parametric VaR. Zymler et al. (2013) use a robust optimization approach to investigate the VaR-based derivative portfolio optimization problem when the required complete distribution information is unavailable. Faias and Santa-Clara (2017) note that the option returns are asymmetrically distributed and propose a utility maximization framework for the portfolio selection involving European options held to maturity. They conclude that the effective performance of the approach is mostly obtained by exploiting mis-pricing between options. Research on portfolio optimization tends to focus on normal risk even for those on the derivative portfolio optimization. There seems to be no research directly addressing the issue of portfolio optimization using derivatives to hedge the disaster caused by a market crash.

The rest of the paper is organized as follows. In Section 2, we propose a tractable formulation of the hedged portfolio with crash risk control. In Section 3, we conduct a simulation analysis and an empirical study to test the proposed approach. Concluding remarks are given in Section 4.

2. Problem formulation

In this section, we discuss parametric approximation of the value change of a hedged portfolio and investigate the problem of measuring risk of a hedged portfolio in normal market conditions and in a crash, respectively. We then propose a tractable convex conic programming approach to solve the hedged portfolio optimization problem with crash risk control.

2.1. Parametric approximation of value change of hedged portfolio

We suppose there are \( m \) risky underlying assets and base on this are \( n \) derivative assets. Denote \( d' = (d_{1}', d_{2}', \ldots, d_{n}') \) and \( u' = (u_{1}', u_{2}', \ldots, u_{m}') \) as the value vectors of the derivative assets and their underlying assets at time \( t \), respectively. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m \) denote the vector of holding amount of derivative assets and their underlying assets, respectively. Then the value change of portfolio \((x, y)\) over time period \([0, \Delta t]\), the investment period we consider, can be expressed as

\[
\Delta v(x, y) = \sum_{i=1}^{n} x_i \Delta d_i + \sum_{i=1}^{m} y_i \Delta u_i = \Delta d'x + \Delta u'y
\]  

(1)

where \( \Delta d_i \) and \( \Delta u_i \) denote the value change of derivative \( i \) and its underlying asset over time period \([0, \Delta t]\), respectively. Generally, the price of a derivative is a function of factors, such as the prices and volatilities of the underlying assets, the risk-free interest rate and the time interval. We reasonably assume that derivative price is a sufficiently smooth function of some corresponding factors. Recall some definitions of “Greeks” (see, e.g., Glasserman, 2004; Hull, 2009):

\[
\delta_i = \frac{\partial d_i}{\partial u} = \left( \frac{\partial d_i}{\partial u_1} \right)_{m \times 1} = \left( \frac{\partial d_i}{\partial u_1}, \ldots, \frac{\partial d_i}{\partial u_m} \right)'
\]

\[
\Gamma_i = \frac{\partial^2 d_i}{\partial u'^2} = \left( \frac{\partial^2 d_i}{\partial u_1^2} \right)_{m \times m} = \left( \frac{\partial^2 d_i}{\partial u_1^2}, \ldots, \frac{\partial^2 d_i}{\partial u_m^2} \right)
\]

\[
\theta_i = \frac{\partial d_i}{\partial t},
\]

where we omit the notation of time \( t \) for simplicity when it is unnecessary. Then we can reasonably approximate the value change of derivative \( i \) over time period \([0, \Delta t]\) by Taylor expansion using Greeks as follows (see, e.g., Glasserman, 2004; Hull, 2009):

\[
\Delta d_i \approx \tilde{\Delta} d_i = \delta_i' \Delta u + \frac{1}{2} \Delta u' \Gamma_i \Delta u + \theta_i \Delta t
\]  

(2)

where the Greeks are all calculated at time 0. Of course, we can use additional terms of Taylor expansion with respect to volatility of underlying asset and/or risk-free interest rate to produce a
more accurate approximation, but the most influential factor is the price of the underlying asset. Approximation (2) could be accurate enough (see Cui et al., 2013 and the reference therein).

Combining (1) and (2), we can approximate the value change of portfolio \( (x, y) \) as

\[
\Delta v(x, y) \doteq \sum_{i=1}^{n} x_i \Delta d_i + \sum_{i=1}^{m} y_i \Delta u_i
\]

\[
= \left( \sum_{i=1}^{n} x_i \delta_i + y \right) \Delta u + \frac{1}{2} \Delta u^\top \Gamma \Delta u + \theta \Delta t,
\]

where \( \delta = \sum_{i=1}^{n} x_i \delta_i + y \). In the sequel, the value change of portfolio \( (x, y) \) is defined by (3).

Remark 1. Since the “more accurate approximation” of \( v(x, y) \) with additional terms associated with volatility of underlying asset or/and risk-free interest rate remains a quadratic form in \( u \) and a linear form in \( x \) and \( y \); just as same as approximation (3), the problem reformulation done in following according to approximation (3) can be similarly applied to “more accurate approximation”. That is to say, the following derivation does not lose its generality in deriving a “more accurate approach”.

2.2. Measuring risk of hedged portfolio

Because of the unusual nature of a crash, traditional risk measures are not suitable for portfolio management. In this subsection, we discuss the measure and calculation of hedged portfolio risk in a normal market and in a crash, respectively. Fig. 2, an example of a jointed distribution of value changes of two assets, illustrates the intuitive idea of distinguishing these two types of measures of risk.

One thing deserving special mention is that a market boom can also cause large losses in a portfolio with a large short position. Actually, a portfolio involving derivatives, such as the butterfly strategy, may encounter large loss even when there is no crash or boom. Thus the concept of “crash risk” in this paper is best interpreted in the broader sense according to CrashMetrics (Wilmott, 2007): the worst-case loss, not only the loss occurred in a real market crash.

2.2.1. Risk measure under normal market condition

In a normal market, there are no extreme asset price movements and therefore the value changes of underlying assets can be effectively modeled with a normal distribution. In such a market, we can assume that the value change vector \( \Delta u \) follows a multivariate normal distribution, i.e.,

\[ \Delta u \sim \mathcal{N}(\mu, \Sigma), \]

where \( \mu \) and \( \Sigma \) denote the mean vector and the covariance matrix, respectively. This assumption will be verified by empirical tests in Section 3. The following useful results can be obtained on calculation of mean and variance of the value change of hedged portfolio \( \Delta v(x, y) \).

**Proposition 1.** Suppose \( \Delta u \sim \mathcal{N}(\mu, \Sigma) \) and \( \Sigma \) is nonsingular. Then the mean and variance of \( \Delta v(x, y) \) are given as

\[ \mu(x, y) = \eta x + \mu' y \]

\[ \sigma^2(x, y) = (x', y') \Psi (xy) + \frac{1}{2} x' \Phi x, \]

respectively, where

\[ \eta = (\eta_1, \ldots, \eta_n)' = \left( \delta_i^\top \mu + \frac{1}{2} \mu^\top \Gamma_i \mu + \frac{1}{2} \text{tr}(\Gamma_i \Sigma) + \theta_i \Delta t \right)' \]

\[ \Psi = \left( \Gamma_1 \mu + \delta_1, \ldots, \Gamma_n \mu + \delta_n, 1 \right)' \times \Sigma \left( \Gamma_1 \mu + \delta_1, \ldots, \Gamma_n \mu + \delta_n, 1 \right) \]

\[ \Phi = \left( \phi_i \right)_{n \times n} = \left( \text{tr}(\Gamma_1 \Sigma \Gamma_1) \right)_{n \times n}. \]

Here, \( \text{tr}(\cdot) \) denotes the trace of a matrix and \( I \) denotes the \( m \)-dimensional identity matrix.

**Proof.** See Appendix A. \( \square \)

With the results of Proposition 1, we can analyze portfolios under the mean-variance framework. Although variance might be the most popular risk measure adopted in portfolio management, it is only regarded as suitable for symmetrical return distributions. Usually return of a derivative such as an option is asymmetrical. Whether variance is a suitable risk measure for hedged portfolio with derivatives will be addressed in the following.

Denote \( \xi = \Sigma^{-1/2} \Delta u \), where \( \Sigma^{-1/2} \) is the inverse of \( \Sigma^{1/2} \) satisfying \( \Sigma = \Sigma^{1/2} \Sigma^{1/2} \). Recall that \( \Delta u \sim \mathcal{N}(\mu, \Sigma) \), which implies \( \xi \sim \mathcal{N}(\Sigma^{-1/2} \mu, I) \), then we can reformulate (3) as:

\[ \Delta v(x, y) = \delta^{\top} \Sigma^{1/2} \xi + \frac{1}{2} \xi^\top \Sigma^{1/2} \Gamma \Sigma^{1/2} \xi + \theta \Delta t. \]

Defining \( A = \Sigma^{1/2} \Gamma \Sigma^{1/2} \), we can decompose the symmetrical matrix as \( A = C' \Lambda C \), where \( \Lambda \) is the diagonal matrix constructed by the eigenvalues \( \lambda_1, \ldots, \lambda_m \) of \( A \), and \( C \) is an orthogonal matrix consisting of the corresponding eigenvectors.

Denoting \( \eta = C' \xi \), we have \( \eta \sim \mathcal{N}(\Sigma^{1/2} \mu, I) \). We can then express the change of portfolio value as

\[ \Delta v(x, y) = c' \eta + \frac{1}{2} \eta^\top \Lambda \eta + \theta \Delta t \]

where \( c = (c_1, \ldots, c_m)' = C \Sigma^{1/2} \delta \). And by assuming \( \lambda_1, \ldots, \lambda_h \neq 0 \) and \( \lambda_{h+1}, \ldots, \lambda_m = 0, h \leq m \) without loss of generality, we can have

\[ \Delta v(x, y) = \frac{1}{2} \sum_{i=1}^{h} (\lambda_i \eta_i^2 + 2c_i \eta_i) + \sum_{i=h+1}^{m} c_i \eta_i + \theta \Delta t. \]

Denote \( z_i = \eta_i + c_i/\lambda_i, i = 1, \ldots, h \). Then the first part of the right side of above equation can be written as

\[ \frac{1}{2} \sum_{i=1}^{h} (\lambda_i \eta_i^2 + 2c_i \eta_i) = \frac{1}{2} \sum_{i=1}^{h} \lambda_i z_i^2 + c_0 \]
where $c_0$ is a constant. Furthermore, it is independent with normal random variable $\sum_{i=1}^{m} c_i \xi_i$.

From the above discussion, it can be seen that $\Delta \nu(x, y)$ is actually the sum of a linear combination of some independent $\chi^2$ random variables, an independent normal random variable and a constant. Now to clarify the normality of $\Delta \nu(x, y)$, we only need to verify the normality of a linear combination of independent $\chi^2$ random variables. We can prove the following result:

**Proposition 2.** Suppose $z_1, \ldots, z_n$ are independent normal random variables with means $\zeta_i, i = 1, \ldots, n$ and unit variances, i.e., $z_i \sim \mathcal{N}(\zeta_i, 1)$. Denote $Z_n = \sum_{i=1}^{n} \lambda_i z_i^2$ and $a_i = \sqrt{\sum_{i=1}^{n} \lambda_i^2 (1 + 2z_i^2)}$. Then we have

$$Z_n \xrightarrow{\mathcal{L}} \mathcal{N}(\zeta, 1)$$

where $E(Z_n)$ and $V(Z_n)$ denote the mean and variance of $Z_n$, and the symbol $\xrightarrow{\mathcal{L}}$ means convergence in distribution.

**Proof.** See Appendix B. $\square$

For a clear understanding of the condition of Proposition 2, let’s consider the special case with $\lambda_i = \lambda$ and $\zeta_i = 0$ for $i = 1, \ldots, n$. The condition holds for this special case since

$$\lim_{n \to \infty} \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{\frac{1}{2}} = \lim_{n \to \infty} \left( \frac{n^{\frac{1}{2}}}{(\sqrt{n})^k} \right)^{\frac{1}{2}} = \lim_{n \to \infty} \sqrt{2n\left(1 + \frac{1}{n}\right)} = 0 \quad \forall k \geq 3.$$

Essentially, except for the assumption of normal distribution on the returns of underlying assets, the condition of Proposition 2 indicates that there are two key points for $Z_n = \sum_{i=1}^{n} \lambda_i z_i^2$ to be close to a normal distribution: Sufficiently large $n$ and no dominant $\lambda_i$. Intuitively, these two points can be satisfied for a sufficiently diversified portfolio. In plain language, Proposition 2 says that the return of a sufficiently diversified hedged portfolio is close to a normal distribution if the returns of underlying assets are normally distributed. Now assume that $\Delta \nu(x, y)$ is normally distributed. Then several other well-known risk measures, such as MAD (Mean-absolute Deviation, see Konno and Yamazaki, 1991), VaR (see Jorion, 2007), CVar (see Rockafellar and Uryasev, 2000), can be uniformly defined as follows:

$$\rho_{\text{risk}}(x, y) = \tau_1 \mu(x, y) + \tau_2 \sqrt{\sigma^2(x, y)}$$

where $\tau_1$ and $\tau_2$ are two constant parameters independent of decision variables $x$ and $y$. Thus, portfolio optimization using these risk measures is equivalent to that using variance as a risk measure (Some mild condition is needed for VaR and CVar model, see Rockafellar and Uryasev, 2000).

Artzner et al. (1999) proposed four axioms to qualify risk measures and called any risk measure satisfying these four axioms a coherent risk measure. Of the four popular risk measures mentioned above, only CVar qualifies as a coherent risk measure. Fortunately, in the situation of a normal distribution, portfolio optimization under a return-risk framework with risk measures other than variance is almost equivalent to that under the mean-variance framework.

We can conclude from the above discussion that, under a normal market condition, variance is a suitable risk measure for diversified hedged portfolios. But downside risk measures, such as VaR and CVar, are better choices for non-diversified hedged portfolios even in a normal market. While could include downside risk measures in the model, we will focus on the mean-variance formulation as it is sufficient for diversified hedged portfolios.

### 2.2.2. Risk measure under crash

In this paper, a crash is identified as the situation where almost all asset prices fall suddenly. In such a situation, almost all asset returns are corrected perfectly and the change in portfolio value can not be well modeled as a random variable. Since a risk measure is usually defined as a moment (e.g., variance and MAD), a quantile (e.g., VaR) or a quantile-based moment (e.g., CVar) of the random return/loss, it is not suitable for measuring crash risk.

When a crash happens, loss of a portfolio is almost an inevitable and can be interpreted, to a large extent, as the realization of portfolio return in the worst-case market condition. Noting its speciality, we follow Wilmott (2007) and define the crash risk measure as

$$\rho_{\text{crash}}(x, y) \equiv \min_{\Delta \nu(x, y)}$$

where $\Delta \nu(x, y)$ is calculated according to (9) and (11) by omitting $e_i$’s and $\hat{\nu}$ is defined as

$$\hat{\nu} = \{ f : || \Delta \nu(x, y) - \hat{f} || \leq 1 \}.$$
Generally, how to choose \( \lambda_1/\lambda_2 \) and \( \Delta u_0/f_0 \) should be problem-oriented. In this paper, we suggest an optimization approach to determine these parameters. More specifically, we can determine them by solving an ellipsoid covering problem, which can be equivalently formulated as a semidefinite program. The details will be discussed in empirical test in Section 3.

2.3. Optimization of hedged portfolio with crash risk control

As we have noted above, crash risk is completely different from the normal risk. We propose a portfolio analysis framework that takes crash risk into account as this can help investor avoid large losses. Such an analysis can be used to deal with portfolio risk more flexibly and effectively.

Let \( \mathcal{X} \) denote the set of all admissible portfolios:

\[
\mathcal{X} = \left\{ (x, y) \mid p_a^\text{ask} x + p_b^\text{bid} y \leq 1, \; l_i \leq p_a x_i \leq u_i, \; l_j \leq p_b y_j \leq u_j \right\},
\]

where \( p_a \) and \( p_b \) are the prices of the derivative assets and their underlying assets at time 0, \( l_i \), \( l_j \) and \( u_i \), \( u_j \) are the lower and upper bounds of the investment amount of the derivative assets and the underlying assets, respectively. The price of an option is different when it is bought and sold. The difference between the two prices is exactly the ask-to-bid spread which is a nonnegligible transaction cost in derivative market. Incorporating the ask-to-bid spread, the portfolio set \( \mathcal{X} \) has the following form:

\[
\mathcal{X} = \left\{ (x, y) \mid \begin{pmatrix} p_a^\text{ask} x^+ + p_b^\text{bid} x^- \leq 1, \\
l_i \leq p_a x_i \leq u_i, \\
l_j \leq p_b y_j \leq u_j \end{pmatrix} \right\}
\]

where \( p_a^\text{ask} \) and \( p_b^\text{bid} \) are vectors of the ask and bid prices, respectively, and \( x^+ \) and \( x^- \) are the vectors with \( \max(x_i, 0) \) and \( \min(x_i, 0) \) being the \( i \)th element, respectively. For the ask-to-bid spreads in option strategies, please refer to Santa-Clara and Saretto (2009), Eraker (2013) and Faias and Santa-Clara (2017).

Following the framework of mean-variance analysis (Markowitz, 1952), we propose a mean-variance model with crash risk control as follows:

\[
(P) \quad \max_{x,y} \mu(x, y) \\
\text{s.t.} \quad \sigma(x, y) \leq \hat{\sigma} \quad \rho_{\text{crash}}(x, y) \geq \hat{\rho} \\
(x, y) \in \mathcal{X}
\]

where \( \hat{\sigma} \) and \( \hat{\rho} \) are pre-determined risk tolerance parameters with respect to the risk under normal condition and the crash risk under extreme condition.

Remark 2. In accordance with Wilmott (2007), the crash risk is defined as the “worst-case return”. Thus crash risk constraint is defined as \( \rho_{\text{crash}}(x, y) \geq \hat{\rho} \) in (P) to control the extreme loss under a crash.

As compared to the bicriteria mean-variance model, problem (P) is a tricriteria decision model integrating crash risk. The model (P) we propose is different from the existing tri-criteria portfolio selection models, such as Mean-Variance-Skewness model (Briere et al., 2007) and Mean-Variance-CVaR model (Gao et al., 2016). In our model, the two risk criteria are defined according to portfolio value distributions under normal v.s. crash market situations whereas in other models risks have been defined in terms of a common portfolio value distribution. Optimizing a utility function, for example, constant relative risk aversion (CRA) function, is also a method widely used for portfolio construction, especially for option strategies (Bliss and Panigirtzoglou, 2004; Faias and Santa-Clara, 2017). The expected utility maximization framework integrates all distribution information in a clear way. However, it is inconvenient for investors to specify their utility functions. Even if the utility function is determined, the limitation may remain. As for the CRRA utility, the existence of expectation of CRRA utility is extremely fragile with respect to distribution assumption (Geweke, 2001). In contrast, the return-risk framework does not take into account all the distribution information. But it captures the key point of return-risk tradeoff and facilitates the practical application.

In problem (P), we can use (9) or (12) to define crash risk \( \rho_{\text{crash}}(x, y) \), and regardless of which definition of crash risk is adopted, it can be translated into a semidefinite program. In the following, to avoid unnecessary repetition, we only show the details of reformulation of problem (P) with crash risk defined by (9) and the details of using (12) is proposed in Appendix B.

First, we show that the first constraint on normal risk in (P) can be formulated as a second-order cone constraint. Following Cui et al. (2013), we give the derivation below. Attention needs to be paid to the fact that we can rewrite the second term in the right side of equation (5) as

\[
\frac{1}{2} x' x = \frac{1}{2} \sum_{i=1}^{n} x_i \sigma_i \geq 0
\]

Since \( \Sigma \) is a covariance matrix, it is positive semidefinite and can be decomposed as \( \Sigma = \Sigma^\frac{1}{2} \Sigma^\frac{1}{2} \) where \( \Sigma^\frac{1}{2} \) is symmetrical. Notice that \( \text{tr}(ABC) = \text{tr}(BCA) \). Then we further have

\[
\frac{1}{2} x' x = \text{tr} \left( \sum_{i=1}^{n} x_i \Gamma_i \right) \geq 0
\]

where the last inequality is from the fact that \( \text{tr}(A^2) \geq 0 \) holds for any real symmetrical matrix \( A \). Thus, \( \Phi \) in (14) is a positive semidefinite matrix.

Since both \( \Phi \) and \( \Sigma \) are positive semidefinite, we can decompose them as

\[\Phi = M'M \text{ and } \Sigma = L'L.\]

Denote

\[H = L(\Gamma_1 \mu + \delta_1, \cdots, \Gamma_n \mu + \delta_n) + \left( \begin{array}{c} 1/\sqrt{2}M_0 \\ 0 \end{array} \right).\]

By (5), we can rewrite \( \sigma(x, y) \leq \hat{\sigma} \) as the following second-order cone constraint

\[
\left[ \begin{array}{c} x' y' \\ \hat{\sigma} \end{array} \right] \geq \text{socp} 0 \quad \text{(15)}
\]

which means that \( \left[ \begin{array}{c} x' y' \\ \hat{\sigma} \end{array} \right] \leq \hat{\sigma} \).

Second, we show that the constraint on crash risk defined by (9) can be reformulated as a semidefinite constraint. Notice the constraint is defined as

\[
\min_{\Delta u} \delta' \Delta u + \frac{1}{2} \Delta u' \Gamma \Delta u + \theta' \Delta t \geq \hat{\rho}.
\]

Now we invoke the following well-known S-Lemma (see, e.g., Boyd et al., 1994) to demonstrate that constraint (16) can be equivalently formulated as a semidefinite constraint.
Lemma 1. Let \( F_k(z) = x^TA_kz + 2b_k^Tz + c_k \), \( k = 0, \ldots, m \), be quadratic functions of \( z \in \mathbb{R}^n \). Then \( F_0(z) \leq 0 \) for all \( z \) such that \( F_k(z) \leq 0 \), \( k = 1, \ldots, m \), if there exist, \( \lambda_i \geq 0, i = 1, \ldots, m \), such that

\[
\lambda_i c_k^T b_k < 0 \quad \text{for} \quad k = 1, \ldots, m,
\]

where \( \lambda_i \geq 0 \) means that \( \lambda_i \) is positive semidefinite. Moreover, if \( m = 1 \), then the converse holds if there exists \( z_0 \in \mathbb{R}^n \) satisfying \( F_1(z_0) < 0 \).

Denote

\[
A_0(x) = \frac{\Gamma}{Z}, \quad b_0(x, y) = \frac{\delta}{Z}, \quad c_0(x) = \bar{\rho} - \theta \Delta t, \quad A_1 = \Lambda_1^{-1} \Lambda_1, \quad b_1 = -\Lambda_1 \Delta u_0, \quad c_1 = \Delta u_0 \Delta u_0 - 1.
\]

Then, according to the notations of Lemma 1, \( \delta \Delta u + \frac{1}{Z} \Delta u' \Gamma \Delta u + \theta \Delta t \geq \bar{\rho} \) and \( \Delta u \in \mathcal{U} \) are \( F_0(\Delta u) \leq 0 \) and \( F_1(\Delta u) \leq 0 \), respectively. Noting that \( F_1(\Lambda_1^{-1} \Delta u_0) < 0 \). by Lemma 1, the crash risk constraint is satisfied if and only if there exists a real number \( \lambda \geq 0 \) such that

\[
\lambda \left( \begin{array}{c} c_1 \\ b_1 \\ A_1 \end{array} \right) - \left( \begin{array}{c} c_0(x) \\ b_0(x, y) \\ A_0(x) \end{array} \right) \geq_{\text{SDP}} 0.
\]

Note that \( A_0(x), b_0(x, y) \) and \( c_0(x) \) are linear in \( x \) and/or \( y \).

Turning to the portfolio set \( \mathcal{X} \), we notice that the term \( (p_{hi}^b)^Tx \) in the budget constraint in \( \mathcal{X} \) is not convex with \( x \). We rewrite the budget constraint:

\[
\sum_{i=1}^n \left( p_{hi}^a x_i + p_{hi}^b x_i \right) + p_{hi}^y y \leq 1.
\]

Due to the fact that the ask price is not lower than the bid price for an option, the constraint above can be transformed in the form:

\[
\sum_{i=1}^n \max \left( p_{hi}^a x_i, p_{hi}^b x_i \right) + p_{hi}^y y \leq 1.
\]

which is equivalent to a group of linear constraints

\[
\sum_{i=1}^n z_i + p_{hi}^y y \leq 1,
\]

\[
p_{hi}^a x_i \leq z_i, \quad i = 1, \ldots, n,
\]

\[
p_{hi}^b x_i \leq z_i, \quad i = 1, \ldots, n.
\]

Thus, portfolio set \( \mathcal{X} \) is a polyhedral set formed by linear constraints.

Using (4), (15), (17) and (18), we get the following proposition which means that the portfolio optimization problem with crash risk control can be solved by a tractable convex programming approach.

Proposition 3. Problem (P) with crash risk defined by (9) can be transformed to the following equivalent semidefinite program (SDP):

\[
\begin{aligned}
\text{(P)} \quad & \max_{x, y, \lambda} \quad \eta \cdot x + \mu \cdot y \\
\text{s.t.} \quad & \left( \begin{array}{c} H(x) \\ y \end{array} \right) \geq_{\text{SOCP}} 0,
\end{aligned}
\]

\[
\lambda \left( \begin{array}{c} c_1 \\ b_1 \\ A_1 \end{array} \right) - \left( \begin{array}{c} c_0(x) \\ b_0(x, y) \\ A_0(x) \end{array} \right) \geq_{\text{SDP}} 0,
\]

\[
\sum_{i=1}^n z_i + p_{hi}^y y \leq 1,
\]

\[
p_{hi}^a x_i \leq z_i, \quad i = 1, \ldots, n,
\]

\[
p_{hi}^b x_i \leq z_i, \quad i = 1, \ldots, n.
\]

\[
\lambda \geq 0.
\]

If we omit the second constraint in (P), then it is referred as a second-order cone program (SOCP). SOCP and SDP are both instances of linear conic program, as they are a linear optimization problem under constraints represented by second-order cones or constraints represented by cones of positive semidefinite matrices. Both of them can be regarded as an extension of linear program (LP). The interior point methods applied in LP can be easily extended to SOCP and SDP, which has been extensively investigated over the last two decades. LP is in fact a special case of SOCP, and SOCP is a special case of SDP, i.e., LP \( \subseteq \) SOCP \( \subseteq \) SDP. The reader is referred to Lobo et al. (1998) and Alizadeh and Goldfarb (2003) for details of SOCP problems, and Vandenberghe and Boyd (1996) for details of SDP problems. The wide applications of conic programming in finance can be found in Cornuejols and Tütüncü (2006).

Remark 3. The portfolio optimization problem (P) can also be defined as one of minimizing normal risk (under constraints placed on expected returns) as well as crash risk, and in this way can also be reformulated as an SDP. Reformulation (P3) of (P) in terms of SDP with crash risk defined by (12) using a factor model is given in Appendix C.

3. Empirical analysis

In this section, we mainly conduct the simulation/empirical test to compare the performance of portfolio strategies generated by different models. The following four portfolio strategies are considered:

1) Crash strategy without factor generated by model (P1) without using factor model;
2) Crash strategy generated by model (P2) using factor model;
3) No crash strategy generated by model (P2) without crash risk constraint;
4) No option strategy generated by model (P2) without options.

Simulation/empirical test is based on historical data related to the index of Dow Jones Industrial Average (DJS), the constituent stocks of DJS and the options written on these stocks and index. In the following, we first investigate the characteristics of value/price change of DJS constituent stocks, which is used to verify the motivation of this paper. We then conduct simulation and empirical analysis to compare in-sample efficient frontiers and out-of-sample performance of different strategies by using data sets of the DJS constituents and options written on them. Finally, we conduct empirical test on the performance of different strategies based on data sets of DJS index and the associate index options in Section 3.3.

All the calculations are conducted on a personal computer using Matlab R2015b and Stata 11.0, and the SDP problem is solved via CVX, which is a Matlab-based modeling package for convex optimization problems.
3.1. Characteristics of value change of stocks

In this subsection, we investigate the characteristics of the value/price change of DJ3 constituents. We perform a test for one dimensional normality of stock returns using samples from January 2000 to December 2018. After deleting 2 constituents with missing data, 28 constituents are retained in the empirical study. Notice that $\Delta u_i = u_{i+1} - u_i = r_i u_i^0$ where $r_i$ denotes the return over time period $[0, \Delta t]$. Thus it can be inferred that the properties of $\Delta u_i$ are characterized equivalently by $r_i$ since $u_i^0$ is a pre-given parameter.

We perform statistical test on the normality of stock returns by using different data samples associated with different time frequencies. Since the test of joint normality in a high-dimensional case is a hard thing, we just perform the test for one-dimensional case in this paper.

We apply both Kolmogorov-Smirnov (K-S) test and Jarque-Bera (J-B) test of the normality of price change for each constituent of DJ3 index. The data samples are divided into three categories: (i) full samples, (ii) full samples after removing those three times standard deviation above or below the mean and (iii) full samples after removing those two times standard deviation above or below the mean. Confidence levels for statistical tests are set as $\alpha = 5\%$ and $1\%$.

The results of the test are summarized in Table 1. From the table, we can see that for daily data with relatively high frequency, price changes of a large part of stocks are non-normal whether or not the extreme data has been removed. But for weekly data and monthly data, after removing extreme cases with two standard deviations above or below the mean, the normality assumption on the price changes cannot be rejected for more than 95% of the stocks.

As a stock market index is a weighted sum of prices of individual stocks, sharp rises or falls of market index signal that the market is abnormal. Thus, according to a market index, we can divide stock data samples into three types: boom samples, crash samples and normal samples. Based on the results of normality test, we cluster the historical returns of the corresponding index as follows: returns with two standard deviations above the mean, returns with two standard deviations below the mean, and the rest. Accordingly, we classify the individual stock data samples as boom samples, crash samples and normal samples, respectively.

Factor model is usually used to describe the common features of a group of stocks. By statistical analysis, two-factor models are enough to model the returns of the constituents of DJ3 index. Fig. 3 illustrates the distribution of two standardized factors for the historical daily returns of DJ3 constituents. Typically, the most typical characteristic is that the factor values during both a boom and a crash scatter near the two poles of the ellipsoid which covers all the realizations of factor values. This finding confirms that it is reasonable to use the worst-case realization of the portfolio returns within an ellipsoid covering all the possible returns to measure the crash risk.

| Table 1 |
| --- |
| Percentage of DJ3 constituents passed normality test. | |
| | Full samples | Out 3 Std removed | Out 2 Std removed |
| | | K-S test | J-B test | K-S test | J-B test | K-S test | J-B test |
| $\alpha$ | Frequency | | | | | | |
| daily | 0.00% | 0.00% | 0.00% | 0.00% | 14.28% | 42.86% |
| monthly | 82.14% | 7.14% | 100.00% | 57.14% | 100.00% | 100.00% |
| daily | 0.00% | 0.00% | 0.00% | 0.00% | 28.57% | 53.57% |
| 1% | weekly | 7.14% | 0.00% | 75.00% | 10.71% | 100.00% | 100.00% |
| monthly | 96.42% | 17.85% | 100.00% | 71.42% | 100.00% | 100.00% |

3.2. Performance analysis of strategies using stock options

3.2.1. Data sets and model calibration

The details of data sets used in the analysis and test in this subsection are as follows:

Data set 1: (DJ3 2015) This data set consists of constituents of Dow Jones Industrial Average and corresponding American options written on these constituents. The price data of the 28 constituents are from January 2000 to December 2015 after removing two constituents with missing data. For each constituent, we choose 4 American options available in January 5, 2015, with a maturity of January 2016, including an ATM call, an ATM put, a 20% OTM call and a 20% OTM put (see Faias and Santa-Clara, 2017; Santa-Clara and Saretto, 2009). This data set contains 40 options after options with missing data and bid prices smaller than $0.3 are excluded. Weekly ask and bid price data of these options are for the period from January 2015 to January 2016.

Data set 2: (DJ3 2018) This data set consists of 28 constituents of Dow Jones Industrial Average and corresponding American options. The price data of the 28 constituents are from January 2000 to December 2018. Again for each constituent, we choose 4 American options available in January 8, 2018, with the expiration date of January 2019, including an ATM call, an ATM put, a 20% OTM call and a 20% OTM put, respectively. This data set contains 49 options after options with missing data and bid prices smaller than $0.3 are excluded. Weekly ask and bid price data of these options are for the period from January 2018 to December 2018.

All the data for stocks and options are from Bloomberg Database. We divide the time period of each data set into in-sample period and out-of-sample period. For each data set, the out-of-sample period is the same as the time period of option data, while the in-sample period is the time period of the stock data after removing the out-of-sample period.

For problem formulation related to Data sets 1 and 2, we use LSM (Least-Square Monte Carlo Simulation) method given by Longstaff and Schwartz (2001) to calculate or estimate the prices and “Greeks” of American options. Specifically, we simulate 20,000 different 5-period paths of the underlying stock price under the risk-neutral measure. By the end of last period, the option will be exercised if it is in the money. The option will be exercised prior to the last period if the value of immediate exercise is more than the expected value of continuation. LSM uses least-square method to calculate the expected value of continuation which can sharply decrease the scale of simulation. When the exercise time has been determined on each path, we can calculate the price of the American option using the mean of the discounted exercising value on these paths. The “Greeks” of American option can be calculated using finite difference method. For example, to calculate $\delta$, we change the price of the stock for one unit and calculate the difference of the option price as an approximation of $\delta$. $\gamma$ can be similarly calculated by using second order difference.

Normal risk in the model is estimated only by normal samples. As to the ellipsoid involved in (10) or (13), we adopt the Löwner-
John ellipsoid, i.e., the minimum volume ellipsoid containing all the samples, as illustrated in Fig. 2. Calibrating this ellipsoid is a convex optimization problem (see, e.g., Boyd et al., 1994), which can be solved by CVX when the number of data points is relatively small. We first calculate the center (mean) of the samples, then pick out the top ten percent data farthest from the center and reconstruct the minimum volume ellipsoid containing these data. Finally, we add the data out of the ellipsoid and reconstruct the minimum volume ellipsoid containing all the samples. All the portfolio optimization models are constructed with the weekly data facilitating description and comparison.

3.2.2. In-sample analysis: efficient frontiers

In this subsection, we compare the efficient frontiers of crash strategy without factor, crash strategy, no crash strategy and no option strategy. Investment period is set as one week, the risk free interest rate r is set to be 5% per year, the total amount of derivatives is restricted within 30%, the upper and lower bounds for holding any assets are 10% and −10%, respectively.

To calculate the efficient frontier of crash strategy without factor by model (P1), we first determine the lower and upper bounds of parameter $\hat{\sigma}$ by minimizing and maximizing $\sigma(x, y)$ under the constraint $(x, y) \in \mathcal{X}$, respectively. Then we choose 10 values of $\hat{\sigma}$ uniformly in this interval. For each choice of $\hat{\sigma}$, we set the lower and upper bounds of parameter $\hat{\rho}$ by minimizing and maximizing $\rho_{\text{crash}}(x, y)$ with constraints $\sigma(x, y) \leq \hat{\sigma}$ and $(x, y) \in \mathcal{X}$. Again we choose 10 values of $\hat{\rho}$ uniformly in this interval. Finally, for each pair of $(\hat{\sigma}, \hat{\rho})$, we solve the optimization problem (P1) to get the maximum $\mu$, denoted by $\bar{\mu}$, and consequently we get the efficient frontier of crash strategy without factor sketched by the triples of $(\bar{\mu}, \hat{\sigma}, \hat{\rho})$. The calculations of efficient frontiers of crash strategy and no option strategy according to model (P2) are basically the same as that according to model (P1). It should be mentioned that the efficient frontier of no crash strategy is with some speciality since it does not consider the parameter $\hat{\rho}$. However, it can be regarded as the one generated by the model with crash risk control where the parameter $\hat{\rho}$ is set sufficiently small.

Fig. 4 shows the efficient frontiers of the portfolio strategies under consideration for a one week investment. Comparing sub-figures (a) and (b), and (d) and (e), we can see that there are no significant differences between the efficient frontiers of strategies that utilize a factor model and those that do not. Since computational efficiency is nonetheless improved when applying a factor model the remaining simulation analysis and empirical test are based on a model that utilizes factor model (P2).

As no crash strategy does not take into account the constraint on crash risk, the corresponding efficient frontier is a two-dimensional curve lying exactly on the boundary (associated with highest crash risk) of the three-dimensional efficient frontier corresponding to crash strategy. Thus, when an extreme event happens, the possible loss of a strategy without considering the crash risk could be very large. Another interesting finding from these figures is that crash risk is positively correlated with normal risk. Hence, the traditional minimum variance strategy is usually accompanied by a relatively small crash risk, which might be the reason why the minimum variance strategy performs well in practice.

Sub-figures (c) and (f) in Fig. 4 show the efficient frontiers corresponding to no option strategy. We find that the efficient frontier of no option strategy is much narrower than that of crash strategy using options. Furthermore, the former is totally dominated by the latter. This means that using option can not only bring a more flexible control on crash risk, while also enhancing the portfolio performance. Using the leverage of option can nonetheless increase risk.

3.2.3. In-sample analysis: performance under different scenarios

In this subsection, we test the performance of different strategies in different market scenarios ranging from a crash to a normal and a boom market simulated using data set 1 and 2.

The investment time horizon is set as one week and parameter values are set $\hat{\sigma} = 0.15$ and $\hat{\rho} = -0.15$, $-0.3$ for the two data sets. The portfolio strategies are denoted as low risk, and high risk according to different choices of crash risk parameter $\hat{\rho}$. The other parameters in model (P2) are set the same as in Section 3.2.2.

We first generate the optimal crash strategy and no crash strategy with different parameters. Then we simulate 100 different market scenarios, which gradually change from crash scenarios to normal and boom scenarios. Specifically, we calculate the ellipsoid associated with factors defined by (13), and choose 100 different points uniformly on the longest axis of the ellipsoid. Attention should be paid to the fact that two factors extracted by statistic method are used in our test. According to Section 3.1, the left bot-
Fig. 4. Efficient frontiers of different portfolio strategies.

tom of the ellipsoid represents the crash scenarios, the top right corner represents the boom scenarios, and the center represents the normal scenarios. Finally, we can use these simulated values of the factors to construct different markets and calculate the portfolio value after one week.

Fig. 5 shows the performance of the two strategies at different parameter settings. The horizontal axis represents different scenarios whereby from the left to the right the market gradually changes from a crash to a boom. The vertical axis represents the portfolio value in different market conditions.

The subfigures show that when an extreme crash happens, no crash strategy suffers a large loss, while high risk strategy bears a relatively small loss and low risk strategy basically maintains its initial value. In a normal market, the performance of crash strategy and no crash strategy are similar. In a boom market, no crash strategy performs better than crash strategy. These results suggest that crash strategy helps to avoid losses in a market crash but only at the cost of losing opportunities under a boom market.

3.2.4. Out-of-sample analysis: rebalancing strategy

In this subsection, we compare the out-of-sample performance of crash strategies with no crash strategy in a rebalancing manner with real data.

The data sets described in Section 3.1 are used to construct portfolio strategies. Note that the out-of-sample period is the same as the time period of option data. For two data sets, we set \( \bar{\rho} = -0.05, -0.15 \) and \( \bar{\sigma} = 0.15 \). The portfolio strategies are denoted as low risk and high risk according to different choices of crash risk parameter \( \bar{\rho} \), respectively. The other parameters required by model \( (P_2) \) are set the same as in Section 3.2.2.

We start with an initial wealth \( P_0 = 1 \) and update the portfolio strategies every week. We estimate the required parameters at the beginning of each week using the samples taken prior to the port-
folio rebalance, and reconstruct the portfolio strategies with the updated parameters. We calculate the portfolio value at the beginning of every week during the out-of-sample period using the real data. As trading costs have a significant influence on the performance of option portfolios during the rebalancing strategy we take them into account in the form of ask-to-bid spreads in the rebalancing process.

The detailed process is described as follows. At the initial time, we solve the problem \( (p_x) \) and generate an initial portfolio \((x^0, y^0)\). Denote the cash \( k^0 = 1 - (p_x^{ask})' (x^0)' + (p_x^{bid})' (y^0)' - p_y^{ask} y^0 \), where \( p_x^{ask} \) and \( p_x^{bid} \) are the ask and bid prices of options, \( p_y \) is the constituent price at week \( t+1 \), and \( r \) is the risk-free rate. At the beginning of week \( t+1 \), we rebalance the portfolio strategy based on the existing portfolio \((x', y')\) and the cash \( k' \). The admissible portfolio \( \hat{x} \) is

\[ \hat{x} = \left\{ (x, y) \mid (p_x^{ask})' (x-x')' + (p_x^{bid})' (x-x')' + p_y (y'-y') \right\}, \]

where \( p_x^{ask}, p_x^{bid} \) are the ask and bid prices of options at week \( t+1 \), \( p_y \) is the stock price at week \( t+1 \), and \( a' \) is the vector with \( \max(a, 0) \) and \( \min(a, 0) \) being the \( i \)th elements, respectively. Solving problem \((\hat{p}_x)\) with set \( \hat{x} \) generates portfolio \((x' + 1, y' + 1)\), and the cash \( k' = k' (1 + r \Delta t) - (p_x^{ask})' (x' + 1 - x')' - (p_x^{bid})' (x' + 1 - x')' - p_y (y' + 1 - y') \). Repeat the process until the last week of the out-of-sample period, and record the trends of portfolio values.

Fig. 6 shows the portfolio value performance for different strategies. We can see from the figure that for data sets 1 and 2 there is no significant gap between the out-of-sample period performances of crash strategy and no crash strategy under a relatively stable market condition.

We also calculate the weekly returns of these strategies during out-of-sample period. Table 2 summarizes the results: mean return (MeanRet), standard deviation (Std), minimum return (MinRet), average drawdown (ADD), upside potential ratio (UP ratio) and downside Sharpe ratio (DS ratio). Here, we just recall the details of the average drawdown and upside potential ratio, because they are not as common as others in the literature. The average drawdown is defined as (Alexander and Baptista, 2006; Chekhlov et al., 2005):

\[ \text{Average drawdown} = \frac{1}{N} \sum_{t=1}^{N} d_t, \quad d_t = \max \{ (W_t - W_{t-1}) / W_t \} \]

where \( W_t \) is the portfolio value at time \( t \) during the out-of-sample period. The upside potential ratio (see Sortino and Van Der Meer, 1991) measures the ratio of higher partial moment of order 1 with the lower partial moment of order 2 under a target return level \( t \):

\[ \text{Uspide potential ratio} = \frac{\sum_{t=1}^{N} \max(\tau_t - r, 0) / N}{\sqrt{\sum_{t=1}^{N} (\min(\tau_t - r, 0))^2 / N}} \]
where $r_1, \cdots, r_N$ are weekly returns during the out-of-sample period. Here, we set $\tau$ the mean return of DJS index. The downside Sharpe ratio (see Ziemba, 2005) is defined as the ratio of the mean return to the square root of lower semivariance. According to the listed 6 performance measures, among the three type of strategies, the low risk strategies have slight outperformance when compared with high risk strategies and no crash strategies.

To detect the performance of the strategies during the 2007 – 2008 crisis, we also conduct an analysis over a time period from January 2007 through December 2018. Unfortunately, we cannot obtain real option data during the crisis. Thus, we simulate a series of European options based on the constituents of DJS index. In the test, the set of candidate assets contains the constituents of the DJS index and simulated European options based on these constituents. We use the historical prices of the constituents of the DJS index from January 2000 through December 2018 as the basis. For every month from January 2007 through December 2018, we construct 4 one-month-to-maturity options for each constituent including four options: an ATM call, an ATM put, a 5% OTM call and a 5% OTM put. All the prices and “Greeks” of these options are calculated under the Black-Scholes pricing formula.

Again we start with the initial wealth of 1 in January 2007, calculate the portfolio value by delivering the options every month and calculate new portfolios with a new group of one-month-to-maturity options. In the model, we set $\sigma = 0.1$ and $\rho = -0.05$, $-0.1$, and the resulting strategies are denoted as low risk strategy and high risk strategy, respectively.

Fig. 7 shows portfolio value of different strategies from January 2007 to December 2018. We can see that, during 2007 to 2009, the strategy with low crash risk could resist a crash and gain the best portfolio value. On the other hand, during the boom period, no crash strategy has an evident advantage of portfolio value when compared with crash strategies at both required risk levels as well as DJS index. Table 3 summarizes the monthly returns of these strategies from 2007 to 2018. The strategy with low crash risk has the minimum standard deviation, the highest minimum return, the best downside Sharpe ratio and the minimum average drawdown among these three strategies.

### 3.3. Performance analysis of strategies using index options

In this subsection, we analyze the performance of different portfolio strategies constructed using index and index options, including no crash strategy and two crash strategies with two different required crash risk levels. Compared with the data set related to constituents of DJS index, we can collect more real data of the DJS index related options, so we can conduct relatively long-term test.

Firstly, we conduct the out-of-sample performance analysis on these strategies with DJS index and the corresponding European options written on it, during June 2012 to December 2018. We collect the historical prices of the DJS index from January 2000 through December 2018. As for options, we choose 4 six-month-to-maturity options with different levels of moneyness: an ATM call, an ATM put, a 10% OTM call, and a 10% OTM put; in each group. The first group contains four options available in June 22, 2012, with an expiration at the third Friday of December 2012. Then we have another four options which are expired at the third Friday of June 2013. Other groups of options are also expired at the third Friday of December or June in a similar pattern. All the data including the ask and bid prices of the options are from Bloomberg. The “Greeks” of these options are calculated under the Black-Scholes pricing formula (See Appendix D).

We start with the initial wealth of 1 in June 2012, rebalance the strategies and evaluate the portfolio value with ask and bid prices in a similar way in Section 3.2.4 every month except June and December. In June and December, we calculate the portfolio value by delivering the options and then construct strategies with a new group of six-month-to-maturity options. The parameters of the underlying index required in the model ($P_1$) and the Black-Scholes formula are estimated based on their historical data from January

---

### Table 2

Comparison of performance of different strategies using data set 1 and 2.

| Strategy | MeanRet (%) | Std (%) | MinRet (%) | ADD (%) | UT ratio | DS ratio |
|----------|-------------|---------|------------|---------|----------|----------|
| Low risk | 0.800       | 7.700   | -19.136    | 11.208  | 0.521    | 0.163    |
| High risk| 0.990       | 8.579   | -27.052    | 11.333  | 0.480    | 0.172    |
| DJS index| -0.036      | 1.906   | -5.823     | 3.616   | 0.508    | -0.026   |
| Low risk | 1.270       | 7.433   | -17.809    | 11.131  | 0.549    | 0.281    |
| High risk| 1.260       | 7.557   | -17.687    | 11.832  | 0.542    | 0.262    |
| DJS index| -0.155      | 2.605   | -6.888     | 6.080   | 0.503    | -0.078   |

---

### Table 3

Comparison of performance of different strategies from January 2007 to December 2018.

| Strategy | MeanRet (%) | Std (%) | MinRet (%) | ADD (%) | UT ratio | DS ratio |
|----------|-------------|---------|------------|---------|----------|----------|
| Low risk | 0.796       | 2.580   | -16.060    | 3.200   | 0.526    | 0.408    |
| High risk| 0.978       | 4.032   | -27.539    | 7.534   | 0.516    | 0.309    |
| No crash | 1.159       | 5.519   | -31.289    | 12.858  | 0.570    | 0.278    |
| DJS index| 0.531       | 4.280   | -26.007    | 13.688  | 0.427    | 0.151    |
2000 to the month when the model is solved. In the model, we set $\bar{\sigma} = 0.1$ and $\bar{\rho} = -0.15, -0.2$. Again, the portfolio strategies are denoted as low risk and high risk according to different choices of crash risk parameter $\bar{\rho}$. Here, the upper and lower bounds for option are 0.5 and -0.5, respectively.

Fig. 8 shows portfolio value of different strategies from June 2012 to December 2018. According to the figure, all the strategies including options perform obviously better than the index. Another observation is that the option strategies are more volatile with larger volatilities, especially in the months when the options are delivered. Table 4 summarizes the monthly returns of these strategies from June 2012 to December 2018. The strategy with low crash risk has the minimum standard deviation, the highest minimum return and the best upside potential ratio among these three strategies.

Due to the novel coronavirus epidemic, the financial market experienced erratic fluctuations in the first half year of 2020, especially from February to May. We use a data set of European options written on the DJS index during May 2019 to May 2020 to test the performance of these strategies. The historical prices of the DJS index from January 2000 to May 2020 are collected. The data set includes 12 groups of options and each group has 4 one-month-to-maturity options with moneyness levels: an ATM call, an ATM put, a 5% OTM call and a 5% OTM put. The first group has four
options available in May 19, 2019, with an expiration at the third Friday of June 2019. The option groups afterwards are chosen in a similar manner.

We start with an initial wealth of 1 in May 2019, and rebalance the strategies every week. The portfolio value is evaluated the same way as before, and the only difference is that we evaluate the portfolio value by delivering the options in the third Friday of every month. In the model, we set $\sigma = 0.02$ and $\rho = -0.05, -0.15$. Other parameters are set in the same way. Fig. 9 displays the portfolio value of these strategies from May 2019 to May 2020. It is obvious that the low risk strategy performs better than other strategies, especially during the volatile months. Table 5 lists the different performance measures of these strategies. The strategy with low crash risk performs best in average return, upside potential ratio, downside Sharpe ratio and average drawdown among the six performance measures.

Figs. 10 and 11 show the distributions of returns of the various strategies over the out-of-sample periods. The distribution of returns of the indices is added to the comparison. We can intuitively observe that crash strategy can flexibly reshape the return-risk profile of a portfolio, and perform better than no crash strategy in balancing the return and risk.

![Fig. 11. Out-of-sample return distribution from May 2019 to May 2020.](image_url)
From these results, it seems that the model with crash risk control can generate a stable strategy by reducing the opportunities of a boom market. This is not, however, the case in practice. In the test, call and put options are both used in portfolio construction, and crash and boom data samples are both covered by the ellipsoid adopted to measure crash risk. In this setting both crash and boom could cause heavy losses. Therefore only stable strategies can satisfy the crash risk constraint. The model allows for flexibility to suit different purposes. If one wants to hedge the risk resulting from a crash but also wants to avoid losing the opportunities of a rising market, then it is possible, for example, to hold a put option while shrinking the ellipsoid for measuring crash risk covering only the crash down data samples.

4. Conclusions

In this paper, we have investigated performance measures, hedging and optimization of portfolio selection involving crash risk. The basic idea for dealing with crash risk has been to distinguish the modeling of return in normal market conditions from that in the extreme situation of a market crash. Following Wilmott (2007), we define the crash risk as the maximum potential loss in an extreme situation and show that it is reasonable to use an worst-case realization of the portfolio return within an ellipsoid to measure it. We have also clarified that the return (value change) of a sufficiently diversified hedged portfolio in normal market conditions approximated with “Greeks” is close to a normal distribution, thus its risk can be well captured by variance. We have further revealed that the optimal portfolio selection problem with crash control can be translated into an efficiently solvable semidefinite program.

In addition, we have compared the performance of a crash strategy with that of a no crash strategy via both a simulation analysis and an empirical test. The results have shown that the crash strategy can resist crash, while the no crash strategy cannot. Nonetheless the crash strategy may lose its opportunity in certain situations, for example, a stable strategy with crash control in any market conditions will bear heavy losses in a boom market. This is to be expected as the crash strategy cannot completely dominate the no crash strategy. In fact, the two-dimensional efficient frontier of no crash strategy lies exactly on the boundary of the three-dimensional efficient frontier of the crash strategy. That’s to say, the no crash strategy can be regarded as a crash strategy with a sufficiently loose parameter for crash risk control.

Finally, it should be mentioned that the simulation analysis and empirical test done in Section 3 exhibit only the outcomes for the case that the ellipsoid used to measure crash risk is selected as the one covering all the historical returns, which is clearly the most conservative but not the necessary choice. Investment is not only a “science”, but also an “art”. Although our focus has been on the science aspect, the art aspect (e.g., predicting the future and determining the location and size of the ellipsoid used to capture the crash risk) is critical to successful practice and should be problem-oriented, and to this end we suggest that the model with crash risk control offers greater flexibility.

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Appendix A. Proof of Proposition 1

Recall that \( \Delta u \sim N(\mu, \Sigma) \) and

\[
\Delta v(x, y) = \delta' \Delta u + \frac{1}{2} \Delta u' \Gamma \Delta u + \theta \Delta t
\]

where \( \delta = \sum_{i=1}^{m} s_i \delta_i + y \), \( \Gamma = \sum_{i=1}^{m} \lambda_i \Gamma_i \), and \( \theta = \sum_{i=1}^{m} \lambda_i \theta_i \).

Since \( \Sigma \) is a covariance matrix, thus it is positive semidefinite and can be decomposed as \( \Sigma = \Sigma^T \Sigma^T \) where \( \Sigma^T \) is symmetrical. Notice that \( \Sigma \) is assumed to be nonsingular. Define

\[
\xi = \Sigma^{-\frac{1}{2}} \Delta u \quad \text{and} \quad \Omega = \Sigma^{-\frac{1}{2}} \Gamma \Sigma^T.
\]

Then the change of portfolio value can be further reformulated as

\[
\Delta v(x, y) = \delta' \Sigma^{-\frac{1}{2}} \xi + \frac{1}{2} \xi' \Omega \xi + \theta \Delta t
\]

(20)

where \( \xi \sim N\left(\Sigma^{-\frac{1}{2}} \mu, I\right) \).

We can decompose the real symmetrical matrix \( \Omega = \Lambda \Lambda^T \), where \( \Lambda \) is the diagonal matrix constructed by the eigenvalues \( \lambda_1, \ldots, \lambda_m \) of \( \Omega \), and \( \Lambda \) is an orthogonal matrix consisting of the corresponding eigenvectors. Let \( z = \Lambda^T \xi \), we can express \( \frac{1}{2} \xi' \Omega \xi \) as a linear combination of independent \( \chi^2_1 \) random variables:

\[
\frac{1}{2} \xi' \Omega \xi = \frac{1}{2} \xi' \Lambda \Lambda^T \xi = \frac{1}{2} \sum_{i=1}^{m} \lambda_i z_i^2.
\]

(21)

Recall the fact that if \( \zeta \) is a normally distributed random variable with mean \( \mu \) and variance \( \sigma^2 \), then we have that the random variable \( (\frac{\zeta}{\sigma})^2 \) follows noncentral \( \chi^2 \)-distribution with mean \( 1 + (\frac{\mu}{\sigma})^2 \) and variance \( 2 + 4(\frac{\mu}{\sigma})^2 \). By (20) and (21), we have that

\[
\mu(x, y) = E(\Delta v(x, y))
\]

\[
= \frac{1}{2} \sum_{i=1}^{m} \lambda_i E(z_i^2) + \delta' \mu + \theta \Delta t
\]

\[
= \frac{1}{2} \sum_{i=1}^{m} \lambda_i \left(1 + \left(C \Sigma^{-\frac{1}{2}} \mu\right)_i^2\right) + \delta' \mu + \theta \Delta t
\]

\[
= \frac{1}{2} \sum_{i=1}^{m} \lambda_i + \frac{1}{2} \mu C \Sigma^{-\frac{1}{2}} \Lambda \Lambda^T \mu + \delta' \mu + \theta \Delta t
\]

\[
= \frac{1}{2} \text{tr}(\Gamma \Sigma) + \frac{1}{2} \mu' \Gamma \mu + \delta' \mu + \theta \Delta t
\]

\[
= \eta' x + \mu' y
\]

where \( \eta \) is defined by (6), and the fifth equation is based on the two facts that \( \text{tr}(\Gamma \Sigma) = \text{tr}\left(\Sigma^{-\frac{1}{2}} \Gamma \Sigma^T \right) \) and the trace of a matrix equals the sum of its eigenvalues.

By (20), the variance of \( \Delta v(x, y) \) is given as

\[
\sigma^2(x, y) = var(\Delta v(x, y))
\]

\[
= var\left(\delta' \Sigma^{-\frac{1}{2}} \xi\right) + 2cov\left(\delta' \Sigma^{-\frac{1}{2}} \xi, \frac{1}{2} \xi' \Omega \xi\right)
\]

(22)

CRediT authorship contribution statement

Shushang Zhu: Conceptualization, Methodology, Investigation. Wei Zhu: Methodology, Software, Writing - original draft. Xi Pei: Data curation, Writing - review & editing. Xueling Cui: Data curation, Writing - review & editing. Software.
It is easy to see that the first term of (22) equals \( \delta^2 \Sigma \delta \). Let's consider the second and the third terms in the sequel.

\[
\text{var}\left( \frac{1}{2} \xi \Omega \xi \right) = \frac{1}{4} \sum_{i=1}^{m} \lambda_i^2 \text{var}(z_i)
\]

\[
= \frac{1}{4} \sum_{i=1}^{m} \lambda_i^2 \left( 2 + 4 \left( \frac{C}{\epsilon} + 1 \right) \mu \right)^2
\]

\[
= \frac{1}{2} \sum_{i=1}^{m} \lambda_i^2 + \mu^2 \Sigma_{1}^{-1} \Xi \Sigma_{1}^{-1} \mu
\]

\[
= \frac{1}{2} \text{tr}(\Gamma \Sigma \Gamma) + \mu^2 \Sigma \Gamma \mu
\]

where the fourth equation is based on the fact that 
\( \text{tr}(\Gamma \Sigma \Gamma) = \text{tr}(\Sigma_{1} \Sigma_{1} \Sigma_{1} \Sigma_{1} \Sigma_{1}) \) and the eigenvalues of the square of a matrix are the squares of the eigenvalues of the original matrix.

Let \( \xi = \xi - \Sigma^{-1} \mu \). Then

\[
\text{cov}\left( \delta^2 \Sigma \xi, \frac{1}{2} \xi \Omega \xi \right) = \text{cov}\left( \delta^2 \Sigma \xi + \delta \mu, \frac{1}{2} \xi \Omega \xi \right)
\]

\[
= \text{cov}\left( \delta^2 \Sigma \xi, \frac{1}{2} \xi \Omega \xi \right) + \text{cov}\left( \delta^2 \Sigma \xi, \frac{1}{2} \xi \Omega \xi \right)
\]

\[
= \delta^2 \Sigma^2 \Omega \xi \xi + \delta^2 \Sigma^2 \Omega \xi \xi
\]

\[
= \delta^2 \Sigma^2 \Omega \xi
\]

\[
= \delta^2 \Sigma \Omega \xi
\]

where the third equation is based on the fact that \( \text{cov}\left( \delta^2 \Sigma \xi, \frac{1}{2} \xi \Omega \xi \right) = 0 \) due to the expectation of \( \xi \) equals zero (see Lemma 2 of Stein (1981) and page 175 of Britten-Jones and Schaefer (1999)).

By (22), we have that

\[
\sigma^2(x, y) = \delta \Sigma \delta + \frac{1}{2} \text{tr}(\Gamma \Sigma \Gamma) + \mu^2 \Gamma \Sigma \Gamma \mu + 2 \delta \Sigma \Gamma \mu
\]

\[
= (x', y') \Psi(x') + \frac{1}{2} x' \Phi x.
\]

Proof. The proof is completed. □

Remark 4. Britten-Jones and Schaefer (1999) derive the similar results for the special case of \( \Delta u \sim N(0, \Sigma) \). Cui et al. (2013) derive the same results under the additional condition that \( \Gamma \) is nonsingular.

Appendix B. Proof of Proposition 2

We first introduce the following Lévy’s continuity lemma (Van der Vaart, 1998):

Lemma 2. For random variables \( Y_n \) and \( Y \), \( Y_n \rightarrow Y \) if and only if \( \lim_{n \rightarrow \infty} \text{E}(e^{itY_n}) = \text{E}(e^{itY}) \) for every \( t \in \mathbb{R} \). Here the symbol ‘\( \rightarrow \)’ means convergence in distribution, \( \text{E}(e^{itY_n}) \) and \( \text{E}(e^{itY}) \) are the characteristic functions of \( Y_n \) and \( Y \).

Now we use Lemma 2 to prove Proposition 2. Denote \( Y_n = \frac{2n\sigma^2(x, y)}{\sqrt{n(\delta^2(l) + \lambda)}^2} \). Since the characteristic function of standard normal distribution is \( e^{-\frac{1}{2}t^2} \), according to Lemma 2, to prove \( Y_n \sim N(0, 1) \), we only need to prove \( \lim_{n \rightarrow \infty} \text{E}(e^{itY_n}) = e^{-\frac{1}{2}t^2} \) or equivalently \( \lim_{n \rightarrow \infty} \text{ln}(\text{E}(e^{itY_n})) = -\frac{1}{2}t^2 \). Recall that the mean and variance of \( z_i^2 \) are \( 1 + \lambda_i^2 \) and \( 2 + 4\lambda_i^2 \). Let \( M = \sqrt{\sum_{j=1}^{n} \lambda_i^2 (1 + 2\xi_j^2)} \). Notice the fact that the characteristic function of \( z_j^2 \) is \( (1 - 2it)^{-\frac{1}{2} e^{\frac{\delta^2(l) + \lambda}{2M}} t} \), then we have

\[
\text{E}(e^{itY_n}) = \text{E}\left( e^{\frac{\sum_{j=1}^{n} \lambda_i^2 (1 + 2\xi_j^2)(1 + 2\xi_j^2)}{M}} \right)
\]

\[
= e^{\frac{\sum_{j=1}^{n} \lambda_i^2 (1 + 2\xi_j^2)}{M}}
\]

\[
= \prod_{j=1}^{n} \left( 1 - it(2\xi_j^2) \right) = e^{\frac{\sum_{j=1}^{n} \lambda_i^2 (1 + 2\xi_j^2)}{M}}
\]

Denoting \( a_j = \sqrt{2\xi_j^2} \) and taking logarithm to the above equation yields

\[
\text{ln}(\text{E}(e^{itY_n})) = -\frac{1}{2} \sum_{j=1}^{n} \text{ln}(1 - ita_j^2) + \frac{1}{2} \sum_{j=1}^{n} \frac{ita_j^2}{1 - ita_j^2} \left( 1 + \frac{\xi_j^2}{1 + \xi_j^2} \right)
\]

Notice that the assumption \( \lim_{n \rightarrow \infty} \left( \sum_{j=1}^{n} a_j^2 \right)^{\frac{1}{2}} = 0 \) for each \( k \geq 3 \) implies \( \lim_{n \rightarrow \infty} a_j = 0 \). Thus for any given \( t \) there is a sufficient large \( n \) such that \( |ita_j| < 1 \). According to Taylor’s expansion

\[
\text{ln}(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} x^n \quad \text{and} \quad \frac{1}{1 + x} = \sum_{n=1}^{\infty} x^n \quad \text{for} \ |x| < 1
\]

we have for sufficient large \( n \) that

\[
\text{ln}(\text{E}(e^{itY_n})) = \frac{1}{2} \sum_{j=1}^{n} ita_j^2 - \frac{1}{2} \sum_{j=1}^{n} \frac{ita_j^2}{1 - ita_j^2} \left( 1 + \frac{\xi_j^2}{1 + \xi_j^2} \right)
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \frac{ita_j^2}{1 - ita_j^2} \left( 1 + \frac{\xi_j^2}{1 + \xi_j^2} \right)
\]

\[
= \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{\infty} \left( \frac{1}{1 + \xi_j^2} \right) a_j^2 t^{k} \epsilon t^k
\]

where the first and the second lines are from the first and the second orders of Taylor’s expansion, respectively.

Since \( \lim_{n \rightarrow \infty} \left( \sum_{j=1}^{n} a_j^2 \right)^{\frac{1}{2}} = 0 \) for each \( k \geq 3 \), for any given \( t \) and \( \epsilon \), there exists a sufficiently large \( N \) such that \( \left| \left( \sum_{j=1}^{n} a_j^2 \right)^{\frac{1}{2}} \right| < \epsilon \). Noting \( \xi_j \) is bounded, there is a bounded constant \( c \) satisfying \( \epsilon + \xi_j^2 < c \). Now we have

\[
\sum_{k=1}^{\infty} \sum_{j=1}^{n} \left( \frac{1}{1 + \xi_j^2} \right) a_j^2 t^{k} \epsilon t^k < \sum_{k=1}^{\infty} c t^{k} \epsilon t^k
\]

which means \( \lim_{n \rightarrow \infty} \left( \sum_{j=1}^{n} a_j^2 \right)^{\frac{1}{2}} \left( 1 + \xi_j^2 \right) a_j^2 t^{k} \epsilon t^k = 0 \). This further implies \( \lim_{n \rightarrow \infty} \text{ln}(\text{E}(e^{itY_n})) = -\frac{1}{2}t^2 \).

Proof. The proof is completed. □

Appendix C. Semidefinite programming reformulation of (P) with crash risk defined by (12)

We just need to show that the constraint on crash risk defined by (12) can be reformulated as a convex semidefinite constraint
similar to that defined by (9). According to (12), we have
\[
\Delta P(x, t) = \delta^T (\beta_0 + Bf) + \frac{1}{2} (\beta_0 + Bf)^T \Gamma (\beta_0 + Bf) + \theta \Delta \tau
\]
where \( \beta_0 = (\beta_{10}, \ldots, \beta_{m0})^T \) and \( B = (\beta_1, \ldots, \beta_m)^T \) and \( f = (f_1, \ldots, f_l)^T \).
Then the constraint on crash risk is equivalent to
\[
\min_{f \in I} (\delta^T (\beta_0 + Bf) + \frac{1}{2} (\beta_0 + Bf)^T \Gamma (\beta_0 + Bf) + \theta \Delta \tau) \geq \tilde{\rho}.
\]
Now we use Lemma 1 to demonstrate that (24) can be reformulated as a semidefinite constraint.

\[
A_0(x) = -\frac{1}{2} B^T \Gamma B
\]
\[
b_0(x, y) = -\frac{1}{2} (B^T \delta + B^T \beta_0)
\]
\[
c_0(x, y) = \tilde{\rho} - \delta^T \beta_0 - \frac{1}{2} \beta_0^T \Gamma \beta_0 - \theta \Delta \tau t
\]
\[
A_1 = \Lambda_2^T \Lambda_2
\]
\[
b_1 = -\Lambda_2^T f_0
\]
\[
c_1 = f_0^T - 1.
\]
According to the notations of Lemma 1, \((\delta^T (\beta_0 + Bf) + \frac{1}{2} (\beta_0 + Bf)^T \Gamma (\beta_0 + Bf) + \theta \Delta \tau) \geq \tilde{\rho}\) is actually \(F_0(f) \geq 0,\) and \(f \in I\) is actually \(F_1(f) \leq 0,\) Noting that \(F_1(\Lambda_2^T f_0) < 0,\) by Lemma 1, the crash risk constraint is satisfied if and only if there exists a real number \( \lambda \geq 0 \) such that
\[
\lambda (c_1 b_1^T A_1) - (c_0(x, y) b_0(x, y)) A_0(x)) \geq SPD 0.
\]
By using (4), (15) and (25), we get the following equivalent SDP reformulation of (P2):

\[
(P_2) \max_{x, y, \lambda} \eta x + \mu y
\]
\[
\text{s.t.} \quad \begin{pmatrix} H(x, y) \\ \lambda c_1 b_1^T A_1 - (c_0(x, y) b_0(x, y)) A_0(x) \end{pmatrix} \geq SPD 0
\]
\[
(x, y) \in \chi, \quad \lambda \geq 0.
\]

Appendix D. “Greeks” European options

The values of European call and put options in terms of Black-Scholes formula are given as:
\[
d^e(S, t) = SN(d_1) - Ke^{-r(T-t)} N(d_2),
\]
\[
d^p(S, t) = Ke^{-r(T-t)} N(-d_2) - SN(-d_1),
\]
where
\[
d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}
\]
\[
d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}
\]
\(S\) – the underlying asset price at current time \(t;\)
\(K\) – the exercise price;
\(T\) – the expiry time;
\(\sigma\) – the volatility of the underlying asset return;
\(r\) – the riskless return;
\(N(x)\) – cumulative probability distribution function of standard normal distribution.

The “Greeks” under Black-Scholes pricing formula are summarized in Table 6.

| Call option | Put option |
|-------------|------------|
| Delta \(N(d_1)\) | \(N(-d_1) - 1\) |
| Gamma \(\frac{N(d_1)}{\sigma \sqrt{T-t}}\) | \(-\frac{N(-d_1)}{\sigma \sqrt{T-t}}\) |
| Theta \(-\frac{N(d_1)}{\sigma T} + rKe^{-r(T-t)} N(-d_2)\) | \(-\frac{N(-d_1)}{\sigma T} - rKe^{-r(T-t)} N(d_2)\) |

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