Abstract

We present the results of the computation of a twisting type N solution to vacuum Einstein equations following an iterative approach. Our results show that the higher order terms fail to provide a full exact solution with non-vanishing twist. Nevertheless, our fourth-order solution still represents a regular and twisting type N solution.
1. Introduction

There is an interesting unsolved problem of finding solutions of vacuum Einstein equations that represent gravitational radiation outside a bounded source. The non-twisting solutions of type N, either approximate or exact, have always singularities in space at infinity, hence they are not good candidates for this purpose. Also, there have been various different efforts to find twisting type N solutions [1, 2] and, so far, there is only one exact solution with non-vanishing twist reported by Hauser [1]. Hauser’s solution also has singularities which extend to spatial infinity. Thus, it fails to provide an exact description of the gravitational radiation.

Recently, Stephani [3] studied a linearised form of twisting type N solutions and claimed that either there will always exist singular lines in space for twisting solutions or that twisting type N fields do not describe a radiation field outside a bounded source. Later on, in [6] an iterative approach has been proposed in an attempt to solve this puzzle.

Using this iterative approach, the linear solution in [3] has been reproduced and higher order terms of this iteration process have been described in [6]. In particular, the authors have used a first-order solution, flat and non-twisting, to derive an approximate non-flat twisting solution. This solution is given by regular functions for sufficiently large spheres (i.e. spheres defined by the ($\zeta, \bar{\zeta}$) coordinates for large values of the affine parameter $r$). This is an interesting example of twisting solutions, where behaviour is contrary to the expectations generated by Stephani’s paper.

After all this, one is wondering whether this iterative approach could shed light on the general behaviour of solutions describing gravitational radiation from a bounded source. In this note, we present the solution obtained by iterating the third-order solution in [6] up to higher orders. Unfortunately, nothing can be said about the existence of a regular exact solution with a non-vanishing twist parameter.

2. The vacuum field equations

Following the notation in [6], we write the metric $g$ in terms of a complex null
tetrad, $e^\mu$ as follows:

$$g = g_{\mu\nu} e^\mu \otimes e^\nu = 2e_1 \otimes e^2 + 2e_3 \otimes e^4, \quad \overline{e^1} = e^2, \text{ and } e^3, e^4 \text{ real}, \quad (1)$$

the overbar indicates complex conjugation. For any vacuum, type-N space-time with non-vanishing complex expansion, i.e., where $Z \equiv -\Gamma_{421} \neq 0$, the null tetrad $e^\mu$ can always be written as [4]

$$e^1 = \frac{1}{PZ}d\zeta, \quad e^2 = \frac{1}{PZ}d\overline{\zeta}, \quad -e^3 = du + Ld\zeta + \overline{L}d\overline{\zeta}, \quad e^4 = dr + Wd\zeta + \overline{W}d\overline{\zeta} - H e^3, \quad (2)$$

where the metric functions are given by

$$Z^{-1} = (r - i\Sigma), \quad 2i\Sigma = P^2(\overline{DL} - DL)$$

$$W = -\overline{Z}^{-1}L_u + iD\Sigma, \quad D \equiv \partial_\zeta - L \partial_u \quad (3)$$

$$H = -r \partial_u(\ln P) + \frac{1}{2}K, \quad K = 2P^2 \Re[D(\overline{DL} - DL_u)]$$

where the subscripts indicate partial differentiation. Within this tetrad, setting $P \equiv V_u$, the remaining Einstein vacuum field equations take the form

$$\overline{D}\left\{P^{-1}D_uDV\right\} = 0, \quad \overline{DDDV} = DDD\overline{DV} \quad (4)$$

The curvature and twist are given below and we must require not to be zero

$$\frac{1}{2}C^{(1)} \equiv \Psi_4 = -ZP^2\partial_u\partial_u \left\{P^{-1}\overline{DDV}\right\} \neq 0, \quad 2i\Sigma = P^2 \left(\overline{DL} - DL\right) \neq 0 \quad (5)$$

The particular form of this (non-zero) curvature component is however not an invariant of this problem, since it may be changed, remaining non-zero, by a choice of
transformation to new (allowed) coordinates. It is known that for type N spacetimes there are no non-zero invariants of the curvature tensor, which requires us to use the gauge invariants of the problem. Therefore, we should use the two (gauge) invariants of this problem, the ratios $\Sigma/r$ and $K/r^2$ [5] with the condition that $\Sigma$ and $K$ be regular functions of their arguments, (see also [7]).

Fixing the gauge such that

$$P = 1 + \frac{1}{2} \zeta \bar{\zeta}; \quad V = Pu$$

(6)

and defining

$$\Phi \equiv L, \zeta + \frac{\zeta}{1 + \frac{1}{2} \zeta \bar{\zeta}} - \zeta L - L \partial_u L$$

(7)

then the remaining Einstein vacuum field equations (4) can be written as (see [6] for details)

$$\partial_\zeta \partial_u \Phi - \bar{L} \partial_u^2 \Phi = 0$$

(8)

$$\text{Im} \left\{ \partial_\zeta \partial_\zeta \Phi + \frac{\zeta}{1 + \frac{1}{2} \zeta \bar{\zeta}} \partial_\zeta \Phi - \bar{L} \partial_u \Phi - L \partial_\bar{u} \Phi \right\} = 0.$$  

(9)

The N-th order approximation will then be one that approximates the complete solution, up to that order of iteration, as a sum of all the preceding orders

$$\Phi \approx \sum_{j=1}^{N} \Phi^{(j)}, \quad L \approx \sum_{j=1}^{N} L^{(j)}$$

(10)

where $L^{(0)} = 0$ and $\Phi^{(0)} = 0$ give the flat space-time. The case $N = 1$ accounts for the linearised version in [3] where $\Phi^{(1)}$ and $L^{(1)}$ can be obtained from the equations

$$\partial_\zeta \partial_u \Phi^{(1)} = 0 \quad \text{and} \quad P^{-2} \left\{ P^2 L^{(1)} \right\}_{\zeta} = \Phi^{(1)}.$$  

(11)
At the n-th level of iteration, $n \geq 2$ we have

$$\partial_\zeta \partial_u \Phi^{(n)} = \sum_{j=1}^{n-1} \mathcal{L}^{(n-j)} \partial_u^2 \Phi^{(j)},$$

$$(12)$$

$$P^{-2} \left\{ P^2 \mathcal{L}^{(n)} \right\} = \partial_\zeta \mathcal{L}^{(n)} + \left( \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \right) \mathcal{L}^{(n)} = \Phi^{(n)} + \sum_{j=1}^{n-1} \mathcal{L}^{(n-j)} \partial_u \mathcal{L}^{(j)}.$$  

whose solutions are obtained from straightforward calculations up to some arbitrary functions that can be fixed with the eqn. (9) constraint. Using (5), the curvature may be explicitly re-written in terms of the current variables:

$$r \Psi_4 = P^2 (1 + i \Sigma/r)^{-1} \partial_u^2 \Phi.$$  

$$(13)$$

the n-th level of the gauge invariants $\Sigma = \sum \Sigma^{(n)}$ and $K = \sum K^{(n)}$ at the lowest order terms are $\Sigma^{(0)} = 0$ and $K^{(0)} = 1$. The higher order terms are given by

$$\Sigma^{(1)} = P^2 \text{Im} \left[ \partial_\zeta \mathcal{L}^{(1)} \right]$$

$$(14)$$

$$K^{(1)} = -2P^2 \text{Re} \left[ \partial_\zeta \partial_u \mathcal{L}^{(1)} \right]$$

$$(15)$$

and for $n \geq 2$

$$\Sigma^{(n)} = P^2 \left\{ \text{Im} \left[ \partial_\zeta \mathcal{L}^{(n)} \right] - \sum_{j=1}^{n-1} \text{Im} \left[ \mathcal{L}^{(n-j)} \partial_u \mathcal{L}^{(j)} \right] \right\}$$

$$(16)$$

$$K^{(n)} = -2P^2 \left\{ \text{Re} \left[ \partial_\zeta \partial_u \mathcal{L}^{(n)} \right] - \sum_{j=1}^{n-1} \text{Re} \left[ \mathcal{L}^{(n-j)} \partial_u^2 \mathcal{L}^{(j)} \right] \right\}$$

$$(17)$$

Next, given a solution at first order one could in principle compute a series that may converge or not.
3. Fourth-order solution

Starting with the ansatz,

\[
L^{(1)} = \frac{1}{\zeta} \quad \text{and} \quad \Phi^{(1)} = \frac{a_1}{\zeta} \left( \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \right)
\]

and following the iterative approach described above, the authors of [6] computed a twisting solution to the third level of iteration. Now, using (12) and solving for \( L^{(4)} \) and \( \Phi^{(4)} \) we have the fourth-order terms given as

\[
L^{(4)} = \frac{a^{(4)}}{\zeta} + \frac{\bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} f^{(4)}(u) + \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \left( \frac{a^{(2)}}{\zeta} \frac{df^{(2)}}{du} + \frac{a^{(1)}}{\zeta} \frac{df^{(3)}}{du} \right) - \frac{\bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \left( 2 f^{(2)}(u) \frac{df^{(2)}}{du} \right) - \frac{1}{\zeta} \left( 1 + \frac{(\zeta \bar{\zeta})^2}{2} \right) \left( \frac{a^{(1)}}{\zeta} \frac{d^2 f^{(2)}}{du^2} (a^{(1)} - a^{(1)}) \right) + \frac{a^{(1)}}{\zeta} \frac{d^2 f^{(2)}}{du^2} \left[ (1 - \frac{(\zeta \bar{\zeta})^2}{2}) \ln(\zeta \bar{\zeta}) - \frac{\zeta \bar{\zeta}}{2} (\ln(\zeta \bar{\zeta}))^2 \right]
\]

\[
\Phi^{(4)} = -\frac{a^{(2)}}{\zeta^2} \frac{df^{(2)}}{du} - \frac{a^{(1)}}{\zeta^2} \frac{df^{(3)}}{du} + \frac{a^{(4)}}{\zeta} \left( \frac{\bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} - \frac{1}{\zeta} \right) - \frac{a^{(1)}}{\zeta} \frac{a^{(1)}}{\zeta^2} \frac{d^2 f^{(2)}}{du^2} \ln(\zeta \bar{\zeta})
\]

that satisfies the constraint (9). Now, one can compute \( \Sigma^{(4)} \) and \( K^{(4)} \) from (16) and (17), thus

\[
\Sigma^{(4)} = \frac{(1 - \frac{1}{2} \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \text{Im} \left( f^{(4)}(u) \right) - 2 \text{Re} \left( a^{(1)} \right) \left( \text{Im} \left( \frac{df^{(3)}}{du} \right) - 2 \text{Re} \left( a^{(2)} \right) \left( \text{Im} \left( \frac{df^{(2)}}{du} \right) \right) \right) - \frac{(1 - \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \text{Im} \left( \frac{d(f^{(2)})^2}{du} \right) + 2 \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \text{Re} \left( a^{(1)} \frac{d^2 f^{(2)}}{du^2} \right) \left( \text{Im} \left( a^{(1)} \right) \right) - \frac{\zeta \bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \text{Im} \left( f^{(2)} \frac{df^{(2)}}{du} \right) - a^{(1)} a^{(1)} \text{Im} \left( \frac{d^2 f^{(2)}}{du^2} \right) \left[ 2 \ln(\zeta \bar{\zeta}) + \frac{1}{2} \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} (\ln(\zeta \bar{\zeta}))^2 \right]
\]
\[ K^{(4)} = -2 \frac{(1 - \frac{1}{2} \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \Re \left( \frac{df^{(4)}}{du} \right) + 4 (\Re a^{(2)}) (\Re \frac{d^2 f^{(2)}}{du^2}) + 4 (\Re a^{(1)}) (\Re \frac{d^2 f^{(2)}}{du^2}) \]
\[ + 2 \frac{(1 - \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \Re \left( \frac{d^2 (f^{(2)})^2}{du^2} \right) + \frac{2 \zeta \bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \Re \left( f^{(2)} \frac{d^2 f^{(2)}}{du^2} \right) \]
\[ + 4 \frac{(1 - \frac{1}{2} \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \Re \left( \frac{(\zeta \bar{\zeta}^2) d^3 f^{(2)}}{du^3} \right) \]
\[ + 2a^{(1)} \Re \left( \frac{d^3 f^{(2)}}{du^3} \right) \left[ 2 \ln(\zeta \bar{\zeta}) + \frac{1}{2} \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} (\ln(\zeta \bar{\zeta}))^2 \right] \]

(22)

Next, adding all the preceding orders of interaction we have the fourth-order approximation as follows

\[ L \approx \frac{a_1}{\zeta} + \frac{\zeta}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} f_2(u) + \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \left( \frac{a_1 df_2}{\zeta du} \right) \]
\[ - \frac{\zeta}{(1 + \frac{1}{2} \zeta \bar{\zeta})^3} (2f^{(2)}(u) \frac{df^{(2)}}{du}) - \frac{1 + \frac{1}{2} \zeta^2}{\zeta(1 + \frac{1}{2} \zeta \bar{\zeta})^2} a^{(1)} \frac{d^2 f^{(2)}}{du^2} (a^{(1)} - \bar{a}^{(1)}) \]
\[ + \frac{a^{(1)} \bar{a}^{(1)}}{\zeta(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \frac{d^2 f^{(2)}}{du^2} \left[ (1 - \frac{\zeta \bar{\zeta}}{2}) \ln(\zeta \bar{\zeta}) - \frac{\zeta \bar{\zeta}}{2} (\ln(\zeta \bar{\zeta}))^2 \right] \]

(23)

\[ \Phi \approx -\frac{a_1}{\zeta^2} \frac{df_2}{du} + \frac{a_1}{\zeta} \left( \frac{\zeta}{1 + \frac{1}{2} \zeta \bar{\zeta}} - \frac{1}{\zeta} \right) - \frac{a^{(1)} \bar{a}^{(1)}}{\zeta^2} \frac{d^2 f^{(2)}}{du^2} \ln(\zeta \bar{\zeta}) \]

(24)

\[ \Sigma \approx \frac{(1 - \frac{1}{2} \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \Im \left( f_2(u) \right) - 2 (\Re a_1) (\Im \frac{df_2}{du}) \]
\[ - \frac{(1 - \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \Im \left( \frac{d(f^{(2)})^2}{du} \right) + 2 \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} \Re \left( a^{(1)} \frac{d^2 f^{(2)}}{du^2} \right) (\Im a^{(1)}) \]
\[ - \frac{\zeta \bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \Im \left( f^{(2)} \frac{df^{(2)}}{du} \right) - a^{(1)} \bar{a}^{(1)} \Im \left( \frac{d^2 f^{(2)}}{du^2} \right) \left[ 2 \ln(\zeta \bar{\zeta}) + \frac{1}{2} \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} (\ln(\zeta \bar{\zeta}))^2 \right] \]

(25)
\[ K \approx 1 - 2 \frac{(1 - \frac{1}{2} \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \text{Re} \left( \frac{df_2}{du} \right) + 4(\text{Re} \ a_1)(\text{Re} \frac{d^2 f_2}{du^2}) \]
\[ + 2 \frac{(1 - \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \text{Re} \left( \frac{d^2 (f^{(2)})}{du^2} \right) + \frac{2 \zeta \bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} \text{Re} \left( f^{(2)} \frac{d^2 f^{(2)}}{du^2} \right) \]
\[ + 4 \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right) \text{Re} \left( \frac{d^3 f^{(2)}}{du^3} \right) \]
\[ + 2 \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \text{Re} \left( \frac{d f^{(2)}}{du} \right) \]
\[ + 2 \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \text{Im} \left( \frac{d f^{(2)}}{du} \right) \]
\[ + 4 \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \text{Re} \left( \frac{d^2 f^{(2)}}{du^2} \right) \]
\[ + 2 \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \text{Im} \left( \frac{d^2 f^{(2)}}{du^2} \right) \]
\[ + 2 \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \text{Re} \left( \frac{d^3 f^{(2)}}{du^3} \right) \]
\[ + 2 \left( 1 - \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \left( 1 + \frac{1}{2} \zeta \bar{\zeta} \right) \text{Im} \left( \frac{d^3 f^{(2)}}{du^3} \right) \]

with
\[ a_1 = \sum_{j=1}^{4} a^{(j)}; \quad f_2 = \sum_{j=2}^{4} f^{(j)} \]

In order to avoid the singularities deriving from the logarithmic terms, we can choose the first term of the function \( f_2 \) as \( f^{(2)} = iku \) with \( k \) a real constant and leaving \( f^{(3)} \) and \( f^{(4)} \) as arbitrary functions of \( u \). Then, we have a regular fourth-order solution given by
\[ L \approx \left( \frac{2k^2}{1 + \frac{1}{2} \zeta \bar{\zeta}} + ik \right) \frac{\bar{\zeta} u}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} + a_1 \bar{\zeta} + \frac{\bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} f_3(u) + \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}} a_1 \frac{df_3}{du} + ik \] (27)

\[ \Phi \approx -ik a_1 \frac{\bar{\zeta}}{\zeta^2} - a_1 \frac{df_3}{du} + a_1 \frac{\bar{\zeta}}{\zeta} \frac{d^2 f_3}{du^2} - 1 \zeta \] (28)

\[ \Sigma \approx k [\frac{(1 - \frac{1}{2} \zeta \bar{\zeta}) u}{(1 + \frac{1}{2} \zeta \bar{\zeta})} - 2(\text{Re} \ a^{(1)})] + \frac{(1 - \frac{1}{2} \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \text{Im} \ (f_3(u)) - 2(\text{Re} a_1)(\text{Im} \frac{df_3}{du}) \] (29)

\[ K \approx -4 \frac{(1 - \zeta \bar{\zeta}) k^2}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} + 1 - 2 \frac{(1 - \frac{1}{2} \zeta \bar{\zeta})}{(1 + \frac{1}{2} \zeta \bar{\zeta})} \text{Re} \left( \frac{df_3}{du} \right) + 4(\text{Re} a_1)(\text{Re} \frac{d^2 f_3}{du^2}) \] (30)

and
\[ \Psi_4 \approx \bar{a}_1 \frac{1 + \frac{1}{2} \zeta \bar{\zeta}}{\zeta} 2 \frac{d^3 f_3}{du^3} \] (31)
with

\[ a_1 = \sum_{j=1}^{4} a^{(j)}; \quad f_3 = f^{(3)} + f^{(4)} \]

At this point we have a non-flat twisting and regular solution of type N. We have assumed that there are no non-zero invariants of the curvature tensor and used only the gauge invariants. Further calculations indicate that from the ansatz (18) we have started with it is not possible to generate well behaved higher order corrections of this solution. A new ansatz is needed in order to continue the study of the twisting type N solutions using this iterative method.

We can only conclude that there is not yet exact and regular twisting type N solutions. We would need to improved our methods to seek for regular twisting type N solutions or to find a consistent proof of the non-existence of regular type N solutions with non vanishing twist.

NOTE ADDED: After this work was completed, we learned of the work by J. Bičák and V. Pravda [7], in which the scalar curvature invariants for type N are studied.

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