A STRONG MINIMUM PRINCIPLE AND LARGE TIME ASYMPTOTICS FOR VISCOSITY SOLUTIONS TO A CLASS OF DOUBLY NONLINEAR POSSIBLY DEGENERATE PARABOLIC EQUATIONS

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Abstract. We study a version of the strong minimum principle, and large time asymptotics of positive viscosity solutions to classes of doubly nonlinear parabolic equations of the form

\[ H(Du, D^2 u) - u^{k-1}u_t = 0, \quad k \geq 1, \quad \text{in } \Omega \times [0,T), \]

where Ω ⊂ ℝ^n is a bounded domain and 0 < T ≤ ∞. The spatial operator H is homogeneous with power k.

1. Introduction

Let Ω ⊂ ℝ^n, n ≥ 2, be a bounded domain, and ̄Ω be its closure. For 0 < T ≤ ∞, define \( \Omega_T = \Omega \times (0,T) \). If T = ∞, we write \( \Omega_\infty = \Omega \times (0,\infty) \). Let \( P_T = (\overline{\Omega} \times \{0\}) \cup (\partial \Omega \times [0,T)) \), and \( P_\infty = P_T \) with \( T = \infty \). These are the parabolic boundaries of \( \Omega_T \) and \( \Omega_\infty \) respectively. Let \( u = u(x,t) : \Omega_T \to [0,\infty) \). For \( k \geq 1 \), set

\[ \Gamma_k[u] := H(Du, D^2 u) - u^{k-1}u_t, \]

where H is an operator that satisfies certain conditions, described later in this section, and k represents the homogeneity of H.

We introduce notation for a more precise formulation of the questions studied in this work. The letters x, y and z will often denote points in ℝ^n. We reserve o for the origin. There will be occasions where we write \( x = (x_1, x_2, \cdots, x_n) \). The notation \( S^n \) is for the set of all \( n \times n \) real symmetric matrices, I is the identity matrix and O is the zero \( n \times n \) matrix. The letters e and ω will often stand for unit vectors in ℝ^n.

In this work, we study a version of the strong minimum principle and large time asymptotic behaviour of continuous positive viscosity solutions to the following doubly nonlinear parabolic equation

\[ \Gamma_k[u] = 0, \quad \text{in } \Omega_T \text{ and } u = h \text{ on } P_T, \]

where \( \Gamma_k \) is as in (1.1) and \( h = h(x,t) \in C(P_T) \). We allow \( T = \infty \) in what follows.

The function \( h = h(x,t) \), for \((x,t) \in P_T\), comprises the initial and side conditions and is as given below:

\[ h(x,t) = \begin{cases} 
  h(x,0) & \forall x \in \overline{\Omega}, \; t = 0, \\
  h(x,t) & \forall (x,t) \in \partial \Omega \times [0,T).
\end{cases} \]
The function $h \in C(P_T)$ in the sense that

(i) If $(x, t) \in \partial \Omega \times (0, T)$ and $(x, t) \to (y, 0)$, $y \in \partial \Omega$, then $\lim_{(x, t) \to (y, 0)} h(x, t) = h(y, 0)$.

(ii) If $x \in \overline{\Omega}$ and $x \to y$, $y \in \overline{\Omega}$, then $\lim_{(x, 0) \to (y, 0)} h(x, 0) = h(y, 0)$.

We assume throughout that

(1.4) $0 < \inf_{P_T} h(x, t) \leq \sup_{P_T} h(x, t) < \infty$.

We describe now the conditions satisfied by $H$. These apply throughout the work.

**Condition A (Monotonicity):** Assume that $H : \mathbb{R}^n \times S^n \to \mathbb{R}$ is continuous. Moreover, we require that $H(\varphi, O) = 0$, for any $\varphi \in \mathbb{R}^n$. For any $X, Y \in S^n$ with $X \leq Y$,

$H(\varphi, X) \leq H(\varphi, Y), \ \forall \varphi \in \mathbb{R}^n$. □

**Condition B (Homogeneity):** There is a constant $k_1 \geq 0$ such that $\forall (\varphi, X) \in \mathbb{R}^n \times S^n$,

$H(\theta \varphi, X) = |\theta|^{k_1} H(\varphi, X), \ \forall \theta \in \mathbb{R}$, \ and \ $H(\varphi, \theta X) = \theta H(\varphi, X), \ \forall \theta > 0$. □

Note that we do not assume that $H$ is odd in $X$. Also, if $k_1 = 0$ then $H(\varphi, X) = H(X)$.

Set $k = k_1 + 1$. Clearly,

(1.5) $H(\theta \varphi, \theta X) = \theta^k H(\varphi, X)$ where $k = k_1 + 1$ and $\theta > 0$.

Let $\Lambda \in \mathbb{R}$ and $e \in \mathbb{R}^n$ be a unit vector. Define

$m(\Lambda) = \min_{|e| = 1} \left( \min H(e, I - \Lambda e \otimes e), -\max_{|e| = 1} H(e, \Lambda e \otimes e - I) \right)$, and

$M(\Lambda) = \max_{|e| = 1} \left( \max H(e, I - \Lambda e \otimes e), -\min_{|e| = 1} H(e, \Lambda e \otimes e - I) \right)$.

Clearly, $m(\Lambda) \leq M(\Lambda)$. By Condition A, if $\Lambda \leq 1$ then $m(\Lambda) \geq 0$, since $I - \Lambda e \otimes e \geq 0$.

However, if $\Lambda > 1$ then no definite statement can be made about $I - \Lambda e \otimes e$.

**Condition C (Coercivity):** We require that $H$ satisfy

(1.7) $C(i) \ m(\Lambda) > 0$, $\forall \Lambda < 1$, \ and \ $C(ii) \ M(\Lambda) < 0$, $\forall \Lambda \geq \Lambda_1$,

for some $\Lambda_1 \geq 1$. □

We make a simple observation. If $\Lambda = 0$ then $C(i)$ implies that

(1.8) (i) $H(e, I) \geq m(0) > 0$, \ and \ $H(e, -I) \leq -m(0) < 0$. 

One of the major origins of motivation for this work is the class of parabolic equations studied in [9]: see Chap II. In particular, we refer to equation (1.1) and the conditions in (A1), (A2) and (A3) therein. The example of the parabolic equation

\[(*) \quad \text{div}(|Du|^{p-2}Du) + |Du|^p = u_t, \quad p > 1,\]

is included in [9]. If we use the change of variables \(v = e^u\) then we obtain the well-known doubly nonlinear parabolic equation

\[(**e) \quad \text{div}(|Du|^{p-2}Du) = v^{p-2}v_t.\]

See Subsection 2.2 for more details. The operator \(H(Du, D^2u) := \text{div}(|Du|^{p-2}Du)\) is quasilinear, \(k = p - 1\), and odd in second derivatives. It is easy to see that Conditions A and B are satisfied, if \(p \geq 2\). Also,

\[H(e, I - \Lambda e \otimes e) = (n + p - 2) - (p - 1)\Lambda.\]

If \(n \geq 2\), Condition C is satisfied. Thus, many of our results would hold for (*), for \(p \geq 2\), suitably modified by the application of the change of variable \(v = e^u\).

The monograph [9] studies equations like (*), in greater generality, and in the context of weak solutions. It contains significant results regarding local behaviour including regularity. Our context in our current work is is the setting of viscosity solutions. We study equations like (**e) in this context and also operators \(H\) that may be fully nonlinear. Examples such as the Pucci operators, including their degenerate versions, are instances included here. While our focus is on equations of the type in (**e), we do utilize versions of the kind (*) (in our context it would be \(H(Du, D^2u + Du \otimes Du))\) to show that a version of the comparison principle holds. In the context of viscosity solutions, this property has great utility.

Further examples of operators \(H\) that satisfy Conditions A, B and C include, the pseudo \(p\)-Laplacian \((p \geq 2)\), the infinity-Laplacian and the Pucci operators. See [6]: Section 3 and [7]: Section 3 for a detailed discussion.

As indicated above, doubly nonlinear parabolic equations are of great interest and there are many works that address them. The ones that are directly related to this work are [2, 3, 4, 5, 7, 15]. The works in [1, 12, 16] are also of interest in this context. As discussed above, the work in [9] has also close connections with this topic.

The results in this work are of two kind. The first addresses the strong minimum principle for \(\Gamma_k\). For \(k = 1\), we show that the results known for the linear case appear to hold even though \(H\) may be fully nonlinear, see [10, 14]. For \(k > 1\), however, there appears to be a departure from the linear case, as our results will show. Many of the results, known for \(k = 1\), fail to hold.

In this context, it is well-known there is a close connection between the strong minimum principle and the Harnack inequality. The latter is known for many of the examples of \(H\) listed above. However, we have been unable to provide a unified proof of a version that holds
for the entire class of operators $\Gamma_k$ being addressed in this work. To better appreciate the importance of Harnack’s inequality, we direct the reader to the references [9, 10, 13, 14]. The texts [10, 14] address the linear case; [9] develops techniques for studying classes of nonlinear, degenerate parabolic equations. These provide deep insights into the properties of solutions to such equations. A further expanded version of these topics can be found in [11].

The second set of results address large time asymptotic behaviour of positive solutions to (1.2) (equations like (**) are included here). We provide a general result and follow it up by proving a result that applies to the case when the side condition $h$ is a constant. In the latter, the case $k = 1$ appears to be different from $k > 1$. In this connection see also [1, 5, 12].

We provide additional definitions. Also, from hereon, we take $k \geq 1$.

Let $U \subset \mathbb{R}^{n+1}$ be a domain. By $usc(lsc)(U)$, we mean the set of all upper semi-continuous (lower semi-continuous) functions defined on the set $U$.

We provide a definition of a viscosity solution of

\[(1.9) \quad \Gamma_k[u] \equiv H(Du, D^2u) - u^{k-1}u_t = 0, \quad \text{in } \Omega_T \quad \text{and} \quad u = h \quad \text{on } P_T.\]

A function $u \in usc(\Omega_T)$, $u > 0$, is said to be a viscosity sub-solution of the differential equation in (1.9) in the set $\Omega_T$ (or solves $\Gamma_k[u] \geq 0$ in $\Omega_T$), if, for any $\psi$, $C^2$ in $x$ and $C^1$ in $t$, such that $u - \psi$ has a maximum at some point $(y, t) \in \Omega_T$, we have

\[H(D\psi, D^2\psi)(y, t) - u(y, t)^{k-1}\psi_t(y, t) \geq 0.\]

We say $u$ is a sub-solution of the problem in (1.9), if $u \in usc(\Omega_T \cup P_T)$, $\Gamma_k[u] \geq 0$ in $\Omega_T$, and $u \leq h$ on $P_T$.

Similarly, $u \in lsc(\Omega_T)$, $u > 0$, is said to be a viscosity super-solution of the differential equation in (1.9) in $\Omega_T$ (or solves $\Gamma_k[u] \leq 0$, in $\Omega_T$), if, for any $\psi$, $C^2$ in $x$ and $C^1$ in $t$, such that $u - \psi$ has a minimum at some $(y, t) \in \Omega_T$, we have

\[H(D\psi, D^2\psi)(y, t) - u(y, t)^{k-1}\psi_t(y, t) \leq 0.\]

We say $u$ is a super-solution of the problem in (1.9), if $u \in lsc(\Omega_T \cup P_T)$, $u > 0$, $\Gamma_k[u] \leq 0$ in $\Omega_T$, and $u \geq h$ on $P_T$.

A function $u \in C(\Omega_T)$ is a solution of $\Gamma_k[u] = 0$ in $\Omega_T$, if it is both a sub-solution and a super-solution. Similarly, $u \in C(\Omega_T \cup P_T)$ is a solution of the problem in (1.9), if it is both a sub-solution and a super-solution of (1.9). The above definitions can be extended to the case $T = \infty$.

We state next the main results. Theorems 1.1 and 1.2 address issues related to the strong minimum principle. Such results are well-known for many equations, see, for instance, [10, 14]. Our goal is to provide a unified proof that works for a large class of equations and in the viscosity framework.
In order to clarify the role of the hypotheses, described earlier, the operator $H$ will be assumed to satisfy Conditions A, B and C unless otherwise mentioned.

For the rest of the work, $B_\rho(x)$ is the $\mathbb{R}^n$ ball centered at $x \in \mathbb{R}^n$ with radius $\rho$. In what follows, a vector $\gamma \in \mathbb{R}^{n+1}$, will sometimes be written as $\gamma = (\gamma_1, \cdots, \gamma_{n+1}) = (\tilde{\gamma}_n, \gamma_{n+1})$, where $\tilde{\gamma}_n \in \mathbb{R}^n$.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $T > 0$. Suppose that $k = 1$. Assume that $u \in lsc(U_T)$, satisfies

$$\Gamma_1[u] \equiv H(D^2u) - u_t \leq 0, \text{ in } \Omega_T.$$ 

Set $m = \inf_{\Omega_T} u$.

(a) **Hopf Boundary principle.** Assume that $\partial \Omega \in C^2$. Let $(p, \tau) \in \partial \Omega \times (0, T)$ be such that $u(p, \tau) = m$ and $u(x, t) > m$, near $(p, \tau)$. Suppose that $\gamma \in \mathbb{R}^{n+1}$ is such that $\gamma_{n+1} \leq 0$, $\tilde{\gamma}_n$ is not tangential to $\partial \Omega$ and points towards the interior of $\Omega$. Suppose that, for some $\theta_0 > 0$, small, $(p, \tau) + \theta \gamma \in \Omega_T$, for every $\theta \in (0, \theta_0)$, then

$$\lim_{\theta \to 0^+} \inf_{\theta \gamma} \frac{u(p + \theta \gamma, \tau + \theta \gamma_{n+1}) - u(p, \tau)}{\theta} > 0.$$ 

(b) **Strong Minimum principle.** Let $(p, \tau) \in \Omega_T$ be such that $u(p, \tau) > m$. Then $u(x, t) > m$ for every $(x, t) \in \Omega \times (\tau, T)$. The result holds without any assumptions on the smoothness of $\partial \Omega$. As a consequence, we get that if $u(p, \tau) = m$ then $u(x, t) = m$ in $\Omega \times (0, \tau)$.

There are no restrictions on the sign of $u$. A proof is presented in Section 3. The proof of part (b) is achieved by using slanted cylinders, see also [14].

We address $k > 1$ in the next result.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $T > 0$. Suppose that $k > 1$. Assume that $u \in lsc(\Omega_T)$, $u > 0$, is super-solution, i.e.,

$$\Gamma_k[u] \equiv H(Du, D^2u) - u^{k-1}u_t \leq 0 \text{ in } \Omega_T.$$ 

Set $m = \inf_{\Omega_T} u$. The following hold.

(a) Suppose that $m > 0$. If for some $(p, \tau) \in \Omega_T$, $u(p, \tau) > m$ then there is a $\rho > 0$ such that $u > m$ in the cylinder $B_\rho(p) \times [\tau, T)$. As a consequence, if $u(p, \tau) = m$ then $u(p, s) = m$ for all $0 < s < \tau$.

(b) Suppose that $m = 0$ and $(p, \tau) \in \Omega_T$ is such that $u(p, \tau) = 0$. Assume that $u \in C(\Omega_T)$. Then there is a sequence of points $\{(x_\ell, t_\ell)\}_{\ell=1}^\infty \subset \Omega_T$, such that $t_\ell < \tau$, $u(x_\ell, t_\ell) = 0$ and $(x_\ell, t_\ell) \to (p, \tau)$.

A proof appears in Section 4. An example shows that the result in Part (a) can not be improved. Also, the Hopf boundary principle may not hold if $k > 1$.

The next two results address large time asymptotic behaviour. See [11, 5, 12].
Theorem 1.3. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, and $h \in C(P_{\infty})$ satisfy (1.3) and (1.4). Suppose that $k \geq 1$.

(a) Let $u \in lsc(\Omega_{\infty} \cup P_{\infty}), u > 0$, be a super-solution to (1.2), i.e., $\Gamma_k[u] \leq 0$. Assume that $u = h$ on $\partial \Omega \times [T, \infty)$. If $\nu_{\text{inf}} = \lim_{t \to \infty} \left( \inf_{\Omega \times [t, \infty)} h \right)$ then

$$\lim_{t \to \infty} \left( \inf_{\Omega \times [t, \infty)} u \right) = \nu_{\text{inf}}.$$ 

(b) Let $u \in usc(\Omega_{\infty} \times P_{\infty}), u > 0$, be a sub-solution to (1.2), i.e., $\Gamma_k[u] \geq 0$. Assume that $u = h$ on $\partial \Omega \times [T, \infty)$. If $\mu_{\text{sup}} = \lim_{t \to \infty} \left( \sup_{\Omega \times [t, \infty)} h \right)$ then

$$\lim_{t \to \infty} \left( \sup_{\Omega \times [t, \infty)} u \right) = \mu_{\text{sup}}.$$ 

See Section 5 for a proof.

The next result provides a slight refinement in the special case where $h \equiv$ constant.

Theorem 1.4. Let $\Omega$ be a bounded domain that satisfies an uniform outer ball condition. Suppose that, for some $\nu \in \mathbb{R}$, $h = \nu$, on $\partial \Omega \times [T, \infty)$ for some $T \geq 0$.

For parts (a) and (b), assume that $\nu > 0$ and $k > 1$. Assume that the sub(super)-solution $u = \nu$ on $\partial \Omega \times [T, \infty)$. The following holds for any $\alpha < 1/(k - 1)$.

(a) If $u > 0$ is a subsolution then $\lim_{t \to \infty} t^{\alpha} \left( \sup_{\Omega \times [t, \infty)} u - \nu \right) = 0$.

(b) If $u > 0$ is a supersolution then $\lim_{t \to \infty} t^{\alpha} \left( \nu - \inf_{\Omega \times [t, \infty)} u \right) = 0$.

(c) Suppose that either $\nu = 0$ and $k \geq 1$, or $\nu > 0$ and $k = 1$. Let $\lambda_{\Omega}$ be the first eigenvalue of $H$ on $\Omega$. If $u \geq 0$ is a sub-solution then

$$\lim_{t \to \infty} \left( \sup_{x \in \Omega} \log \frac{u(x, t)}{t} \right) \leq -\lambda_{\Omega}.$$ 

The result does not appear to hold for super-solutions. For more details and a proof of the theorem, see Section 6.

In this work, we do not address existence issues for the parabolic problems (1.2). See [7]: Theorems 1.2 and 1.3] for a discussion of such issues, see also [2, 3].

2. Preliminaries

We present some elementary calculations that will be useful in the work. Included here are also a few results related to a comparison principle for parabolic equations.

For the rest of the work, $B_{\rho}(x)$ is the $\mathbb{R}^n$ ball centered at $x \in \mathbb{R}^n$ with radius $\rho$. 
2.1. Radial functions. Let \( z \in \mathbb{R}^n \) and \( r = |x - z| \). Suppose that \( v(x) = v(r), \ r \geq 0, \) is \( C^2 \) in \( r > 0 \). Set \( e = (x - z)/r, \) in \( r > 0 \). Then for \( x \neq z, \)
\[
(2.1) \quad H(Dv, D^2v) = H\left(v'(r)e, \frac{v'(r)}{r} (I - e \otimes e) + v''(r)e \otimes e\right),
\]
where \( I \) is the \( n \times n \) identity matrix.

If \( v(r) = r^\alpha, \ \alpha > 0, \) then
\[
H(Dv, D^2v) = \alpha^k r^{\alpha - (k+1)} H(e, I + (\alpha - 2)e \otimes e).
\]

2.2. Change of variable formula. See Lemma 2.3 in [7]: Section 3 for a more general statement. This implies that if \( u \in \text{usc}(\text{lsc})(\Omega_T), \ u > 0, \) satisfies
\[
H(Du, D^2u) - u^{k-1}u_t \geq (\leq)0 \quad \text{in } \Omega,
\]
and \( w = \log u, \) then \( w \in \text{usc}(\text{lsc})(\Omega_T) \) and
\[
H(Dw, D^2w + Dw \otimes Dw) - w_t \geq (\leq)0 \quad \text{in } \Omega_T.
\]
These are in the sense of viscosity. The elliptic counterpart appears in [6]: Section 5. See also [2, 3].

2.3. Parabolic Comparisons. We state versions of the comparison principle used in this work. Note that in all the results stated here, \( \Omega \subset \mathbb{R}^n \) is a bounded domain and \( 0 < T < \infty \). However, many of these continue to hold for \( T = \infty, \) by letting \( T \to \infty. \)

We begin with a well known general principle about sub-solutions that we state without proof.

Lemma 2.1. Suppose that \( H \) satisfies Condition A. Let \( u \in \text{usc}(\Omega_T \cup P_T), \ u \geq 0, \) solve \( \Gamma_k[u] \geq 0 \) in \( \Omega_T, \) and \( \varepsilon \geq 0. \) Then the function \( u_\varepsilon = \max\{u, \varepsilon\} \) solves
\[
\Gamma_k[u_\varepsilon] \geq 0 \quad \text{in } \Omega_T.
\]
Similarly, if \( v \) is super-solution i.e., \( \Gamma_k[v] \leq 0 \) then \( v_\varepsilon = \min\{v, \varepsilon\} \) is a super-solution.

Suppose that \( F : \mathbb{R}^+ \times \mathbb{R}^n \times S^n \to \mathbb{R} \) is continuous and satisfies
\[
(2.2) \quad F(t, \varphi, X) \leq F(t, \varphi, Y), \ \forall (t, \varphi) \in (0, T) \times \mathbb{R}^n, \text{ with } X \leq Y,
\]

Lemma 2.2. (Comparison principle) Let \( F \) be as in (2.2), and \( f : (0, T) \to (0, T) \) be continuous. Suppose that \( u \in \text{usc}(\Omega_T \cup P_T) \) and \( v \in \text{lsc}(\Omega_T \cup P_T) \) satisfy
\[
F(t, Du, D^2u + Du \otimes Du) - f(t)u_t \geq 0 \quad \text{and} \quad F(t, Dv, D^2v + Dv \otimes Dv) - f(t)v_t \leq 0
\]
in \( \Omega_T. \) If \( \sup_{P_T} v < \infty \) and \( u \leq v \) on \( P_T \) then \( u \leq v \) in \( \Omega_T.
\]

See [7]: Lemma 4.1, Section 4. See also [8], for a more general result. We apply the above to obtain:
Theorem 2.3. (Comparison principle) Let $H$ satisfy Conditions A and B. Suppose that $u \in \text{usc}(\Omega_T \cup P_T)$, $u > 0$, and $v \in \text{lsc}(\Omega_T \cup P_T)$, $v > 0$, satisfy

$$\Gamma_k(u) \geq 0, \quad \text{and} \quad \Gamma_k(v) \leq 0, \quad \text{in} \ \Omega_T.$$

Let $k \geq 1$, then the following quotient type comparison result holds in $\Omega_T$:

$$\frac{u}{v} \leq \sup_{P_T} \left( \frac{u}{v} \right).$$

Additionally, if $k = 1$ then $u - v \leq \sup_{P_T} (u - v)$. This holds without any sign restrictions.

A proof can be found in [7]: Theorem 4.3, Section 4 (in Theorem 4.3, take $\phi(t) = e^t$). The functions $\phi = \log u$ and $\psi = \log v$ satisfy the equations of the kind in Lemma 2.2. It follows that $\phi - \psi \leq \sup_{P_T} (\phi - \psi)$. The conclusion of Theorem 2.3 follows. See [2, 3] for a related version. If $k = 1$, Lemma 2.2 may be applied directly to prove the claim. □

We now present a version that is a slight extension of Theorem 2.3.

Lemma 2.4. Let $u \in \text{usc}(\Omega_T \cup P_T)$ and $v \in \text{lsc}(\Omega_T \cup P_T)$. Assume $u \geq 0$ and $v > 0$ in $\Omega_T \cup P_T$. Suppose that

$$\Gamma_k[u] \geq 0, \quad \text{and} \quad \Gamma_k[v] \leq 0 \quad \text{in} \ \Omega_T.$$

If $v > 0$ on $U_T$ then $u/v \leq \max_{P_T} (u/v)$. In particular, if $u = 0$ on $P_T$, then $u \equiv 0$ in $\Omega_T$.

Proof. If $u > 0$ on $P_T$ then the conclusion follows from Theorem 2.3. Suppose that $u \geq 0$ on $\Omega_T \cup P_T$. Thus,

$$0 \leq \max_{P_T} (u/v) < \infty.$$  \hspace{1cm} (2.3)

For a fixed, small $\varepsilon > 0$, set $u_\varepsilon = \max \{u, \varepsilon\}$. By Lemma 2.1, $u_\varepsilon$ is a sub-solution, $u_\varepsilon \geq \varepsilon$, and, hence, by Theorem 2.3 and (2.5),

$$\frac{u}{v} \leq \frac{u_\varepsilon}{v} \leq \sup_{P_T} \frac{u_\varepsilon}{v} = \max \left\{ \sup_{\{u \leq \varepsilon\} \cap P_T} \frac{\varepsilon}{v}, \sup_{\{u > \varepsilon\} \cap P_T} \frac{u}{v} \right\} \leq \max \left\{ \sup_{\{u \leq \varepsilon\} \cap P_T} \frac{\varepsilon}{v}, \sup_{P_T} \frac{u}{v} \right\}.$$

If $\sup_{P_T} (u/v) > 0$, we take $\varepsilon$ small enough to conclude the result. If not, we let $\varepsilon \to 0$ to conclude that $u = 0$ in $\Omega_T$.

□

Corollary 2.5. Let $\bar{u} \in \text{usc}(\Omega_T \cup P_T)$ and $\bar{v} \in \text{lsc}(\Omega_T \cup P_T)$, $\bar{v} > -\infty$. Assume that $\inf_{\Omega_T} \bar{u} > -\infty$ with, possibly, $\inf_{U_T} \bar{u} = -\infty$. If

$$H(D\bar{u}, D^2\bar{u} + D\bar{u} \otimes \bar{u}) - \bar{u}_t \leq 0 \quad \text{and} \quad H(D\bar{v}, D^2\bar{v} + D\bar{v} \otimes \bar{v}) - \bar{v}_t \geq 0, \quad \text{in} \ \Omega_T,$$

then, $\bar{u} - \bar{v} \leq \max_{P_T} (\bar{u} - \bar{v})$.

Proof. For $\varepsilon \in \mathbb{R}$, $\bar{u}_\varepsilon = \max \{\bar{u}, \varepsilon\}$ is a sub-solution. Apply Subsection 2.2 and Lemma 2.2.

□
3. Proof of Theorem 1.1 \( k = 1 \)

We take \( k = 1 \). Then \( \Gamma_k = \Gamma_1 \) and \( u > 0 \) solves

\[
\Gamma_1[u] \equiv H(D^2u) - u_t \leq 0 \quad \text{in } \Omega_T.
\]

Note that \( H \) can be a fully nonlinear operator, as described in Section 1, see discussion following Condition C.

We recall for ease of reference the conditions satisfied by \( H \). From Condition A, \( H = H(X), X \in S^n \), is continuous and non-decreasing in \( X \). Condition B imposes that

\[
(3.1) \quad H(\theta X) = \theta H(X), \; \forall \theta > 0.
\]

In addition, \( H \) satisfies Condition C, see (1.6) and (1.7). From (1.7) (C(ii)),

\[
(3.2) \quad L(\Lambda) \equiv \min_{|e|=1} H(\Lambda e \otimes e - I) \geq -M(\Lambda) > 0, \; \forall \Lambda \geq \Lambda_1 \geq 1.
\]

Let \( z \in \mathbb{R}^n \). Set \( r = |x - z| \) and \( e = (x - z)/r \). Suppose that \( \phi(x,t) = \phi(r,t) \) is \( C^2 \) in \( x \).

From (2.1), in \( r > 0 \),

\[
(3.3) \quad H(D^2\phi) = H \left( \frac{\phi_r}{r} (I - e \otimes e) + \phi_{rr} e \otimes e \right).
\]

We introduce additional notation. Given two points \( x \) and \( y \) in \( \mathbb{R}^n \), the vector \( \vec{xy} \), in \( \mathbb{R}^n \), is the directed segment with initial point \( x \) and terminal point \( y \). Set \( \vec{x} = \vec{x} \); then \( \vec{xy} = \vec{y} - \vec{x} \).

**Proof of Theorem 1.1.** Set \( m \equiv \inf_{\Omega_T} u \). Since \( \Gamma_1[u - m] \leq 0 \), we may assume that \( m = 0 \) and \( u \geq 0 \) in \( \Omega_T \).

3.1. (a) **Hopf Boundary Principle:** The proof is a slight modification of the standard proof. We provide details.

Let \( (p, \tau) \in P_T, \; \tau < T \), be such that \( u(p, \tau) = 0 \). Suppose that there is a \( \mathbb{R}^{n+1} \) neighborhood \( N \) of \( (p, \tau) \) such that \( u > 0 \) in \( N \cap \Omega_T \).

**Side boundary:** Since \( \partial \Omega \) has interior \( \mathbb{R}^n \) ball property at \( p \), there is a \( 0 < \rho_0 < \tau \) and \( q = q(\rho) \in \Omega \), such that for \( 0 < \rho \leq \rho_0 \),

\[
B_\rho(q) \subset \Omega \quad \text{and} \quad p \in \partial B_\rho(q) \cap \partial \Omega.
\]

We take \( \rho > 0 \), small enough so that \( B_\rho(q) \times [\tau - \rho, \tau] \subset N \), and, hence,

\[
(3.4) \quad u > 0 \quad \text{in} \quad B_\rho(q) \times [\tau - \rho, \tau].
\]

Set \( r = |x - q| \), and the \( \mathbb{R}^{n+1} \) partial spherical shell

\[
S = \left\{ (x,t) : \frac{\rho^2}{4} \leq r^2 + (\tau - t)^2 \leq \rho^2, \text{ and } \tau - \rho/4 \leq t \leq \tau \right\}.
\]
Let $S_{in}$ and $S_{out}$ be the inner and the outer boundaries respectively. Clearly,

$$S_{in} = \{(x,t) : r^2 + (\tau - t)^2 = \rho^2/4, \quad \tau - \rho/4 \leq t \leq \tau\} \subset B_{\rho/2}(q) \times [\tau - \rho/4, \tau],$$

$$S_{out} = \{(x,t) : r^2 + (\tau - t)^2 = \rho^2, \quad \tau - \rho/4 \leq t \leq \tau\} \subset B_{\rho}(q) \times [\tau - \rho/4, \tau].$$

Set $U = (B_{\rho}(q) \setminus B_{\rho/2}(q)) \times [\tau - \rho/4, \tau]$. Clearly, $S_{out} \subset U$ and $S_{in}$ is outside $U$.

We observe that the intersection of the $\mathbb{R}^n$ plane $t = \tau - \rho/4$ with:

(i) $S_{out}$ is the $\mathbb{R}^n$ sphere of radius $\sqrt{15}\rho/4$, and

(ii) $S_{in}$ is the $\mathbb{R}^n$ sphere of radius $\sqrt{3}\rho/4$.

Thus,

$$r \geq \sqrt{3}\rho/4, \quad \text{in } S.$$

Moreover, recalling that $U \subset N$, by (3.4), there is an $\varepsilon > 0$ such that

$$u(x,t) > \varepsilon, \quad \text{if } 0 \leq r \leq \frac{\sqrt{15}\rho}{4} \text{ and } \tau - \frac{\rho}{4} \leq t \leq \tau.$$

We construct an auxiliary function $\psi$ in $S$ as follows:

$$\eta(r,t) = \exp(-a(r^2 + (\tau - t)^2)), \quad \text{and} \quad \psi(x,t) := \psi(r,t) = \eta(r,t) - \eta(\rho,\tau),$$

where $a > 0$ is to be determined.

Observe that $\psi = 0$ on $S_{out}$. Choose $a > 0$ large, so that $\forall (x,t) \in S_{in}, \quad 0 < \psi(x,t) = \exp(-a\rho^2/4) - \exp(-a\rho^2) < \varepsilon$. This ensures that $0 < \psi \leq \varepsilon$ in $S$. Summarizing,

$$0 < \psi \leq \varepsilon, \quad \text{in } S, \quad \psi = 0 \text{ on } S_{out} \text{ and } \psi \leq \varepsilon \text{ on } S_{in}.$$

Employing (3.3) and (3.6), we obtain in $r \geq 0$,

$$H(D^2\psi) = H(4a^2r^2\eta e \otimes e - 2a\eta I) = 2a\eta H(2ar^2e \otimes e - I).$$

We get from above,

$$\Gamma_1[\psi] = H(D^2\psi) - \psi_t = 2a\eta \left[ H(2ar^2e \otimes e - I) - (\tau - t) \right], \quad \text{in } S.$$

Recalling (3.2) and that $r \geq \sqrt{3}\rho/4$, we choose $a$ large enough so that

$$2ar^2 \geq \frac{3a\rho^2}{8} \geq \Lambda \geq \Lambda_1.$$
Noting further that $\tau - t \leq \rho/4$, (3.2) leads to
\[ \Gamma_1[\psi] \geq 2\alpha \eta \left[ L(\lambda) - \frac{\rho}{4} \right]. \]
By choosing $\rho$ small enough, $\psi > 0$ is a sub-solution in $S$.

We now apply the comparison principle in Lemma 2.4, see Theorem 2.3 in $U$. Note that $S_{out} \subset U$ and $S_{in}$ is outside $U$. Extending $\psi$ by zero in $r^2 + (t - \tau)^2 \geq \rho^2$, we get a sub-solution in $U$ (a proof is provided below). Next, recall (3.7): $u \geq \psi$ on the parabolic boundary of $U$ since $u \geq \psi = 0$ on $r = \rho$, $u > \varepsilon \geq \psi$ on $r = \rho/2$ (see (3.5)), and $\psi \leq \varepsilon \leq u$ on $t = \tau - \rho/4$. Thus, by the comparison principle,
\[ \psi \leq u \quad \text{in} \quad U, \]
and, hence, in $S$. We note that one could also apply the comparison principle directly to $S_{\rho q}$ as $S_{out}$, $S_{in}$ and the flat base form its parabolic boundary.

Recall (3.6). Since $u(p, \tau) = \psi(p, \tau) = 0$, for any $(x, t) \in S$,
\[
\begin{align*}
\frac{u(x, t) - u(p, \tau)}{|(x, t) - (p, \tau)|} &= \frac{u(x, t) - u(p, \tau)}{|\gamma|} \\
&\geq \frac{\psi(x, t) - \psi(p, \tau)}{|\gamma|} = \exp(-a \rho^2) \left\{ \exp \left( a \left[ \rho^2 - r^2 - (t - \tau)^2 \right] \right) - 1 \right\} \\
&\geq \exp(-a \rho^2) \left[ a \left\{ \rho^2 - r^2 - (t - \tau)^2 \right\} \right].
\end{align*}
\]
(3.8)

Let $\gamma \in \mathbb{R}^{n+1}$ and $\theta_0$ be as in the statement of the lemma. Choose $(x, t) = (p, \tau) + \theta \gamma = (p + \theta \gamma_n, \tau + \theta \gamma_{n+1})$. Note that $(\vec{p} \vec{q}, \gamma_n) > 0$ and $\gamma_{n+1} \leq 0$.

Since $\vec{p} x = \theta \gamma_n$ and $\vec{x} q = \vec{p} q - \vec{p} x = \vec{p} q - \theta \gamma_n$, we have
\[ \rho^2 - r^2 \geq (\rho - r) \rho \geq \rho (\rho - |\vec{x} q|) \geq \rho \theta |\gamma_n|, \]
where $c = c(\gamma_n, \vec{p} q) > 0$.

Clearly, (3.8) implies that for any $0 < \theta \leq \theta_0$,
\[
\frac{u(x, t)}{|(x, t) - (p, \tau)|} \geq a \exp(-a \rho^2) \left( c \theta \rho |\gamma_n| - \theta^2 |\gamma_{n+1}| \right) > 0,
\]
if $\theta_0$ is small enough.

Finally, we check that $\psi$ is a sub-solution. It is enough to check the definition at points on $S_{out}$. Suppose that $\phi$, $C^2$ in $x$ and $C^1$ in $t$, is such that $\psi - \phi$ has a maximum at a point $(y, s) \in S_{out}$, i.e., $(\psi - \phi)(x, t) \leq (\psi - \phi)(y, s)$. Since $\psi \geq 0$ and $\psi(y, s) = 0$, we get that $\phi(y, s) \leq \phi(x, t) - \psi(x, t) \leq \phi(x, t)$. Thus, $\phi$ has a minimum at $(y, s)$ and so $D \phi(y, s) = 0$, $\phi_t(y, s) = 0$, and $D^2 \phi(y, s) \geq 0$. Thus, $H(D^2 \phi(y, s)) - \phi_t(y, s) \geq 0$. $\square$

**Corollary 3.1.** Suppose that $u > 0$ and $m > 0$. By using $v = \log u$, the Hopf principle holds for
\[ H(Dv, D^2 v + Dv \otimes Dv) - v_t \leq 0. \]

See Subsection 2.1.
3.2. (b) Strong Minimum Principle: We continue to assume that \( k = 1 \) and 
\[
\Gamma_1[u] = H(D^2 u) - u_t \leq 0 \quad \text{in} \quad \Omega_T.
\]
Suppose that \( u \geq 0 \) and \( m = 0 \). Suppose that \( u(p, \tau) > 0 \), for some \((p, \tau) \in \Omega_T\).

We make an observation. Let \( \phi = \phi(x, t) \in C^2 \), \( \phi > 0 \) and \( \beta \geq 2 \). Then
\[
H(D^2 \phi^\beta) = \beta \phi^{\beta - 1} H \left( D^2 \phi + \left( \frac{\beta - 1}{\phi} \right) D\phi \otimes D\phi \right).
\]

Let \((q, s) \in \Omega_T\) with \( s > \tau \). We comment on \( q \) and \( s \) later. Set 
\[
\delta = s - \tau, \quad \vec{n} = \vec{pq}, \quad \text{and} \quad \vec{\gamma} = (\vec{n}, \delta).
\]
Clearly, \( \vec{q} = \vec{p} + \vec{n} \gamma \). Vectors with lower case letters are in \( \mathbb{R}^n \), except for \( \gamma \).

The points \( P_t \) on the segment \( S \) (in \( \mathbb{R}^{n+1} \)) with end points \((p, \tau)\) and \((q, s)\), are parametrized by \( t \) as
\[
P_t := \left( \vec{p} + \left( \frac{t - \tau}{\delta} \right) \vec{n}, t \right), \quad \tau \leq t \leq \tau + \delta.
\]
The notation \( P_t \) is a vector in \( \mathbb{R}^{n+1} \).

We call 
\[
d(x, t) = (\vec{x}, t) - P_t = \vec{x} - \left( \vec{p} + \left( \frac{t - \tau}{\delta} \right) \vec{n} \right) \quad \text{and} \quad d(x, t) = |d(x, t)|.
\]
We will often write \( d \) and \( \vec{d} \) in place of \( d(x, t) \) and \( \vec{d}(x, t) \) respectively.

Let \( 0 < \Delta \leq \delta \). Define the slanted cylinder 
\[
C_{\rho, \Delta} = C_{\rho, \Delta}(\vec{p}, \tau) = \{ (x, t) : d(x, t) \leq \rho, \tau \leq t \leq \tau + \Delta \}.
\]
Its axis is along the segment \( S \), see (3.10). Also, at \( t = \tau + \Delta \), the point in \( S \) is 
\[
P_{\tau + \Delta} = \left( \vec{p} + \frac{\Delta}{\delta} \vec{n}, \tau + \Delta \right).
\]
Define
\[
\phi(d) = \phi(x, t) = \rho^2 - d(x, t)^2 \quad \text{and} \quad \eta(t) = \frac{\tau + 2\Delta - t}{2\Delta}.
\]
Choose \( 0 < \rho \leq 1 \), and set
\[
\psi(x, t) = \phi(d)^2 \eta(t), \quad \text{in} \quad C_{\rho, \Delta}(p, \tau).
\]
We show that for an appropriate \( \Delta > 0 \) and \( |\vec{n}| \neq 0 \), \( \psi \) is a sub-solution in \( C_{\rho, \Delta} \).

We compute \( H(D^2 \psi) - \psi_t \). Recalling (3.9), 
\[
H(D^2 \psi) = \eta H(D^2 \phi^2) = 2\eta \phi H \left( D^2 \phi + \frac{D\phi \otimes D\phi}{\phi} \right).
\]
We note that 
\[ d^2 = \langle \vec{d}, \vec{d} \rangle, \quad D\vec{d} = I \quad \text{and} \quad \frac{\partial \vec{d}}{\partial t} = -\frac{\vec{\gamma}_n}{\delta}. \]

Next, we write \( \vec{d} = d\vec{e} = de \), where \( e = e(x,t) \) is a unit vector. Differentiating (3.11),
\[
D\phi = -2\vec{d}, \quad D^2\phi = -2I \quad \text{and} \quad D\phi \otimes D\phi = 4\vec{d} \otimes \vec{d} = 4d^2 e \otimes e.
\]

Hence, from (3.12),
\[
H(D^2\psi) = 4\phi \eta H(-I + \frac{2d^2}{\phi} e \otimes e), \quad \text{and}
\]
\[
\psi_t = -\frac{\phi^2}{2\Delta} + 4\eta \phi \frac{\langle \vec{d}, \vec{\gamma}_n \rangle}{\delta} = \phi \left( \frac{4\eta \langle \vec{\gamma}_n, \vec{d} \rangle}{\delta} - \frac{\phi}{2\Delta} \right).
\]

Combining the two expressions, we get
\[
\Gamma_1[\psi] = H(D^2\psi) - \psi_t = 4\phi \eta H(-I + \frac{2d^2}{\phi} e \otimes e) + \varepsilon \phi \left( \frac{\phi}{2\Delta} - \frac{4\eta \langle \vec{d}, \vec{\gamma}_n \rangle}{\delta} \right)
\]
\[
= \phi \left[ 4\eta H(-I + \frac{2d^2}{\phi} e \otimes e) + \frac{\phi}{2\Delta} - \frac{4\eta \langle \vec{d}, \vec{\gamma}_n \rangle}{\delta} \right].
\]

Noting that \( \eta \leq 1 \) in \( C_{\rho,\Delta} \), a rearrangement leads to
\[
\Gamma_1[\psi] \geq \phi \left[ \frac{\phi}{2\Delta} + 4\eta H(-I + \frac{2d^2}{\phi} e \otimes e) - \frac{4d|\vec{\gamma}_n|}{\delta} \right].
\]
(3.13)

Next, noting (3.2), we choose \( \Lambda \geq \Lambda_1 \) and set
\[
\nu = \sqrt{\frac{\Lambda}{\Lambda + 2}}, \quad \nu_0 = \sqrt{1 - \nu^2} = \sqrt{\frac{2}{\Lambda + 2}}, \quad \text{and} \quad R = v\rho.
\]

Then, \( \phi(R) = \nu_0^2 \rho^2 \), and
\[
in R \leq d < \rho: \quad \frac{2d^2}{\phi(d)} \geq \frac{2R^2}{\phi(R)} = \frac{2v^2}{\nu_0^2} = \Lambda.
\]
Recalling (3.2),
\[
H \left( \frac{2d^2}{\phi} e \otimes e - I \right) \geq H(\Lambda e \otimes e - I) \geq L(\Lambda) > 0, \quad \forall \ R \leq d < \rho.
\]
(3.15)

We divide \( 0 \leq d < \rho \) into two intervals: \( 0 \leq d \leq R \) and \( R \leq d < \rho \), where \( R \) is as in (3.14).

Recalling Conditions A, (3.2) and (3.15), we estimate, for some \( M < 0 \) (see (1.8)),
\[
H \left( -I + \frac{2d^2}{\phi} e \otimes e \right) \geq \begin{cases} 
L(\Lambda) > 0, & \text{in } R \leq d < \rho, \\
H(-I) \geq -|M|, & \text{in } 0 \leq d \leq R.
\end{cases}
\]
(3.16)

Next, we derive conditions under which \( \psi \) is sub-solution in \( C_{\rho,\Delta} \).
**Interval** \((R \leq d < \rho)\): We use \(1/2 \leq \eta \leq 1\), \((3.13)\) and \((3.16)\) to obtain
\[
\Gamma_1[\psi] \geq \phi \left[ 4\eta H \left( -I + \frac{2d^2}{\phi} e \otimes e \right) - \frac{4d|\vec{\gamma}_n|}{\delta} \right] \geq \phi \left[ 2L(\Lambda) - \frac{4\rho|\vec{\gamma}_n|}{\delta} \right].
\]
in \(\tau \leq t \leq \tau + \Delta\).

Then \(\psi\) is a sub-solution in \(R \leq d < \rho\) and \(\tau \leq t \leq \tau + \Delta\), if
\[
(3.17) \quad |\vec{\gamma}_n| \leq \frac{\delta L(\Lambda)}{2\rho}.
\]

**Interval** \((0 \leq d \leq R)\): We use the estimates \(\phi(R) \leq \phi(d) \leq \rho^2\) and \(1/2 \leq \eta \leq 1\). From \((3.13), (3.14)\) and \((3.16)\), we obtain
\[
\Gamma_1[\psi] \geq \phi \left[ \frac{\phi(R)}{2\Delta} + 4\eta H(-I) - \frac{4\rho|\vec{\gamma}_n|}{\delta} \right] \geq \phi \left[ \frac{\nu_0^2\rho^2}{2\Delta} - \frac{4\rho|\vec{M}|}{\rho} - \frac{4\rho|\vec{\gamma}_n|}{\delta} \right].
\]

First, we choose \(\Delta\) and \(\gamma_n\) such that
\[
(3.18) \quad \Delta = \nu_0^2\rho^2 \equiv K_1\rho^2 \quad \text{and} \quad |\vec{\gamma}_n| \leq \frac{|\vec{M}|}{\rho}.
\]
where \(K_1 = K_1(M, \Lambda)\). Next, using \((3.17)\) and \((3.18)\), we select
\[
(3.19) \quad 0 < |\vec{\gamma}_n| \leq \delta \left( \min \left\{ \frac{L(\Lambda)}{2\rho}, \frac{|\vec{M}|}{\rho} \right\} \right) = \frac{K_2\delta}{\rho},
\]
where \(K_2 = K_2(L, M, \Lambda)\). With these selections, \(\psi\) is a sub-solution in \(C_{\rho, \Delta}\). Note that \(\Delta\) depends on \(\rho\) but is independent of \(\delta\). However, \(|\vec{\gamma}_n|\) is dependent on \(\rho\) and \(\delta\).

From the estimate for \(\vec{\gamma}_n\), it is clear that we can allow \(\vec{pq}\) large by selecting \(\rho\) small. However, this makes \(\Delta\) small. Thus, iterations may be needed to reach the time level \(s\). We present details of the argument below.

Observe that the strip \(\Omega_{\tau} \setminus \Omega_{\tau} = \Omega \times [\tau, \tau + \Delta]\). Our goal is to show that if \(u(p, \tau) > 0\) then \(u(q, t) > 0\) for \(t \in [\tau, \tau + \Delta]\).

Let \(\vec{\gamma}_n = \vec{pq}\). Noting \((3.19)\), choose \(0 < \rho < 1\) so that
\[
u(\tau) \geq \frac{u(p, \tau)}{2}, \quad \text{in} \quad |x - p| \leq \rho, \quad \text{and} \quad 0 < |\vec{\gamma}_n| \leq \frac{K_2\delta}{\rho}.
\]

Fix \(\rho\). Recalling \((3.12)\) and \((3.18)\), we choose
\[
\Delta = K_1\rho^2 \quad \text{and} \quad \hat{\psi}(x, t) = \frac{u(p, \tau)}{2} \left( \frac{\psi(d, t)}{\rho^4} \right) = \frac{u(p, \tau)}{2} \left( \frac{\phi(d)^2}{\rho^4} \right) \eta(t).
\]
Note that \(\Gamma_1[c\psi] = c\Gamma_1[\psi] \geq 0\), if \(c > 0\), see Condition B. Thus, \(\Gamma_1 \hat{\psi} \geq 0\).

Observe that \(0 \leq \hat{\psi}(x, \tau) \leq u(p, \tau)/2\), in \(B_{\rho}(p)\), and \(\hat{\psi} = 0\) along the slanted cylindrical side. Thus, \(\hat{\psi}\), with the selections made above, is a sub-solution in \(C_{\rho, \Delta}\), and, by an extension by zero, in \(\Omega_{\tau} \setminus \Omega_{\tau}\). That this extension of \(\hat{\psi}\) results in a sub-solution follows closely the argument presented in Subsection 3.1.
Using the comparison principle in Theorem 2.3, we obtain that $u \geq \hat{\psi}$ in $\Omega_{\tau + \Delta} \setminus \Omega_{\tau}$. If $\delta < \Delta$, then, noting (3.10) and (3.12), and taking $s = \tau + \delta$

\begin{equation}
(3.20) \quad u(q,s) \geq \hat{\psi}(q,s) = \hat{\psi}(0,\tau + \delta) = u(p,\tau) > 0.
\end{equation}

The claim follows.

Suppose that $\Delta \leq \delta$. Let $j = 2, 3, \cdots$, be such that $(j - 1)\Delta \leq \delta < j\Delta$. We use a chain of $j$ slanted $\mathbb{R}^{n+1}$ cylinders of the same $t$ height and shape, with their axes along the segment $S$, see (3.10). Let $\Delta_s < \Delta$ be such that $j\Delta_s > \delta$. This adjustment is needed as $u$ is lsc$(\Omega_T)$ and the relation $u \geq \psi$ (see above) may not extend to $t = \tau + \Delta$. As a result, we use the level $t = \tau + \Delta_s$ to bound $u$ from below by $\psi$. This modification is applied at every step. However, at every step, the $t$-heights of the cylinders, in which we define the auxiliary functions, continues to be $\Delta$.

We use an iterative process and apply the comparison principle at every step. We describe the individual cylinders and the auxiliary functions. The quantities $\rho$, $\gamma_n$, $\Delta$, $\Delta_s$ and $\phi$ stay the same at every step. The function $\eta$ will change. For $\ell = 0, 1, 2, \cdots$, set

\begin{equation}
(3.21) \quad \Delta = K_1 \rho^2, \quad \eta_\ell(t) = \frac{\tau_\ell + 2\Delta - t}{2\Delta} \quad \text{and} \quad \phi(d) = \rho^2 - d^2,
\end{equation}

where $d = |(\bar{x},t) - \mathcal{P}_t|$ (see (3.10)), and the quantity $\tau_\ell$ is defined below. Set for $\ell = 1, 2, \cdots$,

\begin{equation}
\begin{split}
\tau_0 &= \tau \quad \text{and} \quad \tau_\ell = \tau + \ell\Delta_s; \quad \bar{p}_0 = \bar{p} \quad \text{and} \quad \bar{p}_\ell = \bar{p} + \frac{\ell\Delta_s \gamma_n}{\delta}; \\
\bar{P}_0 &= (\bar{p},\tau) \quad \text{and} \quad \bar{P}_\ell = \mathcal{P}_{\tau_\ell} = (\bar{p}_\ell,\tau_\ell) = \left(\bar{p} + \frac{\ell\Delta_s \gamma_n}{\delta}, \tau + \ell\Delta_s\right). 
\end{split}
\end{equation}

Next, we set

\begin{equation}
(3.22) \quad \beta = \frac{2\Delta - \Delta_s}{2\Delta} \quad \text{and} \quad \varepsilon_\ell = \frac{\phi(d)^2 \eta_\ell(t)}{\rho^4} \quad \text{in} \quad C_{\rho,\Delta}(p_\ell,\tau_\ell).
\end{equation}

Applying the above procedure and taking $\ell = j$, we get that $u(q,s) \geq \hat{\psi}_j(q,s)$. Thus,

\begin{equation}
\begin{split}
u(q,s) &\geq \frac{\beta^j \phi(d)^2 u(p,\tau) \eta_j(s)}{2\rho^4} \geq \frac{u(p,\tau) \phi(d)^2}{2^{j+2} \rho^4} \geq \frac{2^{-\delta/\Delta} u(p,\tau) \phi(R)^2}{4\rho^4} \geq 0,
\end{split}
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{Vector $(\bar{p},\tau)$ to $(\bar{q},\tau + \delta)$}
\end{figure}
since $\Delta \geq 1/2$, $\eta_\ell \geq 1/2$ and $\tau_{j-1} < s \leq \tau_j$.

Next, suppose that $\mathcal{P}$ is a polygonal path in $\Omega$ connecting $p$ to $q$. Let $\{(p_i)_{i=0}^\ell\}$ be the end points of the segments that comprise $\mathcal{P}$, and are such that $p_0 = p$, $p_\ell = q$, and, for every $i$, $p_i p_{i+1}$ is a segment. Let $\gamma = \max\{|p_i - p_{i+1}|\}$. We choose $\{\tau_i\}_{i=0}^\ell$ such that $\tau = \tau_0 < \tau_1 < \cdots < \tau_\ell = s$ and $\delta = \tau_{i+1} - \tau_i = (\tau - s)/\ell$. Choose

$$
\rho \leq \max \left\{ \frac{K_2 \delta}{\gamma}, \text{dist}\{\mathcal{P}, \partial \Omega\} \right\}.
$$

Further adjust $\rho$ so that $u(x, \tau) \geq u(p, \tau)/2$ in $|x - p| \leq \rho$. Next, mount $\ell$ slanted cylinders each having width $\rho$ and axis along the $\mathbb{R}^{n+1}$ segment with endpoints $(p_i, \tau_i)$ and $(p_{i+1}, \tau_{i+1})$. The value of $\Delta$ does not change and $|p_i - p_{i+1}| \leq \gamma$. We iterate the previously described process $\ell$ times to obtain an estimate $u(q, s) \geq c u(p, \tau)$, where $0 < c = c(\ell, \delta, \Delta, \gamma) < 1$. The claim holds. □

4. Proof of Theorem 1.1

Let $m = \inf_{\Omega_T} u$. We show that the case $k > 1$ differs quite markedly from $k = 1$. This occurs even when $m > 0$. In this case, our work appears to provide a complete description. However, things are not clear in the case $m = 0$, and we provide what appears, to us, to be a partial result. One of the difficulties seems to be that the quotient version of the comparison principle (see Theorem 2.3) becomes unclear at places where both the sub-solution and the super-solution are small.

4.1. Part (a) $m > 0$: If $u = m$ somewhere in $\Omega_T$ then it appears that, in general, the strong minimum principle may fail to hold. The same appears to be the case for the Hopf boundary principle at points on $P_T$ where $u = m$. However, we show a weaker version of the strong minimum principle does hold. Before presenting the proof, we discuss an example that supports this assertion.

**Example:** Let $m > 0$, $T > 0$ and $k > 1$. We construct a super-solution $\xi$, in an appropriate $\Omega_T$, such that its minimum $m$ is attained along a $t$-segment $(p, t)$, $0 < t \leq T$, for some $p \in \Omega$, and some $T > 0$. However, $\xi > m$ in the rest of $\Omega_T$. Note that our construction produces a super-solution in $\mathbb{R}^n \times (0, T)$.

Set $r = |x|$ and $\phi(r) = r^{(k+1)/(k-1)}$. Using (2.11) (see Subsection 2.1) and (1.7) i.e, Condition C(i),

$$
H(D\phi, D^2\phi) = cr^{(k+1)/(k-1)} H\left(e, I - \frac{k - 3}{k - 1} e \otimes e\right) \leq c\phi(r)L,
$$

for some constant $c = c(k) > 0$ and $L = \max_{|e| = 1} H(e, I)$. 


We take \( \Omega_T = B_R(o) \times [0,T) \), where \( R > 0 \). Define
\[
\xi(x,t) = m + \phi(r)\eta(t), \quad \text{where} \quad \eta(t) = \left( \frac{1}{E(2T-t)} \right)^{1/(k-1)} \quad \text{and} \quad E = \frac{c(k-1)L}{m^{k-1}}.
\]
Note that
\[
\eta'(t) = E\eta^k/(k-1) \geq 0.
\]
Using (4.1), we get in, \( 0 < r < R \)
\[
\Gamma_k[\xi] = H(D\xi, D^2\xi) - \xi^{k-1}\xi_t \leq c\phi\eta^k L - (m + \phi\eta)^{k-1} \phi\eta'
\]
\[
\leq \phi\eta^k \left[ cL - \frac{m^{k-1}E}{k-1} \right] \leq 0.
\]
We verify below that \( \xi \) is a super-solution in \( \Omega_T \). Observe that
\[
\xi(o,t) = m, \quad 0 < t \leq T, \quad \text{and} \quad \xi(x,t) > m, \quad x \neq o.
\]
This shows that \( u \) does not attain its minimum value anywhere except along \((o,t), 0 < t < T\).

Let \( \nabla \) be the \( \mathbb{R}^{n+1} \) gradient. Then \( \nabla \xi(o,t) = 0, \quad 0 < t < T \). We modify the example slightly to show that, in general, the Hopf boundary principle does not hold. Let \( z \neq o \) and \( \rho = |z| \). Consider the domain \( U = B_p(z) \times [0,T] \) and \( r = |x| \), as defined above. Thus, \( \xi > m \) is a super-solution in \( U \) and \( \xi(o,t) = m, \quad 0 < t < T \). This is a segment on the parabolic boundary of \( U \). Since \( \nabla \xi(o,t) = 0 \), the Hopf principle fails.

We now show that \( \xi \) is a super-solution in \( \Omega_T \). Firstly, \( \xi \geq m \) in \( \Omega_T \) and \( \xi(o,t) = m, \quad 0 < t \leq T \). It is sufficient to prove that \( \xi \) is a super-solution at \( r = 0 \).

Let \( \zeta, C^2 \) in \( x \) and \( C^1 \) in \( t \), be such that \( \xi - \zeta \) has a minimum at \((o,s)\) for some \( 0 < s < T \). Then \( \xi(x,t) - \xi(o,s) \geq \zeta(x,t) - \zeta(o,s) \). Note that \( \xi \) is \( C^1 \) in \( x \), and \( C^1 \) in \( t \). At \( r = 0 \), we get that \( D\xi(o,s) = D\zeta(o,s) = 0 \) and \( \xi_t(o,s) = \zeta_t(o,s) = 0 \). Since \( k > 1 \), we get that
\[
H(D\xi(o,s), D^2\xi(o,s)) - \xi(o,s)^{k-1}\zeta_t(o,s) = 0.
\]
This finishes the proof. \( \Box \)

**Proof of Part (a):** We now show that if \( u > 0 \) satisfies \( \Gamma_k[u] \leq 0 \), in \( \Omega_T \), and \( u(p, \tau) > m \), for some \((p, \tau) \in \Omega_T \), then there is a cylinder \( C \equiv B_p(p) \times [\tau,T) \) such that \( u(x,t) > m \) in \( C \).

As a result, if \( u(p, \tau) = m \) then \( u(p,t) = m \) for all \( 0 < t < \tau \). As the above example shows, this result can not be improved.

Suppose that \( u(p, \tau) > m \). Let \( \varepsilon > 0 \) and \( 0 < \rho < 1 \) be such that
\[
u(x, \tau) \geq m + \varepsilon, \quad \text{in} \quad B_\rho(p).
\]
Set \( \delta = T - \tau, \quad r = |x - p| \) and \( S = B_\rho(p) \times [\tau,T) \). Define in \( S \)
\[
\psi(x,t) = m + \varepsilon \phi(r)^2\eta(t) \quad \text{where} \quad \phi(r) = \rho^2 - r^2 \quad \text{and} \quad \eta(t) = \frac{\tau + 2\delta - t}{2\delta}.
\]
(4.2)
Using (2.1) and Condition B, we get
\[ H(D\psi, D^2\psi) = (\varepsilon \eta)^k H \left( -4r\phi e, -4\phi(I - e \otimes e) + (-4\phi + 8r^2)e \otimes e \right) \]
\[ = (4\varepsilon \phi \eta)^k r^{k-1} H \left( e, \frac{2r^2}{\phi} e \otimes e - I \right). \]
Hence,
\[ (4.3) \quad H(D\psi, D^2\psi) - \psi^{k-1}\psi_t = (4\varepsilon \phi \eta)^k r^{k-1} H \left( e, \frac{2r^2}{\phi} e \otimes e - I \right) + \frac{\varepsilon \psi^{k-1}\phi^2}{2\delta}. \]
We now recall (3.14) and (3.15) and divide the interval 0 ≤ r < ρ into the sub-interval 0 ≤ r < R and R ≤ r < ρ. As argued in (3.15), \( \Gamma_k[\psi] \geq 0 \), in R ≤ r < ρ and \( \tau < t < T \).

We consider 0 ≤ r ≤ R; use (3.16), and the three estimates: \( 1/2 \leq \eta \leq 1, \phi(r) \geq \phi(R) = (\nu_0 \rho)^2 \) and \( m \leq \psi \leq m + \varepsilon \), to obtain
\[ (4.4) \quad \Gamma_k[\psi] \geq - (4\varepsilon \phi \eta)^k r^{k-1}|M| + \frac{\varepsilon \psi^{k-1}\phi^2}{2\delta} \geq \frac{\varepsilon m^{k-1}\phi(R)^2}{2\delta} - (4\varepsilon)^k \phi(0)^k r^{k-1}|M| \]
If \( \varepsilon > 0 \) is small enough then \( \psi \) is a sub-solution in \( B_\rho(p) \times [\tau, T) \).

Next, we observe that \( u \geq \psi = m \) on \( \partial B_\rho(p) \times [\tau, T) \) and \( u(x, \tau) \geq m + \varepsilon \geq \psi(x, \tau) \), for \( x \in B_\rho(p) \). By using the comparison principle Theorem 2.3 we get \( \psi \leq u \) in \( S \). Thus, for any \( (x, t) \in S \) we get that
\[ u(x, t) \geq \psi(x, t) = m + \varepsilon(p^2 - |x - p|^2) \left( \frac{T - t + \delta}{2\delta} \right) > m. \]

The claim holds. □

4.2. Part (b) \( m = 0 \): We consider the case where \( u \geq 0 \) and \( m = \inf_{\Omega_T} u = 0 \). We assume that \( u \in C(\Omega_T) \).

Proof of Part (b): We show that the zeros are not isolated. Assume to the contrary. Let \( C = B_\rho(p) \times (\tau - \delta, \tau) \subset \Omega_T \), for some \( \rho > 0 \) and \( \delta > 0 \), such that \( u > 0 \) in \( \overline{C} \setminus \{(p, \tau)\} \).

Let \( P \) be the parabolic boundary of \( C \). Since \( u > 0 \) on \( P \), there is a \( \nu > 0 \) such that \( u \geq \nu \) on \( P \). Recall the calculations done in (3.20), (4.1) and (4.3). Define in \( S \),
\[ \psi(x, t) = \frac{\nu}{2} + \varepsilon(p^2 - r^2)^2 \left( \frac{\tau - t}{2\delta} \right), \]
where \( 0 < \varepsilon \leq \nu/(2\rho^4) \). As done in Sub-section (4.1), by choosing \( \varepsilon \) small enough, \( \psi \) is a sub-solution in \( C \). Moreover, \( \psi \leq \nu \leq u \) on \( P \). Hence, by Lemma 2.4 (see Theorem 2.3), \( u \geq \psi \) in \( S \), and it is clear that by choosing points \( (p, t), t < \tau \), close to \( (p, \tau) \), \( u(p, \tau) \geq \nu/2 > 0 \), a contradiction. The claim holds. □
5. Proof of Theorem 1.3

The proof generalizes the result Theorem 1.2 in [5], and is based on the use of auxiliary functions. We recall a few items and introduce two auxiliary functions before presenting the proof.

We recall that \( \Omega^\infty = \Omega \times (0, \infty) \) and \( P^\infty = (\Omega \times \{0\}) \cup (\partial \Omega \times (0, \infty)) \). For \( t > 0 \), set
\[
Q_t = \overline{\Omega} \times [t, \infty) \quad \text{and} \quad S_t = \partial \Omega \times [t, \infty).
\]

Let \( T > 0 \) be as in the statement of the theorem. We assume that \( u = h \) on \( S_T \). Set
\[
m = \min_{S_T} h \quad \text{and} \quad M = \sup_{S_T} h.
\]

Thus, Theorem 2.3 implies that

\[
\text{If } u > 0 \text{ is a sub-solution then } u \leq \max_{\overline{\Omega}} \{ \max_{\Omega} u(x, T), M \} \text{ in } Q_T,
\]

\[
\text{If } u > 0 \text{ is a super-solution then } u \geq \min_{\overline{\Omega}} \{ \min_{\Omega} u(x, T), m \} \text{ in } Q_T.
\]

(5.1)

First, apply the comparison principle in \( \Omega \times (T, s) \), for \( s > T \), and then let \( s \to \infty \) to get the claim.

Recall the notation
\[
\Gamma_k[w] := H(Dw, D^2w) - w^{k-1}w_t.
\]

Let \( z \in \mathbb{R}^n \setminus \overline{\Omega} \) and set \( r = |x - z| \). In what follows, \( D, E, F \) and \( a \) are positive constants. Our calculations, done next, show \( a \) depends on \( E \), see below. The constants \( D, E \) and \( F \) are chosen in the proof of the theorem.

Set
\[
R = \sup_{x \in \Omega} |x - z|, \quad \mathcal{R} = \inf_{x \in \Omega} |x - z|, \quad \text{and} \quad \mathcal{D} = \text{diam}(\Omega).
\]

Clearly, \( \mathcal{R} > 0 \), and \( r \geq \mathcal{R} > 0 \), if \( x \in \Omega \). Also,
\[
R \leq \mathcal{R} + \mathcal{D} \quad \text{and} \quad \Omega \subset B_{\mathcal{R} + \mathcal{D}}(z) \setminus B_{\mathcal{R}}(z).
\]

**Auxiliary Function 1 (Sub-solution):** Let \( z \) and \( r \) be as defined above. For constants \( D, E, F \), and \( a \), we define the function \( \xi \in C^2(\Omega^\infty) \) as follows:

\[
\xi(x, t) = \alpha(r) \tau(t), \quad \text{where} \quad \alpha(r) = De^{Er^2} \quad \text{and} \quad \tau(t) = \frac{e^{at}}{e^{at} + F}.
\]

Thus,
\[
\alpha'(r) = (2Er)\alpha, \quad \alpha''(r) = 2E\alpha (1 + 2Er^2) \quad \text{and} \quad \tau'(t) = \tau \left( \frac{aF}{e^{at} + F} \right).
\]
Calling $\omega = (x - z)/|x - z|$, we get

$$D\xi = 2Er\xi \omega, \quad \text{and} \quad \xi_t = \xi \left( \frac{aF}{e^{at} + F} \right).$$

Using (2.1), we get

$$D^2\xi = \tau \left[ \alpha' \left( I - \omega \otimes \omega \right) + \alpha'' \omega \otimes \omega \right] = 2E\xi \left( I + 2Er^2\omega \otimes \omega \right).$$

Using the above observations and Conditions A, B and C, we get

$$\Gamma_k[\xi] = \left( 2E\xi \right)^{k-1}H(\omega, I + 2Er^2\omega \otimes \omega) - \xi \left( \frac{aF}{e^{at} + F} \right)$$

$$\geq \xi \left( (2E)^{k-1}H(\omega, I) - \frac{aF}{e^{at} + F} \right).$$

Recalling (5.2),

$$\Gamma_k[\xi] \geq \xi \left[ (2E)^{k-1}H(\omega, I) - a \right].$$

Thus, $\xi$ is sub-solution in $\Omega_\infty$ if we choose (see (1.7) C(i))

$$0 < a < (2E)^{k-1} \min_{|\omega|=1} H(\omega, I).$$

\[\square\]

**Auxiliary Function 2 (Super-solution):** Let $z$ and $r$ be as above. For positive constants $D, E, F$, and $a > 0$, we set

$$\zeta(x,t) = \beta(r)\theta(t), \quad \text{where} \quad \beta(r) = De^{-Er^2} \quad \text{and} \quad \theta(t) = 1 + Fe^{-at}.$$

We impose a condition on $E$ and $a$ for $\zeta$ to be a super-solution. Rest are chosen in the proof of the theorem. Clearly,

$$\beta' = (-2Er)\beta, \quad \beta'' = 2E\beta(2Er^2 - 1), \quad \text{and} \quad \theta' = -aFe^{-at} = -\theta \left( \frac{aFe^{-at}}{1 + Fe^{-at}} \right).$$

Letting $\omega = (x - z)/|x - z|$, we have

$$D\zeta = (-2Er)\zeta \omega, \quad \zeta_t = -\zeta \left( \frac{aFe^{-at}}{1 + Fe^{-at}} \right),$$

$$D^2\zeta = \theta \left[ \frac{\beta'}{r} \left( I - \omega \otimes \omega \right) + \beta'' \omega \otimes \omega \right]$$

$$= 2E\zeta \left( 2Er^2\omega \otimes \omega - I \right).$$

Thus,

$$\Gamma_k[\zeta] = H \left( -2E\zeta \omega, 2E\zeta \left( 2Er^2\omega \otimes \omega - I \right) \right) + \zeta \left( \frac{aFe^{-at}}{1 + Fe^{-at}} \right)$$

$$= (2E\zeta)^{k-1}H \left( \omega, 2Er^2\omega \otimes \omega - I \right) + \zeta \left( \frac{aFe^{-at}}{1 + Fe^{-at}} \right)$$

$$= \zeta \left[ (2E)^{k-1}H \left( \omega, 2Er^2\omega \otimes \omega - I \right) + a \left( \frac{Fe^{-at}}{1 + Fe^{-at}} \right) \right].$$
By (5.2), \( R \leq r \leq R + D \). We choose \( E \) (see (1.7) C(i)) so that
\[
0 < \kappa \equiv 2E(R + D)^2 < 1 \text{ and } J \equiv \max_{|\omega|=1} H(\omega, \kappa \omega \otimes \omega - I) < 0.
\]

Next, select
\[
(5.7) \quad 0 < a < (2E)^k R^{k-1} |J|.
\]

With the above choice for \( E \) and \( a \), we get
\[
\Gamma_K[\zeta] \leq \zeta^k \left[ a + (2E)^k R^{k-1} H(\omega, \kappa \omega \otimes \omega - I) \right] \leq \zeta^k \left[ a - (2E)^k R^{k-1} |J| \right] \leq 0.
\]

With these values, \( \zeta \) is a super-solution in \( \Omega_\infty \). Note that \( E \) depends on \( R \). \( \square \)

Let \( t \geq T \). Define in \( \mathcal{Q}_t \) and \( \mathcal{S}_t \),
\[
(5.8) \quad (i) \mu_{\inf}(t) = \inf_{\mathcal{Q}_t} u, \quad (ii) \mu_{\sup}(t) = \sup_{\mathcal{Q}_t} u, \quad (iii) \nu_{\inf}(t) = \inf_{\mathcal{S}_t} h, \quad \text{and} \quad (iv) \nu_{\sup}(t) = \sup_{\mathcal{S}_t} h.
\]

Since \( u = h \) on \( \mathcal{S}_T \), \( \mu_{\inf}(t) \leq \nu_{\inf}(t) \), and \( \nu_{\sup}(t) \leq \mu_{\sup}(t) \). Set
\[
(5.9) \quad \nu_{\sup} = \lim_{t \to \infty} \nu_{\sup}(t) \quad \text{and} \quad \nu_{\inf} = \lim_{t \to \infty} \nu_{\inf}(t).
\]

**Proof of Part (a) of Theorem 1.3**: Recall the notation in (5.8), and (5.9). We take \( k \geq 1 \). Recall that \( u > 0 \) is a super-solution, and since (1.4) holds, \( \mu_{\inf}(t) < \infty, \forall t > 0 \).

Note that \( \mu_{\inf}(t) \leq \nu_{\inf}(t) \). Thus, the claim follows if we show that
\[
\lim_{t \to \infty} \mu_{\inf}(t) \geq \nu_{\inf}.
\]

Recall that \( u = h \) on \( \mathcal{S}_T \) and \( u \geq \min\{\min_{\Omega} u(x, T), m\} \equiv m_0 \). Since \( \nu_{\inf} \geq \mu_{\inf}(t) \geq m_0 \), if \( \nu_{\inf} = m_0 \), the claim follows. Assume from here on that \( \nu_{\inf} > m_0 \).

Let \( \varepsilon > 0 \) be small, and \( T_0 \geq T \), large, so that for \( t \geq T_0 \) (see (5.9))
\[
\nu_{\inf}(t) \geq \nu_{\inf} - \varepsilon > m_0 > 0.
\]

Fix \( z \in \mathbb{R}^n \setminus \Omega \); set
\[
r = |x - z|, \quad \mathcal{R} = \inf_{x \in \Omega} |x - z| \quad \text{and} \quad D = \text{diam } \Omega.
\]

We employ Auxiliary Function 1, see (5.3), and recall the condition (5.4):
\[
\xi(x, t) = H e^{E x^2} \left( \frac{e^{a(t-T_0)}}{e^{a(t-T_0)} + F} \right), \quad \text{where} \quad 0 < a < (2E)^k R^{k-1} \min_{|\omega|=1} H(\omega, I).
\]

We select
\[
(5.10) \quad D = m_0, \quad E = \frac{1}{(\mathcal{R} + D)^2} \log \left( \frac{\nu_{\inf} - \varepsilon}{m_0} \right), \quad \text{and} \quad F = \frac{\nu_{\inf} - \varepsilon}{m_0} - 1.
\]

Observe that \( e^{E(\mathcal{R} + D)^2} = 1 + F = (\nu_{\inf} - \varepsilon)/m_0 \).
Our aim is to show that \( u \geq \xi \) in \( Q_{T_0} \). Use (5.10) and that \( R \leq r \leq R + D \). Thus,

\[
\frac{m_0 e^{FR^2}}{1 + F} \leq \xi(x, T_0) \leq \frac{m_0 e^{E(R+D)^2}}{1 + F} = m_0 \leq u(x, T_0), \quad \forall x \in \Omega,
\]
and

\[
\xi(x, t) \leq m_0 e^{E(R+D)^2} \left( \frac{e^{a(t-T_0)}}{e^{a(t-T_0)} + F} \right) \leq \nu \inf - \varepsilon \leq h(x, t), \quad \forall (x, t) \in S_{T_0}.
\]

Employing the comparison principle, \( u \geq \xi \) in \( Q_{T_0} \). Using (5.10), we have

\[
uh \geq m_0 \left( \frac{\nu \inf - \varepsilon}{m_0} \right)^{R^2/(R+D)^2} \left( \frac{e^{a(t-T_0)}}{e^{a(t-T_0)} + F} \right), \quad \forall (x, t) \in Q_{T_0}.
\]

Since \( u(x, t) \geq \mu \inf(t) \geq \inf Q_t, \xi, t \geq T_0 \), we get that

\[
\mu \inf(t) \geq m_0 \left( \frac{\nu \inf - \varepsilon}{m_0} \right)^{R^2/(R+D)^2} \left( \frac{e^{a(t-T_0)}}{e^{a(t-T_0)} + F} \right).
\]

Letting \( t \to \infty \), and then letting \( R \to \infty \),

\[
\lim_{t \to \infty} \mu \inf(t) \geq \nu \inf - \varepsilon.
\]

The claim follows since the above is true for any small \( \varepsilon \). \( \square \)

**Proof of Part (b):** We assume that \( u \) is a sub-solution. Recall that \( M = \sup_{S_t} h(x, t) \). Set \( M_0 = \max \{ u(x, T), M \} \). As noted in (5.1), \( u(x, t) \leq M_0 \) in \( Q_T \). Since \( \nu \sup \leq \mu \sup(t) \leq M_0 \), if \( \nu \sup = M_0 \), the statement follows.

For the proof, we assume that \( \nu \sup < M_0 \) and we will show that \( \lim_{t \to \infty} \mu \sup(t) \leq \nu \sup \).

Let \( \varepsilon > 0 \), small, and \( T_0 > 0 \) be such that

\[
\nu \sup \leq \nu \sup(t) \leq \nu \sup + \varepsilon < M_0, \quad \text{for any } t \geq T.
\]

This ensures that \( h(x, t) \leq \nu \sup + \varepsilon \) on \( S_{T_0} \).

We employ the function in (5.10): let \( z \in \mathbb{R}^n \setminus \Omega \) and \( r = |x - z| \). Define

\[
\zeta(x, t) = \zeta(r, t) = De^{-Er^2} \left( 1 + F e^{-a(t-T_0)} \right), \quad \forall (x, t) \in Q_{T_0},
\]
where \( D, E, F \) and \( a \) are positive constants. Recalling (5.2) and (5.7), we choose

\[
0 < a < (2E)^k R^{k-1}|J|, \quad \text{where } J = \max_{|\omega|=1} H(\omega, \kappa \omega \otimes \omega - I) < 0,
\]

(5.14) and \( \kappa \equiv 2E(R + D)^2 < 1 \).

Choose \( \kappa > 0 \), small (\( E \) small), so that \( J < 0 \) (see (1.7) C(i)), and as a result, \( \zeta \) is a super-solution in \( Q_{T_0} \).
For a fixed $\kappa$, we choose 

$$D = e^{\kappa/2(\nu_{\text{sup}} + \varepsilon)}, \quad E = \frac{\kappa}{2(R + D)^2} \quad \text{and} \quad F = \frac{M_0}{\nu_{\text{sup}} + \varepsilon} - 1.$$ 

Thus, in $Q_{T_0}$,

$$\zeta(x, t) = \left(\nu_{\text{sup}} + \varepsilon\right) \exp\left(\frac{\kappa}{2} \left[1 - \frac{r^2}{(R + D)^2}\right]\right) \left(1 + F e^{-a(t-T_0)}\right).$$

Observe that if $x \in \Omega$ then $R \leq r \leq R + D$. Hence, by (5.13),

$$\zeta(x, T_0) \geq \left(\nu_{\text{sup}} + \varepsilon\right) \exp\left(\frac{\kappa}{2} \left[1 - \frac{r^2}{(R + D)^2}\right]\right) \geq \nu_{\text{sup}} + \varepsilon \geq h(x, t), \quad \forall (x, t) \in S_{T_0}.$$ 

Thus, $\zeta \geq u$ on the parabolic boundary of $Q_{T_0}$, and Theorem 2.3 implies that $\zeta \geq u$ in $Q_{T_0}$. Thus, for any $s \geq t > T_0$, $u(x, s) \leq \zeta(x, s)$, and

$$\mu_{\text{sup}}(t) \leq \sup_{Q_t} \zeta \leq \left(\nu_{\text{sup}} + \varepsilon\right) \exp\left(\frac{\kappa}{2} \left[1 - \frac{R^2}{(R + D)^2}\right]\right) \left(1 + F e^{-a(t-T_0)}\right), \quad \text{in} \ Q_t,$$

for any $t > T_0$.

Let $t \to \infty$ and then let $R \to \infty$ to obtain that $\lim_{t \to \infty} \mu_{\text{sup}}(t) \leq \nu_{\text{sup}} + \varepsilon$. The claim holds. \(\square\)

6. Proof of Theorem 1.4

Before presenting the proof, we record the following. See Appendix A.1 for existence and comparison principles.

**Lemma 6.1.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain that satisfies an outer ball condition. Let $k \geq 1$, $\delta \neq 0$ and $\theta \in \mathbb{R}$. Then there is a $\psi$ in $C(\Omega)$ such that

$$H(D\psi, D^2\psi) = \delta, \quad \text{in} \ \Omega, \quad \text{with} \ \psi = \theta \ \text{on} \ \partial \Omega.$$

If $\delta > 0$ then $\psi \leq 0$, and if $\delta < 0$ then $\psi \geq 0$. Also, $\psi = \theta + |\delta|^{1/k} \eta(x)$, where $H(D\eta, D^2\eta) = \delta/|\delta|$, and $\eta = 0$ on $\partial \Omega$.

**Proof of Theorem 1.4.** For parts (a) and (b), we assume that $\nu > 0$ and $k > 1$. Part (c) addresses the cases $\nu = 0$ and $k \geq 1$. Also, we include a comment about $k = 1$.

**Proof of Part (a):** Assume that $u > 0$ is a sub-solution and $u = \nu$ on $S_T$.

Let $\varepsilon > 0$ be small. By Theorem 1.3, there is a $T_0 \geq T$ such that

$$\nu \leq \sup_{x \in \Omega} u(x, t) \leq \nu + \varepsilon, \quad \text{for any} \ t \geq T_0. \quad (6.1)$$

By Lemma 6.1, there is a function $\psi \geq 1$ in $C(\Omega)$ such that

$$H(D\psi, D^2\psi) = -1 \quad \text{in} \ \Omega \ \text{and} \ \psi = 1 \ \text{on} \ \partial \Omega. \quad (6.2)$$
Observe that $\psi \geq 1$ in $\Omega$.

Let $T_1 \geq T_0$, to be determined later. With $\psi$ as in (6.2), set in $Q_{T_1}$,
\[ \phi(x,t) = \nu + \varepsilon \psi(x) \tau(t) \quad \text{in } Q_{T_1}, \]
where $\tau(t) = \left( \frac{T_1}{t} \right)^{1/(k-1)}$.

Define $M_\psi = \sup_{\Omega} \psi$. Clearly,
\[ (6.3) \quad 1 \leq \psi \leq M_\psi \quad \text{and} \quad \nu \leq \phi \leq \nu + \varepsilon M_\psi. \]

Using that $\tau \leq 1$, $\tau' = -\tau/[(k-1)t]$ and (6.2),
\[ \Gamma_k[\phi] = H(D\phi, D^2\phi) - \phi^{k-1} \phi_t = -[\varepsilon \tau]^k + \phi^{k-1} \left( \frac{\varepsilon \psi}{k-1} \right) \left( \frac{\tau}{t} \right). \]

Since $\tau^{k-1} = T_1/t$, using (6.3),
\[ \Gamma_k[\phi] = \varepsilon \tau \left[ \psi^{k-1} - [\varepsilon \tau]^{k-1} \right] \leq \frac{\varepsilon \tau}{t} \left[ \frac{M_\psi (\nu + \varepsilon M_\psi)^{k-1}}{k-1} - \varepsilon^{k-1} T_1 \right]. \]

Hence, $\phi$ is super-solution in $Q_{T_1}$ if
\[ T_1 \geq \max \left\{ \frac{M_\psi (\nu + \varepsilon M_\psi)^{k-1}}{(k-1)\varepsilon^{k-1}}, \; T_0 \right\}. \]

Next, from (6.1) and (6.3),
\[ u(x,T_1) \leq \nu + \varepsilon \leq \phi(x,T_1) \quad \text{and} \quad u(x,t) = \nu \leq \phi(x,t), \; \forall (x,t) \in S_{T_1}. \]

By the comparison principle in Theorem 2.3 and (6.1),
\[ \nu \leq \sup_{\Omega} u(x,t) \leq \sup_{\Omega} \phi(x,t) \leq \nu + \frac{\varepsilon M_\psi T_1^{1/(k-1)}}{t^{1/(k-1)}} = \nu + \frac{K}{t^{1/(k-1)}} \quad \text{in } Q_{T_1}, \]
where $K = K(k, \nu, T, M_\psi)$. Thus,
\[ \lim_{t \to \infty} \left[ t^\alpha \left( \sup_{\Omega} u(x,t) - \nu \right) \right] = 0, \quad \text{for any} \; 0 < \alpha < \frac{1}{k-1}. \]

The claim holds. □

**Proof of Part (b):** We assume that $u > 0$ is a super-solution.

In Lemma 6.1 take $\delta = 1$ and $\theta = -1$. Let $\psi$ be the solution. Set $M_\psi = \max_{\Omega} |\psi|$; thus,
\[ -M_\psi \leq \psi \leq -1. \]

Define
\[ (6.4) \quad T_\varepsilon = \frac{\nu^{k-1} M_\psi}{(k-1)\varepsilon^{k-1}}, \quad \text{where} \; 0 < \varepsilon \leq \varepsilon_0 \quad \text{and} \quad \nu - \varepsilon_0 M_\psi > 0. \]

Fix $0 < \varepsilon \leq \varepsilon_0$, small so that $T_\varepsilon \geq T$.

By Theorem 1.3 let $T_0 \geq T_\varepsilon$ be such that
\[ (6.5) \quad 0 < \nu - \varepsilon \leq \inf_{\Omega} u(x,t) \leq \nu, \quad \forall (x,t) \in Q_{T_0}. \]
Set
\[ \phi(x,t) = \nu + \varepsilon \psi(x) \left( \frac{T_0}{t} \right)^{1/(k-1)} = \nu - \varepsilon |\psi(x)| \left( \frac{T_0}{t} \right)^{1/(k-1)}, \quad \forall (x,t) \in \mathcal{Q}_{T_0}. \]

By (6.4), \( 0 < \phi \leq \nu \). Also, since \( \psi \leq -1 \),
\[ \phi(x,T_0) \leq \nu - \varepsilon, \quad \text{in } \Omega, \quad \text{and } \phi(x,t) \leq \nu \text{ in } \mathcal{S}_{T_0}. \]

Since \( H(D\psi, D^2\psi) = 1 \), \( \psi \leq 0 \), \( 0 < \phi \leq \nu \) and \( T_0 \geq T_\varepsilon \), we have that
\[ \Gamma_k[\phi] = \varepsilon k \left( \frac{T_0}{t} \right)^{k/(k-1)} + \phi(x,t)^{k-1} \left( \frac{\varepsilon \psi}{k-1} \right) \left( \frac{T_0}{t} \right)^{1/(k-1)} \geq \frac{\varepsilon T_0^{1/(k-1)}}{t^{k/(k-1)}} \left( \varepsilon^{k-1} T_0 - \frac{\nu^{k-1} M\psi}{k-1} \right) \geq 0. \]

The last line follows from (6.4).

Since \( \phi \) is sub-solution in \( \mathcal{Q}_{T_0} \) and, by (6.3), \( u \geq \phi \) on its parabolic boundary, using Theorem 2.3 we obtain that
\[ u(x,t) \geq \phi(x,t) = \nu + \varepsilon \psi(x) \left( \frac{T_0}{t} \right)^{1/(k-1)}, \quad \forall (x,t) \in \mathcal{Q}_{T_0}. \]

Observe that \( \inf_\Omega \phi(x,t) \leq \inf_\Omega u(x,t) \leq \nu \).

If \( 0 < \sigma < 1/(k-1) \) we have
\[ \liminf_{t \to \infty} \left[ \sigma \left( \inf_\Omega u(x,t) - \nu \right) \right] = 0. \]

This proves the claim. \( \square \)

**Comment:** Let \( \nu > 0 \) and \( k = 1 \). Since \( H(e, X) = H(X) \), \( u \) is a sub-solution of
\[ H(D^2u) - u_t = 0, \quad \text{in } \Omega_\infty. \]

Clearly, \( v \equiv u - \nu \) is a sub-solution and \( v = 0 \) on \( \Omega \times [T, \infty) \). Parts (a) and (b) do not apply as \( k = 1 \). The decay rate of \( v \) turns out be exponential in \( t \). See Part (c) below. \( \square \)

**Proof of Part (c):** Let \( \nu = 0 \) and \( k \geq 1 \). We continue to assume that \( u \geq 0 \) in \( \Omega_\infty \). Let \( T_0 \geq 0 \) be such that \( h(x,t) = 0 \) on \( \mathcal{S}_{T_0} \). Define
\[ M = \sup_{\Omega} u(x,T_0). \]

We refer to Appendix A.2 for details, and in particular the definition of \( \lambda_\Omega \). Choose \( \lambda < \lambda_\Omega \), close to \( \lambda_\Omega \). Let \( \psi_\lambda = \psi_\lambda(x) > 0 \) solve
\[ H(D\psi_\lambda, D^2\psi_\lambda) + \lambda \psi_\lambda^k = 0 \text{ in } \Omega \text{ and } \psi_\lambda = M \text{ on } \partial \Omega. \]

Set
\[ \phi_\lambda(x,t) = e^{-\lambda(t-T_0)} \psi_\lambda(x) \quad \text{in } \mathcal{Q}_{T_0}. \]
Since \( \psi_{\lambda} \geq M \),
\[
\phi_{\lambda}(x, T_0) \geq u(x, T_0), \quad \forall x \in \Omega \quad \text{and} \quad \phi_{\lambda}(x, t) > u(x, t), \quad \forall (x, t) \in S_{T_0}.
\]
Also, (6.7) yields
\[
H(D\phi_{\lambda}, D^2\phi_{\lambda}) - \phi_{\lambda}^{k-1}(\phi_{\lambda})_t = e^{-\lambda k(t-T_0)} \psi_{\lambda}^k(\lambda - \lambda) = 0.
\]
The comparison principle in Lemma 2.4 in \( Q_{T_0} \) implies that for any \( T > T_0 \),
\[
0 \leq u(x, t) \leq \phi_{\lambda} = \psi_{\lambda}(x)e^{-\lambda(t-T_0)} \quad \text{in} \quad \Omega \times (T_0, T).
\]
Clearly, the above holds in any large \( T \) and so the estimate holds in \( Q_{T_0} \). Thus, for any \( t \geq T_0 \),
\[
\sup_{\Omega} u(x, t) \leq \max_{\Omega} \psi_{\lambda}(x)e^{-\lambda(t-T_0)}.
\]
Applying logarithm to both sides and letting \( t \to \infty \), we obtain
\[
\lim_{t \to \infty} \left( \frac{\sup_{\Omega} \log u}{t} \right) \leq -\lambda.
\]
The statement in the theorem now holds as \( \lambda < \lambda_{\Omega} \) is arbitrary. □

To see that the above may not hold for super-solutions, consider the classical heat equation
\[
\Delta u - u_t = 0.
\]
If we take \( u(x), \lambda_1 \) to be the first eigenfunction, eigenvalue pair of \( \Delta \), with \( u > 0 \), and define \( u(x, t) = u(x) \), we get that \( \Delta u - u_t = -\lambda_1 u \leq 0 \) and \( u = 0 \) on \( \partial \Omega \times (0, \infty) \).
It is well-known that \( u \in C^\infty \) and is a viscosity solution. Clearly, \( u \) does not decay in \( t \).

**Appendix A. Existence for the auxiliary elliptic problem and the Eigenvalue Problem**

We begin with a version of the comparison principle that will be used in this section. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. We recall a result proven in [6].

**Lemma A.1.** Let \( f_i : \Omega \times \mathbb{R} \to \mathbb{R}, \ i = 1, 2, \) be continuous as in (2.2). Suppose that \( u \in \text{usc}(\Omega) \) and \( v \in \text{lsc}(\Omega) \) are solutions to
\[
H(Du, D^2u) \geq f_1(x, u(x)) \quad \text{and} \quad H(Dv, D^2v) \leq f_2(x, v(x)), \quad \text{in} \ \Omega.
\]
If \( \sup_{\Omega} (u - v) > \sup_{\partial \Omega} (u - v) \) then there is a point \( z \in \Omega \) such that \( (u - v)(z) = \sup_{\Omega} (u - v) \) and \( f_1(z, u(z)) \leq f_2(z, v(z)) \).

**Proof.** A proof can be worked out as in Theorem 4.1 in [6; Section 4]. □

**Corollary A.2.** (Comparison Principle) Suppose that \( s, t \in \mathbb{R} \) are such that \( |s| + |t| > 0 \), and \( s \leq t \). Let \( u \in \text{usc}(\Omega) \) and \( v \in \text{lsc}(\Omega) \) satisfy
\[
H(Du, D^2u) \geq t, \quad \text{and} \quad H(Dv, D^2v) \leq s \quad \text{in} \ \Omega.
\]
Then \( u - v \leq \sup_{\partial \Omega} (u - v) \).
Proof. Consider $s < t$. By taking $f_1 = t$ and $f_2 = s$, Lemma A.1 implies that $u - v \leq \sup_{\partial \Omega} (u - v)$.

Assume now that $t = s$. We take $\theta > 1$ if $t > 0$, and $0 < \theta < 1$ if $t < 0$. The function $u_\theta = \theta u$ solves $H(Du_\theta, D^2u_\theta) = \theta^k H(Du, D^2u) \geq t \theta^k > s$. Thus,

$$u_\theta - v \leq \sup_{\partial \Omega} (u_\theta - v).$$

The conclusion follows by letting $\theta \to 1$. \qed

We obtain also the following quotient form of the comparison principle, see [6]: Theorem 1.2, see Sections 1 and 5.

**Lemma A.3.** Let $\lambda > 0$. Suppose that $u \in \text{usc}(\overline{\Omega})$, $u > 0$, and $v \in \text{lsc}(\overline{\Omega})$, $v > 0$ in $\overline{\Omega}$, are solutions to

$$H(Du, D^2u) + \lambda u^k \geq 0 \quad \text{and} \quad H(Dv, D^2v) + \lambda v^k \leq 0, \quad \text{in} \ \Omega.$$

Then $u/v \leq \sup_{\partial \Omega} (u/v)$.

A.1. **Existence for Lemma 6.1.** Let $\delta > 0$ and $\theta \in \mathbb{R}$. In this appendix we show existence of viscosity solutions to the following problems by using the Perron method.

(a) $H(Du, D^2u) = \delta$, in $\Omega$, $u = \theta$ on $\partial \Omega$, and

(A.1)

(b) $H(Du, D^2u) = -\delta$, in $\Omega$, $u = \theta$ on $\partial \Omega$.

We construct suitable sub-solutions and super-solutions. Corollary A.2 provides the necessary comparison principle. Define

(A.2) \hspace{1cm} d = \text{diam}(\Omega).

Observe that for any $y \in \partial \Omega$, there is a $\rho > 0$ and a $q \in \mathbb{R}^n \setminus \Omega$ such that

(A.3) \hspace{1cm} B_\rho(q) \subset \mathbb{R}^N \setminus \Omega \quad \text{and} \quad y \in \partial \Omega \cap \overline{B_\rho(q)}.

**Sub and Super solutions to** (A.1)(a): We note that, for any $\theta$, $w(x) = \theta$ is a supersolution of (A.1)(a). Our effort is to construct sub-solutions.

Let $y \in \partial \Omega$. With $d$ as in (A.2), and $\rho$ and $q_y$ as in (A.3), set $r = |x - q|$. Define

$$v_y(x) = \theta + E \left( \frac{1}{r^\alpha} - \frac{1}{\rho^\alpha} \right), \quad \forall x \in \Omega.$$ 

where $E > 0$ and $\alpha > 0$ are to be determined. Using (2.1), we get, in $r \geq \rho$,

\begin{align}
H(Dv_y, D^2v_y) &= E^k H \left( -\frac{\alpha}{p^{\alpha+1}} e, -\frac{\alpha}{p^{\alpha+2}} (I - e \otimes e) + \frac{\alpha(\alpha + 1)}{p^{\alpha+2}} e \otimes e \right) \\
&= \frac{(E\alpha)^k}{p^{\alpha k + k + 1}} H \left( e, (\alpha + 2)e \otimes e - I \right),
\end{align}

(A.4)
Setting $\Lambda = \alpha + 2$, and recalling (1.6) and Condition C(ii) in Section 1 (see (1.7)),
\[
\min_{|e| = 1} H(e, \Lambda e \otimes e - I) \geq -M(\Lambda) > 0, \quad \text{if } \Lambda > \Lambda_1.
\]
Choose $\Lambda > \Lambda_1$ and $\alpha = \Lambda - 2$. Next, observing that if $x \in \Omega$ then $\rho \leq r \leq \rho + d$, (A.4) yields in $\Omega$,
\[
H_k[v_y] \geq \frac{(E\alpha)^k|M(\Lambda)|}{(\rho + d)^{k\alpha+k+1}} > 0.
\]
We now select $E$ such that
\[
\frac{(E\alpha)^k|M(\Lambda)|}{(\rho + d)^{k\alpha+k+1}} \geq \delta.
\]
With this choice, we obtain that
\[
H(Dv_y, D^2v_y) \geq \delta, \quad v_y(y) = \theta, \quad \text{and } v_y \leq \theta \text{ on } \partial\Omega.
\]
For every $y \in \partial\Omega$, we have constructed a sub-solution $v_y$ that attains the boundary value $\theta$ at $y$. The Perron Method leads to a solution $v_y \leq u \leq w = \theta$ of (A.1)(a).

**Sub and Super solutions to (A.1)(b):** Observe that $v(x) = \theta$ is a sub-solution. Our effort is to construct super-solutions.

Let $y \in \partial\Omega$. With $d$ as in (A.2), and $\rho$ and $q$ as in (A.3), set $r = |x - q|$. Define
\[
w_y(x) = \theta + E \left( \frac{1}{\rho^\alpha} - \frac{1}{r^\alpha} \right), \quad \forall x \in \Omega,
\]
where $E > 0$ and $\alpha > 0$ are to determined. Using (2.1), we get, in $r > 0$,
\[
H(Dw_y, D^2w_y) = E^k H \left( \frac{\alpha}{r^{\alpha+1}} e, \frac{\alpha}{r^{\alpha+2}} (I - e \otimes e) - \frac{\alpha(\alpha + 1)}{r^{\alpha+2}} e \otimes e \right)
\]
\[
= \frac{(E\alpha)^k}{r^{\alpha k + k+1}} H(e, I - (\alpha + 2)e \otimes e).
\]
Set $\Lambda = \alpha + 2$. Recalling (1.6) and Condition C(ii), we see that
\[
\max_{|e| = 1} H(e, I - \Lambda e \otimes e) \leq M(\Lambda) < 0,
\]
if $\Lambda > \Lambda_1$. Choose $\Lambda > \Lambda_1$ and $\alpha > \Lambda - 2$. Since, $\rho \leq r \leq \rho + d$, we see that
\[
H(Dw_y, D^2w_y) = \frac{(E\alpha)^k}{r^{\alpha k + k+1}} H(e, I - (\alpha + 2)e \otimes e) \leq \frac{(E\alpha)^k M(\Lambda)}{(\rho + d)^{\alpha k + \alpha + 1}} < 0.
\]
Choose $E > 0$ such that
\[
\frac{(E\alpha)^k|M(\Lambda)|}{(\rho + d)^{\alpha k + \alpha + 1}} \geq \delta.
\]
Thus, $H(Dw_y, D^2w_y) \leq -\delta$, in $\Omega$, $w_y(y) = \theta$, and $w_y \geq \theta$ on $\partial\Omega$.

By the Perron method, there is a solution $u$ such that $\theta = v \leq u \leq w_y$. □

Next, we discuss the results needed for Theorem 1.4. We refer to the work [6].
Recall the hypothesis that $\Omega \in C^2$. In [6] a distinction is made between the cases $1 \leq \Lambda_1 < 2$ and $\Lambda_1 \geq 2$. This is not required here.
A.2. Eigenvalue Problem. We show here that $\lambda$ used in (6.7) is bounded, see [6]: (1.10), Section 1. Let $k \geq 1$ and $\delta > 0$. Let the differential operator $H$ satisfy conditions A, B and C. Consider the problem of the existence of a pair $\lambda \in \mathbb{R}$ and $u > 0$ satisfying

$$(A.5) \quad H(Du, D^2u) + \lambda u^k = 0, \quad \text{in } \Omega, \quad u = \delta \quad \text{on } \partial \Omega.$$ 

Define $S = \{ \lambda : \text{Problem (A.5) has a positive solution } u \}$. It is shown in [6]: Theorem 1.5, Sections 1 and 8] that $S$ is an interval, and $\lambda_\Omega = \sup S < \infty$. This is shown in [6]: Theorem 1.7, Sections 1 and 9]. The proof uses domain monotonicity of $\lambda_\Omega$. This is shown in [6]: Lemma 8.2, Section 8].

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