Spectral dimension of spheres

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Abstract

In this paper, we associate a growth graph and a length operator to a quotient space of a semisimple compact Lie group. Under certain assumptions, we show that the spectral dimension of a homogeneous space is greater than or equal to summability of the length operator. Using this, we compute spectral dimensions of spheres.

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1 Introduction

Motivated by Connes’ definition of dimension of a spectral triple, Chakraborty and Pal ([1]) introduced an invariant called spectral dimension, for an ergodic C*-dynamical system or equivalently for a homogeneous space of a compact quantum group. Ergodicity of a C*-dynamical system gives a unique state invariant under group action. Chakraborty and Pal considered all finitely summable equivariant spectral triples on the GNS space of the invariant state and defined the spectral dimension of the homogeneous space to be the infimum of the summability of the associated Dirac operators. Some questions naturally arise about this invariant.

1. For classical homogeneous spaces, is it some known quantity associated with the space?

2. Is spectral dimension of q-deformation $G_q$ of a semisimple simpy connected compact Lie group $G$ is same as $G$?

3. Given a Poisson Lie subgroup $H$ of $G$, is spectral dimension of $G_q/H_q$ equal to that of $G/H$?

To answer all these questions, we need to compute the invariant in several cases. Chakraborty and Pal computed spectral dimension of many homogeneous spaces, both in classical and quantum situations and it was conjectured that the spectral dimension of a homogeneous space of a (classical) compact Lie group is same as its dimension as a differentiable manifold. The spectral dimension of $SU(2)$ points towards this conjecture. Saurabh [5] considered the quaternion
sphere $SP(2n)/SP(2n-2)$ with canonical $SP(2n)$ action and proved that its spectral dimension is equal to the dimension of quaternion spheres as a real manifold which strengthens the conjecture of Chakraborty and Pal. So far, we have only these two instances which one can cite as a strong evidence in support of the conjecture. To gain more confidence about proving the conjecture, it is important to explore more examples. In this article, we take up three cases. First one is odd dimensional sphere $SU(n+1)/SU(n)$ with natural $SU(n+1)$ action. It is a type $A$ homogeneous space. Second one is even dimensional sphere $SO(2n+1)/SO(2n)$ with canonical $SO(2n+1)$ action. It is an example of type $B$ homogeneous spaces. And third one is the type $D$ homogeneous space $SO(2n)/SO(2n-1)$ with canonical $SO(2n)$ action. At this moment, it must be pointed out that though $SU(n+1)/SU(n)$ and $SO(2n+2)/SO(2n+1)$ are homeomorphic as a topological space, one can not conclude that their spectral dimensions are equal because the $C^*$-dynamical system in both cases are not isomorphic. Moreover, for $q \in (0,1)$, the spectral dimension of $SU_q(n+1)/SU_q(n)$ is computed in [1] but for $q = 1$, one can not follow that method as the behaviour of these classical spaces can be quite different from their quantum analog. Therefore it is worthwhile to compute the invariant in these cases. In this paper, we prove that the spectral dimensions in all the three cases are equal to their dimensions as a real manifold. So, our results support the conjecture of Chakraborty and Pal.

For a subset $S$ of a $C^*$-algebra $A$, $\overline{S}$ is the closed linear span of $S$ in $A$. We will sometimes write a spectral triple $(A, H, D)$ as $(H, \pi, D)$ where $\pi$ is the representation of $A$ in the Hilbert space $H$. We denote by $S^n$ the $n$-dimensional sphere.

## 2 Spectral dimension

In this section, we recall from [1] the definition of spectral dimension of a $C^*$-dynamical system and then give some conditions on a homogeneous space to get a lower bound on its spectral dimension. Let us begin with the definition of a homogeneous space.

**Definition 2.1.** A compact quantum group $G$ acts on a $C^*$-algebra $A$ if there exists a $*$-homomorphism $\tau : A \to A \otimes C(G)$ such that

1. $(\tau \otimes id)\tau = (id \otimes \Delta)\tau$,
2. $
\{ (I \otimes b)\tau(a) : a \in A, b \in C(G) \} = A \otimes C(G).
$

where $\Delta$ is the comultiplication map of $G$. An action $\tau$ is called homogeneous or ergodic if the fixed point subalgebra $\{ a \in A : \tau(a) = a \otimes I \}$ is $C(G)$. In that case, the triple $(A, G, \tau)$ is called an ergodic $C^*$-dynamical system and the associated $C^*$-algebra $A$ is called a homogeneous space of $G$.

A covariant representation of a $C^*$-dynamical system $(A, G, \tau)$ is a pair $(\pi, U)$ consisting of a representation $\pi : A \to \mathcal{L}(H)$ and a unitary representation of $G$ on $\mathcal{H}$ such that for all $a \in A$, 

one has
\[(\pi \otimes \text{id})\tau(a) = U(\pi(a) \otimes I)U^*.\]

**Definition 2.2.** Let \((\pi, U)\) be a covariant representation of a \(C^*\)-dynamical system \((A, G, \tau)\) and \((\mathcal{H}, \pi, D)\) be a spectral triple for a dense \(*\)-subalgebra \(\mathcal{A}\) of \(A\). We say the spectral triple \((\mathcal{H}, \pi, D)\) equivariant with respect to \((\pi, U)\) if \(D \otimes I\) commutes with \(U\).

Given a homogeneous action \(\tau\) of \(G\), there is a unique invariant state \(\rho\) on the homogeneous space \(A\) that satisfies
\[(\rho \otimes \text{id})\tau(a) = \rho(a)I, \quad a \in A.\]

Consider the GNS representation \((\mathcal{H}_\rho, \pi_\rho, \eta_\rho)\) of \(A\) associated with the state \(\rho\). Using the invariance property of \(\tau\), one can show that the action \(\tau\) induces a unitary representation \(U_\tau\) of \(G\) on \(\mathcal{H}_\rho\) and the pair \((\pi_\rho, U_\tau)\) is a covariant representation of the system \((A, G, \tau)\). Let \(\mathcal{O}(G)\) be the dense \(*\)-Hopf subalgebra of \(C(\mathcal{G})\) generated by matrix entries of irreducible unitary representations of \(G\). Define
\[
\mathcal{A} := \{a \in A : \tau(a) \in A \otimes_{\text{alg}} \mathcal{O}(G)\}.
\]

From part (1) of Theorem 1.5 in [4], it follows that \(\mathcal{A}\) is a dense \(*\)-subalgebra of \(A\). Let \(\xi\) be the class of spectral triples of \(\mathcal{A}\) equivariant with respect to the covariant representation \((\pi_\rho, U_\tau)\). The spectral dimension denoted by \(S\dim(A, G, \tau)\) of the \(C^*\)-dynamical system \((A, G, \tau)\) is defined as follows.

\[
S\dim(A, G, \tau) := \inf\{p > 0 : \exists D \text{ such that } (\mathcal{A}, \mathcal{H}_\rho, D) \in \xi \text{ and } D \text{ is } p\text{-summable}\}.
\]

Let \(G\) be a semisimple compact Lie group and \(H\) be a closed Poisson Lie subgroup of \(G\). Let \(\phi : C(G) \to C(H)\) be a \(C^*\)-epimorphism obeying \(\Delta \phi = (\phi \otimes \phi)\Delta\) where \(\Delta\) is the co-multiplication map of \(C(G)\). In such a case, one defines the quotient space \(C(G/H)\) by,
\[
C(G/H) = \{a \in C(G) : (\phi \otimes \text{id})\Delta(a) = I \otimes a\}.
\]

Consider the following \(G\)-action on the quotient space \(G/H\):
\[
\tau : C(G/H) \to C(G/H) \otimes C(G)
\]
\[
a \mapsto \Delta a.
\]

where \(\Delta\) is the co-multiplication map of the compact group \(G\). By theorem 1.5 of [4], we get,
\[
C(G/H) = \bigoplus_{\lambda \in \hat{G}} \bigoplus_{i \in I_\lambda} W_{(\lambda, i)} \tag{2.1}
\]

where \(\lambda\) represents the highest weight of a finite-dimensional irreducible co-representation \(u_\lambda\) of \(C(G)\), \(I_\lambda\) is the multiplicity of \(u_\lambda\) and \(W_{(\lambda, i)}\) corresponds to \(u_\lambda\) in the sense of Podles (see
Let $G \in \gamma$ be the unbounded positive operator on $O_{\gamma}$ such that for all $i \in I_{\lambda}$. We will reparametrize the index appearing in the equation (2.1) as follows.

$$\Gamma := \{ (\lambda, j) : \lambda \in \hat{G}, 1 \leq j \leq I_{\lambda} \}.$$ 

Therefore

$$C(G/H) = \bigoplus_{\gamma \in \Gamma} W_\gamma.$$ 

We will denote by $N_\gamma$ the dimension of $W_\gamma$. Define

$$O(G/H) := \bigoplus_{\gamma \in \Gamma} W_\gamma.$$ 

Then $O(G/H)$ is a dense Hopf $*$-algebra consisting of all $a \in C(G/H)$ such that $\tau(a) \in C(G/H) \otimes_{alg} O(G)$. It is not difficult to verify that the system $(C(G/H), G, \tau)$ is an ergodic $C^*$-dynamical system. Assume that $\rho$ is the invariant state of $\tau$ and $(H_\rho, \pi_\rho, \eta_\rho)$ be the associated GNS representation of $C(G/H)$. The Hilbert space $H_\rho$ has a basis of the form

$$\{ e_{(\gamma, i)} := \eta_\rho(a_{(\gamma, i)}) : a_{(\gamma, i)} : 1 \leq i \leq N_\gamma \}$$ 

is a basis of $W_\gamma, \gamma \in \Gamma$.

Let $R = \{ r_1, r_2, \ldots, r_k \} \subset O(G/H)$ and $c > 0$. Define a directed graph $G^c_R$ as follows. Take the vertex set to be $\Gamma$. We write $\gamma \leadsto_{r_i} \gamma'$ if $e_{(\gamma', i)} = r_i e_{(\gamma, i)}$ and $\| e_{(\gamma', i)} \| < c$ for some $1 \leq i \leq N_{\gamma'}$ and $1 \leq i' \leq N_{\gamma}$. Define edge set of $G^c_R$ to be

$$E := \{ (\gamma, \gamma') : \gamma \leadsto_{r_i} \gamma' \text{ for some } 1 \leq i \leq k \}.$$ 

We write $\gamma \rightarrow \gamma'$ if $(\gamma, \gamma') \in E$. We call the directed graph $G^c_R = (\Gamma, E)$ a growth graph of $O(G/H)$. We say that the graph $G^c_R$ has a root if there exists a vertex $\gamma_0$ such that for any $\gamma \in \Gamma$, there is a directed path from $\gamma_0$ to $\gamma$. The vertex $\gamma_0$ will be called a root of the graph $G^c_R$. In such a case, define a length function $\ell_{\gamma_0} : \Gamma \rightarrow \mathbb{N}$ as follows;

$$\ell_{\gamma_0}(\gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_0, \\ \text{length of a shortest path from } \gamma_0 \text{ to } \gamma & \text{otherwise.} \end{cases}$$

Let $L_{\gamma_0}$ be the unbounded positive operator on $H_\rho$ with dense domain $O(G/H)$ sending $e_{(\gamma, i)}$ to $\ell_{\gamma_0}(\gamma)e_{(\gamma, i)}$. We call $L_{\gamma_0}$ the length operator associated with the length function $\ell_{\gamma_0}$ or the root $\gamma_0$. The following proposition says that if the graph $G^c_R$ has a root then the growth of eigenvalues of Dirac operators which have bounded commutators with the elements of the algebra $O(G/H)$ is less than or equal to the growth of eigenvalues of the length operator associated with the root.

**Proposition 2.3.** Let $D : e_{\gamma, i} \mapsto d_\gamma e_{\gamma, i}$ be a selfadjoint unbounded operator with compact resolvent acting on the Hilbert space $H_\rho$ such that the triple $(O(G/H), H_\rho, D)$ is a spectral triple. Moreover, assume that there exists a finite set $R \subset O(G/H)$ and $c > 0$ such that the graph $G^c_R$ has a root $\gamma_0$. Then we have

$$|d_{\gamma}| = O(\ell_{\gamma_0}(\gamma)).$$
In this section, we will take $G$ completes the proof.

By Proposition 2.3, we get $a$ continuous map $\{\}$

Then following the arguments in propositions 5.1-5.3 leading to the statement (5.22) in [1], we get

This proves the claim.

Proposition 2.4. Assume that there exists a finite set $R$ in $O(G/H)$ and $c > 0$ such that the graph $G^*_R$ has a root $\gamma_0$. Define $l = \inf\{p : Tr(L^p_{\gamma_0}) < \infty\}$. Then one has

$$Sdim(C(G/H), G, \tau) \geq l.$$

Proof: Let $(O(G/H), H, D)$ be an equivariant spectral triple of the system $(C(G/H), G, \tau)$. Then following the arguments in propositions 5.1-5.3 leading to the statement (5.22) in [1], we can assume that $D$ must be of the form

$$De_{(\gamma,i)} = d_{\gamma} e_{(\gamma,i)}, \quad i \in \{1, 2, \cdots, N_{\gamma}\}, \gamma \in \Gamma.$$

By Proposition 2.3, we get $|d_{\gamma}| = O(\ell_{\gamma_0}(\gamma))$. Therefore $D$ is $p$-summable for any $p > l$. This completes the proof.

3 \quad $SU(n + 1)$ action on $S^{2n+1}$

In this section, we will take $G$ to be $SU(n + 1)$ and $H$ to be $SU(n)$. For $1 \leq i, j \leq n + 1$, define a continuous map

$$u^i_j : SU(n + 1) \to C; \quad A \mapsto a^i_j$$

where $a^i_j$ is the $ij^{th}$ entry of $A \in SU(n + 1)$. The $C^*$-algebra $C(SU(n + 1))$ is generated by elements of the set $\{u^i_j : 1 \leq i, j \leq n + 1\}$. In the same way, define the generators $\{v^i_j : 1 \leq i, j \leq n\}$ of $C(SU(n))$. Define the map $\Phi : C(SU(n + 1)) \to C(SU(n))$ as follows.

$$\Phi(u^i_j) = \begin{cases} v^i_j, & \text{if } i \neq n + 1 \text{ or } j \neq n + 1, \\ \delta_{ij}, & \text{otherwise.} \end{cases}$$
The quotient space $SU(n+1)/SU(n)$ can be realized as the $2n+1$-dimensional sphere $S^{2n+1}$. Also, each of the generators $\{u_j^{n+1} : 1 \leq j \leq n+1\}$ can be viewed as projection on to a fixed complex coordinate of a point in $S^{2n+1} \subset \mathbb{C}^{n+1}$. To describe the set of highest weights of all finite-dimensional irreducible co-representation of $C(SU(n+1))$, define

$$X = \{(\lambda_1, \lambda_2, \cdots, \lambda_{n+1}) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}, \lambda_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n + 1\}$$

we say that two tuple $(\lambda_1, \lambda_2, \cdots, \lambda_{n+1})$ and $(\lambda'_1, \lambda'_2, \cdots, \lambda'_{n+1})$ are equivalent ($\sim$) if $\lambda_1 - \lambda'_1 = \lambda_2 - \lambda'_2 = \cdots = \lambda_{n+1} - \lambda'_{n+1}$. Define $A := X/\sim$. Then $A$ is the set of highest weights of all finite-dimensional irreducible co-representation of $C(SU(n+1))$. Using Zhelobenko branching rule (see Theorem 9, page 74, [6] and Theorem 1.7 in [4]), we get

$$I_{\lambda} = \begin{cases} 1 & \text{if } \lambda_1 \geq 0, \lambda_i = 0 \text{ for all } 2 \leq i \leq n \text{ and } \lambda_{n+1} \leq 0, \\ 0 & \text{otherwise}. \end{cases}$$

We will now find a highest weight vector for each irreducible co-representation of highest weight $(\lambda_1, 0, \cdots, 0, \lambda_{n+1})$ which belongs to $C(SU(n+1)/SU(n))$. Let $U(\mathfrak{su}(n))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{su}(n)$. We will view $\mathfrak{su}(n)$ as a subset of $U(\mathfrak{su}(n))$. Then $U(\mathfrak{su}(n))$ is generated by $H_i, E_i, F_i \in \mathfrak{su}(n), i = 1, 2, \cdots, n$, satisfying the relations given in page 160, [3]. Hopf *-structure of $U(\mathfrak{su}(n))$ comes from the following maps (see page 18 and page 21 of [3]):

$$\Delta(r) = r \otimes 1 + 1 \otimes r, \quad S(r) = -r, \quad \epsilon(r) = 0, \quad r = r^* \quad \forall r \in \mathfrak{su}(n).$$

Denote by $T_1$ the finite dimensional irreducible representation of $U(\mathfrak{su}(n))$ with highest weight $(1, 0, \cdots, 0)$. There exists unique nondegenerate dual pairing $\langle \cdot, \cdot \rangle$ between the Hopf *-algebras $U(\mathfrak{su}(n))$ and $\mathcal{O}(SU(n+1)/SU(n))$ such that

$$\langle f, u^k_i \rangle = t_{kl}(f); \quad \text{for } k = n + 1 \text{ and } 1 \leq l \leq n + 1,$$

where $t_{kl}$ is the matrix element of $T_1$. Using this, one can give the algebra $\mathcal{O}(SU(n+1)/SU(n))$ a $U(\mathfrak{su}(n))$-module structure in the following way.

$$f(a) = (1 \otimes \langle f, \cdot \rangle) \Delta a,$$

where $f \in U(\mathfrak{su}(n))$ and $a \in \mathcal{O}(SU(n+1)/SU(n))$. We call an element $b \in \mathcal{O}(SU(n+1)/SU(n))$ a highest weight vector with highest weight $(\lambda_1, 0, \cdots, 0, \lambda_{n+1})$ if

$$H_1(b) = \lambda_1 b, \quad H_i(b) = 0 \text{ for } 2 \leq i \leq n - 1, \quad H_n(b) = -\lambda_{n+1} b$$

and

$$E_i(b) = 0 \quad \text{for } 1 \leq i \leq n.$$

The following proposition describes a highest weight vector with highest weight $(\lambda_1, 0, \cdots, 0, \lambda_{n+1})$ explicitly.
Proposition 3.1. Let \( y = u_n^{n+1} \) and \( z = (u_n^{n+1})^* \). For \( \lambda_1, \lambda_{n+1} \in \mathbb{N} \), define \( b^{(\lambda_1, \lambda_{n+1})} = y^{\lambda_1} z^{\lambda_{n+1}} \). Then \( b^{(\lambda_1, \lambda_{n+1})} \) is a highest weight vectors in the algebra \( \mathcal{O}(SU(n+1)/SU(n)) \) with highest weight \( (\lambda_1, 0, \cdots, 0, \lambda_{n+1}) \).

Proof: It is not difficult to see that
\[
E_i(y) = E_i(z) = 0 \quad \text{for} \quad 1 \leq i \leq n.
\]
Further
\[
H_1(y) = y \quad \text{and} \quad H_i(y) = 0, \quad \text{for} \quad i > 1,
\]
and
\[
H_n(z) = -z \quad \text{and} \quad H_i(z) = 0, \quad \text{for} \quad i < n.
\]
Using this and properties of Hopf * algebra pairing (see page 21 of [3]), one can check that \( b^{(\lambda_1, \lambda_{n+1})} \) is a highest weight vectors with highest weight \( (\lambda_1, 0, \cdots, 0, \lambda_{n+1}) \).

For convenience, we will reparametrize the index. Define
\[
\Gamma = \{(\gamma_1, \gamma_2) : \gamma_1, \gamma_2 \in \mathbb{N}\}.
\]
Hence we have
\[
\mathcal{O}(G/H) := \oplus_{\gamma \in \Gamma} W_{\gamma}.
\]
where \( W_{\gamma} \) corresponds to \( u_{(\gamma_1, 0, \cdots, 0, -\gamma_2)} \) in the sense of Podles (see page 4, [4]). The invariant state \( \rho \) of \( \tau \) is the faithful Haar state \( h \) of \( C(SU(n+1)) \) restricted to \( C(SU(n+1)/SU(n)) \).
Hence \( \mathcal{H}_\rho \) has a basis of the form
\[
\{e_{(\gamma, i)} : e_{(\gamma, i)} : 1 \leq i \leq N_{\gamma}\} \quad \text{is a basis of} \quad W_{\gamma}, \gamma \in \Gamma\}.
\]
We will take \( e_{(\gamma, 1)} \) as the highest weight vector \( b^\gamma \). Define the set
\[
\Theta = \{(y, z) \in \mathbb{R}^2 : 0 \leq y, z \leq 1, y^2 + z^2 = 1\}.
\]
For \( \gamma = (\gamma_1, \gamma_2) \in \Gamma \), define the function
\[
g^{(\gamma_1, \gamma_2)} : \Theta \to \mathbb{R}
\]
sending \((y, z)\) to \( y^{\gamma_1} z^{\gamma_2}\). Applying rotations on the co-ordinates appropriately, we get
\[
\|b^{(\gamma_1, \gamma_2)}\| = \sup_{(y, z) \in \Theta} g^{(\gamma_1, \gamma_2)}(y, z).
\]
Let us state one result of [5] (see Proposition 3.2 in [5]).
Proposition 3.2. Let $\Theta$ be a compact subset of $\mathbb{R}^n$ and $f$ and $h$ are two real valued continuous functions define on $\Theta$. Let $x_0 \in \Theta$ be a point such that $|f(x_0)| = \|f\| = \sup_{x \in \Theta} |f(x)| \neq 0$ and $h(x_0) \neq 0$. Then one has
\[ \frac{\|h f\|}{\|h^{m+1} f\|} \leq \frac{1}{|h(x_0)|}. \]

Lemma 3.3. Let $\epsilon_1 = (1, 0)$ and $\epsilon_2 = (0, 1)$. Then one has

1. $\sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_1 + \epsilon_2\|} < \infty$.
2. $\sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_1\|} < \infty$.
3. $\sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_2\|} < \infty$.

Proof: It is not difficult to check that $g^{(1,1)}$ attains its maximum value at $y = 1/\sqrt{2}$ and $z = 1/\sqrt{2}$. Let $M = \sup_{(y,z) \in \Theta} yz$.

1. If $\gamma_1 = \gamma_2$ then $g^{(\gamma_1,\gamma_2)} = (yz)^{\gamma_1}$ and $g^{\gamma_1+\epsilon_1+\epsilon_2} = (yz)^{\gamma_1+1}$. Therefore,
\[ \sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_1 + \epsilon_2\|} = \sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_1\|} \sup_{(y,z) \in \Theta} g^{(\gamma_1,\gamma_1)} = \sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{M^{\gamma_1}}{M^{\gamma_1+1}} = \frac{1}{M} < \infty. \]

2. We have
\[ \sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_1\|} = \sup_{(y,z) \in \Theta} g^{(\gamma_1,\gamma_2)} \sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_1\|} \sup_{(y,z) \in \Theta} g^{(\gamma_1+1,\gamma_2)} = \sup_{\gamma \in \Gamma_{1=\gamma_2}} y^{\gamma_1-\gamma_2} g^{(\gamma_1,\gamma_2)} = \sqrt{2} < \infty \quad \text{(by Proposition 3.2).} \]

3. We have
\[ \sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_2\|} = \sup_{(y,z) \in \Theta} g^{(\gamma_1,\gamma_2)} \sup_{\gamma \in \Gamma_{1=\gamma_2}} \frac{\|b^\gamma\|}{\|b^\gamma + \epsilon_2\|} \sup_{(y,z) \in \Theta} g^{(\gamma_1,\gamma_2+1)} = \sup_{(y,z) \in \Theta} z^{\gamma_2-\gamma_1} g^{(\gamma_1,\gamma_1)} = \sqrt{2} < \infty \quad \text{(by Proposition 3.2).} \]

Let $c > 0$ be an upper bound in all the three inequalities of the Lemma 3.3. Take $R = \{y, z, yz\}$.

The following lemma says that $G_R^c$ has a root $(0, 0)$.

Lemma 3.4. Let $\gamma \in \Gamma$. Then there is a path in $G_R^c$ joining $(0, 0)$ and $\gamma$ and of length less than or equal to $\max\{\gamma_1, \gamma_2\}$. 

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Proof: If $\gamma_1 \geq \gamma_2$, then one possible path would be as follows.

$$(0,0) \rightarrow (1,1) \rightarrow (2,2) \rightarrow \cdots \rightarrow (\gamma_2, \gamma_2)$$

(by part(1) of the Lemma 3.3)

$$(\gamma_2, \gamma_2) \rightarrow (\gamma_2 + 1, \gamma_2) \rightarrow \cdots \rightarrow (\gamma_1, \gamma_2)$$

(by part(2) of the Lemma 3.3)

If $\gamma_1 \leq \gamma_2$, then one possible path would be as follows.

$$(0,0) \rightarrow (1,1) \rightarrow (2,2) \rightarrow \cdots \rightarrow (\gamma_1, \gamma_1)$$

(by part(1) of the Lemma 3.3)

$$(\gamma_1, \gamma_1) \rightarrow (\gamma_1 + 1, \gamma_1) \rightarrow \cdots \rightarrow (\gamma_1, \gamma_2)$$

(by part(3) of the Lemma 3.3)

Moreover, the length of the path in first case is $\gamma_1$ and in second case, it is $\gamma_2$. This settles the claim. \hfill \Box

Lemma 3.5. For $1 \leq m \leq n + 1$, $l = n + 1$ and $\gamma \in \Gamma$, one has

$$u_m^l e_{(\gamma, i)} \subset \text{span}\{e_{(\beta, j)} : \beta \in \Gamma, \gamma_1 \leq \beta_1 \leq \gamma_1 + 1 \text{ and } \gamma_2 \leq \beta_2 \leq \gamma_2 + 1\}.$$  

Proof: Let $\xi = (0, \ldots, 0, \frac{1}{i^n-\text{place}}, 0, \ldots, 0)$. Then from equation ((13), page 210, [3]), we get

$$u_{(1,0,\ldots,0),\gamma_2} \otimes u_{(\gamma_1,0,\ldots,0,\gamma_2)} = \oplus_{i=1}^{n} u_{(\gamma_1,0,\ldots,0,\gamma_2)+e_i}$$

Hence $u_m^l e_{(\gamma, i)}$ is in the span of matrix entries of the irreducible representations of highest weight $(\delta_1, \delta_2, \ldots, \delta_{n+1})$ such that $\delta_1 = \gamma_1$ or $\gamma_1 + 1$ and $\delta_{n+1} = \gamma_2$ or $\gamma_2 + 1$. Since $u_m^l e_{(\gamma, i)} \in \mathcal{O}(SU(n+1)/SU(n))$ and $\{e_{(\alpha, j)} : \alpha \in \Gamma, 1 \leq j \leq N_\alpha\}$ is a basis of $\mathcal{O}(SU(n+1)/SU(n))$, we get the claim. \hfill \Box

Theorem 3.6. Let $L_{(0,0)}$ be the Dirac operator $e_{(\gamma, i)} \mapsto \max\{\gamma_1, \gamma_2\} e_{(\gamma, i)}$ acting on the Hilbert space $\mathcal{H}_\rho$. Then the triple $(\mathcal{O}(SU(n+1)/SU(n)), \mathcal{H}_\rho, L_{(0,0)})$ is a $(2n+1)$-summable equivariant spectral triple of the system $(C(SU(n+1)/SU(n)), SU(n+1), \tau)$. The operator $L_{(0,0)}$ is optimal, i.e. if $D$ is any equivariant Dirac operator of the $C^*$-dynamical system $(C(SU(n+1)/SU(n)), SU(n+1), \tau)$ acting on $\mathcal{H}_\rho$ then there exist positive reals $a$ and $b$ such that

$$|D| \leq a|L_{(0,0)}| + b.$$

Proof: Clearly $L_{(0,0)}$ is a selfadjoint operator with compact resolvent. That $L_{(0,0)}$ has bounded commutators with the generators $\{u_m^1 : m \in \{1, 2, \cdots n+1\}\}$ of $\mathcal{O}(SU(n+1)/SU(n))$ follows
from Lemma 3.5. This proves that the triple \((\mathcal{O}(SU(n+1)/SU(n)), \mathcal{H}_\rho, L_{(0,0)})\) is an equivariant spectral triple of the system \((\mathcal{C}(SU(n+1)/SU(n)), SU(n+1), \tau)\). From Weyl dimension formula, we have

\[ N_\gamma = O(\gamma_1^{n-1}\gamma_2^{n-1}(\gamma_1 + \gamma_2)). \]

This shows that \(L_{(0,0)}\) is \((2n+1)\)-summable. Optimality follows from Proposition 2.4. \(\Box\)

**Theorem 3.7.** Spectral dimension of the odd dimensional sphere \(SU(n+1)/SU(n)\) is \(2n+1\).

**Proof:** It is a direct consequence of Theorem 3.6. \(\Box\)

4 \hspace{1em} \textbf{SO}(2n + 1) action on \(S^{2n}\)

For \(1 \leq i, j \leq 2n + 1\), define a continuous map

\[ u^i_j : SO(2n + 1) \to \mathbb{C}; \quad A \mapsto a^i_j \]

where \(a^i_j\) is the \(ij\)th entry of \(A \in SO(2n + 1)\). The \(C^*\)-algebra \(C(SO(2n + 1))\) is generated by elements of the set \(\{u^i_j : 1 \leq i, j \leq 2n + 1\}\). In the same way, define the generators \(\{v^i_j : 1 \leq i, j \leq 2n\}\) of \(C(SO(2n))\). Define the map \(\Phi : C(SO(2n + 1)) \to C(SO(2n))\) as follows.

\[ \Phi(u^i_j) = \begin{cases} v^{i-1}_{j-1}, & \text{if } i \neq 1 \text{ or } j \neq 1, \\ \delta_{ij}, & \text{otherwise}. \end{cases} \]

The quotient space \(SO(2n + 1)/SO(2n)\) can be realized as the \(2n\)-dimensional sphere \(S^{2n}\). Also, each of the generators \(\{u^i_j : 1 \leq j \leq 2n + 1\}\) can be viewed as projection on to a fixed real coordinate of a point in \(S^{2n} \subset \mathbb{R}^{2n+1}\). The set of highest weights of all finite-dimensional irreducible co-representation of \(C(SO(2n + 1))\) can be described as follows.

\[ \Lambda = \{(\lambda_1, \lambda_2, \cdots, \lambda_n) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, \lambda_i's \text{ are all either integers or half integers} \}. \]

Using Zhelobenko branching rule (see Theorem 9, page 74, [6] and Theorem 1.7 in [4]), we get

\[ I_\lambda = \begin{cases} 1 & \text{if } \lambda_i = 0 \text{ for all } 2 \leq i \leq n, \\ 0 & \text{otherwise}. \end{cases} \]

The Hopf \(*\)-algebra \(\mathcal{O}(SO(2n + 1)/SO(2n))\) has a \(\mathfrak{so}(2n + 1)\) module structure induced by the pairing between \(\mathcal{O}(SO(2n + 1)/SO(2n))\) and \(\mathfrak{so}(2n + 1)\) associated with the finite dimensional irreducible co-representation of highest weight \((1,0,\cdots,0)\) (similar to that defined in previous subsection). We call an element \(b \in \mathcal{O}(SO(2n + 1)/SO(2n))\) a highest weight vector with highest weight \((\lambda_1,0,\cdots,0)\) if

\[ H_1(b) = 2\lambda_1 b, \quad H_i(b) = 0 \text{ for } 2 \leq i \leq n, \]
and

\[ E_i(b) = 0 \quad \text{for } 1 \leq i \leq n. \]

The following proposition describes a highest weight vector for highest weight \((\lambda_1, 0, \cdots, 0)\) explicitly.

**Proposition 4.1.** Let \(y = u^1_{2n+1}\). For \(\gamma \in \mathbb{N}\), define \(b^\gamma = y^{2\gamma}\). Then \(b^\gamma\) is a highest weight vector in the algebra \(\mathcal{O}(SO(2n+1)/SO(2n))\) with highest weight \((\gamma, 0, \cdots, 0)\).

**Proof:** It follows from a straightforward calculation. \(\square\)

Let \(\Gamma = \{\gamma : \gamma \in \mathbb{N}\}\) and \(R = \{y^2\}\). Define \(e_{(\gamma,1)} = b^\gamma\). Since \(\|b^\gamma\| = 1\), one can show that for \(c < 1\), one has \(\gamma \rightarrow \gamma + 1\) and hence the graph \(\mathcal{G}_R^\gamma\) has a root 0. Further \(\ell_0(\gamma) = \gamma\) and hence associated length operator \(L_0\) maps \(e_{(\gamma,i)}\) to \(\gamma e_{(\gamma,i)}\). To show that \(L_0\) has bounded commutator with the generators of \(C(SO(2n+1)/SO(2n))\), we need the following result.

**Lemma 4.2.** For \(1 \leq m \leq 2n+1\) and \(l = 1\), one has

\[ u^l_m e_{(\gamma,i)} \subset \text{span}\{e_{(\beta,j)} : \beta \in \Gamma, \gamma - 1 \leq \beta \leq \gamma + 1\}. \]

**Proof:** Proof follows by applying equation ((15), page 210, [3]) and taking similar steps as in Lemma 3.5. \(\square\)

**Theorem 4.3.** Let \(L_0\) be the Dirac operator \(e_i^\gamma \mapsto \gamma e_i^\gamma\) acting on the Hilbert space \(\mathcal{H}_\rho\). Then the triple \((\mathcal{O}(SO(2n+1)/SO(2n)), \mathcal{H}_\rho, L_0)\) is a \((2n)\)-summable equivariant spectral triple of the system \((C(SO(2n+1)/SO(2n)), SO(2n+1), \tau)\). The operator \(L_0\) is optimal, i.e. if \(D\) is any equivariant Dirac operator of the \(C^*\)-dynamical system \((C(SO(2n+1)/SO(2n)), SO(2n+1), \tau)\) acting on \(\mathcal{H}_\rho\) then there exist positive reals \(a\) and \(b\) such that

\[ \|D\| \leq a|L_0| + b. \]

**Proof:** Clearly \(L_0\) is a selfadjoint operator with compact resolvent. That \(L_0\) has bounded commutators with the generators \(\{u^m_n : m \in \{1, 2, \cdots, 2n+1\}\}\) of \(\mathcal{O}(SO(2n+1)/SO(2n))\) follows from Lemma 4.2. This proves that the triple \((\mathcal{O}(SO(2n+1)/SO(2n)), \mathcal{H}_\rho, L_0)\) is an equivariant spectral triple of the system \((C(SO(2n+1)/SO(2n)), SO(2n+1), \tau)\). From Weyl dimension formula, we have

\[ N_\gamma = O(\gamma^{2n-1}). \]

This shows that \(L_0\) is \((2n)\)-summable. Optimality follows from Proposition 2.4. \(\square\)

**Theorem 4.4.** Spectral dimension of the even dimensional sphere \(S^{2n} = SO(2n+1)/SO(2n)\) is \(2n\).

**Proof:** It is a direct consequence of Theorem 4.3. \(\square\)
5 \(SO(2n)\) action on \(S^{2n-1}\)

For \(1 \leq i, j \leq 2n\), define a continuous map

\[ u^i_j : SO(2n) \to \mathbb{C}; \quad A \mapsto a^i_j \]

where \(a^i_j\) is the \(ij^{th}\) entry of \(A \in SO(2n)\). The C*-algebra \(C(SO(2n))\) is generated by elements of the set \(\{u^i_j : 1 \leq i, j \leq 2n\}\). In the same way, define the generators \(\{v^i_j : 1 \leq i, j \leq 2n - 1\}\) of \(C(SO(2n - 1))\). Define the map \(\Phi : C(SO(2n)) \to C(SO(2n - 1))\) as follows.

\[
\Phi(u^i_j) = \begin{cases} 
  v_{j-1}^{i-1}, & \text{if } i \neq 1 \text{ or } j \neq 1, \\
  \delta_{ij}, & \text{otherwise}.
\end{cases}
\]

The quotient space \(SO(2n)/SO(2n - 1)\) can be realized as the \(2n - 1\)-dimensional sphere \(S^{2n-1}\). Also, each of the generators \(\{u^i_j : 1 \leq j \leq 2n\}\) can be viewed as projection on to a fixed real coordinate of a point in \(S^{2n-1} \subset \mathbb{R}^{2n}\). The set of highest weights of all finite-dimensional irreducible co-representation of \(C(SO(2n))\) can be described as follows.

\[
A = \{ (\lambda_1, \lambda_2, \ldots, \lambda_n) : \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|, \lambda_n \in \mathbb{N} \cup \frac{\mathbb{N}}{2} \text{ and } \lambda_i - \lambda_{i+1} \in \mathbb{N} \text{ for } 1 \leq i < n \}.
\]

Using Zhelobenko branching rule (see Theorem 10, page 77, [6] and Theorem 1.7 in [4]), we get

\[
I_{\lambda} = \begin{cases} 
  1 & \text{if } \lambda_i = 0 \text{ for all } 2 \leq i \leq n, \\
  0 & \text{otherwise}.
\end{cases}
\]

The Hopf \(^*\)-algebra \(\mathcal{O}(SO(2n)/SO(2n - 1))\) has a \(\mathfrak{so}(2n)\) module structure induced by the pairing between \(\mathcal{O}(SO(2n)/SO(2n - 1))\) and \(\mathfrak{so}(2n)\) associated with the finite dimensional irreducible co-representation of highest weight \((1,0,\cdots,0)\) (similar to that defined in previous subsection). We call an element \(b \in \mathcal{O}(SO(2n)/SO(2n - 1))\) a highest weight vector with highest weight \((\lambda_1,0,\cdots,0)\) if

\[
H_i(b) = \lambda_i b, \quad H_i(b) = 0 \text{ for } 2 \leq i \leq n,
\]

and

\[
E_i(b) = 0 \quad \text{for } 1 \leq i \leq n.
\]

The following proposition describes a highest weight vector for highest weight \((\lambda_1,0,\cdots,0)\) explicitly.

**Proposition 5.1.** Let \(y = u^1_{2n+1}\). For \(\gamma \in \mathbb{N}\), define \(b^\gamma = y^\gamma\). Then \(b^\gamma\) is a highest weight vector in the algebra \(\mathcal{O}(SO(2n)/SO(2n - 1))\) with highest weight \((\gamma,0,\cdots,0)\).
Proof: It follows from a straightforward calculation. 

Let $\Gamma = \{ \gamma : \gamma \in \mathbb{N} \}$ and $R = \{ y \}$. Define $e_{(\gamma,1)} = b^\gamma$. Since $\|b^\gamma\| = 1$, one can show that for $c < 1$, one has $\gamma \to \gamma + 1$ and hence the graph $G^c_R$ has a root 0. Moreover $\ell_0(\gamma) = \gamma$ and hence associated length operator $L_0$ maps $e_{(\gamma,i)}$ to $\gamma e_{(\gamma,i)}$. To show that $L_0$ has bounded commutator with the generators of $C(SO(2n)/SO(2n-1))$, we need the following result.

Lemma 5.2. For $1 \leq m \leq 2n$ and $i = 1$, one has

$$u^1_m u^\gamma_i \subset \text{span}\{ u^2_1 : \gamma_{1} - 1 \leq \beta_1 \leq \gamma_{1} + 1 \}$$

Proof: Proof follows by applying equation ((14), page 210, [3]) and taking similar steps as in Lemma 3.5.

Theorem 5.3. Let $L_0$ be the Dirac operator $e^\gamma_i \to \gamma e^\gamma_i$ acting on the Hilbert space $\mathcal{H}_\rho$. Then the triple $(\mathcal{O}(SO(2n)/SO(2n-1)), \mathcal{H}_\rho, L_0)$ is a $(2n-1)$-summable equivariant spectral triple of the system $(C(SO(2n)/SO(2n-1)), SO(2n), \tau)$. The operator $L_0$ is optimal, i.e. if $D$ is any equivariant Dirac operator of the $C^*$-dynamical system $(C(SO(2n)/SO(2n-1)), SO(2n), \tau)$ acting on $\mathcal{H}_\rho$ then there exist positive reals $a$ and $b$ such that

$$|D| \leq a|L_0| + b.$$ 

Proof: Clearly $L_0$ is a selfadjoint operator with compact resolvent. That $L_0$ has bounded commutators with the generators $\{ u^1_m : m \in \{ 1, 2, \cdots, 2n \} \}$ of $\mathcal{O}(SO(2n)/SO(2n-1))$ follows from Lemma 5.2. This proves that the triple $(\mathcal{O}(SO(2n)/SO(2n-1)), \mathcal{H}_\rho, L_0)$ is an equivariant spectral triple of the system $(C(SO(2n)/SO(2n-1)), SO(2n), \tau)$. From Weyl dimension formula, we have

$$N_\gamma = O(\gamma^{2n-2}).$$

This shows that $L_0$ is $(2n-1)$-summable. Optimality follows from Proposition 2.13.

Theorem 5.4. Spectral dimension of the odd dimensional sphere $SO(2n)/SO(2n-1)$ is $2n - 1$.

Proof: It is a direct consequence of Theorem 5.3.

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