Resolutivity and invariance for the Perron method for degenerate equations of divergence type

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A R T I C L E   I N F O

Article history:
Received 8 May 2021
Available online 17 December 2021
Submitted by E. Saksman

Keywords:
Capacity
Degenerate quasilinear elliptic equation of divergence type
Perron problem
Perron solution
Quasicontinuous function
Resolutive

A B S T R A C T

We consider Perron solutions to the Dirichlet problem for the quasilinear elliptic equation \( \text{div } A(x, \nabla u) = 0 \) in a bounded open set \( \Omega \subset \mathbb{R}^n \). The vector-valued function \( A \) satisfies the standard ellipticity assumptions with \( 1 < p < \infty \) and a \( p \)-admissible weight \( w \). We show that arbitrary perturbations on sets of \((p, w)\)-capacity zero of continuous (and certain quasicontinuous) boundary data \( f \) are resolutive and that the Perron solutions for \( f \) and such perturbations coincide. As a consequence, we prove that the Perron solution with continuous boundary data is the unique bounded solution that takes the required boundary data outside a set of \((p, w)\)-capacity zero.

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1. Introduction

We consider the Dirichlet problem for quasilinear elliptic equations of the form

\[
\text{div } A(x, \nabla u) = 0
\]

in a bounded open subset \( \Omega \) of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). The mapping \( A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies the standard ellipticity assumptions with a parameter \( 1 < p < \infty \) and a \( p \)-admissible weight as in Heinonen–Kilpeläinen–Martio [10, Chapter 3].

The Dirichlet problem amounts to finding a solution of the partial differential equation in \( \Omega \) with prescribed boundary data on the boundary of \( \Omega \). One of the most useful approaches to solving the Dirichlet problem in \( \Omega \) with arbitrary boundary data \( f \) is the Perron method. This method was introduced by Perron [14] and independently Remak [15] in 1923 for the Laplace equation \( \Delta u = 0 \) in a bounded domain.

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https://doi.org/10.1016/j.jmaa.2021.125937
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\( \Omega \subset \mathbb{R}^2 \). It gives an upper and a lower Perron solution (see Definition 3.2) and when the two coincide, we get a suitable solution \( Pf \) of the Dirichlet problem and \( f \) is called resolutive.

The Perron method for the Laplace equation in Euclidean domains was further studied by Brelot [6], where a complete characterization of resolutive functions was given in terms of the harmonic measure. The Perron method was later extended to other linear and nonlinear equations. Granlund–Lindqvist–Martio [7] were the first to use the Perron method to study the nonlinear equation

\[
\text{div} (\nabla_q F(x, \nabla u)) = 0
\]

(where \( \nabla_q F \) stands for the gradient of \( F \) with respect to the second variable). This is a special type of equation (1.1), including the \( p \)-Laplace equation

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) = 0. \tag{1.2}
\]

Lindqvist–Martio [13] studied boundary regularity of (1.1) in the unweighted case and also showed that continuous boundary data \( f \) are resolutive when \( p > n - 1 \). (For the historically inclined, [13] was published before [7], but submitted after and cites [7] but not vice versa.) Kilpeläinen [11] extended the resolutivity to general \( p \), which in turn was extended to weighted \( \mathbb{R}^n \) by Heinonen–Kilpeläinen–Martio [10]. More recently, the Perron method was used to study \( p \)-harmonic functions in the metric setting, see [1–4].

In this paper, we consider the weighted equation

\[
\text{div} \mathcal{A}(x, \nabla u) = 0
\]

and show that arbitrary perturbations on sets of \((p,w)\)-capacity zero of continuous boundary data \( f \) are resolutive and that the Perron solution for \( f \) and such perturbations coincide, see Theorem 3.9. In Proposition 3.8 we also obtain, as a by-product, that Perron solutions of perturbations of Lipschitz boundary data \( f \) are the same as the Sobolev solution of \( f \). This perturbation result, as well as the equality of the Perron and Sobolev solutions, holds also for quasicontinuous representatives of Sobolev functions, see Theorem 4.2. Moreover, we prove the following uniqueness result. A somewhat weaker uniqueness result is proved for quasicontinuous Sobolev functions in Corollary 4.5.

**Theorem 1.1.** Let \( \Omega \) be a nonempty bounded open set. Let \( f \in C(\partial \Omega) \). Then there exists a unique bounded \( \mathcal{A} \)-harmonic function \( u \) in \( \Omega \) such that

\[
\lim_{\Omega \ni y \to x} u(y) = f(x) \quad \text{for q.e. } x \in \partial \Omega. \tag{1.3}
\]

Moreover \( u = Pf \).

Because of this uniqueness result, condition (1.3) can be used to uniquely define solutions of the Dirichlet problem for the equation \( \text{div} \mathcal{A}(x, \nabla u) = 0 \) in \( \Omega \) with arbitrary continuous boundary data. It has also been used in a similar way in Björn–Mwasas [5] to solve a mixed boundary value problem for \( \text{div} \mathcal{A}(x, \nabla u) = 0 \) in an infinite half-cylinder.

For \( p \)-harmonic functions, i.e. solutions of the \( p \)-Laplace equation (1.2), most of the results in this paper follow from Björn–Björn–Shanmugalingam [2], [3], where this was proved for \( p \)-energy minimizers in metric spaces. However, even in unweighted \( \mathbb{R}^n \) and for the general equation \( \text{div} \mathcal{A}(x, \nabla u) = 0 \), several of them have not been formulated before, including Theorem 1.1.

Our main resolutive and invariance results are Proposition 3.8 and Theorem 4.2. Versions of those results were already obtained in [10, Theorem 9.25 and Corollary 9.29], but the invariance parts are new here. This invariance is the key to deducing Theorem 1.1.
Much as we use Heinonen–Kilpeläinen–Martio [10] as the principal literature for this paper, our proofs of Proposition 3.8 and Theorem 4.2 are quite different from the ones in [10, Theorem 9.25 and Corollary 9.29]. In particular, we do not use exhaustions by regular domains.

Instead, our approach is based on an extensive use of obstacle problems, which gives resolutivity and invariance at the same time. A price for this is that our results are for bounded Ω, while [10] covers also unbounded Ω. Obstacle problems on unbounded sets are much less studied and provide additional difficulties, see Hansevi [8], [9]. Depending on the definition, it can happen that the Perron solution differs from the solution of the corresponding obstacle problem, see [9, comment after Theorem 7.5]. Obstacle problems are also a fundamental tool in [10], but to less extent when dealing with resolutivity results.

Since we do not rely on the resolutivity results in [10], our approach gives alternative proofs of Theorem 9.25 and Corollary 9.29 in [10] for bounded Ω.

The proofs here have been inspired by [2] and [3], but have been adapted to the usual Sobolev spaces to make them more accessible for people not familiar with the nonlinear potential theory on metric spaces and Sobolev spaces based on upper gradients. They also apply to the more general $A$-harmonic functions, defined by equations rather than by minimization problems.

Acknowledgment. A. B and J. B. were supported by the Swedish Research Council grants 2016-03424 resp. 621-2014-3974 and 2018-04106. A. M. was supported by the SIDA (Swedish International Development Cooperation Agency) project 316-2014 “Capacity building in Mathematics and its applications” under the SIDA bilateral program with the Makerere University 2015–2020, contribution No. 51180060.

2. Notation and preliminaries

In this section, we present the basic notation and definitions that will be needed in this paper. Throughout, we assume that Ω is a nonempty bounded open subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$, and $1 < p < \infty$. We use $\partial \Omega$ and $\overline{\Omega}$ to denote the boundary and the closure of Ω, respectively.

We write $x$ to mean a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and for a function $v$ which is infinitely many times continuously differentiable, i.e. $v \in C^\infty(\Omega)$, we write $\nabla v = (\partial_1 v, \ldots, \partial_n v)$ for the gradient of $v$. We follow Heinonen–Kilpeläinen–Martio [10] as the primary reference for the material in this paper.

First, we give the definition of a weighted Sobolev space, which is crucial when studying degenerate elliptic differential equations, see [10] and Kilpeläinen [12].

**Definition 2.1.** The *weighted Sobolev space* $H^{1,p}(\Omega, w)$ is defined to be the completion of the set of all $v \in C^\infty(\Omega)$ such that

$$\|v\|_{H^{1,p}(\Omega, w)} = \left(\int_\Omega (|v|^p + |\nabla v|^p) w \, dx\right)^{1/p} < \infty$$

with respect to the norm $\|v\|_{H^{1,p}(\Omega, w)}$, where $w$ is a $p$-admissible weight, see Definition 2.4 below.

The space $H^{1,p}_0(\Omega, w)$ is the completion of $C^\infty_0(\Omega)$ in $H^{1,p}(\Omega, w)$ while a function $v$ is in $H^{1,p}_0(\Omega, w)$ if and only if it belongs to $H^{1,p}(\Omega', w)$ for every open set $\Omega' \subset \Omega$. As usual, $E \subset \Omega$ if $\overline{E}$ is a compact subset of Ω and

$$C^\infty_0(\Omega) = \{v \in C^\infty(\mathbb{R}^n) : \text{supp } v \subset \Omega\}.$$

Throughout the paper, the mapping $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, defining the elliptic operator (1.1), satisfies the following assumptions with a parameter $1 < p < \infty$, a $p$-admissible weight $w(x)$ and for some constants $\alpha, \beta > 0$, see [10, (3.3)–(3.7)]:
First, assume that \( A(x, q) \) is measurable in \( x \) for every \( q \in \mathbb{R}^n \), and continuous in \( q \) for a.e. \( x \in \mathbb{R}^n \). Also, for all \( q \in \mathbb{R}^n \) and a.e. \( x \in \mathbb{R}^n \), the following hold

\[
A(x, q) \cdot q \geq \alpha w(x)|q|^p \quad \text{and} \quad |A(x, q)| \leq \beta w(x)|q|^{p-1},
\]

(2.1)

\[
(A(x, q_1) - A(x, q_2)) \cdot (q_1 - q_2) > 0 \quad \text{for} \quad q_1, q_2 \in \mathbb{R}^n, \; q_1 \neq q_2,
\]

\[
A(x, \lambda q) = \lambda |\lambda|^{p-2} A(x, q) \quad \text{for} \quad \lambda \in \mathbb{R}, \; \lambda \neq 0.
\]

**Definition 2.2.** A function \( u \in H^{1,p}_{\text{loc}}(\Omega, w) \) is said to be a (weak) solution of (1.1) in \( \Omega \) if for all (test) functions \( \varphi \in C^\infty_0(\Omega) \), the following integral identity holds

\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = 0.
\]

(2.2)

A function \( u \in H^{1,p}_{\text{loc}}(\Omega, w) \) is said to be a supersolution of (1.1) in \( \Omega \) if for all nonnegative functions \( \varphi \in C^\infty_0(\Omega) \),

\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0.
\]

A function \( u \) is a subsolution of (1.1) if \( -u \) is a supersolution of (1.1).

The sum of two (super)solutions is in general not a (super)solution. However, if \( u \) and \( v \) are two (super)solutions, then \( \min\{u, v\} \) is a supersolution, see [10, Theorem 3.23]. If \( u \) is a supersolution and \( a, b \in \mathbb{R} \), then \( au + b \) is a supersolution provided that \( a \geq 0 \).

It is rather straightforward that \( u \) is a solution if and only if it is both a sub- and a supersolution, see [10, bottom p. 58]. By [10, Theorems 3.70 and 6.6], every solution \( u \) has a Hölder continuous representative \( v \) (i.e. \( v = u \) a.e.).

**Definition 2.3.** A function \( u \) is \( A \)-harmonic in \( \Omega \) if it is a continuous weak solution of (1.1) in \( \Omega \).

We remark that \( A \)-harmonic functions do not in general form a linear space. However, if \( u \) is \( A \)-harmonic and \( a, b \in \mathbb{R} \), then \( au + b \) is also \( A \)-harmonic. Nonnegative \( A \)-harmonic functions \( u \) in a connected open set \( \Omega \) satisfy Harnack’s inequality \( \sup_K u \leq c \inf_K u \) whenever \( K \subset \Omega \) is compact, with the constant \( c \) depending on \( K \), see [10, Section 6.2].

**Definition 2.4.** A weight \( w \) on \( \mathbb{R}^n \) is a nonnegative locally integrable function. We say that a weight \( w \) is \( p \)-admissible with \( p \geq 1 \) if the associated measure \( d\mu = w \, dx \) is doubling and supports a \( p \)-Poincaré inequality, see [10, Chapters 1 and 20].

For instance, weights belonging to the Muckenhoupt class \( A_p \) are \( p \)-admissible as exhibited for example by Heinonen–Kilpeläinen–Martio [10] and Kilpeläinen [12]. By a weight \( w \in A_p \) we mean that there exists a constant \( C > 0 \) such that for all balls \( B \subset \mathbb{R}^n \),

\[
\left( \int_B w(x) \, dx \right) \left( \int_B w(x)^{1/(1-p)} \, dx \right)^{p-1} \leq C|B|^p, \quad \text{if} \; 1 < p < \infty,
\]

\[
\int_B w(x) \, dx \leq C|B| \text{ess inf } w, \quad \text{if} \; p = 1,
\]
where $|B|$ is the $n$-dimensional Lebesgue measure of $B$.

We follow [10, Section 2.35] defining the Sobolev capacity as follows.

**Definition 2.5.** Let $E$ be a subset of $\mathbb{R}^n$. The Sobolev $(p,w)$-capacity of $E$ is

$$C_{p,w}(E) = \inf_{E} \int_{\mathbb{R}^n} (|u|^p + |\nabla u|^p) \, dw \, dx,$$

where the infimum is taken over all $u \in H^{1,p}(\mathbb{R}^n, w)$ such that $u = 1$ in an open set containing $E$.

The Sobolev $(p, w)$-capacity is a monotone, subadditive set function. It follows directly from the definition that for all $E \subset \mathbb{R}^n$,

$$C_{p,w}(E) = \inf_{G \supset E} C_{p,w}(G). \quad (2.3)$$

In particular, if $C_{p,w}(E) = 0$ then there exist open sets $U_j \supset E$ with $C_{p,w}(U_j) \to 0$ as $j \to \infty$. For details, we refer the interested reader to [10, Section 2.1]. A property is said to hold *quasieverywhere* (abbreviated q.e.), if it holds for every point outside a set of Sobolev $(p, w)$-capacity zero.

### 3. Perron solutions and resolutivity

In order to discuss the Perron solutions for (1.1), we first recall the following basic results from Heinonen–Kilpeläinen–Martio [10, Chapters 7 and 9].

**Definition 3.1.** A function $u : \Omega \to (-\infty, \infty]$ is $\mathcal{A}$-superharmonic in $\Omega$ if

(i) $u$ is lower semicontinuous in $\Omega$,

(ii) $u$ is not identically $\infty$ in any component of $\Omega$,

(iii) for each open set $\Omega' \subset \Omega$ and all functions $v \in C(\overline{\Omega'})$ which are $\mathcal{A}$-harmonic in $\Omega'$, we have $v \leq u$ in $\Omega'$ whenever $v \leq u$ on $\partial \Omega'$.

A function $u : \Omega \to [-\infty, \infty)$ is $\mathcal{A}$-subharmonic in $\Omega$ if $-u$ is $\mathcal{A}$-superharmonic in $\Omega$.

Let $u$ and $v$ be $\mathcal{A}$-superharmonic. Then $au + b$ and $\min\{u, v\}$ are $\mathcal{A}$-superharmonic whenever $a \geq 0$ and $b$ are real numbers, but in general $u + v$ is not $\mathcal{A}$-superharmonic, see [10, Lemmas 7.1 and 7.2].

We briefly state how supersolutions and $\mathcal{A}$-superharmonic functions are related. It is proved in [10, Theorem 7.16] that if $u$ is a supersolution of (1.1) and

$$u^*(x) := \operatorname{ess lim inf} u(y) \quad \text{for every } x \in \Omega, \quad (3.1)$$

then $u^* = u$ a.e. and $u^*$ is $\mathcal{A}$-superharmonic. Conversely, if $u$ is an $\mathcal{A}$-superharmonic function in $\Omega$, then $u^* = u$ in $\Omega$. If moreover, $u$ is locally bounded from above, then $u \in H^{1,p}_{\text{loc}}(\Omega, w)$ and $u$ is a supersolution of (1.1) in $\Omega$, see [10, Corollary 7.20]. That is, every supersolution has an $\mathcal{A}$-superharmonic representative and locally bounded $\mathcal{A}$-superharmonic functions are supersolutions.

**Definition 3.2.** For a function $f : \partial \Omega \to [-\infty, \infty]$, let $\mathcal{U}_f$ be the set consisting of all functions $u$ on $\Omega$, which are $\mathcal{A}$-superharmonic in $\Omega$, bounded from below and such that

$$\liminf_{\Omega \ni y \to x} u(y) \geq f(x) \quad \text{for all } x \in \partial \Omega.$$
The upper Perron solution $\overline{P}f$ of $f$ is then defined by

$$\overline{P}f(x) = \inf_{u \in \mathcal{U}_f} u(x), \quad x \in \Omega.$$  

The lower Perron solution $\underline{P}f$ of $f$ is defined analogously as

$$\underline{P}f(x) = \sup_{v \in \mathcal{L}_f} v(x), \quad x \in \Omega,$$

where the set $\mathcal{L}_f(\Omega)$ consists of all $\mathcal{A}$-subharmonic functions bounded from above and such that

$$\limsup_{\Omega \ni y \to x} v(y) \leq f(x) \quad \text{for all } x \in \partial \Omega.$$  

We remark that if $\mathcal{U}_f = \emptyset$, then $\overline{P}f \equiv \infty$ and if $\mathcal{L}_f = \emptyset$, then $\underline{P}f \equiv -\infty$. In every component $\Omega'$ of $\Omega$, $\overline{P}f$ (and $\underline{P}f$) is either $\mathcal{A}$-harmonic or identically $\pm \infty$ in $\Omega'$, see [10, Theorem 9.2].

If $\overline{P}f = \underline{P}f$ is $\mathcal{A}$-harmonic, then $f$ is called resolutive with respect to $\Omega$. In this case, we write $Pf := \overline{P}f$.

Continuous functions $f$ are resolutive by [10, Theorem 9.25]. Observe that Perron solutions are denoted by the letter $H$ in [10].

The following comparison principle shows that $Pf \leq \overline{P}f$.

**Theorem 3.3. ([10, Comparison principle 7.6])** Assume that $u$ is $\mathcal{A}$-superharmonic and that $v$ is $\mathcal{A}$-subharmonic in $\Omega$. If

$$\infty \neq \limsup_{\Omega \ni y \to x} v(y) \leq \liminf_{\Omega \ni y \to x} u(y) \neq -\infty \quad \text{for all } x \in \partial \Omega,$$

then $v \leq u$ in $\Omega$.

We follow [10, Chapter 3] giving the following definition.

**Definition 3.4.** Given $\psi : \Omega \to [-\infty, \infty]$ and $f \in H^{1,p}(\Omega, w)$, let

$$\mathcal{K}_{\psi,f}(\Omega) = \{v \in H^{1,p}(\Omega, w) : v - f \in H_0^{1,p}(\Omega, w) \text{ and } v \geq \psi \text{ a.e. in } \Omega\}.$$  

A solution of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem is a function $u \in \mathcal{K}_{\psi,f}(\Omega)$ such that

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla (v - u) \, dx \geq 0 \quad \text{for all } v \in \mathcal{K}_{\psi,f}(\Omega).$$

In particular, a solution $u$ of the $\mathcal{K}_{\psi,u}(\Omega)$-obstacle problem with $\psi \equiv -\infty$ is a solution of (1.1). By considering $v = u + \varphi$ with $0 \leq \varphi \in C_0^\infty(\Omega)$, it is easily seen that the solution $u$ of the obstacle problem is always a supersolution of (1.1) in $\Omega$. Conversely, a supersolution $u$ in $\Omega$ is always a solution of the $\mathcal{K}_{u,u}(\Omega')$-obstacle problem for all open sets $\Omega' \Subset \Omega$. Moreover, a solution $u$ of (1.1) is a solution of the $\mathcal{K}_{\psi,u}(\Omega')$-obstacle problem with $\psi \equiv -\infty$ for all $\Omega' \Subset \Omega$, see [10, Section 3.19].

By [10, Theorem 3.21], there is an almost everywhere (a.e.) unique solution $u$ of the $\mathcal{K}_{\psi,f}(\Omega)$-obstacle problem whenever $\mathcal{K}_{\psi,f}(\Omega)$ is nonempty. Furthermore, by defining $u^*$ as in (3.1) we get a (unique) lower semicontinuously regularized solution belonging to the same equivalence class as $u$, see [10, Theorem 3.63].
We call \( u^* \) the lower semicontinuous (lsc) regularization of \( u \). Furthermore, with \( \psi \equiv -\infty \), the lsc-regularization of the solution of the \( K_{\psi,f}(\Omega) \)-obstacle problem provides us with the \( \mathcal{A} \)-harmonic extension \( Hf \) of \( f \) in \( \Omega \), that is, \( Hf - f \in H^1_0(\Omega, w) \) and \( Hf \) is \( \mathcal{A} \)-harmonic. The continuity of \( Hf \) in \( \Omega \) is guaranteed by [10, Theorem 3.70].

**Definition 3.5.** A point \( x \in \partial \Omega \) is Sobolev regular if, for every \( f \in H^{1,p}(\Omega, w) \cap C(\overline{\Omega}) \), the \( \mathcal{A} \)-harmonic function \( Hf \) in \( \Omega \) satisfies

\[
\lim_{\Omega \ni y \to x} Hf(y) = f(x).
\]

Furthermore, \( x \in \partial \Omega \) is regular if

\[
\lim_{\Omega \ni y \to x} Pf(y) = f(x) \quad \text{for all} \quad f \in C(\partial \Omega).
\]

If \( x \in \partial \Omega \) is not (Sobolev) regular, then it is (Sobolev) irregular.

By [10, Theorem 9.20], \( x \) is regular if and only if it is Sobolev regular, we will therefore just say “regular” from now on. By the Kellogg property [10, Theorems 8.10 and 9.11], the set of irregular points on \( \partial \Omega \) has Sobolev \((p, w)\)-capacity zero.

The following result is due to Björn–Björn–Shanmugalingam [2, Lemma 5.3]. Here it is slightly modified to suite our context. For completeness and the reader’s convenience, the proof is included.

**Lemma 3.6.** Assume that \( \{U_k\}_{k=1}^{\infty} \) is a decreasing sequence of open sets in \( \mathbb{R}^n \) satisfying \( C_{p,w}(U_k) < 2^{-kp} \). Then there is a decreasing sequence \( \{\psi_j\}_{j=1}^{\infty} \) of nonnegative functions such that for all \( j, m = 1, 2, \ldots \),

\[
\|\psi_j\|_{H^{1,p}(\mathbb{R}^n, w)} < 2^{-j} \quad \text{and} \quad \psi_j \geq m \quad \text{in} \quad U_{j+m}.
\]

In particular, \( \psi_j = \infty \) on \( \bigcap_{k=1}^{\infty} U_k \).

**Proof.** Since \( C_{p,w}(U_k) < 2^{-kp} \), by Definition 2.5 there exist \( \varphi_k \in H^{1,p}(\mathbb{R}^n, w) \) such that \( \varphi_k = 1 \) in \( U_k \) and \( \|\varphi_k\|_{H^{1,p}(\mathbb{R}^n, w)} < 2^{-k} \). Replacing \( \varphi_k \) by its positive part \( \max\{\varphi_k, 0\} \), we can assume that each \( \varphi_k \) is nonnegative. Define

\[
\psi_j = \sum_{k=j+1}^{\infty} \varphi_k, \quad j = 1, 2, \ldots.
\]

Then

\[
\|\psi_j\|_{H^{1,p}(\mathbb{R}^n, w)} \leq \sum_{k=j+1}^{\infty} \|\varphi_k\|_{H^{1,p}(\mathbb{R}^n, w)} < \sum_{k=j+1}^{\infty} 2^{-k} = 2^{-j}.
\]

Since \( \varphi_k \geq 1 \) on each \( U_k \) and \( U_k \supset U_{j+m} \) when \( j + 1 \leq k \leq j + m \), it follows that \( \psi_j \geq m \) in \( U_{j+m} \). \( \square \)

We will need the following convergence theorem due to Heinonen–Kilpeläinen–Martio [10, Theorem 3.79] in order to prove the next proposition.

**Theorem 3.7.** Let \( \Omega \) be a nonempty bounded open set. Assume that \( \{\psi_j\}_{j=1}^{\infty} \) is an a.e. decreasing sequence of functions in \( H^{1,p}(\Omega, w) \) such that \( \psi_j \to \psi \) in \( H^{1,p}(\Omega, w) \). Let \( u_j \in H^{1,p}(\Omega, w) \) be a solution of the \( K_{\psi_j,\psi}(\Omega) \)-obstacle problem. Then the sequence \( u_j \) decreases a.e. to a function \( u \in H^{1,p}(\Omega, w) \), which is a solution of the \( K_{\psi,\psi}(\Omega) \)-obstacle problem.
Proposition 3.8. Let $\Omega$ be a nonempty bounded open set. Let $f$ be Lipschitz on $\overline{\Omega}$ and $h : \partial \Omega \to [-\infty, \infty]$ be such that $h = 0$ q.e. on $\partial \Omega$. Then both $f$ and $f + h$ are resolutive and
\[
P(f + h) = Pf = Hf. \tag{3.2}
\]

The equality $Pf = Hf$ was shown already in [10, Corollary 9.29]. It is the first equality in (3.2) that is new. However, the main part of the proof below amounts to showing that $P(f + h) = Hf$. Then (3.2) follows directly (by letting $h \equiv 0$).

Proof. Since $f$ is Lipschitz and $\Omega$ bounded, we get that $f \in H^{1,p}(\Omega, w)$. First, we assume that $f \geq 0$. Let $I_\rho \subset \partial \Omega$ be the set of all irregular points. Let $E = \{ x \in \partial \Omega : h(x) \neq 0 \}$. Then by the Kellogg property [10, Theorem 8.10], we have $C_{p,w}(I_\rho \cup E) = 0$. Using (2.3), we can find a decreasing sequence $\{ U_k \}_{k=1}^\infty$ of bounded open sets in $\mathbb{R}^n$ such that $I_\rho \cup E \subset U_k$ and $C_{p,w}(U_k) < 2^{-kp}$. Let $\{ \psi_j \}_{j=1}^\infty$ be the corresponding decreasing sequence of nonnegative functions, given by Lemma 3.6.

Consider the lsc-regularized solutions $u_j$ of the $K_{f_j,h_j}$-obstacle problems in $\Omega$ with $f_j = Hf + \psi_j$, see [10, Theorems 3.21 and 3.63]. Let $m$ be a positive integer. By the comparison principle [10, Lemma 3.18], we have that $Hf \geq 0$ and hence by Lemma 3.6,
\[
f_j = Hf + \psi_j \geq \psi_j \geq m \quad \text{in } U_{j+m} \cap \Omega.
\]

In particular, $u_j \geq f_j \geq m$ a.e. in $U_{j+m} \cap \Omega$ and since $u_j$ is lsc-regularized, we have that
\[
u_j \geq m \quad \text{everywhere in } U_{j+m} \cap \Omega. \tag{3.3}
\]

Let $\varepsilon > 0$ and $x \in \partial \Omega$ be arbitrary. If $x \notin U_{j+m}$, then $x$ is a regular point and thus $Hf$ is continuous at $x$. Hence, there is a neighbourhood $V_x$ of $x$ such that
\[
Hf(y) \geq f(x) - \varepsilon = (f + h)(x) - \varepsilon \quad \text{for all } y \in V_x \cap \Omega.
\]

As $\psi_j \geq 0$, we have that $f_j = Hf + \psi_j \geq Hf$. So,
\[
u_j(y) \geq f_j(y) \geq (f + h)(x) - \varepsilon \quad \text{for a.e. } y \in V_x \cap \Omega.
\]

Since $u_j$ is lsc-regularized, we get
\[
u_j(y) \geq (f + h)(x) - \varepsilon \quad \text{for all } y \in V_x \cap \Omega.
\]

And if $x \in U_{j+m}$, we instead let $V_x = U_{j+m}$. Then $u_j \geq m$ in $V_x \cap \Omega$ by (3.3).

Consequently, for all $x \in \partial \Omega$, we have
\[
u_j(y) \geq \min\{(f + h)(x) - \varepsilon, m\} \quad \text{for all } y \in V_x \cap \Omega.
\]

Thus,
\[
\liminf_{\Omega \ni y \to x} \nu_j(y) \geq \min\{(f + h)(x) - \varepsilon, m\} \quad \text{for all } x \in \partial \Omega. \tag{3.4}
\]

Letting $\varepsilon \to 0$ and $m \to \infty$ yields
\[
\liminf_{\Omega \ni y \to x} \nu_j(y) \geq (f + h)(x) \quad \text{for all } x \in \partial \Omega.
\]
Since $u_j$ is $A$-superharmonic and nonnegative, we conclude that $u_j \in \mathcal{U}_{f+h}(\Omega)$, and thus $u_j \geq \overline{P}(f + h)$. As $Hf$ is the solution of the $K_{Hf,Hf}(\Omega)$-obstacle problem, we get by Theorem 3.7 that the sequence $u_j$ decreases a.e. to $Hf$ in $\Omega$. Thus, $Hf \geq \overline{P}(f + h)$ a.e. in $\Omega$. But $Hf$ and $\overline{P}(f + h)$ are continuous, so we have that for all Lipschitz functions $f \geq 0$,\begin{equation}
abla Hf \geq \overline{P}(f + h) \quad \text{everywhere in } \Omega. \tag{3.5}\end{equation}

Next, let $f$ be an arbitrary Lipschitz function on $\overline{\Omega}$. Since $f$ is bounded, there exists a constant $c \in \mathbb{R}$ such that $f + c \geq 0$. By the definition of $Hf$ and of Perron solutions we see that

$$H(f + c) = Hf + c \quad \text{and} \quad \overline{P}(f + h + c) = \overline{P}(f + h) + c.$$\n
This together with (3.5) shows that

$$Hf = H(f + c) - c \geq \overline{P}(f + h + c) - c = \overline{P}(f + h),$$\n
i.e. (3.5) holds for arbitrary Lipschitz functions $f$. Applying it to $-f$ and $-h$ gives us that

$$Hf = -H(-f) \leq -\underline{P}(-f - h) = \underline{P}(f + h).$$\n
Together with the inequality $\underline{P}(f + h) \leq \overline{P}(f + h)$, implied by Theorem 3.3, we get that

$$Hf \leq \underline{P}(f + h) \leq \overline{P}(f + h) \leq Hf,$$\n
and thus $P(f + h) = Hf$ and $f + h$ is resolutive. Finally, letting $h \equiv 0$, it follows directly that $f$ is resolutive and $Pf = Hf$. □

It is now possible to extend the resolutivity results to continuous functions. This gives us an alternative way of solving the Dirichlet problem with prescribed continuous boundary data.

**Theorem 3.9.** Let $\Omega$ be a nonempty bounded open set. Let $f \in C(\partial \Omega)$ and $h : \partial \Omega \to [-\infty, \infty]$ be such that $h = 0$ q.e. on $\partial \Omega$. Then both $f$ and $f + h$ are resolutive and $P(f + h) = Pf$.

**Proof.** Since continuous functions can be approximated uniformly by Lipschitz functions, we have that there exists a sequence $\{f_k\}_{k=1}^\infty$ of Lipschitz functions such that

$$f_k - 2^{-k} \leq f \leq f_k + 2^{-k} \quad \text{on } \partial \Omega. \tag{3.6}$$\n
From Definition 3.2 it follows that

$$\overline{P}f_k - 2^{-k} \leq \overline{P}f \leq \overline{P}f_k + 2^{-k} \quad \text{in } \Omega,$$\n
i.e. the functions $\overline{P}f_k$ converge uniformly to $\overline{P}f$ in $\Omega$ as $k \to \infty$. Using (3.6), we also obtain similar inequalities for $\underline{P}f, \overline{P}(f + h)$ and $P(f + h)$ in terms of $\underline{P}f_k$, $\overline{P}(f_k + h)$ and $P(f_k + h)$, respectively. By Proposition 3.8, we have that $f_k$ and $f_k + h$ are resolutive and moreover $P(f_k + h) = Pf_k$. Using the resolutivity of $f_k + h$, we have

$$\overline{P}(f + h) - 2^{-k} \leq \overline{P}(f_k + h) = P(f_k + h) \leq \overline{P}(f + h) + 2^{-k} \quad \text{in } \Omega,$$\n
from which it follows that
\[ 0 \leq |f(x) + h(x) - f(x)| \leq 2^{1-k} \text{ in } \Omega. \]

Letting \( k \to \infty \) shows that \( f + h \) is resolutive. In the same way, \( f \) is resolutive. Next, we have from (3.6) that
\[ P(f + h) - 2^{-k} \leq P(f_k + h) = Pf_k \leq Pf + 2^{-k} \text{ in } \Omega, \]
from which we get
\[ P(f + h) - Pf \leq 2^{1-k}. \]

Similarly,
\[ Pf - P(f + h) \leq 2^{1-k}. \]

Letting \( k \to \infty \) shows that \( P(f + h) = Pf \). \( \square \)

If \( u \) is a bounded \( \mathcal{A} \)-harmonic function in \( \Omega \) such that
\[ f(x) = \lim_{y \to x} u(y) \text{ for all } x \in \partial \Omega, \]
then \( u \in U_f \cap L_f \). Thus,
\[ u \leq Pf \leq Pf \leq u, \]
and so \( f \) is resolutive and \( u = Pf \), see [10, p. 169]. Using Theorem 3.9, we can now generalize this fact and deduce the following uniqueness result.

**Corollary 3.10.** Let \( \Omega \) be a nonempty bounded open set. Assume that \( f \in C(\partial \Omega) \) and that \( u \) is a bounded \( \mathcal{A} \)-harmonic function in \( \Omega \) such that
\[ \lim_{y \to x} u(y) = f(x) \text{ for q.e. } x \in \partial \Omega. \]
Then \( u = Pf \) is the Perron solution of \( f \) in \( \Omega \).

**Proof.** As both \( f \) and \( u \) are bounded, we can by rescaling and adding constants assume without loss of generality that \( 0 \leq f \leq 1 \) and \( 0 \leq u \leq 1 \). Since \( u \) is bounded and \( \mathcal{A} \)-harmonic in \( \Omega \), we have that \( u \in L_{f+\chi_E} \) and \( u \in U_{f-\chi_E} \), where \( E \) consists of those \( x \in \partial \Omega \) for which (3.7) fails. Thus, by Theorem 3.9, we get that
\[ u \leq P_f = Pf = Pf \leq Pf \leq u \text{ in } \Omega. \] \( \square \)

**Remark 3.11.** The word *bounded* is essential for the above uniqueness result to hold. Otherwise it fails. For instance, the Poisson kernel
\[ \frac{1 - |z|^2}{|1 - z|^2} \]
with a pole at \( 1 \) is a harmonic function in the unit disc \( B(0,1) \subset \mathbb{C} = \mathbb{R}^2 \), which is zero on \( \partial B(0,1) \setminus \{1\} \).

**Proof of Theorem 1.1.** By the Kellogg property [10, Theorem 9.11] and Theorem 3.9, we have that \( u = Pf \) satisfies (3.7). On the other hand, if \( u \) satisfies (3.7), then Corollary 3.10 shows that \( u = Pf \). \( \square \)
4. Quasicontinuous functions

One of the useful properties of the Sobolev space $H^{1,p}(\Omega, w)$ is that every function in $H^{1,p}(\Omega, w)$ has a $(p, w)$-quasicontinuous representative which is unique upto sets of $(p, w)$-capacity zero, see [10, Theorem 4.4].

**Definition 4.1.** Let $A \subset \mathbb{R}^n$. A function $v : A \to [-\infty, \infty]$ is $(p, w)$-quasicontinuous in $A$ if for every $\varepsilon > 0$ there is an open set $G$ such that $C_{p,w}(G) < \varepsilon$ and the restriction of $v$ to $A \setminus G$ is finite valued and continuous.

It follows from the outer regularity (2.3) of $C_{p,w}$ that if $v$ is quasicontinuous and $\tilde{v} = v$ q.e. then $\tilde{v}$ is also quasicontinuous. Refining the techniques in Section 3, we can obtain the following result.

**Theorem 4.2.** Let $\Omega$ be a nonempty bounded open set. Let $f : \overline{\Omega} \to [-\infty, \infty]$ be a $(p, w)$-quasicontinuous function in $\overline{\Omega}$ such that $f \in H^{1,p}(\Omega, w)$. Let $h : \partial\Omega \to [-\infty, \infty]$ be such that $h = 0$ q.e. on $\partial\Omega$. Then $f + h$ and $f$ are resolutive and $P(f + h) = Pf = Hf$.

For $f \in H^{1,p}(\Omega, w) \cap C(\overline{\Omega})$, the equality $Pf = Hf$ was shown already in [10, Corollary 9.29]. Here, $f$ is only assumed to be quasicontinuous. In $f + h$ we can interpret $\pm \infty$ arbitrarily in $[-\infty, \infty]$. Before the proof of Theorem 4.2, we give the following two lemmas which may be of independent interest.

**Lemma 4.3.** Let $\Omega$ be a nonempty bounded open set. Let $f : A \to [-\infty, \infty]$ be a $(p, w)$-quasicontinuous function in $A \supset \Omega$ such that $f \in H^{1,p}(\Omega, w)$. Then its $A$-harmonic extension $Hf$, extended by $f$ outside $\Omega$, is $(p,w)$-quasicontinuous in $A$.

**Proof.** Define $v := Hf - f$ and extend it by zero outside $\Omega$. Then $v \in H^{1,p}_0(\Omega, w)$. By [10, Theorem 4.5], there is a $(p,w)$-quasicontinuous function $\tilde{v}$ in $\mathbb{R}^n$ such that $\tilde{v} = v$ a.e. in $\Omega$ and $\tilde{v} = 0$ q.e. in the complement of $\Omega$. Recall that $Hf$ is a continuous function in $\Omega$ and $f$ is assumed to be $(p,w)$-quasicontinuous in $\Omega$. This clearly means that $v$ is also $(p,w)$-quasicontinuous in $\Omega$. It then follows from [10, Theorem 4.12] that $v = \tilde{v}$ q.e. in $\Omega$. We know that $v = 0$ outside the set $\Omega$. Thus, we can conclude that $\tilde{v} = v$ q.e. in $\mathbb{R}^n$. Finally, by (2.3), since $\tilde{v}$ is $(p, w)$-quasicontinuous in $A$, so is $v$ and hence also $f + v$, which concludes the proof. □

**Lemma 4.4.** Let $\Omega$ be a nonempty bounded open set. Let $\{f_j\}_{j=1}^{\infty}$ be a decreasing sequence of functions in $H^{1,p}(\Omega, w)$ such that $f_j \to f$ in $H^{1,p}(\Omega, w)$. Then the sequence $Hf_j$ decreases to $Hf$ in $\Omega$.

**Proof.** By the comparison principle [10, Lemma 3.18], we have for all $j = 1, 2, \ldots$,

$$u_j := Hf_j \geq Hf_{j+1} \geq \ldots \geq Hf \quad \text{in} \quad \Omega.$$  

Thus $u(x) = \lim_{j \to \infty} u_j(x)$ exists for all $x \in \Omega$ and $u(x) \geq Hf(x)$. Note that $Hf$ is continuous in $\Omega$ and so the sequence $u_j$ is locally bounded from below in $\Omega$. By [10, Theorem 3.77], $u$ is a supersolution in $\Omega$. Similarly, [10, Theorem 3.75] applied to $-u_j$ shows that $u$ is a subsolution. Hence $u$ is a solution of (1.1) in $\Omega$, see [10, bottom p. 58].

To conclude the proof, we need to show that $u - f \in H^{1,p}_0(\Omega, w)$. We know that $u_j - f_j \to u - f$ pointwise a.e. and $u_j - f_j \in H^{1,p}_0(\Omega, w)$. Because of [10, Lemma 1.32], it is sufficient to show that $u_j - f_j$ is a bounded sequence in $H^{1,p}(\Omega, w)$.
Using the Poincaré inequality [10, (1.5)] we have

\[ \|u_j - f_j\|_{H^{1,p}(\Omega, w)} \leq C_\Omega \left( \int_{\Omega} |\nabla u_j - \nabla f_j|^p w \, dx \right)^{1/p} \]

\[ \leq C_\Omega \left( \int_{\Omega} |\nabla u_j|^p w \, dx \right)^{1/p} + C_\Omega \left( \int_{\Omega} |\nabla f_j|^p w \, dx \right)^{1/p}, \]

where \( C_\Omega \) is a constant which depends on \( \Omega \). Since \( u_j \) is a solution and \( \mathcal{A} \) satisfies the ellipticity conditions (2.1), testing (2.2) with \( \varphi = u_j - f_j \) yields

\[ \left( \int_{\Omega} |\nabla u_j|^p w \, dx \right)^{1/p} \leq C \left( \int_{\Omega} |\nabla f_j|^p w \, dx \right)^{1/p}, \]

where \( C \) is a constant depending on the structure constants \( \alpha \) and \( \beta \) in (2.1). Therefore,

\[ \|u_j - f_j\|_{H^{1,p}(\Omega, w)} \leq C' \left( \int_{\Omega} |\nabla f_j|^p w \, dx \right)^{1/p} \leq C'\|f_j\|_{H^{1,p}(\Omega, w)} \leq M < \infty, \]

since the sequence \( \{f_j\}_{j=1}^{\infty} \) is bounded in \( H^{1,p}(\Omega, w) \). This shows that \( u_j - f_j \) is bounded in \( H^{1,p}(\Omega, w) \). Consequently, by [10, Lemma 1.32 and Theorem 3.17], \( u - f \in H_0^{1,p}(\Omega, w) \) and \( u = Hf \) a.e.

Finally, \( u \) is continuous by Harnack’s convergence theorem [10, Theorem 6.14]. As \( Hf \) is continuous, we conclude that \( u \equiv Hf \) in \( \Omega \). \( \Box \)

We now prove Theorem 4.2 and refer the reader to closely look at the proof of Proposition 3.8 to fill in details where needed.

**Proof of Theorem 4.2.** First assume that \( f \geq 0 \) and so \( Hf \geq 0 \). Define \( u := Hf \) extended by \( f \) outside \( \Omega \). By Lemma 4.3, \( \mathcal{A} \) is \((p, w)\)-quasicontinuous in \( \overline{\Omega} \). Hence, there is a decreasing sequence \( \{U_k\}_{k=1}^{\infty} \) of bounded open sets in \( \mathbb{R}^n \) such that \( C_{p,w}(U_k) < 2^{-kp} \), \( h = 0 \) outside \( U_k \) and \( u \) restricted to \( \overline{\Omega} \setminus U_k \) is continuous. Consider the \( \text{isc-regularized} \) solutions \( u_j \) of the \( \mathcal{K}_{f_j} \)-obstacle problems \( \Omega \) with \( f_j = u + \psi_j \), where \( \psi_j \) are as in Lemma 3.6, \( j = 1, 2, \ldots \). As in the proof of Proposition 3.8 we get

\[ u_j \geq m \quad \text{everywhere in } U_{j+m} \cap \Omega. \quad (4.1) \]

Let \( \varepsilon > 0 \) and \( x \in \partial \Omega \) be arbitrary. If \( x \in \partial \Omega \setminus U_{j+m} \), then by quasicontinuity, \( u \) restricted to \( \overline{\Omega} \setminus U_{j+m} \) is continuous at \( x \). Thus, there exists a neighbourhood \( V_x \) of \( x \) such that

\[ Hf(y) = u(y) \geq u(x) - \varepsilon = f(x) - \varepsilon = (f + h)(x) - \varepsilon \quad \text{for all } y \in (V_x \cap \Omega) \setminus U_{j+m}. \]

Since \( \psi_j \geq 0 \), we get \( f_j(y) \geq u(y) = Hf(y) \) and so,

\[ u_j(y) \geq f_j(y) \geq (f + h)(x) - \varepsilon \quad \text{for a.e. } y \in (V_x \cap \Omega) \setminus U_{j+m}. \quad (4.2) \]

If \( x \in U_{j+m} \), let \( V_x = \varnothing \). Then by (4.1) and (4.2), we get for all \( x \in \partial \Omega \),

\[ u_j(y) \geq \min\{(f + h)(x) - \varepsilon, m\} \quad \text{for a.e. } y \in (V_x \cup U_{j+m}) \cap \Omega. \]

Since \( u_j \) is \text{isc-regularized}, we have
\[ u_j(y) \leq \min \{(f + h)(x) - \varepsilon, m\} \quad \text{for all } y \in (V_x \cup U_{j+m}) \cap \Omega, \]

and consequently, (3.4) follows. Letting \( \varepsilon \to 0 \) and \( m \to \infty \), we conclude that \( u_j \in \mathcal{U}_{f+h}(\Omega) \). Continuing as in Proposition 3.8, we can conclude that

\[ \mathcal{P}(f + h) \leq Hf \quad \text{in } \Omega \]

holds for all quasicontinuous \( f : \Omega \to [-\infty, \infty] \) in \( H^{1,p}(\Omega, w) \) that are nonnegative (or merely bounded from below).

Now if \( f \in H^{1,p}(\Omega, w) \) is arbitrary, then by (4.3) together with Lemma 4.4 we have that

\[ \mathcal{P}(f + h) \leq \lim_{k \to -\infty} \mathcal{P}(\max\{f, k\} + h) \leq \lim_{k \to -\infty} H \max\{f, k\} = Hf \quad \text{in } \Omega. \]

Thus, (4.3) holds for any \( f \in H^{1,p}(\Omega, w) \) and applying it to \( -f \) and \(-h\), together with the inequality \( P(f + h) \leq \mathcal{P}(f + h) \), concludes the proof. \( \Box \)

Unlike for continuous boundary data in Theorem 3.9, for quasicontinuous boundary data it is in general impossible to have \( \lim_{\Omega \ni y \to x} P_f(y) = f(x) \) for q.e. \( x \in \partial \Omega \), see Example 4.6 below. However, we get the following uniqueness result as a consequence of Theorem 4.2.

**Corollary 4.5.** Let \( \Omega \) be a nonempty bounded open set. Let \( f : \overline{\Omega} \to [-\infty, \infty] \) be a \((p, w)\)-quasicontinuous function in \( \overline{\Omega} \) such that \( f \in H^{1,p}(\Omega, w) \). If \( u \) is a bounded \( \mathcal{A}\)-harmonic function in \( \Omega \) such that

\[ \lim_{\Omega \ni y \to x} u(y) = f(x) \quad \text{for all } x \in \partial \Omega \setminus E, \quad \text{where } C_{p,w}(E) = 0, \]

then \( u = P_f \) in \( \Omega \).

**Proof.** Since \( u \) is a bounded \( \mathcal{A}\)-harmonic function in \( \Omega \), we have that \( u \in \mathcal{L}_{f+\infty \chi_E} \) and \( u \in \mathcal{U}_{f-\infty \chi_E} \). Thus by Theorem 4.2, we get that

\[ u \leq P(f + \infty \chi_E) = P_f = \overline{P}(f - \infty \chi_E) \leq u \quad \text{in } \Omega. \quad \Box \]

The following example shows that in many situations there is a bounded quasicontinuous \( f \in H^{1,p}(\mathbb{R}^n, w) \) such that no function \( u \) satisfies

\[ \lim_{\Omega \ni y \to x} u(y) = f(x) \quad \text{for q.e. } x \in \partial \Omega. \]

In particular it is impossible for the Perron solution \( P_f \) to attain these quasicontinuous boundary data q.e.

**Example 4.6.** Assume that \( \partial \Omega \) contains a dense countable sequence \( \{x_j\}_{j=1}^{\infty} \) of points with \( C_{p,w}(\{x_j\}) = 0 \), \( j = 1, 2, \ldots \). As \( \Omega \) is bounded it follows from [10, Corollary 2.39 and Lemma 2.46] that \( C_{p,w}(\partial \Omega) > 0 \). Using (2.3), we can then find \( r_j > 0 \) so small that \( C_{p,w}(B(x_j, r_j)) < 3^{-j}C_{p,w}(\partial \Omega), \ j = 1, 2, \ldots \).

By [10, Corollary 2.39], each \( x_j \) has zero variational \((p, w)\)-capacity, and hence, by the definition of the variational capacity [10, p. 27] there is \( f_j \in C_0^\infty(B(x_j, r_j)) \) such that \( f_j(x_j) = 1 \) and \( \|f_j\|_{H^{1,p}(\mathbb{R}^n, w)} < 2^{-j} \). Then

\[ f := \sum_{j=1}^{\infty} \max\{f_j, 0\} \in H^{1,p}(\mathbb{R}^n, w). \]
Since the partial sums of $f$ are continuous and coincide with $f$ outside the open sets $\bigcup_{j \geq k} B(x_j, r_j)$, $k = 1, 2, \ldots$, with arbitrarily small $(p,w)$-capacity, we see that $f$ is quasicontinuous. For each $j$ there is $r_j' < r_j$ such that $f_j \geq \frac{1}{2}$ in $B(x_j, r_j')$. Thus

$$f \geq \frac{1}{2} \text{ in } G' = \bigcup_{j=1}^{\infty} B(x_j, r_j') \text{ and } f = 0 \text{ outside } G = \bigcup_{j=1}^{\infty} B(x_j, r_j).$$

Note that $C_{p,w}(G) < \sum_{j=1}^{\infty} 3^{-j} C_{p,w}(\partial \Omega) < C_{p,w}(\partial \Omega)$. Also let

$$S = \{ x \in \partial \Omega : \text{there is } r > 0 \text{ such that } C_{p,w}(B(x, r) \cap \partial \Omega) = 0 \},$$

which is the largest relatively open subset of $\partial \Omega$ with $C_{p,w}(S) = 0$.

Finally, assume that $u : \Omega \to \mathbb{R}$ is such that

$$\bar{u}(x) := \lim_{\Omega \ni y \to x} u(y) = f(x) \text{ for q.e. } x \in \partial \Omega. \quad (4.4)$$

In particular $\bar{u} \geq \frac{1}{2}$ q.e. in $G' \cap \partial \Omega$, and thus in a dense subset of $\partial \Omega \setminus S$. It follows that

$$\lim sup_{\Omega \ni y \to x} u(y) \geq \frac{1}{2} \text{ for all } x \in \partial \Omega \setminus S.$$