Semi-Markov Model of a Technical System with Maintenance and Time Reserve

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Abstract. Ensuring high reliability of technical systems requires the construction of adequate mathematical models of their functioning, taking into account not only individual factors that increase reliability (maintenance, redundancy, control, etc.), but also their combinations. In this paper, a semi-Markov model of a technical system with calendar maintenance and time reserve is constructed. For an approximate determination of the stationary characteristics of the reliability of the system, the phase merging scheme algorithm is used. The effect of the time reserve on the reliability characteristics of the system under consideration is analyzed.

1. Introduction

Modern technology is characterized by a high level of automation, wide use of automated and automatic control, flexible systems and complexes. Due to the ever-increasing complexity of various-purpose technical systems and their maintenance, the degree of responsibility of the tasks they solve, the price of the error (failure) of their operation, special attention is paid to improving and analyzing their reliability.

Maintenance [1 – 6, 18] is an important method for solving the problem of increasing the reliability and efficiency of technical systems. Along with technical maintenance, one of the approaches to solving this problem is time redundancy [7 – 12, 17]. It has received quite a wide application, as it requires relatively less cost and allows significantly improve the reliability and flexibility of the technical system.

In [18], on the basis of the theory of semi-Markov processes with a common phase space of states [13 - 16], a semi-Markov model of a technical system was constructed taking into account calendar maintenance. In this paper, the semi-Markov model is considered taking into account both the calendar maintenance and the availability of time instantly replenished with a reserve. Based on phase merging algorithms developed in the works of V.S Korolyuk, A.F. Turbin, A.V. Swishchuk [13, 14], approximate formulas for the stationary reliability characteristics of the system under consideration are obtained. The analysis of the time reserve size influence on the reliability of the system is given.

2. Description of the system

Let us describe the order of operation of the system under consideration. System uptime is a random variable (RV) \( \alpha \) with a distribution function (DF) \( F_\alpha(t) = P(\alpha \leq t) \), system recovery time is RV \( \beta \) with DF \( G_\beta(t) = P(\beta \leq t) \). At random instants of time (through intervals of time \( \alpha_2 \) with DF
Calendar maintenance (CM) is conducted, CM time is RV $\beta_2$ with DF $G_2(t) = P(\beta_2 \leq t)$. The maintenance is carried out if the moment of the beginning of the CM has got for the period of the system operation. After CM is performed, the system begins work anew (the operating properties of the system are completely updated). Used instantly replenished time reserve is a RV $\tau$ with DF $R(t) = P(\tau \leq t)$. RVs $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ are assumed to be independent in aggregate, having finite mathematical expectations and variances. Functions $F_1(t)$, $F_2(t)$, $G_1(t)$, $G_2(t)$ have distribution densities $f_1(t)$, $f_2(t)$, $g_1(t)$, $g_2(t)$. The failure of this system occurs at the point in time when the recovery or maintenance time after the next violation of its operability or stop for maintenance, respectively, will exceed the time reserve $\tau$.

3. Construction of a Semi-Markov System Model and Finding the Stationary Characteristics

The functioning of the system $S$ is described by a semi-Markov process (SMP) \([13\,\text{-}\,16]\) $\xi(t)$ with a discrete-continuous phase state space. We introduce the following set of semi-Markov states of the system:

$$E = \{221, 121, 210, 132x, 102x, 112x, 230x, 200x\}$$

where the meaning of state codes is:

- $221$ – maintenance started, work of the system is interrupted, a time reserve is used;
- $121$ – time reserve has finished, the system failure came, maintenance is continuing;
- $210$ – maintenance is complete, the system has started working;
- $132x$ – the system has completed work, recovery has begun, a time reserve is used; time remaining before the maintenance is $x > 0$;
- $102x$ – time reserve has finished, time remaining before the maintenance is $x > 0$;
- $112x$ – system restoration has finished, time remaining before the maintenance is $x > 0$;
- $230x$ – maintenance is not carried out, because the maintenance moment was at the time of system recovery, a time reserve is used, time remaining until the end of the time reserve is $x > 0$;
- $200x$ – maintenance is not carried out, because the maintenance moment was at the time of system recovery, time reserve has finished, the remaining time until the end of the restoration is $x > 0$.

Time diagram of the functioning of the system is shown in figure 1.

![Time diagram](image)

**Figure 1.** Time diagram of the functioning of the system.

We proceed to determine stationary characteristics of the system reliability. For this, we use phase merging algorithms developed in the works \([13,\,14]\).

Suppose that the kernel of the embedded Markov chain (EMC) $\{x_n; n \geq 0\}$ of a semi-Markov process $\xi(t)$ is close to the kernel of the EMC $\{x_n^{(0)}; n \geq 0\}$ of supporting system $S_0$ with a unique stationary distribution $\rho(dx)$. Then for an approximate calculation of the mean stationary operating time of the
system to failure $T_s$, the mean stationary restoration time $T_r$ and stationary availability factor $K_a$ of the initial system $S$ we can use the following approximate formulas [16]:

$$\begin{align}
T_r &\approx \int_{E_0} m(x) \rho(dx), \\
T_s &\approx \int_{E_0} \rho(dx) P(x, E_0) m(y), \\
K_a &= \frac{T_r}{T_s + T_r},
\end{align}$$

(1)

where $\rho(dx)$ is stationary distribution of supporting EMC $\{z^{(0)}; n \geq 0\}$; $m(x)$ is mean residence time of the SMP in the state $x \in E$ of the initial system; $P(x, dy)$ – transition probabilities of the EMC $\{\xi_n; n \geq 0\}$ of the initial system.

Choose a supporting system $S_0$. Let's assume that the system's uptime is significantly longer than the restoration time, the time of maintenance and the time reserve. Consequently, the supporting system will be the system $S_0$, which is instantly restored and has an instant maintenance.

Time diagram of the functioning of the supporting system $S_0$ is shown in figure 2.

![Time diagram of the functioning of the supporting system $S_0$.](image)

Figure 2. Time diagram of the functioning of the supporting system $S_0$.

We define the transition probabilities of the embedded Markov chain $\{\xi_n^n; n \geq 0\}$ of the supporting system:

$$P^{221}_{210} = \int_{0}^{\infty} F_1(y) dF_2(y), \quad P^{210}_{221} = 1, \quad P^{12x}_{112} = \int_{0}^{x} f_1(y) dF_2(x+y),$$

$$P^{132}_{112} = f_1(x-y), \quad 0 < y < x, \quad P^{132}_{132} = 1, \quad P^{112}_{12x} = F_1(x).$$

We denote by $\rho_0$ and $\rho_1$ stationary distribution of supporting system EMC $\{z^{(0)}_n; n \geq 0\}$ on states 210 and 221 and assume the existence of a stationary density $\rho_{132}(x)$ and $\rho_{112}(x)$ for states 132x and 112x.

We compose for them a system of integral equations:

$$\begin{align}
\rho_1 &= \rho_0 \int_{0}^{\infty} F_1(y) dF_2(y) + \int_{0}^{\infty} F_1(y) \rho_{132}(y) dy; \\
\rho_0 &= \rho_1; \\
\rho_{132}(x) &= \rho_0 \int_{0}^{x} f_1(y) dF_2(x+y) + \int_{0}^{\infty} f_1(y-x) \rho_{112}(y) dy, x > 0; \\
\rho_0 + \rho_1 + \int_{0}^{\infty} \rho_{132}(x) dx + \int_{0}^{\infty} \rho_{112}(x) dx &= 1.
\end{align}$$

(2)
The solution of system (2) has the form:

$$\rho_{112}(x) = \rho_0 \int_0^\infty h(y)dF_x(x+y),$$  \hspace{1cm} (3)

where the value of constant $\rho_0$ is obtained from the normalization requirement.

For the initial system, the sets of operable $E_+$ and faulty states $E_-$ have the form:

$$E_+ = \{210, 221, 132x, 230x, 112x\}, \quad E_- = \{121, 200x, 102x\}.$$

We proceed to determine stationary characteristics of reliability of the initial system. Let the time reserve $\tau = \text{const}$, $h \geq 0$, $R(t) = 1(t-h)$. In this case, we determine the transition probabilities of EMC $\xi_n, n \geq 0$ of the initial system:

$$p_{221}^{121} = \int_0^\infty \frac{g_2(y)dR_y}{G_x(y)\varrho}\, dy; \quad p_{221}^{120} = 1; \quad p_{221}^{122} = \int_0^\infty F_1(y)dF_2(y)$$

$$p_{210}^{121} = \int_0^\infty f_1(y)dF_2(x+y), \quad p_{212}^{121} = f_1(x-y), \quad 0 < y < x; \quad p_{121}^{121} = F_1(x), \quad \beta < \tau;$$

$$p_{210}^{122} = g_1(x-y), \quad 0 < y < x; \quad p_{212}^{122} = g_1(x+y), \quad y > 0; \quad p_{220}^{121} = F_2(x+dy), \quad y > 0, \quad \beta > \tau;$$

$$p_{210}^{123} = R(x-dy), \quad 0 < y < x; \quad p_{210}^{230} = R(x+dy), \quad y > 0; \quad p_{220}^{230} = F_2(x-dy), \quad 0 < y < x;$$

$$p_{210}^{230} = F_2(x+dy), \quad y > 0; \quad p_{210}^{230} = F_2(x-dy), \quad 0 < y < x; \quad p_{210}^{230} = F_2(x+dy), \quad y > 0;$$

$$p_{210}^{122} = g_1(h+x+y), \quad y > 0; \quad p_{210}^{122} = g_1(h-x-y), \quad 0 < y < x. \hspace{1cm} (4)$$

The mean values of sojourn times in the states of the initial system are represented by formulas:

$$\theta_{221} = \beta_2 \wedge h, \quad \theta_{211} = [\beta_2 - h]^+; \quad \theta_{210} = \alpha_1 \wedge \alpha_2, \quad \theta_{132} = \alpha_1 \wedge h \wedge x,$$

$$\theta_{230} = \alpha_2 \wedge h, \quad \theta_{132} = \alpha_1 \wedge x, \quad \theta_{200} = \alpha_2 \wedge x \quad \text{and} \quad \theta_{102} = [\beta_1 - h]^+ \wedge x,$$

where $\wedge$ is the minimum sign. Consequently,

$$E\theta_{221} = \int_0^h \frac{G_2(t)dt}{G_2(h)}; \quad E\theta_{211} = \int_0^\infty \frac{G_2(h+t)dt}{G_2(t)}; \quad E\theta_{210} = \int_0^\infty \frac{F_1(t)F_2(t)dt}{G_2(t)}; \quad E\theta_{132} = \int_0^{x+h} \frac{G_2(h)dt}{G_2(t)}$$

$$E\theta_{230} = \int_0^h \frac{F_2(t)dt}{G_2(t)}; \quad E\theta_{132} = \int_0^h \frac{F_1(t)dt}{G_2(t)}; \quad E\theta_{200} = \int_0^h \frac{F_2(t)dt}{G_2(t)}; \quad E\theta_{102} = \frac{1}{G_2(h)} \int_0^{x+h} \frac{G_2(h+t)dt}{G_2(t)}. \hspace{1cm} (5)$$

We find mean stationary operating time of the system to failure $T_+$ and mean stationary restoration time $T_-$ by using the formula (1).

We define the expressions in (1). Using (3), (4), (5), we obtain:

$$\int_{E_-} \rho(dx)P(x,E_-) = \rho(221)P(221,E_-) + \int_0^\infty \rho(132x)P(132x,E_-)dx = \rho_0 \frac{G_2(h)}{G_2(t)} + \rho_x \int_0^h \frac{h(t)F_2(h+t)dt}{G_2(t)}. \hspace{1cm} (6)$$
\[ \int_{E} m(x) \rho(dx) = \rho(210)m(210) + \rho(221)m(221) + \int_{0}^{\infty} \rho(132)x)m(132x)dx + \int_{0}^{\infty} \rho(112)x)m(112x)dx = \]
\[ \rho_0 E_{\alpha_2} + \rho_0 E(\beta_2 \wedge h) + \rho_0 \int_{0}^{h} \tilde{G}_1(t)dt \int_{0}^{\infty} h_1(y)\tilde{F}_2(t + y)dy. \]  

(7)

\[ \int_{E} \rho(dx) \int_{E} P(x, dy)m(y) = \rho(221)P(221, 121)m(121) + \int_{0}^{\infty} \rho(132)x)P(132x, 102)y)m(102x)dx = \]
\[ \rho_0 \int_{0}^{\infty} \tilde{G}_2(h + t)dt + \frac{\rho_0}{\tilde{G}_1(h)} \int_{0}^{\infty} \tilde{G}_1(h + z)dz \int_{0}^{\infty} h_1(t)\tilde{F}_2(h + z + t)dt. \]  

(8)

Consequently, mean stationary operating time to failure \( T_+ \), mean stationary restoration time \( T_- \) and stationary availability factor \( K_a \) of the system under consideration are approximately calculated from formulas:

\[ T_+ \approx \left( \rho_0 E_{\alpha_2} + \rho_0 E(\beta_2 \wedge h) + \int_{0}^{h} \tilde{G}_1(t)dt \int_{0}^{\infty} h_1(y)\tilde{F}_2(t + y)dy \right) \left( \tilde{G}_2(h) + \int_{0}^{\infty} \tilde{h}_1(t)\tilde{F}_2(h + t)dt \right)^{-1}. \]  

(9)

\[ T_- \approx \left( \int_{0}^{\infty} \tilde{G}_2(h + t)dt + \frac{1}{\tilde{G}_1(h)} \int_{0}^{\infty} \tilde{G}_1(h + z)dz \int_{0}^{\infty} h_1(t)\tilde{F}_2(h + z + t)dt \right) \left( \tilde{G}_2(h) + \int_{0}^{\infty} \tilde{h}_1(t)\tilde{F}_2(h + t)dt \right)^{-1}. \]  

(10)

\[ K_a(h) \approx \left( \rho_0 E_{\alpha_2} + \rho_0 E(\beta_2 \wedge h) + \int_{0}^{h} \tilde{G}_1(t)dt \int_{0}^{\infty} h_1(y)\tilde{F}_2(t + y)dy \right) \]
\[ \times \left( \int_{0}^{\infty} \tilde{G}_2(h + t)dt + \frac{1}{\tilde{G}_1(h)} \int_{0}^{\infty} \tilde{G}_1(h + z)dz \int_{0}^{\infty} h_1(t)\tilde{F}_2(h + z + t)dt \right)^{-1}. \]  

(11)

We obtain the formula for \( K_a \) under the condition of a nonrandom distribution of the time intervals between maintenance \( \delta = const \). Taking into account that \( \tilde{F}_2(t) = 1(t - \delta) \), we obtain:

\[ K_a(\delta, h) \approx \frac{\delta + E(\beta_2 \wedge h) + \int_{0}^{\delta} \tilde{H}_1(\delta - t)\tilde{G}_1(t)dt}{\delta + E\beta_2 + \int_{0}^{\delta} \tilde{H}_1(\delta - t)\tilde{G}_1(t)dt}. \]  

(12)

We give a formula for the stationary availability factor \( K_a \) of a system with an instantly replenished time reserve without conducting maintenance:

\[ K_a(h) = \frac{E\alpha_1 + \int_{0}^{h} \tilde{G}_1(t)dt}{E\alpha_1 + E\beta_2}. \]  

(13)

As an example of the use of formulas (12) and (13), let us consider a system in which operating time \( E\alpha_1 = 20 \) h, the recovery time \( E\beta_1 = 4 \) h, duration of maintenance \( E\beta_2 = 1 \) h, RV \( \alpha_1, \beta_1, \beta_2 \) have 5th order Erlang distribution. For this case, the optimum maintenance moments were determined at the time reserve \( h \) in the range 0 ... 5 h and the corresponding values of the stationary availability factor \( K_a(\delta, h) \)
for the specified distribution were calculated. The values of the stationary availability factor \( K_a(h) \) for a system with an instantly replenished time reserve were found without maintenance (13) for the same initial data. The results are summarized in Table 1.

| Reserve \( h \), hour | Formula (12) | Formula (13) |
|-----------------------|--------------|--------------|
| \( \delta_{opt} \), hour | \( K_a(\delta_{opt}, h) \) | \( K_a(h) \) |
| 0                     | 12.131       | 0.906        | 0.833        |
| 1                     | 10.278       | 0.895        | 0.856        |
| 2                     | 9.831        | 0.894        | 0.866        |
| 3                     | 11.186       | 0.906        | 0.877        |
| 4                     | 12.606       | 0.916        | 0.896        |
| 5                     | 13.733       | 0.924        | 0.923        |

Based on the results of Table 1, it can be concluded that the stationary availability factor \( K_a \) for a system with an instantly replenished time reserve increases during maintenance compared to a case where maintenance is not performed.

4. Conclusion

A large number of systems for various purposes allow the construction of a semi-Markov model. When constructing a semi-Markov model, it is necessary to build a rather complex phase space of states that reflect the physical states of the system, and ensure the correctness of the construction of the semi-Markov model. Phase merging algorithms not only allow reducing the dimensionality of the phase space of states, but also approximately calculate the values of various characteristics of the functioning of the system.

In this paper, using the theory of semi-Markov processes with a common phase space of states, a semi-Markov model of a technical system with calendar maintenance and time reserve is constructed. Approximate the stationary reliability characteristics of the system are determined under consideration with an instantly replenished constant time reserve, using the phase merging algorithms. On the example considered, the optimal moments of maintenance for a different time reserve values were determined; the effect of influence of the time reserve and the maintenance on the stationary availability factor of the technical system was analyzed.

In the future, this approach is proposed to analyze the functioning of energy systems, control and maintenance of technical systems. The obtained results can be used for engineering calculations and solving optimization problems related to the time reserve and the periodicity of maintenance.

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