Langton’s Type Theorem on Algebraic Orbifolds

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Abstract  In this paper, we prove the Langton’s type theorem on separatedness and properness for the moduli functor of torsion free semistable sheaves on algebraic orbifolds over an algebraically closed field $k$.

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1 Introduction

Let $X$ be a smooth algebraic orbifold (Def. 2.2 and Remark 2.6) over an algebraically closed field $k$. We consider the moduli functor $M$ of modified slope (Def. 2.19) semistable torsion free sheaves on $X$. Following [9, 11, 18], we define the following functor:

$$\widehat{M} : (\text{Sch}/k)^o \to (\text{Sets})$$

as follows. Let $T$ be a $k$-scheme and let $\widehat{M}(T)$ be the set of isomorphism classes of $T$-flat families of torsion free semistable sheaves on $X$. If $f : T' \to T$ is a morphism of schemes, let $\widehat{M}(f)$ be the morphism obtained by pulling back sheaves via the morphism $f_X = \text{id}_X \times f$, i.e.,

$$\widehat{M}(T) \to \widehat{M}(T'), \quad [E] \mapsto [f_X^*E].$$

Then, the moduli functor $M$ is defined to be the quotient functor of $\widehat{M}$ by equivalence relation $\sim$:

$$E \sim E', \text{ for } E, E' \in M'(T) \text{ if and only if there is a line bundle } L \text{ on } T$$

such that $E' = p_2^*L \otimes E$,

where $p_2 : X \times T \to T$ is the projection onto $T$. In general, the moduli functor $M$ is not representable. In fact, if $X$ is a projective scheme and there is a properly semistable sheaf on $X$, then the moduli functor $M$ can not be represented (Lemma 4.1.2 in [9]). In the case that $M$ is representable, Nironi has shown that the corresponding moduli scheme is proper over $k$ (Theorem 6.22 in [15]). But, by the Grothendieck’s valuative criteria, we can also consider the separatedness and properness of $M$ directly. Indeed, Langton [11] has showed that the moduli functor of slope semistable torsion free sheaves on smooth projective varieties over $k$
is separated and proper. Maruyama [13], Mehta and Ramanathan [14] generalised Langton’s results to Gieseker stability. In recent years, many problems about the moduli functor of semistable sheaves on algebraic orbifolds are concerned. There is not a similar result on algebraic orbifolds. For the researchers’ convenience, we generalize the result of Langton and prove that the moduli functor of slope semistable torsion free sheaves on algebraic orbifolds is separated and proper. For the case of Gieseker stability, the similar result can be obtained following the line of Maruyama [13], Mehta and Ramanathan [14], for the sake of the key Lemma 3.3 on algebraic orbifolds (which corresponds to the Proposition 6 in [11]). In the next paragraph, we give the precise description of the problem. Let $R$ be a discrete valuation ring over $k$ with maximal ideal $(\pi)$ and residue field $k$. The quotient field of $R$ is $K$. Consider the following cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{X}_K & \xrightarrow{i} & \mathcal{X}_R \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \xrightarrow{j} & \text{Spec}(k)
\end{array}
$$

where $\mathcal{X}_R = \mathcal{X} \times \text{Spec}(R)$, $\mathcal{X}_K = \mathcal{X} \times \text{Spec}(K)$ and $\mathcal{X}_k = \mathcal{X} \times \text{Spec}(k) = \mathcal{X}$.

Consequently, we have:

1. $M$ is separated if and only if two families $E_R, E'_R$ of torsion free semistable sheaves over $\text{Spec}(R)$ agreeing on the generic fiber $\mathcal{X}_K$, then they agree on $\mathcal{X}_R$;
2. $M$ is proper if and only if every torsion free semistable sheaves $E_K$ on $\mathcal{X}_K$ can be uniquely extend to a flat family of torsion free sheaves on $\mathcal{X}_R$, under isomorphism.

We state our main results:

**Theorem 1.1** Assume that $E_K$ is a torsion free sheaf on $\mathcal{X}_K$. Then

1. If $E_1$ and $E_2$ are two coherent subsheaves of $i_*E_K$ on $\mathcal{X}_R$ such that $i^*E_1 = i^*E_2 = E_K$ and $j^*E_1, j^*E_2$ are semistable torsion free sheaves on $\mathcal{X}_k$, at least one of which is stable, then there is an integer $p$ such that $E_1 = \pi^pE_2$.
2. If $E_K$ is semistable, then there exists a coherent subsheaf $E \subseteq i_*E_K$ such that $i^*E = E_K$ and $j^*E$ is torsion free and semistable on $\mathcal{X}_k$.

As an application of the Theorem 1.1, in a forthcoming paper [8], we use it to show that the Hitchin map on the moduli space of Higgs bundles on Deligne–Mumford curves is proper.

## 2 Torsion Free Sheaves on Algebraic Orbifolds

Throughout this paper, we work over a fixed algebraically closed field $k$. All schemes, algebraic spaces and stacks and morphisms among them are of finite type. In the following, we recall some basic knowledge about torsion free sheaves on algebraic orbifolds. For more details, we refer the reader to [1, 6, 10, 15] and [20].

**Definition 2.1** (Tame Deligne–Mumford stacks) Let $\mathcal{X}$ be a Deligne–Mumford stack with coarse moduli space $p : \mathcal{X} \to X$. Then, $\mathcal{X}$ is tame if the pushforward functor $p_* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X)$ is exact, where $\text{QCoh}(-)$ is the category of quasicoherent sheaves.

**Definition 2.2** (Algebraic orbifolds) Let $\mathcal{X}$ be a Deligne–Mumford tame stack over $k$, which is isomorphic to a separated global quotient $[Z/G]$, where $Z$ is an algebraic space over $k$ and $G$
is a subgroup scheme (a locally closed subscheme which is a subgroup) of some \( \text{GL}_{N,k} \). If the generic stabilizer of \( X \) is trivial, then \( X \) is called an algebraic orbifold over \( k \).

**Definition 2.3** A Deligne–Mumford stack \( \mathcal{X} \) is called irreducible if it is not the union of two proper closed subsets, where the closed sets in \( \mathcal{X} \) mean reduced closed substacks of \( \mathcal{X} \). It is called integral if it is both irreducible and reduced.

**Remark 2.4** A Deligne–Mumford stack \( \mathcal{X} \) is irreducible if and only if its coarse moduli space is irreducible. In fact, there is a bijection between the closed subsets of \( \mathcal{X} \) and the closed subsets of \( X \), as pointed out by Conrad in [5].

Nironi [15] introduces the notion of projective (quasi-projective) Deligne–Mumford stack:

**Definition 2.5** (Projective (quasi-projective) Deligne–Mumford stack) Let \( \mathcal{X} \) be a Deligne–Mumford stack over a field \( k \). We say \( \mathcal{X} \) is projective (quasi-projective) over \( k \) if it is a tame separated global quotient with projective (quasi-projective) coarse moduli scheme.

**Remark 2.6** In this paper, we only consider \( \mathcal{X} \) to be a projective algebraic orbifold, which is irreducible and its coarse moduli space is a projective scheme over \( k \).

**Example 2.7** (Weighted projective line) The weighted projective lines \( \mathbb{P}(n, m) \) are algebraic orbifolds, when \( m \) and \( n \) are coprime.

For more examples, the reader can consult [10]. As point out by Nironi, for a stack, there are no very ample invertible sheaves unless it is an algebraic space. However, under certain hypothesis, there exist locally free sheaves, called generating sheaves, which behave like very ample sheaves.

**Definition 2.8** (Generating sheaf) Let \( \mathcal{X} \) be a tame Deligne–Mumford stack and let \( \pi : \mathcal{X} \rightarrow X \) be the coarse moduli space of \( \mathcal{X} \). A locally free sheaf \( E \) of \( \mathcal{X} \) is said to be a generating sheaf if for any quasi-coherent sheaf \( F \), the following map
\[
\pi^*(\pi^*(E^\vee \otimes F)) \otimes E \rightarrow F
\]
is surjective.

Olsson and Starr proved the existence of the generating sheaves. Also, the generating sheaf is stable for arbitrary base change on the coarse moduli space.

**Proposition 2.9** ([17])

1. Let \( \mathcal{X} \) be a separated Deligne–Mumford tame stack which is a global quotient over \( k \), then there is a locally free sheaf \( E \) over \( \mathcal{X} \) which is a generating sheaf for \( \mathcal{X} \).

2. Let \( \pi : \mathcal{X} \rightarrow X \) be the moduli space of \( \mathcal{X} \) and \( f : X' \rightarrow X \) a morphism of algebraic spaces over \( k \). Moreover, we have the following cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{X} \\
p \downarrow & & \pi \\
X' & \longrightarrow & X
\end{array}
\]

and \( p^*E \) is a generating sheaf for \( \mathcal{X}' \).

For a Deligne–Mumford stack \( \mathcal{X} \) with projective coarse moduli scheme over a field of characteristic zero, the existence of the generating sheaf is equivalent to \( \mathcal{X} \) is a global quotient stack.
Proposition 2.10 ([10]) For a Deligne–Mumford stack $X$ over $k$ and $\text{char} k = 0$, the following are equivalent.

1. $X$ has a projective coarse moduli space and is a quotient stack.
2. $X$ has a projective coarse moduli space and possesses a generating sheaf.
3. $X$ can be embedded into a smooth Deligne–Mumford stack with projective coarse moduli space.

As the case of schemes, the support of coherent sheaves on Deligne–Mumford stacks can be defined in the following way.

Definition 2.11 (Support of coherent sheaf) Let $X$ be a Deligne–Mumford stack over $k$ and let $F$ be a coherent sheaf on $X$. The support $\text{supp}(F)$ of $F$ is the closed substack defined by the sheaf of ideals

$$0 \to \mathcal{I}_F \to \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(F, F) \to 0.$$

Definition 2.12 (Torsion free sheaf) Let $X$ be an integral projective Deligne–Mumford stack over $k$. A coherent sheaf $F$ is said to be a torsion free sheaf if for every nonzero subsheaf $G \subseteq F$, the dimension of the support $\text{supp}(G)$ is $\dim X$.

The torsion freeness of a coherent sheaf on a Deligne–Mumford stack is equivalent to its restriction to an étale covering. (Remark 3.3 in [15]).

Lemma 2.13 ([15]) With the same hypothesis as above, $F$ is a torsion free sheaf if and only if there is an étale covering $f : U \to X$ such that the restriction of $F$ to $U$ is torsion free.

Proposition 2.14 Assume that $X$ is an integral projective Deligne–Mumford stack over $k$ and $F$ is a coherent sheaf on $X$. Then, there exists an open substack $X^o$, such that the restriction $F|_{X^o}$ to $X^o$ of $F$ is locally free.

Proof Take an étale covering $f : U \to X$ such that $U$ is finite type over $k$. We have the following cartesian diagram:

$$
\begin{array}{ccc}
U \times_X U & \xrightarrow{pr_2} & U \\
pr_1 \downarrow & & \downarrow f \\
U & \xrightarrow{f} & X
\end{array}
$$

Denote $U \times_X U$ by $R$. Then $R \xrightarrow{t} U$ is an algebraic groupoid, where $s = pr_1$ and $t = pr_2$. Denote $f^* F$ by $F'$. By the 2-commutativity of above diagram, there is an isomorphism

$$
\phi : s^* F' \to t^* F'.
$$

Because $U$ is reduced, there exists unique maximal nonempty open subset $U' \subset U$ such that $F'|_{U'}$ is locally free. By the flatness of morphism $s$, $s^{-1}(U')$ is the unique maximal open subset on which $s^* F$ is locally free. Similarly, $t^{-1}(U')$ is the unique maximal open subset such that the restriction of $t^* F$ is locally free. Thus, $s^{-1}(U') = t^{-1}(U')$, i.e., $U' \subset U$ descents to an open substack $X^o$ of $X$ such that $F|_{X^o}$ is locally free.

Definition 2.15 (Rank of coherent sheaf) Under the hypothesis of Proposition 2.14, we can define the rank $\text{rk}(F)$ of $F$ to be the rank of $F|_{X^o}$. 

In order to define a notion of Gieseker stability on projective Deligne–Mumford stacks, Nironi introduced the modified Hilbert polynomial in [15]. First of all, we recall the notion of polarization on projective Deligne–Mumford stacks.

**Definition 2.16 (Polarization)** For a Projective Deligne–Mumford stack $\mathcal{X}$, the polarization of $\mathcal{X}$ is a pair $(\mathcal{E}, \mathcal{O}_X(1))$, where $\mathcal{E}$ is a generating sheaf and $\mathcal{O}_X(1)$ is a very ample invertible sheaf on $X$.

**Definition 2.17 (Modified Hilbert polynomial)** Fix a polarization $(\mathcal{E}, \mathcal{O}_X(1))$ on a projective Deligne–Mumford stack $X$. For a coherent sheaf $F$ on $X$, the modified Hilbert polynomial $P_F$ of $F$ is defined by

$$P_F(m) = \chi(\pi_*(F \otimes \mathcal{E}^\vee) \otimes \mathcal{O}_X(m)),$$

where $\chi(\pi_*(F \otimes \mathcal{E}^\vee) \otimes \mathcal{O}_X(m))$ is the Euler characteristic of $\pi_*(F \otimes \mathcal{E}^\vee) \otimes \mathcal{O}_X(m)$.

**Remark 2.18** In general, the modified Hilbert polynomial $P_F(m) = \sum_{i=0}^{d} \frac{a_i(F)}{i!} \cdot m^i$, where $d$ is the dimension of $F$ and $a_i(F)$ are rationals. In the special case: $F$ is a torsion free sheaf (2.12) on a projective algebraic orbifold $X$ of dimension $n$, then the coefficient $a_n(F)$ of the leading term is $\text{rk}(F)\text{rk}(\mathcal{E})\deg(\mathcal{O}_X(1))$, by the sake of Grothendieck–Riemann–Roch formula in [19].

**Definition 2.19 (Modified slope)** Let $\mathcal{X}$ be an integral projective Deligne–Mumford stack over $k$. The modified Hilbert polynomial of $F$ is $P_F(m) = \sum_{i=0}^{d} \frac{a_i(F)}{i!} \cdot m^i$. The modified slope $\mu(F)$ of $F$ is

$$\mu(F) = \frac{a_{d-1}(F)}{a_d(F)}.$$  

Using the modified slope, we can introduce the notions of semistable (stable) torsion free sheaves.

**Definition 2.20 (Stability)** A torsion free sheaf $E$ is said to be semistable (resp., stable) if for all coherent subsheaves $F \subseteq E$ and $\text{rk}(F) < \text{rk}(E)$, we have

$$\mu(F) \leq \mu(E) \quad \text{(resp., } \mu(F) < \mu(E)).$$

If $E$ is not semistable, $E$ is called unstable.

**Definition 2.21 (Subbundle of torsion free sheaf)** Let $E$ be a coherent subsheaf of a torsion free sheaf $F$. If the quotient sheaf $F/E$ is also a torsion free sheaf, we say $E$ is a subbundle of $F$.

Indeed, for every coherent subsheaf of a torsion free sheaf, there is a unique minimal subbundle containing it. We have the following proposition.

**Proposition 2.22** Let $\mathcal{X}$ be an integral projective Deligne–Mumford stack over $k$ and let $F$ be a torsion free sheaf on $\mathcal{X}$. For a coherent subsheaf $G$ of $F$, there is a unique coherent subsheaf $G' \subseteq F$, such that

1. $G \subseteq G'$ and $\text{rk}(G') = \text{rk}(G)$;
2. if $F/G'$ is not zero sheaf, then $F/G'$ is a torsion free sheaf.
Proof. We have the following two exact sequences:

\[
\begin{align*}
0 & \longrightarrow G \longrightarrow F \\
0 & \longrightarrow T(F/G) \longrightarrow F/G \longrightarrow Q \longrightarrow 0,
\end{align*}
\]

where \(T(F/G)\) is the maximal torsion subsheaf of \(F/G\). Then, \(G' = j^{-1}(T(F/G))\) and \(F/G' = Q\). We have to check the uniqueness of \(G'\). Suppose there are two such sheaves \(G_1\) and \(G_2\). Then \(\text{rk}(G_1 \cap G_2) = \text{rk}(G)\). Also, there are two exact sequences

\[
\begin{align*}
0 & \longrightarrow G_1 \cap G_2 \longrightarrow G_1 \longrightarrow (G_1 + G_2)/G_1 \longrightarrow 0, \\
0 & \longrightarrow G_1 \cap G_2 \longrightarrow G_2 \longrightarrow (G_1 + G_2)/G_2 \longrightarrow 0.
\end{align*}
\]

If \((G_1 + G_2)/G_1\) and \((G_1 + G_2)/G_2\) are torsion free, then

\[
\text{rk}(G_1 \cap G_2) < \text{rk}(G_1), \quad \text{rk}(G_1 \cap G_2) < \text{rk}(G_2).
\]

This is impossible. Hence, we have \(G_1 + G_2 = G_1\) and \(G_1 + G_2 = G_2\). So, \(G_1 = G_2\).

Following [11], if \(X\) is a smooth algebraic orbifold, there is an explicit construction of the sheaf \(G'\) in Proposition 2.22.

**Proposition 2.23** For a smooth algebraic orbifold \(X\), there is an open dense substack \(X^o\) of \(X\) such that \(X^o\) is an irreducible smooth variety over \(k\). Let \(\xi\) be the generic point of \(X^o\) and let \(\gamma : X^o \rightarrow X\) be the open immersion. Assume that \(G_\xi\) and \(F_\xi\) are the stalks of \(\gamma^*G\) and \(\gamma^*F\) at \(\xi\), respectively. \(G_\xi\) and \(F_\xi\) can be regarded as quasicoherent sheaves on \(X^o\). Then, we have:

\[
G' = \gamma_*G_\xi \cap F.
\]

**Proof.** Indeed, \(\gamma_*G_\xi \subseteq \gamma_*F_\xi\) and \(F \subseteq \gamma_*\gamma^*F \subseteq \gamma_*F_\xi\). By \(\gamma^*G \subseteq G_\xi\), we have \(G \subseteq \gamma_*\gamma^*G \subseteq \gamma_*G_\xi\). Thus, \(G \subseteq \gamma_*G_\xi \cap F\). Obviously, \(\text{rk}(\gamma_*G_\xi \cap F) = \text{rk}(G)\). Assume that \(F/\gamma_*G_\xi \cap F\) is not zero. Let \(\alpha : U \rightarrow X\) be an étale morphism. Without loss of generality, we can assume that \(U\) is an irreducible smooth affine variety \(\text{Spec}(A)\). We have the following cartesian diagram:

\[
\begin{array}{ccc}
U^o & \xrightarrow{\gamma^*} & U \\
\downarrow{\alpha^o} & & \downarrow{\alpha} \\
X^o & \xrightarrow{\gamma} & X
\end{array}
\]

By the flat base change theorem (Corollary A.2.2 in [3]), we only need to consider the case: \(X = \text{Spec}(A)\). Following the Proposition 1 in [11], if \(F(\text{Spec}(A)) = M\), \(G(\text{Spec}(A)) = N\) and \(K\) is the quotient field of \(A\), then \(G'\) is the coherent sheaf associated to the \(A\)-module \(M \cap N \otimes_A K\).

**Remark 2.24** Under the above hypotheses, the subbundle \(G'\) is uniquely determined by the vector subspace \(G_\xi\) of \(F_\xi\) over the field of rational functions on \(X^o\).

**Definition 2.25** (Join of sheaves) Suppose \(F_1\) and \(F_2\) are two coherent subsheaves of a torsion free sheaf \(F\) on an \(n\)-dimensional integral projective Deligne–Mumford stack \(X\) over \(k\). The unique subbundle \(F_1 \vee F_2\) of \(F\) in Proposition 2.22 containing \(F_1 + F_2\), is called the join of \(F_1\) and \(F_2\).
**Proposition 2.26** If $F_1$ and $F_2$ are two subbundles of a torsion free sheaf $E$ on an $n$-dimensional integral Deligne–Mumford stack $X$ over $k$, then the coefficients of the modified Hilbert polynomials satisfy:

$$a_{n-1}(F_1 \vee F_2) + a_{n-1}(F_1 \cap F_2) \geq a_{n-1}(F_1) + a_{n-1}(F_2).$$

**Proof** By the two exact sequences

$$0 \to F_1 \cap F_2 \to F_1 \xrightarrow{(F_1 + F_2)/F_2} 0$$

and

$$0 \to F_1 \cap F_2 \to F_2 \xrightarrow{(F_1 + F_2)/F_1} 0,$$

we have

$$P_{F_1 \cap F_2} + P_{(F_1 + F_2)/F_1} = P_{F_2}, \quad P_{F_1 \cap F_2} + P_{(F_1 + F_2)/F_2} = P_{F_1}.$$

So,

$$a_{n-1}(F_1 \cap F_2) + a_{n-1}((F_1 + F_2)/F_2) = a_{n-1}(F_1)$$

and

$$a_{n-1}(F_1 \cap F_2) + a_{n-1}((F_1 + F_2)/F_1) = a_{n-1}(F_2).$$

Also, there is an exact sequence

$$0 \to (F_1 + F_2)/F_1 \to (F_1 \vee F_2)/F_1 \to (F_1 \vee F_2)/(F_1 + F_2) \to 0.$$

Hence, we have

$$P_{(F_1 + F_2)/F_1} + P_{(F_1 \vee F_2)/(F_1 + F_2)} = P_{(F_1 \vee F_2)/F_1}.$$

Therefore,

$$a_{n-1}((F_1 + F_2)/F_1) + a_{n-1}((F_1 \vee F_2)/(F_1 + F_2)) = a_{n-1}((F_1 \vee F_2)/F_1).$$

And also, $a_{n-1}((F_1 \vee F_2)/(F_1 + F_2)) \geq 0$, because $(F_1 \vee F_2)/(F_1 + F_2)$ is a torsion sheaf. So,

$$a_{n-1}(F_1 \cap F_2) + a_{n-1}((F_1 \vee F_2)/F_1) \geq a_{n-1}(F_2).$$

By the exact sequence

$$0 \to F_1 \to F_1 \vee F_2 \to (F_1 \vee F_2)/F_1 \to 0,$$

we have

$$a_{n-1}(F_1 \vee F_2) - a_{n-1}(F_1) = a_{n-1}((F_1 \vee F_2)/F_1).$$

Then,

$$a_{n-1}(F_1 \vee F_2) + a_{n-1}(F_1 \cap F_2) \geq a_{n-1}(F_1) + a_{n-1}(F_2). \square$$

As [11], we introduce the $\beta$-invariant.

**Definition 2.27** Let $E$ be a fixed torsion free sheaf on an $n$-dimensional integral projective Deligne–Mumford stack $X$ over $k$. For every torsion free sheaf $F$ on $X$, we can define the $\beta$-invariant as

$$\beta(F) = a_n(E)a_{n-1}(F) - a_{n-1}(E)a_n(F).$$
Remark 2.28  By Proposition 2.22, if every proper subbundle $F \subset E$ satisfies $\beta(F) \leq 0$, then $E$ is semistable.

Proposition 2.29  Let $X$ be an integral projective Deligne-Mumford stack over $k$.

1. If $F_1$ and $F_2$ are two subbundles of $E$ on $X$, then
   \[ \beta(F_1) + \beta(F_2) \leq \beta(F_1 \vee F_2) + \beta(F_1 \cap F_2), \]
   with equality if and only if the codimension of the sheaf $(F_1 \vee F_2)/(F_1 + F_2) \geq 2$.

2. If $0 \rightarrow F \rightarrow G \rightarrow K \rightarrow 0$ is an exact sequence of torsion free sheaves on $X$, then
   \[ \beta(F) + \beta(K) = \beta(G). \]

Proof  For the first statement, we have
   \[
   \begin{align*}
   \beta(F_1) + \beta(F_2) &= a_n(E)a_{n-1}(F_1) - a_{n-1}(E)a_n(F_1) + a_n(E)a_{n-1}(F_2) - a_{n-1}(E)a_n(F_2) \\
   &= a_n(E)(a_{n-1}(F_1) + a_{n-1}(F_2)) - a_{n-1}(E)(a_n(F_1) + a_n(F_2)) \\
   &\leq a_n(E)(a_{n-1}(F_1 \vee F_2) + a_{n-1}(F_1 \cap F_2)) - a_{n-1}(E)(a_n(F_1) + a_n(F_2)).
   \end{align*}
   \]
   By the exact sequence $0 \rightarrow F_1 \cap F_2 \rightarrow F_1 \oplus F_2 \rightarrow F_1 + F_2 \rightarrow 0$, we have
   \[ P_{F_1 \oplus F_2} = P_{F_1 \cap F_2} + P_{F_1 + F_2}. \]
   So,
   \[ a_n(F_1 + F_2) + a_n(F_1 \cap F_2) = a_n(F_1) + a_n(F_2). \]
   Also, $a_n(F_1 + F_2) = a_n(F_1 \vee F_2)$. Then,
   \[ \beta(F_1) + \beta(F_2) \leq \beta(F_1 \vee F_2) + \beta(F_1 \cap F_2). \]
   The second statement is obvious. \(\square\)

Following [11], we consider the set $\Gamma(E)$ of proper subbundles of $E$, which have the following property:
\[ \Gamma(E) = \{ F : F \text{ is a proper subbundle of } E \text{ such that for every subsheaf } G \subset F, \beta(G) < \beta(F) \}. \]

Remark 2.30  The set $\Gamma(E)$ is nonempty. In fact, the zero sheaf is in $\Gamma(E)$. In addition, if $E$ is semistable, there is only one element in the set $\Gamma(E)$, i.e., the zero sheaf.

Proposition 2.31  Let $F$ be a maximal element of $\Gamma(E)$. For every subbundle $G \supseteq F$, we have $\beta(G) \leq \beta(F)$.

Proof  Suppose $\beta(G) > \beta(F)$. Let $H \subset G$ be the minimal subbundle such that $\beta(H) > \beta(F)$ and $F \subseteq H$. For every proper subbundle $I$ of $H$ and $F \nsubseteq I$, we have
   \[ \beta(I \vee F) - \beta(I) \geq \beta(F) - \beta(F \cap I) > 0. \]
   By the minimality of $H$, $\beta(H) \geq \beta(I \vee F)$. So, $\beta(H) > \beta(I)$. Therefore, $H \in \Gamma(E)$. Contradiction! \(\square\)

Corollary 2.32  There is a unique maximal subbundle $F \in \Gamma(E)$. Also, for every subbundle $B \subseteq E$, $\beta(B) \leq \beta(F)$ with equality only if $B \supseteq F$. 

Proof If there are two maximal subbundles $F_1$ and $F_2$ in $\Gamma(E)$, then
\[ \beta(F_1 \cup F_2) - \beta(F_1) \geq \beta(F_2) - \beta(F_1 \cap F_2). \]

By Proposition 2.31, $\beta(F_1 \cup F_2) \leq \beta(F_1)$. Thus, $\beta(F_2) \leq \beta(F_1 \cap F_2)$. On the other hand, $\beta(F_1 \cap F_2) \leq \beta(F_2)$. Then, $\beta(F_1 \cap F_2) = \beta(F_2)$. So, $F_1 \cap F_2 = F_2$. Similarly, $F_1 \cap F_2 = F_1$. Then, $F_2 = F_1$. Hence, there is a unique maximal subbundle $F \in \Gamma(E)$. By $\beta(F \cup B) - \beta(B) \geq \beta(F) - \beta(F \cap B) \geq 0$ and $\beta(F) \geq \beta(F \cup B)$, we get $\beta(F) = \beta(B)$ with equality only if $B \supseteq F$. \qed

Remark 2.33 In the above corollary, the $\beta(F)$ is the maximum value of $\beta$-invariant for subbundles in $E$. Also, $\text{Hom}_{O_X}(F, E/F) = 0$.

At the end of this section, we show that the torsion free semistable sheaves is stable under the extension of the base field $k$ ($k$ is not necessarily algebraic closed).

**Proposition 2.34** Let $k'$ be an extension field of $k$. We have the following cartesian diagram:
\[
\begin{array}{ccc}
\mathcal{X} \times \text{Spec}(k') & \xrightarrow{p_1} & \mathcal{X} \\
p_2 \downarrow & & \downarrow \\
\text{Spec}(k') & \xrightarrow{i} & \text{Spec}(k)
\end{array}
\]

Assume that the field $k$ is infinite when $k'/k$ is not algebraic. Then $E' = p_1^* E$ is semistable if and only if $E$ is semistable.

**Proof** The proof can be proved as Proposition 3 of [11], or can be found in [15]. \qed

3 The Main Results

From now on, $\mathcal{X}$ is an $n$-dimensional smooth algebraic orbifold with a fixed polarization $(\mathcal{E}, O_X(1))$ over $k$. Let $R \supseteq k$ be a discrete valuation ring with maximal ideal $m = (\pi)$ and residue field $k$. $K$ is the quotient field of $R$. Consider the following cartesian diagram:
\[
\begin{array}{ccc}
\mathcal{X}_K & \xrightarrow{i} & \mathcal{X}_R \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \xrightarrow{j} & \text{Spec}(R)
\end{array}
\]

where $\mathcal{X}_R = \mathcal{X} \times \text{Spec}(R)$, $\mathcal{X}_K = \mathcal{X} \times \text{Spec}(K)$ and $\mathcal{X}_k = \mathcal{X} \times \text{Spec}(k) = \mathcal{X}$. $i : \mathcal{X}_K \to \mathcal{X}_R$ is the natural open immersion and $j : \mathcal{X}_k \to \mathcal{X}_R$ is the natural closed immersion. Our goal is to prove the following result:

**Theorem 3.1** Assume that $E_K$ is a torsion free sheaf on $\mathcal{X}_K$. Then

1. If $E_1$ and $E_2$ are two coherent subsheaves of $i_* E_K$ on $\mathcal{X}_R$ such that $i^* E_1 = i^* E_2 = E_K$ and $j^* E_1, j^* E_2$ are semistable torsion free sheaves on $\mathcal{X}_k$, at least one of which is stable, then there is an integer $p$ such that $E_1 = \pi^p E_2$.

2. If $E_K$ is semistable, then there exists a coherent subsheaf $E \subseteq i_* E_K$ such that $i^* E = E_K$ and $j^* E$ is torsion free and semistable on $\mathcal{X}_k$.

We first state a lemma, which corresponds to Proposition 5 in [11].

**Lemma 3.2** If $E_1$ and $E_2$ are two torsion free sheaves on $\mathcal{X}_R$ such that $i^* E_1 = i^* E_2$, then the modified Hilbert polynomials $P_{j^* E_1}(m) = P_{j^* E_2}(m)$. In particular, $a_{n-1}(j^* E_1) = a_{n-1}(j^* E_2)$. 

Proof  Since the field $k$ is algebraically closed and $R$ is a regular local ring, $\mathcal{X}_R$ is integral and smooth over $\text{Spec}(R)$. Then, the torsion free sheaf on $\mathcal{X}_R$ is flat over $\text{Spec}(R)$, since the torsion free modules over valuation rings are flat. By Lemma 3.16 in [15], $P_j^*\mathcal{E}_1(m) = P_j^*\mathcal{E}_2(m)$.

$\mathcal{X}$ has an open dense substack $\mathcal{X}^\circ$ such that it is an irreducible smooth variety over $k$. Let $\gamma : \mathcal{X}^\circ \to \mathcal{X}$ be the corresponding open immersion. $\mathcal{X}^\circ_R = \mathcal{X}^\circ \times \text{Spec}(K)$ and $\mathcal{X}^\circ_R = \mathcal{X} \times \text{Spec}(k)$ are also irreducible and smooth. Let $\Xi$ be the generic point of $\mathcal{X}^\circ_R$ and $\xi$ be the generic point of $\mathcal{X}^\circ_R$. And, we have the following cartesian diagram:

$\mathcal{X}^\circ_R \quad \mathcal{X}^\circ_R \quad \mathcal{X}^\circ_R$

\[\mathcal{X}_R \quad \mathcal{X}_R \quad \mathcal{X}_R\]

Let $E_K$ be a torsion free sheaf of rank $r$ on $\mathcal{X}_R$. Since $\mathcal{X}^\circ$ is an integral scheme, the stalk $(E_K)_\Xi$ of $(E_K)|_{\mathcal{X}_R}$ at $\Xi$ is a free $\mathcal{O}_\Xi$ module. Denote the stalks of $\mathcal{O}_{\mathcal{X}_R}$ at $\Xi$ and $\xi$ by $\mathcal{O}_\Xi$ and $\mathcal{O}_\xi$, respectively.

Lemma 3.3  Suppose $M \subset (E_K)_\Xi$ is a free rank $r$ $\mathcal{O}_\xi$-submodule of $(E_K)_\Xi$. Then there exists a unique torsion free sheaf $E \subseteq i_*E_K$ on $\mathcal{X}_R$ such that $i^*E = E_K$, $E_\xi = M$, and $j^*E$ is a torsion free sheaf on $\mathcal{X}_K$.

Proof  As above, $\mathcal{O}_\xi$ is the stalk of $\mathcal{O}_{\mathcal{X}_R}$ at the generic $\xi$ of $\mathcal{X}^\circ_R$ in $\mathcal{X}^\circ_R$. Then, there is a natural morphism $\beta_1 : \text{Spec}(\mathcal{O}_\xi) \to \mathcal{X}^\circ_R$. Besides, $\Xi$ is the generic point of $\mathcal{X}^\circ_R$. So, $\Xi$ is also the generic point of $\mathcal{X}^\circ_R$. So, there are two natural morphisms $\alpha : \Xi \to \text{Spec}(\mathcal{O}_\xi)$ and $\beta_2 : \Xi \to \mathcal{X}^\circ_R$. Let $i^\circ : \mathcal{X}^\circ_R \to \mathcal{X}^\circ_R$ be the open immersion obtained through base change from the open immersion $\text{Spec}(K) \hookrightarrow \text{Spec}(R)$. And also, they form the following cartesian diagrams:

$\text{Spec}(\mathcal{O}_\xi) \quad \mathcal{X}^\circ_R \quad \mathcal{X}^\circ_R$

\[\Xi \quad \mathcal{X}_R \quad \mathcal{X}_R\]

Denote the torsion free sheaf on $\text{Spec}(\mathcal{O}_\xi)$ corresponding to the modules $\mathcal{M}$ by $\mathcal{M}$. Similarly, $\mathcal{N}$ is the free sheaf on $\Xi$ corresponding to $(E_K)_\Xi$.

Claim: $E = i_*E_K \cap (\gamma_R \circ \beta_1)_*\mathcal{M}$ satisfies the conditions in the conclusion of Lemma 3.3.

First step  We need to explain the intersection of $i_*E_K$ and $(\gamma_R \circ \beta_1)_*\mathcal{M}$ in $(\gamma_R \circ \beta_1 \circ \alpha)_*\mathcal{N}$. By the inclusion $M \subseteq (E_K)_\Xi$, we have the inclusion:

$(\gamma_R \circ \beta_1)_*\mathcal{M} \subseteq (\gamma_R \circ \beta_1)_*(\alpha_*\mathcal{N})$.  (a)

In addition, $\Xi$ is the generic point of $\mathcal{X}^\circ_R$, there is another inclusion:

$(i \circ \gamma_R)_*\mathcal{K}^*E_K) \subseteq (i \circ \gamma_R)_*(\beta_2_*\mathcal{N})$.  (b)

By the diagram (A), we get $(i \circ \gamma_R \circ \beta_2)_*\mathcal{N} = (\gamma_R \circ \beta_1 \circ \alpha)_*\mathcal{N}$. In the following, we show that the morphism:

$E_K \longrightarrow \gamma_K^*(\gamma_K^*E_K)$
obtained by adjunction formula is injective. Indeed, the coarse moduli space of $\mathcal{X}_K$ is $X_K = X \times \text{Spec}(K)$ and $X_K$ is irreducible. By Remark 2.4, $\mathcal{X}_K$ is irreducible. Also, $\mathcal{X}_K$ is reduced. Then, $\mathcal{X}_K$ is integral. For every étale morphism $f : U \to \mathcal{X}$ from an irreducible smooth variety $U$ over $k$ to $\mathcal{X}$, we have the cartesian diagram:

\[
\begin{array}{ccc}
U_K^o & \xrightarrow{\gamma_K} & U_K \\
\downarrow f_K^o & & \downarrow f_K \\
\mathcal{X}_K^o & \xrightarrow{\gamma_K'} & \mathcal{X}_K \\
\end{array}
\]

where $U_K = U \times \text{Spec}(K)$. Pulling back the homomorphism (1) to $U_K$, we get

\[
f_K^*E_K \longrightarrow f_K^*\gamma_K^*E_K.
\]  

By the flat base change theorem of stacky version (Corollaries A.2.2 in [3] and A.3.4 in [4]), we have

\[
f_K^*\gamma_K^*E_K = \gamma_K'^*f_K^*\gamma_K^*E_K.
\]  

On the other hand, $\gamma_K'^*f_K^*E_K = f_K^*\gamma_K^*E_K$. Then, the homomorphism (2) is

\[
f_K^*E_K \longrightarrow \gamma_K'^*f_K^*E_K.
\]  

Because $U$ is integral and $f_K^*E_K$ is torsion free, the homomorphism (4) is injective. Thus, the homomorphism (1) is injective. So, $i_*E_K \longrightarrow i_*\gamma_K^*E_K$ is injective. Hence, by (a), (b) and the diagram (A), we have the following two short exact sequences with the same middle terms:

\[
0 \longrightarrow \gamma_K^*E_K \longrightarrow \gamma_K'^*f_K^*E_K.
\]  

Thus, $E = i_*E_K \cap (\gamma_1 \circ \beta_1)_*\mathcal{M}$ is a quasicoherent sheaf on $\mathcal{X}_R$. We accomplished the first part of the proof.

**Second step** We have to check the sheaf $E$ which we have defined is a torsion free coherent sheaf. We only need to check this locally in the étale topology. Suppose $\theta : \text{Spec}(A) \to \mathcal{X}$ is an étale morphism and $\text{Spec}(A)$ is a smooth irreducible variety over $k$. We have the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}^o & \xrightarrow{\phi} & \mathcal{X} \\
\downarrow \theta & & \downarrow \theta \\
V & \xrightarrow{\phi} & \text{Spec}(A)
\end{array}
\]

Since $\phi$ is an étale morphism of finite type between irreducible smooth varieties, $\phi$ is generically finite dominant map, i.e., $\phi^{-1}(\xi)$ is a finite set. By Exercise 3.7 in page 91 of [7], there is an
open dense subset \(i_W : \mathcal{X}^o \to \mathcal{X}\) such that the morphism \(\phi' : \phi^{-1}(W) \to W\) is finite and

\[
\begin{array}{c}
\Xi \xrightarrow{\alpha} \text{Spec}(\mathcal{O}_\xi) \xrightarrow{\beta_3} W \times \text{Spec}(R) \xrightarrow{i_{W,R}} \mathcal{X}_R^o \xrightarrow{\gamma_1} \mathcal{X}_R \\
\Xi_1 \xrightarrow{\alpha'} \text{Spec}(\mathcal{O}_{V_R,\xi'}) \xrightarrow{\beta'_3} W' \times \text{Spec}(R) \xrightarrow{i'_{W',R}} V \times \text{Spec}(R) \xrightarrow{\gamma'_1} \text{Spec}(A \otimes_k R)
\end{array}
\]

is a cartesian diagram. Denote \(\phi^{-1}(W)\) by \(W'\). By the base change, we have

\[
\begin{array}{c}
\Xi \xrightarrow{\phi_{\Xi}} W \times \text{Spec}(R) \xrightarrow{i_{W,R}} \mathcal{X}_R^o \xrightarrow{\phi_R} \mathcal{X}_R \\
\Xi_1 \xrightarrow{\phi'_{\Xi}} W' \times \text{Spec}(R) \xrightarrow{i'_{W',R}} V \times \text{Spec}(R) \xrightarrow{\phi'_R} \text{Spec}(A \otimes_k R)
\end{array}
\]

where \(\Xi'\) is the generic point of \(V_R = V \times \text{Spec}(R)\) and \(\xi'\) is the generic point of the close subscheme \(W' \times \text{Spec}(k) \hookrightarrow W' \times \text{Spec}(R)\). The first square and the second square are cartesian. Indeed, we may assume \(W = \text{Spec}(B)\) and \(W' = \text{Spec}(C)\). Then \(\phi_{\Xi} : B \to C\) is an injective finite map. \(\xi\) and \(\xi'\) are the prime ideals \(B \otimes_k (\pi)\) and \(C \otimes_k (\pi)\) respectively. Denote the quotient fields of \(B\) and \(C\) by \(K_B\) and \(K_C\) respectively. Since the field \(k\) is algebraically closed, it follows that \(\mathcal{O}_\xi = K_B \otimes_k R\) and \(\mathcal{O}_{V_R,\xi'} = K_C \otimes_k R\). Then

\[
\mathcal{O}_\xi \otimes_{B \otimes_k R} (C \otimes_k R) = (K_B \otimes_k R) \otimes_{B \otimes_k R} (C \otimes_k R) = (K_B \otimes_B (B \otimes_k R)) \otimes_{B \otimes_k R} (C \otimes_k R) = K_B \otimes_B (C \otimes_k R) \otimes_{B \otimes_k R} (C \otimes_k R) = K_C \otimes_k R = \mathcal{O}_{V_R,\xi'},
\]

where \(K_C = K_B \otimes_B C\) (\(C\) is integral over \(B\)). Thus, the second square is cartesian. So, the morphism \(\phi_{\Xi}\) is finite. Then the first square is cartesian. By the flat base change formula of stacky version and cartesian diagram

\[
\begin{array}{c}
\Xi \xrightarrow{\phi_{\Xi}} W \times \text{Spec}(K) \xrightarrow{i_{W,K}} \mathcal{X}_K^o \xrightarrow{\phi_K} \mathcal{X}_K \\
\Xi' \xrightarrow{\phi'_{\Xi}} W' \times \text{Spec}(K) \xrightarrow{i'_{W,K}} V \times \text{Spec}(K) \xrightarrow{\phi'_K} \text{Spec}(A \otimes_k K)
\end{array}
\]

By the last square in diagram (D), we have the equation:

\[
\theta^*_RE = \theta^*_R(i_*E_K \cap (\gamma_1 \circ \beta_1)_*\mathcal{M}).
\]

From the last three square in diagram (C), we get the equation:

\[
\theta^*_RE_K \cap \theta^*_R((\gamma_1 \circ \beta_1)_*\mathcal{M}) = i'_*\theta^*_R(i_*E_K \cap (\gamma'_1 \circ i_{W',R} \circ \beta'_3)_*\phi^*_\xi E_K) \oplus \phi^*_\Xi \mathcal{N}.
\]

Let \(\phi^*_\Xi \mathcal{M} = \mathcal{M}', \theta^*_RE_K = E'\) and \(\phi^*_\Xi \mathcal{N} = N'\). Then,

1. \(E'\) is a torsion free sheaf of rank \(r\) and \(E'|_{\Xi} = N'\);
2. \(\alpha'_1*\mathcal{M}' = N'\);
3. \(\mathcal{M}'\) and \(\mathcal{N}'\) are free sheaves of rank \(r\).

Therefore, we only consider the case: \(\mathcal{X} = \text{Spec}(A)\), where \(\text{Spec}(A)\) is an irreducible smooth affine varieties over \(k\). In this case, all the properties of \(E\) can be checked through commutative algebra, just as Proposition 6 of [11].

\(\square\)
Remark 3.4 Assume that $M_1$ and $M_2$ are two free rank $r$ $O_\xi$ submodules of $(E_K)_\Xi$. Denote the corresponding coherent sheaves in Lemma 3.3 by $E_1$ and $E_2$, respectively. If $M_1 \subseteq M_2$, by the proof of Lemma 3.3, we have $E_1 \subseteq E_2$.

In the following, we show the first part of Theorem 3.1 as [11].

Proof of the first part in Theorem 3.1 Suppose $E_1$ and $E_2$ are two coherent subsheaves of $i_*E_K$ such that $i^*E_1 = i^*E_2 = E_K$ and $j^*E_1$, $j^*E_2$ are torsion free semistable sheaves on $X_k$, at least one of which is stable. Since $O_\xi$ is a principal ideal domain, $E_1,\xi$ and $E_2,\xi$ are free $O_\xi$ modules of rank $r$. Also, $E_{1,\xi} \otimes_{O_\xi} O_\Xi = E_{2,\xi} \otimes_{O_\xi} O_\Xi = (E_K)_\Xi$. By the elementary divisor theorem (Theorem 7.8 in [12]), there is a basis $\{e_1, \ldots, e_r\}$ of $E_{1,\xi}$ over $O_\xi$ such that $\{\pi^{a_1}e_1, \ldots, \pi^{a_r}e_r\}$ is a basis of $E_{2,\xi}$. Since we are trying to prove that $E_1 = \pi^pE_2$ for some $p$, we may multiply $E_{2,\xi}$ by $\pi^m$ for some integer $m$, so that all the $q_i$ are nonnegative, and at least one of the $q_i = 0$. If all the $q_i = 0$, we are done; hence we may also assume that some $q_i$ is positive. By $E_{2,\xi} \subseteq E_{1,\xi}$ and Remark 3.4, we have $E_2 \subseteq E_1$. This inclusion induces a homomorphism $\alpha : j^*E_2 \rightarrow j^*E_1$ on $X_k$. Also, $rk(j^*E_1) = rk(j^*E_2)$ and $a_{n-1}(j^*E_1) = a_{n-1}(j^*E_2)$, for the sake of Lemma 3.2. Hence, $j^*E_1$ and $j^*E_2$ have the same modified slope. By the construction of $\alpha$, the map $\alpha$ is not zero and not isomorphism in codimension one. Therefore, we have $E_1 = \pi^pE_2$, for some integer $p$. 

We state a lemma about the torsion free modules on a discrete valuation ring.

Lemma 3.5 Suppose $M$ is a finitely generated torsion free module on a discrete valuation ring. Then $M$ is a free module of finite rank.

On analogy with [11], we introduce Bruhat–Tits complex of the $E_K$. Assume that $\mathfrak{M}$ is the set of all free rank $r$ $O_\xi$ submodules of $(E_K)_\Xi$. For every $M \in \mathfrak{M}$, there is a unique torsion free sheaf $E_R$ on $X_R$, which is the extension of $E_K$, for the sake of Lemma 3.3. An equivalence relation $\sim$ is defined in $\mathfrak{M}$ by

$$\text{For } M, M' \in \mathfrak{M}, \text{ then } M \sim M' \text{ if and only if } M = \pi^pM', \text{ for some } p \in \mathbb{Z}. \quad (E)$$

Let $\Theta$ be the set of equivalence classes in $\mathfrak{M}$. Obviously, every equivalence class in $\Theta$, defines an extension of $E_K$ to coherent sheaf on $X_R$, modulo isomorphism. We now define the structure of an $r$-dimensional simplicial complex on $\Theta$, which we will call the Bruhat–Tits complex. The dimension of $\Theta$ will be less than or equal to $r$. Two equivalence classes $[M]$ and $[M']$ in $\Theta$ are said to be adjacent if $M$ has a direct decomposition $M = N \oplus P$ such that $M' = N + \pi M$. Since $O_\xi$ is a discrete valuation ring, $M$ has a basis $\{e_1, e_2, \ldots, e_r\}$ over $O_\xi$ such that $\{e_1, \ldots, e_s\}$ and $\{e_{s+1}, \ldots, e_r\}$, are bases of $N$ and $P$, respectively, by the sake of Lemma 3.5. So, $\{e_1, \ldots, e_s, \pi e_{s+1}, \ldots, \pi e_r\}$ is a basis of $M'$ over $O_\xi$. Then, $M$ is adjacent to $M'$ if and only if there is a basis $\{e_1, \ldots, e_r\}$ of $M$ such that $\{e_1, \ldots, e_s, \pi e_{s+1}, \ldots, \pi e_r\}$ is a basis of $M'$. A chain $0 \subset N_1 \subset N_2 \subset \cdots \subset N_i \subset M$ of submodules such that each $N_i$ is a direct factor of $M$ and $M_i = N_i + \pi M$, then the $i + 1$ mutually adjacent vertices $[M], [M_1], \ldots, [M_i]$ are said to form an $i$-simplex in $\Theta$. In other words, the vertices $[M], [M_1], \ldots, [M_i]$ are said to form an $r$-simplex in $\Theta$ if there is a basis $\{e_1, e_2, \ldots, e_r\}$ of $M$ such that $N_k = (e_1, \ldots, e_{s_k})$ and $M_k = (e_1, \ldots, e_{s_k}, \pi e_{s_k+1}, \ldots, \pi e_r)$, for $1 \leq k \leq i$. From the above argument, it is clear that the proof of the part 2 in Theorem 3.1 is equivalent to finding a vertex $[E_\xi]$ of $\Theta$ such that the reduction $E_k$ of the corresponding extension $E_R$ is semistable. Start with any vertex $[E_\xi]$ in $\Theta$. 


We have the following Proposition, which is the orbifold version of Proposition 7 in [11].

**Proposition 3.6** Assume that \([E_ξ]\) is a vertex in \(Ω\) and \(E_k\) is the corresponding sheaf on \(X_k\). Then, there is a natural one-to-one correspondence between edges in \(Ω\) at \([E_ξ]\) and proper subbundles of \(E_k\). Furthermore, if \(F \subset E_k\) is a subbundle corresponding to the edge \([E_ξ]\) \([- [E_ξ']\), and if \(Q' \subset E_k'\) is the subbundle corresponding to the edge \([E_ξ]\) \([- [E_ξ']\), then there are a homomorphism \(E_k \to E_k'\) with kernel \(F\) and image \(Q'\), and a homomorphism \(E_k' \to E_k\) with kernel \(Q'\) and image \(F\).

**Proof** First, let \(E_ξ = (e_1, \ldots, e_r)\) be a representative of the given vertex \([E_ξ]\) and let \(E'_ξ = (e_1, \ldots, e_s, \pi e_{s+1}, \ldots, \pi e_r)\) be a representative of an adjacent vertex. By the Remark 3.4, we have a natural inclusion of the corresponding extensions \(E'_R\) into \(E_R\). If \(\hat{E}_ξ\) and \(\hat{E}'_ξ\) are the coherent sheaves on \(\text{Spec}(O_ξ)\), defined by \(E_ξ\) and \(E'_ξ\), respectively. In the proof of Lemma 3.3, we have showed that \(E_R = i_∗E_K \cap (γ_R \circ β_1)\widehat{E}_ξ\) and \(E'_R = i_∗E_K \cap (γ_R \circ β_1)\widehat{E}'_ξ\), where the morphisms \(γ_R\) and \(β_1\) are the same as in diagram (A). Let \(Q_ξ\) be the cokernel of inclusion \(E'_ξ \to E_ξ\) and let \(0 \to \widehat{E}_ξ \to \widehat{E}'_ξ \to \widehat{Q}_ξ \to 0\) be the associated exact sequence of coherent sheaves on \(\text{Spec}(O_ξ)\). By the cartesian diagram (C), the following sequence:

\[
0 \to (γ_R \circ β_1)_∗\widehat{E}'_ξ \to (γ_R \circ β_1)_∗\widehat{E}_ξ \to (γ_R \circ β_1)_∗\widehat{Q}_ξ \to 0
\]

is exact. Thus, the cokernel \(Q\) of \(E'_R \to E_R\) admits an injection \(Q \to (γ_R \circ β_1)_∗\widehat{Q}_ξ\). So, \(Q\) is a coherent \(O_{X_k}\) module. Restricting to \(X_k\), we get right exact sequence:

\[
E'_k \to E_k \to Q \to 0 .
\]

And also, \(Q\) is torsion free on \(X_k\). Indeed, as the proof of Lemma 3.3, we only need to check this, when \(X\) is an irreducible smooth affine variety. Assume that \(X = \text{Spec}(A)\). Then, \((γ_R \circ β_1)_∗\widehat{Q}_ξ\) is isomorphic to the quasicoherent sheaf that associated to the direct sum of \((r - s)\)-copies \(K_A\), where \(K_A\) is the quotient field of \(A\). Thus, the image \(F = \text{Im}(E'_k \to E_k)\) is a subbundle of \(E_k\), with an exact sequence:

\[
0 \to F \to E_k \to Q \to 0 .
\]

Now, we have constructed a subbundle \(F\) of \(E_k\), from an edge at \([E_ξ]\).

Conversely, if \(F\) is a subbundle of \(E_k\) and \(Q = E_k/F\), then we have an exact sequence of torsion free sheaves:

\[
0 \to F \to E_k \to Q \to 0 .
\]

On the other hand, there is a natural surjective homomorphism \(E_R \to E_k\). Composing this morphism with the last morphism in the above exact sequence, we get a surjective homomorphism \(E_R \to Q\) of coherent sheaves and an exact sequence:

\[
0 \to E'_R \to E_R \to Q \to 0 . \tag{8}
\]

We have to show that the above two procedures are invertible to each other. In fact, by the
exact sequence (8), we have:

$$0 \rightarrow (E'_R)_\xi \rightarrow (E_R)_\xi \rightarrow Q_\xi \rightarrow 0$$

Suppose that $(E_k)_\xi$ is generated by \{\pi_1, \ldots, \pi_r\} and $E_\xi$ is generated by \{\pi_1, \ldots, \pi_s\}. Also, \{\pi_1, \ldots, \pi_r\} lifts to a basis \{e_1, \ldots, e_r\} of $(E_R)_\xi$. Then, $(E'_R)_\xi$ is generated by \{e_1, \ldots, e_s, \pi e_{s+1}, \ldots, \pi e_r\}. And, $(E'_R)_\xi$ represents a vertex of $\mathfrak{Q}$, which is adjacent to $[E_\xi]$. Pulling back the exact sequence
to $\mathcal{X}_k$, we get

$$0 \rightarrow E'_R \rightarrow E_R \rightarrow Q \rightarrow 0$$

where $Q' = \text{Tor}_1^{\mathcal{O}_{\mathcal{X}_R}}(Q, \mathcal{O}_{\mathcal{X}_k})$. Tensoring the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{X}_R} \xrightarrow{\pi} \mathcal{O}_{\mathcal{X}_R} \rightarrow \mathcal{O}_{\mathcal{X}_k} \rightarrow 0$$

with $Q$, we have

$$0 \rightarrow Q' \rightarrow Q \xrightarrow{\pi} Q \xrightarrow{id} Q \rightarrow 0 ,$$

whence $Q' \cong Q$. Thus, we get two exact sequences

$$0 \rightarrow F \rightarrow E_k \rightarrow Q \rightarrow 0 ;$$

$$0 \rightarrow Q \rightarrow E'_k \rightarrow F \rightarrow 0 .$$

Since $Q$ and $F$ are torsion free sheaves, $E'_k$ is torsion free. Hence, $E'_R$ is the extension of $E_K$ to $\mathcal{X}_k$, corresponding to the vertex $[(E'_R)_\xi]$ of $\mathfrak{Q}$. On the other hand, we have the following exact sequence:

$$0 \rightarrow \pi E_R \rightarrow E'_R \rightarrow F \rightarrow 0 .$$

Also, $E_R \xrightarrow{\pi} \pi E_R$ is an isomorphism. We get a homomorphism $E_R \xrightarrow{\pi} \pi E_R \hookrightarrow E'_R$. Pulling back to $\mathcal{X}_k$, we get a homomorphism $E_k \rightarrow E'_k$ and the image $Q'$ of it is the subbundle corresponding to the edge $[E'_\xi] - [E_\xi]$ at vertex $[E'_k]$.

The subbundle $F$ in Proposition 2.31, is called the $\beta$-subbundle of the bundle $E$. For the convenience, in the following, the $\beta$-subbundle of $E$ should be denoted by $B$. Now assume that we are given a vertex $[E_\xi]$ of $\mathfrak{Q}$ such that the corresponding $E_k$ on $\mathcal{X}_k$ is unstable. Let $B \subset E_k$ be the $\beta$-subbundle of $E_k$. Then, $\beta(B) > 0$. By Proposition 3.6, there is an edge in $\mathfrak{Q}$ at $[E_\xi]$ corresponding to $B$. Let $[E'_k]$ be the vertex in $\mathfrak{S}$ determined by the edge, which corresponds to the subbundle $B$. Let $F_1 \subseteq E'_k$ be the image of the canonical homomorphism $E_k \rightarrow E'_k (= \text{the kernel of the homomorphism } E'_k \rightarrow E_k)$.  

**Lemma 3.7** If $G \subset E'_k$ is a subbundle of $E'_k$, then $\beta(G) \leq \beta(B)$, with equality possible only if $G \cap F_1 = E'_k$.

**Proof** By the argument of Proposition 3.6, there are two exact sequences:

$$0 \rightarrow B \rightarrow E_k \rightarrow F_1 \rightarrow 0 ;$$
If \( G \subseteq F_1 \), then there is a subbundle \( W \subseteq E_k \), such that
\[
0 \to F_1 \to E_k(1) \to B \to 0.
\]
Thus, \( \beta(G) = \beta(W) - \beta(B) \leq 0 \) (Proposition 2.29 and Proposition 2.31). If \( F_1 \subset G \), then there is a subbundle \( W' \subseteq B \), such that
\[
0 \to F_1 \to G \to W' \to 0.
\]
So, \( \beta(G) = \beta(F_1) + \beta(W') = \beta(W') - \beta(B) \leq 0 \) (\( \beta(B) + \beta(F_1) = \beta(E_k) = 0 \) and Proposition 2.29). For the other case, we have \( \beta(G) = \beta(G \cap F_1) + \beta(G \cap F_1) - \beta(F_1) \leq \beta(B) \) and equality possible only if \( G \cap F_1 = E_k^{(1)} \).

Following [11], we are now going to define a path \( \mathcal{P} \) in \( \Omega \) which starts at a given vertex \([E_k^{(\ell)}]\) such that the corresponding \( E_k^{(\ell)} \) is unstable. Let the succeeding vertex be the vertex determined by the edge corresponding to the \( \beta \)-subbundle \( B \) of \( E_k \). If \( \mathcal{P} \) reaches a vertex \([E_k^{(m)}]\) such that the corresponding bundle \( E_k^{(m)} \) is semistable, then the process stops automatically. If the path \( \mathcal{P} \) never reaches a vertex corresponding to a semistable reduction, then the process continuous indefinitely. In the following, we will show that the second alternative is impossible.

Denote the \( \beta \) subbundle of \( E_k^{(m)} \) by \( B^{(m)} \) and let \( \beta_m = \beta(B^{(m)}) \). By Lemma 3.7, \( \beta_{m+1} \leq \beta_m \) and we must have \( \beta_m > 0 \) unless \( E_k^{(m)} \) is semistable. Thus, if the path \( \mathcal{P} \) is continuous indefinitely we have \( \beta_m = \beta_{m+1} = \cdots \) for sufficiently large \( m \). Also, by Lemma 3.7, for sufficiently large \( m \), \( B^{(m)} \cap E^{(m)} = E_k^{(m)} \), where \( E^{(m)} = \text{Im}(E_k^{(m-1)} \to E_k^{(m)}) \) (\( \text{Ker}(E_k^{(m)} \to E_k^{(m-1)}) \)). So, rank(\( B^{(m)} \)) + rank(\( F^{(m)} \)) \( \geq r \). On the other hand, rank(\( B^{(m-1)} \)) + rank(\( F^{(m)} \)) = \( r \). Therefore, rank(\( B^{(m)} \)) \( \geq \) rank(\( B^{(m-1)} \)), for sufficiently large \( m \). Since rank(\( B^{(m)} \)) \( \leq r \), we must have rank(\( B^{(m)} \)) = rank(\( B^{(m+1)} \)) = \( \cdots \), for sufficiently large \( m \). Thus, rank(\( B^{(m)} \)) + rank(\( F^{(m)} \)) = \( r \), \( B^{(m)} \cap F^{(m)} = 0 \). Consequently, the canonical homomorphism \( E_k^{(m)} \to E_k^{(m-1)} \) induces an injection \( B^{(m)} \to B^{(m-1)} \). Also, the canonical homomorphism \( E_k^{(m-1)} \to E_k^{(m)} \) induces an injection \( F^{(m-1)} \to F^{(m)} \). Also, \( B^{(m)} \) and rank(\( B^{(m)} \)) are both constant. It implies that \( \beta(F^{(m)}) = \beta(F^{(m+1)}) = \cdots \), for \( m \) sufficiently large.

**Lemma 3.8** Let \( R \) be a complete discrete valuation ring and \( \mathcal{P} \) be an infinite path in \( \Omega \), with vertices \([E_k], [E_k^{(1)}], [E_k^{(2)}], \ldots \). Let \( F^{(m)} = \text{Im}(E_k^{(m+1)} \to E_k^{(m)}) \). Assume that rank(\( F^{(1)} \)) = rank(\( F^{(2)} \)) = rank(\( F^{(3)} \)) = \( \cdots = r \), the canonical homomorphism \( E_k^{(m+1)} \to E_k^{(m)} \) induces injection \( F^{(m+1)} \to F^{(m)} \), for each \( m \), and \( a_{n-1}(F) = a_{n-1}(F^{(1)}) = a_{n-1}(F^{(2)}) = \cdots \). Then \( \beta(F) \leq 0 \).

**Proof** By Lemma 3.3, there is a sequence of extensions of \( E_K \) to \( X_R \), i.e
\[
\cdots \subset E^{(m)} \subset \cdots \subset E^{(1)} \subset E.
\]
Restricting the above inclusions to the special fiber \( X_k \), we get homomorphisms
\[
\cdots \to E_k^{(m)} \to \cdots \to E_k^{(1)} \to E_k
\]
and \( F^{(m)} = \text{Im}(E_k^{(m+1)} \to E_k^{(m)}) \), for each \( m \geq 0 \). Let \( Q^{(m+1)} = \text{Ker}(E_k^{(m+1)} \to E_k^{(m)}) \), for \( m \geq 0 \). Then the hypothesis that \( F^{(m+1)} \to F^{(m)} \) is injective implies that \( Q^{(m)} \cap F^{(m)} = 0 \). Let \( \cdots \leftarrow E_k^{(m)} \leftarrow \cdots \leftarrow E_k^{(1)} \leftarrow E_k \) be the reverse homomorphisms. By
Proposition 3.6. \( Q^{(m)} = \text{Im}(E^{(m-1)} \to E^{(m)}) \) and \( F^{(m)} = \text{Ker}(E^{(m)} \to E^{(m+1)}) \). Since \( F^{(m)} \cap Q^{(m)} = (0) \), the induced map \( Q^{(m)} \to Q^{(m+1)} \) is injective. By the exact sequence \( 0 \to F^{(m-1)} \to E^{(m)} \to Q^{(m)} \to 0 \), we have \( a_{n-1}(F^{(m-1)}) + a_{n-1}(Q^{(m)}) = a_{n-1}(E^{(m)}) = a_{n-1}(E_K) \), for \( m \geq 1 \). Since \( a_{n-1}(F) = a_{n-1}(F^{(1)}) = a_{n-1}(F^{(2)}) = \cdots \), we have \( a_{n-1}(Q^{(1)}) = a_{n-1}(Q^{(2)}) = a_{n-1}(Q^{(3)}) = \cdots \). Hence, the injections \( Q^{(m)} \hookrightarrow Q^{(m+1)} \) are isomorphisms in codimension one. Also, \( Q^{(m)**} \) are reflexive sheaves, then \( Q^{(m)**} \) are determined by their restriction on the codimension one open substack. Thus, we have isomorphisms

\[
Q^{(1)**} \to Q^{(2)**} \to Q^{(3)**} \to Q^{(m)**} \to \cdots .
\]

So there is a sequence of inclusions:

\[
Q^{(1)} \hookrightarrow Q^{(2)} \hookrightarrow Q^{(3)} \hookrightarrow Q^{(m)} \hookrightarrow \cdots \hookrightarrow Q^{(1)**}.
\]

On the other hand, \( Q^{(1)**} \) is a coherent sheaf on \( \mathcal{X}_k \), it follows that

\[
Q^{(m)} \hookrightarrow Q^{(m+1)} \hookrightarrow Q^{(m+3)} \hookrightarrow \cdots .
\]

are isomorphisms, for sufficiently large \( m \). Thus, we may assume without loss of generality that

\[
Q^{(1)} \hookrightarrow Q^{(2)} \hookrightarrow Q^{(3)} \hookrightarrow Q^{(m)} \hookrightarrow \cdots
\]

are isomorphisms. Also, we may assume that there is a subbundle \( Q \subset E_k \) such that \( Q \hookrightarrow Q^{(1)} \) is an isomorphism. Therefore, the exact sequence \( 0 \to F^{(m)} \to E^{(m)} \to Q^{(m+1)} \to 0 \) splits, for each \( m \geq 0 \), i.e., \( E^{(m)}_k = F^{(m)} \oplus Q^{(m)} \). So, the exact sequence

\[
0 \to Q^{(m+1)} \to E^{(m+1)}_k \to F^{(m)} \to 0
\]

yields \( F^{(m+1)} \hookrightarrow F^{(m)} \) is an isomorphism, for each \( m \geq 0 \).

Consider the completion \( \hat{\mathcal{X}}_R \) of \( \mathcal{X}_R \) with respect to the special fiber \( \mathcal{X}_k \). Let \( \mathcal{X}_m = \mathcal{X}_R \times \text{Spec}(R/(\pi^m)) \), for each \( m \geq 0 \). For a coherent sheaf \( G \) on \( \mathcal{X}_R \), we denote the restriction of \( G \) to \( \mathcal{X}_m \) by \( G_m \). Following [11], we will construct a coherent subsheaf \( \hat{F}_R \) of \( \hat{E} = \lim \hat{E}_m \) on \( \hat{\mathcal{X}}_R \).

For each \( m \), we will construct a coherent subsheaf \( F_m \) of \( E_m \) as following: Pulling back the inclusion \( E^{(m)} \to E \) to \( \mathcal{X}_m \), we get a homomorphism \( E^{(m)}_m \to E_m \) and let \( F_m \) be the image of this homomorphism. Let \( j_{m,m'} \) be the closed immersion \( \mathcal{X}_m \hookrightarrow \mathcal{X}_m', \) for \( m' \leq m \). Pulling back the homomorphism \( E^{(m)}_m \to E_m \) to \( \mathcal{X}_m' \), we get homomorphism \( E^{(m)}_m \to j_{m,m'}^* F_m \to E_{m'} \), which fit into a commutative diagram:

\[
\begin{array}{c}
E^{(m)}_m \\
\downarrow \quad \downarrow \\
j_{m,m'}^* F_m \\
\end{array} \quad \begin{array}{c}
E^{(m')}_{m'} \\
\end{array}
\]

So, there is a natural homomorphism \( j_{m,m'}^* F_m \to F_{m'} \). We can show this homomorphism is an isomorphism, step by step as the proof of Lemma 2 in [11]. Thus, we get an inverse system of sheaves \( \{ F_m \} \) and the inverse limit is a coherent subsheaf \( \hat{F}_R \) of \( \hat{E} \) on \( \hat{\mathcal{X}}_R \). By the Grothendieck’s existence theorem for tame stacks in appendix A of [2], there exists a coherent subsheaf \( F_R \) of \( E \) such that \( \hat{F}_R = \lim F_m \). Also, \( j^* F_R = F \). Therefore, \( a_{n-1}(F) = a_{n-1}(F_K) \), where \( F_K = i^* F_R \). Since \( E_K \) is semistable, we have \( \beta(F) = \beta(F_K) \leq 0 \). \( \Box \)
Proof of the second part in Theorem 3.1 For the case: $R$ is a complete discrete valuation ring, we have complete the proof of Theorem 3.1. As in [11], the general case can be reduced to above case by considering the completion $\hat{R}$ of $R$. There is the following commutative diagram:

\[
\begin{array}{ccc}
X_k & \xrightarrow{j} & X_{\hat{R}} \\
\downarrow{id} & & \downarrow{p} \\
X_k & \xrightarrow{j} & X_{\hat{R}} \\
\end{array}
\]

Suppose that $E_K$ is a torsion free semistable sheaf on $X_K$. Then the pullback $p'^* E_K$ is a torsion free semistable sheaf on $X_{\hat{R}}$ (Proposition 2.34), where $\hat{R}$ is the quotient field of $R$. Denote the Bruhat–Tits complexes corresponding to $E_K$ and $p'^* E_K$ by $\Omega_1$ and $\Omega_2$ respectively. For a vertex $[E_\xi]$ in the Bruhat–Tits complex $\Omega_1$, $[E_{\hat{R},\xi}]$ is the vertex in the Bruhat–Tits complex $\Omega_2$, where $E_{\hat{R},\xi} = E_\xi \otimes \hat{R}$. If $E$ is the torsion free sheaf on $X_{\hat{R}}$ corresponding to $[E_\xi]$, then $p^* E$ is the torsion free sheaf on $X_K$ corresponding to $[E_{\hat{R},\xi}]$. When $E_k$ is unstable, denote the $\beta$-subbundle of $E_k$ by $F_k$. By Lemma 3.6, the edge $[E_\xi] - [E'_\xi]$ in $\Omega_1$ corresponding to $F_k$ is constructed as following:

\[
\begin{array}{ccc}
0 & \rightarrow & F_k \\
\rightarrow & & \rightarrow \\
& Q_k & \rightarrow & 0
\end{array}
\]

And, the edge $[p^* E_\xi] - [p^* E'_\xi]$ in $\Omega_2$ corresponding to $F_k$ is given by

\[
\begin{array}{ccc}
0 & \rightarrow & F_k \\
\rightarrow & & \rightarrow \\
& E'_k & \rightarrow & E & \rightarrow & Q_k & \rightarrow & 0
\end{array}
\]

In $\Omega_2$, there is a finite path leading to a vertex whose corresponding torsion free sheaf on $X_k$ is semistable, so it is in $\Omega_1$.

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References

[1] Abramovich, D., Olsson, M., Vistoli, A.: Tame stacks in positive characteristic. Ann. Inst. Fourier (Grenoble), 58(3), 1057–1091 (2008)

[2] Abramovich, D., Vistoli, A.: Compactifying the space of stable maps. J. Am. Math. Soc., 15(1), 27–75 (2002)

[3] Brochard, S.: Finiteness theorems for the Picard objects of an algebraic stack. Adv. Math., 229, 1555–1585 (2012)

[4] Brochard, S.: Foncteur de Picard d’un champ algébrique. Math. Ann., 343, 541–602 (2009)

[5] Conrad, B.: Keel–Mori theorem via stacks, unpublished manuscript, 2005

[6] Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. Publ. Math. IHES, 36, 75–109 (1969)

[7] Hartshorne, R.: Algebraic Geometry, Springer-Verlag, New York, 1977
[8] Huang, Y. H., Jiang, Y. H.: Mirror symmetry and Hitchin system on Deligne-Mumford curves, in preparation

[9] Huybrechts, D., Lehn, M.: The Geometry of Moduli Spaces of Sheaves (Second Edition), Cambridge University Press, Cambridge, 2010

[10] Kresch, A.: On the geometry of Deligne–Mumford stacks. In: Abramovich, D.; Bertram, A.; Katzarkov, L.; Pandharipande, R.; Thaddeus, M., Algebraic Geometry–Seattle, Amer. Math. Soc., Providence, Rhode Island: 259–271, 2005

[11] Langton, S. G.: Valuative criteria for families of vector bundles on algebraic varieties. Ann. Math., 101(1), 88–110 (1975)

[12] Lang, S.: Algebra (Revised Third Edition), Springer-Verlag, New York, 2002

[13] Maruyama, M.: Moduli of stable sheaves II. J. Math. Kyoto Univ., 18, 557–614 (1978)

[14] Mehta, V., Ramanathan, A.: An analogue of Langton’s theorem on valuative criteria for vector bundles. P. Roy. Soc. Edinb. A., 96(1–2), 39–45 (1984)

[15] Nironi, F.: Moduli spaces of semistable sheaves on Projective Deligne–Mumford Stack. arXiv:0811.1949v2

[16] Olsson, M.: Algebraic spaces and stacks, Amer. Math. Soc. Colloquium Publication, Volume 62, Amer. Math. Soc., Providence, RI, 2016

[17] Olsson, M., Starr, J.: Quot functors for Deligne–Mumford stacks. Comm. Algebra, 31(8), 4069–4096 (2003)

[18] Simpson, C.: Moduli of representations of the fundamental group of a smooth projective variety I. Publ. Math. IHES., 79, 47–129 (1994)

[19] Toen, B.: Théorèmes de Riemann–Roch pour les champs de Deligne–Mumford. K-Theory, 18, 33–76 (1999)

[20] Vistoli, A.: Intersection theory on algebraic stacks and on their moduli spaces. Invent. Math., 97, 613–670 (1989)