SINGULARITIES OF ENERGY-MINIMIZING MAPS
FROM THE BALL TO THE SPHERE

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We study maps \( \varphi \) from the unit ball \( B \) in \( \mathbb{R}^3 \) to the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) which minimize Dirichlet's energy integral

\[
\mathcal{E}(\varphi) = \int_B |\nabla \varphi|^2 dV.
\]

If such a \( \varphi \) minimized Dirichlet’s integral among mappings into \( \mathbb{R}^3 \) rather than being constrained to lie in \( S^2 \) it would then be a classical smooth harmonic function. A minimizing constrained \( \varphi \), however, sometimes has isolated point discontinuities. We here announce several new estimates on the number and arrangement of such singular points [AL]. The \( \varphi \)'s we consider have well defined values \( \psi \) on the boundary \( \partial B \) of \( B \), and the boundary Dirichlet’s energy integral is

\[
\partial \mathcal{E}(\psi) = \int_{\partial B} |\nabla_T \psi|^2 dA,
\]

where \( \nabla_T \psi \) denotes the tangential gradient. In our theorems and examples below each \( \psi \) has finite energy. One of our principal results is

**Main Theorem.** Suppose \( \varphi \) minimizes Dirichlet’s integral among all functions mapping \( B \) to \( S^2 \) and having boundary value function \( \psi \) on \( \partial B \). Then the number of points of discontinuity of \( \varphi \) is bounded by a constant times \( \mathcal{E}(\psi) \).

This linear law is noteworthy because examples illustrate linear growth of the number of singularities with \( \partial \mathcal{E}(\psi) \) while other examples show that the number of singularities cannot be bounded by \( \mathcal{E}(\varphi) \). This shows that the number and location of singularities cannot be inferred from simple energy comparisons alone. The subtlety of this estimate is further illustrated by

**Examples.** There are boundary value functions \( \psi \) for which the minimizing \( \varphi \)'s are unique and have an arbitrarily large number of singular points stacked arbitrarily high near the boundary—like bubbles in a pan of water that is almost ready to boil. The number of stacks is also arbitrarily large.

Such examples show the necessity of an analysis containing several different length scales in proving the principal result above—the length scale of a singular point is its distance to the boundary.