A FURTHER QUANTIFICATION OF THE UNIQUE CONTINUATION PROPERTIES OF EIGENFUNCTIONS OF THE MAGNETIC SCHröDINGER OPERATOR

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ABSTRACT. We prove quantitative unique continuation results for solutions of $\Delta w - k^2 w = Vw + W \cdot \nabla w$ in a neighborhood of infinity, where $k > 0$, and $V$ and $W$ are complex-valued decaying potentials that satisfy $|V(x)| \lesssim |x|^{-N}$ and $|W(x)| \lesssim |x|^{-P}$ for some $N, P > 1$. For $M(R, 4n/k) = \inf \left\{ ||w||_{L^2(B_{w,\alpha}(x_0))} : |x_0| = R \right\}$, we show that if the solution $w$ is non-zero, bounded, and normalized, then $M(R, 4n/k) \gtrsim \exp(-kR - G \log R)$, where $G > \frac{2}{k}$ is a constant. An examination of radial solutions to $\Delta w - k^2 w = Vw + W \cdot \nabla w$ shows that this new estimate for $M(R, 4n/k)$ is sharp up to logarithmic terms.

1. INTRODUCTION

In this paper, we continue the study of the unique continuation properties of eigenfunctions of the magnetic Schrödinger operator. In particular, this article presents a quantitative version of a qualitative unique continuation theorem due to Meshkov from [6]. This work was motivated, in part, by the fact that quantitative versions of unique continuation theorems have found useful applications. In [7], Meshkov established the following qualitative unique continuation result: if $w$ solves $\Delta w + Vw = 0$ in $\mathbb{R}^n$, where $V$ is bounded and $|w(x)| \lesssim \exp(-c|x|^{4/3})$, then $w$ must equal zero. In [1], [4] and [5], it was shown that a quantitative version of this theorem is also true: if $w$ solves $\Delta w + Vw = 0$ in $\mathbb{R}^n$, with $w$ bounded and normalized so that $w(0) = 1$, then $M(R) := \inf \left\{ ||w||_{L^2(B_1(x_0))} : |x_0| = R \right\} \gtrsim \exp(-cR^{4/3} \log R)$. Bourgain and Kenig first proved this quantitative estimate in [1] through the use of Carleman inequalities, then applied it to a problem in Anderson localization. This result from [1] was generalized in [2], where the author established quantitative versions of the time-independent results from [3]. Specifically, if $w$ solves $\Delta w + \lambda w = Vw + W \cdot \nabla w$ in $\mathbb{R}^n$, then sharp estimates for $M(R)$ were proved. These estimates for $M(R)$ depend on the decay properties of the electric and magnetic potentials, $V$ and $W$. The aim of this article is to demonstrate that under additional assumptions, the estimates can be significantly improved. In particular, we show that when the eigenvalue $\lambda = -k^2$, where $k > 0$, and the electric and magnetic potentials both decay at a sufficient rate, then the leading constant in the estimate for $M(R)$ may be precisely determined.

Let $w$ be a solution to

$$\Delta w - k^2 w = Vw + W \cdot \nabla w$$

in $\Omega \subset \mathbb{R}^n$, a neighborhood of infinity, where $V$ and $W$ decay at least as fast as $\frac{1}{r}$, and $k > 0$. In [6], Meshkov showed that under suitable assumptions (see [4] for the specific statement),

$$w(x) = e^{-kr} r^{\frac{N-1}{2}} \left( f(\theta) + \phi(r, \theta) \right),$$

where $r = |x|$, $f \in L^2(S^{n-1})$, and $\lim_{r \to \infty} \phi(r, \theta) = 0$. In this paper, a quantitative version of Meshkov’s result is established by employing techniques similar to the ones that appeared in [2]. Specifically, Carleman inequalities are used to determine $L^2$-estimates for the solution over balls. Then a finite iteration argument, based on the $L^2$-estimates, gives the result. Let

$$M(R, \alpha) = \inf \left\{ ||w||_{L^2(B_{\alpha}(x_0))} : |x_0| = R \right\}$$

be a solution to

(2) $$M(R, \alpha) = \inf \left\{ ||w||_{L^2(B_{\alpha}(x_0))} : |x_0| = R \right\}$$

The main result is the following.
**Theorem 1.** Suppose \( w \) is a solution to (1) in \( \Omega \), where \( \Omega_{R_0} \subset \Omega \), \( k > 0 \) and

\[
|V(x)| \leq A_1 |x|^{-N} \\
|W(x)| \leq A_2 |x|^{-P},
\]

for some \( N, P > 1 \), \( A_1, A_2 \geq 0 \). Assume that \( w \) is bounded in the sense of \( \| \cdot \|_\infty \). Assume that \( w \) is normalized in the sense of (14) and (15). Let \( C_2 > 0 \). If \( R \geq (1 + C_2) \max \{ T, \tilde{T} \} \), (where \( T \) and \( \tilde{T} \), specified below, are constants that depend on the PDE) then

\[
M(R, 4n/k) \geq C_5 \exp (-kR - G \log R),
\]

where \( G = G(C_2, n) \), \( C_5 = C_5(n) \).

**Remark.** An examination of the proof of Theorem 1 shows that \( G = \frac{1}{4} (n - 1 + F) \), where \( F \geq 2 \) is a positive constant that depends on the dimension \( n \), and the constant \( C_2 \) that we choose in the proof. In particular, \( G > \frac{1}{2} (n - 1) \), which is the exact constant that appears in the radial constructions presented in Theorem 2.

Under certain largeness conditions on \( k \), we may estimate the \( L^2 \) norm in the 1-ball.

**Corollary 1.** Assume the hypotheses of Theorem 1. If \( k \geq 4n \), then

\[
M(R, 1) \geq C_5 \exp (-kR - G \log R),
\]

where \( G = G(C_2, n) \), \( C_5 = C_5(n) \).

In [2], it was shown that if \( w \) satisfies similar assumptions, then

\[
M(R, 1) \geq C_5 \exp \left( -C_7 R (\log R)^{C_6 \log \log R} \right),
\]

where \( C_5 = C_5(n) \), \( C_6 = C_6(n, N, P) \) and \( C_7 = C_7(n, N, P, A_1, A_2) \). Therefore, Theorem 1 is an improvement over the original result.

To prove Theorem 1, we first transform the equation (1) to an equation in \( u \), where \( u(x) = e^{kr} r^{(n-1)/2} w(x) \) is a scaling of \( w \). We then prove a Carleman estimate for \( L_k \), the second order linear differential operator that appears in the equation for \( u \). Once the Carleman estimate has been established, we establish \( L^2 \) lower bounds for the function \( u \). In Proposition 1 information about the solution over an entire sphere of sufficiently large radius (which is determined from the normalization of the solution) is used to establish an \( L^2 \) lower bound for the solution \( u \) on a ball of radius \( 4n/k \) centered at a specific point that is further from the origin than the sphere. Proposition 2 uses a similar technique to relate the \( L^2 \) norms on balls of radius \( 4n/k \) centered at points on the same sphere with a fixed distance separating them. A finite iteration argument establishes an \( L^2 \) lower bound estimate for the function \( u \) in a ball of radius \( 4n/k \) centered at a point, \( z_0 \), that is sufficiently far away from the origin: First, Proposition 1 is applied to give an \( L^2 \) lower bound at a specific point, \( x_0 \), where \( |x_0| = |z_0| \). Then Proposition 2 is repeatedly applied to points on the same sphere as \( x_0 \) until we reach \( z_0 \). This iteration argument is presented in Proposition 3. To establish the main theorem, Proposition 3 is applied to \( u \), and we recall the relationship between \( u \) and \( w \) to establish the desired estimate for \( w \).

As noted above, Theorem 1 is far stronger than the results obtained previously in [2]. The following theorem (which is a specific case of a construction from [2]) shows that Theorem 1 is in fact sharp up to logarithmic factors.

**Theorem 2.** For any \( m \in \mathbb{N} \), there exists a radial function \( w_m : \mathbb{R}^n \to \mathbb{R} \) of the form

\[
w_m(r) = \exp \left( -kr - \frac{n-1}{2} \log r + c_3 r^{-1} + \ldots + c_m r^{-m} \right).
\]
such that

\[ \Delta w_m - k^2 w_m = \left[ a_m r^{-m} + \mathcal{O} \left( r^{-(m+1)} \right) \right] w_m \]

\[ = \left[ b_m r^{-m} + \mathcal{O} \left( r^{-(m+1)} \right) \right] \hat{r} \cdot \nabla w_m, \]

for some constants \( c_3, \ldots, c_m, a_m, b_m \in \mathbb{R} \), where \( \hat{r} \) denotes the unit vector in the direction of \( r \).

For any \( m > 1 \), \( w_m(R) = \exp \left( -kR - \frac{a}{2R} \log R + \mathcal{O} \left( R^{-1} \right) \right) \), which is comparable to estimate (3) from Theorem 1.

This paper is organized as follows. In §2 we use the equation for \( w \) to determine the PDE that \( u(x) = e^{kr} r^{(n-1)/2} w(x) \) satisfies. We will write this new equation as \( L_k u = \hat{V} u + \hat{W} \cdot u \), where \( L_k \) is a second order linear differential equation that transforms in a desirable way under scaling. The proof of the Carleman estimate for \( L_k \) is contained in §3. Meshkov’s qualitative result is presented in §4. In §5 Propositions 1, 2 and 3 are presented and proved. That is, the Carleman estimate is applied to establish lower bounds for the solution function \( u \). In §6 Proposition 3 is applied to \( u \). By rewriting \( u \) as a scaling of \( w \), we are able to prove Theorem 1. Finally, §7 discusses the sharpness of Theorem 1.

2. TRANSFORMING THE EQUATION

Let \( u(x) = e^{kr} r^{-\frac{n-1}{2}} w(x) \) so that \( w(x) = e^{-kr} r^{-\frac{n-1}{2}} u(x) \). We want to determine the equation that \( u \) satisfies. We first differentiate \( w \)

\[ \Delta w = e^{-kr} r^{-\frac{n-1}{2}} \left\{ \Delta u - 2 \left( k + \frac{n-1}{2r} \right) \partial_r u + \left[ k^2 + \frac{(n-1)^2}{4r^2} - \frac{(n-1)^2}{2r^2} + \frac{n-1}{2r^2} \right] u \right\}, \]

then use the equation (1) for \( w \) to get:

\[ k^2 w + W \cdot \nabla w + V w = e^{-kr} r^{-\frac{n-1}{2}} \left[ \Delta u - 2k \partial_r u - \frac{n-1}{r} \partial_r u + k^2 u + a \right] \]

\[ \Rightarrow W \cdot \nabla w + V w = e^{-kr} r^{-\frac{n-1}{2}} \left[ \Delta u - 2k \partial_r u - \frac{n-1}{r} \partial_r u + a \right], \]

where \( a = \frac{(n-1)^2}{4r^2} - \frac{(n-1)^2}{2r^2} + \frac{n-1}{2r^2} \). Now we rearrange and rewrite \( \nabla w \) in terms of \( u \) to get

\[ \Delta u - 2k \partial_r u = e^{kr} r^{-\frac{n-1}{2}} \left[ W \cdot \nabla w + V w \right] + \frac{n-1}{r} \partial_r u - \frac{a}{r^2} u \]

\[ = W \cdot \left[ -\left( k + \frac{n-1}{2r} \right) \frac{x}{r} + \nabla u \right] + Vu + \frac{n-1}{r} \partial_r u - \frac{a}{r^2} u \]

\[ = \left[ V - \frac{a}{r^2} - \left( k + \frac{n-1}{2r} \right) \frac{x}{r} \right] W \cdot \frac{x}{r} u + \left[ W + \frac{n-1}{r} \frac{x}{r} \right] \cdot \nabla u \]

If we set

(4) \[ \bar{V} = V - \frac{a}{r^2} - \left( k + \frac{n-1}{2r} \right) \frac{x}{r}, \]

(5) \[ \bar{W} = W + \frac{n-1}{r} \frac{x}{r}, \]

(6) \[ L_k = \Delta - 2k \partial_r, \]

then we have that

(7) \[ L_k u = \bar{V} u + \bar{W} \cdot \nabla u. \]
We now make the observation that $L_k$ transforms appropriately when applied to a function that has been shifted and scaled. For example, if $v(x) = u(x_0 + Sx)$ and $\hat{k} = Sk$, then $L_k^3v(x) = S^2L_ku(x_0 + Sx)$. This fact is important in establishing the inequalities that appear in §5.

3. A Carleman Estimate

In this section, we establish a useful Carleman estimate for this new operator, $L_k$. The additional term, $-2k\partial_r$, is crucial to this new Carleman estimate.

**Lemma 1.** Suppose $f \in C_0^\infty (B_R (0) \setminus B_{r_0} (0))$, where $kr_0 \geq 2(n - 1 + a)$, $a > 0$. Let $w(r) = e^r$. Then there exists a constant $C_1 = C_1 (n, a)$ such that whenever $\alpha \leq 4k$,

$$
(8) \quad k \int r^{-(a+1)}w^{-2\alpha} |\nabla f|^2 \, drd\theta + \alpha k^2 \int r^{-(a+1)}w^{-2\alpha} |f|^2 \, drd\theta \leq C_1 \int r^{-a}w^{-2\alpha} |L_k f|^2 \, drd\theta.
$$

**Proof.** Set $g = w(r)^{-\alpha}f$ so that $f = w(r)^\alpha g$ for some constant $\alpha$ to be determined. In contrast to other Carleman estimates, we will want $\alpha$ to be sub linear with respect to the scaling constant. However, $k$ will be replaced with $kS$, where $S$ the scaling constant, so $k$ may become large.

We collect some computations:

\[
\begin{align*}
\partial_r f &= w^\alpha \partial_r g + \frac{w'}{w} w^\alpha g \\
\partial_{rr} f &= w^\alpha \partial_{rr} g + 2\alpha \frac{w'}{w} w^\alpha \partial_r g + \alpha \left( \frac{w'}{w} \right)' w^\alpha g + \left( \frac{w'}{w} \right)^2 w^\alpha g \\
\Delta_\theta f &= w^\alpha \Delta_\theta g
\end{align*}
\]

If $w(r) = e^r$, then

\[
\begin{align*}
w^{-\alpha}L_k f &= \partial_{rr} g - 2k\partial_r g + 2\alpha \partial_r g + \frac{n-1}{r} \partial_r g + \alpha^2 g - 2k \alpha g + \frac{n-1}{r} \alpha g + r^{-2} \Delta_\theta g \\
&= -2k\partial_r g + Qg.
\end{align*}
\]

Assume that $f \in C_0^\infty (B_R (0) \setminus B_{r_0} (0))$ for some $0 < r_0 < R < \infty$, and that $a > 0$.

\[
\begin{align*}
&\int r^{-a}w^{-2\alpha} |L_k f|^2 \, drd\theta \\
&\geq -2k \int r^{-a} 2\partial_r g Qg \, drd\theta + 4k^2 \int r^{-a} |\partial_r g|^2 \, drd\theta \\
&= -2k \int r^{-a} 2\partial_r g \left[ \partial_{rr} g + 2\alpha \partial_r g + \frac{n-1}{r} \partial_r g + \alpha^2 g - 2k \alpha g + \frac{n-1}{r} \alpha g + r^{-2} \Delta_\theta g \right] \, drd\theta \\
&\quad + 4k^2 \int r^{-a} |\partial_r g|^2 \, drd\theta \\
&\geq 2k \int \left( 2k - ar^{-1} - 4\alpha - 2 \frac{n-1}{r} \right) r^{-a} |\partial_r g|^2 \, drd\theta + 2k(a+2) \int r^{-(a+3)} |\nabla_\theta g|^2 \, drd\theta \\
&\quad + 2aak \int \left( 2k - \alpha - \frac{a+1}{a} n - \frac{1}{r} \right) r^{-(a+1)} |g|^2 \, drd\theta
\end{align*}
\]

If $\alpha$ and $r_0$ are chosen so that $4\alpha \leq k$ and $kr_0 \geq 2(n - 1 + a)$, then

\[
\begin{align*}
&\int r^{-a}w^{-2\alpha} |L_k f|^2 \, drd\theta \geq 2ka \int r^{-(a+1)} |\nabla g|^2 \, drd\theta + 2aak \int \left( 2k - \alpha - \frac{a+1}{a} n - \frac{1}{r} \right) r^{-(a+1)} |g|^2 \, drd\theta
\end{align*}
\]
Since $|\nabla g|^2 \geq \frac{1}{1+\mu} w^{-2a} |\nabla f|^2 - \frac{\alpha^2}{\mu} |g|^2$, for any $\mu > 0$, then

$$
\int r^{-a} w^{-2a} |L_{k,f}|^2 \, dr d\theta \geq \frac{2a}{1+\mu} k \int r^{-(a+1)} w^{-2a} |\nabla f|^2 \, dr d\theta
$$

$$
+ 2a \alpha k \int (2k - \alpha \left(1 + \frac{1}{\mu}\right) - k a + 1 - \frac{n-1}{2a} n-1 + a) r^{-(1+a)} |g|^2 \, dr d\theta,
$$

where we used the bound on $r_0$ to estimate the second term. If we choose $a$ and $\mu = \mu (a)$ appropriately (for example, $a = 1$ and $\mu = 3$), then this completes the proof. \qed

4. Meshkov’s Result

In this section, we will quote the qualitative result of Meshkov.

**Theorem 3.** Let $w$ be a non-zero solution to \( (1) \), where $\Omega \subset \mathbb{R}^n$ is a neighborhood of infinity, $V$ and $W$ decay like $1/r$ as $r \to \infty$, and $k > 0$. If

$$
\limsup_{r \to \infty} e^{kr} r^{(n-1)/2} \left( \int_{S^{n-1}} |w(r, \theta)|^2 \, d\theta \right)^{1/2} < \infty
$$

then

$$w(x) = e^{-kr} r^{-(n-1)/2} (f(\theta) + \varphi(r, \theta)),
$$

where $f \in L^2 (S^{n-1})$, $f \neq 0$, \( \int_{S^{n-1}} |\varphi(r, \theta)| \, d\theta \)^{1/2} = \mathcal{O}(r^{-\gamma}) \) as $r \to \infty$, $\gamma < \min \{ \frac{1}{2}, \max \{ N, P \} - 1 \}$.

5. L² Inequalities

Let $\Omega_{R_0} = \mathbb{R}^n \setminus B_{R_0}$. Suppose $u$ is a solution to

$$L_{k} u = \tilde{V} u + \tilde{W} \cdot \nabla u \quad \text{in} \quad \Omega_{R_0},
$$

where

$$|\tilde{V}(x)| \leq A_3 |x|^{-M},
$$

$$|\tilde{W}(x)| \leq A_4 |x|^{-1},
$$

for some $M > 1$, $A_3, A_4 \geq 0$. Assume also that $u$ is bounded in the following sense,

$$\limsup_{r \to \infty} \left( \int_{S^{n-1}} |u(r, \theta)|^2 \, d\theta \right)^{1/2} < \infty.
$$

By the result of Meshkov, Theorem 3 above, we may write $u(r, \theta) = f(\theta) + \varphi(r, \theta)$. Assume that $u$ is normalized in the following sense:

$$\|f\|_{L^2(S^{n-1})} = 2,
$$

and for any $R \geq 4R_0$,

$$\|\varphi(R)\|_{L^2(S^{n-1})} \leq 1.
$$

**Proposition 1.** Assume that conditions (10)-(15) above hold. Let $C_2 > 0$ and let $R$ be sufficiently large in the sense that $R \geq T (R_0, C_2, n, k, A_3, A_4, M)$. There exists $x_0 \in \mathbb{R}^n$ with $|x_0| = (1 + C_2) R$, such that

$$
\int_{B_{|x_0|/2}(x_0)} |u|^2 \geq C_0 |x_0|^{-2}.
$$
Proof of Proposition [7] A constant $N \geq 4$, depending on dimension, will be specified below. Let $\{x_j\}_{j=1}^{C_n} \subset \mathcal{S}_R = \{x \in \mathbb{R}^n : |x| = R\}$ be a collection of points chosen so that

$$\bigcup_{j=1}^{C_n} B_{R/N}(x_j) \supset A_R = \left\{ x \in \mathbb{R}^n : R \left(1 - \frac{1}{2N}\right) \leq |x| \leq R \left(1 + \frac{1}{2N}\right) \right\}.$$ 

By invariance of scale, $C_n$ depends on the dimension, $n$, and $N$.

If we assume that $R \geq \frac{8NR_0}{2N+1}$, then by (14) and (15), we have

$$||u||_{L^2(A_R)} = \int_{A_R} |u(r, \theta)|^2 r^{n-1} dr d\theta$$

$$= \int_{A_R} |f(\theta) + \varphi(r, \theta)|^2 r^{n-1} dr d\theta$$

$$\geq 2 \int_{A_R} f(\theta)^2 r^{n-1} dr d\theta - \int_{A_R} \varphi(r, \theta)^2 r^{n-1} dr d\theta$$

Therefore, there exists some $J \in \{1, 2, \ldots, C_n\}$ such that

$$\text{on } \mathcal{S}_R \cap B_{R/N}(x_j) \geq \frac{R^n}{NC_n}.$$

We will set $S = C_2R$. Choose $x_0 \in \mathcal{S}_R \cap B_{R/N}(x_j)$ to lie along the ray that passes through the origin and $x_j$. Then $S = \text{dist}(x_0, \partial \Omega_R)$. Note that $|x_0| \geq |x_j|$.

Let $K_1 = \left[ \frac{3n}{Sk}, 1 + \frac{R}{2S} \right]$, $K_2 = \left[ \frac{2n}{Sk}, \frac{3n}{Sk} \right]$, $K_3 = \left[ 1 + \frac{R}{2S}, 1 + \frac{2R}{3S} \right]$, where $[a, b]$ denotes the annulus with inner radius $a$ and outer radius $b$.

Choose a smooth cutoff function $\zeta$ so that $\zeta \equiv 1$ on $K_1$ and $\zeta \equiv 0$ on $(K_1 \cup K_2 \cup K_3)^c$. Then

$$|\nabla \zeta| \leq \begin{cases} S & \text{on } K_2 \\ \frac{S}{R} & \text{on } K_3 \end{cases}$$

Let $v(x) = u(x_0 + Sx)$, $\tilde{v} = Sk$.

If $R \geq 4R_0$, and $v$ is defined on $\Omega_{R_0}$, then $\tilde{v}$ is defined on $[0, 1 + \frac{3R}{4S}]$.

Note that

$$|x_0 + Sx| \geq |x_0| - S|x| = R + S - S|x| \geq \begin{cases} \frac{R}{2} & \text{on } K_1 \\ \frac{R}{3} & \text{on } K_2 \\ \frac{R}{6} & \text{on } K_3 \end{cases}$$

Therefore,

$$|L_\tilde{v}v| \leq S^2 |\tilde{v}(x_0 + Sx)||v(x)| + S|\tilde{v}(x_0 + Sx)||\nabla v(x)|$$

$$\leq \begin{cases} A_3S^2(R/2)^{-M} |v| + A_4S(R/2)^{-1} |\nabla v| & \text{on } K_1 \\ A_3S^2R^{-M} |v| + A_4SR^{-1} |\nabla v| & \text{on } K_2 \\ A_3S^2(R/3)^{-M} |v| + A_4SR^{-1} |\nabla v| & \text{on } K_3 \end{cases}.$$
Since \( \tilde{k}r_0 = Sk\frac{2n}{S_k} = 2n \), then, assuming \( 4\alpha \leq \tilde{k} \), we may now apply Lemma 1 with \( a = 1 \) and \( L_\tilde{k} \) to \( f = \zeta v \), and use the above estimate on \( K_1 \).

\[
\begin{align*}
ks & \int r^{-2} w^{-2\alpha} |\nabla f|^2 \, drd\theta + \alpha k^2 S^2 \int r^{-2} w^{-2\alpha} |f|^2 \, drd\theta \\
& \leq C_1 \int_{K_1} r^{-1} w^{-2\alpha} \left| L_{\tilde{k}} v \right|^2 \, drd\theta + C_1 \int_{K_2 \cup K_3} r^{-1} w^{-2\alpha} \left| L_{\tilde{k}} f \right|^2 \, drd\theta \\
& \leq 2C_1 \left( 1 + \frac{R}{2S} \right) \left( 2^M A_3^2 \frac{S^4}{R^{2M}} \right) \int_{K_1} r^{-2} w^{-2\alpha} |v|^2 \, drd\theta + 8C_1 A_4^2 \left( 1 + \frac{R}{2S} \right) \int_{K_1} r^{-2} w^{-2\alpha} \left| \nabla v \right|^2 \, drd\theta \\
& \quad + C_1 \int_{K_2 \cup K_3} r^{-1} w^{-2\alpha} \left| L_{\tilde{k}} f \right|^2 \, drd\theta \\
& = 2C_1 C_4^2 \left( 1 + \frac{1}{2C_2} \right) 2^M A_3^2 R^{4-2M} \int_{K_1} r^{-2} w^{-2\alpha} |v|^2 \, drd\theta + 8C_1 C_2^2 A_4^2 \left( 1 + \frac{1}{2C_2} \right) \int_{K_1} r^{-2} w^{-2\alpha} \left| \nabla v \right|^2 \, drd\theta \\
& \quad + C_1 \int_{K_2 \cup K_3} r^{-1} w^{-2\alpha} \left| L_{\tilde{k}} f \right|^2 \, drd\theta,
\end{align*}
\]

where we used that \( S = C_2 R \) to get to the final line. If

\[
2C_1 C_4^2 \left( 1 + \frac{1}{2C_2} \right) 2^M A_3^2 R^{4-2M} \leq \frac{\alpha}{2} k^2 S^2 \\
8C_1 C_2^2 A_4^2 \left( 1 + \frac{1}{2C_2} \right) \leq kS,
\]

or

\[
R \geq \max \left\{ \left( \frac{4C_1 C_2^2 \left( 1 + \frac{1}{2C_2} \right) 2^M A_3^2}{\alpha k^2} \right)^{1/2(M-1)}, \frac{8C_1 C_2}{k} \right\}
\]

then we may absorb the first terms on the right into the lefthand side. Thus, we get

\[
(17) \quad \frac{\alpha}{2} S^2 k^2 \int_{K_1} r^{-2} w^{-2\alpha} |v|^2 \, drd\theta \leq I_2 + I_3,
\]

where \( I_j = C_1 \int_{K_j} r^{-1} w^{-2\alpha} \left| L_{\tilde{k}} f \right|^2 \, drd\theta \) for \( j = 2, 3 \).

Since \( L_{\tilde{k}} f = (L_{\tilde{k}} v) \zeta + (2 \partial_{\zeta} v - 2 \tilde{k} v + \frac{n-1}{r} v) \partial_r \zeta + v \partial_r \zeta \), then

\[
|L_{\tilde{k}} f| \lesssim \begin{cases} 
\left( \frac{S^2}{R^{2M}} + S^2 \right) |v| + \left( \frac{S}{R} + S \right) |\nabla v| & \text{on } K_2 \\
\left( \frac{S^2}{R^{2M}} + \frac{S^2}{R^2} + \frac{S^2}{R} + \frac{S^2}{R^2} \right) |v| + \frac{S}{R} |\nabla v| & \text{on } K_3
\end{cases}
\]

By Caccioppoli,

\[
\int_{K_j} |\nabla v|^2 \lesssim S^2 \int_{K_j} |v|^2
\]

7
Now we look for a lower bound for the term on the left hand side of (17). There exists

\[ I_2 + I_3 \leq \hat{c}_1 C_1 S^4 \max_{K_2} \{ r^{-n} w(r)^{-2\alpha} \} \int_{K_2} |v|^2 + \hat{c}_2 C_1 \frac{S^4}{R^2} \max_{K_2} \{ r^{-n} w(r)^{-2\alpha} \} \int_{K_2^{+}} |v|^2 \]

\[ \leq c_1 C_1 S^{4+n} \exp \left( -2\alpha \frac{n}{Sk} \right) \int_{B_{\frac{3R}{2}(0)}} |u(x_0 + Sx)|^2 \, dx \]

\[ + \hat{c}_2 C_1 S^4 \left( 1 + \frac{R}{4S} \right)^{-n} \exp \left[ -2\alpha \left( 1 + \frac{R}{4S} \right) \right] \int_{[1 + \frac{R}{4S}, 1 + \frac{3R}{4S}]} |u(x_0 + Sx)|^2 \, dx. \]

Since

\[ \int_{[1 + \frac{R}{4S}, 1 + \frac{3R}{4S}]} |u(x_0 + Sx)|^2 \, dx \leq S^{-n} \int_{\left[ \frac{3}{2} S^{2} + \frac{2n}{4} \right]} |u(y)|^2 \, dy \]

\[ = S^{-n} \int_{\left[ \frac{3}{2} S^{2} + \frac{2n}{4} \right]} |f(\theta) + \varphi(r, \theta)|^2 r^{n-1} dr d\theta \]

\[ \leq 10 S^{-n} \int_{\frac{3}{4}}^{\frac{3}{2} S^{2} + \frac{2n}{4}} r^{n-1} dr \quad \text{(by (14) and (15))} \]

\[ \leq \frac{10}{n} 2^n \left( 1 + \frac{1}{C_2} \right)^n, \]

then

\[ I_2 + I_3 \leq c_1 C_1 S^4 \exp \left( -2\alpha \frac{n}{Sk} \right) \int_{B_{\frac{3R}{2}(x_0)}} |u|^2 + c_2 C_1 \frac{S^4}{R^2} \exp \left[ -2\alpha \left( 1 + \frac{R}{4S} \right) \right]. \]

Now we look for a lower bound for the term on the left hand side of (17). There exists \( N(n) \geq 4 \) such that \( A_R \cap B_{R/N}(x_j) \subset B_{\frac{3}{2} S^{2} + \frac{2n}{4}}(x_0) \). If \( R \geq \frac{6Nn}{k(2NC_2 - 1)} \), then \( A_R \cap B_{R/N}(x_j) \subset B_{\frac{3}{2} S^{2} + \frac{2n}{4}}(x_0) \setminus B_{\frac{3n}{4}}(x_0) \). Therefore,

\[ \frac{\alpha}{2} S^2 k^2 \int_{K_2} r^{-2} w^{-2\alpha} |v|^2 \, dr d\theta \]

\[ \geq \frac{\alpha}{2} S^2 k^2 \int_{[\frac{3}{4} S^{2} + \frac{3n}{4}]} r^{-2} w^{-2\alpha} |v|^2 \, dr d\theta \]

\[ \geq \frac{\alpha}{2} S^2 k^2 \left( 1 + \frac{R}{4S} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{R}{4S} \right) \right] \int_{[\frac{3}{4} S^{2} + \frac{3n}{4}]} |u(x_0 + Sx)|^2 \, dx \]

\[ \geq \frac{\alpha}{2} S^2 k^2 \left( 1 + \frac{R}{4S} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{R}{4S} \right) \right] S^{-n} \int_{A_R \cap B_{R/N}(x_j)} |u(y)|^2 \, dy \]

\[ \geq \frac{k^2 C_3^3}{2NC_n} \left( C_2 + \frac{1}{4} \right)^{-(n+1)} \alpha R^2 \exp \left[ -2\alpha \left( 1 + \frac{1}{4C_2} \right) \right]. \]

where the last line follows from condition (16). Combining (19), (17) and (18) gives

\[ \frac{k^2 C_3^3}{2NC_n} \left( C_2 + \frac{1}{4} \right)^{-(n+1)} \alpha R^2 \exp \left[ -2\alpha \left( 1 + \frac{1}{4C_2} \right) \right] \]

\[ \leq c_1 C_1 S^4 \exp \left( -2\alpha \frac{n}{Sk} \right) \int_{B_{\frac{3R}{2}(x_0)}} |u|^2 + c_2 C_1 \frac{S^4}{R^2} \exp \left[ -2\alpha \left( 1 + \frac{R}{4S} \right) \right]. \]
The inequality
\[ c_2 C_1 S^4 \exp \left[ -2\alpha \left( 1 + \frac{R}{4S} \right) \right] \leq \frac{k^2 C_3^2}{4NC_n} (C_2 + \frac{1}{4})^{-(n+1)} \alpha R^2 \exp \left[ -2\alpha \left( 1 + \frac{1}{4C_2} \right) \right] \]

is equivalent to

\[ \alpha \geq \frac{4NC_2 C_1 C_2 C_n}{k^2} (C_2 + \frac{1}{4})^{(n+1)}. \tag{21} \]

If (21) holds, then we may absorb the second term on the right of (20) into the lefthand side to get

\[ c_1 C_1 S^4 \exp \left( -2\alpha \frac{n}{Sk} \right) \int_{B_{3n/(x_0)}} |u|^2 \geq \frac{k^2 C_3^2}{4NC_n} (C_2 + \frac{1}{4})^{-(n+1)} \alpha R^2 \exp \left[ -2\alpha \left( 1 + \frac{1}{4C_2} - \frac{n}{5k} \right) \right] \]

Substituting (21) and simplifying, we get

\[ \int_{B_{3n/(x_0)}} |u|^2 \geq \frac{C_2}{c_1} \exp \left[ -\frac{8NC_2 C_1 C_n}{k^2} (C_2 + \frac{1}{4})^{(n+2)} \right] R^{-2} \]

\[ = C_0 |x_0|^{-2}, \]

where \( C_0 = \frac{c_2 (1+C_2)^2}{c_1} \exp \left[ -\frac{8NC_2 C_1 C_n}{k^2} (C_2 + \frac{1}{4})^{(n+2)} \right]. \]

**Proposition 2.** Assume that conditions \((10)\) - \((13)\) above hold. Let \( C_2 > 0 \) and let \( R \geq \tilde{R} (R_0, C_2, C_5, n, k, A_3, A_4, M) \). Assume also that for some \( x_0 \in \mathbb{R}^n \) with \( |x_0| = (1+C_2) R \), there exists a constant \( D > 0 \) so that

\[ (22) \int_{B_{3n/(x_0)}} |u|^2 \geq C_5 |x_0|^{-D}. \]

If \( y_0 \in \mathbb{R}^n \) is such that \( |y_0| = |x_0| \) and \( |x_0 - y_0| = C_2 R \), then

\[ \int_{B_{3n/(y_0)}} |u|^2 \geq |y_0|^{-E}, \]

where \( E = C_3 D + C_4; C_3 = C_3 (C_2), C_4 = C_4 (C_2, n) \).

The proof of this proposition will be very similar to that of Proposition \[1\].

**Proof of Proposition 2** Let \( y_0 \in \mathbb{R}^n \) be such that \( |y_0| = |x_0| \). Let \( S = \text{dist} \{ x_0, y_0 \} \). By assumption, \( S = C_2 R \) for some \( C_2 \in \mathbb{R} \).

Notice that \( |x_0| = |y_0| = S + R = (1+C_2) R \).

Choose \( K_1, K_2, K_3 \) as in the proof of Proposition \[1\].

Let \( v(x) = u(y_0 + Sx), \tilde{k} = Sk \).

As in the proof of Proposition \[1\] if \( R \) is sufficiently large, we get

\[ \frac{\alpha}{2} S^2 k^2 \int_{K_1} r^{-2} w^{-2\alpha} |v|^2 drd\theta \]

\[ \leq c_1 C_1 S^4 \exp \left( -2\alpha \frac{n}{Sk} \right) \int_{B_{3n/(y_0)}} |u|^2 + c_2 C_1 \frac{S^4}{R^2} \exp \left[ -2\alpha \left( 1 + \frac{R}{4S} \right) \right]. \tag{23} \]
Now we look for a lower bound for the term on the left hand side of (23). If $R \geq \max \left\{ \frac{8n}{\kappa}, \frac{7n}{c_5 \kappa} \right\}$, then $\left[ 1 - \frac{4n}{3\kappa}, 1 + \frac{4n}{3\kappa} \right] \subset K_1$ and

\[
\frac{\alpha}{2} S^2 k^2 \int_{K_1} r^{-2} w^{-2\alpha} |v|^2 \, dr \, d\theta \\
\geq \frac{\alpha}{2} S^2 k^2 \int_{\left[ 1 - \frac{4n}{3\kappa}, 1 + \frac{4n}{3\kappa} \right]} r^{-2} w^{-2\alpha} |v|^2 \, dr \, d\theta \\
\geq \frac{\alpha}{2} S^2 k^2 \left( 1 + \frac{4n}{Sk} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{4n}{Sk} \right) \right] \int_{\left[ 1 - \frac{4n}{3\kappa}, 1 + \frac{4n}{3\kappa} \right]} |u(y_0 + Sx)|^2 \, dx \\
\geq \frac{\alpha}{2} S^2 k^2 \left( 1 + \frac{4n}{Sk} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{4n}{Sk} \right) \right] S^{-n} \int_{B_{\delta_0}(x_0)} |u(y)|^2 \, dy \\
\geq C_5 \frac{\alpha}{2} S^2 k^2 \left( 1 + \frac{4n}{Sk} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{4n}{Sk} \right) \right] S^{-n} |S + R|^{-D}
\]  

(24)

where the last line follows from the hypothesis (22). Combining (24) with (23) gives

\[
C_5 \frac{\alpha}{2} S^2 k^2 \left( 1 + \frac{4n}{Sk} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{4n}{Sk} \right) \right] S^{-n} |S + R|^{-D}
\]

(25)

\[
\leq c_1 C_1 S^4 \exp \left( -2\alpha \frac{n}{Sk} \right) \int_{B_{\delta_0}(y_0)} |u|^2 + c_2 C_1 \frac{S^4}{R^2} \exp \left( -2\alpha \left( 1 + \frac{R}{4S} \right) \right).
\]

If

\[
c_2 C_1 \frac{S^4}{R^2} \exp \left( -2\alpha \left( 1 + \frac{R}{4S} \right) \right) \leq C_5 \frac{\alpha}{4} S^2 k^2 \left( 1 + \frac{4n}{Sk} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{4n}{Sk} \right) \right] S^{-n} |S + R|^{-D}
\]

or

\[
\frac{4c_2 C_1 C_2^{2+n}}{C_5 \left( 1 + C_2 \right)^n k^2} \left( 1 + \frac{4n}{Sk} \right)^{n+1} |S + R|^{n+D} \leq \alpha \exp \left[ \alpha \left( \frac{1 - \frac{16n}{2C_2}}{1 - \frac{16n}{R \kappa}} \right) \right],
\]

(26)

then we may absorb the second term on the right of (25) into the left. Take $\alpha = (n + D) \frac{2C_2}{1 - \frac{16n}{R \kappa}} \log (R + S)$ and

\[
R \geq \left( 1 + C_2 \right)^{-1} \exp \left[ \frac{2c_2 C_1 C_2^{1+n} \left( 1 - \frac{16n}{R \kappa} \right)}{C_5 \left( 1 + C_2 \right)^n k^2 \left( n + D \right)} \left( 1 + \frac{4n}{Sk} \right)^{n+1} \right]
\]

to satisfy (26). Then we may absorb the second term on the right of (25) into the left hand side to get

\[
\int_{B_{\delta_0}(y_0)} |u|^2 \geq C_5 \frac{\alpha}{4} S^2 k^2 \left( 1 + \frac{4n}{Sk} \right)^{-(n+1)} \exp \left[ -2\alpha \left( 1 + \frac{4n}{Sk} \right) \right] S^{-n} |S + R|^{-D}.
\]

Substituting $\alpha = (n + D) \frac{2C_2}{1 - \frac{16n}{R \kappa}} \log (R + S)$ and simplifying, we get

\[
\int_{B_{\delta_0}(y_0)} |u|^2 \geq \frac{C_5 k^2}{4c_1 C_1} \left( 1 + \frac{4n}{Sk} \right)^{-(n+1)} \alpha \exp \left[ -2\alpha \left( 1 + \frac{3n}{Sk} \right) \right] S^{-\left(2+n\right)} |S + R|^{-D}
\]
\[
\geq \frac{C_5 k^2 C_2 (n + D) \log (R + S)}{2c_1 C_1 \left( 1 - \frac{16n}{R \kappa} \right) \left( 1 + \frac{4n}{Sk} \right)^{n+1}} \exp \left[ -\left( 4(n + D) \frac{1 - \frac{16n}{R \kappa}}{1 - \frac{16n}{R \kappa}} + n + D + 2 \right) \log (R + S) \right]
\]
\[
\geq |y_0|^{-\left[(4C_2(1+\varepsilon)+1)(n+D)+2\right]},
\]
where $\varepsilon > 0$, but small and we assumed that $R \geq (1 + C_2)^{-1} \exp \left[ \frac{2\varepsilon_1 C_1 (1 - \frac{16\varepsilon}{\varepsilon^2})}{C_3 k^2 C_2 (n + D)} \left( 1 + \frac{4n}{Sk} \right)^{n+1} \right]$. Let $C_3 = 4C_2 (1 + \varepsilon) + 1$, $C_4 = 2 + [4C_2 (1 + \varepsilon) + 1] n$. Then $E = C_3 D + C_4$. \hfill \Box

We will now combine the propositions above to get an $L^2$ estimate for an arbitrary point that is sufficiently far away from the origin.

**Proposition 3.** Assume that conditions (10)-(15) above hold. Let $C_2 > 0$ and let $R$ be sufficiently large in the sense that $R \geq \max \{ T, \bar{T} \}$. Let $z_0 \in \mathbb{R}^n$ be such that $|z_0| = (1 + C_2) R$. Then

$$
\int_{B_{4n/|z_0|}} |u|^2 \geq |z_0|^{-F},
$$

where $F = F (C_2, n)$.

**Proof.** By Proposition 1 there exists $x_0 \in \mathbb{R}^n$ such that $|x_0| = |z_0|$ and

$$
(27) \quad \int_{B_{4n/|x_0|}} |u|^2 \geq C_0 |x_0|^{-2}.
$$

Let $\{x_0, x_1, \ldots, x_m\} \subset \mathcal{S}_{|z_0|}$ be a collection of elements such that $x_m = z_0$ and $\text{dist} \{x_i, x_j\} = C_2 R$ for $i = 1, \ldots, m$.

If we apply Proposition 2 with (27) as the hypothesis, we get

$$
(28) \quad \int_{B_{4n/|x_1|}} |u|^2 \geq |x_1|^{-D_1},
$$

where $D_1 = 2C_3 + C_4$. If we now apply Proposition 2 with (28) as the hypothesis, we get

$$
\int_{B_{4n/|x_2|}} |u|^2 \geq |x_2|^{-D_2},
$$

where $D_2 = C_3 D_1 + C_4$. Continuing on, we see that

$$
(29) \quad \int_{B_{4n/|z_0|}} |u|^2 \geq |z_0|^{-D_m},
$$

where

$$
D_m = C_3 D_{m-1} + C_4 = 2C_3^m + C_4 (1 + C_3 + \ldots + C_3^{m-1})
$$

Since $\text{dist} \{x_{i-1}, x_i\} = C_2 R$ and $|z_0| = (1 + C_2) R$, then $m$ is a constant that depends on dimension and $C_2$. The result follows. \hfill \Box

**Remark.** By Proposition 1 it is clear that $F \geq 2$. However, determining an upper bound for $F$ is more complicated. For example, if $x_0$ and $z_0$ are antipodal points, then $F$ may increase substantially. Suppose $C_2$ is chosen so that $4C_2 (1 + \varepsilon) = 1$, then $C_3 = 2$, $C_4 = 2 (n + 1)$, $m \approx 5\pi$ and

$$
D_m = 2^{m+1} + 2 (n + 1) (1 + 2 + \ldots + 2^{m-1}) \approx 2^{m+1}.
$$

Thus, $F$ is rather large.
6. Main Result

We are now prepared to prove Theorem 1. We will use the three propositions from the previous section to accomplish this.

Proof. We will begin by showing that \( u(x) := e^{kr} r^{(n-1)/2} w(x) \) satisfies (10)-(15).

As was shown in 2 if \( w \) solves \((1)\), then \( u \) solves \((7)\), which is equivalent to \((10)\).

By (4), \( \bar{V} = V - \frac{a}{r^2} - \left( \frac{k}{2} + \frac{n-1}{2r} \right) W \cdot \frac{x}{r} \), so

\[
\left| \bar{V}(x) \right| \leq \frac{A_1}{|x|^N} + \frac{a}{|x|^2} + \frac{kA_2}{|x|^P} + \frac{A_2^{n-1}}{|x|^{P+1}} \leq \frac{A_3}{|x|^M}
\]

where \( M = \min \{ 2, N, P \} > 1 \), giving \((11)\).

Similarly, by (5), \( \bar{W} = W + \frac{n-1}{r^2} \), so

\[
\left| \bar{W}(x) \right| \leq \frac{A_2}{|x|^P} + \frac{n-1}{|x|^4} \leq \frac{A_4}{|x|^4},
\]

giving \((12)\).

Assumption (9) immediately gives condition \((13)\), while \((14)\) and \((15)\) hold by hypothesis.

Choose \( y_0 \in \mathbb{R}^n \) so that \( M(R, 4n/k) = \| u \|_{L^2(B_{aR}(y_0))} \).

Since \( R \gg 1 \) by assumption, then we may apply Proposition 3 to \( u \) at \( y_0 \) to get that

\[
\int_{B_{aR}(y_0)} |u|^2 \geq |y_0|^{-F}.
\]

Recalling the definition of \( w \) in terms of \( u \),

\[
\int_{B_{aR}(y_0)} \left| e^{kr} r^{(n-1)/2} w(x) \right|^2 \geq |y_0|^{-F}
\]

\[
\Rightarrow \int_{B_{aR}(y_0)} |w(x)|^2 \geq e^{-2k(4n/k) + 4n/k} \left( |y_0| + 4n/k \right)^{-n-1} |y_0|^{-F}
\]

\[
= e^{-8n} \left( 1 + \frac{4n}{|y_0| k} \right)^{1-n} e^{-2k|y_0|} |y_0|^{-F-n+1}
\]

\[
\geq e^{-8n} (1 + C_5)^{1-n} e^{-2k|y_0|} |y_0|^{-F-n+1}.
\]

If we let \( G = \frac{1}{2} (F + n - 1) \), \( C_5 = e^{-4n} (1 + C_6)^{(1-n)/2} \), then the result follows.

\[\square\]

7. Sharpness of the estimate

To establish that Theorem 1 is sharp up to logarithmic factors, we refer to Theorem 2. It is shown that radial functions of the form \( w_m(r) = \exp \left( -kr - \frac{n-1}{2} r + \mathcal{O}(r) \right) \) solve PDEs of type \((1)\), where the decay of \( V \) and \( W \) depends on \( m \). The proof of Theorem 2 proceeds by induction on \( m \) and the details may be found in 2, Lemma 6.3. Given the relationship between estimate (4) and the constructions from Theorem 2 we may interpret Theorem 1 in the following way: On balls of size \( k^{-1} \), solutions to \((1)\) with potentials that decay sufficiently fast behave, in \( L^2 \)-average, like radial solutions.

Acknowledgement: I would like to thank my advisor, Carlos Kenig, for telling me about \([6]\) and suggesting this problem. I am very grateful for his support and encouragement.
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