Beam-Shaping PEC Mirror Phase Corrector Design

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Abstract—The Perfect Electric Conductor (PEC) mirror phase corrector plays an important role in the beam-shaping mirror system design for Quasi-Optical (QO) mode converter (launcher) in the sub-THz high-power gyrotron. In this article, both the Geometry Optical (GO) method and the phase gradient method have been presented for the PEC mirror phase corrector design. The advantages and disadvantages are discussed for both methods. An efficient algorithm has been proposed for the phase gradient method.

I. Introduction

The PEC mirror phase corrector is essential to shape the input beam from the QO mode converter (launcher) into the desired Fundamental Gaussian Beam (FGB) in the sub-THz high-power gyrotron [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Figure 1 shows the diagram of such beam-shaping mirror system, in which the 4 pieces of PEC mirrors (M1, M2, M3 and M4) serve as the phase correctors, aiming at shaping the input beam from the QO launcher into the desired FGB output beam. During the iterative beam-shaping mirror system design [1, 6], phase unwrapping is commonly required in the PEC mirror phase corrector design, which can effectively suppress the edge diffraction due to the discontinuities if otherwise the wrapped phase is used. In this article, both the GO method and the phase gradient method are discussed. An FFT-based efficient algorithm is also proposed to speed up the PEC mirror phase corrector design for the phase gradient method. The time dependence $e^{j\omega t}$ is assumed.

II. The Problem of Phase Unwrapping

The phase correction requires the knowledge of the unwrapped 2-Dimensional (2D) phases of the incident electric field and the reflected electric field ($\theta^i, \theta^r$). However, the phase obtained from the electric field $E$ through $\tilde{\theta} = \arctan\left[\frac{\Im(E)}{\Re(E)}\right]$ (\Re and \Im denote the real part and the imaginary part respectively) is the wrapped 2D phase, which contains discontinuities of $2n\pi$ (n is an integer).
So, in order to ensure the smoothness of the PEC mirror surface, the wrapped 2D phases ($\tilde{\theta}_i$, $\tilde{\theta}_r$) must be unwrapped through the 2D phase unwrapping methods \[21, 22\].

Mathematically, in the ideal situation where there is no residues in the wrapped 2D phase $\tilde{\theta}$, the discreet phase gradient $\nabla \theta = \nabla \tilde{\theta}$ (assuming that $\nabla \theta < \pi$) and the 2D phase unwrapping can be expressed as,

$$\theta = \int_C \nabla \tilde{\theta} \cdot dr + \theta(r_0) \quad (1)$$

where, $\theta$ denotes the 2D unwrapped phase along the integration path $C$ and $r_0$ denotes the starting point of the path integration. Note that the unwrapped phase $\theta$ obtained through (1) should not depend on the integration path $C$. However, due to the residues in practice, the discrete phase gradient should be written as $\nabla \tilde{\theta} = (\nabla g + \nabla \times R)$ and the unwrapped phase $\theta$ is obtained as follows,

$$\theta = \int_C (\nabla g + \nabla \times R) \cdot dr + \theta(r_0) \quad (2)$$

From (2), it can be seen that $\nabla \times R \neq 0$ is caused by the existence of residues and the unwrapped phase $\theta$ depends on the integration path $C$. There are many 2D phase unwrapping algorithms to deal with the residues in the literatures \[21, 22\]. For example, the path following algorithm (e.g., “quality-guided” method and “mask-cut” method) gives faithful congruent unwrapped phase (with $2n\pi$ difference from the wrapped phase). However, path following algorithm is time-consuming and the unwrapped phase contains many discontinuities due to the existence of residues. Another commonly-used algorithm, the minimum norm method unwraps the wrapped phase by minimizing the $r$-norm phase difference between the gradients of the wrapped phase and the desired unwrapped phase \[22\],

$$Q = \sum_{x} \sum_{z} \left[ w_x \left| \frac{\partial \theta}{\partial x} - \frac{\partial \tilde{\theta}}{\partial x} \right| + w_z \left| \frac{\partial \theta}{\partial z} - \frac{\partial \tilde{\theta}}{\partial z} \right| \right] \quad (3)$$

where, $w_x$ and $w_z$ are weights for $\hat{x}$ and $\hat{z}$ directions respectively. When $r = 2$, it is called the Least Mean Square (LMS) method.

### III. The GO Method

In the sub-THz QO regime, it is reasonable to assume that the intensity or magnitude of the electric field is locally constant and the local phase change can be evaluated through the GO approximation:

$$\alpha' = \alpha$$

$$\delta \hat{y} = \frac{\theta^r - \theta^i}{2k[\cos \alpha']} (\hat{y} \cdot \hat{n})$$

$$\delta \hat{n} = \frac{\theta^r - \theta^i}{2k[\cos \alpha']}$$

Figure 2: The PEC mirror surface correction in the sub-THz QO regime: $k^i$ and $k^r$ are wave vectors for the local incident beam (with incident angle $\alpha^i$) and the local reflection beam (with reflected angle $\alpha^r$). $\delta \hat{n}$ is the PEC mirror surface correction in $\hat{n}$ direction and $\delta \hat{y}$ is the PEC mirror surface correction in $\hat{y}$ direction.
For fixed computational grid given on x-z plane (in favor of FFT operation), $\delta_y$ is preferred, which is rewritten as follows ($\cos \alpha_i = \frac{k_i \cdot \hat{n}}{k}$),

$$
\delta_y = \frac{\delta \theta}{2 (k_i \cdot \hat{n}) (\hat{y} \cdot \hat{n})}, \quad \delta \theta = \theta^r - \theta^i
$$

There are two approaches to calculate the local wave vector $k$ (incident wave vector $k^i$ and reflected wave vector $k^r$), i.e., 1) the Poynting vector approach; and 2) the phase gradient approach. The Poynting vector approach assumes that the local beam propagates in the direction given by the Poynting vector,

$$
k \propto E \times (H)^* \propto E \times \nabla \times (E)^* \tag{5}
$$

The phase gradient approach approximates the local wave vector as the gradient of the phase,

$$
k \propto \nabla \theta \tag{6}
$$

It is not difficult to show that the two approaches are equivalent in the far-field limit.

### IV. The Phase Gradient Method

Instead of (4), the expression of the PEC mirror surface correction $\delta_y$ in the phase gradient method is given as

$$
\delta_y = \frac{\delta \theta}{\nabla (\delta \theta)} = \frac{\delta \theta}{\nabla \theta^r - \nabla \theta^i} \tag{7}
$$

The phase gradient $\nabla \theta$ for the electric field $E = |E| e^{i\theta}$ can be found as

$$
\nabla E = \nabla \left\{ |E| e^{i\theta} \right\} = \nabla \{ |E| \} e^{i\theta} + |E| \nabla \left\{ e^{i\theta} \right\}
\quad = \left[ \nabla \{ |E| \} + i |E| \nabla \theta \right] e^{i\theta}
$$

$$
\rightarrow \nabla \theta = \Im \left[ \frac{\nabla E}{E} \right] = \Im \left[ \nabla \ln E \right] \tag{8}
$$

$$
\rightarrow \nabla \{ |E| \} = \Re \left[ \nabla E e^{-i\theta} \right] \tag{9}
$$

From (8) and (9), the expression for $\nabla (\delta \theta)$ in (7) is obtained,

$$
\nabla (\delta \theta) = \Im \left[ \frac{\nabla E^r}{E^r} - \frac{\nabla E^i}{E^i} \right] = \Im \left[ \nabla \ln \frac{E^r}{E^i} \right] \tag{10}
$$

### V. An Efficient Algorithm for Phase Gradient Method

By slicing the PEC mirror phase corrector into many subdomains, as shown in Fig. 3, the FFT can be used [3, 6] to compute the electric field $E$ and it’s derivatives,

$$
E(y) = \text{IFT} \left\{ F(k_x, k_z) e^{-ik_y y} \right\}, \quad F(k_x, k_z) = \text{FT} \left\{ E(y = 0) \right\} \tag{11}
$$
Figure 3: Illustration of the FFT-based efficient algorithm for the phase gradient method. $E^i$ and $E^r$ are the incident electric field and reflected electric field respectively. $S$ is the PEC mirror phase corrector. $y$ is the coordinate of the PEC mirror phase corrector and $y_r$ is the coordinate of the slicing reference plane. $\delta$ is the spacing between two adjacent slicing reference planes.

$$\frac{\partial E(y)}{\partial v} = \text{IFT} \left\{ -ik_v F(k_x, k_z)e^{-ik_v y} \right\}, \quad v = x, z$$

where, the Fourier Transform (FT) and the Inverse Fourier Transform (IFT) are defined as follows,

$$\text{FT} \{ \cdot \} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ik_x x} \int_{-\infty}^{\infty} \{ \cdot \} e^{ik_z z} dz$$

$$\text{IFT} \{ \cdot \} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_x e^{-ik_x x} \int_{-\infty}^{\infty} \{ \cdot \} e^{-ik_z z} dk_z$$

The wrapped phase difference $\hat{\delta} \theta_x = (\hat{\theta}_x - \hat{\theta}_r)$ is obtained from (11). Due to similarity, only x-component $\hat{x}E_x$ is considered here,

$$\delta \hat{\theta}_x = \arctan \left[ \frac{\Im \left( \text{IFT} \left\{ F^r_x(k_x, k_z)e^{-ik_z y} \right\} \right)}{\Re \left( \text{IFT} \left\{ F^i_x(k_x, k_z)e^{-ik_z y} \right\} \right)} \right] - \arctan \left[ \frac{\Im \left( \text{IFT} \left\{ F^i_x(k_x, k_z)e^{-ik_z y} \right\} \right)}{\Re \left( \text{IFT} \left\{ F^r_x(k_x, k_z)e^{-ik_z y} \right\} \right)} \right]$$

With the help of (11)-(14), the gradient of the phase difference $\nabla (\delta \theta_x)$ on the slicing reference plane $y_r$ in Fig. 3 can be obtained from (10),

$$\nabla (\delta \theta_x) = \nabla (\delta \hat{\theta}_x) = \Re \left[ \text{IFT} \left\{ k F^r_x(k_x, k_z)e^{-ik_z y} \right\} \right] - \Re \left[ \text{IFT} \left\{ k F^i_x(k_x, k_z)e^{-ik_z y} \right\} \right]$$

To obtain the PEC mirror surface correction $\hat{\delta} \hat{y}$ through (7), $\delta \hat{\theta}_x$ has to be unwrapped. Here, an FFT-based phase unwrapping algorithm is presented for the $r$-norm minimum problem given in (3). Suppose that $\delta \theta_x$ can be expressed in the Fourier series,
\[ \delta \theta_x = \text{IFT}\{f(k_x, k_z)\} \]  

(17)

Then,  
\[ \frac{\partial (\delta \theta_x)}{\partial v} = \text{IFT}\{-ik_v f(k_x, k_z)\}, \quad v = x, z \]  

(18)

To obtain \( \delta \theta_x \), the Fourier coefficient \( f(k_x, k_z) \) is chosen to minimize the cost function \( Q \) given in (3), with \( w_x = w_z = 1 \). For LMS method where \( r = 2 \), it can be shown that \( f(k_x, k_z) \) takes the following form,

\[ Q(f + \delta f) - Q(f) \delta f = 0 \rightarrow f(k_x, k_z) = \frac{k_x \text{FT} \{ \nabla (\delta \theta_x) \cdot \hat{x} \} + k_z \text{FT} \{ \nabla (\delta \theta_x) \cdot \hat{z} \}}{k_x^2 + k_z^2} \]  

(19)

Now, the PEC mirror surface correction \( \delta \hat{y} \) can be obtained from (7), with the help of (16)-(19).

VI. Discussion

It has been shown that both the GO method and the phase gradient method can be used in the PEC mirror phase corrector design. Both methods have their advantages and disadvantages, e.g., the GO method is simple and easy to use, but it is time-consuming; the phase gradient method is efficient (due to the use of FFT), but its application is limited by the sampling theorem. In general, the GO method is suitable for problems of significant side lobes; and the phase gradient method is suitable for problems of smooth phase front with negligible side lobes.

VII. Conclusion

In this article, both the GO method and the phase gradient method have been presented for the PEC mirror phase corrector design. The FFT-based efficient algorithm has been proposed for the phase gradient method to speed up the design procedure.

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