Abstract. Agol has conjectured that minimally twisted $n$–chain links are the smallest volume hyperbolic manifolds with $n$ cusps, for $n \leq 10$. In his thesis, Venzke mentions that these cannot be smallest volume for $n \geq 11$, but does not provide a proof. In this paper, we give a proof of Venzke’s statement for a number of cases. For $n \geq 60$ we use a formula from work of Futer, Kalfagianni, and Purcell to obtain a lower bound for volume. The proof for $n$ between 12 and 25 inclusive uses a rigorous computer computation that follows methods of Moser and Milley. Finally, we prove that the $n$–chain link with $2m$ or $2m + 1$ half–twists cannot be the minimal volume hyperbolic manifold with $n$ cusps, provided $n \geq 60$ or $|m| \geq 8$, and we give computational data indicating this remains true for smaller $n$ and $|m|$.

1. Introduction

An $n$–chain link consists of $n$ unknotted circles embedded in $S^3$, linked together in a closed chain. Notice that links of a chain can be connected with an arbitrary amount of twisting. In particular, if we embed the first link in the plane of projection, the next perpendicular to the plane of projection, the next again in the plane of projection, and so on, then the last link may include any integer number of half–twists. See, for example, Figure 1.

Hyperbolic structures on $n$–chain link complements have been studied, for example, by Neumann and Reid [8]. They show any $n$–chain link complement with $n \geq 5$ admits a hyperbolic structure. In this paper, we are primarily interested in hyperbolic manifolds, so we restrict our attention to $n \geq 5$.

A minimally twisted $n$–chain link is an $n$–chain link such that, if $n$ is even, each link component alternates between lying embedded in the projection plane and lying perpendicular to the projection plane. If $n$ is odd, the link may be arranged such that each component alternates between lying in the projection plane and perpendicular to it, except a single link component which connects a link which is embedded in the projection plane to one which is perpendicular, with no twisting. See Figure 1.

Notice that there are actually two choices for the minimally twisted $n$–chain link for $n$ odd, depending on which way the last links are connected. However, these are isometric by an orientation reversing isometry, so we will not distinguish between them.
In [1], Agol conjectures that minimally twisted \( n \)-chain link complements, for \( n \leq 10 \), are the smallest volume hyperbolic 3–manifolds with exactly \( n \) cusps, but notes that Venzke has pointed out they cannot be smallest for \( n \geq 11 \), as the \((n - 1)\)-fold cyclic cover over one component of the Whitehead link has smaller volume. This statement is included in Venzke’s thesis [10]. However, Venzke does not give a proof. In this paper, we give a rigorous proof for \( n \geq 60 \). The following theorem is the main result of this paper.

**Theorem 3.3.** For \( n \geq 60 \), the minimally twisted \( n \)-chain link complement has volume strictly greater than that of the \((n - 1)\)-fold cyclic cover over one component of the Whitehead link. Hence the minimally twisted \( n \)-chain link complement cannot be the smallest volume hyperbolic 3–manifold with \( n \) cusps, \( n \geq 60 \).

For \( n \) between 11 and 59, inclusive, we present computer tabulation of volumes, compared with the volumes of the \((n - 1)\)-fold cyclic cover of the Whitehead link. See Table 1. By inspection, Theorem 3.3 also holds for these manifolds. In Section 4, we explain how these computations can be made completely rigorous — at least for those values of \( n \) for which the minimally twisted \( n \)-chain link complement is triangulated with fewer than 100 tetrahedra. In this case, this includes \( n \) between 12 and 25, inclusive.

Finally, in Section 5 we evaluate volumes of arbitrarily twisted \( n \)-chain link complements. The main result of that section is Theorem 5.3, which states that no \( n \)-chain link complement can be the minimal volume \( n \)-cusped hyperbolic 3–manifold, provided either \( n \geq 60 \), or the chain link contains at least 17 half–twists. We present computational data to show that similarly, for \( 11 \leq n \leq 59 \), no \( n \)-chain link complement can be minimal volume. When \( 5 \leq n \leq 10 \), we rigorously prove, by computer, that no \( n \)-chain link which is not minimally twisted can be the minimal volume \( n \)-cusped hyperbolic 3–manifold.

**Remark 1.1.** Since a version of this paper was made public, Hidetoshi Masai has pointed out to us that for even chain links, more can be said. In [9, Chapter 6], Thurston finds a formula for the volumes of minimally twisted chain links with an even number of link components. Namely,

\[
\text{vol}(S^3 \setminus C_{2n}) = 8n \left( \frac{\Lambda}{4} + \frac{\Lambda}{2n} \right) \left( \frac{\pi}{4} + \frac{\pi}{2n} \right),
\]

where \( \Lambda \) is the Lobachevsky function. Masai notes that the difference of this volume and the volume of the \((2n - 1)\)-cyclic cover over a component of the Whitehead link is an increasing function in \( n \), for \( n \geq 6 \). This result will give a rigorous proof that minimally twisted \( n \)-chain links are not minimal volume for 17 additional chain links, namely for \( n = 26, 28, 30, \ldots, 56, 58 \). In addition, this gives an alternate proof that minimally twisted \( 2n \)-chain links are not minimal volume for larger \( n \). However, the result for odd links requires other techniques, for instance those in this paper.

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2. **Slope lengths on covers**

To prove the main theorems of this paper, for \( n \geq 60 \), we will obtain the complement of the minimally twisted \( n \)-chain link by Dehn filling a manifold \( \widehat{W}_n \) which is geometrically explicit,
constructed by gluing together manifolds isometric to the Whitehead link complement, cut along 2–punctured disks.

We work with the diagram of the Whitehead link as in Figure 2 left, with a link component denoted $K$. Note that by switching the direction of a pair of crossings, we obtain a link whose complement is isometric to that of the Whitehead link complement by an orientation reversing isometry. The isometry takes $K$ to a link component we denote $\overline{K}$, as in Figure 2 right.

**Lemma 2.1.** The shape of the cusp of $K$ is a parallelogram with meridian and longitude meeting at angle $-\pi/4$ (measured from meridian to longitude).

Similarly, the shape of the cusp of $\overline{K}$ is a parallelogram with meridian and longitude meeting at angle $\pi/4$ (measured from meridian to longitude).

When we take a maximal horocusp about $K$ or $\overline{K}$, the meridian has length $\sqrt{2}$, and the longitude has length 4.

Lemma 2.1 is illustrated in Figure 3.

**Proof.** The first part of the lemma, and lengths of slopes on $K$, are well known for the Whitehead link. See, for example [8].

As for $\overline{K}$, the orientation reversing isometry taking the Whitehead link to its reflection takes a meridian of $K$ to a meridian of $\overline{K}$, and reflects the longitude. Since this is an isometry, the meridian and longitude of $\overline{K}$ have the same lengths as those of $K$, but the angle from the meridian to longitude is reflected across the meridian, to be $\pi/4$. $\square$

Now, the manifold $\hat{W}_n$ can be described as the minimally twisted $n$–chain link embedded in a standard solid torus. Therefore, to obtain the complement of the minimally twisted $n$–chain link, we will Dehn fill $\hat{W}_n$ along a standard longitude of the solid torus boundary component.

To build $\hat{W}_n$ from the Whitehead link complement, proceed as follows. First, cut the Whitehead link complement along the 2–punctured disk bounded by $K$ to get a clasp in a cylinder, which we call $W_1$, shown second from left in Figure 4. Similarly, cut the reflected
Whitehead link complement along the 2–punctured disk bounded by $K$ to get a clasp in the opposite direction in a cylinder, which we call $W_1$, shown third from left in Figure 4. In particular, boundary components are glued as shown without twisting. Call the resulting link in a solid cylinder $W$.

For $n$ even, glue $n/2$ copies of $W$ together end to end, without twisting, followed by gluing the remaining two ends. For $n$ odd, glue $(n - 1)/2$ copies of $W$ together without twisting, then glue a single copy of $W_1$, and attach the ends without twisting. This completes the construction of $\hat{W}_n$.

**Lemma 2.2.** Let $\epsilon = n \mod 2$. The minimally twisted $n$–chain link in a solid torus, $\hat{W}_n$, has solid torus boundary component comprised of $\lfloor n/2 \rfloor + \epsilon$ copies of the cusp $K$ coming from $W_1$, and $\lfloor n/2 \rfloor$ copies of the cusp $\overline{K}$ coming from $\overline{W}_1$. The standard longitude of the solid torus follows a meridian of each copy of $K$ and $\overline{K}$, where the meridians of each copy of $K$ are orthogonal to the meridians of each copy of $\overline{K}$. The length of the longitude of the solid torus boundary component is $\sqrt{n^2 + \epsilon}$.

**Proof.** This follows from the construction of $\overline{W}_n$ and Lemma 2.1.

The boundary component corresponding to the solid torus comes from $n/2$ copies of the cusp $K$ and $n/2$ copies of the cusp $\overline{K}$ for $n$ even, and $(n - 1)/2 + 1$ copies of the cusp $K$ and $(n - 1)/2$ copies of the cusp $\overline{K}$, for $n$ odd. These are glued together along their respective longitudes. Since a copy of the cusp of $K$ is glued to one of $\overline{K}$ along the longitude of each, the meridians meet at right angles. See Figure 5.

The longitude of the solid torus of $\overline{W}_n$ is given by following each meridian of the copies of $K$ and $\overline{K}$ that glue to give the solid torus boundary component. Since these meridians always meet at right angles, the length of the longitude of the solid torus may be determined by the Pythagorean theorem. By Lemma 2.1, the length of the meridian of the cusp $K$, and that of the cusp $\overline{K}$, is $\sqrt{2}$. We see that the length of the longitude of the solid torus boundary component of $\overline{W}_n$ is $\sqrt{\left(\sqrt{2} \left(\lfloor n/2 \rfloor + \epsilon \right)\right)^2 + \left(\sqrt{2} \left\lfloor n/2 \right\rfloor\right)^2} = \sqrt{n^2 + \epsilon}$. \hfill \Box

Notice that while the construction of $\overline{W}_n$ as described above uses $\lfloor n/2 \rfloor + \epsilon$ copies of $W_1$ and $\lfloor n/2 \rfloor$ copies of $\overline{W}_1$, we could have constructed it using $\lfloor n/2 \rfloor$ copies of $W_1$ and $\lfloor n/2 \rfloor + \epsilon$
copies of \( \hat{W}_1 \) instead. The result of this modified method of construction would be isometric to that of the original construction, by an orientation reversing isometry.

To obtain \( C_n \) from \( \hat{W}_n \) we Dehn fill a slope on the solid torus boundary component of \( \hat{W}_n \) that follows one standard longitude of the solid torus. Hence we Dehn fill along a slope of length \( \sqrt{n^2 + (n \mod 2)} \).

3. Volumes, large minimally twisted chains

Using the information on slopes above, we may deduce geometric information on minimally twisted chain links by applying appropriate theorems bounding change in geometry under Dehn filling. This will give the desired result when \( n \geq 60 \).

3.1. Dehn filling and volume. We use the following theorem, which is a slightly simpler version of the main theorem in [4].

**Theorem 3.1** (Futer–Kalfagianni–Purcell [4]). Let \( M \) be a complete, finite–volume hyperbolic manifold with (at least one) cusp, with horoball neighborhood \( C \) about that cusp, and let \( s \) be a slope on \( \partial C \) with length \( \ell(s) > 2\pi \). Then the manifold \( M(s) \) obtained by Dehn filling \( M \) along \( s \) is hyperbolic, with volume:

\[
\text{vol}(M(s)) \geq \left( 1 - \left( \frac{2\pi}{\ell(s)} \right)^2 \right)^{3/2} \text{vol}(M).
\]

Putting this theorem together with Lemma 2.2 we obtain the following.

**Theorem 3.2.** For \( n \geq 7 \), the volume of the complement of the minimally twisted \( n \)-chain link \( C_n \) is at least:

\[
\text{vol}(S^3 \setminus C_n) \geq n v_8 \left( 1 - \frac{4\pi^2}{n^2 + \epsilon} \right)^{3/2},
\]

where \( v_8 = 3.66386 \ldots \) is the volume of a hyperbolic regular ideal octahedron, and \( \epsilon = n \mod 2 \).

**Proof.** The chain link complement is obtained by Dehn filling the longitude of \( \hat{W}_n \). This is obtained by gluing copies of the Whitehead link and its reflection along totally geodesic 3–punctured spheres, hence has volume \( n \) times the volume of the Whitehead link, \( n \cdot v_8 \).

By Lemma 2.2 we know the length of the Dehn filling slope is \( \sqrt{n^2 + \epsilon} \). The result follows by putting this data into Theorem 3.1. \( \square \)
We now give a proof of the main theorem.

**Theorem 3.3.** For \( n \geq 60 \), the minimally twisted \( n \)-chain link complement has volume strictly greater than that of the \((n-1)\)-fold cyclic cover over one component of the Whitehead link. Hence the minimally twisted \( n \)-chain link complement cannot be the smallest volume hyperbolic \( \mathcal{3} \)-manifold with \( n \) cusps, \( n \geq 60 \).

**Proof.** The volume of the \((n-1)\)-fold cyclic cover of the Whitehead link is \((n-1)v_8\). By Theorem 3.2, the volume of the complement of the minimally twisted \( n \)-chain link is

\[
\text{vol}(S^3 \setminus C_n) \geq n v_8 \left(1 - \frac{4\pi^2}{n^2 + \epsilon}\right)^{3/2} \geq n v_8 \left(1 - \frac{4\pi^2}{n^2}\right)^{3/2}.
\]

We want to find \( n \) for which the following inequality holds:

\[
n v_8 \left(1 - \frac{4\pi^2}{n^2}\right)^{3/2} > (n-1) v_8,
\]

or

\[
\left(\frac{n}{n-1}\right) \left(1 - \frac{4\pi^2}{n^2}\right)^{3/2} - 1 > 0.
\]

Let \( f(n) \) be the function on the left side of inequality (1). Using calculus, one sees that \( \lim_{n \to \infty} f(n) = 0 \), \( f \) is increasing between \( n = 7 \) and \( n = 6\pi^2 + 2\pi\sqrt{9\pi^2 - 2} \approx 117.8 \), and decreasing for larger \( n \), which implies \( f \) has at most one root for \( n \geq 7 \), and that \( f \) is positive to the right of any root. The Intermediate Value Theorem implies that there is a root between \( n = 59 \) and \( n = 59.1 \). Hence the inequality is satisfied for \( n \geq 60 \). \( \Box \)

4. **Computations of volume, smaller minimally twisted chains**

Now we analyze volumes of minimally twisted \( n \)-chain links for \( n \) between 11 and 59, since the main method of proof of Theorem 3.3 will not apply to these manifolds.

For \( n \) between 11 and 59 inclusive, in Table 1 we present computational data using SnapPea (SnapPy) [11, 3] that shows that the minimally twisted \( n \)-chain link complement cannot be the minimal volume hyperbolic \( 3 \)-manifold with \( n \) cusps. In particular, \( W_{n-1} \), the \((n-1)\)-fold cyclic cover over one component of the Whitehead link, has smaller volume. The volume of \( W_{n-1} \) is always \((n-1)v_8\), where \( v_8 = 3.66386 \ldots \) is the volume of a hyperbolic regular ideal octahedron, which is the volume of the Whitehead link complement. Notice that for \( n \geq 11 \), the volume of \( S^3 \setminus C_n \) is strictly larger than that of \( W_{n-1} \).

It would be nice to turn this data into a rigorous proof that the minimally twisted \( n \)-chain links for \( n \) between 11 and 59 cannot be minimal volume. One way to do this would be to use the methods of Moser [7] and Milley in [6]. Milley has written a program to rigorously prove that a hyperbolic \( 3 \)-manifold with hyperbolic structure computed by Snap [5] has volume greater than some constant. This program, which is available as supplementary material with [6], is in theory exactly what we need for these chain link examples.

However, in practice, making Moser and Milley’s programs work with the chain links has proven to be difficult, due to the computational complexity of the chain links. While Milley worked with small manifolds, for example with less than 10 tetrahedra, and Moser’s largest manifold included 57 tetrahedra, our triangulations of minimally twisted \( n \)-chain link complements include between 40 and 236 tetrahedra. We were successfully able to run Moser’s algorithm for \( n \) between 11 and 25, inclusive, which gives results for those manifolds triangulated with up to 100 tetrahedra, but then the program failed. We were able to run...
Table 1. Volumes of the complement of the minimally twisted $n$–chain link $C_n$, compared to volumes of $W_{n−1}$, the $(n−1)$–fold cyclic cover over a component of the Whitehead link, for $5 \leq n \leq 60$. Note that $S^3\setminus C_n$ has greater volume for $n \geq 11$.

\begin{tabular}{|c|c|c|}
\hline
$n$ & $\text{vol}(S^3\setminus C_n)$ & $\text{vol}(W_{n−1})$
\hline
5 & 10.14941606 & 14.65544951 \\
6 & 14.6544951 & 18.31931188 \\
7 & 19.7965462 & 21.98317426 \\
8 & 24.09218408 & 25.64703664 \\
9 & 28.47566906 & 29.31089901 \\
10 & 32.55154031 & 32.97476139 \\
11 & 36.64918655 & 36.63862377 \\
12 & 40.59766426 & 40.30248614 \\
13 & 44.5536682 & 43.96634852 \\
14 & 48.42519197 & 47.63021090 \\
15 & 52.29990219 & 51.29407327 \\
16 & 56.12184477 & 54.95793565 \\
17 & 59.94533184 & 58.62179803 \\
18 & 63.7354269 & 62.8566041 \\
19 & 67.5257845 & 66.9452278 \\
20 & 71.28681886 & 69.6338516 \\
21 & 75.05153335 & 73.27724753 \\
22 & 78.708638 & 76.94110991 \\
23 & 82.45919211 & 80.60497229 \\
24 & 86.27825885 & 84.28683466 \\
25 & 90.0118157 & 87.93290704 \\
26 & 93.73455871 & 91.59655942 \\
27 & 97.45755771 & 95.26042179 \\
28 & 101.1722364 & 98.24284165 \\
29 & 104.8869804 & 102.58248615 \\
30 & 108.60599062 & 106.2520089 \\
31 & 112.3032167 & 109.9158713 \\
32 & 116.0059062 & 113.5797337 \\
33 & 119.708638 & 117.2435961 \\
34 & 123.4068675 & 120.9074584 \\
35 & 127.1051279 & 124.5713208 \\
36 & 130.7996249 & 128.2351832 \\
37 & 134.494145 & 131.8990456 \\
38 & 138.1854868 & 135.5629079 \\
39 & 141.8768462 & 139.2267703 \\
40 & 145.5654969 & 142.8906327 \\
41 & 149.2541611 & 146.5544951 \\
42 & 152.9404979 & 150.2183574 \\
43 & 156.6268643 & 153.8822198 \\
44 & 160.3111779 & 157.5460822 \\
45 & 163.995519 & 161.2099446 \\
46 & 167.6781044 & 164.8738070 \\
47 & 171.3606965 & 168.5376693 \\
48 & 175.0417493 & 172.2015317 \\
49 & 178.7228075 & 175.8653941 \\
50 & 182.4025087 & 179.5292565 \\
51 & 186.0822143 & 183.1931188 \\
52 & 189.7607173 & 186.8569812 \\
53 & 193.4392239 & 190.5208436 \\
54 & 197.1166599 & 194.1847060 \\
55 & 200.7940988 & 197.8485683 \\
56 & 204.4705802 & 201.5460822 \\
57 & 208.1470642 & 205.1762931 \\
58 & 211.8226885 & 208.8401555 \\
59 & 215.4983149 & 212.5040178 \\
60 & 219.1731666 & 216.1678802 \\
\hline
\end{tabular}

Milley’s algorithm for all values of $n$ for which Moser’s algorithm applied. However, Milley’s algorithm only returned a positive result for $n$ between 12 and 25, inclusive. Therefore, we have the following result.

**Theorem 4.1.** For $n$ between 12 and 25, inclusive, the minimally twisted $n$–chain link complement has volume strictly greater than that of the $(n−1)$–fold cyclic cover over one component of the Whitehead link, hence cannot be the smallest volume hyperbolic 3–manifold with $n$ cusps.

**Proof.** The proof is identical to that of Milley [6], and uses his code, included as supplementary material with that reference [6], modified to read in minimally twisted $n$–chain links rather than Dehn fillings of census manifolds. The first step is to feed the triangulations of the minimally twisted $n$–chain links into Snap, and ensure that the triangulations used in the computation are geometric, that is, all tetrahedra are positively oriented. This is true for all minimally twisted $n$–chain links, $11 \leq n \leq 59$.

Next, use Moser’s algorithm [7] to find a value $δ$ which measures the maximal error between Snap’s computed solution and the true solution. Moser’s algorithm gave us such a value for
11 \leq n \leq 25, but failed thereafter, presumably due to computational complexity of the chain link complements.

Finally, for each $n$ between 12 and 25, inclusive, input the Snap triangulation data and Moser’s value $\delta$ into Milley’s program `rigorous_volume.C`, along with the constant value $(n - 1) \times 3.66386237670888$. The program checks rigorously whether the volume of the given $n$–chain link is larger than the given constant. For $12 \leq n \leq 25$, the program definitively proved that the volumes of the minimally twisted $n$–chain link complement are strictly larger than that of the $(n - 1)$–fold cyclic cover over a component of the Whitehead link. \hfill \Box

**Remark 4.2.** Note that the above theorem does not hold for $n = 11$. Although the triangulation of the minimally twisted 11–chain link is positively oriented, and Moser’s algorithm returns a value of $\delta$ for this link, Milley’s program `rigorous_volume.C` is unable to verify that its volume is larger than that of the 10–fold cyclic cover of the Whitehead link. When $n = 11$, the volumes of these manifolds are too close for rigorous checking.

What about the volumes output by SnapPea for $26 \leq n \leq 59$? Note in Table 1 that the minimally twisted $n$–chain link for these $n$ has volume greater than 2 plus the volume of $W_{n-1}$. It is highly unlikely that SnapPea’s computation would be so far off as to make the theorem untrue for any of these values of $n$. However, since we do not have a rigorous proof at this time, we do not include the result as a theorem.

## 5. Arbitrary Chain Links

In this section, we extend our results to chain links with an arbitrary amount of twisting.

Consider again the manifold $\overleftarrow{W}_n$, which is a minimally twisted $n$–chain link in a solid torus. Let $\lambda_n$ denote the standard longitude of the solid torus, and let $\mu_n$ denote the meridian. In the previous section, we performed Dehn filling along the slope $\lambda_n$ to obtain the complement of the minimally twisted $n$–chain link in $S^3$. Notice that Dehn filling along any slope of the form $\lambda_n + m \mu_n$ will also yield the complement of a chain link in $S^3$, where the resultant chain has $|m|$ additional full twists (or $2|m|$ additional crossings). The twisting will be positive or negative depending on the sign of $m$.

**Lemma 5.1.** For any integer $m$, the slope $\lambda_n + m \mu_n$ on the solid torus boundary component of $\overleftarrow{W}_n$ has length $\sqrt{n^2 + 16m^2 + (n \mod 2)(1 + 8m)}$.

**Proof.** The solid torus boundary component of $\overleftarrow{W}_n$ is tiled by regular ideal octahedra coming from the Whitehead link, and these appear as squares of side length $\sqrt{2}$ by Lemma 2.1. When we place the corner of one such square at $(0, 0)$ in the Euclidean plane, we see that the slope $\lambda_n$ runs from $(0, 0)$ to $(\sqrt{2}([n/2] + (n \mod 2)), \sqrt{2}[n/2])$, as in Lemma 2.2.

The slope $\mu_n$ runs along exactly one of the 2–punctured disks we sliced in the Whitehead link complement (or its reflection) to build $\overleftarrow{W}_n$. Hence by Lemma 2.1, it runs from $(0, 0)$ to $(2\sqrt{2}, -2\sqrt{2})$.

Thus the slope $\lambda_n + m \mu_n$ runs from $(0, 0)$ to $(\sqrt{2}([n/2] + (n \mod 2) + 2m, \sqrt{2}([n/2] - 2m)$, hence has length as follows.

For $n$ even, $n = 2k$,

$$\ell(\lambda_n + m \mu_n) = (2(k + 2m)^2 + 2(k - 2m)^2)^{1/2} = \sqrt{n^2 + 16m^2}. $$

For $n$ odd, $n = 2k + 1$,

$$\ell(\lambda_n + m \mu_n) = (2(k + 1 + 2m)^2 + 2(k - 2m)^2) = \sqrt{n^2 + 16m^2 + (1 + 8m)}. $$

\hfill \Box
In order to obtain any $n$–chain link, in addition to considering Dehn filling on the manifold $\hat{W}_n$, we must also consider Dehn filling on an $n$–chain link in the solid torus which differs from $\hat{W}_n$ by the insertion of a single crossing, or half–twist, at a 2–punctured disk. We call this manifold $\overline{W}_n$. Recall that we constructed $\hat{W}_n$ by gluing together alternating copies of $W_1$ and $\widehat{W}_1$ along their 2–punctured disk boundaries, without twisting, to form a link in a solid cylinder, and then gluing the cylinder end to end without twisting. To form $\overline{W}_n$, we may glue by a single half–twist when we connect the final solid cylinder end to end. Equivalently, if $n$ is even, replace the last copy of $\overline{W}_1$ with $W_1$ and glue end to end without twisting. If $n$ is odd, replace the last $W_1$ with $\overline{W}_1$ and glue end to end without twisting. This gives the desired half–twist in both cases.

Now, denote the standard longitude of the solid torus boundary component of $\overline{W}_n$ by $\lambda_n$, and the meridian by $\mu_n$. Dehn filling along a slope of the form $\lambda_n + m \mu_n$ yields the complement of an $n$–chain link in $S^3$, which differs from the minimally twisted $n$–chain link by the insertion of $2m+1$ half–twists, where the direction of half–twist is determined by the sign of $m$.

**Lemma 5.2.** For any integer $m$, the slope $\lambda_n + m \mu_n$ on the solid torus boundary component of $\hat{W}_n$ has length $\sqrt{n^2 + 4(1 + 2m)^2}$, if $n$ is even, and $\sqrt{n^2 + 16m^2 + (1 - 8m)}$, if $n$ is odd.

*Proof.* Again the solid torus boundary component of $\overline{W}_n$ is tiled by squares of side length $\sqrt{2}$, by Lemma 2.1, which we view with sides parallel to the $x$ and $y$ axes in the Euclidean plane. The slope $\mu_n$ still runs once along a 2–punctured disk, which came from $W_1$ or $\overline{W}_1$, hence runs from $(0, 0)$ to $(2\sqrt{2}, -2\sqrt{2})$, on the Euclidean plane, also by Lemma 2.1.

First suppose $n$ is even, $n = 2k$. The slope $\lambda_n$ will be formed by stepping $k + 1$ times horizontally (following the meridian of $K$ in the cusp tiling), and stepping $k - 1$ times vertically (following the meridian of $\overline{K}$). Hence it runs from $(0, 0)$ to $(\sqrt{2}(k + 1), \sqrt{2}(k - 1))$.

Thus in the case $n = 2k$, $\lambda_n + m \mu_n$ runs from $(0, 0)$ to $(\sqrt{2}(k + 2m), \sqrt{2}(k - 1 - 2m))$, so has length

$$ (2(k + 1 + 2m)^2 + 2(k - 1 - 2m)^2)^{1/2} = \sqrt{n^2 + 4(1 + 2m)^2}. $$

When $n$ is odd, $n = 2k + 1$, the slope $\lambda_n$ is formed by stepping $k$ times horizontally and $k + 1$ times vertically, hence runs from $(0, 0)$ to $(\sqrt{2}k, \sqrt{2}(k + 1))$. So the slope $\lambda_n + m \mu_n$ runs from $(0, 0)$ to $(\sqrt{2}(k + 2m), \sqrt{2}(k + 1 - 2m))$, and thus has length

$$ (2(k + 2m)^2 + 2(k + 1 - 2m)^2)^{1/2} = \sqrt{n^2 + 16m^2 + (1 - 8m)}. $$

$\square$

**Theorem 5.3.** For $n \geq 7$, or $n \geq 5$ and $|m| \geq 1$, the volume of the complement of the $n$–chain link with $r$ (signed) half–twists is at least:

$$ n \frac{v_8}{4\pi^2} \left(1 - \frac{4\pi^2}{n^2 + 16m^2}\right)^{3/2} $$

if $n$ is even and $r = 2m$ is even,

$$ n \frac{v_8}{4\pi^2} \left(1 - \frac{4\pi^2}{n^2 + 16m^2 + 16m + 4}\right)^{3/2} $$

if $n$ is even and $r = 2m + 1$ is odd,

$$ n \frac{v_8}{4\pi^2} \left(1 - \frac{4\pi^2}{n^2 + 16m^2 + (1 + 8m)}\right)^{3/2} $$

if $n$ is odd and $r = 2m$ is even,

$$ n \frac{v_8}{4\pi^2} \left(1 - \frac{4\pi^2}{n^2 + 16m^2 + (1 - 8m)}\right)^{3/2} $$

if $n$ is odd and $r = 2m + 1$ is odd.
In all cases, the volume of the complement of that \(n\)–chain link is larger than the volume of \(W_{n-1}\) whenever \(n \geq 60\) or \(|m| \geq 8\).

When \(n\) lies between 5 and 10, inclusively, the volume of the complement of that \(n\)–chain link is larger than the volume of \(W_{n-1}\) whenever \(|m| \geq 8\).

Proof. The volume estimates come from combining Lemmas 5.1 and 5.2 with Theorem 3.1, using the fact that the volumes of \(\hat{W}_n\) and \(\overline{W}_n\) are both \(nv_8\), as they are both obtained by gluing \(n\) copies of manifolds isometric to the Whitehead link complement along totally geodesic 3–punctured spheres.

Note that in all cases, the bound on the volume is minimized for the integer \(m\) when \(m = 0\), in which case the argument of Theorem 3.3 still shows that the volume is greater than that of \(W_{n-1}\) for \(n \geq 60\).

Let \(\ell(n, m)\) denote the length of the Dehn filling slope, that is, \(\ell(n, m)\) is one of the four functions of \((n, m)\) in Lemmas 5.1 and 5.2. The volume of the chain link is guaranteed to be strictly greater than that of \(W_{n-1}\) whenever we have

\[
nv_8 \left(1 - \frac{4\pi^2}{\ell(n, m)^2}\right)^{3/2} \geq (n - 1) v_8,
\]

or whenever the function

\[
f(n, m) = \frac{n}{n - 1} \left(1 - \frac{4\pi^2}{\ell(n, m)^2}\right)^{3/2} - 1
\]

is strictly greater than 0. Notice that in all four cases for \(\ell(n, m)\), the function \(f\) is increasing with \(|m|\). Hence for fixed \(n\), to find where this function is greater than 0, it suffices to set the function equal to zero and solve for \(m\).

We do so, and after a calculation, find that the zeros for \(m\) can all be computed in terms of the function

\[
R(n) = \frac{1}{2} \sqrt{\frac{\pi^2}{1 - (n-1)^2/3} - \frac{n^2}{4}}.
\]

That is, the zeros of \(f\) for \(m\) are given by:

- \(\pm R(n)\) if \(n\) and \(r\) are even,
- \(-1/2 \pm R(n)\) if \(n\) is even and \(r = 2m + 1\) is odd,
- \(-1/4 \pm R(n)\) if \(n\) is odd and \(r = 2m\) is even,
- \(1/4 \pm R(n)\) if \(n\) is odd and \(r = 2m + 1\) is odd.

Now, check that \(R(n) > 0\) for any integer \(n\) between 1 and 59, inclusively, and that \(R(n)\) is maximized at \(n \approx 29.6104\), with maximum value approximately 7.36. Thus the maximum and minimum possible values for the zeros of \(f\) lie strictly between \(-8\) and \(8\).

As for the \(n\)–chain links with \(n\) between 5 and 10, inclusively, substituting the particular value of \(n\) (\(n = 5, 6, \ldots, 10\)) into \(R(n)\) and finding the zeros of \(f\) for \(m\), we see that all such zeros lie strictly between \(-6\) and 6. \(\square\)

In fact, for particular values of \(n\) between 5 and 59, inclusively, one can check that often the zeros of \(m\) lie in a more restrictive region than between \(-8\) and \(8\). However, the power of Theorem 5.3 is that it reduces the problem of determining whether any \(n\)–chain link may have volume smaller than that of \(W_{n-1}\) to the task of checking only finitely many examples.

As in Section 4, we may now check the volumes of only finitely many examples using the computer. These finitely many examples are checked by performing Dehn filling along slopes of the form \((1, m), m = 0, \pm 1, \pm 2, \ldots, \pm 7\), on 98 initial manifolds, namely the manifolds...
\( \hat{\mathcal{W}}_n \) and \( \mathcal{W}_n \) for \( n = 11, 12, \ldots, 59 \). In fact, when \( n \) is odd, \( \hat{\mathcal{W}}_n \) and \( \mathcal{W}_n \) are isometric, by an orientation reversing isometry, and so it suffices to check volumes of Dehn fillings of \( \hat{\mathcal{W}}_n \) alone in this case. In the even case, \( \hat{\mathcal{W}}_n \) and \( \mathcal{W}_n \) must be checked separately. However, in this case the Dehn fillings of \( \hat{\mathcal{W}}_n \) along slopes \((1, m)\) and \((1, -m)\) are isometric, by an orientation reversing isometry, and Dehn fillings of \( \mathcal{W}_n \) along \((1, m)\) and \((1, -(m - 1))\) are isometric, by an orientation reversing isometry. Finally, since \((1, 0)\) Dehn filling on \( \hat{\mathcal{W}}_n \) gives the minimally twisted \( n \)-chain link, which was examined in the previous section, we omit that case. In total, this leaves 14 Dehn fillings to check on the 25 initial manifolds \( \hat{\mathcal{W}}_n \) for \( n \) odd, 7 Dehn fillings to check on the 24 initial manifolds \( \hat{\mathcal{W}}_n \) for \( n \) even, and 8 Dehn fillings to check on the 24 initial manifolds \( \mathcal{W}_n \) for \( n \) even, or 710 volumes to compute by computer.

We automated the process of computing volumes as follows. Triangulations for initial manifolds \( \hat{\mathcal{W}}_n \) and \( \mathcal{W}_n \) were generated by Schleimer using the program Twister [2], which computes SnapPea triangulations for manifolds described from the viewpoint of the mapping class group.

We then ran the triangulations through Snap [5], to find volumes of the manifolds under appropriate Dehn fillings. These volumes were compared with those of \( \mathcal{W}_{n-1} \). In all cases, the volume of \( \mathcal{W}_{n-1} \) was strictly smaller. The data generated is shown in Tables 2, 3, and 4.

Again in order to convert these results into a rigorous proof, the algorithms of Moser [7] and Milley [6] could be applied directly. However, as the triangulations of general \( n \)-chain link complements are of similar complexity as those of the minimally twisted chain links, which were too complex for the computer implementation of these algorithms for \( n \) larger than 25, we omitted this step.

5.1. Chain links with 5 through 10 link components. Our methods can be used to show that of all \( n \)-chain links, only the minimally twisted \( n \)-chain link can possibly be the minimal volume manifold with \( n \) cusps for \( n \) between 5 and 10, inclusively.

In fact, because the complexity of these manifolds was comparatively small, we ran them through Milley’s algorithm [6], to rigorously check this fact. The algorithm successfully applied, and we have the following theorem.

**Theorem 5.4.** Let \( n \) be an integer between 5 and 10, inclusively. If \( C_n \) is an \( n \)-chain link that is not minimally twisted, then the complement \( S^3 \setminus C_n \) cannot be the minimal volume \( n \)-cusped hyperbolic manifold.

**Proof.** Theorem 5.3 implies that for these \( n \), and those chain links with at least 11 half-twists, the volume is strictly greater than that of the \((n-1)\)-fold cyclic cover over a component of the Whitehead link, which is known to have larger volume than that of the minimally twisted \( n \)-chain links in these cases.

The remaining cases to check are Dehn fillings \((1, \pm 1), (1, \pm 2), \ldots, (1, \pm 5)\) on manifolds \( \hat{\mathcal{W}}_n \) for \( n \) odd, Dehn fillings \((1, 1), \ldots, (1, 5)\) on manifolds \( \hat{\mathcal{W}}_n \) for \( n \) even, and Dehn fillings \((1, 0), (1, 1), \ldots, (1, 5)\) on manifolds \( \mathcal{W}_n \) for \( n \) even. These cases were run through algorithms of Moser [7] and Milley [6], and their programs rigorously proved that the volumes of these chain links were larger than that of the corresponding minimally twisted \( n \)-chain link. Programs are available from Milley [6] or the second author. 

In Table 5, we show the volumes of \( n \)-chain link complements, \( n \) between 5 and 10, whose volumes are not automatically larger than the minimally twisted \( n \)-chain link by Theorem 5.3 These are compared with the volume of the minimally twisted chain link.
Table 2. Volumes of chain links obtained by Dehn filling \( \widehat{W}_n \) along slope \( s = (1, m) \), where \( m \) is the integer at the top of the column, compared with \( \text{vol}(W_{n-1}) \).
Table 3. Volumes of chain links obtained by Dehn filling \( \hat{W}_n \) along slope \( s = (1, m) \), where \( m \) is the integer at the top of the column, compared with \( \text{vol}(W_{n-1}) \).

| \( n \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | \( \text{vol}(W_{n-1}) \) |
|-------|----|----|----|----|----|----|----|-------------------|
| 11    | 37.340 | 38.184 | 38.621 | 39.249 | 39.531 | 39.721 | 39.851 | 36.639 |
| 13    | 44.982 | 45.598 | 46.126 | 46.792 | 46.985 | 47.122 | 47.327 | 43.966 |
| 15    | 52.581 | 53.034 | 53.467 | 54.072 | 54.263 | 54.403 | 54.531 | 51.294 |
| 17    | 60.139 | 60.478 | 60.829 | 61.131 | 61.370 | 61.553 | 61.693 | 58.622 |
| 19    | 67.662 | 67.919 | 68.205 | 68.465 | 68.682 | 68.855 | 69.091 | 66.991 |
| 21    | 75.155 | 75.354 | 75.586 | 75.810 | 76.005 | 76.166 | 76.296 | 73.727 |
| 23    | 82.623 | 82.780 | 82.971 | 83.162 | 83.336 | 83.484 | 83.607 | 80.605 |
| 25    | 90.073 | 90.197 | 90.355 | 90.520 | 90.673 | 90.809 | 90.925 | 87.933 |
| 27    | 97.506 | 97.607 | 97.738 | 97.879 | 98.015 | 98.139 | 98.247 | 95.260 |
| 29    | 104.926 | 105.009 | 105.119 | 105.240 | 105.361 | 105.473 | 105.573 | 102.588 |
| 31    | 112.335 | 112.404 | 112.497 | 112.602 | 112.708 | 112.810 | 112.902 | 109.916 |
| 33    | 119.735 | 119.792 | 119.872 | 119.963 | 120.057 | 120.149 | 120.234 | 117.244 |
| 35    | 127.155 | 127.244 | 127.323 | 127.402 | 127.489 | 127.567 | 127.657 | 124.571 |
| 37    | 134.513 | 134.554 | 134.613 | 134.682 | 134.757 | 134.831 | 134.903 | 131.899 |
| 39    | 141.893 | 141.928 | 141.979 | 142.040 | 142.106 | 142.174 | 142.240 | 139.227 |
| 41    | 149.268 | 149.299 | 149.343 | 149.456 | 149.517 | 149.577 | 149.635 | 146.554 |
| 43    | 156.639 | 156.665 | 156.704 | 156.805 | 156.860 | 156.916 | 156.962 | 153.882 |
| 45    | 164.006 | 164.029 | 164.064 | 164.156 | 164.204 | 164.254 | 164.304 | 161.210 |
| 47    | 171.370 | 171.390 | 171.421 | 171.501 | 171.547 | 171.594 | 171.635 | 168.538 |
| 49    | 178.731 | 178.749 | 178.776 | 178.810 | 178.849 | 178.890 | 178.933 | 175.865 |
| 51    | 186.089 | 186.106 | 186.130 | 186.160 | 186.195 | 186.233 | 186.272 | 183.193 |
| 53    | 193.446 | 193.460 | 193.482 | 193.509 | 193.541 | 193.576 | 193.612 | 190.521 |
| 55    | 200.800 | 200.813 | 200.832 | 200.857 | 200.886 | 200.918 | 200.951 | 197.849 |
| 57    | 208.152 | 208.164 | 208.181 | 208.204 | 208.230 | 208.259 | 208.290 | 205.176 |
| 59    | 215.503 | 215.513 | 215.529 | 215.550 | 215.574 | 215.601 | 215.629 | 212.504 |

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| $n$  | -7     | -6     | -5     | -4     | -3     | -2     | -1     | 0      | $\text{vol}(W_{n-1})$ |
|------|--------|--------|--------|--------|--------|--------|--------|--------|------------------------|
| 12   | 43.513 | 43.389 | 43.214 | 42.959 | 42.586 | 42.049 | 41.349 | 40.709 | 40.302                 |
| 14   | 45.790 | 45.661 | 45.484 | 45.237 | 44.894 | 44.433 | 43.914 | 43.389 | 43.214                 |
| 16   | 50.876 | 50.715 | 50.539 | 50.234 | 49.896 | 49.433 | 48.914 | 48.492 | 48.049                 |
| 18   | 54.370 | 54.241 | 54.073 | 53.858 | 53.514 | 53.171 | 52.871 | 52.586 | 52.049                 |
| 20   | 57.972 | 57.852 | 57.687 | 57.453 | 57.119 | 56.771 | 56.471 | 56.049 | 55.586                 |
| 22   | 61.979 | 61.859 | 61.687 | 61.453 | 61.119 | 60.771 | 60.471 | 60.049 | 59.586                 |
| 24   | 66.370 | 66.241 | 66.073 | 65.858 | 65.514 | 65.171 | 64.871 | 64.471 | 64.049                 |
| 26   | 71.979 | 71.859 | 71.687 | 71.453 | 71.119 | 70.771 | 70.471 | 70.049 | 69.586                 |
| 28   | 78.076 | 77.945 | 77.771 | 77.539 | 77.234 | 76.858 | 76.458 | 76.058 | 75.586                 |
| 30   | 84.790 | 84.661 | 84.484 | 84.237 | 83.896 | 83.433 | 82.914 | 82.492 | 82.049                 |
| 32   | 91.979 | 91.859 | 91.687 | 91.453 | 91.119 | 90.771 | 90.471 | 90.049 | 89.586                 |
| 34   | 99.979 | 99.859 | 99.687 | 99.453 | 99.119 | 98.771 | 98.471 | 98.049 | 97.586                 |
| 36   | 108.076| 107.945| 107.771| 107.539| 107.234| 106.858| 106.458| 106.058| 105.586               |
| 38   | 116.370| 116.241| 116.073| 115.858| 115.514| 115.171| 114.871| 114.471| 114.049               |
| 40   | 125.979| 125.859| 125.687| 125.453| 125.119| 124.771| 124.471| 124.049| 123.586               |
| 42   | 135.979| 135.859| 135.687| 135.453| 135.119| 134.771| 134.471| 134.049| 133.586               |
| 44   | 146.370| 146.241| 146.073| 145.858| 145.514| 145.171| 144.871| 144.471| 144.049               |
| 46   | 157.979| 157.859| 157.687| 157.453| 157.119| 156.771| 156.471| 156.049| 155.586               |
| 48   | 169.979| 169.859| 169.687| 169.453| 169.119| 168.771| 168.471| 168.049| 167.586               |
| 50   | 182.979| 182.859| 182.687| 182.453| 182.119| 181.771| 181.471| 181.049| 180.586               |
| 52   | 196.979| 196.859| 196.687| 196.453| 196.119| 195.771| 195.471| 195.049| 194.586               |
| 54   | 211.979| 211.859| 211.687| 211.453| 211.119| 210.771| 210.471| 210.049| 209.586               |

Table 4. Volumes of chain links obtained by Dehn filling $W_n$ along slope $s = (1, m)$, where $m$ is the integer at the top of the column, compared with $\text{vol}(W_{n-1})$. 

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Table 5. Volumes of small chain links obtained by Dehn filling $\hat{W}_n$ or $W_n$ along slope $s = (1, m)$, where $m$ is the integer at the top of the column, compared with the volume of the minimally twisted chain link.