ON THE MARKUS CONJECTURE IN CONVEX CASE

KYEONGHEE JO AND INKANG KIM

Abstract. In this paper, we show that any convex affine domain with a nonempty limit sets on the boundary under the action of the identity component of the automorphism group cannot cover a compact affine manifold with a parallel volume, which is a positive answer to a Markus conjecture for convex case. We also show that the Markus conjecture is true for convex affine manifolds when the dimension is $\leq 5$ without any further assumption.

1. Introduction

A topological manifold $M$ can be equipped with a $(G, X)$-structure where $X$ is a model space and $G$ is a group acting on $X$ so that $M$ has an atlas $(\phi_i, U_i)$ from open sets $U_i$ in $M$ into open sets in $X$ and the transition maps $\phi_i \circ \phi_j^{-1}$ are restrictions of elements in $G$. Depending on the choice of $(G, X)$, many interesting geometric structures can arise. For instance, if $M$ is a closed surface with genus at least 2, a hyperbolic structure corresponds to $(\text{PSL}(2, \mathbb{R}), \mathbb{H}^2)$, a real projective structure to $(\text{PGL}(3, \mathbb{R}), \mathbb{RP}^2)$, a complex projective structure to $(\text{PSL}(2, \mathbb{C}), \mathbb{CP}^1)$. In this paper, we are concerned with an affine structure $(\text{Aff}(n, \mathbb{R}), \mathbb{R}^n)$ where $\text{Aff}(n, \mathbb{R})$ is the affine group $\text{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

Given a geometric structure, there exist a developing map $D : \tilde{M} \to X$ and a holonomy homomorphism $\rho : \pi_1(M) \to G$ so that $D$ is $\rho$-equivariant. When the developed image $D(\tilde{M}) = X$, it is said that $M$ has a complete $(G, X)$ structure. Depending on the topology of $M$, only specific complete geometric structures are allowed. For example, if $M$ is a torus, $M$ cannot have a complete hyperbolic structure. This fact can be seen also via Gauss-Bonnet theorem.

Sometimes we are concerned with a complete affine structure with a specific holonomy group. It is said that an affine manifold has parallel volume if the linear part of the holonomy group lies in $\text{SL}(n, \mathbb{R})$.

In 1962, Markus conjectured that a compact affine manifold $M$ with parallel volume is complete, i.e., $M = \mathbb{R}^n/\Gamma$ where $\Gamma$ is a discrete subgroup of $\text{Aff}(n, \mathbb{R})$, see [19].

When the developed image $D(\tilde{M})$ is a convex domain $\Omega$ in $\mathbb{R}^n$, it is said that $M$ has a convex affine structure. Markus conjecture says that such an affine manifold cannot have a parallel volume if $\Omega \neq \mathbb{R}^n$. In this paper, we prove this under the condition that the limit set $\Lambda_{\text{Aut}^o}(\Omega)$, the set of all limit points on the boundary under the action of the identity...
component of Aut(Ω), is nonempty. Here Aut(Ω) = \{g ∈ GL(n, R) × R^n | g(Ω) = Ω\}, is the group of automorphisms of Ω.

**Theorem 1.** Let M be a compact convex affine manifold whose developing image Ω is a proper subset of \( \mathbb{R}^n \). Then M cannot have parallel volume if \( \Lambda_{Aut^0(Ω)} \) is nonempty.

Furthermore we will see in section §7 that the Markus conjecture is true for convex case when its dimension is \( \leq 5 \) through the classification of their developing images.

**Theorem 2.** Let \( Ω \) be a divisible convex affine domain in \( \mathbb{R}^n \) with \( n \leq 5 \). Then \( \Lambda_{Aut^0(Ω)} \neq \emptyset \) if \( Ω \neq \mathbb{R}^n \), hence the Markus conjecture is true.

Though there are several partial results concerning this conjecture \([4, 6, 7, 9, 11]\), yet it is far from being completely resolved. Goldman and Hirsch \([10]\) showed that the affine holonomy \( Γ \) of a compact affine \( n \)-manifold with parallel volume preserves no proper (semi)-algebraic subset of \( \mathbb{R}^n \) and in fact the algebraic hull \( A(Γ) \) of \( Γ \) acts transitively on \( \mathbb{R}^n \). We will use this fact in the proof of our main theorem. We also proved the conjecture under the some suitable assumption on projective automorphism of the domain \([16]\) to which the current paper is a sequel. We expect to resolve the conjecture in a near future without the assumption about the limit set. See Proposition 13.

2. PRELIMINARIES

In this section, we review the basic concepts and properties for convex domains in projective space. To begin with, let us be precise about our terminology. When we speak of a simplex in this paper, we mean the domain which consists of all the points in the interior of the simplex. The same is true for a polyhedron, an ellipsoid, a paraboloid, an elliptic cone etc.

**Definition 1.** A subgroup \( G \) of Aff\((n, \mathbb{R})\) is said to be irreducible if \( G \) preserves no proper affine subspace of \( \mathbb{R}^n \).

Sometimes we look at domains and their automorphisms in the projective space \( \mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^* \) and PGL\((n + 1, \mathbb{R})\), where \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \). Naturally PGL\((n + 1, \mathbb{R})\) acts on \( \mathbb{R}P^n \) as projective transformations. Denote the projectivization of \( n \) by \( n \) matrices by PM\((n, \mathbb{R})\). Recall that a domain in \( \mathbb{R}^n \) can be viewed as a domain in \( \mathbb{R}P^n \) whose automorphism group preserves the set of points at infinity, \( \mathbb{R}P^n_{\infty} \), by identifying \( \mathbb{R}^n \) with the affine space given by \( x_{n+1} = 1 \) in \( \mathbb{R}^{n+1} \) so that \( \mathbb{R}P^n \) becomes a compactification of \( \mathbb{R}^n \).

Since PM\((n + 1, \mathbb{R})\) is a compactification of PGL\((n + 1, \mathbb{R})\), any infinite sequence of non-singular projective transformations contains a convergent subsequence in PM\((n + 1, \mathbb{R})\). Note that the limit projective transformation \( g \) of a sequence of non-singular projective transformations \( g_i \) may be singular. We will denote the projectivization of the kernel and range of \( g \) by \( K(g) \) and \( R(g) \). Then \( g \) maps \( \mathbb{R}P^n \setminus K(g) \) onto \( R(g) \), and the \( g_i \)-images of any compact set in \( \mathbb{R}P^n \setminus K(g) \) converges uniformly to the image under the limit transformation \( g \) of \( g_i \) (see [3]). This implies that \( g_i(p_i) \) converges to \( g(p) \) if \( p_i \) converges to \( p \notin K(g) \).

**Definition 2.** Let \( Ω \) be a convex projective domain and \( \overline{Ω} \) be the closure of \( Ω \) in the projective space.
(i) $\Omega$ is called properly convex if it does not contain any complete line.
(ii) A face of $\Omega$ is an equivalence class with respect to the equivalence relation given as follows. $x \sim y$ if either $x = y$ or $\overline{\Omega}$ has an open line segment containing both $x$ and $y$. Then a face is a relatively open convex subset of $\overline{\Omega}$ and $\overline{\Omega}$ is a disjoint union of faces.
(iii) A support of a face $F$ is the subspace generated by $F$. We will denote the support of $F$ by $\langle F \rangle$.
(iv) The dimension of a face $F$ is the dimension of the support.
(v) A supporting subspace $V$ of $\Omega$ is a subspace containing every support of a face intersecting $V$.
(vi) Zero dimensional faces are called extreme points.

We say that $E$ is a closed face of $\Omega$ if $E = \overline{F}$, for a face $F$ of $\Omega$.

**Definition 3.** Let $\Omega$ be a properly convex domain in $\mathbb{R}P^n$.

(i) Let $\Omega_1$ and $\Omega_2$ be convex domains in $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ respectively. $\Omega$ is called a convex sum of $\Omega_1$ and $\Omega_2$, which will be denoted by $\Omega = \Omega_1 + \Omega_2$, if $\langle \Omega_1 \rangle \cap \langle \Omega_2 \rangle = \emptyset$ and $\Omega$ is the union of all open line segments joining points in $\Omega_1$ to points in $\Omega_2$. We say that $\Omega$ is decomposable if $\Omega$ has such a decomposition. Otherwise, $\Omega$ is called indecomposable.

(ii) A $k$-dimensional face $F$ of an $n$-dimensional convex domain $\Omega$ is called conic if there exist $n-k$ supporting hyperplanes $H_1, H_2, \ldots, H_{n-k}$ such that

$$ H_1 \supseteq H_1 \cap H_2 \supseteq \cdots \supseteq H_1 \cap \cdots \cap H_{n-k} = \langle F \rangle. $$

Especially, a codimension one face is conic.

(iii) We say that $\Omega$ has an osculating ellipsoid at $p \in \partial \Omega$ if there exist a suitable affine chart and a basis such that the local boundary equation on some neighborhood of $p = (0, \ldots, 0)$ is expressed by $x_n = f(x_1, \ldots, x_{n-1})$ and

$$ \lim_{(x_1, \ldots, x_{n-1}) \to 0} \frac{f(x_1, \ldots, x_{n-1})}{x_1^2 + \cdots + x_{n-1}^2} = 1. $$

**Remark 1.**

(i) If $\partial \Omega$ is twice differentiable on a neighborhood of a strictly convex boundary point $p$ of $\Omega$, then $\Omega$ has an osculating ellipsoid at $p$. See [11].

(ii) We will see in section [4] that when $\Omega$ is quasi-homogeneous, the property that a face $F$ is conic is equivalent to the property that $F$ is a convex summand of $\Omega$. See Theorem [8].

Let $\text{Aut}_{\text{proj}}(\Omega)$ be the set of all projective transformations which preserves a domain $\Omega \subset \mathbb{R}P^n$. Since there is a natural surjection $\pi : \text{SL}^\pm(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$ with its kernel $\{\text{Id}\}$ or $\{\text{Id}, -\text{Id}\}$, where $\text{SL}^\pm(n+1, \mathbb{R})$ is the group of linear transformations of $\mathbb{R}^{n+1}$ with determinant 1 or $-1$, we will regard $\text{Aut}_{\text{proj}}(\Omega)$ as a subgroup of the group $\text{SL}^\pm(n+1, \mathbb{R})$ from now on. Note that if $\Omega$ is a domain in $\mathbb{R}^n$ then we see that $\text{Aut}(\Omega)$ is a subgroup of $\text{Aut}_{\text{proj}}(\Omega)$ and $\text{Aut}(\Omega) = \text{Aut}_{\text{proj}}(\Omega) \cap \text{Aff}(n, \mathbb{R})$.

It is well-known that if $\Omega$ is properly convex, then there is a complete continuous metric which is invariant under the action of $\text{Aut}_{\text{proj}}(\Omega)$. This metric is called the Hilbert metric.
and defined as follows: Any properly convex projective domain is projectively equivalent to a bounded convex domain in an affine space, we may assume that $\Omega$ is a bounded convex domain in $\mathbb{R}^n$.

**Definition 4.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$. For any two different points $p_1, p_2 \in \Omega$, we define $d_\Omega(p_1, p_2)$ to be the logarithm of the absolute value of the cross ratio of $(s_1, s_2, p_1, p_2)$, where $s_1$ and $s_2$ are the points in which the line $\overrightarrow{p_1p_2}$ intersects $\partial \Omega$. That is, if the four points are in a sequence $s_1, p_1, p_2, s_2$ then

$$d_\Omega(p_1, p_2) = \ln \frac{d_{\mathbb{R}^n}(s_1, p_2)d_{\mathbb{R}^n}(p_1, s_2)}{d_{\mathbb{R}^n}(s_1, p_1)d_{\mathbb{R}^n}(p_2, s_2)}.$$ 

For $p_1 = p_2$, we define $d_\Omega(p_1, p_2) = 0$.

So when $\Omega \subset \mathbb{RP}^n$ is properly convex, any element $\gamma$ of $\text{Aut}_{\text{proj}}(\Omega)$ is an isometry of the Hilbert metric. We say $\gamma$ is *elliptic* if it fixes a point in $\Omega$. If $\gamma$ acts freely on $\Omega$ it is called *parabolic* if every eigenvalue has modulus 1 and *hyperbolic* otherwise. The *translation length* of $\gamma$,

$$t(\gamma) = \inf_{x \in \Omega} d_\Omega(x, \gamma(x)),$$

is equal to the logarithm of the absolute value of the ratio of the eigenvalues of $\gamma$ of maximum modulus $\lambda$ and minimum modulus $\mu$, i.e.,

$$t(\gamma) = \ln |\lambda/\mu|.$$ 

Let $p$ be a boundary point of $\Omega$ and $H$ a supporting hyperplane to $\Omega$ at $p$. Define $S_0$ to be the subset of boundary of $\Omega$ obtained by deleting $p$ and all the line segments in $\partial \Omega$ with end points at $p$. A *generalized horosphere centered on $(H, p)$* is the image of $S_0$ under the action of $G(H, p)$, the set of all the projective transformations which translate towards $p$ and fix $H$ pointwise, which is equal to the set of all the affine translations towards $p$ in the affine space $\mathbb{RP}^n \setminus H$. We simply call a generalized horosphere a *horosphere* in $\Omega$. Then $\Omega$ is foliated by horospheres and this foliation is preserved by an automorphism $A \in \text{Aut}_{\text{proj}}(\Omega)$ if $A$ preserves $p$ and $H$. In particular, each horosphere centered on $(H, p)$ is preserved if $A$ is a parabolic isometry fixing $(H, p)$. (See [3] or [20] for a reference.)

The following figures show two kinds of horospheres of the triangle under the action of a hyperbolic isometry $A \in \text{PGL}(3, \mathbb{R})$ with eigenvalues 2, 2, 1/4. Figure 1 shows the horospheres in $\mathbb{RP}^2$ and Figure 2 shows them in the affine space which is the complement of the supporting hyperplane $H$.

The next theorem will be used later.

**Theorem 3** (Jo, [13]). Let $\Omega$ be a domain in $\mathbb{RP}^n$. Then $\Omega$ is an ellipsoid if and only if $\Omega$ has a locally strictly convex point $p$ in the boundary (that is, there exists a connected open neighborhood $U$ of $p$ such that $U \cap \Omega$ is a strictly convex domain) such that

(i) $\partial \Omega$ is $C^2$ near $p$,

(ii) the Hessian is non-degenerate at $p$,

(iii) $\text{Aut}_{\text{proj}}(\Omega)x$ accumulates at $p$ for some $x \in \Omega$.
Figure 1. Horospheres of a triangle in $\mathbb{R}P^2$

Figure 2. Horospheres of a triangle in the affine space $\mathbb{R}P^2 \setminus H$

3. QUASI-HOMOGENEOUS DOMAINS

A domain $\Omega$ is called homogeneous if $\text{Aut}(\Omega)$ acts transitively on $\Omega$ and quasi-homogeneous if there exists a compact set $K \subset \Omega$ and $G \subset \text{Aut}(\Omega)$ so that $GK = \Omega$. In this case we also say that $G$ acts on $\Omega$ syndetically. $\Omega$ is called divisible if there exists a discrete subgroup $\Gamma \subset \text{Aut}(\Omega)$ so that $\Omega/\Gamma$ is a compact manifold.

Note that both homogeneous and divisible domains are quasi-homogeneous and any compact convex affine $n$-manifold $M$ has a divisible domain $\Omega$ in $\mathbb{R}^n$ and a discrete subgroup $\Gamma$ of $\text{Aff}(n, \mathbb{R})$ acting on $\Omega$ such that $M = \Omega/\Gamma$. Furthermore the following surprising theorem is well-known.

Theorem 4 (Vey, [25]). A divisible properly convex affine domain is a cone.

We list out some useful facts.

Proposition 1 (Vey, [25]). Let $\Omega$ be a quasi-homogeneous properly convex affine domain. Then

1. For any $x \in \Omega$ and extreme point $\xi$, there exist $g_i \in G$ such that $g_i x \to \xi$.
2. $\Omega = CH(Gx)$ for any $x \in \Omega$,
(3) If $L$ is a $G$-invariant proper affine subspace of $\mathbb{R}^n$, then
\[ L \cap \Omega = \emptyset \text{ and } L \cap \partial \Omega \neq \emptyset. \]

Here $CH(Gx)$ means the convex hull of $Gx$.

**Proposition 2** (Jo, [11, 12]).
(i) A quasi-homogeneous convex affine domain cannot have any bounded face with non-zero dimension.
(ii) Simplices are the only quasi-homogeneous polyhedron in $\mathbb{R}P^n$.

For a strictly convex quasi-homogeneous projective domain $\Omega \subset \mathbb{R}P^n$, the following is proved in [11].

**Proposition 3.** Let $\Omega$ be a strictly convex quasi-homogeneous domain in $\mathbb{R}P^n$. Then

(i) $\partial \Omega$ is at least $C^1$,
(ii) $\Omega$ is an ellipsoid if $\partial \Omega$ is twice differentiable,
(iii) if $\partial \Omega$ is $C^\alpha$ on an open subset of $\partial \Omega$, then $\partial \Omega$ is $C^\alpha$ everywhere,
(iv) if a boundary point of $\Omega$ is fixed by $\text{Aut}_{\text{proj}}(\Omega)$, then $\Omega$ is an ellipsoid.

Every quasi-homogeneous convex affine domain contains a cone invariant under the action of linear parts of their automorphism groups, which is called an asymptotic cone. This terminology was originally introduced by Vey in [25].

**Definition 5.** Let $\Omega$ be a convex domain in $\mathbb{R}^n$. The asymptotic cone of $\Omega$ is defined as follows:
\[ AC(\Omega) = \{ u \in \mathbb{R}^n | x + tu \in \Omega, \text{ for all } x \in \Omega, t \geq 0 \}. \]

By the convexity of $\Omega$, for any $x_0 \in \Omega$,
\[ AC(\Omega) = AC_{x_0}(\Omega) := \{ u \in \mathbb{R}^n | x_0 + tu \in \Omega, \text{ for all } t \geq 0 \}. \]

Note that $AC(\Omega)$ is a properly convex closed cone in $\mathbb{R}^n$ if $\Omega$ is properly convex. If we denote the interior of $AC(\Omega)$ relative to its affine hull by $AC^\circ(\Omega)$, then it is proved in [13] that $AC^\circ(\Omega)$ is a (quasi)-homogeneous domain if $\Omega$ is (quasi)-homogeneous.

Even though the asymptotic cone $AC(\Omega)$ of a properly convex affine domain $\Omega$ is possibly empty, it is nonempty if $\Omega$ is quasi-homogeneous because there is no bounded quasi-homogeneous convex domain (see Proposition 2). Vey proved in [25] that a quasi-homogeneous properly convex affine domain is itself a cone if the dimension of its asymptotic cone is equal to the dimension of $\Omega$. More generally, we get the following.

**Theorem 5** (Jo, [11]). Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^n$. Then

(i) $\Omega$ admits a parallel foliation by cosets of the asymptotic cone $AC^\circ(\Omega)$ of $\Omega$, we call this asymptotic foliation of $\Omega$,
(ii) the set of all the asymptotic cone points of $\Omega$ is equal to the set of all the extreme points of $\Omega$, where an asymptotic cone point is a boundary point $\xi$ such that $\xi + AC^\circ(\Omega)$ is a leaf of the above foliation.
Note that if we consider $\Omega$ as a projective domain in $\mathbb{RP}^n$, then the set of all extreme points of $\Omega$ is the union of the set of all asymptotic cone points and infinite extreme points of $\Omega$.

**Theorem 6** (Jo, [15]). Let $\Omega$ be a properly convex affine domain in $\mathbb{R}^n$ and $G$ be a closed subgroup of $\text{Aut}(\Omega)$ acting syndetically on $\Omega$. Then $G$ acts transitively on the set $S(\Omega)$ of all asymptotic cone points of $\Omega$.

**Lemma 1.** Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^n$. Then any boundary point which is a limit point of a sequence of extreme points is again an extreme point. Especially, for any $\gamma \in \text{Aut}(\Omega)$ and an extreme point $\xi$, $\gamma(\xi)$ is an extreme point if it is in $\mathbb{R}^n$ (it is an infinite extreme point if it is in $\partial_\infty \Omega$).

**Proof.** Since every extreme point is an asymptotic cone point and an asymptotic foliation is preserved by $\text{Aut}(\Omega)$, any limit point of a sequence of extreme points must be an extreme point.

The following corollary will be used later in this paper.

**Corollary 1.** Let $\Omega$ be a quasi-homogeneous properly convex affine domain in $\mathbb{R}^n$ and $S$ be the set of extreme points of $\Omega$. Then the following holds.

(i) For any extreme point $\xi$,

$$(\xi + \text{AC}(\Omega)) \cap \Omega = \xi + \text{AC}^c(\Omega),$$

that is, $\xi + \text{AC}^c(\Omega)$ cannot be a face of $\Omega$ unless $\Omega$ itself is a cone.

(ii) If $\Omega = F_1 + F_2$, then $\bar{F}_i \cap \partial_\infty \Omega \neq \emptyset$ for $i = 1, 2$, and $S$ is contained entirely in one of $F_1$ and $F_2$. So either $S \subset F_1$ and $F_2 \subset \partial_\infty \Omega$ holds or $S \subset F_2$ and $F_1 \subset \partial_\infty \Omega$ holds.

(iii) If $\overline{\Omega} = \overline{CH(S)}$, then $\Omega$ cannot have any conic face and $\text{Aut}(\Omega)$ is irreducible.

**Proof.**

(i) For any point $x \in \Omega$, the cone point $\xi_x$ of the leaf of asymptotic foliation of $\Omega$, which contains $x$, is an extreme point by Theorem 5 and thus $\xi_x + \text{AC}(\Omega)$ is included in $\Omega$. Since $\text{Aut}_{\text{aff}}(\Omega)$ acts transitively on $S$, $\xi_x + \text{AC}(\Omega)$ should consist of interior points of $\Omega$ for all $x \in S$.

(ii) Both $\bar{F}_1$ and $\bar{F}_2$ have an infinite boundary point since $\Omega$ cannot have any bounded face by Proposition 2. By Theorem 5 and Lemma 4, $S$ is connected and thus either $S \subset \bar{F}_1$ or $S \subset \bar{F}_2$ holds.

(iii) Suppose $\Omega$ has a conic face $F_1$. Then there is a face $F_2$ of $\Omega$ such that $F_2 \subset \partial_\infty \Omega$ and $\Omega = F_1 + F_2$ (see (iii) of Theorem 8 in the next section). This implies that $S \subset F_1$ and $\Omega$ is affinely equivalent to $F_1 \times C(F_2)$, where $C(F_2)$ is a cone over $F_2$. But this is a contradiction because $\overline{\text{CH}(S)} \subset \bar{F}_1 \neq \overline{\Omega}$. By Theorem 6, $\text{Aut}(\Omega)$ cannot have any invariant proper subspace.

□
4. Benzécri’s result

The structure of quasi-homogeneous domains have been studied a lot in convex case since Benzécri. Here are some results about (quasi-homogeneous) convex domains which are needed later in this paper.

**Theorem 7** (Benzécri, [3]). Let $\Omega$ be a properly convex domain in $\mathbb{RP}^n$.

(i) If $\Omega$ has an osculating ellipsoid $Q$, then there exists a sequence $\{g_n\} \subset PGL(n+1, \mathbb{R})$ such that $g_n \Omega$ converges to $Q$.

(ii) Let $F$ be a conic face of $\Omega$. Then there exist a projective subspace $L$ of $\mathbb{RP}^n$ and projective automorphisms $\{h_i\} \subset \mathbb{RP}^n$ such that $\{h_i \Omega\}$ converges to $F + B$ for some properly convex domain $B$ in $L$.

(iii) If $\Omega = \Omega_1 + \Omega_2$, then $\Omega$ is quasi-homogeneous (respectively, homogeneous) if and only if $\Omega_i$ is quasi-homogeneous (respectively, homogeneous) for each $i$.

**Theorem 8** (Benzécri, [3]). Let $\Omega$ be a quasi-homogeneous properly convex domain in $\mathbb{RP}^n$.

(i) If $W$ is a $q$-dimensional section of $\Omega$ and $W'$ is a $q$-dimensional properly convex domain which is a limit of the sequence of $q$-dimensional properly convex domains $\{g_i(W)\}$ for projective transformations $\{g_i\}$, then $W'$ is a section of $\Omega$.

(ii) If $\Omega$ has an osculating ellipsoid, then $\Omega$ is projectively equivalent to a ball.

(iii) If $\Omega$ has a conic face $F$ of $\Omega$, then there exists another conic face $B$ of $\Omega$ such that $\Omega = F + B$.

From Benzécri’s result, we get the following lemmas, which are proved in [11].

**Lemma 2.** Let $\Omega$ be a properly convex domain in $\mathbb{RP}^n$. Suppose a sequence $g_i \in Aut_{proj}(\Omega)$ converges to a singular projective transformation $g \in PM(n+1, \mathbb{R})$. Then $K(g)$ and $R(g)$ do not meet $\Omega$ and the following holds:

(i) $K(g)$ is a supporting subspace of $\Omega$

(ii) $R(g)$ is a support of a proper face $F$ of $\Omega$

(iii) If $x_0 \in \Omega$ and $\lim_{i \to \infty} g_i x_0 = \xi \in \partial \Omega$, then $R(g)$ is the support of the face $F$ containing $\xi$.

(iv) $g(\Omega) = F$, that is, for any point $\eta \in F$ there exists $x \in \Omega$ so that $\lim_{i \to \infty} g_i x = \eta$.

**Lemma 3** (Jo, [11]). Let $\{f_i\}$ be a sequence in $Aff(n, \mathbb{R})$. Suppose that $f_i$ converges to $f \in PM(n+1, \mathbb{R})$ with $R(f) \cap \mathbb{R}^n \neq \emptyset$. Then $K(f) \cap \mathbb{R}^n = \emptyset$.

**Lemma 4** (Jo, [11]). Let $\Omega$ be a quasi-homogeneous properly convex domain in $\mathbb{RP}^n$ and $G$ a subgroup of $Aut(\Omega)$ acting on $\Omega$ syndetically. Then for each point $p \in \partial \Omega$, there exists a sequence $\{g_i\} \subset G$ and $x \in \Omega$ such that $g_i(x)$ converges to $p$.

**Lemma 5.** Let $\Omega$ be a properly convex projective domain and $\{\phi_n\} \subset Aut(\Omega)$ a sequence converging to $\phi \in PM(n+1, \mathbb{R})$ with kernel $K(\phi)$ and range $R(\phi)$. Suppose $K(\phi) = \langle K(\phi) \cap \partial \Omega \rangle$, i.e., $K(\phi)$ is a support of some face of $\Omega$. Then $K(\phi) \cap \partial \Omega$ is a closed conic face of $\Omega$. 
Proof. Let \( k \) be the dimension of \( K(\phi) \) and \( F \) be the \( k \)-dimensional face of \( \Omega \) such that \( \overline{F} = K(\phi) \cap \partial \Omega \). We can choose a complementary subspace \( L \) of dimension \( (n-k-1) \), that is, the space generated by \( K(\phi) \) and \( L \) is the whole \( \mathbb{R}P^n \). Let \( \rho \) be a projection from \( \mathbb{R}P^n \) to \( L \). Then \( \phi \) can be considered as a projective transformation from \( L = \rho(\mathbb{R}P^n) \) to \( R(\phi) \), since \( \phi \) maps the projective space generated by \( K(\phi) \) and \( y \in L \) to one point \( \phi(y) \) in \( R(\phi) \). The fact that \( K(\phi) \) does not intersect \( \Omega \) implies that \( \phi(x) = \lim_{n \to \infty} \phi_n(x) \) must be contained \( R(\phi) \cap \partial \Omega \) for all \( x \in \Omega \). Since \( \Omega \) is properly convex, \( R(\phi) \cap \partial \Omega \) is also properly convex and so \( \rho(\Omega) \) is properly convex in \( L \). Therefore \( \rho(\Omega) \) is bounded by \( (n-k) \) number of \( (n-k-2) \)-planes \( \{H_L^1, \ldots, H_L^{n-k}\} \) of \( L \) which bound a \( (n-k-1) \)-simplex. If we let \( H_i \) be the \( (n-1) \)-plane generated by \( K(\phi) \) and \( H_L^i \) for each \( i \), then \( \{H_i\} \) are hyperplanes of \( \mathbb{R}P^n \) which satisfies (2.1). Therefore \( F \) is a conic face of \( \Omega \).

\[ \square \]

5. Limit set

**Definition 6.** Let \( \Omega \neq \mathbb{R}^n \) be a domain in \( \mathbb{R}^n \) and \( G < \text{Aut}(\Omega) \). A limit set \( \Lambda_G \subset \mathbb{R}^n \) of \( G \) is

\[ \bigcup_{x \in \Omega} (\overline{Gx} \cap \partial \Omega) \]

**Lemma 6.** Let \( \Omega \) be a properly convex affine domain in \( \mathbb{R}^n \). Suppose \( H \) is a normal subgroup of \( \text{Aut}(\Omega) \subset \text{Aff}(n, \mathbb{R}) \). Then \( \text{Aut}(\Omega) \) leaves \( \Lambda_H \) invariant.

**Proof.** Suppose \( \xi \) is a point of \( \Lambda_H \) and \( g \) is an element of \( \text{Aut}(\Omega) \). Then there exists a sequence \( \{h_i\} \) of \( H \) and a point \( x_0 \) of \( \Omega \) such that \( h_i(x_0) \) converges to \( \xi \). If \( y_0 = g(x_0) \), then

\[ \lim_{i \to \infty} gh_i(g^{-1}(y_0)) = ghg^{-1}(y_0) = gh(x_0) = g(\xi) \]

This implies that \( g(\xi) \) is contained in \( \Lambda_H \) and thus we can conclude that \( \text{Aut}(\Omega) \) leaves \( \Lambda_H \) invariant. \( \square \)

Note that by Lemma 2 the limit set \( \Lambda_H \) is the disjoint union of the boundary face of \( \Omega \) whose support is the range space of some \( h \in \overline{H - H} \).

**Lemma 7.** Let \( \Omega \) be a properly convex quasi-homogeneous affine domain in \( \mathbb{R}^n \). Suppose that \( G = \text{Aut}(\Omega) \) is irreducible and the limit set \( \Lambda_{G^0} \) of the identity component of \( G \) is nonempty. Then \( \overline{CH(\Lambda_{G^0})} = \overline{CH(\Lambda_{G^0})} = \overline{\Omega} \) and \( \overline{\Lambda_{G^0}} \) contains all the extreme points.

**Proof.** Since \( G \) is irreducible and preserves \( \Lambda_{G^0} \neq \emptyset \), \( \overline{CH(\Lambda_{G^0})} \cap \Omega \neq \emptyset \). Now we get from Proposition 1

\[ \overline{CH(\Lambda_{G^0})} = \overline{CH(\Lambda_{G^0})} = \overline{\Omega} \]

by considering \( G \)-orbit of a point \( x \in CH(\Lambda_{G^0}) \cap \Omega \). Hence every extreme point is in \( \overline{\Lambda_{G^0}} \), that is, \( S \subset \overline{\Lambda_{G^0}} \). \( \square \)

We will see in Proposition 4 that under the assumptions in Lemma 7, the domain is homogeneous. Hence the limit set \( \Lambda_{G^0} \) is \( \partial \Omega \) containing all the extreme points.
To prove theorem 1, we first show the following two propositions.

Proposition 4. Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^n$ which has an irreducible $G = \text{Aut}(\Omega)$ and a nonempty $\Lambda_{G^0}$. Then $\Omega$ is a homogeneous affine domain.

Proof. Consider an orbit $G^0x$ of a point $x \in \Omega$. Since $G^0x$ accumulates at every face of $\Omega$ which is in $\Lambda_{G^0}$, we can find a one-parameter subgroup $g_E(t)$ for each face $E \subset \Lambda_{G^0}$ which satisfies the following: $g_E(t)$ is parabolic or hyperbolic, $E$ is fixed by $g_E(t)$, $g_E(t)$ preserves a supporting hyperplane $H^E$ which contains $E$ for all $t$, and $g_E(t)x$ converges to a point of $E$ as $t$ goes to $\infty$.

We denote the set of fixed points of $g_E(t)$ by $F_E$ for each face $E \subset \Lambda_{G^0}$, and especially if $g_E(t)$ is hyperbolic, we denote $F_E^+ \ (F_E^−$, respectively) the set of attracting fixed points (repelling fixed points, respectively) and $F^n_E$ is the set of the remaining fixed points. We call the horospheres centered on $(H^E, p), p \in E$ as $\{g_E(t)\}$-horospheres. Note that these horospheres make a foliation of $\Omega$.

Such a one-parameter subgroup $g_E(t)$ can be constructed as follows. Choose $g_n \in G^0$ such that $g_n x \to p \in E$. Let $g_n(t) = \exp(t\eta_n), \eta_n \in g^0$, be a one-parameter subgroup in $G^0$ so that $g_n(t_n) = g_n$ for $t_n \to \infty$. Here we normalize $\eta_n$ so that its norm is 1 with respect to a Killing form on $\text{SL}(n+1, \mathbb{R})$. Since the set of directions $g^0 \cap S$, where $S$ is a unit sphere in $\mathfrak{sl}(n+1, \mathbb{R})$, is compact, we can pass to a subsequence so that $\eta_n \to \eta$. Set $g_E(t) = \exp(t\eta)$. Then the one-parameter subgroups $g_n(t)$ converges to $g_E(t)$ and

$$p = \lim_{n \to \infty} g_n x = \lim_{n \to \infty} g_n(t_n)x = \lim_{t \to \infty} \exp(t\eta_n)x = \lim_{t \to \infty} \exp(t\eta)x = \lim_{t \to \infty} g_E(t)x.$$

Obviously $g_E(t)$ is not elliptic for each $t$ and thus it preserves a supporting hyperplane $H^E_t$ to $\Omega$ at $p$ by Lemma 2.3 of [5]. Since one parameter subgroup is abelian, it preserves a common supporting hyperplane $H^E$.

By Lemma 7, we can choose two distinct faces $E_1, E_2$ in $\Lambda_{G^0}$ any of which is not contained in the closure of the other. Let $H_{E_1E_2}$ be the subgroup of $G^0$ which is generated by $g_{E_1}(t)$ and $g_{E_2}(s)$. If $\Omega$ is a 2-dimensional domain, then we can prove $H_{E_1E_2}x = \Omega$ as follows: If both $g_{E_1}(t)$ and $g_{E_2}(s)$ are hyperbolic and an arbitrary pair of two orbits $\{g_{E_1}(t)x \mid t \in \mathbb{R}\}$ and $\{g_{E_2}(s)y \mid s \in \mathbb{R}\}$ intersect transversely, then $H_{E_1E_2}x$ must be equal to

$$S_{12} = \bigcup_s g_{E_2}(s)\{g_{E_1}(t)x \mid t \in \mathbb{R}\} = \{g_{E_2}(s)g_{E_1}(t)x \mid t \in \mathbb{R}, s \in \mathbb{R}\},$$

and fully covers $\Omega$. For the case that a one-parameter subgroup $g_{E_1}(t)$ is hyperbolic and another one $g_{E_2}(s)$ is parabolic with $(F_{E_1}^+ \cup F_{E_1}^-) \cap F_{E_2} \neq \emptyset$, we get $S_{12} = H_{E_1E_2}x = \Omega$ since each $g_{E_1}(t)$-orbit meets every $g_{E_2}(s)$-horospheres.

If the case is not the above two, we can show that

$$S_{123} = \{g_{E_1}(u)g_{E_2}(s)g_{E_1}(t)x \mid t, s, u \in \mathbb{R}\}$$

is equal to $H_{E_1E_2}x = \Omega$, considering various possibilities. For example, if $g_{E_1}(t)$ is hyperbolic and $g_{E_2}(s)$ is parabolic with $(F_{E_1}^+ \cup F_{E_1}^-) \cap F_{E_2} = \emptyset$, then there is a $g_{E_2}(s)$-horosphere,
$S_z$, which meets tangentially an orbit $g_{E_1}(t)x$ and thus $S_{12}$ becomes one of two components of $\Omega - S_z$. This implies that

$$\Omega = \bigcup_{u \in \mathbb{R}} g_{E_1}(u)S_{12} = S_{123}.$$

If the dimension of $\Omega$ is greater than 2, $H_{E_1E_2}x$ may not be equal to $\Omega$, but $\partial(H_{E_1E_2}x) \subset \partial\Omega$. In this case, we can find another face $E_3 \subset \Lambda_{G^0}$ outside $\overline{H_{E_1E_2}x}$ since $\Omega = CH(\Lambda_{G^0})$ by Lemma 7. Let $H_{E_1E_2E_3}$ be the subgroup of $G^0$ which is generated by $g_{E_1}(t)$, $g_{E_2}(s)$, and $g_{E_3}(u)$. Now considering the action of the one parameter subgroup $\{g_{E_3}(u)\}$ we can show that

$$\Omega = H_{E_1E_2E_3}x,$$

if the dimension of $\Omega$ is 3. Actually $\Omega = H_{E_1E_2E_3}x$ is equal to

$$\bigcup_{u} g_{E_3}(u)H_{E_1E_2}x \text{ or } \bigcup_{h \in H_{E_1E_2}} \bigcup_{u,v} h(\Lambda)g_{E_3}(u)H_{E_1E_2}x.$$

For the case that the dimension of $\Omega$ is greater than 3 and $\Omega \neq H_{E_1E_2E_3}x$, we can choose another face $E_4 \subset \Lambda_{G^0}$ outside $\overline{H_{E_1E_2E_3}x}$. By repeating this process, we can conclude that $G^0x = H_{E_1E_2...E_k}x = \Omega$ for some natural number $k \leq n$ and thus $G^0$ acts on $\Omega$ transitively.

Proposition 5. Let $\Omega(\neq \mathbb{R}^n)$ be a convex quasi-homogeneous affine domain in $\mathbb{R}^n$ which has an irreducible $G = \text{Aut}(\Omega)$ and a nonempty $\Lambda_{G^0}$. Then $\Omega$ is a semi-algebraic subset of $\mathbb{R}^n$.

Proof. Note that if $\Omega$ has a complete line then there exist a natural number $k < n$ and a $n - k$ dimensional properly convex domain $\Omega'$ such that $\Omega = \mathbb{R}^k \times \Omega'$, which comes from the convexity of $\Omega$. We see immediately that $G = \text{Aut}(\Omega)$ is irreducible if and only if $G' = \text{Aut}(\Omega')$ is irreducible, $\Lambda_{G^0} \neq \emptyset$ if and only if $\Lambda_{G_0} \neq \emptyset$, and $\Omega$ is a semi-algebraic subset of $\mathbb{R}^n$ if and only if $\Omega'$ is a semi-algebraic subset of $\mathbb{R}^{n-k}$. So we just need to prove our proposition for a properly convex domain $\Omega$.

Goldman and Hirsch proved in Proposition 2.15 of [10] that the developing map of a homogeneous affine manifold is a covering onto a semi-algebraic open set. So we can conclude that $\Omega$ is a semi-algebraic subset of $\mathbb{R}^n$, because $\Omega$ is a homogeneous affine domain by Proposition 4.

Now we prove our main theorem.

Proof of Theorem 4. For a compact affine $n$-manifold $M$ with parallel volume, it is well-known that the affine holonomy $\Gamma$ is irreducible and it preserves no (semi)-algebraic subset of $\mathbb{R}^n$. (See [9] and [10] for references.) So if a compact convex affine manifold $M = \Omega/\Gamma$ has a parallel volume, the affine automorphism group $G = \text{Aut}(\Omega)$ of the developing image $\Omega \subset \mathbb{R}^n$ is irreducible. Hence either $\Omega$ has no boundary, i.e., $\Omega = \mathbb{R}^n$, or $\Lambda_{G^0}$ must be empty by Proposition 4 and Proposition 5, which completes the proof for Theorem 1.

□
7. Markus conjecture in dimension $\leq 5$

In the previous section, we proved the Markus conjecture in the affirmative for a convex affine manifold $M$ under the condition of $\Lambda_{\text{Aut}^0(D(\tilde{M}))} \neq \emptyset$. In this section, we will see that the Markus conjecture is true for dimension $\leq 5$, by proving that the condition of $\Lambda_{\text{Aut}^0(D(\tilde{M}))} \neq \emptyset$ holds if the affine manifold is not complete. To see this we need to prove that every divisible convex affine domain has a nonempty limit set by the identity component of its affine automorphism group if it is not the whole affine space.

**Theorem 9.** Let $\Omega$ be a divisible convex affine domain in $\mathbb{R}^n$ with $n \leq 5$. Then $\Lambda_{\text{Aut}^0(\Omega)} \neq \emptyset$ if $\Omega \neq \mathbb{R}^n$.

**Proof.** By Theorem 4, every divisible convex affine domain $\Omega$ is a cone if it has no complete line, which implies that $\Lambda_{\text{Aut}^0(\Omega)} \neq \emptyset$ because the cone point must in $\Lambda_{\text{Aut}^0(\Omega)} \neq \emptyset$.

If $\Omega$ has a complete line, then $\Omega$ can be decomposed into $\mathbb{R}^k$ and a properly convex quasi-homogeneous affine domain $\Omega'$ in $\mathbb{R}^{n-k}$, that is,$\Omega = \mathbb{R}^k \times \Omega'$.

Note that $\Omega'$ may not be divisible, but it is still quasi-homogeneous. Therefore we need to show $\Lambda_{\text{Aut}^0(\Omega')} \neq \emptyset$ to complete the proof. We will see in Propositions 6, 7, 8, 9, 10, 11 that $\Omega'$ is affinely equivalent to one of the following 19 types of domains:

(i) $\{x \in \mathbb{R} \mid x > 0\}$
(ii) $\{(x, y) \in \mathbb{R}^2 \mid x > 0, \ y > 0\}$
(iii) $\{(x, y) \in \mathbb{R}^2 \mid y > x^2\}$
(iv) $\{(x, y, z) \in \mathbb{R}^3 \mid z > x^2 + y^2\}$
(v) $\{(x, y, z) \in \mathbb{R}^3 \mid y > x^2, \ z > 0\}$
(vi) $\{(x, y, z) \in \mathbb{R}^3 \mid x > 0, \ y > 0, \ z > 0\}$
(vii) a 3-dimensional elliptic cone,
(viii) a 3-dimensional non-elliptic strictly convex cone,
(ix) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 > x_1^2 + x_2^2 + x_3^2\}$
(x) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 > x_1^2, \ x_3 > 0, \ x_4 > 0\}$
(xi) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid (x_2 - x_1^2)x_3 > x_4^2\}$
(xii) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 > x_1^2 + x_2^2, \ x_4 > 0\}$
(xiii) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 > x_1^2, \ x_4 > x_3^2\}$

and

(xiv) a 4-dimensional elliptic cone,
(xv) a 4-dimensional non-elliptic strictly convex cone,
(xvi) a double cone over a triangle, i.e.,

\[\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_i > 0 \text{ for } i = 1, 2, 3, 4\},\]

(xvii) a double cone over an ellipse,
(xviii) a double cone over a non-elliptic strictly convex domain,
(xix) a cone over a 3-dimensional non-strictly convex indecomposable projective domain.
(i) is for $n - k = 1$, (ii) and (iii) are for $n - k = 2$, (iv) and (viii) are for $n - k = 3$, (ix)-(xiii) and (xiv)-(xix) are for $n - k = 4$. Each of the above 19 types of domains is either homogeneous or a cone. If $\Omega$ is homogeneous then $\Lambda_{\text{Aut}^0(\Omega')} = \partial \Omega'$, and if $\Omega$ is a cone then its cone point is in the limit set $\Lambda_{\text{Aut}^0(\Omega')}$. □

**Proposition 6** ([3], [11]). Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^2$. Then $\Omega$ is affinely equivalent to either a quadrant or a parabola.

**Proposition 7** (Jo, [11]). Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^3$. Suppose $\Omega$ is not a cone. Then $\Omega$ is affinely equivalent to one of the following:

(i) A 3-dimensional paraboloid, i.e.,
$$\Omega = \{(x, y, z) \in \mathbb{R}^3 | z > x^2 + y^2\}.$$

(ii) A parabola $\times \mathbb{R}^+$, i.e.,
$$\Omega = \{(x, y) \in \mathbb{R}^2 | y > x^2\} \times \{z \in \mathbb{R} | z > 0\} = \{(x, y, z) \in \mathbb{R}^3 | y > x^2, \ z > 0\}.$$

**Proposition 8** (Jo, [11]). Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^4$ with 1-dimensional asymptotic cone. Then $\Omega$ is a 4-dimensional paraboloid, i.e., $\Omega$ is affinely equivalent to $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_4 > x_1^2 + x_2^2 + x_3^2\}$.

**Proof.** This is an immediate consequence of Theorem 5.9 in [11]. □

**Proposition 9.** Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^4$ with 3-dimensional asymptotic cone. Then $\Omega$ is affinely equivalent to one of the following:

(i) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_2 > x_1^2, \ x_3 > 0, \ x_4 > 0\}$

(ii) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | (x_2 - x_1^2)x_3 > x_4^2\}$

**Proof.** The set of asymptotic cone points $S$ is a curve in $\mathbb{R}^4$ by Theorem 5 because $\text{AC}(\Omega)$ is 3-dimensional. Let $S_\infty$ is the set of limit points of $S$ in $\mathbb{RP}^4$. Then $S_\infty$ consists of at most 2-points. Since $\text{Aut}(\Omega)$ acts on $S$ transitively, $S_\infty$ must be contained in the infinite boundary.

Consider the convex hull $\text{CH}(S)$ of $S$. Since all points in $S$ are extreme points, the dimension of the minimal projective subspace $V$ containing $S$ is greater than 1. Let $F$ be the relative interior of the closure of $\text{CH}(S)$. Then $F$ is an invariant face of $\Omega$ with dimension greater than 1.

By [13], $\text{AC}(\Omega)$ is quasi-homogeneous. So there are two cases: $\text{AC}(\Omega)$ is either a simplex cone or a strictly convex cone.

(i) Assume that
$$\text{AC}(\Omega) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 > 0, \ x_2 > 0, \ x_3 > 0\}.$$ 

Let $\{a, b, c\}$ be the set of infinite extreme points of $\Omega$. Then $\text{AC}(\Omega)$ is a tetrahedron with vertices $a, b, c$ and the origin in $\mathbb{R}^4$, when we consider it as a projective domain in $\mathbb{RP}^4$. We denote the infinite face of $\Omega$ by $\triangle_{abc}$. If we let $G$ be the set of
all the element of Aut(Ω) fixing each of the infinite extreme points a, b, c, then G
also acts on Ω syndetically.

First, we show that \( S_\infty \) cannot have an interior point of \( \triangle_{abc} \). This follows from
the fact that the projectivization of the linear parts of \( G < \text{Aut}(\Omega) \) acts on the
infinite boundary of \( \Omega \) syndetically and the fact that if \( G \) acts on a domain \( D \)
syndetically then \( G \)-invariant set cannot intersect \( D \).

Now we know that \( S_\infty \subset \partial \triangle_{abc} \). If \( s' \) is a point in \( S_\infty \) which is not in \( \{a, b, c\} \),
say \( s' \) is in the open line segment \((a, b)\), then another point in \( S_\infty \) should be also in
the closed line segment \([a, b]\) if exists. Suppose \( s'' \in S_\infty \) is not in \([a, b]\). Then the
line \((s', s'')\) is preserved by the action of the projectivization of the linear parts of
\( \text{Aut}(\Omega) \), which is again a contradiction. Hence we may assume that \( S_\infty \subset [a, b] \).

Since \([a, b]\) is preserved under the action of \( G \), there is a sequence of projective
transformation \( g_i \in G < \text{Aut}_{\text{proj}}(\Omega) \) whose limit singular projective transformation
is \( g \) and the range of \( g \) is \( \{a\} \). Then
\[ S \text{ and } \xi \text{ is in } K(g), \] which shows \( \text{CH}(S) \subset K(g) \). So the dimension of \( F \) cannot be
4, i.e., \( \text{CH}(S) \neq \Omega \), since \( K(g) \cap \Omega = \emptyset \).

If the dimension of \( F \) is 3, then \( F \) is a conic face and thus there is an (infinite)
boundary point \( z \) such that
\[ \Omega = F \dot{+} \{z\}, \]
which contradicts that $AC(\Omega)$ is an elliptic cone.

Since the dimension of $F$ is greater than 1, it is 2, and so $F$ is a strictly convex 2-dimensional invariant face of $\Omega$. Note that $\partial F = S$ is twice-differentiable, since the smooth Lie group $\text{Aut}(\Omega)$ acts on $S$ transitively. By Lemma 1 there is a sequence of affine transformation $\{g_i\} \subset \text{Aut}(\Omega)$ converging to a singular projective transformation $g$ whose range is an extreme point $p$. Since the kernel of $g$ does not intersect $\mathbb{R}^4$ by Lemma 3, $g_i(x)$ converges to $p$ for any $x \in \mathbb{R}^4$. Since $F$ is an invariant face of $\Omega$, the restriction of $g_i$ on the support of $F$ is an element of $\text{Aut}(F)$ and $g_i(x) \in F$ converges to $p \in \partial F$ for $x \in F$. This implies that $F$ is a parabola because a strictly convex domain with a twice differentiable boundary is a parabola if an orbit accumulates at a boundary point. See [14].

We have shown up to now that $F$ is a parabola and its infinite boundary is a fixed point $\xi \in \partial_{\infty}\Omega$. So we can choose a basis $\{x_1, x_2, x_3, x_4\}$ of $\mathbb{R}^4$ such that $x_2$ is the invariant direction going to $\xi$, that is, $\lim_{r \to \infty} r x_2 = \xi$,

$$F = \{(x_1, x_2, 0, 0) \in \mathbb{R}^4 \mid x_2 > x_1^2\}. $$

and

$$AC(\Omega) = \{(0, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 x_3 \geq x_4^2\}. $$

Since $\bar{\Omega}$ is the union of all the asymptotic cones whose cone points are extreme points of $\Omega$, that is,

$$\Omega = \bigcup_{s \in S} s + AC^o(\Omega). $$

and each ray $s + r x_2 \subset \bar{F}$ for $s \in S$, we can conclude that

$$\Omega = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid (x_2 - x_1^2)x_3 > x_4^2\}. $$

\[\square\]

**Proposition 10.** Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^4$ with 2-dimensional asymptotic cone. Then $\Omega$ is affinely equivalent to one of the following:

(i) A paraboloid $\times \mathbb{R}^+$, i.e.,

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > x_1^2 + x_2^2\} \times \{x_4 \in \mathbb{R} \mid x_4 > 0\}$$

$$= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 > x_1^2 + x_2^2, \ x_4 > 0\}. $$

(ii) A parabola $\times$ a parabola :

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 > x_1^2, \ x_4 > x_3^2\}. $$

**Proof.** In this case $S$ is a 2-dimensional hypersurface in $\mathbb{R}^4$ and $\Omega$ has two extreme points in the infinite boundary. Let $z$ and $w$ be the infinite extreme points of $\Omega$. Then $AC(\Omega)$ is a triangle with three vertices $z$, $w$ and the origin. The dimension of $F = CH(S)^o$ is either 3 or 4. Let $G$ be the set of all the element of $\text{Aut}(\Omega)$ fixing both $z$ and $w$. Then $G$ also acts on $\Omega$ sydnetically and thus acts on $S$ transitively by Theorem 6.

(i) If $\dim F = 3$, then $F$ is a conic face of $\Omega$ and thus there is a face $B$ of $\Omega$ such that $\Omega = F \dot{\cup} B$ by Theorem 8 (iii). So one of $z$ or $w$ is in $\overline{F}$ and the other is in $\overline{B}$ by Corollary
Suppose that $w$ is in $\overline{F}$. Then we see that $F$ is a 3-dimensional quasi-homogeneous affine domain with a 1-dimensional asymptotic cone by Theorem 7. By Proposition 7, $\partial F$ is a paraboloid and as a projective domain in $\mathbb{RP}^4$

\[ \Omega = F + B = F + \{z\}. \]

Therefore $\Omega$ is affinely equivalent to \{(x₁, x₂, x₃) ∈ $\mathbb{R}^3 | x₃ > x₁² + x₂² \} × \{x₄ ∈ \mathbb{R} | x₄ > 0\}.

(ii) If $\dim F = 4$, then $\Omega = F$ and thus $\overline{\Omega} = CH(S)$. Note that $\Omega$ cannot have any conic face in this case by (iii) of Corollary 1. Let $z$ and $w$ be the unit vectors of $\mathbb{R}^4$ in the direction of $z$ and $w$ respectively. Suppose that all the rays $\{e + rw | r > 0, e \in S\}$ are one dimensional faces of $\Omega$. Choose a 3-dimensional affine space $V$ in $\mathbb{R}^4$ which is transversal to $w$, and project $\Omega$ into $V$ along $w$. If we denote the projection by $p : \Omega \rightarrow V$, then $p(\Omega) \subset V$ is a 3-dimensional convex quasi-homogeneous affine domain. Since every extreme point $e$ of $\Omega$ is projected to an extreme point $p(e)$ of $p(\Omega)$ if $\{e + rw | r > 0\}$ is a one dimensional face, $p(\Omega)$ cannot contain any complete line. (For, if there is a complete line $l$ in $\overline{p(\Omega)}$ then $p(e) + l$ should be also contained in $\overline{p(\Omega)}$ by convexity.) So there are linearly independent hyperplanes $H'_1, H'_2, H'_3$ in $V$ which support $p(\Omega)$. If we denote the hyperplanes of $\mathbb{R}^4$ which contain $p^{-1}(H'_1), p^{-1}(H'_2), p^{-1}(H'_3)$ by $H_1, H_2, H_3$ respectively in $\mathbb{R}^4$, then $H_1, H_2, H_3$ are linearly independent supporting hyperplanes of $\Omega$. Now if we see $\Omega$ in $\mathbb{RP}^4$, then $H_1, H_2, H_3$ and $\partial_{\overline{\mathbb{R}}^4}$ are linearly independent supporting hyperplanes of the extreme point $w$. This implies $w$ is a conic face of $\Omega$, which is a contradiction. In a similar manner, we can show the same thing for $z$.

So we can choose faces $A$ and $B$ whose closures contain properly a ray $\{\xi + rw | r > 0\}$ and a ray $\{\xi' + rz | r > 0\}$ respectively for some $\xi, \xi' \in S$. If either $A$ or $B$ is 3-dimensional, then it is a conic face, which is not allowed. Hence both $A$ and $B$ are properly convex 2-dimensional affine domains with a 1-dimensional asymptotic cone. This implies that every asymptotic cone point $\xi$ is contained in the boundary of two 2-dimensional faces $A_\xi$ and $B_\xi$ whose asymptotic directions are $w$ and $z$ respectively, since the subgroup $G$ of $Aut(\Omega)$ acts on $S$ transitively.

For each extreme point $\xi$, we see that there does not exist any other proper face except $A_\xi$ and $B_\xi$ whose closure contains $\{\xi\}$ properly, because any line segment in $\partial \Omega$ must be contained in the closure of a face having a ray with direction $z$ or $w$. So $A_\xi$ and $B_\xi$ are maximal and the only faces of $\Omega$ whose dimension is neither 0 nor 4. By Proposition 2 and Corollary 1, $\partial A_\xi \subset S$ and thus $A_\xi$ is a strictly convex 2-dimensional face of $\Omega$ whose boundary is a part of $S$.

Since there are no other nonzero-dimensional boundary faces of $\Omega$ except $A_\xi$’s and $B_\xi$’s, either $A_\xi \cap A_{\xi'} = \emptyset$ or $A_\xi = A_{\xi'}$ holds and the same is true for $B_\xi$’s. Note that $A_\xi = A_{\xi'}$ if and only if $\xi' \in \partial A_\xi$. Let $S_\xi$ be a subset of $S$ defined as follows:

\[ S_\xi = \bigcup_{\zeta \in \partial A_\xi} \partial B_\xi \]

Then $S_\xi \cap S_{\xi'} = \emptyset$ for $\xi' \notin S_\xi$. Therefore $S = S_\xi$ for any extreme point $\xi$ and thus $\partial B_{\xi_1} \cap \partial A_{\xi_2} = \emptyset$ for any pair of extreme points $\xi_1$ and $\xi_2$, since $S$ is connected. Similary
one can prove

\[ S = \bigcup_{\eta \in \partial B_{\xi}} \partial A_{\eta}. \]

If we consider the subgroups \( G_{A_{\xi}} \) (\( G_{B_{\xi}} \), respectively) of \( G \) preserving \( A_{\xi} \) (\( B_{\xi} \), respectively), then \( G_{A_{\xi}}(G_{B_{\xi}}, \text{respectively}) \) acts on \( \partial A_{\xi}(\partial B_{\xi}, \text{respectively}) \) transitively and thus the boundary of \( A_{\xi} \) and \( B_{\xi} \) are both twice-differentiable, which means each of \( A_{\xi} \) and \( B_{\xi} \) has an osculating ellipsoid at every boundary point by Remark [1]. There are two continuous maps \( \rho_A \) and \( \rho_B \) from \( \mathbb{R} \) to \( G_{A_{\xi}} \) and \( G_{B_{\xi}}, \text{respectively} \), such that \( \{\rho_A(t)(\xi) \mid t \in \mathbb{R}\} = \partial A_{\xi} \) and \( \{\rho_B(t)(\xi) \mid t \in \mathbb{R}\} = \partial B_{\xi} \). Then each of \( \rho_A(t) \)-orbits and \( \rho_B(t) \)-orbits gives a foliation on \( \Omega \cap \mathbb{R}^4 \) whose leaves are homeomorphic to \( \partial A_{\xi} \) and \( \partial B_{\xi}, \text{respectively} \).

Now we claim here that \( A_{\xi} \) is a parabola. Since \( \Omega \) is quasi-homogeneous and \( \xi \) is an extreme point, there is a sequence of affine transformation \( \{g_i\} \subset G \) such that the range of its limit singular projective transformation \( g \) is \( \{\xi\} \) by Lemma [4]. Since the kernel of \( g \) does not intersect \( \mathbb{R}^n \) by Lemma [3], \( g_i(p) \) converges to \( \xi \) for any point \( p \) in \( A_{\xi} \). Let \( \xi' \) be the boundary point of \( B_{\xi} \) such that \( g_i(\xi') \subset A_{\xi} \). Then \( \xi' \) is guaranteed by the property \( S = \bigcup_{\eta \in \partial B_{\xi}} \partial A_{\eta} \). Then after each \( g_i \) there is \( t_i \in \mathbb{R} \) such that \( \rho_B(t_i)(\xi') = \xi' \). Then \( \rho_B(t_i)^{-1}g_i(\xi) \) is in \( A_{\xi} \) for all \( i \). Since the sequence of leaves \( \{\rho_B(t)(\rho_B(t_i)^{-1}g_i)(p) \mid t \in \mathbb{R}\} = \{\rho_B(t)(g_i(p)) \mid t \in \mathbb{R}\} \) converges to \( \{\rho_B(t)(\xi) \mid t \in \mathbb{R}\} = \partial B_{\xi} \), the leaf containing \( \xi, \rho_B(t_i)^{-1}g_i(p) \) must converge to \( \xi \). So we found a sequence of projective transformations \( f_i = \rho_B(t_i)^{-1}g_i \) preserving \( A_{\xi} \) such that an orbit \( f_i(\xi') \) converges to \( \xi' \). This implies that \( A_{\xi} \) is a parabola due to Theorem [3] stating that a strictly convex domain with a twice differentiable boundary is a parabola if there is an orbit accumulating at a boundary point.

Similarly, we can show that \( B_{\xi} \) is also a parabola, by taking a sequence of projective transformations \( \{\rho_A(t_i)\} \subset G_{B_{\xi}} \) such that \( \rho_A(t_i)^{-1}g_i(B_{\xi}) = B_{\xi} \) and \( \rho_A(t_i)^{-1}g_i(p') \) converges to \( \xi \) for \( p' \in B_{\xi} \).

Now we will show that \( \Omega \) is affinely equivalent to

\[ \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 > x_1^2, \ x_3 > x_4^2\}; \]

by proving that any other \( A_{\xi'} \) is obtained from \( A_{\xi} \) by just an affine translation.

We first show that the isotropy subgroup \( G_{\xi} < G \) fixing \( \xi \) is not trivial. If we suppose that \( G_{\xi} \) is trivial, then \( G \) acts on \( S \) simply transitively and thus \( G \) is connected, which implies that \( \Omega \) is homogeneous by Lemma 2.5 of [1]. But the isotropy subgroup of a homogeneous properly convex affine domain cannot be trivial because any element of \( G \) sending a point of \( \xi + AC(\Omega) \) to another point of the cone fixes \( \xi \), which contradicts our assumption.

Choose two vectors \( a \) and \( b \) such that \( \xi + a \) is in the support of \( A_{\xi} \) and \( \xi + b \) is in the support of \( B_{\xi} \), and \( \{a, w, z, b\} \) is a basis for \( \mathbb{R}^4 \). Then there is a non-trivial element \( g \in G_{\xi} \)
such that
\[
g = \begin{pmatrix}
\delta & 0 & 0 & 0 \\
0 & \delta^2 & 0 & 0 \\
0 & 0 & \theta^2 & 0 \\
0 & 0 & 0 & \theta
\end{pmatrix}
\]
with \( \delta > 0 \) and \( \theta > 0 \), if we normalize \( \xi = (0, 0, 0, 0) \) and the parabolas \( \partial A_\xi \) and \( \partial B_\xi \) by \( x_2 = x_1^2 \) and \( x_3 = x_4^2 \) respectively, since \( g \) must preserve both \( A_\xi \) and \( B_\xi \).

If \( \delta = 1 \), then \( g \in G_\zeta \) for all \( \zeta \in \partial A_\xi \), that is,
\[(7.1) \quad g(\zeta) = \zeta, \quad g(B_\zeta) = B_\zeta \quad \text{for all} \quad \zeta \in \partial A_\xi.
\]
So we can prove that \( B_\zeta \) must be just a translation of \( B_\xi \) for any \( \zeta \in \partial A_\xi \). Suppose not. Then there is \( \zeta \in \partial A_\xi \) such that for some \( r > 0 \) and \( c, d_1, d_2 \in \mathbb{R} \)
\[
\partial B_\zeta = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = r(x_4 - c)^2 - rc^2, \ x_4 = d_1x_1 + d_2x_2 \} + \zeta,
\]
which is not equal to
\[
\partial B_\zeta = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = x_4^2 \}.
\]
But this implies that
\[
g(\partial B_\zeta) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = r(x_4 - \theta c)^2 - r\theta c^2, \ x_4 = \theta(d_1x_1 + d_2x_2) \} + \zeta,
\]
which contradicts (7.1). Similarly, we also get that \( A_\zeta \) must be just a translation of \( A_\xi \) for any \( \zeta \in \partial B_\xi \) if \( \theta = 1 \). Hence we may assume that \( \delta \neq 1 \) and \( 0 < \theta < 1 \) by taking the inverse \( g^{-1} \) if necessary.

Since \( G_{B_\xi} \) is a closed Lie subgroup of \( G \) and \( G_{B_\xi}(\xi) = \partial B_\xi \), we can take an element \( \eta \) in the Lie-subalgebra \( g_{B_\xi} \) of \( g^0 \) such that the one parameter group \( f_t = e^{t\eta} \) is non-trivial.

**Case 1. \( f_t \) is hyperbolic:** If \( f_t \) is hyperbolic, then we may assume that \( f_t(\xi) = \xi \) and thus we get a one-parameter subgroup \( g(t) \) of \( G_\xi \),
\[
f_t = g(t) = e^{t\mu}
\]
and another element \( g' \) of \( G_{B_\xi} \) such that \( g'(\xi) = (0, 0, 1, 1) \) by the transitivity of the action of \( G \) on \( S \). We can choose an element \( f \) of \( G_{B_\xi} \) among \( g(t)g' \) acting on the face \( B_\xi \) as a parabolic transformation as in the following lemma.

**Lemma 8.** The linear part and translation part of \( f \) can be represented by
\[
L_f = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 \\
\beta_1 & \alpha_2 & 0 & 0 \\
\beta_2 & 0 & 1 & 2d \\
\beta_3 & 0 & 0 & 1
\end{pmatrix}, \quad t_f = \begin{pmatrix}
0 \\
0 \\
d^2 \\
d
\end{pmatrix}
\]
for some positive real number \( d \).
Proof. Since \( g' \) preserves \( B_\xi \) and \( \mathbf{w}, \mathbf{z} \) directions, its linear part is of the form
\[
\begin{pmatrix}
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & 0 & * & * \\
* & 0 & 0 & *
\end{pmatrix}
\]
Then by choosing \( t \) properly, the \((3,3)\) component of linear part \( L_f \) of the product \( f = g(t)g' \) is 1. Since \( f \) also preserves \( B_\xi \) and \( \mathbf{w}, \mathbf{z} \) directions
\[
L_f = \begin{pmatrix}
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & 0 & 1 & a \\
* & 0 & 0 & b
\end{pmatrix}
\]
Since \( g' \) sends \( \xi \) to \((0, 0, 1, 1)\) the translation part \( t_f \) of \( f \) is \((0, 0, e^{2t\nu}, e^{t\nu}) = (0, 0, d^2, d)\) for some \( d \). Now using \( f(\partial B_\xi) = \partial B_\xi = \{ (0, 0, x^2, x) \} \) we get
\[
X_3 = x^2 + ax + d^2, \quad X_4 = bx + d, \quad X_4^2 = X_3.
\]
Hence \( b = 1, a = 2d \) as desired. \( \square \)

If \( \mu, \nu > 0 \) or \( \mu, \nu < 0 \) then \( \xi \in \Lambda_{G^0} \) and thus \( \Omega \) is homogeneous by Proposition 4, since \( \text{Aut}(\Omega) \) is irreducible by (iii) of Corollary 1, which implies that \( \Omega \) must be affinely equivalent to
\[
\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 > x_1^2, \ x_3 > x_4^2\}
\]
by the well-known classification of homogeneous projective convex domains in \( \mathbb{R}^4 \). (See p.282 of [21] for a reference.) So we may assume that \( \mu > 0 \) and \( \nu < 0 \). Then by considering
(7.2) \[
g(t_n)f(\xi) = f^n(\xi) = (0, 0, n^2d^2, nd)
\]
and
(7.3) \[
g(t_n)f^{-1}(\xi) = f^{-n}(\xi) = (0, 0, n^2d^2, -nd)
\]
for the sequence \( t_n = \frac{\ln n}{\nu}, n > 0 \), we can show that
\[
\alpha_1^2 = \alpha_2^2, \quad \beta_1 = \beta_2 = \beta_3 = 0
\]
as follows. Since the linear part of \( f^n \) is
\[
L_{f^n} = \begin{pmatrix}
\alpha_1^n & 0 & 0 & 0 \\
\beta_1^* & \alpha_2^n & 0 & 0 \\
\beta_2^* & 0 & 1 & 2nd \\
\beta_3^* & 0 & 0 & 1
\end{pmatrix}
\]
where
\[
\beta_1^* = \beta_1(\alpha_1^{n-1} + \alpha_1^{n-2} + \cdots + \alpha_1^{2n-4} + \alpha_2^{2n-2})
\]
\[
\beta_2^* = \beta_2(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{n-1}) + 2d\beta_3(n - 1 + (n - 2)\alpha_1 + \cdots + 2\alpha_1^{n-3} + \alpha_1^{n-2})
\]
\[
\beta_3^* = \beta_3(1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{n-1})
\]
we get

\[ f^n(x, x^2, 0, 0) = (\alpha_1^n x, \beta_1^n x + \alpha_2^n x^2, \beta_2^n x, \beta_3^n x) + (0, 0, n^2 d^2, nd) \]

and thus \( f^n(\partial A_\xi) \) are the set of all the points \((X_1, X_2, X_3, X_4) + (0, 0, n^2 d^2, nd)\) such that

\[
X_2 = \frac{\beta_1}{\alpha_1} (1 + \frac{\alpha_2}{\alpha_1} + \frac{\alpha_4}{\alpha_1^2} + \cdots + \frac{\alpha_{2n-2}}{\alpha_1^{n-1}} + \frac{\alpha_{2n}}{\alpha_1^n}) X_1 + \frac{\alpha_2}{\alpha_1^n} X_1^2 \\
X_3 = \frac{\beta_2}{\alpha_1} (1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} + \cdots + \frac{1}{\alpha_1^{n-1}}) X_1 + \frac{2d\beta_3}{\alpha_1} (\frac{1}{\alpha_1} + \frac{2}{\alpha_1^2} + \cdots + \frac{n-1}{\alpha_1^{n-1}}) X_1 \\
X_4 = \frac{\beta_3}{\alpha_1} (1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} + \cdots + \frac{1}{\alpha_1^{n-1}}) X_1.
\]

On the other hand, the elements of \( g(t_n) f(\partial A_\xi) \) are

\[
g(t_n) f(x, x^2, 0, 0) = (Y_1, Y_2, Y_3, Y_4) + (0, 0, n^2 d^2, nd) \\
Y_2 = \frac{\beta_1}{\alpha_1} e^{t_n \mu} Y_1 + \frac{\alpha_2}{\alpha_1^n} Y_1^2 \\
Y_3 = \frac{\beta_2}{\alpha_1} e^{t_n (2\nu - \mu)} Y_1 \\
Y_4 = \frac{\beta_3}{\alpha_1} e^{t_n (\nu - \mu)} Y_1.
\]

Since \( g(t_n) f(A_\xi) \) equals \( f^n(A_\xi) \) by \( 7.2 \), we see \( \alpha_1^2 = \alpha_2^2 \), and if \( \beta_1 \neq 0 \)

\[ e^{t_n \mu} = 1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{n-1}. \]

But \( \lim_{n \to \infty} e^{t_n \mu} = 0 \) and \( 1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{n-1} \) cannot converges to 0, which implies \( \beta_1 = 0 \).

Now we get

\[
L_f = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1^2 & 0 & 0 \\ \beta_2 & 0 & 1 & 2d \\ \beta_3 & 0 & 0 & 1 \end{pmatrix}, \quad t_f = \begin{pmatrix} 0 \\ 0 \\ d^2 \\ d \end{pmatrix}
\]

and

\[
L_{f^{-1}} = \begin{pmatrix} 1/\alpha_1 & 0 & 0 & 0 \\ 0 & 1/\alpha_1^2 & 0 & 0 \\ (2d\beta_3 - \beta_2)/\alpha_1 & 0 & 1 & -2d \\ -\beta_3/\alpha_1 & 0 & 0 & 1 \end{pmatrix}, \quad t_{f^{-1}} = \begin{pmatrix} 0 \\ 0 \\ d^2 \\ -d \end{pmatrix}
\]

Thus

\[
L_{f^{-n}} = \begin{pmatrix} 1/\alpha_1^n & 0 & 0 & 0 \\ 0 & 1/\alpha_1^{2n} & 0 & 0 \\ \beta_2 & 0 & 1 & -2nd \\ \beta_3 & 0 & 0 & 1 \end{pmatrix}, \quad n > 0
\]
where

\[ \beta_2 = \frac{-\beta_2}{\alpha_1} \left( 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} + \ldots + \frac{1}{\alpha_1^{n-1}} \right) + \frac{2d\beta_3}{\alpha_1^2} \left( n - \frac{n-1}{\alpha_1} + \frac{2}{\alpha_1^2} + \ldots + \frac{1}{\alpha_1^{n-1}} \right) \]

\[ \beta_3 = \frac{-\beta_3}{\alpha_1} \left( 1 + \frac{1}{\alpha_1^2} + \ldots + \frac{1}{\alpha_1^{n-1}} \right). \]

So we have

\[ f^{-n}(x, x^2, 0, 0) = \left( \frac{1}{\alpha_1^n} x, \frac{1}{\alpha_1^n} x^2, \tilde{\beta}_2 x, \tilde{\beta}_3 x \right) + (0, 0, n^2 d^2, -nd) \]

and thus \( f^{-n}(\partial A_\xi) \) are the set of all the points \((W_1, W_2, W_3, W_4) + (0, 0, n^2 d^2, -nd)\) such that

\[ W_2 = W_1^2 \]

\[ W_3 = -\beta_2(1 + \alpha_1 + \alpha_1^2 + \ldots + \alpha_1^{n-1})W_1 + 2d\beta_3(1 + 2\alpha_1 + 3\alpha_1^2 + \ldots + n\alpha_1^{n-1})W_1 \]

\[ W_4 = -\beta_3(1 + \alpha_1 + \alpha_1^2 + \ldots + \alpha_1^{n-1})W_1. \]

Note that

\[ g(t_\gamma) f^{-1}(x, x^2, 0, 0) = \begin{pmatrix} e^{t_\gamma \mu} & 0 & 0 & 0 \\ 0 & e^{2t_\gamma \mu} & 0 & 0 \\ 0 & 0 & e^{2t_\gamma \nu} & 0 \\ 0 & 0 & 0 & e^{t_\gamma \nu} \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha_1^n} x \\ \frac{1}{\alpha_1^n} x^2 \\ 2d\beta_3 - \beta_2 x + d^2 \\ \frac{\alpha_1^n}{\alpha_1} \beta_3 x - d \end{pmatrix} = (Z_1, Z_2, Z_3, Z_4) + (0, 0, n^2 d^2, -nd) \]

where

\[ Z_2 = Z_1^2 \]

\[ Z_3 = (2d\beta_3 - \beta_2)e^{t_\gamma (2\nu - \mu)} Z_1 \]

\[ Z_4 = -\beta_3 e^{t_\gamma (\nu - \mu)} Z_1. \]

Suppose \( \beta_3 \neq 0 \). Since

\[ g(t_\gamma) f(A_\xi) = f^n(A_\xi), \ g(t_\gamma) f^{-1}(A_\xi) = f^{-n}(A_\xi) \]

by (7.2) and (7.3),

\[ e^{t_\gamma (\nu - \mu)} = 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} + \ldots + \frac{1}{\alpha_1^{n-1}} \]

and

\[ e^{t_\gamma (\nu - \mu)} = 1 + \alpha_1 + \alpha_1^2 + \ldots + \alpha_1^{n-1} \]

must hold simultaneously, which implies \( \alpha_1 = 1 \). But this is impossible because

\[ e^{t_\gamma (\nu - \mu)} = ne^{-t_\gamma \mu} \]

and \( 1 + \alpha_1 + \alpha_1^2 + \ldots + \alpha_1^{n-1} = n \) if \( \alpha_1 = 1 \).
\( \beta_2 = 0 \) is proved similarly, since if we suppose \( \beta_2 \neq 0 \) then
\[
e^{\nu \mu}(2) = 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} + \cdots + \frac{1}{\alpha_1^{n-1}}\]
and
\[
e^{\nu \mu}(2) = 1 + \alpha_1 + \alpha_1^2 + \cdots + \alpha_1^{n-1}\]
must hold simultaneously, which is impossible.

Up to now we have shown that if \( f_t \) is hyperbolic then
\[
L_f = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_1^2 & 0 & 0 \\
0 & 0 & 1 & 2d \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
which implies that \( A_{f_t(\xi)} \) and \( A_{f_{t-1}(\xi)} \) are just translations of \( A_\xi \). Since for any point \( \xi \neq \xi \) of \( \partial B_\xi \), \( A_\xi \) is either \( g(t)(A_{f_t(\xi)}) \) or \( g(t)(A_{f_{t-1}(\xi)}) \) for some \( t \in \mathbb{R} \), \( A_\xi \)'s are all translations of \( A_\xi \). Hence we can conclude that \( \Omega \) is affinely equivalent to
\[
\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 > x_1^2, \ x_3 > x_4^2\}
\]
in this case.

**Case 2.** \( f_t \) is parabolic: If \( f_t \) is parabolic, then the linear part and the translation part of \( f_t \) can be represented by
\[
L_{f_t} = \begin{pmatrix}
\alpha_1(t) & 0 & 0 & 0 \\
\beta_1(t) & \alpha_2(t)^2 & 0 & 0 \\
\beta_2(t) & 0 & 1 & 2t \\
\beta_3(t) & 0 & 0 & 1
\end{pmatrix}, \quad t_{f_t} = \begin{pmatrix}
0 \\
0 \\
t \\
t \end{pmatrix}.
\]

Firstly, we show that \( \alpha_1(t)^2 = \alpha_2(t)^2 \) for all \( n \): Since \( f_{\theta t}(\xi) = g^n(f_t(\xi)) = (0, 0, \theta^{2n}t^2, \theta^n t) \), there is an element
\[
h_n(t) = \begin{pmatrix}
\delta_n(t) & 0 & 0 & 0 \\
0 & \delta_n(t)^2 & 0 & 0 \\
0 & 0 & \theta_n(t)^2 & 0 \\
0 & 0 & 0 & \theta_n(t)
\end{pmatrix} \in G_\xi
\]
such that \( f_{\theta t}h_n(t) = g^n f_t \). Hence we get that for all \( t \in \mathbb{R} \)
\[
\alpha_1(\theta^n t) \delta_n(t) = \delta^n \alpha_1(t), \quad \alpha_2(\theta^n t)^2 \delta_n(t)^2 = \delta^{2n} \alpha_2(t)^2, \quad \theta_n(t) = \theta^n.
\]
From
\[
\lim_{n \to \infty} \alpha_1(\theta^n t) = \alpha_1(0) \in \mathbb{R}^*, \quad \lim_{n \to \infty} \alpha_2(\theta^n t) = \alpha_2(0) \in \mathbb{R}^*,
\]
we get
\[
\lim_{n \to \infty} \frac{\delta^n}{\delta_n(t)} = \frac{\alpha_1(0)}{\alpha_1(t)} = \frac{1}{\alpha_1(t)}, \quad \lim_{n \to \infty} \frac{\delta^{2n}}{\delta_n(t)^2} = \frac{\alpha_2(0)^2}{\alpha_2(t)^2} = \frac{1}{\alpha_2(t)^2}
\]
and thus \( \alpha_1(t)^2 = \alpha_2(t)^2 \) for all \( t \).
Suppose that $\delta_n(t) = \delta_n(0)$ for all $t$ and all $n$. Then $\alpha_1(t) = \alpha_1(0) = 1$ because

$$1 = \frac{\alpha_1(0)}{\alpha_1(0)} = \lim_{n \to \infty} \frac{\delta_n}{\delta_n(0)} = \lim_{n \to \infty} \frac{\delta_n}{\delta_n(t)} = \frac{\alpha(0)}{\alpha(t)}$$

by (7.5). That is, if

$$h_n(t) = \begin{pmatrix}
\delta_n(0) & 0 & 0 & 0 \\
0 & \delta_n(0)^2 & 0 & 0 \\
0 & 0 & \theta^{2n} & 0 \\
0 & 0 & 0 & \theta^n
\end{pmatrix}$$

for all $t$, then

$$L_{f_t} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\beta_1(t) & 1 & 0 & 0 \\
\beta_2(t) & 0 & 1 & 2t \\
\beta_3(t) & 0 & 0 & 1
\end{pmatrix}.$$  

So from the fact that $f_{t+\nu} = f_t f_{\nu}$, we get

Lemma 9.

$$\beta_1(t) = t\beta_1(1), \quad \beta_3(t) = t\beta_3(1).$$

Proof. $f_{t+\nu} = f_t f_{\nu}$ implies that

$$\beta_1(t) + \beta_1(s) = \beta_1(t + s), \beta_1(0) = 0,$$

$$\beta_3(t) + \beta_3(s) = \beta_3(t + s), \beta_3(0) = 0.$$  

We show that $\beta_1$ and $\beta_3$ are linear. Note that

$$q\beta_1(t) = \beta_1(p\frac{q}{p}t) = p\beta_1(q\frac{t}{p}), \beta_1(t) + \beta_1(-t) = \beta_1(0) = 0.$$  

Hence $\beta_1(\frac{2}{p}t) = \frac{2}{p}\beta_1(t)$ and $\beta_1(rt) = r\beta_1(t)$ for any rational number $r$. By continuity of $\beta_1$ and the density of rational numbers in real numbers implies

$$\beta_1(Rt) = R\beta_1(t)$$

for any real number $R$, hence $\beta_1(t) = t\beta_1(1)$. The same argument gives $\beta_3(t) = t\beta_3(1)$.  

Since $g^n f_1(\xi) = f_{g^n}(\xi)$, the face $g^n f_1(A_\xi)$ equals the face $f_{g^n}(A_\xi)$. But

$$g^n f_1(\partial A_\xi) = \{g^n f_1(x, x^2, 0, 0) | x \in \mathbb{R}\}$$

$$= \{(\delta^n x, \delta^n x^2 + \delta^n \beta_1(1)x, \theta^{2n} \beta_2(1)x, \theta^n \beta_3(1)x) | X \in \mathbb{R}\} + (0, 0, \theta^{2n}, \theta^n)$$

and

$$f_{g^n}(\partial A_\xi) = \{f_{g^n}(x, x^2, 0, 0) | x \in \mathbb{R}\}$$

$$= \{(x, x^2 + \beta_1(\theta^n)x, \beta_2(\theta^n)x, \beta_3(\theta^n)x) | x \in \mathbb{R}\} + (0, 0, \theta^{2n}, \theta^n)$$

$$= \{(x, x^2 + \theta^n \beta_1(1)x, \beta_2(\theta^n)x, \theta^n \beta_3(1)x) | x \in \mathbb{R}\} + (0, 0, \theta^{2n}, \theta^n).$$
This implies $\delta = \theta = 1$, which is a contradiction.

Now we have a non-constant continuous function $\delta_n(t)$ from $\mathbb{R}$ to $\mathbb{R}$ for some $n$. So there is an element $h_n(t_0) \neq g^n$ of $G_\xi$ such that $\delta_n(t_0) \neq \delta^n$ and

$$h_n(t_0) = \begin{pmatrix}
\delta_n(t_0) & 0 & 0 & 0 \\
0 & \delta_n(t_0)^2 & 0 & 0 \\
0 & 0 & \theta^{2n} & 0 \\
0 & 0 & 0 & \theta^n
\end{pmatrix}$$

by (7.4), and thus we get an element $h = g^{-n}h_n(t_0) \in G_\xi$ such that

$$h = \begin{pmatrix}
\delta^{-n}\delta_n(t_0) & 0 & 0 & 0 \\
0 & \delta^{-2n}\delta_n(t_0)^2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \neq \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

From the existence of such an element of $G_\xi$, using a similar argument as before we can conclude that $\Omega$ is affinely equivalent to

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2 > x_1^2, \ x_3 > x_4^2\}.$$  

□

**Proposition 11.** Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^4$ with 4-dimensional asymptotic cone. Then $\Omega$ is a cone and affinely equivalent to one of the following:

(i) an elliptic cone,
(ii) a non-elliptic strictly convex cone,
(iii) a double cone over a triangle, i.e.,

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_i > 0 \text{ for } i = 1, 2, 3, 4\},$$
(iv) a double cone over an ellipse,
(v) a double cone over a non-elliptic strictly convex domain,
(vi) a cone over a 3-dimensional non-strictly convex indecomposable projective domain.

**Proof.** By Vey [25] or Theorem [5], $\Omega$ is a cone onto a 3-dimensional properly convex quasi-homogeneous projective domain $P\Omega$ in $\mathbb{RP}^3$. If $P\Omega$ is strictly convex, then $\Omega$ is either (i) or (ii). If $P\Omega$ has a 2-dimensional face $F$, then $F$ is a conic face of $P\Omega$ and thus quasi-homogeneous projective domain. Since $F$ is an ellipse if it is strictly convex and its boundary is twice differentiable and $F$ is a triangle if it is not strictly convex, $\Omega$ is (iii) or (iv) or (v). (vi) is the case when $\Omega$ is neither strictly convex nor has no 2-dimensional face.

Cones in (vi) are all actually divisible cones, which is immediate from the following proposition.

**Proposition 12.** Let $\Omega$ be a properly convex quasi-homogeneous projective domain in $\mathbb{RP}^3$ which is indecomposable and non-strictly convex. Then $\text{Aut}_{\text{proj}}(\Omega)$ is irreducible and discrete.
Proof. Suppose $\text{Aut}_{\text{proj}}(\Omega)$ is reducible and $L$ is a stable projective subspace of $\mathbb{RP}^3$. Then

$$L \cap \Omega = \emptyset \text{ and } L \cap \overline{\Omega} \neq \emptyset,$$

by Vey [25]. Since $\Omega$ has no 2-dimensional face, $L \cap \overline{\Omega}$ is a point or a closed line segment.

If $L \cap \Omega$ is a closed line segment $l$, then we can choose a one dimensional projective subspace $L'$ such that $L' \cap \overline{\Omega}$ is a line segment $l'$ such that $l \cap l' = \emptyset$. Since $\Omega$ is not strictly convex, there are infinitely many 1-dimensional faces. They cannot intersect in their interior because $\Omega$ cannot have 2-dimensional face and only two faces can meet at their end points because any extreme point cannot be a conic point by the indecomposability of $\Omega$. Now we consider a sequence of projective transformation $\{g_i\}$ in $\text{Aut}_{\text{proj}}(\Omega)$ which converges to a singular projective transformation $g$ whose range $R(g)$ is $L'$. By stability of $L$ and $l \cap l' = \emptyset$, the kernel $K(g)$ must be $L$, which implies that $l$ is a conic face of $\Omega$ by Lemma 5. This contradicts that $\Omega$ is indecomposable.

So $L \cap \overline{\Omega}$ is a point $\xi$ in $\partial \Omega$. Similarly we can find infinitely many maximal closed line segments which do not contain $\xi$. This time we consider a sequence of projective transformation $\{g_i\}$ in $\text{Aut}_{\text{proj}}(\Omega)$ which converges to a singular projective transformation $g$ whose range $R(g)$ is $\{\xi\}$. Since $\xi$ is a fixed point, every maximal closed line segment which is disjoint from $\{\xi\}$ should be in the kernel $K(g)$, which is a contradiction.

Up to now we’ve proved that $\text{Aut}_{\text{proj}}(\Omega)$ is irreducible. The discreteness of $\text{Aut}_{\text{proj}}(\Omega)$ follows immediately from the irreducibility of $\text{Aut}_{\text{proj}}(\Omega)$ by Proposition 4.2 of [1], since $\Omega$ is not homogeneous. □

Indecomposable non-strictly convex projective divisible domains in $\mathbb{RP}^3$ were studied by Y. Benoist in [2].

8. REMARKS

We have seen in the previous section that the Markus conjecture is true for convex affine manifolds when the dimension is $\leq 5$. Theorem 1 implies that if it is true that the limit set $\Lambda_{\text{Aut}^0(\Omega)}$ for any quasi-homogeneous domain $\Omega \neq \mathbb{R}^n$ is nonempty, then Markus conjecture is completely solved in convex case. So far, we have the following observation.

**Proposition 13.** Let $\Omega$ be a properly convex quasi-homogeneous domain in $\mathbb{R}^n$. Let $G^0$ be the identity component of $G = \text{Aut}(\Omega)$. Then $G^0$ is noncompact.

**Proof.** If $G^0$ is compact, it will have a fixed point $x_0$ in $\Omega$. Since $G^0$ is normal in $G$, $G^0$ will fix $Gx_0$ pointwise. This implies that $G^0$ is trivial because $CH(Gx_0) = \Omega$ by Proposition 1 so $\text{Aut}(\Omega)$ is a discrete group. But $\text{Aut}(\Omega)$ cannot be discrete because every properly convex divisible affine domain is a cone by Theorem 4 and the automorphism group of a cone is not discrete. So we conclude that $G^0$ is noncompact. □
Acknowledgements

This work was done at Korea Institute for Advanced Study (KIAS) while the first author was a visiting professor in 2016-2017. The first author is grateful for the warm hospitality during her stay.

References

[1] Y. Benoist, Convex divisible II, Duke Math. J. 120 (2003), no.1, 97-120.
[2] Y. Benoist, Convex divisible IV: Structure du bord en dimension 3, Invent. math. 164 (2006), 249-278.
[3] J. P. Benoist, Sur les varietes localement affines et projectives, Bull. Soc. Math. Fr. 88 (1960), 229-332.
[4] Y. V. Carrière, Autour de la conjecture de L. Markus sur les variétés affines, Invent. Math. 95 (1989), 615–628.
[5] D. Cooper, D. Long and S. Tillmann, On convex projective manifolds and cusps, Adv. Math. 277(4) (2015), 181–251.
[6] D. Fried, Distality, completeness and affine structures, J. Diff. Geom., 24 (1986), 265-273.
[7] D. Fried, W. M. Goldman and M. W. Hirsch, Affine manifolds with nilpotent holonomy, Comment. Math. Helv. 56 (1981), 487–523.
[8] W. M. Goldman, Two examples of affine manifolds, Pacific J. Math. 94 (1981), no. 2, 327–330.
[9] W. M. Goldman and M. W. Hirsch, The radiance obstruction and parallel forms on affine manifolds, Trans. Amer. Math. Soc. 286 (1984), 629-649.
[10] W. M. Goldman and M. W. Hirsch, Affine manifolds and orbits of algebraic groups, Trans. Amer. Math. Soc. 295 (1986), no. 1, 175–198.
[11] K. Jo, Quasi-homogeneous domains and convex affine manifolds, Topology Appl. 134 (2003), no. 2, 123–146.
[12] K. Jo, Homogeneity, Quasi-homogeneity and differentiability of domains, Proc. Japan Acad. Ser. A Math. Sci. 79 (2003), no, 9, 150-153
[13] K. Jo, Differentiability of quasi-homogeneous convex affine domains, J. Korean Math. Soc. 42 (2005), No. 3, 485-498.
[14] K. Jo, A rigidity result for domains with a locally strictly convex point, Adv. Geom. 8 (2008), No. 3, 315-328.
[15] K. Jo, Asymptotic foliation of quasi-homogeneous convex affine domains, Commun. Korean Math. Soc. 32 (2017), No. 1, 629-649.
[16] K. Jo and I. Kim, Convex affine domains and Markus Conjecture, Math. Z. 248 (2004), no. 1, 173–182.
[17] J. L. Koszul, Deformation des connexions localement plats, Ann. Inst. Fourier 18 (1968), 103–114.
[18] N. H. Kuiper, On convex locally projective spaces, Convegno Intern. Geom. Diff. Italy (1953), 200–213.
[19] L. Markus, Cosmological models in differential geometry, Mimeographed notes, Univ. of Minnesota, 1962, p. 58.
[20] L. Marquis, Around groups in Hilbert geometry, In Handbook of Hilbert geometry, IRMA Lectures in Mathematics and Theoretical Physics Vol. 22, 207–261.
[21] C. A. Rogers, Some problems in the Geometry of convex bodies, The Geometric Vein: The Coxeter Festschrift, Springer-Verlag (1981), 279–284.
[22] W. Rudin, Functional analysis, International series in Pure and Applied Mathematics, p.75.
[23] J. Vey, Une notion d’hyperbolicite sur les varietes localement plates, C. R. Acad. Sci. Paris Ser. AB 266 (1968), A622–A624.
[24] J. Vey, Sur les automorphismes affines des ouverts convexes dans les espaces numeriques, C. R. Acad. Sci. Paris Ser. AB 270 (1970), A249–A251.
[25] J. Vey, Sur les automorphismes affines des ouverts convexes saillants, Ann. Scuola Norm. Sup. Pisa 24 (3) (1970), 641–665.
[26] E. B. Vinberg and V. G. Kats, *Quasi-homogeneous cones*, Math. Notes 1 (1967), 231–235. (translated from Math. Zametki 1 (1967), 347–354)

[27] E. B. Vinberg, *The theory of convex homogeneous cones*, trans. Moscow math. Soc. 12 (1963), 340–403.

Division of Liberal Arts and Sciences, Mokpo National Maritime University, Mokpo, Chonnam, 58628, Korea

*E-mail address*: khjo@mmu.ac.kr

School of Mathematics, Korea Institute for Advanced Study, Seoul, 02455, Korea

*E-mail address*: inkang@kias.re.kr