Neutrino masses and tribimaximal mixing in the minimal renormalizable SUSY SU(5) Grand Unified Model with $A_4$ Flavor symmetry

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We analyze all possible extensions of the recently proposed minimal renormalizable SUSY SU(5) grand unified model with the inclusion of an additional $A_4$ flavor symmetry. We find that there are 5 possible Cases but only one of them is phenomenologically interesting. We develop in detail such Case and we show how the fermion masses and mixing angles come out. As prediction we obtain the neutrino masses of order of 0.1 eV with an inverted hierarchy.

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I. INTRODUCTION

The flavor puzzle is one of the most intriguing problem in particle physics. Questions like whether there is any reason behind the charged fermion mass hierarchy, why the quark mixings are so small while two of the lepton mixings are large or why one lepton mixing is so close to be maximal while the other large angle is not are still far to be satisfactorily solved.

These problems have usually been approached all together at the same time and a single mechanisms have been suggested for their solution. Recently \[1\]-\[76\] it has become manifest that the three flavor problems have to be approached in a different way and must be solved by introducing different mechanisms \[3\]-\[15\] for each of them. In particular, while hierarchies are described by continuous symmetries, the mixings can be explained by introducing discrete symmetries \[1\], \[2\]. For example, in \[3\]-\[15\] several attempts have been done to face the flavor puzzle by introducing flavor symmetries based in discrete finite groups such as $S_3$, $S_4$, $A_4$, $T'$, and so on.
Moreover some attempts \cite{[1, 16, 17, 18, 19]} have been done to embed the discrete symmetry, as the $A_4$ one, into larger, continuous groups to explain also the hierarchy among the 3rd and the other two generations. It is has been shown there how the discrete symmetry $A_4$ can help in solving both aspects of the flavor problem: lepton-quark mixing hierarchy and family mass hierarchy. The flavor symmetry $A_4$, as shown for example in \cite{[1, 15, 16, 17, 20]}, is also a very promising option if extended to a larger flavor group compatible with $SO(10)$-like gauge grand unification. For example, by embedding $A_4$ into a group like $SU(3) \times U(1)$ \cite{[1]}, it is possible to explain both large neutrino mixing and fermion mass hierarchy in a $SO(10)$ Grand Unified Theory of Flavor (GUTF).

The embedding of discrete symmetries in a grand unified theories is somehow simpler in $SU(5)$ GUTs \cite{[13, 20, 21, 22]}. In fact in this case matter fields belong to two different representations $10_T$ and $\bar{5}_T$ under the gauge group. So there is one degree of freedom more available in the flavor transformations.

Recently a supersymmetric renormalizable grand unified theory has been proposed in \cite{[77]} based on the $SU(5)$ gauge symmetry where the neutrino masses are generated through type I and type III seesaw mechanisms. Within this model it is possible to generate all fermion masses with the requested minimal Higgs sector, the existence of one massless neutrino is predicted and the leptogenesis mechanism can successfully be realized. Moreover it has been shown that the predicted decay of the proton and neutralino properties are in agreement with experimental evidence. This theory can be considered a simple renormalizable supersymmetric grand unified theory based on the $SU(5)$ gauge symmetry since it has the minimal number of superfields and free parameters.

In the framework of SUSY $SU(5)$ GUT models, we will investigate in this work the possibility of adding an $A_4$ flavor symmetry to constrain the flavor structure of the coupling constants. We will verify the predictions on the quark and lepton mixing angles obtained from it as well as the possibility to accommodate all the fermion masses and mixing.

We start considering a class of models requested to satisfy the following conditions:

- minimal, in the sense that they contain the minimal number of superfields (without any gauge singlet) compatible with experimental evidence;
- renormalizable, in the sense that we include all the operators of dimension less or equal to four compatible with our symmetry;
- supersymmetric, where SUSY is helpful in fixing the vacuum alignments;
- Grand Unified, where the gauge group is $SU(5)$, and flavor symmetric, where the flavor
group is $A_4$. Remarkably, there are only a few “Cases”, according to the field transformations, which are compatible with these requirements. Realistic models are selected by imposing that the actual fermion masses can be reproduced, i.e. there are not charged fermions with zero mass and at least two neutrinos are massive; the CKM mixing matrix is not the identity at tree level; the PMNS mixing matrix is close to the tribimaximal one. We will end up with the non trivial result that there is only one possible model or “Case” which is compatible with the experimental evidence.

In the next section we will present the field content of the model and detail the transformation properties of the fields according to the gauge and flavor groups for each of the initial five cases.

II. FIELD CONTENT AND $SU(5) \times A_4$ INVARIANCE

We start considering the field content of a simple renormalizable supersymmetric grand unified theory based on the $SU(5)$ gauge symmetry (the supersymmetric adjoint $SU(5)$ proposed in [77]). This comprises the following chiral superfields: \( 10_T, \ 5_T, \ 24_T, \ 5_H, \ 45_H, \ 5_H, \ 45_H, \ 24_H \) where the subindex \( T \) refers to matter fields, defined respect to R-parity, and \( H \) to Higgs; the index number is the representation dimension under $SU(5)$. The renormalizable operators allowed under $SU(5)$ invariance appearing in the supersymmetric superpotential are [77]:

\[
W_0 = y_1 \ 10_T \ 5_T \ 5_H + y_2 \ 10_T \ 5_T \ 45_H + y_3 \ 10_T \ 10_T \ 5_H + y_4 \ 10_T \ 10_T \ 45_H, \quad (1a)
\]

\[
W_1 = \gamma 5_T \ 24_T \ 45_H + \beta 5_T \ 24_T \ 5_H, \quad (1b)
\]

\[
W_2 = m_\Sigma \ 24_H \ 24_H + \lambda_\Sigma \ 24_H \ 24_H \ 24_H + m \ 24_T \ 24_T + \lambda \ 24_T \ 24_T \ 24_T, \quad (1c)
\]

\[
W_3 = m_5 5_H 5_H + \lambda_H 5_H 24_H 5_H + c_H 5_H 24_H 45_H +
+b_H 45_H 24_H 5_H + m_{45} 45_H 45_H + a_H 45_H 45_H 24_H, \quad (1d)
\]

where $\gamma, \beta, a_H, b_H, c_H$ and the $y/\lambda$'s, $m/\lambda$'s and $m'/\lambda$'s are coupling constants. The usual decomposition of the fields under the Standard Model gauge group is reported in [43]. Let us go now to study the transformation properties with respect the flavor discrete symmetry. We need to fix the $A_4$ representations assigned to the fields and the $A_4$ directions of the vevs of the Higgs scalars. For the sake of minimality, we impose these simple assumptions: a) the fields are either singlet or triplets of $A_4$, b) $24_H$ is flavor singlet, and c) all the operators in eqs. [1] are allowed by the flavor symmetry. Under these, there are only 5 possibilities and they are listed in table [I].

The trivial Case 1, where $A_4$ does not play any role, is the one discussed in [77] and in this case there is no symmetry in the flavor structure of the mass matrices of the fermions. In all the
other cases $10_T$ must be a triplet under $A_4$. Then there is a case where there are three $5_T$ singlets (possibly corresponding to different $1$ representations) and all the other fields, with the exception of the $24_H$ that is assumed to be singlet always, are triplets (Case 2); the case where $24_T$ is a singlet and all the other fields are triplets (Case 3); the case where all the Higgs fields are singlets while the matter fields are triplets (Case 4); finally we have the case where both matter and Higgs fields are triplet (Case 5). An extra freedom in the definition of the models corresponds to the

| Case | $10_T$ | $5_T$ | $24_T$ | $5_H$ | $45_H$ | $5_H$ | $45_H$ | $24_H$ |
|------|--------|-------|--------|-------|--------|-------|--------|--------|
| Case 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Case 2 | 3 | 1 | 3 | 3 | 3 | 3 | 3 | 1 |
| Case 3 | 3 | 3 | 1 | 3 | 3 | 3 | 3 | 1 |
| Case 4 | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 1 |
| Case 5 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 1 |

TABLE I: $A_4$ transformation assignments for matter and Higgs fields in the cases where all the operators in eq. (1) are allowed. Moreover we assume minimality in the Higgs sector, i.e. if a given Higgs transforms as 1 than there is only one of it. For the fields $10_T$ and $5_T$ “1” possibly means three fields, each one transforming as the inequivalent $1, 1'$, or $1''$ representations.

$A_4$-direction of the vevs of the Higgs scalars. It is well known [10] that the $A_4$ symmetry in a triplet representation can get a vev either by respecting $Z_3$ or $Z_2$, i.e. in the $(1, 1, 1)$ direction in the first case or in a direction where only one component is not zero like $(1, 0, 0)$ in the second case. As we will show, the only phenomenologically viable scenario is when both the $45_H$ and $5_H$ fields are $A_4$-triplets and their vevs are in two different, $T$ or $S$ compatible, directions, i.e. $\hat{n}_{111} = (1, 1, 1)$ and $\hat{n}_1 \simeq (1, 0, 0)$ respectively. The full study of the problem of vacuum alignment in the Higgs sector is beyond the scope of this work and will be investigated in detail in a forthcoming paper [78] by minimizing the Higgs potential.

Our objective in this work is mainly to check all different $A_4$ assignment for the fields (Cases 1-5 of the table) and study which of them can be phenomenologically interesting. In sec. III A we will investigate the predictivity of the model and we show that, despite the higher number of fields introduced, the flavor symmetry strongly constraints the physical observables even in the broken phase. In sec. III B we will first investigate the general structure of the mass matrices coming from the flavor symmetry invariance properties of the mass operators and then we investigate the fermion mass and mixing structure for each case. In section IV we consider the flavor predictions for the quark mixing and charged leptons while section V is specifically dedicated to the neutrinos.
and the full leptonic mixing matrix. Our results concerning the viability of the different cases can be summarized as follows:

**Case 1.** In this trivial case $A_4$ does not play any role and the flavor structure is not constrained by the flavor symmetry.

**Case 2.** Depending on the transformation of the three $5_T$ fields, we have different situations. If they all transform in the same way, i.e. as 1, then the down sector mass matrix $M_D$ and charged lepton mass matrix $M_E$ have one zero eigenvalue, for any vev direction. If two or three $5_T$ transform differently each other (i.e. as $1, 1', 1''$), then $M_D$ and $M_E$ have three non-zero eigenvalues and we get in principle a phenomenologically interesting case.

**Case 3.** The $M_D$, $M_E$ and $M_U$ matrices have three independent eigenvalues if and only if one vev is in the direction $(1, 1, 1)$ and another vev is in the direction $(1, 0, 0)$, however the predicted lepton mixing matrix is not of phenomenological interest.

**Case 4.** The CKM mixing matrix is diagonal, the neutrino mass matrix, $M_\nu$, has two zero eigenvalues.

**Case 5.** The three up sector quark masses can be reproduced only if the $45_H$ and $5_H$ acquire vev in the direction $(1, 0, 0)$ apart from a small correction for the $45_H$ vev like $(1, \epsilon, \epsilon)$. $M_\nu$ has at least two non-zero eigenvalues while $M_D$, $M_E$ have three independent eigenvalues. The CKM matrix contains the Cabibbo angle and the lepton mixing matrix is almost tribimaximal.

We conclude that only Cases 2 and 5 are of phenomenological interest. Of these two the mass matrices for the Case 2, where the three $5_T$ fields transform differently, have been already extensively discussed in literature [1, 15, 16, 17, 18, 20, 23, 24, 25, 26] in similar, although not identical settings. So the only case which present promising novelties and should further investigated is Case 5.

**III. FLAVOR STRUCTURES FROM SYMMETRY: Masses and Mixings**

In this section we will first concentrate our attention on the charged fermion mass matrices. As we will see in sec. III B, there are three possible flavor structures for the charged fermion masses, depending on the transformation properties under $A_4$ of the Higgs fields.

Then we will investigate the resulting phenomenology in each of the Cases listed in table (I). Any model should at least accomplish the following conditions to be phenomenologically relevant:
a) the charged fermion masses are not degenerated, b) at least two neutrino masses are not zero, c) the lepton mixing is almost tribimaximal and the quark mixing matrix is not diagonal and compatible with the Cabibbo angle. These statements strongly constrain the $A_4$ assignments.

First we will eliminate Case 4 by using the Cabibbo angle constraint. Then we will investigate Cases 2 and 3: by asking non zero charge fermion masses, we obtain that the vev of the $A_4$ triplets must be in two different directions. In the Case 2 it is not possible to obtain the non degenerate spectrum of the leptons and quarks with the exception when the three matter fields $\mathbf{5}_T$ transform as $1$, $1'$, and $1''$ under the $A_4$ flavor symmetry. In the Case 3 we obtain a reasonable fermion spectrum, a good quark mixing matrix, but a wrong lepton mixing matrix.

Finally we will study the remaining Case 5 and we will show that, to have two non zero neutrino masses, the vev of the $A_4$ triplets must be in two different directions. This constraint automatically guarantees non degenerated charged fermion masses, quark mixing compatible with the Cabibbo structure, and a lepton mixing matrix almost tribimaximal.

A. Predictivity and degrees-of-freedom counting

The coupling constants with three $A_4$ indices, i.e. $\gamma^{abc}$, $\beta^{abc}$, $\lambda^{abc}_\Sigma$, $\lambda^{abc}_H$, $c^{abc}_H$, and $a^{abc}_H$, have $27 = 3^3$ elements each. However, as shown in the Appendix of ref. [8], eq. (A2), any of the coupling constants with three $A_4$ indices contains only two independent elements. For example, in the so-called $S$-diagonal base, we have

$$
\gamma^{abc} \mathbf{5}_T^a \mathbf{24}_T^b \mathbf{45}_H^c = \gamma_1 \left( \mathbf{5}_T^2 \mathbf{24}_T^3 \mathbf{45}_H^1 + \mathbf{5}_T^3 \mathbf{24}_T^1 \mathbf{45}_H^2 + \mathbf{5}_T^1 \mathbf{24}_T^2 \mathbf{45}_H^3 \right) \\
+ \gamma_2 \left( \mathbf{5}_T^3 \mathbf{24}_T^2 \mathbf{45}_H^1 + \mathbf{5}_T^1 \mathbf{24}_T^3 \mathbf{45}_H^2 + \mathbf{5}_T^2 \mathbf{24}_T^1 \mathbf{45}_H^3 \right),
$$

and similarly for $\beta^{abc}$, $\lambda^{abc}_\Sigma$, $\lambda^{abc}_H$, $c^{abc}_H$, and $a^{abc}_H$. The 27 coupling constants have been reduced to only two corresponding to odd and even $S_3$ permutations.

One could worry about the fact that we are introducing in our model extra Higgs and fermion fields $\mathbf{24}_T$ with respect to the minimal SUSY $SU(5)$. Notice however that these extra fields do not reduce the model predictivity because of the inclusion of the $A_4$ symmetry. In fact the "Supersymmetric Adjoint $SU(5)$" contains four 3-by-3 matrices, two 3-vectors, and nine other constants, with a total of 51 parameters (most of them cannot be observed at low energy). Any of the models considered here contains less parameters, as a consequence of the general feature of the flavor symmetries. For example, as we have seen above, the 3-by-3 matrix of $\mathbf{5}_T$ for the operator in eq. (2) translate into the 2 coefficients $\gamma_1$ and $\gamma_2$. In the same way, it should be observed that
the number of Higgs vevs in our model is the same as the number in corresponding model without flavor symmetry \([77]\). This is because \(A_4\) triplets can get a vev either by respecting \(Z_3\) or \(Z_2\), i.e. each vev will only depend on one parameter.

B. Charged fermion mass matrices

We investigate in this section the general flavor structure of the coupling constants resulting in the mass matrices for the charged matter fermions after symmetry breaking.

It is instructive to see how we can extract important information about the mass matrices simply by examining all the possible situations of Higgs-matter coupling with respect to the \(A_4\) assignments and how is the structure of Higgs vacuum alignments. The case 5 will be investigated in further detail and finally its explicit mass matrices will be obtained.

1. Mass matrices general structures from \(A_4\)

After symmetry breaking, we have three possible type of flavor structures of the coupling constants, depending on the \(A_4\) transformation properties of the initial cubic Higgs-Matter (TTH) operators. It is instructive to list all the cases:

**Type I. The Higgs is an \(A_4\)-singlet.** Here, the mass matrix results from the coefficients of the product of a \(A_4\) singlet (the Higgs) with two \(A_4\) triplets (matter fields). In the Case 4, for example, the following operators appear:

\[
y_1 \, 10_T^a \, \bar{5}_T^a \, 5_H, \quad y_2 \, 10_T^a \, \bar{5}_T^a \, 45_H, \quad y_3 \, 10_T^a \, 10_T^a \, 5_H, \quad y_4 \, 10_T^a \, 10_T^a \, 45_H. \quad (3)
\]

In this situation, the resulting mass matrices are diagonal with elements proportional to 1, \(\omega\), and \(\omega^2\) (where \(\omega^3 = 1\)) depending on the singlet \(A_4\) properties.

**Type II. The Higgs is a triplet and one fermion field contains three \(A_4\)-singlets.** The mass matrix comes from the coefficients of the product of two triplets (one Higgs and one matter field) with a set of singlets (the other matter fields). This situation appears in Case 2, where we have operators as:

\[
y_i \, 10_T^a \, \bar{5}_T^i \, \bar{5}_H^a, \quad y_i \, 10_T^a \, \bar{5}_T^i \, 45_H^a. \quad (4)
\]

We can distinguish three subcases for the resulting mass matrix.
In the first subcase (IIA), under the hypothesis that the three singlets transform in the same way, the corresponding contribution to charged fermion mass matrix is of the form:

\[
M_f \sim \begin{pmatrix}
v_1 y^1 & v_1 y^2 & v_1 y^3 \\
v_2 y^1 & v_2 y^2 & v_2 y^3 \\
v_3 y^1 & v_3 y^2 & v_3 y^3
\end{pmatrix},
\]

where \(v_1, v_2, \) and \(v_3\) are the vevs of the Higgs \(A_4\)-triplet, while the \(y^i\) are the three coupling constants. Notice that the charged lepton and quark mass matrices are transpose of each other, due to the generic \(SU(5)\) properties.

In the second subcase (IIB), supposing that only two of the three singlets transform in the same way, we have a contribution of the form:

\[
M_f \sim \begin{pmatrix}
v_1 y^1 & v_1 y^2 & v_1 y^3 \\
v_2 y^1 & v_2 y^2 & \omega^2 v_2 y^3 \\
v_3 y^1 & \omega v_3 y^2 & \omega v_3 y^3
\end{pmatrix}.
\]

Finally, in the third subcase (IIC), supposing that three singlets transform in different ways, i.e. as 1, 1’ and 1” , the mass matrix is of the form:

\[
M_f \sim \begin{pmatrix}
v_1 y^1 & v_1 y^2 & v_1 y^3 \\
v_2 y^1 & \omega v_2 y^2 & \omega^2 v_2 y^3 \\
v_3 y^1 & \omega^2 v_3 y^2 & \omega v_3 y^3
\end{pmatrix}.
\]

**Type III. The Higgs and matter fields are \(A_4\)-triplets.** The mass matrix comes from the coefficients of the product of three triplets. For example in the Cases 3 and 5 we have

\[
y_1^{abc} \; \mathbf{10}^a \; \mathbf{5}^b \; \mathbf{5}^c_H, \quad y_2^{abc} \; \mathbf{10}^a \; \overline{\mathbf{5}}^b \; \overline{\mathbf{45}}^c_H,
\]

and the resulting mass matrix contributions from these operators for the down and charged leptons sectors are of the form:

\[
\begin{pmatrix}
0 & v_3 \gamma_2 & v_2 \gamma_1 \\
v_3 \gamma_1 & 0 & v_1 \gamma_2 \\
v_2 \gamma_2 & v_1 \gamma_1 & 0
\end{pmatrix},
\]

where \(v_i\) are the vevs of the Higgs \(A_4\)-triplet, while the \(\gamma\)'s come from the two independent \(A_4\) contractions of eq. [2].
Similarly, in Cases 2, 3 and 5, we have contributions coming from the operators

\[ y_3^{abc} 10_T^a 10_T^b 5_H^c, \quad y_4^{abc} 10_T^a 10_T^b 45_H^c, \]  

and the up sector mass matrix is of the form:

\[
\begin{pmatrix}
0 & v_3 \gamma & \pm v_2 \gamma \\
\pm v_3 \gamma & 0 & v_1 \gamma \\
v_2 \gamma & \pm v_1 \gamma & 0
\end{pmatrix},
\]

where the first (second) operator gives a symmetric (an antisymmetric) contribution.

2. Higgs vevs and the $A_4$ symmetry. Mixing and masses

Let us turn now to the relation between vacuum alignment and flavor symmetry. The $A_4$ symmetry allows a scalar Higgs $A_4$ triplet ($H_1, H_2, H_3$) to get the vev in only two possible directions (selected by the form of the Higgs potential \[73\]). In the so-called $S$-diagonal base, they are:

- $\langle H_i \rangle = v$ for every $i$ (i.e. the direction invariant under the operator $T$);
- $\langle H_i \rangle = v \delta_{i\bar{k}}$ for a given $\bar{k}$ or small corrections around it (i.e. the direction invariant under the operator $S$, for example $\langle H_i \rangle = v(1,0,0))$.

For this reason each contribution to the mass matrices can have a defined generic form.

For Type (II) (one matter field is a $A_4$-singlet): the mass matrices would become as:

\[
M_{f}^{\text{II}} \sim v \begin{pmatrix} y^1 & y^2 & y^3 \\ y^1 & y^2 & y^3 \\ y^1 & y^2 & y^3 \end{pmatrix}
\]

or

\[
M_{f}^{\text{II}} \sim v \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

or similar variants for the second form (i.e. matrices with only one non zero row).

For Type (III) (matter and Higgs fields are $A_4$-triplets): the mass matrices would become:

\[
M_{f}^{\text{III}} \sim v \begin{pmatrix} 0 & \gamma_2 & \gamma_1 \\ \gamma_1 & 0 & \gamma_2 \\ \gamma_2 & \gamma_1 & 0 \end{pmatrix}
\]
or

\[
M_{ff}^{III} \sim v \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \gamma_2 \\
0 & \gamma_1 & 0
\end{pmatrix},
\]

(13)

where again similar variants for the second form are allowed (i.e. matrices with only one non zero transposed element).

Any of the \(\mathbf{5}, \mathbf{45}, \bar{\mathbf{45}}, \bar{\mathbf{5}}\) Higgs fields appearing in the theory which are \(A_4\)-triplets can have a \(T\) or \(S\) invariant vev. We have a number, according to this, of the order of \(\sim 2^4\) different scenarios; we investigate the three most promising between them:

**Scenario (A):** \(\langle \mathbf{5} \rangle \propto \langle \mathbf{5} \rangle = \{v_5, v_5, v_5\}\) and \(\langle \mathbf{45} \rangle \propto \langle \mathbf{45} \rangle = \{v_{45}, 0, 0\}\).

**Scenario (B):** \(\langle \mathbf{5} \rangle \propto \langle \mathbf{5} \rangle = \{v_5, 0, 0\}\) and \(\langle \mathbf{45} \rangle \propto \langle \mathbf{45} \rangle = \{v_{45}, v_{45}, v_{45}\}\).

**Scenario (C):** A mixed case with \(\langle \mathbf{5} \rangle \propto \langle \mathbf{5} \rangle = \{v_5, 0, 0\}\), \(\langle \mathbf{45} \rangle = \{v_{45}, v_{45}, v_{45}\}\), and \(\langle \mathbf{45} \rangle = \{v_{45}, \delta v_{45}, \delta v_{45}\}\).

As it will appear in the next sections, scenarios (A) and (B) will turn out unsatisfactory because do not allow enough freedom to reproduce the three masses in the up sector while Scenario (C) will be phenomenologically interesting. We can now write more explicit expressions for the charged fermion mass matrices for each of these scenarios. They have the same form in all the scenarios but with distinct parameters.

For Type (II) the mass matrices in eqs. (5) have the form:

\[
\begin{pmatrix}
\tilde{Y}^1 & \tilde{Y}^2 & \tilde{Y}^3 \\
\tilde{y}^1 & \tilde{y}^2 & \tilde{y}^3 \\
\tilde{y}^1 & \tilde{y}^2 & \tilde{y}^3
\end{pmatrix},
\begin{pmatrix}
\tilde{Y}^1 & \tilde{Y}^2 & \tilde{Y}^3 \\
\tilde{y}^1 & \tilde{y}^2 & \omega \tilde{y}^3 \\
\tilde{y}^1 & \tilde{y}^2 & \omega \tilde{y}^3
\end{pmatrix}
\text{ or }
\begin{pmatrix}
\tilde{y}^1 & \omega \tilde{y}^2 & \omega^2 \tilde{y}^3 \\
\tilde{y}^1 & \omega \tilde{y}^2 & \omega \tilde{y}^3 \\
\tilde{y}^1 & \omega^2 \tilde{y}^2 & \omega \tilde{y}^3
\end{pmatrix}.
\]

(14)

For Type (III) the mass matrix in eq. (7) has the form:

\[
M_{ff}^{III} \sim \begin{pmatrix}
0 & \tilde{\gamma}_2 & \tilde{\gamma}_1 \\
\tilde{\gamma}_1 & 0 & \tilde{\Gamma}_2 \\
\tilde{\gamma}_2 & \tilde{\Gamma}_1 & 0
\end{pmatrix}.
\]

(15)

The parameters are given by different expressions in each of the vacuum alignment scenarios.

In the Scenario (A):

\[
\tilde{y}^i = \tilde{y}^i(v_5), \quad \tilde{Y}^i = \tilde{Y}^i(v_{45}, v_{45});
\]

\[
\tilde{\gamma}_i = \tilde{\gamma}_i(v_5), \quad \tilde{\Gamma}_i = \tilde{\Gamma}_i(v_{45}, v_{45});
\]

(16)
in Scenarios (B), (C):

\[ \tilde{y}^i = \tilde{y}^i(v_{45}), \quad \tilde{Y}^i = \tilde{Y}^i(v_5, v_{45}); \]

\[ \tilde{\gamma}^i = \tilde{\gamma}^i(v_{45}), \quad \tilde{\Gamma}^i = \tilde{\Gamma}^i(v_5, v_{45}). \]  \hfill (17)

The first mass matrix in eq. (14) have a zero eigenvalue and cannot be compatible with the experimental data. Only the second and the third mass matrix in eq. (14) have three non zero eigenvalues. We conclude that in our models there is at least a viable \( A_4 \) flavor solution when the field \( S_T \) composes of three singlets and they do not transform equivalently.

We will focus now in the mass matrix in eq. (15). This matrix has the attractive feature of having three independent eigenvalues. We will study it in two opposite limits. This matrix is diagonalized in general by left and right matrices as \( M^{III}_f V \) \( M^{diag}_f W^\dagger \). First, in the limit \( \tilde{\Gamma}^i \to \tilde{\gamma}^i \), the right and left mixing matrices become the same:

\[ V \sim W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \] \hfill (18)

(for the charged lepton the mixing matrices are the Hermitian conjugate of the quark ones) and the masses are given by (in the same limit):

\[ M^{diag}_f = \text{diag}(\tilde{\gamma}_1 + \tilde{\gamma}_2, \tilde{\gamma}_2 + \omega \tilde{\gamma}_1, \tilde{\gamma}_2 + \omega^2 \tilde{\gamma}_1). \] \hfill (19)

On the other side, in the limit \( \tilde{\Gamma}^i \gg \tilde{\gamma}^i \), one can use on the right side the mixing matrix

\[ W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \] \hfill (20)

and the three eigenvalues, or masses, are \( (0, |\tilde{\Gamma}_1|, |\tilde{\Gamma}_2|) \).

IV. FLAVOR PREDICTIONS IN THE QUARK MIXING AND FITS OF THE CHARGED FERMION MASSES

A. Considerations for cases 2 and 4

\textit{Cabibbo angle eliminates Case 4:} First of all we observe that in Case 4 all the charged fermion masses are exactly diagonal, so they cannot generate the CKM matrix at tree level. This is an
enough reason for considering this case of no phenomenological interest and it will not be discussed any further.

In Case 2 charged fermion masses require that the $\overline{5}_T$ transforms differently each other: the mass matrices $M_D$ and $M_E$ are of the Type (III) discussed before, they originate from coupling constants associated to operators where the Higgs is a triplet and one kind of fermion fields contains three $A_4$-singlets, i.e. they are of the form in eq. (14). The mass matrix $M_U$ is originated from a $A_4$-triplet Higgs, i.e. it is of the form in eq. (15). As discussed in the sec. IIIA the mass matrices $M_D$ and $M_E$ have three non zero eigenvalues only if the three $\overline{5}_T$ transforms as 1, 1’, and 1”.

As already mentioned, the mass and quark mixing matrices in this sub-case has been well studied in literature [1, 15, 16, 17, 18, 20, 23, 24, 25, 26] in other phenomenological scenarios and their detailed study in our concrete scenario will appear elsewhere [78].

B. Case 5: realistic charged fermion masses and quark mixing

Let us focus now in the most phenomenologically attractive case. The Case 5 includes the following explicit terms:

$$W_0 = Y_1^{abc} \, 10_T^a \, \overline{5}_T^b \, \overline{5}_H^c + Y_2^{abc} \, 10_T^a \, \overline{5}_T^b \, 45_H^c,$$
$$+ Y_3^{abc} \, 10_T^a \, 5H^b \, 5H^c + Y_4^{abc} \, 10_T^a \, 10_T^b \, 45H^c,$$  

$$W_1 = \gamma^{abc} \, \overline{5}_T^a \, 24_T^b \, 45_H^c + \beta^{abc} \, \overline{5}_T^a \, 24_T^b \, 5H^c,$$  

$$W_2 = m_\Sigma \, Tr(24_H \, 24_H) + \lambda_\Sigma \, Tr(24_H \, 24_H \, 24_H) +$$
$$m \, 24_T^a \, 24_T^a + \lambda \, 24_T^a \, 24_T^a \, 24_H,$$  

$$W_3 = \lambda_H \, \overline{5}_H^a \, 24_H \, 5H^a + c_H \, \overline{5}_H^a \, 24_H \, 45_H^a + b_H \, \overline{45}_H^a \, 24_H \, 5H^a +$$
$$+ m_5 \, \overline{5}_H^a \, 5H^a + m_{45} \, \overline{45}_H^a \, 45_H^a + a_H \, \overline{45}_H^a \, 45_H^a \, 24_H.$$  

The mass matrices for $M_D$, $M_E$, and $M_U$ are given here by coupling constants associated to $A_4$-triplet Higgs operators and the charged fermion masses are of the form as in eq. (15). As we will show below, there is room for three independent eigenvalues and the phenomenological observed charged fermion hierarchy can be easily reproduced.

Let us write in detail the operators $y_1 \, 10_T \, \overline{5}_T \, \overline{5}_H$ and $y_2 \, 10_T \, \overline{5}_T \, \overline{45}_H$ which generate both the down and charged lepton mass matrices. Under SM gauge group invariance, with the field notation
given in [43], the operators read as:

\[
y_1 \mathbf{10}_T \bar{\mathbf{5}}_T \mathbf{5}_H \to Q_i^\alpha M_{ij} d_j \overline{\Pi}^\alpha + e_i M_{ij} L_j^\alpha \overline{\Pi}^\alpha; \tag{22a}
\]

\[
y_2 \mathbf{10}_T \bar{\mathbf{5}}_T \mathbf{45}_H \to Q_i^\alpha (2 M_{ij}) d_j \overline{\Pi}_{(1,2)}^\alpha + e_i (-6 M_{ij}) L_j^\alpha \overline{\Pi}_{(1,2)}^\alpha, \tag{22b}
\]

where \(\alpha\) is the \(SU(2)\) index. The two other operators \(y_3 \mathbf{10}_T \mathbf{10}_T \mathbf{5}_H\) and \(y_4 \mathbf{10}_T \mathbf{10}_T \mathbf{45}_H\) generate the up mass matrix in a similar way; in particular the operator proportional to \(\mathbf{5}_H\) generate a symmetric term, while the operator proportional to \(\mathbf{45}_H\) generate an antisymmetric term:

\[
y_3 \mathbf{10}_T \mathbf{10}_T \mathbf{5}_H \to Q_i^\alpha M_{ij} u_j H^\alpha + u_i M_{ij} Q_j^\alpha H^\alpha; \tag{23a}
\]

\[
y_4 \mathbf{10}_T \mathbf{10}_T \mathbf{45}_H \to Q_i^\alpha M_{ij} u_j H_{(1,2)}^\alpha - u_i M_{ij} Q_j^\alpha H_{(1,2)}^\alpha. \tag{23b}
\]

We consider next the three possible scenarios with respect to the Higgs vacuum alignments in the \(A_4\) structure we mentioned previously and write in detail the mass matrices in each of them.

### 1. Matrices for Scenarios (A) and (B)

In Scenario (A) we impose \(\langle \mathbf{5}_H \rangle \propto \langle \mathbf{5}_H \rangle = \{v_5, v_5, v_5\}\) and \(\langle \mathbf{45}_H \rangle \propto \langle \mathbf{45}_H \rangle = \{v_{45}, 0, 0\}\). Under these assumptions the mass matrices in the up sector are too constrained. In particular there are only two free parameters and it is not possible to fit the huge hierarchy among the masses.

In Scenario (B), the vacuum expectation values are as \(\langle \mathbf{5}_H \rangle \propto \langle \mathbf{5}_H \rangle = \{v_5, v_5, 0\}\) and \(\langle \mathbf{45}_H \rangle \propto \langle \mathbf{45}_H \rangle = \{v_{45}, v_{45}, v_{45}\}\). Under the assumptions \(v_5, v_{45} \gg v_{45}\) and \(\gamma_1^1 \gg \gamma_2^1\) we obtain that the 3rd generation is much more heavy than the other two, the relation \(\tilde{\Gamma}_c^1 \simeq \tilde{\Gamma}_d^1\) holds and we get the bottom-tau unification. However, also in this scenario the up sector mass matrix is too constrained to fit the quark masses in detail. In particular, as in scenario (A), there are only two free parameters and it is not possible at all to fit the three quark masses.

### 2. Scenario (C): \(\langle \mathbf{5}_H \rangle \propto \langle \mathbf{5}_H \rangle = \{v_5, 0, 0\}\), \(\langle \mathbf{45}_H \rangle = \{v_{45}, v_{45}, v_{45}\}\), and \(\langle \mathbf{45}_H \rangle = \{v_{45}, \delta v_{45}, \delta v_{45}\}\)

In this scenario the down and charged lepton mass matrices are given by:

\[
M_D = \begin{pmatrix}
0 & 2\gamma_1^1 v_{15} & 2\gamma_2^1 v_{15} \\
2\gamma_2^1 v_{15} & 0 & 2\gamma_1^1 v_{15} + \gamma_1^1 v_5 \\
2\gamma_1^1 v_{15} & 2\gamma_2^1 v_{15} + \gamma_1^1 v_5 & 0
\end{pmatrix} \equiv \begin{pmatrix}
0 & \gamma_1^1 \tilde{\Gamma}_d^1 & \gamma_2^1 \tilde{\Gamma}_d^1 \\
\gamma_1^2 \tilde{\Gamma}_d^1 & 0 & \tilde{\Gamma}_d^2 \\
\gamma_2^1 \tilde{\Gamma}_d^1 & \tilde{\Gamma}_d^2 & 0
\end{pmatrix}, \tag{24a}
\]

\[
M_E = \begin{pmatrix}
0 & -6\gamma_2^1 v_{15} & -6\gamma_1^1 v_5 \\
-6\gamma_1^1 v_{15} & 0 & -6\gamma_2^1 v_{15} + \gamma_1^2 v_5 \\
-6\gamma_2^1 v_{15} & -6\gamma_1^1 v_{15} + \gamma_1^1 v_5 & 0
\end{pmatrix} \equiv \begin{pmatrix}
0 & -3\gamma_2^1 \tilde{\Gamma}_e^1 & -3\gamma_1^1 \tilde{\Gamma}_e^1 \\
-3\gamma_1^2 \tilde{\Gamma}_e^1 & 0 & \tilde{\Gamma}_e^2 \\
-3\gamma_2^1 \tilde{\Gamma}_e^1 & \tilde{\Gamma}_e^2 & 0
\end{pmatrix}. \tag{24b}
\]
where we introduced the short hand notations \((i = 1, 2)\):

\[
\tilde{\gamma}_d^i = 2 \gamma_2^i v_{\tau 5}, \\
\tilde{\Gamma}_d^i = \gamma_1^i v_\tau + 2 \gamma_2^i v_{\tau 5}, \\
\tilde{\Gamma}_e^i = \gamma_1^i v_\tau - 6 \gamma_2^i v_{\tau 5}.
\]

(25a) \hspace{1cm} (25b) \hspace{1cm} (25c)

A relation between the masses for charged leptons and down quarks is given by:

\[
M_D - M_E = v_{\tau 5} Y_2,
\]

(26)

where the “effective” Yukawa matrix \(Y_2\) is given by:

\[
Y_2 = 8 \begin{pmatrix} 0 & \gamma_1^1 & \gamma_2^1 \\
\gamma_2^1 & 0 & \gamma_2^1 \\
\gamma_2^1 & \gamma_2^1 & 0 \end{pmatrix}.
\]

(27)

The relation given by eq. (26) is a particular case of the D-E relation in SUSY SU(5) theories. In our case the texture of the “Yukawa” matrix \(Y_2\) is predicted from the \(A_4\) symmetry and its coefficients are given by the \(A_4\) coupling constants. If we want to keep the bottom-tau unification at GUT scale, the matrix \(Y_2\) must only modify the relation between first and second quark and lepton masses.

Other matrix relations can be easily obtained. One gets directly from the expressions of matrices \(M_D, M_E\):

\[
3M_D + M_E^\dagger = v_{\tau 5} Y_5,
\]

(28)

where the matrix \(Y_5\) is given by:

\[
Y_5 = 4 \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & \gamma_1^1 \\
0 & \gamma_1^1 & 0 \end{pmatrix}.
\]

(29)

In the limit \(v_5 \sim 0\) we obtain, instead of \(M_D \sim M_E^\dagger\) in minimal SU(5), the relation:

\[
M_D \sim -M_E^\dagger/3.
\]

It is possible to obtain also some useful “sum rules” for the squared masses of quarks and charged leptons by taking the trace of the matrices \(9M_D M_5^\dagger \pm M_E M_E^\dagger\):

\[
9 \sum_{\text{down}} m_q^2 - \sum_l m_l^2 = 48 Re(\gamma_1^1 \gamma_2^1 + \gamma_1^2 \gamma_2^2) v_{\tau 5} v_{\tau 5} + 8 (|\gamma_1^1|^2 + |\gamma_1^2|^2) v_{\tau 5}^2,
\]

(30a)

\[
9 \sum_{\text{down}} m_q^2 + \sum_l m_l^2 = 216 (|\gamma_2^1|^2 + |\gamma_2^2|^2) v_{\tau 5}^2 + B v_{\tau 5} v_{\tau 5} + C v_{\tau 5}^2,
\]

(30b)
where \( A, B \) are simple expressions depending on the \( \gamma_i^j \) constants. Note that the first sum rule, eq. (30a) would become specially simple if the the \( 2 \times 2 \) matrix of constants \( (\gamma_i^j) \) would be unitary. In such a case the term proportional to \( v_{45} \) would vanish. Expressions in eqs. (30) will prove themselves to be useful in the numerical fits of the next section. Let us proceed now to the up quark mass matrix. In this scenario we obtain the following form:

\[
M_U = \begin{pmatrix}
0 & -8(\gamma_4^1 - \gamma_4^2)\delta v_{45} & 8(\gamma_4^1 - \gamma_4^2)\delta v_{45} \\
8(\gamma_4^1 - \gamma_4^2)\delta v_{45} & 0 & -8(\gamma_4^1 - \gamma_4^2)v_{45} + 4(\gamma_3^1 + \gamma_3^2)v_5 \\
-8(\gamma_4^1 - \gamma_4^2)\delta v_{45} & 8(\gamma_4^1 - \gamma_4^2)v_{45} + 4(\gamma_3^1 + \gamma_3^2)v_5 & 0 \\
\end{pmatrix}
\]

\[
\equiv \begin{pmatrix}
0 & -\tilde{\gamma}_u & \tilde{\gamma}_u \\
-\tilde{\gamma}_u & 0 & \tilde{\Gamma}_u^1 \\
-\tilde{\gamma}_u & \tilde{\Gamma}_u^2 & 0 \\
\end{pmatrix},
\]

(31)

where we have defined the following coefficients:

\[
\tilde{\gamma}_u = 8(\gamma_4^1 - \gamma_4^2)\delta v_{45},
\]

(32a)

\[
\tilde{\Gamma}_u^1 = 4(\gamma_3^1 + \gamma_3^2)v_5 - 8(\gamma_4^1 - \gamma_4^2)v_{45},
\]

(32b)

\[
\tilde{\Gamma}_u^2 = 4(\gamma_3^1 + \gamma_3^2)v_5 + 8(\gamma_4^1 - \gamma_4^2)v_{45}.
\]

(32c)

It is straightforward to derive a sum rule for the up squared masses from the trace of the matrix \( M_U M_U^\dagger \); we obtain:

\[
\sum_{up} m_q^2 = |\tilde{\gamma}_u|^2 + |\tilde{\Gamma}_u^1|^2 + |\tilde{\Gamma}_u^2|^2,
\]

(33)

which reduces quickly to the expected sum of the squares of the three vevs. The matrix \( M_U \) turns out to be phenomenologically viable in this case. We can reproduce the three up masses by imposing the hierarchy \( \tilde{\Gamma}_u^1 \gg \tilde{\Gamma}_u^2 \gg \tilde{\gamma}_u \), i.e. with a fine tuning in \( \tilde{\Gamma}_u^2 \), and the eigenvalues of the matrix are then approximately:

\[
\{m_u, m_c, m_t\} \simeq \left\{ \left(\frac{(\tilde{\gamma}_u)^2}{\tilde{\Gamma}_u^2}\right), |\tilde{\Gamma}_u^1|, |\tilde{\Gamma}_u^2| \right\}
\]

(34)

and the experimental values can be easily accommodated.

3. **Numerical Masses and CKM in Case 5, scenario (C).**

Information about the free parameters of the model can be obtained by performing a fit to the experimental values of the masses of up and down quarks and charged leptons at the GUT
scale. For our purpose it will be enough to consider the run quark and lepton masses of \[40\].

The run masses, for example by assuming an unification scale \(\mu = 2 \times 10^{16} \text{ GeV}\), a SUSY scale \(M_S = 1 \text{ TeV}\) and \(\tan \beta (M_S) = 10\), are given in table [II] (from \[40\]). We have computed the

\[
\begin{array}{ccc}
  m_u &=& 0.72^{+0.14}_{-0.15} \\
  m_d &=& 1.5^{+0.4}_{-0.2} \\
  m_e &=& 0.3585 \pm 0.0003 \\
  m_c &=& 210^{+19}_{-21} \\
  m_s &=& 29 \pm 4 \\
  m_t &=& 82^{+30}_{-15} \times 10^3 \\
  m_\mu &=& 75.67 \pm 0.05 \\
  m_\tau &=& 1292.2^{+1.3}_{-1.2} \\
\end{array}
\]

TABLE II: Running masses (MeV) for unification scale \(\mu = 2 \times 10^{16} \text{ GeV}\), SUSY scale \(M_S = 1 \text{ TeV}\) and \(\tan \beta (M_S) = 10\).

exact eigenvalues of our mass matrices, eqs. (24a), (24b) and (31). The best fit four parameters \(\tilde{\gamma}_d^i\), \(\tilde{\Gamma}_d^i\) have been obtained by a standard \(\chi^2\) minimization of the error-weighted distance from the theoretical eigenvalues to running masses of table [II] computed from low scale experimental values.

The resulting parameters in the down and charged lepton sectors, are:

\[
\tilde{\Gamma}_d^2 = (1277.9 - 0.0145 i) \text{ MeV}, \quad \tilde{\Gamma}_d^1 = (-26.83 - 0.494 i) \text{ MeV},
\]

(35)

\[
\tilde{\gamma}_d^1 = (11.849 - 1.069 i) \text{ MeV}, \quad \tilde{\gamma}_d^2 = (-3.32 + 0.37 i) \text{ MeV}.
\]

(36)

The fit is acceptable, \(\chi^2/ndf \sim 1.5\) and the pull out of the best fit masses obtained in return is small:

\[
\{m_d, m_s, m_b\}_{\text{best fit}} = \{0.3, 27, 1280\} \text{ MeV},
\]

(37)

\[
\{m_e, m_\mu, m_\tau\}_{\text{best fit}} = \{0.359, 75.67, 1292.2\} \text{ MeV}.
\]

(38)

We notice that there is a preference for the mass \(m_d\) to be lower and \(m_b\) to be higher than expected. Additional information about the size of the free parameters and vevs of the model can be obtained by using the sum rules in eqs. \[39\]. The experimental value for the quantities:

\[
9 \sum_{\text{down}} m_q^2 - \sum_{l} m_l^2 = (2.9 \pm 0.1 \text{ GeV})^2,
\]

(39a)

\[
9 \sum_{\text{down}} m_q^2 + \sum_{l} m_l^2 = (3.4 \pm 0.1 \text{ GeV})^2,
\]

(39b)

and the quotient:

\[
\left(9 \sum_{\text{down}} m_q^2 - \sum_{l} m_l^2\right) / \left(9 \sum_{\text{down}} m_q^2 + \sum_{l} m_l^2\right) \simeq 0.7 \text{ (exp.)},
\]

(39c)
can be used to extract information about the \( v_5, v_{45} \) vevs. The three parameters of the up sector \( \tilde{\gamma}_u \) and \( \tilde{\Gamma}_u^{i=1,2} \) are obtained numerically directly from the three run masses and the result is:

\[
\begin{align*}
\tilde{\Gamma}_u^1 &= 82.0 \text{ MeV}, \\
\tilde{\Gamma}_u^2 &= 0.20 \text{ MeV}, \\
\tilde{\gamma}_u &= 0.02 \text{ MeV}.
\end{align*}
\]  

(40)

These numerical values are in agreement with the expectations, \( \tilde{\gamma}_u \propto \delta v_{45} \) and it is much smaller. Moreover we can write, directly from the definitions:

\[
\left| \frac{\tilde{\gamma}_u}{\Gamma_u^2 - \Gamma_u^1} \right| = \frac{1}{2} \frac{\delta v_{45}}{v_{45}}; \tag{41}
\]

this means, using the fitted values above:

\[
\frac{\delta v_{45}}{v_{45}} = 5 \times 10^{-4}. \tag{42}
\]

This estimation can be considered a direct fit to the experimental masses.

We can estimate the CKM at this stage. Any mass matrix can be diagonalized as:

\[
M_U = U_U M_U^{\text{diag}} V_U^\dagger, \quad M_D = U_D M_D^{\text{diag}} V_D^\dagger, \quad M_E = U_E M_E^{\text{diag}} V_E^\dagger,
\]  

(43)

from them we can obtain the usual CKM matrix as

\[
U_{\text{CKM}} = U_U^\dagger U_D. \tag{44}
\]

We first build the matrices \( M_U, M_E, M_D \) from the best fit of the parameters obtained above and then numerically find the diagonalizing matrices. The resulting CKM mixing matrix is obtained from eq.\((44)\):

\[
|U_{\text{CKM}}| = \begin{pmatrix}
0.984 & 0.180 & 0.0091 \\
0.180 & 0.984 & 0.0006 \\
0.009 & 0.001 & 0.9999
\end{pmatrix}.
\]  

(45)

We notice that our model reproduce the main features of the CKM structure, i.e. a relatively large Cabibbo angle and very small \((1,3)\) and \((3,1)\) entries. Other characteristics as the size of the entries \((2,3)\) and \((3,2)\) might be improved by performing a more refined global fit of masses and CKM mixing matrix together which is beyond the scope of this work.

V. NEUTRINO MASS AND LEPTON MIXING MATRICES

We investigate now the neutrino mass matrices, their masses and the resulting PMNS mixing matrix. We know, as conclusion of the study of the charged fermion masses, that from the five
initial Cases only two, the 3 and 5, remain phenomenologically interesting. The vacuum alignment scenario C is needed in both cases.

Let us present first a summary of the results of the Case 3. Here the adjoint matter field $24_T$ is singlet under the $A_4$ symmetry while all the Higgs fields are triplets, except $24_H$ according to table (I). The neutrino masses corresponding to the scenarios (A) and (B) (see sec. III B2) are quite general, only one of them is zero. As a general feature, the neutrino masses can be accommodated well in both scenarios. Moreover since $U_E$ is close to the identity, it turns out that the observed tribimaximal mixing structure of PMNS lepton matrix can be obtained ligated to the existence of an inverted neutrino hierarchy. The situation is not the same in scenario (C). In this scenario the neutrino hierarchy results to be the normal one and as a consequence the desired lepton mixing matrix cannot be obtained. Scenario (C), as explained in the previous section, is the only one which reproduce the up sector masses. We arrive to the conclusion that, for a $A_4$-singlet $24_T$, is impossible at the same time to reproduce charged fermion and neutrino masses.

We will study next, now in detail, the much more interesting Case 5.

A. Neutrino masses in Case 5: $24_T$ is a $A_4$-triplet.

Here all the the Higgs fields $5_H, 45_H, 5_H, 45_H$ are flavor triplets (table I). Once the Higgs multiplets acquire a vev, the quadratic mass term between the $5_T$ and the $24_T$ fields, i.e. those originating from $W_1 = \gamma 5_T 24_T 45_H + \beta 5_T 24_T 5_H$, becomes of the form:

$$\bar{5}^a M_{ab} 24^b,$$

(46a)

where the mass matrix is given by:

$$M = (M_{ab}) = \begin{pmatrix} 0 & \gamma_1 v_{45}^3 + \beta_1 v_5^3 & \gamma_2 v_{45}^2 + \beta_2 v_5^2 \\ \gamma_2 v_{45}^3 + \beta_2 v_5^3 & 0 & \gamma_1 v_{45}^1 + \beta_1 v_5^1 \\ \gamma_1 v_{45}^2 + \beta_1 v_5^2 & \gamma_2 v_{45}^1 + \beta_2 v_5^1 & 0 \end{pmatrix},$$

(46b)

where the $A_4$ coupling constants $\gamma_i, \beta_i$ are defined as before.

The SM neutrino masses are generated through a type I+III SeeSaw mechanism mediated by the $SU(2)_{weak}$ singlet and the neutral component of $SU(2)_{weak}$ triplet in the $24_T$ matter field, respectively $\rho_0, \rho_3$ with masses $M_{\rho_0}, M_{\rho_3}$ (see I for the SM decomposition of the fields in our
model). The neutrino mass matrix is of the form:

\[ M_\nu = \frac{1}{M_\rho_1} \begin{pmatrix}
  a_{13}^2 + a_{22}^2 & a_{22}a_{11} & a_{13}a_{21} \\
  a_{22}a_{11} & a_{23}^2 + a_{11}^2 & a_{23}a_{12} \\
  a_{21}a_{13} & a_{12}a_{23} & a_{22}^2 + a_{13}^2
\end{pmatrix} + \frac{1}{M_\rho_0} \begin{pmatrix}
  b_{13}^2 + b_{22}^2 & b_{22}b_{11} & b_{13}b_{21} \\
  b_{22}b_{11} & b_{23}^2 + b_{11}^2 & b_{23}b_{12} \\
  b_{21}b_{13} & b_{12}b_{23} & b_{22}^2 + b_{13}^2
\end{pmatrix}, \tag{47a}
\]

where \((p = 1, 2, j = 1, 2, 3)\):

\[ a_{pj} = -3\gamma_p v_{45}^j + \beta_p v_5^j, \tag{47b} \]
\[ b_{pj} = \frac{\sqrt{15}}{2} \left( \gamma_p v_{45}^j + \frac{1}{5} \beta_p v_5^j \right). \tag{47c} \]

We assume that the vevs are in the following directions (Scenario (C)): \( v_5^j = \delta_{j1} v_5 = (1, 0, 0)^T v_5 \) and \( v_{45}^j = (1, \epsilon, \epsilon) v_{45} \), we get:

\[ a_{p1} = -3\gamma_p v_{45} + \beta_p v_5, \tag{48a} \]
\[ a_{pj} = -3 \epsilon \gamma_p v_{45} \quad (j = 2, 3), \tag{48b} \]
\[ b_{p1} = \frac{\sqrt{15}}{2} \left( \gamma_p v_{45} + \frac{1}{5} \beta_p v_5 \right), \tag{48c} \]
\[ b_{pj} = \frac{\sqrt{15}}{2} \epsilon \gamma_p v_{45} \quad (j = 2, 3). \tag{48d} \]

With these parameters the low energy neutrino mass matrix is given by:

\[ M_\nu = \begin{pmatrix}
  \epsilon^2 (A_{11} + A_{22}) & \epsilon (B_{21} + A_{21}) & \epsilon (B_{12} + A_{12}) \\
  \epsilon (B_{21} + A_{21}) & \epsilon^2 A_{22} + A_{11} + 2B_{11} + C_{11} & \epsilon^2 A_{12} \\
  \epsilon (B_{12} + A_{12}) & \epsilon^2 A_{12} & \epsilon^2 A_{11} + A_{22} + 2B_{22} + C_{22}
\end{pmatrix}, \tag{49}\]

where we have considered explicitly the dependence on \( \epsilon \) of each entry, the \( A \)'s and \( B \)'s are simple combinations of the \( a_{pj}, b_{pj} \) parameters.

Our objective is to study under which conditions the matrix \([19]\) is diagonalized by a tribimaximal unitary matrix. Let us note that for a generic matrix, with arbitrary \( a, b, c \) coefficients:

\[ M^{TBM} = \begin{pmatrix}
  c & b & b \\
  b & c + a & b - a \\
  b & b - a & c + a
\end{pmatrix}, \tag{50}\]

the diagonalizing matrix \( M^{TBM} = U_\nu M^d U_\nu^\dagger \), as a matter of fact, is the tribimaximal matrix \([79]\):

\[ U_\nu^{TBM} = \begin{pmatrix}
  \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
  -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
  \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \tag{51}\]
The PMNS lepton mixing matrix is given by:

$$ U_{PMNS} = U_E^* U_\nu $$  \hspace{1cm} (52) 

where $U_E$ is defined in eq. \[43\] and $U_\nu$ is defined by:

$$ M_\nu = U_\nu M_\nu^{\text{diag}} U_\nu^t. $$ \hspace{1cm} (53)

We conclude from here that if we impose the condition that the matrix \[49\] is of the form of the matrix \[50\] the resulting lepton mixing will be compatible with the experimental evidence \[41\], since we know from the previous section that, in this case and with this vev scenario, the charged lepton matrix $U_E$ is a small perturbation of the same order of CKM mixing matrix. We thus arrive to a “complementarity” relation $U_{PMNS} \approx U_\nu^{TBM} \times U_{CKM}$ \[41\]. The condition $M_\nu \sim M^{TBM}$ implies the following equalities for the coefficients:

$$ M_{\nu,12} = M_{\nu,13}, \quad M_{\nu,22} = M_{\nu,33}, \quad M_{\nu,22} + M_{\nu,23} = M_{\nu,11} + M_{\nu,12}. $$ \hspace{1cm} (54)

A solution for these equations is given, at leading order in $\epsilon$, by:

$$ \beta_2 = \beta_1 \epsilon f_1(\alpha), $$ \hspace{1cm} (56a)

$$ \gamma_2 = \frac{v_5}{v_{45}} \beta_1 \epsilon f_2(\alpha), $$ \hspace{1cm} (56b)

$$ \gamma_1 = \frac{v_5}{v_{45}} \beta_1 f_3(\alpha). $$ \hspace{1cm} (56c)

where the parameter $\beta_1$ is left free, $\alpha = M_{\rho_3}/M_{\rho_0}$ and $f_1 = \frac{16 \sqrt{\alpha} (8 \sqrt{15} + 16 \sqrt{\alpha} - 1 \sqrt{15})}{(20 + 3 \alpha)(12 + 5 \alpha)}$, $f_2 = \frac{16 \sqrt{\alpha}}{\sqrt{15(12 + 5 \alpha)}}$, $f_3 = \frac{60 + 16 \sqrt{15 \alpha} - 15 \alpha}{180 + 75 \alpha}$. We have $f_i \sim o(1)$ for $\alpha \sim 1$, on the other hand $(f_1, f_2, f_3) \to (0, 0, 1/3)$ in the limit $\alpha \to 0$.

We insert these solution and next proceed to diagonalize the neutrino mass matrix; we are specially interested on the resulting solar and atmospheric $\delta m^2$’s, that are:

$$ \delta m^2_{\text{sun}} = \frac{v_5^4 e^4 \beta_1^4}{25 M_{\rho_0}^2} h_1(\alpha), $$ \hspace{1cm} (57)

$$ \delta m^2_{\text{atm}} = \frac{v_5^4 e^4 \beta_1^4}{25 M_{\rho_0}^2} h_2(\alpha); $$ \hspace{1cm} (58)

with the functions $h_1(\alpha) = \frac{(288 (5 \alpha^2 - 168 \alpha + 80))}{(\alpha(5 \alpha + 12)^2)}$, $h_2(\alpha) = \frac{(64 (15 \alpha^2 - 376 \alpha + 240))}{(\alpha(5 \alpha + 12)^2)}$. The experimental values \[42\] $\delta m^2_{\text{sun}} \approx 8 \times 10^{-5} \text{ eV}^2$, $\delta m^2_{\text{atm}} \approx 2.5 \times 10^{-3} \text{ eV}^2$ are reproduced if we take:

$$ M_{\rho_0} \approx 25 v_5 e^2 \beta_1^2, \quad \alpha \approx 1/2. $$ \hspace{1cm} (59)
Most interestingly, the hierarchy between the solar and atmospheric scales only depends on the ratio of the triplet to singlet \( 24_T \) fields, \( M_{\rho 3}/M_{\rho 0} \) as:

\[
\frac{\delta m_{\text{sun}}^2}{\delta m_{\text{atm}}^2} = (\simeq 3 \times 10^{-2} \text{ (exp)}) \sim 1 - 2\alpha,
\]

\[
\frac{M_{\rho 3}}{M_{\rho 0}} \sim \frac{1}{2} \left( 1 - \frac{\delta m_{\text{sun}}^2}{\delta m_{\text{atm}}^2} \right).
\]

We also “predict” the value of the neutrino masses by inserting the fitted values for \( M_{\rho} \) and \( \alpha \) in eq.(59) in the neutrino mass matrix, we obtain the following:

\[
m_1 \simeq m_2 \simeq 0.10 \text{eV},
\]

\[
m_2^2 - m_1^2 \simeq 8 \times 10^{-5} \text{eV}^2,
\]

\[
m_3 = 0.089 \text{eV};
\]

thus an inverted hierarchy is predicted with an absolute mass scale \( \sim 10^{-1} \text{eV} \).

With a quasi degenerated inverted hierarchy the renormalization group running is especially critical: the neutrino masses can change up to a factor 2, and the mixing angles can receive sizable contributions \[80\]. However the full study of the RGE is beyond the scope of this paper and it will be treated in full detail in a next publication \[78\].

VI. CONCLUSIONS

We have analyzed all the possible extensions of the recently proposed minimal renormalizable SUSY \( SU(5) \) grand unified model \[77\] with the inclusion of an additional \( A_4 \) flavor symmetry. We have found that there are 5 possible \( A_4 \) charge assignment cases for the superfields compatible which \( A_4 \) invariance. Among these Cases we found one that is phenomenologically interesting for both charged fermion and neutrinos masses and mixings, despite the highly non triviality of the \( A_4 \) constraints. In this case all, matter and Higgs, fields (except the \( 24_H \)) are triplets under \( A_4 \), the field content and charge assignment are given by:

| \( SU(5) \) | 10_T | \( \overline{5}_T \) | \( 24_T \) | \( \overline{5}_H \) | \( 45_H \) | \( 5_H \) | \( 45_H \) | \( 24_H \)
|-------------|--------|---------|---------|--------|--------|--------|--------|--------|
| \( A_4 \)   | 3      | 3       | 3       | 3      | 3      | 3      | 3      | 1      |

We have studied in detail such Case and showed how the fermion masses and mixing angles come out. In particular we arrived to the conclusion that to reproduce the observed masses and mixing angles the Higgs fields must acquire their vevs along some particular directions basically (described in detail in sec. \[IV.B.2\] of the form \( \langle \overline{5}_H \rangle \propto \langle 5_H \rangle \propto (1,0,0) \), \( \langle 45_H \rangle = (1,1,1) \), and
\langle \mathbf{45}_H \rangle = (1, \epsilon, \epsilon), \text{i.e the } \mathbf{45}_H \text{ vevs breaks down the } A_4 \text{ symmetry to } Z_3, \text{ while the } \mathbf{5}_H, \mathbf{45}_H, \mathbf{5}_H \text{ spontaneous breaking reduce it further down to } Z_2.

We have obtained that all the experimental charged fermion masses, quark and lepton mixing angles can be easily fitted. The absolute scale of neutrino masses is obtained as a prediction, from the experimental TBM mixing neutrino matrix and from the values of solar and atmospheric squared mass differences, they are all nearly degenerate with \( m \sim 0.1 \text{ eV} \) with an inverted hierarchy.

Let us now compare our model with two other recently proposed SUSY SU(5) models with \([13]\) and without flavor symmetry \([77]\).

Let us note first that in the model introduced here, in contraposition to \([13]\), it is necessary to introduce ad-hoc \( U(1) \) and \( Z_N \) symmetries to fit together the discrete and grand unified symmetries.

On the other hand, in the SUSY “Adjoint” SU(5) of \([77]\) a full study of the neutrino sector masses and mixing has not been performed as the one performed here, so it is not possible to compare directly both works. It is simply argued there that it is possible to generate all fermion masses including neutrino masses because the Yukawa matrices are arbitrary are the number of free parameters is very large.

With respect to that model, which serves of basis of the one presented here, the inclusion of an extra flavor symmetry of type \( A_4 \) implies the following distinctive features:

a) In the D-E sector, down quarks and charged leptons, can be fitted with only four parameters, while in a general SUSY SU(5) model we would have 18 parameters. This becomes very much explicit in the combination of matrices: each of the linear combination \( M_D - M^E_t, 3M_D + M^E_t \) depends disjointly on only two of the four possible parameters.

b) The up sector contains three parameters which fit well the three masses. Overall, the \( D - E \) and \( U \) sectors contains 7 parameters which fits well the 9 masses and the CKM matrix. Moreover, the \( D - E, U \) and “\( \nu \)” sectors contain 9 effective parameters which seems to fit well the 12 masses, the CKM and the PMNS matrices.

c) In both models neutrino masses are coming from a I+III SeeSaw Mechanism. However here all the neutrinos are massive, while in the other model there is one massless neutrino. This is due to the form of the low energy mass matrix, in our model the presence of Higgs \( A_4 \) triplets implies the equation given by Eq.\([47]\).

d) The Tribimaximal neutrino mixing can be easily accommodated. The absolute scale of neutrino masses (they are all nearly degenerate with \( m \sim 0.1 \text{ eV} \) with an inverted hierarchy) is obtained as a prediction, from the experimental TBM mixing neutrino matrix and from the values of solar and atmospheric squared mass differences.
So, in conclusion, the introduction of the $A_4$ flavor symmetry seems to be a “good” idea:

- it restricts very much the number of free parameters, giving simple expressions for the Yukawa matrices.

- The number of Higgs is larger (Higgses are $A_4$ triplets) but their vevs are also very much restricted by the $A_4$ symmetry, so the number of free parameters coming from this sector does not increase (only 3 parameters).

- the number of remaining parameters and the texture given to the yukawa matrices is big enough to be able to fit the low energy fermion sector including the largely different structures of masses and mixing.

Two features, at least, of the $A_4$ symmetry could be tested at low energies in present and future experiments. The predicted absolute mass scale and in particular the mass of the electron neutrino lies well within the sensitivity of near future kinematical and double $\beta$ experiments. The observation of a degenerate neutrino mass spectrum, as the one predicted here, could help to differentiate this model with respect other SUSY $SU(5)$ $A_4$ models, in particular with [13].

In summary, we have showed that a simple, TB-Compatible, flavor symmetry exclusively based on $A_4$ is compatible with a Grand Unification scenario. Moreover, the model considered here can be considered as the simplest, realistic, renormalizable supersymmetric grand unified theory of flavor based on the SU(5) gauge symmetry and $A_4$ flavor symmetry since it has the minimal number of fields and space-time dimensions. The details of Higgs vacuum alignment seem to be important for the predictivity of the model, and specific to the discrete flavor symmetry, they will be treated in full detail in a next publication [78]. In particular it might be interesting to delucidate which features of this vacuum alignment are specific to the $A_4$ symmetry and which ones to the mere existence of a discrete symmetry.

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