The Landscape of Nonconvex-Nonconcave Minimax Optimization

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Abstract

Minimax optimization has become a central tool for modern machine learning with applications in robust optimization, game theory and training GANs. These applications are often nonconvex-nonconcave, but the existing theory is unable to identify and deal with the fundamental difficulties posed by nonconvex-nonconcave structures. We break this historical barrier by identifying three regions of nonconvex-nonconcave bilinear minimax problems and characterizing their different solution paths. For problems where the interaction between the agents is sufficiently strong, we derive global linear convergence guarantees. Conversely when the interaction between the agents is fairly weak, we derive local linear convergence guarantees. Between these two settings, we show that limiting cycles may occur, preventing the convergence of the solution path.

1 Introduction

Many important problems in modern machine learning can be formulated as a minimax optimization problem with the form

\[ \min_x \max_y L(x, y) , \tag{1} \]

and often the objective \( L(x, y) \) is neither convex in \( x \) nor concave in \( y \). For example, such structure arises in generative adversarial networks (GANs), robust optimization, reinforcement learning, etc. More specifically, in training GANs [1], while the discriminative network (parameterized by \( x \)) minimizes the average loss, the generative network (parameterized by \( y \)) generates more samples trying to increase such loss. In robust optimization, while the decision maker tries to minimize the loss, there may be certain uncertainty nature in the data so that we instead minimize the loss over the worst-case data in the uncertainty set [2, 3, 4]. In reinforcement learning, the solution to Bellman equations can be obtained by having a dual critic seeking a solution satisfying the Bellman equation and a primal actor seeking state-action pairs to break this satisfaction [5, 6].

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Although there have been extensive studies on convex-concave minimax problems and on nonconvex-concave minimax problems, the understanding of nonconvex-nonconcave minimax problems is surprisingly limited. In this paper, we focus on the unconstrained minimax problems with a bilinear interaction term of the form

$$\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} L(x, y) = f(x) + x^T Ay - g(y), \quad (2)$$

and break through this historical difficulty by identifying three regions of $(2)$ depending on the scale of the interaction term $A$ and characterizing their different solution paths individually. Nonconvex-nonconcave structure and bilinear structure both appear in reinforcement learning [6, 7], and serve as the first step to understand the landscape of general nonconvex-nonconcave minimax problems [1].

Our goal is to find an approximately stationary point $(x, y)$ of $L$:

$$\|\nabla L(x, y)\| \leq \varepsilon,$$

where $\|\cdot\|$ refers to the $\ell_2$ norm throughout the paper. By viewing the problem $(2)$ as a simultaneous zero-sum game, a stationary point can be thought of as a first-order Nash Equilibrium. That is, neither player tends to deviate from their position based on their first-order information. One can instead view the minimax problem as a sequential zero-sum game (where the minimizing player selects $x$ and then the maximizing player exploits that choice in choosing $y$). Unlike the convex-concave case, the solutions between these two types of games no longer coincide and the optimal (sequential) minimax solution need not be a stationary point. In this case, a different asymmetric measure of optimality may be called for [8, 9], but that is beyond the scope of this paper.

The most natural algorithm for solving the minimax problems $(2)$ is perhaps Gradient Descent Ascent (GDA) given by the update rule:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + s \begin{bmatrix} -\nabla_x L(x_k, y_k) \\ -\nabla_y L(x_k, y_k) \end{bmatrix}, \quad (3)$$

with stepsize parameter $s > 0$. However, GDA is known to work only for strongly convex-strongly concave minimax problems, and it may diverge even for simple convex-concave problems [8]. To avoid this issue of divergence, we consider the more stable algorithm given by the following damped Proximal Point Method (PPM) with damp parameter $\lambda \in [0, 1]$ and proximal parameter $\eta > 0$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = (1 - \lambda) \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \lambda \text{prox}_\eta(x_k, y_k), \quad (4)$$

where

$$\text{prox}_\eta(x_k, y_k) := \arg \min_{x_+, y_+} \max_{x, y} L(x_+, y_+) + \frac{\eta}{2}\|x_+ - x_k\|^2 - \frac{\eta}{2}\|y_+ - y_k\|^2. \quad (5)$$

(See [10] for a deeper discussion into the different dynamics between GDA’s and PPM’s for solving convex-concave minimax problems.) Note that when $\eta > \rho$, the subproblem of
\[ \min \max L(x_+, y_+) + \frac{\eta}{2} \|x_+ - x\|^2 - \frac{\eta}{2} \|y_+ - y\|^2 \]

becomes strongly convex-strongly concave and thus has a unique solution. Although the update (4) may not have a closed form solution, it can be quickly solved/estimated as a strongly convex-strongly concave smooth minimax problem\(^1\). Another classic approach in minimax optimization literature is Extragradient Method \([11, 12]\), which can be viewed as first-order approximation of PPM so that avoids such proximal computation, but that is beyond the scope of this paper.

**Related Work.** There is a long history of researches on the convex-concave minimax problem. Rockafellar \([13]\) studies PPM for solving monotone variational inequality, and shows that, as a special case, PPM converges to the stationary point linearly when \(L(x, y)\) is strongly convex-strongly concave or when \(L(x, y)\) is bilinear. Later on Tseng \([11]\) shows that EGM converges linearly to the stationary point under similar conditions. Nemirovski \([14]\) shows that EGM approximates PPM and present the sublinear rate of EGM. Recently, minimax problem gains the attention in the machine learning community, perhaps mainly due to the thrive of research on GANs. See the literature review in \([10]\) for a more detailed description on recent developments of convex-concave minimax problems.

There are also extensive studies on concave-concave minimax problems when the interaction term is bilinear (similar to our setting (2)). Some influential algorithms include Nesterov’s smoothing \([15]\), Douglas-Rachford splitting (a special case is Alternating Direction Method of Multipliers (ADMM)) \([16, 17]\) and Primal-Dual Hybrid Gradient Method (PDHG) \([18]\).

Very recently, a number of works have been undertaken considering nonconvex-concave minimax problems. The basic technique is to first turn the minimax problem (1) to a minimization problem on \(\Phi(x) = \max_y L(x, y)\), which is well-defined since \(L(x, y)\) is concave in \(y\), and then utilize the recent development on nonconvex optimization \([19, 20, 21, 22]\).

Unfortunately, the above technique cannot be extended to nonconvex-nonconcave setting, because \(\Phi(x)\) is now longer tractable to compute (even approximately) as its a nonconcave maximization problem itself. Indeed, the current understanding on nonconvex-nonconcave minimax problems is fairly limited, in particular compared with the growing literature on nonconvex optimization. The recent research on nonconvex-nonconcave minimax problems mostly rely on some form of convex-concave-like assumptions, such as Minty’s Variational Inequality \([23]\) and Polyak-Lojasiewicz conditions \([24, 25]\), which are strong in general and successfully bypass the inherent difficulty in the nonconvex-nonconcave setting. Such theory, unfortunately, falls far short of practice since they presuppose the existence of a globally attractive solution. As such, fundamental nonconvex-nonconcave structures like local solutions and cycling are prohibited.

In contrast to the above lines of research, we do not impose any strong assumptions, and shows that the local solutions and attractive limit circles are a fundamental part of the landscape for solving general nonconvex-nonconcave minimax problems (See Appendix A for sample paths from a variety of different first-order methods where these occur).

**Our Contributions.** The contributions of the paper can be summarised as follow: We identify that there can be three regions in the nonconvex-nonconcave bilinear minimax problems – interaction dominate, interaction middle, and interaction weak – where the solution paths of PPM can have different structures. A simple example showing these three

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\(^1\)For example, applying GDA will linearly converge to the unique solution for such problems. See Theorem B.6 in the appendix for one such result.
regions as $A$ increases is presented in Figure 1, and more examples for different algorithms are presented in Appendix A.

- For interaction dominate problems (i.e., when $A$ is large enough), we show PPM has **global, linear convergence to a stationary point**.

- For interaction weak problems (i.e., when $A$ is small enough), we show PPM has **local linear convergence to a stationary point**.

- For interaction middle problems (i.e., when $A$ has middling size), we show cycling is possible and so the above regimes cannot be extended. Despite the existence of cycles, we show PPM has gradient norm converge down to the order of nonconvexities present. Moreover, we show a lower bound on the average norm of the gradient when cycling based on the size and coarseness of the given cycle.

- We introduce a new concept, **saddle envelope**, which serves as the main analytical machinery for obtaining the above results.

![Figure 1: Sample paths of PPM from different initial solutions applied to (2) with $f(x) = (x + 3)(x + 1)(x - 1)(x - 3)$ and $g(y) = (y + 3)(y + 1)(y - 1)(y - 3)$ and different scalar $A$. As scalar $A \geq 0$ increases, the solution path transitions from having four locally attractive stationary points, to a globally attractive cycle, and finally to a globally attractive stationary point.](image)

## 2 Saddle envelope: Main Machinery for Analyzing PPM

Throughout the paper, we assume that $f$ and $g$ are $\beta \geq 0$-smooth and $\rho$-weakly convex, i.e.,

**Assumption 2.1.** We assume $f$ and $g$ are both twice differentiable, and it holds for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ that

$$-\rho I \preceq \nabla^2 f(x) \preceq \beta I , \quad -\rho I \preceq \nabla^2 g(y) \preceq \beta I .$$
Notice that the objective $L(x, y)$ is convex-concave when $\rho = 0$, and strongly convex-strongly concave when $\rho < 0$. Here our primary interest is in the regime where $\rho > 0$.

In this section, we develop a new concept, saddle envelope, in order to analyze damped PPM for solving nonconvex-nonconcave minimax optimization [2]. Formally, the saddle envelope (with parameter $\eta$) $L_\eta(x, y) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$L_\eta(x, y) := \min_{u \in \mathbb{R}^n} \max_{v \in \mathbb{R}^m} L(u, v) + \frac{\eta}{2} \|u - x\|^2 - \frac{\eta}{2} \|v - y\|^2.$$  \hspace{1cm} (6)

The proximal parameter $\eta$ is selected with $\eta > \rho \geq 0$, which ensures the the minimax problem in (6) is strongly convex-strongly concave and hence the saddle envelop is well-defined and efficiently computable.

The saddle envelope is a generalization of Moreau envelope in optimization literature to minimax problems. In optimization literature, Moreau envelope [26] and its relationship to damped PPM has been key to recent developments in theory for nonconvex nonsmooth minimization [27, 28]. Despite this similarity in definition, it’s important to note that our saddle envelope differs from Moreau envelopes in the following important ways. The Moreau envelope of any nonconvex function has the following nice pair of properties (provided $\eta$ is selected appropriately): (i) it has the same local minimum value and local minimizers as the original function, and (ii) it lower-bounds on the original function everywhere. Unfortunately, neither of these properties carry over to the saddle envelope of the nonconvex-nonconcave problems considered here.

But luckily, the next proposition presents two basic properties of saddle envelope, which connects the saddle envelop $L_\eta$ to the original objective function $L$. The first part shows that $L_\eta$ and $L$ share the same stationary points, thus we can instead focus on finding the stationary points of $L_\eta$. The second part shows the relation of the optimal objective of the two minimax problems, i.e., the minimax objective of $L_\eta$ is upper bounded by that of $L$.

**Proposition 2.2.** 1. (Equivalence of Stationary Points) $(x^*, y^*)$ is a stationary point of $L$ iff it is a stationary point of $L_\eta$.

2. (Optimal Objective Bound) The saddle envelope has optimal objective value bounded by

$$\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} L_\eta(x, y) \leq \min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} L(x, y).$$

Even more importantly, the next proposition shows that damped damped PPM on the original problem [2] is equivalent to GDA on the saddle envelope problem [7] with proper step-size.

**Proposition 2.3.** A step of PPM [4] on the original objective $L(x, y)$ is equivalent to a step of GDA [3] on the saddle envelope $L_\eta(x, y)$ with $s = \lambda/\eta$.

The above two Propostions build up the connection between the original objective $L$ and the saddle envelop $L_\eta$, which allows us to instead study the related minimax optimization problem given by the saddle envelope of

$$\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} L_\eta(x, y).$$ \hspace{1cm} (7)
Indeed, as we will show in later sections, $L_\eta$ has nicer structures than the original objective $L$. For example, in the interaction dominant case, $L_\eta$ can be strongly convex-strongly concave even though $L$ is not. This new perspective is key to allowing us to pass the historical barriers surrounding nonconvex-nonconcave problems.

3 Interaction Dominate Regime

In this section, we study the interaction dominate regime of the nonconvex-nonconcave minimax problem \((2)\) when the matrix $A$ is large enough (to compensate for nonconvexities in $f$ and $g$). Formally, the interaction dominance of $L(x,y)$ is defined as follow:

**Definition 3.1.** We say a saddle-point function $L(x,y)$ is $\alpha$-interaction dominate in $x$ (or in $y$) with $\alpha > 0$ if it holds for all $x$ (or $y$) that

$$\nabla^2 f(x) + \frac{AA^T}{2(\beta + \rho)} \succeq \alpha I \quad \text{or} \quad \nabla^2 g(y) + \frac{A^T A}{2(\beta + \rho)} \succeq \alpha I.$$  

Here we examine when a SPP is interaction dominate. Clearly all strongly convex-strongly concave problems are interaction dominate since the Hessians $\nabla^2 f$ and $\nabla^2 g$ themselves are strictly positive semidefinite. For nonconvex-nonconcave problems, interaction dominance requires $AA^T$ and $A^T A$ to be large enough to dominate any negative curvature in $\nabla^2 f$ and $\nabla^2 g$ respectively. When $A$ is square (that is, $n = m$), bounding the minimum eigenvalue of $AA^T$ is sufficient to show dominance. Namely, having

$$\lambda_{\min}(AA^T) \geq 2(\beta + \rho)(\alpha + \rho)$$

implies $\alpha$-interaction dominance in both $x$ and $y$ since

$$\nabla^2 f(x) + \frac{AA^T}{2(\beta + \rho)} \succeq \nabla^2 f(x) + (\alpha + \rho)I \succeq \alpha I.$$  

For simplicity in stating our results, we fix our choice of the proximal parameter $\eta > \rho > 0$ to be $\eta = 2\rho$. The next proposition show that the saddle envelope $L_\eta(x,y)$ is (i) smooth, and (ii) strongly convex in $x$ (or strongly concave in $y$) if $L(x,y)$ is interaction dominate in $x$ (or in $y$).

**Proposition 3.2.** 1. (Smoothness of $L_\eta$) The saddle envelope $L_\eta(x,y)$ with $\eta = 2\rho$ is $2\eta$-smooth. 

2. (Strong-convexity of $L_\eta$) Suppose $L(x,y)$ is $\alpha$-interaction dominance in $x$ (in $y$), then the saddle envelope $L_\eta(x,y)$ with $\eta = 2\rho$ is $(\eta^{-1} + \alpha^{-1})^{-1}$-strongly convex in $x$ (-strongly concave in $y$).

As a result, the saddle envelope $L_\eta(x,y)$ is a much more structured object than the original objective function $L(x,y)$. The next two subsections present how the saddle envelope helps to obtain the computational guarantees for interaction dominant cases. In particular, damped PPM has linear convergence when $L(x,y)$ is two-side interaction dominant, and sublinear convergence when $L(x,y)$ is one-side interaction dominant, both of which extend the regime of the nonconvex-noncave minimax problems one can solve in the literature.
3.1 Two-Sided Interaction Dominant Problems

Proposition 3.2 establish that whenever interaction dominance holds for both \(x\) and \(y\) the saddle envelope problem (7) is a strongly convex-strongly concave smooth optimization problem. Hence this problem has condition number

\[
\kappa = \frac{2\eta}{(\eta^{-1} + \alpha^{-1})^{-1}} = 2 \left(1 + \frac{2\rho}{\alpha}\right).
\]

The classic literature on minimax optimization guarantees that applying GDA to strongly convex-strongly concave smooth problems will converge linearly based on this condition number (for completeness of the paper, we state it in Theorem B.6). Recall from Proposition 2.3, PPM (4) is equivalent to running GDA on the saddle envelope \(L_\eta\). Therefore, we can conclude that damped PPM linearly converges to the global minimax point of \(L_\eta\), which is the unique stationary point of \(L\) by Proposition 2.2:

**Theorem 3.3.** For any problem \(L\) that is \(\alpha\)-interaction dominate in \(x\) and \(y\), damped PPM with \(\eta = 2\rho\) and \(\lambda = 1/2\kappa\) linearly converges to the unique stationary point \((x^*, y^*)\) of (2) with

\[
\left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|_2^2 \leq \left(1 - \frac{1}{4(1 + 2\rho/\alpha)^2}\right)^k \left\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \right\|_2^2.
\]

**Remark 3.4.** Theorem 3.3 is valid even if \(\alpha\)-interaction dominate only holds locally. That is, as long as \(\alpha\)-interaction dominate holds within an \(l_2\)-ball around a local stationary point, and the initial point is within this ball, then PPM converges linearly to this local stationary point. Of course in this case there can be many local stationary points.

**Remark 3.5.** For \(\mu\)-strongly convex-strongly concave problems, this theorem recovers the standard proximal point convergence rate for any choice of \(\eta > 0\). Note we can set the weak convexity parameter as \(\rho = \eta/2 > 0\). Then lower bounding \(\alpha\) by \(\mu\) gives a \(O(\eta^2/\mu^2 \log(1/\varepsilon))\) convergence rate matching [13].

**Remark 3.6.** The interaction dominance condition is tight to obtain global linear convergence. In Appendix D.3, we give a family of minimax problems where closely describes when global convergences ceases to hold.

3.2 One-Sided Interaction Dominant Problems

If we only have interaction dominance with \(y\), then the saddle envelope \(L_\eta\) is still much more structured than the original objective \(L\). In this case, Proposition 3.2 ensures (7) is a nonconvex-strongly concave problem. Then our saddle envelope theory allows us to extend existing convergence guarantees for nonconvex-concave problems to the larger class of \(y\) interaction dominate problems.

Nonconvex-concave problems has been considered by numerous recent works giving first-order methods that converge to stationary points [19, 20, 21, 22]. For example, Lin et al. [19] recently showed that GDA with different stepsize parameters for \(x\) and \(y\) will converge to a
stationary point at a rate of $O(\varepsilon^{-2})$. We find that running the following damped proximal point method is equivalent to running their variant of GDA on the saddle envelope

$$
\begin{bmatrix}
x_{k+1} \\
y_{k+1}
\end{bmatrix} = \begin{bmatrix}
\lambda x^k + (1 - \lambda)x_k \\
\gamma y^k + (1 - \gamma)y_k
\end{bmatrix} \text{ where } \begin{bmatrix}
x^k \\
y^k
\end{bmatrix} = \text{prox}_\eta(x_k, y_k)
$$

(8)

for proper choice of the parameters $\lambda, \gamma \in [0, 1]$. From this, we derive the following sublinear convergence rate for nonconvex-nonconcave problems whenever $y$ interaction dominance holds.

**Theorem 3.7.** Consider PPM (8) for solving (2) with $\eta = 2\rho$, $\lambda = \Theta(1/\kappa^2)$ and $\gamma = \Theta(1)$. Suppose $L(x, y)$ is $\alpha$-interaction dominate in $y$, and the sequence $y_k$ is bounded$^\ddagger$ then damped PPM finds an $\varepsilon$-stationary point of (2) satisfying $\|\nabla L(x^+_T, y^+_T)\| \leq \varepsilon$ by iteration $T \leq O(\varepsilon^{-2})$.

**Remark 3.8.** ($x$-Interaction Dominate Problems.) Symmetrically, this theorem can be extended to guarantee convergence in stationarity when only $x$-interaction dominance is assumed. Instead one would consider solving the maximin problem of

$$
\max_y \min_x L(x, y) = -\min_y \max_x -L(x, y),
$$

which is now interaction dominate with respect to the inner maximization variable. Notice that this reduction works since although the original minimax problem and this maximin problem need not have the same solutions, they always have the same stationary points.

### 4 Interaction Weak Regime

In this section, we consider the case where $A$ is relatively small. Intuitively, when $A$ is small enough, the interaction term does not contribute too much to the minimax solutions, and we should be able to find a local minimax solution $(x^*, y^*)$ of $L$, such that $x^*$ is close to a local minimizer of $f(x)$ and $y^*$ is close to a local maximizer of $g(y)$. To formalize such observation, we present a Two-Phase Proximal Point Method (2P-PPM), described in Algorithm 1, and show that the algorithm converges to a local stationary point linearly under certain regularity conditions.

**Algorithm 1:** Two Phase Proximal Point Method (2P-PPM)

**Data:** Given $(x', y')$, $f, g$ nonconvex, $A \in \mathbb{R}^{n \times m}$

1. **Phase One (Separable Optimization):**
2. Compute a local minimizer $x_0$ of $\min f(x) + x^T Ay'$;
3. Compute a local maximizer $y_0$ of $\max -g(y) + x'^T Ay$;
4. **Phase Two (Proximal Point Method):**
5. Run PPM starting at $(x_0, y_0)$ with $\eta = 2\rho$ and $\lambda = 1/4\kappa$.

In Phase one, we ignore the interaction term and find the local optimizer over $x$ and $y$ separately using any existing nonconvex minimization (maximization) algorithms. Specifically for a given $(x', y')$, we can compute a local minimizer $x_0$ of $\min f(x) + x^T Ay'$ and compute a local maximizer $y_0$ of $\max -g(y) + x'^T Ay$. These problems amount to smooth nonconvex

$\ddagger$We do not believe this boundedness condition is fundamentally needed, but we make it to leverage the results of [19] which utilize compactness.
minimization, which is well-studied (see for example [29]) and so we take it as a blackbox in this work. Although the resulting point \((x_0, y_0)\) is not a stationary point of \(L\) since we ignored the interaction term, we expect that \((x_0, y_0)\) is near a stationary point of \(L(x, y)\) when the interaction term is relatively small.

In Phase two, we run damped PPM starting from \((x_0, y_0)\). Indeed, under an appropriate local regularity condition, we can show that \(L(x, y)\) is interaction dominant in a ball around \((x_0, y_0)\), which turns out to contain a saddle point of \(L(x, y)\). Consequently, we are able to guarantee that damped PPM convergences linearly to this point. Formally, we define the resulting solution \((x_0, y_0)\) from Phase one to be \(\alpha_0\)-interaction dominant of \(L(x, y)\) as follow:

**Definition 4.1.** We say a point \((x_0, y_0)\) is \(\alpha_0\)-interaction dominant to \(L(x, y)\) if
\[
\nabla^2 f(x_0) + \frac{AA^T}{2(\beta + \rho)} \succeq \alpha_0 I \quad \text{and} \quad \nabla^2 g(y_0) + \frac{A^TA}{2(\beta + \rho)} \succeq \alpha_0 I .
\]

Moreover, we say \((x_0, y_0)\) is strictly interaction dominant to \(L(x, y)\) if \(\alpha_0 > 0\).

Phase one always produces a point where \(\alpha_0 \geq 0\)-interaction dominance holds at both \(x_0\) and \(y_0\). Since these points are local minimizers and maximizers, we know that \(\nabla^2 f(x_0) \succeq 0\) and \(\nabla^2 g(y_0) \succeq 0\). Hence it immediately follows that (9) holds with \(\alpha_0 = 0\).

Moreover, interaction dominance holds at \(x_0\) and \(y_0\) with \(\alpha_0 > 0\) if these local solutions are have strictly positive Hessians, which is a natural condition for ensuring \(x_0\) and \(y_0\) are strict minimizers/maximizers, or if \(A\) is square and full rank which would ensure interaction dominance at \(x_0\) and \(y_0\) holds with \(\alpha_0 \geq \lambda_{\min}(AA^T) / (2(\beta + \rho)) > 0\). Indeed, we can show that interaction dominance holds at \((x_0, y_0)\) if \((x', y')\) are chosen randomly and the methods for computing \(x_0\) and \(y_0\) exhibit reasonable local continuity properties. This is formalized in Appendix E.1.

To present local convergence in this setting, we additionally assume continuity of the Hessians \(\nabla^2 f\) and \(\nabla^2 g\).

**Assumption 4.2.** The Hessians \(\nabla^2 f\) and \(\nabla^2 g\) are \(H\)-Lipschitz continuous.

Since the Hessians of \(f\) and \(g\) are Lipschitz, this \(\alpha_0\)-interaction dominance at \((x_0, y_0)\) extends to \(\alpha_0/2\)-interaction dominance in a ball around \((x_0, y_0)\) of radius \(\alpha_0 / 2H\).

The next assumption provides a sufficient condition of the local convergence of PPM, which holds when the interaction term \(\|A\|\) is small enough or the initialization \((x', y')\) is sufficiently close to the corresponding separable optimization solution \((x_0, y_0)\).

**Assumption 4.3.** The solution \((x_0, y_0)\) obtained in Phase One is strictly interaction dominant to \(L(x, y)\) with \(\alpha_0 > 0\), and furthermore, it holds that
\[
\| [AA^T(x_0 - x')] \| \leq \frac{\alpha_0}{\sqrt{2H} \left( \frac{3}{2} + \frac{\beta + \|A\|}{\rho} \right) \left( \frac{\rho}{\frac{3}{2} + \frac{\alpha_0}{\rho}} \right)}.
\]

The next theory present our local linear convergence results of Algorithm I.
Theorem 4.4. Consider the Two Phase Proximal Point Method (Algorithm 1) with initial solution \((x', y')\). Suppose the function \(f, g\) satisfies Assumption 4.2, and the obtained solution to Phase One satisfies Assumption 4.3. Then iterates \((x_k, y_k)\) from phase two of 2P-PPM will linearly converge to a stationary point \((x^*, y^*)\) of (2) with

\[
\left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|^2 \leq \left( 1 - \frac{1}{4(1 + 4\rho/\alpha_0)^2} \right)^k \left\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \right\|^2.
\]

5 Interaction Moderate Regime

Between the interaction dominate and interaction weak regimes, the proximal point method may diverge or cycle indefinitely. An example of an interaction moderate problem with an attractive limit cycle was given in Figure 1 (b). In this section, we focus on this regime.

The standard analysis of gradient descent on nonconvex optimization relies on the fact that the function value monotonically decays at the level of gradient norm square every iteration. Thus as long as the gradient is large, the function value has sufficient decay. Consequently, the iterates of gradient descent either have gradient norm converge to 0 or objective value approach \(-\infty\). However, such arguments no longer holds in the nonconvex-nonconcave minimax setting: the objective is neither monotonically decreasing nor increasing, and worse yet, methods may cycle indefinitely despite having large gradients, as shown in Figure 1 (b).

In order to obtain a similar analysis as the startard nonconvex optimization, we herein propose to study the following “Lyapunov” function:

\[
\mathcal{L}(x, y) := f(x) - \frac{\eta}{2} \|x\|^2 + \left( f + \frac{\eta}{2} \cdot \|\cdot\|^2 \right)^* (-Ay + \eta x) + g(y) - \frac{\eta}{2} \|y\|^2 + \left( g + \frac{\eta}{2} \cdot \|\cdot\|^2 \right)^* (A^T x + \eta y),
\]

where \((\cdot)^*\) is the (Fenchel) conjugate function (see Appendix B.1 for the definition). The following theorem establishes a number of nice structural properties supporting our consideration of \(\mathcal{L}(x, y)\).

Theorem 5.1. The Lyapunov function \(\mathcal{L}(x, y)\) has the following structural properties:

1. \(\mathcal{L}(x, y) \geq 0\),
2. \(\mathcal{L}(x, y) = 0\) if and only if \((x, y)\) is a stationary point to \(L(x, y)\),
3. When \(\eta = 0\), \(\mathcal{L}(x, y)\) recovers the well-known primal-dual gap of the function \(L(x, y)\)

\[
\mathcal{L}(x, y) = \max_{y^*} L(x, y^*) - \min_{x^*} L(x^*, y).
\]

Despite the fact that damped PPM may not converge to a stationary point, this Lyapunov allows us to derive meaningful upper bounds on its performance. The following theorem shows the average gradient will converge down to have size \(O(\rho)\), matching the level of nonconvexity present in the problem.
Theorem 5.2. For any problem \( L(x, y) \), PPM \((4)\) with \( \eta = 3\rho \) and \( \lambda = 1 \) has

\[
\frac{1}{T} \sum_{k=0}^{T} \| \nabla L(x_{k+1}, y_{k+1}) \|^2 \leq 2 \max \left\{ \frac{36\rho C}{T}, (36\rho D)^2 \right\}
\]

where \( C \) is a constant depending on the initial iterate \((x_0, y_0)\) and any chosen center \((\hat{x}, \hat{y})\), which is specified in Appendix F.2, and \( D \) bounds the average deviation of the iterates from this center given by

\[
D = \sqrt{\frac{1}{T} \sum_{k=0}^{T} \left\| \begin{bmatrix} x_{k+1} - \hat{x} \\ y_{k+1} - \hat{y} \end{bmatrix} \right\|^2}.
\]

Remark 5.3. For problems with a low-level of nonconvex-nonconcave structure (i.e., \( \rho \) is small), this theorem guarantees the average iterate will have similarly small gradient. Hence the cycling behavior that prevents us from finding stationary points fades away as problems approach being convex-concave.

We can also prove a lower-bound on average gradient norm, which follows from the isoperimetric inequality. It says that, if PPM "coarsely" converges to a cyclic attractor, that winds around a large ball, then the average of the squared gradient will be large. Details and definitions in the appendix.

Theorem 5.4. For any problem \( L(x, y) \), if the limiting behaviour of PPM \((4)\), converges uniformly to a cyclic attractor \( C \), such that there exists a point on the minimal surface bounded by \( C \) that is at (geodesic) distance at least \( R \) from every point on \( C \) then, small enough choice of step size \( s \) and a large enough choice of \( T \) and \( S \) (where \( T \gg S \)), we have

\[
\frac{1}{T - S} \sum_{k=S}^{T} \| \nabla L(x_{k+1}, y_{k+1}) \|^2 \geq \frac{C \cdot R^2}{s^2 N^2}
\]

where \( C \) is a constant depending on the properties of the minimal surface \( S \), and \( N \) measures the "coarseness" of the algorithm.

6 Conclusion and Future Works

We gave a three phase characterization for the sample path of the proximal point method when applied to nonconvex-nonconcave bilinear problems. As the interaction terms grows, the dynamics change from local linear convergence to stationary points, to potential cycling with bounded gradients, to global linear convergence to a unique stationary point. These results are the first convergence rate guarantees to address the fundamental nonconvex-nonconcave issues of local solutions and cycling.

Important areas for future work lie in extending this characterization to constrained and nonbilinear nonconvex-nonconcave problems. It will also be important for many modern learning applications to extend these results to understand the solution paths produced by stochastic algorithms.
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A Sample Paths From Other First-Order Methods

Figure 2 plots the solution paths of four common first-order methods for minimax problem for solving a two-dimensional nonconvex-nonconcave minimax problem:

$$\min_x \max_y L(x, y) = (x + 3)(x + 1)(x - 3) + Axy - (y + 3)(y + 1)(y - 3), \quad (12)$$

with four different levels of interaction term, $A = 1, 10, 100, 1000$. This problem is globally $\rho = 20$-weakly convex and $\beta = 172$-smooth on the box $[-4, 4] \times [-4, 4]$.

Each plot in Figure 2 shows the sample paths generated by running 100 iterations of the given method from the twelve different initial solutions around the boundary of the plot $(4, 0), (0, 4), (-4, 0), (0, -4), (4, 2), (2, 4), (4, -2), (2, -4), (-4, 2), (-4, -2), (-2, -4)$ and four initial solutions towards the center of the plot $(1, 0), (0, 1), (-1, 0), (0, -1)$.

Plots (a)-(d) show the behavior of the Proximal Point Method (PPM) \(^4\) with $\eta = 2\rho = 40$ and $\lambda = 1$. These figures match the landscape described by our theory: $A = 1$ is small enough to have local convergence to four different stationary points (each around $\{\pm 2\} \times \{\pm 2\}$), $A = 10$ has moderate size and every sample path is attracted into a limit cycle, and finally $A = 100$ and $A = 1000$ give a large enough interaction term to create a globally attractive stationary point (moreover, comparing plots (c) and (d) shows as $A$ becomes larger the rate of convergence increases).

Plots (e)-(h) show the behavior of the Extragradient Method (EG), which is defined by

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + s \begin{bmatrix} -\nabla_x L(x_k, y_k) \\ -\nabla_y L(x_k, y_k) \end{bmatrix}$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + s \begin{bmatrix} -\nabla_x L(\bar{x}, \bar{y}) \\ -\nabla_y L(\bar{x}, \bar{y}) \end{bmatrix} \quad (13)$$

with stepsize chosen as $s = 1/2(\beta + A) = 1/(344 + 2A)$. This stepsize was chosen since the objective function has a $\beta + A$-Lipschitz gradient. These figures show that the extragradient method follows the same general trajectory as described by our theory for the proximal point method. For small $A = 1$, local convergence occurs. For moderate sized $A = 10$ and $A = 100$, the algorithm falls into an attractive limit cycle, never converging. For large enough $A = 1000$, the method globally converges to a stationary point. The extragradient method only differs from the proximal point method’s landscape in that it requires a larger $A$ to transition into the interaction dominate regime.

Plots (i)-(l) show the behavior of Gradient Descent Ascent (GDA) \(^3\) with $s = 1/2(\beta + A) = 1/(344 + 2A)$. This method is know to be unstable and diverge even for convex-concave problems. The same behavior carries over to our nonconvex-nonconcave example. For small $A$, we still see local convergence. However for $A = 10, 100, 1000$, we find that GDA falls into a limit cycle with increasingly large radius as $A$ grows.

Lastly, plots (m)-(p) show the behavior of Alternating Gradient Descent Ascent (AGDA), which is defined by

$$\begin{align*}
x_{k+1} &= x_k - s \nabla_x L(x_k, y_k) \\
y_{k+1} &= y_k + s \nabla_y L(x_{k+1}, y_k)
\end{align*} \quad (14)$$

with $s = 1/2(\beta + A) = 1/(344 + 2A)$. Again for small $A$, we still see local convergence, but for larger $A = 10, 100, 1000$, AGDA always falls into a limit cycle.
B Preliminary/Review of Existing Results

B.1 Properties of Convex Conjugates and Moreau Envelopes

Denote the convex (Fenchel) conjugate of a function $h$ as

$$h^*(z) = \sup_w z^T w - h(w).$$

Denote the Moreau envelope of a function $h$ with proximal parameter $\eta > 0$ by

$$e_\eta \{h\} (z) = \min_w h(w) + \frac{\eta}{2} \|w - z\|^2.$$

In this section, we state and give brief proofs (for completeness sake) of a number of standard properties of these two classic objects when applied to smooth, strongly convex functions.

Lemma B.1 (Gradients and Hessians of Convex Conjugates). For any twice differentiable, $\gamma$-smooth, $\mu$-strongly convex function $h$, the convex conjugate $h^*$ is $\mu^{-1}$-smooth and $\gamma^{-1}$-strongly convex. Moreover, letting $w^* = \arg\max_w z^T w - h(w)$, the gradient of the convex conjugate is given by

$$\nabla h^*(z) = w^*$$

and the Hessian of the convex conjugate is given by

$$\nabla^2 h^*(z) = (\nabla^2 h(w^*))^{-1}.$$

Proof. Strong convexity gives the lower bound $h(w) \geq h(w^*) + \nabla h(w^*)^T (w - w^*) + \frac{\mu}{2} \|w - w^*\|^2$ and so the convex conjugate $h^*$ is upper bounded by

$$h^*(z') = \sup_w z'^T w - h(w) \leq \sup_w z'^T w - h(w^*) - \nabla h(w^*)^T (w - w^*) - \frac{\mu}{2} \|w - w^*\|^2$$

$$= z'^T w^* - h(w^*) + \frac{1}{2\mu} \|z' - \nabla (w^*)\|^2$$

$$= h^*(z) + (z' - z)^T w^* + \frac{1}{2\mu} \|z' - \nabla (w^*)\|^2.$$

Symmetrically, smoothness gives the upper bound $h(w) \leq h(w^*) + \nabla h(w^*)^T (w - w^*) + \frac{\gamma}{2} \|w - w^*\|^2$ and so the convex conjugate $h^*$ is lower bounded by

$$h^*(z') = \sup_w z'^T w - h(w) \geq \sup_w z'^T w - h(w^*) - \nabla h(w^*)^T (w - w^*) - \frac{\gamma}{2} \|w - w^*\|^2$$

$$= z'^T w^* - h(w^*) + \frac{1}{2\gamma} \|z' - \nabla (w^*)\|^2$$

$$= h^*(z) + (z' - z)^T w^* + \frac{1}{2\gamma} \|z' - \nabla (w^*)\|^2.$$

Thus we have quadratic upper and lower bounds on $h^*$ at $z$ that agree at first order. Therefore its gradient is given by $\nabla h^*(z) = w^*$. The first order optimality condition of $w^* = \arg\max_z z^T w - h(w)$ ensures $\nabla h(w^*) = z$. From this, we have that $z = \nabla h(\nabla h^*(z))$. Differentiating this gives

$$I = \nabla^2 h(\nabla h^*(z)) \nabla^2 h^*(z).$$

Hence the Hessian of the convex conjugate is given by $\nabla^2 h^*(z) = (\nabla^2 h(w^*))^{-1}$. \hfill \Box
Lemma B.2 (Gradients of Moreau Envelopes). For any twice differentiable, $\gamma$-smooth, $\rho$-weakly convex function $h$, let $z_+ = \text{argmin} \ h(w) + \frac{\eta}{2} \|w - z\|^2$. Then the gradient of the Moreau envelope with $\eta > \rho$ is given by
\[
\nabla e_\eta \{h\}(x) = \eta(z - z_+)
\]
and its Hessian is given by
\[
\nabla^2 e_\eta \{h\}(x) = \eta I - \eta^2 \left(\nabla^2 h(z_+) + \eta I\right)^{-1}.
\]
Proof. Observe that the Moreau envelop can be describe as the following convex conjugate
\[
e_\eta \{h\}(z) = \min_w h(w) + \frac{\eta}{2} \|w - z\|^2
\]
\[
= \min_w h(w) + \frac{\eta}{2} \|w\|^2 - \eta w^T z + \frac{\eta}{2} \|z\|^2
\]
\[
= \frac{\eta}{2} \|z\|^2 - \max_w \eta w^T z - (h(w) + \frac{\eta}{2} \|w\|^2)
\]
\[
= \frac{\eta}{2} \|z\|^2 - \left(h + \frac{\eta}{2} \|\cdot\|^2\right)^*(\eta z).
\]
The gradient formula from Lemma B.1 then gives
\[
\nabla e_\eta \{h\}(x) = \eta(z - z_+)
\]
and the Hessian formula from Lemma B.1 gives
\[
\nabla^2 e_\eta \{h\}(x) = \eta I - \eta^2 \left(\nabla^2 h(z_+) + \eta I\right)^{-1}.
\]
\]

Lemma B.3 (Strong Convexity of Moreau Envelopes). For any twice differentiable, $\gamma$-smooth, $\mu$-strongly convex function $h$, the Moreau envelope $e_\eta \{h\}$ is $(\eta^{-1} + \mu^{-1})^{-1}$.

Proof. This follows from the Hessian formula given by Lemma B.2. Namely, we have
\[
\nabla^2 e_\eta \{h\}(x) = \eta I - \eta^2 \left(\nabla^2 h(z_+) + \eta I\right)^{-1}
\]
\[
ge \eta I - \eta^2 \left(\mu I + \eta I\right)^{-1}
\]
\[
= (\eta - \eta^2 / (\eta + \mu)) I
\]
\[
= (\eta^{-1} + \mu^{-1})^{-1} I.
\]

B.2 Properties of (Locally) Strongly Convex-Strongly Concave Functions

Lemma B.4 presents a basic property of a locally strongly convex-strongly concave function $M(x, y)$. In the monotone operator language, this corresponds to showing $F(x, y) := (\nabla_x M(x, y), -\nabla_y M(x, y))$ is locally strongly monotone (or coercive).

Lemma B.4. Suppose $M(x, y)$ is $\mu$-strongly convex-strongly concave on a convex set $S = S_x \times S_y$, then it holds for any $(x, y), (x', y') \in S$ that
\[
\mu \left\| \begin{bmatrix} x - x' \\ y - y' \end{bmatrix} \right\|^2 \leq \left( \begin{bmatrix} \nabla_x M(x, y) \\ -\nabla_y M(x, y) \end{bmatrix} - \begin{bmatrix} \nabla_x M(x', y') \\ -\nabla_y M(x', y') \end{bmatrix} \right)^T \begin{bmatrix} x - x' \\ y - y' \end{bmatrix}.
\]
In particular, when $\nabla M(x', y') = 0$, it holds that
\[
\left\| \begin{bmatrix} x - x' \\ y - y' \end{bmatrix} \right\|^2 \leq \frac{\nabla \| M(x, y) \|}{\mu}.
\]
Applying Lemma B.4 and utilizing (15), we conclude that

\[ M(x', y') \leq M(x, y') - \nabla_x M(x', y')^T (x - x') - \frac{\mu}{2} \|x - x'\|^2 \]

\[ \leq M(x, y) + \nabla_y M(x, y)^T (y' - y) - \nabla_x M(x', y')^T (x - x') - \frac{\mu}{2} \|y - y'\|^2 - \frac{\mu}{2} \|x - x'\|^2 \]

where the first inequality uses strong convexity of \( M \) over \( x \) and the second uses strong concavity of \( M \) over \( y \). Symmetrically,

\[ M(x', y') \geq M(x', y) - \nabla_y M(x', y')^T (y - y') + \frac{\mu}{2} \|y - y'\|^2 \]

\[ \geq M(x, y) + \nabla_x M(x, y)^T (x' - x) - \nabla_y M(x', y')^T (y' - y) + \frac{\mu}{2} \|x - x'\|^2 + \frac{\mu}{2} \|y - y'\|^2. \]

Combining the above two inequalities gives the first claimed inequality

\[ \frac{\mu}{2} \left\| \begin{bmatrix} x - x' \\ y - y' \end{bmatrix} \right\|^2 \leq \left( \begin{bmatrix} \nabla_x M(x, y) \\ -\nabla_y M(x, y) \end{bmatrix} - \begin{bmatrix} \nabla_x M(x', y') \\ -\nabla_y M(x', y') \end{bmatrix} \right)^T \begin{bmatrix} x - x' \\ y - y' \end{bmatrix}. \]

Furthermore, when \( \nabla M(x', y') = 0 \), we have

\[ \frac{\mu}{2} \left\| \begin{bmatrix} x - x' \\ y - y' \end{bmatrix} \right\|^2 \leq \left\| \nabla M(x, y) \right\| \left\| \begin{bmatrix} x - x' \\ y - y' \end{bmatrix} \right\|, \]

which finishes the proof of the second inequality. \( \square \)

From this we conclude that if the set \( S \) is large enough, \( M \) must have a stationary point in \( S \). Now define \( B(z, r) = \{ z' : \|z - z'\| \leq r \} \) as the closed Euclidean ball centered as \( a \) with radius \( r \). The above claim can be formalized as follows

**Lemma B.5.** Suppose \( M \) is \( \mu \)-strongly convex-strongly concave in a set \( B(x, r) \times B(y, r) \) for some fixed \((x, y)\) and \( r > \|\nabla M(x, y)\|/\mu \), then there exists a stationary point of \( M \) in \( B((x, y), r) \).

**Proof.** Consider the following minimax problem

\[ \min_{x' \in B(x, r)} \max_{y' \in B(y, r)} M(x, y). \]

Since \( M(x, y) \) is strongly convex-strongly concave, it must have a unique solution \((x^*, y^*)\). The first-order optimality condition for \((x^*, y^*)\) ensures

\[ \nabla_x M(x^*, y^*) = -\lambda (x^* - x) \]

\[ -\nabla_y M(x^*, y^*) = -\gamma (y^* - y) \]

for some constants \( \lambda, \gamma \geq 0 \) that are nonzero only if \( x^* \) or \( y^* \) are on the boundary of \( B(x, r) \) and \( B(y, r) \) respectively. Taking an inner product with \((x^* - x, y^* - y)\) gives

\[ \begin{bmatrix} \nabla_x M(x^*, y^*) \\ -\nabla_y M(x^*, y^*) \end{bmatrix}^T \begin{bmatrix} x^* - x \\ y^* - y \end{bmatrix} = - \left\| \begin{bmatrix} \sqrt{\lambda} (x^* - x) \\ \sqrt{\gamma} (y^* - y) \end{bmatrix} \right\|^2 \leq 0. \] (15)

Applying Lemma B.4 and utilizing (15), we conclude that

\[ \frac{\mu}{2} \left\| \begin{bmatrix} x^* - x \\ y^* - y \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} \nabla_x M(x, y) \\ -\nabla_y M(x, y) \end{bmatrix} \right\| \left\| \begin{bmatrix} x^* - x \\ y^* - y \end{bmatrix} \right\| \leq 0. \] (16)
Then GDA
stationary point
Theorem B.6. Consider any minimax problem
The following lemma allows us to characterize the saddle envelope in terms of two classical objects:
whenever
s
whereby
s
Theorem B.6. Consider any minimax problem \( \min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} M(x, y) \) where \( M(x, y) \) is \( \gamma \)-smooth and \( \mu \)-strongly convex-strongly concave on a set \( B(x_0, r) \times B(y_0, r) \) with \( r > \|\nabla M(x_0, y_0)\|/\mu \).

Then GDA \( \text{(3)} \) with initial solution \((x_0, y_0)\) and step-size \( s \in (0, 2\mu/\gamma^2) \) linearly converges to a stationary point \((x^*, y^*) \in B((x_0, y_0), r)\) with

\[
\left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|_2 \leq \left( 1 - 2\mu s + \gamma^2 s^2 \right)^{k} \left\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \right\|_2^2.
\]

Proof. Note Lemma B.5 gives the existence of a nearby stationary point \((x^*, y^*)\). Then the standard proof of strongly monotone (from Lemma B.4) and Lipschitz operators give rise to a contraction whenever \( s \in (0, 2\mu/\gamma^2) \):

\[
\left\| \begin{bmatrix} x_{k+1} - x^* \\ y_{k+1} - y^* \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|_2^2 - 2s \left[ \nabla_M(x_k, y_k) \right]^T \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} + s^2 \left\| \nabla_M(x_k, y_k) \right\|_2^2
\leq \left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|_2^2 - 2\mu s \left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|_2^2 + \gamma^2 s^2 \left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|_2^2
\leq \left( 1 - 2\mu s + \gamma^2 s^2 \right)^{k} \left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|_2^2,
\]

where the inequality utilizes \( [16] \) and the smoothness of \( M(x, y) \).

C Proofs for Properties of Saddle Envelope in Section 2

The following lemma allows us to characterize the saddle envelope in terms of two classical objects: convex conjugates and Moreau envelopes. Leveraging this structure facilitates deriving a gradient formula in Lemma C.2 and subsequently the relationships between the original objective \( L \) and the saddle envelope \( L_\eta \) in Propositions 2.2 and 2.3.

Lemma C.1. For any \((x, y)\), the saddle envelope can be characterized as a Moreau envelope in \( x \) as

\[
L_\eta(x, y) = e_\eta \{ h_y \}(x) - \frac{\eta}{2} \| y \|^2
\]

where \( h_y(u) = f(u) + \left( g(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^*(A^T u + \eta y) \)

or as a Moreau envelope in \( y \) as

\[
L_\eta(x, y) = -e_\eta \{ h_x \}(y) + \frac{\eta}{2} \| x \|^2
\]

where \( h_x(v) = g(v) + \left( f(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^*(-A v + \eta x) \).
Proof. Each of these results follow from direct algebraic manipulation of the definition of the saddle envelope:

\[
L_\eta(x, y) = \min_{u \in \mathbb{R}^n} \max_{v \in \mathbb{R}^m} L(u, v) + \frac{\eta}{2} \|x - u\|^2 - \frac{\eta}{2} \|y - v\|^2
\]

\[
= \min_{u \in \mathbb{R}^n} \max_{v \in \mathbb{R}^m} f(u) + \frac{\eta}{2} \|x - u\|^2 + u^T Av - g(v) - \frac{\eta}{2} \|y - v\|^2
\]

\[
= \min_{u \in \mathbb{R}^n} f(u) + \frac{\eta}{2} \|x - u\|^2 + \max_{v \in \mathbb{R}^m} \left( u^T Av - g(v) - \frac{\eta}{2} \|v\|^2 \right)
\]

\[
= \min_{u \in \mathbb{R}^n} f(u) + \frac{\eta}{2} \|x - u\|^2 + \left( \langle A^T u + \eta y \rangle^T v - g(v) - \frac{\eta}{2} \|v\|^2 \right) - \frac{\eta}{2} \|y\|^2
\]

\[
= \min_{u \in \mathbb{R}^n} \left( f(u) + \left( g(\cdot) + \frac{\eta}{2} \|\cdot\|^2 \right)^* (A^T u + \eta y) - \frac{\eta}{2} \|y\|^2 \right)
\]

\[
e_{\eta} \left\{ f(u) + \left( g(\cdot) + \frac{\eta}{2} \|\cdot\|^2 \right)^* (A^T u + \eta y) \right\} (x) - \frac{\eta}{2} \|y\|^2.
\]

Noting that the minimax problem defining the saddle envelope is convex-concave, we can apply the standard minimax theorem to exchange the order of operations. Then a symmetric argument gives the claimed formula based on the Moreau envelope with respect to y.

\[\square\]

**Lemma C.2.** The gradient of the saddle envelope \(L_\eta(x, y)\) is given by

\[
\begin{bmatrix}
\nabla_x L_\eta(x, y) \\
\nabla_y L_\eta(x, y)
\end{bmatrix} = \begin{bmatrix}
\eta (x - x_+) \\
\eta (y + y_+)
\end{bmatrix} = \begin{bmatrix}
\nabla_x L(x_+, y_+) \\
\nabla_y L(x_+, y_+)
\end{bmatrix}
\]

where \((x_+, y_+) = \text{prox}_\eta(x, y)\) is given by the proximal operator.

**Proof.** Recalling the gradient formula for Moreau envelopes in Lemma C.1, our first equality follows from taking the gradient with respect to \(x\) in the first formula from Lemma C.1 and with respect to \(y\) in the second formula from Lemma C.1.

The second equality is precisely the first-order optimality condition for (5). That is,

\[
\nabla f(x_+) + \eta(x_+ - x) + Ay_+ = 0
\]

\[
\nabla g(y_+) + \eta(y_+ - y) - A^T x_+ = 0.
\]

\[\square\]

**C.1 Proof of Proposition 2.2**

1. Both directions follow from Lemma C.2. First consider any stationary point \((x, y)\) of \(L\). Then we have that \(x_+ = x\) and \(y_+ = y\) by Fermat’s rule. Hence \((x, y)\) must be a stationary point of \(L_\eta\) as well since \(\nabla L_\eta(x, y) = \nabla L(x_+, y_+) = \nabla L(x, y) = 0\). Conversely consider a stationary point \((x, y)\) of \(L_\eta\). Then \(\eta(x - x_+) = \nabla_x L_\eta(x, y) = 0\) and \(\eta(y - y_+) = \nabla_y L_\eta(x, y) = 0\). Hence \((x, y) = (x_+, y_+)\) must be a stationary point of \(L\) as well since \(\nabla L(x, y) = \nabla L(x_+, y_+) = \nabla L_\eta(x, y) = 0\).

2. Consider an augmented version of (2) with the addition of two dummy variables \(u\) and \(v\) as follows

\[
\min_x \max_y L(x, y) = \min_x \min_u \max_y \max_v L(u, v) + \frac{\eta}{2} \|u - x\|^2 - \frac{\eta}{2} \|v - y\|^2.
\]

Equality holds here works since the minimum value over \(x\) always occurs at \(x = u\) and the maximum value over \(y\) always occurs at \(y = v\). Then interchanging the middle minimization
and maximization operations can only decrease the objective value. Hence we have the claimed inequality

\[
\min_x \max_y L(x, y) = \min_x \min_u \max_y \max_v L(u, v) + \frac{\eta}{2} \|u - x\|^2 - \frac{\eta}{2} \|v - y\|^2 \\
\geq \min_x \max_y \min_u \max_v L(u, v) + \frac{\eta}{2} \|u - x\|^2 - \frac{\eta}{2} \|v - y\|^2 \\
= \min_x \max_y L_\eta(x, y).
\]

C.2 Proof of Proposition 2.3

Let \((x^+, y^+) = \text{prox}_\eta(x_k, y_k)\) and let \((x_{k+1}, y_{k+1})\) be a step of GDA on \(L_\eta(x, y)\) from \((x_k, y_k)\) with step-size \(s = \lambda/\eta\), then it follows from Lemma C.2 that

\[
\begin{bmatrix}
x_{k+1} \\
y_{k+1}
\end{bmatrix} = \begin{bmatrix}
x_k \\
y_k
\end{bmatrix} + s \begin{bmatrix}
-\nabla_x L_\eta(x_k, y_k) \\
\eta L_\eta(x_k, y_k)
\end{bmatrix} \\
= \begin{bmatrix}
x_k \\
y_k
\end{bmatrix} + \frac{\lambda}{\eta} \begin{bmatrix}
-\eta(x_k - x^+) \\
\eta(y_k^+ - y_k)
\end{bmatrix} \\
= (1 - \lambda) \begin{bmatrix}
x_k \\
y_k
\end{bmatrix} + \lambda \text{prox}_\eta(x_k, y_k),
\]

which finishes the proof.

\[\square\]

D Proofs for Results on the Interaction Dominate Regime in Section 3

D.1 Proof of Proposition 3.2

1. Consider two points \((x, y)\) and \((\bar{x}, \bar{y})\) and denote one proximal step from each of them by \((x^+, y^+) = \text{prox}_\eta(x, y)\) and \((\bar{x}^+, \bar{y}^+) = \text{prox}_\eta(\bar{x}, \bar{y})\).

Define the function underlying the computation of the saddle envelope at \((x, y)\) as

\[
M(u, v) = L(u, v) + \frac{\eta}{2} \|u - x\|^2 - \frac{\eta}{2} \|v - y\|^2.
\]

Our choice of \(\eta > \rho\) ensures that \(M\) is \((\eta - \rho)\)-strongly convex-strongly concave.

First we compute the gradient of \(M\) at \((\bar{x}^+, \bar{y}^+)\) which is given by

\[
\begin{bmatrix}
\nabla_x M(\bar{x}^+, \bar{y}^+) \\
\nabla_y M(\bar{x}^+, \bar{y}^+)
\end{bmatrix} = \begin{bmatrix}
\nabla_x L(\bar{x}^+, \bar{y}^+) + \eta(\bar{x}^+ - x) \\
\nabla_y L(\bar{x}^+, \bar{y}^+) - \eta(\bar{y}^+ - y)
\end{bmatrix} \\
= \begin{bmatrix}
\nabla_x L(\bar{x}^+, \bar{y}^+) + \eta(\bar{x}^+ - \bar{x} + \bar{x} - x) \\
\nabla_y L(\bar{x}^+, \bar{y}^+) - \eta(\bar{y}^+ - \bar{y} + \bar{y} - y)
\end{bmatrix} \\
= \eta \begin{bmatrix}
\bar{x} - x \\
\bar{y} - y
\end{bmatrix}.
\]
Then applying Lemma [B.4] and noticing by definition of \((x_+, y_+)\) that \(\nabla_x L(x, y)M(x_+, y_+) = 0\) and \(\nabla_y L(x, y)M(x_+, y_+) = 0\), we conclude that
\[
\left\| \begin{bmatrix} \bar{x}_+ - x_+ \\ \bar{y}_+ - y_+ \end{bmatrix} \right\|^2 \leq \frac{\eta}{\eta - \rho} \left[ \begin{bmatrix} \bar{x} - x \\ \bar{y} - y \end{bmatrix}^T \begin{bmatrix} \bar{x}_+ - x_+ \\ \bar{y}_+ - y_+ \end{bmatrix} \right].
\]

Recall from Lemma [C.2] the gradients of the saddle envelope are given by
\[
\nabla L_\eta(x, y) = \eta \begin{bmatrix} x - x_+ \\ y_+ - y \end{bmatrix} \quad \text{and} \quad \nabla L_\eta(\bar{x}, \bar{y}) = \eta \begin{bmatrix} \bar{x} - \bar{x}_+ \\ \bar{y}_+ - \bar{y} \end{bmatrix}.
\]

Then we can upper bound the difference in gradients of the saddle envelope by
\[
\frac{1}{\eta^2} \left\| \nabla L_\eta(x, y) - \nabla L_\eta(\bar{x}, \bar{y}) \right\|^2 = \left\| \begin{bmatrix} x - x_+ \\ y_+ - y \end{bmatrix} - \begin{bmatrix} \bar{x} - \bar{x}_+ \\ \bar{y}_+ - \bar{y} \end{bmatrix} \right\|^2 \\
= \left\| \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix} \right\|^2 + \frac{1}{2} \left\| \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix}^T \begin{bmatrix} \bar{x}_+ - x_+ \\ \bar{y}_+ - y_+ \end{bmatrix} \right\|^2 \\
\leq \left\| \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix} \right\|^2 + \left( \frac{\eta}{\eta - \rho} - \frac{1}{2} \right) \left\| \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix}^T \begin{bmatrix} \bar{x}_+ - x_+ \\ \bar{y}_+ - y_+ \end{bmatrix} \right\|^2.
\]

Noting \(\left\| \begin{bmatrix} \bar{x}_+ - x_+ \\ \bar{y}_+ - y_+ \end{bmatrix} \right\| \leq \frac{\eta}{\eta - \rho} \left\| \begin{bmatrix} \bar{x} - x \\ \bar{y} - y \end{bmatrix} \right\|\) gives the following Lipschitz gradient inequality
\[
\left\| \nabla L_\eta(x, y) - \nabla L_\eta(\bar{x}, \bar{y}) \right\|^2 \leq \eta^2 \left( 1 + \left( \frac{\eta}{\eta - \rho} - \frac{1}{2} \right) \frac{\eta}{\eta - \rho} \right) \left\| \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix} \right\|^2.
\]

From our choice of \(\eta = 2\rho\), we arrive at our the claimed gradient Lipschitz constant of \(2\eta\) as
\[
\eta \sqrt{1 + \left( \frac{\eta}{\eta - \rho} - \frac{1}{2} \right) \frac{\eta}{\eta - \rho}} = \eta \sqrt{1 + \left( \frac{\eta}{\eta/2} - \frac{1}{2} \right) \frac{\eta}{\eta/2}} = 2\eta.
\]

2. Notice that for fixed \(y\), the saddle envelope \(L_\eta\) is strongly convex in \(x\) if the Moreau envelope \(e_\eta \{ h_y \} (x)\) from Lemma [C.1] is strongly convex. Recall Lemma [B.3] showed that a Moreau envelope \(e_\eta \{ h \}\) is \((\eta^{-1} + \alpha^{-1})^{-1}\)-strongly convex whenever the given function \(h\) is \(\alpha\)-strongly convex. Then it suffices to show the inner function
\[
h_y(u) = f(u) + \left( g(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^* (A^T u + \eta y)
\]
is \(\alpha\)-strongly convex in \(u\). Letting \((x_+, y_+) = \text{prox}_\eta(x, y)\), we find this does hold as
\[
\nabla^2 h_y(u) = \nabla^2 f(u) + A \left( \nabla^2 g(y_+) + \eta I \right)^{-1} A^T \\
\geq \nabla^2 f(u) + \frac{AA^T}{\beta + \eta} \\
= \nabla^2 f(u) + \frac{AA^T}{\beta + 2\rho} \geq \alpha I
\]
where the first equality uses the Hessian formula for convex conjugates from Lemma [B.1] the first inequality uses the smoothness of \(g\) and the second inequality uses interaction dominance. A symmetric argument using the other formula from Lemma [C.1] shows strong concavity in \(y\).
\[\square\]
D.2 Proof of Theorem 3.3

First, it follows from Proposition 2.3 that PPM (4) on L with \( \eta = 2\rho \) and \( \lambda = 1/2\kappa = (\eta^{-1} + \alpha^{-1})^{-1}/4\eta \) is equivalent to GDA (3) with \( s = \eta/\lambda = (\eta^{-1} + \alpha^{-1})^{-1}/(2\eta)^2 \) on \( L_\eta \). Moreover, Proposition 3.2 shows that \( L_\eta \) is \((\eta^{-1} + \alpha^{-1})^{-1}\)-strongly convex-strongly concave and \( 2\eta \)-smooth by noticing \( L \) is \( \alpha \)-interaction dominant in both \( x \) and \( y \), thus \( L_\eta \) has a unique stationary point \((x^*, y^*)\). Then applying Theorem 3.6 we conclude that running GDA (3) with step-size \( s \) produces iterates \((x_k, y_k)\) converging to the saddle envelope’s unique stationary point \((x^*, y^*)\) with

\[
\Bigg\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \Bigg\|^2 \leq \left( 1 - 2(\eta^{-1} + \alpha^{-1})^{-1}s + (2\eta)^2s^2 \right)^k \Bigg\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \Bigg\|^2
\]

\[
= \left( 1 - \frac{1}{4\eta^2(\eta^{-1} + \alpha^{-1})^2} \right)^k \Bigg\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \Bigg\|^2
\]

\[
= \left( 1 - \frac{1}{4(1 + 2\rho/\alpha)^2} \right)^k \Bigg\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \Bigg\|^2,
\]

where the first equality uses our choice of \( s \) and the second uses our choice of \( \eta = 2\rho \). Finally, it follows from Proposition 2.2 that \((x^*, y^*)\) is also the unique stationary solution to \( L \), which finishes the proof. \( \square \)

D.3 Example showing interaction dominate boundary is tight.

Consider the following nonconvex-nonconcave quadratic minimax problem of

\[
\min_{x \in \mathbb{R}^2} \max_{y \in \mathbb{R}^2} \frac{-\rho}{2} \|x\|^2 + ax^Ty - \frac{-\rho}{2} \|y\|^2
\]

where \( a \in \mathbb{R} \) controls the size of the interaction term. Note this problem is \( \beta = 0 \)-smooth (since the Hessians of \( f \) and \( g \) are always negative definite) and \( \rho \)-weakly convex. The proximal point method with \( \eta = 2\rho \) then corresponds to the following matrix multiplication

\[
\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} (1 - \rho/\eta)I & aI/\eta \\ -aI/\eta & (1 - \rho/\eta)I \end{bmatrix}^{-1} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \left( \frac{1}{2} \begin{bmatrix} I & aI/\rho \\ -aI/\rho & I \end{bmatrix} \right)^{-1} \begin{bmatrix} x_k \\ y_k \end{bmatrix}
\]

\[
= \frac{2}{(a/\rho)^2 + 1} \begin{bmatrix} I & -aI/\rho \\ aI/\rho & I \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}.
\]

This iteration will globally converge to the origin exactly when the matrix \( \frac{2}{(a/\rho)^2 + 1} \begin{bmatrix} I & -aI/\rho \\ aI/\rho & I \end{bmatrix} \) has spectral norm less than one, which happens happens exactly when \( a/\rho > 1 \). Our interaction dominance condition agrees with this bound weakened by factor of two, guaranteeing linear convergence whenever \( a/\rho > 2 \) as

\[
\nabla^2 f(x) + \frac{AA^T}{2(\rho + \beta)} = \left( -\rho + \frac{a^2}{2\rho} \right) I \succeq 0.
\]

\( \square \)

D.4 Proof of Theorem 3.7

Proposition 3.2 show that whenever interaction dominance holds for \( y \) the saddle envelope problem (7) is a nonconvex-strongly concave smooth optimization problem. Recently, Lin et al. [19] considered
such nonconvex-strongly concave problems with a compact constraint \( y \in D \). They analyzed the following variant of GDA

\[
\begin{bmatrix}
x_{k+1} \\
y_{k+1}
\end{bmatrix} = \text{proj}_{\mathbb{R}^n \times D} \left( \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} -\nabla_x L(x_k, y_k)/\eta_x \\ -\nabla_y L(x_k, y_k)/\eta_y \end{bmatrix} \right)
\]

(17)

which projects onto the feasible region \( \mathbb{R}^n \times D \) each iteration and has different stepsize parameters \( \eta_x \) and \( \eta_y \) for \( x \) and \( y \). Lin et al. prove the following theorem showing a sublinear guarantee.

**Theorem D.1** (Theorem 4.4 of [19]). For any \( \beta \)-smooth, nonconvex-\( \mu \)-strongly concave problem, let \( \kappa = \beta/\mu \) be the condition number for \( y \). Then for any \( \varepsilon > 0 \), GDA with stepsizes \( \eta_x = \Theta(1/\kappa^2 \beta) \) and \( \eta_y = \Theta(1/\beta) \) will find a point satisfying \( \|\nabla L(x_T, y_T)\| \leq \varepsilon \) by iteration

\[
T \leq O \left( \frac{\kappa^2 \beta + \kappa \beta^2}{\varepsilon^2} \right).
\]

Assuming that the sequence \( y_k \) above stays bounded, this projected gradient method is equivalent to running GDA on our unconstrained problem by setting the domain of \( y \) as a sufficiently large compact set to contain all the iterates. Consider setting the averaging parameters as \( \lambda = \Theta(1/\kappa^2) \) and \( \gamma = \Theta(1) \). Then using the gradient formula from Lemma C.2 we see that the damped proximal point method is equivalent to running GDA on the saddle envelope with stepsizes \( \eta_x = \lambda/\eta \) and \( \eta_y = \gamma/\eta \):

\[
\begin{bmatrix}
x_{k+1} \\
y_{k+1}
\end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} -\nabla_x L_\eta(x_k, y_k)/\eta_x \\ -\nabla_y L_\eta(x_k, y_k)/\eta_y \end{bmatrix}
\]

\[
= \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} -\lambda(x_k - x_+) \\ -\gamma(y_+ - y_k) \end{bmatrix}
\]

\[
= \begin{bmatrix} \lambda x_+ + (1 - \lambda)x_k \\ \gamma y_+ + (1 - \gamma)y_k \end{bmatrix}.
\]

Then the above theorem guarantees that running our damped proximal point method on \( L \) (or equivalently, applying GDA to the saddle envelope) will converge to a stationary point with \( \|\nabla L_\eta(x_T, y_T)\| \leq \varepsilon \) within \( T \leq O(\varepsilon^{-2}) \) iterations. Then its immediate from the gradient formula that \( (x_+^T, y_+^T) = \text{prox}_\eta(x_T, y_T) \) is also approximately stationary as \( \|\nabla L(x_+^T, y_+^T)\| = \|\nabla L_\eta(x_T, y_T)\| \leq \varepsilon \).

**E Proofs for Results on the Interaction Weak Convergence**

**E.1 Random Initialization of 2P-PPM Almost Surely Gives Interaction Dominance**

Let \( x_0(y') \) be the local minimizer of \( f(x) + x^T Ay' \) computed when given the initialization \( y' \) and \( y_0(x') \) be the local maximizer of \( x^T Ay - g(y) \) computed when given the initialization \( x' \). Consider drawing \( x' \) and \( y' \) randomly from normal distributions supported on \( \mathbb{R}^n \) and \( \mathbb{R}^m \). We ensure our underlying problem and our minimization subroutine are stable under perturbations by assuming the following regularity conditions hold:

With probability one, \( x_0(\cdot) \) is continuous at \( y' \) and \( y_0(\cdot) \) is continuous at \( x' \).
Consider one sample for \( y' \) where this continuity condition holds. Let \( B \) denote a convex neighborhood around \( x_0(y') \) where it is a minimizer. Then we consider the following restricted minimization problem

\[
\min_x h(x) = f(x) + \delta_B(x) + x^T Ay'
\]

where \( \delta_B(x) \) is the indicator function for our neighborhood around \( x_0(y') \). Note \( x_0(y') \) is the global minimizer of this problem. Notice then that this can be written as a Fenchel conjugate at \( y = 0 \) given by

\[
h^*(y) = \sup_x x^T (-Ay' + y) - f(x) - \delta_B(x).
\]

This Fenchel conjugate is convex (as all Fenchel’s conjugates are) and finite valued everywhere since \( h \) has compact support. Then Alexandrov’s Theorem ensures that \( h^* \) almost surely has a quadratic upper bound \( u(y) \) locally around 0.

Consequently using the continuity of \( x_0(\cdot) \) and taking the dual of this upper bound, we arrive at a strictly convex quadratic lower bound \( l(x) \) on \( (f + \delta_B)^\ast \) locally around \( x_0(y') \). Then noting that the biconjugate of any function is a lower bound on any function itself, we conclude \( f \) has the same strictly convex quadratic lower bound \( l(x) \) locally as

\[
f(x) + x^T Ay' = h(x) \geq h^{\ast\ast}(x) \geq l(x)
\]

for any \( x \) near \( x_0(y') \). However at \( x_0(y') \), this holds with equality as

\[
f(x_0(y')) + x_0(y')^T Ay' = h(x_0(y')) = h^{\ast\ast}(x_0(y')) = l(x_0(y'))
\]

since \( x_0(y') \) is the global minimizer of \( h \). Hence we must have \( \nabla^2 f(x_0(y')) > 0 \) as desired.

Symmetric reasoning ensures that \( \nabla^2 g(y_0(x')) > 0 \).

### E.2 Proof of Theorem 4.4

Consider two sets centered at \((x_0, y_0)\): An inner regime \( B_{\text{inner}} = \{(x, y) | x \in B(x_0, r), y \in B(y_0, r)\} \) with radius \( r \) given by

\[
r := \left( \frac{2}{\rho} + \frac{8}{\alpha_0} \right) \| \nabla L(x_0, y_0) \|
\]

and an outer ball \( B_{\text{outer}} = B((x_0, y_0), R) \) with radius \( R \) given by

\[
R = \sqrt{2} \left( \frac{3}{2} + \frac{\beta + \| A \|}{\rho} \right) \left( \frac{2}{\rho} + \frac{8}{\alpha_0} \right) \| \nabla L(x_0, y_0) \| > \sqrt{2}r.
\]

Thus \( B_{\text{inner}} \subset B_{\text{outer}} \), and moreover, the local assumption \(|\alpha| \) combined with the \( H \)-Lipschitz continuity of \( \nabla^2 f \) and \( \nabla^2 g \) ensure \( \alpha_0/2 \)-interaction dominance holds on the outer ball, because it holds that \( HR \leq \frac{\alpha_0}{2} \).

First we observe that taking a proximal step from the inner square will always stay within the outer ball.

**Lemma E.1.** For any \((x, y) \in B_{\text{inner}}\), we have \((x_+, y_+) = \text{prox}_\eta(x, y) \in B_{\text{outer}}\).

**Proof.** Define the function underlying the computation of the proximal step at \((x, y)\) as

\[
M(u, v) = L(u, v) + \frac{\eta}{2} \| u - x \|^2 - \frac{\eta}{2} \| v - y \|^2.
\]
Our choice of $\eta > \rho$ ensures that $M$ is $(\eta - \rho)$-strongly convex-strongly concave. Thus Lemma B.4 implies

$$\left\| \frac{x - x_+}{y - y_+} \right\| \leq \frac{\| \nabla M(x, y) \|}{\eta - \rho} = \frac{\| \nabla L(x, y) \|}{\rho}.$$ 

Since the gradient of the original objective $L$ is $\beta + \| A \|$-Lipschitz, any $(x, y) \in B_{inner}$, $\| \nabla L(x, y) \|$ is at most $\| \nabla L(x_0, y_0) \| + (\beta + \| A \|) \sqrt{2r}$. Hence

$$\left\| \frac{x - x_+}{y - y_+} \right\| \leq \frac{\| \nabla L(x_0, y_0) \| + (\beta + \| A \|) \sqrt{2r}}{\rho}.$$ 

Thus $(x_+, y_+)$ must lie in the outer ball as

$$\left\| \frac{x_0 - x_+}{y_0 - y_+} \right\| \leq \left\| \frac{x_0 - x}{y_0 - y} \right\| + \left\| \frac{x - x_+}{y - y_+} \right\| \leq \sqrt{2r} + \frac{\| \nabla L(x_0, y_0) \| + (\beta + \| A \|) \sqrt{2r}}{\rho} \leq \left( \frac{3}{2} + \frac{\beta + \| A \|}{\rho} \right) \sqrt{2r} = R,$$

which the last inequality uses $r \geq \frac{2}{\rho} \| \nabla L(x_0, y_0) \|$. \hfill \Box

From this, we find that interaction dominance on the outer ball suffices to ensure the saddle envelope is strongly convex-strongly concave on the inner square.

**Lemma E.2.** The saddle envelope is $(\eta^{-1} + (\alpha_0/2)^{-1})^{-1}$-strongly convex-strongly concave on $B_{inner}$.

**Proof.** Consider any $(x, y) \in B_{inner}$ and let $(x_+, y_+) = \text{prox}_\eta(x, y) \in B_{outer}$ (where the inclusion follows from Lemma E.1). Recall Lemma C.1 ensures $L_\eta(x, y) = e_\eta \{ h_y \}(x) - \frac{\eta}{2} \| y \|^2$ where $h_y(u) = f(u) + (g(\cdot) + \frac{\eta}{2} \| \cdot \|^2)(A^T u + \eta y)$. Hence

$$\nabla^2_{xx} L_\eta(x, y) = \nabla^2 e_\eta \{ h_y \}(x) = \eta I + \eta^2 (\nabla^2 h_y(x_+) + \eta I)^{-1}.$$ 

Then mirroring the proof of Lemma B.3, we have $(\eta^{-1} + (\alpha_0/2)^{-1})^{-1}$-strong convexity holding at $(x, y)$ (that is, $\nabla^2 L_\eta(x, y) \succeq (\eta^{-1} + (\alpha_0/2)^{-1})^{-1} I$) if $\nabla^2 h_y(x_+) \succeq \alpha_0/2 I$. Checking the condition $\nabla^2 h_y(x_+) \succeq \alpha_0/2 I$ mirrors the proof of part 2 of Proposition 3.2 as

$$\nabla^2 h_y(x_+) = \nabla^2 f(x_+) + A \left( \nabla^2 g(y_+) + \eta I \right)^{-1} A^T \succeq \nabla^2 f(x_+) + \frac{AA^T}{\beta + \eta} \succeq \frac{\alpha_0}{2} I$$

where the first inequality uses the smoothness of $g$ and the second uses the $\alpha_0/2$-interaction dominance at $(x_+, y_+)$. Symmetric reasoning shows $(\eta^{-1} + (\alpha_0/2)^{-1})^{-1}$-strong concavity also holds at $(x, y)$. \hfill \Box

Now let us go back to the proof of Theorem 4.4. Observe that the gradient of the saddle envelope at $(x_0, y_0)$ is bounded by Lemma C.2 and Lemma B.4 as

$$\| \nabla L_\eta(x_0, y_0) \| \leq \frac{\eta}{\eta - \rho} \| \nabla L(x_0, y_0) \| = 2 \| \nabla L(x_0, y_0) \|.$$

25
Then using Lemma B.5 the local $(\eta^{-1} + (\alpha_0/2)^{-1})^{-1}$-strong convexity-strong concavity of $L_\eta$ ensures that there is a stationary point $(x^*, y^*)$ within $B_{inner}$. Moreover its distance from $(x_0, y_0)$ is bounded by
\[
\left\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \right\| \leq \frac{2}{(\eta^{-1} + (\alpha_0/2)^{-1})^{-1}} \| \nabla L(x_0, y_0) \| = \left( \frac{1}{\rho} + \frac{4}{\alpha_0} \right) \| \nabla L(x_0, y_0) \| = r/2.
\]

Now that we have shown all of the conditions necessary to apply Theorem B.6. The saddle envelope is $(\eta^{-1} + (\alpha_0/2)^{-1})$-strongly convex-strongly concave and $2\eta$-smooth on the square $B(x^*, r/2) \times B(y^*, r/2)$ and the initial iterate $(x_0, y_0)$ is contained in the ball $B((x^*, y^*), r/2)$. Hence applying GDA with $s = (\eta^{-1} + (\alpha_0/2)^{-1})^{-1}/(2\eta)^2$ produces iterates $(x_k, y_k)$ converging to $(x^*, y^*)$ with
\[
\left\| \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \right\|^2 \leq \left( 1 - 2(\eta^{-1} + (\alpha_0/2)^{-1})^{-1}s + (2\eta)^2 s^2 \right)^k \left\| \begin{bmatrix} x_0 - x^* \\ y_0 - y^* \end{bmatrix} \right\|^2.
\]

By Proposition 2.2 $(x^*, y^*)$ must also be a stationary point of $L$. Further, by Proposition 2.3 this sequence $(x_k, y_k)$ converging to $(x^*, y^*)$ is the same as the sequence generated by running damped PPM on (2). \(\square\)

# F Analysis for Interaction Moderate Region

## F.1 Proof of Theorem 5.1

1. Recall Fenchel’s inequality ensures for any $w^*, z$, $h(w^*) + h^*(z) \geq z^T w^*$ and equality holds exactly when $w^* = \text{argmax}_w z^T w - h(w)$. Then it follows that our Lyapunov $L(x, y)$ must always be nonnegative since
\[
L(x, y) + \eta\|x\|^2 + \eta\|y\|^2 \geq x^T (-Ay + \eta x) + y^T (A^T x + \eta y) = \eta\|x\|^2 + \eta\|y\|^2.
\]

2. Equality holds in the previous argument exactly when $x$ attains the maximum defining $(f + \frac{\eta}{2} \cdot \| \cdot \|^2)^* (-Ay + \eta x)$ and $y$ attains the maximum defining $(g + \frac{\eta}{2} \cdot \| \cdot \|^2)^* (A^T x + \eta y)$. This is equivalent to requiring the following first-order optimality conditions to hold
\[
-Ay + \eta x - \nabla \left( f + \frac{\eta}{2} \cdot \| \cdot \|^2 \right)(x) = 0
\]
\[
A^T x + \eta y - \nabla \left( g + \frac{\eta}{2} \cdot \| \cdot \|^2 \right)(y) = 0,
\]

which simplifies to be exactly the condition that $(x, y)$ is a stationary point
\[
Ay + \nabla f(x) = 0
\]
\[
A^T x - \nabla g(y) = 0.
\]
3. When $\eta = 0$, this Lyapunov simplifies to be the primal-dual gap for $L(x, y)$ as

$$L(x, y) = f(x) + \max_{x^*} \{-x^T A y - f(x^*)\} + g(y) + \max_{y^*} \{x^T A y^* - g(y^*)\}$$

$$= \max_{x^*} \{g(y) - x^T A y - f(x^*)\} + \max_{y^*} \{f(x) + x^T A y^* - g(y^*)\}$$

$$= \max_{x^*} L(x, y^*) - \min_{x^*} L(x^*, y).$$

F.2 Proof of Theorem 5.2

To simplify our notation, we begin with a change of variables to translate the center $(\bar{x}, \bar{y})$ to the origin. Consider the problem

$$\min_{\bar{x}} \max_{\bar{y}} \bar{L}(\bar{x}, \bar{y}) = \bar{f}(\bar{x}) + \bar{x}^T A \bar{y} - \bar{g}(\bar{y}).$$

where

$$\bar{f}(\bar{x}) = f(\bar{x} + \bar{x}) + \bar{x}^T A \bar{y}$$

$$\bar{g}(\bar{y}) = f(\bar{y} + \bar{y}) - \bar{x}^T A \bar{y}.$$

This is a translation of our original objective as $\bar{L}(\bar{x}, \bar{y}) = L(\bar{x} - \bar{x}, \bar{y} - \bar{y})$ and so the proximal point method applied to $\bar{L}(\bar{x}, \bar{y})$ will generate the translated points $(\bar{x}_k, \bar{y}_k) = (x_k, y_k) - (\bar{x}, \bar{y})$.

Then the Lyapunov [11] for this translated problem is given by

$$\bar{L}(\bar{x}, \bar{y}) = \frac{1}{2} \left\| \bar{x} \right\|^2 + \left( -A \bar{y} + \eta \bar{x} \right)$$

$$\bar{g}(\bar{y}) = \frac{1}{2} \left\| \bar{y} \right\|^2 + \left( \bar{g}(\bar{y}) + \eta \bar{y} \right).$$

The value of $\bar{L}(\bar{x}_k, \bar{y}_k)$ as the proximal point method runs is bounded by the following inductive lemma:

Lemma F.1. Consider proximal point method [4] with $\eta = 3\rho$ and $\lambda = 1$. Then it holds for any iteration $k$ that

$$\bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) - \bar{L}(\bar{x}_k, \bar{y}_k) \leq -\frac{1}{12\eta} \left\| \bar{x}_{k+1} - \bar{y}_{k+1} \right\|^2 - \frac{\eta}{2} \left\| \bar{x}_k - \bar{y}_k \right\|^2.$$

Proof. At each iteration, $\bar{x}_{k+1}$ is computed as the minimizer of the following convex function in $u$

$$\max_u \bar{f}(u) + u^T A v - \bar{g}(v) + \frac{\eta}{2} \left\| u - \bar{x}_k \right\|^2 - \frac{\eta}{2} \left\| v - \bar{y}_k \right\|^2$$

$$= \bar{f}(u) + \frac{\eta}{2} \left\| u - \bar{x}_k \right\|^2 + \max_u u^T A v - \bar{g}(v) - \frac{\eta}{2} \left\| v - \bar{y}_k \right\|^2$$

$$= \bar{f}(u) + \frac{\eta}{2} \left\| u - \bar{x}_k \right\|^2 + \max (A^T u + \eta \bar{y}_k)^T v - \bar{g}(v) - \frac{\eta}{2} \left\| v \right\|^2 - \frac{\eta}{2} \left\| \bar{y}_k \right\|^2$$

$$= \bar{f}(u) + \frac{\eta}{2} \left\| u - \bar{x}_k \right\|^2 + \left( \bar{g}(\bar{\cdot}) + \frac{\eta}{2} \left\| \cdot \right\|^2 \right)^* (A^T u + \eta \bar{y}_k) - \frac{\eta}{2} \left\| \bar{y}_k \right\|^2.$$

In fact, this function is $\eta - \rho$-strongly convex in $u$. Therefore we have that

$$\bar{f}(\bar{x}_{k+1}) + \left( \bar{g}(\bar{\cdot}) + \frac{\eta}{2} \left\| \cdot \right\|^2 \right)^* (A^T \bar{x}_{k+1} + \eta \bar{y}_k) + \frac{\eta}{2} \left\| \bar{x}_{k+1} - \bar{x}_k \right\|^2$$

$$\leq \bar{f}(\bar{x}_k) + \left( \bar{g}(\bar{\cdot}) + \frac{\eta}{2} \left\| \cdot \right\|^2 \right)^* (A^T \bar{x}_k + \eta \bar{y}_k) - \frac{\eta - \rho}{2} \left\| \bar{x}_{k+1} - \bar{x}_k \right\|^2.$$
Rearranging terms gives the following
\[
\begin{align*}
\hat{f}(\bar{x}_{k+1}) + \left( \bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^* (A^T \bar{x}_{k+1} + \eta \bar{y}_k) &\leq \left( \bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^* (A^T \bar{x}_k + \eta \bar{y}_k) \\
&\leq -\frac{2\eta - \rho}{2} \| \bar{x}_{k+1} - \bar{x}_k \|^2.
\end{align*}
\]

Since \( \bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \) is \( \eta - \rho \)-strongly convex, \( (\bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2)^* \) is \( 1/(\eta - \rho) \)-smooth. Then since \( (\bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2)^* \) at \( A^T \bar{x}_{k+1} + \eta \bar{y}_k \) has gradient \( \bar{y}_{k+1} \), it follows that
\[
\begin{align*}
\left( \bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^* (A^T \bar{x}_{k+1} + \eta \bar{y}_k) &+ \frac{\eta^2}{2(\eta - \rho)} \| \bar{y}_{k+1} - \bar{y}_k \|^2 \\
\geq \left( \bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^* (A^T \bar{x}_{k+1} + \eta \bar{y}_{k+1}).
\end{align*}
\]

Combining this with our previous result shows
\[
\begin{align*}
\hat{f}(\bar{x}_{k+1}) + \left( \bar{g}(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^* (A^T \bar{x}_{k+1} + \eta \bar{y}_{k+1}) &\leq -\frac{2\eta - \rho}{2} \| \bar{x}_{k+1} - \bar{x}_k \|^2 + \frac{\eta^2}{2(\eta - \rho)} \| \bar{y}_{k+1} - \bar{y}_k \|^2 + \eta \bar{y}_{k+1}^T (\bar{y}_{k+1} - \bar{y}_k).
\end{align*}
\]

Symmetric reasoning on \( y \) yields
\[
\begin{align*}
\bar{g}(\bar{y}_{k+1}) + \left( \hat{f}(\cdot) + \frac{\eta}{2} \| \cdot \|^2 \right)^* (-A \bar{y}_{k+1} + \eta \bar{x}_{k+1}) &\leq -\frac{2\eta - \rho}{2} \| \bar{y}_{k+1} - \bar{y}_k \|^2 + \frac{\eta^2}{2(\eta - \rho)} \| \bar{x}_{k+1} - \bar{x}_k \|^2 + \eta \bar{x}_{k+1}^T (\bar{x}_{k+1} - \bar{x}_k).
\end{align*}
\]

Summing these two bounds gives
\[
\begin{align*}
\bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) - \bar{L}(\bar{x}_k, \bar{y}_k) &\leq \left( \frac{2\eta - \rho}{2} - \frac{\eta^2}{2(\eta - \rho)} \right) \| \bar{x}_{k+1} - \bar{x}_k \|^2 + \left[ \frac{\eta}{2} \| \bar{x}_{k+1} \|^2 - \frac{\eta}{2} \| \bar{y}_{k+1} \|^2 \right] - \eta \left( \| \bar{x}_{k+1} \|^2 - \| \bar{x}_k \|^2 \right) - \eta \left( \| \bar{y}_{k+1} \|^2 - \| \bar{y}_k \|^2 \right).
\end{align*}
\]

From lemma \ref{lem:no-regret}, this can be restated as
\[
\begin{align*}
\bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) - \bar{L}(\bar{x}_k, \bar{y}_k) &\leq \left[ \bar{x}_{k+1} \right]^T \left( \frac{\eta}{2} \| \bar{x}_{k+1} \|^2 - \frac{\eta}{2} \| \bar{y}_{k+1} \|^2 \right) - \eta \left( \| \bar{x}_{k+1} \|^2 - \| \bar{x}_k \|^2 \right) - \eta \left( \| \bar{y}_{k+1} \|^2 - \| \bar{y}_k \|^2 \right).
\end{align*}
\]

Plugging in our choice of \( \eta = 3\rho \) gives the claimed inductive lemma.

Now let us go back to the proof of Theorem \ref{thm:main}. Inductively applying Lemma \ref{lem:inductive} and using
that $\bar{L}(x, y) \geq 0$ from Theorem \ref{thm:bar-L-bound} implies
\[
\bar{L}(\bar{x}_0, \bar{y}_0) + \frac{\eta}{2} (\|\bar{x}_0\|^2 + \|\bar{y}_0\|^2) \geq \bar{L}(\bar{x}_0, \bar{y}_0) - \bar{L}(\bar{x}_T, \bar{y}_T) + \frac{\eta}{2} (\|\bar{x}_0\|^2 + \|\bar{y}_0\|^2 - \|\bar{x}_T\|^2 - \|\bar{y}_T\|^2)
\]
\[
\geq \sum_{k=0}^{T} \left( \frac{1}{12\eta} \left\| \nabla \bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) \right\|^2 - \left\| \frac{\bar{x}_{k+1}}{\bar{y}_{k+1}} \right\|^T \left[ \begin{array}{c} -\nabla_x \bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) \\ \nabla_y \bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) \end{array} \right] \right)
\]
\[
\geq \frac{1}{12\eta} \left( \sum_{k=0}^{T} \left\| \nabla \bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) \right\|^2 \right) - D\sqrt{T} \sqrt{\sum_{k=0}^{T} \left\| \nabla \bar{L}(\bar{x}_{k+1}, \bar{y}_{k+1}) \right\|^2}.
\]

Multiplying through by $12\eta/T$ and changing variables back to $(x, y)$ from $(\bar{x}, \bar{y})$ gives the claimed gradient bound as
\[
\frac{1}{T} \sum_{k=0}^{T} \left\| \nabla {L}(x_{k+1}, y_{k+1}) \right\|^2
\leq \frac{12\eta L(x_0 - \bar{x}, y_0 - \bar{y}) + 6\eta^2 (\|x_0 - \bar{x}\|^2 + \|y_0 - \bar{y}\|^2)}{T} + 12\eta D \sqrt{\frac{1}{T} \sum_{k=0}^{T} \left\| \nabla L(x_{k+1}, y_{k+1}) \right\|^2}
\leq 2 \max \left\{ \frac{12\eta L(x_0 - \bar{x}, y_0 - \bar{y}) + 6\eta^2 (\|x_0 - \bar{x}\|^2 + \|y_0 - \bar{y}\|^2)}{T}, 12\eta D \sqrt{\frac{1}{T} \sum_{k=0}^{T} \left\| \nabla L(x_{k+1}, y_{k+1}) \right\|^2} \right\}
\leq 2 \max \left\{ \frac{12\eta L(x_0 - \bar{x}, y_0 - \bar{y}) + 6\eta^2 (\|x_0 - \bar{x}\|^2 + \|y_0 - \bar{y}\|^2)}{T}, (12\eta D)^2 \right\}.
\]

\[\square\]

**F.3 Proof of Theorem 5.4**

We introduce necessary terminology for the proof. Let $C$ be a rectifiable Jordan curve in $\mathbb{R}^n$, and let $S$ be the solution to Plateau’s problem with respect to $C$, i.e., $S$ is a simply connected minimal surface with boundary $C$ (for details, see for example \cite{30}).

While our proof applies to the discrete time setting, cycling behavior is best defined with respect to a continuous time setting. Therefore, we need to take the limit of the dynamical system corresponding to our PPM algorithm (Equation \ref{eq:PPM}), as step size goes to zero. The limiting dynamical system is a system of ODEs with the property that the solution (assuming the same initial condition as the discrete time system) will be a rectifiable closed curve in $\mathbb{R}^n$. Moreover, for any given positive $\varepsilon$, if we choose a small enough step-size $s(\varepsilon)$ then the path, denoted $C'$, taken by the discrete dynamical will lie within a tube of radius $\varepsilon$ around the path $C$.

Note that \cite{10} already studies such limits of dynamical systems. There it is shown, that in general, for any loss function, and any positive $\varepsilon$, there exists a step-size $s(\varepsilon, T)$, such that PPM will remain with a tube of radius $\varepsilon$ around $C$ for at least $T$ iterations.

We note that for our purposes, it’s better if the choice of step-size $s$ does not depend upon $T$ – uniform convergence. The lower bound below is non-trivial if $s$ does not depend upon $T$, therefore we will assume that from here on. The above discussion explains what we mean by uniform convergence to a cyclic attractor.

Finally, suppose the PPM in Equation \ref{eq:PPM} runs for $T$ iterations and traverses the curve $C'$. Observe that there is a natural map $\pi : C \mapsto C'$ which maps points in $C'$ to the corresponding closest point in
C. Therefore, for a small enough choice of \( s(\varepsilon) \), one can think of \( C' \) as a curve which stays within \( \varepsilon \) of \( C \), and the former winds around the latter. Assume that \( C' \) winds around \( C \) for \( \kappa \) times, where \( \kappa \) can be a fraction. Define \( N \) to be the average number of iterations in a single traversal of \( C \). In other words, \( N \) is essentially measures the “coarseness” of our PPM algorithm.

**Theorem F.2.** For any problem \( L(x, y) \), if the limiting behaviour of PPM \(^4\), converges uniformly to a cyclic attractor \( C \), such that there exists a point on the minimal surface bounded by \( C \) that is at (geodesic) distance at least \( R \) from every point on \( C \) then, small enough choice of step size \( s \) and a large enough choice of \( T \) and \( S \) (where \( T \gg S \)), we have

\[
\frac{1}{T-S} \sum_{k=S}^{T} \| \nabla L(x_{k+1}, y_{k+1}) \|_2^2 \geq \frac{C \cdot R^2}{s^2 N^2}
\]

(18)

where \( C \) is a constant depending on the properties of the minimal surface \( S \).

**Proof.** Let \( \ell \) and \( \ell' \) be the arc lengths of \( C \) and \( C' \), respectively. Then \( \ell' \simeq \kappa \ell \), where \( \simeq \) denotes asymptotic equivalence as \( \varepsilon \to 0 \). It suffices to choose \( \varepsilon \ll R \). Additionally, if \( T - S \) is large enough, for a given choice of \( \varepsilon \), then \( \frac{\kappa}{\kappa} \to 1 \), and we have \( \ell' \simeq [\kappa] \ell \) for a small enough \( \varepsilon \) and large enough \( T - S \).

We know from the properties of PPM that

\[
\ell' = s \sum_{k=S}^{T} \| \nabla L(x_{k+1}, y_{k+1}) \|_2.
\]

(19)

Moreover, by Cauchy-Schwarz inequality,

\[
\sum_{k=S}^{T} \| \nabla L(x_{k+1}, y_{k+1}) \|_2^2 \geq \frac{1}{T-S} \left( \sum_{k=S}^{T} \| \nabla L(x_{k+1}, y_{k+1}) \|_2 \right)^2.
\]

(20)

Therefore, we have for a small enough \( \varepsilon \),

\[
\frac{1}{T-S} \sum_{k=S}^{T} \| \nabla L(x_{k+1}, y_{k+1}) \|_2^2 \geq \frac{1}{(T-S)^2 s^2} \cdot ([\kappa] \ell)^2 \to \ell^2 \frac{([\kappa] \ell)^2}{s^2 N^2}.
\]

(21)

where the last simplification above uses that \( T - S \) is large enough so that \( \frac{T-S}{\kappa} \to \frac{T-S}{\kappa} = N \).

However, the isoperimetric inequality for minimal surfaces, see for example Theorem 4.2 in [30], allows us to lower bound \( \ell^2 \) in terms of the area of the minimal surface that is bounded by \( C \). Furthermore, by our assumption, there exists points on the minimal surface that is at distance \( R \) from each point on \( C \), we can lower bound the area of \( S \) enclosed by \( C \) by \( C \cdot R^2 \), for some constant \( C \) depending on the curvature of \( S \). Hence the proof follows.

---

3Here it is helpful that we have uniform convergence. Since the choice of \( s \) does not depend on the number of iterations \( T - S \), otherwise we need to ensure the simultaneous existence of a small enough \( \varepsilon \) and large enough \( T - S \) such that RHS of Equation 18 is greater than zero, i.e., the result remains non-trivial.
Figure 2: Sample paths of 100 iterations of four common first-order methods for minimax optimization, Proximal Point Method (PPM), Extragradient Method (EG), Gradient Descent Ascent (GDA) and Alternating Gradient Descent Ascent (AGDA), for solving (12) with different levels of interaction term $A = 1, 10, 100, 1000$. 

(a) PPM with $A=1$  (b) PPM with $A=10$  (c) PPM with $A=100$  (d) PPM with $A=1000$

(e) EG with $A=1$  (f) EG with $A=10$  (g) EG with $A=100$  (h) EG with $A=1000$

(i) GDA with $A=1$  (j) GDA with $A=10$  (k) GDA with $A=100$  (l) GDA with $A=1000$

(m) AGDA with $A=1$  (n) AGDA with $A=10$  (o) AGDA with $A=100$  (p) AGDA with $A=1000$