Normal scaling in globally conserved interface-controlled coarsening of fractal clusters

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Globally conserved interface-controlled coarsening of fractal clusters exhibits dynamic scale invariance and normal scaling. This is demonstrated by a numerical solution of the Ginzburg-Landau equation with a global conservation law. The sharp-interface limit of this equation is volume preserving motion by mean curvature. The scaled form of the correlation function has a power-law tail accommodating the fractal initial condition. The coarsening length exhibits normal scaling with time. Finally, shrinking of the fractal clusters with time is observed. The difference between global and local conservation is discussed.

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Dynamics of growth of order from disorder in coarsening systems with long-range correlations is an intriguing problem which appears in phase ordering [1] and in many other applications. Systems with long-range correlations are often characterizable by fractal geometry [2,3]. Therefore, a lot of attention has been devoted recently to a variety of “fractal coarsening” problems [4–13]. A typical (though not the only) setting for fractal coarsening is the following. At an earlier stage of the dynamics a fractal cluster (FC) develops due to an instability of growth of the “minority phase” [4–13]. When the mass (or heat) source is depleted, coarsening by surface tension becomes dominant. The main question concerns possible scaling relations and universality classes in this type of coarsening.

A major simplifying assumption in the analysis of a coarsening process is dynamic scale invariance (DSI) (see review [1]). The DSI hypothesis was first applied to fractal coarsening by Toyoki and Honda [4] who considered systems with non-conserved order parameter. Implications of (mass) conservation in fractal coarsening were considered more recently [14,15]. Most remarkable of them is predicted shrinking of the FCs in the process of coarsening. However, there has been no convincing evidence (neither in experiment, nor in simulations) in favor of DSI in conserved fractal coarsening. Moreover, anomalous scaling and breakdown of DSI were observed in recent simulations of locally conserved edge-diffusion- and bulk-diffusion-controlled fractal coarsening [14,15]. This communication reports our finding that DSI and normal scaling hold in the process of interface-controlled fractal coarsening with a globally conserved order parameter. This system is apparently the first realistic conserved fractal coarsening system where this simplifying and beautiful concept is found to work.

Globally conserved interface-controlled coarsening is accessible in experiment. Consider the sublimation/deposition dynamics of a solid and its vapor in a small closed vessel kept at a (constant) low temperature [16–17]. As the acoustic time in the gas phase is short compared to the coarsening time, the gas pressure (and, consequently, density) remains uniform in space, changing only in time. This character of mass transport makes the coarsening dynamics conserved globally rather than locally, which leads to a different kinetics. Another example appears in the context of attachment/detachment-controlled nanoscale fluctuations at solid surfaces [18,19]. There is also a strong recent evidence in favor of interface-controlled transport during the cluster coarsening in driven rapid granular flows [20].

Here is an outline of the rest of the paper. We shall work with the Ginzburg-Landau equation with a global conservation law. The corresponding sharp interface theory is reducible, at large times, to volume preserving motion by mean curvature. Assuming DSI, one can then predict scaling behavior of the correlation function, shrinking of FCs with time and normal scaling of the coarsening length $l(t)$. Our extensive numerical simulations of the coarsening of two-dimensional (2d) diffusion-limited aggregates (DLAs) support all these predictions. We shall conclude by pointing out the main difference between global and local conservation.

We adopt a simple Landau free energy functional:

$$F[u] = \int \left[ (1/2)(\nabla u)^2 + V(u) + Hu \right] \, d^d r, \quad (1)$$

where $V(u) = (1/4)(1 - u^2)^2$ is a double-well potential, $u(r,t)$ is the order parameter and fluctuations are neglected. The effective “magnetic field” $H = H(t)$ changes in time so as to impose a global conservation law: $< u > = \text{const}$, where $< ... >=$ denotes a spatial average:

$$< ... >= L^{-d} \int \langle ... \rangle \, d^d r, \quad (2)$$

$L$ is the system size and the integration is over the whole system. The dynamics is described by a simple gradient descent:

$$\frac{\partial u}{\partial t} = -\frac{\delta F}{\delta u} = \nabla^2 u + u - u^3 - H(t). \quad (3)$$
It follows from Eq. (3) and the conservation law that \( H(t) = \langle u - \bar{u}^3 \rangle \) (either no-flux, or periodic boundary conditions are assumed), so Eq. (3) is a nonlocal reaction-diffusion equation [21, 24]. In the context of phase ordering it can be called the Ginzburg-Landau equation with a global conservation law. To make a theoretical progress, one should work in the sharp-interface limit [24] valid at late times, when the system already consists of large domains of “phase 1” and “phase 2” divided by a sharp interface. At this stage \( H(t) \) is both small, \( H(t) \ll 1 \), and slowly varying in time. The phase field in the phases 1 and 2 is uniform and rapidly adjusts to the current value of \( H(t) \), so \( u = -1 - H(t)/2 \) and \( 1 - H(t)/2 \), respectively. For brevity, we will consider 2d-case. The normal velocity of the interface is

\[
v_n(s, t) = \frac{3}{\sqrt{2}} H(t) - \kappa(s, t) ,
\]

where \( s \) is the coordinate along the interface and \( \kappa \) is the local curvature of the interface. The positive sign of \( v_n \) corresponds to the interface moving toward phase 2, while \( \kappa \) is positive when the interface is convex towards phase 2.

An equation for \( H(t) \) follows from the global conservation law which takes the form

\[
\frac{4A(t)}{L^2} - H(t) = \text{const} ,
\]

where \( A(t) = \int_{u(r, t) > 0} d^2 r \) is the cluster area.

Eqs. (4) and (3) make a closed set and provide a general sharp-interface formulation to the Ginzburg-Landau equation with a global conservation law. Often they can be simplified further. Compute the area loss rate of the cluster:

\[
\dot{A}(t) = \int_0 v_n(s, t) ds = \Lambda(t) \left[ \frac{3}{\sqrt{2}} H(t) - \kappa(s, t) \right] ,
\]

where \( \kappa(s, t) \) is the curvature averaged over the whole interface:

\[
\kappa(s, t) = \frac{1}{\Lambda(t)} \int_0 \kappa(s, t) ds ,
\]

and \( \Lambda(t) \) is the cluster perimeter. If the cluster area is conserved, then \( H(t) = (\sqrt{2}/3) \kappa(s, t) \) which yields

\[
v_n(s, t) = \kappa(s, t) - \kappa(s, t) .
\]

This is area-preserving motion by curvature (in 2d), or volume-preserving motion by mean curvature (in 3d) [22, 24, 25]. Dynamics [2] shortens the interface length (in 2d), or area (in 3d) [22, 25]. This nonlocal coarsening model is simpler than the better known “Laplacian coarsening model” (derivable from the Cahn-Hilliard equation [12, 29]) which describes the late-time asymptotics of the locally-conserved bulk-diffusion-controlled coarsening.

Assuming DSI and using Eq. (3), one obtains normal scaling for the characteristic coarsening length: \( l(t) \sim t^{1/2} \). Therefore, global conservation does not change the scaling law. The same result (again, when assuming DSI) follows from dynamic renormalization group arguments applied to Eq. (3) (with a Gaussian white noise term) [27]. For short-range correlations this result was supported by particle simulations of critical [28] and off-critical [29] quench, and by a numerical solution of Eq. (3) for both critical, and off-critical quench [30].

Let us return to fractal coarsening. The initial conditions represent FCs that are characterizable by their fractal dimension \( D \) on an interval of scales between the lower cutoff \( l_0 \) and the upper cutoff \( L_0 \). The DSI-based coarsening scenario [23, 25] assumes that the fractal dimension of the cluster remains constant on a shrinking interval of distances between the lower cutoff \( l(t) \) (which has the same dynamic scaling as the coarsening length), and the upper cutoff \( L(t) \). Now, the perimeter \( \Lambda \) and area \( A \) of the FC can be estimated as

\[
A \sim l(L/l)^D \quad \text{and} \quad A \sim l^2(L/l)^D ,
\]

respectively. Then area conservation yields \( L \sim l^{(D-2)/D} \sim t^{-(2-D)/2D} \) which implies that the FC shrinks with time [31]. This follows \( \Lambda(t) \sim l^{-3}(t) \sim t^{-1/2} \). One can also predict the asymptotic shape of the equal-time pair correlation function at large times: \( C(r, t) \rightarrow g(|r/l(t)|) \). At distances \( r \ll l(t) \) from a typical reference point inside the cluster the correlation function should obey the Porod law: \( g(\xi) = 1 - k \xi \) with a constant \( k \) of order unity. At \( l(t) \ll r \ll L(t) \), \( g(\xi) \sim \xi^{D-2} \) (see Ref. [1]), a power-law tail with the same exponent as in \( C(r, t) \).

In order to check these predictions, we solved Eq. (3) numerically on a domain \( 2048 \times 2048 \) with no-flux boundary conditions. The accuracy of the numerical scheme was monitored by checking the (approximate) conservation law (5) which was found to hold with an accuracy better than 0.2% for \( t > 3 \).

We used 10 different DLA clusters [31] as the initial conditions. These clusters (like the one shown in Fig. 1, upper left) had a radius of order \( 10^3 \). To prevent fragmentation at an early stage of coarsening, the clusters were reinforced by an addition of peripheral sites, similar to Ref. [1]. The average fractal dimension of the initial clusters, determined from the averaged pair correlation function, was 1.75.

Introducing the density \( \rho(r, t) = (1/2)(u(r, t) + 1) \), we identified the cluster as the locus where \( \rho(r, t) \geq 1/2 \). Typical snapshots of the coarsening process are shown in Fig. 1. One can see that larger features of the FC grow at the expense of smaller ones. At late times the cluster shrinks, in agreement with the prediction of DSI [22]. A similar shrinking is evident in the pictures obtained in Monte Carlo simulations of area-preserving interface-
controlled coarsening [7], although the authors of Ref. [7] did not comment on it.

To characterize the coarsening process, several quantities were sampled and averaged over the 10 initial conditions:

1. The cluster area.
2. The (circularly averaged) correlation function, normalized at r=0:
\[
C(r, t) = \frac{\langle \rho(r' + r, t)\rho(r', t) \rangle}{\langle \rho^2(r', t) \rangle}.
\]
(10)

3. The coarsening length scale \( l(t) \), computed from the equation \( C(l, t) = 1/2 \). \( l(t) \) can be interpreted as the typical width of the cluster branches.
4. The cluster perimeter \( \Lambda(t) \) computed by a standard algorithm [33].

The cluster area was found to be constant with an accuracy better than 0.5% for \( t > 10 \), and better than 0.15% for \( t > 100 \). Hence, area preserving motion by curvature, Eq. (8), provides an accurate description to this regime. Figure 2 shows that, at late times (\( t > 100 \)), \( C(r, t) \) approaches a scaled form. The scaled function has a long-range power-law tail with an exponent \( D - 2 \) (the same as in the initial condition), see the inset of Fig. 2. Noticeable is the absence of any additional dynamic length scales, in a striking contrast to the locally conserved fractal coarsening [11]. The dynamics of \( l(t) \) is shown in Fig. 3. The same figure shows a pure \( t^{1/2} \) power-law line (serving as a reference for the expected late-time behavior) and a corrected power-law fit \( l(t) = l_0 + bt^\alpha \) with \( \alpha = 0.49 \), \( l_0 = 5.0 \) and \( b = 1.2 \). The dotted line represents a pure \( t^{1/2} \) power law. The inset shows the time-dependent effective dynamic exponent \( \alpha_0(t) \) versus \( 1/l(t) \). The solid line is a linear fit.

The inset of Fig. 3 illustrates a different method [34] of determining the dynamic exponent. One defines a (time-dependent) effective exponent:
\[
\alpha_0(t) = d\ln l(t)/d\ln t.
\]
Under the normal scaling assumption, one can determine the “true” dynamic exponent by plotting \( \alpha_0(t) \) ver-
sus $l^{-1}(t)$ and extrapolating it to $t \to \infty$, that is to $l^{-1}(t) \to 0$. The numerical value of $\alpha_0(t)$ is computed from $\alpha_0(t) = \log_{10} \left[l(10t)/l(t)\right]$. This procedure yields $\alpha = 0.50$. Therefore, $l(t)$ exhibits normal scaling.

The same procedures were used for an analysis of the dynamic behavior of the cluster perimeter $\Lambda(t)$. We found that $\Lambda^{-1}(t)$ exhibits the same normal scaling: $\Lambda^{-1}(t) \sim t^{1/2}$. Irisawa et al.\cite{irisawa} reported an exponent 0.38 for the inverse perimeter. Their graph shows, however, that the effective exponent increases at late times. We believe that a careful analysis of their data would also lead to an exponent of $1/2$.

Thus, all predictions following from the DSI hypothesis: the normal scaling of $l(t)$, shrinking of the FC and scaling behavior of correlation function (including its power-law tail), are confirmed by numerical simulations. We therefore conclude that globally-conserved interface-controlled fractal coarsening exhibits DSI and normal scaling. This behavior stands in contrast to interface-controlled fractal coarsening exhibits DSI and normal scaling. This behavior stands in contrast to

The mechanism of scaling violations in locally-conserved systems is not entirely clear at present, therefore a comparison between the two types of systems can be instructive. We relate the difference in scaling behavior to an important (and simple) difference in the character of transport. Global transport, characteristic for interface-controlled systems, is uninhibited by Laplacian screening effects typical for locally-conserved systems. Therefore, large-scale dynamics is always present in globally-conserved systems, but is suppressed in locally-conserved ones. This difference is observed already in a very simpler setting of an area-preserving coarsening (shrinking) of long slender bars. In the locally-conserved case the bar acquires a dumbbell shape, while its initial width remains (almost) constant and represents a relevant length scale until late times\cite{rubinstein}. On the contrary, in the globally-conserved case the shrinking bar has a finger-like shape, and its dimensions are changing on the same time scale\cite{rubinstein}.

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\begin{thebibliography}{99}
\bibitem{bray} A.J. Bray, Adv. Phys. 43, 357 (1994).
\bibitem{mandelbrot} B.B. Mandelbrot, \textit{The Fractal Geometry of Nature} (Freeman, San Francisco, 1982); J. Feder, \textit{Fractals} (Plenum, New York, 1988).
\bibitem{vicsek} T. Vicsek, \textit{Fractal Growth Phenomena} (World Scientific, Singapore, 1992).
\bibitem{toyoki} H. Toyoki and K. Honda, Phys. Lett. 111A, 367 (1985).
\bibitem{sempre} R. Semperek, D. Bourret, T. Wognier, J. Phalippou, and R. Jullien, Phys. Rev. Lett. 71, 3307 (1993).
\bibitem{irisawa} T. Irisawa, M. Uwaha, and Y. Saito, Europhys. Lett. 30, 139 (1995).
\bibitem{irisawa96} T. Irisawa, M. Uwaha, and Y. Saito, Fractals 4 251 (1996).
\bibitem{olivi} N. Olivi-Tran, R. Thouy and R. Jullien, J. Phys. I 6, 557 (1996).
\bibitem{meerson} B. Meerson and P.V. Sasorov, cond-mat/9708036.
\bibitem{conti} M. Conti, B. Meerson, and P.V. Sasorov, Phys. Rev. Lett. 80 4693 (1998).
\bibitem{peleg} A. Peleg and B. Meerson, Phys. Rev. E 59, 1238 (1999); 62 (2000).
\bibitem{crooks} G.E. Crooks, B. Ostrovsky, and Y. Bar-Yam, Phys. Rev. E 60, 4559 (1999).
\bibitem{kalinin} S.V. Kalinin et al., Phys. Rev. E 61, 1189 (2000).
\bibitem{streitenberger} P. Streitenberger, in \textit{Paradigms of Complexity. Fractals and Structures in Sciences}, edited by M.N. Novak (World Scientific, Singapore, 2000), p. 135.
\bibitem{meakin} P. Meakin, \textit{Fractals, Scaling and Growth Far from Equilibrium} (Cambridge University Press, Cambridge, 1997).
\bibitem{toroczkai} Z. Toroczkai and E. Williams, Phys. Today 52 (12), 24 (1999).
\bibitem{zinke} M. Zinke-Allmang, L.C. Feldman, and M.H. Grabow, Surface Sci. Reports 16, 277 (1992).
\bibitem{aranson} I. S. Aranson et al., Phys. Rev. Lett. 84, 3306 (2000).
\bibitem{schimansky} L. Schimansky-Geier, Ch. Zülicke, and E. Schöll, Z. Phys. B 84, 433 (1991).
\bibitem{rubinstein} J. Rubinstein and P. Sternberg,IMA J. Appl. Math. 48, 249 (1992).
\bibitem{mikhailov} A.S. Mikhailov, \textit{Foundations of Synergetics I. Distributed Active Systems} (Springer-Verlag, Berlin, 1993).
\bibitem{meerson96} B. Meerson and P.V. Sasorov, Phys. Rev. E 53, 3491 (1996).
\bibitem{gage} M. Gage, Contemporary Math. 51, 51 (1986).
\bibitem{pego} R.L. Pego, Proc. R. Soc. Lond. A 422, 261 (1989).
\bibitem{bray91} A.J. Bray, Phys. Rev. Lett. 66, 2048 (1991), and references therein.
\bibitem{annett} J.F. Annett and J.R. Banavar, Phys. Rev. Lett. 68, 2941 (1992); L.L. Moseley, P.W. Gibbs, and N. Jan, J. Stat. Phys. 67, 813 (1992); A.D. Rutenberg, Phys. Rev. E 54, 972 (1996).
\bibitem{sire} C. Sire and S.N. Majumdar, Phys. Rev. E 52, 244 (1995).
\bibitem{conti99} M. Conti, B. Meerson, A. Peleg, and P.V. Sasorov, in preparation.
\bibitem{witten} T.A. Witten, Jr. and L.M. Sander, Phys. Rev. Lett. 47, 1400 (1981).
\bibitem{rubinstein92} The predicted “shrinking exponent” $(D - 2)/(2D) \approx -0.07$ is too small to be measured accurately.
\bibitem{parker} J.R. Parker, \textit{Practical Computer Vision Using C} (Wiley, New York, 1993), p. 51.
\bibitem{huse} D.A. Huse, Phys. Rev. B, 34, 7845 (1986).
\bibitem{peleg} A. Peleg, M. Conti, B. Meerson, and A.J. Vilenkin, in preparation.
\end{thebibliography}