Rotations and Statistics

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Abstract. The way a field transforms under rotations determines its statistics—as is easy to see for scalar, Dirac, and vector fields.
1. Introduction

At a Solvay conference more than 40 years ago, Eugene Wigner [1] said that one should be able to derive the connection between spin and statistics [2, 3] directly from the properties of fields under rotations. He noted that a suitable rotation $U$ by $\pi$ about the origin transforms the mean value in the vacuum of the product of two scalar fields $\phi(x, t)\phi(-x, t)$ at the same time into that of the fields in the opposite order

$$
\langle 0 | \phi(x, t)\phi(-x, t)|0\rangle = \langle 0 | U^{-1}U\phi(x, t)U^{-1}U\phi(-x, t)U^{-1}U|0\rangle
$$

so that the vacuum value of the commutator vanishes $\langle 0 | [\phi(x, t), \phi(-x, t)]|0\rangle = 0$.

What follows is an attempt to follow his suggestion; I hope it will be of use to students.

My basic assumptions are listed in Section 2: that quantum fields transform suitably under Lorentz transformations and are linear combinations of annihilation and creation operators that satisfy either commutation or anti-commutation relations [4]. In Sections 3–5, I show that scalar, Dirac, and vector fields respectively commute, anti-commute, and commute at equal times because of how they transform under rotations. That their statistics extend to space-like separations then follows from their properties under Poincaré transformations. In Sec. 6 I show that a field $\psi_{ab}(x, t)$ that transforms under the $(A, B)$ representation of the Lorentz group will commute or anti-commute with the field $\psi_{ab}(y, t)$ of the same index at equal times according to whether $2(A+B)$ is an even or an odd integer.

2. Basic Assumptions

2.1. About Particles

Since the particles of a single species are identical, the normalized states $|..., p, s; p', s'; ...\rangle$ and $|..., p', s'; p, s; ...\rangle$ describe the same particles with the same sets of momenta $..., p, p'; ...$ and spin indices $..., s, s'; ...$. Thus, these states differ at most by a phase factor

$$
|..., p, s; p', s'; ...\rangle = e^{i\alpha}|..., p', s'; p, s; ...\rangle.
$$

In a space of three (but not two [5, 6]) spatial dimensions, one may show [4] that the square of this phase factor is unity so that

$$
e^{i\alpha} = \pm 1.
$$

Thus [4] the (suitably normalized) annihilation and creation operators that relate these states to each other and to the vacuum satisfy either commutation

$$[a(p, s), a(p', s')] = 0 \quad \text{and} \quad [a(p, s), a^\dagger(p', s')] = \delta(p - p')\delta_{ss'}.
$$

or anti-commutation

$$\{a(p, s), a(p', s')\} = 0 \quad \text{and} \quad \{a(p, s), a^\dagger(p', s')\} = \delta(p - p')\delta_{ss'}
$$

relations.
2.2. About Fields

A field is a linear combination of these annihilation and creation operators

\[ \psi_\ell(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \left[ u_\ell(p, s)e^{ipx}a(p, s) + v_\ell(p, s)e^{-ipx}a^\dagger(p, s) \right] \]  

that transforms under a Lorentz transformation \( L \) followed by a translation by \( a \) as

\[ U(L, a) \psi_\ell(x) U^{-1}(L, a) = \sum_{\ell'} D_{\ell\ell'}(L^{-1}) \psi_{\ell'}(Lx + a) \]  

and under a rotation \( R \) as

\[ U(R) \psi_\ell(x) U^{-1}(R) = \sum_{\ell'} D_{\ell\ell'}(R^{-1}) \psi_{\ell'}(Rx) \]

in which the operators \( U(L) \) and \( U(R) \) and the matrix \( D(R) \) are unitary [7].

The mean value in the vacuum of the product of two components \( \psi_\ell(x) \) and \( \psi_{\ell'}(y) \) of the same field at two space-like points \( x \) and \( y \) is not identically zero

\[ \langle 0 | \psi_\ell(x) \psi_{\ell'}(y) | 0 \rangle = \sum_s \int \frac{d^3p}{(2\pi)^3} u_\ell(p, s)v_{\ell'}(p, s)e^{ip(x-y)} = \langle 0 \rangle \]

as follows explicitly from Eqs.\( [4] \) & \( [5] \) for scalar

\[ \langle 0 | \psi_\ell(x) \psi_{\ell'}(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} = \Delta_+(x - y) \neq 0 \]  

Dirac

\[ \langle 0 | \psi_\ell(x) \psi_{\ell'}(y) | 0 \rangle = i [(m- \partial)\gamma^0]_{\ell\ell'} \Delta_+(x - y) \neq 0. \]  

and vector

\[ \langle 0 | \psi_\ell(x) \psi_{\ell'}(y) | 0 \rangle = \left( \delta_{\ell\ell'} - \frac{\partial_\ell\partial_{\ell'}}{m^2} \right) \Delta_+(x - y) \neq 0 \]

fields [7]. For \( (x - y)^2 = r^2 > 0 \), the invariant function \( \Delta_+(x - y) \) is [7] the modified Bessel function

\[ \Delta_+(x - y) = \frac{m}{4\pi^2r} K_1(mr) > 0 \]

which is positive for \( r > 0 \).

3. Scalar Fields

Let us consider a scalar field \( \psi(x) \) at two space-like points \( x_1 \) and \( x_2 \). We may choose an inertial coordinate system in which these points have the same time coordinate and lie on the \( x \)-axis with the spatial origin midway between them. If the two coordinate systems are related by the Lorentz transformation \( L \) followed by the translation by \( a \), then the points are

\[ Lx_1 + a = x_+ = (r/2, 0, 0, t) \quad \& \quad Lx_2 + a = x_- = (-r/2, 0, 0, t). \]

For a scalar field \( \psi(x) \), the matrix \( D_{\ell\ell'}(L^{-1}) \) is just the number 1, and so the Poincaré transformation [7] reduces to

\[ U(L, a) \psi(x) U^{-1}(L, a) = \psi(Lx + a) \]  

Using the abbreviation \( U_{L,a} = U(L, a) \), inserting three factors of unity in the form \( I = U_{L,a}^{-1}U_{L,a} \), and using the invariance of the vacuum state \( |0\rangle \) under Poincaré
transformations \( (U_{L,a}[0] = |0\rangle) \), we see that the mean value in the vacuum of \( \psi(x_1)\psi(x_2) \) is the same as that of \( \psi(x_+)^\dagger \psi(x_-) \)

\[
0\rangle\psi(x_1)\psi(x_2)|0\rangle = (0|U_{L,a}^{-1}U_{L,a}\psi(x_1)U_{L,a}^{-1}U_{L,a}\psi(x_2)U_{L,a}^{-1}U_{L,a}|0\rangle
= (0|U_{L,a}\psi(x_1)U_{L,a}\psi(x_2)U_{L,a}^{-1}|0\rangle
= (0|\psi(x_+)^\dagger \psi(x_-)|0\rangle.
\]

\hspace{2cm} (16)

Thus we easily may switch back and forth between the two coordinate systems.

For a scalar field, the matrix \( D(R) \), like the matrix \( D(L) \), is unity, and so the general form \( k \), of a rotation is simply

\[
U(R) \psi(x) U^{-1}(R) = \psi(Rx).
\]

A rotation by angle \( \pi \) about the \( z \)-axis takes \( x_+ \) into \( x_- \) and \( x_- \) into \( x_+ \). So such a rotation interchanges \( \psi(x_+)^\dagger \) and \( \psi(x_-) \)

\[
U_{\pi} \psi(x_+) U_{\pi}^{-1} = \psi(x_-).
\]

\hspace{2cm} (17)

We can use \( U_{\pi} \) to relate \( (0|\psi(x_+)^\dagger \psi(x_-)|0\rangle \) to \( (0|\psi(x_-)^\dagger \psi(x_+)|0\rangle \) by thrice inserting unity in the form \( I = U_{\pi}^{-1}U_{\pi} \) and by using the invariance of the vacuum state under rotations \( (U_{\pi}|0\rangle = |0\rangle) \)

\[
(0|\psi(x_+) \psi(x_-)|0\rangle = (0|U_{\pi}^{-1}U_{\pi}\psi(x_+)U_{\pi}^{-1}U_{\pi}\psi(x_-)U_{\pi}^{-1}U_{\pi}|0\rangle
= (0|U_{\pi}\psi(x_+) U_{\pi}^{-1}U_{\pi}\psi(x_-)U_{\pi}^{-1}|0\rangle
= (0|\psi(x_-)^\dagger \psi(x_+)|0\rangle.
\]

\hspace{2cm} (18)

Thus the vacuum matrix element of the commutator \( [\psi(x_+),\psi(x_-)] \) vanishes as Wigner noted \( k \),

\[
(0|[\psi(x_+),\psi(x_-)]|0\rangle = 0
\]

\hspace{2cm} (20)

but that of the anti-commutator \( \{\psi(x_+),\psi(x_-)\} \) is twice that of the product

\[
(0|\{\psi(x_+),\psi(x_-)\}|0\rangle = 2(0|\psi(x_+) \psi(x_-)|0\rangle \neq 0
\]

\hspace{2cm} (21)

and does not vanish by \( [10] & [13] \). So the field \( \psi \) does not anti-commute with itself at space-like separations.

What about the creation and annihilation operators? If they obeyed anti-commutation relations, the commutator \( [\psi(x_+),\psi(x_-)] \) would be quadratic in the creation and annihilation operators, but its mean value in the vacuum still would vanish. So we can’t immediately conclude that they must obey commutation relations just because the mean value \( [20] \) is zero.

To see whether \( a \) and \( a^\dagger \) obey commutation or anti-commutation relations, we apply the preceding argument \( [15] & [20] \) to any product of scalar fields at points that can be mapped by a Poincaré transformation to \( x_\pm \) and points \( w_k = (0,0,z_k,t_k) \) on the \( z \)-axis at arbitrary times. We thus find that mean value in the vacuum of the commutator \( [\psi(x_+),\psi(x_-)] \) sandwiched between any two such products of fields \( \psi(w_k) \)

\[
(0|\psi(w_1) \ldots \psi(w_N) \{\psi(x_+),\psi(x_-)\} \psi(w_{N+1}) \ldots \psi(w_{N+M})|0\rangle = 0
\]

\hspace{2cm} (22)

must vanish. If the creation and annihilation operators obeyed anti-commutation relations, then the commutator \( [\psi(x_+),\psi(x_-)] \) would be quadratic in them, and the Wightman \( k \), functions \( [22] \) would not vanish identically for arbitrary \( w_k, N, \) and \( M \). So the creation and annihilation operators can’t obey anti-commutation relations. Since by \( [4] & [5] \) they must obey either commutation or anti-commutation relations,
it follows that they must obey commutation relations. In this case, the commutator 
\[ [\psi(x_+), \psi(x_-)] \] is a number, not an operator, and the vanishing of its vacuum mean value \[ (20) \] means that the commutator itself must vanish
\[ [\psi(x_+), \psi(x_-)] = 0 \] (23)

at the equal-time points \( x_\pm \).

Using the inverse of the Poincaré transformation \[ (16) \], we find that the commutator \[ [\psi(x_1), \psi(x_2)] \] also vanishes for arbitrary space-like points \( x_1 \) and \( x_2 \)
\[ [\psi(x_1), \psi(x_2)] = \left[ U_{L,a}^{-1} \psi(x_+)U_{L,a}, U_{L,a}^{-1} \psi(x_-)U_{L,a} \right] 
= U_{L,a}^{-1} \left[ \psi(x_+), \psi(x_-) \right] U_{L,a} = U_{L,a}^{-1}0U_{L,a} = 0. \] (24)

Thus our basic assumptions \[ (2–13) \] about creation and annihilation operators and about how scalar fields transform under rotations (and Lorentz transformations) imply that spin-zero creation and annihilation operators satisfy commutation relations \[ (4) \] and that scalar fields commute at space-like separations \[ (24) \].

4. Majorana and Dirac Fields

A Lorentz transformation \( L \) followed by a translation by \( a \) maps a four-component Dirac or Majorana field into
\[ U(L, a) \psi_\ell(x) U^{-1}(L, a) = \sum_{\ell' = 1}^4 D_{\ell\ell'}^{(1/2)}(L^{-1}) \psi_{\ell'}(Lux + a) \] (25)
in which (with suitable \( \gamma \) matrices \[ [7] \]) the matrix \( D(L^{-1})^{(1/2)} \) is a 4 \( \times \) 4 block-diagonal matrix
\[ D(L^{-1})^{(1/2)} = \begin{pmatrix} \exp(\vec{z} \cdot \vec{\sigma}) & 0 \\ 0 & \exp(-\vec{z}^\ast \cdot \vec{\sigma}) \end{pmatrix}. \] (26)

Here the \( \sigma \)-matrices are the Pauli matrices. Thus under Poincaré transformations, the upper two components of the field mix and the lower two mix, but the upper two components do not mix with the lower two. This holds in particular for the transformation \( L, a \) that takes the space-like points \( x_1 \) and \( x_2 \) to the equal-time points \( x_\pm \). So to show that the upper or the lower two field components anti-commute at space-like separations, we need only show that the upper or the lower two components anti-commute at equal times. We need not show that the upper components anti-commute with the lower components at equal times.

Under a rotation \( R \), the rule \[ (8) \] is
\[ U(R) \psi_\ell(x) U^{-1}(R) = \sum_{\ell' = 1}^4 D_{\ell\ell'}^{(1/2)}(R^{-1}) \psi_{\ell'}(Rx) \] (27)
in which \( \vec{z} \) is imaginary, and so the two 2 \( \times \) 2 matrices \( r \) that appear in the 4 \( \times \) 4 block-diagonal matrix
\[ D(R^{-1})^{(1/2)} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \] (28)
are the same. The matrix \( r \) that represents a right-handed active rotation \( R^{-1} \) of angle \( \theta = |\vec{\theta}| \) about the axis \( \hat{\theta} \) is
\[ r = \exp(-i \frac{\vec{\sigma}}{2} \cdot \vec{\theta}) = \cos(\theta/2) - i \vec{\sigma} \cdot \hat{\theta} \sin(\theta/2). \] (29)
We'll need two special rotations. The matrices that represent rotations of \( \pi \) about the \( z \)- and \( y \)-axes are

\[
\begin{align*}
    r(\pi, z) &= -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
    r(\pi, y) &= -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

Let \( \psi_\ell \) be a four-component spin-one-half Majorana or Dirac field. The transformation rule tells us that under a left-handed rotation \( U_{\pi,z} \) of angle \( \pi \) about the \( z \)-axis, the first and third components of this field will transform as

\[
U_{\pi,z}\psi_\ell(x_\pm)U^{-1}_{\pi,z} = -i\psi_\ell(x_\mp)
\]

since the rotation \( R^{-1} \) is a right-handed active rotation about the \( z \)-axis by angle \( \pi \). The second and fourth components will transform as

\[
U_{\pi,z}\psi_\ell(x_\pm)U^{-1}_{\pi,z} = i\psi_\ell(x_\mp)
\]

Similarly, under a left-handed rotation \( U_{\pi,y} \) of angle \( \pi \) about the \( y \)-axis, the first and third components of this field will transform as

\[
U_{\pi,y}\psi_\ell(x_\pm)U^{-1}_{\pi,y} = \psi_{\ell+1}(x_\mp)
\]

since the rotation \( R^{-1} \) is a right-handed active rotation about the \( y \)-axis by angle \( \pi \). The second and fourth components will transform as

\[
U_{\pi,y}\psi_\ell(x_\pm)U^{-1}_{\pi,y} = -\psi_{\ell-1}(x_\mp)
\]

Let us first consider the case in which the two indices \( \ell \) and \( \ell' \) are the same. The vacuum state is invariant under rotations, so if we thrice insert unity in the form \( I = U_{\pi,z}^{-1}U_{\pi,z} \), and use Eqs. (32 & 33), then (not summing over \( \ell \)) we get

\[
\langle 0 | \psi_\ell(x_+) | \psi_\ell(x_-) \rangle = \langle 0 | U_{\pi,z}^{-1}U_{\pi,z}\psi_\ell(x_+)U_{\pi,z}^{-1}U_{\pi,z}\psi_\ell(x_-)U_{\pi,z}^{-1}U_{\pi,z}|0 \rangle
\]

\[
= \langle 0 | U_{\pi,z}\psi_\ell(x_+)U_{\pi,z}^{-1}U_{\pi,z}\psi_\ell(x_-)U_{\pi,z}^{-1}|0 \rangle
\]

\[
= -\langle 0 | \psi_\ell(x_-) \psi_\ell(x_+)|0 \rangle.
\]

Thus the vacuum matrix element of the anti-commutator \( \{\psi_+, \psi_-\} \) vanishes

\[
\langle 0 | \{\psi_\ell(x_+), \psi_\ell(x_-)\}|0 \rangle = 0.
\]

But that of the commutator \( [\psi_\ell(x_+), \psi_\ell(x_-)] \) is twice that of the product

\[
\langle 0 | [\psi_\ell(x_+), \psi_\ell(x_-)]|0 \rangle = 2\langle 0 | \psi_\ell(x_+) \psi_\ell(x_-)|0 \rangle
\]

which by (11 & 13) does not vanish identically. So the vacuum matrix element of the commutator \( [\psi_\ell(x_+), \psi_\ell(x_-)] \) does not vanish; the fields \( \psi_\ell(x_+) \) and \( \psi_\ell(x_-) \) do not commute.

What about unequal indices? If we thrice insert unity in the form \( I = U_{\pi,y}^{-1}U_{\pi,y} \), and use the invariance of the vacuum under rotations as well as Eqs. (34 & 35), then we find

\[
\langle 0 | \psi_1(x_+) \psi_2(x_-)|0 \rangle = \langle 0 | U_{\pi,y}\psi_1(x_+)U_{\pi,y}^{-1}U_{\pi,y}\psi_2(x_-)U_{\pi,y}^{-1}|0 \rangle
\]

\[
= -\langle 0 | \psi_2(x_-) \psi_1(x_+)|0 \rangle.
\]

So the mean value in the vacuum of the anti-commutator

\[
\langle 0 | \{\psi_1(x_+), \psi_2(x_-)\}|0 \rangle = 0.
\]
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vanishes, but by (11 & 13) not that of the commutator

\[ \langle 0 | [\psi_1(x_+), \psi_2(x_-)] | 0 \rangle \neq 0. \]  
(41)

So the fields \( \psi_1(x_+) \) and \( \psi_2(x_-) \) do not commute.

An identical argument shows that the vacuum mean value of the anti-commutator

\[ \langle 0 | \{ \psi_3(x_+), \psi_4(x_-) \} | 0 \rangle = 0. \]  
(42)

vanishes, but not that of the commutator

\[ \langle 0 | [\psi_3(x_+), \psi_4(x_-)] | 0 \rangle \neq 0. \]  
(43)

So the fields \( \psi_3(x_+) \) and \( \psi_4(x_-) \) do not commute.

What about the creation and annihilation operators? To see whether they obey commutation or anti-commutation relations, we apply the rotation \( U_{\pi,z} \) and the preceding argument (33, 36, & 37) to any product of Majorana or Dirac fields at points that can be mapped by a Poincaré transformation to \( x_\pm \) and points \( w_k = (0,0,0,\pm t_k) \) on the z axis at arbitrary times. We thus find that the anti-commutator \( \{ \psi_\ell(x_+), \psi_\ell(x_-) \} \) of fields of the same index sandwiched between any two such products of fields \( \psi_\ell(w_k) \)

\[ \langle 0 | \left( \prod_{k=1}^{N} \psi_\ell_k(w_k) \right) \{ \psi_\ell(x_+), \psi_\ell(x_-) \} \left( \prod_{k=N+1}^{N+M} \psi_\ell_k(w_k) \right) | 0 \rangle = 0 \]  
(44)

vanishes as long as the numbers \( N_\ell_k + M_\ell_k \) of fields of index \( \ell_k \) (not including \( \psi_\ell(x_+) \) & \( \psi_\ell(x_-) \)) satisfy

\[ N_1 + M_1 + N_3 + M_3 = N_2 + M_2 + N_4 + M_4 \pm 4n \]  
(45)

for some integer \( n \). If the creation and annihilation operators obeyed commutation relations, then the anti-commutator \( \{ \psi(x_+), \psi(x_-) \} \) would be quadratic in them, and so the Wightman functions (44) could not vanish for arbitrary \( w_k, N, \) and \( M \) satisfying (45). So they can’t obey commutation relations. Since by (4 & 5) they must obey either commutation or anti-commutation relations, it follows that they must obey anti-commutation relations. In this case, the anti-commutator \( \{ \psi(x_+), \psi(x_-) \} \) is a number, not an operator, and the vanishing of its vacuum mean value (37) means that the anti-commutator itself must vanish

\[ \{ \psi_\ell(x_+), \psi_\ell(x_-) \} = 0. \]  
(46)

Finally, by using the relation (24) (adapted via (26) to Dirac fields), we conclude that at space-like separations, the upper two field components anti-commute, and so do the lower two.

What about the anti-commutator \( [\psi_1(x_+), \psi_3(x_-)]_+ \)? Parity relates fields with indices 1 and 2 to those with indices 3 and 4. Under parity, a free Majorana field transforms as

\[ P\psi(x)P^{-1} = \eta^* \beta \psi(Px) \]  
in which \( \beta = i\gamma^0 \). Its intrinsic parity

\[ \eta = \pm i \]  
(48)

is imaginary [7]. So for \( \ell = 1 \) or 2,

\[ P\psi_\ell(x_\pm)P^{-1} = \eta^* \psi_{\ell+2}(x_\mp) \]  
(49)
and for $\ell = 3$ or 4,

$$P\psi_\ell(x_\pm)P^{-1} = \eta^* \psi_{\ell-2}(x_\mp)$$  \hspace{1cm} (50)

Thus if we thrice insert unity in the form $PP^{-1}$ and use the invariance under parity of the vacuum of the free field theory, then we get

$$\langle 0|\psi_1(x_+)\psi_3(x_-)|0\rangle = \langle 0|P\psi_1(x_+)P^{-1}P\psi_3(x_-)P^{-1}|0\rangle$$

$$= (\eta^*)^2 \langle 0|\psi_3(x_-)\psi_1(x_+)|0\rangle$$

$$= - \langle 0|\psi_3(x_-)\psi_1(x_+)|0\rangle$$  \hspace{1cm} (51)

since $\eta$ is imaginary. So the mean value in the vacuum of the anti-commutator

$$\langle 0|\{\psi_1(x_+),\psi_3(x_-)\}|0\rangle = 0$$  \hspace{1cm} (52)

vanishes. And since the creation and annihilation operators obey anti-commutation relations, the anti-commutator $\{\psi_1(x_+),\psi_3(x_-)\}$ is a number, not an operator. So the vanishing of its mean value in the vacuum implies that the anti-commutator itself vanishes

$$\{\psi_1(x_+),\psi_3(x_-)\} = 0.$$  \hspace{1cm} (53)

An identical argument shows that the anti-commutator vanishes

$$\{\psi_2(x_+),\psi_4(x_-)\} = 0.$$  \hspace{1cm} (54)

Since Dirac fields are complex linear combinations of two Majorana fields of the same mass, Eqs (53 & 54) apply also to Dirac fields.

Thus our basic assumptions (2) about creation and annihilation operators and about how Majorana and Dirac fields transform under rotations (and Lorentz transformations and parity) imply that spin-one-half creation and annihilation operators satisfy anti-commutation relations (5) and that Majorana and Dirac fields anti-commute at space-like separations.

5. Vector Fields

A vector field transforms like a 3-vector under rotations

$$U(R)\psi_\ell(x)U^{-1}(R) = \sum_{\ell'=1}^3 D^{(1)}_{\ell\ell'}(R^{-1}) \psi_{\ell'}(Rx) = \sum_{\ell'=1}^3 R^{-1}_{\ell\ell'} \psi_{\ell'}(Rx)$$  \hspace{1cm} (55)

since the matrix $D(R^{-1})^{(1)}$ that appears in the transformation rule (5) is simply the $3 \times 3$ matrix $R^{-1}$. The matrix of a right-handed rotation $R^{-1}$ by angle $\theta$ about the axis $\hat{n}$ is [6]

$$R^{-1}_{ij}(\theta \hat{n}) = \delta_{ij} \cos \theta - \sin \theta \sum_{k=1}^3 \epsilon_{ijk} \hat{n}_k + (1 - \cos \theta) \hat{n}_i \hat{n}_j$$  \hspace{1cm} (56)

in which $\epsilon_{ijk}$ is totally anti-symmetric in $i, j, k$.

To deal with the various pairs of indices $\ell$ and $\ell'$, we rotate the reference points from $x_\pm = (\pm x, 0, 0, t)$ to $y_\pm = (\pm y, \pm y, \pm y, t)$ where $y = x/\sqrt{3}$. The space parts of the 4-vectors $y_\pm$ are parallel or anti-parallel to the 3-vector $r = (1, 1, 1)$. Any rotation by $\pi$ about any axis $\hat{n}$ that is perpendicular to the vector $r$ will interchange $y_\pm$ with $y_\mp$. Any such axis satisfies $\hat{n} \cdot r = 0$ and so is of the form

$$\hat{n} = (a - b, b - c, c - a)$$  \hspace{1cm} (57)
with \((a - b)^2 + (b - c)^2 + (c - a)^2 = 1\). A rotation \(R(\pi \hat{n}) = R^{-1}(\pi \hat{n})\) by \(\pi\) about the axis \(\hat{n}\) is represented by the \(3 \times 3\) matrix

\[
R^{-1}(\pi \hat{n}) = \begin{pmatrix}
2(a - b)^2 - 1 & 2(a - b)(b - c) & 2(a - b)(c - a) \\
2(b - c)(a - b) & 2(b - c)^2 - 1 & 2(b - c)(c - a) \\
2(c - a)(a - b) & 2(c - a)(b - c) & 2(c - a)^2 - 1
\end{pmatrix}
\] (58)

according to the general formula [54].

We shall use three special cases. A rotation by \(\pi\) about the axis \(\hat{n}_1 = (1/\sqrt{2}, -1/\sqrt{2}, 0)\) is represented by the matrix

\[
R^{-1}(\pi \hat{n}_1) = \begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\] (59)

By the rule [5], the unitary operator \(U_{\pi \hat{n}_1}\) rotates the fields \(\psi_1(\pm y)\) and \(\psi_2(\pm y)\) into \(-\psi_2(\mp y)\) and \(-\psi_1(\mp y)\), and the field \(\psi_3(\pm y)\) into \(-\psi_3(\mp y)\)

\[
U_{\pi \hat{n}_1} \psi_1(y \pm) U_{\pi \hat{n}_1}^{-1} = R_{1\ell}^{-1}(\pi \hat{n}_1) \psi(y \mp) = -\psi_2(y \mp) \\
U_{\pi \hat{n}_1} \psi_2(y \pm) U_{\pi \hat{n}_1}^{-1} = R_{2\ell}^{-1}(\pi \hat{n}_1) \psi(y \mp) = -\psi_1(y \mp) \\
U_{\pi \hat{n}_1} \psi_3(y \pm) U_{\pi \hat{n}_1}^{-1} = R_{3\ell}^{-1}(\pi \hat{n}_1) \psi(y \mp) = -\psi_3(y \mp).
\] (60)

A rotation by \(\pi\) about the axis \(\hat{n}_2 = (0, 1/\sqrt{2}, -1/\sqrt{2})\) is represented by the matrix

\[
R^{-1}(\pi \hat{n}_2) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix}
\] (61)

By the rule [5], the unitary operator \(U_{\pi \hat{n}_2}\) rotates the fields \(\psi_2(\pm y)\) and \(\psi_3(\pm y)\) into \(-\psi_3(\mp y)\) and \(-\psi_2(\mp y)\), and the field \(\psi_1(\pm y)\) into \(-\psi_1(\mp y)\)

\[
U_{\pi \hat{n}_2} \psi_1(y \pm) U_{\pi \hat{n}_2}^{-1} = R_{1\ell}^{-1}(\pi \hat{n}_2) \psi(y \mp) = -\psi_1(y \mp) \\
U_{\pi \hat{n}_2} \psi_2(y \pm) U_{\pi \hat{n}_2}^{-1} = R_{2\ell}^{-1}(\pi \hat{n}_2) \psi(y \mp) = -\psi_3(y \mp) \\
U_{\pi \hat{n}_2} \psi_3(y \pm) U_{\pi \hat{n}_2}^{-1} = R_{3\ell}^{-1}(\pi \hat{n}_2) \psi(y \mp) = -\psi_2(y \mp).
\] (62)

A rotation by \(\pi\) about the axis \(\hat{n}_3 = (1/\sqrt{2}, 0, -1/\sqrt{2})\) is represented by the matrix

\[
R^{-1}(\pi \hat{n}_3) = \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\] (63)

By the rule [5], the unitary operator \(U_{\pi \hat{n}_3}\) rotates the fields \(\psi_1(\pm y)\), \(\psi_2(\pm y)\), and \(\psi_3(\pm y)\) into \(-\psi_3(\mp y)\) and \(-\psi_2(\mp y)\), and \(-\psi_1(\mp y)\)

\[
U_{\pi \hat{n}_3} \psi_1(y \pm) U_{\pi \hat{n}_3}^{-1} = R_{1\ell}^{-1}(\pi \hat{n}_3) \psi(y \mp) = -\psi_3(y \mp) \\
U_{\pi \hat{n}_3} \psi_2(y \pm) U_{\pi \hat{n}_3}^{-1} = R_{2\ell}^{-1}(\pi \hat{n}_3) \psi(y \mp) = -\psi_2(y \mp) \\
U_{\pi \hat{n}_3} \psi_3(y \pm) U_{\pi \hat{n}_3}^{-1} = R_{3\ell}^{-1}(\pi \hat{n}_3) \psi(y \mp) = -\psi_1(y \mp).
\] (64)

The transformations [56, 57] and the invariance of the vacuum under rotations imply these relations:

\[
\langle 0 | \psi_1(y \pm) \psi_1(y \mp) | 0 \rangle = \langle 0 | U_{\pi \hat{n}_2} \psi_1(y \pm) U_{\pi \hat{n}_2}^{-1} U_{\pi \hat{n}_3} \psi_1(y \mp) U_{\pi \hat{n}_3}^{-1} | 0 \rangle = \langle 0 | \psi_1(y \mp) \psi_1(y \pm) | 0 \rangle
\]
\[ (0|\psi_1(y_\pm)\psi_2(y_\mp)|0) = (0|U_{\pi,n_1}\psi_1(y_\pm)U^{-1}_{\pi,n_1}U_{\pi,n_1}\psi_2(y_\mp)U^{-1}_{\pi,n_1}|0) = (0|\psi_2(y_\mp)\psi_1(y_\pm)|0) \]

Thus the mean value in the vacuum of the commutator vanishes

\[ (0|[\psi_i(y_\pm),\psi_j(y_\mp)]|0) = 0 \] (66)

for all pairs, \( i, j = 1, 2, 3 \), but by \([12] \& [13]\) that of the anti-commutator does not. So the fields \( \psi_i(y_\pm) \) and \( \psi_j(y_\mp) \) do not anti-commute.

To see whether the creation and annihilation operators for vector fields obey commutation or anti-commutation relations, we use a rotation \( U_{\pi,z} \) by angle \( \pi \) about the \( z \) axis with rotation matrix

\[ R^{-1}(\pi \hat{z}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (67)

We apply this rotation and the preceding argument \([55] \& [56]\) to the fields \( \psi_i(x_+)\psi_i(x_-) \) sandwiched between products of fields at points that can be mapped by a Poincaré transformation to \( x_+ \) and points \( w_k = (0,0,z_k,t_k) \) on the \( z \) axis at arbitrary times.

Thus we find that the commutator \([\psi_i(x_+),\psi_i(x_-)]\) of vector fields of the same index sandwiched between any two such products of fields \( \psi_{jk}(w_k) \)

\[ (0| \prod_{k=1}^{N} \psi_{jk}(w_k) |\psi_i(x_+),\psi_i(x_-)| \prod_{k=N+1}^{N+M} \psi_{jk}(w_k) |0) = 0 \] (68)

vanishes as long as the numbers of fields of index 1 or 2 is even

\[ N_1 + M_1 + N_2 + M_2 = \pm 2n. \] (69)

If the creation and annihilation operators obeyed anti-commutation relations, then the commutator \([\psi_i(x_+),\psi_i(x_-)]\) would be quadratic in them, and so the Wightman functions \([68]\) could not vanish for all \( N_k + M_k \) that satisfy \([69]\) and for arbitrary \( w_k \). So they can’t obey anti-commutation relations. Since by \([4] \& [5]\), they must obey either commutation or anti-commutation relations, they must obey commutation relations. In this case, the equal-time commutators \([\psi_i(y_\pm),\psi_j(y_\mp)]\) are numbers, not operators, and so the vanishing of their mean values in the vacuum implies that these commutators vanish

\[ [\psi_i(y_\pm),\psi_j(y_\mp)] = 0. \] (70)

To extend this relation to points at arbitrary space-like separations, we need to know how the vector field transforms under Lorentz boosts.
The response of a vector field to a Lorentz boost depends upon the mass of the particle the field describes. If the particle is massive, then under a Lorentz transformation \( L \) followed by a translation by \( a \) the field transforms as
\[
U(L, a) \psi_\mu(x) U^{-1}(L, a) = L'_{\nu\mu} \psi_\nu(L x + a).
\]
(71)
In this case, there is no difficulty applying the argument \([24]\) to the vector field \( \psi_\mu \) and so extending the vanishing of the equal-time commutator \([70]\) to points at arbitrary space-like separations.

But if the field \( \psi_\mu \) represents a massless particle, then under a Lorentz boost the preceding equation is augmented by a gauge transformation
\[
U(L, a) \psi_\mu(x) U^{-1}(L, a) = L'_{\nu\mu} \psi_\nu(L x + a) + \partial_\mu \Omega(x + a, L)
\]
(72)
in which \( \Omega \) is a linear combination of annihilation and creation operators \([7]\), and I do not know of a simple way of extending Eq.\(\(70\)\) to space-like separations.

6. More General Fields

A field that transforms according to the \((A, B)\) representation of the homogeneous Lorentz group transforms under rotations as \([7]\)
\[
U(R) \psi_{ab}(x) U^{-1}(R) = \sum_{a', b'} D^{A}_{a a'}(R^{-1}) D^{B}_{b b'}(R^{-1}) \psi_{a'b'}(Rx)
\]
(73)
in which the sums run in integer steps from \(-A\) to \(A\) and from \(-B\) to \(B\). This field can represent a particle of spin \( j \) where \(|A - B| \leq j \leq A + B\) and \( j \) differs from \(A + B\) by an integer \([7]\). If the rotation \( R \) is a right-handed rotation of angle \( \theta \) about the \( z \)-axis, then
\[
D^{(A)}_{a a'}(R^{-1}) = \delta_{a a'} e^{i a \theta} \quad \text{and} \quad D^{(B)}_{b b'}(R^{-1}) = \delta_{b b'} e^{i b \theta}.
\]
(74)
Thus under a right-handed rotation by \( \pi \) about the \( z \)-axis, the field \( \psi_{ab}(x_\pm) \) at \( x_\pm = (\pm x, 0, 0, t) \) transforms into
\[
U_{\pi, z} \psi_{ab}(x_\pm) U^{-1}_{\pi, z} = e^{i(a+b)\pi} \psi_{ab}(x_\mp).
\]
(75)
Again inserting unity in the form \( I = U_{\pi, z} U^{-1}_{\pi, z} \) and invoking the invariance of the vacuum under rotations, we find
\[
\langle 0 | \psi_{ab}(x_\pm) \psi_{ab}(x_\mp) | 0 \rangle = \langle 0 | U_{\pi, z} \psi_{ab}(x_\pm) U^{-1}_{\pi, z} U_{\pi, z} \psi_{ab}(x_\mp) U^{-1}_{\pi, z} | 0 \rangle
= e^{i2(a+b)\pi} \langle 0 | \psi_{ab}(x_\mp) \psi_{ab}(x_\pm) | 0 \rangle.
\]
(76)
Now the indices \( a \) and \( b \) are related to \( A \) and \( B \) by \( a = A - n \) and \( b = B - m \) in which \( n \) and \( m \) are both integers. Thus \(2(a + b) = 2A + 2B - 2n - 2m\), that is, \(2(a + b)\) differs from \(2A + 2B\) by an even integer, and so, \(2(a + b)\) differs from \(2j\) by an even integer. Thus
\[
\langle 0 | \psi_{ab}(x_\pm) \psi_{ab}(x_\mp) | 0 \rangle = e^{i2(a+b)\pi} \langle 0 | \psi_{ab}(x_\mp) \psi_{ab}(x_\pm) | 0 \rangle
= e^{i2j\pi} \langle 0 | \psi_{ab}(x_\mp) \psi_{ab}(x_\pm) | 0 \rangle
= (-1)^{2j} \langle 0 | \psi_{ab}(x_\mp) \psi_{ab}(x_\pm) | 0 \rangle.
\]
(77)
Thus for particles of integral spin, bosons, the mean value in the vacuum of the commutator vanishes
\[
\langle 0 | [\psi_{ab}(x_\pm), \psi_{ab}(x_\mp)] | 0 \rangle = 0
\]
(78)
while that of the anti-commutator does not vanish
\[ \langle 0 | \{ \psi_{ab}(x^+), \psi_{ab}(x^-) \} | 0 \rangle = 2 \langle 0 | \psi_{ab}(x^+) \psi_{ab}(x^-) | 0 \rangle \neq 0. \] (79)

Also, for particles of half-odd-integral spin, fermions, the mean value in the vacuum of the anti-commutator vanishes
\[ \langle 0 | \{ \psi_{ab}(x^+), \psi_{ab}(x^-) \} | 0 \rangle = 0 \] (80)
while that of the commutator does not vanish
\[ \langle 0 | [ \psi_{ab}(x^+), \psi_{ab}(x^-) ] | 0 \rangle = 2 \langle 0 | \psi_{ab}(x^+) \psi_{ab}(x^-) | 0 \rangle \neq 0. \] (81)

So the fields do not obey the wrong statistics. To see whether the creation and annihilation operators for these general fields obey commutation or anti-commutation relations, we apply the rotation \( U_{\pi,z} \) to the fields \( \psi_{ab}(x^+) \psi_{ab}(x^-) \) sandwiched between products of fields at points \( w_k = (0, 0, z_k, t_k) \) on the \( z \) axis at arbitrary times. Thus we find that the \( j \)-commutator
\[ [\psi_{ab}(x^+), \psi_{ab}(x^-)]_j = \psi_{ab}(x^+) \psi_{ab}(x^-) - (-1)^j \psi_{ab}(x^-) \psi_{ab}(x^+) \] (82)
of two general fields of the same index sandwiched between any two such products of fields \( \psi_{akbk}(w_k) \)
\[ \langle 0 | \left( \prod_{k=1}^{N} \psi_{akbk}(w_k) \right) [\psi_{ab}(x^+), \psi_{ab}(x^-)]_j \left( \prod_{k=N+1}^{N+M} \psi_{akbk}(w_k) \right) | 0 \rangle = 0 \] (83)
vanishes as long as
\[ \sum_{k=1}^{N+M} a_k + b_k = 2n. \] (84)

If the creation and annihilation operators obeyed the wrong commutation relations, then the \( j \)-commutator \([\psi_i(x^+), \psi_i(x^-)]_j\) would be quadratic in them, and so these Wightman functions (83) could not vanish for all \( N, M \) and all \( a_k \) and \( b_k \) that satisfy (84) and for arbitrary \( w_k \). So they can’t obey the wrong commutation relations. Since by (4 & 5), they must obey either commutation or anti-commutation relations, they must obey the right commutation relations. In this case, the equal-time \( j \)-commutators \([\psi_i(x^+), \psi_i(x^-)]_j\) are numbers, not operators, and so the vanishing of their mean values in the vacuum implies that these commutators vanish
\[ [\psi_{ab}(x^+), \psi_{ab}(x^-)]_j = 0 \] (85)

It should be possible to extend this argument to arbitrary space-like separations and to all pairs of indices \( ab, a'b' \).

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