THE WAVELET TRANSFORMS IN GELFAND-SHILOV SPACES

STEVAN PILIPOVIĆ, DUŠAN RAKIĆ, NENAD TEOFANOV, AND JASSON VINDAS

Abstract. We describe local and global properties of wavelet transforms of ultradifferentiable functions. The results are given in the form of continuity properties of the wavelet transform on Gelfand-Shilov type spaces and their duals. In particular, we introduce a new family of highly time-scale localized spaces on the upper half-space. We study the wavelet synthesis operator (the left-inverse of the wavelet transform) and obtain the resolution of identity (Calderón reproducing formula) in the context of ultradistributions.

1. Introduction

One of the most useful concepts in time-frequency analysis for signal analysts and engineers is the wavelet series expansion of a signal. The coefficients in such series, representing the discrete version of a signal, are then used in the signal analysis, processing and synthesis. The continuous versions of these discrete representations lead to the wavelet (analysis) transform \( \mathcal{W}_\psi \) and the wavelet synthesis operator \( \mathcal{M}_\phi \), cf. Section 3. The authors were studied both transforms in several papers, [24]-[28], [32]. Although the continuous transforms in the literature are less popular then their discrete counterparts, from our point of view studying the intrinsic properties of continuous transforms is also important. In particular, continuous transforms potentially may serve well in the study of microlocal aspect of a signal, cf. [27]. These aspect are recently studied by different authors by the use of shearlet transforms instead of the wavelet transforms, see e.g. [8, 13].

It is well known that smooth orthonormal wavelets cannot have exponential decay, cf. [7, 9, 13]. In this paper we study the wavelet transform defined by wavelets with almost exponential decay. In this context it is natural to work with Gelfand-Shilov spaces as a background for the
study. We shall prove continuity theorems for the wavelet transform and the wavelet synthesis operator on spaces of Gelfand-Shilov type, see Section 2 for definitions. In contrast to known results [14, 22, 24, 28], we introduce a new family of (semi-)norms with additional parameters in the corresponding wavelet image space. These parameters measure fast decay or growth orders of the wavelet transform and wavelet synthesis operator. Roughly speaking, our considerations are able to detect Gevrey ultradifferentiability properties (such as analyticity) via appropriate decay of the wavelet transform.

The Gelfand-Shilov spaces were originally introduced in [10] as a tool to treat existence and uniqueness questions for parabolic initial-value problems. Exponential decay and holomorphic extension of solutions to globally elliptic equations in terms of Gelfand-Shilov spaces have been recently studied in [2, 3], see also [1]. We refer to [21] for an overview of the results in this direction and for applications in quantum mechanics and traveling waves. On the other hand, in the context of time-frequency analysis, the Gelfand-Shilov spaces have recently captured much attention in connection with modulation spaces [11, 12], localization operators [6], and the corresponding pseudodifferential calculus [30, 31].

We remark that the wavelet transform in the context of Gelfand-Shilov spaces was already studied in [22, 23] in dimension $n = 1$. In the present article we propose and develop an intrinsically different approach, which also covers the multidimensional case. We employ here wavelets with all vanishing moments. The advantage of this condition is that one is able to translate ultradifferentiability and subexponential decay of functions into sharper localization properties in the scale variable of the wavelet transform. Our approach also enables to provide interpretation to the resolution of the identity (Calderón reproducing formula) for ultradistributions. As a matter of fact, the inversion formula for the wavelet transform of ultradistributions seems to be out of reach of the results from [22, 23].

We note that the number of vanishing moments (called cancellations in [20]) of a wavelet $\psi$ is intimately related to the order of approximation of the corresponding wavelet series via the so-called Strang-Fix condition. In particular, wavelets with many vanishing moments are appropriate when dealing with objects which are very regular except for a few isolated singularities, cf. [7, 19]. It is also well known [13] that as soon as an orthogonal wavelet belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ then all its moments must vanish. In [14] wavelets with all vanishing moments were used to develop a distributional framework for the wavelet transform in the context of Lizorkin spaces.
The paper is organized as follows. In Section 2 we explain some facts about Gelfand-Shilov type spaces. In particular, we introduce a new four-parameter family $S_{t_1, t_2, \tau_1, \tau_2}(\mathbb{H}^{n+1})$ of function spaces on the upper-half space and study its properties (see Subsection 2.1). Section 3 contains our main continuity results, Theorems 1 and 2, which imply Calderón reproducing formulas for ultradistributions (Theorem 3 and Corollary 1). Finally, in Section 4 we collect the proofs of the main results.

1.1. Notation and notions. We denote by $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ the upper half-space and $\mathbb{N} = \{0, 1, 2, \ldots\}$. The unit sphere in $\mathbb{R}^n$ is denoted by $S^{n-1}$. When $x, y \in \mathbb{R}^n$ and $m \in \mathbb{N}^n$, $|x|$ denotes the Euclidean norm, $\langle x \rangle = (1 + |x|^2)^{1/2}$, $xy = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$, $x^m = x_1^{m_1} \cdots x_n^{m_n}$, $m! = m_1! m_2! \cdots m_n!$, $\partial^m = \partial_{x_1}^{m_1} \cdots \partial_{x_n}^{m_n}$, and $\triangle$ denotes the Laplacian: $\triangle = \triangle_x = \partial_x^2 + \cdots + \partial_{x_n}^2$. By a slight abuse of notation, the length of a multi-index $m \in \mathbb{N}^n$ is denoted by $|m| = m_1 + \cdots + m_n$, and the meaning of $|\cdot|$ shall be clear from the context of its use. We denote by $C, h, \ldots$ positive constants which may be different in various occurrences; $A \lesssim B$ means that $A \leq C \cdot B$ for some positive constant $C$. If $A \lesssim B$ and $B \lesssim A$ we write $A \asymp B$. The dual pairing between a test function space $\mathcal{A}$ and its dual $\mathcal{A}'$ is denoted by $\mathcal{A}(\cdot, \cdot)_{\mathcal{A}}$.

When $\alpha$ and $\beta$ are multi-indices and $n$ is the space dimension, we have

$$|\alpha|! \leq n^{|\alpha|} \alpha! , \quad \alpha! \beta! \leq (\alpha + \beta)! \leq 2^{|\alpha| + |\beta|} \alpha! \beta! .$$

2. Gelfand-Shilov type spaces

In this section we discuss definitions and properties of the test function spaces that will be employed in our study of the wavelet transform.

For the reader’s convenience, and in order to be self-contained, we first recall various spaces of rapidly decreasing functions that were considered in the context of wavelet transform in e.g. [14, 24].

The Schwartz space of rapidly decreasing smooth test functions is denoted as $\mathcal{S}(\mathbb{R}^n)$. The moments of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ are denoted by $\mu_m(\varphi) = \int_{\mathbb{R}^n} x^m \varphi(x) dx$, $m \in \mathbb{N}^n$. We fix constants in the Fourier transform as $\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-ix \cdot \xi} dx$, $\xi \in \mathbb{R}^n$.

The Lizorkin space $\mathcal{S}_0(\mathbb{R}^n) = \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \mu_m(\varphi) = 0, \forall m \in \mathbb{N}^n \}$ is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$ provided with the relative topology inherited from $\mathcal{S}(\mathbb{R}^n)$, [14, 29].
The space $\mathcal{S}(\mathbb{H}^{n+1})$ of “highly localized functions over the half-space” \cite{14} consists of those $\Phi \in C^\infty(\mathbb{H}^{n+1})$ such that the seminorms

$$p_{\alpha,\beta}^{l,k}(\Phi) = \sup_{(b,a)\in\mathbb{H}^{n+1}} \left( a^l + \frac{1}{a^l} \right) \langle b \rangle^k \left| \partial_x^\alpha \partial_y^\beta \Phi(b, a) \right|$$

(1)

are finite for all $l, k, \alpha \in \mathbb{N}$ and for all $\beta \in \mathbb{N}^n$.

When $(b, a) \in \mathbb{H}^{n+1}$ and $k, l \in \mathbb{N}$, then $\left( a^l + \frac{1}{a^l} \right) \asymp (a + \frac{1}{a})^l$ and $\langle b \rangle^k \asymp |b|^k$ when $|b|$ is large enough, see also Remark 1 below.

We introduce spaces of Gelfand-Shilov type in an analogy to $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}_0(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{H}^{n+1})$. The family of spaces $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ was introduced by I. M. Gelfand and G. E. Shilov in the study of uniqueness of the Cauchy problems and systematically studied in \cite{10}, see \cite{21} for a recent survey.

Recall that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ belongs to the Gelfand-Shilov space $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$, $\rho_1, \rho_2 > 0$, if there exists a constant $h > 0$ such that

$$|x^{\alpha} \varphi^{(\beta)}(x)| \lesssim h^{\rho_2 - |\alpha + \beta|} \alpha^{\rho_2} \beta! \rho_1, \quad x \in \mathbb{R}^n, \quad \alpha, \beta \in \mathbb{N}^n.$$

The space $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ is nontrivial if and only if $\rho_1 + \rho_2 \geq 1$. The family of norms

$$p^h_{\rho_1, \rho_2}(\varphi) = \sup_{x \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}^n} \frac{h^{\rho_2 - |\alpha + \beta|}}{\alpha! \beta!} |x^{\alpha} \varphi^{(\beta)}(x)|, \quad h > 0,$$

(2)

defines the canonical inductive topology of $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$.

It is well known \cite{3} that $\varphi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ if and only if there exists $h > 0$ such that

$$\sup_{x \in \mathbb{R}^n} e^{h|x|^{1/\rho_2}} |\varphi(x)| < \infty \quad \text{and} \quad \sup_{\xi \in \mathbb{R}^n} e^{h|\xi|^{1/\rho_1}} |\hat{\varphi}(\xi)| < \infty.$$

Hence, the Fourier transform is an isomorphism between $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ and $\mathcal{S}^{\rho_2}_{\rho_1}(\mathbb{R}^n)$.

The space $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ is non-quasianalytic, i.e. it contains compactly supported functions, if and only if $\rho_1 > 1$. When $\rho_1 = 1$, all elements of $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ are real analytic and when $0 < \rho_1 < 1$, its elements are actually entire functions.

Remark 1. We will often use equivalent sequences of norms where in (2) (and in other similar situations) $x^{\alpha} \varphi^{(\beta)}(x)$ is replaced by $\langle x \rangle^{\alpha} \varphi^{(\beta)}(x)$, $(\langle x \rangle^{\alpha} \varphi(x))^{(\beta)}$ or $(x^{\alpha} \varphi(x))^{(\beta)}$. Moreover, instead of the supremum norm any $L^p$ norm ($1 \leq p < \infty$) gives rise to an equivalent topology on $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ (cf. \cite{11} Ch 2.5).

We denote by $(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$ the closed subspace of $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ given by

$$(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n) = \left\{ \varphi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n) : \mu_m(\varphi) = 0, \forall m \in \mathbb{N}^n \right\}.$$
The wavelet transforms in Gelfand-Shilov spaces

One can show that \((S_{\rho_1}^0(\mathbb{R}^n), \rho_1, \rho_2 > 0, \) is nontrivial if and only if \(\rho_2 > 1\) (cf. [10]).

2.1. Gelfand-Shilov type spaces on the upper half-space. In this subsection we introduce a new scale of function which describes sharp subexponential localization over the upper half-space. Our spaces refine the space \(S(\mathbb{H}^{n+1})\) discussed above. To this end, we employ parameters which measure the decay properties of a function with respect to the scaling variable \(a > 0\) at zero and at infinity as well as their Gevrey ultradifferentiability and decay properties in the localization variable \(b\). While the seminorms in (1) measure polynomial decay of a certain order with respect to the scaling parameter \(a > 0\) at zero and at infinity, the seminorms in (3) may detect (super- and sub-) exponential decay of different orders at zero and at infinity.

Definition 1. Let \(s, t, \tau_1, \tau_2 > 0\). A smooth function \(\Phi\) belongs to \(S_{s,t,\tau_1,\tau_2}(\mathbb{H}^{n+1})\), if for every \(\alpha \in \mathbb{N}\) there exists a constant \(h > 0\) such that

\[
p_{s,t,\tau_1,\tau_2}^{\alpha}(\Phi) = \sup_{(b,a), (k,l_1,l_2), \beta} \left| \frac{h^{|\beta|+k+l_1+l_2}}{\beta! k! l_1! l_2! \tau_1 \tau_2} \left( a^{l_1} + \frac{1}{a^{l_2}} \right) \langle b \rangle^k \partial_a^{\alpha} \partial_b^{\beta} \Phi(b,a) \right| < \infty,
\]

where the supremum is taken over

\[((b,a), (k,l_1,l_2), \beta) \in \Lambda = \mathbb{H}^{n+1} \times \mathbb{N}^3 \times \mathbb{N}^n.\]

The topology of \(S_{s,t,\tau_1,\tau_2}(\mathbb{H}^{n+1})\) is defined via the family of seminorms (3), as inductive limit with respect to \(h\) and projective limit with respect to \(\alpha\).

The space \(S_{s,t,\tau_1,\tau_2}(\mathbb{H}^{n+1})\) is nontrivial if and only if \(s + t \geq 1\), which can be proved as follows.

Consider the set of smooth functions in \(\mathbb{H}^{n+1}\) of the form \(\Phi(b,a) = g(b)f(a), b \in \mathbb{R}^n, a \in \mathbb{R}_+.\) Then \(p_{s,t,\tau_1,\tau_2}(\Phi) < \infty\) is equivalent to \(p_h(g) < \infty\) and

\[
\sup_{a > 0, l_1, l_2 \in \mathbb{N}} \left| \frac{h^{l_1+l_2}}{l_1! l_2! \tau_1 \tau_2} \left( a^{l_1} + \frac{1}{a^{l_2}} \right) |\partial_a^{l_1} f(a)| \right| < \infty.
\]

Thus, if \(s + t \geq 1\), then \(S_{s,t,\tau_1,\tau_2}(\mathbb{H}^{n+1})\) is non-trivial, \(\tau_1, \tau_2 > 0\). For example, if \(g \in S_t(\mathbb{R}^n), then \mathbb{H}^{n+1} \ni (b,a) \mapsto e^{-a^{1/\tau_1}-a^{-1/\tau_2}} g(b) \in S_{s,t,\tau_1,\tau_2}(\mathbb{H}^{n+1}).\)

Since for fixed \(a \in \mathbb{R}_+\) and \(l_1 = l_2 = \alpha = 0\), it follows from (3) that \(\Phi(\cdot,a) \in S_t(\mathbb{R}^n), we see that the condition \(s + t \geq 1\) is also necessary for the non-triviality of \(S_{s,t,\tau_1,\tau_2}(\mathbb{H}^{n+1}).\)
Obviously, the family \( \mathcal{S}_{s_1, s_2}^s (\mathbb{H}^{n+1}) \) is increasing with respect to parameters \( s, t, \tau_1, \tau_2 \). The parameters \( \tau_1 \) and \( \tau_2 \) measure the behavior of \( \Phi \in \mathcal{S}_{s_1, s_2}^{s} (\mathbb{H}^{n+1}) \), with respect to \( a > 0 \) at infinity and at zero, respectively.

It can be shown that all these spaces of test functions are closed under multiplication by polynomials, partial differentiation, translation and dilation, cf. [10] for \( \mathcal{S}_{\rho}^p (\mathbb{R}^n) \). The following lemma can be proved in the same way as it is done in Chapter IV 6.2 for \( \mathcal{S}_{\rho}^p (\mathbb{R}^n) \), we therefore omit its proof.

**Lemma 1.** Let \( \Phi \in C^\infty (\mathbb{H}^{n+1}) \) and let \( \mathcal{F}_1 \Phi \) denote its Fourier transform with respect to the first variable:

\[
\mathcal{F}_1 \Phi (\xi, a) = \int_{\mathbb{R}^n} e^{-i \xi b} \Phi (b, a) \, db, \quad (\xi, a) \in \mathbb{H}^{n+1}.
\]

Then \( \Phi \in \mathcal{S}_{s_1, s_2}^{s} (\mathbb{H}^{n+1}) \) if and only if \( \mathcal{F}_1 \Phi \in \mathcal{S}_{s_1, s_2}^{s} (\mathbb{H}^{n+1}) \). Furthermore, \( \mathcal{F}_1 \) is a topological isomorphism between \( \mathcal{S}_{s_1, s_2}^{s} (\mathbb{H}^{n+1}) \) and \( \mathcal{S}_{s_1, s_2}^{s} (\mathbb{H}^{n+1}) \).

Next, we show that (3) precisely describes the rate of decay of the derivatives of \( \Phi \).

**Proposition 1.** Let \( \Phi \in \mathcal{S}_{s_1, s_2}^{s} (\mathbb{H}^{n+1}) \) and \( \alpha \in \mathbb{N} \). Set

\[
q_{\alpha, \rho}^{s_1, \rho} (\Phi) := \sup_{(b, \rho) \in \mathbb{H}^{n+1} \times \mathbb{R}^n} \frac{|b|^{1/1+s_1+a_1+1/2 + |b|^{1/1+s_1}}} {\rho^{1/(a_2 + 1/2 + |b|^{1/1+s_1})}} \left| \partial_{a_2}^\alpha \partial_{b}^\beta \Phi (b, a) \right|.
\]

Then \( p_{\alpha, \rho}^{s_1, \rho} (\Phi) < \infty \) for some \( \rho > 0 \), if and only if \( q_{\alpha, \rho}^{s_1, \rho} (\Phi) < \infty \) for some \( \rho > 0 \).

**Proof.** Assume that \( p_{\alpha, \rho}^{s_1, \rho} (\Phi) < \infty \) for some \( \rho > 0 \). Then, for any given \( l_1, l_2, k \in \mathbb{N} \),

\[
\frac{|b|^{1/1+s_1+a_1+1/2 + |b|^{1/1+s_1}}} {\rho^{1/(a_2 + 1/2 + |b|^{1/1+s_1})}} \left| \partial_{a_2}^\alpha \partial_{b}^\beta \Phi (b, a) \right|
\]

is uniformly bounded on \( \mathbb{H}^{n+1} \). This implies that appropriate summations over \( l_1, l_2 \) and \( k \) are also uniformly bounded. Indeed, the estimate

\[
C^{-1} e^{(r-\varepsilon) \eta^{1/r}} \leq \sum_{j=0}^{\infty} \frac{\eta^j} {j!} \leq C e^{(r+\varepsilon) \eta^{1/r}}, \quad \forall \eta \geq 0,
\]

which holds for every \( r, \varepsilon > 0 \) and for some \( C = C (r, \varepsilon) > 0 \), yields

\[
|\partial_{a_2}^\alpha \partial_{b}^\beta \Phi (b, a)| \lesssim \frac{\eta^j |b|^{1/1+s_1+a_1+1/2 + |b|^{1/1+s_1}}} {\rho^{1/(a_2 + 1/2 + |b|^{1/1+s_1})}}, \quad (b, a) \in \mathbb{H}^{n+1}, \beta \in \mathbb{N}^n,
\]
for some $\tilde{h} > 0$. By taking the corresponding supremum, we obtain that $q^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi)$ is finite for some $h > 0$.

Conversely, assume that $q^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) < \infty$ for some $h > 0$. Employing the same estimate as above, we conclude that

$$(1 + a^1) |\partial_\alpha^a \partial_\beta^b \Phi(b,a)| \lesssim h^{[\beta]+l_1} \beta! l_1! \tau_1,$$

$$(1 + \frac{1}{a^{l_2}}) |\partial_\alpha^a \partial_\beta^b \Phi(b,a)| \lesssim h^{[\beta]+l_2} \beta! l_2! \tau_2,$$

and

$$\langle b \rangle^k |\partial_\alpha^a \partial_\beta^b \Phi(b,a)| \lesssim h^{[\beta]+l_1+l_2+k} \beta! l_1! l_2! \tau_1 \tau_2 k!,$$

for every $((b,a), (k,l_1,l_2), \beta) \in \Lambda$. Hence,

$$\left( a^{l_1} + \frac{1}{a^{l_2}} \right) \langle b \rangle^k |\partial_\alpha^a \partial_\beta^b \Phi(b,a)|^3 \lesssim h^{3[\beta]+l_1+l_2+k} \beta! l_1! l_2! \tau_1 \tau_2 k!),$$

i.e.

$$\tilde{h}^{[\beta]+l_1+l_2+k} \beta! l_1! l_2! \tau_1 \tau_2 k! \left( a^{l_1} + \frac{1}{a^{l_2}} \right) \langle b \rangle^k |\partial_\alpha^a \partial_\beta^b \Phi(b,a)| < C$$

for some $\tilde{h}, C > 0$. By taking the supremum over $\Lambda$, we obtain that $p^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi)$ is finite for some $h > 0$.

□

3. Wavelet transform of ultradifferentiable functions and ultradistributions

In this section we study the continuity properties of the wavelet transform in different contexts. In particular we derive the resolution of identity formula in a class of tempered ultradistributions. As mentioned in the introduction, the most technical proofs are postponed to Section 4.

3.1. Continuity theorems. A function $\psi \in S^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ is called a wavelet if $\mu_0(\psi) = 0$. The wavelet transform of a tempered ultradistribution $f \in (S^{\rho_1}_{\rho_2}(\mathbb{R}^n))'$ with respect to the wavelet $\psi \in S^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ is defined via

$$\mathcal{W}_\psi f(b,a) = \left< f(x), \frac{1}{a^n} \bar{\psi} \left( \frac{x-b}{a} \right) \right> = \frac{1}{a^n} \int_{\mathbb{R}^n} f(x) \bar{\psi} \left( \frac{x-b}{a} \right) dx,$$

where $(b,a) \in \mathbb{H}^{n+1}$. In fact, if $\psi$ is a test function and the dual pairing in (5) makes sense, then we call $\mathcal{W}_\psi f$ the wavelet transform of $f$ with respect to $\psi$.

We first focus our attention on the continuity properties of the wavelet transform when the analyzing function belongs to a space of ultradifferentiable functions.
Theorem 1. Let $\rho_1 > 0$, $\rho_2 > 1$ and let $s > 0$, $t > \rho_1 + \rho_2$, $\tau_1 > \rho_1$ and $\tau_2 > \rho_2 - 1$. Then the wavelet mapping 

$$\mathcal{W} : (S_{\rho_2}^{\rho_1})_0(\mathbb{R}^n) \times (S_{1-\rho_1,1-\rho_2,\tau_1}^{s-\rho_2-\rho_1})_0(\mathbb{R}^n) \to S_{t,\tau_1,\tau_2}^s(\mathbb{H}^{n+1}),$$

given by $\mathcal{W} : (\psi, \varphi) \mapsto \mathcal{W}_\psi \varphi$, is continuous.

Remark 2. If we choose $\tau_1 = t - \rho_2$ and $\tau_2 = s + \rho_2 - 1$ in Theorem 1 then 

$$\mathcal{W} : (S_{\rho_2}^{\rho_1})_0(\mathbb{R}^n) \times (S_{t-\rho_2}^s)_0(\mathbb{R}^n) \to S_{t,\rho_2,s+\rho_2-1}^s(\mathbb{H}^{n+1})$$

is continuous.

In view of Plancherel’s theorem, we have 

$$\mathcal{W}_\psi \varphi(b, a) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ib\xi} \overline{\psi}(a\xi) \hat{\varphi}(\xi) d\xi, \quad (b, a) \in \mathbb{H}^{n+1}.$$ 

This implies that $\mathcal{F}_1 \mathcal{W}_\psi \varphi(\xi, a) = \overline{\psi}(a\xi) \hat{\varphi}(\xi)$, $(\xi, a) \in \mathbb{H}^{n+1}$ and, for $(b, a) \in \mathbb{H}^{n+1}$,

$$\partial^\beta \mathcal{W}_\psi \varphi(b, a) = \int_{\mathbb{R}^n} \varphi^{(\beta)}(ax + b) \overline{\psi}(x) dx = i^{\|\beta\|} \int_{\mathbb{R}^n} e^{ib\xi} \hat{\varphi}(\xi) \overline{\psi}(a\xi) d\xi.$$ 

Note that $\psi \in (S_{\rho_2}^{\rho_1})_0(\mathbb{R}^n)$ implies $\int b^\gamma \mathcal{W}_\psi \varphi(b, a) db = 0$, $\gamma \in \mathbb{N}^n$.

In order to construct the left-inverse for the wavelet transform, we proceed as follows. The wavelet synthesis transform of $\Phi \in S_{t,\tau_1,\tau_2}^s(\mathbb{H}^{n+1})$, $s, t, \tau_1, \tau_2 > 0$, $s + t \geq 1$, with respect to $\phi \in (S_{\rho_2}^{\rho_1})_0(\mathbb{R}^n)$, $\rho_1 > 0$, $\rho_2 > 1$, is defined by

$$\mathcal{M}_\phi \Phi(x) = \int_0^\infty \left( \int_{\mathbb{R}^n} \Phi(b, a) \frac{1}{a^n} \phi \left( \frac{x - b}{a} \right) db \right) \frac{da}{a}, \quad x \in \mathbb{R}^n.$$ 

Theorem 2. Let $\rho_1 > 0$, $\rho_2 > 1$ and let $s > 0$, $t > \rho_2$ and $\tau > 0$. Then the bilinear mappings 

a) $\mathcal{M} : (S_{\rho_2}^{\rho_1})_0(\mathbb{R}^n) \times S_{t-\rho_2,s-\rho_1}^s(\mathbb{H}^{n+1}) \to (S_t^s)_0(\mathbb{R}^n)$, when $s > \rho_1$;

b) $\mathcal{M} : (S_{\rho_2}^{\rho_1})_0(\mathbb{R}^n) \times S_{t,\tau_2}^s(\mathbb{H}^{n+1}) \to (S_t^s)_0(\mathbb{R}^n)$,

given by $\mathcal{M} : (\phi, \hat{\Phi}) \mapsto \mathcal{M}_\phi \Phi$, are continuous.

Remark 3. 1. It will be seen from the proof of Theorem 2 that a more general statement holds true. In fact, $\mathcal{M}$ can actually be extended to a continuous mapping from $S_{\rho_2}^{\rho_1}(\mathbb{R}^n) \times S_{t,\tau_2}^s(\mathbb{H}^{n+1})$ or from $S_{\rho_2}^{\rho_1}(\mathbb{R}^n) \times S_{t,\tau_2}^s(\mathbb{H}^{n+1})$ to $S_t^s(\mathbb{R}^n)$. However, we will only use wavelets with all vanishing moments in the rest of this article.

2. The continuity properties from Theorem 2 a) and b) provide information about high regularity and the decay properties of $\mathcal{M}_\phi \Phi$. In the notation of Gelfand-Shilov spaces the upper index is related
to Gevrey ultradifferentiability while the lower index is related to the
decay of a function. Note that when \( s = 1 \), the function \( \mathcal{M}_\psi \Phi \) is real
analytic and if \( 0 < s < 1 \), it extends to an entire function on \( \mathbb{C}^n \). The
index \( t \) gives subexponential decay at rate \( e^{-h|x|^{1/t}} \), for some \( h > 0 \). In
Theorem 2 a) the regularity of the image \( \mathcal{M}_\psi \Phi \) is measured in terms
of the regularity of the wavelet \( \phi \) and the decay of \( \Phi \) when \( a > 0 \) tends
to zero, while Theorem 2 b) shows that the regularity of \( \Phi \) is preserved
under the action of the synthesis operator. Similarly, the decay of \( \mathcal{M}_\psi \Phi \)
at infinity is related to the corresponding decays of \( \phi \) and \( \Phi \) in both
Theorem 2 a) and b).

The importance of the wavelet synthesis operator follows from the
fact that it can be used to construct a left inverse for the wavelet
transform, whenever the wavelet possesses nice reconstruction prop-
erties. We end this subsection with a necessary and sufficient condi-
tion for such property to hold in the context of Gelfand-Shilov spaces.

We start with some terminology. We say that a wavelet \( \psi \in S_0(\mathbb{R}^n) \)
admits a reconstruction wavelet \( \phi \in S_0(\mathbb{R}^n) \) if

\[
c_{\psi,\phi}(\omega) = \int_0^\infty \overline{\hat{\psi}(r\omega)} \hat{\phi}(r\omega) \frac{dr}{r}, \quad \omega \in S^{n-1},
\]

is finite, non-zero, and independent of the direction \( \omega \in S^{n-1} \). In such
a case we write \( c_{\psi,\phi} := c_{\psi,\phi}(\omega) \).

For example, if \( \psi \in S_0(\mathbb{R}^n) \) is non-trivial and rotation invariant,
then it is its own reconstruction wavelet. In fact, the existence of a
reconstruction wavelet is equivalent to non-degenerateness in the sense
of the following definition (see [26, Proposition 5.1]).

**Definition 2.** ([25, 26]) A test function \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) is said to be non-degenerate
if for any \( \omega \in S^{n-1} \) the function \( R_\omega(r) = \hat{\varphi}(r\omega) \), \( r \in [0, \infty) \)
is not identically zero, that is, \( \supp R_\omega \neq \emptyset \), for each \( \omega \in S^{n-1} \). If in
addition \( \mu_0(\varphi) = 0 \), then \( \varphi \) is called a non-degenerate wavelet.

We can now state the reconstruction formula for the wavelet trans-
form (cf. [14, Theorem 14.0.2]). If \( \psi \in \mathcal{S}_0(\mathbb{R}^n) \) is non-degenerate and
\( \phi \in \mathcal{S}_0(\mathbb{R}^n) \) is a reconstruction wavelet for it, then

\[
\varphi = \frac{1}{c_{\psi,\phi}} \mathcal{M}_\psi \mathcal{W}_\psi \varphi, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).
\]

We are interested in wavelets in Gelfand-Shilov spaces. The ensuing
lemma shows that if the non-degenerate wavelet \( \psi \) possesses higher reg-
ularity properties, then it is possible to choose a reconstruction wavelet
with the same regularity as \( \psi \).
Lemma 2. Let \( \psi \in (S_{\rho_2}^0(\mathbb{R}^n), \rho_1 > 0, \rho_2 > 1 \), be non-degenerate. Then, it admits a reconstruction wavelet \( \phi \) such that \( \phi \in (S_{\rho_2}^* \rho_1(\mathbb{R}^n) \), \( \forall \tau > 0 \).

Proof. The proof is similar to that of [26, Proposition 5.1], but taking care of the regularity properties of the reconstruction wavelet.

Since \( \psi \) is non-degenerate, there are \( 0 < r_1 < r_2 \) such that supp \( R_\omega \cap [r_1, r_2] \neq \emptyset \), \( \forall \omega \in \mathbb{S}^{n-1} \), where \( R_\omega \) are the functions from Definition 2. Let \( \eta \in \cap_{\tau > 0} S_{\tau}^\rho(\mathbb{R}^n) \) be a compactly supported nonnegative rotation-invariant function with \( 0 \notin \text{supp} \eta \) and \( \eta(\xi) = 1 \) for \( r_1 \leq |\xi| \leq r_2 \).

We write \( g(\omega) = \int_0^\infty \eta(r) |\hat{\psi}(r\omega)|^2 r^{-1} dr > 0 \). A straightforward computation shows that \( g \in \mathcal{E}((p!)^{\rho_2}) (\mathbb{S}^{n-1}) \), the Gevrey class of \( \{(p!)^\rho \} \)-ultradifferentiable functions on the unit sphere \( \mathbb{S}^{n-1} \) (see [16] for the definition of Gevrey classes). Then \( 1/g \in \mathcal{E}((p!)^{\rho_2}) (\mathbb{S}^{n-1}) \), see [17, Lemma 1]. Set \( \hat{\phi}(\xi) = \eta(\xi) \hat{\psi}(\xi)/g(\xi/|\xi|) \). Since the function \( \omega : \xi \mapsto \xi/|\xi| \) is analytic off the origin, it follows that \( 1/g(\xi/|\xi|) \in \mathcal{E}((p!)^{\rho_2}) \) away from the origin, cf. [15, Theorem 8.2.4]. Then \( \hat{\phi} \in \cap_{\tau > 0} S_{\tau}^\rho(\mathbb{R}^n) \), it is compactly supported, and all of its partial derivatives vanish at the origin. Thus \( \phi \in (S_{\rho_2}^\rho(\mathbb{R}^n) \}, \forall \tau > 0 \), the inverse Fourier transform of \( \hat{\phi} \). Clearly, \( c_{\psi, \phi} = c_{\psi, \phi}(\omega) = 1 \).

We give an example of a non-degenerate wavelet from \( (S_{\rho_2}^0(\mathbb{R}^n) \).

Example 1. Assume that \( \rho_1 > 0 \) and \( \rho_2 > 1 \). Let \( e_j = (0, 0, \ldots, 1, \ldots, 0) \), with 1 at the \( j \)-th coordinate, and let \( B_{\pm_j} = B(\pm \frac{1}{2} e_j, \frac{1}{2}) \), \( j = 1, 2, \ldots, n \) denote the closed balls centered at \( \pm \frac{1}{2} e_j \) with radius \( \frac{1}{2} \). Since the class \( S_{\rho_1}^\rho(\mathbb{R}^n) \) is non-quasianalytic, it contains compactly supported functions. Set \( \hat{\psi} = \sum_{j=-n}^{n} \hat{\phi}_j \), where the \( \hat{\phi}_{\pm_j} \in S_{\rho_1}^\rho(\mathbb{R}^n) \) are functions supported by \( B_{\pm_j} \), \( j = 1, 2, \ldots, n \), respectively, and positive in its interior. Then the function \( \psi \), the inverse Fourier transform of \( \hat{\psi} \), is an example of a non-trivial non-degenerate wavelet from \( (S_{\rho_2}^\rho(\mathbb{R}^n) \).

3.2. Wavelet transform of tempered ultradistributions. We start with a useful growth estimate for the wavelet transform of an ultradistribution. Recall, the wavelet transform of an ultradistribution \( f \) with respect to the test function \( \psi \) is given by (5) whenever the dual pairing is well defined.

Proposition 2. Let \( \rho_1, \rho_2 > 0, \rho_1 + \rho_2 \geq 1, s > \rho_1 \) and \( t > \rho_2 \). If \( \psi \in S_{\rho_2}^\rho(\mathbb{R}^n) \) and \( f \in (S_{\rho_1}^\rho(\mathbb{R}^n))' \), then for every \( k > 0 \),

\[
|\mathcal{W}_\psi f(b, a)| \lesssim e^{k \left( a^{1-\rho_2} + \frac{1}{s-\rho_1} + |b|^\frac{1}{s} \right)}, \quad (b, a) \in \mathbb{R}^{n+1}.
\]
Remark 4. Naturally, if \( \psi \in (S_{\rho_2}^s(\mathbb{R}^n))_0 \), then Proposition 2 remains valid for \( f \in (S_{\rho_2}^s(\mathbb{R}^n))' \). Furthermore, if \( B' \subset (S_{\rho_2}^s(\mathbb{R}^n))' \) is a bounded set (resp. \( B' \subset (S_{\rho_2}^s(\mathbb{R}^n))' \) when \( \psi \in (S_{\rho_2}^s(\mathbb{R}^n))_0 \)), then the conclusion of Proposition 2 holds uniformly for \( f \in B' \), as follows from the Banach-Steinhaus theorem.

Next, we give an alternative definition of the wavelet transform of an ultradistribution via duality.

**Definition 3.** Let \( \rho_1 > 0, t > \rho_2 > 1, s > 0 \) and \( \tau > 0 \). If \( \psi \in (S_{\rho_2}^s(\mathbb{R}^n))_0 \) and \( f \in ((S_{\rho_2}^s(\mathbb{R}^n))' \) then the wavelet transform \( \mathcal{W}_\psi f \) of \( f \) with respect to the wavelet \( \psi \) is defined as

\[
\langle \mathcal{W}_\psi f(b,a), \Phi(b,a) \rangle := \langle f(x), M_\psi \Phi(x) \rangle, \quad \Phi \in S_{t-\rho_2,\tau}(\mathbb{R}^{n+1}).
\]

Thus, \( \mathcal{W}_\psi : ((S_{\rho_2}^s(\mathbb{R}^n))' \rightarrow (S_{t-\rho_2,\tau}(\mathbb{R}^{n+1}))' is continuous for the strong dual topologies.
By Theorem 2(b), the transposition in (7) is well defined. Note that we have freedom of the choice of \( \tau \). This fact will be crucial below. If we assume that \( s > \rho_1 \), then the choice \( \tau = s - \rho_1 \) leads to the continuous mapping \( \mathcal{W}_\psi : ((S^s_{\rho_1})_0(\mathbb{R}^n))' \to (S^s_{t, t - \rho_2, s - \rho_1}(\mathbb{H}^{n+1}))' \). The next result shows the consistency between Definition 3 and (5) for this choice of \( \tau \).

**Proposition 3.** Assume that \( s > \rho_1 > 0 \) and \( t > \rho_2 > 1 \). Let \( f \in ((S^s_{\rho_1})_0(\mathbb{R}^n))' \) and \( \psi \in (S^s_{\rho_2})_0(\mathbb{R}^n) \). Then, for every \( \Phi \in S^s_{t, t - \rho_2, s - \rho_1}(\mathbb{H}^{n+1}) \),

\[
\langle f(x), \mathcal{M}_\psi \Phi(x) \rangle = \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi f(b, a) \Phi (b, a) \frac{dbda}{a}.
\]

Proof. Fix \( \Phi \in S^s_{t, t - \rho_2, s - \rho_1}(\mathbb{H}^{n+1}) \). Proposition 1 implies that there is \( h > 0 \) such that

\[
|\Phi(b, a)| \leq e^{-h \left(a^{\frac{1}{\rho_2}} b + a^{\frac{1}{\rho_1}} b^{\frac{1}{s}}\right)}, \quad (b, a) \in \mathbb{H}^{n+1}.
\]

Let \( \{f_j\}_{j=0}^\infty \) be a sequence such that \( f_j \to f \) in \(((S^s_{\rho_1})_0(\mathbb{R}^n))' \) and \( f_j \in \mathcal{S}_0(\mathbb{R}^n) \), for every \( j \in \mathbb{N} \). In view of Proposition 2 (cf. Remark 3),

\[
|\mathcal{W}_\psi f_j(b, a)| \leq e^{\frac{a^{\frac{1}{\rho_2}} + (\frac{1}{a^{\frac{1}{\rho_1}} + |b|^{\frac{1}{s}})}}}, \quad (b, a) \in \mathbb{H}^{n+1},
\]

uniformly in \( j \in \mathbb{N} \). Fubini’s theorem and the regularity of \( f_j \) imply

\[
\int_{\mathbb{R}^n} f_j(x) \mathcal{M}_\psi \Phi(x) dx = \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi f_j(b, a) \Phi (b, a) \frac{dbda}{a}.
\]

Noticing that \( \mathcal{W}_\psi f_j(b, a) \to \mathcal{W}_\psi f(b, a) \) pointwisely, the estimates (9) and (10) allow us to use the Lebesgue dominated convergence theorem in (11) to conclude (8). □

We also introduce the the wavelet synthesis transform of an ultradistribution on \( \mathbb{H}^{n+1} \) via a duality approach. The consistency of the following definition is ensured by Theorem 1 (cf. Remark 2).

**Definition 4.** Let \( \rho_1 > 0, \rho_2 > 1, s > 0 \) and \( t > \rho_1 + \rho_2 \). Let \( F \in (S^s_{t, t - \rho_2, s + \rho_2 - 1}(\mathbb{H}^{n+1}))' \) and \( \phi \in (S^s_{\rho_2})_0(\mathbb{R}^n) \). The wavelet synthesis transform \( \mathcal{M}_\phi F \) of \( F \) with respect to the wavelet \( \phi \) is defined by

\[
\langle \mathcal{M}_\phi F(x), \varphi(x) \rangle := \langle F(b, a), \mathcal{W}_\phi \varphi(b, a) \rangle, \quad \varphi \in (S^s_{t, t - \rho_2, s + \rho_2 - 1}(\mathbb{H}^{n+1}))'.
\]

Thus, \( \mathcal{M}_\phi : (S^s_{t, t - \rho_2, s + \rho_2 - 1}(\mathbb{H}^{n+1}))' \to ((S^s_{t + 1 - \rho_1 - \rho_2})_0(\mathbb{R}^n))' \) is continuous.

We derive the following resolution of the identity mapping \( \mathrm{Id} \) as an easy consequence of our previous results. In the next theorem we implicitly use the choice \( \tau = s + \rho_2 - 1 \) in Definition 3.
Theorem 3. Let \( \rho_1 > 0, \rho_2 > 1, s > 0 \) and \( t > \rho_1 + \rho_2 \). Let \( \psi \in (S^{\rho_1}_{\rho_2})_0(\mathbb{R}^n) \) be a non-degenerate wavelet and let \( \phi \in (S^{\rho_1}_{\rho_2})_0(\mathbb{R}^n) \) be a reconstruction wavelet for it. Then the Calderón reproducing formula

\[
\text{Id} = \frac{1}{c_{\psi, \phi}} \mathcal{M}_\phi \mathcal{W}_\psi
\]

holds in \(((S^s)^0(\mathbb{R}^n))'\).

Proof. Let \( f \in ((S^s)^0(\mathbb{R}^n))' \). Since \((S^s)^0(\mathbb{R}^n)\) is dense in the space \((S^s)^0(\mathbb{R}^n)\), it is enough to prove the identity for test functions \( \varphi \in (S^s_{t-\tau})_0(\mathbb{R}^n) \). Then, by Definitions 4 and 3 and the reconstruction formula (3), it follows that

\[
\langle \mathcal{M}_\phi \mathcal{W}_\psi f, \varphi \rangle = \langle \mathcal{W}_\psi f, \mathcal{W}_\psi \varphi \rangle = \langle f, \mathcal{M}_\psi \mathcal{W}_\varphi \rangle = c_{\psi, \varphi} \langle f, \varphi \rangle. \tag{12}
\]

Combining Remark 2, Proposition 3, and the relation (12), we obtain an extension of the desingularization formula now in the context of ultradistributions (cf. [14, 26] for the case of distributions).

Corollary 1. In addition to the assumptions of Theorem 3 suppose that \( \sigma := \rho_1 + \rho_2 - 1 < s \). Then,

\[
\langle f, \varphi \rangle = \frac{1}{c_{\psi, \phi}} \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi f(b, a) \mathcal{W}_\psi \varphi(b, a) d\varphi(b, a), \quad \forall \varphi \in (S^s_{t-\tau})_0(\mathbb{R}^n).
\]

As an immediate consequence of Theorem 3 and Theorem 2 b) we have the following regularity theorem for ultradistributions.

Corollary 2. Let \( \rho_1 > 0, \rho_2 > 1, s > 0 \) and \( t > \rho_1 + \rho_2 \). Let \( \psi \in (S^s_{\rho_2})_0(\mathbb{R}^n) \) be non-degenerate and let \( f \in ((S^s)^0(\mathbb{R}^n))' \). If \( \mathcal{W}_\psi f \in S^s_{t, t-\rho_2, \tau}(\mathbb{H}^n, +1) \) for some \( \tau > 0 \) then \( f \in (S^s)^0(\mathbb{R}^n) \).

4. Proofs of main results

This section collects the proofs of Theorems 1 and 2.

Remark 5. On several occasions we will use the following facts. If \( \varphi \in S^s_{\rho_2}(\mathbb{R}^n) \), \( \rho_1 + \rho_2 \geq 1 \), so that \( p_{h_0}^{\rho_1, \rho_2}(\varphi) < \infty \) for some \( h_0 > 0 \), then there exists \( h_1 > 0 \) such that

\[
\sup_{\alpha, \beta \in \mathbb{N}^n} \frac{h_1^{\alpha + \beta}}{\alpha! \rho_2 \beta! \rho_1} \int |x|^{\alpha} |\varphi^{(\beta)}(x)| dx \lesssim p_{h_0}^{\rho_1, \rho_2}(\varphi). \tag{13}
\]

In addition, if (13) holds, then there exists \( h_2 > 0 \) such that

\[
\sup_{\alpha, \beta \in \mathbb{N}^n} \frac{h_2^{\alpha + \beta}}{\alpha! \rho_2 \beta! \rho_1} |x|^{\alpha} |\varphi^{(\beta)}(x)| < \infty.
\]
We will omit the parts of the proofs where these arguments appear. We will often make use of the fact that multiplication by $| \cdot |^{[\alpha]}$ (or by $\langle \cdot \rangle_{[\alpha]}$) simply enlarges the corresponding constants $h > 0$. We will also use, without explicit reference, the estimate

$$
\sup_{\beta \in \mathbb{N}^n, x \in \mathbb{R}^n} \frac{h_0|\beta| (fg)^{(\beta + \alpha)}(x)}{\beta!} \lesssim \sup_{\beta \in \mathbb{N}^n, x \in \mathbb{R}^n} \frac{h_1|\beta| f^{(\beta)}(x)}{\beta!} \sup_{\beta \in \mathbb{N}^n, x \in \mathbb{R}^n} \frac{h_1|\beta| g^{(\beta)}(x)}{\beta!}
$$

for some $h_1 = h_1(\alpha) > 0$. Finally, we shall need the following form of the reminder term in the Taylor formula

$$(R_{\alpha,m} f)(x,y) = \sum_{|\alpha|=m} \frac{m(y-x)^{\alpha}}{\alpha!} \int_0^1 (1 - \theta)^{m-1} f^{(\alpha)}(x + \theta(y-x)) d\theta.$$ 

4.1. **Proof of Theorem 1** Let $\varphi$ and $\psi$ satisfy

$$p_h^{\min\{s,\tau_2-\rho_2+1\},1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)p_h^{\rho_1,\rho_2}(\psi) < \infty. \tag{14}$$

We will show that there exists $h_0 > 0$ such that $p_{\alpha,h_0}^{s,t,\tau_1,\tau_2}(\mathcal{W}_\psi \varphi) < \infty$, that is, the supremum of

$$J = \frac{h_0|\beta|+k+l_1+l_2(a^{l_1} + a^{-l_2}) (b)^{k}|\partial_a^\alpha \partial_b^\beta \mathcal{W}_\psi \varphi(b,a)|}{\beta!k!l_1!l_2!}$$

over

$$((b,a),(k,l_1,l_2),\beta) \in \Lambda = \mathbb{H}^{n+1} \times \mathbb{N}^3 \times \mathbb{N}^n$$

is finite for some $h_0 > 0$. Without loss of generality we assume from now on that $\alpha = 0$.

**Remark 6.** In the following steps of the proof we start with $h_0 > 0$ and in every step determine a new constant $h_1 \leq h_2 \leq \ldots \leq h_7$ which successively depends on the previous one, that is, $h_m$ depends on $h_{m-1}$, $m = 1, 2, \ldots, 7$ and $h_7$ should be equal to $h > 0$ in (14). Then, by going in the opposite direction, we determine $h_6$ from $h_7$, $h_5$ from $h_6$, $\ldots$ and $h_0$ from $h_1$. In such way for the constant $h > 0$ given in (14) we find $h_0 > 0$ so that

$$p_{0,h_0}^{s,t,\tau_1,\tau_2}(\mathcal{W}_\psi \varphi) = \sup_{\Lambda} J \lesssim p_h^{\min\{s,\tau_2-\rho_2+1\},1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)p_h^{\rho_1,\rho_2}(\psi),$$

which will prove the Theorem.

We will estimate

$$J_1 = \frac{h_0|\beta|+2k+2l_1(1+a^{2l_1}) (b)^{2k}|\partial_a^\beta \mathcal{W}_\psi \varphi(b,a)|}{\beta!k!l_1!}$$
over \(((b, a), (k, l_1), \beta) \in \Lambda_1 = \mathbb{H}^{n+1} \times \mathbb{N}^2 \times \mathbb{N}^n\), and

\[ J_2 = \frac{h_0^{[\beta]+2l_2} (1 + a^{-2l_2}) |\partial_\beta^l \mathcal{W}_\varphi(b, a)|}{\beta! l_2 ! 2^{2r_2}} \]

over \(((b, a), (k, l_2), \beta) \in \Lambda_2 = \mathbb{H}^{n+1} \times \mathbb{N}^2 \times \mathbb{N}^n\).

Since

\[ h_0^{2([\beta]+k+l_1+l_2)} (a^{l_1} + a^{-l_2})^2 (b)^{2k} |\partial_\beta^l \mathcal{W}_\varphi(b, a)|^2 \lesssim \sup_{\Lambda_1} J_1 \sup_{\Lambda_2} J_2 \]

we would have

\[ p_{0, h_0}^{s,t,\tau_1,\tau_2} (\mathcal{W}_\varphi \varphi) \lesssim \sqrt[\min]{} \sup_{\Lambda_1} J_1 \sup_{\Lambda_2} J_2. \]

We will show that there exists \( h_7 \) which depends on \( h_0 > 0 \) such that

\[ \sup_{\Lambda_1} J_1 \lesssim p_{h_7}^{s,1+\rho_1+\min\{t-\rho_2,\tau_1\}} (\varphi) p_{h_7}^{\rho_1,\rho_2} (\psi) \]

and

\[ \sup_{\Lambda_2} J_2 \lesssim p_{h_7}^{\min\{s,\tau_2-\rho_2+1\},t} (\varphi) p_{h_7}^{\rho_1,\rho_2} (\psi). \]

We first estimate \( \sup_{\Lambda_1} J_1 \). There exists \( h_1 = h_1(h_0) \) such that

\[ J_1 \lesssim h_1^{[\beta]+2k+2l_1} (1 + a^{2l_1}) (b)^{2k} |\int_{\mathbb{R}^n} e^{i\xi b} \left| 1 - \triangle_\zeta \right| \left( (\xi^\beta \hat{\varphi}(\xi) \hat{\psi}(a\xi) ) d\xi \right) | \]

\[ = \frac{h_1^{[\beta]+2k+2l_1}}{\beta! s (2k)! l_1 ! n} |\int_{\mathbb{R}^n} e^{i\xi b} \sum_{|\gamma| \leq 2k} c_\gamma \partial_\gamma^l (\xi^\beta \hat{\varphi}(\xi) (1 + a^{2l_1}) \hat{\psi}(a\xi)) d\xi | \]

\[ \lesssim \frac{h_1^{[\beta]+2k+2l_1}}{\beta! s (2k)! l_1 ! n} \sum_{|\gamma| \leq 2k} |c_\gamma| \sum_{i+j \leq \gamma} |\tilde{c}_{i,j}| \int_{\mathbb{R}^n} |\partial^i (\xi^\beta \hat{\varphi}(\xi)) a^{lj} (1 + a^{2l_1}) \hat{\psi}(a\xi)| d\xi, \]

where \( c_\gamma \) and \( \tilde{c}_{i,j} \) are correspondent binomial coefficients. As already noticed, by the use of Leibniz rule, the binomial coefficients simply increase the constant \( h_1 \) so that

\[ J_1 \lesssim \sup h_2^{[\beta]+2k+2l_1} \frac{1}{\beta! s (2k)! l_1 ! n} \sum_{i+j \leq 2k} (I_1 + I_2), \]

for some \( h_2 = h_2(h_1) > 0 \) which does not depend on \( \beta, k \) and \( l_1 \), where

\[ I_1 = \int_{|\xi| \leq 1} |\partial^i (\xi^\beta \hat{\varphi}(\xi)) a^{lj} (1 + a^{2l_1}) \hat{\psi}(a\xi)| d\xi, \]

and

\[ I_2 = \int_{|\xi| \geq 1} |\partial^i (\xi^\beta \hat{\varphi}(\xi)) a^{lj} (1 + a^{2l_1}) \hat{\psi}(a\xi)| d\xi, \]
and the supremum is taken over $\beta, k$ and $l_1$.

By Remarks 1 and 5 it follows that there exists $h_3 = h_3(h_2) > 0$, which does not depend on $\beta, i, j$ and $l_1$, such that

$$h_2^{[\beta]+i+j+2l_1} \lesssim \frac{h_3^{[\beta]+i+j+2l_1}}{\beta!^s|\beta|!} (2l_1)! I_2$$

$$\lesssim \frac{h_3^{[\beta]+i+j+2l_1}}{\beta!^s|\beta|!} \int_{\xi \geq 1} |\partial^\beta (\xi^{\beta} \hat{\varphi}(\xi))| |\xi|^n (1 + |\xi|)|\hat{\varphi}(\xi)(a\xi)| d\xi$$

$$\lesssim \sup_{h_3} \frac{h_3^{[\beta]+i+j}}{\beta!^s|\beta|!} \int_{\xi \geq 1} |\partial^\beta (\xi^{\beta} \hat{\varphi}(\xi))| \sup_{|j|!}(2l_1)! \int_{\mathbb{R}^n} |x||x|^{-n} |\hat{\varphi}(\xi)(x)| dx$$

$$\lesssim p_{h_3}(\varphi) p_{h_3}^{\min{t-\rho_2, \tau_1}, \rho_2}(\psi),$$

where the suprema are taken over $\beta, i, j, l_1$ and $\xi$, and we have split $|j|!$ into $|j|!^{t-\rho_2}$ and $|j|!^{\rho_2}$.

From the above calculations we conclude that there exists $h_3 > 0$ such that

$$\sup_{\Lambda_1} \frac{h_3^{[\beta]+2k+2l_1}}{\beta!^s(2k)!} (2l_1)! I_2 \lesssim p_{h_3}(\varphi) p_{h_3}^{\min{t-\rho_2, \tau_1}, \rho_2}(\psi).$$

Next, we estimate the term involving $I_1$:

$$I_1 \lesssim \sum_{p \leq i} \left( \begin{array}{c} i \\ p \end{array} \right) \frac{\beta!}{p!} \int_{\xi \geq 1} |\xi|^{\beta-p} |\hat{\varphi}(\xi)(a\xi)| d\xi$$

$$= \sum_{p \leq i} \left( \begin{array}{c} i \\ p \end{array} \right) \frac{\beta!}{p!} \int_{\xi \geq 1} |\xi|^{\beta-p} \sum_{|r|=2l_1+j-\rho} (2l_1+|j|) \frac{|x|^r}{r!} I \cdot a|j| (1 + a^{2l_1}) \hat{\varphi}(\xi)(a\xi) d\xi$$

where $I = \int_0^1 (1 - \theta)^{2l_1+j-\rho-1} \varphi^{(p+r)}(\theta \xi) d\theta$, and we have used Taylor’s formula for $\varphi$ and the vanishing moments of $\varphi$.

Since $\sum_{|r|=2l_1+j-\rho} \frac{1}{r!} \lesssim \frac{c^{2l_1+j}}{(2l_1)!|j|!}$ for some $c > 0$ and binomial coefficients just increase the constant $h_2$, we obtain

$$h_2^{[\beta]+2k+2l_1} \lesssim \frac{h_4^{[\beta]+i+j+2l_1}}{\beta!^s|\beta|!} (2l_1)! I_1$$

$$\lesssim \sup_{h_4} \frac{h_4^{[\beta]+i+j+2l_1}}{\beta!^s|\beta|!} \int_{\xi \geq 1} |\xi|^{\beta}(x) \int_{|x| \leq 1} a^n (1 + a\xi) |\hat{\varphi}(\xi)(a\xi)| d\xi$$

$$\lesssim \sup_{h_4} \frac{h_4^{[\beta]+i+j+2l_1}}{\beta!^s|\beta|!} \int_{\xi \geq 1} |\xi|^{\beta}(x) \int_{|x| \leq 1} a^n (1 + a\xi) |\hat{\varphi}(\xi)(a\xi)| d\xi$$

with $|r| = 2l_1 + |j|$, the suprema taken over $\beta, i, j, l_1$ and $x$, and where $h_4 > 0$ does not depend on $\beta, i, j, l_1$. Moreover, we may choose $h_4 \geq h_3$. 
By Remarks 1 and 5 and similar arguments to those used in the estimates of $I_2$ it follows that there exists $h_5 = h_5(h_4) > 0$ (which does not depend on $\beta, k$ and $l_1$) such that

$$
\frac{h_2^{\beta+2k+2l_1}}{\beta!(2k)!^l(2l_1)!^\tau} I_1 \lesssim \rho^{s_1,1-\rho_1+\min\{t,\rho_2,\tau_1\}}(\varphi) \rho^{\rho_1,\rho_2}(\psi).
$$

Since the sequence $h_0, h_1, \ldots, h_5$ is non-decreasing, we conclude that

$$
\sup_{A_1} J_1 \lesssim \rho^{s_1,1-\rho_1+\min\{t,\rho_2,\tau_1\}}(\varphi) \rho^{\rho_1,\rho_2}(\psi).
$$

It remains to estimate $\sup_{A_2} J_2$. We now use Taylor’s formula for $\varphi$ and the vanishing moments of $\psi$ to obtain, for $a < 1$,

$$
J_2 = \frac{h_0^{\beta+2l_2}(1 + a^{-2l_2})}{\beta!(2l_2)!^\tau_2} \left| \int_{\mathbb{R}^n} \left( \sum_{|r|=2l_2} \frac{2l_2}{r!} \left( \int_0^1 (1 - \theta)^{2l_2-1} \varphi(\beta+r)(b + \theta ax)d\theta \right) a^{2l_2} \varphi(\psi(x)) dx \right) \right|
$$

$$
\lesssim \frac{h_0^{\beta+2l_2}}{\beta!(2l_2)!^\tau_2} \sup_{x \in \mathbb{R}^n} \max_{|r|=2l_2} |\varphi(\beta+r)(x)| \sup_{x \in \mathbb{R}^n} |\langle x \rangle^{r+n+1} \varphi(\psi(x))|
$$

$$
\lesssim \rho^{\min\{s_2,\tau_2+1\}}(\varphi) \rho^{\rho_1,\rho_2}(\psi),
$$

for some $h_0 \geq h_5$ and $h_7 = h_7(h_0)$. When $a \geq 1$, we employ a similar argument (Taylor’s formula is not needed for this case).

Thus, we choose $h_7 = h > 0$ for which

$$
\rho^{\min\{s_2,\tau_2+1\},1-\rho_1+\min\{t,\rho_2,\tau_1\}}(\varphi) \rho^{\rho_1,\rho_2}(\psi) < \infty.
$$

Now, reasoning as in Remark 5, we determine for given $h_7 = h$ the corresponding $h_0 > 0$ so that

$$
P_{0,h_0}^{s,t,\tau_2}(\mathcal{W}_h \varphi) \lesssim P_{h_7}^{\min\{s_2,\tau_2+1\},1-\rho_1+\min\{t,\rho_2,\tau_1\}}(\varphi) \rho^{\rho_1,\rho_2}(\psi) < \infty,
$$

which proves the Theorem.

4.2. Proof of Theorem 2. We may again assume that $\alpha = 0$.

a) Let $h > 0$ be chosen so that

$$
P_{0,h}^{t,t,\tau_2,\rho_1}(\Phi) \rho^{\rho_1,\rho_2}(\phi) < \infty.
$$

By Lemma 1 it is enough to prove that there exists $h_0 > 0$ such that

$$
P_{0,h_0}^{s,t}(\mathcal{M}_\Phi \phi) \lesssim P_{h_0}^{t,t,\tau_2,\rho_1}(\mathcal{F}_1 \Phi) \rho^{\rho_1,\rho_2}(\phi).
$$

Let $\beta \in \mathbb{N}^n$ and $k \in \mathbb{N}$. We may assume that $k$ is even. Then

$$
\langle x \rangle^k \left| \partial_x^\beta (\mathcal{M}_\Phi \phi(x)) \right| = \langle x \rangle^k \left| \partial_x^\beta \left( \int_{\mathbb{R}^n} \Phi(b,a) \frac{1}{a^n} \phi \left( \frac{x-b}{a} \right) db \frac{da}{a} \right) \right|
$$

\[\]
which implies (15).

Theorem 1. In the corresponding steps of the proof we enlarge $h$ through $h_0 > 0$ and regroup the integrands in an appropriate way. In fact, by taking the supremum over $h_0$, one can show that there exist $h_1 > 0$ and $h_2 = h_2(h_1) > 0$, which do not depend on $\beta, q$ and $r$, such that

\[
\frac{h_0^{\beta+1-k}}{\beta!k!} \langle x \rangle^k | \partial_x^{\beta} (M_\phi \Phi(x)) | \lesssim \frac{h_0^{\beta+1-k}}{\beta!k!} \sum_{|r|+|q| \leq k} I \tag{16}
\]

where

\[
I = \int \int_{\mathbb{R}^n} a^{\beta} | \xi |^{\beta} | \partial_x^{\beta} \hat{\Phi}(\xi, a) | a^{\beta} \Phi(q) (a) | d\xi a \frac{da}{a}.
\]

We use again Remark 3 in a similar way as it was done in the proof of Theorem 1. In the corresponding steps of the proof we enlarge $h_0 > 0$ and regroup the integrands in an appropriate way. In fact, by taking the corresponding suprema, one can show that there exist $h_1 = h_1(h_0) > 0$ and $h_2 = h_2(h_1) > 0$, which do not depend on $\beta, q$ and $r$, such that

\[
\sup_{\beta \in \mathbb{N}_n, k \in \mathbb{N}_n} \frac{h_0^{\beta+1-k}}{\beta!k!} \langle x \rangle^k | \partial_x^{\beta} (M_\phi \Phi(x)) | \lesssim \frac{h_0^{\beta+1-k}}{\beta!k!} \sum_{|r|+|q| \leq k} I \tag{16}
\]

where the supremum is taken over $\xi, \eta, \lambda$ and $\theta$ (we have also used $a^{\beta} r^{\beta} q^{\beta} \Phi(q) (a) \lesssim a^{\beta} r^{\beta} q^{\beta} \Phi(q) (a)$).

By Remark 6 for $h_2 = h$, there exists $h_0 > 0$ such that

\[
\sup_{\beta \in \mathbb{N}_n, k \in \mathbb{N}_n} \frac{h_0^{\beta+1-k}}{\beta!k!} \langle x \rangle^k | \partial_x^{\beta} (M_\phi \Phi(x)) | \lesssim \frac{h_0^{\beta+1-k}}{\beta!k!} \sum_{|r|+|q| \leq k} I \tag{16}
\]

which implies (15).

b) Here we bound (16) by

\[
\sup_{\beta \in \mathbb{N}_n, k \in \mathbb{N}_n} \frac{h_0^{\beta+1-k}}{\beta!k!} \langle x \rangle^k | \partial_x^{\beta} (M_\phi \Phi(x)) | \lesssim \frac{h_0^{\beta+1-k}}{\beta!k!} \sum_{|r|+|q| \leq k} I \tag{16}
\]

where

\[
I = \int \int_{\mathbb{R}^n} a^{\beta} | \xi |^{\beta} | \partial_x^{\beta} \hat{\Phi}(\xi, a) | a^{\beta} \Phi(q) (a) | d\xi a \frac{da}{a}.
\]
for some $h_3 = h_3(h_0)$, and $h_4 = h_4(h_3)$, where the supremum is taken over $\xi, a, \beta, r$ and $q$. Once again by Remark 6 for $h_4 = h$, it follows that there exists $h_0 > 0$ such that

$$\sup_{\beta \in \mathbb{N}^n, k \in \mathbb{N}} \frac{h_0^{\beta + k}}{\beta! k! t} \sum_{|\tau| + |q| \leq k} I \lesssim p_{0, h}^{t, s, t - \rho_2, \tau} (\mathcal{F}_1 \Phi) p_{h}^{\rho_1, \rho_2} (\phi) < \infty,$$

which completes the proof.

**Acknowledgement**

S. Pilipović and N. Teofanov are supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia through Project 174024. D. Rakić is supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia through Project III44006 and by PSNTR through Project 114-451-2167.

**References**

[1] J. L. Bona, Y. A. Li, *Decay and analyticity of solitary waves*, J. Math. Pures Appl. (9) 76 (1997), 377–430.

[2] M. Cappiello, T. Gramchev, L. Rodino, *Sub-exponential decay and uniform holomorphic extensions for semilinear pseudodifferential equations*, Comm. Partial Differential Equations 35 (2010), 846–877.

[3] M. Cappiello, T. Gramchev, L. Rodino, *Entire extensions and exponential decay for semilinear elliptic equations*, J. Anal. Math. 111 (2010), 339–367.

[4] R. D. Carmichael, A. Kamiński, S. Pilipović, *Boundary values and convolution in ultradistribution spaces*, Series on analysis, applications and computation, Vol. 1, World Scientific Publishing Company Pte. Ltd., Hackensack, NJ, 2007.

[5] J. Chung, S.-Y. Chung, D. Kim, *Characterizations of the Gelfand-Shilov spaces via Fourier transforms*, Proc. Am. Math. Soc. 124 (1996), 2101–2108.

[6] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov, *Quasiaalytic Gelfand-Shilov spaces with application to localization operators*, Rocky Mountain J. Math. 40 (2010), 1123–1147.

[7] I. Daubechies, *Ten lectures on wavelets*, SIAM, Philadelphia, Pennsylvania, 1992.

[8] D. Donoho, G. Kutyniok, *Microlocal Analysis of the Geometric Separation Problem*, Comm. Pure Appl. Math. LXVI (1), 1–47 (2013)

[9] J. Dziubański, E. Hernández, *Band-limited wavelets with subexponential decay* Canad. Math. Bull. 41 (1998), 398–403.

[10] I. M. Gelfand, G. E. Shilov, *Generalized functions, Vols. II and III*, Academic Press, 1967.

[11] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[12] K. Gröchenig, G. Zimmermann, *Spaces of test functions via the STFT*, J. Funct. Spaces Appl. 2 (2004), 25–53.
[13] E. Hernández, G. Weiss, *A first course on wavelets*, CRC Press, Boca Raton, 1996.
[14] M. Holschneider, *Wavelets. An analysis tool*, The Clarendon Press, Oxford University Press, New York, 1995.
[15] L. Hörmander *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, 1983, 1990.
[16] H. Komatsu, *Ultradistributions I, structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA 20 (1973) 25–105.
[17] H. Komatsu, *Linear ordinary differential equations with Gevrey coefficients*, J. Differential Equations 45 (2) (1982), 272-306.
[18] G. Kutyniok, D. Labate (editors), *Shearlets: Multiscale Analysis for Multivariate Data* (Applied and Numerical Harmonic Analysis), (Birkhäuser, Springer Basel, 2012).
[19] S. Mallat, *A wavelet tour of signal processing*, Academic Press, London, 1999.
[20] Y. Meyer, *Wavelets and operators*, Cambridge University Press, Cambridge, 1992.
[21] F. Nicola, L. Rodino, *Global Pseudo-differential calculus on Euclidean spaces*, Pseudo-Differential Operators. Theory and Applications, 4, Birkhäuser Verlag, Basel, 2010.
[22] R. S. Pathak, S. K. Singh, *The wavelet transform on spaces of type S*, Proc. Roy. Soc. Edinburgh Sect. A 136 (2006), 837–850.
[23] R. S. Pathak, *The wavelet transform*, Atlantic Press/World Scientific, Paris, 2009.
[24] S. Pilipović, D. Rakić, J. Vindas, *New classes of weighted Hölder-Zygmund spaces and the wavelet transform*, J. Funct. Spaces Appl. Vol. 2012 (2012), Art. ID 815475, 18 pages.
[25] S. Pilipović, J. Vindas, *Multidimensional Tauberian theorems for wavelet and non-wavelet transforms*, preprint (arXiv:1012.5090v2).
[26] S. Pilipović, J. Vindas, *Multidimensional Tauberian theorems for vector-valued distributions*, Publ. Inst. Math. (Beograd) (N.S.) 95 (109) (2014), 1–28.
[27] S. Pilipović, M. Vuletić, *Characterization of wave front sets by wavelet transforms*, Tohoku Math. J. (2) 58 (2006), no. 3, 369-391.
[28] D. Rakić, N. Teofanov, *Progressive Gelfand-Shilov spaces and wavelet transforms*, J. Funct. Spaces Appl. Vol. 2012 (2012), Article ID 951819, 19 pages.
[29] S. G. Sankó, *Hypersingular integrals and their applications*, Taylor and Francis, New York, 2002.
[30] J. Toft, *The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators*, J. Pseudo-Differ. Oper. Appl. 3 (2012), 145–227.
[31] J. Toft, *Multiplication properties in Gelfand-Shilov pseudo-differential calculus*, in: Pseudo-differential operators, generalized functions and asymptotics, pp. 117-172, Oper. Theory Adv. Appl., 231, Birkhäuser/Springer Basel AG, Basel, 2013.
[32] J. Vindas, S. Pilipović, D. Rakić, *Tauberian theorems for the wavelet transform*, J. Fourier Anal. Appl. 17 (2011), 65–95.
S. Pilipović, University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: pilipovics@yahoo.com

D. Rakić, University of Novi Sad, Faculty of Technology, Bul. cara Lazara 1, 21000 Novi Sad, Serbia
E-mail address: drakic@tf.uns.ac.rs

N. Teofanov, University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: nenad.teofanov@dmi.uns.ac.rs

J. Vindas, Department of Mathematics, Ghent University, Krijgslaan 281 Gebouw S22, B 9000 Gent, Belgium
E-mail address: jvindas@cage.ugent.be