ROBUST MULTI-PERIOD AND MULTI-OBJECTIVE PORTFOLIO SELECTION

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ABSTRACT. In this paper, a multi-period multi-objective portfolio selection problem with uncertainty is studied. Under the assumption that the uncertainty set is ellipsoidal, the robust counterpart of the proposed problem can be transformed into a standard multi-objective optimization problem. A weighted-sum approach is then introduced to obtain Pareto front of the problem. Numerical examples will be presented to illustrate the proposed method and validate the effectiveness and efficiency of the model developed.

1. Introduction. Portfolio selection is to optimally allocate investors’ capital to a number of candidate securities. Traditionally, this problem has been formulated as an optimization problem through Markowitz mean-variance model [8, 13]. In this model, the mean is used as the measurement of return investment and the variance is leveraged to measure risk. Once the expected unit return of each of the securities and their covariance matrix are given, the portfolio selection problem can be formulated as a quadratic optimization problem subject to linear constraints if the weight on the mean and the risk are given. Thus, before establishing a mean-variance model, we need to: 1) estimate the input parameters, including the expected unit return of each security and the covariance matrix and 2) determine the weights of mean and variance. In practice, the input parameters are usually estimated through empirical observations or subjective studies [5]. However, a small perturbation of the input parameters may lead to a large deviation of the selected portfolio performances. In addition, how to determine the weights of mean and variance is also challenging [5].

There are many existing works to alleviate the aforementioned problems. In [7], the minimum transaction lots is considered in Markowitz’s model for portfolio selection to make the problem more practical. This problem is then formulated as a combinatorial optimization problem and a genetic algorithm is proposed to solve the problem. In [12], minimum transaction lots is further discussed with a cardinality constraint where the cardinality constraint is to constrain the number of portfolios to be selected. The problem is also formulated as a mixed-integer optimization problem and a customized genetic algorithm is introduced to solve the problem. In [2], the minimum transaction lots has been further discussed with four different

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models: mean-variance model, mean absolute deviation model, minimax model and combinational Value-at-Risk model. These four different models are of the forms of four different discrete optimization problems. The published results show that the mean-variance model performs better than the others.

How to handle uncertainties in the input parameters is also another important problem to be considered [19, 18]. In the mean-variance model, the future returns and variances are usually obtained through classic point-estimation [16]. The uncertain nature of risk and return often leads to that the results obtained are unreliable and sensitive to perturbations in parameters. There are extensive works developed to address data uncertainty in an optimization problem and this research topic is now called robust optimization. In general, there are two different types of methods to cope with uncertainty: stochastic-based methods and deterministic-based methods [1]. Stochastic-based methods usually require the statistical properties of the uncertainty. Deterministic-based methods use minimax criteria to optimize the worst case scenario and thus only the range of uncertainty needs to be known. For a quadratic programming problem with ellipsoidal uncertainties, it has been shown in [1] that it can be transformed to an equivalent conic quadratic optimization problem. If the problem is a Semi-Definite programming (SDP) and uncertainties are in a ellipsoid, the corresponding robust optimization problem is still an SDP [15]. Robust optimization is also introduced to handle the uncertainties in portfolio selection. In [10], a robust portfolio selection model with a combined worst-case conditional value-at-risk and multi-factor model is studied. The authors have shown that the probability distributions in the definition of WC VaR can be determined by specifying the mean vectors under the assumption of multivariate normal distribution with a fixed variance-covariance matrix [10]. In [4], robust mean-variance portfolio selection problem is studied. Through introducing uncertainty structures, the authors show that the robust counterpart is a second order cone program. This problem has been further discussed in [11] where two different uncertainty sets are introduced for the uncertainties of input parameters.

In the standard mean-variance model, only one period of investment is considered. However, in practice, investors are willing to adjust their investment from time to time based on real time information from a financial market. Therefore, multi-period portfolio selection has attracted much research interest. In [17], a multi-period portfolio selection problem with an uncertain investment horizon is studied. Under the assumption that the exit time follows a given distribution, the problem is transformed into one with deterministic exit time. An analytical expression of the mean-variance efficient frontier is derived. In [14], a multi-period portfolio selection problem with fixed and proportional transaction costs is considered. The optimal solution and the boundaries of the no-transaction region are obtained through introducing Lagrange multiplier and dynamic programming approach. Although extensive results are discussed in literature, there are still lack of results on the portfolio selection with transaction costs under uncertainty of input parameters. In this paper, we shall fill this gap and extend Markowitz mean-variance model to multi-period portfolio selection with transaction costs under input parameters uncertainty. We will first formulate this problem as a bi-objective optimization problem and then the robust counterpart of this problem is derived. The weighted-sum approach is introduced to present the efficient Pareto front of the formulated bi-objective optimization problem. Numerical results will be presented to demonstrate the efficiency and effectiveness of the proposed method.
2. **Problem statement.** Consider portfolio of \( n \) financial assets which can be traded at discrete times \( 1, 2, \cdots, T \). Suppose that at the initial time \( 0 \), the investor has already chosen a portfolio \( w = (w_1, 0, w_2, \cdots, w_n, 0) \), where \( w_{i,0} \) is the investor’s investment in the \( i \)-th asset for \( i = 1, 2, \cdots, n \). At the beginning of each period \( t, t = 1, 2, \cdots, T \), the investor needs to adjust the investment in each asset by increasing or decreasing the capital amount:

\[
\Delta w_t = (\Delta w_{1,t}, \Delta w_{2,t}, \cdots, \Delta w_{n,t}).
\]

Let \( \xi_{i,t} \) be the uncertain return rate of the \( i \)-th asset during period \( t \). Then, at the end of period \( t \), the investor’s total wealth becomes

\[
r_t = \sum_{i=1}^{n} (w_{i,t-1} + \Delta w_{i,t})(1 + \xi_{i,t}).
\]

The wealth increment is thereby calculated as

\[
\Delta r_t = r_t - r_{t-1}.
\]

Suppose that each transaction incurs a transaction cost for asset \( i \) which is calculated as \( \varsigma_i |\Delta w_{i,t}| \), where \( \varsigma_i \) is the unit transaction cost. The total transaction cost is thereby computed as

\[
\sum_{i=1}^{n} \varsigma_i |\Delta w_{i,t}|
\]

Let \( r_{t,f} \) be the risk-free interest rate at the time \( t \). Then, the excess return is defined as \( r_t - r_{t,f} \). Suppose that the covariance matrix of the excess returns is \( \Sigma_t \). The variance \( (w_{t-1} + \Delta w_{t-1})^T \Sigma_t (w_{t-1} + \Delta w_t) \) can be used to describe the risk of the wealth. Using these quantities, the portfolio selection at the time \( t \) can be formulated as the following optimization problem:

**Problem PS.**

\[
\begin{align*}
\text{min} & \quad ((w_{t-1} + \Delta w_t)^T \Sigma_t (w_{t-1} + \Delta w_t))^{\frac{1}{2}}, \\
\text{max} & \quad r_t - \sum_{i=1}^{n} \varsigma_i |\Delta w_{i,t}|, \\
\text{s.t.} & \quad w_{i,t-1} + \Delta w_{i,t} \geq 0, \\
& \quad \sum_{i=1}^{n} \Delta w_{i,t} + \sum_{i=1}^{n} \varsigma_i |\Delta w_{i,t}| \leq 0.
\end{align*}
\]

Here the constraint (5) means that the investment is self-financing, i.e., the investment must be equal to or less than the net income of the sales of the assets minus the total transaction cost.

To solve Problem PS, \( \Sigma_t \) and \( \xi_{i,t}, i = 1, \cdots, n \), are required which are usually estimated by sampling historical data. Their estimates \( \hat{\Sigma}_t \) and \( \hat{\xi}_{i,t}, i = 1, \cdots, n \), vary depending on the samples chosen and the method to be used for calculating them. Thus, \( \hat{\Sigma}_t \) and \( \hat{\xi}_{i,t}, i = 1, \cdots, n \), are uncertain. Let \( \xi_t = [\xi_{1,t}, \cdots, \xi_{n,t}]^T \). If the samples are i.i.d and satisfy \( \xi_t \sim N(\bar{\xi}_t, \bar{\Sigma}_t) \), then

\[
\hat{\xi}_t = \frac{1}{t} \sum_{j=0}^{t-1} \xi_j \sim N(\bar{\xi}_t, \frac{1}{t} \bar{\Sigma}_t).
\]
\[ \hat{\Sigma}_t = \frac{1}{t-1} \sum_{j=0}^{t-1} (\xi_t - \hat{\xi}_t)(\xi_t - \hat{\xi}_t)^T \sim \mathcal{W}\left( \frac{1}{t-1} \Sigma_t, t-1 \right), \]

where \( \mathcal{W}(G,v) \) denotes the Wishart distribution with scale matrix \( G \) and \( v \) degrees of freedom.

In the open literature, there are several different confidence ellipsoids introduced to achieve a robust solution. A particular one is the following separated elliptical uncertainty set proposed in [4]:

\[ S_\xi = \{ \xi_{t,i} = \xi_{i,t} + \varepsilon_{i,t}, |\varepsilon_{i,t}| \leq \gamma_i \text{ for } i = 1, \ldots, n \}, \]

\[ S_\Sigma = \{ \Sigma_t = \Sigma_0 + \Theta_t, \|\Theta_t\| \leq \rho \}, \]

where \( \| \cdot \| \) denotes the Euclidean norm, \( \rho \) and \( \gamma_i \)'s are proper bounds estimated using historical data. A robust portfolio selection problem based on this confidence set has been proved to achieve a robust solution with the given confidence. Instead of considering the above separated uncertainty sets, we consider a joint confidence ellipsoid as follows:

\[ S_\delta(\hat{\xi}_t, \hat{\Sigma}_t) = \left\{ (\xi_t, \Sigma_t) \in \mathbb{R}^n \times \mathcal{S}_n \mid t(\xi_t - \hat{\xi}_t)^T \hat{\Sigma}_t^{-1}(\xi_t - \hat{\xi}_t) + \frac{t-1}{2} \| \hat{\Sigma}_t^{-1/2}(\Sigma_t - \hat{\Sigma}_t)\hat{\Sigma}_t^{-1/2} \|_F^2 \leq \delta^2 \right\}, \]

where \( \| A \|_F^2 = tr(A^T A) \) for a matrix \( A \) and \( \delta \) is a parameter characterizing the desired confidence. In this paper, the joint confidence ellipsoid uncertainty set (6) is used and the corresponding multi-objective optimization problem can be formally stated as:

**Problem RPS.**

\[
\begin{align*}
\min_{\Delta \mathbf{w}_t, (\xi_t, \Sigma_t) \in S_\delta(\hat{\xi}_t, \hat{\Sigma}_t)} & \quad \left\{ \left( (w_{t-1} + \Delta \mathbf{w}_t)^T \Sigma_t (w_{t-1} + \Delta \mathbf{w}_t) \right)^{1/2}, \left( \sum_{i=1}^{n} \left| \xi_i \right| |\Delta \mathbf{w}_{i,t}| - r_t \right) \right\} \\
\text{s.t.} & \quad w_{t-1} + \Delta \mathbf{w}_{t} \geq 0 \\
& \quad \sum_{i=1}^{n} \Delta \mathbf{w}_{i,t} + \sum_{i=1}^{n} \left| \xi_i \right| |\Delta \mathbf{w}_{i,t}| \leq 0
\end{align*}
\]

(Recall \( r_t \) is a function of \( \Delta \mathbf{w}_t \) and \( \xi_t \) as in (1).) In Problem RPS, there are two objectives. One is to minimize the worst risk and the other one is to maximize the worst wealth return over the given uncertainty set.

3. **Problem transformation.** To solve Problem RPS, we need to transform the objective function (7) into a more tractable form. Note that the confidence ellipsoid uncertainty set in (4) is for both \( \xi_t \) and \( \Sigma_t \). In this joint set, \( \hat{\xi}_t \) and \( \hat{\Sigma}_t \) are calculated through the available samples. As in [11], we introduce a dummy variable \( \kappa \) and define the following two sets:

\[ S_\delta(\hat{\xi}_t) = \{ \xi_t \in \mathbb{R}^n \mid t(\xi_t - \hat{\xi}_t)^T \hat{\Sigma}_t^{-1}(\xi_t - \hat{\xi}_t) \leq \kappa \delta^2 \}, \]

\[ S_\delta(\hat{\Sigma}_t) = \{ \Sigma_t \in \mathbb{R}^{n \times n} \mid \| \hat{\Sigma}_t^{-1/2}(\Sigma_t - \hat{\Sigma}_t)\hat{\Sigma}_t^{-1/2} \|_F^2 \leq (1 - \kappa) \delta^2 \}. \]

Then, we can easily verify that Problem RPS is equivalent to the following one:

\[
\begin{align*}
\min_{\Delta \mathbf{w}_t, \kappa \in [0,1]} & \quad \max_{\xi_t \in S_\delta(\hat{\xi}_t)} \left\{ \max_{\Sigma_t \in S_\delta(\hat{\Sigma}_t)} \left( (w_{t-1} + \Delta \mathbf{w}_t)^T \Sigma_t (w_{t-1} + \Delta \mathbf{w}_t) \right)^{1/2}, \max_{\xi_t \in S_\delta(\hat{\xi}_t)} \left( \sum_{i=1}^{n} \left| \xi_i \right| |\Delta \mathbf{w}_{i,t}| - r_t \right) \right\}, \quad (12)
\end{align*}
\]
Using the above optimal solution, we have
\[ f_t \leq \xi_t \equiv \hat{\xi}_t - \delta \sqrt{\frac{n}{\kappa (w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)} \hat{\Sigma}_t (w_{t-1} + \Delta w_t)}. \]

To further simplify the problem, we need to find an analytical solution to the inner maximization problem in (7). More specifically, the maximization problem
\[
\max_{\xi_t \in \mathcal{S}_t(\hat{\xi}_t)} \left( \sum_{i=1}^{n} \xi_t |\Delta w_{i,t}| - r_t \right)
\]
in (7) has the solution
\[
\xi_t^* = \hat{\xi}_t - \delta \sqrt{\frac{\kappa}{n (w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)} \hat{\Sigma}_t (w_{t-1} + \Delta w_t)}. \]

Using the above optimal solution, we have
\[
\max_{\xi_t \in \mathcal{S}_t(\hat{\xi}_t)} \left( \sum_{i=1}^{n} \xi_t |\Delta w_{i,t}| - r_t \right) = \xi^T |\Delta w_t| - (w_{t-1} + \Delta w_t)^T 1_n - (w_{t-1} + \Delta w_t)^T \hat{\xi}_t + \delta \sqrt{\frac{\kappa}{n (w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)}},
\]
where \(1_n = [1, \ldots]^T \in \mathbb{R}^n\).

Now we consider the maximization problem \(\max_{\Sigma_t \in \mathcal{S}_t(\hat{\Sigma}_t)} (w_{t-1} + \Delta w_t)^T \Sigma_t (w_{t-1} + \Delta w_t)\). As done in [11], we are able to show that
\[
\max_{\Sigma_t \in \mathcal{S}_t(\hat{\Sigma}_t)} \sqrt{(w_{t-1} + \Delta w_t)^T \Sigma_t (w_{t-1} + \Delta w_t)} = \sqrt{\left(1 + \delta \sqrt{\frac{2}{n-1} (1 - \kappa)} \right) (w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)},
\]

To simplify the notation, we denote
\[
f_1(\Delta w_t, \kappa) = \left(1 + \delta \sqrt{\frac{2}{n-1} (1 - \kappa)} \right) \frac{1}{2} \sqrt{(w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)},
\]

\[
f_2(\Delta w_t, \kappa) = \xi^T |\Delta w_t| - (w_{t-1} + \Delta w_t)^T 1_n - (w_{t-1} + \Delta w_t)^T \hat{\xi}_t + \delta \sqrt{\frac{\kappa}{n (w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)}},
\]

Using \(f_1(\Delta w_t, \kappa)\) and \(f_2(\Delta w_t, \kappa)\) defined above, we rewrite Problem RPS as follows.

**Problem ERPS.**
\[
\min_{\Delta w_t, \kappa \in [0,1]} \max_{[0,1]} \left\{ f_1(\Delta w_t, \kappa), f_2(\Delta w_t, \kappa) \right\}, \quad \text{s.t.} \quad w_{i,t-1} + \Delta w_{i,t} \geq 0, \quad \sum_{i=1}^{n} \Delta w_{i,t} + \sum_{i=1}^{n} \xi_t |\Delta w_{i,t}| \leq 0.
\]
4. Solution strategy. Problem ERPS is a minimax and bi-objective optimization problem. Classical methods to solve multi-objective optimization problems include weighted-sum approach [9], ε−constraint approach [6], evolutionary multi-objective optimization method [3], decomposition-based approach [20], etc. The weighted-sum approach is to transform a multi-objective optimization problem into a single-objective optimization problem through a given set of weights. The Pareto-front is obtained through diversified weights. The ε−constraint approach is to transform a multi-objective optimization problem into a single-objective optimization problem through optimizing one objective and putting all the other objectives as ε−constraints. Evolutionary based methods are using non-dominated sorting Genetic Algorithm to find solutions in Pareto-front. Decomposition-based methods are also using weighting to find Pareto solutions, which is different from weighted-sum approach in which maximization weight, rather than summation of the weight, is used. Note that the objective function in (7) contains two sub-objectives and maximization is involved in the inner level. Thus, the weighted-sum approach is ideal for the problem. We now proposed such a weighted-sum method below.

4.1. Weighted-sum approach. Traditional weighted-sum approach is to balance the conflict objectives through weighted method. Suppose that the two weights for the two objectives in Problem ERPS are λ1 and λ2, respectively. Then, the original two objectives are transformed into a single one as below:

\[ f_\lambda(\Delta w_1, \kappa) = \lambda_1 f_1(\Delta w_1, \kappa) + \lambda_2 f_2(\Delta w_1, \kappa), \]  

where \( \lambda_1 \) and \( \lambda_2 \) are the given weights. \( \lambda_2 \) is usually defined as \( \lambda_2 = 1 - \lambda_1 \).

We comment that, in practice, the values of the two objectives might not be in the same magnitude, and thus one may be much smaller than the other in magnitude. In this case, we need to adjust the weights so that the points in the Pareto front can be determined evenly. Let \( (\Delta w_1^1, \kappa_1^1) \) and \( (\Delta w_1^2, \kappa_2^1) \) be the optimal solutions obtained for the individual optimization problems involving \( f_1(\Delta w_1, \kappa) \) and \( f_2(\Delta w_1, \kappa) \), respectively. Then, the adjusted weights \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) can be defined as:

\[ \tilde{\lambda}_1 = \frac{\lambda f_1(\Delta w_1^2, \kappa_2^1) - f_1(\Delta w_1^1, \kappa_1^1)}{f_1(\Delta w_1^2, \kappa_2^1) - f_1(\Delta w_1^1, \kappa_1^1) + \frac{1-\lambda}{\lambda} f_2(\Delta w_1^2, \kappa_2^1) - f_2(\Delta w_1^1, \kappa_1^1)} \]  

(20)

and \( \tilde{\lambda}_2 = 1 - \tilde{\lambda}_1 \).

The benefit through this manipulation is that the normalized weights can balance the two objectives to avoid one dominates the other in the optimization process.

4.2. Sub-optimization solution. Solving Problem ERPS is therefore transformed into solving a series of standard optimization problems. Let us study the inner optimization problem

\[
\min_{\Delta w_1} \max_{\kappa \in [0,1]} f_\lambda(\Delta w_1, \kappa) \text{ s.t. (17) and (18)}. 
\]  

(21)

This sub-optimization problem (21) is still hard to solve as both minimization and maximization are involved. To address this difficulty, we will examine the properties of the function \( f_\lambda(\Delta w_1, \kappa) \) with respect to \( \kappa \). In fact, for each given \( \lambda \in [0, 1] \), we have the following lemma:

Lemma 4.1. For any given \( \lambda \in [0, 1] \), the function \( f_\lambda(\Delta w_1, \kappa) \) is concave with respect to \( \kappa \).
Proof. For each given $\lambda_1 = \lambda$, let $\lambda_2 = 1 - \lambda$. Then, $f_\lambda(\Delta w_t, \kappa)$ can be rewritten as:

$$f_\lambda(\Delta w_t, \kappa) = \left( \lambda \left( 1 + \delta \sqrt{\frac{2}{n-1}(1-\kappa)} \right)^{\frac{1}{2}} + \delta(1-\lambda)\sqrt{\frac{\kappa}{n}} \right) \sqrt{(w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)}$$

$$+ \zeta^T [\Delta w_t] - (w_{t-1} + \Delta w_t)^T 1_n - (w_{t-1} + \Delta w_t)^T \xi_t.$$  

Define $g(\kappa) = \lambda \left( 1 + \delta \sqrt{\frac{2}{n-1}(1-\kappa)} \right)^{\frac{1}{2}} + \delta(1-\lambda)\sqrt{\frac{\kappa}{n}}.$

To prove $f_\lambda(\Delta w_t, \kappa)$ is concave in $\kappa$, we only need to prove $g(\kappa)$ is concave in terms of $\kappa$. For any $\kappa \in (0, 1)$, we have

$$\frac{d^2 g(\kappa)}{d\kappa^2} = -\frac{1}{2} \frac{\delta \lambda}{(n-1)^2} \left( 1 + \delta \frac{2}{n-1}(1-\kappa) \right)^{-\frac{1}{2}} \left( \frac{2}{n-1}(1-\kappa) \right)^{-\frac{1}{2}}$$

$$-\frac{1}{4} \frac{\lambda \delta^2}{(n-1)^2} \left( 1 + \delta \frac{2}{n-1}(1-\kappa) \right)^{-\frac{3}{2}} \left( \frac{2}{n-1}(1-\kappa) \right)^{-1}$$

$$-\frac{\delta(1-\lambda)}{4n^2} \left( \frac{\kappa}{n} \right)^{-\frac{3}{2}}$$

$$\leq 0, \text{ for all } \lambda \in [0, 1] \text{ and } \kappa \in (0, 1).$$

Thus, $g(\kappa)$ is concave. We complete the proof.

For given $\Delta w_t$, $\delta$ and $\lambda$, Lemma 4.1 shows that the inner maximization problem will be maximized at its equilibrium if it is within $(0, 1)$. Otherwise, it will be maximized at either $\kappa = 0$ or $\kappa = 1$. Note that

$$\frac{dg(\kappa)}{d\kappa} = -\frac{1}{2} \frac{\delta \lambda}{n-1} \left( 1 + \delta \frac{2}{n-1}(1-\kappa) \right)^{-\frac{1}{2}} \left( \frac{2}{n-1}(1-\kappa) \right)^{-\frac{1}{2}}$$

$$-\frac{1}{2n} \delta(1-\lambda) \left( \frac{\kappa}{n} \right)^{-\frac{3}{2}} = 0. \quad (22)$$

Thus, for given $\lambda$, $\delta$ and $\Delta w_t$, to solve $\max_{\kappa \in [0, 1]} f_\lambda(\Delta w_t, \kappa)$, we first need to check whether (22) has a solution within $(0, 1)$. If it has a solution $\kappa^*$, then this solution also solves the problem $\max_{\kappa \in [0, 1]} f_\lambda(\Delta w_t, \kappa)$ as $g(\kappa)$ is concave within $[0, 1]$. Otherwise, $\kappa^* = \arg \max \{ f_\lambda(\Delta w_t, 0), f_\lambda(\Delta w_t, 1) \}$. Then, the sub-optimization problem (21) becomes

$$\min_{\Delta w_t} f_\lambda(\Delta w_t, \kappa^*) \text{ s.t. (17) and (18),} \quad (23)$$

where

$$f_\lambda(\Delta w_t, \kappa^*) = g(\kappa^*) \sqrt{(w_{t-1} + \Delta w_t)^T \hat{\Sigma}_t (w_{t-1} + \Delta w_t)} + \zeta^T [\Delta w_t]$$

$$- (w_{t-1} + \Delta w_t)^T 1_n - (w_{t-1} + \Delta w_t)^T \xi_t.$$

5. Numerical experiments. In this section, several examples are solved by the proposed method. We will use the numerical solutions to study the impact of uncertainties of the return rate and the variance on the multi-objective mean-variance model.
5.1. **Convex of** \( g(\kappa) \). We note that Problem PS is equivalent to Problem ERPS. In Problem ERPS, in addition to the original variable \( \Delta w_t \), there is one more variable \( \kappa \) which only appears in \( g(\kappa) \). Thus, \( g(\kappa) \) plays a crucial role in the numerical solution of Problem ERPS. In the previous section, we have already showed that \( g(\kappa) \) is concave for fixed \( \lambda \) and \( \delta \). We now verify this property computationally. To show this we plot \( g(\kappa) \) for \( \lambda = 0.1 \) and \( \delta = 0.1, 0.5 \), and 0.9. The results are depicted in Figures 1-3. From the three figures, we clearly see that the maximum point of \( g(\kappa) \) moves from the right end-point to the middle and then to the left end-point of the interval as \( \lambda \) increases. This phenomenon is also observed for different values of \( \delta \).

![Figure 1. The variation of \( g(\kappa) \) in terms of \( \kappa \) when \( \lambda = 0.1 \)](image)

5.2. **Impact of the input parameters.** We first study the impact of the uncertainty \( \delta \). For this purpose, we choose various values of \( \lambda \) and plot the optimal cost function \( f_\lambda(\Delta t^*, \kappa^*) \) against \( \delta \) in Figures 4–6. From these figures, we see clearly that as \( \lambda \) increases, the weighted optimal objective function value decreases. In fact, this phenomenon is consistent with our intuition that the larger the weight \( \lambda \) is, the more the investor is pursuing the return. To reduce the weight \( \lambda \) will lead to a more conservative investment option.

Figure 5 and 6 depict the variation of the optimal objective function value \( f_\lambda(\Delta w_t, \kappa^*) \) with respect to the uncertainty parameter \( \delta \). From the two figures, we can clearly observe that the return decreases with the increase of \( \lambda \). Figure 5 demonstrates that \( f_\lambda(\Delta w_t, \kappa^*) \) increases from \(-6.4598 \times 10^4\) to \(-6.4543 \times 10^4\) as \( \delta \) goes from 0 to 0.1 when \( \lambda = 0.5 \), while in Figure 6, \( f_\lambda(\Delta w_t, \kappa^*) \) increases from \(-8.41842 \times 10^4\) to \(-8.417 \times 10^4\) when \( \delta \) moves from 0 to 0.1 and \( \lambda = 0.9 \).
Figure 2. The variation of $g(\kappa)$ in terms of $\kappa$ when $\lambda = 0.5$

Figure 3. The variation of $g(\kappa)$ in terms of $\kappa$ when $\lambda = 0.9$
Figures 7 shows the influence of the transaction cost on the portfolio selection return. From this figure we see that, with the increase of $\zeta$ from $10^{-3}$ to $10^{-2}$, the return investment $f_3(\Delta w_t, \kappa^*)$ decreases from $-6.625 \times 10^4$ to $-6.46 \times 10^4$. Thus, the return of the portfolio is a decreasing function of the transaction cost $\zeta$, as expected.

5.3. **Numerical solutions with different parameters.** In the above, we have analyzed the variation of the objective function with respect to different input parameters. We now look into the numerical solutions in different scenarios. An investment with 10 portfolios are selected in this numerical experiment. When $\delta = 0.01$ and $\zeta = 0.01$, the obtained solutions with a 5-period investment are presented in Table 1. Tables 2 and 3 show the scenarios with respect to $(\delta, \zeta) = (0.1, 0.01)$ and $(\delta, \zeta) = (0.1, 0.001)$, respectively. From Table 1 and Table 2, we can clearly observe that transaction cost has a significant impact on the portfolio selection. For example, in the second period, selling $w_1$ will be changed to buying. The perturbation has also a significant impact on the portfolio selection. For example, for the first portfolio, it has been changed from selling to buying once we increase $\delta$ from 0.01 to 0.1.

5.4. **Pareto-front analysis.** In this subsection, we present some results on the one-period portfolio selection under different values of the weight $\lambda$. Figure 8 plots the Pareto front with $\delta = 1$ and $\zeta = 0.001, 0.002, 0.005$, in which the horizontal axis represents $f_1(\Delta w_t, \kappa^*)$ and the vertical one represents $-f_2(\Delta w_t, \kappa^*)$. From this figure, we see clearly that $-f_2(\Delta w_t, \kappa^*)$ increases as $f_1(\Delta w_t, \kappa^*)$ decreases. From the figure we also see that when the transaction cost increases, the investment return decreases.
Figure 5. The variation of $f_{\lambda}(\Delta w_t, \kappa^*)$ in terms of $\delta$ when $\lambda = 0.5$.

Figure 6. The variation of $f_{\lambda}(\Delta w_t, \kappa^*)$ in terms of $\delta$ when $\lambda = 0.9$. 
Figure 7. The variation of $f_{\lambda}(\Delta w_1, \kappa^*)$ in terms of $\varsigma$ when $\lambda = 0.5$

Table 1. Solution with five period; $\varsigma = 0.001, \delta = 0.01$

| $\Delta w_1$ | $\Delta w_2$ | $\Delta w_3$ | $\Delta w_4$ | $\Delta w_5$ | $\Delta w_6$ | $\Delta w_7$ | $\Delta w_8$ | $\Delta w_9$ | $\Delta w_{10}$ |
|---------------|---------------|---------------|--------------|--------------|---------------|---------------|---------------|---------------|---------------|
| 1.6062540e+002 | -4.3718793e+001 | -1.0000000e+003 | 1.9724480e+003 | -1.3675740e+002 | 1.5230230e+003 | -3.9555914e+001 | 1.6005381e+002 | -4.348311e+001 | 1.604819e+002 |

Table 2. Solution with five period; $\varsigma = 0.01, \delta = 0.01$

| $\Delta w_1$ | $\Delta w_2$ | $\Delta w_3$ | $\Delta w_4$ | $\Delta w_5$ | $\Delta w_6$ | $\Delta w_7$ | $\Delta w_8$ | $\Delta w_9$ | $\Delta w_{10}$ |
|---------------|---------------|---------------|--------------|--------------|---------------|---------------|---------------|---------------|---------------|
| 2.1589294e+002 | -2.8283011e+001 | -3.0382696e+002 | 7.0391477e+002 | -1.3859574e+002 | -1.380908e+001 | 2.1891752e+003 | -2.8203111e+002 | -3.209527e+002 | -7.4650111e+002 |

Table 1. Solution with five period; $\varsigma = 0.001, \delta = 0.01$

Table 2. Solution with five period; $\varsigma = 0.01, \delta = 0.01$
Table 3. Solution with five period; \( \varsigma = 0.01, \delta = 0.1 \)

| \( \Delta w_1 \) | \( k_1 \) | \( k_2 \) | \( k_3 \) | \( k_4 \) | \( k_5 \) |
|---|---|---|---|---|---|
| -5.4467604e-002 | -7.0948978e-007 | -2.2561076e-004 | -1.0737564e-002 | -1.6291721e-004 | -1.6291721e-004 |
| \( \Delta w_2 \) | -1.6419802e-003 | -5.6357371e-007 | -7.4291123e-003 | 4.3365254e-003 | -2.9140832e-004 |
| \( \Delta w_3 \) | -1.0000000e+003 | -1.0000000e+003 | -1.0000000e+003 | -1.0000000e+003 | -1.0000000e+003 |
| \( \Delta w_4 \) | 1.7420318e+003 | 1.7397785e+003 | 1.7370980e+003 | 1.7349762e+003 | 1.7353447e+003 |
| \( \Delta w_5 \) | -9.7249593e-008 | -3.9907242e-002 | -1.8502745e-007 | -4.0113434e-002 | -3.8173914e-004 |
| \( \Delta w_6 \) | -7.9037071e+002 | -7.9581203e+002 | -7.9336516e+002 | -7.8689932e+002 | -7.9102920e+002 |
| \( \Delta w_7 \) | -2.1545195e+002 | -2.0339581e+002 | -2.0451219e+002 | -2.0476930e+002 | -1.9763968e+002 |
| \( \Delta w_8 \) | 1.2043583e+003 | 1.2000782e+003 | 1.2014231e+003 | 1.1974970e+003 | 1.1941434e+003 |

Figure 9 plots the Pareto front for \( \delta = 5 \) and \( \varsigma = 0.001, 0.002 \) and 0.005. From Figure 9, we again see that the more risk the investors take, the more return they will have. Comparing Figure 9 with Figure 8, we observe that the impact of the transaction cost on the investment return becomes less significant when \( \delta \) is close to 1. This trend has been further observed in Figure 10.

![Figure 8. The pareto front with \( \delta = 1 \)](image)

6. Conclusion. In this paper, we have studied the portfolio selection problem with uncertain input parameters. Under the assumption that the uncertainty is in a ellipsoid, the original robust bi-objective optimization problem is transformed into an optimization problem with the weighted sum approach which is easy to solve. The numerical results obtained demonstrate that the uncertainties of the input parameters affect the portfolio selection significantly. More specifically, the numerical results suggest that the more uncertain the input parameters are, the less the portfolio return is.

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Figure 9. The pareto front with $\delta = 5$

Figure 10. The pareto front with $\delta = 10$

REFERENCES

[1] A. Ben-Tal and A. Nemirovski, Robust optimization—methodology and applications, *Math. Program.*, 92 (2002), 453–480.

[2] X. Cui, X. Sun and D. Sha, An empirical study on discrete optimization models for portfolio selection, *J. Ind. Manag. Optim.*, 5 (2009), 33–46.

[3] K. Deb, A. Pratap, S. Agarwal and T. A. Meyarivan, A fast and elitist multiobjective genetic algorithm: NSGA-II, *IEEE Transactions on Evolutionary Comput.*, 6 (2002), 182–197.

[4] D. Goldfarb and G. Iyengar, Robust portfolio selection problems, *Math. Oper. Res.*, 28 (2003), 1–38.

[5] D. Huang, S. Zhu, F. J. Fabozzi and M. Fukushima, Portfolio selection under distributional uncertainty: A relative robust CVaR approach, *European J. Oper. Res.*, 203 (2010), 185–194.
[6] K. Khalili-Damghani and M. Amiri, Solving binary-state multi-objective reliability redundancy allocation series-parallel problem using efficient epsilon-constraint, multi-start partial bound enumeration algorithm, and DEA, Reliability Engineering and System Safety, 103 (2012), 35–44.

[7] C. C. Lin and Y. T. Liu, Genetic algorithms for portfolio selection problems with minimum transaction lots, European J. Oper. Res., 185 (2008), 393–404.

[8] H. Markowitz, Portfolio Selection: Efficient Diversification of Investments, John Wiley & Sons, Inc., New York, 1959.

[9] R. T. Marler and J. S. Arora, The weighted sum method for multi-objective optimization: New insights, Struct. Multidiscip. Optim., 41 (2010), 853–862.

[10] K. Ruan and M. Fukushima, Robust portfolio selection with a combined WCVaR and factor model, J. Ind. Manag. Optim., 8 (2012), 343–362.

[11] K. Schottle and R. Werner, Robustness properties of mean-variance portfolios, Optimization, 58 (2009), 641–663.

[12] H. Soleimani, H. R. Golmakani and M. H. Salimi, Markowitz-based portfolio selection with minimum transaction lots, cardinality constraints and regarding sector capitalization using genetic algorithm, Expert Systems with Appl., 36 (2009), 5058–5063.

[13] Q. Wang and H. Sun, Sparse Markowitz portfolio selection by using stochastic linear complementarity approach, J. Ind. Manag. Optim., 14 (2018), 541–559.

[14] Z. Wang and S. Liu, Multi-period mean-variance portfolio selection with fixed and proportional transaction costs, J. Ind. Manag. Optim., 9 (2013), 643–657.

[15] C. Wu, K. L. Teo and S. Wu, Min-max optimal control of linear systems with uncertainty and terminal state constraints, Automatica J. IFAC, 49 (2013), 1809–1815.

[16] P. Xidonas, G. Mavrotas, C. Hassapis and C. Zopounidis, Robust multiobjective portfolio optimization: A minimax regret approach, European J. Oper. Res., 262 (2017), 299–305.

[17] L. Yi, Z. F. Li and D. Li, Multi-period portfolio selection for asset-liability management with uncertain investment horizon, J. Ind. Manag. Optim., 4 (2008), 535–552.

[18] N. Zhang, A symmetric Gauss-Seidel based method for a class of multi-period mean-variance portfolio selection problems, J. Ind. Manag. Optim., (2018).

[19] P. Zhang, Chance-constrained multiperiod mean absolute deviation uncertain portfolio selection, J. Ind. Manag. Optim., 15 (2019), 537–564.

[20] C. Zhao, C. Wu, J. Chai, X. Wang, X. Yang, J. M. Lee and M. J. Kim, Decomposition-based multi-objective firefly algorithm for RFID network planning with uncertainty, Appl. Soft Comput., 55 (2017), 549–564.

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