ON ADMISSIBLE LIMITS OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. The aim of the present article is to establish the connection between the existence of the limit along the normal and an admissible limit at a fixed boundary point for holomorphic functions of several complex variables.

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1. Introduction

The connection between the existence of a radial limit and an angular limit for a holomorphic function defined on the unit disc is described by Lehto and Virtanen [6, Theorem 5] in terms of the growth of the spherical derivative.

For a precise description we introduce several terms and notation.

Let \( \{ z \in \mathbb{C} : |z| < 1 \} \) be a unit disc in \( \mathbb{C} \). Let \( \alpha > 1 \).

A non-tangential region \( \Gamma_\alpha(\xi) \) for \( \alpha > 1 \) and an angular region \( A_\theta(\xi) \) for \( \theta \in (0, 2\pi) \) at \( \xi \in \partial U \) are defined as follows:

\[
\Gamma_\alpha(\xi) = \{ z \in U : |1 - z\xi| < \frac{\alpha}{2}(1 - |z|^2) \},
\]

\[
A_\theta(\xi) = \{ z \in U : \pi - \theta < \arg(z - \xi) < \pi + \theta \}.
\]

It is to be noted that non-tangential regions and angular regions are equivalent: For every \( \alpha > 1 \) there is a \( \theta \in (0, \frac{\pi}{2}) \) such that \( \Gamma_\alpha(\xi) \subset A_\theta(\xi) \) and for every \( \theta \in (0, \frac{\pi}{2}) \) there is an \( \alpha > 1 \) and a disk \( d \) centered at \( \xi \) such that \( A_\theta(\xi) \cap d \subset \Gamma_\alpha(\xi) \).

To see this let \( d_1 \) be the unit disk with center \( \xi \), \( z \in U \) and \( \varphi = \pi - \arg(z - \xi) \). From the law of cosines

\[
|z|^2 = 1 - 2\cos \varphi |\xi - z| + |\xi - z|^2.
\]

Since \( |\xi| = 1 \) we have \( |\xi - z| = |1 - z\xi| \) and

\[
\frac{|1 - z\xi|}{1 - |z|^2} = \frac{1}{2\cos \varphi - |1 - z\xi|}.
\]

Thus,

\[
\frac{1}{2\cos \varphi} \leq \frac{1 - |z\xi|}{1 - |z|^2} \quad \text{for} \quad z \in U,
\]

and

\[
\frac{|1 - z\xi|}{1 - |z|^2} \leq \frac{2}{2\cos \varphi} \quad \text{for} \quad z \in U \cap d_1.
\]

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We say that a holomorphic function $f$ in $U$ (notation $f \in \mathcal{O}(U)$) has the non-tangential limit $L$ at $\xi \in \partial U$ if $f(z) \to L$ as $z \to \xi$, $z \in \Gamma_a(\xi)$; has radial limit $L$ at $\xi$ if $\lim_{t \to 1} f(t\xi) = L$.

Define the spherical derivative of $f(z)$ to be

$$f^s(z) = \frac{|f(z)|}{1 + |f(z)|^2}.$$

Now we can reformulate Theorem 5 in [9] as follows:

**Theorem 1.1.** If $f \in \mathcal{O}(U)$ has a radial limit at the point $\xi \in \partial U$, then it has an non-tangential limit at this point if and only if for any fixed $\alpha > 1$ in the non-tangential region $\Gamma_a(\xi)$

\[
(1.1) \quad f^s(z) \leq O\left(\frac{1}{1 - |z|}\right)
\]

Let $B^n = \{z \in \mathbb{C}^n : |z| < 1\}$ be a unit ball in $\mathbb{C}^n$, $n \geq 1$. Consider the set $D_n(\xi) \subset B^n$ such that

$$|1 - (z, \xi)| < \frac{\alpha}{2}(1 - |z|^2),$$

where $(z, \xi) = z_1 \overline{\xi}_1 + z_2 \overline{\xi}_2$ and $|z|^2 = (z, z)$.

Following Koranyi [4], we say that a holomorphic function $f$ in $B^n$ (henceforth, in symbols, $f \in \mathcal{O}(B^n)$) has admissible limit $L$ at $\xi$ if for every $\alpha > 1$ for every sequence $\{z^j\}$ in $D_n(\xi)$ that converges to $\xi$, $f(z^j) \to L$ as $j \to \infty$. (The case $L = \infty$ is not excluded.)

It is clear that the notions of admissible limit and non-tangential limit coincides when $n = 1$.

The real tangent space to $\partial B^n$ at point $\xi$ contains the complex tangent space $T^c(\partial B^n)$ and $\mathbb{C}^n$ can be splitting $\mathbb{C}^n = N_\xi(\partial B^n) \oplus T^c(\partial B^n)$. The complex line $N_\xi(\partial B^n)$ is called the complex normal to $\partial B^n$ at point $\xi$.

For each $z$ near $\partial B^n$ denote by $\zeta(z)$ the point on $\partial B^n$ closest to $z$. Choose the coordinate system $\zeta_1, \ldots, \zeta_n$ in $\mathbb{C}^n$ such that $\zeta(z) = 0$, $T^c_0(\partial B^n) = \{(0, \zeta_1, \ldots, \zeta_n)\}$, and $N_0 = \{(\zeta_1, 0, \ldots, 0)\}$ and $\nu_0 = (i, 0, \ldots, 0)$ is the inner normal to $\partial B^n$ at $\zeta(z)$.

Set $\overline{\zeta} = (\overline{\zeta}_2, \ldots, \overline{\zeta}_n)$. Then $B^n = \{\overline{z} \in \mathbb{C}^n : |\overline{\zeta}_2 - i|^2 + |\overline{\zeta}_2|^2 < 1\}$.

Let polydisc $P_c(z)$ is defined to be the set of all $\overline{z} \in \mathbb{C}^n$ whose coordinates $\overline{\zeta}_1, \ldots, \overline{\zeta}_n$ in $\mathbb{C}^n$ satisfy the inequalities $|\overline{\zeta}_1 - |z|| < c(1 - |z|), |\overline{\zeta}_\mu| < c\sqrt{1 - |z|},$ $\mu = 2, \ldots, n$, where $c < 1/\sqrt{2}$. For every $\overline{z} \in P_c(z)$ we have $|\overline{\zeta}_1 - i|^2 + |\overline{\zeta}_2|^2 \leq 2|\overline{\zeta}_1 - |z||^2 + 2||z| - 1|^2 + 2|\overline{\zeta}_2|^2 < 2c^2(1 - |z|)^2 + [(n - 1)c^2 + 4](1 - |z|) < [4 + (n + 1)c^2(1 - |z|) < 1$ for all $z$ sufficiently close to $\partial B$. It follows $P_c(z) \subset B$ for all $z$ sufficiently close to $\partial B$. The one variable Cauchy’s estimate shows that

\[
\left|\frac{\partial f}{\partial \overline{\zeta}_1}(z)\right| \leq \frac{\sup_{w \in P_c(z)}|f(w)|}{c(1 - |z|)},
\]

\[
\left|\frac{\partial f}{\partial \overline{\zeta}_2}(z)\right| \leq \frac{\sup_{w \in P_c(z)}|f(w)|}{c\sqrt{1 - |z|}}.
\]

This shows that in several variables the complex normal and complex tangential directions are not equivalent, therefore we will distinct the spherical derivative of $f$ in point $z$ in the complex normal direction ($= \left|\frac{\partial f}{\partial \overline{\zeta}_1}(z)/(1 + |f(z)|^2)\right|$) and the complex tangential directions ($= \left|\frac{\partial f}{\partial \overline{\zeta}_\mu}(z)/(1 + |f(z)|^2)\right|, \mu = 2, \ldots, n$).
Theorem 1.1 fail to be true in several variables. Look at the function \( f(z_1, z_2) = \frac{z_2^2}{1 - z_1} \). It is holomorphic and bounded in \( B^2 \), since \( |f(z)| < (1 - |z_1|)^2/(1 - |z_1|) \leq 2 \). From (1.2) follows that spherical derivative of \( f \) in the complex normal and complex tangential direction grows no faster than \( 2c/(1 - |z|) \) and \( 2c/\sqrt{1 - |z|^2} \) respectively. But this is not sufficient in order that the existence of a limit along the normal for the function \( f \) should imply the existence of an admissible limit.

Indeed, put \( z^j = (1 - 1/j, 1/\sqrt{j}) \) for \( j = 4, 5, \ldots \). It is clear that \( z^j \to \zeta = (1, 0) \) as \( j \to \infty \). A simple calculation shows that \( z^j \in D_0(\zeta) \) if \( j \) is sufficiently large. Notice that \( \lim_{r \to 1-} f(\nu^j_\zeta) = \lim_{r \to 1-} 0 = 0 \) and \( f(z^j) = \frac{1/j^2}{j^2} = 1 \), and so \( f \) does not have admissible limit at \( \zeta \).

However, for \( n = 1 \) in the estimate (1.1) (if a limit along the normal exists) we can replace the right-hand side by \( o(1) \). It turns out that this refined estimate solves the problem for \( n > 1 \).

It was proved in [3] that if \( f \in \mathcal{O}(B^2) \), the spherical derivatives of \( f \) in normal direction increases like \( o(1/(1 - |z|)) \) and spherical derivatives of \( f \) in complex tangential direction increases like \( o(1/\sqrt{1 - |z|^2}) \) then the existence of a limit along the normal for the function \( f \) should imply the existence of an admissible limit.

The main result of the article is the analogous result for arbitrary domains with \( C^2 \)-smooth boundary in \( C^n \), \( n > 1 \).

Montel [5] used normal families in a simple but ingenious way to investigate boundary behavior of holomorphic functions in angular domains. We apply his method to investigate boundary behavior of holomorphic functions of several complex variables in admissible domains.

2. A CRITERION OF EXISTENCE OF ADMISSIBLE LIMITS

If \( D \) is a bounded domain in \( C^n, n > 1 \), with \( C^2 \)-smooth boundary \( \partial D \), then at each \( \xi \in \partial D \) the tangent space \( T_\xi(\partial D) \) and the unit outward normal vector \( \nu_\xi \) are well-defined. We denote by \( T_\xi(\partial D) \) and \( N_\xi(\partial D) \) the complex tangent space and the complex normal space, respectively. The complex tangent space at \( \xi \) is defined as the \((n-1)\) dimensional complex subspace of \( T_\xi(\partial D) \) and given by \( T_\xi(\partial D) = \{ z \in C^n : (z, w) = 0, \forall w \in N_\xi(\partial D) \} \), where \( (\cdot, \cdot) \) denotes canonical Hermitian product of \( C^n \). Let \( \delta(z) \) denotes the Euclidean distance of \( z \) from \( \partial D \) and \( p(z, T_\xi(\partial D)) \) is the Euclidean distance from \( z \) to the real tangent plane \( T_\xi(\partial D) \).

An admissible approach domain \( A_\alpha(\xi) \) with vertex \( \xi \in \partial D \) and aperture \( \alpha > 0 \) is defined as follows [8]:

\[
A_\alpha(\xi) = \{ z \in D : |(z - \xi, \nu_\xi)| < (1 + \alpha)\delta_\xi(z), |z - \xi|^2 < \alpha \delta_\xi(z) \},
\]

\[
\delta_\xi(z) = \min\{\delta(z, \partial D), p(z, T_\xi(\partial D))\}.
\]

It is well known that the introduction of \( \delta_\xi(z) \) and the second condition in (2.1), i.e. \( |z - \xi|^2 < \alpha \delta_\xi(z) \) only serves to rule out the pathological case when \( \partial D \) has flat or concave points. For a ball \( B^n = \{ z \in C^n : |z| < 1 \} \) the set \( D_\alpha(\xi) \) essentially coincides with (2.1).

Definition 2.1. The function \( f \), defined in a domain \( D \) in \( C^n \) has a limit \( L, L \in C \), along the normal \( \nu_\xi \) to \( \partial D \) at the point \( \xi \) iff \( \lim_{t \to 0} f(\xi - t \nu_\xi) = L \); \( f \) has an admissible limit \( L \), at \( \xi \in \partial D \) iff

\[
\lim_{A_\alpha(\xi) \ni z \to \xi} f(z) = L
\]
for every $\alpha > 0$; $f$ is admissible bounded at $\xi$ if $\sup_{z \in A_\alpha(\xi)} |f(z)| < \infty$ for every $\alpha > 0$.

Let $x_j, y_j$ be the real coordinates of $z \in \mathbb{C}^n$ such that $z_j = x_j + iy_j$. At times it will be convenient to use real variable notations by identifying $z$ with $(x_1, \zeta) \in \mathbb{R}^{2n}$, where $\zeta = (y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n-1}$. After a unitary transformation of $\mathbb{C}^n$, if necessary, we may assume the inner normal to $\partial D$ at $0$ points the positive $x_1$ direction, $T_0(\partial D) = \{z \in \mathbb{C}^n : z_1 = 0\}$. Let $\pi : \mathbb{C}^n \to N_0$ be an orthogonal projection, i.e., if $z = (z_1, \ldots, z_n)$ then $\pi(z) = (z_1, 0, \ldots, 0)$.

Without loss of generality, there is a real valued $C^2$ function $\psi$ defined on $T_0(\partial D) = \{(0, \zeta), \zeta \in \mathbb{R}^{2n-1}\}$ so that $\partial D = \{((\psi(\zeta), \zeta), \zeta \in \mathbb{R}^{2n-1}\}$ and $D = \{(x_1, \zeta), x_1 > \psi(\zeta)\}$. (This is certainly true in the neighborhood of $0$ by the implicit function theorem, and our concerns are purely local here.) The fact that $T_0(\partial D)$ is tangent to $\partial D$ at $0$ implies $\nabla \psi(0) = 0$.

For $z = (x_1, \zeta) \in D$ we set

$$d(z) = \min\{x_1, x_1 - \psi(\zeta)\},$$

and define an approach region

$$A_\alpha(\xi) = \{z \in D : |z|^2 < \alpha d(z), |y_1| < \alpha x_1\}.$$

The regions $A_\alpha(\xi)$ are “equivalent” to the admissible approach regions (see \cite[Lemma 5.2]{[1]}) in the sense that

$$A_{\beta(\alpha)}(\xi) \subseteq A_\alpha(\xi) \subseteq A_{\gamma(\alpha)}(\xi).$$

Set

$$(\nabla F)^2 = d^2(z)|\nabla_1 F(z)|^2 + d(z)|\nabla_{2,n} F(z)|^2,$$

where

$$|\nabla_1 F(z)|^2 = \frac{\partial F}{\partial z_1}(z)^2, \quad |\nabla_{2,n} F(z)|^2 = \sum_{j=2}^n \left|\frac{\partial F}{\partial z_j}(z)\right|^2.$$

We begin with proposition.

**Proposition 2.2.** Let $D$ be a domain in $\mathbb{C}^n$, $n > 1$, with $C^2$-smooth boundary. Suppose that the function $f \in \mathcal{O}(D)$ has a limit $L$ along the normal $\nu_\xi$ to $\partial D$ at the point $\xi$ equal to $L$, $L \neq \infty$. If

$$(1 + |f(z)|^2)^{-1}\nabla f(z)$$

is admissible bounded at $\xi$, then $f$ admissible bounded at $\xi$.

**Proof.** Assume $\xi = 0$. Since the domain $D$ has $C^2$-smooth boundary, then there is a constant $r > 0$ such that the ball $B_r(-rv_0) \subset D$ and $\partial B_r(-rv_0) \cap \partial D = \{0\}$.

Let the function $f$ has the finite limit $L$ along the normal $v_0$ to $\partial D$ at the point $0$. Since $d(z) \geq |r - z_1| \geq r - |z_1| \geq \frac{1}{2r}(r^2 - |z_1|^2)$ for all $z \in B_r(-rv_0)$ sufficiently close to $0$ we have

$$(r^2 - |z_1|^2) \left| \frac{\partial F}{\partial z_1}(\pi(z)) \right| < \frac{\nabla f(\pi(z))}{1 + |f(\pi(z))|^2} < O(1), \quad z \in A_\alpha(0) \cap N_0'(\partial D).$$

Therefore $f(\pi(z))$ fulfills all the hypotheses of Theorem \[1\] Hence $f(\pi(z)) \to L$ as $z \to 0$, $z \in A_\alpha(0) \cap N_0'(\partial D)$.

Assume, to reach a contradiction, that $f$ is not admissible bounded at $0$. Let $\{z^m\}$ be any sequence of points from $A_\alpha(0)$ such that $z^m \to 0$ as $m \to \infty$ and $f(z^m) \to \infty$ as $m \to \infty$. 
For the biholomorphic mapping $\Phi_b(z) = (w_1(z), \ldots, w_n(z))$, where $w_1(z) = \frac{z - b_1}{2\alpha (b_1)}$, $w_\mu(z) = \frac{z - b_\mu}{2\alpha \sqrt{d(b_\mu)}}$, $\mu = 2, \ldots, n$, the polydisc

$$P(b, c) = \{ z \in \mathbb{C}^n : |z_1 - b_1| < cd(b), |w_\mu - b_\mu| < c\sqrt{d(b)}, \mu = 2, \ldots, n, \}$$

is mapped to the unit polydisc $U^n = \{ w \in \mathbb{C}^n : |w_\mu| < 1, \mu = 1, \ldots, n \}$. By [7, Lemma 7.2] there exists $c = c(\alpha)$ such that $P(b, c) \subset A_{2\alpha}(0)$ for all sufficiently small $b \in A_{\alpha}(0)$. Therefore with each point $b \in A_{\alpha}(0)$ sufficiently close to 0 we can associate a function $g_b = f(\Psi_b^{-1}(w))$ which is well defined and holomorphic in polydisc $U^n$.

By [7, Lemma 5.2] there exists $c = c(\alpha)$ so that if $z = (x_1, \zeta)$ sufficiently small and $|z| < cd(z)$ we have $d(z) \geq cx_1$. Let $t$ be an arbitrary point of the interval $[z^m, \pi(z^m)]$. Note that $x_1^m \geq d(t) \geq cx_1^m$.

Choose an integer $N$ such that $\alpha < cN/2$. From the definitions of the set $A_{\alpha}(0)$ it follows that $|z^m - \pi(z^m)|^2 < cN x_1^m/2$. Then any interval $[z^m, \pi(z^m)]$ may be covered by $k_m$ polydiscs, where $k_m < N + 1$,

$$P_{m,k}(c) = P(b^m, k, c) = \{ z \in \mathbb{C}^n : |z_1 - b^m_1| < cd(b^m, k), |z_\mu - b^m_\mu| < c\sqrt{d(b^m, k)}, \mu = 2, \ldots, n \}$$

such that $b^m_1 = (z^m_1, 0)$, $b^m_\mu = (z^m_\mu, \pi(z^m))$, $\mu = 2, \ldots, m - 1, k = 2, \ldots, k_m - 1$, $P_{m,k}(c/2) \supset b^m_{k+1}$ (and hence $P_{m,k}(c/2) \cap P_{m,k+1}(c/2) \neq \emptyset$) for all $m, k$. To each point $b^m_\mu$ we associate a function $g_{m,k} = g_{b^m,k}$ as above.

Set $G^m = g_{m,k_m}$, $m \geq 1$. Since $f(z^m) \to \infty$ as $m \to \infty$ and $P_{m,k_m}(c) \supset z^m$ we have $g_{m,k_m}(0) = f(z^m) \to \infty$ as $m \to \infty$. Suppose that there is a sequence of points $\{ w^m \}$ which belongs to some polydisc $P_2 \supset U_2 \subset U^n$, such that $G^m(w^m) \to \infty$ as $m \to \infty$. It follows that the family $\{ G^m \}$ is not normal in $U^n$ and by Marty’s criterion (see, e.g., [3]) there are points $\theta^m \in \partial P_2$ and vectors $v^m \in \mathbb{C}^n$ with $|v^m| = 1$ such that

$$\frac{(dG^m_{\theta v^m}(v^m), dG^m_{v^m}(v^m))}{(1 + |G^m(v^m)|^2)^2} > m, \quad (m = 1, 2, \ldots), \quad (2.3)$$

where

$$dG^m_{v^m}(v^m) = \sum_{\mu=1}^{n} \frac{\partial G^m_{\theta v^m}(v^m)}{\partial w_\mu} v_\mu.$$

According to the rule of differentiation of composite functions

$$\frac{\partial G^m_{v^m}(v^m)}{\partial w_1}(v^m) = cd(b^m_1) \frac{\partial f(t^m)}{\partial z_1}(t^m)$$

$$\frac{\partial G^m_{v^m}(v^m)}{\partial w_\mu}(v^m) = c\sqrt{d(b^m_\mu)} \frac{\partial f(t^m)}{\partial z_\mu}(t^m), \quad (\mu = 2, \ldots, n),$$

where $t^m = \Psi^{-1}_{b^m,1}(p^m) \in P_{m,1}(c) \subset A_{2\alpha}(0)$. By [7, Lemma 5.2] there exists $c_1 = \min\{1/2, 1/2K\alpha\}$ so that if $z = (x_1, \zeta) \in A_{2\alpha}(0)$ is sufficiently small then $x_1 > d(z) \geq c_1 x_1$. Since $b^m_1 = x_1^m$ and $(1 - c)x_1^m \leq Re t^m \leq (1 + c)x_1^m$ we have

$$c_1 \left( 1 + \frac{d(b^m_1)}{d(t^m)} \right) \leq \frac{1}{c_1(1 - c)}.$$

This, together with the Bunyakovskiï-Schwarz inequality, implies from (2.3) that

$$O(1) \frac{(\nabla f(t^m))^2}{(1 + |f(t^m)|^2)^2} > m.$$
It follows that $\frac{(|\nabla f(z)|^2)}{1 + |f(z)|^2}$ is not admissible bounded in 0, a contradiction with hypothesis of the theorem. Therefore sequence $\{G^m\}$ converges uniformly on compact subsets of $U^n$ to $\infty$. Put now $G^m = g_m \{k_m - 1\}$, $m \geq 1$. (Note that we set $g_m \{k_m - 1\} = g_m k_m$ if $k_m - 1 \leq 0$.) Since $P_{m,k_m - 1}(c/2) \cap P_{m,k_m}(c/2) \neq \emptyset$ we have $G^m(0) \to \infty$ as $m \to \infty$ and we may repeat the above argument. After finite number of steps the proof will be completed since $P_{m,1}(c) \ni \pi(z^m)$ and $f(\pi(z^m)) \to L$ as $m \to \infty$. We get $|\nabla f(z)|^2/(1 + |f(z)|^2)$ is not admissible bounded in 0, contrary to the hypothesis on $\nabla f(z)/(1 + |f(z)|^2)$. This contradiction proves our claim. \hfill \Box

**Theorem 2.3.** Let $D$ be a domain in $\mathbb{C}^n$, $n > 1$, with $C^2$-smooth boundary. If a function $f$ holomorphic in $D$ has a limit along the normal $\nu_\xi$ at a point $\xi \in \partial D$, then it has an admissible limit at this point if and only if for every $\alpha > 0$

$$
(2.4) \quad (1 + |f(z)|^2)^{-1}\nabla f(z) \to 0
$$

as $z \to \xi$, $z \in A_\alpha(\xi)$.

**Proof.** Assume $\xi = 0$, without loss of generality. First, let $f$ has finite admissible limit $L$ at 0. Without loss of generality, assume $L = 0$ at 0. Let $P_{1}(z)$ denote the polydisc centered at $z$, whose radii are essentially $cx_1, c\sqrt{x_1}, \ldots, c\sqrt{x_1}$, with $c$ sufficiently small. By \[7\, Lemma 7.2\] exists $c = c(\alpha)$ such that $P_{1}(z) \subset A_{2\alpha}(\xi)$. Let $P(z)$ denote the polydisc centered at $z$, whose radii are essentially $cd(z), c \sqrt{d(z)}, \ldots, c \sqrt{d(z)}$. Since $d(z) = \min\{x_1, x_1 - \psi(\zeta)\} \leq x_1$ we have $P(z) \subset P_{1}(z) \subset D$. The one variable Cauchy’s estimate shows that

$$
|\nabla_1 f(z)| \leq \sup_{w \in P(z)} |f(w)| \frac{cd(z)}{c \sqrt{d(z)}},
$$

$$
|\nabla_2 f(z)| \leq \sup_{w \in P(z)} |f(w)| \frac{c \sqrt{d(z)}}{c \sqrt{d(z)}}.
$$

Since $f(z) \to 0$ as $z \to 0$, $z \in A_{\alpha}(0)$, we have

$$
\nabla f(z) \to 0
$$

as $z \to 0$, $z \in A_{\alpha}(0)$. It remains to observe that $\nabla f(z) \geq (1 + |f(z)|^2)^{-1}\nabla f(z)$.

If the function $f$ has an admissible limit at the point 0 equal to infinity, then for any $\alpha > 0$ there is a $\varepsilon > 0$ such that $1/f \in \mathcal{O}(A_{\alpha}(0) \cap B_{\varepsilon}(0))$. The function $F = 1/f$ has an admissible limit equal to zero at the point 0, so, as we have proved, $F$ satisfies (2.4). It remains to observe that outside the zeros of $f$ we obviously have $|1/F(z)|^2^{-1}\nabla F(z) = (1 + |f(z)|^2)^{-1}\nabla f(z)$.

**Sufficiency.** (a) Suppose that the function $f$ has a limit $L$ along the normal $\nu_0$ to $\partial D$ at the point 0 equal to $L$, $L \neq \infty$.

We may assume, without loss of generality, that $L = 0$. Write

$$
f(z) = \{f(z) - f(z_1, 0, \ldots, 0)\} + f(z_1, 0, \ldots, 0).
$$

The first term on the right side is dominated by $|z(1) - z(0)| \sup_{0 < t < 1} |\nabla_2, n f(z(t))|$, where $z(t) = (z_1, z_2 t, \ldots, z_n t)$, $t \in [0, 1]$. If $z \in A_{\alpha}(0)$, then by \[7\, Lemma 7.3\] $z(t) \in A_{\alpha}(0)$, $t \in [0, 1]$, and there $d(z(t)) \approx d(z)$ while $|z(1) - z(0)| < \alpha d(z)$.

(The expression $A \approx B$ means that there are positive constants $c_1$ and $c_1$ such that $c_1 A < B < c_2 A$.) By Proposition \[22\] $f$ is admissible bounded in 0 and therefore

$$
|z(1) - z(0)| \sup_{0 < t < 1} |\nabla_2, n f(z(t))| \leq O(1) \frac{\nabla f(z(t_0))}{1 + |f(z(t_0))|^2},
$$
where \(0 \leq t_0 \leq 1\). Since \(\nabla f(z(t_0))/(1 + |f(z(t_0))|) \to 0\) as \(z(t_0) \to 0\) we have that \(f(z) - f(z_1,0,\ldots,0) \to 0\) as \(z \to 0\) in \(A_\alpha(0)\). Since \(f(z_1,0,\ldots,0) \to 0\) as \(z \to 0\) in \(A_\alpha(0)\) we conclude that

\[
\lim_{A_\alpha(0) \ni z \to 0} f(z) = 0.
\]

The above proof is quite analogous to the proof in [8, p. 68].

(b) Let the function \(f\) has the infinite limit along the normal \(\nu_0\) to \(\partial D\) at the point 0. Let \(\{z^m\}\) be any sequence of points from \(A_\alpha(0)\) such that \(z^m \to 0\) as \(m \to \infty\). As in the proof of Proposition 2.2 let \(\{G^m\}\), be a sequence of function defined on \(U^m\). Then as in Proposition 2.2 we obtain \(f(z^m) \to \infty\) as \(m \to \infty\). Since the sequence of points \(\{z^m\}\) was arbitrary, by definition this means that \(f\) has the admissible limit equal to infinity at the point 0. The theorem is proved. \(\square\)

For each \(z\) near \(\partial D\) denote by \(\zeta(z)\) the point on \(\partial D\) closest to \(z\). Choose the coordinate system \(\tilde{z}_1,\ldots,\tilde{z}_n\) such that \(\zeta(z) = 0\), and \(\{\tilde{z} \in C^n : (\tilde{z}_1,0,\ldots,0) = N_0(\partial D)\), and \(\{\tilde{z} \in C^n : (0,\tilde{z}_2,\ldots,\tilde{z}_n) = T_0(\partial D)\), and \(\nu_0 = (1,0,\ldots,0)\). Denote by

\[\text{grad}_C F = \left(\frac{\partial F}{\partial \tilde{z}_1},\ldots,\frac{\partial F}{\partial \tilde{z}_n}\right)\]

the complex gradient of function \(F\). Write also

\[
|\nabla_1 F|^2 = \left|\frac{\partial F}{\partial \tilde{z}_1}\right|^2,
\]

\[
|\nabla_{2,n} F|^2 = \sum_{j=2}^n \left|\frac{\partial F}{\partial \tilde{z}_j}\right|^2.
\]

Then \(|\text{grad}_C F|^2 = |\nabla_1 F|^2 + |\nabla_{2,n} F|^2\) but this splitting varies (with the decomposition \(C^n = N_\zeta(\xi) \oplus T_\zeta(\xi)\)) as \(z\) varies in \(A_\alpha(\xi)\).

We need to observe that (the proof is the same as in [8, pp. 61-62])

\[
2.5 \quad d^2(\zeta) |\nabla_1 F|^2 + d(\zeta) |\nabla_{2,n} F|^2 \approx d^2(\zeta) |\nabla_1 F|^2 + d(\zeta) |\nabla_{2,n} F|^2 \quad (z \in A_\alpha(\xi)).
\]

We write \(A \approx B\) if the ratio \(|A|/|B|\) is bounded between two positive constants. We call

\[
\frac{\nabla_{\tilde{z}_1} f(\zeta)}{1 + |f(\zeta)|^2} \text{ and } \frac{\nabla_{\tilde{z}_\mu} f(\zeta)}{1 + |f(\zeta)|^2} \quad (\mu = 2,\ldots,n)
\]

the spherical derivative of \(f(z)\) in the normal and complex tangent direction, respectively. From (2.5) follows that Theorem 2.3 is actually equivalent to:

**Theorem 2.4.** Let \(D\) be a domain in \(C^n\), \(n > 1\), with \(C^2\)-smooth boundary. If a holomorphic function \(f\) has a limit along the normal to \(\partial D\) at the point \(\xi\), then at the point \(\xi \in \partial D\) the function \(f\) has an admissible limit if and only if in every admissible domain with vertex \(\xi\) the spherical derivative of \(f\) in the normal and complex tangent directions increases like \(o(1/d(\zeta))\) and \(o(1/\sqrt{d(\zeta)})\), respectively.

The example in the beginning of this article shows that the Lindelöf principle for bounded functions – formulated in terms of admissible convergence – fails. However the following refinement of Lindelöf’s theorem holds.

**Theorem 2.5.** Let \(D\) be a domain in \(C^n\), \(n > 1\), with \(C^2\)-smooth boundary. If a function \(f\) in \(D\) has a limit \(L, L \in C\), along the normal \(\nu_\zeta\) at a point \(\xi \in \partial D\), and in every admissible domain with vertex \(\xi\) the function \(f\) is holomorphic, \(L\) is its omitted value and the spherical derivative of \(f\) in the normal and complex tangent directions grows no faster than \(K/d(z)\) and \(K/\sqrt{d(z)}\), respectively, then \(f\) has an admissible limit \(L\) at \(\xi\).
that Using the notation introduced in the proof of Proposition 2.2, the Bunyakovskiĭ-
Schwarz inequality and the fact that a little calculation shows that for all grows no faster than for all .

\( \text{Proof.} \) By hypothesis of the theorem \( f(D) \) then \( f(z) - L \) is holomorphic on \( D \) and has a radial limit at \( \xi \) equal to \( \infty \). It is thus sufficient to consider the case \( L = \infty \).

By Theorem \[1\] and hypothesis on \( f \) we have \( f(\pi(z)) \to \infty \) as \( z \to \xi, z \in A_\alpha(\xi) \cap N(\partial D). \) Let \( \{z^m\} \) be any sequence of points from \( A_\alpha(\xi) \) such that \( z^m \to \xi \) as \( m \to \infty \). Since the spherical derivative of \( f \) in the normal and complex tangent directions grows no faster than \( K/d(z) \) and \( K/\sqrt{d(z)} \), respectively, from \( (2.3) \) follows

\[ d^2(z) | \nabla_1 F(z) |^2 + d(z) | \nabla_{2,n} F(z) |^2 \leq O(1) \quad (z \in A_\alpha(\xi)). \]

Using the notation introduced in the proof of Proposition \[2\] the Bunyakovskiĭ-
Schwarz inequality and the fact that \( d(b^{m,1}) \approx d(z) \) for all \( z \in P_{m,1} \) it follows that

\[ \frac{(dG_p^m(v), dG_p^m(v))}{(1 + |G_p^m(v)|^2)^2} \leq O(1) \quad (m = 1, 2, \ldots) \]

for all \( p \in P \) and all \( v \in C^n, |v| = 1 \).

By Marty’s criterion (see, e.g., \[2\]) the family \( \{G^m\} \) are normal in \( U^n \). Since \( G^m(\pi(z^m)) = g_{m,1}(0) \to \infty \) as \( m \to \infty \) it follows that the sequence \( \{G^m\} \) converges uniformly on compact subsets of \( U^n \) to \( \infty \). Then as in Theorem \[2.3\] we obtain \( f(z^m) \to \infty \) as \( m \to \infty \).

Since the sequence of points \( \{z^m\} \) chosen from \( A_\beta(0) \) is arbitrary, this completes the proof that the function \( f \) has the admissible limit \( L \) at the point \( \xi \). The theorem is proved.

\[ \square \]

**Theorem 2.6.** Let \( D \) be a domain in \( \mathbb{C}^n, n > 1 \), with \( C^2 \)-smooth boundary. Let in every admissible domain with vertex \( \xi \) the function \( f \) is holomorphic and its spherical derivative in the normal and complex tangent directions grows no faster than \( K/d(z) \) and \( K/\sqrt{d(z)} \), respectively. If

\[ \lim_{A_\beta(\xi) \ni z \to \xi} f(z) = L \text{ for some } \beta > 0, \]

then \( f \) has an admissible limit at \( \xi \).

**Proof.** Fix \( \alpha > \beta \). Let \( \{z^m\} \) be an arbitrary sequence of \( A_\alpha(\xi) \). Let \( G^m = g_{m,1}, m \geq 1, \) be the sequence of function defined as in proof of Proposition \[2.2\]. The family \( \{g_{m,1}\} \) is normal on \( D \) (this was proved in Theorem \[2.3\]). Since \( f(z) \to L \) as \( z \to 0 \) in \( A_\beta(0) \), without lost a generality, we may assume that \( P_{m,1}(c) \subset A_\beta(0) \) for all \( m = 1, 2, \ldots \). Hence \( G^m \) tends to \( L \) uniformly on every compact subset of \( P \).

By \[2\] Lemma 5.2] there exists \( c_1 = \min\{1/2, 1/2K_\alpha\} < 1/2 \) so that if \( z = (x_1, \zeta) \in A_\alpha(0) \) is sufficiently small then \( |x_1| > d(z) \geq c_1x_1 \). Since \( b_1^{m,1} = b_1^{m,2} = x_1^m \)
we have

\[ c_1 \leq \frac{d(b^{m,2})}{d(b^{m,1})} \leq \frac{1}{c_1}. \]

Since

\[ \Psi_{b_1^{m,2}}(w) = (cd(b^{m,2})w + b_1^{m,2}, c\sqrt{d(b^{m,2})}w + b_2^{m,2}, \ldots, c\sqrt{d(b^{m,2})}w + b_n^{m,2}), \]

\[ |b_1^{m,2} - b_1^{m,1}| < c/2 \cdot d(b^{m,1}), \text{ and } |b_i^{m,2} - b_i^{m,1}| < c/2\sqrt{d(b^{m,1}), \mu = 1, 2, \ldots, n}, \]

the little calculation shows that for for all \( w \in P(0, c_1/4) \subset P \)

\[ |w_1cd(b^{m,2}) - b_1^{m,2}| < \frac{cc_1}{4} \frac{d(b^{m,2})}{d(b^{m,1})}d(b^{m,1}) + \frac{c}{2}d(b^{m,1}) < \frac{3c}{4}d(b^{m,1}) \]

and
and
\[ |w_{\mu} c d(b_{m}^{\mu}) - b_{m}^{\mu}| < \left( \frac{cc_{1}}{4\sqrt{c_{1}}} + \frac{c}{2} \right) \sqrt{d(b_{m}^{\mu.1})} < \frac{3c}{4} d(b_{m}^{\mu.1}) \quad \mu = 1, 2, \ldots, n, \]

It follows \( g_{m,2} \) takes the same values on \( P(0, c_{1}/4) \) as \( f \) on \( \Psi_{0}^{-1}(P(0, c_{1}/4)) \subset P_{m,1}(c) \) hence \( g_{m,2} \rightarrow L \) on \( P(0, c_{1}/5) \subset P \).

The family \( \{g_{m,2}\} \) is normal on \( P \) (this was proved in Theorem 2.4) hence the family \( \{g_{m,2}\} \) also tends to \( L \) uniformly on compact subsets of \( P \). After finite steps we obtain that \( f(z^{m}) \rightarrow L \) as \( m \rightarrow \infty \). Since the sequence of points \( \{z^{m}\} \) chosen from \( A_{\beta}(0) \) is arbitrary, this completes the proof that the function \( f \) has the admissible limit \( L \) at the point \( \xi \). The theorem is proved. \( \square \)

For bounded holomorphic functions this theorem appears in Chirka’s paper [1], with the proof sketched there relying on certain estimates on harmonic measures. A proof based on a different method was given by Ramey [7, Theorem 2].

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