A SIMPLE PROOF OF BROWN’S DIAGONALIZABILITY THEOREM

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We present here a simple proof of Brown’s diagonalizability theorem for certain elements of the algebra of a left regular band \([1, 2]\), including probability measures. Brown’s theorem also provides a uniform explanation for the diagonalizability of certain elements of Solomon’s descent algebra, since the descent algebra embeds in a left regular band algebra \([1, 2]\). Recall that a left regular band is a semigroup satisfying the identities \(x^2 = x\) and \(xyx = xy\). In this paper all semigroups are assumed finite.

Let \(S\) be a left regular band with identity (there is no loss of generality in assuming this) and let \(L\) be the lattice of principal left ideals of \(S\) ordered by inclusion. We view \(L\) as a monoid via its meet, which is just intersection. There is a natural surjective homomorphism \(\sigma : S \rightarrow L\), called the support map, given by \(\sigma(s) = Ss\). A key fact that we shall exploit is that \(\sigma(s) \leq \sigma(t)\) if and only if \(st = s\), that is, \(s \in St\) if and only if \(st = s\). Indeed, let \(S\) act on the right of itself. Because \(t\) is an idempotent, it acts as the identity on its image; but this is just \(St\).

Let \(k\) be a field and let

\[
\sum_{t \in S} w_t t \in kS.
\]

(1)

For \(X \in L\), define

\[
\lambda_X = \sum_{\sigma(t) \geq X} w_t.
\]

(2)

Brown [1, 2] showed that \(k[w]\) is split semisimple provided that \(X > Y\) implies \(\lambda_X \neq \lambda_Y\). We give a new proof of this by showing that if \(\lambda_1, \ldots, \lambda_k\) are the distinct elements of \(\{\lambda_X \mid X \in L\}\), then

\[
0 = \prod_{i=1}^{k} (w - \lambda_i).
\]

(3)

This immediately implies that the minimal polynomial of \(w\) has distinct roots and hence \(k[w]\) is split semisimple.

Everything is based on the following formula for \(sw\).

\[\text{Date: October 10, 2009.}\]
\[\text{The author was supported in part by NSERC.}\]
\[\text{1Brown calls the dual of this lattice the support lattice.}\]
Lemma 1. Let $s \in S$. Then
$$sw = \lambda_{\sigma(s)} s + \sum_{\sigma(t) \not\supset \sigma(s)} w_{tst}$$
and moreover, $\sigma(s) > \sigma(st)$ for all $t$ with $\sigma(t) \not\supset \sigma(s)$.

Proof. Using that $\sigma(t) \geq \sigma(s)$ implies $st = s$, we compute
$$sw = \sum_{\sigma(t) \geq \sigma(s)} w_{tst} + \sum_{\sigma(t) \not\supset \sigma(s)} w_{tst}$$
$$= \sum_{\sigma(t) \geq \sigma(s)} w_{tst} + \sum_{\sigma(t) \not\supset \sigma(s)} w_{tst}$$
$$= \lambda_{\sigma(s)} s + \sum_{\sigma(t) \not\supset \sigma(s)} w_{tst}.$$ 

It remains to observe that $\sigma(t) \not\supset \sigma(s)$ implies $\sigma(st) = \sigma(s)\sigma(t) < \sigma(s)$. □

The proof of Lemma 2 proceeds via an induction on the support. Let us write $\hat{0}$ for the bottom of $L$ and $\hat{1}$ for the top. If $X \in L$, put
$$\Lambda_X = \{\lambda_Y \mid Y \leq X\} \quad \text{and} \quad \Lambda'_X = \{\lambda_Y \mid Y < X\}.$$ 

Our hypothesis says exactly that $\Lambda_X = \{\lambda_X\} \cup \Lambda'_X$ (disjoint union). Define polynomials $p_X(z)$ and $q_X(z)$, for $X \in L$, by
$$p_X(z) = \prod_{\lambda_i \in \Lambda_X} (z - \lambda_i)$$
$$q_X(z) = \prod_{\lambda_i \in \Lambda'_X} (z - \lambda_i) = \frac{p_X(z)}{z - \lambda_X}. $$

Notice that, for $X > Y$, we have $\Lambda_Y \subseteq \Lambda'_X$, and hence $p_Y(z)$ divides $q_X(z)$, because $\lambda_X \not\in \Lambda_Y$ by assumption. Also observe that
$$p_{\hat{1}}(z) = \prod_{i=1}^{k} (z - \lambda_i)$$
and hence establishing $p_{\hat{1}}$ is equivalent to proving $p_{\hat{1}}(w) = 0$.

Lemma 2. If $s \in S$, then $s \cdot p_{\sigma(s)}(w) = 0$.

Proof. The proof is by induction on $\sigma(s)$ in the lattice $L$. Suppose first $\sigma(s) = \hat{0}$; note that $p_{\hat{0}}(z) = z - \lambda_{\hat{0}}$. Then since $\sigma(t) \geq \sigma(s)$ for all $t \in S$, Lemma 1 immediately yields $s(w - \lambda_{\sigma(s)}) = 0$. In general, assume the lemma holds for all $s' \in S$ with $\sigma(s') < \sigma(s)$. Then by Lemma 1
$$s \cdot p_{\sigma(s)}(w) = s \cdot (w - \lambda_{\sigma(s)}) \cdot q_{\sigma(s)}(w) = \sum_{\sigma(t) \not\supset \sigma(s)} w_{tst} \cdot q_{\sigma(s)}(w) = 0.$$ 

Here the last equality follows because $\sigma(t) \not\supset \sigma(s)$ implies $\sigma(s) > \sigma(st)$ and so $p_{\sigma(st)}(z)$ divides $q_{\sigma(s)}(z)$, whence induction yields $st \cdot q_{\sigma(s)}(w) = 0$. □
Applying the lemma to the identity element of $S$ yields $p_1(w) = 0$ and hence we have proved:

**Theorem 3.** Let $w$ be as in (1) and let $\lambda_X$ be as in (2) for $X \in L$. If $X > Y$ implies $\lambda_X \neq \lambda_Y$, then $k[w]$ is split semisimple.

If $k = \mathbb{R}$, and $w$ is a probability measure, then $X > Y$ implies $\lambda_X > \lambda_Y$ provided the support of $w$ generates $S$ as a monoid. If this is not the case, then semisimplicity of $\mathbb{R}[w]$ follows by considering $\mathbb{R}[w] \subseteq \mathbb{R}T \subseteq \mathbb{R}S$ where $T$ is the submonoid generated by the support of $w$.

**References**

[1] K. S. Brown. Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938, 2000.

[2] K. S. Brown. Semigroup and ring theoretical methods in probability. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 3–26. Amer. Math. Soc., Providence, RI, 2004.

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