Existence of a positive solution for a logarithmic Schrödinger equation with saddle-like potential

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Abstract

In this article we use the variational method developed by Szulkin [21] to prove the existence of a positive solution for the following logarithmic Schrödinger equation

\[
\begin{cases}
-\epsilon^2 \Delta u + V(x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

where \( \epsilon > 0, N \geq 1 \) and \( V \) is a saddle-like potential.

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1 Introduction

In recent years, the nonlinear Schrödinger equation

\[
\begin{equation}
\tag{1.1}
\begin{array}{c}
-\epsilon^2 \Delta \psi + (V(x) + E) \psi - f(\psi), \\
\end{array}
\end{equation}
\]

where \( N \geq 2, \epsilon > 0, V, f \) are continuous functions, has been studied by many researchers. It is very important to seek the standing wave solutions of (1.1), which are solutions of the type \( \Phi(z, t) = \exp(iEt)u(z), u : \mathbb{R}^N \to \mathbb{R} \), where \( u \) is a solution of the elliptic equation

\[
\begin{cases}
-\epsilon^2 \Delta u + V(x)u = f(u), & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

(P_\epsilon)

The existence and concentration of positive solutions for general semilinear elliptic equation \((P_\epsilon)\), for the case \( N \geq 2 \), have been extensively studied, we refer to [1, 5, 7, 10, 11, 13, 17, 18, 23] for the advances in this area.

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In [11], del Pino, Felmer and Miyagaki considered the case where potential \( V \) has a geometry like saddle, essentially they assumed the following conditions on \( V \): First of all, they fixed two subspaces \( X, Y \subset \mathbb{R}^N \) such that \( \mathbb{R}^N = X \oplus Y \). By supposing that \( V \) is bounded, they fixed \( c_0, c_1 > 0 \) satisfying

\[
 c_0 = \inf_{z \in \mathbb{R}^N} V(z) > 0 \quad \text{and} \quad c_1 = \sup_{x \in X} V(x).
\]

Furthermore, they also assumed that \( V \in C^2(\mathbb{R}^N) \) and it verifies the following geometric conditions:

\[
 (V1) \quad c_0 = \inf_{R > 0} \sup_{x \in \partial B_R(0) \cap X} V(x) < \inf_{y \in Y_\lambda} V(y),
\]

for some \( \lambda \in (0, 1) \) and \( Y_\lambda = \{ z \in \mathbb{R}^N : |z \cdot y| > \lambda |z||y|, \text{for some } y \in Y \} \)

\[
 (V2) \quad \text{The functions } V, \frac{\partial V}{\partial x_i} \text{ and } \frac{\partial^2 V}{\partial x_i \partial x_j} \text{ are bounded in } \mathbb{R}^N \text{ for all } i, j \in \{1, \ldots, N\}.
\]

\[
 (V3) \quad V \text{ satisfies the Palais-Smale condition, that is, if } (x_n) \subset \mathbb{R}^N \text{ is a sequence such that } (V(x_n)) \text{ is convergent and } \nabla V(x_n) \to 0, \text{ then } (x_n) \text{ possesses a convergent subsequence in } \mathbb{R}^N.
\]

Using the above conditions on \( V \), and supposing that

\[
 c_1 < 2^{\frac{2(p-1)}{N+2-p(N-2)}} c_0,
\]

del Pino, Felmer and Miyagaki used the variational method to show the existence of positive solutions for the following problem

\[
 -\epsilon^2 \Delta u + V(z)u = |u|^{p-2}u, \text{ in } \mathbb{R}^N,
\]

where \( p \in (2, 2^*) \) if \( N \geq 3 \) and \( p \in (2, +\infty) \) if \( N = 1, 2 \), for \( \epsilon > 0 \) small enough.

Under the same assumptions on the potential \( V \), Alves [1] considered the existence of a positive solution for the following elliptic equation with exponential critical growth in \( \mathbb{R}^2 \)

\[
 -\epsilon^2 \Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^2.
\]

After, in [5], Alves and Miyagaki studied the following critical nonlinear elliptic equation with saddle-like potential in \( \mathbb{R}^N \)

\[
 \epsilon^{2s}(-\Delta)^s u + V(z)u = \lambda |u|^{q-2}u + |u|^{2^*_s-2}u, \text{ in } \mathbb{R}^N,
\]

where \( \epsilon, \lambda > 0 \) are positive parameters, \( q \in (2, 2^*_s) \), \( 2^*_s = \frac{2N}{N-2s} \), \( N > 2s \), \( s \in (0, 1) \), \( (-\Delta)^s \) is fractional Laplacian. Under similar assumptions on \( V \), the authors obtained the existence of a positive solution by using the variational method.

Recently, the logarithmic Schrödinger equation given by

\[
 i\epsilon \partial_t \Psi = -\epsilon^2 \Delta \Psi + (W(x) + w)\Psi - \Psi \log |\Psi|^2, \Psi : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}, \ N \geq 1,
\]
has also been received considerable attention. This equation appears in interesting physical applications, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein condensation (see [24] and the references therein). In its turn, standing waves solution, \( \Psi \), for this logarithmic Schrödinger equation is related to solutions of the equation

\[
- \epsilon^2 \Delta u + V(x)u = u \log u^2, \quad \text{in } \mathbb{R}^N.
\]

Besides the importance in applications, this last equation also raises many difficult mathematical problems. The natural candidate for the associated energy functional would formally be the functional

\[
\hat{I}_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (\epsilon^2 |\nabla u|^2 + (V(x) + 1)|u|^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx.
\]  

(1.2)

It is easy to see that each critical point of \( \hat{I}_\epsilon \) is a solution of (1.2). However, this functional is not well defined in \( H^1(\mathbb{R}^N) \) because there is \( u \in H^1(\mathbb{R}^N) \) such that \( \int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty \).

In order to overcome this technical difficulty some authors have used different techniques, for more details see [2], [3], [4], [8], [9], [12], [14], [19], [20], [22] and their references.

Motivated by studies found in the above-mentioned papers and the results obtained in [1, 5, 11], in the present paper, our main goal is to show the existence of a positive solution for the following logarithmic Schrödinger equation

\[
- \epsilon^2 \Delta u + V(x)u = u \log u^2, \quad \text{in } \mathbb{R}^N,
\]  

(1.3)

where \( N \geq 1 \), \( \epsilon > 0 \) is a positive parameter and the potential \( V \) satisfies the conditions (V1) – (V3). In our case, different from [11], we can assume \( c_0 > -1 \).

By a change of variable, we know that problem (1.3) is equivalent to the problem

\[
\begin{cases}
-\Delta u + V(\epsilon x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases}
\]  

(1.4)

We shall use the variational method found in Szulkin [21] to study the problem (1.4). Firstly notice that, a weak solution of (1.4) in \( H^1(\mathbb{R}^N) \) is a critical point of the associated energy functional

\[
J_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (V(\epsilon x) + 1)|u|^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx.
\]

Definition 1.1. For us, a positive solution of (1.4) means a positive function \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that \( u^2 \log u^2 \in L^1(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(\epsilon x)u \cdot v) dx = \int_{\mathbb{R}^N} uv \log u^2 dx, \quad \text{for all } v \in C_0^\infty(\mathbb{R}^N).
\]  

(1.5)

Following the approach explored in [2] [14] [19], due to the lack of smoothness of \( J_\epsilon \), let us decompose it into a sum of a \( C^1 \) functional plus a convex lower semicontinuous functional, respectively. For \( \delta > 0 \), let us define the following functions:
\[
F_1(s) = \begin{cases} 
0, & s = 0 \\
-s^2 \log s^2 - \frac{1}{2} s^2 (\log \delta^2 + 3) + 2\delta |s| - \frac{1}{2} \delta^2, & |s| \geq \delta 
\end{cases}
\]

and

\[
F_2(s) = \begin{cases} 
0, & |s| < \delta \\
\frac{1}{2} s^2 \log (s^2 / \delta^2) + 2\delta |s| - \frac{3}{2} s^2 - \frac{1}{2} \delta^2, & |s| \geq \delta 
\end{cases}
\]

Therefore

\[
F_2(s) - F_1(s) = \frac{1}{2} s^2 \log s^2, \quad \forall s \in \mathbb{R},
\]

and the functional \( J_\epsilon : H^1(\mathbb{R}^N) \rightarrow (-\infty, +\infty] \) may be rewritten as

\[
J_\epsilon(u) = \Phi_\epsilon(u) + \Psi(u), \quad u \in H^1(\mathbb{R}^N)
\]

where

\[
\Phi_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\epsilon x + 1))|u|^2 \, dx - \int_{\mathbb{R}^N} F_2(u) \, dx,
\]

and

\[
\Psi(u) = \int_{\mathbb{R}^N} F_1(u) \, dx.
\]

It was proved in [14] and [19] that \( F_1 \) and \( F_2 \) verify the following properties:

\[
F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R}).
\]

If \( \delta > 0 \) is small enough, \( F_1 \) is convex, even, \( F_1(s) \geq 0 \) for all \( s \in \mathbb{R} \) and

\[
F_1'(s)s \geq 0, \quad s \in \mathbb{R}.
\]

For each fixed \( p \in (2, 2^*) \), there is \( C > 0 \) such that

\[
|F_2'(s)| \leq C|s|^{p-1}, \quad \forall s \in \mathbb{R}.
\]

Using the above information, it follows that \( \Phi_\epsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R}) \), \( \Psi \) is convex and lower semicontinuous, but \( \Psi \) is not a \( C^1 \) functional, since we are working on \( \mathbb{R}^N \).

Before to state our main result, we need to fix some notations. If potential \( V \) in (1.4) is replaced by a constant \( A > -1 \), we have the following problem

\[
\begin{cases} 
-\Delta u + Au = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases}
\]

The corresponding energy functional associated to (1.13) will be denoted by \( J_A : H^1(\mathbb{R}^N) \rightarrow (-\infty, +\infty] \) and defined as

\[
J_A(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (A + 1)|u|^2) \, dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx.
\]
In [19] is proved that problem (1.13) has a positive ground state solution given by

\[ m(A) := \inf_{u \in N_A} J_A(u) = \inf_{u \in D(J_A) \setminus \{0\}} \left\{ \max_{t \geq 0} J_A(tu) \right\}, \tag{1.14} \]

where

\[ N_A = \left\{ u \in D(J_A) \setminus \{0\}; J_A(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \right\} \]

and

\[ D(J_A) = \left\{ u \in H^1(\mathbb{R}^N) : J_A(u) < +\infty \right\}. \]

The main result of this paper is the following:

**Theorem 1.1.** Suppose that \( V \) satisfies (V1) – (V3). If

\[ (V4) \quad m(V(0)) \geq 2m(c_0) \quad \text{and} \quad c_1 \leq c_0 + \frac{3}{10} c_2, \]

where \( c_2 = \min\{1, c_0\} \), then there is \( \epsilon_* > 0 \) such that equation (1.3) has a positive solution for all \( \epsilon \in (0, \epsilon_*) \).

Theorem 1.1 is inspired from [1, 5, 11], however we are working with the logarithmic Schrödinger equation, whose the energy functional associated is not continuous, for this reason, some estimates for this problem are also very delicate and different from those used in the Schrödinger equation (1.1). The reader is invited to see that the way how we apply the deformation lemma in Section 3 is different of that explored in [1, 5, 11], because in our paper the Nehari set is not a manifold, and so, new arguments have been developed to overcome this difficulty. The plan of the paper is as follows: In Section 2 we show some estimates that do not appear in [1, 5, 11] and prove a technical result. In Section 3 we apply the deformation lemma to prove Theorem 1.1.

**Notation:** From now on in this paper, otherwise mentioned, we use the following notations:

- \( B_r(u) \) is an open ball centered at \( u \) with radius \( r > 0 \), \( B_r = B_r(0) \).
- If \( g \) is a measurable function, the integral \( \int g(z) \, dz \) will be denoted by \( \int g(z) \, dz \).
- \( C \) denotes any positive constant, whose value is not relevant.
- \( | \cdot |_p \) denotes the usual norm of the Lebesgue space \( L^p(\mathbb{R}^N) \), for \( p \in [1, +\infty] \).
- \( H^1_c(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) : u \text{ has compact support} \} \).
- \( o_n(1) \) denotes a real sequence with \( o_n(1) \to 0 \) as \( n \to +\infty \).
- \( 2^* = \frac{2N}{N-2} \) if \( N \geq 3 \) and \( 2^* = +\infty \) if \( N = 1, 2 \).
2 Technical results

We begin this section recalling some definitions that can be found in [21].

Definition 2.1. Let $E$ be a Banach space, $E'$ be the dual space of $E$ and $\langle \cdot , \cdot \rangle$ be the duality paring between $E'$ and $E$. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and $\Psi$ is convex and lower semicontinuous. Let us list some definitions:

(i) The sub-differential $\partial J(u)$ of the functional $J$ at a point $u \in E$ is the following set

$$\{ w \in E' : \langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u \rangle, \forall v \in E \}. \quad (2.1)$$

(ii) A critical point of $J$ is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$\langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \forall v \in E. \quad (2.2)$$

(iii) A Palais-Smale sequence at level $d$ for $J$ is a sequence $(u_n) \subset E$ such that $J(u_n) \to d$ and there is a numerical sequence $\tau_n \to 0^+$ with

$$\langle \Phi'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n ||v - u_n||, \forall v \in E. \quad (2.3)$$

(iv) The functional $J$ satisfies the Palais-Smale condition at level $d$ ($(PS)_d$ condition, for short) if all Palais-Smale sequences at level $d$ has a convergent subsequence.

(v) The effective domain of $J$ is the set $D(J) = \{ u \in E : J(u) < +\infty \}$.

To proceed further we gather and state below some useful results that leads to a better understanding of the problem and of its particularities. In what follows, for each $u \in D(J_e)$, we set the functional $J'_e(u) : H^1_e(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\langle J'_e(u), z \rangle = \langle \Phi'_e(u), z \rangle - \int F'_e(u)z \, dx, \quad \forall z \in H^1_e(\mathbb{R}^N)$$

and define

$$||J'_e(u)|| = \sup \{ \langle J'_e(u), z \rangle : z \in H^1_e(\mathbb{R}^N) \text{ and } ||z||_e \leq 1 \}.$$ 

If $||J'_e(u)||$ is finite, then $J'_e(u)$ may be extended to a bounded operator in $H^1(\mathbb{R}^N)$, and so, it can be seen as an element of $(H^1(\mathbb{R}^N))'$.

Lemma 2.1. Let $J_e$ satisfy (L?), then:

(i) If $u \in D(J_e)$ is a critical point of $J_e$, then

$$\langle \Phi'_e(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \forall v \in H^1(\mathbb{R}^N),$$

or equivalently

$$\int \nabla u \nabla (v-u) \, dx + \int (V(ex)+1)u(v-U) \, dx + \int F_1(v) \, dx - \int F_1(u) \, dx \geq \int F'_e(u)(v-u) \, dx, \forall v \in H^1(\mathbb{R}^N).$$

(ii) For each $u \in D(J_e)$ such that $||J'_e(u)|| < +\infty$, we have $\partial J_e(u) \neq \emptyset$, that is, there is $w \in (H^1(\mathbb{R}^N))'$, which is denoted by $w = J'_e(u)$, such that

$$\langle \Phi'_e(u), v - u \rangle + \int F_1(v) \, dx - \int F_1(u) \, dx \geq \langle w, v - u \rangle, \forall v \in H^1(\mathbb{R}^N), \text{ (see [21], [22])}$$
(iii) If a function \( u \in D(J_\epsilon) \) is a critical point of \( J_\epsilon \), then \( u \) is a solution of (1.4) (i) in Lemma 2.4, [14].
(iv) If \( (u_n) \subset H^1(\mathbb{R}^N) \) is a Palais-Smale sequence, then
\[
\langle J'_\epsilon(u_n), z \rangle = o_n(1)\|z\|_\epsilon, \quad \forall z \in H^1_\epsilon(\mathbb{R}^N).
\] (see (ii) in Lemma 2.4, [14]).
(v) If \( \Omega \) is a bounded domain with regular boundary, then \( \Psi \) (and hence \( J_\epsilon \)) is of class \( C^1 \) in \( H^1(\Omega) \) (Lemma 2.2 in [19]). More precisely, the functional
\[
\Psi(u) = \int_\Omega F_1(u) \, dx, \quad \forall u \in H^1(\Omega)
\]
belongs to \( C^1(H^1(\Omega), \mathbb{R}) \).

As a consequence of the above properties, we have the following results whose the proofs can be found in [2].

**Lemma 2.2.** If \( u \in D(J_\epsilon) \) and \( \|J'_\epsilon(u)\| < +\infty \), then \( F_1'(u)u \in L^1(\mathbb{R}^N) \).

An immediate consequence of the last lemma is the following.

**Corollary 2.1.** For each \( u \in D(J_\epsilon) \setminus \{0\} \) with \( \|J'_\epsilon(u)\| < +\infty \), we have that
\[
J'_\epsilon(u) = \int (|\nabla u|^2 + V(\epsilon x)|u|^2) \, dx - \int u^2 \log u^2 \, dx
\]
and
\[
J_\epsilon(u) - \frac{1}{2} J'_\epsilon(u)u = \frac{1}{2} \int |u|^2 \, dx.
\]

**Corollary 2.2.** If \( (u_n) \subset H^1(\mathbb{R}^N) \) is a (PS) sequence for \( J_\epsilon \), then \( J'_\epsilon(u_n)u_n = o_n(1)\|u_n\|_\epsilon \). If \( (u_n) \) is bounded, we have
\[
J_\epsilon(u_n) = J_\epsilon(u_n) - \frac{1}{2} J'_\epsilon(u_n)u_n + o_n(1)\|u_n\|_\epsilon = \frac{1}{2} \int |u_n|^2 \, dx + o_n(1)\|u_n\|_\epsilon, \quad \forall n \in \mathbb{N}.
\]

**Corollary 2.3.** If \( u \in H^1(\mathbb{R}^N) \) is a critical point of \( J_\epsilon \) and \( v \in H^1(\mathbb{R}^N) \) verifies \( F_1'(u)v \in L^1(\mathbb{R}^N) \), then \( J'_\epsilon(u)v = 0 \).

Next we will prove some results that will be useful in the proof of Theorem 1.1.

**Lemma 2.3.** For any \( \epsilon > 0 \), all (PS) sequences of \( J_\epsilon \) are bounded in \( H^1(\mathbb{R}^N) \).

**Proof.** Let \( (u_n) \) be a (PS) sequence. Then
\[
\int |u_n|^2 \, dx = 2J_\epsilon(u_n) - J'_\epsilon(u_n)u_n = 2d + o_n(1) + o_n(1)\|u_n\| \leq C + o_n(1)\|u_n\|,
\]
for some \( C > 0 \). Consequently
\[
|u_n|^2 \leq C + o_n(1)\|u_n\|. \tag{2.5}
\]
Now, let us employ the following logarithmic Sobolev inequality found in [16],

$$\int u^2 \log u^2 dx \leq \frac{a^2}{\pi} |\nabla u|^2 + (\log |u|^2 - N(1 + \log a))|u|^2$$  \hspace{1cm} (2.6)

for all $a > 0$. Fixing $\frac{a^2}{\pi} = \frac{1}{4}$ and $\xi \in (0, 1)$, the inequalities (2.5) and (2.6) yield

$$\int u_n^2 \log u_n^2 dx \leq \frac{1}{4} |\nabla u_n|^2 + C(\log |u_n|^2 + 1)|u_n|^2$$

$$\leq \frac{1}{4} |\nabla u_n|^2 + C_1 (1 + \|u_n\|)^{1+\xi}.$$  \hspace{1cm} (2.7)

Since $d_o(n(1) = J_\epsilon(u_n) = \frac{1}{2} |\nabla u_n|^2 + \int (V(\epsilon x) + 1) |u_n|^2 dx - \frac{1}{2} \int u_n^2 \log u_n^2 dx$

assertion (2.7) assures that

$$d + o_n(1) \geq C \left[\|u_n\|^2 - (1 + \|u_n\|)^{1+\xi}\right].$$  \hspace{1cm} (2.8)

showing that the sequence $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$.

**Lemma 2.4.** Under the assumptions (V1) – (V4), for each $\sigma > 0$, there is $\epsilon_0 = \epsilon_0(\sigma) > 0$, such that $J_\epsilon$ satisfies the $(PS)_c$ condition for all $c \in (m(c_0) + \sigma/2, 2m(c_0) - \sigma)$, for all $\epsilon \in (0, \epsilon_0)$.

**Proof.** We shall prove the lemma arguing by contradiction, by supposing that there are $\sigma > 0$ and $\epsilon_n \to 0$, such that $J_{\epsilon_n}$ does not satisfy the $(PS)$ condition. Therefore, there is $c_n \in (m(c_0) + \sigma/2, 2m(c_0) - \sigma)$ such that $J_{\epsilon_n}$ does not satisfy the $(PS)_{c_n}$ condition. Then, there exists a sequence $(u_{n_m})$ such that

$$\lim_{m \to +\infty} J_{\epsilon_n}(u_{n_m}) = c_n \quad \text{and} \quad \lim_{m \to +\infty} \|J'_{\epsilon_n}(u_{n_m})\| = 0. \hspace{1cm} (2.9)$$

By Lemma 2.3 the sequence $(u_{n_m})$ is bounded in $H^1(\mathbb{R}^N)$, then

$$u_{n_m} \rightharpoonup u_n \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{but} \quad u_{n_m} \not\rightarrow u_n \quad \text{in} \quad H^1(\mathbb{R}^N). \hspace{1cm} (2.10)$$

Now, we claim that there is $\delta > 0$, such that

$$\liminf_{m \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_{n_m}|^2 dx \geq \delta, \quad \forall n \in \mathbb{N}.$$  

Indeed, on the contrary, there is $(n_j) \subset \mathbb{N}$ satisfying

$$\liminf_{m \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_{n_j}|^2 dx \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$
Arguing as in Lions [15],

\[ \limsup_{m \to +\infty} |u_m^{n_j}|_p = o_j(1), \quad \forall p \in (2, 2^*). \]

Then, by (1.12), we would get

\[ \limsup_{m \to +\infty} \int F'_2(u_m^{n_j})u_m^{n_j} \, dx = o_j(1). \]

On the other hand, by (2.9), we obtain

\[ \|u_m^{n_j}\|_2^{\epsilon} + \int F'_1(u_m^{n_j})u_m^{n_j} \, dx = J'_{\epsilon n_j}(u_m^{n_j})u_m^{n_j} \quad \text{or} \quad J'_{\epsilon n_j}(u_m^{n_j})u_m^{n_j} = o_m(1)\|u_m^{n_j}\|_\epsilon + \int F'_2(u_m^{n_j})u_m^{n_j} \, dx, \]

and

\[ \limsup_{m \to +\infty} \|u_m^{n_j}\|_\epsilon^{2} = o_j(1). \]

It is easy to see that \( u_m^{n_j} \not\to 0 \) as \( m \to \infty \) in \( H^1(\mathbb{R}^N) \), otherwise we get

\[ \int F'_1(u_m^{n_j})u_m^{n_j} \, dx \to 0, \]

which together with convexity of \( F_1 \) yields

\[ \int F_1(u_m^{n_j}) \, dx \to 0. \]

This limit combined with \( u_m^{n_j} \to 0 \) implies that \( J_{\epsilon n_j}(u_m^{n_j}) \to 0 \), which contradicts with \( c_n > 0 \). Thus

\[ \liminf_{m \to +\infty} \|u_m^{n_j}\|_\epsilon^{2} > 0. \]

Without loss of generality, we can assume that \( \{u_m^{n_j}\} \subset H^1(\mathbb{R}^N) \setminus \{0\} \). Thereby, there exists \( t_m^{n_j} \in (0, +\infty) \) such that

\[ t_m^{n_j} u_m^{n_j} \in \mathcal{N}_{\epsilon n_j}. \]

Since

\[ J_{\epsilon n_j}(u_m^{n_j}) = \frac{2 \log t_m^{n_j} + 1}{2} \int |u_m^{n_j}|^2 \, dx \tag{2.11} \]

and

\[ J_{\epsilon n_j}(u_m^{n_j}) = \frac{1}{2} \int |u_m^{n_j}|^2 \, dx + o_m(1), \tag{2.12} \]

it is easy to see that

\[ \lim_{m \to +\infty} t_m^{n_j} = 1. \]

Recalling that

\[ J_{\epsilon n_j}(t_m^{n_j} u_m^{n_j}) = \frac{1}{2} \int |t_m^{n_j} u_m^{n_j}|^2 \, dx, \]

we have that

\[ \lim_{m \to +\infty} J_{\epsilon n_j}(t_m^{n_j} u_m^{n_j}) = \lim_{m \to +\infty} J_{\epsilon n_j}(u_m^{n_j}). \]
From the above limits, there is $r^{n_j}_{m} \in (0, 1)$ such that
\[ r^{n_j}_{m}(t^{n_j}_{m} u^{n_j}_{m}) \in \mathcal{N}_{c_0}. \]

Hence,
\[
m(c_0) \leq \limsup_{m \to +\infty} J_{c_0}(r^{n_j}_{m}(t^{n_j}_{m} u^{n_j}_{m})) \leq \limsup_{m \to +\infty} J_{c_{n_j}}(t^{n_j}_{m} u^{n_j}_{m}) = \limsup_{m \to +\infty} J_{c_{n_j}}(u^{n_j}_{m}) \]
\[
= \frac{1}{2} \limsup_{m \to +\infty} \int |u^{n_j}_{m}|^2 dx \leq C \limsup_{m \to +\infty} \|u^{n_j}_{m}\|^2,
\]
that is
\[ 0 < m(c_0) \leq o_j(1), \]
which is a contradiction.

From the above study, for each $m \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that
\[ \int_{B_R(z^{n}_{m_n})} |u_{n}^{n}||2 dx \geq \delta, \quad |\epsilon_{n}z^{n}_{m_n}| \geq n, \quad \|J'_{\epsilon_{n}}(u_{n}^{n})\| \leq \frac{1}{n} \quad \text{and} \quad |J_{c_{n}}(u_{n}^{n}) - c_n| \leq \frac{1}{n}. \]
In what follows, we denote by $(z_n)$ and $(u_n)$ the sequences $(z^{n}_{m_n})$ and $(u_{n}^{n})$ respectively. Thus,
\[ \int_{B_R(z_n)} |u_{n}|^2 dx \geq \frac{\delta}{2}, \quad |\epsilon_{n}z_{n}| \geq n, \quad \|J'_{\epsilon_{n}}(u_n)\| \leq \frac{1}{n} \quad \text{and} \quad |J_{c_{n}}(u_n) - c_n| \leq \frac{1}{n}. \]

Arguing as in the proof of Lemma 2.3, the sequence $(u_n)$ is a bounded in $H^1(\mathbb{R}^N)$. Then, for some subsequence, there exists $u \in H^1(\mathbb{R}^N)$ such that
\[ u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^N). \]
We claim that $u = 0$, because if $u \neq 0$, the limit $\|J'_{\epsilon_{n}}(u_n)\| \to 0$ together with (V2) yield $u$ is a nontrivial solution of the problem
\[ -\Delta u + V(0)u = u \log u^2, \quad \text{in} \quad \mathbb{R}^N. \]
Then, combining the definition of $m(V(0))$ with (V4), we get
\[ J_{V(0)}(u) \geq m(V(0)) \geq 2m(c_0). \]
On the other hand, using the Fatou’s lemma, one has
\[ J_{V(0)}(u) \leq \frac{1}{2} \liminf_{n \to +\infty} \int |u_n|^2 dx = \liminf_{n \to +\infty} (J_{\epsilon_{n}}(u_n) - \frac{1}{2} J'_{\epsilon_{n}}(u_n)u_n) = \liminf_{n \to +\infty} J_{c_{n}}(u_n), \]
and so,
\[ J_{V(0)}(u) \leq \liminf_{n \to +\infty} c_n \leq 2m(c_0) - \sigma, \]
which is a contradiction, from where it follows that $u_n \to 0$ in $H^1(\mathbb{R}^N)$. 

Considering $\omega_n = u_n(\cdot + z_n)$, we have that $(\omega_n)$ is bounded in $H^1(\mathbb{R}^N)$. Therefore, there is $\omega \in H^1(\mathbb{R}^N)$ such that

$$\omega_n \rightharpoonup \omega \quad \text{in} \quad H^1(\mathbb{R}^N)$$

and

$$\int_{B_R(0)} |\omega|^2 \, dx = \liminf_{n \to +\infty} \int_{B_R(0)} |\omega_n|^2 \, dx = \liminf_{n \to +\infty} \int_{B_R(z_n)} |u_n|^2 \, dx \geq \frac{\delta}{2},$$

showing that $\omega \neq 0$.

Now, for each $\phi \in C^\infty_0(\mathbb{R}^N)$, we have

$$\int \nabla \omega_n \nabla \phi \, dx + \int V(\epsilon_n z_n + \epsilon_n x) \omega_n \phi \, dx - \int \omega_n \phi \log \omega_n^2 \, dx = o_n(1)\|\phi\|,$$

which implies that $\omega$ is a nontrivial solution of the problem

$$-\Delta u + \alpha_1 u = u \log u^2 \quad \text{in} \quad \mathbb{R}^N,$$  \hspace{1cm} (2.14)

where $\alpha_1 = \lim_{n \to +\infty} V(\epsilon_n z_n)$. By \cite{8}, $\omega \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.

For each $k \in N$, there is $\phi_k \in C^\infty_0(\mathbb{R}^N)$ such that

$$\|\phi_k - \omega\| \to 0 \quad \text{as} \quad k \to +\infty,$$

that is,

$$\|\phi_k - \omega\| = o_k(1).$$

Using $\frac{\partial \phi_k}{\partial x_i}$ as a test function, we obtain

$$\int \nabla \omega_n \nabla \frac{\partial \phi_k}{\partial x_i} \, dx + \int V(\epsilon_n z_n + \epsilon_n x) \omega_n \frac{\partial \phi_k}{\partial x_i} \, dx - \int \omega_n \frac{\partial \phi_k}{\partial x_i} \log \omega_n^2 \, dx = o_n(1).$$

Now, using well known arguments,

$$\int \nabla \omega_n \nabla \frac{\partial \phi_k}{\partial x_i} \, dx = \int \nabla \omega \nabla \frac{\partial \phi_k}{\partial x_i} \, dx + o_n(1)$$

and

$$\int \omega_n \frac{\partial \phi_k}{\partial x_i} \log \omega_n^2 \, dx = \int \omega \frac{\partial \phi_k}{\partial x_i} \log \omega^2 \, dx + o_n(1).$$

Combining the above limit with (2.14), we derive that

$$\limsup_{n \to +\infty} \left| \int (V(\epsilon_n z_n + \epsilon_n x) - V(\epsilon_n z_n)) \omega_n \frac{\partial \phi_k}{\partial x_i} \, dx \right| = 0.$$

As $\phi_k$ has compact support, the above limit gives

$$\limsup_{n \to +\infty} \left| \int (V(\epsilon_n z_n + \epsilon_n x) - V(\epsilon_n z_n)) \omega \frac{\partial \phi_k}{\partial x_i} \, dx \right| = 0.$$
Now, recalling that $\frac{\partial \omega}{\partial x_i} \in L^2(\mathbb{R}^N)$, we have that $\left(\frac{\partial \phi_k}{\partial x_i}\right)$ is bounded in $L^2(\mathbb{R}^N)$. Hence,

$$\limsup_{n \to +\infty} \left| \int (V(\epsilon_n z_n + \epsilon_n x) - V(\epsilon_n z_n)) \phi_k \frac{\partial \phi_k}{\partial x_i} \, dx \right| = o_k(1),$$

and so,

$$\limsup_{n \to +\infty} \left| \frac{1}{2} \int (V(\epsilon_n z_n + \epsilon_n x) - V(\epsilon_n z_n)) \left( \frac{\partial (\phi_k^2)}{\partial x_i} \right) \, dx \right| = o_k(1).$$

Using Green’s Theorem together with the fact that $\phi_k$ has compact support, we have the following limit

$$\limsup_{n \to +\infty} \left| \int \frac{\partial V}{\partial x_i}(\epsilon_n z_n + \epsilon_n x) \phi_k^2 \, dx \right| = o_k(1),$$

which combined with $(V2)$ yields

$$\limsup_{n \to +\infty} \left| \frac{\partial V}{\partial x_i}(\epsilon_n z_n) \int |\phi_k|^2 \, dx \right| = o_k(1).$$

As

$$\int |\phi_k|^2 \, dx \to \int |\omega|^2 \, dx \quad \text{as} \quad k \to +\infty,$$

it follows that

$$\limsup_{n \to +\infty} \left| \frac{\partial V}{\partial x_i}(\epsilon_n z_n) \right| = o_k(1), \quad \forall \{1, \cdots, N\}.$$

Since $k$ is arbitrary, we obtain that

$$\nabla V(\epsilon_n z_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $(\epsilon_n z_n)$ is a $(PS)_{\alpha_1}$ sequence for $V$, which is a contradiction, because by assumption $V$ satisfies the $(PS)$ condition and $(\epsilon_n z_n)$ does not have any convergent subsequence in $\mathbb{R}^N$.

The next lemma will be crucial in our study to show an important estimate, see Lemma 3.3 in Section 3.

**Lemma 2.5.** Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{c_0}$ such that $J_{\epsilon_n}(u_n) \to m(c_0)$. Then, there are $(z_n) \subset \mathbb{R}^N$ with $|z_n| \to +\infty$ and $u_1 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$u_n(\cdot + z_n) \to u_1 \quad \text{in} \ H^1(\mathbb{R}^N).$$

Moreover, $\liminf_{n \to +\infty} |\epsilon_n z_n| > 0$.

**Proof.** Since $u_n \in \mathcal{N}_{c_0}$, we have that $J'_{c_0}(u_n) u_n \leq 0$ and $J_{c_0}(u) \leq J_{\epsilon_n}(u)$ for all $u \in H^1(\mathbb{R}^N)$ and $n \in \mathbb{N}$. From this, there is $t_n \in (0, 1]$ such that

$$(t_n u_n) \subset \mathcal{N}_{c_0} \quad \text{and} \quad J_{c_0}(t_n u_n) \to m(c_0).$$
Since \((t_n)\) is bounded, by \([2\text{ Section }6]\), there are \((z_n) \subset \mathbb{R}^N\), \(u_1 \in H^1(\mathbb{R}^N) \setminus \{0\}\), and a subsequence of \((u_n)\), still denote by \((u_n)\), verifying
\[
u_n(\cdot + z_n) \to u_1 \quad \text{in } H^1(\mathbb{R}^N).
\]

Now, we claim that \(\liminf_{n \to +\infty} |\epsilon_n z_n| > 0\). Indeed, as \(u_n \in \mathcal{N}_{\epsilon_n}\) for all \(n \in \mathbb{N}\), the function \(u_n^1 = u_n(\cdot + z_n)\) must verify
\[
\int (|\nabla u_n^1|^2 + V(\epsilon_n z_n + \epsilon_n x)|u_n^1|^2) dx + \int F'_1(u_n^1)u_n^1 dx = \int F'_2(u_n^1)u_n^1 dx. \tag{2.15}
\]

Since \(F_1\) is convex, even and \(F_1(t) \geq F_1(0) = 0\) for all \(t \in \mathbb{R}\), we derive that \(0 \leq F_1(t) \leq F'_1(t) t\) for all \(t \in \mathbb{R}\). Supposing by contradiction that for some subsequence \(\lim_{n \to +\infty} \epsilon_n z_n = 0\), taking the limit of \(n \to +\infty\) in \((2.15)\), we derive the following inequality
\[
\int (|\nabla u_1|^2 + V(0)|u_1|^2) dx + \int F'_1(u_1)u_1 dx \leq \int F'_2(u_1)u_1 dx.
\]

Thus, there exists \(t_1 \in (0,1]\) such that \(t_1 u_1 \in \mathcal{N}_{V(0)}\). Therefore, we have
\[
J_{V(0)}(t_1 u_1) \geq m(V(0)) > m(c_0) > 0. \tag{2.16}
\]

On the other hand, one has
\[
\lim_{n \to +\infty} J_{\epsilon_n}(u_n) = \lim_{n \to +\infty} \frac{1}{2} \int |u_n^1|^2 dx = \frac{1}{2} \int |u_1|^2 dx \\
\geq \frac{1}{2} \int |t_1 u_1|^2 dx \\
= J_{V(0)}(t_1 u_1),
\]
that is,
\[
m(c_0) \geq J_{V(0)}(t_1 u_1),
\]
which contradicts \((2.16)\), finishing the proof.

### 3 A special minimax level

In order to prove Theorem 1.1, we shall consider a special minimax level. At first, we fix the barycenter function by
\[
\beta(u) = \frac{\int \frac{1}{|x|} |u|^2 dx}{\int |u|^2 dx}, \quad \forall u \in H^1(\mathbb{R}^N) \setminus \{0\}.
\]

In what follows, \(u_0\) denotes a positive ground state solution for \(J_{c_0}\), that is,
\[
J_{c_0}(u_0) = m(c_0) \quad \text{and} \quad J'_{c_0}(u_0) = 0.
\]
Moreover, by \([\square]\), \(u_0\) is also radial. For each \(z \in \mathbb{R}^N\) and \(\epsilon > 0\), we set the function

\[
\phi_{\epsilon,z}(x) = t_{\epsilon,z}u_0\left(x - \frac{z}{\epsilon}\right),
\]

where \(t_{\epsilon,z} > 0\) is such that \(\phi_{\epsilon,z} \in \mathcal{N}_\epsilon\). In what follows, we set \(\Phi_{\epsilon}(z) = \phi_{\epsilon,z}\) for all \(z \in \mathbb{R}^N\).

**Lemma 3.1.** The function \(\Phi_{\epsilon} : \mathbb{R}^N \rightarrow \mathcal{N}_\epsilon\) is a continuous function.

**Proof.** Let \((z_n) \subset \mathbb{R}^N\) and \(z \in \mathbb{R}^N\) with \(z_n \rightarrow z\) in \(\mathbb{R}^N\). We must prove that \(\Phi_{\epsilon}(z_n) \rightarrow \Phi_{\epsilon}(z)\) in \(H^1(\mathbb{R}^N)\).

Here, the main point is to prove that \(t_{\epsilon,z_n} \rightarrow t_{\epsilon,z}\) in \(\mathbb{R}\).

By definition of \(t_{\epsilon,z_n}\) and \(t_{\epsilon,z}\), they are the unique numbers that satisfy

\[
J_{\epsilon}(t_{\epsilon,z_n}u_0(\cdot - \frac{z_n}{\epsilon})) = \frac{1}{2} \int |t_{\epsilon,z_n}u_0(x - \frac{z_n}{\epsilon})|^2 \, dx
\]

and

\[
J_{\epsilon}(t_{\epsilon,z}u_0(\cdot - \frac{z}{\epsilon})) = \frac{1}{2} \int |t_{\epsilon,z}u_0(x - \frac{z}{\epsilon})|^2 \, dx
\]

that is,

\[
\frac{1}{2} \int \left(|t_{\epsilon,z_n}\nabla u_0|^2 + (V(\epsilon x + z_n) + 1)|t_{\epsilon,z_n}u_0|^2 \right) \, dx + \int F_1(t_{\epsilon,z_n}u_0) \, dx - \int F_2(t_{\epsilon,z_n}u_0) \, dx = \frac{1}{2} \int |t_{\epsilon,z_n}u_0|^2 \, dx \tag{3.1}
\]

and

\[
\frac{1}{2} \int \left(|t_{\epsilon,z}\nabla u_0|^2 + (V(\epsilon x + z) + 1)|t_{\epsilon,z}u_0|^2 \right) \, dx + \int F_1(t_{\epsilon,z}u_0) \, dx - \int F_2(t_{\epsilon,z}u_0) \, dx = \frac{1}{2} \int |t_{\epsilon,z}u_0|^2 \, dx \tag{3.2}
\]

A simple calculation gives that \((t_{\epsilon,z_n})\) is bounded, thus for some subsequence, we can assume that \(t_{\epsilon,z_n} \rightarrow t_*\). Since \(F_1\) is increasing in \([0, +\infty)\) and \(F_1(lu_0) \in L^1(\mathbb{R}^N)\) for all \(l > 0\), taking the limit of \(n \rightarrow +\infty\) in \((3.1)\) and applying the Lebesgue Theorem, we can conclude that

\[
\frac{1}{2} \int \left(|t_*\nabla u_0|^2 + (V(\epsilon x + z) + 1)|t_*u_0|^2 \right) \, dx + \int F_1(t_*u_0) \, dx - \int F_2(t_*u_0) \, dx = \frac{1}{2} \int |t_*u_0|^2 \, dx \tag{3.3}
\]

By uniqueness of \(t_{\epsilon,z}\), it follows that \(t_{\epsilon,z} = t_*\), and so, \(t_{\epsilon,z_n} \rightarrow t_{\epsilon,z}\). Now, since

\[
u_0\left(\cdot - \frac{z_n}{\epsilon}\right) \rightarrow u_0\left(\cdot - \frac{z}{\epsilon}\right)\quad \text{in}\quad H^1(\mathbb{R}^N),
\]

we get the desired result. \(\square\)

From definition of \(\beta\), we have the following result.
Lemma 3.2. For each \( r > 0 \), \( \lim_{\epsilon \to 0} \left( \sup \left\{ \left| \beta(\Phi_{\epsilon}(z)) - \frac{z}{|z|} \right| : |z| \geq r \right\} \right) = 0 \).

Proof. The proof follows by showing that for any \( (z_n) \subset \mathbb{R}^N \) with \( |z_n| \geq r \) and \( \epsilon_n \to 0 \), we have that

\[
\left| \beta(\Phi_{\epsilon_n}(z_n)) - \frac{z_n}{|z_n|} \right| \to 0 \quad \text{as} \quad n \to +\infty.
\]

By change of variables,

\[
\left| \beta(\Phi_{\epsilon_n}(z_n)) - \frac{z_n}{|z_n|} \right| = \frac{\int \left| \frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|} \right| u_0(x)^2 \, dx}{\int |u_0(x)|^2 \, dx}.
\]

As for each \( x \in \mathbb{R}^N \),

\[
\left| \frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|} \right| \to 0 \quad \text{as} \quad n \to +\infty,
\]

by the Lebesgue Dominated Convergence Theorem, we have that

\[
\int \left| \frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|} \right| u_0(x)^2 \, dx \to 0 \quad \text{as} \quad n \to +\infty,
\]

proving (3.4).

As a by-product of Lemma 3.2, we have the following corollary.

Corollary 3.1. Fixed \( r > 0 \), there is \( \epsilon_0 > 0 \) such that

\[
(\beta(\Phi_{\epsilon}(z)), z) > 0, \quad \forall |z| \geq r \quad \text{and} \quad \forall \epsilon \in (0, \epsilon_0).
\]

Proof. By last lemma, for fixed \( r > 0 \), there exists \( \epsilon_0 > 0 \) such that

\[
\left| \beta(\Phi_{\epsilon}(z)) - \frac{z}{|z|} \right| < \frac{1}{2}, \quad \forall |z| \geq r \quad \text{and} \quad \forall \epsilon \in (0, \epsilon_0).
\]

On the other hand, notice that

\[
(\beta(\Phi_{\epsilon}(z)), z) = (\beta(\Phi_{\epsilon}(z)) - \frac{z}{|z|}, z) + (\frac{z}{|z|}, z) = (\beta(\Phi_{\epsilon}(z)) - \frac{z}{|z|}, z) + |z|, \quad \forall z \in \mathbb{R}^N \setminus \{0\}.
\]

Therefore, for \( |z| \geq r \),

\[
(\beta(\Phi_{\epsilon}(z)), z) \geq |z| \left( 1 - \left| \beta(\Phi_{\epsilon}(z)) - \frac{z}{|z|} \right| \right) > \frac{|z|}{2} \geq \frac{r}{2} > 0,
\]

which completes the proof.

Now, we define the set

\[
\mathcal{B}_\epsilon = \{ u \in \mathcal{N}_\epsilon : \beta(u) \in Y \}.
\]

Since \( \beta(\phi_{\epsilon,0}) = 0 \in Y \), for all \( \epsilon > 0 \), we have that \( \mathcal{B}_\epsilon \neq \emptyset \). Associated with the above set, let us consider the real number \( D_\epsilon \) given by

\[
D_\epsilon = \inf_{u \in \mathcal{B}_\epsilon} J_\epsilon(u).
\]

The next lemma establishes an important relation between the levels \( D_\epsilon \) and \( m(c_0) \).
Lemma 3.3. (i) There exist $\epsilon_0, \sigma > 0$ such that

$$D_\epsilon \geq m(c_0) + \sigma, \quad \forall \epsilon \in (0, \epsilon_0).$$

(ii) $\limsup_{\epsilon \to 0} \left\{ \sup_{x \in X} J_\epsilon(\Phi_\epsilon(x)) \right\} < 2m(c_0) - \sigma.$

Proof. 
(i): From the definition of $D_\epsilon$, we know that

$$D_\epsilon \geq m(c_0) \quad \forall \epsilon > 0.$$ 

Supposing by contradiction that the lemma does not hold, there exists $\epsilon_n \to 0$ satisfying

$$D_{\epsilon_n} \to m(c_0).$$ 

Hence, there is $u_n \in N_{\epsilon_n}$ with $\beta(u_n) \in Y$ such that

$$J_{\epsilon_n}(u_n) \to m(c_0).$$ 

Applying Lemma 2.5, there are $u_1 \in H^1(\mathbb{R}^N) \setminus \{0\}$ and a sequence $(z_n) \subset \mathbb{R}^N$ with

$$\liminf_{n \to +\infty} |\epsilon_n z_n| > 0$$

verifying

$$u_n(\cdot + z_n) \to u_1 \quad \text{in} \quad H^1(\mathbb{R}^N),$$

that is

$$u_n = u_1(\cdot - z_n) + \omega_n \quad \text{with} \quad \omega_n \to 0 \quad \text{in} \quad H^1(\mathbb{R}^N).$$

From the definition of $\beta$,

$$\beta(u_1(\cdot - z_n)) = \frac{\int_{\mathbb{R}^N} \frac{\epsilon_n x + \epsilon_n z_n}{|\epsilon_n x + \epsilon_n z_n|} |u_1|^2 \, dx}{\int |u_1|^2 \, dx}.$$

Repeating the same arguments explored in the proof of Lemma 3.2, we see that

$$\beta(u_1(\cdot - z_n)) = \frac{z_n}{|z_n|} + o_n(1),$$

and so,

$$\beta(u_n) = \beta(u_1(\cdot - z_n)) + o_n(1) = \frac{z_n}{|z_n|} + o_n(1).$$

Since $\beta(u_n) \in Y$, we infer that $\frac{z_n}{|z_n|} \in Y_\lambda$ for $n$ large enough. Consequently, $z_n \in Y_\lambda$ for $n$ large enough, implying that

$$\liminf_{n \to \infty} V(\epsilon_n z_n) > c_0.$$ 

Assuming $A = \liminf_{n \to \infty} V(\epsilon_n z_n)$, the last inequality together with the Fatou’s lemma yields,

$$m(c_0) = \liminf_{n \to \infty} J_{\epsilon_n}(u_n) \geq \liminf_{n \to \infty} J_{\epsilon_n}(tu_n) \geq J_A(tu_1) \geq m(A) > m(c_0),$$
which is a contradiction. Here \( t \in (0, 1] \) such that \( J'_A(tu_1)tu_1 = 0 \) and \( u_1 \neq 0 \).

(ii): By condition (V4) and the fact that \( u_0 \) is a ground state solution associated with \( J_{c_0} \), we deduce that

\[
\limsup_{\epsilon \to 0} \left\{ \sup_{x \in X} J_\epsilon(\Phi_\epsilon(x)) \right\} \leq \frac{1}{2} \int (|\nabla u_0|^2 + (c_1 + 1)|u|^2) \, dx - \frac{1}{2} \int u_0^2 \log u_0^2 \, dx \\
\leq J_{c_0}(u_0) + \frac{3}{10} c_2 \int |u_0|^2 \, dx \\
\leq J_{c_0}(u_0) + \frac{3}{5} J_{c_0}(u_0) = m(c_0) + \frac{3}{5} m(c_0) < 2m(c_0), \quad \forall \epsilon \in (0, \epsilon_0),
\]

where \( c_2 = \min\{1, c_0\} \).

Now, we are ready to show the minimax level. In what follows, we fix \( \epsilon \in (0, \epsilon_0) \), \( \Phi = \Phi_\epsilon, J = J_\epsilon \) and the following sets

\[
J_d = \{ u \in H^1(\mathbb{R}^N) : J(u) \leq d \}, \quad Q = \overline{B}_R(0) \cap X \quad \text{and} \quad \partial Q = \partial \overline{B}_R(0) \cap X.
\]

Using the above notations, we define the class of the functions

\[
\Gamma = \left\{ h \in C(Q, K_r) : h(x) = \Phi(x), \quad \forall x \in \partial Q \right\}
\]

where \( \approx \) denotes the homotopy relation, \( r > 0, K = \Phi(Q) \) and

\[
K_r = \{ u \in H^1(\mathbb{R}^N) : \text{dist}(u, K) < r \}.
\]

Note that \( \Gamma \neq \emptyset \), because Lemma 3.1 ensures that \( \Phi \in \Gamma \).

In what follows we set

\[
\Upsilon_r = \{ u \in K_r : \beta(u) \in Y \},
\]

which is not empty because \( B_\epsilon \subset K_r \).

**Lemma 3.4.** There is \( r_0 > 0 \) such that such that

\[
\Theta_r = \inf_{u \in \Upsilon_r} J(u) > m(c_0) + \sigma/2, \quad \forall r \in (0, r_0).
\]

Moreover, there exists \( R > 0 \) such that

\[
J_\epsilon(\Phi_\epsilon(x)) \leq \frac{1}{2}(m(c_0) + \Theta_r), \quad \forall x \in \partial B_R(0) \cap X.
\]

**Proof.** Assume by contradiction that the lemma does not hold. Then, there is \( r_n \to 0 \) and \( u_n \in \Upsilon_{r_n} \) such that \( J(u_n) \leq m(c_0) + \sigma/2 \). From definition of \( \Upsilon_{r_n} \), there is \( v_n \in K \) such that \( \|u_n - v_n\| \leq r_n \). Since \( K \) is compact, there is a subsequence of \( (v_n) \), still denoted by itself, and \( v \in K \) such that \( v_n \to v \) in \( H^1(\mathbb{R}^N) \), then \( u_n \to v \) in \( H^1(\mathbb{R}^N) \) and \( \beta(v) \in Y \), from where it follows that \( v \in B_\epsilon \), then by Lemma 3.3(i), \( J(v) \geq m(c_0) + \sigma \). On the other hand, since \( J_\epsilon \) is l.s.c., we have that \( \liminf_{n \to +\infty} J(u_n) \geq J(v) \), which is absurd.
By (V1), given $\delta > 0$, there are $\epsilon_0, R > 0$ such that
\[
\sup \left\{ J_\epsilon(\Phi_\epsilon(x)) : x \in \partial B_R(0) \cap X \right\} \leq m(c_0) + \delta, \ \forall \epsilon \in (0, \epsilon_0).
\]
Fixing $\delta = \frac{\sigma}{4}$, where $\sigma$ was given in (i), we have that
\[
\sup \left\{ J_\epsilon(\Phi_\epsilon(x)) : x \in \partial B_R(0) \cap X \right\} \leq \frac{1}{2} \left( 2m(c_0) + \frac{\delta}{2} \right) < \frac{1}{2} \left( m(c_0) + \Theta_r \right), \ \forall \epsilon \in (0, \epsilon_0).
\]

**Lemma 3.5.** If $h \in \Gamma$, then $h(Q) \cap \Upsilon_r \neq \emptyset$ for all $r \in (0, r_0)$.

**Proof.** It is enough to show that for all $h \in \Gamma$, there is $x_* \in Q$ such that
\[
\beta(h(x_*)) \in Y.
\]
For each $h \in \Gamma$, we set the function $g : Q \to \mathbb{R}^N$ given by
\[
g(x) = \beta(h(x)) \ \forall x \in Q,
\]
and the homotopy $\mathcal{F} : [0, 1] \times Q \to X$ as
\[
\mathcal{F}(t, x) = tP_X(g(x)) + (1 - t)x,
\]
where $P_X$ is the projection onto $X = \{(x, 0) : x \in \mathbb{R}^N\}$. By Corollary 3.1 fixed $R > 0$ and $\epsilon > 0$ small enough, we have that
\[
(\mathcal{F}(t, x), x) > 0, \ \forall (t, x) \in [0, 1] \times \partial Q.
\]
Using the homotopy invariance property of the Topological degree, we derive
\[
d(g, Q, 0) = 1,
\]
implying that there exists $x_* \in Q$ such that $\beta(h(x_*)) = 0$. \hfill $\Box$

Now, define the minimax value
\[
C_\epsilon = \inf_{h \in \Gamma} \sup_{x \in Q} J(h(x)).
\]
From Lemmas 3.4 and 3.5
\[
C_\epsilon \geq \Theta_r = \inf_{u \in \Upsilon_r} J(u) \geq m(c_0) + \sigma/2. \quad (3.5)
\]
On the other hand,
\[
C_\epsilon \leq \sup_{x \in Q} J(\Phi(x)).
\]
Then, by Lemma 3.3 ii),
\[
C_\epsilon \leq \sup_{x \in Q} J(\Phi(x)) < 2m(c_0) - \sigma. \quad (3.6)
\]
From (3.5) and (3.6),
\[ C \epsilon \in (m(c_0) + \sigma/2, 2m(c_0) - \sigma). \]

Now, by Lemma 2.4, we know that the \( J \) satisfies the \((PS)\) condition for all \( c \in (m(c_0) + \sigma/2, 2m(c_0) - \sigma) \), hence \( J \) satisfies the \((PS)\) condition for \( \epsilon \) small enough. Using this fact we can ensure that \( C \epsilon \) is a critical level for \( J \). To see why, we will follow the same type of ideas found in the proof of [21, Theorem 3.4]. Have this in mind, by Lemma 3.4 we can fix for \( \tau > 0 \) small enough such that
\[ C \epsilon - \tau/2 > \frac{1}{2}(m(c_0) + \Theta_r), \]
and we set
\[ \Gamma_1 = \{ h \in C(Q, K_r) : h|_{\partial Q} \approx \Phi|_{\partial Q} \text{ in } J_{C \epsilon - \tau/4}, \sup_{x \in \partial Q} J(h(x)) \leq C \epsilon - \tau/2 \} \]
where \( \approx \) denotes the homotopy relation and the number
\[ C^* = \inf_{h \in \Gamma_1} \sup_{x \in Q} J(h(x)). \]

Arguing as in [21, Theorem 3.4] we have that \( C^* = C \epsilon \), and so, it is enough to prove that \( C^* \) is a critical level. In order to show this, we argue by contradiction by supposing that \( C^* \) is not a critical point and fixing \( \tau > 0 \) small enough and \( h \in \Gamma_1 \) such that
\[ \Pi(h) \leq C^* + \tau \text{ and } \Pi(g) - \Pi(h) \geq -\tau d(g, h), \forall g \in \Gamma_1 \]
where
\[ \Pi(g) = \sup_{x \in Q} J(g(x)), \forall g \in \Gamma_1 \]
and
\[ d(g, h) = \sup_{x \in Q} \|g(x) - h(x)\|. \]

Now, we apply [21, Proposition 2.3] with \( A = h(Q) \) to find a closed subset \( W \) containing \( A \) in it interior and a deformation \( \alpha_s : W \to H^1(\mathbb{R}^N) \) having the following properties:
\[ \|u - \alpha_s(u)\| \leq s, \forall u \in W \text{ and } s \approx 0^+, \]
\[ J(\alpha_s(u)) - J(u) \leq 2s, \forall u \in W, \]
\[ J(\alpha_s(u)) - J(u) \leq -2\tau s \forall u \in W \text{ with } J(u) \geq C^* - \tau \]
and
\[ \sup_{u \in A} J(\alpha_s(u)) - \sup_{u \in A} J(u) \leq -2\tau s. \]

Now, it is easy to see that \( g = \alpha_s \circ h \in \Gamma_1 \) for \( s \) small enough. However, from (3.7), (3.8) and (3.11)
\[ -2\tau s \geq \Pi(g) - \Pi(h) \geq -\tau d(g, h) \geq -\tau s, \]
which is a contradiction. This contradiction shows that \( C^* \) is a critical level and the proof is completed. The fact that \( C^* \in (m(c_0), 2m(c_0)) \) permits to conclude that the solutions with energy equal to \( C^* \) do not change sign, and as \( f(t) = t \log t^2 \) is an odd function, we can assume that they are nonnegative. Now, the positivity follows by maximum principle.
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