Precise light quark masses from lattice QCD in the RI/SMOM scheme

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Mass of the strange quark

WEIGHTED AVERAGE
100.6+2.1-1.8 (Error scaled by 2.4)

DAVIES 10  LATT 29.8  HPQCD 2+1+1 dyn. flavours, rooted staggered
ALLTON  08  LATT  0.3  RBC&UKQCD2+1 dyn. flavours, domain wall ferm.
BLOSSIER 08  LATT  0.2  ETM, 2 dyn. flavours, twisted mass fermions
DOMINGUEZ 08A THEO 0.0  QCD sum rule
ISHIKAWA 08  LATT 17.4  CP-PACS&JLQCD 2+1 dyn. flavours, Wilson quarks
NAKAMURA 08  LATT  4.1  CP-PACS quenched lattice
BLUM     07  LATT  0.8  QCD sum rule
CHETYRKIN 06 THEO  0.2  QCD sum rule
GOCKELEN  06  LATT  0.9  QCD sum rule
GOCKELEN 06A LATT  3.8
JAMIN     06 THEO  5.1
MASON     06  LATT  5.1
NARISON   06  THEO
BAIKOV    05  THEO
GAMIZ     05  THEO
GORBUNOV  05  THEO
NARISON   05  THEO
AUBIN     04  LATT  9.4
AOKI      04  LATT  6.5
AOKI      03B LATT 18.1
BECIREVIC 03  LATT  0.4
CHIU      03  LATT
GAMIZ     03  THEO
GAMIZ     03  THEO

fundamental parameter
(->Yukawa coupling) in SM

- enters predictions for nonleptonic
decay matrix elements
- probes of Yukawa unification

(Confidence Level < 0.0001)

s-QUARK MASS (MeV)  PDG 2010
Quantum field theory

• correlation functions given by path integrals
  \[ \langle 0 | O_1(x_1) \cdots O_n(x_n) | 0 \rangle = \]
  \[ \int \left( \prod_x dA(x) \right) \left( \prod_x d\psi(x) d\bar{\psi}(x) \right) O(x_1) \cdots O(x_n) e^{(i/\hbar) \int d^4x \mathcal{L}_{QCD}} \]

• \( O_i \) local operators constructed from quark and gluon fields
  either gauge invariant; or one has to fix a gauge and the
  correlation functions depend on the gauge fixing

• perturbation theory (small \( g_s \) expansion): Feynman diagrams

• this does not produce confinement, chiral SB, etc, which are
  non-perturbative phenomena
Lattice QCD

- spacetime replaced by discrete lattice of points
  - gives well-defined path integral

- numerical evaluation including non-perturbative physics

- continuum (a->0) and infinite volume limits (extrapolation)

- due to relatively recent progress, chiral symmetry can be preserved by the lattice regularisation
  - all symmetries of QCD are then preserved
Quark mass on the lattice

- general idea: mass spectrum depends on quark masses
  \[ m_{\pi, K, \ldots} = f(m_u, m_d, m_s; g_s) \]
- r.h.s numerically calculated on the lattice (by studying suitable 2-point functions)
  use measured meson mass spectrum to determine \( m_u, m_d, m_s \)
- These parameters are ‘bare’ and need to be renormalized to be of any use outside this particular lattice calculation
Renormalization

• bare parameters depend on details of regularization (lattice), diverge in continuum limit if physical quantities (meson masses) held fixed

• renormalize: \( m = Z_m m_{\text{bare}} \)

• properly defining \( Z_m(g_s;a) \) gives a finite continuum limit for \( m \)

• many ways to specify a renormalization scheme, e.g.

  - physical renormalization scheme (e.g. mass parameter = observed particle mass) 
    not possible for confined quarks
  - minimal subtraction \( Z_m = 1 - g_s^2/(2\pi^2) \ln(a) + \ldots \) divergent terms only

    preferred in perturbation theory (\( \overline{\text{MS}} \)), not defined beyond
  - Schroedinger functional method 
    difficult/impractical to implement perturbatively
    possible in principle to determine RGI quark mass from step scaling process
Momentum-space subtraction

- Renormalization conditions imposed on Green’s functions
- consider two-point function

\[ -iS(p) = \int dx e^{ipx} \langle T[\Psi(x)\overline{\Psi}(0)] \rangle = \frac{i}{\not{p} - m + i\epsilon - \Sigma(p)} \]

- The RI-MOM and RI’-MOM schemes renormalize the fields and masses by requiring, in Landau gauge,

\[
\lim_{m_R \to 0} \frac{1}{12m_R} \Tr[S_R^{-1}(p)] \bigg|_{p^2 = -\mu^2} = 1
\]

\[
\lim_{m_R \to 0} \frac{1}{48} \Tr \left[ \gamma^\mu \frac{\partial S_R^{-1}(p)}{\partial p^\mu} \right] \bigg|_{p^2 = -\mu^2} = -1 \quad \text{RI-MOM}
\]

\[
\lim_{m_R \to 0} \frac{1}{12p^2} \Tr[S_R^{-1}(p) \not{\psi}] \bigg|_{p^2 \to -\mu^2} = -1 \quad \text{RI’-MOM}
\]

[Martinelli et al 1995]
Conversion to $\overline{\text{MS}}$ scheme

- The MOM renormalization prescription can be implemented in continuum perturbation theory, most conveniently in dimensional regularization. Then the quark mass can be converted from a MOM scheme to e.g. MS-bar

\[
C_{m \text{scheme}} = \frac{m_{\overline{\text{MS}}}}{m_{\text{scheme}}} = \frac{Z_{\overline{\text{MS}}}}{Z_{m \text{scheme}}}
\]

- In practice, the conversion has been done for RI’-MOM up to three loops, and the perturbation expansion does not behave well:

\[
C_{m}^{(\text{RI’})} = 1 - 0.127 \ [\text{NLO}] - 0.069 \ [\text{NNLO}] - 0.046 \ [\text{NNNLO}]
\]

| Loops   | Correction  |
|---------|-------------|
| NLO     | $\uparrow$ ok  |
| NNLO    | $\uparrow$ sizable |
| NNNLO   | $\uparrow$ large |

[Chetyrkin & Retey 1999, Gracey 2003]
Ward identity

- By the (non-singlet) axial-vector ward identity,

\[-iq_\mu \Lambda_{A,B}^\mu (p_1, p_2) = 2m_B \Lambda_{P,B} (p_1, p_2) - i\gamma_5 S_B^{-1}(p_1) - S_B^{-1}(p_2) i\gamma_5\]

where $\Lambda_{A,B}$ and $\Lambda_{P,B}$ are the bare three-point Green’s functions involving the axial current and pseudoscalar density, respectively, the RI/MOM mass renormalization can alternatively be computed as

\[Z_{m}^{RI' / MOM} = \frac{-p^2 \text{tr} [\Lambda_{P,B} (p, p') \gamma_5]_{p^2=p'^2=-\mu^2}}{\text{tr} [S_B^{-1} \gamma_5]}\]

where $q^2 = (p-p')^2 = 0$
RI-SMOM

- Origin of the bad perturbative behaviour unclear. However, nonperturbatively, at \( q^2 = 0 \) there are \( 1/p^2 \) power corrections, and also the chiral limit does not exist because of a “pseudo-goldstone pole” term

\[
\Lambda(p, p'; q^2 = 0) = \frac{\text{const}}{m_K^2} \langle K^+ | s(p) \bar{u}(p') | 0 \rangle + \ldots
\]

(the pseudoscalar density \( P \) has the correct quantum numbers to create pions or kaons, which become massless in the chiral limit)

- a practical issue in lattice simulations involving light quarks [Aoki et al 2008]

- the nonperturbative issues can be addressed by going to more general kinematics

\[
p^2 = (p')^2 = -\mu^2, \quad q^2 = \omega \ p^2
\]

SMOM (“symmetric MOM”) : \( \omega = 1 \) [Sturm et al 2009]
RI-SMOM to \( \overline{\text{MS}} \)

- one-loop conversion factor at the SMOM point

\[ C_m^{(\text{RI-SMOM})} = 1 - 0.015 \quad \text{[\text{NLO}]} \]

a tiny one-loop correction

(recall \( C_m^{(\text{RI}')} = 1 - 0.127 \quad \text{[\text{NLO}]} - 0.069 \quad \text{[\text{NNLO}]} - 0.046 \quad \text{[\text{NNNLO}]} \) )

- as a function of \( \omega = q^2/p^2 \) and the gauge parameter

[Sturm et al 2009]
Two-loop (NNLO) calculation

- straightforward evaluation of traces over numerators
- express numerators as polynomials of the denominators
- Feynman integrals with general propagator powers

\[ \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \prod_i (P_i^n)^{-a_i} \equiv \left( \frac{i}{16\pi^2} \right)^2 \left( \frac{\mu^2}{4\pi} e^\gamma \right)^{2\epsilon} k_n(a_1, \ldots, a_m; \{s_k\}) \]
Integral reduction

- integration by part identities via Laporta’s algorithm [S Laporta 2001]
  use the public Mathematica implementation FIRE [A V Smirnov 2008]
- two two-loop master integrals - known in terms of higher polylogarithms [Davydychev and Usyukina 1994]
- recursively one-loop diagrams with spurious poles -> sensitivity to higher orders in $\epsilon$
only unknown “master” ingredient is one-loop integral !

\[ j(d; \nu_1, \nu_2, \nu_3; p_1^2, p_2^2, p_3^2) \equiv \]

\[
\left( \frac{i}{16\pi^2} \right)^{-1} \left( \frac{\mu^2}{4\pi} e^\gamma \right)^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{[-k^2]^{\nu_3}[-(k + p_1)^2]^{\nu_2}[-(k - p_2)^2]^{\nu_1}}
\]

in particular, need \( j(d; 1,1,2+\epsilon; \ p^2, \ x \ p^2, \ y \ p^2) \), proportional to:

\[
\frac{1}{xy} \left( -\frac{1}{\epsilon} + 2 \ln x + 2 \ln y - \epsilon(2 \ln^2 x + 2 \ln^2 y + \ln x \ln y + 3(1-x-y)\Phi^{(1)}(x,y) - \frac{\pi^2}{6}) + \epsilon^2 \beta(x,y) + O(\epsilon^2) \right)
\]

(actually needed only for \( y=1 \) or \( y=x \))

Need \( O(\epsilon^2) \) term \( \beta(x,y) \) - not known and difficult to compute [at least for us]

can avoid computation - reducing the known 2-loop masters & using an identity from rotating the triangle by 120 degrees, one can obtain sufficient (algebraic) constraints on it
Analytical result

\[ C_{m}^{\text{RI/SMOM}}(\omega) = 1 + \frac{\alpha_s}{4\pi} C_F \left( \frac{3 + \xi}{2} \Phi^{(1)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right) - 4 - \xi + 3 \ln r \right) + \left( \frac{\alpha_s}{4\pi} \right)^2 C_F \left\{ N_c \left( -\frac{2513}{48} - \frac{3\xi}{2} - \frac{\xi^2}{4} + 12\zeta(3) \right) + \frac{307 + 6\xi^2}{12} \ln r - \frac{13}{4} \ln^2 r + \left[ \frac{301}{24} + \frac{3\xi}{4} - \frac{\xi^2}{8} - \frac{13 + \xi^2}{4} \ln r - \frac{7 + 3\xi}{4} \ln \omega \right] \Phi^{(1)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right) + \frac{9 + 6\xi + \xi^2}{8} \Phi^{(1)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right)^2 + \omega \Phi^{(2)} \left( 1, \omega \right) - \frac{3 + \xi}{2} \Phi^{(2)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right) \right\} + n_f \left( \frac{83}{12} + \ln r - \frac{5}{3} \right) \Phi^{(1)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right) - \frac{13}{3} \ln r + \ln^2 r \right\} + \frac{1}{N_c} \left( -\frac{19}{16} - 2\xi - \frac{\xi^2}{2} + \left[ \frac{7}{2} + \xi + \frac{\xi^2}{2} - \frac{9 + 3\xi}{4} \ln r + \frac{5 + 3\xi}{4} \ln \omega \right] \Phi^{(1)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right) + \frac{21 + 6\xi}{4} \ln r - \frac{9}{4} \ln^2 r \right) + \frac{1 + \xi}{2} \Phi^{(2)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right) + \frac{1}{2} \Omega^{(2)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right) - \Omega^{(2)} \left( 1, \omega \right) - \left[ \frac{5}{8} + \frac{3\xi}{4} + \frac{\xi^2}{8} + \frac{1}{\omega} \right] \Phi^{(1)} \left( \frac{1}{\omega}, \frac{1}{\omega} \right)^2 \right\} \]

- The functions \( \Phi^{(1)}, \Phi^{(2)}, \Psi^{(2)}, \Omega^{(2)} \) are all given in terms of polylogarithms up to fourth order
- full \( \omega \) dependence: can interpolate RI/MOM - RI/SMOM

[Gorbahn, SJ, PRD82 (2010) 114001, arXiv:1004.3997]
NNLO result: $\omega$ dependence

\[ C_m^{(\text{SMOM})} = 1 - 0.015 \ [\text{NLO}] - 0.006 \ [\text{NNLO}] \]

\[ C_m^{(\text{RI}') \ prime} = 1 - 0.127 \ [\text{NLO}] - 0.069 \ [\text{NNLO}] - 0.046 \ [\text{NNNLO}] \]

RI’ point: large loop corrections
SMOM point: tiny corrections

confirmed by Almeida, Sturm
[at $\omega = 1$]

[Gorbahn, SJ, PRD82 (2010) 114001, arXiv:1004.3997]
Residual scale dependence

- construct a formally RG-invariant quantity
- e.g. convert to \(\overline{\text{MS}}\) mass from fixed MOM scale 2 GeV, varying dim reg scale used in conversion and RG-evolving back to 2 GeV
- alternatively, consider “RGI” mass, similar picture

\[\text{[Gorbahn, SJ, PRD82 (2010) 114001, arXiv:1004.3997]}\]
NNLO result with error

- take the range of the NNLO band as theoretical range
- symmetrizing around the midpoint, obtain

\[ m_{\overline{\text{MS}}} \left( 2 \text{ GeV} \right) = \left( 0.978^{+0.024}_{-0.010} \mid \text{h.o.} \mid 0.001 \mid \alpha_s \right) m_{\text{RI/SMOM}} \left( 2 \text{ GeV} \right) \]

\[ m_{\text{RGI}} = \left( 2.53^{+0.052}_{-0.014} \mid \text{h.o.} \mid 0.02 \mid \alpha_s \right) m_{\text{RI/SMOM}} \left( 2 \text{ GeV} \right) \]

- 2 percent error!
- error dominated by unknown higher orders, \( \alpha_s \) uncertainty subleading
- new RBC/UKQCD result: \( m_{\overline{\text{MS}}} \left( 2 \text{ GeV} \right) = (96.2 \pm 2.7) \text{ MeV} \)
- similar stability for other quantities? (\( B_K/\epsilon_K \), \( \epsilon'/\epsilon \), ...)

[Gorbahn, S], PRD82 (2010) 114001, arXiv:1004.3997

[Aoki et al, PRD83(2011)074508, arXiv:1011.0892]