Approach to equilibrium for the phonon Boltzmann equation

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Abstract

We study the asymptotics of solutions of the Boltzmann equation describing the kinetic limit of a lattice of classical interacting anharmonic oscillators. We prove that, if the initial condition is a small perturbation of an equilibrium state, and vanishes at infinity, the dynamics tends diffusively to equilibrium. The solution is the sum of a local equilibrium state, associated to conserved quantities that diffuse to zero, and fast variables that are slaved to the slow ones. This slaving implies the Fourier law, which relates the induced currents to the gradients of the conserved quantities.

1 Introduction

If a piece of solid is heated locally and left to cool, the initial temperature distribution will diffusively relax to a constant temperature. The process is accompanied by a heat flow that is proportional to the local temperature gradient, according to the Fourier law. A mathematical understanding of this phenomenon starting from a microscopic model of matter is a considerable challenge [2].

A simple classical mechanical system modeling heat transport in solids is given by a system of coupled oscillators organized on a \(d\)-dimensional cubic lattice \(\mathbb{Z}^d\). The oscillators are indexed by lattice points \(x \in \mathbb{Z}^d\), and carry momenta and coordinates \((p_x, q_x)\). In the

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simplest model $p_x$ and $q_x$ take real values and their dynamics is generated by a Hamiltonian which is a perturbation of a harmonic system:

$$
H(q, p) = \frac{1}{2} \sum_x p_x^2 + \frac{1}{2} \sum_{xy} q_x q_y \omega^2(x - y) + \frac{\lambda}{4} \sum_x q_x^4. \tag{1.1}
$$

In (1.1), the oscillators are coupled by harmonic forces generated by the coupling matrix $\omega^2$ which is taken short range. The parameter $\lambda$ describing the strength of the anharmonicity is assumed to be small. The classical dynamics is given by Hamilton’s equations

$$
\dot{q}_x = p_x, \quad \dot{p}_x = -\frac{\partial H}{\partial q_x}. \tag{1.2}
$$

The dynamics (1.2) preserves the Gibbs measures in the phase space, formally given by

$$
Z^{-1} e^{-\beta H(q, p)} dq dp \tag{1.3}
$$

and describing an equilibrium state with constant inverse temperature $\beta = 1/T$. One expects initial states that agree with (1.3) “at infinity” to be attracted by (1.3) under the flow (1.2).

While such a result is well beyond current techniques, one approach is to study this problem in an appropriate small coupling limit, the kinetic limit. In that limit, one takes $\lambda \approx R \epsilon$, one rescales space and time by $\epsilon$ and, in the limit as $\epsilon \to 0$, one formally arrives at an evolution equation for the covariance of a Gaussian measure of the form $\mathcal{S}$:

$$
\dot{W}(x, k, t) + \frac{1}{2\pi} \nabla_x \omega(k) \cdot \nabla_x W(x, k, t) = RC(W)(x, k, t), \tag{1.4}
$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$. The relation of $W$ to the microscopic model is

$$
\int e^{-iky} W(x, k, t) dk = \lim_{\epsilon \to 0} \sum_z \omega(y - z) \langle q_{x/\epsilon + z} q_{x/\epsilon - z} \rangle_\epsilon + i \langle q_{x/\epsilon + y} p_{x/\epsilon - y} \rangle_\epsilon, \tag{1.5}
$$

where $\langle - \rangle_\epsilon$ is the state at time $t/\epsilon$. $W$ is real and positive. Since the function $W(Rx, k, Rt)$ satisfies (1.4) with $R = 1$ we may, with no loss of generality, make that assumption.

Equation (1.4) is a Boltzmann type equation where the collision term is given by

$$
C(W)(x, k, t) = \frac{9\pi}{4} \int_{\mathbb{T}^d} dk_1 dk_2 dk_3 (\omega \omega_1 \omega_2) \omega_3^{-1} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(k + k_1 - k_2 - k_3) [W_1 W_2 W_3 - W(W_1 W_2 + W_1 W_3 - W_2 W_3)] \tag{1.6}
$$

with $\omega = \omega(k)$, $W = W(x, k, t)$, $\omega_i = \omega(k_i)$, $W_i = W(x, k_i, t)$, $i = 1, 2, 3$. The sum $k + k_1 - k_2 - k_3$ is mod $2\pi\mathbb{Z}$. We normalize $\int_{\mathbb{T}^d} dk = 1$. 

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Analogously to the collision term of the standard Boltzmann equation for gases, the collision term (1.6) describes a scattering process. The scattered "particles" are phonons, i.e. vibrational modes, and the scattering process in (1.6) involves four phonons. The phonons carry momenta, \( k, k_1, k_2, k_3 \) and energies \( \omega(k), \omega(k_1), \omega(k_2), \omega(k_3) \) and the scattering process conserves the total energy and momentum.

We may pose the problem stated in the beginning, i.e. the question of relaxation to equilibrium, for the equation (1.4). Thus, starting from some initial \( W \) that coincides ”at infinity” with an equilibrium state we would like to prove that \( W(t) \) tends diffusively to equilibrium and the process is accompanied by heat fluxes governed by Fourier’s law. In this paper, we prove such a theorem (Theorem 1 in Section 3), for an initial condition that is close to equilibrium. The equation (1.4) becomes a coupled system for "fast" and "slow" variables (see (3.6), (3.7) below). The slow variables correspond to the temperature and the chemical potential of another conserved quantity, defined in (3.16). The slow variables flow under a nonlinear diffusion equation whereas the fast ones, that include the heat currents, are "slaved" to the slow ones by the Fourier law, which is discussed after stating Theorem 1 in Section 3.

There is a rich literature on the derivation of hydrodynamics from the Boltzmann equation for gases. In particular in [1], it is proven that, in an appropriate limit, the Boltzmann equation gives rise to the Navier-Stokes equations (see also [6, 5, 10] for more results and background on the link between Boltzmann and Navier-Stokes). In Theorem 2 (Section 3), we consider the diffusive scaling limit, i.e. the limit where one scales the spatial variable by \( \epsilon \), and the time variable by \( \epsilon^2 \), and we show that a suitably rescaled solution of the Boltzmann equation (1.4) solves, in the limit \( \epsilon \to 0 \), a nonlinear heat equation.

Compared with the Boltzmann equation for gases, eq. (1.4) has some nice features. In particular, its long time dynamics is described by hydrodynamic equations that are considerably simpler than the Navier-Stokes-Fourier system for the former (in the Euler scaling, where space and time are both scaled by \( \epsilon \), eq. (1.4) reduces to the Fourier law). A mathematical theory of (1.4) however appears to be lacking. This paper should be considered as a step in that direction. For an attempt to prove the Fourier law for phonons, beyond the Boltzmann approximation, see [3, 4].

2 The collision term

The collision term (1.6) involves integration over the subset determined by the constraints in the delta functions. Some assumptions on \( \omega \) are needed for regularity of the resulting measure. The standard nearest neighbor coupling between the oscillators corresponds to
\( \omega = \omega_0 \) with

\[
\omega_0^2 = 2 \sum_{j=1}^{d} (1 - \cos k_j) + r
\]

(2.1)

where \( r \geq 0 \) is the \textit{pinning} parameter. The regularity properties stated in Proposition 2.1 should hold for this \( \omega \) as long as \( r > 0 \). The proof of this seems, however, quite tedious and, therefore, we take

\[
\omega(k) = \omega_0^2(k),
\]

(2.2)

with \( r > 0 \), which is simpler to analyze. Moreover, we need to take the spatial dimension \( d \geq 2 \). \textit{From now on, we'll make that assumption.}

For \( r = 0 \) and \( \omega(k) = \omega_0(k) \) the regularity problems are more serious and we expect to need \( d \geq 3 \). Moreover, the spectrum of the linearized collision operator has no gap and the nature of the flow of eq. (1.4) is quite different since also the slaved modes are diffusive. Progress in that case is so far hampered by the complexity of the integrals appearing in (1.6).

The basic properties of the collision term are summarized in Proposition 2.1. Since the \( x \) variable plays no role in \( C \) we consider \( C \) as a map in

\[
\mathcal{B} = C^0(\mathbb{T}^d)
\]

equipped with the sup norm, denoted by \( \| \cdot \| \). We also need the Hilbert space

\[
\mathcal{H} = L^2(\mathbb{T}^d, \omega^2(k)dk),
\]

(2.3)

whose norm will be denoted by \( \| \cdot \|_{\mathcal{H}} \). In this paper, we shall use \( C \) or \( c \) to denote constants that may vary from place to place.

**Proposition 2.1** (a) \( C : \mathcal{B} \to \mathcal{B} \) with \( \| C(W) \| \leq C\| W \|^3 \), for some constant \( C < \infty \).

(b) The equation \( C(W) = 0 \) has, for \( W \geq 0 \), exactly a two parameter family of solutions in \( \mathcal{B} \)

\[
W_{T,A} = \frac{T}{\omega(k) + A}
\]

(2.4)

where \( W \geq 0 \) for \( A \geq -r^2 \).

(c) For all \( W \in \mathcal{B} \)

\[
\int dkC(W)(k) = \int dk\omega(k)C(W)(k) = 0.
\]

(2.5)

(d) Let \(-L \) be the linearization of \( C(W) \) around \( W_0 = \omega(k)^{-1} \) i.e.

\[
L = -DC(W_0).
\]

(2.6)
Then, $L$ is bounded in $\mathcal{B}$ and bounded and positive in $\mathcal{H}$. It has two zero modes

$$ L\omega^{-1} = L\omega^{-2} = 0. \quad (2.7) $$

and the rest of its spectrum in $\mathcal{B}$ and $\mathcal{H}$ is contained in $\Re \lambda \geq a > 0$.

Moreover $L = M + K$ where $M$ is a multiplication operator by a strictly positive continuous function and $K$ is an integral operator compact in $\mathcal{H}$ and $\mathcal{B}$ and satisfying

$$ \sup_k \int |K(k, k')|dk' < \infty. \quad (2.8) $$

**Proof.** (b) We use an argument given in [8], for a similar kernel. First, writing the $[-]$ in (1.6) as

$$ \prod_{i=0}^{3} W(k_i) \left[ W(k_0)^{-1} - W(k_3)^{-1} - W(k_2)^{-1} + W(k_1)^{-1} \right], \quad (2.9) $$

we have:

$$ C(W)(k) = \frac{9\pi}{4} \int_{T^4} \prod_{i=1}^{3} (dk_i\omega_i^{-1}W(k_i))\delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) $$

$$ \cdot \delta(k_0 + k_1 - k_2 - k_3) \left[ W(k_0)^{-1} - W(k_3)^{-1} - W(k_2)^{-1} + W(k_1)^{-1} \right], \quad (2.10) $$

which vanishes, for $W(k_i) \geq 0$, if $f(k) = W(k)^{-1}$ is a collisional invariant, i.e. a function satisfying

$$ f(k_0) + f(k_1) = f(k_2) + f(k_3), \quad (2.11) $$

on the set of vectors $k_i$ in the support of the delta functions in (2.10). In [9] it is proven that all $L^1$-solutions of (2.11) are of the form $a\omega(k) + b$. Hence, the functions in (2.4) satisfy $C(W) = 0$.

To show that these functions are the only solutions of $C(W) = 0$, write, for $g$ bounded,

$$ \int dk g(k)C(W)(k) = \frac{9\pi}{4} \int_{T^4} \prod_{i=0}^{3} (dk_i\omega_i^{-1}W(k_i))\delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) $$

$$ \cdot \delta(k_0 + k_1 - k_2 - k_3) g(k_0) \left[ W(k_0)^{-1} - W(k_3)^{-1} - W(k_2)^{-1} + W(k_1)^{-1} \right], \quad (2.12) $$

and use the symmetry $0 \leftrightarrow 1$ of the rest of the integrand, to replace $g(k_0)$ in front of $[-]$ in (2.12) by $\frac{1}{2}(g(k_0) + g(k_1))$, and, using the antisymmetry of the rest of the integrand under the exchange $0 \leftrightarrow 3, 1 \leftrightarrow 2$, we may replace $\frac{1}{2}(g(k_0) + g(k_1))$ by $-\frac{1}{2}(g(k_3) + g(k_2))$. This implies that (2.12) is proportional to

$$ \int_{T^4} \prod_{i=0}^{3} (dk_i\omega_i^{-1}W(k_i))\delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\delta(k_0 + k_1 - k_2 - k_3) $$

$$ \cdot [g(k_0) + g(k_1) - g(k_2) - g(k_3)] \left[ W(k_0)^{-1} - W(k_3)^{-1} - W(k_2)^{-1} + W(k_1)^{-1} \right] \quad (2.13) $$
Taking $g(k) = W(k)^{-1}$, we obtain

$$\int_{\mathbb{T}^d} \prod_{i=0}^3 (dk_i \omega_i^{-1} W(k_i)) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(k_0 + k_1 - k_2 - k_3)$$

$$\cdot [W(k_0)^{-1} - W(k_3)^{-1} - W(k_2)^{-1} + W(k_1)^{-1}]^2,$$

(2.14)

which vanishes, for $W(k_i) \geq 0$, only if $W(k)^{-1}$ is a collisional invariant.

(c) Since $\int dk g(k) C(W)(k)$ is proportional to \([2.13]\), it vanishes, for any $W$, whenever $g$ is a collisional invariant, in particular for $g(k) = 1$ and $g(k) = \omega(k)$.

(d) Differentiating $C(W_{T,A}) = 0$ with respect to $T$ and $A$ at $T = 1$ and $A = 0$, we obtain

$$L \omega^{-1} = L \omega^{-2} = 0.$$

(2.15)

Explicitly, we have (note the $-$ sign in \([2.6]\))

$$L f = \frac{9\pi}{4\omega_0^2} \int_{\mathbb{T}^d} \prod_{i=1}^3 dk_i \omega_i^{-2} \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(k_0 + k_1 - k_2 - k_3)$$

$$(\omega(k_0)^2 f(k_0) + \omega(k_1)^2 f(k_1) - \omega^2(k_2)f(k_2) - \omega^2(k_3)f(k_3)) \cdot$$

(2.16)

To obtain \((2.16)\) from \((1.6)\), write the $[-]$ in \((1.6)\) as in \((2.9)\) and expand

$$W(k_i)^{-1} = (\omega(k_i)^{-1} + f(k_i))^{-1} = \omega(k_i) - \omega(k_i)^2 f(k_i) + ...$$

(2.17)

If we take the first term in \((2.17)\) in the $[-]$, we get 0 because of the delta function in \((1.6)\).

So, the only linear terms in $f$ correspond to taking the second term in \((2.17)\) inside the $[-]$, and replacing $\prod_{i=0}^3 W(k_i)$ by $\prod_{i=0}^3 \omega_i^{-1}$ which implies \((2.16)\). Using the symmetries that led to \((2.14)\), we see that $(f, Lf)$ is proportional to

$$\int_{\mathbb{T}^d} \prod_{i=0}^3 dk_i \omega_i^{-2} \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(k_0 + k_1 - k_2 - k_3)$$

$$(\omega(k_0)^2 f(k_0) + \omega(k_1)^2 f(k_1) - \omega^2(k_2)f(k_2) - \omega^2(k_3)f(k_3))^2 \geq 0.$$

(2.18)

Thus, the zero modes of $L$ are of the form $f = \omega^{-2}g$, where $g$ are collisional invariants, i.e. of the form $a\omega(k) + b$ (see part (b)) and therefore \((2.7)\) give the only zero modes in $\mathcal{H}$ and in $\mathcal{B}$.

From \((2.16)\), we get the decomposition $L = M + K$ with

$$M(k) = \frac{9\pi}{4\omega_0^2} \int_{\mathbb{T}^d} \prod_{i=1}^3 dk_i \omega_i^{-2} \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(k_0 + k_1 - k_2 - k_3) \omega^2(k_0),$$

(2.19)

where $k = k_0$, and $K$ is an integral operator

$$(Kf)(k) = \int_{\mathbb{T}^d} K(k, k') f(k') \omega(k')^2 dk'.$$
given by the last three terms in (2.16), i.e.

$$K(k, k') = -\frac{9\pi}{4}(\omega(k)\omega(k'))^{-2}(I^{(1)}(k, k') + I^{(2)}(k, k'))$$

(2.20)

where

$$I^{(1)}(k, k') = 2\int_{\mathbb{T}^d} dk_1(\omega(k_1)\omega(k_1 + k - k'))^{-2}\delta(\omega(k_1) - \omega(k_1 + k - k') + \omega(k) - \omega(k')),$$  

(2.21)

and

$$I^{(2)}(k, k') = -\int_{\mathbb{T}^d} dk_1(\omega(k_1)\omega(k_1 + k - k'))^{-2}\delta(-\omega(k_1) - \omega(k_1 + k - k') + \omega(k) + \omega(k')).$$  

(2.22)

The singularities of $I^{(1)}$ and $I^{(2)}$ are similar, and we will study $I^{(1)}$ first; let

$$r = \frac{1}{2}(k - k')$$

and change variables in (2.21) to $k_1 = q - r - (\frac{\pi}{2}, \ldots, \frac{\pi}{2})$ and write $\frac{\pi}{2}$ for $(\frac{\pi}{2}, \ldots, \frac{\pi}{2})$. Recalling (2.1, 2.2), we get

$$I^{(1)}(k, k') = \int_{\mathbb{T}^d} dq \ \delta(\Omega(q, r, k')) \ g(q, r)$$

(2.23)

with $g = (\omega(q - r - \frac{\pi}{2}\pi))\omega(q + r - \frac{\pi}{2}\pi))^{-2}$, and

$$\Omega(q, r, k') = \sum_{j=1}^{d}(\sin(q_j + r_j) - \sin(q_j - r_j)) + \omega(k' + 2r) - \omega(k').$$  

(2.24)

The summand equals $2\cos q_j \sin r_j$. We change variables to $\cos q_j = s_j(1 - y_j)$ with $s_j = 1$ in the region $q_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $s_j = -1$ in the region $|q_j| > \frac{\pi}{2}$. In both cases, $y_j \in [0, 1]$. Our integral becomes a sum of integrals of the form

$$I_s(k, k') = \int_{[0,1]^d} \delta(2\sum_j y_j s_j \sin r_j - m(k', r))G(y, r) \prod_j y_j^{-\frac{1}{2}} h(y_j)dy_j,$$  

(2.25)

with $h(y) = (2 - y)^{-\frac{1}{2}}$, $G$ smooth, and

$$m(k', r) = \omega(k' + 2r) - \omega(k') + 2\sum_j s_j \sin r_j.$$  

(2.26)

Let $\sin r_j \neq 0$ for all $j$ i.e. $k_j - k'_j \neq 0$ modulo $2\pi$ and $m(k', r) \neq 0$. Given $k$, the set of $k'$ such that these conditions hold is of full measure. Then,

$$I_s(k, k') = \prod_j |\sin r_j|^{-\frac{1}{2}} J_s(k, k'),$$

(2.27)
where

\[ J_s(k, k') = \int_0^{\lceil \sin r_j \rceil} \prod_j \frac{dy_j}{\sqrt{y_j}} h(\frac{y_j}{\sin r_j}) \delta(2 \sum_j y_j s_j - m(k', r)) G(y, r), \quad (2.28) \]

with \( y_{rj} = y_j/|\sin r_j| \) and \( s_j = \pm 1 \).

Consider e.g. \( d = 2 \). Integrating the delta function over the variable \( y_2 \) shows that the singularities of \( J \), given by (2.20), is compact in \( H \). From (2.16), we see that \( L \) is self-adjoint in \( H \).

To show compactness in \( B \), we will use Hölder continuity properties of \( I_s(k, k') \). We want to bound

\[ \sup_{k, \tilde{k}} \int dk' |I_s(k, k') - I_s(\tilde{k}, k')|/|k - \tilde{k}|^{\alpha}, \]

for some \( \alpha > 0 \). We will analyze the dependence in \( k \) of each factor in (2.27), (2.28) separately. For \( u, v \in (0, 1] \) we have, for any \( n \),

\[ |u^{-\frac{1}{2}} - v^{-\frac{1}{2}}| = |u^\frac{1}{2n} - v^\frac{1}{2n}| \left| \sum_{k=1}^n u^{-\frac{k}{2n}} v^{-\frac{n+1-k}{2n}} \right| \leq C_n |u - v|^\frac{1}{2n} \max\{u^{-\frac{n+1}{2n}}, v^{-\frac{n+1}{2n}}\} \quad (2.31) \]

since \( u^\frac{1}{2n} \in C^{\alpha}_{\infty}([0, 1]) \). We can use this, with \( n > 1 \), to control the \( k \) dependence of the factor \( \prod_j |\sin r_j|^{-\frac{1}{2}} \) in (2.27). For \( J_s \), let us, for simplicity, again study \( j \). Let \( \tilde{m} = m(k', \tilde{r}) \), with \( \tilde{r} = \frac{1}{2}(\tilde{k} - k') \). Applying (2.31) to (2.29) we get, with \( \alpha = \frac{1}{2n} \),

\[ \left| \int_0^{\lceil \sin r_j \rceil} \chi(sy + m) \in [0, |\sin r_2|] \right| \left( \frac{1}{\sqrt{y\sqrt{s}y + m}} - \frac{1}{\sqrt{y\sqrt{s}y + \tilde{m}}\sqrt{\tilde{m}}} \right) dy \]

\[ \leq C_{\alpha} |m - \tilde{m}|^{\alpha} (1 + \max\{|m|^{-\alpha}, |\tilde{m}|^{-\alpha}\}) \]

\[ \leq C_{\alpha} |k - \tilde{k}|^{\alpha} (1 + \max\{|m|^{-\alpha}, |\tilde{m}|^{-\alpha}\}), \quad (2.32) \]

where the last inequality holds because \( m \) is analytic in \( k \). Next, let \( |\sin \tilde{r}_1| = |\sin r_1| + \delta \), assuming \( \delta > 0 \). Then, using the fact that

\[ \int_{|\sin r_1|}^{\lceil |\sin r_1| + \delta \rceil} \chi(sy + m) \in [0, |\sin r_2|] dy \sqrt{y\sqrt{s}y + m} \leq C \min(1 + |\log |m||, \log(1 + \frac{\delta}{|\sin r_1|})) \]
we get the following crude estimate:

\[ |j(r_1, r_2, k', m) - j(\tilde{r}_1, r_2, k', m)| \leq C\delta^{\frac{1}{2}} + \chi(|\sin r_1| \leq \delta^{\frac{1}{2}}(1 + |\log |m||)). \quad (2.33) \]

A similar estimate holds for the dependence with respect to \( r_2 \), and for the one with respect to \( m \) in \( \chi(sy + m) \in [0, |\sin r_2|] \). Since \( m \) is real analytic, \( \prod_j |\sin r_j|^{-\frac{1}{2}} |m(k', \frac{1}{2}(k-k'))|^{-\alpha} \) is integrable in \( k' \) uniformly in \( k \) for \( \alpha > 0 \) small enough. Moreover, the integral of \( \prod_j |\sin r_j|^{-\frac{1}{2}} |m(k', \frac{1}{2}(k-k'))|^{-\alpha} \chi(|\sin r_1| \leq \delta^{\frac{1}{2}}) \) over \( k' \) is of order \( \delta^{\alpha} \), for \( \alpha \) small.

We get, by combining all these estimates, that for some \( \alpha > 0 \) and all \( f \in \mathcal{B} \),

\[ \| \int I_s(k, k') f(k') dk' \|_\alpha \leq C \| f \| \quad (2.34) \]

where \( \| - \|_\alpha \) is the \( C^\alpha \) norm. We may repeat this analysis for \( I^{(2)} \), writing:

\[ I^{(2)}(k, k' + \pi) = - \int_{\mathbb{T}^d} dk_1 (\omega(k_1 - \pi)\omega(k_1 + k - k')^{-2} \delta(\omega(k_1) - \omega(k_1 + k - k') + \omega(k) - \omega(k')) \quad (2.35) \]

where we changed variables \( k_1 \to k_1 + (\pi, \ldots, \pi) \) in the integrand and used formula (2.21), (2.22) for \( \omega(k) \) and \( \cos(k_j + \pi) = -\cos k_j \). Since the factor \( (\omega(k_1 - \pi)\omega(k_1 + k - k'))^{-2} \) is smooth, we can do exactly the same analysis as above and obtain the bound (2.34) for \( I^{(2)} \). Then, the same bound holds for the kernel \( K \), given by (2.20). This shows that \( K \) is a continuous map from \( \mathcal{B} \) into the space \( C^\alpha \) of Hölder continuous functions. Since \( C^\alpha \) is compactly embedded in \( \mathcal{B} \) (by Ascoli’s theorem), this proves that \( K \) is compact. Since, using (2.20), (2.21), (2.22), (2.35),

\[ M(k) = \int dk' K(k, k') \omega(k')^2, \]

we also get \( M \in \mathcal{B} \) (actually \( M \) is Hölder continuous) and that \( L \) is bounded (since \( K \) is compact).

The case \( d > 2 \) is similar.
(a) We proceed as in (d) with the delta functions and get

\[ |C(W)(k)| \leq C\|W\|^3 \int dk' \int_{[0,1]^d} \delta(2 \sum_j y_j s_j \sin r_j - m(k', r)) \prod_j y_j^{-\frac{1}{2}} h(y_j) dy_j \quad (2.36) \]

which, using (2.27), (2.29), proves the claim. \( \square \)

The identities (2.5) yield two conservation laws. To show this, define, for \( \alpha = 1, 2 \),

\[ j_\alpha(x, t) = -\frac{1}{2\pi} (\omega^{-\alpha}, \nabla \omega \ W), \quad (2.37) \]
where the scalar product is in $\mathcal{H}$, see (2.3); $j_1$ is the thermal current and $j_2$ can be called the phonon number current. Similarly, set

$$T_\alpha(x, t) = (\omega^{-\alpha}, W).$$ (2.38)

$T_1$ is the temperature and $T_2$ is related to the phonon chemical potential. Equations (1.4) and (2.5) give then the conservation laws

$$\dot{T}_\alpha = \nabla \cdot j_\alpha.$$ (2.39)

$T_\alpha$ are the slow modes that will diffuse. The currents $j_\alpha$ will be related to their gradients via the Fourier law, see (3.17) below.

### 3 Results

Let $\hat{W}(p, k, t)$ denote the Fourier transform of $W$ in the $x$ variable. We shall look for solutions of (1.4) of the form:

$$\hat{W}(p, k, t) = W_0(k)\delta(p) + w(p, k, t).$$ (3.1)

where, we recall, $W_0 = \omega^{-1}$. The equation (1.4) becomes then,

$$\dot{w} = -Dw + n(w),$$ (3.2)

where the linear operator is given by

$$D = L + \frac{i}{2\pi} p \cdot \nabla \omega(k)$$ (3.3)

and the nonlinear term is $n(w) = C(W_0 + w) + Lw = C(W_0 + w) - DC(W_0)w$. Written as an integral equation, (3.2) becomes

$$w(t) = e^{-tD}w(0) + \int_0^t ds e^{-(t-s)D}n(w(s)).$$ (3.4)

We need to decompose this in terms of the slow and the fast variables. Let $P$ be the orthogonal projection in the Hilbert space $\mathcal{H}$ on $E = \text{span} \{\omega^{-1}, \omega^{-2}\}$ and let $Q$ be the one on the complement of $E$. The identities (2.5) can then be written as:

$$Pn = 0,$$ (3.5)

or $n = Qn$, since, by differentiation, (2.5) implies the same identities with $C(W)$ replaced by $DC(W_0)w$. Let

$$Pw = T, \quad Qw = v.$$
Then, $w(t) = T(t) + v(t)$, and eq. (3.2) becomes

\[ T(t) = Pe^{-tD}w(0) + \int_0^t dsPe^{-(t-s)D}Qn(w(s)) \] (3.6)
\[ v(t) = Qe^{-tD}w(0) + \int_0^t dsQe^{-(t-s)D}Qn(w(s)) \] (3.7)

Define

\[ e(p, t) := (1 + (t + 1)p^2)^{-n} \] (3.8)

with $n > d/2$ and let $\mathcal{E}_t$ be the space of continuous functions $f(p, k)$ equipped with the norm

\[ \|f\|_t \equiv \sup_{k, p} e(p, t)^{-1}|f(p, k)| = \sup_p e(p, t)^{-1}\|f(p, \cdot)\| \] (3.9)

and let $\mathcal{E}$ be the space of functions $w(t) \in \mathcal{E}_t$, with

\[ \|w\|_\mathcal{E} \equiv \sup_t \|w(t)\|_t. \] (3.10)

Let $\kappa : E \to E$ be the linear operator

\[ \kappa = (2\pi)^{-2}P\partial_1\omega L^{-1}\partial_1\omega P, \] (3.11)

where $\partial_1$ denotes the derivative with respect to the first argument. $\kappa$ is strictly positive by Proposition 2.1.d, since the set $\partial_1\omega E$ forms a two dimensional subspace in $E_\perp$ (by symmetry: $\partial_1\omega$ is odd in $k$, while the functions in $E$ are even). Let

\[ T_0(p, t) = e^{-tp^2\kappa}T(p, 0)\chi(||p|| \leq 1), \] (3.12)

and

\[ v_0(p, t) = \frac{-i}{2\pi}L^{-1}p \cdot \nabla \omega T_0(p, t) \] (3.13)

(These lie in $E$ and $E_\perp$ respectively). Then, we have our main result:

**Theorem 1.** There exists $\delta > 0$ such that, for $\|w(0)\|_0 \leq \delta$, the equation (3.2) has a unique solution in $\mathcal{E}$, $w(t) = T(t) + v(t)$, satisfying, for $t \geq 1$,

\[ \|T(t) - T_0(t)\|_t \leq C(t)t^{-\frac{1}{2}}\|w(0)\|_0, \] (3.14)
\[ \|v(t) - v_0(t)\|_t \leq C(t)t^{-1}\|w(0)\|_0, \] (3.15)

where $C(t) = C \log(1 + t)$ for $d = 2$ and is constant for $d > 2$.

**Remark.** The norm (3.9) and (3.14) imply that, in $x$ space, $T(x, t)$ is given by the usual diffusive Gaussian term, which, for bounded $x$, decays like $t^{-\frac{d}{2}}$, plus a correction bounded,
uniformly in $x$, by $O(t^{-\frac{d+1}{2}})$. Eq. (3.15) says that the fast mode $v$ is slaved to the slow one $T$, i.e. it decays like an explicit term, the Fourier transform of $v_0(p,t)$, plus a correction uniformly bounded by $O(C(t)t^{-\frac{d+2}{2}})$.

Eq. (3.15) also implies the Fourier law for the leading terms of the solution. Write $T$ in the basis $T(p,k,t) = \sum_{\beta=1,2} \omega^{-\beta}(k)\tilde{T}_\beta(p,t)$, (3.16)

(since the basis is not orthogonal, $\tilde{T}_\beta$ does not coincide with the Fourier transform of $T_\beta$ in (2.38)). Then, in $x$-space, the currents (2.37) (where, by symmetry, only the $v$ part of $W$ contributes) become, up to terms $O(C(t)t^{-\frac{d+2}{2}})$,

$$j_\alpha = \sum_\beta \kappa_{\alpha\beta} \nabla T_\beta, \quad (3.17)$$

with the positive conductivity matrix

$$\kappa_{\alpha\beta} = (\omega^{-\alpha}, \kappa\omega^{-\beta}), \quad (3.18)$$

since, by symmetry, $(\omega^{-\alpha}, \partial_i\omega L^{-1}\partial_j\omega^{-\beta})$ equals zero for $i \neq j$, and, for $i = j$, $i$ can be chosen equal to 1.

Finally, we can also derive a nonlinear heat equation as the hydrodynamic scaling limit of the Boltzmann equation (1.4). We scale

$$\tilde{W}(x,k,t) = W(\epsilon x, k, \epsilon^2 t),$$

where $\tilde{W}(x,k,t)$ satisfies (1.4)). Thus, we obtain, for $W$, the equation:

$$\dot{W}(x,k,t) + (2\pi \epsilon)^{-1} \nabla \omega(k) \cdot \nabla W(x,k,t) = \epsilon^{-2} C(W)(x,k,t). \quad (3.19)$$

We shall solve it with initial data $W|_{t=0} = \omega(k)^{-1} + w(x,k,0)$ and

$$w(x,k,0) = T(x,k,0) + \epsilon v(x,k,0), \quad (3.20)$$

with $T(x,\cdot,0) \in E$, $v_0(x,\cdot,0) \in E^\perp$, $\forall x$. $\tilde{W}(x,k,0) = W(\epsilon x, k, 0)$ has spatial variations at scale $\epsilon^{-1}$.

We have then

**Theorem 2.** There exists $\delta > 0$ such that, for all $\epsilon \leq 1$ and $\|T(0)\|_0$, $\|v(0)\|_0 \leq \delta$, the equation (3.19) has a unique solution $W^\epsilon = T^\epsilon + \epsilon v^\epsilon \in E$. Moreover, $T^\epsilon(t) \to T(t)$ and $v^\epsilon(t) \to v(t)$ in $E_t$, for all $t > 0$, as $\epsilon \to 0$, where $T$ and $v$ are the unique solutions of

$$DC(\omega^{-1} + T)v = (2\pi)^{-1} \nabla \omega \cdot \nabla T \quad (3.21)$$

$$\dot{T} = -(2\pi)^{-1} P \nabla \omega \cdot \nabla v \quad (3.22)$$
Remark. Since we have $DC(\omega^{-1} + T)$ instead of $DC(\omega^{-1}) = -L$ in (3.22), we get a nonlinear heat equation for $T$:

$$\dot{T} = \nabla \cdot (K(T) \nabla T)$$

with

$$K(T) = -(2\pi)^{-2} P \partial_1 \omega DC(\omega^{-1} + T)^{-1} \partial_1 \omega P,$$

(remembering that $L$ is positive, we see that $DC$ is negative and that $K(T)$ is a positive matrix). One can show that the long-time behavior of (3.23) is similar to the one in Theorem 1.

4 Proofs

We start by proving some results on the spectrum of the operator $D$ defined in (3.3).

Recall that, by Proposition 2.1 d, $\Re(\sigma(L) \setminus \{0\}) \geq a$.

**Proposition 4.1** There exists $p_0 > 0$, and $b > 0$, such that, both in $\mathcal{B}$ and in $\mathcal{H}$, the following holds:

a) If $|p| < p_0$ then 

$$\sigma(D) = \{\lambda_1, \lambda_2\} \cup \tilde{\sigma}$$

with $\inf \Re \tilde{\sigma} > a/2$ and

$$\lambda_i = p^2 \mu_i + \mathcal{O}(|p|^3)$$

where $\mu_i > 0$ are the eigenvalues of the operator $K$ defined in (3.11).

b) If $|p| \geq p_0$ then $\inf \Re \sigma(D) \geq b$.

**Proof.** a) Let

$$L(\beta) = L + \beta \rho$$

where $\rho = (2\pi)^{-1} |p|^{-1} p \cdot \nabla \omega(k)$ so that $D = L(i|p|)$. $\rho$ is a bounded operator in $\mathcal{B}$ and in $\mathcal{H}$. Hence, for $|\beta| < p_0$ small enough, the spectrum of $L(\beta)$ consists of two eigenvalues close to the origin, with the rest of the spectrum having real part larger than $a/2$. Moreover, $L(\beta)$ is self adjoint in $\mathcal{H}$ for $\beta$ real and thus forms an analytic Kato family (see [7]). Therefore, there is a $p_0$ such that, for $|\beta| < p_0$, the two eigenvalues of $L(\beta)$, $\lambda_1$, $\lambda_2$, and the associated eigenfunctions $\psi_i$, are analytic in $|\beta| < p_0$. To compute their Taylor series, use the decomposition $w = T + v$ and project the equation $(L + \beta \rho)w = \lambda w$ onto $Q\mathcal{H}$ and $P\mathcal{H}$. Then, using $P\rho T = 0$ (since $\rho T$ is odd in $k$), we get a pair of equations:

$$Lv + \beta Q \rho (T + v) = \lambda w,$$

$$\beta P \rho v = \lambda T,$$
whereby, since $\Re(\sigma(L) \setminus \{0\}) \geq a$, we get, for $\beta$ small and $\lambda = O(\beta)$,

$$v = -\beta(L - \lambda + \beta Q \rho)^{-1}Q \rho T = -\beta L^{-1}Q \rho T + O(\beta^2)T,$$

where $O(\beta^2)$ denotes a bound on the operator norm in $\mathcal{B}$. Since $Q \rho T = \rho T$ and $T = PT$, we get

$$-\beta^2 P \rho L^{-1} \rho PT + O(\beta^3)T = \lambda T.$$  

The $2 \times 2$ matrix $P \rho L^{-1} \rho P$ is strictly positive, by Proposition 2.1 (d). The result follows by letting $\beta = |p|$, and by observing that, since the functions in $E$ are invariant under reflections and permutations of the coordinates of $k$, the matrix $\kappa(p) = P \rho L^{-1} \rho P$ equals $\kappa(e_1)$ i.e. the one given in (3.11).

b) Write $D = \mathcal{M} + i|p|\rho + K$ as a sum of a multiplication and a compact operator. Since $\Re(\mathcal{M} + i|p|\rho) \geq \inf\mathcal{M} > 0$ we need only to show that all eigenvalues $\lambda$ satisfy $\Re\lambda \geq b$, if $|p| \geq p_0$. We decompose $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o$ into even and odd subspaces. Let $\lambda$ be an eigenvalue. Then, since $\rho$ is odd, and the operator $L$ is parity-preserving,

$$Lw_e + i|p|\rho w_o = \lambda w_e$$
$$Lw_o + i|p|\rho w_e = \lambda w_o$$

(4.2)

Write $\lambda = x + iy$ and suppose that $x < \frac{a}{2}$ (where $a = \inf\sigma(L|\mathcal{H}_e)$). Then $(L - \lambda)|\mathcal{H}_o$ is invertible, and (4.2) becomes

$$(L + p^2 \rho(L - \lambda)^{-1} \rho)w_e = \lambda w_e.$$  

(4.3)

Let $\|w_e\|_{\mathcal{H}} = 1$. Then, taking the scalar product of (4.3) in $\mathcal{H}$ with $w_e$, and writing separately the real and imaginary parts, one gets:

$$x = (w_e, (L + p^2 \rho(L - x)((L - x)^2 + y^2)^{-1} \rho)w_e) \geq \frac{a}{2}p^2(w_e, \rho((L - x)^2 + y^2)^{-1} \rho w_e),$$

(4.4)

since $L \geq 0$ and, as operators in $\mathcal{H}_o$, $(L - x)((L - x)^2 + y^2)^{-1} \geq \frac{a}{2}((L - x)^2 + y^2)^{-1}$. For the imaginary part, we get:

$$y = yp^2(w_e, \rho((L - x)^2 + y^2)^{-1} \rho w_e).$$

(4.5)

If $y \neq 0$ (4.4) and (4.5) imply $x \geq a/2$, against our assumption. Thus $y = 0$, i.e. all eigenvalues $\lambda$ of $D$ with $\Re\lambda < \frac{a}{2}$ are real, and, by (4.4), positive.

Let

$$l(r, \lambda) = \inf_{\|w\| = 1} (w, (L + r^2 \rho(L - \lambda)^{-1} \rho)w).$$

Since, for $\lambda \leq a/2$, $(L - \lambda)^{-1} \geq c > 0$, we get

$$(w, (L + r^2 \rho(L - \lambda)^{-1} \rho)w) \geq (w, (L + cr^2 \rho^2)w).$$

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To bound this from below, observe that both terms in \( L + cr^2 \rho^2 \) are positive, that 
\((w, LW) \geq c\|Qw\|^2\), and that 
\((Pw, \rho^2 Pw) \geq c\|Pw\|^2\), by inspection of the functions 
in \( PH \) and of \( \rho \). So, if \( \|Qw\|^2 \geq c\|w\|^2\), we can use the first lower bound, while, if \( \|Qw\|^2 \leq c'\|w\|^2\), i.e. \( \|Pw\|^2 \geq (1 - c')\|w\|^2\), we can use the second one, if \( c' \ll c \), to get \( L + cr^2 \rho^2 > c''r^2 \) for \( r < p_0 \), and \( p_0 \) small enough. So, if 
\( \|Qw\|^2 \geq c'\|w\|^2 \), we can use the first one, while, if 
\( \|Qw\|^2 \leq c'\|w\|^2 \), i.e. 
\( \|Pw\|^2 \geq (1 - c')\|w\|^2 \), we can use the second one, if \( c' \ll c \), to get 
\( L + cr^2 \rho^2 > c''r^2 \) for \( r < p_0 \), and \( p_0 \) small enough. So, we can take 
\( b = \min(\frac{a}{2}, c''p_0^2) \).

We are now ready to state the estimates for the heat kernels in equations (3.6) and (3.7). Let

\[ R(z) = (z - D)^{-1} \]

be the resolvent of the operator \( D \) and \( \gamma \) be a circle of radius \( a/4 \) around the origin. Then

\[ \tilde{P} = \oint_\gamma R(z) \frac{dz}{2\pi i}, \tag{4.6} \]

is a projection onto the 2-dimensional eigenspace of \( D \) introduced in Proposition 4.1 a. Let \( \tilde{Q} = 1 - \tilde{P} \). We have:

**Proposition 4.2** For all \( t \geq 0 \) we have

\[ \|e^{-tD}\| \leq C(e^{-ctp^2} + e^{-ct}) \tag{4.7} \]

where \( \| \cdot \| \) is the operator norm in \( \mathcal{B} \). Moreover, there exists \( p_0 > 0 \) so that for \( |p| \leq p_0 \)

\[ \|P e^{-tD} Q\| + \|Q e^{-tD} P\| \leq C|p|(e^{-ctp^2} + e^{-ct}) \tag{4.8} \]

\[ \|Q e^{-tD} Q\| \leq C(p^2 e^{-ctp^2} + e^{-ct}) \tag{4.9} \]

\[ \|e^{-tD} \tilde{Q}\| \leq Ce^{-ct} \tag{4.10} \]

**Proof.** Since \( D = L + \frac{2\pi}{i} p \cdot \nabla \omega(k) \), we have, by the resolvent expansion:

\[ \tilde{P} = \oint_\gamma (z - L)^{-1} \frac{dz}{2\pi i} + \oint_\gamma (z - L)^{-1} \frac{i}{2\pi} p \cdot \nabla \omega(z - L)^{-1} \frac{dz}{2\pi i} + \mathcal{O}(p^2) \]

\[ = P - \frac{i}{2\pi} L^{-1} p \cdot \nabla \omega P - \frac{i}{2\pi} P p \cdot \nabla \omega L^{-1} Q + \mathcal{O}(p^2) \]

\[ = P + AP + PBQ + \mathcal{O}(p^2) \tag{4.11} \]

where, here and below, \( \mathcal{O}(p^2) \) denotes a bound on the norm of operators in \( \mathcal{B} \), and we define

\[ A = -\frac{i}{2\pi} L^{-1} p \cdot \nabla \omega, \quad B = -\frac{i}{2\pi} p \cdot \nabla \omega L^{-1}. \tag{4.12} \]
Hence, for $|p|$ small,
\[ \|P\bar{Q}\| + \|\bar{Q}P\| + \|\bar{P}Q\| + \|QP\| \leq C|p|. \] (4.13)

Now, write
\[ e^{-tD} = \tilde{P}e^{-tD} + \bar{Q}e^{-tD} \] (4.14)
(since $\tilde{P}$ and $\bar{Q}$ project on invariant subspaces of $D$, we have $e^{-tD} = \tilde{P}e^{-tD}$, $e^{-tD} = \bar{Q}e^{-tD}$), and
\[ e^{-tD} = \int_{\gamma} e^{-tz} R(z) \frac{dz}{2\pi i} \] (4.15)
where the curve $\gamma$ goes around the part of the spectrum of $D$ that lies in the complement of a ball of radius $a/4$, centered at the origin. Since $L$ is bounded and $|p|$ small, the length of $\gamma$ is $O(1)$, and we get (4.10). Then, the other claims follow, for $|p|$ small, from (4.13), (4.14) and Proposition 4.1 a.

To get (4.7) for $|p| > p_0$, we note that by Proposition 4.1, $e^{-tD}$ is given by the right hand side of (4.15), where the curve $\gamma$ can be chosen to be a rectangle in $\Re z > b$, with vertical sides of length $\text{const } |p|$, and horizontal ones of length $O(1)$ (since $L$ is bounded).

Hence, on the curve $\gamma$, we have $|e^{-tz}| \leq e^{-bt}$, but the length of the contour of integration is not bounded as $|p| \to \infty$. To control the integral, recall that $D = M + i|p|\rho + K$, and let
\[ R_0(z) = (z - M - i|p|\rho)^{-1}. \]

Then, by the resolvent formula,
\[ R = R_0 + R_0KR_0 + R_0KR_0KR. \] (4.16)

The first term in (4.16), inserted in (4.15), gives the multiplication operator $e^{-t(M+i|p|\rho)}$, which satisfies the bound b) with $c = \inf M$. For the second term, we use $|e^{-tz}| \leq e^{-bt}$ in the integral and write
\[ |\int_{\gamma} dk' dz R_0(z, k)K(k, k')R_0(z, k')| \leq \int dk' |K(k, k')| \int_{\gamma} dz \frac{1}{|z - z_1||z - z_2|} \] (4.17)
where $z_1 = (M + i|p|\rho)(k)$, $z_2 = (M + i|p|\rho)(k')$. Since $|z - z_i| \geq c$ on $\gamma$, we have
\[ \int_{\gamma} dz \frac{1}{|z - z_1||z - z_2|} \leq C, \] (4.18)
uniformly in $k, k'$. Since, moreover, $\int |K(k, k')|dk' \leq C$, uniformly in $k$ (which holds by (2.8)), we get the bound in b) for the second term in (4.16). The third term in (4.16) is similar, since $KR$ is bounded.

**Proof of Theorem 1.** Let us write eq. (3.4) as
\[ w(t) = w_e(t) + N(w, t) \] (4.19)
where \( w(t) \) is the solution of the linear problem:

\[
w(t) = e^{-tD}w(0).
\] (4.20)

We will solve (4.19) in the space \( E \).

The linear term is bounded, using (4.7), and recalling the definition (3.8) of the weight function \( e \), by

\[
\|w\|_E \leq C \sup_{p,t} e(p,t)^{-1}(e(0)(e^{-ct} + e^{-ctp^2})\|w\|_0 \leq C\|w\|_0.
\] (4.21)

Consider then the nonlinear term in eq. (3.4). By an easy extension of Proposition 2.1 a (see (2.36)),

\[
\|n(w(s)(p))\| \leq C(\|w\|_E^2 e(s)^2(p) + \|w\|_E^3 e(s)^3(p)).
\] (4.22)

We need the simple estimate

\[
e(s)^2(p) \leq C(1 + s)^{-2}e(p,s),
\] (4.23)

which follows from

\[
e(s)^2(p) \leq 2 \int_{|p-q| \geq \frac{1}{2}|p|} dq e(p-q,s)e(q,s) \leq Ce(p,s) \int dq e(q,s),
\] (4.24)

(the first inequality holds because either \(|q| \geq \frac{1}{2}|p| \) or \(|p-q| \geq \frac{1}{2}|p| \)). The last integral yields the power of \( 1 + s \) since \( n > \frac{1}{2}d \) in (3.8). The last term in (4.22) is smaller, for \( \|w\|_E \leq 1 \).

Thus, the second term in equation (3.6) and the second term in (3.7) are bounded as by

\[
\|RN(w,t,p)\| \leq C\|w\|^2_E \int_0^t ds\|Re^{-(t-s)D}Q\|(1 + s)^{-\frac{d}{2}}e(s,p)
\] (4.25)

where \( R \) equals \( P \) in (3.6) and \( Q \) in (3.7). Proposition 4.2 implies

\[
\|Re^{-(t-s)D}Q\| \leq C((1 + t - s)^{-\frac{1}{2}}m e^{-\frac{1}{2}(t-s)p^2} + e^{-c(t-s)}),
\] (4.26)

with \( m = 1 \) if \( R = P \) and \( m = 2 \) if \( R = Q \) (using \(|p|^me^{-\frac{1}{2}p^2(t-s)} \leq C(1 + t - s)^{-\frac{1}{2}m} \), for \( t - s \geq 1 \)). Bounding

\[
e^{-\frac{1}{2}p^2(t-s)}e(s,p) \leq Ce(t,p)
\]

and

\[
e^{-\frac{1}{2}(t-s)}e(s,p) \leq Ce(t,p)
\]
we may bound the remaining $s$-integral in (4.25), for $m = 1, 2$, i.e., for $m \leq d$, by $C(t)(1 + t)^{-\frac{1}{2}m}$ where $C(t)$ may be taken to be constant, except for $d = 2$, where $C(t) = C \log(1 + t)$. Hence we end up with
\[
\|RN(w, t)\|_t \leq C(t)(1 + t)^{-\frac{1}{2}m}\|w\|_t^2.
\]  
(4.27)

In particular, we have
\[
\|N(w)\|_\epsilon \leq C\|w\|_t^2,
\] so that $N$ in (4.19) maps a ball of radius $\epsilon$ in $E$ into itself, for $\epsilon$ small enough and is obviously a contraction there. The existence of a solution for eq. (4.19) then follows from (4.21) and the Banach fixed point theorem and, since $m = 1$ if $R = P$, i.e., for (3.6), and $m = 2$ if $R = Q$ i.e., for (3.7), we obtain the bounds
\[
\|T(t) - T_\ell(t)\|_t \leq C(t)(1 + t)^{-\frac{1}{2}}\|w(0)\|_0^2, \tag{4.28}
\]
\[
\|v(t) - v_\ell(t)\|_t \leq C(t)(1 + t)^{-1}\|w(0)\|_0^2, \tag{4.29}
\]
where we wrote
\[
w_\ell = Pw_\ell + Qw_\ell \equiv T_\ell + v_\ell. \tag{4.30}
\]

To conclude the proof of Theorem 1 we need to relate $T_0, v_0$ to $T_\ell, v_\ell$. For this we need to write the leading terms of $w_\ell(t)$ more explicitly. Let us formulate this a little more generally, which will be useful also in the proof of Theorem 2. Let us denote
\[
K(t) = e^{-\lambda t^2}p^\lambda P \tag{4.31}
\]
where $\kappa$ is given in eq. (3.11). We have then the following Lemma, proven at the end:

**Lemma 4.3** Let $|p| \leq p_0$. Then the semigroup $e^{-\lambda tD}$ can be written with respect to the decomposition $E \oplus E_\perp$, as
\[
e^{-\lambda tD} = \begin{pmatrix} K(t) & K(t)B \\ AK(t) & AK(t)B + R(t) \end{pmatrix} + \begin{pmatrix} O(|p|e^{-\lambda t^2} + p^2e^{-\lambda t}) & O(p^2e^{-\lambda t^2} + |p|e^{-\lambda t}) \\ O(p^2e^{-\lambda t^2} + |p|e^{-\lambda t}) & O(|p|^3e^{-\lambda t^2}) \end{pmatrix}
\]
where $O$ is with respect to the operator norm in $B$,\[
R(t) = Q\tilde{Q}e^{-\lambda tD}Q \tag{4.32}
\]
and the operators $A$ and $B$ are defined in equation (4.12).

Returning to the proof of Theorem 1 and recalling the definition (3.12) of $T_0$ and (4.20), (4.30) of $T_\ell$, we get from Lemma 4.3, for $|p| \leq p_0$,
\[
\|T_\ell(t, p) - T_0(t, p)\| \leq C|p|(e^{-\lambda t^2} + e^{-\lambda t})e(p, 0)\|w(0)\|_0. \tag{4.33}
\]
Since \( v_0 \) given in eq. (3.13) equals, see (4.12), \( v_0 = AK(t)T(0) \) for \( |p| \leq p_0 \leq 1 \), we have, from Lemma 4.3,

\[
\|v_\ell(t,p) - v_0(t,p)\| \leq C(p^2 e^{-c t p^2} + e^{-c t})e(p,0)\|w(0)\|_0 \tag{4.34}
\]

where we used the bound, coming from (4.10),

\[
\|R(t)\| \leq C e^{-c t}. \tag{4.35}
\]

For \(|p| \geq p_0\), by (4.7),

\[
\|w_0(t,p)\| + \|w_\ell(t,p)\| \leq C e^{-c t} e(p,0)\|w(0)\|_0.
\]

Using now the simple estimates

\[
e(p,t)^{-1}(e^{-c t p^2/2} + e^{-c t/2}) \leq C e(p,0)^{-1},
\]

for \( t \geq 1 \), \( e(p,t)^{-1} \leq C e(p,0)^{-1} \) for \( t \leq 1 \), and

\[
|p|^m e^{-c t p^2/2} + e^{-c t/2} \leq C(m)t^{-\frac{1}{2} m},
\]

for \( t \geq 1 \), \( m \geq 0 \), we conclude that

\[
\|T_\ell(t) - T_0(t)\|_t \leq C(1 + t)^{-\frac{1}{2}}\|w(0)\|_0, \tag{4.36}
\]

\[
\|v_\ell(t) - v_0(t)\|_t \leq C(1 + t)^{-1}\|w(0)\|_0. \tag{4.37}
\]

The estimates (3.14) and (3.15) follow now from (4.28), (4.29), (4.36) and (4.37). \( \square \)

**Proof of Theorem 2.** The proof goes along the lines of the one of Theorem 1 and we will be brief. We expand

\[
\frac{1}{\epsilon^2} C(W) = -\frac{1}{\epsilon^2} Lw + \frac{1}{\epsilon} m(T,v) + n(T,v),
\]

where \( m(T,v) \) is the term quadratic in \( T \) but linear in \( v \), and \( n(T,v) \) collects the terms that are quadratic and cubic in \( v \). Note that, since \( LT = 0 \), \( \frac{1}{\epsilon^2} Lw = \frac{1}{\epsilon} Lv \) and thus

\[
DC(\omega^{-1} + T)v = -Lv + m(T,v). \tag{4.38}
\]

We write (3.19) as

\[
w(t) = \exp\left(\frac{-t}{\epsilon^2} D_\epsilon\right)w(0) + \frac{1}{\epsilon} \int_0^t ds \exp\left(\frac{-t - s}{\epsilon^2} D_\epsilon\right)(m(s) + \epsilon n(s))ds,
\]

where

\[
D_\epsilon = L + \epsilon \frac{i}{2\pi} p \cdot \nabla \omega, \tag{4.39}
\]
i.e. it equals $\frac{3.3}{\epsilon}$ evaluated at $\epsilon p$.

The decomposition $w = T + \epsilon v$ yields, as in (3.6), (3.7):

$$T(t) = P e^{-tD_{\epsilon}/\epsilon^2} PT(0) + \epsilon P e^{-tD_{\epsilon}/\epsilon^2} Qv(0) + \frac{1}{\epsilon} \int_0^t ds P e^{-(t-s)D_{\epsilon}/\epsilon^2} Q(m(s) + \epsilon n(s))$$

(4.40)

$$v(t) = \frac{1}{\epsilon} Q e^{-tD_{\epsilon}/\epsilon^2} PT(0) + Q e^{-tD_{\epsilon}/\epsilon^2} Qv(0) + \frac{1}{\epsilon^2} \int_0^t ds Q e^{-(t-s)D_{\epsilon}/\epsilon^2} Q(m(s) + \epsilon n(s))$$

(4.41)

Consider first the case $|p| \leq p_0/\epsilon$. We use Lemma 4.3, with $t$ replaced by $t/\epsilon^2$ and $p$ by $\epsilon p$. This leads to

$$T = T^* + T_1$$

(4.42)

$$v = AT^* + \frac{1}{\epsilon^2} \int_0^t ds Q \tilde{Q} e^{-(t-s)D_{\epsilon}/\epsilon^2} \tilde{Q} Qm(s) + v_1$$

(4.43)

where

$$T^* = e^{-tp^2/\epsilon} T(0) + \int_0^t ds e^{-(t-s)p^2/\epsilon} PBm(s)$$

(4.44)

and

$$T_1 = \epsilon P e^{-tD_{\epsilon}/\epsilon^2} Qv(0) + \int_0^t ds P e^{-(t-s)D_{\epsilon}/\epsilon^2} Qn(s) + \int_0^t ds O(\epsilon p^2 e^{-c(t-s)/\epsilon^2} + |p| e^{-c(t-s)/\epsilon^2}) m(s),$$

$$v_1 = Q e^{-tD_{\epsilon}/\epsilon^2} Qv(0) + \frac{1}{\epsilon} \int_0^t ds Q e^{-(t-s)D_{\epsilon}/\epsilon^2} Qn(s) + \epsilon \int_0^t ds O(|p|^3 e^{-c(t-s)/\epsilon^2}) m(s).$$

Then, following the proof of (4.36), (4.37), (4.28), (4.29), we get

$$\|T_1(t, p)\| \leq C \epsilon (1 + t)^{-1} e(p, t)(\|T(0)\| + \|v(0)\|),$$

(4.45)

$$\|v_1(t, p)\| \leq C(t) \epsilon (1 + t)^{-1} e(p, t)(\|T(0)\| + \|v(0)\|),$$

(4.46)

where $C(t) = C \max(t^{-1/2}, \log(1 + t)) < \infty$, for $t > 0$. We need to study the second term on the RHS of eq. (4.43). Write $m(s) = m(t) + (m(s) - m(t))$ and consider the first term

$$\frac{1}{\epsilon^2} \int_0^t ds Q \tilde{Q} e^{-(t-s)D_{\epsilon}/\epsilon^2} \tilde{Q} Qm(t) = \frac{1}{\epsilon^2} \int_0^\infty ds Q \tilde{Q} e^{-sD_{\epsilon}/\epsilon^2} \tilde{Q} Qm(t) + O(e^{-ct/\epsilon^2})$$

$$= Q \tilde{Q} D_{\epsilon}^{-1} \tilde{Q} Qm(t) + O(e^{-ct/\epsilon^2})$$

$$= L^{-1} m(t) + O(\epsilon) + O(e^{-ct/\epsilon^2})$$

(4.47)

where we used $Q \tilde{Q} D_{\epsilon}^{-1} \tilde{Q} Q = Q L^{-1} Q + O(\epsilon)$ and where $O(\cdot)$ denotes a bound on the norm $\sup_{|p| \leq p_0/\epsilon} e(p, t)^{-1} \|\cdot\|$ of the remainders, since $\sup_{|p| \leq p_0/\epsilon} e(p, t)^{-1} \|m(t)\|$ is bounded, uniformly in $\epsilon$, because of the bounds on $T$, $v$, coming from (4.42)-(4.46).
For the second term, we use the equations (4.40) and (4.41), to show that \( \|m(s) - m(t)\|_t \leq C |t - s| \), for \(|t - s|\) small; whence

\[
\frac{1}{\epsilon^2} \int_0^t ds \tilde{Q} \tilde{Q} e^{-(t-s)/\epsilon^2} \tilde{Q}(m(s) - m(t)) = O(\epsilon). \tag{4.48}
\]

Altogether we obtain

\[
v(t) = AT^*(t) + L^{-1}m(t) + v_2 \tag{4.49}
\]

and, for any \( t > 0 \),

\[
\sup_{|p| \leq p_0/\epsilon} e(p, t)^{-1}(\|T_1\| + \|v_2\|) \to 0 \tag{4.50}
\]
as \( \epsilon \to 0 \). For \( |p| > p_0/\epsilon \) we have, from Proposition 4.2,

\[
\|e^{-tD_\epsilon/\epsilon^2}\| \leq e^{-ct/\epsilon^2},
\]

which implies that

\[
\sup_{|p| \geq p_0/\epsilon} e(p, t)^{-1}(\|T\| + \|v\|) \to 0, \tag{4.51}
\]
as \( \epsilon \to 0 \). Hence, in the space \( \mathcal{E}_t \), we have

\[
\lim_{\epsilon \to 0}(T(t), v(t)) = (T^*(t), AT^*(t) + L^{-1}m(t)) \equiv (T^*(t), v^*(t)) \tag{4.52}
\]

Let us finally check that \( T^* \) and \( v^* \) satisfy the equations (3.21) and (3.22). From (4.38) and (4.52), we get

\[
AT^* = -L^{-1}DC(\omega^{-1} + T^*)v^* \tag{4.53}
\]

which is (3.21) if we recall the definition (4.12) of \( A \).

From (4.44) and the definition (4.12) of \( B \), we see that \( T^* \) satisfies

\[
\dot{T}^* = -p^2\kappa T^* - \frac{i}{2\pi} Pp \cdot \nabla \omega L^{-1}m. \tag{4.54}
\]

From (4.52) and the definition (4.12) of \( A \), we have

\[
L^{-1}m = v^* - AT^* = v^* + i(2\pi)^{-1}L^{-1}P \cdot \nabla \omega T^*.
\]

Substituting this into (4.54) and recalling the definition of \( \kappa \) in (3.11), which implies that

\[-p^2\kappa T^* + p^2(2\pi)^{-2}P \cdot \nabla \omega L^{-1} \cdot \nabla \omega T^* = 0, \]

we get equation (3.22). □
Proof of Lemma 4.3. We need to calculate $Re^{-tD}R'$ for $R$ and $R'$ either $P$ or $Q$. We use the representation (4.14) for $e^{-tD}$ and the formula (4.11) for $\tilde{P}$. We need the following expansions

\[
P\tilde{P} = P + PBQ + \mathcal{O}(p^2), \quad \tilde{P}P = P + AP + \mathcal{O}(p^2)
\]
\[
\tilde{P}Q = PBQ + \mathcal{O}(p^2), \quad Q\tilde{P} = AP + \mathcal{O}(p^2)
\]
\[
P\tilde{Q} = P(1 - \tilde{P}) = -PBQ + \mathcal{O}(p^2), \quad \tilde{Q}P = -QAP + \mathcal{O}(p^2)
\]
\[
\tilde{Q}Q = (1 - \tilde{P})Q = Q - PBQ + \mathcal{O}(p^2), \quad Q\tilde{Q} = Q - AP + \mathcal{O}(p^2)
\]

where we used repeatedly $PAP = 0$ ($A$ maps $E$ into $E^\perp$, because $\nabla\omega$ is odd in $k$). Thus we get, using the identities on the left,

\[
P e^{-tD} Q = P \tilde{P} e^{-tD} \tilde{Q} Q + P \tilde{Q} e^{-tD} \tilde{Q} Q
\]
\[
= (P + PBQ)e^{-tD}PBQ - PBQe^{-tD}(Q - PBQ) + \mathcal{O}(p^2e^{-ctp^2}) + \mathcal{O}(p^2e^{-ctp^2})
\]
\[
= e^{-tp^2\kappa} PBQ + \mathcal{O}(p^2e^{-ctp^2}) + \mathcal{O}(|p|e^{-ct})
\]

where we use the bounds of Proposition 4.2, $B = \mathcal{O}(|p|)$, and $\lambda_\alpha = \mu_\alpha p^2 + \mathcal{O}(p^3)$, with $\mu_\alpha$ being the eigenvalues of $\kappa$, see (4.1).

Similarly, we get

\[
Q e^{-tD} P = A e^{-tp^2\kappa} PBQ + Q \tilde{Q} e^{-tD} \tilde{Q} Q + \mathcal{O}(|p|^3e^{-ctp^2})
\]

and

\[
P e^{-tD} P = e^{-tp^2\kappa} P + \mathcal{O}(|p|e^{-ctp^2}) + \mathcal{O}(p^2e^{-ct})
\]
\[
Q e^{-tD} P = A e^{-tp^2\kappa} P + \mathcal{O}(p^2e^{-ctp^2}) + \mathcal{O}(|p|e^{-ct}).
\]

\[\square\]

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References

[1] C. Bardos, S. Ukai, The classical incompressible Navier-Stokes limit of the Boltzmann equation, Math. Models and Methods in Applied Sciences 1, 235-257 (1991).

[2] F. Bonetto, J. L. Lebowitz, and L. Rey-Bellet, Fourier Law: A challenge to Theorists. In: Mathematical Physics 2000, Imp. Coll. Press, London 2000, pp. 128–150.

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[3] J. Bricmont, A. Kupiainen, Towards a derivation of Fourier’s law for coupled anharmonic oscillators, to appear in Commun. Math. Phys.

[4] J. Bricmont, A. Kupiainen, Fourier’s law from closure equations, preprint.

[5] R. Esposito, M. Pulvirenti. From particles to fluids, in *Handbook of Mathematical Fluid Dynamics*, Vol. III, S. Friedlander and D. Serre, eds, Elsevier Science, 2004.

[6] F. Golse, L. Sant-Raymond, The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels, Invent. Math. **155**, 81-161 (2004).

[7] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. Vol IV. Analysis of Operators*, New York, Academic Press, 1978.

[8] H. Spohn, The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics, J. Stat. Phys. **124**, 1041-1104 (2006).

[9] H. Spohn, Collisional invariants for the phonon Boltzmann equation, J. Stat. Phys. **124**, 1131-1135, (2006).

[10] C. Villani, A review of mathematical topics in collisional kinetic theory. In: *Handbook of Mathematical Fluid Dynamics*, Vol. I, S. Friedlander and D. Serre, Eds, Elsevier Science, 2002.