The von Neumann entropy and information rate for integrable quantum Gibbs ensembles, 2

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Abstract

This paper considers the problem of data compression for dependent quantum systems. It is the second in a series under the same title which began with [6] and continues with [12]. As in [6], we are interested in Lempel–Ziv encoding for quantum Gibbs ensembles. Here, we consider the canonical ideal lattice Bose- and Fermi-ensembles. We prove that as in the case of the grand canonical ensemble, the (limiting) von Neumann entropy rate \( h \) can be assessed, via the classical Lempel–Ziv universal coding algorithm, from a single eigenvector \( \psi \) of the density matrix \( \rho \).

1 Introduction

This paper continues paper [6] under the same title and extends results established there for grand canonical ensembles to canonical ensembles of ideal (free) quantum systems, bosonic or fermionic. This will allow us to analyse the question of the Bose-Einstein condensation (in the case of a bosonic ensemble) and extend results to a number of other integrable models (see [12]).

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The reader is referred to [6] for a general introduction into the subject; here we only provide a background formally needed for exposition of the results. We consider ideal quantum systems on a cubic lattice $\mathbb{Z}^d$. The starting point here is a given function $\omega: y \in [0,1]^d \mapsto \omega(y)$ (more precisely, a family of functions $\omega_\mu$ depending on the chemical potential $\mu$), with non-negative values for bosons and real for fermions, describing the energy of normal mode $y$ (in a single-particle momentum space $[0,1]^d$). An example which we will follow closely is where

$$\omega_\mu(y) = \frac{1}{d} \sum_{1 \leq j \leq d} [1 - \cos(2\pi y_j)] - \mu, \quad y = (y_1, \ldots, y_d) \in [0,1]^d, \quad (1.1)$$

and $\mu < 0$ for bosons, $\mu \in \mathbb{R}$ for fermions. Function (1.1) determines the Fourier transform of the operator $-\frac{1}{2}\Delta - \mu I$ acting in $l_2(\mathbb{Z}^d)$ where $\Delta$ stands for the discrete Laplacian on $\mathbb{Z}^d$.

Associated with $\omega_\mu(y)$ are two important integrals:

$$h_{\beta,\mu}^\pm = \int_{[0,1]^d} \left( \mp \log \left( 1 \mp e^{-\beta \omega_\mu(y)} \right) + \frac{\beta \omega_\mu(y)}{\log e \frac{e^{-\beta \omega_\mu(y)}}{1 \mp e^{-\beta \omega_\mu(y)}}} \right) dy \quad (1.2)$$

and

$$m_{\beta,\mu}^\pm = \int_{[0,1]^d} \frac{e^{-\beta \omega_\mu(y)}}{1 \mp e^{-\beta \omega_\mu(y)}} dy \quad (1.3)$$

giving, respectively, the value of the von Neumann entropy rate and the particle density, in the thermodynamic limit. (In the left-hand side we put plus for bosons and minus for fermions; we will follow this convention throughout the paper). Parameter $\beta > 0$ is the inverse temperature.

To simplify matters, we will assume that $d = 1$, though all technicalities can be extended to the case of a general $d$ in a straightforward way.\(^1\) The free Gibbs grand-canonical Bose- or Fermi-ensemble in a finite volume $\Lambda_\ell = \{0, 1, \ldots, \ell-1\}$ (represented by a segment of the integer lattice $\mathbb{Z}$) is described by a density matrix $\rho_\ell^\pm$ acting in the Fock Hilbert space $\mathcal{H}_\ell^\pm$. It is of the form

$$\rho_\ell^\pm = \frac{1}{\Xi_\ell^\pm} \exp(-\beta H_\ell^\pm)$$

\(^1\)The issue of Bose-Einstein condensation arises of course for $d \geq 3$, unless one uses particular boundary conditions. This will be the subject of forthcoming research.
where $\Xi_\ell^\pm = \tr_{H_\ell^\pm} \exp(-\beta H_\ell^\pm)$ and $H_\ell^\pm$ is the Hamiltonian of the free Bose- or Fermi-system in $\Lambda_\ell$ which is the (bosonic or fermionic) second quantisation of the single-particle energy operator $H_{\ell,1}$ in $l_2(\Lambda_\ell) \simeq \mathbb{C}^\ell$. In the above example,

$$H_{\ell,1} = -\frac{1}{2} \Delta_\ell - \mu I_\ell$$

(1.4)

where $\Delta_\ell$ is the lattice Laplacian in $\Lambda_\ell$ and $I_\ell$ the unit matrix in $l_2(\Lambda_\ell)$. If we impose periodic boundary conditions then the eigenvectors $\psi_\ell$ and eigenvalues $\kappa_\ell$ of $H_{\ell,1}$ are naturally labelled by $j = 0, 1, \ldots, \ell - 1$:

$$\varphi_\ell(j) = \frac{1}{\sqrt{\ell}} \exp(2\pi ij/\ell), \quad \kappa_\ell(j) = 1 - \cos(2\pi ij/\ell) - \mu.$$  

(1.5)

In particular, the eigenvalues $\kappa_\ell$ in (1.5) are described as $\omega_\mu(j/\ell)$ where function $\omega$ was defined in (1.1).

We retain the notation $\psi_\ell(j)$ and $\kappa_\ell(j)$ for the eigenvectors and eigenvalues of a general single-particle energy operator $H_{\ell,1}$ in $l_2(\Lambda_\ell)$, and assume the form $\kappa_\ell(j) = \omega_\mu(j/\ell)$ ($j = 0, 1, \ldots, \ell - 1$), where $\omega_\mu(y)$, is a ‘nice’ function figuring in (1.1) and (1.2). See Assumption 1.1 below.

The condition that $\kappa_\ell(j)$ is of the form $\omega_\mu(j/\ell)$ is quite restrictive (although it holds in the example of (1.3) with periodic boundary conditions). We consider it as a first step in studying more general situations.

Returning to the multi-particle Hamiltonian $H_\ell^\pm$, its eigenvectors $\phi_\ell$ and eigenvalues $\lambda_\ell$ are naturally labelled by occupancy number configurations $k = (k_0, \ldots, k_{\ell-1})$ where entry $k_j \in \{0, 1\}$ for fermions and $k_j \in \mathbb{Z}_+$ for bosons:

$$k \in \{0, 1\}^\ell \text{ or } k \in \mathbb{Z}^\ell.$$

Space $\mathcal{H}_\ell^-$ has dimension $2^\ell$ and $\mathcal{H}_\ell^+$ infinite dimension.

More precisely, the eigenvectors and eigenvalues of $H_\ell^\pm$ have the form

$$\phi_\ell^\pm(k) = \left( \prod_{j \in \Lambda_\ell} \psi_\ell(j)^{\otimes k_j} \right)_{S/A}, \quad \lambda_\ell^\pm = \sum_{j \in \Lambda_\ell} k_j \kappa_\ell(j).$$

(1.6)

Here, subscript $S$ means symmetrisation and $A$ antisymmetrisation of the corresponding tensor product $\prod_{j \in \Lambda_\ell} \psi_\ell(j)^{\otimes k_j}$.  

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In probabilistic terms, the density matrix \( \rho^\pm_\ell \) generates a probability distribution where an eigenvector \( \phi_\ell(k) \) ‘occurs’ with probability \( \frac{1}{\Xi^\pm_\ell} \exp (-\beta \lambda_\ell(k)) \), with \( \Xi^\pm_\ell = \sum_{k \in \Lambda_\ell} \exp (-\beta \lambda_\ell(k)) \). It is convenient to assign the above probability to occupancy number configuration \( k \) i.e. consider a probability measure \( P^\pm_\ell \) on space \( \mathbb{Z}^\ell \) (bosons) or \( \{0, 1\}^\ell \) (fermions)

\[
P^\pm_\ell(k) = \frac{1}{\Xi^\pm_\ell} \exp (-\beta \lambda_\ell(k)).
\]

As follows from Equation (1.6), the entries \( K_j, j \in \Lambda_\ell \), of the random configuration \( K = (K_0, \ldots, K_{\ell-1}) \) are independent variables, each with a two-point distribution for fermions and geometric for bosons:

\[
P^\pm_\ell(K_j = k) = (1 \mp e^{-\beta \kappa_\ell(j)})^{\pm1} e^{-k \beta \kappa_\ell(j)}, \quad k \in \mathbb{Z}_+, \quad k = 0, 1.
\] (1.7)

The above independence property is a feature of the grand canonical ensemble for free particles. Paper [6] focused on properties of the Lempel-Ziv encoding for (the sequence of) probability measures \( P_\ell \) as \( \ell \to \infty \) (the thermodynamic limit). The main result of [6] was that if (i) eigenvalues \( \kappa_\ell(j) \) of the single-particle energy operator \( H_\ell,1 \) are of the form \( \omega(j/\ell) \) (which is the case in the example under consideration) and (ii) function \( \omega \) satisfies a certain condition (see Assumption 1 from Section 2 of [6]), the Lempel-Ziv parsing of the random string \( K = (K_0, \ldots, K_{\ell-1}) \) identifies integral (1.2) as the data compression limit.

Replacing the grand canonical with the canonical ensemble means fixing values \( n_\ell, \ell = 1, 2, \ldots \) for the number of particles. Probabilistically, we have to pass to conditional distributions \( P^\pm_{\ell,n_\ell} = P^\pm_\ell \left( \cdot \left| \sum_{0 \leq j \leq n-1} K_j = n_\ell \right. \right) \) which do not have the independence property.

It is convenient to specify the integrands in (1.2) and (1.3) as

\[
g^\pm_{\beta,\mu}(y) = \left( \pm \log (1 \mp e^{-\beta \omega_\mu(y)}) \right) + \frac{\beta \omega_\mu(y)}{\log e} \frac{e^{-\beta \omega_\mu(y)}}{1 \mp e^{-\beta \omega_\mu(y)}}, \quad 0 \leq y \leq 1,
\] (1.8)

and

\[
l^\pm_{\beta,\mu}(y) = \frac{e^{-\beta \omega_\mu(y)}}{1 \mp e^{-\beta \omega_\mu(y)}}, \quad 0 \leq y \leq 1,
\] (1.9)
and interpret the values $l_{\beta,\mu}^{\pm}(j/\ell)$ and $g_{\beta,\mu}^{\pm}(j/\ell)$ as the mean and entropy of the marginal probability distribution (two-point or geometric) of variable $K_j$ in (1.7), $j \in \Lambda_\ell$. [Of course, these values can be expressed in terms of each other.] Here, in agreement with (1.1), we set:

$$\omega_\mu(y) = \omega_0(y) - \mu,$$

although a more general form of dependence can be considered. We also assume that $\omega_0$ is a non-negative function for bosons and real for fermions. Observe that while parameter $\mu$ figures in the definition of the measure $P_\ell^\pm$, it does not in that of the conditioned measure $P_{\ell,n_\ell}^\pm$.

From now on, we state conditions that we need in terms of the mean-value functions $l_{\beta,\mu}^{\pm}(y)$, and the whole exposition is purely probabilistic. We follow the notation and definitions from Sections 1 and 2 of [6].

**Assumption 1.1** Functions $l_{\beta,\mu}^{\pm}$ in (1.9) take

\[ y \in [0, 1] \mapsto l_{\beta,\mu}^{+}(y) \in (0, \infty) \]
\[ y \in [0, 1] \mapsto l_{\beta,\mu}^{-}(y) \in (0, 1) \]

are continuous in $y$ and for all $\beta > 0$ and $\mu < 0$ for bosons and $\mu \in \mathbb{R}$ for fermions. Moreover, for all $\beta > 0$, the function $\mu \mapsto m_{\beta,\mu}^{\pm}$ is monotone increasing with $\mu$ so that, for all $r$ in the range of this function, there exists a unique $\mu$ such that $m_{\beta,\mu}^{\pm} = r$. Here, and below

$$m_{\beta,\mu}^{\pm} = \int_{[0,1]} l_{\beta,\mu}^{\pm}(y)dy. \quad (1.10)$$

Furthermore, functions

\[ y \in [0, 1] \mapsto g_{\beta,\mu}^{\pm}(y) \in (0, \infty) \]

are continuous in $y$ for all $\beta > 0$ and $\mu < 0$ for bosons and $\mu \in \mathbb{R}$ for fermions.

**Assumption 1.2** Sequence of particle numbers $n_\ell$ satisfies:

$$\left| \frac{n_\ell}{\ell} - r \right| = o \left( \frac{1}{\sqrt{\ell}} \right) \quad (1.11)$$

for some $r$ from the range of function $\mu \mapsto m_{\beta,\mu}^{\pm}$.
Assumptions 1.1 and 1.2 are henceforth presumed to hold. The main result of the paper is the following:

**Theorem 1.3** Consider a triangular array of independent random variables $K_j^{(\ell)}$, $0 \leq i \leq \ell - 1$, $\ell = 2, 3, \ldots$, (either geometric or 0,1-valued), where $K_j^{(\ell)}$ has mean $l_{\beta, \mu}^\pm (j/\ell)$ and entropy $g_{\beta, \mu}^\pm (j/\ell)$. From $K_j^{(\ell)}$, define random variables $Y_j^{(\ell)}$, distributed as $K_j^{(\ell)} | \left( \sum_{0 \leq i \leq \ell - 1} K_i^{(\ell)} = n_\ell \right)$.

Then for all $\beta > 0$, in probability, almost surely and in mean the number of words $C(Y^{(\ell)})$ in the Lempel-Ziv parsing of the string $Y^{(\ell)} = \{Y_1^{(\ell)}, \ldots Y_\ell^{(\ell)}\}$ satisfies:

$$\lim_{\ell \to \infty} \frac{\log \ell}{\ell} C(Y^{(\ell)}) = h_{\beta, \mu}^\pm := \int_{[0,1]} g_{\beta, \mu}^\pm (y) dy,$$

(1.12)

where $\mu$ is the unique value for which $m_{\beta, \mu}^\pm = r$. Here, almost surely is understood with respect to the product-measure $\times \mathcal{P}_{\ell, n_\ell}$ (see [7]).

The almost sure form of convergence is the most subtle, so we treat it as principal. The logic behind the proof is as follows. By Shannon’s Noiseless Coding Theorem (see for example Theorem 5.3.1 of [2]), the limit of the Shannon entropy of probability distributions $\mathcal{P}_{\ell, n_\ell}$

$$\lim_{\ell \to \infty} \frac{1}{\ell} H \left( K_0, \ldots, K_{\ell - 1} \left| \sum_{0 \leq i \leq \ell - 1} K_i = n_\ell \right. \right)$$

(if it exists) represents the data compression limit for the sequence of canonical ensemble distributions $\mathcal{P}_{\ell, n_\ell}$. In terms of the Lempel-Ziv encoding, repeating arguments from Chapter II of [11] yields:

**Lemma 1.4** Almost surely with respect to $\times \mathcal{P}_{\ell, n_\ell}$:

$$\liminf_{\ell \to \infty} \frac{\log \ell}{\ell} C(Y^{(\ell)}) \geq \liminf_{\ell \to \infty} \frac{1}{\ell} H \left( K_0, \ldots, K_{\ell - 1} \left| \sum_{0 \leq i \leq \ell - 1} K_i = n_\ell \right. \right).$$

(1.13)

On the other hand, we establish two lemmas:
Lemma 1.5  Almost surely with respect to $\times P^{\pm}_{\ell,n\ell}$:
\[
\limsup_{\ell \to \infty} \frac{\log \ell}{\ell} C(Y^{(\ell)}) \leq h^{\pm}_{\beta,\mu} = \lim_{\ell \to \infty} \frac{1}{\ell} H(K_0, \ldots, K_{\ell-1}), \tag{1.14}
\]

the averaged Shannon entropy of probability measure $P$, where $\mu$ has been specified in Theorem 1.3.

Lemma 1.6  The entropies obey the bound:
\[
H \left( K_0, \ldots, K_{\ell-1} \bigg| \sum_{0 \leq i \leq \ell-1} K_i = n_{\ell} \right) - H(K_0, \ldots, K_{\ell-1}) \geq \delta(\ell, n_{\ell}), \tag{1.15}
\]
where
\[
\lim_{\ell \to \infty} \frac{1}{\ell} \delta(\ell, n_{\ell}) = 0, \tag{1.16}
\]

Together (1.13)–(1.16) imply (1.12). In Sections 2 and 3 we develop an argument that proves Lemma 1.5 and in Section 4 we prove Lemma 1.6. To simplify the notation, we will often omit superscripts $\pm$ and subscripts $\beta, \mu$.

2  Properties of the typical set

Throughout the rest of the paper, Assumptions 1.1 and 1.2 are presumed valid. In this section we concentrate on the fermionic case of 0,1-valued variables – the proofs adapt to the geometric case as in [6]. As in [6], we will use the idea of a typical set – firstly in Proposition 2.1 we shall show that the added restriction of being in the typical set forces extra useful properties to hold. Then in Proposition 3.5 we shall show that we will ‘nearly always’ be in the typical set, so we can exploit these extra properties.

We write $e_a$ for the entropy of the random variable under consideration (geometric or 0,1-valued) with mean $a$. Given $\epsilon \in (0,1)$ and integer $M$ and $j$, define the typical set of realisations $T^{(\ell)}_{j,M}$ by
\[
T^{(\ell)}_{j,M} = \left\{ k : \sum_{i=j}^{j+M-1} (k_i - l(i/\ell)) \leq M e' \right\}, \text{ where } e' = \epsilon \epsilon L/(2L) \tag{2.1}
\]
and \( L = \sup_{0 \leq y \leq 1} l(y) \).

Suppose the \( r \)th word in the Lempel-Ziv parsing begins at \( t(r) \), has length \( s(r) \) and ensemble-entropy defined to be \( E^{(\ell)}(r) = \sum_{u = t(r)}^{t(r) + s(r) - 1} g(u/\ell) \). We set \( N = N(Y^{(\ell)}) = \{ t(r) : E^{(\ell)}(r) \leq \log \ell (1 - \epsilon) \} \) (the set of start-points of low ensemble-entropy words). For any sequence \( k \) we can write:

\[
N \subseteq \left\{ t(r) : E^{(\ell)}(r) \leq \log \ell (1 - \epsilon), k \in T^{(\ell)}_{t(r), s(r)} \right\} 
\bigcup \left\{ t(r) : k \notin T^{(\ell)}_{t(r), s(r)} \right\}.
\]

We bound the size of the first set in Proposition 2.1 to find that it is less than \( K_1 \ell^{1-\epsilon^2} \), since these parsed words are short distinct strings in the typical set.

**Proposition 2.1** Given \( \epsilon \), if \( l(y) \) is uniformly continuous on \([0, 1]\) and bounded above by \( L \) there exists a constant \( K_1(\epsilon) \) such that:

\[
\# \left\{ r : E^{(\ell)}(r) \leq \log \ell (1 - \epsilon), k \in T^{(\ell)}_{t(r), s(r)} \right\} \leq K_1 \ell^{1-\epsilon^2}.
\]

**Proof** We can find a finite number of intervals \( J_i \) in which our variables have their means close together. The key property is that \( l(y) \) is (uniformly) continuous, so given \( \epsilon \), we can calculate \( N = N(\epsilon) \) and \( u_1, \ldots, u_N \) with \( u_1 = 0 \), \( u_N = 1 \) such that for \( i = 1, \ldots, N - 1 \):

\[
\sup_{x, y \in [u_i, u_{i+1}]} |l(x) - l(y)| \leq \epsilon'.
\]

where \( \epsilon' \) is from Equation (2.1). Define \( J_i = \{ j / \ell \in (u_i, u_{i+1}) \} \), \( L_i = \sup_{x \in J_i} l(x) \).

As in [3], we compare \( T^{(\ell)}_{j, M} \) with \( D_{a,M} \), a set which we can count and control more easily. Given \( a > 0 \) and integer \( M \), define:

\[
D_{a,M} = \left\{ x^M_1 = (x_1, \ldots, x_M) \in \{0, 1\}^M : \sum_{i=1}^{M} x_i \leq M \left( a + \frac{\epsilon e a}{a} \right) \right\}.
\]

For \( x^M_1 \in D_{a,M} \), writing \( \mathbb{P}_a \) for product measure for independent 0,1-valued random variables with mean \( a \):

\[
\mathbb{P}_a(x) = \exp \left( M \log(1 - a) + \log a \sum_{i=1}^{M} x_i \right) \geq \exp(-Ma(1 + \epsilon)).
\]
If $k \in T_{j,M}$, where $j \in J_i$:

$$
\sum_{u=j}^{j+M-1} k_u \leq \sum_{u=j}^{j+M-1} l(u/\ell) + M\epsilon' \\
\leq M (L_i + 2\epsilon') \leq M (L_i + \epsilon e_{L_i}/L_i),
$$

so the sub-string $k_{j+M-1}^j = (k_j, \ldots, k_{j+M-1}) \in D_{L_i,M}$. We therefore know that if $k \in T_{l(j),t(j)+s(j)-1}^{(l)}$ and $s(j) \leq (\log \ell)(1 - \epsilon)/e_j^{(l)} = M(j)$ then

$$
P_{L_i}(k_{t(j)}, \ldots, k_{t(j)+l(j)-1}) \geq P_{L_i}(k_{t(j)}, \ldots, k_{t(j)+M(j)-1}) \\
\geq \exp(-M(j)\epsilon e_{L_i}(1 + \epsilon)) = \ell^{1-\epsilon^2}.
$$

Since these finite strings are distinct, the number of strings in $J_i$ such that these two conditions hold is less than $\ell^{1-\epsilon^2}$. Summing over intervals $J_i$, the total number of such strings is less than $\ell^{1-\epsilon^2}N$. \hfill \Box

## 3 Negative association and the typical set

In this section, we develop necessary technical tools to work with the canonical Gibbs distribution, and then finish the proof of Lemma 1.5. The key property that we shall use is that our variables $K$ are negatively associated. That is, under the condition that $\sum_{i=0}^{\ell-1} K_i = n_\ell$, since $K_i$ are non-negative, if one variable is large, then the others are forced to be smaller. Formally:

**Definition 3.1** A collection of real-valued random variables $(U_k)$ is negatively associated (NA) if the covariance

$$
\text{Cov} (f(U_i : i \in A), g(U_j : j \in B)) \leq 0,
$$

for all increasing functions $f$ and $g$, taking arguments over disjoint sets of indices $A$ and $B$.

We require a result that gives a class of variables with conditional distributions that are negatively associated. This comes via the idea of logarithmic concavity:
Definition 3.2 A random variable $V$ taking values in $\mathbb{Z}_+$ with probabilities $p(s) = \mathbb{P}(V = s)$ satisfies logarithmic concavity (LC) if for all $s \geq 1$, $p(s)^2 \geq p(s-1)p(s+1)$.

Notice that the 0, 1-valued and geometric distributions have this property. This is sometimes referred to as Newton’s inequality (see Niculescu [7]). Further, we can use the following fact, going back to Hoggar (see [4]).

Theorem 3.3 If $V$ and $W$ are independent LC random variables, then their sum $V + W$ is also LC.

We also rely on a result of Joag-Dev and Proschan [5], which is itself based on Efron [3]. We reproduce its proof here since the proofs in [3] and [5] only describe the case of random variables with densities.

Proposition 3.4 Let $V_i, i = 1, 2 \ldots$ be independent $\mathbb{Z}_+$-valued LC random variables, with sum $S_\ell = \sum_{i=1}^\ell V_i$. Then for any $\ell$ and $n$, the conditional variables $W_i \sim (V_i | S_\ell = n), i = 1, \ldots \ell$, form an NA family.

Proof First, we establish an assertion similar to the main theorem of [3]: if $V_1, \ldots V_\ell$ are LC random variables then for any increasing function $\phi$ the conditional expectation

$$\mathbb{E}(\phi(V_1, \ldots V_\ell) | S_\ell = s)$$

is an increasing function of $s$. (3.1)

We prove (3.1) by induction on $\ell$.

By log-concavity, $p_2(s+1-x)/p_2(s+1-y) \leq p_2(s-x)/p_2(s-y)$ for integer $0 \leq x \leq y \leq s$. Then for any $1 \leq t \leq x - 1$:

$$\frac{\sum_{x=t}^{s+1} p_1(x)p_2(s+1-x)}{\sum_{x=t}^{s+1} p_1(x)p_2(s+1-x)} = \frac{\sum_{x=t}^{s+1} p_1(x)(p_2(s+1-x)/p_2(s+1-t))}{\sum_{x=t}^{s+1} p_1(x)(p_2(s+1-x)/p_2(s+1-t))} \leq \frac{\sum_{x=t}^{s} p_1(x)(p_2(s-x)/p_2(s-t))}{\sum_{x=t}^{s} p_1(x)(p_2(s-x)/p_2(s-t))} = \frac{\sum_{x=t}^{s} p_1(x)p_2(s-x)}{\sum_{x=t}^{s} p_1(x)p_2(s-x)}.$$

Now, since $a/b \leq c/d$ implies that $a/(a+b) \leq c/(c+d)$, this gives us that for any $t$, $\mathbb{P}(V_1 \leq t| V_1 + V_2 = s+1) \leq \mathbb{P}(V_1 \leq t| V_1 + V_2 = s)$, which implies Equation (3.1) for $\ell = 2$. 

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For $\ell > 2$ and an increasing function $\phi$ define

$$
\Phi(t,u) = \mathbb{E}(\phi(V_1, \ldots V_\ell) | T = t, V_\ell = u), \text{ where } T = \sum_{i=1}^{\ell-1} V_i.
$$

We know that $\Phi$ is increasing in $t$ by the inductive hypothesis for Equation (3.1) for $\ell - 1$, and in $u$ by the monotonicity of $\phi$. Then

$$
\mathbb{E}(\phi(V_1, \ldots V_\ell) | S_\ell = s) = \mathbb{E}(\Phi(T, V_\ell) | T + V_\ell = s),
$$

which is increasing in $s$ by the inductive hypothesis for $\ell = 2$. This concludes the proof of (3.1).

Next, as in [5], we use (3.1), relying on two further results. Firstly, Chebyshev’s rearrangement Lemma: for $F_+ \text{ increasing and } F_- \text{ decreasing},$

$$
\text{Cov}(F_+(X), F_-(X)) \leq 0 \quad (3.2)
$$

(as $\mathbb{E}F_+(X)\mathbb{E}F_-(X) - \mathbb{E}F_+(X)F_-(X) = \sum_{i\neq j} (p(i)p(j)F_+(i)F_-(j) - p(i)p(j)F_+(i)F_-(i)) = \sum_{j<i} p(i)p(j)(F_+(i) - F_+(j))(F_-(j) - F_-(i)) \geq 0,$

where $i, j \in \mathbb{Z}_+).$

Secondly, by expanding with conditioning, for any random variables $U, V, W:$

$$
\text{Cov}(U, V) = \mathbb{E}\text{Cov}(U, V | W) + \text{Cov}(\mathbb{E}(U | W), \mathbb{E}(V | W)).
$$

Now, taking $U = f(V_i, i \in A) | S_\ell, V = g(V_j, j \in B) | S_\ell$ and $W = (S_A, S_B) = (\sum_{i \in A} V_i, \sum_{j \in B} V_j)$ where $A, B \subset \{1, \ldots \ell\}$ are disjoint sets:

$$
\text{Cov}(f, g | S) = \mathbb{E}(\text{Cov}(f, g | S, S_A, S_B)) + \text{Cov}(\mathbb{E}(f | S_A, S_B), \mathbb{E}(g | S_A, S_B) | S).
$$

The first term is zero. As for the second term; as $S_A$ increases, (3.1) implies that $\mathbb{E}(f | S_A)$ increases. At the same time, since $S_A + S_B = S_\ell, S_B$ decreases, so again by (3.1), $\mathbb{E}(g | S_B)$ decreases, so we can apply (3.2). This completes the proof of Proposition 3.4. \(\square\)

In Proposition 3.5 we will work with a general family of $\mathbb{Z}_+-$valued variables, but assume that functions $l$ and $g$ satisfy Assumption 1.1.

**Proposition 3.5** Consider a triangular array of $\mathbb{Z}_+-$valued random variables $Y_j^{(\ell)}, j \in \Lambda_\ell = \{0, \ldots \ell - 1\}$, with $Y_j^{(\ell)}$ forming an NA family for each $\ell$. Assume $Y_j^{(\ell)}$ have mean $l(j/\ell)$ and entropy $g(j/\ell)$ and a uniform bound
on their centred fourth moment: $\mathbb{E}((Y_j^{(\ell)} - l(j/\ell))^4 \leq b$. Then for any $\epsilon$ and any $\eta \in (0,1)$, there exists a constant $K_2 = K_2(\epsilon, \eta, b)$ such that for any $\ell$ large and for $C$ the number of words in the Lempel-Ziv parsing:

$$\mathbb{P}\left( \frac{1}{C} \sum_{r=1}^{C} I(Y \notin T_{t(r),s(r)}^{(\ell)}) \geq \eta \right) \leq \frac{K_2^2}{C^2}. \quad (3.3)$$

Here $T_{i,M} \subseteq \mathbb{Z}_+^\ell$ is defined in \[2.1\].

**Proof** By NA, for any $i \neq j \neq k \neq m$ from $\Lambda_\ell$:

$$0 \geq \mathbb{E}(Y_i^{(\ell)} - l(i/\ell))(Y_j^{(\ell)} - l(j/\ell))^3,$$
$$0 \geq \mathbb{E}(Y_i^{(\ell)} - l(i/\ell))(Y_j^{(\ell)} - l(j/\ell))(Y_k^{(\ell)} - l(k/\ell))^2,$$
$$0 \geq \mathbb{E}(Y_i^{(\ell)} - l(i/\ell))(Y_j^{(\ell)} - l(j/\ell))(Y_k^{(\ell)} - l(k/\ell))(Y_m^{(\ell)} - l(m/\ell)).$$

Hence, for any set $A \subseteq \Lambda_\ell$:

$$\mathbb{E}\left( \sum_{j \in A} (Y_j^{(\ell)} - l(j/\ell))^4 \right) \leq \sum_{j \in A} (Y_j^{(\ell)} - l(j/\ell))^4 + 3 \sum_{i,j \in A, i \neq j} (Y_i^{(\ell)} - l(i/\ell))^2 (Y_j^{(\ell)} - l(j/\ell))^2$$
$$\leq 3b|A|^2.$$

Define $Z_r = I(Y^{(\ell)} \notin T_{t(r),s(r)}^{(\ell)})$. Note that by Chebyshev’s inequality, for any $s$:

$$\mathbb{E}Z_r = \mathbb{P}(Y^{(\ell)} \notin T_{t(r),s(r)}^{(\ell)}) \leq \mathbb{E}\left( \frac{\sum_{j=t(r)}^{t(r)+s-1} (Y_j^{(\ell)} - l(j/\ell)))}{(s\epsilon)^4} \right)^4 \leq \frac{3b}{s^2 \epsilon^4}.$$

Hence, it is sufficient to show that $\sum_{r=1}^{C} s(r)^{-2}/C \rightarrow 0$ almost surely, so that

if $E_C = \sum_{r=1}^{C} \mathbb{E}Z_r$, then $E_C/C \rightarrow 0. \quad (3.4)$

However, the counting argument described in Chapter II of \[11\] shows that this will hold. Specifically, in the finite alphabet case, the number of possible parsed words of length less than $(\log \ell)/100$ is $c \log \ell 1/100$. In the infinite alphabet case, we can truncate as in \[6\].

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Then, by NA, writing $Z_j$ for a random variable with the same marginal distribution as $Z_j$, but independent of the other $Z_k$: 

$$
P \left( \sum_{r=1}^{C} (Z_r - \mathbb{E}Z_r) \geq \nu C \right) \leq P \left( \sum_{r=1}^{C} (\overline{Z}_r - \mathbb{E} \overline{Z}_r) \geq \nu C \right) \leq \exp \left( -2C \nu^2 \right)
$$

by Hoeffding’s inequality (see for example Note 2.6.2 of [9]). Hence, writing $\eta C = \nu C + E_C$

$$
P \left( \sum_{r=1}^{C} Z_r \geq \eta C \right) \leq \exp \left( -2C(\eta - E_C/C)^2 \right) \leq \frac{1}{2C^2(\eta - E_C/C)^4},
$$

so by (3.4), we are done. \qed

We can now complete the proof of Lemma 1.5:

**Proof of Lemma 1.5** We need to bound the $|N|$, by controlling the two sets in the RHS of (2.2). Proposition 2.1 gives that the size of the first set is $O(\ell^{1-\epsilon^2})$.

Proposition 3.4 shows that our variables have the NA property and hence we can apply Proposition 3.5 to the size of the second set in the RHS of (2.2). Specifically, if $C(Y^{(\ell)})$ grows more slowly than linearly in $\ell/\log \ell$, then Theorem 1.3 holds. Otherwise, the upper bound $O(1/C^2)$ provided by of Proposition 3.5 becomes summable in $\ell$. Hence by the Borel-Cantelli Lemma, almost surely the proportion of words in the second set becomes smaller than $\eta$, for any $\eta$. Overall then, the $|N| \log \ell/\ell \to 0$.

Then considering the entropy present in the parsed words, we deduce that:

$$
\sum_{j=1}^{\ell} g(j/\ell) = \sum_{r} E_{(\ell)}(r) \geq \log \ell(1 - \epsilon)(C(Y^{(\ell)}) - |N|).
$$

On rearranging we obtain that

$$
\limsup_{\ell \to \infty} \frac{\log \ell}{\ell} C(Y^{(\ell)}) \leq \frac{1}{\ell} \sum_{j} g(j/\ell) + \epsilon.
$$

\qed
4 The von Neumann entropy of the ensemble

One might expect that the conditioning might have only a small effect on the entropy of the ensemble, and that hence the entropy of the canonical and grand canonical ensembles will be very close to one another. This is confirmed in this section, in which we prove Lemma 1.6.

In the IID case of $0,1$-valued random variables, it is clear that Assumption 1.2 is the right condition, since there the conditioned variables $Y_{\ell j} \sim (K_j^{(\ell)} | \sum_{0\leq i \leq \ell-1} K_i = n_{\ell})$ is equiprobable on the $\binom{\ell}{n_{\ell}}$ possible values, so

$$H \left( K_0, \ldots, K_{\ell-1} \biggm| \sum_{0\leq i \leq \ell-1} K_i = n_{\ell} \right) - H(K_0, \ldots, K_{\ell-1})$$

$$= \log \left( \binom{\ell}{n_{\ell}} \right) - \ell (-p \log p - (1 - p) \log(1 - p))$$

$$\simeq \left( \frac{\ell}{2p(1-p)} \right) \left( \frac{n_{\ell}}{\ell} - p \right)^2 \to 0 \text{ as } \ell \to \infty.$$ (using Stirling’s formula, and expanding in a series in $n_{\ell}/\ell$ close to $p$).

In the non-IID case we exploit a relationship between the mode and mean described by Bottomley [1]. Specifically, for a unimodally distributed random variable $S$:

$$|\text{mode}(S) - \mathbb{E}S| \leq \sqrt{3 \text{Var } S}.$$ Hence if $S$ is the sum of $n$ ‘approximately IID’ variables, we expect that the mean will be of the order of $n$, and within $\sqrt{n}$ of the mode.

As elsewhere in this paper (and in [6]), we will use the fact that along small enough intervals the variables are ‘nearly IID’. That is, by continuity of the mean-value function $l$, we can find intervals such that $\sup_{y, y' \in I_j} |l(y) - l(y')|$ is arbitrarily close to zero.

Lemma 4.1 Given $\epsilon > 0$, we can find a partition of $[0,1]$ by intervals $I_j = [u_j, u_{j+1}]$ such that defining

$$k^*_{j,\ell} = \frac{1}{(u_{j+1} - u_j)\ell} \sum_{i/\ell \in I_j} \frac{l(i/\ell)}{1 - l(i/\ell)},$$
then
\[ l_j \leq \lim_{\ell \to \infty} \frac{k^*_j}{1 + k^*_j} \leq l_j(1 + \epsilon). \]

where \( l_j = \int l(x) dx \).

In the case of 0,1-variables, we require an extra statement, as follows (the case of geometric variables is actually simpler, and discussed at the end of the section).

**Lemma 4.2** For the sum \( S^{(M)} \) of independent 0,1-variables \( X_1, \ldots X_M \), with \( \mathbb{E}X_j = p_j \), define \( k^*_M = \frac{\left( \sum_{i=1}^M p_i / (1 - p_i) \right)}{M} \), then:

\[ \frac{\mathbb{P}(S^{(M)} = n - 1)}{\mathbb{P}(S^{(M)} = n)} \geq \frac{n}{(M - n + 1)k^*_M}. \]

**Proof** Following Niculescu [7], write \( E_i \) for the elementary symmetric functions of degree \( i \), and \( E_i = \frac{E_i^{(M)}}{\binom{M}{i}} \) for the averaged version. By Newton’s inequalities, discussed in [7], the ratio \( \mathcal{E}_n / \mathcal{E}_1 \geq \mathcal{E}_0 / \mathcal{E}_1 \). Hence, we deduce that, writing \( u_j = p_j / (1 - p_j) \):

\[
\begin{align*}
\frac{\mathbb{P}(S^{(M)} = n - 1)}{\mathbb{P}(S^{(M)} = n)} &= \frac{E_{n-1}(u_1, \ldots u_M)}{E_n(u_1, \ldots u_M)} = \frac{\binom{M}{n-1} \mathcal{E}_{n-1}(u_1, \ldots u_M)}{\binom{M}{n} \mathcal{E}_n(u_1, \ldots u_M)} \\
&= \frac{n \mathcal{E}_{n-1}(u_1, \ldots u_M)}{(M - n + 1) \mathcal{E}_n(u_1, \ldots u_M)} \\
&\geq \frac{n \mathcal{E}_0(u_1, \ldots u_M)}{(M - n + 1) \mathcal{E}_1(u_1, \ldots u_M)} = \frac{n}{(M - n + 1)k^*_M}.
\end{align*}
\]

\[ \square \]

We will now use a local lattice Central Limit Theorem, the main result of Petrov [8] (see Theorem 4.4 below). Note that this result extends a very similar one of Prohorov [10], which only holds for uniformly bounded variables, thus ruling out the geometric case.

**Assumption 4.3** Consider integer-valued random variables \( X_1, X_2, \ldots \) satisfying the following conditions:
1. The highest common factor of the integers \( j \) such that
\[
\frac{1}{\log n} \left( \sum_{i=1}^{n} \mathbb{P}(X_i = 0) \mathbb{P}(X_i = j) \right) \to \infty
\]
is 1, where for each \( X_i \), by shifting, \( \mathbb{P}(X_i = j) \) has its largest value at \( j = 0 \).

2. As \( n \to \infty \), \( \sigma_n^2 \to \infty \), and \( \sup_m L_m \sigma_m < \infty \) where \( \sigma_n^2 = \sum_{i=1}^{n} \text{Var} X_i \) and
\[
L_n = \frac{1}{\sigma_n^3} \sum_{i=1}^{n} \mathbb{E} |X_i - \mathbb{E}X_i|^3.
\] (4.1)

**Theorem 4.4 (Petrov)** Let \( X_1, X_2, \ldots \) be independent integer-valued random variables and write \( S^{(n)} = \sum_{i=1}^{n} X_i \), \( a_n = \mathbb{E} S^{(n)} \) and \( \sigma_n^2 = \text{Var} S^{(n)} \). Then under Assumption 4.3 there exists a constant \( C \) such that for all \( q \in \mathbb{Z} \):
\[
\left| \sigma_n \mathbb{P}(S_n = q) - \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(q - a_n)^2}{2\sigma_n^2} \right) \right| \leq CL_n,
\]
where \( L_n \) is defined in (4.1) above.

To apply Theorem 4.4 we need to check that Assumption 4.3.2 holds in both the fermionic and bosonic case. For a 0,1-variable \( X \) with \( \mathbb{E}X = p \), \( \mathbb{E}|X - p|^3 = p(1 - p)(2p^2 - 2p + 1) \leq 2\text{Var} X \). Hence in the fermion case, variables \( K_0, \ldots K_{\ell-1} \) satisfy:
\[
\sum_{0 \leq i \leq \ell-1} \mathbb{E}|K_i - \mathbb{E}K_i|^3 \leq c \sum_{0 \leq i \leq \ell-1} \text{Var} K_i
\] (4.2)
with \( c = 2 \) and Assumption 4.3.2 holds.

For \( X \) geometric, we can distinguish two cases. (a) Parameter \( p < 1/2 \), so that \( \mathbb{E}X = p/(1 - p) < 1 \) and \( \mathbb{E}|X - \mathbb{E}X|^3 = \mathbb{E}(X - \mathbb{E}X)^3 + 2\mathbb{E}X^3 \mathbb{P}(X = 0) = p(1 + p + 2p^2 - 2p^3)/(1 - p)^3 \leq 7\text{Var} X \). (b) Value \( p \geq 1/2 \), so \( \mathbb{E}X \geq 1 \) and \( \mathbb{E}|X - \mathbb{E}X|^3 \leq \mathbb{E}X^3 + \mathbb{E}X^3 = (p + 4p^2 + 2p^3)(1 - p)^3 \leq 7/(1 - p)^3 \leq 28\mathbb{E}X\text{Var} X \). In either case the bound (4.2) is fulfilled with \( c = 28L = 28 \max_i \mathbb{E}K_i \) and again Assumption 4.3.2 holds.

**Proof of Lemma 1.6** Again, let us first consider the case of 0,1-valued variables. We again break the interval \([0,1] \) down into a collection of smaller
ones as described in Lemma 4.1, such that the variables are approximately IID on each interval. Refer to the sum over interval $I_j$ as $U_j$, and set $p_j = l(j/\ell)$. Further, for technical reasons, we insist that the point $1/2$ occurs at the boundary of an interval (so no interval contains points where $p_j > 1/2$ and $p_j < 1/2$).

Writing $S^{(\ell)}$ for $\sum_{0 \leq i \leq \ell-1} K_i$, we formally expand $H(K_0, \ldots, K_{\ell-1}|S^{(\ell)} = n_{\ell}) - H(K_0, \ldots, K_{\ell-1})$ as

$$
\sum_{k_0 + \ldots + k_{\ell-1} = n_{\ell}} \frac{\mathbb{P}(K_0 = k_0) \ldots \mathbb{P}(K_{\ell-1} = k_{\ell-1})}{\mathbb{P}(S^{(\ell)} = n_{\ell})} \left( \log \mathbb{P}(S^{(\ell)} = n_{\ell}) - \sum_{0 \leq i \leq \ell-1} \log \mathbb{P}(K_i = k_i) \right) 
+ \sum_{0 \leq i \leq \ell-1} \sum_{k_i} \mathbb{P}(K_i = k_i) \log \mathbb{P}(K_i = k_i)
= \sum_{0 \leq i \leq \ell-1} \sum_{k_i} \left( \mathbb{P}(K_i = k_i) - \frac{\mathbb{P}(K_i = k_i) \mathbb{P}(S_i^{(\ell)} = n_{\ell} - k_i)}{\mathbb{P}(S^{(\ell)} = n_{\ell})} \right) \log \mathbb{P}(K_i = k_i) \quad (4.3)
+ \log \mathbb{P}(S^{(\ell)} = n_{\ell}), \quad (4.4)
$$

where $S_i^{(\ell)} = \sum_{j \neq i} K_j = S^{(\ell)} - K_i$.

We deal with each of (4.3) and (4.4) separately. The second term, (4.4), is bounded directly using Theorem 4.4, since $(n_{\ell} - r)^2 / (\sum_{0 \leq i \leq \ell-1} \text{Var} K_i) \to 0$ as $\ell \to \infty$ (see (1.9)).

Next, we split the sum over $i$ up into subintervals: given a value $i$, where $i/\ell \in I_j$, we define

$$
U_{j,i}^{(\ell)} = \sum_{r:r/\ell \in I_j} K_r, \quad \overline{U}_{j,i}^{(\ell)} = \sum_{r:r/\ell \in I_j, r \neq i} K_r = U_{j,i}^{(\ell)} - K_i, \quad S_j^{(\ell)} = \sum_{r:r/\ell \notin I_j} K_r = S^{(\ell)} - U_{j,i}^{(\ell)}.
$$

Then the first term, Equation (4.3) can be rearranged to give the sum over $i$ of:

$$
\sum_t \frac{\mathbb{P}(U_{j,i}^{(\ell)} = t) \mathbb{P}(S_j^{(\ell)} = n - t)}{\mathbb{P}(S^{(\ell)} = n_{\ell})} \sum_k \left( 1 - \frac{\mathbb{P}(\overline{U}_{j,i}^{(\ell)} = t - k)}{\mathbb{P}(U_{j,i}^{(\ell)} = t)} \right) \times \mathbb{P}(K_i = k) \log \mathbb{P}(K_i = k). \quad (4.5)
$$

Now, for any value of $t$, this inner sum can be rearranged to give:

$$
\log(1/p_i - 1)p_i(1 - p_i) \left( \frac{\mathbb{P}(U_{j,i}^{(\ell)} = t - 1)/\mathbb{P}(U_{j,i}^{(\ell)} = t) - 1}{1 - p_i + p_i \mathbb{P}(U_{j,i}^{(\ell)} = t - 1)/\mathbb{P}(U_{j,i}^{(\ell)} = t)} \right). \quad (4.6)
$$
Notice that if $t$ is close to the mode of $U_{j,i}^{(\ell)}$, then this is close to zero. Hence, if the mean and mode are ‘close together’ (as \[1\] ensures), we can produce sensible bounds, using Lemma 4.2.

If $p_i \leq 1/2$, log$(1/p_i - 1)$ is positive, and so $0 \leq (1 - p_i)p_i\log(1/p_i - 1) \leq 1$. Writing $N$ for the number of summands in $U_{j,i}^{(\ell)}$, Lemma 4.2 implies that $\mathbb{P}(U_{j,i}^{(\ell)} = t - 1)/\mathbb{P}(U_{j,i}^{(\ell)} = t) \geq t/(k^*(N - t))$, so since $f(v) = (v - 1)/(1 - p + pv)$ is an increasing function in $v$, we can deduce that (4.6) is at least:

$$\frac{t(k^* + 1) - k^* N}{k^* N(1 - p) - t(k(1 - p) - p)} \geq \frac{(1 + k^*)^2}{k^* N} \left( t - \frac{k^* N}{k^* + 1} \right) \geq \frac{4}{N} \left( t - \frac{k^* N}{k^* + 1} \right),$$

where the first inequality follows by concavity in $t$.

Thus we control (4.3) through a bound on $\mathbb{E} \left( U_{j,i}^{(\ell)} - \mathbb{E} U_{j,i}^{(\ell)} \middle| S^{(\ell)} = n_{\ell} \right)$ provided by Theorem 4.4. That is, we know that $U_{j,i}^{(\ell)}$, $S_j^{(\ell)}$ and $S^{(\ell)}$ are all close to normal, so the expectations will be close to the values they take in the normal case. That is, writing $Z_1$, $Z_2$ and $Z_3$ for normal variables, with densities $\phi_1$, $\phi_2$ and $\phi_3$ of mean and variance matching $U_{j,i}^{(\ell)}$, $S_j^{(\ell)}$ and $S^{(\ell)}$ respectively:

$$\begin{align*}
&\left| \sum_y \frac{y \mathbb{P}(U_{j,i}^{(\ell)} = y) \mathbb{P}(S_j = n_{\ell} - y)}{\mathbb{P}(S^{(\ell)} = n_{\ell})} - \frac{y \phi_1(y) \phi_2(n_{\ell} - y)}{\phi_3(n_{\ell})} \right| \\
&\leq \sum_y \frac{|y \mathbb{P}(U_{j,i}^{(\ell)} = y) - \mathbb{P}(S_j = n_{\ell} - y)|}{\mathbb{P}(S_j = n_{\ell})} \phi_3(n_{\ell}) + \frac{|y \phi_2(n_{\ell} - y)|}{\mathbb{P}(S^{(\ell)} = n_{\ell})} \mathbb{P}(S^{(\ell)} = n_{\ell}) - \phi_3(n_{\ell}) \\
&+ \sum_y \frac{|y \phi_1(y)|}{\mathbb{P}(S^{(\ell)} = n_{\ell})} \phi_2(n_{\ell} - y) \mathbb{P}(S_j = n_{\ell}) - \phi_3(n_{\ell}) \\
&\leq \frac{\epsilon}{\mathbb{P}(S^{(\ell)} = n_{\ell})} \left( \sqrt{\text{Var} U_{j,i}^{(\ell)}} + |n_{\ell} - \mathbb{E} S_j^{(\ell)}| + \sqrt{\text{Var} S_j^{(\ell)}} + \sqrt{\text{Var} U_{j,i}^{(\ell)}} \right) \\
&= O \left( \frac{1}{\sqrt{\ell}} \right),
\end{align*}$$

where $\epsilon$ is the largest of the bounds given by Theorem 4.4 and hence is $O(1/\ell)$.
The result follows, using the fact that if $Z_i$ are independent $N(\mu_i, \sigma_i^2)$ random variables (for $i = 1, 2$) then $Z_1 \mid Z_1 + Z_2 = \mu_1 + \mu_2 + r$ has a $N(\mu_1 + r\sigma_1^2/(\sigma_1^2 + \sigma_2^2), \sigma_1^2\sigma_2^2/(\sigma_1^2 + \sigma_2^2))$ distribution.

For $p_i > 1/2$, the same argument works, only replacing the lower bound on $\mathbb{P}(n-1)/\mathbb{P}(n)$ from Lemma 4.2 with the corresponding upper bound that $\mathbb{P}(n-1)/\mathbb{P}(n) \leq l^*/(N-n+1)$, where $l^* = (\sum(1-p_i)/p_i)/N$. The argument goes through in a similar way.

We can use a similar idea in the case of geometric distributions. Again we can bound the term (4.4) using Theorem 4.4. Then the first term, Equation (4.3) can be rearranged to give the sum over $i$ of:

$$\sum_t \frac{\mathbb{P}(U_{j,i}^{(t)} = t)\mathbb{P}(S_j = n - t)}{\mathbb{P}(S^{(t)} = n)} \sum_x x \left( \frac{\mathbb{P}(X_i = x)\mathbb{P}(T_j' = t - x)}{\mathbb{P}(U_{j,i}^{(t)} = t)} - \frac{\mathbb{P}(X_i = x)\mathbb{P}(T_j' = t - x)}{\mathbb{P}(U_{j,i}^{(t)} = t)} \right),$$

which is the same term previously bounded, and the same arguments apply.

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