Self-similarities in the frequency-amplitude space of a loss-modulated \CO_2 laser

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We show the standard two-level continuous-time model of loss-modulated \CO_2 lasers to display the same regular network of self-similar stability islands known so far to be typically present only in discrete-time models based on mappings. For class B laser models our results suggest that, more than just convenient surrogates, discrete mappings in fact could be isomorphic to continuous flows.

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Lasers with modulated parameters are arguably among the simplest and most accessible laser systems of interest for applications in science and engineering and for theoretical investigations. The intrinsic interest in practical applications and in the nonlinear dynamics of modulated lasers has spurred a wide range of studies after the remarkably influential work of Arecchi et al. reporting the first measurement of subharmonic bifurcations, multistability, and chaotic behavior in a Q-switched \CO_2 laser. Since then, \CO_2 lasers have been fruitfully exploited in many situations. Recent applications include studies of stochastic bifurcations in modulated \CO_2 laser \cite{5}, multistability induced by periodic modulations \cite{4}. Rich nonlinear response of \CO_2 lasers with current modulation and cavity detuning \cite{14}, and self-focusing effects in nematic liquid crystals \cite{5}.

In the last 20 years the \CO_2 laser was extensively studied theoretically, numerically and experimentally \cite{6, 7, 8, 9}, but focusing mainly on the characterization of dynamical behaviors in phase-space for specific parameters. While a detailed description of phase-space dynamics is already available in the literature \cite{6, 7, 8, 9}, no equivalent description exists for the parameter space, except for works by Goswami \cite{13} who investigated analytically the first few period-doubling bifurcations for the Toda model of the \CO_2 laser \cite{14}.

The present Letter reports an investigation of the parameter space of a paradigmatic model of class B lasers, the \CO_2 laser. More specifically, we study a popular two-level model of a \CO_2 laser with modulated losses, focusing on the global stability of the laser with respect to the modulation, not the intensity. The remarkable discovery reported here is that stability islands of the continuous-time laser model emerge organized in a very regular network of self-similar structures called shrimps \cite{15}, illustrated in Figs. 1 and 2, and previously known to exist only in the parameter space of discrete-time dynamical systems \cite{13, 16, 17, 18}. Thus far, all attempts to uncover shrimps in flows, i.e. in continuous-time dynamical systems modeled with sets of differential equations, have failed to produce them \cite{14}.

The single-mode dynamics of the loss-modulated \CO_2 laser involves two coupled degrees of freedom and a time-dependent parameter which we write, as usual \cite{6, 7, 8},

\begin{align}
\frac{du}{dt} &= \frac{1}{\tau} (z - k)u, \quad (1a) \\
\frac{dz}{dt} &= (z_0 - z)\gamma - uz. \quad (1b)
\end{align}

Here, \(u\) is proportional to the radiation density, \(z\) and \(z_0\) are the gain and unsaturated gain in the medium, respectively, \(\tau\) denotes the transit time of the light in the laser cavity, \(\gamma\) is the gain decay rate, and \(k \equiv k(t)\) represents the total cavity losses. The losses are modulated periodically as follows,

\[k(t) = k_0 (1 + a \cos 2\pi ft),\]

where \(k_0\) is the constant part of the losses and \(a\) and \(f\), the amplitude and frequency of the modulation, are the main bifurcation parameters. The remaining parameters are fixed at \(\tau = 3.5 \times 10^{-9}\) s, \(\gamma = 1.978 \times 10^5\) s\(^{-1}\), \(z_0 = 0.175\) and \(k_0 = 0.1731\). These are realistic values, used in recent theoretical and experimental investigations \cite{6}. Integrations were done using the standard fourth-order Runge-Kutta scheme with fixed time-step, equal to \(h = 4 \times 10^{-8}\). Phase diagrams in \(a \times f\) space are obtained by computing Lyapunov exponents for a mesh of \(600 \times 600\) equally spaced parameters. Starting from an arbitrary initial condition, we “followed the attractor” that is, after increasing parameters we initiated iterations using the last obtained values as the new initial conditions. The largest exponents were codified into a bitmap with a continuous color scale ranging from the maximum positive (green) to maximum negative (blue) exponents. Zero exponents were codified in black. One of the exponents is always zero since it is simply related to the time evolution. Three illustrative bitmaps for the laser model are shown in Fig. 1.

Figure 1 displays a global view of the parameter space. The most prominent features, the broad curved structures in Fig. 1a, show that the parameter space of the laser model above, Eqs. (1a-b), agrees qualitatively quite well with the description of Goswami \cite{13} for the Toda model of the \CO_2 laser. For the parameters chosen, the
FIG. 1: (Color online) Structure of the frequency-amplitude phase-diagram of the laser showing a regular network of stability islands (a) Global view, (b) Zoom of the box in (a). Numbers indicate the main period of each stability island; (c) Magnification of the period-11 stability island (indicated in (b) by the encircled 11), displaying the generic shape of all stability islands [16]. Color intensities are proportional to Lyapunov exponents: blue for negative exponents (periodic oscillations), black for zero and green for positive exponents (chaotic oscillations). Frequencies are in kHz.

FIG. 2: (Color online) Structure of the parameter space of the Hénon map. (a) The organization of shrimps here coincides with that of the laser (compare with Fig. 1b). The fine structure observed around period-8 here reproduces the laser period-8 sequence along the curve passing by the encircled numbers in Fig. 1b; (b) Magnification of the box in (a); (c) Magnification of the box in (b). Numbers indicate the main period of each stability island. Points mark windows, not doubly superstable crossings. Download ALL figures high-res PDF, http://www.if.ufrgs.br/∼jgallas/jg_papers.html

The relaxation frequency of our laser model is 50 kHz. From Fig. 1a it is possible to see that there is a minimum amplitude threshold \( a \) beyond which subharmonic bifurcations start to occur, corresponding to about 100 kHz, the harmonic of the relaxation frequency. In addition, for certain parameter values new stability domains are created by saddle-node bifurcations, each of them undergoing then its own cascade of period doublings. So, in certain parameter ranges more than one stable mode coexist, giving rise to generalized multistability. This feature may be recognized in Fig. 1 from the apparent sudden discontinuities in the coloring, due only to the impossibility of plotting two distinct colors in the same place.

The most interesting feature in Fig. 1a is the remarkably regular structuring which appears in the region containing the box, shown magnified in Fig. 1b. This figure shows that embedded in the wide domain of parameters leading to chaotic laser oscillations there is a regular structuring of self-similar parameter windows, shrimps, containing cascades of stable periodic oscillations, the main period of a few of the larger shrimps indicated by the number near to them. The period-11 shrimp seen in Fig. 1b is shown magnified in Fig. 1c. Starting from the main period-11 body, it displays two distinct doubling cascades as well as an infinite number of additional period-doubling cascades, as thoroughly described for discrete-time systems in Refs. [15, 16].

The computation of bitmaps for the laser model is very computer demanding. To alleviate this problem and to manifest the isomorphism between flows and maps, we display the generic fine and hyperfine structure of stability islands typically present in multidimensional systems using the two-parameter Hénon map as a paradigm:

\[
x_{t+1} = a - x_t^2 + b y_t, \quad y_{t+1} = x_t. \tag{3}
\]
The nonlinearity parameter $a$ (forcing) represents the bifurcation or control parameter. The damping parameter $b$ varies between $-1 \leq b \leq 1$, with $b = 1$ representing the conservative limit and $b = 0$ the limit of strong damping. While for $b = 0$ there exists just a single chaotic attractor over a wide range of the parameter $a$, several periodic and chaotic attractors coexist when $b \neq 0$.

Since all these attractors evolve in the same generic manner, we consider here the strongly dissipative limit, focusing on slightly negative values of $b$. In this domain, Pando et al. [21] found that a much more sophisticated four-level model of the CO$_2$ laser with modulated losses behaves qualitatively similar to the Hénon map. The CO$_2$ laser dynamics, as of any class B laser, is characterized by a time-delay between the intensity and the modulation, a fact that nicely matches the delayed character of the Hénon map when written as a one-dimensional recurrence relation.

Figure 5 shows how regularly shrimps organize themselves along very specific directions in parameter space. The ordering along the main diagonal of Fig. 5, is the same found for the laser, in Fig. 4a, along the direction containing the encircled periods. Analogously, the secondary diagonal in Fig. 4b displays the same ordering that the parabolic arc in the middle of Fig. 4c.

The laser-Hénon agreement in parameter space perme-
ates also to the phase space as corroborated by Fig. 3 comparing return maps for the laser (on the left column) with those for the Hénon map (right column). The laser return maps were constructed using the sequence \( u_{\ell}(t) \), \( \ell = 1, 2, 3, \ldots \) of relative maxima of \( u(t) \). As it is easy to see, both sets of return maps agree remarkably well [21].

How easy is to detect experimentally the regular structure reported here? Figure 4 illustrates a representative laser signal in two scales. Although waveforms and underlying periodicities are easy to recognize in logarithmic scale, their experimental detection may become troublesome, particularly as the period increases. For instance, contemplating the six period-16 stability islands in Fig. 1, two of them arising from period-8 via saddle-node bifurcations, one may ask what sort of differences should be expected in their measurement. The answer is depicted in Fig. 4. In a real experimental setup, the difficulties to surmount are mainly to access narrow high-period windows, and to have a wide enough detection range. Modulated losses are usually obtained with an intracavity polarizer and an electro-optical modulator. Recent progress in low-voltage electro-optical modulators have considerably improved their performances [22]. To detect large and small peaks simultaneously one can use a logarithmic preamplifier [23]. Thus, detection and discrimination of the laser signals in Fig. 4 is experimentally feasible with existing technology.

To uncover isomorphisms between continuous-time (flows) and discrete-time (maps) in dynamical systems is an important event from a physical and dynamical point of view and immediately raises interesting questions. For instance, is an isomorphism to be expected also for more refined laser models such that, e.g., of Ciofini et al. [24], involving two rate-equations derived for a single-mode CO\(_2\) laser using center manifold theory for a four-levels model? Their model is interesting because, as they say, it agrees well with experiments. Do networks of shrimps exist for the laser model proposed very recently by Meucci et al. [21]. Do they exist for other laser models too? Is it possible to find them in the parameter space of the laser parameters, not modulation? In autonomous systems? Which laser system presents more restricted amplitude variations, making life easier for experimentalists? What sort of mechanism generates shrimps in differential equations? How are basins of attraction entangled in multistable domains where self-similar structures are abundant [21]? We will report on this elsewhere.

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