On unified-entropy characterization of quantum channels

A E Rastegin
Department of Theoretical Physics, Irkutsk State University, Gagarin Boulevard 20, Irkutsk 664003, Russia
E-mail: rast@api.isu.ru

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Abstract
We consider properties of quantum channels with the use of unified entropies. Extremal unravelings of quantum channel with respect to these entropies are examined. The concept of map entropy is extended in terms of the unified entropies. The map \((q,s)\)-entropy is naturally defined as the unified \((q,s)\)-entropy of a rescaled dynamical matrix of given quantum channel. Inequalities of Fannes type are obtained for introduced entropies in terms of both the trace and Frobenius norms of difference between corresponding dynamical matrices. Additivity properties of introduced map entropies are discussed. The known inequality of Lindblad with the entropy exchange is generalized to many of the unified entropies. For the tensor product of a pair of quantum channels, we derive a two-sided estimate on the output entropy of a maximally entangled input state.

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1. Introduction

The concept of entropy is one of most important notions in both statistical physics and information theory. New applications of this concept are connected with those advantages that can be reached by using quantum resources to process and transmit the information [1]. In addition to the Shannon and von Neumann entropies, which both are fundamental, other entropic measures were found to be useful. Among them, the Rényi and Tsallis \(q\)-entropic functionals are well known [2]. A general treatment of these and some other entropies in terms of unified \((q,s)\)-entropies was given in [3]. For many or even all values of parameters \(q\) and \(s\), the quantum unified \((q,s)\)-entropy enjoy features similar to properties of the standard von Neumann entropy [4]. The unified entropies were applied for treatment of quantum entanglement and monogamy [5].

Entropic measures have widely been adopted in studying features of quantum channels [1, 2]. Different characteristics are properly ensured by entropies of different kinds and forms.
The entropy exchange [6] and the map entropy [7] have been put to describe entanglement transmission and decoherence induced by a quantum channel. Additivity properties of map entropies with respect to the tensor product of two channels are of interest [8]. For the minimum output entropy, this question is considered to be even more relevant [9, 10]. Together with the von Neumann entropy, the quantum Rényi entropy has been applied for these purposes. In this work, we treat characteristics of quantum channels with use of the unified \((q, s)\)-entropies.

This paper is organized as follows. The main definitions and the notation are introduced in section 2. In section 3, we examine those channel unravelings that are extremal with respect to the unified entropies. Continuity estimates of Fannes type are derived for the map \((q, s)\)-entropies in section 4. A distance between rescaled dynamical matrices is quantified by means of both the trace and Frobenius norms. Section 5 is devoted to properties of the map \((q, s)\)-entropies with respect to the tensor product of two quantum channels. Using an extension of Lindblad’s inequality, we derive a two-sided estimate on the output \((q, s)\)-entropy for the tensor product of two channels acting on a maximally entangled input state. In section 6, we conclude the paper.

2. Definitions and notation

In this section, we introduce terms and conventions that will be used through the text. First, the background material is presented. Second, the notions of \((q, s)\)-entropy exchange and map \((q, s)\)-entropy are defined.

2.1. Notation

Let \(\mathcal{H}\) be the \(d\)-dimensional Hilbert space. We denote the space of linear operators on \(\mathcal{H}\) by \(\mathcal{L}(\mathcal{H})\) and the set of positive semidefinite operators on \(\mathcal{H}\) by \(\mathcal{L}_+(\mathcal{H})\). The support of an operator is defined as the vector space orthogonal to its kernel. A density operator \(\rho \in \mathcal{L}_+(\mathcal{H})\) has unit trace, i.e. \(\text{tr}(\rho) = 1\). For \(X, Y \in \mathcal{L}(\mathcal{H})\), we define the Hilbert–Schmidt inner product by [11]

\[
\langle X, Y \rangle_{\text{hs}} := \text{tr}(X^\dagger Y).
\]

(1)

The Schatten norms, which form an important class of unitarily invariant norms, are defined in terms of singular values. Recall that the singular values \(\varsigma_j(X)\) of operator \(X\) are put as the eigenvalues of \(|X| = \sqrt{X^\dagger X}\). For \(q \geq 1\), the Schatten-\(q\)-norm of \(X \in \mathcal{L}(\mathcal{H})\) is then defined by [11, 12]

\[
\|X\|_q = \left(\sum_{j=1}^d \varsigma_j(X)^q\right)^{1/q}.
\]

(2)

This definition is closely related to the \(q\)-mean \(M_q(x) = \left(\frac{1}{n} \sum_{j=1}^n x_j^q\right)^{1/q}\). The properties of such means are extensively considered in [13]. The family (2) includes the trace norm \(\|X\|_1\) for \(q = 1\), the Frobenius norm \(\|X\|_2\) for \(q = 2\), and the spectral norm \(\|X\|_{\infty} = \max\{\varsigma_j(X) : 1 \leq j \leq d\}\) for \(q = \infty\). The trace norm distance is one of most frequently used distances. Its partitioned varieties are also defined on the base of the Fan norms [14, 15]. In some respects, however, we will prefer the Frobenius norm distance. Since the Frobenius norm is induced by the Hilbert–Schmidt inner product, i.e. \(\|X\|_2^2 = \langle X, X \rangle_{\text{hs}}\), this distance is often named the Hilbert–Schmidt distance. Recently, it has been found to be fruitful in studying two-dimensional projections of the set of mixed states [16].

The formalism of quantum operations provides a unified treatment of possible state change in quantum theory [1, 11]. Consider a linear map \(\Phi\) that takes elements of \(\mathcal{L}(\mathcal{H})\) to elements
of $\mathcal{L}(\mathcal{H})$ and also satisfies the condition of complete positivity. Let $\text{id}^\mathcal{H}$ be the identity map on $\mathcal{L}(\mathcal{H}_n)$, where the space $\mathcal{H}_n$ is assigned to an ancillary reference system. The complete positivity implies that $\Phi \otimes \text{id}^\mathcal{H}$ transforms a positive operator into a positive operator again for each dimension of the extended space. Such linear maps are typically called ‘quantum operations’ [1] or ‘super-operators’ [11]. Each completely positive map can be written in the operator-sum representation. Namely, for any $X \in \mathcal{L}(\mathcal{H})$, we have

$$\Phi(X) = \sum_j A_j X A_j^\dagger,$$

where the Kraus operators $A_j$ map the input space $\mathcal{H}$ to the output space $\mathcal{H}'$ [1, 11]. In general, the normalization condition implies that

$$\sum_j A_j^\dagger A_j \leq 1,$$

where $1$ is the identity operator on $\mathcal{H}$. In most applications, the input and output spaces are the same. When the physical process is deterministic, the equality in (4) holds and $\text{tr}(\Phi(\rho)) = 1$.

In this case, the map $\Phi$ is usually referred to as ‘quantum channel’ [1]. Probabilistic operations, such as probabilistic cloning [17] or quantum state separation [18], are of interest. But, in this paper, we will deal only with deterministic quantum operations.

As an entropic measure, we will use the unified $(q,s)$-entropy introduced in [3] and further studied in [4]. The $(q,s)$-entropies form a family of two-parameter entropic functionals continuous with respect to both the parameters [3]. Many generalized entropies including the Rényi and Tsallis ones are contained in this family. The quantum unified $(q,s)$-entropy of density operator $\rho$ is defined as [3]

$$H_q^{(s)}(\rho) := \frac{1}{1-q} \ln \left[\text{tr}(\rho^q)^s - 1\right]$$

for $q > 0$, $q \neq 1$ and $s \neq 0$. For $q = 1$, this entropy is defined as the von Neumann entropy $H_1(\rho) = -\text{tr}(\rho \ln \rho)$. For $s = 1$, we obtain the quantum Tsallis $q$-entropy

$$T_q(\rho) := \frac{1}{1-q} \text{tr}(\rho^q - \rho) = \text{tr}(\eta_q(\rho)),$$

where $\eta_q(x) := (x^q - x)/(1-q) = -x^q \ln_q x$ in terms of the $q$-logarithm $\ln_q x = (x^{1-q} - 1)/(1-q)$. In the limit $s \to 0$, definition (5) leads to the quantum Rényi $q$-entropy of density matrix $\rho$ defined as

$$R_q(\rho) := \frac{1}{1-q} \ln[\text{tr}(\rho^q)].$$

For $q = 1$, both expressions (6) and (7) recover the von Neumann entropy. The classical entropies can all be obtained by replacing the traces with the proper sums over a probability distribution. Let $X$ be a random variable, taking $m$ possible values with probabilities $p_X(i)$ (where $i = 1, \ldots, m$). Its $(q,s)$-entropy is

$$H_q^{(s)}(X) := \frac{1}{1-q} s \left[\left(\sum_{i=1}^m p_X(i)^q\right)^s - 1\right]$$

for $q > 0$, $q \neq 1$ and $s \neq 0$, and the Shannon entropy $H_1(X) = -\sum_i p_X(i) \ln p_X(i)$ for $q = 1$. For $s = 1$, the right-hand side of (8) gives the classical Tsallis $q$-entropy $T_q(X)$. In the limit $s \to 0$, the unified $(q,s)$-entropy recovers the Rényi $q$-entropy

$$R_q(X) = \frac{1}{1-q} \ln \left(\sum_{i=1}^m p_X(i)^q\right).$$
2.2. Channel characteristics based on \((q, s)\)-entropies

Two specific entropic measures are adopted throughout the paper. The first is the \((q, s)\)-entropy exchange [6]. For its description, we have to separate explicitly the principal system \(Q\) from an imagined reference system \(R\) and an environment \(E\). The Hilbert spaces are denoted by \(\mathcal{H}_Q\), \(\mathcal{H}_R\) and \(\mathcal{H}_E\), respectively. Under the action of quantum channel \(\Phi^{Q}\), the initial state \(\rho^{Q}\) of system \(Q\) is mapped into \(\Phi^{Q}(\rho^{Q})\). To see the entanglement transmission, we consider a purification \(|\psi^{QR}\rangle\in\mathcal{H}_Q\otimes\mathcal{H}_R\), which is transformed into the final state
\[
\rho^{QR} = \Phi^{Q} \otimes \text{id}^{R}(|\psi^{QR}\rangle\langle\psi^{QR}|)
\]
of the system \(QR\). The system \(R\) itself is not altered, i.e. \(\text{tr}_E(\rho^{QR}) = \text{tr}_Q (|\psi^{QR}\rangle\langle\psi^{QR}|)\).

Putting an environment \(E\), we can re-express the quantum channel \(\Phi^{Q}\) in terms of the unitary operator \(\tilde{U}\) on \(\mathcal{H}_E\otimes\mathcal{H}_Q\) as
\[
\Phi^{Q}(\rho^{Q}) = \text{tr}_E (\tilde{U} (|e_0\rangle\langle e_0| \otimes \rho^{Q}) \tilde{U}^\dagger).
\]

Since the final state \((\tilde{U} \otimes 1^E)|e_0\rangle\otimes |\psi^{QR}\rangle\) of the triple system \(EQR\) is obviously pure, the final density operators \(\rho^{E} \in \mathcal{L}_{+}(\mathcal{H}_E)\) and \(\rho^{QR} \in \mathcal{L}_{+}(\mathcal{H}_Q \otimes \mathcal{H}_R)\) have the same nonzero eigenvalues. We define the \((q, s)\)-entropy exchange as
\[
\bar{H}^{(q)}_{\eta}(\rho^{Q}, \Phi^{Q}) := H^{(q)}_{\eta}(\rho^{QR}) = H^{(q)}_{\eta}(\rho^{E}).
\]
This quantity characterizes an amount of \((q, s)\)-entropy introduced by the quantum channel \(\Phi^{Q}\) into an initially pure environment \(E\). Definition (12) is a direct extension of the Schumacher entropy exchange [6] to the considered entropic measure. It can be shown that the right-hand side of (12) depends only on the initial state \(\rho^{Q}\) of the principal system and the quantum channel \(\Phi^{Q}\) (for details, see subsection 12.4.1 in [1]). So further we can leave out the superscript \(Q\) of the principal system and merely write \(\bar{H}^{(q)}_{\eta}(\rho, \Phi)\).

Another specific entropic measure is the map \((q, s)\)-entropy. It is defined within the Jamiołkowski–Choi representation of \(\Phi\) [19, 20]. Let \(\mathcal{H}_Q = \mathcal{H}_R = \mathcal{H}\) and \(|\nu\rangle\) be an orthonormal basis in \(\mathcal{H}\). To this basis, we assign the normalized pure state
\[
|\phi_+\rangle := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |v\rangle \otimes |v\rangle,
\]
where \(d\) is the dimensionality of \(\mathcal{H}\). We put the operator \(\sigma(\Phi) := \Phi \otimes \text{id}(|\phi_+\rangle\langle\phi_+|)\), acting on the doubled space \(\mathcal{H}^ {\otimes 2}\). The matrix \(D(\Phi) = d \sigma(\Phi)\) is called ‘dynamical matrix’ or ‘Choi matrix’ [20]. For each \(X \in \mathcal{L}(\mathcal{H})\), the action of super-operator \(\Phi\) can be recovered from \(D(\Phi)\) by means of the relation [11]
\[
\Phi(X) = \text{tr}_R(D(\Phi)(1 \otimes X^T)),
\]
where \(X^T\) denotes the transpose operator to \(X\). The map \(\Phi\) is completely positive, if and only if the matrix \(D(\Phi)\) is positive. The condition \(\text{tr}_R(D(\Phi)) = 1\) is equivalent to the map \(\Phi\) being trace preserving and the rescaled matrix \(\sigma(\Phi)\) of unit trace. Writing positive semidefinite \(D(\Phi)\) as a sum of one-rank matrices from \(\mathcal{L}_{+}(\mathcal{H}^ {\otimes 2})\), we can obtain a set of Kraus operators (for details, see [21] or section 5.2 in [11]). For the given channel \(\Phi\), we define the map \((q, s)\)-entropy by
\[
M^{(q)}_{\eta}(\Phi) := H^{(q)}_{\eta}(\sigma(\Phi)).
\]
This is an extension of the standard map entropy introduced in [7] and further examined in [8]. The map entropy is used to characterize the decoherent behavior of the given channel.
3. Extremal unravelings of a quantum channel

In this section, we study extremality of unravelings of a quantum channel with respect to unified entropies. Recall that representations of the form (3) are not unique [11]. For the given map \( \Phi \), there are many sets \( \mathcal{A} = \{A_j\} \) that enjoy (3). In [22], each concrete set \( \mathcal{A} = \{A_j\} \) resulting in (3) is named an ‘unraveling’ of the map \( \Phi \). This terminology is due to Carmichael [23] who introduced this word for a representation of the master equation (for a review of this topic, see [24]). Following the method of [25], we introduce the matrix

\[
\Pi(\mathcal{A}|\rho) := [[|A_j\sqrt{\rho}|, A_j\sqrt{\rho}]_{\text{hs}}] = [[\text{tr}(A_j^\dagger A_j|\rho)]].
\]  

(16)

for the given density operator \( \rho \) and channel unraveling \( \mathcal{A} = \{A_j\} \). The diagonal element \( p_i = \text{tr}(A_i^\dagger A_i|\rho) \) is clearly positive and gives the \( i \)th effect probability. Then, the entropy \( H_q^{(s)}(\mathcal{A}|\rho) \) is defined by (8). It is well known that two operator-sum representations of the same completely positive map are related as

\[
B_i = \sum_j A_j u_{ji},
\]

(17)

where the matrix \( U = [|u_{ij}] \) is unitary [1]. Recall the known unitary freedom in the ensemble for density matrices (see [26] or theorem 2.6 in [1]). To obtain relation (17), this freedom should be applied to corresponding decompositions of the dynamical matrix. If the two sets \( \mathcal{A} = \{A_j\} \) and \( \mathcal{B} = \{B_j\} \) fulfill (17), then we have

\[
(B\sqrt{\rho}, B_k\sqrt{\rho})_{\text{hs}} = \sum_{jl} u_{jk}^* u_{lk} (A_j\sqrt{\rho}, A_l\sqrt{\rho})_{\text{hs}},
\]

(18)

or merely \( \Pi(B|\rho) = U^\dagger \Pi(\mathcal{A}|\rho) U \). In other words, the matrices \( \Pi(\mathcal{A}|\rho) \) and \( \Pi(\mathcal{B}|\rho) \) are unitarily similar [25]. By Hermiticity, all such matrices assigned to the same channel are unitarily similar to a unique (up to permutations) diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots) \), where \( \lambda_j \) denotes the eigenvalues of each of these matrices. Hence, any \( \Pi(\mathcal{A}|\rho) \) is positive semidefinite. For the given unraveling \( \mathcal{A} = \{A_j\} \), we obtain the concrete matrix \( \Pi(\mathcal{A}|\rho) \) and diagonalize it through a unitary transformation \( V^\dagger \Pi(\mathcal{A}|\rho) V = \Lambda \). Let us define a specific unraveling \( A_{\rho}^{(\text{ex})} \) related to the given \( \mathcal{A} \) as

\[
A_{\rho}^{(\text{ex})} = \sum_j A_j v_{ji},
\]

(19)

where the unitary matrix \( V = [|v_{ij}] \) diagonalizes \( \Pi(\mathcal{A}|\rho) \). We shall show that the unraveling (19) enjoys the extremality property with respect to almost all of the \((q, s)\)-entropies.

**Theorem 1.** Let \( \rho \) be a density operator on \( \mathcal{H} \), and \( \mathcal{A} \) be an unraveling of quantum channel. For all \( q > 0 \) and \( s \neq 0 \), there holds

\[
H_q^{(s)}(A_{\rho}^{(\text{ex})}|\rho) \leq H_q^{(s)}(\mathcal{A}|\rho),
\]

(20)

where the extremal unraveling \( A_{\rho}^{(\text{ex})} \) is defined by (19).

**Proof.** We first suppose that \( q \neq 1 \). By construction, we have \( \Pi(A_{\rho}^{(\text{ex})}|\rho) = \Lambda \) with the probabilities \( \lambda_j \) of effects. Due to \( \Pi(\mathcal{A}|\rho) = V \Lambda V^\dagger \), the probabilities \( p_i = \text{tr}(A_i^\dagger A_i|\rho) \) of different effects of \( \mathcal{A} \) is related to \( \lambda_j \) by

\[
p_i = \sum_j v_{ij} \lambda_j v_{ij}^* = \sum_j w_{ij} \lambda_j.
\]

(21)

Here, numbers \( w_{ij} = v_{ij} v_{ij}^* \) are the elements of unistochastic matrix, whence \( \sum_j w_{ij} = 1 \) for all \( j \) and \( \sum_i w_{ij} = 1 \) for all \( i \). The function \( x \mapsto x^q \) is concave for \( 0 < q < 1 \) and convex for \( 1 < q \). Applying Jensen’s inequality to this function, we obtain

\[
\sum_j p_i^q = \sum_j (\sum_j w_{ij} \lambda_j)^q \quad \left\{ \begin{array}{ll} \geq & 0 < q < 1 \\ \leq & 1 < q \end{array} \right\}, \quad \sum_j \sum_i w_{ij} \lambda_j^q = \sum_j \lambda_j^q
\]

for the given unraveling \( \mathcal{A} = \{A_j\} \).
in view of the above unistochasticity. The function \( y \mapsto \frac{y}{s} \) monotonically increases for all \( s \neq 0 \), whence
\[
\frac{1}{s} \left( \sum_i p_i^q \right)^y \begin{cases} \leq, & 0 < q < 1 \\ \geq, & 1 < q \end{cases} \frac{1}{s} \left( \sum_j \lambda_j^q \right)^y.
\] (22)
Since the term \((1 - q)\) is positive for \( q < 1 \) and negative for \( 1 < q \), relations (22) are merely combined as
\[
\frac{1}{s} \left( \sum_i p_i^q \right)^y \geq \frac{1}{(1 - q)s} \left( \sum_j \lambda_j^q \right)^y.
\] (23)
Due to definition (8), inequality (23) provides (20). In the case \( q = 1 \), we deal with the Shannon entropy. Applying Jensen’s inequality to the concave function \( x \mapsto -x \ln x \) completes the proof of this case. □

For the prescribed state \( \rho \) and given unraveling of quantum channel, all the unified \((q, s)\)-entropies with the parameter \( s \neq 0 \) are minimized by the extremal unraveling, which is built in line with (19). For the Shannon entropy, a question with ‘minimal’ unraveling was considered in [27] and later in [22]. It was noted in [25] that diagonalizing the matrix \( \Pi(A|\rho) \) is formally equivalent to the extreme condition obtained in [22]. As was also shown in [25], the Rényi entropies enjoy the extremality property with unraveling operators (19) only for the order \( q \in (0; 1) \). In this regard, the Rényi entropies differ from other considered entropies.

A quantum state is characterized by the probabilities of the outcomes of every conceivable test [28]. So the measurements are quantum operations of special conceptual interest. A general measurement is described by the set \( \{ M_i \} \) of measurement operators [1]. Suppose that \( \rho \) is the density operator of the system right before the measurement. Separate terms of the sum \( \sum_i M_i \rho M_i^\dagger \) are related to different outcomes of the measurement. Since the probabilities \( \text{tr}(M_i^\dagger M_i \rho) \) are summed to 1, the measurement operators follow the equality in (4). A standard measurement is described by the set \( \{ P_j \} \) of mutually orthogonal projectors. As a rule, projective measurements are easier to realize experimentally. One of the basic properties of the von Neumann entropy is that it cannot be decreased by a projective measurement (see, e.g., theorem 11.9 in [1]). For the given measurement \( \{ P_j \} \), the output density operator is expressed as [1]
\[
C(\rho) = \sum_j P_j \rho P_j.
\] (24)
In matrix analysis, an operation of such a kind is referred to as ‘pinching’ (for details, see section IV.2 of [12]). The non-decreasing of the von Neumann entropy is posed as \( H_1(\rho) \leq H_1(C(\rho)) \). As shown in [4], all the unified entropies are non-decreasing under projective measurements, namely
\[
H_q^{(s)}(\rho) \leq H_q^{(s)}(C(\rho)).
\] (25)
So, we may see that \( H_q^{(s)}(\rho) \leq H_q^{(s)}(C(\rho)|\rho) \) for any measurement with one-rank projectors only. In this case, the entropy \( H_q^{(s)}(\rho) \) is a lower bound for the extremal unraveling entropy \( H_q^{(s)}(C(\rho)|\rho) \). The latter inequality is always saturated, when the projective measurement is carried out in the basis of the eigenstates of \( \rho \). Such treatment can be extended to other quantum channels, when the Kraus operators of extremal unraveling fulfill a certain condition.

**Theorem 2.** Suppose that the Kraus operators \( A_i^{(\text{ex})} \) of extremal unraveling of a quantum channel satisfy
\[
\text{tr}(A_i^{(\text{ex})} A_i^{(\text{ex})}) = 1
\] (26)
for all values of index $i$. For $0 < q$ and $s \neq 0$ as well as for $0 < q < 1$ and $s = 0$, we then have
\begin{equation}
H_q^{(s)}(\rho) \leq H_q^{(s)}(A^{(ex)}_p|\rho).
\end{equation}

**Proof.** We first suppose that $q \neq 1$. By definition, the matrix $\Pi(A^{(ex)}_p|\rho)$ is diagonal. Taking the trace in the eigenbasis $\{|j\rangle\}$ of $\rho$, we rewrite the diagonal elements as
\begin{equation}
\lambda_i = \text{tr}(A^{(ex)}_i A^{(ex)}_i \rho) = \sum_j \varsigma_j |j| A^{(ex)}_i A^{(ex)}_i |j\rangle,
\end{equation}
where $\varsigma_j$ denotes the eigenvalues of $\rho$. The numbers $t_{ij} = \langle j | A^{(ex)}_i A^{(ex)}_i | j \rangle$ form a doubly stochastic matrix due to
\begin{equation}
\sum_j t_{ij} = \text{tr}(A^{(ex)}_i A^{(ex)}_i) = 1, \quad \sum_i t_{ij} = \langle j | j \rangle = 1.
\end{equation}

Here, the first follows from (26), the second follows from equality (4). Combining both the relations in (29) with the Jensen inequality, we obtain
\begin{equation}
\sum_i \lambda_i^q = \sum_i \left( \sum_j t_{ij} \varsigma_j \right)^q \left\{ \begin{array}{ll}
\geq, & 0 < q \leq 1 \\
< & 1 < q
\end{array} \right\} \sum_i \sum_j t_{ij} \varsigma_j^q = \sum_j \varsigma_j^q.
\end{equation}

As the function $y \mapsto y^q/s$ monotonically increases for $s \neq 0$, these inequalities lead to
\begin{equation}
\frac{1}{s} \left( \sum_j \varsigma_j^q \right)^q \left\{ \begin{array}{ll}
\geq, & 0 < q \leq 1 \\
< & 1 < q
\end{array} \right\} \frac{1}{s} \left( \sum_j \varsigma_j^q \right)^q.
\end{equation}

Multiplying (30) by the factor $(1 - q)^{-1}$, which is positive for $0 < q < 1$ and negative for $1 < q$, we complete the proof for $q \neq 1$ and $s \neq 0$. In the case $q = 1$, when the von Neumann and Shannon entropies are dealt with, we merely combine (29) with the concavity of the function $x \mapsto -x \ln x$. When $s = 0$, we apply the relations
\begin{equation}
R_q(\rho) = \frac{1}{(1 - q)} \ln \left( 1 + (1 - q) T_q(\rho) \right),
\end{equation}
\begin{equation}
R_q(A^{(ex)}_p|\rho) = \frac{1}{(1 - q)} \ln \left( 1 + (1 - q) T_q(A^{(ex)}_p|\rho) \right)
\end{equation}
which follow from the definitions of the Rényi and Tsallis $q$-entropies. Since the bound (27) holds for the Tsallis case ($s = 1$) and the function $y \mapsto (1 - q)^{-1} \ln(1 + (1 - q) y)$ is increasing for $q < 1$, the Rényi $q$-entropies also obey (27) for $0 < q < 1$.

So, if condition (26) holds, then the extremal unraveling entropy $H_q^{(s)}(A^{(ex)}_p|\rho)$ is bounded from below by the entropy of the input state $H_q^{(0)}(\rho)$. This property holds for all the unified $(q, s)$-entropies, except for the Rényi $q$-entropies of order $q > 1$. Inequality (27) may also be regarded as an estimate on the entropy of the input state of a channel, when the extremal unraveling entropy is known (exactly or approximately) from other reasons.

### 4. Continuity estimates on the map $(q, s)$-entropies

In this section, we derive inequalities of Fannes type for the map $(q, s)$-entropies. First, the known continuity bounds in terms of the trace norm distance are applied to the considered entropic measures. Second, we obtain upper continuity bounds in terms of the Frobenius norm distance.
4.1. Estimates based on the known bounds

One of essential properties of von Neumann entropy is its continuity first stated by Fannes [29].
A sharp refinement of Fannes’ original inequality was presented independently by Audenaert
[30] and by Petz, in theorem 3.8 of [31]. The method of Audenaert is rather algebraic in
nature, whereas the proof of theorem 3.8 in [31] is based on the classical Fano inequality (and
is attributed to Csiszár). Fannes’ inequality has been generalized to the Tsallis entropy [32,
33] and its partial sums [34]. Continuity estimates of Fannes type have also been derived for
the quantum conditional entropy [35] as well as for the standard quantum relative entropy
[36, 37] and its

\[
\sum_{s, \mu, \nu} \langle \xi | \mu \rangle \langle \mu | \xi \rangle = 0
\]

under the condition \(\| \sigma (\Phi) - \sigma (\Psi) \|_1 = 2t \leq q^{1/(1-q)}\), there holds

\[
\| M_{\Phi}^{(t)} (\Phi) - M_{\Psi}^{(t)} (\Psi) \| \leq (2t)^{q} \ln q d + \eta_q (2t).
\]  

Define the factor \( \kappa_s = d^2(q-1) \) for \( s \in [-1; 0] \) and \( \kappa_s = 1 \) for \( s \in [+1; +\infty) \). For the parameter range

\[
\{(q, s) : 1 < q < 1, s \in (-\infty; -1] \cup [0; +1]\}
\]  

under the condition \((1/2) \| \sigma (\Phi) - \sigma (\Psi) \|_1 = t \leq (d - 1)/d\), there holds

\[
\| M_{\Phi}^{(1)} (\Phi) - M_{\Psi}^{(1)} (\Psi) \| \leq \kappa_s \left[ t^d \ln q (d - 1) + T_q (t, 1 - t) \right].
\]  

Here, \( T_q (t, 1 - t) \) is the binary Tsallis entropy. The validity ranges \( 0 \leq 2t \leq q^{1/(1-q)} \) for (34)
and \( 0 \leq t \leq (d - 1)/d \) for (36) are essential and obtained as intervals of non-decreasing
\( \eta_q (2t) \) and the right-hand side of (36), respectively.

In principle, the continuity property is established by means of inequalities (34) and (36).
At the same time, the norm \( \| \sigma (\Phi) - \sigma (\Psi) \|_1 \) is sufficiently difficult for calculation in general form. The Frobenius norm distance \( \| \sigma (\Phi) - \sigma (\Psi) \|_2 \) is easy to estimate. By the definition of \( \sigma (\cdot) \), there holds

\[
\| \sigma (\Phi) - \sigma (\Psi) \|^2 = \frac{1}{d^2} \sum_{\mu, \nu, \xi} \langle \xi | \mu \rangle \langle \mu | \xi \rangle (\Phi(|\mu\rangle \langle \mu|) - \Psi(|\mu\rangle \langle \mu|)) \otimes |\mu\rangle \langle \xi|.
\]

Hence, in view of \( \text{tr}(X \otimes Y) = \text{tr}(X) \text{tr}(Y) \) and the linearity of the trace, we obtain

\[
\text{tr} (\| \sigma (\Phi) - \sigma (\Psi) \|^2) = \frac{1}{d^2} \sum_{\mu, \nu} \| \Phi(|\mu\rangle \langle \mu|) - \Psi(|\mu\rangle \langle \mu|) \|_2^2.
\]  

In other words, the Frobenius norm distance \( \| \sigma (\Phi) - \sigma (\Psi) \|_2 \) is expressed as the 2-mean

\[
\| \sigma (\Phi) - \sigma (\Psi) \|_2 = M_1 (\| \Phi - \Psi \| (|\mu\rangle \langle \mu|))
\]  

of the particular distances \( \| \Phi - \Psi \| (|\mu\rangle \langle \mu|)\|_2 \) taken with equal weights. Other questions, in
which the Frobenius norm distance is very useful, are treated in [16]. So, it is of interest to
pose continuity estimates on the map entropies in terms of the distance (38).
4.2. Estimates in terms of the Frobenius norm distance

Inequalities of Fannes type are naturally formulated in terms of the trace norm distance. To reformulate these inequalities, we need relations between the trace and Frobenius norms. This issue is considered in the appendix. Taking \( q = 2 \) in (A.5), we obtain \( \|X\|_1 \leq \sqrt{d} \|X\|_2 \) for each \( X \in \mathcal{L}(\mathcal{H}) \). In the mentioned intervals of non-decreasing, the upper bounds (34) and (36) are recast with larger \( \sqrt{d} \|\sigma(\Phi) - \sigma(\Psi)\|_2 \) instead of \( \|\sigma(\Phi) - \sigma(\Psi)\|_1 \) as follows. For the parameter range (33), there holds

\[
|M_\Phi^{(s)} - M_\Psi^{(s)}| \leq d^{1/2} (2\tau)^q \ln\eta_q(2d\tau) + \eta_q(2d\tau),
\]

provided that \( \|\sigma(\Phi) - \sigma(\Psi)\|_2 = 2\tau < q^{1/2}d^{-1/2} \). For the parameter range (35), there holds

\[
|M_\Phi^{(s)} - M_\Psi^{(s)}| \leq \eta_q(d^{1/2} + q^{1/2}d^{-1/2} \ln\eta_q(1 - \sqrt{d}\tau)),
\]

provided that \( (1/2)\|\sigma(\Phi) - \sigma(\Psi)\|_2 = \tau < (d - 1)^{-3/2} \). So, we have arrived at upper bounds in terms of the Frobenius norm distance between the two rescaled dynamical matrices. However, the validity ranges of the bounds (39) and (40), particularly the former, seem to be too restrictive for sufficiently large \( d \). Since the operator \( \sigma(\Phi) - \sigma(\Psi) \) is traceless Hermitian and \( \text{tr}(\sigma(\Phi)) = \text{tr}(\sigma(\Psi)) = 1 \), the trace and Frobenius norms obey \( \|\sigma(\Phi) - \sigma(\Psi)\|_1 \leq 2 \) and \( \|\sigma(\Phi) - \sigma(\Psi)\|_2 \leq \sqrt{2} \), respectively. Below, we will derive upper bounds that hold for all \( \|\sigma(\Phi) - \sigma(\Psi)\|_2 \leq \sqrt{2} \).

As was noted above, the classical Fano inequality leads to an improvement of the original Fannes’ bound [31]. Developing this point for the Tsallis entropies [39], we obtain the inequality

\[
|T_q(\rho) - T_q(\omega)| \leq t^q \ln\eta_q(d(d - 1)) + t^q \ln\eta_q(d - 1) + T_q(t, 1 - t)
\]

for \( 0 < q < 1 \) and all \( (1/2)\|\rho - \omega\|_1 = t \leq 1 \), and the inequality

\[
|T_q(\rho) - T_q(\omega)| \leq t^q \ln\eta_q(d - 1) + T_q(t, 1 - t)
\]

for \( 1 < q \) and \( (1/2)\|\rho - \omega\|_1 = t \leq (d - 1)/d \). The second is exactly the bound that was derived immediately in [33]. In comparison with (34), the upper bound (41) is valid for all acceptable values of the norm \( \|\rho - \omega\|_1 \), including its maximal value 2. We shall now modify (42) in this regard. The bound (42) was derived in [39] with use of the relation

\[
T_q(X) - T_q(Y) \leq T_q(X|Y) \leq P_e^q \ln\eta_q(m - 1) + T_q(P_e, 1 - P_e).
\]

Here, random variables \( X \) and \( Y \) take the same \( m \) possible values, \( T_q(X|Y) \) is the conditional \( q \)-entropy, and \( P_e \) is the probability of error, i.e. the probability that \( X \neq Y \). The inequality on the left follows from the chain rule for the conditional \( q \)-entropy (for its properties, see [40]). The right-hand side of (43) is an extension of the classical Fano inequality. We rewrite the right-hand side of (43) in the form

\[
\frac{1 - (1 - P_e)^q - (m - 1)^{1-q}P_e^q}{q - 1} \leq qP_e - (m - 1)^{1-q}P_e^q,
\]

which follows from the inequality \( 1 - (1 - P_e)^q = \int_0^P q(1 - x)^{q-1}dx \leq \int_0^P q\ dx = qP_e \) in view of \( q > 1 \). As a function of \( P_e \), the left-hand side of equation (44) monotonically increases up to the point \( P_e = (m - 1)/m \), where the derivative is zero. But the derivative of the right-hand side of (44) vanishes at the point \( P_e = m - 1 \). So, the interval of non-decreasing becomes wider, though the bound with the right-hand side of (44) will somewhat weaker. Setting \( m = d \), we consider the probabilities of \( X \) and \( Y \) as the eigenvalues of \( \rho \) and \( \omega \).
respectively. The joint probability mass function of $X$ and $Y$ can be built in such a way that 

$P_r = (1/2) \sum_i |p_r(i) - p_l(i)| \leq (1/2) ||\rho - \omega||_1$ (the equality on the left is a part of the coupling inequality [41] and the inequality on the right follows from lemma 11.1 of [31]). Combining this with (43) leads to the inequality of Fannes type in the form

$$|T_q(\rho) - T_q(\omega)| \leq \frac{qt - (d - 1)^{1 - q}q}{q - 1},$$  \hspace{1cm} (45)

where $t = (1/2) ||\rho - \omega||_1$ may take all possible values from the range $0 \leq t \leq 1$. Replacing $t$ with larger $\sqrt{d} \tau$, we obtain the upper bound

$$|T_q(\rho) - T_q(\omega)| \leq \frac{q\sqrt{d} \tau - (d - 1)^{1 - q}q\sqrt{d} \tau}{q - 1},$$  \hspace{1cm} (46)

where $\tau = (1/2) ||\rho - \omega||_2$. Such replacing is valid under the constraint $\sqrt{d} \tau \leq d - 1$, i.e. $||\rho - \omega||_2 \leq 2(d - 1)/\sqrt{d}$. The latter always holds, since $||\rho - \omega||_2 \leq \sqrt{d}$ for two density matrices and $\sqrt{d}/2 \leq d - 1$ for all $d \geq 2$. In other words, the upper bound (46) is dealt for all possible values of the Frobenius norm distance between states (and $1 < q$). In the same manner, from (41), we obtain

$$|T_q(\rho) - T_q(\omega)| \leq \frac{d^{1 - q/2}(d - 1)^{1 - q}d^{q/2} \tau q}{1 - q}.$$  \hspace{1cm} (47)

The right-hand side of (41) monotonically increases up to the point $t = d(d - 1)$, whence our replacing is formally valid under the constraint $\sqrt{d} \tau \leq d - 1$, i.e $||\rho - \omega||_2 \leq 2\sqrt{d} (d - 1)$. So, the upper bound (47) is also dealt for all possible values of the Frobenius norm distance between states (and $0 < q < 1$). Repeating the reasons of section 3 in [4] with new bounds on the Tsallis entropy, we obtain the following statement.

**Theorem 3.** Let $\rho$ and $\omega$ be the density operators on the $d$-dimensional Hilbert space $\mathcal{H}$. For the parameter range (33) and all possible values of the distance $\tau = (1/2) ||\rho - \omega||_2$, there holds

$$|H_q^{(s)}(\rho) - H_q^{(s)}(\omega)| \leq \frac{d^{1 - q/2}(d - 1)^{1 - q}d^{q/2} \tau q}{1 - q}.$$  \hspace{1cm} (48)

For the parameter range (35) and all possible values of the distance $\tau = (1/2) ||\rho - \omega||_2$, there holds

$$|H_q^{(s)}(\rho) - H_q^{(s)}(\omega)| \leq \chi_s \frac{q\sqrt{d} \tau - (d - 1)^{1 - q}q\sqrt{d} \tau}{q - 1},$$  \hspace{1cm} (49)

where the factor $\chi_s = d^{2(q - 1)}$ for $s \in [-1; 0]$ and $\chi_s = 1$ for $s \in [+1; +\infty)$.

Hence, we obtain another upper bounds on the map $(q, s)$-entropies in terms of the quantity (38). Designating $||\sigma(\Phi) - \sigma(\Psi)||_2 = 2\tau$, we have

$$|M_q^{(s)}(\Phi) - M_q^{(s)}(\Psi)| \leq \frac{d^{1 - q/2}(d - 1)^{1 - q}d^{q/2} \tau q}{1 - q},$$  \hspace{1cm} (50)

$$|M_q^{(s)}(\Phi) - M_q^{(s)}(\Psi)| \leq \chi_s \frac{q\sqrt{d} \tau - (d - 1)^{1 - q}q\sqrt{d} \tau}{q - 1},$$  \hspace{1cm} (51)

in the parameter ranges (33) and (35), respectively. In comparison with the results (39) and (40), these bounds are somewhat weaker, but their validity intervals for $||\sigma(\Phi) - \sigma(\Psi)||_2$ do not depend on the dimensionality of the Hilbert space. However, for sufficiently small values of this distance, we prefer the bounds (39) and (40).
5. Some additivity properties of the map \((q, s)\)-entropies

In this section, we examine properties of the map \((q, s)\)-entropies with respect to the tensor product of a pair of quantum channels. Broad use of the Tsallis entropy in non-extensive statistical mechanics stems from the fact that it does not share the additivity in the following sense [42]. If two random variables \(X\) and \(Y\) are independent, then the Shannon entropy of the joint distribution \(H(X, Y) = H(X) + H(Y)\). In the quantum regime, the von Neumann entropy enjoys \(H_1(\rho^\otimes \otimes \rho^B) = H_1(\rho^\otimes) + H_1(\rho^B)\). In general, we have the subadditivity [2]

\[
H_1(\rho^{QR}) \leq H_1(\rho^Q) + H_1(\rho^R),
\]

where the reduced densities \(\rho^Q\) and \(\rho^R\) are obtained from \(\rho^{QR}\) by taking the partial trace. The equality in equation (52) holds only for the above case of product states. The Tsallis entropies do not enjoy the additivity with the product states, though the subadditivity of quantum \(q\)-entropy is still obeyed for \(q > 1\), namely

\[
H_q(\rho^{QR}) \leq H_q(\rho^Q) + H_q(\rho^R).
\]

This fact has been conjectured in [43] and later proved in [44]. Concerning additivity properties, the unified entropies succeed to the Tsallis entropies [3]. In particular, we have the subadditivity of the quantum \((q, s)\)-entropy for \(q > 1\) and \(s > 1/q\) [4]. Additivity properties of the map \((q, s)\)-entropies with respect to the tensor product of quantum channels are posed as follows.

**Theorem 4.** Let \(\Phi_1\) and \(\Phi_2\) be the quantum channels. For \(q > 0\) and all real \(s\), the map \((q, s)\)-entropy satisfies

\[
M_q^{(s)}(\Phi_1 \otimes \Phi_2) = M_q^{(s)}(\Phi_1) + M_q^{(s)}(\Phi_2) + (1 - q)s M_q^{(s)}(\Phi_1) M_q^{(s)}(\Phi_2).
\]

**Proof.** The claim is based on the two points. The first is the expression for the unified \((q, s)\)-entropy of a product state, which is formulated as [3]

\[
H_q^{(s)}(\rho \otimes \omega) = H_q^{(s)}(\rho) + H_q^{(s)}(\omega) + (1 - q)s H_q^{(s)}(\rho) H_q^{(s)}(\omega).
\]

The second is the important result that the dynamical matrix \(D(\Phi_1 \otimes \Phi_2)\) is unitarily similar to the product matrix \(D(\Phi_1) \otimes D(\Phi_2)\) [8], whence

\[
M_q^{(s)}(\Phi_1 \otimes \Phi_2) = H_q^{(s)}(\sigma(\Phi_1) \otimes \sigma(\Phi_2)).
\]

Combining equalities (55) and (56) finally leads to (54). \(\square\)

For the case \(s = 0\), relation (54) gives the additivity of the Rényi map entropies. The latter has been proved in [8]. It is known that the rank of the dynamical matrix is equal to the minimal number of terms required in the operator-sum representation [7]. So, if one of channels \(\Phi_1\) and \(\Phi_2\) represents a unitary evolution, then its dynamical matrix is of rank 1 and its map \((q, s)\)-entropy is 0. Then, relation (54) claims the additivity of the map \((q, s)\)-entropy. If neither of \(\Phi_1\) and \(\Phi_2\) represents a unitary evolution, then both the dynamical matrices are of rank \(\geq 2\) and both the map entropies are nonzero. In this case, the map \((q, s)\)-entropy is strictly subadditive for \(0 < q < 1, s < 0\) \(\cup\) \(\{1 < q, 0 < s\}\), i.e.

\[
M_q^{(s)}(\Phi_1 \otimes \Phi_2) < M_q^{(s)}(\Phi_1) + M_q^{(s)}(\Phi_2).
\]

The map \((q, s)\)-entropy is strictly superadditive for \(0 < q < 1, 0 < s\) \(\cup\) \(\{1 < q, s < 0\}\):

\[
M_q^{(s)}(\Phi_1 \otimes \Phi_2) > M_q^{(s)}(\Phi_1) + M_q^{(s)}(\Phi_2).
\]

Using the map \((q, s)\)-entropies, we can herewith separate quantum channels that represent a unitary evolution. In contrast, the map Rényi entropies are additive irrespective of channel features [8].
Theorem 5. Let $\rho$ be a density operator and $\Phi$ be a quantum channel. For $q > 1$ and $s \geq q^{-1}$, the inequality
\begin{equation}
\left| H_q^{(s)}(\rho) - \tilde{H}_q^{(s)}(\rho, \Phi) \right| \leq H_q^{(s)}(\Phi(\rho)) \leq H_q^{(s)}(\rho) + \tilde{H}_q^{(s)}(\rho, \Phi),
\end{equation}
including permutations of the three entropies, holds.

Proof. Within the proof, we recall the meaning of the systems $E, Q, R$ from (10)–(12) and the respective notation. Using one result of [44], both the subadditivity and the triangle inequality have been stated for the quantum $(q, s)$-entropies for $q > 1, s \geq q^{-1}$ [4]. For the reduced densities
\begin{equation}
\rho^{Q^*} = \text{tr}_E(\rho^{E Q^*}), \quad \rho^{Q^*} = \text{tr}_Q(\rho^{E Q^*}),
\end{equation}
obtained from the output density operator $\rho^{E Q^*}$, and the parameter values $q > 1, s \geq q^{-1}$, we have the relations
\begin{equation}
\left| H_q^{(s)}(\rho^{Q^*}) - \tilde{H}_q^{(s)}(\rho^{Q^*}, \Phi) \right| \leq H_q^{(s)}(\Phi(\rho^{Q^*})) \leq H_q^{(s)}(\rho^{Q^*}) + \tilde{H}_q^{(s)}(\rho^{Q^*}, \Phi).
\end{equation}
The inequality on the left is the triangle inequality, and the inequality on the right expresses the subadditivity. We then rewrite (61) as
\begin{equation}
\left| H_q^{(s)}(\rho^{Q^*}, \Phi(\rho^{Q^*})) - H_q^{(s)}(\Phi(\rho^{Q^*})) \right| \leq H_q^{(s)}(\rho^{Q^*}) \leq H_q^{(s)}(\Phi(\rho^{Q^*})) + \tilde{H}_q^{(s)}(\Phi(\rho^{Q^*}), \Phi) + \tilde{H}_q^{(s)}(\rho^{Q^*}, \Phi).
\end{equation}
in view of definition (12) and $\rho^{Q^*} = \Phi(\rho^{Q^*})$. It was also used that the systems $Q$ and $R$ initially share the same pure state and the system $R$ itself is not altered. By performing some simple algebra, we further obtain
\begin{equation}
\left| H_q^{(s)}(\Phi(\rho^{Q^*})) - \tilde{H}_q^{(s)}(\rho^{Q^*}, \Phi) \right| \leq H_q^{(s)}(\rho^{Q^*}) \leq H_q^{(s)}(\Phi(\rho^{Q^*})) + \tilde{H}_q^{(s)}(\rho^{Q^*}, \Phi)
\end{equation}
and inequality (59) (in which the superscript $Q$ is already left out).

For the von Neumann entropy, an analog of inequality (59) plus permutations were presented by Lindblad [45]. The authors of [8] used Lindblad’s inequality for the estimation of the entropy of an output state arising from a maximally entangled input state. We shall develop this issue with respect to the unified $(q, s)$-entropies.
in view of \( \bar{H}_q^{(s)}(\rho_*, \Phi_1) = \bar{H}_q^{(s)}(\sigma(\Phi^*_1)), \Phi_1 \otimes \text{id} \). Definition (12) merely gives \( \bar{H}_q^{(s)}(\rho_*, \Phi_1) = H_q^{(1)}(\sigma(\Phi^*_1)) \), since any purification of \( \rho_* \) is a maximally entangled state. Combining this with (64) completes the proof. \( \square \)

As in theorem 5, the inequalities with permutations of the three entropies also hold. Thus, the output entropy of a maximally entangled input state satisfies the two-sided estimate (62) for many values of the parameters \( q \) and \( s \). Estimation of such a kind seems to be important in the context of studies of subadditivity conjecture for a tensor product of two quantum channels [8]. The proved bounds (62) are expressed in terms of the two map \( (q, s) \)-entropies, which characterize the decoherent behavior of involved quantum channels. For the von Neumann entropy, inequality (62) was derived in [8]. Some remarks concerning the quantum Tsallis \( q \)-entropy are contained therein. The authors in [8] also provide another characterization of a product channel in terms of the minimum output entropies. We do not consider such entropies here.

6. Conclusions

We have discussed some important properties of quantum channels in terms of the unified \((q, s)\)-entropies, which form a family of two-parametric extensions of the standard Shannon and von Neumann entropies. In many respects, these entropies are similar to the standard ones. For a given input state, different effects of each unraveling of a channel result in some probability distribution at the output. Except for Rényi’s \( q \)-entropies of order \( q \geq 1 \), the unified \((q, s)\)-entropies of this distribution are all minimized by the same unraveling of a quantum channel. We have also specified some class of extremal unravelings such that their \((q, s)\)-entropies are bounded from below by the quantum \((q, s)\)-entropy of the input state. Several upper bounds of Fannes type have been derived for the introduced map \((q, s)\)-entropies. The Frobenius norm distance between two rescaled dynamical matrices is easy to express than the trace norm one. So, we have given new continuity estimates on the unified \((q, s)\)-entropies in terms of the Frobenius norm of density operators. If none of two quantum channels represents a unitary evolution, then the map \((q, s)\)-entropy of their tensor product is strictly subadditive in the range \( \{ 0 < q < 1, s < 0 \} \cup \{ 1 < q, 0 < s \} \) and strictly superadditive in the range \( \{ 0 < q < 1, 0 < s \} \cup \{ 1 < q, s < 0 \} \). Extending Lindblad’s inequality, we have obtained a two-sided estimate on the output \((q, s)\)-entropy for the tensor product of two channels acting on a maximally entangled input state. Overall, the map entropies of the considered kind enjoy useful properties and may be applied in the context of quantum information processing.

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Appendix. Some relations between the Schatten norms

In this appendix, we examine relations between different Schatten norms of the same operator. A few results of such a kind were presented in section IV of [36]. The statement of Lemma 3 therein allows one to give the upper bound on a unitarily invariant norm \( \| X \| \) of the traceless Hermitian \( X \) in terms of the trace norm \( \| X \|_1 \). In particular, we have \( \sqrt{2} \| X \|_2 \leq \| X \|_1 \) for all traceless Hermitian \( X \in \mathcal{L}(\mathcal{H}) \) [36]. In this work, however, we rather needed an upper
bound on the trace norm in terms of the Frobenius one. In general, the following statement is obtained.

**Lemma A.1.** Let $X$ be an operator with $d$-dimensional support. For $p, q \geq 1$, there holds

$$
\|X\|_p \leq d^{(q-1)/pq} \|X\|_{pq},
$$

(A.1)

with equality if and only if the $X$ acts on its support as a multiple of unitary operator.

**Proof.** We can restrict our consideration to the support of $X$, in which its singular values are nonzero. In line with the Hölder inequality, we have [13]

$$
|\langle u, v \rangle| \leq \|u\|_q \|v\|_r,
$$

(A.2)

where the conjugate indices $q$ and $r$ obey $1/q + 1/r = 1$ and the vector norms are

$$
\|u\|_q = \left(\sum_{j=1}^d |u_j|^q\right)^{1/q}, \quad \|v\|_r = \left(\sum_{j=1}^d |v_j|^r\right)^{1/r}.
$$

(A.3)

Putting $u_j = \varsigma_j(X)^p$ and $v_j = 1$ for all $j$, we then obtain

$$
\sum_{j=1}^d \varsigma_j(X)^p \leq \left(\sum_{j=1}^d \varsigma_j(X)^{pq}\right)^{1/q} d^{1-1/q}.
$$

(A.4)

Raising both sides to the power $1/p$, we finally obtain (A.1). Equality (A.2) holds if and only if the $d$-tuples $u$ and $v$ are linearly related (see, e.g., theorem 14 in [13]). Hence equality (A.4) is equivalent to that $\varsigma_j = \varsigma_k$ for all $j \neq k$. In the basis of its eigenstates, $|X|$ is then rewritten as $|X| = \varsigma_1 |I|$, whence the operator $\varsigma_1^{-1}X$ is unitary. □

Relation (A.1) gives an upper bound on the Schatten $p$-norm in terms of other Schatten norms with larger values of the parameter. This bound is not trivial, because the Schatten $p$-norm is non-increasing in $p$ [11]. In particular, for $p = 1$, we have an estimate on the trace norm expressed as

$$
\|X\|_1 \leq d^{(q-1)/q} \|X\|_q.
$$

(A.5)

We also have

$$
\|X\|_p \leq d^{1/p} \|X\|_\infty
$$

for $q \to \infty$ and finite $p$, with the same conditions on the equality.

**References**

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[2] Bengtsson I and Życzkowski K 2006 *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge: Cambridge University Press)

[3] Hu X and Ye Z 2006 *J. Math. Phys.* 47 023502

[4] Rastegin A E 2011 *J. Stat. Phys.* 143 1120

[5] Kim J S and Sanders B 2011 *J. Phys. A: Math. Theor.* 44 295303

[6] Schumacher B 1996 *Phys. Rev. A* 54 2614

[7] Życzkowski K and Bengtsson I 2004 *Open Syst. Inf. Dyn.* 11 3

[8] Roga W, Życzkowski K and Fannes M 2011 *Int. J. Quantum Inf.* 9 1031

[9] Shor P W 2004 *Commun. Math. Phys.* 246 453

[10] Brandão F G S L and Horodecki M 2010 *Open Syst. Inf. Dyn.* 17 31

[11] Watrous J 2008 *CS 798: Theory of Quantum Information* (University of Waterloo, Ontario, Canada) http://www.cs.uwaterloo.ca/~watrous/quant-info/lecture-notes/)

[12] Bhattacharyya R 1997 *Matrix Analysis* (New York: Springer)

[13] Hardy G H, Littlewood J E and Polya G 1934 *Inequalities* (London: Cambridge University Press)

[14] Rastegin A E 2010 *Quantum Inf. Process.* 9 61

[15] Rastegin A E 2011 *Quantum Inf. Process.* 10 123
[16] Dunkl C F et al 2011 J. Phys. A: Math. Theor. 44 335301
[17] Duan L-M and Guo G-C 1998 Phys. Lett. A 243 261
[18] Chefles A and Barnett S M 1998 J. Phys. A: Math. Gen. 31 10097
[19] Jamiolkowski A 1972 Rep. Math. Phys. 3 275
[20] Choi M-D 1975 Linear Algebra Appl. 10 285
[21] Miszczak J A 2011 Int. J. Mod. Phys. C 22 897
[22] Il’ichev L V 2003 J. Exp. Theor. Phys. 96 982
[23] Carmichael H J 1993 An Open Systems Approach to Quantum Optics (Lecture Notes in Physics vol m18) (Berlin: Springer)
[24] Wiseman H M and Milburn G J 2010 Quantum Measurement and Control (Cambridge: Cambridge University Press)
[25] Rastegin A E 2011 J. Phys. A: Math. Theor. 44 095303
[26] Hughston L P, Jozsa R and Wootters W K 1993 Phys. Lett. A 183 14
[27] Breslin J K and Milburn G J 1997 J. Mod. Opt. 44 2469
[28] Peres A 1993 Quantum Theory: Concepts and Methods (Dordrecht: Kluwer)
[29] Fannes M 1973 Commun. Math. Phys. 31 291
[30] Audenaert K M R 2007 J. Phys. A: Math. Theor. 40 8127
[31] Petz D 2008 Quantum Information Theory and Quantum Statistics (Berlin: Springer)
[32] Furuchichi S, Yanagi K and Kuriyama K 2007 J. Inequal. Pure Appl. Math. 8 5
[33] Zhang Z 2007 Lett. Math. Phys. 80 171
[34] Rastegin A E 2010 Lett. Math. Phys. 94 229
[35] Alicki R and Fannes M 2004 J. Phys. A: Math. Gen. 37 L55
[36] Audenaert K M R and Eisert J 2005 J. Math. Phys. 26 102104
[37] Audenaert K M R and Eisert J 2011 Continuity bounds on the quantum relative entropy: part II arXiv:1105.2656 [math-ph]
[38] Rastegin A E 2011 J. Math. Phys. 52 062203
[39] Rastegin A E 2011 Continuity estimates on the Tsallis relative entropy arXiv:1102.5154 [math-ph]
[40] Furuchichi S 2006 J. Math. Phys. 47 023302
[41] Lindvall T 1992 Lectures on the Coupling Method (New York: Wiley)
[42] Tsallis C 2001 Nonextensive statistical mechanics and thermodynamics: historical background and present status Nonextensive Statistical Mechanics and Its Applications (Lecture Notes in Physics vol 560) ed S Abe and Y Okamoto (Berlin: Springer) p 3
[43] Raggio G A 1995 J. Math. Phys. 36 4785
[44] Audenaert K 2007 J. Math. Phys. 48 083507
[45] Lindblad G 1991 Quantum entropy and quantum measurements Quantum Aspects of Optical Communications (Lecture Notes in Physics vol 378) ed C Bendjaballah, O Hirota and S Reynaud (Berlin: Springer) p 71