Entanglement between two scalar fields in an expanding spacetime

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We study the evolution of the two scalar fields entangled via a mutual interaction in an expanding spacetime. We compute the logarithmic negativity to leading order in perturbation theory and show that for lowest order in the coupling constants, the mutual interaction will give rise to the survival of the quantum correlations in the limit of the smooth expansion. The results suggest that interacting fields can codify more information about the underlying expansion spacetime and lead to interesting observable effects.

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Introduction - Entangled states as firstly described in the seminal paper of Einstein, Podolsky and Rosen [1] has been the subject of many studies, principally due to the fact that they has emerged as indispensable physical resource for the performance of present-day quantum information tasks, such as quantum communication [2], quantum teleportation [3], quantum cryptography [4], superdense coding [5] and quantum computation [6]. Recently, much attention has been directed to understanding how these correlations behave in a relativistic setting [7]. From the practical viewpoint a good example is Relativistic Quantum Metrology, which exploits non-inertial effects on quantum entanglement to develop extremely high-precision parameter estimation protocols, with signal-to-noise ratios that may achieve the Heisenberg limit [8]. These metrology protocols have been employed to conceive precision measurements of Unruh temperatures and effects of gravity on entanglement [9][10][11], as well as to conceive novel schemes for gravitational wave detection which may provide feasible alternatives to experiments such as LIGO (Laser Interferometer Gravitational-Wave Observatory) and may be within technological reach in the near future [12][13].

However, the interest on relativistic effects on entanglement does not arise only from its role as a resource for quantum information tasks. Sometimes entanglement itself may actually encode the parameters of interest in relativistic settings. One example is given by the parameters of the large-scale spacetime metric. It has long been known that in the context of expanding spacetimes in general, pairs of entangled particles are dynamically created into modes of opposite momenta of a free scalar field [15] for instance. The amount of entanglement generated by a period of expansion may be determined by the spacetime metric. Therefore [16], [17], and [18] suggest that measurement of these quantum correlations offers a tool to estimate the parameters that characterize the scale factor. Moreover in [19][20] it is discussed that cosmic neutrinos may encode entanglement generated in the early universe epoch which could possibly survive to be detected, since they interact very weakly with other sources of energy and matter.

The role of interactions is evidently important in this proof-of-principle level as new phenomena occur, such as a competition between multiparticle production from the vacuum and thermalization [21][22]. In this context the interaction leads the system towards equilibrium, while the spacetime expansion deviate the system from equilibrium because of entropy production and particle creation. Interacting processes over this type of spacetime background can lead, depending on statistics, either to gravitational amplification or attenuation of particle creation [23], and the exact impact these effects will have on the amount of entropy and quantum correlations generated has many subtleties [24][25]. Moreover, in the context of inflationary theory [26][27], there are several works which point out how the different aspects of the quantum-to-classical transition of quantum inflaton fluctuations are realized and favored when the inflaton participates in interacting processes in general [28][30]. Therefore, the ubiquitous interactions between fields could possibly suppress the modewise entanglement initially present at one of them. If strong enough, they could render it impossible to use as suggested above.

Thus, it is interesting to quantify and understand the effect of self-interactions and interactions between quantum fields on particle creation, entropy generation and quantum entanglement during a period of spacetime expansion. Of course, treating interactions in quantum field theory over expanding spacetimes faces several technical difficulties, which tend to obscure the analysis of basic qualitative features of quantum information measures.

In this short letter we study a simple toy model with two scalar fields mutually interacting and generating bipartite correlations in an expanding spacetime. In particular, we investigate the effects that an expanding spacetime has on the interaction between the modes of a massless scalar field $\phi$ and of a massive scalar field $\psi$. We
evaluate the logarithmic negativity to leading order in perturbation theory and investigate the expansion effect on the entanglement between the fields $\phi$ and $\psi$. Our results suggest a possible competition between entanglement production by interaction and thermalization generated by expansion. This means that in the regime of smooth expansion $\frac{d\eta}{a}\ll 1$ the interaction provide an important contribution to the survival of the quantum correlations in the distant future.

The $\phi\psi$ model - Let us consider two real scalar fields $\phi$ and $\psi$ in a spatially flat Robertson-Walker spacetime with metric

$$ds^2 = a^2(\eta) \left( d\eta^2 - dx^2 \right),$$

where $a(\eta)$ is the scaler factor and $\eta = \int \frac{dt}{a(t)}$ is the conformal time ranging from $-\infty$ to $\infty$. The action of the system reads

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left( \partial_{\mu} \phi \partial^{\mu} \phi + \partial_{\mu} \psi \partial^{\mu} \psi + m^2 \phi \psi^2 + 2 \lambda \phi \psi \right),$$

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $\lambda$ is the coupling parameter normalized such that, $|\lambda| \ll 1$. The dynamics of the fields $\phi$ and $\psi$ in the interaction picture are governed by the covariant from Klein-Gordon equations in the curved spacetime

$$\Box \phi(\eta, x) = 0, \quad (2)$$
$$\Box + m^2)\psi(\eta, x) = 0, \quad (3)$$

and the state vector $|\Psi\rangle$ of the system satisfies the Schrodinger’s equation

$$H_I |\Psi\rangle = i\partial_\eta |\Psi\rangle, \quad (4)$$

where $H_I$ is the normal ordered interaction Hamiltonian

$$H_I = \lambda \int dx \sqrt{-g} \phi(\eta, x) \psi(\eta, x). \quad (5)$$

The canonical quantization of the fields $\phi$ and $\psi$ are identical to that in the free field case. Thus one has that

$$\phi(\eta, x) = \int dk (a_k \phi_k + a_k^\dagger \phi_k^*), \quad (6)$$
$$\psi(\eta, x) = \int dk (b_k \psi_k + b_k^\dagger \psi_k^*), \quad (7)$$

with

$$\phi_k(\eta, x) = \frac{e^{ikx}}{\sqrt{2\pi}} a_k(\eta), \quad (8)$$
$$\psi_k(\eta, x) = \frac{e^{ikx}}{\sqrt{2\pi}} v_k(\eta), \quad (9)$$

where the operators $a_k$, $a_k^\dagger$, $b_k$, and $b_k^\dagger$ satisfy the usual commutation relations $[a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = \delta_{k,k'}$, and $u_k$ and $v_k$ are solutions of the equations

$$u_k''(\eta) + k^2 u_k(\eta) = 0, \quad (10)$$
$$v_k''(\eta) + (k^2 + a^2(\eta)m^2) v_k(\eta) = 0. \quad (11)$$

We suppose that $a^2(\eta) = 1 + \epsilon(1 + \tanh(\rho \eta))$, where $\epsilon$ and $\rho$ controlling the volume and rapidity of the expansion, respectively. The spacetime becomes flat since $a^2(\eta)$ is sufficiently smooth and approaches constant values in the distant past $a^2(\eta \to -\infty) = 1$ and far future $a^2(\eta \to \infty) = 1 + 2\epsilon$, as illustrated in figure (1). In such asymptotic regions Poincaré invariance guarantees the existence of a time-like Killing vector field $\partial_\eta$ orthogonal to all spacelike hypersurfaces of constant conformal time, and therefore there is an unambiguous way to distinguish positive- and negative-frequency modes solution of the field equations (10) and (11).

![FIG. 1: Conformal factor for a toy model universe which possesses asymptotic regions.](image)

In this scenario in which spacetime possesses stationary asymptotic regions, the solutions of the equation (10) in the asymptotic regions are equivalent, i.e.,

$$u_k^{\text{in}} = u_k^{\text{out}} = \frac{e^{-ik\eta}}{\sqrt{2k}}. \quad (12)$$

Notice that in the particular case of two spacetime dimensions the theory of massless scalar field is conformally invariant, and as a consequence no particles are present in the asymptotic future. On the other hand, the solutions of the equation (11) in the asymptotic regions are

$$v_k^{\text{in}} = \frac{e^{-i\omega_k^{\text{in}} \eta}}{\sqrt{2\omega_k^{\text{in}}}}, \quad \text{in-region,} \quad (13)$$
$$v_k^{\text{out}} = \frac{e^{-i\omega_k^{\text{out}} \eta}}{\sqrt{2\omega_k^{\text{out}}}}, \quad \text{out-region,} \quad (14)$$

where $\omega_k^{\text{in}} = \sqrt{k^2 + m^2}$ and $\omega_k^{\text{out}} = \sqrt{k^2 + (1 + 2\epsilon)m^2}$. These asymptotic solutions are connected by a Bogoliubov transformation that only mixes modes of the same momentum $k$:

$$v_k^{\text{in}}(\eta) = \alpha_k v_k^{\text{out}}(\eta) + \beta_k v_k^{\text{out}*}(\eta), \quad (15)$$

where the Bogoliubov coefficients $\alpha_k$ and $\beta_k$ satisfy the normalization condition $|\alpha_k|^2 - |\beta_k|^2 = 1$. In this partic-
ular case they are readily evaluated to
\[
\alpha_k = \sqrt{\frac{\omega_k^{\text{out}}}{\omega_k^{\text{in}}}} \frac{\Gamma(1 - \frac{i\omega_k^{\text{in}}}{\rho})}{\Gamma(1 - \frac{i\omega_k^{\text{out}}}{\rho})},
\]
\[
\beta_k = \sqrt{\frac{\omega_k^{\text{out}}}{\omega_k^{\text{in}}}} \frac{\Gamma(1 - \frac{i\omega_k^{\text{in}}}{\rho})}{\Gamma(1 + \frac{i\omega_k^{\text{out}}}{\rho})},
\]
where \(\omega_{\pm} = \frac{1}{2}(\omega_k^{\text{out}} \pm \omega_k^{\text{in}})\). In the interaction picture, the Bogolyubov coefficients carry information only about \(\omega\) where later case they are readily evaluated to

**Entanglement state** - Now, let us assume that the composite system \(\phi\) and \(\psi\) in the distant past, is a global vacuum (initial condition) for a given mode \(|0_k^\phi;0_p^\psi\rangle\). This vacuum state is annihilated by both \(a_k\) and \(b_p\). We denote the \(\phi\) vacuum by \(|0_k^\phi\rangle\) and \(\psi\) vacuum by \(|0_p^\psi\rangle\). Thus the particle creation due to the mutual interaction can be calculated by evaluating the \(S\)-matrix to leading order in the interaction picture. Thus to lowest order in \(\lambda\),

\[
S = 1 - i \int_{-\infty}^{\infty} H_I d\eta = 1 - i\lambda \int d^2x \sqrt{-g} \phi \psi. \tag{17}
\]

To this order such an interaction produces particles in pairs as depicted in figure 2, where the probability amplitude is given by

\[
\langle 1_k^\phi;1_p^\psi |S|0_k;0_p\rangle = -2\lambda |\lambda| \int d^2x \sqrt{-g} \phi_k^* \psi_p,
\]

\[
= \lambda\delta(k + p) A(k, p), \tag{18}
\]

where

\[
A(k, p) = \frac{-i}{\pi} \int_{-\infty}^{\infty} d\eta a^2(\eta) n_k^\phi(\eta) n_p^\psi(\eta). \tag{19}
\]

**FIG. 2:** Particle creation out of the vacuum by mutual interaction.

Inserting the asymptotic mode functions \([12]\) and \([13]\) in \(A(k, p)\), we obtain \([15]\)

\[
A(k, p) = \frac{-i}{2\pi \sqrt{k \omega_p}} \int_{-\infty}^{\infty} d\eta a^2(\eta) e^{-i(k + \omega_p)} \pi(1 + \epsilon) \delta(k + \omega_p)
\]

\[
= \frac{\epsilon}{\sqrt{2k \omega_p}} \frac{1}{k + \omega_p} \sinh(\frac{\pi}{2\rho}(k + \omega_p)), \tag{20}
\]

The first term proportional to the delta function gives no contribution to pair creation process, as it amounts to a shift in the scale factor \([15]\). Thus we find

\[
A(k, p) = \frac{\epsilon}{\sqrt{2k \omega_p}} \frac{1}{k + \omega_p} \sinh(\frac{\pi}{2\rho}(k + \omega_p)). \tag{21}
\]

This term expresses a thermal like profile of the vacuum generated by the interaction over the spacetime evolution.

The initial vacuum state of the total system \(|0_k^\phi;0_p^\psi\rangle\) to lowest order in \(\lambda\) reads

\[
|\Psi\rangle = N|0_k^\phi;0_p^\psi\rangle \]

\[
+ \frac{1}{2!} \int dk dp |1_k^\phi;1_p^\psi S|0_k;0_p\rangle|1_k^\phi;1_p^\psi + ...|, \tag{22}
\]

where \(2!\) is the symmetry factor and \(N\) is the normalization factor

\[
N^{-2} = 1 + \frac{1}{2!} \int dk dp |1_k^\phi;1_p^\psi S|0_k;0_p\rangle|^2. \tag{23}
\]

Note that \([22]\) is a bona fide entangled state described by the Hilbert space \(H = H_\phi \otimes H_\psi\). This means that the mutual interaction generates bipartite quantum correlations between the fields \(\phi\) and \(\psi\).

Consider that the vacuum state \(|0_p^\psi\rangle\) in the asymptotic past correspond to a two-mode squeezed state from the point of view of an inertial observer in the asymptotic future

\[
|0_p^\psi\rangle = \sqrt{1 - \gamma_p} \sum_{n=0}^{\infty} \gamma_p^n |n_p^\psi, n_{-p}^\psi\rangle, \tag{24}
\]

where \(\gamma_p = \frac{\beta_p}{\sqrt{\alpha_p}}\frac{1}{2}. \) Since that we are working a single mode, we will drop the frequency index \(k\).

Similarly, the one-particle excitation in the in-vacuum \(|1_p^\psi\rangle\) evolves as

\[
|1_p^\psi\rangle = (1 - \gamma_p) \sum_{n=0}^{\infty} \gamma_p^n \sqrt{n + 1 + n + 1, n_{-p}^\psi}. \tag{25}
\]

However, note that for the field \(\phi\), \(|0_k^\phi\rangle = |0_k^\phi\rangle\).

Using the equations \([24]\) and \([25]\), we can rewrite the equation \([22]\) in terms of out-region Fock states for the field \(\psi\)

\[
|\Psi\rangle = N[\sqrt{1 - \gamma_p} \sum_{n=0}^{\infty} \gamma_p^n |0_k^\phi;0_p^\psi\rangle + \lambda \int dp A(p, -p) \times (1 - \gamma_p) \sum_{n=0}^{\infty} \gamma_p^n \sqrt{n + 1 + n + 1, n_{-p}^\psi} + ...],
\]

This state enable us to construct the density matrix of whole tripartite state \(\rho_{k,p,-p} = |\Psi\rangle \langle \Psi|\) which includes modes of the two fields. Since an inertial observer in the
out-region has no access to modes $-p$, the state $\hat{\rho}^{\phi\psi}_{k_{p}}$ will be projected into a mixed state by tracing over all states with modes $-p$

$$\hat{\rho}^{\phi\psi}_{k} = \text{Tr}_{-p}[\hat{\rho}^{\phi\psi}_{k_{p}}] = (1 - \gamma_{k}) \sum_{n=0}^{\infty} \gamma_{k}^{n} \hat{\rho}^{\phi\psi}_{n},$$

where

$$\hat{\rho}^{\phi\psi}_{n} = |0_{k_{n}}^{\phi}; n_{k}^{\psi}| \langle 0_{k_{n}}^{\phi}; n_{k}^{\psi}| + \lambda A(k) \sqrt{1 - \gamma_{k} \sqrt{n + 1}} |1_{k_{n}}^{\phi}; n + 1_{k}^{\psi}| \langle 1_{k_{n}}^{\phi}; n + 1_{k}^{\psi}| + \lambda A^{*}(k) \sqrt{1 - \gamma_{k} \sqrt{n + 1}} |0_{k_{n}}^{\phi}; n_{k}^{\psi}| \langle 0_{k_{n}}^{\phi}; n_{k}^{\psi}| + \lambda^{2}|A(k)|^{2} (1 - \gamma_{k})(n + 1) |1_{k_{n}}^{\phi}; n + 1_{k}^{\psi}| \langle 1_{k_{n}}^{\phi}; n + 1_{k}^{\psi}| + ...$$

(27)

Logarithmic negativity - The most adequate separability criterion to estimate the quantum correlation in mixed quantum states is the partial transpose criterion of Peres-Horodecki [31, 32]. This criterion states that, if a density matrix is entangled, then its partial transpose has some negative eigenvalues and hence lacks the positivity required by all density matrices. It follows that the positivity of the partial transpose (PPT) is a necessary condition for system, specifically bipartite systems of dimensionality $2 \times 2$ and $2 \times 3$. In higher dimensional systems it has been shown in [32] that there are entangled states with positive partial transpose. These states are known as bound entangled states. Thus, to quantify entanglement we use the logarithmic negativity defined as:

$$E_{N} = \log_{2}[1 + 2N],$$

(28)

where $N = \max\{0, -\sum_{j} \nu_{j}\}$ and $\nu_{j}$s are the negative eigenvalues of the partial transpose density matrix. Thus, the negative eigenvalues qualifies and quantifies the entanglement of quantum in a mixed state. It has been proved that this quantity exhibits monotonic behavior under Local Operation and Classical Communication (LOCC) operations or operators that conserves PPT [33].

The partial transpose of (27) is obtained by exchanging $|n_{k_{n}}^{\phi}; n_{k}^{\psi}\rangle \langle m_{k_{n}}^{\phi}; m_{k}^{\psi}| \rightarrow |m_{k_{n}}^{\phi}; m_{k}^{\psi}\rangle \langle n_{k_{n}}^{\phi}; n_{k}^{\psi}|$

$$\hat{\rho}^{\phi\psi^{T}}_{k} = (1 - \gamma_{k}) \sum_{n=0}^{\infty} \gamma_{k}^{n} \hat{\rho}^{\phi\psi^{T}}_{n},$$

(29)

with

$$\hat{\rho}^{\phi\psi^{T}}_{n} = |0_{k}^{\phi}; n_{k}^{\psi}| \langle 0_{k}^{\phi}; n_{k}^{\psi}| + \lambda A(k) \sqrt{1 - \gamma_{k} \sqrt{n + 1}} |1_{k}^{\phi}; n + 1_{k}^{\psi}| \langle 1_{k}^{\phi}; n + 1_{k}^{\psi}| + \lambda A^{*}(k) \sqrt{1 - \gamma_{k} \sqrt{n + 1}} |0_{k}^{\phi}; n_{k}^{\psi}| \langle 0_{k}^{\phi}; n_{k}^{\psi}| + \lambda^{2}|A(k)|^{2} (1 - \gamma_{k})(n + 1) |1_{k}^{\phi}; n + 1_{k}^{\psi}| \langle 1_{k}^{\phi}; n + 1_{k}^{\psi}| + ...$$

(30)

This matrix is infinite dimensional, however it has a block-diagonal structure which allows us to calculate the eigenvalues analytically block by block. Note that the eigenvalues corresponding to the first and last diagonal entries of the matrix are always positive. Therefore, we must simply diagonalize the matrix

$$\hat{\rho}^{\phi\psi^{T}}_{n} = \left( \lambda^{2}|A(k)|^{2} \frac{1 - \gamma_{k} \sqrt{n + 1}}{\gamma_{k}} \lambda A(k) \sqrt{1 - \gamma_{k} \sqrt{n + 1}} \right)$$

It follows that the eigenvalues of the density matrix $\hat{\rho}^{\phi\psi^{T}}_{n}$ in the $(n, n + 1)$ sector are

$$\nu_{\pm} = \frac{(1 - \gamma_{k}) \gamma_{n}^{\nu} \left[ \lambda^{2}A^{2}(k)n(1 - \gamma_{k}) \pm \sqrt{Z_{n}} \right]}{2}$$

with

$$Z_{n} = \left( \frac{\lambda^{2}A^{2}(k)n(1 - \gamma_{k})}{\gamma_{k}} \right)^{2} + 4\lambda^{2}A^{2}(k)(1 - \gamma_{k}).$$

Note that the eigenvalues depend on the values of $\lambda, \epsilon$ and $\rho$. In particular, for $\lambda, \epsilon$ and $\rho$ finites, one of the eigenvalues is always negative. Only in the limit $\epsilon, \rho \rightarrow \infty$ could the negative eigenvalue vanishing ($\nu \rightarrow 0$). It follows that the logarithmic negativity is given by

$$E_{N} = \log_{2}[1 + \lambda^{2}A^{2} + \gamma_{k} + \sum_{n=0}^{\infty} (1 - \gamma_{k}) \gamma_{n}^{\nu} \sqrt{Z_{n}}].$$

(32)

By a numerical analysis of this expression, summarized in figure 3, our first observation is that a degradation of the entanglement generated by the interaction occurs during the period of expansion. Figure 3 shows that if $\lambda$ is fixed, the degree of entanglement is reduced as the parameter $\rho$ increases. On the other hand, we observe that for small values of $\rho$ there is an enhancement in the amount of the quantum correlation as $\lambda$ increases. This is due to the fact that the interaction destroys the conformal symmetry of the theory.

FIG. 3: Logarithmic negativity as function of the $\rho$ for different coupling constants $0.0005 \leq \lambda < 0.001$ with $k = m = 1$ and $\epsilon = 40$, where higher spectral peaks correspond to strong couplings.

Notice that the effects of degradation, counteracting entanglement production by interaction, is dominant in
the fast expansion regime, so that entanglement sudden death is expected in the distant future. However, in the limit of smooth expansion $\frac{\rho}{\omega} \ll 1$ the quantum correlation between the quantum fields in the early universe could survive up to the distant future despite decoherence effects due to interactions. This means that in principle these quantum correlations are robust enough to be detectable. Since recent researches has discussed that an initially entangled state between two free massive scalar fields in de Sitter space might affect cosmological observables, such as the power spectrum and other correlation functions of the inflaton $\lambda$.

**Conclusion** - In summary, we have studied a simple toy model of two scalar fields interacting in an expanding spacetime. We applied a $S$-matrix scheme in the interaction picture to investigate the effect of the dynamics of spacetime expansion in quantum entanglement generated by mutual interaction. In addition, we computed the logarithmic negativity to leading order in the coupling constant $\lambda$. Our results show that an increase in the expansion parameter produces a decreasing in the quantum entanglement between two scalar fields whereas increasing the coupling constant within the limit of perturbation theory enhances quantum entanglement.

These results suggest that during the period of cosmic expansion, the interaction is important to the survival of the quantum correlations. More realistic extensions of the ideas explored here may lead to interesting observable effects. This is interesting, since entanglement and quantum coherence are affected by the dynamics spacetime. Another important aspect of this problem that deserves further study is related to the nature of the interaction. One possible avenue for further research along this line is to study the effect of other types of interactions, for instance, weak interaction responsible by radioactive decay, Yukawa interaction $(g\phi\psi\bar{\psi})$, pion-proton scattering $(g_\pi^2\psi\bar{\psi})$, electromagnetic interaction, and a number of other decay process.

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