Statistics of incomplete quotients of continued fractions of quadratic irrationalities

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Abstract

V.I. Arnold has experimentally established that the limit of the statistics of incomplete quotients of partial continued fractions of quadratic irrationalities coincides with the Gauss–Kuz’min statistics. Below we briefly prove this fact for roots of the equation \(rx^2 + px = q\) with fixed \(p\) and \(r\) \((r > 0)\), and with random \(q\), \(q \leq R\), \(R \to \infty\). In Section 3 we estimate the sum of incomplete quotients of the period. According to the obtained bound, prior to the passage to the limit, incomplete quotients in average are logarithmically small. We also upper estimate the proportion of the “red” numbers among those representable as a sum of two squares.

Keywords: periodic continued fractions, Arnold’s conjecture, Gauss–Kuz’min statistics, Bykovskii’s theorem, Farey fraction, Pell’s equation.

MSC classes: 11T06; 11T24; 37E15.

1 The statement of the main results and their consideration

The papers, monographs, and reports of V.I. Arnold dedicated to the statistics of periods of continued fractions and their multidimensional generalizations (see [1]–[7]) contain a vast number of experimental facts and outline prospects for further investigations. The first step in this direction is the proof of the following assertion (established by V.I. Arnold): the statistics of incomplete quotients of the continued fraction of solutions to a random quadratic equation with integer coefficients, proceeding to the “thermodynamic” limit, turns into the Gauss–Kuz’min statistics. Unfortunately, due to circumstances the proof of this assertion performed by V.A. Bykovskii and his followers was not published. Below we adduce an elementary proof of the one-parameter version of this assertion.

Any number \(x \in \mathbb{R}\) is representable as a continuous fraction in the form

\[
x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = [a_0; a_1, a_2, \ldots],
\]

where \(a_0 \in \mathbb{Z}, a_i \in \mathbb{N}\) for all \(i \geq 1\).

Following Khinchin [8], we denote by \(E(s_1, s_2, \ldots, s_k; A_1, A_2, \ldots, A_k)\) the set of real numbers which satisfy the conditions

\[a_{s_1} = A_1, a_{s_2} = A_2, \ldots, a_{s_k} = A_k;\]

here, certainly, all \(A_i\) and \(s_i\) are positive integers, and all \(s_i\) are different. This set is the union of a countable number of intervals [8].

Let \(I_{A_1, \ldots, A_k}^{s_1, \ldots, s_k}(x)\) stand for the indicator function of the corresponding set \(E:\)

\[
I_{A_1, \ldots, A_k}^{s_1, \ldots, s_k}(x) = \begin{cases} 
1, & \text{if } x \in E(s_1, \ldots, s_k; A_1, \ldots, A_k), \\
0, & \text{otherwise},
\end{cases}
\]

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Let
\[ \mu(s_1, \ldots, s_k) = \int_0^1 f_{A_1, \ldots, A_k}(x) \, dx \]  
(1)

be the Lebesgue measure of the set of points of the interval [0, 1] which belong to \( E_{A_1, \ldots, A_k} \).

Let \( x_+(q) \) "+" stand for a root of the quadratic equation \( rx^2 + px = q \), i.e., \( x_+(q) = \frac{-p + \sqrt{p^2 + 4rq}}{2r} \), where \( \Delta = p^2 + 4rq \). In what follows we understand the random choice as the equiprobable sampling from a finite set.

**Theorem 1.** Let \( r \in \mathbb{N}, p \in \mathbb{Z}, q \in \{1, \ldots, R\} \); here \( r \) and \( p \) are fixed, \( q \) is randomly chosen. Let \( P_{R}^{r,p}(s_1, \ldots, s_k) \) stand for the probability that \( x_+(q) \in E_{s_1, \ldots, s_k} \). Then
\[ \lim_{R \to \infty} P_{R}^{r,p}(s_1, \ldots, s_k) = \mu(s_1, \ldots, s_k). \]

**Remark 1.** One can replace the condition \( q \in \{1, \ldots, R\} \) with that \( q \in \mathbb{Z}, q \leq R, \, \Delta \geq 0; \) evidently, this does not affect the limit probability.

Using the Kuz'min theorem on the exponential convergence to \( \ln(1 + \frac{1}{A(A+2)})/\ln 2 \) one can easily obtain the following assertion.

**Corollary of Theorem 1.** Let \( r, p, \) and \( q \) be chosen in the same way as in Theorem 1; let the number \( s, \, s \in \{1, \ldots, n\} \), be chosen randomly. Then as \( R \to \infty \) the probability that \( a_s = A \) tends to \( \ln(1 + \frac{1}{A(A+2)})/\ln 2 + O(1/n) \).

**Remark 2.** The probability \( \sum_{s=1}^n P_{R}^{r,p}(a_s)/n \) (considered in the corollary) with fixed \( R \) and \( n \to \infty \) tends to the probability that a number chosen randomly from the period of the continued fraction \( x_+(q) \) equals \( A \).

The proof of Theorem 1 is based on the following fundamental idea: one can treat integral (1) not only as the Lebesgue integral, but also as the Riemann integral (considered in the school course on mathematics). A certain special choice of integral sums actually leads to simplified versions of Theorem 1. In a general case, it is convenient to perform the proof on the base of the theory of divergent series.

The idea to apply the integral Riemann sums is not new; it was used implicitly in the proof of a similar correlation in the case of rational numbers with a fixed denominator [9]. However, usually one uses another technique for this purpose. The asymptotic fairness of the Gauss–Kuz’min statistics in the case of a fixed denominator follows from results obtained by L. Heilbronn and J.W. Porter (see [10]). Not long time ago, V.A. Bykovskii, M.O. Avdeeva, A.V. Ustinova [11], [12], [13] obtained the limit statistics for finite fractions, whose numerators and denominators belong to a sector or to an arbitrary expanding domain.

Let us explain the (mentioned in Remark 2) connection with the statistics of the period of a continued fraction. In accordance with the Lagrange theorem, a continued fraction for the quadratic irrationality (and only for it) is periodic. Therefore the continued fraction for \( x_+(q) \) takes the form
\[ \frac{-p + \sqrt{p^2 + 4rq}}{2r} = [a_0; \ldots, a_m, a_{m+1}, \ldots, a_{m+T}, a_{m+1}, \ldots, a_{m+T}], \]
\[ = [a_0; \ldots, a_m, [a_{m+1}, \ldots, a_{m+T}]], \]
\[ \text{(2)} \]
where \( T = T(r, p, q) \) is the length of the period of the continued fraction, \( m = m(r, p, q) \) is the length of the preperiod. Evidently, one can represent the probability that with random \( q, \, q \leq R, \) a number randomly chosen from the period \( a_{m+1}, \ldots, a_{m+T} \) coincides with \( A \) as the limit \( \lim_{n \to \infty} \sum_{s=1}^n P_{R}^{r,p}(a_s)/n \). Unfortunately, we did not succeed in proving that this probability tends to \( \ln(1 + \frac{1}{A(A+2)})/\ln 2 \) as \( R \to \infty \).

In papers [2], [3] V.I. Arnold experimentally studies \( T(1, p, q) \). He establish that in average this period increases proportionally to the square root of the discriminant:
\[ \hat{T} \sim \text{const} \sqrt{\Delta}. \]
\[ \text{(3)} \]
Let \( T_0(q) = T(1, 0, q) \) be the period of the continued fraction of the square root of \( q \). Let \( \hat{T}'(Q) = \frac{\sum_{q=1}^{Q} T_0(q) / Q}{Q} \). This kind of an average was studied experimentally by experts in the theory of numbers, who stated the conjecture

\[
\hat{T}_0(Q) \sim \text{const} \sqrt{Q} \log^\alpha(Q), \quad \text{where } \alpha < 0
\]

(see [14] and references therein).

One can easily obtain the following upper bound for \( \hat{T}_0(Q) \) [15]:

\[
\hat{T}'(Q) < \text{const} \sqrt{Q}.
\]

In [16] (see also [14]) one proves that the left-hand side of this inequality is asymptotically small in comparison with the right-hand one, moreover, if the extended Riemann hypothesis is fulfilled, then the order of their difference is not less than \( \log(Q) \log 2 - \varepsilon \).

The well-known lower estimate for \( \hat{T}_0(Q) \) significantly differs from the right-hand side of (4). In [14] one only proves that \( \hat{T}_0(Q) > \text{const} \log(Q) \).

One can construct a continued fraction for a quadratic irrationality, using the algorithm of successive approximations by mediants (the Farey fractions). Based on the connection of this algorithm with explicit representations of values of an integer-valued quadratic form (see [23]), in Section 3 we obtain the following bound for the sum of partial quotients of the quadratic irrationality within one its period.

**Theorem 2.** Let \( rx^2 + px = q \) be an arbitrary quadratic equation with integer coefficients and real irrational roots. In denotations of (2), (5) we have the inequality

\[
\sum_{i=1}^{\lfloor \sqrt{Q} \rfloor} \tau(Q - u^2) + f(\Delta/4) \leq f(Q), \quad \text{where}
\]

\[
f(Q) = 2D(Q) + \tau(Q), \text{ if } Q \text{ is integer; }
\]

\[
f(Q) = 2 \sum_{i \in \{1, 3, \ldots\}, \; i^2 < 4Q} \tau(Q - i^2/4), \text{ otherwise.}
\]

This inequality was first obtained by Hickerson in 1973 [18] (see also [19]). Later Western researchers [20], [21] succeeded in proving the correlation

\[
T_0(Q) = O(\sqrt{Q} \ln Q).
\]

Earlier mathematicians of the Leningrad scientific school treated it as a direct corollary of the Dirichlet formula and results of [19] (see [24]). Moreover, if the extended Riemann hypothesis is fulfilled, then one can analogously obtain the double logarithmic bound

\[
T(r, p, q) = O(\sqrt{\Delta} \ln \Delta).
\]

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\sum_{i=1}^{T(r, p, q)} a_{m(r, p, q) + i} \leq f(\Delta/4), \quad \text{where}
\]

\[
f(Q) = 2D(Q) + \tau(Q), \text{ if } Q \text{ is integer; }
\]

\[
f(Q) = 2 \sum_{i \in \{1, 3, \ldots\}, \; i^2 < 4Q} \tau(Q - i^2/4), \text{ otherwise.}
\]
In the case of an odd period $T(r, p, q)$ one can improve this bound, namely, divide the right-hand side of inequality (8) by two.

According to the results of computer experiments, for almost all roots of prime numbers in the form $4k + 3$, as well as the roots of the doubled numbers in the same form, the left-hand side of inequality (8) is half as large as the right-hand one.

Problem 5 in the list of problems stated by Arnold in [7] implies the estimation of rate of growth of the average value of elements of the period of a continued fraction. To put it more precisely, the problem is to solve the alternative between the power rate of growth and the logarithmic one. Comparing inequality (8) with formula (6) and the results of the numerical tests (3, 4), we conclude that the mean value $\hat{a}$ is logarithmically small as against $\Delta$.

Note that the geometric method used for the proof of Theorem 2 enables one to easily obtain assertions on the palindromicity of continued fractions in cases when $p = 0$ or $r = 1$ (see remarks 5 and 6). We are going to dedicate a separate paper to the study of problem 12 in [7].

Studying periods lengths $T_0(Q)$, V.I. Arnold stated the problem of the constructive definition of the set $K$ of the so-called “red” numbers: $K = \{ Q : Q \in \mathbb{N}, T_0(Q) \text{ is odd}\}$.

The set $K$ is a subset of the totality $M$ of positive integers representable as a sum of two squares. The explicit representation of the set $M$ is well known (see, for example, [25]). In particular, prime numbers $Q$ of the set $M$ obey the formula: $Q \mod 4 \neq 3$.

Let $K_n$ (or $M_n$) stand for the part of the set $K$ (the set $M$) which consists of numbers not greater than $n$. According to the Arnold conjecture, the following limit exists:

$$\lim_{n \to \infty} \frac{K_n}{M_n} = c.$$

The sets $K$ and $M$ are closely connected with the generalized Pell equation:

$$x^2 - Qy^2 = -1. \quad (11)$$

**Theorem 3.**

1. $Q \in K \iff$ equation (11) is solvable in integer numbers.
2. $Q \in M \iff$ equation (11) is solvable in rational numbers.
3. Let $P$ be the set of prime numbers. With sufficiently large $n$,

$$|K_n| < |M_n| \prod_{p:p \in P, \ p \mod 4 \neq 1} (1 - \frac{1}{p^2}).$$

4. $Q \in K \cap P \iff Q \in P, \ Q \mod 4 \neq 3 \iff Q \in M \cap P$.

One can easily calculate that

$$\prod_{p:p \in P, \ p \mod 4 \neq 1} (1 - \frac{1}{p^2}) = 0.64208 \ldots.$$

This gives a rough bound for $c$; computer tests performed for all $n$ from $10^5$ to $10^6$ show that $0.47 < K_{M_n} < 0.48$.

Theorem 3.1 is well known for experts in the theory of numbers (see, for example, [17, Theorem 12]), as well as the absence of simple solvability criteria for equation (11) (i.e., for the membership of $K$).

The assertion about the sufficiency in Theorem 3.2 is evident. The proof of the necessity is reduced to the proof of the following property: if a positive integer is representable as a sum of two squares of rational numbers, then it is representable as a sum of two squares of integers. The proof for the case of two squares verbatim coincides with that adduced in [24, Appendix to Chapter 4] for the case of three squares.

Theorem 3.3 follows from two evident corollaries of the previous items of Theorem 3. Apparently, Theorem 3.2 implies that the set $M$ contains sets of all numbers in the form $4K, 9K, 49K$, etc (here we
restrict ourselves only with the multiplication by squares of prime numbers, for which the residue of division by 4 differs from 1). On the other hand, Theorem 3.1 implies that \( K \) does not intersect with these sets. To put it more precisely, if \( Q \in K \), then \( Q \) is neither divisible by 4 nor by all prime \( p \) in the form \( 4k + 3 \). Otherwise, considering equality (11) modulo 4 or modulo \( p \), we get the equation \( x^2 \equiv -1 \); it is well known that the latter equation is neither solvable modulo 4 nor modulo \( p \).

Now Theorem 3.4 is reduced to the assertion that all primes in the form \( 4k + 1 \) belong to \( K \). This fact was proved by Legendre (see \[25\]. See \[17\]) for the proof. He used it in order to solve the inverse problem on the representation of such a number as a sum of two squares.

Note that Theorem 3 is absent in author’s article with the same title submitted to the journal “Functional Analysis and Its Applications.”

This paper is written under the bright impression of (video) lectures delivered by V.I. Arnold \[4\], \[5\], \[6\]. It resulted from the addition to these lectures which, in turn, appeared as a result of the performed numerical experiments and the study of remarkable books \[9\], \[23\], \[17\].

### 2 How to apply the condition “accurate to a zero-measure set” in a discrete case (the proof of Theorem 1)

Let us first prove Theorem 1 in the case of the simplest quadratic irrationality \( \sqrt{q} \).

**Lemma 1.** Let \( q \) be arbitrarily chosen from the set \( \{n^2 + 1, n^2 + 2, \ldots, n^2 + 2n\}, \ n \in \mathbb{N} \). Denote by \( P_n(\sum_{i=1}^{k} \frac{A_i}{B_i}; B_i \neq A_i) \) the probability that \( \sqrt{q} \in E(\sum_{i=1}^{k} \frac{A_i}{B_i}) \). Then

\[
\lim_{n \to \infty} P_n(\sum_{i=1}^{k} \frac{A_i}{B_i}) = \mu(\sum_{i=1}^{k} \frac{A_i}{B_i}),
\]

**Proof of Lemma 1.** Let us first note that under the assumptions of the lemma \( |q| = n \). Therefore, by definition we have

\[
P_n(\sum_{i=1}^{k} \frac{A_i}{B_i}) = \frac{1}{2n} \sum_{i=1}^{2n} E(\sum_{i=1}^{k} \frac{A_i}{B_i}) \equiv \frac{1}{2n} \sum_{i=1}^{2n} + 1 \frac{B_i}{B_i} \in E(\sum_{i=1}^{k} \frac{A_i}{B_i}); B_i \neq A_i) \equiv \frac{1}{2n} \sum_{i=1}^{2n} + 1 \frac{B_i}{B_i} \in E(\sum_{i=1}^{k} \frac{A_i}{B_i}); B_i \neq A_i), \text{ i.e., as the union of a countable number of intervals. Consequently, the function } I_{A_1,\ldots,A_k}(E(\sum_{i=1}^{k} \frac{A_i}{B_i})) \text{ is countably continuous, therefore, by the Lebesgue theorem one can treat integral } I_{A_1,\ldots,A_k}(E(\sum_{i=1}^{k} \frac{A_i}{B_i})) \text{ as the Lebesgue integral, but also as the Riemann integral. In order to calculate it, we divide the segment } [0, 1) \text{ onto } 2n \text{ equal parts. We choose } \sqrt{n^2 + i} - n \text{ as the point, where we calculate the integrand on the interval } \left(\frac{i-1}{2n}, \frac{i}{2n}\right) \text{ (one can easily make sure that } \frac{i-1}{2n} < \sqrt{n^2 + i} - n < \frac{i}{2n} \text{ for all } i = 1, \ldots, 2n). \text{ Then the value of the Riemann integral sum coincides with the right-hand side of equality } (12). \text{ Lemma 1 is proved.}

Now the proof of Theorem 1 in the case \( r = 1, p = 0 \) follows from several remarks.

**Remark 3.** For an arbitrary sequence \( s_n \) and any sequence of nonnegative numbers \( t_n \) such that \( \sum_{n=1}^{\infty} t_n = \infty \), the convergence \( s_n \to \mu \) implies that

\[
S_n = \frac{t_1s_1 + \ldots + t_ns_n}{t_1 + \ldots + t_n} \to \mu.
\]

In the theory of divergent series this property is known as the regularity of the Riesz summation method [26, Chap. 3, Theorem 12].

**Remark 4.** The assertion of Theorem 1 on the convergence of a sequence \( P_n \) to \( \mu \) follows from the convergence of a subsequence \( P_{n_i} \) to \( \mu \), if \( \lim_{i \to \infty} \frac{n_{i+1}}{n_i} \to 1 \).

Really, it is evident that

\[
\frac{n_i}{R} P_{n_i} \leq P_R \leq \frac{n_{i+1}}{R} P_{n_{i+1}} \text{ with } R \in \{n_i, n_i + 1, \ldots, n_{i+1}\}.
\]
In Remark 4 and in what follows we omit fixed parameters of functions $P$, $\mu$, and $P'$.

In order to complete the proof of Theorem 1 in the case $r = 1$, $p = 0$, it remains to note that by definition

$$P^1_{R^2 + 2R} = \sum_{n=1}^{R} \frac{2n}{R^2 + 2R} P_n', \quad \text{i.e.,}$$

$$\frac{R^2 + R}{R^2 + 2R} P_{1,0}^{R^2 + 2R} = \sum_{n=1}^{R} \frac{2n}{R^2 + R} P_n'$$

(14)

According to Lemma 1 and Remark 3, the limit of the right-hand side of equality (14) for $R \to \infty$ coincides with the corresponding value of $\mu$, and the limit of the left-hand side (due to Remark 4) is $\lim_{R \to \infty} P_R^{1,0}$.

One can prove Theorem 1 analogously in the case $r = p = 1$. According to [3], one can reduce the general proof with $r = 1$ to these two cases. Unfortunately, partitions of the integration interval $[0, 1)$ onto equal parts do not enable us to prove Theorem 1 with $r > 1$. More intricate techniques are necessary.

**Proof of Theorem 1 in the general case.** Let $n \in \mathbb{N}$. In order to calculate integral (1), let us divide the segment $[0, 1)$ onto $p + r + nr$ parts with left ends at the points $x_+(q) - n$; here $q = i - 1 + n(p + nr)$, $i$ stands for the number of an interval. Let us calculate the value of the integrand on the interval at the same point $x_+(q) - n$ (further $x(q) \equiv x_+(q)$). We obtain

$$\mu = \lim_{n \to \infty} \sum_{q=n(p+nr)}^{(n+1)(p+(n+1)r)-1} (x(q+1) - x(q)) I(x(q)).$$

(15)

In accordance with Remark 3 and due to the boundedness of the sums in the right-hand side of (15), we have

$$\mu = \lim_{n \to \infty} \sum_{q=1}^{n(p+nr)} \frac{x(q) - x(q-1)}{n} I(x(q-1)) = \lim_{N \to \infty} \sum_{q=0}^{N-1} \frac{x(q+1) - x(q)}{[x(N)]} I(x(q)).$$

Evidently, in the denominator of the latter expression we can write $x(N)$ in place of $[x(N)]$. As a result, we get the limit in form (13), where $t_i = x(i) - x(i-1)$. It is well known that (see [26, Chap. 3, theorem 14])

$$S_N = \frac{t_1 I_1 + \ldots + t_N I_N}{t_1 + \ldots + t_N} \to \mu \quad \Leftrightarrow \quad S'_N = \frac{t'_1 I_1 + \ldots + t'_N I_N}{t_1 + \ldots + t_N} \to \mu,$$

provided that the following conditions hold:

$$\frac{t_{N+1}}{t_N} \leq \frac{t'_{N+1}}{t'_N},$$

$$\sum_{q=1}^{N} t_q/t_N \leq \text{const} \sum_{q=1}^{N} t'_q/t'_N.$$

Let us verify them in the case $t_q = x(q) - x(q-1)$, $t'_q \equiv 1$. The first condition $x(q-1) + x(q+1) \leq 2x(q)$ follows from the convexity of the function $x(q)$, the second one does from the existence of the limit

$$\lim_{N \to \infty} N \left(1 - \frac{x(N-1)}{x(N)}\right) = 1/2.$$ 

Therefore, we get

$$\mu = \lim_{N \to \infty} \sum_{q=0}^{N-1} I(x(q))/N = \lim_{R \to \infty} \sum_{q=1}^{R} I(x(q))/R.$$ 

By definition $\sum_{q=1}^{R} I(x(q))/R = P_R^{p,0}$. Theorem 1 is proved.

3 The gradual “Nose-Hoover” algorithm (the proof of Theorem 2)

"... and before he knew what he was doing he lifted up his trunk and hit that fly dead with the end of it."

R. Kipling. “The Elephant’s Child”

In what follows, without loss of generality, we consider only continued fractions of positive irrational numbers.
The standard “Nose-Hoover” algorithm for finding a continued fraction of a real number $x$ was proposed by Klein in 1895 [25] (the term was introduced by B.N. Delone). It is well known [11] that this algorithm is reduced to the geometric method which constructs the boundary of the convex shell of the set of integer nonnegative points $(u, v)$ located above (below) the straight line $v = xu$. Let $e_0$ stand for the vector $(0, 1), e_1 = (1, 0)$. Put

$$e_{n+1} = e_{n-1} + a_{n-1} e_n, \quad n = 1, 2, \ldots,$$

(16)

where $a_{n-1}$ is the maximal integer such that the vector $e_{n+1}$ lies below (for odd $n$) or above (for even $n$) the straight line $v = xu$. Klein noted that $a_n, n = 0, 1, \ldots$, coincide with partial quotients of the continued fraction of the number $x$. The geometric algorithm results in two polylines which represent parts of sails of the continued fraction.

We need a slightly modified algorithm which results in one infinite polyline $L$, originating at the point $(1, 1)$, whose segments are the vectors $a_{n-1} e_n, n = 1, 2, \ldots$. Evidently, $e_{n+1} + e_n = e_n + e_{n+1}$. Consequently, if in the standard “Nose-Hoover” algorithm (16) we add the extra vector $e_n$ and thus go out of the line $v = xu$ (i.e., we consider the vector $e_{n-1} + (a_{n-1} + 1)e_n$), then we get the first integer point on the segment of the polyline of another sail constructed at step $n + 1$. This fact justifies a simple geometric algorithm which constructs the polyline $L$.

We begin the construction process with the point $(1, 1)$; it is convenient to connect it with the origin of coordinates by the segment which does not enter in $L$. As the main “constructive” term at the first (n-th) step we choose the vector $e_1$ (the vector $e_n$). We add this vector till we go out of the line $v = xu$. The newly added vector $e_{n+1}$ is directed from the origin of coordinates to the point obtained as a result of the latter addition up to the step out of the line. See Fig. 1 for the first segments of the polyline for $x = \sqrt{2}$.

For convenience, we assume (if the contrary is not specified) that the zero partial quotient (i.e., $[x]$) differs from zero. In order to provide the correspondence to the indices of partial quotients, it is also convenient to begin the enumeration of segments of the polyline $L$ with zero.

Let us first consider the approximation of a number $x$ by mediants. The study of series of mediants is usually connected with the names of the German mathematician Stern, the French watchmaker Brocot (he described this series and applied it in clock manufacturing) or (more often) the English geologist Farey. The latter paid attention to the “curious property of vulgar fractions” (this was the title of a short letter of Farey published in “Philosophical Magazine” in 1816). Actually, the Farey sequence was described by Haros (see [27] p. 36–37); he considered its application in 1802. Even ancient Greek mathematicians were able to enumerate rational numbers with the help of mediants.

Let $f_1 = \frac{u_1}{v_1}, f_2 = \frac{u_2}{v_2}$ be irreducible fractions such that $v_1, u_1, v_2, v_2 \geq 0$ and $f_1 < f_2$. Let us define the operation $\downarrow$ of finding the mediant (the “insertion” operation) by the formula $f_1 \downarrow f_2 = \frac{u_1 + u_2}{v_1 + v_2}$, where $v = v_1 + v_2, u = u_1 + u_2$. Earlier we used the denotation introduced by A.A. Kirillov. Evidently, in the geometric representation of the fraction $\frac{u}{v}$ as the integer vector $(u, v)$ the operation $\downarrow$ corresponds to the addition of vectors $(u_1, v_1)$ and $(u_2, v_2)$. The obtained diagonal of the parallelogram appears to be “inserted” between its sides.

Let $x \in (f_1, f_2), f_3 = f_1 \downarrow f_2$. Since $f_3 \in (f_1, f_2)$, one can treat $f_3$ as an approximation of the number $x$. If $x \in (f_1, f_3)$, then we put $f_4 = f_1 \downarrow f_3$, otherwise we do $f_4 = f_3 \downarrow f_2$. The process of the approximation of the number $x$ by mediants consists in the repetition of these operations for the corresponding intervals.

The condition $x > 0$ means that $x \in \left(\frac{u}{v}, \frac{u+1}{v+1}\right)$. The initial approximation of $x$ for such an interval is the fraction $\frac{u}{v}$. One can easily see that for the mapping

$$\frac{v}{u} \quad \leftrightarrow \quad (u, v)$$

the geometric representation of the algorithm for approximating the number $x$ by mediants is completely identical to the algorithm for constructing the polyline $L$. The technical distinction consists in the following fact: earlier we constructed each segment of the polyline $L$ “at once”, but now it “grows gradually” due to the stepwise addition of the next vector $e_i$. We call a simplified version of this algorithm as applied to quadratic irrationalities the gradual “Nose-Hoover” algorithm.

Let $(u_n, v_n)$ stand for the $n$th integer point on the polyline $L, n = 0, 1, 2, \ldots$. We obtain it at the $n$th step of the gradual “Nose-Hoover” algorithm. In Fig. 1, for example, $(u_0, v_0) = (1, 1), (u_1, v_1) = (1, 2), (u_2, v_2) = (2, 3), (u_3, v_3) = (3, 4), (u_4, v_4) = (5, 7)$, etc. The number $i$ of the vector $e_i$ which we add at the $n$th step, generally speaking, does not exceed $n$; the identity $i \equiv n$ takes place only for the golden ratio.
Figure 1: The gradual “Nose-Hoover” algorithm for $x = \sqrt{2}$
The main inconvenience of the standard “Nose-Hoover” algorithm consists in the following fact: the polylines very quickly tend to the straight line \( u = xv \) and the geometric illustration requires the high accuracy (“all noses become longer”). In the case of a quadratic irrationality we can use only conventional illustrations which allow us to “visualize” the topology of polylines \( L \) (and therefore the continued fraction) without its exact representation on the integer lattice. In essence, these illustrations coincide with the approach (described in [23]) to the visualization of values of a quadratic form on the banks of its river (we use the terminology of [23]). Below we show the geometric juxtaposition.

Let us draw one more vector which originates from the point \((u_n, v_n)\); denote it by \( e'_n \). The vector \( e'_n \) is defined by the condition \( e'_n + e_i + e''_n = 0 \), where \( e_i \) is the vector which originates from the point \((u_n, v_n)\) and goes along the polyline \( L \), \( e''_n = -(u_n, v_n) \) (we considered this vector earlier). The collection of vectors \(\{e'_n, e_i, e''_n\} \), where each one is defined accurate to the multiplier \((-1)\), is a superbasis [23]. This means that any pair of these vectors generates the whole integer lattice. The transition from the point \((u_n, v_n)\) to that \((u_{n+1}, v_{n+1})\) corresponds to the replacement of one superbasis with another one; the latter differs from the initial superbasis only in one of three its elements. Thus, for the polyline \(L\) in Fig. 1 the initial superbasis is \(\{(0,1), (1,0), (1,1)\}\), the 1st one is \(\{(1,1), (1,0), (1,2)\}\), the 2nd one is \(\{(1,2), (1,1), (2,3)\}\), the 3rd one is \(\{(1,1), (2,3), (3,4)\}\), etc.

Let us extend vectors \(e'_n\) up to the rays which originate at the points \((u_n, v_n)\). As a result, the first quadrant appears to be divided onto disjoint connected domains (see Fig. 1). Let us associate each vector \((u_n, v_n)\) with a domain, whose boundary contains all points of polylines which include the vector \((u_n, v_n)\) in the corresponding superbasis. In the picture these domains look like narrow “crevices” located at the north-east of the points \((u_n, v_n)\). We associate the domains which border on the coordinate axes with the unit vectors of these axes.

Without loss of generality, in what follows we assume that in this statement of Theorem 2 the coefficient \(r\) is positive.

If, in addition, the equation has a unique positive root, then

\[
\text{the point}(u_n, v_n)\text{ is located above the line } v = x_+(q)u \iff r v_n^2 + pv_n u_n - q u_n^2 > 0.
\]

If both roots are positive, then this equivalence does not necessarily take place for all \(n\). The quadratic form is also positive at the points located below the line \(v = x_+(q)u\). However, since the polyline \(L\) tends to the straight line \(v = x_+(q)u\) (it is known that \(\frac{1}{u_{n-1}} - x < 1/u_n^2\)), correlation (17) is true for all \(n \geq n_0\). Let \(l_0\) stand for the number of the segment of the polyline which contains the point \((u_{n_0}, v_{n_0})\). In the case of one positive root, \(l_0 = 0\).

Let us write the values of the quadratic form \(rv^2 + pvu - q u^2\) at the points \((u_n, v_n)\) in the corresponding domain. See Fig. 1 for the result obtained in the case \(r = 1\), \(p = 0\), \(q = 2\) (the upper right corner of the figure contains the value calculated at the point \((5, 7)\)). From (17) it follows that with all \(n \geq n_0\) the points of the polyline \(L\) belong to boundaries of domains with positive and negative values. The collection of superbases which correspond to these points is called the river of the quadratic form [23].

Let us adduce the main result of the theory of quadratic forms which enables one to make the calculation of the period of a continued fraction much easier. Using the values of the quadratic form at three vectors from any superbasis, one can easily restore the continued fraction. To put it more precisely, the following correlation [23] is true. Let domains associated with values \(a, b, c, d\) of the quadratic form be located in accordance with Fig. 2. Then values \(d, a + b, c\) form an arithmetic progression. Moreover, if \(h\) is the common difference of this progression, then

\[
h^2/4 - ab = \Delta/4.
\]
Moreover, according to the arithmetic progression rule, one can restore all previous values of partial quotients (whose numbers exceed $l_0$).

**Remark 5.** If $h = 0$, then the subsequent values and the previous ones coincide. In this case the periodic part of the continued fraction is palindromic. In particular, this takes place for $p = 0$, i.e., for the root of a rational number (see the latter paragraph of this section).

**Remark 6.** The uniqueness of the river of a quadratic form (see [23]) implies that indefinitely restoring the previous values, we obtain (beginning with certain $n_0'$) the basic (infinite) part of the continued fraction of the associated irrationality $x_-(q)$. Hence, one can obtain the period of the continued fraction $x_-(q)$ from the period $a_{m(r,p,q)+1}, \ldots, a_{m(r,p,q)+T}$ by operations of the inversion and the cyclic shift. In particular, for $r = 1$, when $x_-(q)$ is connected with $x_+(q)$ by the evident unimodular transform (it is well known that the latter preserves the period of a continued fraction), we observe the palindromicity.

Let us now immediately prove the theorem. Its assertion directly follows from equality [18]. Really, as was mentioned above, one can perfectly restore partial quotients $a_i$ with $i > l_0$, using the gradual “Nose-Hoover” algorithm with $n \geq n_0$. This can be done not only with the help of the exact geometric representation in the coordinate plane, but also by a “caricature” representation of the motion along the river of the quadratic form. Assume that at step $n_1$ we get the same parameters of the arithmetic progression as those obtained at step $n_0$. Let $l_1$ stand for the number of the polyline segment which contains the point $(u_{n_1}, v_{n_1})$. Then a part of the continued fraction $a_{l_0+1}, a_{l_0+2}, \ldots, a_{l_1}$ becomes periodic. In addition, the number $l_1 - l_0$ is even, because the points $(u_{n_0}, v_{n_0})$ and $(u_{n_1}, v_{n_1})$ are located to one side of the polyline $L$. In the case of an odd period the sequence $a_{l_0+1}, a_{l_0+2}, \ldots, a_{l_1}$ contains at least two repeating subsequences. In accordance with the algorithm we have $\sum_{i=l_0+1}^{l_1} a_i = n_1 - n_0$. Therefore, suffice it to estimate the number of all possible triplets $(a, b, h)$ which satisfy [18]. The obtained bound obeys formulas [9], [11]. The multiplier 2 in the right-hand sides of these formulas appears, because we have to take into account various signs of $h$, and the term $\tau(\Delta/4)$ does when we consider the case $h = 0$. Theorem 2 is proved.

Let us consider the case of one positive (and one negative) root, which is equivalent to $rq > 0$. If $[x_+(q)] = 0$, i.e., $q_0 = 0$, then the numeration of segments of the polyline shifts by one. The above considerations imply that only three alternatives are possible. Namely, a continued fraction can have no preperiod $(m(r, p, q) = 0)$, or $m(r, p, q) = 1$, or $m(r, p, q) = 2$ and $a_0 = 0$. In the second and third cases the latter partial quotient of the preperiod is less than the latter partial period of the continued fraction.

### Table 1: All possible values of $(a, b, h)$ for the polyline represented in Fig. 1

| the number of a segment of the polyline | 0 | 1 | 2 | 3 | 4 |
|-----------------------------------------|---|---|---|---|---|
| $a$ is the value located at the north west of $L$ | 1 | 2 | 1 | 1 | 1 |
| $b$ is the value located at the south east of $L$ | -1 | -1 | -1 | -2 | -1 |
| $h$ is the common difference of the arithmetic progression | 2 | 0 | -2 | 0 | 2 |

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