Supersymmetric $t$-$J$ Gaudin Models and KZ Equations

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Abstract

Supersymmetric $t$-$J$ Gaudin models with both periodic and open boundary conditions are constructed and diagonalized by means of the algebraic Bethe ansatz method. Off-shell Bethe ansatz equations of the Gaudin systems are derived, and used to construct and solve the KZ equations associated with $sl(2|1)^{(1)}$ superalgebra.

I introduction

In the study of one dimensional long-range interacting systems, Gaudin type models occupied an important place, due to their role in establishing the integrability of the Seiberg-Witten theory and diagonalizing the BCS hamiltonian of ultrasmall metallic grains. They also served as a testing ground for ideas such as the functional Bethe ansatz and general procedure of separation of variables.

The $t$-$J$ model was proposed in an attempt to understand high-$T_c$ superconductivity. It is a correlated electron system with nearest-neighbor hopping ($t$) and antiferromagnetic exchange ($J$) of electrons. Using the nested algebraic Bethe ansatz method, Essler and Korepin obtained the eigenvalues of the periodic system. Soon after the open boundary case was studied.

In the periodic $t$-$J$ Gaudin model was investigated and its eigenvalues were obtained. In this paper, we study both the periodic and open boundary $t$-$J$ Gaudin models by a method different from that used in [14].

The Knizhnik-Zamolodchikov (KZ) equations were first proposed as a set of differential equations satisfied by correlation functions of the Wess-Zumino-Witten models. The connection between Gaudin type magnets and the KZ equations has been studied by many authors. We are interested in the super KZ equations associated with $sl(2|1)^{(1)}$ superalgebra. We will construct and solve these KZ equations with the help of the $t$-$J$ Gaudin models.

The outline of this paper is as follows: In the first part of this paper (section 2-4), we study the periodic $t$-$J$ Gaudin model and its corresponding KZ equation. The $t$-$J$ model hamiltonian is constructed in section 2, and diagonalized in section 3, by using the algebraic Bethe ansatz method. We also derive its off-shell Bethe ansatz equations, and use them, in section 4, to construct solution to the corresponding KZ equation. The second part of this paper is devoted to dealing with the open boundary...
This R-matrix satisfies the graded Yang-Baxter equation (YBE)
\[ R(\lambda - \mu)_{a_1 a_2} R(\lambda)_{b_1 b_2} R(\mu)_{c_1 c_2} = R(\lambda)_{a_1 b_2} R(\mu)_{b_1 c_2} R(\lambda - \mu)_{c_1 a_2}, \]
where \( \epsilon_a \) is the Grassman parity: \( \epsilon_a = 0 \) for bosons and \( \epsilon_a = 1 \) for fermions. The R-matrix satisfies the unitarity and cross-unitarity relations,
\[ R_{12}(\lambda) R_{21}(-\lambda) = \rho(\lambda) \cdot id, \quad \rho(\lambda) = -\sinh(\lambda + \eta) \sinh(\lambda - \eta), \]
\[ R_{12}^{st}(\lambda - \eta) M_1 R_{21}^{str} M_1^{-1} = \bar{\rho}(\lambda) \cdot id, \quad \bar{\rho}(\lambda) = \sinh(\lambda) \sinh(\lambda - \eta), \]
where \( M \) is a diagonal matrix \( \text{diag}(e^{2\eta}, 1, 1) \) and \( st \) is the super-transposition defined by
\[ (A^{st})_{ij} = A_{ji}(-1)^{\epsilon_i + 1}\epsilon_j. \]

Consider the L-operator
\[ L_{aq}(\lambda) \equiv R_{aq}(\lambda), \]
where $a$ represents the auxiliary space and $q$ represents the quantum space. The L-operator also obeys the (graded) YBE
\[ R_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda - \mu). \] (II.7)

The tensor product is graded, namely,
\[ (F \otimes G)_{ac}^{bd} = F_a^b G_c^d(-)^{(\epsilon_a + \epsilon_b)\epsilon_c}. \] (II.8)

The row-to-row monodromy matrix $T_N(\lambda)$ is defined as the product of $N$ operators,
\[ T_a(\lambda) = L_{a1}(\lambda - z_1)L_{a2}(\lambda - z_2)\cdots L_{aN}(\lambda - z_N), \] (II.9)

In matrix form,
\[ \{[T(\lambda)]^{ab}_{\beta_1 \cdots \beta_N} \}_{\alpha_1 \cdots \alpha_N} = L_1(\lambda - z_1)^{c_1 \beta_1}_{\alpha_1} L_2(\lambda - z_2)^{c_2 \beta_2}_{c_1 \alpha_2} \cdots L_N(\lambda - z_N)^{c_N \beta_N}_{c_{N-1} \alpha_N} \]
\[ (-1)^{\sum_{j=1}^{N-1}(\epsilon_{\alpha_j} + \epsilon_{\beta_j})}\sum_{i=j+1}^{N} \epsilon_{\alpha_i}. \] (II.10)

By repeatedly using the YBE, one can easily check that the monodromy matrix satisfies
\[ R(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda - \mu). \] (II.11)

The transfer matrix $t(\lambda)$ is defined as the supertrace of the monodromy matrix over the auxiliary space:
\[ t(\lambda) = \text{str}T(\lambda) = \sum (-1)^{\epsilon_a}T(\lambda)_{aa}. \] (II.12)

Using the YBE, one can show that the transfer matrix $t(\lambda)$ constitutes a one-parameter commuting family, i.e.
\[ [t(\lambda), t(\mu)] = 0. \] (II.13)

Therefore, the $t$-$J$ model is integrable.

### II.2 Supersymmetric $t$-$J$ Gaudin model

Supersymmetric $t$-$J$ Gaudin model can be obtained by taking the quasi-classical limit $\eta \to 0$ of the transfer matrix at the point $\lambda = z_j$ \([22]\). So we expand the R-matrix, L-operator and transfer matrix around the point $\eta = 0$ to get
\[ R(\lambda) = 1 + \eta \hat{R}(\lambda) + O(\eta^2), \] (II.14)
\[ L(\lambda) = 1 + \eta \hat{L}(\lambda) + O(\eta^2), \] (II.15)
\[ t(z_j) = -1 + \eta \hat{t}(z_j) + O(\eta^2). \] (II.16)

Then the Hamiltonian of periodic $t$-$J$ Gaudin model can be obtained from the second term of (II.10).
From (II.14), we have
\[
\hat{R}_{ij}(\lambda) = 2 \coth(\lambda) e_{33}^{i} \otimes e_{33}^{j} + \coth(\lambda) \left[ e_{11}^{i} \otimes e_{22}^{j} + e_{11}^{j} \otimes e_{22}^{i} \right] \\
+ e_{22}^{i} \otimes e_{11}^{j} + e_{33}^{i} \otimes e_{33}^{j} + e_{33}^{j} \otimes e_{22}^{i} \right] \\
+ e^{-\lambda} \frac{1}{\sinh(\lambda)} \left[ e_{13}^{i} \otimes e_{31}^{j} + e_{23}^{i} \otimes e_{32}^{j} - e_{12}^{i} \otimes e_{21}^{j} \right] \\
+ e^{\lambda} \frac{1}{\sinh(\lambda)} \left[ e_{31}^{i} \otimes e_{13}^{j} + e_{32}^{i} \otimes e_{23}^{j} - e_{21}^{i} \otimes e_{12}^{j} \right],
\]
(II.17)
where \( e_{ij}^{k} \) is a matrix acting on the \( k \)-th space with elements \((e_{ij})_{\alpha\beta} = \delta_{\alpha\beta}\). Denote by \( S, S^{\dagger}, S^{z}, Q_{\pm1}, Q_{\pm1}^{\dagger} \) and \( T^{z} \) the generators of \( sl(2|1) \), which satisfy, among others,
\[
[S^{\dagger}, S] = S^{z}, \quad \{Q_{1}^{\dagger}, Q_{1}\} = T^{z}, \quad \{Q_{-1}^{\dagger}, Q_{-1}\} = S^{z} + T^{z}.
\]
(II.18)

In the fundamental representation, they take the form,
\[
S^{z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^{z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
S = e_{21}, \quad S^{\dagger} = e_{12}, \quad Q_{1} = e_{32}, \\
Q_{1}^{\dagger} = e_{23}, \quad Q_{-1} = e_{31}, \quad Q_{-1}^{\dagger} = e_{13}.
\]
(II.19)

Thus,
\[
\hat{R}_{ij}(\lambda) = 2 \coth(\lambda) \left[ (1 - (S^{z})^{2})(1 - (S^{z})^{2}) \right] \\
+ \coth(\lambda) \left[ (1 - T^{z}_{i})(2 - T^{z}_{j} - S^{z}_{j} - (S^{z}_{j})^{2}) \right] + (2 - T^{z}_{i} - S^{z}_{i} - (S^{z}_{i})^{2})(1 - T^{z}_{i}) \\
+ (1 - T^{z}_{i} - S^{z}_{i})(1 - (S^{z}_{j})^{2}) + (1 - (S^{z}_{j})^{2})(1 - T^{z}_{j} - S^{z}_{j}) \right] \\
+ \frac{e^{-\lambda}}{\sinh(\lambda)} \left[ -S^{\dagger}_{i}S_{j} + \sum_{\sigma=\uparrow,\downarrow} Q^{\dagger}_{i,\sigma}Q_{i,\sigma} \right] + \frac{e^{\lambda}}{\sinh(\lambda)} \left[ -S_{i}S^{\dagger}_{j} + \sum_{\sigma=\uparrow,\downarrow} Q_{i,\sigma}Q^{\dagger}_{i,\sigma} \right]
\]
(II.20)

Using the standard fermionic representation [II]
\[
S_{j} = c_{j,1}^{\dagger}c_{j,-1}, \quad S_{j}^{\dagger} = c_{j,-1}^{\dagger}c_{j,1}, \quad S^{z}_{j} = n_{j,-1} - n_{j,1}, \\
T^{z}_{j} = 1 - n_{j,-1}, \quad Q_{j,\sigma}^{\dagger} = c_{j,\sigma}^{\dagger}(1 - n_{j,-\sigma}), \quad Q_{j,\sigma} = c_{j,\sigma}(1 - n_{j,-\sigma}),
\]
(II.21)
where \( n_{j,\pm} = c_{j,\pm}^{\dagger}c_{j,\pm} \) and \( n_{j,+} + n_{j,-} = n_{j} \), we have
\[
\hat{R}_{ij}(\lambda) = 2 \coth(\lambda) \left[ (1 - (n_{i,\downarrow} - n_{i,\uparrow})^{2})(1 - (n_{j,\downarrow} - n_{j,\uparrow})^{2}) \right] \\
+ \coth(\lambda) \left[ n_{i,\downarrow}(1 + n_{j,\uparrow} - (n_{j,\downarrow} - n_{j,\uparrow})^{2}) + (1 + n_{i,\uparrow} - (n_{i,\downarrow} - n_{i,\uparrow})^{2})n_{j,\downarrow} \right. \\
+ n_{i,\uparrow}(1 - (n_{i,\downarrow} - n_{i,\uparrow})^{2}) + (1 - (n_{i,\downarrow} - n_{i,\uparrow})^{2})n_{j,\uparrow}
\]
III Bethe ansatz for the Hamiltonian (II.24) can be diagonalized by using the algebraic Bethe ansatz method.

Finally, the Hamiltonian of the $t$-$J$ Gaudin model can be written as

$$
H_j = \frac{dt(\lambda = z_j)}{d\eta} \bigg|_{\eta=0} = \sum_{k=1}^{N} \frac{1}{\sinh(z_j - z_k)} \left\{ \cosh(z_j - z_k) [-n_{j,-1}n_{k,-1} - n_{j,1}n_{k,1} + (1 - n_j)(1 - n_k)] + e^{-(z_j - z_k)} \left[ \sum_{\sigma = \pm 1} c^\dagger_{j,\sigma} (1 - n_{j,-\sigma}) c_{k,\sigma} (1 - n_{k,-\sigma}) - S^\dagger_j S_k \right] \right. \\
+ \left. e^{z_j - z_k} \left[ \sum_{\sigma = \pm 1} c_{j,\sigma} (1 - n_{j,-\sigma}) c^\dagger_{k,\sigma} (1 - n_{k,-\sigma}) - S_j S^\dagger_k \right] \right\}.
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$$
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+ \left. e^{z_j - z_k} \left[ \sum_{\sigma = \pm 1} c_{j,\sigma} (1 - n_{j,-\sigma}) c^\dagger_{k,\sigma} (1 - n_{k,-\sigma}) - S_j S^\dagger_k \right] \right\}.
$$

(II.23)

III Bethe ansatz for the $t$-$J$ Gaudin model

Hamiltonian (II.24) can be diagonalized by using the algebraic Bethe ansatz method. To simplify calculation, we use the gauge transformation

$$
R(\lambda) \to \text{diag} \left( e^{\lambda/2}, e^{\lambda/2}, e^{-\lambda/2} \right) \otimes 1 \cdot R(\lambda) \cdot \text{diag} \left( e^{\lambda/2}, e^{-\lambda/2}, e^{\lambda/2} \right) \otimes 1
$$

(III.1)

to get

$$
R(\lambda) = \begin{pmatrix}
a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b(\lambda) & 0 & -c_{-}(\lambda) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b(\lambda) & 0 & 0 & d(\lambda) & 0 & 0 & 0 \\
0 & -c_{+}(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & d(\lambda) & 0 \\
0 & 0 & d(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d(\lambda) & 0 & b(\lambda) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w(\lambda)
\end{pmatrix}
$$

(III.2)
where \( d(\lambda) = \sinh(\eta)/\sinh(\lambda - \eta) \). The corresponding \( L \)-matrix can be written as

\[
L_n(\lambda) = \begin{pmatrix}
  b(\lambda) - (b(\lambda) - a(\lambda))e_{11} \varepsilon_1 & -c_-(\lambda)e_{21} \varepsilon_1 & d(\lambda)e_{31} \\
  -c_+(\lambda)e_{12} \varepsilon_2 & b(\lambda) - (b(\lambda) - a(\lambda))e_{22} & d(\lambda)e_{32} \\
  d(\lambda)e_{13} & d(\lambda)e_{23} & b(\lambda) - (b(\lambda) - w(\lambda))e_{33}
\end{pmatrix}.
\]

We expand the row-to-row monodromy matrix \([I.3]\) around \( \eta = 0 \):

\[
T(\lambda) = \begin{pmatrix}
  A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\
  A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\
  C_1(\lambda) & C_2(\lambda) & D(\lambda)
\end{pmatrix}
= 1 + \eta \tilde{T}(\lambda) + \mathcal{O}(\eta^2)
= 1 + \eta \begin{pmatrix}
  \hat{A}_{11}(\lambda) & \hat{A}_{12}(\lambda) & \hat{B}_1(\lambda) \\
  \hat{A}_{21}(\lambda) & \hat{A}_{22}(\lambda) & \hat{B}_2(\lambda) \\
  \hat{C}_1(\lambda) & \hat{C}_2(\lambda) & \hat{D}(\lambda)
\end{pmatrix} + \mathcal{O}(\eta^2).
\]

By the graded YBE \([I.11]\), one finds

\[
[\hat{T}_1(\lambda), \hat{T}_2(\mu)] = [\hat{T}_1(\lambda) + \hat{T}_2(\mu), \hat{R}_{12}(\lambda - \mu)],
\]

and the commutation relations,

\[
\hat{C}_{s_1}(\mu_1)\hat{C}_{s_2}(\mu_2) = -\hat{C}_{s_2}(\mu_2)\hat{C}_{s_1}(\mu_1),
\]

\[
\hat{D}(z)\hat{C}(\mu) = -\coth(z - \mu)\hat{C}(\mu)\hat{D}(z)|_{\eta=0} + \hat{C}(\mu)\hat{D}(z) + \frac{1}{\sinh(\mu - z)}E_j(s)\hat{D}(\mu) - \frac{1}{\sinh(\mu - z)}\hat{C}(z)\hat{D}(\mu)|_{\eta=0},
\]

\[
\hat{A}_{s_1s_2}(z)\hat{C}_{p_1}(\mu) = (r_{s_1s_2}^{p_1}(z,j - \mu)|_{\eta=0}) \hat{C}_{p_1}(\mu)A_{s_1s_2}(z)|_{\eta=0} + \hat{C}_{p_1}(\mu)\hat{A}_{s_1s_2}(z) + \frac{1}{\sinh(\mu - z)}E_j(s)\hat{A}_{s_1s_2}(\mu) - \frac{1}{\sinh(\mu - z)}\hat{C}_{p_1}(z)A_{s_1s_2}(\mu)|_{\eta=0},
\]

where \( j \) indicates the lattice position and \( E_j(s) \) acts on the quantum space with \( E_j(1) = e_{13} \) for \( s = 1 \) and \( E_j(2) = e_{23} \) for \( s = 2 \); the \( r \)-matrix \( r(\lambda) \) is defined by

\[
r(\lambda) = \frac{1}{\sinh(\lambda)} \begin{pmatrix}
  \sinh(\lambda - \eta) & 0 & 0 & 0 \\
  0 & \sinh(\lambda) & -e^{-\lambda}\sinh(\eta) & 0 \\
  0 & -e^{\lambda}\sinh(\eta) & \sinh(\lambda) & 0 \\
  0 & 0 & 0 & \sinh(\lambda - \eta)
\end{pmatrix}.
\]
Define the vacuum state:

\[ |0\rangle_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |0\rangle = \bigotimes_{k=1}^{N} |0\rangle_k. \tag{III.10} \]

Then,

\[ \hat{B}_a(\lambda)|0\rangle = 0, \quad \hat{C}_a(\lambda)|0\rangle \neq 0, \]
\[ \hat{D}(\lambda)|0\rangle = \sum_{i=1}^{N} 2 \coth(\lambda - z_i)|0\rangle \]
\[ \hat{A}_{ab}(\lambda)|0\rangle = \begin{cases} 0 & \text{for } \lambda = z_j, \text{ and } a \neq b \\ \sum_{i=1}^{N} \coth(\lambda - z_i) & \text{for } \lambda \neq z_j, \text{ and } a = b \end{cases}. \tag{III.11} \]

where \( j = 1, 2, \ldots, N. \)

Define the Bethe state

\[ \phi = \hat{C}_{d_1}(\mu_1)\hat{C}_{d_2}(\mu_2) \cdots \hat{C}_{d_n}(\mu_n)|0\rangle > F_{d_1 \cdots d_n}, \tag{III.12} \]

where \( F_{d_1 \cdots d_n} \) is a function of the spectral parameters \( \mu_j. \) We moreover define state

\[ \phi^{(1)} = \hat{C}^{(1)}(\mu^{(1)}_1)\hat{C}^{(1)}(\mu^{(1)}_2) \cdots \hat{C}^{(1)}(\mu^{(1)}_n). \tag{III.13} \]

Then \( \phi^{(1)} \) spans a subspace of the space spanned by \( \phi. \) Applying the transfer matrix to these states and keeping in mind of the commutation relation (III.7), we can find the eigenvalues \( E_j \) of \( t(z_j) \) and the Bethe ansatz equations. This is done as follows.

Firstly, from (III.11) we have

\[ \hat{D}(z_j)\phi = \left[ \sum_{\alpha=1}^{n} \coth(z_j - \mu_\alpha) - 2 \sum_{k=1,\neq j}^{N} \coth(z_j - z_k) \right] \phi + \sum_{\alpha=1}^{n} \frac{(-1)^{\alpha-1}}{\sinh(\mu_\alpha - z_j)} \left[ 2 \sum_{k=1}^{N} \coth(\mu_\alpha - z_k) - \sum_{\beta=1,\neq \alpha}^{n} \coth(\mu_\alpha - \mu_\beta) E_j^{-}(d_\alpha) \phi_\alpha \right. \]
\[ - \sum_{\alpha=1}^{n} \frac{(-1)^{\alpha-1}}{\sinh(\mu_\alpha - z_j)} \hat{C}_{d_\alpha}(z_j) \phi_\alpha, \tag{III.14} \]

where \( \phi_\alpha = \prod_{\beta=1,\neq \alpha}^{n} \hat{C}_{d_\beta}(\mu_\beta). \) Secondly, the action of \( \hat{A}(z_j) \) on \( \phi \) is given by

\[ \hat{A}_{ab}(z_j)\phi = \sum_{\alpha=1}^{n} \frac{(-1)^{\alpha-1}}{\sinh(\mu_\alpha - z_j)} \left[ \sum_{k=1}^{N} \coth(\mu_\alpha - z_k) + \Lambda^{(1)}(\mu_\alpha) \right] E_j^{-}(d_\alpha) \phi_\alpha \]
\[ + \sum_{\alpha=1}^{n} \frac{(-1)^{\alpha-1}}{\sinh(\mu_\alpha - z_j)} \hat{C}_{d_\alpha}(z_j) \phi_\alpha, \tag{III.15} \]
where $\Lambda^{(1)}(\mu_\alpha)$ is the operator

$$t^{(1)}(\lambda) = \frac{d}{d\eta} \text{str} \left[ r^{a_1 d_1}_{m_1 n_1} (\lambda - \mu_1) r^{m_1 d_2}_{m_2 n_2} (\lambda - \mu_2) \cdots r^{m_n d_M}_{n_1 n_2 \cdots n_M} (\lambda - \mu_n) \right]_{\eta=0}. \quad (\text{III.16})$$

Using results in the appendix, we obtain

$$\Lambda^{(1)}(\mu_\alpha) = \sum_{\beta=1, \neq \alpha}^{n} \coth(\mu_\alpha - \mu_\beta) - \sum_{\gamma=1}^{m} \coth(\mu_\alpha - \mu^{(1)}_\gamma), \quad (\text{III.17})$$

where $\mu^{(1)}_1, \cdots, \mu^{(1)}_m$ satisfy the constraints

$$f^{(1)}_\gamma \equiv \sum_{\beta=1, \neq \alpha}^{n} \coth(\mu_\beta - \mu^{(1)}_\gamma) + 2 \sum_{\delta=1, \neq \gamma}^{m} \coth(\mu^{(1)}_\gamma - \mu^{(1)}_\delta) = 0. \quad (\text{III.18})$$

Thus, from (II.24), (III.14)-(III.15) and (III.17)-(III.18), we obtain the off-shell Bethe ansatz equations

$$H_j \phi = E_j \phi + \sum_{\alpha=1}^{M} \frac{(-1)^{a-1}}{\sinh(\mu_\alpha - z_j)} f_\alpha E^-_j(d_\alpha) \phi_\alpha, \quad (\text{III.19})$$

where $\mu_\alpha$ satisfy the condition $f^{(1)}_\gamma = 0$ and

$$E_j = \sum_{\alpha=1}^{n} \coth(z_j - \mu_\alpha) - 2 \sum_{k=1, \neq j}^{N} \coth(z_j - z_k), \quad (\text{III.20})$$

$$f_\alpha = - \sum_{\gamma=1}^{m} \coth(\mu_\alpha - \mu^{(1)}_\gamma) + \sum_{k=1}^{N} \coth(\mu_\alpha - z_k), \quad (\text{III.21})$$

$$\phi = \prod_{\alpha=1}^{n} \left( \sum_{k=1}^{N} \frac{1}{\sinh(\mu_\alpha - z_k)} E^-_k(d_\alpha) \right) |0 > F^{d_1 \cdots d_n}, \quad (\text{III.22})$$

$$\phi_\alpha = \prod_{\beta=1, \neq \alpha}^{n} \left( \sum_{k=1}^{N} \frac{1}{\sinh(\mu_\beta - z_k)} E^-_k(d_\beta) \right) |0 > F^{d_1 \cdots d_n}. \quad (\text{III.23})$$

### IV Super KZ equation

As a set of partial differential equations, the KZ equations take the form

$$\nabla_j \Psi = 0 \quad \text{for} \quad j = 1, 2, \cdots, N, \quad (\text{IV.1})$$

where the differential operator $\nabla_j$ is defined by the Gaudin Hamiltonian $H_j$:

$$\nabla_j = \kappa \frac{\partial}{\partial z_j} - H_j \quad (\text{IV.2})$$
with \( \kappa \) being a parameter. Substituting (II.24) into (IV.2), we can check

\[
[\nabla_j, \nabla_k] = 0,
\]

which ensures the integrability of the KZ equation.

To simplify our calculation, we make the following transformation

\[
H_j \rightarrow H_j + 2 N \sum_{k=1, k \neq j}^{N} \coth(z_j - z_k),
\]

\[
E_j \rightarrow E_j + 2 N \sum_{k=1, k \neq j}^{N} \coth(z_j - z_k) = \sum_{a=1}^{n} \coth(z_j - \mu_a).
\]

Under this transformation, the form of the off-shell Bethe ansatz equations is invariant.

The function \( \Psi(z) \) can be constructed by the hypergeometric function \( \chi(z, \mu) \) which obeys the equations

\[
\kappa \frac{\partial}{\partial z} \chi = E_j \chi,
\]

\[
\kappa \frac{\partial}{\partial \mu} \chi = f_j \chi,
\]

and the constraint \( f_j^{(1)} = 0 \). The solution to the above equations is given by

\[
\chi(z, \mu) = \prod_{\beta < \alpha} \left[ \sinh(\mu_\alpha - \mu_\beta^{(1)}) \right]^{-1/\kappa} \prod_{a=1}^{n} \prod_{j=1}^{N} [\sinh(z_j - \mu_a)]^{1/\kappa}
\]

with \( \mu_\alpha \) satisfying the condition \( f_j^{(1)} = 0 \). With the help of \( \chi(z, \mu) \), the function \( \Psi(z) \) is given by

\[
\Psi(z) = \oint_C \cdots \oint_C d\mu_1 \cdots d\mu_n \chi(t, z) \phi(t, z),
\]

where the integration path \( C \) is a closed contour in the Riemann surface such that the integrand resumes its initial value after \( t_\alpha \) has described it. Substituting the expressions of \( \nabla_j \) and \( \Psi(z) \) into (IV.1), we can show that the KZ equation is satisfied. The proof is as follows

\[
\kappa \frac{\partial}{\partial z_j} \Psi(z) = \oint_C \cdots \oint_C d\mu_1 \cdots d\mu_n \left( \kappa \frac{\partial \chi}{\partial z_j} \phi + \kappa \chi \frac{\partial \phi}{\partial z_j} \right)
\]

\[
= \oint_C \cdots \oint_C d\mu_1 \cdots d\mu_n \left( \chi E_j \phi + \kappa \chi \frac{\partial \phi}{\partial z_j} \right)
\]

\[
= \oint_C \cdots \oint_C d\mu_1 \cdots d\mu_n \left[ \chi H_j \phi - \chi \sum_{a=1}^{n} W(\mu_\alpha, z_j) f_\alpha E_j^{-} (d_\alpha) \phi_\alpha
\right.

\[
- \kappa \chi \sum_{a} \frac{\partial}{\partial \mu_\alpha} \left( W(\mu_\alpha, z_j) \phi_\alpha E_j^{-} (d_\alpha) \right)
\]

\[
= \oint_C \cdots \oint_C d\mu_1 \cdots d\mu_n \left[ \chi H_j \phi
\right.

\[
- \kappa \sum_{a} \frac{\partial}{\partial \mu_\alpha} \left( \chi W(\mu_\alpha, z_j) \phi_\alpha E_j^{-} (d_\alpha) \right)
\]

\[
= H_j \Psi,
\]
where \( W(\mu, z) = (-1)^{\alpha-1}/\sinh(\mu - z) \).

### V Open boundary \( t-J \) Gaudin model

In this and the next section, we discuss the open boundary \( t-J \) Gaudin system. We start with the graded reflection relation [12],

\[
R(\lambda - \mu)_{a_1a_2}^{b_1b_2} K(\lambda)_{b_1}^{c_1} R(\lambda + \mu)_{b_2c_1}^{c_2d_1} K(\mu)_{c_2}^{d_2} (-1)^{(\epsilon_1, \epsilon_1)\epsilon_2} \\
= K(\mu)_{a_2}^{b_2} R(\lambda + \mu)_{a_1b_2}^{b_1c_2} K(\lambda)_{b_1}^{c_1} R(\lambda - \mu)_{c_2c_1}^{d_2d_1} (-1)^{(\epsilon_1, \epsilon_1)\epsilon_2}, \tag{V.1}
\]

where the \( K(\lambda) \) is the reflection K-matrix. The diagonal solutions of the reflection equation were found in [12]. In the present paper, we only consider a special case in which \( K(\lambda) = 1 \).

Following the standard procedure, we define the double-row monodromy matrix

\[
\mathcal{T}(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda). \tag{V.2}
\]

Here \( T(\lambda) \) is same as in the periodic case. One can check that the following relation is satisfied:

\[
R(\lambda - \mu)_{a_1a_2}^{b_1b_2} \mathcal{T}(\lambda)_{b_1}^{c_1} R(\lambda + \mu)_{b_2c_1}^{c_2d_1} \mathcal{T}(\mu)_{c_2}^{d_2} (-1)^{(\epsilon_1, \epsilon_1)\epsilon_2} \\
= \mathcal{T}(\mu)_{a_2}^{b_2} R(\lambda + \mu)_{a_1b_2}^{b_1c_2} \mathcal{T}(\lambda)_{b_1}^{c_1} R(\lambda - \mu)_{c_2c_1}^{d_2d_1} (-1)^{(\epsilon_1, \epsilon_1)\epsilon_2}. \tag{V.3}
\]

The dual reflection relation reads

\[
R_{12}(\mu - \lambda)K^+(\lambda)M_1^{-1}R_{21}(\eta - \lambda - \mu)K^+(\mu)M_2^{-1} \\
= K^+(\mu)M_2^{-1}R_{12}(\eta - \lambda - \mu)K^+(\lambda)M_1^{-1}R_{21}(\mu - \lambda), \tag{V.4}
\]

where \( M \) is a diagonal matrix \( M = \text{diag}(e^{2\eta}, 1, 1) \). Solution \( K^+ \) associated with \( K(\lambda) = 1 \) is given by

\[
K^+(\lambda) \equiv \text{diag}(\lambda, \lambda + \eta/2) = \text{diag}(e^{2\eta}, 1, 1). \tag{V.5}
\]

Define the boundary transfer matrix,

\[
t^b(\lambda) = \text{str} K^+(\lambda)\mathcal{T}(\lambda). \tag{V.6}
\]

By (V.4) and (V.3), one can show the commutativity of the transfer matrix for different \( \lambda \) values.

Similar to the periodic case, the boundary \( t-J \) Gaudin system can be obtained by expanding the boundary transfer matrix at the point \( \lambda = z_j \) around \( \eta = 0 \):

\[
t^b(\lambda = z_j) = 1 + \eta H_j^b + \mathcal{O}(\eta^2). \tag{V.7}
\]

The second term on the right hand side gives the Hamiltonian of the open boundary \( t-J \) Gaudin model. Explicitly,

\[
H_j = \left. \frac{dt(z_j)}{d\eta} \right|_{\eta=0}
\]
VI Bethe ansatz for boundary $t$-$J$ Gaudin model

As in the periodic case, we write the double-monomodrphy matrix as

$$
\mathcal{T}(\lambda) = \begin{pmatrix}
A_{11}(\lambda) & A_{12}(\lambda) & B_{1}(\lambda) \\
A_{21}(\lambda) & A_{22}(\lambda) & B_{2}(\lambda) \\
C_{1}(\lambda) & C_{2}(\lambda) & D(\lambda)
\end{pmatrix}. \tag{VI.1}
$$

Around $\eta = 0$:

$$
\mathcal{T}(\lambda) = 1 + \eta \hat{\mathcal{T}}(\lambda) + \mathcal{O}(\eta^2)
$$

$$
= 1 + \eta \begin{pmatrix}
\hat{A}_{11}(\lambda) & \hat{A}_{12}(\lambda) & \hat{B}_{1}(\lambda) \\
\hat{A}_{21}(\lambda) & \hat{A}_{22}(\lambda) & \hat{B}_{2}(\lambda) \\
\hat{C}_{1}(\lambda) & \hat{C}_{2}(\lambda) & \hat{D}(\lambda)
\end{pmatrix} + \mathcal{O}(\eta^2). \tag{VI.2}
$$

Applying $\hat{\mathcal{T}}(\lambda = z_j), \ j = 1, 2, \cdots, N$, to the vacuum state $\mid 0 \rangle$, we have

$$
\hat{B}_a(z_j)|0\rangle = 0, \quad \hat{C}_a(z_j)|0\rangle \neq 0
$$

$$
\hat{D}(z_j)|0\rangle = \sum_{i=1}^{N} 2(\coth(z_j - z_i) + \coth(z_j - z_i))|0\rangle >
$$

$$
\hat{A}_{ab}(\lambda)|0\rangle = \begin{cases}
0 & \text{for } \lambda = z_j, \ \text{and } a \neq b \\
\sum_{i=1}^{N} (\coth(\lambda + z_i) + \coth(\lambda - z_i)) & \text{for } \lambda \neq z_j, \ \text{and } a = b
\end{cases}.
\tag{VI.3}
$$

The Bethe state of the boundary system can still be taken as

$$
\phi_b = \hat{C}_{d_1}(\mu_1)\hat{C}_{d_2}(\mu_2)\cdots\hat{C}_{d_n}(\mu_n)|0\rangle > F^{d_1\cdots d_n}. \tag{VI.4}
$$
Write
\[
\mathcal{A}(\lambda)_{ab}|_{\eta=0} = \mathcal{A}(\lambda)_{ab}|_{\eta=0} + \delta_{ab} \frac{1}{\sinh(2\lambda)} D(\lambda)_{\eta=0},
\] (VI.5)
where
\[
\mathcal{A}(\lambda)_{ab}|_{0} = \sum_{i=1}^{N}(\coth(\lambda + z_i) + \coth(\lambda - z_i)) - \frac{\delta_{ab}}{\sinh(2\lambda)} D(\lambda)_{\eta=0}.
\] (VI.6)

Then
\[
\hat{t}^b(\lambda = z_j) \equiv H_j^b = \frac{d}{d\eta} (K^+ T(z_j))_{\eta=0}
= -\hat{A}_{aa}(z_j) + \mathcal{D}(z_j)
- \left(k_a^+\right)'_{\eta=0} A(z_j)_{\eta=0} - U D(\lambda)_{\eta=0},
\] (VI.7)
where \(a = 1, 2\) and \(U = 2/\sinh(2z_j)\). The last term in (VI.7) corresponds to the boundary condition.

We now find commutation relations between \(\hat{A}_{ab}(\lambda), \mathcal{D}(\lambda)\) and \(\hat{C}_d(\mu)\). After a tedious but direct computation, we get
\[
\hat{C}_d(\mu_1)\hat{C}_d(\mu_2) = -\hat{C}_c(\mu_2)\hat{C}_c(\mu_1),
\] (VI.8)
\[
\mathcal{D}(z_j)\hat{C}_d(\mu) = \hat{C}_d(\mu)\mathcal{D}(z_j) - \frac{\sinh(2z_j)}{\sinh(z_j - \mu) \sinh(z_j + \mu)} \hat{C}_d(\mu) D(z_j)|_{\eta=0}
\quad + \frac{1}{\sinh(z_j - \mu)} \left(-E_j^-(d)\mathcal{D}(\mu) + \hat{C}_d(z_j) D(\mu)|_{\eta=0}\right)
\quad - \frac{1}{\sinh(z_j + \mu)} \left(-E_j^-(d)\hat{A}_{bd}(\mu) + \hat{C}_b(\mu) \hat{A}_{bd}(\mu)|_{\eta=0}\right)
\quad + \frac{2\coth(2\mu)}{\sinh(z_j - \mu)} E_j^-(b)\mathcal{D}(\mu)|_{\eta=0} - \frac{2\cosh(z_j + \mu)}{\sinh^2(z_j + \mu)} E_j^-(b)\hat{A}_{bd}(\mu)|_{\eta=0},
\] (VI.9)
\[
\hat{A}_{a_1d_1}(z_j)\hat{C}_d(\mu) = \hat{C}_d(\mu) \hat{A}_{a_1d_1}(z_j)
\quad + \left(r_{12}(z_j + \mu + \eta)\right)^{c_1b_2}_{a_1c_2} r_{21}(z_j - \mu) \delta_{b_1b_2} \left(\hat{C}_c(\mu) \hat{A}_{c_1b_1}(\lambda)|_{\eta=0}\right)
\quad + \frac{1}{\sinh(z_j + \mu)} \delta_{a_1b_2} \delta_{b_1d_1} \left(-E_j^-(b_1) \hat{A}_{a_1d_2}(\mu) + \hat{C}_{d_1}(z_j) \hat{A}_{a_1d_2}(\mu)|_{\eta=0}\right)
\quad - \frac{1}{\sinh(z_j + \mu)} \delta_{a_1d_2} \delta_{b_2d_1} \left(-E_j^-(b_2) \mathcal{D}(\mu) + \hat{C}_{b_2}(z_j) \mathcal{D}(\mu)|_{\eta=0}\right)
\quad - \left(\frac{\sinh(\eta) r_{12}(2z_j + \mu + \eta) \delta_{b_1d_1}}{\sinh(z_j - \mu)}\right)^n_{\eta=0} E_j^-(b_1) \hat{A}_{b_2d_2}(\mu)|_{\eta=0}
\quad + \left(\frac{\sin(2\mu) \sinh(\eta) r_{12}(2z_j + \mu + \eta) \delta_{b_1d_1}}{\sinh(z_j + \mu + \eta) \sinh(2\mu + \eta)}\right)^n_{\eta=0} E_j^-(b_2) \mathcal{D}(\mu)|_{\eta=0},
\] (VI.10)
\[ \mathcal{D}(z_j)_{\eta=0} \hat{\mathcal{C}}_d(\mu) = \hat{\mathcal{C}}_d(\mu) \mathcal{D}(z_j)_{\eta=0} + \frac{1}{\sinh(z_j - \mu)} E_j^- (d) \mathcal{D}(\mu)_{\eta=0} \]

\[ - \frac{1}{\sinh(z_j + \mu)} E_j^- (b) \tilde{A}_{bd}(\mu)_{\eta=0}, \quad \text{(VI.11)} \]

\[ \tilde{A}_{a_1 d_1} (z_j)_{\eta=0} \hat{\mathcal{C}}_{d_2} (\mu) = \hat{\mathcal{C}}_{d_2} (\mu) \tilde{A}_{a_1 d_1} (z_j)_{\eta=0} + \frac{1}{\sinh(z_j - \mu)} E_j^- (d_1) \tilde{A}_{a_1 d_2} (\mu)_{\eta=0} \]

\[ - \frac{1}{\sinh(z_j + \mu)} \delta_{a_1 d_2} E_j^- (d_1) \mathcal{D}(\mu)_{\eta=0} \quad \text{(VI.12)} \]

Then, applying (VI.7) to the Bethe state and using the above commutation relations repeatedly, we obtain the off-shelled Bethe ansatz equations

\[ H_j^{b|b} = \frac{dt(\lambda = z_j)}{d\eta} \bigg|_{\eta=0} \phi^b = E_j^b \phi^b - \sum_{\alpha=1}^{n} W^b(\mu, z_j) f^b_{\alpha} E_j^- \phi^b, \quad \text{(VI.13)} \]

where

\[ \phi^b = \prod_{\alpha=1}^{n} \left( \frac{2 \sinh(\mu_{\alpha}) \cosh(z_k)}{\sinh(\mu_{\alpha} - z_k) \sinh(\mu_{\alpha} + z_k)} E_k^- (d_{\alpha}) \right) |0 > F^{d_1 \cdots d_n}, \quad \text{(VI.14)} \]

\[ \phi^b_{\alpha} = \prod_{\beta=1, \neq \alpha}^{n} \left( \frac{2 \sinh(\mu_{\beta}) \cosh(z_k)}{\sinh(\mu_{\beta} - z_k) \sinh(\mu_{\beta} + z_k)} E_k^- (d_{\beta}) \right) |0 > F^{d_1 \cdots d_n}. \quad \text{(VI.15)} \]

\[ E_j^b = \frac{2}{\sinh(z_j)} - 2 \sum_{k=1, \neq j}^{N} \left[ \coth(z_j + z_k) + \coth(z_j - z_k) \right] \]

\[ + \sum_{\alpha=1}^{n} \left[ \coth(z_j + \mu_{\alpha}) + \coth(z_j - \mu_{\alpha}) \right] \quad \text{(VI.16)} \]

\[ f^b_{\alpha} = - \frac{2}{\sinh(2\mu_{\alpha})} - \sum_{\beta=1, \neq \alpha}^{n} \left( \coth(\mu_{\alpha} - \mu_{\beta}) + \coth(\mu_{\alpha} + \mu_{\beta}) \right) \]

\[ + \sum_{k=1}^{N} \left( \coth(\mu_{\alpha} - z_k) + \coth(\mu_{\alpha} + z_k) \right) + \Lambda^{(1)}_b(\mu_{\alpha}), \quad \text{(VI.17)} \]

\[ W^b(\mu_{\alpha}, z_j) = \frac{(-1)^{\alpha-1} 2 \sinh(z_j) \cosh(\mu_{\alpha})}{\sinh(z_j + \mu_{\alpha}) \sinh(z_j - \mu_{\alpha})}. \quad \text{(VI.18)} \]

Here \( \Lambda^{(1)}_b \) is the eigenvalue of the nested transfer matrix

\[ t^{(1)}_b(\lambda) = \left. \frac{d}{d\eta} \right|_{\eta=0} \left[ \text{str} K^{(1)+} r(\lambda + \mu_1 + \eta)_{a_1 c_1} r(\lambda + \mu_2 + \eta)_{a_2 c_2} \cdots r(\lambda + \mu_n + \eta)_{a_n c_n} \right. \]

\[ \left. K^{(1)} r_{21}(\lambda - \mu_n)_{b_n e_n} \cdots r_{21}(\lambda - \mu_2)_{b_2 e_2} r_{21}(\lambda - \mu_1)_{b_1 e_1} \right], \quad \text{(VI.19)} \]
where
\[ K^{(1)+} = \text{diag} \left( e^{2\eta}, 1 \right), \quad K^{(1)} = \left( 1 - \sinh(\eta)/\sinh(2\lambda + \eta) \right) \cdot \text{id}. \]

The detailed calculation of the nested eigenvalue is discussed in the appendix. Here we write the result
\[
\Lambda_b^{(1)}(\mu_\alpha) = \frac{e^{-\mu_\alpha}}{\sinh(\mu_\alpha)} + \sum_{\beta=1,\neq \alpha}^n \left[ \coth(\mu_\alpha - \mu_\beta) + \coth(\mu_\alpha + \mu_\beta) \right] - \sum_{\gamma=1}^m \left[ \coth(\mu_\alpha - \mu_\gamma^{(1)}) + \coth(\mu_\alpha + \mu_\gamma^{(1)}) \right], \tag{VI.20}
\]
where \( \mu_\gamma^{(1)} \) satisfies the nested Bethe ansatz equation
\[
f_{\gamma}^{(1)} = \sum_{\beta=1}^n \left( \coth(\tilde{\mu}_\gamma^{(1)} + \tilde{\mu}_\beta) + \coth(\tilde{\mu}_\gamma^{(1)} - \tilde{\mu}_\beta) \right) - 2 \sum_{\delta=1,\neq \gamma}^m \left( \coth(\tilde{\mu}_\gamma^{(1)} + \tilde{\mu}_\delta^{(1)}) + \coth(\tilde{\mu}_\gamma^{(1)} - \tilde{\mu}_\delta^{(1)}) \right) = 0. \tag{VI.21}
\]

Substituting the nested eigenvalue into (VI.17), we obtain the Bethe ansatz equations for the boundary \( t-J \) Gaudin model
\[
f_{\alpha}^b = -\frac{e^{-\mu_\alpha}}{\cosh(\mu_\alpha)} + \sum_{k=1}^N \left( \coth(\mu_\alpha - z_k) + \coth(\mu_\alpha + z_k) \right) - \sum_{\gamma=1}^m \left( \coth(\mu_\alpha - \mu_\gamma^{(1)}) + \coth(\mu_\alpha + \mu_\gamma^{(1)}) \right) = 0. \tag{VI.22}
\]

\section{VII \hspace{1em} Super KZ equation in the boundary case}

As in the periodic case, the KZ equations are
\[
\nabla_j \Psi = 0, \quad \nabla_j = \kappa \frac{\partial}{\partial z_j} - H_j^b, \quad j = 1, 2, \ldots, N, \tag{VII.1}
\]
but now \( H_j^b \) is the hamiltonian of the boundary \( t-J \) Gaudin model. We make the transformation
\[
H_j \rightarrow H_j - 2/\sinh(2z_j) + 2 \sum_{k=1,\neq j}^N \left[ \coth(z_j + z_k) + \coth(z_j - z_k) \right],
\]
\[
E_j^b \rightarrow E_j^b - 2/\sinh(2z_j) + 2 \sum_{k=1,\neq j}^N \left[ \coth(z_j + z_k) + \coth(z_j - z_k) \right] = \sum_{\alpha=1}^n \left[ \coth(z_j + \mu_\alpha) + \coth(z_j - \mu_\alpha) \right]
\]
This transformation leaves invariant the form of the off-shell Bethe ansatz equations.

To construct \( \Psi(z) \), we introduce a hypergeometric function \( \chi(z, \mu) \) which satisfies the following equations

\[
\kappa \frac{\partial}{\partial z_j} \chi = E^b_j \chi,
\]
\[
\kappa \frac{\partial}{\partial \mu_i} \chi = f^b_i \chi,
\]
and the constraint \( f^{b(1)}_\gamma = 0 \). Solving these two equations, one gets

\[
\chi(z, \mu) = \prod_{\alpha=1}^{n} \left(1 + e^{-2\mu_\alpha}\right)^{1/\kappa} \prod_{\alpha=1}^{n} \sinh(\mu_\alpha - \mu^{(1)}_\gamma) \sinh(\mu_\alpha - \mu^{(1)}_\gamma)^{-1/\kappa}
\times \prod_{\alpha=1}^{n} \prod_{j=1}^{N} \sinh(z_j + \mu_\alpha) \sinh(z_j - \mu_\alpha)^{1/\kappa},
\]

(VII.3)

where \( \mu^{(1)}_\gamma \) satisfies the nested Bethe ansatz equation \( f^{b(1)}_\gamma = 0 \). With the help of \( \chi(z, \mu) \), the function \( \Psi(z) \) is given by

\[
\Psi(z) = \int_C \cdots \int_C \mu_1 \cdots \mu_M \chi(t, z) \phi(t, z),
\]

(VII.4)

where the integration path \( C \) is a closed contour in the Riemann surface such that the integrand resumes its initial value after \( t_\alpha \) has described it. Substituting the expressions of \( \nabla_j \) and \( \Psi(z) \) into (IV.1), we can show that the KZ equation is satisfied. The proof is as follows

\[
\kappa \frac{\partial}{\partial z_j} \Psi(z) = \int_C \cdots \int_C \mu_1 \cdots \mu_M \left( \kappa \frac{\partial}{\partial z_j} \phi + \kappa \chi \frac{\partial}{\partial z_j} \phi \right)
\]
\[
= \int_C \cdots \int_C \mu_1 \cdots \mu_M \left( \chi E^b_j \phi + \kappa \chi \frac{\partial}{\partial z_j} \phi \right)
\]
\[
= \int_C \cdots \int_C \mu_1 \cdots \mu_M \left[ \chi H^b_j \phi + \chi \sum_{\alpha=1}^{M} W(\mu_\alpha, z_j) f^b_\alpha E^-_j (d_\alpha) \phi^b_\alpha \right.
\]
\[
+ \kappa \chi \sum_{\alpha} \frac{\partial}{\partial \mu_\alpha} \left( W(\mu_\alpha, z_j) \phi^b_\alpha E^-_j (d_\alpha) \right)
\]
\[
= \int_C \cdots \int_C \mu_1 \cdots \mu_M \left[ \chi H^b_j \phi
\right.
\]
\[
+ \kappa \sum_{\alpha} \frac{\partial}{\partial \mu_\alpha} \left( \chi W(\mu_\alpha, z_j) \phi^b_\alpha E^-_j (d_\alpha) \right)
\]
\[
= H^b_j \Psi,
\]

(VII.5)

where

\[
W(\mu_\alpha, z) = \frac{(-1)^{n-1} \cosh(z_j) \sinh(\mu_\alpha)}{\sinh(\mu_\alpha + z_j) \sinh(\mu_\alpha - z_j)}.
\]

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A Details on the nested Bethe ansatz

A.1 Periodic case

The nested Bethe ansatz was used to obtain eigenvalues of the transfer matrix constructed from the following $r$-matrix:

$$r(\lambda) = \frac{1}{\sinh(\lambda)} \begin{pmatrix}
\sinh(\lambda) & 0 & 0 & 0 \\
0 & \sinh(\lambda) & -e^\lambda \sinh(\eta) & 0 \\
0 & -e^\lambda \sinh(\eta) & \sinh(\lambda) & 0 \\
0 & 0 & 0 & \sinh(\lambda - \eta)
\end{pmatrix}.$$  \hfill (A.6)

One can check that this $r$-matrix satisfies the unitarity and cross-unitarity relations

$$r_{12}(\lambda)r_{21}(-\lambda) = -\sinh(\lambda + \eta) \sinh(\lambda + \eta) \cdot \text{id.},$$

$$r_{12}^{s_1}(2\eta - \lambda)r_{21}^{s_2}(\lambda) = \sinh(\lambda) \sinh(2\lambda - \eta) \cdot \text{id.},$$  \hfill (A.7)

and the YBE

$$r_{12}(\lambda - \mu)L_1^{(1)}(\lambda)L_2^{(1)}(\mu) = L_2^{(1)}(\mu)L_1^{(1)}(\lambda)r_{12}(\lambda - \mu),$$  \hfill (A.8)

where

$$L_j^{(1)}(\lambda) = \frac{1}{\sinh(\lambda)} \begin{pmatrix}
\sinh \left( \frac{\lambda - \eta}{2} - \frac{1 + \sigma_j^z}{2} \right) & -\sigma_j^- \sinh(\eta) \\
-\sigma_j^+ \sinh(\eta) & \sinh \left( \frac{\lambda - \eta}{2} + \frac{1 + \sigma_j^z}{2} \right)
\end{pmatrix}.$$  \hfill (A.9)

with $\sigma^\pm, \sigma^z$ being the usual Pauli matrices. The monodromy matrix is

$$T^{(1)}(\lambda) = L_m^{(1)}(\lambda - \mu_m) \cdots L_2^{(1)}(\lambda - \mu_2)L_1^{(1)}(\lambda - \mu_1)$$

$$\equiv \begin{pmatrix}
A^{(1)}(\lambda) & B^{(1)}(\lambda) \\
C^{(1)}(\lambda) & D^{(1)}(\lambda)
\end{pmatrix},$$  \hfill (A.10)

which satisfies

$$r_{12}(\lambda - \mu)T_1^{(1)}(\lambda)T_2^{(1)}(\mu) = T_2^{(1)}(\mu)T_1^{(1)}(\lambda)r_{12}(\lambda - \mu).$$  \hfill (A.11)

The transfer matrix of the nested system is defined by

$$t^{(1)}(\lambda) = \text{str} T^{(1)}(\lambda).$$ \hfill (A.12)

One can easily prove the commutativity of the transfer matrix: $[t^{(1)}(\lambda), t^{(1)}(\mu)] = 0$.

Expanding $r, L^{(1)}, T^{(1)}$ and $T^{(1)}$ around $\eta = 0$:

$$r(\lambda) = 1 + \eta \hat{r}(\lambda)|_{\eta=0} + \mathcal{O}(\eta^2),$$

$$L^{(1)}(\lambda) = 1 + \eta \hat{L}^{(1)}(\lambda)|_{\eta=0} + \mathcal{O}(\eta^2),$$

$$T^{(1)}(\lambda) = 1 + \eta \hat{T}^{(1)}(\lambda)|_{\eta=0} + \mathcal{O}(\eta^2),$$

$$t^{(1)}(\lambda) = -2 + \eta \hat{t}^{(1)}(\lambda)|_{\eta=0} + \mathcal{O}(\eta^2).$$  \hfill (A.13)
we obtain

\[
\hat{r}(\lambda) = \begin{pmatrix}
-\coth(\lambda) & 0 & 0 & 0 \\
0 & 0 & -e^{-\lambda}/\sinh(\lambda) & 0 \\
0 & -e^{\lambda}/\sinh(\lambda) & 0 & 0 \\
0 & 0 & 0 & -\coth(\lambda)
\end{pmatrix}.
\]

\[
\hat{L}^{(1)}_k(\lambda) = \begin{pmatrix}
-\frac{1+\sigma^2_k}{2} \coth(\lambda) & -\sigma_k e^{-\lambda}/\sinh(\lambda) \\
-\sigma_k e^{\lambda}/\sinh(\lambda) & -\frac{1-\sigma^2_k}{2} \coth(\lambda)
\end{pmatrix}.
\]

\[
\hat{T}^{(1)}(\lambda) = \begin{pmatrix}
\hat{A}^{(1)}(\lambda) & \hat{B}^{(1)}(\lambda) \\
\hat{C}^{(1)}(\lambda) & \hat{D}^{(1)}(\lambda)
\end{pmatrix},
\]

\[
t^{(1)}(\lambda) = \text{str} \begin{pmatrix}
\hat{A}^{(1)}(\lambda) & \hat{B}^{(1)}(\lambda) \\
\hat{C}^{(1)}(\lambda) & \hat{D}^{(1)}(\lambda)
\end{pmatrix} = -\hat{A}^{(1)}(\lambda) - \hat{D}^{(1)}(\lambda).
\]

By the nested YBE (A.11), we find the following commutation relations

\[
\hat{C}^{(1)}(\lambda)\hat{C}^{(1)}(\mu^{(1)}) = \hat{C}^{(1)}(\mu^{(1)})\hat{C}^{(1)}(\lambda),
\]

\[
\hat{A}^{(1)}(\lambda)\hat{C}^{(1)}(\mu^{(1)}) = -\coth(\lambda - \mu^{(1)})\hat{C}^{(1)}(\mu^{(1)}) + \hat{C}^{(1)}(\mu^{(1)})\hat{A}^{(1)}(\lambda) \\
+ \frac{e^{-\lambda+\mu^{(1)}}}{\sinh(\lambda - \mu^{(1)})} \left( \hat{C}^{(1)}(\lambda)\hat{A}^{(1)}(\mu^{(1)}) + \hat{C}^{(1)}(\lambda)\hat{D}^{(1)}(\mu^{(1)}) \right)_{n=0},
\]

\[
\hat{D}^{(1)}(\lambda)\hat{C}^{(1)}(\mu^{(1)}) = \coth(\lambda - \mu^{(1)}) \hat{C}^{(1)}(\mu^{(1)})_{n=0} + \hat{C}^{(1)}(\mu^{(1)})\hat{D}^{(1)}(\lambda) \\
+ \frac{e^{-\lambda+\mu^{(1)}}}{\sinh(\lambda - \mu^{(1)})} \left( \hat{C}^{(1)}(\lambda)\hat{D}^{(1)}(\mu^{(1)}) + \hat{C}^{(1)}(\lambda)\hat{A}^{(1)}(\mu^{(1)}) \right)_{n=0}.
\]

For the nest transfer matrix, we choose its vacuum state as

\[
|0 >^{(1)}_k = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad |0 >^{(1)} = \otimes_{k=1}^{n} |0 >^{(1)}_k.
\]

Applying the \(T\)-operator to the vacuum state, we have

\[
\hat{A}^{(1)}(\lambda)|0 >^{(1)} = \hat{B}^{(1)}(\lambda)|0 >^{(1)} = 0, \quad \hat{C}^{(1)}(\lambda)|0 >^{(1)} \neq 0, \\
\hat{D}^{(1)}(\lambda)|0 >^{(1)} = \sum_{i=1}^{m} \coth(\lambda - \mu_i) \quad \lambda \neq \mu_k \ (k = 1, 2, \ldots, m).
\]

The eigenvector of the nested transfer matrix (A.14) can be constructed from

\[
\phi^{(1)} = \hat{C}^{(1)}(\mu^{(1)}_1)\hat{C}^{(1)}(\mu^{(1)}_2)\cdots\hat{C}^{(1)}(\mu^{(1)}_m).
\]

Applying the nested transfer matrix to the above state and keeping in mind \(\lambda = \mu_{\alpha}\), we obtain

\[
\Lambda^{(1)}(\mu_{\alpha}) = \sum_{\beta=1, \neq \alpha}^{n} \coth(\mu_{\alpha} - \mu_{\beta}) - \sum_{\gamma=1}^{m} \coth(\mu_{\alpha} - \mu_{\gamma}^{(1)}).
\]
where $\mu^{(1)}_{1}, \ldots, \mu^{(1)}_{m}$ satisfy the constraints
\[
    f^{(1)}_{\gamma} \equiv \sum_{\beta=1, \beta \neq \alpha}^{n} \coth(\mu_{\beta} - \mu^{(1)}_{\gamma}) + 2 \sum_{\delta=1, \delta \neq \gamma}^{m} \coth(\mu^{(1)}_{\gamma} - \mu^{(1)}_{\delta}) = 0. \tag{A.22}
\]

### A.2 Open boundary case

In the open boundary case, we have boundary reflection and dual reflection equations
\[
    r_{12}(\lambda - \mu)K^{(1)}_{1}(\mu)K^{(1)}_{2}(\mu) = K^{(1)}_{2}(\mu)K^{(1)}_{1}(\mu)K^{(1)}_{2}(\mu), \tag{A.23}
\]
\[
    r_{12}(\mu - \lambda)K^{(1)+}_{1}(\lambda)M^{-1}_{1}r_{21}(2\eta - \lambda - \mu)K^{(1)+}_{2}(\mu)M^{-1}_{2}
    = K^{(1)+}_{2}(\mu)M^{-1}_{2}r_{12}(2\eta - \lambda - \mu)K^{(1)+}_{1}(\lambda)M^{-1}_{1}r_{21}(\mu - \lambda). \tag{A.24}
\]

One can check $K^{(1)} = 2 \cosh(\eta + \lambda) \sinh(\lambda)/\sinh(2\lambda + \eta)$ and $K^{(1)+} = \text{diag}(e^{2\eta}, 1)$ are solutions to the first and second equations, respectively.

Using the nested monodromy matrix \([A.10]\), we define the double-row monodromy matrix for the open boundary system
\[
    T^{(1)}(\lambda) \equiv T^{(1)}(\tilde{\lambda})K^{(1)}(\lambda)T^{(1)-1}(\tilde{\lambda})
    = \left( \begin{array}{cc}
        A^{(1)}(\lambda) & B^{(1)}(\lambda) \\
        C^{(1)}(\lambda) & D^{(1)}(\lambda) \end{array} \right), \tag{A.25}
\]

where $T^{(1)}$ and $T^{(1)-1}$ are defined by
\[
    T^{(1)}_{aa_{n}}(\tilde{\lambda})e_{c_{1} \cdots c_{n}} = r(\tilde{\lambda} + \tilde{\mu}_{1})a_{1}e_{1} \cdots r(\tilde{\lambda} + \tilde{\mu}_{n})a_{n}e_{n}
    \equiv L^{(1)}_{1}(\tilde{\lambda} + \tilde{\mu}_{1})L^{(1)}_{2}(\tilde{\lambda} + \tilde{\mu}_{2}) \cdots L^{(1)}_{n}(\tilde{\lambda} + \tilde{\mu}_{n}), \tag{A.26}
\]
\[
    T^{(1)-1}(\tilde{\lambda}) = r(\tilde{\lambda} - \tilde{\mu}_{n})b_{a_{n-1}}d_{a_{n}} \cdots r(\tilde{\lambda} - \tilde{\mu}_{2})b_{a_{2}}d_{a_{1}}r(\tilde{\lambda} - \tilde{\mu}_{1})d_{a_{1}}
    \equiv L^{(1)-1}_{n}(\tilde{\lambda} + \tilde{\mu}_{n}) \cdots L^{(1)-1}_{2}(\tilde{\lambda} + \tilde{\mu}_{2})L^{(1)-1}_{1}(\tilde{\lambda} + \tilde{\mu}_{1}), \tag{A.27}
\]

respectively, and the $L$-operator takes the form
\[
    L^{(1)}_{k}(\lambda) = \begin{pmatrix}
        b(\lambda) & (b(\lambda) - a(\lambda))e_{k}^{11} & -c_{-}(\lambda)e_{k}^{21} \\
        -c_{+}(\lambda)e_{k}^{12} & b(\lambda) & (b(\lambda) - a(\lambda))e_{k}^{22} \end{pmatrix}. \tag{A.28}
\]

Let $\tilde{\lambda} = \lambda + \eta/2$, $\tilde{\mu} = \mu - \eta/2$, one sees that the above definitions coincide with \([VI.19]\).

The double-row monodromy matrix satisfies the reflection equation
\[
    r_{12}(\lambda - \mu)T^{(1)}_{1}(\lambda)r_{21}(\lambda + \mu)T^{(1)}_{2}(\mu) = T^{(1)}_{2}(\mu)r_{12}(\lambda + \mu)T^{(1)}_{1}(\lambda)r_{21}(\lambda - \mu). \tag{A.29}
\]

Thus, we can define the transfer matrix as
\[
    t^{(1)}_{b}(\lambda) = \text{str} K^{(1)+} T^{(1)}(\lambda). \tag{A.30}
\]
Around \( \eta = 0 \), we have the expansions

\[
\mathcal{T}^{(1)}(\lambda) = 1 + \eta \left( \hat{A}^{(1)}(\lambda) \hat{B}^{(1)}(\lambda) \right)_{\eta=0} + \mathcal{O}(\eta^2), \\
\hat{t}_b^{(1)}(\lambda) = -2 + \eta \hat{t}_b^{(1)}(\lambda)_{\eta=0} + \mathcal{O}(\eta^2).
\]

Write

\[
\hat{A}^{(1)}(\lambda)|_{\eta=0} = \hat{A}^{(1)}(\lambda)_{\eta=0} - \frac{e^{-2\lambda}}{\sinh(2\lambda)} \hat{D}^{(1)}(\lambda)|_{\eta=0}. \\
\begin{align*}
\hat{A}^{(1)}(\lambda)\hat{B}^{(1)}(\mu) &= \hat{C}^{(1)}(\mu)\hat{C}^{(1)}(\lambda), \\
\hat{D}^{(1)}(\lambda)\hat{C}^{(1)}(\mu) &= \frac{\sinh(2\lambda)}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \hat{C}^{(1)}(\mu)\hat{D}^{(1)}(\lambda) + \hat{C}^{(1)}(\mu)\hat{D}^{(1)}(\lambda) \\
&\quad - \frac{e^{-(\lambda+\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \left( \hat{C}^{(1)}(\lambda)\hat{D}^{(1)}(\mu) + \hat{C}^{(1)}(\mu)\hat{D}^{(1)}(\mu) \right) \\
&\quad + \frac{e^{-(\lambda-\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \left( \hat{C}^{(1)}(\lambda)\hat{A}^{(1)}(\mu) + \hat{C}^{(1)}(\mu)\hat{A}^{(1)}(\mu) \right), \\
\hat{C}^{(1)}(\lambda)\hat{C}^{(1)}(\mu) &= \frac{\sinh(2\lambda)}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \hat{C}^{(1)}(\mu)\hat{C}^{(1)}(\lambda) + \hat{C}^{(1)}(\mu)\hat{C}^{(1)}(\lambda) \\
&\quad - \frac{e^{-(\lambda+\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \left( \hat{C}^{(1)}(\lambda)\hat{C}^{(1)}(\mu) + \hat{C}^{(1)}(\mu)\hat{C}^{(1)}(\mu) \right) \\
&\quad + \frac{e^{-(\lambda-\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \left( \hat{C}^{(1)}(\lambda)\hat{A}^{(1)}(\mu) + \hat{C}^{(1)}(\mu)\hat{A}^{(1)}(\mu) \right), \\
\hat{D}^{(1)}(\lambda)\hat{C}^{(1)}(\mu)|_{\eta=0} &= \hat{C}^{(1)}(\mu)\hat{D}^{(1)}(\lambda)|_{\eta=0} + \frac{e^{-(\lambda-\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \hat{C}^{(1)}(\lambda)\hat{D}^{(1)}(\mu) \\
&\quad - \frac{e^{-(\lambda+\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \hat{C}^{(1)}(\lambda)\hat{A}^{(1)}(\mu)|_{\eta=0}, \\
\hat{A}^{(1)}(\lambda)\hat{C}^{(1)}(\mu)|_{\eta=0} &= \hat{C}^{(1)}(\mu)\hat{A}^{(1)}(\lambda)|_{\eta=0} + \frac{e^{-(\lambda-\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \hat{C}^{(1)}(\lambda)\hat{D}^{(1)}(\mu) \\
&\quad - \frac{e^{-(\lambda+\mu)}}{\sinh(\lambda + \mu) - \sinh(\lambda - \mu)} \hat{C}^{(1)}(\lambda)\hat{A}^{(1)}(\mu)|_{\eta=0}. 
\end{align*}
\]
Substituting $K^{(1)}(\lambda) = 2 \cosh(\eta + \lambda) \sinh(\lambda)/\sinh(2\lambda + \eta)$ and $K^{(1)+}(\lambda) = \text{diag}(e^{2\eta}, 1)$ into the nested transfer matrix, one obtains the transfer matrix $\hat{t}_b^{(1)}(\tilde{\lambda})$ of the $t$-$J$ Gaudin model

$$
\hat{t}_b^{(1)}(\tilde{\lambda}) \equiv \frac{d}{d\eta} \text{str} \left( \begin{array}{cc} e^{2\eta} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} A^{(1)}(\tilde{\lambda}) & B^{(1)}(\tilde{\lambda}) \\ C^{(1)}(\tilde{\lambda}) & D^{(1)}(\tilde{\lambda}) \end{array} \right) = -\hat{A}^{(1)}(\lambda) - \hat{D}^{(1)}(\lambda) - 2\hat{A}^{(1)}(\tilde{\lambda})_{\eta=0} \\
\hat{D}^{(1)}(\lambda)_{\eta=0} = -\hat{A}^{(1)}(\lambda) - \hat{D}^{(1)}(\lambda) - 2\hat{A}^{(1)}(\tilde{\lambda})_{\eta=0} - \frac{e^{-2\tilde{\lambda}}}{\sinh(2\tilde{\lambda})} \hat{D}^{(1)}(\tilde{\lambda})_{\eta=0}.
$$

(A.39)

As in the periodic case, define the vacuum state for the nested open boundary system

$$
|0 >^{(1)} = \bigotimes_{k=1}^{n} |0 >^{(1)}_k
$$

(A.40)

with $|0 >^{(1)}_k \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then, applying the elements of $\hat{T}(\lambda)$ to the vacuum, we obtain

$$
\hat{B}^{(1)}(\lambda)|0 >= 0, \quad \hat{C}^{(1)}(\lambda)|0 >= 0,
$$

(A.41)

$$
\hat{D}^{(1)}(\lambda)|0 > = -\left( \frac{1}{\sinh(2\lambda)} + \sum_{i=1}^{n} (\coth(\lambda + \mu_i) + \coth(\lambda - \mu_i)) \right)|0 >^{(1)} \lambda \neq \mu_k (k = 1, 2, \ldots, m).
$$

(A.42)

Using the nested YBE (A.11) and the transformation (A.33), we obtain

$$
\hat{A}^{(1)}(\lambda)|0 >^{(1)} = -e^{-\lambda}/\sinh(\lambda)|0 >^{(1)}.
$$

(A.43)

The eigenvalues of the nested open boundary Gaudin system can be obtained by applying $\hat{r}^{(1)}(\lambda)$ to the eigenvector

$$
\phi^{(1)}_b = \hat{C}^{(1)}(\mu^{(1)}_1)\hat{C}^{(1)}(\mu^{(1)}_2) \cdots \hat{C}^{(1)}(\mu^{(1)}_m) |0 >.
$$

(A.44)

The result is

$$
\Lambda^{(1)}(\tilde{\lambda}) = e^{-\lambda}/\sinh(\lambda) + \sum_{i=1}^{n} (\coth(\lambda + \tilde{\mu}_i) + \coth(\lambda - \tilde{\mu}_i)) \\
- \sum_{i=1}^{m} (\coth(\lambda + \mu^{(1)}_i) + \coth(\lambda - \mu^{(1)}_i)),
$$

(A.45)

where $\mu^{(1)}_i$ satisfy the constraints

$$
f^{(1)} = \sum_{i=1}^{n} \left( \coth(\mu^{(1)}_j + \mu_i) + \coth(\mu^{(1)}_j - \mu_i) \right) \\
- 2 \sum_{l=1, l \neq j}^{m} \left( \coth(\mu^{(1)}_j + \mu^{(1)}_l) + \coth(\mu^{(1)}_j - \mu^{(1)}_l) \right)
$$

(A.46)
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