Ergodic properties of the quantum geodesic flow on tori

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Abstract

We study ergodic averages for a class of pseudodifferential operators on the flat N-dimensional torus with respect to the Schrödinger evolution. The later can be consider a quantization of the geodesic flow on $T^N$. We prove that, up to semi-classically negligible corrections, such ergodic averages are translationally invariant operators.

1 Introduction

In this paper we study time averages for a class of pseudodifferential operators on the N- dimensional torus $T^N = \mathbb{R}^N/(2\pi\mathbb{Z})^N$ equipped with the natural flat Riemannian metric. More generally, instead of the torus one can consider a compact manifold $X$ and a Riemannian metric $g$ on $X$. Denote by $\Delta$ the corresponding Laplace operator (on functions).
Let $Q$ be a quantization of the cotangent space $T^*X$ of $X$, i.e. a linear correspondence $F \to Q(F)$, that to a class of functions $F : T^*X \to \mathbb{C}$ associates operators $Q(F)$ in $L^2(X, d\mu)$. Let $\mathcal{A}$ be an algebra generated by all the operators $Q(F)$. The most popular choices for $Q(F)$ are pseudo-differential operators on $X$ with symbols $F$.

The averages are taken with respect to the time evolution given by a unitary group $\{ e^{itH} \}$ of operators in $L^2(X, d\mu)$. There are two common choices for the generator $H$ of the evolution: $H = \sqrt{-\Delta}$ or $H = -\frac{1}{2} \Delta$. The later is the choice in this paper. This operator has discrete spectrum, $H\phi_n = \mu_n \phi_n$. We can arrange $\mu_n$ so that $\mu_1 \leq \mu_2 \ldots \to \infty$.

The ergodic averages are, by definition, the following expressions:

$$<Q(F)> := \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-itH} Q(F) e^{itH} \, dt.$$  

(1)

The problem considered in this paper is to compute such averages and in particular to relate them to the geodesic flow on $T^*X$.

This problem belongs to the area of quantum chaos, which studies semiclassical asymptotics of eigenfunctions and eigenvalues of quantum systems. Here $H$ is the quantum hamiltonian of the free particle on $X$, and can be considered a quantization of the geodesic flow on $T^*X$. Detailed survey of this area is contained in [11] covering both the motivation and the overview of the results.

One of the main problems in that theory is to study possible limit points of the sequence of functionals

$$\{ F \to (\phi_n, Q(F)\phi_n) \}.$$  

(2)

The point is that the ergodic averages (1) may be more tractable than general operators while not changing the limit structure of (2) since:

$$(\phi_n, Q(F)\phi_n) = (\phi_n, <Q(F) > \phi_n),$$

which gives a motivation for our study.

There are numerous results and conjectures concerning those limit points which depend on whether the corresponding geodesic flow is ergodic or not. The case of ergodic geodesic flow is more interesting and will be briefly reviewed in Section 2. If the geodesic flow is not ergodic but completely integrable, as is the case for $T^N$ considered in this paper, the limits of (2) are complicated, possibly reflecting different ergodic components (invariant tori) of the geodesic flow. Some results in that direction are described in [10].
and \[2\]. The main outcome of our analysis is that the ergodic averages are, up to semiclassically negligible correction, equal to their classical averages:

\[ < Q(F) >= Q(\bar{F}) + a_F, \]

where \(\bar{F}(p) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} F(x,p) d^N x\), and \(a_F\) is a semi-classically negligible operator - see below. Moreover, if \(N=1\), then \(a_F\) is a compact operator.

The paper is organized as follows. In section \(2\) we shortly discuss ergodic averages in different situations. Section \(\bar{3}\) describes the class of pseudodifferential operators used for our analysis. In section \(\bar{4}\) we study the ergodic averages. That section contains the main result.

## 2 Semi-classical Ergodicity

In what follows we give a short overview of quantum ergodic theorems. Let \(\mathcal{A}\) be an algebra of operators acting on a Hilbert space \(\mathcal{H}\), and let \(\rho_t\) be a one-parameter group of \(*\)-automorphisms of \(\mathcal{A}\) of the form

\[ \rho_t(a) = e^{itH}ae^{-itH}, \quad a \in \mathcal{A}, \]

where \(H\) is a self-adjoint, usually unbounded operator on \(\mathcal{H}\). The main object of ergodic theory are time averages

\[ < a > := \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_t(a) dt, \]

where the limit above is taken with respect to a suitable topology. Under convenient assumptions, this limit exists and is \(\rho_t\)-invariant. The dynamical system \((\mathcal{A}, \rho_t)\) is then called **ergodic** if the following statement holds (ergodic theorem):

\[ < a >= \tau(a)I \]

where \(\tau\) is a \(\rho_t\)-invariant state on \(\mathcal{A}\). For examples of this situation see \[4\], \[5\], \[6\].

In many important quantum mechanical problems this scenario does not apply \[11\], \[3\]. Let us assume that the spectrum of \(H\) is purely discrete:

\[ H\phi_n = \mu_n\phi_n, \]

and \(\mu_n \to \infty\). One considers the following special, invariant state on \(\mathcal{A}\):

\[ \tau(a) = \lim_{E \to \infty} \tau_E(a) := \lim_{E \to \infty} \frac{1}{\#\{\mu_n \leq E\}} \sum_{\mu_n \leq E} (\phi_n, a\phi_n). \]
Intuitively this state captures the information from very large eigenvalues of $H$ (=high energies) and is usually taken as the starting point of the semiclassical ergodic theory. An operator $a$ is called semi-classically negligible if

$$\tau(a^*a) := \lim_{E \to \infty} \tau_E(a^*a) = 0.$$  

The dynamical system $(\mathcal{A}, \rho_t)$ is then called \textit{semiclassically ergodic} if the following statement holds (semiclassical ergodic theorem):

$$<a> = \tau(a)I + C_a,$$  

(4)

where $C_a$ is semi-classically negligible. A theorem of Zelditch [9] gives sufficient conditions for (4) to hold. Let $\pi$ be the GNS representation of $\mathcal{A}$ associated with $\tau$, where $\tau$ is given by (3). The theorem says that if we assume that the algebra $\pi(\mathcal{A})$ is commutative, and that $\pi(\rho_t)$ is ergodic (in the classical sense), then the system $(\mathcal{A}, \rho_t, \tau)$ is semiclassically ergodic i.e. (4) holds.

We now describe the main example of a semiclassically ergodic system. Let $Q(f)$ be a $\psi$DO of order zero on $X$ with symbol $f : T^*X \mapsto \mathbb{C}$. Let $f_P$ be the principal symbol of $Q(f)$. It is a homogeneous function on the cotangent bundle so, effectively, it is a function on the unit sphere bundle of the cotangent bundle $f_P : ST^*X \mapsto \mathbb{C}$. Let $\gamma_t$ be the geodesic flow on $ST^*X$ and denote by $dL$ the Liouville measure on $ST^*X$. This measure is invariant with respect to the geodesic flow $\gamma_t$. Moreover, if $(X,g)$ has constant negative curvature then $\gamma_t$ is ergodic.

Take $H := \sqrt{-\Delta}$. Using $\psi$DO theory one shows that

$$\tau(Q(f)) := \lim_{E \to \infty} \frac{1}{\#\{\mu_n \leq E\}} \sum_{\mu_n \leq E} (\phi_n, Q(f)\phi_n) = \int_{ST^*X} f_P dL.$$  

Let $\mathcal{A}$ be the $\mathbb{C}^*$-algebra obtained as the norm closure of the algebra of order zero $\psi$DO on $X$. It is well known, and not difficult to prove, that $\mathcal{A}$ has the following structure:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{\sigma} C(ST^*X) \longrightarrow 0.$$  

Here $\mathcal{K}$ is the ideal of compact operators in $L^2(X,d\mu)$ and $\sigma$ is called the symbol map. It follows that if $\pi_\tau$ the GNS representation of $\mathcal{A}$ associated with $\tau$, we have $\pi_\tau(\mathcal{A}) = C(ST^*X)$, which is abelian. The Egorov’s theorem implies now that $\pi_\tau(\rho_t) = \gamma_t$, the geodesic flow on $ST^*X$. So, if $X$ has
constant negative curvature, the ergodic theorem follows. From here it is not difficult to see that the following theorem, due to Colin de Verdière and Schnirelman, is true: if $X$ has constant negative curvature, there is a subset $S \subset \mathbb{N}$ of density 1 such that

$$\lim_{n \in S, n \to \infty} (\phi_n, Q(f)\phi_n) = \int_{ST^*X} f_p \, dL.$$  \hspace{1cm} (5)

It was conjectured in that the remainder term $C_a$ in (4) is compact so that the limit in (5) exists for every subsequence. If true, this would imply that no “scars” exist for such systems.

Finally, when $X = T^N$, the geodesic flow is not ergodic but is completely integrable and the corresponding quantum system is not semiclassically ergodic. The main result of this paper establishes an analog of (4) in this case.

### 3 Operators

First we introduce a class of pseudodifferential operators suitable for our analysis. In what follows $T^N$ is the $N$-dimensional torus considered as the quotient $T^N = \mathbb{R}^N/(2\pi\mathbb{Z})^N$, and equipped with the induced flat Riemannian metric. The cotangent space $T^*(T^N)$ of $T^N$ is naturally identified with $T^N \times \mathbb{R}^N$. The operators we are interested in are constructed using symbols, which are functions on $T^N \times \mathbb{R}^N$. To specify the class of functions we need the following notation: for a bounded function $F$ on $T^N \times \mathbb{R}^N$, continuous in the $T^N$ direction, let $\hat{F}$ be its partial Fourier transform along $T^N$ i.e.

$$\hat{F}(k, p) = \frac{1}{(2\pi)^N} \int_{T^N} F(x, p) e^{-ikx} \, d^N x.$$  \hspace{1cm} (6)

Define $C_r(T^N \times \mathbb{R}^N)$, $r > N$, to be the space of functions $F$ on $T^N \times \mathbb{R}^N$ such that

$$||F||_r := \sup_{(k, p) \in \mathbb{Z}^N \times \mathbb{Z}^N} \left| (1 + |k|^2)^{r/2} \hat{F}(k, p) \right| < \infty.$$  \hspace{1cm} (6)

It should be noted that in the above definition only the values of $F(x, p)$ on $T^N \times \mathbb{Z}^N$ matter as justified below by the formula (5) for the operator $Q(F)$ defined by $F$. Also $C_r(T^N \times \mathbb{R}^N)$ is a Banach space and linear combinations of functions of the form $f(x)g(p)$, where $f$ is a trigonometric polynomial, form a dense subspace in $C_r(T^N \times \mathbb{R}^N)$. Moreover we have:
Proposition 1

\[
\frac{1}{C_r} \sup_{(x,p) \in \mathbb{T}^N \times \mathbb{Z}^N} |F(x,p)| \leq \|F\|_r \leq \sup_{(x,p) \in \mathbb{T}^N \times \mathbb{Z}^N} |(1 - \Delta)^{r/2} F(x,p)|
\]

where \(\Delta\) is the laplacian on \(\mathbb{T}^N\) and

\[
C_r = \sum_{k \in \mathbb{Z}^N} (1 + |k|^2)^{-r/2}.
\]

(As stated above we assume \(r > N\) throughout the paper).

Proof. We have:

\[
\sup_{(x,p) \in \mathbb{T}^N \times \mathbb{Z}^N} |F(x,p)| = \sup_{(x,p) \in \mathbb{Z}^N \times \mathbb{Z}^N} \left| \sum_{k \in \mathbb{Z}^N} \hat{F}(k,p) e^{ikx} \right| \leq \sup_{p \in \mathbb{Z}^N} \left| \sum_{k \in \mathbb{Z}^N} \hat{F}(k,p) \right| \leq \sup_{(k,p) \in \mathbb{Z}^N \times \mathbb{Z}^N} \left| (1 + |k|^2)^{r/2} \hat{F}(k,p) \right| \sum_{k \in \mathbb{Z}^N} (1 + |k|^2)^{-r/2}.
\]

which proves the first inequality. On the other hand we have:

\[
\sup_{(k,p) \in \mathbb{Z}^N \times \mathbb{Z}^N} \left| (1 + |k|^2)^{r/2} \hat{F}(k,p) \right| \leq \sup_{(x,p) \in \mathbb{T}^N \times \mathbb{Z}^N} \left| (1 - \Delta)^{r/2} F(x,p) \right| \leq \sup_{(x,p) \in \mathbb{T}^N \times \mathbb{Z}^N} |(1 - \Delta)^{r/2} F(x,p)|,
\]

because \(\hat{f}(k) \leq \sup_{x \in \mathbb{T}^N} |f(x)|\). □

To each symbol \(F \in C_r(\mathbb{T}^N \times \mathbb{R}^N)\) we associate a bounded operator \(Q(F)\) in \(L^2(\mathbb{T}^N, d^N x)\) by the following, global version of the usual formula:

\[
Q(F)\psi(x) = \frac{1}{(2\pi)^N} \sum_{k \in \mathbb{Z}^N} \int_{\mathbb{T}^N} F(x,k) e^{ik(x-y)} \psi(y) d^N y.
\]

Notice also that writing \(\psi \in L^2(\mathbb{T}^N, d^N x)\) in the Fourier series:

\[
\psi(x) = \sum_{k \in \mathbb{Z}^N} \psi_k e^{ikx},
\]
we have
\[ Q(F)\psi(x) = \sum_{k \in \mathbb{Z}^N} \psi_k F(x, k) e^{ikx}. \]  

(8)

Lemma 2 We have
\[ ||Q(F)|| \leq C_r ||F||_r, \]
where \( C_r \) is given by (4).

Proof. Note that in the formulas below the summation indexes run over \( \mathbb{Z}^N \).
We study \((\psi, Q(F)\psi)\) with \(\psi(x) = \sum \psi_k e^{ikx}\). Using (8) we obtain:
\[
(\psi, Q(F)\psi) = \sum_{k,l,m} \tilde{\psi}_k \psi_m \hat{F}(l, m) \int_{\mathbb{T}^N} e^{i(l+m-k)x} \frac{d^N x}{(2\pi)^N} = \\
= \sum_{k,m} \tilde{\psi}_k \psi_m \hat{F}(k - m, m) \sum_{k,m} \tilde{\psi}_k \psi_m \hat{F}(k - m, m) \frac{(1 + (k - m)^2)^{r/2}}{1 + (k - m)^2}.
\]
It follows that:
\[
| (\psi, Q(F)\psi) | \leq ||F||_r \sum_{k,m} |\psi_k| |\psi_m| (1 + (k - m)^2)^{-r/2} = \\
= ||F||_r \sum_{k,l,m} |\psi_k| |\psi_m| (1 + l^2)^{-r/2} \int_{\mathbb{T}^N} e^{i(l+m-k)x} \frac{d^N x}{(2\pi)^N}
\]
Let \(\tilde{\psi}(x) = \sum_k |\psi_k| e^{ikx}\). Notice that \(||\tilde{\psi}|| = ||\psi||\). We can use \(\tilde{\psi}\) to write the above estimate as
\[
| (\psi, Q(F)\psi) | \leq ||F||_r (\tilde{\psi}, \sum_l (1 + l^2)^{-r/2} e^{ilx} \tilde{\psi}) \leq \\
\leq ||F||_r \sup_{x \in \mathbb{T}^N} \left| \sum_l (1 + l^2)^{-r/2} e^{ilx} \right| ||\tilde{\psi}||^2 \leq C_r ||F||_r ||\psi||^2,
\]
and the claim follows. \(\square\)

Proposition 3 If \(F \in C_r(\mathbb{T}^N \times \mathbb{R}^N)\) and \(|F(x,p)| \to 0\) as \(p \to \infty\), then \(Q(F)\) is a compact operator.
Proof. First notice that if \(|F(x, p)| \to 0\) as \(p \to \infty\) then also \(|\hat{F}(k, p)| \to 0\) as \(p \to \infty\). Indeed:

\[
|\hat{F}(k, p)| = \left| \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} F(x, p) e^{-ikx} \, d^N x \right| \leq \sup_{x \in \mathbb{T}^N} |F(x, p)| \to 0.
\]

Consider now \(F_L(x, p) := \sum_{|k| \leq L} \hat{F}(k, p) e^{ikx}\). It follows that \(|F_L(x, p)| \to 0\) as \(p \to \infty\). Additionally \(F_L \to F\) in \(C_r(\mathbb{T}^N \times \mathbb{R}^N)\), which by Lemma 2 implies \(Q(F_L) \to Q(F)\) in norm. Finally \(Q(F_L)\) is a compact operator. Indeed, for each Fourier coefficient \(\hat{F}(k, \cdot)\) we have

\[
Q(\hat{F}(k, \cdot))e^{inx} = F(k, n)e^{inx}.
\]

This means that \(Q(\hat{F}(k, \cdot))\) has discrete spectrum equal to \(F(k, n), n \in \mathbb{Z}^N\) which goes to 0 as \(n \to \infty\), implying that \(Q(\hat{F}(k, \cdot))\) and also \(Q(F_L)\) are compact operators. Finally, \(Q(F)\) is compact as a norm limit of compact operators. □

The following observation is worth a notice even though it is not used in the remainder of the paper. It gives more information on the topological properties of the quantization map \(F \to Q(F)\).

**Proposition 4** If \(F_n \to F\) pointwise (a.e.) and if \(\{F_n\}\) have uniformly bounded norms (in \(C_r(\mathbb{T}^N \times \mathbb{R}^N)\)) then \(Q(T_n) \to Q(T)\) weakly.

**Proof.** First notice that Proposition 1 and uniform boundedness of \(C_r(\mathbb{T}^N \times \mathbb{R}^N)\) norms implies that \(\{F_n\}\) are uniformly bounded, that is, their sup norms are uniformly bounded. It follows using Lebesque dominated convergence theorem that \(\hat{F}_n \to \hat{F}\) pointwise. To show weak convergence of \(Q(F_n)\) we study \((\psi, Q(F_n)\psi)\) just as in Lemma 2. We have

\[
(\psi, Q(F_n)\psi) = \sum_{k, m} \tilde{\psi}_k \psi_m \hat{F}_n(k - m, m),
\]

and by the previous remark the inside of the sum converges pointwise to \(\tilde{\psi}_k \psi_m \hat{F}(k - m, m)\) as \(n \to \infty\). Also

\[
|\tilde{\psi}_k \psi_m \hat{F}_n(k - m, m)| = \left| \tilde{\psi}_k \psi_m \hat{F}_n(k - m, m) \frac{1 + (k - m)^2 r/2}{1 + (k - m)^2 r/2} \right| \\
\leq (\sup_n ||F_n||_r) |\psi_k| |\psi_m| (1 + (k - m)^2)^{-r/2}.
\]

The last expression is summable just as in the proof of Lemma 2 so using Lebesque dominated convergence theorem again concludes the proof. □
4 Ergodic averages

Let $H$ be the following operator in $L^2(\mathbb{T}^N, d^N x)$:

$$H := -\frac{1}{2} \Delta = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1} + \ldots + \frac{\partial^2}{\partial x_N} \right),$$

and let $\phi_n = e^{inx} \in L^2(\mathbb{T}^N, d^N x)$. Then $H \phi_n = \mu_n \phi_n$ with $\mu_n = n^2/2$. We are interested in studying the norm limits:

$$\langle Q(F) \rangle := \lim_{T \to \infty} \langle Q(F) \rangle_T = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-itH} Q(F) e^{itH} \, dt,$$

called ergodic averages.

The following general lemma deals with ergodic averages of compact operators.

**Lemma 5** If $C$ is a compact operator and $H$ has discrete spectrum with finite multiplicities, then $\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-itH} C e^{itH} \, dt$ exists in norm and is a compact operator.

**Proof.** We study the weak limit first:

$$\langle C \rangle = w - \lim_{T \to \infty} \langle C \rangle_T = w - \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-itH} C e^{itH} \, dt.$$

The matrix elements with respect to the basis of eigenvectors of $H$ are

$$(\phi_i, \langle C \rangle_T \phi_j) = (\phi_i, C \phi_j) \frac{1}{T} \int_0^T e^{it(\lambda_i - \lambda_j)} \, dt.$$

Consequently,

$$(\phi_i, \langle C \rangle \phi_j) = \begin{cases} 0 & \text{if } \lambda_i = \lambda_j \\ (\phi_i, C \phi_j) & \text{otherwise.} \end{cases} \quad (9)$$

We need the following notation. Writing the spectral decomposition of the Hilbert space $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ with respect to operator $H$, denote by $P_i$ the orthogonal projection onto $\mathcal{H}_i$, the eigenspace corresponding to eigenvalue $\lambda_i$. Projections $P_i$ are mutually orthogonal $P_i P_j = \delta_{ij} P_i$. We will also need $P_{\leq n} := \sum_{i \leq n} P_i$, and $P_{\geq n} := \sum_{i \geq n} P_i$. We have

$$P_i P_{\geq n} = P_i,$$
if $i \geq n$. In this notation we have

$$<C> = \sum_i P_i C P_i,$$

where the limit defining the series is in the weak topology. We claim that $<C>$ is compact. To prove it we show that

$$\left\| \sum_{i \geq n} P_i C P_i \right\| \to 0 \text{ as } n \to \infty.$$

This implies that $<C>$ is compact as a norm limit of compact operators. We compute:

$$\left\| \sum_{i \geq n} P_i C P_i x \right\|^2 = \sum_{i \geq n} \left\| P_i C P_i x \right\|^2 = \sum_{i \geq n} \left\| P_i P_{\geq n} C P_i x \right\|^2 \leq \left\| P_{\geq n} C \right\|^2 \sum_{i \geq n} \left\| P_i x \right\|^2 \leq \left\| P_{\geq n} C \right\|^2 \left\| x \right\|^2.$$

Consequently,

$$\left\| \sum_{i \geq n} P_i C P_i \right\| \leq \left\| P_{\geq n} C \right\| \to 0 \text{ as } n \to \infty,$$

since $C$ is compact.

So far we have

$$w - \lim_{T \to \infty} <C>_T = <C>,$$

with compact $<C>$. Observe also that the averaging is a contraction:

$$\left\| <C>_T \right\| \leq \left\| C \right\|.$$

To prove that $<C>_T$ converges to $<C>$ in norm we set up an approximation argument. To this end we introduce $C_n := P_{\leq n} C P_{\leq n}$. We have:

- $C_n \to C$ in norm (by compactness of $C$).
- $<C>_T \to <C> \implies P_{\leq n} <C> P_{\leq n}$ weakly, but since $C_n$ are finite dimensional, it means in norm.
- $<C>_n \to <C>$ in norm (by compactness of $<C>$).
It follows that in the estimate

\[ || < C > T - < C > || \leq || < C - C_n > T || + \\
+ || < C_n > T - < C > || + || < C > - < C > ||, \]

all three terms are small. □

**Corollary 6** If \( F \in C^r(\mathbb{T}^N \times \mathbb{R}^N) \) and \( |F(x,p)| \to 0 \) as \( p \to \infty \), then \( < Q(F) > \) is a compact operator.

**Proof.** This immediately follows from Proposition 3 and Lemma 5. □

Next we study ergodic averages of arbitrary operators \( Q(F) \). To this end we obtain an explicit formula for \( e^{itH}Q(F)e^{-itH} \).

**Lemma 7** If \( F \in C^r(\mathbb{T}^N \times \mathbb{R}^N) \) then

\[ e^{itH}Q(F)e^{-itH} = Q(F_t), \tag{10} \]

where \( F_t \in C^r(\mathbb{T}^N \times \mathbb{R}^N) \) and

\[ F_t(x,p) = e^{-it\Delta_x / 2} (F(x + tp, p)), \tag{11} \]

where \( \Delta_x = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_N^2} \) is the Laplace operator in the \( x \)-coordinate. In Fourier transform:

\[ \hat{F}_t(k,p) = e^{it(k^2 / 2 + kp)} \hat{F}(k,p). \tag{12} \]

Notice that in (11) the argument shift \( x \to x + tp \) is precisely the geodesic flow on the cotangent space of \( \mathbb{T}^N \) with respect to the flat metric. The action of \( e^{-it\Delta_x / 2} \) on \( F \) in (11) is a quantum effect, and, as we will see, it does influence the ergodic properties of the operators.

**Proof.** Since \( \hat{F} \) and \( \hat{F}_t \) differ by a phase it follows that \( F \) and \( F_t \) have the same norm:

\[ ||F_t||_r = ||F||_r. \]

Because of this, the usual 2\( \varepsilon \) argument shows that it is enough to verify (10) for a dense set of \( F \)'s. Indeed, if \( F_n \to F \) and (10) holds for \( F_n \) then in the following estimate all terms are small:

\[ ||e^{itH}Q(F)e^{-itH} - Q(F_t)|| \leq ||e^{itH}Q(F) - Q(F_n)||e^{-itH}|| + \\
+ ||e^{itH}Q(F_n)e^{-itH} - Q((F_n)t)|| + ||Q((F_n)t) - Q(F_t)||. \]
Also, both sides of (10) are linear in $F$ so it is enough to verify the hypothesis for functions in the following form: $F(x,p) = e^{ikx} g(p)$, since linear combinations of them form a dense set in $C_r(T^N \times \mathbb{R}^N)$. Applying $e^{itH} \tilde{Q}(F) e^{-itH}$ to $\psi(x) = \sum_l \psi_l e^{ilx}$ we obtain

$$e^{itH} \tilde{Q}(F) e^{-itH} \psi(x) = \sum_l e^{it(k+l)^2/2} e^{ikx} g(l) e^{-itl^2/2} \psi_l e^{ilx} =$$

$$= e^{itk^2/2} \sum_l e^{ik(x+tl)} g(l) \psi_l e^{ilx} =$$

$$= \tilde{Q}(e^{itk^2/2} e^{ik(x+tp)} g(p)) \psi(x) = \tilde{Q}(F_t) \psi(x),$$

which concludes the proof. □

Now we use Lemma 7 to explicitly calculate ergodic averages of $\tilde{Q}(F)$. It turns out that long time averaging simply drops most frequencies out from $F$ as described by the following lemma.

**Lemma 8** The norm limit $<\tilde{Q}(F)> := \lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{Q}(F) e^{itH} dt$ exists and $<\tilde{Q}(F)> = \tilde{Q}(<F>)$, where

$$<F> = \sum_{\{k \in \mathbb{Z}^N, kp+k^2/2=0\}} \widehat{F}(k,p) e^{ikx}. \quad (13)$$

**Proof.** The following observations:

- $< \cdot >$ is a contraction: $\|<\tilde{Q}(F)>_T\| \leq \|\tilde{Q}(F)\|,$
- norm estimate: $\|\tilde{Q}(F)\| \leq C_r \|F\|_r$, and
- $<\tilde{Q}(F)>_T = \tilde{Q}\left(\frac{1}{T} \int_0^T F_t dt\right),$

imply that in order to prove the lemma we need to show that the norm limit of $\frac{1}{T} \int_0^T F_t dt$ in $C_r(T^N \times \mathbb{R}^N)$ is $<F>.$

Let us write down the formulas for the Fourier components of $F_t$ and $<F>.$ Formula (12) gives:

$$\widehat{F}_t(k,p) = e^{it(k^2/2+kp)} \widehat{F}(k,p).$$

It follows that

$$\frac{1}{T} \int_0^T \widehat{F}_t(k,p) dt = \begin{cases} \widehat{F}(k,p) \left(e^{iT(kp+k^2/2)}-1\right) / iT(kp+k^2/2) & \text{if } kp+k^2/2 = 0 \\ \widehat{F}(k,p) & \text{otherwise.} \end{cases}$$
Formula (13) implies:

\[
\left\langle F \right\rangle(k,p) = \begin{cases} 
\hat{F}(k,p) & \text{if } kp + k^2/2 = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

We can now estimate the norm of the difference:

\[
\left| \frac{1}{T} \int_0^T \hat{F}_t \, dt - \left\langle F \right\rangle \right|_r = \sup_{\{k,p\in\mathbb{Z}^N; \ kp+2=0\}} (1 + k^2)^{r/2} \left| \frac{e^{iT(kp+k^2/2)} - 1}{iT(kp+k^2/2)} \right| \hat{F}(k,p) \leq \frac{2}{T} ||F||_r \sup_{\{k,p\in\mathbb{Z}^N; \ kp+2=0\}} |(kp + k^2/2)^{-1}| \leq \frac{4}{T} ||F||_r,
\]

where in the last step we used the fact that \( kp + k^2/2 \) is half-integer. Clearly this implies that \( \frac{1}{T} \int_0^T F_t \, dt \) converges in norm to \( \left\langle F \right\rangle \) and the thesis follows.

Let, as before, \( \mu_n = n^2/2 \) be the eigenvalues of \( H \). We use the notation \( N(E) := \#\{\mu_n \leq E\} \) for the number of eigenvalues of \( H \) which are less or equal \( E \). Recall [11] that an operator \( a \) is called semi-classically negligible (SN) with respect to \( H \) if

\[
\tau(a^*a) := \lim_{E\to\infty} \tau_E(a^*a) := \lim_{E\to\infty} \frac{1}{N(E)} \sum_{\mu_n \leq E} \left( \phi_n, a^* a \phi_n \right) = 0.
\]

**Proposition 9** Semi-classically negligible operators form a closed left-sided ideal in the algebra of all bounded operators.

**Proof.** We need to verify the following three things:

1. If \( a, b \) are SN so is \( a + b \)
2. If \( a_k \to a \) and \( a_k \) are SN then \( a \) is also SN
3. If \( a \) is SN and \( b \) bounded then \( ba \) is SN

Since \( \left( \phi_n, a^* a \phi_n \right) = ||a \phi_n||^2 \), part 1 follows from the estimate:

\[
||(a + b) \phi_n||^2 \leq 2(||a \phi_n||^2 + ||b \phi_n||^2).
\]
To prove part 2 we set up a $2\varepsilon$ estimate. Using (14) we get:

$$
\frac{1}{N(E)} \sum_{\mu_n \leq E} ||a\phi_n||^2 \leq \frac{2}{N(E)} \sum_{\mu_n \leq E} ||(a - a_k)\phi_n||^2 + \\
+ \frac{2}{N(E)} \sum_{\mu_n \leq E} ||a_k\phi_n||^2 \leq 2||a - a_k||^2 + \frac{2}{N(E)} \sum_{\mu_n \leq E} ||a_k\phi_n||^2,
$$

and both terms are small. Finally, part 3 is obvious:

$$
||ba\phi_n||^2 \leq ||b||^2 ||a\phi_n||^2.
$$

□

The following is the main result of our paper:

**Theorem 10** With the above notation

$$
<Q(F) >= Q(\hat{F}) + a_F,
$$

where

$$
\hat{F}(x,p) = \hat{F}(0,p) = \frac{1}{(2\pi)^N} \int_{T^N} F(x,p) d^N x,
$$

and $a_F$ is a semi-classically negligible operator. Moreover, if $N=1$, then $a_F$ is a compact operator.

**Proof.** We can rephrase the theorem as saying that all but zero Fourier component of $F$ give rise to operators with semi-classically negligible ergodic averages. Also notice that typically if $\hat{F}(0,p)$ does not go to 0 as $p \to \infty$ then $Q(\hat{F})$ is not SN.

Items 1 and 2 in Proposition 9 let us use an approximation argument. Indeed it is enough to verify that for functions in the following form:

$$
F(x,p) = e^{ikx}g(p),
$$

$k \neq 0$, the corresponding ergodic averages $<Q(F) >= Q(<F>)$ are SN.

It follows from (13) that for such $F$ we have

$$
<F>(x,p) = \begin{cases} 
  e^{ikx}g(p) & \text{if } kp + k^2/2 = 0 \\
  0 & \text{otherwise.}
\end{cases}
$$

Since $Q(<F>)(x) = <F>(x,n) e^{inx}$ we have

$$
Q(<F>)(x) = \begin{cases} 
  e^{i(k+n)x}g(n) & \text{if } kn + k^2/2 = 0 \\
  0 & \text{otherwise.}
\end{cases}
$$

(15)
We can now estimate $\tau_E(Q(<F>^*Q(<F>))$ as follows:

$$\frac{1}{N(E)} \sum_{\mu_n \leq E} ||Q(<F>)\phi_n||^2 = \frac{1}{N(E)} \sum_{n \in \mathbb{Z}^N: \mu_n \leq E, kn+k^2/2=0} |g(n)|^2 \leq ||F|| \frac{\# \{n \in \mathbb{Z}^N: \frac{1}{2}n^2 \leq E, kn+k^2/2 = 0 \}}{\# \{n \in \mathbb{Z}^N: \frac{1}{2}n^2 \leq E \}}.$$ 

In the above expression the denominator $\# \{n \in \mathbb{Z}^N: \frac{1}{2}n^2 \leq E \}$ behaves like $E^{N/2}$ for large $E$, while the numerator, because of the constraint $kn+k^2/2 = 0$ grows at most like $E^{N/2-1}$, if $N > 1$. Consequently,

$$\frac{1}{N(E)} \sum_{\mu_n \leq E} ||Q(<F>)\phi_n||^2 = O(1/E),$$

which proves that $Q(<F>)$ is SN. Consider now the case of $N = 1$. We can again use an approximation argument and, just as before, it is enough to verify that for functions in the following form: $F(x,p) = e^{ikx}g(p)$, $k \neq 0$, the corresponding ergodic averages $<Q(F)> = Q(<F>)$ are compact. Examining formula (15) we see that $Q(<F>)\phi_n(x)$ is nonzero only when $kn+k^2/2 = 0$. But this equation has no solutions if $k$ is odd, and one solution $n = -k/2$ if $k$ is even. In any case we obtain a finite dimensional operator. Since norm limits of finite dimensional operators are compact, the theorem is proved. □

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