2–DIMENSIONAL KÄHLER-EINSTEIN METRICS INDUCED BY FINITE DIMENSIONAL COMPLEX PROJECTIVE SPACES

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Abstract. We give a complete list of non-isometric bidimensional rotation invariant Kähler-Einstein submanifolds of a finite dimensional complex projective space endowed with the Fubini-Study metric. This solves in the aforementioned case a classical and long-standing problem addressed among others in [5] and [24].

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1. INTRODUCTION

1.1. Description of the problem and state of the art. Holomorphic and isometric immersions (from now on Kähler immersions) into complex space forms (i.e. Kähler manifolds with constant holomorphic sectional curvature) are a classical topic in complex differential geometry. Even though it has been extensively studied starting from S. Bochner’s work [3] and E.
Calabi's seminal paper [4], a complete classification of Kähler manifolds admitting such type of immersions does not exist, even for Kähler manifolds of great interest, such as Kähler-Einstein manifolds and homogeneous Kähler ones.

In [25], M. Umehara classified Kähler-Einstein manifolds that are Kähler immersed into a finite dimensional complex space form with non-positive holomorphic sectional curvature: they are the totally geodesic submanifolds of either the complex Euclidean space or the complex hyperbolic one. In the case when the space form has positive holomorphic curvature, i.e., the complex projective space $\mathbb{C}P^n$ (endowed with the Fubini–Study metric $g_{FS}$), only some partial results exist (see for instance [21, 5, 24, 7, 9, 10]). Motivated by this, in the present paper we consider the problem to list those complex manifolds admitting a projectively induced Kähler-Einstein metric.

**Definition 1.** We say that a Kähler metric on a connected complex manifold $M$ is projectively induced, if $M$ can be Kähler immersed into a finite dimensional complex projective space $\mathbb{C}P^n$ endowed with the Fubini–Study metric $g_{FS}$, namely the metric associated to the Kähler form given in homogeneous coordinates by

$$i \frac{1}{2} \partial \bar{\partial} \log (|Z_0|^2 + \ldots + |Z_n|^2).$$

The most relevant facts known so far about complex manifolds admitting projectively induced Kähler-Einstein metrics can be summarized by the following theorems:

**Theorem A** (S. S. Chern [5], K. Tsukada [24]). Let $(M, g)$ be a complete $n$-dimensional Kähler–Einstein manifold ($n \geq 2$). If $(M, g)$ admits a Kähler immersion into $(\mathbb{C}P^{n+2}, g_{FS})$, in particular $g$ is projectively induced, then $M$ is either totally geodesic or the complex quadric in $(\mathbb{C}P^{n+1}, g_{FS})$.

**Theorem B** (D. Hulin [10]). If a compact Kähler-Einstein manifold is projectively induced then its Einstein constant is positive.

Considering the previous results and taking also into account that all the explicit examples hitherto known are homogeneous manifolds (cfr. [22]), it has been proposed the following conjecture (see e.g. [16] Chap. 4):

\footnote{Often in the literature, the definition of projectively induced metric does not exclude that ambient complex projective space may be infinite dimensional. Our choice is dictated by purely practical reasons, indeed we are going to study a conjecture that cannot be extended to the infinite dimensional setting (see Remark [11]).}
Conjecture 1. If \((M, g)\) is a Kähler-Einstein manifold endowed with a projectively induced metric, then it is an open subset of a complex flag manifold\(\mathbb{C}P^n\).

Remark 1.1. The conjecture cannot be extended to Kähler-Einstein manifolds embedded into the infinite dimensional complex projective spaces\(\mathbb{C}P^n\), indeed explicit examples of such non-homogeneous Kähler-Einstein manifolds can be found in [9, 11, 13].

1.2. Description of the main result. The present paper is a first step toward a more ambitious research plan aimed at approaching the problem described in Section 1.1 (in particular, Conjecture 1) from a different perspective compared to the past: we do not give any assumption about the codimension of the studied immersions (cfr. [21, 5, 24, 20]). Our only assumption involves the group of symmetries of the metric. Indeed, our goal will be to test the above mentioned conjecture in the case of rotation invariant Kähler metrics (see also [13] for a list of projectively induced extremal metrics in the radial case).

Definition 2. A Kähler metric \(g\) on a connected complex manifold \(M\) is said to be rotation invariant if there exist a point \(p \in M\), a local coordinate system \((z_1, \ldots, z_n)\) centered at \(p\) and a (local) Kähler potential \(\Phi\) for \(g\) such that \(\Phi\) only depends on \(|z_1|^2, \ldots, |z_n|^2\).

Since complex projective spaces are the only irreducible rotation invariant flag manifolds (cfr. [11, 12]) and since only the integer multiples of the Fubini-Study metric are projectively induced (see [3, 16]), in the specific case of rotation invariant Kähler metrics Conjecture 1 reads as:

Conjecture 2. The only projectively induced and rotation invariant Kähler-Einstein manifolds are open subsets of \(\mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_k}\) endowed with the Kähler metric

\[
q (c_1g_{FS} \oplus \ldots \oplus c_kg_{FS}),
\]

where \(k\) and \(q \in \mathbb{Z}^+, \ c_i = \frac{1}{G_{k-1}} \prod_{j \neq i} (n_j + 1)\) for \(i = 1, \ldots, k\) and \(G = \gcd(n_1 + 1, \ldots, n_k + 1)\), namely the greatest common divisor between \(n_1 + 1, \ldots, n_k + 1\).

\(\text{2A compact simply-connected Kähler manifold acted upon transitivity by its holomorphic isometry group.}\)

\(\text{3The classification of Kähler-Einstein manifolds admitting an immersion into an infinite dimensional complex space form is an open problem in all three cases (for some partial results see e.g. [6, 13, 15, 17]).}\)

\(\text{4I.e. those Kähler metrics admitting a local potential depending only on the sum of the moduli of certain local coordinates.}\)
Remark 1.2. The homogeneous spaces \((\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}, q_{c_1g_{FS}} \oplus \cdots \oplus c_kg_{FS})\) are fully embedded into \(\mathbb{C}P^{(n_1+q_{c_1}) \cdots (n_k+q_{c_k})-1}\). A Kähler embedding can be explicitly described through a composition of suitable normalizations of the Veronese embeddings:

\[
(\mathbb{C}P^n, cg_{FS}) \to (\mathbb{C}P^{(n+c)-1}, g_{FS})
\]

\[
[Z_i]_{0 \leq i \leq n} \mapsto \sqrt{\frac{(c-1)!}{c^2-2}} \left[ \frac{Z_i^{c_0} \cdots Z_i^{c_n}}{\sqrt{c_0! \cdots c_n!}} \right]_{c_0 + \cdots + c_n = c},
\]

together with a Segre embedding (cfr. [4, 16]).

Our main result is contained in the following theorem, that solves Conjecture 2 in the 2-dimensional case.

**Theorem 1.3.** If \((M, g)\) is a 2-dimensional Kähler-Einstein manifold whose metric is rotation invariant and projectively induced, then \((M, g)\) is an open subset of either \((\mathbb{C}P^2, qg_{FS})\) or \((\mathbb{C}P^1 \times \mathbb{C}P^1, q(g_{FS} \oplus g_{FS}))\), where \(q \in \mathbb{Z}^+\).

## 2. Proof of Theorem 1.3

The proof of Theorem 1.3 is organized in three subsections, described below.

In Section 2.1, we recall the definition of Calabi’s diastasis function and Bochner’s coordinates.

In Section 2.2, on account of the results recalled in Section 2.1, by proving several auxiliary lemmas, we rephrase in Proposition 2.5 the statement of Theorem 1.3 in terms of existence and uniqueness of polynomial solutions of a particular family of real Monge-Ampère equations, where the unknown function is the Calabi’s diastasis function and the independent variables are the moduli of the Bochner’s coordinates. The existence of polynomial solutions is a part of Proposition 2.5 whereas the proof of the uniqueness of such solutions is the core of Section 2.3.

In fact, in Section 2.3, we find a set of suitable initial conditions for the aforementioned family of Monge-Ampère equations: an arbitrary polynomial solution to a Monge-Ampère equation of this family needs to satisfy one and only one initial condition of such set. Taking this into account, in the end of the section, we prove that the solutions we listed in Proposition 2.5 are actually unique, thus getting the statement of Theorem 1.3.

### 2.1. Calabi’s diastasis function.

In order to prove Theorem 1.3 we need to recall the definition of Calabi’s diastasis function and some of its properties.
Let $(M, g)$ be a Kähler manifold with a local Kähler potential $\Phi$, namely $\omega = \frac{i}{2} \partial \bar{\partial} \Phi$, where $\omega$ is the Kähler form associated to $g$. If $g$ (and hence $\Phi$) is assumed to be real analytic, by duplicating the variables $z$ and $\bar{z}$, $\Phi$ can be complex analytically extended to a function $\tilde{\Phi}$ defined in a neighbourhood $U$ of the diagonal containing $(p, \bar{p}) \in M \times \tilde{M}$ (here $\tilde{M}$ denotes the manifold conjugated to $M$). Thus one can consider the power expansion of $\Phi$ around the origin with respect to $z$ and $\bar{z}$ and write it as

$$\Phi(z, \bar{z}) = \sum_{j,l=0}^{\infty} a_{jl} z^{m_j} \bar{z}^{m_l},$$

where we arrange every $n$-tuple of nonnegative integers as a sequence $m_j = (m_{j,1}, \ldots, m_{j,n})$ and order them as follows: $m_0 = (0, \ldots, 0)$ and if $|m_j| = \sum_{\alpha=1}^{n} m_{j,\alpha}$, $|m_j| \leq |m_{j+1}|$ for all positive integer $j$. Moreover, $z^{m_j}$ denotes the monomial in $n$ variables $\prod_{\alpha=1}^{n} z_{\alpha}^{m_{j,\alpha}}$.

A Kähler potential is not unique, but it is defined up to an addition of the real part of a holomorphic function. The diastasis function $D_0$ for $g$ is nothing but the Kähler potential around $p$ such that each matrix $(a_{jk})$ defined according to equation (1) with respect to a coordinate system $z = (z_1, \ldots, z_n)$ centered in $p$, satisfies $a_{j0} = a_{0j} = 0$ for every nonnegative integer $j$.

Moreover, for any real analytic Kähler manifold there exists a coordinates system, in a neighbourhood of each point, such that

$$D_0(z) = \sum_{\alpha=1}^{n} |z_{\alpha}|^2 + \psi_{2,2},$$

where $\psi_{2,2}$ is a power series with degree $\geq 2$ in both $z$ and $\bar{z}$. These coordinates, uniquely determined up to unitary transformation (cfr. [3, 4]), are called Bochner’s coordinates (cfr. [3, 4, 9, 10, 19, 23]).

Notice that throughout this paper we will consider either projectively induced metrics or Kähler-Einstein metrics. In both cases these metrics are real analytic and hence diastasis functions and Bochner’s coordinates are defined. Moreover, in the particular case of rotation invariant metrics, the diastasis function around the origin of the Bochner’s coordinates system is a rotation invariant Kähler potential.

2.2. Real Monge-Ampère equations. The lemmas contained in this section hold for manifolds of arbitrary dimension. By applying them to the bidimensional case, we show how the property of the projectively induced metrics to be rotation invariant, allows us to address Conjecture 2 through
real analysis’ techniques. Indeed, we prove the equivalence of the statement of Theorem 1.3 to a uniqueness problem in a class of solutions of a family of real Monge-Ampère equations (Proposition 2.5).

Lemma 2.1. Let $V$ be an open subset of $\mathbb{C}^n$ where it is defined a rotation invariant potential for a Kähler metric $g$. Let $f : (V, g) \to (\mathbb{C}P^N, g_{FS})$ be a full Kähler immersion. Then $D_0(z)$ can be written as

$$D_0(z) = \log (P(z)),$$

where

$$P(z) = 1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=n+1}^{N} a_j |z_j^{m_{h_j}}|^2$$

with $a_j > 0$ and $h_j \neq h_l$ for $j \neq l$.

Proof. Recall that $Z_0, \ldots, Z_N$ are the homogeneous coordinates on $\mathbb{C}P^N$ (see Definition 1). Up to a unitary transformation of $\mathbb{C}P^N$ and by shrinking $V$ if necessary we can assume $f(p) = [1, 0, \ldots, 0]$ and $f(V) \subset U_0 = \{Z_0 \neq 0\}$. Since the affine coordinates on $U_0$ are Bochner’s coordinates for the Fubini–Study metric $g_{FS}$, by [4, Theorem 7], $f$ can be written as:

$$f : V \to \mathbb{C}^N, \quad z = (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, f_{n+1}(z), \ldots, f_N(z)),$$

where

$$f_j(z) = \sum_{l=n+1}^{\infty} \alpha_{jl} z_l^{m_j}, \quad j = n + 1, \ldots, N.$$

Since the diastasis function is hereditary (see [4, Prop. 6] and that of $\mathbb{C}P^n$ around the point $[1,0,\ldots,0]$ is given on $U_0$ by $\Phi(z) = \log(1 + \sum_{j=1}^{N} |z_j|^2)$, where $z_j = \frac{Z_j}{Z_0}$, one gets

$$D_0(z) = \log \left(1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=n+1}^{N} |f_j(z)|^2\right).$$

The rotation invariance of $D_0(z)$ and the fact that $f$ is full imply that the $f_j$’s are monomials of $z$ of different degree and formula (3) follows. □

By setting

$$x = (x_1, \ldots, x_n) = (|z_1|^2, \ldots, |z_n|^2),$$

the diastasis function $D_0$ of a rotation invariant Kähler metric $g$ can be viewed as a function of the real variables $x_i$.

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5A holomorphic immersion $f : U \to \mathbb{C}P^n$ is said to be full provided $f(U)$ is not contained in any $\mathbb{C}P^h$ for $h < n$. 
From now on we set, with a little abuse of notation,

\[ P(x) = P(z(x)), \]

where \( P(z) \) is given by (4) and \( x \) by (5).

A diastasis function of a rotation invariant Kähler-Einstein metric satisfies the following lemma.

**Lemma 2.2.** If \( g \) is a rotation invariant Kähler-Einstein metric, its diastasis \( D_0(x) \), where \( x \) is given by (5), is a solution of the real Monge-Ampère equation

\[ \det \left( \frac{\partial^2 D_0}{\partial x_\alpha \partial x_\beta} x_\alpha + \frac{\partial D_0}{\partial x_\alpha} \delta_{\alpha\beta} \right) = e^{-\frac{\lambda}{2} D_0} \]

where \( \delta_{\alpha\beta} \) is the Kronecker delta and \( \lambda \) is the Einstein constant.

**Proof.** A Kähler metric \( g \) with diastasis function \( D_0(z) \) is Einstein (see e.g. [18]) if and only if there exists \( \lambda \in \mathbb{R} \) such that

\[ \lambda i \frac{1}{2} \partial \bar{\partial} D_0 = -i \partial \bar{\partial} \log \det(g_{\alpha\bar{\beta}}). \]

Hence, by the \( \partial \bar{\partial} \)-lemma, there exists a holomorphic function \( \varphi \) such that

\[ \det(g_{\alpha\bar{\beta}}) = e^{-\frac{i}{2}(D_0 + \varphi + \bar{\varphi})}. \]

Once Bochner’s coordinates are set, by comparing the series expansions of both sides of the previous equation, we get that \( \varphi + \bar{\varphi} \) is forced to be zero (cfr. [2, 10, 20]). The PDE (8), in coordinates (5), coincides with (7). \( \square \)

**Lemma 2.3.** The Einstein constant \( \lambda \) of a projectively induced and rotation invariant Kähler-Einstein manifold of dimension \( n \) is a positive rational number less than or equal to \( 2(n + 1) \).

**Proof.** By Lemma 2.1, the diastasis of a rotation invariant and projectively induced Kähler metric can be written as \( D_0(x) = \log(P(x)) \), where \( P \) is a polynomial of type (6). By Lemma 2.2, we have

\[ D_n(P) = P^{-\frac{1}{2} + n + 1}, \]

where we denote by \( D_n \) the following differential operator

\[ D_n(P) = \frac{\det \left[ \left( P \frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - P \frac{\partial P}{\partial x_\alpha} \frac{\partial P}{\partial x_\beta} \right) x_\alpha + P \frac{\partial P}{\partial x_\alpha} \delta_{\alpha\beta} \right]}{P^{n-1}}. \]

By multilinearity of determinants and by considering that \( \left( \frac{\partial P}{\partial x_\alpha} \frac{\partial P}{\partial x_\beta} x_\alpha \right) \) is a rank-1 matrix, we get that left side of (9) is a polynomial. Therefore \( \lambda \) needs to be a rational number satisfying the inequality \(-\frac{1}{2} + n + 1 \geq 0, \)
Then we obtain the upper bound for the Einstein constant \( \lambda \). Furthermore, by comparing the degrees of both sides of (9), we get \( \lambda \geq 2\frac{n}{\deg P} > 0 \).  

**Remark 2.4.** It is worth pointing out that Conjectures 1 and 2 are of local nature, i.e. there are no the topological assumptions on projectively induced manifolds and the immersions are not required to be injective. Moreover, we notice that Lemma 2.3 has important topological consequences supporting Conjecture 2, namely a rotation invariant and projectively induced Kähler-Einstein manifold is an open subset of a complete, compact and simply-connected Kähler-Einstein manifold globally embedded into a finite dimensional complex projective space. Indeed, every Kähler-Einstein manifold embedded into a (possibly infinite dimensional) complex projective space can be extended to a complete Kähler-Einstein manifold \( M \) (see [9]). Since the Einstein constant of \( M \) is positive by Lemma 2.3, then \( M \) is compact by Myers’ theorem. Moreover, \( M \) is a simply connected by a well-known theorem of Kobayashi [11] and every local immersion of a simply-connected manifold into a complex space form can be extended to a global one (see [11]).

Now, let \( \lambda \) be the Einstein constant of a projectively induced and rotation invariant Kähler-Einstein manifold of dimension \( n \). In view of Lemma 2.3, \( \lambda = 2\frac{\alpha}{q} \), where \( \gcd(s, q) = 1 \). Since \( \gcd(2nq, s) = 1 \), a polynomial solution of type (6) to (9), is forced to be the \( q \)-th power of a polynomial \( R(x) \). After the change of variables \( z = \frac{x}{q} \), we easily check that \( R(\tilde{x}) \) is a solution for (9) with \( q = 1 \). Vice versa, every solution \( R(\tilde{x}) \) of (9) for \( q = 1 \) gives rise to a solution of (9) for \( q \neq 1 \) by taking the \( q \)-th power of \( R(\tilde{x}) \) and by considering the same changing of variables \( \tilde{x} = qx \). Hence, we are going to study from now on the real Monge-Ampère equations (9) just when \( q = 1 \).

By restricting (9) to the case \( n = 2 \), by recalling that, for our purposes, we consider only solutions belonging to the polynomial class (6) and that the upper bound for the above parameter \( s \) can be obtained by Lemma 2.3 we have that the statement of Theorem 1.3 can be get by proving the following proposition.

**Proposition 2.5.** The only solutions of type

\[
P(x) = P(x_1, x_2) = 1 + x_1 + x_2 + \xi(x_1, x_2),
\]

where \( \xi \) is a polynomial with positive coefficients and no terms of degree less than 2, to the real Monge-Ampère equation

\[
\mathcal{D}_2(P) = P^{3-s}
\]
for some integer $s \in \{1, 2, 3\}$, are

\[
\begin{align*}
1 + x_1 + x_2, & \quad \text{when } s = 3; \\
(1 + x_1)(1 + x_2) & \quad \text{when } s = 2; \\
(1 + x_1 + x_2)^3 \text{ and } (1 + \frac{x_1}{2})^2(1 + \frac{x_2}{2})^2 & \quad \text{when } s = 1.
\end{align*}
\]

2.3. Proof of Proposition 2.5. As a first step towards the proof of Proposition 2.5, we characterize the initial conditions that an arbitrary polynomial solution of type (10) to the Monge-Ampère equation (11) needs to satisfy on the coordinate axes. These conditions will be given by the Corollary 2.7 of the following lemma, that holds true for any dimension.

Lemma 2.6. The restriction $p$ on a coordinate axis of a polynomial solution of type (6) to the Monge-Ampère equation (9) reads as:

\[
\begin{align*}
p(t) &= 1 + t, \quad \text{when } s = n + 1; \\
p(t) &= (1 + \frac{t}{k})^k, \quad \text{with } k \in \{1, 2\} \quad \text{when } s = n; \\
p(t) &= (1 + \frac{t}{k})^k, \quad \text{with } k \in \mathbb{Z}^+ \quad \text{when } 1 \leq s \leq n - 1.
\end{align*}
\]

Proof. Let $p$ be the restriction on the $i$-th coordinate axis (i.e. the line $x_j = 0$, for $j \neq i$) of a polynomial solution $P$ of type (6) to the Monge-Ampère equation (9). Hence, we have that

\[
D_1(p(t))q(t) = p(t)^{n-s+1},
\]

where the polynomial $q(t)$ is the restriction on the $i$-th coordinate axis of $\prod_{j \neq i} \frac{\partial P}{\partial x_j}$. Let $\{-r_1, \ldots, -r_R\}$ be the (possibly complex) distinct roots of $p$, namely

\[
p(t) = \frac{1}{\prod_{i=1}^{R} r_i^{k_i}} \prod_{i=1}^{R} (t + r_i)^{k_i}.
\]

Considering that

\[
D_1 \left( \prod_{i=1}^{R} (t + r_i)^{k_i} \right) = \prod_{i=1}^{R} (t + r_i)^{2k_i - 2} \sum_{i=1}^{R} k_i r_i \prod_{j=1}^{R} (t + r_j)^2,
\]

the equation (14) can be written as

\[
\left( \sum_{i=1}^{R} k_i r_i \prod_{j=1}^{R} (t + r_j)^2 \right) q(x) = \frac{1}{\prod_{i=1}^{R} r_i^{k_i(n-s-1)}} \prod_{i=1}^{R} (t + r_i)^{k_i(n-s-1)+2}.
\]

Notice that the constant term of $p(x)$ and $q(x)$ are fixed to be equal to 1 by the definition of (6).
Therefore we get

\[ q(t) = \frac{1}{\prod_{i=1}^{R} r_i^{k_i(n-s-1)+2}} \prod_{i=1}^{R} (t + r_i)^{k_i(n-s-1)+2} \]

and

\[ \sum_{i=1}^{R} k_i r_i \prod_{j=1}^{R} (t + r_j)^2 - \prod_{i=1}^{R} r_i^2 = 0. \]

Let now consider (16) as a linear system in the variables \( k_1, \ldots, k_R \). If \( R = 1 \), such system consists of just one equation, which has a unique solution:

\[ k_1 = r_1. \]

If \( R \geq 2 \) it cannot be compatible for any \( t \). Indeed, being the left hand side of (16) a polynomial in \( t \) of degree \( 2R - 2 \), in particular its first \( R \) higher order coefficients have to vanish. Therefore, \( k_1, \ldots, k_R \) need to satisfy a homogeneous system, whose determinant of the coefficients matrix can be easily computed:

\[ \frac{R!}{R!} \prod_{i=1}^{R} r_i \prod_{1 \leq i < j \leq R} (r_i - r_j). \]

In view of our hypotheses, such determinant is always different from zero. Therefore our system admits only the trivial solution, leading to a contradiction, since \( k_i \) represent the multiplicity of a root of a polynomial, so they should be positive.

\[ \square \]

**Corollary 2.7.** Any arbitrary polynomial solution of type (10) to the Monge-Ampère equation (11) satisfies one and only one of the following initial conditions on the coordinate axis \( x_2 = 0 \):

\[ P(x_1, 0) = 1 + x_1, \quad \frac{\partial P}{\partial x_2}(x_1, 0) = 1 \quad \text{when } s = 3; \]

\[ P(x_1, 0) = 1 + x_1, \quad \frac{\partial P}{\partial x_2}(x_1, 0) = 1 + x_1 \quad \text{when } s = 2; \]

\[ P(x_1, 0) = (1 + \frac{x_1}{2})^2, \quad \frac{\partial P}{\partial x_2}(x_1, 0) = (1 + \frac{x_1}{2})^2 \quad \text{or} \]

\[ P(x_1, 0) = (1 + \frac{x_1}{2})^3, \quad \frac{\partial P}{\partial x_2}(x_1, 0) = (1 + \frac{x_1}{2})^2 \quad \text{when } s = 1. \]

**Proof.** Let \( P \) be a solution of type (10) to (11). By Lemma 2.6, \( P(x_1, 0) = \left(1 + \frac{x_1}{k}\right)^k \) and \( P(0, x_2) = \left(1 + \frac{x_2}{h}\right)^h \) for suitable \( k, h \in \mathbb{Z}^+ \). Moreover, by (15), \( \frac{\partial P}{\partial x_2}(x_1, 0) = \left(1 + \frac{x_1}{k}\right)^{k(1-s)+2} \) and \( \frac{\partial P}{\partial x_1}(0, x_2) = \left(1 + \frac{x_2}{h}\right)^{h(1-s)+2} \). By
computing $\frac{\partial^2 P}{\partial x_1 \partial x_2}(0,0)$, we get $k = h$. Therefore, $P$ reads as:

$$
(18) \quad \left(1 + \frac{x_1}{k}\right)^k + \left(1 + \frac{x_2}{k}\right)^k - 1 + x_1 \left(1 + \frac{x_2}{k}\right)^{k(1-s)+2} + x_2 \left(1 + \frac{x_1}{k}\right)^{k(1-s)+2} - x_1 - x_2 - \left(1 - s + \frac{2}{k}\right) x_1 x_2 + \frac{x_1^2 x_2^2}{k} \eta(x_1, x_2),
$$

where $\eta$ is a polynomial. By putting (18) in (11), by differentiating both sides of the equation by $\frac{\partial^2}{\partial x_1 \partial x_2}$ and by evaluating at $(0,0)$, we straightforwardly get the following Diophantine equation

$$
s^2 k^2 - 5sk + 6 = 0.
$$

Therefore, by solving the previous equation, we easily get our statement. □

Since each solution (12) satisfies the correspondent initial condition (17), we conclude the proof of Proposition 2.5 by showing that

**Lemma 2.8.** If there exists a polynomial solution to (11) satisfying an initial condition of type (17), then it is unique.

**Proof.** Let $F_s$ be a function whose zero defines the PDE (11), i.e., $F_s := D_2(P) - P^{3-s}$. Then, from a straightforward computation, we get the following formula

$$
(19) \quad \frac{\partial^h F_s}{\partial x_2^h}(x_1, 0) = \left( h \left( P \frac{\partial^2 P}{\partial x_1^2} x_1 - \left( \frac{\partial P}{\partial x_1} \right)^2 x_1 + P \frac{\partial P}{\partial x_1} \right) \frac{\partial^{h+1} P}{\partial x_1^{h+1}} + T^h \right)(x_1, 0),
$$

where $T^h(x_1, 0)$ is a polynomial expression in $x_1$, $P(x_1, 0)$ and derivatives of $P$ up to order $h + 1$ (computed in $(x_1, 0)$), that does not contain $\frac{\partial^h P}{\partial x_2^h}(x_1, 0)$ and $\frac{\partial^{h+1} P}{\partial x_2^{h+1}}(x_1, 0)$. If $P$ is a polynomial solution to (11) satisfying an initial condition of type (17), $P(x_1, 0) = \left(1 + \frac{x_1}{k}\right)^k$ for a suitable integer $k$, hence we have

$$
\left( P \frac{\partial^2 P}{\partial x_1^2} x_1 - \left( \frac{\partial P}{\partial x_1} \right)^2 x_1 + P \frac{\partial P}{\partial x_1} \right)(x_1, 0) = \left( \frac{x_1}{k} + 1 \right)^{2k-2} \neq 0.
$$

By considering formula (19) when $h = 1$, we realize that initial conditions (17) uniquely determine $\frac{\partial^2 P}{\partial x_2^2}(x_1, 0)$, from which one obtains $\frac{\partial^{h+1} P}{\partial x_1^h \partial x_2^h}(x_1, 0)$ for every $h \in \mathbb{N}$. By iteration, we get the whole Taylor expansion of $P$ on the line $x_2 = 0$. Therefore, we get the statement of the lemma. □
References

[1] D. Alekseevsky, A. Perelomov, *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funct. Anal. Appl. 20 (1986), no. 3, 171–182.

[2] C. Arezzo, A. Loi, *A note on Kähler-Einstein metrics and Bochner’s coordinates*, Abh. Math. Sem. Univ. Hamburg 74 (2004), 49–55.

[3] S. Bochner, *Curvature in Hermitian metric*, Bull. Amer. Math. Soc. 53 (1947), no. 2, 179–195.

[4] E. Calabi, *Isometric Imbedding of Complex Manifolds*, Ann. of Math. 58 (1953), no. 1, 1–23.

[5] S. S. Chern, *Einstein hypersurfaces in a Kähler manifold of constant sectional curvature*, J. Differential Geom. 553 (1947), no. 2, 179–195.

[6] E. Calabi, *Isometric Imbedding of Complex Manifolds*, Ann. of Math. 58 (1953), no. 1, 1–23.

[7] J. Hano, *Einstein complete intersections in complex projective space*, Math. Ann. 216 (1975), no. 3, 197–208.

[8] Y. Hao, A. Wang, L. Zhang, *On holomorphic isometric immersions of nonhomogeneous Kähler-Einstein manifolds into the infinite dimensional complex projective space*, J. Math Anal. Appl. 423 (2015), no. 1, 547–560.

[9] D. Hulin, *Sous-variétés complexes d’Einstein de l’espace projectif*, Bull. Soc. Math. France, 124 (1996), no. 2, 277–298.

[10] D. Hulin, *Kähler–Einstein metrics and projective embeddings*, J. Geom. Anal. 10 (2000), no. 3, 525–528.

[11] S. Kobayashi, *Compact Kähler manifolds with positive Ricci tensor*, Bull. Amer. Math. Soc. 67 (1961), no. 4, 412–413.

[12] A. Loi, F. Salis, F. Zuddas *A characterization of complex space forms via Laplace operators*, Abh. Math. Semin. Univ. Hambg. 90 (2020), 99–109.

[13] A. Loi, F. Salis, F. Zuddas *Extremal Kähler metrics induced by finite or infinite complex space forms*, to appear in J. Geom. Anal.

[14] A. Loi, F. Salis, F. Zuddas *Two conjectures on Ricci-flat Kähler metrics*, Math. Z. 290 (2018), 599–613.

[15] A. Loi, M. Zedda *Kähler-Einstein submanifolds of the infinite dimensional projective space*, Math. Ann. 350 (2011), 145–154.

[16] A. Loi, M. Zedda *Kähler immersions of Kähler manifolds into complex space forms*, Lecture notes of the Unione Matematica Italiana 23, Springer (2018).

[17] A. Loi, M. Zedda, F. Zuddas *Ricci flat Calabi’s metric is not projectively induced*, to appear in Tohoku Math. J.

[18] A. Moroianu, *Lectures on Kähler Geometry*, London Mathematical Society student Texts 69.

[19] W. D. Ruan *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom. 6 (1998), no. 3, 589–631.

[20] F. Salis, *Projectively induced rotation invariant Kähler metrics*, Arch. Math. 109 (2017), 285–292.

[21] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. (2) 85 (1967), no. 2, 246–266.
[22] M. Takeuchi, *Homogeneous Kähler submanifolds in complex projective spaces*, Jpn. J. Math 4 (1978), no. 1, 171–219.

[23] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. 32 (1990), no. 1, 99–130.

[24] K. Tsukada, *Einstein-Kähler Submanifolds with codimension two in a Complex Space Form*, Math. Ann. 274 (1986), 503–516.

[25] M. Umehara, *Einstein-Kähler submanifolds of complex linear or hyperbolic space*, Tohoku Math. J. 39 (1987), no. 3, 385–389.

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