Fractional dynamics on directed networks

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This paper introduces the fractional dynamics over multi-agent systems on directed networks through the construction of a fractional version of a nonsymmetric Laplacian for weighted directed graphs. The main property of the fractional-based dynamics is the possibility of exploring the network employing stochastic jumps of arbitrary length. We provide some examples of the applicability of the proposed dynamics.

I. INTRODUCTION

Systems made of highly interconnected units, where the connection stands for a kind of directed interaction between the different nodes, are a ubiquitous modeling approach for several natural phenomena. Examples include, e.g., social interactions in the real and digital world, gene regulatory networks, networks of chemical reactions, phone call networks, and many others. An efficient way for representing these complex networks is the use of graphs models. One of the main goals in this framework is to develop techniques and measures able to characterize the topology of real networks, i.e., of graphs whose structure is irregular, complex and, possibly, evolving in time. Here we investigate the behavior of certain non-local dynamical processes evolving on the network. This particular choice can be interpreted in terms of multi-hop exploration strategies on the network [1], and specifically, as a probability distribution that makes the hopper perform Lévy flights, a phenomenon that is naturally linked to the fractional calculus [2]. In some aspects, this framework was already addressed in the case of undirected graphs; see [3]. These are cases in which the underlying dynamics is governed by a symmetric operator implying that the information flows both ways along the edges of the graph, i.e., the relation between the nodes constituting the network is symmetric.

Here we investigate how the network structure affects the properties of a dynamical system evolving on it while accounting for the orientation of the connections. The method we propose can be formulated as the problem of evolving a system of ordinary differential equations in time using as coefficient matrix the fractional power of a nonsymmetric Laplacian of the underlying graph. With this approach, we can study the dynamics of networks representing different applications such as the ability of a random walker to explore these networks in a faster way Section III A, and consensus models for vehicle control, Section III B.

A. Preliminaries and notation on graphs

We recall here some basic notions on graphs that will be used in the following discussions. A directed graph, or digraph, is a pair $G = (V, E)$, where $V = v_1, \ldots, v_n$ is a set of nodes (or vertices), and $E \subseteq V \times V$ is a set of ordered pairs of distinct nodes called edges. A weighted directed graph $G = (V, E, W)$ is then obtained by considering a weight matrix $W$ with non-negative entries $(W)_{i,j} = w_{i,j} \geq 0$ and such that $w_{i,j} > 0$ if and only if $(v_i, v_j)$ is an edge of $G$. If all the nonzero weights have value 1 we omit the weighted specification. For every node $v \in V$, the degree $\deg(v)$ of $v$ is the number of edges leaving or entering $v$ taking into account their weights,

$$\deg(v_i) = \sum_{j : (v_i, v_j) \in E} w_{i,j}. \quad (1)$$

A vertex is isolated if its degree is zero.

The degree matrix $D \equiv D(G)$ is then the diagonal matrix whose entries are given by the degrees of the nodes,
i.e.,
\[
D = \text{diag}(\deg(v_1), \ldots, \deg(v_n)) = \text{diag}(d_1, \ldots, d_n).
\] (2)

In light of the fact that we want to consider dynamical processes on directed graphs, it is useful to separate the degrees also between the incoming and outgoing edges with respect to the node \(v_i\), i.e., to consider
\[
\deg_{\text{in}}(v_i) = \sum_{j : (v_j, v_i) \in E} w_{j,i}, \quad \deg_{\text{out}}(v_i) = \sum_{j : (v_j, v_i) \in E} w_{i,j},
\]

with the related diagonal matrices
\[
D_{\text{in}} = \text{diag}(\deg_{\text{in}}(v_1), \ldots, \deg_{\text{in}}(v_n)) = \text{diag}(d_{1}\text{in}, \ldots, d_{n}\text{in}),
\]
and
\[
D_{\text{out}} = \text{diag}(\deg_{\text{out}}(v_1), \ldots, \deg_{\text{out}}(v_n)) = \text{diag}(d_{1}\text{out}, \ldots, d_{n}\text{out}).
\]

Moreover, we assume from now on that no vertex of the graph has degree zero, and that all the graphs are loop-less, i.e., that there is no edge going from a vertex to itself. Given a weighted directed graph \(G = (V, E, W)\) with \(V = \{v_1, \ldots, v_n\}\), the incidence matrix \(B\) of \(G\) is the \(n \times m\) matrix whose entries \(b_{i,j}\) are given by
\[
b_{i,j} = \begin{cases} 
+\frac{w_{i,j}}{\sqrt{w_{i,j}}}, & \text{if } e_j = (v_i, v_k) \text{ for some } k, \\
0, & \text{if } e_j = (v_k, v_i) \text{ for some } k, \\
\end{cases}
\] (3)

Observe that the choice of the sign in \(B\) is purely conventional.

If the ordering of the vertices in the edges in \(E\) is not relevant, i.e., if each edge can be traversed both ways, we move from directed graphs to undirected graphs, i.e., an undirected graph is a pair \(G = (V, E)\), where \(V = \{v_1, \ldots, v_n\}\) is a set of nodes or vertices, and \(E \subseteq V \times V\) is a set of two–element subsets of \(V\) called edges such that if \((v_i, v_j) \in E\), then \((v_j, v_i) \in E\) for all \(i, j\). A weighted undirected graph \(G = (V, E, W)\) is then obtained by considering the weight matrix \(W\) with non-negative entries \((W)_{i,j} = w_{i,j} \geq 0\) and such that \(w_{i,j} > 0\) if and only if \((v_i, v_j)\) is an edge of \(G\). If all the nonzero weights have value 1 we omit the weighted specification. For any two nodes \(u, v\) in a graph \(G = (V, E)\), a walk from \(u\) to \(v\) is an ordered sequence of nodes \((v_0, v_1, \ldots, v_k)\) such that \(v_0 = u, v_k = v,\) and \((v_i, v_{i+1}) \in E\) for all \(i = 0, \ldots, k - 1\). The integer \(k\) is the length of the walk. The walk is closed if the initial and terminal nodes coincide, i.e., \(u = v\). A cycle in a graph is a non-empty closed walk in which the only repeated vertices are the first and last. An undirected graph \(G\) is connected if for any two distinct nodes \(u, v \in V\), there is a walk between \(u\) and \(v\). A directed graph \(G\) is strongly connected if for any two distinct nodes \(u, v \in V\), there is a directed walk from \(u\) to \(v\).

The definition of an incidence matrix is then simplified since now the distinction between a starting and an ending node for an edge is no longer relevant. For both a directed and an undirected graph \(G\) we introduce the adjacency matrix \(A\) as the \(n \times n\) matrix with elements
\[
(A)_{i,j} = a_{i,j} = \begin{cases} 
1, & \text{if } (u, v_j) \in E, \\
0, & \text{otherwise.}
\end{cases}
\]

Observe that the adjacency matrix \(A\) of an undirected graph \(G\) will be always symmetric. In particular, if \(G = (V, E)\) is a graph, given two nodes \(u, v \in V\), we say that \(u\) is adjacent to \(v\) and write \(u \sim v\), if \((u, v) \in E\). The above binary relation is symmetric if \(G\) is an undirected graph, while in general it is not for a directed graph. Note that for an unweighted graph, \(W = A\).

1. Graph Laplacian

Finally, let us introduce the matrix representing the focus of our interest here, i.e., the Laplacian, for either a undirected or a directed graph.

Definition 1.1 (Graph Laplacian). Let \(G = (V, E)\) be a weighted graph with weight matrix \(W\) the relative (weighted) degree matrix \(D\) and \(B\) the weighted incidence matrix \(B\), then the graph Laplacian \(L\) of \(G\) is
\[
L = D - W = BB^T.
\]
The normalized random walk version of the graph Laplacian
\[
D^{-\frac{1}{2}}L = I - D^{-\frac{1}{2}}W = D^{-\frac{1}{2}}BB^T,
\]
with \(I\) the identity matrix and \(D^{-\frac{1}{2}}W\) a row–stochastic matrix. The normalized symmetric version is
\[
D^{-\frac{1}{2}}LD^{-\frac{1}{2}} = I - D^{-\frac{1}{2}}WD^{-\frac{1}{2}}.
\]
If \(G\) is unweighted then \(W = A\) in the above definitions. Here we assume that every vertex has nonzero degree.

In the case of a directed graph the situation is more intricate since many non-equivalent definitions of the Laplacian exists. We can easily define, mimicking Definition 1.1 the non–normalized version with respect to the in– and out degrees, in both the simple and weighted case.

Definition 1.2 (Directed graph Laplacian). Let \(G = (V, E, W)\) be a weighted directed graph, with degree matrices \(D_{\text{out}}\) and \(D_{\text{in}}\) The non–normalized directed graph Laplacian \(L_{\text{out}}\) and \(L_{\text{in}}\) of \(G\) are
\[
L_{\text{out}} = D_{\text{out}} - W, \quad L_{\text{in}} = D_{\text{in}} - W.
\]

To define the normalized versions, we need to invert either the \(D_{\text{in}}\) or the \(D_{\text{out}}\) matrices, but the absence of isolated vertices is no longer sufficient to ensure this. One could have a node with only outgoing or ingoing edges. A first way of overcoming this issue could be imposing that every vertex has at least one outgoing and one incoming edges which is rather restrictive. Otherwise, we
could restrict ourselves to the set of nodes having an out, respectively in, degree different from zero, as in [4]. Another approach, that avoids reducing the size of the graph, is instead mimicking the recipe for the PageRank algorithm [5] and simply substituting any diagonal zeros in the $D_{\text{in}}$ or $D_{\text{out}}$ matrix with a one while substituting to the relative column, respectively row, the vector with entries $1/n$.

The last approach we briefly mention is the one in [6]. In this case a symmetric Laplacian is built also for the directed case. It is easy to see that this kind of approach returns the same Laplacian matrix for non isomorphic graphs, as it happens also for the case in which we define the digraph Laplacian by using the incidence matrix in Definition I.1 and build $L$ as in Definition I.1. In the rest of the paper we focus principally on the nonsymmetric Laplacian $L_{\text{out}}$ and its normalized version.

II. FRACTIONAL LAPLACIANS OF A DIRECTED GRAPH

To justify the use of a fractional Laplacian for exploring the structure of the network, let us first consider a simple diffusion problem in the case in which $G$ is an undirected graph. Let $u : V \rightarrow \mathbb{R}$ describe a “heat” distribution on the nodes of the graph with heat diffusivity $\kappa$. We can express the variation of heat in the nodes as

$$\frac{du_i}{dt} = -\kappa \sum_{j : (v_j, v_i) \in E} (u_i - u_j)$$

$$= -\kappa \left( u_i \sum_{j : (v_j, v_i) \in E} 1 - \sum_{j : (v_j, v_i) \in E} u_j \right)$$

$$= -\kappa \left( u_i \deg(v_i) - \sum_{j : (v_j, v_i) \in E} u_j \right)$$

$$= -\kappa \sum_{j : (v_j, v_i) \in E} (\delta_{i,j} \deg(v_i) - 1) u_j$$

$$= -\kappa \sum_{j : (v_j, v_i) \in E} (L)_{i,j} u_j,$$

which in matrix form reads

$$\text{find } u : [0, T] \rightarrow \mathbb{R}^n$$

$$\text{s.t. } \begin{cases} \frac{du(t)}{dt} = -\kappa L u(t), & t \in (0, T], \\ u(0) = u_0, \end{cases}$$

(4)

where now $L$ is the unweighted Laplacian from Definition I.1. Since $L$ is still a normal (symmetric) matrix, one can apply the process of “fractionalization” considered in [7] for the continuous Laplace operator. Following [2], we then exploit the spectral decomposition of $L$ to obtain its fractional extension as

$$L^\alpha \triangleq U \Lambda^\alpha U^T, \quad U^T U = I,$$

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad \alpha \in (0, 1].$$

The definition become significantly different when the case of the (non–normalized) digraph Laplacian from Definition [2] is considered. In general we lose the normality of the operator, and thus we cannot define the fractional power as in [5]. We need to define the $\alpha$th power of a non–normal matrix.

Without loss of generality, we focus the analysis on the out–degree Laplacian $L_{\text{out}} = D_{\text{out}} - A$ since it remains essentially the same in the case of the in–degree Laplacian. We recall first a suitable definition for the matrix function $f(A)$ for a generic $A$ that extends the one based on the diagonalization in [5]. The latter can be stated in terms of the Jordan canonical form of the underlying matrix [9 Section 1.2.2].

**Definition II.1** (Jordan canonical form). Any matrix $A \in \mathbb{C}^{n \times n}$ can be expressed in Jordan canonical form as

$$Z^{-1}AZ = J = \text{diag}(J_1, \ldots, J_p),$$

where

$$J_k = J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 \\ \vdots & \ddots & \ddots \end{bmatrix}_{m_k \times m_k},$$

$Z$ is nonsingular and $m_1 + m_2 + \ldots + m_p = n$. If each block in which the eigenvalue $\lambda_k$ appears is of size 1 then $\lambda_k$ is said to be a semisimple eigenvalue.

Let us denote by $\lambda_1, \ldots, \lambda_p$ the distinct eigenvalues of $A$, and by $m_i$ the order of the largest Jordan block in which the $\lambda_i$ appears, i.e., the index of the eigenvalue $\lambda_i$. We have the following definition.

**Definition II.2.** The function $f$ is defined on the spectrum of $A$ if the values

$$f^{(j)}(\lambda_i), \quad j = 0, 1, \ldots, m_i - 1, \quad i = 1, \ldots, p,$$

exist, where $f^{(j)}$ denotes the $j$th derivative of $f$ with $f^{(0)} = f$.

We can define the matrix function $f(A)$ for a generic matrix $A$ by using the Jordan canonical form.

**Definition II.3.** Let $f$ be defined on the spectrum of $A \in \mathbb{C}^{n \times n}$, which is represented in Jordan canonical form as in Definition II.1. Then,

$$f(A) \triangleq Z f(J) Z^{-1} = Z \text{diag}(f(J_1), \ldots, f(J_p)) Z^{-1},$$

where

$$f(J_k) \triangleq \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & f^{(m_k-1)}(\lambda_k) \\ \vdots & \ddots & \ddots & \vdots \\ f(m_k-1)(\lambda_k) & \cdots & \ddots & f'(\lambda_k) \\ f(\lambda_k) & \cdots & \cdots & f(\lambda_k) \end{bmatrix}_{m_k \times m_k}.$$
Moreover, let \( f \) be a multivalued function and suppose some eigenvalues occur in more than one Jordan block. If the same choice of branch of \( f \) is made in each block, then a primary matrix function is obtained.

Note that in the real symmetric case, given in \([5]\), the Jordan canonical form reduces to the diagonalization of the underlying matrix.

Hence, in order to ensure that \( f(L_{\text{out}}) = L_{\text{out}}^\alpha \), \( \alpha \in (0,1] \) is well defined, we need to check first that \( f(x) = x^\alpha \) is defined on the spectrum of \( L_{\text{out}} \) (Definition II.2). This depends on the spectral properties of the matrix \( L_{\text{out}} \). As in the symmetric case, the matrix \( L_{\text{out}} \) is a singular \( M \)-matrix.

**Definition II.4** (\( M \)-matrix, \([10]\)). A matrix \( A \in \mathbb{R}^{n \times n} \) is an \( M \)-matrix if \( A = sI - B \) for some non-negative matrix \( B \), where \( s \geq \rho(B) \), the spectral radius of \( B \). It is a singular \( M \)-matrix if \( s = \rho(B) \).

Note that the real part of a nonzero eigenvalue of a \( M \)-matrix is positive, and that the \( M \)-matrices form what is called a closed positive cone. This means that they form a closed subset of the vector space of real matrices \( \mathbb{M}_n \), and that the matrix \( \eta A + \zeta B \in \mathbb{M}_n \) if \( A,B \in \mathbb{M}_n \) and \( \eta, \zeta \in \mathbb{R}_+ \). This means that the real part of any eigenvalue of a \( M \)-matrix is positive.

**Proposition II.1** (Properties of \( L_{\text{out}} \)).

- \( L_{\text{out}} \) is a singular \( M \)-matrix,
- \( L_{\text{out}} \mathbf{1} = \mathbf{0} \),
- \( \mathbf{0} \) is a semisimple eigenvalue of \( L_{\text{out}} \).

As a consequence we have the following Theorem.

**Theorem II.1.** Given a weighted graph \( G = (V,E,W) \) and its Laplacian with respect to the out degree \( L_{\text{out}} \) (Definition II.3), the function \( f(x) = x^\alpha \) is defined on the spectrum of \( L_{\text{out}} \) and induces a matrix function for all \( \alpha \in (0,1] \).

Under the same hypothesis of Proposition II.1 we can say more about the structure of \( L_{\text{out}}^\alpha \). Indeed, this is a result that is already known for the case of the matrix \( p \)-th root.

**Theorem II.2** (Guo \([11]\)). If \( A \) is a singular \( M \)-matrix, then so is \( A^{1/p} \) for every \( p \in \mathbb{N} \).

Similarly, we get the following useful result; for the proof see the Appendix.

**Theorem II.3.** If \( A \) is a singular \( M \)-matrix with semisimple zero eigenvalues, then so is \( A^\alpha \) for every \( \alpha \in (0,1] \).

### III. APPLICATIONS

We discuss here some applications of the dynamics generated by fractional digraph Laplacians.

#### A. Random walk on the directed graph

Having defined the fractional \( \alpha \)-th power of the matrix \( L_{\text{out}} \), we consider the normalized version of \( \bar{L}_{\text{out}}^\alpha \) with entries \( (\bar{L}_{\text{out}}^\alpha)_{i,j} \). It can then be exploited to generate the discrete time dynamics of a random walk on a directed graph by considering the transition matrix \( P_{\text{out}}^\alpha = I - \bar{L}_{\text{out}}^\alpha \). As in the symmetric case discussed in \([5]\), this matrix is a row-stochastic matrix, and the standard transition matrix for the Laplacian \( L_{\text{out}} \) is recovered as \( \alpha \to 1 \).

To completely describe the behavior in a fully analytical setting we consider two test cases, the directed path \( \mathcal{P}_n \), and the directed cycle graph \( \mathcal{C}_n \).

The directed path \( \mathcal{P}_n \) is the graph with adjacency matrix \( A = (a_{ij}) \) with \( a_{ij} = 1 \) if \( i = 1, \ldots, n - 1 \), and whose outdegree Laplacian \( L_{\text{out}} \) is

\[
L_{\text{out}} = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 1 & -1 & \cdots & 0 \\
0 & -1 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 1 \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}.
\]

This is a nonsymmetric, non–diagonalizable matrix, thus we cannot apply decomposition \([5]\), and we need to use Definition II.3. Therefore, we first need to compute the Jordan canonical form of \( L_{\text{out}} = ZJZ^{-1} \), that reads as

\[
Z = \begin{bmatrix}
1 & -1 & \cdots & 0 & \cdots & 0 \\
1 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & -1 & \cdots & 1 \\
0 & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

Thus, the resulting matrix function can be expressed by computing

\[
J^\alpha = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\alpha_0 & \cdots & 0 \\
\alpha_1 & \cdots & 0 \\
\vdots & \ddots & \ddots \\
\alpha_{n-1} & \cdots & 0
\end{bmatrix}, \quad \left(\begin{array}{l}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{n-1}
\end{array}\right) = \frac{\alpha \cdots \alpha_{k+1}}{k!},
\]

and by expressing \( L_{\text{out}}^\alpha = ZJ^\alpha Z^{-1} \), and thus, for \( h \geq 1 \) and \( k < n \) we can express its \((h,k)\) element as

\[
(L_{\text{out}}^\alpha)_{h,k} = \begin{cases}
0, & \text{if } k < h \text{ or } h = n, \\
-1, & \text{if } (h,k) = (n-1,n), \\
(-1)^{h+k+1}(\alpha_{k-1}), & \text{if } 1 \leq h < n-1, \\
\alpha_{k-1}, & \text{if } h \leq k \leq n, \\
\alpha_{n-1}, & \text{if } h < k \neq (n-1,n).
\end{cases}
\]
Therefore, the probability \( p_{h \to k}^{(\alpha)} \) of a transition \( h \to k \) on the directed path graph is given by
\[
p_{h \to k}^{(\alpha)} = \begin{cases} 
0, & h = n, \\
\delta_{h,k} - \frac{(L_{\text{out}}^{\alpha})_{h,k}}{(L_{\text{out}}^{\alpha})_{h,h}} = \delta_{h,k} - (L_{\text{out}}^{\alpha})_{h,k}, & \text{otherwise.}
\end{cases}
\]
If we let the size of the graph \( n \) grow to infinity, and consider the decay of the transition probability for large values of \( k > h \) we observe that
\[
p_{h \to k}^{(\alpha)} = e^{i\pi(k+1)+O(\frac{1}{k})}\sin(\pi(k-\alpha))\frac{\Gamma(\alpha+1)}{\pi k} + O\left(\frac{1}{k^2}\right),
\]
\[
\approx -e^{i\pi k} \Gamma(\alpha+1) k^{-\alpha-1} \sin(\pi(k-\alpha)),
\]
i.e., a polynomial decay parameterized by \( \alpha \). Note that the associated chain has an absorbing state (the last vertex), which is always reached. Therefore, the effect of the nonlocality is reflected by the fact that we have a higher probability of transitioning to a far away node without exploring the network. In Figure 1, we observe the simulated behavior for 10 steps on a directed path with \( n = 20 \) nodes, always starting from the first one. Moreover, decreasing the value of \( \alpha \) resolves in faster absorption. To compute the average number of steps needed to reach the absorbing state starting from the first node we partition the matrix \( L_{\text{out}}^{\alpha} \) into the block form
\[
L^{\alpha} = \begin{bmatrix} I - Q & r \\ 0 & 0 \end{bmatrix},
\]
to extract the fundamental matrix \( Q \). Then we can compute the expected number of steps \( n_{\text{step}} \) as \( n_{\text{step}} = [I - (I - Q)^{-1}1]_t \). By reusing the computation done for \( J^{\alpha} \), it is easy to prove that \( (I - Q)^{-1} \) is the upper triangular Toeplitz matrix with first row \( (t)_\ell = t_\ell = (-1)^{\ell-1}(\frac{\alpha}{\ell-1}) \),
\[
\ell = 1, \ldots, n-1. \text{ Therefore, the expected number of steps needed to reach the absorbing state starting from the first node is}
\]
\[
n_{\text{step}} = \sum_{\ell=1}^{n-1} (-1)^{\ell-1}\left(\frac{-\alpha}{\ell-1}\right) = \frac{\left(-1\right)^{n+1}(n-1)(-\alpha)}{\alpha},
\]
which is a monotonically increasing function with respect to \( \alpha \).

For the case of the directed cycle graph \( C_n \), i.e., of the graph with \( n \) nodes \( j = 1, \ldots, n \) whose directed edges are given by \( E = \{(j,j+1), j = 1, \ldots, n-1 \} \cup \{(n,1)\} \) the out-degree Laplacian is then the circulant matrix of size \( n \) with first row \( c = [1,-1,0,\ldots,0] \), i.e., \( (L_{\text{out}}^{\alpha})_{i,j} = c_{j-i \mod n} \). This is a normal matrix which is diagonalized by the discrete Fourier matrix of size \( n \), \( F_n \), and whose eigenvalues are given by \( \lambda \ell = 1 - \exp(-2i\pi/\ell) \). By using (5) for this particular out-degree Laplacian we find
\[
(L_{\text{out}}^{\alpha})_{h,k} = \frac{1}{n} \sum_{\ell=1}^{n} \left(1 - e^{-2i\pi \ell - h} \right)^{\alpha} e^{i2\pi \ell(h-k)}.
\]
Taking the limit for \( n \to +\infty \) we can then express the \( (h,k) \) element of \( L_{\text{out}}^{\alpha} \) as
\[
(L_{\text{out}}^{\alpha})_{h,k} = \frac{1}{2\pi} \int_0^{2\pi} (1 - e^{-i\theta})^{\alpha} e^{ih\theta} d\theta = \frac{\Gamma(d_{h,k} - \alpha)}{d_{h,k} \Gamma(-\alpha)},
\]
where \( d_{h,k} = h - k \mod n \). Therefore, the probability \( p_{h \to k}^{(\alpha)} \) of a \( h \to k \) transition on the cycle graph is given by
\[
p_{h \to k}^{(\alpha)} = \delta_{h,k} - \frac{(L_{\text{out}}^{\alpha})_{h,k}}{(L_{\text{out}}^{\alpha})_{h,h}} = \delta_{h,k} - \frac{\Gamma(d_{h,k} - \alpha)}{d_{h,k} \Gamma(-\alpha)}.
\]
For \( h, k \) such that \( d_{h,k} >> 1 \) we can expand this transition probability, for \( \alpha \in (0,1) \), as
\[
p_{h \to k}^{(\alpha)} = - \frac{d_{h,k} - \alpha}{d_{h,k} \Gamma(-\alpha)} + O\left(\frac{1}{d_{h,k}^{\alpha-1}}\right) \approx - \frac{d_{h,k} - \alpha}{\Gamma(-\alpha)},
\]
thus showing that, for a large enough cycle, the transition probability behaves as a Lévy distribution with respect to \( \alpha \). In this case the underlying graph is strongly connected, therefore we do not have any absorbing states in the chain. In the local dynamics case, we are certain that in a number of steps equal to the number of nodes of the network we completely explore it, while, on the other hand, the possibility of performing longer jumps increases the probability of returning to certain states while leaving others untouched. See, e.g., the example in Figure 2 for a directed cycle graph with \( n = 20 \) nodes in which 10 jumps are performed.
These simple examples seem to suggest that, in the presence of a strong directionality in the network, the possibility of performing long-distance jumps does not necessarily lead to better (i.e., faster) exploration of the network compared to the classical, local random walk (or, in the case of continuous time, diffusion) dynamics. Real world directed networks, however, are very different from these simple “uni-directional” graphs, and allow for far richer exploration dynamics. To understand what we have gained in moving from the standard random walk on the network to its fractional extension consider the efficiency of the new dynamics in exploring the underlying directed graph compared to the classical dynamic. To measure it, we consider the average return probability at time $t$, $p(\alpha, t)(i, 0)$, for a continuous-time random walker described by the master equation for the probability $p(i, t|0, 0)$ of being at node $i$ at time $t$ having started from node $i_0$ at time $t = 0$, for the dynamics induced by the normalized version of $\tilde{L}^{(\alpha)}$. The latter reads as

$$\partial_t p(i, t|0, 0) = -\sum_j (\tilde{L}^{(\alpha)}),_{i,j} p(j, t|0, 0),$$

with initial condition $p(i, 0|0, 0) = \delta_{i,0}$. The desired average return probability is obtained as $p_0^{(\alpha)}(t) = 1/n \sum_{i=1}^{n} \exp(-\lambda_i(\tilde{L}^{(\alpha)}))$. Observe that even if $\tilde{L}^{(\alpha)}$ has complex eigenvalues, they always appear in conjugate pairs. Therefore, $p_0^{(\alpha)}(t)$ is always a real number. The following examples demonstrate that in the case of real-world complex digraphs, the use of non-local diffusion processes (or random walks) display similar advantages to those observed in the undirected case. We report in Figure 2 the quantity $p_0^{(\alpha)}(t)$ while highlighting the value of the first nonzero eigenvalue of associated integer Laplacian. As we can observe, the higher the spectral gap, i.e., the larger the modulus of the second smallest eigenvalue of the Laplacian matrix $L_{out}$ is, the more efficient the fractional exploration of the associated network is. This is an expected behavior since the average return probability is directly linked to the whole spectral distribution of the associated normalized Laplacian matrix. In particular, it is well known that sparse networks with larger spectral gap can be explored more efficiently than those having a smaller spectral gap.

We also observe that the behavior shown in Figure 3 is similar to that observed for the fractional dynamics on undirected networks in [3].

### B. Consensus models for control of vehicle motions

Consider having an ensemble of $N$ vehicles moving in an $m$th dimensional space. We denote the initial positions by $x_i \in \mathbb{R}^m$, $i = 1, \ldots, N$, and the initial velocities by $v_i \in \mathbb{R}^m$, $i = 1, \ldots, N$. We are interested in steering the vehicles from their initial position to a prefixed end state, $\{(x^*(t), v^*(t)) \in \mathbb{R}^{N \times m} : x^*(t) = v^*(t), \ \forall t \geq T_{final}\}$, while maintaining fixed the geometric configuration between them. Of the many available approaches for this task, we focus on the class of consensus algorithms for systems modeled by a second-order dynamics in which the communication among the various vehicles is described in terms of the Laplacian of the graph of their connections. Specifically, we consider the
following consensus model from [13]:

\[
\begin{align*}
\dot{x}_i &= v_i, \\
\dot{v}_i &= x_i - \beta (x_i - x_j^*) - \gamma \beta (v_i - x_j^*) - \sum_{j=1}^{N} L_{i,j} (x_i - x_j^*) - \gamma \sum_{j=1}^{N} L_{i,j} (v_i - x_j^*) - (x_j - x_j^*),
\end{align*}
\]  

(7)

which can be expressed in matrix form as

\[
\frac{d}{dt}[\tilde{x}] = \left(\begin{bmatrix} O_{n \times N} \\ -\nu \beta I_N \end{bmatrix} \otimes I_m \right) [\tilde{x}] - \left(\left(\begin{bmatrix} L_\alpha & 0 \\ 0 & L_\beta \end{bmatrix} \right) \otimes I_m \right) [\tilde{v}],
\]

where \( \tilde{x} = x^* - x \), and \( \tilde{v} = v^* - v \). From Ren [13 Theorem 3.3] we can extract the following limit result for (7).

**Theorem III.1.** Let \( L \) be the Laplacian of the graph of the connections in (7). Let \( \mu_i \) be the \( i \)-th eigenvalue of \(-L\). Then, \( x \to x^*, v \to v^* \) if

\[
\gamma > \max_i \sqrt{2} \left( |\nu_i| \cos \left( \frac{\pi}{2} \right) - \tan^{-1} \left( \frac{\text{Re}(\nu_i)}{\text{Im}(\nu_i)} \right) \right)^{-\frac{1}{2}},
\]

(8)

\( \nu_i = -\beta + \mu_i \).

The model (7) is extended here by considering a fractional power of the graph Laplacian, i.e., \( L_\alpha \), \( \alpha \in (0,1) \) instead of \( L \). The convergence analysis can be performed with the same tools used in [13] and gives a result analogous to Theorem [11,11] but with \( \mu_i \) the eigenvalues of \(-L_\alpha\).

The notable differences are that now the dynamics is faster as \( \alpha \) approaches 0, together with the fact that increasing the amount of communication helps the vehicles in maintaining their formation; see the numerical experiment in Figure 4 in which it can be seen that the position at each time step of the vehicles resembles the initial one faster as \( \alpha \) is smaller.

IV. CONCLUSIONS

In this paper we have investigated non-local diffusion dynamics (both discrete and continuous in time) on directed networks using fractional powers of a suitable (nonsymmetric) version of the graph Laplacian. We have obtained analytical solutions for two simple directed graphs (a periodic one and an absorbing one) and highlighted some differences and similarities with fractional diffusion on related undirected graphs. Experiments on a few real-world examples indicate that, similar to the undirected case, non-local (fractional) diffusion and related random walks on directed graphs result in more efficient navigation of complex directed networks than using the standard (local) counterparts. Finally, we have extended an existing consensus models for vehicle motions on directed networks to one driven by a fractional nonsymmetric Laplacian and observed that the system displays faster convergence to consensus than the standard (non-fractional) model.

In conclusion, the dynamics of nonlocal fractional diffusion appears to be a promising tool in the study of several problems involving directed networks.

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Appendix: Proofs of some of the stated results

**Proof of Theorem II.1.** By Proposition II.1 we know that 0 is a semisimple eigenvalue of \( L_{\text{out}} \), then all the Jordan blocks related to the eigenvalue \( \lambda_1 = 0 \) have index 1 and \( f(\lambda_1) = f(0) \) exists. Since \( L_{\text{out}} \) is a singular \( M \)-matrix, \( \text{Re}(\lambda_k) > 0 \), for all \( \lambda_k \neq 0 \), and \( f^{(j)}(\lambda_k) \) exist for all \( j \). Thus, by Definition II.3 \( f \) is defined on the spectrum of \( L_{\text{out}} \). Moreover, let \( \lambda \) be any nonzero eigenvalue of \( L_{\text{out}} \). Then, \( \lambda = re^{i\theta} \) with \( \theta \in [-\pi, \pi] \) since \( \text{Re}(\lambda) > 0 \) and thus \( \lambda^\alpha = r^\alpha e^{i\alpha\theta} \) with \( \alpha \theta \in [-\pi, \pi] \). Therefore, we can always select the branch of \( x^\alpha \) preserving the positivity of the real part of the eigenvalues, thus
ensuring the choice of a primary matrix function.

Proof of Theorem II.3. Let \( A(\varepsilon) = A + \varepsilon I \), then \( A(\varepsilon) \) is a nonsingular M–matrix and so is \( A(\varepsilon)^\alpha \) [14 Corollary 3.7]. By looking at the Jordan canonical forms of the matrices \( A(\varepsilon)^\alpha \) and \( A^\alpha \) (Proposition II.1), we get \( A(\varepsilon)^\alpha \to A^\alpha \) for \( \varepsilon \to 0 \). Therefore, \( A^\alpha \) is a singular M-matrix, since the M–matrices form a closed positive cone whose boundary consists of the singular M–matrices.

Moreover, note that the matrix produced in this way is a primary matrix function since we selected the same branch of the \( f(x) = x^\alpha \) for every matrix of the sequence.

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