Abstract

Starting from the standard form of the five discrete Painlevé equations we show how one can obtain (through appropriate limits) a host of new equations which are also the discrete analogues of the continuous Painlevé equations. A particularly interesting technique is the one based on the assumption that some simplification takes place in the autonomous form of the mapping following which the deautonomization leads to a new $n$-dependence and introduces more new discrete Painlevé equations.
1. Introduction.

The recent intense activity in the domain of integrable discrete systems [1] has led to the discovery of these most interesting entities that are the Painlevé mappings [2]. They are similar in essence to their continuous counterparts. In fact, for each property of the continuous Painlevé equations there exists a discrete analog [3]. However the discrete Painlevé equations (d-P’s) are richer. This becomes manifest when one examines all their possible forms. In the continuous case, there exist just one canonical form for each Painlevé equation, written as a second order differential equation of the type

\[ w'' = f(w', w, t) \]

with \( f \) rational in \( w' \), algebraic in \( w \) and analytic in \( t \). In the discrete case, on the other hand, there exists a profusion of d-P’s. This is true even when we make the restriction to three-point rational mappings, resulting from the de-autonomization of a Quispel [4] form

\[ x_{n+1} = f_1(x_n) - x_{n-1}f_2(x_n) \]

(where \( f_4 = f_2 \) at the autonomous limit). No canonical form for the d-P’s are known, neither does one know how to classify them. For historical reasons the basic forms of the first five d-P’s (the form of d-PVI is unknown to date) are [5]:

\[ x_{n+1} + x_{n-1} = -x_n + \frac{z}{x_n} + a \]  
\[ x_{n+1} + x_{n-1} = \frac{zx_n + a}{1 - x_n^2} \]  
\[ x_{n+1}x_{n-1} = \frac{ab(x_n - p)(x_n - q)}{(x_n - a)(x_n - b)} \]  
\[ (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z)^2 - c^2} \]  
\[ (x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = \frac{pq(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(x_n - p)(x_n - q)} \]

where \( z = \alpha n + \beta, p = p_0\lambda^n, q = q_0\lambda^n \) and \( a, b, c \) constants.

In the continuous case, the six Painlevé equations are known to form a coalescence cascade [6]. This means that by taking the appropriate limits of the dependent and independent variables \((w, t)\) as well as the parameters of the equation, we can recover a ‘lower’ equation starting form a ‘higher’ in the following pattern:

\[ P_{VI} \rightarrow P_V \rightarrow \{P_{IV}, P_{III}\} \rightarrow P_{II} \rightarrow P_I \]

We can easily show that this is true for the discrete equations as well (see Section 2). However, the d-P’s are related by more than just the coalescence cascade. In order to fix
the ideas let us summarize here what we mean by coalescences, limits and degeneracies. Coalescence is a limiting procedure performed on the parameters of the equation but also on the dependent variable as well as on the explicitly \(n\)-dependent ones. In that way one gets an equation that has fewer parameters than the equation one starts with. The limits correspond to taking some parameters of the equation to zero or infinity. The remaining parameters have the same \(n\)-dependence as in the initial equation. In this way one obtains either a special case of the initial equation (a trivial case) or a new equation with fewer parameters. Finally, the degenerate forms are obtained if one assumes that some simplification occurs to the initial equation prior to deautonomization. Due to this simplification, we may find new deautonomizations (corresponding to equations with fewer parameters than the initial one) leading to an equation the form of which cannot be retrieved from the original neither as a limit nor as a coalescence. This procedure allows us to generate new discrete Painlevé equations.

In what follows, we shall study the limits and degeneracies of the five ‘standard’ d-P’s (1.2). Whenever an interesting degenerate form is obtained in the autonomous case, the deautonomization is performed on the basis of the singularity confinement criterion [7]. Namely, we accept only those nonautonomous forms that satisfy this discrete integrability detector that we have developed in the recent past and which has proven already to be an efficient and valuable tool [8].

2. Coalescence cascade of the discrete Painlevé equations.

As we have explained in the introduction, the d-P’s form a coalescence cascade allowing one to obtain ‘lower’ ones starting from a ‘higher’ one by taking the appropriate limits of dependent variables as well as of the parameters (and also the explicitly \(n\)-dependent variables). The analogy with the continuous Painlevé equations is perfect. In this case the coalescence chain is:

\[
\text{d-P}_V \to \{\text{d-P}_{IV}, \text{d-P}_{III}\} \to \text{d-P}_{II} \to \text{d-P}_I
\]

In what follows, we will present the result for the five standard forms (1.2a-e). The following conventions will be used. The variables and parameters of the ‘higher’ equation will be given in capital letters \((X, Z, P, Q, A, B, C)\), while those of the ‘lower’ equation are given in lowercase letters \((x, z, p, q, a, b, c)\). The small parameter that will introduce the coalescence limit will be denoted by \(\delta\).

In order to illustrate the process, let us work out in full detail the case d-P_{II} \to
d-P$_1$. We start with the equation:

$$X_{n+1} + X_{n-1} = \frac{ZX_n + A}{1 - X_n^2} \quad (2.1)$$

We put $X = 1 + \delta x$ whereupon the equation becomes:

$$4 + 2\delta(x_{n+1} + x_{n-1} + x_n) = -\frac{Z(1 + \delta x_n) + A}{\delta x_n} \quad (2.2)$$

Now, clearly, $Z$ must cancel $A$ up to order $\delta$ and this suggests the ansatz $Z = -A - 2\delta^2 z$. Moreover, the $O(\delta^0)$ term in the rhs must cancel the 4 of the lhs and we are thus led to $A = 4 + 2\delta a$. Using these values of $Z$ and $A$ we find (at $\delta \to 0$):

$$x_{n+1} + x_{n-1} + x_n = \frac{z}{x_n} + a \quad (2.3)$$

i.e. precisely d-P$_1$.

The coalescence d-P$_{III}$ to d-P$_{II}$ requires a more delicate limit since the independent variable of d-P$_{III}$ enters in an exponential way. In order to perform the limit we take $\lambda = 1 + \gamma \delta^r$ for some $r$, whereupon $\lambda^n$ becomes $1 + n\gamma \delta^r + O(\delta^{2r})$ and thus, at the limit, $p, q$ are of the form $\alpha + \beta n + O(\delta^{2r})$ with $\beta = \alpha \gamma \delta^r$. We start from:

$$X_{n+1}X_{n-1} = \frac{AB(X_n - P)(X_n - Q)}{(X_n - A)(X_n - B)} \quad (2.4)$$

The ansatz for $X$ is here, too, $X = 1 + \delta x$. For the remaining quantities we find:

$$A = 1 + \delta, \quad B = 1 - \delta$$

$$P = 1 + \delta + \delta^2(z + a)/2 + O(\delta^3), \quad Q = 1 - \delta + \delta^2(z - a)/2 + O(\delta^3) \quad (2.5)$$

so in fact $r = 2$, and at the limit $\delta \to 0$, d-P$_{III}$ reduces exactly to d-P$_{II}$:

$$x_{n+1} + x_{n-1} = \frac{zx_n + a}{1 - x_n^2} \quad (2.6)$$

As we saw above, d-P$_{IV}$ also reduces to d-P$_{II}$. Here we start from:

$$(X_{n+1} + X_n)(X_n + X_{n-1}) = \frac{(X_n^2 - A^2)(X_n^2 - B^2)}{(X_n - Z)^2 - C^2} \quad (2.7)$$

and put $X = 1 + \delta x$. We take:

$$A = 1 + \delta, \quad B = 1 - \delta$$
\[ C = \delta - \delta^2 a/2, \quad Z = 1 - \delta^2 z/4 \]  

(2.8)

The result at \( \delta \to 0 \) is precisely d-P_{II} given by (2.6).

In the case of d-P_V:

\[(X_{n+1}X_n - 1)(X_nX_{n-1} - 1) = \frac{PQ(X_n - A)(X_n - 1/A)(X_n - B)(X_n - 1/B)}{(X_n - P)(X_n - Q)} \]  

(2.9)

two different limits exist. In order to obtain d-P_{IV} we put \( X = 1 + \delta x \) and take:

\[ A = 1 + \delta a, \quad B = 1 - \delta b \]

\[ P = 1 + \delta(z + c), \quad Q = 1 + \delta(z - c) \]

(2.10)

i.e. \( \lambda = 1 + \alpha \delta \), such that \( z = \alpha n + \beta \). At the limit \( \delta \to 0 \) we find d-P_{IV} (1.2d) in terms of the variable \( x \). The case of the coalescence d-P_V to d-P_{III} requires a different ansatz. Here we put \( X = x/\delta \). Moreover we take:

\[ P = \frac{p}{\delta}, \quad Q = \frac{q}{\delta}, \quad A = \frac{a}{\delta}, \quad B = \frac{b}{\delta} \]

(2.11)

We find then at the limit \( \delta \to 0 \):

\[ x_{n+1}x_{n-1} = \frac{pq(x_n - a)(x_n - b)}{(x_n - p)(x_n - q)} \]

(2.12)

While this is not exactly the form of d-P_{III} (1.2c or 2.4) it is very easy to reduce it to the latter. We introduce \( y \) through \( x = y\lambda^n \) (recall \( p = p_0\lambda^n, \quad q = q_0\lambda^n \)) and find with \( \mu = 1/\lambda \):

\[ y_{n+1}y_{n-1} = \frac{p_0q_0(x_n - a\mu^n)(x_n - b\mu^n)}{(x_n - p_0)(x_n - q_0)} \]

(2.13)

that is obviously of the form (2.4).

While, as far as coalescence is concerned, the discrete Painlevé equations follow closely the behaviour of the continuous ones, this will not be the case of limits and degeneracies. As we will see in the following sections, the d-P’s have a very rich structure.

3. Limits and Degenerate Forms of the d-P_1/d-P_{II} Equations.

In this section we shall examine the possible forms of discrete Painlevé equations that have the same \( x_{n+1}, x_{n-1} \) dependence as d-P_1 and d-P_{II}, namely:

\[ x_{n+1} + x_{n-1} = \frac{\beta x_n^2 + \epsilon x_n + \zeta}{\alpha x_n^2 + \beta x_n + \gamma} \]

(3.1)
In (3.1) the standard notations of the Quispel mapping have been used. The singularity confinement integrability criterion can be used on (3.1) in order to determine its possible deautonomizations. Two case must be distinguished from the outset.

a) d-P$_I$, corresponding to $\alpha = 0$ in which case we can take $\beta = 1$ and $\gamma = 0$ through a simple translation, (the case $\alpha = \beta = 0$ being linear, thus trivial). The deautonomization of this case leads simply to:

$$x_{n+1} + x_{n-1} + x_n = \frac{z}{x_n} + a$$

(3.2)

where $z$ is linear in $n$, $z = \lambda n + \kappa$, and it is just the ‘standard’ d-P$_I$. The limit $a = 0$ of (3.2) was examined in [9] and dubbed d-P$_0$. The latter is an equation that does not possess any interesting continuous limit.

b) d-P$_{II}$, corresponding to $\alpha \neq 0$, in which case we can take $\alpha = 1$ and $\beta = 0$ by translation. Two cases can be distinguished, both with $\epsilon$ linear in $n$. The first, $\gamma = -1$ is just d-P$_{II}$:

$$x_{n+1} + x_{n-1} = \frac{zx_n + a}{1 - x_n^2}$$

(3.3)

while the limit $\gamma \to 0$ corresponds to a known form [10] of d-P$_1$:

$$x_{n+1} + x_{n-1} = \frac{z}{x_n} + \frac{a}{x_n^2}$$

(3.4)

The analysis above has dealt with the limits of d-P$_{1-11}$. However, another possibility exists. Suppose that the numerator and denominator in the rhs of (3.1) have a common factor. This is what we call degenerate case. This case is of interest only when $\alpha \neq 0$ (otherwise the degenerate equation becomes linear). In this case we obtain:

$$x_{n+1} + x_{n-1} = \frac{\epsilon}{x_n + \rho}$$

(3.5)

We can translate $\rho$ to zero and deautonomizing (3.5), using singularity confinement, we obtain:

$$x_{n+1} + x_{n-1} = \frac{z}{x_n} + a$$

(3.6)

with again $z$ linear in $n$ and $a$ constant, which is another form of d-P$_1$ [5]. Its continuous limit is obtained through $x = 1 + \epsilon^2 w$, $a = 4$, $z = -2 - \epsilon^5 n$, leading at $\epsilon \to 0$ to

$$w'' + 2w^2 + t = 0,$$

with $t = \epsilon n$. (The same convention $t = \epsilon n$ will be used throughout this paper).

4. LIMITS AND DEGENERATE FORMS OF THE d-P$_{III}$ EQUATION.
Let us start with the form of d-P\(_{III}\) obtained by deautonomization of the Quispel mapping with \(f_2 = 0\):

\[
x_{n+1}x_{n-1} = -\frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \beta x_n + \gamma}
\] (4.1)

The full d-P\(_{III}\) corresponds to \(\gamma \neq 0\). We find (through application of the singularity confinement criterion) that \(\zeta = \zeta_0 \lambda^n\) and \(\mu = \mu_0 \lambda^{2n}\). Special values of \(\alpha, \beta, \zeta\) and \(\mu\) just lead to special forms of d-P\(_{III}\) with less than the full complement of free parameters.

For instance, the limit \(\alpha = 0\) in (4.1) does not present any particular interest and the transformation \(x \to 1/x\) reduces the equation to d-P\(_{III}\) with \(\mu = 0\). In the case \(\alpha = \beta = 0\) we find (through \(x \to 1/x\)) a d-P\(_{III}\) with \(\mu = \zeta = 0\) which, moreover, is strictly autonomous. The limits can be readily obtained. When \(\gamma = 0\) we find the equation (still for \(\zeta \propto \lambda^n, \mu \propto \lambda^{2n}\)):

\[
x_{n+1}x_{n-1} = \frac{\zeta x_n + \mu}{(x_n + \beta)x_n}
\] (4.2)

This is a novel form of d-P\(_{II}\). Its continuous limit can be obtained through \(x = 1 + \epsilon w, \beta = -2 + \epsilon^3 g, \zeta = -2\lambda^n, \mu = \lambda^{2n}\) where \(\lambda = 1 + \epsilon^3/2\), leading to \(w'' = 2w^3 + wt + g\). A complete study of this equation: special solutions, Bäcklund and Miura transforms etc. is reserved for a future publication [11]. A further limit can be obtained, starting from (4.1), by taking \(\beta = 0\), in addition to \(\gamma = 0\). In this case we find:

\[
x_{n+1}x_{n-1} = \frac{\zeta x_n + \mu}{x_n^2}
\] (4.3)

where, by the gauge \(x \to x\lambda^{n/2}\), \(\mu\) can be taken as a constant and \(\zeta\) of the form \(\zeta_0 \lambda^{n/2}\).

Equation (4.3) is a discrete P\(_I\) [12] as can be seen from the continuous limit obtained through \(x = 1 + \epsilon^2 w, \zeta = 4\kappa^n, \mu = -3\) and \(\kappa(\equiv \lambda^{1/2}) = 1 - \epsilon^5/4\), leading to \(w'' = 6w^2 + t\).

Let us now consider the degenerate forms of (4.1). They are obtained when the numerator and the denominator in the rhs of (4.1) have a common factor. We have in this case:

\[
x_{n+1}x_{n-1} = \frac{ax_n + b}{cx_n + d}
\] (4.4)

The deautonomization of this equation yields \(a = a_0 \lambda^n\) and \(d = d_0 \lambda^n\). Unless \(c = 0\), we can always take \(c = 1\), through division, and a proper gauge allows us to take \(b = 1\).

Equation (4.4) in its nonautonomous form is a novel form of discrete P\(_II\) [11]. The limit \(d = 0\) in (4.4) leads to the equation \((c = 1)\):

\[
x_{n+1}x_{n-1} = a + \frac{1}{x_n}
\] (4.5)
where $a = a_0 \lambda^n$. This is another form of d-P$_1$. The continuous limit is obtained through $x = x_0 (1 + \epsilon^2 w)$ where $x_0^3 = -1/2$, $a = 3x_0^2 \lambda^n$, with $\lambda = 1 - \epsilon^5/3$, leading to $w'' + 3w^2 + t = 0$. An equivalent equation can be obtained from (4.4) by taking $a = 0$:

$$x_{n+1}x_{n-1} = \frac{1}{x_n + d}$$  \hfill (4.6)

Equation (4.6) is transformed into (4.5) by taking $x \to 1/x$ and exchanging $a, d$. Another limit, leading to another d-P$_1$, is $c = 0$. We find:

$$x_{n+1}x_{n-1} = ax_n + b$$  \hfill (4.7)

where $a$, here, is a constant and $b = b_0 \lambda^n$ with continuous limit $w'' + 6w^2 + t = 0$ obtained through $x = 1 + \epsilon^2 w$, $a = 2$, $b_0 = -1$ and $\lambda = 1 + \epsilon^5$. An equivalent equation can be obtained also by taking $b = 0$ in (4.4). We find:

$$x_{n+1}x_{n-1} = \frac{ax_n}{x_n + d}$$  \hfill (4.8)

Equations (4.8) and (4.7) are related through the transformation $x \to 1/x$, with the appropriate relations of the parameters.

We remark that d-P$_{III}$ is particularly rich, since it has yielded two new multiplicative d-P$_{II}$’s and five d-P$_1$’s, three of which are genuinely independent forms.

5. LIMITS AND DEGENERATE FORMS OF THE d-P$_{IV}$ AND d-P$_{34}$ EQUATIONS.

The fact that we treat d-P$_{34}$ as a fundamental equation should not be considered a curiosity. Just as in the continuous case (number 34 in the Gambier classification), this equation is of capital importance. It is, in fact, the ‘modified’ d-P$_{II}$, in the sense that it is related to d-P$_{II}$ through a Miura transformation [13] in perfect analogy to the continuous case. Given the $x_{n+1}, x_{n-1}$ dependence of d-P$_{34}$ (we have in fact $f_2 = -xf_3$ in the Quispel notations (1.1)) it is quite natural to consider this equation together with d-P$_{IV}$.

Both d-P$_{IV}$ and d-P$_{34}$ start from a Quispel form that can be written as:

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{\alpha x_n^4 + \kappa x_n^2 + \mu}{\alpha x_n^2 + \beta x_n + \gamma}$$  \hfill (5.1)

a) Let us start with the study of d-P$_{34}$ which corresponds to $\alpha = 0$ [10]. Its standard form is obtained for $\beta \neq 0$ (and we take $\beta = 1$):

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{\kappa x_n^2 + \mu}{x_n + \gamma}$$  \hfill (5.2)
The deautonomization yields $\gamma = z$, and constant $\kappa, \mu$ as only singularity confining case. Special limits can be obtained. The simplest is the one for $\kappa = 0$:

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{\mu}{x_n + z}$$

(5.3)

This is a form of d-P$_1$. However this is not a new d-P$_1$ as can be seen through the following transformation [10]. We put $y = 1/(x_n + x_{n-1})$ and finally obtain for $y$ the d-P$_1$ (3.4). A much more interesting limit is the one corresponding to $\alpha = \beta = 0$ in (5.1). We have now:

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = a_n(x_n^2 - b^2)$$

(5.4)

The singularity confinement criterion leads to $b$ constant and $a$ a free function of $n$!

From our experience on integrable mappings we expect (5.4) to be integrable through linearization [14]. This turns out to be true. It can be shown that the solution of (5.4) is obtained by solving first $(y_{n+1} + 1)(y_n - 1) = -4/a_n$ where $y$ is related to $x$ through $y_n(x_{n-1} + x_n) + x_n - x_{n-1} - 2b = 0$. Thus this mapping comes from the coupling of a linear to a Riccati (homographic) mapping. The continuous limit is consistent with this result. Putting $x = 1 + \epsilon^2 w$, $b^2 = \epsilon^3/4$, $a = 4 + 2\epsilon^2 f(\epsilon n)$, (we recall that we have a standing convention $t = \epsilon n$), at the $\epsilon \to 0$ limit we find that (5.4) is a discretization of (a particular case of) the Gambier equation (i.e. equation number 27 in the Painlevé-Gambier classification) [15]: $w'' = \frac{w'^2}{2w} + wf(t) - \frac{1}{2w}$.

A degenerate case of (5.2) exists when the numerator and the denominator of the rhs of (5.2) have a common factor. In this case we find:

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = ax_n + b$$

(5.5)

The deautonomization of (5.5) leads to a constant $a$ and $b = z$. This equation is also a d-P$_1$ although not a new one. Putting $y_n = x_{n+1} + x_n$ we find indeed for $y$ the equation $y_{n+1} + y_{n-1} = a + (z_n + z_{n+1})/y_n$ i.e. equation (3.6).

b) we now turn to the full d-P$_{IV}$ i.e. $\alpha \neq 0$ that we rewrite as:

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - p)(x_n - q)}$$

(5.6)

The full d-P$_{IV}$ corresponds to $p = z + c$, $q = z - c$. The limiting cases of d-P$_{IV}$ do not present any interest. Thus, we look directly at the degenerate cases. First we have the
case of a rhs of (5.1) with cubic numerator and linear denominator. The application of singularity confinement criterion yields two different d-P’s. We have:

\[
(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n + z)(x_n^2 - b^2)}{(x_n - z)}
\]  

(5.7)

which is yet another novel form of d-P_I. Its continuous limit can be obtained through \(x = 5 + \epsilon^2 w, b^2 = -375\) and \(z = -3 + \epsilon^5 n\) leading to \(4w'' + 3w^2 - 25t = 0\). But, besides (5.7), we find also:

\[
(x_{n+1} + x_n)(x_n + x_{n-1}) = (x_n + k)(x_n^2 - b^2)
\]  

(5.8)

which turns out to be a novel form of d-P_{II}. Its continuous limit is given by \(x = 6 + \epsilon w, b = 18, k = -9 + g\epsilon^3\) and \(z = 7 + \epsilon^3 n\) leading to \(32w'' = w^3 - 36wt + 96g\). Finally the doubly degenerate case leads, after deautonomization, to:

\[
(x_{n+1} + x_n)(x_n + x_{n-1}) = (x_n - z)^2 - c^2
\]  

(5.9)

another form of d-P_I with continuous limit \(x = 1 + \epsilon^2 w, c^2 = -12\), and \(z = -3 + \epsilon^5 n\) leading to \(w'' + 3w^2/2 + 4t = 0\).

6. LIMITS AND DEGENERATE FORMS OF THE d-P_V EQUATION.

We shall conclude our study with d-P_V. Since this is the equation with the largest number of parameters (among the ones studied here) we expect its limits to be particularly rich. In order to study these limits it is more convenient to start with a form:

\[
(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\gamma(x_n^4 + 1) + \kappa x_n (x_n^2 + 1) + \mu x_n^2}{\alpha x_n^2 + \beta x_n + \gamma}
\]  

(6.1)

From [5,12] we know that the nonautonomous form of d-P_V corresponds to \(\alpha=\text{constant}, \beta \propto \lambda^n\) and \(\gamma, \kappa, \mu \propto \lambda^{2n}\). The limits of \(\alpha\) or \(\beta\) equal to zero with \(\gamma \neq 0\) do not present any interest: they correspond to particular cases of d-P_V. The interesting case is \(\gamma = 0\). We have then:

\[
(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\kappa(x_n^2 + 1) + \mu x_n}{\alpha x_n + \beta}
\]  

(6.2)

This equation is a novel form of d-P_{IV} as can be seen through the continuous limit \(x = 1 + \epsilon w, \alpha = 1 + \epsilon^2 a, \beta = -2\lambda^n, \kappa = -4\lambda^{2n}(1 - \epsilon^4 g), \mu = 8\lambda^{2n}\) where \(\lambda = 1 - \epsilon^2\), leading to \(w'' = \frac{w'^2}{2w} + \frac{3}{2}w^3 + 4w^2t + 2w(t^2 + a) + \frac{2}{w}\).

If, moreover we put another parameter to zero the equation reduces further to:
(6.3)

\[(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \kappa (x_n - a)(x_n - 1/a)\]

where \(\kappa \propto \lambda^n\) and \(a\) is a constant. This is a new form of d-P\(_{34}\) (continuous limit: \(w'' = \frac{w'^2}{2w} - 2w^2 - wt - 2g^2\) obtained through \(x = 1 + \epsilon^2 w, \ a = 1 + \epsilon^3 g, \ \kappa = 4\lambda^n, \ \lambda = 1 - \epsilon^3/2\)).

ii) If \(\kappa = 0\)

\[(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\mu x_n}{\alpha x_n + \beta}\]  

(6.4)

which is a new d-P\(_{II}\) (continuous limit: \(w'' + 2w^3 - 2wt + p = 0\) obtained through \(x = i + \epsilon w, \ \alpha = 1 + \epsilon^3 p/2, \ \beta = -2i\lambda^n, \ \mu = -4\lambda^n, \ \lambda = 1 - \epsilon^3/2\)).

iii) Finally, taking \(\alpha = \kappa = 0\) we find \((c = \frac{\mu}{\beta})\):

\[(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = c x_n\]  

(6.5)

which turns out to be another form of d-P\(_1\). Its continuous limit is obtained through \(x = x_0(1 + \epsilon^2 w)\) where \(x_0^2 = -1/3, \ c = -16x_0/3\lambda^n, \ \lambda = 1 - \epsilon^5/4\) leading to \(w'' + 3w^2 + t = 0\).

In order to study the degenerate cases of d-P\(_V\) it is more convenient to go back to the autonomous form (where \(p\) and \(q\) are constants):

\[(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{pq(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(x_n - p)(x_n - q)}\]  

(6.6)

We assume first that the numerator and denominator of the rhs of (6.6) have one common factor e.g. \(p = a\). This gives:

\[(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{(1 - ax_n)(x_n - b)(x_n - 1/b)}{(1 - x_n/q)}\]  

(6.7)

In order to deautonomize (6.7) we use the singularity confinement criterion and we find that two solutions exist. The first corresponds to \(a \propto \lambda^n\) and \(q \propto \lambda^{2n}\). In this case (6.7) is a new form of d-P\(_{IV}\) with continuous limit obtained through \(x = 1 + \epsilon w, \ b = 1 + \epsilon^2 g, \ a = -2\lambda^n, \ q = 4\lambda^{2n}(1 - 3\epsilon^2 c), \ \lambda = 1 + \epsilon^2\) leading to \(w'' = \frac{w'^2}{2w} + \frac{1}{6} w^3 - \frac{4}{3} w^2 t + 2w(t^2 + c) - \frac{2g^2}{w}\).

The second case corresponds to \(a = q \propto \lambda^n\). In this case the continuous limit is a P\(_{34}\) equation \(w'' = \frac{w'^2}{2w} - \frac{4}{5} w^2 - wt - \frac{2g^2}{w}\) obtained through \(x = 1 + \epsilon^2 w, \ b = 1 + \epsilon^3 g, \ a = -4\lambda^n, \ \lambda = 1 - \epsilon^3/2\).
Having obtained the degenerate form (6.7) we can first perform the limit \( q \to \infty \). In this case we have:

\[
(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = (1 - ax_n)(x_n - b)(x_n - 1/b)
\]  

(6.8)

with \( a \propto \lambda^n \) as before. Equation (6.8) is still another discrete form of \( P_{34} \) (continuous limit \( w'' = \frac{w^2}{2w} - \frac{1}{2}w^2 - wt - \frac{2w^2}{w} \) obtained through \( x = 1 + \epsilon^2 w, b = 1 + \epsilon^3 g, a = -3\lambda^n, \lambda = 1 - 2\epsilon^3/3 \)).

The second possibility is to consider a double degeneracy where \( q = b \). We find in this case:

\[
(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = (1 - ax_n)(1 - bx_n)
\]  

(6.9)

The deautonomization of (6.9) gives \( a \propto \lambda^n, b \propto \lambda^n \) and the resulting equation is a novel d-P\(_\text{II} \) (continuous limit \( w'' = w^3 + 4wt + \sqrt{2}g \) obtained through \( x = \epsilon w, a = \sqrt{2}\lambda^n, b = -(1 - \epsilon^3 g)a, \lambda = 1 + \epsilon^3 \)).

One last limit can be performed on this equation by taking \( b = 0 \) while \( a \) is always proportional to \( \lambda^n \):

\[
(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = 1 - ax_n
\]  

(6.10)

This simplified equation is now a d-P\(_\text{I} \) and its continuous limit is given by \( x = x_0(1 + \epsilon^2 w), a = 4x_0\lambda^n/3, \lambda = 1 + \epsilon^5/4 \) leading to \( x'' = 6x^2 + t \).

Before closing this section, one remark is in order. In all the cases examined above we found relations of d-P\(_V \) to d-P\(_\text{IV} \) and the equations related to the latter i.e. d-P\(_{34} \) and d-P\(_\text{II} \). Thus, the question arises naturally: “is there any relation of d-P\(_V \) to d-P\(_\text{III} \)”?

There exists of course a Miura transform between d-P\(_\text{III} \) and d-P\(_V \), but this is not what we have in mind here. (Neither do we look for a limiting process of the coalescence type that we discussed in section 2). From the theory of the continuous Painlevé equations it is known that for a special value of the parameters of P\(_V \) the latter reduces to a particular P\(_\text{III} \) for some new dependent variable. Implementing this condition to our discrete P\(_V \) we would expect the equation:

\[
(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = \frac{\gamma(x_n^2 + 1)^2}{ax_n^2 + \beta x_n + \gamma}
\]  

(6.11)

to be equivalent to some d-P\(_\text{III} \). The variable \( y \) of the latter would be related to the variable \( x \) of d-P\(_V \) in a complicated way which, at the continuous limit, should reduce to \( x = \frac{1}{2}(y + \frac{1}{y}) \). However there is no indication as to what this transformation should
be in the fully discrete case. Thus the question of the relation between d-P_{V} and d-P_{III} remains open for the time being.

7. Conclusion.

The aim of this paper was to show the extreme richness of the discrete Painlevé equations and of the relations that exist between them. We have restricted ourselves here to just the six ‘standard’ forms of the d-P’s (where the count of six is reached when we include d-P_{34}, the discrete form of d-P_{VI} being still unknown). Even so, we have been able to show that many more equations than the ones initially obtained were ‘hidden’ in the latter as limits or degenerate forms.

In every case examined in this paper we have only considered the non trivial equations. Whenever the limiting or degenerate case led to a linear equation we have omitted it altogether. The same was true for multiplicative equations, since the latter can be linearized in a straightforward way. For example, starting with \( x_{n+1}x_{n-1} = f(n)x_{n} \), an equation in the d-P_{III} family, we can reduce it to a linear equation by just taking logarithms.

The analysis presented in this paper is only part of the story. As is well known (and as this study amply illustrates) there exist many ‘alternate’ forms of the d-P’s. One could, in principle, study their coalescences, limits and degeneracies as well. Given the sheer volume that this work represents, we prefer to leave it for some future publication.

This study has added several new entries to the list of the discrete Painlevé equations (represented by 3-point mappings of one variable). What remains to be done now is to apply the arsenal we have developed in order to show that these new d-P’s have all the special properties that characterize the Painlevé transcendents and which are encountered, in a perfectly parallel way, in both continuous and discrete equations.

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