Calculation of a Class of Three-Loop Vacuum Diagrams with Two Different Mass Values

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Abstract

Using the method of Chetyrkin, Misiak, and Münz we calculate analytically a class of three-loop vacuum diagrams with two different mass values, one of which is one-third as large as the other. In particular, this specific mass ratio is of great interest in relation to the three-loop effective potential of the $O(N) \phi^4$ theory. All pole terms in $\epsilon = 4 - D$ ($D$ being the space-time dimensions in a dimensional regularization scheme) plus finite terms containing the logarithm of mass are kept in our calculation of each diagram. It is shown that three-loop effective potential calculated using three-loop integrals obtained in this paper agrees, in the large-$N$ limit, with the overlap part of leading-order (in the large-$N$ limit) calculation of Coleman, Jackiw, and Politzer [Phys. Rev. D 10, 2491 (1974)].

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I. INTRODUCTION

Although quantum field theory has a long history and there are a number of different approaches to it, Feynman diagrams are still the main source of its dynamical information. The necessity to know more exactly about the characteristics of specific physical processes and corresponding quantum-field-theoretic functions themselves stimulates the calculation of radiative corrections of ever higher order. The main purpose of this work is to calculate a class of three-loop diagrams shown in Figs. 1 to 4 in a dimensional regularization scheme [1]. Most of the diagrams carry two kinds of propagator lines: A-type lines and B-type lines. The mass parameter of the B-type line is one-third as large as that of the A-type line. These diagrams are interesting because they are genuine three-loop integrals — genuine in the sense that they cannot be factorized into lower-loop integrals — appearing in the three-loop effective potential of the massless $O(N) \phi^4$ theory.

The calculation of two-loop effective potential of the same theory can be found in Jackiw’s classic paper [2]. Very recently, the three-loop effective potential was calculated using three-loop integrals obtained in this paper: Details of calculation itself of three-loop integrals are described here and details of effective potential calculation itself were reported elsewhere [3]. Our final expression of three-loop effective potential in Ref. [3] is appended to the end of the summary section of this paper in order to compare the result of Coleman, Jackiw, and Politzer [4], since this comparison was not done in Ref. [3]. The diagrams $J(a)$ in Fig. 1, $K(a)$ in Fig. 2, $L(a)$ in Fig. 3, and $M(a)$ in Fig. 4 were already calculated in the literature [5,6,7]. The calculation of these diagrams are reproduced in this paper using the method of Chetyrkin, Misiak, and Münz [6], with which we are also going to calculate other diagrams.

The organization of this paper is as follows: In Sec. II we decompose each three-loop integral in Figs. 1 to 3 into a three-loop integral yet to be calculated and the known two- and one-loop integrals by separating the pole part and the finite part of the two-loop integral inside each three-loop integral, and discuss the calculation of diagrams in Fig. 4. Decomposed three-loop parts are calculated in Sec. III. We summarize our results of three-loop calculations by listing them in the $\epsilon$-expanded form and simply give the result of the three-loop effective potential of massless $O(N) \phi^4$ theory in order to compare it with earlier result obtained in the large-$N$ limit [1] in Sec. IV. In the Appendix, we list one- and two-loop integrals which are needed in the calculation of three-loop integrals with the method of Chetyrkin, Misiak, and Münz.

II. PRELIMINARY: DECOMPOSITION OF THREE-LOOP INTEGRALS

It is convenient to use the Euclidean metric in our discussion. Thus throughout the paper the momenta appearing in the formulas are all (Wick-rotated) Euclidean ones and the abbreviated integration measure is defined as

$$\int_k = \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D},$$

where $D = 4 - \epsilon$ is the space-time dimension in the framework of dimensional regularization [1] and $\mu$ is an arbitrary constant with mass dimension.

First let us define $P_n$ as follows:
Given as Fig. 3 are the two-loop integrals given in Eq. (A1) and \( S_n \)'s \((n = 1, 3)\) are the one-loop integrals in Eq. (A2). Likewise, let us define \( Q_n \) as

\[
Q_n \equiv \frac{1}{k_{pq}} \frac{1}{(p^2 + \sigma^2)((p + k)^2 + \sigma^2)[(q^2 + \sigma^2/3][(q + k^2 + \sigma^2/3)]]
\]

and use the following separation of the pole term

\[
\int_p \frac{1}{(p^2 + \sigma^2)((p + k)^2 + \sigma^2)[(q^2 + \sigma^2/3][(q + k^2 + \sigma^2/3)]]}
\]

\[
= \frac{1}{(4\pi)^2} \frac{1}{(4\pi^2\mu^2)^{-\epsilon/2}} \left[ \frac{2}{\epsilon} + G(k^2) \right],
\]

we readily see that three-loop integrals \( J(a) \) in Fig. 1, \( K(a) \) in Fig. 2, and \( L(a) \) in Fig. 3 are given as

\[
J(a) \equiv P_0 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon/2} \int_k \left( F(k^2) \right)^2 + \frac{4W_1}{(4\pi)^2}\left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon/2},
\]

\[
K(a) \equiv P_1 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon/2} \int_k \left( F(k^2) \right)^2 + \frac{4W_4}{(4\pi)^2}\left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon/2} - \frac{4S_1}{(4\pi)^4}\left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon},
\]

\[
L(a) \equiv P_2 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon/2} \int_k \left( F(k^2) \right)^2 + \frac{4W_6}{(4\pi)^2}\left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon/2} - \frac{4S_3}{(4\pi)^4}\left( \frac{\sigma^2}{4\pi^2\mu^2} \right)^{-\epsilon},
\]

where \( W_n \)'s \((n = 1, 2, 4, 5, 6, 8)\) are the two-loop integrals given in Eq. (A1) and \( S_n \)'s \((n = 1, 3)\) are the one-loop integrals in Eq. (A2). Similarly, defining \( R_n \) as
\[ R_n \equiv \int_{kpq} \frac{1}{(k^2 + \sigma^2/3)^n(p^2 + \sigma^2)((p + k)^2 + \sigma^2/3)(q^2 + \sigma^2)((q + k)^2 + \sigma^2/3)} , \]

and using the following separation of the pole part

\[ \int_p \frac{1}{(p^2 + \sigma^2)((p + k)^2 + \sigma^2/3)} = \frac{1}{(4\pi)^2} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon/2} \left[ \frac{2}{\epsilon} + H(k^2) \right] , \tag{5} \]

we see that three-loop integrals \( J(b) \) in Fig. 1, \( K(c) \) in Fig. 2, and \( L(c) \) in Fig. 3 are given as

\[ J(b) \equiv R_0 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k (H(k^2))^2 + \frac{4W_3}{(4\pi)^2\epsilon} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon/2} , \]

\[ K(c) \equiv R_1 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[H(k^2)]^2}{k^2 + \sigma^2/3} + \frac{4W_5}{(4\pi)^2\epsilon} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon/2} - \frac{4S_2}{(4\pi)^4\epsilon^2} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} , \]

\[ L(c) \equiv R_2 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[H(k^2)]^2}{k^2 + \sigma^2/3} + \frac{4W_7}{(4\pi)^2\epsilon} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon/2} - \frac{4S_4}{(4\pi)^4\epsilon^2} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} , \tag{6} \]

where \( W_n \)'s \((n = 3, 5, 7)\) are the two-loop integrals given in Eq. (A1) and \( S_n \)'s \((n = 2, 4)\) are the one-loop integrals in Eq. (A2). Finally, we define \( T_n \) as

\[ T_n \equiv \int_{kpq} \frac{1}{(k^2 + \sigma^2)^n(p^2 + \sigma^2/3)((p + k)^2 + \sigma^2/3)(q^2 + \sigma^2/3)((q + k)^2 + \sigma^2/3)} . \]

With the decomposition, Eq. (B), we obtain the following decomposed expressions for three-loop integrals \( J(c) \) in Fig. 1, \( K(d) \) in Fig. 2, and \( L(d) \) in Fig. 3:

\[ J(c) \equiv T_0 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon} \int_k (G(k^2))^2 + \frac{4W_2}{(4\pi)^2\epsilon} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon/2} , \]

\[ K(d) \equiv T_1 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[G(k^2)]^2}{k^2 + \sigma^2/3} + \frac{4W_5}{(4\pi)^2\epsilon} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon/2} - \frac{4S_1}{(4\pi)^4\epsilon^2} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon} , \]

\[ L(d) \equiv T_2 = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[G(k^2)]^2}{k^2 + \sigma^2/3} + \frac{4W_7}{(4\pi)^2\epsilon} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon/2} - \frac{4S_4}{(4\pi)^4\epsilon^2} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon} , \tag{7} \]

where \( W_n \)'s \((n = 2, 5, 8)\) are the two-loop integrals given in Eq. (A1) and \( S_n \)'s \((n = 1, 3)\) are the one-loop integrals in Eq. (A2).

Three integrals in Fig. 4 are relatively simple, though their three integration variables \((k, p, \text{and } q)\) are overlapped in a most complicated way. We investigate first the three-loop integral \( M(a) \) in Fig. 4:

\[ M(a) \equiv \int_{kpq} \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2)((k - p)^2 + \sigma^2)((p - q)^2 + \sigma^2)((q - k)^2 + \sigma^2)} \]

\[ = \int_k \frac{I(k^2)}{k^2 + \sigma^2} , \]

where

\[ I(k^2) = \int_{pq} \frac{1}{(p^2 + \sigma^2)((k - p)^2 + \sigma^2)((q - k)^2 + \sigma^2)((p - q)^2 + \sigma^2)} . \]
The calculation of $M(a)$ can be found in the literature [1,2]. The two-loop integration $I(k^2)$ above is finite because it has negative degree of divergence and no subdivergences. The divergences of $M(a)$ can only arise from integration over large $k^2$. Thus it is sufficient to know the behavior of $I(k^2)$ at large $k^2$. Details can be found in Ref. [5]. In the Appendix B of Ref. [4] the method of Kotikov [8] is used for the integration of $M(a)$. The result is

$$M(a) = \frac{1}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-3\epsilon/2} \left[ \frac{4\zeta(3)}{\epsilon} \right].$$

Other two three-loop integrals $M(b)$ and $M(c)$ in Fig. 4 are equal to $M(a)$ within the desired order. This can be seen easily by expanding the propagators with one third of $\sigma^2$ in the following fashion:

$$\frac{1}{p^2 + \sigma^2/3} = \frac{1}{p^2 + \sigma^2 - (2/3)\sigma^2} = \frac{1}{p^2 + \sigma^2} \sum_{\ell=0}^{\infty} \left( \frac{2\sigma^2}{3} \right)^\ell \left( \frac{1}{p^2 + \sigma^2} \right)^\ell.$$

Consequently we have

$$M(b) = \int_{k_{pq}} \frac{1}{(k^2 + \sigma^2/3)(p^2 + \sigma^2/3)(q^2 + \sigma^2/3)((k-p)^2 + \sigma^2)((p-q)^2 + \sigma^2)((q-k)^2 + \sigma^2)} \equiv M(a) + O(\epsilon^0),$$

$$M(c) = \int_{k_{pq}} \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2/3)(q^2 + \sigma^2/3)((k-p)^2 + \sigma^2/3)((p-q)^2 + \sigma^2)((q-k)^2 + \sigma^2/3)} \equiv M(a) + O(\epsilon^0).$$

### III. INTEGRALS CONTAINING $F$, $G$, AND $H$

In order to calculate the $k$ integrals containing $F(k^2)$, $G(k^2)$, and $H(k^2)$, let us first investigate the following decomposition of a propagator-type one-loop integral:

$$\int_p \frac{1}{(p^2 + \sigma^2)_{k^2}} \frac{1}{(p+k)^2 + \sigma^2} = \frac{(\sigma^2)^{4-2(k_1+k_2)}}{(4\pi)^2} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon/2} \left[ -\frac{2}{\epsilon} \delta_{1k_1}\delta_{1k_2} + F^{\text{ren}}_{k_1k_2}(k^2) \right].$$

The asymptotic behavior of $F^{\text{ren}}_{k_1k_2}(k^2)$ is given as [3,4]

$$F^{\text{ren}}_{k_1k_2}(k^2) = \sum_{r=0}^{\infty} \left[ \left( \frac{\sigma^2}{k^2} \right)^r a^{\text{ren}}(k_1, k_2, r) + \left( \frac{\sigma^2}{k^2} \right)^{r+\epsilon/2} b(k_1, k_2, r) \right],$$

where

$$a^{\text{ren}}(k_1, k_2, r) = \begin{cases} -\frac{2}{\epsilon} \delta_{1k_1}\delta_{1k_2}\delta_{0r} & \text{(when } r < k_1 \text{ or } \frac{k_1+k_2}{2} \leq r < k_2 \text{)}, \\ -\frac{2}{\epsilon} \delta_{1k_1}\delta_{1k_2}\delta_{0r} + \frac{(-1)^{r-k_1}(r-1)!(k_2-r-1)!\Gamma(k_1+k_2-r-2+\epsilon/2)}{(k_1-1)!(k_2-1)!(r-k_1)!(k_1+k_2-2r-1)!} & \text{(when } k_1 \leq r < \frac{k_1+k_2}{2} \text{)}, \\ -\frac{2}{\epsilon} \delta_{1k_1}\delta_{1k_2}\delta_{0r} + \frac{2(r-1)!(2r-k_1-k_2)!\Gamma(k_1+k_2-r-2+\epsilon/2)}{(k_1-1)!(k_2-1)!(r-k_1)!(r-k_2)!} & \text{(when } r \geq k_2 \text{)}, \end{cases}$$
Using this equation together with Eq. (10), we obtain

\[ b(k_1, k_2, r) = \begin{cases} 
0 & \text{when } r < k_1 + k_2 - 2 \\
\frac{(-1)^{r-k_1-k_2+2}}{(k_1-1)!(k_2-1)!} \frac{\Gamma(k_1 - r - \epsilon/2) \Gamma(k_2 - r - \epsilon/2) \Gamma(r + \epsilon/2)}{(k_1 - k_2)!} & \text{when } r \geq k_1 + k_2 - 2 
\end{cases}. \]

Note that \( a^{\text{ren}}_{k_1, k_2, 0}(1, 1, 0) \) does contain a simple pole which cancels the pole of \( b(1, 1, 0) \) in the expression \( F^{\text{ren}}_{k_1, k_2, 0}(k^2) \). Thus we see that the asymptotic behavior of \( F(k^2) \) in Eq. (11) is given as

\[ F(k^2) = \sum_{r=0}^{\infty} \left[ \left( \frac{\sigma^2}{k^2} \right)^{-\epsilon} a^{\text{ren}}_{k_1, k_2, 0}(1, 1, r) + \left( \frac{\sigma^2}{k^2} \right)^{\epsilon/2} b(1, 1, r) \right]. \]  

(10)

Using this equation we obtain

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k [F(k^2)]^2 = \frac{\sigma^2}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-3\epsilon/2} \frac{2}{\epsilon \Gamma(2 - \epsilon/2)} \times \sum_{n=0}^{2-n} \sum_{m=0}^{2-n-r} \left( 2 - n - r - m \right) \left[ a^{\text{ren}}_{k_1, k_2, 0}(1, 1, r_1)a^{\text{ren}}_{k_1, k_2, 0}(1, 1, r_2) + \frac{1}{3} b(1, 1, r_1)b(1, 1, r_2) + \frac{1}{2} \right] + \text{[finite terms]},
\]  

(11)

where the symbol \( \binom{a}{b} \) is the expansion coefficient of \( (1+x)^a = \sum_{b=0}^{\infty} \binom{a}{b} x^b \). Details of the step leading to Eq. (11) can be found in Ref. [6]. Therefore, the calculations of three integrals in \( \mathcal{P}_0, \mathcal{P}_1, \) and \( \mathcal{P}_2 \) can be regarded as simple quotations of Ref. [6]:

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k [F(k^2)]^2 = \Omega_2 \left[ \frac{1}{\epsilon^2} \left\{ \frac{44}{3} - 8\gamma \right\} + \frac{1}{\epsilon} \left\{ 23 - 30\gamma + 10\gamma^2 + \frac{\pi^2}{3} \right\} + O(\epsilon^0) \right],
\]

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[F(k^2)]^2}{k^2 + \sigma^2} = \Omega_1 \left[ \frac{8}{\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{28}{3} - 8\gamma \right\} + \frac{1}{\epsilon} \left\{ -\frac{14}{3} + 2\gamma^2 + \frac{\pi^2}{3} \right\} + O(\epsilon^0) \right],
\]

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[F(k^2)]^2}{(k^2 + \sigma^2)^2} = \Omega_0 \left[ \frac{8}{3\epsilon^3} - \frac{4}{3\epsilon^2} - \frac{2}{3\epsilon} + O(\epsilon^0) \right].
\]  

(12)

In the above equation the overall multiplying factors are defined as

\[
\Omega_0 = \frac{1}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-3\epsilon/2}, \quad \Omega_1 = \frac{\sigma^2}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-3\epsilon/2}, \quad \Omega_2 = \frac{\sigma^4}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-3\epsilon/2},
\]

and \( \gamma \) is the usual Euler constant, \( \gamma = 0.5772156649 \cdots \).

Likewise, the asymptotic behavior of \( G(k^2) \) in Eq. (12) is given as

\[ G(k^2) = \sum_{r=0}^{\infty} \left[ \left( \frac{\sigma^2}{k^2} \right)^{\epsilon} a^{\text{ren}}_{k_1, k_2, 0}(1, 1, r) \left( \frac{1}{3} \right)^r + \left( \frac{\sigma^2}{k^2} \right)^{\epsilon/2} b(1, 1, r) \left( \frac{1}{3} \right)^{r/2} \right]. \]  

(13)

Using this equation together with Eq. (12), we obtain
From this equation three integrals corresponding to hypergeometric function \( F \) see Eq. (21) of this reference. Applying directly this formula expressed in terms of Appell’s with arbitrary two different masses in propagators obtained by Boos and Davydychev [9].

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^{-\epsilon/2} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \int \frac{F(k^2)G(k^2)}{(k^2 + \sigma^2)^n} = \frac{(\sigma^2)^{2-n}}{(4\pi)^6} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \frac{2}{\epsilon \Gamma(2 - \epsilon/2)} \\
\times \sum_{r_1=0}^{2-n} \sum_{r_2=0}^{2-n-r_1} \left( \begin{array}{c} -n \\ 2 - n - r_1 - r_2 \end{array} \right) \left[ a^{\text{ren}}(1,1,r_1)a^{\text{ren}}(1,1,r_2) \left( \frac{1}{3} \right)^{r_2} \right] \\
+ \frac{1}{3} b(1,1,r_1)b(1,1,r_2) \left( \frac{1}{3} \left[ \left( \frac{1}{3} \right)^{r_2+\epsilon/2} \right] \\
+ \frac{1}{2} \left[ a^{\text{ren}}(1,1,r_1)b(1,1,r_2) \left( \frac{1}{3} \right)^{r_2+\epsilon/2} + b(1,1,r_1)a^{\text{ren}}(1,1,r_2) \left( \frac{1}{3} \right)^{r_2} \right] \right]
\]  
+ \text{finite terms} . \quad (14)

From this equation three integrals corresponding to \( n = 0, 1, 2 \) are evaluated as follows:

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^{-\epsilon/2} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \int \frac{F(k^2)G(k^2)}{(k^2 + \sigma^2)^2} = \Omega_2 \left[ -\frac{64}{27\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{92}{27} - \frac{8}{9} \gamma - \frac{16}{9} \ln 3 \right\} \right] \\
+ \frac{1}{\epsilon} \left\{ \frac{185}{27} - \frac{86}{9} \gamma + \frac{26}{9} \pi^2 + \frac{11}{9} \ln 3 + \frac{2}{9} \gamma \ln 3 - \frac{4}{9} \ln^2 3 \right\} + O(\epsilon^0) , \quad (15)
\]

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^{-\epsilon/2} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \int \frac{F(k^2)G(k^2)}{(k^2 + \sigma^2)^2} = \Omega_1 \left[ \frac{40}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{20}{3} - \frac{16}{3} \gamma + \frac{10}{3} \ln 3 \right\} \right] \\
+ \frac{1}{\epsilon} \left\{ \frac{26}{9} + \frac{4}{3} \gamma + \frac{2}{9} \pi^2 + 7 \ln 3 - \frac{14}{3} \gamma \ln 3 + \frac{5}{6} \ln^2 3 \right\} + O(\epsilon^0) , \quad (15)
\]

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^{-\epsilon/2} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \int \frac{F(k^2)G(k^2)}{(k^2 + \sigma^2)^2} = \Omega_0 \left[ \frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left\{ -\frac{4}{3} + 2 \ln 3 \right\} \right] \\
+ \frac{1}{\epsilon} \left\{ -\frac{2}{3} - \ln 3 + \frac{1}{2} \ln^2 3 \right\} + O(\epsilon^0) . \quad (15)
\]

Similarly the asymptotic behavior of \( H(k^2) \) is given as

\[
H(k^2) = \sum_{\alpha=0}^{\infty} \sum_{r=0}^{\infty} \left( \frac{2\sigma^2}{3} \right)^{\alpha} \left[ \left( \frac{\sigma^2}{k^2} \right)^r a^{\text{ren}}(1,1+\alpha,r) + \left( \frac{\sigma^2}{k^2} \right)^{r+\epsilon/2} b(1,1+\alpha,r) \right] . \quad (16)
\]

This can be shown readily by writing

\[
\int \frac{1}{(p^2 + \sigma^2)[(p+k)^2 + \sigma^2/3]} = \sum_{\alpha=0}^{\infty} \left( \frac{2\sigma^2}{3} \right)^{\alpha} \int \frac{1}{(p^2 + \sigma^2)[(p+k)^2 + \sigma^2]^{1+\alpha}} , \quad (17)
\]

and using Eq. (8). Note that there exists a one-loop integration formula for the diagrams with arbitrary two different masses in propagators obtained by Boos and Davydychev [9]. See Eq. (21) of this reference. Applying directly this formula expressed in terms of Appell’s hypergeometric function \( F_4 \) of two variables to our calculation is more difficult than using the expansion of Eq. (17) and the formulas of Eqs. (8) and (9). Using Eq. (16) we arrive at

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^{-\epsilon} \int \frac{[H(k^2)]^2}{(k^2 + \sigma^2/3)^n} = \frac{(\sigma^2)^{2-n}}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^{-3\epsilon/2} \frac{2}{\epsilon \Gamma(2 - \epsilon/2)} \\
\times \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{2-n-\alpha} \sum_{r_2=0}^{2-n-\beta} \left( \frac{1}{3} \right)^{2-n-r_1-r_2} \left( \frac{-n}{2 - n - r_1 - r_2} \right)
\]
from which three integrals for $n = 0, 1, 2$ are calculated as follows:

\[
\frac{1}{(4\pi)^3} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k (H(k^2))^2 = \Omega_2 \left[ \frac{32}{27\epsilon^3} + \frac{1}{\epsilon^2} \left( \frac{188}{27} - \frac{40}{9}\gamma + \frac{4}{9}\ln 3 \right) \right] + \frac{1}{\epsilon} \left( \frac{257}{27} - \frac{118}{9}\gamma + \frac{14}{3}\gamma^2 + \frac{5}{27}\pi^2 + \frac{31}{9}\ln 3 - 2\gamma\ln 3 + \frac{1}{3}\ln^2 3 \right) + O(\epsilon^0),
\]

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[H(k^2)]^2}{k^2 + \sigma^2/3} = \Omega_1 \left[ \frac{56}{9\epsilon^3} + \frac{1}{\epsilon^2} \left( \frac{52}{9} - \frac{16}{3}\gamma + \frac{4}{3}\ln 3 \right) \right] + \frac{1}{\epsilon} \left( \frac{10}{3} + \frac{4}{3}\gamma^2 + \frac{2}{9}\pi^2 - \frac{2}{3}\gamma\ln 3 + \frac{1}{3}\ln^2 3 \right) + O(\epsilon^0),
\]

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[H(k^2)]^2}{k^2 + \sigma^2/3} = \Omega_0 \left[ \frac{8}{3\epsilon^3} - \frac{4}{3\epsilon^2} - \frac{2}{3}\epsilon + O(\epsilon^0) \right].
\]

In dealing with the infinite sums in Eq. (18) ($n = 0, 1$) we have used the following relations:

\[
\sum_{\alpha=1}^{\infty} \left( \frac{2}{3} \right)^\alpha \frac{\psi(\alpha)}{\alpha(\alpha + 1)} = - \frac{1}{2} \sum_{\alpha=1}^{\infty} \left( \frac{2}{3} \right)^\alpha \frac{\psi(\alpha)}{\alpha} + 1 - \gamma - \frac{1}{2}\ln 3,
\]

\[
\sum_{\alpha=1}^{\infty} \left( \frac{2}{3} \right)^\alpha \frac{\psi(\alpha)}{\alpha(\alpha + 1)(\alpha + 2)} = \frac{1}{8} \sum_{\alpha=1}^{\infty} \left( \frac{2}{3} \right)^\alpha \frac{\psi(\alpha)}{\alpha} - \frac{1}{4} + \frac{3}{16}\ln 3,
\]

with

\[
\sum_{\alpha=1}^{\infty} \left( \frac{2}{3} \right)^\alpha \frac{\psi(\alpha)}{\alpha} = -\gamma \ln 3 + \frac{1}{2}\ln^2 3.
\]

Finally, using the asymptotic form of $G(k^2)$, Eq. (13), we obtain

\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2/3}{4\pi\mu^2} \right)^{-\epsilon} \int_k \frac{[G(k^2)]^2}{k^2 + \sigma^2/2} = \frac{(\sigma^2/3)^{-\epsilon}}{(4\pi)^6} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^{-\epsilon/2} \frac{2}{\epsilon^2} \left( \frac{8}{3\epsilon^3} - \frac{4}{3\epsilon^2} - \frac{2}{3}\epsilon + O(\epsilon^0) \right)
\]

\[
\times \sum_{r_1=0}^{2-n} \sum_{r_2=0}^{2-n-r_1} \left( 2 - n - r_1 - r_2 \right) \left[ a^{\text{ren}}(1, 1, r_1)a^{\text{ren}}(1, 1, r_2)(\frac{1}{3})^{r_1+r_2} + \frac{1}{3} b(1, 1, r_1)b(1, 1, r_2)(\frac{1}{3})^{r_1+r_2+\epsilon} 
\]

\[
+ \frac{1}{2} \left( a^{\text{ren}}(1, 1, r_1)b(1, 1, r_2) + b(1, 1, r_1)a^{\text{ren}}(1, 1, r_2) \right) (\frac{1}{3})^{r_1+r_2+\epsilon/2} \right] + [\text{finite terms}],
\]

and after substitutions of the appropriate values for $a^{\text{ren}}(1, 1, r)$ and $b(1, 1, r)$ we end up with
\[
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^\epsilon \int_k (G(k^2))^2 = \Omega_2 \left[ \frac{1}{\epsilon^2} \left( \frac{44}{27} - \frac{8}{9} \gamma \right) \right. \\
+ \frac{1}{\epsilon} \left\{ \frac{23}{9} - \frac{10}{3} \gamma + \frac{10}{9} \gamma^2 + \frac{\pi^2}{27} + \frac{22}{9} \ln 3 - \frac{4}{3} \gamma \ln 3 \right\} + O(\epsilon^0) \], \\
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^\epsilon \int_k \frac{(G(k^2))^2}{k^2 + \sigma^2} = \Omega_1 \left[ \frac{8}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ 4 - \frac{8}{3} \gamma + \frac{4}{3} \ln 3 \right\} \\
+ \frac{1}{\epsilon} \left\{ -\frac{10}{9} + \frac{2}{3} \gamma^2 + \frac{\pi^2}{9} + 6 \ln 3 - 4 \gamma \ln 3 + \ln^2 3 \right\} + O(\epsilon^0) \], \\
\frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi\mu^2} \right)^\epsilon \int_k \frac{(G(k^2))^2}{(k^2 + \sigma^2)^2} = \Omega_0 \left[ \frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left\{ -\frac{4}{3} + 4 \ln 3 \right\} \\
+ \frac{1}{\epsilon} \left\{ -\frac{2}{3} - 2 \ln 3 + 3 \ln^2 3 \right\} + O(\epsilon^0) \].
\]

IV. SUMMARY

We summarize all results calculated in the previous sections by listing them in the \(\epsilon\)-expanded forms, but leaving the \(\Omega_{0,1,2}\) factors unexpanded. Our desired accuracy of three-loop integrals is to calculate up to the order of \(\epsilon^{-1}\) apart from an overall multiplying factor \((\sigma^2/(4\pi\mu^2))^{-3\epsilon/2}\). When this overall multiplying factor is expanded in powers of \(\epsilon\), the resulting accuracy is up to finite terms containing the logarithm of mass squared, \(\ln(\sigma^2/(4\pi\mu^2))\). The list reads as follows:

\[
J(a) = \Omega_2 \left[ \frac{16}{\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{92}{3} - 24 \gamma \right\} \right. \\
+ \frac{1}{\epsilon} \left\{ 35 - 46 \gamma + 18 \gamma^2 + \pi^2 \right\} \], \\
J(b) = \Omega_2 \left[ \frac{176}{27\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{332}{27} - \frac{88}{9} \gamma + \frac{28}{9} \ln 3 \right\} \\
+ \frac{1}{\epsilon} \left\{ \frac{365}{27} - \frac{1}{2} \gamma^2 + \frac{22}{3} \gamma + \frac{11}{27} \pi^2 + \frac{55}{9} \ln 3 - \frac{14}{3} \gamma \ln 3 + \ln^2 3 \right\} \], \\
J(c) = \Omega_2 \left[ \frac{16}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{92}{27} - \frac{8}{3} \gamma + \frac{8}{3} \ln 3 \right\} \\
+ \frac{1}{\epsilon} \left\{ \frac{35}{9} - \frac{46}{9} \gamma + 2 \gamma^2 + \frac{\pi^2}{9} + \frac{46}{9} \ln 3 - 4 \gamma \ln 3 + 2 \ln^2 3 \right\} \], \\
K(a) = \Omega_1 \left[ -\frac{8}{\epsilon^3} + \frac{1}{\epsilon^2} \left\{ -\frac{68}{3} + 12 \gamma \right\} \right. \\
+ \frac{1}{\epsilon} \left\{ \frac{134}{3} - 12 A + 34 \gamma - 9 \gamma^2 - \frac{\pi^2}{2} \right\} \], \\
K(b) = \Omega_1 \left[ -\frac{56}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ \frac{52}{3} + \frac{28}{3} \gamma - \frac{4}{3} \ln 3 \right\} \\
+ \frac{1}{\epsilon} \left\{ -\frac{302}{9} + 6 A - \frac{2}{3} B + 26 \gamma - 7 \gamma^2 - \frac{7}{8} \pi^2 - 4 \ln 3 + 2 \gamma \ln 3 \right\} \], \\
K(c) = \Omega_1 \left[ -\frac{40}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ -\frac{116}{9} + \frac{20}{3} \gamma - \frac{8}{3} \ln 3 \right\} \\
+ \frac{1}{\epsilon} \left\{ -26 + \frac{4}{3} B + \frac{58}{3} \gamma - 5 \gamma^2 - \frac{5}{18} \pi^2 + 4 \gamma \ln 3 - \frac{22}{3} \ln 3 \right\} \], \\
K(d) = \Omega_1 \left[ -\frac{40}{9\epsilon^3} + \frac{1}{\epsilon^2} \left\{ -12 + \frac{20}{3} \gamma - \frac{8}{3} \ln 3 \right\}
\]

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A parameter calculation in Ref. [2]. (Note that in this reference though regularization method differs where \( \tilde{\lambda} \) involved, but straightforward [3]: calculated using our integration results of three-loop diagrams. The renormalization is highly up to three-loop order. In order to see that three-loop part of our calculation

\[
M(a) = M(b) = M(c) = \Omega_0 \left[ \frac{4\zeta(3)}{\epsilon} \right].
\]

In the above equation, \( A, B, \) and \( C \) are constants whose numerical values are given in Eq. (A3).

Now we close this section with three-loop effective potential of massless \( O(N) \phi^4 \) theory, calculated using our integration results of three-loop diagrams. The renormalization is highly involved, but straightforward [3]:

\[
V_{\text{eff}}(\phi_a) = \left[ \frac{\lambda}{4!} \phi^4 + \frac{\lambda^2 \phi^4}{(4\pi)^2} \left[ -\frac{25}{96} + \frac{1}{16} \ln\left( \frac{\phi^2}{M^2} \right) + \tilde{N} \left\{ -\frac{25}{864} + \frac{1}{144} \ln\left( \frac{\phi^2}{M^2} \right) \right\} \right] \\
+ \frac{\lambda^3 \phi^4}{(4\pi)^4} \left[ \frac{55}{24} - \frac{13}{16} \ln\left( \frac{\phi^2}{M^2} \right) + \frac{3}{32} \ln^2\left( \frac{\phi^2}{M^2} \right) + \tilde{N} \left\{ \frac{635}{1296} - \frac{19}{108} \ln\left( \frac{\phi^2}{M^2} \right) \right\} \right] \\
+ \frac{1}{48} \ln^2\left( \frac{\phi^2}{M^2} \right) \right] + \tilde{N}^2 \left\{ \frac{85}{3888} - \frac{11}{1296} \ln\left( \frac{\phi^2}{M^2} \right) + \frac{1}{864} \ln^2\left( \frac{\phi^2}{M^2} \right) \right\} \\
+ \frac{\lambda^4 \phi^4}{(4\pi)^6} \left[ -\frac{27035}{1152} - \frac{25}{32} A - \frac{25}{24} \zeta(3) + \left( \frac{1957}{192} + \frac{3}{16} A + \frac{\zeta(3)}{4} \right) \ln\left( \frac{\phi^2}{M^2} \right) \right] \\
- \frac{359}{192} \ln^2\left( \frac{\phi^2}{M^2} \right) + \frac{9}{64} \ln^3\left( \frac{\phi^2}{M^2} \right) + \tilde{N}^2 \left\{ -\frac{228725}{31104} - \frac{125}{864} A - \frac{125}{648} \zeta(3) \right\} \\
+ \frac{175}{5184} \ln 3 + \frac{16769}{5184} A + \frac{5}{144} \zeta(3) + \frac{7}{864} \ln 3 \right] \ln\left( \frac{\phi^2}{M^2} \right) \\
- \frac{523}{864} \ln^2\left( \frac{\phi^2}{M^2} \right) + \frac{3}{64} \ln^3\left( \frac{\phi^2}{M^2} \right) \right] + \tilde{N}^2 \left\{ -\frac{62105}{93312} - \frac{25}{3888} A + \frac{175}{15552} \ln 3 \right\} \\
+ \left( \frac{4775}{15552} + \frac{A}{648} - \frac{7}{2592} \ln 3 \right) \ln\left( \frac{\phi^2}{M^2} \right) - \frac{319}{5184} \ln^2\left( \frac{\phi^2}{M^2} \right) + \frac{1}{192} \ln^3\left( \frac{\phi^2}{M^2} \right) \right\} \\
+ \tilde{N}^3 \left\{ -\frac{4655}{279936} + \frac{395}{46656} \ln\left( \frac{\phi^2}{M^2} \right) - \frac{5}{2592} \ln^2\left( \frac{\phi^2}{M^2} \right) + \frac{1}{5184} \ln^3\left( \frac{\phi^2}{M^2} \right) \right\},
\]

where \( \tilde{N} = N - 1 \). Up to two-loop level, Eq. (24) completely coincides with the existing calculation in Ref. [2]. (Note that in this reference though regularization method differs from ours, the same renormalization conditions are used.) The above result is exact in parameter \( N \) up to three-loop order. In order to see that three-loop part of our calculation
is not erroneous, we compare the leading order part of Eq. (20), when written down in $1/N$ expansion formalism with a simple replacement $\lambda \rightarrow \lambda/N$, with the following extracted overlap part of the large-$N$ limit calculation [4]:

$$V_{\text{CJP}} = \frac{\lambda}{4!N} \phi^4 + \frac{\lambda^2 \phi^4}{(4\pi)^2 N} \left[ -\frac{1}{288} + \frac{1}{144} \ln \left( \frac{\phi^2}{M^2} \right) \right]$$

$$+ \frac{\lambda^3 \phi^4}{(4\pi)^4 N} \left[ \frac{1}{864} \ln^2 \left( \frac{\phi^2}{M^2} \right) \right] + \frac{\lambda^4 \phi^4}{(4\pi)^6 N} \left[ \frac{1}{5184} \ln^2 \left( \frac{\phi^2}{M^2} \right) + \frac{1}{5184} \ln^3 \left( \frac{\phi^2}{M^2} \right) \right].$$

(21)

Shifting an arbitrary parameter $M^2$ in Eq. (21) as $M^2 \rightarrow M^2 \exp[(11/3) - (49/54)\lambda/(4\pi)^2 + (563/1944)\lambda/(4\pi)^4]$, we can readily see that leading order part of Eq. (20) in $1/N$ agrees with overlap part of leading-order approximation calculation of Coleman, Jackiw and Politzer [4].

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APPENDIX: ONE- AND TWO-LOOP INTEGRALS

We list one- and two-loop integrals which appear in our calculation of the three-loop integrals. The one-loop integrations are quite elementary and for the two-loop integrations one may refer to Ref. [10]. We need to calculate one-loop integrals up to the order of $\epsilon$ except an overall multiplying factor $(\sigma^2/(4\pi\mu^2))^{-\epsilon/2}$, two-loop integrals up to the order of $\epsilon^0$ except an overall multiplying factor $(\sigma^2/(4\pi\mu^2))^{-\epsilon}$, in conformity with our desired accuracy of three-loop integrals, i.e., the order of $\epsilon^{-1}$ except an overall multiplying factor $(\sigma^2/(4\pi\mu^2))^{-3\epsilon/2}$.

One-loop integrals, $S_1$ to $S_4$:

$$S_1 \equiv \int \frac{1}{k^2 + \sigma^2} = \frac{\sigma^2}{(4\pi)^2} \frac{1}{4\pi \mu^2} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^{-\epsilon/2} \Gamma \left( \frac{\epsilon}{2} - 1 \right),$$

$$S_2 \equiv \int \frac{1}{k^2 + \sigma^2/3} = \frac{\sigma^2}{3(4\pi)^2} \frac{1}{4\pi \mu^2} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \Gamma \left( \frac{\epsilon}{2} - 1 \right),$$

$$S_3 \equiv \int \frac{1}{k^2 + \sigma^2/3} = \frac{1}{(4\pi)^2} \frac{1}{4\pi \mu^2} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \Gamma \left( \frac{\epsilon}{2} \right),$$

$$S_4 \equiv \int \frac{1}{k^2 + \sigma^2/3} = \frac{1}{(4\pi)^2} \frac{1}{4\pi \mu^2} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \Gamma \left( \frac{\epsilon}{2} \right).$$

(A1)
Two-loop integrals, $W_1$ to $W_8$:  

$$ W_1 \equiv \int_{k_p} \frac{1}{(p^2 + \sigma^2)((p + k)^2 + \sigma^2)} = \frac{\sigma^4}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^\epsilon \Gamma^2 \left( \frac{\epsilon}{2} - 1 \right), $$

$$ W_2 \equiv \int_{k_p} \frac{1}{(p^2 + \sigma^2/3)((p + k)^2 + \sigma^2/3)} = \frac{\sigma^4}{9(4\pi)^4} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^\epsilon \Gamma^2 \left( \frac{\epsilon}{2} - 1 \right), $$

$$ W_3 \equiv \int_{k_p} \frac{1}{(p^2 + \sigma^2)((p + k)^2 + \sigma^2/3)} = \frac{\sigma^4}{3(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^\epsilon/2 \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^{-\epsilon/2} \Gamma^2 \left( \frac{\epsilon}{2} - 1 \right), $$

$$ W_4 \equiv \int_{k_p} \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2)((p + k)^2 + \sigma^2)} $$

$$ = \frac{\sigma^2}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^\epsilon \Gamma^2 \left( 1 + \frac{\epsilon}{2} \right) \left[ \frac{6}{\epsilon^2} - 3A + O(\epsilon) \right], $$

$$ W_5 \equiv \int_{k_p} \frac{1}{(k^2 + \sigma^2)(p^2 + \sigma^2/3)((p + k)^2 + \sigma^2/3)} $$

$$ = \frac{\sigma^2}{3(4\pi)^4} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^\epsilon \Gamma^2 \left( 1 + \frac{\epsilon}{2} \right) \left[ \frac{10}{\epsilon^2} + \frac{6}{\epsilon} \ln 3 - \frac{3}{2} \ln^2 3 - B + O(\epsilon) \right], $$

$$ W_6 \equiv \int_{k_p} \frac{1}{(k^2 + \sigma^2)^2(p^2 + \sigma^2)((p + k)^2 + \sigma^2)} $$

$$ = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^\epsilon \left[ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( 1 - 2\gamma \right) + \frac{1}{2} - \gamma + \gamma^2 + \frac{\pi^2}{12} + A + O(\epsilon) \right], $$

$$ W_7 \equiv \int_{k_p} \frac{1}{(k^2 + \sigma^2/3)^2(p^2 + \sigma^2/3)((k + p)^2 + \sigma^2)} $$

$$ = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2/3}{4\pi \mu^2} \right)^\epsilon \left[ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( 1 - 2\gamma \right) + \frac{1}{2} - \gamma + \gamma^2 + \frac{\pi^2}{12} + B + O(\epsilon) \right], $$

$$ W_8 \equiv \int_{k_p} \frac{1}{(k^2 + \sigma^2)^2(p^2 + \sigma^2/3)((k + p)^2 + \sigma^2/3)} $$

$$ = \frac{1}{(4\pi)^4} \left( \frac{\sigma^2}{4\pi \mu^2} \right)^\epsilon \left[ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( 1 - 2\gamma \right) + \frac{1}{2} - \gamma + \gamma^2 + \frac{\pi^2}{12} + C + O(\epsilon) \right]. \quad (A2) $$

In the above equation, $\gamma$ is the usual Euler constant, $\gamma = 0.5772156649 \cdots$, and numerical values of the constants, $A$, $B$, and $C$ in Eq. (A2) are

$$ A = f(1, 1) = -1.1719536193 \cdots, $$

$$ B = f(1, 3) = -2.3439072387 \cdots, $$

$$ C = f \left( \frac{1}{3}, \frac{1}{3} \right) = 0.1778279325 \cdots. \quad (A3) $$

where

$$ f(a, b) \equiv \int_0^1 dx \int_0^{1-x} dy \left( -\ln \left( 1 - \frac{y}{z} \right) \right)^a \left( \frac{z \ln z}{1 - z} \right)^b, \quad z = \frac{ax + b(1 - x)}{x(1 - x)}. $$

These constants $A$, $B$, and $C$ can be analytically integrated [11]. The results are expressed in terms of Clausen function.
\[ A = \frac{B}{2} = -\frac{3}{2}C - \frac{3}{4}\ln^2 3 = -\frac{2}{\sqrt{3}} \text{Cl}_2\left(\frac{\pi}{3}\right), \]

where

\[ \text{Cl}_2(\theta) = \int_0^\theta \ln[2\sin(\theta'/2)]d\theta'. \]
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FIGURES

FIG. 1.

FIG. 2.

FIG. 3.

FIG. 4.