A note on the Huq-commutativity of normal monomorphisms

James Richard Andrew Gray and Tamar Janelidze-Gray

June 28, 2022

Abstract

We give an alternative criteria for when a pair of Bourn-normal monomorphisms Huq-commute in a unital category. We use this to prove that in a unital category, in which a morphism is a monomorphism if and only if its kernel is zero morphism, a pair of Bourn-normal monomorphisms with the same codomain Huq-commute as soon as they have trivial pullback. As corollaries we show that several facts known only in the protomodular context are in fact true in more general contexts.

1 Introduction

It is well known and easy to prove that if $K$ and $L$ are normal subgroups of a group $G$ and $K \cap L = 0$, then each element in $K$ commutes with each element of $L$. This fact has several known generalizations to categories.

An immediate generalization is obtained in the context where there is a suitable notion of a commutator $[-,-]$ defined for normal subobjects, (which is commutative and) satisfying the property that if $K, L$ are normal subobjects then $[K, L] \leq K$. In this context, if $K$ and $L$ are normal then $[K, L] \leq K \wedge L$. Therefore if $K$ and $L$ are trivial it immediately follows that $K \wedge L$ is trivial, which implies that $K$ and $L$ commute. This is the case for the Huq commutator in a normal unital category.

An alternative generalization was obtained by D. Bourn (Theorem 11 [4]) in the context of pointed protomodular category [3] (also introduced by D. Bourn): he proved that if the meet of $k$ and $l$ is 0, and if $k$ and $l$ are Bourn-normal with the same codomain, then $k$ and $l$ Huq-commute [10]. Recall that in a pointed finitely complete category, a Bourn-normal monomorphism is essentially the zero class of an internal equivalence relation.

We show (Corollary 3.11) that this latter fact is true in the wider context of a unital category [2] (introduced by F. Borceux and D. Bourn) satisfying Condition 3.4 which simply requires a morphism to be a monomorphism as soon as its kernel is zero. This context is sufficiently wide so that it includes every normal unital category which implies that the former result also becomes a special case. In doing so we produce an alternative criteria (Theorem 3.2) for when a pair of Bourn-normal monomorphisms
commute in a unital category, which closely resembles Proposition 2.6.13 of [2].

We briefly study Condition 3.4 and in particular: (i) we explain that it is a special case of a known condition (see Remark 4.5) and that it together with regularity is easily equivalent to normality; (ii) we give examples of categories satisfying it as well as our other conditions (some of which are not normal categories); (iii) we characterize it in terms of the fibration of points (Proposition 3.7). Using in part this characterization, we show that in a pointed Mal’tsev category [6] satisfying Condition 3.4, the join of Bourn-normal monomorphisms, with the same codomain and trivial meet, exists and is Bourn-normal. In addition, we show that the characterization of abelian objects, via the normality of their diagonal in the product, lifts from pointed protomodular categories to strongly unital categories [2] satisfying Condition 3.4.

2 Preliminaries

In this section we recall the necessary definitions and preliminary facts, and introduce the notation we will use.

For a pointed category $C$ we write 0 for the zero object as well as for each zero morphism between each pair of objects. For objects $X$ and $Y$ we will often write $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ for the first and second product projections (when they exists), and for morphisms $f : W \to X$ and $g : W \to Y$ we will write $(f, g) : W \to X \times Y$ for the unique morphism with $\pi_1(f, g) = f$ and $\pi_2(f, g) = g$. Recall that a category $C$ is unital if $C$ is pointed, finitely complete, and for each pair of objects $X$ and $Y$ the unique morphisms $(1, 0) : X \to X \times Y$ and $(0, 1) : Y \to X \times Y$ are jointly strongly epimorphic.

A pair of morphisms $f : X \to A$ and $g : Y \to A$ in a unital category $C$ are said to Huq-commute, if there exists a unique morphism $\varphi : X \times Y \to A$ making the diagram

\[ \begin{array}{ccc}
X & \overset{(1, 0)}{\longrightarrow} & X \times Y \\
\downarrow f & & \downarrow (0, 1) \\
A & \underset{\varphi}{\longleftarrow} & Y \\
\end{array} \]

commute. The morphism $\varphi$ is called the cooperator of $f$ and $g$. A morphism $f : X \to A$ is called a central monomorphism if it is a monomorphism and it Huq-commutes with $1_A$.

We will also need the following lemmas (see e.g [2] and the references there):

**Lemma 2.1.** For $u : X' \to X$, $v : Y' \to Y$, $f : X \to A$ and $g : Y \to A$ morphisms in $C$ and $m : A \to B$ and monomorphism.

(i) The morphisms $f$ and $g$ Huq-commute if and only if the morphisms $g$ and $f$ Huq-commute;

(ii) The morphisms $mf$ and $mg$ Huq-commute if and only if the morphisms $f$ and $g$ Huq-commute;

2
(iii) If the morphisms \( f \) and \( g \) Huq-commute, then so do the morphisms \( fu \) and \( gv \).

**Lemma 2.2.** For \( f : X \to A \), \( g : Y \to A \), \( f' : X' \to A' \) and \( g' : Y' \to A' \) in \( \mathcal{C} \), the morphisms \( f \times f' \) and \( g \times g' \) Huq-commute if and only if both the morphisms \( f \) and \( g \), and the morphisms \( f' \) and \( g' \) Huq-commute.

**Lemma 2.3.** For \( f : X \to A \), \( g : Y \to A \), \( f' : X' \to A' \) and \( g' : Y' \to A' \) in \( \mathcal{C} \), the morphisms \( \langle f, f' \rangle \) and \( \langle g, g' \rangle \) Huq-commute if and only if \( f \) and \( g \) Huq-commute, and \( f' \) and \( g' \) Huq-commute.

3 The results

Throughout this section we assume that \( \mathcal{C} \) is a unital category. Let \( k : X \to A \) and \( l : Y \to A \) be monomorphisms, let \( r_1, r_2 : R \to A \) and \( s_1, s_2 : S \to A \) be equivalence relations and let \( \kappa \) and \( \lambda \) be morphisms such that the diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{\kappa} & R \\
\downarrow{k} & & \downarrow{(r_1, r_2)} \\
A & \xrightarrow{(1,0)} & A \times A \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
Y & \xrightarrow{\lambda} & S \\
\downarrow{l} & & \downarrow{(s_1, s_2)} \\
A & \xrightarrow{(0,1)} & A \times A
\end{array}
\]

are pullbacks. Note that in pointed context this amounts to saying \( k \) and \( l \) are Bourn-normal. In particular, this includes the case when \( k \) and \( l \) are the kernels of some morphisms \( f \) and \( g \); in this case, \( r_1, r_2 \) and \( s_1, s_2 \) can be constructed as the kernel pairs of \( f \) and \( g \) respectively, and \( \kappa \) and \( \lambda \) are the unique morphisms with \( r_1 \kappa = k \), \( r_2 \kappa = 0 \), \( s_1 \lambda = 0 \), and \( s_2 \lambda = 0 \).

We will need the relation \( R \boxtimes S \), which is a pointed counter part to \( R \boxtimes S \) introduced by A. Carboni, M.C. Pedicchio and N. Pirovano in \([7]\). In the context of pointed sets has elements

\[ \{ (x, a, y) \in X \times A \times Y \mid (k(x), a) \in S \text{ and } (a, l(y)) \in R \}. \]

Note that an element \( (x, a, y) \) in \( R \boxtimes S \) can, after identifying \( k(x) \) and \( x \), and \( l(y) \) and \( y \), be displayed as follows

\[
\begin{array}{ccc}
x & \xrightarrow{r} & 0 \\
\downarrow{s} & & \downarrow{s} \\
a & \xrightarrow{r} & y
\end{array}
\]

Categorically this relation can be built via the pullbacks

\[
\begin{array}{ccc}
P & \xrightarrow{p_1} & S \\
\downarrow{s_1} & & \downarrow{s_2} \\
A & & \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
P & \xrightarrow{p_2} & R \\
\downarrow{r_1} & & \downarrow{r_2} \\
A & & A
\end{array}
\]

3
\[ R \Box S \xrightarrow{\theta} X \times Y \]
\[ P \xrightarrow{(s_1, p_1, r_2, p_2)} A \times A \]

or directly as the limit of the outer arrows of what is easily seen to be a limiting cone

\[ \begin{array}{c}
A \\
\downarrow s_1 \\
S \\
\downarrow s_2 \\
A \\
\end{array} \xleftarrow{k} \begin{array}{c}
X \\
\downarrow p_1 \\
R \Box S \\
\downarrow p_2 \\
A \\
\end{array} \xrightarrow{\pi_2} \begin{array}{c}
Y \\
\downarrow r_1 \\
R \\
\downarrow r_2 \\
A \\
\end{array} \xrightarrow{i} \]

(5)

Let \( \alpha : X \to R \Box S \) and \( \beta : Y \to R \Box S \) be the unique cone morphisms induced by the cones

\[ \begin{array}{c}
A \\
\downarrow s_1 \\
S \\
\downarrow s_2 \\
A \\
\end{array} \xleftarrow{k} \begin{array}{c}
X \\
\downarrow 1_X \\
0 \\
\downarrow \kappa \\
Y \\
\downarrow i \\
A \\
\end{array} \]

(6)

Note that, in particular, it follows that \( \alpha \) and \( \beta \) are morphisms making the two triangles in the diagram

\[ \begin{array}{c}
R \Box S \\
\downarrow \theta \\
X \xleftarrow{(1,0)} X \times Y \xrightarrow{(0,1)} Y, \\
\end{array} \]

where \( \theta \) is defined as in Diagram 4. The morphism \( \theta \) in (3) is a strong epimorphism.

Proposition 3.1. The morphism \( \theta \) in (3) is a strong epimorphism. \( \square \)
Using in part the previous fact, we are now ready to state and prove our alternative criteria for when a pair of Bourn-normal monomorphisms commute.

**Theorem 3.2.** The following conditions are equivalent:

(a) \( k : X \to A \) and \( l : Y \to A \), as defined in (2), Huq-commute;

(b) \( \alpha : X \to R\Box^1 S \) and \( \beta : Y \to R\Box^1 S \), as defined in (6), Huq-commute;

(c) \( \theta : R\Box^1 S \to X \times Y \) is a split epimorphism of cospans with domain \((R\Box^1 S, \alpha, \beta)\) and \((X \times Y; (1,0), (0,1))\).

**Proof.** Let \( m : R\Box^1 S \to X \times X \) be the morphism defined by \( m = \langle \pi_1\theta, s_2p_1\psi, \pi_2\theta \rangle \). An easy calculation shows that \( m \) is a monomorphism. Noting that \( m\alpha = (1,k,0) \) and \( m\beta = (0,l,1) \), it follows from Lemma 2.1 that \( \alpha \) and \( \beta \) Huq-commute if and only if \( (1,k,0) \) and \( (0,l,1) \) Huq-commute. However, by Lemma 2.3 this latter condition is equivalent to requiring \( k \) and \( l \) to Huq-commute. This proves \((a) \Leftrightarrow (b)\). To prove that \((b) \Rightarrow (c)\) we note that \((b)\) is equivalent to requiring that there is a morphism \( \sigma : X \times Y \to R\Box^1 S \) making the upper part of the diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\sigma} & (0,1) \\
\downarrow^{(1,0)} & & \downarrow^{(0,1)} \\
X & \overset{\alpha}{\lhr} & Y \\
\downarrow^{(1,0)} & & \downarrow^{(0,1)} \\
X \times Y & \xleftarrow{\beta} & (0,1)
\end{array}
\]

commute. However, since \((1,0)\) and \((0,1)\) are jointly epimorphic any such morphism must satisfy \( \theta\sigma = 1_{X \times Y} \) and so \((c)\) holds. The converse is immediate, since \((c)\) implies that there is a morphism \( \sigma \) making the upper part of (7) commute, and as mentioned \((b)\) is equivalent to the existence of such a morphism.

**Lemma 3.3.** The objects \( \text{Ker}(\theta) \) and \( X \times_A Y \), where \( X \times_A Y \) is the pullback of \( k : X \to A \) and \( l : Y \to A \), are isomorphic.

**Proof.** Note that since (4) is a pullback, it follows that \( \text{Ker}(\theta) \cong \text{Ker}\left((s_1p_1, r_2p_2)\right) \).
Now consider the diagram

![Diagram image](image_url)

consisting of the diagram (8) and in which:
- $i$ and $j$ are the unique morphism such that $\lambda j = p_1 \ker(s_1p_1)$ and $\kappa i = p_2 \ker(r_2p_2)$;
- $u$ and $v$ are the unique morphisms making (*) in the diagram above, commute.

Since each diamond in (8) is a pullback and $r_1\kappa = k$ and $s_2\lambda = l$, it follows that the diagram

![Diagram image](image_url)

is a also a pullback, and therefore, $\text{Ker}(\theta) \cong X \times_A Y$ as desired. □

Let $X$ be a pointed category. Consider the condition:

**Condition 3.4.** A morphism $f : A \to B$ in $X$ is a monomorphism if and only if the kernel of $f$ is 0.

**Remark 3.5.** Note that a pointed category $X$ satisfies Condition 3.4 if and only if each reflexive relation in $X$ satisfies what was called Condition $(\ast \pi_0)$ in [8], with respect to the ideal of zero morphisms.

Recall that a regular category [1] is normal [12] if and only if every regular epimorphism is a normal epimorphism. The following proposition follows from Corollary 2.3 of [5], however we give a direct proof in order to avoid introducing notation and terminology that would not otherwise be needed in this paper.

**Proposition 3.6.** A regular category $X$ with cokernels is normal if and only if it satisfies Condition 3.4.
Proof. It is immediate that a normal category satisfies Condition [3.4]. It remains to prove the converse. Suppose \( f : A \to B \) is a regular epimorphism and consider the diagram

\[
\begin{array}{ccc}
\text{Ker}(f) & \xrightarrow{\text{ker}(f)} & A \\
\downarrow u & & \downarrow f \\
\text{Ker}(r) & \xrightarrow{\text{ker}(r)} & Q
\end{array}
\]

in which \( q \) is cokernel of \( \text{ker}(f) \), \( r \) is the unique morphism with \( rq = f \), and \( u \) the unique morphism with \( \text{ker}(r)u = \text{qker}(f) \). Since the left hand square is a pullback it follows that \( u \) is a regular epimorphism. Since \( \text{ker}(r)u = \text{qker}(f) = 0 \), it follows that \( \text{ker}(r) = 0 \), and therefore \( r \) is a monomorphism. Since \( r \) is also a regular epimorphism, the latter implies that \( r \) is an isomorphism.

Recall that for a category \( \mathcal{X} \) and an object \( B \) in \( \mathcal{X} \), the category \( \text{Pt}_{\mathcal{X}}(B) \) of points, in the sense of D. Bourn, has objects triples \((A, \alpha, \beta)\), where \( A \) is an object in \( \mathcal{X} \), and \( \alpha : A \to B \) and \( \beta : B \to A \) are morphisms in \( \mathcal{X} \) such that \( \alpha \beta = 1_B \). A morphism \( f \) from \((A, \alpha, \beta)\) to \((A', \alpha', \beta')\) in \( \text{Pt}_{\mathcal{X}}(B) \) is a morphism \( f : A \to A' \), such that \( \alpha'f = \alpha \) and \( f\beta = \beta' \). Furthermore, a morphism \( p : E \to B \) in \( \mathcal{X} \) determines a pullback functor \( p^* : \text{Pt}_{\mathcal{X}}(B) \to \text{Pt}_{\mathcal{X}}(E) \) which sends \((A, \alpha, \beta)\) in \( \text{Pt}_{\mathcal{X}}(B) \) to \((E \times_B A, \pi_1, (1, \beta p))\) in \( \text{Pt}_{\mathcal{X}}(E) \), with objects and morphism defined as in the following commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{(1, \beta p)} & E \times_B A \\
\downarrow 1_E & & \downarrow 1_A \\
E & \xrightarrow{p} & B
\end{array}
\]

in which \( \square \) is a pullback. When \( \mathcal{X} \) is a pointed category, pullback functors along morphisms of the form \( 0 \to B \) are essentially the same as kernel functors \( \text{Ker}_B : \text{Pt}_{\mathcal{X}}(B) \to \mathcal{X} \).

**Proposition 3.7.** For a pointed finitely complete category \( \mathcal{X} \) the following are equivalent:

(a) The category \( \mathcal{X} \) satisfies Condition [3.4].
(b) For each object \( B \) in \( \mathcal{X} \) the functor \( \text{Ker}_B \) reflects terminal objects;
(c) For each object \( B \) in \( \mathcal{X} \) the functor \( \text{Ker}_B \) reflects monomorphisms;
(d) For each object \( B \) in \( \mathcal{X} \) the category \( \text{Pt}_{\mathcal{X}}(B) \) satisfies Condition [3.4];
(e) For each morphism \( p : E \to B \) in \( \mathcal{X} \) the functor \( p^* : \text{Pt}_{\mathcal{X}}(B) \to \text{Pt}_{\mathcal{X}}(E) \) reflects terminal objects;
(f) For each morphism \( p : E \to B \) in \( \mathcal{X} \) the functor \( p^* : \text{Pt}_{\mathcal{X}}(B) \to \text{Pt}_{\mathcal{X}}(E) \) reflects monomorphisms.

7
Proof. For a morphism \( f : A \to B \) in \( X \), note that:

(i) \( f : A \to B \) is a monomorphism if and only if in the pullback diagram

\[
\begin{array}{ccc}
A \times_B A & \xrightarrow{\pi_2} & A \\
\downarrow \pi_1 & & \downarrow f \\
A & \rightarrow & B
\end{array}
\]

\( \pi_1 \) is an isomorphism.

(ii) The morphism \( \pi_1 \) is an isomorphism whenever \( (A \times_B A, \pi_1, (1, 1)) \) is a terminal object in \( \text{Pt}_X(A) \);

(iii) The kernel of \( f \) is isomorphic to the kernel of \( \pi_1 \).

Combining these observations we see that (a) \( \Rightarrow \) (b). For any functor \( F \) between pointed categories which preserves terminal objects, since morphisms into the terminal object are necessarily split epimorphisms, one easily shows that if \( F \) reflects monomorphisms, then \( F \) reflects terminal objects. Therefore (f) \( \Rightarrow \) (e) and (c) \( \Rightarrow \) (b). Recalling that a composite of functors \( FG \) reflects some property and \( F \) preserves it, then \( G \) reflects it, and noting that kernel functors certainly preserve terminal objects, one easily sees that (b) \( \Rightarrow \) (e) (just note that for each morphism \( p : E \to B \) the functor \( \text{Ker}_E \circ p^* \) is isomorphic to \( \text{Ker}_B \)). Since each pullback functor between points along a morphism in a category of points of \( X \) is up to isomorphism a pullback functor between points for \( X \) it follows that (e) \( \Rightarrow \) (d). For a functor \( F \) between pointed finitely complete categories satisfying Condition 3.4, preserving limits and reflecting terminal objects, if \( F(f) \) is a monomorphism then \( F(\text{Ker}(f)) \cong \text{Ker}(F(f)) \cong 0 \) and hence \( \text{Ker}(f) \cong 0 \) which forces \( f \) to be a monomorphism. This proves (e) \( \Rightarrow \) (f) since we already know that (e) \( \Rightarrow \) (d). The proof is completed by noting that trivially (f) \( \Rightarrow \) (c) and (d) \( \Rightarrow \) (a).

Proposition 3.8. Let \( V \) be a (quasi)-variety of universal algebras considered as a category, and let \( X \) be a category with finite limits. If \( V \) satisfies Condition 3.4, then \( V(X) \) satisfies Condition 3.4.

Proof. Since the Yoneda embedding \( Y : X \to \text{Set}^{\text{op}} \) preserves and reflects limits and \( V(\text{Set}^{\text{op}}) = V^{\text{op}} \), taking internal \( V \) algebras we obtain a functor \( \tilde{Y} : V(X) \to V^{\text{op}} \) which preserves and reflects limits. The claim now follows by noting that Condition 3.4 lifts to functor categories.

Example 3.9. Recall that an implication algebra is a triple \((X, \cdot, 1)\) where \( X \) is a set, \( \cdot \) is a binary operation and 1 is constant satisfying the axioms: \((xy)x = x, (xy)y = (yx)x, x(yz) = y(xz), 11 = 1\). H. P. Gumm and A. Ursini showed in \footnote{[9]} that the variety of implication algebras form an ideal determined variety of universal algebras which is not congruence permutable. This means that the category of implication algebras is ideal determined but not Mal’tsev \footnote{[11]}. Since the two element boolean algebra \( 2 = (2, \to, 1) \) forms an implication algebra and \( \{(0, 1), (1, 0), (1, 1)\} \) is a sub-algebra of \( 2 \times 2 \), we see that it is not a unital category. However adding an independent binary operation \( \ast \) satisfying \( x \ast 1 = 1 \ast x = x \) will
produce a unital ideal determined category, and hence a strongly unital normal category. We leave as open problems whether this latter variety is Mal’tsev or not and if there exists a normal strongly unital variety which is not Mal’tsev. On the other hand the previous proposition tells us that internal such algebras in a category with finite limits always produce a category which is strongly unital and satisfies Condition 3.4.

Example 3.10. It is easy to show that the quasi-variety $V$ of universal algebras, with terms $p(x, y)$ and $s(x, y)$ satisfying $p(x, 0) = p(0, x) = x$, $s(x, 0) = x$, $s(x, x) = 0$, and $s(x, y) = 0 \Rightarrow x = y$, is a normal strongly unital category. In fact, it turns out that this quasi-variety is almost exact (i.e. every regular epimorphism is an effective descent morphism) and is not Mal’tsev. As before, by the previous proposition, we obtain that internal such algebras in a finitely complete category will produce strongly unital categories satisfying Condition 3.4. In particular, if the base category is the product of the category of sets with the quasi-variety $W$ of abelian groups satisfying $4x = 0 \Rightarrow 2x = 0$, then resulting category will on the one hand not be Mal’tsev since $V$ is not, and on the other hand not be regular (and hence not normal) since $V(W) = W$ which is not regular.

Corollary 3.11. Let $k$ and $l$ be Bourb-normal monomorphisms in a unital category $C$ satisfying Condition 3.4. If $k$ and $l$ have trivial pullback, then $k$ and $l$ commute.

Proof. If $k$ and $l$ have trivial pullback, then by Lemma 3.3 the morphism $\theta$ has trivial kernel and hence is a monomorphism. Moreover, since by Proposition 3.1 the morphism $\theta$ is strong epimorphism it follows that it is an isomorphism. The claim now follows from Theorem 3.2.

Lemma 3.12. Let $k : X \to A$ and $l : Y \to A$ be monomorphisms in a strongly unital category $C$ which commute, and let $\varphi : X \times Y \to A$ be their cooperator. If $(u, v) : W \to X \times Y$ is the kernel of $\varphi$, then $u$ and $v$ are central monomorphisms and $ku = k(-v)$.

Proof. Since each of the squares in the following diagram

```
\begin{array}{ccc}
W & \xrightarrow{\varphi} & A \\
\downarrow & & \downarrow \\
X \times Y & \xrightarrow{(u,v)} & Y
\end{array}
```

are pullbacks, we see via Lemma 2.2 that $u$ and $v$ are central. To complete the proof just note that $(u, 0) = (u, v) + (0, -v)$ and therefore

\[
ku = \varphi(u, 0) = \varphi(u, v) + \varphi(0, -v) = \varphi(u, v) + \varphi(0, -v) = \varphi(0, -v) = l(-v).
\]
Proposition 3.13. Let \( k \) and \( l \) be Bourn-normal monomorphisms in a strongly unital category \( C \) satisfying Condition 3.4. If \( k \) and \( l \) have trivial pullback, then \( k \) and \( l \) commute and their cooperator \( \varphi : X \times Y \to A \) is a monomorphism, which is also their join.

Proof. By Corollary 3.11 we know that \( k \) and \( l \) commute. It remains to show that their cooperator is a monomorphism and is their join. The first point follows from Condition 3.4 since Lemma 3.12 implies the kernel of \( \varphi \) is zero. The final point follows immediately from the fact that \( (1, 0) \) and \( (0, 1) \) are jointly strongly epimorphic.

Recall that in the Mal’tsev context an equivalence relation \( r_1, r_2 : R \to A \) is essentially the same thing as a monomorphism

\[
\begin{array}{ccc}
R & \xrightarrow{(r_1, r_2)} & A \times A \\
\downarrow{r_1} & & \downarrow{\pi_1} \\
A & & A
\end{array}
\]

in the category \( \text{Pt}(A) \). Moreover such a monomorphism \( (r_1, r_2) \) is necessarily Bourn-normal. To see why, consider the pullback diagram

\[
\begin{array}{ccc}
R \times A & \xrightarrow{p_2} & R \\
\downarrow{r_1} & & \downarrow{r_1} \\
R & & A
\end{array}
\]

It follows that \( \langle r_1 p_1, r_2 p_1 \rangle, \langle r_1 p_1, r_2 p_2 \rangle : (R \times A, r_1 p_1, (e, e)) \to (A \times A, \pi_1, (1, 1)) \) (where \( e \) is the splitting of \( r_1 \) and \( r_2 \)) is an equivalence relation and the diagrams

\[
\begin{array}{ccc}
A \times (A \times A) & \xrightarrow{1 \times \pi_2} & A \times A \\
\downarrow{1 \times \pi_1} & & \downarrow{\pi_1} \\
A \times A & & A
\end{array}
\quad
\begin{array}{ccc}
R & \xrightarrow{(1, r_1 e)} & R \times_A R \\
\downarrow{\pi_1} & & \downarrow{\langle r_1, r_2 \rangle} \\
A \times A & \xrightarrow{1 \times (p_2, p_1)} & A \times (A \times A)
\end{array}
\]

are pullbacks.

Theorem 3.14. Let \( k \) and \( l \) be Bourn-normal monomorphisms in a Mal’tsev category \( C \) satisfying Condition 3.4. If \( k \) and \( l \) have trivial pullback, then \( k \) and \( l \) commute and their cooperator \( \varphi : X \times Y \to A \) is a Bourn-normal monomorphism which is also their join.

Proof. By Proposition 3.7 we see that \( X \land Y = 0 \) implies \( R \land S = 0 \) when considered as subobjects of \( (A \times A, \pi_1, (1, 1)) \) in the category of points over \( A \). It now follows from Proposition 3.13 that \( R \times S \) (in \( \text{Pt}(A) \)) is a subobject of \( (A \times A, \pi_1, (1, 1)) \), and hence is an equivalence relation with zero class the cooperator of \( k \) and \( l \).
Corollary 3.15. Let \( C \) be strongly unital category satisfying Condition 3.4. For an object \( X \) in \( C \) the following conditions are equivalent:

\begin{itemize}
  \item[(a)] \( X \) is abelian;
  \item[(b)] \( (1, 1) : X \to X \times X \) is a normal monomorphism;
  \item[(c)] \( (1, 1) : X \to X \times X \) is a Bourn-normal monomorphism.
\end{itemize}

Proof. The implications \( (a) \Rightarrow (b) \Rightarrow (c) \) are immediate. It is therefore sufficient to show that \( (a) \) follows from \( (c) \). Suppose \( X \) is object in \( C \) and \( (1, 1) \) is Bourn-normal. Since \( (1, 0) \) and \( (1, 1) \) have trivial pullback, it follows that they commute and hence we obtain a morphism \( \psi \) making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(1,0)} & X \\
\downarrow \downarrow & & \downarrow \downarrow \\
X & \xleftarrow{(0,1)} & X \\
\end{array}
\]

commute. The claim now follows from Corollary 1.8.20 of [2], since \( \pi_1 \psi \) is a cooperador for \( 1_X \) and \( 1_X \).

Remark 3.16. Given that abelianess is a property in a subtractive category, and abelianization is obtained by forming the cokernel of the diagonal in a regular subtractive category (provided the cokernel exists) [3], one expects that the above corollary is true in a wider context.

References

[1] M. Barr, Exact categories, in: Lecture Notes in Mathematics 236, 1–120, 1971.
[2] F. Borceux and D. Bourn, Mal’cev, protomodular, homological and semi-abelian categories, Kluwer Academic Publishers, 2004.
[3] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, Lecture Notes in Mathematics, Category theory (Como, 1990) 1488 , Springer, Berlin, 43–62, 1991.
[4] D. Bourn, Normal subobjects and abelian objects in protomodular categories, Journal of Algebra 228(1), 143–164, 2000.
[5] D. Bourn and Z. Janelidze, A note on the abelianization functor, Communications in Algebra 44(5), 2009–2033, 2016.
[6] A. Carboni, J. Lambe, and M. C. Pedicchio, Diagram chasing in Mal’cev categories, Journal of Pure and Applied Algebra 69(3), 271–284, 1991.
[7] A. Carboni, M. C. Pedicchio, and N. Pirovano, Internal graphs and internal groupoids in Mal’cev categories, CMS Conf. Proc., Category theory 1991 (Montreal, PQ, 1991) 13 , Amer. Math. Soc., Providence, RI, 97–109, 1992.
[8] M. Gran and Z. Janelidze, Star-regularity and regular completions, Journal of Pure and Applied Algebra 218(10), 1771–1782, 2014.
[9] H. P. Gumm and A. Ursini, *Ideals in universal algebras*, Algebra Universalis **19**(1), 45–54, 1984.

[10] S. A. Huq, *Commutator, nilpotency and solvability in categories*, Quarterly Journal of Mathematics **19**(1), 363–389, 1968.

[11] G. Janelidze, L. Márki, W. Tholen, and A. Ursini, *Ideal determined categories*, Cahiers de topologie et géométrie différentielle catégoriques **51**, 115–127, 2010.

[12] Z. Janelidze, *The pointed subobject functor, 3 x 3 lemmas, and subtractivity of spans*, Theory and Applications of Categories **23**(11), 221–242, 2010.