ON REGULAR MAPS AND PARALLEL LINES

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Abstract. Let $f : \mathbb{R}^{m+1} \to \mathbb{R}^{m+2^r}$, where $2^{r-1} \leq m + 1 < 2^r$, be a continuous map. Improving a recent result of Frick and Harrison, we show that there are 4 points $x_0, x_1, y_0, y_1$ in $\mathbb{R}^m$, which are distinct if $m+1 \neq 2^{r-1}$, and satisfy $x_0 \neq x_1, y_0 \neq y_1, \{x_0, x_1\} \neq \{y_0, y_1\}$ if $m+1 = 2^{r-1}$, such that the vectors $f(x_1) - f(x_0)$ and $f(y_1) - f(y_0)$ are parallel.

1. Introduction

We improve a recent result [6] Theorems 1 and 3 of Frick and Harrison using methods (perhaps more conceptual than those in [6]) from [3, 1].

Two vectors in a vector space will be called parallel if they lie in a 1-dimensional vector subspace; this allows one or both vectors to be zero.

Theorem 1.1. Suppose that $m$ is a non-negative integer and let $r \geq 1$ be the integer such that $2^{r-1} \leq m + 1 < 2^r$. Let $f : \mathbb{R}^{m+1} \to \mathbb{R}^{m+2^r}$ be a continuous map. Then

(a) if $m + 1 \neq 2^{r-1}$, there exist 4 distinct points $x_0, x_1, y_0, y_1 \in \mathbb{R}^{m+1}$ such that the vectors $f(x_1) - f(x_0)$ and $f(y_1) - f(y_0)$ are parallel;

(b) if $m + 1 = 2^{r-1}$, there exist 4 points $x_0, x_1, y_0, y_1 \in \mathbb{R}^{m+1}$ such that $x_0 \neq x_1$, $y_0 \neq y_1, \{x_0, x_1\} \neq \{y_0, y_1\}$, and the vectors $f(x_1) - f(x_0)$ and $f(y_1) - f(y_0)$ are parallel.

At the extremes, when $m + 1 = 2^{r-1}$ we are considering a map $\mathbb{R}^{2^{r-1}} \to \mathbb{R}^{2^{r-1}+1}$ – the first few cases with $r = 2, 3, 4, 5$ were established in [6] Corollaries 2 and 4 – and when $m + 1 = 2^{r-1} - 1$ a map $\mathbb{R}^{2^{r-1}-1} \to \mathbb{R}^{2^{r-1}+1}$.

The assertion is trivial if $f$ is not injective. For, if there are distinct points $x_0$ and $x_1$ such that $f(x_0) = f(x_1)$, we can choose any two points $y_0$ and $y_1$ in the complement of $\{x_0, x_1\}$, and then $f(x_1) - f(x_0) = 0$ is parallel to $f(y_1) - f(y_0)$.

As a corollary we have the following special case of [1] Theorem 6.16.

Corollary 1.2. Let $f : \mathbb{R}^{m+1} \to \mathbb{R}^{m+2^r}$ be a continuous map, where $2^{r-1} \leq m + 1 < 2^r$.

Then there exist 4 distinct points $x_0, x_1, x_2, x_3 \in \mathbb{R}^{m+1}$ such that the vectors $f(x_0), f(x_1), f(x_2), f(x_3)$ lie in some 2-dimensional affine subspace of $\mathbb{R}^{m+2^r}$.
For, if say $x_0 = y_0$ in (b), the points $f(x_0), f(x_1)$ and $f(y_1)$ lie on an affine line and we can then take $x_2 = y_1$ and choose an arbitrary point $x_3$ in the complement of \(\{x_0, x_1, x_2\}\).

2. Prelude

The method will be illustrated by proving the following classical result. (See, for example, [1].)

**Proposition 2.1.** Suppose that $m \geq 0$ is a non-negative integer and that $f : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ is a continuous map. Then there exist distinct points $x, y \in \mathbb{R}^{m+1}$ such that the vectors $f(x)$ and $f(y)$ are parallel.

**Remark 2.2.** The map $f : \mathbb{R}^{m+1} \to \mathbb{R} \oplus \mathbb{R}^{m+1}, x \mapsto (1, x)$ has the property that $f(x), f(y) \in \mathbb{R} \oplus \mathbb{R}^{m+1}$ are parallel only if $x = y$.

Writing $V = \mathbb{R}^{m+1}$ and $W = \mathbb{R}^n$ where $n \geq 1$, we shall consider a continuous map $f : V \to W$. The unit sphere in the Euclidean space $V$ is denoted by $S(V)$. We denote the real projective space on $V$ by $P(V) = S(V)/\{\pm 1\}$, and write $H_V$ (or simply $H$ if the meaning is clear) for the Hopf line bundle.

**Proof.** The assertion is trivially true if $m = 0$. So assume that $m \geq 1$. We have a map

$$\varphi : X = S(V) \to V \oplus V, \quad x \mapsto (x, -x).$$

So $\varphi(x)$ determines 2 distinct points $\pm x$ of $V$. The quotient of $X$ by the free action of the group $\{\pm 1\}$ is the real projective space $Y = P(V)$. The mod 2 Euler class $\lambda = H_Y$ over $P(V)$ determines the cohomology ring

$$H^*(Y; \mathbb{F}_2) = \mathbb{F}_2[t]/(t^{m+1}).$$

Now the map $f$ determines a $\{\pm 1\}$-equivariant map

$$S(V) \to W \oplus W, \quad x \mapsto (f(x) + f(-x), f(x) - f(-x)),$$

with $\{\pm 1\}$ acting trivially on the first factor and as $\pm 1$ on the second, and so a section of $(\mathbb{R} \oplus \lambda) \otimes W$ over $Y$. Lifting to $Y \times P(W)$ and projecting from the trivial bundle $W$ to $W/H_W$, we get a section $s$ of the $2n$-dimensional real vector bundle $(\mathbb{R} \oplus \lambda) \otimes (W/H_W)$ over $Y \times P(W)$. A zero of $s$ gives a point $x \in S(V)$ and a line $L \subset P(W)$ such that $f(x) + f(-x), f(x) - f(-x) \in L$, or, equivalently, $f(x), f(-x) \in L$.

Now, as we recollect in the Appendix, the direct image homomorphism (evaluation on the fundamental class of $P(V)$)

$$\pi_1 : H^{2n}(Y \times P(W); \mathbb{F}_2) \to H^n(Y; \mathbb{F}_2)$$

maps the Euler class $e((\mathbb{R} \oplus \lambda) \otimes (W/H_W))$ to $w_n(-(\mathbb{R} \oplus \lambda))$.

This Stiefel-Whitney class is the degree $n$ term of $(1 + t)^{-1} = \sum_{0 \leq i \leq m} t^i$, and so is non-zero if $n \leq m$. Hence the Euler class of $((\mathbb{R} \oplus \lambda) \otimes (W/H_W)$ is zero, and the section $s$ has a zero.

**Remark 2.3.** A section of $((\mathbb{R} \oplus \lambda) \otimes W$ can be thought of as a vector bundle homomorphism $\mathbb{R} \oplus \lambda \to P(V) \times W$ to the trivial bundle over $P(V)$. If this bundle homomorphism is injective in each fibre, then $\mathbb{R} \oplus \lambda$ is included as a subbundle of $P(V) \times W$ with orthogonal complement of dimension $n - 1$. So $w_n(-(\mathbb{R} \oplus \lambda))$, the Stiefel-Whitney class of this orthogonal complement, is zero. This argument,
going back to [2] Proposition 2.1], could be used to replace the reference to Lemma A.1 in the proof above. However, the argument involving the Euler class has the advantage that it provides some information on the solution set, as is explained in Appendix B.

Remark 2.4. Taking a dual viewpoint, we may lift from $Y$ to the projective bundle $P(\mathbb{R} \oplus \lambda)$ of the vector bundle $\mathbb{R} \oplus \lambda$ and, by restricting to the fibrewise Hopf bundle $H_{\mathbb{R} \oplus \lambda} \subseteq \mathbb{R} \oplus \lambda$, get a section $s^*$ of $H_{\mathbb{R} \oplus \lambda} \otimes W$.

The direct image homomorphism

$$H^{n+1}(P(\mathbb{R} \oplus \lambda); \mathbb{F}_2) \to H^n(Y; \mathbb{F}_2)$$

maps $e(H_{\mathbb{R} \oplus \lambda} \otimes W)$ to $w_n(-(\mathbb{R} \oplus \lambda))$. (See Lemma A.2) A zero of $s^*$ gives a point $x \in S(V)$ and $(\alpha, \beta) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $\alpha f(x) + \beta f(-x) = 0$, or, in other words, $f(x)$ and $f(-x)$ are parallel.

3. Parallel lines

The case in which $m = 0$ and $f$ is a map $\mathbb{R} \to \mathbb{R}^2$ is special and elementary.

Proposition 3.1. Suppose that $f: \mathbb{R} \to \mathbb{R}^2$ is a continuous map. Then there exist 4 points $x_0 < y_0 < y_1 < x_1$ in $\mathbb{R}$ such that $f(x_1) - f(x_0)$ and $f(y_1) - f(y_0)$ are parallel.

Proof. If the image of $f$ is contained in some (affine) line, we can choose the 4 points arbitrarily. If not, choose 3 points $x_0 < z < x_1$ in $\mathbb{R}$ such that $f(x_0), f(z), f(x_1)$ are not collinear. Apply the Intermediate Value theorem to the function $\varphi: [x_0, x_1] \to \mathbb{R}$ given by the inner product $\rho(y) = (f(y) - f(x_0), e)$ for some unit vector $e \in \mathbb{R}^2$ perpendicular to $f(x_1) - f(x_0)$. Since $\rho(x_0) = 0 = \varphi(x_1)$ and $\rho(z) \neq 0$, there are points $y_0, y_1$ such that $x_0 < y_0 < z < y_1 < x_1$ such that $\rho(y_0) = \rho(y_1)$.

Assume now that $m \geq 1$. We consider as in Section 2 a continuous map $f: V = \mathbb{R}^{m+1} \to W = \mathbb{R}^{n+1}$.

The manifold $\tilde{X} = S(V) \times S(V \oplus V)$, of dimension $3m + 1$, has a free action of the group $G = \{\pm 1\} \times \mathbb{Z}/2\mathbb{Z}$ generated by the commuting involutions $(-1, 0): (x, (u, v)) \mapsto (x, (-u, -v))$ and $(1, 1): (x, (u, v)) \mapsto (-x, (v, u))$.

The orbit space $\tilde{Y} = \tilde{X}/G$ can be identified with the real projective bundle $P(V \oplus (H \otimes V))$ by mapping the orbit of $(x, (u, v))$ to $[u + x, x \otimes (u - v)]$ in the fibre over $[x] \in P(V)$.

There are two important line bundles over $\tilde{Y}$: the pullback $\lambda$ of the Hopf bundle $H$ over $P(V)$ and the Hopf bundle $\mu$ of the projective bundle. The cohomology ring of $\tilde{Y}$ is described in terms of the mod 2 Euler classes $t = e(\lambda)$ and $x = e(\mu)$ as

$$H^*(\tilde{Y}; \mathbb{F}_2) = F_2[t, x]/(t^{m+1}, (x(t + x))^{m+1})$$

(because $e(\mu \otimes (\lambda \otimes V)) = e(\mu \otimes \lambda)^{m+1}$). Put $y = x(t + x) = e((\lambda \otimes \mu) \oplus \mu)$. As $\mathbb{F}_2$-vector space, $H^*(\tilde{Y}; \mathbb{F}_2)$ has a basis $t^iy^j$, $0 \leq i, j \leq m$.

The bundles $\lambda$ and $\mu$ are associated with the representations $G \to O(\mathbb{R})$:

$$(a, c) \mapsto (-1)^{k}, \quad (a, c) \mapsto a,$$

respectively. (Isomorphisms are given by $\tilde{X} \times_G \mathbb{R} \to \lambda$: $(x, (u, v), t) \mapsto tx$ and $\tilde{X} \times_G \mathbb{R} \to \mu$: $((x, (u, v)), t) \mapsto t(u + v, x \otimes (u - v))$.)
Choose $\delta$, $0 < \delta < \frac{1}{2}$. There is a basic map (introduced, implicitly, in [3])
\[
\phi : \hat{X} = S(V) \times S(V \oplus V) \to V \oplus V \oplus V \oplus V,
\]
\[
(x, (u, v)) \mapsto (x + \delta u, x - \delta u, -x + \delta v, -x - \delta v).
\]

**Proof of Theorem [1.1]** The 4-tuple $\phi(x, (u, v))$ determines a pair of 2-element subsets of $V$:
\[
\{x + \delta u, -x + \delta v\}, \{x - \delta u, -x - \delta v\},
\]
with one point in common if $u = 0$ or $v = 0$, and otherwise disjoint.

Now $f$ determines sections of $\lambda \otimes W$ and $\lambda \otimes \mu \otimes W$ by the equivariant maps
\[
(x, (u, v)) \mapsto f(x + \delta u) - f(-x - \delta v) + f(x - \delta u) - f(-x + \delta v)
\]
and
\[
(x, (u, v)) \mapsto f(x + \delta u) - f(-x + \delta v) - f(x - \delta u) + f(-x - \delta v).
\]
Taking their sum and projecting from $W$ to the quotient $W/H$, we get a section $s$ of $(\lambda \otimes (\mathbb{R} \oplus \mu)) \otimes (W/H)$ over $\hat{Y} \times P(W)$.

At a zero $[(x, (u, v)), L] \in \hat{Y} \times P(W)$ of $s$, the vectors $f(x + \delta u) - f(-x + \delta v) + f(x - \delta u) - f(-x - \delta v)$ and $f(x + \delta u) - f(-x + \delta v) - f(x - \delta u) + f(-x - \delta v)$ lie in the 1-dimensional subspace $L$ of $W$. So their sum $2(f(x + \delta u) - f(-x + \delta v))$ and difference $2(f(x - \delta u) - f(-x - \delta v))$ lie in $L$. Thus the vectors $f(x + \delta u) - f(-x + \delta v)$ and $f(x - \delta u) - f(-x - \delta v)$ are parallel.

Now the image of the Euler class
\[
\epsilon(\lambda \otimes (\mathbb{R} \oplus \mu)) \in H^{2n}(\hat{Y} \times P(W); \mathbb{F}_2)
\]
in $H^n(\hat{Y}; \mathbb{F}_2)$ under the direct image homomorphism is $w_n(-\lambda \otimes (\mathbb{R} \oplus \mu))$.

To compute this Stiefel-Whitney class, note that
\[
1 + t + y = w((\lambda \otimes \mu) \oplus \mu) = w(\lambda \otimes \mu)w(\mu) = (1 + t + x)(1 + x).
\]
So
\[
w(-\lambda \otimes (\mathbb{R} \oplus \mu)) = (1 + t)^{-1}(1 + t + x)^{-1} = (1 + t)^{-1}(1 + t + y)^{-1}(1 + x)
\]
\[
(1 + t)^{-2}(1 + y(1 + t)^{-1})^{-1}(1 + x) = \sum_{j \geq 0}(1 + t)^{-2-j-1}y^j(1 + x)
\]
\[
\sum_{0 \leq i,j \leq m}(\begin{pmatrix} -j - 2 \\ i \end{pmatrix}t^iy^j(1 + x) = \sum_{i,j \leq m}(\begin{pmatrix} 2r - j - 2 \\ i \end{pmatrix})t^iy^j(1 + x).
\]
The coefficient of $t^{2r-m-2}y^m x$ is 1, and $2r - m - 2 \leq m$.

With $n = m + 2r - 1$, we have shown that the Euler class is non-zero, so that the section $s$ has a zero and the assertion (b) has been proved.

We also have a section $z$ of $\lambda$ determined by the equivariant map
\[
(x, (u, v)) \mapsto \|u\|^2 - \|v\|^2.
\]
At a zero of $z$, $u$ and $v$ are both non-zero.

The section $z \oplus s$ of $\lambda \oplus ((\mu \oplus (\lambda \otimes \mu)) \otimes (W/H))$ will (by Lemma [A.1]) have a zero if
\[
\epsilon(\lambda)w_n(-\mu - (\lambda \otimes \mu)) = tw_n(-\mu - (\lambda \otimes \mu)) \in H^{n+1}(\hat{Y}; \mathbb{F}_2)
\]
is non-zero. From the calculation above this is the case if $t(t^{2r-m-2}y^m x) \neq 0$, which is true provided that $2r - m - 1 \leq m$, that is, $m + 1 \neq 2r - 1$. This completes the proof of assertion (a).
We discuss next the relation between these techniques and the method used in [6]. Writing \( O(V) \) for the orthogonal group of \( V \), let \( G = \{\pm 1\} \times \mathbb{Z}/2\mathbb{Z} \) act on \( \tilde{X}_0 = O(V) \times S(V \oplus V) \) by the involutions \((g, (u, v)) \mapsto (g, (u, -v))\) and \((g, (u, v)) \mapsto (-g, (u, v))\). The orbit space \( \tilde{Y}_0 = \tilde{X}_0/G \) is the product \( PO(V) \times P(V \oplus V) \) of the projective orthogonal group of \( V \) and the projective space.

Choosing a vector \( e \in S(V) \) we have a \( G \)-map

\[
\tilde{X}_0 = O(V) \times S(V \oplus V) \to \tilde{X} = S(V) \times S(V \oplus V) : \]

\[
(g, (u, v)) \mapsto (ge, \frac{1}{\sqrt{2}}(u + gv, u - gv)).
\]

Under the quotient map \( \tilde{Y}_0 \to \tilde{Y} \), the line bundles \( \lambda \) and \( \mu \) pullback to the line bundle \( \lambda_0 \) associated with the double cover \( O(V) \to PO(V) \) and the Hopf line bundle \( \mu_0 \) over \( P(V \oplus V) \).

**Proposition 3.2.** Let \( q \geq 0 \) be the largest integer such that \( 2^q \) divides \( m + 1 \). Then \( w_n(-\langle \lambda_0 \otimes (\mathbb{R} \oplus \mu_0) \rangle) \neq 0 \) for \( n \leq 2m + 2^q \).

**Proof.** It is classical that the Euler class \( t_0 = c(\lambda_0) \) satisfies \( t_0^{2^q} \neq 0 \), \( t_0^q = 0 \); see, for example, [4] Proposition A.1. And \( x_0 = c(\mu_0) \) satisfies \( x_0^{2^q} \neq 0 \), \( x_0^q = 0 \).

Thus \( t_0 \) and \( t_0 + x_0 \) generate a subalgebra of \( H^*(\tilde{Y}_0; \mathbb{F}_2) \) with the only relations \( t_0^q = 0 \) and \( (t_0 + x_0)^{2^q} = 0 \). (Notice that \( (t_0 + x_0)^{m+1} = (t_0^q + x_0^{2^q})^{(m+1)/2^q} = x_0^{m+1} \))

Now

\[
w(-\langle \lambda_0 \otimes (\mathbb{R} \oplus \mu_0) \rangle) = (1 + t_0)^{-1}(1 + t_0 + x_0)^{-1} = \sum_{i,j \geq 0} t_0^i(t_0 + x_0)^j.
\]

So the assertion follows, because \( 2m + 2^q = (2^q - 1) + (2m + 1) \). \( \square \)

**Remark 3.3.** The argument in [6] Theorems 1 and 3] proceeds, in effect, by using, instead of the fact that \( t_0^{2^q} \neq 0 \), a result of the form \( t_0^{2^q} \neq 0 \), where \( \alpha(q) \leq q \) is expressed in terms of Hurwitz-Radon numbers. For \( q \geq 1 \), \( \alpha(q) \geq 1 \) is chosen so that one has a Clifford module multiplication \( \mathbb{R}^{2\alpha(q)} \otimes V \to V \), and this determines an embedding \( \mathbb{P}(\mathbb{R}^{2\alpha(q)} \otimes V) \hookrightarrow PO(V) \), under which \( \lambda_0 \) restricts to the Hopf line bundle. Hence \( t_0^{2\alpha(q) - 1} \neq 0 \). Now we recall a general \( H \)-space argument. Consider the group multiplication \( \mu : PO(V) \times PO(V) \to PO(V) \). Since \( \mu^*(t_0) = t_0 \otimes 1 + 1 \otimes t_0 \),

\[
\mu^*(t_0^{2\alpha(q) - 1}) = (t_0 \otimes 1 + 1 \otimes t_0)^{2\alpha(q) - 1} = \sum_{i+j = 2\alpha(q) - 1} t_0^i \otimes t_0^j,
\]

which is non-zero, because it contains the term \( t_0^{2\alpha(q)-1} \otimes t_0^{\alpha(q)-1} \), and hence \( t_0^{2\alpha(q) - 1} \neq 0 \).

### 4. Regular maps

Let \( X \) be the codimension one submanifold

\[
X = S(V) \times (S(V) \times S(V)) \subseteq \tilde{X}
\]

of \( \tilde{X} \). The quotient of \( X \) by the group \( G = \{\pm 1\} \times \mathbb{Z}/2\mathbb{Z} \) is a submanifold \( Y \) of \( \tilde{Y} \). The quotient of \( X \) by the free action of the group \( K = \{\pm 1\} \times \mathbb{Z}/2\mathbb{Z} \) generated by the involutions \( ((-1,1),0), ((1,-1),0) \) and \( ((1,1),1): (x,(u,v)) \mapsto (-x,-u,-v) \).
(x, (−u, v)), (x, (u, −v)), (−x, (v, u)) is denoted by \( Z \). It fibres over the real projective space \( P(V) \) on \( V \) by the map induced by the projection \( (x, (u, v)) \mapsto x \). We have covers \( X \to Y \to Z \).

There are various natural real vector bundles over \( Z \): the pull-back \( λ \) of the Hopf line bundle over \( \mathbb{R}x \), determined by \( (u, v) \) is the ideal generated by \( a \) over \( \mathbb{R}x \). The determinant bundle \( Λ^2 \) splits as \( \mathbb{R}u \oplus \mathbb{R}v \). They are associated with the representations:

\[
(a, b; c) \mapsto (−1)^c;
\]

\[
(a, b; c) \mapsto ab;
\]

\[
(a, b; 0) \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad (a, b; 1) \mapsto \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix},
\]

The determinant bundle \( Λ^2 \) is isomorphic to \( λ \otimes \alpha \).

The mod 2 Euler classes \( e(λ) \) and \( y = e(ζ) \) generate the cohomology ring

\[
H^*(Z; \mathbb{F}_2) = \mathbb{F}_2[t, y, a]/(t^{m+1}, y^{m+1}, ta, y^m a, \ldots, y^{m+1−i} a^i, \ldots, a^{m+1}).
\]

The pullback of \( α \) to \( Y \) is trivial. Indeed, \( Y = S(α) \). The pullback of \( ζ \) to \( Y \) splits as \( (λ \otimes μ) \oplus μ \). The kernel of the homomorphism

\[
H^*(Z; \mathbb{F}_2) \to H^*(Y; \mathbb{F}_2)
\]

is the ideal generated by \( a \).

Let \( ϕ : X \to V ⊕ V ⊕ V \) be the restriction of \( ϕ \). Then \( ϕ(x, (u, v)) \) determines 4 distinct points \( x ± δu, −x ± δv \) of \( V \), and a pair of disjoint 2-element subsets

\[
\{x + δu, x − δu\}, \{−x + δv, −x − δv\}
\]

of \( V \).

Second proof of Theorem 1.1(a). The map \( f \) determines a section \( s \) of \( ζ \otimes (W/H) \) over \( Z \times P(W) \) by the equivariant map

\[
(x, (u, v)) \mapsto (f(x + δu) − f(x − δu) + f(x + δv) − f(x − δv),
\]

\[
f(x + δu) − f(x − δu) − f(x + δv) + f(x − δv).
\]

The Euler class will be non-zero if \( w_n(−ζ) \in H^n(Z; \mathbb{F}_2) \) is non-zero. We lift to \( H^n(Y; \mathbb{F}_2) \) and compute

\[
(1 + t + y)^{-1} = (1 + t)^{-1}(1 + y(1 + t)^{-1})^{-1} = \sum_{j ≥ 0} (1 + t)^{(−j+1)} y^j
\]

\[
= \sum_{0 ≤ i, j ≤ m} \left(\begin{array}{c} j−1 \\ i \end{array}\right) t^i y^j = \sum_{0 ≤ i, j ≤ m} \left(\begin{array}{c} 2^r − j−1 \\ i \end{array}\right) t^i y^j.
\]

The coefficient of \( t^{2r−m−1} y^m \) is 1, and \( 2^r − m − 1 ≤ m \), because \( m + 1 \neq 2^{r−1} \).

Second proof of Corollary 1.2. We shall, in effect, reproduce the proof in [1] for this special case.

The equivariant map

\[
(x, (u, v)) \mapsto f(x + δu) − f(x + δv) + f(x − δu) − f(x − δv),
\]

determines a section of \( λ \otimes W \); and, combined with the section of \( ζ \otimes W \) already constructed, a section of the \( 3(n−1) \)-dimensional vector bundle \( (λ \otimes ζ) \otimes (W/γ) \)
over $Z \times G_2(W)$, where $\gamma$ is the canonical 2-plane bundle over the Grassman manifold $G_2(W)$ of 2-dimensional subspaces of $W$. The Euler class will be non-zero if $w_{n-1}(-\lambda \oplus \zeta) \neq 0$ (by [3, Lemma 6.4] or [2, Proposition 2.1]). Passing to the quotient of $H^*(Z; \mathbb{F}_2)$ by the ideal generated by $a$, we look at the degree $n - 1$ term in

$$(1 + t)^{-1}(1 + t + y)^{-1} = \sum_{j \geq 0} (1 + t)^{-(j+2)}y^j = \sum_{0 \leq i, j \leq m} \left(\frac{2^r - j - 2}{i}\right) t^i y^j.$$  

The coefficient of $t^{2r - m - 2}y^m$ is equal to 1. Take $n - 1 = 2^r - m - 2 + 2m$. 

There is an equivalent spherical version of Corollary 1.2 (which, in fact, is the form given in [1, Theorem 6.16]).

**Proposition 4.1.** Let $\tilde{f} : \mathbb{R}^{m+1} \to S(\mathbb{R}^{m+2r+1}) \subseteq \mathbb{R}^{m+2r+1}$ be a continuous map, where $2^{r-1} \leq m + 1 < 2^r$.

Then there exist 4 distinct points $x_0, x_1, x_2, x_3 \in \mathbb{R}^{m+1}$ such that the vectors $\tilde{f}(x_0), \tilde{f}(x_1), \tilde{f}(x_2), \tilde{f}(x_3)$ are linearly dependent.

We give two proofs, one using the equivalence and one by a direct argument. Consider a general continuous map $\tilde{f} : V \to S(\tilde{W})$ to the unit sphere in a Euclidean vector space $\tilde{W}$ of dimension $(n + 1) + 1$.

**First proof.** Let $i : W \to S(\mathbb{R} \oplus W)$ be the inclusion $w \mapsto (1, w)/(1 + \|w\|^2)^{1/2}$ of $W$ as the open northern hemisphere of the sphere. It has the property that $k + 1$ vectors $w_0, \ldots, w_k$ in $W$ are affinely independent, that is, span a $k$-dimensional affine subspace, if and only if $i(w_0), \ldots, i(w_k)$ are linearly independent in $\mathbb{R} \oplus W$, that is, span a vector subspace which intersects $S(\mathbb{R} \oplus W)$ in a $k$-dimensional sphere with centre 0.

Now $\tilde{f}$ maps a sufficiently small open neighbourhood of $0 \in W$ into an open hemisphere in $S(\mathbb{R} \oplus W)$. So, by restricting $\tilde{f}$ to a neighbourhood homeomorphic to $W$ and choosing an appropriate splitting of $\tilde{W} = \mathbb{R} \oplus W$, we may assume that $\tilde{f}$ has the form $i \circ f$ for a continuous map $f : V \to W$. So the result follows from Corollary 1.2.

**Second proof.** The constructions in the second proof of Proposition 1.2 applied to $\tilde{f}$ instead of $f$ give sections of $\zeta \otimes \tilde{W}$ and $\lambda \otimes \tilde{W}$ and the map

$$(x, (u, v)) \mapsto f(x + \delta u) + f(-x + \delta v) + f(x - \delta u) + f(-x - \delta v)$$

gives a section of the trivial bundle with fibre $\tilde{W}$. Combining the three maps we get a section of $(\mathbb{R} \oplus \zeta \oplus \lambda) \otimes (\tilde{W}/\gamma)$ over $Z \times G_3(\tilde{W})$, where $\gamma$ is now the canonical 3-dimensional bundle over the Grassmanian. Its Euler class will be non-zero if $w_{n+1-2}(-(\mathbb{R} \oplus \zeta \oplus \lambda)) \neq 0$. This is the same condition as in the affine case.

**Appendix A.**

We give an elementary proof of a special case of [3, Lemma 6.4].

**Lemma A.1.** Let $\xi$ be a 2-dimensional real vector bundle over a compact ENR $B$. Write $W = \mathbb{R}^{n+1}$. Then the image of the Euler class $e(\xi \otimes (W/H))$ under the direct image homomorphism

$$\pi_1 : H^{2n}(B \times P(W); \mathbb{F}_2) \to H^n(B; \mathbb{F}_2)$$

is equal to $w_n(-\xi)$.
Proof. By universality, it suffices to verify the formula when \( \xi = \lambda_1 \oplus \lambda_2 \) is a sum of line bundles. Write \( t_i = e(\lambda_i) \in H^1(B; \mathbb{F}_2) \) and \( x = e(H) \in H^1(P(W); \mathbb{F}_2) \). Thus \( w_{n-j}(W/H) = x^{n-j} \) and

\[
e((\lambda_1 \oplus \lambda_2) \otimes (W/H)) = \prod_{i=1}^{2} e(\lambda_i \otimes (W/H)) = \prod_{i=1}^{2} (x^n + \ldots + t_i^j x^{n-j} + \ldots + t_i^n)
\]

in \( H^*(B; \mathbb{F}_2)[x]/(x^{n+1}) \). We have to determine the coefficient of \( x^n \) in this truncated polynomial ring, or, equivalently, the coefficient of \( T^{2n} \) in the formal power series

\[
\prod_{i=1}^{2} (1 + \ldots + t_i^j T^j + \ldots + t_i^n T^n) = \prod_{i=1}^{2} \frac{1 - t_i^{n+1} T^{n+1}}{1 - t_i T} \in H^*(B; \mathbb{F}_2)[[T]].
\]

Now

\[
\prod (1 - t_i T)^{-1} = (1 + w_1(\xi) T + w_2(\xi) T^2)^{-1} = 1 + w_1(-\xi) T + \ldots + w_i(-\xi) T^i + \ldots.
\]

So the coefficient of \( T^n \) is \( w_n(-\xi) \). \( \square \)

And here is the dual version, a special case of the discussion preceding Proposition 4.2 in [5].

Lemma A.2. Let \( \xi \) be a 2-dimensional real vector bundle over a compact ENR \( B \). Write \( W = \mathbb{R}^{n+1} \). Consider the projective bundle \( \pi : P(\xi) \to B \). Then the image of the Euler class \( e(H \otimes W) \) under the direct image homomorphism

\[
\pi_! : H^{n+1}(P(\xi); \mathbb{F}_2) \to H^n(B; \mathbb{F}_2)
\]

is equal to \( w_n(-\xi) \).

Proof. The cohomology ring of the projective bundle is

\[
H^*(P(\xi); \mathbb{F}_2) = H^*(B)[t]/(t^2 + w_1(\xi) t + w_2(\xi)),
\]

where \( t = e(H) \), so that \( e(H \otimes W) = t^{n+1} \). Induction on \( n \), using the identity \( w_{n+1}(-\xi) + w_1(\xi) w_n(-\xi) + w_2(\xi) w_{n-1}(-\xi) = 0 \) for \( n \geq 1 \), shows that \( t^{n+1} = w_n(-\xi) t + w_2(\xi) w_{n-1}(-\xi) \). The Umkehr \( \pi_! \) picks out the coefficient of \( t \). \( \square \)

APPENDIX B.

Consider a continuous map \( f : V = \mathbb{R}^{m+1} \to W = \mathbb{R}^{n+1} \) and write \( \text{Sing}(f) \) for the subspace of \( V^4 \) consisting of those 4-tuples \( (x_0, x_1, x_2, x_3) \) such that \( x_i \neq x_j \) for \( i \neq j \) and the points \( f(x_0), f(x_1), f(x_2), f(x_3) \) lie in some 2-dimensional affine subspace of \( W \). We shall establish a lower bound for the covering dimension of this singularity set \( \text{Sing}(f) \).

Proposition B.1. Suppose that \( n + 1 = m + 2^r \) where \( 2^{r-1} \leq m + 1 < 2^r \). Then \( \text{Sing}(f) \) contains a compact subspace with covering dimension greater than or equal to \( 4(m+1) - (n-1) \).

Proof. We follow the notation of Section 4 where we used the inclusion \( (x, (u, v)) \mapsto (x + \delta u, x - \delta u, -x + \delta v, -x - \delta v) \):

\[
X = S(V) \times (S(V) \times S(V)) \hookrightarrow V^4.
\]

Choose a \( K \)-equivariant closed tubular neighbourhood \( \tilde{X} \) of \( X \) in \( V^4 \), so small that, for all \( (x_0, x_1, x_2, x_3) \in \tilde{X} \), we have \( x_i \neq x_j \) for \( i \neq j \). Then \( \tilde{Z} = \tilde{X}/K \) is a compact
manifold with boundary and the inclusion $Z \hookrightarrow \tilde{Z}$ is a homotopy equivalence. Let $\lambda$ and $\zeta$ denote the natural extensions of $\lambda$ and $\zeta$ to $\tilde{Z}$. The construction in the second proof of Corollary 1.2 in Section 4 gives a section, say, of the vector bundle $(\tilde{\lambda} \oplus \tilde{\zeta}) \otimes (W/\gamma)$ over $\tilde{Z} \times G_2(W)$. And the zero-set $\text{Zero}(s)$ projects under $\pi: \tilde{Z} \times G_2(W) \to \tilde{Z}$ to the image of the compact space $\text{Sing}(f) \cap \tilde{X}$ under the 8-fold covering map $\tilde{X} \to \tilde{Z}$.

We know that the Stiefel-Whitney class $w_{n-1}(-(\tilde{\lambda} \oplus \tilde{\zeta}))$ is non-zero. By Poincaré duality for the manifold $\tilde{Z}$ of dimension $4(m+1)$, there is a class $a \in H^{4(m+1)-(n-1)}(\tilde{Z}, \partial \tilde{Z}; \mathbb{F}_2)$ such that

$$a \cdot w_{n-1}(-(\tilde{\lambda} \oplus \tilde{\zeta})) = \pi_1((\pi^* a) \cdot e((\tilde{\lambda} \oplus \tilde{\zeta}) \otimes (W/\gamma)))$$

is non-zero. Thus $(\pi^* a) \cdot e((\tilde{\lambda} \oplus \tilde{\zeta}) \otimes (W/\gamma))$ is non-zero. It follows by standard arguments, as, for example, in [5] Proposition 2.7, that $\pi^* a$ restricts to a non-zero class in the (representable) cohomology of $\text{Zero}(s)$. Hence $a$ must restrict to a non-zero class in the cohomology of the image $S$ of $\text{Sing}(f) \cap \tilde{X}$ in $\tilde{Z}$. Therefore the covering dimension of the compact space $S$ is at least $4(m+1)-(n-1)$, and so, too, is the covering dimension of its 8-fold cover $\text{Sing}(f) \cap \tilde{X}$. □

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