The Donaldson equation

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1 Introduction

S. Donaldson [3] introduced a Weil-Peterson type metric on the space of volume forms (normalized) on any Riemannian manifold \((X, g)\) with fixed total volume. This infinite dimensional space can be parameterized by all smooth functions such that
\[ \mathcal{H} = \{ \phi \in C^\infty(X) : 1 + \triangle_g \phi > 0 \}. \]
This is a locally Euclidean space. The tangent space is exactly \(C^\infty(X)\) up to addition of some constants. The metric is defined by
\[ \| \delta \phi \|^2 = \int_X (\delta \phi)^2 (1 + \triangle_g \phi) dg. \]
The energy function on a path \(\Phi : [0,1] \rightarrow \mathcal{H}\) is defined as
\[ E(\Phi(t)) = \int_0^1 \int_X |\dot{\Phi}|^2 (1 + \Delta \Phi) dg. \]
Then, the geodesic equation is
\[ \Phi_{tt}(1 + \Delta \Phi) - |\nabla \Phi_t|^2_g = 0. \quad (1.1) \]
This is a degenerated elliptic equation. To approach this equation, Donaldson introduced a perturbed of the geodesic equation
\[ \Phi_{tt}(1 + \Delta \Phi) - |\nabla \Phi_t|^2_g = \epsilon, \quad (1.2) \]
for any \(\epsilon > 0\). The equation (1.2) can be also formulated as the other two equivalent free boundary problems according to [3]. In joint work with X. Chen [2], we get a smooth solution of the equation (1.2) and a weakly \(C^2\) solution of the geodesic equation (1.1), where the a priori estimates on \(|\Phi|_{C^1}, \Delta \Phi, \Phi_{tt}, \nabla \Phi_t\) are independent of \(\inf \epsilon\). Using these solutions, we prove that \(\mathcal{H}\) is a non-positively curved metric space, parallel to the result of the space of Kähler metrics [4].

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From the PDE point of view, the equations (1.1) and (1.2) are relevant to the operator

\[ Q(D^2\Phi) = \Phi_{tt}(1 + \Delta \Phi) - |\nabla \Phi_t|^2. \]  

(1.3)

In this short note, we want to solve the following Dirichlet problem

\[ Q(D^2\Phi) = f, \]  

(1.4)

with boundary condition

\[ \Phi(\cdot, 0) = \phi_0, \Phi(\cdot, 1) = \phi_1, \]

where \( f \in C^\infty(X \times [0,1]) \) is a positive function and \( \phi_0, \phi_1 \in \mathcal{H} \). We call the equation (1.4) Donaldson equation and the operator \( Q \) Donaldson operator. In the paper [2], the fact that \( f = \epsilon \) is a constant is used crucially to get a priori estimates. We notice that the equation can be still solved provided \( f > 0 \), while inf \( f > 0 \) is actually used crucially to get a uniform \( C^1 \) bound below. We obtain

**Theorem 1.1.** Let \((X, g)\) be a compact Riemannian manifold and \( f \in C^k(X \times [0,1]) \) with \( k \geq 2 \) is a positive function. The Dirichlet problem (1.4) has a unique solution \( \Phi(x, t) \in C^{k+1, \beta}(X \times [0,1]) \) for any \( \beta \in [0,1) \). Moreover,

\[ 1 + \Delta \Phi > 0 \]

for any \( t \in [0,1] \).

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### 2 A priori estimates

In this section we derive the *a priori* estimates for the Donaldson equation

\[ Q(D^2\Phi) = f, \]  

(2.1)

with boundary condition

\[ \Phi(\cdot, 0) = \phi_0, \Phi(\cdot, 1) = \phi_1, \]

where \( f \) is a positive smooth function on \( X \times [0,1] \). The linearized operator is given by

\[ dQ(h) = \Phi_{tt}\Delta h + (1 + \Delta \Phi)h_{tt} - 2(\nabla h_t, \nabla \Phi_t). \]

Recall the concavity for the Donaldson equation.

**Lemma 2.1.** (Donaldson [3]) 1. If \( A > 0 \), then \( Q(A) > 0 \) and if \( A \geq 0 \), \( Q(A) \geq 0 \).

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2. If $A, B$ are two matrices with $Q(A) = Q(B) > 0$, and if the entries $A_{00}, B_{00}$ are positive then for any $s \in [0, 1],$

$$Q(sA + (1-s)B) \geq Q(A), Q(A - B) \leq 0.$$ 

Moreover, strict inequality holds if the corresponding arguments are not the same.

We have its equivalent form.

**Lemma 2.2.** Consider the function

$$f(x, y, z_1, \cdots, z_n) = \log \left( xy - \sum z_i^2 \right).$$

Then $f$ is concave when $x > 0, y > 0, \ xy - \sum z_i^2 > 0$.

We will use the following notations. At any point $p \in X \times [0, 1]$, take local coordinates $(x_1, \cdots, x_n, t)$. We can always diagonalize the metric tensor $g$ as $g_{ij}(p) = \delta_{ij}, \partial_k g_{ij}(p) = 0$. We will use, for any smooth function $f$ on $X \times [0, 1]$, the following notations

$$\triangle f_i = \triangle (f_i), \ \triangle f_{ij} = \triangle (f_{ij}), \ \triangle f,_{i} = (\triangle f),_{i} \ \text{and} \ \triangle f,_{ij} = (\triangle f),_{ij}.$$ 

For any function $f, f_i, f_{ij}$ etc are covariant derivatives. By Weitzenbock formula, we have

$$\triangle f_i = \triangle f,_{i} + R_{ij} f_{j}, \quad \text{(2.2)}$$

where $R_{ij}$ is the Ricci tensor of the metric $g$.

### 2.1 $C^0$ estimates

Denote

$$\Phi_a = at(1 - t) + (1 - t)\phi_0 + t\phi_1$$

for any number $a$. The $C^0$ estimate is similar as in [2]. For the sake of the completeness, we include the proof here.

**Lemma 2.3.** If $\Phi$ satisfies (2.1), then for some $a$ big enough,

$$\Phi_a \leq \Phi \leq (1 - t)\phi_0 + t\phi_1.$$ 

**Proof.** First we have

$$\Phi_{tt} > 0.$$ 

It follows that

$$\frac{\Phi(\cdot, t) - \Phi(\cdot, 0)}{t - 0} < \frac{\Phi(\cdot, 1) - \Phi(\cdot, t)}{1 - t}.$$ 

Namely

$$\Phi(t) < (1 - t)\phi_0 + t\phi_1.$$
Note $\Phi = \Phi_{-a}$ on the boundary. If $\Phi < \Phi_{-a}$ for some point, then $\Phi - \Phi_{-a}$ obtains its minimum in the interior, say at $p$. Then $D^2\Phi \geq D^2\Phi_{-a}$ at $p$. Note $Q(D^2\Phi) = f$, and $Q(D^2\Phi_{-a}) = 2a((1 - t)\triangle\phi_0 + t\triangle\phi_1) - |\nabla\phi_1 - \nabla\phi_0|^2$.

If $a$ is sufficiently big, we know that

$$Q(D^2\Phi) < Q(D^2\Phi_{-a}).$$

Let $A$ be a $(n+2) \times (n+2)$ symmetric matrix such that the $(n+1) \times (n+1)$ block of $A$ is $D^2\Phi_{-a}$, and $A_{(n+2)} = A_{(n+2)i} = 0$ for $1 \leq i \leq n + 1$, $A_{(n+2)(n+2)} = 1$. Let $B$ be a $(n+2) \times (n+2)$ symmetric matrix such that the $(n+1) \times (n+1)$ block of $B$ is $D^2\Phi$ and $B_{i(n+2)} = B_{(n+2)i} = 0$ for $1 \leq i \leq n + 1$, $B_{(n+2)(n+2)} = \lambda$. $\lambda$ is a constant satisfying

$$Q(B) = \Phi_{tt}(\lambda + \triangle\Phi) - \Phi_{ik}^2 = Q(A) = Q(D^2\Phi_{-a}).$$

We know that $\lambda > 1$ by (2.3). It follows that $Q(B - A) < 0$. But $B - A$ is semi-positive definite, $Q(B - A) \geq 0$. Contradiction.

2.2 $C^1$ estimates

To get a $C^1$ estimate independent of $\epsilon$, in particular when $\epsilon \to 0$, the fact that $\epsilon$ is a constant is used heavily in [2]. In general, the required estimates can be obtained depending on $\inf f > 0$.

**Lemma 2.4.** Suppose that $\Phi$ satisfies (2.7), then there is a uniform constant $C$ depending on $\inf f > 0, |f|_{C^1}$ and the boundary data, such that

$$|\nabla\Phi| \leq C, |\Phi_t| \leq C.$$

**Proof.** Since $\Phi_{tt} > 0$, $\Phi_t$ obtains its maximum on the boundary. By Lemma 2.3 it is easy to see that $|\Phi_t|$ is bounded on the boundary. To bound $\nabla\Phi$, take

$$h = \frac{1}{2} (|\nabla\Phi|^2 + b\Phi^2),$$

where $b$ is a constant determined later. We want to show that $h$ is bounded. Namely, there exists a constant $C$ depending only on $\inf f$, $|f|_{C^1}$ and the boundary data such that

$$\max h \leq C.$$

Since $h$ is uniformly bounded on the boundary, we assume $h$ takes its maximum at $(p, t_0) \in X \times (0, 1)$. Taking derivative, we get that

$$h_t = \Phi_{tk}\Phi_k + b\Phi_t\Phi, \quad h_k = \Phi_{ik}\Phi_i + b\Phi_k\Phi,$$

$$h_{tt} = \Phi_{tik}\Phi_k + \Phi_{tk}^2 + b(\Phi_{tt}\Phi + \Phi_t^2),$$

$$h_{tk} = \Phi_{tik}\Phi_i + \Phi_{ti}\Phi_{ik} + b(\Phi_{tk}\Phi + \Phi_t\Phi_k),$$

$$\triangle h = \Phi_{tik}\Phi_i + \Phi_{ik}^2 + b(\triangle\Phi\Phi + \Phi_k^2),$$

$$= \triangle\Phi_{i}\Phi_i + \Phi_{ik}^2 + b(\triangle\Phi\Phi + \Phi_k^2) + R_{ij}\Phi_i\Phi_j,$$

$$h_{kk} = \Phi_{ik}\Phi_i + \Phi_{i}^2 + b\Phi_k\Phi_k.$$ (2.4)
where $R_{ij}$ is the Ricci curvature of $(X, g)$. It follows that

\[
\begin{align*}
\frac{dQ(h)}{dt} &= \Phi_{tt} \Delta h + (1 + \Delta \Phi) h_{tt} - 2 \Phi_{tk} h_{tk} \\
&= \Phi_{tt} (\Delta \Phi_i \Phi_i + \Phi^2 + b(\Delta \Phi + \Phi^2_i)) + \Phi_{tt} R_{ij} \Phi_i \Phi_j \\
&\quad + (1 + \Delta \Phi) (\Phi_{tkk} \Phi_k + \Phi^2_{tk} + b(\Phi_{tt} \Phi_i + \Phi^2_i)) \\
&\quad - 2 \Phi_{tk} (\Phi_{ttk} \Phi_t + \Phi^2_{tk} + b(\Phi_{ttk} \Phi_i + \Phi^2_i)) \\
&= \Phi_{tt} \Phi^2_{ik} + (1 + \Delta \Phi) \Phi^2_{ik} - 2 \Phi_{ttk} \Phi_{tk} \\
&\quad + b(\Phi_{tt} \Phi^2_k + (1 + \Delta \Phi) \Phi^2_i - 2 \Phi_{tt} \Phi_i \Phi_k) \\
&\quad + \Phi_{tk} (\Phi_{tt} \Delta \Phi_k + (1 + \Delta \Phi) \Phi_{ttk} - 2 \Phi_{ttk} \Phi_k) \\
&\quad + b \Phi (\Phi_{tt} \Delta \Phi + \Phi_{tt} - 2 \Phi^2_k) + \Phi_{tt} R_{ij} \Phi_i \Phi_j. \\
\end{align*}
\]

Taking derivative of (2.1), we can get that

\[
\begin{align*}
\Phi_{ttk} (1 + \Delta \Phi) + \Phi_{tt} \Delta \Phi_k - 2 \Phi_{ttk} \Phi_{tt} &= f_k, \\
\Phi_{tt} (1 + \Delta \Phi) + \Phi_{tt} \Delta \Phi - 2 \Phi_{tt} \Phi_{tt} &= f_t. \\
\end{align*}
\]

By (2.5) and (2.6), we have

\[
\begin{align*}
\frac{dQ(h)}{dt} &= \Phi_{tt} \Phi^2_{ik} + (1 + \Delta \Phi) \Phi^2_{ik} - 2 \Phi_{ttk} \Phi_{tk} \\
&\quad + b(\Phi_{tt} \Phi^2_k + (1 + \Delta \Phi) \Phi^2_i - 2 \Phi_{tt} \Phi_i \Phi_k) \\
&\quad + \Phi_{tk} f_k - b \Phi_{tt} + \Phi_{tt} R_{ij} \Phi_i \Phi_j. \\
\end{align*}
\]

Note at the point $(p, t_0)$, $h_t = h_k = 0$, it follows that

\[
\Phi_{tk} \Phi_t = -b \Phi \Phi_t.
\]

We can get from (2.8) that

\[
\begin{align*}
\frac{dQ(h)}{dt} &= \Phi_{tt} \Phi^2_{ik} + (1 + \Delta \Phi) \Phi^2_{ik} - 2 \Phi_{ttk} \Phi_{tk} \\
&\quad + b(\Phi_{tt} \Phi^2_k + (1 + \Delta \Phi) \Phi^2_i - 2 \Phi_{tt} \Phi_i \Phi_k) \\
&\quad + \Phi_{tk} f_k - b \Phi_{tt} + \Phi_{tt} R_{ij} \Phi_i \Phi_j \\
&> \Phi_{tt} \left( \frac{1}{2} b |\nabla \Phi|^2 - b \Phi - C_0 |\nabla \Phi|^2 \right) \\
&\quad + \frac{1}{2} b \Phi_{tt} |\nabla \Phi|^2 + (1 + \Delta \Phi) |\Phi_t|^2 + \Phi_{tk} f_k \\
&> \Phi_{tt} \left( \frac{1}{2} b |\nabla \Phi|^2 - b \Phi - C_0 |\nabla \Phi|^2 \right) \\
&\quad + (\sqrt{2bf} |\Phi_t| - |\nabla f|) |\nabla \Phi|,
\end{align*}
\]

where $C_0 = 1 + \max |R_{ij}|$ is a constant. If $\Phi$ solves (2.1) with boundary condition

\[
\Phi(\cdot, 0) = \phi_0, \Phi(\cdot, 1) = \phi_1,
\]

then $\Phi = \Phi + At$ solves (2.1) with boundary condition

\[
\Phi(x, 0) = \phi_0, \quad \Phi(x, 1) = \phi_1 + A.
\]
where $A$ is any constant. Since $|\Phi_t|$ and $|\Phi|$ are bounded, we can choose normalization ($A$ big enough) such that for any $(x,t)$, $|\Phi_t| \geq 1$. Choose $b$ such that

$$b = \max \left( \frac{|\nabla f|}{\sqrt{f}}, 4C_0 \right).$$

At the point $(p, t_0)$, $dQ(h) \leq 0$, it follows from (2.9) that

$$|\nabla \Phi|^2(p) \leq \frac{b\Phi}{C_0}.$$

\[\square\]

### 2.3 $C^2$ estimates

The $C^2$ estimates are only slight different with the case $f = \epsilon$. First we have the following interior estimates.

**Lemma 2.5.** Suppose that $\Phi$ satisfies (2.1), then there is a uniform positive constants $C_1$ depending on $\inf f > 0, |f|_{C^1}, |f|_{C^2}$ and the boundary data, such that

$$0 < \Phi_{tt} + 1 + \Delta \Phi \leq C_1 (1 + \max_{\partial(X \times [0,1])} |\Phi_{tt}|).$$

**Proof.** It is clear that

$$\Phi_{tt} + 1 + \Delta \Phi > 0.$$

Take

$$F = \frac{1}{2} bt^2 - b\Phi, \quad h = \Phi_{tt} + 1 + \Delta \Phi, \quad \tilde{h} = \exp(F)h,$$

where $b$ is some constant determined later. We want to show that $\tilde{h}$ obtains its maximum on the boundary. If not, suppose $h$ obtains its maximum at the point $(p, t_0) \in X \times (0, 1)$. Taking derivative,

$$\tilde{h}_t = \exp(F)(F_t h + h_t), \quad \tilde{h}_k = \exp(F)(F_k h + h_k)$$

and

$$\tilde{h}_{tt} = \exp(F)(h_{tt} + F_{tt}h + 2F_th_t + hF_t^2), \quad \tilde{h}_{kk} = \exp(F)(h_{kk} + F_{kk}h + 2F_kh_k + hF_k^2).$$

Also we have

$$\tilde{h}_{tk} = \exp(F)(h_{tk} + h_tF_k + hF_{tk} + hF_tF_t + h_kF_t).$$

Note at the point $(p, t_0)$, $\tilde{h}_t = \tilde{h}_k = 0$. It follows that

$$h_t + hF_t = 0, \quad h_k + hF_k = 0.$$
We can calculate that at the point \((p, t_0)\)
\[
\begin{align*}
dQ(\hat{h}) &= \Phi_{tt} \Delta \hat{h} + (1 + \Delta \Phi) \hat{h}_{tt} - 2\Phi_{tk} \hat{h}_{tk} \\
&= \Phi_{tt} \exp(F)(\Delta h + h \Delta F - h F_t^2) \\
&\quad + (1 + \Delta \Phi) \exp(F)(h_{tt} + h F_{tt} - h F_t^2) \\
&\quad - 2\Phi_{tk} \exp(F)(h_{tk} + h F_{tk} - h F_k F_t) \\
&= \exp(F)(dQ(h) + hdQ(F) - P(h, F)), \quad (2.10)
\end{align*}
\]
where
\[
P(h, F) = h(\Phi_{tt} F_k^2 + (1 + \Delta \Phi) F_t^2 - 2\Phi_{tk} F_t F_k).
\]
Now we carry out \(dQ(F), dQ(h)\). It is clear that
\[
dQ(F) = b(1 + \Delta \Phi + \Phi_{tt} - 2f).
\]
Taking derivative, we have
\[
\begin{align*}
\dot{h}_t &= \Phi_{ttt} + \Delta \Phi_t, \quad \dot{h}_{tt} = \Phi_{tttt} + \Delta \Phi_{tt} \\
\dot{h}_k &= \Phi_{ttk} + \Delta \Phi_{tk}, \quad \Delta = \Delta \Phi_{tt} + \Delta^2 \Phi, \quad \ddot{h}_{tk} = \Phi_{ttkk} + \Delta \Phi_{ttk}.
\end{align*}
\]
We calculate
\[
\begin{align*}
dQ(h) &= \Phi_{tt} \Delta h + (1 + \Delta \Phi) \dot{h}_{tt} - 2\Phi_{tk} \dot{h}_{tk} \\
&= (1 + \Delta \Phi)(\Phi_{tttt} + \Delta \Phi_{tt}) + \Phi_{tt} (\Delta \Phi_{tt} + \Delta^2 \Phi) \\
&\quad - 2\Phi_{tk} (\Phi_{ttkk} + \Delta \Phi_{ttk}). \\
&= \exp(F)(dQ(h) + h dQ(F) - P(h, F)), \quad (2.11)
\end{align*}
\]
Taking derivative of (2.6) and (2.7), we have
\[
\begin{align*}
\Phi_{tt} \Delta \Phi_{tt} + (1 + \Delta \Phi) \Phi_{tttt} - 2\Phi_{tk} \Phi_{ttk} + 2\Phi_{tt} \Delta \Phi_t - 2\Phi_{ttk}^2 &= f_{tt}, \quad (2.12) \\
\Phi_{tt} \Delta^2 \Phi + (1 + \Delta \Phi) \Delta \Phi_{tt} - 2\Phi_{ti} \Delta \Phi_{ti} + 2\Phi_{ttk} \Delta \Phi_{tk} - 2\Phi_{ttk}^2 &= \Delta f. \quad (2.13)
\end{align*}
\]
It follows that
\[
dQ(h) = 2\Phi_{ttk}^2 + 2\Phi_{tki}^2 - 2\Phi_{ttt} \Delta \Phi_t - 2\Phi_{ttk} \Delta \Phi_{tk} + 2R_{ij} \Phi_{ti} \Phi_{tj} + f_{tt} + \Delta f. \quad (2.14)
\]
Denote
\[
L = \Phi_{ttk}^2 - \Phi_{ttt} \Delta \Phi_t, \quad M = \Phi_{tki}^2 - \Phi_{ttk} \Delta \Phi_{tk}.
\]
By (2.6) and (2.7), we get that
\[
\Phi_{tt} L = \Phi_{tt} \Phi_{ttk}^2 + (1 + \Delta) \Phi_{tttt} - 2\Phi_{tk} \Phi_{ttk} \Phi_{ttt},
\]
and
\[
\Phi_{tt} M = \Phi_{tt} \Phi_{tki}^2 + (1 + \Delta) \Phi_{ttk}^2 - 2\Phi_{tk} \Phi_{ttk} \Phi_{tti}.
\]
It follows that \(L, M \geq 0\). It is clear that
\[
R_{ij} \Phi_{ti} \Phi_{tj} \geq -C_0 (\Phi_{tt} + 1 + \Delta \Phi)^2,
\]
where \( C_0 = 1 + \max |R_{ij}| \). It follows that
\[
dQ(h) \geq -C_0(\Phi_{tt} + 1 + \Delta \Phi)^2 - |f|_{C^2}.
\]
It is also easy to get that
\[
P(h, f) \leq C_2(\Phi_{tt} + 1 + \Delta \Phi)^2,
\]
where \( C_2 \) is constant depending on \( |\Phi|_{C^1} \). We can get that
\[
dQ(\tilde{h}) > \exp(F)((b - C_0 - C_2)(\Phi_{tt} + 1 + \Delta \Phi)^2 - 2bf(\Phi_{tt} + 1 + \Delta \Phi) - |f|_{C^2}).
\]
Note at the point \((p, t_0)\), \( dQ(\tilde{h}) \leq 0 \). Take
\[
b = C_0 + C_2 + 1,
\]
we have at the point \((p, t_0)\)
\[
\Phi_{tt} + 1 + \Delta \Phi \leq C_3(|f|_{C^2}, \inf f).
\]
Since \( \exp(F)(\Phi_{tt} + 1 + \Delta \Phi) \) obtain its maximum at \((p, t_0)\), it follows that
\[
\Phi_{tt} + 1 + \Delta \Phi \leq C_4.
\]
It means that either \( \exp(F)(\Phi_{tt} + 1 + \Delta \Phi) \) obtains its maximum on the boundary, or \( \Phi_{tt} + 1 + \Delta \Phi \) is uniformly bounded. In any case, we have
\[
0 < \Phi_{tt} + 1 + \Delta \Phi \leq C_1(1 + \max_{\partial(X \times [0, 1])} |\Phi_{tt}|).
\]

The boundary \( C^2 \) estimates follow exactly the same as in \[2\].

**Lemma 2.6.** If \( \Phi \) is a solution of (2.1), then \( \Phi \) satisfies the following a priori estimate
\[
|\Delta \Phi| \leq C, \quad |\Phi_{tt}| \leq C, \quad |\Phi_{tk}| \leq C,
\]
where \( C \) is a universal constant depending on \( \inf f, |f|_{C^2} \) and the boundary data.

The Hölder estimate of \( D^2 \Phi \) follows from Evans-Krylov theory using the concavity of \( \log Q \). Once we get the Hölder estimates of \( D^2 \Phi \), the standard boot-strapping argument gives all higher order derivatives of \( \Phi \).

### 3 Solve the equation

To solve the Donaldson equation for general \( f \), we consider the following continuity family for \( s \in [0, 1] \)
\[
Q(D^2 \Phi) = (1 - s)Q(D^2 \Phi - a) + sf, \quad (3.1)
\]
with the boundary condition
\[ \Phi(\cdot, 0, s) = \phi_0, \Phi(\cdot, 1, s) = \phi_1, \]
where \( \Phi_{-a} = -at(1-t) + (1-t)\phi_0 + t\phi_1 \). When \( a \) is big enough, \( Q(D^2\Phi_{-a}) \) is positive and bounded away from 0. We shall now prove that if \( f \in C^k(X \times [0, 1]) \) with \( k \geq 2 \) then we can find of solution of (2.1) such that \( \Phi \in C^{k+1,\beta}(X \times [0, 1]) \) for any \( 0 \leq \beta < 1 \). Consider the set
\[ S = \{ s \in [0, 1] : \text{the equation (3.1) has a solution in } C^{k-1,\beta}(X \times [0, 1]) \} \]
Obviously \( 0 \in S \). Hence we need only show that \( S \) is both open and close. It is clear that \( Q : C^{k+1,\beta} \rightarrow C^{k-1,\beta} \) is open if
\[ 1 + \Delta \Phi > 0 \quad \text{and} \quad Q(D^2\Phi) > 0. \]
In this case \( dQ \) is an invertible elliptic operator and openness follows. The closeness of \( S \) follows from the a prior estimates derived in Section 2. Hence Theorem 1.1 holds.

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