A UNIFIED APPROACH TOWARDS THE IMPOSSIBILITY OF FINITE TIME VANISHING DEPTH FOR INCOMPRESSIBLE FREE BOUNDARY FLOWS

ZHIYUAN GENG AND RAFAEL GRANERO-BELINCHÓN

Abstract. In this paper we study the motion of an internal water wave and an internal wave in a porous medium. For these problems we establish that, if the free boundary and, in the case of the Euler equations, also the tangential velocity at the interface are sufficiently smooth, the depth cannot vanish in finite time. This result holds regardless of gravity and surface tension effects or, if applicable, the stratification in multiphase flows.

1. Introduction

This paper studies the motion of an incompressible fluid with a free boundary in presence of an impervious bottom. Example of such a flow are the sea near the shoreline or the case of a bounded porous aquifer. For these problems we establish a number of conditions on the interface and the tangential velocity of the interface such that if they hold, the depth cannot vanish (see Figure 1) in finite time. These results hold regardless of the stratification, gravity or surface tension effects. Let us also remark that, in the case of two phase Euler, a similar result was proved using Lagrangian coordinates in a very interesting paper by Daniel Coutand in [13] (see also the related results [14, 16]). However, our work presents a different and unified approach to this question. Our approach is based on a careful study of the contour equation formulation.

![Free boundary evolution](image)

**Figure 1.** The situation that is discarded by our results.

The study of incompressible fluids with a free surface is a long-standing research problem for physicists, engineers and mathematicians since the XVIIIth century. In particular, the possible formation of finite time singularities where certain geometrical quantity of the free boundary blows up have been studied since the end of the XVIIIth century [24]. There are a number of well-known physical situations that we can fit in this framework. For instance, we can consider drop formation [15, 26, 25] and pinch-off singularities [8, 1], corner singularities [22, 23] or wetting [4, 6, 5] to give just some examples.
In this paper we consider two different problems. On the one hand we consider the case of an internal wave in a porous medium with an impervious bottom.

\[
\begin{align*}
\mu_\pm u_\pm &= -\nabla p_\pm - \rho_\pm (0, 1)^T g & \text{in } \Omega_\pm(t), \\
\nabla \cdot u_\pm &= 0 & \text{in } \Omega_\pm(t), \\
p_+ - p_- &= \gamma \kappa & \text{on } \Gamma(t), \\
u_- \cdot (0, 1)^T &= 0 & \text{on } \{y = 0\}, \\
u_- \cdot (\partial_\alpha z)^\perp &= u_+ \cdot (\partial_\alpha z)^\perp & \text{on } \Gamma(t).
\end{align*}
\]

Here \(g > 0\) is the acceleration due to gravity force, \(\gamma\) is the surface tension strength and \(\kappa\) is the curvature of the free surface \(\Gamma(t)\), which, for a curve, is given by

\[
\kappa = \frac{\partial_\alpha z_1 \partial^2_\alpha z_2 - \partial_\alpha z_2 \partial^2_\alpha z_1}{[((\partial_\alpha z_1)^2) + (\partial_\alpha z_2)^2]^{3/2}}.
\]

Similarly, \(p_\pm\) is the pressure, \(\mu_\pm\) and \(\rho_\pm\) are the viscosity and density of the fluids involved. This problem is known in the literature as the (confined) two-phase Muskat problem [11, 12, 19, 17].

On the other hand, we consider the motion of a water wave bounded below by a fixed impervious bottom. In this second situation the appropriate PDE’s are

\[
\begin{align*}
\rho_\pm (\partial_t u_\pm + (u_\pm \cdot \nabla) u_\pm) &= -\nabla p_\pm - \rho_\pm (0, 1)^T g & \text{in } \Omega_\pm(t), \\
\nabla \cdot u_\pm &= 0 & \text{in } \Omega_\pm(t), \\
\nabla \times u_\pm &= 0 & \text{in } \Omega_\pm(t), \\
p_+ - p_- &= \gamma \kappa & \text{on } \Gamma(t), \\
u_- \cdot (0, 1)^T &= 0 & \text{on } \{y = 0\}, \\
u_- \cdot (\partial_\alpha z)^\perp &= u_+ \cdot (\partial_\alpha z)^\perp & \text{on } \Gamma(t).
\end{align*}
\]

This situation has been studied both theoretically and numerically by many different authors (see for instance [2, 20, 21]). In the case of two-phase Euler, the impossibility of vanishing depth for smooth enough initial data was obtained by Daniel Coutand in [13, Theorem 2]. In his results, the author established that, if

\[
||\partial_\alpha \tau||_{L^\infty(\mathbb{R})} + \int_0^T ||\nabla u_-(s)||_{L^\infty(\Omega_-(s))} ds < \infty,
\]

where \(\tau\) is the tangent vector to \(\Gamma(t)\), then the interface cannot reach the impervious bottom.

In both cases, we assume that the fluids lie above the \(x\)-axis, i.e.

\[
\Omega_+(t) \cup \Omega_-(t) = \{(x, y, t) \in \mathbb{R}_+^2 \times \mathbb{R}_+\},
\]

where \(\mathbb{R}_+^2\) is the upper half plane \(\{(x, y) \in \mathbb{R}^2 : y > 0\}\), and \(\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}\), the interface is described as

\[
\Gamma(t) = \{(z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)), \alpha \in \mathbb{R}),
\]

for some functions \(z_i : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}, i = 1, 2\), and the impervious bottom is given by

\[
\Gamma_{\text{bot}} = \{(\alpha, 0), \alpha \in \mathbb{R}\}.
\]

Then, both problems can be written in terms of nonlinear and nonlocal evolution equations [9, 10]. Namely,

\[
\partial_\alpha z(\alpha) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \left[ \frac{\left(\frac{z(\alpha)}{z(s)}\right)^\perp - \left(\frac{z(\alpha) - (z_1(s), z_2(s))}{|z(\alpha) - (z_1(s), z_2(s))|^2}\right)\tilde{\omega}(s)}{|\frac{z(\alpha)}{z(s)}|^2 - \left(\frac{z(\alpha) - (z_1(s), z_2(s))}{|z(\alpha) - (z_1(s), z_2(s))|^2}\right)^2} \right] ds + c(\alpha, t) \partial_\alpha z,
\]

where \(c(\alpha, t)\) is a constant, \(\tilde{\omega}(s)\) is a given function and \(p.v.\) denotes the Cauchy principal value.
where $\tilde{\omega}$ denotes the strength of the vorticity (or equivalently, the difference between the tangential components of the velocities $[3]$) and $c$ is an arbitrary function that reflects the invariance of the problem by reparametrization of the free surface. The difference between the case of water waves and the case of the Muskat problem relies on the unknown $\tilde{\omega}$. In the case of the Muskat problem $\tilde{\omega}$ in general satisfies a nonlinear integral equation $[10]$, while in the case of water waves $\tilde{\omega}$ satisfies another nonlinear and nonlocal partial differential equation $[9]$ (see below for more details).

Our main results can be stated as follows

**Theorem 1.1.** Let $(z, u_+, p_+)$ be a smooth solution of the two-phase Muskat problem (equations $(5)-(12)$) such that $z$ does not have any self-intersection and verifying

$$\sup_{\alpha, \beta \in \mathbb{R}, t \in [0, T]} |\alpha - \beta| |z(\alpha) - z(\beta)| < \infty.$$  

Then we have that

- In the case $\mu_+ = \mu_-$, if $z \in C^{2+2\text{sgn}(\gamma)}([0, T] \times \mathbb{R})$,
  
  the interface $z$ cannot touch the impervious bottom in $[0, T]$.

- In the case $\mu_+ \neq \mu_-$, if $z \in C^{2+2\text{sgn}(\gamma)}([0, T] \times \mathbb{R})$
  
  and

$$(u_+ - u_-) \cdot \partial_\alpha z \in C^1([0, T] \times \Gamma(t)),$$

the interface $z$ cannot touch the impervious bottom in $[0, T]$.

We do not expect the regularity assumptions in this result to be sharp. For instance, in the case with $\mu_+ = \mu_-$, we think that it is enough to have

$$z \in C^{1, 2+2\text{sgn}(\gamma)}$$

or even a less strict condition. However, we prefer this statement for the sake of simplicity.

Furthermore, in addition to the previous result, we are actually able to establish the decay of the interface under certain hypothesis on the physical setting (see Proposition $[4,1]$) which seems as a first step towards a finite time singularity result (see also $[7]$).

**Theorem 1.2.** Let $(z, u_+, p_+)$ be a smooth solution of the two-phase irrotational Euler problem (equations $(5)-(13)$) such that $z$ does not have any self-intersection and verifying

$$\sup_{\alpha, \beta \in \mathbb{R}, t \in [0, T]} |\alpha - \beta| |z(\alpha) - z(\beta)| < \infty.$$  

Then we have that if $z \in C^2([0, T] \times \mathbb{R})$

and

$$(u_+ - u_-) \cdot \partial_\alpha z \in C^1([0, T] \times \Gamma(t)),$$

the interface $z$ cannot touch the impervious bottom in $[0, T]$.

As before, we do not think that these regularity assumptions are sharp.

These theorems can be understood as *continuation criteria* for the free boundary problems under consideration. Our results generalize the one by Daniel Coutand in $[13]$ in several ways. On the one hand, we give a unitary approach that serves to every free boundary problem in an incompressible and irrotational flow. In particular, it also covers the case of a Muskat flow that, to the best of authors’ knowledge, was open. On the other hand, our results improve the criterion on $[13]$ in the
sense that the quantities that prevent the vanishing depth scenario are related to the interface and not the bulk of the fluids. In other words, while previously the whole velocity in $\Omega_-$ must satisfy certain smoothness condition, our result only requires the jump of the tangential velocity at the interface to be smooth.

2. Derivation of the governing equation for the interface

Let us briefly sketch how to obtain the contour equation formulation of the previous systems. The free boundary is transported by the normal velocity of the fluids. In particular, the evolution of the curve $\Gamma(t)$ is governed by

\[ \partial_t z \cdot (\partial_\alpha z)^\perp = u_\pm \cdot (\partial_\alpha z)^\perp. \]  

Then we introduce the global velocity,

\[ v = u_+ \chi_{\Omega_+} + u_- \chi_{\Omega_-}. \]

The vorticity is defined in the sense of distributions as a scalar function

\[ \omega = \partial_x v^2 - \partial_y v^1. \]

Note that $\omega$ is a measure supported only on $\Gamma(t)$, and we can compute it explicitly. Take any $\phi \in C_0^1(\mathbb{R}_+^2)$,

\[ \int_{\mathbb{R}_+^2} \omega \phi \, dx \, dy = \int_{\mathbb{R}_+^2} (v^1 \partial_y \phi - v^2 \partial_x \phi) \, dx \, dy = \int_{\Omega_+} (v^1 \partial_y \phi - v^2 \partial_x \phi) \, dx \, dy + \int_{\Omega_-} (v^1 \partial_y \phi - v^2 \partial_x \phi) \, dx \, dy \]

\[ = \int_{\mathbb{R}} \phi (v_- - v_+) \cdot (\partial_\alpha z_1, \partial_\alpha z_2) \, d\alpha \]

with $\tilde{\omega}$ is set as

\[ \tilde{\omega} = -[v \cdot \partial_\alpha z], \]

where $[F] = F_+ - F_-$ denotes the jump of $F$.

Using the Biot-Savart formula for the upper-half plane, we have

\[ u(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \left[ \frac{(-y - z_2(s), x - z_1(s))}{(x - z_1(s))^2 + (y - z_2(s))^2} - \frac{(-y + z_2(s), x - z_1(s))}{(x - z_1(s))^2 + (y + z_2(s))^2} \right] \tilde{\omega}(s) \, ds \]

Then on $\Gamma(t)$, using the Plemelj formula, we have

\[ u_\pm(z(\alpha), t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left[ \frac{(z(\alpha) - z(s))^\perp}{|z(\alpha) - z(s)|^2} - \frac{(z(\alpha) - (z_1(s), -z_2(s)))^\perp}{|z(\alpha) - (z_1(s), -z_2(s))|^2} \right] \tilde{\omega}(s) \, ds \pm \frac{1}{2} \frac{\partial_\alpha z}{|\partial_\alpha z|^2} \tilde{\omega}(\alpha) \]

Substituting (9) into (6) gives

\[ \partial_t z(\alpha, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left[ \frac{(z(\alpha) - z(s))^\perp}{|z(\alpha) - z(s)|^2} - \frac{(z(\alpha) - (z_1(s), -z_2(s)))^\perp}{|z(\alpha) - (z_1(s), -z_2(s))|^2} \right] \tilde{\omega}(s) \, ds + c(\alpha, t) \partial_\alpha z. \]

Here the quantity $c(\alpha, t)$ represents the change of parametrization along $z$. In particular,

\[ \partial_t z_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \left[ \frac{z_1(s) - z_3(s)}{|z(\alpha) - z(s)|^2} - \frac{z_1(s) - z_3(s)}{|z(\alpha) - (z_1(s), -z_2(s))|^2} \right] \tilde{\omega}(s) \, ds + c(\alpha, t) \partial_\alpha z_2. \]

The particular expression for $\tilde{\omega}$ depends on the model under consideration. Namely,
• in the case of the Muskat problem, $\tilde{\omega}$ satisfies the following nonlinear integral equation \[ \int_{\mathbb{R}} \frac{z_1(\alpha) - z_1(s)}{|z(\alpha) - z(s)|^2} \tilde{\omega}(s) \, ds \cdot \partial_\alpha z(\alpha) \]

(12) \[ + \left( \frac{\mu_- + \mu_+}{2} \right) \tilde{\omega}(\alpha) = \gamma \partial_\alpha \kappa + [\rho] g \partial_\alpha z_2(\alpha). \]

If we assume that the viscosities are equal, the previous equation simplifies to

\[ \mu \tilde{\omega}(\alpha) = \gamma \partial_\alpha \kappa + [\rho] g \partial_\alpha z_2(\alpha). \]

In this latter case and without lossing generality we will take $[\mu] = 0$.

• in the case of the irrotational Euler equations, following [10], we have that $\tilde{\omega}$ satisfies

\[
\partial_t \tilde{\omega}(\alpha) = -\partial_\alpha \left[ \frac{[\rho]}{\rho_+ + \rho_-} \int_{\mathbb{R}} \left( \frac{(z(\alpha) - z(s))^\perp}{|z(\alpha) - z(s)|^2} - \frac{(z(\alpha) - (z_1(s), -z_2(s)))^\perp}{|z(\alpha) - (z_1(s), -z_2(s))|^2} \right) \tilde{\omega}(s) \, ds \right] \partial_\alpha z(\alpha) \]

\[ + \frac{[\rho]}{\rho_+ + \rho_-} \int_{\mathbb{R}} \left( \frac{(z(\alpha) - z(s))^\perp}{|z(\alpha) - z(s)|^2} - \frac{(z(\alpha) - (z_1(s), -z_2(s)))^\perp}{|z(\alpha) - (z_1(s), -z_2(s))|^2} \right) \tilde{\omega}(s) \, ds \cdot \partial_\alpha z(\alpha) c(\alpha) \]

\[ - c(\alpha) \tilde{\omega}(\alpha) - \frac{2\gamma \kappa}{\rho_+ + \rho_-} g z_2 \]

(13) \[ + \frac{[\rho]}{\rho_+ + \rho_-} \partial_t \left[ \int_{\mathbb{R}} \left( \frac{(z(\alpha) - z(s))^\perp}{|z(\alpha) - z(s)|^2} - \frac{(z(\alpha) - (z_1(s), -z_2(s)))^\perp}{|z(\alpha) - (z_1(s), -z_2(s))|^2} \right) \tilde{\omega}(s) \, ds \cdot \partial_\alpha z(\alpha) \right]. \]

3. The cornerstone

The results in this paper are mainly obtained as a consequence of the following cornerstone theorem:

**Theorem 3.1.** Let $(z, \omega)$ be a couple satisfying (5) on some time interval $[0, T]$. Assume also that there is a constant $A > 0$ such that

(14) \[ \|z(\alpha, t) - (\alpha, 1)\|_{C^2(\mathbb{R} \times [0, T])} \leq A, \]

(15) \[ \|\omega\|_{C^1(\mathbb{R} \times [0, T])} \leq A, \]

(16) \[ \lim_{|\alpha| \to \infty} |z(\alpha, t) - (\alpha, 1)| = 0, \forall t \in [0, T], \]

(17) \[ \sup_{\alpha \in \mathbb{R}, t \in [0, T]} \frac{|\alpha - \beta|}{|z(\alpha) - z(\beta)|} \leq A. \]

Then there is a constant $C(A)$ such that

\[ \min_{\alpha \in \mathbb{R}} |z_2(\alpha, t)| \geq e^{-Ce^{Ct}}, \forall t \in [0, T], \]

which means $z(\alpha, t)$ cannot touch the bottom in finite time.

**Proof of Theorem 3.1.** The proof has the same flavour as the one in [18]. The argument works by contradiction. Let us assume that the solution satisfies

\[ z \in C^2([0, T] \times \mathbb{R}) \]

and define

(18) \[ m(t) = z_2(\alpha_t, t) = \min_{\alpha \in \mathbb{R}} z_2(\alpha, t). \]
Using the regularity of $z$, we have that $m(t)$ is a Lipschitz function and, as a consequence, it is almost everywhere differentiable with derivative given by

$$\frac{d}{dt} m(t) = \partial_t z_2(\alpha_t, t).$$

From (11) and $\partial_\alpha z_2(\alpha_t, t) = 0$ we obtain that

$$(19) \quad \frac{d}{dt} m(t) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \left[ \frac{z_1(\alpha_t) - z_1(s)}{|z(\alpha_t) - z(s)|^2} - \frac{z_1(\alpha_t) - z_1(s)}{|z(\alpha_t) - (z_1(s), -z_2(s))|^2} \right] \tilde{\omega}(s) \, ds$$

We define

$$J := \text{p.v.} \int_{\mathbb{R}} \left[ \frac{z_1(\alpha) - z_1(s)}{|z(\alpha) - z(s)|^2} - \frac{z_1(\alpha) - z_1(s)}{|z(\alpha) - (z_1(s), -z_2(s))|^2} \right] \tilde{\omega}(s) \, ds$$

The goal is to show that

$$|J| \leq Cm \log \left( \frac{1}{m} \right).$$

We decompose it as follows:

$$J = J_m + J_1 + J_\infty,$$

with

$$J_m = \text{p.v.} \int_{|s| < m} \left[ \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - z(\alpha_t - s)|^2} - \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - (z_1(\alpha_t - s), -z_2(\alpha_t - s))|^2} \right] \, ds$$

$$J_1 = \int_{m \leq |s| < 1} \left[ \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - z(\alpha_t - s)|^2} - \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - (z_1(\alpha_t - s), -z_2(\alpha_t - s))|^2} \right] \, ds$$

$$J_\infty = \text{p.v.} \int_{|s| \geq 1} \left[ \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - z(\alpha_t - s)|^2} - \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - (z_1(\alpha_t - s), -z_2(\alpha_t - s))|^2} \right] \, ds.$$

For any function $g$ and points $a, b \in \mathbb{R}$, we introduce the following abbreviations

$$\delta_g(a, b) := g(a) - g(b), \quad \sigma_g(a, b) := g(a) + g(b).$$

Here $g$ is either a scalar function or a vector function.

For the term $J_m$, we first write

$$J_m = J_m^1 + J_m^2,$$

with

$$J_m^1 = \text{p.v.} \int_{0 < |s| < m} \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - z(\alpha_t - s)|^2} \, ds$$

$$J_m^2 = - \text{p.v.} \int_{0 < |s| < m} \frac{\tilde{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - (z_1(\alpha_t - s), -z_2(\alpha_t - s))|^2} \, ds.$$
Then we estimate

\begin{equation}
(20)\quad J_m^1 = \lim_{\varepsilon \to 0} \int_{-m}^{m} \left[ \frac{\bar{\omega}(\alpha_t - s)(z_1(\alpha_t) - z_1(\alpha_t - s))}{|z(\alpha_t) - z(\alpha_t - s)|^2} + \frac{\bar{\omega}(\alpha_t + s)(z_1(\alpha_t) - z_1(\alpha_t + s))}{|z(\alpha_t) - z(\alpha_t + s)|^2} \right] ds
\end{equation}

\begin{equation}
= \lim_{\varepsilon \to 0} \int_{-m}^{m} \left[ \frac{|\delta z(\alpha_t, \alpha + s)|^2}{|\delta z(\alpha_t, \alpha - s)|^2} |\delta z_1(\alpha_t, \alpha + s)|^2 |\bar{\omega}(\alpha_t - s)|^2 + \frac{|\delta z(\alpha_t, \alpha - s)|^2}{|\delta z(\alpha_t, \alpha + s)|^2} |\delta z_1(\alpha_t, \alpha - s)|^2 |\bar{\omega}(\alpha_t + s)|^2 \right] ds
\end{equation}

Due to \((14), (17)\) together with the regularity of \(\bar{\omega}\) and the fact that

\[ \partial_\alpha \bar{z}_2(\alpha, t) = 0, \]

we have

\[ |\delta z(\alpha_t - s, \alpha_t + s)| \leq 2\|\bar{\omega}\|_{C^1, s}, \]
\[ |\delta z(\alpha_t, \alpha + s)|^2 \geq \frac{1}{A^2 s^2}, \]
\[ |\delta z(\alpha_t, \alpha - s)|^2 \geq \frac{1}{A^2 s^2}, \]
\[ |\delta z_1(\alpha_t, \alpha + s)| \leq \|z_1\|_{C^1, s}, \]
\[ |\delta z_1(\alpha_t, \alpha - s)| \leq \|z_1\|_{C^1, s}, \]
\[ |\delta z_2(\alpha_t, \alpha + s)| \leq \frac{1}{2}\|z_2\|_{C^2, s^2}, \]
\[ |\delta z_2(\alpha_t, \alpha - s)| \leq \frac{1}{2}\|z_2\|_{C^2, s^2}, \]
\[ |\delta z_1(\alpha, \alpha + s) + \delta z_1(\alpha, \alpha - s)| \leq \|z_1\|_{C^2, s^2}. \]

Furthermore, we compute

\[ |\delta z(\alpha, \alpha + s)|^2 \delta z_1(\alpha, \alpha - s) + |\delta z(\alpha, \alpha - s)|^2 \delta z_1(\alpha, \alpha + s) \]
\[ = \delta z_1(\alpha, \alpha + s) \delta z_1(\alpha, \alpha - s) \delta z_1(\alpha, \alpha + s) + \delta z_1(\alpha, \alpha - s) + \delta z_2(\alpha, \alpha + s) + \delta z_2(\alpha, \alpha - s) + \delta z_2(\alpha, \alpha + s). \]

Substituting all these estimates into \(20\), it follows that

\begin{equation}
(21)\quad |J_m^1| \leq C \int_{0}^{m} \left[ \frac{(\|\bar{\omega}\|_{C^1, s^3} \leq z_1 \|C^1, s^3\| \leq z_1 \|C^1, s^5\|)}{A^2 s^2} + \frac{\|\bar{\omega}\|}{A^2 s^2} \right] ds
\leq C(A, \|\bar{\omega}\|_{C^1, s^3} \|C^2, s^2\| m).
\end{equation}
Similarly, for $J_m^2$ we compute

\[
J_m^2 = \lim_{\varepsilon \to 0} \int_0^m \left[ \frac{\tilde{\omega}(\alpha_t - s)\delta_{z_1}(\alpha_t, \alpha_t - s) + \tilde{\omega}(\alpha_t + s)\delta_{z_1}(\alpha_t, \alpha_t + s)}{\delta_{z_1}(\alpha_t, \alpha_t - s)^2 + \sigma_{z_2}(\alpha_t, \alpha_t - s)^2 + \delta_{z_1}(\alpha_t, \alpha_t + s)^2 + \sigma_{z_2}(\alpha_t, \alpha_t + s)^2} \right] \, ds
\]

\[
= \lim_{\varepsilon \to 0} \int_0^m \left[ \frac{\tilde{\omega}(\alpha_t - s)\delta_{z_1}(\alpha_t, \alpha_t + s) + \tilde{\omega}(\alpha_t + s)\delta_{z_1}(\alpha_t, \alpha_t - s)}{\sigma_{z_2}(\alpha_t, \alpha_t + s)^2 + \sigma_{z_2}(\alpha_t, \alpha_t - s)^2} \right] \, ds,
\]

(22)

We can write

\[
\delta_{z_1}(\alpha_t, \alpha_t - s)\delta_{z_1}(\alpha_t, \alpha_t + s)^2
+ \sigma_{z_2}(\alpha_t, \alpha_t + s)^2 + \delta_{z_1}(\alpha_t, \alpha_t + s)(\delta_{z_1}(\alpha_t, \alpha_t - s)^2 + \sigma_{z_2}(\alpha_t, \alpha_t - s)^2)
= (\delta_{z_1}(\alpha_t, \alpha_t - s) + \delta_{z_1}(\alpha_t, \alpha_t + s))\delta_{z_1}(\alpha_t, \alpha_t - s)\delta_{z_1}(\alpha_t, \alpha_t + s)
+ \delta_{z_1}(\alpha_t, \alpha_t - s)(\sigma_{z_2}(\alpha_t, \alpha_t + s)^2 - (2\sigma_2(\alpha_t))^2)
+ \delta_{z_1}(\alpha_t, \alpha_t + s)(\sigma_{z_2}(\alpha_t, \alpha_t - s)^2 - (2\sigma_2(\alpha_t))^2)
+ (\delta_{z_1}(\alpha_t, \alpha_t - s) + \delta_{z_1}(\alpha_t, \alpha_t + s))(2\sigma_2(\alpha_t))^2.
\]

(23)

We also need the following estimates, valid for $0 < s < m$,

\[
|\sigma_{z_2}(\alpha_t, \alpha_t \pm s)| \geq 2m,
\]

(24)

\[
|\sigma_{z_2}(\alpha_t, \alpha_t \pm s)^2 - (2\sigma_2(\alpha_t))^2| \leq \|z_2\|_{C^2}s^2(3m + \frac{1}{2}\|z_2\|_{C^2}s^2) \leq \|z_2\|_{C^2}4s^2m,
\]

(25)

when $m$ is sufficiently small. Therefore, substituting (23), (24) and (25) into (22), we have

\[
|J_m^2| \leq \left| \int_0^m \frac{\|\tilde{\omega}\|_{C^1}\|z_1\|_{C^1}s^2}{4m^2} \, ds \right|
+ \left| \int_0^m \|\tilde{\omega}\|_{C^0}(\|z_1\|^2_{C^2}s^4 + 8\|z_1\|_{C^1}\|z_2\|_{C^2}ms^3) + 4\|z_1\|_{C^2}s^2m^2 \, ds \right|
\]

\[
\leq C(\|\tilde{\omega}\|_{C^1}, \|z\|_{C^2})m.
\]

(26)

Collecting (21) and (26) we conclude that

\[
|J_m| \leq C(A, \|\tilde{\omega}\|_{C^1}, \|z\|_{C^2})m.
\]

(27)

For the term $J_1$, for the sake of convenience, we define the following quantity

\[
P(\alpha, s) := |z(\alpha) - z(s)|^2|z(\alpha) - (z_1(s), -z_2(s))|^2
\]

(28)
Then we write

\[
J_1 = \int_{m < |\alpha_t - s| < 1} \left[ \frac{1}{|z(\alpha_t) - z(s)|^2} - \left( \frac{1}{(z_1(\alpha_t) - z_1(s))^2 + (z_2(\alpha_t) + z_2(s))^2} \right) \right] \tilde{\omega}(s) \delta_{z_1} (\alpha_t, s) \, ds
\]

\[
= \int_{m \leq |s| \leq 1} 4m z_2(\alpha_t - s) \delta_{z_1} (\alpha_t, \alpha_t - s) \tilde{\omega}(\alpha_t - s) \, ds
\]

\[
= 4m \int_{m}^{1} \frac{z_2(\alpha_t - s) \delta_{z_1} (\alpha_t, \alpha_t - s)}{P(\alpha_t, \alpha_t - s)} \tilde{\omega}(\alpha_t - s, \alpha_t + s) \, ds
\]

\[
+ 4m \int_{m}^{1} \tilde{\omega}(\alpha_t + s) \left[ \frac{z_2(\alpha_t - s) \delta_{z_1} (\alpha_t, \alpha_t - s)}{P(\alpha_t, \alpha_t - s)} + \frac{z_2(\alpha_t + s) \delta_{z_1} (\alpha_t, \alpha_t + s)}{P(\alpha_t, \alpha_t + s)} \right] \, ds
\]

\[
= J_1^1 + J_1^2.
\]

We have the following simple estimates

\[
(30) \quad z_2(\alpha - s) \leq m + \|z_2\|_{C_1} s \leq (1 + \|z_2\|_{C_1}) s,
\]

\[
(31) \quad \frac{1}{A^4} s^4 \leq P(\alpha, \alpha \pm s) \leq C(\|z\|_{C_1}) s^4,
\]

when \( m \leq s \leq 1 \). It follows that

\[
(32) \quad |J_1| \leq 4m(1 + \|z_2\|_{C_1}) \|z_1\|_{C_1} \|\tilde{\omega}\|_{C_1} A^4 \int_{m}^{1} \frac{s^3}{s^4} \, ds \leq C(A, \|z\|_{C_1}, \|\tilde{\omega}\|_{C_1}) m \log \frac{1}{m}.
\]

For \( J_1^2 \), as

\[
P(\alpha_t, \alpha_t \pm s) \geq \frac{1}{A^4} s^4,
\]

it suffices to estimate

\[
z_2(\alpha_t - s) \delta_{z_1} (\alpha_t, \alpha_t - s) P(\alpha_t, \alpha_t + s) + z_2(\alpha_t + s) \delta_{z_1} (\alpha_t, \alpha_t + s) P(\alpha_t, \alpha_t - s).
\]

We have that

\[
(33) \quad |z_2(\alpha_t - s) \delta_{z_1} (\alpha_t, \alpha_t - s) P(\alpha_t, \alpha_t + s) + z_2(\alpha_t + s) \delta_{z_1} (\alpha_t, \alpha_t + s) P(\alpha_t, \alpha_t - s)|
\]

\[
\leq |\delta_{z_1} (\alpha_t, \alpha_t - s) + \delta_{z_1} (\alpha_t, \alpha_t + s)| z_2(\alpha_t - s) P(\alpha_t, \alpha_t + s) |
\]

\[
+ |\delta_{z_1} (\alpha_t, \alpha_t + s)| z_2(\alpha_t + s) P(\alpha_t, \alpha_t - s) - z_2(\alpha_t - s) P(\alpha_t, \alpha_t + s)|
\]

\[
\leq C(\|z\|_{c_2}) s^7 + |\delta_{z_1} (\alpha_t, \alpha_t + s) \delta_{z_2} (\alpha_t + s, \alpha_t - s) P(\alpha_t, \alpha_t - s)|
\]

\[
+ |\delta_{z_1} (\alpha_t, \alpha_t + s) z_2(\alpha_t - s)(P(\alpha_t, \alpha_t - s) - P(\alpha_t, \alpha_t + s))|
\]

\[
\leq C(\|z\|_{c_2}) s^7 + C(\|z\|_{c_2}) s^2 |P(\alpha, \alpha - s) - P(\alpha, \alpha + s)|,
\]

where we have used \( m < s \). Now the only problem left becomes the estimation of

\[
|P(\alpha_t, \alpha_t - s) - P(\alpha_t, \alpha_t + s)|.
\]

By definition

\[
|P(\alpha_t, \alpha_t - s) - P(\alpha_t, \alpha_t + s)| = |(\delta_{z_1} (\alpha_t, \alpha_t - s)^2 + \delta_{z_2} (\alpha_t, \alpha_t - s)^2)(\delta_{z_1} (\alpha_t, \alpha_t - s)^2 + \sigma_{z_2} (\alpha_t, \alpha_t - s)^2)
\]

\[
- (\delta_{z_1} (\alpha_t, \alpha_t + s)^2 + \delta_{z_2} (\alpha_t, \alpha_t + s)^2)(\delta_{z_1} (\alpha_t, \alpha_t + s)^2 + \sigma_{z_2} (\alpha_t, \alpha_t + s)^2)|
\]

\[
=P_1 + P_2 + P_3 + P_4.
\]
with
\[ P_1 = |\delta_{z_1}(\alpha, \alpha - s)^4 - \delta_{z_1}(\alpha, \alpha + s)^4| \]
\[ P_2 = |\delta_{z_1}(\alpha, \alpha - s)^2 \delta_{z_2}(\alpha, \alpha - s)^2 - \delta_{z_1}(\alpha, \alpha - s)\delta_{z_2}(\alpha, \alpha + s)| \]
\[ P_3 = |\delta_{z_1}(\alpha, \alpha - s)^2 \delta_{z_2}(\alpha, \alpha - s)^2 - \delta_{z_1}(\alpha, \alpha + s)| \]
\[ P_4 = |\delta_{z_1}(\alpha, \alpha - s)^2 \delta_{z_2}(\alpha, \alpha - s)^2 - \delta_{z_2}(\alpha, \alpha + s)|. \]

For the term \( P_2 \) and \( P_4 \), since they contain the term \( \delta_{z_1}(\alpha, \alpha \pm s)^2 \) which is of order \( s^4 \), we get that
\[ P_2 + P_4 \leq C(\|z_2\|_{C^2}, \|z_1\|_{C^1})s^6. \]

For \( P_1 \) we have that
\[ P_1 = |\delta_{z_1}(\alpha, \alpha - s)^2 - \delta_{z_1}(\alpha, \alpha + s)^2| \cdot |\delta_{z_1}(\alpha, \alpha - s)\delta_{z_2}(\alpha, \alpha + s)| \leq C(\|z_1\|_{C^2})s^5. \]

Similarly, for \( P_3 \) we estimate that
\[ P_3 = |(\delta_{z_1}(\alpha, \alpha - s)^2 - \delta_{z_1}(\alpha, \alpha + s)^2)\delta_{z_2}(\alpha, \alpha - s)^2 - \delta_{z_2}(\alpha, \alpha + s)^2| \leq C(\|z_1\|_{C^2}, \|z_2\|_{C^1})s^5 + C(\|z_2\|_{C^2}, \|z_1\|_{C^1})s^5. \]

Substituting all the estimates of \( P_i \) into (33) and using (29), we conclude that
\[ |J|^2 \leq 4m\|\tilde{w}\|_{C^0}C(\|z\|_{C^2})A^8 \int_m^1 \frac{s^7}{s^8} ds \leq C(A, \|z\|_{C^2}, \|\tilde{w}\|_{C^0})m \log \frac{1}{m}. \]

Estimate (34) together with (32) leads to
\[ |J_1| \leq C(A, \|z\|_{C^2}, \|\tilde{w}\|_{C^1})m \log \frac{1}{m}. \]

Finally, for the term \( J_\infty \), we compute
\[ |J_\infty| = \left| p.v. \int_{|\alpha - s| > 1} \frac{4\delta_{z_1}(\alpha, s)\tilde{w}(s)z_2(\alpha, s)z_2(s)}{P(\alpha, s)} ds \right| \leq 4m\|z_1\|_{C^1}\|\tilde{w}\|_{C^0}\|z_2\|_{L^\infty} A^4 \int_{|\alpha - s| > 1} \frac{|\alpha - s|}{|\alpha - s|^4} ds \leq (2A^4\|z_1\|_{C^1}\|\tilde{w}\|_{C^0}\|z_2\|_{L^\infty})m. \]

Finally, (27), (35) and (36) together imply that
\[ \left| \frac{d}{dt}m \right| \leq C(\|z\|_{C^2}, \|\tilde{w}\|_{C^1}, A)m \log \frac{1}{m}, \]
where the constant \( C \) only depends on \( \|z\|_{C^2}, \|\tilde{w}\|_{C^1} \) and the constant \( A \) in the chord-arc condition \( (17) \). It follows that
\[ m(t) \geq e^{-Ce^Ct}, \]
which completes the proof of the theorem. \( \square \)
4. The Muskat problem

We start this section proving that the solution to the Muskat problem in the Rayleigh-Taylor unstable regime where the heavier fluid is on top of the lighter fluid can actually approach the impervious bottom. Namely, we have the following result

**Proposition 4.1.** Let \((z, \tilde{\omega})\) be an analytical solution of the Muskat problem \((5)-(12)\) with

\[
\gamma = 0, [\mu] = 0
\]

in the Rayleigh-Taylor unstable case

\[
[\rho] > 0.
\]

Assume that

\[
\partial_s z_1(s) \geq 0.
\]

Then

\[
\frac{d}{dt} m(t) \leq 0,
\]

where

\[
m(t) = \min_{s \in \mathbb{R}} z_2(s, t).
\]

**Proof.** We observe that even if in the RT unstable the Muskat problem is ill-posed in Sobolev spaces in absence of capillary forces, the solution exists and is unique in the analytic setting \([12]\). We define \(\alpha_t\) as in \((18)\). Due to the regularity of \(z\) we have that \(m\) satisfies the equation \((19)\). By \((19)\) and \(\gamma = 0\) it suffices to show that, after taking \([\rho] = 2\pi\) without loss of generality,

\[
0 \geq I := \text{p.v.} \int_{\mathbb{R}} \frac{\partial_s z_2(s)(z_1(\alpha_t) - z_1(s))}{|z(\alpha_t) - z(s)|^2} - \frac{\partial_s z_2(s)(z_1(\alpha_t) - z_1(s))}{|z(\alpha_t) - (z_1(s), -z_2(s))|^2} ds.
\]

We define

\[
\tilde{I} := \text{p.v.} \int_{\mathbb{R}} \frac{\partial_s z_1(s)(z_2(\alpha_t) - z_2(s))}{|z(\alpha_t) - z(s)|^2} + \frac{\partial_s z_1(s)(z_2(\alpha_t) + z_2(s))}{|z(\alpha_t) - (z_1(s), -z_2(s))|^2} ds.
\]

We first show that

\[
\tilde{I} - I = \pi.
\]

To prove this, we use the basic tools from complex integrals. Write

\[
z(s) := z_1(s) + iz_2(s).
\]

\(\Gamma(t)\) is a curve in the complex plane. Direct calculation implies that

\[
\tilde{I} - I = \lim_{r \to 0, R \to \infty} \text{Im} \left( \int_{\Gamma \cap \{r \leq |z - z(\alpha_t)| \leq R\}} \frac{dz}{z - z(\alpha_t)} + \frac{dz}{z - z(\alpha_t) - z} \right),
\]

where \(\text{Im}\) means the imaginary part of a complex number. We define

\[
\Gamma_{r, R} := (\Gamma(t) \cap \{|r \leq |z - z(\alpha_t)| \leq R\}) \cap(|z - z(\alpha_t)| = r) \cap \Omega_+(t) \cap (\{|z - z(\alpha_t)| = R\} \cap \Omega_+(t)),
\]
for $r \ll 1$, $R \gg 1$. According to (16) and the regularity assumption of $\Gamma(t)$, we know that $\Gamma(t)$ is very flat near the minimal point $z(\alpha)$ and at infinity, which further implies that $\Gamma_{r,R}$ is a well-defined simple closed curve when $r$ is suitably small and $R$ is appropriately large. It is obvious that the functions

$$\frac{1}{z - z(\alpha_t)}$$

and

$$\frac{1}{\bar{z}(\alpha_t) - z}$$

do not contain any poles in the region enclosed by $\Gamma_{r,R}$. We can use Cauchy’s integral theorem to derive that

$$0 = \lim_{r \to 0, R \to \infty} \int_{\Gamma_{r,R}} \frac{dz}{z - z(\alpha_t)}$$

$$= \lim_{r \to 0, R \to \infty} \int_{\Gamma \cap \{r \leq |z - z(\alpha_t)| \leq R\}} \frac{dz}{z - z(\alpha_t)} + \lim_{R \to \infty} \int_{\{|z - z(\alpha_t)| = R\} \cap \Omega_+} \frac{dz}{z - z(\alpha_t)}$$

$$+ \lim_{r \to 0} \int_{\{|z - z(\alpha_t)| = r\} \cap \Omega_+} \frac{dz}{z - z(\alpha_t)}$$

(42)

$$= \lim_{r \to 0, R \to \infty} \int_{\Gamma \cap \{|z - z(\alpha_t)| \leq R\}} \frac{dz}{z - z(\alpha_t)} + \pi i - \pi i$$

Similarly,

$$0 = \lim_{r \to 0, R \to \infty} \int_{\Gamma_{r,R}} \frac{dz}{\bar{z}(\alpha_t) - z}$$

$$= \lim_{r \to 0, R \to \infty} \int_{\Gamma \cap \{r \leq |z - z(\alpha_t)| \leq R\}} \frac{dz}{\bar{z}(\alpha_t) - z} + \lim_{R \to \infty} \int_{\{|z - z(\alpha_t)| = R\} \cap \Omega_+} \frac{dz}{\bar{z}(\alpha_t) - z}$$

$$+ \lim_{r \to 0} \int_{\{|z - z(\alpha_t)| = r\} \cap \Omega_+} \frac{dz}{\bar{z}(\alpha_t) - z}$$

$$= \lim_{r \to 0, R \to \infty} \int_{\Gamma \cap \{|z - z(\alpha_t)| \leq R\}} \frac{dz}{\bar{z}(\alpha_t) - z} - \pi i.$$  

(43)

Equation (40) then follows immediately from (41), (42) and (43).
Now we only need to prove $\tilde{I} \leq \pi$. We compute

$$\pi - \tilde{I}$$

$$= \int_{\mathbb{R}} \frac{2\partial_s z_1(s)z_2(\alpha_t)}{|z_1(\alpha_t) - z_1(s)|^2 + (2z_2(\alpha_t))^2} ds$$

$$- \text{p.v.} \int_{\mathbb{R}} \frac{\partial_s z_1(s)(z_2(\alpha_t) - z_2(s))}{|z(\alpha_t) - z_2(s)|} + \frac{\partial_s z_1(s)(z_2(\alpha_t) + z_2(s))}{|z(\alpha_t) - (z_1(s), -z_2(s))|^2} ds$$

$$= \left( \int_{\mathbb{R}} \frac{2\partial_s z_1(s)z_2(\alpha_t)}{|z_1(\alpha_t) - z_1(s)|^2 + (2z_2(\alpha_t))^2} ds - \int_{\mathbb{R}} \frac{2\partial_s z_1(s)z_2(\alpha_t)}{|z(\alpha_t) - (z_1(s), -z_2(s))|^2} ds \right)$$

$$+ \left( \int_{\mathbb{R}} \frac{2\partial_s z_1(s)(z_2(\alpha_t) - z_2(s))}{|z(\alpha_t) - z_2(s)|} + \frac{\partial_s z_1(s)(z_2(\alpha_t) + z_2(s))}{|z(\alpha_t) - (z_1(s), -z_2(s))|^2} ds \right)$$

Using the hypothesis $\partial_s z_1(s) \geq 0$, then the fact that $\pi - \tilde{I} \geq 0$ follows immediately since every term in the final two integrands is non-negative. □

We observe that this proposition generalizes the result in [12] to the case where the curve has a vertical tangent.

**Proof of Theorem 4.1** The case with $[\mu] = 0$

follows from Theorem 3.1 observing that

$$\tilde{\omega}(\alpha) = \gamma \partial_\alpha \kappa + [\rho] \rho \partial_\alpha z_2(\alpha),$$

so

$$\|\tilde{\omega}\|_{C^1} \leq C \|z\|_{C^4}$$

if $\gamma > 0$ and

$$\|\tilde{\omega}\|_{C^1} \leq C \|z\|_{C^2}$$

if $\gamma = 0$. The case with $[\mu] \neq 0$

is an application of Theorem 3.1 once we recall that

$$\|\tilde{\omega}\|_{C^1([0,T] \times \Gamma(\iota))} = \|[v \cdot \partial_\alpha \mathbf{Z}]\|_{C^1([0,T] \times \mathbb{R})} \leq A,$$

by the hypotheses of the theorem. □

5. The Internal Waves Problem

**Proof of Theorem 4.2** This theorem follows from an application of Theorem 3.1 noticing that

$$\|\tilde{\omega}\|_{C^1([0,T] \times \Gamma(\iota))} = \|[v \cdot \partial_\alpha \mathbf{Z}]\|_{C^1([0,T] \times \mathbb{R})} \leq A,$$

by the hypotheses of the theorem. □
Acknowledgments

Z.G. was supported by the Basque Government through the BERC 2022-2025 program and by the Spanish State Research Agency through BCAM Severo Ochoa excellence accreditation SEV-2017-0718 and through project PID2020-114189RB-I00 funded by Agencia Estatal de Investigación (PID2020-114189RB-I00 / AEI / 10.13039/501100011033). R.G-B was supported by the project ”Mathematical Analysis of Fluids and Applications” Grant PID2019-109348GA-I00 funded by MCIN/AEI/ 10.13039/501100011033 and acronym ”MAFyA”. This publication is part of the project PID2019-109348GA-I00 funded by MCIN/ AEI /10.13039/501100011033. R.G-B is also supported by a 2021 Leonardo Grant for Researchers and Cultural Creators, BBVA Foundation. The BBVA Foundation accepts no responsibility for the opinions, statements, and contents included in the project and/or the results thereof, which are entirely the responsibility of the authors. Part of this research was performed when R.G-B was Visiting Fellow of the Basque Center for Applied Mathematics. R.G-B is grateful to Basque Center for Applied Mathematics for their hospitality during this visit.

References

[1] E. Alvarez-Lacalle, J. Casademunt, and J. Eggers. Pinch-off singularities in rotating Hele-Shaw flows at high viscosity contrast. Physical Review E, 80(5):056306, 2009.
[2] D. M. Ambrose, R. Camassa, J. L. Marzuola, R. M. McLaughlin, Q. Robinson, and J. Wilkening. Numerical algorithms for water waves with background flow over obstacles and topography. arXiv preprint arXiv:2108.01786, 2021.
[3] C. Arthur, R. Granero-Belinchón, S. Shkoller, and J. Wilkening. Rigorous asymptotic models of water waves. Water Waves, 1(1):71–130, 2019.
[4] H. Bae and R. Granero-Belinchón. Singularity formation for the Serre-Green-Naghdi equations and applications to abcd-Boussinesq systems. Monatshefte für Mathematik, pages 1–14, 2021.
[5] R. Camassa, G. Falqui, G. Ortenzi, M. Pedroni, and G. Pitton. Singularity formation as a wetting mechanism in a dispersionless water wave model. Nonlinearity, 32(10):4079, 2019.
[6] R. Camassa, G. Falqui, G. Ortenzi, M. Pedroni, and C. Thomson. Hydrodynamic models and confinement effects by horizontal boundaries. Journal of Nonlinear Science, 29(4):1445–1498, 2019.
[7] Á. Castro, D. Córdoba, C. Fefferman, and F. Gancedo. Breakdown of smoothness for the Muskat problem. Archive for Rational Mechanics and Analysis, 208(3):805–909, 2013.
[8] P. Constantin, T. Elgindi, H. Nguyen, and V. Vicol. On singularity formation in a Hele-Shaw model. Communications in Mathematical Physics, 363(1):139–171, 2018.
[9] A. Córdoba, D. Córdoba, and F. Gancedo. Interface evolution: water waves in 2-d. Advances in Mathematics, 223(1):120–173, 2010.
[10] A. Córdoba, D. Córdoba, and F. Gancedo. Interface evolution: the Hele-Shaw and Muskat problems. Annals of mathematics, pages 477–542, 2011.
[11] D. Córdoba and F. Gancedo. Contour dynamics of incompressible 3-d fluids in a porous medium with different densities. Communications in Mathematical Physics, 273(2):445–471, 2007.
[12] D. Córdoba Gazolaz, R. Granero-Belinchón, and R. Orive-Illera. The confined Muskat problem: Differences with the deep water regime. Communications in Mathematical Sciences, 12(3):423–455, 2014.
[13] D. Coutand. Finite-time singularity formation for incompressible Euler moving interfaces in the plane. Archive for Rational Mechanics and Analysis, 232(1):337–387, 2019.
[14] D. Coutand and S. Shkoller. On the impossibility of finite-time splash singularities for vortex sheets. Arch. Rational Mech. Anal., 221:987–1033, 2016.
[15] J. Eggers. Nonlinear dynamics and breakup of free-surface flows. Reviews of modern physics, 69(3):865, 1997.
[16] C. Fefferman, A.D. Ionescu, and V. Lie, On the absence of “splash” singularities in the case of two-fluid interfaces. Duke Math. J. 165:417?462, 2016.
[17] F. Gancedo, R. Granero-Belinchón, and S. Scrobogna. Surface tension stabilization of the Rayleigh-Taylor instability for a fluid layer in a porous medium. 37(6):1299–1343, 2020.
[18] F. Gancedo and R. M. Strain. Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem. Proceedings of the National Academy of Sciences, 111(2):635–639, 2014.
[19] R. Granero-Belinchón and O. Lazar. Growth in the Muskat problem. *Mathematical Modelling of Natural Phenomena*, 15:7, 2020.

[20] D. Lannes. Well-posedness of the water-waves equations. *Journal of the American Mathematical Society*, 18(3):605–654, 2005.

[21] D. Lannes. *The water waves problem: mathematical analysis and asymptotics*, volume 188. American Mathematical Soc., 2013.

[22] J.-G. Liu and R. L. Pego. On local singularities in ideal potential flows with free surface. *Chinese Annals of Mathematics, Series B*, 40(6):925–948, 2019.

[23] J.-G. Liu and R. L. Pego. In search of local singularities in ideal potential flows with free surface. *arXiv preprint arXiv:2108.00445* 2021.

[24] E. Mariotte. *Traité de mouvement des eaux et des autres corps fluides... Mis en lumière par les soins de M. de La Hire... Nouvelle édition corrigée*. Jean Jombert, 1700.

[25] M. Moseler and U. Landman. Formation, stability, and breakup of nanojets. *Science*, 289(5482):1165–1169, 2000.

[26] A. Oron, S. H. Davis, and S. G. Bankoff. Long-scale evolution of thin liquid films. *Reviews of modern physics*, 69(3):931, 1997.

*Email address: zgeng@bcamath.org*

BCAM–Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Basque Country, Spain

*Email address: rafael.granero@unican.es*

Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, Santander, Spain