Convergence Criteria for Dynamic Integer Systems

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Abstract

Criteria are presented for testing whether every trajectory (integer sequence) of a dynamic system converges to the same fixed point. The criteria are sufficient and applicable to a large variety of systems on positive integer values. The variety includes the (suitably reduced) Collatz $3n+1$ dynamic system. A system converges according to a well-known old criterion if every trajectory value has a smaller successor. This criterion is not general enough for application to more complex systems. It is therefore generalized for this purpose by suitable system reduction, e.g., by successively removing branches (values with all their own predecessors) without influencing the convergence property of the system in question until the old criterion becomes applicable. A system also converges if it has the same structure as another system which is already known to converge. Moreover, it converges if it has a self-similar structure where all values have either the same or an infinite given number of direct predecessors. Every system is mainly represented and investigated by a graph of connected knots identified by the trajectory values. The system structure is then obtained by removing all these values, but never the knots themselves. The structure alone determines the system convergence.

1 Introduction

Natural systems (imagine a fountain or a waterfall) develop dynamically and irreversibly from a present state to following states. A realistic or fictitious mathematical model of such a system is called a dynamic(al) system. Similarly to nature, the model can show convergent, divergent, periodic, stable, random or chaotic behavior. A dynamic system on real values $n$ can be generated by iterative application of a unique function $f(n)$ to form a sequence of successors or values $n_{i+1} = f(n_i)$ of a starting value $n = n_1$. Values $n_j$ with $j < i$ are predecessors of $n_i$. An application of the generating function $f(n)$ is called a step and the sequence of successors is called a trajectory $T(n) = (n_1 = n, n_2, \ldots, n_i, \ldots)$. A trajectory is
also called an integer sequence if all values $n_i$ are integers. The aim of the present study is to construct sufficient criteria for stating that every trajectory of a dynamic system on a set $\forall$ of admitted integers $n > 0$ converges to the same reoccurring (or stopping) value $\tilde{n} = f(\tilde{n})$ called the fixed point.

An outstanding example is the (original) Collatz dynamic system which is generated on all integers $n > 0$ by the function

$$f_C(n) = \begin{cases} 
3n + 1, & \text{if } n \text{ is odd;} \\
n/2, & \text{if } n \text{ is even.}
\end{cases}$$

(1)

The famous Collatz conjecture (which was posed in 1937 and is also called the 3n+1 conjecture) refers to the observed, but seemingly yet unproven property of the Collatz system that every trajectory leads to a periodically reoccurring value 1. The function $f_C(n)$ according to Equation (1) can be simplified by excluding the needless and hindering values $n$ divisible by 2 or 3 (see Section 8). The result of this reduction is

$$n' = f_C(n) = (3n + 1)/2^n, \quad \text{if } n \text{ and } n' \text{ are indivisible by 2 and 3}$$

(2)

where the integer $\nu > 0$ is chosen for each $n$ so that $n' = f_C(n)$ becomes odd. Then, all successors of the value 1 are 1 since $f_C(1) = 1$. This fact means $T(1) = (1, 1, \ldots)$ and is also taken as a trajectory stop. The reoccurring (or stopping) value $\tilde{n} = 1$ is the fixed point where all trajectories are conjectured to stop.

In Section 2, a general dynamic (integer) system is introduced and represented by a graph of connected knots. These knots are either arbitrarily, but uniquely identified or not at all. If not, then the graph is called the system structure. The structure alone determines the system convergence. There is a large variety of possibly convergent systems (including the Collatz system) with the same, isomorphic structure. Section 3 explains why knot identifications should be removed to form the system structure that can finally lead to a conjectured convergence. Section 4 deals with system isomorphism and transformation needed for using the convergence criteria defined in Section 5. These criteria are then applied in Sections 6 to 8 to some systems which are already known or not known to converge. In particular, the Collatz system turns out to converge (see Section 8).

There is already an “old” convergence criterion. Accordingly, a system converges to the fixed point $\tilde{n} = 1$ if every value $n > 1$ has a smaller successor. Then, this successor can repeatedly represent and replace $n$ until $n = \tilde{n} = 1$ is obtained. Another approach could show that cycles or divergent trajectories (see Section 2) do not exist, but it was tried in vain to the Collatz conjecture. Therefore, essentially other, much more general criteria are presented. Then, a system converges if it has, e.g., a self-similar structure (see Sections 2 and 5).

In this study and if not otherwise stated, the term “number” mainly means a result of counting or marking objects, whereas a number itself is called by a more informative term like “integer”, “prime” or “real value”. This use avoids phrases like “... a number of numbers ...”. A lower case italic Latin or Greek letter always denotes an element of the
set \( \mathbb{N}^+ \) of all positive integers. Every used set or infinity means a countable one. Phrases similar to “finite or (countably) infinite” or “set or subset” are abbreviated by “(in)finite” or “(sub)set”, respectively; a reference to the Collatz case by “CC”.

For more details about approaches to the Collatz problem (CC), see, e.g., References [1] to [5] and the literature cited there. Wirsching’s book [1] is a comprehensive overview. The articles in Wikipedia [2] and by Pöppe [3] (in German) are first introductions to the matter. Feinstein [4] and Opfer [5] already presented interesting, but rather complex and seemingly unconfirmed proofs stating that the Collatz conjecture “is not provable” and “holds true”, respectively (see Section 9).

2 Dynamic system, its graph and structure

A dynamic (integer) system is usually generated, represented and investigated by integers \( n' = f(n) \). But this approach can be complemented impressively by knots of a graph introduced as follows.

As well known, a (countable) set contains a counted number 0, 1, 2, ..., or \( \infty \) of elements. An element is any object or may itself be a (sub)set.

Similarly, a (countable) graph is a set of a counted number 0, 1, 2, ..., or \( \infty \) of knots. A knot is any object or may itself be a (sub)graph. The knots are (dis)connected and uniquely identified or not. A graph of unidentified knots is called a structure.

The structure is a very important essential property of a dynamic system (say, the system skeleton). It is in general defined and represented by the system graph. Such a graph consists of an (in)finite number of identical knots which are arbitrarily (in)directly connected with one another or not, and (only for the structure) without any knot identification by numbers, names or other marks. It is stressed that the graph and its structure have to be understood purely topologically without any dimensions like space, time, measure, dynamics.

In the following, only dynamic systems are considered where each of them is generated by iterative application of a unique function \( n' = f(n) \) on positive integers \( n, n' \in \mathbb{V} \). This generating function may not only be given by arithmetic formulas, but also by an algorithm or a list of connections \( n \rightarrow n' \). The system structure is represented by a directed graph of connected, identical knots \( \bigcirc \rightarrow \), each with a single connection pointer to only one following knot, the direct successor. As an exception, a knot \( \bigcirc \) without a pointer may be used for representing a fixed point. Although knot numbering could diminish the transparency of the system structure, the values \( n \) of the dynamic system are later on used as knot numbers (see Section 4).

Trajectories of a dynamic system are connected if they have common values. If all trajectories are connected with one another, then the entire system is connected. A value \( \tilde{n} = f(\tilde{n}) \) is a fixed point. If all trajectories converge to the same fixed point, then the system is convergent. A periodically reoccurring sub-sequence of values of a trajectory is called a cycle. A fixed point could be taken as a cycle with period 1, but this is avoided by the present study. Every trajectory ends at its root, that is, it converges either to a fixed point or to a cycle or it diverges to infinity. Every root characterizes a subsystem disconnected
from others. Every value can act as a delegate (even as the fixed point) of all its predecessors. Only these predecessors together with their delegate form a branch. The delegate is the only root of a branch. This fact will become important in Section 4.

As already described in Section 1, the aim of the present study is to introduce criteria for stating that an entire dynamic system, generated by a function \( f(n) \), is connected and converges to a single fixed point \( \tilde{n} = f(\tilde{n}) \). Then, more roots do not exist. Before a convergence criterion can be applied, the system graph has to be constructed by using the function \( f(n) \) with \( n \) acting as the knot identifying numbers. A following removal of all these numbers cleans the graph, but does not change the system structure at all. The structure alone determines the roots of the system and should therefore be investigated, whereas \( f(n) \) itself with all values and trajectories will play a minor role only.

It is stressed again that a structure only deals with knots and their connections, but yet not with the knot identification, e.g., by values \( n \) of the dynamic system in question. Systems with the same structure are isomorphic. A set of them is called a family. Two structures are isomorphic if there is a complete one-to-one (bijective) correspondence between them, concerning all knots (unidentified), connections, roots, trajectories.

There are related structures (typed by \( \eta \)) which can be used as valuable tools in convergence investigation. Every chosen one of these structures defines an isomorphic and convergent infinite system family. And every knot \( X \) of the chosen structure points to its own direct successor and has a given, for all knots the same fixed counted number \( \eta > 0 \) or \( \eta = \infty \) of other knots which are the direct predecessors of knot \( X \). Only a single knot does not have a successor, the fixed point of the chosen structure. All knots are (in)directly connected with one another at least at the fixed point. Therefore, other roots (fixed points, cycles and divergent trajectories) do not exist.

Self-similarity is another important essential property of every graph (or system) of a family with the just chosen particular structure. One can easily see by inspecting the graph that every knot acts as the fixed point of a branch and every branch has exactly the same structure as the chosen structure of the entire graph. This property is referred to in the following by the adjective “self-similar (of type \( \eta \)).”

More generally, only infinite, connected dynamic systems with a single fixed point and no other roots can be self-similar provided that the fixed point and every knot have the same number \( \eta > 0 \) or \( \eta = \infty \) of direct predecessors. Otherwise, the structure of the entire system and that one of every branch cannot be isomorphic (see Sections 6 to 8 for examples).

3 Interchanging or removing knot identifications

The following story shall explain that knots of a system graph do not necessarily need to be identified and available to determine system roots, e.g., convergence to a fixed point. Any marks used to identify knots can arbitrarily be interchanged or removed without modifying the system structure with all its knots, connections, roots.

Consider a town with many sights and other important objects such as places, large and small streets, crooked lanes, churches, monuments and more. Every day, many tourists arrive
at the railway station and want to return in the evening with the last train. A problem is that many tourists get lost in the town and cannot be back in time at the hidden station. And there are not enough hotels. What should be done? Two alternative solutions are discussed by the town council:

(a) Every tourist gets a detailed town map when he starts at the station for sight-seeing.
(b) At every important object of the town there is a signpost “To the station →”.

Walking to the station can be taken as a dynamic system with the station as a fixed point and the important objects of the town as the values or knots. The Solutions (a) and (b) are equivalent alternatives. With Solution (a), every object must be uniquely identified (e.g., by a name) itself and on the map as well. Solution (b) represents the structure of the system, a directed graph of the object connections. Object identification is not needed, but it must be required that a pointed way (trajectory) does not end at a cycle or anywhere off the station. This can best be achieved by setting up the signposts (knots) from the station back to the objects. There is no need to identify the signposts, but if they are identified (by names or numbers for maintenance or other purposes), then they can arbitrarily be interchanged or not without any modifying the structure. This sentence is most essential. It expresses a new important finding stated here in general for short: Structure only determines convergence and roots at all!

4 System isomorphism and transformation

As just stated above, only the structure can determine convergence of a dynamic system A to a single fixed point. System A is proven to converge if it has the same structure as another system B which is already known to converge (see Section 5 Criterion 1). This isomorphism of A and B is often missing, but can possibly achieved by system transformation which is equivalent with respect to the main intention, e.g., to prove a conjecture or to reduce or to expand a system.

Pay first attention to several important general building blocks for equivalently transforming (reducing or, reversely, expanding as well) a dynamic system which shall be tested for convergence and is represented by a directed graph of knots $\textcircled{\small 0}$ and $\textcircled{\small 1}$. Let these knots be uniquely identified by admitted numbers $n \in V$ and each knot be connected and pointing at most to one single direct successor knot $n' \in V$ obtained by $n' = f(n)$. The main question is whether the system converges to a single fixed point $n \rightarrow f(n)$. This equation must have just one single solution $\tilde{n} \in V$. Otherwise, the system does not converge. The system structure is formed by removing all knot identifying numbers. But all knots with all their directed connections remain.

Every knot can not only be arbitrarily and uniquely identified by a number $n$ or otherwise or not at all. It can itself, if suitable, also be a connected (sub)graph. Reversely, any connected (sub)graph can also act as a single knot. This slightly generalized knot definition easily allows equivalent system transformations, preferably reductions with respect to convergence or, as well, expansions to make systems isomorphic.

Notice that such a transformation changes not only the graph and the structure, but also
the generating function \( f(n) \) and the set \( V \) of admitted knot numbers.

Examples of transformation building blocks for direct or reversal application:

**Block (1):** A chain \( \circ \rightarrow \circ \rightarrow \ldots \rightarrow \circ \rightarrow \) with all its inputs (connections from the direct predecessors) can be replaced by a single knot \( \circ \rightarrow \) with all inputs from outside the chain.

**Block (2):** A cycle can be replaced by a fixed point \( \circ \) with all inputs from outside the cycle.

**Block (3):** A knot \( \circ \rightarrow \) with no input at all can be removed.

**Block (4):** A branch \( B(n) \) (knot \( n \circ \rightarrow \) together with all its predecessors, see also Section 2) can be taken as a subsystem converging to knot \( n \) and can completely be removed since the branch has only the single root knot \( n \) (delegate). But attention! Knot \( n \) itself must not be a fixed point, a cycle member, or infinite. Otherwise, the due (possibly questionable) subsystem would disappear and could thus cause a wrong proof of convergence. If \( n' > n \), it can sometimes be sufficient to interchange the knot numbers \( n \) and \( n' \) only instead of removing the whole branch \( B(n) \).

**Block (5):** Knot numbers, names, or other knot identifiers can arbitrarily be interchanged (by permutation), replaced or removed without any modifying the structure.

**Block (6):** If two trajectories are connected at some knot, then an arbitrarily other knot of both trajectories can instead be chosen for the connection.

A short example of using the building blocks is the equivalent transformation of CC from Equation (1) to Equation (2). Blocks (1) and (3) are used to exclude values \( n \) divisible by 2 and 3, respectively. For a more elaborate transformation, see CC in Section 8.

Let now again an (in)finite set \( V \) be given, the elements of which are all the admitted values \( n \) of a dynamic system generated by the unique function \( f(n) \). The values \( n \) shall serve for uniquely identifying the knots of the system graph. If a person A needs all \( n \in V \) for the identification, then a person B also needs all \( n \in V \) provided that these elements \( n \) differ from one another and each \( n \) is used once and only once. Naturally, the distributions of the values \( n \) to the knots by person A and by person B differ in general by a permutation.

Consider again a system as just described with an (in)finite number of knots. Let every knot be uniquely identified by a value \( n \in V \) and have an own given (in)finite number \( \eta(n) \geq 0 \) or \( \eta(n) = \infty \) of direct predecessors forming a set \( U_n \). All these \( U_n \) are disjoint. This fact allows altogether a unique serial numbering of all knots since it is well-known that the union of (in)finately many (in)finite, disjoint sets \( U_n \) becomes a single (in)finite set \( U = \bigcup_n U_n \). It is stated again that all the admitted \( n \in V \) have to be different and used once and only once for uniquely identifying the knots.

Although knots are usually numbered by iterative application of the unique generating function \( n' = f(n) \), there is also another, possibly more practical way to generate numbers for the knots of a conjectured convergent structure. This way is an iterative application of the reverse generating function \( n = f^*(n') \) with start at the already known fixed point \( n' = \tilde{n} \). The reverse function is not unique. Every \( n' \) can have an own (in)finite number of results \( n \), the direct predecessors of \( n' \). These different results \( n \) should also be identified themselves by a suitable new parameter \( \mu \). The described way identifies all knots of the convergent subsystem with \( \tilde{n} \) as its root. If all admitted \( n \in V \) are needed, then no \( n \) at
all remains for other, disconnected subsystems and, thus, the entire system is proven to converge.

5 Convergence criteria

After laying the foundations in Sections 1 to 4, six convergence criteria can now easily be constructed and understood. In every system, all the admitted \( n \in \mathcal{V} \) have to be used once and only once for a unique knot identification. The criteria are mainly based on the idea of strictly separating structure from knot identification. For applications of the criteria, see Sections 6 to 8. The criteria are to be used to test the convergence of a dynamic system to the fixed point \( \tilde{n} \). The criteria are sufficient. This means that in case of a negative test result, one cannot decide whether or not the system converges. Before a criterion is applied, the system should be suitably reduced, e.g., by using the transformation building blocks of Section 4. The following criteria together with system transformations can turn out to be related to one another, e.g., Criteria 3 and 4. It is also stressed again that the system to be tested for convergence must have one single fixed point only.

**Criterion 1:** If two dynamic systems A and B are isomorphic (have the same structure) and system B is known to converge, then system A also converges.

— Comment: Structure alone determines roots (see Sections 2 and 3).

This convergence Criterion 1 is quite simple and clear, most basic, very general, and does not need any knot identification. But a convergent system B must be available for the structure comparison. The criterion could even be more generalized for other structural system properties, e.g., cycles or divergences.

The systems A and B are isomorphic if there is a complete one-to-one correspondence between them, concerning all knots (unidentified), connections, roots, trajectories. If A and B are not isomorphic, then they can possibly equivalently transformed to become isomorphic, e.g., if the structure of A (or B) contains that of B (or A) with corresponding fixed points. In particular, this applies to the case of a self-similar A where the entire A and every branch are isomorphic. The branch converges and can act as B. The best way to test A and B for being isomorphic seems trying to generate the graph of A step by step with the reverse function \( n = f^*(n') \) on the known graph of B from the known fixed points \( \tilde{n} \) of A and B up to all predecessors of every knot \( n' \) of A. Then, system A converges if all \( n \in \mathcal{V} \) are needed. (See also Section 4 and Criterion 2.)

**Criterion 2:** A system converges if all its knots are (in)directly connected with one another and need all \( n \in \mathcal{V} \) for a unique identification.

— Comment: No disconnected knots remain for trajectories to roots besides the single fixed point (For application, see the last paragraph of Section 4 and also Criterion 1).

**Criterion 3:** An infinite system converges if its structure is self-similar.

— Comment: Systems which are finite or have roots besides the single fixed point are never self-similar (see end of Section 2).
Criterion 4: An infinite system converges if all knots have the same number \( \eta > 0 \) or \( \eta = \infty \) of direct predecessors.
— Comment: The system is self-similar (see Criterion 3). Moreover, systems with the same \( \eta \) of all knots are always isomorphic.

Criterion 5: A system converges if it completely consists of branches with all their delegates (in)directly connected with one another.
— Comment: The branches can successively be removed without influencing the convergence to the fixed point (see Section 4 Blocks (1),(3),(4)).

Criterion 6: A system converges to the single fixed point \( \tilde{n} = \min n \in V \) if every knot \( n > \tilde{n} \) has a successor knot \( m < n \).
— Comment: This is the old criterion (see Section 1).

The old criterion can only be applied if every case \( n' = f(n) > n \) can be removed by a suitable equivalent system transformation, e.g., by removing the branch \( B(n) \) or by interchanging the knot numbers \( n \) and \( n' \) (only these numbers, never the knots themselves with their directed connection!). Cases \( n' > n \) of cycles or divergent trajectories can never be removed at all. (See also Section 4 Blocks (4) and (5).) But such cases need no attention. This is an advantage of the old criterion. A shortcoming seems to be that structure is not taken into account.

6 A simple dynamic system

The convergence criteria are first applied to the simple (but not trivial) dynamic system with the following generating function \( f_S(n) \) and its reverse \( f_S^*(n') \):

\[
n' = f_S(n) = \begin{cases} 
n, & \text{if } n = 1 \\
(n - 1)/2, & \text{if } n > 1 \text{ is odd} \\
n/2, & \text{if } n \text{ is even} \end{cases}
\]

(3)

\[
n = f_S^*(n') = 2n' \text{ and } = 2n' + 1
\]

(4)

with all values \( n, n' > 0 \) admitted and forming the set \( V = \mathbb{N}^+ \). Since always \( n' < n \), except \( n' = n = \tilde{n} = 1 \) for the fixed point \( \tilde{n} \), the system is connected and converges to \( \tilde{n} \) according to the old Criterion 6. Every \( n' \) has \( \eta = 2 \) direct predecessors. This fact makes the system structure self-similar. Accordingly, the system also turns out to converge by using Criteria 3 or 4.

For a test of convergence to \( \tilde{n} \), the knots of a system graph can best also be generated and numbered by \( n = f^*(n') \) reversely from \( \tilde{n} \). If all \( n \in V \) are needed once and only once, then the system converges since no values \( n \) remain for disconnected subsystems of other roots. Indeed, \( f_S^*(n') \) generates \( \eta = 2 \) knots \( n \) for every \( n' \) already given. Every \( n \) is needed once and only once and obtains an own trajectory \( T(n) \) different from others. See reversely, e.g., \( 1 \Rightarrow 2, 3 \Rightarrow 4, 5, 6, 7 \Rightarrow ... \Rightarrow 2^{\nu}, ..., 2^{\nu+1}-1 \Rightarrow ... \) (\( \Rightarrow \) denotes a bundle of reverse connections from ... to ... by \( f^* \)). No \( n \) is lost and knots with the same number do not exist. The system thus converges.
7 Families of convergent dynamic systems

The application of the most general convergence Criterion 1 to a system A requires an available, possibly isomorphic, convergent system B. Such a system B can easily be constructed because every integer \( n > 1 \) can uniquely be represented by multiplied primes or, similarly, by a sum of powers \( 2^\nu \) with exponents \( \nu \in \mathbb{N} \), or by proceeding one of many other ways.

In particular, the generating function \( f_P(n) \) of a large family of isomorphic systems on multiplied primes (MP) and their reverse function \( f_P^*(n') \) read

\[
n' = f_P(n) = n/p(n) ; \quad n = f_P^*(n') = n'p ; \quad (p, p(n) > 3)
\]

where \( p \) and \( p(n) \) are arbitrarily chosen prime factors (of \( n \) for \( p(n) \)). The particular choice \( p, p(n) > 3 \) of Equation \( (5) \) is suitable only for use in Section 8. The systems of the family differ only by the sequences of chosen primes used for generating trajectories, that is, by knot numbering. Accordingly, the systems are isomorphic (all have the same structure). Since always \( n' < n \) and by application of Criterion 6, all the systems converge to the only fixed point \( \tilde{n} = 1 \). A more detailed example follows.

The knot numbers \( n \) of the family graphs are obtained by \( n = f_P^*(n') \) with an infinity \( \eta = \infty \) of direct predecessors \( n \) of every given \( n' \), beginning with \( n' = \tilde{n} = 1 \). For example with all \( p > 3 \), the direct predecessors of the fixed point \( n' = \tilde{n} = 1 \) are all single primes \( n = p_1 = 5, 7, 11, \ldots \) in arbitrary order; those ones of \( n' = 5 \) are all the products \( n = p_1p_2 = 25, 35, \ldots \) of two primes, also in arbitrary order, and so on. Every direct predecessor \( n \) of \( n' \) has one prime factor more than \( n' \) and is indivisible by 2 and 3 so that \( n = 6k \pm 1 > 1 \) for every integer \( k > 0 \) (see also CC in Section 8). All these \( n \) and \( \tilde{n} = 1 \) form the infinite set \( \mathbb{V} \) of admitted values. Every \( n \) has to be used as a knot number, but only once for a unique knot numbering. Indeed, all direct predecessors \( n \) of some given \( n' \) differ from one another due to different prime factors \( p \). They also differ from all other values \( n \) because of a different number of prime factors. And every knot \( n \) can be arrived by a reverse trajectory from \( \tilde{n} = 1 \) successively multiplied with all prime factors of \( n \).

Quite similarly, the generating function \( f_2(n) \) of another large system family on sums of powers \( 2^\nu \) and the reverse \( f_2^*(n') \) read

\[
n' = f_2(n) = n - 2^\nu ; \quad n = f_2^*(n') = n' + 2^\nu
\]

where \( \nu \) is a suitably chosen exponent and \( n > \tilde{n} = 0 \). All systems converge according to Criterion 6 since always \( n' < n \). Again, every knot \( n \) can be arrived by a reverse trajectory from \( \tilde{n} = 0 \) successively added with all powers \( 2^\mu \) of \( n \).

All systems of both and related other families are self-similar since \( \eta = \infty \) of every knot. They also converge according to Criteria 3 or 4. Moreover, the systems are isomorphic, i.e., all have the same structure independent from any unique knot numbering.

8 Collatz dynamic system and conjecture

The convergence criteria of Section 5 together with the graphic properties, tools and transformations of Sections 2 and 4 are now applied to easily prove the convergence of a suitably
reduced Collatz system for confirming the Collatz conjecture (CC, see Section 1).

The original CC system graph is first generated by \( n' = f_C(n) \) of Equation (1) on all admitted knot numbers \( n, n' > 0 \). The graph has a cycle \( T(1) = (1, 4, 2, 1, \ldots) \) with \( n = 1 \) conjectured to be also a value of every trajectory \( T(n) \). The graph is then reduced suitably and equivalently with respect to the CC conjecture by using the building blocks of Section 1 for transformations until a criterion can work.

1st reduction: According to Block (1), every chain of knots, generated by some even \( m \) divided \( \nu \) times repeatedly by 2, is replaced by the single knot with odd \( m/2^\nu \). This transformation removes all knots with even \( n \) and leads to \( f_C(n) \) in Equation (2) with \( \nu > 0 \). The cycle disappears, but its member knot \( n = 1 \) remains as a conjectured fixed point \( \tilde{n} = 1 \) of every trajectory. This fixed point \( \tilde{n} = 1 \) together with \( \nu = 2 \) is the only solution of \( \tilde{n} = f_C(\tilde{n}) = (3\tilde{n} + 1)/2^\nu \) transformed to \( (2^\nu - 3)\tilde{n} = 1 \).

2nd reduction: Every knot with \( n \) divisible by 3 has no predecessor \( m \) and can thus also be removed according to Block (3). No \( n = (3m + 1)/2^\nu \) is divisible by 3.

3rd reduction: Block (4) allows to also remove branches \( B(n) \), e.g., if \( n' > n \) obtained from \( n' = (3n + 1)/2^\nu > n \) for \( \nu = 1 \) only (see Equation (2)). Knot \( n' \) and all its direct predecessors \( n \) for \( \nu > 1 \) remain with \( n' < n \) since they do not belong to \( B(n) \).

Notice that the reductions remove some at first admitted \( n \) and change the original cycle to a fixed point, but do not remove any possibly existing roots. Similarly, this applies also to the branches. The generating function \( f_C(n) \) of the three times reduced CC system graph and its reverse \( f_C^*(n') \) now read

\[
n' = f_C(n) = (3n + 1)/2^\nu \quad ; \quad n = f_C^*(n') = (2^\mu n' - 1)/3
\]

with admitted \( n \) and \( n' \) indivisible by 2 and 3 (or \( n, n' = 6k \pm 1 > 0 \), see below) and with the single fixed point \( \tilde{n} = 1 \). The parameter \( \nu > 1 \) (in contrast to \( \nu > 0 \) in Equation (2)!) serves for always \( n' < n \) and makes the old Criterion 6 thus applicable. Reversely, the parameter \( \mu > 1 \) uniquely identifies all the infinite number \( \eta = \infty \) of admitted results (direct predecessors) \( n = f_C^*(n') \) for every admitted \( n' \) given. Although \( \eta = \infty \), by far not all values of \( \nu \) and \( \mu \) have been used in Equation (7) to connect admitted \( n \) and \( n' \). But this fact does not matter.

The CC system is now already proven to converge to the fixed point \( \tilde{n} = 1 \) according to the old Criterion 6 since always \( n' < n \). It also converges according to Criteria 3 and 4 since it is self-similar because of \( \eta = \infty \). See also the self-similar particular structure chosen in Section 2 and the MP and other structures discussed in Section 7. All these structures and the Collatz one are isomorphic to one another. They are taken as B and A, respectively, for Criterion 1. Each of the results here obtained already confirms the Collatz conjecture.

Nevertheless, someone may remain unsatisfied possibly by lack of numeric calculations similar to those vain ones in the past. More elaborate arguments shall convince him. Read paragraph “The knot numbers ...” of Section 7 and choose any MP system according to Equation (5) with all the prime factors \( p, p(n) > 3 \). These factors are needed to generate the entire convergent MP graph reversely by \( n = f_p(n') \) from the fixed point \( \tilde{n} = 1 \) to all knots uniquely numbered by using the set \( V \) with all the admitted \( n, n', \tilde{n} \) indivisible by
Every $n'$ turns out to have $\eta = \infty$ direct predecessor knots. The entire CC graph is now reversely generated by $n = f_C(n')$ strictly in parallel to MP with the same set $V$. The same result $\eta = \infty$ of MP and CC then leads with Criteria 3 and 4 to the same self-similar and convergent structures. Criterion 1 confirms the convergence of CC as system A and MP as system B. These systems are isomorphic, the knot number distributions differ only by a permutation. This fact becomes clear since every knot number of CC can as well be uniquely expressed by a knot number of MP and vice versa. This one-to-one correspondence between $q = 6k \pm 1 > 0$ of CC and “$\Pi > 0$ indivisible by 2 and 3” of MP is (as required by a criticism) shown as follows.

Every $q$ is obviously indivisible by 2 and 3. But can every $\Pi$ also uniquely be represented by $q$? Here, $\Pi$ is a product of prime factors $p > 3$. Then, $\Pi = q = 6k \pm 1 > 0$ must apply for every $\Pi$, that is, the unknown $k$ must have a unique solution. With other words, either only $\Pi + 1$ or only $\Pi - 1$ must be divisible by 6. Both are already even, divisible by 2. Every triple of succeeding integers $a - 1, a, a + 1$ has always one integer divisible by 3. But $a \neq \Pi$ since $\Pi$ is already indivisible by 3. Thus, either only $\Pi - 1$ or only $\Pi + 1$ is divisible by 3 and 2. The division by 6 then leads uniquely to $k$. Accordingly, CC can be transformed to MP without modifying the structure and vice versa. CC and MP are isomorphic.

This fact can also be stated more easily since every knot $n'$ has an infinity $\eta = \infty$ of direct predecessors $n$. This results for CC from $n = f_C(n') = (2^\nu n' - 1)/3$ for an infinity of values $\nu$ and all $n$ and $n'$ indivisible by 2 and 3; for MP from $n = f_P(n') = n'p$ for an infinity of primes $p > 3$ and all $n$ and $n'$. But these values do not matter at all. They only show a one-to-one correspondence between the knots of CC and MP. Much more easily, if each of all knots of two systems A and B has the same number $\eta > 0$ or $\eta = \infty$ of direct predecessors, then A and B are isomorphic.

9 Concluding remarks

As the main result of the present study, sufficient criteria are established for testing whether a dynamic system on positive integers is connected and converges to a fixed point. The criteria are based on a quite simple idea. Let, e.g., the structure of a dynamic system in question be represented by an infinite, directed graph of identical knots $\circ \rightarrow$, each with a single connection pointer to only one following knot. If this structure is self-similar or the same as that of another system already known to be convergent, then cycles and divergent trajectories do not exist and the system converges to just a single fixed point. Every knot identification, e.g., by numbers does not influence the system structure. The criteria thus are general enough for application to a large variety of related systems and problems. In particular, the Collatz conjecture is easily confirmed. The criteria also allow to avoid knot numbers at all to easily overcome the high barrier between very many logical steps and an infinity of them.

One may ask why the properties of the huge number of previously investigated individual trajectories of the Collatz dynamic system (see Wikipedia [2] or Pöppe [3]) were not taken into account. The reason is that the present approach only considers the structure and one
characteristic problem of the system in its entirety, namely, its convergence to the fixed point 1. Consider, e.g., the related system generated by the function \( f(n) = (n + 1)/2^n \) \((n \text{ and } f \text{ are odd})\). Here, every trajectory is proven to converge to the fixed point 1, easily and merely by the general and inherent property \( n' = f(n) < n \) of \( f(n) \) itself \((n > 1)\), but not by taking into account infinities of trajectories.

Many experienced mathematicians on number theory, inspired laymen, and, previously also this author, tried in vain to prove the Collatz conjecture. Why did they not succeed? Possibly, they followed a mainstream approach in which too much attention was paid to all of the trajectories (e.g., by relying on computer experiments) rather than paying attention to the problem in its entirety, say, its infinite structure.

Similarly, Feinstein [4] may be right in that an infinity of program lines or computing time is needed to test whether or not all the individual trajectories converge to 1. However, to solve a problem it may be sufficient to investigate an essential property, such as the system structure or a permanent increase of entropy (or loss of information), which allows for a completely different approach that avoids difficulties. One example is the irrational number \( \pi \) that can never be determined exactly by numerical calculations, but many essential facts about \( \pi \) are already known and always used. The human brain can think better than a computer about abstractions, continuities, infinities and irrationals. Therefore, there are not necessarily contradictions between Feinstein’s proof [4], Opfer’s proof [5] and the present approach. Problems can have several solutions, impractical ones needing an infinity of logical steps or others requiring a large or short finite number of them. See, for instance, \( \sum_{i=0}^{\infty} x^i = 1/(1 - x) \) with \(|x| < 1\). Then, one can say that the present effort to confirm the Collatz conjecture is short compared with other approaches and could finally finish the research of more than 80 years.

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The present paper describes results of the author’s private theoretical research. Some applied concepts, facts, and methods are well-known in mathematics, especially in number theory of dynamic systems. They are combined or used as tools to establish new, simple convergence criteria for dynamic systems.

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11 A motto for dynamic systems

The following poem, “The Roman Fountain” by Conrad Ferdinand Meyer (Swiss novelist and epic poet, 1825–1898), could be a motto for the dynamic systems, each appearing as a fountain of integers.

Der römische Brunnen
Auf steigt der Strahl und fallend gießt
Er voll der Marmorschale Rund,
Die, sich verschleiernd, überfließt
In einer zweiten Schale Grund;
Die zweite gibt, sie wird zu reich,
Der dritten wallend ihre Flut,
Und jede nimmt und gibt zugleich
Und strömt und ruht.

References

[1] G. J. Wirsching, The Dynamical System on the Natural Numbers Generated by the 3n+1 Function, *Lecture notes in mathematics* 1681, Springer, 1998.

[2] Wikipedia, Collatz conjecture, [http://en.wikipedia.org/wiki/Collatz_conjecture](http://en.wikipedia.org/wiki/Collatz_conjecture).

[3] C. Pöppe, Das Schicksal einer Zahlenfolge, *Spektrum der Wissenschaft*, (Feb. 2014) 72–75, in German, [www.spektrum.de/artikel/1216443](http://www.spektrum.de/artikel/1216443).

[4] C. A. Feinstein, The Collatz 3n+1 conjecture is unprovable, preprint version 19, 2011, [http://arxiv.org/pdf/math/0312309v19.pdf](http://arxiv.org/pdf/math/0312309v19.pdf).

[5] G. Opfer, An analytical approach to the Collatz 3n+1 problem, *Hamburger Beiträge zur Angewandten Mathematik*, University of Hamburg, Faculty for Mathematics, Informatics, and Natural Sciences, 2011, [http://preprint.math.uni-hamburg.de/public/papers/hbam/hbam2011-09.pdf](http://preprint.math.uni-hamburg.de/public/papers/hbam/hbam2011-09.pdf).

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