Exploration via design and the cost of uncertainty in keyword auctions

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Abstract

We present a deterministic exploration mechanism for sponsored search auctions, which enables the auctioneer to learn the relevance scores (Click-Through-Rates) of advertisers, and allows advertisers to estimate the true value of clicks generated at the auction site. This exploratory mechanism deviates only minimally from the mechanism being currently used by Google and Yahoo! in the sense that it retains the same pricing rule, similar ranking scheme, as well as, similar mathematical structure of payoffs. In particular, the estimations of the relevance scores and true-values are achieved by providing a chance to lower ranked advertisers to obtain better slots. This allows the search engine (the auctioneer) to potentially test a new pool of advertisers, and correspondingly, enables new advertisers to estimate the value of clicks/leads generated via the auction. Both these quantities are unknown a priori, and their knowledge is necessary for the auction to operate efficiently. We show that such an exploration policy can be incorporated without any significant loss in revenue for the auctioneer. We compare the revenue of the new mechanism to that of the standard mechanism (i.e., without exploration) at their corresponding symmetric Nash equilibria (SNE) and compute the cost of uncertainty, which is defined as the relative loss in expected revenue per impression. We also bound the loss in efficiency (i.e. social welfare), as well as, in user experience due to exploration, under the same solution concept (i.e. SNE). Thus the proposed exploration mechanism learns the relevance scores while incorporating the incentive constraints from the advertisers who are selfish and are trying to maximize their own profits, and therefore, the exploration is essentially achieved via mechanism design. We also discuss variations of the new mechanism such as truthful implementations.

1 Introduction

1.1 Preliminary Background

With the growing popularity of web search for obtaining information, sponsored search advertising, where advertisers pay to appear alongside the algorithmic/organic search results, has become a significant business model today and is largely responsible for the success of Internet Search giants such as Google and Yahoo!. In this form of advertising, the Search Engine allocates the advertising space using an auction. Advertisers bid upon specific keywords. When a user searches for a keyword, the search engine (the auctioneer) allocates the advertising space to the bidding merchants based on their bid values and quality scores/factors, and their ads are listed accordingly. Usually, the sponsored search results appear in a separate section of the page.
be in higher ranked slots and compete for them.

From the above description, we can note that after merchants have bid for a specific keyword, when that keyword is queried, the auctioneer follows two steps. First, she allocates the slots to the advertisers depending on their bid values. Normally, this allocation is done using some ranking function. Secondly, she decides, through some pricing scheme, how much a merchant should be charged if the user clicks on her ad and in general this depends on which slot she got, on her bid and that of others. In the auction formats for sponsored search, there are two ranking functions namely rank by bid (RBB) and rank by revenue (RBR) and there are two pricing schemes namely generalized first pricing (GFP) and generalized second pricing (GSP) which have been used widely. In RBB, bidders are ranked according to their bid values. The advertiser with the highest bid gets the first slot, that with the second highest bid get the second slot and so on. In RBR, the bidders are ranked according to the product of their bid value and quality score. The quality score represents the merchant’s relevance to the specific keyword, which can basically be interpreted as the possibility that her ad will be viewed if given a slot irrespective of what slot position she is given. In GFP, the bidders are essentially charged the amount they bid and in GSP they are charged an amount which is enough to ensure their current slot position. For example, under RBB allocation, GFP charges a bidder an amount equal to the bid value of the bidder just below her.

Formal analysis of such sponsored search advertising model has been done extensively in recent years, from algorithmic as well as from game theoretic perspective[5, 11, 8, 1, 17, 9, 10]. In a formal setup, there are $K$ slots to be allocated among $N$ ($\geq K$) bidders. A bidder $i$ has a true valuation $v_i$ (known only to the bidder $i$) for the specific keyword and she bids $b_i$. The expected click through rate of an ad put by bidder $i$ when allocated slot $j$ has the form $c_{i,j} = \gamma_j e_i$, i.e., separable into a position effect and an advertiser effect. $\gamma_j$’s can be interpreted as the probability that an ad will be noticed when put in slot $j$ and it is assumed that $\gamma_1 > \gamma_2 > \cdots > \gamma_K > 0$. $e_i$ can be interpreted as the probability that an ad put by bidder $i$ will be clicked on if noticed and is referred to as the relevance of bidder $i$. This is the quality score used in the RBR allocation rule mentioned earlier. The payoff/utility of bidder $i$ when given slot $j$ at a price of $p$ is given by $e_i \gamma_j (v_i - p)$ and they are assumed to be rational agents trying to maximize their payoffs. Further, in typical slot auctions, bidders can adjust their bids up or down at any time and therefore the auction can be viewed as a continuous-time process in which bidders learn each other’s bids. If the process stabilizes, the result can then be modeled as solution of the static one-shot game of complete information, since each bidder will be playing a best-response to others’ bids.

As of now, Google as well as Yahoo! use schemes that can be accurately modeled as RBR with GSP. The bidders are ranked according to $e_i b_i$ and the slots are allocated as per these ranks. For simplicity of notation, assume that the $i$th bidder is the one allocated slot $i$ according to this ranking rule, then $i$ is charged an amount equal to $\frac{e_i b_i}{e_i b_i}$. The revenue and incentive properties of this model has been thoroughly analyzed in the above mentioned articles.

### 1.2 The need for exploration

In the standard model described above, it is implicitly assumed that the auctioneer knows the relevance $e_i$’s, but in practice, this is not entirely true as new advertisers do also join the game and the estimates for the advertisers getting lower ranked slots is also generally poor as they hardly get any clicks. Further, it is also assumed that the bidders know their true valuations accurately and bid accordingly, and high budget advertisers and low budget advertisers (e.g., mom-and-pop businesses) have similar awareness and risk levels. In reality, an advertiser might not know her true value and what to bid, and in particular a low
budget advertiser might be loss-averse and may not be able to bid high enough to explore, due to the potential risks involved. Furthermore, in the sponsored search auctions, the value is derived from the clicks themselves (i.e. rate of conversion or purchase given a click), and therefore, unless she actually obtains a slot and receives user clicks, there is essentially no means for her to estimate her true value for the keyword. Certainly, a model that automatically allows one to estimate these key parameters (i.e. CTRs and true values) is desirable.

1.3 Results in this paper and related work

Our goal in this paper is to study the problem of learning relevance scores and valuations in a mechanism design framework while deviating only minimally from the mechanism being currently used by Google and Yahoo!. The problem of learning CTRs has also been addressed in [12, 6, 13, 7]. Our result is different from [12] in that the latter disregards the advertisers’ incentives. The result in [6] does consider the advertisers’ incentive; however, its goal is not to study exploration in the mechanisms currently being used by search engines, but to implement a truthful mechanism that also learns the CTRs, and therefore, it had to deviate from the current pricing scheme. Our mechanism can also be easily adapted for truthful implementation via a new pricing scheme, and in fact, all the revenue analysis remains the same as we shall discuss later in the paper. Study in [13] is empirical and that in [7] is not exploration based, and restricts itself to a single slot case and does not consider advertisers’ incentives.

We recently learned about an independent study by Wortman et al. [18] along lines similar to ours, i.e., designing mechanisms for exploration that deviate minimally from the standard model without exploration and then comparing their respective incentive properties. Our mechanisms for exploration are, however, quite different and they originated from a different set of approaches. Indeed, a preliminary draft that includes all the main results presented in the current paper (although motivated a little differently) was posted in early July 2007[14], well before the work in [18] was made publicly available. As discussed in greater detail in the following, here are some of the distinctive features of our independent work: (i) Our exploration mechanism is a deterministic one, unlike a randomized one analyzed in [18]; (ii) We explicitly discuss how advertisers could estimate their true valuations under our exploration based mechanism. As argued before, true valuation is often unknown a priori, and has to be accurately estimated; (iii) Besides studying the loss in revenue due to exploration, we also explicitly discuss the loss in efficiency, as well as, loss in user experience due to exploration; (iv) The tools and approaches used in the analysis of our mechanism are very different from those presented in [18], and they highlight several interesting features of mechanism design and incentive analysis. For example, we show that the mathematical structure of payoffs in our exploration mechanism is the same as in the standard mechanism without exploration, which allows us to utilize results from the latter. Thus, our approach represents an instance where reduction among mechanism design problems is being successfully used as an analytical tool.

Moreover, as we discuss later in Section 8 the problem of designing a family of optimal exploratory mechanisms, which for example would provide the most information while minimizing expected loss in revenue is far from being solved. The work in [18] and in this paper provide just two instances of mechanism design which do provably well, but more work that analyze different aspects of exploratory mechanisms are necessary in this emerging field. Thus, to the best of our knowledge, we are one of the first groups to formally study the problem of estimating relevance and valuations from incentive as well as learning theory perspective without deviating much from the current settings of the mechanism currently in place.

In the following we summarize our results as well as the organization of the rest of the paper:

1. We design a deterministic exploration mechanism to learn the relevance scores by deviating minimally from the mechanism being currently used by Google and Yahoo! in the sense that it retains the same pricing rule, as well as, similar ranking scheme. In particular, the estimation of the relevance scores is
achieved by providing a chance to lower ranked advertisers to obtain better slots. Qualitatively, some top slots are designated for exploration purposes and each of the advertisers whose relevance is to be estimated, is given an equal chance to appear in those slots. In Section 2, we formally introduce this exploration mechanism which we call Exp-GSP and the standard RBR with GSP mechanism without exploration is referred to as GSP.

2. In Section 3 we study the incentive properties of Exp-GSP mechanism by modeling it as one shot static game of complete information, like in the case of GSP[5][17]. We show that the mathematical structure of the payoffs of the bidders in Exp-GSP is the same as in GSP, and therefore all the incentive analysis from GSP can be adopted for Exp-GSP. This further corroborates our claim that our exploration mechanism deviates only minimally from GSP and indeed our approach can also be understood as reduction among mechanism design problems. Furthermore, another interesting feature of our exploration mechanism is that the attention or the quality of service (in terms of position based CTRs i.e. probability of being noticed) provided to advertisers is still in the same relative order as in standard mechanism without exploration.

3. It is clear that any exploration mechanism will incur some cost in terms of revenue compared to the case when we do not need an exploration. We formalize this cost via cost of uncertainty which is defined as the relative loss in expected revenue of the auctioneer per impression. To this end, we compare the revenue of the Exp-GSP to that of GSP at their corresponding symmetric Nash equilibria(SNE) and bound the cost of uncertainty. Our analysis confirms the intuition that a higher cost is incurred for better exploration i.e. there is a tradeoff between quality of exploration/estimation and the revenue. Nevertheless, the associated parameters can be tuned to ensure a suitable balance between these two conflicting needs- minimizing the loss in revenue while allowing for sufficient exploration to be able to estimate parameters such as the relevance scores. These revenue properties are studied in the Section 4.

4. Section 5 discusses the loss in efficiency in Exp-GSP compared to GSP. As in the case of revenue, there is a tradeoff between efficiency (i.e. social welfare) and the quality of exploration/estimation. Additionally, our analysis also suggests that closer we are to the optimal efficiency (i.e. the case when the auctioneer knows true values of relevance scores and the advertisers know that of their valuations), lesser we lose in the efficiency due to exploration. This means that during several phases of the exploration the loss in the efficiency degrades. Similar observations can also be obtained for user experience which can be defined as the total clickability of all ads.

5. In Section 6 we discuss how our exploration mechanism i.e. Exp-GSP can be used to estimate relevance scores and valuations, as well as, the quality of such estimation using Chernoff bound arguments.

6. In all the Sections from 2 through 6 we restrict ourselves to a standard assumption in literature that the CTRs are separable. In Section 7 we remove this assumption and study some other variations of Exp-GSP. In particular, by imposing a new pricing rule we can turn our exploration mechanism to a truthful one. Moreover, a similar upper bound on the cost of uncertainty is established as in the case of Exp-GSP with separable CTRs.

2 An exploration based Generalized Second Price mechanism

In this section, we formally introduce our exploration mechanism. First we setup some notations and definitions.
Notation: There are \( N \) advertisers/bidders bidding for a specific keyword and this keyword appears several times during a day. There are \( K \leq N \) slots to be allocated among the bidders for this keyword. A bidder \( i \) has a true valuation \( v_i \) for this keyword and she bids \( b_i \). The expected click through rate of an ad put by bidder \( i \) when allocated slot \( j \) has the form \( CTR_{i,j} = \gamma_j e_i \), i.e., separable into a position effect and an advertiser effect wherein \( e_i \) is the relevance of the bidder \( i \). Further, it is assumed that \( \gamma_j \geq \gamma_{j+1} \) for all \( j = 1, 2, \ldots, K \) and \( \gamma_j = 0 \) for all \( j > K \). The search engines’ estimate of relevance \( e_i \) of bidder \( i \) is denoted by \( q_i \) and bidder \( i \)’s estimate of her relevance \( e_i \) is denoted by \( f_i \). There are no budget constraints.

Explore slots and tuning parameters: Auctioneer chooses two parameters \( n \leq N \) and \( L \leq K \). Auctioneer designates top \( L \) slots for exploratory purpose. Let us call these slots as explore slots and slots \( L + 1 \) through \( K \) will be called non-explore. Auctioneer decides a set \( F \) of \( n \) bidders whose relevance, she wants to estimate. As described in the mechanism below, these \( n \) bidders will be the top \( n \) bidders according to auctioneer’s ranking rule. If auctioneer wants to just improve the estimate for some bidders, she chooses \( n \leq K \) and if she also wants to estimate the relevance of some new bidder or some left-out bidder, she chooses \( n \geq K + 1 \). The parameters \( n \) and \( L \) are publicly known. Further, as we shall see below, the mechanism has \( n \) steps and during these \( n \) steps, the bidders in set \( F \) will be given equal chance to appear in the explore slots in the sense that they appear exactly once in each explore slot. During a step, when a bidder does not appear in one of the explore slots, she competes for non-explore slots with all the bidders who do not appear in the explore slots. Now we are ready to formally describe the new mechanism which we call Exp-GSP (Exploratory-Generalized Second Price).

The Exp-GSP Mechanism:

- **Bidders** report their bids \( b_1, b_2, \ldots, b_N \).
- **Ranking Bidders**: Auctioneer uses RBR to rank the bidders i.e. she ranks the bidders in the decreasing order of \( q_i b_i \). For clarity of notation, let us rename the bidders according to this ranking, i.e., bidder \( m \) is the one ranked \( m \) in this ranking.
- **Allocating Explore Slots**: There are \( n \) steps in the mechanism and the \( n \) bidders in \( F \) are ordered in each step as follows. The ordering at step 1 is the above mentioned RBR ranking i.e. \([1, 2, \ldots, L | (L + 1), \ldots, n]\). This order is cyclicly shifted towards left for \( n - 1 \) more steps. Thus the ordering in step 2 is \([2, 3, \ldots, L, (L + 1) | (L + 2), \ldots, n, 1]\) and that in step 3 is \([3, \ldots, (L + 2), (L + 3), \ldots, n, 1, 2]\) and so on. In a particular step, for \( j \leq L \), the \( j \)th slot is assigned to the bidder having rank \( j \) in this cyclicly rotating ordering at that step. For example, in step 1, the slot \( j \leq L \) is allocated to the bidder \( j \). In step 2, the slot \( j \leq L \) is allocated to the bidder \( j + 1 \) and in step \( n \), first slot is assigned to the bidder \( n \), and for \( 2 \leq j \leq L \), the \( j \)th slot is allocated to the bidder \( j - 1 \). In a particular step, a bidder will be called explore-active if she is assigned one of explore slots in that step. Note that this cyclicly shifting rule ensures that during total of \( n \) steps, each of the \( n \) bidders in \( F \) gets to each explore slot exactly once, thus each one is explore-active for exactly \( L \) steps out of the \( n \) steps. Also, in each step there are exactly \( L \) explore-active bidders.
- **Allocating non-Explore Slots**: Bidders from \( F \) who are not explore-active at a particular step along with bidders not in \( F \), are allocated to non-explore slots as follows. Let \( i_1 < i_2 < \cdots < i_{N-L} \) be the bidders who are not explore-active in this particular step. Recall that we renamed the bidders according to the RBR ranking. Now the slot \( L + j \) for \( 1 \leq j \leq K - L \) is assigned to the bidder \( i_j \). For example, in step 1, we have \( i_j = L + j \); in step 2 we have \( i_1 = 1 \) and \( i_j = L + j \) otherwise, and in step \( n \) we have \( i_j = L + j - 1 \).
- **Payments based on GSP**: A bidder \( i \) is charged an amount equal to \( \frac{q_{i+1} b_{i+1} + 1}{q_i} \) per-click.
Nomenclatures: For the rest of the paper, we fix some nomenclatures. The standard one step mechanism with RBR ranking and GSP pricing will be referred to as GSP and the new exploration based mechanism described above (all the $n$ steps together) will be referred to as Exp-GSP. Further, we will refer $\gamma_j$’s to as position based click-through rates. Let $I_i$ denote all the information about the bidder $i$ i.e. $I_i$ includes bidder $i$’s true relevance $e_i$, auctioneer’s estimate of her relevance $q_i$, her estimate of her relevance $f_i$, her true value $v_i$ and her estimate of her true value $\tilde{v}_i$, all the knowledge of bidder $i$ about the auction game etc. An instance of the GSP is represented by $(N, K, (\gamma_j), (I_i))$ and that of Exp-GSP by $(N, K, n, L, (\gamma_j), (I_i))$. Clearly, any given instance $(N, K, (\gamma_j), (I_i))$ of GSP is equivalent to an instance $(N, K, n, L, (\gamma_j), (I_i))$ of Exp-GSP where $n = 1, L = 0$. Further, as we show in Section 3 a large class of instances of Exp-GSP of our interest can also be mapped to instances of GSP with properly defined position based click-through rates. This corroborates our claim that we deviate minimally from the mechanism currently in place.

3 Incentive properties

In this section, we study the incentives properties of $n$-step Exp-GSP mechanism modeling it as one shot static game of complete information, where the advertisers know others’ bids, and play the best response to others’ bids given their current estimates of their CTR’s and their true valuations. This is reasonable as the bidding process can be thought of as a continuous process, where bidders learn each other’s bids[5, 17, 8, 9]. As we explain in the following, a large class of the instances of Exp-GSP can be mapped to instances of GSP with properly defined click-through rates and therefore will allow us to use the results on GSP. This corroborates our claim that we deviate minimally from the mechanism currently in place. The solution concept we will use is Symmetric Nash Equilibria(SNE)/locally envy-free equilibria studied in [5,17]. First, we define effective CTR which will help us mapping instances of Exp-GSP to that of GSP.

Definition 1 Effective Click-Through Rates: Let $l_1, l_2, \ldots, l_n$ be the slot positions that a bidder $j$ is assigned in the steps 1, 2, $\ldots, n$ of Exp-GSP respectively, then the effective CTR of a bidder $i$ for slot $j \leq N$ denoted as $\hat{\gamma}_{i,j}$ is defined as $\sum_{l=1}^{n} e_i(l_m)$. Thus for the separable case, the effective position based CTR for slot $j \leq N$ denoted $\theta_j$ is $\sum_{l=1}^{n} \gamma_{l,m}$.

Intuitively, the effective CTR of a bidder $i$ for slot $j$ is the sum of the expected CTR of bidder $i$ for each of the $n$ step in Exp-GSP if he would have been ranked $j$. It is not hard to derive the following lemma.

Lemma 2 Let $\gamma = \sum_{j=1}^{L} \gamma_j$ then

$$\theta_m = \begin{cases} \gamma + d_m & \text{if } m \leq n \\ n\gamma_m & \text{if } m > n \end{cases}$$

where

$$d_m = \begin{cases} (n - L - (m - 1))\gamma_{L+m} + \\ \gamma L+1 + \gamma L+2 + \cdots + \gamma_{L+m-1} & \text{if } m \leq L \\ (m - L)\gamma_m + \gamma_{m+1} + \cdots + \gamma_{m+L} & \text{if } L \leq m \leq n - L \\ (m - L)\gamma_m + \\ \gamma_{m+1} + \cdots + \gamma_{n} & \text{if } m \geq n - L \end{cases}$$
In the above lemma, $\gamma$ basically represents the effective position based click through that a bidder obtains from the explore slots (in $n$ steps) and $d_m$ represents the effective position based click through that the bidder $m$ obtains from the non-explore slots (in $n$ steps). In particular, the $d_m$ indicates how many steps the bidder $m$ spends in specific non-explore slots. For example, $d_1 = (n - L)\gamma_{L+1}$ indicates that the bidder 1 spends $(n - L)$ steps in the slot numbered $(L + 1)$, $d_2 = (n - L - 1)\gamma_{L+2} + \gamma_{L+1}$ indicates that the bidder 2 spends $(n - L - 1)$ steps in the slot $(L + 2)$ and one step in the slot $(L + 1)$, and so on for other bidders. In the following lemma we observe that these effective position based CTRs are in fact strictly monotonically decreasing like $\gamma_j$’s. The proof is provided in the Appendix.

**Lemma 3** Let $\bar{K} = \max\{K, n\}$, $n \leq \min\{K + 1, K + L\}$, and $L \leq \frac{1}{2}(n - 1)$ then

$$\theta_1 > \theta_2 \cdots > \theta_{\bar{K}} > 0$$

and $\theta_i = 0$ for all $i > \bar{K}$.

Now under Exp-GSP the payoff of the bidder $m$ is

$$u_m = \theta_m e_m (v_m - \frac{q_{m+1}b_{m+1}}{q_i}).$$

(3)

which has exactly the same functional form as in GSP where $\theta_m$’s takes the place for $\gamma_m$’s and therefore our name for $\theta_m$’s makes sense. Thus an instance $(N, K, n, L, (\gamma_j), (I_i))$ of Exp-GSP where $n \leq K + 1$, and $L \leq \frac{1}{2}(n - 1)$, can be mapped to an instance $(N, \max\{K, n\}, (\theta_j), (I_i))$ of GSP. We formalize this in the following theorem.

**Theorem 4** For each instance $(N, K, n, L, (\gamma_j), (I_i))$ of Exp-GSP with $n \leq K + 1$, and $L \leq \frac{1}{2}(n - 1)$, there is an instance $(\bar{N}, \bar{K}, (\bar{\gamma}_j), (\bar{I}_i))$ of GSP such that the game induced by $(N, K, n, L, (\gamma_j), (I_i))$ is equivalent to the game induced by $(\bar{N}, \bar{K}, (\bar{\gamma}_j), (\bar{I}_i))$. In particular, $\bar{N} = N, \bar{K} = \max\{n, K\}, \bar{\gamma}_j = \theta_j, \bar{I}_i = I_i$ where $\theta_j$’s are defined by Equations $[7]$ and $[2]$.

It is interesting to note that even though we allowed lower ranked bidders to obtain top slots, the competition for the non-explore slots keeps the effective position based CTRs still in the same relative order. The highest ranked bidder still gets the best service compared to others although her effective payoff might have decreased. A lower ranked bidder still gets relatively lower quality of service than the bidders above her although her payoff might have improved. This same structural form of payoffs allows us to derive Theorem $[4]$ and therefore to utilize the results on GSP studied in $[5\ 17\ 8\ 9\ 2\ 11]$ and in particular the following theorem on existence of pure Nash equilibria for Exp-GSP. Thus our approach can also be understood as reduction among mechanism design problems.

**Theorem 5** There always exist a pure Nash equilibrium bid profile for the Exp-GSP.

As noted in the above theorem, there always exist pure strategy Nash equilibria for the Exp-GSP auction game. However, this existential proof does not give much insight about what equilibria might arise in practice. Edelman et al $[5]$ proposed a class of Nash equilibria which they call as locally envy-free equilibria and argue that such an equilibrium arises if agents are raising their bids to increase the payments of those above them, a practice which is believed to be common in actual keyword auctions. Varian$[17]$ independently proposed this solution concept which he calls as symmetric Nash equilibria (SNE) and provided some empirical evidence that the Google bid data agrees well with the SNE bid profile. In a similar way we can obtain the following observation.
Theorem 6 An SNE bid profile $b_i$’s for Exp-GSP satisfies

$$\begin{align*}
(\theta_i - \theta_{i+1})v_{i+1}q_{i+1} + \theta_{i+1}q_{i+2}b_{i+2} &\leq \theta_{i+1}q_{i+1}b_{i+1} \\
&\leq (\theta_i - \theta_{i+1})v_iq_i + \theta_{i+1}q_{i+2}b_{i+2}
\end{align*}$$

(4)

for all $i = 1, 2, \ldots, N$.

Note that the Theorem 6 assumes that the bidders know their true valuations $v_i$’s, however the theorem holds even if it is not the case by replacing $v_i$ by bidder $i$’s current estimate of her true valuation.

Now, recall that in the Exp-GSP, the bidder $i$ pays an amount $q_i q_{i+1} b_{i+1}$ per-click, therefore the expected payment $i$ makes under Exp-GSP (in $n$ steps) is $\theta_i e_i q_{i+1} b_{i+1} = \theta_i q_i q_{i+1} b_{i+1}$. Thus the best SNE bid profile for advertisers (worst for the auctioneer) is minimum bid profile possible according to Theorem 6 and is given by

$$\theta_i q_{i+1} b_{i+1} = \sum_{j=1}^{K} (\theta_j - \theta_{j+1})v_{j+1}q_{j+1}. \quad (5)$$

For the revenue comparison in the next section, we fix this minimum SNE bid profile as the solution concept. The same result essentially hold for the maximum SNE bid profile as well.

4 Revenue comparison and the cost of uncertainty

In this section we study the revenue properties of Exp-GSP and compare it to that of GSP. We first define the cost of uncertainty to formalize the loss of revenue due to exploration.

Definition 7 Cost of uncertainty: Let $R_0$ be the expected revenue of the auctioneer for GSP at its minimum SNE and $R$ be her expected revenue for Exp-GSP at the corresponding minimum SNE, then “cost of uncertainty” associated with the exploration is defined as $\frac{R_0 - R}{R_0}$ i.e. the expected relative loss in the revenue per impression and is denoted as $\rho$.

Using Equation , we have

$$R_0 = \sum_{s=1}^{K} \sum_{j=s}^{K} \frac{e_s}{q_s} (\gamma_j - \gamma_{j+1})q_{j+1}v_{j+1}$$

and

$$R = \sum_{s=1}^{K} \sum_{j=s}^{K} \frac{e_s}{q_s} (\theta_j - \theta_{j+1})q_{j+1}v_{j+1}$$

∴ $\frac{R_0 - R}{R_0} = \sum_{s=1}^{K} \sum_{j=s}^{K} \frac{e_s}{q_s} \left[(\gamma_j - \gamma_{j+1}) - \frac{1}{n}(\theta_j - \theta_{j+1})\right] v_{j+1}q_{j+1}$.

By utilizing the relationship among $\gamma_j$’s and $\theta_j$’s we can obtain the following theorem which provides a nice upper bound on the cost of uncertainty. The proof this theorem is provided in the Appendix.
Theorem 8. Let $R^l_0$ be the revenue of auctioneer from top $l$ bidders and $R_0$ be her total revenue in GSP and let
\[ c = \min_{1 \leq j < n - L} \frac{\gamma_j + L - \gamma_{j+1} + L}{\gamma_j - \gamma_{j+1}}. \tag{6} \]
then
\[ \rho(L,n) \leq \left\{ 1 - \min\{1, c\} \left( 1 - \frac{2L}{n} \right) \right\} \left( \frac{R_0^\min(n,K)}{R_0} \right) \tag{7} \]

First, note that the above bound is 0 when $L = 0$, indicating no revenue loss when there is no exploration. Further, given an $n$, as $L$ increases the bound deteriorates confirming our intuition that higher cost is incurred for better exploration. Also for a given $L$, we can note that the factor $\frac{R_0^\min(n,K)}{R_0}$ is dominant and increases as $n$ increases and therefore the bound deteriorates as $n$ increases. We see that auctioneer can tune parameters $L$ and $n$ so as to improve revenue, smaller the $L$ and $n$, better off the auctioneer is. But as the auctioneer also wants to get some valuable information so as to estimate parameters such as relevance of the advertisers and do also want to give flexibility to lower ranked bidders to figure out their valuations, she would like to keep $L$ and $n$ to be large. Therefore, the auctioneer can choose a suitable $L$ and $n$ to balance between these two conflicting needs. Furthermore, it is clear that a finer analysis will reveal much better revenue guarantee i.e. even smaller $\rho$. For example, usually the expression on right hand side of Equation 6 in the above theorem is dominated by $j = 1$, however if we look at the expression for revenue the $j = 1$ term appears only once unlike all other $j$’s and neglecting $j = 1$ does not noticeably change the difference in the revenues and therefore a better $c$ might be achievable with this fine tuning.

We can also note that Theorem 8 still holds true when we replace the RBR ranking rule in GSP and Exp-GSP by any weighted ranking rule (i.e. in the decreasing order of $w_i b_i$’s) and change the payment rules accordingly (i.e. $\frac{w_i + b_{i+1}}{w_i}$ per-click to the $i$th ranked bidder).

5 Efficiency comparison

Revenue is a natural yardstick for comparing different auction forms from the viewpoint of the seller (the auctioneer), however from a social point of view yet another yardstick that is natural and may be important is efficiency, that is, the social value of the object. The object should end up in the hands of the people who value it the most. The efficiency in the adword auction model is therefore the total valuation, and turns out to be the combined profit of the auctioneer and all the bidders. Let us denote the efficiency for the Exp-GSP as $E$ and that for GSP as $E_0$ then,
\[ E = \sum_{m=1}^{K} \theta_m c_m v_m \tag{8} \]
\[ E_0 = \sum_{m=1}^{K} \gamma_m c_m v_m. \tag{9} \]

Using Lemma 2 and rearranging the terms in $E$ we get,
Lemma 9

\[ E = \sum_{m=1}^{K} \gamma_m y_m \]  \hspace{1cm} (10)

where

\[ y_m = \begin{cases} 
\sum_{i=1}^{n} e_i v_i & \text{if } m \leq L \\
(n-m+1)e_{m-L}v_{m-L} + \sum_{i=m-L+1}^{m-1} e_i v_i & \text{if } L < m \leq n \\
nev_m & \text{if } m > n 
\end{cases} \]  \hspace{1cm} (11)

The above lemma allows us to bound the loss in efficiency due to exploration as we note in the following theorem whose proof is deferred to Appendix.

Theorem 10 Let \( E_0^e = \sum_{i=1}^{L} \gamma_m e_m v_m \), \( E_0^{ne} = \sum_{i=L+1}^{n} \gamma_m e_m v_m \) then the relative loss in efficiency per impression is

\[
\frac{E_0^e - \frac{1}{n} E_0^{ne}}{E_0} \leq \left\{ (1 - \beta) \left( \frac{E_0^e}{E_0} \right) + \eta \left( \frac{E_0^{ne}}{E_0} \right) \right\}
\]  \hspace{1cm} (13)

where

\[
\beta = \frac{1}{n} \sum_{i=1}^{n} e_i v_i, \quad \eta = \max_{L < m \leq n} \left\{ \max_{m-L \leq i \leq m} \left( \frac{1 - e_i v_i}{e_m v_m} \right) \right\}.
\]  \hspace{1cm} (14)

First, note that the above bound is 0 when \( L = 0 \), indicating no efficiency loss when there is no exploration. Further, given an \( n \), as \( L \) increases the bound deteriorates and similarly for a given \( L \), the bound deteriorates as \( n \) increases. Apart from the tuning parameters \( n \) and \( L \), note that there is another interesting parameter \( \eta \) which actually depends on the true relevance and the true values of the advertisers. In particular, it indicates that how far the current estimates are from the true ones. For example, in the extreme case when the auctioneer knows the true relevances, then the ordering by \( q_m v_m \), will be equivalent to the ordering by \( e_m v_m \) and \( \eta \) will in fact be 0, improving the bound. Thus closer we are to the optimal efficiency, lesser we lose in efficiency due to exploration. The proof of Theorem 10 includes the following observation in the case when the ordering by \( q_m v_m \) is same as the ordering by \( e_m v_m \).

Corollary 11 Under the assumption that \( e_m v_m \geq e_{m+1} v_{m+1} \) for all \( 1 \leq m \leq n \) the upper bound in Theorem 10 can be improved to

\[
\left\{ (1 - \alpha) \left( \frac{E_0^e}{E_0} \right) - \frac{L}{n} \omega \left( \frac{E_0^{ne}}{E_0} \right) \right\}
\]

where \( \alpha = \frac{1}{n} \sum_{i=1}^{n} e_i v_i \), \( \omega = \min_{L < m \leq n} \left( \frac{e_{m-1} v_{m-1}}{e_m v_m} - 1 \right) \).

Now let us consider the effect on the user experience due to exploration. Following [9], the user experience can be defined as the total clickability of all the ads i.e. how likely an user is to click on the ads altogether. Therefore, for GSP it is \( \sum_{m=1}^{K} \gamma_m e_m \) and that for Exp-GSP it is \( \sum_{m=1}^{K} \theta_m e_m \). Clearly, similar observations in the loss of user experience due to exploration can be obtained as in the case of efficiency.
6 Estimating the relevance and valuations

Let \( M_i \) be the number of clicks that the advertiser \( i \) receives in Exp-GSP then her relevance \( e_i \) is estimated as \( \frac{M_i}{\theta_i} \) and the deviation will not be high as can be argued using Chernoff bound arguments. Formally, let \( M_{i,j} \) be a 0–1 random variable indicating whether the advertiser \( i \) gets a click in the \( j \)th impression (i.e. \( j \)th step in Exp-GSP) or not and \( M_i = \sum_{j=1}^{n} M_{i,j} \). Clearly, \( E[M_i] = \sum_{j=1}^{n} E[M_{i,j}] = \theta_i e_i \). Then by Chernoff bound, for any \( 0 < \delta < 1 \), we have

\[
Pr(|e_i - \frac{M_i}{\theta_i}| \geq \delta e_i) \leq 2e^{-\theta_i \delta^2 e_i^2}. \tag{15}
\]

A simple calculation implies that, we can get an estimate of \( e_i \) within a \( \delta \) fraction with probability \( 1 - \epsilon \) as long as we have,

\[
\theta_i \geq \frac{3}{\delta^2 \epsilon^2} \ln(\frac{\epsilon}{2}). \tag{16}
\]

Normally we will be interested in estimating the relevance of lower ranked advertisers and clearly for them the value of \( \theta_i \) increase as we increase the value of \( L \) and we can guarantee a better estimation. In particular, given a value of \( L \) and \( n \), we can have reliable estimation with probability \( 1 - \epsilon \) within a fraction of \( \sqrt{\frac{3}{\epsilon^2 \theta_i} \ln(\frac{\epsilon}{2})} \) and an additive estimation within \( \sqrt{\frac{3}{\epsilon^3} \ln(\frac{\epsilon}{2})} \). The above estimation can be improved even further by sampling from many phases of Exp-GSP. Note that even if we consider the \( l \) phases of Exp-GSP as a single shot game, the results of the sections 3 and 4 remains unchanged and in particular the cost of uncertainty does not change. As above using Chernoff-bounds arguments, we can obtain an additive estimation within \( \delta \) with probability \( 1 - \epsilon \) if we use \( l \) phases where

\[
l \geq \frac{3}{\delta^2 \theta_i} \ln(\frac{\epsilon}{2}). \tag{17}
\]

Thus we can obtain an estimation negligibly (i.e. inverse polynomially in parameter \( n, L \)) close to the true value with probability exponentially close to 1 in polynomially many phases of Exp-GSP. We summarize the above observation in the following theorem.

**Theorem 12** The relevance of the advertiser \( i \) can be estimated within \( \delta \) with probability \( 1 - \epsilon \) by using \( l \) phases of Exp-GSP where,

\[
l \geq \frac{3}{\delta^2 \theta_i} \ln(\frac{\epsilon}{2}).
\]

Even a single phase of Exp-GSP can provide pretty good estimate with probabilty \( 1 - \epsilon \) within \( \sqrt{\frac{3}{\epsilon^2 \theta_i} \ln(\frac{\epsilon}{2})} \) of her true relevance.

In a similar way, the advertisers can estimate their valuations. A reasonable way an advertiser can estimate her value is via tracking conversions i.e. which clicks lead to a purchase or an activity of the advertiser’s interest. Let \( x_i \) be the value advertiser \( i \) derives from a single conversion and \( a_i \) be the conversion probability per click and \( Q_i \) be the total number of conversions she obtains in Exp-GSP then she can estimate her value to be \( \frac{Q_i}{a_i} x_i \) per click and using Chernoff-bound as above and union bound we can argue that this estimation is very good. Here \( \hat{f}_i \) is her updated estimate of her relevance using the current phase of Exp-GSP. In reality, it might be difficult to track conversions but it is not clear how can the advertiser estimate without the knowledge of her conversion rate. Further, it is also possible that she derives some values from impressions and clicks even though it does not lead to a conversion. For example, an impression gives some branding value and a click improves her relevance score even when they do not lead to a conversion. In this general case, let \( x_i, x_i^C, x_i^A \) be the values advertiser \( i \) derives from an impression, a click and a conversion respectively then she can estimate her value to be \( \frac{nx_i^I + M_i x_i^C + Q_i x_i^A}{\theta_i f_i} \) per click.
7 Variations of Exp-GSP:

Truthful Implementation and non-separable Click-through rates

Recall from Section 5 that the effective CTR of a bidder $i$ for slot $j$ denoted $\tilde{c}_{i,j}$ is the sum of the expected CTR of bidder $i$ for each of the $n$ steps in Exp-GSP if he would have been ranked $j$ and in a similar way as for $\theta_i$’s we can derive the following lemmas.

**Lemma 13** Let $\beta_i = \sum_{j=1}^{L} c_{i,j}$ then

$$\tilde{c}_{i,m} = \begin{cases} 
\beta_i + d_{i,m} & \text{if } m \leq n \\
nc_{i,m} & \text{if } m > n
\end{cases} \quad (18)$$

where

$$d_{i,m} = \begin{cases} 
(n-L-(m-1))c_{i,L+m} + \\
c_{i,L+1} + c_{i,L+2} + \cdots + c_{i,L+m-1} & \text{if } m \leq L \\
(m-L)c_{i,m} + c_{i,m+1} + \cdots + c_{i,m+L-1} + \\
(n-m-L+1)c_{i,m+L} & \text{if } L \leq m \leq n-L \\
(m-L)c_{i,m} + \\
c_{i,m+1} + \cdots + c_{i,n} & \text{if } m \geq n-L
\end{cases} \quad (19)$$

**Lemma 14** Let $\tilde{K} = \max\{K, n\}$, $n \leq \min\{K+1, K+L\}$, and $L \leq \frac{1}{2}(n-1)$ then for all $1 \leq i \leq N$

$$\tilde{c}_{i,1} > \tilde{c}_{i,2} \cdots > \tilde{c}_{i,\tilde{K}} > 0$$

and $\tilde{c}_{i,j} = 0$ for all $j > \tilde{K}$.

Consider any ranking based mechanism and the corresponding exploration based generalization as described in Section 2 with payment rule modified accordingly then the instances of the two mechanisms are given by $(N, K, (c_{i,j}), (I_i))$ and $(N, K, n, L, (c_{i,j}), (I_i))$ respectively. Therefore, using the Lemmas 13, 14 we can obtain a reduction similar to Theorem 3 for each instance $(N, K, n, L, (c_{i,j}), (I_i))$ of exploration based mechanism with $n \leq K+1$, and $L \leq \frac{1}{2}(n-1)$, there is the instance $(N, \max\{n, K\}, (\tilde{c}_{i,j}), (I_i))$ of corresponding one step mechanism without exploration such that the game induced by $(N, K, n, L, (c_{i,j}), (I_i))$ is equivalent to the game induced by $(N, \max\{n, K\}, (\tilde{c}_{i,j}), (I_i))$, where $\tilde{c}_{i,j}$ is given by the Equations 18, 19. Therefore, we can use all the results from one step mechanism without exploration. In the following we consider two variations of Exp-GSP - (i) for the given ranking mechanism the goal is to design a truthful mechanism and even allowing non-separable CTRs and we do so by introducing a new payment rule and utilizing results from [1] via the above reduction, and (ii) where we restrict ourselves to the same ranking and payment rules but allow CTRs to be non-separable utilizing results from [2] via the above reduction.

It is known that the GSP is not truthful[1,5,8] and clearly this holds true for Exp-GSP as well. And as we mentioned in the Section 1, there is a result 6 with a goal towards implementing a truthful mechanism while learning the CTRs, and to achieve this goal it had to deviate from the current pricing scheme. Our exploration based mechanism described in Section 2 can also be made truthful by changing the payment rule. All the description of the mechanism remains the same except the following:

- The bidders are ranked by $\tilde{q}_i b_i$, where $\tilde{q}_i$ is the quality score the search engines defines for the bidders $i$. For example, usual choices of $\tilde{q}_i$ are search engines’ estimate of $c_{i,1}$ or that of $\sum_{j=1}^{K} c_{i,j}$.
• The bidder $i$ is charged an amount per-click $p_i$ given by,

$$p_i = \sum_{j=1}^{\tilde{K}} \left( \frac{\tilde{c}_{i,j} - \tilde{c}_{i,j+1}}{\tilde{c}_{i,i}} \right) \tilde{q}_{j+1} b_{j+1} \tilde{q}_i.$$  

In spirit of [11], we call this variation of our exploration mechanism as **Exp-Laddered** and it can be proved to be truthful by adopting the proof in [11]. We refer the usual one step truthful mechanism without any exploration to as **Laddered**. Now let us compute the cost of uncertainty in this truthful implementation and as will see below we can obtain a similar upper bound as in Section 4. Let $R_0$ be the expected revenue of the auctioneer for **Laddered** and $R$ be her expected revenue for **Exp-Laddered** then

$$R_0 = \sum_{i=1}^{K} \sum_{j=1}^{K} \left( c_{i,j} - c_{i,j+1} \right) \tilde{q}_{j+1} b_{j+1} \tilde{q}_i$$  

$$R = \sum_{i=1}^{\tilde{K}} \sum_{j=1}^{\tilde{K}} \left( \tilde{c}_{i,j} - \tilde{c}_{i,j+1} \right) \tilde{q}_{j+1} b_{j+1} \tilde{q}_i$$

Performing calculations as in Section 4, we can obtain the following theorem.

**Theorem 15** Let

$$c = \min_{1 \leq i \leq \min\{n,K\}} \min_{i \leq j < n-L} \frac{c_{i,j} + L \cdot c_{i,j+1}}{c_{i,j} - c_{i,j+1} + L}$$  

then the “cost of uncertainty” associated with truthful implementation is upper bounded by

$$\left( 1 - \min\{1, c\} \left( 1 - \frac{2L}{n} \right) \right).$$

Note that the Theorem 15 is consistent with Theorem 8 when we assume CTRs to be separable i.e. $c_{i,j} = \gamma_j e_i$.

Now we consider the variation of **Exp-GSP** where we restrict ourselves to the same ranking and payment rules but allow CTRs to be non-separable. If there were no restrictions on the ranking rule, following [15, 4, 3] we could argue that there would always exist Walrasian equilibria and in particular such an equilibrium where every bidder pays her opportunity cost. This equilibrium is called **MP pricing** equilibrium as at this equilibrium every bidder obtains her marginal product as her payoff. But there exists ranking rules for which there is no **MP pricing** equilibrium [11]. As **Laddered** is unique truthful mechanism given a weighted ranking rule, whenever **MP pricing** equilibrium exists which is compatible with the ranking rule in **Exp-GSP**, every bidder’s payment is the same as in **Exp-Laddered** and therefore the expected revenue of the auctioneer at minimum SNE of **GSP** and **Exp-GSP** are same as for **Laddered** and **Exp-Laddered** respectively. Thus the cost of uncertainty is the same as in the case of truthful implementation and is given by Theorem 15. The existence of Walrasian equilibria (not necessarily the **MP pricing**) can be explicitly proven for the ranking used in **Exp-GSP** utilizing the results from [2], but unfortunately it does not have a nice analytical form unlike in the separable CTRs case or in the truthful case and analytical computation of cost of uncertainty does not seem feasible. However, intuition from the earlier section indicates that similar results should hold as in Section 4.

It is clear that the estimation results from Section 6 can easily be extended for both the variations of **Exp-GSP** discussed above and we omit the detailed discussion.
8 Concluding remarks

We proposed a deterministic exploration mechanism to learn the relevance scores by deviating minimally from the mechanism being currently used by Google and Yahoo! in the sense that it retains the same pricing rule, as well as, similar ranking scheme. We show that such an exploration policy can be incorporated without any significant loss in revenue for the auctioneer. An independent work reported in [18] introduces a randomized exploratory mechanism and analyzes its incentive properties. We demonstrate that the mathematical structure of the payoffs in our proposed exploratory mechanism (EXP-GSP) is identical to that in the standard mechanism (i.e., without exploration), allowing us to compare and contrast the various metrics at the corresponding SNES. We show that while the actual bid profiles of Exp-GSP and GSP may differ at the corresponding SNES, the macroscopic measures, such as revenue, efficiency etc. do not differ significantly, allowing auctioneers to limit the cost of uncertainty. The approach in [18], on the other hand, centers around showing that both the mechanisms (i.e., the standard GSP and the proposed exploratory randomized mechanism ) would share almost-identical equilibrium bid profiles; of course, the auctioneer still pays a price for learning the quality factors (as in our case). These two different approaches to the design of exploratory mechanisms raise an important topic for future work: what other exploratory mechanisms can one design, and are their lower bounds on the cost or price of uncertainty? That is, can one design mechanisms that have the optimal characteristics when it comes to revenue loss vs. the information gathered about quality factors and valuations. Clearly, more work is necessary and more mechanisms such as those proposed herein and in [18] need to be studied.

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Appendix

Proof of Lemma 3: Let $m < L$, then

$$d_m = (n - L - (m - 1))\gamma_{L+m} + \gamma_{L+1} + \gamma_{L+2} + \cdots + \gamma_{L+m-1}$$

$$d_{m+1} = (n - L - m)\gamma_{L+m+1} + \gamma_{L+1} + \gamma_{L+2} + \cdots + \gamma_{L+m}$$

$$\therefore d_m - d_{m+1} = (n - L - m)(\gamma_{m+L} - \gamma_{m+1+L})$$

As we have $\gamma_j > \gamma_{j+1}$ for all $1 \leq j \leq K$, we get

$$d_m > d_{m+1}$$

whenever $m < n - L$ and $m \leq K - L$ and therefore we have

$$d_1 > d_2 > \cdots > d_{L-1} > d_L$$

whenever $L \leq \frac{1}{2} \min \{n, K + 1\}$.

For $L \leq m < n - L$,

$$d_m = (m - L)\gamma_m + \gamma_{m+1} + \cdots + \gamma_{m+L-1} + (n - m - L + 1)\gamma_{m+L}$$

$$d_{m+1} = (m + 1 - L)\gamma_{m+1} + \gamma_{m+2} + \cdots + \gamma_{m+L} + (n - m - L)\gamma_{m+L+1}$$

$$\therefore d_m - d_{m+1} = (m - L)(\gamma_m - \gamma_{m+1}) + (n - m - L)(\gamma_{m+L} - \gamma_{m+1+L})$$

$$d_L - d_{L+1} = (n - 2L)(\gamma_{2L} - \gamma_{2L+1})$$

$$> 0 \text{ whenever } n > 2L \text{ and } 2L \leq K.$$

For, $L < m < n - L$, clearly $(n - m - L)(\gamma_{m+L} - \gamma_{m+1+L}) \geq 0$, and $(m - L)(\gamma_m - \gamma_{m+1}) > 0$ whenever $m \leq K$ and therefore $d_m > d_{m+1}$ whenever $n \leq K + L + 1$.

$$\therefore d_L > d_{L+1} > \cdots > d_{n-L}$$

whenever $L \leq \frac{1}{2} \min \{n - 1, K\}$ and $n \leq K + L + 1$. 

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Further, for \( n - L \leq m \leq n - 1 \),
\[
\begin{align*}
d_m &= (m - L)\gamma_m + \gamma_{m+1} + \cdots + \gamma_n \\
d_{m+1} &= (m + 1 - L)\gamma_{m+1} + \gamma_{m+2} + \cdots + \gamma_n \\
d_m - d_{m+1} &= (m - L)(\gamma_m - \gamma_{m+1}) \\
\therefore d_m > d_{m+1} &\text{ whenever } m \leq K \\
\therefore d_{n-L} > d_{n-L+1} > \cdots > d_n &\text{ whenever } n \leq K + 1.
\end{align*}
\]

Combining the above relations and noting that \( \theta_j = \gamma + d_j \) for all \( 1 \leq j \leq n \), we obtain
\[
\theta_j > \theta_{j+1} \text{ for all } 1 \leq j \leq n - 1
\]
whenever \( L \leq \frac{1}{2} \min\{n - 1, K\} \) and \( n \leq K + 1 \).

Now, \( \theta_n - \theta_{n+1} = \gamma + (n - L)\gamma_n - n\gamma_{n+1} > 0 \) whenever \( L > 0 \) or \( n \leq K \) and for \( j > n \), \( \theta_j - \theta_{j+1} = n(\gamma_j - \gamma_{j+1}) > 0 \) whenever \( j \leq K \) and \( \theta_j - \theta_{j+1} \) is 0 otherwise. This completes the proof.

**Proof of Theorem**

Now from proof of Lemma 2 we can observe that
\[
\theta_j - \theta_{j+1} = \begin{cases} (n - j - L)(\gamma_{j+L} - \gamma_{j+1+L}) & : j < L \\ (j - L)(\gamma_j - \gamma_{j+1}) + (n - j - L)(\gamma_{j+L} - \gamma_{j+1+L}) & : L \leq j < n - L \\ (j - L)(\gamma_j - \gamma_{j+1}) + (n - L)(\gamma_n - \gamma_{n+1}) & : j = n \\ n(\gamma_j - \gamma_{j+1}) & : j > n \end{cases}
\]

\[
\therefore \frac{\theta_j - \theta_{j+1}}{n(\gamma_j - \gamma_{j+1})} = \begin{cases} (1 - \frac{j+L}{n})(\frac{\gamma_{j+L} - \gamma_{j+1+L}}{\gamma_j - \gamma_{j+1}}) & : j < L \\ \frac{j - L}{n} + (1 - \frac{j+L}{n})(\frac{\gamma_{j+L} - \gamma_{j+1+L}}{\gamma_j - \gamma_{j+1}}) & : L \leq j < n - L \\ \frac{1}{n} (j - L) & : n - L \leq j < n \\ \frac{1}{n} (\gamma_n - \gamma_{n+1}) + (1 - \frac{L}{n}) & : j = n \\ 1 & : n < j \leq K \end{cases}
\]

Let
\[
c = \min_{1 \leq j < n-L} \frac{\gamma_{j+L} - \gamma_{j+1+L}}{\gamma_j - \gamma_{j+1}}
\]

then
\[
\frac{\theta_j - \theta_{j+1}}{n(\gamma_j - \gamma_{j+1})} \geq \begin{cases} (1 - \frac{j+L}{n})c & : j < L \\ \frac{j - L}{n} + (1 - \frac{j+L}{n})c & : L \leq j < n - L \\ \frac{1}{n} (j - L) & : n - L \leq j < n \\ 1 - \frac{L}{n} & : j = n \\ 1 & : n < j \leq K \end{cases}
\]

16
\[
\begin{align*}
&\left\{ \begin{array}{ll}
(1 - \frac{2L}{n})c & ; j < L \\
(1 - \frac{2L}{n}) \min\{1, c\} & ; L \leq j < n - L \\
(1 - \frac{2L}{n}) & ; n - L \leq j < n \\
1 - \frac{L}{n} & ; j = n \\
n & ; n < j \leq K
\end{array} \right.
\end{align*}
\]

Therefore, for all \(1 \leq j \leq K\), we have
\[
(\gamma_j - \gamma_{j+1}) - \frac{1}{n}(\theta_j - \theta_{j+1}) 
\leq (1 - \min\{1, c\}(1 - \frac{2L}{n})) (\gamma_j - \gamma_{j+1}).
\]

\[
\therefore R_0 - \frac{1}{n}R = 
\sum_{s=1}^{K} \sum_{j=s}^{K} \frac{e_s}{q_s} \left[ (\gamma_j - \gamma_{j+1}) - \frac{1}{n}(\theta_j - \theta_{j+1}) \right] q_{j+1}v_{j+1}
\leq \sum_{s=1}^{\min\{n,K\}} \sum_{j=s}^{\min\{n,K\}} \frac{e_s}{q_s} \left(1 - \min\{1, c\}(1 - \frac{2L}{n})\right) (\gamma_j - \gamma_{j+1})q_{j+1}v_{j+1}
\leq (1 - \min\{1, c\}(1 - \frac{2L}{n})) R_{0}^{\min\{n,K\}},
\]

where \(R_l^l\) denotes the revenue of auctioneer from top \(l\) bidders in GSP
\[
\therefore \frac{R_0 - \frac{1}{n}R}{R_0} \leq (1 - \min\{1, c\}(1 - \frac{2L}{n})) \left( \frac{R_{0}^{\min\{n,K\}}}{R_0} \right)
\leq (1 - \min\{1, c\}(1 - \frac{2L}{n})).
\]

**Proof of Theorem 10:**
Using Lemma 9 we have,
\[
E_0 - \frac{1}{n}E = \sum_{m=1}^{K} \gamma_m e_m v_m - \frac{1}{n} \sum_{m=1}^{K} \gamma_m y_m
= \sum_{m=1}^{K} \gamma_m e_m v_m \left(1 - \frac{1}{n} \frac{y_m}{e_m v_m}\right).
\]

Let us first assume that \(e_m v_m \geq e_{m+1} v_{m+1}\) for all \(1 \leq m \leq n\). (25)
For $m \leq L$, we have
\[
\frac{1}{n} y_m = \frac{1}{n} \sum_{i=1}^{n} e_i v_i
\]
\[
\geq \frac{1}{n} \sum_{i=1}^{n} e_i v_i
\]
\[
\therefore 1 - \frac{1}{n} y_m \leq (1 - \alpha)
\]
where \(\alpha = \frac{1}{n} \sum_{i=1}^{n} e_i v_i\).

For $L < m \leq n$,
\[
\frac{1}{n} y_m = \frac{1}{n} \left[ (n - m + 1) \left( \frac{e_{m-L} v_m - L}{e_m v_m} \right) + \sum_{i=m-L+1}^{m-1} \left( \frac{e_i v_i}{e_m v_m} \right) + (m - L) \right]
\]
\[
\geq \frac{1}{n} \left[ (n - m + L) \left( \frac{e_{m-L} v_m - L}{e_m v_m} \right) + (m - L) \right]
\]
\[
\therefore 1 - \frac{1}{n} y_m \leq \frac{1}{n} \left[ (n - m + L) \left( 1 - \frac{e_{m-L} v_m - L}{e_m v_m} \right) \right]
\]
\[
= -\frac{1}{n} \left[ (n - m + L) \left( \frac{e_{m-L} v_m - L}{e_m v_m} - 1 \right) \right]
\]
\[
\leq -\frac{L}{n} \left( \frac{e_{m-L} v_m - L}{e_m v_m} - 1 \right)
\]
where
\[
\omega = \min_{L < m \leq n} \left( \frac{e_{m-L} v_m - L}{e_m v_m} - 1 \right).
\]

For $n < m \leq K$,
\[
\frac{1}{n} y_m = 1.
\]

Therefore,
\[
E_0 - \frac{1}{n} E = \sum_{m=1}^{L} \gamma_m e_m v_m \left( 1 - \frac{1}{n} y_m \right)
\]
\[
+ \sum_{m=L+1}^{n} \gamma_m e_m v_m \left( 1 - \frac{1}{n} y_m \right)
\]
\[
+ \sum_{m=n+1}^{K} \gamma_m e_m v_m \left( 1 - \frac{1}{n} y_m \right)
\]
\[
\leq (1 - \alpha) \sum_{m=1}^{L} \gamma_m e_m v_m - \frac{L}{n} \omega \sum_{m=L+1}^{n} \gamma_m e_m v_m
\]
\[
= (1 - \alpha) E_0 - \frac{L}{n} \omega E_0^{ne}
\]
where $E_0^e = \sum_{i=1}^{L} \gamma_m e_m v_m$ and $E_0^{ne} = \sum_{i=L+1}^{n} \gamma_m e_m v_m$.

But it might be the case that the Equation 25 does not hold. In this case, we have for $L < m \leq n$,

$$1 - \frac{1}{n} \frac{y_m}{e_m v_m} \leq (1 - \beta)$$

where $\beta = \frac{1}{n} \frac{\sum_{i=1}^{n} e_i v_i}{\max_{1 \leq m \leq L} e_m v_m}$

and for $L < m \leq n$,

$$1 - \frac{1}{n} \frac{y_m}{e_m v_m} \leq \eta$$

where $\eta = \max_{L < m \leq n} \left\{ \max_{m-L \leq i \leq m} \left( 1 - \frac{e_i v_i}{e_m v_m} \right) \right\}$.

\[ \therefore E_0 - \frac{1}{n} E \leq (1 - \beta) E_0^e + \eta E_0^{ne} \]