A REMARK ON BEREZIN’S QUANTIZATION AND CUT LOCUS

Stefan Berceanu
Institute for Physics and Nuclear Engineering
Department of Theoretical Physics
PO BOX MG-6, Bucharest-Magurele, Romania
E-mail: Berceanu@theor1.ifa.ro; Berceanu@Roifa.Ifa.Ro

Abstract
The consequences for Berezin’s quantization on symmetric spaces of the identity of the set of coherent vectors orthogonal to a fixed one with the cut locus are stated precisely. It is shown that functions expressing the coherent states, the covariant symbols of operators, the diastasis function, the characteristic and two-point functions are defined when one variable does not belong to the cut locus of the other one.

1. INTRODUCTION
Recently I have emphasized [1] the deep relationship between geodesics and coherent states [2]. For homogeneous manifolds in which the exponential from the Lie algebra to the Lie group equals the geodesic exponential, and in particular for symmetric spaces, it was proved that the cut locus (CL) of a fixed point in the manifold is equal to the so called polar divisor (denoted Σ), i.e. the set of coherent vectors orthogonal to the coherent vector corresponding to the fixed point. The coherent states are a powerful tool in global differential geometry and algebraic geometry [3].

In this talk I comment the physical relevance of the result \( CL_0 = Σ_0 \). The coherent states offer a straightforward recipe [4, 5, 6] for geometric quantization [7]. Mainly, I point out the consequences of the results obtained in Ref. [1] for Berezin’s quantization, especially for symmetric spaces (see firstly the second reference in [8]). Also I clear up the significance of the identity \( CL_0 = Σ_0 \) for the works of Onofri [9], Rawnsley [4], Moreno [10] and Cahen, Gutt and Rawnsley [11].

The main message of the present contribution is that on symmetric spaces the cut locus is present everywhere one speaks about coherent states. Indeed, the functions expressing the coherent states, the covariant symbols of operators, the diastasis function, the characteristic and two-point functions are defined at a fixed \( x \) for \( y \) not in \( CL_x \).

The paper is laid out as follows: some mathematical results are collected in §2. The notation in §3 for the coherent states differs from that used earlier in Refs. [12, 13]. The main results of Ref. [1] on the relationship between coherent states and cut locus are briefly recalled in §4, while the last Section is devoted to the announced comments. I have also included in §5 a remark on the significance of the polar divisor as divisor in the meaning of algebraic geometry [14], not exhibited in my talk, which was generated by a discussion during the Workshop. An illustration of all presented results on the complex Grassmann manifold \( G_n(C^{m+n}) \) is available in Ref. [15].
2. MATHEMATICAL PRELIMINARIES

1). Let $\chi$ be a representation of the group $K$ on the Hilbert space $\mathcal{K}$ and let us consider the principal bundle

$$K \ni g \leadsto G \ni \tilde{M},$$

where $\tilde{M}$ is diffeomorphic with $G/K$, $i$ is the inclusion and $\lambda$ is the natural projection $\lambda(g) = gK$. We recall [10] the definition of the $G$-homogeneous vector bundle $M_\chi$ associated by the character $\chi$ to the principal $K$-bundle (2.1): $M_\chi := \tilde{M} \times_K \mathcal{K}$, or simply $M := \tilde{M} \times_K \mathcal{K}$. The total space of $M_\chi$ consists of all equivalence classes $[g, l]$ of elements $(g, l)$ under the equivalence relation $(gp, l) \sim (g, \chi(p)l), \ g \in G, \ p \in K, \ l \in \mathcal{K}$. If $U \subset \tilde{M}$ is open, let the notation

$$(G)^U = \{ g \in G | \pi(g)\psi_0 \in U \} ,$$

where $\pi$ is a representation of $G$ whose restriction to $K$ is $\chi$ and $\psi_0 \in \mathcal{K}$ corresponds to the base point $o \in \tilde{M}$. Then the continuous sections of $M_\chi$ over $U$ are precisely the continuous maps $\sigma : U \rightarrow G \times_K \mathcal{K}$ of the form

$$\sigma(\pi(g)\psi_0) = [g, e_\sigma(g)], \ e_\sigma : (G)^U \rightarrow \mathcal{K},$$

where $e_\sigma$ satisfies the “functional equation”:

$$e_\sigma(gp) = \chi(p)^{-1}e_\sigma(g), \ g \in (G)^U, p \in K. \quad (2.4)$$

For homogeneous holomorphic line bundles [17] ($\mathcal{K} = \mathbb{C}$) the functions in eq. (2.4) are holomorphic.

2). Borel-Weil theorem (cf. Ref. [13]) For every irreducible representation $\pi_j$ of dominant weight $j$ of the compact connected semisimple Lie group $G$ corresponds on every homogenous Kählerian space $G/K \simeq G^c/P_j$ a complete linear system $|D|$. The representation space $\mathcal{K}_j$ of the representation $\pi_j$ is the dual of $\mathcal{L}(D)$. The associated line bundle $M'$ is ample iff the space $G/K \simeq G^c/P_j$ is strictly associated to the representation $\tilde{\pi}_j$. Here $G^c$ is the complexification of $G$, $P_j$ is the parabolic group corresponding to the dominant weight $j$ of the representation $\pi_j$ and $|D| \simeq P(\mathcal{L}(D))$.

3). The following theorem summarises some properties of flag manifolds with significance for the present paper [19]. Let $X = G^c/P$ be a complex manifold, where $G^c$ is a complex semisimple Lie group and $P$ is a parabolic subgroup. The following conditions are equivalent: a) $X = G^c/P$ is compact; b) $X$ is a compact simply connected Kähler manifold; c) $X$ is a projective variety; d) $X$ is a closed $G^c$-orbit in a projective representation; e) $X$ is a Hodge manifold and all homogeneous Hodge manifolds are of this type.

4). A holomorphic line bundle $M'$ on a compact complex manifold $\tilde{M}$ is said very ample [20] if: a) the set of divisors is without base points, i.e. there exists a finite set of global sections $s_1, \ldots, s_N \in \Gamma(\tilde{M}, M')$ such that for each $m \in \tilde{M}$ at least one $s_j(m)$ is not zero; b) the holomorphic map $\iota_{M'} : \tilde{M} \hookrightarrow \mathbb{C}P^{N-1}$ given by

$$\iota_{M'} = [s_1(m), \ldots, s_N(m)] \quad (2.5)$$

is a holomorphic embedding.

The line bundle $M'$ is said to be ample if there exists a positive integer $r_0$ such that $M'^r$ is very ample for all $r \geq r_0$. Note that if $M'$ is an ample line bundle on $\tilde{M}$, then $\tilde{M}$ must be projective-algebraic by Chow’s theorem, hence $\tilde{M}$ is Kähler.
The concepts of ampleness and positivity for line bundles coincide. The following theorem summarises the properties of ample line bundles that are needed in this paper [20, 16, 14]. Below [1] denotes the hyperplane line bundle on the projective space \( \mathbb{P}(\mathcal{H}) \) and the \( C \)-spaces are the simply connected compact homogeneous manifolds.

Let \( \mathcal{M}' \) be a holomorphic line bundle on a compact complex manifold \( \widetilde{\mathcal{M}} \) of complex dimension \( D \). The following conditions are equivalent: a) \( \mathcal{M}' \) is positive; b) the zero section of \( \mathcal{M}'^* \) can be blown down to a point; c) for all coherent analytic sheaves \( \mathcal{S} \) on \( \widetilde{\mathcal{M}} \) there exists a positive integer \( m_0(\mathcal{S}) \) such that \( H^i(\widetilde{\mathcal{M}}, \mathcal{S} \otimes \mathcal{M}'^m) = 0 \) for \( i > 0, m \geq m_0(\mathcal{S}) \) (the vanishing theorem of Kodaira); d) there exists a positive integer \( m_0 \) such that for all \( m \geq m_0 \), there is an embedding \( \iota_{\mathcal{M}} : \widetilde{\mathcal{M}} \hookrightarrow \mathbb{C}P^{N-1} \) for some \( N \geq D \) such that \( \mathcal{M} = \mathcal{M}'^m \) is projectively induced, i.e. \( \mathcal{M} = \iota^*[1] \); e) \( \widetilde{\mathcal{M}} \) is a Hodge manifold (the embedding theorem of Kodaira); f) the fundamental two-form of \( \widetilde{\mathcal{M}} \), the curvature matrix and the first Chern class of \( \mathcal{M}' \) are related by the relations \( \omega = \sqrt{-1}/2\Theta_{\mathcal{M}'}, c_1(\mathcal{M}') = \omega/\pi \); g) moreover, if \( \widetilde{\mathcal{M}} \) is a Kählerian \( C \)-space, then \( \widetilde{\mathcal{M}} \) is a flag manifold.

5). We shall be concerned with manifolds \( \widetilde{\mathcal{M}} \) which admit an embedding in some projective Hilbert space \( \iota : \widetilde{\mathcal{M}} \hookrightarrow \mathbb{P}(\mathcal{H}). \) (2.6)

In this paper we shall restrict ourselves to biholomorphic embeddings \( \iota \). Because \( \iota \) in formula (2.6) is injective and holomorphic, then it is a kählerian embedding, i.e.

\[ \omega_{\widetilde{\mathcal{M}}} = \iota^* \omega(\mathcal{H}), \] (2.7)

where \( \omega \) is the fundamental two-form (i.e. closed, (strongly) non-degenerate) of the Kähler manifold and \( \iota^* \) is the pull-back of the mapping \( \iota \). Equivalently, \( \iota \) is an isometric embedding.

3. THE COHERENT STATE AND COHERENT VECTOR MANIFOLDS

Let \( \xi : \mathcal{H}^* = \mathcal{H} \setminus \{0\} \to \mathbb{P}(\mathcal{H}), \xi(z) = [z] \) be the mapping which associates to the point \( z \) in the punctured Hilbert space the linear subspace \( [z] \) generated by \( z \), where \( [\lambda z] = [z], \lambda \in \mathbb{C}^* \). The hermitian scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H} \) is linear in the second argument.

Let us consider the principal bundle (2.1) and let us suppose the existence of a map \( e : G \to \mathcal{H}^* \) as in eq. (2.3) with the property (2.4) but globally defined, i.e. on the neighbourhood (2.2) \( (G)\widetilde{\mathcal{M}} \). Then \( e(G) \) is called family of coherent vectors [4]. If there is a morphism of principal bundles [21], i.e. the following diagram is commutative,

\[
\begin{array}{ccc}
G & \xrightarrow{e} & \mathcal{H}^* \\
\lambda \downarrow & & \downarrow \xi \\
\widetilde{\mathcal{M}} & \xrightarrow{\iota} & \mathbb{P}(\mathcal{H})
\end{array}
\] (3.1)

then \( \iota(\widetilde{\mathcal{M}}) \) is called family of coherent states corresponding to the family of coherent vectors \( e(G) \) [4].

We restrict ourselves to the case where the mapping \( \iota \) is an embedding of the homogeneous manifold \( \widetilde{\mathcal{M}} \) [22]. The manifold \( \widetilde{\mathcal{M}} \) is called coherent state manifold and the \( G \)-homogeneous line bundle \( \mathcal{M}_\chi \) is called coherent vector manifold [12].

Let now \( \tilde{\pi} \) be a projective (in physical literature [23] “ray”) representation associated to the unitary irreducible representation \( \pi \) and \( G \) the group of transformations
which leaves invariant the transition probabilities in the complex separable Hilbert space $\mathcal{H}$. If we use the projection $\xi' = \xi|\mathcal{S}(\mathcal{H})$; i.e. $\xi' : \mathcal{S}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$, $\xi'(\psi) = \tilde{\psi} = \{e^{i\phi}\psi | \phi \in \mathbb{R}\}$, where $\mathcal{S}(\mathcal{H})$ is the unit sphere in $\mathcal{H}$, then $\tilde{\pi} \circ \xi' = \xi' \circ \pi$.

The triplet $(\tilde{\pi}, \tilde{G}, \mathcal{H})$ is a quantum system with symmetry in the sense of Wigner and Bargmann [24, 23]. Then the manifold $\tilde{M} \approx G/K$ can be realized as the orbit $\tilde{M} = \{\tilde{\pi}(g)e_0 | g \in G\}$, where $K$ is the stationary group of $e_0$ and $e_0 \in \mathcal{H}^\ast$ is fixed. For a compact connected simply connected Lie group $G$, the existence of the representation $\tilde{\pi}$ implies the existence of the unitary irreducible representation $\pi$ (cf. the theorem of Wigner and Bargmann [24, 23]). This implies the existence of cross sections $\sigma : \tilde{M} \to \mathcal{S}(\mathcal{H})$. However, the (Hopf) principal bundle $[\pi, \xi, \mathcal{P}(\mathcal{H})]$ is a $U(1)$-bundle and in the construction of coherent vector manifold we need line bundles. But the principal line bundle $\xi'$ is obtained from the (tautological) line bundle $[-1] = \xi = (\mathcal{H}^\ast, \xi, \mathcal{P}(\mathcal{H}))$ reducing the group structure from $\tilde{G}$ to $G$.

Here we also stress that the theorem of Wigner and Bargmann is essentially [25] the (first) fundamental theorem of projective geometry [26].

In order to have the physical interpretation of the “classical system” obtained by dequantizing the quantum one [24, 27], we restrict to Kähler manifolds $\tilde{M}$. For example, for a compact connected simply connected Lie group $G$, $\tilde{M} \approx G/K \approx G^c/P$ is a Kähler manifold and the Borel-Weil theorem assures the geometrical realisation of the (first) fundamental theorem of projective geometry [26].

The representation $\pi_j$ can be uniquely extended to the group homomorphism $\pi_j^* : G^c \to \pi_j^*(G^c)$, and respectively, Lie algebra isomorphism $\pi_j^* : \mathfrak{g}^c \to \pi_j^*(\mathfrak{g}^c)$ by

$$\pi_j^*(\exp(Z)) = e^{\pi_j^*(Z)}, Z \in \mathfrak{g}^c, \quad (3.2)$$

where $\exp : \mathfrak{g}^c \to G^c$ and $e : \pi_j^* \to \pi_j^*(G)$ are exponential maps, while $\pi_j^*$ is the complexification of the Lie algebra $\pi_j(\mathfrak{g})$. We use also the notation $F_{\alpha} = \pi_j^*(f_{\alpha})$, where $\alpha$ is in the set $\Delta$ of the roots of the Lie algebra $\mathfrak{g}$ of $G$ with generators $f_{\alpha}$ of the Cartan-Weyl base of $\mathfrak{g}^c$ (see also [13]).

Then $e_g := e(g) := \pi_j^*(g)e_0, g \in G^c$ is the family of coherent vectors, while $\{e\}_{g \in G^c}$ is the family of coherent states. The relation $e_g = e^{i\alpha(g)}e_{\alpha(g)}$ defines a fibre bundle with base $\tilde{M}$ and fibre $U(1)$ [22]. More precisely, the function

$$\Upsilon(g) = (\Upsilon, e_g) \quad (3.3)$$

is holomorphic on $G^c$ and defines holomorphic sections on the homogeneous holomorphic line bundle $M'$ associated to the principal line bundle $P \to G^c \to G^c/P$ by the holomorphic character $\chi$

$$\pi^*(p)e_0 = \chi^{-1}(p)e_0, \quad p \in P, \quad \chi(p) = e^{-i\alpha(p)}. \quad (3.4)$$

Indeed, the function $\Upsilon(g)$ verifies $\Upsilon(gp) = \chi^{-1}(p)\Upsilon(g), g \in G^c, p \in G$, i.e. eq. (3.3), and the corresponding holomorphic sections are associated via eq. (3.4).

Let also the function

$$\Upsilon'(g) = \Upsilon'(gP) := \frac{\Upsilon(g)}{(e_0, e_g)}, \quad (3.5)$$

defined on the set

$$(e_0, e_g) \neq 0. \quad (3.6)$$

Then

$$\Upsilon' : \mathcal{V}_0 \to \mathbb{C}, \quad \Upsilon'(Z) = (\Upsilon, e_{\pi_j^*}), \quad (3.7)$$
where the Perelomov’s coherent vectors are
\[ e_{Z,j} = \exp \sum_{\varphi \in \Delta^+} (Z_\varphi F_\varphi^+) j, \quad e_{Z,j} = (e_{Z,j}, e_{Z,j})^{-1/2} e_{Z,j}, \quad (3.8) \]
\[ e_{B,j} = \exp \sum_{\varphi \in \Delta^+} (B_\varphi F_\varphi^+ - B_\varphi F_\varphi^-) j, \quad e_{B,j} = e_{Z,j}. \quad (3.9) \]
Here \( \Delta^+ \) denotes the positive non-compact roots, \( Z := (Z_\varphi) \in \mathbb{C}^D \) are local coordinates in the maximal neighbourhood \( \mathcal{V}_0 \subset \tilde{M} \). In eqs. (3.8), (3.9) \( F_\varphi^+ j \neq 0, F_\varphi^- j = 0, \varphi \in \Delta^+ \).

The system \( \{e_g\}, g \in G^C \) is overcomplete \[ 8, 22, 9 \] and \( (e_g, e_{g'}) \), up to a factor, is a reproducing kernel for the holomorphic vector bundle \( \xi_0 : M \rightarrow \tilde{M} [28] \).

4. CUT LOCUS AND COHERENT STATES

Let \( X \) be complete Riemannian manifold. The point \( q \) is in the cut locus \( \text{CL}_p \) of \( p \in X \) if \( q \) is the nearest point to \( p \) on the geodesic emanating from \( p \) beyond which the geodesic ceases to minimize his arc length (cf. [29], see also Ref. [1] for more references).

**Remark 1** \( \text{codim}_C \text{CL}_p \geq 1 \).

We call polar divisor of \( e_0 \) the set \( \Sigma_0 = \{e \in e(G)| (e_0, e) = 0\} \). This denomination is inspired after Wu [30], who used this term in the case of the complex Grassmann manifold \( G_n(\mathbb{C}^{m+n}) \).

Let \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \) be the orthogonal decomposition of \( \mathfrak{g} \) with respect to the \( B \)-form, \( \text{Exp}_p : M_p \rightarrow \tilde{M} \) the the geodesic exponential map and \( o = \lambda(e)_0 \), where \( e \) is the unit element in \( G \). Then \( \mathfrak{m} \) is identified with the tangent space at \( o, M_o, \) and \( \tilde{M} \approx \exp \mathfrak{m} \).

Let us consider the following two conditions
A) \( \text{Exp}_o = \lambda \circ \exp |\mathfrak{m}| \).
B) On the Lie algebra \( \mathfrak{g} \) of \( G \) there exists an \( \text{Ad}(G) \)-invariant, symmetric, non-degenerate bilinear form \( B \) such that the restriction of \( B \) to the Lie algebra \( \mathfrak{t} \) of \( K \) is likewise non-degenerate.

Note that the symmetric spaces have property A) and if \( \tilde{M} \approx G/K \) verifies B), then it also verifies A) (cf. [29]).

In [1] it was proved the following

**Theorem 1** Let \( \tilde{M} \) be a homogeneous manifold \( \tilde{M} \approx G/K \). Suppose that there exists a unitary irreducible representation \( \pi_j \) of \( G \) such that in a neighbourhood \( \mathcal{V}_0 \) around \( Z = 0 \) the coherent states are parametrized as in eq. (3.8). Then the manifold \( \tilde{M} \) can be represented as the disjoint union
\[ \tilde{M} = \mathcal{V}_0 \cup \Sigma_0. \quad (4.1) \]
Moreover, if the condition B) is true, then
\[ \Sigma_0 = \text{CL}_0. \quad (4.2) \]

**Corollary 1** Suppose that \( \tilde{M} \) verifies B) and admits the embedding (2.6). Let \( 0, Z \in \tilde{M} \). Then \( Z \in \text{CL}_0 \) iff the Cayley distance between the images \( \iota(0), \iota(Z) \in \mathbb{P}(\mathfrak{H}) \) is \( \pi/2 \)
\[ d_c(\iota(0), \iota(Z)) = \pi/2. \quad (4.3) \]

Here \( d_c \) denotes the the hermitian elliptic Cayley distance on the projective space
\[ d_c(\omega', \omega) = \arccos \frac{|(\omega', \omega)|}{||\omega'||||\omega||}. \quad (4.4) \]
5. DISCUSSION

1. We now state precisely the consequences of theorem 1 for Berezin’s quantization on symmetric compact complex manifolds (see firstly the second reference in [3]).

Expressing the supercompleteness of the system of coherent vectors \( \{ e_q \} \) by the Parseval identity, Berezin introduces the Hilbert space \( \mathcal{F}_h \) of holomorphic functions \([3.7]\) on \( \mathcal{V}_0 \), denoted \( \mathcal{V}(z) \), with the scalar product

\[
(\mathcal{V}_1, \mathcal{V}_2) = \hat{c}(h) \int_{\mathcal{V}_0} \mathcal{V}_1(z) \overline{\mathcal{V}_2(z)} F^\dagger(z, \overline{z}) d\mu(z, \overline{z}),
\]

(5.1)

Here \( - \ln F \) is the Kähler potential on \( \tilde{\mathcal{M}} \), \( d\mu(z, \overline{z}) = \pi^{-n} F(z, \overline{z}) d\mu_L(z, \overline{z}) \), and \( \mu_L \) denotes the Lebesgue measure on \( \tilde{\mathcal{M}} \). In Berezin’s terminology, \( h \) belongs to the admissible set, \( z \) are special coordinates on \( \mathcal{V}_0 \subset \tilde{\mathcal{M}} \) and \( z = 0 \) is a distinguished point.

Here we just comment that, due to Remark 1, the integration in eq. (5.1) can be extended to all \( \tilde{\mathcal{M}} \).

In Berezin’s formulation, the quantization algebra \( \mathfrak{A} \), which is a special quantization with correspondence principle in the weak form, is restricted also only on \( \mathcal{V}_0 \). We remember Berezin’s definition of the *-product of covariant symbols

\[
(A_1 * A_2)(z, \overline{z}) = \int A_1(z, \overline{z}) A_2(v, \overline{z}) G_h(z, \overline{z}|v, \overline{v}) d\mu(v, \overline{v}),
\]

(5.2)

where the kernel and the covariant symbol attached to the operator \( \hat{A} \) are, respectively

\[
G_h(z, \overline{z}|v, \overline{v}) = c(h) \frac{\mathcal{V}_1(z) \mathcal{V}_2(\overline{z})}{\mathcal{V}_1(\overline{z}) \mathcal{V}_2(z)},
\]

(5.3)

\[
A(z, \overline{z}) = \left( \frac{\hat{A} \mathcal{V}_1, \mathcal{V}_2}{\mathcal{V}_1, \mathcal{V}_2} \right).
\]

(5.4)

Here \( \mathcal{V}_1(z) = L_h(z, \overline{v}) = F^{-1/h}(z, \overline{v}) \) and \( (f, \mathcal{V}_1) = f(v) \). If \( f_k(z) \) is an orthonormal basis in \( \mathcal{F}_h \), then \( L_h(z, \overline{v}) = \sum f_k(z) \overline{f}_k(v) \), and \( L_h \) is the kernel of the (Bergman) projector \( P_B : L^2(F^{-1/h} d\mu) \rightarrow \mathcal{F}_h : (P_B) f(z) = \int L_h(z, \overline{v}) f(v) \overline{f}(v, \overline{v}) d\mu(v, \overline{v}). \) We can explicitly write down the domain of definition of the functions \( \mathcal{V}_1(z) \in \mathcal{F}_h, \) at fixed \( v, z \notin \mathbb{C} \mathcal{L}_v \).

2. Referring to Onofri’s paper [9] (see also the papers of Rawnsley [1] and Moreno [10], we also comment that, due to eqs. (3.6), (3.7) and (4.1), the functions expressing the coherent states are not defined on all the manifold and their domain of definition has the geometrical significance given by the relation (4.2) in theorem 1.

3. Cahen, Gutt and Rawnsley [11] formulated globally Berezin’s construction of covariant symbols of operators in terms of sections of the prequantization bundle \( (\mathcal{M}, h, \nabla) \) of the Kähler manifold \( (\tilde{\mathcal{M}}, \omega) \). Berezin’s definition (5.2) of the *-product is modified by the presence of the function \( \epsilon \) as

\[
(A_1 * \epsilon_k A_2)(x) = \int_{\tilde{\mathcal{M}}} A_1(x, y) A_2(y, x) \Psi^k(x, y) e(k) \frac{k^n \omega^n}{n!}.
\]

(5.5)

We remember the notation. Let \( q \in \mathcal{M}_x = \xi_0^{-1}(x) \) be a fixed frame field over \( \tilde{\mathcal{M}} \) and the holomorphic section \( s \in \Gamma(\tilde{\mathcal{M}}, \mathcal{M}) \). The evaluation of section at \( x \) gives \( s(x) = \mathcal{V}_q(s)q \) and the continuous coordinate function corresponds to \( \mathcal{V} \in \mathcal{F}_h \) in Berezin’s notation. The unique element \( e_q \in \mathcal{H} \) determined by Riesz theorem from the relation \( \mathcal{V}_q(s) = (s, e_q) \) verifies the definition (3.1), (2.3) of coherent states with \( \chi(e) = c \).

The Berezin’s symbol in eq. (5.5) is defined in terms of sections (its analogue for functions is given by eq. (5.4)) under the restriction (3.6), \((e_q', e_q) \neq 0) :

\[
A(x, y) = \frac{(\hat{A} e_q', e_q)}{(e_q', e_q)}, \quad s_0(q) = x, \quad s_0(q') = y.
\]

(5.6)
So, the Berezin symbol $A(x, y)$ in eq. (5.4) is defined at a fixed $x$ for $y \notin \text{CL}_x$.

The characteristic function is defined in Ref. [14] in a neighbourhood $U \times U$ of the diagonal of $\tilde{M} \times \tilde{M}$ as

$$\tilde{\Psi}(x, y) = \frac{|s|^2(x, \bar{s})|s|^2(y, \bar{s})}{||s|^2(x, \bar{s})||^2},$$

and it is related to the Calabi’s diastasis [31] by the relation $D = -2 \log \tilde{\Psi}$. In eq. (5.7) $|s|^2(x, \bar{s}) = h_x(s(x), s(x))$ and $|s|^2(x, \bar{s})$ is its analytic continuation in $U \times U$. In fact, due to theorem 1, $\tilde{\Psi}(x, y)$ and $D(x, y)$ are defined at a fixed $x$ for $y \notin \text{CL}_x$.

The 2-point function, whose local analogue is given by eq. (5.3), is globally defined in terms of sections as

$$\Psi(x, y) = \frac{|(e_{q'}, e_q)|^2}{||e_{q'}||^2||e_q||^2}, \quad \xi_0(q) = x, \quad \xi_0(q') = y.$$ (5.8)

If the quantization is regular, i.e. $\epsilon = ct$, then $\tilde{\Psi} = \Psi$ and eq. (5.3) can be put in the form (5.2). We remember also that $\epsilon(x) = ||e_q||^2h(q, q), \xi_0(q) = x, x \in \tilde{M}$.

4). Finally, let me mention that during the Workshop I have received a positive answer [32] to my question: is the polar divisor in theorem 1 a divisor in the sense of algebraic geometry? A more precise statement relative to the advanced question can be given. Indeed, let $[\ ]$ be the functorial homomorphism $[\ ] : \text{Div}(\tilde{M}) \rightarrow H^1(\tilde{M}, \mathcal{O}^*)$ between the group of divisors of a complex manifold and the Picard group of equivalence classes of $\mathcal{C}\infty$ line bundles $[14]$. Then [33] : Let $\tilde{M}$ be a simply connected Hodge manifold admitting the embedding (2.4). Let $M = \nu[1]$ be the unique, up to equivalence, projectively induced line bundle with a given admissible connection. Then $M = \Sigma_0$. Moreover, if the homogeneous manifold $\tilde{M}$ verifies condition B), then $M = [\text{CL}_0]$. In particular, the first relation is true for Kählerian $C$-spaces, while the second one for hermitian symmetric spaces.

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