THERE IS NO HIERARCHY OF CANONICAL SYSTEM FLOWS SIMILAR TO THE KDV HIERARCHY

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Abstract. The KdV hierarchy is a family of evolutions on a Schrödinger operator that preserves its spectrum. Canonical systems are a generalization of Schrödinger operators, that nevertheless share many features with Schrödinger operators. Since this is a very natural generalization, one would expect that it would also be possible to build a hierarchy of isospectral evolutions on canonical systems analogous to the KdV hierarchy. Surprisingly, we show that constructing a hierarchy of flows on canonical systems that obeys the standard assumptions of the KdV hierarchy is in some sense impossible. This suggests that we need a more sophisticated approach to develop such a hierarchy, if it is indeed possible to do so.

1. Introduction

The Korteweg-de Vries (or KdV) equation

\[
\frac{\partial}{\partial t} V(x, t) = \frac{1}{4} \frac{\partial^3}{\partial x^3} V(x, t) - \frac{3}{2} V(x, t) \frac{\partial}{\partial x} V(x, t)
\]

is a well-known mathematical model for waves on shallow water surfaces. This KdV equation has numerous applications in mathematics and physics. For example, if \( V(x, t) \) is considered as a potential function of a Schrödinger operator

\[
L = -\frac{d^2}{dx^2} + V,
\]

evolving this potential via the KdV equation (thought of as an evolution in time \( t \)) will leave the spectrum of the Schrödinger operator invariant. In other words, the KdV equation is associated to an isospectral evolution of the Schrödinger operator.

This concept may extend to a whole family of smooth evolutions of \( V \), known as the KdV hierarchy, all of which preserve the spectrum of the Schrödinger operator containing \( V \) as a potential. The KdV hierarchy is of crucial importance in the inverse spectral theory of Schrödinger operators, and the KdV equation (1.1) is the simplest nontrivial member of this hierarchy. Please see [2] for a detailed explanation of the KdV hierarchy.
In this paper, we would like to see what happens if we apply ideas from the KdV hierarchy to \emph{canonical systems}. Canonical systems can be thought of as a generalization of eigenvalue equations of Schrödinger operators. See [10] for a thorough discussion of the spectral theory of canonical systems. This generalization is best understood from the perspective of Weyl-Titchmarsh \emph{m}-functions. These are Herglotz functions, that is, functions that holomorphically map the upper half plane to itself. The spectrum of a Schrödinger operator on the nonnegative half line $[0, \infty)$ can be derived from the limiting behavior of its corresponding \emph{m}-function (or its corresponding pair of \emph{m}-functions, if we consider a Schrödinger operator on the whole line $\mathbb{R}$ instead). Each Schrödinger operator on $[0, \infty)$ corresponds to a Herglotz function in this way, whereas a Herglotz function might not have a Schrödinger operator associated with it. On the other hand, every Herglotz function corresponds to a canonical system on $[0, \infty)$, and it is in this sense that canonical systems are the “highest generalization” of the Schrödinger spectral problem.

Since the generalization from Schrödinger operators to canonical systems is so natural in spectral theory, naively, one would expect that we can find a family of isospectral canonical system flows that behave like KdV flows. Indeed, many of the tools that we use for the spectral analysis of Schrödinger operators and the development of the KdV hierarchy are also present in the canonical system setting. For example, canonical systems also have a transfer matrix formalism.

Surprisingly, in our paper we show that it is almost impossible to construct a family of isospectral flows on canonical systems that is similar to the KdV hierarchy. More precisely, as a consequence of the zero-curvature equation (3.8) for canonical systems we derive some conditions for the existence of a hierarchy of flows. We then demonstrate that there are difficult obstructions to satisfying these conditions in the canonical system setting, which are not present in the Schrödinger operator setting even though KdV flows also have to satisfy their pertinent zero curvature equation.

To be perfectly clear, we are not asserting that non-trivial isospectral flows on canonical systems are impossible to find. The best way to clarify our impossibility result is as follows: in a very natural way, each member of the KdV hierarchy corresponds to a polynomial with real coefficients. Our result demonstrates that for canonical systems, it is more or less impossible to create isospectral flows corresponding to polynomials of degree two or higher.

The main takeaway from this paper is that a direct approach to constructing an isospectral flow hierarchy for canonical systems is probably not viable. It is, however, possible that a more sophisticated approach might be fruitful, perhaps by incorporating the “twisted shift” idea introduced in [11].

This paper is organized as follows. In Section 2 the KdV hierarchy is reviewed via its zero-curvature equation in order to examine the tools to achieve the KdV hierarchy. The similar tools will be employed on canonical
system flows and then the related zero-curvature equation is obtained in Section 3. It is in the last section that we realize the condition to canonical system flows similar to isospectral KdV flows, and we then see why these isospectral canonical system flows are hard to construct.

Acknowledgement: The authors would like to deeply thank Christian Remling for valuable discussions. The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF-2016R1D1A1B03931764), and the second author was supported by a Xiamen University Malaysia Research Fund (Grant No: XMUMRF/2018-C1/IMAT/0001) and a grant from the Fundamental Research Grant Scheme from the Malaysian Ministry of Education (Grant No: FRGS/1/2018/STG06/XMU/02/1).

2. Review for the KdV hierarchy

Let us review KdV flows and their hierarchy. There are two equivalent ways to construct the KdV hierarchy: the Lax pair formalism and the zero-curvature equation. In this section we almost exclusively employ the latter approach, since it will be easier to generalize to canonical systems later. For related work, see [2] for an exposition of the KdV hierarchy by means of the Lax pair formalism and [7, 8] for a treatment of Toda flows (a discrete analogue of KdV flows) using a zero-curvature equation.

We define Schrödinger operators $L$ as in (1.2), where the potentials $V$ are assumed to be real-valued functions of two variables $x$ and $t$ such that they are $C^\infty$-functions in $x$ (in $\mathbb{R}$) and $C^1$-functions in $t$. To emphasize $t$-dependence, we sometimes think of an $\mathbb{R}$-group action of $t$ acting on $L$, which we denote as $t \cdot L$ instead. That is, $t \cdot L$ also refers to $L$ after $t$ units of time.

We then introduce transfer matrices from 0 to $x$ by

$$T(x, t, z; L) = \begin{pmatrix} u(x, t, z; L) & v(x, t, z; L) \\ u_x(x, t, z; L) & v_x(x, t, z; L) \end{pmatrix}. \tag{2.1}$$

Here the subscript $x$ means the derivative in $x$, and $u$ and $v$ are the solutions to the eigenvalue equation $Ly = zy$ of (1.2), where $z$ is a spectral parameter in $\mathbb{C}$ independent of $x$ and $t$, such that for all $t$,

$$u(0, t, z; L) = v_x(0, t, z; L) = 1 \quad \text{and} \quad u_x(0, t, z; L) = v(0, t, z; L) = 0. \tag{2.2}$$

Note that, since the Wronskian of two solutions to $Ly = zy$ is constant, (2.2) implies that $\det T = 1$ for all $x$ and $t$.

The reason why the $T$’s are called transfer matrices is that for any solutions $y$ to $Ly = zy$ and each $t$,

$$T(\tilde{x}, t, z; L) \begin{pmatrix} y(0, t, z; L) \\ y_x(0, t, z; L) \end{pmatrix} = \begin{pmatrix} y(\tilde{x}, t, z; L) \\ y_x(\tilde{x}, t, z; L) \end{pmatrix}.$$

In other words, $T$’s “transfer” the solutions from 0 to $\tilde{x}$. Actually, we may choose any initial point, say $x_0$, when replacing 0 by $x_0$ in $u$ and $v$ in (2.2).
We can also view this transfer property as an action of the transfer matrix on the Weyl-Titchmarsh $m$-functions associated with the Schrödinger equation $Ly = zy$. These $m$-functions are defined as

$$m_{\pm}(z) = \mp \frac{\tilde{y}_+'(0, z)}{\tilde{y}_\pm(0, z)},$$

where $\tilde{y}_\pm(0, z)$ are solutions of the Schrödinger equation that are $\ell^2$ at $\pm\infty$. (Note that choosing different initial conditions at $x = 0$ a bit change the $m$-functions.) Using this perspective, the $T$’s act on these $m$-functions as linear fractional transformations. See [4, 6, 11] for more details.

The KdV hierarchy is a family of evolutions that obey a

- cocycle property,
- commutativity with the shift in $x$ and
- a polynomial recursion formalism.

We will clarify three items above shortly. For convenience let us ignore the $x$, $t$ or $z$-dependence when it is unambiguous.

**Definition 2.1.** We say that $T$’s are cocycles, if they satisfy the equality

$$(2.3) \quad T(x, s + t; L) = T(x, t; s \cdot L)T(x, s; L),$$

where $s \cdot L$ refers to $L$ after $s$ units of time (which will be clear on the section 3.3 but via canonical system flows). See also [11] for more details.

This means that we should update Schrödinger operators at $s$, when moving from $0$ to $s + t$ by passing through the middle point $s$.

**Figure A.** Cocycle Property

In fact, (2.3) is equivalent to the property that $T$’s satisfy the following first-order autonomous equation in $t$. 
Proposition 2.1. For a cocycle $\mathcal{T}$ we have the differential equation

$$
\frac{d}{dt} \mathcal{T}(t; L) = B(t \cdot L) \mathcal{T}(t; L),
$$

where $B$ is a map from the set of Schrödinger operators (1.2) to the space of $2 \times 2$ complex matrices.

Conversely, if $\mathcal{T}$ obeys the equation (2.4) for some trace zero matrix $B$ dependent on $t \cdot L$ but not on $t$ or $L$ individually, then $\mathcal{T}$ itself is a cocycle.

Note that $\text{tr } B = 0$, due to the fact that $\det T = 1$.

Proof. By differentiating (2.3) with respect to $t$ and treating $s$ as a constant, we have that

$$
\frac{d}{dt} \mathcal{T}(s + t; L) = \left( \frac{d}{dt} \mathcal{T}(t; s \cdot L) \right) \mathcal{T}(s; L)
$$

and

$$
\left( \frac{d}{dt} \mathcal{T}(s + t; L) \right) \mathcal{T}^{-1}(s + t; L) = \left( \frac{d}{dt} \mathcal{T}(t; s \cdot L) \right) \mathcal{T}^{-1}(t; s \cdot L).
$$

Now setting $t = 0$ for this last equality indicates that

$$
B(s \cdot L) = \left( \frac{d}{dt} \mathcal{T}(s; L) \right) \mathcal{T}^{-1}(s; L) = \left( \frac{d}{dt} \mathcal{T}(0; s \cdot L) \right) \mathcal{T}^{-1}(0; s \cdot L).
$$

To prove the converse, we again set $s$ as a constant, and notice that the left and right sides of (2.3) both solve the same differential equation $df/dt = B((s + t) \cdot L)f$, and that they share the same initial conditions at $t = 0$. Then the uniqueness of a solution of such a differential equation implies (2.4). \hfill \Box

Similarly, differentiating $\mathcal{T}$ with respect to $x$ we can directly show that

$$
\frac{d}{dx} \mathcal{T}(x, t, z; L) = \begin{pmatrix} 0 & 1 \\ V(x, t) - z & 0 \end{pmatrix} \mathcal{T}(x, t, z; L) \quad (=: \mathcal{M} \mathcal{T}).
$$

Let us denote the first factor matrix on the right-hand side by $\mathcal{M}(x, t, z; L)$. The reason for the equation (2.5) is due to the fact that

$$
\frac{d}{dx} \begin{pmatrix} y \\ y_x \\ y_{xx} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V - z & 0 \end{pmatrix} \begin{pmatrix} y \\ y_x \end{pmatrix},
$$

where $y$ is any solution to the eigenvalue equation $Ly = sy$.

We have explained the cocycle property for evolutions in time. We can derive a cocycle property for shifts in space too, and in fact it is true that the cocycle property holds for evolutions in time and shifts in space jointly. This will be explained in more detail and in a more general setting in Subsection 3.3.

To construct the zero-curvature equation, we additionally assume that $\mathcal{T}$’s commute with the shift in $x$, in the sense that shifting in $x$ first and
then evolving by $t$ have the same effect on the flows as the reverse order (shifting in $x$ after evolving by $t$). See the figure below.

\[
\begin{align*}
T(x, 0; t \cdot L) &
\rightarrow (x, t) \\
T(x, t; L) &
\rightarrow (x, 0)
\end{align*}
\]

\[
\begin{align*}
T(0, t; L) &
\rightarrow (0, t) \\
T(0, 0; L) &
\rightarrow (0, 0)
\end{align*}
\]

**Figure B.** Commutativity with the shift in $x$.

Since two (blue and red) paths on the figure should have the same effect,

\[(2.6) \quad T(x, t; L)T(x, 0; L) = T(x, 0; t \cdot L)T(0, t; L).\]

This commutativity (2.6) implies the zero-curvature equation

\[
\begin{align*}
\frac{d}{dt} M(L(0)) - \frac{d}{dx} B(L(0)) &= -M(L(0))B(L(0)) + B(L(0))M(L(0)),
\end{align*}
\]

or simply

\[(2.7) \quad \partial_t M - \partial_x B = -[M, B] = -(MB - BM).\]

Indeed, by differentiating (2.6) in $x$ and $t$, we see that

\[
\begin{align*}
\partial_t \partial_x T(x, t; L)T(x, 0; L) + \partial_t T(x, t; L)\partial_x T(x, 0; L) \\
= \partial_x \partial_t T(x, 0; t \cdot L)T(0, t; L) + \partial_x T(x, t; t \cdot L)\partial_t T(0, t; L).
\end{align*}
\]

After exchanging the order of derivatives of the first term on the left-hand side (which is OK due to the smoothness condition on $V$ and therefore on solutions to $Ly = zy$) and then putting $x = 0$ and $t = 0$, both (2.4) and (2.5) indicate (2.7), as desired. (Recall that $T(0, 0, z; L) = I_{2 \times 2}$.)

Lastly, we describe the polynomial recursion formalism on $\mathcal{B}$. For this, put

\[
\mathcal{B} = \begin{pmatrix} A & C \\ -D & -A \end{pmatrix}
\]

due to $\text{tr} \mathcal{B} = 0$ and then insert this and $\mathcal{M}$ into the zero-curvature equation (2.7). By comparing entries, the following three equations should be satisfied:

\[
\begin{align*}
2A + C_x &= 0 \\
(V - z)C + D + A_x &= 0 \\
2(V - z)A + D_x + V_t &= 0.
\end{align*}
\]
In order to see the KdV hierarchy, assume that the entries $A$, $C$ and $D$ are polynomials in $z$. Due to three equations (2.8) above, we can recursively construct the polynomial entries up to integral constants.

More precisely, plugging the first two equations into the third and expressing it in terms of $C$ show that

$$V_t = -\frac{1}{2}C_{xxx} + 2(V - z)C_x + V_x C.$$  

(2.9)

It turns out that the polynomials $C$ in $z$ (therefore, $A$ and $D$) are differential polynomials in $V$, i.e., these are polynomials of $V$ and their derivatives in $x$ ($V_x$, $V_{xx}$ and so on), and they can be constructed recursively by

$$C_{xxx} - 4(V - z)C_x - 2V_x C = 0.$$  

(2.10)

To get rid of some confusion on both (2.9) and (2.10), we clarify what we mean with “recursively”: first fix the degree of $C$, say $n \in \mathbb{N}$, and then construct any (homogeneous) polynomials of degree at most $n - 1$ satisfying (2.10) and some polynomial of the largest degree (which is $n$) should satisfy (2.9). Any linear combinations of these two polynomials then become those of the (infinite) family of KdV hierarchy. See Section 1 of [2] for more details.

Then (2.9) gives the whole family of the KdV hierarchy when we vary our choice of the polynomial $C$. For example, when $C = 1$, (2.9) gives the transport equation $V_t = V_x$, and when $C = z + \frac{1}{2}V$ we get the classical KdV equation (1.1).

3. Canonical system flows

We now would like to apply a similar construction in Section 2 to canonical system flows.

3.1. Canonical systems. Canonical systems are the equations

$$(3.1) \quad J \tilde{u}_x(x, z) = z \mathcal{H}(x) \tilde{u}(x, z), \quad x \in \mathbb{R},$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{H}$ (known as a Hamiltonian) is a (symmetric) positive semi-definite $2 \times 2$ matrix whose entries are real-valued and locally integrable functions. Here $z$ is a spectral parameter in $\mathbb{C}$, as in the Schrödinger (eigenvalue) equations. In particular, canonical systems are called trace-normed if $\operatorname{Tr} \mathcal{H}(x) = 1$ for almost all $x$ in $\mathbb{R}$.

These canonical systems are in some sense the highest generalization of many self-adjoint spectral problems. For instance, eigenvalue equations involving Jacobi matrices, Schrödinger operators (more generally, Sturm-Liouville operators) and Dirac operators can be written as canonical systems. See Proposition 8 in [9] or Proposition 4.1 in [4] to convert Schrödinger eigenvalue equations to (unique trace-normed) canonical systems and vice versa. For discrete cases such as Jacobi operators, see [3].

In the viewpoint of (inverse) spectral theory, this generalization is important for three reasons. First, rewriting a Jacobi, Schrödinger, Dirac, etc. equation as a canonical system does not change its spectrum. Second, there
is one-to-one correspondence between Herglotz functions and (trace-normed and half line) canonical systems \[1\] \[9\] \[12\]. Lastly, the set of all canonical systems becomes a compact topological space with the local uniform convergence of their related Herglotz functions (or Hamiltonians \(\mathcal{H}\) in \(3.1\)) \[1\] \[4\] \[6\] \[9\].

In the next subsection we will try to find isospectral canonical system flows which look like KdV flows by the zero-curvature equation for canonical system flows (which is \(3.8\)). It will turn out that this is very difficult to achieve.

Note that unlike the Schrödinger equation, there is in general no (self-adjoint) operator associated with a canonical system. Instead of an operator we have a (self-adjoint) relation \[3\]. This explains the difficulty to apply a Lax pair formalism via operators on canonical system flows \[3.2\]. Thus, we favor the zero-curvature equation approach.

### 3.2. Canonical system flows and their zero-curvature equation.

To consider a (smooth) evolution of canonical systems \(3.1\), let us impose a \(t\)-dependence on \(3.1\), i.e.,

\[
\mathcal{J} \bar{u}_x(x, t, z; \mathcal{H}) = z\mathcal{H}(x, t)\bar{u}(x, t, z; \mathcal{H}), \quad x \in \mathbb{R}
\]

This \(t\)-evolution on the canonical system is what we call a canonical system flow. Denote \(\mathcal{H}\) and its determinant respectively by

\[
\mathcal{H}(x, t) = \begin{pmatrix}
    f(x, t) & g(x, t) \\
    g(x, t) & h(x, t)
\end{pmatrix}
\]

and

\[
\Delta(x, t) := \det \mathcal{H}(x, t) = f(x, t)h(x, t) - (g(x, t))^2.
\]

Due to the positive semi-definiteness condition on \(\mathcal{H}\), the functions \(f, h\) and \(\Delta\) are non-negative at all times.

Similar to the smooth condition on the potentials \(V\) for KdV flows, assume the following:

**Hypothesis 3.1.** The entries \(f, g\) and \(h\) are smooth functions (jointly) on both \(x\) and \(t\).

As in the construction of the KdV hierarchy, let us impose three conditions on evolution in \(t\): a cocycle property on transfer matrices, commutativity with the shift in \(x\) and a polynomial recursion formalism. Since our approach comes from KdV setting, we will use \(\tilde{\cdot}\) for canonical system flows to compare them to KdV flows. For convenience we may drop the \(x, t, z\)- or \(\mathcal{H}\)-dependence when it is clear.

Similar to \(2.1\), let us first introduce transfer matrices \(\tilde{T}\) (equivalent to \(\rho\)-matrix functions in \[9\]) by

\[
\tilde{T}(x, t, z; \mathcal{H}) = \begin{pmatrix}
    u_1(x, t, z; \mathcal{H}) & v_1(x, t, z; \mathcal{H}) \\
    u_2(x, t, z; \mathcal{H}) & v_2(x, t, z; \mathcal{H})
\end{pmatrix}.
\]
where two columns $\vec{u} = (u_1^1 u_2^1)$ and $\vec{u} = (v_1^1 v_2^1)$ are the solutions to (3.2), such that, for all $t$,

$$u_1(0, t, z) = v_2(0, t, z) = 1 \quad \text{and} \quad u_2(0, t, z) = v_1(0, t, z) = 0. \quad (3.5)$$

Since the Wronskian of two solution vectors to (3.2) is constant, (3.5) implies that $\det \bar{T} = 1$ for all $x$ and $t$.

Due to a similar proof of Proposition 2.1, the cocycle property for \( \bar{T} \) enables us to introduce the matrices \( \tilde{B} \) so that

$$\frac{d}{dt} \bar{T}(t; \mathcal{H}) = \bar{B}(t \cdot \mathcal{H}) \bar{T}(t; \mathcal{H}), \quad (3.6)$$

where $\bar{B}$ is a map from the set of Hamiltonians $\mathcal{H}$ to the space of $2 \times 2$ complex matrices, and $t \cdot \mathcal{H}$ refers to $\mathcal{H}$ after a canonical system flow of $t$ units of time, which will be clarified as a group action on the next subsection.

Note that $\text{tr} \; \bar{B} = 0$, due to the fact that $\det \bar{T} = 1$.

With respect to $x$, the systems (3.2) and the fact that $\mathcal{J}^{-1} = -\mathcal{J}$ lead to the differential equations for $\bar{T}$ in $x$

$$\frac{d}{dx} \bar{T}(x, t, z) = -z \mathcal{J}(x, t) \bar{T}(x, t, z) \quad (=: \tilde{M} \bar{T}). \quad (3.7)$$

Similar to (2.5) for KdV flows, put

$$\tilde{M} := -z \mathcal{J} = z \begin{pmatrix} g & h \\ -f & -g \end{pmatrix}. \quad (\text{3.9})$$

By assuming that our evolution in $t$ commutes with the shift in $x$ and using (3.6) and (3.7), we can construct the zero-curvature equation for (3.2), which is almost (2.7) except tildes, that is,

$$\frac{d}{dt} \tilde{M} - \frac{d}{dx} \tilde{B} = \bar{B} \tilde{M} - \tilde{M} \bar{B}. \quad (3.8)$$

As the last condition, assume that $\tilde{B}$ have polynomial entries (in $z$), and denote them by

$$\tilde{B} = \begin{pmatrix} \tilde{A} & \tilde{C} \\ -\tilde{D} & -\tilde{A} \end{pmatrix}. \quad (\text{3.10})$$

Plugging the matrix above and $\tilde{M}$ into the zero-curvature equation (3.8) then leads to three conditions for entries of $\tilde{B}$ and $\mathcal{H}$:

$$zh_t - C_x = 2zh\tilde{A} - 2zg\tilde{C} \quad (3.9)$$

$$-zf_t + D_x = 2zf\tilde{A} - 2zg\tilde{D}$$

$$zg_t - A_x = -zf\tilde{C} + zh\tilde{D}.$$

So far we have found the conditions (3.9), which are equivalent to the zero-curvature equation (3.8) for the canonical system flows (3.2).
3.3. Canonical system flows as a group action. We may also describe an integrable flow as a group action. In this perspective, we will regard our flow as a group action on $\mathbb{R} \times \mathbb{R}$, where the first component represents an action on time $t$, and the second component represents a shift on the $x$-axis. We define a group action of time $t$ and shifts $x$ acting on $\mathcal{H}$ as follows: $(0, t) \cdot \mathcal{H}$ refers to $\mathcal{H}$ after a canonical system flow of $t$ units of time, whereas $(x, 0) \cdot \mathcal{H}$ refers to a shift in the $x$-direction, that is, $(\tilde{x}, 0) \cdot \mathcal{H}(x) = \mathcal{H}(x + \tilde{x})$.

It is clear from (3.6) that $\mathcal{T}$ obeys the $t$-cocycle property expressed in Definition 2.1. We can in a similar way deduce from (3.7) that $\mathcal{T}$ also obeys the cocycle property in terms of shifts. For an $(x, t) \in \mathbb{R} \times \mathbb{R}$, we define

$$\mathcal{T}(x, t; \mathcal{H}) = \mathcal{T}((0, t); (x, 0) \cdot \mathcal{H}) \mathcal{T}((0, 0); \mathcal{H}).$$

We now have to prove that $\mathcal{T}$ obeys the joint cocycle property,

$$\mathcal{T}(g + h; \mathcal{H}) = \mathcal{T}(g; h \cdot \mathcal{H}) \mathcal{T}(h; \mathcal{H}), \text{ for } g, h \in \mathbb{R} \times \mathbb{R}.$$

**Proposition 3.2.** The zero-curvature equation (3.8) and the joint cocycle condition (3.11) are equivalent.

**Proof.** Showing that the joint cocycle condition implies the zero-curvature equation is straightforward. Obviously the existence of the joint cocycle on $x$ and $t$ implies the cocycle property for $x$ and $t$ individually; and this in turn implies (3.6) and (3.7). We can demonstrate this implication in the same way as the proof of Proposition 2.1. Then (3.8) follows immediately.

The other direction is harder. First, observe that it suffices to prove

$$\mathcal{T}((0, t); (x, 0) \cdot \mathcal{H}) \mathcal{T}((0, 0); \mathcal{H}) = \mathcal{T}((x, 0); (0, t) \cdot \mathcal{H}) \mathcal{T}((0, t); \mathcal{H}).$$

This is due to the commutativity of the group $\mathbb{R} \times \mathbb{R}$. More precisely we have that

$$\mathcal{T}((0, t); (x + y, 0) \cdot \mathcal{H}) \mathcal{T}((0, 0); (y, 0) \cdot \mathcal{H}) = \mathcal{T}((x, 0); (y, t) \cdot \mathcal{H}) \mathcal{T}((0, t); (y, 0) \cdot \mathcal{H}).$$

(3.13)

Note that the cocycle property applies for $x$ and $t$ individually (although we don’t know this yet for $x$ and $t$ jointly). Thus for arbitrary $(x, s)$ and $(y, t)$ in $\mathbb{R} \times \mathbb{R},$

$$\mathcal{T}((x + y, s + t); \mathcal{H})$$

$$= \mathcal{T}((0, s + t); (x + y, 0) \cdot \mathcal{H}) \mathcal{T}((x + y, 0); \mathcal{H})$$

$$= \mathcal{T}((0, s); (x + y, t) \cdot \mathcal{H}) \mathcal{T}((0, t); (x + y, 0) \cdot \mathcal{H}) \mathcal{T}((x, 0); (y, 0) \cdot \mathcal{H}) \mathcal{T}((0, 0); \mathcal{H})$$

$$= \mathcal{T}((x, s); (y, t) \cdot \mathcal{H}) \mathcal{T}((0, t); (y, 0) \cdot \mathcal{H}) \mathcal{T}((y, 0); \mathcal{H})$$

$$= \mathcal{T}((x, s); (y, t) \cdot \mathcal{H}) \mathcal{T}((0, t); (y, 0) \cdot \mathcal{H} \mathcal{T}((y, 0); \mathcal{H})$$

$$= \mathcal{T}((x, s); (y, t) \cdot \mathcal{H}) \mathcal{T}((y, 0); \mathcal{H}) \mathcal{T}((y, t); \mathcal{H}).$$

Here we have used (3.13) to go from the third line to the fourth. The last equality follows from (3.10).
It remains to prove (3.12). For convenience the left and right sides of (3.12) are denoted by $\ell$ and $r$ respectively. It is easy to check that their initial conditions at $x = t = 0$ are the same.

Note that by (3.6) we have

$$
\frac{\partial}{\partial t} \ell = \tilde{B}(\ell \cdot \mathcal{H})
$$

and

$$
\frac{\partial}{\partial t} r = \left( \frac{\partial}{\partial t} \tilde{T}((x,0); (0,t) \cdot \mathcal{H}) \right) \tilde{T}((0,t); \mathcal{H}) + \tilde{T}((x,0); (0,t) \cdot \mathcal{H}) \tilde{B}((0,t) \cdot \mathcal{H}) \tilde{T}((0,t); \mathcal{H}).
$$

Our goal is to have $\ell$ and $r$ solve the same differential equation, so we are done if we can express the right hand side of (3.15) in the form $\tilde{B}(\ell \cdot \mathcal{H}) r$. This will obviously follow from

$$
\left( \frac{\partial}{\partial t} \tilde{T}((x,0); (0,t) \cdot \mathcal{H}) \right) \tilde{T}^{-1}((x,0); (0,t) \cdot \mathcal{H}) + \tilde{T}((x,0); (0,t) \cdot \mathcal{H}) \tilde{B}((0,t) \cdot \mathcal{H}) \tilde{T}^{-1}((x,0); (0,t) \cdot \mathcal{H}) = \tilde{B}(\ell \cdot \mathcal{H}).
$$

Replacing $(0,t) \cdot \mathcal{H}$ with $\mathcal{H}$, this is equivalent to

$$
\frac{\partial}{\partial t} \tilde{T}((x,0); (0,t) \cdot \mathcal{H})|_{t=0} = \tilde{B}((x,0) \cdot \mathcal{H}) \tilde{T}((x,0); \mathcal{H}) - \tilde{T}((x,0); \mathcal{H}) \tilde{B}(\mathcal{H}).
$$

We perform the same trick again. We denote the left hand side of (3.15) as $\tilde{\ell}$ and the right hand side as $\tilde{r}$. Observe that they have the same initial conditions at $x = 0$. Note that

$$
\frac{\partial}{\partial x} \tilde{\ell} = \tilde{M}((x,0) \cdot \mathcal{H}) \ell(x) + \frac{\partial}{\partial t} \tilde{M}((x,0) \cdot \mathcal{H}) \tilde{T}((x,0); H)
$$

and

$$
\frac{\partial}{\partial x} \tilde{r} = \frac{\partial}{\partial x} \tilde{B}((x,0) \cdot \mathcal{H}) \tilde{T}((x,0); H)
$$

$$
+ \tilde{B}((x,0) \cdot \mathcal{H}) \tilde{M}((x,0) \cdot \mathcal{H}) \tilde{T}((x,0); \mathcal{H}) - \tilde{M}((x,0) \cdot \mathcal{H}) \tilde{T}((x,0); \mathcal{H}) \tilde{B}(\mathcal{H}).
$$

The zero-curvature equation (3.8) together with (3.17) implies that (3.16) still holds if we replace $\tilde{\ell}$ with $\tilde{r}$. Thus $\tilde{\ell} = \tilde{r}$ and this concludes our proof. □

4. Main result

In this section we would like to see how difficult it is to find canonical system flows (3.2) satisfying the three equations (3.9).
Theorem 4.1. Let $f, g, h$ be the entries of $\mathcal{H}$, as in (3.3) and assume that they are smooth on $x$ and $t$ and that they are differential polynomials in $z$ of degree at least two. Then there is no canonical system flow (3.2) satisfying the three equations (3.9), unless $(\det \mathcal{H})^{-1/2}$ is a differential polynomial in $f$, $g$ and $h$.

Remarks

(1) The condition that the entries of $\mathcal{B}$ are differential polynomials of $f$, $g$ and $h$ is reasonable, since the same restriction applies for the corresponding $\mathcal{B}$ for KdV flows. For KdV setting, the potentials $V$ are expressed as

$$V = \frac{1}{4} \det \mathcal{H}_x^V,$$

where $\mathcal{H}_x^V$ are the Hamiltonians of the canonical systems which are re-written as the Schrödinger equations $Ly = zy$. As discussed before, the entries of $\mathcal{B}$ are differential polynomials in $V$, and they are therefore differential polynomials of the entries of $\mathcal{H}_x^V$. In other words, since $V$ is a differential polynomial of the entries of $\mathcal{H}_x^V$ and the entries of $\mathcal{B}$ are those of $V$, the entries of $\mathcal{B}$ are differential polynomials of the entries of $\mathcal{H}_x^V$. This is exactly the same condition as the one in our main theorem. For (4.1), it turns out that

$$\mathcal{H}_x^V = \begin{pmatrix} u_0^2 & u_0 v_0 \\ u_0 v_0 & v_0^2 \end{pmatrix}$$

where $u_0$ and $v_0$ are the solutions for zero energy, i.e., $Ly = 0$ with $V$ satisfying (2.2) with $u_0$ and $v_0$ instead. Due to the fact that $u_0,xx = V u_0$ and $v_0,xx = V v_0$, direct computation shows the expression

$$\det(\mathcal{H}_x^V) = 4V(u_0 v_0, x - u_0, x v_0)^2 = 4V.$$

(2) It is notable that, even if we assume that $\Delta$ would be constant (maybe this is the easiest case), the canonical system flows (3.2) or three equations (3.9) would not be significantly easier. For example, the zero $\Delta$ can reduce three equations (3.9) to two equations. However, it does not give any better information about the existence of the global solutions to (3.2) satisfying these two conditions.

(3) Even though we expect that it would be very difficult to have canonical system flows (3.2) like KdV flows, this does not contradict to the existence of KdV flows and the fact that canonical systems are generalization of any eigenvalue equations of Schrödinger operators. This is because the shift in $x$ on KdV flows is different from the one on canonical system flows. More precisely, when we express a Schrödinger equation as a canonical system, the $x$-shift in the Schrödinger equation does not correspond to the $x$-shift in the canonical system. This observation also gives some evidence to consider twisted canonical system flows on [11], whose $x$-shift is matched to
the one for KdV flows during the generalization by canonical systems.

(4) Our result shows that there are no canonical system flows corresponding to polynomials of degree two or more (unless perhaps \((\det H)^{-1/2}\) is a differential polynomial for every \(t\), which is a rather implausible situation). Flows of degree zero correspond to shifts in \(x\). Finding flows of degree one requires us to solve an existence problem involving a system of nonlinear PDEs. We consider this an interesting open problem.

To prove Theorem 4.1, the following lemma is easy but crucial, and it gives insight to the reasons for the conditions in Theorem 4.1.

**Lemma 4.2.** For the three equations (3.9) to be consistent, the determinants \(\Delta\) of Hamiltonians \(H\), must satisfy the condition

\[
(4.2) \quad z\Delta_t = f\tilde{C}_x + h\tilde{D}_x - 2g\tilde{A}_x.
\]

The condition (4.2) strongly suggests that it is difficult to have canonical system flows similar to KdV flows, since the independence of \(\Delta\) on \(z\) tells that the left-hand side is (at most) linear in \(z\). However, the right-hand side consists of polynomials of any degree which can be chosen at our disposal. As polynomials in \(z\), the equality (4.2) would thus be very difficult to satisfy.

**Proof.** The idea is to express three equations (3.9) as a matrix equation:

\[
(4.3) \quad z \begin{pmatrix} 2h & -2g & 0 \\ 2f & 0 & -2g \\ 0 & -f & h \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{C} \\ \tilde{D} \end{pmatrix} = \begin{pmatrix} zh_t - \tilde{C}_x \\ -zf_t + \tilde{D}_x \\ zg_t - \tilde{A}_x \end{pmatrix}.
\]

Observe that the coefficient matrix above has determinant 0. By applying elementary row operations or multiplying some vector in the kernel of the coefficient matrix from the left, (4.3) reads

\[
\begin{pmatrix} 2fh & -2fg & 0 \\ 0 & 0 & 0 \\ -fg & gh \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{C} \\ \tilde{D} \end{pmatrix} = \begin{pmatrix} zfh_t - f\tilde{C}_x \\ -z\Delta_t + f\tilde{C}_x + h\tilde{D}_x - 2g\tilde{A}_x \\ zgg_t - g\tilde{A}_x \end{pmatrix}.
\]

Thus (4.3) has a solution only if (4.2) is satisfied. \(\square\)

**Remark.** A similar process can be applied to the KdV setting. A version of the three equations in (2.8) for KdV flows can be written as the matrix equation

\[
\begin{pmatrix} 2 & 0 & 0 \\ 0 & V - z & 1 \\ 2(V - z) & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ C \\ D \end{pmatrix} = \begin{pmatrix} -C_x \\ -A_x \\ -V_t - D_x \end{pmatrix}.
\]

By the Gauss elimination method, we obtain another reduced matrix equation

\[
\begin{pmatrix} 2 & 0 & 0 \\ 0 & V - z & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ C \\ D \end{pmatrix} = \begin{pmatrix} -C_x \\ -A_x \\ -V_t - D_x + (V - z)C_x \end{pmatrix},
\]
which implies that

\[ V_t = -D_x + (V - z)C_x. \]

This can be considered as a "prototype" of the KdV hierarchy. This is because two first equations on (2.8), which are

\[ A = -\frac{1}{2}C_x \quad \text{and} \quad D = -A_x - (V - z)C, \]

imply that

\[ V_t = -D_x + (V - z)C_x \implies V_t = -\frac{1}{2}C_{xxx} + 2(V - z)C_x + V_x C, \]

where the last equation is the KdV hierarchy (2.9). All this means that (4.2) can be thought of as a prototype of the hierarchy for canonical system flows (3.2) satisfying three equations (3.9).

**Proof of Theorem 4.1.** Assume the entries of \( \tilde{B} \) are polynomials in \( z \) of degree at least two, and write \( \tilde{A} = A_0 + A_1 z + \cdots + A_n z^n \) with similar notation for \( \tilde{C} \) and \( \tilde{D} \).

Since the left-hand sides on (3.9) are of degree at most \( n \geq 2 \), comparing the coefficients for the greatest power \( z^{n+1} \) shows that

\[ hA_n = gC_n, \quad fA_n = gD_n \quad \text{and} \quad fC_n = hD_n. \]

Then we can introduce a \( K \) such that \( A_n = gK, \quad C_n = hK \) and \( D_n = fK \) solve (4.4). Note that \( K \) must be a differential polynomial in \( f, g, h \). Observe that (4.2) shows

\[ fC_{k,x} + hD_{k,x} - 2gA_{k,x} = 0 \quad \text{for} \quad 2 \leq k \leq n. \]

By plugging (4.4) on the condition above when \( k = n \) (\( \geq 2 \)), it reads the equation for \( K \) and \( \Delta \)

\[ \Delta_x K + 2\Delta K_x = 0, \]

where \( \Delta = \det \mathcal{H} = fh - g^2 \) (which is (3.4)). Solving the ODE (4.5) gives us that, for some constant \( \omega \),

\[ K = \frac{\omega}{\sqrt{\Delta}}. \]

Our conditions on \( K \) then imply that \( \Delta^{-1/2} \) has to be a differential polynomial in \( f, g \) and \( h \).

\[ \square \]

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