UNIQUENESS OF SOLUTIONS FOR A NONLOCAL ELLIPTIC EIGENVALUE PROBLEM

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Abstract. We examine equations of the form

\[
\begin{cases}
(-\Delta)^{\frac{1}{2}} u = \lambda g(x)f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\lambda > 0\) is a parameter and \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N, N \geq 2\). Here \(g\) is a positive function and \(f\) is an increasing, convex function with \(f(0) = 1\) and either \(f\) blows up at 1 or \(f\) is superlinear at infinity. We show that the extremal solution \(u^*\) associated with the extremal parameter \(\lambda^*\) is the unique solution. We also show that when \(f\) is suitably supercritical and \(\Omega\) satisfies certain geometrical conditions then there is a unique solution for small positive \(\lambda\).

1. Introduction

We are interested in the following nonlocal eigenvalue problem

\[(P)_\lambda \quad \begin{cases}
(-\Delta)^{\frac{1}{2}} u = \lambda g(x)f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}\]

where \((-\Delta)^{\frac{1}{2}}\) is the square root of the Laplacian operator, \(\lambda > 0\) is a parameter, \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\) where \(N \geq 2\), and where \(0 < g(x) \in C^{1,\alpha}(\overline{\Omega})\) for some \(0 < \alpha\). The nonlinearity \(f\) satisfies one of the following two conditions:

(R) \(f\) is smooth, increasing and convex on \(\mathbb{R}\) with \(f(0) = 1\) and \(f\) is superlinear at \(\infty\) (i.e., \(\lim_{t \to \infty} \frac{f(t)}{t} = \infty\)), or

(S) \(f\) is smooth, increasing, convex on \([0, 1]\) with \(f(0) = 1\) and \(\lim_{t \to 1^-} f(t) = +\infty\).

In this paper, we prove there is a unique solution of \((P)_\lambda\) for two parameter ranges: for small \(\lambda\) and for \(\lambda = \lambda^*\) where \(\lambda^*\) is the so called extremal parameter associated with \((P)_\lambda\). First, let us to recall various known facts concerning the second order analog of \((P)_\lambda\).

Some notations: Let \(F(t) := \int_0^t f(\tau)d\tau\) and \(C_f := \int_0^{a_f} f(t)^{-1}dt\) where \(a_f = \infty\) (resp. \(a_f = 1\)) when \(f\) satisfies (R) (resp. \(f\) satisfies (S)). We say a positive function \(f\) defined on an interval \(I\) is logarithmically convex (or log convex) provided \(u \mapsto \log(f(u))\) is convex on \(I\). Also, \(\Omega\) will always denote a smooth bounded domain in \(\mathbb{R}^N\) where \(N \geq 2\).
1.1. The local eigenvalue problem. For a nonlinearity $f$ which satisfies (R) or (S), the following second order analog of $(P)_\lambda$ with the Dirichlet boundary conditions

$$(Q)_\lambda \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

is by now quite well understood whenever $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [2–5, 14–16, 18, 20, 21]. We now list the properties one comes to expect when studying $(Q)_\lambda$.

It is well known that there exists a critical parameter $\lambda^* \in (0, \infty)$ such that for all $0 < \lambda < \lambda^*$ there exists a smooth, minimal solution $u_\lambda$ of $(Q)_\lambda$. Here the minimal solution means in the pointwise sense. In addition for each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing in $(0, \lambda^*)$. This allows one to define the pointwise limit $u^* := \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ which can be shown to be a weak solution, in a suitably defined sense, of $(Q)_{\lambda^*}$. It is also known that for $\lambda > \lambda^*$ there are no weak solutions of $(Q)_\lambda$. Also, one can show that the minimal solution $u_\lambda$ is a semi-stable solution of $(Q)_\lambda$ in the sense that

$$\int_{\Omega} \lambda f'(u_\lambda) \psi^2 \leq \int_{\Omega} |\nabla \psi|^2, \quad \forall \psi \in H_0^1(\Omega).$$

We now come to the results known for $(Q)_\lambda$ that we are interested in extending to $(P)_\lambda$. In [18] it was shown that the extremal solution $u^*$ is the unique weak solution of $(Q)_{\lambda^*}$. Some of the techniques involve using concave cut offs which do not seem to carry over to the nonlocal setting. Here, we use some techniques developed in [1] that were used in studying a fourth order analogue of $(Q)_\lambda$. In [11] the uniqueness of the extremal solution for $\Delta^2 u = \lambda e^u$ on radial domains with Dirichlet boundary conditions was shown and this was extended to log convex (see below) nonlinearities in [17]. Some of the methods used in [17] were inspired by the techniques of [1] and so will ours in the case where $f$ satisfies (R). In [8] it was shown that the extremal solution associated with $\Delta^2 u = \lambda(1 - u)^{-2}$ on radial domains is unique and our methods for nonlinearities satisfying (S) use some of their techniques.

In [19] and [23] a generalization of $(Q)_\lambda$ was examined. They showed that if $f$ is suitably supercritical at infinity and if $\Omega$ is a star-shaped domain, then for small $\lambda > 0$ the minimal solution is the unique solution of $(Q)_\lambda$. In [13] this was done for a particular nonlinearity $f$ which satisfies (S). One can weaken the star-shaped assumption and still have uniqueness, see [22], but we do not pursue this approach here. In Section 3, we extend these results to $(P)_\lambda$. For more results on the uniqueness of solutions for various elliptic problems involving parameters, see [12].

For questions on the regularity of the extremal solution in fourth order problems, we direct the interested reader to [10]. We also mention the recent preprint [9] which examines the same issues as this paper but for equations of the form $\Delta^2 u = \lambda f(u)$ in $\Omega$ with either the Dirichlet boundary conditions $u = |\nabla u| = 0$ on $\partial \Omega$ or the Navier boundary conditions $u = \Delta u = 0$ on $\partial \Omega$. Elliptic systems of the form $-\Delta u = \lambda f(v)$, $-\Delta v = \gamma g(u)$ in $\Omega$ with $u = v = 0$ on $\partial \Omega$ are also examined.

1.2. The nonlocal eigenvalue problem. We first give the needed background regarding $(-\Delta)^{\frac{1}{2}}$ to examine $(P)_\lambda$, for a more detailed background see [6]. In [7] they examined the problem $(P)_\lambda$ with $(-\Delta)^{s}$ replacing $(-\Delta)^{\frac{1}{2}}$ and with $g(x) = 1$. They
did not investigate the questions, we are interested in but they did develop much of
the needed theory to examine \((P)\lambda\) and so we will use many of their results.

There are various ways to make sense of \((-\Delta)^{\frac{1}{2}}u\). Suppose that \(u(x)\) is a smooth
function defined in \(\Omega\) which is zero on \(\partial\Omega\) and suppose that \(u(x) = \sum_k a_k \phi_k(x)\) where
\((\phi_k, \lambda_k)\) are the eigenpairs of \(-\Delta\) in \(H_0^1(\Omega)\) which are \(L^2\) normalized. Then one defines
\[ (-\Delta)^{\frac{1}{2}}u(x) = \sum_k a_k \sqrt{\lambda_k} \phi_k(x). \]

Another way is to suppose we are given \(u(x)\) which is zero on \(\partial\Omega\) and we let \(u_e = u_e(x, y)\) denote a solution of
\[
\begin{aligned}
\Delta u_e &= 0 \quad \text{in } C := \Omega \times (0, \infty) \\
u u_e &= 0 \quad \text{on } \partial_L C := \partial \Omega \times (0, \infty) \\
u u_e &= u(x) \quad \text{in } \Omega \times \{0\}.
\end{aligned}
\]
Then we define
\[ (-\Delta)^{\frac{1}{2}}u(x) = \partial_{\nu} u_e(x, y) \big|_{y=0}, \]
where \(\nu\) is the outward pointing normal on the bottom of the cylinder, \(C\). We call \(u_e\)
the harmonic extension of \(u\). We define \(H_0^1, L(C)\) to be the completion of \(C_c^\infty(\Omega \times [0, \infty))\)
under the norm \(\|u\|^2 = \int_C |\nabla u|^2\). When working on the cylinder generally we will
write integrals of the form \(\int_{\Omega \times \{y=0\}} \gamma(u_e)\) as \(\int_{\Omega} \gamma(u)\).

Some of our results require one to examine quite weak notions of solutions to \((P)\lambda\)
and so we begin with our definition of a weak solution.

**Definition 1.** Given \(h(x) \in L^1(\Omega)\) we say that \(u \in L^1(\Omega)\) is a weak solution of
\[
\begin{aligned}
(-\Delta)^{\frac{1}{2}}u &= h(x) \quad \text{in } \Omega \\
u u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]
provided that
\[ \int_{\Omega} u \psi = \int_{\Omega} h(x) (-\Delta)^{-\frac{1}{2}} \psi \quad \forall \psi \in C_c^\infty(\Omega). \]

Here \((-\Delta)^{-\frac{1}{2}}\psi\) is given by the function \(\phi\) where
\[
\begin{aligned}
(-\Delta)^{\frac{1}{2}} \phi &= \psi \quad \text{in } \Omega \\
\phi &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]

The following is a weakened special case of a lemma taken from [7].

**Lemma 1.** Suppose that \(h \in L^1(\Omega)\). Then there exists a unique weak solution \(u\) of
\((1.1)\). Moreover if \(0 \leq h \text{ a.e. then } u \geq 0 \text{ in } \Omega\).

**Definition 2.** Let \(f\) be a nonlinearity satisfying (R).

- We say that \(u(x) \in L^1(\Omega)\) is a weak solution of \((P)\lambda\) provided \(g(x) f(u) \in L^1(\Omega)\), and
\[ \int_{\Omega} u \psi = \lambda \int_{\Omega} g(x) f(u) (-\Delta)^{-\frac{1}{2}} \psi \quad \forall \psi \in C_c^\infty(\Omega). \]
• We say $u$ is a regular energy solution of $(P)_\lambda$ provided that $u$ is bounded, the harmonic extension $u_e$ of $u$, is an element of $H^1_{0,L}(C)$ and satisfies

$$\int_C \nabla u_e \cdot \nabla \phi = \lambda \int_\Omega g(x)f(u)\phi,$$

for all $\phi \in H^1_{0,L}(C)$.

• We say $\overline{u}$ is a regular energy supersolution of $(P)_\lambda$ provided that $0 \leq \overline{u}$ is bounded, the harmonic extension of $\overline{u}$ is an element of $H^1_{0,L}(C)$ and satisfies

$$\int_C \nabla \overline{u}_e \cdot \nabla \phi \geq \lambda \int_\Omega g(x)f(u)\phi,$$

for all $0 \leq \phi \in H^1_{0,L}(C)$.

In the case where $f$ satisfies (S) a few minor changes are needed in the definition of solutions. For a weak solution $u$ one requires that $u \leq 1$ a.e. in $\Omega$. For $u$ to be a regular energy solution one requires that $\sup_\Omega u < 1$.

We will need the following monotone iteration result, see [7]. Suppose that $u$ and $\overline{u}$ are regular energy sub and supersolutions of $(P)_\lambda$. Then there exists a regular energy solution $u$ of $(P)_\lambda$ and $u \leq \overline{u} \leq \overline{u}$ in $\Omega$. By a regular energy subsolution, we are using the natural analog of regular energy supersolution.

We define the extremal parameter

$$\lambda^* := \sup \{0 \leq \lambda : (P)_\lambda \text{ has a regular energy solution} \},$$

and we now show some basic properties.

**Lemma 2.**

1. Then $0 < \lambda^*$.
2. Then $\lambda^* < \infty$.
3. For $0 < \lambda < \lambda^*$ there exists a regular energy solution $u_\lambda$ of $(P)_\lambda$ which is minimal and semi-stable.
4. For each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$ and hence the pointwise limit $u^*(x) := \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ is well defined. Then $u^*$ is a weak solution of $(P)_{\lambda^*}$ and satisfies $\int_\Omega g(x)f(u^*)^2dx < \infty$.

In this paper, we do not need the notion of a semi-stable solution other than for the proof of (4). For the definition of a semi-stable solution one can either use a nonlocal notion, see [7] or instead work on the cylinder which is what we choose to do. We say that a regular energy solution $u$ of $(P)_\lambda$ is semi-stable provided that

$$\int_C |\nabla \phi|^2 \geq \lambda \int_\Omega g(x)f(u)^2\phi^2 \quad \forall \phi \in H^1_{0,L}(C).$$

We now prove the lemma.

**Proof:**

1. Let $\overline{u}$ denote a solution of $(-\Delta)^{\frac{1}{2}} \overline{u} = tg(x)$ with $\overline{u} = 0$ on $\partial \Omega$ where $t > 0$ is small enough such that $\sup_\Omega \overline{u} < 1$. One sees that $\overline{u}$ is a regular energy supersolution of $(P)_\lambda$ provided $t \geq \lambda \sup_\Omega f(\overline{u})$ which clearly holds for small positive $\lambda$. Zero is clearly a regular energy subsolution and so we can apply the monotone iteration procedure to obtain a regular energy solution and hence $\lambda^* > 0$.

2. Suppose that either $f$ satisfies (R) and $C_f < \infty$ or $f$ satisfies (S) and so trivially $C_f < \infty$. 

Let $u$ denote a regular energy solution of $(P)_\lambda$ and let $u_\epsilon$ denote the harmonic extension. Let $\phi$ denote the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ and let $\phi_\epsilon$ be its harmonic extension; so $\phi_\epsilon(x,y) = \phi(x)e^{-\sqrt{\lambda}ty}$. Multiply $0 = -\Delta u_\epsilon$ by $\frac{\phi_\epsilon}{f(u_\epsilon)}$ and integrate this over the cylinder $C$ to obtain
\[
\int_\Omega \lambda g(x)\phi = \int_C \frac{\nabla u_\epsilon \cdot \nabla \phi_\epsilon}{f(u_\epsilon)} - \int_C \frac{\nabla u_\epsilon |\phi_\epsilon|^2 f'(u_\epsilon)}{f(u_\epsilon)^2},
\]
and note that the second integral on the right is nonpositive and hence we can rewrite this as
\[
\int_\Omega \lambda g(x)\phi \leq \int_C \nabla \phi_\epsilon \cdot \nabla h(u_\epsilon),
\]
where $h(t) = \int_0^t \frac{1}{f(\tau)}d\tau$. Integrating the right-hand side by parts, we have that it is equal to $\int_\Omega (-\Delta)^{\frac{3}{2}} \phi h(u)$ which is equal to $\sqrt{\lambda} \int_\Omega \phi h(u)$. So $h(u) \leq C_f$ and hence we have
\[
\lambda \int_\Omega g(x)\phi \leq \sqrt{\lambda} C_f \int_\Omega \phi.
\]
This shows that $\lambda^* < \infty$. The case where $f$ satisfies (R) and where $C_f = \infty$ needs a separate proof, see the proof of (4). Note that there are examples of $f$ which satisfy (R) and for which $C_f = \infty$, for example $f(t) := (t+1)\log(t+1) + 1$.

(3) The proof in the case where $g(x) = 1$ also works here, see [7].

(4) Again the proof used in the case where $g(x) = 1$ works to show the monotonicity of $u_\lambda$, see [7], and hence $u^*$ is well defined. One should note that our notion of a weak solution is more restrictive than what is typically used, i.e., we require $g(x)f(u_\lambda) \in L^1(\Omega)$ where typically one would only require that $\delta(x)g(x)f(u_\lambda) \in L^1(\Omega)$ where $\delta(x)$ is the distance from $x$ to $\partial \Omega$. Hence here our proof will differ from [7].

Claim: There exist some $C < \infty$ such that
\[
(1.4) \quad \int_\Omega g(x) f'(u_\lambda)f(u_\lambda) \leq C,
\]
for all $0 < \lambda < \lambda^*$ (at this point we are allowing for the possibility of $\lambda^* = \infty$). We first show that the claim implies that $\lambda^* < \infty$. Note that if $(-\Delta)^{\frac{3}{2}} \phi = g(x)$ with $\phi = 0$ on $\partial \Omega$ then an application of the maximum principle along with the fact that $f(u_\lambda) \geq 1$ gives $u_\lambda \geq \lambda^* \phi$ in $\Omega$. This along with (1.4) rules out the possibility of $\lambda^* = \infty$. Using a proof similar to the one in [7] one sees that $u^*$ is a weak solution to $(P)_\lambda$ except for the extra integrability condition $g(x)f(u^*) \in L^1(\Omega)$ that we require. But sending $\lambda \nearrow \lambda^*$ in (1.4) gives us the desired regularity and we are done.

We now prove the claim. Let $u = u_\lambda$ denote the minimal solution of $(P)_\lambda$ and let $u_\epsilon$ denote its harmonic extension. Take $\psi := f(u_\epsilon) - 1$ in (1.3) ($\psi$ can be shown to be an admissible test function) and write the right-hand side as
\[
\int_C \nabla (f(u_\epsilon) - 1)f'(u_\epsilon) \cdot \nabla u_\epsilon,
\]
and integrate this by parts. Using $(P)_\lambda$ and after some cancellation one arrives at
\[
(1.5) \quad \int_C (f(u_\epsilon) - 1)f''(u_\epsilon)|\nabla u_\epsilon|^2 \leq \lambda \int_\Omega g(x)f'(u)f(u).
\]
Define $H(t) := \int_0^t f''(\tau)(f(\tau) - 1)d\tau$ and so the left-hand side of (1.5) can be written as $\int_C \nabla H(u_e) \cdot \nabla u_e$ and integrating this by parts gives

$$\lambda \int_{\Omega} g(x)f(u)H(u).$$

Combining this with (1.5) gives

$$(1.6) \int_{\Omega} g(x)f(u)H(u) \leq \int_{\Omega} g(x)f(u)f'(u).$$

To complete the proof, we show that $H(u)$ dominates $f'(u)$ for big $u$ (resp. $u$ near 1) when $f$ satisfies (R) (resp. (S)). If $0 < T < t$ then one easily sees that

$$H(t) \geq (f(T) - 1)(f'(t) - f'(T)).$$

Using this along with (1.6) and dividing the domain of $\Omega$ into regions $\{u \geq T\}$ and $\{u < T\}$ one obtains the claim.

### 2. Uniqueness of the extremal solution

**Theorem 1.** Suppose that either $f$ satisfies (R) and is log convex or satisfies (S) and is strictly convex. Then the followings hold.

1. There are no weak solutions for $\lambda > \lambda^*$.
2. The extremal solution $u^*$ is the unique weak solution of $(P)_{\lambda^*}$.

The following are some properties that the nonlinearity $f$ satisfies.

**Proposition 1.** (1) Let $f$ be a log convex nonlinearity which satisfies (R).

(i) For all $0 < \lambda < 1$ and $\delta > 0$ there exists $k > 0$ such that

$$f(\lambda^{-1}t) + k \geq (1 + \delta)f(t) \quad \text{for all } 0 \leq t < \infty.$$

(ii) Given $\varepsilon > 0$ there exists $0 < \mu < 1$ such that

$$\mu^2(f(\mu^{-1}t) + \varepsilon) \geq f(t) + \frac{\varepsilon}{2} \quad \text{for all } 0 \leq t < \infty.$$

(iii) Then $f$ is strictly convex.

(2) Let $f$ be a nonlinearity which satisfies (S).

(i) Given $\varepsilon > 0$ there exists $0 < \mu < 1$ such that

$$\mu(f(\mu^{-1}t) + \varepsilon) \geq f(t) + \frac{\varepsilon}{2} \quad \text{for all } 0 \leq t \leq \mu.$$

(ii) Then $\lim_{t \to 1} \frac{f(t)}{F(t)} = \infty$ where $F(t) := \int_0^t f(\tau)d\tau$.

**Proof.** See [1,17] for the proof of (1)-(i) and (1)-(ii). Part (1)-(iii) is trivial.

(2)-(i) Set $h(t) := \mu\{f(\mu^{-1}t) + \varepsilon\} - f(t) - \frac{\varepsilon}{2}$ and note that $h'(t) \geq 0$ for all $0 \leq t \leq \mu$ and that $h(0) > 0$ for $\mu$ sufficiently close to 1, which gives us the desired result.

(2)-(ii) Let $0 < t < 1$ and we use a Riemann sum with right-hand endpoints to approximate $F(t)$. So for any positive integer $n$ we have

$$F(t) \leq \frac{t}{n} \sum_{k=1}^n f\left(\frac{kt}{n}\right) \leq \frac{t(n-1)}{n} f\left(\frac{(n-1)t}{n}\right) + \frac{t}{n}f(t),$$

and it is clear that passing to the limit as $n \to \infty$ yields $F(t) = \int_0^t f(\tau)d\tau$.
Applying Lemma 3, there exists a regular energy solution $u$.

A straightforward computation shows that Proposition 1, we have $0$.

Note that by the strict convexity of $f$.

$(z) := (\varepsilon \tau$ denote the extremal solution of $(P)$. Suppose that $h \neq 0$ and note that $|\tau|_{\Omega}$.

Set $w := \chi \tau$. Define $h_{\varepsilon}$.

Note that by the strict convexity of $f$, which we obtain either by hypothesis or by Proposition 1, we have $0 \leq h$ in $\Omega$ and $h > 0$ in $\Omega_0$. Also note that $h \in L^1(\Omega)$. Define $z := u^* + v$. Since $u^*$ and $v$ are weak solutions of $(P)_{\lambda^*}$, $z$ is a weak solution of

$$(\Delta)^{\frac{1}{2}} z = g(x) f(z) + g(x) h(x) \quad \text{in } \Omega,$$

with $z = 0$ on $\partial \Omega$. From now on we omit the boundary values since they will always be zero unless otherwise mentioned. Let $\chi$ and $\phi$ denote weak solutions of $(-\Delta)^{\frac{1}{2}} \chi = g(x) h(x)$ and $(-\Delta)^{\frac{1}{2}} \phi = g(x)$ in $\Omega$. By taking $\varepsilon > 0$ small enough one has that $\chi \geq \varepsilon \phi$ in $\Omega$. Set $\tau := z + \varepsilon \phi - \chi$ and note that $\tau$ is a weak solution of

$$(-\Delta)^{\frac{1}{2}} \tau = g(x)(f(\tau) + \varepsilon) \geq 0 \quad \text{in } \Omega,$$

and by Lemma 1, we have that $0 \leq \tau$. Moreover, from the fact that $\tau \leq z$ in $\Omega$ we have $g(x)(f(\tau) + \varepsilon) \leq (-\Delta)^{\frac{1}{2}} \tau \in L^1(\Omega)$.

Applying Lemma 3, there exists a regular energy solution $u$ of

$$(-\Delta)^{\frac{1}{2}} u = g(x) (f(u) + \frac{\varepsilon}{2}) \quad \text{in } \Omega.$$

Set $w := u + \alpha u - \varepsilon \phi$ where $\alpha > 0$ is chosen small enough such that $\alpha u \leq \frac{\varepsilon}{2} \phi$ in $\Omega$. A straightforward computation shows that $w$ is a regular energy solution of

$$(-\Delta)^{\frac{1}{2}} w = (1 + \alpha) g(x) f(u) + \frac{\varepsilon}{2} \alpha g(x) \quad \text{in } \Omega.$$
and that $w \leq u$ in $\Omega$. By Lemma 1, we also have $0 \leq w$ in $\Omega$. From this, we see that $w$ is a regular energy supersolution of

$(-\Delta)^{\frac{1}{2}} w \geq (1 + \alpha)g(x)f(w) \quad \text{in } \Omega,$

with zero boundary conditions. We now apply the monotone iteration argument to obtain a regular energy solution $\tilde{u}$ of $(-\Delta)^{\frac{1}{2}} \tilde{u} = (1 + \alpha)g(x)f(\tilde{u})$ in $\Omega$ which contradicts the fact that $\lambda^* = 1$. So, we have shown that $|\Omega_0| = 0$ and so $u^* = v$ a.e. in $\Omega$.

**Proof of Lemma 3:** Let $\varepsilon > 0$ and suppose that $0 \leq \tau \in L^1(\Omega)$ is a weak solution of $(-\Delta)^{\frac{1}{2}} \tau = l(x)$ in $\Omega$ where $0 \leq g(x) (f(\tau) + \varepsilon) \leq l(x)$ in $\Omega$. As in the proof of Theorem 1, we omit the boundary values since they will always be Dirichlet boundary conditions and we also assume that $\lambda^* = 1$. First, assume that $f$ is a log convex nonlinearity, which satisfies (R). Let $u_0 := \tau$ and let $u_1, u_2, u_3$ be weak solutions of

$(-\Delta)^{\frac{1}{2}} u_1 = \mu g(x)(f(u_0) + \varepsilon) \quad \text{in } \Omega,$

$(-\Delta)^{\frac{1}{2}} u_2 = \mu g(x)(f(u_1) + \varepsilon) \quad \text{in } \Omega,$

$(-\Delta)^{\frac{1}{2}} u_3 = \mu g(x)(f(u_2) + \varepsilon) \quad \text{in } \Omega,$

where $0 < \mu < 1$ is the constant given in Proposition 1 such that $\mu^2(f(\frac{\tau}{\mu}) + \varepsilon) \geq f(t) + \frac{\varepsilon}{2}$ for all $t \geq 0$. One easily sees that $u_2 \leq u_1 \leq \mu u_0$. Now note that

$$(-\Delta)^{\frac{1}{2}} u_1 = \mu g(x)(f(u_0) + \varepsilon) \geq \mu g(x) \left( f \left( \frac{u_1}{\mu} \right) + \varepsilon \right).$$

By Proposition 1 with $\delta := 2N - 1 > 0$ and $0 < \lambda = \mu < 1$ there exist some $k > 0$ such that

$$f \left( \frac{u_1}{\mu} \right) \geq 2N f(u_1) - k,$$

hence one can rewrite (2.1) as

$$(-\Delta)^{\frac{1}{2}} u_1 \geq \mu g(x)(2N f(u_1) - k + \varepsilon).$$

We let $\phi$ be as in the proof of Theorem 1 and examine $u_1 + t\phi$ where $t > 0$ is to be picked later. Note that

$$(-\Delta)^{\frac{1}{2}} (u_1 + t\phi) = (-\Delta)^{\frac{1}{2}} u_1 + tg(x) \geq 2N \mu g(x)(f(u_1) + \varepsilon) + mg(x),$$

where $m := t - \mu k + \mu \varepsilon(1 - 2N)$ and we now pick $t > 0$ big enough such that $m = 0$. Therefore, from the definition of $u_2$ we have that

$$(-\Delta)^{\frac{1}{2}} (u_1 + t\phi) \geq 2N (-\Delta)^{\frac{1}{2}} u_2 \quad \text{in } \Omega.$$
but
\[ \beta(\mu u_0 + t\phi) = \beta \left( \mu u_0 + (1 - \mu) \frac{t\phi}{1 - \mu} \right) \leq \mu \beta(u_0) + (1 - \mu) \beta \left( \frac{t\phi}{1 - \mu} \right). \]

From this, we can conclude
\[ f(u_2)^{2N} \leq e^{\mu \beta(u_0)} e^{(1-\mu)\beta \left( \frac{t\phi}{1-\mu} \right)} \leq f(u_0) f \left( \frac{t\phi}{1-\mu} \right)^{1-\mu}. \]

So, we see that \( g(x) f(u_2)^{2N} \leq C g(x) f(u_0) \in L^1(\Omega) \) for some large constant \( C \).

Since \( g(x) \) is bounded, we conclude that \( g(x) f(u_2) \in L^{2N}(\Omega) \). But \( u_3 \) satisfies \((-\Delta)^{\frac{1}{2}} u_3 = \mu g(x)(f(u_2) + \varepsilon) \) in \( \Omega \) and so by elliptic regularity we have that \( u_3 \) is bounded (since the right-hand side is an element of \( L^p(\Omega) \) for some \( p > N \)) and now we use the fact that \( 0 \leq u_3 \leq u_2 \) and the monotone iteration argument to obtain a regular energy solution \( w \) to \((-\Delta)^{\frac{1}{2}} w = \mu g(x)(f(w) + \varepsilon) \) in \( \Omega \).

Now, set \( \xi := \mu w \) and note that \( \xi \) is a regular energy solution of
\[ (-\Delta)^{\frac{1}{2}} \xi = \mu^2 g(x) \left( f \left( \frac{\xi}{\mu} \right) + \varepsilon \right) \text{ in } \Omega, \]
and from Proposition 1, we have
\[ (-\Delta)^{\frac{1}{2}} \xi \geq g(x) \left( f(\xi) + \frac{\varepsilon}{2} \right) \text{ in } \Omega, \]
and so by an iteration argument, we have the desired result.

Now, assume that \( f \) satisfies \( (S) \). In this case, the proof is much simpler. Define \( w := \mu \tau \) where \( 0 < \mu < 1 \) is from Proposition 1. Then note that \( 0 \leq w \leq \mu \) a.e. and
\[ (-\Delta)^{\frac{1}{2}} w = \mu l(x) \geq \mu g(x) \left( f \left( \frac{w}{\mu} \right) + \varepsilon \right) \]
\[ \geq g(x) \left( f(w) + \frac{\varepsilon}{2} \right). \]
Hence, \( w \) is a regular energy supersolution of
\[ (-\Delta)^{\frac{1}{2}} w \geq g(x) \left( f(w) + \frac{\varepsilon}{2} \right), \]
and we have the desired result after an application of the monotone iteration argument.

3. Uniqueness of solutions for small \( \lambda \)

In this section, we prove uniqueness theorems for equation \((P)_\lambda\) for small enough \( \lambda \).
Throughout this section, we assume that \( g = 0 \) on \( \partial \Omega \). We need the following regularity result.

**Proposition 2.** [6] Let \( \alpha \in (0, 1) \), \( \Omega \) be a \( C^{2,\alpha} \) bounded domain in \( \mathbb{R}^N \) and suppose that \( u \) is a weak solution of \((-\Delta)^{\frac{1}{2}} u = h(x) \) in \( \Omega \) with \( u = 0 \) on \( \partial \Omega \).

1. Suppose that \( h \in L^\infty(\Omega) \). Then \( u_\varepsilon \in C^{0,\alpha}(\overline{\Omega}) \) hence \( u \in C^{0,\alpha}(\overline{\Omega}) \).
2. Suppose that \( h \in C^{k,\alpha}(\overline{\Omega}) \) where \( k = 0 \) or \( k = 1 \) and \( h = 0 \) on \( \partial \Omega \). Then \( u_\varepsilon \in C^{k+1,\alpha}(\overline{\Omega}) \) hence \( u \in C^{k+1,\alpha}(\overline{\Omega}) \).
Using this one easily obtains the following:

**Corollary 1.** For each $0 < \lambda < \lambda^*$ the minimal solution of $(P)_{\lambda}$, $u_{\lambda}$, belongs to $C^{2,\alpha}(\overline{\Omega})$. In addition $u_{\lambda} \to 0$ in $C^1(\overline{\Omega})$ as $\lambda \to 0$.

We now come to our main theorem of this section.

**Theorem 2.** Suppose that $\Omega$ is a star-shaped domain with respect to the origin and set $\gamma := \sup_{\Omega} \frac{x \cdot \nabla g(x)}{g(x)}$.

1. Suppose that $f$ satisfies (R) and that

\[
\limsup_{t \to \infty} \frac{F(t)}{f(t)t} < \frac{N-1}{2(N+\gamma)}.
\]

Then for sufficiently small $\lambda$, $u_{\lambda}$ is the unique regular energy solution of $(P)_{\lambda}$.

2. Suppose that $f$ satisfies (S). Then for sufficiently small $\lambda$, $u_{\lambda}$ is the unique regular energy solution $(P)_{\lambda}$.

**Proof:** Let $f$ satisfy (R) and (3.1) or let $f$ satisfy (S) and suppose that $u$ is a second regular energy solution of $(P)_{\lambda}$ which is different from the minimal solution $u_{\lambda}$. Set $v := u - u_{\lambda}$ and note that $v \geq 0$ by the minimality of $u_{\lambda}$ and $v \neq 0$ since $u$ is different from the minimal solution.

A computation shows that $v$ satisfies the equation

\[
(-\Delta)^\frac{3}{2} v = \lambda g(x) \{ f(u_{\lambda} + v) - f(u_{\lambda}) \}.
\]

Applying Proposition 2 to $u$ and $u_{\lambda}$ separately shows that $v_e \in C^{2,\alpha}(\overline{\Omega})$.

A computation shows the following identity holds:

\[
\text{div} \left\{ (z, \nabla v_e) \nabla v_e - z \frac{\left| \nabla v_e \right|^2}{2} \right\} + \frac{N-1}{2} \left| \nabla v_e \right|^2 = (z, \nabla v_e) \Delta v_e,
\]

where $z = (x, y)$. Integrating this identity over $\Omega \times (0, R)$ we end up with

\[
\frac{1}{2} \int_{\partial \Omega \times (0, R)} \left| \nabla v_e \right|^2 x \cdot v + \int_{\Omega} x \cdot \nabla v_e \partial_v v_e + \frac{N-1}{2} \int_{\Omega \times (0, R)} \left| \nabla v_e \right|^2 + \varepsilon(R) = 0,
\]

where

\[
\varepsilon(R) := \int_{\Omega \times \{y=R\}} (x \cdot \nabla v_e + R \partial_y v_e) \partial_y v_e - \frac{R}{2} \left| \nabla v_e \right|^2.
\]

One can show that $\varepsilon(R) \to 0$ as $R \to \infty$, for details on this and the above calculations see [24]. Sending $R \to \infty$ and since $\Omega$ is star-shaped with respect to the origin, we have

\[
\frac{N-1}{2} \int_{\Omega} \left| \nabla v_e \right|^2 \leq - \int_{\Omega} x \cdot \nabla x v \partial_v v_e,
\]

and after using (3.2) one obtains

\[
\frac{N-1}{2} \int_{\Omega} \left| \nabla v_e \right|^2 \leq \lambda \int_{\Omega} x \cdot \nabla x v g(x) \{ f(u_{\lambda} + v) - f(u_{\lambda}) \}.
\]

We now compute the right-hand side of (3.4). Set $h(x, \tau) := f(u_{\lambda}(x) + \tau) - f(u_{\lambda}(x))$ and let $H(x, t) = \int_{0}^{t} h(x, \tau) d\tau$. For this portion of the proof, we are working on $\Omega$ and
hence all gradients are with respect to the $x$-variable. To clarify our notation note that the chain rule can be written as

$$\nabla H(x, v) = \nabla_x H(x, v) + h(x, v) \nabla v,$$

where we recall $v = v(x)$. Some computations now show that

$$H(x, t) = F(u_\lambda + t) - F(u_\lambda) - f(u_\lambda) t,$$

and

$$\nabla_x H(x, t) = \{f(u_\lambda + t) - f(u_\lambda) - f'(u_\lambda) t\} \nabla u_\lambda,$$

and so the right-hand side of (3.4) can be written as

$$-\lambda \int_{\Omega} g(x) \{f(u_\lambda + v) - f(u_\lambda)\} x \cdot \nabla v = -\lambda \int_{\Omega} g(x) h(x, v) x \cdot \nabla v$$

$$= -\lambda \int_{\Omega} g(x) \{\nabla H(x, v) - \nabla_x H(x, v)\}$$

$$= \lambda \int_{\Omega} g(x) x \cdot \nabla_x H(x, v) + \lambda N \int H(x, v) g(x)$$

$$+ \lambda \int H(x, v) x \cdot \nabla g(x).$$

Therefore, (3.4) can be written as

$$\frac{N - 1}{2} \int_{C} \|\nabla v_e\|^2 \leq \lambda \int_{\Omega} x \cdot \nabla u_\lambda g(x) \{f(u_\lambda + v) - f(u_\lambda) - f'(u_\lambda) v\}$$

$$+ N \lambda \int_{\Omega} g(x) \{F(u_\lambda + v) - F(u_\lambda) - f(u_\lambda) v\}$$

$$+ \lambda \int_{\Omega} x \cdot \nabla g(x) \{F(u_\lambda + v) - F(u_\lambda) - f(u_\lambda) v\}.$$

(3.5)

We now assume that we are in case (1). Let $\alpha$ be such that

$$\limsup_{\tau \to \infty} \frac{F(\tau)}{\tau f(\tau)} < \alpha < \frac{N - 1}{2(N + \gamma)},$$

so there exist some $\tau_0 > 0$ such that $F(\tau) < \alpha \tau f(\tau)$ for all $\tau \geq \tau_0$. Let $0 < \theta < 1$ be such that $\frac{\theta(N - 1)}{2} - \alpha(N + \gamma) > 0$ and we now decompose the left-hand side of (3.5) into the convex combination

$$\frac{\theta(N - 1)}{2} \int_{C} \|\nabla v_e\|^2 + \frac{(N - 1)(1 - \theta)}{2} \int_{C} \|\nabla v_e\|^2.$$

(3.6)

Using the following trace theorem: there exist some $\tilde{C} > 0$ such that

$$\int_{C} \|\nabla w\|^2 \geq \tilde{C} \int_{\Omega} w^2, \quad \forall w \in H^1_{0,L}(C),$$

(3.7)

one sees that (3.6) is bounded below by

$$\frac{\theta(N - 1)}{2} \int_{C} \|\nabla v_e\|^2 + C \int_{\Omega} v^2.$$
By taking $C > 0$ smaller if necessary one can bound this from below by
\[
\frac{\theta(N - 1)}{2} C \int_\Omega |\nabla v_\epsilon|^2 + C \int_\Omega g(x)v^2,
\]
and after using (3.2), this last quantity is equal to
\[
\frac{\lambda \theta(N - 1)}{2} \int_\Omega g(x)\{f(u_\lambda + v) - f(u_\lambda)\}v + C \int_\Omega g(x)v^2.
\]
Substituting (3.8) into (3.4) and rearranging one arrives at an inequality of the form
\[
\int_\Omega g(x)T_\lambda(x, v) \leq 0,
\]
where
\[
T_\lambda(x, \tau) = \frac{\theta(N - 1)}{2} \{f(u_\lambda + \tau) - f(u_\lambda)\}\tau + \frac{C}{\lambda} \tau^2
- N\{F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda)\tau\}
- x \cdot \nabla g \{F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda)\tau\}
- x \cdot \nabla u_\lambda \{f(u_\lambda + \tau) - f(u_\lambda) - f'(u_\lambda)\tau\}.
\]
To obtain a contradiction, we show that for sufficiently small $\lambda > 0$ that $T_\lambda(x, \tau) > 0$ on $(x, \tau) \in \Omega \times (0, \infty)$ and hence we must have that $v = 0$. Define
\[
S_\lambda(x, \tau) = \frac{\theta(N - 1)}{2} \{f(u_\lambda + \tau) - f(u_\lambda)\}\tau + \frac{C}{\lambda} \tau^2
- (N + \gamma)\{F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda)\tau\}
- \varepsilon_\lambda \{f(u_\lambda + \tau) - f(u_\lambda) - f'(u_\lambda)\tau\}.
\]
where $\varepsilon_\lambda := \|\nabla u_\lambda \cdot x\|_{L^\infty}$. Note that since $f$ is increasing and convex that $T_\lambda(x, \tau) \geq S_\lambda(x, \tau)$ for all $\tau \geq 0$. We now show the desired positivity for $S_\lambda$ and to do this we examine large and small $\tau$ separately.

Large $\tau$: Take $\tau \geq \tau_0$ and $0 < \lambda \leq \frac{\lambda^*}{2}$. Since $f$ is convex and increasing
\[
S_\lambda(x, \tau) \geq \frac{\theta(N - 1)}{2} \{f(u_\lambda + \tau)\}\tau - (N + \gamma)F(u_\lambda + \tau)
- \varepsilon_\lambda f(u_\lambda + \tau) + \frac{C}{\lambda} \tau^2
- \frac{\theta(N - 1)}{2} f(u_\lambda)\tau,
\]
but $F(u_\lambda + \tau) < \alpha(u_\lambda + \tau)f(u_\lambda + \tau)$ for all $\tau \geq \tau_0$ and so the right-hand side of (3.9) is bounded below by
\[
f(u_\lambda + \tau) \left\{ \frac{\theta(N - 1)}{2} - (N + \gamma)\alpha \right\} - \varepsilon_\lambda - (N + \gamma)\alpha u_\lambda
- \frac{\theta(N - 1)}{2} f(u_\lambda)\tau + \frac{C}{\lambda} \tau^2.
\]
Using the fact that \( f \) is superlinear at \( \infty \) there exist some \( \tau_1 \geq \tau_0 \) such that \( S_\lambda(x, \tau) > 0 \) for all \( \tau \geq \tau_1 \) and \( 0 < \lambda \leq \frac{\lambda_0}{2} \).

Small \( \tau \): Let \( 0 < \lambda_0 < \frac{\lambda}{2} \) be such that \( \|u_\lambda\|_{L^\infty} \leq 1 \). Using the convexity and monotonicity of \( f \) and Taylor’s Theorem there exist some \( C_1 > 0 \) such that

\[
F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda)\tau \leq C_1 \tau^2, \\
F(u_\lambda + \tau) - f(u_\lambda) - f'(u_\lambda)\tau \leq C_1 \tau^2,
\]

for all \( 0 \leq \tau \leq \tau_0 \), \( 0 < \lambda \leq \lambda_0 \) and \( x \in \Omega \). Noting that the first term of \( S_\lambda(x, \tau) \) is positive for \( \tau > 0 \) one sees that for all \( 0 < \tau \leq \tau_0 \), \( x \in \Omega \) and \( 0 < \lambda < \lambda_0 \) one has the lower bound

\[
S_\lambda(x, \tau) \geq \frac{C}{\lambda} \tau^2 - (N + \gamma + \varepsilon_\lambda) C_1 \tau^2,
\]

and hence by taking \( \lambda \) smaller if necessary we have the desired result.

(2) We now assume that \( f \) satisfies (S). One uses a similar approach to arrive at an inequality of the form

\[
\int_{\Omega} T_\lambda(x, v) \leq 0,
\]

where as before \( v = u - u_\lambda \geq 0 \) and where we assume that \( v \neq 0 \). To arrive at a contradiction we show that for sufficiently small \( \lambda \) that \( T_\lambda(x, \tau) > 0 \) for all \( x \in \Omega \) and for all \( 0 < \tau < 1 - u_\lambda(x) \). Again the idea is to break the interval into two regions. For \( \tau \) such that \( \tau + u_\lambda(x) \) close to 1, we use Proposition 1, 2 (ii) to see the desired positivity. For the remainder of the interval, we again use Taylor’s Theorem.

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