THE HOMOMORPHISM OF PRESHEAVES $K_*^{MW} \to \pi_*^{\ast,\ast}$ OVER A BASE.

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ABSTRACT. We construct the homomorphism of presheaves $K_*^{MW} \to \pi_*^{\ast,\ast}$ over an arbitrary base scheme $S$, where $K_*^{MW}$ is the (naive) Milnor-Witt $K$-theory presheave.

Also we discuss some partly alternative proof (or proofs) of the isomorphism of sheaves $K_*^{MW} \cong \pi_*^{n,n}$, $n \in \mathbb{Z}$, over a filed $k$ originally proved in [31] and [33].

1. Introduction

The presheaf of the (naive) Milnor-Witt $K$-theory $K_*^{MW}$ is defined as a graded ring with generators $[a] \in K_*^{MW}$, $\forall a \in G_m$ and $\eta \in K_{-1}^{MW}$ and relations

\begin{align}
& \text{(Steinberg relation)} \quad [x][1-x] = 0, \quad \forall x \in (G_m - \{1\}), \\
& \eta[x][y] = [xy] - [x] - [y], \quad \forall x, y \in G_m \\
& \eta \in G_m \\
& \eta[\eta - 1] + 2 = 0.
\end{align}

As shown in [12, section 4.2.1] the result [23, theorem 6.3] implies that the Zariski sheafification $K_*^{MW}$ of the presheaf $K_*^{MW}$ over an infinite field $k$ of odd characteristic is equal to the unramified Milnor-Witt $K$-theory sheaf $K_*^{MW}$ defined in [33, section 3], which is by defined as an unramified sheaf that is equal to the (naive) $K_*^{MW}$ on fields. The stable version of the Morel’s theorem [33, theorem 19, cor. 21] states isomorphism of sheaves $K_*^{MW} \cong \pi_*^{n,n}$ for a (perfect) base field $k$ of an arbitrary characteristic.

The result of the paper is the following

Theorem 1.2. The assignment

\begin{align}
& [x] \mapsto [pt \mapsto x] \\
& \eta \mapsto \Sigma^{-1}_G \{m - p_1 - p_2\} \in [pt, G_m, G_m, G_m, \ldots, G_m]_{SH(S)},
\end{align}

where $m: G_m \to G_m$, $(x, y) \mapsto xy$, and $p_1, p_2: G_m \to G_m$ are the projections, induces the homomorphism of presheaves $K_*^{MW} \to \pi_*^{\ast,\ast}$ for any base scheme $S$.

Since there is a canonical endomorphism $K_*^{MW}(S) \to K_*^{MW}(S \times G_m)$ given by $\phi \mapsto [t] \phi$, and $[G_m \times X, G_m, G_m, \ldots, G_m]$ in $SH(S)$, the theorem follows directly form

Proposition 1.4. The following equalities hold in the stable motivic homotopy category $SH(S)$ for all $S$

\begin{align}
& [x, 1-x] = 0 \\
& \Sigma^{[x]}[y] = \Sigma^{[x]} \in (G_m, G_m, G_m, \ldots, G_m)_{SH(S)}, \\
& \Sigma^{[x]}[y] = \Sigma^{[x]} \in (G_m, G_m, G_m, \ldots, G_m)_{SH(S)}, \\
& (\Sigma^{[x]} - 1) + 2\Sigma^{[x]} = 0 \\
& (\Sigma^{[x]} - 1) + 2\Sigma^{[x]} = 0
\end{align}

where the products in equalities are the external product with respect to the monoidal structure, and

- $\Sigma^{[x]}$ denotes $\Sigma^{[x]}$, $\Sigma^{[x]}$ denotes $\Sigma^{[x]}$,
- $(1-x, x): (A^1 \times \{0, 1\}) \times S \to G_m \times G_m$ is a regular map, and $[(1-x, x)]$ is its class in $SH(S)$,
- $(x), (y): G_m \to G_m$ denote two copies on the identity map, $[x], [y] \in [G_m, G_m]$ denote their classes, and $[-1]: [pt, G_m, G_m]$ denotes the class of $pt \to G_m$ given by $-1$,
- $m, p_1, p_2: G_m \to G_m$ are the product map and the projection maps, and $[m], [p_1], [p_2]: [G_m, G_m]$ are the induced homomorphisms.

We should say that the Steinberg relation in $SH$ (the first one in the list above) is proven originally by Po Hu and Igor Kriz in [27]. The proof is written for the base filed base but it works as well for an arbitrary base. To keep the text being complete we present here the short alternative argument. From what the author understands this argument is essentially equivalent to the original proof.

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1.1. Strategy of the proof.

1.1.1. Steinberg relation. The Steinberg relation follows from that the class of a morphism \( c: U \to G_m \times G_m \) in the group \([U_+, G_m^{\Lambda^2}]_{\text{SH}(S)}\) is equal to a composition \( U_+ \to pt_+ \to G_m^{\Lambda^2} \), if \( c \) fits into a diagram

\[
\begin{array}{ccc}
U & \longrightarrow & X \\
\downarrow & & \downarrow \\
G_m^2 & \longrightarrow & \mathbb{A}^2 \\
\end{array}
\]

such that \( X \simeq \text{pt} \in \text{SH}(S) \). Then applying this to \( U = (\mathbb{A}^1 - \{0, 1\}) \amalg G_m = Z(x+y-1) \amalg Z(x-1) \subset G_m^2 \) we see that the classes of the maps \((\mathbb{A}^1 - \{0, 1\})_+ \to G_m^{\Lambda^2}: (t) \mapsto (t,1-t), \text{and } (G_m)_+ \to G_m^{\Lambda^2}: (t) \mapsto (1,t)\) both are equal to the same constant. But the class of the second one is trivial, hence the class of the first one too.

In non-stable case our the first argument requires the stabilisation by one \(S^1\) suspension.

1.1.2. Other relations. The image of the homomorphism \( K_{n}^{\text{MW}} \to \pi_{n,n}(S) \) are the sum of class \( \Sigma^{-1}_G f \) for a regular morphisms \( f \in G_m^2 \to G_m^{\Lambda^2} \). Let \( \bullet \) denotes the external product (composition) of morphisms with respect to the monoidal structure, and \( \circ \) denotes the composition of morphisms in the categorical sense. The last three relations from (1.7) follows from the following observations:

\[
\text{(prop 3.11)} \quad \Sigma_G^2 [f \circ g] = (-1)^{mn} f \bullet g = \Sigma_G^2 (-1)^{m(n+1)} (g \circ \Sigma_G^m f)
\]

for any \( f \in \text{Map}(G_m^2, G_m^2), g \in \text{Map}(G_m^2, G_m^2) \) for some \( r_1, r_2 \in \mathbb{Z} \);

\[
\text{(lm 3.10)} \quad [T] = \Sigma_G^2 (-1) \in [G_m^2 \amalg G_m^2, G_m \amalg G_m]_{\text{SH}(S)},
\]

where \( T \) is a permutation on \( G_m^2 \) and \( \Sigma_G^2 (-1) \in [G_m^{\Lambda^1}, G_m^{\Lambda^1}]_{\text{SH}(S)} \) is the class of the map \( G_m \to G_m: t \mapsto -t \).

\[
\text{(rem 3.9)} \quad m \circ T = m, \text{where } m: G_m^2 \to G_m \text{ is the product.}
\]

The first equality is a variant of the fact that two groupoid operations satisfying the property \((f_1 \circ f_2) \bullet (g_1 \circ g_2) = (f_1 \bullet g_1) \circ (f_2 \bullet g_2)\) are equal and commutative. Actually the first equality follows from the second one from the list above. The second equality can be proven either as a consequence of the fact that elementary transformation over \( S \) which permutes coordinates. In the nonstable case the argument with elementary transformations uses stabilisation by one \(S^2\). So the proof of other three relations holds after the smash with \( S^2 \).

Note that alternatively the equality \( T = \Sigma_G^2 (-1) \) can be obtained using the framed permutation homotopy on \( G_m^{\Lambda^2} \) form (2), but this argument requires \(P^1\) stabilisation.

1.2. Proofs for the case of a base field.

1.2.1. (The universal strongly homotopy invariant theory). In the Morel’s book \[33\] the homomorphism \( K_{n}^{\text{MW}} \to \pi_{n,n}(S) \), where \( K_{n}^{\text{MW}} \) is the sheaf of the unramified Milnor-Witt K-theory, follows from the universal property of the sheaf \( K_{n}^{\text{MW}} \) in the class of strongly homotopy invariant sheaves over a field. In the case of non-zero dimensional base it is unknown does the sheaves \( \pi_{n,n} \) are strongly homotopy invariant sheaves. In the same time some key inner arguments from \[33\] in the proof of the last three relations (1.1) looks being general and should work over an arbitrary base. So the author doesn’t know entirely is it possible to prove this relations in \( \pi_{n,n} \) over any \( S \) using the arguments from \[33\].

1.2.2. (The Steinberg relation). The Steinberg relation in \( \text{SH}(k) \) for an arbitrary filed \( k \) was proven originally by Po Hu and Igor Kriz in \[27\], and reproven by Geoffrey M.L. Powell in \[30\]. The arguments of both proofs can be word by word repeated in the case of an arbitrary base.

The author apologize for the doubts in the correctness of the arguments \[27\], which he had wroted in previous version, now he had understand the original proof.

Nevertheless the author still do not understand the alternative proof in \[30\]. In \[30\] definition 3.0.7, proposition 3.0.8 nothing is mentioned about the base point in the \( \mathbb{A}^n \) for the morphism \( X \to \mathbb{A}^n \), and is the cone considered in \[30\] proposition 3.0.8 is just the cone of the morphism the unpointed varieties \( X \to \mathbb{A}^n \) then in \( \text{SH} \) it is equivalent to \( S^1 \wedge \text{Fib}_S(X \to pt_+) \), but not to the suspension of \( X_+ \). The author would appreciate if some one can explain what is meant there.
1.2.3. (Framed correspondences). Alternatively the relations of Milnor-Witt K-theory in the stable motivic homotopy groups over fields were proven in [34] by A. Neshitov. The prove is given by precise framed homotopies and the relations are proven in $H^q(\mathcal{ZF}(\Delta^*, G^{red}_m))$, which is formally stronger than relations in $\pi^{n,n}(k)$. In the same time the proof of Steinberg relation requires assumption that the base field is of characteristic different form 2 and 3. From what the author understands at least some of the framed homotopies used in the proof could be lifted at least to henselian local bases. If this is true for all homotopies, then this would imply the homomorphism of Nisnevich sheaves $K^M_{/\mathbb{A}} \to \pi_{/\mathbb{A}}^{n,n}$ over $\mathbb{Z}[1/6]$.

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1.4. Notation: All products, points, and schemes are considered relatively over the base scheme $S$.

2. Proof of the Steinberg relation

2.1. The reduced curve. In the subsection we prove the Steinberg relation up to some constant, i.e we prove that the morphism $(1 - x, x): (\mathbb{A}^1 - \{0, 1\}) \to G^m_n$ can be passed in $\mathbf{SH}(S)$ throw $((\mathbb{A}^1 - \{0, 1\}))_+ \to pt_+ \to G^m_n$ in $\mathbf{SH}(S)$. We refer reader to [24] Appendix C for the definition of the stable motivic homotopy category $\mathbf{SH}(S)$ over an arbitrary scheme $S$.

Notation 2.1. For any $X \in Sm_S$ denote by $X/pt$ the fibre of the morphism $X \to pt$ in $\mathbf{SH}(S)$

$$X/pt \to X_+ \to pt_+ \to (C/pt) \land S^1.$$  

So any morphism $X \to Y$ for $X, Y \in \mathbf{SH}(S)$ induces the morphism $C/pt \to Y$ via the composition $C/pt \to C \to Y$.

Lemma 2.2. Let $X$ be a scheme over $S$, and assume that $X$ is $\mathbb{A}^1$ contractible, i.e. the canonical morphism $X \to pt$ is equivalence in $SH(S)$. Let $\zeta: X \to \mathbb{A}^2$ be a morphism of schemes such that $X \times_{\mathbb{A}^2} (Z(xy) - Z(x+y-1)) = \emptyset$, where $x$ and $y$ denotes coordinate functions on $\mathbb{A}^2$ (and so $Z(xy) - Z(x+y-1)$ $\simeq (\mathbb{A}^1 - \{0\}) \coprod (\mathbb{A}^1 - \{1\})$).

Denote $U = X \times_{\mathbb{A}^2} G^m_n$ and $e: U \to G^2_n$. Then the class of the morphism $e$ in $[U/pt, G^m \land G_m]$ is trivial.

Proof. It follows form the assumption on $\zeta$ that $X \times \mathbb{A}^2 \{(0, 0)\} = \emptyset$. Consider the diagram of the triangles in $\mathbf{SH}(S)$

$$(2.3) \quad \begin{array}{ccc}
\mathbf{G}_m \times \{1\} & \hookrightarrow & (\mathbb{A}^1 \times \{0\}) \cup (\{0\} \times \mathbb{A}^1) \\
\mathbf{G}_m \land \mathbf{G}_m & \xrightarrow{\beta} & (\mathbb{A}^2 - 0) \\
\mathbf{G}_m \land \mathbf{G}_m & \xrightarrow{\gamma} & \text{Cone}(\alpha) \\
\mathbf{G}_m \land \mathbf{G}_m & \xrightarrow{\delta} & \text{Cone}(\gamma) \\
U/pt & \xrightarrow{\zeta} & X/pt \\
U/pt & \xrightarrow{\xi} & \text{Cone}(\zeta) \\
\end{array}$$

where $X/U = \text{Cone}(U \to X) = \text{Cone}(U_+ \to X_+)$. It follows form the assumption $\xi$ that $X/U \simeq (X - Z_1)/(U - Z_1) \lor (X - Z_2)/(U - Z_2)$, where $Z_1 = (X \times \mathbb{A}^2 \{0\})_{red} = X \times \mathbb{A}^2 \{(1, 0)\}$, $Z_2 = X \times \mathbb{A}^2 \{0\} \times \mathbb{A}^1 = X \times \mathbb{A}^2 \{(0, 1)\}$, and $U_1 = X - Z_2$, $U_2 = X - Z_1$. 

$\mathbf{G}_m \land \mathbf{G}_m \simeq X/pt \land S^1$
Let \( \xi_1, \xi_2 \to X \to \mathbb{A}^1, \xi = (\xi_1, \xi_2) \). The homotopy
\[
(X - Z_l)/(U - Z_l) \times \mathbb{A}^1 \to (k^2 - 0)/(G_m \times \mathbb{A}^1) \ni (p) \mapsto (\xi_1(p), (1 - \lambda)\xi_2(p) + \lambda),
\]
implies that the morphism \( (X - Z_l)/(U - Z_l) \to \text{Cone}(\beta) \) induced by \( \xi \) in \( \text{SH}(S) \) is trivial. Similarly the morphism \( (X - Z_l)/(U - Z_l) \to \text{Cone}(\beta) \) is trivial. Hence the vertical arrow \( X/U \to \text{Cone}(\beta) \) in the diagram is trivial.

Since \( X \to pt \) is an isomorphism by assumption, it follows that the last arrow in the last row in the diagram is isomorphism; then since the second last vertical arrow in the diagram is isomorphism, it follows that the composition \( S^1 \cup U/pt \to S^1 \cup (U_+) \simeq G_m \land G_m \) is trivial. The claim follows.

**Proposition 2.4.** The class of the morphism \((\mathbb{A}^1 - \{0, 1\}) \to G_m; (t) \mapsto (t, 1-t) \) in \([\mathbb{A}^1 - \{0, 1\}], G_m^2 \text{SH}(S)\) is trivial.

**Proof.** Applying lemma 2.2 to the closed subscheme \( X = Z((x - 1)(x + y - 1)) \subset \mathbb{A}^2 \) we see that the morphism \( \iota: (\mathbb{A}^1 - \{0, 1\}) \to G_m \to G_m^2 \) in \([\mathbb{A}^1 - \{0, 1\}], G_m^2 \) is trivial. Hence the class \([\mathbb{A}^1 - \{0, 1\}], G_m^2 \text{SH}(S)\) is trivial.

Now since the class of the composition \( c' \circ 1 \), where \( 1: pt_+ \to (G_m)_+ \) is given by the point \( \{1\} \), defines the zero morphism in the group \([pt_+, G_m^2 \text{SH}(S)], (X, Y, Z) \subset \text{SH}(S), \).

\[\circ:\text{ Then if } Y = X', \text{ we can define the composition morphism in } \text{Map}(X, Y') \text{ (or } [X, Y']_{\text{SH}} \text{) which we denote by } g \circ f.\]

\[\bullet:\text{ Denote by } f \cdot g \in \text{Map}(X \times X', Y \times Y') \text{ the (external) product, which we also call an external composition, the same notation we use for } f \cdot g \in \text{Hom}(X \times X', Y \times Y').\]

\[\circ:\text{ Let us note that we use both notations } fg \text{ and } f \cdot g \text{ for the external products in } SH, \text{ but only } f \cdot g \text{ for } Sm \text{ and } ZSm^{id}.\]

\[\Sigma g_m^l: \text{ denote } \Sigma g_m^l f = \text{id}_{G_m} \circ f \text{, } f \circ \Sigma g_m^l = f \circ \text{id}_{G_m}.\]

\[\Sigma G_m^l: \text{ let us write } f \sim G_m^{l'} g \text{ iff } \Sigma G_m^{l'} = f \circ \Sigma G_m^{l} g \text{ for some } l, l' \in Z.\]

**Remark 3.2.** For any \( f \in \text{Map}(G_m^{n}, G_m^{n'}) \) \( g \in \text{Map}(G_m^{m}, G_m^{m'}) \), \( f \circ \Sigma G_m^{m} g = f \cdot g \in \text{Map}(G_m^{m+n'}, G_m^{n+m}) \).

**Remark 3.3.** If \( f \sim G_m^{l'} f', g \sim G_m^{l''} g', f' = g' \text{ then } f = g.\)

**Definition 3.4.** Define regular maps
- \( m: G_m \times G_m \to G_m; (x, y) \mapsto xy, \)
- \( m_a: G_m \to G_m; x \mapsto ax.\)

**Definition 3.5.** Let \( G_m \simeq G_m^1 \otimes pt \in SH \) be the isomorphism given by the point \( 1 \in G_m. \) For a regular map \( f \in \text{Map}(G_m^{m}, G_m^{m}) \) denote by \( f \in \text{Hom}(ZSm^{id}, G_m^{m}) \) the induced morphism in the Karoubi envelope of the linearisation of \( Sm. \)

For any morphism \( f \in ZSm^{id}, f \in \text{Hom}(X, Y) \) denote by \([f] \in [X, Y]_{\text{SH}}\) the class of the morphism in \( SH. \)
Example 3.6. Let \( m : G_m \times G_m \to G_m \) \((a, b) \mapsto ab\) be the multiplication morphism,\n\[
\begin{align*}
  & m = \Sigma_G^2 \eta \in [G_m^2, G_m^1], \\
  & [(a)] = [a] \in [pt, G_m^{\Lambda^1}]_{SH}, \\
  & [m_a] = \Sigma_G^2(a) \in [G_m^{\Lambda^2}, G_m^{\Lambda^1}]_{SH}. 
\end{align*}
\]

Remark 3.7. Computing the composition of morphisms \( \overrightarrow{f} \circ \overrightarrow{g} \); \( f \in \text{Map}(G_m^n, G_m^m) \), \( g \in \text{Map}(G_m^m, G_m^l) \), it is suitable to think about the morphisms induced by \( \overrightarrow{f} \) (and \( \overrightarrow{g} \)) in \([G_m^n, G_m^m]_{SH}\). This is given by the formula \( P_n \circ f \circ P_m \) where \( P_n = \prod_{i=1}^n (id_{G_m^i} - 1 \circ p_i) \), \( P_l = pt \to G_m^l, p_l : G_m^n \to G_m^{n-l} \) is the projection along the \( i\)-th multiplicand.

Definition 3.8. Let \( T \) be the permutation on \( G_2^2 \);
\[
H = \text{id}_G + m \in \mathbb{Z} Sm(G_m, G_m),
\]
\[
h = \Sigma_G^{-1} \{ \overrightarrow{f} \} = (1) + (-1) \in [pt, pt]_{SH}.
\]

Remark 3.9. By commutativity we have \( m \circ T = m \).

Lemma 3.10. For a permutation \( P \in \text{Aut}_m(G_m^n) \) with the sign \( s \)
\[
\{ \overrightarrow{f} \} = \Sigma_G^2(-1)^s = \Sigma_G^1(-1)^s \Sigma_G^{N-1}.
\]

Proof. Since \( G_m^n \cong G_m^1 \wedge S^1 = T \wedge \mathbb{A}^1 = \mathbb{A}^1 / (\mathbb{A}^1 - 0) \) the claim follows from the fact that any permutation defines the matrix in the subgroup in \( \text{GL}(\mathbb{Z}) \) generated by elementary matrices and the matrix the diagonal matrix \((-1, 1, \ldots, 1) \). Let us note in addition that the general case follows from the case of the twist on \( G_m^2 \). Linear homomorphism is equal to \( \square \)

Proposition 3.11. For any \( f \in [G_m^n, G_m^m]_{SH}, g \in [G_m^m, G_m^n]_{SH} \)
we have \( f \circ g \sim^G (1 - 1)^{m \cdot n} f \circ g \sim^G (-1)^{m \cdot n} f \).

Proof. The first equivalence follows form
\[
f \circ g \sim^G \Sigma_G^{m+n} f \circ \Sigma_G^{n+m} g = \Sigma_G^m f \circ \hat{P} \circ \Sigma_G^m g \circ \hat{P} = \Sigma_G^m f \circ (-1)^{n(m+1)} \Sigma_G^m \circ \Sigma_G^m g \circ (-1)^{n(m-1)} \Sigma_G^m = (-1)^{m \cdot n} f \circ g
\]
where \( \hat{P} : G_m^n \wedge G_m^{m+1} \to G_m^{n+1} \wedge G_m^n \) and \( \hat{P} : G_m^n \wedge G_m^{m+1} \to G_m^{n+1} \wedge G_m^n \). Note that the sign of the permutation \((1, \ldots, l, l+1, \ldots, l+k) \to (l+1, \ldots, l+k, 1, \ldots, l) \) is equal to \( l(l+k+1) = l^2 + l(k+1) = l(1+k+1) = l(k+1) \text{ mod } 2, \forall l, k \in \mathbb{Z}_{\geq 0} \). The second equivalence follows form the first one applied to \( \Sigma_G^{m \cdot n} f \) and \( g \). \( \square \)

Remark 3.12. The sign in \( \Sigma_G^{m \cdot n} \) is the only one sign which we essentially use in the proof of relation of Milnor-Witt K-theory in \( SH \).

The first relation in the list \( (1) \) follows almost tautologically form proposition \( 2.2 \) and the definition of \( \eta \).

Lemma 3.13. The following equality for morphisms in \( SH \) holds
\[
\Sigma \eta \bullet [a] \bullet [b] = [p \circ (m - \overrightarrow{m} - \overrightarrow{n})] \in [G_m \wedge G_m, G_m^{\Lambda_1}]_{SH},
\]
where \( p_1, p_2 : G_m^2 \to G_m \) are projections, and where \( a, b, p : G_m \to G_m^{\Lambda_1} \) denotes three copies of the canonical projection.

Proof. It follows from prop \( 8.1(1) \) that
\[
\Sigma \eta \bullet [a] \bullet [b] = \Sigma \eta \circ ([a] \bullet [b]) = [\overrightarrow{m} \circ ((a) \bullet (b))] = p \circ (m - \overrightarrow{m} - \overrightarrow{n})
\]
where \( \Sigma \eta = \Sigma_G \eta \). \( \square \)

Now we prove the other two relation.

Proposition 3.14. The following equality holds
\[
[a] \Sigma \eta = \Sigma \eta [a] \in [G_m^{\Lambda_2}, G_m^{\Lambda_1}]_{SH}.
\]
Proof.

\[[a] \eta = [(\overline{m} \bullet m) = T \circ (m \bullet \overline{m})]^{prop. 3.11} \circ [m \bullet (a) \circ T] = ([m \circ T] \bullet (a))^{ren \frac{3.9}{m \bullet (a)}} = \eta[a] \]

□

Proposition 3.15. The following equality holds

\[\eta^2[-1] + 2\eta = 0 \in [pt, G_m^{\wedge -2}]_{SH}. \]

Proof. Recall \(\Sigma \eta = \Sigma^2 G \eta\). Using prop 3.11 we have

\[\Sigma \eta(\Sigma \eta[-1]) = [m \bullet (m \bullet (-1))]^{con.(3,2)} \circ [m \circ (m \bullet (-1))],\]

and the straightforward computation in the Karoubi envelope of the linearisation of \(S_m \) in view of rem 3.7 shows that

\[m \circ (m \bullet [-1]): G_m^2 \rightarrow G_m: (x, y) \mapsto (-xy) - (-x) - (-y) - (xy) + (x) + (y) - (-1).\]

So \(\eta \bullet (\eta \bullet [-1] + 2iG_{m,2}) = \Sigma_G^2 [m \circ (m \bullet (-1)) + 2m]\), and

\[m \circ (m \bullet (-1)) + 2m: G_m^2 \rightarrow G_m: \]

\[(x, y) \mapsto (-xy) + (xy) - ((x) + (x)) - ((-y) + (y)) - ((-1) + (1)) = \]

\[\overline{H}(xy) - (x) - (y) + (1)) = \overline{H}(m(x, y)).\]

Thus we have got

\[\eta \bullet \eta \bullet [-1] + 2\eta = \Sigma^2_G [m \circ m]^{prop. 3.11} h_m.\]

Now we see

\[\overline{H} [m \circ m]^{prop. 3.11} \circ [m \circ \Sigma_G^1 \overline{H}]^{in. 3.10} \circ [m \circ (iG_{m,2} - T)]^{ren \frac{3.9}{m \circ (iG_{m,2} - T)}} \]

□

4. THE HOMOMORPHISM \(K_{n}^{MW}(S) \rightarrow H^0(\Delta S, G_m^n)\).

In the section we lift the homomorphism \(K_{n}^{MW}(S) \rightarrow \pi^{n,n}_S\) to the level of framed correspondences.

Let us briefly recall definition of the category \(SH^{fr}(S)\), see [10]. Consider the infinite category of additive presheaves of \(S^1\)-spectra with framed transfers \(Pre^{\Sigma}(\text{Corr}^{fr}_S)\) over \(S\). Let \(SH^{fr}_{A^1,S} = Pre^{\Sigma}_{A^1}(\text{Corr}^{fr}_S)\) denotes the localisation with respect to morphisms \(A^1 \rightarrow X\). Let \(SH^{fr}_{A^1}\) be the stabilisation of \(SH^{fr}_{A^1,S}\) with respect to \(G_m\). Then it follows form the usual (simplicial or topological) Hurevich isomorphism that

\[\text{[pt, Y]}_{SH^{fr}_S} = H^0(\Delta S, Y).\]

Since any regular map gives us a framed correspondences and since by the Cancellation theorem [2] we have \([X \wedge G_m, Y \wedge G_m]_{SH^{fr}(S)}\) we can consider the right side of the assignment (1.3)

\[x \mapsto [pt \mapsto x] \in [pt, G_m^1]_{SH^{fr}(S)}, \eta \in [pt, G_m^{\wedge -1}]_{SH^{fr}(S)}, \]

where \(m: G_m^2 \rightarrow G_m: (x, y) \mapsto xy, \) and \(p_1, p_2: G_m^2 \rightarrow G_m\) are the projections, as morphisms in \(SH^{fr}(S)\).

Proposition 4.1. The similar assignment as (1.3) induces the homomorphism

\[K_{n}^{MW}(S) \rightarrow H^0(\Delta S, G_m^n)\].

Proof. (Steinberg relation) In the proof of the Steinberg relation in \(SH(S)\) sublemma 2.2 we have essentially uses Zariski excision isomorphisms with respect to

\[A_{S \times A^1} \equiv (A_{S \times A^1} - Z, A_{S \times A^1} - (0_S \Pi Z))\]

in the second two last rows of the diagram (2.3) applied to the morphism \((A^1 - (0,1)) \rightarrow G_m: t \mapsto (t, 1-t)\) as in prop 2.4 and

\[(A^2 - 0, A^2 - (1 \times A^1)) \equiv (A^2 - (A^1 \times 1), A^2 - (1 \times A^1 \Pi A^1 \times 1))\]

in the second row. Now let us see that lemma 4.3 yields that 4.2 and 4.3 are equivalences in \(SH^{fr}_{A^1}\). Actually the first case is immediate. In the case of 4.3 it is enough to turn the picture and the consider the projection \(A^2 \rightarrow A^1: (x, y) \mapsto (x, y)\). Then 4.3 becomes the particular case of the 4.2.
(Other relations) Other relations in $\text{SH}^T$ follow by the same arguments as in section 3, all what we need that it follows form 2 the permutation morphism on $G_m \times \mathbb{Z}$ is equal to $(-1)$, $A^1$-homotopy, where $(-1)$ denotes the class of the framed corr. $(0, -t, pr)$ in $F_{r_1}(pt, pt)$.

Lemma 4.4. For any homotopy invariant presheaf $F$ over a base $S$ and closed subschemes $Z_1, Z_2 \subset A^1_S$ finite surjective over $S$, $Z_1 \cap Z_2 = \emptyset$ the canonical morphism $F(A^1_S - (Z_1 \cup Z_2))/F(A^1_S - Z_2) = F(A^1_S - Z_1)/F(A^1_S)$

In other words the canonical morphism $i: A^1_S - Z_1/A^1_S - (Z_1 \cup Z_2) \to A^1_S - (Z_1 \cup Z_2)$ is an equivalence in $\text{SH}^T$, where $A^1_S - Z_1/A^1_S - (Z_1 \cup Z_2)$ and $A^1_S - (Z_1 \cup Z_2)$ denotes the cones.

Proof. For any scheme $X$ over $S$ a function $\phi \in O(A^1_X)$ such that $Z(\phi)$ is finite over $X$. We can define a framed correspondence $(Z(\phi), A^1_X, \phi, pr) \in Fr((X, A^1_X), \text{pr})$, where $A^1_X \times X \to A^1_X$. Then for a given section $s \in \Gamma(A^1_X, O(n))$ such that $s|_{\infty \times X}$ is invertible we can apply the construction to the function $s/t_n^\infty$. Denote the resulting correspondence by $\langle s \rangle$.

Moreover if $E \subset X D_1, D_2 \subset A^1_S$ are a closed subschemes, and $s|_{X \times D_1}$ and $s|_{(X - E) \times D_2}$ are invertible then the construction $\langle s \rangle$ gives the correspondence between pairs, i.e. an element in $Fr((X, X - E))$, $(A^1_S - D_1, A^1 - (D_1 \cup D_2))$.

Let $\delta \in \Gamma(P^1_{A^1 \times S}, O(1))$, $Z(\delta)$ is the diagonal in $A^1_{A^1 \times S}$. Then in view of the described construction $\langle \delta \rangle$ is equal to the $\sigma$-suspension of the identity element in $Fr(A^1_S, A^1)$ and consequently the identity elements in $Fr_1(A^1_S - Z_1, A^1 - (Z_1 \cup Z_2))$, $(A^1_S - Z_1, A^1 - (Z_1 \cup Z_2))$ and $Fr_1(A^1_S, A^1_S - Z_2, (A^1_S, A^1_S - Z_2))$.

By Serre theorem for large enough $n$ we find a sections $s \in \Gamma(P^1_{A^1 \times S}, O(n)), s' \in \Gamma(P^1_{A^1 \times S}, O(n - 1))$ such that

$s|_{\infty \times S \times A^1} = t_n^0|_{\infty \times S \times A^1}$,
$s|_{Z_1 \times A^1} = t_n^\infty|_{Z_1 \times A^1}$,
$s|_{Z_2 \times A^1} = \delta t_n^{n-1}$,
$s'|_{\infty \times S \times A^1} = t_n^0|_{\infty \times S \times A^1}$,
$s'|_{Z_1 \times A^1} = t_n^\infty|_{Z_1 \times A^1}$,
$s'|_{Z_2 \times A^1} = \delta t_n^{n-1}$. Then $\langle s' \rangle \in Fr_1((A^1_S, A^1_S - Z_2), (A^1_S, A^1_S - Z_2))$ is equal to zero, and

$(s) \in Fr_1((A^1_S, A^1_S - Z_2), (A^1_S - Z_1, A^1 - (Z_1 \cup Z_2)))$ is a left inverse up to a suspension to the canonical morphism in $i: (A^1_S - Z_1, A^1 - (Z_1 \cup Z_2)) \to (A^1_S, A^1_S - Z_2)$, where the homotopy between $\sigma id_{(A^1_S, A^1_S - Z_2)}$ and $i \circ (s)$ is given by

$(as + (1 - \alpha)\delta s') \in Fr_1((A^1_S, A^1_S - Z_2) \times A^1, (A^1_S, A^1_S - Z_2)).$

On other side by Serre theorem for a large enough $n$ we find a sections $s \in \Gamma(P^1_{A^1 \times S}, O(n)), s' \in \Gamma(P^1_{A^1 \times S}, O(n - 1))$ such that

$s|_{\infty \times S \times A^1} = t_n^0|_{\infty \times S \times A^1}$,
$s|_{Z_1 \times A^1} = t_n^\infty|_{Z_1 \times A^1}$,
$s|_{Z_2 \times A^1} = \delta t_n^{n-1}$,
$s'|_{\infty \times S \times A^1} = t_n^0|_{\infty \times S \times A^1}$,
$s'|_{Z_1 \times (A^1_S - Z_1)} = t_n^\infty|_{Z_1 \times (A^1_S - Z_1)}$, $s'|_{Z_2 \times (A^1_S - Z_2)} = \delta t_n^{n-1}$. Then $\langle s' \rangle \in Fr_1((A^1_S - Z_1, A^1_S - (Z_1 \cup Z_2)), (A^1_S - Z_1, A^1_S - (Z_1 \cup Z_2)))$ is equal to zero, and

$(s) \in Fr_1((A^1_S, A^1_S - Z_2), (A^1_S - Z_1, A^1 - (Z_1 \cup Z_2)))$ is a right inverse up to a suspension to the canonical morphism in $i: (A^1_S - Z_1, A^1 - (Z_1 \cup Z_2)) \to (A^1_S, A^1_S - Z_2)$, where the homotopy between $\sigma id_{(A^1_S - Z_1, A^1_S - (Z_1 \cup Z_2))}$ and $\langle s \rangle \circ i$ is given by

$(as|_{A^1_S - Z_1} + (1 - \alpha)\delta|_{A^1_S - Z_1} s') \in Fr_1((A^1_S - Z_1, A^1_S - (Z_1 \cup Z_2)) \times A^1, (A^1_S - Z_1, A^1_S - (Z_1 \cup Z_2))).$

\[\square\]

5. The isomorphisms $K^n_{\text{MW}} \to \pi^{n,n}_* \text{and } K_n^{\text{MW}} \to \mathbb{L}(\text{DF}(\Delta \times -, G^n_{\text{MW}}))$, where $k \neq 2$. It is proven in 31 and 33 that there is a canonical isomorphism of sheaves

\[(5.1) \quad K^n_{\text{MW}} \to \pi^{n,n}_* \text{over a base field } k \text{ for all } n \in \mathbb{Z}. \text{ Precisely the proofs are written for the case of a prefect field } k \text{ and as mentioned in the remark in 33 the result for the non-perfect filed follows by the general base change argument.} \]
In the section we present the proof of the isomorphism based on the theory of framed motives [20] and theory of Chow-Witt groups (see the recent works [19], [8] for the char k = 2 case).

5.1. The proof using the theory of framed motives and Chow-Witt groups. The results of the Garkusha Panin theory of framed motives [20] implies in particular that it is proven that \( \pi^{n,n}_k \to H^0(ZF(\Delta \times -, G^n_m)) \) for a perfect field \( k \). As shown in [16] the general base change argument like as above extends the result to the case of a base schemes \( S \) that are essentially smooth (and even pro-smooth) over some perfect \( k \).

Combining the methods of framed correspondences and homotopies with the theory of Chow-Witt groups [3], [17] it is proven in [34] that \( K_{MW}(k) \cong \pi^{n,n}_k \) in the case char \( k \neq 2 \) the result is extended to the case of a perfect fields char \( k \neq 2 \).

Let us note that the Chow-Witt groups the are used to prove the injectivity of the map.

Here we improve the argument that the argument for the proof of the surjectivity of the map \( K_{MW}(k) \cong \pi^{n,n}_k \) in the case char \( k \neq 2 \) using the moving lemma proved in the next section. Then we deduce the isomorphism (5.1) for an arbitrary base field \( k \). Let us note that similar as above and to [10] the arguments implies the result for an arbitrary pro-smooth base scheme.

**Lemma 5.2.** For an arbitrary field \( k \) the homomorphism \( K_{MW}^n(k) \to H^0(ZF(\Delta \times -, G^n_m)) \) is surjective.

**Proof.** The claim follows similarly to [34] using proposition 6.12 (moving lemma) proven in the next section, and separable field extension transfers for \( K_{MW}^n \) form [33] section 4.5 or [8]. \( \square \)

**Lemma 5.3.** Assume one that one of the following conditions holds for a base scheme \( S \) (a) \( S = Spec \  k \), \( k \) is perfect, or (b) the unramified Milnor-Witt K-theory \( K_{MW}^n \) of \( S \) is strictly homotopy invariant for \( n \geq 0 \).

Then there is a homomorphism of sheaves \( H^0(ZF(\Delta^*, G^n_m)) \to K_{MW}^n \), for all \( n \geq 0 \), that takes a correspondences \( a \in Fr_{1}(pt, G^n_m) \) by invertible \( a \in k \) to the symbol \([a] \).

The proof for (a). The claim follows immediate form the universal property of the sheaf \( H^0(ZF(\Delta^*, G^n_m)) \) since \( K_{MW}^n \) is a homotopy invariant linear presheaf with framed transfers.

Actually, is \( K_{MW}^n \) is the basic example of a presheaf with Milnor-Witt transfers, see [9], in detail it is provided by the fact that \( K_{MW}^n \) is a zeroth homotopy groups of the complexes \( C(X, G^n) \) \( [17], [18] \), the pushforwards for the homologies of the complexes \( C(X \times \mathbb{P}^d, G^n, O(1)) \) \( Z \to C(X, G^{n+d}) \) \( Z \subset X \times \mathbb{P}^n \) is closed finite over \( X \), and the ring structure on the cohomologies of \( C(X, G^{n+d}) \). Hence \( K_{MW}^n \) is stable framed presheaf because of the functor form the category of framed correspondences to the category of Chow-Witt correspondences constructed in [11] or [13].

Let us note that the homotopy invariance of \( K_{MW}^n \) follows form the isomorphism \( K_{MW}^n(A^n_{K}) \cong K_{MW}^n(K) \) due to the injectivity property for the framed stable linear homotopy invariant presheaves. \( \square \)

The proof for (b). By the lemma 5.4 the assumption implies that the \( G^n_m \)-spectrum of Nisnevich sheaves \( K_{MW}^n \) represents in \( D_{Nis,k}^{(\infty)}(Sh_{Nis})[G^n_m] \) the sheaves \( K_{MW}^n \). Hence the sheaves \( K_{MW}^n \) are a homotopy invariant stable framed presheaves like as any SH-representable presheaf. Now by the universal property of the sheaf \( H^0(ZF(\Delta^* \times -, G^n_m)) \) there is a homomorphism \( H^0(ZF(\Delta^* \times -, G^n_m)) \to K_{MW}^n \) induced by the map \( G^n_m \to K_{MW}^n \): \((a_1, \ldots, a_n) \to [a_1, \ldots, a_n] \).

It follows form the definitions that the composition \( K_{MW}^n(k) \to H^0(ZF(\Delta^* \times -, G^n_m)) \to K_{MW}^n(k) \) is identity for \( n \neq 0 \). This proves the injectivity of the map \( K_{MW}^n(k) \to K_{MW}^n(k) \to H^0(ZF(\Delta^* \times -, G^n_m)) \) for \( n \geq 0 \).

**Lemma 5.4.** (1) Assume that \( K_{MW}^n \) is strictly homotopy invariant for all integer \( n \in \mathbb{Z} \) over some base \( S \); then the canonical homomorphisms \( K_{MW}^n \to \pi^{n,n}(K_{MW}) \) are isomorphisms for \( n \in \mathbb{Z} \), where \( K_{MW}^n \) is the spectrum of Nisnevich sheaves of abelian groups

\[
K_{MW}^n = (K_{MW}^1, K_{MW}^2, \ldots, K_{MW}^n, \ldots), \quad K_{MW}^n \times G_m \to K_{MW}^{n+1}: (\phi, a) \mapsto \phi \cdot a.
\]

(2) Assume that \( K_{MW}^n \) is strictly homotopy invariant for all integer \( n \) larger some \( n_0 \) over some base \( S \); then \( K_{MW}^n \) is strictly homotopy invariant for all integer \( n \).

**Proof.** (1) It follows from the strictly homotopy invariance of the sheaf \( K_{MW}^n \) that the fibrant replacement of \( K_{MW}^n \) with respect to the injective Nisnevich local model structure on the category of simplical spectra of Nisnevich sheaves is \( A^1 \)-local. So \( |X, K_{MW}^n|_{Sh_{Nis}}(k) = K_{n} \). (Actually, in the case of a filed base case \( K_{MW}^n \)
defines an element in hart of $\text{SH}_{S^1}(k)$ with respect to the homotopy t-structure on $\text{SH}_{S^1}(k)$ [29 section 6.2].)

Then the claim follows form the isomorphisms

$$K^\text{MW}_n(-) \simeq K^\text{MW}_{n+1}(- \land G_m)$$

given by the canonical isomorphisms $K^\text{MW}_n(X \times G_m) \simeq K^\text{MW}_n(X) \oplus K^\text{MW}_{n-1}(X)$.

(2) The claim follows immediate form (5.5). $\square$

Remark 5.6. Let us recall that the last isomorphism (5.5) follows form the homotopy invariance of $K^\text{MW}_n$.

Consider the homomorphism $K^\text{MW}_n(X \times G_m) \simeq K^\text{MW}_n(X) \oplus K^\text{MW}_{n-1}(X)$ defined by the sub of the inverse image along the unit section $i_1: X \to X \times G_m$, and the residue map at zero section $\delta_0: K^\text{MW}_n(X \times G_m) \to K^\text{MW}_{n-1}(X)$. The inverse image $p^*$ along the projection $p: X \times G_m \to X$ and $i_1^*$ induces the splitting $K^\text{MW}_n(X \times G_m) \simeq K^\text{MW}_n(X) \oplus \text{Coker}(p^*)$; on other side the morphism $K^\text{MW}_{n-1}(X) \to K^\text{MW}_n(X \times G_m)$: $[a_1, \ldots, a_{n-1}] \mapsto [t, a_1, \ldots, a_{n-1}]$ induces the left inverse to $\delta_0$, so we have the splitting $\text{K^\text{MW}_n(X \times G_m) \simeq Ker(\delta_0) \oplus K^\text{MW}_{n-1}(X)}$. Now the claim follows since $\text{Im}(j^*) = \text{Ker}(\delta_0)$, where $j: X \times G_m \to X \times A^1$, and $\text{Im}(p^*) = \text{Im}(j^*) \simeq K^\text{MW}_n(X)$.

Remark 5.7. Let us note that the isomorphism $K^\text{MW}_n(X) \simeq K^\text{MW}_n(X \land G_m)$ follows form the case of $X = pt$ and homotopy invariance of $K^\text{MW}_n$ due to the injectivity for the framed linear stable homotopy invariant presheaves.

Theorem 5.8. Assume one of the following (a) the base filed $k$ is perfect, (b) the base field $k$ is of characteristic different form 2. The homomorphisms of sheaves $K^\text{MW}_n \to I^n_\ast(ZF(\Delta^1 \times -, G^n_m)) \to \pi_*^{n,n}$ are an isomorphsism and $n \in \mathbb{Z}$.

Proof. By the above lemmas we have the isomorphism $K^\text{MW}_n(k) \simeq I^n_\ast(ZF(\Delta^1 \times -, G^n_m))$ is an isomorphism for $n \geq 0$ due to the injectivity property for a homotopy invariant stable linear framed presheaves [21].

Finally, the isomorphism $K^\text{MW}_n \simeq I^n_\ast(ZF(\Delta^1 \times -, G^n_m))$ for all $n$ follows from the the isomorphisms $K^\text{MW}_n(X) \simeq K^\text{MW}_{n+1}(X \land G_m)$, and the isomorphisms $I^n_\ast(ZF(\Delta^1 \times -, G^n_m) \simeq \pi_*^{n,n}$ and $\pi_*^{n,n}(G_m) \simeq \pi_*^{n-1,n-1}(pt)$.

5.2. The strictly homotopy invariance of $K^\text{MW}_n$. In the subsection we summarise known arguments for the strictly homotopy invariance of $K^\text{MW}_n$.

5.2.1. Morel’s pullback. Firstly we recall the argument from [31] for the case of a field $k$, char $k \neq 2$.

Lemma 5.9. There are isomorphisms of sheaves $K^\text{MW}_n \simeq I^n_\ast \times I^{n+1}_\ast K^\text{MW}_n$ for all $n \in \mathbb{Z}$.

Proof. We refer to [3, 30] and [22] for the case of the sections on fields of odd characteristic. (Nevertheless the author haven’t found a reference for the proof of the pullback of sheaves) To get the claim for the sheaves firstly we need to note that all maps in the pullback square commutes with the residue morphisms, where by the residue morphism on Witt groups we mean the homomorphism $W(k(X)) \to W(x)$ for $x \in X(1)$ constructed by Schmid in [38]. Namely this is provided by the formulas [31 theorem 2.15] for the residues on Milnor-Witt K-theory, the similar formular of residue homomorphism on Milnor-K-theory [28], and for the residue map constructed by Schmid [38] section 2.2, $D\tilde{W}_3$, formula bottom of page 21]. Now it is enough to note that $W(U) \to W(k(U))$ is injective for a essential smooth $U$ and $W(U) \subset \text{Ker}(W(k(U)) \to \bigoplus_{x \in X^0(1)} W(x))$. $\square$

Remark 5.10. Let $Z \subset X$ be a closed subscheme of codimension one in a locally essentially smooth $k$-scheme $X$, $Z = Z(t)$, $t \in O(X)$. Let $\delta: W(X - Z) \to W(Z)$ with respect to the equation $t$. To check that the morphism $K^\text{MW}_n \to I^n_\ast$ is agreed wit the differentials it is enough to prove that $\delta(t) = (1)$. In the case of a smooth scheme $Z$ it is given by the standard formula of the Gysin map. In an arbitrary case the claim follows form the regular one due to the rigidity along non-smooth closed embeddings for $W(-)$ proven by S. Gille, since any such $X$ there is an embedding $X \subset X'$ and $Z \subset Z'$, $Z' \to X'$ is a closed smooth subscheme of codimension one.

Proposition 5.11. Let the base be a filed $k$, char $k \neq 2$. Then the sheaves $K^\text{MW}_n$ are strictly homotopy invariant for $n \geq 0$. 
Proof. Let us repeat some of the arguments from [31, section 6, steps 1-4].

The claim follows from the lemma above and from the strictly homotopy invariance of underline K\textsuperscript{M} and I\textsuperscript{n} and I\textsuperscript{n+1}/I\textsuperscript{n}. The sheaves K\textsuperscript{M} and K\textsuperscript{M}/2 are Rost’s cyclomodules it follows form [37] proposition 8.6, proposition 2.2(H)] and [37] Theorem 6.1] that K\textsuperscript{MW} and K\textsuperscript{MW}/2 are strictly homotopy invariant. So it follows form Milnor’s conjecture I\textsuperscript{n}/I\textsuperscript{n+1} = K\textsuperscript{M}/2, [32, 35], that the sheaves I\textsuperscript{n}/I\textsuperscript{n+1} are strictly homotopy invariant. Since as proven in [26] that the sheaves W\textsuperscript{M}(−) are SH(k)-representable, they are strictly homotopy invariant. Hence by induction we get strictly homotopy invariance of the sheaves I\textsuperscript{n}.

5.2.2. Unramified sheaves and Milnor-Witt cyclomodules. The next proof in the case of a perfect field of an arbitrary characteristic was given in [33] Chapter 5. The idea is to combine the theory of Rost cyclomodules and its adaptation for Witt-groups to get the precise construction of one complex, so called Rost-Smidt complex, defined over an arbitrary perfect field and equal in the odd characteristic base field case to the fibred product of the Rost complex for K\textsuperscript{MW} and the similar complex for Witt groups constructed by Schmid. In [33] the starting object which gives a rise to the complex is are unramified sheaves, and the main example is the Milnor-Witt K-theory.

Recently, the idea was revisited and deeply studied in works [19] and [8]. So called Milnor-Witt cyclomodules are defined, and the main example is the Milnor-Witt K-theory. It is proven in particular that the unramified sheaf corresponding to the Milnor-Witt cyclomodule is strictly homotopy invariant over a perfect field.

5.2.3. Chow-Witt correspondences. Also one proof for the case of a perfect field is provided by the strictly homotopy invariance theorem for a homotopy invariant sheaves with Milnor-Witt transfers. Actually, it follows from the definitions that the sheaves K\textsuperscript{MW} are sheaves with Chow-Witt correspondences, see [9], so by [15] the homotopy invariance of K\textsuperscript{MW} implies the strictly homotopy invariance.

6. Neshitov’s moving lemma.

Definition 6.1. Let c = (Z, V; φ, g) ∈ Fr\textsubscript{n}(X, G\textsubscript{m}), V → A\textsuperscript{n} is an etale neighbourhood of a closed subscheme Z in A\textsuperscript{n}, Z is finite over S, φ: V → A\textsuperscript{n}, g: V → A\textsuperscript{n}, Z = V ×₁ A\textsuperscript{n} 0. We say that c is simple iff Z is smooth over S.

Definition 6.2. Fr\textsubscript{n}(pt, G\textsubscript{m}) denotes the factor group of Fr\textsubscript{n}(pt, G\textsubscript{m}) up to A\textsuperscript{l}-homotopy equivalence.

Lemma 6.3. For any c ∈ Fr\textsubscript{n}(pt, G\textsubscript{m}) over an affine base scheme S and for all large enough d\textsubscript{i}, i = 1, . . . , n, and r\textsubscript{j}, j = 1, . . . , l, there is a correspondence c′ ∈ Fr\textsubscript{n}(pt, G\textsubscript{m}) such that [c′] = [c] ∈ Fr\textsubscript{n}(pt, G\textsubscript{m}), and such that

c′ = (Z, A\textsuperscript{n} − ((Z(s) − Z) ∪ Z(e)); s\textsubscript{1}/t\textsubscript{d\textsubscript{1}}, . . . , s\textsubscript{n}/t\textsubscript{d\textsubscript{n}}; e\textsubscript{1}/t\textsubscript{r\textsubscript{1}}, . . . , e\textsubscript{l}/t\textsubscript{r\textsubscript{l}})

for some sections s\textsubscript{i} ∈ Γ(P\textsubscript{n}, O(d\textsubscript{i})), d\textsubscript{i} ∈ Z, i = 1, . . . , n, e\textsubscript{j} ∈ Γ(P\textsubscript{n}, O(r\textsubscript{j})), r\textsubscript{j} ∈ Z, j = 1, . . . , l.

Proof. By Serre’s theorem [25, theorem 5.2] we can choose integers d\textsubscript{i} and sections s\textsubscript{i} = (s\textsubscript{ij}), 1 ≤ i ≤ n, s\textsubscript{i}/t\textsubscript{d\textsubscript{i}} = φ\textsubscript{i}|_{Z(I(Z))}\textsuperscript{p}, where P\textsubscript{n−1} ⊂ P\textsubscript{n} is the subspace at infinity and t\textsubscript{∞} ∈ O(1), Z(t\textsubscript{∞}) = P\textsubscript{n−1}. Similarly we can choose sections e\textsubscript{j} ∈ Γ(P\textsubscript{n}, O(d\textsubscript{j})), 1 ≤ j ≤ k, e\textsubscript{j}/t\textsubscript{r\textsubscript{j}} ∈ Z(I(Z)) = g\textsubscript{j}|_{Z(I(Z))}\textsuperscript{p}, where the g\textsubscript{j}’s are the coordinates of the composition V → Y → A\textsuperscript{c}. The functions λ\textsuperscript{u}(s\textsubscript{i}/t\textsubscript{d\textsubscript{i}}) + (1 - λ)(φ\textsubscript{i}) and λ\textsuperscript{v}(e\textsubscript{j}/t\textsubscript{r\textsubscript{j}}) + (1 - λ)g\textsubscript{j} gives a homotopy from c to the framed correspondence c′ = (Z, A\textsuperscript{n} − (Z(s) − Z); s\textsubscript{1}/t\textsubscript{d\textsubscript{1}}, . . . , s\textsubscript{n}/t\textsubscript{d\textsubscript{n}}; e\textsubscript{1}/t\textsubscript{r\textsubscript{1}}, . . . , e\textsubscript{l}/t\textsubscript{r\textsubscript{l}}).

Notation 6.4. Denote by (F)\textsubscript{x} the denote the fibre of the coherent sheaf F on the scheme X at a point x ∈ X, i.e. (F)\textsubscript{x} = \textit{i}\textsubscript{x}∗(F), where i\textsubscript{x}: x → X is the canonical embedding.

Denote Ω\textsubscript{X}/X the canonical sheaf of the closed subscheme Y ⊂ X.

Lemma 6.5. Let p: X → Y be a finite morphism of schemes. Then there is a closed subscheme X\textsuperscript{ns} ⊂ X such that x ∈ X\textsuperscript{ns} if x ∈ \textit{Supp} Ω\textsubscript{p}, or the residue file at x is not separable over the residue filed of the image of x in Y; Consequently if p: X → Y is flat then x ∈ X\textsuperscript{ns} iff p is not etale at x.

Proof. Consider the projection p\textsubscript{1}: \tilde{X} ×\textsubscript{X} Y → X, which is finite morphism as well. Denote by ∆\textsubscript{X} ⊂ \tilde{X} ×\textsubscript{X} Y the diagonal subscheme, and fro any point x ∈ X denote by δ\textsubscript{x} ∈ ∆ the corresponding point under the canonical isomorphism ∆\textsubscript{X} ∼ X. Then

X\textsuperscript{ns} = p\textsubscript{1}(\textit{Supp} Ω\textsubscript{p}\textsubscript{x} ∩ ∆) ⊂ X.
Actually, let $x \in X$, denote $\tilde{X} = X \times_Y x$, $x^2 = x \times_Y x$, and let $\delta_x \subset x \times x$ be the diagonal. Then the claim follows form the short exact sequence

$$0 \to (\tilde{N}_{x^2/\tilde{X}})_x \to (\Omega_{p_1})_{s_x} \to \tilde{N}_{b_x/x^2},$$

and isomorphisms

$$\tilde{N}_{x^2/\tilde{X}} \simeq p_2^*(\Omega_{p_2}), \quad \tilde{N}_{b_x/x^2} \simeq \Omega_{x^2 \to x}.$$

where $p_2 : x^2 \to x$ is the projection onto the second multiplicant.  

**Corollary 6.6.** Let $p : X \to Y$ be flat finite surjective morphism, and $X$ is irreducible. Assume that there is a point $x \in X$ such that the residue field extension $\mathcal{O}(x)/\mathcal{O}(p(x))$ is separable, and $f$ is unramified at $x$. Then there is a non-empty open subscheme $U \subset Y$ such that $X \times_Y U \to U$ is etale.

**Proof.** It follows form lemma [6,3] and from assumption that there is a proper closed $X^{ns} \subset X$ such that $X - X^{ns} \to Y$ is etale. Since $X \to Y$ is finite and X is irreducible, then so is $Y$. Let $d = \dim Y$. Since $X^{ns} \subset X$ is proper and since $X \to Y$ is finite, it follows that $\dim X^{nc} < d$. Hence $Y - p(X^{ns}) \neq \emptyset$. Thus the claim is true for $U = Y - p(X^{ns})$. □

**Lemma 6.7.** Let $S$ be a noetherian scheme of a finite type over $\mathbb{Z}$; let $s_1, \ldots, s_n \in \Gamma(\mathbb{P}^n_S, \mathcal{O}(d))$ be a set of sections, $s_i|_{\mathbb{P}^{n-1}} = t_i^d$. Then the vanishing locus $Z(s_1, \ldots, s_n)$ is finite surjective and flat over $S$.

**Proof.** Consider the morphism $f : \mathbb{A}^n \times \Gamma_d \to \mathbb{A}^n \times \Gamma_d$ defined by the regular functions $s_i/d/t_i^d \in \mathcal{O}(\mathbb{A}^n \times \Gamma_d)$. Since $s_i/d|_{\mathbb{P}^{n-1}} = t_i^d$, it follows that $f$ is quasi-finite. In the same time the condition provides that the graph of $f$ is equal to the vanishing locus $Z(s_1 - \alpha_1 t_1^d, \ldots, s_n - \alpha_n t_n^d) \subset \mathbb{P}^n \times \mathbb{A}^n_S$ where $(\alpha_1, \ldots, \alpha_n)$ denotes coordinates on $\mathbb{A}^n$. Hence $f$ is projective. Thus $f$ is finite, and since dimensions of the domain and the co-domain of $f$ are equal it follows that $f$ is finite.

Now let $x \in S$ be a point, $U \subset S$ is affine Zariski neighbourhood of $x$. Since $U$ is affine there is a closed embedding $U \subset \text{Spec} R$ be a regular ring $R$. Consider a lift $\hat{s}_i \in \Gamma(\mathbb{P}^n_{\text{Spec} R}, \mathcal{O}(d))$ of the sections $s_i$, and the morphism $\hat{f} : \mathbb{A}^n_{\text{Spec} R} \to \mathbb{A}^n_{\text{Spec} R}$ defined by $\hat{s}_i/t_i^d$. Then by the same reason as for $f$ the morphism $\hat{f}$ is finite and surjective. Hence $\hat{f}$ is flat by [11 Corollary 3.6]. Thus $Z(\hat{s}_i)$ is flat over $\Gamma_d$. □

**Lemma 6.8.** Let $S$ be a scheme, and denote by $\mathcal{O}(1)$ the ample bundle on $\mathbb{P}^n_S$ over $S$ and denote by $t_1, \ldots, t_{n+1}$ the coordinate section of $\mathcal{O}(1)$, in particular $Z(t_{n+1}) = \mathbb{P}^n - 1$ is the infinite hypersurface. Assume that $Z \subset \mathbb{A}^n_S$ is a closed subscheme finite over $S$, $e \in \mathbb{Z}$, and $\beta_i \in \Gamma(Z(I^2(Z)), \mathcal{O}(e))$, $i = 1, \ldots, n$ are sections such that $Z(\beta_1, \ldots, \beta_n) = Z$.

Denote by $\Gamma_d$ the affine space over $S$ that $S$-points is the set

$$\Gamma_d(S) = \{(s_1, \ldots, s_n) \in \Gamma(\mathbb{P}^n_S, \mathcal{O}(d)) \mid s_i|_{Z(I^2(Z))} = \beta_i t_i^d, s_i|_{\mathbb{P}^{n-1}} = t_i^d\}.$$

Let $s_d = (s_1, \ldots, s_n, d) \in \Gamma(X \times \Gamma_d, \mathcal{O}(d) \otimes \mathcal{O}(d))$ be the universal section. Denote by $\mathbb{Z}_d$ the closed subscheme $\mathbb{Z}_d = Z(s_d) - Z(\Gamma_d) \subset X \times \Gamma_d$.

Then there exist $N$ such that $\forall d > N$ the vanishing locus $\mathbb{Z}_d$ is connected and smooth over $S$, and $\mathbb{Z}_d$ is flat finite surjective over $\Gamma_d$.

**Proof.** It follows form the relative version of the Serre’s theorem [25 theorem 8.8] that there is $N$ such that $\forall d > N$ the homomorphisms $\Gamma(X, \mathcal{L}_i \otimes \mathcal{O}(d)) \to \Gamma(Z(I^2(Z))) II x_1 II x_2, \mathcal{L}_i \otimes \mathcal{O}(d))$, are surjective for all $i = 1, \ldots, n$, $x_1, x_2 \in X$ is a pair of different closed points.

Then for all $d > N$ the universal vanishing locus $Z(s_d)$ is smooth over $S$. Actually, let $s = (s_1, \ldots, s_n) \in \Gamma_d$ be an $S$-point, and let $x \in Z(s_1, \ldots, s_n) \subset X$. By assumption there is a section $s' \in \Gamma(Z(I^2(Z)), \mathcal{O}(d))$ such that $s'|_{Z(I^2(Z))} = 0$, and $s'|_x$ is invertible. Denote by

$$v_i = (0, \ldots, 0, s', 0, \ldots)$$

where $s'$ is located at the $i$-th slot, the vectors in the tangent space $T_{x, x}s$. Now on the one side we have

$$Z(s_d) \times_{\mathbb{P}^n_S} (\mathbb{P}^n_S - Z(s')) = Z(s_1, s'/s_1, \ldots, s_n, s'/s_n) \times_{\mathbb{P}^n_S} (\mathbb{P}^n_S - Z(s')).$$

On the other side we see that differentials of the functions $s_1/s_1', i = 1, \ldots, n$, at the point $(x, s) \in X \times \Gamma_d$ in the directions defined by vectors $v_j, j = 1, \ldots, n$, are linearly independent, namely

$$d_{v_i}(s_1, s'/s_1') = 1, d_{v_i}(s_1, s'/s_1') = 0, i \neq j.$$  

Thus the conormal cone of $Z(s_d)$ in $X \times \Gamma_d$ is a vector bundle of the dimension $n$. So $Z(s_d)$ is smooth over $S$, since $X$ and $\Gamma_d$ are smooth.
Now we need to show that $Z_d$ is connected. By lemma 6.7 $Z_d$ is flat finite and surjective over $\Gamma_d$. So it is enough to show that for any $s \in \Gamma_d(S)$ and $x_1, x_2 \in Z(s)$ there is a subspace $E \subset \Gamma_2$ such that $x_1$ and $x_2$ are in the same connected component of $Z_d \times E$. Consider the section $s' \in \Gamma(P^n_d, \mathcal{O}(d))$, $s'|_{P^n_d - 1|U_d(Z_d)} = 0$, $s'|_{x_1, x_2}$ is invertible. Define $E$ as the subspace of $\Gamma_2$ spanned by the point $s$ and tangent vectors $v_i$. Then we see from (6.10) that $Z(s_d) \times_{P^n_d} P^n_d = Z(s')$ is equal to the graph of the map $(P^n_d - Z(s)) \to A^n_d$ given by regular functions $s_i/s'$. So it is connected. And by assumption on $s'$, we have $x_1, x_2 \in P^n_d - Z(s')$. Thus the claim follows. □

**Corollary 6.11.** Let $Z \subset P^n_d$ be a closed subscheme in the projective space over a semi-local base scheme $S$ with infinite residue fields. Let $\beta_i \in \Gamma(Z(F_i(Z)), \mathcal{O}(e_i))$, $i = 1, \ldots, n$, be a set of sections for some $e \in Z$. Then for all large enough $d$ there is a vector of sections $(s_1, \ldots, s_n)$, $s_i \in \Gamma(P^n_d, \mathcal{O}(d))$ such that $s_i|_{Z(F_i(Z))} = \beta_i t^{d - e}$, $s_i|_{P^n_d - 1} = t^d$ and such that $Z(s_1, \ldots, s_n) = Z$ is etale over $S$.

**Proof.** Consider the universal section $s_d$ on $P^n_d \times \Gamma_d$ as in lemma 6.8. By Serre’s theorem [5.2] for all large enough $d$ there is a vector $s = (s_1, \ldots, s_n) \in \Gamma_d(S)$ such that $s_i|_{Z(F_i(0_e))} = t_i t^{d - 1}$ where $0_e \subset P^n_d$ denotes the zero point-section. Then the morphism $Z(s_1, \ldots, s_n) - Z \to S$ is etale on $0_e$. Hence by corollary 6.6 there is a non-empty open subscheme $U \subset \Gamma_d$ such that $Z_d \cap_{\Gamma_d} U \to U$ is etale. Now since $S$ is semi-local with infinite residue fields, there is an $S$-point $s: S \to U$. So $s$ is a vector of sections $(s_1, \ldots, s_n)$ such that $Z(s_1, \ldots, s_n) = Z \# Z$ is etale over $S$. □

**Proposition 6.12.** For any $c \in \text{Fr}_{n}(pt, G^n_m)$ over a semi-local base scheme $S$ there are simple correspondences $c^+, c^- \in \text{Fr}_{n}(pt, G^n_m)$ such that $[c^+] - [c^-] = [c] \in Z\text{Fr}_{n}(pt, G^n_m)$.

**Proof.** We can assume that the residue fields of $S$ is infinite due to the finite descent for framed correspondences, see Appendix A, lemma 7.2. In details, assume the result for local schemes with infinite residue fields; then for any $S$ we can consider extensions $S_{1,1} \to S$ and $S_{2,1} \to S$ defined by equations $x^d - 1$ and $x^d - 1$ on $S$, where $q_{1,2}$ are prime integers coprime to parameters of $S$, and $n \in \text{mathbbN}$. Let $S_1 = \lim_{\rightarrow} S_{1,1}$, $S_2 = \lim_{\rightarrow} S_{2,1}$. Then $S_1$ and $S_2$ are semi-local schemes with infinite residue fields, so by assumption there are simple correspondences $c_1^+, c_1^- \in \text{Fr}(S_1, G^n_m)$, $[p_i^1(e)] = [c_1^+] - [c_1^-]$. By $c_1^+$ are defined for all $s_1, \ldots, s_n$ for some $l \subset Z$. By assumption on $q_1$ and $q_2$ the schemes $S_{1,1}$ and $S_{2,1}$ are etale over $S$. Hence the correspondences $c^+ = (c_1^+ \# c_2^+)$ $\cap L$ defined by the finite descent, where $L \in \text{Fr}(S, S_{1,1} \# S_{2,1})$ is defined in lemma 7.2 are simple, and by lemma 7.2 we have $[c] = [c^+] - [c^-]$.

By lemma 6.3 we can assume
$$c = (Z, k^n - ((Z(s) - Z) \cup Z(e)); s_1/t_1^\infty, \ldots, s_n/t_1^\infty; e_1/t_1^q, \ldots, e_l/t_1^q)$$
where $s_i \in \Gamma(P^n_d, \mathcal{O}(p))$, $p \in Z$, $i = 1, \ldots, n$, $e_j \in \Gamma(P^n_d, \mathcal{O}(q))$, $q \in Z$, $j = 1, \ldots, l$, and $Z(e) = Z(e)$. Denote $\tilde{Z} = Z(s) - Z$, then $Z(s) = Z \# \tilde{Z}$.

By corollary 6.11 we see that there is $N$ such that for all $d > N$ there are sections of vectors that
$$s^+ = (s_1^+, \ldots, s_n^+) \in \Gamma(P^n_d, \mathcal{O}(d)^n)$$
$$s^- = (s_1^-, \ldots, s_n^-) \in \Gamma(P^n_d, \mathcal{O}(d)^n)$$
such that
$$s_i^+|_{Z(F_i(Z))} = s_i|_{Z(F_i(Z))} t^{d - p}$$
$$s_i^-|_{Z(F_i(Z))} = s_i|_{Z(F_i(Z))} t^{d - p}$$
and such that $Z(s^+) - \tilde{Z}$ and $Z(s^-) - Z(s)$ are etale over $S$.

In the same times by Serre’s theorem for all large enough $d$ there are sections $r_i \in \Gamma(P^n_d, \mathcal{O}(d - p))$, $r_i|_{Z(F_i(Z))} = t_i^d$, $r_i|_{P^n_d - 1} = t_i^d$.

Denote $s' = (s_1', \ldots, s_n')$, $s_i' = s_i r_i$. Then the affine homotopy of framed correspondences given by $\lambda s_i r_i + (1 - \lambda) s_i t_i^d$ implies that
$$c k^l c' = (Z, k^n - ((Z(s') - Z) \cup Z(e)); s_1 r_1/t_1^\infty, \ldots, s_n r_n/t_l^\infty; e_1/t_1^q, \ldots, e_l/t_1^q).$$
On other side
$$c' = c^+ - c^- \in Z\text{Fr}_{n}(pt, G^n_m),$$
$$c'^+ = (Z(s') - \tilde{Z}, k^n - ((Z(s') - \tilde{Z}) \cup Z(e)), s_1/t_1^q, \ldots, s_n/t_l^q; e_1/t_1^q, \ldots, e_l/t_1^q),$$
$$c'^- = (Z(s') - Z(s), k^n - ((Z(s') - Z(s)) \cup Z(e)); s_1/t_1^q, \ldots, s_n/t_l^q; e_1/t_1^q, \ldots, e_l/t_1^q).$$
So the claim follows since by the above we have
\[ c^+ L_\lambda (Z(s^+ - Z), \quad \mathbb{A}^n - ((Z(s^+ - Z) \cup Z(e)); \quad s_1/t_{\infty}, \ldots, s_n/t_{\infty}; \quad e_1/t_{\infty}, \ldots, e_l/t_{\infty}) \]
\[ c^- L_\lambda (Z(s) - Z(s), \quad \mathbb{A}^n - ((Z(s) - Z(s) \cup Z(e)); \quad s_1/t_{\infty}, \ldots, s_n/t_{\infty}; \quad e_1/t_{\infty}, \ldots, e_l/t_{\infty}). \]

7. Appendix A: The finite descent over a base.

In this section we recall the $\mathbb{A}^1$-homotopy finite descent for framed correspondences and representable presheaves with framed transfers presented originally simultaneously and independently in [13] (see also [14, Appendix], and [10] Appendix B). We refer to [39] and [20] the theory of framed correspondences and framed motives, see [20] definition 2.1, definition 8.4 for the definition of framed correspondences. We will use the functor form the category $\mathbb{Z}Fr(S) \to \mathbb{SH}(S)$ induced by the composition map $Fr_n(X, Y) \to Sh_{nis}(X \times \mathbb{P}^{n}/X \times \mathbb{P}^{n-1}, Y \times \mathbb{A}^n/ Y \times (\mathbb{A}^n - 0)) \to [X, Y]_{\mathbb{SH}(S)}$.

**Proposition 7.1.** Let $S_1 = Z(f_1) \subset \mathbb{A}^1_S$ and $S_2 = Z(f_2) \subset \mathbb{A}^1_S$, where $f_1, f_2$ are polynomials of coprime degrees with coefficients in the ring of regular functions on a scheme $S$ and leading coefficients being equal to 1. Suppose that $S_1 \rightarrow S$ and $S_2 \rightarrow S$ are etale. Then the homomorphism $e : [X, Y]_{\mathbb{SH}(S)} \to [X \times S, Y \times S]_{\mathbb{SH}(S)}$ is injective for any dotted smooth schemes $X$ and $Y$ over $S$.

**Proof.** To get the claim it is enough to construct the left inverse $e^{-1}$.

Let $L$ be the morphism in $[S, S]_{\mathbb{SH}(S)}$ given by the sum of framed correspondence $(A^1_{S_1} - (S_1 \times S_1 - \Delta_{S_1/J_S}), f, pr_{S_1}) \in Fr_1(S, S_1)$, via the functor $ZFr(S) \to \mathbb{SH}(S)$, where $\Delta_{S_1/J_S} \to S_1 \times S_1$ is the diagonal, $pr_{S_1} : A^1_{S_1} \to S_1$. Let $p : \tilde{S} \rightarrow S$ and $pr : Y \times \tilde{S} \rightarrow Y$ be the canonical projections. Let’s denote $L_X = id_X = pr \circ p^{-1} \circ p^{-1} (L)$, where $p : X \times S \rightarrow S$ is the structural morphism.

Then the explicit framed correspondence $(A^1_{\tilde{S}}, \mathbb{A}^1 - (S_1 \times S_1 - \Delta_{S_1/J_S}) \times 0, Z(h))$, where $h = (1 - \lambda)_i f_i + \lambda \deg f_i$, $i = 1, 2$, gives the homotopy between $p \circ L$ and the morphism defined by framed correspondence $A_{deg f}$, defined by the framed correspondence $(A_{S_1} \times Z(\deg f_i, \deg f_i, pr_{S_1})), where pr : A^1_{\tilde{S}} \rightarrow S$ is canonical projection. In the same time the homotopy given by $(A^1_{\tilde{S}}, Z((1 - \lambda)_i x^1 + \lambda x^1, pr_{S_1}))$ gives the homotopy between $A_1$ and $A_{1-1} + ((-1)^{i-1})$, where $(a)$ denotes the element in $[pt, pt]_{\mathbb{SH}(S)}$ defined by the multiplication $G^1 \rightarrow G^1 : x \mapsto ax$ for any invertible regular function $a$ on $S$.

Then the left inverse is given by
\[ [X \times S, Y \times S]_{\mathbb{SH}(S)} \rightarrow [X, Y]_{\mathbb{SH}(S)} \quad a \mapsto p_Y \circ a \circ L_X \]

Actually let $\tilde{a} = p^*(a)$ be the base change of $a$ along the morphism $p : \tilde{S} \rightarrow S$ for $a \in [X, Y]_{\mathbb{SH}(S)}$, then we have
\[ a = a \circ p_X \circ L_X = p_Y \tilde{a} \circ p_Y \]

since $p \circ L$ is $\mathbb{A}^1$ homotopy equivalent to the identity morphism $id_{S}$. □

We see from the above proof the following.

**Lemma 7.2.** For any base scheme $S$ and etale coverings $S_1 \rightarrow S$, $S_2 \rightarrow S$ defined by two separable polynomials over $S$ with unit leading terms and of a coprime degrees, there is framed correspondences $L : Fr_1(S, S_1 \amalg S_2)$ such that $[p \circ L] = [id_{S}] \in \mathbb{Z}F(S, S)$, where $\mathbb{Z}F(S, S)$ is a factor sheaf of $\mathbb{Z}F(S, S)$ with respect to $\mathbb{A}^1$-homotopies.

8. Appendix B: The sign for the compositions in $[G^m_n, G^m_n]$.  

**Proposition 8.1.** Let $f \in [G^m_n, G^m_n]_{\mathbb{SH}} \in [G_\wedge ^m_n, G_\wedge ^m_n]_{\mathbb{SH}}$, then
\[ (1) \quad \Sigma_{G_m} f \circ \Sigma_{G_m} g = \Sigma_{G_m} (\Sigma^s_{G_m} f \circ \Sigma^s_{G_m} g), \quad \text{where} \quad s = (m' + n')(m' + n); \]
\[ (2) \quad \Sigma_{G_m} f \circ \Sigma_{G_m} g = \Sigma_{G_m} (-1)^s (\Sigma^s_{G_m} f \circ \Sigma^s_{G_m} g), \quad \text{where} \quad s = (n' + m)(m' + n) + n' + m \]

**Proof of the proposition.**

1The finite descent for framed correspondences is written for the case of fields and representable presheaves but the finite descent we use here is given by the same formulas.
The equality is provided by the permutation on the middle term of the composition and \( \text{lm}^{3.10} \)
\[
\Sigma^{1+n+n'}(\Sigma_{G_m}^f \circ \Sigma_{G_m}^g) = (\Sigma_{G_m}^{1+n+n'}) \circ (\Sigma_{G_m}^{1+n+n'}) = \\
(\Sigma_{G_m}^{1+n+n'} \circ P \circ \Sigma_{G_m}^{1+n+n'}) \circ (\Sigma_{G_m}^{1+n+n'}) = \\
(\Sigma_{G_m}^{1+n+n'} \circ P \circ \Sigma_{G_m}^{1+n+n'}) = (\Sigma_{G_m}^{1+n+n'}) = (\Sigma_{G_m}^{1+n+n'}) = (\Sigma_{G_m}^{1+n+n'})
\]
where \( P: G_m^{n+n+1} \times G_m^{n+n+1} \to G_m^{n+n+1} \times G_m^{n+n+1} \to G_m^{n+n+1} \times G_m^{n+n+1} \) are the permutations which replace the multiplicands, sign \( P = (n'+n)(n+n'+1) = 0 \), sign \( P = (n+n')(n+n'+1) \).

(2) Since
\[
\hat{G} \circ (\Sigma_{G_m}^f \circ \Sigma_{G_m}^g) \circ \hat{G} = \Sigma_{G_m}^{n'} \circ \Sigma_{G_m}^{n'} f, \\
\]
where
\[
\hat{G}: G_m^{E} \to G_m^{E} \to G_m^{E} \to G_m^{E}, \\
\hat{G}: G_m^{E} \to G_m^{E} \to G_m^{E} \to G_m^{E},
\]
and sign(\( \hat{G} \)) sign(\( \hat{G} \)) = (n+n')(m+n'+1) the claim follows from point (1) and \( \text{lm}^{3.10} \) □

Corollary 8.2. Let \( f \in [G_m^{E}, G_m^{E}]_{SH} \) \( g \in [G_m^{E}, G_m^{E}]_{SH} \), then

(0) \( f \circ g \sim^{G} (\Sigma_{G_m}^f) \circ g \sim^{G} (\Sigma_{G_m}^f) \circ g \\
(1) \{1\}^{1+n+n'}(\Sigma_{G_m}^{k+n'}) \circ k+n' = (\Sigma_{G_m}^{k+n'}) \circ k+n' = (\Sigma_{G_m}^{k+n'}) \circ k+n' = (\Sigma_{G_m}^{k+n'}) \circ k+n'
\]
for any \( k \in \mathbb{Z} \) such that all terms in the formula are defined.

Proof. (0) The claim follows from lemma \( \text{lm}^{3.10} \). (1) The claim follow from point (0) and prop \( \text{lm}^{3.10} \). □

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