Mixed Anomalies: Chiral Vortical Effect and the Sommerfeld Expansion

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Abstract

We discuss the connection between the integer moments of the Fermi distribution function that occur in the Sommerfeld expansion and the coefficients that occur in anomalous conservation laws for chiral fermions. As an illustration we extract the chiral magneto-thermal energy current from the mixed gauge-gravity anomaly in the 3+1 dimensional energy-momentum conservation law. We then use a similar method to confirm the conjecture that the $T^2/12$ thermal contribution to the chiral vortical effect (CVE) current arises from the gravitational Pontryagin term in the 3+1-dimensional chiral anomaly.

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I. INTRODUCTION

A recent experiment \cite{1} and its widespread coverage in the media \cite{2} have focussed attention on the idea that the physics of a system containing chiral fermions can be influenced by effects of gravitational origin even in flat space-time \cite{3}. These effects occur because the coefficients in certain constitutive relations for transport currents are related to the coefficients in corresponding anomalous conservation laws. As anomalies are not renormalized by interactions, these anomaly-induced, non-dissipative, contributions to transport currents should take the same values in both strongly-coupled and free theories. In the free case the currents can be computed without reference to any anomaly, and the free-theory computations reduce to the evaluation of integer moments of the Fermi function that turn out to be polynomials in the temperature and chemical potential. One is left with a sense that these Sommerfeld-expansion integrals somehow know about anomalies. This impression was made concrete by Loganayagam and Surówka \cite{4} who observed that a generating function for the integer moments of the Fermi function bears a close resemblance to the the product of the A-roof genus and the total Chern character that occurs in the general-dimension Dirac index theorem — and which, via the Bardeen-Zumino descent equations \cite{5}, is the ultimate source of the anomalies. Their observation lead them to a “replacement rule” that allowed them to compute anomaly-induced contributions to transport and fluid dynamics in $N$ space-time dimensions directly from the anomaly polynomial in $N + 2$ dimensions \cite{4, 6–12}.

In this paper we illustrate some of ideas by computing two of these currents — the thermomagnetic current that plays a central role in \cite{1}, and the thermal contribution to the chiral vortical effect (CVE) current that arises when a chiral fermion is in thermal equilibrium in a rotating frame — both from the free theory and from the corresponding anomaly. The first example is merely a repackaging of the gravitational-anomaly derivation of Hawking radiation \cite{13–15} but it serves to set the stage for an explicit confirmation of the conjecture \cite{3} that the thermal component of the CVE is related to the gravitational anomaly. These two derivations also help explicate the geometric origin of the replacement-rule mapping that takes the first Pontryagin class of the Riemann curvature tensor to minus the square of the temperature.

In section \[\text{II}n\] we introduce the specific currents whose anomaly-driven origin we seek to illustrate. In section \[\text{III}n\] we will construct gedanken spacetimes in which the currents are
created \textit{ex nihilo} by tidal forces in the vicinity of black-hole event horizons. In section \textbf{IV} we use similarity of the their generating functions and the observation that often only one of the formal eigenvalues of the curvature tensor will be non-zero to link the anomalies with the Sommerfeld integrals. A final section \textbf{V} will put these ideas into context.

II. ANOMALIES AND ANOMALY-INDUCED CURRENTS

The "mixed axial-gravitational anomalies" that are invoked in the condensed-matter context in \cite{1} and also \cite{16} are the (3+1)-dimensional anomalous conservation equation

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_{N,\lambda} - \frac{1}{384\pi^2\sqrt{-g}} \nabla_\mu [F_{\rho\sigma} R^{\nu\mu}_{\alpha\beta}]$$ (1)

for the energy-momentum tensor $T^{\mu\nu}$ of a unit-charge right-handed Weyl fermion, and the anomalous conservation equation

$$\nabla_\mu J^\mu_N = -\frac{1}{32\pi^2\sqrt{-g}} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{768\pi^2\sqrt{-g}} R^\alpha_{\beta\mu\nu} R^\beta_\alpha \rho\sigma$$ (2)

for the particle-number current $J^\mu_N$. The right-hand-side of Eq (1) shows that energy-momentum is delivered to the fermion from two sources: the first is the expected Lorentz force $F^{\nu\lambda} J_{N,\lambda}$; the second is the gravitational anomaly which requires a cooperation between the gauge field $F_{\mu\nu}$ and the tidal forces encoded in the Riemann tensor $R^{\nu\mu}_{\alpha\beta}$ of the background space-time geometry. The right-hand-side of the gauge-current conservation law Eq (2) contains two anomalous source terms: a gauge field Chern-character density and a geometric Pontryagin-class density. The two equations, (1) and (2), display mixed anomalies because the anomalous sources for both the geometry-related energy-momentum tensor $T^{\mu\nu}$ and the gauge-field-related particle-number current $J^\mu_N$ contain expressions involving the background field that couples to the other.

Anomaly-induced currents appear in solid-state systems \cite{1, 16} and also in relativistic fluid dynamics \cite{17} where (in the $[-, +, +, +]$ metric convention) we have \cite{51}

$$T^{\mu\nu} = pg^{\mu\nu} + (\epsilon + p)u^\mu u^\nu + \xi_{TB} (B^\mu u^\nu + B^\nu u^\mu) + \xi_{T\omega} (\omega^\mu u^\nu + \omega^\nu u^\mu),$$ (3)

$$J^\mu_N = n u^\mu + \xi_{JB} B^\mu + \xi_{J\omega} \omega^\mu;$$ (4)

$$J^\mu_S = s u^\mu + \xi_{SB} B^\mu + \xi_{S\omega} \omega^\mu.$$ (5)

Here $T^{\mu\nu}$ and $J^\mu_N$ are the energy-momentum tensor and number current that we have already met, while $J^\mu_S$ is the entropy current. The first terms on the RHS of each of these expressions
are the usual expressions for a relativistic fluid where \( u^\mu \) denotes the 4-velocity of the fluid, and \( \epsilon, p, n, \) and \( s \) are respectively the energy density, pressure, particle-number density, and entropy density. The remaining anomaly-induced terms involve the angular-velocity 4-vector defined by
\[
\omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\sigma\tau} u_\nu \partial_\sigma u_\tau.
\]
(6)

With \( \epsilon^{0123} = +1 \) and \( \epsilon_{0123} = -1 \), and in the \( u^\mu = (1, 0, 0, 0) \) rest frame we have \( \omega^\mu = (0, \Omega) \) where \( \Omega = \frac{1}{2} \nabla \times u \) is the local 3-vector angular velocity. The extra currents also involve the magnetic field \( B \) as it appears to an observer moving at velocity \( u^\mu \). We have
\[
E^\mu = F^{\mu\nu} u_\nu, \quad B^\mu = \frac{1}{2} \epsilon^{\mu\nu\sigma\tau} u_\nu F_{\sigma\tau}.
\]
(7)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, A^\mu = (\phi, A) \). Again, in the \( u^\mu = (1, 0, 0, 0) \) rest frame, we have \( E^\mu = (0, E), B^\mu = (0, B) \) and
\[
\frac{1}{4\pi^2} E \cdot B = \frac{1}{4\pi^2} E_\mu B^\mu = -\frac{1}{32\pi^2} \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau}.
\]
(8)

In relativistic fluid dynamics the notion of the “velocity of the fluid” requires further specification. We will take \( u^\mu \) to be the 4-velocity of the no-drag frame introduced in [18, 19]. This is the frame in which the \( \xi \) coefficients take their simplest form, and is usually the frame in which the fluid is in local thermodynamic equilibrium.

Demanding that no entropy production be associated with the anomaly-induced currents requires [19] that the six coefficients \( \xi_{TB}, \xi_{T\omega}, \xi_{JB}, \xi_{J\omega}, \xi_{SB}, \xi_{S\omega} \) depend at most three underlying parameters through
\[
\begin{align*}
\xi_{JB} &= C\mu, \\
\xi_{J\omega} &= C\mu^2 + X_B T^2, \\
\xi_{SB} &= X_B T, \\
\xi_{S\omega} &= 2\mu T X_B + X_\omega T^2, \\
\xi_{TB} &= \frac{1}{2} \left(C\mu^2 + X_B T^2\right), \\
\xi_{T\omega} &= \frac{2}{3} \left(C\mu^3 + 3X_B\mu T^2 + X_\omega T^3\right).
\end{align*}
\]
(9)

Here \( T \) is the temperature and \( \mu \) the chemical potential associated with the U(1) particle-number current \( J_N^{\mu N} \). For the ideal Weyl gas the three parameters \( C, X_B, \) and \( X_\omega \) take the values
\[
C = \frac{1}{4\pi^2}, \quad X_B = \frac{1}{12}, \quad X_\omega = 0.
\]
(10)
It is clear from the derivation in [19] that $C$ is the coefficient of the term (8) in the chiral anomaly (2). It was conjectured in [3] that the parameter $X_B$ is the coefficient appearing before the Pontryagin density in the same equation. This conjecture was originally motivated by the simple observation that both $X_B$ and the Pontryagin coefficient depend on the same physical data (spin, chirality, but not charge), but it has gained support from consideration of global anomalies [20, 21] and from calculations using AdS-CFT formalism [22]. It is not, however, straightforward to confirm the conjecture by extending the flat-space considerations in [19] to curved space.

For an ideal gas of right-handed Weyl fermions at rest in flat space (so $u^\mu = (1, 0, 0, 0)$) the term with $\xi_{TB}$ in (3) leads to an anomaly-induced magneto-thermal energy flux
\[
J_\epsilon = B \left( \frac{\mu^2}{8\pi^2} + \frac{1}{24} T^2 \right),
\]
which plays a central role in [1].

A similar gas in thermal equilibrium in a frame rotating at angular velocity $\Omega$ (so that $u^\mu = (1, 0, 0, 0)$ on the rotation axis) acquires from the $\xi_{J_\omega}$ term in (4) a chiral vortical effect (CVE) number current that (again on the rotation axis) is given by [23]
\[
J_N = \Omega \left( \frac{\mu^2}{4\pi^2} + \frac{|\Omega|^2}{48\pi^2} + \frac{1}{12} T^2 \right).
\]

We do not need the gravitational anomaly to understand the origin of the magneto-thermal current in (11). It is well-known that solving the for the eigenvalues of the Weyl Hamiltonian in the presence of a magnetic field $B$ yields a set of energy levels
\[
\epsilon_l(k) = \pm \sqrt{2|B|l + k^2},
\]
where $k$ is the component of momentum parallel to $B$. The levels have degeneracy $|B|/2\pi$ per unit area in a plane transverse to $B$, and all levels are are gapped except for $l = 0$. The $l = 0$ level is special in that only one sign of the square-root is allowed and we effectively have an array of gapless one-dimensional chiral fermions with
\[
\epsilon(k) = +k.
\]

Each one-dimensional chiral fermion contributes a current of
\[
J_\epsilon = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \epsilon \left\{ \frac{1}{1 + e^{\beta(\epsilon - \mu)}} - \theta(-\epsilon) \right\} = 2\pi \left( \frac{\mu^2}{8\pi^2} + \frac{1}{24} T^2 \right),
\]
where $\beta = 1/T$, and the $-\theta(-\epsilon)$ counter-term effects a normal-ordering vacuum subtraction of the contribution of Dirac/Fermi sea, ensuring that there is no current when $\mu = T = 0$. Combining (15) with the $|B|/2\pi$ areal degeneracy leads immediately to (11).

The free fermion computation of the CVE current is rather lengthier [23] but it also reduces to a Sommerfeld integral, in this case

$$
J_N = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \epsilon^2 d\epsilon \left( \frac{1}{1 + e^{\beta(\epsilon-(\mu+\Omega/2))}} - \frac{1}{1 + e^{\beta(\epsilon-(\mu-\Omega/2))}} \right),
$$

$$
= \frac{\mu^2 \Omega}{4\pi^2} + \frac{\Omega^3}{48\pi^2} + \frac{1}{12\Omega T^2}.
$$

Although we do not need the mixed anomalies to obtain these currents, we can use them to do so, and in doing so gain insight in the physical origin of the anomalies. We will devote the next section to the anomaly derivation of (11) and (12). We will see that a number of deep ideas are combined in these derivations.

III. CURRENTS FROM ANOMALIES

In this section we will construct spacetimes in which the thermal contributions to the anomaly-induced currents arise from the gravitational source terms in the associated anomalous conservations laws.

A. magneto-thermal current

To derive the thermal part of (11) from the anomaly we will take for granted the $4 \to 2$ dimensional reduction provided by the magnetic field, and consider the current as that of our array of 1+1 dimensional right-going fermions. We imagine a gedanken experiment in which we heat each right-going Fermi field to temperature $T$ by terminating its space-time on the left by a 1+1 dimensional black hole whose Hawking temperature is $T$. The $T^2$ contribution to (11) is then the Fermi field’s contribution to the outgoing Hawking radiation. To relate this interpretation to the anomaly we review how [13–15] Hawking radiation arises from 1+1 dimensional version of the energy-momentum anomaly

$$
\nabla_\mu T^{\mu\nu} = -\frac{c}{96\pi} \frac{e^{\nu\sigma}}{\sqrt{|g|}} \partial_\sigma R,
$$

(16)
to which (1) reduces in a uniform $\mathbf{B}$ field. Here $e^{01} = 1$, $R = R^{\alpha\beta}_{\alpha\beta} = 2R^{01}_{01} = 2R^r_r$ is the Ricci scalar, and $c = c_R - c_L$ is the difference between the conformal central charges of the right-going and left going massless fields. As our magnetic field leaves us with only right-going fields, we have $c = 1$.

As the black hole is an externally imposed background space-time, we do not need its metric to satisfy the Einstein equations and a suitable metric is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2,$$

where all that is required of $f(r)$ is that it tends to unity at large $r$ and vanishes linearly as $r$ approaches the event horizon $r = r_H$. In this metric the Ricci scalar is given by

$$R = -f''.$$  

A covariant energy-momentum conservation equation does not, on its own, lead to conserved energy and momentum. For that we need a space-time symmetry i.e. a Killing-vector field $\eta^\mu$ which obeys the isometry condition

$$\nabla_\mu \eta_\nu + \nabla_\nu \eta_\mu = 0.$$  

Combining the isometry equation with (16) then gives us

$$\nabla_\mu (T^{\mu\nu} \eta_\nu) = -\frac{c}{96\pi} \frac{e^{\nu\sigma} \eta_\nu}{\sqrt{|g|}} \partial_\sigma R,$$

in which

$$\sqrt{|g|} \nabla_\mu (T^{\mu\nu} \eta_\nu) = \partial_\mu (\sqrt{|g|} T^{\mu\nu} \eta_\nu)$$

involves a conventional total divergence.

Our Schwarzschild-metric posses a Killing vector $\eta = \partial_t$ whose covariant components are $(\eta_t, \eta_r) = (-f(r), 0)$. From this we find that

$$\nabla_\mu T^{\mu\nu} \eta_\nu = (\partial_r \sqrt{|g|} T^{r}_t) / \sqrt{-g}$$

We then have

$$\frac{\partial}{\partial r} (\sqrt{|g|} T^{r}_t) = -\frac{c}{96\pi} f \partial_r f'' = -\frac{c}{96\pi} \partial_r \left( f f'' - \frac{1}{2} (f')^2 \right),$$

and integrating from $r_H$ to $r = \infty$ gives

$$\sqrt{|g|} T^{r}_t \bigg|_{r_H}^\infty = -\frac{c}{96\pi} \left( f f'' - \frac{1}{2} (f')^2 \right) \bigg|_{r_H}^\infty.$$
According to [15], the appropriate boundary condition is that \( T^r_t \) be zero at the horizon. The RHS of (24) by contrast is zero at infinity, and contributes \( \left( c/96\pi \right) (f')^2/2 \) at the horizon. As \( \sqrt{|g|} \to 1 \) at large \( r \), we see that an energy current of magnitude

\[
T^{rt}(z \to \infty) = -T^r_t(z \to \infty) = \frac{c\kappa^2}{48\pi}, \quad \kappa = f'(r_H)/2,
\]

has been built up by the anomaly as we move away from the horizon. The quantity \( \kappa \) is the \textit{surface gravity} of the black hole.

To complete the derivation of (11) we recall the argument [24, 25] that the geometry of the Euclidean section of our black-hole metric shows that the Hawking temperature is given by \( T_H = \kappa/2\pi \). We begin by setting \( t = -i\tau \) and see that in imaginary-time our Schwarzschild space metric becomes

\[
d\sigma^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2.
\]

If we introduce a new radial co-ordinate

\[
\rho = \int_{r_H}^{r} \frac{dr'}{\sqrt{f(r')}} \approx \frac{2}{\sqrt{f'(r_H)}}\sqrt{r - r_H},
\]

where the approximation holds for \( r \) just above \( r_H \). Then, in this same region,

\[
d\sigma^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2
= f(r)d\tau^2 + d\rho^2,
\approx f'(r_H)(r - r_H)d\tau^2 + d\rho^2
= \kappa^2 \rho^2 d\tau^2 + d\rho^2.
\]
Comparison with the metric of plane polar coordinates now shows that if there is to be no conical singularity at \( r_H \) we must identify \( \kappa \tau \) with the polar angle \( \theta \). Thus the smooth euclidean manifold described by (26) looks like the skin of a salami sausage in which the circumferential coordinate \( \theta \) is identified with \( \theta + 2\pi \), or equivalently \( \tau \) is identified with \( \tau + \beta \) where \( \beta = 2\pi/\kappa \). Green functions \( G(r,t) \) in Minkowski signature spacetime will be periodic in imaginary time with period \( \beta \) and are therefore \( \text{[24, 25]} \) thermal Green functions with temperature \( T_H = \beta^{-1} = \kappa/2\pi \) (or \( k_B T_H = \hbar \kappa/2\pi c \) if we include dimensionful constants).

This derivation seems quite straightforward, but there are two subtleties that need to be discussed. Firstly, anomalies are usually presented as being of two types: consistent and covariant \( \text{[5]} \). Following \( \text{[15]} \) we have exclusively used the covariant form of the anomaly. Secondly, it is well known that Hawking radiation is observer-dependent. These two issues are not unrelated. To illuminate this point we will repeat the Hawking radiation calculation using the two-dimensional version of Kruskal-Szekeres coordinates.

We begin by defining a tortoise coordinate \( r_* \) by solving

\[
\frac{dr_*}{dr} = \frac{1}{f(r)}.
\]

and taking as boundary condition that \( r_* \) coincides with \( r \) at large positive distance. In \((t, r_*)\) coordinates the metric becomes

\[
ds^2 = e^\phi (-dt^2 + dr_*^2)
\]

where

\[
\phi(r_*) = \ln f(r(r_*))
\]

and the event horizon lies at \( r_* = -\infty \). On setting \( z = r_* + i \tau \) and \( \bar{z} = r_* - i \tau \) the euclidean version of this metric takes the isothermal (conformal) form

\[
d\sigma^2 = e^\phi d\bar{z} dz.
\]

In \( \bar{z}, z \) coordinates the only non-vanishing Christoffel symbols are

\[
\Gamma^z_{zz} = \partial_z \phi, \quad \Gamma^\bar{z}_{\bar{z}z} = \partial_{\bar{z}} \phi,
\]

and the Ricci scalar is

\[
R = -4e^{-\phi} \partial^2_{zz} \phi.
\]
In two-dimensional conformal field theory we are used to defining energy-momentum operators $\hat{T}(z)$ and $\hat{\bar{T}}(\bar{z})$ where, for a free $c = 1$ boson field $\hat{\phi}(z, \bar{z}) = \hat{\phi}(z) + \hat{\phi}(\bar{z})$ for example, we have

$$\hat{T}(z) = \hat{\partial}_z \hat{\phi}(z) \hat{\partial}_z \hat{\phi}(z) : \lim_{\delta \to 0} \left( \hat{\partial}_z \hat{\phi}(z + \delta/2) \hat{\partial}_z \hat{\phi}(z - \delta/2) + \frac{1}{4\pi \delta^2} \right).$$  \((35)\)

(Note that conformal field theory papers often define $\hat{T}(z)$ to be $-2\pi$ times \((35)\) so as to simplify the operator product expansion.) The operator $\hat{T}(z)$ has been constructed to be explicitly holomorphic in $z$, but at a price of tying its definition to the $z, \bar{z}$ coordinate system — both in the normal ordered expression in the first line and by the explicit counterterm in the second line. It is not surprising, therefore, that under a change of co-ordinates the operator $\hat{T}(z)$ does not transform as a tensor but instead acquires an inhomogeneous Schwarzian-derivative c-number part \([26]\). If we want a genuine energy-momentum tensor we must define

$$\hat{T}_{zz} = \hat{T}(z) + \frac{c}{24\pi} \left( \partial_{zz}^2 \phi - \frac{1}{2} (\partial_z \phi)^2 \right),$$

$$\hat{T}_{\bar{z}\bar{z}} = \hat{T}(\bar{z}) + \frac{c}{24\pi} \left( \partial_{\bar{z}\bar{z}}^2 \phi - \frac{1}{2} (\partial_{\bar{z}} \phi)^2 \right),$$

$$\hat{T}_{\bar{z}z} = -\frac{c}{48\pi} \partial_{z\bar{z}}^2 \phi,$$ \((36)\)

in which the c-number Schwarzians in the operator transformation are cancelled by Schwarzians from the transformation of the c-number additions.

A direct computation, using the holomorphicity and anti-holomorphicity of the operators $\hat{T}(z)$ and $\hat{T}(\bar{z})$ together with the formulae for the Christoffel symbols, shows that

$$\nabla^z \hat{T}_{zz} + \nabla^\bar{z} \hat{T}_{\bar{z}z} = 0.$$ \((37)\)

If, however, we keep only the right-going field, the chiral energy momentum tensor becomes

$$\hat{T}_{zz} = \hat{T}(z) + \frac{c}{24\pi} \left( \partial_{zz}^2 \phi - \frac{1}{2} (\partial_z \phi)^2 \right),$$

$$\hat{T}_{\bar{z}\bar{z}} = 0,$$

$$\hat{T}_{\bar{z}z} = -\frac{c}{48\pi} \partial_{z\bar{z}}^2 \phi.$$ \((38)\)

A similar computation shows that the chiral tensor obeys

$$\nabla^z \hat{T}_{zz} + \nabla^\bar{z} \hat{T}_{\bar{z}z} = -\frac{c}{96\pi} \partial_z R,$$

$$\nabla^z \hat{T}_{\bar{z}z} + \nabla^\bar{z} \hat{T}_{z\bar{z}} = +\frac{c}{96\pi} \partial_{\bar{z}} R.$$ \((39)\)
where the second term on the left hand side of the second equation is identically zero. In our $z, \bar{z}$ co-ordinates system we have $\sqrt{g} = \sqrt{-g_{zz}g_{\bar{z}\bar{z}}} = -ie^{\phi}/2$ (perhaps more clearly, we can express this as $e^{\phi}dt \wedge dr_* = (e^{\phi}/2i)d\bar{z} \wedge dz$), and we can write these last two equations in a covariant manner as

$$\nabla^z T_{zz} + \nabla^{\bar{z}} T_{\bar{z}\bar{z}} = i \frac{c}{96\pi} \sqrt{g} \epsilon_{zz} \partial^z R,$$

$$\nabla^z T_{\bar{z}z} + \nabla^{\bar{z}} T_{\bar{z}\bar{z}} = i \frac{c}{96\pi} \sqrt{g} \epsilon_{\bar{z}z} \partial^{\bar{z}} R. \quad (40)$$

In general euclidean co-ordinates we therefore have

$$\nabla_{\mu} T_{\mu\nu} = i \frac{c}{96\pi} \sqrt{g} \epsilon_{\nu\sigma} \partial^\sigma R. \quad (41)$$

The factor “$i$” appears in (42) because it is only the imaginary part of the Euclidean effective action that can be anomalous. It is absent when we write the equation in Minkowski signature space-time where it becomes

$$\nabla_{\mu} T_{\mu\nu} = -\frac{c}{96\pi} \frac{1}{\sqrt{|g|}} \epsilon^{\nu\sigma} \partial_\sigma R. \quad (42)$$

Because we have used a covariantly-transforming energy momentum tensor, we find the covariant form of the anomaly. In this calculation we also see that the anomaly arises solely from the c-number terms.

Now define Euclidean Kruskal coordinates $U, V$ by setting

$$Z = U + iV = \exp\{\kappa(r_* + i\tau)\} = \exp\{\kappa z\},$$

$$\bar{Z} = U - iV = \exp\{\kappa(r_* - i\tau)\} = \exp\{\kappa \bar{z}\}, \quad (43)$$

so that

$$|Z|^2 = U^2 + V^2 = \exp\{2\kappa r_*\}. \quad (44)$$

In terms of these coordinates we have

$$d\sigma^2 = f(r)\kappa^{-2}e^{-2\kappa r_*}(dU^2 + dV^2). \quad (45)$$

With $\kappa$ being the surface gravity, this goes to the non-singular metric

$$d\sigma^2 = \text{const.}(dU^2 + dV^2) \quad (46)$$

(where the constant is determined by the exact form of $f(r)$ near the horizon point at $U^2 + V^2 = 0$) and to

$$d\sigma^2 = \kappa^{-2}(U^2 + V^2)^{-1}(dU^2 + dV^2) \quad (47)$$
at large distance. This last expression is the metric of a cylinder of circumference $2\pi/\kappa$, confirming the time periodicity.

For a general conformal “salami sausage” metric we need $ds^2 = e^\phi dZd\bar{Z}$ with $\phi = 0$ at $|Z| = 0$, and $\phi \approx -2 \ln \kappa |Z|$ at large $|Z|$ where the circumference of the sausage becomes constant. The coefficient “$-2$” is required by the Gauss-Bonnet theorem as the end-cap is topologically a hemisphere. At short distance the sausage looks like a spherical cap, and we have

$$e^\phi = 1 - \frac{1}{4} \bar{z}z R + O(|z|^4), \quad \phi = \frac{1}{4} \bar{z}z R + O(|z|^4),$$

(48)

where $R$ is the Ricci scalar (twice the Gaussian curvature) at the horizon.

As Kruskal $Z$, $\bar{Z}$ coordinates are again isothermal, the chiral energy momentum tensor is of the form

$$\hat{T}_{ZZ} = \hat{T}(Z) + \frac{c}{24\pi} \left( \frac{\partial^2_{ZZ} \phi}{2} - \frac{1}{2} (\partial_Z \phi)^2 \right),$$

(49)

where $\hat{T}(Z)$ is the normal-ordered operator part that transforms inhomogeneously under conformal maps. The second term is the c-number counterterm whose transformation cancels that of $\hat{T}(Z)$ so as to make $\hat{T}_{ZZ}$ transform as a tensor.

At short distance the c-number part in $\hat{T}_{ZZ}$ vanishes — in fact it vanishes identically on a sphere with conformal coordinates. Consequently, as $\hat{T}_{ZZ}$ is zero at the horizon, the expectation value of $\hat{T}(Z)$ is zero there, and hence everywhere. At large distance, however, we will have

$$\phi(Z, \bar{Z}) \sim -\ln \kappa Z - \ln \kappa \bar{Z}$$

(50)

and so the c-number part gives us

$$T_{ZZ} \sim \frac{c}{24\pi} \left( \frac{1}{Z^2} - \frac{1}{2} \frac{Z^2}{Z^2} \right)
= \frac{c}{48\pi} \frac{1}{Z^2}.$$ 

(51)

Now let us shift to the tortoise light-cone coordinates $z = r_* + i\tau$, $\bar{z} = r_* - i\tau$. Then

$$\hat{T}_{zz} = \left( \frac{\partial Z}{\partial z} \right)^2 \hat{T}_{ZZ}
= \kappa^2 Z^2 \hat{T}_{ZZ}
\rightarrow \frac{ck^2}{48\pi}, \quad \text{as} \ r_* \rightarrow \infty.$$ 

(52)
In the asymptotic Minkowski space \( r_* = r \), and with the speed of light equal to one and \( \pm \) denoting \( r \pm t \) components, we have

\[
\hat{T}^{++} = \frac{1}{4}(\hat{T}_{rr} + \hat{T}_{tt} - 2\hat{T}_{rt}),
\]

\[
\hat{T}^{--} = \frac{1}{4}(\hat{T}_{rr} + \hat{T}_{tt} + 2\hat{T}_{rt}),
\]

\[
\hat{T}^{+-} = \frac{1}{4}(\hat{T}_{rr} - \hat{T}_{tt}),
\]

with \( \hat{T}^{+-} = \hat{T}^{--} = 0 \). Consequently \( \hat{T}^{++} = \hat{T}^{tt} = \hat{T}^{rt} \) and the Kruskal coordinate energy density and flux coincide with those from the Schwarzschild coordinate calculation. The break-up between operator and c-number is different however. The c-number part in \( \hat{T}_{zz} \) vanishes at large \( r \) so the large-distance contribution to the Schwarzschild energy flux comes entirely from the expectation-value of the operator \( \hat{T}(z) \). In other words, the Schwarzschild observer sees the asymptotic energy being carried by actual particles. In Kruskal coordinates the \( \hat{T}_{ZZ} \) operator part has vanishing expectation value everywhere and the asymptotic energy flux comes entirely from the c-number term. Thus \( \hat{T}(Z) \) and \( \hat{T}(z) \) record very different particle content and the Schwarzschild \( r, t \) coordinate observer’s zero-particle state is not the same as the Kruskal observer’s zero-particle state.

The physical interpretation should now be clear: Both the Schwarzschild time and the Kruskal time coordinate lines correspond to the flows of time-like Killing vectors. In each coordinate system the field’s mode expansions have well-defined yet different positive-frequency modes whose coefficients are annihilation operators. The operators \( \hat{T}_{ZZ} \) and \( \hat{T}_{zz} \) are normal ordered so that the annihilators are all to the right. It is the positive frequencies that can excite a detector from its ground state, and the normal-ordered operators keep track of what a detector at fixed spatial coordinate would record in each coordinate system. Close to the horizon a detector at fixed Schwarzschild coordinate \( r \) sees a high-temperature thermal distribution of outgoing particles. However, at the horizon, their contribution to the expectation value of \( \hat{T}_z \) is exactly cancelled by the c-number term. As we move away from the horizon this c-number term decreases and allows the total covariantly-defined energy current to grow to its asymptotic value. On the other hand the Kruskal observer never sees any particles, and all their energy flux comes from the c-number contribution which grows from zero at the horizon to the same asymptotic value as the Schwarzschild flux. Presumably the source term in the Einstein equations will be the expectation value of a covariantly-defined energy-momentum tensor. Therefore it is independent of the observer’s motion — but as
we are not investigating the back reaction of the emitted radiation on the geometry, this is not our present concern.

### B. Chiral Vortical current

We now seek an analogous derivation of \( \text{(12)} \) from the Pontryagin term in \( \text{(2)} \). To do this we need to modify our toy black hole to acquire a non-zero Pontryagin density. A suggestion of how to proceed comes from the Kerr metric of a rotating black hole. In Boyer-Lindquist coordinates \((t, r, \phi, \theta)\), and with \( \cos \theta = \chi \), this metric is

\[
 ds^2 = - \left(1 - \frac{2mr}{r^2 + a^2 \chi^2}\right) dt^2 + \left(\frac{r^2 + a^2 \chi^2}{r^2 + a^2 - 2mr}\right) dr^2 - \frac{4amr(1 - \chi^2)}{r^2 + a^2 \chi^2} dt d\phi
\]

\[
 (1 - \chi^2) \left(a^2 + r^2 + \frac{2a^2 mr (1 - \chi^2)}{r^2 + a^2 \chi^2}\right) d\phi^2 + \frac{r^2 + a^2 \chi^2}{1 - \chi^2} d\chi^2,
\]

where \( m \) is the mass and \( J = ma \) the angular momentum of the black hole.

This metric has two special horizon surfaces at the roots

\[
 r_{\pm} = m \pm \sqrt{m^2 - a^2}
\]

of \( r^2 + a^2 - 2mr = 0 \). The outer horizon \( r = r_+ \) is the causal event horizon on which trapped photons are forced to orbit at fixed \( r, \theta \) and angular velocity

\[
 \Omega_+ \overset{\text{def}}{=} \frac{d\phi}{dt} = \frac{a}{r_+^2}.
\]

Both \( \Omega_+ \) and the surface gravity

\[
 \kappa_+ = \frac{1}{4m} - m \Omega_+^2
\]

are constant over the horizon. As in the Schwarzschild case, the absence of a conical singularity in the Euclidean section requires \( \tau \sim \tau + \beta_H \) where

\[
 \beta_H = \frac{1}{T_H} = \frac{2\pi}{\kappa_+}.
\]

The Kerr black hole is therefore both rotating and hot.

What is important for us is that the numerical coefficient

\[
 \frac{1}{4} \epsilon^{\lambda \mu \rho \sigma} R^a_{\lambda \beta \mu \nu} R^b_{\alpha \rho \sigma} = -\frac{48am^2r \chi(r^2 - 3a^2 \chi^2)(3r^2 - a^2 \chi^2)}{(r^2 + a^2 \chi^2)^5}
\]

\[
 (59)
\]
of $dt \wedge dr \wedge d\phi \wedge d\chi$ in the Pontryagin-density 4-form $\text{tr} \{ R^2 \}$ is non-zero. For small $a/m$ the coefficient is largest near the poles of rotation at $\chi = \pm 1$.

The Kerr metric can be conveniently written in terms of the functions

\begin{align}
\Delta &= r^2 + a^2 - 2mr, \\
\rho^2 &= r^2 + a^2 \cos^2 \theta,
\end{align}

and two mutually orthogonal one forms

\begin{align}
\omega &= \frac{r^2 + a^2}{\rho^2} (dt - a \sin^2 \theta d\phi), \\
\tilde{\omega} &= \frac{r^2 + a^2}{\rho^2} \left( d\phi - \frac{a}{r^2 + a^2} dt \right),
\end{align}

as

\begin{equation}
ds^2 = -\frac{\Delta \rho^2}{(r^2 + a^2)^2} \omega^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta \tilde{\omega}^2.
\end{equation}

Motivated by this rewriting, we consider a 3 + 1 space with metric

\begin{equation}
ds^2 = -f(z) \left( \frac{dt - \Omega r^2 d\phi}{(1 - \Omega^2 r^2)} \right)^2 + \frac{1}{f(z)} dz^2 + dr^2 + \frac{r^2 (d\phi - \Omega dt)^2}{(1 - \Omega^2 r^2)}.
\end{equation}

The metric has been constructed so that at large $z$, where $f(z) = 1$, it reduces to

\begin{equation}
ds^2 \to -dt^2 + dz^2 + dr^2 + r^2 d\phi^2
\end{equation}

where $r$, $z$, and $\phi$ and $z$ are the cylindrical-coordinate radial, axial, and azimuthal coordinates of an asymptotically flat spacetime. Thus, in contrast to previous usage, $z$ is a real coordinate that provides a measure of the distance from the horizon at $f(z_H) = 0$, and $r$ is a measure of distance from the rotation axis. Replacing the complicated Kerr-metric coefficients by the function $f(z)$ allows us to ensure that the space-time curvature is concentrated near the horizon, which now appears to be planar and rotating at angular velocity $\Omega$. We anticipate that outgoing Fermi fields in this space will be in asymptotic thermal equilibrium at temperature $T_H = f'(z_H)/4\pi$ in a frame rotating about the $z$-axis with the horizon angular velocity $\Omega$. They should therefore acquire a CVE current as they move though the curved and twisted near-horizon geometry.

The numerical coefficient of the Pontryagin density in this space-time is

\begin{equation}
\frac{1}{4} \epsilon_{\mu
u\rho\sigma} R^\alpha_{\beta\mu\nu} R^\beta_{\alpha\rho\sigma} = \frac{2r \Omega f'(z)(8\Omega^2(1 - f(z)) + (1 - \Omega^2 r^2)^2 f''(z))}{(1 - \Omega^2 r^2)^3} \\
\sim 2r \Omega f'(z)f''(z) \\
= \frac{\partial}{\partial z}(\Omega r[f'(z)]^2).
\end{equation}
In the last two lines we have kept only the leading term in $\Omega$. The error is $O[\Omega^3]$.

When we divide by $\sqrt{|g|} = r$ to get the Pontryagin-density scalar we see that we have created in the region abutting the horizon an $r$-independent source term for the axial current of our anomalous relativistic fluid. We assume that this planar source drives a current only in the $z$ direction. In that case, we find that to leading order in $\Omega$ the anomalous conservation law (2) becomes

$$\frac{\partial}{\partial z} (\sqrt{|g|} J_N^z) = -\frac{\Omega}{192\pi^2} \frac{\partial}{\partial z} (\sqrt{|g|} [f'(z)]^2).$$

(68)

With boundary condition $J^z(z_H) = 0$, we can again integrate up with respect to $z$ to find

$$J_N^z(z \to \infty) = \frac{1}{12} \Omega T_H^2.$$  

(69)

This is the expected thermal contribution to the CVE current (12).

If we retain terms of order $\Omega^3$, we do get a contribution to the on-axis current similar to that in (12), but with with coefficient $1/24\pi^2$ rather than $1/48\pi^2$. Trying slightly modified metrics indicates that this correction to the current is sensitive to how the metric varies away from the axis of rotation. For example, omitting the $(1 - \Omega^2 r^2)$ factors in the denominators in (65) does not alter the on-axis asymptotic metric, and does not affect the coefficient of the $T^2$ term in the current. It does, however, lead to the coefficient of $\Omega^3$ becoming zero. Perhaps we should not be surprised by this as the notion of a rigidly rotating coordinate system such as that used by [23] is bound to be problematic away from the rotation axis.

Note that our CVE current (69) is not, as suggested in [31], simply proportional to the Chern-Simons current associated with the Pontryagin class. The latter current

$$J_{\text{CS}}^\lambda = 2 \frac{\lambda_{\mu\nu\sigma}}{\sqrt{|g|}} \left( \Gamma^\alpha_{\beta\mu} \partial_\rho \Gamma^\beta_{\alpha\sigma} + \frac{2}{3} \Gamma^\alpha_{\beta\mu} \Gamma^\beta_{\gamma\rho} \Gamma^\gamma_{\alpha\sigma} \right)$$

(70)

has (to leading order in $\Omega$) two non-zero components

$$J_{\text{CS}}^z = \Omega [f'(z)]^2, \quad J_{\text{CS}}^r = 2\Omega f'(z)/r.$$  

(71)

It does satisfy

$$\partial_\lambda \sqrt{|g|} J_{\text{CS}}^\lambda = \frac{1}{4} \epsilon_{\mu\nu\rho} R^\mu_{\beta\lambda\nu} R^\beta_{\alpha\rho\sigma},$$

(72)

but obeys different boundary conditions in that $J_{\text{CS}}^\lambda(z)$ vanishes at $z = \infty$ rather than at the horizon. Our derivation in this section was, however, motivated by the discussion in [31].
IV. SOMMERFELD INTEGRALS AND ANOMALIES

In the introduction we made the claim that the Fermi-distribution moment integrals that appear in the higher-order terms of the Sommerfeld expansion somehow know about anomalies. In this section we try to explain how this knowledge comes about by combining the ideas in [4] with the geometry behind our gedanken trick of using the Hawking effect as our heat source.

The first few such moment integrals are

\[ \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left\{ \frac{1}{1 + e^{\beta(\epsilon - \mu)}} - \theta(-\epsilon) \right\} = \left( \frac{\mu}{2\pi} \right) , \]

\[ \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left( \frac{\epsilon}{2\pi} \right)^2 \left\{ \frac{1}{1 + e^{\beta(\epsilon - \mu)}} - \theta(-\epsilon) \right\} = \frac{1}{2} \left( \frac{\mu}{2\pi} \right)^2 + T^2 \frac{4!}{4!} , \]

\[ \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left( \frac{\epsilon}{2\pi} \right)^3 \left\{ \frac{1}{1 + e^{\beta(\epsilon - \mu)}} - \theta(-\epsilon) \right\} = \frac{1}{3} \left( \frac{\mu}{2\pi} \right)^3 + \left( \frac{\mu}{2\pi} \right) T^2 \frac{4!}{4!} , \]

\[ \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left( \frac{\epsilon}{2\pi} \right)^4 \left\{ \frac{1}{1 + e^{\beta(\epsilon - \mu)}} - \theta(-\epsilon) \right\} = \frac{1}{5} \left( \frac{\mu}{2\pi} \right)^4 + \frac{1}{2} \left( \frac{\mu}{2\pi} \right)^2 T^2 \frac{4!}{4!} + \frac{7}{8} T^4 \frac{6!}{6!} , \]

\[ \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left( \frac{\epsilon}{2\pi} \right)^5 \left\{ \frac{1}{1 + e^{\beta(\epsilon - \mu)}} - \theta(-\epsilon) \right\} = \frac{1}{6} \left( \frac{\mu}{2\pi} \right)^6 + \frac{1}{2} \left( \frac{\mu}{2\pi} \right)^4 T^2 \frac{4!}{4!} + \frac{1}{2} \left( \frac{\mu}{2\pi} \right)^2 \frac{7}{8} T^4 \frac{6!}{6!} + \frac{31}{24} T^6 \frac{8!}{8!} . \] (73)

These are all polynomials in the temperature and the chemical potential. It is essential for the simplicity of these results that the \( \epsilon \) integral runs from \(-\infty\) to \(+\infty\). If we had kept only the positive energy part of the integrals we would have instead

\[ \int_{0}^{\infty} \frac{d\epsilon}{2\pi} \left( \frac{\epsilon}{2\pi} \right)^k \frac{1}{1 + e^{\beta(\epsilon - \mu)}} = -\frac{1}{(2\pi\beta)^{k+1}} \text{Li}_{k+1}(-e^{\beta\mu}) , \] (74)

where the polylogarithm function \( \text{Li}_k(x) \) is defined by analytic continuation from the series

\[ \text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad |z| < 1. \] (75)

The polynomial form of the full-range integral arises from the identity

\[ \text{Li}_k(-e^{\beta\mu}) + (-1)^k \text{Li}_k(-e^{-\beta\mu}) = -\frac{(2\pi i)^k}{k!} B_k \left( \frac{1}{2} + \frac{\beta\mu}{2\pi i} \right) \] (76)

which holds for integer \( k \), and where \( B_k(x) \) are the Bernoulli polynomials defined by

\[ \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x) . \] (77)
The identity (76) is a special case of a general identity for the polylogarithm due to Hurwitz. A compact generating function
\[
\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} e^{\tau \epsilon/2\pi} \left\{ \frac{1}{1 + e^{(\epsilon-\mu)}} - \theta(-\epsilon) \right\} = \frac{1}{\tau} \left\{ \left( \frac{\tau T}{2} \right) \sin\left( \frac{\tau T}{2} \right) e^{\tau \mu/2\pi} - 1 \right\}, \quad 0 < \tau T/2\pi < 1, \quad (78)
\]
for the Fermi-distribution moments encapsulates these facts. Expanding both sides of (78) in powers of \( \tau \) and comparing coefficients reveals the equalities in (73), and also explains the reason for the inclusion of the factors of \( 1/n!(2\pi)^n \) in the left hand side integrals of (73).

The generating function identity (78) is easily established by substituting \( x = \exp\{\beta(\epsilon-\mu)\} \) and then using the standard integral
\[
\int_{0}^{\infty} dx \frac{x^{\alpha-1}}{1 + x} = \frac{\pi}{\sin(\pi\alpha)}, \quad 0 < \alpha < 1. \quad (79)
\]

The authors of [4] point out that the generating function (78) is strongly reminiscent of the general formula
\[
\text{Index}[D] = \int_{\mathcal{M}} \hat{A}[R] \text{ch}[F] \quad (80)
\]
for the index of the Dirac operator on a euclidean manifold \( \mathcal{M} \). Here
\[
\text{ch}[\tau F] = \exp\{\tau F/2\pi i\} = 1 + \tau\{F/2\pi i\} + \frac{\tau^2}{2}\{F/2\pi i\}^2 + \ldots \quad (81)
\]
is the total Chern character involving the gauge-field curvature \( F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu \), and
\[
\hat{A}[\tau R] \overset{\text{def}}{=} \sqrt{\text{det} \left( \frac{\tau R/4\pi i}{\sinh \tau R/4\pi i} \right)}
\]
\[
= 1 + \frac{\tau^2}{(4\pi)^2} 2\text{tr} \{R^2\} + \frac{\tau^4}{(4\pi)^4} \left[ \frac{1}{288}(\text{tr} \{R^2\})^2 + \frac{1}{360} \text{tr} \{R^4\} \right] + \ldots \quad (82)
\]
is the A-roof genus involving the Riemann curvature matrix-valued two-form
\[
R_{ij} = \frac{1}{2}R_{ij\mu}dx^\mu dx^\nu. \quad (83)
\]

In the last line of (82) the 4N-forms \( p_n(R) \) are the Pontryagin classes normalized as is customary in the mathematics literature. It is tacitly understood that in the product of \( \text{ch}[F] \) and \( \hat{A}(R) \) in (80) we only retain those terms whose total form degree matches that of the manifold \( \mathcal{M} \).
To derive the equalities in (82) and see the connection with (78) we make use of the algebraic trick that underlies the *splitting principal* from the general theory of characteristic classes. We regard the curvature two-form of the \( N \)-dimensional manifold \( M \) as an \( n \)-by-\( n \) skew-symmetric matrix that can be reduced to the canonical form

\[
\frac{1}{2\pi} R_{ij} \equiv \frac{1}{4\pi} R_{ij\mu\nu} dx^\mu dx^\nu = \begin{bmatrix}
0 & -x_1 \\
x_1 & 0 \\
& & \ddots \\
& & & 0 & -x_{N/2} \\
& & & x_{N/2} & 0
\end{bmatrix}_{ij}.
\]

(84)

Here the \( x_i \) are formal objects (Chern roots) which become real numbers when we evaluate the curvature two-form at a point on some chosen vectors, and only then perform the canonical-form reduction. In terms of the \( x_i \), the A-roof genus and the total Pontryagin class are given by

\[
\hat{A}[\tau R] = \sqrt{\det \left( \frac{\tau R/4\pi i}{\sinh \tau R/4\pi i} \right)} = \prod_{i=1}^{n/2} \frac{\tau x_i/2}{\sinh \tau x_i/2},
\]

(85)

\[
p(\tau R) = \det (1 - \tau R/2\pi) = \prod_i (1 + \tau^2 x_i^2),
\]

(86)

and

\[
p(\tau R) = 1 + \tau^2 p_1(R) + \tau^4 p_2(R) + \ldots.
\]

(87)

For the Pontryagin classes the expressions

\[
p_1(R) = \sum_i x_i^2 = -\frac{1}{(2\pi)^2} \left[ \frac{1}{2} \text{tr} \{ R^2 \} \right],
\]

\[
p_2(R) = \sum_{i<j} x_i^2 x_j^2 = \frac{1}{(2\pi)^4} \left[ \frac{1}{8} (\text{tr} \{ R^2 \})^2 - \frac{1}{4} \text{tr} \{ R^4 \} \right],
\]

(88)

account for the equality of the last two lines in (82). A similar expansion of (85) leads to the equality of the first two lines.

The discussion in [4] combines a general solution [32] to the constraints imposed by demanding absence of entropy creation by the anomaly induced currents with the generating function (78) to obtain an effective action for the ideal Weyl gas from the anomaly polynomial \( \mathfrak{P}[R, F] \equiv \hat{A}[R] \text{ch}[F] \). A key ingredient is the replacement rule [4, 6–10, 12]

\[
F \rightarrow \mu
\]
The replacement rule result is very striking but one is left wondering whether the similarity of the Sommerfeld-integral generating function’s factor

\[
\frac{\tau T/2}{\sin(\tau T/2)}
\]

(90)
to the anomaly polynomial’s factor

\[
\prod_{i=1}^{n/2} \frac{\tau x_i/2}{\sinh(\tau x_i/2)}
\]

(91)
is anything more than a mere coincidence. The question of how the \( T^2 \) contributions to the currents are linked the gravitational anomaly is also raised in [4] but was left unanswered because they work only in flat space. We believe that the illustrative examples in our section III go some of the way to explaining that the similarity is not a coincidence. The essential idea is that when we generate our temperature from the 1+1 dimensional Schwarzschild sausage we need only to curve together the radial and time dimensions. As a consequence only one \( x_i \) is non-zero, so only one non-trivial factor appears in the A-roof generating function. This also means that when expressed in terms of the Pontryagin classes only one of the \( p_n \) can be non-zero. This will be \( p_1(R) = x_1^2 = -T^2 \), where the minus sign accounts for the difference between \( \sinh \tau x/2 \) and \( \sin \tau T/2 \).

V. DISCUSSION

In 1967 Sutherland [33] and Veltman [34] argued that PCAC and current algebra require the decay \( \pi_0 \to \gamma \gamma \) to be strongly suppressed — a result contrary both to the experimental fact that this is principal decay mode of the neutral pion, and to the fact that the observed decay rate had been accurately calculated by Steinberger in 1949 [35] from a Pauli-Villars regulated AVV triangle diagram. Two years later the contradiction was resolved by Adler [36] and by Bell and Jackiw [37] who showed that the Sutherland-Veltman argument fails because it requires an illegitimate shift of integration variable in the triangle diagram, which is only conditionally convergent. As a consequence they found that even for massless fermions the axial current is not conserved.
The early understanding of such an anomalous failure of conservation laws was mostly of a formal mathematical character. The subtle issue of conditionally-convergent Feynman integrals was followed by Kiskis, Nielsen and Schroer and others making a connection with the mathematically deep Atiyah-Singer index theorem \[38, 39\]. Fujikawa \[40\] then showed that the index-theorem mandated difference between the number of left- and right-handed Dirac eigenmodes led to the path-integral measure failing to be invariant under chiral transformations. It was only around 1982 that Peskin \[41\] and others realized that in the massless case the physical source of the the \(\mathbf{E} \cdot \mathbf{B}/4\pi^2\) chiral anomaly is that the \(|\mathbf{B}|/2\pi\) density of gapless modes in the \(\mathbf{B}\) field allows a steady \(\dot{N} = \dot{k}_\|/2\pi = E_\|/2\pi\) flow of eigenstates out of the infinitely deep Dirac sea, which is acting as a Hilbert hotel. At about the same time Nielsen and Ninomiya \[42\] showed that in crystals, where there are necessarily equal numbers of left- and right-handed Weyl nodes, the Hilbert-hotel picture is not needed because the Dirac seas of left and right-handed fermions pass eigenstates to one another at their common seabed. This ambichiral traffic is the basis for our present understanding of Weyl semimetals. Later Callan and Harvey \[43\] showed that, in the case of an uncanceled net anomaly, charge is supplied to the bottom of the Dirac sea \(\text{via}\) inflow from higher dimensions. Their bulk-edge and bulk-surface connection is central to the physics of the quantum Hall effect and topological insulators. In the latter the picture is particularly clear because top and bottom of the branches of gapless boundary-modes merge with, and emerge from, the lower and upper edges of the higher-dimensional bulk states’ energy gap. It is now also understood \[44, 45\] that the spectral flow of eigenstates can be be computed by including a Berry-phase induced anomalous velocity in semiclassical dynamics.

Today we have a good mathematical understanding of gravitational anomalies \[27, 28\], but a comparable physical explanation, analogous to the \(\mathbf{E} \cdot \mathbf{B}\) spectral flow mechanism, does not seem to exist. In \[46\] an attempt was made to generalize the semiclassical Berry-phase picture to motion in curved space, but the generalization was frustrated by the unusual Lorentz transformation properties of massless particles with spin \[47–50\]. While the main result of the present work is the explicit derivation of the \(T^2\) contribution to the CVE from the anomaly, we hope the simple gedanken spacetime that we have constructed to do this will be useful for developing a physical understanding in the gravitational case also.
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