Note on Spectral Factorization
Results of Krein and Levin

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Abstract

Bohr proved that a uniformly almost periodic function $f$ has a bounded spectrum if and only if it extends to an entire function $F$ of exponential type $\tau(F) < \infty$. If $f \geq 0$ then a result of Krein implies that $f$ admits a factorization $f = |s|^2$ where $s$ extends to an entire function $S$ of exponential type $\tau(S) = \tau(F)/2$ having no zeros in the open upper half plane. The spectral factor $s$ is unique up to a multiplicative factor having modulus 1. Krein and Levin constructed $f$ such that $s$ is not uniformly almost periodic and proved that if $f \geq m > 0$ has absolutely converging Fourier series then $s$ is uniformly almost periodic and has absolutely converging Fourier series. We derive necessary and sufficient conditions on $f \geq m > 0$ for $s$ to be uniformly almost periodic, we construct an $f \geq m > 0$ with non absolutely converging Fourier series such that $s$ is uniformly almost periodic, and we suggest research questions.

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1 Notation

:= means ‘is defined to equal’ and iff means ‘if and only if’. $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are the natural, integer, rational, real, and complex numbers. For $z \in \mathbb{C}$, $x := \Re z$; $y := \Im z$ are its real; imaginary coordinates. $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1 \}$ is the closed unit disk, $\mathbb{D}^\circ := \{z \in \mathbb{C} : |z| < 1 \}$ is the open unit disk, and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1 \}$ is the circle. $\mathbb{U} := \{z \in \mathbb{C} : \Re z \geq 0 \}$ is the closed upper half–plane and $\mathbb{U}^\circ := \{z \in \mathbb{C} : \Re z > 0 \}$ is the open upper half plane. For $z \in \mathbb{C}$, $\chi_z(x) := e^{ixz}$. For closed $K \subset \mathbb{C}$, $C_b(K)$ is the $C^*$–algebra of bounded continuous complex–valued functions on $K$ with norm $\|f\| := \sup_{z \in K} |f(z)|$. For $\rho > 0$, $\mathcal{D}_\rho : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is the dilation operator $(\mathcal{D}_\rho f)(x) := f(\rho x)$. Zeros of nonzero entire functions are denoted by sequences $z_n, n \geq 1$ of finite (possibly zero) or infinite length.

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Sections 2 and 3 records classical results and derives preliminary results. Section 4 presents two new results. Section 5 suggests areas for research.

2 Entire Functions of Exponential Type

Lemma 1 If $E : U \to \mathbb{C}$ is continuous, holomorphic on $U^o$, $\Re E$ is bounded above, and $\Re E|_R = 0$, then $E(z) = i(a + bz)$ for a real and $b \geq 0$.

Proof For some $c > 0$ the function $E_1 : D\{−1\} \to \mathbb{R}$ defined by

$$E_1(z) := c - \Re E\left(\frac{i - iz}{1 + z}\right), \quad z \in D\{−1\}$$

is positive and continuous and its restriction to $D$ is harmonic. Herglotz [14], (23), (Theorem 11.30) proved that there exists a unique positive Borel measure $\mu$ on $T$ such that

$$E_1(z) := \Re \int_{w \in T} \frac{w + z}{w - z} d\mu(w), \quad z \in D\{−1\}.$$

Since $\Re E|_R = 0$ implies $E_1(z) = c$ for $z \in T\{−1\}$, it follows that there exists $b \geq 0$ with $d\mu(w) = cd\sigma + b\delta_1$ where $\sigma$ is normalized Haar measure on $T$ and $\delta_1$ is the point measure at $1$. Therefore

$$E_1(z) = c + b \Re \frac{1 - z}{1 + z}, \quad z \in D\{−1\} \implies \Re E(z) = \Re ibz, \quad z \in U$$

and the result follows from the Cauchy–Riemann equations.

Throughout this section $F$ denotes a nonconstant entire function. Its order ([4], Definition 2.1.2), defined by

$$\rho(F) := \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}$$

where

$$M(r) := \max\{|F(z)| : |z| \leq r\}, \quad 0 \leq r,$$

is in $[0, \infty]$. For integers $k \geq 1$ the entire function $z^k, e^{zk}$ has order $0; k$. The convergence exponent ([4], Definition 2.5.2) $\rho_1 \in [0, \infty]$ of the zeroes $z_j, j \geq 1$ of $F$ is

$$\rho_1 := \inf \left\{ \alpha \in (0, \infty) : \sum_n |z_n|^\alpha < \infty \right\}.$$

If $\rho_1 < \infty$ the genus $p$ of its zeros is the smallest nonnegative integer with

$$\sum_n |z_n|^{p+1} < \infty.$$

Clearly $p \in [\rho_1, \rho_1 - 1]$. Finite sequences have $\rho_1 = p = 0$. The infinite sequence $\frac{1}{n}; \frac{1}{n \log^2(n+1)}$ has $\rho_1 = 1; 1$ and $p = 1; 0$.

The following result shows how to construct certain entire functions with finite order ([4], (2.6.3), Theorem 2.6.4).
Lemma 2 If \( z_j, j \geq 1 \) has finite convergence exponent \( \rho_1 \) and genus \( p \), then the canonical product of genus \( p \) defined by

\[
P(z) := \prod_{0<|z_n|} E(z/z_n, p)
\]

where

\[
E(z, p) := (1 - z) \exp \left[ z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right]
\]
is the Weierstrass primary factor, converges uniformly on compact subsets to an entire function \( P \) that has order \( \rho_1 \) and zeros \( \{z_j : 0 < |z_n|\} \).

Hadamard’s factorization ([4], Theorem 2.7.1) gives

Lemma 3 If \( F \) has finite order \( \rho(F) \) then its zeros \( z_j, j \geq 0 \) have finite convergence exponent \( \rho_1 \leq \rho(F) \).

If \( m \geq 0 \) is the multiplicity of \( 0 \) as a root of \( F \), then there exists a polynomial \( Q \) having degree \( q \leq \rho(F) \) such that

\[
F(z) = z^m e^{Q(z)} P(z)
\]

where \( P \) is the canonical product of genus \( p \). Furthermore \( \rho(F) = \max\{q, \rho_1\} \).

If \( F \) has finite order \( \rho \) we define its type

\[
\tau(F) := \limsup_{r \to \infty} r^{-\rho} \log M(r)
\]

and say it has finite type if \( \tau(F) < \infty \). Define

\[
n_F(r) := \text{cardinality of } \{j : |z_j| \leq r\}.
\]

Lindelöf [20], ([4], Theorem 2.10.1) proved

Lemma 4 If \( F \) is an entire function whose order \( \rho \) is a positive integer, then \( F \) has finite type iff both \( n_F(r) = O(r^\rho) \) and

\[
\sup_{r \geq 0} \left| \sum_{0<|z_n| \leq r} z_n^{-\rho} \right| < \infty.
\]

We say \( F \) has exponential type if \( \rho(F) = 1 \) and \( \tau(F) < \infty \). For \( \alpha \in \mathbb{C}\{0\} \), the function \( e^{\alpha z} \) has exponential type \( \tau = |\alpha| \).

Krein [17] proved

Lemma 5 If \( F \) is of exponential type and \( f := |S|^2 \) is bounded and nonnegative, then \( f \) admits a factorization \( f = |s|^2 \) where \( s \) extends to an entire function \( S \) of exponential type that has no zeros in \( \mathbb{U}^\circ \). Moreover \( \tau(S) = \tau(F)/2 \) and \( s \) is unique up to multiplication by a constant having modulus 1.

Levin ([19], p. 437) said that Krein used approximation of \( f \) by Levitan trigonometric polynomials ([19], Appendix I, Section 4). We observe that \( F = S\overline{S} \) where \( \overline{S}(z) := \overline{S(\overline{z})} \). Levin ([19], Chapter V) defines

\[
A := \left\{ F \text{ entire : } \sum_{0<|z_n|} \left| \frac{1}{z_n} \right| < \infty \right\}.
\]

Boas ([1], p. 134) proved:
Lemma 6  If $F$ is of exponential type, then $F \in A$ iff

$$\sup_{L \geq 1} \int_{1}^{L} x^{-2} \log |f(x)f(-x)| \, dx < \infty.$$ 

Moreover, this condition holds whenever

$$\int_{-\infty}^{\infty} (1 + x^2)^{-1} \max \{ 0, \log |f(x)| \} dx < \infty.$$ 

Ahiezer \[1\] extended Krein’s result by proving:

Lemma 7  If $f = F|_{\mathbb{R}} \geq 0$, then $F \in A$ iff $f$ admits a factorization $f = |s|^2$ where $s$ extends to an entire function $S$ of exponential type $\tau (S) = \tau (F)/2$ having no zeros in $\mathbb{U}^o$.

We summarize Levin’s proof in ([19], p. 437). The if part follows from Lemma 4. The roots of $F$ occur in conjugate pairs so we order $\{z_n\}$ so $n$ odd $\implies \exists n_n \leq 0$ and $z_{n+1} = \bar{z}_n$. Lemma 3 implies there exists an integer $m \geq 0$ and $a, b \in \mathbb{R}$ such that

$$F(z) = z^{2m} e^{2az + 2b} \prod_{n \geq 1} \left( 1 - \frac{z}{z_n} \right) e^{\frac{i\gamma}{z_n}}.$$ 

Define $\gamma : = - \sum_{n \ odd} \frac{1}{z_n}$, and $s : = S|_{\mathbb{R}}$ where

$$S(z) := z^m e^{az + b + iz\gamma} \prod_{n \ odd} \left( 1 - \frac{z}{z_n} \right) e^{\frac{i\gamma}{z_n}}.$$ 

Then $F = SS$ and $S$ has no zeros in $\mathbb{U}^o$. Since $S \in A$, Lemma 4 implies that $S$ is of exponential type. The rest of the proof shows that $\tau (S) = \tau (F)/2$.

$\xi (\mathbb{R})$ is the space of smooth complex valued functions on $\mathbb{R}$ with the topology of uniform convergence of derivatives of every order on compact subsets. Its dual space $\xi' (\mathbb{R})$ is the space of compactly supported distributions ([15], Theorem 1.5.2). For $u \in \xi' (\mathbb{R})$ we define $\alpha (u), \beta (u) \in \mathbb{R}$ so that $[\alpha (u), \beta (u)]$ is the smallest closed interval containing the support of $u$, and we define its Fourier-Laplace transform by $\hat{u} (z) : = u (\chi_{-z}), z \in \mathbb{C}$.

Lemma 8  Conditions 1 and 2 are equivalent and imply condition 3

1. $F$ is of exponential type and $\exists c > 0, N \in \mathbb{Z}$ with $|F(x)| \leq c \left( 1 + |x| \right)^N$, $x \in \mathbb{R}$.

2. $F = \hat{u}$ for $u \in \xi' (\mathbb{R})$ with $\tau (F) = \max \{ |\alpha (u)|, |\beta (u)| \}$.

3. $\lim_{r \to \infty} r^{-1} n_F (r) = \frac{\beta (u) - \alpha (u)}{2\pi}$.

Schwartz \[24\] proved the equivalence of conditions 1 and 2 and Titchmarsh \[25\] proved they imply condition 3.
3 Uniformly Almost Periodic Functions

$T(\mathbb{R})$ is the algebra of trigonometric polynomials spanned by $\{\chi_\omega : \omega \in \mathbb{R}\}$, and the $C^*$-algebra $U(\mathbb{R})$ of uniformly almost periodic functions is its closure with respect to $\|\cdot\|$. Bohr [5] proved that if $f \in U(\mathbb{R})$ its mean value

$$M(f) := \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} f(x)dx$$

exists, defined its Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(\omega) := M(f \chi_{-\omega}), \ \omega \in \mathbb{R},$$

and proved that its spectrum $\Omega(f) := \{\omega \in \mathbb{R} : \hat{f} \neq 0\}$ is countable. For $f \in U(\mathbb{R})$ its Fourier series is the formal sum

$$f \sim \sum_{\omega \in \Omega(f)} \hat{f}(\omega) \chi_\omega,$$

its bandwidth $b(f) := \sup \Omega(f) - \inf \Omega(f) \in [0, \infty]$, and

$$||f||_A := \sum_{\omega \in \Omega(f)} |\hat{f}(\omega)| \in [0, \infty].$$

We define the following subsets of $U(\mathbb{R})$:

1. Bandlimited algebra $B(\mathbb{R}) := \{ f \in U(\mathbb{R}) : b(f) < \infty \}$.

2. Wiener Banach algebra $A(\mathbb{R}) := \{ f \in U(\mathbb{R}) : ||f||_A < \infty \}$.

3. Hardy Banach algebra $H(\mathbb{R}) := \{ f \in U(\mathbb{R}) : \Omega(f) \subset [0, \infty) \}$.

4. Invertible Hardy functions $IH(\mathbb{R}) := \{ f \in H(\mathbb{R}) : \frac{1}{f} \in H(\mathbb{R}) \}$.

Bohr [5], ([19], Chapter VI, p. 268, Corollary to Theorem 1) proved:

**Lemma 9** If $f \in U(\mathbb{R})$ then $f \in B(\mathbb{R})$ iff $f$ extends to an entire function $F$ of exponential type. Then $\tau(F) = \max\{|\inf \Omega(f)|, |\sup \Omega(f)|\}$.

The Fourier series for $f \in U(\mathbb{R})$ converges absolutely iff $f \in A(\mathbb{R})$. Cameron [7] and Pitt [21], ([10], Section 29, Corollary 2 to Theorem 2) proved

**Lemma 10** If $f \in A(\mathbb{R})$ and $\Phi$ is holomorphic in an open region containing the closure of $f(\mathbb{R})$, then the composition $\Phi \circ f \in A(\mathbb{R})$.

If $f \in B(\mathbb{R})$ and $f \geq 0$ then Lemma 9 implies that $f$ extends to an entire function $F$ of exponential type $\tau(F) = b(f)/2$ so Lemma 9 implies that $f = |s|^2$ where $s$ extends to an entire function $S$ of exponential type $\tau(S) = b(f)/4$. Levin ([19], Appendix 2, Theorem 2) used Lemma 10 to prove

**Lemma 11** If $f \in A(\mathbb{R}) \cap B(\mathbb{R})$ and there exists $m > 0$ with $f \geq m$, then the spectral factor $s \in A(\mathbb{R}) \cap B(\mathbb{R})$. 
For $\Delta > 0$ let $[\Delta]$ denote the set of entire functions $F$ of exponential type $\tau(F) = \Delta$ such that $f := F|_\mathbb{R} \in B(\mathbb{R})$ and $-\Delta, \Delta \in \Omega(f)$. Krein and Levin obtained a precise characterization of the zeros of functions in $[\Delta]$ and published these results without proofs in [16]. In ([19], Appendix VI) for the first time they gave proofs for these results and used them ([19], Appendix VI, p. 463) to prove

**Lemma 12** There exists $\Delta > 0$ and $F \in [\Delta]$ with $f \geq 0$ whose spectral factor $s \notin U(\mathbb{R})$.

Levin’s result ([19], Chapter V1, Section 2, Lemma 3) implies:

**Lemma 13** If $h \in U(\mathbb{R})$ then $-\infty < \Delta := \inf \Omega(h) \iff h$ extends to a continuous function $H$ on $\mathbb{U}$ which is holomorphic on $\mathbb{U}^o$ and satisfies

$$\lim_{y \to \infty} e^{-i\Delta(x+iy)} H(x + iy) = \hat{h}(-\Delta)$$

where convergence is uniform in $x$. Therefore $h \in H(\mathbb{R}) \iff H \in C_b(\mathbb{U})$.

The Poisson kernel functions $P_y : \mathbb{R} \to \mathbb{R}$, $y > 0$ are

$$P_y(x) := \frac{1}{\pi} \frac{1}{x^2 + y^2}, \quad x + iy \in \mathbb{U}^o.$$ 

For $f \in C_b(\mathbb{R})$ its Poisson integral $P[f] : \mathbb{U} \to \mathbb{C}$ is $P[f](x) := f(x)$ and

$$P[f](x + iy) := \begin{cases} f(x), & y = 0 \\ \int_{-\infty}^{\infty} P_y(x - s) f(s) \, ds, & y > 0 \end{cases}$$

**Lemma 14** If $f \in C_b(\mathbb{R})$, then $P[f] \in C_b(\mathbb{U})$, its restriction $P[f]|_{\mathbb{U}^o}$ is harmonic, and

$$\sup_{x \in \mathbb{R}} \Re f(x) \geq \Re P[f](z) \geq \inf_{x \in \mathbb{R}} \Re f(x), \quad z \in \mathbb{U}^o.$$ 

If $f \in U(\mathbb{R})$ then $f \in H(\mathbb{R}) \iff P[f]|_{\mathbb{U}^o}$ is holomorphic.

**Proof** The first assertion follows since $P_y(x)$ is harmonic, positive valued, and $\int P_y(x) \, dx = 1$, $y > 0$. The second assertion follows since

$$P[\chi_{\omega}](z) = \begin{cases} e^{i\omega z}, & \omega \geq 0 \\ e^{i\omega x}, & \omega < 0. \end{cases}$$

Bohr [5], ([19], Chapter VI, Theorem 2) proved:

**Lemma 15** If $h \in H(\mathbb{R})$ is nonzero, $H := P[h]$, and $z_n, n \geq 1$ are the zeros of $H$, then $\{\Im z_n\}$ is bounded iff $\inf \Omega(h) \in \Omega(h)$.

Bohr [6], ([19], p. 274, footnote) proved:

**Lemma 16** If $h \in U(\mathbb{R})$ and $|h|^2 \geq m$ for some $m > 0$, then there exists $c \in \mathbb{R}$ and $\theta \in U(\mathbb{R})$ such that

$$(\arg h)(x) = cx + \theta(x), \quad x \in \mathbb{R}.$$
Lemma 17 If \( h \in IH(\mathbb{R}) \) then \(|P[h]|\) is bounded below by a positive number, \( \Re \log P[h] \in C_b(\mathbb{U}) \), and \( \hat{h}(0) \neq 0 \).

Proof Lemma 13 implies \( P[h], P[1/h], P[h]P[1/h] \in C_b(\mathbb{U}) \). Lemma 14 implies \( P[h]P[1/h] \) is holomorphic on \( \mathbb{U}^\circ \). Since \( P[h]P[1/h]|_\mathbb{R} = 1 \), the Schwarz reflection principle (\cite{23}, Theorem 11.14) implies \( P[h]P[1/h] = 1 \). Therefore \( |P[h]| \) is bounded below by a positive number, so \( \log P[h] \) exists, is unique up to addition by an integer multiple of \( 2\pi i \), and \( \Re \log P[h] = \log |P[h]| \in C_b(\mathbb{U}) \).

Since \( P[h] \) and \( P[1/h] \) have no zeros, Lemma 15 implies \( \inf_\Omega(h) \in \Omega(h) \) and \( \inf_\Omega(1/h) \in \Omega(1/h) \). Since \( \{0\} = \Omega(1) \subset \Omega(h) + \Omega(1/h) \) it follows that \( \inf_\Omega(h) = \inf_\Omega(1/h) = 0 \) so \( \hat{h}(0) \neq 0 \).

Lemma 18 If \( h \in IH(\mathbb{R}) \), \( f := |h|^2 \), and \( f \in B(\mathbb{R}) \), then \( h \in B(\mathbb{R}) \) and \( \chi_{-b(f)/4} \) is a spectral factor of \( f \).

Proof Lemma 5 implies \( f = |s|^2 \) where the spectral factor \( s \) extends to an entire function \( S \) of exponential type \( \tau(S) = b(f)/4 \) which has no zeros in \( \mathbb{U}^0 \). Define \( S_1(z) := e^{b(f)/4} S(z)|_\mathbb{U} \) and \( H := P[h] \). Then \( S_1 \in C_b(\mathbb{U}) \) is bounded and holomorphic with no zeros in \( \mathbb{U}^0 \) and Lemma 17 implies that \( H \) is holomorphic on \( \mathbb{U}^0 \) and \(|H| \) is bounded below by a positive number. Therefore \( G := S_1/H \in C_b(\mathbb{U}) \) is holomorphic with no zeros on \( \mathbb{U}^0 \), and \(|G(z)| = |s(x)/|h(x)|| = 1 \) for \( x \) real. Therefore \( E := \log G \) exists and satisfies the hypothesis of Lemma 11 hence \( E(z) = i(a + bz) \) for some \( a \) real and \( b \geq 0 \). Therefore \( e^{b(f)/4} S(z) = e^{i(a+bx)} H(z) \) hence \( s = e^{a} \chi_{b-f/4}h \) so \( h \in B(\mathbb{R}) \). Lemma 17 implies \( \inf_\Omega(h) = 0 \). Since \( \inf_\Omega(s) = -b(f)/4 \) it follows that \( b = 0 \) hence \( \chi_{-b(f)/4} \) is a spectral factor.

Boas (\cite{4}, Theorem 11.1.2) proved this generalization of Sergei Bernstein’s classic theorem \( \cite{2}, \cite{4}, \) Theorem 11.1.1) for polynomials:

Lemma 19 If \( F \) is an entire function of exponential type and \( f := F|_\mathbb{R} \) is bounded, then \( f' := \frac{dF}{dx} \) is bounded and \(||f'||| \leq \tau(F) ||f|| \).

Lemma 20 If \( f \in B(\mathbb{R}) \), \( m > 0 \), \( f \geq m \), \( g := \frac{1}{2} \log f \), \( u := P[g] \), \( u_o := u|_{\mathbb{U}^0} \), and \( v_o : \mathbb{U}^0 \to \mathbb{R} \) is a harmonic function conjugate to \( u_o \), then \( v_o \) is uniformly continuous on each horizontal line and on each vertical ray in \( \mathbb{U}^0 \) so extends to a continuous function \( v : \mathbb{U} \to \mathbb{R} \). Furthermore \(|v(z)| = O(|z|)| \).

Proof Let \( \gamma := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|s|^2-1}{(s^2+1)^2} ds \). \( \frac{\partial u_o}{\partial x}(x+iy) \) is bounded on each horizontal line since

\[
\left| \frac{\partial u_o}{\partial x}(x+iy) \right| = \left| \frac{\partial u_o}{\partial y}(x+iy) \right| \leq \frac{\gamma}{2y} \max\{|\log m|, |\log ||f||| \} < \infty, \quad x+iy \in \mathbb{U}^0.
\]

\( \frac{\partial u_o}{\partial y} \) is bounded on each vertical ray since Lemma 19 implies

\[
\left| \frac{\partial v_o}{\partial y}(x+iy) \right| = \left| \frac{\partial u_o}{\partial x}(x+iy) \right| \leq \frac{b(f) ||f||}{4m} < \infty, \quad x+iy \in \mathbb{U}^0.
\]
4 New Results

Theorem 1 If \( f, g, u \) and \( v \) are as in Lemma 20, then \( f \) has a spectral factor \( s \in B(\mathbb{R}) \) iff \( v|_\mathbb{R} \) has the form in Lemma 16.

Proof Let \( H := e^{u+i\nu}, h := H|_\mathbb{R}. \) If \( v|_\mathbb{R} = \arg h \) has the form in Lemma 16, then \( h = \chi_\omega \sqrt{f} e^{\bar{\theta}} \in U(\mathbb{R}). \) Since \( P[h] = H \) is holomorphic on \( \mathbb{U}^\rho, \) Lemma 14 implies \( h \in H(\mathbb{R}). \) A similar argument implies \( \frac{1}{h} \in H(\mathbb{R}) \) so \( h \in IH(\mathbb{R}). \) Since \( |h|^2 = f, \) Lemma 18 implies \( h \in B(\mathbb{R}) \) and \( s := \chi_{-\delta} e^{i\phi} \) is a spectral factor of \( f \) in \( B(\mathbb{R}). \) Conversely, if \( f \) has a spectral factor \( s \in B(\mathbb{R}) \) then \( h := \chi_{-\delta} e^{i\phi} \in H(\mathbb{R}) \) hence \( P[h] = e^{u+i\nu}. \) Since \( |h|^2 = f \geq m > 0, \) Lemma 16 implies that \( v|_\mathbb{R} = \arg h \) has the form in Lemma 16.

The following result shows that the assumption \( f \in A(\mathbb{R}) \) in Lemma 11 is not necessary to ensure that \( f \) has a spectral factor \( s \in B(\mathbb{R}). \)

Theorem 2 For \( m > 0 \) there exists \( f \in B(\mathbb{R}) \setminus A(\mathbb{R}) \) with \( f \geq m \) whose spectral factor \( h \in B(\mathbb{R}) \setminus A(\mathbb{R}). \)

Proof Rudin (23, Theorem 5.12) gave a proof, based on the Banach-Steinhaus theorem (23, Theorem 5.8), that the subset of \( U(\mathbb{R}) \) of period \( 2\pi \) functions with non absolutely convergent Fourier series is nonempty.

Zygmund (24, Chapter VI, 3.7) gave the example

\[
\phi(x) \sim \sum_{n=2}^{\infty} \frac{\sin nx}{n \log n},
\]

Fejér (24, Chapter III, Theorem 3-4) proved its Cesàro sums

\[
p_n(x) = \sum_{k=2}^{n} \frac{n + 1 - k \sin kx}{n \log k}, \quad n \geq 2
\]

converge uniformly to \( \phi \) therefore there exists an integer sequence \( 2 \leq n_1 < n_2 < \cdots \) with \( ||\phi - p_{n_j}|| \leq 2^{-j}/3. \) Define \( q_1 = p_{n_1} \) and \( q_j = p_{n_{j+1}} - p_{n_j}, j \geq 2. \) Then \( ||q_j|| \leq 2^{-j} \) and \( \min \Omega(q_j) = -n_{j+1} \) and \( \max \Omega(q_j) = n_{j+1}. \) Construct a sequence \( \rho_j \in (0, n_{j+1}) \) with \( \{\rho_j\} \) linearly independent over \( \mathbb{Q} \) and define

\[
g_j := \Omega(\rho_j q_j), \quad j \geq 1.
\]

Then \( ||g_j|| = ||q_j||, ||g_j||_A = ||q_j||_A, \) and \( \Omega(g_j) = \rho_j \Omega(q_j) \) are pairwise disjoint subsets of \((-1, 1). \) Therefore

\[
g := \sum_{j=1}^{\infty} g_j.
\]

satisfies \( g \in B(\mathbb{R}), \Omega(g) \subset (-1, 1), \) and

\[
||g||_A = \lim_{k \to \infty} \sum_{j=1}^{k-1} ||q_j||_A \geq \lim_{k \to \infty} \sum_{j=1}^{k-1} ||q_j||_A = \lim_{k \to \infty} ||p_{n_k}||_A = \infty
\]

so \( g \in B(\mathbb{R}) \setminus A(\mathbb{R}), \) \( g \) is real valued, and \( \Omega(g) \subset (-1, 1). \) Define \( \Delta = -\inf \Omega(g), h_1 := \chi_{\Delta} g, \) and \( H_1 = P[h]. \) Lemma 13 implies that \( H_1 \) is bounded and \( h_1 \in B(\mathbb{R}) \cap H(\mathbb{R}). \) Define \( c := \sqrt{m} - \inf \Re H, h := c + h_1, f := |h|^2, \) and \( H := P[h]. \) Lemma 13 implies that \( |H(z)| \geq |\Re H(z)| \geq \sqrt{m} \) so \( H \) has no zeros in \( \mathbb{U}. \) Then \( s := \chi_{-\Delta} h \) is a spectral factor of \( f. \) Since \( s \notin A(\mathbb{R}), \) Lemma 11 implies that \( f \notin A(\mathbb{R}). \)
5 Research Questions

We suggest the following questions for future research:

1. If the hypothesis $f \geq m > 0$ in Lemma 11 and Theorem 1 is replaced by the weaker hypothesis $f \geq 0$, what conclusions about the spectral factor $s$ of $f$ can be deduced?

2. If $f \in B(\mathbb{R})$ is nonzero and $f \geq 0$, Bohr showed that it lifts to a function $\tilde{f} \in C(\mathbb{R}_B)$, where $\mathbb{R}_B$ is the Bohr compactification of $\mathbb{R}$, and we proved in [18] that $\log \tilde{f} \in L^1(\mathbb{R})$. Helson and Lowdenslager [13] proved that $\tilde{f} = |\tilde{h}|^2$ where $\tilde{h}$ is an outer function in the Hardy space $H^2(\mathbb{R}_B)$ (with respect to the linear order on the Pontryagin dual of $\mathbb{R}_B$ which equals the discrete real group). What is the relationship between $\tilde{h}$ and the lift $\tilde{s}$ of the spectral factor $s$ of $f$ given by Lemma 5?

3. Ahiezer’s result in Lemma 7 holds for operator valued and matrix valued functions [8, 9, 11, 12, 22]. What analogues do the results in this paper have in this context?

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