Physics of the Ideal Semion Gas: Spinons and Quantum Symmetries of the Integrable Haldane-Shastry Spin Chain.

F. D. M. Haldane

Department of Physics, Princeton University, Princeton, New Jersey 08544, USA

Various aspects of the Haldane-Shastry spin chain with $1/r^2$ exchange, and its various generalizations, are reviewed, with emphasis on its Yangian quantum group structure, and the interpretation of the model as the generalization of an ideal gas (of “spinons”) to the case of fractional statistics. Some recent results on its dynamical correlation function are discussed, and conjectured extensions of these remarkably simple results to the $SU(n)$ model, and to the related Calogero-Sutherland model with integer coupling constant, are presented.

1. Introduction

Most of our understanding of quantum many-body physics is based on solvable models – the ideal Bose and Fermi gases, and harmonic oscillators. These are the paradigms, solvable in full detail, and treatments of more complicated, interacting systems usually aim to find a description of the system that is close to one of these paradigms. There are very few other fully-solvable models; the only general class of non-trivial solvable quantum models are the integrable one-dimensional models that can be traced back to the $S = 1/2$ Heisenberg chain solved by Bethe [1] in 1931. Starting with Bethe, there has been success in calculating the energy eigenvalues and thermodynamics of these models, and the algebraic structures that make them solvable have been identified as the quantum Yang-Baxter equation and “quantum groups”, but progress in explicit calculation of their correlation functions has so far been very limited.

Bethe’s model and its generalizations generally involve contact or delta-function interactions. A different class of integrable models based on inverse-square interactions was introduced around 1970 by Calogero [2] and Sutherland [3]. However, this class of models
involved only gases of spinless impenetrable particles, and did not have the richness and lattice generalizations of the Bethe family. On the other hand, certain (static) correlations functions were explicitly found [3]. A few years ago, spin-chain relatives of these models were discovered (the “Haldane-Shastry” model [4, 5]), and it was subsequently found how to put internal spin degrees of freedom [6, 7, 8, 9] into the Calogero-Sutherland model (CSM).

In this paper, I will focus on the inverse-square-exchange spin-chain, and its trigonometric \((1/\sin^2)\) variants [4, 5], mentioning in passing also its hyperbolic \((1/\sinh^2)\) and elliptic \((1/\sin^2)\) variants [10], which provide a continuous interpolation to Bethe’s model. The trigonometric models are remarkable because they are explicitly solvable in much greater detail than Bethe’s model, and share many of the characteristics of the ideal gas, but with non-standard or fractional statistics, which can occur in spatial dimensions below three. In particular, it seems that the action of the physically-relevant local operators (such as the spin operator on a given site of the spin chain) on their ground states excites only a finite number of elementary excitations. This parallels the action of a one-particle operator on an ideal gas ground state, and greatly simplifies the calculation of correlation functions in terms of the “form factors”, the matrix elements of local operators between the ground state of the system and eigenstates characterized by finite numbers of elementary excitations.

In the rational limit (pure \(1/r^2\) interactions), it has proved possible to obtain simple explicit closed-form expressions for the thermodynamic potentials [11, 12], and very recently, for ground-state dynamical correlation functions [13], which are again simple, but non-trivial. I believe that these models will finally be solved in full detail, approaching that with which the ideal gas can be solved, including, for example, full correlation functions at all finite temperatures. This should provide the first fully-developed extension to the standard paradigms.

In this presentation, Section (2) will provide a self-contained introduction to the Yangian “quantum group” [14, 15, 16] symmetries of these models; I will mainly restrict the discussion to the \(S = 1/2\) spin-chain models with \(SU(2)\) symmetry, and try to avoid the more esoteric mathematical characterizations (Hopf algebras, coproducts, etc.), disguising them in a more pedestrian physicists’ terminology. Section (3) describes the application of this to the eigenspectrum of the trigonometric models, and Section (4) describes the recent remarkably simple results [13] for some dynamical correlations of the \(S = 1/2\) chain (and the \(\lambda = 2\) CSM), and presents new conjectured (and certainly correct) generalizations to the \(SU(n)\) version of the chain, and to the CSM with arbitrary integer coupling constant \(\lambda\).

2. The Yangian “Quantum Group” and Integrable Heisenberg Chains

In this section, I will present a self-contained outline of the Yangian “quantum group” algebra, and its application to the Haldane-Shastry model (HSM) and its variants, pointing out a number of open questions. The HSM contains a novel realization of the Yangian
algebra, originally characterized by Drinfeld [14] in the context of the algebraic Bethe Ansatz [14], but a mathematical structure in its own right. I will not give a comprehensive history of this elegant mathematical edifice; a concise account by Kirillov and Reshetikin [18] references many of the original works. An account of the representation theory of the Yangian is given by Chari and Pressley [19].

The original integrable model is the $S = 1/2$ Heisenberg chain, solved by Bethe in 1931 [1]:

$$H = \sum_i P_{i,i+1}, \quad P_{ij} = \frac{1}{4} + 2\vec{S}_i \cdot \vec{S}_j.$$  \hspace{1cm} (2.1)

The integrability of Bethe’s model derives from an underlying “quantum group” algebra called the $Y(gl_2)$ Yangian [14]: let $T(u)$ be a $2 \times 2$ matrix with non-commuting operator-valued entries that depend on a “spectral parameter” $u$, act in the Hilbert space of the spin chain, and obey the algebra

$$R^{12}T^1T^2 = T^2T^1R^{12},$$  \hspace{1cm} (2.2)

where $T^1$ is the $4 \times 4$ operator-valued matrix $(T(u_1) \otimes 1)$, $T^2$ is $(1 \otimes T(u_2))$, and $R^{12}(u_1, u_2)$ is a $4 \times 4$ c-number matrix defined in the direct product space $V^1 \otimes V^2$ of two $2 \times 2$ c-number matrices. Consistency of the algebra requires that $R$ satisfies the quantum Yang-Baxter equation (QYBE) $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ in $V_1 \otimes V_2 \otimes V_3$. The Yangian $Y(gl_2)$ corresponds to the rational QYBE solution

$$R^{12} = (u_1 - u_2)1 + hP^{12};$$  \hspace{1cm} (2.3)

here $P^{12}$ is the exchange matrix where (for c-number matrices) $P^{12}(A \otimes B)P^{12} = (B \otimes A)$, and $h$ is the quantum parameter, which in this case sets the scale of the spectral parameter.

The Yangian algebra can be written as the commutation relation

$$(u - v)[T^{\alpha\beta}(u), T^{\gamma\delta}(v)] = h(T^{\gamma\beta}(v)T^{\alpha\delta}(u) - T^{\gamma\beta}(u)T^{\alpha\delta}(v))$$  \hspace{1cm} (2.4)

It is commutative in the “classical” limit $h \to 0$. The algebra implies that

$$[t(u), t(v)] = 0, \quad t(u) \equiv \text{Tr}[T(u)],$$  \hspace{1cm} (2.5)

so the $t(u)$ are an infinite set of commuting operators. Another consequence is that $[T(u), \text{Det}_Q|T(v)|] = 0$, where $\text{Det}_Q|T(v)|$ is the quantum determinant [20]

$$\text{Det}_Q|T(u)| = T^{11}(u) T^{22}(u-h) - T^{12}(u) T^{21}(u-h).$$  \hspace{1cm} (2.6)

The quantum determinant commutes with all elements of the algebra and is analogous to a Casimir operator. In the “classical limit” $h \to 0$, the quantum determinant reduces to the usual determinant of c-number-matrices.

It is consistent to impose the asymptotic condition $T(u) \to 1$ for $u \to \infty$. An asymptotic expansion can then be defined:

$$T(u) = \phi(u) \left(1 + \frac{h}{u} \left(\vec{J}_0 \cdot \vec{\sigma} + \sum_{n=1}^{\infty} \left(\frac{\vec{J}_n \cdot \vec{\sigma} + J_0^n}{u^n}\right)\right)\right); \quad \phi(u) = \left(1 + \sum_{n=1}^{\infty} a_n u^n\right),$$  \hspace{1cm} (2.7)
where \( \vec{\sigma} \) are Pauli matrices, and \( \{a_n\} \) are the commuting generators of an infinite set of Abelian subalgebras of \( Y(gl_2) \): \([a_m, a_n] = [\vec{J}_m, a_n] = [J^0_m, a_n] = 0\); they alone determine the quantum determinant:

\[
\text{Det}_Q(T(u)) = \phi(u)\phi(u - h).
\]

(2.8)

The infinite-dimensional non-Abelian \( Y(sl_2) \) Yangian subalgebra, is completely generated by \( \vec{J}_0 \), the generator of the \( sl_2 \) Lie algebra (of \( SU(2) \) generators, so \([J^0_0, J^0_n] = i\epsilon^{abc}J^c_n \)) and the additional generator \( \vec{J}_1 \). All other operators can be expressed in terms of these: for example,

\[
\begin{align*}
J^0_1 &= \frac{1}{2}h\vec{J}_0 \cdot \vec{J}_0; & J^0_2 &= h\vec{J}_0 \cdot \vec{J}_1; & \ldots, \\
\vec{J}_2 &= -i\vec{J}_1 \times \vec{J}_1 + \frac{1}{2}h^2(\vec{J}_0 \cdot \vec{J}_0)\vec{J}_0; & \ldots.
\end{align*}
\]

(2.9a)

(2.9b)

Note that, as a consequence of the Yang-Baxter relation, \([J^0_0, J^0_n] = 0\); also \([\vec{J}_0, J^0_n] = 0\), but \([\vec{J}_1, J^0_n] \neq 0\).

The requirement that \( T(u) \) obeys the \( Y(gl_2) \) algebra imposes consistency conditions on the \( Y(sl_2) \) generators; the first non-trivial condition is the Serre relation

\[
[J^0_1, J^0_2] = [J^0_2, J^0_1].
\]

(2.10)

In fact, this relation is sufficient to ensure that \( \vec{J}_1 \) generates \( Y(sl_2) \), and can be used to compute the value of \( h^2 \); either sign of \( h = \pm \sqrt{h^2} \) can be consistently chosen. If the “classical limit” \( h \to 0 \) is taken, the \( \vec{J}_n \) obey the infinite-dimensional Lie algebra \( sl_{2+} \):

\[
[J^0_n, J^0_m] = i\epsilon^{abc}J^c_{m+n}, \quad m, n \geq 0.
\]

(2.11)

(A familiar example of this algebra is the subalgebra of any \( SU(2) \) Kac-Moody algebra defined by its non-negative modes).

The fundamental representation of \( Y(gl_2) \) (called the “evaluation” in the mathematical literature [14, 19]) is

\[
T_1(u) = 1 + \frac{hP_1}{u}; \quad P_1 = \frac{1}{2}[1 + \vec{\sigma} \cdot \vec{S}_1]
\]

(2.12)

where \( \vec{S}_1 \) is a \( S = 1/2 \) spin. In this irreducible representation, \( \vec{J}_1 = 0 \), and the quantum determinant is \((u + h)/u\).

If \( T(u) \) is a representation of \( Y(gl_2) \), so is \( f(u)T(u - a) \), where \( f(u) \) is a c-number function \((f(u) \to 1 \text{ as } u \to \infty) \) and \( a \) is a spectral parameter shift. The corresponding change in \( Y(sl_2) \) is \( \vec{J}_1 \to \vec{J}_1 + a\vec{J}_0 \).

Like Lie algebras, “quantum group” algebras have a fundamental property that allows larger representations to be constructed from tensor-products of smaller representations (the “coproduct”); however, unlike Lie algebras, the tensor-product operation is “non-cocommutative”, which means that the result of a tensor product depends on the order in which it is carried out. In the case of \( Y(gl_2) \), if \( T_1(u) \) is a representation acting on a Hilbert space \( \mathcal{H}_1 \) and \( T_2(u) \) is another representation acting on another Hilbert space \( \mathcal{H}_2 \), then the matrix product

\[
T(u) = T_1(u)T_2(u)
\]

(2.13)
is also a representation, acting on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, and

$$\text{Det}_Q[T(u)] = (\text{Det}_Q[T_1(u)])(\text{Det}_Q[T_2(u)]). \quad (2.14)$$

For the $Y(sl_2)$ subalgebra, the tensor product representation is

$$\vec{J}_0 = \vec{J}_0(1) + \vec{J}_0(2) \quad (2.15a)$$

(the usual “law of addition of angular momentum”) and

$$\vec{J}_1 = \vec{J}_1(1) + \vec{J}_1(2) + i\hbar \vec{J}_0(1) \times \vec{J}_0(2). \quad (2.15b)$$

Note that reversing the ordering of $\vec{J}_0(1)$ and $\vec{J}_0(2)$ defines an alternative tensor product with $\hbar \to -\hbar$; either definition can be used, which is why only $\hbar^2$ is fixed by the Serre relation. In $Y(gl_2)$ the two alternative tensor products are right and left matrix multiplication.

A (Yangian) highest-weight state (YHWS), is a state that, for all $u$, is annihilated by $T^{21}(u)$, and is an eigenstate of $T^{11}(u)$ and $T^{22}(u)$ with eigenvalues $\phi^+_1(u)$ and $\phi^+_2(u)$, where

$$(\text{Det}_Q[T(u)])|\text{YHWS}⟩ = \phi^+_1(u)\phi^+_2(u - \hbar)|\text{YHWS}⟩. \quad (2.16)$$

Every finite-dimensional representation has at least one highest-weight state. It is irreducible if there is only one YHWS; conversely, an irreducible representation can be derived from each YHWS contained in a reducible representation. To each such YHWS can be associated an invariant subspace of all the states that can be generated from it by successive action of elements of the algebra. In contrast to what happens in the unitary representation theory of Lie groups and their associated Lie algebras, the representation of $Y(gl_2)$ within this invariant subspace may still be reducible, as the subspace may contain other YHWS (and their invariant subspaces) which are eigenstates of the quantum determinant with the same eigenvalue.

An irreducible representation containing (only) a given YHWS is obtained by constructing its invariant subspace, and then projecting out the invariant subspaces of all other YHWS contained within it. This is not necessary in the important special case of full reducibility, associated with “quantum symmetries”, when the invariant subspaces associated with each independent YHWS are always irreducible, and orthogonal to each other. Reducibility without full reducibility, where the YHWS $|2⟩$ is contained within the invariant subspace generated by YHWS $|1⟩$, means that there are non-vanishing matrix elements of the form $⟨2|f(T(u))|1⟩$, but all matrix elements of the form $⟨1|f(T(u))|2⟩$ vanish.

The representation theory of $Y(sl_2)$ has been described in detail in [19]. The basic theorem is that all finite-dimensional irreducible representations are isomorphic to an irreducible representation derived from the maximal-$J^z_0$ YHWS of some tensor-product of fundamental representations. This means that the eigenvalues of $T^{11}(u)$ and $T^{22}(u)$ acting on a YHWS which generates a finite-dimensional invariant subspace have the form

$$\phi^+_1(u) = f(u)P^+(u + \hbar); \quad \phi^+_2(u) = f(u)P^+(u), \quad (2.17)$$
where $P^+(u)$ is a finite-dimensional polynomial called the Drinfeld polynomial. Let

$$P^+(u) = \prod_{n \geq 1} \prod_{i=1}^{N_n} \left( \prod_{\nu=0}^{n-1} \left( u - u_i^{(n)} + \nu \hbar \right) \right),$$

be the unique decomposition of the roots of $P^+(u)$ into “strings” of consecutive roots with spacing $\hbar$, obtained by successively factoring out strings, starting with the longest ones present (this fixes how multiple roots are treated). The dimension and $SU(2)$-representation content of the irreducible representation of $Y(gl_2)$ derived from the YHWS can immediately be identified from this form: it is equivalent to a tensor product of $SU(2)$ representations where $S = n/2$ occurs $N_n$ times, with a total dimension $(2^N_1)(3^N_2)\ldots((k+1)^N_k)$, where the YHWS has $J_0^z = k/2 = \sum_n nN_n/2$. If only 1-strings are present in the Drinfeld polynomial of a YHWS, the invariant subspace generated from it must be irreducible.

The Algebraic Bethe Ansatz [17] is the solution to the problem of finding the eigenvectors of the commuting operators $t(u)$ within an irreducible finite-dimensional representation of $Y(gl_2)$: the eigenstates with eigenvalue $t_\nu(u)$ which are $SU(2)$ highest-weight states with $J_0^z = (N/2) - M_\nu$, are given by

$$|\nu\rangle = \left( \prod_{i=1}^{M_\nu} T^{12}(v_i^{(\nu)}) \right) |\text{YHWS}\rangle$$

where the $M^{(\nu)}$ “rapidities” $\{v_i^{(\nu)}\}$ are solutions of the Bethe Ansatz equations (BAE)

$$\phi_1^+(u)Q_\nu(u - h) + \phi_2^+(u)Q_\nu(u + h) = t_\nu(u)Q_\nu(u)$$

and $Q_\nu(u)$ is the polynomial

$$Q_\nu(u) = \prod_{i=1}^{M_\nu} (u - v_i^{(\nu)}).$$

The eigenstates of $t(u)$ form an orthogonal basis only if $[(\vec{J}_0 \cdot \vec{J}_1), (\vec{J}_0 \cdot \vec{J}_1)^\dagger] = 0$.

The Bethe model on a periodic chain of $N$ sites with $S = 1/2$ spins corresponds to a $Y(gl_2)$ representation that is a simple tensor product of fundamental representations with no spectral parameter shifts:

$$T(u) = T_1(u)T_2(u)\ldots T_N(u).$$

It is irreducible, with $\phi_1^+(u) = ((u + h)/u)^N$ and $\phi_2^+(u) = 1$. The limit as $u \to 0$ of $u^N t(u)$ is $\hbar^N \exp iK$, where $\exp iK$ is the lattice translation operator, and we may write

$$u^N t(u) = \hbar^N \exp iK \exp \left( \sum_{n \geq 1} u^n H_{n+1} \right),$$

where $\{H_n\}$ are a set of local Hamiltonians. Then $[H_m, H_n] = [H_m, \vec{J}_0] = [H_m, \exp iK] = 0$, and $H \propto \vec{H}_2$. The $Y(sl_2)$ generator may be written

$$\vec{J}_1 = i\hbar \sum_{i < j} \vec{S}_i \times \vec{S}_j.$$
In the conventional Bethe parameterization, $u$ is the rapidity parameter if the choice $h = i$ is made; in this case $\vec{J}_1$ is Hermitian.

For finite $N$, $[\exp iK, \vec{J}_1] \neq 0$, and $[H_m, \vec{J}_1] \neq 0$. However, in the thermodynamic limit $N \to \infty$, the model acquires a full non-Abelian quantum symmetry: the Yangian generator $\vec{J}_1$ now commutes with the lattice translation operator, and with all the $H_n$. The infinite-dimensional representation of the Yangian becomes fully reducible into orthogonal subspaces in which the $H_n$ are diagonal. Furthermore, the antiferromagnetic ground state becomes the unique state that is singlet under the action of the Yangian. In the thermodynamic limit a spectral parameter “boost” operator $B$ [21], where $[\vec{J}_1, B] = \hbar \vec{J}_0$, is given by

$$B = \sum_i (i + \frac{1}{2}) P_{ii+1}. \quad (2.25)$$

This operator, which is incompatible with periodic boundary conditions, also generates the family of commuting Hamiltonians with quantum symmetry from the lattice translation operator: $[B, \exp iK] = \hbar H_2 \exp iK, [B, H_n] = n\hbar H_{n+1}$.

The most general representation of the $Y(sl_2)$ Yangian based on a simple tensor product of fundamental representations (including spectral parameter shifts) is

$$\vec{J}_1 = \sum_{i} \gamma_i \vec{S}_i + \frac{1}{2} \sum_{i \neq j} w_{ij} \vec{S}_i \times \vec{S}_j \quad (2.26)$$

where $w_{ij} = i\hbar \text{sgn}(i - j)$, and the $\{\gamma_i\}$ are arbitrary. It is an instructive exercise to substitute a form of this type into the Serre relation, without making any assumptions about $w_{ij}$, except that it can generally be taken to be odd. The result [15] is that this is a representation of the Yangian generator if

$$w_{ij} w_{jk} + w_{jk} w_{ki} + w_{ki} w_{ij} = \hbar^2, \quad i \neq j \neq k. \quad (2.27)$$

The general solution of this is

$$w_{ij} = \hbar \left( \frac{z_i + z_j}{z_i - z_j} \right) \quad (2.28)$$

where $\{z_i\}$ are a set of $N$ arbitrary, but distinct, complex parameters. The simple tensor-product representation is recovered in a limit where $|z_i/z_j| \to \infty$ for $i > j$.

Note that $Y(sl_2)$ representations with an invariant subspace where groups of $2S$ consecutive $S = 1/2$ spins remain in the symmetric spin-$S$ state can be obtained from the simple tensor product by suitably choosing the associated spectral parameter shifts $\gamma_i$ to be a $2S$-string, and this is the basis for the extension of Bethe’s model to chains of spins with $S > 1/2$ [22]. However, this requires the stronger property $(w_{ij})^2 + \hbar^2 = 0$, which eliminates more general representations based on (2.26) and (2.28) for such spins.

The $Y(sl_2)$ representation based on (2.28) is evidently derived from a more general class of representations of $Y(gl_2)$ than those obtained by a simple tensor product. Instead, following Ref. [13], they can be obtained as follows. Let $\{\gamma_i\}$ and $\{z_i\}$ be variables that commute with each other, and with the spin variables, and let $K_{ij}$ be a permutation...
operator that permutes their labels: $K_{ij} \gamma_i = \gamma_j K_{ij}$, etc. These permutation operators do not affect the spins: $[K_{ij}, \hat{S}_k] = 0$. Now consider the quantities

$$\hat{\gamma}_i = \gamma_i + h \sum_{j(\neq i)}' \left( \frac{z_j \theta(j - i) + z_i \theta(i - j)}{z_i - z_j} \right) K_{ij}. \quad (2.29)$$

These variables commute, and obey a “degenerate affine Hecke algebra” [16]:

$$K_{ii+1} \hat{\gamma}_i - \hat{\gamma}_{i+1} K_{ii+1} = h; \quad [\hat{\gamma}_i, \hat{\gamma}_j] = 0. \quad (2.30)$$

(The role of Hecke algebras in this context has been stressed in Ref. [7].) This implies that

$$[K_{ij}, \Delta(u)] = 0, \quad \Delta(u) \equiv \prod_i (u - \hat{\gamma}_i), \quad (2.31)$$

and hence that $\Delta(u)$ is a c-number function of $u$ and the $\{\gamma_i, z_i\}$. Since the $\hat{\gamma}_i$ commute with each other and with the spins, a representation of $Y(gl_2)$ is clearly given by

$$T(u) = T_{1}(u - \hat{\gamma}_1) \ldots T_{N}(u - \hat{\gamma}_N), \quad (2.32)$$

where $\Delta(u)/T(u)$ is a polynomial of degree $N$ in $u$.

Now let $\Pi^+$ be a projection operator into the subspace that is fully symmetric under simultaneous permutation of spins and the parameters $\{\gamma_i, z_i\}$, so $K_{ij} \Pi^+ = P_{ij} \Pi^+$. The Hecke algebra guarantees the property

$$\Pi^+ T(u) \Pi^+ = T(u) \Pi^+. \quad (2.33)$$

This ensures that the totally-symmetric subspace is an invariant subspace of $T(u)$, and hence that the projection of (2.32) into this subspace is also a representation of $Y(sl_2)$. The representation can now be evaluated [16] within the totally-symmetric subspace as

$$T(u) = 1 + h \sum_{ij} P_i ((u - L)^{-1})_{ij}, \quad (2.34a)$$

where $L_{ij}$ are elements of a $N \times N$ “quantum Lax matrix”

$$L_{ij} = \gamma_i \delta_{ij} + h(1 - \delta_{ij})(1 - (z_i/z_j))^{-1} P_{ij}. \quad (2.34b)$$

The $2^N$ states of this representation are all eigenstates of the quantum determinant with the same eigenvalue $\Delta(u + h)/\Delta(u)$ where $\Delta(u)$ is the polynomial of degree $N$ given by $\det|u - L^0|$ and $L^0_{ij}$ is the eigenvalue of $L_{ij}$ acting on the fully-spin-symmetric YHWS with $J^z_0 = N/2$ and $P_{ij} = 1$. This YHWS has the Drinfeld polynomial $P^+(u) = \Delta(u)$.

For generic values of the $\{\gamma_i\}$ and $\{z_i\}$, the representation is irreducible. We can now ask whether for some special choices of these parameters, the representation becomes fully-reducible as a direct sum of smaller representations, which would imply that there is a non-trivial Hermitian operator $H$ that commutes with both $\hat{J}_0$ and $\hat{J}_1$. We can postulate
such an operator to be of the form \( H = \sum_{i<j} h_{ij} P_{ij} \), and look for solutions of the condition 
\[ H, \vec{J}_i ] = 0. \] This is satisfied provided
\[ \gamma_i - \gamma_j = 0; \quad h_{ij} \propto (w_{ij}^2 + h^2); \quad \sum_{j(\neq i)}' w_{ij} h_{ij} = 0. \] (2.35)

We also require that \( H \) is Hermitian (\( h_{ij} \) real). We find [13] two families of translationally-

invariant Hamiltonians with quantum symmetry, where \([\exp iK, \vec{J}_i ] = [\exp iK, H] = 0\); the

Hamiltonian is even parity under spatial reflection, while \( \vec{J}_1 \) is odd-parity, with \((\vec{J}_1)^\dagger \propto \vec{J}_1\),

which ensures that any reducibility is full reducibility. The solutions are \( \gamma_i = 0 \), and \( h_{ij} \propto 1/d(i-j)^2 \), where in the hyperbolic models, defined on an infinite chain,

\[ w_{ij} = i\hbar \coth(\kappa(i-j)); \quad d(j) = \kappa^{-1} \sinh(\kappa j) \] (2.36a)

with real \( z_{i+1}/z_i = \exp 2\kappa \), which can be either positive (\( \kappa \) real) or negative (\( \kappa - i\pi/2 \) real). In the limit \( \kappa \to \infty \), the hyperbolic model becomes the \( N = \infty \) Bethe model; in

the limit \( \kappa \to 0 \), and \( i\hbar = 2\kappa \), so \( \hbar \to 0 \), we get a “classical limit” where \( Y(sl_2) \) becomes

the infinite-dimensional Lie algebra \( sl_{2+} \), and we obtain the rational model

\[ w_{ij} = (i-j)^{-1}; \quad d(j) = j. \] (2.36b)

The other family of solutions are the trigonometric models, obtained from the hyper-
bolic models by the replacement \( \kappa \to i\pi/N \), with

\[ w_{ij} = \hbar \cot(\pi(i-j)/N); \quad d(j) = (N/\pi) \sin(\pi j/N), \] (2.37c)

defined on a finite periodic chain on \( N \) sites, with \( d(j) \) being the chord distance between

lattice sites equally-spaced on a circle. For the choice of real \( \hbar \), the \( N \) roots of \( \Delta(u) \) form

an \( N \)-string along the real axis, centered on the origin. The trigonometric model then has

\((L_{ij})^\dagger = L_{ji}\), and hence \((T^{\alpha\beta}(u))^\dagger = T^{\beta\alpha}(u^*)\).

In all these models, \( H \propto H_2 \) is the first member of a family of commuting constants

of the motion \( \{H_2, H_3, H_4, \ldots\} \) that commute with \( \vec{J}_0, \vec{J}_1 \), and \( \exp iK \). The eigenstates

of these operators form irreducible representations of \( Y(sl_2) \). For example [14],

\[ H_3 \propto \sum_{i\neq j \neq k} z_{ij} z_{jk} z_{kl} \vec{S}_i \cdot \vec{S}_j \times \vec{S}_k, \quad z_{ij} \equiv z_i - z_j, \] (2.37)

but the systematic construction beyond \( H_4 \) [15] has not yet been elucidated. So far, no

generalization of the spectral-parameter “boost” operator from the Bethe limit to the

hyperbolic and trigonometric models has been found. The origin of the Hamiltonian

constants of the motion in the models with quantum symmetry is conceptually rather

different from that in the finite-\( N \) Bethe model. They must commute with all elements

of \( T(u) \), but the only elements of \( Y(gl_2) \) with this property derive from the quantum

determinant, which is a trivial c-number in these representations. Thus the Hamiltonians

with quantum symmetry cannot be expressed as functions of \( T(u) \), and are independent

objects.
The Yangian symmetry of the hyperbolic models is essentially similar to that of the Bethe model: it occurs in the thermodynamic limit, and $\vec{J}_1$ is made Hermitian by the usual BAE choice $h = i$. On the other hand, the realization of Yangian symmetry in the trigonometric models is essentially new, as it occurs in a finite periodic system, which has discrete, highly-degenerate energy levels that can be classified as finite-dimensional irreducible representations of $Y(sl_2)$. In this case $\vec{J}_1$ is made Hermitian by choosing $h$ real. The two variants are only connected via the “classical” rational model with $h = 0$.

The consequences of Yangian quantum symmetry are very different in the two cases; there is an analogy to the difference between the non-compact Lorentz group in $(1+1)$-dimensions and $SU(2)$, which have symmetry algebras corresponding to different real forms of the same complex $sl_2$ Lie algebra. In the hyperbolic case, the representation theory involves the Bethe “rapidity strings” with spacing $h = i$ in the imaginary rapidity direction, which are related to bound states. In the trigonometric case, the string spacing is in the real direction, and is related to the quantization of the momentum of free particles with periodic boundary conditions in units of $2\pi/N$, and to a generalization of the Fock space structure of such a system.

The trigonometric models, and their rational limit, were independently discovered by Haldane [4] and Shastry [5]. The extension to the hyperbolic model, was found by Inozemtsev [10], who also proposed the integrability of an elliptic variant with $h_{ij} \propto \wp(i - j)$, where $\wp(u)$ is the doubly-periodic Weierstrass function with periods $\kappa$ and $i\pi/\kappa$. This is a version of the hyperbolic model, made periodic on a finite chain of $N$ sites, and (like the finite-$N$ Bethe model ($\kappa \to \infty$)) does not have quantum symmetry. A proof of integrability is missing, but two odd-parity operators that commute with the Hamiltonian are given in [10], making integrability very plausible. One linear combination of these corresponds to $H_3$ in the trigonometric and hyperbolic limits, the other, which corresponds to $\vec{J}_0 \cdot \vec{J}_1$, is (where $\zeta(u)$ is the elliptic zeta function)

$$\sum_{i\neq j\neq k} (\zeta(i - j) + \zeta(j - k) + \zeta(k - i)) \vec{S}_i \times \vec{S}_j \cdot \vec{S}_k.$$  (2.38)

I have not been able to find a $Y(sl_2)$ generator $\vec{J}_1$ where $\vec{J}_1 \cdot \vec{J}_0$ gives (2.38). I therefore conjecture that the integrability of this model may involve a yet more general “quantum group” algebra with two “quantum parameters” $h = 2i\kappa$ and $h' = 2\pi/N$ (which would be scale parameters for two distinct spectral parameters), which only degenerates to the Yangian when either of them vanishes or becomes infinite (the Bethe limit). This would correspond to a “double quantization” of $\hat{sl}_{2+}$, or a further “quantization” of $Y(sl_2)$. Indeed, Inozemtsev [10] suggests that two spectral parameters play a role in this model. A better understanding of the elliptic model is clearly needed.

To conclude the formal discussion of algebraic aspects of these models, I note that, as in the case of Bethe’s model [23], there is a straightforward extension from the $Y(gl_2)$ quantum group with $SU(2)$ symmetry, to the $Y(gl_n)$ quantum group, where $T(u)$ is an $n \times n$ matrix, with $SU(n)$ symmetry. A further extension is to the graded or supersymmetric generalization, $Y(gl_{m|n})$, with $SU(m|n)$ supersymmetry, where a site can be in one of $m$ states with even fermion number or $n$ states with odd fermion number. Let $c_{i\alpha}^\dagger$ create on
site \(i\) a particle of species \(\alpha\), which may be fermionic or bosonic, and impose the constraint
\[
\sum_\alpha c^\dagger_{\alpha i} c_{\alpha i} = 1, \quad \text{all } i.
\] (2.39)

The exchange operator becomes
\[
P_{ij} = \sum_{\alpha \beta} c^\dagger_{\alpha i} c^\dagger_{\beta j} c_{\beta i} c_{\alpha j},
\] (2.40)
and the generators of the \(sl_{m|n}\) graded Lie algebra and its Yangian extension can be given as the traceless operator-valued matrices
\[
J^0_{\alpha \beta} = \sum_i \left( c^\dagger_{\alpha i} c_{\beta i} - (m + n)^{-1} \delta_{\alpha \beta} \right),
\] (2.41a)
\[
J^1_{\alpha \beta} = \frac{1}{2} \sum_{i \neq j, \gamma} w_{ij} c^\dagger_{\alpha i} c^\dagger_{\gamma j} c_{\gamma i} c_{\alpha j}.
\] (2.41b)
In particular, the graded trigonometric exchange model with \((m|n) = (1|2)\) corresponds to the “supersymmetric t-J model” variant found by Kuramoto [24].

3. Spectrum of the Trigonometric Haldane-Shastry Model, and its Interpretation as a Generalized Ideal Gas

In this section I will focus on the remarkable properties of the trigonometric model, with its realization of Yangian quantum symmetry in a form compatible with periodic boundary conditions. The Hamiltonian, normalized so the spin-wave velocity is \(v_s\) in units where the lattice spacing \(a = \hbar = 1\), is
\[
H = \frac{\pi v_s}{N^2} \sum_{i<j} \frac{P_{ij}}{\sin^2(\pi(i-j)/N)}. \tag{3.1}
\]
Let us first consider the case where the symmetry group is \(SU(1|1)\), when the model is equivalent to a lattice gas of free spinless fermions, with creation operators \(a^\dagger_i = c^\dagger_{i2} c_{i1}\), so
\[
P_{ij} = 1 - (a^\dagger_i - a^\dagger_j)(a_i - a_j) = -1 + (a_i - a_j)(a^\dagger_i - a^\dagger_j).
\] (3.2)
The energy levels of this model are characterized by a sequence of \(N - 1\) Bloch-orbital occupation numbers \(\{n_1, n_2, \ldots, n_{N-1}\}\), taking values 0 or 1, so
\[
E = \sum_{m=1}^{N-1} \epsilon_m (n_m - \frac{1}{2}); \quad e^{iK} = \prod_{m=1}^{N-1} e^{i k_m n_m} \tag{3.3}
\]
where
\[
\epsilon_m = \left( \frac{2\pi v_s}{N^2} \right) m(m - N); \quad k_m = 2\pi m/N. \tag{3.4}
\]
All eigenstates have a two-fold degeneracy, corresponding to the two possible occupation states of the translationally-invariant Bloch orbital, which has zero energy. This is the supersymmetry: all states are $sl_{1|1}$ doublets, and there is no non-trivial Yangian extension in this case. The ground state of $H$ is the state with maximum occupancy, $\{n_m\} = \{111\ldots11\}$, and the ground state of $-H$ has minimum occupancy $\{000\ldots00\}$; The replacement $1 \leftrightarrow 0$ in any configuration maps $E$ to $-E$ and $K$ to $\pi(N-1) - K$.

It is a remarkable fact that this set of energy-momentum levels is the complete set of levels contained in the general $SU(m|n)$ exchange model, but in the general case they have different and much larger Yangian multiplicities. Specializing to the $SU(n|0)$ and $SU(0|n)$ cases, some of these multiplicities are zero, and the energy level is absent. The selection rule for allowed multiplets in the case $SU(n|0)$ is that occupation patterns containing a sequence of $n$ or more consecutive 1’s are forbidden; in the $SU(0|n)$ case, $n$ or more consecutive 0’s are forbidden. Each allowed sequence corresponds to an irreducible Yangian multiplet. In general, the spectrum of $H$ with $SU(m|n)$ symmetry is the same as that of $-H$ with $SU(n|m)$ symmetry.

The above rule was found empirically by examination of numerical diagonalization results [13], but I now specialize to the simplest case, $SU(2|0) \equiv SU(2)$, where the rule is that occupation patterns with consecutive 1’s are forbidden, and make contact with the $Y(sl_2)$ representation theory [13].

Recall that each YHWS labeled by $\nu$ has an associated Drinfeld polynomial, $P^{\nu}_+(u)$, and that $\Delta(u)T(u)$ is a polynomial, with quantum determinant $\Delta(u-h)\Delta(u+h)$. Thus there is a polynomial $f_\nu(u)$ where $\phi^+_\nu(u)$, (the eigenvalue of $T^{11}(u)$), has the form $f_\nu(u)P^{\nu}_+(u+h)$, $\phi^+_\nu(u) = f_\nu(u)P^{\nu}_+(u)$, and

$$f_\nu(u)f_\nu(u-h)P^{\nu}_+(u+h)P^{\nu}_+(u-h) = \Delta(u+h)\Delta(u-h). \quad (3.5)$$

Since the roots of $\Delta(u)$ are an $N$-string, it is easily seen that $P^{\nu}_+(u)$ must be a factor of $\Delta(u)$, so $\Delta(u) = P^{\nu}_+(u)g_\nu(u)$, where $g_\nu(u)$ is a polynomial with roots at the roots of $\Delta(u)$ not contained in $P^{\nu}_+(u)$, and

$$f_\nu(u)f_\nu(u+h) = g_\nu(u-h)g_\nu(u+h). \quad (3.6a)$$

This is a polynomial equation with the elementary solution

$$g(u) = (u-a)(u-a+h), \quad f(u) = (u-a-h)(u-a+h). \quad (3.6b)$$

Any product of such solutions is a solution. This shows that $g_\nu(u)$ must be a product of 2-strings.

We now have the recipe [13] for constructing the possible Drinfeld polynomials (in fact, there is one YHWS in the spectrum of the trigonometric model corresponding to each allowed Drinfeld polynomial [13]). First partition the $N$-string of roots of $\Delta(u)$ into 1-strings and 2-strings. Between each consecutive pair of roots place a 1 if they belong to the same 2-string, and 0 otherwise. This gives a binary sequence of length $N - 1$, with the constraint that there are no consecutive 1’s, as found empirically. The 1-strings are the roots of the Drinfeld polynomial; if an extra 0 is added at each end of the binary
sequence, the 1-strings are located between each consecutive pair of 0’s. From the Yangian representation theory, we conclude (in agreement with the empirical findings [15]) that a sequence of \(M+1\) consecutive 0’s represents an independent \(S = M/2\) degree of freedom.

There is a simple physical interpretation: each root of the Drinfeld polynomial represents the presence of a \(S = 1/2\) “spinon” excitation. A \(m\)-string of such roots represents \(m\) spinons “in the same orbital” with a rule that their spin state must be totally symmetric. If the total number of spinons present is \(N_{sp}\), there are \(M = (N - N_{sp})/2\) 1’s in the occupation pattern, which serve to separate the \(N_{orb} = M + 1\) “orbitals” into which the spinons are distributed. The change in the number of available “orbitals” as the spinon number is changed obeys \(\Delta N_{orb}/\Delta N_{sp} = -1/2\), which allows spinons to be identified as excitations with “semionic” fractional statistics, in between Bose and Fermi statistics [25].

If \(\{I_i\}\) are the positions of the \(M\) non-zero entries in the length-\((N-1)\) binary sequence \(\{n_j\}\), “spinon orbital occupations” \(\{n_{i\sigma}\}\), \(1 \leq i \leq M + 1\), \(\sigma = \pm 1/2\), are given by \(I_{i+1} - I_i = 2 + n_{i\uparrow} + n_{i\downarrow}\), with \(I_0 \equiv -1\), and \(I_{M+1} \equiv N + 1\). We can then treat the \(\{n_{i\sigma}\}\) as independent variables subject only to the constraint

\[
N_{sp} \equiv \sum_{i\sigma} n_{i\sigma} = N - 2M. \tag{3.7}
\]

Each distinct configuration \(\{n_{i\sigma}\}\) satisfying this constraint corresponds to an eigenstate of the system. The energy and momentum can now be expressed in terms of the “spinon orbital occupations”: it is convenient to relabel \(n_{k\sigma}\), with \(1 \leq k \leq M+1\), as \(n_{k\sigma}\), where \(k\) is a crystal momentum in the range \(-k_0 \leq k \leq k_0\), \(k_0 = \pi M/N\), then if

\[
\epsilon(k) = \frac{v_s}{\pi}(k_0^2 - k^2); \quad V(k) = v_s(k_0 - |k|); \tag{3.8}
\]

the spectrum is given by

\[
E = E_{MN} + \sum_{k\sigma} \epsilon(k)n_{k\sigma} + \frac{1}{2N} \sum_{k\sigma,k'\sigma'} V(k - k'n_{k\sigma}n_{k'\sigma'}), \tag{3.9a}
\]

\[
e^{iK} = (-1)^M \prod_{k\sigma} e^{ikn_{k\sigma}}; \quad J^z_0 = \sum_{k\sigma} \sigma n_{k\sigma}. \tag{3.9b}
\]

Here \(E_{MN} = \pi v_s(2N(N^2 - 1) + 4M(M^2 + 2) - 3MN^2)/6N^2\). From this spectrum, it is straightforward to derive the thermodynamic potentials [11] in the limit \(N \to \infty\), including a Zeeman coupling \(-hJ^z_0\).

The spinon orbitals can be parameterized by a rapidity \(x\) in the range \(-1 < x < 1\), (where the spinons in that orbital have velocity \(v_s x\)), and have mean occupation numbers

\[
\bar{n}_\sigma(x) = \exp(-\beta[\varepsilon(x) - \sigma h(1 + \mu(x))]), \tag{3.10}
\]

where \(\varepsilon(x) = (\pi v_s/4)(1 - x^2)\) and

\[
\frac{\sinh(\beta h \mu(x)/2)}{\sinh(\beta h/2)} = \exp(-\beta \varepsilon(x)). \tag{3.11}
\]
This gives an easily-solved quadratic equation for $\exp(\beta h \mu(x)/2)$. The free energy per site is given by

$$-\beta f(\beta, h) = \frac{1}{2} \int_{-1}^{1} dx \ln \left( \frac{\sinh(\beta h(1 + \mu(x))/2)}{\sinh(\beta h/2)} \right). \quad (3.12)$$

In the absence of a magnetic field, the entropy per site is

$$s(\beta, h = 0) = k_B \int_{-1}^{1} dx \ln \left[ 2 \cosh \beta \varepsilon(x) \right] - \beta \varepsilon(x) \tanh \beta \varepsilon(x),$$

which unexpectedly is even in $\beta$, so is the same for the ferromagnetic and antiferromagnetic models.

The identification of spinons as semions is supported when the wavefunctions of the trigonometric $SU(2)$ models are examined. A class of polynomial wavefunctions of the Calogero-Sutherland model of a non-relativistic gas with $1/\sin^2$ interactions was discovered \[11\] to also give a class of “fully-spin-polarized spinon gas” \[11\] wavefunctions of the trigonometric model, and these are now identified with the YHWS states. If $Z_i$ (with $(Z_i)^N = 1$) are the complex coordinates of the $N$ sites with $\sigma_i = -1/2$, the wavefunctions have the form

$$\Psi_\nu(\{Z_i\}) = \Phi_\nu(\{Z_i\}) \Psi_0(\{Z_j\}), \quad (3.14)$$

where $\Phi_\nu(\{Z_i\})$ is a symmetric polynomial with degree $N_{sp}$ in each $Z_j$, and $\Psi_0(\{Z_j\})$ is

$$\Psi_0(\{Z_i\}) = \prod_{j<k} (Z_j - Z_k)^2 \prod_j Z_j. \quad (3.15)$$

The is essentially the same as the $m=2$ Laughlin fractional quantum Hall effect state for bosons \[26\], and for $N_{sp} = 0$, this is the unique Yangian singlet state. The symmetric polynomials $\Phi_\nu$ are solutions of the eigenvalue equation

$$\left( \sum_j x_j^2 \frac{\partial^2}{\partial x_j^2} + \lambda \sum_{j \neq k} \left( \frac{x_j^2}{x_j - x_k} \right) \frac{\partial}{\partial x_j} \right) \Phi(\lambda)(\{x_i\}) = \mu \Phi(\lambda)(\{x_i\}), \quad (3.16)$$

with $\lambda = 2$. (The solutions are known in the Mathematical Literature as the Jack Polynomials \[27\]). If $\{I_i\}$ are the positions of the non-zero entries in the binary “occupation number” sequence, the Taylor series expansion of the corresponding YHWS wavefunction has the form

$$\Psi = \sum_{\{m_i\}} C(\{m_i\}) \sum_P \left( \prod_i (Z_{P(i)})^{m_i} \right), \quad \{m_i\} \leq \{I_i\} \quad (3.17)$$

where $P(i)$ is a permutation, and where $m_i \leq m_{i+1}$, and $\{m_i\} < \{m'_i\}$ means that $\{m_i\}$ can be reached from $\{m'_i\}$ through a sequence of pairwise “squeezing” operations $m_i \to m_i + 1$, $m_j \to m_j - 1$, with $m_i < m_j - 1$, and $m_k$ unchanged for $k \neq i, j$.

The $N_{sp} = 0$ Yangian singlet state may also be written in terms of the azimuthal spin variables $\sigma_i$: it occurs only for even $N$, and is the $n = 2$ case of the $SU(n)$ singlet wavefunction where the $\sigma_i$ can take one of $n$ ordered values $\{\alpha\}$:

$$\Psi_0^{(n)}(\{z_i, \sigma_i\}) = \prod_{i<j} (z_i - z_j)^{\delta(\sigma_i, \sigma_j)} (i)^{\text{sgn}(\sigma_i, \sigma_j)} \prod_\alpha \delta(N, nN(\alpha)), \quad (3.18)$$
where $N(\alpha) \equiv \sum_i \delta(\alpha, \sigma_i)$. Here the $\{z_i\}$ are a complete set of the $N$’th roots of unity. Note that this state is essentially the same as the wavefunction for a filled Landau level of $SU(n)$ fermions, and is hence explicitly $SU(n)$-singlet for arbitrary $\{z_i\}$.

The $N_{sp} = 1$ states are also particularly simple: they occur only for odd $N = 2M + 1$, and are generated by

$$
\Psi(z; \{Z_j\}) = \prod_i (z - Z_i) \prod_{i<j} (Z_i - Z_j)^2 \prod_i Z_i.
$$

(3.19)

Expanding this in powers of $z$ gives a band of Bloch states with crystal momentum $-\pi/2 < K + M\pi < \pi/2$, so the spinon band covers half a Brillouin zone. The localized spinon wavefunction $\Psi(z; \{Z_i\})$ is essentially the quasi-hole excitation of the $m = 2$ bosonic Laughlin state, and from this perspective, clearly describes a semionic excitation. If the parameter $z$ is chosen to be a lattice coordinate $z_i$, the localized spinon wavefunction with $\sigma_i = \sigma$ can be rewritten in terms of spin variables as

$$
\delta(\sigma, \sigma_i)\Psi_0^{(2)}(\{z_j, \sigma_j; j \neq i\}).
$$

(3.20)

This shows that the spinon is completely localized on the lattice site, and induces no spin polarization of its local environment. While there are $N$ such fully-localized spinon states, they are fundamentally non-orthogonal, since the expansion in orthogonal Bloch states shows there are only $(N + 1)/2$ independent $N_{sp}=1$ states.

To end this section, I discuss the extension of the state-counting from $SU(2)$ to the general $SU(n|m)$ case. The Yangian counting rules for the $SU(2)$ model are very simple, and allowed the thermodynamic functions to be explicitly obtained [11] in closed-form in the thermodynamic limit, but become much more complicated in the general $SU(m|n)$ case. Recently Sutherland and Shastry [12] showed how to recover the $SU(2)$ results for the thermodynamics, and generalize them to $SU(m|n)$, from a strong-coupling limit of the exchange-generalization of the Calogero-Sutherland model [3, 4, 5, 6].

The spin chain degrees of freedom are essentially those of the spin-1/2 Fermi gas with the charge degrees of freedom removed. From this viewpoint, the spinon is the spin-1/2 fermion with the charge degrees of freedom factored out, leaving “half a fermion”, and hence a semion. I will interpret Sutherland and Shastry’s result [12] as showing how to “put back” charge degrees of freedom into the spin chain to recover a spectrum with the familiar degeneracies of the ideal gas with internal degrees of freedom. This facilitates the computation [12] of the thermodynamics in the general $SU(m|n)$ case.

Let $b_k^\dagger$, $k = 1, \ldots N - 1$, be a set of harmonic oscillator creation operators, and add these degrees of freedom to the spin chain as follows:

$$
H' = \left(\frac{\sqrt{\pi}}{N^2}\right) \left(\sum_{i<j} \frac{1 - P_{ij}}{\sin^2(\pi(i - j)/N)} + \sum_{k=1}^{N-1} 2k(N - k)b_k^\dagger b_k\right),
$$

(3.21a)

$$
e^{iK'} = e^{iK_s} e^{iK_c}, \quad K_c = \frac{2\pi}{N} \left(\sum_{k=1}^{N-1} k b_k^\dagger b_k\right),
$$

(3.21b)
where \( \exp iK_s \) is the spin-chain translation operator, and \( P_{ij} \) is the \( SU(m|n) \) exchange operator.

Now let \( n_{k\alpha} = c_{k\alpha}^\dagger c_{k\alpha} \) be occupation numbers of a set of orbitals for particles of a bosonic or fermionic species \( \alpha \), with \( k = 0, \pm 1, \ldots, \pm \infty \), subject to the constraint that

\[
N = \sum_{k\alpha} n_{k\alpha}
\]

is fixed. From the result of Sutherland and Shastry, the spectrum of \( H' \) (with the constraint (3.22)) is identical to that of

\[
H'' = \left( \frac{\nu \pi}{N^2} \right) \sum_{kk'} \sum_{\alpha\beta} |k - k'| n_{k\alpha} n_{k'\beta},
\]

\[
K'' = \frac{2\pi}{N} \sum_{k\alpha} k n_{k\alpha}; \quad J_{0}^{\alpha\beta} = \sum_{k} (c_{k\alpha}^\dagger c_{k\beta} - (m + n)^{-1} \delta_{\alpha\beta}).
\]

The spectrum of \( H'' \) is periodic in \( K'' \rightarrow K'' + 2\pi \), corresponding to a shift \( n_{k\alpha} \rightarrow n_{k+1\alpha} \) of the occupation number pattern. The precise statement is that the spectrum of \( H'' \) in one period of \( K'' \) coincides with that of \( H' \). In the thermodynamic limit, the free energy of \( H' \) can thus be easily calculated, as can the free energy contribution from the extra oscillator modes; the difference is the spin-chain free energy [12]. However, this does not give a simple method for identifying the Yangian degeneracies of the discrete (finite \( N \)) spectrum of \( H \) without the oscillator modes. The explicit expressions for the free energy are in fact only obtained in the limit \( N \rightarrow \infty \), when the Yangian algebra has degenerated to its “classical” \( \widehat{sl}_{m|n+} \) limit.

4. Dynamical Correlation functions

Let us consider the state \( S_i^+ |0\rangle \), where \( |0\rangle \) is the Yangian singlet ground state of the trigonometric chain. It is a spin-1 state, and is easily seen to be a linear combination of Yangian highest weight states with \( N_{sp} = 2 \). The action of \( S_i^+ \) is to remove a down-spin coordinate at site \( i \). Thus \( M = N/2 - 1 \) and

\[
\Psi(z_i; \{Z_j\}) = \prod_{j=1}^{M} (z_i - Z_j)^2 \Psi_0(\{Z_j\}).
\]

The polynomial prefactor has degree 2, confirming that this state is composed purely of two-spinon eigenstates. Thus if we wish to compute the dynamical correlation function

\[
C(i - j, t - t') = \langle 0 | S_i^- (t) S_j^+ (t') | 0 \rangle,
\]

the only intermediate states that contribute to its spectral function are the \( N_{sp}=2 \) YHWS.
Recently, it has become possible to compute certain dynamical correlation functions of the Calogero-Sutherland model \[2\,3\]

\[ H = \sum_{i=1}^{N} \frac{\frac{\not{p}_i^2}{2m}}{2m} + \frac{\hbar^2}{m} \sum_{i<j} \lambda(\lambda - 1) d(\vec{x}_i - \vec{x}_j)^2, \]  

with \(d(x) = (L/\pi) \sin(\pi x/L)\), in the thermodynamic limit at fixed density \( \rho = N/L \). The eigenfunctions of the CSM have the form

\[ \Phi^{(\lambda)}(\{Z_i\}) \Psi_J(\{Z_i\}); \quad \Psi_J = \prod_{i<j} (Z_i - Z_j)^\lambda \prod_i (Z_i)^J (\lambda \geq 0) \]  

where \(\{\Psi_J\}\) is the family of states that includes the ground state and Galilean boosts of it, \(Z_i = \exp 2\pi i x_i/L\), and \(\Phi^{(\lambda)}(\{Z_i\})\) is a symmetric polynomial solution of (3.16). While the apparent statistics can be modified with a singular gauge transformation, the “natural” statistics of this model are, from (4.4), evidently fractional, with statistical parameter \(\theta = \pi \lambda\) (i.e., particles carry charge 1 and flux \(\pi \lambda\)). The elementary excitations are particles with velocities greater than the speed of sound \(v_s = \bar{\hbar} \lambda \rho/m\), and holes with velocities less than \(v_s\). The holes carry charge \(-1/\lambda\) and flux \(-\pi\), and have statistical parameter \(\theta_h = \pi/\lambda\).

The calculations can be carried out at one of three special couplings (only two of which are non-trivial), and involve a mapping to the \(N \to \infty\) limit of a Gaussian dynamical \(N \times N\) matrix-model, with either orthogonal (\(\lambda = 1/2\)), unitary (\(\lambda = 1\)), or symplectic (\(\lambda = 2\)) symmetry. This reduces the problem to a complicated, but tractable Gaussian problem \([28]\).

At integer couplings \(\lambda = q\), the natural particle statistics are Bose (even \(q\)) or Fermi (odd \(q\)). The ground state is \(\Psi_J\) with \(J = -q(N - 1)/2\). The wavefunction of the state \(\Psi(x)|J\rangle\) is

\[ \prod_i (z - Z_i)^q \Psi_J(\{Z_i\}), \quad z = \exp(2\pi i x/L), \]  

which is composed only of eigenstates with just \(q\) hole excitations, so the spectral function of the retarded single-particle Greens function will only contain contributions from intermediate states of that type. This correlation function will thus have the form

\[ \langle 0|\Psi^\dagger(x, t)\Psi(0, 0)|0\rangle = \rho \left(\prod_{i=1}^{q} \int_{-v_s}^{v_s} dv_i\right) |f_q(\{v_i\})|^2 e^{i(Px - Et)}, \]  

\[ P = \sum_i m_h v_i, \quad E = \sum_i \frac{1}{2} m_h v_i^2, \quad m_h = -m/q, \]  

where \(f_q(\{v_i\})\) is a form factor that must be calculated.

Using the mapping to the symplectic matrix model, it has been possible \([13]\) to calculate this form factor for \(\lambda = q = 2\). The result is remarkably simple:

\[ |f_2(v_1, v_2)|^2 = \frac{1}{8v_s} \left( \frac{(v_1 - v_2)^2}{v_s^2 - v_1^2} \right)^{1/2}. \]
The significance of this in the context of the \( SU(2) \) trigonometric Haldane-Shastry chain is that the YHWS wavefunctions with \( Z_i = \exp(2\pi ix_i/L) \) are also eigenstates of the \( \lambda = 2 \) CSM, and the matrix elements of \( S_i^+(t) \) between two YHWS are equivalent to those of \( \Psi(x_i,t) \) between the corresponding CSM states. The CSM result can immediately be translated to the results for the spin chain in the rational limit, where the Yangian symmetry algebra becomes classical.

While the calculation \[13\] is complicated and indirect, the simplicity of the result suggests there should be a simple derivation, not based on the “accident” of a mapping to a Gaussian matrix model. It is very tempting to conjecture the extension of the result to the \( SU(n) \) chain, for which no such mapping is known. Examination of the action of the operator \( c_i^\dagger \alpha c_i^\beta \) on the singlet ground state of the \( SU(n) \) chain leads to the conclusion that it produces a two-parameter family of YHWS states with one spinon (of “color” \( \bar{\beta} \)) moving with rapidity \( x_1 \), and a complex of \( n-1 \) spinons (with net “color” \( \alpha \)) moving together (but not as a bound state) with rapidity \( x_2 \). In the singlet ground state of the rational \( SU(n|0) \) chain state, the correlation function \( \langle 0| X_0^\alpha \beta j(t) X_0^\gamma \delta (0) |0 \rangle \), where \( X_0^\alpha \beta i \equiv c_i^\dagger \alpha c_i^\beta \), will thus have the form

\[
\left( -1 \right)^j \frac{1}{n} \delta^{\alpha \delta} \delta^{\bar{\beta} \bar{\gamma}} C(j, t) + \left( -1 \right)^j \frac{1}{n^2} \delta^{\alpha \bar{\beta}} \delta^{\bar{\gamma} \delta} \left( 1 - C(j, t) \right),
\]

(4.8)

where

\[
C(m, t) = \frac{1}{4} \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2 |F_n(x_1, x_2)|^2 \left( e^{i(q(x_1)m-\epsilon(x_1)t)} \right) \left( e^{i(q(x_2)m-\epsilon(x_2)t)} \right)^{n-1} \tag{4.9a}
\]

\[
q(x) = \frac{\pi x}{n}; \quad \epsilon(x) = \frac{\pi v_s}{2n}(1 - x^2). \tag{4.9b}
\]

Here \( F_n(x_1, x_2) \) is the form factor that must be found.

The asymptotic form of the correlations are straightforward to compute from bosonization (they are free-fermion correlations with the factor coming from charge degrees of freedom divided out), or from conformal field theory \[29\], and can be fit to a simple Ansatz based on (4.7). I therefore present the conjecture for the form factor of the rational \( SU(n|0) \) chain (which is a rigorous result for \( n=2 \)):

\[
|F_n(x_1, x_2)|^2 = A_n \left( \frac{4(x_1 - x_2)^2}{(1 - x_1^2)(1 - x_2^2)} \right)^{1/n}, \quad A_n = \frac{\left( \frac{n-1}{n} \right) \Gamma \left( \frac{n+1}{n} \right)}{\Gamma \left( \frac{n-1}{n} \right) \Gamma \left( \frac{n+2}{n} \right)}.
\]

(4.10)

It has been verified \[30\] that this form fits the numerically computed static structure factor of the trigonometric \( SU(3) \) chain with \( N \leq 18 \) extremely well, with very small finite-size corrections, leaving no doubt that this conjecture for \( n > 2 \) is indeed correct.

As another test of the reliability of generalizing (4.7) “by conjecture”, based on its remarkable simplicity, the same type of arguments can be used to obtain the form factor for the retarded Greens function (4.6) of the CSM at a general integer coupling \( \lambda = q \).
In this case, the states contributing to the spectral function have \( q \) hole excitations with independent velocities. The resulting conjecture is

\[
|f_q(\{v_i\})|^2 = \frac{1}{2v_s} B_q \left( \prod_{i=1}^{q} (v_s^2 - v_i^2) \right)^{-1+1/q} \left( \prod_{i<j} (v_i - v_j)^2 \right)^{1/q},
\]

where the multi-dimensional integral fixing the normalization \( B_q \) is hard to do. I have just learned of recent work by Forrester [31], who calculates the equal-time limit of this correlation function using quite different methods based on the celebrated Selberg trace formula. His expression is precisely the equal-time limit of the formula conjectured here! Moreover, Forrester’s result provides the normalization as

\[
B_q = \prod_{j=1}^{q} \left( \frac{\Gamma \left( \frac{1+q}{q} \right)}{\Gamma \left( \frac{2}{q} \right)} \right)^{2q}.
\] (4.11b)

It is striking that the form factors are essentially Laughlin-type wavefunctions for anyons with the same statistics as the holes, now with the rapidities as coordinates.

The remarkable property of the rational and trigonometric models is that the local operators such as the spin on a given site act on the ground state to produce only a very restricted class of excitations. There is a general selection rule, verified empirically [32], that the local spin operator \( \vec{S}_i \) acting on any eigenstate cannot change the spinon number \( N_{sp} \) by more than \( \pm 2 \). Such properties are very reminiscent of an ideal gas, and the most natural interpretation of the trigonometric and rational models is as generalizations of the ideal gas Fock-space structure to non-trivial statistics.

### 5. Conclusion

I have attempted to present, from my perspective, the main results associated with the still-unfolding properties of the Haldane-Shastry spin chain model, stressing the simplest \( SU(2) \) or \( S = 1/2 \) model. I have clearly omitted many aspects, and there is clearly much more yet to emerge.

A direct algebraic treatment in the rational limit, when the “quantum group” becomes classical, would be particularly desirable. It can be no accident that this is the limit in which all the explicit results are obtained, but (disappointingly) so far a direct use of its algebraic properties such as the infinite \( \hat{sl}_2^+ \) symmetry not yet been made. An algebraic construction of the highest weight eigenstates (fully spin-polarized spinon gas states) in terms of particle creation operators (“vertex operators”) acting on the vacuum is also needed; this would closely parallel a similar treatment needed for the Calogero-Sutherland model.

Other possible lines of investigation are whether a “quantum deformation” \( \hat{sl}_2^+ \rightarrow Y(\hat{sl}_2) \) can be used to calculate the hyperbolic model form factors, and what is the origin of integrability of the elliptic models. There is clearly much more work to be done!
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