Nonlinear diagnostic of a shallow sea stratified by density with weak dispersion and weak nonlinearity

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Abstract. The paper discusses the methodology for determining the parameters of the hydrophysical background of a shallow sea stratified by density. The background of this sea is characterized by quadratic nonlinearity - $\alpha$, high-frequency dispersion - $\beta$ and phase velocity of linear internal gravitational waves - $c_0$. In the framework of the theory of weakly-intensity shock waves in media with weak dispersion, expressions are obtained for calculating the parameters: $\alpha$, $\beta$ and $c_0$. The data for calculations are in-situ measurement of the spatial-temporal characteristics of the internal undular bores, traveling on the pycnocline of the shallow sea.

1. Introduction
The hydrodynamic model of a shallow sea stratified by density with weak dispersion and nonlinearity is widely used in numerical modeling of the dynamics of nonlinear internal gravitational waves (IGW) in the shelf zones of the ocean. To study the nonlinear dynamics of the IGW in a specific area of the shelf, it is necessary to set the hydrophysical background along which these waves propagate. The original method for determining the parameters of the hydrophysical background is described. The method is formulated using data from in-situ measurement of the spatial and temporal characteristics of internal undular bores on a shallow sea pycnocline, wave trains of nonlinear IGW resulting from the breaking of low-frequency IGW with small amplitude. Packets of such waves are regularly observed on the pycnocline of the ocean shelf waters. These are the so-called internal undular bores (IUB).

A consistent hydrodynamic interpretation of a weak-intensity IUB is constructed using the theory of weakly dispersing shock waves. The theory of such undular bores is constructed on the basis of a cnoidal solution of the Korteweg de Vries equation (KdV). Its use for the interpretation of internal undular bores was performed in [1]. In our work, in the framework of this theory, based on the analytical model of the IUB on the pycnocline of the shallow sea, we obtained relations for calculating the parameters of the hydrophysical background of this sea. When calculating these parameters, the data of field measurements of IUB characteristics are used.

2. Undular bore model on a shallow sea pycnocline
In order to obtain the expressions for calculating the parameters $\alpha$, $\beta$, $c_0$, according to the field measurements of the parameters of the IUB, we use its model based on the cnoidal solution of the KdV equation. We write it in the form
\[ \frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0, \]

where \( \eta(x, t) \) is the vertical displacement of the shallow sea pycnocline, \( x \) is the horizontal coordinate, \( t \) is time, \( c_0 \) is the phase velocity of linear IGWs, and \( \alpha, \beta \) are the coefficients of quadratic non-linearity and high-frequency dispersion, respectively.

A simple, model idea of the spatial and temporal structure of the undular bore can be obtained by using the cnoidal solution of equation (1) with the initial condition \( \eta(x,0) = \eta_0 H(-x) \), where it is assumed that the perturbation amplitude \( \eta_0 \) is positive and \( H(x) = 1 \) if \( x > 0 \) and \( H(x) = 0 \) if \( x < 0 \), i.e. \( H(x) \) is the Heaviside function. To solve the problem, we use the approach formulated in [2, 3] for analyzing the evolution of such formations in weakly dispersing media and adapted for the study of weakly nonlinear internal waves in the shallow sea described in [1].

Within the framework of this approach, the spatial and temporal structure of the IUB is determined by the analytical solution of the KdV equation in the form of a cnoidal traveling wave of the following form

\[ \eta(x,t) = \eta_0 \left[ 2dn^2 \left( k_0(x-Vt)/2 \right) - (1-s^2) \right] + \eta_m. \]

Here, \( \eta_0 \) is the amplitude factor of the IUB, \( dn_s(y) \) is the Jacobi elliptic function with the module \( s \) defined on the interval \( 0 < s < 1 \), and \( \eta_m \) is the average level. According to the properties of elliptic functions, the expressions for the wave vector \( k \) of the cnoidal wave (2) and its velocity \( V \) depend on the parameters of the hydrophysical background: \( \alpha, \beta \) and \( c_0 \), as follows

\[ k = \frac{\pi k_0}{2K(s)} = \frac{\pi}{K(s)} \sqrt{\frac{\alpha \eta_0}{6\beta}}, \quad V = c_0 + \frac{1+s^2}{3} \alpha \eta_0. \]

In relations (2) and (3), the parameter \( s = s(\tau) \) is a slow function of the self-similar variable \( \tau \equiv (x - c_0 t) / (\alpha \eta_0 c_0 t) \) given in the interval \( 1 \leq \tau \leq 2/3 \) which satisfies the equation

\[ \tau = \frac{1}{3} \left( 1 + s^2 \right) - \frac{2}{3} \frac{s^2(1-s^2)K(s)}{E(s) - (1-s^2)K(s)}. \]

In expressions (3), (4), \( K(s) \), \( E(s) \) is full elliptic integrals of the first and second kind, respectively. Relations (2), (4) in parametric form give an asymptotically exact solution as \( t \to \infty \) to the problem of decay of the initial jump of the pycnocline elevation. An analysis of these relationships shows that the vertical displacements of the isopycnic or isothermal line caused by the IUB at a given point in time occupy a limited area in space.

The front boundary of this region is determined by the condition \( \tau \to 2/3 \). At this boundary, the nonlinearity parameter tends to its maximum value \( s^2 \to 1 \), the wave vector \( k \to \Delta_k^{-1} \), and \( dn_s(x) \to \text{sech}^2(x) \). Thus, cnoidal wave (2) takes the form of a solitary wave or KdV soliton with a wave profile \( \sim \eta_k \text{sech}^2[\Delta_k^{-1}(x - V_k t)] \), amplitude \( \eta_k = 2\eta_0 \), nonlinear velocity \( V_k \) and half-width \( \Delta_k \), which are expressed through the parameters of the hydrophysical background of the shallow sea: \( \alpha, \beta, c_0 \) according to the relations:

\[ V_k = c_0 + \frac{1}{3} \alpha \eta_k, \quad \Delta_k^2 = \frac{\alpha}{12\beta} \eta_k. \]

In the rear of the IUB, i.e. at its trailing edge (it is determined by the condition \( \tau \to -1 \)), the nonlinearity parameter \( s \to 0 \), the wave vector \( k \to \infty \), and its velocity \( V \to c_0 \), i.e. the cnoidal wave (2) collapses to the linear harmonic wave \( \sim s \cos[k_0(x - c_0 t)] \) with the disappearing amplitude \( \sim s \) and propagates with the speed \( c_0 \).

We note an important feature of the cnoidal wave (2) for the proposed technique. In this wave, the parameter \( k_0 \) is, firstly, the wavenumber of the linear harmonic wave \( \sim s \cos(k_0 x) \) in the region of the
trailing edge of the IUB, and secondly, this parameter is connected by a simple ratio \( k_0 / 2 = \Delta_K^{-1} \) with the half-width leading soliton – \( \Delta_K \). From this, taking into account the definitions of the wave number \( k_0 = 2\pi/(c_0 T_W) \) and the width of the soliton \( 2\Delta_K = T_K V_K \), it follows that the duration of the leading soliton \( T_K \) and the ripple period in the region of the trailing edge of the IUB are related by a one-to-one dependence \( V_K T_K = 2c_0 T_W / \pi \), where \( T_W \) is the period of oscillations of the pycnocline in the rear zone of the IUB.

In conclusion, the following should be mentioned the coefficients of the KdV equation \((c_0, \alpha, \beta)\) characterize the hydrophysical background of the shallow sea. Along with this, they also determine the spatial and temporal characteristics of the IUB: the nonlinear velocity – \( V_K \) and the duration of the leading bore soliton – \( T_K \), as well as the wave number \( k_0 \) of the linear wave in the rear zone of the IUB. Thus, by in-situ measuring the listed characteristics of IUB, it is easy to calculate

(*) the quadratic nonlinearity parameter \( \alpha \), according to direct measurements of the expansion velocity of the IUB wave zone \( \mu_K \equiv (V_K - c_0) \), and the amplitude \( \eta_K \), calculated by the formula

\[
\alpha = \frac{3 \mu_K}{\eta_K} \quad (6)
\]

(*) the high-frequency dispersion parameter is – \( \beta^+ \) and \( \beta^- \), calculated from direct in-situ measuring \( \mu_K, V_K, T_K, T_W \) using the expressions

\[
\beta^+ = \mu_K (V_K T_K/4)^2, \quad \beta^- = \mu_K (c_0 T_W/2\pi)^2 \quad (7)
\]

(*) and finally, parameter \( c_0 \) is the phase velocity of long linear IGWs using the data of direct measurements of \( V_K \) and \( \mu_K \), or to obtain an indirect estimate of this parameter using the data of measurements of \( V_K, T_K, T_W \), and determine it using one of two expressions

\[
c_0 = V_K - \mu_K, \quad c_0 = (\pi T_K/2T_W) V_K \quad (8)
\]

Thus, relations (6, 7 and 8) solve the problem of determining the parameters characterizing the hydrophysical background of a shallow, weakly dispersed sea stratified by density with a quadratic nonlinearity using in-situ measurements of the characteristics of the spatiotemporal structure of the internal undular bore on a pycnocline of such a sea.

3. Discussion and conclusion

Using the described methodology, we calculate the parameters of the hydrophysical background at the marine polygon of the POI FEB RAS. We will calculate the parameters according to the data of a field experiment, delivered at the marine range in the autumn of 2013. The description of the experiment and its results were published in [4, 5, 6]. In this paper, we use the measurement data that were performed at the buoy stations N. 1 and 2, installed on the isobath – 40 m and spaced 1827 m apart.

Let us turn to a brief description of the temporal structure of the wave train during its passage through the buoy stations. Figure 1 shows the waveform of a packet consisting of vertical displacements of isotherms (8, 9, 10 and 11 °C marked by lines of various styles) with maximum deviations from the unperturbed state recorded during its successive passage through Mr1 and Mr2. The numbers 1, 2, and 3 in the figure numbered soliton-like pulsations of depression isotherm (further pulsations of negative polarity), which are part of the leading group of the wave train. The structure of the presented wave train, characteristic of IUB, is noteworthy. It is preceded by a low-frequency depression with duration of ~ 15 min with a steep trailing edge of considerable amplitude ~ 4 m, which simultaneously represents the leading edge of the wave packet. This is followed by the actual wave packet, consisting of several pulsations; the shape of the first three of them is close to the form of a soliton solution of the KdV equation.
Figure 1. Wave records of the undular bore during its passage through Mr 1 (figure left) and Mr 2 (figure right). Numbers 1, 2, 3 indicate soliton-like pulsations in the leading group of the bore.

We found that during the propagation time between stations, the number of soliton-like pulsations in the packet increased to four. Its length increased by about 10%, and the height of the leading pulsation increased to 7 m. The average period of pulsations at the trailing edge was ~ 360 s. Thus, the high-frequency packet of internal waves recorded on Mr 2, which was formed as a result of the decay of a low-frequency nonlinear depression at its trailing, steeping front, is a prototype of IUB, which was formed as a result of the decay of a low-frequency internal wave.

In accordance with the IUB cnoidal model, a soliton with maximum amplitude and speed is located at its leading edge. Table 1 presents the results of the calculation performed according to direct measurements of the amplitude $A_s$, the velocity of the leading soliton $V_s$, and the dispersion parameter $\beta^+$ and $\beta^-$ calculated from the data of the IUB parameters recorded in October 2013. According to the calculations, the hydrophysical background at this time of year at the marine range is characterized by the following parameter values: quadratic nonlinearity $\alpha \approx 0.023$ s$^{-1}$, dispersion $\beta \approx 31.4$ m$^3$s$^{-1}$, phase velocity of linear internal waves $c_0 \approx 0.428$ m/s. In subsequent calculations, we used the velocity value $c_0 \approx 0.428$ m/s refined using relation (8).

Further, when calculating the parameters of the hydrophysical background, we used the velocity value $c_0 \approx 0.428$ m/s, refined using relation (8). Using this value of the velocity of linear internal waves, as well as the average velocity $V_s$ and amplitude $A_s$ of the first three pulsations, using the relation $3\mu_r/A_s$, we calculated the quadratic nonlinearity parameter $\alpha$ at the polygon. Table 2 presents the results of these calculations. It follows from it that the average over three values of the quadratic nonlinearity parameter at the polygon is 0.023 s$^{-1}$.

| $A_s$, m | $M_1$, m/s$^{-1}$ | $T_s$, s | $V_s$, m/s | $c_0$, m/s | $T_w$, s | $c_ex$, m/s | $\alpha$, s$^{-1}$ | $\beta^+$, m$^3$s$^{-1}$ | $\beta^-$, m$^3$s$^{-1}$ | $c_0$, m/s |
|----------|------------------|---------|-----------|-----------|---------|------------|-----------|----------------|----------------|-----------|
| 6.8      | 1380             | 203     | 0.485     | 0.428     | 360     | 0.432      | 0.025     | 31.6           | 31.2           | 0.428     |
| 5.9      | 0.472            | 0.022   | 31.3      | 31.2      | 0.428   |
| 4.6      | 0.463            | 0.023   | 31.5      | 31.2      | 0.428   |

Table 1. The values of the amplitude is $A_s$, the «mass» is $M_1$, the duration is $T_s$, the speed is $V_s$ of the leading IUB soliton, the phase velocity is $c_{ex}$, the wave period is $T_w$ at its trailing edge, and the phase velocity is $c_0$ obtained from experimental data, quadratic nonlinearity parameter $\alpha^+$, high-frequency dispersion $\beta^+$, calculated with help by the corresponding formulas.

| $A_s$, m | $V_s$, m/s$^{-1}$ | $\alpha$, s$^{-1}$ | $\beta^+$, m$^3$s$^{-1}$ | $\beta^-$, m$^3$s$^{-1}$ | $c_0$, m/s |
|----------|------------------|------------------|----------------|----------------|-----------|
| 1        | 6.8              | 0.485            | 0.025          | 31.6           | 31.2      | 0.428     |
| 2        | 5.9              | 0.472            | 0.022          | 31.3           | 31.2      | 0.428     |
| 3        | 4.6              | 0.463            | 0.023          | 31.5           | 31.2      | 0.428     |
Let us compare the parameters of the hydrophysical background at the marine polygon of the POI FEB RAS presented in Table 2 with the parameters obtained using the traditional technique. A year earlier, in October 2012, hydrological work was carried out at the landfill to determine its hydrophysical background. To this end, according to the data obtained at the daily hydrological station and hydrological section, along the expected propagation of internal waves, the background distribution of the buoyancy frequency: N(z), was determined at the marine polygon using a standard technique that included daily averaging of vertical soundings of temperature and salinity fields performed at the station with 30 minutes. Then, using the obtained daily average buoyancy frequency profile, the boundary value eigenvalue problem, was numerically solved. Basic parameters: c₀, α and β were calculated according to the average daily density stratification at the test site. The results of this calculation, borrowed from [6], are as follows: quadratic nonlinearity, dispersion and phase velocities of linear explosives at the test site in 2012, calculated by formulas (3), (4) amounted to $\alpha \sim 0.023 \text{ s}^{-1}$, $\beta \sim 36.7 \text{ m}^3\text{s}^{-1}$ and $c_0 \sim 0.38 \text{ m s}^{-1}$.

The paper presents an original method for determining the parameters of the hydrophysical background, by which solitary internal waves of the KdV type propagate. The method is based on direct measurements of the spatial temporal characteristics of internal undular bores: the expansion velocity of its wave zone, the velocity and amplitude of the leading IUB soliton, and the phase velocity of a linear internal wave representing its trailing edge.

In the framework of the theory of undular bore of KdV type, it was found that the quadratic nonlinearity of solitary waves of this type linearly depends on the expansion speed of the wave zone of the IUB; the high-frequency dispersion parameter is proportional to the square of the wavelength that closes this bore, and its phase velocity is equal to the difference between the velocities of the leading IUB soliton and its expansion speed.

According to an experiment conducted in the fall of 2012, using the proposed method, the hydrophysical background parameters were determined at the marine polygon of the POI FEB RAS. The values of these parameters were: $\alpha \sim 0.023 \text{ s}^{-1}$, $\beta \sim 36.7 \text{ m}^3\text{s}^{-1}$ and $c_0 \sim 0.38 \text{ m s}^{-1}$. Comparison of these values and values obtained by hydrological data using standard methods showed a satisfactory agreement between them.

Acknowledgments
This work was performed under the program "Priority research in the interests of the integrated development of the Far Eastern Branch of the Russian Academy of Sciences" (no. 18-1-010) and its theoretical part was supported by the POI FEBRAS Program 'Mathematical simulation and analysis of dynamical processes in the ocean (no. 0271-2019-0001).

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