On the Cramér-Rao Lower Bound for Frequency Correlation Matrices of Doubly Selective Fading Channels for OFDM Systems

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Abstract—In this paper, the Cramér-Rao lower bound (CRLB) of the sample frequency correlation matrices (SFCM) is derived based on a rigorous model of the doubly selective fading channel for orthogonal frequency division multiplexing (OFDM) systems with pilot-symbol-aided modulation. By assuming a fixed pilot sequence and independent samples, SFCM is complex Wishart distributed. Then, the maximum likelihood estimator (MLE) and the exact expression of CRLB are obtained. From CRLB, the lower bounds of total mean squared error (TMSE) and average mean squared error (AvgMSE) independent of the pilot sequence are deduced, which reveal that the amount of samples is the dominant factor affecting AvgMSE while the signal-to-noise ratio and the maximum Doppler spread have negligible effect. Numerical simulations demonstrate the analytic results.

Index Terms—CRLB, Frequency correlation matrix, Doubly selective fading channels, OFDM.

I. INTRODUCTION

Playing a key role in the channel estimation for orthogonal frequency division multiplexing (OFDM) systems, the frequency correlation matrix (FCM) is utilized by many statistics-based channel estimation algorithms, e.g., the linear minimum mean-squared error (LMMSE) estimator and its optimal low-rank approximations [1], the MMSE estimator exploring both time and frequency correlations [2], the two-dimensional Wiener filtering [3], and those algorithms based on parametric channel model [4] [5] [6]. In real applications, the sample FCM (SFCM) is used in stead of the true one, and usually the sample frequency correlation matrices (SFCM) is derived for FCM to evaluate the performance of maximum likelihood estimator (MLE) and uncover the factors influencing the estimation accuracy.

Numerical results appear in Section IV. Finally, Section V concludes the paper.

Notation: Lowercase and uppercase boldface letters denote column vectors and matrices, respectively. (·)*, (·)H, and ||·||F denote conjugate, conjugate transposition, and Frobenius norm, respectively. ⊗ denotes the Kronecker product. E(·) represents expectation. [A]_{i,j} and [a]_{i} denotes the (i,j)-th element of A and the i-th element of a, respectively. diag(a) is a diagonal matrix by placing a on the diagonal.

II. SYSTEM MODEL

Consider an OFDM system with a bandwidth of BW = 1/T Hz (T is the sampling period). N denotes the total number of tones, and a cyclic prefix (CP) of length Lcp is inserted before each symbol to eliminate inter-block interference. Thus the whole symbol duration is Ts = (N + Lcp)T.

The complex baseband model of a linear time-variant mobile channel with L paths can be described by [11]

\[ h(t, \tau) = \sum_{l=1}^{L} h_l(t) \delta(\tau - \tau_l T) \]  

where \( \tau_l \in \mathbb{R} \) is the normalized non-sample-spaced delay of the l-th path, and \( h_l(t) \) is the corresponding complex amplitude. According to the wide-sense stationary uncorrelated scattering (WSSUS) assumption, \( h_l(t) \)'s are modeled as uncorrelated narrowband complex Gaussian processes.

Furthermore, by assuming the uniform scattering environment introduced by Clarke [12], \( h_l(t) \)'s have the identical
normalized time correlation function (TCF) for all l’s, thus the TCF of the l’s path is

\[ r_{l,t}(\Delta t) = \sigma_l^2 J_0(2\pi f_d \Delta t) \tag{2} \]

where \( \sigma_l^2 \) is the power of the l-th path, \( f_d \) is the maximum Doppler spread, and \( J_0(\cdot) \) is the zeroth order Bessel function of the first kind. Additionally we assume the power of channel is normalized, i.e., \( \sum_{l=0}^{L-1} \sigma_l^2 = 1 \).

Assuming a sufficient CP, i.e., \( L_{cp} \geq L \), the discrete signal in the frequency domain is written as

\[ y_f(n) = H_f(n)x_f(n) + n_f(n) \tag{3} \]

where \( x_f(n), y_f(n), n_f(n) \in \mathbb{C}^{N \times 1} \) are the n-th transmitted and received signal and additive white Gaussian noise (AWGN) vectors, respectively, and \( H_f(n) \in \mathbb{C}^{N \times N} \) is the channel transfer matrix with \((k + \nu, k)\)-th element as

\[ [H_f(n)]_{k+\nu,k} = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{l=1}^{L} h_l(n, m) e^{-j2\pi (\nu m + k \tau_l) / N} \tag{4} \]

where \( h_l(n, m) = h_l(nT_s + (L_{cp} + m)T) \) is the sampled complex amplitude of the l-th path. \( \tau_l \) and \( \nu \) denote frequency and Doppler spread, respectively. Apparently, as \( H_f(n) \) is non-diagonal, ICI is present. In fact, when the normalized maximum Doppler spread \( f_d T_s \leq 0.1 \), the signal-to-interference ratio (SIR) is over 17.8 dB [13].

### III. CRLB FOR FREQUENCY CORRELATION MATRICES

Usually SFCM is obtained through the LS channel estimation. We consider OFDM systems adopting pilot-symbol-assisted modulation (PSAM) [1], hence only pilot symbols, denoted as \( y_p(n) \in \mathbb{C}^{N \times 1} \), are extracted and used to perform LS channel estimation. In addition, the pilot sequence is assumed to be invariant along the time. Therefore,

\[ h_{p,l,s}(n) = X_p^{-1}y_p(n) = X_p^{-1}H_p(n)x_p + X_p^{-1}n_p(n) \tag{5} \]

where \( X_p = \text{diag}(x_p) \) is a diagonal matrix consisting of pilot symbols, and the noise term is \( n_p(n) \sim \mathcal{CN}(0, \sigma_p^2 I_N) \).

Denote the instantaneous channel impulse response (CIR) vector as \( h_t(n, m) = [h_1(n, m), \ldots, h_L(n, m)]^T \), \( m = 0, \ldots, N - 1 \), according to the assumptions of WSSUS and uniform scattering, \( h_t(n, m) \) is complex normal, i.e., \( h_t(n, m) \sim \mathcal{CN}_L(0, D) \).

\[ \Omega \in \mathbb{C}^{N \times N} \] is a Toeplitz time correlation matrix (TCM), defined as

\[ [\Omega]_{m_1, m_2} = J_0(2\pi f_d (m_1 - m_2) T) \tag{6} \]

Then according to [7], the channel transfer matrix \( H_f(n) = F_r H_t(n) \), where \( F_r \in \mathbb{C}^{N \times L} \) is the unbalanced Fourier transform matrix, defined as \( [F_r]_{l,j} = e^{-j2\pi l \tau_j / N} \). Thus

\[ H_f(n) \sim \mathcal{CN}_{N \times N}(0, \Omega \otimes (F_r D F_r^H)) \tag{7} \]

Assuming CIR is independent of the thermal noise, with [5] and [7], we have

\[ h_{p,l,s}(n) \sim \mathcal{CN}_N(0, \Sigma) \tag{8} \]

where the covariance matrix \( \Sigma \) is defined as

\[ \Sigma = \omega X_p^{-1} R_p + \frac{\sigma_p^2}{\omega} I_N X_p^H \tag{9} \]

where \( \omega = x_p^H \Omega x_p \) and \( R_p = F_r D F_r^H \) is the true FCM.

When the LS estimated CFR’s, i.e., \( h_{p,l,s}(n) \), are available, SFCM is constructed as

\[ \hat{R}_{p,l,s} = \frac{1}{N_t} \sum_{n=1}^{N_t} h_{p,l,s}(n) h_{p,l,s}^H(n) \tag{10} \]

where \( N_t \) is the amount of samples. To derive the probability density function (PDF) of SFCM, we assume that samples are independent of each other, which may be a strict constraint. However, when the maximum Doppler spread is large and the spacing between two contiguous pilot symbols is comparatively small, the correlation between them is rather low, which alleviates the effect of model mismatch. Then, based on the assumption of independence and (8), we know that SFCM has the complex central Wishart distribution with \( N_t \) degrees of freedom and covariance matrix \( \Sigma' = \Sigma / N_t \) [14], denoted as

\[ \hat{R}_{p,l,s} \sim \mathcal{CN}_N(\hat{\Sigma}', \Sigma') \tag{11} \]

and its PDF is

\[ f(\hat{R}_{p,l,s}) = \frac{\text{etr}(-\Sigma'^{-1} \hat{R}_{p,l,s}) (\text{det}(\hat{R}_{p,l,s}))^{N_t - N}}{\text{CT}_N(N_t) \text{det}(\Sigma')^{N_t}} \tag{12} \]

where \( \text{etr}(\cdot) = \exp(\text{tr}(\cdot)) \) and \( \text{CT}_N(\cdot) \) is the complex multivariate gamma function, defined as

\[ \text{CT}_N(N_t) = \pi^{N(N-1)/2} \prod_{k=1}^{N} \Gamma(N_t - k + 1) \]

Then, from (12), the likelihood function is written as

\[ L(R_p) = \text{tr}(-\Sigma'^{-1} \hat{R}_{p,l,s}) + (N_t - N) \ln(\text{det}(\hat{R}_{p,l,s})) \]

\[ - \ln(\text{CT}_N(N_t)) - N_t \ln(\text{det}(\Sigma')) \]

Therefore, the score function with respect to the parameter matrix \( R_p \) is

\[ \frac{\partial L(R_p)}{\partial \text{vec}(\Sigma')} = \frac{\partial \text{vec}(\Sigma')^T}{\partial \text{vec}(R_p)} \times \frac{\partial L(R_p)}{\partial \text{vec}(\Sigma')} \tag{13} \]

where the first term on the right-hand side of (13) is

\[ \frac{\partial \text{vec}(\Sigma')^T}{\partial \text{vec}(R_p)} = \frac{\omega}{N_t} (X_p^H \otimes X_p^{-1}) \tag{14} \]

and the second term is

\[ \frac{\partial L(R_p)}{\partial \text{vec}(\Sigma')} = \text{vec}([\Sigma'^{-1} \hat{R}_{p,l,s} \Sigma'^{-1} - N_t \Sigma'^{-1}]) \tag{15} \]

By letting the score function equal zero and with (9), the MLE of FCM is derived as

\[ \frac{\text{MLE}(R_p)}{X_p X_p^H \Sigma'} = \frac{x_p^H \hat{R}_{p,l,s} - \sigma_p^2 I_N}{x_p^H \Omega x_p} \tag{16} \]
Accordingly, (19) is rewritten as

$$J(R_p) = \frac{\omega^2}{N_t} (X_p^{-H} \otimes X_p^{-1})(\Sigma'^{-H} \otimes \Sigma'^{-T}) (X_p^{-1} \otimes X_p^{-H})$$

From the Fisher Information matrix, the CRLB of $R_p$ can be derived as [16] [17]

$$\text{CRLB}(R_p) = J^{-1}(R_p)$$

$$= \frac{N_t}{\omega^2} (X_p \otimes X_p^H)(\Sigma'^H \otimes \Sigma'^T) (X_p^H \otimes X_p)$$

$$= \frac{1}{N_t} \left( \frac{1}{\omega} X_p \Sigma'^H X_p^H \otimes \left( \frac{1}{\omega} X_p^H \Sigma'^T X_p \right) \right) \tag{21}$$

With (9), (21) can be further written as

$$\text{CRLB}(R_p) = \frac{1}{N_t} (R_p + \frac{\sigma_n^2}{\omega} I_N) \otimes (R_p + \frac{\sigma_n^2}{\omega} I_N)^T \tag{22}$$

Based on (22), a lower bound of the total mean squared error (TMSE) for MLE$(R_p)$ is

$$\text{TMSE}_{LB}(R_p) = \text{tr}(\text{CRLB}(R_p))$$

$$= \frac{1}{N_t} \text{tr}^2(R_p + \frac{\sigma_n^2}{\omega} I_N)$$

$$= \frac{N^2}{N_t} (1 + \frac{1}{\omega})^2 \tag{23}$$

where $\gamma = \sigma_n^{-2}$ is the signal-to-noise ratio (SNR). And, accordingly, the lower bound of the average mean squared error (avgMSE) is

$$\text{AvgMSE}_{LB}(R_p) = \frac{\text{TMSE}_{LB}(R_p)}{N_t^2} = \frac{1}{N_t} (1 + \frac{1}{\omega})^2 \tag{24}$$

(24) verifies the common sense that the more samples collected, the more accurate estimation acquired. And it also reveals that increasing SNR can reduce the estimation error. Furthermore, since

$$\omega = X_p^H \Omega x_p = \|x_p\|^2_2 \times \frac{X_p^H \Omega x_p}{x_p^H X_p} = \|x_p\|^2_2 \times R_x(p)(\Omega) \tag{25}$$

where $R_x(p)(\Omega)$ is the Rayleigh quotient of $\Omega$ associated with the pilot sequence $x_p$ and $R_x(p)(\Omega) \leq \lambda_{max}$ where $\lambda_{max}$ is the maximum eigenvalue of $\Omega$. Besides, when the power of pilot symbol is normalized, $\|x_p\|^2_2 = N$. Hence (24) is further lower bounded by

$$\frac{\text{AvgMSE}_{LB}(R_p)}{N_t} = \frac{1}{N_t} (1 + \frac{1}{N \lambda_{max}^2})^2 \tag{26}$$

To further look into the relationship between $f_d T_s$ and $\lambda_{max}$, we examine the extreme eigenvalues of $\Omega$ for different $f_d T_s$’s and $N$’s numerically, and the results are plotted in Fig. 1. Moreover, we find a simple function fitting the maximum eigenvalues of all cases very well. The function is

$$\lambda_{max}(\Omega) = N J_0(2\pi c f_d T_s) \tag{27}$$
ICI within a tolerable range \[13\].

The current applied OFDM systems have decreasing with respect to the collected samples are apart from each others far enough sequences are QPSK modulated and randomly chosen. And according to (28), we know that the amount of samples, i.e., \(N_t\), effects the estimation accuracy dominantly but SNR and maximum Doppler spread do not, since \(N_t^2\) is sufficiently large for most of current systems.

**IV. Numerical Results**

The OFDM system in simulations is of \(BW = 1.25\) MHz \((T = 1/BW = 0.8\) ms\), \(N = 128\), and \(L_{cp} = 16\). Two 3GPP E-UTRA channel models are adopted: Extended Vehicular A model (EVA) and Extended Typical Urban model (ETU) \[18\]. The excess tap delay of EVA is \([0, 30, 150, 310, 370, 710, 1090, 1730, 2510]\) ns, and its relative power is \([0.0, -1.5, -1.4, -3.6, -0.6, -9.1, -7.0, -12.0, -16.9]\) dB. For ETU, they are \([0, 50, 120, 200, 230, 500, 1600, 2300, 5000]\) ns and \([-1.0, -1.0, -1.0, 0.0, 0.0, 0.0, 0.0, -3.0, -5.0, -7.0]\) dB, respectively. The classic Doppler spectrum, i.e., Jakes’ spectrum \[11\], is applied to generate the Rayleigh fading channel.

In Fig.2 we compare the analytic results \[24\] and the numerical results over a range of \(N_t\)’s for EVA and ETU channels, respectively, when \(\gamma = 20\) dB and \(f_d = 200\) Hz. The pilot sequences are QPSK modulated and randomly generated. And the collected samples are apart from each others far enough to guarantee the assumption of independence. Apparently, the analytic results meet the numerical ones quite well.

In Fig.3, we compare the analytic results \[28\] and the numerical results for EVA and ETU channels, respectively, when \(\gamma = 20\) dB and \(f_d = 200\) Hz. The pilot sequences are QPSK modulated. In order to examine the effect of different pilot sequences on \(\omega\), one hundred different sequences randomly generated are tested and their MSE’s are averaged and plotted. From the figure, we find that \[28\] is a tight bound even for an arbitrary pilot sequence.

The distributions of avgMSE for different SNR’s and Doppler’s are plotted in Fig.4 through ten thousands estimations for EVA and ETU channels, respectively. The amount of samples of each test is 200, and the pilot sequences are QPSK modulated and randomly generated. Clearly, avgMSE’s are centered around zero and most of them are within the range of zero to CRLB, which follows that \[16\] is an unbiased estimator. Moreover, it is also obvious that the distributions of avgMSE for EVA and ETU channels are negligibly influenced by \(\gamma\) and \(f_d\), which follows the analytic lower bound \[28\].

**V. Conclusion**

In this paper, the maximum likelihood estimator and CRLB of the frequency correlation matrix for OFDM systems in doubly selective fading channels are derived and analyzed. Through the analyses, we obtain an insightful lower bound of average MSE, i.e., \[28\], and according to which, the amount of samples shows a dominant impact on the accuracy of estimation while SNR and maximum Doppler spread have relatively small effect when the number of subcarriers are sufficiently large, although increasing SNR and decreasing maximum Doppler spread can help to reduce MSE slightly.

\[
J(R_p) = \frac{\omega^2}{N_t} (X_{p}^{-H} \otimes X_{p}^{-1}) E \{vec[(\Sigma^{-1} \hat{R}_{p,j} \Sigma^{-1} - N_t \Sigma^{-1})^{T}]vec[(\Sigma^{-1} \hat{R}_{p,j} \Sigma^{-1} - N_t \Sigma^{-1})^{H}] (X_{p}^{-H} \otimes X_{p}^{-1})^{H} \} (18)
\]
Fig. 4. Distributions of average MSE for EVA and ETU channels when $N_t = 200$.

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