ON SOME PROPERTIES OF WEAK SOLUTIONS TO THE MAXWELL EQUATIONS

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Abstract. This paper is concerned with weak solutions \( \{e, h\} \in L^2 \times L^2 \) of the time-dependent Maxwell equations. We show that these solutions obey an energy equality. Our method of proof is based on the approximation of \( \{e, h\} \) by its Steklov mean with respect to time \( t \). This approximation technique is well-known for establishing integral estimates for weak solutions of parabolic equations. In addition we prove the uniqueness of \( \{e, h\} \).

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1. Introduction

1.1. Field equations. Constitutive laws

Let $\Omega \in \mathbb{R}^3$ be a bounded domain, let $0 < T < +\infty$ and put $Q_T = \Omega \times ]0, T[$. The evolution of an electromagnetic field in a medium at rest occupying the region $\Omega$, is governed by the system of PDEs

(1.1) \[ \partial_t d + j = \text{curl} h \]  
Ampère-Maxwell law,

(1.2) \[ \partial_t b = - \text{curl} e \]  
Faraday law.

The meaning of the vector fields $\{d, b; e, h\}$ is

- $d$ electric displacement,
- $b$ magnetic induction,
- $e$ electric field,
- $h$ magnetic field.

The scalar fields

- $d \cdot e$ electric density,
- $b \cdot h$ magnetic density,
- $j \cdot e$ electric power density

are the basic energy densities for the electromagnetic field modelled by $\{d, b; e, h\}$.

Applying the div-operator to both sides of (1.1) and (1.2) and integrating over the interval $[0, t]$ ($0 < t \leq T$) gives

\[
(\text{div} \ d)(x, t) + \int_0^t (\text{div} \ j)(x, s) \, ds = (\text{div} \ d)(x, 0),
\]

\[
(\text{div} \ b)(x, t) = (\text{div} \ b)(x, 0)
\]

for all $(x, t) \in Q_T$. The scalar function

\[
\rho(x, t) = -\int_0^t (\text{div} \ j)(x, s) \, ds
\]

is called electric charge. For details see, e.g., [2, Chap. 1], [9, Chap. 1], [10, Chap. 18; 27], [12, Chap. 6] and [17, Teil I, §§3,4].

In this paper, we consider the following constitutive laws for the vector fields $\{d, b; e, h\}$

(1.3) \[ d = \varepsilon e, \quad b = \mu h \]

where the symmetric non-negative matrices $\varepsilon = \varepsilon(x) = \{\varepsilon_{kl}(x)\}_{k,l=1,2,3}$ and $\mu = \mu(x) = \{\mu_{kl}(x)\}_{k,l=1,2,3}$ ($x \in \Omega$) characterize the electric permittivity and magnetic permeability, respectively, of the medium under consideration. More general constitutive laws are discussed, e.g., in [9, Section 1.4], and [17, S. 20-22].

Substituting (1.3) into (1.1), (1.2) gives

(1.4) \[ \varepsilon \partial_t e + j = \text{curl} h, \]

(1.5) \[ \mu \partial_t h = - \text{curl} e. \]

Here, the vector fields $e$ and $h$ are the unknowns.

---

1) For $a = \{a_{kl}\}_{k,l=1,2,3}$ and $\xi = \{\xi_k\}_{k=1,2,3}$ ($a_{kl}, \xi_k \in \mathbb{R}$) we write $a\xi = \{a_{kl}\xi_k\}_{k=1,2,3}$ (summation over repeated indices).
Remark 1  The following structure of $j$ is widely considered

$$j = j_0 + j_1,$$

where

$$j_0 = \sigma e \quad \text{Ohm law.}$$

The matrix $\sigma = \sigma(x,t) = \{\sigma_{kl}(x,t)\}_{k,l=1,2,3}$ ($(x,t) \in Q_T$) characterizes the electrical conductivity of the medium. By physical reasons, $\sigma$ has to satisfy the conditions

$$(\sigma(x,t)\xi) \cdot \xi \geq 0 \quad \forall (x,t) \in Q_T, \ \forall \xi \in \mathbb{R}^3$$

The vector field $j_1$ represents a given current density (see, e.g., [2, pp. 10-11], [17, S. 19-20]).

1.2. Balance of electromagnetic energy

Let $\{e,h\}$ be a classical solution of (1.4), (1.5) in $Q_T$. We multiply scalarly (1.4) and (1.5) by $e$ and $h$, respectively, and add the equations. We get

$$\frac{1}{2} \frac{\partial}{\partial t} \left( (\varepsilon e) \cdot e + (\mu h) \cdot h \right) + \text{div } S + j \cdot h = 0$$

where

$$S = e \times h$$

denotes the Poynting vector of $\{e,h\}$. Integrating (1.6) over $\Omega$ yields

$$\frac{d}{dt} \mathcal{E}(t) + \int_{\Omega} (\text{div } S)(x,t) \, dx + \int_{\Omega} j(x,t) \cdot e(x,t) \, dx = 0$$

for all $t \in [0,T]$, where

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} [ (\varepsilon(x)e(x,t)) \cdot e(x,t) + (\mu(x)h(x,t)) \cdot h(x,t) ] \, dx, \quad t \in [0,T].$$

The scalar $\mathcal{E}(t)$ represents the electromagnetic energy of $\{e,h\}$ at the time $t$. Equation (1.7) is called balance of electromagnetic energy (or Poynting’s theorem).

Throughout our further discussion in this section, we suppose that the boundary $\Gamma = \partial \Omega$ of $\Omega$ is sufficiently smooth. An application of the divergence theorem to the integral involving div $S$ in (1.7) gives

$$\frac{d}{dt} \mathcal{E}(t) + \int_{\Gamma} n(x) \cdot S(x,t) \, d\Gamma + \int_{\Omega} j(x,t) \cdot e(x,t) \, dx = 0,$$

where $n(x)$ denotes the outward directed unit normal at $x \in \Gamma$. The boundary integral in (1.8) characterizes the outgoing flux of electric power through $\Gamma$ (see, e.g., [2, Chap. 1.3.1], [12, Chap. 6.8], and [17, Teil I, §5]).

2) Notice $\text{div}(a \times b) = b \cdot \text{curl } a - a \cdot \text{curl } b$. 3
We note that (1.8) is formally equivalent to the following energy equality

\begin{equation}
\mathcal{E}(t) + \int_{t}^{s} \int_{\Gamma} n(x) \cdot S(x, \tau) \, d\Gamma \, d\tau + \int_{s}^{t} \int_{\Omega} j(x, \tau) \cdot e(x, \tau) \, dx \, d\tau = \mathcal{E}(s)
\end{equation}

for all \( s, t \in [0, T], \ s < t. \)

We now consider a classical solution \( \{e, h\} \in C^1(\overline{Q}_T)^3 \times C^1(\overline{Q}_T)^3 \) that satisfies the conditions

\begin{align}
\text{(1.10)} & \quad n \times e = 0 \quad \text{on} \ \Gamma \times [0, T], \\
\text{(1.11)} & \quad e = e_0, \quad h = h_0 \quad \text{in} \ \Omega \times \{0\},
\end{align}

where \( \{e_0, h_0\} \) are given data in \( \Omega. \) We obtain

\begin{align*}
\int_{\Gamma} n \cdot S \, d\Gamma &= \int_{\Gamma} n \cdot (e \times h) \, d\Gamma \quad \text{(definition of} \ S) \\
&= \frac{1}{2} \int_{\Gamma} (-n \times h) \cdot e + (n \times e) \cdot h \, d\Gamma \quad \text{3)} \\
&= 0.
\end{align*}

Thus, energy equality (1.9) takes the form

\begin{equation}
\mathcal{E}(t) + \int_{s}^{t} \int_{\Omega} j(x, \tau) \cdot e(x, \tau) \, dx \, d\tau = \mathcal{E}(s) \quad \forall s, t \in [0, T], \ s \leq t,
\end{equation}

or, equivalently,

\begin{equation}
\mathcal{E}(t) + \int_{0}^{t} \int_{\Omega} j(x, \tau) \cdot e(x, \tau) \, dx \, d\tau = \mathcal{E}(0) \quad \forall t \in [0, T],
\end{equation}

where

\[ \mathcal{E}(0) = \frac{1}{2} \int_{\Omega} \left[ (\varepsilon(x) e_0(x)) \cdot e_0(x) + (\mu(x) h_0(x)) \cdot h_0(x) \right] \, dx. \]

1.3. An integral identity for classical solutions of (1.4), (1.5), (1.10), (1.11)

Let \( \{e, h\} \in C^1(\overline{Q}_T)^3 \times C^1(\overline{Q}_T)^3 \) be a classical solution of (1.4), (1.5). Given \( \{\phi, \psi\} \in C^1(\overline{Q}_T)^3 \times C^1(\overline{Q}_T)^3 \) such that \( \phi(x, T) = \psi(x, T) = 0 \) for all \( x \in \Omega, \) we multiply (1.4) and (1.5) scalarly by \( \phi \) and \( \psi, \) respectively, add the obtained equations, integrate over \( Q_T \) and observe

\[ e \cdot (h \times n) = n \cdot (e \times h) = h \cdot (n \times e) \quad \text{(cf. footnote 2)} \]

Observe

\[ e \cdot (h \times n) = n \cdot (e \times h) = h \cdot (n \times e) \quad \text{(cf. footnote 2)} \]
carry out an integration by parts with respect to $t$ over the interval $[0,T]$. It follows

\[- \int_{Q_T} ((\varepsilon e) \cdot \partial_t \phi + (\mu h) \cdot \partial_t \psi) \, dx \, dt + \int_{Q_T} ((-\text{curl } h) \cdot \phi + (\text{curl } e) \cdot \psi) \, dx \, dt \]

\[
(1.14) \quad + \int_{Q_T} j \cdot \phi \, dx \, dt = \int_{\Omega} \left( (\varepsilon e) \cdot \phi + (\mu h) \cdot \psi \right) \, dx \bigg|_{t=0}.
\]

To proceed further, we will need the following Green formula

\[
\int_{\Omega} (\text{curl } a) \cdot b \, dx - \int_{\Omega} a \cdot (\text{curl } b) \, dx = \int_{\Gamma} (n \times a) \cdot b \, d\Gamma, \quad a, b \in C^1(\overline{\Omega})^3.
\]

We make use of this formula with $a = -h(\cdot, t)$, $b = \phi(\cdot, t)$ resp. $a = e(\cdot, t)$, $b = \psi(\cdot, t)$ ($t \in [0,T]$), and integrate then over the interval $[0,T]$. We obtain

\[
\int_{Q_T} (-(\text{curl } h) \cdot \phi + (\text{curl } e) \cdot \psi) \, dx \, dt = \int_0^T \int_\Gamma ((-n \times h) \cdot \phi + (n \times e) \cdot \psi) \, d\Gamma \, dt + \int_{Q_T} (-(h \cdot (\text{curl } \phi) + e \cdot (\text{curl } \psi)) \, dx \, dt.
\]

Substituting this into (1.14) gives

\[
- \int_{Q_T} ((\varepsilon e) \cdot \partial_t \phi + (\mu h) \cdot \partial_t \psi) \, dx \, dt + \int_{Q_T} T \int_0^T (-(n \times h) \cdot \phi + (n \times e) \cdot \psi) \, d\Gamma \, dt
\]

\[
+ \int_{Q_T} (-(h \cdot (\text{curl } \phi) + e \cdot (\text{curl } \psi)) \, dx \, dt + \int_{Q_T} j \cdot \phi \, dx \, dt
\]

\[
= \int_{\Omega} \left( (\varepsilon e) \cdot \phi + (\mu h) \cdot \psi \right) \, dx \bigg|_{t=0}.
\]

Thus, since $-(n \times h) \cdot \phi = (n \times \phi) \cdot h$ on $\Gamma$, it follows that every classical solution $\{e, h\}$ of (1.4), (1.5), (1.10), (1.11) satisfies the integral identity

\[
- \int_{Q_T} ((\varepsilon) \cdot \partial_t \phi + (\mu) \cdot \partial_t \psi) \, dx \, dt + \int_{Q_T} T \int_0^T (-(n \times h) \cdot \phi + (n \times e) \cdot \psi) \, d\Gamma \, dt
\]

\[
+ \int_{Q_T} (-(h \cdot (\text{curl } \phi) + e \cdot (\text{curl } \psi)) \, dx \, dt + \int_{Q_T} j \cdot \phi \, dx \, dt
\]

\[
= \int_{\Omega} \left[ ([\varepsilon(x)e_0(x)] \cdot \phi(x,0) + (\mu(x)h_0(x)) \cdot \psi(x,0) \right] \, dx
\]

\[
(1.15)
\]

for all $\{\phi, \psi\} \in C^1(\overline{Q_T})^3 \times C^1(\overline{Q_T})^3$ such that

\[
\square
\]
\[ \begin{align*}
\bullet \phi &= \psi = 0 \quad \text{in } \Omega \times \{T\}, \\
\bullet \ n \times \phi &= 0 \quad \text{on } \Gamma \times [0, T].
\end{align*} \]

Integral identity (1.15) evidently continues to make sense for \( \{e, h\} \in L^2(Q_T)^3 \times L^2(Q_T)^3 \). This motivates the definition of the notion of weak solution of (1.4), (1.5), (1.10), (1.11) we will give in the following section.

2. Definition and basic properties of weak solutions of (1.4), (1.5), (1.10), (1.11)

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain. We introduce the space

\[ V := \left\{ u \in L^2(\Omega)^3; \, \text{curl } u \in L^2(\Omega)^3 \right\}. \]

\( V \) is a Hilbert space with respect to the scalar product

\[ (u, v)_V := \int_{\Omega} (u \cdot v + (\text{curl } u) \cdot (\text{curl } v)) \, dx \]

We next define

\[ V_0 := \left\{ u \in V; \, \int_{\Omega} (\text{curl } u) \cdot z \, dx = \int_{\Omega} u \cdot (\text{curl } z) \, dx \quad \forall \, z \in V \right\}. \]

To our knowledge, this space has been introduced for the first time in [15], and then used in other papers, e.g., [13], [18]. The vector fields \( u \in V_0 \) satisfy the condition \( n \times u = 0 \) on the boundary \( \Gamma = \partial \Omega \) of \( \Omega \) in a generalized sense. More specifically, if \( \Gamma \) is Lipschitz continuous, then there exists a linear continuous mapping \( \gamma_\tau : V \to H^{-1/2}(\Gamma)^3 \) such that

\[ \gamma_\tau(u) = n \times u|_\Gamma \quad \forall u \in C^1(\Omega)^3, \]

\[ \int_{\Omega} (\text{curl } u) \cdot \varphi \, dx - \int_{\Omega} u \cdot (\text{curl } \varphi) \, dx = \langle \gamma_\tau(u), \varphi \rangle_{(H^{1/2}(\Gamma))^3} \quad \forall u \in V, \, \forall \varphi \in H^1(\Omega)^3 \]

(cf., e.g., [4, Chap. IX, 2.], [11, Chap. I, Th. 2.11]). It follows

\[ V_0 := \left\{ u \in V; \, \gamma_\tau(u) = 0 \text{ in } H^{-1/2}(\Gamma)^3 \right\}. \]

Based upon this result, in Appendix II we give an equivalent characterization of \( V_0 \).

For our discussion of weak solutions of (1.4), (1.5), (1.10), (1.11) we will need spaces of functions from the interval \([0, T]\) into a real normed vector space \( X \).

Let \( | \cdot |_X \) denote the norm in \( X \). By \( L^p(0, T; X) \) \( (1 \leq p < +\infty) \) we denote the vector space of all equivalence classes of strongly (Bochner) measurable functions \( u : [0, T] \to X \)

\[ 4) \langle \cdot, \cdot \rangle_{(H^{1/2}(\Gamma))^3} = \text{dual pairing between } H^{-1/2}(\Gamma)^3 \text{ and } H^{1/2}(\Gamma)^3 \text{ (cf. also below).} \]
such that $t \mapsto |u(t)|_X$ is in $L^p(0, T)$. A norm on $L^p(0, T; X)$ is given by

$$
\|u\|_{L^p(0, T; X)} := \begin{cases}
\left( \int_0^T |u(t)|_X^p \right)^{1/p} & \text{if } 1 \leq p < +\infty, \\
\text{ess sup} \{u(t)|_{X} \} & \text{if } p = +\infty.
\end{cases}
$$

For more details see, e.g., [3, Chap. 4], [6, Chap. 1] and [19, Chap. 23.2, 23.3].

If $H$ denotes a real Hilbert space with scalar product $(\cdot, \cdot)_H$, then $L^2(0, T; H)$ is a Hilbert space for the scalar product

$$(u, v)_{L^2(0, T; H)} := \int_0^T (u(t), v(t))_H \, dt.$$

Given $u \in L^p(Q_T)$ $(1 \leq p < +\infty)$, we define

$$[u](t) := u(\cdot, t) \text{ for a.e. } t \in [0, T].$$

By Fubini’s theorem, $[u](\cdot) \in L^p(0, T)$ and

$$\int_{Q_T} |u(x, t)|^p \, dx \, dt = \int_0^T \|u(t)\|_{L^p(\Omega)}^p \, dt.$$

It is easy to prove that the mapping $u \mapsto [u]$ is a linear isometry from $L^p(Q_T)$ onto $L^p(0, T; L^p(\Omega))$. Throughout our paper we identify these spaces.

For what follows, we suppose that

\begin{equation}
(2.1) \quad \begin{cases}
\text{the entries of the matrices } \varepsilon = \varepsilon(x), \mu = \mu(x) \\
\text{are bounded measurable functions in } \Omega;
\end{cases}
\end{equation}

\begin{equation}
(2.2) \quad j \in L^2(Q_T)^3, \, e_0, h_0 \in L^2(\Omega)^3.
\end{equation}

The following definition extends the integral identity (1.15) to the $L^2$-framework.

**Definition 1** Assume (2.1), (2.2). The pair

$$\{e, h\} \in L^2(Q_T)^3 \times L^2(Q_T)^3$$

is called weak solution of (1.4), (1.5), (1.10), (1.11) if
\[
\begin{aligned}
  &- \int_{Q_T} ((\varepsilon e) \cdot \partial_t \phi + (\mu h) \cdot \partial_t \psi) \, dx \, dt \\
  &+ \int_{Q_T} (-h \cdot (\text{curl } \phi) + e \cdot (\text{curl } \psi)) \, dx \, dt + \int \, j \cdot \phi \, dx \, dt \\
  &= \int_{\Omega} [(\varepsilon(x)e_0(x)) \cdot \phi(x,0) + (\mu(x)h_0(x)) \cdot \psi(x,0)] \, dx
\end{aligned}
\]

(2.3)

\[
\begin{cases}
  \text{for all } \{\phi, \psi\} \in L^2(0,T;V_0) \times L^2(0,T;V) \text{ such that } \\
  \partial_t \phi, \partial_t \psi \in L^2(Q_T)^3, \; \phi(\cdot, T) = \psi(\cdot, T) = 0 \text{ a.e. in } \Omega.
\end{cases}
\]

From our discussion in Section 1 it follows that every classical solution of (1.4), (1.5), (1.10), (1.11) is a weak solution of this problem. We notice that (2.3) basically coincides with the definition of weak solutions of initial-boundary value problems for the Maxwell equations that is introduced in [7, Chap. VII,4.2], [9, p. 326] and [13]. Existence theorems for weak solutions of (1.4), (1.5), (1.10), (1.11) are presented in [7, Chap. VII,4.3], [8], [9, Section 7.8.3] and [13].

Existence of the distributional derivatives \((\varepsilon e)\prime, (\mu h)\prime\). We introduce more notations.

Let \(X\) be a real normed vector space with norm \(|\cdot|_X\). By \(X^\prime\) we denote the dual space of \(X\), and by \(\langle x^\ast, x \rangle_X\) the dual pairing between \(x^\ast \in X^\ast\) and \(x \in X\). Let \(H\) be a real Hilbert space with scalar product \((\cdot, \cdot)_H\) such that \(X \subset H\) continuously and densely. Identifying \(H\) with its dual space \(H^\prime\) via the Riesz representation theorem, we obtain

\[
H \subset X^\ast \text{ continuously, } \langle z, x \rangle_X = (z, x)_H \quad \forall z \in H, \forall x \in X
\]

(cf., e.g., [19, Chap. 23.4]). If \(X\) is reflexive, then \(H \subset X^\ast\) densely.

Next, let \(X\) and \(Y\) be two real normed vector spaces such that \(X \subset Y\) continuously and densely. Given \(u \in L^1(0,T;X)\), we identify \(u\) with an element in \(L^1(0,T;Y)\) and denote this element again by \(u\). If there exists \(U \in L^1(0,T;Y)\) such that

\[
\int_0^T \zeta(t)u(t) \, dt \overset{\text{in } Y}{=} -\int_0^T \zeta(t)U(t) \, dt \quad \forall \zeta \in C_c^\infty([0,T]),
\]

then \(U\) will be called the \textit{derivative of \(u\ in the sense of distributions from }[0,T][ into } Y\text{ and denoted by}

\[
u' := U
\]

(see, e.g., [4, Appendice], [6, Chap. 21], [16, Chap. 1.3] and [19, Chap. 23.5, 23.6]). \(u'\) is uniquely determined by \(u\).

The existence of the distributional derivative \(u' \in L^1(0,T;Y)\) is equivalent to the existence of a function \(\tilde{u} : [0,T] \rightarrow Y\) such that

- \(\tilde{u}(t) \overset{\text{in } Y}{=} u(t)\text{ for a.e. } t \in [0,T],\)
- \(\tilde{u}(t) = \tilde{u}(0) + \int_0^t u(s) \, ds \quad \forall t \in [0,T],\)

i.e., \(\tilde{u}\) is the absolutely continuous representative of equivalence class \(u\) (cf. [4, Appendice]).
Let $X$ and $H$ be as above, $X \subset H$ continuously and densely. Let be $u \in L^2(0,T;H)$ such that $u' \in L^2(0,T;X^*)$. For the proof of Theorem 1 we need the following formula of integration by parts

$$\left\langle \int_0^T \alpha(t)u'(t) \, dt, \varphi \right\rangle_X = \langle \alpha(T)\tilde{u}(T), \varphi \rangle_X - \langle \alpha(0)\tilde{u}(0), \varphi \rangle_X - \int_0^T (\tilde{u}(t), \dot{\alpha}(t)\varphi)_{H} \, dt$$

for all $\varphi \in X$ and all $\alpha \in C^1([0,T])$ ($\tilde{u}$ denotes the absolutely continuous representative of $u$ with values in $X^*$). This formula is easily seen by combining the equation

$$\int_0^T (\alpha(t)u'(t) + \dot{\alpha}(t)u(t)) \, dt \overset{\text{in } X^*}{=} \alpha(T)\tilde{u}(T) - \alpha(0)\tilde{u}(0)$$

and the equation in (2.4).

We consider the following special cases for $X$ and $H$:

$X = V$ (resp. $X = V_0$), $H = L^2(\Omega)^3$.

It follows

$$L^2(\Omega)^3 \subset V^* \quad \text{(resp. } L^2(\Omega)^3 \subset V_0^*) \quad \text{continuously, densely}$$

$$\langle z, u \rangle_V = \langle z, u \rangle_{L^2(\Omega)^3} \quad \forall z \in L^2(\Omega)^3, \ \forall u \in V$$

$$\langle z, u \rangle_{V_0} = \langle z, u \rangle_{L^2(\Omega)^3} \quad \forall z \in L^2(\Omega)^3, \ \forall u \in V_0).$$

Without any further reference, in what follows we suppose that $V$ is separable.

**Theorem 1** For every weak solution $\{e, h\}$ of (1.4), (1.5), (1.10), (1.11) there exist the distributional derivatives

$$\langle \varepsilon e \rangle' \in L^2(0,T;V_0^*), \quad \langle \mu h \rangle' \in L^2(0,T;V^*)$$

and there holds

$$\begin{align*}
\int_0^T \langle \varepsilon e \rangle'(t), \varphi \rangle_{V_0} + \int_\Omega (-h(x,t) \cdot \text{curl} \varphi(x) + j(x,t) \cdot \varphi(x)) \, dx &= 0 \\
&\text{for a.e. } t \in [0,T] \text{ and all } \varphi \in V_0,
\end{align*}$$

$$\begin{align*}
\int_0^T \langle \mu h \rangle'(t), \psi \rangle_{V} + \int_\Omega e(x,t) \cdot \text{curl} \psi(x) \, dx &= 0 \\
&\text{for a.e. } t \in [0,T] \text{ and all } \psi \in V.
\end{align*}$$

---

5) The Lipschitz continuity of the boundary $\Gamma$ is sufficient for the density of $C^1(\overline{\Omega})^3$ in $V$. 

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Moreover,
\[(2.9)\quad (\varepsilon e) \in C([0, T]; V_0^*), \quad (\mu h) \in C([0, T]; V^*),\]
\[(2.10)\quad (\varepsilon e)(0) = \varepsilon e_0 \text{ in } V_0^*, \quad (\mu h)(0) = \mu h_0 \text{ in } V^*.\]

**Remark 2**
1. Let \(\mathcal{N}\) denote the set of those \(t \in [0, T]\) for which the equation in (2.7) fails. Then \(\text{mes}\mathcal{N} = 0\), and \(\mathcal{N}\) does not depend on \(\varphi \in V_0\). This follows from the separability of \(V_0\). An analogous observation is true for (2.8).

2. In (2.9) and (2.10), the absolutely continuous representatives of \(\varepsilon e\) and \(\mu h\) with respect to the norms in \(V_0^*\) and \(V^*\), respectively, are understood.

**Proof of Theorem**
We identify \(\varepsilon e\) with an element in \(L^2(0, T; V_0^*)\) and prove the existence of the distributional derivative \((\varepsilon e)' \in L^2(0, T; V_0^*)\).

We define \(F \in (L^2(0, T; V_0^*))^*\) by
\[
\langle F, \phi \rangle_{L^2(V_0^*)} := \int_{Q_T} (-h \cdot \text{curl } \phi + j \cdot \phi) \, dx \, dt, \quad \phi \in L^2(0, T; V_0^*).\]

The linear isometry \((L^2(0, T; V_0^*))^* \cong L^2(0, T; V_0^*)\) permits to identify \(F\) with its isometric image in \(L^2(0, T; V_0^*)\) which we again denote by \(F\). Thus,
\[
\langle F, \phi \rangle_{L^2(V_0^*)} = \int_0^T \langle F(t), \phi(t) \rangle_{V_0^*} \, dt \quad \forall \phi \in L^2(0, T; V_0^*).\]

Given any \(\varphi \in V_0\) and \(\zeta \in C_\infty([0, T])\), in (2.3) we take \(\phi(x, t) = \varphi(x)\zeta(t), \psi(x, t) = 0\) for a.e. \((x, t) \in Q_T\). It follows
\[
\left\langle \int_0^T \zeta(t)(\varepsilon e)(t), \varphi \right\rangle_{V_0^*} = \int_0^T \left\langle \zeta(t)(\varepsilon e)(t), \varphi \right\rangle_{V_0^*} \, dt \quad \text{(cf. [19], p. 421)}
\]
\[
= \int_0^T \left( \zeta(t)(\varepsilon e)(t) \right)_{L^2(\Omega)^3} \, dt \quad \text{(by (2.4))}
\]
\[
= \int_{Q_T} (-h \cdot \text{curl } \phi + j \cdot \phi) \, dx \, dt \quad \text{(by (2.3))}
\]
\[
= \left\langle \int_0^T \zeta(t) F(t) \, dt, \varphi \right\rangle_{V_0^*},
\]
i.e., \(\varepsilon e \in L^2(0, T; L^2(\Omega)^3)\) (when identified with an element in \(L^2(0, T; V_0^*)\)) possesses the distributional derivative
\[(\varepsilon e)' = -F \in L^2(0, T; V_0^*),\]

---

\(6^*) If there is no danger of confusion, for indices we write \(L^p(X)\) in place of \(L^p(0, T; X)\).
and there holds
\[ \int_0^T \langle (\varepsilon e)'(t), \varphi \rangle_{V_0} \zeta(t) \, dt = - \int_0^T \langle \mathcal{F}(t), \varphi \zeta(t) \rangle_{V_0} \, dt \]
\[ = \int_0^T \left( \int_{Q_T} (h(x, t) \cdot \text{curl} \varphi(x) - j(x, t) \cdot \varphi(x)) \, dx \right) \zeta(t) \, dt \]

(2.11)

for any \( \varphi \in V_0 \) and \( \zeta \in C^\infty_c([0, T]) \). Thus, by a routine argument, (2.11) is equivalent to the equation in (2.7) where the set of measure zero of those \( t \in [0, T] \) for which this equation is not true, does not depend on \( \varphi \) (cf. Remark 2.1). By an analogous reasoning one proves \( (\mu h)' \in L^2(0, T; V^*) \).

To prove the first equation in (2.10), fix \( \alpha \in C^1_0([0, T]) \) with \( \alpha(0) = 1 \), \( \alpha(T) = 0 \). We multiply the equation in (2.7) by \( \alpha(t) \) and integrate over \([0, T] \). Thus, for any \( \varphi \in V_0 \),
\[ \int_0^T \langle (\varepsilon e)'(t), \varphi \alpha(t) \rangle_{V_0} \, dt = \int_{Q_T} (h(x, t) \cdot \text{curl} \varphi(x) - j(x, t) \cdot \varphi(x)) \alpha(t) \, dx \, dt \]
\[ = - \int_{Q_T} (\varepsilon e) \cdot \varphi \alpha \, dx \, dt - \int_\Omega (\varepsilon(x)e_0(x)) \cdot \varphi(x) \, dx \]

(2.12)

On the other hand, the formula of integration by parts (2.5) reads
\[ \int_0^T \langle (\varepsilon e)'(t), \varphi \alpha(t) \rangle_{V_0} \, dt = - \langle (\tilde{\varepsilon} e)(0), \varphi \rangle_{V_0} - \int_{Q_T} (\varepsilon e) \cdot \varphi \alpha \, dx \, dt. \]

(2.13)

Combining (2.12) and (2.13) gives
\[ \langle \varepsilon e_0, \varphi \rangle_{V_0} = \langle (\tilde{\varepsilon} e)(0), \varphi \rangle_{V_0}, \]
i.e., the first equation in (2.10) holds in the sense of \( V_0^* \). An analogous argument gives the second equation in (2.10).

3. An energy equality

For \( \{e, h\} \in L^2(Q_T)^3 \times L^2(Q_T)^3 \) we define
\[ E(t) = \frac{1}{2} \int_\Omega [(\varepsilon(x)e(x, t)) \cdot e(x, t) + (\mu(x)h(x, t)) \cdot h(x, t)] \, dx \quad \text{for a.e.} \ t \in [0, T] \]
(c.f. Section 2.2). Then \( E \in L^1(0, T) \).

The main result of our paper is the following
Theorem 2  Assume (2.1), (2.2) and 
\[ \varepsilon(x), \mu(x) \text{ are symmetric non-negative matrices for all } x \in \Omega. \]

Then, for every weak solution \( \{ e, h \} \) of (1.4), (1.5), (1.10), (1.11), there exists an absolute continuous function

\[ \tilde{E} : [0, T] \to [0, +\infty[ \]

such that

\[ \tilde{E}(t) = E(t) \text{ for a.e. } t \in [0, T], \]

\[ \tilde{E}(t) = \tilde{E}(s) - \int_s^t \int_{\Omega} j \cdot e \, dx \, d\tau \quad \forall s, t \in [0, T], \; s < t \]

\[ \max_{t \in [0, T]} \tilde{E}(t) \leq \tilde{E}(0) + \| \varepsilon \|_{L^2(Q_T)} \| j \|_{L^2(Q_T)}^3, \]

\[ E(t) = E(s) - \int_s^t \int_{\Omega} j \cdot e \, dx \, d\tau \text{ for a.e. } s, t \in [0, T], \; s < t. \]

Proof. We make use of a well-known technique for proving energy inequalities for weak solutions of parabolic equations by regularizing these solutions in time by the Steklov mean (see, e.g., [14, Chap. III, §2]). We divide the proof into three parts.

1° Integral identities for the Steklov mean of \( \{ e, h \} \).

We extend \( \{ e, h \} \) by zero for a.e. \( (x, t) \in \Omega \times (\mathbb{R} \setminus [0, T]) \) and denote this extension by \( \{ e, h \} \) again.

Let be \( \alpha \in C^\infty(\mathbb{R}) \) with \( \text{supp}(\alpha) \subset [0, T] \), i.e. there exists \( 0 < t_0 < t_1 < T \) such that \( \alpha(t) = 0 \) for all \( t \in ]-\infty, t_0]\cup]t_1, +\infty[ \). Given \( \varphi \in V_0 \), for \( 0 < \lambda < T - t_1 \) we consider the function

\[ \phi(x, t) = \varphi(x) \int_{t-\lambda}^t \alpha(s) \, ds \text{ for a.e.} (x, t) \in Q_T. \]

Then

\[ \phi(\cdot, t) \in V_0 \text{ for all } t \in [0, T], \quad \phi(x, 0) = \phi(x, T) = 0 \text{ for a.e. } x \in \Omega, \]

\[ \partial_t \phi(x, t) = \varphi(x)(\alpha(t) - \alpha(t - \lambda)) \text{ for a.e. } (x, t) \in Q_T. \]

Inserting this \( \phi \) and the function \( \psi = 0 \) into [23] yields

\(^7\) Cf. [15,2].
\[
\int_{Q_T} \left[ (\varepsilon e)(x, t + \lambda) - (\varepsilon e)(x, t) \right] \cdot \varphi(x) \alpha(t) \, dx \, dt \\
- \int_{Q_T} h(x, t) \cdot \text{curl} \varphi(x) \left( \int_{t-\lambda}^{t} \alpha(s) \, ds \right) \, dx \, dt \\
+ \int_{Q_T} j(x, t) \cdot \varphi(x) \left( \int_{t-\lambda}^{t} \alpha(s) \, ds \right) \, dx \, dt \\
= 0
\] (3.6)

We divide each term of this equation by \(\lambda\) (\(0 < \lambda < T - t_1\)) and make use of Prop. I.1.2, 1.3 (Appendix I). Then (3.6) reads

\[
\int_{Q_T} \left( \left( \frac{\partial}{\partial t} (\varepsilon e)_{\lambda}(x, t) \right) \cdot \varphi(x) - h_{\lambda}(x, t) \cdot \text{curl} \varphi(x) + j_{\lambda}(x, t) \cdot \varphi(x) \right) \alpha(t) \, dx \, dt = 0,
\] (3.7)

where

\[
f_{\lambda}(x, t) = \frac{1}{\lambda} \int_{t}^{t+\lambda} f(x, s) \, ds, \quad \lambda > 0, \; (x, t) \in Q_T
\]

denotes the Steklov mean of \(f \in L^p(Q_T)\) (\(1 \leq p < +\infty\)) (cf. [14, Chap. II, §4] and Appendix I below for details).

By an analogous reasoning we conclude from (2.3) that

\[
\int_{Q_T} \left( \left( \frac{\partial}{\partial t} (\mu h)_{\lambda}(x, t) \right) \cdot \psi(x) + e_{\lambda}(x, t) \cdot \text{curl} \psi(x) \right) \alpha(t) \, dx \, dt = 0,
\] (3.8)

for all \(\psi \in V\) and all \(0 < \lambda < T - t_1\).

To proceed further, we take any sequence \((\lambda_m)_{m \in \mathbb{N}}\) (\(0 < \lambda_m < T - t_1\)) such that \(\lambda_m \to 0\) as \(m \to \infty\). By a routine argument (cf. Section 1.3), from (3.7) and (3.8) with \(\lambda = \lambda_m\) \((m \text{ fixed})\) it follows that

\[
\int_{\Omega} \left( \left( \frac{\partial}{\partial t} (\varepsilon e)_{\lambda_m}(x, t) \right) \cdot \varphi(x) - h_{\lambda_m}(x, t) \cdot \text{curl} \varphi(x) + j_{\lambda_m}(x, t) \cdot \varphi(x) \right) \, dx = 0
\]

for a.e. \(t \in [0, T]\), for all \(\varphi \in V_0\) and all \(m \in \mathbb{N}\),

\[
\int_{\Omega} \left( \left( \frac{\partial}{\partial t} (\mu h)_{\lambda_m}(x, t) \right) \cdot \psi(x) + e_{\lambda_m}(x, t) \cdot \text{curl} \psi(x) \right) \, dx = 0
\]

for a.e. \(t \in [0, T]\), for all \(\psi \in V\) and all \(m \in \mathbb{N}\),

\[13\]
respectively. We note that the set of measure zero of those \( t \in [0, T] \) for which both (3.9) and (3.10) fail, is independent of \( \varphi \in V_0, \psi \in V \) and \( m \in \mathbb{N} \).

2° \( e_\lambda(\cdot, t) \in V_0, h_\lambda(\cdot, t) \in V \) for all \( 0 < \lambda < T - t_1 \), for a.e. \( t \in [0, T] \) and all \( m \in \mathbb{N} \).

Indeed, (3.10) implies

\[
\left| \int_\Omega e_\lambda(x, t) \cdot \text{curl} \psi(x) \, dx \right| \leq \| \partial_t (\mu h)_\lambda(\cdot, t) \|_{L^2(\Omega)^3} \| \psi \|_{L^2(\Omega)^3}
\]

for all \( \psi \in V \). Thus, by App. II, Prop. II.1, \( e_\lambda(\cdot, t) \in V_0 \), i.e., \( \text{curl} e_\lambda(\cdot, t) \in L^2(\Omega)^3 \) and

\[
(3.11) \quad \int_\Omega (\text{curl} e_\lambda(x, t)) \cdot \psi(x) \, dx = \int_\Omega e_\lambda(x, t) \cdot \text{curl} \psi(x) \, dx \quad \forall \psi \in V
\]

(for a.e. \( t \in [0, T] \) and all \( m \in \mathbb{N} \)). The claim \( \text{curl} h_\lambda(\cdot, t) \in L^2(\Omega)^3 \) is readily seen by using Riesz’ representation theorem.

We now insert \( \psi = h_\lambda(\cdot, t) \) into (3.11) and obtain

\[
\int_\Omega (\text{curl} e_\lambda(x, t)) \cdot h_\lambda(x, t) \, dx = \int_\Omega e_\lambda(x, t) \cdot \text{curl} h_\lambda(x, t) \, dx.
\]

On the other hand, since

\[
(\varepsilon e)_\lambda(x, t) = \varepsilon(x)e_\lambda(x, t), \quad (\mu h)_\lambda(x, t) = \mu(x)h_\lambda(x, t)
\]

for a.e. \( (x, t) \in Q_T \), we obtain by virtue of the symmetry of \( \varepsilon(x) \) and \( \mu(x) \)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega [(\varepsilon(x)e_\lambda(x, t)) \cdot e_\lambda(x, t) + (\mu(x)h_\lambda(x, t)) \cdot h_\lambda(x, t)] \, dx
+ \int_\Omega j_\lambda(x, t) \cdot e_\lambda(x, t) \, dx
\]

(3.12)

for a.e. \( t \in [0, T] \) and all \( m \in \mathbb{N} \).

Finally, given any \( \zeta \in C_c^\infty([0, T]) \), we multiply (3.12) by \( \zeta(t) \) and carry out an integration by parts of the first integral on the left hand side. Thus,

\[
-\frac{1}{2} \int_0^T \left( \int_\Omega [(\varepsilon(x)e_\lambda(x, t)) \cdot e_\lambda(x, t) + (\mu(x)h_\lambda(x, t)) \cdot h_\lambda(x, t)] \, dx \right) \dot{\zeta}(t) \, dt
+ \int_0^T \left( \int_\Omega j_\lambda(x, t) \cdot e_\lambda(x, t) \, dx \right) \zeta(t) \, dt
\]

(3.13)

for all \( m \in \mathbb{N} \).

8) For notational simplicity, in the present part 2° of our proof we omit the index \( m \) at \( \lambda_m \).
3º Passing to limits \( m \to \infty \).

Observing that

\[
\begin{align*}
e_{\lambda_m} & \to e, \quad h_{\lambda_m} \to h, \quad j_{\lambda_m} \to j \quad \text{in } L^2(Q_T)^3 \quad \text{as } m \to \infty \\
\end{align*}
\]

(cf. App. I, Prop. I.2), the passage to limits \( m \to \infty \) in (3.13) (with \( \lambda = \lambda_m \)) gives

\[
-\int_0^T E(t) \dot{\zeta}(t) \, dt + \int_0^T \left( \int_{\Omega} j(x,t) \cdot e(x,t) \, dx \right) \zeta(t) \, dt = 0 \quad \forall \zeta \in C_c^\infty([0,T]).
\]

It follows that the equivalence class \( E \in L^1(0,T) \) possesses an absolutely continuous representative \( \tilde{E} : [0,T] \to [0,+\infty[ \) such that

\[
\tilde{E}(t) = \tilde{E}(s) - \int_s^t \int_{\Omega} j(x,t) \cdot e(x,t) \, dx \, dt \quad \forall s,t \in [0,T], \ s < t,
\]

i.e., (3.3) holds.

Estimate (3.4) and the equality in (3.5) are direct consequences of (3.2) and (3.3). The proof of Theorem 2 is complete.

4. Uniqueness of weak solutions

Let be \( \varepsilon(x), \mu(x) \ (x \in \Omega) \) as in (2.1) and (3.1). In addition, suppose that

\[
\exists \varepsilon_* = \text{const} > 0, \ \mu_* = \text{const} > 0, \ \text{such that}
\]

\[
(\varepsilon(x)\xi) \cdot \xi \geq \varepsilon_* |\xi|^2, \ (\mu(x)\xi) \cdot \xi \geq \mu_* |\xi|^2 \quad \forall x \in \Omega, \ \forall \xi \in \mathbb{R}^3.
\]

We consider equ. (1.4) with Ohm law

\[
j = \sigma e,
\]

where the entries of the matrix \( \sigma = \{\sigma_{kl}(x)\}_{k,l=1,2,3} \ (x \in \Omega) \) are bounded measurable functions in \( \Omega \).

**Theorem 3** Suppose that the matrices \( \varepsilon(x), \mu(x) \ (x \in \Omega) \) satisfy (2.1), (3.1) and (4.1).

Let \( \{e,h\} \in L^2(Q_T)^3 \times L^2(Q_T)^3 \) be a weak solution of (1.4), (1.5), (1.10), (1.11) with initial data

\[
e_0 = h_0 = 0 \quad \text{a.e. in } \Omega.
\]

Then

\[
e = h = 0 \quad \text{a.e. in } Q_T.
\]

To prove this theorem, we first derive an energy inequality for the primitives of the functions \( t \mapsto e(x,t), \ t \mapsto \mu(x,t) \ (x \in \Omega) \) (cf. the proof of Theorem 2 above; see also [9 Sect 7.8.2]). From this inequality the claim \( e = h = 0 \) a.e. in \( Q_T \) follows easily (cf. also [7 Chap. VII, 4.3] for a different argument). A uniqueness result for solutions of linear second order evolution equations in Hilbert spaces has been proved in [10 Chap. 3, 8.2].
Proof of Theorem 3. Given any $t \in [0, T]$, we consider (2.7) for a.e. $s \in [0, t]$, multiply this identity by $t - s$, integrate over the interval $[0, t]$ and use an integration by parts in the integral
\[ \int_0^t \langle (\varepsilon e)'(s), (t - s)\varphi \rangle_{V_0} \, ds \]
(observe $(\varepsilon e)(0) = \varepsilon e_0 = 0$ in $V_0^*$; cf. (2.10)). It follows
\[ \int_0^t \int_\Omega ((\varepsilon e)(x, s)) \cdot \varphi(x) \, dx \, ds \]
\[ + \int_0^t \int_\Omega (-h(x, s) \cdot \text{curl} \varphi(x) + (\sigma e)(x, s) \cdot \varphi(x))(t - s) \, dx \, ds \]
(4.2) \[= 0 \]
for all $\varphi \in V_0$. Differentiating each term on the left hand side with respect to $t$ we find
\[ \int_\Omega ((\varepsilon e)(x, t)) \cdot \varphi(x) \, dx \]
\[ + \int_0^t \int_\Omega (-h(x, s) \cdot \text{curl} \varphi(x) + (\sigma e)(x, s) \cdot \varphi(x))(t - s) \, dx \, ds \]
(4.3) \[= 0 \]
for all $\varphi \in V_0$ and for a.e. $t \in [0, T]$. From (2.8) we obtain analogously
\[ \int_\Omega (\mu \dot{h})(x, t) \cdot \psi(x) \, dx + \int_0^t \int_\Omega e(x, s) \cdot \text{curl} \psi(x) \, dx \, ds = 0 \]
(4.4) for all $\psi \in V$ and for a.e. $t \in [0, T]$.

For $t \in [0, T]$ and for a.e. $x \in \Omega$ we define
\[ \hat{e}(x, t) := \int_0^t e(x, s) \, ds, \quad \hat{h}(x, t) := \int_0^t h(x, s) \, ds. \]
Then $\hat{e}, \hat{h} \in L^2(Q_T)^3$, and the weak time-derivatives of these functions are $\partial_t \hat{e} = e$, $\partial_t \hat{h} = h$ a.e. in $Q_T$ (cf. App. I, Prop. I.1.2). Using Fubini’s theorem, (4.3) and (4.4) can
be rewritten in the form

\[ (4.5) \quad \int_{\Omega} \hat{h}(x,t) \cdot \text{curl} \varphi(x) \, dx = \int_{\Omega} \left( (\varepsilon e)(x,t) + (\sigma \hat{e})(x,t) \right) \cdot \varphi(x) \, dx \quad \forall \varphi \in V_0, \]

\[ (4.6) \quad \int_{\Omega} \hat{e}(x,t) \cdot \text{curl} \psi(x) \, dx = -\int_{\Omega} (\mu h)(x,t) \cdot \psi(x) \, dx \quad \forall \psi \in V \]

for a.e. \( t \in [0,T] \), respectively.

From (4.5) and (4.6) we conclude

\[ \text{curl} \hat{h}(\cdot,t) \in V \quad [\text{by Riesz’ representation theorem}], \]

\[ \text{curl} \hat{e}(\cdot,t) \in V_0 \quad [\text{by App. II, Prop. II.1}] \]

(i.e., \( \text{curl} \hat{e}(\cdot,t) \in V \), and

\[ (4.7) \quad \int_{\Omega} (\text{curl} \hat{e}(x,t)) \cdot \psi(x) \, dx = \int_{\Omega} \hat{e}(x,t) \cdot \text{curl} \psi(x) \, dx \quad \forall \psi \in V. \]

respectively. Thus, \( \varphi = \hat{e}(\cdot,t) \) and \( \psi = \hat{h}(\cdot,t) \) are admissible test functions in (4.5) and (4.6). Adding then these equations and observing (4.7) we find

\[ (4.8) \quad \int_{\Omega} ((\varepsilon e)(x,t)) \cdot \hat{e}(x,t) \, dx + \int_{\Omega} (\sigma \hat{e})(x,t) \cdot \hat{e}(x,t) \, dx + \int_{\Omega} (\mu h)(x,t) \cdot \hat{h}(x,t) \, dx = 0 \]

for a.e. \( t \in [0,T] \).

To proceed, we notice that for every \( x \in \Omega \) the functions \( t \mapsto \hat{e}(x,t) \), \( t \mapsto \hat{h}(x,t) \) are Hölder continuous with exponent \( \frac{1}{2} \) on the interval \([0,T]\). By (3.1) (symmetry of \( \varepsilon(x) \), \( \mu(x) \)),

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} ((\varepsilon \hat{e})(x,t)) \cdot \hat{e}(x,t) + (\mu \hat{h})(x,t) \cdot \hat{h}(x,t) \, dx \]

\[ = \int_{\Omega} ((\varepsilon \hat{e})(x,t)) \cdot \hat{e}(x,t) + (\mu \hat{h})(x,t) \cdot \hat{h}(x,t) \, dx \]

for a.e. \( t \in [0,T] \). From (4.8) it follows by integration that

\[ \frac{1}{2} \int_{\Omega} ((\varepsilon \hat{e})(x,t)) \cdot \hat{e}(x,t) + (\mu \hat{h})(x,t) \cdot \hat{h}(x,t) \, dx \]

\[ = -\int_{0}^{t} \int_{\Omega} ((\sigma \hat{e})(x,s)) \cdot \hat{e}(x,s) \, dx \, ds \]

(4.9)
for all $t \in [0,T]$. Observing (4.11) we derive from (4.9) by the aid of Gronwall’s inequality

$$\int_\Omega (|\hat{e}(x,t)|^2 + |\hat{h}(x,t)|^2) \, dx = 0 \quad \forall t \in [0,T].$$

Thus, by Fubini’s theorem,

$$\int_0^t e(x,s) \, ds = \int_0^t h(x,s) \, ds = 0 \quad \text{for a.e. } (x,t) \in Q_T.$$

We extend $e, h$ by zero onto $\Omega \times [T, +\infty[$ and denote this extension by these letters again. Hence, for all $\lambda > 0$,

$$\int_t^{t+\lambda} e(x,s) \, ds = \int_t^{t+\lambda} h(x,s) \, ds = 0 \quad \text{for a.e. } (x,t) \in Q_T,$$

i.e., the Steklov means $e_\lambda, h_\lambda$ vanish a.e. in $Q_T$. Whence

$$e = h = 0 \quad \text{a.e. in } Q_T$$

(see App. I, Prop. I.2).

\[\square\]

**Appendix I. The Steklov mean of an $L^p$ function**

Let $\Omega \subseteq \mathbb{R}^N \ (N \geq 2)$ be any open set, let $0 < T < +\infty$ and put $Q_T = \Omega \times ]0,T[$.

Let $f \in L^p(Q_T) \ (1 \leq p < +\infty)$. We extend $f$ by zero a.e. onto $\Omega \times ]T, +\infty[$ and denote this function by $f$ again. For $\lambda > 0$, the function

$$f_\lambda(x,t) = \frac{1}{\lambda} \int_t^{t+\lambda} f(x,s) \, ds \quad \text{for a.e. } (x,t) \in Q_T.$$ is called **Steklov mean** of $f$.

**Proposition I.1** For every $f \in L^p(Q_T)$ and every $\lambda > 0$ there holds

1. $\|f_\lambda\|_{L^p(Q_T)} \leq \|f\|_{L^p(Q_T)}$;
2. \[
\int_{Q_T} f_\lambda(x,t) \partial_t \zeta(x,t) \, dx \, dt = - \frac{1}{\lambda} \int_{Q_T} (f(x,t+\lambda) - f(x,t)) \zeta(x,t) \, dx \, dt
\]

for every $\zeta \in C^\infty_c(Q_T)$, i.e., $f_\lambda$ possesses the weak derivative

$$\frac{\partial}{\partial t} f_\lambda(x,t) = \frac{1}{\lambda} (f(x,t+\lambda) - f(x,t))$$

for a.e. $(x,t) \in Q_T$.

3. \[
\frac{1}{\lambda} \int_{Q_T} f(x,t) \left( \int_t^{t+\lambda} \alpha(s) \, ds \right) \, dx \, dt = \int_{Q_T} f_\lambda(x,t) \alpha(t) \, dx \, dt
\]
for any \( \alpha \in L^\infty(\mathbb{R}) \) such that \( \alpha(t) = 0 \) for all \( t \in \mathbb{R} \setminus [t_0, t_1] \) (\( 0 < t_0 < t_1 < T \) depending on \( \alpha \)).

**Proposition I.2** For every \( f \in L^p(Q_T) \),

\[
\lim_{\lambda \to 0} \| f_\lambda - f \|_{L^p(Q_T)} = 0.
\]

Prop. I.1.1 and Prop. I.2 are special cases of well-known results about the mollification of \( L^p \)-functions. The properties of the Steklov mean presented in Prop. I.1.2, I.1.3 are used in [14, Chap. III, §2, Lemma 2.1] to establish energy inequalities for weak solutions of parabolic equations. For reader’s convenience we present the proofs.

**Proof of Proposition I.1.2** Let \( \zeta \in C^\infty_0(Q_T) \). Then

\[
- \int_{Q_T} f_\lambda(x,t) \frac{\partial \zeta}{\partial t}(x,t) \, dx \, dt = \lim_{h \to 0} \int_{Q_T} f_\lambda(x,t) \frac{1}{h} (\zeta(x,t-h) - \zeta(x,t)) \, dx \, dt.
\]

On the other hand, for \( 0 < h < \lambda \) we have

\[
\int_{Q_T} f_\lambda(x,t) \frac{1}{h} (\zeta(x,t-h) - \zeta(x,t)) \, dx \, dt
\]

\[
= \frac{1}{\lambda h} \int_{Q_T} \left( \int_{t+\lambda}^{t+h} f(x,s) \, ds - \int_{t}^{t+h} f(x,s) \, ds \right) \zeta(x,t) \, dx \, dt
\]

\[
= \frac{1}{\lambda h} \int_{Q_T} \left( \int_{t+\lambda}^{t+h} (f(x,s) - f(x,t+\lambda)) \, ds \right) \zeta(x,t) \, dx \, dt
\]

\[
- \frac{1}{\lambda h} \int_{Q_T} \left( \int_{t}^{t+h} (f(x,s) - f(x,t)) \, ds \right) \zeta(x,t) \, dx \, dt
\]

\[
+ \frac{1}{\lambda} \int_{Q_T} (f(x,t+\lambda) - f(x,t)) \zeta(x,t) \, dx \, dt.
\]

Here, the first and the second term on the right hand side converge to zero when \( h \to 0 \). This can be easily seen by combinig Fubini’s theorem and continuity of \( f \) with respect to the integral mean.

Whence the claim.

**Proof of Proposition I.1.3** Let \( \lambda > 0 \). We introduce a function \( \xi_\lambda : \mathbb{R}^2 \to \{0\} \cup \{1\} \) as follows: given \( t \in \mathbb{R} \), define

\[
\xi_\lambda(s,t) = \begin{cases} 
1 & \text{if } s \in [t - \lambda, t], \\
0 & \text{if } s \in \mathbb{R} \setminus [t - \lambda, t], 
\end{cases}
\]
or, equivalently, given \( s \in \mathbb{R} \), define

\[
\xi_\lambda(s, t) = \begin{cases} 
1 & \text{if } t \in [s, s + \lambda], \\
0 & \text{if } t \in \mathbb{R} \setminus [s, s + \lambda].
\end{cases}
\]

We obtain for a.e. \( x \in \Omega \)

\[
\int_0^T f(x, t) \left( \int_{t-\lambda}^t \alpha(s) \, ds \right) \, dt \\
= \int_0^T \int_{\mathbb{R}} f(x, t) \xi_\lambda(s, t) \alpha(s) \, ds \, dt \\
= \int_{\mathbb{R}} \int_0^T f(x, t) \xi_\lambda(s, t) \alpha(s) \, dt \, ds \quad \text{(by Fubini’s theorem)} \\
= \int_0^T \left( \int_s^{s+\lambda} f(x, t) \, dt \right) \alpha(s) \, ds \quad \text{(since } \alpha = 0 \text{ on } \mathbb{R} \setminus [t_0, t_1]).
\]

Integrating over \( \Omega \) and dividing by \( \lambda \) gives the claim.

**Appendix II. Equivalent characterization of the space \( V_0 \)**

Let \( \Omega \subset \mathbb{R}^3 \) be an open set. We recall the definition of the spaces \( V \) and \( V_0 \) introduced in Section 2

\[
V := \left\{ u \in L^2(\Omega^3); \; \text{curl } u \in L^2(\Omega)^3 \right\}, \\
V_0 := \left\{ u \in V; \; \int_\Omega \left( \text{curl } u \right) \cdot z \, dx = \int_\Omega u \cdot (\text{curl } z) \, dx \; \forall z \in V \right\}.
\]

**Proposition II.1** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz boundary \( \Gamma = \partial \Omega \). Then the following are equivalent

1° \( u \in V_0 \);

2° \( u \in L^2(\Omega)^3, \exists c = \text{const} > 0 \text{ such that } \left| \int_\Omega u \cdot \text{curl } z \, dx \right| \leq c \| z \|_{(L^2)^3} \; \forall z \in V. \)

The implication 1° \( \Rightarrow \) 2° is easily seen. To prove the reverse implication, we will use the following

**Lemma 1** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz boundary \( \Gamma \). Let \( w^* \in H^{-1/2}(\Gamma) \) satisfy

\[
(\text{II.1}) \quad \exists c_0 = \text{const} > 0 \text{ such that } \left| \langle w^*, u \rangle_{H^{1/2}} \right| \leq c_0 \| u \|_{L^2} \; \forall u \in H^1(\Omega).
\]

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Then

\[ w^* = 0. \]

Indeed, if \( w^* \neq 0 \), then there would exist \( u_0 \in H^1(\Omega) \) such that

\[ \langle w^*, u_0 \rangle_{H^{1/2}} \neq 0. \]

We then take an open set \( \Omega' \subset \Omega \) and a function \( \zeta \in C^1(\Omega) \) with the following properties

\[ \Omega' \subset \Omega, \quad \int_{\Omega \setminus \Omega'} u_0^2 \, dx \leq \left( \frac{1}{2c_0} \left| \langle w^*, u_0 \rangle_{H^{1/2}} \right| \right)^2, \]

\[ 0 \leq \zeta \leq 1 \text{ in } \Omega, \quad \zeta = 0 \text{ in } \Omega', \quad \zeta = 1 \text{ on } \Gamma. \]

It follows

\[ \left| \langle w^*, u_0 \rangle_{H^{1/2}} \right| = \left| \langle w^*, \zeta u_0 \rangle_{H^{1/2}} \right| \]

\[ \leq c_0 \left( \int_{\Omega \setminus \Omega'} u_0^2 \, dx \right)^{1/2} \quad \text{(by (III.1))} \]

\[ \leq \frac{1}{2} \left| \langle w^*, u_0 \rangle_{H^{1/2}} \right|, \]

a contradiction.

**Proof of 2° ⇒ 1°.** From 2° one concludes by the aid of Riesz’ representation theorem that \( \text{curl } u \in L^2(\Omega)^3 \). We obtain

\[ \int_{\Omega} (\text{curl } u) \cdot z \, dx - \int_{\Omega} u \cdot (\text{curl } z) \, dx = \langle \gamma_\tau(u), z \rangle_{(H^{1/2})^3} \quad \forall z \in H^1(\Omega)^3 \]

(cf., e.g., [5] p. 207]; recall \( \gamma_\tau(u) = n \times u|_\Gamma \) if \( u \in C^1(\overline{\Omega})^3 \), cf. Section 2). Thus, for all \( z \in H^1(\Omega)^3 \),

\[ \left| \langle \gamma_\tau(u), z \rangle_{(H^{1/2})^3} \right| \leq \left( \| \text{curl } u \|_{(L^2)^3} + c \right) \| z \|_{(L^2)^3} \quad \text{(by 2°).} \]

By the above lemma,

\[ \gamma_\tau(u) = 0 \text{ in } H^{-1/2}(\Gamma)^3. \]

The density of \( C^1(\overline{\Omega})^3 \) in \( V \) (cf. [5] p. 204], [7] Chap. VII, Lemme 4.1]) implies

\[ \int_{\Omega} (\text{curl } u) \cdot z \, dx - \int_{\Omega} u \cdot (\text{curl } z) \, dx = 0 \quad \forall z \in V, \]

i.e., \( u \in V_0. \)

\[ \Box \]
From Prop. II.1 we conclude

For \( \{u, v\} \in L^2(\Omega)^3 \times L^2(\Omega)^3 \) the following are equivalent

(a) \( \{u, v\} \in V \times V_0 \);

(b) \( \exists c = \text{const} > 0 \) such that

\[
\left| \int_{\Omega} (-u \cdot \text{curl} \varphi + v \cdot \text{curl} \psi) \, dx \right| \leq c \left( \|\varphi\|_{L^2(\Omega)^3}^2 + \|\psi\|_{L^2(\Omega)^3}^2 \right)^{1/2} \quad \forall \{\varphi, \psi\} \in V_0 \times V.
\]

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