Three-point correlation function of a scalar mixed by an almost smooth random velocity field

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Abstract

We demonstrate that if the exponent $\gamma$ that measures non-smoothness of the velocity field is small then the isotropic zero modes of the scalar’s triple correlation function have the scaling exponents proportional to $\sqrt{\gamma}$. Therefore, zero modes are subleading with respect to the forced solution that has normal scaling with the exponent $\gamma$.

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INTRODUCTION

Kraichnan’s model of passive scalar advection by white-in-time velocity field has became a paradigm within which analytic theory of anomalous scaling in turbulence starts to appear. The instrument is the perturbation theory around three limiting cases where scalar statistics is Gaussian: i) infinite space dimensionality, ii) extremely irregular velocity field which corresponds to a smooth scalar field, and iii) smooth velocity field (the Batchelor-Kraichnan limit). The perturbation theory is regular in the first two cases and the sets of the exponents obtained agree when the limits intersect. The perturbation theory around the Batchelor-Kraichnan limit is singular, only dipole part of three-point correlation function has been found so far. In this paper, motivated by, we find the isotropic part of the triple correlation function and show that the leading term has a normal scaling at small scales.

It is instructive to discuss first the physics involved to understand the significant difference between the first two limits on the one hand and the third limit on the other. Since the scalar field at any point is the superposition of fields brought from $d$ directions then it follows from a central limit theorem that scalar’s statistics approaches Gaussian when space dimensionality $d$ increases. In the case ii), an irregular velocity field acts like Brownian motion so that turbulent diffusion is much like linear diffusion: statistics is Gaussian provided the input is Gaussian. What is general in both limits is that Gaussianity is rather uniform over the scales, the degree of Gaussianity (say, flatness) is independent of the ratio $r/L$ where $r$ is a typical distance in the correlation function and $L$ is an input scale. Quite contrary, $\ln(L/r)$ is the parameter of Gaussianity in the Batchelor-Kraichnan limit so that statistics is getting Gaussian at small scales whatever the input statistics. The mechanism of Gaussianity is temporal rather than spatial in this case: since the stretching is exponential in a smooth velocity field then the cascade time grows logarithmically as the scale decreases. This is opposite to what one expects from intermittency and anomalous scaling (anticipated beyond the Gaussian limits): the degree of non-Gaussianity has to grow downscales. Already
that simple reasoning shows that the way from the Batchelor-Kraichnan limit towards an anomalous scaling at non-smooth velocity field is not to be easy. The formal reason for this perturbation theory to be singular is that, at the limit, the many-point correlation functions have singularity (smeared by molecular diffusion only) at the collinear geometry – smooth velocity provides for homothetic transformation that does not break collinearity [17]. Even weak non-smoothness of the velocity smears the singularity i.e. strongly influences the solution in the narrow region near collinearity; such a situation calls for a boundary layer approach introduced into this problem by Shraiman and Sigcia [4,11].

1. GENERAL RELATIONS

The triple correlation function of the scalar $F(r_1, r_2, r_3)$ advected by white-in-time velocity field satisfies the closed balance equation [2]

$$(\hat{\mathbf{L}} + \hat{\mathbf{L}}_d)F_3 = -\chi_3. \quad (1.1)$$

Here, $\chi_3(r_1, r_2, r_3)$ is the triple correlation function of the (non-Gaussian) pumping force. Actually it depends on differences $r_{ij} = r_i - r_j$. If $|r_{ij}| \ll L$ (where $L$ is the pumping length) then $\chi_3 \simeq P_3$ where $P_3$ is the third-order flux. At growing $|r_{ij}|$ the function $\chi_3$ tends to zero on distances larger than $L$. The quantity $\hat{\mathbf{L}}_d$ in (1.1) is the operator of molecular diffusion

$$\hat{\mathbf{L}}_d = \kappa(\nabla_1^2 + \nabla_2^2 + \nabla_3^2) \quad (\kappa \text{ is the diffusion coefficient}),$$

and

$$\hat{\mathbf{L}} = -(1/2) \sum_{i,j=1}^3 K_{\alpha\beta}(r_{ij})\nabla_i^\alpha \nabla_j^\beta \quad (1.2)$$

is the operator of turbulent diffusion. Here

$$K_{\alpha\beta}(r) = Dr^{-\gamma} \left[(r^2 \delta_{\alpha\beta} - r^\alpha r^\beta) + \frac{d-1}{2} r^2 \delta_{\alpha\beta} \right] \quad (1.3)$$

is the eddy diffusivity related to the velocity pair correlation function. Parameter $\gamma$ in (1.3) is a measure of velocity non-smoothness, $0 \leq \gamma \leq 2$.

At $\gamma = 0$, the operator $\hat{\mathbf{L}}$ is singular for collinear geometry — see (2.1) below. That leads to an angular singularity in the correlation functions which is smoothed only by diffusion
which is therefore relevant at all scales \[17\]. Contrary, at \(\gamma > 0\) the operator \(\hat{L}\) is not singular at the collinear geometry and therefore the angular singularity is absent as was pointed out in \[15\]. Therefore we can omit the diffusive term \(\hat{L}_d\) in \(\[1.1\]\) in comparison with \(\hat{L}\). This is possible as long as \(\kappa \ll \gamma D r^2\).

In the following we believe \(\chi_3\) to be an isotropic function of \(r_{ij}\) what dictates the symmetry of the solution of \(\[1.1\]\). In this case \(F_3\) can be treated as a function of three distances \(r_{12}, r_{13}\) and \(r_{23}\) only. Then the operator \(\hat{L}\) \(\[1.2\]\) can also be rewritten in terms of the separations \[14\]

\[
\hat{L} = \frac{D(d-1)}{2-\gamma} \sum_{i>j} r_{ij}^{1-d} \partial_{ij} r_{ij}^{1+d-\gamma} \partial_{ij} + \ldots \tag{1.4}
\]

where dots stand for the terms with cross derivatives \(\partial_{ij} \partial_{kl}\). Since \(\chi_3 = P_3\) at \(r_{ij} \ll L\) we can easily find a solution of \(\[1.1\]\) in the region \(r_{ij} \ll L\) (cf. \[14\]). Using \(\[1.4\]\) we get

\[
F_{\text{forc}} = \frac{(2-\gamma)P_3 L^\gamma}{3D(d-1)d\gamma} \left[ C - \left(\frac{r_{12}}{L}\right)^\gamma - \left(\frac{r_{13}}{L}\right)^\gamma - \left(\frac{r_{23}}{L}\right)^\gamma \right],
\]

which we will call the forced solution. Here \(C\) is a constant of the order unity. That solution satisfies the equation \(r_{ij} \ll L\) but not necessarily matching conditions at \(r_{ij} \gtrsim L\). The solution of \(\[1.1\]\) that satisfies the condition can be written at \(r_{ij} \ll L\) as follows:

\[
F = F_{\text{forc}} + Z_0, \tag{1.6}
\]

where \(Z_0\) is a zero mode of the operator of the turbulent diffusion: \(\hat{L}Z_0 = 0\). We examine here different solutions of the equation \(\hat{L}Z = 0\). Whether the given mode \(Z\) contributes the correlation function has to be determined from the matching at \(r_{ij} \sim L\) which is beyond the scope of our paper. Note that we consider isotropic zero modes while dipole zero modes for the anisotropic problem with an imposed mean gradient were treated in \[13\].

It is convenient to introduce instead of \(r_{ij}\) the following set of variables

\[
x_1 = r_{13} \cos \theta, \quad x_2 = r_{13} \sin \theta, \quad s = r_{12} r_{13} \sin \theta, \tag{1.7}
\]

where \(\theta\) is the angle between \(r_{12}\) and \(r_{13}\) and \(-\infty < x_1 < \infty, 0 < x_2 < \infty, 0 < s < \infty\). Note that the variable \(s\) (which is the doubled area of the triangle) is the only dimensional parameter among \(s, x_1, x_2\). The expressions inverse to \(\[1.7\]\) are
The operator $\hat{L}$ and both correlation functions $\chi_3$ and $F_3$ should be invariant under permutation of points $r_1$, $r_2$ and $r_3$, that is under permutations of $r_{12}$, $r_{13}$ and $r_{23}$. In terms of the variable $z = x_1 + ix_2$ these transformations can be written as follows

$$1 \leftrightarrow 2 : \quad z \to 1 - z^*, \quad 2 \leftrightarrow 3 : \quad z \to \frac{1}{z^*}, \quad 1 \leftrightarrow 3 : \quad z \to 1 + \frac{1}{z^* - 1},$$

$$1 \to 2 \to 3 : \quad z \to \frac{1}{1 - z}, \quad 1 \to 3 \to 2 : \quad z \to 1 - \frac{1}{z},$$

where $z^*$ is complex conjugated to $z$. The doubled area of the triangle $s$ is obviously invariant under the transformations.

2. THE CASE $\gamma = 0$

Here, we start with the case $\gamma = 0$. Below we treat the dimensionality $d = 2$. The results can be generalized for an arbitrary dimensionality $d$. The operator (1.2) for $d = 2$ and $\gamma = 0$ is rewritten in terms of the variables (1.7) as follows

$$\hat{L}_0 = 2Dx_2^2(\partial_1^2 + \partial_2^2).$$

Let us stress that derivatives with respect to $s$ are absent in $\hat{L}_0$. Then a solution of the equation $\hat{L}_0F_3 = -\chi_3$ can be written as

$$F_3(s, x_1, x_2) = \frac{1}{8\pi D} \int_{-\infty}^{+\infty} \int_0^{\infty} \frac{dx'_1 dx'_2}{x_2^2} \ln \left[ \frac{(x'_1 - x_1)^2 + (x'_2 + x_2)^2}{(x'_1 - x_1)^2 + (x'_2 - x_2)^2} \right] \chi_3(s, x'_1, x'_2),$$

where we used the explicit expression for the resolvent of Laplacian (cf. [17]).

We are interested in the behavior of the correlation functions at $r_{ij} \ll L$ which is not sensitive to the particular form of the pumping $\chi_3$. Thus we can choose any convenient form of $\chi_3$ supplying the convergence of the integral in (2.2). We get the following function

$$\chi_3 = \frac{P_3}{1 + (r_{12}^2 + r_{13}^2 + r_{23}^2)/L^2}.$$
Then the correlation function is given by

\[ F_3 = \frac{P_3}{8\pi D} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{dx'_1dx'_2}{x'_2 x_2^2} \frac{x'_2}{x_2^2 + 2sL^2(x_1^2 + x_2^2 + 1 - x'_1)} \ln \left[ \frac{(x'_1 - x_1)^2 + (x'_2 + x_2)^2}{(x_1 - x'_1)^2 + (x'_2 - x_2)^2} \right]. \]  

(2.4)

Performing the integration over \( x'_1 \) one obtains

\[ F_3 = \frac{L^2P_3}{16sD} \int_0^{\infty} dt \frac{dtw}{tw} \ln \left[ \frac{(x_2(1+t) + w)^2 + (x_1 - 1/2)^2}{(x_2|1-t| + w)^2 + (x_1 - 1/2)^2} \right], \]  

(2.5)

where \( w = \sqrt{3/4 + x_2^2t^2 + L^2x_2t/(2s)} \) (cf. [17]). The asymptotics of \( F_3 \) at \( r_{ij} \ll L \) is

\[ F_3 \sim \frac{P_3}{2D} \ln \left[ \frac{2L^2}{r_{ij}^2 + r_{ij}^2 + r_{ij}^2 + 2\sqrt{3}s} \right] + \text{const}. \]  

(2.6)

The expression (2.6) can be rewritten as

\[ F_3 \sim \frac{P_3}{3D} \ln \frac{L^3}{r_{ij}r_{ij}r_{ij}} + \frac{P_3}{6D} \ln \frac{(x_1^2 + x_2^2)((x_1 - 1/2)^2 + x_2^2)}{[(x_1 - 1/2)^2 + (x_2 + \sqrt{3}/2)^2]^3} + \text{const}. \]  

(2.7)

Here, the first term is the forced solution \( F_{\text{forc}} \) which can be obtained from (1.5) at \( \gamma \to 0, d = 2, C = 3 \). The second term in (2.7) is the zero mode \( Z_0 \) which has logarithmic singularities in the points \( z = 0, z = 1 \) and \( z = \infty \) that is where one of \( r_{ij} \) tends to zero. Nevertheless the function \( F_3 \) has no singularity where one of \( r_{ij} \) tends to zero as it is obvious from the expression (2.6). Further, as follows from (2.7) the zero mode \( Z_0 \) has the linear term in the expansion over \( x_2 \). Remembering (1.7) we conclude that at small angles \( \theta \) there is the term proportional to \( |\theta| \) in \( F_3 \). That is just the angular singularity noted above. Repeat again that the singularity is smoothed only by diffusion [17].

We see that the zero mode \( Z_0 \) at \( \gamma = 0 \) does not depend on the dimensional parameter \( s \). Besides, any function of \( s \) is a zero mode of the operator \( \hat{L}_0 \) since it does not contain the derivative over \( s \) what is a remarkable property of the case \( d = 2 \) and \( \gamma = 0 \). Nevertheless all the zero modes do not contribute to \( Z_0 \).

3. ASYMPOTICS AT SMALL ANGLES

Because of the scaling properties of \( \hat{L} \), it is possible to seek the zero mode in the form \( Z \propto s^\Delta \). It is convenient to introduce the following function
\[
Z = \left(\frac{s}{x_2}\right)^\Delta \left\{ 1 + (x_1^2 + x_2^2)^\Delta + \left[ (1 - x_1)^2 + x_2^2 \right]^\Delta \right\} X(x_1, x_2), \tag{3.1}
\]

The function \(X(x_1, x_2)\) should be invariant under all transformations \((1.9, 1.10)\) and have no angular singularities since the proportionality coefficient between \(Z\) and \(X\) is equal to \(r_{12}^{2\Delta} + r_{13}^{2\Delta} + r_{23}^{2\Delta}\) as follows from \((1.8)\).

The equation \(\hat{\mathcal{L}} Z = 0\) can be rewritten as \(\hat{\mathcal{L}} X X = 0\) where \(\hat{\mathcal{L}} X\) is a differential operator of the second order over \(\partial_1 \equiv \partial/\partial x_1\) and \(\partial_2 \equiv \partial/\partial x_2\). Coefficients at the derivatives are the functions of \(x_1, x_2, \Delta\) which can be found from \((1.1, 1.8)\). The functions are quite complicated. Fortunately only particular parts of the operator \(\hat{\mathcal{L}} X\) will be needed for us.

At \(\gamma = 0\) the operator \(\hat{\mathcal{L}} X\) is determined by \((2.1)\). The operator tends to zero at \(x_2 \to 0\). Therefore at small \(x_2\) besides \((2.1)\) we should take into account also the residue. The term leading at small \(x_2\) can be written as

\[
\hat{\mathcal{L}} X \propto \hat{\mathcal{L}}_2 = [2x_2^2 + c_0(x_1)]\partial_2^2 - 4\Delta x_2 \partial_2 + 2\Delta(\Delta + 1), \tag{3.2}
\]

\[
c_0(x) = -\frac{3\gamma}{4}(1 - x)x [x \ln |x| + (1 - x) \ln |1 - x|]. \tag{3.3}
\]

Note that \(c_0 > 0\). The expression \((3.3)\) is correct if \(x_2 \ll |x_1|, |x_1 - 1|; |x_1|, |x_1 - 1| \gg \exp(-1/\gamma); |x_1|, |x_1 - 1| \ll \exp(1/\gamma)\). The asymptotic behavior of a solution of \(\hat{\mathcal{L}}_2 X = 0\) at \(x_2 \ll \sqrt{c_0}\) is

\[
X = A_1(x_1) + A_2(x_1)x_2, \tag{3.4}
\]

where \(A_1(x)\) and \(A_2(x)\) are arbitrary functions. The analyticity of \(X\) at small angles excludes the second term in \((3.4)\) since it supplies the contribution to \(X\) which behaves \(\propto |\vartheta|\). The equation \(\hat{\mathcal{L}}_2 X = 0\) can be solved explicitly, a solution having the asymptotics \((3.4)\) with \(A_2 = 0\) is

\[
X_0 = A_1(x_1)F\left(-\frac{1 + \Delta}{2}, \frac{\Delta}{2}; \frac{1}{2}; -\frac{2x_2^2}{c_0(x_1)}\right). \tag{3.5}
\]

Here, \(F(\alpha, \beta; \gamma; z)\) is the hypergeometric function.

The expression \((3.5)\) gives the behavior of the zero mode in the vicinity of the boundary layer \(x_2 \sim \sqrt{c_0}\). If we are interested in the behavior of the zero mode outside the boundary
layer then it is more convenient to return to \( Z \) since \( \hat{\mathcal{L}} \) can be approximated as (2.1) and consequently \( Z \) is a harmonic function there. The asymptotics of (3.5) valid at \( x_2 \gg \sqrt{c_0} \) gives

\[
Z \propto (\Delta + 1) \cos(\pi \Delta/2) - x_2 \sin(\pi \Delta/2) \left( \frac{2}{c_0(x_1)} \right)^{1/2}.
\]

(3.6)

The behavior (3.6) occurs outside the boundary layer but at small \( x_2 \). Since we are interested in small \( \Delta \) we can suggest \( \Delta \ll 1 \). Thus we come to the following problem: find the harmonic function \( Z(x_1, x_2) \) in the upper half-plane \( x_2 > 0 \) at the boundary condition

\[
\frac{\sqrt{2c_0}}{\pi \Delta} \partial_2 Z + Z = 0,
\]

(3.7)

which is imposed on the function \( Z \) at \( x_2 = 0 \) since at small \( \gamma \) the width of the boundary layer is negligible.

Let us show that there is no zero mode with \( \Delta \ll \sqrt{\gamma} \). A harmonic function \( Z(x_1, x_2) \) inside the region can be presented as an integral of its normal derivative \( \partial Z/\partial n \) along the contour which is the boundary of the region:

\[
Z(z) = \frac{1}{\pi} \oint |dt| \frac{\partial Z(t)}{\partial n} \ln |z - t|,
\]

(3.8)

where \( z = x_1 + i x_2 \) and \( t \) is the complex variable going along the contour. Let us consider the contour consisting of the semi-circles going around the singular points 0, 1, \( \infty \) and the parts near the real axis (outside the boundary layer but at small \( x_2 \)) that link the semicircles. If \( \Delta \ll \sqrt{\gamma} \) then the condition (3.7) tells us that in this case only contributions to (3.8) from the semicircles are relevant since the contributions from the parts of the real axis are negligible. Separate consideration of the vicinities of the singular points 0, 1, \( \infty \) (see below) shows that the logarithmic derivative of \( Z \) has to be bounded there. Thus the only possible contribution to the zero mode associated, say, with the singular point \( z = 0 \) is \( \propto \Re \ln z = \ln \sqrt{x_1^2 + x_2^2} \).

The complete zero mode should be symmetric under transformations (1.9). Performing all the transformations to \( \ln \sqrt{x_1^2 + x_2^2} \) and summing the results we obtain zero. That means that the function possessing the required symmetry does not exist.
Another (equivalent) way of showing that there is no zero mode with $\Delta \ll \sqrt{\gamma}$ is to continue $Z$ to negative $x_2$ by $Z(x_1, x_2) = Z(x_1, -x_2)$. Because in our case we can believe $\partial_2 Z(x_1, x_2) = 0$ the function should be harmonic in semicircles surrounding the singular points $0, 1, \infty$. Then one can use the properties of the analytical functions in the circles to exclude the existence of zero modes with bounded logarithmic derivatives near the singular points. Note the difference with the dipole case where such mode has been found [15].

4. VICINITIES OF SINGULAR POINTS

Here, we describe the set of zero modes that do not have additional smallness of $\Delta$ relative to $\sqrt{\gamma}$ so that the whole boundary condition (3.7) is to be accounted. Necessary information about the structure of the modes can be extracted from the analysis of the vicinities of the singular points $z = 0$, $z = 1$ and $z = \infty$ where one needs a separate consideration. Using the symmetry properties ($1.9$, $1.10$) we can reduce the consideration to the vicinity of one of the points, say $z = 1$. At $x_2 \ll 1$ and $|x_1 - 1| \ll 1$ the operator $\hat{L}_X$ acquires the following form

$$\hat{L}_X = \mu \left[ \rho^2 \partial^2_\rho + 3 \rho \partial_\rho + 3 \partial^2_\varphi + 2 \sin^2 \varphi \left[ \rho^2 \partial^2_\rho + \rho \partial_\rho + \partial^2_\varphi \right] \right]$$

$$-4 \Delta \left[ \sin^2 \varphi \rho \partial_\rho + \cos \varphi \sin \varphi \partial_\varphi \right] + 2 \Delta (\Delta + 1),$$

(4.1)

$$x_1 - 1 = \rho \cos \varphi, \quad x_2 = \rho \sin \varphi, \quad \mu = \frac{1}{2} (\rho^{-\gamma} - 1).$$

(4.2)

In the exponentially narrow vicinity of the singular point where $\rho \ll \exp(-1/\gamma)$ one has $\mu \gg 1$ and the equation $\hat{L}_X X = 0$ is reduced to

$$\left[ \rho^2 \partial^2_\rho + 3 \rho \partial_\rho + 3 \partial^2_\varphi \right] X = 0.$$  

(4.3)

Solutions of the equation (4.3) can be expanded into the Fourier series over the angular harmonics

$$X_m \propto \sin(m \varphi) \rho^{\lambda_m}.$$  

(4.4)

Substituting (4.4) into (4.3) we obtain
\[ \lambda_m = -1 + \sqrt{1 + 3m^2}, \quad (4.5) \]

where only nonnegative \( \lambda_m \) are taken since \( X \) should remain finite at \( \rho \to 0 \). Thus besides a constant corresponding to \( m = 0 \) we obtain the next term corresponding to \( m = 1 \) that behaves \( \propto \rho \). Since inside the exponentially narrow vicinity a zero mode \( X \) is a linear combination of (4.4) we come to the conclusion that the matching condition on the boundary of the vicinity should be imposed on the logarithmic derivative of \( X \) (or \( Z \)) which remains constant there.

Now, let us consider the region \( 1 \gg \rho \gg \exp(-1/\gamma) \) where

\[ \mu = \frac{1}{2} \gamma \ln \frac{1}{\rho} \ll 1. \quad (4.6) \]

Here, we can consider separately small angles \( \varphi \ll 1 \) where \( \hat{L}_X X = 0 \) is reduced to

\[ \left[(3\mu + 2\varphi^2)\partial_{\varphi}^2 - 4\Delta \partial_{\varphi} + 2\Delta(\Delta + 1)\right] X = 0, \quad (4.7) \]

what exactly corresponds to \( \hat{L}_2 X = 0 \). Again an appropriate solution of (4.7) is as follows

\[ X \propto F \left(-\frac{1 + \Delta}{2}, -\frac{\Delta}{2}; \frac{1}{2}; -\frac{2\varphi^2}{3\mu}\right), \quad (4.8) \]

that gives the asymptotics

\[ Z \propto 1 - \frac{\pi \Delta}{\sqrt{6\mu}} \varphi, \quad (4.9) \]

at \( 1 \gg \varphi \gg \sqrt{\mu} \). Substituting now (4.6) we come to the boundary condition

\[ \partial_{\varphi} Z = -\frac{\pi \Delta}{\sqrt{3\gamma \ln(1/\rho)}} Z, \quad (4.10) \]

imposed on the harmonic function \( Z \) at \( \varphi = 0 \). Note that (4.10) is nothing but the limit of (3.7) at \( \rho \ll 1 \). That boundary condition is simple enough and permits explicit expression for \( Z \) near the singularity.

Let us represent \( Z \) in the following form

\[ Z = \text{Re} \left\{ f \left( \ln \frac{1}{\rho} + i\varphi \right) + f \left( \ln \frac{1}{\rho} + i\pi - i\varphi \right) \right\}, \quad (4.11) \]
which is obviously harmonic and invariant under $\varphi \to \pi - \varphi$. Taking into account $\ln(1/\rho) \gg 1$ we obtain from (4.11)

$$Z(\rho, \varphi = 0) = 2\text{Re} \left\{ f \left( \ln \frac{1}{\rho} \right) \right\},$$

(4.12)

$$\frac{\partial Z}{\partial \varphi}(\rho, \varphi = 0) = \pi \text{Re} \left\{ f'' \left( \ln \frac{1}{\rho} \right) \right\}.$$  

(4.13)

Thus we see that $f(x)$ satisfies the following equation

$$f''(x) + \frac{2\Delta}{\sqrt{3}\gamma x} f(x) = 0.$$  

(4.14)

Asymptotic behavior of the solution at $x \gg 1$ is

$$f(x) = \exp \left[ \pm \frac{4}{3} i \left( \frac{4\Delta^2}{3\gamma} \right)^{1/4} x^{3/4} \right].$$  

(4.15)

Expanding (4.11) with (4.15) over $\ln^{-1}(1/\rho)$ we obtain

$$Z \propto \cos \left[ \frac{4}{3} \left( \frac{4\Delta^2}{3\gamma} \right)^{1/4} \left( \ln \frac{1}{\rho} \right)^{3/4} + \phi_0 \right] \left\{ 1 - \frac{\Delta}{\sqrt{3\gamma}} \frac{\varphi(\pi - \varphi)}{\sqrt{\ln(1/\rho)}} \right\},$$

(4.16)

where $\phi_0$ is some phase. We can believe $|\phi_0| < \pi$, its actual value has to be determined by the matching at $-\ln \rho \sim \gamma$.

Symmetry requirement with respect to $x_1 \to 1 - x_1$ leads to the condition $\partial_1 Z(1/2, x_2) = 0$ which can be used as the quantization rule for the zero modes having the asymptotics (4.16). They can be classified in accordance with the number of zeros $n$ which the function $Z$ have where $x_1$ goes from $1/2$ to $1 - \exp(-\gamma)$. Using the expression (4.16) we conclude that $\Delta_{\text{min}} = \alpha \sqrt{\gamma}$ and for $n \gg 1$

$$\Delta_n = \beta \sqrt{\gamma n^2}.$$  

(4.17)

with yet unknown numerical factors $\alpha, \beta$ which are of order unity. Note that nonsymmetric zero modes $Z$ (with another values of $\alpha, \beta$) may exist yet they cannot contribute to $Z_0$. For all the modes, the dependence $\Delta(\gamma)$ obtained here has an infinite slope at zero which has been also observed in numerics [18]. We conclude that the set of zero modes thus found at small $\gamma$ has exponents larger than the exponent $\gamma$ of forced solution. Therefore, the isotropic
part of the triple correlation function is shown here to have a normal scaling for sufficiently small $\gamma$. Since at $\gamma = 2$ the lowest zero mode has $\Delta = 4$, it is likely that the scaling of the isotropic part of the triple correlation function is normal for all $\gamma$.

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