Jarden’s Property and Hurwitz Curves

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Abstract

The purpose of this article is twofold. Firstly we show that the action of the absolute Galois group on certain classes of algebraic curves (Hurwitz curves, Hurwitz translation surfaces) and dessins d’enfants (regular dessins, classical and higher, with fixed signature) is faithful. Secondly we introduce a property of profinite groups, called Jarden’s property, and show this property for certain étale fundamental groups. Combining the latter result with faithfulness of the Galois action on these groups, which was known before, we obtain the results on curves and dessins.

I. Introduction and Statement of Results

In this introduction we first present the two main themes of this article and then explain how they go together. Proofs will be provided in the later sections.

Hurwitz Curves and Translation Surfaces. By a well-known theorem of Hurwitz [13] a (smooth projective) curve of genus \( g \geq 2 \) over \( \mathbb{C} \) has no more than \( 84(g-1) \) automorphisms. Curves which attain this bound are called Hurwitz curves. They are relatively rare: Conder computed [5] that there are only 92 Hurwitz curves of genus less than one million, with only 32 different genera occurring. Furthermore, the series \( \sum_X g(X)^{-s} \), where \( X \) runs over all Hurwitz curves, converges precisely for \( \Re(s) > \frac{1}{3} \), see [19]. And yet:

**Theorem 1.** The absolute Galois group \( \Gamma_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \) operates faithfully on the set of isomorphism classes of Hurwitz curves.

This is to be understood as follows: every Hurwitz curve has a unique model over \( \overline{\mathbb{Q}} \), and conjugating it by an automorphism of \( \overline{\mathbb{Q}} \) will yield another, possibly different, Hurwitz curve.

Theorem 1 can be understood as a special case of a more general result about the Galois action on dessins d’enfants[1].

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[1] For dessins d’enfants see [24].
**Theorem 2.** Let \( p, q, r \in \mathbb{N} \) with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \). Then \( \Gamma_\mathbb{Q} \) acts faithfully on the set of all regular\(^2\) dessins d’enfants where the white vertices have degree dividing \( p \), the black vertices degree dividing \( q \) and the cells are \( 2r' \)-gons with \( r' \mid r \).

Even more generally, this theorem also holds mutatis mutandis for regular higher dessins d’enfants (see [18]), with the same proof.

One can ask similar questions for translation surfaces\(^3\); this has been initiated in [22]. There it is shown that a translation surface of genus \( g \geq 2 \) has at most \( 4(g-1) \) automorphisms, and surfaces achieving this bound are named Hurwitz translation surfaces. They are more common than Hurwitz curves; for example, a Hurwitz translation surface exists in genus \( g \) if and only if \( g \equiv 1, 3, 4, 5 \mod 6 \), see [22, Theorem 2]. We show similarly:

**Theorem 3.** The absolute Galois group \( \Gamma_\mathbb{Q} \) operates faithfully on the set of isomorphism classes of Hurwitz translation surfaces.

For the precise definition of this operation see below.

Finally we can deduce consequences for the mod \( \ell \) Galois representations associated with Hurwitz curves:

**Theorem 4.** Fix an element \( \sigma \in \Gamma_\mathbb{Q} \) other than the identity. Then there exists a Hurwitz curve \( Y \) with moduli field \( \mathbb{Q} \) such that for any model\(^4\) \( Y \) of \( Y \) over \( \mathbb{Q} \) and for every odd prime \( \ell \), the image of \( \sigma \) under the representation \( \rho_{Y,\ell} : \Gamma_\mathbb{Q} \to \text{GL}(2g, \mathbb{F}_\ell) \) is not the identity.

Here \( \rho_{Y,\ell} \) is the usual Galois representation on the \( \ell \)-torsion of the Jacobian, \( (\text{Jac}Y)[\ell] \cong \mathbb{F}_\ell^{2g} \). A similar statement holds for Hurwitz translation surfaces, where “with moduli field \( \mathbb{Q} \)” must be replaced by “admitting a model over \( \mathbb{Q} \)”.

Theorems 1 to 4 are proved, in this order, from page 10 onwards. To obtain these results we use Jarden’s property for certain étale fundamental groups. Since we believe this to be of independent interest, we now give a short introduction to Jarden’s property.

**Jarden’s Property.** Let \( G \) be a profinite group and let \( F \) be an open normal subgroup of \( G \). An automorphism\(^5\) \( \varphi \) of \( G \) is called \( F \)-normal if \( \varphi(N) = N \) for every open subgroup \( N \subseteq F \subseteq G \) which is normal in \( G \) (not necessarily in \( F \)). Inner automorphisms are evidently \( F \)-normal. Instead of “\( G \)-normal”, we simply say “normal”\(^6\).

**Definition 5.** A pair of profinite groups \( (G, F) \) with \( F \subseteq G \) an open subgroup has Jarden’s property if every \( F \)-normal automorphism of \( G \) is inner. A profinite group \( G \) has Jarden’s property if every normal automorphism of \( G \) is inner.

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\(^2\)A dessin is called regular if the canonical morphism to \( \mathbb{CP}^1 \) is a Galois covering.

\(^3\)A translation surface is a closed Riemann surface with a nonzero holomorphic one-form; for more geometric descriptions, see [12].

\(^4\)Every Hurwitz curve admits a model over its moduli field, see [8].

\(^5\)In this work, homomorphisms between profinite groups are always tacitly assumed to be continuous.

\(^6\)The notion of a normal automorphism dates back to [3] and is used throughout the literature; the more general notion of an \( F \)-normal automorphism is introduced explicitly for the first time in this work, but was used implicitly in [9].
The first discussion of this property is in [14]: free profinite groups on at least two (possibly infinitely many) generators have Jarden’s property. In [15], two further results were shown: [15, Theorem A] states that the absolute Galois group $G_K$ has Jarden’s property for every finite extension $K$ of $\mathbb{Q}_p$, and [15, Theorem B] contains as a special case:

**THEOREM 6** (Jarden–Ritter). *Let $\Gamma$ be a finitely presented group on $e$ generators and $d$ relations, with $e \geq d + 2$. Then the profinite completion of $\Gamma$ has Jarden’s property.*

The technical heart of the present article is Jarden’s property for étale fundamental groups of projective hyperbolic curves and a slight generalisation:

**DEFINITION 7.** *Let $k$ be a field. A closed Fuchsian orbifold over $k$ is a smooth Deligne-Mumford stack $X$ over $k$ with trivial generic stabilisers, admitting a finite étale covering $Y \to X$ with $Y$ a smooth projective geometrically connected curve of genus at least two over $k$. For such Deligne-Mumford stacks we can define an étale fundamental group in the usual way, and on page 8 we prove after a sequence of lemmas:*

**THEOREM 8.** *Let $k$ be a separably closed field and let $X_1 \to X$ be an étale covering map between closed Fuchsian orbifolds over $k$. Then the pair of étale fundamental groups $(\pi_1^\text{ét}(X), \pi_1^\text{ét}(X_1))$ has Jarden’s property.*

For instance for $k = \mathbb{C}$ we get Jarden’s property for any pair $(\hat{\Gamma}, \hat{\Gamma}_1)$ where $\Gamma_1 \subseteq \Gamma$ are cocompact lattices in $\text{PSL}(2, \mathbb{R})$ (with $X(\mathbb{C}) = \Gamma\backslash \mathbb{H}$). But Theorem 8 is more general since it also holds in positive characteristic where the isomorphism types of such fundamental groups vary wildly, see [21].

Theorem 8 was proved in [9, Theorem 27] for the following special case: $k = \mathbb{C}$ and the analytification of $X_1 \to X$ is the orbifold quotient of the upper half plane by a triangle group; in particular, $\pi_1^\text{ét}(X)$ is the profinite completion of that triangle group. Our proof of Theorem 8 basically follows [9], but we translate their methods, which work partly with the profinite group and partly with the discrete triangle group, into the language of $\ell$-adic cohomology, thereby simplifying and generalising the argument.

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### II. Jarden’s Property: The Proof

Let $X$ and $X_1$ be as in the statement of Theorem 8 and choose some basepoints with trivial stabilisers $\bar{x} \in X(k)$ and $\bar{x}_1 \in X_1(k)$ such that $\bar{x}_1$ maps to $\bar{x}$. Set $G = \pi_1^\text{ét}(X, \bar{x})$ and $G_1 = \pi_1^\text{ét}(X_1, \bar{x}_1)$. Finally let $\varphi: G \to G$ be a $G_1$-normal automorphism; we have to show that $\varphi$ is an inner automorphism.

This is done by character theory of profinite groups with special consideration of those characters of $G$ that appear in the $\ell$-adic cohomology of finite Galois covers of $X$. We begin by explaining the required notions from character theory.
Unless otherwise noted, we fix a rational prime \( \ell \neq p = \text{char } k \) and an algebraic closure \( \overline{Q}_\ell \) of the field \( Q_\ell \) of \( \ell \)-adic numbers.

**Definition 9.** Let \( \Gamma \) be a profinite group. A finite representation of \( \Gamma \) is a continuous group homomorphism \( \rho : \Gamma \to \text{GL}(V) \) with finite image, where \( V \) is a finite-dimensional \( \overline{Q}_\ell \)-vector space.

The function \( \chi : \Gamma \to \overline{Q}_\ell, \gamma \mapsto \text{tr} \rho(\gamma) \), is called the character associated with \( \rho \); every function arising this way for some finite representation is called a finite character of \( \Gamma \).

Note that the definition of finite representations and characters makes no use of the \( \ell \)-adic topology on \( \overline{Q}_\ell \); we arrive at exactly the same notion if we endow it with the discrete topology — or in fact choose a field isomorphism \( \overline{Q}_\ell \cong \mathbb{C} \) and demand that \( \rho \) be continuous for the complex topology on \( \mathbb{C} \). From this we deduce that the category of finite representations of \( \Gamma \) is semi-simple.

**Lemma 10.** Let \( \rho : \Gamma \to \text{GL}(V) \) and \( \rho' : \Gamma \to \text{GL}(V') \) be two finite representations whose associated characters agree as functions on \( \Gamma \). Then \( \rho \cong \rho' \).

**Proof.** There exists an open normal subgroup \( \Delta \subseteq \Gamma \) such that both \( \rho \) and \( \rho' \) factor through the quotient \( \Gamma / \Delta \). They induce the same characters of \( \Gamma / \Delta \), hence the two representations of \( \Gamma / \Delta \) are isomorphic, hence also those of \( \Gamma \). \( \square \)

By virtue of this lemma we may speak of the representation \( V_\chi \) associated with a finite character \( \chi : \Gamma \to \overline{Q}_\ell \).

**Lemma 11.** Let \( \Delta \) be an open normal subgroup of a profinite group \( \Gamma \) and let \( \chi : \Gamma \to \overline{Q}_\ell \) and \( \psi : \Delta \to \overline{Q}_\ell \) be irreducible characters. Then the following are equivalent:

(i) \( V_\psi \) is a subrepresentation of \( V_\chi \big|_\Delta \) (i.e. of \( V_\chi \) regarded as a finite representation of \( \Delta \));

(ii) the induced representation \( \text{Ind}^\Gamma_\Delta V_\psi \) of \( \Gamma \) (defined as usual) contains \( V_\chi \) as a subrepresentation.

If these conditions are fulfilled, we say that \( \chi \) lies above \( \psi \) and that \( \psi \) lies below \( \chi \).

**Proof.** By definition of the induced representation,

\[
\text{Hom}_\Delta(V_\psi, V_\chi) \cong \text{Hom}_\Gamma(\text{Ind}_\Delta^\Gamma V_\psi, V_\chi);
\]

but as \( \psi \) and \( \chi \) are irreducible, the left hand side is nonzero if and only if (i) is satisfied; the right hand side is nonzero if and only if (ii) is satisfied. \( \square \)

In this case \( \Gamma \) operates on the set \( \text{Irr}(\Delta) \) of irreducible finite characters of \( \Delta \): if \( \psi \in \text{Irr}(\Delta) \) and \( \gamma \in \Gamma \), then

\[
\psi^\gamma : \Delta \to \overline{Q}_\ell, \quad \delta \mapsto \psi(\gamma \delta \gamma^{-1})
\]

is again an irreducible finite character of \( \Delta \).

**Theorem 12 (Clifford).** Let \( \chi : \Gamma \to \overline{Q}_\ell \) be an irreducible finite character. Then the set of irreducible characters \( \Delta \to \overline{Q}_\ell \) lying below \( \chi \) is precisely one \( \Gamma \)-orbit in \( \text{Irr}(\Delta) \).

\(^7\text{see [7, Corollary 30.14]}\)
Proof. The corresponding statement for finite groups, from which our generalisation directly follows, is proved in \[4\] Theorem 1.

We shall apply these concepts for \( \Gamma = G = \pi_1^\text{et}(X, \bar{x}) \) as in the beginning of this section. For this, recall that continuous finite-dimensional \( \overline{\mathbb{Q}}_\ell \)-representations of an étale fundamental group are equivalent to smooth \( \overline{\mathbb{Q}}_\ell \)-sheaves on the corresponding variety. We will not apply this to \( G \) directly but to its open normal torsion-free subgroups which are fundamental groups of honest algebraic curves.

To be more technical, let \( S \) be a connected noetherian scheme over \( k \) and \( \bar{s} \) a geometric point of \( S \). The fibre functor at \( \bar{s} \) then provides an equivalence of categories

\[
\{ \text{locally constant sheaves of finite abelian groups on } S \} \xrightarrow{\sim} \{ \text{finite abelian groups with continuous } \pi_1^\text{et}(S, \bar{s})\text{-action} \}
\]

(1)

and its more elaborate version

\[
\{ \text{smooth } \overline{\mathbb{Q}}_\ell\text{-sheaves on } S \} \xrightarrow{\sim} \{ \text{finite-dim. cont. representations of } \pi_1^\text{et}(S, \bar{s}) \text{ over } \overline{\mathbb{Q}}_\ell \}
\]

(2)

for the notion of \( \overline{\mathbb{Q}}_\ell \)-sheaves and the proof of this equivalence see \[16\] Appendix A. A finite representation in our sense then corresponds to a smooth \( \overline{\mathbb{Q}}_\ell \)-sheaf which becomes trivialised on some finite étale cover of \( S \).

In the case relevant for us, these correspondences extend to cohomology:

**Proposition 13.** Let \( Y \) be a smooth curve of genus at least two over a separably closed field \( k \) of characteristic \( p \geq 0 \), and let \( \mathcal{F} \) be a locally constant sheaf of finite abelian groups on \( X \) without \( p \)-torsion. Let \( \mathcal{F}_{\bar{y}} \) be its fibre at \( \bar{y} \). Then there is a natural isomorphism

\[
H^1(Y, \mathcal{F}) \cong H^1(\pi_1^\text{et}(Y, \bar{y}), \mathcal{F}_{\bar{y}})
\]

(continuous group cohomology). Similarly, let \( \mathcal{F} \) be a smooth \( \overline{\mathbb{Q}}_\ell \)-sheaf on \( Y \) corresponding via (2) to the representation \( V \) of \( \pi_1^\text{et}(Y, \bar{y}) \). Then there is a natural isomorphism of \( \overline{\mathbb{Q}}_\ell \)-vector spaces

\[
H^1(Y, \mathcal{F}) \cong H^1(\pi_1^\text{et}(Y, \bar{y}), V).
\]

**Proof.** This is a folklore result, see e.g. \[25\] p. 510].

Now we have all technical ingredients at hand to begin with the proof of Theorem 8.

Consider the following scenario: \( Y \to X \) is an étale covering which is also normal and which factors over \( X_1 \), and such that \( Y \) is a curve (and not merely a stack). This corresponds to an open normal subgroup \( F \subseteq G \) which is torsion-free and contained in \( G_1 \). (To see that such an \( F \) exists, recall that by definition of a Fuchsian orbifold there exists a torsion-free open subgroup of \( G \); by intersecting it with its conjugates and with \( G_1 \) we arrive at a suitable subgroup.) Then \( G \) operates via its quotient \( G/F \) on \( Y \) and therefore on its étale cohomology.

**Lemma 14.** Let \( q \neq p \) be an odd prime. Then \( G/F \) operates faithfully on the étale cohomology group \( H^1(Y, \mu_q) \).

**Proof.** This follows from the main result of \[23\], noting that \( G/F \) operates faithfully on \( Y \).
Similarly, $G$ operates on $H^1(Y, \mathbb{Q}_\ell) = H^1(F, \mathbb{Q}_\ell)$ which is the dual of $F^{ab} \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell$; the action of $G$ can be understood as derived from that on $F^{ab}$ via conjugation on $F$.

Recall that $\varphi$ is a $G_1$-normal automorphism of $G$, hence $\varphi(F) = F$, and $\varphi$ induces a linear automorphism of $H^1(Y, \mathbb{Q}_\ell)$ which we denote by $\varphi_\ell$.

**Lemma 15.** Let $\chi : G \to \mathbb{Q}_\ell$ be a finite irreducible character contained in $H^1(Y, \mathbb{Q}_\ell)$. Then $\chi \circ \varphi = \chi$.

To prove this lemma we need to modify $\ell$ conveniently, making use of:

**Theorem 16.** Let $Y$ be a smooth proper curve over a separably closed field of characteristic $p \geq 0$ and $f$ an automorphism of $Y$. Then the trace of $f$ acting on $H^1(Y, \mathbb{Q}_\ell)$ is a rational integer independent of $\ell \neq p$.

**Proof of Theorem 16.** On the other nonzero cohomology groups $H^0$ and $H^2$, $f$ acts as the identity. Therefore our statement follows from the well-known corresponding facts for the Lefschetz number and for $H^1(Y, \mathbb{Q}_\ell)$.

**Proof of Lemma 15.**

By Theorem 16 this statement is independent of $\ell$. By this we mean the following:

Let $\ell' \neq p$ be some other prime, and choose a field isomorphism $\mathbb{Q}_\ell \cong \mathbb{Q}_{\ell'}$. This isomorphism induces a bijection between finite characters (i.e. between isomorphism classes of finite representations, see Lemma 10) of $G$ with values in these two fields. We identify these two sets of characters by this bijection. Then by Theorem 16 the characters of $G$ operating on $H^1(Y, \mathbb{Q}_\ell)$ and $H^1(Y, \mathbb{Q}_{\ell'})$ agree, hence an irreducible character occurs in the former if and only if it occurs in the latter. So the statement of the lemma is independent of $\ell$.

Now assume that $\chi$ is defined on a finite quotient of $G$ of order $m$; then for $\ell' \equiv 1 \mod m$, which can always be found by Dirichlet’s theorem on primes in arithmetic progressions, $\mathbb{Q}_{\ell'}$ contains all $m$-th roots of unity, hence all values of $\chi$. To sum up, $\chi$ can be assumed to have values in $\mathbb{Q}_\ell$.

Hence $\chi$ occurs in $H^1(Y, \mathbb{Q}_\ell)$ and therefore also in its dual $F^{ab} \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell$, which is then a finitely generated free $\mathbb{Z}_\ell$-module on which $G$ acts via $G/F$, and on which $\varphi$ again defines a linear automorphism $\varphi_\ell$. We claim that $\varphi_\ell(M) = M$ for every $\mathbb{Z}_\ell[G]$-submodule $M \subseteq F_\ell$.

Namely, $M = N/[F,F]$ with some closed normal subgroup $N$ of $G$; since $\varphi$ is normal and $N$ is the intersection of the finite normal subgroups it is contained in, we find that $\varphi(N) = N$, whence $\varphi_\ell(M) = M$.

Now we use a trick from 14: let $M \subseteq F_\ell$ be a $\mathbb{Z}_\ell[G]$-submodule with $M \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell \cong V_\chi$. Then $\varphi_\ell$ is an automorphism of this module, but also $G$ operates on $M$ by conjugation (denoted, as usual, by exponentiation). Now unravelling of definitions yields that

$$(\varphi_\ell(m))^{\varphi(g)} = \varphi_\ell(m^g)$$

for $m \in M$ and $g \in G$. That is, $\varphi(g) \circ \varphi_\ell = \varphi_\ell \circ g$ as automorphisms of $M$; in other words, $g$ and $\varphi(g)$ as automorphisms of $M$ are conjugate. Therefore they have the same trace, i.e. $\chi(g) = \chi(\varphi(g))$. □

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8 following the proof of [9, Lemma 25]
Lemma 17. Let $F \subseteq G$ as before, corresponding to $Y \to X$. Let furthermore $E \subseteq F$ be another open subgroup which is normal both in $F$ and in $G$; this corresponds to $Z \to Y \to X$. Let $\psi : F \to \Ql$ be an irreducible finite character factoring through $F/E$. Then there exists an irreducible finite character $\chi : G \to \Ql$ lying above $\psi$ with $\chi$ contained in $H^1(Z, \Ql)$.

Proof. Let $V_\psi$ be the $F$-module corresponding to $\psi$, and

$$W = \text{Ind}_F^G V_\psi$$

the associated $G$-module. Now, $\chi$ lies above $\psi$ if and only if it is contained in $W$. So what we have to show is that some irreducible $G$-submodule of $W$ is also contained in $H^1(Z, \Ql)$ or, equivalently, in its dual $E \otimes \Ql$; since the category of finite $G$-representations is semi-simple, this amounts to showing that

$$\text{Hom}_G(E \otimes \Ql, W) \neq 0.$$  

We can rewrite the left hand side:

$$\text{Hom}_G(E \otimes \Ql, W) \cong H^1(E, W)^G \cong H^1(G, W)$$

(continuous cohomology). The second identification is justified by the inflation-restriction exact sequence

$$H^1(G/E, W^E) \to H^1(G, W) \to H^1(E, W)^G \to H^2(G/E, W^E)$$

and the observation that $G/E$ is a finite group and $W^E$ a $\Ql$-vector space, so the first and the last cohomology groups vanish. Now by Shapiro’s lemma (see e.g. [26, p. 172]), $H^1(G, W) \cong H^1(F, V_\psi)$ since $W$ is induced from $V$ and $F$ has finite index in $G$. Now let $\mathcal{V}$ be the $\Ql$-sheaf on $Z$ corresponding to $V_\psi$; we can then identify $H^1(F, V_\psi)$ with $H^1(Y, \mathcal{V})$. Finally by Theorem 18 below,

$$h^0(Y, \mathcal{V}) - h^1(Y, \mathcal{V}) + h^2(Y, \mathcal{V}) = e(Y, \mathcal{V}) = e(Y) \cdot \text{rank} \mathcal{V} < 0,$$

whence $h^1(F, V_\psi) = h^1(Y, \mathcal{V}) > 0$.  

Theorem 18 (Raynaud). Let $Y$ be a proper smooth algebraic curve over an algebraically closed base field. Let $\mathcal{F}$ be a lisse $\Ql$-sheaf over $Y$ of rank $d$. Then the following relation holds for the Euler–Poincaré characteristics:

$$e(Y, \mathcal{F}) = e(Y) \cdot \text{rank} \mathcal{F}.$$  

Proof. See [20].  

Lemma 19. Let $F, G$ and $G_1$ as above and let $\varphi$ be a $G_1$-normal automorphism of $G$. Then $\varphi$ induces an inner automorphism of $G/F$.

Proof. Choose some odd prime $q$ larger than both $(G : F)$ and $p$. Recall that $G$ acts on $H^1(Y, \mu_q)$. By Proposition 13 this cohomology group can be identified with $H^1(F, \mu_q) = \text{Hom}(F, \mu_q)$. Thus

\footnotesize{9}following the proof of [9, Lemma 26]  
\footnotesize{10}following the proof of [9, Theorem 27]
interpreting cohomology classes in $H^1(F, \mu_q)$ as irreducible characters $F \to \mu_q \subset \overline{Q}_l$, we set for every $g \in G$:

$$M^{\phi, \psi} = \{ \psi \in H^1(F, \mu_q) \mid \psi \circ \phi = \psi^g \}.$$ 

We claim that

$$H^1(F, \mu_q) = \bigcup_{g \in G} M^{\phi, \psi}:$$

(3)

let $\psi: F \to \mu_q$ be an element of this cohomology group, then by Lemmas [15] and [17] there exists an irreducible finite character $\chi: G \to \overline{Q}_l$ of $G$ above $F$ with $\chi \circ \phi = \chi$. By Theorem [12] this means that there exists a $g \in G$ with $\psi \circ \phi = \psi^g$, since both $\psi$ and $\psi \circ \phi$ lie below $\chi$. This proves (3).

Now $M^{\phi, \psi}$ only depends on the residue class of $g$ modulo $F$, and therefore there are at most $(G:F)$ distinct subspaces on the right hand side in (3). But all these spaces are finite-dimensional $\mathbf{F}_q$-vector spaces, and $q > (G:F)$. So there must be at least one of them which is already equal to $H^1(F, \mu_q)$; let us assume that $H^1(F, \mu_q) = M^{\phi, \psi_0}$, i.e.

$$\psi \circ \phi = \psi^g_0 \text{ for all } \psi \in H^1(F, \mu_q).$$

(4)

Next, set for every $g \in G$:

$$M^g = \{ \psi \in H^1(F, \mu_q) \mid \psi = \psi^g \}.$$ 

Since $G/F$ operates faithfully on $H^1(F, \mu_q)$ by Lemma [14] $M^g$ must be a proper $\mathbf{F}_q$-subspace of $H^1(F, \mu_q)$ whenever $g \in G \setminus F$. Again, $M^g$ only depends on the coset of $g$ modulo $F$, so there are only $(G:F) - 1 < q$ distinct subspaces $M^g$ for $g \in G \setminus F$; hence they cannot cover the entire space, and there exists a $\psi_0 \in H^1(F, \mu_q)$ not contained in any of them, i.e. satisfying

$$\psi_0 \neq \psi^g_0 \text{ for all } g \in G \setminus F.$$ 

(5)

Combining (4) and (5), we obtain

$$(\psi_0)^{g_0} = (\psi_0^g)^{g_0} = \psi_0^{g_0} \circ \phi = (\psi_0 \circ \phi)^{\phi(g)} = (\psi_0^{g_0})^{\phi(g)} = \psi_0^{g_0\phi(g)},$$

whence $(\psi_0)^{g_0\phi(g)}^{-1}g_0^{-1} = \psi_0$, and by (5) this yields $g_0g_0\phi(g)^{-1}g_0^{-1} \in F$. That is, $\phi$ operates as conjugation by $g_0$ on $G/F$. \hfill \Box

Proof of Theorem [15] Recall that $G = \pi_1^{et}(\overline{X}, \overline{x})$ and $G_1 = \pi_1^{et}(\overline{X}_1, \overline{x}_1)$. The set $\mathcal{N}$ of those open normal subgroups $F \subseteq G$ that are contained in $G_1$ is cofinal in the directed set of all open normal subgroups of $G$, that is

$$G = \lim_{F \in \mathcal{N}} G/F.$$ 

Now by assumption $\phi(F) = F$ for every $F \in \mathcal{N}$, and $\phi$ operates as an inner automorphism on each $G/F$. Choose, for every $F \in \mathcal{N}$, an element $g_F \in G$ such that $\phi$ acts as conjugation by $g_FF$ on $G/F$. Since $G$ is compact, the net $(g_F)_{F \in \mathcal{N}}$ must have a convergent subnet $(g_{F'})_{F' \in \mathcal{M}}$. By definition of “subnet”, $\mathcal{M}$ is again cofinal in all open normal subgroups, so that

$$G = \lim_{F \in \mathcal{M}} G/F.$$ 

Let $g = \lim_{F \in \mathcal{M}} g_F \in G$. This means that for every $F \in \mathcal{M}$ there exists some $E \in \mathcal{M}$ with $\phi$ operating as conjugation by $g_E$ on $G/E$; taking the limit over all these $E$ we see that $\phi$ is indeed conjugation by $g$ on $G$. \hfill \Box
III. Galois Actions

Let $k$ be a number field and let $\mathcal{X}$ be a closed Fuchsian orbifold over $k$. Denote the base change $\mathcal{X} \times_{\text{Spec} \, k} \text{Spec} \, \overline{Q}$ by $\mathcal{X}_{\overline{Q}}$. Then for a geometric point $\bar{x}$ of $\mathcal{X}$ with generic stabiliser, for simplicity assumed to lie over some point $x \in \mathcal{X}(k)$, we obtain a natural split short exact sequence of profinite groups (see [1, IX.6.1]):

$$1 \longrightarrow \pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}}, \bar{x}) \longrightarrow \pi^\text{ét}_{1}(\mathcal{X}, \bar{x}) \longrightarrow \Gamma_{k} \longrightarrow 1. \quad (6)$$

This yields an action of $\Gamma_{k}$ the “geometric fundamental group” $\pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}}, \bar{x})$ and hence, after forgetting the basepoint, an outer action of $\Gamma_{k}$ on $\pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}})$. The latter action also exists if $\mathcal{X}(k) = \emptyset$ and can be constructed by Galois descent.

**Proposition 20.** The exterior Galois action of $\Gamma_{k}$ on $\pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}})$ is faithful.

**Proof.** Choose a normal étale covering $f : Y \to \mathcal{X}$ where $Y$ is a geometrically connected curve (i.e. an “honest” curve and not merely a stack) over $k$. Choose further a point $\bar{y} \in Y(\overline{Q})$ which is mapped to a point with generic stabilisers under $f$, and consider the corresponding action of $\Gamma_{k}$ on $\pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}}, f(\bar{y}))$; denote the latter group by $G$ and the subgroup $\pi^\text{ét}_{1}(Y_{\overline{Q}}, \bar{y})$ by $F$.

Now consider the closed subgroup

$$\Delta = \{ \sigma \in \Gamma_{k} \mid \sigma \text{ operates by an inner automorphism on } G \} \quad (7)$$

of $\Gamma_{k}$. Let $\mathcal{Z}(G)$ be the centre of $G$; it is a closed normal subgroup of $G$.

**Proposition 21.** The action of $\Gamma_{k}$ on $\text{GC}(\mathcal{X}_{\overline{Q}})$ is faithful.

**Proof.** Assume that $\sigma \in \Gamma_{k}$ operates trivially on $\text{GC}(\mathcal{X}_{\overline{Q}})$.

Choose some normal étale covering $X_{1} \to \mathcal{X}$ where $X_{1}$ is a geometrically connected curve over $k$, and choose convenient basepoints as above (suppressed in the notation). Every open normal subgroup of $\pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}})$ contained in $\pi^\text{ét}_{1}(X_{1})$ defines an element of $\text{GC}(\mathcal{X}_{\overline{Q}})$; this amounts to an $\Gamma_{k}$-equivariant injection from the set of such subgroups to $\text{GC}(\mathcal{X}_{\overline{Q}})$. Since $\sigma$ operates trivially on the image, it has to operate trivially on the domain. But by Jarden’s property for the pair $(\pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}}), \pi^\text{ét}_{1}(X_{1}))$ (Theorem [8]), $\sigma$ operates on $\pi^\text{ét}_{1}(\mathcal{X}_{\overline{Q}})$ as an inner automorphism. By Proposition 20 this implies $\sigma = \text{id}$. 

\[11^1\text{In fact it is finite since it cannot meet } F : F \text{ is centrefree by [2, Proposition 18]. We conjecture that it is trivial.} \]
We now deduce Theorems 1 to 3 from Corollary 21 by suitable choices of $X_{\mathbb{Q}}$.

**Proof of Theorem 1.** If $Y$ is a Hurwitz curve over $\overline{\mathbb{Q}}$, then $Y/\text{Aut}(Y)$ is isomorphic to the projective line $\mathbb{P}^1$, and the projection map $X \to \mathbb{P}^1$ has precisely three ramification points, which can be taken as $0, 1, \infty$ after a suitable change of coordinates. Further, the orders of ramifications at these points are 2, 3 and 7. Vice versa, if $Y \to \mathbb{P}^1$ is a normal ramified covering with ramification points $0, 1, \infty$ and orders 2, 3, 7 respectively, then $Y$ is a Hurwitz curve and the Deck transformation group of this covering is the full automorphism group of $Y$.

That said, we consider the following Fuchsian orbifold $X$ over $\mathbb{Q}$: its underlying coarse moduli space is $\mathbb{P}^1_{\mathbb{Q}}$, and it has trivial point stabilisers except for the points $0, 1$ and $\infty$ where the stabilisers are $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/7\mathbb{Z}$, respectively. Then the elements of $\text{GC}(X_{\mathbb{Q}})$ and the isomorphism classes of Hurwitz curves are in canonical $\Gamma_{\mathbb{Q}}$-equivariant bijection, so Theorem 1 follows from Corollary 21. \[ \square \]

This result should be compared with the relative rarity of Hurwitz curves as mentioned in the introduction. From [5] we read that the only $g \leq 100$ such that there exist Hurwitz curves of genus $g$ are 3, 7, 14 and 17, and the tables in [6] tell us about their behaviour under $\Gamma_{\mathbb{Q}}$:

(i) The only Hurwitz curve in genus three is *Klein’s quartic curve* with homogeneous equation $x^3 y + y^3 z + z^3 x = 0$, hence fixed by $\Gamma_{\mathbb{Q}}$.

(ii) The only Hurwitz curve in genus seven is the *Macbeath curve* which is therefore again fixed by $\Gamma_{\mathbb{Q}}$. However, no simple defining equations over $\mathbb{Q}$ are known; there is a simple model over $\mathbb{Q}(\zeta_7)$, and in [10] an extremely complicated model over $\mathbb{Q}$ was found.

(iii) In genus fourteen there are three Hurwitz curves known as the *first Hurwitz triplet*. They are defined over $k = \mathbb{Q}(\cos \frac{2\pi}{7})$ and permuted simply transitively by $\text{Gal}(k|\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$.

(iv) Finally, in genus seventeen there are two Hurwitz curves, defined over $\mathbb{Q}(\sqrt{-3})$ and exchanged by this field’s nontrivial automorphism.

**Proof of Theorem 2.** This is analogous to the proof of Theorem 1 with the ramification indices $(2, 3, 7)$ replaced by $(p, q, r)$. \[ \square \]

**Proof of Theorem 3.** Theorem 1 in [22] can be reinterpreted as follows: Hurwitz translation surfaces are precisely the normal translation coverings of a torus with one ramification point and ramification order two at this point. To define a Galois action, we have to fix the algebraic structure on the covered torus (actually a model over $\overline{\mathbb{Q}}$). It does not matter for our proof which one we take, but it is convenient to take the standard model of the square torus, $T: y^2 = x^3 - x$. Then let $\mathcal{T}$ be the Fuchsian orbifold over $\mathbb{Q}$ which has $T$ as its coarse moduli space and precisely one point with nontrivial stabiliser; that point is the point at infinity, and its stabiliser is $\mathbb{Z}/2\mathbb{Z}$. Then Hurwitz translation surfaces are in canonical $\Gamma_{\mathbb{Q}}$-equivariant bijection with the elements of $\text{GC}(\mathcal{T}_{\overline{\mathbb{Q}}})$. \[ \square \]
Proof of Theorem 4. Let $X$ be as in the proof of Theorem 1, so that Hurwitz curves correspond to elements of $G C(X, \mathbb{Q})$, and set $G = \pi_1(X, \mathbb{Q})$. Every open normal subgroup $N$ of $G$ contains one which is stable under $\Gamma_X$: the setwise stabiliser of $N$ in $\Gamma_X$ has finite index in $\Gamma_X$, therefore

$$\tilde{N} = \bigcap_{\sigma \in \Gamma_X} \sigma(N)$$

is an open normal subgroup of $G$ contained in $N$. This means that $G$ can also be described as the projective limit of all $G/N$ with $N$ open, normal and stable under $\Gamma_X$. We conclude (using the compactness of $G$ as in the proof of Theorem 8) that $\sigma$ operates by a non-trivial outer automorphism on some such $G/N$. Now $N$ corresponds to a Hurwitz curve $Y$ with moduli field $\mathbb{Q}$; we claim that $Y$ has the desired properties.

The Hurwitz group $H = G/N = \text{Aut}_\mathbb{Q}Y$ sits in a short exact sequence:

$$1 \longrightarrow \text{Aut}_\mathbb{Q}Y \longrightarrow \text{Aut}_\mathbb{Q}Y \longrightarrow \Gamma_X \longrightarrow 1.$$  \hfill (9)

Here the middle term means the group of all automorphisms of $Y$ as a $\mathbb{Q}$-scheme (or, which amounts to the same, as a scheme without any further structure). A choice of a model $\mathcal{Y}$ over $\mathbb{Q}$ yields a splitting $s$ of this sequence.

Now $\text{Aut}_\mathbb{Q}Y$ acts naturally on the étale cohomology group $H^1(Y, F_\ell)$; by Lemma 13 the subgroup $H = \text{Aut}_\mathbb{Q}Y$ operates faithfully on this cohomology group. But $Y$ was chosen in such a way that $s(\sigma)hs(\sigma)^{-1} \neq h$ for some $h \in H = \text{Aut}_\mathbb{Q}Y$, hence also these elements operate differently on $H^1(Y, F_\ell)$. But this means that $s(\sigma)$ has to operate nontrivially on this cohomology group. Finally, the $\ell$-torsion points of the Jacobian are canonically identified with the dual of $H^1(Y, F_\ell)$, so $\sigma$ also operates nontrivially there. \hfill \Box

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