The Chabauty space of $\mathbb{Q}_p^\times$

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Abstract

Let $\mathcal{C}(G)$ denote the Chabauty space of closed subgroups of the locally compact group $G$. In this paper, we first prove that $\mathcal{C}(\mathbb{Q}_p^\times)$ is a proper compactification of $\mathbb{N}$, identified with the set $N$ of open subgroups with finite index. Then we identify the space $\mathcal{C}(\mathbb{Q}_p^\times) \setminus N$ up to homeomorphism: e.g. for $p = 2$, it is the Cantor space on which 2 copies of $\overline{\mathbb{N}}$ (the 1-point compactification of $\mathbb{N}$) are glued.

1 Introduction

In 1950, Chabauty [Cha] introduced a topology on the set $\mathcal{F}(X)$ of closed subsets of a locally compact space $X$, turning $\mathcal{F}(X)$ into a compact space, see Definition 1 below; for $X$ discrete, this is nothing but the product topology on $2^X$. When $G$ is a locally compact group, the set $\mathcal{C}(G)$ of closed subgroups of $G$ is a closed subset of $\mathcal{F}(G)$, so $\mathcal{C}(G)$ is a compact set canonically associated with $G$: we call it the Chabauty space of $G$.

Definition 1.1. For a locally compact space $X$, the Chabauty topology on $\mathcal{F}(X)$ has as open sets finite intersections of subsets of the form

$$\mathcal{O}_K = \{F \in \mathcal{F}(X) : F \cap K = \emptyset\}$$

with $K$ compact in $X$, and

$$\mathcal{O}_U = \{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$$

with $U$ open in $X$.

Let us give some examples of Chabauty spaces, first for additive groups of some locally compact fields:
Example 1.2. 1. For $G = \mathbb{R}$, the Chabauty space $C(\mathbb{R})$ is homeomorphic to a closed interval, say $[0, +\infty]$, with the subgroup $\lambda \mathbb{Z}$ (with $\lambda > 0$) being mapped to $\lambda$, the subgroup $\{0\}$ being mapped to $+\infty$, and the subgroup $\mathbb{R}$ being mapped to $0$.

2. For $G = \mathbb{C}$, the situation is already much more subtle, and it was proved by Hubbard and Pourezza [HP] that $C(\mathbb{C})$ is homeomorphic to the 4-sphere.

3. For $G = \mathbb{Q}_p$, the field of $p$-adic numbers, every non-trivial closed subgroup is of the form $p^k \mathbb{Z}_p$ for some $k \in \mathbb{Z}$, so $C(\mathbb{Q}_p)$ is homeomorphic to the 2-point compactification of $\mathbb{Z}$, namely $\mathbb{Z} \cup \{\pm \infty\}$, with the subgroup $p^k \mathbb{Z}_p$ being mapped to $k \in \mathbb{Z}$, the subgroup $\{0\}$ being mapped to $+\infty$ and the subgroup $\mathbb{Q}_p$ being mapped to $-\infty$.

Let us turn to multiplicative groups of some locally compact fields:

Example 1.3. 1. For $G = \mathbb{R}^\times$, since $G \simeq \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$, we get 3 types of closed subgroups:

- Closed subgroups of $\mathbb{R}$, contributing a copy of $[0, +\infty]$.
- Subgroups which are products $H \times \mathbb{Z}/2\mathbb{Z}$, with $H$ a closed subgroup of $\mathbb{R}$. They contribute another copy of $[0, +\infty]$, with origin the subgroup $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$ and extremity the subgroup $\{0\} \times \mathbb{Z}/2\mathbb{Z}$.
- Infinite cyclic subgroups which are not contained in $\mathbb{R}$; those are of the form $\langle (\lambda, 1) \rangle$, for some $\lambda > 0$. So they contribute a copy of $]0, +\infty[$.

As $\langle (\lambda, 1) \rangle$ converges to $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$ for $\lambda \to 0$, and to $\{(0, 0)\}$ for $\lambda \to +\infty$, we see that subgroups of the 3rd type “connect” subgroups of the first and the second type, so that $C(\mathbb{R}^\times)$ is homeomorphic to a closed interval.

2. For $G = \mathbb{C}^\times \simeq \mathbb{R} \times \mathbb{T}$, the structure of $C(G)$ was elucidated by Haettel [Hae]: it is path connected but not locally connected, and with uncountable fundamental group.

In this paper, we deal with the multiplicative group $G = \mathbb{Q}_p^\times$ of the field $\mathbb{Q}_p$. We enjoy the general results by Y. Cornulier [Cor] on $C(H)$ for $H$ a locally compact abelian group. Applied to $G = \mathbb{Q}_p^\times$, they yield that $C(G)$ is totally disconnected and uncountable (Theorems 1.5 and 1.6 in [Cor]), that $\{1\}$ is not isolated in $C(G)$ while $\{G\}$ is isolated in $C(G)$ (Lemmas 4.1 and 4.2 in [Cor]). More generally, a closed subgroup $H$ defines an isolated point in $C(G)$ if and only if $H$ is open with finite index in $G$ (Theorem 1.7 in [Cor]).
To describe our main result, recall that a compact metric space \( Y \) is a *proper compactification* of \( N \) if \( Y \) contains an open, countable, dense, discrete subset \( N \). It is known (see e.g. Propositions 2.1 and 2.3 in [Tsa]) that every non-empty compact metric space can be written as \( Y \setminus N \), with \( Y \) and \( N \) as above, in a unique way up to homeomorphism.

We will use the following notations: \( \overline{N} = N \cup \{ \infty \} \) is the one-point compactification of \( N \), \( C \) is the Cantor space, \( [k] \) is the set \( \{1,2,...,k\} \), and \( d(n) \) is the number of divisors of \( n \).

**Theorem 1.4.** Let \( p \) be a prime. For \( G = \mathbb{Q}_p^\times \):

1. \( C(G) \) is a proper compactification of \( N \), viewed as the set \( N \) of open subgroups of finite index.

2. For \( p \) odd, the space \( C(G) \setminus N \) is homeomorphic to the space obtained by glueing \( [d(p-1)] \times \overline{N} \) on \( C \), with the \( d(p-1) \) accumulation points of \( [d(p-1)] \times \overline{N} \) being identified to \( d(p-1) \) pairwise distinct points of \( C \).

3. For \( p = 2 \), the space \( C(G) \setminus N \) is homeomorphic to the space obtained by glueing \( [2] \times \overline{N} \) on \( C \), with the 2 accumulation points of \( [2] \times \overline{N} \) being identified to 2 pairwise distinct points of \( C \).

In the above picture, the isolated points of \( C(G) \setminus N \) correspond to the closed infinite subgroups of \( \mathbb{Z}_p^\times \), the invertible group of the ring \( \mathbb{Z}_p \) of \( p \)-adic integers; and (for \( p \) odd) the glueing points are the finite subgroups of \( \mathbb{Q}_p^\times \), which are exactly the cyclic groups \( C_d \) of \( d \)-roots of unity, for \( d \) dividing \( p-1 \). It follows from the result that every non-isolated point of \( C(G) \setminus N \) is a condensation point, and the Cantor-Bendixson rank of \( C(G) \setminus N \) is 1.

To prove Theorem 1.4, we use the decomposition as topological groups \( \mathbb{Q}_p^\times \simeq \mathbb{Z}_p \times C_{p-1} \times \mathbb{Z} \) (for \( p \) odd), and \( \mathbb{Q}_2^\times \simeq \mathbb{Z}_2 \times C_2 \times \mathbb{Z} \) (for all this, see section 3 in Chapter II of [Ser]). For a locally compact abelian group \( H \) with Pontryagin dual \( \hat{H} = \text{hom}(H, \mathbb{T}) \), the space \( C(H) \) identifies with \( C(\hat{H}) \) for \( H \): see Proposition 2.5 below for the precise statement of this result of Cornulier [Cor]. So, actually we work with the Pontryagin dual of \( \mathbb{Q}_p^\times \), which for \( p \) odd identifies with \( C_{p^\infty} \times C_{p-1} \times \mathbb{T} \), where \( C_{p^\infty} \) denotes the \( p \)-Prüfer group.

At the end of the Introduction in [Cor], Cornulier mentions as a non-trivial question to determine the homeomorphism type of \( C(G) \), when \( G \) is a totally disconnected locally compact abelian group that fits into a short exact sequence

\[ 0 \to A \times P \to G \to D \to 0, \]
where $A$ is finitely generated abelian, $D$ is Artinian (i.e. it has no infinite decreasing chain of subgroups), and $P$ is a finite direct product of finite groups and groups isomorphic to $\mathbb{Z}_p$ (for various $p$’s, multiplicities allowed). Since $Q_p^\infty$ is of that form, Theorem 1.4 and its extension Theorem 4.7 can be seen as a contribution to Cornulier’s question.

The structure of the paper is as follows: in section 2, we gather some preliminary material about the Chabauty space $C(G)$. In section 3, we determine the Chabauty space of $C(C_n \times \mathbb{Z})$: the Chabauty space of any finitely generated abelian group has been determined in Theorem C of [CGP2], but we give a direct proof for completeness in the case of $C_n \times \mathbb{Z}$; this will be needed in the proof of our main result. Although we do not need it, we see that the determination of the Chabauty space of $\mathbb{Z}^2$ follows easily from the one of $C_n \times \mathbb{Z}$. Finally, in section 4 we prove Theorem 1.4 along the lines sketched above.

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2 Preliminaries

2.1 General locally compact groups

In this subsection $G$ will denote a locally compact second countable group. First, we recall a result on convergence in $C(G)$ that we will use throughout the paper without further reference (see [Pau] for a proof):

Proposition 2.1. A sequence $(H_n)_{n>0}$ in $C(G)$ converges to $H$ if and only if

1. $\forall h \in H, \exists h_n \in H_n$ such that $h = \lim h_n$.

2. For every sequence $(h_{n_k})_{k>0}$ in $G$ with $h_{n_k} \in H_{n_k}$, $n_{k+1} > n_k$, converging to some $h \in G$, we have $h \in H$. \hfill $\square$

The following is lemma 1.3(1) in [CGP1].

Lemma 2.2. Let $N \triangleleft G$ be a closed normal subgroup, let $p : G \to G/N$ denote the quotient map. The map $p^* : C(G/N) \to C(G) : H \mapsto p^{-1}(H)$

is a homeomorphism onto its image. \hfill $\square$

The proof of the next lemma is an easy exercise.
Lemma 2.3. Let $N \triangleleft G$ be a compact normal subgroup. If $(H_k)_{k>0}$ converges to $H$ in $\mathcal{C}(G/N)$, then $(H_kN)_{k>0}$ converges to $HN$ in $\mathcal{C}(G)$. \hfill \Box

If $L$ is a closed subgroup of $G$, the map $\mathcal{C}(G) \to \mathcal{C}(L) : H \mapsto H \cap L$ is in general not continuous (take e.g. $G = \mathbb{R}^2$, and $L$ a 1-dimensional subspace). We single out a case where it is continuous.

Lemma 2.4. Let $G = D \times L$ be a locally compact group, with $D$ discrete. Identify $L$ with $\{1\} \times L$. The map $\mathcal{C}(G) \to \mathcal{C}(L) : H \mapsto H \cap L$ is continuous.

Proof: Assume that the sequence $(H_n)_{n>0}$ converges to $H$ in $\mathcal{C}(G)$, let us check that $(H_n \cap L)_{n>0}$ converges to $H \cap L$ in $\mathcal{C}(L)$.

- If $(1,l)$ is in $H \cap L$, we can write $(1,l) = \lim_{n \to \infty} (d_n, l_n)$, with $(d_n, l_n) \in H_n$ for every $n > 0$. Since $D$ is discrete, we have $d_n = 1$ for $n$ large enough, so that we have also $(1,l) = \lim_{n \to \infty} (1, l_n)$, with $(1,l_n) \in H_n \cap L$ for every $n$.

- If a sequence $((1,l_n))_{n>0}$, with $(1, l_n) \in H_n \cap L$, converges to $(1,l) \in L$, then by assumption we have $(1,l) \in H$, i.e. $(1,l) \in H \cap L$. \hfill \Box

If $G$ is a locally compact abelian group, recall that $\hat{G} = \text{hom}(G, \mathbb{T})$ denote its Pontryagin dual. For $H$ a closed subgroup of $G$, denote by $H^\perp$ the orthogonal of $H$:

$$H^\perp = \{ \chi \in \hat{G} : \chi|_H = 1 \}.$$ 

The following result is due to Cornulier (Theorem 1.1 in [Cor]):

Proposition 2.5. The orthogonal map $\mathcal{C}(G) \to \mathcal{C}(\hat{G}) : H \mapsto H^\perp$ is an inclusion-reversing homeomorphism. \hfill \Box

2.2 Discrete groups

The symbol $k \gg 0$ means “for $k$ large enough”. In this section, $G$ denotes a discrete group. Let $(P_k)_{k>0}$, $P$ denote subsets of $G$. We say that $P$ is locally contained in $(P_k)_{k>0}$ if, for every finite subset $S \subseteq P$, we have $S \subseteq P_k$ for $k \gg 0$. In other words, $P$ is locally contained in $(P_k)_{k>0}$ if every finite subset of $P$ is contained in all but finitely many $P_k$’s.

Lemma 2.6. Let $(H_k)_{k>0}, H$ be subgroups of $G$.

1. The sequence $(H_k)_{k>0}$ converges to $H$ in $\mathcal{C}(G)$ if and only if $H$ is locally contained in $(H_k)_{k>0}$ and $G \setminus H$ is locally contained in $(G \setminus H_k)_{k>0}$.

2. If $(H_k)_{k>0}$ converges in $\mathcal{C}(G)$ to $H$ and $H$ is finitely generated, then $H \subset H_k$ for $k \gg 0$. 

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3. The sequence \((H_k)_{k>0}\) converges to the trivial subgroup \(\{e\}\) of \(G\), if and only if for every finite subset \(F \subset G \setminus \{e\}\), we have \(F \cap H_k = \emptyset\) for \(k \gg 0\).

**Proof:**

1. Assume that \((H_k)_{k>0}\) converges to \(H\). If \(S\) is a finite subset of \(H\), then for any \(g \in S\) we find a sequence \((h_k)_{k>0}\) in \(G\), with \(h_k \in H_k\) for every \(k\), such that \(g = \lim_{k \to \infty} h_k\). As \(G\) is discrete, this means \(g = h_k\) for \(k \gg 0\), i.e. \(g \in H_k\) for \(k \gg 0\). As \(S\) is finite, we have \(S \subset H_k\) for \(k \gg 0\). Similarly, if \(T\) is finite and disjoint from \(H\), we must show that \(T\) is disjoint from all but finitely many \(H_k\)'s. Suppose not, i.e. \(T\) intersects infinitely many \(H_k\)'s. As \(T\) is finite, we find \(g \in T\) and a subsequence \((k_i)_{i>0}\) such that \(g \in H_{k_i}\) for every \(i > 0\). Setting \(g_{k_i} := g\), and writing \(g = \lim_{i \to \infty} g_{k_i}\), because \((H_k)_{k>0}\) converges to \(H\) this implies \(g \in H\), which contradicts \(T \cap H = \emptyset\).

Conversely, if \(H\) is locally contained in \((H_k)_{k>0}\) and \(G \setminus H\) is locally contained in \((G \setminus H_k)_{k>0}\), let us check that \((H_k)_{k>0}\) converges to \(H\). For \(g \in H\), set \(h_k = g\) so that \(h_k \in H_k\) for \(k \gg 0\); so we write \(g\) as a limit of elements in the \((H_k)\)'s. Now, let \((k_i)_{i>0}\) be a subsequence and \(h_{k_i} \in H_{k_i}\) such that \((h_{k_i})_{i>0}\) converges to \(g \in G\). This means \(g = h_{k_i}\) for \(i \gg 0\), so that \(g\) belongs to infinitely many \(H_k\)'s. As \(G \setminus H\) is locally contained in \((G \setminus H_k)_{k>0}\), this forces \(g \in H\), hence \(\lim_{k \to \infty} H_k = H\).

2. Apply the previous point to a finite generating \(S\) of \(H\): it is contained in \(H_k\) for \(k \gg 0\), so the same holds for \(H\).

3. The condition is equivalent to \(G \setminus \{e\}\) being locally contained in \((G \setminus H_k)_{k>0}\).

**Lemma 2.7.** Let \(G\) be a finitely generated group.

1. Any finite index subgroup defines an isolated point in \(\mathcal{C}(G)\).

2. The index of a subgroup is continuous on \(\mathcal{C}(G)\). More precisely, the map \(\mathcal{C}(G) \rightarrow \overline{\mathbb{N}} : H \mapsto [G : H]\) is continuous.

**Proof:**

1. Let \(H\) be a finite index subgroup of \(G\) and let \((H_k)_{k>0}\) be a sequence in \(\mathcal{C}(G)\) converging to \(H\). Let us show that eventually \(H_k = H\). As \(H\) is finitely generated, by Lemma 2.6 we already know that \(H \subset H_k\) for \(k \gg 0\). But there are finitely many distinct subgroups containing \(H\), as \(H\) has finite index. Passing to a subsequence we may assume that \(H_k = K\) for \(k \gg 0\). So the sequence \((H_k)_{k>0}\) converges both to \(H\) and \(K\), so \(H = K\).
2. Let \((H_k)_{k>0} \subset \mathcal{C}(G)\) be a sequence converging to \(H \in \mathcal{C}(G)\); let us prove that \([G : H_k] \to [G : H]\). We have to consider two cases:

- Let assume that \([G : H] = n < \infty\). By the first part, \(H\) is isolated in \(\mathcal{C}(G)\), so \(H_k = H\) for \(k \gg 0\), and the result follows.

- Now assume that \([G : H] = \infty\). If \([G : H_k] = \infty\) for \(k \gg 0\), the result follows. Else, we may assume that \([G : H_k] = n_k < \infty\). If \(n_k \not\to \infty\), passing to a subsequence, we may assume that \(n_k = n\) for all \(k\). As \(G\) is finitely generated, the number of subgroups of index \(n\) is finite. By the pigeonhole principle, we may assume that the sequence \((H_k)_{k>0}\) is constant. Hence, \([G : H] < \infty\) and this is a contradiction. \(\square\)

Let \(IC(G)\) denote the set of infinite cyclic subgroups of a group \(G\).

**Lemma 2.8.** Let \(G = \mathbb{Z}^d \oplus F\) be a finitely generated abelian group, with \(F\) a finite abelian group, written additively.

1. The closure of \(IC(G)\) in \(\mathcal{C}(G)\) is \(IC(G) \cup \{(0,0)\}\).
2. Let \((g_k)_{k>0}\) be a sequence in \(G\). The sequence \(\langle g_k \rangle_{k>0}\) converges to \(\{(0,0)\}\) if and only if \(\lim_{k \to \infty} g_k = \infty\) in \(G\).
3. The topology induced by \(\mathcal{C}(G)\) on \(IC(G)\) is discrete.

**Proof:**

1. Assume that the sequence \((g_k)_{k>0}\) in \(\mathcal{C}(G)\) converges to a subgroup \(H\). We know that every subgroup of \(G\) is finitely generated and by Lemma 2.6, \(H \subset \langle g_k \rangle\) for \(k \gg 0\). So \(H\) is a subgroup of the cyclic infinite group \(\langle g_k \rangle\). So \(H\) is either trivial or infinite cyclic.

2. This follows immediately from the last part of lemma 2.6.

3. Let \((\langle g_k \rangle)_{k>0} \subset IC(G)\) be a sequence converging to \(\langle g \rangle \in IC(G)\). Let us show that \(\langle g_k \rangle = \langle g \rangle\) for \(k \gg 0\). By Lemma 2.6, we may assume that \(\langle g \rangle \subset \langle g_k \rangle\) for \(k > 0\). So, there exists non-zero integers \(m_k \in \mathbb{Z}\) such that \(g = g_k m_k\). We may assume that \(m_k > 0\) up to replacing \(g_k\) by \(-g_k\). If \(m_k \to \infty\), then \(g\) is divisible by infinitely many integers, a contradiction. So the \(m_k\)'s are bounded and, passing to a subsequence, we may assume that \(m_k = m\) for all \(k > 0\), i.e. \(g = mg_k\). Since the equation \(g = mx\) has finitely many solutions in \(G\), by the pigeonhole principle we may assume that the sequence \((g_k)_{k>0}\) is constant. Because of the convergence of \((\langle g_k \rangle)_{k>0}\) to \(\langle g \rangle\), this implies \(g_k \in \langle g \rangle\), i.e. \(m = 1\), and \(g_k = g\). \(\square\)
3 The case $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}^2$

3.1 The group $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

In this subsection, we set $G = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Recalling that the rank of a finitely generated group is the minimal number of generators, by viewing $G$ as a quotient of $\mathbb{Z}^2$ we see that every subgroup of $G$ has rank at most 2. Now $G$ has three types of subgroups:

- Finite subgroups, i.e. subgroups of $\{0\} \times \mathbb{Z}/n\mathbb{Z}$: there are exactly $d(n)$ of them, where $d(n)$ is the number of divisors of $n$;

- Infinite cyclic subgroups;

- Infinite subgroups of rank 2.

We describe the structure of infinite subgroups more precisely.

**Lemma 3.1.** Let $H$ be an infinite subgroup of $G$. Set $F := H \cap (\{0\} \times \mathbb{Z}/n\mathbb{Z})$. There exists $(a, b) \in H$, with $a > 0$, such that $H = F \oplus \langle (a, b) \rangle$. In particular, $H$ is cyclic if $F$ is trivial and $H$ has rank 2 if $F$ is non-trivial.

**Proof:** Let $p_1 : G \to \mathbb{Z}$ be the projection onto the first factor. As $p_1(H)$ is infinite, it is of the form $p_1(H) = a\mathbb{Z}$ for some $a > 0$. Let $(a, b) \in H$ be any element such that $p_1(a, b) = a$. Then clearly $F \cap \langle (a, b) \rangle = \{(0, 0)\}$, and for $(x, y) \in H$, writing $x = ma$ for some integer $m$, we get $(x, y) = m(a, b) + (0, y - mb)$ so that $H = F \oplus \langle (a, b) \rangle$. □

The group $G$ has the feature that every infinite subgroup has finite index, so defines an isolated point in $C(G)$, by Lemma 2.7. By Lemma 2.8, the only accumulation point of infinite cyclic groups is the trivial subgroup. So it remains to study accumulation points of rank 2 subgroups, which are necessarily finite subgroups. Every non-trivial finite subgroup $H$ of $\mathbb{Z}/n\mathbb{Z}$ is the limit of the sequence $(k\mathbb{Z} \times H)_{k>0}$ of rank 2 subgroups. The converse is provided by:

**Proposition 3.2.** Let $m$ be a divisor of $n$. Consider a sequence $(H_k)_{k>0}$ of infinite subgroups of rank 2 of $G$. It converges in $C(G)$ to $\{0\} \times \langle m \rangle$ if and only if, for $k \gg 0$, there exists $g_k \in G$ with $\lim_{k \to \infty} g_k = \infty$, such that $H_k$ is generated by $(0, m)$ and $g_k$.

**Proof:** For the sufficient condition: if $H_k = \langle (0, m), g_k \rangle = \langle (0, m) \rangle \oplus \langle g_k \rangle$, as $\lim_{k \to \infty} \langle g_k \rangle = \{(0, 0)\}$ by Lemma 2.8, we have $\lim_{k \to \infty} H_k = \langle (0, m) \rangle$ by Lemma 2.3.
For the necessary condition: assume \((H_k)_{k>0}\) converges to \(\{0\} \times \langle m \rangle\). By Lemma 3.1, we have \(H_k = F_k \oplus \langle g_k \rangle\) with \(F_k = H_k \cap (\{0\} \times \mathbb{Z}/n\mathbb{Z})\) and \(g_k = (a_k, b_k)\) with \(a_k > 0\). Because of the assumed convergence, we have \(F_k = \langle (0, m) \rangle\) for \(k \gg 0\), and \(\lim_{k \to \infty} g_k = \infty\). □

Recall that we denote by \([k]\) the set \(\{1, 2, ..., k\}\), and by \(d(n)\) the number of divisors of \(n\). From Proposition 3.2, we get immediately the following special case of Theorem C in [CGP1]:

**Corollary 3.3.** The Chabauty space \(C(\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})\) is homeomorphic to \(\mathbb{N} \times [d(n)]\), the accumulation points corresponding to the subgroups \(\{0\} \times \langle m \rangle\), with \(m\) a divisor of \(n\). □

### 3.2 The group \(\mathbb{Z}^2\)

The list of subgroups of \(G = \mathbb{Z}^2\) is as follows:

- The trivial subgroup \(\{(0, 0)\}\);
- Subgroups of rank 1, i.e. infinite cyclic subgroups;
- Subgroups of rank 2, i.e. finite index subgroups in \(G\) (which define isolated points in \(C(G)\), by Lemma 2.7).

In each infinite subgroup \(H\) of \(G\), pick a minimal vector \(m_H\) (so \(m_H\) has minimal norm among all non-zero vectors in \(H\)). From the 3rd part of Lemma 2.6, we immediately get:

**Proposition 3.4.** Let \((H_k)_{k>0}\) be a sequence of infinite subgroups of \(G\). This sequence converges to \(\{(0, 0)\}\) in \(C(G)\) if and only if \(\lim_{k \to \infty} \|m_{H_k}\| = +\infty\). □

It remains to see how a rank 1 subgroup \(\langle h \rangle\) can be a limit in \(C(G)\).

**Proposition 3.5.** A sequence \((H_k)_{k>0}\) in \(C(G)\) converges to the rank 1 subgroup \(H = \langle h \rangle\) if and only if, for \(k \gg 0\), there exists \(g_k \in G\) such that \(H_k = \langle h, g_k \rangle\) and the sequence \((g_k + H)_{k>0}\) goes to infinity in \(G/H\).

**Proof:** Write \(h = (a, b)\), let \(n > 0\) be the GCD of \(a\) and \(b\), set \(p = (\frac{a}{n}, \frac{b}{n})\), so that \(p\) is a primitive vector proportional to \(h\). Let \(c, d \in \mathbb{Z}\) be integers such that \(ad - bc = n\), so that, with \(q = (c, d)\), the set \(\{p, q\}\) is a basis of \(G\), and every vector in \(G\) may be written uniquely \(\alpha p + \beta q\), for \(\alpha, \beta \in \mathbb{Z}\). The map

\[
\pi : \mathbb{Z}^2 \to (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z} : \alpha p + \beta q \mapsto (\alpha \mod n, \beta)
\]
is then a surjective homomorphism with kernel $H$.

If $(H_k)_{k>0}$ is a sequence of subgroups converging to $H$, we have $H \subset H_k$ for $k \gg 0$, by Lemma 2.6. So to study convergence to $H$, we may as well assume that $H \subset H_k$ for every $k > 0$. By Lemma 2.2, such a sequence $(H_k)_{k>0}$ converges to $H$ if and only if the sequence $(\pi(H_k))_{k>0}$ converges to the trivial subgroup in $\mathcal{C}((\mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z})$. By Proposition 3.2, this happens if and only if, for $k \gg 0$, the subgroup $\pi(H_k)$ is infinite cyclic, say $\pi(H_k) = \pi(\langle g_k \rangle)$ for some $g_k \in \pi^{-1}\pi(H_k) = H_k$, with the property that $\lim_{k \to \infty} \pi(g_k) = \infty$. This concludes the proof. □

As a consequence, we get another special case of Theorem C in [CGP1]:

**Corollary 3.6.** The Chabauty space $\mathcal{C}(\mathbb{Z}^2)$ is homeomorphic to $\mathbb{N}^2$. □

### 4 The proof of Theorem 1.4

Let $p$ be a prime. Recall that $C_k$ denotes the cyclic group of order $k$ (viewed as the group of $k$-th roots of 1 in $\mathbb{T}$). It is classical (see [Ser]) that

$$Q_p^\times \simeq \mathbb{Z}_p \times C_{p-1} \times \mathbb{Z} \quad (p \text{ odd});$$

$$Q_2^\times \simeq \mathbb{Z}_2 \times C_2 \times \mathbb{Z}.$$  

Hence by Pontryagin duality

$$\hat{Q}_p^\times \simeq C_{p^\infty} \times C_{p-1} \times \mathbb{T} \quad (p \text{ odd});$$

$$\hat{Q}_2^\times \simeq C_{2^\infty} \times C_2 \times \mathbb{T}$$

where $C_{p^\infty} = \bigcup_{\ell \geq 1} C_{p^\ell}$ denotes the Prüfer $p$-group. By Proposition 2.3, $Q_p^\times$ and $\hat{Q}_p^\times$ have canonically isomorphic Chabauty spaces. We choose to work with $\hat{Q}_p^\times$ as it is a 1-dimensional Lie group. More generally, for $k > 0$ an integer, we set $G_{p,k} := C_{p^\infty} \times C_k \times \mathbb{T}$ and we aim to determine $\mathcal{C}(G_{p,k})$.

We will need some notation. We will denote by $\pi_1, \pi_2, \pi_3$ the projections of $G_{p,k}$ onto $C_{p^\infty}$ (resp. $C_k$, resp. $\mathbb{T}$). Set also $\pi = (\pi_1, \pi_2) : G_{p,k} \to C_{p^\infty} \times C_k$.

We first give a list of closed subgroups of $G_{p,k}$: as it is a 1-dimensional Lie group, closed subgroups are either discrete, or 1-dimensional.

**Lemma 4.1.** Every closed 1-dimensional subgroup of $G_{p,k}$ is of the form $H_D^{(1)} := D \times \mathbb{T}$, where $D$ is any subgroup of $C_{p^\infty} \times C_k$. The set of 1-dimensional subgroups is closed in $\mathcal{C}(G_{p,k})$ and identifies with $\mathcal{C}(C_{p^\infty} \times C_k)$ via $\pi^*$ (defined as in Lemma 2.2).
**Proof:** A 1-dimensional subgroup of $G_{p,k}$ has the same connected component of identity as $G_{p,k}$, namely $\{1\} \times \{1\} \times T$. The second statement follows immediately from Lemma 2.2. □

**Remark 4.2.** Note that $C(C_p^\infty \times C_k)$ is homeomorphic to $N \times [d(k)]$, with accumulation points corresponding to the subgroups $C_p^\infty \times C_d$, for $d$ a divisor of $k$: this follows from Proposition 3.2 by dualizing.

We now turn to infinite discrete subgroups of $G_{p,k}$. For $F$ a finite subgroup of $C_k \times T$, we denote by $q_F : G_{p,k} \to G_{p,k}/(\{1\} \times F)$ the quotient map. As we will need homomorphisms $C_p^\infty \to (C_k \times T)/F$ in Proposition 4.4 below, we start by describing those homomorphisms. We denote by $T^0$ the connected component of identity of the 1-dimensional compact abelian Lie group $(C_k \times T)/F$. We choose some identification $T^0 \cong T$, and we denote by $\iota : T^0 \to (C_k \times T)/F$ the inclusion.

**Lemma 4.3.** The map
\[
\hat{C}_p^\infty \to \text{hom}(C_p^\infty, (C_k \times T)/F) : f \mapsto \iota \circ f
\]
is an isomorphism of compact groups.

**Proof:** It is enough to see that any homomorphism $f : C_p^\infty \to (C_k \times T)/F$ takes values in $T^0$. First observe that, being a 1-dimensional compact abelian group, $(C_k \times T)/F$ is isomorphic to $A \times T^0$, for some finite abelian group $A$. Second, $C_p^\infty$ is a divisible group, and so is every homomorphic image of $C_p^\infty$. This applies in particular to the projection of $f(C_p^\infty)$ to the first factor $A$. But a finite divisible group must be trivial, hence $f(C_p^\infty) \subset T^0$. □

**Proposition 4.4.** For a finite subgroup $F$ of $C_k \times T$ and a homomorphism $f : C_p^\infty \to (C_k \times T)/F$, set $H_{F,f} := q_F^{-1}(\text{Graph}(f))$. Then $H_{F,f}$ is an infinite discrete subgroup of $G_{p,k}$ and every infinite discrete subgroup is of that form.

**Proof:** It is clear that $H_{F,f}$ is an infinite discrete subgroup of $G_{p,k}$. Conversely, let $H$ be an infinite discrete subgroup. We first claim that $\pi_1(H) = C_p^\infty$: otherwise we would have that $\pi_1(H) = C_{p^\ell}$ for some $\ell > 0$, so that $H$ appears as a discrete subgroup of the closed subgroup $\pi_1^{-1}(C_{p^\ell})$ of $G_{p,k}$: as this is a compact subgroup, this forces $H$ to be finite, a contradiction.

Set then $F := \ker(\pi_1|_H) = H \cap (\{1\} \times C_k \times T)$: as a discrete subgroup in a compact group, $F$ is finite. We consider two cases.

Special case: $F$ is trivial. Then $\pi_1|_H$ is injective, hence $f := (\pi_2, \pi_3) \circ (\pi_1|_H)^{-1}$ is a homomorphism $C_p^\infty \to C_k \times T$, and $H$ is exactly the graph of $f$.  

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**General case:** Let $F$ be arbitrary. We will reduce to our special case. We consider the infinite discrete subgroup $q_F(H)$ in $G_{p,k}/(\{1\} \times F)$. By the previous remark, for some divisor $d$ of $k$, we may identify $(C_k \times T)/F$ with $C_d \times T$, hence also $G_{p,k}/(\{1\} \times F)$ with $G_{p,d}$. By construction, the kernel $\ker(\pi_1|_{q_F(H)})$ is trivial, so we are back to the special case: therefore there exists a morphism $f: C_p^\infty \to (C_k \times T)/F$ such that $q_F(H) = \text{Graph}(f)$. Then $H = q_F^{-1}(q_F(H)) = H_{F,f}$ as claimed. □

**Example 4.5.** (with $k = 1$) Let $F = C_p$ viewed as a subgroup of $T$. The quotient map $q_F$ identifies with the map $T \to T: z \mapsto z^p$. Let $\iota$ denote the inclusion of $C_p^\infty$ into $T$. Then

$$H_{F,\iota} = \{(w, z) \in C_p^\infty \times T : w = z^p\}.$$ 

Observe that $\iota$ does not lift, i.e. there is no $\tilde{\iota}: C_p^\infty \to T$ such that $\iota = q_F \circ \tilde{\iota}$.

To summarize, from Proposition 4.4 and Lemma 4.1 we have now the complete list of closed subgroups of $G_{p,k}$:

- Finite groups;
- Discrete infinite subgroups $H_{F,f}$, as described by Proposition 4.4;
- 1-dimensional closed subgroups $H^{(1)}_D$, as described by Lemma 4.1.

The first part of Theorem 1.4 follows from:

**Proposition 4.6.** $C(G_{p,k})$ is a proper compactification of $N$, identified with the set $N$ of finite subgroups of $G_{p,k}$.

**Proof:** By Theorem 1.7 in [Cor], every finite subgroup defines an isolated point in $C(G_{p,k})$. So $N$ is open and discrete in $C(G_{p,k})$, it remains to show that it is dense. This follows from:

- $H^{(1)}_D$ is the limit of the sequence $(D \times C_n)_{n>0}$ of finite subgroups.
- $H_{F,f}$ is the limit of the sequence $(q_F^{-1}(\text{Graph}(f|_{C_p^\infty})))_{\ell>0}$, also consisting in finite subgroups. □

Our aim is now to determine the homeomorphism type of $C(G_{p,k}) \setminus N$. Let $C^{(1)}$ be the set of one-dimensional subgroup, and let $C^{\text{dis}}$ be the set of infinite discrete subgroups, so that $C(G_{p,k}) \setminus N$ is the disjoint union of $C^{(1)}$ and $C^{\text{dis}}$. By lemma 4.1 and the remark following it, $C^{(1)}$ is homeomorphic to $\overline{N} \times [d(k)]$, the accumulation points corresponding to the subgroups $C_p^\infty \times C_d \times T$, with $d$ a divisor of $k$. Note
Theorem 4.7.  

1. The map 

\[ \alpha : \mathcal{C}^\text{fin}(C_k \times \mathbf{T}) \times \overline{C_p}^\infty \rightarrow \mathcal{C}^\text{dis} : (F, f) \mapsto H_{F,f} \]

is a homeomorphism.

2. The closure of \( \mathcal{C}^\text{dis} \) in \( \mathcal{C}(G_{p,k}) \) is the union of \( \mathcal{C}^\text{dis} \) with the \( d(k) \) subgroups \( C_{p\infty} \times C_d \times \mathbf{T} \), with \( d \) a divisor of \( k \).

Proof:

1. \( \alpha \) is onto, by combining lemma [4.3] with Proposition [4.4].

- To show that \( \alpha \) is injective, we observe that \( H_{F,f} \) determines both \( F \) and \( f \): first, \( F \) is obtained as the intersection of \( H_{F,f} \) with \( \{1\} \times C_k \times \mathbf{T} \); second, \( q_F(H_{F,f}) \) is the graph of \( f \), which of course determines \( f \).

- \( \alpha \) is continuous. As \( \mathcal{C}^\text{fin}(C_k \times \mathbf{T}) \) is discrete, it is enough to show that, for each \( F \in \mathcal{C}^\text{fin}(C_k \times \mathbf{T}) \), the map \( \overline{C_p}^\infty \rightarrow \mathcal{C}^\text{dis} : f \mapsto H_{F,f} \) is continuous.

Let fix \( F \in \mathcal{C}^\text{fin}(C_k \times \mathbf{T}) \) and let \((f_n)_{n>0} \subset \overline{C_p}^\infty \) be a sequence converging to \( f \in \overline{C_p}^\infty \). We want to show that \( H_{F,f_n} \rightarrow H_{F,f} \).

- Let \( z \in H_{F,f} \). We have \( q_F(z) = (a, f(a)) \) for a certain \( a \in C_{p\infty} \). As \( f_n \rightarrow f \), we have that \( (a, f_n(a)) \rightarrow (a, f(a)) \). So \( \overline{q_F}^{-1}(a, f(a)) \rightarrow \overline{q_F}^{-1}(a, f(a)) \) in the Chabauty topology of closed subsets. Moreover, \( |F| = |\overline{q_F}^{-1}(a, f_n(a))| = |\overline{q_F}^{-1}(a, f(a))| < \infty \), so we can find \( z_n \in H_{F,f_n} \) such that \( z_n \rightarrow z \).

- Let \( z_{n_i} \in H_{F,f_{n_i}} \) such that \( z_{n_i} \rightarrow z \in G_{p,k} \) and let us show that \( z \in H_{F,f} \). We have \( q_F(z_{n_i}) = (a_{n_i}, f(a_{n_i})) \) for a certain \( a_{n_i} \in C_{p\infty} \). By discreteness of \( C_{p\infty} \), convergence of \((z_{n_i})\) and continuity of \( q_F \), we may assume that \( a_{n_i} = a \) for all \( i > 0 \). As \( f_n \rightarrow f \), \( q_F(z_{n_i}) \rightarrow (a, f(a)) \). This implies that \( z \in \overline{q_F}^{-1}(a, f(a)) \) and \( z \in H_{F,f} \).

- \( \alpha \) is open. To show this, it is enough to show that, for every \( F \in \mathcal{C}^\text{fin}(C_k \times \mathbf{T}) \), the set \( \{H_{F,f} : f \in \overline{C_p}^\infty\} \) is open. This is in turn implied by the continuity of the map \( \mathcal{C}^\text{dis} \rightarrow \mathcal{C}^\text{fin}(C_k \times \mathbf{T}) : H_{F,f} \mapsto F \). Since \( F \) can be expressed as \( H_{F,f} \cap (\{1\} \times C_k \times \mathbf{T}) \), the result follows from lemma [2.3].
2. Let \((H_{F_n, f_n})_{n>0}\) be a sequence in \(\mathcal{C}^{\text{dis}}\) converging to \(H \in \mathcal{C}(G_{p, k})\). If the orders of the \(F_n\)’s remain bounded, passing to a sub-sequence we may assume that \(F_n = F\) for \(n \gg 0\). By compactness of \(\widehat{C}_p^{\infty}\), again passing to a subsequence we may assume that \((f_n)_{n>0}\) converges to \(f \in \widehat{C}_p^{\infty}\). Then \((H_{F_n, f_n})_{n>0}\) converges to \(H_{F, f}\). Assume now that the orders of the \(F_n\)’s are unbounded. Passing to a subsequence, we may assume that \((F_n)_{n>0}\) converges in \(\mathcal{C}(C_k \times T)\) to \(C_d \times T\), for some divisor \(d\) of \(k\). By the dual version of Proposition 3.2, \(F_n\) contains \(C_d \times \{1\}\) for \(n \gg 0\). For every \((\lambda_n, z_n) \in C_d \times T\), the sequence of cosets \(((\lambda_n, z_n)F_n)_{n>0}\) converges to \((\lambda, z)\). So we have expressed \((w, \lambda, z)\) as the limit of the \((w, \lambda_n, z_n)\)’s in \(H_{F_n, f_n}\). Conversely, if \((w_i, \lambda_i, z_i) \in H_{F_{n_i}, f_{n_i}}\) converges to \((w, \lambda, z) \in G_{p, k}\), then \(\lambda_i = \lambda\) for \(i \gg 0\), hence \(\lambda \in C_d\), and \((w, \lambda, z) \in C^{\infty}_p \times C_d \times T\). \(\square\)

The second part of Theorem 4.3 is then a special case of :

**Corollary 4.8.** \(\mathcal{C}(G_{p, k}) \setminus \mathbb{N}\) is homeomorphic to the space obtained by glueing \([d(k)] \times \overline{\mathbb{N}}\) on the Cantor space \(C\), with the \(d(k)\) accumulation points of \([d(k)] \times \overline{\mathbb{N}}\) being identified to \(d(k)\) pairwise distinct points of \(C\).

**Proof:** By Theorem 4.7, the closure of \(\mathcal{C}^{\text{dis}}\) is a metrizable compact space which is totally disconnected and perfect (no isolated point), so it is homeomorphic to the Cantor space \(C\). On the other hand we already observed after Proposition 4.6 that \(\mathcal{C}^{(1)}\) is homeomorphic to \(\overline{\mathbb{N}} \times [d(k)]\), and the \(d(k)\) accumulation points are identified with \(d(k)\) pairwise distinct points of \(C\). \(\square\)

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