FINITE POWERS OF SELECTIVELY PSEUDOCOMPACT GROUPS

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Abstract. A space $X$ is called selectively pseudocompact if for each sequence $(U_n)_{n \in \mathbb{N}}$ of pairwise disjoint nonempty open subsets of $X$ there is a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $X$ such that $\text{cl}_X(\{x_n : n < \omega\}) \setminus (\bigcup_{n<\omega} U_n) \neq \emptyset$ and $x_n \in U_n$, for each $n < \omega$. Countably compact space spaces are selectively pseudocompact and every selectively pseudocompact space is pseudocompact. We show, under the assumption of $CH$, that for every positive integer $k > 2$ there exists a topological group whose $k$-th power is countably compact but its $(k+1)$-st power is not selectively pseudocompact. This provides a positive answer to a question posed in [9] in any model of $ZFC + CH$.

1. Introduction

In this article, every space will be Tychonoff and every topological group will be Hausdorff (hence, they will also be Tychonoff). For an infinite set $X$, $[X]^{<\omega}$ will denote the family of all finite subsets of $X$ and $[X]^{\omega}$ will denote the family of all countable infinite subsets of $X$. A finite set $\{x_0, \ldots, x_l\}$ of elements of an Abelian group $G$ is called independent if does not contain 0 and if $\sum_{i \leq l} n_i x_i \neq 0$, then $n_i x_i = 0$ for each $i \leq l$. A nonempty subset $X$ of $G$ is called independent if every finite subset of $X$ is independent. The continuum will be denoted by $c$. For $A \subseteq c$, the symbol $\chi_A : c \to \{0, 1\}$ stands for the characteristic function of $A$. For each $x \in \{0, 1\}^c$, we define $\sup(x) = \{\xi < c : x(\xi) \neq 0\}$.

The following generalization of countable compactness was introduced in [7] with the name “strong pseudocompactness” (the name was changed it in the paper [4] since the authors noticed that the term “strong pseudocompactness” was already used to name a different topological property). This paper [4] also contains several results on selectively pseudocompact spaces and its relations with some other pseudocompact like properties.

Definition 1.1. A space $X$ is called selectively pseudocompact if for each sequence $(U_n)_{n \in \mathbb{N}}$ of pairwise disjoint nonempty open subsets of $X$ there is a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $X$ and $x \in X$ such that $x_n \in U_n$, for every $n < \omega$, and $x$ is an accumulation point of $\{x_n : n < \omega\}$.

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It is evident that every countably compact space is selectively pseudocompact and every selectively pseudocompact space is pseudocompact. There is a selectively pseudocompact group that is not countably compact and a pseudocompact group that is not selectively pseudocompact (for these two examples the reader is referred to [9]). However, it was pointed out in [7] that pseudocompact groups have a property very similar to selectively pseudocompactness:

**Theorem 1.2.** [7] For a topological group $G$, the following conditions are equivalent.

1. $G$ is pseudocompact.
2. For each family $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint nonempty open subsets of $G$ there is a discrete subset $D$ of $G$ contained in $\bigcup_{n \in \mathbb{N}} U_n$ such that $\text{cl}_G(D) \setminus (\bigcup_{n \in \mathbb{N}} U_n) \neq \emptyset$ and $|D \cap U_n| < \omega$, for every $n \in \mathbb{N}$.
3. For each family $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint nonempty open subsets of $G$ there is a discrete subset $D$ of $G$ contained in $\bigcup_{n \in \mathbb{N}} U_n$ such that $\text{cl}_G(D) \setminus (\bigcup_{n \in \mathbb{N}} U_n) \neq \emptyset$, $|D \cap U_n| \leq \omega$, for every $n \in \mathbb{N}$, and $D \cap U_n \neq \emptyset$ for infinitely many $n$'s.

A countably compact space whose square is not pseudocompact (for one of these spaces see [11]) is an example of a selectively pseudocompact space whose square is not pseudocompact. In the realm of topological groups, it is well-known that “the product of pseudocompact groups is pseudocompact”, this fact was established by W. W. Comfort and K. A. Ross in their classical paper [1]. All these remarks suggest naturally the following question listed in [9] and in [8].

**Question 1.3.** Is selectively pseudocompactness productive in the class of topological groups?

Let us make some comments about a possible solution to Question 1.3 inside of a model of ZFC. It is an old open problem posed by W. W. Comfort whether or not the product of two countably compact groups is countably compact. In 1980, E. K. van Douwen [5] assuming the existence of a countably compact Boolean group without non-trivial convergent sequences constructed, in ZFC, two countably compact groups whose product is not countably compact. Thus Comfort’s problem reduces to the existence of a countably compact Boolean group without non-trivial convergent sequences. The first such a topological group was constructed, assuming $CH$, by A. Hajnal and I. Juhász [13]. Other known examples of such topological groups are constructed either by using some set-theoretic axiom compatible with ZFC (see for instance [2], [3], [14], [17] and [18]), by assuming the existence of selective ultrafilters on $\omega$ ([10]) or in a Random model (see [15]). The existence of a countably compact group without non-trivial convergent sequences is still unknown in ZFC.

Our purpose in this paper is the construction for every positive integer $k > 0$, under the assumption of $CH$, of a topological group whose $k$-th power is countably compact but its $(k + 1)$-th power is not selectively pseudocompact (concerning spaces, Z. Frolík (see [6]) constructed for each $1 < k < \omega$, a space $X$ such that $X^k$ is countably compact and $X^{k+1}$ is not pseudocompact.). Unfortunately, we could not answer Question 1.3 by assuming only the axioms of ZFC.

For the construction of the required countably compact groups we shall follow some basic ideas from [16]. In particular, the following notions and proposition that were used in that paper.
Definition 1.4. Let $\lambda$ be an infinite cardinal and let $Y$ be an infinite subset of a topological product $X = \prod_{\alpha < \lambda} X_\alpha$.

(1) $Y$ is called finally dense in $X$ if there exists $\beta < \lambda$ such that $\pi_{[0, \beta]}[Y]$ is dense in $\prod_{\beta \leq \alpha < \lambda} X_\alpha$, where $\pi_{[0, \beta]}$ is the projection from $X$ onto $\prod_{\beta \leq \alpha < \lambda} X_\alpha$.

(2) If every infinite subset of $Y$ is finally dense in $X$, then we say that $Y$ is hereditarily finally dense (HFD) in $X$.

(3) $Y$ has property (P) if the projection $\pi_I : \prod_{\alpha < \lambda} X_\alpha \to \prod_{\alpha \in I} X_\alpha$ satisfies that $\pi_I[Y] = \prod_{\alpha \in I} X_\alpha$ for all $I \subseteq [\lambda]^\omega$.

Property (P) was first considered in [13] and HFD-spaces were introduced in [12]. It was proved in [13] that a HFD-subspace of $\{0, 1\}^\omega$ with property (P) is countably compact. Because of the diagonal, the powers of a HFD cannot be HFD space.

Proposition 1.5. [3, Prop. 3.2] Let $\lambda$ be an uncountable regular cardinal and let $X = \prod_{\alpha < \lambda} X_\alpha$ be a product of compact metrizable spaces $X_\alpha$ each of which contains at least two points. Suppose that $Y$ is a subset of $X$ such that $\pi_{[0, \beta]}[Y] = \prod_{\alpha < \beta} X_\alpha$ for each $\beta < \lambda$. If $S \subseteq Y$ is an HFD set in $X$, then $S$ has a cluster point in $Y$, but no sequence in $S$ converges. In particular, if $S$ is HFD in $X$, then it is a countably compact dense subspace of $X$ which does not contain non-trivial convergent sequences.

The following result in combination with the Proposition above has been used in [16] to make finite powers countably compact.

Lemma 1.6. Suppose that $G$ is the group generated by $\{x_\mu : \mu \in c\} \subseteq 2^{\omega_1}$ and $\{\sum_{\mu \in h(i, n)} x_\mu : i \in m\} : n \in C$ has an accumulation point in $G^m$ for each $h : m \times C \to [c]^{<\omega}$, where $1 \leq m \leq k$ and $C \in [\omega]^{\omega}$, satisfying that the set $\{h(i, n) : i < m$ and $n \in C\}$ is linearly independent. Then $G_k$ is countably compact.

Given a positive integer $k$, we shall prove, under the assumption of CH, the existence of an independent set $\{x_\xi : \xi < c\} \subseteq \{0, 1\}^c$ so that the group generated by this set, $G = \{x_\xi : \xi < c\}$ is HFD with property (P), $G_k$ is countably compact and $G^{k+1}$ is not selectively pseudocompact. In the second section, we list the conditions that the generators must have and, based on these conditions, we shall give the details of the topological properties of the group $G$. In the last section, we proceed to the technical constructions which allows to define such generators with the required properties.

2. The topological group

The family of open sets that will guarantee the destruction of selectively pseudocompactness of our topological group will be the one constructed in the next lemma, which is an extension of Lemma 2.1 from [9].

Lemma 2.1. Let $G$ be an non-discrete topological group of order 2 with identity 0 and let $k < \omega$ be positive. Then there is a countable family $\{U_n : n < \omega\}$ of nonempty open subsets of $G^{k+1}$ such that if $x_n \in U_n$, for an arbitrary $n < \omega$, then the set $\{x_n(i) : i \leq k$ and $n < \omega\}$ is a linearly independent set in $G$.

Proof. By Lemma 2.1 of [9], we can find a sequence of nonempty open sets $\{W_n : n < \omega\}$ of $G$ such that $0 \notin \sum_{n \in F} W_n$ for each nonempty finite subset $F$ of $\omega$. Then, for each $n < \omega$, we define $U_n = \prod_{i \leq k} W_{n(k+1)+i}$. For every $n < \omega$, choose $x_n \in U_n$. By the definition and the
properties of the family \( \{W_n : n < \omega \} \), it is easy to see that \( \{x_n(i) : i \leq k \text{ and } n < \omega \} \) is linearly independent in \( G \).

For each \( \xi < c \), the coordinates of \( x_\xi \in \{0,1\}^\xi \) will be defined inductively. To do that we shall need to consider the set \( [c]^{<\omega} \) as a vector space over the field \( \{0,1\} \) with the symmetric difference as its group operation and, for each \( \alpha < c \), we shall construct (see Theorem 3.3) suitable homomorphisms \( \psi_\alpha : [c]^{<\omega} \to \{0,1\} \) which shall determine the coordinates of the \( x_\xi \)'s by defining \( x_\xi(\alpha) = \psi_\alpha(\{\xi\}) \) for each \( \alpha, \xi < c \). The following easy lemma will be fundamental to guarantee the linear independence of the generators.

**Lemma 2.2.** Let \( \{x_\xi : \xi < c\} \) be an independent subset of \( \{0,1\}^\xi \) and let \( \{F_n : n < \omega \} \subseteq [c]^{<\omega} \setminus \{\emptyset\} \). Then the set \( \{\sum_{\xi \in F_n} x_\xi : n < \omega \} \) is independent iff \( \{F_n : n < \omega \} \) is a linearly independent subset of the vector space \( [c]^\omega \) over the field \( \{0,1\} \).

First of all, we fix some suitable enumerations:

- For each \( \xi \in I \), fix \( t_\xi \in \{0,1\}^\xi \) so that for every \( \alpha < c \) and for every \( x \in \{0,1\}^\alpha \) there exists \( \xi \in I \) such that \( \xi > \alpha \) and \( t_\xi|_\alpha = x \).
- Let \( \{h_\xi : \xi \in I\} \) be an enumeration of all functions \( h : m \times C \to [c]^{<\omega} \), where \( 1 \leq m \leq k \) and \( C \subseteq [\omega]^{<\omega} \), satisfying that the set \( \{h_\xi(i,n) : i < m \text{ and } n \in C \} \) is linearly independent, and \( \bigcup_{(i,n) \in m \times C} h_\xi(i,n) \subseteq \xi \) for each \( \xi \in I \). Put \( \text{dom}(h_\xi) = m \times C_\xi \) for each \( \xi \in I \).
- Let \( \{(f_\xi, \vec{F}_\xi) : \xi \in I\} \) and \( \bar{F}_\xi = (F_\xi^0, \ldots, F_\xi^k) \) be an enumeration of all pairs \((f, \vec{F})\), where \( \vec{F} = (F_0^1, \ldots, F_0^k) \in ([c]^{<\omega})^{k+1} \) and \( f : \omega \to [c]^{<\omega} \) is a function such that \( \{f(n) : n < \omega \} \) is linearly independent, the dimension of the quotient of \( [\xi]^{<\omega} \) by the subgroup \( \langle \{f_\xi(n) : n < \omega \} \rangle \) is infinite, and \( \bigcup_{n < \omega} f_\xi(n) \cup \bigcup_{i \leq k} F_\xi^i \subseteq \xi \).

To carry out the construction, let us state the conditions that the homomorphisms \( \psi_\alpha \)'s should have when \( \alpha \in I \):

\begin{itemize}
  \item[a)] \( \psi_\alpha(F_\alpha^0) = 1 \) if \( F_\alpha^0 \neq \emptyset \) for each \( \alpha \in I \);
  \item[b)] \( \psi_\alpha(\{\xi\}) = t_\xi(\alpha) \) for each \( \alpha \in I \) and \( \xi \in I \) with \( \alpha < \xi \); and
  \item[c)] \( \{n < \omega : \forall i \leq k (\psi_\alpha(F_\alpha^i) = \psi_\alpha(f_\alpha(n(k+1)+i)))\} \) is finite for each \( \alpha \in I \).
\end{itemize}

For every \( \gamma < \beta \in I \), we shall consider the homomorphism \( \Psi_{[\gamma,\beta]} : [c]^{<\omega} \to \{0,1\}^{[\gamma,\beta]} \) defined by \( \Psi_{[\gamma,\beta]}(F) = (\psi_\xi(F))_{\gamma \leq \xi < \beta} \) for each \( F \in [c]^{<\omega} \). Then, we have that:

\begin{itemize}
  \item[d)] \( \{\Psi_{[\gamma,\beta]}(h_\gamma(0,n)), \ldots, \Psi_{[\gamma,\beta]}(h_\gamma(m_\gamma - 1, n)) : n \in C_\gamma \} \) is dense in \( \{(0,1)^{[\gamma,\beta]}\}^{m_\gamma} \) for each \( \gamma < \beta \in I \).
\end{itemize}

Let us see in the next theorem that all these four conditions will give us the desired topological group.

**Theorem 2.3.** [CH] Fix a positive integer \( k \) and consider the homomorphisms \( \{\psi_\alpha : [c]^{<\omega} \to \{0,1\} : \alpha < c\} \) satisfying the conditions a) – d) from above. For each \( \xi < c \), we define \( x_\xi \in \{0,1\}^\xi \) by \( x_\xi(\alpha) = \psi_\alpha(\{\xi\}) \) for each \( \alpha < c \). \( G = \{x_\xi : \xi < c\} \) is HFD with property \((P)\), \( G^k \) is countably compact and \( G^{k+1} \) is not selectively pseudocompact.

**Proof.** Lemma 2.2 and condition a) imply directly that the generators \( \{x_\xi : \xi < c\} \) are linearly independent. Fix \( 0 < m \leq k \). Next, we shall show that \( \{(F(i,n))_{i \in m} : n \in C\} \) has an accumulation point in the topological group \( G^m \) whenever \( \{F(i,n) : i < m \text{ and } n \in C\} \) is a linearly independent of \( G \).
Then, choose the function \( h : m \times C \to [c]^{<\omega} \) defined by \( F(i, n) = \sum_{\mu \in h(i, n)} x_\mu \), for each \((i, n) \in m \times C\). This \( h \) is equal to \( h_\gamma \) for some \( \gamma \in I \). Observe that
\[
\Psi|_{[\gamma, \beta)}(F) = (\psi_\xi(F))_{\gamma \leq \xi < \beta} = \pi_{[\gamma, \beta)}(\sum_{\xi \in F} x_\xi)
\]
for every \( F \in [c]^{<\omega} \) and \( \gamma \leq \beta < c \). So, we have that
\[
(\sum_{\xi \in F^0} x_\xi, \ldots, \sum_{\xi \in F^n} x_\xi) = (\Psi_{[\gamma, \beta)}[h_\gamma(0, n)], \ldots, \Psi_{[\gamma, \beta)}[h_\gamma(m - 1, n)])
\]
for every \( n \in C \). According to clause \( d \), we know that \( \{(\sum_{\xi \in F^0} x_\xi, \ldots, \sum_{\xi \in F^n} x_\xi) : n \in C\} \) is dense in \( ([0, 1]^{[\gamma, \beta)})^m \) for every \( \gamma < \beta < \alpha \). Thus, we obtain that this sequence is finally dense in \( ([0, 1]^{\gamma})^k \). Now, all its infinite subsets are also linearly independent, hence the sequence is \( HFD \). To prove that \( G_m \) has property \( (P) \) in \( ([0, 1]^{\omega_1})^m \) consider the projection map \( \pi_\alpha : ([0, 1]^{\omega_1})^m \to ([0, 1]^{\alpha})^m \) for an arbitrary \( \omega + \omega < \alpha < \omega_1 \). Fix \( y = (y_0, \ldots, y_m) \in ([0, 1]^{\alpha})^m \). Then, for every \( i \leq m \) we can find \( \xi_i \in I \) such that \( \xi_i > \alpha \) and \( t_{\xi_i}|_\alpha = y_i \). By condition \( b \), we have that \( t_{\xi_i}(\beta) = \psi_\beta(\xi_i) = x_{\xi_i}(\beta) = y_i(\beta) \) for each \( i \leq m \) and for each \( \beta < \alpha \). This means that \( \pi_\alpha(\sum_{i \leq m} x_{\xi_i}) = y \). Thus, we have proved that \( G_m \) has property \( (P) \). By Proposition \ref{prop:accumulation} we conclude that this sequence has an accumulation point in \( G_m \). By Lemma \ref{lem:countablyCompact} it follows that \( G_k \) is countably compact. Therefore, \( G_m \) is strongly pseudocompact for all positive integer \( m \leq k \).

It remains to show that \( G^{k+1} \) cannot be selectively pseudocompact. Choose a countable family \( \{U_n : n < \omega\} \) consisting of open subsets of \( G^{k+1} \) given by Lemma \ref{lem:countablyCompact}. For each \( n < \omega \) fix \( y_n \in U_n \). Then the set \( \{y_n(i) : i \leq k \text{ and } n < \omega\} \) is linearly independent in \( G \). Fix \( x = (\sum_{\xi \in F^0} x_\xi, \ldots, \sum_{\xi \in F^k} x_\xi) \in G^{k+1} \). Define \( f : \omega \to [c]^{<\omega} \) so that \( y_n(i) = \sum_{\xi \in f(n(k+1)+i)} x_\xi \) for each \( i \leq k \) and \( n \in \mathbb{N} \). Now, choose \( \beta \in I \) for which \( (f, (F^0, \ldots, F^k)) = (f_\beta, (F^0_\beta, \ldots, F^k_\beta)) \).

By clause \( c \), for every \( i \leq k \) we know that
\[
\pi_\beta(\sum_{\xi \in F^i} x_\xi) = \sum_{\xi \in F^i} x_\xi(\beta) = \sum_{\xi \in F^i} \psi_\beta(\xi) = \psi_\beta(F^i) = \psi_\beta[f(n(k+1)+i)] = y_n(i)(\beta) = \pi_\beta(y_n(i))
\]
holds for finitely many \( n \in \mathbb{N} \). It then follows that \( \{y_n(k+1)+i : n < \omega\} \) does not have an accumulation point in \( G^{k+1} \). Therefore, the family of open sets \( \{U_{(k+1)n+i} : n < \omega\} \) witnesses that \( G^{K+1} \) is not selectively pseudocompact.

\[ \square \]

3. The construction

In this section, we shall construct the homomorphisms \( \psi_\alpha \)’s that will allow us to define the generators of the desired topological group.

**Notation:** In what follows, for a positive \( k < \omega \), the italic letter \( \mathfrak{B} \) will stand for the \( c \)-sized set of functions of the form \( f : m_f \times C_f \to [c]^{<\omega} \), where \( 1 \leq m_f \leq k \) and \( C_f \in [\omega]^{\omega} \), such that the set \( \{f(i, n) : i < m_f \text{ and } n \in C_f\} \) is linearly independent.

**Lemma 3.1.** Let \( \omega \leq \alpha < c \) and \( k \) a positive integer. If \( \{F^i : i \leq k\} \subseteq [\alpha]^{<\omega} \) and \( \mathfrak{B} \) is a countable subset of \( \mathfrak{B} \) such that \( \bigcup_{(i,n) \in m_f \times C_f} f(i, n) \subseteq \alpha \) for every \( f \in \mathfrak{B} \), then there exists a homeomorphism \( \psi : [\alpha]^{<\omega} \to \{0, 1\} \) such that:

1. Here, \( \pi_{[\gamma, \beta)} : \{0, 1\}^\gamma \to \{0, 1\}^{[\gamma, \beta)} \) is the projection map.
\[ i) \psi(F^0) = 1 \text{ if } F^0 \neq \emptyset; \]
\[ ii) \{n < \omega : \forall i \leq k(\psi(F^i) = \psi(\{n(k + 1) + i\}))\} \text{ is finite; and} \]
\[ iii) \{n < \omega : \forall i < m_f(\psi(f(i, n)) = \sigma(i))\} \text{ is infinite for every } f \in B \text{ and } \sigma \in \{0, 1\}^{m_f}. \]

Proof. Enumerate the set of all pairs \((f, \sigma)\) such that \(f \in B\) and \(\sigma \in \{0, 1\}^{m_f}\) as \(\{(f_n, \sigma_n) : n < \omega\}\) so that each \((f, \sigma)\) appears infinitely often. Choose a positive integer \(N_0\) for which 
\[ \omega \cap (F^0 \cup \ldots \cup F^k) \subseteq N_0(k+1). \]
Set \(E_0 := F^0 \cup \ldots \cup F^k \cup N_0(k+1)\) and fix an homomorphism 
\[ \phi_0 : [E_0]^{\omega} \rightarrow \{0, 1\} \text{ so that } \phi_0(F^0) = 1 \text{ whenever } F^0 \neq \emptyset. \]
Suppose that for a positive integer \(l\) we have defined the sets \(\{N_s : s \leq l\}\), \(\{n_s : s < l\}\), \(\{E_s : s \leq l\}\) and \(\{\phi_s : s \leq l\}\) so that the following hols:

a) \(\{N_s : s \leq l\}\) is a strictly increasing sequence of positive integers;

b) \(\{n_s : s < l\}\) is a strictly increasing sequence of positive integers;

c) \(E_s := F^0 \cup \ldots \cup F^k \cup N_s(k+1) \cup (\bigcup \{f_r(i, n_r) : r < s \text{ and } i < m_r\})\);

d) for each \(s \leq l\), \(\phi_s : [E_s]^{\omega} \rightarrow \{0, 1\}\) is a homomorphism that extends \(\phi_r\) for every \(r < s\);

e) if \(N_s \leq n < N_{s+1}\), then there exists \(i < k+1\) such that 
\[ \phi_s(F^i) \neq \phi_s(\{n(k + 1) + i\}) \]
for each \(s < l\);

f) \((\bigcup \{f_r(i, n_r) : r < s \text{ and } i < m_r\}) \cap \omega \subseteq N_s(k+1)\) for each \(s \leq l\); and

g) \(\phi_s(f_s(i, n_s)) = \sigma_s(i)\) for each \(s < l\) and \(i < m_{f_s}\).

We will construct \(n_l, N_{l+1}, E_{l+1}\) and \(\phi_{l+1}\).

Claim. For every \(E \in [\omega]^{\omega}\) there is \(n < \omega\) such that the set \(\{\{\beta : \beta \in E\} \cup \{f_l(i, n) : i < m_{f_l}\}\}\) is linearly independent.

Proof of the Claim: In fact, suppose that this is not the case. Then for each \(n < \omega\) there exists a non-empty set \(F_n\) of \(m_{f_l}\) such that 
\[ \sum_{k \in F_n} f_l(k, n) \in [E]^{\omega}. \]
As the group \([E]^{\omega}\) is finite, it follows that there exist two distinct integers \(m_0\) and \(m_1\) such that 
\[ \sum_{i \in F_{m_0}} f_l(i, m_0) = \sum_{i \in F_{m_1}} f_l(i, m_1), \]
but this contradicts the assumption that \(f_l(i, n) : i < m_{f_l} \text{ and } n < \omega\) is linearly independent.

By the Claim, we can find \(n_l < \omega\) such that \(n_{l-1} < n_l\) and \(\{\{\beta : \beta \in E_l\} \cup \{f_l(i, n_l) : i < m_{f_l}\}\}\) is linearly independent. Choose a positive integer \(N_{l+1}\) so that \(N_{l+1} > N_l\) and 
\[ \omega \cap (\bigcup_{i \leq m_{f_l}} f_l(i, n_l)) \subseteq N_{l+1}(k+1). \]
By condition \(f)\) for \(l\), we obtain that condition \(f)\) for \(l + 1\) is satisfied. Define
\[ E_{l+1} := F^0 \cup \ldots \cup F^k \cup N_{l+1}(k+1) \cup (\bigcup \{f_s(i, n_s) : s < l + 1 \text{ and } i < m_{s}\}). \]

It is straightforward to see that conditions \(a)\), \(b)\) and \(c)\) are satisfied for \(l + 1\).

Recall that the set \(\{\{\beta : \beta \in E_l\} \cup \{f_l(i, n_l) : i < m_{f_l}\}\}\) was chosen to be independent and the set \(\{\{\beta : \beta \in E_l\} \cup \{N_l(k + 1) + i : i < k + 1\}\}\) is independent because of clause \(f)\) for \(l\). Since \(|\bigcup_{i \leq m_{f_l}} f_l(i, n_l)| < |\{N_l(k + 1) + i : i < k + 1\}|\), it follows that there exists \(i_{N_l}\) for which the set
\[ \{\{\beta : \beta \in E_l\} \cup \{f_l(i, n_l) : i < m_{f_l}\} \cup \{N_l(k + 1) + i_{N_l}\}\} \]
is linearly independent. Thus, we may proceed inductively to find \(i_n < k + 1\), for each \(n \in [N_l, N_{l+1})\), such that \(\{\{\beta : \beta \in E_l\} \cup \{f_l(i, n_l) : i < m_{f_l}\} \cup \{n(k + 1) + i_n\} : N_l \leq n < N_{l+1}\}\) is linearly independent. Hence, it is possible to define a homomorphism 
\[ \phi_{l+1} : [E_{l+1}]^{\omega} \rightarrow \{0, 1\} \]
that satisfies the following:
A) $\phi_{l+1}(\{\beta\}) = \phi_l(\{\beta\})$ for each $\beta \in E_l$;
B) $\phi_{l+1}(\{n(k+1)+i_n\}) = 1 - \phi_{l+1}(F^{i_n})$ for each $N_l \leq n < N_{l+1}$ and
C) $\phi_{l+1}(f(i, n)) = \sigma_l(i)$ for each $i < m_f$.

It follows from condition A) that $\phi_{l+1}$ extends $\phi_l$ and so condition d) holds. Condition e) follows directly from B) and condition C) implies g).

Let $\psi$ be an extension of $\bigcup_{f \in w} \phi_f$. Condition i) follows from the fact that $\psi$ extends $\phi_0$. According to e), if $N_l \leq n$, then there exists $i < k + 1$ such that $\phi_l(F^i) \neq \phi_s(\{n(k+1)+i\})$. This clearly implies that the set $\{n < \omega : \forall i \leq k(\psi(F^i) = \psi(\{n(k+1)+i\}))$ is finite and so condition $ii)$ is satisfied. Let $f \in B$ and $\sigma \in \{0, 1\}^{m_f}$. Choose $s < \omega$ so that $f_s = f$ and $\sigma_s = \sigma$. By condition g), we have that $\phi_s(f_s(i, n_s)) = \sigma_s(i)$ for each $i < m_f$, and since the set $\{n < \omega : (f_s, \sigma_s) = (f_n, \sigma)\}$ is infinite, we conclude that $\{n < \omega : (f(i, n) = \sigma(i))\}$ is also infinite. Then, condition iii) holds.

Lemma 3.2. Suppose that $k$ is a positive integer and $\omega + \omega \leq \alpha < \omega_1$. Let $F^0, \ldots, F^k \in [\alpha]^{<\omega}$ and let $B$ be a countable subset of $\mathcal{B}$ such that $\bigcup_{f(i, n) \in m_f \times C_f} f(i, n) \subseteq \alpha$ for each $f \in \mathcal{B}$.

If $\{H_n : n < \omega\}$ is a linearly independent subset of $[\alpha]^{<\omega}$ such that the quotient $[\alpha]^{<\omega}$ by the group generated by $\{H_n : n < \omega\}$ is infinite, then there exists a homomorphism $\psi : [\alpha]^{<\omega} \to \{0, 1\}$ such that

I) $\psi(F^0) = 1$ whenever $F^0 \neq \emptyset$;
II) $\{n < \omega : \forall i \leq k(\psi(F^i) = \psi(H_{n(k+1)+i}))$ is finite; and
III) $\{n \in C_f : \forall i < m_f(\psi(f(i, n) = \sigma(i))\}$ is infinite for every $f \in \mathcal{B}$ and $\sigma \in \{0, 1\}^{m_f}$.

Proof. By the hypothesis, there exists $\{H_\xi : \omega \leq \xi < \alpha\}$ such that $\{H_n : n < \omega\} \cup \{H_\xi : \omega \leq \xi < \alpha\}$ is a basis for $[\alpha]^{<\omega}$. Consider the homomorphism $\Phi : [\alpha]^{<\omega} \to [\alpha]^{<\omega}$ defined by $\Phi(H_\xi) = \{\xi\}$ for each $\xi < \alpha$.

For each $f \in \mathcal{B}$ choose $h_f : m_f \times C_f \to [\alpha]^{<\omega}$ so that $h_f(i, n) = \Phi(f(i, n))$ for all $i < m_f$ and $n \in C_f$. Put $\mathcal{A} = \{h_f : f \in \mathcal{B}\}$. Notice that $\{h_f(i, n) : i < m_f \land n \in C_f\}$ is linearly independent for every $f \in \mathcal{B}$, for every $f \in \mathcal{B}$. Now, we apply Lemma 3.1 to obtain a homomorphism $\tilde{\psi} : [\alpha]^{<\omega} \to [\alpha]^{<\omega}$ such that:

i) $\tilde{\psi}(\Phi(F^0)) = 1$ if $\Phi(F^0) \neq \emptyset$;
ii) $\{n < \omega : \forall i < k + 1(\tilde{\psi}(\Phi(F^i)) = \tilde{\psi}(\{n(k+1)+i\})\}$ is finite; and
iii) $\{n \in C_f : \forall i < m_f(\tilde{\psi}(h_f(i, n)) = \sigma(i))\}$ is infinite for every $f \in \mathcal{B}$ and $\sigma \in \{0, 1\}^{m_f}$.

Define $\psi = \tilde{\psi} \circ \Phi$. From clause i) it follows that $\psi(F^0) = 1$ if $\Phi(F^0) \neq \emptyset$ which is equivalent to $F^0 \neq \emptyset$ since $\Phi$ is an isomorphism. Thus, I) holds. According to condition ii) and the fact $\Phi(H_n) = \{n\}$ for every $n < \omega$, we have that $\{n < \omega : \forall i < k + 1(\psi(F^i) = \psi(H_{n(k+1)+i}))\}$ is finite. Thus, II) is satisfied. By condition iii) and the definitions of $\mathcal{B}$ and $\Phi$, we obtain that $\{n \in C_f : \forall i < m_f(\psi(f(i, n)) = \sigma(i))\}$ is infinite for each $f \in \mathcal{B}$ and $\sigma \in \{0, 1\}^{m_f}$, which is condition III).

Now, we turn out to the construction of the required homeomorphisms in the next theorem.

To explain the properties of such homomorphisms, it is necessary to use the enumerations from the second section, which will be listed again for the reader’s convenience:

- Set $I = [\omega + \omega, \omega_1]$.
- For each $\xi \in I$, fix $t_\xi \in \{0, 1\}^\xi$ so that for each $\alpha < \omega_1$ and for each $x \in \{0, 1\}^\alpha$ there exists $\xi \in I$ such that $\xi > \alpha$ and $t_\xi|_\alpha = x$. 

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• Let \( \{h_\xi : \xi \in I\} \) be an enumeration of all functions \( h : m \times C \rightarrow [\varepsilon]^{<\omega} \), where \( 1 \leq m \leq k \) and \( C \in [\omega]^{<\omega} \), satisfying that the set \( \{h(i,n) : i < m \text{ and } n \in C\} \) is linearly independent, and \( \bigcup_{(i,n)\in m \times C} h_\xi(i,n) \subseteq \xi \) for each \( \xi \in I \). Put \( \text{dom}(h_\xi) = m_\xi \times C_\xi \) for each \( \xi \in I \).

• Let \( \{(f_\xi, \vec{F}_\xi) : \xi \in I\} \) be an enumeration of all pairs \((f, \vec{F})\), where \( \vec{F} = (F^0, \ldots, F^k) \in [[\varepsilon]^{<\omega}]^{k+1} \) and \( f : \omega \rightarrow [\varepsilon]^{<\omega} \) is a function such that \( \{f(n) : n < \omega\} \) is linearly independent, the dimension of the quotient of \([\varepsilon]^{<\omega}\) by the subgroup \( \{\{f(n) : n < \omega\}\} \) is infinite, and \((\bigcup_{n<\omega} f(n)) \cup (\bigcup_{i\leq k} \vec{F}_\xi) \subseteq \xi \).

**Theorem 3.3. [CH]** For every \( \alpha < \varepsilon \), there is a homomorphism \( \psi_\alpha : [\varepsilon]^{<\omega} \rightarrow \{0,1\} \) such that:

\( a) \ \psi_\alpha(F^0_\alpha) = 1 \) if \( F^0_\alpha \neq \emptyset \) for each \( \alpha \in I \);

\( b) \ \psi_\alpha(\{\xi\}) = t_\xi(\alpha) \) for each \( \alpha \in I \) and \( \xi \in I \) with \( \alpha < \xi \); and

\( c) \ \{n < \omega : \forall i \leq k(\psi_\alpha(F^i_\alpha) = \psi_\alpha(f_\alpha(n(k+1)+i)))\} \) is finite for each \( \alpha \in I \).

For every \( \gamma < \beta \in I \), we let \( \Psi|_{[\gamma,\beta)} : [\varepsilon]^{<\omega} \rightarrow \{0,1\}^{[\gamma,\beta)} \) be the homomorphism defined by \( \Psi|_{[\gamma,\beta)}(F) = (\psi_\xi(F))_{\gamma \leq \xi < \beta} \) for each \( F \in [\varepsilon]^{<\omega} \). Then, we must have that:

\( d) \ \{\Psi|_{[\gamma,\beta)}[h_\beta(0,n)], \ldots, \Psi|_{[\gamma,\beta)}[h_\beta(m_\beta-1,n)] : n \in C_\gamma\} \) is dense in \( \{\{0,1\}^{[\gamma,\beta)}\}^{m_\beta} \) for each \( \gamma < \beta \in I \).

**Proof.** For \( \alpha < \omega + \omega \), we let \( \psi_\alpha : [\varepsilon]^{<\omega} \rightarrow \{0,1\} \) be the homomorphism satisfying \( \psi_\alpha(\{\xi\}) = 0 \) if \( \xi < \omega + \omega \) and \( \psi_\alpha(\{\xi\}) = t_\xi(\alpha) \) for each \( \xi \in I \).

Now suppose that \( \omega + \omega \leq \gamma < \omega_1 \) and that we have defined a suitable homeomorphism \( \psi_\alpha : [\varepsilon]^{<\omega} \rightarrow \{0,1\} \) for each \( \alpha < \gamma \) so that the following inductive conditions are satisfied:

\( A) \ \psi_\beta(F^0_\beta) = 1 \) if \( F^0_\beta \neq \emptyset \) for each \( \beta \in I \cap \gamma \);

\( B) \ \psi_\beta(\{\xi\}) = t_\xi(\beta) \) for each \( \beta \in I \cap \gamma \) and for each \( \xi \in I \) with \( \beta < \xi \);

\( C) \ \{n < \omega : \forall i \leq k(\psi_\beta(f_\beta((k+1)n+i)) = \psi_\beta(F^i_\beta))\} \) is finite for each \( \beta \in I \cap \gamma \); and

\( D) \ \{\Psi|_{[\gamma,\beta)}[h_\beta(0,n)], \ldots, \Psi|_{[\gamma,\beta)}[h_\beta(m_\beta-1,n)] : n \in C_\gamma\} \) is dense in \( \{\{0,1\}^{[\gamma,\beta)}\}^{m_\beta} \) for each \( \beta \in I \cap \gamma \) and for each \( \beta < \alpha < \gamma \).

Now, we proceed to construct the homeomorphism \( \psi_\gamma : [\varepsilon]^{<\omega} \rightarrow \{0,1\} \). By applying Lemma 3.2 to \( \gamma \), \( \{F^0_\gamma, \ldots, F^k_\gamma\}, B = \{h_\beta : \beta \leq \gamma\} \) and \( \{f_\gamma(n) : n < \omega\} \), we can find a homomorphism \( \psi_\gamma : [\gamma]^{<\omega} \rightarrow \{0,1\} \) satisfying:

\( i) \ \psi_\gamma(F^0_\gamma) = 1 \) if \( F^0_\gamma \neq \emptyset \);

\( ii) \ \{n < \omega : \forall i \leq k(\psi_\gamma(f_\gamma(n(k+1)+i)) = \psi_\gamma(F^i_\gamma))\} \) is finite and

\( iii) \ \{n \in C_\beta : \forall i < m_\beta(\psi_\gamma(h_\beta(i,n)) = \sigma(i))\} \) is infinite for each \( \beta \leq \gamma \) and for each \( \sigma \in \{0,1\}^{m_\beta} \).

Next, we extend \( \psi_\gamma \) to a homeomorphism \( \psi_\gamma : [\varepsilon]^{<\omega} \rightarrow \{0,1\} \) such that \( \psi_\gamma(\{\xi\}) = t_\xi(\gamma) \) for each \( \xi \in I \) with \( \gamma < \xi \). Thus, conditions \( a) \) and \( b) \) are clearly satisfied. It follows directly from clause \( D) \) that

\( \text{(*)} \ \{\Psi|_{[\gamma,\beta)}[h_\beta(0,n)], \ldots, \Psi|_{[\gamma,\beta)}[h_\beta(m_\beta-1,n)] : n \in C_\beta\} \) is dense in \( \{\{0,1\}^{[\gamma,\beta)}\}^{m_\beta} \) for each \( \beta \in I \cap \gamma \).

By condition \( iii) \), we know that \( \{\psi_\gamma(h_\beta(0,n)), \ldots, \psi_\gamma(h_\beta(m_\beta-1,n))\} : n \in C_\beta\} = \{0,1\}^{m_\beta} \).

Hence,

\( \text{(**) } \{\Psi|_{[\gamma,\beta)}[h_\beta(0,n)], \ldots, \Psi|_{[\gamma,\beta)}[h_\beta(m_\beta-1,n)] : n \in C_\beta\} \) is dense in \( \{\{0,1\}^{[\gamma,\beta)}\}^{m_\beta} \) for each \( \beta \in I \cap \gamma \).
Thus, condition $d$) holds for $\gamma$. Condition $c$) follows directly from condition $C$) and $iii$).
This shows the theorem. □

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