Line-closed matroids, quadratic algebras, and formal arrangements

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Abstract

Let $G$ be a matroid on ground set $\mathcal{A}$. The Orlik-Solomon algebra $A(G)$ is the quotient of the exterior algebra $E$ on $\mathcal{A}$ by the ideal $I$ generated by circuit boundaries. The quadratic closure $\overline{A}(G)$ of $A(G)$ is the quotient of $E$ by the ideal generated by the degree-two component of $I$. We introduce the notion of nbb set in $G$, determined by a linear order on $\mathcal{A}$, and show that the corresponding monomials are linearly independent in the quadratic closure $\overline{A}(G)$. As a consequence, $A(G)$ is a quadratic algebra only if $G$ is line-closed. An example of S. Yuzvinsky proves the converse false. These results generalize to the degree $r$ closure of $A(G)$.

The motivation for studying line-closed matroids grew out of the study of formal arrangements. This is a geometric condition necessary for $A$ to be free and for the complement $M$ of $A$ to be a $K(\pi, 1)$ space. Formality of $A$ is also necessary for $A(G)$ to be a quadratic algebra. We clarify the relationship between formality, line-closure, and other matroidal conditions related to formality. We give examples to show that line-closure of $G$ is not necessary or sufficient for $M$ to be a $K(\pi, 1)$, or for $A$ to be free.

1 Introduction

Let $\mathbb{K}$ be a field. An arrangement is a finite set $\mathcal{A} = \{H_1, \ldots, H_n\}$ of linear hyperplanes in $V = \mathbb{K}^\ell$. Each $H_i$ is the kernel of a linear form $\alpha_i : V \rightarrow \mathbb{K}$, unique up to nonzero scalar multiple. Let $[n]$ denote the set $\{1, \ldots, n\}$ and $2^{[n]}$ the set of subsets of $[n]$.

A coordinate-free combinatorial model of the arrangement $\mathcal{A}$ is provided by the underlying matroid of $\mathcal{A}$, which we denote by $G(\mathcal{A})$, or simply $G$. This matroid contains the same information as the intersection lattice $L(\mathcal{A})$ — see [DT92]. By definition the matroid $G$ is the collection of dependent subsets of the set of defining forms $\{\alpha_1, \ldots, \alpha_n\}$. We identify these subsets with the corresponding sets of labels. Then it is easy to see that

$$G = \{ S \subseteq [n] \mid \text{codim}(\bigcap_{i \in S} H_i) < |S| \}.$$
Elements of $G$ are called dependent sets, and elements of $2^{[n]} - G$ are independent sets. The projective point configuration $\mathcal{A}^* \subseteq \mathbb{P}(V^*)$ determined by $\{\alpha_1, \ldots, \alpha_n\}$ is called a projective realization of $G$.

There are several other data besides the dependent sets which suffice to determine $G$ uniquely. Among these are the circuits of $G$, which are the minimal dependent sets, and the bases of $G$, which are the maximal independent sets. Besides these, we single out two functions which also uniquely determine $G$.

The rank function $rk : 2^{[n]} \to \mathbb{Z}$ is given by $rk(X) = \text{codim}(\bigcap_{i \in X} H_i)$. In the abstract setting, $rk(X)$ is the (unique) size of a maximal independent subset of $X$. The rank $rk(G)$ of $G$ is $rk([n])$. The closure operator $\text{cl} : 2^{[n]} \to 2^{[n]}$, given by

$$\text{cl}(X) = \{i \in [n] \mid \text{rk}(X \cup \{i\}) = \text{rk}(X)\},$$

also uniquely determines $G$.

We refer the reader to the recent survey [FR00] for more discussion of the role of matroid theory in the study of complex hyperplane arrangements.

A set $S$ is closed if $\text{cl}(S) = S$. Closed sets are also called flats. A flat corresponds to the collection of hyperplanes in $\mathcal{A}$ containing a fixed subspace of $K^t$, or equivalently, the intersection of the point configuration $\mathcal{A}^*$ with a fixed projective subspace of $\mathbb{P}(V^*)$. The set of flats, ordered by inclusion, forms a geometric lattice $L(G)$ isomorphic to the intersection lattice $L(\mathcal{A})$. The flats of rank one are the singletons, called points. Flats of rank two are called lines. This terminology is natural with regard to the dual projective point configuration $\mathcal{A}^*$.

Let $K = \mathbb{C}$. The complement of $\mathcal{A}$ is $V - \bigcup_{i=1}^{n} H_i$, denoted by $M$. The cohomology $H^*(M)$ is isomorphic to the Orlik-Solomon algebra $A(G)$ of the underlying matroid $G$, defined in the next section. Study of the lower central series of $\pi_1(M)$ [Fal88, PY99] leads to the consideration of arrangements for which the cohomology algebra $H^*(M)$, or equivalently, the Orlik-Solomon algebra $A(G)$, is quadratic. Here $A(G)$ is quadratic if it has a presentation in which all relations have degree two. While this condition depends only on $G$, the underlying combinatorial meaning has never been understood.

The best results in this direction involve the notion of formality. An arrangement is formal if it is uniquely determined up to linear isomorphism by the dependence relations yielding dependent sets of rank two in $G$. This is a geometric, non-matroidal condition, and is a necessary condition for $A(G)$ to be quadratic. In looking for a matroidal analogue of formality we were led naturally to the study of line-closed matroids.

**Definition 1.1** A subset $S \subseteq [n]$ is line-closed if $\text{cl}([i, j]) \subseteq S$ for every $i, j \in S$. The matroid $G$ is line-closed if every line-closed set is closed.

In attempting to sort out how this property fits in with other properties related to formality, we were led to the following.

**Conjecture 1.2** $G$ is line-closed if and only if $A(G)$ is quadratic.
In this paper we prove half of this conjecture, that \( A(G) \) quadratic implies \( G \) line-closed, for arbitrary coefficient fields \( \mathbb{K} \).

The author sketched this proof and stated Conjecture 1.2 in a lecture at the workshop “Arrangements in Boston” in 1999 [Fal99]. Subsequently S. Yuzvinsky found a counter-example for the full conjecture (at least for \( \mathbb{K} = \mathbb{C} \)), a line-closed matroid whose Orlik-Solomon algebra is not quadratic. We exhibit Yuzvinsky’s example, and refer the reader to the companion paper [DY00] for details on Yuzvinsky’s approach, and for a stronger condition, also necessary but not sufficient for quadraticity of \( A(G) \).

One can define a quadratic algebra \( \overline{A}(G) \) and a surjection \( \overline{A}(G) \to A(G) \) which is an isomorphism if and only if \( A(G) \) is quadratic; \( \overline{A}(G) \) is called the quadratic closure of \( A(G) \). Our main theorem follows from a more general construction, a partial generalization of the well-known \( \text{nbc} \) (“no broken circuits”) basis for \( A(G) \). We generalize one of the characterizations of \( \text{nbc} \) sets to the lattice of line-closed sets of \( G \). The result is the notion of \( \text{nbb} \) set. We show that the monomials corresponding to \( \text{nbb} \) sets are linearly independent in the quadratic closure \( \overline{A}(G) \). In contrast to the situation for \( \text{nbc} \) sets, the number of \( \text{nbb} \) sets is not independent of the linear ordering of the atoms. But the collection of \( \text{nbb} \) sets will include all of the \( \text{nbc} \) sets, for any given linear ordering. The two collections coincide, for every linear ordering, if and only if \( G \) is line-closed. Thus, if \( G \) is not line-closed, \( \overline{A}(G) \) must be strictly bigger than \( A(G) \), so \( A(G) \) is not quadratic. The entire development generalizes to any degree, with the line-closed sets and quadratic closure replaced by \( r \)-closed sets and degree \( r \) closure. The main theorem and its generalization are developed and proved in Section 2.

The problem of finding a (monomial, or combinatorial) basis for \( \overline{A}(G) \) is an interesting problem with some applications to lower central series calculations. Yuzvinsky’s example shows that there may be no linear ordering for which the \( \text{nbb} \) monomials form a basis of \( \overline{A}(G) \). Our definition of \( \text{nbb} \) set is a special case of the \( \text{NBB} \) (“no bounded below”) sets of A. Blass and B. Sagan [BS97], for the lattice of line-closed sets of \( G \), with a linear ordering on the set of atoms. Blass and Sagan define \( \text{NBB} \) sets for finite atomic lattices with an arbitrary partial order on the set of atoms. Although general \( \text{NBB} \) monomials for the lattice of line-closed sets are not linearly independent in \( \overline{A}(G) \), we present a partial generalization of our main result to non-linear orderings, possibly yielding better lower bounds on \( \dim \overline{A}(G) \) for \( G \) of rank four or greater.

Much of the research in complex hyperplane arrangements focuses on the extent to which properties of the complement \( M \) as a topological space or algebraic variety are determined by the combinatorial structure of \( G \). In particular, two important open problems are whether asphericity of \( M \) [FR86] or freeness of \( \mathcal{A} \) [OT92] are dependent only on \( G \) – see [FR86] and [OT92]. Formality of \( \mathcal{A} \) is also a necessary condition for each of these two properties. Thus attempts were made to replace the definition of formality with some stronger purely combinatorial notion – line-closure is one example. In the last section we give several other natural candidates for combinatorial analogues of formality. Each of them
is stronger than formality. We establish the relationships among these various notions and show by example that in fact none of them have true topological implications. The discussion leads to an interesting conjecture concerning matroids which are determined by their points and lines.

2 Quadratic closure and \( nbb \) sets

We will use the matroid-theoretic terminology developed in the introduction without further comment. The reader is referred to [Whi86, Oxl92] for further background.

We begin with the definition of the Orlik-Solomon algebra \( A(G) \) of a matroid \( G \) on ground set \([n]\). For the remainder of the paper, let \( K \) be any field, or indeed any commutative ring. Assume \( G \) has no loops or multiple points.

Let \( E = \bigwedge(V) \), the exterior algebra generated by 1 and \( \{e_i \mid 1 \leq i \leq n\} \), with the usual grading by degree. If \( S = (i_1, \ldots, i_p) \) is an ordered \( p \)-tuple we denote the product \( e_{i_1} \cdots e_{i_p} \) by \( e_S \). We occasionally use the same notation when \( S \) is an unordered set – in this case \( e_S \) is well-defined up to sign.

Define the linear mapping \( \partial : E^p \to E^{p-1} \) by

\[
\partial(e_{i_1} \cdots e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p},
\]

where \( \hat{\cdot} \) indicates an omitted factor. Then \( \partial \) is a graded derivation, that is,

\[
\partial(x \wedge y) = \partial x \wedge y + (-1)^{\deg(x)} x \wedge \partial y
\]

for homogeneous \( x, y \in E \).

Let \( I \) denote the ideal of \( E \) generated by \( \{\partial e_S \mid S \text{ is dependent}\} \).

**Definition 2.1** The Orlik-Solomon algebra \( A = A(G) \) of \( G \) is the quotient \( E/I \).

Since \( I \) is generated by homogeneous elements, both \( I \) and \( A \) inherit gradings from \( E \). We denote the image of \( e_S \) in \( A \) by \( a_S \). The topological significance of \( A \) is given in the following.

**Theorem 2.2** ([OS80]) If \( A \) is an arrangement in \( \mathbb{C}^f \) with complement \( M \) and underlying matroid \( G \), then \( A(G) \cong H^*(M, \mathbb{K}) \).

**The quadratic closure of \( A(G) \)**

**Definition 2.3** A graded algebra \( U \) is quadratic if \( U \) has a presentation with generators of degree one and relations of degree at most two.

Let \( J \) denote the ideal of \( E \) generated by \( I^2 \), the degree two part of the relation ideal \( I \). Because \( E \) itself is quadratic, the Orlik-Solomon algebra \( A \) will
be quadratic if and only if $J = I$. More generally the quotient $E/J$ is called the \textit{quadratic closure} of $A(G)$, denoted $\overline{A}(G)$, or sometimes $\overline{A}$.

Quadratic Orlik-Solomon algebras appear in the study of complex arrangements, in the rational homotopy theory of the complement $M$ \cite{Fal88} and the Koszul property of $A(G)$ \cite{PY99}. Rational homotopy theory provides a connection with the lower central series of the fundamental group. In that vein an invariant $\phi_3$ of $A(G)$ was introduced in \cite{Fal89}, defined as follows:

$$\phi_3(G) = \text{nullity}\left(\delta : \mathcal{E}^1 \times \mathcal{I}^2 \to \mathcal{E}^3\right),$$

where $\delta$ is multiplication in $\mathcal{E}$. When $\mathbb{K} = \mathbb{C}$ and $G$ is the matroid of an arrangement with complement $M$, $\phi_3(G)$ is the rank of the third factor in lower central series of $\pi_1(M)$. Because the image of $\delta$ is precisely $J^3$, the cokernel of $\delta$ is $\overline{A}^3$, and we find a simple relationship between $\phi_3(G)$ and $\dim(\overline{A}^3)$. The proof of the identity is left as an exercise.

**Theorem 2.4**

$$\phi_3(G) = \dim(\overline{A}^3) + n \dim(\mathcal{I}^2) - \binom{n}{3}$$

$$= 2 \binom{n+1}{3} - n \dim(A^2) + \dim(\overline{A}^3)$$

The preceding identity can be stated in a simpler way, indicating that $\dim(\overline{A}^3/A^3)$ measures of the failure of the LCS (lower central series) formula relating the ranks of lower central series factors of $\pi_1(M)$ to the betti numbers $\dim(A^p)$, $p \geq 0$. Let $\gamma_3 = \dim(A^3) + n \dim(\mathcal{I}^2) - \binom{n}{3}$. Then, according to \cite{Fal89}, $\gamma_3$ is the value of $\phi_3$ predicted by the LCS formula for the given values of $\dim(A^p)$, and $\phi_3 \geq \gamma_3$ with equality if and only if $\mathcal{I}^3 = \mathcal{J}^3$.

**Corollary 2.5**

$$\dim(\overline{A}^3) - \dim(A^3) = \phi_3 - \gamma_3$$

**The line-closure of a matroid** Next we refine further the material on line closure from Section \ref{sec:line_closure}. Let us define an idempotent, order-preserving closure operator on subsets of $S \subseteq [n]$ using line-closed sets: the \textit{line-closure} $\ell c(S)$ is by definition the intersection of the line-closed sets containing $S$. Since closed sets are automatically line-closed, $\ell c(S) \subseteq \text{cl}(S)$. We will consider the combinatorial structure consisting of the set $[n]$ equipped with the closure operator $\ell c : 2^{[n]} \to 2^{[n]}$ to be the line-closure of the matroid $G$, and denote it by $\overline{G}$. This set system $\overline{G}$ will not be a matroid in general, because the operator $\ell c$ fails to satisfy the Steinitz exchange axiom. The arguments and constructions in this section are seriously affected by this defect. Clearly $G$ is line-closed if and only if $\overline{G} = G$.

The collection of line-closed sets, partially-ordered by inclusion, will be denoted by $L(G)$. We call $L(G)$ the \textit{line-closure} of $L(G)$. The poset $L(G)$ is a lattice \cite[Section 2]{Rot64} in which every element is a join of atoms. But $L(G)$ is not a graded lattice, as the example below shows. This is a reflection of the failure of the exchange axiom. Again, $G$ is line-closed if and only if $L(G) = L(G)$. 


Example 2.6 Let $G$ be the rank-three matroid on $[6]$ with rank-two circuits \{1, 2, 3\}, \{3, 4, 5\}, and \{1, 5, 6\}. This example, pictured in Figure 1, is the “rank-three wheel” \[Oxl92\].

Then there are two maximal chains in $\mathcal{L}(G)$ of different lengths, namely

$$\emptyset < 1 < 123 < 123456,$$
$$\emptyset < 2 < 24 < 246 < 123456.$$

\[\Box\]

\textbf{nbc and nbb sets} Fix a linear order of the ground set $[n]$. A broken circuit of $G$ is a set of the form $C - \min(C)$, where $C$ is a circuit of $G$. An \textit{nbc} set of $G$ is a subset of $[n]$ which contains no broken circuits. The collection of \textit{nbc} sets of $G$ will be denoted \textit{nbc}(G). The set of elements of \textit{nbc}(G) of cardinality $p$ is denoted \textit{nbc}$_p$(G). The dependence on the linear order of $[n]$ is suppressed in the notation. For a flat $X$ of $G$, let \textit{nbc}$_X$(G) denote the set of \textit{nbc} sets with closure equal to $X$.

Among the properties of \textit{nbc}(G) we highlight the following. For proofs and a more complete discussion see \[Bjö92\]. Let $\mu : L \rightarrow \mathbb{Z}$ be the Möbius function of $L$.

**Theorem 2.7** For any linear order on $[n]$,

(i) \textit{nbc}(G) is a pure simplicial complex of dimension $\text{rk}(G) - 1$.

(ii) The cardinality of \textit{nbc}$_p$(G) is equal to $w_p(L)$, the $p$th Whitney number of $L$.

(iii) For every flat $X$ of $G$, the cardinality of the set \textit{nbc}$_X$(G) is equal to $(-1)^{\text{rk}(X)}\mu(X)$.

The relevance of \textit{nbc} sets to Orlik-Solomon algebras was established by several authors independently – see \[Bjö92\, Section 7.11, § 7.10\].

**Theorem 2.8** The set \{a$_S$ | $S \in \text{nbc}(G)$\} forms a basis for $A(G)$.
There are several natural ways in which one might attempt to relate $\overline{A}(G)$ directly to $G$, motivated by the various connections between $A(G)$ and $G$. (For instance, independent sets in $G$ correspond to nonzero monomials in $A(G)$.) None of these seem to work; the difficulties can all be traced back to the failure of the exchange axiom. There is at least an indirect connection between $A(G)$ and $G$ obtained by generalizing the following well-known property of $nbc$ sets.

Let us impose the natural linear order on $[n]$, unless otherwise noted.

**Theorem 2.9** ([Bjö92]) An increasing subset $S = \{i_1, \ldots, i_p\} \subseteq [n]$ is $nbc$ if and only if $i_k = \min \cl(S)$ for each $1 \leq k \leq p$.

Replacing matroid closure with line-closure, we propose the following.

**Definition 2.10** An increasing subset $S = \{i_1, \ldots, i_p\} \subseteq [n]$ is $nbb$ if and only if

\[ i_k = \min \lc(S) \]

or each $1 \leq k \leq p$.

The collection of $nbb$ sets of $G$ will be denoted by $nbb(G)$. Of course $nbb(G)$ is dependent only on $\overline{G}$, rather than $G$.

**Theorem 2.11** $nbb(G)$ is a simplicial complex, containing $nbc(G)$ as a subcomplex.

**proof:** The first assertion follows from the monotonicity of the line-closure operator. The second is a consequence of the fact that $\lc(S) \subseteq \cl(S)$ for any subset $S$ of $[n]$. \hfill \Box

We will customarily specify $nbb(G)$ and $nbc(G)$ by listing the facets, or maximal simplices.

Because of the lack of exchange, $nbb(G)$ depends heavily on the linear ordering of the points.

**Example 2.12** Let $G$ be the matroid of Example 2.6. Then, with the natural linear order on $[6]$, the facets of $nbb(G)$ are

\[ 1246, 136, 135, 125, 134, \text{ and } 124. \]

If we adopt the linear ordering $2 < 1 < 3 < 4 < 5 < 6$, the new $nbb$ complex has facets

\[ 246, 236, 216, 235, 215, 234, \text{ and } 214. \]

In fact, for this second linear ordering, $nbb(G) = nbc(G)$. \hfill \Box

We see from this example that the number of $nbb$ sets of a fixed size $p$ is not independent of the linear ordering, and the complex $nbb(G)$ may fail to be pure. Compare with Theorem 2.7(i) and (ii). We also see that $nbb(G)$ may agree with $nbc(G)$ even when $G$ is not line-closed. However, these $nbb$ sets do capture the lack of line-closure, in the following sense.
Theorem 2.13 The matroid $G$ is line-closed if and only if $nbb(G) = nbc(G)$ for every linear ordering of $[n]$.

proof: Suppose $G$ is not line-closed. Then there exists a line-closed set $X$ which is not closed. Let $i \in \text{cl}(X) - X$, and choose a linear order on $[n]$ such that $i$ precedes $\min(X)$. Now, let $S = (i_1, \ldots, i_p)$ be the lexicographically first ordered basis for the flat $\text{cl}(X)$ which is contained in $X$. Then, by the choice of ordering, $S \notin nbc(G)$, by Theorem 2.9. We claim $S \in nbb(G)$. Suppose not. Then, for some $k$, $i := \min \ellc \{i_k, \ldots, i_p\}$ is less than $i_k$. Since $X$ is line-closed, $i \in X$. Also, by the exchange axiom in $G$, $S - \{i_k\} \cup \{i\}$ is a basis for $\text{cl}(X)$, and is lexicographically smaller than $S$. This contradicts the choice of $S$. Thus $S \in nbb(G) - nbc(G)$, so $nbc(G) \neq nbb(G)$. Conversely, if $G$ is line-closed, then $nbb(G) = nbc(G)$ by Theorem 2.9. \hfill \blacksquare

Independence of $nbb$ monomials in $\overline{A}(G)$ We turn now to the analysis of the quadratic closure $\overline{A}(G)$ of the Orlik-Solomon algebra.

For each line-closed set $X \in L(G)$, let $E_X$ be the subspace of $E$ spanned by monomials $e_S$ for which $\ellc(S) = X$. Then we have a grading of $E$ by $L(G)$:

$$E = \bigoplus_{X \in L(G)} E_X.$$ 

Let $J_X = J \cap E_X$ and $\overline{A}_X(G) = E_X/J_X$. Then we have the following analogue of [OT92, Theorem 3.26].

Lemma 2.14 $\overline{A}(G) = \bigoplus_{X \in L(G)} \overline{A}_X(G)$.

proof: The ideal $J$ is generated by elements $\partial e_{ijk}$ where $\{i, j, k\}$ is dependent. Since $G$ has no multiple points, $\{i, j, k\}$ is a circuit. Then $\ellc(\{i, j\}) = \ellc(\{i, k\}) = \ellc(\{j, k\})$, each being equal to $\ellc(\{i, j, k\})$. This shows that $\partial e_{ijk}$ is homogeneous in the grading above. Thus $J = \bigoplus_{X \in L(G)} J_X$, and the result follows. \hfill \blacksquare

We will also use the following elementary observation, proof left to the reader.

Lemma 2.15 The graded derivation $\partial : \mathcal{E} \rightarrow \mathcal{E}$ induces a graded derivation $\overline{\partial} : \overline{\mathcal{A}}(G) \rightarrow \overline{\mathcal{A}}(G)$.

We are now prepared to prove the main result. For $S \subset [n]$ we denote by $\overline{\pi}_S$ the image of $e_S$ in the quadratic closure $\overline{A}(G)$.

Theorem 2.16 The set $\{\overline{\pi}_S \mid S \in nbb(G)\}$ is linearly independent in $\overline{A}(G)$.

proof: With Lemma 2.14 in hand the proof is identical to the argument in the proof of Theorem 2.8. It is enough to prove the result for $nbb$ sets of a fixed size $p$. Then we induct on $p$. Suppose

$$\sum_{S \in nbb^p(G)} \lambda_S \overline{\pi}_S = 0.$$
By Lemma 2.14 we may assume that $\ell(S) = X$ for a fixed element $X \in \overline{L}(G)$ and all $S$ in the sum. Setting $i_0 = \min(X)$, we have $\min(S) = i_0$ for all $S$, by definition of $nbb(G)$. Write $S' = S - \{i_0\}$. Then we have

$$\pi_{i_0} \land \left( \sum_{S \in nbb^p(G) \atop \ell_c(S) = X} \lambda_S \pi_{S'} \right) = 0.$$  

Applying the derivation $\partial$ we obtain

$$\sum_{S \in nbb^p(G) \atop \ell_c(S) = X} \lambda_S \pi_{S'} + \sum_{S \in nbb^p(G) \atop \ell_c(S) = X} \pi_{i_0} \land \partial \pi_{S'} = 0.$$  

Using again the definition of $nbb(G)$, we have that $i_0 \notin \ell_c(S')$ for $\ell_c(S) = X$. Then, applying Lemma 2.14 once more, we have

$$\sum_{S \in nbb^p(G) \atop \ell_c(S) = X} \lambda_S \pi_{S'} = 0.$$  

Since $S \in nbb^p(G)$ implies $S' \in nbb^{p-1}(G)$, we conclude $\lambda_S = 0$ for all $S$ by the inductive hypothesis. \qed

As a consequence we obtain half of Conjecture 1.2.

**Corollary 2.17** Suppose $A(G)$ is quadratic. Then $G$ is line-closed.

**proof:** If $G$ is not line-closed, then by Theorems 2.11 and 2.13 there is a linear ordering of $[n]$ such that the cardinality of $nbb(G)$ is strictly greater than that of $nbc(G)$. Then, by Theorems 2.16 and 2.8, we have $\dim A(G) > \dim A(G)$, so $A(G)$ is not quadratic. \qed

Because of Example 2.6, it is not the case that $\{\pi_S \mid S \in nbb(G)\}$ spans $A(G)$ for every linear order. When we announced Theorem 2.16 in the Boston lecture [Fal99], we expressed some hope that one could show the existence of some linear order for which the $nbb$ monomials span $A(G)$, yielding a proof of the converse of Corollary 2.17 as well as a combinatorial calculation of $\phi_3(G)$ via Corollary 2.3. Subsequently S. Yuzvinsky found a counterexample.

**Example 2.18** (Yuzvinsky [DY00]) Let $G$ be the rank-three matroid on $[8]$ with nontrivial lines

$$123, 148, 257, 3678, \text{ and } 456,$$

pictured in Figure 2 on the next page.

One can use Theorem 3.13 of Section 3 below to check fairly quickly by hand that $G$ is line-closed. On the other hand, one computes $\phi_3(G) = 16$. (We use a Mathematica script available from the author.) Then, by Theorem 2.5, we have $\dim A^3 = 16$. But $\dim A^3(G) = 14$. Thus $A(G)$ is not quadratic.
For us, Example 2.18 shows that there may be no “good” linear order, for which \( \text{nbb}(G) \) spans \( A(G) \), for some matroids \( G \). See [DY00] for further discussion of the converse of Corollary 2.17.

Using Theorem 2.17 we have an alternate proof of [Fal88, Prop. 5.1].

**Corollary 2.19** If \( n > \text{rk}(G) \) and \( G \) has a basis each of whose two-point subsets is closed in \( G \), then \( A(G) \) is not quadratic.

**proof:** Such a basis \( B \) would form a line-closed set by hypothesis, but it cannot be closed in \( G \) since \( |\text{cl}(B)| = n > \text{rk}(G) = |B| \).

We may also use the work on quadratic algebras together with Corollary 2.17 to give a nice sufficient condition for line-closure of matroids. The following assertion is a generalization of [Fal89a, Proposition 3.2], with essentially the same proof.

**Corollary 2.20** Suppose, for every circuit \( S \) of \( G \) with \( |S| \geq 4 \), the closures in \( G \) of two disjoint two-point subsets of \( S \) meet. Then \( G \) is line-closed.

**proof:** We show that \( A(G) \) is quadratic by verifying directly that \( \mathcal{I}^p \subseteq \mathcal{J}^p \) for all \( p \geq 3 \). Let \( S = \{i_1, \ldots, i_p\} \) be a circuit, \( p \geq 4 \), and suppose \( i_0 \in \text{cl}(\{i_1, i_2\}) \cap \text{cl}(\{i_3, i_4\}) \). Then

\[
\partial e_{i_1 i_2 i_3 i_4} = (e_{i_3} - e_{i_4})\partial e_{i_0 i_1 i_2} + (e_{i_1} - e_{i_2})\partial e_{i_0 i_3 i_4}.
\]

Thus \( \partial e_{i_1 i_2 i_3 i_4} \) lies in \( \mathcal{J} \). Then \( e_{i_1 i_2 i_3 i_4} \in \mathcal{J} \), and it follows from the Leibniz rule that \( \partial e_S \) lies in \( \mathcal{J} \).

If \( G \) has rank three, the hypothesis of Theorem 2.20 can be weakened, for in this case it suffices to show that \( \partial e_S \in \mathcal{J} \) for those circuits \( S \) with \( |S| \geq 4 \) and \( 1 \in S \). Arrangements of rank three whose matroids satisfy the weaker hypothesis are called parallel arrangements. See [FR86].

**Generalization to high rank/degree** All of the results of this section on line-closure and quadraticity can be generalized, with essentially identical proofs.
**Definition 2.21** A subset of \([n]\) is \(r\)-closed if it contains the closures of all of its \(p\)-subsets for all \(p \leq r\). The matroid \(G\) is \(r\)-closed if every \(r\)-closed set is closed.

For arbitrary \(G\) the collection \(L_r(G)\) of \(r\)-closed sets forms a lattice, and we have a sequence of surjective order-preserving maps

\[
B_n = L_1(G) \rightarrow \overline{L}(G) = L_2(G) \rightarrow \cdots \rightarrow L_r(G) \rightarrow \cdots \rightarrow L_n(G) = L(G),
\]

where \(B_n\) is the boolean lattice.

**Definition 2.22** The degree \(r\) closure \(\overline{A}_r(G)\) of \(A(G)\) is \(E / \mathcal{J}_r\), where \(\mathcal{J}_r\) is the ideal generated by the elements of \(\mathcal{I}\) of degree less than or equal to \(r\).

Thus we have a sequence of surjective homomorphisms

\[
E = A_1(G) \rightarrow \overline{A}(G) = A_2(G) \rightarrow \cdots \rightarrow A_r(G) \rightarrow \cdots A_n(G) = A(G).
\]

**Definition 2.23** An ordered subset \(S = \{i_1, \ldots, i_p\}\) is \(r\)-nbb if for each \(k\), \(i_k\) is the first element in the \(r\)-closure of \(\{i_k, \ldots, i_p\}\).

**Lemma 2.24** \(A_r(G) = \bigoplus_{X \in \overline{L}(G)} \overline{A}_X(G)\).

**proof:** The crucial points are (i) that \(\mathcal{J}_r\) is generated by boundaries of circuits of size at most \(r + 1\), and (ii) that \(r\)-closure agrees with matroid closure on sets of size at most \(r\). Using these observations the proof of Lemma 2.14 is easy to adapt to the more general setting.

The proof of the following generalization is now identical to the proof of Theorem 2.16.

**Theorem 2.25** The set of monomials in \(A_r(G)\) corresponding to the \(r\)-nbb sets of \(G\) forms a linearly independent set.

**Corollary 2.26** If \(A_r(G) = A(G)\), then \(G\) is \(r\)-closed.

**The nbb complex and a generalization to nonlinear orders** After formulating Definition 2.11 and proving Theorem 2.16, we found that our notion of \(nbb\) set coincides with a special case of the more general notion of \(NBB\) set in a finite lattice with a partial ordering of the atoms, introduced by Blass and Sagan in [BS97]. These results are stated only for the line-closure of \(G\), but again analogous results will hold for \(r\)-closure.

**Definition 2.27** ([BS97]) Suppose \((\overline{L}, \leq)\) is a finite lattice, and \(\preceq\) is a partial ordering of the atoms of \(L\). A set \(T\) of atoms is bounded below if there exists an atom \(a\) such that \(a < \bigwedge T\) and \(a < t\) for all \(t \in T\). A set \(S\) is \(NBB\) if \(S\) does not contain any set \(T\) which is bounded below.
Theorem 2.28 A set $S$ is nbb if and only if $S$ is an NBB set in the lattice $\overline{L}(G)$ for the given linear ordering of the atoms.

**proof:** In our setting the atom ordering $\preceq$ is the natural linear ordering on $[n]$. In this context a set $T \subseteq [n]$ is bounded below if and only if there exists $i \in \ell c(T)$ with $i < \min(T)$. Suppose $S = \{i_1, \ldots, i_p\}$ is nbb and $T \subseteq S$. Let $i_k = \min(T)$. Then $\ell c(T) \subseteq \ell c(\{i_k, \ldots, i_p\})$, so $\min(\ell c(T)) \geq \min(\ell c(\{i_k, \ldots, i_p\}))$. Since $S$ is nbb we conclude $\min(\ell c(T)) = i_k$, so $T$ is not bounded below. Conversely, if $S$ is not nbb then for some $k$, $T = \{i_k, \ldots, i_p\}$ is bounded below by $\min(\ell c(T)) < i_k$, and thus $S$ is not NBB.

This observation yields a numerical result on the number of nbb sets, by one of the main results of [BS97]. Let $\overline{\mu} : \overline{L}(G) \to \mathbb{Z}$ denote the Möbius function of $\overline{L}(G)$.

**Corollary 2.29** The sum $\sum_{S \in \text{nbb}(G)} (-1)^{|S|}$ is equal to $\overline{\mu}(X)$.

Let $\text{NBB}(G, \preceq)$ denote the collection of NBB sets of $\overline{L}(G)$ under the atom-order $\preceq$.

**Theorem 2.30** Suppose $\preceq$ is a partial order on $[n]$ with the property that each line-closed set $X$ has a unique smallest element relative to $\preceq$. Then

$$\{\overline{\mu}_S \mid S \in \text{NBB}(G, \preceq)\}$$

is linearly independent in $\overline{A}(G)$.

**proof:** Assume without loss that the natural order on $[n]$ is a linear extension of $\preceq$. Then the proof of 2.16 goes through without change. \qed

We also have the following analogue of Theorem 2.13. Recall that nbc$(G)$ is determined by a linear order on $[n]$.

**Theorem 2.31** nbc$(G) \subseteq \text{NBB}(G, \preceq)$ for any linear order which extends $\preceq$.

**proof:** Suppose $T$ is a bounded below set. Let $a \preceq \bigvee T$ with $a \prec t$ for all $t \in T$. Then $a$ precedes $\min(T)$ in any linear extension of $\preceq$. Since $\bigvee T = \ell c(T) \subseteq \ell c(T)$, it follows that $T$ contains a broken circuit. \qed

Theorem 2.30 raises the possibility of finding more than $|\text{nbc}(G)|$ linearly independent monomials in $\overline{A}(G)$, even when $G$ is line-closed. At this point we have no examples of this phenomenon.
3 Combinatorial notions of formality

Our research on line-closed matroids was motivated by the study of formal arrangements, and specifically by attempts to describe combinatorially the property of an arrangement being “generic with given codimension two structure.” We start this section by recalling the definition of formal arrangement, and outlining some of the motivation and main results. We then turn to various combinatorial versions of formality. These have been studied to some extent before, but the definitions have never been published. We present some newly rediscovered results and examples, which appeared long ago in the combinatorial literature but not in the context of formal arrangements.

Line-closure turns out to be the strongest among the properties we study here. The remaining notions form a hierarchy descending to formality of a realization, which in itself is not a combinatorial notion. We show by example that none of the combinatorial notions have the nice topological or algebraic consequences that formality affords.

Formal arrangements The notion of formality was introduced in [FR86] and studied further in [Yuz93, BT94, BB97]. The terminology is unfortunate; there is a notion of formality of spaces that is important in rational homotopy theory, and has implications for arrangements, but the definition of formal arrangement is completely unrelated.

We adopt the definition from [BT94]. Henceforth assume $A$ is an essential arrangement, that is, $\text{rk}(G) = \dim(V)$. Let $e_i$ denote the $i$th standard basis vector in $K^n$. The weight of a vector in $K^n$ is the number of nonzero entries.

Definition 3.1 Let $\rho : K^n \to V^*$ be the linear mapping defined by $\rho(e_i) = \alpha_i$. Then $A$ is formal if the kernel of $\rho$ is spanned by elements of weight at most three.

An element of $\ker(\rho)$ of weight three corresponds to a dependent set of $G$ of size three, and thus of rank two. A vector in $\ker(\rho)$ gives the coefficients in a dependence relation among the linear forms. Thus $A$ is formal if all dependence relations among the forms $\alpha_i$ are consequences of “rank two dependence relations.”

There is a natural way to associate a subspace $W$ of $K^n$ of dimension $r$ with a (possibly degenerate) arrangement of $n$ hyperplanes in $K^r$, by considering the set of intersections of $W$ with the $n$ coordinate hyperplanes as an arrangement in $W$. If $K \subseteq K^n$ denotes the kernel of $\rho$, then its orthogonal complement $K^\perp$ returns the original arrangement $A$ under this construction. Indeed, the linear mapping

$$\Phi = (\alpha_1, \ldots, \alpha_n) : V \to K^n$$

carries $V$ isomorphically to $K^\perp$ and $H_i$ to the intersection of $K^\perp$ with \{$x_i = 0$\}. Let $F \subseteq K$ denote the subspace spanned by elements of weight three. The arrangement corresponding to the subspace $F^\perp \subseteq K^n$ is called the formalization of $A$, denoted $A_F$. This construction first appears in [Yuz93].
A section $\overline{B}$ of an arrangement $B$ in $V$ is formed by intersecting the hyperplanes of $B$ with a linear subspace $W$ of $V$. The section $\overline{B}$ is generic if $W$ is transverse to every intersection of hyperplanes of $B$. In this case the combinatorics and topology of $\overline{B}$ depend only on $B$ and $\text{dim}(W)$. A section of $B$ by a 3-dimensional subspace is called a planar section.

**Theorem 3.2 ([Yuz93])** Let $A$ be an essential arrangement. Then

(i) $A_F$ is formal.

(ii) $A$ is a section of $A_F$.

(iii) $A$ and $A_F$ have identical generic planar sections.

(iv) $A$ is formal if and only if $\text{rk}(A) = \text{rk}(A_F)$.

We remark that $A$ need not be a generic section of $A_F$, nor of any other arrangement. Indeed, if there are no nontrivial lines in $G$, then $A_F$ is the boolean arrangement. Then, if $A$ has some nontrivial plane, and has more than four elements, $A$ will not be a generic section of $A_F$. If $A$ is inerectible, it will not be a generic section of any arrangement. Such arrangements are easy to construct.

The interest in formal arrangements is due to the following theorem.

**Theorem 3.3** Let $A$ be an essential arrangement. Then

(i) If $A$ is a $K(\pi, 1)$ arrangement, then $A$ is formal.

(ii) If $A$ is a rational $K(\pi, 1)$ arrangement, then $A$ is formal.

(iii) If $A$ is a free arrangement, then $A$ is formal.

(iv) If $A$ has quadratic Orlik-Solomon algebra, then $A$ is formal.

Assertions (i) and (iv) above are easy consequences of Theorem 3.2, and (ii) is a consequence of (iv), because rational $K(\pi, 1)$ arrangements have quadratic Orlik-Solomon algebras. See [FR85]. Assertion (iii) was proved in [Yuz93].

In [Yuz93], Yuzvinsky presented examples of two arrangements, one formal and the other not, with the same underlying matroid. See [FR00] for diagrams of the dual point configurations. The underlying matroid is the dual of the matroid of complete bipartite graph $K_{3,3}$. This observation yields different realizations and a geometric explanation of this phenomenon.

**Example 3.4** The diagram on the left in Figure 3 on the following page is a representation of the rank-four matroid $G^*$ dual to the graphic matroid of $K_{3,3}$. By considering intersecting planes (in $\mathbb{C}^3$), one can see that the three dotted lines in this, or in any $\mathbb{C}$-representation of $G^*$, must be concurrent. The diagram on the right is a representation of the truncation $T(G^*)$ in which the corresponding lines are not concurrent. And indeed, one can show that the configuration on the right is formal. In fact, it cannot be lifted to a rank four configuration with the same points and lines.
The dual $G^*$ of $K_3,3$. Figure 3: Formality is not matroidal.

Combinatorial formality Example 3.4 shows that formality is not a combinatorial property. Since the discovery of these examples, efforts have been made to strengthen Theorem 3.3 by replacing the formality assumption with some purely combinatorial property. In this subsection we present several reasonable candidates.

Theorem 3.2(ii) suggests a natural combinatorial formulation of formality. We recall for the reader the notion of strong map (or quotient) of matroids. See [Whi86, Section 7.4 and Chaps. 8-9] for more details. Suppose $G'$ and $G$ are two matroids on ground set $[n]$. We say $G$ is a quotient of $G'$ if every closed set in $G$ is closed in $G'$. This is the case precisely when the identity map on $[n]$ is a strong map from $G'$ to $G$. If $G'$ is the matroid of an arrangement $\mathcal{A}$, then the matroid of any section of $\mathcal{A}$ is a quotient of $G$. The matroid of a generic $r$-dimensional section of $\mathcal{A}$ coincides with the truncation $G[r]$ of the matroid $G$ to rank $r$, the matroid whose dependent subsets are those of $G$ together with every subset of size greater than $r$.

Definition 3.5 A matroid $G$ is taut if $G$ is not a quotient of any matroid $G' \neq G$ satisfying $(G')^3 = G^3$.

The condition $(G')^3 = G^3$ says merely that $G$ and $G'$ have the same points and lines. The following assertion is a consequence of Theorem 3.2.

Corollary 3.6 If $G$ is a taut matroid, then every arrangement realizing $G$ is formal.

In a lecture in 1992 Yuzvinsky formulated a definition of “dimension” of a geometric lattice, or matroid, based on line-closure: the dimension of $G$ is the size of the smallest set of points whose line-closure is $[n]$. Corollary 3.8
below was presented in [Yuz], but was never published. The result was already known to matroid theorists [Cra70].

**Theorem 3.7** Suppose $G$ has a basis (of $\text{rk}(G)$ points) whose line-closure is $[n]$. Then $G$ is taut.

**proof:** Suppose $G$ is a quotient of a matroid $G'$ with the same points and lines as $G$. Since the closure of $B$ in $G'$ contains the line-closure of $B$ in $G'$, which agrees with the line-closure of $B$ in $G$, we conclude that $B$ is a basis for $G'$. Thus $\text{rk}(G') = \text{rk}(G)$. It follows that $G' = G$. $\square$

The following criterion is the standard method to prove an arrangement is formal, although it has never appeared in the literature.

**Corollary 3.8** Suppose $G$ has a basis whose line-closure is $[n]$. Then every realization of $G$ is formal.

We now have four notions which might capture the combinatorics of formality, at least in spirit:

(i) $G$ is line-closed,

(ii) $G$ has a basis whose line-closure is $[n],$

(iii) $G$ is taut, that is, $G$ is not a proper quotient preserving points and lines, and

(iv) every realization of $G$ is formal.

We have the string of implications

$$(i) \implies (ii) \implies (iii) \implies (iv).$$

The first of these is trivial, and the others were proved in the preceding paragraphs. We now give counter-examples for the first two of the reverse implications. We note that Example 3.4 provides a formal arrangement whose matroid fails to satisfy (iv).

**Example 3.9** In [Cra70] there appears an example of a matroid $G$ of rank three on nine points, with the property that no set of three points line-closes to the entire ground set $[9]$. See Figure 4 on the next page. Thus (ii) fails. But it is easy to see that $G$ is not a quotient of a rank-four matroid with the same points and lines, so (iii) is satisfied.

**Example 3.10** The rank-three wheel, illustrated in Figure 2.6 on page 6, provides an example of a matroid which satisfies (ii) but is not line-closed.
Finding a counter-example for the implication \((iv) \implies (iii)\) presents a delicate problem. One needs a \(\mathbb{C}\)-representable matroid which is a proper quotient, preserving points and lines, but such that the quotient “mapping” is not representable over \(\mathbb{C}\), either because the larger matroid is not \(\mathbb{C}\)-representable, or because the larger matroid is not realizable in such a way that the original matroid is obtained from it via projection.

**Local formality** In \([Yuz93]\) an arrangement is defined to be *locally formal* if, for every flat \(X\), the arrangement \(A_X := \{H_i \mid i \in X\}\) is formal. This idea can be applied to the combinatorial notions of formality discussed above. We will focus on the local version of tautness, for reasons that will become clear.

**Definition 3.11** A matroid \(G\) is **locally taut** if the restriction of \(G\) to any flat is taut.

By Corollary 3.6, any realization of a locally taut matroid is locally formal. Theorem 3.7 can be used to compare local tautness to line-closure, in the next pair of results.

**Corollary 3.12** Suppose \(G\) is a matroid in which every flat \(X\) has a basis whose line-closure is equal to \(X\). Then \(G\) is locally taut.

**proof:** Thus is an immediate consequence of Theorem 3.7. \(\square\)

By way of contrast, the definition of line-closed matroid may be restated as follows.

**Theorem 3.13** A matroid \(G\) is line-closed if and only if, for every flat \(X\), every basis of \(X\) has line-closure equal to \(X\).

Thus every line-closed matroid is even locally taut, strengthening Theorem 3.7. The rank-three wheel \(W_3\) (Examples 2.6 and 3.10) serves as an example of a locally taut matroid which is not line-closed.

We make one more observation concerning locally taut matroids.
Theorem 3.14 Suppose $G$ is locally taut. Then $G$ is determined by its points and lines.

proof: This is a consequence of part (5) of Theorem 7.5.4 in [Bry86], which asserts that $G$ is determined by its essential flats, along with their ranks. A flat of $G$ is essential if it is a truncation of a matroid of higher rank. Truncation is in particular a quotient map, and preserves points and lines, so long as the image has rank at least two. Thus, if $G$ is locally taut, the only essential flats of $G$ are the nontrivial lines. So $G$ is determined by the number of points and the list of nontrivial lines. $\square$

The converse of this result also holds. For if $G$ is not locally taut, then $G$ is not taut, so $G$ is a quotient of a distinct matroid with the same points and lines. A natural question is evident: what axioms govern the point-line incidence structures of locally taut matroids?

Topology vs. combinatorics One motivation of the present discussion is to find combinatorial conditions with the same consequences as formality, as in Theorem 3.3. Unfortunately, we have no positive results of this sort, aside from the implication “quadratic Orlik-Solomon algebra implies line-closed matroid” strengthening assertions (ii) and (iv) of that theorem.

First of all, it is not the case that $K(\pi,1)$ or free arrangements must have line-closed matroids.

Example 3.15 Consider the arrangement $A$ with defining equation $Q(x, y, z) = z(x + y)(x - y)(x + z)(y + z)(y - z)$, denoted $J_2$ in [FR86, section 2.6]. The underlying matroid is the non-Fano plane, pictured in Figure 5. Then $A$ is not line-closed: apply Corollary 2.19 to the three “edge-midpoints.” But this arrangement is well-known to be both free and $K(\pi,1)$ - see [FR86].

![Figure 5: Not line-closed, with a free, $K(\pi,1)$ realization](image_url)

Line-closure is not sufficient for $K(\pi,1)$-ness or freeness either.

Example 3.16 Let $A$ be the parallel arrangement with defining equation $Q(x, y, z) = z(x + z)(x - z)(y + z)(y - z)(x + y + 2z)(x + y - 2z)$, which appears as $X_2$ in
The underlying matroid $G$ of $\mathcal{A}$ is pictured in Figure 6 on the following page.

Since the hypothesis of Theorem 2.20 is satisfied, $G$ is line-closed. But $\mathcal{A}$ is not a free arrangement, nor is $\mathcal{A}$ a $K(\pi, 1)$ arrangement, nor a rational $K(\pi, 1)$ arrangement (though it has quadratic Orlik-Solomon algebra). The first of these assertions holds because the characteristic polynomial of $G$ has non-integer roots. The second assertion was proved by L. Paris (unpublished), and the last was proved in \cite{PY99}. See \cite{FR00}.

We close with a fascinating conjecture about the combinatorial structure of free or $K(\pi, 1)$ arrangements inspired by the preceding discussion. First we note the stronger version of Theorem 3.3: a free or $K(\pi, 1)$ arrangement must in fact be locally formal \cite{Fal93,Yuz93}.

Now, let us consider the hierarchy \((i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)\) among the combinatorial notions we have introduced. We have seen that the matroids of $K(\pi, 1)$ or free arrangements need not satisfy (i). The question whether such matroids satisfy (iv) is very close to two famous and important problems in the theory of arrangements. Indeed, a matroid which underlies both a free arrangement and also a non-formal arrangement would be a counter-example to Terao’s conjecture, that freeness is a matroidal property. A matroid which underlies both a $K(\pi, 1)$ arrangement and a non-formal arrangement would improve upon Rybnikov’s (non-$K(\pi, 1)$) counter-examples to the homotopy-type conjecture, that the homotopy type of the complement is matroidal. See \cite{FR00} for a discussion of the latter problem.

Finally, we note that the matroid appearing in Example 3.15, while not line-closed, is taut (i.e., satisfies (iii)), in fact locally taut, by Theorem 3.8, and is therefore determined by its points and lines.

**Conjecture 3.17** The underlying matroid of a free or $K(\pi, 1)$ arrangement is determined by its points and lines.

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