Bose-Einstein Condensation Temperature of Homogenous Weakly Interacting Bose Gas in Variational Perturbation Theory Through Seven Loops

Boris Kastening
Institut für Theoretische Physik
Freie Universität Berlin
Arnimallee 14
D-14195 Berlin
Germany
boris.kastening@physik.fu-berlin.de
(Dated: September 2003)

The shift of the Bose-Einstein condensation temperature for a homogenous weakly interacting Bose gas in leading order in the scattering length \( a \) is computed for given particle density \( n \). Variational perturbation theory is used to resum the corresponding perturbative series for \( \Delta \langle \phi^2 \rangle / N \u \) in a classical three-dimensional scalar field theory with coupling \( u \) and where the physical case of \( N = 2 \) field components is generalized to arbitrary \( N \). Our results for \( N = 1, 2, 4 \) are in agreement with recent Monte-Carlo simulations; for \( N = 2 \), we obtain \( \Delta T_c ^{\ast} / T_c = 1.27 \pm 0.11 \, a n^{1/3} \). We use seven-loop perturbative coefficients, extending earlier work by one loop order.

PACS numbers: 03.75.Hh, 05.30.Jp, 12.38.Cy

I. INTRODUCTION

The recent advent of experimental realizations of Bose-Einstein condensation (BEC) in dilute gases has been followed by an intense revival of theoretical investigation of the subject. The diluteness of the gas allows for an approximation where the interactions of the bosons are described by just one parameter, the s-wave scattering length \( a \) corresponding to the two-particle interaction potential. For many questions of interest, mean-field theory may be applied as a further approximation. An exception is the physics around the condensation temperature \( T_c \). Perturbation theory (PT) breaks down close to the transition due to infrared (IR) divergences. The phase transition is second-order and the interactions change the universality class from Gaussian to that of a three-dimensional O(2)-symmetric scalar field theory.

In this work, we are interested in the shift of the condensation temperature of a homogenous gas away from its ideal-gas value

\[
T_0 = \frac{2\pi}{m} \left[ \frac{n}{\zeta(3/2)} \right]^{2/3},
\]

where \( m \) is the mass of the bosons, \( n \) their number density and we work throughout in units where \( k_B = \hbar = 1 \). There have been many attempts to solve this problem [1–23], but the non-perturbative nature of the physics around the phase transition makes the problem highly non-trivial. More precisely, we ask what the value of the constant \( c_1 \) is in the diluteness expansion [7, 8, 15]

\[
\frac{\Delta T_c}{T_0} = c_1 n^{1/3} + [c_2' \ln(n^{1/3}) + c_2''(an^{1/3})^2 + \cdots \]

While \( c_2' \) can be evaluated exactly, \( c_2'' \) involves a non-perturbative calculation similar to the one needed for \( c_1 \). For results for both \( c_2' \) and \( c_2'' \), the reader is referred to Ref. [16].

Results for the constant \( c_1 \) range from \(-0.93 \) [9] to \( 4.7 \) [2] as has been summarized for instance in Refs. [14, 17] and the review [24], see also Fig. 3 below. In [7] it was realized that \( c_1 \) is generated exclusively by long-wavelength fluctuations and that therefore three-dimensional field theory is a convenient tool for the determination of \( c_1 \). Subsequently, schemes have been set up that are geared towards exclusively determining \( c_1 \) without having to disentangle the different contributions in (2). It appears that the most reliable results so far are obtained by Monte-Carlo (MC) simulations [13, 14], as we will explain in Sec. V.

The strategy employed here is to expand \( c_1 \) in a perturbative series, which is subsequently resummed to obtain the physical non-perturbative limit. While naive schemes such as Padé approximants may slowly converge towards the correct value, a scheme suited particularly well for critical phenomena is Kleinert’s field theoretic variational perturbation theory (VPT, see [25–27] and Chapters 5 and 19 of the textbooks [28] and [29], respectively). Improving PT by a variational principle goes back at least to [30]. Several other determinations of \( c_1 \) use variationally resummed PT employing the linear \( \delta \) expansion (LDE) [12, 18–21]. These were criticized in Refs. [22, 31]. In [22] VPT was used through five loops. In [23], we extended the calculation of [22] to six loops, but with a different treatment of the one-loop term, which is absent in [23]; The first non-zero coefficient arises only at three loops. In consequence, we obtained a result about 30% larger than that of Ref. [22]. We comment on the issue at the end of Sec. II below. Here we extend the work in [23] to seven loops, which reduces the error bar of the final result. We also provide many details and a comparison with MC simulation results not only for the case of \( N = 2 \) real field components, relevant for BEC, but also for \( N = 1 \) and \( N = 4 \), which were treated with MC methods in [32].

This work is organized as follows. In Sec. II we set up the field-theoretic framework and repeat arguments why,
for the determination of \( c_1 \), we may work entirely in three dimensions, ignoring the (imaginary) time direction. In Sec. III we discuss which perturbative series we need to resum to obtain \( c_1 \) and provide perturbative coefficients through seven loops. In Sec. IV the resummation procedure is described and carried out and the results are presented. Sec. V concludes with a discussion.

## II. FIELD-THEORETIC CONSIDERATIONS

The theoretical basis for the description of equilibrium quantities of a gas of identical spin-0 bosons in the grand canonical ensemble is a non-relativistic (3 + 1)-dimensional field theory in imaginary time \( \tau \), given by the Euclidean action

\[
S_{3+1} = \int_0^\beta d\tau d^3x \left[ \psi^* \left( \frac{\partial}{\partial \tau} - \frac{1}{2m} \nabla^2 - \mu \right) \psi + \frac{2\pi a}{m} (\psi^* \psi)^2 \right],
\]

where \( \mu \) is the chemical potential. This assumes that the momenta of the particles are small enough so that their interaction potential is well described by just one parameter, the \( s \)-wave scattering length \( a \). It also assumes that three- and more particle interactions are rare, i.e. that the gas is dilute. As we shall see below, the leading perturbative contribution to \( \Delta T_c \) is \( \propto a^2 \). A consequence is that the leading contribution to \( \Delta T_c \) arises exclusively from terms that are infrared divergent in the framework of PT. The leading IR divergent diagrams involve, however, only propagators with zero Matsubara frequencies. Put another way, a diagram with a propagator with a non-zero Matsubara frequency has the same degree of IR divergence as a diagram with no non-zero Matsubara frequencies but less powers of \( a \) and is therefore not contributing in leading order to \( \Delta T_c \). This is why, for the determination of \( c_1 \), we can set all Matsubara frequencies to zero or, equivalently, work with a three-dimensional field theory \[7\]. Denoting the zero-Matsubara modes by \( \phi_0 \), we may define the fields and parameters of this theory by \( \psi_0 = \sqrt{mT}(\phi_0 + i\phi_2) \), \( r_{\text{bare}} = -2m\mu \), \( u = 48\pi amT \) and obtain the three-dimensional Euclidean action

\[
S_3 = \int d^3x \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{r_{\text{bare}}}{2} \phi^2 + \frac{u}{24} (\phi^2)^2 \right].
\]

For the determination of quantities that are not governed by the leading IR divergences, one may still use a three-dimensional field theory which, however, is obtained by a more complicated matching procedure, which introduces corrections to the above relations of the parameters of the \((3+1)\)-dimensional and the three-dimensional theory and also necessitates the inclusion of more interaction terms. This matching is described in detail in \[16\].

It is convenient to generalize the model (4) to an \( O(N) \) field theory, where \( \phi = (\phi_1, \ldots, \phi_N) \), \( \phi^2 \equiv \phi_0 \phi_0 \). This allows to make contact with the exactly known large-\( N \) result for \( c_1 \) \[8\] and with the MC results for \( N = 1 \) and \( N = 4 \) \[32\].

From the form of the ideal-gas result (1) one obtains, in leading order, the relation [7]

\[
\frac{\Delta T_c}{T_0} = -\frac{2}{3} \frac{\Delta n}{n},
\]

where \( \Delta T_c \) is the shift of the condensation temperature for fixed \( n \) and \( \Delta n \) is the shift of the critical particle density for fixed condensation temperature. In our field-theoretic setup, \( \Delta n \) is given by

\[
\Delta n = \Delta(\psi^* \psi) = mT \Delta(\phi^2).
\]

Combining (1), (2), (5) and (6), \( c_1 \) is given by

\[
c_1 = \alpha \frac{\Delta(\phi^2)}{Nu} \Big|_{\text{crit.}},
\]

with \( \alpha = -256\pi^3/(\zeta(3/2))^{1/3} \approx -2206.19 \) and the remaining task is to compute the critical limit of

\[
\frac{\Delta(\phi^2)}{Nu} = \frac{1}{u} \int \frac{d^3p}{(2\pi)^3} (G(p) - G_0(p)),
\]

where \( G \) and \( G_0 \) are the interacting and the free propagator, respectively. The theory (4) is superrenormalizable, which means that only a finite number of primitive diagrams are divergent. Since the divergence in the free energy is of no interest for us here, the only divergences we have to cancel appear in the self-energy \( \Sigma(p) \). They are, however, \( p \)-independent and can be absorbed into \( r_{\text{bare}} \) by defining the renormalized quantity \( r = r_{\text{bare}} - \Sigma(0) \). This renormalization scheme has two major advantages: (i) It allows to work without any regulator to obtain renormalized results. (ii) The critical limit is easily identified as \( r \to 0 \), since the full propagator reads now \( G(p) = 1/[p^2 + r - \Sigma(p) - \Sigma(0)] \). For the free propagator follows \( G_0(p) = 1/(p^2 + r) \) \[20\].

Another popular scheme, used in \[12, 18, 19, 22, 31\], is to take \( G_0(p) = 1/p^2 \). This generates an unnatural non-zero one-loop contribution to \( \Delta(\phi^2) \), even in the absence of interactions. Although this contribution vanishes as \( r \to 0 \), it strongly influences resummation and is responsible for the small value of \( c_1 = 0.92 \pm 0.13 \) \[33\] and partial absence of real variational solutions in Ref. \[22\].

## III. PERTURBATIVE SERIES FOR \( c_1 \)

Dimensional analysis shows that PT organizes \( \Delta(\phi^2)/Nu \) in a power series of \( u/r^{1/2} \). It is convenient to define \( c_1 \) as in (7) but as a function of \( u/r^{1/2} \), without yet taking the critical limit. Specifically, we write

\[
c_1(u_r) = \alpha \sum_{l=1}^{\infty} a_l u_r^{l-2}, \quad u_r = \frac{Nu}{4\pi r^{1/2}}, \quad (9)
\]

As we will see in Sec. IV, we can compute \( c_1(u_r) \) through resummation all the way from the perturbative region around \( u_r = 0 \) to \( u_r \to \infty \), where it assumes the value which we need for the determination of \( \Delta T_c \). The specific choice of the variable \( u_r \) allows to go smoothly to
the $N \to \infty$ limit with finite perturbative coefficients $a_L$. Comparison with the large-$N$ limit is useful since $c_1$ has been determined exactly for this case in [8]. The successful application of VPT to the large-$N$ limit has been demonstrated in [23].

The perturbative series can be represented by Feynman diagrams and then becomes a loop expansion. We have constructed the relevant diagrams through seven loops using the recursive methods of [34] and collect them in Table I.

The subtraction of the zero-momentum part of the self-energy in the full propagator has to be performed recursively for all subdiagrams. Representing this recursive subtraction by an operator $\mathcal{R}$ [23], the perturbative coefficient at $L$ loops is given by

$$a_L u_r^{L-2} = \frac{1}{N u} \sum_{k=1}^{n_L} \mathcal{R} D_{L-k},$$

where $D_{L-k}$ is the $k$-th $L$-loop diagram and the first few $n_L$ are given in Table II. Here we count only diagrams surviving the subtractions of $\Sigma(0)$, i.e. we discard all diagrams containing a momentum-independent self-energy part. The first non-zero perturbative coefficient arises at the three-loop level, since the lowest momentum-dependent self-energy contribution has two loops. The coefficient is given by

$$a_3 u_r = \frac{1}{N u} \mathcal{R} \frac{1}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(k^2 + r)^2} \times \left[ \frac{1}{(k+p)^2 + r} - \frac{1}{p^2 + r} \right] \frac{1}{(p+q)^2 + r} (q^2 + r) \ln \frac{1}{576\pi^2} u_r.$$

As already announced in Sec. II, the recursive subtraction of zero-momentum self-energies in the integrands eliminates all ultraviolet (UV) divergences and we can work in three dimensions without regulator.

The numerical evaluation of Feynman diagrams is performed in momentum space, using analytic results for one-loop subdiagrams (the cases where one needs only one independent external momentum are essentially trivial; see [35] for the more complicated cases of one-loop triangle and box diagrams) and the subtracted sunset diagram (see Eq. (70) of [17] and Eq. (A6) of [20]) and carrying out the remaining integrations on the computer. The necessary diagrams are among a larger set of diagrams that can serve to obtain critical exponents in $O(N)$ field theories. The results for diagrams through six loops in the renormalization scheme employed here can be found in [36]. The seven-loop diagrams have been computed by B. Nickel in 1991 and used for the determination of critical exponents for $N = 0, 1, 2, 3$ in [37]. Nickel’s results for the diagrams through seven loops [38] are displayed in Table III together with their weights. The different group factors are collected in Table IV. The resulting perturbative coefficients $a_L$ for $N = 1, 2, 4$ are displayed in Table V.

**IV. RESUMMATION AND RESULTS**

Being interested in the $u_r \to \infty$ limit of $c_1(u_r)$, we need to resum our perturbative series (9). One of the simplest such resummations is the reexpansion in Padé approximants. Since we are computing a finite quantity, we only consider diagonal approximants $[n, n]$. The $[1, 1]$ and $[2, 2]$ approximants use perturbative coefficients through four and six loops, respectively. They are plotted in Fig. 1 together with their limiting values $c_1 = 0.754$ for $[1, 1]$ and $c_1 = 0.985$ for $[2, 2]$.

We can do better than that, though. First of all, diagonal Padé approximants allow only to work consistently through even loop orders. Second, their large-$u_r$ behavior is

$$c_1(u_r) = \alpha \sum_{m=0}^{\infty} f_m u_r^{-m \omega'},$$

with $\omega' = 1$. However, the interactions cause the phase transition to be second order with critical exponents of the $O(2)$ universality class. The leading class of corrections of a physical quantity that remains finite in the critical limit are integer powers of $t^{\omega'}$ [29, 40, 41], where $t \equiv (T - T_c) / T_c$, $\nu$ is the critical exponent of the correlation length and $\omega = \beta(g')$ in a renormalization group approach. Since, in our renormalization scheme, the propagator obeys $G(p = 0) = 1 / r \propto t^{-\gamma}$, the leading corrections are integer powers of $u_r^{-\omega'}$ with

$$\omega' = \frac{2\omega}{2 - \eta}.$$  

Here we have employed the universal scaling relation $\gamma = \nu(2 - \eta)$, where $\eta$ is the anomalous dimension of the critical propagator, i.e. $G(r = 0) \propto 1 / p^{2-\eta}$ in the small-$p$ limit. Thus an expansion which correctly describes the leading corrections to scaling [40] has the form (12) with $\omega'$ from (13).

The ansatz (12) does not account for so-called confluent singularities which cause the true large-$u_r$ expansion to also contain other negative powers of $u_r^\omega$, which are subleading at least compared to $u_r^{-\omega'}$. We can expect methods that can accommodate the leading behavior in (12) correctly to converge faster to the true result than methods having the wrong leading behavior as e.g. Padé approximants or the LDE, the latter being used extensively for the current problem [12, 18–21]. On the other hand, convergence will be slowed by the fact that we do not make an ansatz reflecting the full power structure in $u_r$, but the expansion (12) will try to mimic the neglected subleading powers.

The alternating signs of the $a_L$ displayed in Table V suggest that the perturbation series for $c_1(u_r)$ is Borel summable. In the context of critical phenomena, such
TABLE I: Diagrams through seven loops. The lines represent free propagators $\delta_{ab}/(p^2 + r)$. The vertices connecting four lines represent the interaction $-u(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})/3$. The vertex connecting only two lines is an insertion $\delta_{ab}$ connecting two free propagators. Indices run from 1 to $N$.

| $L$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|----|
| $n_L$ | 1 | 4 | 5 | 12 | 56 | 230 | 1262 | 7295 |

TABLE II: Numbers of diagrams in low loop orders.

series have been successfully resummed using Kleinert’s VPT. Accurate critical exponents [26, 27, 29] and amplitude ratios [42] have been obtained. For a truncated partial sum $\sum_{l=1}^L a_l u_r^{l-2}$ of (9), the method requires replacing

$$u_r^{l-2} \to (\hat{u}^{l-2}) 1 + t \left[ \left( \frac{\hat{u}}{u_r} \right)^\omega - 1 \right] \left[ \left( \frac{\hat{u}}{u_r} \right)^\omega - 1 \right]^{-l-2/\omega'}$$

(note that this is an identity for $t = 1$), reexpanding the resulting expression in $t$ through $t^{L-2}$, setting $t = 1$ and then optimizing in $\hat{u}$, where optimizing is done in accordance with the principle of minimal sensitivity [43] and in practice means finding appropriate stationary or turning points. That is, we replace

$$u_r^{l-2} \to \hat{u}^{l-2} \sum_{k=0}^{L-l} \left( \frac{-(l-2)/\omega'}{k} \right) \left[ \left( \frac{\hat{u}}{u_r} \right)^\omega - 1 \right]^k$$

and optimize the resulting expression in $\hat{u}$. For $u_r \to \infty$,
we obtain the $L$-loop approximation of $f_0$, \begin{equation}
\begin{aligned}
f_{0}^{(L)} &= \text{opt}_{\hat{u}} \left[ \sum_{l=1}^{L} a_l \hat{u}^{l-2} \sum_{k=0}^{L-l} \left( \frac{-(l-2)}{\omega} \right) (1)^k \right].
\end{aligned}
\end{equation}
Results are only available starting at four loops, since two non-zero perturbative coefficients are necessary for VPT to work. E.g., in four-loop order the optimization yields \begin{equation}
\begin{aligned}
f_{0}^{(4)} &= \text{opt}_{\hat{u}} \left[ a_3 \left( 1 + \frac{1}{\omega^2} \right) \hat{u} + a_4 \hat{u}^2 \right] = -\frac{a_3^2 \left( 1 + \frac{1}{\omega^2} \right)^2}{4a_4},
\end{aligned}
\end{equation}
as the best attempt at determining $f_0$ and thus $c_1 = \alpha f_0$.
For $N = 2$, we have $\omega = 0.79 \pm 0.01, \eta = 0.037 \pm 0.003$ (see, e.g., [29]) and therefore, using (13), $\omega' = 0.805 \pm 0.011$. We have determined $c_1(u_r)$ at loop orders $L = 4, 5, 6, 7$ for arbitrary $u_r$ and plotted the results together.
with their large-\(u_r\) limit in Fig. 1. Also plotted are the unresummed perturbative results. It is instructive to see how VPT converts them to the smooth function \(c_1(u_r)\) and how, after resummation, the different loop orders have apparent converging behavior for any \(u_r\). Table VI and Fig. 2 display the limiting values \(c_1\) for fixed \(\omega'\) for \(N = 1, 2, 4\). From [29] we take \(\omega = 0.79 \pm 0.01, \eta = 0.035 \pm 0.003\) and thus \(\omega' = 0.804 \pm 0.011\) for \(N = 1\) and \(\omega = 0.79 \pm 0.01, \eta = 0.0335 \pm 0.0030\) and thus \(\omega' = 0.803 \pm 0.011\) for \(N = 4\). The uncertainties of the \(\omega'\) contribute negligibly to the errors of the final results that we obtain in (18)-(20) below for \(c_1\).

There is also the possibility to determine \(\omega'\) self-consistently [26, 28, 29]. It relies on the fact that assuming a behavior of \(c_1(u_r)\) as in (12), the quantity \(d \ln c_1(u_r)/d \ln u_r\) has an expansion of the same type with the same \(\omega'\) as \(c_1(u_r)\), but with a vanishing large-\(u_r\) limit (i.e. its large-\(u_r\) expansion starts out with \(f_0 = 0\)). \(\omega'\) is tuned such that the value VPT gives for \(d \ln c_1(u_r)/d \ln u_r\) is zero in a given loop order. This \(\omega'\) is then used as an input for the determination of the approximation of \(c_1(u_r)\) at the same loop order. We have used this method to determine \(c_1\) at loop orders \(L = 5, 6, 7\) for \(N = 1, 2, 4\) with the results for \(c_1\) and \(\omega'\) given in Table VI and displayed in Fig. 2. The \(\omega'\) appear to approach their correct values at best very slowly. This may be interpreted as an indication of the presence of confluent singularities as described above. The self-consistently determined \(\omega'\) values then try to mimic the true mixture of powers. One may speculate that this causes the method of self-consistently determined \(\omega'\) to be ultimately superior than using the correct leading power \(\omega'\), whenever the power structure is more complicated than that in (12).

It appears that, in our treatment, for each \(N\) the correct value for \(c_1\) is approached from below by the \(c_1\) values for fixed \(\omega'\) and from above for self-consistently determined \(\omega'\). We therefore take the difference of the results as a conservative estimate for the full error bar and obtain at seven-loop order

\[
N = 1: \quad c_1 = 1.07 \pm 0.10, \quad (18)
\]
\[
N = 2: \quad c_1 = 1.27 \pm 0.11, \quad (19)
\]
\[
N = 4: \quad c_1 = 1.54 \pm 0.11. \quad (20)
\]
There is a plethora of results for the parameter $c_1$ describing the leading deviation of the BEC temperature due to a small repulsive interaction. A comparison of our results through seven loops with most other results found in the literature is given in Fig. 3. Let us have a closer look concerning their reliability.

The large-$N$ results [8, 10] move towards the region of (19), but their convergence behavior is unclear. The results obtained from a functional Keldysh formalism [2], the leading-order Ursell Operator method [5], the self-consistent method of [7], the canonical ensemble [9] and the renormalization group [11] come without any next order or error bar. The low MC value [4] has found explanations in [14, 15, 17]. The setup of the MC simulation of [6] has been criticized in [14]. The LDE and LDE* results of [12, 18–21] show no apparent converging behavior. Therefore, the closeness of the five-loop results of [19] to the three-dimensional MC data and our VPT result appears somewhat accidental. Use of the LDE in the context of field theoretic calculations with anomalous approach-to-scaling exponents has been criticized on general grounds in [31].

The VPT treatment of [22] has inspired [23] and its extension to seven loops discussed in the current work. However, as argued at the end of Sec. II, the choice of a massless free reference propagator in [22] even before tuning the full propagator to the critical limit causes a non-zero one-loop result that strongly influences resummation.

Our result (19) is in agreement with the apparently most reliable other sources available, the MC results from the three-dimensional theory, $c_1 = 1.29 \pm 0.05$ [13] determined by Kashurnikov, Prokof’ev and Svistunov and $c_1 = 1.32 \pm 0.02$ [14] determined by Arnold and Moore.

MC results for $N = 1$ and $N = 4$ are also available. X. Sun has computed $\Delta(\phi^2)_{\text{crit.}}/u$ for these cases [32] which translate to $c_1 = 1.09 \pm 0.09$ for $N = 1$ and $c_1 = 1.59 \pm 0.10$ for $N = 4$. Our values (18) and (20) agree well with these results, which is also illustrated in Fig. 2. It appears exceedingly unlikely that the agreement between the completely different approaches of MC and VPT for all three values of $N$ is accidental.

**Acknowledgments**

The author is extremely grateful to B. Nickel for providing the results of his seven-loop calculations, without which the extension of our work through seven loops would not have been possible. He thanks H. Kleinert for

| $L$ | $a_L$ for $N = 1$ | $a_L$ for $N = 2$ | $a_L$ for $N = 4$ |
|-----|-----------------|-----------------|-----------------|
| 3   | $-1.51814 \times 10^{-4}$ | $-1.01209 \times 10^{-4}$ | $-7.59070 \times 10^{-5}$ |
| 4   | $8.08989 \times 10^{-5}$ | $2.99626 \times 10^{-5}$ | $1.34832 \times 10^{-5}$ |
| 5   | $-5.88280 \times 10^{-5}$ | $-1.19872 \times 10^{-5}$ | $-3.17492 \times 10^{-6}$ |
| 6   | $5.25790 \times 10^{-5}$ | $5.85519 \times 10^{-6}$ | $9.01670 \times 10^{-7}$ |
| 7   | $-5.45889 \times 10^{-5}$ | $-3.30467 \times 10^{-6}$ | $-2.93269 \times 10^{-7}$ |

**TABLE V:** Perturbative coefficients $a_L$ through seven loops for $N = 1, 2, 4$. $a_1 = a_2 = 0$ for any $N$.

**TABLE VI:** $c_1$ with fixed $\omega^L$ and $c_1$ and $\omega^L$ for self-consistent $\omega^L$ for $N = 1, 2, 4$ through loop order $L = 7$.

![Figure 3: Comparison of $c_1$ from VPT as a function of the number of loops $L$ for $N = 2$ with most results from other sources. Upper dots: self-consistent (sc) $\omega^L$. Lower dots: $\omega^L = 0.805$. The label LDE* indicates use of a resummation method for accelerated convergence. The label VPT* indicates the inclusion of a non-zero one-loop term in Ref. [22], which is absent in the present treatment, labeled by VPT. Not shown are other approximate results of [17], the renormalization group results $c_1 = 4.7$ [2] and $c_1 = 3.42$ [11] and the canonical ensemble result $c_1 = -0.93$ [9].](image-url)
important suggestions, many discussions, and a careful reading of the manuscript. He is grateful to P. Arnold for many helpful communications and for suggesting the extension of our work to \( N = 1, 4 \) to allow for comparison with the results of [32]. He thanks E. Braaten for many valuable suggestions. Useful communications with J. Andersen, M. Holzmann, F. Laloë, D. Murray, M. Pinto and R. Ramos are also acknowledged.

[1] T.D. Lee and C.N. Yang, Phys. Rev. 112, 1419 (1957). K. Huang, in Studies in Statistical Mechanics, edited by J. de Boer and G.E. Uhlenbeck (North-Holland Publishers, Amsterdam, 1964), Vol. II, p. 1; T. Toyoda, Ann. Phys. (N.Y.) 141, 154 (1982).

[2] H.T.C. Stoof, Phys. Rev. A 45, 8398 (1992).

[3] M. Bijlsma and H.T.C. Stoof, Phys. Rev. A 54, 5085 (1996); K. Huang, Phys. Rev. Lett. 83, 3770 (1999) [cond-mat/9904027]; E.J. Mueller, G. Baym and M. Holzmann, J. Phys. B 34, 4561 (2001) [cond-mat/0105359].

[4] P. Grüter, D. Ceperley and F. Laloë, Phys. Rev. Lett. 79, 3549 (1997) [cond-mat/9707205].

[5] M. Holzmann, P. Grüter and F. Laloë, Eur. Phys. J. B 10, 739 (1999) [cond-mat/9809356].

[6] M. Holzmann and W. Krauth, Phys. Rev. Lett. 83, 2687 (1999) [cond-mat/9905198].

[7] G. Baym, J.-P. Blaizot, M. Holzmann, F. Laloë and D. Vautherin, Phys. Rev. A 63, 1703 (1999) [cond-mat/9905430].

[8] G. Baym, J.-P. Blaizot and J. Zinn-Justin, Europhys. Lett. 49, 150 (2000) [cond-mat/9907241].

[9] M. Wilkens, F. Illuminati, M. Kraemer, J. Phys. B 33, L779 (2000) [cond-mat/0001422].

[10] P. Arnold and B. Tomášik, Phys. Rev. A 62, 063604 (2000) [cond-mat/0005197].

[11] G. Alber, Phys. Rev. A 63, 023613 (2001) [cond-mat/0006130].

[12] F.F. de Souza Cruz, M.B. Pinto and R.O. Ramos, Phys. Rev. B 64, 014515 (2001) [cond-mat/0007151].

[13] V.A. Kashurnikov, N.V. Prokof’ev and B.V. Svistunov, Phys. Rev. Lett. 87, 120402 (2001) [cond-mat/0103149]; N.V. Prokof’ev and B.V. Svistunov, cond-mat/0103146v1.

[14] P. Arnold and G. Moore, Phys. Rev. Lett. 87, 120401 (2001) [cond-mat/0103228]; Phys. Rev. E 64, 066113 (2001) [cond-mat/0103227].

[15] M. Holzmann, G. Baym, J.-P. Blaizot and F. Laloë, Phys. Rev. Lett. 87, 120403 (2001) [cond-mat/0103595].

[16] P. Arnold, G. Moore and B. Tomášik, Phys. Rev. A 65, 013606 (2001) [cond-mat/0107124].

[17] G. Baym, J.-P. Blaizot, M. Holzmann, F. Laloë and D. Vautherin, Eur. Phys. J. B 24, 104 (2001) [cond-mat/0107129].

[18] F.F. de Souza Cruz, M.B. Pinto, R.O. Ramos and P. Sena, Phys. Rev. A 65, 053613 (2002) [cond-mat/0112306].

[19] J.-L. Kneur, M.B. Pinto and R.O. Ramos, Phys. Rev. Lett. 89, 210403 (2002) [cond-mat/0207089]; cond-mat/0207295v3, Phys. Rev. A, to appear.

[20] E. Braaten and E. Radescu, Phys. Rev. A 66, 063601 (2002) [cond-mat/0206186].

[21] E. Braaten and E. Radescu, Phys. Rev. Lett. 89, 271602 (2002).

[22] H. Kleinert, cond-mat/0210162v3.

[23] B. Kastening, cond-mat/0303486v3.

[24] J.O. Andersen, cond-mat/0305138v1.

[25] H. Kleinert, Phys. Lett. A 207, 133 (1995) [quant-ph/9507005].

[26] H. Kleinert, Phys. Rev. D 57, 2264 (1998); Phys. Rev. D 58, 107702 (1998) [cond-mat/9803268].

[27] H. Kleinert, Phys. Rev. D 60, 085001 (1999) [hep-th/9812179]; Phys. Rev. A 277, 205 (2000) [cond-mat/9906107].

[28] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics and Polymer Physics, 3rd ed. (World Scientific, Singapore, 2003).

[29] H. Kleinert and V. Schulte-Frohlinde, Critical Properties of \( \phi^4\)-Theories, 1st ed. (World Scientific, Singapore, 2001).

[30] V.I. Yukalov, Moscow Univ. Phys. Bull. 31, 10 (1976).

[31] B. Hamprecht and H. Kleinert, hep-th/0302116v1.

[32] X. Sun, Phys. Rev. E 67, 066702 (2003) [hep-lat/0209144].

[33] The final result \( c_1 = 1.14 \pm 0.11 \) of [22] was obtained by also applying VPT with self-consistent \( \omega' \) to scheme III of [20] (as first done in [23]) through five loops (albeit with the imprecise numerical values for the integrals given in [20]) and combining the result with \( c_1 = 0.92 \pm 0.13 \).

[34] H. Kleinert, A. Pelster, B. Kastening and M. Bachmann, Phys. Rev. E 62, 1537 (2000) [hep-th/9907168]; B. Kastening, Phys.Rev.E 61, 3501 (2000) [hep-th/9908172].

[35] B.G. Nickel, J. Math. Phys. 19, 542 (1978).

[36] M. Muthukumar and B. Nickel, J. Chem. Phys. 80, 5839 (1984).

[37] D.B. Murray and B.G. Nickel, Univ. of Guelph Report, 1991, unpublished.

[38] B.G. Nickel, private communication.

[39] B.G. Nickel, D.I. Meiron and G.A. Baker, Jr., Univ. of Guelph Report, 1977, unpublished.

[40] F. Wegner, Phys. Rev. B 5, 4529 (1972).

[41] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, 4th ed. (Clarendon, Oxford, 2002); A. Pelissetto and E. Vicari, Phys. Rept. 368, 549 (2002) [cond-mat/0112164].

[42] H. Kleinert and B. Van den Bossche, Phys. Rev. E 63, 056113 (2001) [cond-mat/0111329].

[43] P.M. Stevenson, Phys. Rev. D 23, 2916 (1981).