q-Calculus Revisited

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Abstract. In this study, a new representation is obtained for q-calculus, as proposed by Borges [Physica A 340 (2004) 95], and a new dual q-integral is suggested.

Keywords: q-calculus, q-exponential, q-logarithm

1. Introduction

With a generalization of the Boltzmann-Gibbs entropy [1], the q-logarithm and q-exponential functions were first proposed by Tsallis [2].

\[
\ln_q(x) \equiv \frac{x^{1-q} - 1}{1 - q} \quad (x > 0) \quad (1)
\]

\[
e_q(x) \equiv [1 + (1 - q)x]^{1/(1-q)} \quad (x, q \in \mathbb{R}) \quad (2)
\]

where \([A]_+ \equiv \max\{A, 0\}\)

A q-calculus associated with non-extensive statistical mechanics and thermodynamics was developed by Borges in 2004 [3]. He developed a primal q-derivative operator, \(D_q\), for which the q-exponential function is an eigenfunction, as the ordinary exponential function is the eigenfunction of the ordinary derivative operator.

\[
D_q(e_q(x)) = e_q(x) \quad (3)
\]

A primal q-integral operator, \(I_q\), which is the inverse operator of the primal q-derivative operator, was also developed. The primal q-integral of a q-exponential function is a q-exponential function.

\[
I_q(e_q(x)) = e_q(x) + c \quad (4)
\]

In general, the following relationships hold for the primal q-derivative and q-integral.
operator is proposed herein, which satisfies equation (3)

\[ D_{(q)} \left( I_{(q)}^{x}(f(t)) \right) = f(x) \]  (5)

\[ I_{(q)}^{x} \left( D_{(q)} \left( F(t) \right) \right) = F(x) + c \]

where \( I_{(q)}^{x}(f(t)) = F(x) - F(a), F(t) = I_{(q)} \left( f(t) \right) - c \)

For the \( q \)-logarithm function, which is the inverse function of the \( q \)-exponential function, he developed a dual \( q \)-derivative operator, \( D_{(q)} \). The dual \( q \)-derivative of the \( q \)-logarithm function is \( 1/x \), analogous to the fact that the ordinary derivative operator on the logarithm function gives \( 1/x \).

\[ D_{(q)} \left( \ln_{q}(x) \right) = \frac{1}{x} \]  (6)

Finally, he suggested a dual \( q \)-integral operator, \( I_{(q)}^{x} \), which is the inverse operator of the dual \( q \)-derivative. However, the dual \( q \)-integral of \( 1/x \) is found to be \( \ln(x) - \frac{1}{\ln(x)} + c \), and the following relationship does not hold:

\[ I_{(q)}^{x} \left( \frac{1}{x} \right) = \ln_{q}(x) + c \]  (7)

In general, the following relationship does not hold for the dual \( q \)-derivative and \( q \)-integral, and it is a significant weakness of the dual \( q \)-calculus suggested by Borges.

\[ D_{(q)} \left( I_{(q)}^{x}(f(t)) \right) = f(x) \]  (8)

\[ I_{(q)}^{x} \left( D_{(q)} \left( F(t) \right) \right) = F(x) + c \]

where \( I_{(q)}^{x}(f(t)) = F(x) - F(a), F(t) = I_{(q)} \left( f(t) \right) - c \)

To address this issue, a new representation of \( q \)-calculus with a new dual \( q \)-integral operator is proposed herein, which satisfies equation (8) with a modification of ordinary addition to \( q \)-difference (\( \oplus_{q} \)) or \( q \)-addition (\( \ominus_{q} \)) in equations (7) and (8):

\[ D_{[q]} \left( F(x) \right) \equiv \lim_{y \to x} \frac{F(x) - F(y)}{E_{q}(x) - E_{q}(y)} \]  (9)

\[ I_{[q]}^{x} \left( f(t) \right) \equiv \int_{t=x_{0}}^{x} f(t) \, du(t) \]  (10)

where \( u(t) = \ln \left[ \frac{1 + (1 - q)t}{1 - qt} \right] \)

\[ D_{[q]}^{x} F(x) \equiv \lim_{y \to x} \frac{\ln(c_{q}(F(x))) - \ln(c_{q}(F(y)))}{x - y} \]  (11)

\[ I_{[q]}^{x} \left( f(t) \right) \equiv \ln_{q} \left[ \exp \left( \int_{x_{0}}^{x} f(t) \, dt \right) \right] \]  (12)

The new representation of \( q \)-calculus is based on the concept of primal and dual \( q \)-tangent lines, analogous to ordinary tangent lines. The primal and dual \( q \)-derivatives are defined as the slope of the primal and dual \( q \)-tangent lines at each point on the curve \( y = f(x) \); this is analogous to the fact that the ordinary derivative of a function
is the slope of the tangent line at each point on the curve $y = f(x)$. The primal and dual $q$-integrals are defined as the signed primal and dual $q$-area between the curve $y = f(x)$ and the horizontal axis, as the ordinary integral of a function is the signed area between the curve $y = f(x)$ and the horizontal axis.

The remainder of this paper is organized as follows. Section 2 provides some background on $q$-algebra, $q$-calculus of Borges, and ordinary calculus. Section 3 involves the derivation of a new representation of $q$-calculus and the new dual $q$-integral operator. Section 4 presents the relationship between the primal and dual $q$-derivatives and integrals.

2. Review of $q$-Algebra, $q$-Calculus and Ordinary Calculus

2.1. $q$-Algebra

$q$-algebra was first proposed by Borges [3].

\[
x \oplus_q y = x + y + (1-q)xy
\]

\[
x \odot_q y = \frac{x-y}{1+(1-q)y}
\]

\[
x \otimes_q y = \begin{cases} [x^{1-q} + y^{1-q} - 1]_+^{1/(1-q)} & (x, y > 0) \\ [x^{1-q} - y^{1-q} + 1]_+^{1/(1-q)} & (x, y > 0) \end{cases}
\]

\[
x \oslash_q^n x = \underbrace{x \odot_q x \odot_q x \cdots \odot_q x}_{n \text{ times}} = [nx^{1-q} - (n-1)]_+^{1/(1-q)}
\]

\[
n \odot_q x = \underbrace{x \odot_q x \odot_q x \cdots \odot_q x}_{n \text{ times}} = \frac{1}{1-q} \left\{ [1 + (1-q)x]^n - 1 \right\}
\]

The properties of the $q$-logarithm and the $q$-exponential can be expressed as follows.

\[
\ln_q(xy) = \ln_q(x) \oplus_q \ln_q(y), \quad e_q(x) e_q(y) = e_q(x \oplus_q y)
\]

\[
\ln_q(x \odot_q y) = \ln_q(x) + \ln_q(y), \quad e_q(x) \odot_q e_q(y) = e_q(x + y)
\]

\[
\ln_q(x/y) = \ln_q(x) \ominus_q \ln_q(y), \quad e_q(x)/e_q(y) = e_q(x \ominus_q y)
\]

\[
\ln_q(x \otimes_q y) = \ln_q(x) - \ln_q(y), \quad e_q(x) \otimes_q e_q(y) = e_q(x - y)
\]

2.2. $q$-Calculus

Borges [3] defined primal and dual $q$-derivatives and $q$-integrals as follows.

\[
D_{(q)} \left( f(x) \right) = \lim_{y \to x} \frac{f(x) - f(y)}{x \odot_q y} = [1 + (1-q)x] \frac{df(x)}{dx}
\]

\[
I_{(q)} \left( f(x) \right) = \int \frac{f(x)}{1 + (1-q)x} \, dx
\]

\[
D^{(q)} \left( f(x) \right) = \lim_{y \to x} \frac{f(x) \odot_q f(y)}{x - y} = \frac{1}{1 + (1-q)f(x)} \frac{df(x)}{dx}
\]
\[ I^q(f(x)) = \int [1 + (1 - q)f(x)]f(x)dx \quad (26) \]

2.3. Ordinary Calculus

2.3.1. Derivative

Let \( f(x) \) be the ordinary derivative function of \( F(x) \).

\[
\frac{d}{dx}F(x) \equiv \lim_{t \to x} \frac{F(x) - F(t)}{x - t} = f(x) \quad (27)
\]

A derivative operation is a function that takes a function \( F(x) \) as an argument and produces another function \( f(x) \) as an output.

In ordinary calculus, \( f(x_0) \), the value of \( f(x) \) evaluated at \( x_0 \), represents the slope of the tangent line \( T(C, P) : y = T(x; F(x), x_0) = k_{(C,P)}x + c_{(C,P)} \) at the point \( P = (x, y) = (x_0, F(x_0)) \) on the curve \( C : y = F(x) \).

Let the line passing through two points \( P_i = (x_i, F(x_i)) \) and \( P_j = (x_j, F(x_j)) \) on the curve \( C \) be \( L(C, P_i, P_j) : y = L(x; F(x), x_i, x_j) \).

\[
L(x; F(x), x_i, x_j) = \frac{F(x_i) - F(x_j)}{x_i - x_j} (x - x_i) + F(x_i) \quad (28)
\]

The line \( L(C, P, Q) \) passing through \( P = (x_0, F(x_0)) \) and another point \( Q = (t, F(t)) \) on the curve \( C \) tends to \( T(C, P) \), as \( Q \) approaches \( P \), and the slope of \( L(C, P, Q) \) converges to that of \( T(C, P) \).

\[
f(x_0) = \lim_{t \to x_0} \frac{L(x_0; F(x), x_0, t) - L(t; F(x), x_0, t)}{x_0 - t} = k_{(C,P)} \quad (29)
\]
Note that
\[
\frac{d}{dx} H(x) = \frac{d}{dx} F(x) \iff H(x) = F(x) + c \tag{30}
\]

If \( C \) is a line, that is, \( C : F(x) = kx + c \), the curve (line) \( C \) itself is regarded as the tangent line at each point on it, and the slope of the tangent line is constant, \( k \), for all points on \( C \).

2.3.2. Integral  A definite integral operation is a function that takes a function \( f(x) \) and a pair of values \((x_L, x_H)\) as arguments and outputs the difference between the values of primitive function, \( F(x) \), evaluated at \( x_H \) and \( x_L \).

\[
\int_{x_L}^{x_H} f(x)dx \equiv F(x_H) - F(x_L). \tag{31}
\]

Let the signed area between the curve \( C' : y = f(x) \) and the horizontal axis \((y = 0)\) from \( x_L \) to \( x_H \) be \( A(f(x), x_L, x_H) \). The value of \( A(f(x), x_L, x_H) \) is equal to the difference between the two values of the primitive function \( F(x) \) evaluated at \( x_H \) and \( x_L \),

\[
\int_{x_L}^{x_H} f(x)dx \equiv F(x_H) - F(x_L) = A(f(x), x_L, x_H). \tag{32}
\]

The relationship is clear when the primitive function is \( F(x) = kx + c \), and its derivative is a constant; that is, \( \frac{d}{dx} F(x) = k \) (see Figure 2).

\[
A(k, x_L, x_H) = k(x_H - x_L)
\]

\[
y = f(x) = k
\]

\[
y = F(x) = kx + c
\]

**Figure 2.** Definite integral and the signed area when \( \frac{d}{dx} F(x) = k \)
In general, \( A(f(x), x_L, x_H) \) can be evaluated as follows:

Let \( T = \{x_0, x_1, \ldots, x_{n-1}, x_n\} \) be the \( n \)-partitions of \([x_L, x_H]\), \( \Delta t_i = x_i - x_{i-1} \), and the length of the longest sub-interval is the norm of the partition, \(|T|\). The signed area \( A_i = A(f(x), x_{i-1}, x_i) \) between the curve \( C' : y = f(x) \) and the horizontal axis \((y = 0)\) over the \( i \)-th partition \([x_{i-1}, x_i]\) can be approximated by the signed area of a rectangle with height \( f(x_i) \) and base \( \Delta t_i \); that is, \( A_i \approx f(x_i)\Delta t_i \). Note that \( A_i = f(x_i)\Delta t_i \) if \( f(x) \) is constant over \([x_L, x_H]\).

As the norm of the partition approaches zero, the sum of areas of rectangles converges to the signed area between the curve \( C \) and the horizontal axis \((y = 0)\) over \([x_L, x_H]\), and the definite integral is defined as the limit.

\[
\int_{x_L}^{x_H} f(x) \, dx = A(f(x), x_L, x_H) = \lim_{|T| \to 0} \sum_{i=1}^{n} f(x_i)\Delta t_i \quad (33)
\]

In contrast, an indefinite integral is a function that takes a function \( f(x) \) as an argument and produces a family of functions \( F = \{F(x) + c, c \in \mathbb{R}\} \). The codomain of an indefinite integral operation is a set of families of functions, and the image of an indefinite integral is a translation family of functions along the \( y \) axis.

\[
\int f(x) \, dx = F(x) + c = \mathcal{F} \quad (34)
\]

We can recover the exact function \( F(x) \) only if we know a point \( P = (x_0, F(x_0)) \) on the curve \( C \) with \( f(x) \).

\[
F(x) = \int_{x_0}^{x} f(t) \, dt + F(x_0) = A(f(x), x, x_0) + F(x_0) \quad (35)
\]
3. New Representation of $q$-Calculus and New Dual $q$-integral

3.1. Primal

3.1.1. Primal $q$-derivative  Let a family of curves $L_q(k_q, \cdot) = \{y = L_q(x; k_q, c)\}$ have a constant primal $q$-derivative (defined by equation (23)), $k_q$, at every point in their domain.

$L_q(x; k_q, c)$ must satisfy the condition $D_{(q)} \left(L_q(x; k_q, c)\right) = k_q$.

$$D_{(q)} \left(L_q(x; k_q, c)\right) = \{1 + (1 - q)x\} \frac{d}{dx} L_q(x; k_q, c) = k_q$$  \hspace{1cm} (36)

Therefore,

$$L_q(x; k_q, c) = \int \frac{k_q}{1 + (1 - q)x} dx$$

$$= \frac{k_q}{1 - q} \cdot \ln(|1 + (1 - q)x|) + c$$

$$= k_q \cdot \ln(|1 + (1 - q)x|^{1/\alpha}) + c = k_q \cdot \ln[L_q(x)] + c$$  \hspace{1cm} (37)

where

$$L_q(x) = |1 + (1 - q)x|^{1/\alpha}$$  \hspace{1cm} (38)

Note that $E_q(x) = e_q(x)$, where $1 + (1 - q)x > 0$.

Let us call the curve $L_q(k_q, c) : y = L_q(x; k_q, c) = k_q \cdot \ln(E_q(x)) + c$ a primal $q$-line with primal $q$-slope $k_q$ and $y$-intercept $c$.

Note that $L_q(0, c) : y = L_q(x; 0, c)$ is a horizontal line, $y = c$.

The line $L_q(x; k_q, c)$ has an interesting property:

$$\frac{L_q(x_i; k_q, c) - L_q(x_j; k_q, c)}{\ln(E_q(x_i)) - \ln(E_q(x_j))} = k_q, \forall x_i, x_j \in \mathbb{R}, x_i \neq x_j, x_i, x_j \neq \frac{1}{q - 1}$$  \hspace{1cm} (39)

From this finding, we define a new primal $q$-derivative operator $D_{[q]}$,

$$D_{[q]} \left(F(x)\right) = \lim_{y \to x} \frac{F(x) - F(y)}{\ln[E_q(x)] - \ln[E_q(y)]}$$  \hspace{1cm} (40)

Note that

$$D_{[q]} \left(H(x)\right) = D_{[q]} \left(F(x)\right) \iff H(x) = F(x) + c.$$  \hspace{1cm} (41)

$e_q(x)$ is the eigenfunction of $D_{[q]}$.

$$D_{[q]} \left(e_q(x)\right) = \lim_{y \to x} \frac{e_q(x) - e_q(y)}{\ln[E_q(x)] - \ln[E_q(y)]} = e_q(x), \text{ if } 1 + (1 - q)x > 0$$  \hspace{1cm} (42)

In general, $D_{[q]}$ is equivalent to $D_{(q)}$; that is, $D_{[q]} \left(F(x)\right) = D_{(q)} \left(F(x)\right)$, in general, because

$$\lim_{y \to x} \frac{\ln[E_q(x)] - \ln[E_q(y)]}{x \ominus_q y} = 1$$  \hspace{1cm} (43)
Let \( L_q(C, P, P_j) : y = L_q(x; F(x), x_i, x_j) \) be a primal \( q \)-line that passes through the two points \( P_i = (x_i, y_i) = (x_i, F(x_i)) \) and \( P_j = (x_j, y_j) = (x_i, F(x_j)) \) on the curve \( C : y = F(x) \).

\[
L_q(x; F(x), x_i, x_j) = k_q \cdot \ln(E_q(x_i)) + c
\]

\[
k_q = \frac{F(x_i) - F(x_j)}{\ln(E_q(x_i)) - \ln(E_q(x_j))}
\]

\[
c = y_i - k_q \cdot \ln(E_q(x_i))
\]

\( L_q(x; F(x), x_i, x_j) \) can also be expressed as follows:

\[
L_q(x; F(x), x_i, x_j) = k_q \cdot \ln(E_q(x \ominus_q x_i)) + F(x_i)
\]

\[
L_q(x; F(x), x_i, x_j) = k_q \cdot \ln(E_q(x \ominus_q x_j)) + F(x_j)
\]

**Figure 4.** Primal \( q \)-derivative as the slope of the primal \( q \)-tangent line

We call \( T_q(C, P) : y = T_q(x; F(x), x_0) = k_{q(C,P)} \cdot \ln(E_q(x)) + c_{q(C,P)} \) as the primal \( q \)-tangent line of the curve \( C : y = F(x) \) at the point \( P = (x_0, F(x_0)) \) when \( F(x_0) = T_q(x_0) \) and \( D_{[q]} \big(F(x)\big)\big|_{x=x_0} = k_{q(C,P)}. \)

\[
D_{[q]} \big(F(x)\big)\big|_{x=x_0} \text{ is the primal } q \text{-slope of the primal } q \text{-tangent line at point } P \text{ on the curve } C.
\]

The primal \( q \)-line \( L_q(C, P, Q) \) passing through \( P = (x_0, F(x_0)) \) and another point \( Q = (t, F(t)) \) on the curve \( C \) tends to \( T_q(C, P) \) as \( Q \) approaches \( P \), and the primal \( q \)-slope of \( L_q(C, P, Q) \) converges to that of \( T_q(C, P) \).
3.1.2. **Primal q-integral** If \( f(x) = D_q[F(x)] \), the definite primal q-integral of \( f(x) \) from \( x_L \) to \( x_H \), denoted as \( I_{[q]}^{x_H}_{x_L} f(x) \), is equal to \( F(x_H) - F(x_L) \).

\[
I_{[q]}^{x_H}_{x_L} f(x) = F(x_H) - F(x_L) \tag{51}
\]

Equation (39) gives us a hint on how a primal q-integral can be related to a (deformed) signed area.

Consider the most immediate example, the case of primal q-lines. Let \( F(x) = k_q \cdot \ln(E_q(x)) + c \) and \( f(x) = D_q[F(x)] = k_q \). Let us consider the transformation \((x, y) = (x, f(x)) \rightarrow (u, v) = (\ln(E_q(x)), f(x))\).

Figure 5 shows the graph of \( y = F(x) \) at the top and that of \( u \) and \( v \) at the bottom. The bottom left graph shows the case \( x_{L_1}, x_{H_1} < -\frac{1}{1-q} \), and the bottom right graph shows the case \( x_{L_2}, x_{H_2} > -\frac{1}{1-q} \). \( x_{L_2} \) and \( x_{H_2} \) are set as \( x_{L_2} = \frac{-2}{1-q} - x_{H_1} \) and \( x_{H_2} = \frac{-2}{1-q} - x_{L_1} \).
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\[ y_{L_1} = y_{H_2} \text{ and } y_{H_1} = y_{L_2} \text{ because} \]
\[ E_q(x) = |1 + (1 - q)x|^{\frac{1}{1-q}} \]
\[ = | -1 - (1 - q)x|^{\frac{1}{1-q}} = \left| 1 + (1 - q) \left( -\frac{2}{1-q} - x \right) \right|^{\frac{1}{1-q}} \]
\[ = E_q(-\frac{2}{1-q} - x) \]

Let \( A_q(k_q, u_L, u_H) \) be the signed rectangular area formed by \( v = k_q \) and \( v = 0 \) over the range \([u_L, u_H] \); this is called the primal \( q \)-area.

\[ A_q(k_q, u_L, u_H) = k_q \cdot (u_H - u_L) \] (53)

With equation (53), we find that the signed area of each shaded rectangle in the \( u - v \) graph is equal to the corresponding definite integral.

\[ A_q(k_q, u_{L_1}, u_{H_1}) = k_q \cdot (u_{H_1} - u_{L_1}) \]
\[ = k_q \cdot \{ \ln(E_q(x_{H_1})) - \ln(E_q(x_{L_1})) \} \]
\[ = L_q(x_{H_1}; k_q, c) - L_q(x_{L_1}; k_q, c) \]
\[ = \int_{[q]x_{L_1}}^{x_{H_1}} L_q(x; k_q, c) \] (54)

\[ A_q(k_q, u_{L_2}, u_{H_2}) = k_q \cdot (u_{H_2} - u_{L_2}) \]
\[ = k_q \cdot \{ \ln(E_q(x_{H_2})) - \ln(E_q(x_{L_2})) \} \]
\[ = L_q(x_{H_2}; k_q, c) - L_q(x_{L_2}; k_q, c) \]
\[ = \int_{[q]x_{L_2}}^{x_{H_2}} L_q(x; k_q, c) \] (55)

\[ y_{H_1} - y_{L_1} = -(y_{H_2} - y_{L_2}) \text{, because } y_{L_1} = y_{H_2} \text{ and } y_{H_1} = y_{L_2}. \]

From this, we find that

\[ \int_{[q]x_{L}}^{x_{H}} L_q(x; k_q, c) = -\int_{[q]x_{H}}^{x_{L}} E_q(-\frac{2}{1-q} - x) \] (56)

From this, we get

\[ \int_{[q]x_{L}}^{x_{H}} L_q(x; k_q, c) = \int_{[q]x_{L}}^{x_{H}} L_q(x; k_q, c) + \int_{[q]x_{H}}^{x_{L}} E_q(-\frac{2}{1-q} - x) \] (57)

\[ = \int_{[q]x_{L}}^{x_{H}} L_q(x; k_q, c) - \int_{[q]x_{L}}^{x_{H}} E_q(-\frac{2}{1-q} - x) \]
\[ = \int_{[q]x_{L}}^{x_{H}} L_q(x; k_q, c), \]
\[ \text{ when } x_{L} < -\frac{1}{1-q} < x_{H} \]

Based on the above findings, we define the definite primal \( q \)-integral of \( f(x) \) from \( x_L \) to \( x_H \), \( \int_{[q]x_{L}}^{x_{H}} f(x) \), as follows:

\[ \int_{[q]x_{L}}^{x_{H}} f(x) = F(x_{H}) - F(x_{L}) \equiv A_q(f(x), u_L, u_H) \] (58)
We derive the primal $q$-integral of the $q$-exponential function, $I_{[q]}^{x_H} (e_q(x))$, with $A_q(k_q, u_L, u_H)$. We can use $A_q(k_q, u_L, u_H) = k_q \cdot \{ \ln(e_q(x_H)) - \ln(e_q(x_L)) \}$ instead of $A_q(k_q, u_L, u_H) = k_q \cdot \{ \ln(E_q(x_H)) - \ln(E_q(x_L)) \}$ because $k_q = e_q(x) = 0$ when $1 + (1-q)x \leq 0$.

Split $[x_L, x_H]$ into $n$-partitions, $T_q = \{ x_L = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = x_H \}$, where $x_i = x_L \oplus_q (i \odot_q t)$,

$$t = \frac{1}{1 - q} \left[ \frac{1}{n} \cdot (x_H \ominus_q x_L) \right]^{\frac{1}{q-1}} - 1$$

![Figure 6. Definite primal $q$-integral of a $q$-exponential function](image)

$T_q$ transforms into $U_q = \{ \ln[e_q(x_L)] = u_0, u_1, u_2, \ldots, u_{n-1}, u_n = \ln[e_q(x_H)] \}$, where $u_i = \ln[e_q(x_i)]$, with a transformation $(x, y) = (x, f(x)) = (x, e_q(x)) \rightarrow (u, v) = (\ln[e_q(x)], f(x)) = (\ln[e_q(x)], e_q(x))$.

$$u_i = \ln[e_q(x_i)] = \ln[e_q(x_L)] + i \cdot \ln \{ e_q(t) \}$$

$$u_i - u_{i-1} = \ln \{ e_q(t) \}$$

$$e_q(t) = 1 + (1 - q) \cdot (x_H \ominus_q x_L) \left[ \frac{1}{1 - q} \right]^{\frac{1}{q-1}}$$

$$= e_q(x_H \ominus_q x_L) \left[ \frac{1}{1 - q} \right]^{\frac{1}{q-1}}$$
\[
\ln \left( e_q(t) \right) = \frac{1}{n(1-q)} \ln \left[ 1 + (1-q) \cdot (x_H \oplus_q x_L) \right] \\
= \frac{1}{n} \ln \left[ e_q(x_H \oplus_q x_L) \right] 
\]

Equation (63) shows that the norm of the partition \( U_q, \| U_q \| \), approaches zero when the number of partitions approaches infinity.

Let \( A_{qi} \) be the primal \( q \)-area formed by \( v = e_q(x_i) \) and \( v = 0 \) over the \( i \)-th partition \([u_{i-1}, u_i]\).

\[
e_q(x_i) = e_q(x_L \oplus_q (i \oplus_q t)) = e_q(x_L) \cdot e_q(i \oplus_q t) = e_q(x_L) \cdot \{ e_q(t) \}^i 
\]

\[
A_{qi} = A_q(e_q(x_i), u_{i-1}, u_i) = e_q(x_i) \cdot (u_i - u_{i-1}) \\
= \left[ e_q(x_L) \cdot \{ e_q(t) \}^i \right] \ln [e_q(t)] \\
= e_q(x_L) \cdot \{ 1 + (1-q) \cdot (x_H \oplus_q x_L) \}^{\frac{1}{(1-q)}} \cdot \frac{1}{n(1-q)} \ln \left[ 1 + (1-q) \cdot (x_H \oplus_q x_L) \right] \\
= e_q(x_L) \cdot \frac{\ln(z)}{1-q} \cdot \frac{1}{n} \left( \frac{z^{\frac{1}{(1-q)}}}{z^{\frac{1}{(1-q)}} - 1} \right) \\
\text{where } z = 1 + (1-q) \cdot (x_H \oplus_q x_L) 
\]

\[A_q(k_q, u_L, u_H)\) can be evaluated as the sum of \( A_{qi} \) when the norm of the partition \( U_q, \| U_q \| \), approaches zero,

\[
\sum_{i=1}^{n} A_{qi} = \sum_{i=1}^{n} \left\{ e_q(x_L) \cdot \frac{\ln(z)}{1-q} \cdot \frac{1}{n} \left( \frac{z^{\frac{1}{(1-q)}}}{z^{\frac{1}{(1-q)}} - 1} \right) \right\} \\
= e_q(x_L) \cdot \frac{\ln(z)}{1-q} \cdot \frac{1}{n} \left( \frac{z^{\frac{1}{(1-q)}}}{z^{\frac{1}{(1-q)}} - 1} \right) \\
= e_q(x_L) \cdot \frac{\ln(z)}{1-q} \cdot \left( \frac{z^{\frac{1}{(1-q)}} - 1}{z^{\frac{1}{(1-q)}} - 1} \right) \\
\]

\[
A_q(k_q, u_L, u_H) = \lim_{\| U_q \| \to 0} \sum_{i=1}^{n} A_{qi} \\
= e_q(x_L) \cdot \frac{\ln(z)}{1-q} \cdot \left( \frac{z^{\frac{1}{(1-q)}} - 1}{z^{\frac{1}{(1-q)}} - 1} \right) \\
= e_q(x_L) \cdot \frac{\ln(z)}{1-q} \cdot \left( \frac{z^{\frac{1}{(1-q)}} - 1}{\ln(z)} \right) \\
= e_q(x_L) \cdot \left( \frac{z^{\frac{1}{(1-q)}} - 1}{\ln(z)} \right) \\
= e_q(x_L) \cdot \{ e_q(x_H \oplus_q x_L) - 1 \} \\
= e_q(x_L) \cdot \{ e_q(x_H) - e_q(x_L) \} \\
\]

Therefore,

\[
\int_{[q]x_L}^{x_H} e_q(x) = F(x_H) - F(x_L) = e_q(x_H) - e_q(x_L) 
\]
If \( F(x) = e_q(x) \), we know that \( D_{[q]} \left( F(x) \right) = e_q(x) \) and \( F(0) = 1 \), for all \( q \).

\[
F(x) = 1 + \int_{0}^{x} \left( e_q(t) \right) = 1 + \left\{ e_q(x) - e_q(0) \right\} = e_q(x) \tag{69}
\]

Indefinite primal \( q \)-integral of the \( q \)-exponential function can be expressed as follows:

\[
F(x) = I_{[q]} \left( e_q(x) \right) = \int_{0}^{x} \left( e_q(t) \right) = e_q(x) - e_q(x_0) = e_q(x) + c \tag{70}
\]

In general, the signed area between the trajectory formed by \((u, v)\) over the range \([x_L, x_H]\) and \( v = 0 \) can be evaluated using the Riemann–Stieltjes integral, \( \int_{x=x_L}^{x=x_H} f(x) \, du(x) \) \[1\], where \( u(x) = \ln(E_q(x)) \).

Therefore, when \( f(x) = D_{[q]} \left( F(x) \right) \),

\[
I_{[q]}_{x=x_L}^{x=x_H} \left( f(x) \right) = F(x_H) - F(x_L) \tag{71}
\]

\[
= \int_{x=x_L}^{x=x_H} f(x) \, du(x)
= \int_{x=x_L}^{x=x_H} \frac{f(x)}{1 + (1-q)x} \, dx
\]

Equation \(71\) shows that the new primal \( q \)-integral, \( I_{[q]} \), is equivalent to the primal \( q \)-integral \( I_{(q)} \) in equation \(24\).

Equation \(7\) holds for \( D_{[q]} \) and \( I \) because they are equivalent to \( D_{(q)} \) and \( I_{(q)} \), respectively.

When \( f(x) = D_{[q]} \left( F(x) \right) \) and \( F(x_0) = -c \), the new primal indefinite \( q \)-integral can be expressed as follows:

\[
I_{[q]}_{x=x_0}^{x} \left( f(t) \right) = F(x) + c \tag{72}
\]
3.2. Dual

3.2.1. Dual $q$-derivative  Let a family of curves $L^q_{(k^q,)} = \{y = L^q(x; k^q, c)\}$ have a constant dual $q$-derivative (as defined by equation (25)), $k^q$, at every point on their domain.

$L^q(x; k^q, c)$ must satisfy the condition $D^{(q)} L^q(x; k^q, c) = k^q$,

$$D^{(q)} L^q(x; k^q, c) = \frac{1}{1 - q} \frac{d}{dx} L^q(x; k^q, c) = k^q$$ (73)

Therefore,

$$L^q(x; k^q, c) = c' \cdot \exp((1 - q)k^q x) - \frac{1}{1 - q}$$

$$= \frac{c}{1 - q} \exp((1 - q)k^q x) - \frac{1}{1 - q}$$

$$= \left\{ \frac{1}{1 - q} \exp((1 - q)k^q x) + \frac{1}{1 - q} \right\} + \frac{c - 1}{1 - q}$$

$$+ (1 - q) \frac{c - 1}{1 - q} \left\{ \frac{1}{1 - q} \exp((1 - q)k^q x) + \frac{c - 1}{1 - q} \right\}$$

$$= \left\{ \frac{1}{1 - q} \exp((1 - q)k^q x) + \frac{1}{1 - q} \right\} \oplus_q \frac{c - 1}{1 - q}$$

$$= \ln_q (\exp(k^q x)) \oplus_q \frac{c - 1}{1 - q}$$ (74)

Note that $L^q_{(k^q,)}$ is a $\oplus_q$ translation family of a curve along the $y$ axis.

Consider the curve $L^q(k^q, c) : y = L^q(x; k^q, c) = \ln_q (\exp(k^q x)) \oplus_q \frac{c - 1}{1 - q}$ as the dual $q$-line with dual $q$-slope $k^q$ and $y$-intercept $\frac{c - 1}{1 - q}$.

$L^q(x; k^q, c)$ also has the property,

$$\frac{\ln(e_q(L^q(x_i; k^q, c))) - \ln(e_q(L^q(x_j; k^q, c)))}{x_i - x_j} = k^q, \forall x_i, x_j \in \mathbb{R}, x_i \neq x_j.$$ (75)

From this finding, we define a new dual $q$-derivative operator $D[q]$, as

$$D[q] \left( F(x) \right) \equiv \lim_{y \rightarrow x} \ln(e_q(F(x))) - \ln(e_q(F(y))) \over x - y$$ (66)

Note that

$$D[q] \left( H(x) \right) = D[q] \left( F(x) \right) \iff H(x) = F(x) \oplus_q c$$ (77)

It follows that $D[q] \ln_q(x) = \frac{1}{x}$.

$$D[q] \ln_q(x) = \lim_{y \rightarrow x} \ln(e_q(\ln_q(x))) - \ln(e_q(\ln_q(y))) \over x - y$$

$$= \lim_{y \rightarrow x} \ln(x) - \ln(y) \over x - y = \frac{1}{x}$$ (78)

$D[q]$ turns out to be equivalent to $D^{(q)}$; that is, $D[q]F(x) = D^{(q)}F(x)$, in general, because

$$\lim_{y \rightarrow x} \ln(e_q(F(x))) - \ln(e_q(F(y))) \over F(x) \oplus_q F(y) = 1$$ (79)
\[ D_q(F(x)) = \lim_{y \to x} \frac{\ln(e_q(F(x))) - \ln(e_q(F(y)))}{x - y} \]
\[ = \lim_{y \to x} \frac{\ln(e_q(F(x))) - \ln(e_q(F(y)))}{F(x) \ominus_q F(y)} \cdot \frac{F(x) \ominus_q F(y)}{x - y} \]
\[ = \lim_{y \to x} \frac{F(x) \ominus_q F(y)}{x - y} = D_q(F(x)) \quad (80) \]

Let \( L^q(C, P_i, P_j) : y = L^q(x; F(x), x_i, x_j) \) be the dual \( q \)-line that passes through the two points \( P_i = (x_i, y_i) = (x_i, F(x_i)) \) and \( P_j = (x_j, y_j) = (x_j, F(x_j)) \) on the curve \( C : y = F(x) \).

\[ L^q(x; F(x), x_i, x_j) = \ln_q(\exp(k^q x)) \ominus_q \left( \frac{c - 1}{1 - q} \right) \quad (81) \]

\[ k^q = \frac{\ln(e_q(F(x_i))) - \ln(e_q(F(x_j)))}{x_i - x_j} \quad (82) \]

\[ \frac{c - 1}{1 - q} = y_i \ominus_q \ln_q(\exp(k^q x_i)) \quad (83) \]

\( L^q(x; F(x), x_i, x_j) \) can also be expressed as follows:

\[ L^q(x; F(x), x_i, x_j) = \ln_q(\exp(k^q(x - x_i)) \ominus_q F(x_i)) \quad (84) \]

\[ L^q(x; F(x), x_i, x_j) = \ln_q(\exp(k^q(x - x_j)) \ominus_q F(x_j)) \quad (85) \]

We call \( T^q(C, P) : y = T^q(x; F(x), x_0) = \ln_q(\exp(k^q(x, C, p)x)) \ominus_q \left( \frac{c^q_{C, p} - 1}{1 - q} \right) \) as the dual \( q \)-tangent line of the curve \( C : y = F(x) \) at the point \( P = (x_0, F(x_0)) \) when \( F(x_0) = T^q(x_0) \) and \( D_q(F(x)) \big|_{x=x_0} = k^q_{C, P} \).
\[ D[q] \left( F(x) \right) \bigg|_{x=x_0} \text{ is the dual } q\text{-slope of the dual } q\text{-tangent line at the point } P \text{ on the curve } C. \]

The dual \( q \)-line \( L^q(x; C, P, Q) \) passing through \( P = (x_0, F(x_0)) \) and another point \( Q = (t, F(t)) \) on the curve \( C \) tends to \( T^q(C, P) \) as \( Q \) approaches \( P \), and the dual \( q \)-slope of \( L^q(x; C, P, Q) \) converges to that of \( T^q(C, P) \).

\[
D[q] \left( F(x) \right) \bigg|_{x=x_0} = \lim_{t \to x_0} \frac{\ln(c_q(L^q(x_0; F(x), x_0, t))) - \ln(c_q(L^q(t; F(x), x_0, t)))}{x_0 - t} = k^q_{(C, P)} \tag{86}
\]

### 3.2.2. New dual \( q \)-integral

Let the indefinite dual \( q \)-integral operator, \([q]_I\), be the inverse operator of the dual \( q \)-derivative operator, \( D[q] \). Because the \( \oplus_q \)-translation family of a function has the same \( q \)-derivative (see equation \( \tag{77} \)), the indefinite dual \( q \)-integral operator should produce a \( \oplus_q \)-translation family of a function.

\[
f(x) = D[q] \left( F(x) \right) \implies [q]_I \left( f(x) \right) = F(x) \oplus_q c. \tag{87}
\]

Let us denote the definite dual \( q \)-integral of \( f(x) \) from \( x_L \) to \( x_h \) as \([q]^{x_h \rightarrow x_L}_I \left( f(x) \right)\).

It is desirable and natural that a definite dual \( q \)-integral vanishes when the integrating range is zero.

\[
[q]^c_I \left( f(x) \right) = 0, \quad \forall c \in \mathbb{R}. \tag{88}
\]

From equation \( \tag{88} \), the function \( F(x) \), with \( D[q] \left( F(x) \right) = f(x) \) and \( F(x_0) = 0 \), can be represented as follows:

\[
F(x) = [q]^{x}_{x_0} \left( f(t) \right). \tag{89}
\]

If \( H(x) = F(x) \oplus_q c \), \( D[q] \left( H(x) \right) = f(x) \) and \( H(x_0) = F(x_0) \oplus_q c = c \). \( H(x) \) can be represented as

\[
H(x) = F(x) \oplus_q c = [q]^{x}_{x_0} \left( f(t) \right) \oplus_q c. \tag{90}
\]

Therefore,

\[
[q]^{x}_{x_0} \left( f(t) \right) = H(x) \oplus_q H(x_0) = \left( F(x) \oplus_q c \right) \oplus_q \left( F(x_0) \oplus_q c \right) = F(x) \oplus_q F(x_0) \tag{91}
\]

Therefore,

\[
[q]^{x_H \rightarrow x_L}_I \left( f(x) \right) = F(x_H) \oplus_q F(x_L) = \ln_q (c_q(F(x_H))) \tag{92}
\]
From equation (92), we find that the following relationship holds for the definite dual $q$-integral.

$$\int_{x_L}^{x_H} [q] f(x) = \int_{x_L}^{c} [q] f(x) \oplus_q \int_{c}^{x_H} [q] f(x), \quad \forall c \in \mathbb{R} \quad (93)$$

Equation (75) gives us a hint on how a definite dual $q$-integral can be related to a signed area.

Consider the immediate example, the case of dual $q$-lines. Let $F(x) = L^q(x; k^q, c) = \ln_q(\exp(k^q x)) \oplus_q \frac{1}{1-q} 1 - q^{x_0}$ and $f(x) = D[eq](F(x)) = k^q$.

$$\int_{x_L}^{x_H} [q] k^q = L^q(x_H; k^q, c) \oplus_q L^q(x_L; k^q, c) \quad (94)$$

Let us consider the transformation $(x, Y) = (x, F(x)) = (x, L^q(x; k^q, c)) \rightarrow (x, w) = (x, \ln(eq(Y))) = (x, \ln(eq(L^q(x; k^q, c))))$.

Figure 8. Relationship between the dual $q$-integral and signed dual $q$-area.

Figure 8 shows the graph of $Y = F(x)$ at the top, the graph of $x$ and $w$ in the middle, and the graph of $y = f(x)$ at the bottom.

With equation (75), we find that the signed area of the shaded rectangle at the bottom graph is equal to $w_H - w_L$ in the middle graph.

Let $A^q(k^q, x_L, x_H)$ be the ordinary signed rectangular area formed by $y = k^q$ and $y = 0$ over the range $[x_L, x_H]$. 
\[ A(q^q, x_L, x_H) = k^q \cdot (x_H - x_L) \] (95)

\[ w_H - w_L = \ln [e_q(L^q(x_H; k^q, c))] - \ln [e_q(L^q(x_L; k^q, c))] \]

\[ = \ln \left( \frac{e_q(L^q(x_H; k^q, c))}{e_q(L^q(x_L; k^q, c))} \right) \]

\[ = \ln \left( [e_q(L^q(x_H; k^q, c))] [e_q(L^q(x_L; k^q, c))] \right) \]

\[ = \ln \left( e_q(q^q, x_L) \right) = \ln_q(\exp [A(q^q, x_L, x_H)]) = \ln_q(\exp [k^q \cdot (x_H - x_L)]) \]

Let \( A^q(k^q, x_L, x_H) \equiv \ln_q(\exp [A(k^q, x_L, x_H)]) = \ln_q(\exp [k^q \cdot (x_H - x_L)]) \) be a dual \( q \)-area,

\[ \int_{x_L}^{x_H} (k^q) = \ln_q(\exp [A(k^q, x_L, x_H)]) \] (96)

\[ = A^q(k^q, x_L, x_H) \]

For \( f(x) = D[q](F(x)) \), in general, the ordinary signed area, \( A(f(x), x_L, x_H) \), formed by \( y = f(x) \) and \( y = 0 \) over the range \([x_L, x_H]\) is equal to the ordinary definite integral \( \int_{x_L}^{x_H} f(x)dx \). Therefore, the following relationship holds true:

\[ \int_{x_L}^{x_H} (f(x)) = A^q(f(x), x_L, x_H) \] (97)

\[ = \ln_q(\exp [A(f(x), x_L, x_H)]) \]

\[ = \ln_q \left( \exp \left[ \int_{x_L}^{x_H} f(x)dx \right] \right) \]

Using equation (97), \( \int_{x_0}^{x} (\frac{1}{t}) \) is found to be a \( q \)-logarithmic function,

\[ \int_{x_0}^{x} (\frac{1}{t}) = \ln_q (\exp \left[ \int_{x_0}^{x} \frac{1}{t} dt \right]) \] (98)

\[ = \ln_q (\exp [\ln(x) - \ln(x_0)]) \]

\[ = \ln_q (\frac{x}{x_0}) = \ln_q(x) \odot_q \ln_q(x_0) \]

If \( F(x) = \ln_q(x) \), we know that \( D[q](\ln_q(x)) = \frac{1}{x} \) and \( F(1) = 0 \) for all \( q \),

\[ F(x) = \int_{x_0}^{x} (\frac{1}{t}) = \ln_q(x) \odot_q \ln_q(1) = \ln_q(x) \]

(99)

The indefinite dual \( q \)-integral of \( \frac{1}{t} \) can be expressed as follows, and equation (17) holds with a modification of ordinary addition to \( q \)-difference (\( \odot_q \)).

\[ F(x) = \int_{x_0}^{x} (\frac{1}{t}) = \ln_q(x) \odot_q \ln_q(x_0) = \ln_q(x) \odot_q x. \]

(100)
Equation (8) also holds with a modification of ordinary addition to $q$-difference $(\ominus_q)$.

\begin{align}
D_{[q]} \left( \frac{[q]^x}{a} \left( f(x) \right) \right) &= D_{[q]} \left( F(x) \ominus_q F(a) \right) = f(x) \\
\frac{[q]^x}{a} \left( D_{[q]} \left( F(x) \right) \right) &= \frac{[q]^y}{b} \left( f(x) \right) = F(x) \ominus_q c
\end{align}

4. Relationship Between the Primal and Dual

4.1. Relationship Between Primal and Dual $q$-derivatives

Let $G(y)$ be the inverse function of $F(x)$, that is, $y = F(x)$ and $x = G(y)$, and $P = (x_0, y_0)$, $Q = (x_1, y_1)$ be points on the curve $C : y = F(x)$. The curve $C$ can also be represented by $C : x = G(y)$.

Let $L_q(C, P, Q)$ be a primal $q$-line that passes through $P$ and $Q$.

\begin{equation}
L_q(C, P, Q) : y = k_q \cdot \ln(\exp(\frac{1}{k_q}(y - y_1))) \oplus_q x_1
\end{equation}

Equations (103) and (104) show that the primal $q$-line that passes through $P$ and $Q$ is the dual $q$-line passing through $P$ and $Q$, $L^q(C, P, Q)$, and $k^q$; the dual $q$-slope of $L^q(C, P, Q)$ is $\frac{1}{k_q}$.

Let $T_q(C, P)$ and $T^q(C, P)$ be the primal and dual $q$-tangent lines at $P$ on $C$, respectively, and let $k_q^\ast$ and $k^q\ast$ be the primal and dual $q$-slopes of the corresponding $q$-tangent line. As $Q$ approaches $P$, $k_q$ converges to $k_q^\ast$, and $k^q$ converges to $k^q\ast$. Therefore, the primal $q$-derivative and the dual $q$-derivative are inversely related.

\begin{equation}
k^q\ast = \frac{1}{k_q}.
\end{equation}

4.2. Relationship Between Primal and Dual $q$-integrals

Let $f(x) = D_{[q]} \left( F(x) \right)$ and $g(x) = D_{[q]} \left( G(x) \right)$.

\begin{align}
I_{[q]} x_1 \left( f(x) \right) &= y_1 - y_0 \\
I_{[q]} \left( \frac{[g]^y}{y_0} \right) \left( g(y) \right) &= x_1 \ominus_q x_0
\end{align}

\begin{align}
I_{[q]} x_1 \left( f(x) \right) &= y_1 - y_0 \\
I_{[q]} \left( \frac{[g]^y}{y_0} \right) \left( g(y) \right) &= x_1 \ominus_q x_0 = k_q = \frac{1}{k^q}
\end{align}

where $k_q = \frac{y_1 - y_0}{\ln(\exp(\frac{1}{k_q}(y_1))) - \ln(\exp(\frac{1}{k_q}(y_0)))}$.

Therefore, the definite primal and dual $q$-integrals are proportionally related to the $q$-slope of the $q$-line passing through two points $P$ and $Q$. 
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Reference

[1] Tsallis C 1988 Possible generalization of Boltzmann-Gibbs statistics Journal of Statistical Physics, 52, 479-487
[2] Tsallis C 1994 What are the numbers that experiments provide? Quimica Nova, 17, 468-471.
[3] Borges E P 2004 A possible deformed algebra and calculus inspired in nonextensive thermostatistics, Physica A, 340, 95-101.
[4] Lang S 1993 Real and Functional Analysis 3rd edn (New York: Springer)