Abstract

For a diagram automorphism of an affine Kac-Moody algebra such that the folded diagram is still an affine Dynkin diagram, we show that the associated Drinfeld-Sokolov hierarchy also admits an induced automorphism. Then we show how to obtain the Drinfeld-Sokolov hierarchy associated to the affine Kac-Moody algebra that corresponds to the folded Dynkin diagram from the invariant sub-hierarchy of the original Drinfeld-Sokolov hierarchy.

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1 Introduction

Starting from an affine Kac-Moody algebra with a marked vertex of its Dynkin diagram, Drinfeld and Sokolov constructed a hierarchy of integrable Hamiltonian evolutionary PDEs of Korteweg de Vries (KdV) type in [3]. It is called the Drinfeld-Sokolov hierarchy associated to the affine Kac-Moody algebra, which plays important roles in different research fields of mathematical physics. For instance, if the Drinfeld-Sokolov hierarchy is associated to an untwisted simply laced affine Kac-Moody algebra \( g \) with the zeroth vertex marked, then the tau function of its solution selected by the string equation is proved to be the total descendant potential of the FJRW theory of ADE singularities [6, 12, 16]. For the untwisted affine Kac-Moody algebra of BCFG types, Ruan and the first- and the third-named authors of the present paper showed in [13] that the tau function of a particular solution of the associated Drinfeld-Sokolov hierarchy coincides with the total descendant potential of the so-called \( \Gamma \)-sector of the FJRW theory for the corresponding ambient ADE singularities.

In the above mentioned work of [13], the following \( \Gamma \)-reduction theorem plays an important role. Let \( g \) be an affine Kac-Moody algebra of type \( A^{(1)}_{2n-1}, D^{(1)}_{n+1}, D^{(1)}_{4}, G^{(1)}_{2}, F^{(1)}_{4} \) given in Table 1, then the automorphism \( \tilde{\sigma} \) of its Dynkin diagram induces a diagram automorphism \( \sigma \) on \( g \), and the \( \sigma \)-invariant subalgebra \( g^\sigma \) is just an affine Kac–Moody algebra of type \( C^{(1)}_{n}, B^{(1)}_{n}, G^{(1)}_{2}, F^{(1)}_{4} \) respectively [10]. The automorphism \( \sigma \) generates a finite group \( \Gamma \), which yields an action on the Drinfeld-Sokolov hierarchy associated to \( g \).

**Theorem 1.1 (Theorem 1.4 of [13])** The \( \Gamma \)-invariant flows of an ADE Drinfeld-Sokolov hierarchy define the corresponding \( B_n, C_n, F_4, G_2 \) Drinfeld-Sokolov hierarchy. Furthermore, the restriction of the ADE tau function to the \( \Gamma \)-invariant subspace of the big phase space provides a tau function of the corresponding \( B_n, C_n, F_4, G_2 \) Drinfeld-Sokolov hierarchy.

A twisted affine Kac-Moody algebra can be realized as the invariant subalgebra of a twisted automorphism of an untwisted simply laced affine Kac-Moody algebra. So it is natural to ask: is there a \( \Gamma \)-reduction theorem for the Drinfeld-Sokolov hierarchy associated to the twisted affine Kac-Moody algebra? Unfortunately, since the twisted automorphism does not yield an action on the associated Drinfeld-Sokolov hierarchy, there is no such a theorem along this line. Instead, to obtain an analogue of the \( \Gamma \)-reduction theorem for the cases of twisted affine Kac-Moody algebras, we have to study the (untwisted) diagram automorphisms on affine Kac-Moody algebras and their invariant subalgebras.

The Dynkin diagrams of affine type on which there exist automorphisms \( \tilde{\sigma} \) such that the corresponding folded diagrams are still Dynkin diagrams of affine type are listed in Tables 1–3 with the vertices of the Dynkin diagrams labeled as in [10] (the blank node always stands for the zeroth vertex). We observe, in particular, that the Dynkin diagrams of all twisted affine Kac-Moody algebras are contained in Tables 2 and 3. Hence, to answer the above question, we only need to clarify the relationship between the Drinfeld–Sokolov hierarchies of types \( X^{(r)}_{\tilde{\ell}} \) and \( X^{(r)}_{\bar{\ell}} \) listed in Tables 2 and 3.

Let us consider an affine Kac-Moody algebra \( g \) of type \( X^{(r)}_{\bar{\ell}} \) given in Tables 1–3 on which there is a diagram automorphism \( \sigma \) induced by \( \tilde{\sigma} \) [10]. Denote by \( g^\sigma \) the subalgebra of \( g \) that consists of all \( \sigma \)-invariant elements. Note that the subalgebra \( g^\sigma \) may be not the affine Kac-Moody algebra of type \( X^{(r)}_{\tilde{\ell}} \) associated to the folded Dynkin diagram (as we already pointed out, \( g^\sigma \) is indeed such a subalgebra for each case listed in Table 1). Instead, \( g^\sigma \) may be the direct sum of such a
Table 1: Dynkin diagrams with automorphism: \( X_\ell^{(1)} \sim X_\ell^{(1)} \)

\[\begin{array}{c}
\text{(a1)} & \begin{array}{c}
A_{2n-1}^{(1)} \\
\begin{array}{cccccccc}
2n-1 & 2n-2 & n+2 & n+1 & \cdots & n-2 & n-1 & n \\
0 & 1 & 2 & \sigma & \vdots & \sigma & \vdots & \sigma
\end{array}
\end{array}
\end{array} & \begin{array}{c}
C_n^{(1)} \\
\begin{array}{cccccccc}
0 & 1 & 2 & \cdots & n-2 & n-1 & n \\
\sigma & \sigma & \sigma & \vdots & \sigma & \vdots & \sigma
\end{array}
\end{array} \\
\text{(d1)} & \begin{array}{c}
P_{n+1}^{(1)} \\
\begin{array}{cccccccc}
0 & 2 & 3 & \cdots & n-1 & n+1 & n \\
1 & 2 & 3 & \sigma & \vdots & \vdots & \sigma
\end{array}
\end{array} & \begin{array}{c}
B_n^{(1)} \\
\begin{array}{cccccccc}
0 & 2 & 3 & \cdots & n-1 & n+1 & n \\
1 & 2 & 3 & \sigma & \vdots & \vdots & \sigma
\end{array}
\end{array} \\
\text{(d2)} & \begin{array}{c}
D_4^{(1)} \\
\begin{array}{cccc}
0 & 2 & 3 & 4 \\
1 & 2 & 3 & \sigma
\end{array}
\end{array} & \begin{array}{c}
G_2^{(1)} \\
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\sigma & 0 & 1 & \sigma
\end{array}
\end{array} \\
\text{(e1)} & \begin{array}{c}
E_6^{(1)} \\
\begin{array}{cccccccc}
0 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & \sigma
\end{array}
\end{array} & \begin{array}{c}
F_4^{(1)} \\
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & \sigma
\end{array}
\end{array}
\end{array}\]

subalgebra and a certain linear subspace. So what we need is to figure out the difference between these subalgebras. For this purpose, let us decompose \( g \) according to its principal gradation as \( g = \bigoplus_{k \in \mathbb{Z}} g^k \), then its \( \sigma \)-invariant subalgebra is also decomposed as \( g^\sigma = \bigoplus_{k \in \mathbb{Z}} (g^\sigma)^k \). Let \( \bar{g} \) be the subalgebra of \( g \) generated by \( (g^\sigma)^{-1}, (g^\sigma)^0, (g^\sigma)^1 \). Then we have \( \bar{g} \subset g^\sigma \) and the following theorem.

**Theorem 1.2** For any case listed in Tables 1 and 2, the following assertions hold true:

(i) \( \bar{g} \) is an affine Kac-Moody algebra associated to the folded Dynkin diagram \( X_\ell^{(r)} \);

(ii) \( g^\sigma = \bar{g} \) for any case listed in Tables 1 and 2;

(iii) \( g^\sigma = \bar{g} \oplus \tilde{H} \) for any case listed in Table 3, where \( \tilde{H} \) is a nontrivial subspace of the principal Heisenberg subalgebra \( H \) of \( g \) (see Theorem 4.1 below).
Table 2: Dynkin diagrams with automorphism: $X_p^{(r)} \rightsquigarrow X_p^{(r)}$

(a2) $A_{2n}^{(1)}$ $\rightsquigarrow$ $A_{2n}^{(2)}$

(b) $B_{n+1}^{(1)}$ $\rightsquigarrow$ $D_{n+1}^{(2)}$

(c1) $C_{2n}^{(1)}$ $\rightsquigarrow$ $C_{n}^{(1)}$

(d3) $D_{2n+1}^{(1)}$ $\rightsquigarrow$ $B_{n}^{(1)}$

(d6) $D_{(2n+1)+1}^{(2)}$ $\rightsquigarrow$ $D_{n+1}^{(2)}$
Table 3: Dynkin diagrams with automorphism: $X^{(r)}_\ell \xrightarrow{\sigma} X^{(r)}_{\ell'}$

(a4) $A^{(2)}_{2(n+1)} \xrightarrow{\sigma} A^{(2)}_{2n}$

(c2) $C^{(1)}_{2n+1} \xrightarrow{\sigma} A^{(2)}_{2n}$

(d4) $D^{(1)}_{2n} \xrightarrow{\sigma} A^{(2)}_{2n-1}$

(d5) $D^{(1)}_4 \xrightarrow{\sigma} A^{(2)}_2$

(d7) $D^{(2)}_{2n+1} \xrightarrow{\sigma} A^{(2)}_{2n}$

(e2) $E^{(1)}_6 \xrightarrow{\sigma} D^{(3)}_4$

(e3) $E^{(1)}_7 \xrightarrow{\sigma} E^{(2)}_6$
Although this theorem looks very basic in the theory of Lie algebra, we could not find it in the literature (except those cases in Table 1), so we will give a proof of it in the present paper.

Based on Theorem 1.2, we will study the relationship between the Drinfeld-Sokolov hierarchies associated to \( g \) and to \( \bar{g} \). In the original construction of the Drinfeld-Sokolov hierarchy (3), two gradations of \( g \) play important roles: one is the principal gradation \( \Gamma = (1, \ldots, 1) \), and the other one \( s_m \) is given by the marked vertex \( m \):

\[
s_m = (s_0, \ldots, s_t), \quad \text{with} \quad s_i = \delta_{im}.
\]

Observe that the gradation \( s_m \) is not preserved by the automorphism \( \sigma \) for all cases in Tables 1, 3. So in order to obtain an analogue of the \( \Gamma \)-reduction theorem for these cases, we will start from a certain generalization of the Drinfeld-Sokolov hierarchies proposed by de Groot et al. in [9] (cf. [7]).

To construct such a generalized version of the Drinfeld-Sokolov hierarchy we need to fix, instead of certain generalization of the Drinfeld-Sokolov hie- ra- chies proposed by de Groot et al. in [9] (cf. [7]). So in order to obtain an analogue of the \( \Gamma \)-reduction theorem for these cases, we will start from a certain generalization of the Drinfeld-Sokolov hierarchies proposed by de Groot et al. in [9] (cf. [7]).

Then associated to the affine Kac-Moody algebra \( \bar{g} \), we also have the Drinfeld-Sokolov hierarchy with the flows \( \partial/\partial t_j \), \( j \in J^\sigma \), that are associated to the gradations \( \bar{s} \) and \( \bar{\Theta} \) of \( \bar{g} \). The flows \( \partial/\partial t_j \) and \( \partial/\partial t_j \) are defined respectively on the jet space \( \bar{R} \) and \( \bar{R} \) of certain finite dimensional subspaces \( V \subset g \) and \( \bar{V} = V \cap \bar{g} \) that will be defined in (3.20) and (4.21).

The main result of the present paper is given by the following theorem.

**Theorem 1.3** Let \( g = X^{(r)}_\rho \) be an affine Kac-Moody algebra whose Dynkin diagram is given in Tables 1, 3, on which there is a diagram automorphism \( \sigma \) induced by \( \bar{\sigma} \). Fix a gradation \( s \leq \Theta \) on \( g \) being consistent with \( \sigma \), and let \( \bar{s} \) be the gradation it induces on \( \bar{g} \). Denote by \( \Gamma \) the finite group generated by \( \sigma \), then this group acts on the Drinfeld–Sokolov hierarchy associated to \( (g, s, \Theta) \). Furthermore, we have the following assertions:

(i) The flows \( \partial/\partial t_j \) are \( \Gamma \)-invariant if and only if \( j \in J^\sigma_+ \).

(ii) When \( j \in \bar{J}_+ \), the restrictions of \( \partial/\partial t_j \) on \( \bar{R} \) are proportional to the Drinfeld-Sokolov hierarchy \( \partial/\partial t_j \), \( j \in \bar{J}_+ \), associated to \( (g, \bar{s}, \bar{\Theta}) \).

(iii) When \( j \in J^\sigma_+ \setminus J_+ \), the restrictions of \( \partial/\partial t_j \) on \( R \) are zeroes.
(iv) For any case in Tables I and II, suppose that $\tau^s$ is a $\Gamma$-invariant tau function of the Drinfeld–Sokolov hierarchy associated to $(\mathfrak{g}, s, \mathfrak{s})$, then

$$\log \tau^s = \frac{1}{\mu} \log \tau^s \bigg|_{t_j = \sqrt{\mu} t_j, j \in J_+; t_j = 0, j \in J_+ \setminus J_+}$$

(1.1)

gives a tau function $\bar{\tau}^s$ of the Drinfeld–Sokolov hierarchy associated to $(\bar{\mathfrak{g}}, \bar{s}, \bar{\mathfrak{s}})$. Here the notion of tau function is given in Definition 3.14, and $\mu$ is a constant listed in Appendix B.

This theorem generalizes Theorem 1.1. Especially, we conclude that the Drinfeld–Sokolov hierarchies of twisted type can be obtained by reduction of those hierarchies of ADE type.

The paper is arranged as follows. In Section 2 we recall some basic properties of affine Kac–Moody algebras. In Section 3 we give the definitions of Drinfeld–Sokolov hierarchies and their tau functions used in the present paper. In Section 4 we prove Theorems 1.2 and 1.3. In the final section, we give some remarks on the application of Theorem 1.3 to the so-called topological tau functions.

2 Preliminary knowledge on affine Kac–Moody algebras

In this section, we recall the definition and some basic properties of affine Kac–Moody algebras following [3, 10], as a preparation for what follows.

2.1 Affine Kac–Moody algebra and its principal Heisenberg subalgebra

Let $A = (a_{ij})_{0 \leq i, j \leq \ell}$ be a generalized Cartan matrix of affine type $X_{\ell}^{(r)}$ with $r = 1, 2, 3$ (in particular, $\ell = \ell'$ when $r = 1$), and $\{k_0, k_1, \ldots, k_\ell\}$ and $\{k'_0, k'_1, \ldots, k'_\ell\}$ be the sets of Kac labels and dual Kac labels respectively. It is known that $k'_i a_{ij} k_j = k'_j a_{ji} k_i$.

Denote by $\mathfrak{g}(A)$ the complex affine Kac-Moody algebra associated to $A$, with $\mathfrak{h}$ being its Cartan subalgebra, $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ and $\Pi^\vee = \{\alpha^\vee_0, \alpha^\vee_1, \ldots, \alpha^\vee_\ell\}$ being the sets of simple roots and simple coroots respectively. Then $\mathfrak{g}(A)$ has the following root space decomposition:

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha},$$

(2.1)

where $\Delta$ is the root system of $\mathfrak{g}(A)$. In $\mathfrak{g}(A)$ there is a set of Chevalley generators $\{e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i} \mid i = 0, 1, \ldots, \ell\}$, which satisfy the following Serre relations:

$$[e_i, f_j] = \delta_{ij} \alpha^\vee_i, \quad [\alpha^\vee_i, \alpha^\vee_j] = 0,$$

(2.2)

$$[\alpha^\vee_i, e_j] = a_{ij} e_j, \quad [\alpha^\vee_i, f_j] = -a_{ij} f_j,$$

(2.3)

$$(\text{ad}_{e_i})^{1-a_{ij}} e_j = 0, \quad (\text{ad}_{f_i})^{1-a_{ij}} f_j = 0.$$  

(2.4)

Here $\delta_{ij}$ is the Kronecker delta function, $i, j = 0, 1, 2, \ldots, \ell$, and $i \neq j$ in the equations of (2.4).

The Cartan subalgebra of $\mathfrak{g}(A)$ is given by

$$\mathfrak{h} = \mathbb{C} \alpha^\vee_0 \oplus \mathbb{C} \alpha^\vee_1 \oplus \cdots \oplus \mathbb{C} \alpha^\vee_\ell \oplus \mathbb{C} d,$$
where \( d \) is a scaling element satisfying

\[
[d, e_i] = e_i, \quad [d, f_i] = -f_i, \quad i = 0, 1, \ldots, \ell.
\] (2.5)

Note that

\[
c = \sum_{i=0}^{\ell} k_i^\vee \alpha_i^\vee \in \mathfrak{h}
\] (2.6)

is the canonical central element of \( g(A) \). On \( \mathfrak{h} \) there is a nondegenerate symmetric bilinear form such that

\[
(\alpha_i^\vee | \alpha_j^\vee) = a_{ij} \kappa_j \kappa_j^\vee, \quad (d | \alpha_j^\vee) = \frac{k_j}{k_j^\vee}, \quad (d | d) = 0,
\] (2.7)

where \( i, j = 0, 1, \ldots, \ell \). Clearly,

\[
(d | c) = \sum_{i=0}^{\ell} k_i =: h,
\] (2.8)

which is called the Coxeter number of \( g(A) \). This bilinear form can be uniquely extended to the normalized invariant symmetric bilinear form \((\cdot | \cdot)\) on \( g(A) \).

Let \( g = [g(A), g(A)] \) be the derived algebra of \( g(A) \), then we have \( g(A) = g \oplus \mathbb{C}d \). In what follows we will mainly use the derived algebra \( g \) rather than \( g(A) \). Note that \( g \) is generated by the Chevalley generators, and we will also call \( g \) the affine Kac–Moody algebra associated to \( A \).

The adjoint action of \( d \) induces on \( g \) the principal gradation

\[
g = \bigoplus_{k \in \mathbb{Z}} g^k, \quad g^k = \{ X \in g \mid [d, X] = kX \}.
\] (2.9)

Fix the following cyclic element

\[
\Lambda = \sum_{i=0}^{\ell} e_i \in g^1
\]

and consider its adjoin action on \( g \), then we obtain

\[
g = \text{Im ad}_\Lambda + \mathcal{H}, \quad \text{Im ad}_\Lambda \cap \mathcal{H} = \mathbb{C}c,
\] (2.10)

where \( \mathcal{H} = \{ X \in g \mid \text{ad}_\Lambda X \in \mathbb{C}c \} \) is called the principal Heisenberg subalgebra of \( g \). More precisely, the principal Heisenberg subalgebra can be represented as

\[
\mathcal{H} = \bigoplus_{j \in J} \mathbb{C} \Lambda_j \oplus \mathbb{C}c,
\] (2.11)

where

\[
J = \{ m_1, m_2, \ldots, m_\ell' \} + rh\mathbb{Z}, \quad 1 = m_1 < m_2 \leq m_3 \leq \cdots \leq m_{\ell'-1} < m_\ell = rh - 1,
\] (2.12)

is the set of exponents of \( g \) and \( \Lambda_j \in g^j \), and the basis elements can be chosen so that

\[
[\Lambda_i, \Lambda_j] = i\delta_{i,-j} c, \quad i, j \in J.
\] (2.13)

In particular, we have

\[
\Lambda_1 = \nu \Lambda
\]
for a certain constant $\nu$. Note that these generators also have the following properties:

\[
(\Lambda_i \mid \Lambda_j) = \frac{1}{i}(d, \Lambda_i) \mid \Lambda_j) = \frac{1}{i}(d \mid [\Lambda_i, \Lambda_j]) = \delta_{i,j}h.
\] (2.14)

Denote

\[
\mathcal{I}^k = (\text{Im} \, \text{ad}_\Lambda) \cap g^k \text{ for } k \neq 0, \quad \mathcal{I}^0 = \text{ad}_\Lambda \left( \mathcal{I}^{-1} \right), \quad \mathcal{I} = \bigoplus_{k \in \mathbb{Z}} \mathcal{I}^k.
\] (2.15)

Then we know that (see [3] for example) $\dim \mathcal{I}^k = \ell$ and

\[
\text{ad}_\Lambda : \mathcal{I}^k \rightarrow \mathcal{I}^{k+1}, \quad k \in \mathbb{Z},
\] (2.16)

are bijections. We also have

\[
g = \mathcal{I} \oplus \mathcal{H}.
\] (2.17)

2.2 Gradations on an affine Kac–Moody algebra

Let us denote

\[
\Gamma = \{ \mathbf{s} = (s_0, s_1, \ldots, s_\ell) \in \mathbb{Z}^{\ell+1} \mid s_i \geq 0, s_0 + s_1 + \cdots + s_\ell > 0 \}.
\] (2.18)

Then for each \( \mathbf{s} = (s_0, s_1, \ldots, s_\ell) \in \Gamma \), there is an element \( d^s \) in the Cartan subalgebra \( \mathfrak{h} \) of \( g(A) \) fixed by the relations

\[
(d^s \mid \alpha_i) = \frac{k_i}{k_i^\vee} s_i, \quad i = 0, 1, \ldots, \ell; \quad (d^s \mid d^s) = 0.
\] (2.19)

Clearly, for any \( X \in g \) such that \( X|_{g^0} = \sum_{i=0}^{\ell} x_i \alpha_i^\vee \) with respect to the decomposition (2.9), we have

\[
(d^s \mid X) = \sum_{i=0}^{\ell} \frac{x_i k_i^\vee s_i}{k_i^\vee}.
\] (2.20)

We also have the following relation of the canonical central element:

\[
(d^s \mid c) = k_0 s_0 + k_1 s_1 + \cdots + k_\ell s_\ell =: h^s.
\] (2.21)

The fact \( h^s > 0 \) also implies that \( g \oplus \mathbb{C}d^s = g \oplus \mathbb{C}d = g(A) \). In particular, if we denote \( \mathbf{1} := (1, 1, \ldots, 1) \), then we have \( d^\mathbf{1} = d \), and \( h^{\mathbf{1}} = h \) is just the Coxeter number.

The element \( d^s \) satisfies the commutation relations

\[
[d^s, e_i] = s_i e_i, \quad [d^s, f_i] = -s_i f_i, \quad i = 0, 1, \ldots, \ell,
\]

and it induces a gradation

\[
g = \bigoplus_{k \in \mathbb{Z}} g_{k[s]}, \quad g_{k[s]} = \{ X \in g \mid [d^s, X] = kX \}.
\] (2.22)

For instance, when \( s = \mathbf{1} \) we arrive at the principal gradation (2.9) of \( g \). We also have the following relations for any \( X_k \in g_{k[s]}, \ X_l \in g_{l[s]} \):

\[
(d^s \mid [X_k, X_l]) = \delta_{k,-l} k(X_k \mid X_l).
\] (2.23)
Let us recall briefly how to realize $\mathfrak{g}$ starting from a simple Lie algebra, see §8.6 of [10] for details. Let $\mathcal{G}$ be a simple Lie algebra of type $X_\ell'$, on which there is a diagram automorphism of order $r$. The integer vector $s \in \Gamma$ induces on $\mathcal{G}$ a $\mathbb{Z}/rh^s\mathbb{Z}$-gradation $\mathcal{G} = \bigoplus_{k=0}^{rh^s-1} \mathcal{G}_k$. Then $\mathfrak{g}$ with gradation $s$ can be realized as

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} (z^k \otimes \mathcal{G}_{k \mod rh^s}) \oplus \mathbb{C}c. \quad (2.24)$$

More precisely, for any elements $X(k), Y(l) \in z^k \otimes \mathcal{G}_{k \mod rh^s}$ and parameters $\xi, \eta \in \mathbb{C}$, the Lie bracket is defined by

$$[X(k) + \xi \cdot c, Y(l) + \eta \cdot c] = (k + l)[X, Y] + \delta_{k,-l} \frac{k}{rh^s} (X \mid Y)_{\mathcal{G}} \cdot c, \quad (2.25)$$

where $(\cdot \mid \cdot)_{\mathcal{G}}$ is the standard nondegenerate invariant symmetric bilinear form on $\mathcal{G}$. The normalized invariant bilinear form on $\mathfrak{g}$ is then given by

$$(X(k) + \xi \cdot c \mid Y(l) + \eta \cdot c) = \delta_{k,-l} \frac{1}{r} (X \mid Y)_{\mathcal{G}}. \quad (2.26)$$

In fact, one can choose elements $E_i, F_i$ and $H_i$ of $\mathcal{G}$ with $i = 0, 1, \ldots, \ell$ which yield the Weyl generators of $\mathfrak{g}$ as follows (see §8.3 of [10]):

$$e_i = E_i(s_i), \quad f_i = F_i(-s_i), \quad \alpha_i^\vee = H_i(0) + \frac{k_i s_i}{k_i h^s} \cdot c. \quad (2.27)$$

### 3 Drinfeld-Sokolov hierarchies

In this section, we recall the definition of Drinfeld-Sokolov hierarchies of both untwisted and twisted types, and then introduce the notion of their tau functions. To this end, let us consider two gradations on $\mathfrak{g}$: the principal gradation $\mathfrak{1}$, and a gradation $s = (s_0, s_1, s_2, \ldots, s_\ell) \in \Gamma$ with $s_i \leq 1$ ($s \leq \mathfrak{1}$ for short). For convenience we will write

$$\mathfrak{g}^k = \mathfrak{g}_{k[1]}, \quad \mathfrak{g}_k = \mathfrak{g}_{k[s]},$$

and use the notations $\mathfrak{g} \geq l = \bigoplus_{k \geq l} \mathfrak{g}_k$, $\mathfrak{g} \leq l = \bigoplus_{k < l} \mathfrak{g}^k$ etc.

#### 3.1 The dressing lemma and the pre-Drinfeld-Sokolov hierarchy

Let us introduce a Borel subalgebra of $\mathfrak{g}$ as follows:

$$\mathcal{B} = \left\{ X \in \mathfrak{g}_0 \cap \mathfrak{g}^{\leq 0} \mid (d^\mathfrak{g} \mid X) = 0 \right\}, \quad (3.1)$$

and consider operators of the form

$$\mathcal{L} = \frac{d}{dx} + \Lambda + Q, \quad Q = Q(x) \in C^\infty(\mathbb{R}, \mathcal{B}). \quad (3.2)$$

The following lemma plays a crucial role in the sequel, and we will call it the dressing lemma (cf. [3, 9, 17]).
Lemma 3.1  Given an operator \( \mathcal{L} \) of the form (3.2), then there exists a unique function \( U \in C^\infty(\mathbb{R}, g^{<0}) \) satisfying the following two conditions:

\[
\begin{align*}
(\text{i}) \quad & e^{-\text{ad}_U} \mathcal{L} = \frac{d}{dx} + \Lambda + H, \quad H \in C^\infty(\mathbb{R}, \mathcal{H} \cap g^{<0}), \\
(\text{ii}) \quad & (d^s | e^{\text{ad}_U} \Lambda_j) = 0 \quad \text{for any positive exponent } j \in J_+.
\end{align*}
\]

Moreover, both \( U \) and \( H \) are differential polynomials in the components of \( Q \) with respect to a basis of \( \mathcal{B} \) (differential polynomials in \( Q \) for short below) with zero constant terms.

Proof: The existence of \( U \in C^\infty(\mathbb{R}, g^{<0}) \) and \( H \in C^\infty(\mathbb{R}, \mathcal{H} \cap g^{<0}) \) satisfying the condition (3.3) was proved by Drinfeld and Sokolov in [3] based on the decomposition property (2.10) of \( g \). The condition (3.4) is imposed to ensure the uniqueness of \( U \) and \( H \). In the case when \( s = (1, 0, 0, \ldots, 0) \) the conclusion is proved in [17]. For the general case the proof is similar to that of [17], which we give below for the convenience of the readers.

Let us write the equation (3.3) in the form

\[
\begin{align*}
& e^{\text{ad}_{\Sigma_{k\leq -1} U_k}} \left( \frac{d}{dx} + \Lambda + \sum_{k \leq -1} H_k \right) = \frac{d}{dx} + \Lambda + \sum_{k \leq 0} Q_k, 
& \text{where } X_k \text{ take values in } g^k \text{ (note that } Q_k = 0 \text{ when } k \leq -h', \text{ where } h' \text{ is the lowest positive integer that } B \cap g^{-h'} = \{0\}). \text{ By comparing the homogeneous terms of both sides of the above equation, we obtain}
\end{align*}
\]

\[
\begin{align*}
& [U_{-1}, \Lambda] = Q_0, \\
& [U_k, \Lambda] + H_{k+1} + W_k = Q_k, \quad k = -2, -3, -4, \ldots.
\end{align*}
\]

Here \( W_k \in C^\infty(\mathbb{R}, g^{k+1}) \) depends on \( U_i \) and \( H_{i+1} \) with \( i > k \) and their \( x \)-derivatives.

From the decomposition (2.10) of \( g \) and the properties (2.13) of the elements of the Heisenberg subalgebra (2.11), it follows the existence of \( U_{-1} \in C^\infty(\mathbb{R}, g^{-1}) \) satisfying (3.6). The property of the Heisenberg subalgebra also implies that the map \( \text{ad}_\Lambda : g^{-1} \to g^0 \) is injective, so the solution \( U_{-1} \) of the equation (3.6) is unique. Moreover, the condition (3.4) with \( j = 1 \) is satisfied since

\[
(d^s | [U_{-1}, \Lambda]) = (d^s | Q_0) = 0.
\]

For \( k \leq -2 \), suppose that \( U_i \) and \( H_{i+1} \) with \( i > k \) are given, then \( W_k \) is known. By using the decomposition (2.10) of \( g \) we know the existence of solution \( U_k \in C^\infty(\mathbb{R}, g^k) \), \( H_{k+1} \in C^\infty(\mathbb{R}, g^{k+1}) \) of the equation (3.7), and we also know the uniqueness of \( H_{k+1} \) since \( k < -1 \). To prove the uniqueness of \( U_k \), we consider the following two cases:

- If \( k \notin J \), then from the property of the Heisenberg subalgebra \( \mathcal{H} \) we know that the map \( \text{ad}_\Lambda : g^k \to g^{k+1} \) is injective, so \( U_k \) is uniquely determined by the equation (3.7).
- If \( k \in J \), then \( U_k \) is determined by the equation (3.7) up to the addition of an element of \( \mathcal{H} \cap g^k \). By using the property (2.20) of \( d^s \) we know that the condition (3.4) with \( j = -k \) precisely fixes this freedom.

Therefore the lemma is proved. \( \square \)
Remark 3.2 Due to the uniqueness result of the dressing lemma, we will write \( U = U(Q) \) and \( H = H(Q) \) determined by the dressing lemma to emphasize their dependence on \( Q \) when it is needed.

Let the function \( U \) be determined by Lemma 3.1, then from the properties (2.20), (2.23) we know that the following family of evolutionary equations are well defined:

\[
\frac{\partial \mathcal{L}}{\partial t_j} = \left[-(e^{ad_U} \Lambda_j)_{\geq 0}, \mathcal{L}\right], \quad j \in J_+.
\]  

(3.8)

Here and in what follows, we denote by \( X_{\geq 0} \) the projection of \( X \in \mathfrak{g} \) to \( \mathfrak{g}_{\geq 0} \) w.r.t. the gradation (2.22) given by \( s \).

Denote \( \mathcal{L}' = d/dx + \Lambda + H \) and \( V = \partial U/\partial t_j \), then from the identity \( \mathcal{L} = e^{ad_U} \mathcal{L}' \) we obtain

\[
\frac{\partial \mathcal{L}}{\partial t_j} = \sum_{k \geq 0} \frac{1}{(k+1)!} \sum_{l=0}^{k} \text{ad}^l_U \text{ad} V \text{ad}^{k-l} U \mathcal{L}' + e^{ad_U} \frac{\partial H}{\partial t_j}
\]

\[
= \sum_{k \geq 0} \frac{1}{(k+1)!} \sum_{l=0}^{k} \frac{m}{(m+1)!} \text{ad}^m_U V \text{ad}^{k-m} U \mathcal{L}' + e^{ad_U} \frac{\partial H}{\partial t_j}
\]

\[
= \sum_{m \geq 0} \frac{1}{(m+1)!} \text{ad}^m_U V \mathcal{L}' + e^{ad_U} \frac{\partial H}{\partial t_j}
\]

\[
= \left[ \nabla_{t_j U} \mathcal{L}' \right] + e^{ad_U} \frac{\partial H}{\partial t_j}.
\]

(3.9)

Here we use the notation

\[
\nabla_{t_j U} U = \sum_{m \geq 0} \frac{1}{(m+1)!} (\text{ad}^m_U) \frac{\partial U}{\partial t_j}.
\]

(3.10)

Denote

\[
G(\Lambda_j) = e^{-ad_U} \left( \nabla_{t_j U} U - (e^{ad_U} \Lambda_j)_{<0} \right), \quad j \in J_+,
\]

(3.11)

then we have the following lemma.

Lemma 3.3 The functions \( G(\Lambda_j) \) take value in \( \mathcal{H} \cap \mathfrak{g}_{<0} \), and satisfy the equations

\[
\frac{\partial G(\Lambda_j)}{\partial x} = \frac{\partial H}{\partial t_j}, \quad [\Lambda_j, G(\Lambda_j)] = [\Lambda_j, H].
\]

(3.12)

\[
\frac{\partial}{\partial t_j} e^{ad_U} \Lambda_i = \left( e^{ad_U} \Lambda_j \right)_{<0} e^{ad_U} \Lambda_i + [G(\Lambda_j), \Lambda_i], \quad i, j \in J_+.
\]

(3.13)

**Proof:** From the equations (3.8) and (3.9) it follows that

\[
\left[ \nabla_{t_j U} U - (e^{ad_U} \Lambda_j)_{<0}, \mathcal{L}' \right] + e^{ad_U} \frac{\partial H}{\partial t_j} + [e^{ad_U} \Lambda_j, \mathcal{L}] = 0.
\]

(3.14)
Due to the decomposition (2.17), the function $G(\Lambda_j)$ can be uniquely represented as

$$G(\Lambda_j) = A + B,$$

where $A \in \mathcal{I} \cap \mathfrak{g}^{<0}$, $B \in \mathcal{H} \cap \mathfrak{g}^{<0}$.

Then the equation (3.14) can be rewritten as

$$\left[ A + B, \frac{d}{dx} + \Lambda + H \right] + \frac{\partial H}{\partial t_j} + [\Lambda_j, H] = 0. \quad (3.15)$$

We write $A = \sum_{i<0} A_i$ and $B = \sum_{i<0} B_i$ according to the principal gradation $\mathfrak{g}^{<0} = \bigoplus_{i<0} \mathfrak{g}^i$. The degree zero part of (3.15) reads

$$[A_{-1} + B_{-1}, \Lambda] + [\Lambda_j, H] = 0. \quad (3.16)$$

Note $A_{-1} \in \mathcal{I}^{-1}$ so that $[A_{-1}, \Lambda] \in \mathcal{I}^0$, and that $[B_{-1}, \Lambda] + [\Lambda_j, H] \in \mathcal{H}$, then from (2.17) and the bijection (2.16) it follows that

$$A_{-1} = 0, \quad [B_{-1}, \Lambda] + [\Lambda_j, H] = 0. \quad (3.16)$$

The negative degree part of (3.15) yields the following two equations:

$$-\frac{\partial A}{\partial x} + [A, \Lambda + H] = 0, \quad (3.17)$$

$$-\frac{\partial B}{\partial x} + \frac{\partial H}{\partial t_j} = 0. \quad (3.18)$$

By considering the homogeneous terms in the equation (3.17) with respect to the principal gradation and by using the bijection (2.16), we arrive at $A = 0$ and the fact that $G(\Lambda_j) \in \mathcal{H} \cap \mathfrak{g}^{<0}$. Then the two equations given in (3.12) follow from the equation (3.18) and the second equation of (3.16).

Due to the fact that $[G(\Lambda_j), \Lambda_i]$ is a multiple of the canonical center $c$, we have

$$\frac{\partial}{\partial t_j} e^{\text{ad}_U \Lambda_i} = \left[ \nabla_{t_j} U, e^{\text{ad}_U \Lambda_i} \right] = \left[ e^{\text{ad}_U \Lambda_j} e^{\text{ad}_U \Lambda_i}, e^{\text{ad}_U [G(\Lambda_j), \Lambda_i]} \right] = \left[ e^{\text{ad}_U \Lambda_j}, e^{\text{ad}_U \Lambda_i} \right] + [G(\Lambda_j), \Lambda_i],$$

thus $G(\Lambda_j)$ satisfies the equation (3.13). The lemma is proved. \qed

From the above lemma we arrive at the following proposition.

**Proposition 3.4** The flows $\partial/\partial t_j$ defined in (3.8) commute with each other, and they form the so-called pre-Drinfeld-Sokolov hierarchy associated to the quadruple $(\mathfrak{g}, \mathfrak{I}, \Lambda, s)$.

### 3.2 The Drinfeld-Sokolov hierarchy and its Hamiltonian structure

Note that the Borel subalgebra $\mathcal{B}$ contains a nilpotent subalgebra $\mathcal{N} = \mathfrak{g}_0 \cap \mathfrak{g}^{<0}$, which is generated by the generators $f_i$ with $s_i = 0$. According to the Serre relations, we have

$$\text{ad}_\Lambda \mathcal{N} = \text{ad}_\Lambda \mathcal{N} \subset \mathcal{B}, \quad I = \sum_{i|s_i=0} e_i. \quad (3.19)$$
Since $\mathcal{N} \cap \mathcal{H} = \{0\}$ the map $\text{ad}_\Lambda : \mathcal{N} \to \mathcal{B}$ is injective, hence we have a decomposition
\[
\mathcal{B} = \text{ad}_\Lambda \mathcal{N} \oplus \mathcal{V}
\] (3.20)
with $\mathcal{V}$ being some $\ell$-dimensional subspace of $\mathcal{B}$. Clearly (3.19) and (3.21) hold true when $\Lambda$ is replaced by $\Lambda_1 = \nu \Lambda$.

Observe that operators of the form (3.2) admits the following gauge transformations:
\[
\mathcal{L} \mapsto \tilde{\mathcal{L}} = e^{\text{ad}_N} \mathcal{L}, \quad N \in C^\infty(\mathbb{R}, \mathcal{N}).
\] (3.21)

The results of the following two lemmas can be derived by using the method given in [3, 1], based on the decomposition (3.20) of the Borel subalgebra $\mathcal{B}$.

**Lemma 3.5** Let $\mathcal{L}$ be an operator of the form (3.2). Fix a complementary subspace $\mathcal{V} \subset \mathcal{B}$ as in (3.20), then there exists a unique function $N \in C^\infty(\mathbb{R}, \mathcal{N})$ such that $\mathcal{L}^\mathcal{V} = e^{\text{ad}_N} \mathcal{L}$ takes the form
\[
\mathcal{L}^\mathcal{V} = \frac{d}{dx} + \Lambda + Q^\mathcal{V}, \quad Q^\mathcal{V} \in C^\infty(\mathbb{R}, \mathcal{V}).
\] (3.22)
Moreover, both $N$ and $Q^\mathcal{V}$ are differential polynomials in $Q$ with zero constant terms.

We call $Q^\mathcal{V}$ a gauge of the function $Q$, or $\mathcal{L}^\mathcal{V}$ a gauge of the operator $\mathcal{L}$. In fact, if one takes another complementary subspace $\tilde{\mathcal{V}}$, then the components of $Q^\tilde{\mathcal{V}}$ can be represented as differential polynomials in the components of $Q^\mathcal{V}$, i.e. they are related by Miura-type transformations. In what follows, we will fix a basis $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ of the subspace $\mathcal{V}$, then $Q^\mathcal{V}$ takes the form
\[
Q^\mathcal{V} = \sum_{i=1}^{\ell} u_i \gamma_i.
\] (3.23)

We will denote by $\mathcal{R}$ the ring of differential polynomials in $u = (u_1, u_2, \ldots, u_\ell)$, i.e.,
\[
\mathcal{R} = \mathbb{C}[u, u', u'', \ldots]
\] (3.24)
with the prime means taking derivatives with respect to $x$.

**Lemma 3.6** For the operators $\mathcal{L}$ and $\mathcal{L}^\mathcal{V}$ given above, the functions $U$ and $H$ determined in Lemma [3.1] have the following properties:
\[
e^{\text{ad}_U(Q^\mathcal{V})} = e^{\text{ad}_N} e^{\text{ad}_U(Q)}, \quad H(Q) = H(Q^\mathcal{V}).
\] (3.25)
In particular, the second property implies that the function $H(Q)$ is invariant w.r.t. the gauge transformations (3.21).

**Theorem 3.7** The equations (3.8) lead to a system of evolutionary equations on $\mathcal{V}$:
\[
\frac{\partial \mathcal{L}^\mathcal{V}}{\partial t_j} = \left[ -(e^{\text{ad}_U(Q^\mathcal{V})})_{\Lambda_j})_{\geq 0} + R(Q^\mathcal{V}, \Lambda_j), \mathcal{L}^\mathcal{V} \right], \quad j \in J_+,
\] (3.26)
where $R(Q^\mathcal{V}, \Lambda_j)$ takes value in $\mathcal{N}$, and its components belong to $\mathcal{R}$.
Proof: From (3.8) it follows that

$$\frac{\partial L^V}{\partial t_j} = \frac{\partial}{\partial t_j} \left( e^{ad_N} L^V \right) = \left[ \nabla_{t_j,N}, N_j, e^{ad_N} L^V \right] + e^{ad_N} \left[ (e^{ad_U} L^V)_{\geq 0}, L^V \right] = \left[ (e^{ad_U} L^V)_{\geq 0} + \nabla_{t_j,N}, L^V \right].$$

(3.27)

Here in the last equality we have used the relations (3.25). Let us denote $R(Q^V, \Lambda_j) = \nabla_{t_j,N} N_j$, then $R(Q^V, \Lambda_j)$ takes value in $\mathcal{N}$, and we are left to show that the components of $R(Q^V, \Lambda_j)$ are differential polynomials in that of $Q^V$. To this end, let us denote

$$M := \left[ - (e^{ad_U} L^V)_{\geq 0}, L^V \right] = \left[ (e^{ad_U} L^V)_{< 0}, L^V \right] - [\Lambda_j, H(Q^V)],$$

then $M$ takes value in the Borel subalgebra $\mathcal{B}$ whose components belong to $\mathcal{R}$, and from (3.27) we have

$$M + [R(Q^V, \Lambda_j), L^V] \in C^\infty(\mathbb{R}, V).$$

(3.28)

Similar to the proof of Lemma 3.1 we expand $M$ and $R(Q^V, \Lambda_j)$ in the form

$$M = \sum_{k=-h'^1+1}^{0} M_k, \quad R(Q^V, \Lambda_j) = \sum_{k=-h'^1+1}^{-1} R_k$$

such that $M_k$ and $R_k$ belong to $\mathcal{B} \cap \mathfrak{g}^k$. Then from (3.28) we have

$$M_k + [R_k-1, A] + T_k \in C^\infty(\mathbb{R}, V \cap \mathfrak{g}^k), \quad k = 0, -1, -2, \ldots, -h'^2 + 2,$$

(3.29)

where $T_0 = 0, T_k \in C^\infty(\mathbb{R}, \mathcal{B} \cap \mathfrak{g}^k)$ with $k < 0$ depends on $R_{-1}, R_{-2}, \ldots, R_k, Q^V$ and their $x$-derivatives. By using the decomposition (3.20), the relation (3.29) for $k = 0$ implies that

$$R_{-1} = (ad_A)^{-1} \left( M_0 |_{ad_A \mathcal{N}} \right),$$

so the components of $R_{-1}$ indeed belong to $\mathcal{R}$. Similarly, we conclude in a recursive way that the components of $R_k-1$ belong to $\mathcal{R}$ for $k = -1, -2, \ldots, -h'^2 + 2$. Therefore the components of $R(Q^V, \Lambda_j)$ are differential polynomials in $\mathfrak{u}$. The theorem is proved.

From the proof we also have the following corollary that generalizes Lemma 3.8 in [3].

**Corollary 3.8** For any $j \in J_+$, $X = R(Q^V, \Lambda_j)$ is the unique function in $C^\infty(\mathbb{R}, \mathcal{N})$ satisfying the condition that $\left[ -(e^{ad_U} L^V)_{\geq 0} + X, L^V \right]$ takes value in $\mathcal{V}$.

Due to the above theorem, we can represent the equations (3.26) in the form

$$\frac{\partial u_i}{\partial t_j} = X^j_i(u, u', u'', \ldots) \in \mathcal{R}, \quad i = 1, \ldots, \ell; \quad j \in J_+.$$

(3.30)

Note that $\mathcal{R}$ is the set of differential polynomials that are invariant with respect to the gauge transformations (3.21).

**Definition 3.9** The hierarchy of evolutionary PDEs (3.26) or (3.30) is called the Drinfeld-Sokolov hierarchy associated to $(\mathfrak{g}, \mathfrak{s}, 1)$.
By using the fact that $\Lambda_1 = \nu \Lambda$, we know that the first flow $\frac{\partial}{\partial t_1}$ of the Drinfeld-Sokolov hierarchy is given by
\[
\frac{\partial}{\partial t_1} = \nu \frac{\partial}{\partial x}.
\]
So we will identify $t_1$ with $x/\nu$ in what follows.

**Theorem 3.10 ([1, 3])** The Drinfeld-Sokolov hierarchy (3.30) can be written, in terms of a given gauge slice $u_1, \ldots, u_\ell$, as a hierarchy of Hamiltonian systems
\[
\frac{\partial u_i}{\partial t_j} = \{u_i(x), \mathcal{H}_j\}, \quad i = 1, \ldots, \ell; \quad j \in J_+,
\]
with Hamiltonians
\[
\mathcal{H}_j = \int h_j \, dx, \quad h_j = (-\Lambda_j | H).
\]
Here the function $H$ is given by Proposition 3.1 and $h_j \in \mathcal{R}$ due to (3.25).

More precisely, the Hamiltonian systems (3.31) can be represented as
\[
\frac{\partial u_i}{\partial t_j} = \{u_i(x), \mathcal{H}_j\} = \sum_{m=1}^{\ell} \sum_{k \geq 0} P_{1m}^{(i)} \frac{\partial}{\partial x^k} \frac{\delta \mathcal{H}_j}{\delta u_m(x)}, \quad j \in J_+,
\]
where $P_{1m}^{(i)} \in \mathcal{R}$, and the variational derivatives of $\mathcal{H}_j$ are given by
\[
\frac{\delta \mathcal{H}_j}{\delta u_m(x)} = \sum_{k \geq 0} (-1)^k \frac{\partial}{\partial x^k} \frac{\partial h_j}{\partial u_m^{(k)}}.
\]

The above theorem shows that the function $H$ is a generating function of the Hamiltonian densities $h_j$. In fact, from (2.14) it follows that
\[
H = -\frac{1}{h} \sum_{j \in J_+} h_j \Lambda_{-j}.
\]

**Remark 3.11** Let $m_1 \leq m_2 \leq \cdots \leq m_\ell$ be the first $\ell$ elements of $J_+$, then the Hamiltonian densities $h_{m_1}, h_{m_2}, \ldots, h_{m_\ell}$ generate the ring $\mathcal{R}$ of gauge-invariant differential polynomials. So they can also be used as the unknown functions of the Drinfeld-Sokolov hierarchy, which can be written as
\[
\frac{\partial h_{m_i}}{\partial t_j} = Y_j^i(h, h', h'', \ldots), \quad i = 1, \ldots, \ell; \quad j \in J_+,
\]
where $h = (h_{m_1}, h_{m_2}, \ldots, h_{m_\ell})$ and $Y_j^i \in C^\infty(h)[[h', h'', \ldots]]$. In the notion of [4] such Hamiltonian densities are called the normal coordinates of the Drinfeld-Sokolov hierarchy.

**Remark 3.12** The Hamiltonians $\mathcal{H}_j$ are independent of the choice of the gradation $s$, although the densities $h_j$ do depend on it. The reason is that, according to [3], the differences of the densities $h_j$ associated to different choices of the gradation $s$ can be represented as the $x$-derivative of some gauge-invariant differential polynomials.
3.3 The tau functions of the Drinfeld-Sokolov hierarchy

In this subsection, we give the definition of the tau function of the Drinfeld-Sokolov hierarchy based on that of [14, 17].

Let $Q$ be a solution of the pre-Drinfeld-Sokolov hierarchy (3.8) associated to $(\mathfrak{g}, 1, \Lambda, s)$ and $U$ be given as in Lemma 3.1. We introduce a class of differential polynomials in the components of $Q$ as follows:

\[
\Omega_{ij}^s = \frac{1}{h^s} \left( d^s \left| \left[ \left( e^{a_{\mathfrak{g}}U} \Lambda_i \right)_{\geq 0}, \left( e^{a_{\mathfrak{g}}U} \Lambda_j \right)_{< 0} \right] \right) \right), \quad i, j \in J_+.
\]  

Proposition 3.13 The following identities hold true:

\[
\Omega_{ij}^s = \Omega_{ji}^s, \quad \frac{\partial \Omega_{ij}^s}{\partial t_k} = \frac{\partial \Omega_{kj}^s}{\partial t_i}, \quad i, j, k \in J_+.
\]  

Moreover, the differential polynomials $\Omega_{ij}^s$ are invariant with respect to the gauge transformations (3.21), namely, $\Omega_{ij}^s \in \mathcal{R}$.

Proof: Since $(d^s | [X, Y]) = 0$ for any $X, Y \in \mathfrak{g}_0$, it follows from $\left[ e^{a_{\mathfrak{g}}U} \Lambda_i, e^{a_{\mathfrak{g}}U} \Lambda_j \right] = 0$ that

\[
\Omega_{ij}^s = \frac{1}{h^s} \left( d^s \left| \left[ \left( e^{a_{\mathfrak{g}}U} \Lambda_i \right)_{\geq 0}, \left( e^{a_{\mathfrak{g}}U} \Lambda_j \right)_{< 0} \right] \right) \right)
\]
\[
= \frac{1}{h^s} \left( d^s \left| - \left[ \left( e^{a_{\mathfrak{g}}U} \Lambda_i \right)_{< 0}, \left( e^{a_{\mathfrak{g}}U} \Lambda_j \right)_{\geq 0} \right] \right) \right)
\]
\[
= \frac{1}{h^s} \left( d^s \left| \left[ \left( e^{a_{\mathfrak{g}}U} \Lambda_j \right)_{\geq 0}, \left( e^{a_{\mathfrak{g}}U} \Lambda_i \right)_{< 0} \right] \right) \right)
\]
\[
= \Omega_{ji}^s.
\]

To simplify notations, we denote $X = e^{a_{\mathfrak{g}}U} \Lambda_i$, $Y = e^{a_{\mathfrak{g}}U} \Lambda_j$ and $Z = e^{a_{\mathfrak{g}}U} \Lambda_k$. Clearly they commute with each other. By using (3.13) we obtain

\[
\frac{\partial \Omega_{ij}^s}{\partial t_k} = \frac{1}{h^s} \left( d^s \left| \left[ Z_{< 0}, X \right]_{\geq 0}, Y \right] + \left[ X_{\geq 0}, Z_{< 0}, Y \right] \right)\]
\[
= \frac{1}{h^s} \left( d^s \left| - \left[ Z_{\geq 0}, X \right]_{\geq 0}, \left[ X_{\geq 0}, Z_{\geq 0}, Y \right] \right) \right)
\]
\[
= \frac{1}{h^s} \left( d^s \left| \left[ X_{\geq 0}, Z \right]_{\geq 0}, Y \right] - \left[ X_{\geq 0}, Z_{\geq 0}, \left[ X_{\geq 0}, Y \right] \right] \right)
\]
\[
= \frac{1}{h^s} \left( d^s \left| \left[ X_{< 0}, Z_{\geq 0} \right]_{\geq 0}, \left[ X_{\geq 0}, Y \right] \right] - Z_{\geq 0}, \left[ X_{\geq 0}, Y \right] \right)\]
\[
= \frac{1}{h^s} \left( d^s \left| \left[ X_{\geq 0}, Z_{\geq 0} \right]_{\geq 0}, \left[ X_{\geq 0}, Y \right] \right] \right)
\]
\[
= \frac{\partial \Omega_{kj}^s}{\partial t_i}.
\]

Finally, for any $N \in C^\infty(\mathbb{R}, \mathcal{N})$ one has $e^{a_{\mathfrak{g}}N} d^s = d^s$, hence the right hand side of (3.37) is invariant whenever $e^{a_{\mathfrak{g}}U}$ is replaced by $e^{a_{\mathfrak{g}}U(\mathfrak{q}^V)} = e^{a_{\mathfrak{g}}N} e^{a_{\mathfrak{g}}U(\mathfrak{q})}$ (recall Lemma 3.6). Thus the proposition is proved. \(\Box\)
The proposition implies that for any solution \( u(t) \) with \( t = \{t_j\}_{j \in J_+} \) of the Drinfeld-Sokolov hierarchy, there exists a function \( \tau^s = \tau^s(t) \) such that

\[
\frac{\partial^2 \log \tau^s}{\partial t_i \partial t_j} = \Omega_{ij}^s(u, u', u'', \ldots)|_{u=u(t)}, \quad i, j \in J_+.
\] (3.39)

Note that \( \log \tau^s \) is determined up to a linear function of the time variables.

**Definition 3.14** The function \( \tau^s \) is called the tau function of the Drinfeld-Sokolov hierarchy (3.30).

The following proposition shows that the Hamiltonian densities \( h_j \) given in (3.32) are tau symmetric in the sense of [4].

**Proposition 3.15** The tau function and the Hamiltonian densities for the Drinfeld-Sokolov hierarchy (3.30) are related by

\[
\frac{\partial^2 \log \tau^s}{\partial x \partial t_j} = j h_j, \quad j \in J_+.
\] (3.40)

**Proof:** By using (3.4) and (3.13) we obtain

\[
\Omega_{ji}^s = \frac{1}{h^s} (d^s | [G(\Lambda_j), \Lambda_i]).
\] (3.41)

So from (5.12) and \( \partial/\partial t_1 = \nu \partial/\partial x \) it follows that

\[
\frac{\partial^2 \log \tau^s}{\partial x \partial t_j} = \frac{1}{\nu} \Omega_{ji}^s = \frac{1}{\nu h^s} (d^s | [G(\Lambda_j), \Lambda_i]) = \frac{1}{h^s} (d^s | [G(\Lambda_j), \Lambda]) = -\frac{1}{h^s} (d^s | [\Lambda_j, H]).
\] (3.42)

Then by using (3.35) we complete the proof of the proposition. \( \square \)

We emphasize that the tau function \( \tau^s \) depends essentially on the gradation \( s \leq 1 \). Let us illustrate it with the following examples.

**Example 3.16 (Examples 5.1 and A.1 in [17])** Let \( g \) be the affine Kac-Moody algebra of type \( A_1^{(1)} \), with the elements \( \Lambda_j \) chosen as in [3]. For \( s = (1, 0) \), we take the gauge \( Q^V = -u f_1 \), then the first two nontrivial equations of the Drinfeld-Sokolov hierarchy are \( (t_1 = x) \):

\[
u_{t_3} = \frac{3 u u_x}{2} + \frac{u_{xxx}}{4},
\] (3.43)

\[
u_{t_5} = \frac{15 u^2 u_x}{8} + \frac{5}{8} u u_{xxx} + \frac{5 u_x u_{xx}}{4} + \frac{u_{xxxxx}}{16},
\] (3.44)

where the subscripts \( t_j \) mean the partial derivatives with respect to them. As is well known, this integrable hierarchy is just the KdV hierarchy.

For \( s = (1, 1) \), note that the nilpotent subalgebra \( \mathcal{N} \) is trivial, then we take \( Q^V = Q = \frac{1}{2} (\alpha_1 \varphi - \alpha_0 \varphi) \) and have the first two nontrivial equations of the Drinfeld-Sokolov hierarchy as:

\[
u_{t_3} = -\frac{3}{2} v^2 v_x + \frac{1}{4} v_{xxx},
\] (3.45)
\[ v_{t_5} = \frac{15}{8} v^4 v_x - \frac{5}{8} v^2 v_{xxx} - \frac{5}{2} v v_x v_{xx} - \frac{5}{8} v_x^3 + \frac{1}{16} v_{xxxx}. \]  

This integrable hierarchy is also called the modified KdV hierarchy, which is related to the KdV hierarchy by the Miura transformation \( u = -v^2 + v_x \).

By using Definition 3.14 we know that the second-order derivatives of the tau functions for the above two different choices of gradations satisfy the relations:

| s    | \( \frac{\partial^2 \log \tau^s}{\partial t_1^2} \) | \( \frac{\partial^2 \log \tau^s}{\partial t_1 \partial t_3} \) |
|------|-----------------|-----------------|
| (1, 0) | \( \frac{1}{2} u \) | \( \frac{1}{8} (3u^2 + u_{xx}) \) |
| (1, 1) | \( -\frac{1}{2} v^2 \) | \( \frac{1}{8} (3v^4 - 2vv_{xx} + v_x^2) \) |

**Example 3.17** Let \( g \) be the affine Kac-Moody algebra of type \( A_2^{(2)} \), and its elements \( \Lambda_j \) be chosen as in Example 5.6 of [17]. For the Drinfeld-Sokolov hierarchies associated to different gradations \( s \) on \( g \), the gauge \( Q^V \), the first nontrivial equation and the second-order derivatives of tau functions are listed as follows:

| s    | \( Q^V \) | \( t_{5-\text{flow}} \) | \( \frac{\partial^2 \log \tau^s}{\partial t_1^3} \) |
|------|----------|-----------------|-----------------|
| (1, 0) | \( -uf_1 \) | \( u_{t_5} = -\frac{1}{108} (20u^3 + 30uu'' + 3u^{(4)})' \) | \( \frac{1}{3} u \) |
| (0, 1) | \( -\frac{w}{3} f_0 \) | \( w_{t_5} = -\frac{1}{108} (20w^3 + 30ww'' + 45(w')^2 + 3w^{(4)})' \) | \( \frac{1}{3} w \) |
| (1, 1) | \( -\frac{v}{3\sqrt{2}} (\alpha_0 \vee - 4\alpha \vee) \) | \( v_{t_5} = \frac{1}{36} (-16v^5 + 20v(v')^2 + 20v^2v'' + 10v'v'' - v^{(4)})' \) | \( -\frac{2}{3} v^2 \) |

Here we note \( x = \sqrt{2} t_1 \) since \( \nu = \sqrt{2} \), and in the present example the prime means to take derivative with respect to \( t_1 \). Observe that the evolutionary equations of \( u \) and of \( w \) are known as the Sawada-Kotera equation [15] and the Kaup-Kupershmidt equation [11] respectively, and the above three equations are related by the following Miura-type transformations:

\[ u = -2 v^2 + v', \quad w = -2 v^2 - 2 v'. \]

**Remark 3.18** Formulae of the form (3.37) were introduced by Miramontes in [14] to define tau functions for generalized Drinfeld-Sokolov hierarchies (see Equation (3.8) and (4.13) therein). The method of [14] relies on an assumption, in terms of our notations, that the functions \( U \) and \( H \) given by (3.3) must fulfill the following constraints (Equation (2.25) therein):

\[ U \in C^\infty (\mathbb{R}, g_{<0}), \quad H \in C^\infty (\mathbb{R}, (\mathcal{H} \cap g_{<0}) \oplus Cc). \]  

(3.47)

When \( s \) is chosen to be the principal gradation \( 1 \), such constraints are satisfied automatically, and the tau function introduced in [14] coincides with the one defined by (3.37). However, when \( s \neq 1 \), the above constraints are no longer trivial, since \( H \) may contain nontrivial component along the basis vector \( \Lambda_{-1} \in \mathcal{H} \), however in this case \( \Lambda_{-1} \notin g_{<0} \). For this reason, the condition (3.47) was replaced in [17] by the condition (3.44) for the case when \( s = s^0 \) is the homogeneous gradation.
In the present paper, we use the condition (3.4) to fix the functions $U$ and $H$ for general $s \leq 1$, and define the tau function of the Drinfeld-Sokolov hierarchy by (3.39) which generalizes the definition of the tau function given in [17]. In term of the present definition, $\tau_s^0$ coincides with the one defined in [17], and the tau function $\tau_s^1$ coincides with the one given by Miramontes [14] (and also by Erinequez and Frenkel [9]).

4 Proof of the main results

From now on we assume that $\mathfrak{g}$ is one of the affine Kac-Moody algebras that are listed in Tables 1–3 (or listed in Appendix B). Let us proceed to study the diagram automorphism $\sigma$ on $\mathfrak{g}$, and the induced symmetry of the corresponding Drinfeld-Sokolov hierarchy.

4.1 Diagram automorphisms and affine subalgebras

Let $\bar{\sigma}$ be a permutation of the index set $\{0, 1, 2, \ldots, \ell\}$ given in Tables 1–3 such that it preserves the generalized Cartan matrix $A$ of $\mathfrak{g}$, i.e.,

$$a_{\bar{\sigma}(i)\bar{\sigma}(j)} = a_{ij}, \quad i, j = 0, 1, 2, \ldots, \ell.$$ 

Then a diagram automorphism $\sigma$ on the affine Kac-Moody algebra $\mathfrak{g}$ is determined by

$$\sigma(e_i) = e_{\bar{\sigma}(i)}, \quad \sigma(f_i) = f_{\bar{\sigma}(i)}, \quad i = 0, 1, 2, \ldots, \ell. \quad (4.1)$$

Assume that the order of the diagram automorphism $\sigma$ is $p$, then its eigenvalues are given by

$$1, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1}$$

with $\epsilon = \exp(2\pi i/p)$. Since $\sigma$ is compatible with the principal gradation on $\mathfrak{g}$, we have a series of bijections

$$\sigma : \mathfrak{g}^k \to \mathfrak{g}^k, \quad k \in \mathbb{Z},$$

and the eigenspace decompositions

$$\mathfrak{g}^k = \mathfrak{g}^{k,1} \oplus \mathfrak{g}^{k,\epsilon} \oplus \cdots \oplus \mathfrak{g}^{k,\epsilon^{p-1}}.$$ 

Let $\mathfrak{g}^\sigma$ be the subalgebra of $\mathfrak{g}$ that consists of $\sigma$-invariant elements. It can be represented as

$$\mathfrak{g}^\sigma = \bigoplus_{k \in \mathbb{Z}} (\mathfrak{g}^\sigma)^k, \quad (\mathfrak{g}^\sigma)^k = \mathfrak{g}^\sigma \cap \mathfrak{g}^k = \mathfrak{g}^{k,1}. \quad (4.2)$$

Since $\Lambda, c \in \mathfrak{g}^\sigma$, $\sigma$ can be restricted to $\mathcal{I}$ and $\mathcal{H}$ (see their definitions given in Section 2.1), we can choose the basis elements $\Lambda_j$ of $\mathcal{H}$ so that they are eigenvectors of $\sigma$. Denote

$$\mathcal{I}^\sigma = \mathfrak{g}^\sigma \cap \mathcal{I}, \quad \mathcal{H}^\sigma = \mathfrak{g}^\sigma \cap \mathcal{H}, \quad J^\sigma = \{ j \in J \mid \sigma(\Lambda_j) = \Lambda_j \}, \quad (4.3)$$

then we have

$$\mathcal{H}^\sigma = \bigoplus_{j \in J^\sigma} \mathbb{C}\Lambda_j \oplus \mathbb{C}c, \quad (4.4)$$
and the gradation $\mathcal{I}^\sigma = \bigoplus_{k \in \mathbb{Z}} (\mathcal{I}^\sigma)^k$. We also have the bijections

$$\text{ad}_\Lambda: (\mathcal{I}^\sigma)^k \to (\mathcal{I}^\sigma)^{k+1}, \ k \in \mathbb{Z}. \quad (4.5)$$

Now let us consider the following subalgebra of $\mathfrak{g}$:

$$\bar{\mathfrak{g}} = \langle (\mathfrak{g}^\sigma)^{-1}, (\mathfrak{g}^\sigma)^0, (\mathfrak{g}^\sigma)^1 \rangle \quad (4.6)$$

which has the gradation

$$\bar{\mathfrak{g}} = \bigoplus_{k \in \mathbb{Z}} \bar{\mathfrak{g}}^k, \quad \bar{\mathfrak{g}}^k = \bar{\mathfrak{g}} \cap \mathfrak{g}^k. \quad (4.7)$$

In particular, we have $\bar{\mathfrak{g}}^k = \mathfrak{g}^k$ for $k = -1, 0, 1$ and $\Lambda \in \bar{\mathfrak{g}}$. We will show in the following theorem that $\bar{\mathfrak{g}}$ is the derived algebra of an affine Kac-Moody algebra $\bar{\mathfrak{g}}(\bar{\mathfrak{A}})$ for a certain generalized Cartan matrix $\bar{\mathfrak{A}}$. Thus $\bar{\mathfrak{g}}$ has the decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{I}} \oplus \bar{\mathfrak{H}}, \quad (4.8)$$

where $\bar{\mathfrak{I}} = \bigoplus_{k \in \mathbb{Z}} \bar{\mathfrak{I}}^k$ and $\bar{\mathfrak{H}}$ are defined as in (2.11), (2.15). From the bijections

$$\text{ad}_\Lambda: \bar{\mathfrak{I}}^k \to \bar{\mathfrak{I}}^{k+1}, \ k \in \mathbb{Z} \quad (4.9)$$

and the facts $\bar{\mathfrak{I}}^k = (\mathcal{I}^\sigma)^k, \ k = -1, 0, 1$, it follows that $\bar{\mathfrak{I}} = \mathcal{I}^\sigma$. On the other hand, we have $\mathfrak{H} \subset \mathfrak{H}^\sigma \subset \mathfrak{H}$. More precisely, we can modify, if needed, the choice of the basis $\Lambda_j (j \in J)$ of the Heisenberg subalgebra $\mathfrak{H}$ of $\mathfrak{g}$ so that

$$\mathfrak{H} = \bigoplus_{j \in \bar{J}} C\Lambda_j \oplus Cc,$$

where $\bar{J}$ is a subset of $J^\sigma$. Note that the sets $J, J^\sigma$ and $\bar{J}$ of “exponents” satisfy $\bar{J} \subset J^\sigma \subset J$.

The following theorem is a slightly more detailed account of Theorem 1.2.

**Theorem 4.1** Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $X^{(r)}_{\ell'}$ given in Tables 1–3 of Section 1, on which there is a diagram automorphism $\sigma$ induced by $\bar{\sigma}$. Then the following assertions hold true:

(i) The subalgebra $\bar{\mathfrak{g}}$ is an affine Kac-Moody algebra corresponding to the folded Dynkin diagram of type $X^{(r)}_{\ell'}$. Its Chevalley generators are given in Appendix A and its canonical center $\bar{c} = \mu c$ with a constant $\mu$ given in Appendix B.

(ii) For any case listed in Tables 1 and 2, one has $\mathfrak{g}^\sigma = \bar{\mathfrak{g}}$.

(iii) For any case listed in Table 3, one has $\mathfrak{g}^\sigma = \bar{\mathfrak{g}} \oplus \mathfrak{H}$, where $\mathfrak{H} = \bigoplus_{j \in J^\sigma \setminus \bar{J}} C\Lambda_j$ with $J^\sigma \setminus \bar{J}$ nonempty (see Appendix B).

**Proof:** By taking average of the elements of the $\sigma$-orbits of the Chevalley generators of $\mathfrak{g}$, we obtain a system of generators $E_i, F_i, H_i, i = 0, 1, \ldots, \ell$ of $\bar{\mathfrak{g}}$ (see (A.1) of Appendix A). To prove the first assertion of the theorem, we only need to show that these generators satisfy the Serre relations for a certain generalized Cartan matrix $\bar{\mathfrak{A}} = (A_{ij})$ (see its definition given in (A.2)) corresponding
to the folded Dynkin diagram. We present the details of the proof of this fact in Appendix A. In particular, from (A.1) one sees that

$$\Lambda = \sum_{i=0}^{\bar{\ell}} E_i,$$

hence it is also the cyclic element of $\bar{g}$.

In order to prove the second and the third assertions of the theorem, it is sufficient to compare the dimensions of $\bar{g}_k$ and of $(g^\sigma)_k$ for $k \in \mathbb{Z}$ due to the fact that $\bar{g} \subset g^\sigma$. From the realization (2.24) with $s = 1$ it follows that we only need to consider the integers $k$ which lie in an interval of $\mathbb{Z}$ with length $rh$. We do this comparison case by case, and illustrate this procedure for the case (a2) in Table 2 (also listed in Appendix B). To simplify the presentation we assume that the canonical center $c$ is trivial in this proof. In this case $g = A_{2n}^{(1)}$ and

$$\dim g^k = \begin{cases} 2n + 1, & 2n + 1 \nmid k, \\ 2n, & 2n + 1 \mid k. \end{cases} \quad (4.10)$$

We first note that the linear space $g^0$ has a basis $\alpha^y_i = [e_i, f_i]$, $i = 1, \ldots, 2n$, from this fact it is easy to see that

$$\dim(g^\sigma)^0 = n. \quad (4.11)$$

In the case when $k = 1, 2, 3, \ldots, 2n$, the linear space $g^k$ is spanned by $X^k_0, X^k_1, \ldots, X^k_{2n}$ with

$$X^k_i = \ldots [e_i, c_{i+1}], c_{i+2}, \ldots, c_{i+k-1},$$

where $\bar{j} \in \{0, 1, 2, \ldots, 2n\}$ equals $j$ modulo $2n + 1$. We have

$$\sigma(X^k_i) = \ldots [e_{\bar{2n} - i}, c_{\bar{2n} - i + 1}], c_{\bar{2n} - i + 2}, \ldots, c_{\bar{2n} - i - k + 1}]$$

$$= (-1)^{k-1} \ldots [e_{\bar{2n} - i - k + 1}, c_{\bar{2n} - i - k + 2}, c_{\bar{2n} - i - k + 3}], \ldots, c_{\bar{2n} - i}]$$

$$= (-1)^{k-1} X^k_{\bar{2n} - i - k + 1}. \quad (4.12)$$

where the second equality holds true due to the Jacobi identity and the fact that $[e_i, e_j] = 0$ unless $|i - j| \in \{1, 2n\}$. Under the action of $\sigma$, $g^k$ is decomposed into $n$ 2-dimensional orbit spaces and one 1-dimensional orbit space. The 1-dimensional orbit space is spanned by

$$Y^k = \begin{cases} X^2_{\bar{2n} - j + 1}, & k = 2j + 1, \\ X^2_{\bar{2n} - j + 1}, & k = 2j, \end{cases} \quad (4.13)$$

which satisfies the relation

$$\sigma(Y^k) = (-1)^{k-1} Y^k.$$
This together with (4.11) leads to, for general $k \in \mathbb{Z}$, that
\[
\dim(\mathfrak{g}^\sigma)^k = \begin{cases} 
  n + 1, & k \text{ is odd, and } 2n + 1 \nmid k, \\
  n, & \text{other cases.} 
\end{cases}
\] (4.15)

From this fact and the property of the affine Kac-Moody algebra $\tilde{\mathfrak{g}}$ it follows that $\dim(\mathfrak{g}^\sigma)^k = \dim \tilde{\mathfrak{g}}^k$ for $k \in \mathbb{Z}$. Thus the Case (a2) is verified. The other cases can be verified similarly. Therefore the theorem is proved.

The first assertion of the theorem implies that, one can norma lized the generators of the principal Heisenberg subalgebra $\tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{g}}$ as
\[
\tilde{\Lambda}_j = \sqrt{\mu} \Lambda_j \text{ with } j \in \tilde{J}, \text{ such that } [\tilde{\Lambda}_i, \tilde{\Lambda}_j] = \delta_{i,-j}i\tilde{c}.
\] (4.16)

Moreover, one can extend the automorphism $\sigma$ on $\mathfrak{g}$ to $\sigma : \mathfrak{g}(A) \rightarrow \mathfrak{g}(A)$ by setting $\sigma(d) = d$, and see that the standard bilinear form $(\cdot | \cdot)$ on $\mathfrak{g}(A)$ is invariant with respect to $\sigma$. Accordingly, it follows that $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ is the derived algebra of $\tilde{\mathfrak{g}}(\tilde{A}) := \tilde{\mathfrak{g}} \oplus \mathbb{C}d \subset \mathfrak{g}(A)$, on which the standard bilinear form is given by
\[
(\cdot | \cdot)^- = \kappa(\cdot | \cdot)_{\tilde{\mathfrak{g}}(\tilde{A})},
\] (4.17)

with a normalization constant $\kappa$ (see Appendix B). This constant satisfies
\[
\tilde{h} = (\tilde{\Lambda}_j | \tilde{\Lambda}_{-j})^- = \kappa \mu (\Lambda_j | \Lambda_{-j}) = \kappa \mu h,
\]
where $h$ and $\tilde{h}$ are the Coxeter numbers of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ respectively.

\textbf{Remark 4.2} One can also consider composition of automorphisms, and obtain chains of subalgebras like $A^{(1)}_{4n+1} \supset C^{(1)}_{2n+1} \supset A^{(2)}_{2n}, D^{(1)}_{n+2} \supset B^{(1)}_{n+1} \supset D^{(2)}_n, \text{ e.t.c.}$

4.2 The $\Gamma$-reduction theorem

Let us note that the principal gradation $\mathbb{1}$ of $\mathfrak{g}$ is compatible with the diagram automorphism $\sigma$. In what follows we fix a gradation $s \leq \mathbb{1}$ of $\mathfrak{g}$ that is also compatible with $\sigma$, i.e., it satisfies the condition $s_{\mathfrak{g}(i)} = s_i$.

Recall the decomposition (3.20) of the Borel subalgebra $\mathcal{B}$. Since both $\mathcal{B}$ and its subspace $\text{ad}_A \mathcal{N}$ are invariant w.r.t. the action of $\sigma$, we can take a $\sigma$-invariant complementary subspace $\mathcal{V} \subset \mathcal{B}$. Choose a basis $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of $\mathcal{V}$ such that
\[
\sigma \gamma_i = \epsilon_i \gamma_i,
\]
where $\epsilon_i$ are some roots of unity. The automorphism $\sigma : \mathcal{V} \rightarrow \mathcal{V}$ induces a pullback of the coordinate functions of $Q^\vee = \sum_{i=1}^n u_i \gamma_i$, defined in (3.22), as follows:
\[
\sigma^* u_i = \epsilon_i u_i, \quad i = 1, 2, \ldots, \ell.
\] (4.18)

This pullback action extends naturally to the ring $\mathcal{R}$ of differential polynomials $i u_1, u_2, \ldots, u_\ell$. 23
On the other hand, as indicated in Subsection 4.1, the generators $\Lambda_j$ of the principal Heisenberg subalgebra $\mathcal{H}$ can be chosen such that

$$\sigma(\Lambda_j) = \zeta_j \Lambda_j, \quad j \in J,$$

where $\zeta_j^p = 1$ with $p$ being the order of $\sigma$. In particular, by using $\Lambda_1 = \nu \sum_{i=0}^e ll e_i$, (2.14) and (2.24), we have

$$\zeta_1 = 1, \quad \zeta_j = \zeta_j^{-1}, \quad \zeta_j + rh = \zeta_j.$$

With the help of such generators, the Drinfeld-Sokolov hierarchy (3.26) is defined, whose flows $\partial/\partial t_j$ are regarded as vector fields on the jet space of $\mathcal{V}$ acting on the differential polynomial ring $\mathcal{R}$.

**Proposition 4.3** The pullback and the pushforward maps induced by $\sigma : \mathcal{V} \to \mathcal{V}$ have the following properties:

1. $\sigma^*(h_j) = \zeta_j^{-1} h_j$;
2. $\sigma^*(\Omega_{jk}^s) = \zeta_j^{-1} \zeta_k^{-1} \Omega_{jk}^s$;
3. $\sigma^*(X_j^i) = \zeta_j^{-1} \epsilon_i X_j^i$;
4. $d \sigma \left( \frac{\partial}{\partial t_j} \right) = \zeta_j \frac{\partial}{\partial t_j}$.

Here $j,k \in J_+$, the functions $h_j$, $\Omega_{jk}^s$ and the flows $\partial/\partial t_j$ are defined respectively in (3.32), (3.37) and (3.26).

**Proof:** The proof is almost the same as the one for Proposition 4.8 of [13]. For the convenience of the readers, let us write it down briefly here.

We first verify the validity of the second property as follows:

$$\sigma^* \Omega_{ij}^s = \frac{1}{h^s} \left( d^s \left| \left( e^{ad_{U(Q^V)} \Lambda_j} \right)_{\geq 0}, \left( e^{ad_{U(Q^V)} \Lambda_k} \right)_{< 0} \right) \right)$$

$$= \frac{1}{\zeta_j \zeta_k h^s} \left( \sigma d^s | \sigma \left( e^{ad_{U(Q^V)} \Lambda_j} \right)_{\geq 0}, \left( e^{ad_{U(Q^V)} \Lambda_k} \right)_{< 0} \right)$$

$$= \frac{1}{\zeta_j \zeta_k h^s} \left( d^s | \left( e^{ad_{U(Q^V)} \Lambda_j} \right)_{\geq 0}, \left( e^{ad_{U(Q^V)} \Lambda_k} \right)_{< 0} \right)$$

$$= \zeta_j^{-1} \zeta_k^{-1} \Omega_{ij}^s.$$

The first property of the proposition then follows from the second one, the formulae given in (3.39), (3.40) and the fact that $\zeta_1 = 1$. To prove the third property, let us apply $\sigma$ act on both sides of (3.26) to obtain

$$\sigma \frac{\partial \mathcal{L}^V}{\partial t_j} = \sigma \left[ -(e^{ad_{U(Q^V)} \Lambda_j})_{\geq 0} + R(Q^V, \Lambda_j), \mathcal{L}^V \right], \quad j \in J_+.$$

For the above equations we have

$$\text{l.h.s.} = \sum_{i=1}^\ell \frac{\partial \sigma^* u_i}{\partial t_j} \gamma_i,$$
\[
\text{r.h.s.} = \left[-(e^{ad_{\sigma V(Q^V)}}(\sigma \Lambda_j))_{\geq 0} + R(\sigma Q^V, \sigma \Lambda_j), \sigma \mathcal{L}^V\right]
\]
\[
= \zeta_j \left[-(e^{ad_{\sigma V(Q^V)}} \Lambda_j)_{\geq 0} + R(\sigma Q^V, \Lambda_j), \sigma \mathcal{L}^V\right] = \zeta_j \sum_{i=1}^{\ell} \left(\sigma^* \frac{\partial u_i}{\partial t_j}\right) \gamma_i.
\]

Here in computing the right hand side of the equation (4.20) we used the property of linear dependence of \(R(Q^V, \Lambda_j)\) on \(\Lambda_j\), which can be seen from Corollary 3.8 By comparing both sides of (4.20) and by using (4.18), we obtain
\[
\sigma^* \left(\frac{\partial u_i}{\partial t_j}\right) = \frac{\epsilon_i}{\zeta_j} \frac{\partial u_i}{\partial t_j}, \quad d\sigma \left(\frac{\partial}{\partial t_j}\right) = (\sigma^*)^{-1} \frac{\partial}{\partial t_j} \sigma^* = \zeta_j \frac{\partial}{\partial t_j}.
\]

The proposition is proved. \(\square\)

The above proposition implies that, each flow \(\partial/\partial t_j\) with \(j \in J^\sigma_+\) (i.e., \(\zeta_j = 1\), see (4.3)) of the Drinfeld-Sokolov hierarchy associated to \(g\) can be restricted to the jet space
\[
\mathcal{V}^\sigma := \mathcal{V} \cap g^\sigma = \{X \in \mathcal{V} \mid \sigma(X) = X\}.
\]
By using the fact that \(\mathcal{V} \cap \hat{H} = \{0\}\) which can be seen from Theorem 4.1 we know that \(\mathcal{V}^\sigma\) coincides with
\[
\hat{\mathcal{V}} := \mathcal{V} \cap \hat{g}, \quad (4.21)
\]
where the subalgebra \(\hat{g}\) of \(g\) is introduced in the previous subsection. We denote by \(\hat{\mathcal{R}}\) the ring of differential polynomials of the jet space of \(\hat{\mathcal{V}}\), i.e.
\[
\hat{\mathcal{R}} = \mathbb{C} \left[u_i, u_i', u_i'', \ldots \mid 1 \leq i \leq \ell, \sigma^* u_i = u_i\right]. \quad (4.22)
\]

As we have shown in the previous subsection that \(\hat{g}\) is a Kac-Moody algebra, so we can construct the associated Drinfeld-Sokolov hierarchy as we do in Section 3. It can be represented in the form of (3.26), i.e.
\[
\frac{\partial \mathcal{L}^\hat{V}}{\partial t_j} = \left[-(e^{ad_{\hat{U}(\hat{Q}^\hat{V})}} \hat{\Lambda}_j)_{\geq 0} + R(\hat{Q}^\hat{V}, \hat{\Lambda}_j), \mathcal{L}^\hat{V}\right], \quad j \in J_+^\sigma. \quad (4.23)
\]
Here
\[
\mathcal{L}^\hat{V} = \frac{d}{dx} + \hat{\Lambda} + \hat{Q}^\hat{V}, \quad \hat{Q}^\hat{V} \in C^\infty(\mathbb{R}, \hat{\mathcal{V}}), \quad (4.24)
\]
the function \(\hat{U}(\hat{Q}^\hat{V}) \in (\mathbb{R}, \hat{g}^{<0})\) is determined by Lemma 3.1 with \(g\) replaced by \(\hat{g}\) and the principal gradation defined in (4.7), the normalized generators \(\hat{\Lambda}_j\) are defined in (4.16), and the function \(R(\hat{Q}^\hat{V}, \hat{\Lambda}_j)\) takes value in \(\hat{\mathcal{N}} = \mathcal{N} \cap \hat{g}\) (see Corollary 3.8).

We emphasize that the equations (4.23) are well defined since the gradations \(I\) and \(s\) of \(g\) are compatible with \(\sigma\). In fact, they induce two gradations \(\hat{I}\) and \(\hat{s}\) on \(\hat{g}\) respectively, and the adjoint actions of the corresponding derivations \(d^I\) and \(d^s\) on \(\hat{g}\) are equal to that of \(d^\hat{I}\) and \(d^\hat{s}\) respectively. By abusing notations, the principal gradation \(\hat{I}\) on \(\hat{g}\) will also be written as \(I\).

The following theorem is the main result of the present section, and it will be called the \(\Gamma\)-reduction theorem.

**Theorem 4.4** Each flow \(\partial/\partial t_j\) with \(j \in J^\sigma_+\) of the Drinfeld-Sokolov hierarchy (3.26) associated to \((g, s, I)\) can be restricted to \(\hat{\mathcal{R}}\). Moreover,
The second-order derivatives of the logarithm of the tau function are given by

Example 4.5 (Case (a2) of Table 1) Let $\mathfrak{g}$ be the affine Kac-Moody algebra of type $A_2^{(1)}$, and $s = (1, 0, 1)$. The Borel and the nilpotent subsalgebras are given by $\mathcal{B} = \text{span}\{\alpha_0^\vee - \alpha_2^\vee, \alpha_1^\vee, f_1\}$ and $\mathcal{N} = \text{span}\{f_1\}$. Take $\mathcal{V} = \text{span}\{\alpha_0^\vee - \alpha_2^\vee, f_1\}$, then we have the gauge

$$Q^\mathcal{V} = v(\alpha_0^\vee - \alpha_2^\vee) - uf_1.$$ The first three nontrivial systems of equations in the Drinfeld-Sokolov hierarchy are given by $(t_1 = x)$:

$$u_{t_2} = 2uv_x + 4uv_v + u_{xxx}, \quad v_{t_2} = -\frac{u_x}{3} - 2uv_x;$$

$$u_{t_4} = \frac{2}{3}u_{xxx}x + 2uv_x + \frac{8u_xv_x}{3} - \frac{4}{3}u^3x + 4uvux + \frac{8u^2v_x}{3} + 2uv_{xxx} - 8uv^2v_x$$

$$+ \frac{v_{xxxxx}}{3} - 4(v_x)^3 - 2v_{xxx}v^2 - 12uv_vv_{xxx},$$

$$v_{t_4} = -\frac{u_{xx}}{9} + \frac{2^2u_x}{3} - \frac{2uv_x}{3} + \frac{4}{3}uvx - \frac{2}{3}u_{xxx}v - \frac{4v_xv_{xxx}}{3} + \frac{20v^3v_x}{3};$$

$$u_{t_5} = -\frac{5u^2v_x}{9} - \frac{5uv_{xx}}{3} - \frac{5uv_{xxx}}{9} - \frac{u_{xxxxx}}{9} + \frac{5}{3}u_{xxx}v^2 + 10uv_vxv + 10uv_xv_{xxx}$$

$$+ \frac{25}{3}u_xv_x^2 + \frac{5v^4u_x}{3} + 10uv^2u_x + \frac{40}{3}u^2v_vx + 10u_{xxx}v + 10uv_{xxx}v$$

$$+ \frac{20}{3}uv^3v_x + 20(v_x)^3 - 5v_{xxx}v_{xxx} + \frac{5}{3}v_{xxxx}v_x + 10v_{xxx}v^3 + 30v^2v_vxx,$$

$$v_{t_5} = -\frac{5u_{xx}}{9} - \frac{10u_{xx}v_x}{9} - \frac{10v^3u_x}{9} - \frac{10v_{xxx}v_x}{9} - \frac{20v^2v_{xxx}}{9} - \frac{5u^2v_x}{9} - \frac{5v_{xxx}}{9} - \frac{10}{3}uv^2v_x$$

$$- \frac{v_{xxxx}}{9} + \frac{5(v_x)^3}{3} + \frac{5}{3}v_{xxx}v^2 + 20\frac{vv_{xxx}}{3}v_x - \frac{35}{3}v^4v_x.$$ The second-order derivatives of the logarithm of the tau function are given by

$$\Omega_{11}^s = \frac{u}{3} - v^2, \quad \Omega_{12}^s = \frac{4uv}{3} + \frac{v_{xx}}{3} + \frac{4u^3}{3},$$

$$\Omega_{14}^s = \frac{4uv_{xx}}{9} + \frac{2u_xv_x}{3} + \frac{8u^2v}{9} + \frac{2uv_{xx}}{9} - \frac{16u^3}{3} + \frac{v_{xxxx}}{9} - \frac{4}{3}v(v_x)^2 - \frac{8v^5}{3} + \frac{2v^2v_{xx}}{3},$$

$$\Omega_{15}^s = -\frac{5u^3}{81} - \frac{5uv_{xx}}{27} - \frac{u_{xxxx}}{27} + \frac{5uv_{xx}}{27} + \frac{10}{3}uv_{xx}v + \frac{25u^2v^2}{9} + \frac{20}{9}uv_{xxx}v - \frac{5}{9}u(v_x)^2.$$
\[
\begin{align*}
\Omega_{22}^s &= -\frac{u_{xx}}{9} - \frac{2u^2}{9} + \frac{4uv^2}{3} - 2v^4 + \frac{2uv_{xx}}{9} - \frac{4(v_x)^2}{3}, \\
\Omega_{24}^s &= -\frac{u_{xxxx}}{27} - \frac{8u^3}{81} + \frac{2v^2 u_{xx}}{3} + \frac{8u v^2 v_x}{3} - \frac{2u_{xx} v_x}{9} + \frac{8u v_{xx}}{9} + \frac{20}{9} v u v_{xx} \\
&\quad - \frac{4}{3} u (v_x)^2 + \frac{40u^6}{9} + \frac{8}{9} u v x_{xx} - \frac{2(v_x)^2}{9} \frac{-28v^2 v_{xx}}{9} - \frac{10}{9} v_x v_{xxx} + 4v^2 (v_x)^2.
\end{align*}
\]

From (3.40) it follows that \( h_1 = 3 \Omega_{11}^s \) and \( h_2 = \frac{3}{2} \Omega_{12}^s \), hence the map \((u, v) \mapsto (h_1, h_2)\) is a Miura-type transformation, which illustrates Remark 3.11.

On \( \mathfrak{g} \) the diagram automorphism \( \sigma \) is induced by the permutation
\[
\bar{\sigma}(0) = 2, \quad \bar{\sigma}(1) = 1, \quad \bar{\sigma}(2) = 0.
\]

The action of \( \sigma \) on the basis of \( V \) is given by
\[
\sigma(f_1) = f_1, \quad \sigma(\alpha_0^\vee - \alpha_2^\vee) = -(\alpha_0^\vee - \alpha_2^\vee),
\]
which induces the map
\[
\sigma^* u = u, \quad \sigma^* v = -v.
\]

On the other hand, by comparing \( J \) and \( J^\sigma \), we see that the generators \( \Lambda_j \) satisfy
\[
\sigma(\Lambda_j) = (-1)^{j-1} \Lambda_j, \quad j = 3k \pm 1 \text{ with } k \in \mathbb{Z}.
\]

It is easy to see that the evolutionary equations and the differential polynomials given in Example ?? satisfy
\[
\sigma^*(\Omega_{ij}^s) = (-1)^{(i-1)+(j-1)} \Omega_{ij}^s, \quad \frac{d}{d\tau_j} u = (-1)^{j-1} u_{t_j}, \quad \frac{d}{d\tau_j} v = (-1)^{j-1} v_{t_j}.
\]

By setting \( v \equiv 0 \), the flows \( \partial u / \partial t_j \) with \( j \in \{1, 5\} + 6\mathbb{Z}_+ \) are reduced to the Drinfeld-Sokolov hierarchy associated to \( \mathfrak{g} \) of type \( \Lambda_2^{(2)} \) with gradation \( \mathfrak{s} = (1, 0) \) (note \( \mu = 2 \) then \( \partial / \partial t_j = \sqrt{2} \partial / \partial t_j \)). In particular, equation (4.28) is reduced to the Sawada–Kotela equation (see Example 3.17)
\[
\frac{\partial u}{\partial t_5} = -\frac{1}{108} \frac{\partial}{\partial t_1} \left( 20u^3 + 30u \frac{\partial^2 u}{\partial t_1^2} + 3 \frac{\partial^4 u}{\partial t_1^4} \right).
\]

**Example 4.6 (Case (c2) of Table 3)** Let \( \mathfrak{g} \) be the affine Kac-Moody algebra of type \( \mathfrak{c}_3^{(1)} \), with a gradation \( s = (0, 1, 1, 0) \). Let us take a gauge
\[
Q^\vee = v(\alpha_2^\vee - \alpha_1^\vee) - u(f_0 + f_3) - w(f_0 - f_3).
\]

With \( \Lambda_j \) taken as in [3], the first nontrivial equations in the Drinfeld-Sokolov hierarchy are given by
\[
\begin{align*}
\frac{d}{d\tau_5} u_{t_3} &= -6uvv_x - 3v_{xxx}v + 3wv_{xx} + 18v^3 v_x + \frac{3}{2} vw_{xx} - 3ww_x, \\
\frac{d}{d\tau_5} v_{t_3} &= uv_x + uv_x - \frac{1}{2} v_{xxx} + 3v^2 v_x + \frac{3w_{xx}}{4}.
\end{align*}
\]
\[
\begin{align*}
\omega_t &= -\nu u_{xx} - w_{x} + 3uv_{xx} - 2uw_x + \frac{3}{2}v_{xxxx} - 9v^2v_{xx} - 6uvw_x - 18v^2w_x - \frac{5}{4}w_{xxx}; \\
\omega_t &= -\frac{5}{9}u^2u_x - \frac{5}{9}u_xu_{xx} - \frac{5}{9}u_{xxx} - \frac{1}{9}u_{xxxx} - \frac{5}{6}u_{xxx}v^2 - \frac{5}{2}v_{xx}w_x + \frac{5}{2}vw_{xx} \\
&\quad - \frac{25}{6}vuxw_{xx} + \frac{10}{3}uwxv_x - \frac{5}{3}ux^2v_x - \frac{5}{6}vuxw_x + 5v^4u_x - \frac{10}{3}uv^2u_x + \frac{5}{6}w^2u_x \\
&\quad - \frac{5}{2}uw_{xxx}v - \frac{5}{3}uw_{xx}v - 5uv_xw_x + 20uv^3v_x - \frac{20}{3}u^2v_{xx} - \frac{5}{2}uv_{xx}w_x - \frac{10}{3}uvw_{xx} \\
&\quad + \frac{5}{3}uw_{xx} + \frac{5}{6}v_{xxxx}v - \frac{5}{3}w_{xxxx} - \frac{10}{3}v_{xxx}v^3 - \frac{15}{4}v_{xx}w_{xx} + 10v^2w_{xx} \\
&\quad - \frac{5}{3}w_{xxxx}v + 20v^2w_{xx} + 25uw_{xx} + 10vw^2v_x + 15v^3v_x - \frac{5}{3}v_{xxxx}v - \frac{5}{6}w_{xxx}v_x \\
&\quad - \frac{35}{24}vw_{xxxx} + 5v^3w_{xx} + 5v^2ww_x + \frac{5}{3}ww_{xxx} + \frac{5}{4}w_xw_{xx}.
\end{align*}
\]

(4.39)

Here we omit \( \omega_t \) and \( \omega_{5t} \), each term of which contains \( v, w \) or there \( x \)-derivatives. The first several differential polynomials \( \Omega_{jk}^s \) are as follows:

\[
\begin{align*}
\Omega_{11}^s &= \frac{u}{3} - v^2, \\
\Omega_{13}^s &= -2v^2 + wv_x - vw_x - \frac{w^2}{2}, \\
\Omega_{15}^s &= -\frac{u_{xxxx}}{27} - \frac{5u^3}{81} - \frac{5u^2v^2}{3} - \frac{5uv_{xx}}{27} - \frac{5v_{xxx}v}{27} - \frac{5uv_xw_x}{3} + \frac{5uv_{xxx}v}{3} + \frac{5uv_xw_x}{3} - \frac{5uv_{xxx}}{3} - \frac{9v_{xxx}w_x}{9} - \frac{5uv_{xx}}{9} \\
&\quad - \frac{10}{3}uv_{xx}w_x + \frac{5uw_{xx}^2}{18} - \frac{5v^6}{3} - \frac{11}{18}v_{xxxx}v - \frac{5v_{xxx}w_x}{9} - \frac{5v_{xx}w_x}{2} - \frac{5v_{xx}w_x}{9} - \frac{(v_{xx})^2}{6} \\
&\quad - \frac{25v_{xx}w_x}{36} + \frac{5v^3w_{xx}}{18}v_{xxx}v + 5v^2ww_x + 20v^2(v_x)^2 + \frac{5}{36}vw_{xxxx} + \frac{5w_{xx}w_{xx}}{9} - \frac{5}{72}(w_x)^2.
\end{align*}
\]

The diagram automorphism \( \sigma \) on \( g \) is given by

\[
\sigma(i) = 3 - i, \quad i = 0, 1, 2, 3.
\]

It induces

\[
\sigma^*u = u, \quad \sigma^*v = -v, \quad \sigma^*w = -w,
\]

and satisfies, due to \( J^\sigma = J \),

\[
\sigma(\Lambda_j) = \Lambda_j, \quad j \in 2\mathbb{Z} + 1.
\]

Clearly, the above flows \( \partial/\partial t_j \) and the differential polynomials \( \Omega_{jk}^s \) satisfy

\[
\mathrm{d}\sigma \left( \frac{\partial}{\partial t_j} \right) = \frac{\partial}{\partial t_j}, \quad \sigma^*\Omega_{jk}^s = \Omega_{jk}^s,
\]

which are consistent with Proposition 4.3. Moreover, according to Theorem 4.4 and \( J^\sigma = \tilde{J} \cup 3\mathbb{Z}^{\text{odd}} = J \), the flows \( \partial/\partial t_j, j \in 3\mathbb{Z}^{\text{odd}} \), must vanish when restricted to \( \mathcal{R} = \mathbb{C}[u, u_x, u_{xxx}, \ldots] \), which is illustrated above for the case \( j = 3 \). Moreover, the equations \( \partial u/\partial t_j \) with \( j \in (\{1, 5\} + 6\mathbb{Z}_+) \) are reduced to the flows that compose the Drinfeld–Sokolov hierarchy associated to \( g \) of type \( A_2^{(2)} \) with gradation \( \tilde{s} = (1, 0) \). In particular, the flow \( \partial u/\partial t_5 \) is clearly reduced to \( \mu = 2 \) then \( \partial/\partial t_j = \sqrt{2} \partial/\partial t_j \).
4.3 Proof of the $\Gamma$-reduction theorem

Let $\mathcal{L}$ be the operator introduced in (3.2) with $Q \in C^\infty(\mathbb{R}, \mathcal{B})$, and the functions

$$U(Q) \in C^\infty(\mathbb{R}, \mathfrak{g}^\leq), \quad H(Q) \in C^\infty(\mathbb{R}, \mathcal{H} \cap \mathfrak{g}^\leq)$$  \hspace{1cm} (4.40)

be defined by Lemma 3.1.

Note that the Borel subalgebras $\mathcal{B}$ and $\mathcal{B} := \mathcal{B} \cap \mathfrak{g}$ are invariant under the action of the automorphism $\sigma$, and

$$\mathcal{B} = \mathcal{B} \cap \mathfrak{g}^\sigma = \{ X \in \mathcal{B} \mid \sigma(X) = X \}$$

due to the fact that $\mathcal{B} \cap \mathcal{H} = \{0\}$ with $\mathcal{H}$ defined in Theorem 4.1. Then we have the following decomposition of subspaces:

$$\mathcal{B} = \mathbf{\bar{B}} \oplus \hat{\mathbf{B}}$$ \hspace{1cm} (4.41)

where $\hat{\mathbf{B}} \subset \mathcal{B}$ is spanned by the eigenvectors of $\sigma$ for all eigenvalues nonequal to 1.

Let $\bar{Q} = Q|_{\bar{\mathcal{B}}}$, then we have the following analogue of the operator (3.2) associated to the affine Kac-Moody algebra $\mathbf{\bar{g}}$:

$$\bar{\mathcal{L}} = \frac{d}{dx} + \Lambda + \bar{\mathcal{Q}} \in \mathbb{C} \frac{d}{dx} \ltimes C^\infty(\mathbb{R}, \bar{\mathcal{B}}).$$

We denote the functions $U$ and $H$ that are defined by Lemma 3.1 as follows:

$$\bar{\mathbf{U}}(\bar{Q}) \in C^\infty(\mathbb{R}, \bar{\mathfrak{g}}^\leq), \quad \bar{\mathbf{H}}(\bar{Q}) \in C^\infty(\mathbb{R}, \bar{\mathcal{H}} \cap \bar{\mathfrak{g}}^\leq).$$  \hspace{1cm} (4.42)

The operator $\bar{\mathcal{L}}$ admit a group of gauge transformations given by the nilpotent subalgebra $\mathbf{\bar{N}} = \mathbf{N} \cap \mathbf{\bar{g}}$, and a gauge slice $\bar{\mathcal{L}}^{\mathbf{\bar{V}}}$ of it is given in (4.24).

We have the following lemma.

Lemma 4.7 The functions given in (4.40) and (4.42) satisfy the following relations:

$$U(\bar{Q}) = \bar{\mathbf{U}}(\bar{Q}), \quad H(\bar{Q}) = \bar{\mathbf{H}}(\bar{Q}).$$  \hspace{1cm} (4.43)

Proof: We regard $\bar{\mathcal{L}}$ as an operator in $\mathbb{C} d/dx \ltimes C^\infty(\mathbb{R}, \bar{\mathcal{B}})$, and $\bar{\mathcal{L}}(Q), H(Q)$ as functions taking values in $C^\infty(\mathbb{R}, \mathfrak{g}^\leq), C^\infty(\mathbb{R}, \mathcal{H} \cap \mathfrak{g}^\leq)$ respectively. Due to the uniqueness result of Lemma 3.1, we only need to show that

$$\left( d^\mathbf{g} \mid e^{\text{ad}(\bar{Q})} \Lambda_j \right) = 0, \quad j \in J_+.$$  \hspace{1cm} (4.44)

The case when $j \in J_+$ follows immediately from the definition of $\bar{\mathcal{L}}(Q)$ and the property (4.17) of the bilinear form defined on $\mathfrak{g} \oplus \mathbb{C} d^\mathbf{g} = \mathfrak{g}(\hat{A})$. For the case $j \in J_+ \setminus J_+$, since

$$\left( d^\mathbf{g} \mid e^{\text{ad}(\bar{Q})} \Lambda_j \right) = \left( e^{-\text{ad}(\bar{Q})} d^\mathbf{g} \mid \Lambda_j \right),$$

where $e^{-\text{ad}(\bar{Q})} d^\mathbf{g}$ lies in $\mathfrak{g} \oplus \mathbb{C} d^\mathbf{g}$, we only need to show that the relation

$$\left( X \mid \Lambda_j \right) = 0$$  \hspace{1cm} (4.45)

holds true for any $X \in \mathfrak{g} \oplus \mathbb{C} d^\mathbf{g}$ and $j \in J_+ \setminus J_+$. In fact, it is easy to see that $(d^\mathbf{g} \mid \Lambda_j) = 0$, and that $(X \mid \Lambda_j) = 0$ for any $X$ lying in $[\Lambda, \mathfrak{g}]$ or in $\mathcal{H}$ (note $1 \notin J_+ \setminus J_+$). Thus (4.45) follows from the decomposition (4.8) of $\mathfrak{g}$. The lemma is proved.  \hfill $\square$
Lemma 4.8 The Hamiltonians densities

\[ h_j(Q) = - (\Lambda_j \mid H(Q)), \quad \bar{h}_j(Q) = - (\bar{\Lambda}_j \mid \bar{H}(\bar{Q}))^- \]

of the Drinfeld-Sokolov hierarchies associated to \((\mathfrak{g}, \mathfrak{s}, \mathfrak{l})\) and to \((\bar{\mathfrak{g}}, \bar{\mathfrak{s}}, \bar{\mathfrak{l}})\) satisfy the following relations:

\[ h_j(Q) = \begin{cases} \frac{1}{\kappa \sqrt{\mu}} \bar{h}_j(Q), & j \in \bar{J}_+; \\ 0, & j \in J_+ \setminus \bar{J}_+. \end{cases} \quad (4.46) \]

Proof: For \( j \in \bar{J}_+ \), by using (4.16) and (4.17), we obtain

\[ \bar{h}_j(Q) = ( -\bar{\Lambda}_j \mid \bar{H}(\bar{Q}))^- = \kappa \sqrt{\mu} ( -\Lambda_j \mid H(Q) ) = \kappa \sqrt{\mu} h_j(Q). \quad (4.47) \]

For \( j \in J_+ \setminus \bar{J}_+ \), Lemma 4.7 implies that \( H(Q) \) belongs to \( \bar{g} \), then \( h_j(Q) \) vanishes due to (4.45). Thus the lemma is proved. \( \square \)

Proof of Theorem 4.4 Suppose \( Q^V \) is a gauge of \( Q \in C^\infty(\mathbb{R}, \mathcal{B}) \), then \( Q^\bar{V} = Q^V \mid B \) taking value in \( \bar{V} = V \cap \bar{g} \) is a gauge of \( Q^\mid B \). Let \( \mathcal{L}^V \) and \( \mathcal{L}^\bar{V} \) be the corresponding operators defined in (3.22) and (4.41).

By using Lemma 4.7, together with the decompositions (3.20) and (4.41) of the Borel subalgebra \( \mathcal{B} \), the equations (4.23) are recast to

\[
\frac{\partial \mathcal{L}^\bar{V}}{\partial t_j} = \left[ -(e^{ad_{U(Q^\bar{V})}\bar{\Lambda}_j})_{\geq 0} + R(\bar{Q}^\bar{V}, \bar{\Lambda}_j), \mathcal{L}^\bar{V} \right]
= \sqrt{\mu} \left[ -(e^{ad_{U(Q^V\mid B)}\Lambda_j})_{\geq 0} + R(Q^V\mid B, \Lambda_j), \frac{d}{dx} + \Lambda + Q^V\mid B \right]
= \sqrt{\mu} \frac{\partial \mathcal{L}^V}{\partial t_j} \bigg|_{\mathcal{R}} , \quad j \in \bar{J}_+.
\]

(4.48)

So the first assertion of Theorem 4.4 is proved.

Now let us proceed to show the second assertion of Theorem 4.4. We represent the coordinates of \( V \) as \( u = (\bar{u}, \bar{u}) \), where

\[ \bar{u} = \{ u_i \mid \sigma^* u_i = u_i \}, \quad \bar{u} = \{ u_i \mid \sigma^* u_i \neq u_i \}. \quad (4.49) \]

Recall the Hamiltonian representation (3.33) of the flows of the Drinfeld-Sokolov hierarchy. For any \( j \in J_+ \setminus \bar{J}_+ \), according to Proposition 4.3 and Lemma 4.8, the Hamiltonian densities \( h_j = h_j(Q^V) \in \mathcal{R} \) satisfy the relations

\[ \sigma^* h_j = h_j, \quad h_j \mid \mathcal{R} = 0. \]

They imply that each monomial term of the differential polynomial \( h_j \) contain factors in \( \bar{u} \) and their \( x \)-derivatives of total degree no less than 2, hence the variational derivatives of \( \mathcal{H}_j = \int h_j dx \) satisfy

\[
\frac{\delta \mathcal{H}_j}{\delta u_m(x)} \bigg|_{\mathcal{R}} = \sum_{k \geq 0} \left( \frac{\partial}{\partial x} \right)^k \frac{\partial h_j}{\partial u_m^{(k)}} \bigg|_{\mathcal{R}} = 0, \quad m = 1, 2, \ldots, \ell.
\]
Thus we have \((\partial u / \partial t_j)\mid_{\bar{R}} = 0\) for \(j \in J_\sigma^+ \setminus \bar{J}_+.\) The theorem is proved.

With the notations introduced in (4.49), the Drinfeld-Sokolov hierarchy (3.30) associated to \((g, s, 1)\) can be represented as:

\[
\frac{\partial \Bar{u}}{\partial t_j} = \Bar{X}(\Bar{u}, \hat{u}), \quad \frac{\partial \hat{u}}{\partial t_j} = \hat{X}(\bar{u}, \hat{u}), \quad j \in J_+, \tag{4.50}
\]

where \(\Bar{X}, \hat{X} \in \mathcal{R}\). The \(\Gamma\)-reduction theorem means that

\[
\frac{\partial \Bar{u}}{\partial \bar{t}_j} \bigg|_{\bar{R}} = \sqrt{\mu} \Bar{X}(\Bar{u}, 0), \quad \frac{\partial \hat{u}}{\partial \bar{t}_j} \bigg|_{\bar{R}} = \sqrt{\mu} \hat{X}(\hat{u}, 0) = 0, \quad j \in \bar{J}_+, \tag{4.51}
\]

and that the reduced flows form the Drinfeld-Sokolov hierarchy associated to \((\Bar{g}, \Bar{s}, 1)\).

### 4.4 Reduction properties of tau functions

We proceed to study properties of solutions of the Drinfeld-Sokolov hierarchies, and in particular we will concentrate on solutions of formal series in the time variables. Given any solution \(u(t) = (u_1(t), u_2(t), \ldots, u_\ell(t))\) of the Drinfeld-Sokolov hierarchy (3.26) (or (3.30) ) associated to \((g, s, 1)\) such that each \(u_i(t) \in S := \mathbb{C}[t]\) with \(t = \{t_j\}_{j \in J_+}\), there is an action on it induced by (4.18) such that

\[
\sigma^* : u(t) \mapsto (\epsilon_1 u_1(t), \ldots, \epsilon_\ell u_\ell(t)). \tag{4.52}
\]

In general, for any differential polynomial \(f(u) \in \mathcal{R},\) we assign

\[
\sigma^* : f(u(t)) \mapsto f(\sigma^*(u(t))). \tag{4.53}
\]

Furthermore, in consideration of the last item of Proposition 4.3, let \(\sigma^* t\) be defined by

\[
\sigma^* t_j = \zeta_j t_j, \quad j \in J_+, \tag{4.54}
\]

then one sees that

\[
\hat{u}(\sigma^* t) := \sigma^*(u(t)) \tag{4.54}
\]

is also a solution of the Drinfeld-Sokolov hierarchy (at \(\sigma^* t\)). Indeed, by using Proposition 4.3 one has, for any \(j \in J_+,\)

\[
\frac{\partial \hat{u}_j(\sigma^* t)}{\partial (\sigma^* t_j)} = \frac{1}{\zeta_j} \frac{\partial \sigma^*(u_j(t))}{\partial t_j} = \sigma^* \left( \frac{\partial u_j(t)}{\partial t_j} \right) = X_j^\sigma(\sigma^*(u(t))) = X_j^\sigma(\hat{u}(\sigma^* t)). \tag{4.55}
\]

For the solution \(\hat{u}(\sigma^* t),\) let \(\tilde{\tau}^s(\sigma^* t)\) be the tau function defined by (3.39), and we write formally

\[
\sigma^* (\log \tau^s(t)) := \log \tilde{\tau}^s(\sigma^* t). \tag{4.56}
\]

**Proposition 4.9** The following equality holds true (up to a linear function in \(t_j\)):

\[
\sigma^* (\log \tau^s(t)) = \log \tau^s(t). \tag{4.56}
\]
Proof: We only need to compare the second-order derivatives of both sides of (4.56). In fact, by using Proposition 4.3, for any \( j, k \in J_+ \) we have

\[
\frac{\partial^2 \sigma^*(\log \tau^s(t))}{\partial t_j \partial t_k} = \zeta_j \zeta_k \frac{\partial^2 \log \tilde{\tau}^s(\sigma^*t)}{\partial (\sigma^*t_j) \partial (\sigma^*t_k)} = \zeta_j \zeta_k \Omega^s_{jk}(u)|_{u=\sigma^*(u(t))} = \zeta_j \zeta_k \sigma^* \left( \Omega^s_{jk}(u) \right)|_{u=u(t)} = \frac{\partial^2 \log \tau^s(t)}{\partial t_j \partial t_k}. 
\]

Thus the proposition is proved. \( \square \)

The proposition implies that \( \tilde{\tau}^s(t) = \tau^s(t) \) if and only if

\[
\log \tau^s(\sigma^*t) = \log \tau^s(t). 
\]

(4.57)

**Definition 4.10** The tau function \( \tau^s(t) \) for a certain solution \( u(t) \) of the Drinfeld-Sokolov hierarchy (4.23) associated to \((\mathfrak{g}, s, \mathfrak{l})\) is called \( \Gamma \)-invariant if it satisfies (4.57).

Now let us consider the tau functions \( \tilde{\tau}^s \) of the Drinfeld-Sokolov hierarchy (4.23) associated to \((\mathfrak{g}, \mathfrak{s}, \mathfrak{l})\) which is defined by the formulae

\[
\frac{\partial^2 \log \tilde{\tau}^s}{\partial t_i \partial t_j} = \frac{1}{\hbar^s} \left( d^s \mid \left( e^{\text{ad} u(Q^i)} \Lambda_i \right)_{i \geq 0} , e^{\text{ad} u(Q^j)} \Lambda_j \right)^-, \quad i, j \in \tilde{J}_+ 
\]

with

\[
\hbar^s = (d^s \mid \tilde{c})^- = \kappa(d^s \mid \mu c) = \kappa \mu h^s. 
\]

(4.59)

For the purpose of clarifying the relation between \( \tilde{\tau}^s \) and \( \tau^s \), let us introduce the following notations:

\[
\tilde{t} = \{t_j\}_{j \in \tilde{J}_+}, \quad \hat{t} = \{t_j\}_{j \in J_+ \setminus \tilde{J}_+}, \quad \check{t} = \{\check{t}_j\}_{j \in \check{J}_+}. 
\]

(4.60)

**Theorem 4.11** Let the affine Kac-Moody algebras \( \mathfrak{g} \) and \( \mathfrak{g} \) be given as in Tables 1 and 2. Suppose a tau function \( \tau^s(t) \), satisfying the condition \( \log \tau^s(t) \in \mathcal{S} \), of the Drinfeld-Sokolov hierarchy (4.26) associated to \((\mathfrak{g}, \mathfrak{s}, \mathfrak{l})\) is \( \Gamma \)-invariant, then the function \( \tilde{\tau}^s(\check{t}) \) defined by

\[
\log \tilde{\tau}^s(\check{t}) = \frac{1}{\mu} \log \tau^s(t) \bigg|_{t=0; \check{t} = \sqrt{\mu} \check{t}} 
\]

(4.61)
is a tau function of the Drinfeld-Sokolov hierarchy (4.23) associated to \((\mathfrak{g}, \mathfrak{s}, \mathfrak{l})\).

Proof of Theorem 4.11 Since \( \tau^s(t) \) is \( \Gamma \)-invariant, it follows from (3.40) that the Hamiltonian densities \( h_j \) with \( j \in J_+ \) satisfy

\[
\begin{align*}
\hat{h}_j(u(t))|_{t=0} & = \frac{h_j}{j} \frac{\partial \log \tau^s(t)}{\partial x \partial t_j} \bigg|_{t=0} = \frac{h_j}{j} \frac{\partial \log \tilde{\tau}^s(t)}{\partial x \partial t_j} \bigg|_{t=0} = h_j(u(t))|_{t=0} = \sigma^* h_j(u((\sigma^*)^{-1}t))|_{t=0} \\
& = \frac{1}{\zeta_j} h_j(u((\sigma^*)^{-1}t))|_{t=0} = \frac{1}{\zeta_j} h_j(u(t))|_{t=0}, 
\end{align*}
\]

(4.62)
where the last two equalities follow from the first item of Proposition 4.3 and the fact that $\sigma^* \tilde{t} = \tilde{t}$. For any case listed in Tables 1 and 2 we have $\zeta_j \neq 1$ whenever $j \in J_+ \setminus \tilde{J}_+$, hence

$$h_j(u(t))|_{t=0} = 0, \quad j \in J_+ \setminus \tilde{J}_+. \quad (4.63)$$

Let us recall the fact, as we explained in Remark 3.11, that the Hamiltonian densities $h = (h_{m_1}, h_{m_2}, \ldots, h_{m_\ell})$, where $m_1, m_2, \ldots, m_\ell$ are the first $\ell$ exponents in $J_+$, are related to $u = (u_1, u_2, \ldots, u_\ell)$ by a Miura-type transformation. Hence, we can represent $u$ in terms of $h$ as follows:

$$u_i = g_i(h) \in C^\infty(h[[h', h'', \ldots]]), \quad i = 1, \ldots, \ell. \quad (4.64)$$

Moreover, such a representation must be compatible with the action of $\sigma^*$, namely, we have

$$\sigma^* u_i = g_i(\sigma^* h).$$

By using (4.52), (4.63) and that $\sigma^* h_j = h_j$ for $j \in \tilde{J}_+$, we have

$$\epsilon_i u_i(t)|_{t=0} = \sigma^*(u_i(t))|_{t=0} = g_i(\sigma^* h(u(t)))|_{t=0} = g_i(h(u(t)))|_{t=0} = u_i(t)|_{t=0}. \quad (4.65)$$

It follows from (4.39) that $\tilde{u}_i|_{t=0} = 0$, hence $\tilde{u}_i|_{t=0}$ satisfies the equations given in (4.51), which means that $\tilde{u}_i|_{t=0}, \tilde{t}_i = \sqrt{\mu} t$ is a solution of the Drinfeld-Sokolov hierarchy associated to $(\tilde{g}, \tilde{s}, \tilde{f})$. In other words, these solutions of the Drinfeld-Sokolov hierarchies associated to $(\tilde{g}, \tilde{s}, \tilde{f})$ and to $(g, s, f)$ are related by $Q^\tilde{v} = Q^v|_{t=0, \tilde{t}_i = \sqrt{\mu} t}$. Consequently, by using Lemma 4.17 and (4.58) we have, for $i, j \in \tilde{J}_+$, that

$$\frac{\partial^2 \log \tilde{s}^8}{\partial \tilde{t}_i \partial \tilde{t}_j} = \frac{\kappa \mu}{\kappa \mu h^8} \left( d^s \left| \left[ e^{ad_{U(Q^v)}} \Lambda_j \right] \right| \right)_{t=0, \tilde{t}_i = \sqrt{\mu} t} = \frac{1}{h^s} \left( d^s \left| \left[ e^{ad_{U(Q^v)}} \Lambda_j \right] \right| \right)_{t=0, \tilde{t}_i = \sqrt{\mu} t}$$

$$= \frac{\partial^2 \log s^8}{\partial t_i \partial t_j} \bigg|_{t=0, \tilde{t}_i = \sqrt{\mu} t}.$$

Therefore, in consideration of $\partial/\partial t_j = \sqrt{\mu} \partial/\partial \tilde{t}_j$, we complete the proof of the theorem. \hfill \Box

As a conclusion, from Theorems 4.4 and 4.11 we arrive at Theorem 1.3.

**Example 4.12** Let us compute some explicit solutions of the Drinfeld-Sokolov hierarchy associated to $(g = \mathbb{A}_2^{(1)}, s = (1, 0, 1), f)$. The first few flows are given in Example 4.5 with unknown functions $u(t), v(t)$. Note that every solution is determined by its initial value $(u(t), v(t))|_{t>0} = (u^0(x), v^0(x))$. We take $u^0(x) = ax$ and $v^0(x) = b$ with arbitrary constants $a$ and $b$, and we arrive at the following result:

$$u(t)|_{t>0} = 0$$

$$= ax + 2 abst_2 + \frac{4}{3} t_4 (3a^2 bx - ab^3) - \frac{5}{9} t_5 (a^3 x^2 - 18a^2 b^2 x - 3ab^4) - \frac{1}{3} a^2 t_2^2$$

$$- \frac{4}{27} t_4^2 \left( 5a^4 x^2 - 126a^3 b^2 x + 45a^2 b^4 \right) + \frac{25}{81} t_5^2 \left( 2a^5 x^3 - 114a^4 b^2 x^2 + a^4 + 246a^3 b^4 x + 42a^2 b^6 \right)$$

$$- \frac{80}{9} t_5 t_2 (a^3 bx - 2a^2 b^3) - \frac{4}{3} t_4 t_2 (a^3 x - 7a^2 b^2) - \frac{20}{27} t_4^2 t_5 (19a^4 b x^2 - 122a^3 b^3 x + 3a^2 b^5) + \text{h.o.t.}.$$
\[ v(t)_{t_i>0} = -b \frac{2}{3}a^2t_2 - \frac{2}{9}t_4 \left( a^2x - 3ab^2 \right) - \frac{10}{9}t_5 \left( a^2bx + ab^3 \right) - \frac{32}{27}t_4 \left( a^3bx - a^2b^3 \right) \]
\[ + \frac{50}{81}t_2 \left( 3a^4bx - 10a^3b^3x + 3a^2b^5 \right) - \frac{8}{9}a^2bt_2t_4 + \frac{10}{27}t_2 t_5 \left( a^3x - 3a^2b^2 \right) \]
\[ + \frac{10}{27}t_4 t_5 \left( a^4x^2 - 22a^3b^2x - 7a^2b^4 \right) + \text{h.o.t.,} \]

where ‘h.o.t.’ means higher order terms in \( t_2, t_4, t_5 \). The tau function reads

\[
\log \tau^a(t)_{t_i>0} = -\frac{b^2x^2}{2} + \frac{ax^3}{18} + t_2 \left( \frac{2}{3}abx^2 + \frac{4b^3x}{3} \right) + t_4 \left( \frac{8}{27}a^2bx^3 - \frac{8}{9}ab^3x^2 - \frac{8b^5x}{3} \right) \]
\[ + t_5 \left( -\frac{5a^3x^4}{324} + \frac{25}{27}a^2b^2x^3 + \frac{25}{18}ab^4x^2 + \frac{35b^6x}{9} \right) + \frac{1}{2}t_2^2 \left( -\frac{2a^2x^2}{9} + \frac{4}{3}abx - 2b^4 \right) \]
\[ + \frac{1}{2}t_4 \left( -\frac{5a^5x^5}{243} - \frac{25}{9}a^4b^2x^4 + \frac{950}{81}a^3b^4x^3 + x \left( -\frac{350a^3b^2}{27} - \frac{575ab^6}{9} \right) \right) \]
\[ + \frac{1}{2}t_5 \left( -\frac{25a^4}{243} - \frac{50a^2b^6}{81} - \frac{245b^{10}}{9} \right) + t_2 t_5 \left( -\frac{100}{27}a^3b^3x + \frac{20}{27}a^2b^3x^2 - \frac{20}{9}ab^5x - \frac{20b^7}{3} \right) \]
\[ + t_4 t_5 \left( -\frac{8a^3x^3}{9} + \frac{5}{3}a^2b^2x^2 + \frac{8}{9}ab^4x + \frac{40b^6}{9} \right) + t_4 t_5 \left( -\frac{40}{81}a^4b^3x + x \left( \frac{320ab^7}{27} - \frac{20a^3b^3}{27} \right) \right) \]
\[ + \frac{640}{81}a^3b^3x^3 + \frac{80}{27}a^2b^5x^2 + \frac{20a^2b^3}{9} + \frac{1400b^5}{81} + \text{h.o.t.} \]

Observe that \( \sigma^*(u(t), v(t)) = (u(t), -v(t)) \) can be realized by the replacement:

\[ b \mapsto -b, \quad t_j \mapsto -t_j \quad \text{with} \quad j \in \{ 6\mathbb{Z}_+ + \{ 2, 4 \} \}, \]

and moreover, such a replacement does not change the tau function \( \tau^a(t) \). This observation agrees with Proposition 4.9.

On the other hand, such a tau function \( \tau^a(t) \) is \( \Gamma \)-invariant if and only if \( b = 0 \). In this case, the tau function of the Drinfeld-Sokolov hierarchy associated to \( (\mathfrak{g} = A_2^{(2)}, \mathfrak{s} = (1, 0), 1) \), with the \( t_5 \) flow (4.35), is given by (4.61), i.e. it satisfies the relation (with \( x = \sqrt{2}t_1 \))

\[
\log \tau^a(t)_{t_5>0} = \log \tau^a(t)_{b=0, t_5=\sqrt{2}t_5, t_2=t_4=0, t_6=0} = \frac{ax^3}{18} - \frac{5a^3x^4}{162\sqrt{2}}t_5 + \left( \frac{5a^5x^5}{243} + \frac{25a^4x^2}{243} \right) t_5^2 + \text{h.o.t.} \]  
\[ (4.66) \]

Remark 4.13 Theorem 4.11 generalizes the second part of Theorem 1.1 to the affine Kac-Moody algebras listed in Table 2; however, such a result does not hold for the cases listed in Table 3. The reason is that, in the latter case, the equality (4.62) is not sufficient to derive \( h_{m_i} t_i=0 \) when \( m_i \in \mathfrak{J}_+ \setminus \mathfrak{J}_+ \). For instance, considering the Drinfeld-Sokolov hierarchy in Example 4.6, let us compute its solution \( (u, v, w) \) with the initial value \( (u, v, w)_{t_i=0} = (ax, b, c) \), where \( a, b, c \) are constants. We have

\[ u(t)_{t_i>0=0} \]
\[ v(t)\big|_{t>0=0} = b + abt_3 + \frac{10}{9} a^2 bxt_3 + \frac{1}{2} a^2 b t^2 + \frac{1}{2} t^2_5 \left( -\frac{200}{27} a^3 b^3 x + \frac{25 a^3 c}{108} + \frac{100 a^2 b^5}{9} - \frac{25}{27} a^2 b c^2 \right) + \frac{10}{9} a^3 b x t_5 + \text{h.o.t.}, \]

\[ w(t)\big|_{t>0=0} = c - act_3 + \frac{a^2}{18} t_5 \left( \frac{5 a^2 b}{9} + \frac{5 a^2 c x}{3} - \frac{5 a b c}{3} \right) + \frac{1}{2} a^2 c t_3 + \frac{1}{2} t^2_5 \left( -\frac{25}{81} a^4 c x^2 - \frac{275 a^3 b^3}{27} - \frac{25}{9} a^2 b^4 c \right) + \frac{25 a^2 c^3}{27} + x \left( -\frac{25 a^4 b}{27} - \frac{50}{27} a^3 b^2 c \right) + t_5 t_3 \left( \frac{5 a^3 b}{9} - \frac{5}{9} a^3 c x - \frac{5}{3} a^2 b^2 c \right) + \text{h.o.t.;} \]

\[ \log \tau^8(t)\big|_{t>0=0} = -\frac{b^2 x^2}{2} + \frac{a x^3}{18} + t_3 \left( -\frac{a b^2 x^2 - c^2 x}{2} \right) + t_5 \left( -\frac{5}{324} a^3 x^4 - \frac{5}{9} a^2 b^2 x^3 + x^2 \left( \frac{5 a b^4}{6} + \frac{5 a c^2}{36} \right) \right) + \frac{1}{2} \left( -\frac{5 a b c}{3} - \frac{5 b^6}{3} + \frac{5 b^2 c^3}{2} \right) + \frac{1}{2} \left( -\frac{25 a^4 b^2 x^4}{27} + \frac{50}{27} a^3 b^4 x^3 + x \left( \frac{25 a^3 b^2}{81} + \frac{25 a b^8}{3} - \frac{50}{9} a b^4 c^2 + \frac{25 a c^4}{108} \right) \right) - \frac{125 a^2 c^2}{648} + x^2 \left( -\frac{25 a^4}{243} - \frac{50}{27} a^3 b^6 + \frac{50}{27} a^2 b^2 c^2 \right) - \frac{50 b^4}{36} - \frac{25 b^2 c^4}{36} + t_3 t_5 \left( \frac{10 a^2 b^2}{3} - \frac{5 a^2 c^2}{18} - \frac{5 a^2 b^2}{6} - \frac{10 a b^4 x}{6} - \frac{5 b^4 c^2}{2} - \frac{5 c^4}{24} \right) + \text{h.o.t.} \]

Note that \( \sigma^* t_j = t_j \) for all \( j \), so \( \tau^8(t) \) is automatically \( \Gamma \)-invariant. However, the formula \((4.61)\) does not give a tau function of the hierarchy associated to \((\mathfrak{g} = A_2^{(2)}, \mathfrak{s} = (1, 0), \mathbb{I})\), since the reduced tau function and the associated function \( u \) still depends on the parameters \( b \) and \( c \). In fact, if we take \( b = c = 0 \), then \( v(t) = w(t) = 0 \), hence the function \( \tau^8 \) given by

\[ \log \tau^8(t)\big|_{t>0=0} = \log \tau^8(t)|_{b=c=0, t_5=\sqrt{2}, t_3=0, t>0=0} = \frac{a x^3}{18} - \frac{5 a^3 x^4}{162 \sqrt{2}} t_5 + \left( \frac{5 a^3 x^5}{243} + \frac{25 a^4 x^2}{243} \right) t^2 + \text{h.o.t.}, \]

is a tau function for the Drinfeld-Sokolov hierarchy associated to \((\mathfrak{g} = A_2^{(2)}, \mathfrak{s} = (1, 0), \mathbb{I})\), and it coincides with the one given in \((4.66)\).
5 Concluding remarks

In this paper, we prove a $\Gamma$-reduction theorem for the Drinfeld-Sokolov hierarchies associated to affine Kac-Moody algebras which admit diagram automorphisms. We expect that this result is useful to the study of the FJRW theory.

In [13], it was shown that the partition function of the FJRW theory of BCFG types can be obtained from that of the FJRW theory of ADE types, by using the fact that the string equation of the corresponding Drinfeld-Sokolov hierarchy and the integrable hierarchy itself uniquely determine the topological solution, which is exactly the partition function of the corresponding cohomological field theory. The topological solution of the ADE and BCFG Drinfeld-Sokolov hierarchies can be computed explicitly, for instance, via an algorithm proposed in [2]. However, the Drinfeld-Sokolov hierarchy associated to a twisted affine Kac-Moody algebra does not have an analogue of the string equation, so one can not use it to pick up a particular solution. Actually, there is no definition for the topological solution in the twisted cases.

For the untwisted cases, the string equation $L_{-1} \tau = 0$ is the first one of the Virasoro constraints. In the twisted cases, the Virasoro constraints

$$L_m \tau = 0, \quad m \geq 0$$

start from the zeroth one. One can show that these constraints together with the integrable hierarchy itself determine a collection of solutions, which are parametrized by a finite-dimensional space. We observe that these solutions possess certain Painlevé properties, and conjecture that there exists a certain affine Weyl group action on this space. We hope that these observations will be useful to the construction of the FJRW theory corresponding to twisted affine Kac-Moody algebras. This problem will be studied in a separate publication.

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Appendix A Proof of the Serre relations for $\bar{\mathfrak{g}}$

Let $\mathfrak{g}$ be the derived algebra of the Kac-Moody algebra associated to an affine generalized Cartan matrix $(a_{ij})_{i,j=0,\ldots,\ell}$. Suppose

$$\{e_i, f_i, \alpha_i^\vee \mid i = 0, \ldots, \ell\}$$

is a collection of Chevalley generators of $\mathfrak{g}$, then they satisfy the following Serre relations

$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [e_i, f_j] = \delta_{ij} \alpha_i^\vee, \quad [\alpha_i^\vee, e_j] = a_{ij} e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij} f_j,$$

$$(\text{ad}_{e_i})^{1-a_{ij}} (e_j) = 0, \quad (\text{ad}_{f_i})^{1-a_{ij}} (f_j) = 0.$$

Suppose $\bar{\sigma}$ is a diagram automorphism of $A$, i.e. a bijection

$$\bar{\sigma} : \{0, \ldots, \ell\} \rightarrow \{0, \ldots, \ell\}$$

such that

$$a_{ij} = a_{\bar{\sigma}(i) \bar{\sigma}(j)},$$
then one can define a Lie-algebra automorphism $\sigma : g \to g$ such that

$$
\sigma (e_i) = e_{\sigma(i)}, \quad \sigma (f_i) = f_{\sigma(i)}, \quad \sigma (a_i^\vee) = a_{\sigma(i)}^\vee.
$$

For an index $i \in \{0, \ldots, \ell\}$, we denote its $\sigma$-orbit by $\langle i \rangle$, and $N_i = \# \langle i \rangle$, then define

$$
b_i = 3 - \sum_{j \in \langle i \rangle} a_{ij}.
$$

If we choose another representative $i' = \sigma^m(i) \in \langle i \rangle$, then

$$
b_i' = 3 - \sum_{j \in \langle i' \rangle} a_{ij'} = 3 - \sum_{j \in \langle \sigma^m(i) \rangle} a_{\sigma^m(i)j} = 3 - \sum_{\sigma^{-m}(j) \in \langle i \rangle} a_{i\sigma^{-m}(j)} = b_i,
$$

so $b_i$ is independent of the choice of the representative $i \in \langle i \rangle$.

It is easy to see that $b_i$’s are integers satisfying $b_i \geq 1$. Following [8], we say that $\sigma$ satisfies the linking condition if

$$
b_i \leq 2, \quad \forall i \in \{0, \ldots, \ell\}.
$$

Note that in affine cases, all diagram automorphisms satisfy this condition, except the diagram automorphism of $A^{(1)}_\ell$ with $\sigma(i) \equiv i + 1 \pmod{\ell + 1}$ or its power. We assume that $\sigma$ satisfies the linking condition from now on.

For each orbit $\langle i \rangle$, define

$$
E_i = \sum_{k \in \langle i \rangle} e_k, \quad F_i = b_i \sum_{k \in \langle i \rangle} f_k, \quad H_i = b_i \sum_{k \in \langle i \rangle} a_k^\vee,
$$

(A.1)

and define

$$
A_{ij} = b_i \sum_{k \in \langle i \rangle} a_{kj}
$$

(A.2)

for two orbits $\langle i \rangle, \langle j \rangle$. Then it is easy to see that these elements of $g$ and integers are also independent of the choice of the representative $i \in \langle i \rangle$ or $j \in \langle j \rangle$.

Let $I$ be the set of all orbits $I = \{\langle i \rangle \mid i \in \{0, \ldots, \ell\}\}$. It is shown in [8] that $(A_{ij})_{i,j \in I}$ is also an affine generalized Cartan matrix. In this appendix, we show that the elements $\{E_i, F_i, H_i\}_{i \in I}$ of $g$ satisfy the Serre relations associated to this generalized Cartan matrix.

**Lemma A.1** The elements $\{E_i, F_i, H_i\}_{i \in I}$ of $g$ satisfy

$$
[H_i, H_j] = 0,
$$

$$
[E_i, F_j] = \delta_{ij} H_i,
$$

$$
[H_i, E_j] = A_{ij} E_j,
$$

$$
[H_i, F_j] = -A_{ij} F_j,
$$

Proof: We only prove the third one:

$$
[H_i, E_j] = \left[ b_i \sum_{k \in \langle i \rangle} a_k^\vee, \sum_{l \in \langle j \rangle} e_l \right] = b_i \sum_{k \in \langle i \rangle} \sum_{l \in \langle j \rangle} a_{kl} e_l
$$

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\[ b_i \sum_{\alpha=0}^{N_i-1} \sum_{\beta=0}^{N_j-1} a_{\sigma^\alpha(i)\sigma^\beta(j)} e_{\sigma^\beta(j)} = \sum_{\beta=0}^{N_j-1} \left( b_i \sum_{\alpha=0}^{N_i-1} a_{\sigma^\alpha-\beta(i)j} \right) e_{\sigma^\beta(j)} \]

\[ = A_{ij} \sum_{\beta=0}^{N_j-1} e_{\sigma^\beta(j)} = A_{ij} E_j. \]

Proofs for the other three identities are similar, so we omit them here. \[\square\]

**Lemma A.2** If \( b_i = 1 \), then we have

\[ (\text{ad}_{E_i})^{1-A_{ij}} (E_j) = 0, \quad (\text{ad}_{F_i})^{1-A_{ij}} (F_j) = 0. \]

**Proof:** We only prove the first one, since the second one is similar. The condition \( b_i = 1 \) means that \( a_{ij} = 0 \) for all \( j \in \langle i \rangle \) and \( j \neq i \), then it is easy to see that \([e_j, e_k] = 0\) and \([\text{ad}_{e_j}, \text{ad}_{e_k}] = 0\) for all \( j, k \in \langle i \rangle \).

For an arbitrary positive integer \( m \), we have

\[
(\text{ad}_{E_i})^m = \left( \sum_{k \in \langle i \rangle} \text{ad}_{e_k} \right)^m = \left( \sum_{\alpha=0}^{N_i-1} \text{ad}_{e_{\sigma^\alpha(i)}} \right)^m = \sum_{(d_1, \ldots, d_{N_i})} \frac{m!}{d_1! \cdots d_{N_i}!} \left( \text{ad}_{e_{\sigma^0(i)}} \right)^{d_1} \cdots \left( \text{ad}_{e_{\sigma^{N_i-1}(i)}} \right)^{d_{N_i}},
\]

where the summation is taken over integer tuples \((d_1, \ldots, d_{N_i})\) satisfying

\[ d_1, \ldots, d_{N_i} \geq 0, \quad d_1 + \cdots + d_{N_i} = m. \]

On the other hand,

\[
(\text{ad}_{E_i})^m (E_j) = (\text{ad}_{E_i})^m \left( \sum_{l \in \langle j \rangle} e_l \right) = \sum_{\beta=0}^{N_j-1} \sigma^\beta ((\text{ad}_{E_i})^m (e_j)),
\]

so we only need to show that \((\text{ad}_{E_i})^m (e_j) = 0\).

If \( d_k (k = 1, \ldots, N_i) \) satisfies \( d_k \geq 1 - a_{\sigma^{k-1}(i)j} \), then

\[
(\text{ad}_{e_{\sigma^{k-1}(i)}})^{d_k} (e_j) = 0,
\]

so we do not need to consider the tuples \((d_1, \ldots, d_{N_i})\) with such kind of \( d_k \)'s in the summation \([A.3]\) for \((\text{ad}_{E_i})^m (e_j)\).

If all \( d_k \)'s satisfy \( d_k \leq -a_{\sigma^k(i)j} \), then

\[
m = d_1 + \cdots + d_{N_i} \leq - \sum_{\alpha=0}^{N_i-1} a_{\sigma^\alpha(i)j} = -A_{ij}.
\]

So, when \( m \geq 1 - A_{ij} \), we have \((\text{ad}_{E_i})^m (e_j) = 0\). The lemma is proved. \[\square\]
If \( b_i = 2 \), there is exactly one index \( k \in \langle i \rangle \) such that \( a_{ik} = -1 \). Actually, in all the affine cases satisfying the linking condition, the automorphism \( \bar{\sigma} \) with some \( b_i = 2 \) must be an involution, i.e. \( \langle i \rangle = \{ i, \bar{\sigma}(i) \} \). We denote

\[
X = \text{ad}_{e_i}, \quad Y = \text{ad}_{e_{\bar{\sigma}(i)}}, \quad Z = [X, Y],
\]

then \([X, Z] = [Y, Z] = 0\). Denote by

\[
e_j = v, \quad p = -a_{ij}, \quad q = -a_{\bar{\sigma}(i)j},
\]

then we have \( X^{p+1}(v) = 0, Y^{q+1}(v) = 0 \). The Serre relation

\[
(\text{ad}_{E_j})^{1-A_{ij}}(E_j) = 0
\]

is equivalent to

\[
(X + Y)^{2p+2q+1}(v) = 0.
\]

So the problem is reduced to the following conjecture:

**Conjecture A.1** Let \( V \) be a linear space over a field \( k \) with \( \text{char} k = 0 \). Suppose \( X, Y, Z \in \text{End} (V) \) satisfy

\[
Z = [X, Y], \quad [X, Z] = [Y, Z] = 0,
\]

and \( v \in V, p, q \in \mathbb{N} \) satisfy

\[
X^{p+1}(v) = 0, \quad Y^{q+1}(v) = 0,
\]

then we have

\[
(X + Y)^{2p+2q+1}(v) = 0.
\]

**Lemma A.3** Conjecture A.1 is correct when \( p = 0 \) or \( q = 0 \).

**Proof:** Assume that \( q = 0 \), i.e. \( Y(v) = 0 \). The Baker-Campbell-Hausdorff formula implies that

\[
e^{t(X+Y)}(v) = e^{\frac{t^2}{2}Z} e^{tX}(v).
\]

We need to prove that the left hand side is actually a polynomial in \( t \) with degree \( \leq 2p \). The right hand side reads

\[
e^{\frac{t^2}{2}Z} e^{tX}(v) = \sum_{k,l \geq 0} \frac{t^{2k+l}}{k!l!} 2^k Z^k \left( X^l(v) \right).
\]

We are to show that, when \( 2k + l > 2p \), \( Z^k \left( X^l(v) \right) = 0 \).

By using the identity \([X^l, Y] = lX^{l-1}Z\), we have

\[
Z^k X^l(v) = X^l Z^{k-1}(XY - YX)(v) = -X^l Y X Z^{k-1}(v) = -lZ^k X^l(v) - Y X^{l+1} Z^{k-1}(v),
\]

which implies that

\[
Z^k X^l(v) = -\frac{1}{l+1} Y X^{l+1} Z^{k-1}(v) = \cdots = \frac{(-1)^k}{(l+1) \cdots (l+k)} Y^k X^{k+l}(v).
\]

When \( 2k + l > 2p \), we have \( k + l > p \), so \( X^{k+l}(v) = 0 \). The lemma is proved. \( \square \)
Lemma A.4  Conjecture [A.1] is correct when \( p = q = 1 \).

Proof: First, we have

\[
(X + Y)^2(v) = (X^2 + XY + YX + Y^2)(v) = (XY + YX)(v),
\]

then

\[
(X + Y)^3(v) = (X^2Y + XYX + YXY + Y^2X)(v)
= ((XY^2 + 2ZX) + (YX + Z)X + (XY - Z)Y + (XY^2 - 2ZY))(v)
= 3Z(X - Y)(v).
\]

Next

\[
(X + Y)^4(v) = 3Z(X^2 + YX - XY - Y^2)(z) = -3Z^2(v),
\]

and

\[
(X + Y)^5(v) = -3Z^2(X + Y)(v).
\]

Note that

\[
Z^2X(v) = \frac{1}{2}Z[X^2, Y](v) = \frac{1}{2}ZX^2Y(v) = \frac{1}{4}X[X^2, Y]Y(v)
= -\frac{1}{4}XYX^2Y(v) = -\frac{1}{4}XY(X^2Y - Y^2X)(v)
= -\frac{1}{2}XYXZ(v) = \frac{1}{2}Z(XY - YX)X(v) = -\frac{1}{2}Z^2X(v),
\]

so we have \( Z^2X(v) = 0 \). Similarly, \( Z^2Y(v) = 0 \). The lemma is proved. \( \square \)

In all the affine cases satisfying the linking condition, either \( p = 0 \), \( q = 0 \), or \( p = q = 1 \) (since \( |A_{ij}| \leq 4 \)), so the Serre relation for the cases with \( b_i = 2 \) is proved.

Finally, when \( g \) is an affine Kac–Moody algebra of type \( X^{(r)}_\ell \) given in Tables 1–3, then one can check case by case that the generalized Cartan matrix \( (A_{ij})_{i,j} \in I \) is of type \( X^{(r)}_\ell \).

Appendix B  List of Affine Kac–Moody algebras and their subalgebras

In this appendix we use the same notations as those appeared in Sections 2 and 4.1. Let \( g \) be the derived subalgebra of an affine Kac-Moody algebra \( g(A) \) of type \( X^{(r)}_\ell \) (we write \( g = X^{(r)}_\ell \) for short) listed in Tables 1–3. For \( g(A) = g \oplus C\mu \), let \( J \) be the set of exponents, \( h \) be the Coxeter number, \( c \) be the canonical central element, and \( (\cdot \mid \cdot) \) be the standard invariant bilinear form. Similar notations for the subalgebras \( \tilde{g}, g^\sigma \subset g \) introduced in Subsection 4.2 are used. In particular, \( \tilde{g} \) is the derived subalgebra of the Affine Kac–Moody algebras \( g(\bar{A}) \) of type \( X^{(r)}_{\bar{\ell}} \) (\( \tilde{g} = X^{(r)}_{\bar{\ell}} \) for short), whose canonical central element and standard invariant bilinear form are given by

\[
\tilde{c} = \mu c, \quad (X \mid Y)^\kappa = \kappa(X \mid Y) \quad \text{for } X, Y \in \tilde{g}(\bar{A})
\]

with constants \( \mu \) and \( \kappa \). Such data are listed in what follows.
(a1) \[ g = A_{2n-1}^{(1)} \quad (n \geq 2), \]
\[ J = \{1, 2, 3, \ldots, 2n - 1\} + h\mathbb{Z}, \quad h = 2n, \]
• \[ \tilde{\sigma}(0) = 0, \quad \tilde{\sigma}(i) = 2n - i, \quad i = 1, 2, \ldots, 2n - 1, \]
• \[ \tilde{g} = C_n^{(1)}, \]
\[ \tilde{J} = \{1, 3, 5, \ldots, 2n - 1\} + \tilde{h}\mathbb{Z}, \quad \tilde{h} = 2n, \]
\[ \mu = 1, \quad \kappa = 1, \]
• \[ g^\sigma = \tilde{g}, \quad J^\sigma = \tilde{J} \]

(a2) \[ g = A_{2n}^{(1)} \quad (n \geq 1), \]
\[ J = \{1, 2, 3, \ldots, 2n\} + h\mathbb{Z}, \quad h = 2n + 1, \]
• \[ \tilde{\sigma}(i) = 2n - i, \quad i = 0, 1, 2, \ldots, 2n, \]
• \[ \tilde{g} = A_{2n}^{(2)}, \]
\[ \tilde{J} = \{1, 3, \ldots, 2n - 1, 2n + 3, 2n + 5, \ldots, 4n + 1\} + 2\tilde{h}\mathbb{Z}, \quad \tilde{h} = 2n + 1, \]
\[ \mu = 2, \quad \kappa = \frac{1}{3}, \]
• \[ g^\sigma = \tilde{g}, \quad J^\sigma = \tilde{J} \]

(a3) \[ g = A_{2n+1}^{(1)} \quad (n \geq 2), \]
\[ J = \{1, 2, 3, \ldots, 2n + 1\} + h\mathbb{Z}, \quad h = 2n + 2, \]
• \[ \tilde{\sigma}(i) = 2n - i + 1, \quad i = 0, 1, 2, \ldots, 2n + 1, \]
• \[ \tilde{g} = D_{n+1}^{(2)}, \]
\[ \tilde{J} = \{1, 3, 5, \ldots, 2n + 1\} + 2\tilde{h}\mathbb{Z}, \quad \tilde{h} = n + 1, \]
\[ \mu = 2, \quad \kappa = \frac{1}{3}, \]
• \[ g^\sigma = \tilde{g}, \quad J^\sigma = \tilde{J} \]

(a4) \[ g = A_{2n+1}^{(2)} \quad (n \geq 2), \]
\[ J = \{1, 3, 5, \ldots, 4n + 1\} + 2h\mathbb{Z}, \quad h = 2n + 1, \]
• \[ \tilde{\sigma}(0) = 1, \quad \tilde{\sigma}(1) = 0, \quad \tilde{\sigma}(i) = i, \quad i = 2, 3, \ldots, n + 1, \]
• \[ \tilde{g} = A_{2n}^{(2)}, \]
\[ \tilde{J} = \{1, 3, 5, \ldots, 2n - 1, 2n + 3, 2n + 5, \ldots, 4n + 1\} + 2\tilde{h}\mathbb{Z}, \quad \tilde{h} = 2n + 1, \]
\[ \mu = 1, \quad \kappa = 1, \]
• \[ g^\sigma \neq \tilde{g}, \quad J^\sigma = \tilde{J} \cup (2n + 1)\mathbb{Z}^{\text{odd}} \]

(b) \[ g = B_{n+1}^{(1)} \quad (n \geq 2), \]
\[ J = \{1, 3, 5, \ldots, 2n + 1\} + h\mathbb{Z}, \quad h = 2n + 2, \]
• \[ \tilde{\sigma}(0) = 1, \quad \tilde{\sigma}(1) = 0, \quad \tilde{\sigma}(i) = i, \quad i = 2, 3, \ldots, n + 1, \]
• \[ \tilde{g} = D_{n+1}^{(2)}, \]
\[ \tilde{J} = \{1, 3, 5, \ldots, 2n + 1\} + 2\tilde{h}\mathbb{Z}, \quad \tilde{h} = n + 1, \]
\[ \mu = 1, \quad \kappa = \frac{1}{3}, \]
• \[ g^\sigma = \tilde{g}, \quad J^\sigma = \tilde{J} \]
(c1) \( g = C_{2n}(1) \) \((n \geq 2)\),
\[ J = \{1, 3, 5, \ldots, 4n + 1\} + h\mathbb{Z}, \quad h = 4n, \]
\( \bar{\sigma}(i) = 2n - i, \quad i = 0, 1, 2, \ldots, 2n, \)
\( g = C_{2n}^{(1)}, \)
\[ J = \{1, 3, 5, \ldots, 2n - 1\} + \bar{h}\mathbb{Z}, \quad \bar{h} = 2n, \]
\( \mu = 1, \quad \kappa = \frac{1}{2}, \)
\( g^\sigma = \bar{g}, \quad J^\sigma = \bar{J} \)

(c2) \( g = C_{2n+1}^{(1)} \) \((n \geq 1)\),
\[ J = \{1, 3, 5, \ldots, 4n - 1\} + h\mathbb{Z}, \quad h = 4n + 2, \]
\( \bar{\sigma}(i) = 2n - i + 1, \quad i = 0, 1, 2, \ldots, 2n + 1, \)
\( g = A_{2n}^{(2)}, \)
\[ J = \{1, 3, 5, \ldots, 2n - 1, 2n + 3, 2n + 3, \ldots, 4n + 1\} + 2\bar{h}\mathbb{Z}, \quad \bar{h} = 2n + 1, \]
\( \mu = 2, \quad \kappa = \frac{1}{4}, \)
\( g^\sigma \neq \bar{g}, \quad J^\sigma = \bar{J} \cup (2n + 1)\mathbb{Z}^{\text{odd}} \)

(d1) \( g = D_{n+1}^{(1)} \) \((n \geq 3)\),
\[ J = \{1, 3, 5, \ldots, 2n - 3, 2n - 1, n\} + h\mathbb{Z}, \quad h = 2n, \]
\( \bar{\sigma}(n) = n + 1, \quad \bar{\sigma}(n + 1) = n, \quad \bar{\sigma}(i) = i, \quad i = 0, 1, 2, \ldots, n - 1, \)
\( g = B_{n}^{(1)}, \)
\[ J = \{1, 3, 5, \ldots, 2n - 1\} + \bar{h}\mathbb{Z}, \quad \bar{h} = 2n, \]
\( \mu = 1, \quad \kappa = 1, \)
\( g^\sigma = \bar{g}, \quad J^\sigma = \bar{J} \)

(d2) \( g = D_{4}^{(1)}, \)
\[ J = \{1, 3, 3, 5\} + h\mathbb{Z}, \quad h = 6, \]
\( \bar{\sigma}(0, 1, 3, 4, 2) = (0, 3, 4, 1, 2), \)
\( g = G_{2}^{(1)}, \)
\[ J = \{1, 5\} + 6\mathbb{Z}, \quad \bar{h} = 6, \]
\( \mu = 1, \quad \kappa = 1, \)
\( g^\sigma = \bar{g}, \quad J^\sigma = \bar{J} \)

(d3) \( g = D_{2n+1}^{(1)} \) \((n \geq 2)\),
\[ J = \{1, 3, 5, \ldots, 4n - 3, 4n - 1, 2n\} + h\mathbb{Z}, \quad h = 4n, \]
\( \bar{\sigma}(i) = 2n - i + 1, \quad i = 0, 1, 2, \ldots, 2n + 1, \)
\( g = B_{n}^{(1)}, \)
\[ J = \{1, 3, 5, \ldots, 2n - 1\} + \bar{h}\mathbb{Z}, \quad \bar{h} = 2n, \]
\( \mu = 1, \quad \kappa = \frac{1}{2}, \)
\( g^\sigma = \bar{g}, \quad J^\sigma = \bar{J} \)
(d4) \( g = D_{2n}^{(1)} \ (n \geq 2) \),
\( J = \{1, 3, 5, \ldots, 4n - 5, 4n - 3, 2n - 1}\} + h\mathbb{Z} \quad ((2n - 1)\mathbb{Z}^{\text{odd}} \text{ are double exponents})
\( h = 4n - 2, \)
\( \bar{\sigma}(i) = 2n - i, \quad i = 0, 1, 2, \ldots, 2n, \)
\( \bar{g} = A_{2n-1}^{(2)} \),
\( \bar{J} = \{1, 3, 5, \ldots, 4n - 3\} + 2\bar{h}\mathbb{Z}, \quad \bar{h} = 2n - 1, \)
\( \mu = 1, \quad \kappa = \frac{1}{2}, \)
\( g^\sigma \neq \bar{g}, \quad J^\sigma = \bar{J} \sqcup (2n - 1)\mathbb{Z}^{\text{odd}} \)

(d5) \( g = D_{4}^{(1)} \),
\( J = \{1, 3, 3, 5\} + h\mathbb{Z}, \quad h = 6, \)
\( \bar{\sigma}(0, 1, 3, 4, 2) = (1, 3, 4, 0, 2), \)
\( \bar{g} = A_{2}^{(2)} \),
\( \bar{J} = \{1, 5\} + 6\mathbb{Z}, \quad \bar{h} = 3, \)
\( \mu = 1, \quad \kappa = \frac{1}{2}, \)
\( g^\sigma \neq \bar{g}, \quad J^\sigma = \bar{J} \cup 3\mathbb{Z}^{\text{odd}} \)

(d6) \( g = D_{2(n+1)}^{(2)} \ (n \geq 1) \),
\( J = \{1, 3, 5, \ldots, 4n + 3\} + 2h\mathbb{Z}, \quad h = 2n + 2, \)
\( \bar{\sigma}(i) = 2n - i + 1, \quad i = 0, 1, 2, \ldots, 2n + 1, \)
\( \bar{g} = D_{n+1}^{(2)} \),
\( \bar{J} = \{1, 3, 5, \ldots, 2n + 1\} + 2\bar{h}\mathbb{Z}, \quad \bar{h} = n + 1, \)
\( \mu = 1, \quad \kappa = \frac{1}{2}, \)
\( g^\sigma = \bar{g}, \quad J^\sigma = \bar{J} \)

(d7) \( g = D_{2n+1}^{(2)} \),
\( J = \{1, 3, 5, \ldots, 4n + 1\} + 2h\mathbb{Z}, \quad h = 2n + 1, \)
\( \bar{\sigma}(i) = 2n - i, \quad i = 0, 1, 2, \ldots, 2n, \)
\( \bar{g} = A_{2n}^{(2)} \),
\( \bar{J} = \{1, 3, 5, \ldots, 2n - 1, 2n + 3, 2n + 5, \ldots, 4n + 1\} + 2\bar{h}\mathbb{Z}, \quad \bar{h} = 2n + 1, \)
\( \mu = 1, \quad \kappa = 1, \)
\( g^\sigma \neq \bar{g}, \quad J^\sigma = \bar{J} \sqcup (2n + 1)\mathbb{Z}^{\text{odd}} \)

(e1) \( g = E_{g}^{(1)} \),
\( J = \{1, 4, 5, 7, 8, 11\} + h\mathbb{Z}, \quad h = 12, \)
\( \bar{\sigma}(0, 1, 2, 3, 4, 5, 6) = (0, 5, 4, 3, 2, 1, 6), \)
\( \bar{g} = F_{4}^{(1)} \),
\( \bar{J} = \{1, 5, 7, 11\} + 12\mathbb{Z}, \quad \bar{h} = 12, \)
\( \mu = 1, \quad \kappa = 1, \)
\[ \mathfrak{g}^{\sigma} = \bar{\mathfrak{g}}, \quad J^{\sigma} = \bar{J} \]

(e2)

\[ \mathfrak{g} = E_6^{(1)}, \]
\[ J = \{1, 4, 5, 7, 8, 11\} + h\mathbb{Z}, \quad h = 12, \]
\[ \bar{\sigma}(0, 1, 5, 2, 4, 6, 3) = (1, 5, 0, 4, 6, 2, 3), \]
\[ \bar{\mathfrak{g}} = D_4^{(3)}, \]
\[ \bar{J} = \{1, 5, 7, 11\} + 12\mathbb{Z}, \quad \bar{h} = 4, \]
\[ \mu = 1, \quad \kappa = \frac{1}{3}, \]
\[ \bar{\mathfrak{g}}^{\sigma} \neq \bar{\mathfrak{g}}, \quad J^{\sigma} = \bar{J} \]

• \[ \mathfrak{g} = E_7^{(1)}, \]
\[ J = \{1, 5, 7, 9, 11, 13, 17\} + h\mathbb{Z}, \quad h = 18, \]
\[ \bar{\sigma}(7) = 7, \quad \bar{\sigma}(i) = 6 - i, \quad i = 0, 1, 2, 3, 4, 5, 6, \]
\[ \bar{\mathfrak{g}} = E_6^{(2)}, \]
\[ \bar{J} = \{1, 5, 7, 11, 13, 17\} + 18\mathbb{Z}, \quad \bar{h} = 9, \]
\[ \mu = 1, \quad \kappa = \frac{1}{2}, \]
\[ \bar{\mathfrak{g}}^{\sigma} \neq \bar{\mathfrak{g}}, \quad J^{\sigma} = \bar{J} \]

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