Ginsparg-Wilson relation and lattice Weyl fermions

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Abstract

We demonstrate that in the topologically trivial gauge sector the Ginsparg-Wilson relation for lattice Dirac operators admits an exactly gauge invariant path integral formulation of the Weyl fermions on a lattice.
1. The Ginsparg-Wilson relation [1] for chirally noninvariant lattice Dirac operator yields perfect solution to the long-standing problem of maintaining chiral symmetry in lattice vector gauge theories [2]–[6]. In turn, the chiral noninvariance of the lattice Dirac operator is necessary for the operator to be local on a regular lattice without producing species doubling [7], and also to reproduce the Atiyah-Singer index theorem [8] on a finite lattice [9]. The simplest form of the relation is

$$D \gamma_5 + \gamma_5 D = 2rD \gamma_5 D,$$

(1)

where $D$ is the lattice Dirac operator and $r$ is a nonzero real parameter (the lattice spacing $a$ is set to one). Then, if $D$ satisfies (1), the fermion action

$$S_f = \overline{\psi} D \psi,$$

(2)

is invariant under chiral transformations of the following form

$$\psi \rightarrow \exp[i \alpha + i \beta \gamma_5 E(s)] \psi,$$

$$\overline{\psi} \rightarrow \overline{\psi} \exp[-i \alpha + i \beta F(s) \gamma_5],$$

(3)

where $\alpha$ and $\beta$ are the parameters of the vector and the axial transformations and

$$E(s) = 1 - (2r - s)D,$$

$$F(s) = 1 - sD,$$

(4)

is a family of the generators of the axial transformations parametrised by the real parameter $s$. The symmetry (3), (4) at $s = r$, in which case $E = F = 1 - rD$, has been discovered by Lüscher [6], and at $s = 0$ considered in [10], [11]. Obviously, it protects the theory from the additive mass renormalization. At the same time the measure in the fermion path integral

$$Z_f = \int \prod_x d\psi(x) d\overline{\psi}(x) \exp(-S_f) = \det D,$$

(5)

is not invariant under (3), and the corresponding Jacobian produces the anomaly in the divergence of the axial part of the Nöther current associated with these transformation [6]. The anomaly is proportional to [3], [6], [12]

$$q(x) = i \frac{1}{2} \frac{\partial}{\partial \beta(x)} J^{-1}(\beta) = -r \text{tr}(\gamma_5 D).$$

(6)

Another remarkable property of solutions of the Ginsparg-Wilson relation is that [3], [8]

$$\sum_x q(x) = n_+ - n_- \equiv -\text{index } D,$$

(7)
where $n_+$ ($n_-$) is the number of zero modes of the $D$ of positive (negative) chirality. So the Atiyah-Singer index theorem

\[ \text{index } D = Q, \]  

where $Q$ is topological number of the background gauge field, may be reproduced exactly even on a finite lattice.

At present only one explicit solution of the Ginsparg-Wilson equation is known, the operator proposed by Neuberger [4], that indeed reproduces the index theorem [8], [1], [13] and is local at least in sufficiently smooth gauge field background [14].

Since $[\gamma_5E(0)]^2 = [F(0)\gamma_5]^2 = 1$, one can defined two pairs of projecting operators [10]

\[ P_\pm^E = \frac{1}{2}[1 \pm \gamma_5E(0)] = \frac{1}{2}[1 \pm \gamma_5(1 - 2rD)], \]
\[ P_\pm = \frac{1}{2}[1 \pm F(0)\gamma_5] = \frac{1}{2}(1 \pm \gamma_5), \]

and the action (2) can be written as [10]

\[ S_f = \overline{\psi}_+ D\psi_+^E + \overline{\psi}_- D\psi_-^E, \]

where

\[ \psi_+^E = P_+^E \psi, \quad \overline{\psi}_+ = \overline{\psi} P_+. \]

The decomposition (10), (11) itself however does not yet allow one to write down the functional integral (5) as the product of two factors related to the Weyl fermions of opposite chiralities [15], since the fermion measure in (5) is not factorised according to (10), and the fields $\psi_\pm^E$ cannot be treated as the Weyl fields.

We are aiming now to demonstrate that at one condition specified below, a certain change of variables $\psi$ leads to natural definition of the path integral for the Weyl field of positive or negative chirality.

2. To proceed, let us set some properties of the solutions of relation (1). We limit our consideration to the operators $D$ with the property

\[ D^\dagger = \gamma_5D\gamma_5. \]

In the chiral representation of $\gamma$ matrices where $\gamma_5 = \text{diag}(1, -1)$, $\gamma_5^\dagger = \gamma_\mu$, such operators have the form

\[ D = \begin{pmatrix} M & D_- \\ D_+ & M \end{pmatrix}. \]

Matrices $D_+$ and $D_- = -D_+^\dagger$ are lattice transcriptions of the covariant Weyl operators $\sigma_\mu(\partial_\mu + iA_\mu)$ and $\sigma_\mu^\dagger(\partial_\mu + iA_\mu)$, respectively, where $A_\mu$ is the gauge
field, \( \sigma_\mu = (1, i) \) in two dimensions and \( \sigma_\mu = (1, i\sigma_i) \) in four dimensions, and the matrix \( M \) determines chirally noninvariant part of \( D \). In this representation

\[
\psi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \chi_+^\dagger \\ \chi_-^\dagger \end{pmatrix},
\]

where \( \chi_\pm \) are the Weyl fields of the chirality \( \pm \).

Then from the Ginsparg-Wilson relation (1) it follows

\[
D_+ D_- = D_- D_+ = M(M - r^{-1}),
\]

\[
MD_\pm = D_\pm M,
\]

i.e. all the entries of matrix (13) commute with each other. Using these properties it is easy to obtain the following relations

\[
\det D = \det (r^{-1} M),
\]

\[
\det (1 - r D) = \det (1 - r M).
\]

Note that singularity of the matrix \( 1 - r D \), and therefore of the matrix \( 1 - r M \), is the necessary condition [16] for the operator \( D \) to have nonzero index [8].

Consider the combination \( 1 - r M \) in more detail. Using the fact [15], [16] that any solution of eqs. (1), (12) can be presented in the form

\[
D = \frac{1}{2r}(1 + V), \quad V^\dagger = \gamma_5 V \gamma_5,
\]

where \( V \) is a unitary matrix, and taking into account (1) and (15), we find that

\[
\begin{pmatrix}
1 - r M & 0 \\
0 & 1 - r M
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 - V + V^\dagger \\
1 - V - V^\dagger
\end{pmatrix},
\]

from where it follows that the matrix \( 1 - r M \) is nonnegative. Combining now this fact with eq. (8) and the above mentioned necessary condition [16], we conclude that the matrix \( 1 - r M \) is positive definite in the topologically trivial gauge sector \( Q = 0 \), except may be some exceptional configurations on which it gets singular.

Let us now consider the theory in the topologically trivial sector with the exceptional configurations excluded. Then \( 1 - r M \) is positive definite and there exist the unique positive matrix \( (1 - r M)^{1/2} \) and the unitary matrix

\[
U = \begin{pmatrix}
(1 - r M)^{1/2} & -r D_-(1 - r M)^{-1/2} \\
-r D_+(1 - r M)^{-1/2} & (1 - r M)^{1/2}
\end{pmatrix}, \quad U^\dagger U = 1, \quad \det U = 1,
\]

such that the operator \( DU^\dagger \) is chirally invariant:

\[
DU^\dagger = \begin{pmatrix}
0 & D_-(1 - r M)^{-1/2} \\
D_+(1 - r M)^{-1/2} & 0
\end{pmatrix}, \quad DU^\dagger \gamma_5 = \gamma_5 DU^\dagger.
\]
Note that the $U$ diagonalizes the projecting operators $P^E_{\pm}$:

$$UP^E_{\pm}U^\dagger = P_\pm, \quad (22)$$

and that such a diagonalization is possible only in the sector $Q = 0$.

Now it is straightforward to redefine the variables $\psi$. Making the obvious change of variables

$$\psi' \equiv \left( \begin{array}{c} \chi'_+ \\ \chi'_- \end{array} \right) = U\psi, \quad (23)$$

whose Jacobian in view of (20) equals unity, and omitting the primes we get

$$Z_f = Z_+ Z_-,$$

$$Z_\pm = \int \prod_x d\chi_\pm(x) d\chi_\pm^\dagger(x) \exp(-\chi_{\pm}^\dagger W_\pm \chi_{\pm}), \quad (24)$$

where the Weyl operators $W_\pm$ read as

$$W_\pm = \frac{D_\pm}{\sqrt{1 - rM}}. \quad (25)$$

This is an explicit path integral realization of the factorization of $Z_f$ pointed out in [15]. Note that in view of (15) and (17) we do have $\det W_+ \det W_- = \det D$, and in the free fermion case the operators $W_\pm$ have correct continuum limit.

Thus eqs. (24), (25) define the path integrals for the Weyl fermions in the topologically trivial gauge sector $Q = 0$. We should make however few important comments.

3. The most striking feature of this formulation is its exact gauge invariance. Indeed, the fermion measure and the actions in (24), (25) are invariant under the ordinary gauge transformations

$$\chi_\pm(x) \to g_\pm(x) \chi_\pm(x), \quad \chi_{\pm}^\dagger(x) \to \chi_{\pm}^\dagger(x) g_{\pm}^\dagger(x), \quad g_{\pm}^\dagger(x) g_{\pm}(x) = 1. \quad (26)$$

This means that in such a formulation the so called consistent anomaly [17] vanishes for each Weyl fermion. Also obviously, that all the Nöther currents associated with transformations (26) are conserved. We would like to emphasize however that this does not contradict to the index theorem (7), (8), since the above formulation exists only in the topologically trivial sector.

The operators $W_\pm$ in (25) are nonlocal. This follows from the analysis of the Fourier transform of $W_\pm, W_\pm(p)$, in the free fermion limit. Indeed, taking into account the first equation in (15), one can see that $W_\pm(p)$ is discontinuous at the boundary of the Brillouin zone and thus avoids species doubling [7]. An explicit form of the $W_\pm$ can be constructed from Neuberger’s operator [1]. For instance, simple consideration of the chiral Schwinger model on a finite torus
$L \times L$ in the constant gauge field background with such operators shows that $Z_{\pm}$ are real, and the exceptional configurations correspond to $e A_\mu = \pi/L$.

With one exception, all previous gauge invariant formulations with nonlocal actions \cite{18} failed due to non canceling singularities in the interaction vertices \cite{19}. Due to such singularities none of those formulations could reproduce correct absolute value of the fermion determinant without special subtractions \cite{20} even in the perturbation theory. The exception is the non-local fixed point action \cite{21}, \cite{22} corresponding to $r = 0$ in \cite{1}, whose explicit form however is known only in the lowest orders of the perturbation theory. Thus, eqs. (24), (25) present the first example of the nonlocal gauge invariant formulation that succeeds to reproduce correct value of $|Z_{\pm}|$ nonperturbatively.

Of course, due to the limitation only to the topologically trivial sector $Q = 0$, the formulation (24), (23) is not complete. However within this sector the question of principle arises: whether nonlocality of Weyl operators always produces some defects, so far hidden for the $W_{\pm}$ in (24), or the chiral anomalies in the topologically trivial sector is necessary attribute of only local formulations.

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