New Inequalities of Weaving K-Frames in Subspaces

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Abstract: In the present paper, we obtain some new inequalities for weaving K-frames in subspaces based on the operator methods. The inequalities are associated with a sequence of bounded complex numbers and a parameter $\lambda \in \mathbb{R}$. We also give a double inequality for weaving K-frames with the help of two bounded linear operators induced by K-dual. Facts prove that our results cover those recently obtained on weaving frames due to Li and Leng, and Xiang.

Keywords: weaving frame; weaving K-frame; K-dual; pseudo-inverse

MSC: 42C15; 47B40

1. Introduction

This paper adopts the following notations: $\mathbb{J}$ is a countable index set, $\mathbb{H}$ and $\mathbb{K}$ are complex Hilbert spaces, and $\text{Id}_{\mathbb{H}}$ and $\mathbb{R}$ are used to denote respectively the identical operator on $\mathbb{H}$ and the set of real numbers. As usual, we denote by $B(\mathbb{H}, \mathbb{K})$ the set of all bounded linear operators on $\mathbb{H}$ and, if $\mathbb{H} = \mathbb{K}$, then $B(\mathbb{H}, \mathbb{K})$ is abbreviated to $B(\mathbb{H})$.

Frames were introduced by Duffin and Schaeffer [1] in their study of nonharmonic Fourier series, which have now been used widely not only in theoretical work [2,3], but also in many application areas such as quantum mechanics [4], sampling theory [5–7], acoustics [8], and signal processing [9]. As a generalization of frames, the notion of K-frames (also known as frames for operators) was proposed by L. Găvruţa [10] when dealing with atomic decompositions for a bounded linear operator $K$. Please check the papers [11–17] for further information of K-frames.

Recall that a family $\{\psi_j\}_{j \in \mathbb{J}} \subset \mathbb{H}$ is called a K-frame for $\mathbb{H}$, if there exist two positive numbers $A$ and $B$ satisfying

$$A\|K^* f\|^2 \leq \sum_{j \in \mathbb{J}} |\langle f, \psi_j \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathbb{H}.$$

The constants $A$ and $B$ are called K-frame bounds. If $K = \text{Id}_{\mathbb{H}}$, then a K-frame turns to be a frame. In addition, if only the right-hand inequality holds, then we call $\{\psi_j\}_{j \in \mathbb{J}}$ a Bessel sequence.

Inspired by a question arising in distributed signal processing, Bemrose et al. [18] introduced the concept of weaving frames, which have interested many scholars because of their potential applications such as in wireless sensor networks and pre-processing of signals; see [19–24]. Later on, Deepshikha and Vashisht [25] applied the idea of L. Găvruţa to the case of weaving frames and thus providing us the notion of weaving K-frames.

Balan et al. [26] obtained an interesting inequality when they further examined the remarkable identity for Parseval frames deriving from their work on signal reconstruction [27]. The inequality was then extended to alternate dual frames and general frames by P. Găvruţa [28], the results in which have already been applied in quantum information theory [29]. Recently, those inequalities have been extended to some generalized versions of frames such as continuous g-frames [30], fusion frames and continuous fusion frames [31,32], Hilbert-Schmidt frames [33], and weaving frames [34,35].
Motivated by the above-mentioned works, in this paper, we establish several new inequalities for weaving \( K \)-frames in subspaces from the operator-theoretic point of view, and we show that our results can naturally lead to some corresponding results in \([34,35]\).

One says that two frames \( \Psi_1 = \{ \psi_1 \}_j \in J \) and \( \Psi_2 = \{ \psi_2 \}_j \in J \) in \( \mathbb{H} \) are woven, if there are universal constants \( C_\Psi \) and \( D_\Psi \) such that, for any \( \sigma \subset J \), \( \{ \psi_1 \}_j \in \sigma \cup \{ \psi_2 \}_j \in \sigma' \) is a frame for \( \mathbb{H} \) with bounds \( C_\Psi \) and \( D_\Psi \). If \( C_\Psi = D_\Psi = 1 \), then we call \( \Psi_1 \) and \( \Psi_2 \) 1-woven. Each family \( \{ \psi_1 \}_j \in \sigma \cup \{ \psi_2 \}_j \in \sigma' \) is said to be a waving frame, related to which there is an invertible operator \( S_{\Psi_1, \Psi_2} : \mathbb{H} \rightarrow \mathbb{H} \), called the frame operator, given by

\[
S_{\Psi_1, \Psi_2} f = \sum_{j \in \sigma} \langle f, \psi_1 \rangle \psi_1 + \sum_{j \in \sigma'} \langle f, \psi_2 \rangle \psi_2.
\]

Recall also that a frame \( \Psi_3 = \{ \psi_3 \}_j \in J \) is called an alternate dual frame of \( \{ \psi_1 \}_j \in \sigma \cup \{ \psi_2 \}_j \in \sigma' \), if for each \( f \in \mathbb{H} \) we have

\[
f = \sum_{j \in \sigma} \langle f, \psi_1 \rangle \psi_3 + \sum_{j \in \sigma'} \langle f, \psi_2 \rangle \psi_3, \quad \forall f \in \mathbb{H}.
\]

**Lemma 1.** Suppose that \( P, Q, \) and \( K \) are bounded linear operators on \( \mathbb{H} \) and \( P + Q = K \). Then, for each \( f \in \mathbb{H} \),

\[
\| Pf \|^2 + \text{Re}(Qf, Kf) \geq \frac{3}{4} \| Kf \|^2.
\]

**Proof.** We have

\[
\| Pf \|^2 + \text{Re}(Qf, Kf) = \langle (K - Q)f, (K - Q)f \rangle + \frac{1}{2} \langle (Qf, Kf) + (Kf, Qf) \rangle
\]

\[
= \langle (Q^*Q - (K^*Q + Q^*K))f, f \rangle + \frac{1}{2} \langle (K^*Q + Q^*K)f, f \rangle
\]

\[
= \langle Q - \frac{1}{2} K \rangle^*(Q - \frac{1}{2} K)f, f \rangle + \frac{3}{4} \langle K^*Kf, f \rangle \geq \frac{3}{4} \| Kf \|^2
\]

for any \( f \in \mathbb{H} \). \( \square \)

The next two lemmas are collected from the papers \([36]\) and \([32]\), respectively.

**Lemma 2.** If \( \Phi \in B(\mathbb{H}, \mathbb{K}) \) has a closed range, then there is the pseudo-inverse \( \Phi^+ \in B(\mathbb{K}, \mathbb{H}) \) of \( \Phi \) such that

\[
\Phi \Phi^+ \Phi = \Phi, \quad \Phi^+ \Phi \Phi^+ = \Phi^+, \quad (\Phi \Phi^+)^* = \Phi \Phi^+, \quad (\Phi^+ \Phi)^* = \Phi^+ \Phi.
\]

**Lemma 3.** If \( P \) and \( Q \) in \( B(\mathbb{H}) \) satisfy \( P + Q = \text{Id}_{\mathbb{H}} \), then, for any \( \lambda \in \mathbb{R} \), we have

\[
P^*P + \lambda(Q^* + Q) = Q^*Q + (1 - \lambda)(P^* + P) + (2\lambda - 1)\text{Id}_{\mathbb{H}} \geq (2\lambda - \lambda^2)\text{Id}_{\mathbb{H}}.
\]

2. **Main Results**

We start with the definition on weaving \( K \)-frames due to Deepshikha and Vashisht \([25]\).

**Definition 1.** Two \( K \)-frames \( \Psi_1 = \{ \psi_1 \}_j \in J \) and \( \Psi_2 = \{ \psi_2 \}_j \in J \) in \( \mathbb{H} \) are said to be \( K \)-woven, if there are universal constants \( C_\Psi \) and \( D_\Psi \) such that, for any \( \sigma \subset J \), the family \( \{ \psi_1 \}_j \in \sigma \cup \{ \psi_2 \}_j \in \sigma' \) is a \( K \)-frame for \( \mathbb{H} \) with \( K \)-frame bounds \( C_\Psi \) and \( D_\Psi \). In this case, the family \( \{ \psi_1 \}_j \in \sigma \cup \{ \psi_2 \}_j \in \sigma' \) is called a weaving \( K \)-frame.

Given a weaving \( K \)-frame \( \{ \psi_1 \}_j \in \sigma \cup \{ \psi_2 \}_j \in \sigma' \) for \( \mathbb{H} \), recall that a Bessel sequence \( \Phi = \{ \phi_j \}_j \) for \( \mathbb{H} \) is said to be a \( K \)-dual of \( \{ \psi_1 \}_j \in \sigma \cup \{ \psi_2 \}_j \in \sigma' \), if

\[
Kf = \sum_{j \in \sigma} \langle f, \psi_1 \rangle \phi_j + \sum_{j \in \sigma'} \langle f, \psi_2 \rangle \phi_j, \quad \forall f \in \mathbb{H}.
\]
Let $\Psi_1 = \{\psi_1\}_{j \in \mathbb{J}}$ be a given $K$-frame for $\mathbb{H}$. For any $\sigma \subset \mathbb{J}$, we can define a positive operator $S_{\Psi_1}^\sigma$ in the following way:

$$S_{\Psi_1}^\sigma : \mathbb{H} \to \mathbb{H}, \quad S_{\Psi_1}^\sigma f = \sum_{j \in \sigma} \langle f, \psi_1 \rangle \psi_1.$$ 

In the following, we show that, for given two $K$-woven frames, we can get some inequalities under the condition that $K$ has a closed range, which are related to a sequence of bounded complex numbers, the corresponding $K$-dual and a parameter $\lambda \in \mathbb{R}$.

**Theorem 1.** Suppose that $K \in B(\mathbb{H})$ has a closed range and $K$-frames $\Psi_1 = \{\psi_1\}_{j \in \mathbb{J}}$ and $\Psi_2 = \{\psi_2\}_{j \in \mathbb{J}}$ in $\mathbb{H}$ are $K$-woven. Then,

(i) for any $f \in \text{Range}(K)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j \in \mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R},$

$$\left\| \sum_{j \in \sigma} a_j \langle K^t f, \psi_1 \rangle \psi_1 + \sum_{j \in \sigma} a_j \langle K^t f, \psi_2 \rangle \psi_2 \right\|^2
+ \Re \left( \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_2 \rangle \langle \psi_2, f \rangle \right)$$

$$= \left\| \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_1 \rangle \psi_1 + \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_2 \rangle \psi_2 \right\|^2
+ \Re \left( \sum_{j \in \sigma} a_j \langle K^t f, \psi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} a_j \langle K^t f, \psi_2 \rangle \langle \psi_2, f \rangle \right)$$

$$\geq (\lambda - \frac{\lambda^2}{4}) \Re \left( \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_2 \rangle \langle \psi_2, f \rangle \right)$$

$$+ (1 - \frac{\lambda^2}{4}) \Re \left( \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} (1 - a_j) \langle K^t f, \psi_2 \rangle \langle \psi_2, f \rangle \right),$$

where $\Phi = \{\phi_j\}_{j \in \mathbb{J}}$ is a $K$-dual of $\{\psi_1\}_{j \in \sigma} \cup \{\psi_2\}_{j \in \sigma}$.

(ii) for any $f \in \text{Range}(K^\ast)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j \in \mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R},$

$$\left\| \sum_{j \in \sigma} a_j \langle (K^\ast)^t f, \phi_1 \rangle \psi_1 + \sum_{j \in \sigma} a_j \langle (K^\ast)^t f, \phi_2 \rangle \psi_2 \right\|^2$$

$$+ \Re \left( \sum_{j \in \sigma} (1 - a_j) \langle (K^\ast)^t f, \phi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} (1 - a_j) \langle (K^\ast)^t f, \phi_2 \rangle \langle \psi_2, f \rangle \right)$$

$$= \left\| \sum_{j \in \sigma} (1 - a_j) \langle (K^\ast)^t f, \phi_1 \rangle \psi_1 + \sum_{j \in \sigma} (1 - a_j) \langle (K^\ast)^t f, \phi_2 \rangle \psi_2 \right\|^2$$

$$+ \Re \left( \sum_{j \in \sigma} a_j \langle (K^\ast)^t f, \phi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} a_j \langle (K^\ast)^t f, \phi_2 \rangle \langle \psi_2, f \rangle \right)$$

$$\geq (2\lambda - \lambda^2) \Re \left( \sum_{j \in \sigma} a_j \langle (K^\ast)^t f, \phi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} a_j \langle (K^\ast)^t f, \phi_2 \rangle \langle \psi_2, f \rangle \right)$$

$$+ (1 - \lambda^2) \Re \left( \sum_{j \in \sigma} (1 - a_j) \langle (K^\ast)^t f, \phi_1 \rangle \langle \psi_1, f \rangle + \sum_{j \in \sigma} (1 - a_j) \langle (K^\ast)^t f, \phi_2 \rangle \langle \psi_2, f \rangle \right),$$

where $\Phi = \{\phi_j\}_{j \in \mathbb{J}}$ is a $K$-dual of $\{\psi_1\}_{j \in \sigma} \cup \{\psi_2\}_{j \in \sigma}$. 
Proof. We define two bounded linear operators $P_1$ and $P_2$ on $\mathbb{H}$ as follows:

$$P_1 f = \sum_{j \in \sigma} a_j \langle f, \psi_1 \rangle \phi_j + \sum_{j \in \sigma} a_j \langle f, \psi_2 \rangle \phi_j,$$

$$P_2 f = \sum_{j \in \sigma} (1 - a_j) \langle f, \psi_1 \rangle \phi_j + \sum_{j \in \sigma} (1 - a_j) \langle f, \psi_2 \rangle \phi_j. \quad (2)$$

Then, clearly, $P_1 f + P_2 f = K f$ for each $f \in \mathbb{H}$ and thus $P_1 + P_2 = K$. Since $K$ has a closed range, by Lemma 2, we have

$$P_1 K^+ + P_2 K^+ = KK^+ = P_{\text{Range}(K)},$$

where $P_{\text{Range}(K)}$ is the orthogonal projection onto $\text{Range}(K)$. Thus,

$$P_1 K^+ |_{\text{Range}(K)} + P_2 K^+ |_{\text{Range}(K)} = \text{Id}_{\text{Range}(K)}.$$

By Lemma 3 (taking $\frac{\lambda}{2}$ instead of $\lambda$), we get

$$\|P_1 K^+ f\|^2 + \lambda \text{Re}\langle P_2 K^+ f, f \rangle = \|P_2 K^+ f\|^2 + (2 - \lambda) \text{Re}\langle P_1 K^+ f, f \rangle + (\lambda - 1) \|f\|^2,$$

for any $f \in \text{Range}(K)$. Hence,

$$\|P_1 K^+ f\|^2 = \|P_2 K^+ f\|^2 + 2 \text{Re}\langle P_1 K^+ f, f \rangle - \lambda (\text{Re}\langle P_1 K^+ f, f \rangle + \text{Re}\langle P_2 K^+ f, f \rangle) + (\lambda - 1) \|f\|^2$$

$$= \|P_2 K^+ f\|^2 + 2 \text{Re}\langle P_1 K^+ f, f \rangle - \lambda \|f\|^2 + (\lambda - 1) \|f\|^2$$

$$= \|P_2 K^+ f\|^2 + 2 \text{Re}\langle P_1 K^+ f, f \rangle - \text{Re}\langle P_1 K^+ f, f \rangle - \text{Re}\langle P_2 K^+ f, f \rangle.$$

It follows that

$$\|P_1 K^+ f\|^2 + \text{Re}\langle P_2 K^+ f, f \rangle = \|P_2 K^+ f\|^2 + \text{Re}\langle P_1 K^+ f, f \rangle,$$

(3)

from which we arrive at

$$\left\| \sum_{j \in \sigma} a_j \langle K^+ f, \psi_1 \rangle \phi_j + \sum_{j \in \sigma} a_j \langle K^+ f, \psi_2 \rangle \phi_j \right\|^2$$

$$+ \text{Re}\left( \sum_{j \in \sigma} (1 - a_j) \langle K^+ f, \psi_1 \rangle \langle \phi_j, f \rangle + \sum_{j \in \sigma} (1 - a_j) \langle K^+ f, \psi_2 \rangle \langle \phi_j, f \rangle \right)$$

$$= \left\| \sum_{j \in \sigma} (1 - a_j) \langle K^+ f, \psi_1 \rangle \phi_j + \sum_{j \in \sigma} (1 - a_j) \langle K^+ f, \psi_2 \rangle \phi_j \right\|^2$$

$$+ \text{Re}\left( \sum_{j \in \sigma} a_j \langle K^+ f, \psi_1 \rangle \langle \phi_j, f \rangle + \sum_{j \in \sigma} a_j \langle K^+ f, \psi_2 \rangle \langle \phi_j, f \rangle \right).$$

For the inequality in Equation (1), we apply Lemma 3 again,

$$\|P_1 K^+ f\|^2 \geq (\lambda - \frac{\lambda^2}{4}) \|f\|^2 - \lambda \text{Re}\langle P_2 K^+ f, f \rangle$$

$$= (\lambda - \frac{\lambda^2}{4}) \text{Re}\langle P_1 K^+ f + P_2 K^+ f, f \rangle - \lambda \text{Re}\langle P_2 K^+ f, f \rangle$$

$$= (\lambda - \frac{\lambda^2}{4}) \text{Re}\langle P_1 K^+ f, f \rangle - \frac{\lambda^2}{4} \text{Re}\langle P_2 K^+ f, f \rangle.$$
Thus, for any \( f \in \text{Range}(K) \),
\[
\left\| \sum_{j \in \sigma} a_j \langle K^f, \psi_{j1} \rangle \psi_{j1} + \sum_{j \in \sigma^c} a_j \langle K^f, \psi_{j2} \rangle \psi_{j2} \right\|^2 \\
+ \text{Re} \left( \sum_{j \in \sigma} (1 - a_j) \langle K^f, \psi_{j1} \rangle \langle \psi_{j1}, f \rangle + \sum_{j \in \sigma^c} (1 - a_j) \langle K^f, \psi_{j2} \rangle \langle \psi_{j2}, f \rangle \right) \\
\geq (\lambda - \frac{\lambda^2}{4}) \text{Re} \langle P_1 K^f, f \rangle + (1 - \frac{\lambda^2}{4}) \text{Re} \langle P_2 K^f, f \rangle \\
= (\lambda - \frac{\lambda^2}{4}) \text{Re} \left( \sum_{j \in \sigma} a_j \langle K^f, \psi_{j1} \rangle \langle \psi_{j1}, f \rangle + \sum_{j \in \sigma^c} a_j \langle K^f, \psi_{j2} \rangle \langle \psi_{j2}, f \rangle \right) \\
+ (1 - \frac{\lambda^2}{4}) \text{Re} \left( \sum_{j \in \sigma} (1 - a_j) \langle K^f, \psi_{j1} \rangle \langle \psi_{j1}, f \rangle + \sum_{j \in \sigma^c} (1 - a_j) \langle K^f, \psi_{j2} \rangle \langle \psi_{j2}, f \rangle \right). 
\]

(ii) The proof is similar to (i), so we omit the details. \( \square \)

**Corollary 1.** Suppose that two frames \( \Psi_1 = \{ \psi_{1j} \}_{j \in \mathbb{J}} \) and \( \Psi_2 = \{ \psi_{2j} \}_{j \in \mathbb{J}} \) in \( \mathbb{H} \) are woven. Then, for any \( f \in \mathbb{H} \), for all \( \sigma \subset \mathbb{J} \) and all \( \lambda \in \mathbb{R} \), we have
\[
\sum_{j \in \mathbb{J}} |\langle f, \psi_{j1} \rangle|^2 + \sum_{j \in \mathbb{J}} \left| \langle S_{\mathbb{J} \setminus \sigma}^0 \Psi_1, f \rangle, S_{\mathbb{J} \setminus \sigma}^{-1} \Psi_1, \psi_{j1} \rangle \right|^2 + \sum_{j \in \mathbb{J}} \left| \langle S_{\mathbb{J} \setminus \sigma}^0 \Psi_2, f \rangle, S_{\mathbb{J} \setminus \sigma}^{-1} \Psi_2, \psi_{j2} \rangle \right|^2 \\
\geq (\lambda - \frac{\lambda^2}{4}) \sum_{j \in \mathbb{J}} |\langle f, \psi_{j1} \rangle|^2 + (1 - \frac{\lambda^2}{4}) \sum_{j \in \mathbb{J}} |\langle f, \psi_{j2} \rangle|^2.
\]

**Proof.** Letting \( K^f = \text{Id}_{\mathbb{H}} \) and
\[
\phi_j = \begin{cases} 
S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{1j}, & j \in \sigma, \\
S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{2j}, & j \in \sigma^c.
\end{cases}
\]
In addition, taking \( S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{1j}, S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{2j} \) and \( S_{\mathbb{J} \setminus \sigma}^{1/2} f \) instead of \( \psi_{1j}, \psi_{2j} \) and \( f \) respectively in (i) of Theorem 1 leads to
\[
\left\| \sum_{j \in \sigma} a_j \langle f, \psi_{j1} \rangle S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{1j} + \sum_{j \in \sigma^c} a_j \langle f, \psi_{j2} \rangle S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{2j} \right\|^2 \\
+ \text{Re} \left( \sum_{j \in \sigma} (1 - a_j) \langle f, \psi_{j1} \rangle \langle \psi_{j1}, f \rangle + \sum_{j \in \sigma^c} (1 - a_j) \langle f, \psi_{j2} \rangle \langle \psi_{j2}, f \rangle \right) \\
= \left\| \sum_{j \in \sigma} (1 - a_j) \langle f, \psi_{j1} \rangle S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{j1} + \sum_{j \in \sigma^c} (1 - a_j) \langle f, \psi_{j2} \rangle S_{\mathbb{J} \setminus \sigma}^{-1/2} \psi_{j2} \right\|^2 \\
+ \text{Re} \left( \sum_{j \in \sigma} a_j \langle f, \psi_{j1} \rangle \langle \psi_{j1}, f \rangle + \sum_{j \in \sigma^c} a_j \langle f, \psi_{j2} \rangle \langle \psi_{j2}, f \rangle \right) \\
\geq (\lambda - \frac{\lambda^2}{4}) \text{Re} \left( \sum_{j \in \sigma} a_j \langle f, \psi_{j1} \rangle \langle \psi_{j1}, f \rangle + \sum_{j \in \sigma^c} a_j \langle f, \psi_{j2} \rangle \langle \psi_{j2}, f \rangle \right) \\
+ (1 - \frac{\lambda^2}{4}) \text{Re} \left( \sum_{j \in \sigma} (1 - a_j) \langle f, \psi_{j1} \rangle \langle \psi_{j1}, f \rangle + \sum_{j \in \sigma^c} (1 - a_j) \langle f, \psi_{j2} \rangle \langle \psi_{j2}, f \rangle \right). 
\]

(5)
A direction calculation shows that 

\[ \left\| \sum_{j \in \sigma} \langle f, \psi_j \rangle S_{\mathcal{V}^1_{1/2}} \psi_j \right\| \leq \left\| S_{\mathcal{V}^1_{1/2}} \sum_{j \in \sigma} \langle f, \psi_j \rangle \psi_j \right\| = \left\| S_{\mathcal{V}^1_{1/2}} f \right\| \]

and similarly, 

\[ \left\| \sum_{j \in \sigma} \langle f, \psi_j \rangle S_{\mathcal{V}^1_{1/2}} \psi_j \right\| \leq \left\| \sum_{j \in \sigma} \langle S_{\mathcal{V}^1_{1/2}} f, S_{\mathcal{V}^1_{1/2}} \psi_j \rangle \psi_j \right\| \]

The result follows immediately from (ii) in Theorem 1 when taking Proof.

\[ \|贾\| \]

Corollary 3. Suppose that two frames \( \Psi_1 = \{\psi_1\}_{j \in J} \) and \( \Psi_2 = \{\psi_2\}_{j \in J} \) are woven. Then, for any \( \sigma \subseteq J \), for all \( \lambda \in \mathbb{R} \) and all \( f \in \mathcal{H} \), we have

\[ \left\| \sum_{j \in \sigma} \langle f, \phi_j \rangle \psi_j \right\|^2 + \text{Re} \sum_{j \in \sigma} \langle f, \phi_j \rangle \langle \psi_j, f \rangle \]

\[ = \left\| \sum_{j \in \sigma} \langle f, \phi_j \rangle \psi_j \right\|^2 + \text{Re} \sum_{j \in \sigma} \langle f, \phi_j \rangle \langle \psi_j, f \rangle \]

\[ \geq (2\lambda - \lambda^2) \text{Re} \sum_{j \in \sigma} \langle f, \phi_j \rangle \langle \psi_j, f \rangle + (1 - \lambda^2) \text{Re} \sum_{j \in \sigma} \langle f, \phi_j \rangle \langle \psi_j, f \rangle, \]

where \( \Phi = \{\phi_j\}_{j \in J} \) is an alternate dual of \( \{\psi_1\}_{j \in J} \cup \{\psi_2\}_{j \in J} \).

Proof. The result follows immediately from (ii) in Theorem 1 when taking \( K^t = \text{Id}_{\mathcal{H}} \) and

\[ a_j = \begin{cases} 1, & j \in \sigma, \\ 0, & j \in \sigma^c. \end{cases} \]

Thus, Corollary 2 provides us a direct consequence as follows.

Corollary 3. Let the two frames \( \Psi_1 = \{\psi_1\}_{j \in J} \) and \( \Psi_2 = \{\psi_2\}_{j \in J} \) in \( \mathbb{H} \) be 1-woven. Then, for any \( \sigma \subseteq J \) and any \( j \in J \), taking \( \phi_j = \begin{cases} \psi_1_j, & j \in \sigma, \\ \psi_2_j, & j \in \sigma^c. \end{cases} \) Then, obviously, \( \Phi = \{\phi_j\}_{j \in J} \) is an alternate dual of the frame \( \{\psi_1\}_{j \in \sigma} \cup \{\psi_2\}_{j \in \sigma^c} \). Thus, Corollary 2 provides us a direct consequence as follows.
Remark 1. Corollaries 1 and 2 are respectively Theorems 7 and 9 in [34], and Theorem 5 in [34] can be obtained if we put $\lambda = \frac{1}{2}$ in Corollary 3.

Theorem 2. Suppose that $K \in B(\mathbb{H})$ has a closed range and that $K$-frames $\Psi_1 = \{\psi_1\}_{j \in \mathbb{J}}$ and $\Psi_2 = \{\psi_2\}_{j \in \mathbb{J}}$ in $\mathbb{H}$ are $K$-woven. Then, for any $f \in \text{Range}(K)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j \in \mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R}$,

$$
\left\| \sum_{j \in \sigma} a_j (K^t f, \psi_1_j) \psi_j + \sum_{j \in \sigma^c} a_j (K^t f, \psi_2_j) \phi_j \right\|^2 + \left\| \sum_{j \in \sigma} (1 - a_j) (K^t f, \psi_1_j) \psi_j + \sum_{j \in \sigma^c} (1 - a_j) (K^t f, \psi_2_j) \psi_j \right\|^2
\geq (2\lambda - \frac{\lambda^2}{2} - 1) \text{Re} \left( \sum_{j \in \sigma} a_j (K^t f, \psi_1_j) \langle \psi_j, f \rangle + \sum_{j \in \sigma^c} a_j (K^t f, \psi_2_j) \langle \phi_j, f \rangle \right)
\quad + (1 - \frac{\lambda^2}{2}) \text{Re} \left( \sum_{j \in \sigma} (1 - a_j) (K^t f, \psi_1_j) \langle \psi_j, f \rangle + \sum_{j \in \sigma^c} (1 - a_j) (K^t f, \psi_2_j) \langle \phi_j, f \rangle \right),
$$

where $\Phi = \{\psi_j\}_{j \in \mathbb{J}}$ is a $K$-dual of $\{\psi_1\}_{j \in \mathbb{J}} \cup \{\psi_2\}_{j \in \mathbb{J}}$.

Moreover, if $(P_1 K^t)^* P_2 K^t$ is a positive operator, then

$$
\left\| \sum_{j \in \sigma} a_j (K^t f, \psi_1_j) \psi_j + \sum_{j \in \sigma^c} a_j (K^t f, \psi_2_j) \psi_j \right\|^2 + \left\| \sum_{j \in \sigma} (1 - a_j) (K^t f, \psi_1_j) \psi_j + \sum_{j \in \sigma^c} (1 - a_j) (K^t f, \psi_2_j) \psi_j \right\|^2 \leq \|f\|^2
$$

for any $f \in \text{Range}(K)$, where $P_1$ and $P_2$ are given in Equation (2).

Proof. For any $f \in \text{Range}(K)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j \in \mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R}$, we know, by combining Equation (3) and Lemma 3, that

$$
\left\| \sum_{j \in \sigma} a_j (K^t f, \psi_1_j) \psi_j + \sum_{j \in \sigma^c} a_j (K^t f, \psi_2_j) \phi_j \right\|^2 + \left\| \sum_{j \in \sigma} (1 - a_j) (K^t f, \psi_1_j) \psi_j + \sum_{j \in \sigma^c} (1 - a_j) (K^t f, \psi_2_j) \psi_j \right\|^2
= \|P_1 K^t f\|^2 + \|P_2 K^t f\|^2 - 2 \|P_2 K^t f\|^2 + \text{Re} \langle P_1 K^t f, f \rangle - \text{Re} \langle P_2 K^t f, f \rangle
\geq (2\lambda - \frac{\lambda^2}{2}) \|f\|^2 - (4 - 2\lambda) \text{Re} \langle P_1 K^t f, f \rangle + \text{Re} \langle P_1 K^t f, f \rangle - \text{Re} \langle P_2 K^t f, f \rangle
= (2\lambda - \frac{\lambda^2}{2} - 1) \text{Re} \langle P_1 K^t f, f \rangle + (1 - \frac{\lambda^2}{2}) \text{Re} \langle P_2 K^t f, f \rangle
= (2\lambda - \frac{\lambda^2}{2} - 1) \text{Re} \left( \sum_{j \in \sigma} a_j (K^t f, \psi_1_j) \langle \psi_j, f \rangle + \sum_{j \in \sigma^c} a_j (K^t f, \psi_2_j) \langle \phi_j, f \rangle \right)
\quad + (1 - \frac{\lambda^2}{2}) \text{Re} \left( \sum_{j \in \sigma} (1 - a_j) (K^t f, \psi_1_j) \langle \psi_j, f \rangle + \sum_{j \in \sigma^c} (1 - a_j) (K^t f, \psi_2_j) \langle \phi_j, f \rangle \right).
$$

For the “Moreover” part, we have for any $f \in \text{Range}(K)$ that

$$
\|P_1 K^t f\|^2 = \|P_2 K^t f\|^2 - \text{Re} \langle P_2 K^t f, f \rangle + \text{Re} \langle P_1 K^t f, f \rangle
= \text{Re} \langle P_2 K^t f, P_2 K^t f \rangle - \text{Re} \langle P_2 K^t f, f \rangle + \text{Re} \langle P_1 K^t f, f \rangle
= -(\text{Re} \langle P_2 K^t f, P_1 K^t f + P_2 K^t f \rangle - \text{Re} \langle P_2 K^t f, P_2 K^t f \rangle) + \text{Re} \langle P_1 K^t f, f \rangle
= -\text{Re} \langle P_2 K^t f, P_1 K^t f \rangle + \text{Re} \langle P_1 K^t f, f \rangle \leq \text{Re} \langle P_1 K^t f, f \rangle.
$$
With a similar discussion, we can show that \( \|P_2K^tf\|^2 \leq \text{Re}(P_2K^tf,f) \). Thus,

\[
\left\| \sum_{j \in \sigma} a_j(K^tf, \psi_{1j})\phi_j + \sum_{j \in \sigma} a_j(K^tf, \psi_{2j})\phi_j \right\|^2 + \left\| \sum_{j \in \sigma} (1 - a_j)(K^tf, \psi_{1j})\phi_j + \sum_{j \in \sigma} (1 - a_j)(K^tf, \psi_{2j})\phi_j \right\|^2 \\
\leq \text{Re}(P_1K^tf,f) + \text{Re}(P_2K^tf,f) = \text{Re}(P_1K^tf + P_2K^tf,f) = \|f\|^2.
\]

\[
\square
\]

**Corollary 4.** Suppose that two frames \( \Psi_1 = \{\psi_{1j}\}_{j \in J} \) and \( \Psi_2 = \{\psi_{2j}\}_{j \in J} \) in \( \mathbb{H} \) are woven. Then, for any \( \sigma \subseteq J \), for all \( \lambda \in \mathbb{R} \) and all \( f \in \mathbb{H} \), we have

\[
(2\lambda - \frac{\lambda^2}{2} - 1) \sum_{j \in \sigma} |\langle f, \psi_{1j} \rangle|^2 + (1 - \frac{\lambda^2}{2}) \sum_{j \in \sigma} |\langle f, \psi_{2j} \rangle|^2 \\
\leq \sum_{j \in \sigma} \left| \langle S_{\Psi_1^\perp,\Psi_2^\perp}^1 f, S_{\Psi_1^\perp,\Psi_2^\perp}^{1/2} \psi_{1j} \rangle \right|^2 + \sum_{j \in \sigma} \left| \langle S_{\Psi_1^\perp,\Psi_2^\perp}^1 f, S_{\Psi_1^\perp,\Psi_2^\perp}^{1/2} \psi_{2j} \rangle \right|^2 \\
+ \sum_{j \in \sigma} \left| \langle S_{\Psi_1^\perp,\Psi_2^\perp} f, S_{\Psi_1^\perp,\Psi_2^\perp}^{1/2} \psi_{1j} \rangle \right|^2 + \sum_{j \in \sigma} \left| \langle S_{\Psi_1^\perp,\Psi_2^\perp} f, S_{\Psi_1^\perp,\Psi_2^\perp}^{1/2} \psi_{2j} \rangle \right|^2 \\
\leq \sum_{j \in \sigma} |\langle f, \psi_{1j} \rangle|^2 + \sum_{j \in \sigma} |\langle f, \psi_{2j} \rangle|^2.
\]

(8)

**Proof.** Letting \( K^+ = \text{Id}_\mathbb{H} \) and for any \( \sigma \subseteq J \), taking

\[
a_j = \begin{cases} 
1, & j \in \sigma, \\
0, & j \in \sigma',
\end{cases}
\quad \phi_j = \begin{cases} 
S_{\Psi_1^\perp,\Psi_2^\perp}^{-1/2} \psi_{1j}, & j \in \sigma, \\
S_{\Psi_1^\perp,\Psi_2^\perp}^{-1/2} \psi_{2j}, & j \in \sigma'.
\end{cases}
\]

If, now, we replace \( \psi_{1j}, \psi_{2j} \) and \( f \) in the left-hand inequality of Theorem 2 respectively by \( S_{\Psi_1^\perp,\Psi_2^\perp}^{-1/2} \psi_{1j}, S_{\Psi_1^\perp,\Psi_2^\perp}^{-1/2} \psi_{2j} \) and \( S_{\Psi_1^\perp,\Psi_2^\perp}^{1/2} f \), then

\[
\left\| \sum_{j \in \sigma} \langle f, \psi_{1j} \rangle S_{\Psi_1^\perp,\Psi_2^\perp}^{-1/2} \psi_{1j} \right\|^2 + \left\| \sum_{j \in \sigma} \langle f, \psi_{2j} \rangle S_{\Psi_1^\perp,\Psi_2^\perp}^{-1/2} \psi_{2j} \right\|^2 \\
\geq (2\lambda - \frac{\lambda^2}{2} - 1) \text{Re} \sum_{j \in \sigma} \langle f, \psi_{1j} \rangle \langle \psi_{1j}, f \rangle + (1 - \frac{\lambda^2}{2}) \text{Re} \sum_{j \in \sigma} \langle f, \psi_{2j} \rangle \langle \psi_{2j}, f \rangle \\
= (2\lambda - \frac{\lambda^2}{2} - 1) \sum_{j \in \sigma} |\langle f, \psi_{1j} \rangle|^2 + (1 - \frac{\lambda^2}{2}) \sum_{j \in \sigma} |\langle f, \psi_{2j} \rangle|^2.
\]

This along with Equations (6) and (7) gives the left-hand inequality in Equation (8), and the proof of the right-hand inequality is similar and we omit the details. \( \square \)

**Theorem 3.** Suppose that \( K \in B(\mathbb{H}) \) has a closed range and that \( K \)-frames \( \Psi_1 = \{\psi_{1j}\}_{j \in J} \) and \( \Psi_2 = \{\psi_{2j}\}_{j \in J} \) in \( \mathbb{H} \) are \( K \)-woven. Then, for all \( \sigma \subseteq J \), for any \( \{a_j\}_{j \in J} \in l^\infty(J), \lambda \in \mathbb{R} \) and \( f \in \text{Range}(K) \),

\[
\text{Re} \left( \sum_{j \in \sigma} a_j(K^tf, \psi_{1j})\phi_j + \sum_{j \in \sigma} a_j(K^tf, \psi_{2j})\phi_j \right) - \left\| \sum_{j \in \sigma} a_j(K^tf, \psi_{1j})\phi_j + \sum_{j \in \sigma} a_j(K^tf, \psi_{2j})\phi_j \right\|^2 \\
\leq (1 - \frac{\lambda}{2})^2 \text{Re} \left( \sum_{j \in \sigma} a_j(K^tf, \psi_{1j})\phi_j + \sum_{j \in \sigma} a_j(K^tf, \psi_{2j})\phi_j \right) \\
+ \frac{\lambda^2}{4} \text{Re} \left( \sum_{j \in \sigma} (1 - a_j)(K^tf, \psi_{1j})\phi_j + \sum_{j \in \sigma} (1 - a_j)(K^tf, \psi_{2j})\phi_j \right).
\]
Proof. The proof is similar to Corollary 4 by using Theorem 3, so we omit it.

Corollary 5. Let the two frames $\Psi_1 = \{\psi_j\}_{j \in J}$ and $\Psi_2 = \{\psi_j\}_{j \in J}$ in $\mathbb{H}$ be woven. Then, for any $\sigma \subset J$, for all $\lambda \in \mathbb{R}$ and all $f \in \mathbb{H}$, we have

$$0 \leq \sum_{j \in \sigma} |\langle f, \psi_j \rangle|^2 - \sum_{j \in \sigma} |\langle S_{\Psi_1^{-1}}f, \Psi_1^{-1}\psi_j \rangle|^2 - \sum_{j \in \sigma^c} |\langle S_{\Psi_2^{-1}}f, \Psi_2^{-1}\psi_j \rangle|^2 \\
\leq (1 - \frac{\lambda}{2})^2 \sum_{j \in \sigma} |\langle f, \psi_j \rangle|^2 + \frac{\lambda^2}{4} \sum_{j \in \sigma^c} |\langle f, \psi_j \rangle|^2.$$

Proof. The proof is similar to Corollary 4 by using Theorem 3, so we omit it.

Remark 2. Corollaries 4 and 5 are respectively Theorems 15 and 14 in [34].

We conclude the paper with a double inequality for $K$-weaving frames stated as follows.
Theorem 4. Suppose that K-frames $\Psi_1 = \{\psi_{1j}\}_{j \in \mathcal{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j \in \mathcal{J}}$ in $\mathbb{H}$ are K-woven. Then, for any $\sigma \subset \mathcal{J}$, for all $\{a_j\}_{j \in \sigma} \in \ell^\infty(\mathcal{J})$ and all $f \in \mathbb{H}$, we have

$$\frac{3}{4}\|Kf\|^2 \leq \left\| \sum_{j \in \sigma} a_j(f, \psi_{1j})\psi_{1j} + \sum_{j \in \sigma} a_j(f, \psi_{2j})\psi_{2j} \right\|^2 + \text{Re}\left( \sum_{j \in \sigma} (1 - a_j)(f, \psi_{1j})\langle \phi_j, Kf \rangle + \sum_{j \in \sigma} (1 - a_j)(f, \psi_{2j})\langle \phi_j, Kf \rangle \right),$$

where $P_1$ and $P_2$ are given in Equation (2), and $\Phi = \{\phi_j\}_{j \in \mathcal{J}}$ is a K-dual of $\{\psi_{1j}\}_{j \in \sigma} \cup \{\psi_{2j}\}_{j \in \sigma}$.

Proof. For any $\sigma \subset \mathcal{J}$, for all $\{a_j\}_{j \in \sigma} \in \ell^\infty(\mathcal{J})$ and all $f \in \mathbb{H}$, it is easy to check that $P_1 + P_2 = K$.

By Lemma 1, we get

$$\left\| \sum_{j \in \sigma} a_j(f, \psi_{1j})\phi_{1j} + \sum_{j \in \sigma} a_j(f, \psi_{2j})\phi_{2j} \right\|^2 + \text{Re}\left( \sum_{j \in \sigma} (1 - a_j)(f, \psi_{1j})\langle \phi_j, Kf \rangle + \sum_{j \in \sigma} (1 - a_j)(f, \psi_{2j})\langle \phi_j, Kf \rangle \right) = \|P_1f\|^2 + \text{Re}(P_2f, Kf) \geq \frac{3}{4}\|Kf\|^2.$$

We also have

$$\left\| \sum_{j \in \sigma} a_j(f, \psi_{1j})\phi_{1j} + \sum_{j \in \sigma} a_j(f, \psi_{2j})\phi_{2j} \right\|^2 + \text{Re}\left( \sum_{j \in \sigma} (1 - a_j)(f, \psi_{1j})\langle \phi_j, Kf \rangle + \sum_{j \in \sigma} (1 - a_j)(f, \psi_{2j})\langle \phi_j, Kf \rangle \right)$$

$$= \langle P_1f, P_1f \rangle + \frac{1}{4}\langle P_2f, Kf \rangle + \frac{1}{2}\langle Kf, P_2f \rangle$$

$$= \langle P_1f, P_1f \rangle + \frac{1}{2}\langle (K - P_1)f, Kf \rangle + \frac{1}{2}\langle Kf, (K - P_1)f \rangle$$

$$= \langle Kf, Kf \rangle - \frac{1}{2}[(P_1f, Kf) - \langle P_1f, P_1f \rangle] - \frac{1}{2}[(Kf, P_1f) - \langle P_1f, P_1f \rangle]$$

$$= \langle Kf, Kf \rangle - \frac{1}{2}(P_1f, P_2f) - \frac{1}{2}(P_2f, P_1f)$$

$$= \frac{3}{4}\|Kf\|^2 + \frac{1}{4}\langle P_1f, P_2f \rangle + \frac{1}{2}(Kf, P_2f) - \frac{1}{2}(P_2f, P_1f)$$

$$= \frac{3}{4}\|Kf\|^2 + \frac{1}{4}(P_1f, P_2f, Kf) + \frac{1}{2}(P_1f, P_2f) - \frac{1}{2}(P_2f, P_1f)$$

$$\leq \frac{3}{4}\|K\|^2\|f\|^2 + \frac{1}{4}\|P_1 - P_2\|^2\|f\|^2 = \frac{3}{4}\|K\|^2\|f\|^2 + \frac{1}{4}\|P_1 - P_2\|^2\|f\|^2,$$

and the proof is over. \(\square\)

Remark 3. Theorem 3 in [35] can be obtained when taking $K = \text{Id}_\mathbb{H}$ in Theorem 4.

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