Hawking radiation in GHS blackhole, Effective action and Covariant Boundary condition

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Abstract
We exploit the expression for the anomalous (chiral) effective action to obtain the Hawking radiation from the GHS (stringy) blackhole falling in the class of the most general spherically symmetric blackholes \(\sqrt{-g} \neq 1\), using only covariant boundary condition at the event horizon. The connection between the anomalous and the normal energy-momentum tensors is also established from the effective action approach.

Keywords: Hawking radiation, Effective action, Covariant Boundary condition

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Introduction:

There are several derivations of Hawking radiation. The most direct is Hawking’s original one \[1\] \[2\] which computes the Bogoliubov coefficients between in and out states for a body collapsing to form a blackhole. Another elegant derivation based on Euclidean quantum gravity \[3\] has been interpreted as a calculation of tunnelling through classically forbidden trajectories \[4\]. Therefore it remains of interest to consider alternative derivations since although there are many approaches, none is completely clinching or conclusive.

Recently, a new derivation of the Hawking effect has been given in \[5\] \[6\]. It relies on the cancellation of anomalies\(^1\) (gravitational and gauge) at the horizon. It was shown that effective field theories become two dimensional and chiral near the event horizon of a black hole by the process of dimensional reduction. This leads to the occurrence of gravitational and gauge anomalies. The Hawking flux is necessary to cancel these anomalies. Further applications of the approach was made in \[19\]-\[17\].

The approach of \[5\] \[6\] was further generalised in \[18\] where it was shown that unlike in \[5\] \[6\], the complete analysis was feasible in terms of covariant expressions only. The flux from a charged black hole was correctly determined by a cancellation of the covariant anomaly with the boundary condition being the vanishing of the covariant current (and energy-momentum tensor) at the horizon. The method was soon extended in \[19\] to discuss Hawking radiation from the Garfinkle-Horowitz-Strominger (GHS) blackhole spacetime in string theory which is an example of the most general spherically symmetric blackhole \(\sqrt{-g} \neq 1\) \[20\] \[21\].

From the analysis of \[5\] \[6\] \[18\] \[19\] it appears therefore that covariant boundary conditions at the horizon play a fundamental role. Indeed in \[22\], the arguments of \[5\] \[6\] which imply that effective field theories are chiral and two dimensional near the horizon was adopted. Then by exploiting known structures of the two dimensional effective actions, deductions for the currents and the energy-momentum tensors have been made by the imposition of covariant boundary conditions only at the horizon. The Hawking flux from charged black holes is correctly reproduced in this manner.

In this paper, we shall adopt the method in \[22\] to discuss Hawking radiation from the GHS blackhole. However, we shall ignore effects to the Hawking flux due to scatterings by the gravitational potential, for example the greybody factor \[23\].

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\(^1\)For a detailed discussion of anomalies see \[7\] \[8\].
General Setting and Effective Action:

We are interested in discussing Hawking radiation from GHS blackholes defined by the metric

\[ ds^2 = f(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2d\Omega \]  

(1)

where,

\[ f(r) = (1 - \frac{2Me^{\phi_0}}{r})\left(1 - \frac{Q^2e^{3\phi_0}}{Mr}\right)^{-1} \]

\[ h(r) = (1 - \frac{2Me^{\phi_0}}{r})\left(1 - \frac{Q^2e^{3\phi_0}}{Mr}\right) \]

(2)

with \( \phi_0 \) being the asymptotic constant value of the dilaton field. We consider the case when \( Q^2 < 2e^{-2\phi_0}M^2 \) for which the above metric describes a blackhole with a horizon situated at \( r_H = 2Me^{\phi_0} \).

As mentioned earlier, with the aid of dimensional reduction technique, the effective field theory near the horizon becomes a two dimensional chiral theory with a metric given by the “\( r-t \)” sector of the full spacetime metric (1) near the horizon. The important point to note however is that the determinant of the GHS metric \( \sqrt{-g} \neq 1 \). The theory away from the horizon is not chiral and hence is anomaly free. We now adopt the methodology in [22]. For a two dimensional theory the expressions for the anomalous (chiral) and normal effective actions are known [24, 25]. We shall use only the anomalous form of the effective action for deriving the Hawking flux. Also we consider only the gravitational part of the effective action since we have only gravitational anomaly in the region near the horizon in the GHS case. The energy-momentum tensor in the region near the horizon is computed by taking appropriate functional derivative of the chiral effective action. Next, the parameters appearing in the solution is fixed by imposing the vanishing of covariant energy-momentum tensor at the horizon. Once these are fixed, the Hawking flux is obtained by taking the asymptotic (\( r \rightarrow \infty \)) limit of the chiral energy-momentum tensor.

With the above methodology in mind, we write down the gravitational part of the anomalous (chiral) effective action (describing the theory near the horizon) [25]

\[ \Gamma_{(H)} = -\frac{1}{3}z(\omega) \]

(4)

where \( \omega_\mu \) is the spin connection and

\[ z(\nu) = \frac{1}{4\pi} \int d^2x \ d^2y \ \epsilon^{\mu\nu} \partial_\mu v_\nu(x) \Delta_g^{-1}(x,y)\partial_\mu[(\epsilon^{\rho\sigma} + \sqrt{-g}g^{\rho\sigma})v_\sigma(y)] \]

(5)

where \( \Delta_g = \nabla^\mu \nabla_\mu \) is the laplacian in this background.

The energy-momentum tensor is computed from a variation of this effective action. To get their covariant forms in which we are interested, one needs to add appropriate local polynomials [25]. Here we quote the final result for the covariant energy-momentum tensor [23]:

\[ T^\mu_\nu = \frac{1}{48} \left( \frac{1}{48} D^\mu GD_\nu G - \frac{1}{24} D^\mu D_\nu G + \frac{1}{24} \delta^\mu_\nu R \right) \]

(6)

where \( D_\mu \) is the chiral covariant derivative

\[ D_\mu = \nabla_\mu - \tilde{\epsilon}_{\mu\nu} \nabla_\nu = -\tilde{\epsilon}_{\mu\nu}D_\nu \]

(7)
and $\bar{\epsilon}^{\mu\nu} = \epsilon^{\mu\nu}/\sqrt{-g}$, $\bar{\epsilon}_{\mu\nu} = \sqrt{-g} \epsilon_{\mu\nu}$ are two dimensional antisymmetric tensors for the upper and lower cases with $\bar{\epsilon}^{rt} = \epsilon_{rt} = 1$. Also $R$ is the two dimensional Ricci scalar given by

$$R = \frac{h}{f} f'' + \frac{f'''}{2} - \frac{f' f}{2 f^2}$$

and $G$ is given by

$$G(x) = \int d^2 y \Delta_g^{-1}(x, y) \sqrt{-g} R(y)$$

satisfying

$$\nabla^\mu \nabla_\mu G = R.$$ (10)

The solution for $G$ reads

$$G = G_o(r) - 4 p t + q ; \partial_r G_o = - \frac{1}{\sqrt{f h}} \left( \frac{h}{f} f' + z \right)$$ (11)

where $p, q$ and $z$ are constants. Also note that $G_o$ is a function of $r$ only.

By taking the covariant divergence of (6), we get the anomalous Ward identity

$$\nabla^\mu T^{\mu \nu} = \frac{1}{96 \pi} \epsilon_{\nu \rho} \partial^\rho R .$$ (12)

The anomalous term is the covariant gravitational anomaly. This Ward identity was also obtained from different considerations in [18].

In the region away from the horizon, the effective theory is given by the standard effective action $\Gamma$ of a conformal field with a central charge $c = 1$ in this blackhole background [21] and reads:

$$\Gamma = \frac{1}{96 \pi} \int d^2 x d^2 y \sqrt{-g} R(x) \frac{1}{\Delta_g(x, y)} \sqrt{-g} R(y).$$ (13)

The energy-momentum tensor $T_{\mu\nu(o)}$ in the region outside the horizon is given by

$$T_{\mu\nu(o)} = \frac{2}{\sqrt{-g} \delta g^{\mu\nu}} \frac{\delta \Gamma}{\delta g^{\mu\nu}} = \frac{1}{48 \pi} \left( 2 g_{\mu\nu} R - 2 \nabla_\mu \nabla_\nu G + \nabla_\mu G \nabla_\nu G - \frac{1}{2} g_{\mu\nu} \nabla^\rho G \nabla_\rho G \right)$$ (14)

and satisfies the normal Ward identity

$$\nabla_\mu T^{\mu \nu(o)} = 0 .$$ (15)

Energy Flux:

In this section we calculate the energy flux by using the expression for the covariant energy-momentum tensor (6). We will show that the results are the same as that obtained by the anomaly cancellation (consistent or covariant) method [19].

Using (7) and the solution for $G(x)$ (11), the $r - t$ component of the anomalous (chiral) covariant energy-momentum tensor (6) becomes

$$T^{rt}(r) = \frac{1}{12 \pi} \left( \frac{h}{f} \right) \left[ p - \frac{1}{4} \left( \sqrt{\frac{h}{f}} f' + z \right) \right]^2 + \frac{1}{24 \pi} \left( \frac{h}{f} \right) \left[ \sqrt{\frac{h}{f}} f' \left( p - \frac{1}{4} \left( \sqrt{\frac{h}{f}} f' + z \right) \right) + \frac{1}{4} h f'' - \frac{f'}{8} \left( \frac{h}{f} f' - h' \right) \right].$$ (16)
Now implementing the boundary condition namely the vanishing of the covariant energy-momentum tensor at the horizon, $T_{r t}(r_H) = 0$, leads to

$$p = \frac{1}{4} \left[ z \pm \sqrt{f'(r_H)h'(r_H)} \right]; \quad f'(r_H) \equiv f'(r = r_H).$$  \hspace{1cm} (17)

Using either of the above solutions in (16) yields

$$T_{r t} = \frac{1}{192\pi} \sqrt{\frac{h}{f}} \left[ f'(r_H)h'(r_H) - 2h f'^2 + 2hf'' + f'h' \right].$$  \hspace{1cm} (18)

This expression is in agreement with that given in [19].

The energy flux is now given by the asymptotic ($r \to \infty$) limit of the anomaly free energy-momentum tensor (14). Now from (12), we observe that the anomaly vanishes in this limit. Hence the energy flux is abstracted by taking the asymptotic limit of (18). This yields

$$T_{r t}(r \to \infty) = \frac{1}{192\pi} f'(r_H)h'(r_H)$$  \hspace{1cm} (19)

which correctly reproduces the Hawking flux [26, 19].

We now consider the normal (anomaly free) energy-momentum tensor (14) to establish its relation with the chiral (anomalous) energy-momentum tensor (18). The $r - t$ component of $T^\mu_\nu(o)$ is given by

$$T_{r t(o)} = -\frac{1}{12\pi} \sqrt{\frac{h}{f}} zp.$$  \hspace{1cm} (20)

The asymptotic form of the above equation (20) must agree with the asymptotic form of (18)\(^3\). This yields:

$$p = -\frac{z}{4}.$$  \hspace{1cm} (21)

Solving (17) and (21) gives two solutions for $p$ and $z$:

$$p = \frac{1}{8} \sqrt{f'(r_H)h'(r_H)}; \quad z = -\frac{1}{2} \sqrt{f'(r_H)h'(r_H)}$$

$$p = -\frac{1}{8} \sqrt{f'(r_H)h'(r_H)}; \quad z = \frac{1}{2} \sqrt{f'(r_H)h'(r_H)}.$$  \hspace{1cm} (22)

Using either of the above solutions in (16) and (20) yields (18) and

$$T_{r t(o)}(r) = \frac{1}{192\pi} \sqrt{\frac{h}{f}} f'(r_H)h'(r_H).$$  \hspace{1cm} (23)

The above expressions (18) and (23) yields the equation between the chiral (anomalous) and the normal energy-momentum tensors [19].

**Discussions:**

In this paper, we have employed the effective action approach (as was done in [22]) to derive Hawking flux from the GHS blackhole in string theory. As has been stressed in [22], generally such approaches require some other boundary condition apart from conditions at the horizon, as for example, the vanishing of ingoing modes at infinity [27, 28]. In this approach we only need covariant boundary conditions, the importance of which was first stressed in [22]. Another important ingredient in the entire analysis is the expression for the anomalous (chiral) effective action (which yields anomalous Ward identity having covariant gravitational anomaly). The unknown parameters in the covariant energy-momentum tensor derived from this anomalous effective action were fixed by a boundary condition- namely the vanishing of the covariant energy-momentum tensor at the event horizon of the GHS blackhole. Finally, the energy flux was extracted by taking the $r \to \infty$ limit of the chiral covariant energy-momentum tensor. The relation between the chiral and the normal energy-momentum tensors is also established by requiring that both of them match in the asymptotic limit which is possible since the anomaly vanishes in this limit.

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\(^3\)This is true since the anomaly in the asymptotic limit ($r \to \infty$) vanishes as can be readily seen from [22].
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