ON THE VOLUME CONJECTURE FOR SMALL ANGLES

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Dedicated to Louis Kauffman on the occasion of his 60th birthday

ABSTRACT. Given a knot in 3-space, one can associate a sequence of Laurent polynomials, whose nth term is the nth colored Jones polynomial. The Generalized Volume Conjecture states that the value of the nth colored Jones polynomial at \( \exp(2\pi i \alpha/n) \) is a sequence of complex numbers that grows exponentially, for a fixed real angle \( \alpha \). Moreover the exponential growth rate of this sequence is proportional to the volume of the 3-manifold obtained by \((1/\alpha,0)\) Dehn filling. In this paper we will prove that (a) for every knot, the limsup in the hyperbolic volume conjecture is finite and bounded above by an exponential function that depends on the number of crossings. (b) Moreover, for every knot \( K \) there exists a positive real number \( \alpha(K) \) (which depends on the number of crossings of the knot) such that the Generalized Volume Conjecture holds for \( \alpha \in [0, \alpha(K)] \). Finally, we point out that a theorem of Agol-Storm-W.Thurston proves that the bounds in (a) are optimal, given by knots obtained by closing large chunks of the weave.

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1. INTRODUCTION

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1.1. The volume conjecture. The Volume Conjecture (in short, VC) connects two very different approaches to knot theory, namely Topological Quantum Field Theory and Riemannian (mostly Hyperbolic) Geometry.

In its fundamental form, the VC states that for every hyperbolic knot $K$ in $S^3$ we have:

\[
\lim_{n \to \infty} \frac{\log |\text{ev}_{\alpha,n}(J_K(n))|}{n} = \frac{1}{2\pi} V_K,
\]

where

- $\text{ev}_{\alpha,n}(f)$ denotes the evaluation of a rational function $f(q)$ at $q = e^{2\pi i \alpha/n}$,
- $J_K(n) \in \mathbb{Z}[q^\pm]$ is the Jones polynomial of a knot colored with the $n$-dimensional irreducible representation of $\text{SL}_2$, normalized so that it equals to 1 for the unknot (see \cite{H}, \cite{F}), and
- $V_K$ is the volume of the knot complement $S^3 - K$, using the unique complete hyperbolic metric; \cite{Th}.

The VC was formulated in this form by H. and J. Murakami \cite{MM}, who reinterpreted an earlier version due to Kashaev, \cite{K}.

It is natural to ask what happens when we use evaluations $\text{ev}_{\alpha,n}$ of the colored Jones polynomial for other complex numbers $\alpha$. In \cite{Gu}, Gukov proposed a Generalized Volume Conjecture (in short, GVC), which states that for every hyperbolic knot $K$ in $S^3$ and every irrational $\alpha$ near 1 (or $\alpha = 1$), we have:

\[
\lim_{n \to \infty} \frac{\log |\text{ev}_{\alpha,n}(J_K(n))|}{n} = \frac{1}{2\pi} V_K(1/\alpha),
\]

where

- $V_K(1/\alpha)$ is the volume of the $(1/\alpha, 0)$ Dehn filling of the knot complement $S^3 - K$ (that is, the Dehn filling corresponding to $M^{2\pi i \alpha} L^0 = 1$ where $(M, L)$ is the meridian and longitude of the knot); see \cite{Th}.

The GVC is mostly about hyperbolic knots and hyperbolic Dehn fillings. In case a manifold (with or without boundary) is not hyperbolic, we will declare its corresponding Gromov-Thurston volume to be zero.

There are two rather independent parts in the GVC:

(a) To show that the limit exists in (2),

(b) To identify the limit with the volume of the corresponding Dehn filling.

At the moment, the GVC is known for the $4_1$ knot and certain values of $\alpha$; see Murakami, \cite{MM}.

In the following, we will refer to the parameter $\alpha$ in the GVC as the angle, making contact with standard terminology from hyperbolic geometry.

One may further ask what happens to the GVC when the angle $\alpha$ is small. For example, when $\alpha = 0$, then $\text{ev}_{0,n}(J_K(n)) = 1$ for all knots $K$ and integers $n$, thus the corresponding limit on the left hand side of (2) vanishes. To attach a meaning to the right hand side of (2), we need to use the volume of an appropriate $\text{SL}_2(\mathbb{C})$ representation of the knot complement. Unfortunately, for small angles $\alpha$, the corresponding Dehn fillings are not hyperbolic. But fortunately, there is a natural $\text{SL}_2(\mathbb{C})$ representation to consider when $\alpha = 0$, namely the trivial one. This is in agreement with physics, where the case $\alpha = 0$ is a classical limit of a quantum theory, and corresponds to the only $\text{SL}_2(\mathbb{C})$ flat connection on $S^3$, namely the trivial one. When $\alpha$ is small and real, we will define $V_K(1/\alpha, 0)$ to be the volume of the reducible $\text{SL}_2(\mathbb{C})$ representation

\[
\rho_\alpha : \pi_1(S^3 - K) \to \text{SL}_2(\mathbb{C}), \quad \rho_\alpha(M) = \begin{pmatrix} e^{2\pi i \alpha/n} & 0 \\ 0 & e^{-2\pi i \alpha/n} \end{pmatrix}.
\]

(\text{where $M$ is a meridian}). A simple calculation shows that $V_K(1/\alpha, 0) = 0$ for $\alpha$ small and positive.

Now, we can formulate the GVC for small positive angles $\alpha$.

Thus, we have formulated a GVC for $\alpha$ near 0 and $\alpha$ near 1, using representations near the trivial one and a discrete faithful, respectively. How can we connect and explain our choices for other angles $\alpha$? A natural answer to this question requires analyzing asymptotics of solutions of difference equations with a parameter. This is a different subject that we will not discuss here; instead we will refer the curious reader to \cite{GG}, and forthcoming work of the first author.
1.2. Upper bounds and confirmation of the volume conjecture for small angles. Our results are the following:

**Theorem 1.** For every knot $K$ with $c + 2$ crossings and every $\alpha > 0$, we have

$$\limsup_{n \to \infty} \frac{\log |e_{\alpha,n}(J_K(n))|}{n} \leq c \log 4.$$  

**Theorem 2.** For every knot $K$, there exists a positive angle $\alpha(K) > 0$ such that the GVC holds for all $\alpha \in [0, \alpha(K))$.

In fact, the proof of Theorem 2 reveals that we can take $\alpha(K)$ to be a function that depends on the number of crossings of $K$.

Notice that the Hyperbolic Volume Conjecture is the case of $\alpha = 1$, which corresponds to a complete hyperbolic structure. On the other hand, Theorem 2 deals with small cone-angle fillings, which are expected to be spherical structures of zero volume.

Theorem 1 follows easily from a stronger result. Observe that when $|q| = 1$, then

$$|J_K(n)(q)| \leq ||J_K(n)||_1$$

where the $l^1$-norm of a Laurent polynomial $f = \sum_k c_k q^k$ is given by:

$$||\sum_k c_k q^k||_1 = \sum_k |c_k|.$$

Then, we have the following:

**Theorem 3.** For every knot $K$ of $c + 2$ crossings and every $n$ we have:

$$||J_K(n)||_1 \leq n^{c_4}.$$  

1.3. Relation with hyperbolic geometry, and optimal bounds. When $\alpha = 1$, the upper bound in Theorem 1 is not optimal, and does not reveal any relationship between the lim sup and hyperbolic geometry. Our next theorem fills this gap.

**Theorem 4.** For every knot $K$ with $c + 2$ crossings we have

$$2\pi \limsup_{n \to \infty} \frac{\log |e_{\nu,n}(J_K(n))|}{n} \leq v_8 c,$$

where $e_{\nu,n} = e_{\nu,1,n}$, $v_8 = 8\Lambda(\pi/4) \approx 3.66386$ is the volume of the regular ideal octahedron.

Using an ideal decomposition of a knot complement by placing one octahedron per crossing, it follows that for every knot $K$ with $c + 2$ crossings, we have

$$V(K) \leq v_8 c.$$  

On the other hand, if the volume conjecture holds for $\alpha = 1$, then

$$2\pi \limsup_{n \to \infty} \frac{\log |e_{\nu,n}(J_K(n))|}{n} = V(K) \leq v_8 c.$$  

One may ask whether (4) (and therefore, whether the bound in Theorem 4) is optimal. Optimality is at first sight surprising, since it involves all knots (and not just alternating ones) and their number of crossings (which carries little known geometric information). In conversations with I.Agol and D.Thurston, it was communicated to us that the upper bound in (4) is indeed optimal. Moreover a class of knots that achieves (in the limit) the optimal ratio of volume by number of crossings is obtained by taking a large chunk of the...
following weave, and closing it up to a knot:

The complement of the weave has a complete hyperbolic structure associated with the square tessellation of the Euclidean plane:

Optimality follows along similar lines as the Appendix of [La], using a stronger estimate for the lower bound of the volume of Haken manifolds, cut along an incompressible surface. The stronger statement is the following result which will appear in subsequent work of Agol-Storm-W. Thurston, [AST]. Its proof uses, among other things, work of Perelman.

**Theorem 5.** ([AST]) If $M$ is a hyperbolic finite volume 3-manifold containing a properly imbedded orientable, boundary incompressible, incompressible surface $S$, then

$$V(M) \geq V(\text{Guts}(M - \text{int}(\text{nbd}(S)))),$$

where $V$ stands for volume, and the Guts terminology are defined in [Ag].

The reader may compare (4) with the following result of Agol-Lackenby-D. Thurston [La]:

**Theorem 6.** If $K$ is an alternating knot with a planar projection with $t$ twists, then

$$v_3(t - 1)/2 < V(K) < 10v_3(t - 1),$$

where $v_3 = 2\Lambda(\pi/3) \approx 1.01494$ is the volume of the regular ideal tetrahedron. Moreover, the class of knots obtained by Dehn filling on the chain link has asymptotic ratio of volume by twist number equal to $10v_3$. The corresponding tessellation of the Euclidean plane is given by the star of David.

1.4. **The main ideas.** Our results have quick proofs, and require only a small dose of elementary analysis, and an appropriate view of these powerful quantum invariants of knots.

We already saw how Theorem 4 follows from the stronger Theorem 3.

To prove Theorem 3 we will make use of a state-sum definition of the colored Jones polynomials, where the local weights are given by $R$-matrices. In [GL], we used the basic fact that the local weights are $q$-hypergeometric, in order to deduce that the sequence of colored Jones polynomials satisfies a linear recursion relation (the recursion depends on the knot, of course). In our case, we will focus on the fact that the local weights take values in $\mathbb{Z}[q^2]$ in order to give elementary estimates for their $l^1$-norm.
Theorem 2 is trickier. Among other things, the proof uses a key integrality property (due to Habiro) of the cyclotomic transform of the colored Jones function. To the best of our knowledge, this is a first application of the cyclotomic transform. In more detail, we will separate out the dependence of the color in the colored Jones polynomial $J_K(n)$, via the cyclotomic transform. This replaces $J_K(n)$ by a sequence $C_K(n)$ (for $n \in \mathbb{N}$) of rational functions. Habiro proved that $C_K(n)$ are actually polynomials. Using the $l^1$-estimates on $J_K(n)$, one can deduce only a weak estimate for the Mahler measure of $C_K(n)$. Due to the specific structure of the inverse cyclotomic transform, one can prove a useful $l^1$-estimate for $C_K(n)$. Using that estimate, and some elementary analysis, it is easy to finish the proof of Theorem 2.

In Section 5, we exploit the fact that the colored Jones function and the cyclotomic function is a solution to a linear $q$-difference equation. Using elementary methods, we give quadratic degree bounds for solutions of $q$-difference equations, and exponential bounds for the $l^1$ norms of solutions of integral (in the sense of Laurent polynomials) $q$-difference equations. Based on experimental evidence, we conjecture that the cyclotomic function is a solution of an integral $q$-difference equation.

The upper bounds for the limsup in Theorem 1 are not sharp for $\alpha = 1$, since they are obtained by $l^1$ estimates of the local weights. Using the fact that the local weights are given by a ratio of 5 quantum factorials, and the fact that the asymptotics of quantum factorials are governed by the Lobachevsky function, in Section 4 we give better bounds, stated in Theorem 4, which are linear in the number of crossings, and involve the volume of the regular ideal octahedron. In addition, we conjecture that our improved bounds are optimal.

Finally, in two appendices we discuss the Volume Conjecture for the Borromean rings, and the Generalized Volume Conjecture for torus knots.

2. Proof of Theorem 3

As we mentioned above, we will make use of an $R$-matrix state sum definition of the colored Jones polynomial $J_K(n)$, discussed, for example, in [GL, Sec.3]. Consider a long knot $K'$ whose closure is $K$, and fix a positive integer $n$. To compute $J_K(n)$, we follow the following algorithm:

- Assign angle variables $k$ at each crossing, and let $k = (k_1, k_2, \ldots)$.
- Color each part-arc of the knot projection such that around each crossing the color of the part-arcs is given by:

$$\begin{array}{ccc}
\text{b} + k & \text{a} - k & \text{b} - k \\
\text{a} & \text{k} & \text{a} + k \\
\text{b} & \text{k} & \text{b}
\end{array}$$

There is a unique coloring of the part-arcs such that the two broken part-arcs have color 0.
- Assign local weights $f(n; a, b, k) \in \mathbb{Z}[v^{\pm 1/2}]$ at each crossing, and for a fixed coloring $k$, let $F(n, k)$ denote the product over all crossings of the corresponding local weights, times $v$ raised to a linear form on $n, k$.
- Form the sum

\[
J_K(n) = \sum_k F(n, k).
\]

Strictly speaking, in [GL] we discussed the above algorithm for knots which are closures of braids, but a similar algorithm works for planar projections of knots as well. This follows from the following figure:

which moves crossings (positive or negative) into standard upright position, by an isotopy that creates local minima/maxima and no further crossings. The local minima/maxima give rise to an additional multiplicative factor (a monomial in $v$ raised to a linear form in $n, k$) in $F(n, k)$, and does not affect the estimates below.
In [GL] we used the fact that the local weights \( R_\pm : \mathbb{Z}^5 \to \mathbb{Z}[v^{\pm 1/2}] \), are \( q \)-holonomic functions, in order to deduce from first principles that \( J_K \) is \( q \)-holonomic, and thus satisfies a linear recursion relation.

In our case, we will make use of the specific integral form of the local weights in order to deduce our result. Let us recall the specific form of the local weights, from [GL Sec.3],

\[
R_+(n; a, b, k) := (-1)^k v^{-(n-1-2a)(n-1-2b)+k(k-1)/2} \binom{b+k}{k} \{n-1+k-a\}_k,
\]

\[
R_-(n; a, b, k) := v^l(n-1-2a-2b)(n-1-2b+2k+k(k-1)/2) \binom{a+k}{k} \{n-1+k-b\}_k,
\]

where \( v = q^{1/2} \), and for \( a, b \in \mathbb{N} \), we define the \( q \)-integers, \( q \)-factorial and \( q \)-binomial coefficients by:

\[
\{a\} := v^a - v^{-a}, \quad \{a\}! := \prod_{i=1}^{a} \{a\}, \quad \{a\}_b := \frac{\{a\}!}{\{a-b\}!}, \quad \binom{a}{b} := \frac{\{a\}!}{\{b\}!\{a-b\}!}
\]

For a fixed coloring \( k \), the colors at the part-arcs of \( K' \) are linear forms on \( k \) and \( n \), and the contribution is nonzero only when all these linear forms are nonnegative and less than \( n \). In particular, each of the angle variables \( k \) has to be \( 0 \leq k < n \). Observe that the \( l^1 \) norm satisfies the inequalities

\[
\|f + g\|_1 \leq \|f\|_1 + \|g\|_1, \quad \|fg\|_1 \leq \|f\|_1 \|g\|_1,
\]

Moreover, \( \binom{m}{k} \in \mathbb{N}[v^{1/2}] \) and

\[
\binom{m}{k} = \binom{m}{k} \leq 2^m \leq 2^n
\]

if \( m \leq n \). Moreover,

\[
\|\{n-1+k-a\}_k\|_1 \leq 2^k \leq 2^n
\]

for \( k \leq n \). Combining with the above formulas for \( R_\pm \), it follows that \( \|R_\pm (n; a, b, k)\|_1 \leq 4^n \). Since

\[
R_\pm(n; 0, 0, k) = \delta_{k,0},
\]

and since we can always choose a breaking of a knot so that the broken arc ends under the first crossing and out of the last crossing, it follows that we can ignore at least two angle variables corresponding to the broken part-arcs. Thus, \( \|F(n, k)\|_1 \leq 4^n \), where \( c + 2 \) is the number of crossings of \( K \). Since \( k \) has to be \( 0 \leq k < n \), \( \|\cdot\|_1 \) implies that we have at most \( n^c \) choices for the angle variables. The result follows.

\textbf{Remark 2.1.} There are several other formulations of the \( n \)-th colored Jones polynomial, for example coming from the Kauffman bracket skein module. Unfortunately, using the Kauffman bracket skein module formulation, it is hard to prove Theorem 2 since the number of crossings of an \( n \)-parallel of a knot with \( c + 2 \) crossings is \((c + 2)n^2\), a quadratic function of \( n \). Nevertheless, the Kauffman bracket skein module can give good estimates of the min and max degree (and their difference, the \textit{span}) of the \( n \)-th colored Jones polynomial. Compare with [Le, Prop.1.2] of the second author, who observes that

\[
\text{span}(J_K(n)) \leq cn^2 + O(n).
\]

\section{Proof of Theorem 2}

\subsection{A reduction of Theorem 2}

The proof of Theorem 2 will use the cyclotomic expansion of the colored Jones polynomial, introduced by Habiro, and an improved Mahler-type estimated, communicated to us by D. Boyd.

\textbf{Definition 3.1.} Given a function \( f : \mathbb{N} \to \mathbb{Q}(q) \), we define its \textit{cyclotomic transform} \( Cf : \mathbb{N} \to \mathbb{Q}(q) \) by:

\[
Cf(n) = \sum_{k=0}^{\infty} C(n, k) f(k)
\]


where $C(n, k)$ is given by:

$$C(n, k) := \frac{1}{q^{n/2} - q^{-n/2}} \prod_{j=n-k}^{n+k} (q^{j/2} - q^{-j/2})$$

$$= \prod_{j=1}^{k}((q^{n/2} - q^{-n/2})^2 - (q^{j/2} - q^{-j/2})^2)$$

$$= \prod_{j=1}^{k}((q^{n/2} + q^{-n/2})^2 - (q^{j/2} + q^{-j/2})^2).$$

The cyclotomic transform has an inverse $C^{-1} f : \mathbb{N} \to \mathbb{Q}(q)$ defined by:

$$C^{-1} f(n) = \sum_{k=0}^{\infty} R(n, k) f(k)$$

where $R(n, k)$ is given by:

$$R(n, k) = (-1)^{n-k} \frac{\{2k\}}{(2n+1)!2n} \left[ \begin{array}{c} 2n \\ n-k \end{array} \right]$$

where for $a \in \mathbb{N}$, we define:

$$[a] := \{a \over 1\} = \frac{\theta^{a/2} - \theta^{-a/2}}{\theta^{1/2} - \theta^{-1/2}}$$

(and $\theta = q^{1/2}$). Notice that $R(n, k) = C(n, k) = 0$ for $k > n$, so the above sums are finite.

**Definition 3.2.** If $J_K : \mathbb{N} \to \mathbb{Z}[q^{\pm}]$ denotes the colored Jones function of a knot $K$, we define the cyclotomic function of $K$ by $C_K = C J_K$.

A key result of Habiro is that the cyclotomic function of a knot takes values in $\mathbb{Z}[q^{\pm}]$; see [H1].

**Proof.** (of Theorem 2) Let us apply the cyclotomic transform to $J_K(n)$, in order to isolate the dependence of the color. By the above definition, we have:

$$J_K(n) = \sum_{k=0}^{n} C(n, k) C_K(k) = 1 + \sum_{k=1}^{n} C(n, k) C_K(k).$$

where

$$\{2n+1\}!2n|C_K(n) = \sum_{k=0}^{n} (-1)^{n-k} \{2k\} \left[ \begin{array}{c} 2n \\ n-k \end{array} \right] J_K(k)$$

Let us assume for the moment the following:

**Theorem 7.** For every knot $K$ we have:

$$||C_K(n)||_1 \leq e^{Cn + O(\log n)}$$

Here, and below, the $O(f(n))$ notation means that the error is bounded by a constant times $f(n)$.

The reader may wonder what we gained starting from $J_K$, going to $C_K$, and then back to $J_K$. The point of Equation (11) is that it separates the dependence of $J_K(n)$ on the color $n$, and Theorem 7 gives exponential bounds for the $l^1$ norm of the polynomials $C_K(n)$.

For $q = e^{2\pi i \alpha/n}$ and $\alpha$ small and positive, and $0 < k < n$, $C(n, k)$ becomes small. Using Lemma 3.3 below and Theorem 3 it follows that when $0 < k < n$, then

$$|ev_{\alpha,n}(C_K(k)C(n, k))| \leq e^{Ck + O(\log k)} |3 \sin(\pi \alpha)|^{2k} = e^{C\alpha k + O(\log k)},$$
where \( C'(\alpha) := C + 2\log(3|\sin(\pi\alpha)|) \). Now, choose \( \alpha \) small enough so that \( C'(\alpha) < 0 \), and then choose \( k_0 = k_0(\alpha) \) so that \( C'(\alpha)k + O(\log k) < C'(\alpha)/2k \) for \( k \geq k_0 \). It follows that for \( k_0 < k < n \), we have
\[
|\text{ev}_{\alpha,n}(C_K(k)C(n,k))| \leq e^{C'(\alpha)/2k}
\]
and the last term is in absolute value less than 1. Moreover, for \( 0 < k < k_0 \), we have
\[
\lim_{n \to \infty} \text{ev}_{\alpha,n}(C_K(k)C(n,k)) = 0.
\]
This and Equation 11 implies Theorem 2.

**Lemma 3.3.** If \( 0 < \alpha < \pi/6 \), then for every \( 0 \leq k < n \) we have:
\[
|\text{ev}_{\alpha,n}(C(n,k))| \leq |3\sin(\pi\alpha)|^{2k}.
\]

**Proof.** Evaluating at \( q = e^{2\pi i\alpha}/n \), we have:
\[
|\text{ev}_{\alpha,n}(C(n,k))| = \prod_{k=1}^{n} |q^n - q^k||q^n - q^{-k}|.
\]

Since \( 0 < \alpha < \pi/6 \) and \( q^n = e^{2\pi i\alpha} \) it follows that for all \( 0 < k < n \) we have:
\[
|q^n - q^k| \leq |q^n - q| < |q^n - 1| = |2\sin(\pi\alpha)|
\]
and
\[
|q^n - q^{-k}| \leq |q^n - q^{-1}| < |q^{n+1} - 1| = |2\sin(\pi\alpha(n+1)/n)| \leq |3\sin(\pi\alpha)|.
\]
The result follows.

It remains to prove Theorem 7.

**3.2. Proof of Theorem 7.** Consider Equation 12. First of all it gives:
\[
\text{span}C_K(n) \leq C^m n^2.
\]
Since \( ||\left[ \begin{array}{c} 2n \\ n - k \end{array} \right] ||_1 = (n-k) \leq 2n \), Equation 12 and Theorem 3 imply that
\[
||\{2n+1\}[2n]C_K(n)||_1 \leq n^c 4^{(c+1)n}.
\]
A priori, this estimate is weak and implies an exponential upper bound on the Mahler measure of \( C_K(n) \), and a doubly exponential upper bound on the \( l^2 \) norm of \( C_K(n) \). Let us digress a bit and discuss this in detail.

Recall that the \( l^2 \)-norm
\[
||f||_2 := \left( \sum_k |a_k|^2 \right)^{1/2} = \left( \int_0^1 |f(e^{2\pi i t})|^2 dt \right)^{1/2}
\]
of a polynomial \( f = \sum_k a_k q^k \) also satisfies the inequalities of 7. However, neither the \( l^1 \) nor the \( l^2 \) norm are multiplicative. Mahler introduced a *measure*
\[
M(f) = \exp \int_0^1 \log |f(e^{2\pi i t})| dt
\]
which although it is not a norm, it is by definition multiplicative:
\[
M(fg) = M(f)M(g).
\]
The next proposition summarizes how the Mahler measure compares with the \( l^1 \) and \( l^2 \) norms:

**Proposition 3.4.** If \( f \) is a Laurent polynomial of degree \( d \), then
\[
M(f) \leq ||f||_2 \leq ||f||_1
\]
\[
||f||_1 \leq 2^d M(f)
\]
\[
M(f) = 1 \quad \text{if} \quad f \quad \text{is a product of cyclotomic polynomials}.
\]
Returning to the proof of Theorem 2, Equation (15) and Proposition 3 imply that 
\[ \| C_K(n) \|_1 \leq n^{C(c+1)2 \deg C_K(n)} \].
We will see later that \( \deg C_K(n) = O(n^2) \). Thus, the above estimate is exponential in \( n^2 \).

At this point, we will use the following theorem, communicated to us by D. Boyd.

**Theorem 8.** (Boyd) If \( f(q) \) is a polynomial that satisfies
- \( \deg f(q) = Cn^2 + O(n) \)
- \( \| (1 - q)(1 - q^2)\ldots(1 - q^n) f(q) \|_1 \leq e^{C'n + O(\log n)} \) then
  \[ \| f \|_1 \leq e^{C''n + O(\log n)}. \]

**Proof.** Consider the polynomial \( g(q) = f(q)(1 - q)\ldots(1 - q^n) \) of degree at most \( Cn^2 + n(n + 1)/2 + O(n) \).
Thus, we may write
\[
g(q) = \sum_{k=0}^{(C+1/2)n^2 + O(n)} a_k q^k.
\]
Let us expand
\[
\prod_{k=1}^{n} (1 - q^k) = \sum_{k=0}^{\infty} c_k q^k
\]
where \( c_k \in \mathbb{N} \). We can obtain an upper bound for the growth of \( c_k \) as follows. Consider
\[
\prod_{k=1}^{\infty} (1 - q^k) = \sum_{k=0}^{\infty} p_k q^k
\]
where \( p_n \) is the number of partitions of \( n \). Using growth rate of \( p_n \) (see [An]), it follows that
\[ 0 \leq c_k \leq p_k = e^{\pi \sqrt{2/3} + O(1)}. \]
The important thing is that the growth rate of \( p_k \) involves \( \sqrt{k} \). Moreover, we have:
\[
f(q) = \frac{g(q)}{(1 - q)\ldots(1 - q^n)} = \sum_{k=0}^{Cn^2 + O(n)} d_k q^k
\]
where
\[
d_k = \sum_{i=0}^{k} a_i c_{k-i}.
\]
Since \( |a_i| \leq \| g(q) \|_1 \leq e^{C'n + O(\log n)} \) for all \( i \), Equation (16) and the above implies that for all \( 0 \leq k \leq Cn^2 + O(n) \) we have:
\[
|d_k| \leq \sum_{i=0}^{k} |a_i| c_{k-i}
\]
\[
\leq e^{C'n + O(\log n)} \sum_{i=0}^{k} p_{k-i}
\]
\[
\leq e^{C'n + O(\log n)} k p_k
\]
\[
\leq e^{C'n + O(\log n)} p_{Cn^2 + O(n)}
\]
\[
\leq e^{C'n + O(\log n)} e^{\pi \sqrt{2C'/3} + O(\log n)}
\]
\[
\leq e^{C''n + O(\log n)}
\]
where \( C'' = C' + \pi \sqrt{2C'/3} \). Since \( \| f \|_1 = \sum_{k=0}^{Cn^2 + O(n)} |d_k| \), the result follows. \( \square \)
4. Some estimates

4.1. The Lobachevsky function. In this largely independent section we will prove refined (and optimal) estimates for the growth rate of the $R$-matrices. These estimates reveal the close relationship between hyperbolic geometry and the asymptotics of the quantum factorials. The main result in this section is a judicious application of the Euler-MacLaurin summation formula.

As a warm-up, let us consider the Lobachevsky function

$$
\Lambda(z) = -\int_0^z \log |2 \sin x| \, dx = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(2n \pi x)}{n^2}
$$

The Lobachevsky function is odd, with period $\pi$. Its graph for $z \in [0, \pi]$ is given by:

![Graph of the Lobachevsky function](image)

Notice also that $v_8 = 8\Lambda(\pi/4) \approx 3.66386$ is the volume of the regular ideal octahedron.

Recall that if $f(q) \in \mathbb{Z}[q^{\pm 1/2}]$, we denote by $ev_n(f)$ the evaluation of $f$ at $e^{2\pi i/n}$.

Recall also the $q$-factorial $\{a\}!$ from Equation (6). The next lemma discusses the asymptotics of the evaluation of quantum factorials. The notation $O(\log n)$ below is a term which is bounded by $C \log n$ for some constant $C$ independent of $\alpha$.

**Lemma 4.1.** For every $\alpha \in (0, 1)$ we have:

$$
ev_n(\{\lfloor \alpha n \rfloor \}) = \exp \left( -\frac{n}{\pi} \Lambda(\pi \alpha) + O(\log n) \right).
$$

**Remark 4.2.** The proof reveals an asymptotic expansion of the form:

$$
ev_n(\{\lfloor \alpha n \rfloor \}) \sim n^\theta \exp \left( -\frac{n}{\pi} \Lambda(\pi \alpha) \right) \left( C_0 + \frac{C_1}{n} + \frac{C_2}{n^2} + \ldots \right)
$$

for explicitly computable constants $C_i$.

**Proof.** Recall the Euler-MacLaurin summation formula, with error term (see for example, [O, Chapt.8]):

$$
\sum_{k=a}^{b} f(k) = \int_{a}^{b} f(x) \, dx + \frac{1}{2} f(a) + \frac{1}{2} f(b) + \sum_{k=1}^{m-1} \frac{B_{2k}}{(2k)!} \left( f^{(2k-1)}(b) - f^{(2k-1)}(a) \right) + R_m(a, b, f)
$$

where

$$
|R_m(a, b, f)| \leq (2 - 2^{1-2m}) \frac{B_{2m}}{(2m)!} \int_{a}^{b} |f^{(2m)}(x)| \, dx,
$$

and $B_k$ is the $k$th Bernoulli number given by the generating series:

$$
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.
$$
Applying the above formula to for $m = 1$ to $f(x) = \log |e^{2\pi ix/n} - 1|$, we have:

$$\log \left( \prod_{k=1}^{\left\lfloor \alpha n \right\rfloor} |e^{2\pi ik/n} - 1| \right) = \frac{1}{2} (f(1) + f(\{\alpha n\})) + \int_{0}^{\left\lfloor \alpha n \right\rfloor} \log |e^{2\pi it/n} - 1| dt + R_1(1, \alpha n, f)$$

$$= \frac{1}{2} (f(1) + f(\{\alpha n\})) + \int_{0}^{\alpha n} \log |e^{2\pi it/n} - 1| dt + R_1(1, \alpha n, f) + \epsilon(\alpha, n)$$

$$= \frac{1}{2} (f(1) + f(\{\alpha n\})) + \frac{n}{\pi} \int_{\pi/n}^{\alpha n} \log |2\sin(u)| u + R_1(1, \alpha n, f) + \epsilon(\alpha, n)$$

$$= \frac{1}{2} (f(1) + f(\{\alpha n\})) + \frac{n}{\pi} \left(-\Lambda(\pi \alpha) + \Lambda\left(\frac{\pi}{n}\right)\right) + R_1(1, \alpha n, f) + \epsilon(\alpha, n).$$

Now,

$$\frac{1}{2} |f(1) + f(\{\alpha n\})| \leq C \log n,$$

and

$$\epsilon(\alpha, n) \leq C''.$$

Moreover, $f'(x) = -\frac{\pi}{x} \cot(\pi x/n)$ and $f''(x) = \frac{2\pi^2}{n^2} (\csc(\pi x/n))^2$, and $(\csc x)^2 = 1/x^2 + 1/3 + x^2/15 + O(x^3)$.

Thus $f''_{\alpha n} \csc x |dx \leq Cn^2$, and

$$|R_1(1, \alpha n, f)| \leq C'.$$

Furthermore, using the asymptotic expansion of $\Lambda(z)$ for $z \in (0, \pi)$:

$$\Lambda(z) = z - z \log(2z) + \sum_{k=1}^{\infty} \frac{B_k}{2k} \frac{(2z)^{2k+1}}{(2k+1)!},$$

it follows that

$$\frac{n}{\pi} |\Lambda\left(\frac{\pi}{n}\right)| \leq C'' \log n.$$

The result follows. \(\square\)

Consider the $R$-matrix evaluated at $q = e^{2\pi i/n}$, $ev_n(R(n; a, b, k))$. Recall from (1) that $R$ is a ratio of 5 quantum factorials. Let us assume that $a = \lfloor \alpha n \rfloor$, $b = \lfloor \beta n \rfloor$, $k = \lfloor \kappa n \rfloor$, where

(17) \hspace{1cm} \alpha, \beta, \kappa \in [0, 1] \quad 0 \leq \beta + \kappa \leq 1, \quad 0 \leq \alpha - \kappa \leq 1.

Let us define

$$r(n; \alpha, \beta, \kappa) := ev_n(\log |R(n; \alpha n, \beta n, \kappa n)|)$$

$$= ev_n(\{b + k\}) - ev_n(\{b\}) - ev_n(\{k\}) + ev_n(\{a\}) - ev_n(\{a - k\})$$

Clearly,

$$\max_{a, b, k} ev_n(\log |R(n; a, b, k)|) = \max_{\alpha, \beta, \kappa} r(n; \alpha, \beta, \kappa),$$

with the understanding that $\alpha, \beta, \kappa$ satisfy (17).

The next result gives the asymptotics of the $R$-matrix.

**Theorem 9.**

$$\max_{a, \beta, \kappa} r(n; \alpha, \beta, \kappa) = r(n; 3/4, 1/4, 1/2) = \frac{v_8}{2\pi} n + O(\log n)$$

where $v_8 = 8\Lambda(\pi/4) \approx 3.66386$ is the volume of the regular hyperbolic ideal octahedron.

Thus, asymptotically, the winning configuration is given by:

```
3n/4   n/4
\downarrow \quad \uparrow
n/2    n/2
```

```
3n/4   n/4
```

```
3n/4   n/4
```
Proof. Lemma 4.1 and the definition of $r$ imply that
\[ r(n; \alpha, \beta, \kappa) = \frac{n}{\pi} f(\alpha, \beta, \kappa) + O(\log n) \]
where
\[ f(\alpha, \beta, \kappa) = -\Lambda(\pi(\beta + \kappa)) + \Lambda(\pi\beta) + \Lambda(\pi\kappa) - \Lambda(\pi\alpha) + \Lambda(\pi(\alpha - \kappa)) \]
The domain of $f$ is a compact set, thus the maximum of $f$ exists. Moreover, $f$ vanishes on the boundary, thus the maximum is one of the critical points in the interior. To find the critical points in the interior, let us set
\[ z_\alpha = e^{2\pi i \alpha}, \quad z_\beta = e^{2\pi i \beta}, \quad z_\kappa = e^{2\pi i \kappa}. \]
Using the derivative of the Lobachevsky function
\[ \Lambda'(x) = \log |e^{2ix} - 1|, \]
it follows that
\[ \frac{\partial f}{\partial \alpha} = \log |z_\alpha - 1| - \log |z_\alpha z_\kappa^{-1} - 1| = 0 \]
\[ \frac{\partial f}{\partial \beta} = \log |z_\beta z_\kappa - 1| - \log |z_\beta - 1| = 0 \]
\[ \frac{\partial f}{\partial \kappa} = \log |z_\beta z_\kappa - 1| - \log |z_\kappa - 1| - \log |z_\alpha z_\kappa^{-1} - 1| = 0. \]
Thus, the critical points in the interior are given by the solutions of:
\[ |z_\alpha - 1| = |z_\alpha z_\kappa^{-1} - 1| \]
\[ |z_\beta z_\kappa - 1| = |z_\beta - 1| \]
\[ |z_\beta z_\kappa - 1| = |z_\kappa - 1||z_\alpha z_\kappa^{-1} - 1|. \]
Using Lemma 4.2 below, and the fact that $z_\alpha, z_\beta, z_\kappa \neq 1$, the first two equations imply that
\[ z_\alpha = z_\alpha z_\kappa^{-1} = z_\beta z_\kappa^{-1} = z_\beta = \pm z_\beta. \]
Plugging in the third equation gives $(z_\alpha, z_\beta, z_\kappa) = (-i, i, -1)$, i.e., $(\alpha, \beta, \kappa) = (3/4, 1/4, 1/2)$. Since
\[ \Lambda(\pi/2) = 0, \quad \Lambda(3\pi/4) = -\Lambda(\pi/4), \]
it follows that
\[ f(3/4, 1/4, 1/2) = 4\Lambda(\pi/4) = v_8/2. \]
The result follows. \qed

Remark 4.3. The proof also reveals that $r(n; \alpha, \beta, \kappa) = \frac{V(\alpha, \beta, \kappa)}{\pi n} + O(\log n)$, where $V(\alpha, \beta, \kappa)$ is the volume of an ideal octahedron with vertices 0, 1, $\infty$, $z_\alpha, z_\beta, z_\kappa$.

Lemma 4.4. If $z, w$ are complex numbers that satisfy $|z| = |w| = 1$ and $|1 - z| = |1 - w|$, then $z = w^\pm$.

Proof. Let us define
\[ C_{u_0,r} := \{ u \in \mathbb{C} \mid |u - u_0| = r > 0 \}. \]
Then $C_{u_0,r}$ is a circle with center $u_0$ and radius $r$. Fixing $w$, it follows that $z \in C_{0,1} \cap C_{1,|1-w|}$. The intersection of two circles is two points, and since $w$ and $w^{-1} = \bar{w}$ both lie in the intersection, the result follows. \qed

Remark 4.5. The same bound in Theorem 4 holds for the $R_-$ matrix of Equation 6.

4.2. Proof of Theorem 4 Recall that the colored Jones function is given by the state-sum of Equation 6, where the summand $F(n, k)$ is a product of local $R$-matrices, one for each crossing of $K$. Theorem 7 implies that
\[ |ev_n(F(n, k))| \leq e^{v_k/(2\pi)nc + O(\log n)} \]
for each $k$, where the error term is bounded independent of $k$. Since $k$ takes $O(n^4)$ values, Theorem 4 follows.
5. The $q$-holonomic point of view

5.1. Bounds on $l^1$-norm and Mahler measure of $q$-holonomic functions. The main result of [GL] is that for every knot $K$, the functions $J_K$ and $C_K$ are $q$-holonomic. Recall that a sequence $f : \mathbb{N} \rightarrow \mathbb{Q}(q)$ is $q$-holonomic if satisfies a $q$-linear difference equation. In other words, there exists a natural number $d$ and rational functions $a_j(u, v) \in \mathbb{Q}(u, v)$ for $j = 0, \ldots, d$ with $a_d \neq 0$ such that for all $n \in \mathbb{N}$ we have:

$$\sum_{j=0}^{d} a_j(q^n, q) f(n + j) = 0.$$

In this section we observe that $q$-holonomic functions satisfy a priori upper bounds on their degrees and (under an integrality assumption) on their $l^1$-norm. As a simple corollary, we obtain an independent proof of Theorems [1] and [3].

**Definition 5.1.** We say that a sequence $f : \mathbb{N} \rightarrow \mathbb{Z}[q^{\pm}]$ is $q$-integral holonomic if it satisfies an integral $q$-difference equation as above with $a_d = 1$ and $a_j(u, v) \in \mathbb{Z}[u, v]$.

Although $J_K$ takes values in $\mathbb{Z}[q^{\pm}]$, it is known for example that $J_{4_1}$ is not $q$-integral holonomic. On the other hand, it is known that $C_K$ is $q$-integral holonomic for all twist knots; see [GS].

**Question 1.** Is it true that $C_K$ is $q$-integral holonomic for every knot $K$?

For a Laurent polynomial $f(q) = \sum_{k=m}^{M} a_k q^k$, with $a_m a_M \neq 0$, let us define $\deg_{\max}(f) = M$ and $\deg_{\min}(f) = m$.

**Theorem 10.** (a) If $f : \mathbb{N} \rightarrow \mathbb{Z}[q^{\pm}]$ is $q$-holonomic, then for all $n$ we have:

$$\deg_{\max}(f(n)) = O(n^2) \quad \text{and} \quad \deg_{\min}(f(n)) = O(n^2).$$

(a) If $f$ is $q$-integral holonomic, then for all $n$ we have:

$$||f(n)||_1 \leq C^n$$

for some constant $C$. In particular,

$$M(f(n)) \leq C^n$$

and

$$\limsup_{n \to \infty} \frac{\log |ev_{\alpha,n}(f(n))|}{n} \leq C$$

for all $\alpha \in \mathbb{R}$.

**Proof.** For the first claim in (a), let us assume without loss of generality that

$$a_d(q^n, q) f(n + d) = -a_{d-1}(q^n, q) f(n + d - 1) - \ldots - a_0(q^n, q) f(n)$$

where $a_j(Q, q) \in \mathbb{Q}(Q, q)$ are polynomials in $Q, q$. Choose $C'$ so that

- $\deg_{\max}(a_j(q^n, q)) \leq 2C'(n + d)$ for all $j = 0, \ldots, d - 1$, and
- $\deg_{\max}(f(n)) \leq C'(n + 1)^2$ for $n = 0, \ldots, d - 1$.

We will prove by induction on $n$ that $\deg_{\max}(f(n)) \leq C'(n + 1)^2$. By assumption, it is true for $n = 0, \ldots, d - 1$. Then, by induction we have:

$$\deg_{\max}(f(n + d)) \leq \deg_{\max}(a_d(q^n, q) f(n + d))$$

$$\leq \deg_{\max}(a_d(q^n, q)) \deg_{\max}(f(n + d))$$

$$\leq \max_{0 \leq j < d} \deg_{\max}(a_j(q^n, q) f(n + j))$$

$$\leq \max_{0 \leq j < d} 2C'(n + j + 1)^2 + C'(n + j + 1)^2$$

$$= 2C'(n + d)^2 + C'(n + d)^2$$

$$< C'(n + d + 1)^2.$$
The second claim in (a) follows similarly.

For (b), let $c_j = \|\alpha_j(Q, q)\|_1$ for $j = 0, \ldots, d - 1$, and choose $C$ so that

- $C^d \leq c_{d-1}C^{d-1} + \cdots + c_0C^0$, and
- $\|f(n)\|_1 \leq C^n$ for $n = 0, \ldots, d - 1$.

Then, it is easy to see by induction that (b) holds for all $n$. □

As advertised above, Theorem 10 gives an alternative proof of Theorem 1 and 3, under the assumption that Question 1 has a positive answer. However, the explicit upper bounds in terms of the number of crossings cannot be obtained from Theorem 10, unless we know something more about the $q$-difference equation of the colored Jones function. Moreover, Theorem 2 cannot be obtained from general theory of asymptotics of solutions of $q$-difference equations, since a typical solution would be growing exponentially, even for small positive $\alpha$. With additional assumptions on the shape of the $q$-difference equations, the first author can show Theorem 2. The proof was replaced with the one of the present paper.

5.2. Bounds for higher rank groups. In [GL], we considered the colored Jones function $J_{g,K}: \Lambda_w \to \mathbb{Z}[q^\pm]$ of a knot $K$, where $g$ is a simple Lie algebra with weight lattice $\Lambda_w$. In the above reference, the authors proved that $J_{g,K}$ is a $q$-holonomic function, at least when $g$ is not $G_2$. For $g = \mathfrak{sl}_2$, $J_{\mathfrak{sl}_2,K}$ is the colored Jones function $J_K$ discussed earlier.

In [GL], the authors gave state-sum formulas for $J_{g,K}$ similar to (5) where the summand takes values in $\mathbb{Z}[q^{\pm1/2}]$, where $D$ is the size of the center of $g$.

The methods of the present paper give an upper bound for the growth-rate of the $g$-colored Jones function. More precisely, we have:

**Theorem 11.** For every simple Lie algebra $g$ (other than $G_2$) there exists a constant $C_g$ such that for every knot with $c + 2$ crossings, and every $\alpha > 0$ and every $\lambda \in \Lambda_w$, we have:

$$\limsup_{n \to \infty} \log \left| \text{ev}_{\alpha,n}(J_{g,K}(n\lambda)) \right| n \leq C_g c.$$

The details of the above theorem will be explained in a subsequent publication.

5.3. Acknowledgement. The authors wish to thank I. Agol, D. Boyd, N. Dunfield, D. Thurston and D. Zeilberger for many enlightening conversations.

**Appendix A. The volume conjecture for the Borromean rings**

It is well-known that the complement of the Borromean rings $B$ can be geometrically identified by gluing two regular ideal octahedra; [Th]. As a result, the volume $V(B)$ of $B$ is given by $2v_8$.

If $L$ is a link with a distinguished component (to be broken), then one may define the colored Jones function $J_L(n)$ to be the invariant of the $(1,1)$-tangle obtained by breaking the distinguished component of $L$ and coloring all components of the tangle by the $n$-dimensional irreducible representation of $\mathfrak{sl}_2$. In general, $J_L(n)$ depends on the link and its distinguished component. In the case of the Borromean rings $B$ though, due to symmetry, we may choose any component as the distinguished one. Habiro uses the notation $\tilde{J}_L(n)$ for $J_L(n)$.

The next theorem confirms the volume conjecture for the Borromean rings.

**Theorem 12.** If $J_B(n)$ denotes the colored Jones function of the Borromean rings $B$, then

$$\lim_{n \to \infty} \frac{\log |\text{ev}_n(J_B(n))|}{n} = \frac{1}{2\pi} V(B).$$
Proof. Using Habiro’s formula for \( \hat{J}_L \) of the Borromean ring \([2]\), one has
\[
J_B(n) = \sum_{l=0}^{N-1} (-1)^l \left( \prod_{j=1}^{l} \left\{ n \right\} \{ n+j \} \{ n-j \} \right)^3 \left( \prod_{j=l+1}^{2l+1} \left\{ j \right\} \right)^2.
\]
When \( v = e^{i\pi/n} \), on has \( \{ j \} = 2i \sin \frac{j\pi}{n} \), which is 0 exactly when \( j \) is divisible by \( n \). Hence if \( 2l+1 < n \), then the denominator of the term in the above sum is never 0, while the numerator is 0, since it has 2 factors \( \{ n \} \). On the other hand, if \( 2l+1 > n \), then the denominator has 2 factors \( \{ n \} \), which would cancel with the 2 same factors of the numerator. Hence when evaluating at \( v = e^{i\pi/n} \) one can assume that \( 2l+1 \geq n \), or \( l > n/2 - 1 \):
\[
ev_n(J_B(n)) = \sum_{n > l > n/2 - 1} (-1)^l \left( \prod_{j=1}^{l} \left\{ n+j \right\} \left\{ n-j \right\} \right)^3 \left( \prod_{j=l+1}^{2l+1} \left\{ j \right\} \right)^2.
\]
Note that when \( v = e^{i\pi/n} \), one has \( \{ n+j \} = -\{ j \} = -2i \sin(j\pi/n) \). Hence a simple calculation shows that
\[
ev_n(J_B(n)) = \sum_{n > l > n/2 - 1} (\tau_{1,l})^3 \tau_{l+1,n-1} \tau_{n+1,2l+1},
\]
where
\[
\tau_{p,l} := \prod_{j=p}^{l} 4 \sin^2(j\pi/n).
\]
The following properties of \( \tau \) are easy to verify

**Lemma A.1.** We have:

(19) \[ \tau_{1,m} = \tau_{n-m,n-1} \text{ for } 0 < m < n \]
(20) \[ \tau_{n+1,m} = \tau_{2n-m,n-1} \text{ for } n < m < 2n \]
(21) \[ \tau_{1,n-1} = n^2 \]
(22) \[ \tau_{1,m} \tau_{1,n-m-1} = n^2 \text{ for } 0 < m < n. \]

From the Lemma it follows that \( \tau_{1,n-1} = n^2/\tau_{1,1} \) and \( \tau_{n+1,2l+1} = \tau_{2n-l-1,n-1} = n^2/\tau_{1,2n-l-2} \). Hence
\[
ev_n(J_B(n)) = n^4 \sum_{n > l > n/2 - 1} (\tau_{1,l})^4 \tau_{1,n-1} \tau_{1,2n-2l-2}.
\]

Let \( k = 2l+1 - n \). Then \( n > k \geq 0, k + n \equiv 1 \pmod{2} \), and \( l = (n+k-1)/2 \). From the above Lemma one has that
\[
\tau_{1,2n-2l-2} = \frac{n^2}{\tau_{1,k}}, \quad \tau_{1,k} = \frac{n^2}{\tau_{n-l-1}}.
\]

Thus,
\[
(\tau_{1,l})^4 = (\tau_{1,l})^2 \left( \frac{n^2}{\tau_{n-l-1}} \right)^2,
\]
and
\[
ev_n(J_B(n)) = n^{10} \sum_{n > k \geq 0, k + n \equiv 1 \pmod{2}} \frac{(\tau_{1,(n+k-1)/2})^2}{(\tau_{1,(n-k-1)/2} \tau_{1,k})^2}.
\]
Using the notation of Section 4, we have:

\[ \frac{(\tau_1, (n+k-1)/2)}{\tau_1, (n-k-1)/2} = |\text{ev}_n(R(n; a, b, k))|^2 \]

where \( a = [(n+k-1)/2] \) and \( b = [(n-k-1)/2] \). There are less than \( n \) terms, hence using Theorem 9 one has that

\[ \text{ev}_n(J_B(n)) < n^{11/2}e^{2r(n;3/4,1/4,1/2/2)} \]

Therefore the limsup of the left hand side of (18) is \( 2v_8/(2\pi) \).

On the other hand, all the terms in the sum of (23) are positive, hence the sum is at least the biggest term, which is when \( k = [n/2] \). The limit when we retain only this term is easily seen to be equal to \( (2v_8)/(2\pi) \), which is the same as the limsup. \( \Box \)

**Appendix B. The volume conjecture for torus knots**

Let \( T(a, b) \) denote the \((a, b)\)-torus knot for co-prime integers \( a, b \) \((a, b > 1)\). Although \( T(a, b) \) is not a hyperbolic knot, the volume function \( V(l, m) \) on the deformation variety can be defined; see [CCGLS]. Let us discuss this first.

**Lemma B.1.** The restriction of the volume form on the deformation variety, i.e., the zero set of the A-polynomial is equal to 0.

**Proof.** For a torus knots the A-polynomial is either of the form \( l \pm mc \) for some integer \( c \), or the product of 2 factors of that forms. It is easy to check that the restriction of the volume form on any such factor is equal to 0. \( \Box \)

Thus the volume function \( V(l, m) \) on the deformation variety must be constant. Because the Gromov norm of the knot complement is 0, one should define \( V(l, m) = 0 \) for every \( l, m \). Then the generalized volume conjecture can be proved easily in this case:

**Proposition B.2.** For the torus knot \( K = T(a, b) \) and for any real number \( \alpha \), one has

\[ \lim_{n \to \infty} \frac{\log |\text{ev}_{\alpha,n}(J_K(n))|}{n} = 0. \]

**Proof.** There are 2 cases: \( \alpha \) is an integer, or \( \alpha \) is not. The first case is actually much more difficult, this is the usual volume conjecture and it has been proved by Kashaev and Tirkkonen in [KT]. Let us consider the easier case, when \( \alpha \) is not an integer.

Note that \( v = \exp(\pi i \alpha/n) \), and if \( \alpha \) is not an integer one has \( v^n - v^{-n} = 2i \sin \pi \alpha/n \neq 0 \).

The colored Jones polynomial was calculated by Morton in [Mo]:

\[ J_{T(a,b)}(n) = \frac{v^{-ab(n^2-1)/2}}{v^n - v^{-n}} \sum_{k=1-n, k+n \equiv 1 \pmod{2}}^{n-1} (v^{2abk^2+2ka+2kb+1} - v^{2abk^2+2ka-2kb-1}). \]

The sum contains \( n + 1 \) terms, each by absolute value is less than or equal to 2. Hence

\[ |\text{ev}_{\alpha,n}(J_K(n))| < \frac{2(n+1)}{\sin(\pi \alpha/n)}. \]

Thus, the limsup is less than or equal to 0. The argument in Section 3 shows that the lim inf is greater than or equal to 0. The result follows. \( \Box \)

**Remark B.3.** In [M2], H. Murakami discusses the Generalized Volume Conjecture for the torus knots and angles \( \alpha \) with nonzero imaginary part.
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