Rank one tensor completion problem

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Abstract. In this paper we consider the rank-one tensor completion problem. We address the question of existence and uniqueness of the rank-one solution. In particular we show that the global uniqueness over the field of real numbers can be verified in a polynomial time. We give examples showing that there is an essential difference between the question of global uniqueness over the fields of real and complex numbers. Finally we briefly discuss the rank-one approximation problem for noisy observations.

Key Words: tensor completion, tensor rank, local and global uniqueness, noisy observations

1 Introduction

The problem of recovering matrix of low-rank from a few observed entries is known as the matrix completion problem. There is a substantial body of literature on that problem, e.g. [1, 2, 4, 5, 7, 10, 13, 14]. Tensor completion is a natural generalization where the goal is to recover a low-rank tensor from observations of few of its entries (e.g., [6, 12, 15]). The concept of tensor rank is considerably more involved then its matrix counterpart (see, e.g., survey paper [11]). Although the low rank approximation problem has been well studied for matrices, there is not much work on tensors. From a generic point of view the question of existence and uniqueness of low rank tensor decomposition was investigated, e.g., in [3] and references therein.

In this paper we consider the rank-one tensor completion problem. In particular we address the question of uniqueness of rank-one solution. Necessary and sufficient conditions for uniqueness of rank-one matrix completion solution are known and have a combinatorial flavor (cf., [9, 14]). On the other hand, tensor completion problem is considerably more involved and even rank-one tensor completion turns out to be nontrivial (cf., [8]). Among other things we demonstrate that there is a significant difference between considering the problem over real and complex numbers.

2 Tensor completion

Let us consider the following tensor completion problem over the fields of real or complex numbers. Consider a three way tensor $X \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ over the field of complex, $\mathbb{F} = \mathbb{C}$, or real
\( \mathbb{F} = \mathbb{R} \), numbers. Let \( \Omega \) be a (nonempty) set of indexes \((i, j, k)\), that is \( \Omega \subset [n_1] \times [n_2] \times [n_3] \), and let \( Q_{ijk}, (i, j, k) \in \Omega \), be the observed values (by \( [n] \) we denote the set \( \{1, \ldots, n\} \)). In the deterministic setting we want to find tensor \( X = a \circ b \circ c \) of rank one such that \( X_{ijk} = Q_{ijk}, (i, j, k) \in \Omega \). Recall that \( [a \circ b \circ c]_{ijk} = a_ib_jc_k \) with \( a \in \mathbb{F}^{n_1}, b \in \mathbb{F}^{n_2}, c \in \mathbb{F}^{n_3} \). That is we want to find vectors \( a, b, c \) such that

\[
Q_{ijk} = a_ib_jc_k, \ (i, j, k) \in \Omega.
\] (2.1)

We assume throughout the paper that all observed values are nonzero, i.e.,

\[
Q_{ijk} \neq 0, \ (i, j, k) \in \Omega.
\] (2.2)

This implies that the components of vectors \( a, b, c \) are nonzero.

The main goal of this paper is investigation of uniqueness of the rank one solution. Note that vectors \( a, b, c \) of tensor \( a \circ b \circ c \) are defined up to change of scale \( \lambda_1 a \circ \lambda_2 b \circ \lambda_3 c \), where scalars \( \lambda_1 \lambda_2 \lambda_3 = 1 \). Therefore we assume that \( a_1 = 1 \) and \( b_1 = 1 \). It is said that the solution (2.1) is locally unique if the representation (2.1) is unique up to sufficiently small perturbations of vectors \( a, b, c \).

By setting \( q_{ijk} := \log |Q_{ijk}| \) we have the following system of linear equations associated with the representation (2.1),

\[
x_i + y_j + z_k = q_{ijk}, \ (i, j, k) \in \Omega,
\] (2.3)

with respect to variables \( x_i := \log |a_i|, y_j := \log |b_j|, z_k := \log |c_k| \). Since \( a_1 = 1 \) and \( b_1 = 1 \), and hence \( x_1 = 0 \) and \( y_1 = 0 \), we have that the system (2.3) has \( m = |\Omega| \) equations and \( n_1 + n_2 + n_3 - 2 \) unknowns. Consider the following condition.

(A) Setting \( x_1 = 0 \) and \( y_1 = 0 \), the homogeneous linear system

\[
x_i + y_j + z_k = 0, \ (i, j, k) \in \Omega,
\] (2.4)

with \( m = |\Omega| \) equations and \( n_1 + n_2 + n_3 - 2 \) unknowns, is nondegenerate, i.e, has only zero solution.

Note that the rank one solution (2.1) is locally unique iff the rank one solution

\[
|Q_{ijk}| = |a_i||b_j||c_k|, \ (i, j, k) \in \Omega,
\] (2.5)

for the respective absolute values is locally unique. Therefore we have the following result.

**Proposition 2.1** The rank one solution (2.1) is locally unique if and only if condition (A) holds.

The above characterization of local uniqueness is the same for the fields of real and complex numbers. Of course the local uniqueness is necessary for the corresponding global uniqueness of the solution.

• Unless stated otherwise, we assume from now on that condition (A) holds.
As we are going to show, condition (A) is not sufficient for the global uniqueness.

Let us consider the question of (global) uniqueness over complex numbers. We can write complex numbers $a_i, b_j, c_k$ in the following form $a_i = |a_i|e^{i\alpha_i}$, $b_j = |b_j|e^{i\beta_j}$, $c_k = |c_k|e^{i\gamma_k}$, for some $\alpha_i, \beta_j, \gamma_k \in [0, 2\pi)$ where $i^2 = -1$. Consider the following systems of linear equations, in unknowns $\alpha_i, \beta_j, \gamma_k$, $(i, j, k) \in [n_1] \times [n_2] \times [n_3]$,}

\[
\alpha_1 = 0, \ \beta_1 = 0, \ \alpha_i + \beta_j + \gamma_k = \sigma_{ijk}, \ (i, j, k) \in \Omega, \quad (2.6)
\]

where $\sigma_{ijk}$ are either 0, 2$\pi$ or 4$\pi$, $(i, j, k) \in \Omega$. Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be a solution of (2.6) for some $\sigma_{ijk}$. Then

\[
a_i b_j c_k = |a_i||b_j||c_k|e^{i(\alpha_i + \beta_j + \gamma_k)} = |a_i||b_j||c_k|e^{i(\bar{\alpha}_i + \bar{\beta}_j + \bar{\gamma}_k + \gamma_k)}.
\]

giving another solution $a_i e^{i\bar{\alpha}_i}, b_j e^{i\bar{\beta}_j}, c_k e^{i\bar{\gamma}_k}$. That is we have the following.

**Proposition 2.2** The rank one solution is not unique, over the field of complex numbers, if and only if there exist $\alpha_i, \beta_j, \gamma_k \in (0, 2\pi)$ which solve at least one system of equations (2.6) for $\sigma_{ijk}$ being either 0, 2$\pi$ or 4$\pi$, $(i, j, k) \in \Omega$.

Of course there are $3^m$ such systems of equations. Note that the homogeneous counterpart of (2.6) is the same as (2.4). Therefore (under condition (A)) if the system (2.6) has a solution for some $\sigma_{ijk}$, it is unique. Note also that $a_i, b_j, c_k$ are determined by the respective $\alpha_i, \beta_j, \gamma_k$ up to the period of 2$\pi$.

## 3 Rank one solution over real numbers

In this section, we give the criteria for existence and uniqueness of the rank one solution over real numbers. Consider the linear system of equations (2.3) in unknowns $x_i, y_j, z_k$, $(i, j, k) \in [n_1] \times [n_2] \times [n_3]$, with $x_1 = 0$ and $y_1 = 0$. If this system does not have a solution, then the corresponding rank one solution does not exist. Therefore assume that the system (2.3) has a solution. Then existence and uniqueness of the rank one solution comes to verification of the signs in the right hand side of (2.5).

Consider the finite field $GF(2)$ consisting of two elements 0 and 1 (this can be viewed as integers with arithmetic operations modulo 2, where 0 representing the class of even integers and 1 represents the class of odd integers). Consider the following system of equations over the field $GF(2)$:

\[
\varepsilon_i + \nu_j + \eta_k = c_{ijk}, \ (i, j, k) \in \Omega, \ \varepsilon_1 = 0, \ \nu_1 = 0, \quad (3.1)
\]

with unknowns $\varepsilon_i, \nu_j, \eta_k \in GF(2)$, where we set $c_{ijk} = 1$ if $Q_{ijk} > 0$, and $c_{ijk} = 0$ if $Q_{ijk} < 0$. Then the rank one solution does exist iff the system (3.1) has a solution. In the later case the rank one solution is unique iff (3.1) has a unique solution. By running the Gaussian elimination procedure to solve (3.1) over the field $GF(2)$, it is possible to check the above existence and uniqueness conditions. The running time of the algorithm is polynomial in the input size.

3
4 Numerical experiments and counterexamples

In this section, by numerical examples, we show that for some $\Omega$ the global uniqueness over the field of real numbers does not imply the global uniqueness over the field of complex numbers (recall that it is assumed that condition (A) is satisfied). Of course the solutions over the field of real numbers are also the solutions over the field of complex numbers.

In Tables 1, 2, 3 and 4 tensors with partial observations are considered. From the left to the right, there are the first, second, and third slice of the tensor, with “*” denoting the missing value. In all examples the tensors are of size $3 \times 3 \times 3$, $|\Omega| = 7$. In table 1, 2, and 3 all observed values equal to 1, i.e. $Q_{ijk} = 1, \forall (i,j,k) \in \Omega$. Moreover, it can be checked that condition (A) is satisfied in these examples. Recall that our goal is to find vectors $a, b, c$, such that $Q_{ijk} = a_i b_j c_k$, for $(i,j,k) \in \Omega$. Table 4 gives an example with only one solution over both fields $\mathbb{R}$ and $\mathbb{C}$. Table 2 gives an example with two solutions over $\mathbb{R}$. Table 3 gives an example with only one solution over $\mathbb{R}$ but multiple solutions over $\mathbb{C}$. Table 4 gives an example where there exist solutions over $\mathbb{C}$, but there are no solutions over $\mathbb{R}$. This can be verified by a straightforward checking of all relevant combinations. The respective solutions are provided.

| TABLE 1 | Tensor with only one solution | Solution |
|---------|-----------------------------|----------|
|         | a   | b   | c   |
| 1 1 1   | 1 1 | 1 1 |
| 1 * *   | 1 * | 1 * |
| 1 * *   | 1 * | 1 * |

| TABLE 2 | Tensor with two solutions  | Solution 1 | Solution 2 |
|---------|-----------------------------|------------|------------|
|         | a   | b   | c   | a   | b   | c   |
| 1 1 *   | 1 1 | 1 1 | 1 1 | 1 1 |
| 1 * *   | 1 1 | 1 1 | 1 1 | 1 1 |
| * 1     | 1 1 | -1 | -1 | -1 |

5 Rank one approximation of noisy observations

Condition (A) can hold only if the number of unknowns is not bigger than the number of equations, i.e., if $n_1 + n_2 + n_3 - 2 \leq m$. If the strict inequality holds, i.e.,

$$n_1 + n_2 + n_3 - 2 < m,$$

then the set of vectors $[Q]_{(i,j,k) \in \Omega}$, in the respective space of dimension $m$, has Lebesgue measure zero. That is, if values $Q_{ijk}$, $(i,j,k) \in \Omega$, are viewed as random variables with a (joint) continuous distribution, and the inequality (5.1) holds, then with probability one the one rank problem does not have an exact solution.
Table 3: Tensor with globally unique over $\mathbb{R}$, but multiple over $\mathbb{C}$ solutions

|      | Solution 1 |          |          |          |
|------|------------|----------|----------|----------|
|      | $a$        | $b$      | $c$      |          |
| 1    | 1          | 1        | 1        |          |
| 1    | 1          | $e^{4\pi i/3}$ | $e^{2\pi i/3}$ | $e^{4\pi i/3}$ |
| 1    | $e^{2\pi i/3}$ | $e^{4\pi i/3}$ | $e^{2\pi i/3}$ | $e^{4\pi i/3}$ |

Table 4: Tensor with solutions only over $\mathbb{C}$

|      | Solution 1 |          |          |          |
|------|------------|----------|----------|----------|
|      | $a$        | $b$      | $c$      |          |
| -1   | 1          | 1        | -1       |          |
| 1    | 1          | $e^{2\pi i/3}$ | $e^{4\pi i/3}$ | $e^{2\pi i/3}$ |
| *    | $e^{2\pi i/3}$ | $e^{4\pi i/3}$ | $e^{2\pi i/3}$ | $e^{4\pi i/3}$ |

Suppose that the inequality (5.1) holds. As it was pointed above, in that case the exact rank one solution does not exist with probability one. Therefore it makes sense to talk about approximate solutions. Let us consider the following model

$$Q_{ijk} = Q_{ijk}^* + \varepsilon_{ijk}, \quad (i, j, k) \in \Omega,$$

where $Q_{ijk}^* = a_ib_jc_k$, $(i, j, k) \in \Omega$, allow exact one rank solution and $\varepsilon_{ijk}$ are viewed as noise or disturbances (not necessary random). Suppose that $Q_{ijk}^* > 0$, $(i, j, k) \in \Omega$, and $\varepsilon_{ijk}$ are small enough so that $Q_{ijk} > 0$, $(i, j, k) \in \Omega$. Let $q_{ijk} := \log Q_{ijk}$ and consider the least squares problem

$$\min_{x, y, z} \sum_{(i,j,k) \in \Omega} (q_{ijk} - x_i - y_j - z_k)^2.$$ 

Suppose that the condition (A) holds for $Q_{ijk}^*$, and let $\hat{x}, \hat{y}, \hat{z}$ be solutions of (5.3) with $\hat{x}_1 = \hat{y}_1 = 0$ (condition (A) ensures that these solutions are unique).

Note that

$$q_{ijk} = \log Q_{ijk}^* + \log(1 + \varepsilon_{ijk}/Q_{ijk}^*) = q_{ijk}^* + \varepsilon_{ijk}^* + o(\varepsilon_{ijk}^*),$$

where $q_{ijk}^* = \log Q_{ijk}^*$ and $\varepsilon_{ijk}^* = \varepsilon_{ijk}/Q_{ijk}^*$. Therefore if the relative disturbances $\varepsilon_{ijk}^*$ are small, then $\hat{x}_i = \exp(\hat{x}_i)$, $\hat{y}_j = \exp(\hat{y}_j)$, $\hat{z}_k = \exp(\hat{z}_k)$, could give good approximate solution for fitting one rank tensor to the observed values $Q_{ijk}$.

5.1 Example

In this example, the true tensor is of size $3 \times 3 \times 3$ and all the elements of it equal to 1, and $\varepsilon_{ijk} \sim \text{Uniform}(-0.2, 0.2)$. Noisy observations of the tensor are shown in table 5. Table 6 is the approximate solution from the least square problem, eq.(5.3).
Table 5: Tensor with noisy observations

|       |       |       | 0.8193 | 0.8585 |       | 0.9003 | 1.1636 |       |
|-------|-------|-------|--------|--------|-------|--------|--------|-------|
| 1.1718 | 1.1438 | 0.8739 |        |        | *     |        |        |       |
|       | *     | *     | 0.9160 |        | *     | 0.8386 | 1.0515 |       |
| 0.8469 | *     | 1.1119 | 1.0942 | 0.8058 | *     | 0.9664 |        | *     |

Table 6: Rank-one approximate solution

|       |       |       | 0.8976 | 0.9050 | 0.9377 | 0.9925 | 1.0007 | 1.0368 |
|-------|-------|-------|--------|--------|--------|--------|--------|--------|
| 1.0052 | 1.0135 | 1.0501 |        |        |        |        |        |        |
| 0.9448 | 0.9526 | 0.9869 | 0.8436 | 0.8506 | 0.8813 | 0.9328 | 0.9405 | 0.9745 |
| 0.9934 | 1.0016 | 1.0378 | 0.8871 | 0.8944 | 0.9267 | 0.9809 | 0.9889 | 1.0247 |

Solution

| \( \hat{a} \) | \( \hat{b} \) | \( \hat{c} \) |
|-------|-------|-------|
| 1.0052 | 1.0082 | 0.8976 |
| 0.9399 | 0.9925 |

6 Conclusion remarks

It could be mentioned that the approach discussed in this paper can be extended in a straightforward way to an analysis of rank one solutions of tensors of higher order. This also could be applied to matrix completion, of course a two way tensor can be viewed as a matrix. For the rank one matrix completion problem the counterpart of the linear system (2.3) becomes

\[ x_i + y_j = q_{ij}, \quad (i, j) \in \Omega. \]  

(6.1)

By setting \( x_1 = 0 \) we have that the respective rank one solution is locally unique iff the homogeneous counterpart of equations (6.1) is nondegenerate, i.e., has only zero solution. By applying the Gauss elimination procedure it is not difficult to see that this condition is also necessary and sufficient for the global uniqueness and is equivalent to conditions for uniqueness of the rank one matrix completion solution derived in [9] and [14]. On the other hand, as it was demonstrated in Section 4, for the rank one tensor completion problem the situation is different - the local uniqueness does not imply the respective global uniqueness and conditions for global uniqueness over real and complex numbers are different.

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