Criterion of holomorphy with respect to a coupling constant of continuous functions of a perturbed self-adjoint operator

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Abstract. Sufficient and necessary conditions on the spectral measure of a self-adjoint operator $A$, acting in a Hilbert space $\mathcal{H}$, are obtained, under which for any continuous scalar function the operator function $\phi(A + \gamma B)$ is holomorphic with respect to the coupling constant $\gamma$ in a neighborhood of $\gamma = 0$, where $B$ is a self-adjoint operator. The sharpest results are obtained in the case where $B$ is a rank-one operator.

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1. Introduction

In [1] sufficient and necessary conditions on the spectral measure of a self-adjoint operator $A$, acting in a Hilbert space $H$, were obtained, under which any continuous function $\phi$ (without any additional smoothness properties) has a directional operator - derivative

$$\phi'(A)(B) := \frac{\partial}{\partial \gamma} \phi(A + \gamma B)|_{\gamma=0}$$

in the direction of a quite general bounded, self-adjoint operator $B$. Here the operator function $\phi(A + \gamma B)$ is defined on the real axis $\mathbb{R}$. The sharpest results in [1] were obtained in the case where $B$ is a rank-one operator. It turned out that in this case the sufficient condition for the existence of the directional derivative for any continuous function $\phi$ is the membership of the Borel transform of the spectral measure of the operator $A$ to the Hardy class $H_\infty$ (BTB property). Furthermore, in this case the directional derivative is expressed via the commutator of the multiplication operator $M_\phi$ on the function $\phi$ with the Hilbert transform $H$.

It is shown in the present paper by means of Friedrix method [5] that the BTB property is sufficient and necessary for the existence of the holomorphic continuation of the operator function $\phi(A + \gamma B)$ from an interval $(-\delta, \delta)$ to the open disk $D_\delta = \{ \gamma \in \mathbb{C} \mid |\gamma| < \delta \}$ for any continuous function $\phi$. Moreover, under this condition there exists a family of unitary operators $U_\gamma$ ($\gamma \in (-\delta, \delta)$), which is a real-analytic operator function on $(-\delta, \delta)$ and establishes the unitary equivalence between $A + \gamma B$ and $A$ for each fixed $\gamma \in (-\delta, \delta)$. Let us notice that perturbations considered in this paper belong to the class of so called gentle perturbations, studied in [5] and [7].

2. Notations

$\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers;

$\Re z$ and $\Im z$ are the real and the imaginary parts of a number $z \in \mathbb{C}$;

$\mathcal{O}(x_0)$ is a neighborhood of a point $x_0$ belonging to a topological space $\mathcal{T}$;

$\text{mes}(A)$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$;

$\hat{f}$ is the Fourier transform of a function $f$ from $L_1(\mathbb{R})$ or from $L_2(\mathbb{R})$;

$\text{supp}(f)$ is the support of a function $f$;

If $\rho$ is a measure on $\mathbb{R}$, then $\text{supp}(\rho)$ denotes its support;

$M_\phi$ is the operator of multiplication by a function $\phi(t)$ on $\mathbb{R}$;

$\| \cdot \|_E$ is the norm of a Banach space $\mathcal{E}$;

$C[a,b]$ is the Banach algebra of continuous functions on $[a,b]$ with the supremum norm;

$(\cdot, \cdot)_\mathcal{H}$ and $\| \cdot \|_\mathcal{H}$ are the inner product and the norm in a Hilbert space $\mathcal{H}$.

If it is clear what Hilbert space is meant, we shall simply write $(\cdot, \cdot)$ and $\| \cdot \|$;

$\text{span}(\mathcal{M})$ is the closure of the linear span of a subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$.
For Banach spaces $E$ and $F$, $\mathcal{B}(E,F)$ is the Banach space of all bounded linear operators from $E$ into $F$.

$\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ is the Banach algebra of bounded linear operators acting in a Hilbert space $\mathcal{H}$.

If $A$ is a linear operator acting in a Hilbert space $\mathcal{H}$, then:

- $\sigma(A)$ and $\sigma_e(A)$ are the spectrum and essential spectrum of $A$;
- $\sigma_d(A) = \sigma(A) \setminus \sigma_e(A)$; if $A$ is self-adjoint, $\sigma_d(A)$ consists of isolated eigenvalues of finite multiplicity;
- $R_\lambda(A)$ ($\lambda \notin \sigma(A)$) is the resolvent of $A$, i.e., $R_\lambda(A) = (A - \lambda I)^{-1}$;
- $\text{tr}(A)$ is the trace of a linear operator $A$ belonging to the trace class.

### 3. Properties of Holomorphic Unitary Equivalence, Functional Holomorphy and Complex Local Isospectrality

In this section we consider self-adjoint bounded operators $A$ and $B$ acting in a Hilbert space $\mathcal{H}$. Consider also a perturbed operator $A_\gamma = A + \gamma B$, where $\gamma$ is a real coupling constant. First we introduce some definitions. First of all, notice that the holomorphy of an operator function $F : U \to \mathcal{B}(\mathcal{H})$ ($U \subseteq \mathbb{C}$), $U$ is open) in the uniform operator topology is equivalent to its holomorphy in the weak operator topology (see [4]). Hence we shall use simply the term "holomorphic operator function".

#### 3.1. Definitions

**Definition 3.1.** We say that the pair of operators $A$, $B$ has the property of Holomorphic Unitary Equivalence (briefly-HUE property), if there exists $\delta > 0$ and a family of unitary operators $\{U_\gamma\}_{\gamma \in (-\delta, \delta)}$ such that

$$\forall \gamma \in (-\delta, \delta) : \ A_\gamma = (U_\gamma)^{-1}AU_\gamma \quad (3.1)$$

and the operator function $U_\gamma$ admits a holomorphic continuation from $(-\delta, \delta)$ to the open disk $D_\delta = \{\gamma \in \mathbb{C} \mid |\gamma| < \delta\}$.

**Definition 3.2.** We say that the pair of operators $A$, $B$ has the property of Functional Holomorphy (briefly- FH property), if there exists $\delta > 0$ such that for any continuous function $\phi : \mathbb{R} \to \mathbb{C}$ the operator function $\phi(A_\gamma)$ admits a holomorphic continuation $\Phi(\gamma)$ from $(-\delta, \delta)$ to the open disk $D_\delta = \{\gamma \in \mathbb{C} \mid |\gamma| < \delta\}$.

**Definition 3.3.** We say that the pair of operators $A$, $B$ has the property of Complex Local Isospectrality (briefly- CLI property), if there exists $\delta > 0$ such that

$$\forall \gamma \in D_\delta : \ \sigma(A_\gamma) = \sigma(A). \quad (3.2)$$
3.2. Relation between the above properties

**Proposition 3.4.** The HUE property of the pair $A$, $B$ implies its FH and CLI properties.

**Proof.** Let $V_\gamma$ be the holomorphic continuation of the operator function $U_\gamma$ from $(-\delta, \delta)$ to the open disk $D_\delta$. Then there exists $\sigma \in (0, \delta]$ such that for each $\gamma \in D_\sigma$ operator $V_\gamma$ is continuously invertible. Then, by the principle of holomorphic continuation, (3.1) implies that $A_\gamma = (V_\gamma)^{-1}AV_\gamma$ for any $\gamma \in D_\sigma$, that is the operator $A_\gamma$ is similar to $A$ for these values of $\gamma$. This fact implies the CLI property of the pair $A$, $B$. The FH property follows from the evident equality $\phi(A_\gamma) = (V_\gamma)^{-1}\phi(A)V_\gamma$ which is valid for any continuous function $\phi : \mathbb{R} \to \mathbb{C}$. The proposition is proven. □

Before formulating the next theorem, we need the following lemma, which is a slight generalization of Proposition 3.5 from [1].

**Lemma 3.5.** Let $A$ and $B$ be linear operators acting in a Hilbert space $H$, $A$ is self-adjoint and $B$ is compact. Assume that for some $\delta > 0$ \[ \forall \gamma \in D_\delta : \quad \sigma(A_\gamma) \subseteq \sigma(A). \] (3.3)

Then the property (3.2) is valid.

**Proof.** Since $B$ is compact, then by H. Weyl Theorem ([6], Chapter IV, §5, no 6, Theorem 5.35), \[ \forall \gamma \in \mathbb{C} : \quad \sigma_e(A_\gamma) = \sigma_e(A). \] (3.4)

Then, in view of (3.3), in order to prove (3.2), it is enough to show that \[ \sigma_d(A) \subseteq \sigma_d(A_\gamma) \quad \forall \gamma \in D_\delta. \] (3.5)

Take $\lambda_0 \in \sigma_d(A)$. Then, in view of (3.3), there exist $(-\delta, \delta)$ is a neighborhood $D_\varepsilon(\lambda_0) = \{ \lambda \in \mathbb{C} | |\lambda - \lambda_0| < \varepsilon \}$ of the point $\lambda_0$, such that $(D_\varepsilon(\lambda_0) \setminus \{\lambda_0\}) \cap \sigma(A_\gamma) = \emptyset$ for any $\gamma \in D_\delta$. Since the function \[ T(\gamma) = -\frac{1}{2\pi i} \text{tr} \left( \oint_{|\lambda - \lambda_0| = \frac{\varepsilon}{2}} R_{\lambda}(A_\gamma) \, d\lambda \right) \]

is continuous and takes only non-negative integer values and $T(0) > 0$, then $T(\gamma) > 0$ for any $\gamma \in (D_\delta$. This means that for these values of $\gamma$ the point $\lambda_0$ belongs to $\sigma_d(A_\gamma)$. So, (3.5) is valid, hence (3.2) is valid too. □

We now turn to the main result of this section.

**Theorem 3.6.** Let $A$, $B$ be a pair of bounded self-adjoint operators acting in a Hilbert space $H$. If this pair has the FH property and $B$ is compact, then it has the CLI property.

**Proof.** Let $M > 0$ be such a number that $\sigma(A) \subset (-M, M)$. Then for a small enough $\gamma$ $\sigma(A_\gamma) \subset (-M, M)$. In view of the FH property, for any function $\phi \in C[-M, M]$ the operator function $\Phi(\gamma)$, which is the holomorphic continuation of $\phi(A_\gamma)$ from $(-\delta, \delta)$ to the disk $D_\delta$, is differentiable at the point.
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\( \gamma = 0 \) with respect to the uniform operator topology, that is in this topology there exists the limit

\[
\lim_{\gamma \to 0} \frac{\Phi(\gamma) - \phi(A)}{\gamma}
\]

for any \( \phi \in C[-M, M] \). By Banach-Steinhaus Theorem this fact implies that

\[
\limsup_{\gamma \to 0} \|\Sigma(\cdot, \gamma)\|_{B(C[-M, M], B(H))} < \infty,
\]

where

\[
\Sigma(\phi, \gamma) = \frac{\Phi(\gamma) - \phi(A)}{\gamma}
\]

By Lemma 3.5 in order to prove CLI property, it is enough to establish the property:

\[
\exists \sigma > 0 \; \forall \gamma \in D_\sigma : \; \sigma(A_\gamma) \subseteq \sigma(A).
\]

Assume, on the contrary, that this property does not hold. This means that there exists a sequence \( \{\gamma_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} \gamma_n = 0 \), \( \gamma_n \neq 0 \) and

\[
S_n = \sigma(A_{\gamma_n}) \setminus \sigma(A) \neq \emptyset
\]

for all natural \( n \). Since \( B \) is compact, then by H. Weyl Theorem, \([3.4]\) is valid. Hence each set \( S_n \) is contained in \( \sigma_d(A_{\gamma_n}) \), that is it consists of isolated eigenvalues of the operator \( A_{\gamma_n} \) of finite algebraic multiplicity. If we shall prove that

\[
\limsup_{n \to \infty} \|\Sigma(\cdot, \gamma_n)\|_{B(C[-M, M], B(H))} = \infty,
\]

then we shall obtain a contradiction with \([3.6]\) and the theorem will be proved. Denote by \( \mathcal{E} \) the set of all entire functions \( \phi : \; C \to C \) and for any \( \phi \in \mathcal{E} \) denote

\[
\phi_r = \phi|_{[-M, M]}.
\]

Then, in order to prove \([3.10]\), it is enough to show that

\[
\limsup_{n \to \infty} \left( \sup \left\{ \|\Sigma(\phi_r, \gamma_n)\|_{B(H)} : \phi \in \mathcal{E}, \|\phi_r\|_{C[-M, M]} \leq 1 \right\} \right) = \infty.
\]

As is easy to see, the family of spectra \( \{\sigma(A_{\gamma})\}_{\gamma \in D_\delta} \) is uniformly bounded in the complex plane. Then there exists a closed contour \( \Gamma \) lying in the resolvent set \( \mathcal{R}(A_{\gamma}) \) of each operator \( A_{\gamma} \) \( (\gamma \in D_\delta) \) and surrounding all the spectra \( \sigma(A_{\gamma}) \) \( (\gamma \in D_\delta) \). As is known, for any \( \phi \in \mathcal{E} \)

\[
\phi(A_{\gamma}) = -\frac{1}{2\pi i} \oint_{\Gamma} \phi(\lambda) R_{\lambda}(A_{\gamma}) d\lambda.
\]

and for any fixed \( \lambda \in \mathcal{R}(A_{\gamma}) \) the resolvent \( R_{\lambda}(A_{\gamma}) \) is a holomorphic operator function of \( \gamma \) in a neighborhood of \( \gamma = 0 \). These circumstances imply that \( \phi(A_{\gamma}) \) is a holomorphic operator function of \( \gamma \) in a disk \( D_\sigma \) \( (\sigma \in (0, \delta)) \). Thus, by the principle of analytic continuation, for any \( \phi \in \mathcal{E} \) the function \( \Phi(\gamma) \), corresponding to \( \phi_r \), coincide with \( \phi(A_{\gamma}) \) in \( D_\sigma \). Using the assumption \([3.9]\), we can take \( \lambda_n \in S_n \) for each natural \( n \). We can select from this sequence an infinite subsequence (we denote it in the same manner: \( \lambda_n \)) having one of the following properties:

(a) or \( \lambda_n \notin \mathbb{R} \) for any natural \( n \),
(b) or \( \lambda_n \in \mathbb{R} \) for any natural \( n \).

Let us construct a sequence of entire functions \( \phi_n(\lambda) \) in the following manner. In case (a) we put:

\[
\phi_n(\lambda) := \begin{cases} 
  e^{-i\tau_n(\lambda-\lambda_n)}, & \text{if } \Im(\lambda_n) > 0, \\
  e^{i\tau_n(\lambda-\lambda_n)}, & \text{if } \Im(\lambda_n) < 0.
\end{cases}
\] (3.13)

In case (b) we put:

\[
\phi_n(\lambda) := e^{-\tau_n(\lambda-\lambda_n)^2}.
\] (3.14)

In both the cases the sequence \( \tau_n > 0 \) will be specified in the sequel. For each natural \( n \) let us take an eigenvector \( e_n \) of the operator \( A_{\gamma_n} \), corresponding to its eigenvalue \( \lambda_n \), such that \( \|e_n\| = 1 \). As is known, \( \phi_n(A_{\gamma_n})e_n = \phi(\lambda_n)e_n \).

Taking into account that, in view of definitions (3.13), (3.14), \( \phi_n(\lambda_n) = 1 \), we obtain in both the cases (a) and (b):

\[
\Sigma(\phi_{n,r}, \gamma_n)e_n = \frac{\phi_n(A_{\gamma_n})e_n - \phi(A)e_n}{\gamma_n} = \frac{1}{\gamma_n}e_n - \frac{1}{\gamma_n}\phi(A)e_n.
\]

Here \( \phi_{n,r} \) is defined by (3.11) with \( \phi = \phi_n \). Then

\[
\|\Sigma(\phi_{n,r}, \gamma_n)\|_{\mathcal{B}(H)} \geq \frac{1}{|\gamma_n|} - \frac{1}{|\gamma_n|}\|\phi(A)\|_{\mathcal{B}(H)}.
\] (3.15)

On the other hand, since the operator \( A \) is self-adjoint, we get:

\[
\|\phi_n(A)\|_{\mathcal{B}(H)} \leq \max_{\lambda \in \sigma(A)} |\phi_n(\lambda)|.
\] (3.16)

In the case (a) we obtain from (3.13):

\[
\max_{\lambda \in \sigma(A)} |\phi_n(\lambda)| \leq e^{-\tau_n \Im(\lambda_n)},
\] (3.17)

and in the case (b) we obtain from (3.14):

\[
\max_{\lambda \in \sigma(A)} |\phi_n(\lambda)| \leq e^{-\tau_n d_n^2},
\] (3.18)

where \( d_n = \text{dist}(\lambda_n, \sigma(A)) \). Let us choose \( \tau_n \) in the following manner: in case (a) we put \( \tau_n = |\Im(\lambda_n)|^{-1} \) and in case (b) we put \( \tau_n = d_n^{-2} \). Then in both cases (a) and (b) we obtain from (3.13)-(3.18):

\[
\|\Sigma(\phi_{n,r}, \gamma_n)\|_{\mathcal{B}(H)} \geq \frac{1 - e^{-1}}{|\gamma_n|}
\] (3.19)

Observe that, in view of definitions (3.13), (3.14), \( \|\phi_{n,r}\|_{C[-M,M]} \leq 1 \). Then, taking into account that \( \lim_{n \to \infty} \gamma_n = 0 \), we obtain from (3.19) the desired limiting relation (3.12). The theorem is proven. \( \square \)
4. The case of a rank-one perturbation

In the space $\mathcal{H} = L^2(\mathbb{R}, \rho)$ with a compactly supported non-negative Borel measure $\rho$ consider the multiplication operator

$$(Af)(\mu) = \mu f(\mu) \quad (f \in \mathcal{H})$$

and its perturbation $A_\gamma = A + \gamma B$, where $B$ is a rank-one operator: $B = (\cdot, g)g$ and $g(\mu) = 1$ almost everywhere with respect to the measure $\rho$ (hence $g \in \mathcal{H}$). We call $\rho$ the spectral measure of $A$. Let us take $M > 0$ such that $\sigma(A) = \text{supp}(\rho) \subset (-M, M)$. In this section we shall obtain a necessary and sufficient condition for the spectral measure $\rho$ ensuring the UHE property of the pair $A, B$.

4.1. Friedrichs method

We shall use the method of 0. K. Friedrichs ([5], Chapt. II, Sect. 6). Let us describe it briefly. One searches for a pair of operators $U^+$ and $U^-$ satisfying the conditions

$$AU^+ = U^+ A_\gamma \quad (4.1)$$

and

$$A_\gamma U^- = U^- A \quad (4.2)$$

and shows that they can be chosen such that $U^+ U^- = U^- U^+ = I$ and $U^- = (U^+)^\ast$. Let us write the equations (4.1) and (4.2) in the form:

$$[A, U^+] = AU^+ - U^+ A = \gamma U^+ B; \quad (4.3)$$

$$[A, U^-] = AU^- - U^- A = -\gamma BU^- . \quad (4.4)$$

In order to solve these equations, we need be able to solve a simpler equation

$$[A, Z] = AZ - ZA = R \quad (R \in \mathcal{B}(\mathcal{H})) \quad (4.5)$$

in the class $\mathcal{B}(\mathcal{H})$. It is clear that if there exists a solution $Z_0$ of this equation, it is not unique, because $Z = Z_0 + C$, where $C$ commutes with $A$, is also solution of this equation. We shall show that equation (4.5) has a solution, if $R$ belongs to some class of operators $\mathcal{R}$. Denote by $\Gamma$ a transformer which associates a solution $Z$ of equation (4.5) to each $R \in \mathcal{R}$: $Z = \Gamma R$. In the sequel this transformer $\Gamma$ will be chosen in a suitable manner. Following the method of Friedrichs, one searches for solutions of equations (4.3), (4.4) in the form:

$$U^+ = I + \Gamma R^+, \quad U^- = I - \Gamma R^- , \quad (4.6)$$

where $R^+$ and $R^-$ satisfy the equations

$$R^+ = \gamma (I + \Gamma R^+) B, \quad (4.7)$$

$$R^- = \gamma B (I - \Gamma R^-) . \quad (4.8)$$

We see easily that if $R^+$ and $R^-$ are solutions of these equations, then $U^+$ and $U^-$, defined by (4.6), indeed satisfy equations (4.3), (4.4).
4.2. **Transformer** $\Gamma$

In order to define the transformer $\Gamma$, we need to solve the equation (4.5). How to do this? In the book [3] the operator equations of the form

$$AZ - BZ = R \quad (A, B \in \mathcal{B}(\mathcal{H}))$$

(and even of more general form) are considered. As is shown there, if

$$\sigma(A) \cap \sigma(B) = \emptyset,$$  

(4.10)

then the equation (4.9) has a unique solution in $\mathcal{B}(\mathcal{H})$ and it has the form:

$$Z = -\frac{1}{4\pi^2} \oint_{C_A} \oint_{C_B} \frac{R_\lambda(A)RR_\mu(B)}{\lambda - \mu} d\lambda d\mu,$$

where the contours $C_A$ and $C_B$ lie in $\mathcal{R}(A) \cap \mathcal{R}(B)$, $C_A$ surrounds $\sigma(A)$, but does not surround $\sigma(B)$ and $C_B$ surrounds $\sigma(B)$, but does not surround $\sigma(A)$. We see that the condition (4.10) is not satisfied for our equation (4.5). Hence we consider first the following “regularized” equation

$$(A + i\epsilon I)Z - ZA = R,$$  

(4.11)

where $\epsilon > 0$. Since $A$ is self-adjoint, $\sigma(A + i\epsilon I) \cap \sigma(A) = \emptyset$, hence the equation (4.11) has in $\mathcal{B}(\mathcal{H})$ a unique solution

$$Z_\epsilon = \Gamma_\epsilon R = -\frac{1}{2\pi i} \oint_{C_{\epsilon}^1} \left( -\frac{1}{2\pi i} \oint_{C_{\epsilon}^2} \frac{R_\lambda(A + i\epsilon I)}{\lambda - \mu} d\lambda \right) RR_\mu(A) d\mu,$$  

(4.12)

where the contours $C_{\epsilon}^1$ and $C_{\epsilon}^2$ lie in $\mathcal{R}(A) \cap \mathcal{R}(A + i\epsilon I)$, $C_{\epsilon}^2$ surrounds $\sigma(A + i\epsilon I)$, but does not surround $\sigma(A)$ and $C_{\epsilon}^1$ surrounds $\sigma(A)$, but does not surround $\sigma(A + i\epsilon I)$. Hence the expression inside the brackets in (4.12) yields $f_\mu(A + i\epsilon I)$, where $f_\mu(\lambda) = (\lambda - \mu)^{-1}$. Thus, we have:

$$Z_\epsilon = \Gamma_\epsilon R = -\frac{1}{2\pi i} \oint_{C_{\epsilon}^1} R_{\mu - i\epsilon}(A)RR_\mu(A) d\mu.$$  

(4.13)

If for some $R \in \mathcal{B}(\mathcal{H})$ there exists the limit

$$Z = \lim_{\epsilon \downarrow 0} Z_\epsilon$$  

(4.14)

in the strong operator topology, then tending $\epsilon \downarrow 0$ in (4.11) with $Z = Z_\epsilon$, we obtain that $Z$ is a solution of the equation (4.5) belonging to $\mathcal{B}(\mathcal{H})$. As above, we denote it by $\Gamma R$. Denote by $\text{Dom}(\Gamma)$ the domain of definition of the transformer $\Gamma$, i.e. this is the set of all $R \in \mathcal{B}(\mathcal{H})$ such that for the solution $Z_\epsilon$ of the equation (4.11) there exists the limit (4.14) in the strong operator topology. It is clear that $\Gamma$ is a linear transformer.

### 4.3. Transformer $\Gamma$ and Riesz projection

Let us return to the multiplication operator $A$ and its rank-one perturbation described in the beginning of this section. In the sequel we shall describe a class of rank-one operators acting in the space $\mathcal{H} = L_2(\mathbb{R}, \rho)$ and belonging to $\text{Dom}(\Gamma)$. To this end we need the following
Lemma 4.1. Assume that the spectral measure $\rho$ of the operator $A$ is absolutely continuous with the density $\tilde{\rho}$. Let $R$ be a rank-one operator acting in $\mathcal{H}$, that is $R = (\cdot, r_1) r_2$ ($r_1, r_2 \in \mathcal{H}$), and let $\Gamma_\epsilon$ ($\epsilon > 0$) be the transformer in $\mathcal{B}(\mathcal{H})$ defined by (4.13). Then $\Gamma_\epsilon R$ is an integral operator with the kernel

$$
\frac{r_2(x) r_1(t) \tilde{\rho}(t)}{x + i \epsilon - t}.
$$

Proof. Recall that $\sigma(A) = \text{supp}(\rho) \subset (-M, M)$. Let us take in (4.13) $C^1_\epsilon = \{ \lambda \in \mathbb{C} | \text{dist}(\lambda, (-M, M)) = \frac{\epsilon}{4} \}$. We have for $f \in \mathcal{H}$:

$$
\Gamma_\epsilon Rf(x) = -\frac{1}{2\pi i} \int_{C^1_\epsilon} d\mu \int_{-\infty}^{\infty} \frac{r_2(x) f(t) r_1(t) \tilde{\rho}(t) dt}{(x + i \epsilon - \mu)(t - \mu)} = \int_{-M}^{M} f(t) r_1(t) \tilde{\rho}(t) dt \left( \frac{1}{2\pi i} \int_{C^1_\epsilon} d\mu \frac{1}{(x + i \epsilon - \mu)(\mu - t)} \right) r_2(x)
$$

Using Cauchy formula inside of the brackets, we get:

$$
\Gamma_\epsilon Rf(x) = \int_{-\infty}^{\infty} \frac{r_2(x) r_1(t) \tilde{\rho}(t) f(t) dt}{x + i \epsilon - t}.
$$

(4.15)

This proves the lemma. \qed

We can write the formula (4.15) with the help of the following bounded operator acting in the space $L^2(\mathbb{R})$:

$$(P_{+, \epsilon}h)(u) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{h(s) ds}{s - u - i \epsilon} \quad (\epsilon > 0, h \in L^2(\mathbb{R})).
$$

(4.16)

In order to do this, we shall impose the following conditions on the operator $A$ and on a rank-one operator $R$:

(A) The spectral measure $\rho$ of the operator $A$ is absolutely continuous and its density $\tilde{\rho}$ belongs to the class $L^\infty(\mathbb{R})$;

(B) $R = (\cdot, r_1) r_2$, where $r_k \in L^\infty(\mathbb{R})$ ($k = 1, 2$).

We have the following consequence of Lemma 4.1:

Corollary 4.2. If the conditions (A) and (B) are satisfied, then the rank-one operator $R = (\cdot, r_1) r_2$ is bounded in $\mathcal{H}$ and the representation is valid:

$$
\Gamma_\epsilon R = -2\pi i J M_{r_2} P_{+, \epsilon} M_{r_1} \tilde{\rho},
$$

(4.17)

where $J$ is the embedding operator from $L^2(\mathbb{R})$ into $\mathcal{H}$.

Proof. Observe that the assumptions of the corollary and the fact that the measure $\rho$ has a compact support imply that $r_k \in \mathcal{H}$, hence $R$ is a bounded operator. Observe that since $\tilde{\rho} \in L^\infty(\mathbb{R})$, the embedding $J : L^2(\mathbb{R}) \rightarrow \mathcal{H}$ holds and it is continuous. Furthermore, in view of the above assumptions, the multiplication operators $M_{r_1} \tilde{\rho}$ and $M_{r_2}$ act continuously from $\mathcal{H}$ into $L^2(\mathbb{R})$. 

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and from $L_2(\mathbb{R})$ into itself, respectively. These circumstances and formula (4.15) imply the desired representation (4.17).

**Proposition 4.3.** If the conditions (A) and (B) are satisfied, then the rank-one operator $R = (\cdot, r_1)r_2$ belongs to $\text{Dom}(\Gamma)$ and the representation is valid:

$$\Gamma R = -2\pi iJM_{r_2}P_+M_{r_1} \bar{\rho},$$

where $P_+$ is Riesz projection in $L_2(\mathbb{R})$ on the Hardy space

$$\mathcal{H}_+ = \{f \in L_2(\mathbb{R}) | \hat{f}(\omega) = 0 \text{ a.e. on } (-\infty, 0)\}.$$

**Proof.** By Proposition 2.1 of [1], the family of operators $P_{+,\epsilon}$, defined by (4.16), has the property: $\lim_{\epsilon \downarrow 0} P_{+,\epsilon} = P_+$ in the strong operator topology. Hence, in view of (4.17),

$$\lim_{\epsilon \downarrow 0} \Gamma_{\epsilon} R = -2\pi iJM_{r_2}P_+M_{r_1} \bar{\rho}$$

in the strong operator topology. This means that $R \in \text{Dom}(\Gamma)$ and the desired equality (4.18) is valid. The proposition is proven. □

### 4.4. BTB property and solution of basic equations

We now turn to solution of the equations (4.7) and (4.8). In the sequel we need the following notion.

**Definition 4.4.** We say that the spectral measure $\rho$ of the operator $A$ has the property of Borel Transform Boundedness (briefly - BTB property), if its Borel transform

$$B\rho(\lambda) := \int_{\mathbb{R}} \frac{\rho(dt)}{t - \lambda}$$

belongs to the Hardy class $H_+^\infty$ (that is, it is bounded in the upper (hence in the lower) half-plane of the complex plane $\mathbb{C}$).

**Remark 4.5.** BTB property implies that the measure $\rho$ is absolutely continuous and its density $\tilde{\rho}$ belongs to the class $L_\infty(\mathbb{R})$. This fact follows from Stieltjes inversion formula

$$\rho(b) - \rho(a) = -\frac{1}{\pi} \lim_{\tau \downarrow 0} \int_a^b \Im(B\rho(t + i\tau))dt,$$

in which $\rho(\mu)$ is the non-decreasing function defining the measure $\rho$ and $a$, $b$ are points of continuity of this function.

As in the beginning of this section, along with the multiplication operator $A$ acting in $\mathcal{H} = L_2(\mathbb{R}, \rho)$ we consider the rank-one perturbing operator $B = (\cdot, g)g$, where $g(t) = 1$ almost everywhere with respect to the measure $\rho$. The following statement is valid.

**Proposition 4.6.** Assume that the spectral measure $\rho$ of the operator $A$ has the BTB property. Then for a small enough $\gamma \in \mathbb{C}$ the equations (4.7) and (4.8) have unique solutions $R^+$ and $R^-$ in the class of operators satisfying condition (B). Furthermore, they have the form:

$$R^+ = \gamma(\cdot, g)M_{\psi, g}$$

(4.20)
Holomorphy with respect to coupling constant

\[ R^- = \gamma(\cdot, M_{\psi}, g)g, \]  
\[ \psi_\gamma(x) = (1 + 2\pi i \gamma P_+ \bar{\rho}(x))^{-1}. \]

\[ R^+ = \gamma(\cdot, g)(I + \Gamma R^+)g, \]
\[ R^- = \gamma(\cdot, (I - \Gamma R^-)^* g)g, \]

Proof. Let \( R^+ \) and \( R^- \) be solutions of equations (4.7) and (4.8) satisfying condition (B). Then, taking into account that \( B = (\cdot, g)_g \) and using Proposition 4.3, we have:

\[ R^+ = \gamma(\cdot, g)g, \]  
\[ R^- = \gamma(\cdot, g)g, \]

that is \( R^+ \) and \( R^- \) have the form:

\[ R^+ = (\cdot, g)r^+, \]
\[ R^- = (\cdot, r^-)g, \]

where \( r^+, r^- \in H \). Let us find \( r^+ \) and \( r^- \). By Proposition 4.3,

\[ \Gamma R^+ = -2\pi i JM_r + P_+ M_{\bar{\rho}}, \]
\[ \Gamma R^- = -2\pi i JM_r - P_+ M_{\bar{\rho}}. \]

Hence, taking into account that \( J^* = M_{\bar{\rho}} \), we have:

\[ (\Gamma R^-)^* = 2\pi i JM_r - P_+ M_{\bar{\rho}}. \]

Then, after substituting (4.25), (4.26), (4.27) and (4.28) into (4.23) and (4.24), we get:

\[ (\cdot, g)r^+ = \gamma(\cdot, g) (g - 2\pi i JM_r + P_+ (|g|^2 \bar{\rho})) \]
\[ (\cdot, r^-)g = \gamma(\cdot, g - 2\pi i JM_r - P_+ (|g|^2 \bar{\rho})), \]

that is

\[ r^+(x) (1 + 2\pi i \gamma P_+ (|g|^2 \bar{\rho})(x)) = \gamma g(x) \]  
\[ r^-(x) (1 + 2\pi i \gamma P_+ (|g|^2 \bar{\rho})(x)) = \gamma g(x). \]

Observe that \( |g|^2 \bar{\rho} = \bar{\rho} \), because \( g(x) = 1 \) almost everywhere with respect to the measure \( \rho \). Consider the operator \( P_{+,\epsilon} (\epsilon > 0) \) defined by (4.16). In view of BTB property of \( \rho \), the family of functions \( \{P_{+,\epsilon} \bar{\rho}(x)\}_{\epsilon > 0} \) is uniformly bounded on \( \mathbb{R} \). On the other hand, by the property of Hardy class \( H_2 \),

\[ \lim_{\epsilon \downarrow 0} P_{+,\epsilon} \bar{\rho}(x) = P_+ \bar{\rho}(x) \]

for almost all \( x \in \mathbb{R} \). These circumstances mean that \( P_+ \bar{\rho} \in L_\infty(\mathbb{R}) \). Hence, for a small enough \( \gamma \in \mathbb{C} \) the function \( \psi_\gamma \), defined by (4.22), belongs to the class \( L_\infty(\mathbb{R}) \). Then we have from (4.29) and (4.30) that

\[ r^+ = r^- = M_{\psi_\gamma} g. \]

Hence, in view of (4.25) and (4.26), the operators \( R^+ \) and \( R^- \) have the form (4.20) and (4.21). So, we have proved the uniqueness of solution of equations (4.7) and (4.8) in the class of operators satisfying condition (B) (for a small
enough $\gamma$). Carrying out the above arguments in the inverse direction, we can show that for a small enough $\gamma$ the operators $R^+$ and $R^-$, expressed by (4.20) and (4.21), satisfy condition (B) and equations (4.7) and (4.8), respectively. The proposition is proven.

Let us return to formulas (4.6) for the operators $U^+_\gamma$ and $U^-_\gamma$. In this section we shall show that

$$U^+_\gamma U^-_\gamma = U^-_\gamma U^+_\gamma = I$$

for a small enough $\gamma$. Substituting $R^+$ and $R^-$ from (4.20) and (4.21) into (4.6) and taking into account (4.27), (4.28) and (4.32), we obtain the explicit formulas for $U^+_\gamma$ and $U^-_\gamma$:

$$U^+_\gamma = I - 2\pi i\gamma JM\psi_{\gamma,\gamma} g P + M\bar{g} \rho $$

$$U^-_\gamma = I + 2\pi i\gamma JM g P + M\bar{g} \psi_{\gamma,\gamma} \rho,$$

where $\psi_{\gamma,\gamma}$ is defined by (4.22). In the sequel we need the following

**Lemma 4.7.** The following identity is valid for any $R_1, R_2 \in B(H)$ and $\varepsilon > 0$ (see [5], Chapter II, Sect. 6):

$$\Gamma \varepsilon R_1 \cdot \Gamma \varepsilon R_2 = \Gamma_2 \varepsilon (\Gamma \varepsilon R_1 \cdot R_2 + R_1 \cdot \Gamma \varepsilon R_2).$$

**Proof.** Denote $[A, Z]_\varepsilon = (A + i\varepsilon I)Z - AZ$. Then, by the definition of $\Gamma \varepsilon$, the equality $[A, Z]_\varepsilon = R$ is equivalent to the equality $Z = \Gamma \varepsilon R$. Let us take $R_1, R_2 \in B(\mathcal{H})$ and denote $Z_k = \Gamma \varepsilon R_k$ ($k = 1, 2$). We have:

$$[A, Z_1Z_2]_\varepsilon = (A + 2i\varepsilon I)Z_1 Z_2 - Z_1 Z_2 A = (A + i\varepsilon I)Z_1 Z_2 + i\varepsilon Z_1 Z_2 - Z_1 AZ_2 + Z_1 AZ_2 - Z_1 Z_2 A = ((A + i\varepsilon I)Z_1 - Z_1 A)Z_2 + Z_1((A + i\varepsilon I)Z_2 - Z_2 A) = [A, Z_1]\varepsilon Z_2 - Z_1[A, Z_2]_\varepsilon.$$

So, we get:

$$[A, Z_1Z_2]_\varepsilon = [A, Z_1]_\varepsilon Z_2 - Z_1[A, Z_2]_\varepsilon.$$  

Applying $\Gamma \varepsilon$ to both sides of the latter equality, we obtain the desired identity (4.35). The lemma is proven.

On the base of the previous lemma we obtain the following

**Lemma 4.8.** If all the conditions of Proposition 4.6 are satisfied, then the operators $R^+$ and $R^-$, defined by (4.20) and (4.21), have the property:

$$\Gamma R^+ \cdot \Gamma R^- = \Gamma (\Gamma R^+ \cdot R^- + R^- \cdot \Gamma R^+).$$

**Proof.** By Lemma 4.7 we have:

$$\Gamma \varepsilon R^+ \cdot \Gamma \varepsilon R^- = \Gamma_2 \varepsilon (\Gamma \varepsilon R^+ \cdot R^- + R^- \cdot \Gamma \varepsilon R^+).$$

On the other hand, since the operators $R^+$ and $R^-$ satisfy the condition (B), then by Proposition 4.3 they belong to Dom($\Gamma$). These circumstances imply that, in order to prove (4.36), it is enough to show that

$$\lim_{\varepsilon \downarrow 0} \Gamma_2 \varepsilon (\Gamma \varepsilon R^+ \cdot R^-) = \Gamma (\Gamma R^+ \cdot R^-),$$
\[
\lim_{\epsilon \downarrow 0} \Gamma_{2\epsilon}(R^+ \cdot \Gamma_{\epsilon} R^-) = \Gamma(R^+ \cdot \Gamma R^-)
\]
in the strong operator topology. By Lemma 4.1 and Proposition 4.6, \(\Gamma_{\epsilon} R^+\) and \(\Gamma_{\epsilon} R^-\) are integral operators with the kernels
\[
\frac{r_{\gamma}(x)g(s)\tilde{\rho}(s)}{x + i\epsilon - s} \quad \text{and} \quad \frac{g(x)r_{\gamma}(s)\tilde{\rho}(s)}{x + i\epsilon - s},
\]
respectively, where \(r_{\gamma}(x) = \psi_{\gamma}(x)g(x)\) and \(\psi_{\gamma}(x)\) is defined by (4.22). Then \(\Gamma_{\epsilon} R^+ \cdot R^-\) and \(R^+ \cdot \Gamma_{\epsilon} R^-\) are integral operators with the kernels
\[
\frac{r_{\gamma}(x)\int_{\mathbb{R}} \frac{|g(s)|^2 \tilde{\rho}(s) d\tilde{s}}{x + i\epsilon - s} r_{\gamma}(s)\tilde{\rho}(s)}{x + 2i\epsilon - s} \quad \text{and} \quad \frac{r_{\gamma}(x)\int_{\mathbb{R}} \frac{|g(s)|^2 \tilde{\rho}(s) d\tilde{s}}{s + i\epsilon - s} r_{\gamma}(s)\tilde{\rho}(s)}{x + 2i\epsilon - s}
\]
respectively. Then, by Lemma 4.1, \(\Gamma_{2\epsilon}(\Gamma_{\epsilon} R^+ \cdot R^-)\) and \(\Gamma_{2\epsilon}(R^+ \cdot \Gamma_{\epsilon} R^-)\) are integral operators with the kernels
\[
\frac{r_{\gamma}(x)\int_{\mathbb{R}} \frac{|g(s)|^2 \tilde{\rho}(s) d\tilde{s}}{x + i\epsilon - s} r_{\gamma}(s)\tilde{\rho}(s)}{x + 2i\epsilon - s} \quad \text{and} \quad \frac{r_{\gamma}(x)\int_{\mathbb{R}} \frac{|g(s)|^2 \tilde{\rho}(s) d\tilde{s}}{s + i\epsilon - s} r_{\gamma}(s)\tilde{\rho}(s)}{x + 2i\epsilon - s}
\]
respectively. Then, making use of the operator \(P_{+,\epsilon}\), defined by (4.16), we have:
\[
\Gamma_{2\epsilon}(\Gamma_{\epsilon} R^+ \cdot R^-) = (2\pi i)^2 J M_{r_{\gamma}, P_{+,\epsilon}(|g|^2 \tilde{\rho})} P_{+,2\epsilon} M_{\tilde{r}_{\gamma}, \tilde{\rho}}, \quad (4.37)
\]
\[
\Gamma_{2\epsilon}(R^+ \cdot \Gamma_{\epsilon} R^-) = (2\pi i)^2 J M_{r_{\gamma}, P_{+,\epsilon}(|g|^2 \tilde{\rho})} P_{+,2\epsilon} M_{\tilde{r}_{\gamma}, \tilde{\rho} \cdot P_{+,\epsilon}(|g|^2 \tilde{\rho}) \tilde{\rho}}, \quad (4.38)
\]
Recall that \(|g|^2 \tilde{\rho} = \tilde{\rho}\), because \(g(x) = 1\) almost everywhere with respect to the measure \(\rho\). In view of BTB property of \(\rho\), the family of functions \(\{P_{+,\epsilon}(x)\}_{\epsilon > 0}\) is uniformly bounded on \(\mathbb{R}\). On the other hand, by the property of Hardy class \(H_2\), the limiting relation (4.31) is valid for almost all \(x \in \mathbb{R}\). These circumstances and the fact that \(r_{\gamma} \in L_\infty(\mathbb{R})\) for a small enough \(\gamma\), imply that for such \(\gamma\)
\[
\lim_{\epsilon \downarrow 0} M_{r_{\gamma}, P_{+,\epsilon}(|g|^2 \tilde{\rho})} = M_{r_{\gamma}, P_{+}(|g|^2 \tilde{\rho})}
\]
with respect to the strong operator topology. Furthermore, by Proposition 2.1 of [1], \(\lim_{\epsilon \downarrow 0} P_{+,\epsilon} = P_{+}\) with respect to this topology. Then we obtain from (4.37) and (4.38) that
\[
\lim_{\epsilon \downarrow 0} \Gamma_{2\epsilon}(\Gamma_{\epsilon} R^+ \cdot R^-) = (2\pi i)^2 J M_{r_{\gamma}, P_{+}(|g|^2 \tilde{\rho})} P_{+} M_{\tilde{r}_{\gamma}, \tilde{\rho}}, \quad (4.39)
\]
and
\[
\lim_{\epsilon \downarrow 0} \Gamma_{2\epsilon}(R^+ \cdot \Gamma_{\epsilon} R^-) = (2\pi i)^2 J M_{r_{\gamma}, P_{+} M_{r_{\gamma}, P_{+}(|g|^2 \tilde{\rho}) \tilde{\rho} \cdot P_{+}(|g|^2 \tilde{\rho}) \tilde{\rho}}). \quad (4.40)
\]
with respect to the strong operator topology. On the other hand, we can show with the help of Proposition 4.3 that the right hand sides of (4.39) and (4.40) are equal to \(\Gamma(\Gamma R^+ \cdot R^-)\) and \(\Gamma(R^+ \cdot \Gamma R^-)\) respectively. The lemma is proven. \(\square\)
We now turn to main result of this subsection.

**Proposition 4.9.** Assume that the spectral measure \( \rho \) of the operator \( A \) has the BTB property. Then for a small enough \( \gamma \in \mathbb{C} \)

\[
U_\gamma^+ U^-_\gamma = U^-_\gamma U^+_\gamma = I.
\]

**Proof.** Using Lemma 4.8 and \((4.6)\), we have:

\[
U^+_\gamma U^-_\gamma = (I + \Gamma R^+)(I - \Gamma R^-) = (R^+ - R^- - \Gamma R^+ \cdot \Gamma R^-) = (R^+ U^-_\gamma - U^+_\gamma R^-).
\]

But, in view of \((4.7)\) and \((4.8)\), \(R^+ U^-_\gamma = U^+_\gamma BU^-_\gamma = U^+_\gamma R^-\). Hence we have proved that \(U^+_\gamma U^-_\gamma = I\), that is \(U^-_\gamma\) is a right inverse of the operator \(U^+_\gamma\). On the other hand, we obtain from \((4.27)\) and \((4.32)\) that

\[
\Gamma R^+ = -2\pi i \gamma JM \psi_\gamma \cdot g P_+ M \bar{g}.
\]

where \(\psi_\gamma\) is defined by \((4.22)\). Then we see that \(\|\Gamma R^+\| < 1\) for a small enough \(\gamma\), hence the operator \(U^+_\gamma = I + \Gamma R^+\) has a bounded inverse \((U^+_\gamma)^{-1}\), which coincides with the right inverse \(U^-_\gamma\). This means that \(U^-_\gamma U^+_\gamma = I\). The proposition is proven. \(\square\)

**4.5. Final results**

On the base of previous results we can obtain criterion for HUE property for the pair \(A, B\) in the case of the rank-one perturbation \(B\) of the multiplication operator \(A\). The following result is valid.

**Theorem 4.10.** If the spectral measure \(\rho\) of the operator \(A\) has the BTB property, then there exists \(\delta > 0\) such that

(i) the family of operators \(U^+_\gamma\), defined by \((4.33)\), depends holomorphically on \(\gamma\) in the open disk \(D_\delta = \{\lambda \in \mathbb{C} \mid |\lambda| < \delta\}\);

(ii) for each fixed \(\gamma \in (-\delta, \delta)\) the operator \(U^+_\gamma\) is unitary and it establishes a unitary equivalence between \(A\) and \(A_\gamma\), that is

\[
A_\gamma = (U^+_\gamma)^{-1} AU^+_\gamma.
\]  

In other words, the pair of operators \(A, B\) has the HUE property.

**Proof.** Assertion (i) follows immediately from \((4.33)\) and \((4.22)\). Let us prove assertion (ii). We see from \((4.33)\), \((4.34)\) and \((4.22)\) that for a small enough \(\gamma \in \mathbb{R}\) \((U^+_\gamma)^* = U^-_\gamma\). On the other hand, by Proposition 4.9, for a small enough \(\gamma\)

\[
U^-_\gamma = (U^+_\gamma)^{-1}.
\]  

These circumstances mean that for a small enough \(\gamma \in \mathbb{R}\) the operator \(U^+_\gamma\) is unitary. In view of \((4.1)\) and \((4.42)\), the formula \((4.41)\) is valid. The theorem is proven. \(\square\)
Corollary 4.11. If the condition of Theorem 4.10 is satisfied, then for any function \( \phi \in C[-M,M] \) and for a small enough \( \gamma \) the representation is valid:

\[
\phi(A_\gamma) = U_\gamma^- M_{\phi_0} U_\gamma^+ = \left( I + 2\pi i \gamma J M g P_+ M_{\psi_\gamma \bar{g} \rho} \right) M_{\phi_0} \times \left( I - 2\pi i \gamma J M_{\psi_\gamma \bar{g} \rho} P_+ M_{\bar{g} \rho} \right),
\]

where \( \psi_\gamma \) is defined by (4.22) and \( \phi_0 \) is the continuation of the function \( \phi \) by zero out of the interval \([-M,M]\). Recall that \( M > 0 \) is such a number that \( \text{supp}(\rho) \subset (-M,M) \).

Proof. Since \( A \) is the multiplication operator by the independent variable in the space \( H = L_2(\mathbb{R},\rho) \), then \( \phi(A) = M_{\phi_0} \). Hence we have from (4.41) that \( \phi(A_\gamma) = (U_\gamma^+)^{-1} M_{\phi_0} U_\gamma^+ \) for a small enough \( \gamma \). Then, taking into account (4.33), (4.34) and (4.42), we obtain the desired equality (4.43). \( \square \)

As a consequence of Theorem 4.10 and Corollary 4.11 we obtain the following strengthening of Theorem 2.4 from [1]:

Corollary 4.12. If the spectral measure \( \rho \) of the operator \( A \) has the BTB property, then

(i) the pair \( A, B \) has the FH property;

(ii) the formula is valid:

\[
\frac{\partial}{\partial \gamma} \phi(A_\gamma) |_{\gamma=0} = \pi i M_g [H, M_{\phi_0}] M_{\bar{g} \rho},
\]

where \( H \) is the Hilbert transform in \( L_2(\mathbb{R}) \): \( H = P_+ - P_- \) (\( P_- = I - P_+ \)).

Proof. Assertion (i) follows from Theorem 4.10 and Proposition 3.4 Let us prove assertion (ii). We have from (4.43) and (4.22):

\[
\phi(A_\gamma) - \phi(A) = 2\pi i \gamma M_g (P_+ M_{\phi_0} - M_{\phi_0} P_+) M_{\bar{g} \rho} + O(\gamma^2).
\]

The latter representation and the formula \( P_+ = \frac{I+H}{2} \) imply the desired formula (4.44). \( \square \)

The following assertion connects the CLI property with BTB property in the case of the rank-one perturbation \( B \) of the multiplication operator \( A \):

Theorem 4.13. If the pair of operators \( A, B \), defined above, has the CLI property, then the spectral measure \( \rho \) of the operator \( A \) has the BTB property.

Proof. As was shown in [2], a number \( \lambda \notin \sigma(A) \) is an eigenvalue of the operator \( A_\gamma \) if and only if it is a root of the equation

\[
1 + \gamma B \rho(\lambda) = 0.
\]

Recall that \( B \rho \) is the Borel transform of the spectral measure \( \rho \) of the operator \( A \), defined by (4.19). Assume, on the contrary, that the measure \( \rho \) does not have the BTB property. This means that the function \( B \rho \) does not belong to the Hardy class \( H_\infty \). Then there exists such a sequence \( \lambda_n \in \mathbb{C} \) that \( \Im(\lambda_n) > 0 \) and

\[
\lim_{n \to \infty} B \rho(\lambda_n) = \infty.
\]
Denote $\gamma_n = -(B\rho(\lambda_n))^{-1}$. Then $\lim_{n \to \infty} \gamma_n = 0$ and each $\lambda_n$ is a root of the equation (4.45) with $\gamma = \gamma_n$. Since $\Im(\lambda_n) > 0$, then $\lambda_n \notin \sigma(A)$, hence each $\lambda_n$ is an eigenvalue of the operator $A_{\gamma_n}$. These circumstances contradict the CLI property of the pair $A$, $B$. The theorem is proven.

Proposition 3.4 and Theorems 3.6, 4.10 and 4.13 imply the following final result in the case of the rank-one perturbation $B$ of the multiplication operator $A$:

**Theorem 4.14.** If $A$, $B$ are the operators defined above, then the following assertions are equivalent to each other:

(i) The pair $A$, $B$ has the HUE property;
(ii) The pair $A$, $B$ has the FH property;
(iii) The pair $A$, $B$ has the CLI property;
(iv) The spectral measure $\rho$ of the operator $A$ has the BTB property.

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