FROM THE KÄHLER-RICCI FLOW TO MOVING FREE BOUNDARIES AND SHOCKS

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Abstract. We show that the twisted Kähler-Ricci flow on a complex manifold $X$ converges to a flow of moving free boundaries, in a certain scaling limit. This leads to a new phenomenon of singularity formation and topology change which can be seen as a complex generalization of the extensively studied formation of shocks in Hamilton-Jacobi equations and hyperbolic conservation laws (notably, in the adhesion model in cosmology). In particular we show how to recover the Hele-Shaw flow (Laplacian growth) of growing 2D domains from the Ricci flow. As we briefly indicate the scaling limit in question arises as the zero-temperature limit of a certain many particle system on $X$.

1. Introduction

The celebrated Ricci flow

\[ \frac{\partial g(t)}{\partial t} = -2\text{Ric } g(t), \]

(1.1)

can be viewed as a diffusion type evolution equation for Riemannian metrics $g(t)$ on a given manifold $X$. In fact, as described in the introduction of [40], this was one of the original motivations of Hamilton for introducing the flow. The point is that, locally, the principal term of minus the Ricci curvature $g$ of a Riemannian metric is the Laplacian of the tensor $g$ (which ensures the short-time existence of the flow).

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The factor 2 is just a matter of normalization as it can be altered by rescaling the time parameter by a positive number $\beta$. However, this symmetry is broken when a source term $\theta$ is introduced in the equation:

$$\frac{\partial g(t)}{\partial t} = -\frac{1}{\beta} \text{Ric } g(t) + \theta,$$

where $\theta$ is an appropriate symmetric two tensor on $X$ and $\beta$ thus plays the role of the inverse diffusion constant or equivalently, the inverse temperature (according to the “microscopic” Brownian motion interpretation of diffusions). In general terms the main goal of the present paper is to study the corresponding zero-temperature limit $\beta \to \infty$ of the previous equation. An important feature of the ordinary Ricci flow is that it will typically become singular in a finite time, but in some situations (for example when $X$ is a three manifold, as in Perelman’s solution of the Poincaré conjecture) the flow can be continued on a new manifold obtained by performing a suitable topological surgery of $X$. In our setting it turns out that a somewhat analogous phenomenon of topology change appears at a finite time $T^*$ in the limit $\beta \to \infty$, even if one assumes the long time existence of the flows for any finite $\beta$.

More precisely, following [23, 72, 71] we will consider the complex geometric framework where $X$ is a complex manifold, i.e. it is endowed with a complex structure $J$ and the initial metric $g_0$ is Kähler with respect to $J$. We will identify symmetric two-tensors and two-forms of type $(1,1)$ on $X$ using $J$ in the usual way - then the Kähler condition just means that the form defined by $g_0$ is closed. We will also assume that $\theta$ defines a closed (but not necessarily semi-positive) form. Then it is well-known that the corresponding flow $g(\beta)(t)$ emanating from the fixed metric $g_0$ preserves the Kähler property as long as it exists - it is usually called the twisted Kähler-Ricci flow in the literature (and $\theta$ is called the twisting form); see [23, 72, 71, 25, 38] and references therein. For simplicity we will also assume that

$$-\frac{1}{\beta} c_1(X) + [\theta] \geq 0$$

as $(1,1)$–cohomology classes (where $c_1(X)$ denotes the first Chern class of $X$) which ensures that the flows $g^{(\beta)}(t)$ exist for all positive times [71]. Our main result says that $g^{(\beta)}(t)$ admits a unique (singular) limit $g(t)$ as $\beta \to \infty$, where $g(t)$ defines a positive current with $L^\infty$–coefficients:

**Theorem 1.1.** Let $X$ be a compact complex manifold endowed with a smooth form $\theta$. Then

$$\lim_{\beta \to \infty} g^{(\beta)}(t) = P(g_0 + t\theta) \ (:= g(t))$$

in the weak topology of currents, where $P$ is a (non-linear) projection operator onto the space of positive currents. Moreover, the metrics $g^{(\beta)}(t)$ are uniformly bounded on any fixed time interval $[0,T]$.

The definition of the projection operator $P$ will be recalled in Section 2.3.2. The point is that the linear curve $g_0 + t\theta$, which coincides with the limiting flow for short times will, unless $\theta \geq 0$, leave the space of Kähler forms at the time

$$T_* := \sup\{t : g_0 + t\theta \geq 0\}$$

and hence it cannot be the limit of the metrics $g^{(\beta)}(t)$, even in a weak sense, for $t > T_*$. In particular, this means that around the time $T_*$ the Ricci curvatures
Ric $g^{(β)}(t)$ will become unbounded as $β → ∞$ (indeed, otherwise one could neglect the first term in the equation $1.2$ to obtain a linear ODE in the large $β$–limit solved by $g_0 + tθ$). Still we will show that the metrics $g^{(β)}(t)$ do remain uniformly bounded from above as $β → ∞$. However, unless $θ > 0$, the limiting $L^∞$–metrics $g(t)$ will, for $t ≥ T_*$, degenerate on large portions of $X$, i.e. the support

$$X(t) := \text{supp}(dV_{g(t)})$$

of the limiting $L^∞$–volume form $dV_{g(t)}$ is a proper closed subset of $X$ evolving with $t$. Moreover, on the support $X(t)$ the metrics $g(t)$ do evolve linearly, or more precisely

$$g(t) = g_0 + tθ \text{ on } X(t),$$

in the almost everywhere sense. As a consequence, typically the volume form $dV_{g(t)}$ has a sharp discontinuity over the boundary of $X(t)$, showing that the limiting (degenerate) $L^∞$–metrics $g(t)$ are not continuous and hence $C^0$–convergence in the previous theorem cannot hold, in general. In the generic case the evolving open sets $Ω(t) := X − X(t)$, where $dV_{g(t)}$ vanishes identically are increasing and may be characterized as solutions of moving free boundary value problems for the complex Monge-Ampère equation (see Section 2.3.2).

The projection operator $P$ appearing in the theorem, which associates to a given $(1,1)$ current $η$ on $X$ a positive current, cohomologous to $η$, is defined as a (quasi) plurisubharmonic envelope on the level of potentials (Section 2.3.2) and can be viewed as a complex generalization of the convex envelope of a function. Such envelopes play a key role in pluripotential theory (as further discussed in Section 1.1 below). In particular, the previous theorem yields a dynamic PDE construction of the envelopes in question, giving an alternative to previous dynamic constructions appearing in the real convex analytical setting [76, 24] (see the discussion in Section 3.1).

More generally, the weak convergence in Theorem 1.1 will be shown to hold as long as the $θ$ (viewed as a current) has continuous potentials. But then the limit $g(t)$ will, in general, not be in $L^∞$ (unless $θ$ is). Moreover, the support of the corresponding measure $dV_{g(t)}$ may then be a subset of low Hausdorff dimension. For example, in the one dimensional setting appearing in the adhesion model discussed below the conjectures formulated in [62] suggest an explicit formula for the Hausdorff dimension of the support, at any given time $t$, when $θ$ is taken as a random Gaussian distribution with given scaling exponent (see Section 6.1).

We will pay a particular attention to the special case in Theorem 1.1 where the twisting form $θ$ represents the trivial cohomology class, i.e.

$$θ = dd^c f,$$

for a function $f$ on $X$. The large $β$–limit of the corresponding twisted Kähler-Ricci flow turns out to be intimately related to various growth processes appearing in mathematical physics (and hence the Kähler-Ricci flow can be used as a new regularization of such processes):

*Hamilton-Jacobi equations, shock propagation and the adhesion model in cosmology.* In the particular case when $X$ is an abelian variety (or more specifically $X = \mathbb{C}^n/\Lambda + i\mathbb{Z}^n$, for a lattice $\Lambda$ in $\mathbb{R}^n$) and the potential $f$ of the twisting form $θ$ is invariant along the imaginary direction, we will show that the corresponding limiting twisted Kähler-Ricci flow $g(t)$ corresponds, under Legendre transformation in
the space variables, to a viscosity solution $u(x,t)$ of the Hamilton-Jacobi equation in $\mathbb{R}^n$ with periodic Hamiltonian $f$. Under this correspondence the critical time $T^*$ (formula 1.3) corresponds to the first moment of shock (caustic) formation in the solution $u_t(x)$, i.e. the time where $u_t$ ceases to be differentiable. From this point of view the moving domains $\Omega(t)$ correspond, under Legendre duality, to the evolving shock hypersurfaces $S_t$ (i.e. the non-differentiability locus of $u_t$). The evolution and topology change of such shocks plays a prominent role in various areas of mathematical physics (and more generally fit into the general problem of singularity formation in hyperbolic conservation laws [63]). In particular, the evolving shock hypersurface $S_t$ model the concentration of mass density in the cosmological adhesion model describing the formation of large-scale structures during the early expansion of the universe [75, 39, 42, 43]. Our setting contains, in particular, the case when the initial data in the adhesion model is periodic [48, 42, 43]. It should also be pointed out that in this picture the limit $\beta \to \infty$ can be seen as a non-linear version of the classical vanishing viscosity limit [30, 3, 49], which has the virtue of preserving convexity.

We will also study the corresponding large time limits and show that if the set $F$ of absolute minima of the potential $f$ is finite, then the support of the positive current defined by the joint large $\beta$ and large $t$—limit of the twisted Kähler-Ricci flow is a piecewise affine hypersurface whose vertices coincides with $F$ and whose lift to $\mathbb{R}^n$ gives a Delaunay type tessellation of $\mathbb{R}^n$ (which is consistent with numerical simulations appearing in cosmology [48, 42, 43]).

Applications to the Hele-Shaw flow (Laplacian growth). In another direction, allowing $\theta$ to be a singular current of the form

$$\theta = \omega_0 - [E],$$

where $\omega_0$ is the initial Kähler form and $[E]$ denotes the current of integration along a given effective divisor (i.e. complex hypersurface) in $X$ cohomologous to $\omega_0$, we will show that the corresponding domains $\Omega(t)$, which in this setting are growing continuously with $t$, give rise to a higher dimensional generalization of the classical Hele-Shaw flow in a two-dimensional geometry. More precisely, the Hele-Shaw flow appears when $X$ is a Riemann surface, $\omega_0$ is normalized to have unit area and $E$ is given by a point $p$ (in the classical setting $X$ is the Riemann sphere and $p$ is the point at infinity; the general Riemann surface case was introduced in [11]). Then $\Omega(t)$ coincides, up to a time reparametrization, with the Hele-Shaw flow (also called Laplacian growth) injected at the point $p$ in the medium $X$ with varying permeability (encoded in the form $\omega_0$). The latter flow was originally introduced in fluid mechanics to model the expansion of an incompressible fluid $\Omega(t)$ of high viscosity (for example oil) injected at a constant rate in another fluid of low viscosity (such as water) occupying the decreasing region $X(t)$. In more recent times the Hele-Shaw flow has made its appearance in various areas such as random matrix theory, integrable system and the Quantum Hall Effect [78] to name a few (see [73] for a historical overview). In particular, in the latter setting $X(t)$ represents the electron droplet. Special attention has been payed to an interesting phenomenon of topology change in the flow appearing at the time where $\Omega(t)$ becomes singular (which is different from $T^*$ which in this singular setting vanishes). Various approaches have been proposed to regularize the Hele-Shaw flow in order to handle the singularity formation (see [73, Section 5.3]). The present realization of the
Hele-Shaw flow from the limit of the Kähler metrics $\omega^{(\beta)}(t)$ on $X - \{p\}$ suggest a new type of regularization scheme, for example using the corresponding thick-thin decomposition of $X$, as in the ordinary Ricci flow (with $X(t)$ and $\Omega(t)$ playing the role of the limiting thick and thin regions, respectively). But we will not go further into this here.

1.1. Further relations to previous results. There is an extensive and rapidly evolving literature on the Kähler-Ricci flow (and its twisted versions) starting with [23]; see for example [69] and references therein. But as far as we know the limit $\beta \to \infty$ (which is equivalent to scaling up the twisting form and rescaling time) has not been studied before. For a finite $\beta$ there is no major analytical difference between the Kähler-Ricci flow and its twisted version, but in our setting one needs to make sure that the relevant geometric quantities do not blow up with $\beta$ (for example, as discussed above the Ricci curvature does blow up). For a finite $\beta$ the surgeries in the Kähler-Ricci flow have been related to the Minimal Model Program in algebraic geometry in [28, 68], where the final complex-geometric surgery produces a minimal model of the original algebraic variety. In Section 3.3 we compare some of our results with the corresponding long time convergence results on the minimal model (which produces canonical metrics of Kähler-Einstein type) [72, 71, 67].

In the algebro-geometric setting negative twisting currents $\theta$ also appear naturally, when $X$ is the resolution of a projective variety with canonical singularities [68, 36] ($\theta$ is then current of integration along minus the exceptional divisor). Recently, viscosity techniques were introduced in [36] to produce viscosity solutions for the twisted Kähler-Ricci flow (and in particular its singular variants appearing when $\theta$ is singular). But, again, this concerns the case when $\beta$ is finite.

In the case when $(X, \omega)$ is invariant under the action of a suitable torus $T$ (i.e. $X$ is a toric variety or an Abelian variety) the corresponding time dependent convex envelopes (studied in Section 3.1) have recently appeared in [58, 59] in a different complex geometric than the Kähler-Ricci flow, namely in the study of the Cauchy problem for weak geodesic rays in the space of Kähler metrics (see Remark 4.6). Moreover, in [57, 56, 55] the Hele-Shaw flow and the corresponding phenomenon of topology change was exploited to study the singularities of such weak geodesic rays (and solutions to closely related homogeneous complex Monge-Ampère equations) in the general non-torus invariant setting (see Remark 5.8).

We also recall that envelope type constructions as the one appearing in the definition of the projection operator $P$ play a pivotal role in pluripotential theory (and have their origins in the classical work of Siciak and Zakharyuta on polynomial approximations in $\mathbb{C}^n$ (see [37] for the global setting). Moreover, by the results in [5] the corresponding measure $(P\theta)^n$ on $X$ can be characterized as the unique normalized minimizer of the (twisted) pluripotential energy (which generalizes the classical weighted logarithmic energy of a measure in $\mathbb{C}$). The $L^\infty$-regularity of $P\theta$ was first established in [14] in a very general setting (of big cohomology classes), using pluripotential techniques. A new PDE proof of the latter regularity, in the case of nef and big cohomology classes, was then given in the paper [10], which can be seen as the “static” version of the present paper.

1.2. Organization. In Section 2 we state and prove refined versions of Theorem 1.1 (stated above). Then in Section 3 we go on to study the joint large $\beta$ and large $t$–limits of the corresponding flows. In particular, a dynamical construction
of plurisubharmonic (as well as convex) envelopes is given and a comparison with previous work on canonical metrics in Kähler geometry (concerning finite $\beta$) is made. In Sections 4 and 5 the relation to Hamilton-Jacobi equations and Hele-Shaw flows, respectively, is exhibited. The extension to twisting potentials which are merely continuous and the relation to random twistings is discussed in Section 6.1. In the final section we present a (deterministic, as well as stochastic) gradient flow interpretation of our results which will be expanded on elsewhere.

2. The zero-temperature limit of the Kähler-Ricci flow

2.1. Notation and setup. Let $X$ be an $n$-dimensional compact complex manifold. We will identify symmetric two-tensors with two-forms of type $(1, 1)$ on $X$ using $J$ in the usual way: if $g$ is a symmetric tensor, then the corresponding form $\omega := g(\cdot, J\cdot)$, is said to be Kähler if $\omega$ is closed and $g$ is strictly positive (i.e. $g$ is a Riemannian metric). We will assume that $X$ is Kähler, i.e. it admits a Kähler metric and we fix such a reference Kähler metric once and for all. On a Kähler manifold the De Rham cohomology class $[\eta] \in H^2(X, \mathbb{R})$ defined by a given closed real two form $\eta$ of type $(1, 1)$ may (by the “$\partial \bar{\partial}$—lemma”) be written as

\[
[\eta] = \{ \eta + dd^c u : u \in C^\infty(X) \}, \quad dd^c := \frac{i}{2\pi} \partial \bar{\partial}.
\]

In our normalization the Ricci curvature form $Ric$ of a Kähler metric $\omega$ on $X$ is defined, locally, by

\[
Ric \omega := -dd^c \log \frac{\omega^n}{dV(z)}
\]

where $z$ are local holomorphic coordinates on $X$ and $dV(z)$ denotes the corresponding Euclidean volume. The form $Ric$ represents, for any Kähler metric $\omega$, minus the first Chern class $c_1(K_X) \in H^2(X, \mathbb{R})$ of the canonical line bundle $\det(T^*X)$.

2.1.1. Setup. Specifically, our geometric setup is as follows: we assume given a family $\theta_\beta$ of closed real $(1, 1)$–forms (the “twisting forms”) with the asymptotics

\[
\theta_\beta = \theta + o(1),
\]

as $\beta \to \infty$ (in $L^\infty$-norm). We will assume that

\[
c_1(K_X)/\beta + [\theta_\beta] \geq 0
\]

as $(1, 1)$–cohomology classes, i.e. there exists a semi-positive form $\chi_\beta$ in the class $c_1(K_X)/\beta + [\theta_\beta]$ (we will fix one such choice for each $\beta > 0$). This assumption ensures that the corresponding twisted Kähler-Ricci flow

\[
\frac{\partial \omega(t)}{\partial t} = -\frac{1}{\beta} Ric \omega(t) + \theta_\beta, \quad \omega(0) = \omega_0
\]

exist for all $t \geq 0$ and $\beta > 0$. The extra flexibility offered by $\beta$–dependence of $\theta_\beta$ will turn out to be quite useful (for example, taking $\theta_\beta := \theta + \frac{1}{\beta} Ric \omega$ for $\theta$ defining a semi-positive cohomology class, ensures that the semi-positivity condition (2.2) holds).

\[\text{In fact, it is enough to assume that } c_1(K_X)/\beta + [\theta_\beta] \text{ is nef (i.e. a limit of positive classes), which is equivalent to the long time existence of the corresponding KRF [71]. Indeed, the estimates we get will be independent of the choice of reference form } \chi \text{ and hence the nef case can be reduced to the semi-positive case by perturbation of the class } [\theta].\]
More precisely, we will refer to the flow above as the *non-normalized twisted Kähler-Ricci flow* (or simply the *non-normalized KRF*) to distinguish it from its normalized version:

\[
\frac{\partial \omega(t)}{\partial t} = -\frac{1}{\beta} \text{Ric} \omega(t) - \omega(t) + \theta_{\beta}, \quad \omega(0) = \omega_0
\]

As is well-known the two flows are equivalent under a scaling combined with a time reparametrization: denoting by \( \tilde{\omega}(s) \) the non-normalized KRF one has

\[
\frac{1}{s+1} \tilde{\omega}(s) = \omega(t), \quad e^t := s + 1,
\]

(the equivalence follows immediately from the fact that \( \text{Ric}(c\omega) = \text{Ric} \omega \), for any given positive constant \( c \)). In the proofs we will reserve the notation \( \omega(t) \) for the normalized version of the flows.

In order to write the flows in terms of Kähler potentials we represent \( \tilde{\omega}(s) = (\omega_0 + s\chi_{\beta}) + d\bar{d}\tilde{\varphi}(s), \quad \tilde{\varphi}(0) = 0 \) for the fixed semi-positive form \( \chi_{\beta} \) in \( \frac{1}{\beta}c_1(K_X) + [\theta_{\beta}] \) (the first term ensures that the equation holds on the level of cohomology). Then the non-normalized KRF is equivalent to the following Monge-Ampère flow:

\[
\frac{\partial \tilde{\varphi}(s)}{\partial s} = \frac{1}{\beta} \log \left( \frac{\tilde{\omega}_0 + s\chi_{\beta} + d\bar{d}\tilde{\varphi}(s)}{\omega_0} \right)^{n} + f_{\beta}, \quad \tilde{\varphi}(0) = 0
\]

for the smooth function \( \tilde{\varphi}(s) \), which is a Kähler potential of \( \tilde{\omega}(s) \) wrt the Kähler reference metric \( \omega_0 + s\chi_{\beta} \), and where \( f_{\beta} \) is uniquely determined by the equation

\[
\theta_{\beta} - \frac{1}{\beta} \text{Ric} \omega = d\bar{d}f_{\beta} + \chi_{\beta}
\]

together with the normalization condition

\[
\inf_X f_{\beta} = 0.
\]

We will also assume that \( \chi_{\beta} \) is uniformly bounded from above, i.e.

\[
\chi_{\beta} \leq C_0 \omega,
\]

for some constant \( C_0 \) and some fixed Kähler form \( \omega \), and hence \( \chi_{\beta} \) converge smoothly to \( \chi \in [\theta] \) as \( \beta \to +\infty \). Accordingly, \( f_{\beta} \) converge smoothly to \( f \) uniquely determined by \( \chi + d\bar{d}f = \theta \) and \( \inf_X f = 0 \). Since \( \omega_0 \) is smooth, up to enlarging \( C_0 \) we can also assume that

\[
C_0^{-1} \omega_0 \leq \omega_0 \leq C_0 \omega.
\]

Finally, even when the functions \( f_{\beta} \) are not uniformly bounded, Lemma ensures that the envelope \( P_{\omega}(f_{\beta}) \) stays bounded from above if the functions \( f_{\beta} \) do not go uniformly to \( +\infty \). After enlarging \( C_0 \) one more time we can assume that

\[
P_{\omega}(f_{\beta}) \leq C_0, \quad \forall \beta > 0, \quad \forall t \geq 0.
\]

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2The convergence result still holds without an upper bound on \( \chi_{\beta} \) (which does not hold when \( \theta \) is nef but not semi-positive), but the dependence on \( t \) in the estimates will be worse.
Similarly, the normalized KRF is equivalent to the Monge-Ampère flow
\[
\frac{\partial \varphi(t)}{\partial t} = \frac{1}{\beta} \log \left( \frac{\omega_t + dd^c \varphi(t)}{\omega^n} \right) - \varphi(t) + f_\beta,
\]
where
\[
\omega(t) = \omega_t + dd^c \varphi(t), \quad \omega_t := e^{-t} \omega_0 + (1 - e^{-t}) \chi_\beta.
\]
The corresponding scalings are now given by
\[
\tilde{\varphi}(s) = e^s (\varphi(t) + c_\beta(t)), \quad c_\beta(t) = \frac{n}{\beta} \left( t - 1 + e^{-t} \right)
\]
(abusing notation slightly we will occasionally also write \( \omega(t) = \omega_\varphi \)).

**Remark 2.1.** Under the scaling above a curve \( \tilde{\varphi}(s) \) of the form \( \tilde{\varphi}(s) = \varphi_0 + sf \) corresponds to a curve \( \varphi(t) \) of the form \( e^{-t} \varphi_0 + (1 - e^{-t}) f - c_\beta(t) \).

### 2.2. Statement of the main results.

In the following section we will prove the following more precise version of Theorem 1.1 stated in the introduction of the paper.

**Theorem 2.2.** Let \( X \) be a compact complex manifold endowed with a family of twisting form \( \theta_\beta \) as above. Denote by \( \omega^{(\beta)}(t) \) the flow of Kähler metrics evolving by the (non-normalized) twisted Kähler-Ricci flow \( 2.3 \) with parameter \( \beta \), emanating from a given Kähler metric \( \omega_0 \) on \( X \). Then
\[
\lim_{\beta \to \infty} \omega^{(\beta)}(t) = P(\omega_0 + t \theta)
\]
in the weak topology of currents. On the level of Kähler potentials, for any fixed time-interval \([0, T]\), the functions \( \varphi^{(\beta)}(t) \) converge uniformly wrt \( \beta \) in the \( C^{1,\alpha}(X) \)-topology (for any fixed \( \alpha < 1 \)) towards the envelope \( P_{\omega_0 + \theta}(0) \). More precisely, fixing a reference Kähler metric \( \omega \) on \( X \),
\[
0 \leq \omega^{(\beta)}(t) \leq e^{C(1 + \frac{1}{\beta}) t \log(1 + t)} \omega
\]
and
\[
-C - n \log(1 + t))/\beta \leq \frac{\partial \varphi^{(\beta)}(t)}{\partial t} \leq C(1 + \frac{1}{\beta} \log(1 + t))/t
\]
where the constant \( C \) only depends on \( \theta \) through the following quantities: \( \sup_X Tr_\omega \theta \) and \( C_0 \) (as in \( 2.5, 2.4 \) and \( 2.10 \)); it also depends on a lower bound on the holomorphic bisectional curvature of the reference Kähler metric \( \omega \).

The definition of the non-linear projection operators \( P \) and \( P_{\omega_0} \) will be recalled in Section 2.3.2. The dependence of the constants above on the potential \( f \) of \( \theta \) will be crucial in the singular setting of Hele-Shaw type flows where \( f \) blows up on a hypersurface of \( X \), but \( P_{C_\omega}(f) \) is finite (see Section 5).

Under special assumptions on \( X \) we get an essentially optimal bound on \( \omega^{(\beta)}(t) \):

**Theorem 2.3.** Assume that \( X \) admits a Kähler metric \( \omega \) with non-negative holomorphic bisectional curvature. Then the following more precise estimates hold
\[
\left\| \omega^{(\beta)}(t) \right\| \leq (t + 1) \max \left\{ \left\| \omega_0 \right\|, \left\| \theta_\beta - \frac{\text{Ric} \omega}{\beta} \right\| \right\}
\]
in terms of the trace norm defined wrt \( \omega \) (i.e the sup on \( X \) of the point-wise \( L^1 \)-norm wrt \( \omega \)). Moreover, for any Riemann surface (i.e. \( n = 1 \)) the previous estimate holds without any conditions on the Kähler metric \( \omega \).
In particular, letting $\beta \to \infty$ gives that
\[ \|P(\omega_0 + t\theta)\| \leq (t + 1) \max\{\|\omega_0\|, \|\theta\|\}, \]
which is also a consequence of the estimates in the “static” situation considered in [10] (of course, in the case when $\theta$ is semi-positive the latter bound follows directly from the triangle inequality!).

However, it should be stressed that, in general, it is not possible to bound $\omega^{(\beta)}(t)$ by a factor $C_\beta t$, even for a fixed $\beta$ (see Prop 3.4). On the other hand, as we show in Section 3.1 this is always possible if $|\theta| = |\omega_0|$ (and in particular positive).

### 2.3.1. Parabolic comparison/max principles

We will make repeated use of standard parabolic comparison and maximum principles for smooth sub/super solutions of parabolic problems of the form
\[ \frac{\partial u}{\partial t} = D u \]
for a given differential operator $D$ acting on $C^\infty(X)$ (or a subset thereof). We will say that $u$ is a sub (super) solution if $(\frac{\partial}{\partial t} - D)u \leq 0$ ($\geq 0$).

**Proposition 2.4. (Comparison principle)** Let $X$ be a compact complex manifold and consider a second order differential operator $D$ on $C^\infty(X)$ of the form
\[ (Du)(x) = a(t,x)u(x) + F_t((dd^c u)(x)), \]
where $a$ is a bounded function on $[0,\infty[ \times X$ and $F_t(A)$ is a family of increasing functions on the set of all Hermitian matrices. If $u$ and $v$ are smooth sub- and supersolutions, respectively, to the corresponding parabolic problem for $D$ on $X \times [0,T]$, then $u_0 \leq v_0$ implies that $u_t \leq v_t$ for all $t \in [0,T]$. In particular, the result applies to the heat flow of the time-dependent Laplacian $\Delta_{g_t}$, defined wrt a family of Kähler metrics, and to the twisted KRF (normalized as well as non-normalized).

**Proof.** For completeness (and since we shall need a slight generalization) we recall the simple proof. After replacing $u$ with $e^{At}u$ for $A$ sufficiently large we may as well assume that $a_t < 0$. Assume to get a contradiction that it is not the case that $u_t \leq v_t$ on $X \times [0,T]$. Then there exists a point $(x,t) \in X \times [0,T]$ such that
\[ u_t(x) - v_t(x) > 0, \quad \frac{\partial(u_t - v_t)}{\partial t}(x) \geq 0, \quad (\nabla_{g_t} u)(x) = (\nabla_{g_t} v)(x) = 0, \]
and $(dd^c u_t)(x) - (dd^c v_t)(x) \leq 0$. Indeed, one first takes $t$ to be the first time violating the condition $u_t \leq v_t$ on $X$ and then maximize $u_t(x) - v_t(x)$ over $X$ to get the point $x$. In particular, since $a_t(x) > 0$ and $F_t$ is increasing we have that
\[ \frac{\partial(u_t - v_t)}{\partial t}(x) - (D u - D v)(x) > 0. \]
But this contradicts that $u$ and $v$ are sub/super solutions (since this implies the reversed inequality $\leq 0$).

**Remark 2.5.** The condition that $X$ be a complex manifold (and the Kähler condition) have just been included to facilitate the formulation of the proposition. Moreover, exactly the same proof as above shows that any first order term of the $H(t,x,(\nabla u)(x))$ for $H$ smooth can be added to $D$ above (as in the setting of Hamilton-Jacobi equations considered in Section 4).
Proposition 2.6. (Maximum principle) Let $X$ be a compact complex manifold and consider a second order differential operator $D$ on $C^\infty(X)$ of the form

$$(Du)(x) = F_t((dd^c u)(x)),$$

where $F_t(A)$ is a family of increasing functions on the set of all Hermitian matrices.

Given a smooth function $u(x,t)$ on $X \times [0, T]$ we have that

- The following dichotomy holds: either the maximum of $u(x,t)$ is attained at $X \times \{0\}$ or at a point $x \in X \times ]0, T]$ satisfying

$$\left( \frac{\partial u(x,t)}{\partial t} - D(u) \right) \geq -F_t(0),$$

- In particular, if $F_t(0) = 0$ for all $t$ and

$$\left( \frac{\partial}{\partial t} - D \right) \leq 0$$

on $X \times [0, T]$, then the maximum of $u(x,t)$ is attained at $X \times \{0\}$.

Proof. The first property is proved exactly as in the beginning of the proof of the comparison principle. The second point then follows by replacing $u$ with $u - \delta t$ for any number $\delta > 0$. \qed

Remark 2.7. We will need a slight generalization of the comparison principle to functions $u(x,t)$ which are continuous on $X \times [0, T]$ and such that $u(\cdot, t)$ is smooth on $X$ for any fixed $t > 0$ and $u(x, \cdot)$ is quasi-concave on $[0, T]$ for $x$ fixed, i.e. the sum of a concave and a smooth function. Then we simply define $\frac{\partial}{\partial t} u(x,t)$ on $[0, T]$ as the left derivate i.e. $\frac{\partial}{\partial t} u(x,t) := \lim_{h \to 0} (u(x,t+h) - u(x,t))/h$ for $h < 0$. In particular, the notion of a subsolution still makes sense for $u$ and the proof of the comparison principle then goes through word for word. This is just a very special case of the general notion of viscosity subsolution [31] which, by definition, means that the parabolic inequality holds with respect to the super second order jet of $u$ (which in our setting is just the ordinary jet in the space-direction and the interval between the right and the left derivative in the time-direction). See [36] for the complex setting, where very general comparison principles are established for viscosity sub/super solution (which however are not needed for our purposes).

2.3.2. The projection operator $P$. Let $\eta$ be a given closed smooth real $(1, 1)$--form on $X$ and denote by $[\eta]$ the corresponding De Rham cohomology class of currents which may be represented as in formula 2.11 in terms of functions $u \in L^1(X)$. Under this representation the subspace of all positive currents in $[\eta]$ corresponds to the space of all $\eta$--plurisubharmonic (psh) functions $u$, denoted by $PSH(X, \eta)$, i.e. $u$ is an upper semi-continuous (usc) function such that

$$\eta_u := \eta + dd^c u \geq 0$$

in the sense of currents. We will always assume that $PSH(X, \eta)$ is non-empty (which, by definition, means that the class $[\eta]$ is pseudo-effective. This is the weakest notion of positivity of a class $[\eta] \in H^{1,1}(X, \mathbb{R})$, the strongest being that $[\eta]$ is a Kähler class (also called positive), which, by definition, means that it contains a Kähler metric.
Given a lsc bounded function $f$ one obtains an $\eta$-psh function $P_\eta(f)$ as the envelope

$$P_\eta(f)(x) := \sup_{u \in \text{PSH}(X, \eta)} \{ u(x) : u \leq f, \text{on } X \}. \tag{2.11}$$

The operator $P_\eta$ is clearly a projection operator in the sense that $P_\eta(u) = u$ if $u$ is in $\text{PSH}(X, \eta) \cap C^\infty(X)$. We then define

$$P(\eta) := \eta + dd^c(\eta),$$

which thus defines a positive current cohomologous to $\eta$. Equivalently, if one fixes another reference form $\omega$ in $[\eta]$, i.e.

$$\eta = \omega + dd^c f$$

for some function $f$. Then

$$P(\eta) := \omega + dd^c(P_\omega f).$$

If the class $[\eta]$ is semi-positive, i.e. $\text{PSH}(X, \eta) \cap C^\infty(X)$ is non-empty, then it follows immediately from the definition that $P_\eta(f)$ is bounded if $f$ is. However, even if $f$ is smooth $P_\eta(f)$ will in general not be $C^2$-smooth. On the other hand, by [14, 10] $P_\eta(f)$ is almost $C^2$-smooth if the class $[\eta]$ is positive:

**Proposition 2.8.** Let $\omega$ be a Kähler form and $f$ a smooth function on $X$. Then the complex Hessian $dd^c(P_\omega f)$ is in $L^\infty$. Equivalently, given any smooth form $\eta$ defining a positive class $[\eta]$ the corresponding positive current $P(\eta)$ in $[\eta]$ is in $L^\infty$.

As a consequence,

$$P(\eta)^n = 1_C \eta^n, \tag{2.12}$$

in the point-wise almost everywhere sense, where $C$ is the corresponding (closed) coincidence set:

$$C := \{ x \in X : P_\eta(0)(x) = 0 \}.$$

In fact, we will get a new proof of the previous result using the Kähler-Ricci flow (which can be seen as a dynamic version of the proof in [10]); see Section 3.1.

**Remark 2.9.** Setting $u := P_\omega f$ and $\Omega := \{ P_\omega f < f \}$ the previous proposition implies that the pair $(u, \Omega)$ can be characterized as the solution to the following free boundary value problem for the complex Monge-Ampère operator with obstacle $f$, i.e. $u \leq f$ on $X$ and

$$(\omega + dd^c u)^n = 0 \text{ in } \Omega, \quad u = f, \ du = df \text{ on } \partial \Omega$$

and $\omega + dd^c u \geq 0$ on $X$. In the case when $n = 1$ it is well-known that $u$ is even $C^{1,1}$-smooth [21], but the free boundary $\partial \Omega$ may be extremely irregular and even if $\omega$ is real analytic it will, in general, have singularities [61].

A key role in the present paper will be played by parametrized envelopes (where $f$ varies linearly with time).

**Lemma 2.10.** Given functions $\varphi$ and $f$ on $X$ the function $t \mapsto \varphi(t, x) := P_\tau(\varphi + tf)(x)$ on $\mathbb{R}$ is concave for $x$ fixed. Moreover, locally on $]0, \infty[$ the corresponding curve $\varphi(t)$ can be written as a uniform limit $\varphi_\tau(t)$ of concave curves with values in $\text{PSH}(X, \omega) \cap C^\infty(X)$. Furthermore, if $\frac{\partial \varphi_\tau(t)}{\partial t} \leq g$ for a continuous function $g$ (in terms of the left derivative) then we may assume that $\frac{\partial \varphi_\tau(t)}{\partial t} \leq g$. 

Proof. It follows immediately from its definition that the projection operator $P_\omega$ is concave and in particular locally Lip continuous as a function of $t$. As for the approximation property it seems likely that it can be deduced in a much more general setting from an appropriate parametrized version of the approximation schemes for $\omega$-psh function introduced by Demailly. But here we note that a direct proof can be given exploiting that $dd^c \phi = 0(t)$ is in $L^\infty$ and in particular $\phi(t)$ is in $C^1(X)$. Indeed, $\varphi_\epsilon(t)$ can be defined by using local convolutions (which gives local $C^1$-convergence) together with a partition of unity and finally replacing $\varphi_\epsilon(t)$ with $(1 - \delta_1(t))\varphi_\epsilon(t) - \delta_2(t)$ for appropriate sequence $\delta_1(t)$ tending to zero with $\epsilon$. The point is that, by the $C^1$-convergence the error terms coming from the first derivative on the partition of unity are negligible and hence $\varphi_\epsilon(t)$ is $\omega$-psh up to a term of order $o(\epsilon)$. Indeed, setting

$$\phi_\epsilon(t) := \sum_{i=1}^m \rho_i \phi_\epsilon^{(i)}(t), \quad 1 = \sum_{i=1}^m \rho_i, \quad \rho_i \in C^\infty_c(X)$$

and using Leibniz rule gives $dd^c \phi_\epsilon(t) = \sum_{i=1}^m (\rho_i dd^c \phi_\epsilon^{(i)}(t) + R_i(\phi^{(i)}))$ where the second term $R(\varphi_\epsilon)$ only depends on the first order jet of $\phi$. Now, by the local $C^1$-convergence $R(\varphi_\epsilon) = R(\phi) + o(\epsilon)$. But $R(\phi)$ vanishes (since $\phi = \rho_1 \phi + \ldots$ and $dd^c \phi = \rho_1 dd^c \phi + \ldots$) and hence $dd^c \phi_\epsilon(t) = \sum_{i=1}^m \rho_i dd^c \phi_\epsilon^{(i)}(t) + o(\epsilon)$. Finally, from the definition of convolution we have $dd^c \phi_\epsilon(t) + \omega \geq -C\epsilon \omega$ and $\phi^{(i)}_\epsilon(t) \leq g + C\epsilon$ (for some positive constant $C$) and hence we may first take $\delta_1(\epsilon) = C\epsilon$ and then $\delta_2(\epsilon) = C\epsilon (1 + \sup |g|)$. \hfill \Box

Remark 2.11. The parametrized non-coincidence sets $\Omega_t := \{ P_\omega(\varphi + tf) < (\varphi + tf) \}$ are, in fact, increasing in $t$. Indeed, as shown in the proof of Proposition 2.11 $P_\omega(\varphi + tf) - (\varphi + tf)$ is decreasing in $t$.

We will also have use for the following generalized envelope associated to a given compact subset $K$ of a Kähler manifold $(X, \omega)$ and a lsc function $f$ on $X$ :

$$P_{(K,\omega)}(f)(x) := \sup_{u \in PSH(X,\omega)} \{ u(x) : u \leq f \text{ on } K \}$$

(the function $V_{K,\omega} := P_{(K,\omega)}(0)$ is called the global extremal function of $(K,\omega)$ in [37].)

We recall that a subset $K$ in $X$ is said to be non-pluripolar if it is not locally contained in the $-\infty$-set of a local psh function.

Lemma 2.12. Suppose that $f$ is a lsc function on a compact Kähler manifold $(X, \omega)$ taking values in $[0, \infty]$ such that $f$ is bounded from above on $K$, where $K$ is non-pluripolar. Then the function $P_{(K,\omega)}(f)$ is bounded from above. As a consequence, if $X = K$ and $f$ is locally bounded on the complement of an analytic subvariety, then $P_{(X,\omega)}(f)$ is bounded from above.

Proof. By assumption $P_{(K,\omega)}(f) \leq P_{(K,\omega)}(0) + C$ for $C$ a sufficiently large constant. But it is well-known that $P_{(K,\omega)}(0)$ is finite iff $K$ is non-pluripolar [37]. The last statement of the lemma then follows by fixing a coordinate ball $B$ contained in the open subset where $f$ is locally bounded and using that $P_{(X,\omega)}(f) \leq P_{(B,\omega)}(f) < \infty$. \hfill \Box
In general, $P_{(K, \omega)}(f)$ is not upper semicontinuous. But we recall that $K$ is said to be regular (in the sense of pluripotential theory) if $P_{(K, \omega)}(f)$ is continuous (and hence $\omega$-psh) for any continuus function $f$ (see [13] and references therein).

2.4. A priori estimates. The key element in the proof of Theorem 2.2 is the Laplacian estimate which provides a uniform bound on the metrics $(\omega^\beta(t))$ on any fixed time interval. There are various well-known approaches for providing such an estimate for a fixed $\beta$, using parabolic versions of the classical estimate of Aubin and Yau and its variants. However, in our setting one has to make sure that all the estimates are uniform in $\beta$ and that they do not rely on a uniform positive lower bound on $\omega^\beta(t)$ (which is not available).

2.4.1. The Laplacian estimate in the one dimensional case. We start with the one-dimensional case where the Laplacian estimate becomes particularly explicit:

**Proposition 2.13.** When $n = 1$ we have, for any fixed Kähler form $\omega$ on $X$

$$\left\| \omega^\beta(t) \right\| \leq \max \left\{ \left\| \omega_0 \right\|, \left\| \theta_\beta - \frac{Ric \, \omega}{\beta} \right\| \right\}$$

in term of the sup norm defined by $\omega$ (i.e. $\|\eta\| := \sup_X |\eta/\omega|$).

**Proof.** We write the normalized KRF as

$$h := \log \frac{\omega^\beta}{\omega} = \beta \left( \varphi_t - f_\beta + \frac{\partial \varphi_t}{\partial t} \right).$$

Applying the parabolic operator $\frac{1}{\beta} \Delta_t - \frac{\partial}{\partial t}$, where $\Delta_t(=dd^c/\omega^\beta_t)$ denotes the Laplacian wrt the metric $\omega^\beta_t$, to the equation above gives

$$\frac{1}{\beta} \Delta_t h - \frac{\partial}{\partial t} h = \frac{1}{\omega^\beta_t} (dd^c \varphi_t - dd^c f_\beta) + \Delta_t \frac{\partial \varphi_t}{\partial t} - \frac{\partial}{\partial t} h.$$

Now

$$\frac{\partial}{\partial t} h := \frac{\partial}{\partial t} \log \left( \frac{e^{-t} \omega_0 + (1 - e^{-t}) \chi_\beta + dd^c \varphi_t}{\omega} \right) = \frac{1}{\omega^\beta_t} \left( -e^{-t} \omega_0 + e^{-t} \chi_\beta + dd^c \frac{\partial \varphi_t}{\partial t} \right).$$

Hence, the two terms involving $\frac{\partial \varphi_t}{\partial t}$ cancel, giving

$$\frac{1}{\beta} \Delta_t h - \frac{\partial}{\partial t} h = \frac{\omega}{\omega^\beta_t} \left( \Delta_\omega (\varphi_t - f_\beta) \right) - e^{-t}(\chi_\beta - \omega_0)/\omega,$$

i.e.

$$\omega^\varphi_t \left( \frac{1}{\beta} \Delta_t h - \frac{\partial}{\partial t} h \right) + dd^c f_\beta + e^{-t}(\chi_\beta - \omega_0) = dd^c \varphi_t,$$

which in terms of $\omega(t) := \omega_0 + (1 - e^{-t})(\chi_\beta - \omega_0) + dd^c \varphi_t$ becomes

$$\omega^\varphi_t \left( \frac{1}{\beta} \Delta_t h - \frac{\partial}{\partial t} h \right) + dd^c f_\beta + \chi_\beta = \omega(t).$$

Applying the parabolic maximum principle to $h$ concludes the proof. Indeed, there are two alternatives: either $h$ has its maximum on $X \times [0, T]$ (for $T$ fixed) at $t = 0$ which implies that $Tr_{\omega}(h) \leq Tr_{\omega_0}$ on $X \times [0, T]$, or the maximum of $h$ is attained at a point $(x, t)$ in $X \times [0, T]$. In the latter case $Tr_{\omega}(h) \leq \sup_X Tr_{\omega}(dd^c f_\beta + \chi_\beta) \leq C$ (since $dd^c f_\beta + \chi_\beta = \theta - \frac{1}{\beta} Ric \, \omega = \theta + o(1)$).
2.4.2. **The upper bound on \( \varphi_t \).** Next, we come back to the general case. Writing the normalized KRF flow as

\[
\frac{\partial (\varphi_t - f_\beta)}{\partial t} + \frac{1}{\beta} \log \left( \frac{\hat{\omega}_t + dd^c f_\beta + dd^c (\varphi_t - f_\beta)}{\omega^n} \right)^n = \varphi_t - f_\beta,
\]

it follows immediately from the parabolic maximum principle that

\[
\varphi_t(x) - f_\beta(x) \leq \max \left\{ \sup_X (0 - f_\beta), \frac{1}{\beta} \sup_X \log \left( \frac{\hat{\omega}_t + dd^c f_\beta + dd^c (\varphi_t - f_\beta)}{\omega^n} \right) \right\} \leq A/\beta,
\]

where \( A \) only depends on the upper bounds of \( \theta_\beta \). In particular,

\[
\varphi_t \leq P_t(f_\beta) + A/\beta
\]

and, as a consequence,

\[
\leq P_C'(\omega)(f_\beta) + A/\beta
\]

where \( C' \) is any constant satisfying \( \chi_\beta \leq C' \omega \) and \( \omega_0 \leq C' \omega \) (thus ensuring that \( \text{PSH}(X, \hat{\omega}_t) \subset \text{PSH}(X, C' \omega) \)).

2.4.3. **The lower bounds on \( \frac{\partial \hat{\varphi}_s}{\partial s} \) and \( \frac{\partial \varphi_t}{\partial t} \).** Differentiating the non-normalized KRF with respect to \( s \) gives, with \( g(x, s) := -\frac{\partial \hat{\varphi}_s(x)}{\partial s} \),

\[
\frac{\partial g}{\partial s} - \frac{1}{\beta} \Delta_s g = -\frac{1}{\beta} \text{Tr}_s (\chi_\beta) \leq 0.
\]

Hence, by the parabolic maximum principle the sup of \( g \) is attained at \( t = 0 \) which gives

\[
-\frac{\partial \hat{\varphi}_s}{\partial s} \leq C_1, \quad C_1 = \sup_X \left( -\frac{1}{\beta} \log \frac{\omega^n_{\varphi_0}}{\omega^n} - f_\beta \right)
\]

where \( C_1 \) thus only depends on the strict positive lower bound of \( \omega^n_{\varphi_0} \) and on \( \inf_X(f_\beta) \) (which by our normalizations vanishes).

Next, using that

\[
\frac{\partial \hat{\varphi}_s}{\partial s} = \frac{\partial \varphi_t}{\partial t} + \varphi_t + nt/\beta
\]

(2.16)

gives

\[
\frac{\partial \varphi_t}{\partial t} \geq -C_1 - \varphi_t - nt/\beta \geq -C'_1 - nt/\beta
\]

(2.17)

using the previous upper bound on \( \varphi_t \).

2.4.4. **The lower bound on \( \varphi_t \).** It follows immediately from the previous bound that

\[
\varphi_t \geq \varphi_0 - C'_1 t - nt^2/2\beta.
\]

2.4.5. **The Laplacian bound.** We will use Siu’s well-known variant [66, pp. 98-99] of the classical Aubin-Yau Laplacian estimate

**Lemma 2.14.** Given two Kähler forms \( \omega' \) and \( \omega \) such that \( \omega^n = e^F \omega^n \) we have that

\[
\Delta_{\omega'} \log \text{Tr}_{\omega'} \omega' \geq \frac{\text{Tr}_{\omega} dd^c F}{\text{Tr}_\omega \omega'} - B_+ \text{Tr}_{\omega'} \omega,
\]

where the constant \( B_+ \) is a multiple of the absolute value of the infimum on \( X \) of the holomorphic bisectional curvatures of \( \omega \).
Proof. In the original statement in [66, pp. 98-99] it was assumed that $\omega'$ and $\omega$ are cohomologous, but since the proof is local this assumption is not needed. See for example [20, Prop 4.1.2] where it is shown that

$$\Delta_{\omega'} \log \text{Tr}_{\omega'} \geq \frac{\text{Tr}_{\omega}(-\text{Ric}_{\omega'})}{\text{Tr}_{\omega'}} + B \text{Tr}_{\omega},$$

where $B$ is the infimum of the holomorphic bisectional curvatures of $\omega$. In our notations, $-\text{Ric}_{\omega'} = dd^c F - \text{Ric}_{\omega}$ and since $-\text{Tr}_{\omega} \text{Ric}_{\omega} \geq -c_n |B|$ and $\text{Tr}_{\omega'} \text{Tr}_{\omega} \geq n$ we arrive at the inequality in the statement of the lemma. □

We start with the case when $X$ admits a Kähler metric $\omega$ with non-negative holomorphic bisectional curvature. In this case the constant $B$ vanishes.

**Proposition 2.15.** Suppose that $X$ admits a Kähler metric $\omega$ with non-negative bisectional curvature. Then

$$\left\| \omega^{(\beta)}(t) \right\| \leq \max \left\{ \left\| \omega_0 \right\|, \left\| \theta_{\beta} - \frac{Ric \ \omega}{\beta} \right\| \right\}.$$

**Proof.** Setting $h := \log \text{Tr}_{\omega} \omega'$, where $\omega' = \hat{\omega} + dd^c \varphi(t)$, we get, using Siu’s inequality,

$$\left( -\frac{\partial}{\partial t} h + \frac{1}{\beta} \Delta_h \right) \geq \frac{1}{\text{Tr}_{\omega'} \omega'} \Delta_{\omega} \left( \varphi_t - f_\beta \right) + \frac{1}{\text{Tr}_{\omega'} \omega'} \left( \Delta_{\omega} \varphi_t - f_\beta \omega_{\beta} \right) - \frac{\partial}{\partial t} h.$$

The rest of the proof then proceeds precisely as in the Riemann surface case. □

In the general case we get the following

**Proposition 2.16.** There is a constant $C$ such that, for $\beta > \beta_0$

$$\omega^{(\beta)}(t) \leq e^{C(1+1/\beta)(1+t)} e^{t} \omega,$$

where $C$ depends on the same quantities as in the statement of Theorem 2.2.

**Proof.** Recall that by abuse of notation we set $\omega_t = \hat{\omega} + dd^c \varphi^{(\beta)}(t)$. By the Laplacian inequality (Lemma 2.14) we have

$$\frac{1}{\beta} B_+ \text{Tr}_{\omega_t} \omega + \left( -\frac{\partial}{\partial t} \log \text{Tr}_{\omega_t} \omega_t + \frac{1}{\beta} \Delta_t \log \text{Tr}_{\omega_t} \omega_t \right) \geq \Delta_{\omega} (\varphi_t - f_\beta) - e^{-t} \text{Tr}_{\omega} (\chi_\beta - \omega_0)$$

thanks to the cancelation of the terms involving $\partial \varphi_t / \partial t$, just as before. To handle the first term in the left-hand side above we note that

$$\omega \leq C e^t \hat{\omega}_t,$$

where $1/C$ is a positive lower bound for $\omega_0$. Since $\text{Tr}_{\omega} \hat{\omega}_t = n - \Delta_{\omega_t} \varphi$ we thus get, by setting

$$G(x,t) := \log (\text{Tr}_{\omega_t} \omega_t) - B_+ C e^t \varphi_t - f(t),$$
for any given function $f(t)$ of $t$,
\[
- \frac{\partial f(t)}{\partial t} - CB_+ \frac{\partial (e^t \varphi_t)}{\partial t} + \frac{n}{\beta} B_+ C e^t + \left( - \frac{\partial}{\partial t} G + \frac{1}{\beta} \Delta_t G \right) \\
\geq \frac{\Delta_\omega(\varphi_t - f_\beta) - e^{-t} \text{Tr}_\omega(\chi_\beta - \omega_0)}{\text{Tr}_\omega \omega_t}.
\]

Next we note that, thanks to the lower bound on $\frac{\partial \varphi_t}{\partial t}$ above (2.17) we have
\[
\frac{\partial (e^t \varphi_t)}{\partial t} \geq - e^t \left( C + nt/\beta \right).
\]
Hence, taking $f(t) = C' (1 + t)(1 + 1/\beta)e^t$ for $C'$ sufficiently large gives
\[
\left( - \frac{\partial}{\partial t} + \frac{1}{\beta} \Delta_t \right) G \geq \frac{\text{Tr}_\omega (dd^c \varphi_t - dd^c f_\beta - e^{-t}(\chi_\beta - \omega_0))}{\text{Tr}_\omega \omega_t}.
\]
Since $e^{-t}(\chi_\beta - \omega_0) = \chi_\beta - \hat{\omega}_t$ this implies that
\[
\left( - \frac{\partial}{\partial t} + \frac{1}{\beta} \Delta_t \right) G \geq \frac{\text{Tr}_\omega (\omega_t - dd^c f_\beta - \chi_\beta)}{\text{Tr}_\omega \omega_t}.
\]
Finally, using $\chi_\beta + dd^c f_\beta = \theta_\beta - \frac{1}{2} \text{Ric} \omega$ this shows that the estimate on $\text{Tr}_\omega \omega_t$ we get from the parabolic maximum principle applied to $G$ only depends on $\chi_\beta$ through the upper bound on $\varphi_t$ (which in turn depends on $\chi_\beta$ and is of the order $1/\beta$).

2.4.6. The upper bound on $\frac{\partial \varphi}{\partial t}$ and $\frac{\partial \varphi}{\partial s}$. (these upper bounds are not needed for the proof of the convergence in Theorem 1.1). From the upper bound on $\omega(t)$ and the defining equations for the KRFs one directly obtains bounds on $\frac{\partial \varphi}{\partial t}$ and $\frac{\partial \varphi}{\partial s}$. However, better bounds can be obtained by a variant of the proof of the lower bounds on $\frac{\partial \varphi}{\partial t}$ and $\frac{\partial \varphi}{\partial s}$. Indeed, differentiating the normalized and the non-normalized KRFs, respectively gives
\[
\frac{\partial}{\partial s} - \Delta_s \frac{\partial \varphi_s}{\partial s} = \text{Tr}_s \chi_\beta = 0
\]
and
\[
\frac{\partial t e^t \frac{\partial \varphi_t}{\partial t}}{\partial t} - \frac{\Delta t e^t \frac{\partial \varphi_t}{\partial t}}{\partial t} = \text{Tr} (\chi_\beta - \omega_0) = 0.
\]
Using that $\omega(s) = e^s \omega(t)$, $ds/dt = e^{-t} dt/dt$ and $\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{\partial t} + \varphi_t + nt/\beta$ the first equation above becomes
\[
\frac{\partial}{\partial t} (\frac{\partial \varphi}{\partial t} + \varphi_t + nt/\beta) - \frac{\partial t}{\partial t} (\varphi_t + nt/\beta) = - \text{Tr}_t \chi_\beta = 0.
\]
Hence, taking the differences between equations (2.18) and (2.19) gives that $g := e^t \frac{\partial \varphi_t}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi_t - nt/\beta$ satisfies
\[
\frac{\partial g}{\partial t} - \Delta_t \frac{\partial g}{\partial t} = - \text{Tr}_t \omega_0 \leq 0.
\]
Accordingly, the parabolic maximum principle reveals that the sup over $X$ of $e^t \frac{\partial \varphi_t}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi_t - nt/\beta$ is decreasing, thanks to the upper bound on $\varphi_t$,
\[
\frac{\partial \varphi_t}{\partial t} \leq \sup_X P_t (f_\beta) + (A + nt)/\beta
\]
(this is a minor generalization of the estimate in [71]). Finally, this yields
\[
\frac{\partial \tilde{\omega}_s}{\partial s} \leq C'' \frac{1 + \beta^{-1} \log(1 + s)}{s}.
\]

2.5. **Existence and characterizations of the large \( \beta \) limit of the KRF.** By the previous estimates there is a subsequence of \( \varphi^{(\beta)}(t) \) which converges uniformly (and even in \( C^{1,\alpha} \)) to a limiting Lip curve \( \varphi(t) \) with values in \( PSH(X, \hat{\omega}_t) \). As we next show \( \varphi(t) \) is uniquely determined, i.e. the whole family converges to \( \varphi(t) \).

**Proposition 2.17.** The large \( \beta \)-limit of \( \varphi^{(\beta)}(t) \) of the normalized KRF exists: it is equal to the curve defined as the sup over all curves \( \psi(t) \) in \( PSH(X, \omega) \) such that \( \psi(0) = \varphi(0) \) and such that \( \varphi^{(\beta)}(t) \) is locally Lipchitz in \( t \) (for \( t > 0 \)) and in \( C^1(X) \), for a fixed \( t \) and
\[
\frac{\partial \psi(t)}{\partial t} \leq -\psi(t) + f
\]
in the weak sense, or equivalently: such that
\[
(\psi(t) - f)e^t
\]
is decreasing in time.

**Proof.** By the second order a priori estimates we have
\[
\frac{d\varphi^{(\beta)}(t)}{dt} \leq \frac{C(t)}{\beta} - \varphi^{(\beta)}(t) + f_
\]
and hence the limiting Lip curve \( \varphi(t) \) satisfies
\[
\frac{d\varphi(t)}{dt} \leq -\varphi(t) + f
\]
in the weak sense, i.e. \( \varphi(t) \) is a candidate for the sup appearing in the statement of the proposition. Alternatively, we get
\[
\frac{d}{dt} \left( (\varphi^{(\beta)} - f_\beta)(e^t - C_{\beta} e^t) \right) \leq 0,
\]
i.e.
\[
(\varphi^{(\beta)} - f_\beta)(t) \leq \frac{1}{(1 - C/\beta)} e^{-t} g_\beta(x, t),
\]
where \( g_\beta(x, t) \) is decreasing in time. Hence, after passing to a subsequence the limit satisfies
\[
(\varphi - f)(t) \leq e^{-t} g(x, t),
\]
where \( g(x, t) \) is decreasing in time.

Next, by the parabolic maximum principle \( \varphi^{(\beta)}(t) \) is the sup over all smooth curves \( u_\beta(t) \) with values in (the interior of) \( PSH(X, \hat{\omega}_t) \) such that \( u_\beta(0) = \varphi(0) \) and
\[
\frac{du_\beta}{dt} \leq \frac{1}{\beta} \log \left( \frac{\hat{\omega}_t + dd^c u_\beta(t)}{\omega_n} \right) - (u_\beta(t) - f_\beta)
\]
on a fixed time-interval \( [0, T] \). Now take a smooth curve \( v(t) \) from \( [0, T] \) to \( PSH(X, \hat{\omega}_t) \cap C^\infty(X) \) such that and \( v(0) = \varphi(0) \) and such that
\[
\frac{d}{dt} v(t) \leq -(v(t) - f).
\]
We set
\[ v_\varepsilon(t) := (1 - \varepsilon)v(t) - \varepsilon, \]
ensuring that
\[ \frac{d}{dt}v_\varepsilon(t) \leq -v_\varepsilon(t) - f - \varepsilon \]
and
\[ (\hat{\omega}_t + dd^c v_\varepsilon(t))^n \geq e^n\hat{\omega}_t^n \geq e^n C(T)\omega^n. \]
Hence, for \( \beta \) sufficiently large (depending on the lower bound \( C(T) \) of the positivity of \( \hat{\omega}_t^n \) on \([0, T]\) and the convergence speed of \( f_\beta \) towards \( f \)),
\[ \frac{dv_\varepsilon(t)}{dt} \leq \frac{1}{\beta} \log \left( \frac{(\hat{\omega}_t + dd^c v_\varepsilon(t))^n}{\omega^n} \right) - v_\varepsilon(t) - f_\beta. \]
But then it follows from the parabolic maximum principle that \( v_\varepsilon(t) \leq \varphi_{\beta, \varepsilon}(t) \), for \( \beta \gg 1 \), where \( \varphi_{\beta, \varepsilon}(t) \) satisfies the same KRF as \( \varphi^{(\beta)}(t) \), but with initial value \((1 - \varepsilon)\varphi_0 + \varepsilon v_0\). By the maximum principle we have
\[ \left| \varphi_{\beta, \varepsilon}(t) - \varphi^{(\beta)}(t) \right| \leq C\varepsilon \]
and hence letting \( \beta \to \infty \) gives, for any limit \( \varphi(t) \) of \( \varphi^{(\beta)}(t) \)
\[ v_\varepsilon(t) \leq \varphi(t) + C\varepsilon. \]
Since \( \varepsilon \) was arbitrary this gives \( v(t) \leq \varphi(t) \). All that remain is thus to show that the smoothness assumption on \( v(t) \) can be removed. This could be done by working with the notion of viscosity subsolutions [30], but here we will use a more direct approach by first noting that the sup above is realized by \( P_{\omega_0}(e^{-t}\varphi_0 + (1 - e^{-t})f) \), as shown in the next proposition. Then we can use the regularization in Lemma 2.10 together with the slight generalization of the parabolic comparison principle formulated in Remark 2.7 to conclude.

**Proposition 2.18.** The sup in the previous proposition coincides with \( P_{\omega_0}(e^{-t}\varphi_0 + (1 - e^{-t})f) \).

**Proof.** It will be convenient to use the equivalent “non-normalized setting” which means that we replace the convex combination above with \( \varphi_0 + tf \) and have to prove that \( a(t) := P_t(\varphi_0 + tf) - tf \), where \( P_t = P_{\omega_0 + \theta \psi} \), is decreasing, i.e. that \( a(t + s) - a(t) \leq 0 \) for any fixed \( t, s \geq 0 \) (compare Remark 2.1). To this end we rewrite the difference above as
\[ P_{t+s}(\varphi_0 + tf + sf) - P_t(\varphi_0 + tf) - sf = P_{t+s}((1 - \lambda)\varphi_0 + \lambda\psi_t) - P_t(\psi_t) - sf, \]
where
\[ \psi_t := \varphi_0 + tf, \quad \lambda := (t + s)/t. \]
In particular, \( \lambda \geq 1 \) and hence it follows from the very definition of \( P \) (as an upper envelope wrt a convex set) that
\[ P_{t+s}((1 - \lambda)\varphi_0 + \lambda\psi_t) \leq (1 - \lambda)\varphi_0 + \lambda P_t(\psi_t), \]
which gives that \( a(t + s) - a(t) \) can be estimated from above by
\[ (1 - \lambda)\varphi_0 + \lambda P_t(\psi_t) - P_t(\psi_t) - sf = (\lambda - 1)tf - sf = 0 \]
as desired (a similar direct proof can be given for the normalized KRF, but using instead \( \lambda = (1 - e^{-(t+s)})/(1 - e^{-t}) \)).
Remark 2.19. It is possible to prove the uniform large \( \beta \)-convergence of the flows \( \varphi^{(\beta)}(t) \) directly without the Laplacian estimate and without going through the characterization in terms of curves appearing in 2.17. Indeed, an upper bound of the form \( \varphi^{(\beta)}(t) \leq P_t(\varphi_0 + tf_\beta) + C_t/\beta \) can be proved (in the non-normalized setting) by applying the parabolic maximum principle. Indeed, for some uniform constant \( C \) it can be shown that the function

\[
\varphi_0 + tf_\beta + \frac{Ct + nt \log(t+1)}{\beta}
\]

is a super-solution of the parabolic complex Monge-Ampère equation. The lower bound is then proved as before using regularization and the parabolic comparison principle. Alternatively, the lower bound can also be proved directly without regularization using the monotonicity property (Proposition 2.18) and the viscosity principle. Indeed, an upper bound is then proved as before using regularization and the parabolic comparison principle to get

\[
\varphi^{(\beta)}(t) \geq (1-\delta)P_t(\varphi_0 + tf_\beta) - \frac{C_t(1-\log(1-\delta))}{\beta} - (1-\delta)C_t, \ \delta \in (0,1).
\]

More generally, the uniform convergence holds even if \( f \) is merely continuous (see Section 3).

3. LARGE TIME ASYMPTOTICS OF THE FLOWS

In order to study the joint large \( t \) and large \( \beta \)-limit of the non-normalized Kähler-Ricci flows \( \omega^{(\beta)}(t) \) introduced in the previous section we consider, as usual, the normalized Kähler forms \( \omega^{(\beta)}(t)/(t+1) \) (which have uniformly bounded volume) evolving according to the normalized Kähler-Ricci flow 2.4. Our first observation is that the following double limit always exists:

\[
\lim_{t \to \infty} \lim_{\beta \to \infty} \omega^{(\beta)}(t)/(t+1) = P(\theta),
\]

for any initial Kähler metric \( \omega_0 \) (where the large \( t \)-limit holds in the weak topology of currents). This follows immediately from Theorem 2.2 combined with the following

Lemma 3.1. Assume that \( \chi \geq 0 \) and set \( \hat{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\chi \). Then, for any given smooth functions \( \varphi_0 \) and \( f \) on \( X \),

\[
P_{\hat{\omega}_t}(e^{-t}\varphi_0 + (1-e^{-t})f) \to P_{\chi} f
\]

as \( t \to \infty \), in the \( L^1 \)-topology. In particular, if \( [\theta] \geq 0 \) then \( P(e^{-t}\omega_0 + (1-e^{-t})\theta) \to P(\theta) \) in the weak topology of currents.

Proof. Set \( \psi_t := P_{\hat{\omega}_t}(e^{-t}\varphi_0 + (1-e^{-t})f) := P_{\omega_t}(f(t)) \). Since \( \psi_t \in PSH(X,C\omega) \) for \( C \) sufficiently large the family \( \psi_t \) is relatively compact in the \( L^1 \)-topology. We denote by \( \psi_\infty \) a given limit point of \( \psi_t \), which clearly is in \( PSH(X,\chi) \). Moreover, \( \psi_t \leq f(t) \) implies \( \psi_\infty \leq f \) and hence \( \psi_\infty \leq P_{\chi} f \). To prove the converse we set \( \psi := P_{\chi} f \) and fix \( \delta > 0 \). Observe that \( dd^c(1-\delta)\psi + \hat{\omega}_t \geq (1-\delta)dd^c\psi + (1-e^{-t})\chi \geq (1-\delta)(dd^c\psi + \chi) \geq 0 \), for \( t >> 1 \). Hence, since \( \psi \) is bounded we get \( (1-\delta)\psi \leq P_{\omega_t}(f + C\delta) \leq P_{\omega_t}(f_t) + C\delta + C'e^{-t} \). Hence, letting first \( t \to \infty \) gives \( (1-\delta)\psi \leq \psi_\infty + C\delta \). Finally, letting \( \delta \to 0 \) concludes the proof. \( \square \)
In the following two sections we will look closer at the situation appearing in the two extreme cases, where \( c_1(K_X)/\beta + [\theta_\beta] = [\omega_0] \) is positive and trivial, respectively. Then we will make some comments on the intermediate cases and the relations to previous results in complex geometry concerning the case when \( \beta \) is fixed.

3.1. The case when \( c_1(K_X)/\beta + [\theta_\beta] = [\omega_0] \): a dynamic construction of envelopes. In this section we will consider the situation when the normalized KRF preserves the initial cohomology class. Given a volume form \( dV \) on \( X \) and a smooth function \( f \) on \( X \), setting \( \theta = dd^c f + \omega \) and

\[
\theta_\beta = \theta + \frac{1}{\beta} \text{Ric} \ dV
\]

for a fixed choice of Kähler metric \( \omega \in [\theta] \) the normalized KRF in \([\theta]\) on the level of Kähler potentials becomes

\[
(3.2) \quad \frac{\partial \phi^{(\beta)}(t)}{\partial t} = \frac{1}{\beta} \log \left( \frac{\omega + dd^c \phi^{(\beta)}(t)}{\omega} \right) - \phi^{(\beta)}(t) + f
\]

(hence \( f_\beta = f + \frac{1}{\beta} \log(dV/\omega^n) \) and the reference Kähler metric on \( X \) and \( \chi_\beta = \chi \) in \([\theta]\) are both taken as \( \omega \) in this setting).

**Theorem 3.2.** Let \((X, \omega)\) be a Kähler manifold and fix a volume form \( dV \) on \( X \). Given a smooth function \( f \) we denote by \( \phi^{(\beta)}(x, t) \) the solution of the evolution equation \((3.2)\) with initial data \( \phi_0 \) and set \( \phi^{(\infty)}(x, t) := P_{\omega}(e^{-t} \phi_0 + (1 - e^{-t})f) \). Then

\[
\sup_X \left| \phi^{(\beta)}(t) - \phi^{(\infty)}(t) \right| \leq C \frac{\log \beta}{\beta}
\]

and there is a constant \( C \) such that

\[
(3.4) \quad \left| \frac{\partial \phi}{\partial t} \right| \leq Ce^{-t}, \quad |dd^c \phi_t|_{\omega_0} \leq C
\]

**Proof.** Note that \( g := e^t \frac{\partial \phi_t}{\partial t} \) satisfies

\[
\frac{\partial g}{\partial t} - \Delta g = 0
\]

and hence, by the parabolic maximum principle, \( |g(x, t)| \leq \sup_X |g(x, 0)| := C \), i.e. \( \left| \frac{\partial \phi^{(\beta)}}{\partial t} \right| \leq Ce^{-t} \). But then we can (for \( t \) large) employ the function \( f(t) = -Ce^{-t} \) in the proof of the Laplacian estimate in Prop 2.1.10 (since the cohomological term vanishes), which implies the estimate on \( dd^c \phi_t \). Next, the rate of convergence in \((3.3)\) is proved by tracing through the proof of Prop 2.1.17. Indeed, first the upper bound on \( \phi^{(\beta)}_t \) follows from the uniform upper uniform bound of \( dd^c \phi_t \) in formula \((3.4)\), giving an error term of the order \( 1/\beta \). As for the lower bound it is obtained by taking \( \epsilon = C/\beta \) in the proof of Prop 2.1.17 and using that, since \( \chi > 0 \), the constant \( C(T) \) can be taken to be independent of \( T \) (more precisely, one first fixes \( x \in X \) and take \( u(t) \) such that \( u(x, t) \geq \phi^{(\infty)}_t - \delta \) and finally let \( \delta \rightarrow 0 \)).

In particular, by \((3.3)\)

\[
\sup_X \left| \phi^{(\beta)}_t - P_{\omega}(f) \right| \leq C \left( \frac{\log \beta}{\beta} + e^{-t} \right)
\]
and hence the envelope $P_\omega(f)$ can be constructed from the joint large $\beta$ and large $t$–limit of the Monge-Ampère flow 3.2:

$$P_\omega(f) := \lim_{t \to \infty} \varphi^{(\beta_t)}$$

in the $C^0(X)$–norm for any family of $t$–dependent $\beta_t$ such that $\beta_t \to \infty$ as $t \to \infty$.

Interpreting $\beta_t$ as the “inverse temperature” this construction is thus analogous to the method of simulated annealing algorithms used in numerics to find nearly optimal global minima of a given energy type function by cooling down a thermodynamical system (and decreasing the corresponding free energy). The analogy can be made more precise using the gradient flow picture in Section 7 where the energy functional in question is the pluricomplex energy introduced in [5]. It would be interesting to see is numerically useful in concrete situations, for example by adapting the numerical implementations for the Kähler-Ricci flow on a toric manifold introduced in [33] (concerning a finite $\beta$).

It may be illuminating to compare the dynamic construction of the envelope $P_\omega(f)$ above with the dynamic PDE construction of the convex envelope of a given smooth function $f$ on $\mathbb{R}^n$ introduced in [76]:

$$\frac{\partial \psi(t)}{\partial t} = \sqrt{1 + |\partial_x \psi(t)|^2 \min\{0, \lambda_1(\partial^2_x \psi(t))\}} \quad \psi(0) = f,$$

i.e. the graph of the solution $\psi_t$ evolves in the normal direction at each point, with the speed $\min\{0, \lambda_1(\partial^2_x \psi(t))\}$ (expressed in terms of the first eigenvalue of the real Hessian $\partial^2_x \psi(t)$); here $\psi(t)$ is a solution in the viscosity sense. A variant of the latter construction, obtained by removing the first factor in the right-hand side of the evolution equation above, was studied in [24] using stochastic calculus, where exponential convergence was established with a uniform control bound on $\partial^2_x \psi(t))$, which is thus analogous to the result in Theorem 3.2 above. Our approach can also be applied to convex envelopes by imposing invariance in the imaginary directions (as in Section 4.2). But the main difference in our setting is that we start with an arbitrary convex function $\psi(0)$ and the dependence on $f$ instead appears in the evolution equation itself. Moreover, the large parameter $\beta$ appears as a regularization parameter ensuring that the solution remains smooth for positive times.

3.2. The case when the class $\frac{1}{\beta}c_1(K_X) + [\theta_\beta]$ is trivial. Next we specialize to the case when $\frac{1}{\beta}c_1(K_X) + [\theta_\beta]$ is trivial, which is the one relevant for the applications to Hele-Shaw type flows and Hamilton-Jacobi equation (in the latter case $K_X$ is even trivial). Equivalently, this means that the non-normalized KRF preserves the initial cohomology class. In particular, letting $\beta \to \infty$ reveals that $[\theta]$ is trivial and hence we can write

$$\theta = dd^c f, \quad \inf_X f = 0$$

for a unique function $f$ and then take

$$\theta_\beta := dd^c f_\beta + \frac{1}{\beta} \text{Ric } \omega, \quad f_\beta := f.$$

for a fixed Kähler form $\omega$, i.e. by imposing the equation 2.6.

In this setting, the normalized flow always tends to zero as $t \to \infty$ (as the volume of the class does). But, by the seminal result in [23], the non-normalized KRF flow converges to a Kähler form $\omega_\beta$:
Proposition 3.3. For a fixed $\beta > 0$ the non-normalized Kähler-Ricci flow $\omega^{(\beta)}(t)$ emanating from any given form $\omega_0$ converges (in the $C^\infty$-topology), as $t \to \infty$, to the unique solution $\omega_\beta \in [\omega_0]$ of the Calabi-Yau equation

$$\frac{1}{V_0} \omega_\beta^n = \frac{e^{-\beta f}}{\int_X e^{-\beta f} \omega^n},$$

where $V_0$ is the volume of $\omega_0$. More precisely, under the normalizations above the convergence holds on the level of Kähler potentials.

Remark 3.4. By definition the volume form of the limiting Kähler metric is the Boltzmann-Gibbs measure associated to the Hamiltonian function $f$, at inverse temperature $\beta$, which gives a hint of the statistical mechanical interpretation of the large $\beta$-limit (see Section 7 for further hints).

It should be stressed that by the estimates in [23] one has in this setting that

$$\omega^{(\beta)}(t) \leq C \beta$$

independently of $t$, which seemingly improves on the bounds in Theorem 2.2 and Theorem 2.3 for large $t$ (the proof uses a different application of the Laplacian estimate, along the lines of Yau’s original argument, which needs a two-sided bound on the potential). But the point of the estimates in Theorem 2.3 is to get a multiplicative constant that is independent of $\beta$ (at least when $t \to \infty$). In fact, for a generic $f$, it is impossible to get a constant $C \beta$ in formula (3.6) which is independent of $\beta$. Indeed, unless $f$ vanishes identically the Gibbs measure in the right-hand side of the Calabi-Yau equation 3.5 blows up as $\beta \to \infty$, concentrating on the subset of $X$ where $f$ attains its absolute minimum ($= 0$ with our normalizations). Hence, for a generic $f$ any limit point of $\omega^{(\beta)}(\infty)$ is a sum of Dirac measures. Accordingly, the convergence in the previous proposition, motivates (by formally interchanging the large $t$ and large $\beta$-limits) the following

Proposition 3.5. Let $f$ be a smooth function on $X$ and $\omega_0$ a Kähler form on $X$. Then any limit point of the family $(P(\omega_0 + tdd^c f))^n$ (in the weak topology) is supported in the closed set $F$ where $f$ attains its absolute minimum. In particular,

- if $f$ admits a unique absolute minimum $x_0$ then
  $$\lim_{t \to \infty} \lim_{\beta \to \infty} (P(\omega_0 + tdd^c f))^n = V_0 \delta_{x_0}$$
  weakly. Hence, the corresponding non-normalized Kähler-Ricci flows $\omega^{(\beta)}(t)$ emanating from $\omega_0$ satisfy
  $$\lim_{t \to \infty} \lim_{\beta \to \infty} \omega^{(\beta)}(t)^n = V_0 \delta_{x_0}$$
  in the weak topology.
- In general, under the normalization $\inf_X f = 0$,
  $$\lim_{t \to \infty} P_\omega(\varphi_0 + tf) = P_{(\omega,F)}(\varphi_0)$$
  (increasing pointwise) for any initial continuous $\omega$-psh function $\varphi_0$. In particular, if $F$ is not pluripolar, then
  $$\lim_{t \to \infty} \lim_{\beta \to \infty} \omega^{(\beta)}(t) = \omega_\infty$$
  in the weak topology, where $\omega_\infty$ is the positive current defined by $\omega_0 + ddd^c P_{(\omega_0,F)}(0)$. 

Proof: First observe that, under the normalization $f(x_0) = 0$ and $f > 0$ we have $P_X(f) = 0$, as in this case $\chi = 0$ and all psh functions on a compact manifold $X$ are constants. Next, by Lemma 3.1 $P_{\omega_0}((1 - e^{-t})f) := P_t(f_i) \to P_X(f)$ in the $L^1$-topology. Since $P_t(f_i)$ is in $PSH(X, C\omega)$ for $C$ sufficiently large it follows from basic properties of psh functions that $\sup_X P_t(f_i) \to \sup_X P_X(f) = 0$. Fixing $\epsilon > 0$ this means that for $t \geq t_\epsilon$, $\sup_X P_t(f_i) \leq \epsilon/2$ and hence the non-coincidence sets $\Omega_t$ satisfy $\{ f > \epsilon \} \subset \Omega_t$ for $t \geq t_\epsilon$. In particular, $(\hat{\omega} + dd^c P_{\omega}(1 - e^{-t}))$ and hence its non-normalized version $(\omega_0 + dd^c P_{\omega_0}(t))$ is supported in $\{ f \leq \epsilon \}$ for $t \geq t_\epsilon$, which concludes the proof of the first statement. The first point then follows immediately.

To prove the second point we may assume that $\inf_X f = 0$, hence the family $P_\omega(\varphi_0 + tf)$ is increasing in $t$. By assumption $P_\omega(\varphi_0 + tf) \leq \varphi_0 + tf = \varphi_0$ on $F$ and hence $P_\omega(\varphi_0 + tf) \leq P_{(\omega, F)}(\varphi_0)$. To prove the reversed inequality we fix $\epsilon > 0$ and $u \in PSH(X, \omega)$ such that $u \leq \varphi_0$ on $F$. Since the sets $\{ f \leq \epsilon \}$ decrease to the compact set $F$ as $\epsilon \downarrow 0$ and $\varphi_0$ is continuous, there exists $c > 0$ small enough such that $\{ f \leq \epsilon \} \subset \{ v - \epsilon < \varphi_0 \}$. Now, for $t > c^{-1} \sup_X (v - \varphi_0)$ we have $v - \epsilon \leq \varphi_0 + tf$, giving that the limit $\varphi_\infty$ of the increasing family $P_\omega(\varphi_0 + tf)$ is greater than $v - \epsilon$. As $v$ and $\epsilon$ were chosen arbitrarily the conclusion follows.  \[ \square \]

3.3. Comparison with convergence properties for a finite $\beta$ and canonical metrics.

3.3.1. The big case. Let us start by considering the case when $\theta = 0$. Then, up to a scaling, we may as well also assume that $\beta = 1$. When $K_X$ is nef and big, which equivalently means that $K_X$ is semi-positive (by the base point freeness theorem) and with non-zero volume, $K_X^n > 0$, it is well-known that the normalized Kähler-Ricci flow emanating from any given Kähler metric $\omega_0$ on $X$ converges, weakly in the sense of currents, to the unique (possible singular) Kähler-Einstein metric (or rather current) $\omega_{KE}$ on $X$ \cite{[72]} \cite{[71]}. This fact implies the following

Proposition 3.6. Assume that $K_X$ is nef and big, but not ample. Then it is not possible to have an upper bound of the form $\omega^{(\beta)}(t) \leq C\beta t$ along the non-normalized Kähler-Ricci flow, for $t$ large.

Proof. Fixing a semi-positive form $\chi$ in $c_1(K_X)$ and representing $\omega_{KE} = \chi + dd^c \varphi_{KE}$ the potential $\varphi_{KE}$ may be characterized as the unique continuous solution in $PSH(X, \chi)$ to the equation

\[ (\chi + dd^c \varphi)^n = e^\varphi dV_{\chi} \]

(in the sense of pluripotential theory) where $dV_{\chi}$ is the normalized volume form determined by $\chi$ (i.e. Ric $dV_{\chi} = \chi$). In particular, if $K_X$ is not positive (i.e. not ample) then $\omega_{KE}$ is not a bounded current. Indeed, assuming to get a contradiction that $\omega_{KE} \leq C\omega_0$ the previous equation gives that $\omega_{KE}^n \geq \delta \omega_0^n$ for some positive constant $\delta$. But this means that, up to enlarging the constant $C$ we get $\omega_0/C \leq \omega_{KE} \leq C\omega_0$ which forces $K_X$ to be ample (for example, by the Nakai-Moishezon criterion or by a direct regularization argument). \[ \square \]

\[ \footnote{The same result holds even when $F$ is non-pluripolar and $\varphi_0$ is unbounded (using the domination principle in finite energy classes due to Dinew \cite{[15]}), but we are not going further into this here.} \]
More generally, essentially the same arguments apply to any smooth twisting form $\theta$ and parameter $\beta$ as long as $K_X/\beta + [\theta]$ is nef and big.

3.3.2. The non-big case. Again we start with the case when $\theta = 0$ with $K_X$ nef, but now not big. Assuming that the abundance conjecture holds, i.e. that $K_X$ is semi-ample it was shown in [67] that the normalized Kähler-Ricci flow, emanating from any given Kähler metric $\omega_0$, on $X$ converges, weakly in the sense of currents, to a canonical current $\omega_X$ on $X$ defined as follows: by the semi-ampleness assumption there exists a holomorphic map $F$ from $X$ to a variety $Y$ such that $K_X = F^* A$ where $A$ is an ample line bundle on $Y$. In case $Y$ is zero-dimensional the limit $\omega_X$ vanishes identically (as in Section 3.2). Otherwise, denoting by $\kappa$ the dimension of $Y$ (which equals the Kodaira dimension of $X$), picking a Kähler form $\omega_A$ in $c_1(A)$ and taking $\chi := F^* \omega_A$, the limiting current $\omega_X$ obtained in [67] can be realized as $F^*(\omega_A + dd^c \psi)$ where $\psi$ is the unique continuous solution in $PSH(Y, \omega_A)$ of the equation

$$(\omega_A + dd^c \psi)^\kappa = e^\psi F_* (dV_\chi).$$

Next, we make some heuristic remarks about the connection to the double limit in formula [3.11]. We assume that $K_X$ is semi-ample and fix a smooth form $\theta$ in $c_1(K_X)$, a Kähler metric $\omega$ on $X$ and define $\theta_\beta$ and $f$ by

$$\theta_\beta := \theta - \frac{1}{\beta} \text{Ric} \omega, \quad \theta = dd^c f + \chi.$$

In particular, $c_1(K_X)/\beta + [\theta_\beta] = c_1(K_X)$ for all $\beta$. In the light of the result in [67] one would expect that the corresponding twisted normalized KRF $\omega^{(\beta)}(t)$ converges, as $t \to \infty$, to the current $F^*(\omega_A + dd^c \psi_\beta)$, where $\psi_\beta$ is the unique continuous solution in $PSH(X, \chi)$ of the equation

$$(\omega_A + dd^c \psi_\beta)^\kappa = e^{\beta \psi_\beta} F_* (e^{-\beta f} \omega^n).$$

We will make the hypothesis that this is the case. It can be shown that as $\beta \to \infty$ there exist a (mildly singular) volume form $\mu_Y$ on $Y$ such that

$$F_* (e^{-\beta f} dV_\chi) = e^{-\beta (\bar{f} + o(1))} \mu_Y,$$

where $\bar{f}(y) := \inf_{P_{f-1}(y)} f$ (using that the push forward $F_*$ amounts to integration along the fibers of $F$ which thus picks out the infimum of $f$ over the fibers as $\beta \to \infty$; the error term $o(1)$ is uniform away from the branching locus of the map $F$). But then a variant of Theorem 3.2 (see [10]) shows that $\psi_\beta \to \psi_\infty := P_{\omega_A}(\bar{f})$ and hence, under the hypothetical convergence above,

$$\lim_{t \to \infty} \lim_{\beta \to \infty} \omega^{(\beta)}(t) = \chi + dd^c P_{F^* \omega_A} (F^* \bar{f}).$$

Finally, since $K_X = F^* A$ we have $PSH(X, \chi) = F^* PSH(Y, \omega_A)$, forcing $P_{F^* \omega_A} (F^* \bar{f}) = P_\chi (f)$, i.e. the rhs above is equal to the current obtained by interchanging the limits in the lhs (as in Lemma 3.1), i.e. the two limits may be interchanged under the hypothesis above.

4. Applications to Hamilton-Jacobi equations and shocks

4.1. Background. Let $H$ be a smooth function on $\mathbb{R}^n$. The corresponding Hamilton-Jacobi equation (with Hamiltonian $H$) is the following evolution equation
(4.1) \[ \frac{\partial \psi_t(y)}{\partial t} + H(\nabla \psi(y)) = 0, \quad \psi_{t=0} = \psi_0 \]

for a function \( \psi(x,t) \) on \( \mathbb{R}^n \times [0,\infty] \). It is a classical fact that, even if the initial function \( \psi_0 \) is smooth a solution \( \psi_t \) typically develops shock singularities at a finite time \( T^* \), i.e. it ceases to be differentiable in the space-variable (due to the crossing of characteristics). In order to get a solution defined for any positive time the notion of viscosity solution was introduced in [30, 31]. The momentary shock locus \( S_t \) of such a solution \( \psi_t \) is defined by

\[ S_t := \{ x : \psi_t \text{ is not differentiable at } x \}. \]

When \( H \) is convex the classical Hopf-Lax formula provides an explicit envelope expression for a viscosity solution of the Cauchy problem for the HJ-equation 4.1 with any given smooth initial data \( \psi_0 \) (which, for example, appears naturally in optimal control problems):

\[ \psi_t(y) = \inf_{x \in \mathbb{R}^n} \psi_0(x) + t H^* \left( \frac{x - y}{t} \right), \]

expressed in terms of the Legendre transform:

\[ g^*(y) := \sup_{x \in \mathbb{R}^n} x \cdot y - g(x). \]

On the other hand, in the case when \( H \) is non-convex, but the initial data \( \psi_0 \) is assumed convex, the second Hopf formula [3, 49] provides a viscosity solution which may be represented as

(4.2) \[ \psi_t = (\psi_0^* + tH)^*. \]

This was shown in [3] using the theory of differential games and in [30] by a more direct approach. In particular the viscosity solution \( \psi_t \) above remains convex for all positive times and as a consequence its shock locus \( S_t \) is a codimension one hypersurface with singularities (or empty, as is the case for small \( t \)).

We recall that the viscosity terminology can be traced back to the fact that viscosity solutions may often be realized as limits of smooth solutions \( \psi^{(\beta)} \) of the following perturbed (viscous) HJ-equations (where the constant \( \beta^{-1} \) plays the role of the viscosity constant in fluid and gas dynamics):

(4.3) \[ \frac{\partial \psi_t(y)}{\partial t} + H(\nabla \psi_t(y)) = \frac{1}{\beta} \Delta \psi_t(y) \]

as \( \beta \to \infty \). For example, the following result holds:

**Theorem 4.1.** [30, 31] Theorem 3.1. (Vanishing viscosity limit). Assume that \( \psi^{(\beta)} \) are smooth solutions to the previous equation and that a subsequence converges uniformly to \( \psi_t \). Then \( \psi_t \) is a viscosity solution to the HJ-equation (4.1).

In particular, under suitable growth assumptions, ensuring that the viscosity solution \( \psi_t \) is uniquely determined, the whole family converges to \( \psi_t \). Note however that, in general, \( \Delta \psi^{(\beta)}_t \) will not be uniformly bounded (even locally), as this would entail that the limit \( \psi_t \) is differentiable on \( \mathbb{R}^n \).

Next, we make the observation that in the case when the initial data \( \psi_0 \) above is taken to be \( \frac{|y|^2}{2} \) the second Hopf formula is equivalent to the Hopf-Lax formula for the convex Hamiltonian \( \frac{|x|^2}{2} \):

\[ \psi_t = (\psi_0^* + t \frac{|x|^2}{2})^*. \]
Lemma 4.2. Let $\Phi_0$ be a given function on $\mathbb{R}^n$ and denote by $\Phi_t$ the Hopf-Lax viscosity solution to the HJ-equation with convex Hamiltonian $|x|^2/2$ and initial data $\Phi_0$. Then

$$\psi_t(y) := \left(-\Phi_t(y)t + \frac{|y|^2}{2}\right)$$

gives the viscosity solution to the HJ-equation with non-convex Hamiltonian $H := \Phi_0$ and initial data $\psi_0(y) := \frac{|y|^2}{2}$ provided by the second Hopf formula (and conversely). In particular, the shock loci of $\Phi_t$ and $\psi_t$ coincide.

Proof. This follows immediately from comparing the Hopf-Lax formula and the second Hopf formula. □

The previous lemma is consistent (as it must) with the fact that when $\psi_0(y) = \frac{|y|^2}{2}$ the Hamiltonian $H$ can, by the definition of the HJ-equation, be recovered as minus the derivative at $t = 0$ of the corresponding viscosity solution $\psi_t$.

4.1.1. The adhesion model in cosmology. The convex case where $H(x) = |x|^2/2$ is ubiquitous in mathematical physics and appears, in particular, in the adhesion model for the formation of the large-scale structure in the early universe (known as the “cosmic web”) where $\Phi_0$ is proportional to the gravitational potential of the initial fluctuations of the density field and the shock region $S_t$ corresponds to emerging regions of localized mass concentration (the adhesion model is an extension of the seminal Zel’dovich approximation beyond $t \geq T_*$) \[39, 75, 42\]. The corresponding singularities of $S_t$ and their metamorphosis as $t$ evolves have been classified in dimensions $n \leq 3$, for generic initial data, using the catostropy theory of Lagrangian singularities initiated by Arnold \[1, 39, 19, 42, 48\]. In this setting the Legendre transform $\phi_t := \psi_t^*$ of the corresponding function $\psi_t$ appearing in the previous lemma is given by

$$\phi_t(x) = x + t\Phi_0(x)$$

and the corresponding map

$$(4.4) \quad x \mapsto \nabla_x \phi_t(x)$$

describes, in the Zeldovich approximation, the displacement of a particle with initial coordinate $x$ to the position $y$ at a time $t$ (in the physics literature the initial coordinate space $x$ is called the Lagrangian space and the position space $y$ at time $t$ is called the Euler space; accordingly $\phi_t$ is often called the Lagrangian potential). The map above is injective precisely for $t < T_*$. In the next section we will show that the adhesion model can be realized as the zero-temperature limit of the twisted Kähler-Ricci flow (using Lemma 4.2).

Remark 4.3. When $H(x) = |x|^2/2$ the vector field $v_t(y) := \nabla u_t(y)$ determined by a solution $u_t$ of the corresponding HJ-equation satisfies Burger’s equation:

$$\frac{\partial v_t(y)}{\partial t} + \frac{1}{2} \nabla |v_t(y)|^2 = 0,$$

which is the prototype of a hyperbolic conservation law \[63\] and non-linear wave phenomena \[39\].
4.2. Relation to the Kähler-Ricci flow and Theorem 2.2. The relation between the Hamilton-Jacobi equation and the Kähler-Ricci flow, which does not seem to have been noted before, arises when the linear viscosity term in the perturbed HJ-equation 4.3 is replaced by the following non-linear one:

\[
\frac{\partial \psi_t(y)}{\partial t} + H(\nabla \psi_t(y)) = \frac{1}{\beta} \log(\partial^2 \psi_t(y))
\]

for \(\psi_0\) strictly convex (for example, \(\psi_0(y) = |y|^2/2\), as in the adhesion model above). One virtue of the latter evolution equation is that, as will be shown below, the smooth solution \(\psi_t^{(\beta)}\) remains convex (and even strictly so) for positive times.

We will consider the case when the Hamiltonian \(H\) is periodic, i.e. invariant under the action of a lattice \(\Lambda\) on \(\mathbb{R}^n\) by translation. Without loss of generality we may and will assume that a fundamental domain for \(\Lambda\) has unit volume. Since there are no non-constant periodic convex functions on \(\mathbb{R}^n\) the natural condition on the initial function \(\psi_0\) is that it is in the class of all convex functions \(u\) which are quasi-periodic in the sense that \(\psi(y) - |y|^2/2\) is \(\Lambda\)-periodic on \(\mathbb{R}^n\). We denote by \(C_\Lambda\) the space of all quasi-periodic convex functions on \(\mathbb{R}^n\). The point is that for any \(\psi \in C_\Lambda\) the Hessian \(\partial^2 \psi\) is periodic and the gradient map \(\partial \psi\) is \(\Lambda\)-equivariant and hence all terms appearing in the equation 4.5 are \(\Lambda\)-periodic.

**Lemma 4.4.** Equip the space \(C_\Lambda\) with the sup-norm. Then the Legendre transform \(\phi \mapsto \psi := \phi^*\) induces an isometry on \(C_\Lambda\) and for any quasi-periodic function \(f\)

\[\sup_{\phi \leq f} \{\phi\} = f^{**},\]

where the sup, that we shall denote by \(P f\), can be taken either over all convex functions \(\phi\) or over all quasi-periodic convex functions. Moreover, the subspace of all \(\phi\) in \(C_\Lambda\) such that \(\sup_{x \in \mathbb{R}} (\phi(x) - |x|^2/2) = 0\) is compact.

**Proof.** The isometry property follows directly from the relation \((\phi + c)^* = \phi^* - c\). Next, if \(P f\) denotes the sup over all convex \(\phi\) below \(f\) then, by the extremal property, the function \(P' f - \frac{|x|^2}{2}\) has to be \(\Lambda\)-periodic, as \(f - \frac{|x|^2}{2}\) is, i.e. \(P' f\) is quasi-periodic, as desired. Finally, it is well-known that if \(f\) is convex then \(P' f = f^{**}\).

The compactness is a consequence of the Arzela-Ascoli theorem and the fact that if \(\phi\) is in \(C_\Lambda\) then the periodic function \((\phi(x) - |x|^2/2)\) is \(L\)-Lipschitz for a constant \(L\) only depending on the diameter of a fundamental domain of \(\Lambda\); see [45] Lemma 3.14 where further properties of the space \(C_\Lambda\) are also established.

In this setting Theorem 2.2 admits the following dual formulation:

**Theorem 4.5.** Consider the perturbed HJ-equation 4.3 with \(\Lambda\)-periodic smooth Hamiltonian \(H\) and strictly convex and quasi-periodic initial data \(\psi_0\). Denote by \(\psi^{(\beta)}_t\) the unique solution of the corresponding Cauchy problem such that \(\psi^{(\beta)}_t\) is quasi-periodic and strictly convex. Then \(\psi^{(\beta)}_t\) converges, as \(t \to \infty\), uniformly in space, to \(\psi_t\) given by the second Hopf formula 4.2 which is the unique viscosity...
solution of the HJ-equation \(4.1\) with initial data \(\psi_0\). Moreover, \(\psi_t^{(\beta)}\) is strictly convex for any \(t > 0\), uniformly in \(\beta\):

\[
\left\| \partial^2 \psi_t^{(\beta)} \right\| \geq \frac{1}{t+1} \min\{\left\| \partial^2 \psi_0 \right\|, \left\| \partial^2 H \right\|},
\]

in terms of the trace norm defined wrt the Euclidean metric (i.e. the sup on \(\mathbb{R}^n\) of the point-wise \(L^1\)-norm).

To make the connection to the complex geometric setting we let \(X\) be the abelian variety \(X := \mathbb{C}^n/(\Lambda + i\mathbb{Z}^n)\) and consider the following holomorphic \(T\)-action on \(X\):

\[
([x + iy, a]) \mapsto [x + iy + a],
\]

where \(T\) denotes the real \(n\)-torus \(T := \mathbb{R}^n/\mathbb{Z}^n\) and \(\pi(x) := [z]\) denotes the corresponding quotient map. Let \(\omega\) be the standard flat Kähler metric on \(X\) induced from the Euclidean metric \(\omega_0\) on \(\mathbb{C}^n\) normalized so that \(\omega_0 = dd^c|x|^2/2\) and fix a closed \(T\)-invariant \((1,1)\)-form \(\theta\) which is exact, i.e.

\[
\theta = dd^c f
\]

for a \(T\)-invariant function \(f\) on \(X\) (uniquely determined up to an additive constant). Now we can identify \(T\)-invariant elements in \(PSH(X,\omega)\) with convex functions \(\phi(x)\) on \(\mathbb{R}^n\) in the space \(\mathcal{C}_\Lambda\) (by setting \(\phi := |x|^2/2 + \pi^* \varphi\) and using that \(dd^c(|x|^2/2 + \pi^* \varphi) = \omega_0 + dd^c \pi^* \varphi \geq 0\)). Accordingly, the non-normalized KRF in the class \([\omega]\) with twisting form \(\theta\) thus gets identified with the following parabolic equation on \(\mathbb{R}^n\):

\[
(4.6) \quad \frac{\partial \phi_t^{(\beta)}(x)}{\partial t} = \frac{1}{\beta} \log(\partial^2 \phi_t^{(\beta)}(x)) + H(x),
\]

where \(H\) is the \(\Lambda\)-periodic function on \(\mathbb{R}^n\) corresponding to \(f\) and \(\phi_t^{(\beta)} \in \mathcal{C}_\Lambda\). More precisely, \(\phi_t^{(\beta)}\) is smooth and strictly convex. The key observation now is that setting

\[
\psi_t^{(\beta)}(y) := \phi_t^{(\beta)*}(y)
\]

gives a solution in \(\mathcal{C}_\Lambda\) to the perturbed HJ-equation \(4.3\). Indeed, this follows from the following well-known properties of the (involutive) Legendre transform between smooth and strictly convex functions (say with quadratic growth at infinity):

\[
(4.7) \quad \partial^2 \phi(x) = (\partial^2 \psi(y))^{-1}, \quad \frac{\partial}{\partial t} \left( \partial_t (\phi + tv)(x) \right) |_{t=0} = -v(\partial_y \psi(y)), \quad y := \partial_x \phi(x)
\]

(see for example the appendix in \([11]\) for a proof of the latter formula). Now, by Theorem \([22]\) and the previous lemma

\[
\lim_{\beta \to \infty} \phi_t^{(\beta)} = P_\Lambda(\phi_0 + tH) = (\phi_0 + tH)^**
\]

in \(\mathcal{C}_\Lambda\). Since the Legendre transform is an isometry on \(\mathcal{C}_\Lambda\) and in particular continuous this equivalently means that \(\lim_{\beta \to \infty} \psi_t^{(\beta)} = (\phi_0 + tH)^*\), which coincides with the viscosity solution of the HJ-equation provided by the second Hopf formula. Finally, the proof of the previous theorem is concluded by noting that the uniqueness of viscosity solutions in \(\mathcal{C}_\Lambda\) follows from the standard uniqueness argument \([30]\,[31]\), using that for any two functions in \(\mathcal{C}_\Lambda\) the difference \(u - v\) is continuous and attains its maximum and minimum (since it is periodic).
In fact, in this way Theorem 4.2 could be used to give an alternative proof of the fact that the second Hopf formula defines a viscosity solution to the HJ-equation \[4.1\] by adapting the proof of Theorem 4.1 to the present non-linear setting. But we will not go further into this here.

**Remark 4.6.** Convex envelopes of the form \(\psi_t := (\psi_t^0 + tH)^*(= \phi_t^*)\) and the corresponding sets \(X(t)\) also appear in a different Kähler-geometric setting in [53, 59], where it is shown that \(\psi_t\) defines a torus invariant (weak) Kähler geodesic precisely on \([0, T^*]\) (what we call \(T^*\) is called the “convex life span” in [53, 59]). By definition, such a Kähler geodesic \(\phi_t\) is characterized by the homogeneous Monge-Ampère equation \(MA(\phi) = 0\) on the product \(X \times [0, T]\). The relation to \((C^1-\text{smooth})\) solutions of Hamilton-Jacobi equations was also pointed out in Section 6 in [59]. In the light of the results in [53, 59] it seems notable that in our setting \(\phi_t\) has a natural complex geometric interpretation also for \(t > T^*\) (namely, as a limiting Kähler-Ricci flow).

**4.2.1. Remarks on convex duality in the present setting.** By a well-known duality principle in convex analysis differentiability of a convex function \(\psi\) corresponds, loosely speaking, to strict convexity of its Legendre transform \(\phi := \psi^*\). To make this precise we will assume that both \(\phi\) and \(\psi\) are defined on all of \(\mathbb{R}^n\) and have super-linear growth (which is the case when any, and hence both, of the functions are in \(CA_1\)). This ensures that the sub gradient maps \(\partial \phi\) and \(\partial \psi\) are both surjective. We recall that a convex function \(\phi\) is differentiable at \(x\) iff the subgradient \(\partial \phi(x)\) is single valued and then we will write \(\partial \phi(x) = (\nabla \phi)(x)\). The starting point for the duality in question is the following fact (which follows directly from the definitions):

\[
x \in \partial \psi(y) \iff y \in \partial \phi(x) \iff x \cdot y = \phi(x) + \psi(y)
\]

In our setting \(\phi := \phi_t\) (for a fixed time \(t\)) is \(C^{1,1}\)-smooth, i.e. \(\partial \phi(\nabla \phi)\) defines a surjective Lipschitz map \(\mathbb{R}^n \to \mathbb{R}^n\). As a consequence, a point \(y\) is in the shock locus \(S_t\) of \(\psi_t\) iff \(y \in \partial \phi_t(U)\), for an open set \(U\) where the Lipschitz map \(\partial \phi_t\) is not injective (which can be interpreted as a local strict convexity of \(\phi_t\)). Let now \(X_t\) be the support of the Monge-Ampère measure \(\det(\partial^2 \phi_t)dx\) and denote by \(\Omega_t\) its complement. For simplicity we assume that the locus where \(\phi_t\) is in \(C^2_{\text{loc}}\) is dense in \(\mathbb{R}^n\) (which presumably holds for a generic \(H\) using the arguments in [1, 19]). In that case the continuous map \(\partial \phi_t\) maps the interior of \(X_t\) injectively to \(\mathbb{R}^n - S_{\psi_t}\), and \(\overline{\Omega}_t\) non-injectively to \(S_{\psi_t}\) (since a \(C^2\)–convex function \(u\) has an invertible gradient iff \(\det(\partial^2 u) > 0\)). Conversely, \(\nabla \psi_t\) maps \(\mathbb{R} - S_{\psi_t}\) to \(X_t\). See for example [75, Fig 5] for an illustration of this duality.

It may also be illuminating to consider the case when \(\psi\) is piece-wise affine (which, as we will show in the next section, happens when \(t = \infty\)). Then \((\nabla \phi)(\mathbb{R}^n - S_{\phi})\) is contained in the 0--dimensional stratum \(S_{\psi}^{(0)}\) of \(S_{\psi}\) (i.e. in the vertex set). Indeed, if \(y_0 := (\nabla \phi)(x_0)\) is not in \(S_{\phi}^{(0)}\) then there is an open affine segment \(L\) passing through \(y_0\) along which \(\psi\) is affine. One then gets a contradiction to the the differentiability of \(\phi\) at \(x_0\) by noting that \(L \subset \partial \phi(x_0)\). Indeed, since \(x_0 \in \partial \psi(y_0)\) one gets \(\psi(y) = \psi(y_0) + x_0 \cdot (y - y_0)\) along \(L\). But this means that \(x_0 \cdot y = \phi(x_0) + \psi(y)\) and hence \(y \in \partial \phi(x_0)\).

In fact, this argument also shows that \(\phi\) is piecewise affine iff its Legendre transform \(\psi\) is. Indeed, if \(\psi\) is piecewise affine then by the growth assumptions the sup defining \(\phi\) is always attained. Hence, for any \(x \in \mathbb{R}^n - S_{\phi}\) we have that
\( \phi = (\chi_{S_\phi}(\psi))^* \). Since the rhs is also a convex function and the complement of \( \mathbb{R}^n - S_\phi \) is a null set it then follows that \( \phi = (\chi_{S_\phi}(\psi))^* \) everywhere, showing that \( \phi \) is also piece-wise affine, as desired.

4.3. The large time limit and Delaunay/Voronoi tessellations. Next, we specialize the large time convergence result in Prop 4.5 to the present setting, showing, in particular, that the Hessian of the limiting solution vanishes almost everywhere:

**Theorem 4.7.** Denote by \( F_\Lambda \) the closed set in \( \mathbb{R}^n \) where the \( \Lambda \)-periodic Hamiltonian \( H \) attains its minimum, normalized to be 0 and assume that \( F_\Lambda \) is discrete. Then, for any given initial data \( \psi_0 \) in the space \( C_\Lambda \) the unique viscosity solution \( \psi_t \) in \( C_\Lambda \) of the corresponding Hamilton-Jacobi equation converges uniformly to the following convex piecewise affine function:

\[
(4.8) \quad \psi_\infty(y) := \sup_{x \in F_\Lambda} x \cdot y - \psi_0^*(x).
\]

Equivalently, the large \( \beta \)-limit \( \phi_t \) of the Kähler-Ricci flow 4.6 converges to the convex piecewise affine function \( \phi_\infty(x) \) whose graph is the convex hull of the discrete graph of the function \( \phi_0 \) restricted to \( F_\Lambda \).

**Proof.** By the second point in Prop 3.5

\[
\phi_\infty(x) := \sup_{\phi \in C_\Lambda} \{ \phi(x) : \phi \leq \phi_0 \text{ on } F_\Lambda \}.
\]

Indeed, recall that the limit in Prop 3.5 is the supremum over all \( \omega \)-psh functions lying below \( \chi_{F_\Lambda} \phi_0 \). But as \( F_\Lambda \) is non-pluripolar (which follows from the classical fact in pluripotential theory that \( \mathbb{R}^n \) is non-pluripolar in \( \mathbb{C}^n \)), the function \( \phi_\infty \) is convex bounded in \( \mathbb{R}^n \). This together with the maximality property yields that \( \phi_\infty \) is \( T \)-invariant and hence the corresponding function in \( C_\Lambda \) equals the supremum taken over \( C_\Lambda \) as above. Alternatively, the boundedness can also be seen directly in the present setting using the compactness property in Lemma 4.4. Writing this as \( \phi_\infty = P(\chi_{F_\Lambda} \phi_0) \), where \( \chi_{F_\Lambda} = 0 \) on \( F_\Lambda \) and \( +\infty \) on the complement of \( F_\Lambda \) (compare Lemma 4.4) reveals that the previous sup coincides with the relaxed sup \( \phi^* \) obtained by simply requiring that \( \phi \) be convex (but not quasi-periodic), i.e. the graph of \( \phi_\infty(x) \) is the convex hull of the discrete graph of the function \( \phi_0 \) restricted to \( F_\Lambda \), as desired. By Lemma 4.4 this means that \( \phi_\infty = P(\chi_{F_\Lambda} \phi_0) = \left( (\chi_{F_\Lambda} \phi_0)^* \right)^* \) and hence \( \phi_\infty^* = (\chi_{F_\Lambda} \phi_0)^* \). Moreover, since the Legendre transform is a continuous operator on \( C_\Lambda \) it follows from the second Hopf formula that \( \psi_\infty := \lim_{t \to \infty} \psi_t = \phi_\infty^* \), which proves formula 4.8. As a consequence \( \psi_\infty(y) \) is locally the max of a finite number of affine functions (indeed, since \( F_\Lambda \) is locally finite and \( \phi \) has quadratic growth the sup defining \( (\chi_{F_\Lambda} \phi_0)^* \) can, locally wrt \( y \), be taken over finitely points in \( F_\Lambda \)). Hence, \( \psi := \psi_\infty \) is piecewise affine and hence so is \( \phi_\infty \) (compare Remark 4.2.1). □

In particular, if \( \psi_0(y) = |y|^2/2 \), then we can complete the square and rewrite

\[
\psi_\infty(y) = \frac{1}{2} |y|^2 - \inf_{x \in F_\Lambda} \frac{1}{2} |x - y|^2.
\]

Accordingly the non-differentiability \( S_{\psi_\infty} \) locus of \( \psi_\infty \) coincides with the subset of all points \( y \) in \( \mathbb{R}^n \) where the corresponding minimum is non-unique (compare Remark 4.2.1). The latter set is the honeycomb like connected \((n - 1)\)-dimensional piecewise linear manifold obtained as the union of the boundaries of the open sets
\{(O_y)_{y \in F_\Lambda}\} consisting of points in \(\mathbb{R}^n\) for which \(y\) is the unique closest point in \(F_\Lambda\). In the computational geometry literature the sets \(O_y\) are called Voronoi cells (attached to the point set \(F_\Lambda\)) and the corresponding tessellation of \(\mathbb{R}^n\) by convex polytopes is called the Voronoi tessellation (or Voronoi diagram) \([51]\). Similarly, the non-differentiability locus \(S_{\psi_\infty}\) of \(\psi_\infty\) is the \((n-1)\)-dimensional stratum in the Delaunay tessellation of \(\mathbb{R}^n\) whose 0-dimensional stratum is given by the point set \(F_\Lambda\). The Delaunay tessellation can be defined as the dual tessellation of the Voronoi tessellation, in a suitable sense. For example, when \(n = 2\) this simply means that \(S_{\psi_\infty}\) is obtained by connecting any two points in \(F_\Lambda\) which are neighbors in the corresponding Voronoi tessellation by a segment \([51]\).

**Remark 4.8.** Under suitable generality assumptions it is well-known that the corresponding Delaunay tessellation consists of simplices giving a triangulation of \(F\) with remarkable optimality properties \([51]\).

The previous proposition give a rigorous mathematical justification of the Voronoi tessellations appearing in numerical simulations in cosmology, which use periodic boundary conditions \([48, 42, 43]\): for large times Voronoi polytopes form around points where \(H\) has its absolute minimum (the Voronoi polytopes in question are called voids in the cosmology literature, since the mass in the universe is localized on the shock locus \(S_{\psi_\infty}\) between voids). The dual Delaunay tessellation is also frequently used for the numerics \([48, 42, 43]\).

**Remark 4.9.** When \(H\) has a unique minimum \(x_m\) (modulo \(\Lambda\)), the corresponding convex piecewise affine function \(\psi_\infty\) appears naturally in tropical geometry as a tropical theta function with characteristics (in the case when \(x_m\) and \(\Lambda\) are defined over the integers). The tropical subvariety defined by its non-differentiability locus is called the tropical theta divisor and seems to first have appeared in complex geometry in the compactification of the moduli space of abelian varieties (see \([50]\) and references therein).

### 5. Application to Hele-Shaw type flows

**5.1. Background.** The Hele-Shaw flow was originally introduced in fluid mechanics in the end of the 19th century to model the expansion of an incompressible fluid of high viscosity (for example oil) injected at a constant rate in another fluid of low viscosity (such as water) in a two dimensional geometry. Nowadays the Hele-Shaw flow, also called Laplacian growth, is ubiquitous in engineering, as well as in mathematical physics where it appears in various areas ranging from diffusion limited aggregation (DLA) to integrable systems (the dispersionless limit of the Toda lattice hierarchy), random matrix theory and quantum gravity; see \([73, 47]\) and references therein.

To explain the general geometric setup, introduced in \([11]\), we let \(X\) be a compact Riemann surface and fix a point \(p\) (the injection point) together with an area form \(\omega_0\) of total area one (whose density models the inverse permeability of the medium). The classical situation appears when \(X\) is the Riemann sphere and \(p\) is the point at infinity so that \(X - \{p\}\) may be identified with the complex plane \(\mathbb{C}\). A family of increasing domains \(\Omega^{(\lambda)}\) with time parameter \(\lambda \in [0, 1]\) is said to be a classical solution to the Hele-Shaw flow corresponding to \((p, \omega_0)\) if \(\Omega^{(0)} = \emptyset\) and the closure of \(\Omega^{(\lambda)}\) is diffeomorphic to the unit-disc in \(\mathbb{C}\) for \(\lambda > 0\), the point \(p\) is contained in
the interior of $\Omega^{(\lambda)}$, the area grows linearly:

$$\int_{\Omega^{(\lambda)}} \omega_0 = \lambda$$

and the velocity of the boundary $\partial \Omega^{(\lambda)}$ equals minus the gradient (wrt $\omega_0$) of the Green function $g_p$ for $\Omega^{(\lambda)}$ with a logarithmic pole at $p$ (i.e. Darcy’s law holds). Such a solution exists for $\lambda$ sufficiently small (see [41] for the case when $\omega_0$ is real analytic and [54] for the general case). However, typically the boundary of the expanding domains $\Omega^{(\lambda)}$ develop a singularity for some time $\lambda < 1$ and then changes its topology so that the notion of a classical solution breaks down. Still, there is a well-known notion of weak solution of the Hele-Shaw flow, defined in terms of subharmonic envelopes (obstacles) and which exists for any $\lambda \in [0, 1]$ (where $\Omega^{(1)} = X$); see [41] and references therein. In our notations the envelopes in question may be defined as

$$\phi_{\lambda} := \sup_{\phi \in PSH(X, \omega_0)} \left\{ \phi : \phi \leq 0, \phi \leq \lambda \log |z - p|^2 + O(1) \right\},$$

which, for $\lambda$ fixed, is thus a restrained version of the envelope $P_{\omega_0}(0)$ defined in Section 2.3.2, where one imposes a logarithmic singularity of order $\lambda$ at the given point $p$. The weak Hele-Shaw flow is then defined as the evolution of the corresponding increasing non-coincidence sets:

$$\Omega^{(\lambda)} := \{ \phi_{\lambda} < 0 \} \subset X,$$

(which thus is empty for $\lambda = 0$, as it should). We will write

$$X^{(\lambda)} := X - \Omega^{(\lambda)}$$

for the corresponding decreasing “water domains”. When $\omega_0$ is real analytic it follows from the results in [41, 60] (applied to the pull-back of $\omega_0$ to the universal covering $\tilde{X}$ of $X$) that the boundary of $\Omega^{(\lambda)}$ is a piecewise real analytic curve having a finite number of cusp and double points (if moreover $\omega_0$ has negative Ricci curvature then the lifted Hele-Shaw on $\tilde{X}$ exists for any $t > 0$).

**Example 5.1.** The classical situation in fluid mechanics appears when $X$ is the Riemann sphere and $p$ is the point at infinity, so that $X - \{p\}$ may be identified with the complex plane $\mathbb{C}$. Writing $\omega_0 = d\bar{d}\Phi_0$ in $\mathbb{C}$ (where the condition $\Phi_0$ has logarithmic growth, since $\int \omega_0 = 1$), the function $\phi_{\lambda}$ may be identified with the subharmonic function $\Phi_{\lambda} := \Phi_0 + \phi_{\lambda}$ with the property that $\Phi_{\lambda} = (1 - \lambda) \log |z|^2 + O(1)$ as $z \to \infty$. Accordingly, $X^{(\lambda)}$ may, for $\lambda > 0$, be identified with a decreasing family of compact domains in $\mathbb{C}$.

5.2. **A canonical regularization of the Hele-Shaw flow using the Kähler-Ricci flows.** To make the link to the present setting of Kähler-Ricci flows we set

$$\theta = \omega_0 - \delta_p,$$

where $\delta_p$ denotes the Dirac measure at $p$, which defines a trivial cohomology class (this is thus a singular version of the setting in Section 3.2). The corresponding Kähler-Ricci flows will be defined as follows: first fixing a Kähler form $\omega$ on $X$ we set

$$\theta_\beta := \theta + \frac{1}{\beta} \text{Ric}_\omega,$$
for a fixed Kähler form $\omega$, i.e. by imposing the equation \ref{eq:2.6}. Moreover, we will use $\omega_0$ as the initial data in the corresponding Kähler-Ricci flows. We then get the following theorem saying that the corresponding Kähler-Ricci flows concentrate, as $\beta \to \infty$, precisely on the complement $X^{(\lambda)}$ of $\Omega^{(\lambda)}$ (i.e. on the “water domain”) up to a time reparametrization:

**Theorem 5.2.** Consider the non-normalized Kähler-Ricci flow $\omega^\beta(t)$ with twisting current $\theta_\beta$ as above and initial condition $\omega_0$. Then

\begin{equation}
\lim_{\beta \to \infty} \omega^{(\beta)}(t) = 1_{X - \Omega^{(\lambda)}(t + 1)} \omega_0,
\end{equation}

weakly on $X$, where $\Omega^{\lambda}$ is the weak Hele-Shaw flow corresponding to $(p, \omega_0)$ and $\lambda(t) = t/(t + 1)$. Moreover,

$$\sup_X \frac{\omega^{(\beta)}(t)}{\omega} \leq (t + 1) \sup_X \frac{\omega_0}{\omega}.$$ 

**Remark 5.3.** If one instead let $\omega^\beta(t)$ denote the corresponding normalized Kähler-Ricci flow, which has total area $e^{-t}(= 1 - \lambda)$, then the corresponding limiting measure is given by $1_{X - \Omega^{(\lambda)}(t + 1)} \omega_0$ and the last estimate above holds without the factor $(t + 1)$. Moreover, in the canonical case, where $\omega$ is taken as $\omega_0$, setting $\eta_t := \omega_0 - \omega^{(\beta)}(t)$ then yields a family of semi-positive forms of increasing area $1 - e^{-t}$ concentrating on the “oil-domains” $\Omega_t$.

To prove the previous theorem we first need to make the link between the envelopes \ref{eq:5.1} and the ones appearing in our setting. To this end we introduce, as before, the potential $f$ of $\theta$ (wrt the reference semi-positive form $\chi = 0$ in $\theta$), satisfying

$$\theta = dd^c f,$$

which defines a lsc function $f : X \to [0, \infty)$ which is smooth on $X - \{p\}$ and such that $-f$ has a logarithmic singularity of order one at $p$.

**Lemma 5.4.** The following holds

$$\phi_\lambda := (1 - \lambda) P_{\omega_0} \left( \frac{\lambda}{1 - \lambda} f \right) - \lambda f,$$

Equivalently, setting $t = \lambda/(1 - \lambda)$ (i.e. $\lambda := t/(t + 1)$) gives

$$\Omega^{(\lambda)} := \{ P_{\omega_0}(tf) < tf \} := \Omega_t.$$ 

**Proof.** By a simple scaling argument it will be enough to prove that

$$\phi_\lambda = P_{\omega_0(1-\lambda)}(\lambda f) - \lambda f.$$

But the latter identity follows immediately from the fact that a given function $\phi \in PSH(X, \omega_0)$ has a logarithmic pole of order at least $\lambda$ at a point $p$, i.e. it satisfies

$$\phi + \lambda f \leq C$$

on $X$ iff the $\omega_0(1-\lambda)$-psh function $\phi + \lambda f$ on $X - \{p\}$ extends to a unique $\omega_0(1-\lambda)$-psh function on all of $X$ (as follows from the basic local fact that a psh function has a unique psh extension over an analytic subvariety, or more generally over a pluripolar subset). \hfill $\square$
Finally, we need to extend Theorem 2.2 to the present setting. To this end we first recall that, by [68, Theorem 3.2], there is, for $\beta$ fixed, a notion of weak Kähler-Ricci flows on $X$ which applies to any twisting current $\theta$ which is smooth away from a (suitable) divisor $D$ in $X$. In particular, the result applies to any current $\theta$ of the form

$$\theta = \theta_0 - [E],$$

where $\theta_0$ is smooth and $[E]$ denotes the current of integration along an effective divisor, i.e.

$$D = -E := -\sum c_i E_i$$

for $c_i > 0$ and $E_i$ are irreducible hypersurfaces in $X$. The result in [68, Theorem 3.2] yields a unique flow $\omega(\beta)(t)$ of currents in $[\omega_0 + t\theta]$ which are smooth on $X - D$ and such that the corresponding Kähler potentials are in $L^\infty(X)$ (as shown in [36, Section 4.2] this flow coincides with the unique viscosity solution constructed in [36, Section 4.2]).

**Theorem 5.5.** Let $\theta$ be a current of the form $\theta = \theta_0 - [E]$, with $\theta_0$ smooth and $E$ an effective divisor. Then the conclusion in Theorem 2.2 still applies and the constant $C$ only depends on upper bounds on $\theta_0$ (and the oscillation of its potential) and on the divisor $E$. Moreover, the sharp bounds in Theorem 2.3 still hold with $\theta$ replaced by $\theta_0$.

**Proof.** We recall that the weak KRF defined in [68, Theorem 3.2] is constructed by approximating $\theta$ with a suitable sequence $\theta_\epsilon$ of smooth forms. In the present setting this can be done so that $\theta_\epsilon \leq C \omega$ and $\theta_\epsilon$ converges to $\theta$ in $C^\infty_{loc}(X - E)$. Indeed, decomposing $f = f_0 + f_E$ in terms of potentials for $\theta_0$ and $-[E]$, respectively, we have that up to a smooth function $f$ can be written as $-\log \|s_E\|^2$, where $s_E$ is a holomorphic section of the line bundle $O(E)$ cutting out $E$ and $\|\cdot\|$ is a fixed smooth Hermitian metric on $O(E)$. Then the form $\theta_\epsilon$ is simply obtained by replacing $-\log \|s_E\|^2$ with $\log(\|s_E\|^2 + \epsilon)$. The proof of the theorem then follows immediately from Theorem 2.2 applied to $\theta_\epsilon$ by noting that that $P(f) \leq \sup_X f_0 + P(f_E)$, where the second term thus only depends on the divisor $E$, as desired (and is finite, by Lemma 2.12). 

**Example 5.6.** Coming back to the classical setting when $E$ is the point $p$ and $X - \{p\} = \mathbb{C}$ considered in the previous example, the density $\rho(\beta)(t)$ wrt Lebesgue measure on $\mathbb{C}$ of the Kähler form $\omega(\beta)(t)$ on $X - \{p\}$ is a solution of the following logarithmic diffusion equation for the smooth and strictly positive probability densities $\rho(t)$ on $\mathbb{C}$

$$\frac{\partial \rho(t)}{\partial t} = \frac{1}{\pi \beta} \partial^2 \partial_\bar{z} \rho(t) + \rho_0 + O(\frac{1}{\beta}), \quad \rho(0) = \rho_0,$$

where the last term is equal to $\frac{1}{\beta} \Delta \log \rho_0(t)$ (but it could be removed at the expense of slightly worse estimates in $t$ and $\beta$). The equivalence between Ricci flow on Riemann surfaces and logarithmic diffusion is well-known [74], but as far as we know the limit $\beta \to \infty$ has not been investigated before.
5.3. Monge-Ampère growth. There is also a natural higher dimensional generalization of the Hele-Shaw flow/Laplacian growth on a compact Kähler manifold $(X, \omega_0)$ where the higher dimensional viscous “fluid” is injected along a given effective divisor $E$ on $X$. Indeed, one simply defines $\phi_\lambda$ as before, but imposing a singularity of order $\lambda$ along $E$ (i.e. $z - p$ is in formula (5.1) replaced by a local defining equation for $E$). Then one obtains a sequence of increasing domains $\Omega_\lambda$ as before for which the name Monge-Ampère growth was proposed in [2]. The terminology is motivated by the fact that $\Omega_\lambda$ can be characterized as the solution of a free boundary problem for the complex Monge-Ampère operator on $(X, \omega_0)$ with singular obstacle $\lambda f$ (see Remark 2.9), where $f$ is defined by

$$\theta = \omega_0 - |E|, \quad \theta = dd^c f,$$

as before. By the recent results in [53], for $\lambda$ sufficiently small, $\Omega_\lambda$ is diffeomorphic to a ball (and admits a regular foliation, transversal to $E$, by holomorphic discs along which $\phi_\lambda$ is $\omega_0$–harmonic).

Now, by Theorem 5.5, the volume forms $\omega^n(t)$ of the Kähler-Ricci flows with twisting form $\theta$ as above concentrate on $X^{(\lambda)}(\cdot := X - \Omega^{(\lambda)(t)})$

$$\lim_{\beta \to \infty} \omega^n_\beta(t) = \frac{1}{f_X^{(\lambda)}} \frac{\omega^n_0}{\omega^n_0}$$

with uniform upper bounds on the normalized Kähler forms $\omega(t)/(t + 1)$ on $X - E$, as before (in this setting $\int_X \omega^n_0 = [\omega_0 - \lambda(t(E)]^n$).

**Example 5.7.** In the case when $X = \mathbb{P}^n$ equipped with a Kähler form $\omega_0$ of unit volume and $E$ is the hyperplane at infinity the corresponding sets $X(t)$ yield, for $t > 0$, a decreasing family of compact domains in $\mathbb{C}^n$ of volume $1/(t + 1)^n$.

**Remark 5.8.** As shown in [53] performing a Legendre transform of $\phi_\lambda$ with respect to $\lambda$ produces a weak geodesic ray $\hat{\phi}_{\tau}$ in the space of Kähler metrics (compare Remark 4.0). Moreover, topology change in the corresponding Hele-Shaw flow $\Omega^{(\lambda)}$ corresponds (in a certain sense) to singularities of the geodesic $\hat{\phi}_{\tau}$ [56, 57]. In a nutshell, this stems from the the fact (shown in [53]) that $\Omega^{(\lambda)} = \{ h < \lambda \}$ where $h(x) := \frac{d\phi_\lambda}{dt}|_{t=0^+}$.

6. The case of twisting currents with merely continuous potentials

Without loss of generality we may and will in this section, assume that $\varphi_0 = 0$. As will be next explained the weak convergence in Theorem 1.4 can be extended to any twisting form (or rather current) with continuous potentials.

To illustrate this we start with the case $n = 1$ and assume that $\frac{1}{\pi} C_1(K_X) + [\theta_\beta]$ is trivial, i.e. that the non-normalized KRF preserves the initial cohomology class. To simplify the notation we will drop the subscript $\beta$ in the notation $f_\beta$ for the corresponding twisting potential.

**Proposition 6.1.** Assume that $n = 1$ and $f$ is Hölder continuous. Then there is a unique solution $\varphi^{(\beta)}(t)$ to the corresponding non-normalized KRF which is in $C^{2, \alpha}(X)$ for some $\alpha > 0$.

**Proof.** In the following $\beta$ will be fixed and we will not pay attention to the dependence on $\beta$. First assume that $f$ is smooth. Differentiating the non-normalized KRF wrt $t$ reveals that $d\varphi^{(\beta)}(t)/dt$ evolves by the heat equation for the metric
\( \omega_\beta(t) \) and hence, by the parabolic maximum principle, \( |d\varphi^{(\beta)}(t)/dt| \leq C \), where the constant only depends on \( \sup_X |f| \). The defining equation for the KRF then gives that \( C^{\alpha-1} \leq \omega_\beta(t) \leq C' \) for a positive constant \( C' \) only depending on \( \sup_X |f| \). But then applying the parabolic Krylov-Safonov Hölder estimate to the heat equation w.r.t \( \omega_\beta(t) \) gives that there exists a Hölder exponent \( \alpha' \) such that \( \|d\varphi^{(\beta)}(t)/dt\|_{C^{\alpha'}} \leq C'' \). Using again the defining equation for \( \varphi^{(\beta)}(t) \) we deduce that, \( 1 + \Delta \omega \varphi^{(\beta)}(t) = e^{\beta g_\beta(t)} \), where the Hölder norm of \( g_\beta(t) \) is under control, for some Hölder exponent. But then the proof is concluded by invoking the classical Schauder estimates for the Laplacian \( \Delta _\omega \) and approximating \( f \) with smooth functions (note that the limit of the approximate solutions is unique, by the comparison principle).

Given a twisting potential \( f \) we denote by \( P_t^{(\beta)} f \) the solution of the corresponding KRF at time \( t \) and set \( P_t f := P_{\omega_0}(tf) \).

**Lemma 6.2.** The operator \( P_t^{(\beta)} \) is increasing, i.e. if \( f \leq g \), then \( P_t^{(\beta)} f \leq P_t^{(\beta)} g \). Moreover, \( P_t^{(\beta)}(f + c) = P_t^{(\beta)}(f) + ct \) for any \( c \in \mathbb{R} \) and hence

\[
(6.1) \quad \left\| P_t^{(\beta)} f - P_t^{(\beta)} g \right\|_{L^\infty(X)} \leq t \| f - g \|_{L^\infty(X)},
\]

and similarly for the operator \( P_t \).

**Proof.** The increasing property follows directly from the comparison principle and the scaling property from the very definitions of the flows. \( \square \)

**Theorem 6.3.** Let \( X \) be a Riemann surface endowed with the twisting current \( \theta = d\bar{d} f \), where \( f \) is Hölder continuous. Then the corresponding non-normalized KRFs \( \omega_\beta(t) \) defines a family of Hölder continuous Kähler metrics satisfying the weak convergence in Theorem 1.1, as \( \beta \to \infty \) (more precisely, the convergence holds in \( C^0(\bar{X}) \)) on the level of Kähler potentials.

**Proof.** In the following \( t \) will be fixed once and for all. Let \( f_\epsilon \) be a family of smooth functions such that \( \| f_\epsilon - f \|_\infty \leq \epsilon \). By the previous lemma

\[
\left\| P_t^{(\beta)} f - P_t f \right\|_{L^\infty(X)} \leq \left\| P_t^{(\beta)} f_\epsilon - P_t f_\epsilon \right\|_{L^\infty(X)} + 2\epsilon t.
\]

Hence, letting first \( \beta \to \infty \) (using Theorem 2.2) and then \( \epsilon \to 0 \) concludes the proof.

Of course, even if \( \omega_\beta(t) \) is bounded for a fixed \( \beta \) the limiting current \( \omega_\infty(t) \) will, in general, not be bounded unless \( f \) has a bounded Laplacian. The previous theorem also holds when \( f \) is assumed to be merely continuous, but then the corresponding evolution equations have to be interpreted in a generalized sense. More generally, when \( f \) is continuous and the dimension \( n \) of \( X \) is arbitrary the corresponding KRFs are well-defined in the sense of viscosity solutions and satisfy the comparison principle, by 30. Accordingly, the \( C^0 \)-convergence in the previous theorem still holds. However, even if \( f \) is Hölder continuous it does not seem to follow, in general, from existing regularity theory that \( \omega_\beta(t) \) is even bounded, for \( \beta \) fixed. \( \square \)
6.1. An outlook on random twistings. Hölder continuous potentials \( f \) appear naturally when \( f \) is taken to be an appropriate random Gaussian function. For example, in the setting described in Section 4.2, when \( n = 1 \) and \( X = \mathbb{R}/\mathbb{Z} + i\mathbb{R}/\mathbb{Z} \) and the potential \( f \) is assumed invariant along the imaginary direction, we can identify the potential \( f \) with a \( 1 \)-periodic function \( f(x) \) on \( \mathbb{R} \) and expand \( f(x) \) in a Fourier series:

\[
f(x) = \sum_{k \in \mathbb{Z}} A_k \cos(2\pi kx) + B_k \sin(2\pi kx).
\]

Taking the coefficients \( A_k \) and \( B_k \) to be independent Gaussian random numbers with mean zero and variance proportional to \( k^{-3-2h} \), for a given number \( h \in [-1, 1] \), it is well-known that \( f \) is almost surely in the Hölder class \( C^{1,h} \). Indeed, the derivative \( f' \) is a Brownian fractional bridge, whose sample paths are well-known to be almost surely in \( C^h \) (recall that a Brownian bridge is defined as a Brownian motion \( B \) conditioned by \( B(0) = B(1) \) and similarly in the fractional case, with \( h = 1/2 \) corresponding to ordinary Brownian motion). The corresponding limiting convex envelopes \( \phi_t(x) \) have been studied extensively in the mathematical physics literature in the setting of Burger’s equation and the adhesion model, where \( f' \) represents the random initial velocity function (compare Section 4). According to a conjecture in [62], for any fixed positive time \( t \), the support \( X_t \) of the distribution second derivative of the corresponding random function \( \phi_t(x) \) on \( \mathbb{R} \) is almost surely of Hausdorff dimension \( h \) when \( h \in [0, 1] \) and 0 when \( h \in [-1, 0] \) (which, when \( h = 1 \) is consistent with the uniform bound in Theorem 1.1 and formula 2.12 which, in this real setting, holds as long as \( f \in C^{1,1} \)). See [46] for the case when \( h = -1/2 \) and [65] for a proof of the conjecture in the case \( h = 1/2 \) in a non-periodic setting.

In view of the connections to the Kähler-Ricci flow and the Hele-Shaw flow exhibited in Sections 4 and 5 it would be interesting to extend this picture to any complex manifold, or at least to Riemann surfaces. For example, in the latter case one would, at least heuristically, get conformally invariant processes of random metrics \( \omega_\beta(t) \) by taking \( f \) to be a Gaussian free field on \( X \). Heuristically, this means that \( f \) is taken as random function in the corresponding Dirichlet Hilbert space \( H^1(X)/\mathbb{R} \). However, the situation is complicated by the fact that, almost surely, \( f \) only exists as a distribution in a certain Banach completion of \( H^1(X)/\mathbb{R} \). [64]. On the other hand the formal random measure appearing in the static version of the non-normalized KRF, i.e. in the Laplace equation

\[
\omega_\beta + dd^c \varphi_\beta(t) = e^{-\beta f} \omega_0
\]

appears as the Liouville measure of quantum gravity and has been rigorously defined, for \( \beta \in [0, 2] \) in [34] using a regularization procedure. But as far as we know the corresponding stochastic parabolic problem has not been investigated.

7. The gradient flow picture (an outlook)

In this section we introduce a complementary point of view on the convergence result in Theorem 1.1 which in particular leads to a gradient flow type realization of the limiting flows. We also indicate the relations to stochastic interacting particle system and the thermodynamical formalism introduced in [31]. A more complete picture will appear in a separate publication.
Let $X$ be a compact complex manifold endowed with a Kähler class $T \in H^2(X, \mathbb{R})$ and denote by $\mathcal{K}(X, T)$ the space of all Kähler metrics $\omega$ in $T$. Up to a trivial scaling we may and will assume that $T^n = 1$. Fixing a reference Kähler metric $\omega_0$ in $T$ we will identify $\mathcal{K}(X, T)$ with the corresponding space $\mathcal{H}(X, \omega_0)/\mathbb{R}$ of Kähler potentials (modulo constants). Occasionally, it will also be convenient to identify a Kähler metric with its normalized volume form, using the Calabi-Yau map

$$\omega \mapsto \mu := \omega^n, \quad \mathcal{K}(X, T) \rightarrow \mathcal{P}(X)$$

(7.1)

which induces an isomorphism between the space $\mathcal{K}(X, T)$ of all Kähler metrics $\omega$ in $T$ and the subspace $\mathcal{P}^\infty(X)$ of all volume forms in the space $\mathcal{P}(X)$ of all probability measures on $X$ \[7\].

In this section we will focus on the case when the cohomology class is not moving under the corresponding normalized KRF (as in Section 3.1), which equivalently means that

$$T = \frac{1}{\beta} c_1(K_X) + [\theta_\beta].$$

(However, see Section 7.3 for the non-normalized setting). As before we denote by $f$ the potential of $\theta$:

$$\theta = \omega_0 + \bar{d}d^c f$$

Occasionally it will be convenient to pass between Kähler potentials $\varphi$ relative to $\omega_0$ and Kähler potentials $u$ relative to $\theta$ by setting

$$u := \varphi - f$$

ensuring that $\omega_\varphi = \theta_u$.

7.1. The twisted Kähler-Ricci flow as a gradient flow. We recall that the gradient flow of a smooth function $F$ on a Riemannian manifold $Y$ is the flow defined by

$$\frac{dy(t)}{dt} = -\nabla F(y(t)), \quad y(0) = y_0$$

where $\nabla$ denotes the gradient wrt the given Riemannian metric on $Y$. In our infinite dimensional setting we equip the space $\mathcal{K}(X, T)$ with the Riemannian metric defined as follows:

$$\langle u, u \rangle_\varphi := n \int_X du \wedge d^c u \wedge \omega_\varphi^{n-1},$$

(7.2)

where the tangent space of $\mathcal{K}(X, T)$ at $\varphi$ has been identified with $C^\infty(X)/\mathbb{R}$ in the usual way (i.e. using the standard affine structure). In other words, $\langle u, u \rangle_\varphi$ is the $L^2$-norm of the gradient of $u$ wrt the Kähler metric $\omega_\varphi$.

Next, we recall that the $\theta$–twisted (and $\beta$–normalized) version of Mabuchi’s K-energy functional on $\mathcal{H}(X, \omega_0)/\mathbb{R}$ is defined by specifying its differential, viewed as a measure valued operator:

$$-(\delta M^{(\beta)}_\theta)_\varphi = \left(\frac{1}{\beta} \text{Ric} \omega_\varphi - \theta\right) \wedge \omega_\varphi^{n-1} - C \omega^n,$$

where $C$ is the cohomological constant ensuring that rhs above integrates to zero.

Proposition 7.1. The gradient flow of the twisted K-energy $M^{(\beta)}_{\theta_\beta}$ on $\mathcal{K}(X, T)$ coincides with the normalized KRF with twisting form $\theta_\beta$. 
Proof. It will be convenient to use the “thermodynamical formalism” \cite{9} in order to identify $M_{\beta}(\varphi)$ with a free energy type functional $F_{\beta}$ on $\mathcal{P}(X)$:

$$M_{\beta}(\varphi) = F_{\beta}(\mu), \quad \mu = \omega^n_{\varphi}$$

where

$$F_{\beta}(\mu) = E_{\theta}(\mu) + \frac{1}{\beta} H_{\mu_0}(\mu),$$

and where $E_{\theta}(\mu)$ is the pluricomplex energy of $\mu$ relative to $\theta$ and $H_{\mu_0}(\mu)$ is the entropy of $\mu$ relative to a certain fixed normalized volume form $\mu_0$ determined by $\theta$ and $\omega_0$ (see \cite{9}).

Next, we make the following general observation: if $M(\varphi) = F(\mu)$ then the following relation holds

$$\nabla M|_{\varphi} = -\left(\delta F\right)_{|\mu},$$

between the gradient $\nabla M|_{\varphi} \in C^\infty(X)$ of $M$ at $\varphi$ wrt the Dirichlet metric and the differential $\delta F_{|\mu}$ at $\mu$, identified with a function on $X$ (using the standard integration pairing between functions and measures). Indeed, if $\mu(t)$ is a curve such that $\mu(0) = \mu$ then, by the very definition of $\delta F_{|\mu}$, we have

$$\int_X \delta F_{|\mu} d\mu(t) dt \bigg|_{t=0} := \frac{dF(\mu(t))}{dt} \bigg|_{t=0}. $$

In particular, if $\mu(t) = \omega^n_{\varphi(t)}$ then setting $u := \delta F_{|\mu}$ gives

$$\frac{dM(\varphi(t))}{dt} \bigg|_{t=0} = \frac{dF(\mu(t))}{dt} \bigg|_{t=0} = n \int_X u dd^c \left(\frac{d\varphi(t)}{dt}\right) \wedge \omega^{n-1}_{\varphi(0)} =$$

$$= -n \int_X du \wedge dd^c \left(\frac{d\varphi(t)}{dt}\right) \bigg|_{t=0} \wedge \omega^{n-1}_{\varphi(0)},$$

which, by definition, equals $-\left(\varphi, \frac{d\varphi(t)}{dt}\right)_{|t=0, \varphi}$, proving formula (7.4).

Finally, as shown in \cite{9} we have

$$(\delta E_{\theta})_{|\mu} = -\left(\varphi - f\right), \quad (\delta H_{\mu_0})_{|\mu} = \log \left(\frac{\mu}{\mu_0}\right)$$

and hence the gradient flow of $M_{\beta}(\varphi)$ is given by

$$\frac{d\varphi(t)}{dt} = \frac{1}{\beta} \log \left(\frac{\omega^n_{\varphi(t)}}{\mu_0}\right) - \left(\varphi(t) - f\right),$$

as desired (modulo constants). \hfill \Box

Remark 7.2. The metric (7.2) seems to first have appeared in \cite{22}, where it is attributed to Calabi and called Calabi’s gradient metric (not to be confused with another metric usually referred to as the Calabi metric obtained by replacing the gradient of $u$ by the Laplacian of $u$). The metric (7.2) was further studied in \cite{26, 27} where it is called the Dirichlet metric. See also \cite{35} where the metric (7.2) appears from a symplecto-geometric point of view. In Section 7.4 below we will give a new interpretation of the metric (7.2) motivated by probabilistic considerations. The relation between the KRF and gradient flows wrt the Dirichlet metric first appeared in \cite{29} (in the non-twisted setting).

\footnote{From a thermodynamical point of view $F_{\beta}$ is the Gibbs free energy at inverse temperature $\beta$ for a system with internal energy $E$.}
7.2. The zero-temperature limit $\beta \to \infty$. We denote by $\varphi_t$ the following curve of functions in $PSH(X, \omega_0)$:

$$\varphi(t) := P_{\omega_0} (e^{-t} \varphi_0 + (1 - e^{-t})f)$$

and by $\mu(t)$ the corresponding curve of probability measures on $X$. Using that the latter curve arises as the large $\beta$–limit of the corresponding normalized Kähler-Ricci flows (by Theorem 2.2) Proposition 7.1 implies that $E_\theta(\mu(t))$ is decreasing. But, in fact, a direct argument reveals that it even strictly decreasing away from a minimizer:

**Proposition 7.3.** The pluricomplex energy $E_\theta(\mu(t))$ is decreasing wrt $t$, and strictly decreasing unless $\mu(t)$ reaches a minimizer. Moreover, $E_\theta(\mu(t))$ converges to the infimum of $E_\theta$ over $P(X)$ and $\mu(t)$ converges to the corresponding minimizer.

**Proof.** First of all observe that, by Prop 7.1 $M^{(\beta)}$ is decreasing along $\varphi^{(\beta)}(t)$ for $\beta$ fixed. But, by the uniform bound on the Laplacians we have that $H(\mu^{(\beta)}(t)) \leq C$ and $E_\theta(\mu_t) \to E_\theta(\mu)$ as $\beta \to \infty$ and hence $E_\theta(\mu(t))$ is also decreasing, as desired. Alternatively a direct proof using envelopes can be given as follows, which also includes strict monotonicity. Recall that $\mu(t) = (\omega_0 + dd^c \varphi_t)^n$ and $E_\theta$ (acting on the level of potentials) is defined by

$$E_\theta(\varphi) = E(\varphi) - \int_X \varphi \text{AM}(\varphi) + \int_X f \text{MA}(\varphi),$$

where $E = \text{AM}$ is the Aubin-Mabuchi energy. We denote $\varphi_t = P_{\omega_0} (e^{-t} \varphi_0 + (1 - e^{-t})f)$ and we assume for simplicity that $\varphi_0 = 0$. Fix $t \geq 0, s > 0$. By basic properties of the Aubin-Mabuchi functional we have

$$\text{AM}(\varphi) - \text{AM}(\psi) \leq \int_X (\varphi - \psi) \text{MA}(\psi).$$

We will use the $I$ functional in [12]: $I(u, v) = \int_X (u - v)(\text{MA}(v) - \text{MA}(u)) \geq 0$. Using the formula of $E_\theta$ and the inequality above we can write

$$E_\theta(\varphi_{t+s}) - E_\theta(\varphi_t) \leq I(\varphi_{t+s}, \varphi_t) + \int_X (f - \varphi_t)(\text{MA}(\varphi_{t+s} - \text{MA}(\varphi_t)).$$

We claim that

$$\int_X (f - \varphi_t)(\text{MA}_\omega(\varphi_{t+s} - \text{MA}_\omega(\varphi_t)) \leq -\lambda I(\varphi_{t+s}, \varphi_t),$$

where $\lambda = \frac{\omega}{\varphi_t}$. Indeed, let $\Omega_t := \{ \varphi_t < (1 - e^{-t})f \}$ be the non-coincidence set. By the monotonicity result we know that $\Omega_t \subset \Omega_{t+s}$. It suffices to prove that

$$\int_X (f - \lambda \varphi_{t+s} + (\lambda - 1)\varphi_t)(\text{MA}_\omega(\varphi_{t+s} - \text{MA}_\omega(\varphi_t)) \leq 0.$$

The integrand is non-negative thanks to Proposition 2.18 and it vanishes out side $\Omega_{t+s}$. As $\text{MA}_\omega(\varphi_{t+s})$ vanishes in $\Omega_{t+s}$, the inequality (7.9) follows. Now (7.5) and (7.6) give that

$$E_\theta(\varphi_{t+s}) - E_\theta(\varphi_t) \leq \frac{1}{e^s - 1} I(\varphi_{t+s}, \varphi_t).$$

Finally, if $E_\theta(\varphi_{t+s}) = E_\theta(\varphi_t)$ then we must have $\varphi_{t+s} = \varphi_t$, as $I$ is non-degenerate. The next lemma shows that the flow is stationary from $t$. In fact, if $\{f = 0\}$ has Lebesgue measure zero then $E_\theta$ is strictly decreasing. □
Lemma 7.4. Denote by $\varphi_t = P_\omega(e^{-t}\varphi_0 + (1-e^{-t})f)$. If $\varphi_t = \varphi_s$ for some $0 \leq t < s$ then $\varphi_{t+h} = \varphi_t$ for all $h \geq 0$.

Proof. Again, for simplicity we assume that $\varphi_0 = 0$. As $\text{MA}(\varphi_t) = \text{MA}(\varphi_{t+s})$, the measure is concentrated on the set $\{\varphi_t = (1 - e^{-t})f = (1 - e^{t-s})f\}$, which equals $\{\varphi_t = f = 0\}$. Now, as $\text{MA}(\varphi_t)(\varphi_t < 0) = 0$ the domination principle gives that $\varphi_t \geq 0$. It follows that $f \geq 0$, and hence $\varphi_t$ is increasing in $t$. But again we have that $\text{MA}(\varphi_t)$ vanishes on $\{\varphi_t < \varphi_{t+h}\}$. To see this we note that the measure is supported on the coincidence set and on this set, by the monotonicity property (Proposition 2.18), $\varphi_{t+h} \leq e^{-s}\varphi_t + (1 - e^{-s})f = 0$. Thus the domination principle again yields $\varphi_t \geq \varphi_{t+h}$, hence equality holds.

In the light of the previous results one would expect that the curve $\mu_t$ arises as a gradient flow of $E_\theta$ (in the sense of metric spaces [2]) when $\mathcal{P}(X)$ is equipped with the metric induced by the metric $\mathcal{H}$ (under the Calabi-Yau isomorphism). However, in order to make this precise one has to deal with several technical problems related to the geometry of the metric completion of the space $\mathcal{K}(X,T)$ equipped with the Dirichlet metric above, that we leave for the future. Here we just formulate a precise result in the case when $n = 1$ where the metric $\mathcal{H}$ coincides with the classical Dirichlet norm on the Riemann surface $X$. To this end we denote by $H^1(X)/\mathbb{R}$ the Sobolev quotient space obtained by completing the Dirichlet norm on $C^\infty(X)/\mathbb{R}$. Since $H^1(X)/\mathbb{R}$ is a Hilbert space there is a classical notion of gradient flows of lsc convex functionals on $H^1(X)/\mathbb{R}$ which we briefly recall. Given a lsc convex function $F$ on a Hilbert space $H$: a curve $v(t)$, which is absolutely continuous as a map from $]0, \infty[\to H$, is said to be the gradient flow of $F$ emanating from $v_0$ if $v(t) \to v_0$ as $t \to 0$ and for almost any $t$

$$\frac{dv(t)}{dt} \in \partial H|_{v(t)}, \tag{7.7}$$

where the rhs above denotes the subgradient of $H$ at $v(t)$. There are also other equivalent definitions. For example, the differential inclusion may be replaced by the following Evolutionary Variational Inequalities (EVI): for any given $w \in H$

$$\frac{d}{dt} \frac{1}{2} \|v(t) - w\|^2 + F(v(t)) - F(w) \leq 0.\tag{7.7}$$

In turn, this is equivalent to $v(t)$ arising as a limit of a Minimizing Movement, i.e. a variational form of the backward Euler discretization scheme. The virtue of the latter two characterizations is that they can be formulated when the Hilbert space $H$ is replaced by a general metric space (in particular, the corresponding weak gradient flows always exist when $F$ is a lsc convex function on a complete metric space with non-positive sectional curvature; see [2] and references therein).

Theorem 7.5. Let $X$ be a Riemann surface endowed with a smooth two-form $\theta$. Equip the Sobolev space $H^1(X)/\mathbb{R}$ with the classical Dirichlet metric and consider

\footnote{Unfortunately, when $n > 1$, the corresponding metric geometry appears to be more complicated - from the point of view of gradient flows - than the case of the Mabuchi-Semmes-Donaldson metric on $\mathcal{K}(X,T)$ whose metric completion has non-positive sectional curvature and where the corresponding gradient flow of the K-energy functional yields a weak version of the Calabi flow [70] [77].}
the following lower semi-convex functional $F$ on $H^1(X)/\mathbb{R}$:
$$F(u) := \frac{1}{2} \int_X du \wedge d^c u$$
for $u$ in the convex subset $C_\theta$ of $H^1(X)/\mathbb{R}$ defined by the condition $dd^c u + \theta \geq 0$ and let $F = \infty$ on the complement of $C_\theta$. Denote by $u(t)$ the solution of the gradient flow of $F$ emanating from a given element $u_0 \in C_\theta \cap C^\infty(X)$. Then $dd^c u(t) + \theta$ coincides with the curve $\mu(t)$ of probability measures on $X$ defined by the envelope construction above.

Proof. Fix a normalized volume form $dV$ on $X$ and let $F_\beta$ be the corresponding free energy functional (formula (7.3)) on $H^1(X)/\mathbb{R} : F_\beta(u) = F(u) + H_{AV}(dd^c u(t) + \theta)/\beta$. This is a convex lsc functional and hence its gradient flow $u(\beta)(t)$ emanating from the given element $u_0$ is well-defined. It then follows from well-known stability results that $u(\beta)(t) \to u(t)$ in $H^1(X)/\mathbb{R}$. But, by the uniqueness of weak gradient flows in Hilbert spaces, $dd^c u_\beta(t) + \theta$ coincides with the curve of Kähler forms defined by the KRF (compare Proposition 7.1) and hence the proof is concluded by invoking Theorem 2.2. Alternatively, a direct proof can be given as follows: by Proposition 7.4 and the convexity of $F_\beta$ the curve $u(\beta)(t)$ satisfies the Evolutionary Variational Inequalities wrt $F_\beta$. Then, passing to the limit and using Theorem 2.2, reveals that $u(t)$ also satisfies the latter inequalities wrt $F$, which, as recalled above, is equivalent to $u(t)$ being the gradient flow of $F$.

\[ \square \]

Note that one virtue of the EVI formulation of the gradient flow in the previous theorem (used in the end of the proof) is that the gradient flow is intrinsically defined on the convex subset $C_\theta$ (which is a Euclidean complete metric space, but not a Hilbert space).

Remark 7.6. The solution of the gradient flow in the previous theorem can be given by the following description purely in terms of the geometry of the Hilbert space $H := H^1(X)/\mathbb{R}$. Let $F$ be half the squared Hilbert space norm $h^2/2$ on $H$ restricted to a given compact convex subset $C$ (which does not contain the origin) and then extended by $\infty$ to all of $H-C$, i.e. $F = h^2/2 + \chi_C$, where $\chi_C$ is the indicator function of $C$. Given an initial point in $C$ the gradient flow of $h^2/2$ is an affine curve $v(t)$ which leaves the space $C$ after a finite time. However, replacing $v(t)$ with $P(v(t))$, where $P(v)$ is the projection of $v$ onto $C$, gives the weak gradient flow of $F$, which does stay in $C$. To be more precise: $P(v)$ is the point in $C$ which is closest to $v$ wrt the metric defined by the Hilbert norm (which is uniquely determined by standard Hilbert space theory).

7.3. The non-normalized KRF as a gradient flow. Next we briefly consider the setting when the cohomology class $T$ is preserved by the non-normalized KRF, i.e. $\frac{1}{2}c_1(K_X) + [\theta] \in H^{1,1}(X, \mathbb{R})$ is trivial (as in Section 3.2). Then $\theta = dd^c f$ for a smooth function $f$ on $X$. Introducing the functional
$$F(\mu) := \int f \mu$$
on $P(X)$ whose differential at $\mu$ may be identified with the function $f$, all the results above still apply with $E_\theta(\mu)$ replaced by $F(\mu)$. In particular, as we next explain this leads to gradient flow representations of the Hele-Shaw flow, as well as Hamilton-Jacobi equations, which appear to be new.
7.3.1. The Hele-Shaw flow. In particular, we have the following result where $H^{-1}(X)$ denotes the Hilbert space of all signed measures on $X$ with finite (logarithmic) energy equipped with the Dirichlet norm.

**Theorem 7.7.** Let $(X, \omega)$ be a Riemann surface with a normalized area form $\omega$ and $p$ a given point on $X$. Denote by $\Omega(t)$ the corresponding weak Hele-Shaw flow of increasing domains in $X$, injected at $p$. Then the corresponding family

$$
\mu(t) := 1_{X-\Omega(t)}(t+1)\omega
$$

of probability measures on $X$ is the gradient flow of the lsc convex functional $\tilde{F}$ on the Hilbert space $H^{-1}(X)$ obtained by extending $F$ by infinity from $\mathcal{P}(X) \cap H^{-1}(X)$. In particular, $\mathcal{F}(\mu(t))$ is strictly decreasing along the flow.

**Proof.** This is proved precisely in Theorem 7.9. Even if $f$ is not smooth in this setting, the general results about gradient flows in Hilbert spaces still apply as $f$ is lsc and the corresponding functional is convex. \hfill \Box

7.3.2. Hamilton-Jacobi equations. Next we turn to the setting of Hamilton-Jacobi equations, using the notation in Section 4. We will denote by $C^+_\Lambda$ the subspace of all smooth and strictly convex functions $\psi$ in $C\Lambda$ and by $\text{Ent}(\mu|\nu)$ the entropy of a measure $\mu$ relative to another measure $\nu$. We equip the space $C^+_\Lambda$ with the Riemannian metric induced from the Dirichlet type metric.

**Proposition 7.8.** The perturbed Hamilton-Jacobi equation

$$
F_\beta(\psi) := \frac{1}{\beta} \text{Ent}(dy|MA(\psi)) + \mathcal{E}_H(\psi), \quad \mathcal{E}_H(\psi) := \int_{R^n/\Lambda} H(\nabla\psi(y))dy,
$$

where $\nabla\psi$ denotes the $L^\infty$–Brenier gradient map.

**Proof.** As shown in Section 4.2 the solution $\phi_\beta(t)$ is the Legendre transform of the corresponding twisted Kähler-Ricci flow $\phi_\beta(t)$ in $C^+_\Lambda$. Moreover, using formula 4.7 gives

$$
\text{Ent}(MA(\phi)|dx) = \text{Ent}(dy|MA(\psi)), \quad \int_{R^n/\Lambda} MA(\phi)H = \int_{R^n/\Lambda} H(\nabla\psi(y))dy
$$

and hence the result follows from the fact that the twisted KRF is the gradient flow wrt the Dirichlet type metric of the functional $\text{Ent}(MA(\phi)|dx)/\beta + F(MA(\phi))$. \hfill \Box

**Corollary 7.9.** The functional $\mathcal{E}_H$ is decreasing along the viscosity solution of the Hamilton-Jacobi equation with Hamiltonian $H$ and initial data in $C\Lambda$.

Specializing to the one-dimensional case we arrive at the following

**Theorem 7.10.** Denote by $\mu(t) := \partial^2 \psi_t$ the curve in the space of probability measures on $S^1$ defined by the distributional second derivative of the unique viscosity solution $\psi_t$ of the Hamilton-Jacobi equation with Hamiltonian $H$. Then $\mu(t)$ is the gradient flow of the functional corresponding to $\mathcal{E}_H$ on the space $\mathcal{P}(S^1)$ equipped with the Wasserstein $L^2$–metric. In particular, $\mathcal{E}_H$ is strictly decreasing at $\psi_{t_0}$ unless $\psi_{t_0}$ is a minimizer of $\mathcal{E}_H$ (or equivalently: $\mu(t_0)$ is supported in the set where $H$ attains its absolute minimum).
Proof: This is shown as in the proof of Theorem 7.3 using Prop 7.8 and the observation that the Wasserstein $L^2$–metric corresponds under the Legendre transform to the Dirichlet metric on the Legendre transform side. This is well-known in the case of $\mathbb{R}$ and the proof in the $S^1$–case can, for example, be obtained using the transformation properties of the Otto metric (the proof will appear elsewhere). □

7.4. Relations to the Otto metric and stochastic gradient flows. Given a Riemannian manifold $(X, g)$ the Otto metric [52] is defined on the space $P^\infty(X)$ of all volume forms $\mu$ in $P(X)$ as follows. First note that a vector field $V$ on $X$ induces a tangent vector on $P^\infty(X)$:

$$d\mu t |_{t=0} := \frac{d}{dt} |_{t=0} ((F_t^V)_* \mu),$$

where $F_t^V$ denotes the one-parameter group of diffeomorphisms of $X$ defined by the flow of $V$. Now the Otto metric may be defined by

$$\langle \frac{d\mu}{dt} |_{t=0}, \frac{d\mu}{dt} |_{t=0} \rangle_{\mu} := \inf_V \int_X g(V, V) \mu,$$

where the infimum runs over all vector fields $V$ satisfying the equation (7.8). In physical terms, considering a gas of particles on $X$ distributed according to the measure $\mu$, the norm above is the minimal kinetic energy needed to produce the rate of change $d\mu t dt$ of $\mu$;

As explained in [52] the Otto metric on $P^\infty(X)$ is, at least formally, the Riemannian metric underlying the Wasserstein $L^2$–metric $d_2$ on $P(X)$, induced by $g$, which may be expressed as follows on $P^\infty(X)$:

$$d_2(\mu, \nu) := \inf_S \int_X d_g(x, T(x))^2 \mu(x), \ S_* \mu = \nu$$

expressed in terms of the distance function $d_g$ on $X \times X$ determined by the given Riemannian metric $g$, i.e. $d_2(\mu, \nu)^2$ is the minimal cost to transport $\mu$ to $\nu$ (the general formula on $P(X)$ employs transport plans rather than transport maps $S$).

Now, one can envisage a generalization of the Otto metric where $g$ is allowed to depend on $\mu$. In particular, if $X$ is a Kähler manifold with a given Kähler class $T$ then we may simply take $g_\mu$ to be the unique Kähler metric in $T$ furnished by the Calabi-Yau isomorphism, i.e. the metric $g_\mu$ in $T$ with volume $\mu$.

Proposition 7.11. Let $X$ be a Kähler manifold endowed with a Kähler class $T$. Then the corresponding Otto type metric (obtained by replacing $g$ with $g_\mu$ in formula 7.9) coincides with the Dirichlet type metric defined by formula 7.2 above (up to the multiplicative constant n!).

Proof. First recall that, by Hodge theory, the infimum in formula 7.9 is attained precisely for $\nu$ of the form

$$V = \nabla v, \ v \in C^\infty(X),$$

7The argument in [52] uses that the Otto metric can be identified with the quotient metric on $DIFF(X)/SDIFF(X, dV)$ (defined wrt to the non-invariant $L^2$–metric on $DIFF(X)$ induced from $g$ under which the group of volume preserving diffeomorphisms $SDIFF(X, dV)$ acts from the right by isometries) under the submersion $S \mapsto S_* dV_g$. A different argument, motivated by numerical applications, is given in [4].
where $\nabla$ denotes the gradient wrt $g_\mu$ (where $v$ is uniquely determined mod $\mathbb{R}$). Moreover, writing $\mu = \rho dV_g$ the element $v \in C^\infty(X)/\mathbb{R}$ may be characterized as the unique solution to the following continuity equation

$$\frac{d\rho_t}{dt} \big|_{t=0} = -\nabla \cdot (\rho \nabla v).$$

In the Kähler setting above $\rho = 1$ and hence the previous equation is equivalent to

$$\frac{d\mu_t}{dt} \big|_{t=0} = -\frac{d\delta v \cdot \omega_{n-1}}{(n-1)!}.$$ 

Accordingly, writing $\mu_t = \omega^n_{u_t}$ for some curve $u_t$ in $\mathcal{H}$ reveals that

$$v = \frac{du_t}{dt} \big|_{t=0}$$

which concludes the proof.

A remarkable property of the Otto metric (defined wrt a fixed background metric $g$ on $X$) is that the gradient flow of the relative entropy $H_{dV_g}$ is precisely the heat (diffusion) equation. More generally, if $G$ is a functional on $P^\infty(X)$ then the corresponding gradient flow is given by

$$\frac{\partial \rho_t}{\partial t} = \nabla \cdot (\rho V_t), \quad V_t = \nabla(\delta G)_{(\rho_t dx)},$$

where $\delta G_{\mu_t}$ denotes, as before, the differential of $G$ at $\mu_t$ identified with a function on $X$. In particular, if $G$ is a free energy type functional of the form (7.3) then the corresponding gradient flow is the following drift diffusion equation (non-linear Fokker-Planck equation):

$$\frac{\partial \rho_t}{\partial t} = \frac{1}{\beta} \Delta \rho_t + \nabla \cdot (\rho_t V[\rho_t]),$$

where $V[\rho_t]$ is the vector field given by

$$V[\rho_t] = \nabla(\delta E)_{(\rho_t dx)}.$$

Such drift diffusion equations can often be realized as large $N$–limits of stochastic gradient flows on the $N$–particle space $X^N$ of the form

$$dx_i(t) = -\nabla x_i E^{(N)}(x_1, x_2, ..., x_N) dt + \sqrt{\frac{2}{\beta}} dB_i(t),$$

where $B_t$ denotes $N$ independent Brownian motions on the Riemann manifold $(X, g)$ and $E^{(N)}$ is a suitable symmetric “microscopic” version of $E$ (in statistical mechanical terms this expresses the non-equilibrium dynamics of $N$ diffusing particles on $(X, g)$, at inverse temperature $\beta$, interacting by the energy $E^{(N)}$). This is the starting point for the stochastic dynamics approach to the construction of Kähler-Einstein metrics introduced in [15]. In particular, this leads to a new dynamic construction of (twisted) Kähler-Einstein metrics [16]. However, one geometric drawback of this approach is that it requires the choice of a background metric on $X$ and hence the corresponding evolution equation is not canonical (even if its large $t$–limit is). This motivates using the generalized Otto type metric which amounts to coupling the back-ground metric $g_t$ at time $t$ to the measure $\mu_t$. As will be explained elsewhere the latter road leads to a new microscopic stochastic approach to the Kähler-Ricci flow where the individual particles $x_1, ..., x_N$ perform coupled
Brownian motions defined with respect to a changing metric which depends on the location of the whole configuration of particles.

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