GREEN FORMULA IN HALL ALGEBRAS AND CLUSTER ALGEBRAS

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The objective of the present paper is to give a survey of recent progress on applications of the approaches of Ringel-Hall type algebras to quantum groups and cluster algebras via various forms of Green’s formula. In this paper, three forms of Green’s formula are highlighted, (1) the original form of Green’s formula [Gre], (2) the degeneration form of Green’s formula [DXX] and (3) the projective form of Green’s formula [XX2] i.e. Green formula with a $C^*$-action. The original Green’s formula supplies the comultiplication structure on Ringel-Hall algebras. This compatibility theorem for multiplication and comultiplication on Ringel-Hall algebras deals with the symmetric relation between extensions and flags in the module category of a hereditary algebra. It provides the quantum group a Hopf algebra structure and the Drinfeld double in a global way (see [Gre], [X] and [Ka]). The second and third section contribute to these results. The degenerated Green formula can be viewed as the Green formula over the complex field. We found that it holds for any algebra given by quiver with relations, not only hereditary algebras. We write the formula in a geometric version of the original Green formula, although we know that it essentially agrees with the restriction functor given by Lusztig in [Lu1]. It is applied to provide the geometric realization of the comultiplication of universal enveloping algebras. Section 4 is concerned with these results. The projective version of Green formula has its independent interest. We give its expression and explain its meaning in Section 6. It is applied to prove the cluster multiplication theorem which extends the Caldero-Keller formula [CK1]. Section 7 is used to explain the proof in some details. There is a key ingredient contributing to the above application to cluster categories, i.e. 2-Calabi-Yau property for the cluster categories. In the last section, we extend the multiplication formula in [GLS] for preprojective algebras to any 2-Calabi-Yau algebras.

Since our main concern in this article is around various forms of Green formula and their applications, also due to lack of space and knowledge, we do not include many important topics related to cluster algebras and Hall algebras. For cluster algebras we refer to [FZ3] for an excellent survey and we refer to [DX2] and [Sc] for further study on Ringel-Hall algebras.

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The research was supported in part by NSF of China (No. 10631010) and by NKBRPC (No. 2006CB805905)

2000 Mathematics Subject Classification: 16G20, 16G70.
Key words and phrases: Ringel-Hall algebra, Green’s formula, Cluster category.
4. Green formula over complex fields with an application to the realization of universal enveloping algebras
5. Cluster algebras and cluster categories
6. Green formula under $\mathbb{C}^*$-action
7. Cluster multiplication formula
8. 2-Calabi-Yau and Cluster multiplication formula
References

1. Representations of quivers and varieties of modules

1.1. Let $k$ be a field. A *quiver* is a quadruple $Q = (I, Q_1, s, e)$ where $I$ and $Q_1$ are sets with $I$ non-empty, and $s, t : Q_1 \to I$ are maps such that $s^{-1}(i)$ and $e^{-1}(i)$ are finite for all $i \in I$. We call $I$ the set of *vertices* and $Q_1$ the set of *arrows* of $Q$. For an arrow $\alpha \in Q_1$ one calls $s(\alpha)$ the starting vertex and $e(\alpha)$ the end vertex of $\alpha$.

A *path* of length $t$ in $Q$ is a sequence $p = \alpha_1 \alpha_2 \cdots \alpha_t$ of arrows such that $s(\alpha_i) = e(\alpha_{i+1})$ for $1 \leq i \leq t - 1$. Set $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_1)$. Additionally, for each vertex $i \in I$ let $e_i$ be a path of length 0. By $kQ$ we denote the *path algebra* of $Q$, with basis the set of all paths in $Q$ and product given by concatenation. A *relation* for $Q$ is a linear combination $\sum_{i=1}^t \lambda_i p_i$ where $\lambda_i \in k$ and the $p_i$ are paths of length at least two in $Q$ with $s(p_i) = s(p_j)$ and $t(p_i) = t(p_j)$ for all $1 \leq i, j \leq t$. Thus, we can regard a relation as an element in $kQ$. An ideal $J$ in $kQ$ is *admissible* if it is generated by a set of relations for $Q$.

1.2. A map $d : I \to \mathbb{N}$ such that $I \setminus d^{-1}(0)$ is finite is called a *dimension vector* for $Q$. We also write $d_i$ instead of $d(i)$, and we often use the notation $d = (d_i)_{i \in I}$. By $\mathbb{N}^{(I)}$ we denote the semigroup of dimension vectors for $Q$.

A representation $(V, f)$ of the quiver $Q$ over $k$ is a set of vector spaces $\{V(i) \mid i \in I\}$ together with $k$-linear maps $f_\alpha : V(i) \to V(j)$ for each arrow $\alpha : i \to j$. For any dimension vector $d = \sum_i a_i i \in \mathbb{N}I$, we consider the affine space over $k$

$$E_d(Q) = \bigoplus_{\alpha \in Q_1} \text{Hom}_k(k^{a_{s(\alpha)}}, k^{a_{t(\alpha)}}).$$

For a representation $x = (x_\alpha)_{\alpha \in Q_1} \in E_d(Q)$ and a path $p = \alpha_1 \alpha_2 \cdots \alpha_t$ in $Q$ set

$$x_p = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_t}.$$ 

Then $x$ satisfies a relation $\sum_{i=1}^t \lambda_i p_i$ if $\sum_{i=1}^t \lambda_i x_{p_i} = 0$. If $R$ is a set of relations for $Q$, then let $E_d(Q, R)$ be the set of all representations $x \in E_d(Q)$ which satisfy all relations in $R$. This is a closed subvariety of $E_d(Q)$. Let $\Lambda$ be the algebra $kQ/J$, where $J$ is the admissible ideal generated by $R$. We often write $E_d(\Lambda)$ instead of $E_d(Q, R)$. Note that every element in $E_d(\Lambda)$ can be naturally interpreted as an $\Lambda$-module structure on the $I$-graded vector space $k^d : \oplus_{i \in I} k^{d_i}$, where $d = (d_i)_{i \in I}$. A *dimension vector* for $\Lambda$ is by definition the same as a dimension vector for $Q$, that is, an element of $\mathbb{N}^{(I)}$.

We consider the algebraic group:

$$G_d := G_d(Q) = \prod_{i \in I} GL(a_i, k),$$
1.3. In the following three subsections, we assume the base field $k = \mathbb{C}$. The intersection of an open subset and a close subset in $E_d(Q, R)$ is called a locally closed subset. A subset in $E_d(Q, R)$ is called constructible if and only if it is a disjoint union of finitely many locally closed subsets. Obviously, an open set and a close set are both constructible sets. A function $f$ on $E_d(Q, R)$ is called constructible if $E_d(Q, R)$ can be divided into finitely many constructible sets satisfies that $f$ is constant on each such constructible set. Write $M(X)$ for the $\mathbb{Q}$-vector space of constructible functions on some complex algebraic variety $X$.

Let $\mathcal{O}$ be an above constructible set, $1_{\mathcal{O}}$ is called a characteristic function if $1_{\mathcal{O}}(x) = 1$, for any $x \in \mathcal{O}$; $1_{\mathcal{O}}(x) = 0$, for any $x \notin \mathcal{O}$. It is clear that $1_{\mathcal{O}}$ is the simplest constructible function and any constructible function is a linear combination of characteristic functions. For any constructible subset $\mathcal{O}$ in $E_d(Q, R)$, we say $\mathcal{O}$ is $G_d$-invariant if $G_d \cdot \mathcal{O} = \mathcal{O}$.

In the following, the constructible sets and functions will always be assumed $G_d$-invariant unless particular statement.

1.4. Let $\chi$ denote Euler characteristic in compactly-supported cohomology. Let $X$ be an algebraic variety and $\mathcal{O}$ a constructible subset as the disjoint union of finitely many locally closed subsets $X_i$ for $i = 1, \cdots, m$. Define $\chi(\mathcal{O}) = \sum_{i=1}^{m} \chi(X_i)$. We note that it is well-defined. We will use the following properties:

**Proposition 1.1** ([Rie] and [Joy2]). Let $X, Y$ be algebraic varieties over $\mathbb{C}$. Then

1. If an algebraic variety $X$ is the disjoint union of finitely many constructible sets $X_1, \cdots, X_r$, then
   \[ \chi(X) = \sum_{i=1}^{r} \chi(X_i) \]

2. If $\varphi : X \to Y$ is a morphism with the property that all fibers have the same Euler characteristic $\chi$, then $\chi(X) = \chi \cdot \chi(Y)$. In particular, if $\varphi$ is a locally trivial fibration in the analytic topology with fibre $F$, then $\chi(Z) = \chi(F) \cdot \chi(Y)$.

3. $\chi(\mathbb{C}^n) = 1$ and $\chi(\mathbb{P}^n) = n + 1$ for all $n \geq 0$. 

We recall pushforward functor from the category of algebraic varieties over $\mathbb{C}$ and the category of $\mathbb{Q}$-vector spaces (see [Mac] and [Joy2]).

Let $\phi : X \to Y$ be a morphism of varieties. For $f \in M(X)$ and $y \in Y$, define

$$\phi_*(f)(y) = \sum_{c \neq \emptyset} c \chi(f^{-1}(c) \cap \phi^{-1}(y))$$

**Theorem 1.2** ([Di], [Joy2]). Let $X, Y$ and $Z$ be algebraic varieties over $\mathbb{C}$, $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms of varieties, and $f \in M(X)$. Then $\phi_*(f)$ is constructible, $\phi_* : M(X) \to M(Y)$ is a $\mathbb{Q}$-linear map and $(\psi \circ \phi)_* = (\psi)_* \circ (\phi)_*$ as $\mathbb{Q}$-linear maps from $M(X)$ to $M(Z)$.

In fact, Joyce extended this result to algebraic stacks [Joy2].

1.5. In order to deal with orbit spaces, We also need to consider the geometric quotients.

**Definition 1.3.** Let $G$ be an algebraic group acting on a variety $X$ and $\phi : X \to Y$ be a $G$-invariant morphism, i.e. a morphism constant on orbits. The pair $(Y, \phi)$ is called a geometric quotient if $\phi$ is open and for any open subset $U$ of $Y$, the associated comorphism identifies the ring $\mathcal{O}_Y(U)$ of regular functions on $U$ with the ring $\mathcal{O}_X(\phi^{-1}(U))^G$ of $G$-invariant regular functions on $\phi^{-1}(U)$.

The following result due to Rosenlicht [Ro] is essential to us.

**Lemma 1.4.** Let $X$ be a $G$-variety, then there exists an open and dense $G$-stable subset which has a geometric $G$-quotient.

By this Lemma, we can construct a finite stratification over $X$. Let $U_1$ be a open and dense $G$-stable subset of $X$ as in Lemma [1.4]. Then $\dim_\mathbb{C}(X - U_1) < \dim_\mathbb{C}X$. We can use the above lemma again, there exists a dense open $G$-stable subset $U_2$ of $X - U_1$ which has a geometric $G$-quotient. Inductively, we get the finite stratification $X = \bigcup_{i=1}^l U_i$ which $U_i$ is $G$-invariant locally closed subset and has geometric quotient, $l \leq \dim_\mathbb{C}X$. We denote by $\phi_{U_i}$ the geometric quotient map on $U_i$. Define the quasi Euler-Poincaré characteristic of $X/G$ by $\chi(X/G) := \sum_i \chi(\phi_{U_i}(U_i))$. If $\{U'_i\}$ is another choice for the definition of $\chi(X/G)$, then $\chi(\phi_{U_i}(U_i)) = \sum_j \chi(\phi_{U_i \cap U'_j}(U_i \cap U'_j))$ and $\chi(\phi_{U'_j}(U'_j)) = \sum_i \chi(\phi_{U_i \cap U'_j}(U_i \cap U'_j))$. Thus $\sum_i \chi(\phi_{U_i}(U_i)) = \sum_j \chi(\phi_{U'_j}(U'_j))$ and $\chi(X/G)$ is well-defined (see [XXZ]). Similarly, $\chi(\mathcal{O}/G) := \sum_i \chi(\phi_{U_i}(\mathcal{O} \cap U_i))$ is well-defined for any $G$-invariant constructible subset $\mathcal{O}$ of $X$.

We also introduce the following notation. Let $f$ be a constructible function over a variety $X$, it is natural to define

$$\int_{x \in X} f(x) := \sum_{m \in \mathbb{C}} m \chi(f^{-1}(m))$$

Comparing with Proposition [1.1], we also have the following (see [XXZ]).

**Proposition 1.5.** Let $X, Y$ be algebraic varieties over $\mathbb{C}$ under the actions of the algebraic groups $G$ and $H$ respectively. Then
(1) If an algebraic variety $X$ is the disjoint union of finitely many $G$-invariant constructible sets $X_1, \cdots, X_r$, then
\[ \chi(X/G) = \sum_{i=1}^{r} \chi(X_i/G) \]

(2) If $\varphi : X \to Y$ is a morphism induces the quotient map $\phi : X/G \to Y/H$ with the property that all fibers for $\phi$ have the same Euler characteristic $\chi$, then $\chi(X/G) = \chi \cdot \chi(Y/H)$.

Moreover, if there exists an action of algebraic group $G$ on $X$ as Definition 1.3, and $f$ is $G$-invariant constructible function over $X$, We define
\[ \int_{x \in X/G} f(x) := \sum_{m \in C} m \chi(f^{-1}(m)/G) \]

In particular, we frequently use the following Corollary.

**Corollary 1.6.** Let $X,Y$ be algebraic varieties over $\mathbb{C}$ under the actions of the algebraic groups $G$. These actions naturally induce the action of $G$ on $X \times Y$. Then
\[ \chi(X \times G Y) = \int_{y \in Y/G} \chi(X/G y) \]
where $G_y$ is the stable subgroup of $G$ for $y \in Y$ and $X \times G Y$ is the orbit space of $X \times Y$ under the action of $G$.

### 2. Green’s formula over finite fields

In this section, we recall Green’s formula over finite fields (Gre, Rin2). Let $k$ be a finite field and $\Lambda$ a hereditary finitary $k$-algebra. Let $P$ be the set of isomorphism classes of finite $\Lambda$-modules. Given $\alpha \in P$, let $V_\alpha$ be a representative in $\alpha$ (denoted by $V_\alpha \in \alpha$) and $a_\alpha = |\text{Aut}_\Lambda V_\alpha|$. Given $\xi, \eta$ and $\lambda$ in $P$, let $g_{\xi \eta}^\lambda$ be the number of submodules $Y$ of $V_\lambda$ such that $Y$ and $V_\lambda/Y$ are isomorphic to $\eta$ and $\xi$, respectively. The following identity is called Green’s formula.

**Theorem 2.1.** Let $k$ be a finite field and $\Lambda$ a hereditary finitary $k$-algebra. Let $\xi, \eta, \xi', \eta' \in P$. Then
\[ a_\xi a_\eta a_\xi' a_{\eta'} \sum_{\lambda} g_{\xi \eta}^\lambda g_{\xi' \eta'}^{\lambda^{-1}} = \sum_{\alpha, \beta, \gamma, \delta} |\text{Ext}^1(V_\gamma, V_\beta)| g^\lambda_{\gamma \alpha} g^\xi_{\alpha \beta} g^\eta_{\beta \gamma} g_{\alpha \beta} a_\alpha a_\beta a_\gamma \]

Given $X \in \xi, Y \in \eta, E \in \lambda$, let us introduce the notation
\[ h_{\xi \eta}^\lambda := \frac{|\text{Ext}^1(X, Y)_E|}{|\text{Hom}(X, Y)|} \]

where $\text{Ext}^1(X, Y)_E$ is the subset of $\text{Ext}^1(X, Y)$ consisting of equivalence classes of the exact sequence represented whose middle term is isomorphic to $E$. The following identity is called Riedtmann-Peng formula (Rin, Pe).

**Proposition 2.2.** Let $\alpha, \beta, \lambda \in P$. Then
\[ g_{\alpha \beta}^\lambda a_\alpha a_\beta = h_{\lambda}^\alpha \]
Using this proposition, Green’s formula can be rewritten as (DXX, [Hu2])

\[ \sum_{\lambda} g_{\xi}^{\lambda} h_{\eta}^{\xi' \eta'} = \sum_{\alpha, \beta, \gamma, \delta} |\text{Ext}^1(A, D)||\text{Hom}(M, N)| |\text{Hom}(A, D)||\text{Hom}(A, C)||\text{Hom}(B, D)| g_{\gamma}^{\alpha} h_{\xi}^{\alpha \gamma} h_{\eta}^{\delta \beta}. \]

where \( X \in \xi, Y \in \eta, M \in \xi', N \in \eta' \) and \( A \in \gamma, C \in \alpha, B \in \delta, D \in \beta, E \in \lambda \).

Let’s explain Green’s formula by associating some sets to two sides of the formula. For fixed \( kQ \)-modules \( X, Y, M, N \) with \( \dim X + \dim Y = \dim M + \dim N \), we fix the \( I \)-graded \( k \)-space \( E \) such that \( \dim E = \dim X + \dim Y \). Let \( (E, m) \) be a \( kQ \)-module structure on \( E \) given by the algebraic morphism \( m : \Lambda \rightarrow \text{End}_k E \). Let \( Q(E, m) \) be the set of \((a, b, a', b')\) which satisfies the following diagram with the exact row and column:

(4) \[
\begin{array}{cccccc}
0 & \rightarrow & N & \xrightarrow{a} & (E, m) & \xrightarrow{b} & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & X & \xrightarrow{a'} & (E, m) & \xrightarrow{b'} & M & \rightarrow & 0
\end{array}
\]

Let

\[ Q(X, Y, M, N) = \bigcup_{m : \Lambda \rightarrow \text{End}_k E} Q(E, m) \]

It is clear that

\[ |Q(E, m)| = g_{\xi}^{\lambda} g_{\xi'}^{\lambda} a_{\eta} a_{\eta'} \]

where \( \lambda \in P \) such that \( (E, m) \in \lambda \), or simply write \( m \in \lambda \).

\[ |Q(X, Y, M, N)| = |\text{Aut}_k E| \cdot \sum_{\lambda} g_{\xi}^{\lambda} g_{\xi'}^{\lambda} a_{\eta} a_{\eta'} a_{\lambda}^{-1} \]

It is almost left side of Green’s formula.
Let $D(X, Y, M, N)^*$ be the set of $(B, D, e_1, e_2, e_3, e_4)$ satisfying the following diagram with exact rows and exact columns:

\[
\begin{array}{ccc}
0 & \rightarrow & D \\
\downarrow & & \downarrow e_1 \\
0 & \rightarrow & Y \\
\downarrow & & \downarrow e_2 \\
0 & \rightarrow & B \\
\downarrow & & \downarrow x \\
0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & C \\
\downarrow & & \downarrow e_3 \\
0 & \rightarrow & X \\
\downarrow & & \downarrow e_4 \\
0 & \rightarrow & A \\
\downarrow & & \downarrow y \\
0 & \rightarrow & 0 \\
\end{array}
\]

where $B, D$ are submodules of $M, N$, respectively and $A = M/B, C = N/D$. The maps $u', v'$ and $x, y$ are naturally induced. We have

$$|D(X, Y, M, N)^*| = \sum_{\alpha, \beta, \gamma, \delta} g_{\gamma}^{\epsilon} g_{\gamma}^{c} g_{\delta}^{a} g_{\delta}^{a} a_{\alpha} a_{\beta} a_{\delta} a_{\gamma}$$

Fix the above square, let $T = X \times_A M = \{(x \oplus m) \in X \oplus M \mid e_4(x) = y(m)\}$ and $S = Y \sqcup_D N = Y \oplus N/\{e_1(d) \oplus u'(d) \mid d \in D\}$. There is a unique map $f : S \rightarrow T$ (see [Rin2]) such that the natural long sequence

\[
\begin{array}{ccc}
0 & \rightarrow & D \\
\downarrow & & \downarrow f \\
S & \rightarrow & T \\
\downarrow & & \downarrow T \\
A & \rightarrow & 0 \\
\end{array}
\]

is exact.

We define $(c, d)$ to be the pair of maps satisfying $c$ is surjective, $d$ is injective and $cd = f$. The counting of the pair $(c, d)$ can be made in the following process. We have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & S \\
\downarrow & & \downarrow d \ (E, m) d_1 \\
0 & \rightarrow & A \\
\downarrow & & \downarrow c \\
0 & \rightarrow & Imf \\
\end{array}
\]

The exact sequence

\[
0 \rightarrow D \rightarrow S \rightarrow Imf \rightarrow 0
\]

induces the following long exact sequence:

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}(A, D) \\
\downarrow & & \downarrow \text{Hom}(A, S) \\
\text{Hom}(A, Imf) & \rightarrow & \text{Hom}(A, Imf) \\
\downarrow & & \downarrow \phi \\
\text{Ext}^1(A, D) & \rightarrow & \text{Ext}^1(A, S) \\
\end{array}
\]

\[
\rightarrow \text{Ext}^1(A, Imf) \rightarrow 0
\]
We set $\varepsilon_0 \in Ext^1(A, Imf)$ corresponding to the canonical exact sequence

$$0 \to Imf \to T \to A \to 0$$

and denote $\phi^{-1}(\varepsilon_0) \cap Ext^1(A, S)_{(E,m)}$ by $\phi^{-1}_m(\varepsilon_0)$. Let $\mathcal{F}(f; m)$ be the set of $(c, d)$ induced by diagram (7) with center term $(E, m)$. Let

$$\mathcal{F}(f) = \bigcup_{m: \Lambda \to \text{End}_k E} \mathcal{F}(f; m)$$

Then

$$\mathcal{F}(f; m) = |\phi^{-1}_m(\varepsilon_0)| \frac{|\text{Aut}_\Lambda(E, m)|}{|\text{Hom}(A, S)|} |\text{Hom}(A, Imf)|$$

$$\mathcal{F}(f) = |\text{Aut}_\Lambda(E)| \frac{|Ext^1(A, D)|}{|\text{Hom}(A, D)|}$$

Let $\mathcal{O}(E, m)$ be the set of $(B, D, e_1, e_2, e_3, e_4, c, d)$ satisfying the following commutative diagram:

(9)

with exact rows and columns, where the maps $q_X, u_Y$ and $q_M, u_N$ are naturally induced. In fact, the long exact sequence (10) has the following explicit form:

(10)

$$|\mathcal{O}(E, m)| = \sum_{\alpha, \beta, \gamma, \delta} |\phi^{-1}_m(\varepsilon_0)| \frac{|\text{Aut}_\Lambda(E, m)|}{|\text{Hom}(A, S)|} |\text{Hom}(A, Imf)| |g_{\gamma} a_{\beta} a_{\delta} g_{\alpha} a_{\gamma}|$$

Put

$$\mathcal{O}(X, Y, M, N) = \bigcup_{m: \Lambda \to \text{End}_k E} \mathcal{O}(E, m)$$
Then
\[ |O(X, Y; M, N)| = |\text{Aut}_k(E)| \sum_{\alpha, \beta, \gamma, \delta} |\text{Ext}^1(A, D)| \frac{|\text{Ext}^1(A, D)|}{|\text{Hom}(A, D)|} g_{\gamma_0}^{\xi_0} g_{\gamma_1}^{\xi_1} g_{\alpha_0}^{\eta_0} g_{\alpha_1}^{\eta_1} a_\alpha a_\beta a_\delta a_\gamma. \]

It is almost the right side of Green’s formula. Now Green’s formula is equivalent to say

**Proposition 2.3.** There exist bijections \( \Omega : Q(E, m) \rightarrow O(E, m) \).

By the reformulation (3) of Green’s formula, Green’s formula characterizes the following situation. Given \( \varepsilon \in \text{Ext}^1(M, N)_E \) and \( Y \subseteq E \), we count the submodules \( M \subseteq M, N \subseteq N \) induced by \( Y \subseteq E \). The following proposition deals with the converse situation when \( Y \) is not fixed [Hu2].

**Proposition 2.4.**

Given \( \varepsilon \in \text{Ext}^1(M, N)_E \) and \( M \subseteq M, N \subseteq N \). Set
\[ V_{M, N} := \{ Y \subseteq E \mid Y \text{ induces } M, N \} \]
If it is not an empty set, then it is isomorphic to \( \text{Hom}(M, N) \).

As Ringel pointed out [Rin2 [ZW]], let \( \Lambda \) be a finitary \( k \)-algebra, if Green’s formula holds for all finite \( \Lambda \)-modules, then the category of finite \( \Lambda \)-modules is hereditary. In general, if the category of finite \( \Lambda \)-modules is not hereditary, we can extend Green’s formula to non-hereditary case. Given \( \Lambda \)-modules \( A, D \), let \( \text{Ext}^2(A, D) \) denote the subset of \( \text{Ext}^2(\Lambda, A, D) \) consisting of equivalence classes of long exact sequence of the form
\[ 0 \rightarrow D \rightarrow S \rightarrow T \rightarrow A \rightarrow 0 \]

**Lemma 2.5.** The extension \( \varepsilon = 0 \in \text{Ext}^2(\Lambda, A, D) \) if and only if \( \mathcal{F}(f) \neq \emptyset \).

We write \( \varepsilon(\alpha, \beta, \gamma, \delta) = 0 \) if the equivalence class in \( \text{Ext}^2(\Lambda, A, D) \) of the long exact sequence (10) vanishes. By Lemma 2.5, we have [DXX]

**Theorem 2.6.** Let \( k \) be a finite field and \( \Lambda \) a finitary \( k \)-algebra. Let \( \xi, \eta, \xi', \eta' \in \mathcal{P} \). Then
\[ a_\xi a_\eta a_\xi' a_\eta' = \sum_{\lambda} \sum_{\alpha, \beta, \gamma, \delta} \frac{\varepsilon(\alpha, \beta, \gamma, \delta)}{|\text{Hom}(\Lambda, A, D)|} g_{\gamma}^{\xi} g_{\delta}^{\eta} a_\alpha a_\beta a_\gamma. \]

3. Hall algebras and the realization of quantum groups

3.1. The Ringel-Hall algebra of \( \mathcal{H}(A) \) associated to an abelian category \( \mathcal{A} \) is an associative algebra, which, as a vector space, has a basis consisting of the isomorphism classes \( [X] \) of objects \( X \) in \( \mathcal{A} \), and which has the multiplication
\[ [X] \ast [Y] = \sum_{[L]} g_{XY}^L [L], \]
where \( F_{XY}^L \) is the number of subobjects \( L' \) of \( L \) such that \( L' \cong Y, L/L' \cong X \) and is called the Hall number. Ringel-Hall algebra has many important variants. We just list here the following five main types [Joy1]:
- **Counting subobjects over finite fields**, as in Ringel [Rin1].
- **Perverse sheaves on moduli spaces** are used by Lusztig [Lu1].
- **Homology of moduli spaces**, as in Nakajima [Na].
• Constructible functions on moduli spaces are used by Lusztig [Lu1], Nakajima [Na §10], Frenkel, Malkin and Vybornov [FMV], Riedtmann [Rie] and others.

• Constructible functions on Artin stacks are used by Joyce [Joy1], Kapranov and Vasserot [KV].

3.2. Let $\mathcal{A}$ be an $k$-linear abelian category of finite global dimension. The Grothendieck group $G$ of $\mathcal{A}$ is the quotient of the free abelian group with generators the isomorphism classes of objects of $\mathcal{A}$ by the relations $[X] + [Y] - [L]$ whenever there exists an exact sequence $0 \to Y \to L \to X \to 0$. For any two objects $M, N \in \mathcal{A}$, we define the multiplicative Euler form $\langle , \rangle : G \times G \to \mathbb{C}$ by

$$\langle M, N \rangle = \sum_i (-1)^i \dim_k \text{Ext}^i(M, N)$$

Denote by $h_X$ the image of $[X]$ in $G$. We assume $\mathcal{A}$ finitary, i.e., each homomorphism group is finite and, given any $h \in G$, there exist only finitely many isomorphism classes $[X]$ with $h_X = h$. Let $W(X,Y;L)$ denote the set of all exact sequences $0 \to Y \to L \to X \to 0$. The group $\text{Aut}(X,Y) := \text{Aut} X \times \text{Aut} Y$ acts on $W(X,Y;L)$ via

\[
0 \to Y \xrightarrow{f} L \xrightarrow{g} X \xrightarrow{\xi} 0
\]

\[
0 \to Y \xrightarrow{\eta} L \xrightarrow{\tau} X \xrightarrow{\xi} 0
\]

and we denote the quotient set by $V(X,Y;L)$. Since $f$ is monic and $g$ epic this action is free, so

$$F^L_{XY} := |V(X,Y;L)| = \frac{|W(X,Y;L)|}{|\text{Aut}(X,Y)|}.$$ 

By [Rie], $F^L_{XY} = g^L_{XY}$. The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is the free $\mathbb{Z}$-module on generators indexed by the set of isomorphism classes of objects, writing $u_X$ for $u_{[X]}$, the multiplication is given by

$$u_X u_Y := \sum_{[L]} F^L_{XY} u_L.$$ 

This sum is finite and that $u_0$ is a unit for the multiplication.

**Theorem 3.1.** The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is an associative, unital algebra.

The associativity of the multiplication is equivalent to say for any $X, Y, Z \in \mathcal{A}$, we have

$$\sum_{[L]} F^L_{XY} F^M_{LY} = \sum_{[L']} F^M_{XL'} F^L_{YZ}.$$
3.3. Let $k$ be a finite field of order $q_k$, let $\Lambda$ be a $k$-algebra which is hereditary and finitary and $A = \text{mod}\Lambda$. Let $R$ be an integral domain containing the rational field $\mathbb{Q}$ and an invertible element $v$ such that $v^2 = q_k$. The following will consider $\mathcal{H}(A)$ as the free $R$-module with the basis $\{u_\alpha \mid \alpha \in P\}$. Now, we can define the comultiplication on $\mathcal{H}(A)$ by

$$\delta : \mathcal{H}(A) \rightarrow \mathcal{H}(A) \otimes_R \mathcal{H}(A)$$

given by

$$\delta(u_\lambda) = \sum_{\alpha, \beta} h^{\alpha \beta}_\lambda u_\alpha \otimes u_\beta,$$

where $h^{\alpha \beta}_\lambda$ is defined in Section 3. We also consider the tensor product $\mathcal{H}(A) \otimes K \mathcal{H}(A)$ as an algebra with multiplication $*$ as follows:

$$(u_\alpha \otimes u_\beta) * (u_\gamma \otimes u_\delta) = |\text{Ext}^1(V_\alpha, V_\delta)| |\text{Hom}(V_\alpha, V_\delta)| u_\alpha u_\gamma \otimes u_\beta u_\delta$$

It is routine to verify that the following fact is an equivalent version of Green formula in Theorem 2.1. This is fundamental important to Hall algebras and quantum groups.

**Proposition 3.2.** The map $\delta$ is a homomorphism between algebras and then $\mathcal{H}(A)$ is a bialgebra.

The twist of the multiplication of $\mathcal{H}(A)$ is defined by

$$u_X u_Y := v^{\dim X \cdot \dim Y} \sum L \, F^L_{XY} u_L.$$

With respect to this twist multiplication, we write $\mathcal{H}_*(A)$ instead of $\mathcal{H}(A)$, the corresponding comultiplication $\delta_*$ is defined by

$$\delta_*(u_\lambda) = \sum_{\alpha, \beta} v^{\dim X \cdot \dim Y} h^{\alpha \beta}_\lambda u_\alpha \otimes u_\beta.$$

We define a symmetric bilinear form on $\mathcal{H}(A)$ and $\mathcal{H}_*(A)$. For $\alpha \in P$, let $t_\alpha = |V_\alpha|_{a_{\alpha \alpha}}$. Define

$$(u_\alpha, u_\beta) = \begin{cases} t_\alpha, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise}. \end{cases}$$

**Proposition 3.3.** For any $u_\alpha, u_\beta$ and $u_\gamma$ in $\mathcal{H}(A)$ and $\mathcal{H}_*(A)$, we have

$$(u_\alpha, u_\beta u_\gamma) = (\delta(u_\alpha), u_\beta \otimes u_\gamma)$$

and

$$(u_\alpha, u_\beta \ast u_\gamma) = (\delta_*(u_\alpha), u_\beta \otimes u_\gamma).$$
3.4. An Euler form is a pair \((\omega, d)\) consisting of a bilinear form \(\omega\) on \(\mathbb{Z}I\) with values in \(\mathbb{Z}\) and a function \(d : I \to \mathbb{N} \cup \{0\}\) such that the following three properties are satisfied for all \(i, j \in I\):

- \(\omega(i, j)\) is divisible by \(d_i\) and \(d_j\);
- \(\omega(i, j) \leq 0\) for \(i \neq j\);
- \(d_i^{-1}\omega(i, i) \leq 1\).

The Euler form \((\omega, d)\) is called without short cycles if \(\omega(i, i) = d_i\) and \(\omega(i, j)\omega(j, i) = 0\) for \(i \neq j\). Given an Euler form without short cycles, its symmetrization will be a Cartan datum, and any Cartan datum arises in this way \([\text{Rin}2]\). Given a hereditary \(k\)-algebra, its Euler form is defined by

\[\omega(i, j) = (S_i, S_j), d = (d_1, \cdots, d_n)\]

where \(S_i\) are simple \(A\)-modules and \(d_i = \dim_k \text{End}(S_i)\).

**Proposition 3.4.** Let \((\omega, d)\) be an Euler form. Let \(k\) be a finite field. Then there exists a hereditary \(k\)-algebra \(A_k\) with Euler form \((\omega, d)\).

Given an Euler form \((\omega, d)\) without short cycles, let \((I, \cdot, \cdot)\) be the induced Cartan datum and let \(K\) be a set of finite fields \(k\) such that the set \(\{q_k = |k| : k \in K\}\) is infinite and \(R\) an integral domain containing \(\mathbb{Q}\), and also containing an element \(v_k\) such that \(v_k^2 = q_k\) for each \(k \in K\). There exists a hereditary algebra \(A_k\) with Euler form \((\omega, d)\). Consider the direct product

\[\mathcal{H}_*(\omega, d) := \prod_{k \in K} \mathcal{H}_*(A_k)\]

By \(\mathcal{C}(\omega, d)\) we denote the \(\mathbb{Q}\)-subalgebra generated by \(t, t^{-1}\) and \(u_{i,k}\) whose \(k\)-components are \(v, v^{-1}\) and \(u_{i,k}\), respectively. We define the \(\mathbb{Q}(t)\)-algebra

\[\mathcal{C}_*(\omega, d) = \mathbb{Q}(t) \otimes_{\mathbb{Q}[t, t^{-1}]} \mathcal{C}(\omega, d)\]

Let \(g\) be the Kac-Moody algebra of type \((I, \cdot, \cdot)\). Furthermore, let \(\mathbb{Q}(t)\) be the function field in one indeterminate \(t\) over the field \(\mathbb{Q}\) of rational numbers. The Drinfeld-Jimbo quantization \(U\) of the enveloping algebra \(\mathcal{U}(g)\) is by definition the \(\mathbb{Q}(t)\)-algebra generated by \(\{E_i, F_i \mid i \in I\}\) and \(\{K_\mu \mid \mu \in \mathbb{Z}I\}\) with quantum Serre relations. There is a triangular decomposition

\[U = U^- \otimes U^0 \otimes U^+\]

where \(U^+\) (resp. \(U^-\)) is the subalgebra generated by \(E_i\) (resp. \(F_i\)), \(i \in I\) and \(U^0\) is the subalgebra generated by \(K_\mu, \mu \in \mathbb{Z}I\). Depending on Proposition 3.2, the following theorem gives the realization of the positive part of a quantum group.

**Theorem 3.5.** Given an Euler form \((\omega, d)\) without short cycles, let \((I, \cdot, \cdot)\) be the induced Cartan datum and \(U\) be the Drinfeld-Jimbo quantum group of type \((I, \cdot, \cdot)\). Then the correspondence \(u_{i,*} \mapsto E_i\) induces a Hopf algebra isomorphism \(\mathcal{C}_*(\omega, d) \rightarrow U^+\).

4. Green formula over complex fields with an application to the realization of universal enveloping algebras

In this section, we assume the base field \(k = \mathbb{C}\).
4.1. Now we introduce the $\chi$-Hall algebra $\mathcal{S}_\mathcal{Q}$. Assume the base field $k = \mathbb{C}$. Let $\Lambda = \mathbb{C} Q / (R)$ be an associative algebra over $\mathbb{C}$. Let $\mathcal{O}_1, \mathcal{O}_2$ be two constructible sets with dimension vectors $d_1, d_2$ invariant under the action of the corresponding group, respectively, fix $\Lambda$-module $L$, we define:

$$V(\mathcal{O}_2, \mathcal{O}_1; L) = \{0 = X_0 \subseteq X_1 \subseteq X_2 = L \mid X_i \in \text{mod} \Lambda, X_1 \in \mathcal{O}_1, \text{ and } L/X_1 \in \mathcal{O}_2\}$$

It is a constructible subset of Grassmann varieties. Put

$$g_{\alpha, \beta}^\Lambda = \chi(V(X, Y; L))$$

for $X \in \alpha, Y \in \beta$ and $L \in \lambda$. Consider the $\mathbb{C}$-space

$$M_G(\Lambda) = \bigoplus_{d \in \mathbb{N}} M_{G_d}(\Lambda)$$

where $M_{G_d}(\Lambda)$ is the $\mathbb{C}$-space of $G_d$-invariant constructible function on $\mathbb{E}_d(\Lambda)$. We define the convolution multiplication on $M_G(\Lambda)$ as follows. For any $f \in M_{G_d}(\Lambda)$ and $g \in M_{G_{d'}}(\Lambda)$, $f \bullet g \in M_{G_{d+d'}}(\Lambda)$ is given by the formula

$$f \bullet g(L) = \sum_{c, d \in \mathbb{C}} \chi(V(f^{-1}(c), g^{-1}(d); L))cd.$$

We have

**Proposition 4.1.** The space $M_G(\Lambda)$ under the convolution multiplication $\bullet$ is an associative $\mathbb{C}$-algebra with unit element.

Let $\alpha$ be the image of $X$ in $\mathbb{E}_d(Q)/G_d$. We write $X \in \alpha$, sometimes we also use the notation $\overline{X}$ to denote the image of $X$ and the notation $V_\alpha$ to denote a representative of $\alpha$. Instead of $d_\alpha$, we use $\alpha$ to denote the dimension vector of $\alpha$.

4.2. For any $L \in \text{mod} \Lambda$, let $L = \bigoplus_{i=1}^r L_i$ be the decomposition into indecomposables, then an action of $\mathbb{C}^*$ on $L$ is defined by

$$t.(v_1, \cdots, v_r) = (tv_1, \cdots, t^rv_r)$$

for $t \in \mathbb{C}^*$ and $v_i \in L_i$ for $i = 1, \cdots, r$.

It induces an action of $\mathbb{C}^*$ on $V(X, Y; L)$ for any $A$-module $X, Y$ and $L$. Let $(L_1 \subseteq L) \in V(X, Y; L)$ and $t.L_1$ be the action of $\mathbb{C}^*$ on $X_1$ by the decomposition of $L$, then there is a natural isomorphism $t.L_1 : L_1 \cong t.L_1$. Define $t.(L_1 \subseteq L) = (t.L_1 \subseteq L)$. We call this action the first kind of $\mathbb{C}^*$-action.

Let $D(X, Y)$ be the vector space over $\mathbb{C}$ of all tuples $d = (d(\alpha))_{\alpha \in Q_0}$ such that linear maps $d(\alpha) \in \text{Hom}_{\mathbb{C}}(X_{s(\alpha)}, Y_{t(\alpha)})$ and the matrices $L(d)_{\alpha} = \begin{pmatrix} Y_\alpha & d(\alpha) \\ 0 & X_\alpha \end{pmatrix}$ satisfy the relations $R$. Define $\pi : D(X, Y) \rightarrow \text{Ext}^1(X, Y)$ by sending $d$ to the equivalence class of a short exact sequence

$$\varepsilon : 0 \rightarrow Y \xrightarrow{(1 \ 0)} L(d) \xrightarrow{(0 \ 1)} X \rightarrow 0$$

where $L(d)$ is the direct sum of $X$ and $Y$ as a vector space. For any $\alpha \in Q_1$,

$$L(d)_{\alpha} = \begin{pmatrix} Y_\alpha & d(\alpha) \\ 0 & X_\alpha \end{pmatrix}$$
Fixed a vector space decomposition \( D(X, Y) = \text{Ker} \pi \oplus E(X, Y) \), then we can identify \( \text{Ext}^1(X, Y) \) with \( E(X, Y) \) \cite{DXX}. There is a natural \( \mathbb{C}^* \)-action on \( E(X, Y) \) by \( t.d = (t \cdot d) \) for any \( t \in \mathbb{C}^* \). This induces the action of \( \mathbb{C}^* \) on \( \text{Ext}^1(X, Y) \). By the isomorphism of \( \mathbb{C}Q \)-modules between \( L(d) \) and \( L(t.d) \), we have \( t. \xi \) is the following short exact sequence:

\[
\begin{array}{c}
0 \\
\end{array} \xrightarrow{t} Y \xrightarrow{0} L(d) \xrightarrow{(0, 1)} X \xrightarrow{0}
\]

for any \( t \in \mathbb{C}^* \). Let \( \text{Ext}^1(X, Y)_L \) be the subset of \( \text{Ext}^1(X, Y) \) with the middle term isomorphic to \( L \), then \( \text{Ext}^1(X, Y)_L \) can be viewed as a constructible subset of \( \text{Ext}^1(X, Y) \) by the identification between \( \text{Ext}^1(X, Y) \) and \( E(X, Y) \). Put

\[
h_{\alpha \beta} = \chi(\text{Ext}^1_{\Lambda}(X, Y)_L)
\]

for \( X \in \alpha, Y \in \beta \) and \( L \in \lambda \). The following is known, see a proof in \cite{DXX}. The above \( \mathbb{C}^* \)-action on the extensions induces an action on the middle terms. As a vector space, \( L = X \oplus Y \). So we can define \( t.(x, y) = (x, ty) \) for any \( t \in \mathbb{C}^* \) and \( x \in X, y \in Y \). Hence, for any \( L_1 \subseteq L \), we obtain a new flag \( t.L_1 \subseteq L \). We call this action the second kind of \( \mathbb{C}^* \)-action.

**Lemma 4.2.** For \( A, B, X \in \text{mod} \Lambda, \chi(\text{Ext}^1_{\Lambda}(A, B)_X) = 0 \) unless \( X \simeq A \oplus B \).

By considering the \( \mathbb{C}^* \)-action on grassmannians and extensions, we obtain the degeneration form of Green’s formula.

**Theorem 4.3.** For fixed \( \xi, \eta, \xi', \eta' \), we have

\[
g_{\xi \eta}^{\xi' \eta'} = \int_{\alpha, \beta, \gamma, \delta; \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} g_{\gamma \delta}^\alpha g_{\alpha \beta}^\eta
\]

This theorem can be induced by consider the morphism

\[
\bigcup_{\alpha, \beta, \gamma, \delta; \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} V(V_\alpha, V_\beta; V_\eta') \times V(V_\gamma, V_\delta; V_{\xi'}) \xrightarrow{i} V(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})
\]

sending \( (V_\alpha^1 \subseteq V_{\eta'}, V_\beta^1 \subseteq V_{\xi'}) \) to \( V_\xi^1 \oplus V_{\eta'}^1 \subseteq V_{\xi'} \oplus V_{\eta'} \) in a natural way. We set \( \overline{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) := V(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \setminus \text{Im}(i) \). This induces the map

\[
\bigcup_{\alpha, \beta, \gamma, \delta; \dim \nu_{\beta} + \dim \nu_{\delta} = d} V(V_\alpha, V_\beta; V_{\eta'}) \times V(V_\gamma, V_\delta; V_{\xi'}) \xrightarrow{i} \bigcup_{\dim \nu_\alpha = d} V(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})
\]

for fixed \( d \).

Furthermore, we can consider the inverse morphism:

\[
\bigcup_{\dim \nu_\alpha = d} V(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \rightarrow \bigcup_{\alpha, \beta, \gamma, \delta} V(V_\alpha, V_\beta; V_{\eta'}) \times V(V_\gamma, V_\delta; V_{\xi'})
\]

mapping the submodules of \( V_{\xi'} \oplus V_{\eta'} \) to the induced submodules of \( V_{\eta'} \) and \( V_{\xi'} \) as the bijection in Proposition 2.3. This map is a vector bundle by Proposition 2.4 and is analogous to the restriction map in \cite{Lu1}.
4.3. A constructible set is called indecomposable if all points in it correspond to indecomposable \( \Lambda \)-modules. Let \( \mathcal{O} \) be a constructible set. If it has the form:
\[ \mathcal{O} = n_1 \mathcal{O}_1 \oplus \cdots \oplus n_k \mathcal{O}_k, \]
where \( \mathcal{O}_i, 1 \leq i \leq k \) are indecomposable constructible sets, then \( \mathcal{O} \) is called to be of Krull-Schmidt. A constructible set \( \mathcal{Q} \) is called to be of stratified Krull-Schmidt if it has a finite stratification \( \mathcal{Q} = \bigcup_i \mathcal{Q}_i \) where each \( \mathcal{Q}_i \) is locally closed in \( \mathcal{Q} \) and is of Krull-Schmidt. Define \( R(\Lambda) = \bigoplus \mathcal{Q} \mathcal{O} \), where \( \mathcal{Q} \) are of stratified Krull-Schmidt constructible sets. We note that \( R \) is not free generated by \( 1_\mathcal{O} \) for any constructible subset \( \mathcal{O} \). By Proposition 4.4 we have

**Theorem 4.4.** The \( \mathbb{Z} \)-module \( R(\Lambda) \) under the convolution multiplication \( \ast \) is an associative \( \mathbb{Z} \)-algebra with unit element.

Define the \( \mathbb{Z} \)-submodule \( L(\Lambda) = \bigoplus \mathcal{O} \mathcal{O} \mathcal{O} \) of \( R(\Lambda) \), where \( \mathcal{O} \) are indecomposable constructible sets. Then we have the following result.

**Theorem 4.5.** The \( \mathbb{Z} \)-submodule \( L(\Lambda) \) is a Lie subalgebra of \( R(\Lambda) \) with bracket \([x, y] = xy - yx\).

We consider the following tensor algebra over \( \mathbb{Z} \):
\[ T(L) = \bigoplus_{i=0}^{\infty} L^\otimes i \]
where \( L^\otimes 0 = \mathbb{Z}, L^\otimes i = L \otimes \cdots \otimes L \) (i times). Then \( T(L) \) is an associative \( \mathbb{Z} \)-algebra using the tensor as multiplication. Let \( J \) be the two-sided ideal of \( T(L) \) generated by
\[ 1_{\mathcal{O}_1} \otimes 1_{\mathcal{O}_2} - 1_{\mathcal{O}_2} \otimes 1_{\mathcal{O}_1} - [1_{\mathcal{O}_1}, 1_{\mathcal{O}_2}] \]
where \( \mathcal{O}_1, \mathcal{O}_2 \) are any indecomposable constructible sets and \([-, -]\) is Lie bracket in \( L(\Lambda) \). Then \( U(\Lambda) = T(\Lambda)/J \) is the universal enveloping algebra of \( L(\Lambda) \) over \( \mathbb{Z} \).

For a distinguish we write the multiplication in \( U(\Lambda) \) as \( * \) and the multiplication in \( R(\Lambda) \) as \( \ast \). We have the canonical homomorphism \( \varphi : U(\Lambda) \to R(\Lambda) \) satisfying \( \varphi(1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}) = 1_{\mathcal{O}_1} \ast 1_{\mathcal{O}_2} \). Let \( R'(\Lambda) \) be the subalgebra of \( R(\Lambda) \) generated by the elements \( \lambda_1! \cdots \lambda_n! \lambda_1 \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \), where \( \lambda_i \in \mathbb{N} \) for \( i = 1, \cdots, n \) and \( \mathcal{O}_i, i = 1, \cdots, n \), are indecomposable constructible sets and disjoint to each other.

**Theorem 4.6.** The canonical embedding \( \varphi : U(\Lambda) \to R(\Lambda) \) induces the isomorphism \( \varphi : U(\Lambda) \to R'(\Lambda) \). Therefore \( U(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} \cong R(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} \).

This theorem inspires us that there exists a comultiplication structure on \( R'(\Lambda) \). Indeed, it can be realized by the degeneration form of Green’s formula. Let \( \delta : R'(\Lambda) \to R'(\Lambda) \otimes R'(\Lambda) \) be given by \( \delta(1_{\mathcal{O}_1})(A, B) = \chi(\text{Ext}^1_{\mathcal{O}_1}(A, B)_{\mathcal{O}_1}) \) where \( \text{Ext}^1_{\mathcal{O}_1}(A, B)_{\mathcal{O}_1} \) is the set of the equivalence classes of extensions \( B \) by \( A \) with middle terms belonging to \( \mathcal{O}_1 \).

**Theorem 4.7.** The map \( \delta : R'(\Lambda) \to R'(\Lambda) \otimes R'(\Lambda) \) is an algebra homomorphism.
5. Cluster algebras and cluster categories

5.1. Let $CQ$ be a hereditary algebra associated to a connected quiver without oriented cycles, and let $\mathcal{D} = \mathcal{D}^b(\text{mod } CQ)$ be the bounded derived category with the shift functor $M \mapsto M[1]$ for any object $M$ in $\mathcal{D}$. We identify the category $\text{mod}CQ$ with the full subcategory of $\mathcal{D}$ formed by the complexes whose homology is concentrated in degree 0. The Grothendieck group of $\mathcal{D}$ is the same as that of $\text{mod}CQ$, i.e., $G_Q := G(\mathcal{D}) = G(\text{mod}CQ)$.

Let $\tau$ be the Auslander-Reiten translation. It can be characterized by the Auslander-Reiten formula:

$$\text{Ext}^1_{\mathcal{D}}(M, N) \simeq \text{DHom}_{\mathcal{D}}(N, \tau M)$$

where $M, N$ are any objects in $\mathcal{D}$ and where $\text{D}$ is the functor which takes a vector space to its dual. The AR-translation $\tau$ is a triangle equivalence and therefore induces an automorphism of the Grothendieck group of $\mathcal{D}$.

The cluster category is defined as the orbit category $CQ = \mathcal{D}/F$, where $F = \tau^{-1}[1]$ and The category $CQ$ has the same objects as $\mathcal{D}$, but maps are given by

$$\text{Hom}_{C}(X, Y) = \bigoplus_i \text{Hom}_{\mathcal{D}}(F^i X, Y).$$

The category $CQ$ was defined in [CCS] for the $A_n$-case in [BMRT] for general case.

Each object $M$ of $CQ$ can be uniquely decomposed in the following way:

$$M = M_0 \oplus P_{M[1]}$$

where $M_0$ is the $CQ$-module, and where $P_{M[1]}$ is the shift of a projective module.

Theorem 5.1. Let $CQ$ be a hereditary algebra. Let $CQ$ the cluster category of $CQ$ and $\pi : \mathcal{D}Q \rightarrow CQ$ is the canonical functor. Then

(a) $CQ$ is a Krull-Schmidt category;
(b) $CQ$ is triangulated and $\pi$ is exact;
(c) $CQ$ has AR-triangles and $\pi$ preserves AR-triangles.

The statement (b) is due to Keller [K2], while (a) and (c) are proved in [BMRT].

We note that the cluster category is 2-Calabi-Yau which means that the functor $\text{Ext}^1$ is symmetric in the following sense:

$$\text{Ext}^1_{C}(M, N) \simeq \text{DExt}^1_{C}(N, M).$$

5.2. Let $n$ be a positive integer. We fix the ambient field $\mathcal{F} = \mathbb{Q}(u_1, \ldots, u_n)$ with algebraically independent generating set $\underline{u} := (u_1, \ldots, u_n)$. Let $x$ be a free generating set of $\mathcal{F}$ and let matrix $B = (b_{ij})_{1 \leq i, j \leq n} \in M_n(\mathbb{Z})$ is antisymmetric. Such a pair $(x, B)$ is called a seed. The Cartan counterpart of a matrix $B = (b_{ij})$ in $M_n(\mathbb{Z})$ is the matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

Let $(x, B)$ be a seed and let $x_j, 1 \leq j \leq n$, be in $x$. We define a new seed as follows. Let $x'_j$ be the element of $\mathcal{F}$ defined by the exchange relation:

$$x_j x'_j = \prod_{b_{ij} > 0} x^{b_{ij}} + \prod_{b_{ij} < 0} x^{-b_{ij}}.$$
where, by convention, we have \( x_i = u_i \) for \( i > n \). Set \( \mathbf{x}' = \mathbf{x} \cup \{ x'_j \} \setminus \{ x_j \} \). Let \( B' \) be the \( n \times n \) matrix given by

\[
b'_{ik} = \begin{cases} 
  -b_{ik} & \text{if } i = j \text{ or } k = j \\
  b_{ik} + \frac{1}{2}(|b_{ij}| b_{jk} + b_{ij} |b_{jk}|) & \text{otherwise.}
\end{cases}
\]

Then Fomin and Zelevinsky showed that \((\mathbf{x}', B')\) is also a seed. It is called the mutation of the seed \((\mathbf{x}, B)\) in the direction \( x_j \). Dually, the mutation of the seed \((\mathbf{x}', B')\) in the direction \( x'_j \) is \((\mathbf{x}, B)\). We consider all the seeds obtained by iterated mutations. The free generating sets occurring in the seeds are called clusters, and the variables they contain are called cluster variables. By definition, the cluster algebra \( \mathcal{A}(\mathbf{x}, B) \) associated to the seed \((\mathbf{x}, B)\) is the \( \mathbb{Z}[u_1, \ldots, u_n] \)-subalgebra of \( \mathcal{F} \) generated by the set of cluster variables.

An antisymmetric matrix \( B \) defines a quiver \( Q = Q_B \) with vertices corresponding to its rows (or columns) and which has \( b_{ij} \) arrows from the vertex \( i \) to the vertex \( j \) whenever \( b_{ij} > 0 \). The cluster algebra associated to the seed \((\mathbf{x}, B)\) will be also denoted by \( \mathcal{A}(Q) \). The following result is so-called Laurent phenomenon [FZ1].

**Theorem 5.2.** Let \( B \) be an antisymmetrizable matrix in \( M_n(\mathbb{Z}) \), i.e. there exists a diagonal matrix in \( M_n(\mathbb{N}) \) such that \( DB \) is antisymmetric. Then the cluster algebra \( \mathcal{A}(B) \) associated to \( B \) is a subalgebra of \( \mathbb{Q}[u_i^{\pm 1}, 1 \leq i \leq n] \).

A cluster algebra is finite if it has a finite number of cluster variables.

**Theorem 5.3.** A cluster algebra \( \mathcal{A} \) is finite if and only if there exists a seed \((\mathbf{x}, B)\) of \( \mathcal{A} \) such that the Cartan counterpart of the matrix \( B \) is a Cartan matrix of finite type.

In this case, there exists a quiver \( Q \) of simply laced Dynkin type associated to the seed \((\mathbf{x}, B)\).

The following proposition is basic for cluster category associated to a simply laced Dynkin quiver.

**Proposition 5.4.** [BMRRT] [CK1] Let \( Q \) be a Dynkin quiver and \( M, N \) be indecomposable \( \mathbb{C}Q \)-modules. Let \( \mathcal{C}Q \) be the corresponding cluster category. Then

(i) \( \text{Ext}^1_{\mathcal{C}Q}(M, N) = \prod \text{Ext}^1_{\mathcal{C}Q}(N, M) \) and at least one of the two direct factors vanishes.

(ii) any short exact sequence of \( \mathbb{C}Q \)-modules

\[
0 \rightarrow M \xrightarrow{i} Y \xrightarrow{p} N \xrightarrow{} 0
\]

provides a (unique) triangle \( M \xrightarrow{i} Y \xrightarrow{p} N \xrightarrow{} M[1] \) in \( \mathcal{C} \).

5.3. An object \( M \) of \( \mathcal{C}Q \) is called exceptional if it has no selfextensions, i.e., \( \text{Ext}^1_{\mathcal{C}Q}(M, M) = 0 \). An object \( T \) of \( \mathcal{C}Q \) is a tilting object if it is exceptional, multiplicity free, and has the following maximality property: if \( M \) is an indecomposable object such that \( \text{Ext}^1(M, T) = 0 \), then \( M \) is a direct factor of a direct sum of copies of \( T \). Note that a tilting object can be identified with a maximal set of indecomposable objects \( T_1, \ldots, T_n \) such that \( \text{Ext}^1(T_i, T_j) = 0 \) for all \( i, j \). Define the map

\[ X : \text{obj}(\mathcal{C}Q) \rightarrow \mathbb{Q}(x_1, \ldots, x_n) \]

satisfying:

(i) \( X_M \) only depends on the isomorphism class of \( M \);
(ii) \( X_{M \oplus N} = X_M X_N \) for all \( M, N \) of \( \mathcal{C}_Q \);
(iii) \( X_{P_i[i]} = x_i \) for the \( i \)th indecomposable projective \( P_i \);
(iv) if \( M \) is the image in \( \mathcal{C}_Q \) of an indecomposable \( kQ \)-module, then

\[
X_M = \sum_{\varepsilon} \chi(Gr_{\varepsilon}(M)) x^{\tau(\varepsilon) - \dim M + \varepsilon}
\]

where \( \tau \) is the Auslander-Reiten translation on the Grothendieck group \( K_0(D^b(Q)) \) and, for \( v \in \mathbb{Z}^n \), we put

\[
x^v = \prod_{i=1}^n x_i^{(\dim S_i, v)}
\]

and \( Gr_{\varepsilon}(M) \) is the \( \varepsilon \)-Grassmannian of \( M \), i.e. the variety of submodules of \( M \) with dimension vector \( \varepsilon \). This definition is equivalent to \([Hu2]\)

\[
X_M = \int_{\alpha, \beta} g^M_{\alpha \beta} x^{2R + \alpha R' - \dim M}
\]

where the matrices \( R = (r_{ij}) \) and \( R' = (r'_{ij}) \) satisfy \( r_{ij} = \dim \mathbb{C} \text{Ext}^1(S_i, S_j) \) and \( r'_{ij} = \dim \mathbb{C} \text{Ext}^1(S_j, S_i) \) for \( i, j \in Q_0 \). We note that \([Hu2]\)

\[
(\dim P)R = \dim \text{rad} P \quad (\dim I)R' = \dim I - \dim \text{soc} I
\]

**Proposition 5.5.** Let \( Q \) be a Dynkin quiver and \( M \) be a \( \mathcal{C}Q \) module with dimension vector \( \underline{d} \). Then

\[
X_M = \frac{1}{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}} \sum_{\varepsilon \in \mathbb{Z}^n} \chi(Gr_{\varepsilon}(M)) \prod_{i=1}^n x_i^{\sum_{i} \varepsilon_i - \sum_{i-j} (d_j - e_i)}
\]

This formula is called Caldero-Chapoton formula. It induces a bijection from the set of tilting objects to the set of clusters of \( A(Q) \).

**Proposition 5.6.** Let \( M \) be an indecomposable non-projective \( \mathcal{C}Q \)-module. Then

\[
X_M X_{\tau M} = 1 + X_E
\]

where \( E \) is the middle term of the Auslander-Reiten sequence ending in \( M \).

If \( M \) is an indecomposable projective \( \mathcal{C}Q \)-module, then

**Proposition 5.7.**

\[
X_{P_i} x_i = 1 + (X_{\text{rad} P_i}) x^{(\dim S_i) R'} \quad X_{I_i} x_i = 1 + (X_{I_i/\text{soc} I_i}) x^{(\dim S_i) R}
\]

where \( S_i \) is the \( i \)th simple module.

We give an example to confirm the above properties \([CC]\).

**Example 5.8.** Suppose that \( Q \) is the following quiver:

1 \[\rightarrow\] 2 \[\rightarrow\] 3.
Then, the AR-quiver of $C_Q$ has the following shape:

$$\begin{array}{c}
\begin{array}{c}
P_2[1] \\
P_1[1] \
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
P_3 \\
S_2 = P_2 \\
I_2 \\
P_1 \\
S_3 \\
P_2[1] \
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
P_3[1] \\
P_2[1] \
\end{array}
\end{array}\end{array}$$

Using formula (11), we can compute explicitly the $X_M$. The submodules of $S_2$ are $0$ and $S_2$.

$$X_{S_2} = \frac{x_1 x_3}{x_2} + \frac{1}{x_2} = \frac{x_1 x_3 + 1}{x_2},$$

The submodules of $P_3$ are $0$, $S_2$ and $P_3$ and the submodules of $P_1$ are $0$, $S_2$ and $P_1$.

$$X_{P_3} = \frac{x_1}{x_2} + \frac{1}{x_2 x_3} + \frac{1}{x_3} = \frac{1 + x_2 + x_1 x_3}{x_2 x_3},
X_{P_1} = \frac{x_3}{x_2} + \frac{1}{x_2 x_1} + \frac{1}{x_1} = \frac{1 + x_2 + x_1 x_3}{x_2 x_1},$$

It is clear that we have

$$X_{P_3} x_3 = 1 + X_{S_2},
X_{P_1} x_1 = 1 + X_{S_2},$$

which confirm Proposition 5.7. The submodules of $I_2$ are $0$, $P_1$, $P_3$, $S_2$ and $I_2$.

$$X_{I_2} = \frac{1}{x_2} + \frac{1}{x_1 x_3} + \frac{1}{x_1 x_2 x_3} + \frac{x_2}{x_1 x_3} = \frac{1 + 2x_2 + x_2^2 + x_1 x_3}{x_1 x_2 x_3},$$

The submodules of $S_1$ are $0$ and $S_1$; the submodules of $S_3$ are $0$ and $S_3$.

$$X_{S_1} = \frac{1}{x_1} + \frac{x_2}{x_1 x_3} = \frac{1 + x_2}{x_1 x_3},
X_{S_3} = \frac{1}{x_3} + \frac{x_2}{x_3} = \frac{1 + x_2}{x_3}.$$

It is clear that we have

$$X_{S_1} X_{P_3} = 1 + X_{I_2},
X_{S_3} X_{P_1} = 1 + X_{I_2},$$

which confirm Proposition 5.6.

6. Green formula under $\mathbb{C}^*$-action

6.1. For fixed $\xi, \eta$ and $\xi', \eta'$ with $\xi + \eta = \xi' + \eta' = \Delta$, let $V_\lambda \in E_\lambda$ and $Q(V_\lambda)$ be the set of $(a, b, a', b')$ which satisfies the following diagram with the exact row and
We let

\[ Q(\xi, \eta, \xi', \eta') = \bigcup_{V_\lambda \in \mathbb{E}_\lambda} Q(V_\lambda) \]

We remark that \( Q(\xi, \eta, \xi', \eta') \) can be viewed as a constructible subset of module variety \( \mathbb{E}_{(\xi, \eta, \xi', \eta')} \) with \( \xi + \eta = \xi' + \eta' = \lambda \) of the following quiver

\[ \cdots \rightarrow 0 \rightarrow V_\eta \rightarrow a' V_\lambda \rightarrow b V_\xi' \rightarrow 0 \rightarrow V_\eta' \rightarrow b' V_\xi \rightarrow 0 \]

We have the action of \( G_\lambda \) on \( Q(\xi, \eta, \xi', \eta') \):

\[ g.(a, b, a', b') = (ga, bg^{-1}, ga', b'g^{-1}) \]

The orbit space of \( Q(\xi, \eta, \xi', \eta') \) is denoted by \( Q(\xi, \eta, \xi', \eta')^* \) and the orbit in \( Q(\xi, \eta, \xi', \eta')^* \) is denoted by \( (a, b, a', b')^* \). Also we have the action of \( G_\lambda \) on \( W(V_\zeta', V_\eta'; \mathbb{E}_\Delta) \) by \( g.(a, b) = (ga, bg^{-1}) \) where

\[ W(V_\zeta', V_\eta'; \mathbb{E}_\Delta) = \{(f, g) \mid (f, g) \text{ is an exact sequence with the middle term } L \in \mathbb{E}_\Delta\} \]

This induces the orbit space \( \text{Ext}^1(V_\zeta', V_\eta') \) whose orbit is denoted by \( (a, b)^* \). Hence, we have

\[ W(V_\zeta', V_\eta'; \mathbb{E}_\Delta) \times W(V_\zeta', V_\eta'; \mathbb{E}_\Delta) = Q(\xi, \eta, \xi', \eta') \xrightarrow{\phi_2} \text{Ext}^1(V_\zeta', V_\eta') \]

\[ \phi_1 \]

\[ Q(\xi, \eta, \xi', \eta')^* \]

where \( \phi((a, b, a', b')^*) = (a, b)^* \) is well defined.

Let \( (a, b, a', b') \in Q(V_\lambda) \), then we know that the stable group of \( \phi_1 \) at \( (a, b, a', b') \) is

\[ a' c_1 \text{Hom}(\text{Coker } b' a, \text{Ker } ba') e_4 b' \]
which is isomorphic to Hom(Coker $b' a$, Ker $ba'$), where the injection $e_1 : \text{Ker } ba' \to V_{\lambda}$ is induced naturally by $a'$ and the surjection $e_2 : V_{\lambda} \to \text{Coker } b' a$ is induced naturally by $b'$. In the same way. The stable group of $\phi_2$ at $(a, b)$ is $1 + a \text{Hom}(V_{\xi}, V_{\eta}) b$, which is isomorphic to Hom($V_{\xi}', V_{\eta}'$) just for $a$ is injective and $b$ is surjective. The fibre $\phi_1(\phi_2^{-1}((a, b)^*))$ of $\phi$ for $(a, b)^*$ has the Euler characteristic $\chi(W(V_{\xi}, V_{\eta}'; V_{\lambda}))$.

Moreover, consider the action of $G_{\xi} \times G_{\eta}$ on $Q(\xi, \eta, \xi', \eta')^*$ and the induced orbit space is denoted by $Q(\xi, \eta, \xi', \eta')^\sim$. The stable subgroup $G((a, b, a', b')^*)$ at $(a, b, a', b')^*$ is

$$\{(g_1, g_2) \in G_{\xi} \times G_{\eta} \mid g a' = a' g_2, b' g = g_1 b' \text{ for some } g \in 1 + a \text{Hom}(V_{\xi}', V_{\eta}') b\}$$

This determines the group embedding

$$G((a, b, a', b')^*) \to (1 + a \text{Hom}(V_{\xi}', V_{\eta}') b)/(1 + ae_1 \text{Hom}(Coker b' a, \text{Ker } ba') e_2 b) .$$

The group $G((a, b, a', b')^*)$ is isomorphic to a vector space since $ba = 0$. We Know that $1 + a \text{Hom}(V_{\xi}', V_{\eta}') b$ is the subgroup of $\text{Aut } V_{\lambda}$, it acts on $W(V_{\xi}, V_{\eta}; E_{\lambda})$ naturally. The orbit space of $W(V_{\xi}, V_{\eta}; E_{\lambda})$ under the action of $1 + a \text{Hom}(V_{\xi}', V_{\eta}') b$ is denoted by $\tilde{W}(V_{\xi}, V_{\eta}; E_{\lambda})$ and a similar consideration for $V(V_{\xi}, V_{\eta}; E_{\lambda})$. By the above discussion, we have the following commutative diagram of group actions:

\[ 
\begin{array}{ccc}
W(V_{\xi}, V_{\eta}; E_{\lambda}) & \xrightarrow{1 + a \text{Hom}(V_{\xi}', V_{\eta}') b} & \tilde{W}(V_{\xi}, V_{\eta}; E_{\lambda}) \\
\downarrow{G_{\xi} \times G_{\eta}} & & \downarrow{G_{\xi} \times G_{\eta}} \\
V(V_{\xi}, V_{\eta}; E_{\lambda}) & \xrightarrow{1 + a \text{Hom}(V_{\xi}', V_{\eta}') b} & \tilde{V}(V_{\xi}, V_{\eta}; E_{\lambda})
\end{array}
\]

The stable group for the bottom map is

$$\{g \in 1 + a \text{Hom}(V_{\xi}', V_{\eta}') b \mid g a' = a' g_2, b' g = g_1 b' \text{ for some } (g_1, g_2) \in G_{\xi} \times G_{\eta}\}$$

which is isomorphic to a vector space too, it is denoted by $V(a, b, a', b')$. We can construct the map from $V(a, b, a', b')$ to $G((a, b, a', b')^*)$ sending $g$ to $(g_1, g_2)$. It is well-defined since $a'$ is injective and $b'$ is surjective. We have

$$V(a, b, a', b')/\text{Hom}(\text{Coker } b' a, \text{Ker } ba') \cong G((a, b, a', b')^*)$$

We have the following proposition.

**Proposition 6.1.** The surjective map

$$\phi^* : Q(\xi, \eta, \xi', \eta')^* \to \text{Ext}^1(V_{\xi}', V_{\eta}')$$

has fibre isomorphic to $\tilde{V}(V_{\xi}, V_{\eta}; E_{\lambda})$ at $(a, b)^* \in \text{Ext}^1(V_{\xi}', V_{\eta}')$ where $\tilde{V}(V_{\xi}, V_{\eta}; E_{\lambda})$ satisfies that there exists a surjective morphism from $V(V_{\xi}, V_{\eta}; V_{\lambda})$ to $\tilde{V}(V_{\xi}, V_{\eta}; E_{\lambda})$ with fibre isomorphic to the vector space

$$\text{Hom}(V_{\xi}', V_{\eta})/(\text{Hom}(\text{Coker } b' a, \text{Ker } ba') \times G((a, b, a', b')^*)) .$$
6.2. Let \( \mathcal{O}(\xi, \eta, \xi', \eta') \) be the set of \((V_\beta, V_\alpha, e_1, e_2, e_3, e_4, c, d)\) satisfying the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & V_\beta & \rightarrow & V_\gamma & \rightarrow & V_\delta & \rightarrow & 0 \\
0 & \rightarrow & V_\alpha & \rightarrow & V_\lambda & \rightarrow & V_\mu & \rightarrow & 0 \\
0 & \rightarrow & V_\alpha & \rightarrow & V_\xi & \rightarrow & V_\eta & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

where \( V_\beta, V_\alpha \) are submodules of \( V_\gamma, V_\eta \), respectively; \( V_\gamma = V_\gamma'/V_\delta, V_\alpha = V_\alpha'/V_\beta \), \( u', x, v', y \) are the canonical morphisms, and \( V_\lambda \) is the center induced by the above square, \( T = V_\xi \times V_\eta, V_\delta = \{(x \oplus m) \in V_\xi \oplus V_\eta | e_4(x) = y(m)\} \) and \( S = V_\eta \cap V_\eta'/V_\eta'' = V_\eta' \cap V_\eta'' \setminus \{e_1(v_\beta) \oplus u'(v_\beta) | v_\beta \in V_\beta\} \). Then there is unique map \( f : S \rightarrow T \) for the fixed square. We define \((c, d)\) to be the pair of maps satisfying \( c \) is surjective, \( d \) is injective and \( cd = f \). In particular, its subset with four vertexes \( V_\gamma, V_\delta, V_\alpha, V_\beta \) is denoted by \( \mathcal{O}(V_\gamma, V_\delta, V_\alpha, V_\beta) \). There is a natural action of group \( G_{\Delta} \) on \( \mathcal{O}(V_\gamma, V_\delta, V_\alpha, V_\beta) \) as follows:

\[
g(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d) = (V_\delta, V_\beta, e_1, e_2, e_3, e_4, cg^{-1}, gd)
\]

We denote the orbit spaces of \( \mathcal{O}(V_\gamma, V_\delta, V_\alpha, V_\beta) \) and \( \mathcal{O}(\xi, \eta, \xi', \eta') \) by \( \mathcal{O}(V_\gamma, V_\delta, V_\alpha, V_\beta) \) and \( \mathcal{O}(\xi, \eta, \xi', \eta')^* \) respectively under the actions of \( G_{\Delta} \). The following proposition can be viewed as the geometrization of Proposition 2.3.

**Proposition 6.2.** [DXX] There is a homeomorphism

\[
\theta^* : Q(\xi, \eta, \xi', \eta')^* \rightarrow \mathcal{O}(\xi, \eta, \xi', \eta')^*
\]

induced by the map between \( Q(\xi, \eta, \xi', \eta') \) and \( \mathcal{O}(\xi, \eta, \xi', \eta') \).

There is an action of \( G_{\xi} \times G_{\eta} \) on \( \mathcal{O}(\xi, \eta, \xi', \eta')^* \), defined as follows: for \((g_1, g_2) \in G_{\xi} \times G_{\eta} \),

\[
(g_1, g_2)(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d)^* = (V_\delta, V_\beta, g_2e_1, e_2g_2^{-1}, g_1e_3, e_4g_1^{-1}, c', d')^*
\]

Let us determine the relation between \((c', d')\) and \((c, d)\).

Suppose \((V_\delta, V_\beta, g_2e_1, e_2g_2^{-1}, g_1e_3, e_4g_1^{-1})\) induces \( S', T' \) and the unique map \( f' : S' \rightarrow T' \), then it is clear there are isomorphisms:

\[
a_1 : S \rightarrow S' \quad \text{and} \quad a_2 : T \rightarrow T'
\]
induced by isomorphisms:
\[
\begin{pmatrix}
g_2 & 0 \\
0 & \text{id}
\end{pmatrix} : V_\eta \oplus V_{\eta'} \to V_\eta \oplus V_{\eta'} \quad \text{and} \quad \begin{pmatrix}
g_1 & 0 \\
0 & \text{id}
\end{pmatrix} : V_{\xi} \oplus V_{\xi'} \to V_{\xi} \oplus V_{\xi'}
\]
So \( f' = a_2f a_1^{-1} \), we have the following commutative diagram:

\[
\begin{array}{ccc}
S' & \xrightarrow{d'} & V_\lambda \\
\downarrow a_1 & & \downarrow d_1 \\
S & \xrightarrow{d} & V_\lambda \\
\downarrow c & & \downarrow e \\
\text{Im} f & \xrightarrow{\gamma} & T \\
\downarrow a_2 & & \downarrow a_2 \\
\text{Im} f' & \xrightarrow{\gamma} & T'
\end{array}
\]

Hence, \( c' = a_2c a_1^{-1} \) and \( d' = gda_1^{-1} \). In particular, \( c = c' \) and \( d = d' \) if and only if \( g_1 = \text{id}_{V_\eta} \) and \( g_2 = \text{id}_{V_{\eta'}} \). This shows the action of \( G_{\xi} \times G_{\eta} \) is free.

Its orbit space is denoted by \( O(\xi, \eta, \xi', \eta')^\wedge \). The above homeomorphism induces the Proposition

**Proposition 6.3.** There exists a homeomorphism under quotient topology
\[
\theta^\wedge : Q(\xi, \eta, \xi', \eta')^\wedge \to O(\xi, \eta, \xi', \eta')^\wedge.
\]

6.3. Let \( D(\xi, \eta, \xi', \eta')^* \) be the set of \((V_\delta, V_\beta, e_1, e_2, e_3, e_4)\) satisfying the diagram \( \text{[17]} \), in particular, its subset with four vertexes \( V_\gamma, V_\delta, V_\alpha, V_\beta \) is denoted by \( D_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}^* \).

Then we have a projection:
\[
\varphi^* : O(\xi, \eta, \xi', \eta')^* \to D(\xi, \eta, \xi', \eta')^*.
\]

The fibre of this morphism is a affine space with dimension \( \dim_{\mathbb{C}} \text{Ext}^1(V_\gamma, V_\beta) \) for any element in \( D_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}^* \).

There is also an action of group \( G_{\xi} \times G_{\eta} \) on \( D(\xi, \eta, \xi', \eta')^* \) with the stable group isomorphic to the vector space \( \text{Hom}(V_\gamma, V_\alpha) \times \text{Hom}(V_\beta, V_\beta) \). The orbit space is denoted by \( D(\xi, \eta, \xi', \eta')^\wedge \). The projection naturally induces the projection:
\[
\varphi^\wedge : O(\xi, \eta, \xi', \eta')^\wedge \to D(\xi, \eta, \xi', \eta')^\wedge
\]

Its fibre for \((V_\delta, V_\beta, e_1, e_2, e_3, e_4)^\wedge\) is isomorphism to the quotient space of
\[
(\varphi^*)^{-1}(V_\delta, V_\beta, e_1, e_2, e_3, e_4)
\]
under the action of \( \text{Hom}(V_\gamma, V_\alpha) \times \text{Hom}(V_\beta, V_\beta) \). The corresponding stable subgroup is
\[
\{(g_1, g_2) \in 1 + e_3\text{Hom}(V_\gamma, V_\alpha)e_4 \times 1 + e_1\text{Hom}(V_\delta, V_\beta)e_2 \mid g_1a' = a'g, b'g = g_2b' \}
\]
where \( a, b, a', b' \) are induced by diagram \( \text{[17]} \). It is isomorphic to the vector space \( G((a, b, a', b')^\wedge) \). Therefore, we have
Proposition 6.4. There exists a projection
\[ \varphi^\wedge : O(\xi, \eta, \xi', \eta')^\wedge \to D(\xi, \eta, \xi', \eta')^\wedge \]
with the fibre isomorphic to affine space of dimension
\[ \dim_{C}(\text{Ext}^1(V_{\gamma}, V_{\beta}) \times G((a, b, a', b')^\wedge)/\text{Hom}(V_{\gamma}, V_{\alpha}) \times \text{Hom}(V_{\delta}, V_{\beta})). \]

Now we have the following diagram of morphisms:
\[ \text{(19)} \]
\[ \text{Ext}^1(V_{\xi'}, V_{\eta'}) \xrightarrow{\delta^\wedge} Q(\xi, \eta, \xi', \eta')^\wedge \xrightarrow{\varphi^\wedge} O(\xi, \eta, \xi', \eta')^\wedge \xrightarrow{\varphi^\wedge} D(\xi, \eta, \xi', \eta')^\wedge \]

6.4. In this subsection, we consider the action of \( C^* \) on each term in (19).
(1) For \( t \in C^* \) and \( \varepsilon = (a, b)^* \in \text{Ext}^1(V_{\xi'}, V_{\eta'}), \) set \( t.\varepsilon = (t^{-1}a, b)^* \).
(2) For \( t \in C^* \) and \( (a, b, a', b')^\wedge \in Q(\xi, \eta, \xi', \eta')^\wedge, \) we note that \( Q(\xi, \eta, \xi', \eta')^\wedge \) is just
\[ \{ (\varepsilon, L_1 \subseteq L) \mid \varepsilon \in \text{Ext}^1(V_{\xi'}, V_{\eta'}), L_1 \cong V_{\eta}, L_1/L_1 \cong V_{\xi} \}. \]
Set
\[ t.(a, b, a', b')^\wedge = (t^{-1}a, b, t.a'(V_{\eta}), b't_{b'}^{-1})^\wedge \]
where
\[ t.a'(V_{\eta}) : a'(V_{\eta}) \to t.a'(V_{\eta}) \]
and
\[ t_{b'}.V_{\lambda}/a'(V_{\eta}) \to V_{\lambda}/t.a'(V_{\eta}) \]
are under the second kind of \( C^* \)-action in Section 5.2.
(3) For \( t \in C^* \) and \( (V_{\delta}, V_{\beta}, e_1, e_2, e_3, e_4, c, d)^\wedge \in O(\xi, \eta, \xi', \eta')^\wedge, \) set
\[ t.(V_{\delta}, V_{\beta}, e_1, e_2, e_3, e_4, c, d)^\wedge = (t.V_{\delta}, t.V_{\beta}, t^{-1}e_1t_{V_{\beta}}^{-1}, t.V_{e_2}, t^{-1}e_3t_{V_{\alpha}}^{-1}, t.V_{e_4}, e_1, c, d, t,d)^\wedge. \]
(4) For \( t \in C^* \) and \( (V_{\delta}, V_{\beta}, e_1, e_2, e_3, e_4)^\wedge \in D(\xi, \eta, \xi', \eta')^\wedge, \) set
\[ t.(V_{\delta}, V_{\beta}, e_1, e_2, e_3, e_4)^\wedge = (t.V_{\delta}, t.V_{\beta}, t^{-1}e_1t_{V_{\beta}}^{-1}, t.V_{e_2}, t^{-1}e_3t_{V_{\alpha}}^{-1}, t.V_{e_4})^\wedge \]
where \( t.V_{\alpha} = a^{-1}t_{b'}a, t.V_{\beta} = a^{-1}t.a'(V_{\eta})a, t.V_{\gamma} = bt_{b'}b^{-1}, t.V_{\delta} = bt.a'(V_{\eta})b^{-1} \)
and \( t.V_{\delta}, t.V_{\beta} \)
are under the first kind of \( C^* \)-action in Section 5.2.

By considering the Euler characteristic of the orbit space of every diagram \[ \text{(19)} \]
under \( C^* \)-action, we have the following theorem, which can be viewed as a geometric version of Green’s formula under the \( C^* \)-action.

**Theorem 6.5.** For fixed \( \xi, \eta, \xi', \eta' \), we have
\[
\int_{\lambda \neq \xi', \eta'} \chi(P\text{Ext}^1(V_{\xi'}, V_{\eta'}), \lambda)g_{\lambda \eta}^2 = \\
\int_{\alpha, \beta, \gamma, \alpha \neq \gamma, \alpha \neq \gamma} \chi(P(\text{Ext}^1(V_{\gamma}, V_{\alpha}) \times \text{Ext}^1(V_{\delta}, V_{\beta})), \gamma)g_{\beta \gamma}^\xi g_{\alpha \beta}^\eta \\
+ \int_{\alpha, \beta, \gamma, \alpha = \gamma, \beta = \gamma} [d(\xi', \eta') - d(\gamma, \alpha) - d(\delta, \beta) - \langle \gamma, \beta \rangle]g_{\beta \gamma}^\xi g_{\alpha \beta}^\eta \\
- \int_{\alpha, \beta, \gamma, \alpha = \gamma, \beta = \gamma} \chi(P(V_{\xi}, V_{\eta}; V_{\xi'} \oplus V_{\eta'}))
\]
where \( P(V_{\xi}, V_{\eta}; V_{\xi'} \oplus V_{\eta'}) \) is the orbit space of \( \nabla(V_{\xi}, V_{\eta}; V_{\xi'} \oplus V_{\eta'}) \) defined in Section 5.2.
7. Cluster multiplication formula

7.1. Let $Q$ be a simply-laced Dynkin quiver. Let $C_Q$ and $A(Q)$ be the corresponding cluster category and cluster algebra respectively. As Section 6 showed, $A(Q)$ is a cluster categories of finite type. A natural question is how to realize the cluster algebra $A(Q)$ by the cluster category $C_Q$. In [CK1], the authors extended the Caldero-Chapoton formula to give a cluster multiplication formula to answer this question. For a variety $X$, we define $\chi_c(X)$ to be the Euler-Poincaré characteristic of the étale cohomology with proper support of $X$.

**Theorem 7.1.** For any objects $M$, $N$ of $\mathcal{C}$, we have

(i) If $\text{Ext}^1(M, N) = 0$, then $X_MX_N = X_M \oplus X_N$,

(ii) If $\text{Ext}^1(M, N) \neq 0$, then

$$\chi_c(\text{PExt}^1(M, N))X_MX_N = \sum_Y (\chi_c(\text{PExt}^1(M, N)Y) + \chi_c(\text{PExt}^1(N, M)Y))X_Y,$$

where $Y$ runs through the isoclasses of $\mathcal{C}$.

The formula in this theorem is called Caldero-Keller formula. The proof of this theorem depends on the Caldero-Chapoton formula and the fact that the cluster category is 2-Calabi-Yau described as follows under the context of module category. For $M_1 \subseteq M$ and $N_1 \subseteq N$, we consider the map

$$\beta' : \text{Ext}^1(M, N) \oplus \text{Ext}^1(M_1, N_1) \rightarrow \text{Ext}^1(M_1, N)$$

sending $(\varepsilon, \varepsilon')$ to $\varepsilon_{M_1} - \varepsilon_N'$ where $\varepsilon_{M_1}$ and $\varepsilon'_N$ are induced by including $M_1 \subseteq M$ and $N_1 \subseteq N$, respectively as follows:

$$\varepsilon_{M_1} : 0 \rightarrow N \rightarrow L_1 \rightarrow M_1 \rightarrow 0$$

$$\varepsilon : 0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$$

where $L_1$ is the pullback, and

$$\varepsilon' : 0 \rightarrow N_1 \rightarrow L' \rightarrow M_1 \rightarrow 0$$

$$\varepsilon'_N : 0 \rightarrow N \rightarrow L'_1 \rightarrow M_1 \rightarrow 0$$

where $L'_1$ is the pushout. It is clear that $\varepsilon, \varepsilon'$ and $M_1, N_1$ induce the inclusions $L_1 \subseteq L$ and $L' \subseteq L'_1$. Set Let

$$p_0 : \text{Ext}^1(M, N) \oplus \text{Ext}^1(M_1, N_1) \rightarrow \text{Ext}^1(M, N)$$

Using 2-Calabi-Yau property (Auslander-Reiten formula) $\text{Ext}^1(M, N) \cong D\text{Hom}(N, \tau M)$, we can consider the dual of $\beta'$

$$\beta : \text{Hom}(N, \tau M_1) \rightarrow \text{Hom}(N, \tau M) \oplus \text{Hom}(N_1, \tau M_1)$$

Using bilinear form and orthogonality, we know

**Proposition 7.2.**

$$(p_0(Ker\beta'))^\perp = \text{Im}\beta \bigcap \text{Hom}(N, \tau M).$$

We give an example to illustrate Theorem [CK1].
Example 7.3. Assume $Q$ is the quiver of type $A_2$:

\[ 
\begin{array}{c}
1 \\
\end{array} \xrightarrow{f} \begin{array}{c}
2
\end{array}
\]

Set $M = S_2 \oplus S_2$, $N = S_1 \oplus S_1$. The middle term $Y$ is either $S_1 \oplus P_2 \oplus S_2$ or $P_2 \oplus P_2$ if $Y$ is an object such that $\text{Ext}^1(M, N)_Y$ is not empty. The cardinality of $\text{Ext}^1(N, M)_Y$ on $\mathbb{F}q$ is respectively $q^2 + 2q + 1$ and $q(q^2 - 1)$. In a dual way, the middle term $Y$ is either $S_1 \oplus S_2$ or 0 if $\text{Ext}^1(N, M)_Y$ is not empty. The cardinality of $\text{Ext}^1(N, M)_Y$ on $\mathbb{F}q$ is respectively $q^2 + 2q + 1$ and $q(q^2 - 1)$. The cluster multiplication theorem gives:

$$ X_N X_M = X_{S_1 \oplus P_2 \oplus S_2} + X_{S_1 \oplus S_2}. $$

7.2. Let $\Lambda = \mathbb{C}Q$ be a hereditary algebra associated to a connected quiver without oriented cycles. The following theorem generalizes Caldero-Keller’s cluster multiplication theorem for cluster categories of finite type to the following theorem for cluster categories of any type. Define

$$ \text{Hom}(L_1, L_2)|_{Y[1] \oplus X} = \{ g \in \text{Hom}(L_1, L_2) | \text{Ker} g \simeq Y, \text{Coker} g \simeq X \} $$

Theorem 7.4. (1) For any $\Lambda$-modules $V_{\xi'}, V_{\eta'}$ we have

$$ d^1(\xi', \eta')X_{V_{\xi'}} X_{V_{\eta'}} = \int_{\lambda \neq \xi' \oplus \eta'} \chi(\text{Ext}^1(V_{\xi'}, V_{\eta'})_\lambda) X_{V_\lambda} $$

and

$$ + \int_{\gamma, \delta, \iota} \chi(\text{Hom}(V_{\eta'}, V_{\xi'})_{\gamma[1] \oplus \tau V_{\xi'} \oplus I_0}) X_{V_\gamma} X_{V_\delta} x^{\dim \text{soc} I_0} $$

where $I_0$ is injective and $V_\gamma = V_{\eta'} \oplus P_0$, $P_0$ is the projective direct summand of $V_{\xi'}$.

(2) For any $\Lambda$-module $V_{\xi'}$ and $P \in \rho$ is projective. Then

$$ d(\rho, \xi')X_{V_{\xi'}} x^{\dim P/\text{rad} P} = \int_{\delta, \lambda'} \chi(\text{Hom}(V_{\xi'}, I)_{V_{\delta[1] \oplus \lambda'}}) X_{V_\delta} x^{\dim \text{soc} \lambda'} $$

and

$$ + \int_{\gamma, \rho'} \chi(\text{Hom}(P, V_{\xi'})_{P[1] \oplus \tau V_{\xi'}}) X_{V_{\gamma}} x^{\dim P'/\text{rad} P'} $$

where $I = \text{DHom}(P, \Lambda)$, and $\iota' \in \iota'$ injective, $P' \in \rho'$ projective.

Remark. The formula in this theorem varies from the reformulation in [Hu2, Theorem 12]. It extends Theorem 7.2. Indeed, if $\Lambda = \mathbb{C}Q$ for a simply laced Dynkin quiver, then by Proposition 5.4 we can assume $\text{Ext}^1(N, M) = 0$ and $\text{Ext}_C(M, N) = \text{Ext}_A(M, N)$. Furthermore, we have [Hu2, Lemma 13]

$$ \text{Ext}_C(\Lambda, N)_E \simeq \text{Ext}_A(\Lambda, N)_E $$

and

$$ \text{Ext}_C(\Lambda, N)_{K \oplus C[-1]} \simeq \text{Hom}(N, \tau M)_{K[1] \oplus C} $$

The proof of this theorem depends on Theorem 6.3 and the projectivization of “Higher order” associativity described as follows. For any $\Lambda$-modules $X, Y, L_1$ and $L_2$, we define

$$ W(X, Y; L_1, L_2) := \{(f, g, h) | 0 \xrightarrow{f} Y \xrightarrow{g} L_1 \xrightarrow{\rho} L_2 \xrightarrow{h} X \xrightarrow{0} \} $$

is an exact sequence.
Under the actions of $G_\alpha \times G_\beta$, where $\alpha = \dim X$ and $\beta = \dim Y$, the orbit space is denoted by $V(X,Y; L_1, L_2)$. In fact, 
\[ V(X,Y; L_1, L_2) = \{ g : L_1 \to L_2 \mid \text{Ker} g \cong Y \text{ and Coker} g \cong X \} \]

Put
\[ h_{XY}^{L_1 L_2} = \chi(V(X,Y; L_1 L_2)) \]
We have the following “higher order” associativity.

**Theorem 7.5.** For fixed $\Lambda$-modules $X,Y,L_i$ for $i = 1,2$, we have
\[ \int_Y g_Y^X f_{XY}^{L_1 L_2} h_{XY}^{L_1 L_2} = \int_{L_1} g_{L_1 Y_1}^X f_{L_1 Y_1}^{L_1 L_2} h_{L_1 Y_1}^{L_1 L_2} \]
Dually, for fixed $\Lambda$-modules $X_i,Y,L_i$ for $i = 1,2$, we have
\[ \int_X g_X^{X_2 X_1} h_{X_2 X_1}^{X_2 X_1} = \int_{L_2} g_{X_2 L_2}^X h_{X_2 L_2}^X \]
We have
\[ V(X,Y; L_1, L_2) = \text{Hom}(L_1, L_2)_{Y[1] \oplus X} \]
There is a natural $C^*$-action on $\text{Hom}(L_1, L_2)_{Y[1] \oplus X}$ or $V(X,Y; L_1, L_2)$ simply by $t.(f,g,h)^* = (f,tg,ht)^*$ for $t \in C^*$ and $(f,g,h)^* \in V(X,Y; L_1, L_2)$. We also have a projective version of Theorem 7.5 where $P$ indicates the corresponding orbit space under the $C^*$-action.

**Theorem 7.6.** For fixed $\Lambda$-modules $X_i,Y_i,L_i$ for $i = 1,2$, we have
\[ \int_Y g_Y^X \chi(P\text{Hom}(L_1, L_2)_{Y[1] \oplus X}) = \int_{L_1} g_{L_1 Y_1}^X \chi(P\text{Hom}(L_1, L_2)_{Y_2[1] \oplus X}) \]
\[ \int_X g_X^{X_2 X_1} \chi(P\text{Hom}(L_1, L_2)_{Y[1] \oplus X}) = \int_{L_2} g_{X_2 L_2}^X \chi(P\text{Hom}(L_1, L_2')_{Y[1] \oplus X_1}) \]

7.3. We illustrate Theorem 7.2 by the following example for infinite type quiver.

**Example 7.7.** Let $Q$ be the Kronecker quiver $\begin{array}{c} 1 \end{array} \rightarrow \begin{array}{c} 2 \end{array}$. Let $S_1$ and $S_2$ be simple modules of vertex 1 and 2, respectively. Hence,
\[ R = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R' = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \]
\[ X_{S_1} = x^{\dim S_1 R' - \dim S_1} + x^{\dim S_1 R - \dim S_1} = x_1^{-1}(1 + x_2^2) \]
\[ X_{S_2} = x^{\dim S_2 R' - \dim S_2} + x^{\dim S_2 R - \dim S_2} = x_2^{-1}(1 + x_1^2) \]
For $\lambda \in P^1(C)$, let $u_\lambda$ be the regular representation $C \xrightarrow{1/\lambda} C$. Then
\[ X_{u_\lambda} = x^{(1,1) R'-(1,1)} + x^{(1,1) R-(1,1)} + x^{(0,1) R+(1,0) R'-(1,1)} = x_1 x_2^{-1} + x_1^{-1} x_2 + x_1^{-1} x_2^{-1} \]
Let $I_1$ and $I_2$ be indecomposable injective modules corresponding vertex 1 and 2, respectively, then
\[ X_{(I_1 \oplus I_2)[-1]} := x^{\dim_{soc}(I_1 \oplus I_2)} = x_1 x_2 \]
The left side of the identity of Theorem 7.4 is
\[ \dim_C \text{Ext}^1(S_1, S_2) X_{S_1} X_{S_2} = 2(x_1^{-1} x_2^{-1} + x_1^{-1} x_2^{-1} + x_1^{-1} x_2 + x_1 x_2) \]
The first term of the right side is
\[
\int_{\Lambda \in \mathcal{P}(\mathbb{C})} \chi(\mathbb{P} \text{Ext}^1(S_1, S_2)_{\Lambda}) X_{\Lambda} = 2(x_1^{-1}x_2^{-1} + x_1x_2^{-1} + x_1^{-1}x_2)
\]

As for the second term of the right side, we note that for any \( f \neq 0 \in \text{Hom}(S_2, \tau S_1) \), we have the exact sequence:
\[
0 \to S_2 \xrightarrow{f} \tau S_1 \to I_1 \oplus I_2 \to 0
\]

This implies \( \text{Hom}(S_2, \tau S_1) \setminus \{0\} = \text{Hom}(S_2, \tau S_1)_{I_1 \oplus I_2} \). Hence, the second term is equal to \( 2x_1x_2 \).

8. 2-Calabi-Yau and Cluster multiplication formula

A category \( \mathcal{C} \) is 2-Calabi-Yau if there is an almost canonical non-degenerate bifunctorial pairing
\[
\phi : \text{Ext}^1(M, N) \times \text{Ext}^1(N, M) \to \mathbb{C}
\]

for any object \( M, N \in \mathcal{C} \). In particular, if \( \mathcal{C} \) is the category of nilpotent \( \Lambda \)-modules for some algebra \( \Lambda \) over \( \mathbb{C} \), then \( \Lambda \) is called a 2-Calabi-Yau algebra. In particular, if \( \Lambda \) is the preprojective algebra associated to a connected quiver \( Q \) without loops, then \( \Lambda \) is a 2-Calabi-Yau algebra. Let \( \Lambda = \mathbb{C}Q / \langle R \rangle \) be the 2-Calabi-Yau algebra associated to a connected quiver \( Q \) without loops and let \( I \) be the set of vertices of \( Q \).

Let \( \Lambda_d \) be the affine variety of nilpotent \( \Lambda \)-modules of \( \text{dim}. \text{vector} = d \). For \( c = (c_1, \ldots, c_m) \in \{0, 1\}^m \) and \( \mathbf{j} = (j_1, \ldots, j_m) \) a sequence of elements of \( I \), let \( x \in \Lambda_d \) be a \( x \)-stable flag of type \( (\mathbf{j}, c) \) as a composition series of \( x \)
\[
f_x = (V = (\mathbb{C}^d, x) \supseteq V^1 \supseteq \cdots \supseteq V^m = 0)
\]

of \( \Lambda \)-submodules of \( V \) such that \( |V^k - V^{k-1}| = c_k S_{j_k} \) where \( S_{j_k} \) is the simple module at the vertex \( j_k \). Let \( d_{j,c} \) be the constructible functions satisfying \( d_{j,c}(x) = \chi(\Phi_{j,c,x}) \) where \( x \in \Lambda_d \) and \( \Phi_{j,c,x} \) is the variety of \( x \)-stable flags of type \( (\mathbf{j}, c) \). We simply write \( d_j \) if \( c = (1, \ldots, 1) \). Define \( M(d) \) to be the vector space spanned by \( d_j \). Define
\[
\Phi_j(\Lambda_d) = \{ (x, f) \mid x \in \Lambda_d, f \in \Phi_j(x) \}.
\]

We consider a natural projection: \( p : \Phi_j(\Lambda_d) \to \mathbb{Q} \) mapping \( x \) to \( \chi(\Phi_{j,x}) \) is constructible by Theorem 1.2.

**Proposition 8.1.** For a type \( j \), the function \( \Lambda_d \to \mathbb{Q} \) mapping \( x \) to \( \chi(\Phi_{j,x}) \) is constructible.

For fixed \( \Lambda_d \), there are finitely many types \( j \) such that \( \Phi_j(\Lambda_d) \) is not empty. Hence, there exists a finite subset \( S(d) \) of \( \Lambda_d \) such that
\[
\Lambda_d = \bigcup_{M \in S(d)} \langle M \rangle
\]

where \( \langle M \rangle = \{ M' \in \Lambda_d \mid \chi(\Phi_{j,M'}) = \chi(\Phi_{j,M}) \text{ for any type } j \} \).

For any \( M \in \Lambda_d \), we define the evaluation form \( \delta_{M} : M(d) \to \mathbb{C} \) mapping a constructible function \( f \in M(d) \) to \( f(M) \). We have
\[
\langle M \rangle = \{ M' \in \Lambda_d \mid \delta_{M'} = \delta_{M} \}
\]
We have the following multiplication formula \[XX3\].

**Theorem 8.2.** For any \(\Lambda\)-modules \(M\) and \(N\), we have

\[
\chi(\mathbb{P}\text{Ext}_\Lambda^1(M, N))\delta_{M\oplus N} = \sum_{L \in S_\Lambda} (\chi(\mathbb{P}\text{Ext}_\Lambda^1(M, N)) + \chi(\mathbb{P}\text{Ext}_\Lambda^1(N, M)))\delta_L
\]

where \(c = \dim M + \dim N\).

**Remark.** In the case that \(\Lambda\) is a preprojective algebra, the above multiplication formula is given in [GLS, Theorem 1].

The proof of this theorem heavily depends on the fact that the property of 2-Calabi-Yau induces the symmetry of the following two linear maps. Define [GLS, Section 2]

\[
\beta_{j, c', c'', f_M, f_N} : \bigoplus_{k=0}^{m-2} \text{Ext}_\Lambda^1(N_k, M_{k+1}) \rightarrow \bigoplUS_{k=0}^{m-2} \text{Ext}_\Lambda^1(N_k, M_k)
\]

by the following map

\[
\begin{array}{ccc}
N_k & \xrightarrow{\varepsilon_k} & M_{k+1}[1] \\
\downarrow \iota_{N,k} & & \downarrow \iota_{M,k+1} \\
N_{k-1} & \xrightarrow{\varepsilon_{k-1}} & M_k[1]
\end{array}
\]

satisfying

\[
\beta_{j, c', c'', f_M, f_N} = \iota_{M,1} \circ \varepsilon_0 + \sum_{k=1}^{m-2} (\iota_{M,k+1} \circ \varepsilon_k - \varepsilon_{k-1} \circ \iota_{N,k}).
\]

Depending on the 2-Calabi-Yau property of \(\Lambda\), we can write down its dual.

\[
\beta'_{j, c', c'', f_M, f_N} : \bigoplUS_{k=0}^{m-2} \text{Ext}_\Lambda^1(M_k, N_k) \rightarrow \bigoplUS_{k=0}^{m-2} \text{Ext}_\Lambda^1(M_{k+1}, N_k)
\]

by the following map

\[
\begin{array}{ccc}
M_{k+1} & \xrightarrow{\eta_{k+1}} & N_{k+1}[1] \\
\downarrow \iota_{M,k+1} & & \downarrow \iota_{N,k+1} \\
M_k & \xrightarrow{\eta_k} & N_k[1]
\end{array}
\]

satisfying

\[
\beta'_{j, c', c'', f_M, f_N} = \sum_{k=0}^{m-3} (\eta_k \circ \iota_{M,k+1} - \iota_{N,k+1} \circ \eta_{k+1}) + \eta_{m-2} \circ \iota_{M,m-1}.
\]

By the property of 2-Calabi-Yau, we have

\[
\text{Ker}(\beta'_{j, c', c''}) = \text{Im}(\beta_{j, c', c'', f_M, f_N})^\perp.
\]

Of course, the next question is we should try to set up the same formula over more general 2-Calabi-Yau categories.
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