$t$-Resilient $k$-Immediate Snapshot and its Relation with Agreement Problems

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Abstract

An immediate snapshot object is a high level communication object, built on top of a read/write distributed system in which all except one processes may crash. It provides the processes with a single operation denoted $\text{write\_snapshot}_k()$, which allows a process to write a value and obtain a set of pairs (process id, value) satisfying some set containment properties, that represent a snapshot of the values written to the object, occurring immediately after the write step.

Considering an $n$-process model in which up to $t$ processes may crash, this paper introduces first the $k$-resilient immediate snapshot object, which is a natural generalization of the basic immediate snapshot (which corresponds to the case $k = t = n - 1$). In addition to the set containment properties of the basic immediate snapshot, a $k$-resilient immediate snapshot object requires that each set returned to a process contains at least $(n - k)$ pairs.

The paper first shows that, for $k, t < n - 1$, $k$-resilient immediate snapshot is impossible in asynchronous read/write systems. Then the paper investigates a model of computation where the processes communicate with each other by accessing $k$-immediate snapshot objects, and shows that this model is stronger than the $t$-crash model. Considering the space of $x$-set agreement problems (which are impossible to solve in systems such that $x \leq t$), the paper shows then that $x$-set agreement can be solved in read/write systems enriched with $k$-immediate snapshot objects for $x = \max(1, t + k - (n - 2))$. It also shows that, in these systems, $k$-resilient immediate snapshot and consensus are equivalent when $1 \leq t < n/2$ and $t \leq k \leq (n - 1) - t$. Hence, the paper establishes strong relations linking fundamental distributed computing objects (one related to communication, the other to agreement), which are impossible to solve in pure read/write systems.

Keywords: Asynchronous system, Atomic read/write register, Consensus, Distributed computability, Immediate snapshot, Impossibility, Iterated model, $k$-Set Agreement, Linearizability, Process crash failure, Snapshot object, $t$-Resilience, Wait-freedom.

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1 Introduction

Context This article considers the $t$-crash model consisting of $n$ asynchronous processes, among which any subset of at most $t$ processes may crash, and communicate through a shared memory composed of single writer/multi reader (SWMR) atomic registers. The $(n-1)$-crash model is also called wait-free model [12]. We keep the term $t$-resilience for algorithms. Several progress conditions have been associated with $(n-1)$-resilient algorithms: wait-freedom [12], non-blocking [18], or obstruction-freedom [14] (see a unified presentation in Chapter 5 of [23]). This article focuses on the wait-free condition, in the context of tasks: every non-failed process has to produce an output value. A task is defined in terms of (a) possible inputs to the processes, and (b) valid outputs for each assignment of input values (tasks are precisely defined in [6, 15, 17]). Of special importance is the family of $x$-set agreement tasks [8], one for each integer value of $x$, $1 \leq x \leq n$. Set agreement was introduced to show a hierarchy of tasks whose solvability depends on $t$, the number of processes that may crash. In the $x$-set agreement task, processes decide at most $x$ different values, out of their input assignments. When $x = 1$, $x$-set agreement is the celebrated consensus task, which is impossible even in the presence of a single process crash [10, 20]. More generally, $x$-set agreement is solvable if and only if $t < x$, a result proved using algebraic topology [3, 6, 17, 24]. There are characterizations of the solvability of any given task, in the $t$-crash model, and in others (for an overview of results see [13]).

Immediate snapshot object The immediate snapshot (IS) communication object was first introduced in [4, 24], and then further investigated as an “object” in [3]. This object is at the heart of the iterated immediate snapshot (IIS) model introduced in [5, 16], which consists of $n$ asynchronous processes, among which any subset at most $(n-1)$ processes may crash. These processes execute a sequence of asynchronous rounds, and each round is provided with exactly one IS object, which allows the processes to communicate only during this round. More precisely, for any $x > 0$, a process accesses the $x$-th immediate snapshot only when it executes the $x$-th round, and it accesses it only once.

From an abstract point of view, an IS object $IS$, can be seen as an initially empty set, which can then contain up to $n$ pairs (one per process), each made up of a process index and a value. This object provides each process with a single operation denoted $\text{write} \_\text{snapshot}()$, that it can invoke only once. The invocation $\langle IS, \text{write} \_\text{snapshot}(v) \rangle$ by a process $p_i$ adds the pair $(i, v)$ to $IS$ and returns a set of pairs belonging to $IS$ such that the sets returned to the processes that invoke $\text{write} \_\text{snapshot}()$ satisfy specific inclusion properties. It is important to notice that, in the IIS model, the processes access the sequence of IS objects one after the other, in the same order, and asynchronously.

Contribution of the paper As previously said, the IS object was designed for the wait-free model (i.e., $t = n - 1$). This paper considers it in the context of the $t$-crash $n$-process system models where $t < n - 1$. To this end it generalizes the IS object by introducing the notion of a $k$-immediate snapshot ($k$-IS) object. Such an object provides the processes with a single operation denoted $\text{write} \_\text{snapshot}_k()$ which, in addition to the properties of an IS object, returns a set including at least $(n-k)$ pairs. Hence, for $k < n - 1$, due to the implicit synchronization implied by the constraint on the minimal size of the sets it returns, a $k$-IS object allows processes to obtain more information from the whole set of processes than a simple IS object (which may return sets containing less than $(n-k)$ pairs).

The obvious question is then the implementability of a $k$-IS object in the $t$-crash $n$-process asynchronous read/write model. The paper shows first that, differently from the basic IS object which can be implemented in the wait-free model, no $k$-IS object where $k < n - 1$, can be implemented in a 1-crash $n$-process read/write system.

This impossibility result is far from being the first impossibility result in the presence of asynchrony and process crashes. The most famous of them, which concern agreement problems, are the impossibility of Consensus (CONS) in the presence of even a single process crash [10, 20], and the impossibility of
x-set agreement (x-SA) when $x \leq t$ [4, 17, 24] (let us remind that CONS is 1-SA). These objects are at the heart of the theory of fault-tolerant distributed computing.

Hence, a second natural question: Are they relations linking the previous “impossible” objects, namely k-IS and x-SA? The paper provides the following answers to this question.

- Let $1 \leq k \leq t < n$. It is possible to implemented a k-IS object in a t-crash n-process read/write system enriched with consensus objects.
- Let $1 \leq t < n/2$ and $t \leq k \leq (n - 1) - t$. k-IS and Consensus are equivalent in a t-crash n-process read/write system. (A and B are equivalent if A can be implemented in the t-crash n-process read/write system enriched with B, and reciprocally.)
- Let $(n - 1)/2 \leq k \leq n - 1$ and $(n - 1) - k \leq t \leq k$. It is possible to implemented an x-SA object, where $x = t + k - (n - 2)$, in a t-crash n-process read/write system enriched with k-IS objects.

**Roadmap**  The paper develops the previous map. It is made up of 7 sections. Section 2 presents the basic t-crash n-process asynchronous read/write model, and the definitions of the IS and x-SA objects. Section 3 defines the k-IS object and its impossibility in the previous basic model. The other sections are on the power of k-IS with respect to x-SA. Section 4 shows that x-SA can be built in the t-crash n-process asynchronous read/write model enriched with k-IS objects, for $x = \text{max}(1, t + k - (n - 2))$. Section 5 shows that t-IS and CONS are equivalent in the t-crash n-process asynchronous read/write model when $1 \leq t < n/2$. Section 6 shows that CONS is stronger than k-IS when $n/2 \leq t \leq k < n - 1$. Finally, Section 7 concludes the paper. An illustration of the previous results is presented in Table 1, which considers a system of $n = 11$ processes.

| $k \rightarrow$ | 1  | 2  | 3  | ... | ... | ... | n - 4 | n - 3 | n - 2 | n - 1 |
|-----------------|----|----|----|------|------|------|------|------|------|------|
| $t \downarrow$  | 1  | SA | SA | SA  | SA  | 1-SA | 1-SA | 1-SA | 1-SA | 1-SA |
| 2               | 1-SA | SA | SA | SA  | 1-SA | 1-SA | 1-SA | 1-SA | 1-SA | 1-SA |
| 3               | 1-SA | 1-SA | SA | SA  | | | | | | |
| 4               | 1-SA | 1-SA | 1-SA | SA  | | | | | | |
| $< n/2$         | 1-SA | 1-SA | 1-SA | 1-SA | | | | | | |
| $\geq n/2$      | 1-SA | 1-SA | 1-SA | 1-SA | | | | | | |
| $7 = n - 4$     | 1-SA | 1-SA | 1-SA | 1-SA | | | | | | |
| $8 = n - 3$     | 1-SA | 1-SA | 1-SA | 1-SA | | | | | | |
| $9 = n - 2$     | 1-SA | 1-SA | 1-SA | 1-SA | | | | | | |
| $10 = n - 1$    | 1-SA | 1-SA | 1-SA | 1-SA | | | | | | |

Table 1: From k-IS to x-SA with $x = \text{max}(1, t + k - (n - 2))$ ($n = 11$)

2 Basic Model, Immediate Snapshot, and x-Set Agreement

2.1 Basic read/write system model

**Processes**  The computing model is composed of a set of $n \geq 3$ sequential processes denoted $p_1$, ..., $p_n$. Each process is asynchronous which means that it proceeds at its own speed, which can be arbitrary and remains always unknown to the other processes.

A process may halt prematurely (crash failure), but executes correctly its local algorithm until it possibly crashes. The model parameter $t$ denotes the maximal number of processes that may crash in a run. A process that crashes in a run is said to be faulty. Otherwise, it is correct or non-faulty. Let us notice that, as a faulty process behaves correctly until it crashes, no process knows if it is correct or
faulty. Moreover, due to process asynchrony, no process can know if another process crashed or is only very slow.

It is assumed that (a) \(0 < t < n\) (at least one process may crash and at least one process does not crash), and (b) any process, until it possibly crashes, executes correctly the algorithm assigned to it.

**Communication layer** The processes cooperate by reading and writing Single-Writer Multi-Reader (SWMR) atomic read/write registers \([19]\). This means that the shared memory can be seen as a set of arrays \(A[1..n]\) where, while \(A[i]\) can be read by all processes, it can be written only by \(p_i\).

**Notation** The previous model is denoted \(\text{CARW}_{n,t}[\emptyset]\) (which means “Crash Asynchronous Read/Write with \(n\) processes, among which up to \(t\) may crash”). A model constrained by a predicate on \(t\) (e.g. \(t < a\)) is denoted \(\text{CARW}_{n,t}[t < a]\). It is assumed that at least one process does not crash, \(\text{CARW}_{n,t}[t = n - 1]\) is a synonym of \(\text{CARW}_{n,t}[\emptyset]\), which (as always indicated) is called wait-free model. When considering \(t\)-crash models, \(\text{CARW}_{n,t}[t < a]\) is less constrained than \(\text{CARW}_{n,t}[t < a - 1]\). More generally, \(\text{CARW}_{n,t}[P,T]\) denotes the system model \(\text{CARW}_{n,t}[\emptyset]\) restricted by the predicate \(P\), and enriched with any number of shared objects of the type \(T\) (e.g., consensus objects).

Shared objects are denoted with capital letters. The local variables of a process \(p_i\) are denoted with lower case letters, sometimes suffixed by the process index \(i\).

### 2.2 Immediate snapshot

The immediate snapshot (IS) object \([3]\) was informally presented in the introduction. Defined in the context of the wait-free model (i.e., \(t = n - 1\)), it can be seen as a variant of the snapshot object introduced in \([1,2]\). While a snapshot object provides the processes with two operations (write() and snapshot()) which can be invoked separately by a process (usually a process invokes write() before snapshot()), a one-shot immediate snapshot object provides the processes with a single operation write_snapshot() (one-shot means that a process may invoke write_snapshot() at most once).

**Definition** Let \(IS\) be an IS object. It is a set, initially empty, that will contain pairs made up of a process index and a value. Let us consider a process \(p_i\) that invokes \(IS\).write_snapshot\((v)\). This invocation adds the pair \((i, v)\) to \(IS\) (contribution of \(p_i\) to \(IS\)), and returns to \(p_i\) a set, called view and denoted \(view_i\), such that the sets returned to the processes collectively satisfy the following properties.

- **Termination.** The invocation of write\(_\text{snapshot}\)() by a correct process terminates.
- **Self-inclusion.** \(\forall i : (i, v) \in view_i\).
- **Validity.** \(\forall i : ((j, v) \in view_i) \Rightarrow p_j\) invoked write\(_\text{snapshot}\)\((v)\).
- **Containment.** \(\forall i, j : (view_i \subseteq view_j) \lor (view_j \subseteq view_i)\).
- **Immediacy.** \(\forall i, j : ((i, v) \in view_j) \Rightarrow (view_i \subseteq view_j)\).\(^1\)

Implementations of an IS object in the wait-free model \(\text{CARW}_{n,t}[t = n - 1]\) are described in \([3,11]\ [22,23]\). While both a one-shot snapshot object and an IS object satisfy the Self-inclusion, Validity and Containment properties, only an IS object satisfies the Immediacy property. This additional property creates an important difference, from which follows that, while a snapshot object is atomic (operations on a snapshot object can be linearized \([18]\), an IS object is not atomic (its operations cannot always be linearized). However, an IS object is set-linearizable (set-linearizability allows several operations to be linearized at the same point of the time line \([7,21]\)).

\(^1\)An equivalent formulation of the Immediacy property is: \(\forall i, j : ((i, v) \in view_j) \land ((j, v) \in view_i) \Rightarrow (view_i = view_j)\).
2.3 \textit{x}-Set agreement

\textit{x}-Set agreement was introduced by S. Chaudhuri \[8\] to investigate the relation linking the number \(x\) of different values that can be decided in an agreement problem, and the maximal number of faulty processes \(t\). It generalizes consensus which corresponds to the instance \(x = 1\).

An \(x\)-set agreement (\(x\)-SA) object is a one-shot object that provides the processes with a single operation denoted \texttt{propose}_{\!\!x}()\). This operation allows the invoking process \(p_i\) to propose a value, which is called \textit{proposed} value, an is passed as an input parameter. It returns a value, called \textit{decided} value. The object is defined by the following set of properties.

- **Termination.** The invocation of \texttt{propose}_{\!\!x}() by a correct process terminates.
- **Validity.** A decided value is a proposed value.
- **Agreement.** No more than \(x\) different values are decided.

It is shown in \[4, 17, 24\] that the problem is impossible to solve in \(\mathcal{CARW}_{n,t}[x \leq t]\).

3 \textit{k}-Immediate Snapshot and its \(t\)-Resilience Impossibility

3.1 Definition and a property of \(k\)-immediate snapshot

A \(k\)-immediate snapshot (\(k\)-IS) object is an immediate snapshot object with the following additional property.

- **Output size.** The set \(\text{\textit{view}}\) obtained by a process is such that \(|\text{\textit{view}}| \geq n - k\).

This means that in addition to the Self-inclusion, Validity, Containment, and Immediacy properties, the set returned to a process contains at least \((n - k)\) pairs. The associated operation is denoted \texttt{write}_{\!\!\text{\textit{snapshot}}}_{\!\!k}()\).

\textit{k}-Immediate snapshot vs \(x\)-set agreement When considering a \(k\)-IS object and a \(x\)-SA object, we have the following differences.

- **On concurrency.** An \(x\)-SA object is atomic (linearizable), while a \(k\)-SA object is not (it is only set-linearizable \[7, 21\]). In other words, \(k\)-SA objects “accept” concurrent accesses (this is captured by the Immediacy property), while \(x\)-SA objects do not.

- **On the values returned.** When considering an \(x\)-SA object, each process \(p_i\) knows that each other process \(p_j\) (which returns from its invocation of \texttt{propose}_{\!\!x}()) obtains a single value, but it does know which one (uncertainty); \(p_i\) knows only that at most \(k\) values are decided by all processes (certainty).

When considering a \(k\)-IS object, each process \(p_i\) knows that each other process \(p_j\) (which returns from its invocation of \texttt{write}_{\!\!\text{\textit{snapshot}}}_{\!\!k}()) obtains a set of pairs \(\text{\textit{view}}_j\) that is included in, is equal to, or includes its own set of pairs (certainty due to the containment property), but it does not know the size of \(\text{\textit{view}}_j\) (uncertainty).

A property associated with \(k\)-IS objects The next theorem characterizes the power of a \(k\)-IS object in term of its Output size and Containment properties.

**Theorem 1** Let us consider a \(k\)-IS object, and assume that all correct processes invoke \texttt{write}_{\!\!\text{\textit{snapshot}}}_{\!\!k}(). If the size of the smallest view obtained by a process is \(\ell\) \((\ell \geq n - k)\), there is a set \(S\) of processes such that \(|S| = \ell\) and each process of \(S\) obtains the smallest view or crashes during its invocation of \texttt{write}_{\!\!\text{\textit{snapshot}}}_{\!\!k}().
Proof It follows from the Output size property of the $k$-IS object that no view contains less than $\ell \geq n - k$ pairs. Let $\text{min}_\text{view}$ be the smallest view returned by a process; hence $\ell = |\text{min}_\text{view}|$.

Let us consider a process $p_i$ such that $((i, -) \in \text{min}_\text{view})$, which returns a view. Due to (a) the Immediacy property (namely $((i, -) \in \text{min}_\text{view}) \Rightarrow (\text{view}_i \subseteq \text{min}_\text{view})$) and (b) the minimality of $\text{min}_\text{view}$, it follows that $\text{view}_i = \text{min}_\text{view}$. As this is true for each process whose pair participates in $\text{min}_\text{view}$, it follows that there is a set $S$ of processes such that $|S| = \ell \geq n - k$, and each of these processes obtains $\text{min}_\text{view}$, or crashes during its invocation of $\text{write\_snapshot}_p()$. Due to the Containment property, the others processes crash or obtain views which are a superset of $\text{min}_\text{view}$. $\blacksquare$

This theorem establishes the most important property of a $k$-IS object. This property is used in nearly all lemmas and theorems appearing in the paper.

3.2 An impossibility result

Theorem 2 A $k$-IS object cannot be implemented in $\text{CARW}_{n,t}[k < t]$.

Proof To satisfy the output size property, the view obtained by a process $p_i$ must contain pairs from $(n - k)$ different processes. If $t$ processes crash (e.g., initial crashes), a process can obtain at most $(n - t)$ pairs. If $t > k$, we have $n - t < n - k$. It follows that, after it has obtained pairs from $(n - t)$ processes, a process can remain blocked forever waiting for the $(t - k)$ missing pairs. $\blacksquare$

Theorem 3 Let $k < n - 1$. It is impossible to implement a $k$-IS object in $\text{CARW}_{n,t}[1 \leq t \leq k < n - 1]$.

Proof The case where $k < t$ was proved in Theorem 2. Hence, the proof considers the case $1 = t \leq k < n - 1$ (this constraint explains the model assumption $n \geq 3$, Section 2.1). If, for $k \leq n - 1$, there is no implementation of a $k$-IS object in $\text{CARW}_{n,t}[t = 1]$, there is no implementation either for $t \geq 1$. The proof is by contradiction, namely, assuming an implementation of a $k$-IS object, where $k < n - 1$, in $\text{CARW}_{n,t}[t = 1]$, we show that it is possible to solve consensus in $\text{CARW}_{n,t}[t = 1, k$-IS]. As consensus cannot be solved in $\text{CARW}_{n,t}[t = 1]$, it follows that $k$-IS cannot be implemented in $\text{CARW}_{n,t}[1 \leq t \leq k]$.

Let us recall the main property of $k$-IS (captured by Theorem 1). Let $\ell$ be the size of the smallest view ($\text{min}_\text{view}$) returned by a process. There is a set $S$ of $\ell$ processes such that any process of $S$ returns $\text{min}_\text{view}$ or crashes, and $\ell \geq n - k$. As $k < n - 1$ (theorem assumption), we have $\ell \geq 2$, which means that at least two processes obtain $\text{min}_\text{view}$. It follows that, if a process obtains the views returned by the $k$-IS object to $(n - 1)$ processes, one of these views is necessarily $\text{min}_\text{view}$. This constitutes Observation $O$.

Algorithm 1: Solving consensus in $\text{CARW}_{n,t}[t = 1, k$-IS] (code for $p_i$)

Let us now consider Algorithm 1. In addition to a $k$-IS object denoted $\text{IS}$, the processes access an array $\text{VIEW}[1..n]$ of SWMR atomic registers, initialized to $[\bot, \cdots, \bot]$. The aim of $\text{VIEW}[i]$ is to store the view obtained by $p_i$ from the $k$-IS object $\text{IS}$. When it calls $\text{propose}_1(v)$, a process $p_i$ invokes
first the \(k\)-IS object, in which it deposits the pair \((i, v)\), and obtains a view from it (line 1), that it writes in \(V I E W[i]\) to make it publicly known (line 2). Then, it waits until it sees the views of at least \((n - 1)\) processes (line 3). Finally, \(p_i\) extracts from these views the one with the smallest cardinality (line 4), and returns the smallest value contained in this smallest view (line 5).

We show that this reduction algorithm solves consensus in \(C A R W_{n,t}[t = 1, k\text{-IS}]\). As at least \((n - 1)\) processes do not crash, and write in their entry of the array \(V I E W[1..n]\), no correct process can block forever at line 2, proving the Termination property of consensus.

As \(\ell \geq n - k \geq 2\), it follows from Observation O that at least one of the views obtained by a process at line 3 is necessarily \(\text{min\_view}\). It follows that each process that executes line 3 obtains \(\text{min\_view}\) and returns its smallest value at line 4, proving the Agreement property of consensus.

The consensus Validity property follows directly from \(k\)-IS Validity property, and the observation that any set \(\text{view}\) contains only proposed values line 4.

\[\square \text{Theorem 3}\]

Remark When considering the algorithm described in Figure 1, let us observe that, as \(n - k \leq n - t\), the array \(V I E W[1..n]\) can replaced by a second \(k\)-immediate snapshot object \(IS^2\). We obtain then the following algorithm.

\[
\begin{align*}
\text{operation } & \text{propose}_i(v) \text{ is} \\
& \text{view}1, \leftarrow \text{IS.write}\_\text{snapshot}_i(v) ; \\
& \text{view}2, \leftarrow \text{IS}^2\text{.write}\_\text{snapshot}_i(\text{view}1) ; \\
& \text{let } \text{view} \text{ be the smallest view in } \text{view}2 ; \\
& \text{return}(\text{smallest proposed value in } \text{view})
\end{align*}
\]

end operation.

4 From \(k\)-Immediate Snapshot to \(x\)-Set Agreement

This section proves the content of Table 1 namely \(x\)-SA can be implemented in the system model \(C A R W_{n,t}[t \leq k < n - 1]\), for \(x = \max(1, t + k - (n - 2))\). Interestingly, the algorithm providing such an implementation is Algorithm 1 whose operation name is now \(\text{propose}_x()\) (instead of \(\text{propose}_1(v)\)).

Theorem 4 Let \(x = \max(1, k + t - (n - 2))\). Algorithm 1 implements an \(x\)-SA object in \(C A R W_{n,t}[1 \leq t \leq k < n - 1, k\text{-IS}]\).

Proof The consensus Termination follows directly from the Termination property of the underlying \(k\)-IS object \(IS\), the fact that there are at least \((n - t)\) correct processes, and the assumption that all correct processes invoke \(\text{propose}_x()\). The consensus Validity property follows directly from the Validity property of the \(IS\).

As far as the consensus Agreement property is concerned, we have the following. Due to Theorem 1 a set of \(\ell \geq n - k\) processes obtain the smallest possible view \(\text{min\_view}\), which is such that \(|\text{min\_view}| = \ell \geq n - k\). It follows that, at most \(k\) processes obtain a view different from \(\text{min\_view}\). In the worst case, these \(k\) views are different. Consequently, there are at most \(k + 1\) different views, namely \(\text{min\_view}, V(1), \ldots, V(k)\), and due to their Containment property, we have \(\text{min\_view} \subset V(1) \subset \cdots \subset V(k)\). The rest of the proof is a case analysis according to the value of \((n - t)\) with respect to \(k\).

- \(n - t > k\). In this case, a process obtains at line 3 views from \((n - t)\) processes, and in the first case it obtains the views \(V(1), \ldots, V(k)\). But as \(n - t > k\) it also obtains \(\text{min\_view}\) from at least one process. It follows that, all processes see \(\text{min\_view}\), and they consequently decide the same value at line 5. Hence, \((n - t > k) \Rightarrow (x = 1)\).
n − t = k. In this case, it is possible that some processes do not obtain \textit{min\_view} at line[3]. But, if this occurs, they necessarily obtain the views from the \(n − t = k\) processes that deposited \(V(1)\)..., \(V(k)\) in \(\text{VIEW}[1...n]\). Hence, all these processes obtains \(V(1)\) at line[3] and decide consequently the same value from \(V(1)\). As the decided values are decided from the views \textit{min\_view} and \(V(1)\), we have \((n − t = k) \Rightarrow (x = 2)\).

- \(n − t = k − 1\). In this case, it is possible that, at line[3] some processes do not obtain not only \textit{min\_view}, but also \(V(1)\) and decide the smallest value of \(V(2)\). As the decided values are then decided from the views \textit{min\_view}, \(V(1)\), and \(V(2)\), we have \((n − t = k − 1) \Rightarrow (x = 3)\).

- Applying the same reasoning to the general case \(n − t = k − c\), we obtain \((n − t = k − c) \Rightarrow (x = 2 + c)\).

Abstracting the previous case analysis, we obtain \(x = 1\) (consensus) for \(n − t > k\), and \(x = k + t − (n − 2)\), i.e., when \(n − t = k − x + 2\), from which follows that \(x = \max(1, k + t − (n − 2))\), which completes the proof of the theorem. \(\square\) Theorem[4]

The next corollary is a re-statement of Theorem[4] for \(x = 1\).

**Corollary 1** Algorithm[1] implements a consensus object in \(\text{CARW}_{n,t}[1 \leq t < n/2, t \leq k \leq (n − 1) − t, k\text{-IS}]\).

## 5 An Equivalence Between \(k\)-Immediate Snapshot and Consensus

This section shows first that consensus is strong enough to implement a \(k\)-IS object when \(t \leq k\). Combining this result with the fact consensus can be implemented from a \(k\)-IS object in \(\text{CARW}_{n,t}[1 \leq t < n/2, t \leq k \leq (n − 1) − t]\) (Corollary[1]), we obtain that consensus and \(k\)-IS are equivalent in \(\text{CARW}_{n,t}[1 \leq t < n/2, t \leq k \leq (n − 1) − t]\).

### 5.1 From consensus to \(k\)-IS in \(\text{CARW}_{n,t}[t \leq k \leq n − 1]\)

Algorithm[2] describes a reduction of \(k\)-IS to consensus in \(\text{CARW}_{n,t}[0 < t \leq k \leq n − 1]\). This algorithm uses three shared data structures. The first is an array \(\text{REG}[1..n]\) of SWMR atomic registers (where \(\text{REG}[i]\) is associated with \(p_i\)), the second is a consensus objects denoted \(\text{CS}\), and the third is an immediate snapshot object denoted \(\text{IS}\) (let us recall that such an object can be implemented in \(\text{CARW}_{n,t}[t \leq n − 1]\)).

```
operation write_snapshot\(_i\)(v\(_i\)) is
(1) REG\(_i\) ← v\(_i\);
(2) wait (\(j\) such that \(\text{REG}[j] \neq \bot\)) \(\geq n − k\);
(3) aux\(_i\) ← \(\{j, \text{REG}[j]\} \) such that \(\text{REG}[j] \neq \bot\);
(4) view\(_i\) ← CS.propose\(_i\)(aux\(_i\));
(5) if ((\(i, v\(_i\)\)) \in view\(_i\))
(6) then return(view\(_i\));
(7) else aux\(_i\) ← IS.write\(_i\)(v\(_i\));
(8) view\(_i\) ← view\(_i\) \cup aux\(_i\);
(9) return(view\(_i\));
(10) end if
end operation.
```

Algorithm 2: Implementing \(k\)-IS in \(\text{CARW}_{n,t}[0 < t \leq k \leq n − 1, \text{CONS}]\) (code for \(p_i\))

The behavior of a process \(p_i\) can be decomposed in three parts.
• When it invokes write_snapshot_k(v_i), p_i first deposits its value v_i in REG[i], in order all processes can know it, and waits until at least (n - k) processes have deposited their input value in REG[1..n] (lines 12).

• Then p_i proposes to the underlying consensus object CS, the set of all the pairs (j, REG[j]) such that REG[j] ≠ ⊥ (lines 14). Let us notice that this set contains at least (n - k) pairs. Hence, the consensus object returns to p_i a view view_i, which contains at least (n - k) pairs.

• Finally, p_i returns a view (of at least (n - k) pairs).
  - If view_i contains its own pair ⟨i, v_i⟩, p_i returns view_i (line 6).
  - If view_i does not contain ⟨i, v_i⟩, p_i proposes v_i to the underlying immediate snapshot object from which it obtains a set pairs aux_i (line 7). Let us notice that, due to the properties of the immediate snapshot object IS, aux_i contains the pair ⟨i, v_i⟩. Process p_i then adds aux_i to view_i (line 8) and returns it (line 9).

**Theorem 5** Algorithm 2 implements k-IS in CARW_{n,t}[0 < t ≤ k ≤ n − 1, CONS].

**Proof** Proof of k-IS Self-inclusion. If p_i returns at line 3 self-inclusion follows directly from the predicate of line 5. If this predicate is not satisfied, p_i invokes the underlying immediate snapshot object IS with the value v_i it initially proposed (line 7). It then follows from the self-inclusion property of IS that aux_i contains ⟨i, v_i⟩, and due to line 8 the set view_i that is returned at line 9 contains ⟨i, v_i⟩.

Proof of k-IS Validity. This property follows from (a) the fact that a process p_i assigns to REG[i] the value it wants to deposit in the k-IS object, (b) this atomic variable is written at most once (line 11), and (c) the predicate REG[j] ≠ ⊥ is used at line 3 to extract values from REG[1..n].

The Output size property follows from (a) the predicate of line 2 which ensures that the set view_i obtained at line 4 from the underlying consensus object contains at least n − t ≥ n − k pairs, and the fact that a set view_i cannot decrease (line 8).

Proof of k-IS Containment. Let P6 (resp., P9) the set of processes that terminate at line 6 (resp., 9). Let view be the set of pairs decided by the underlying consensus object CS (line 4). Hence, all the processes in P6 return view. Due to line 8 the set view returned by a process that terminates at line 9 includes view. It follows that ∀ p_j ∈ P6, p_i ∈ P9, we have view_j = view ⊆ view_i.

Let us now consider two processes p_i and p_j belonging to P9. It then follows from the IS Containment property of the underlying IS object, that we have aux_i ⊆ aux_j or aux_j ⊆ aux_i (where the value of aux_i and aux_j are the ones at line 7). Consequently, at line 8 we have view_i ⊆ view_j or view_j ⊆ view_i, which completes the proof of the k-IS Containment property.

Proof of k-IS Immediacy. Let p_i and p_j be two processes that return view_i and view_j, respectively, such that ⟨i, v⟩ ∈ view_j. We have to show that view_i ⊆ view_j. Let us considering the sets P6 and P9 defined above. There are three cases.

• Both p_i and p_j belong to P6. In this case, due to line 4 we have view_i = view_j.

• p_i belongs to P6, while p_j belong to P9. In this case, due to line 8 we have view_i ⊆ view_j.

• Both p_i and p_j belong to P9. In this case, due to the IS Immediacy property of IS we have (at line 8) (i, v) ∈ aux_j ⇒ aux_i ⊆ aux_j (and (j, v) ∈ aux_i ⇒ aux_j ⊆ aux_i). Let view the set of pairs returned by the consensus object line 4 As, due to line 9 we have view_i ← view ∪ aux_i and view_j ← view ∪ aux_j, the k-IS Immediacy property follows.
Proof of k-IS Termination. Let $p$ be the number of processes that deposit a value in $REG$. As $t \leq k$, we have $n - k \leq n - t \leq p \leq n$. It follows that no correct process can wait forever at line 2.

The fact that no correct process blocks forever at line 4 and line 7 follows from the termination property of the underlying consensus and immediate snapshot objects.

\[\square\text{Theorem 5}\]

5.2 When consensus and k-IS are equivalent

Let us consider the right triangular matrix defined by the entries are marked “$x$-SA” in Table 1. Theorem 5 states that it is possible to implement $k$-IS from CONS for any entry $(t, k)$ belonging to this triangular matrix. Combined with Corollary 1 we obtain the following theorem.

**Theorem 6** Consensus and $k$-IS are equivalent in $\text{CARW}_{n,t}[0 < t < n/2, t \leq k \leq (n - 1) - t]$.

6 When Consensus is Stronger than $k$-Immediate Snapshot

Section 4 investigated the power of $k$-IS to implement $x$-SA objects, namely $x$-SA can be implemented in $\text{CARW}_{n,t}[1 \leq t \leq k < n - 1, k-IS]$ where $x = \max(1, t + k - (n - 2))$, see Theorem 4. As we have seen, considering the other direction, Section 5 has shown that $k$-IS can be implemented in $\text{CARW}_{n,t}[1 \leq t \leq k < n - 1, \text{CONS}]$ (Theorem 5). The combination of these results showed that Consensus and $k$-IS are equivalent in $\text{CARW}_{n,t}[0 < t = k < n/2]$ (Theorem 6).

This section shows an upper bound on the power of $k$-IS to implement $x$-SA objects, namely, $k$-IS objects are not powerful enough to implement consensus in $\text{CARW}_{n,t}[n/2 \leq t \leq k < n - 1]$.

**Preliminary: a simple lemma** Let us remark that, as immediate snapshot objects that they generalize, $k$-immediate snapshot objects are not linearizable. As a $k$-IS object $IS$ contains values from at least $(n - k)$ processes, at least $(n - k)$ processes must have invoked the operation $IS.write\_snapshot_k()$ for any invocation of $write\_snapshot_k()$ be able to terminate. It follows that there is a time $\tau$ at which $n - k$ processes have invoked $IS.write\_snapshot_k()$ and have not yet returned. We then say that these $(n - k)$ processes are "inside IS". Hence the following lemma.

**Lemma 1** If an invocation of $write\_snapshot_k()$ on a $k$-immediate snapshot object $IS$ terminates, there is a time $\tau$ at which at least $(n - k)$ processes are inside $IS$.

**Theorem 7** There is no algorithm implementing consensus in $\text{CARW}_{n,t}[n/2 \leq t \leq k < n - 1, k-IS]$.

**Proof** To prove the theorem, let us first consider first the case $n = 2t$. The proof is by contradiction. Let us assume that $A$ is a $t$-resilient consensus algorithm for a set of processes $\{p_1, \ldots, p_n\}$ which uses a $k$-IS object in a system where $n = 2t$. The contradiction is obtained by simulating $A$ with two processes $Q_0$ and $Q_1$, such that $Q_0$ and $Q_1$ solve consensus despite the possible crash of one of them. As there is no wait-free consensus algorithm for 2 processes, it follows that such a consensus algorithm $A$ based on $t$-immediate snapshot objects cannot exist. The simulation is described in Algorithm 3.

Let $A_0$ and $A_1$ be a partition of $\{p_1, \ldots, p_n\}$ such that $|A_0| = |A_1| = t$. $Q_0$ simulates the processes in $A_0$, while $Q_1$ simulates the processes in $A_1$. In the simulation, if $Q_i$ is correct, then each simulated process in $A_i$ executes its sequence of operations (it is consequently correct in the simulated run). If $Q_i$ crashes, its crash entails (in the simulated run) the crashes of all the processes in $A_i$. Note that, as at most $t$ simulated processes may crash in a simulated run, no process of $A_{1-t}$ crashes if all processes of $A_t$ crash.

In the following, given a simulated process $p$, and a $k$-IS object $o$, $op(o, v)$ denotes the invocation by $p$ of $write\_snapshot_k(v)$ by $p$ on the $k$-IS object $o$. The underlying idea of the simulation is that a 1-IS
Let $A_0$ and $A_1$ be a partition of $\{p_1, \ldots, p_n\}$;

$|A_0| = |A_1| = t$, $\{p_1, \ldots, p_n\} = A_0 \cup A_1$, and $A_0 \cap A_1 = \emptyset$.

Code for $Q_i$ ($i \in \{0, 1\}$):

1. for all $p_i$ in $A_i$: initialize $v_{p_i}$ with the initial value of $Q_i$;
2. repeat forever
3. for each $p$ in $A_i$ in a round robin way do
   4. if next operation of $p$ is $op(o, v)$ (i.e. write_snapshot($v$) on the $k$-IS object $o$)
      then $prop_i[o] \leftarrow prop_i[o] \cup \{(p, v)\}$;
       6. if $REG[i][o] = \perp$
          then if $REG[1-i][o] \neq \perp$
             then $REG[i][o] \leftarrow REG[1-i][o] \cup \{(p, v)\}$;
          simulation of $op(o, v)$ for $p$ which returns $REG[i][o]$
       (10)
      end if
6. else $REG[i][o] \leftarrow REG[i][o] \cup \{(p, v)\}$;
8. simulation of $op(o, v)$ for $p$ which returns $REG[i][o]$
11. end if
14. else simulate the next operation of $p$;
15. if $p$ decides $v$ in this step then $Q_i$ decides $v$ end if
16. end if;
17. if ($\left|prop_i(o)\right| = t \land (REG[i][o] = \perp)$)
18. then $REG[i][o] \leftarrow IS[o].write\_snapshot(prop_i(o))$
19. end if
20. end for
21. end repeat.

Algorithm 3: Simulation of $\mathcal{A}$ by $Q_i$ ($i \in \{0, 1\}$) for $n = 2t$

The 1-IS object associated with the simulated $k$-IS object $o$, is denoted $IS[o]$. Hence, in the following “write_snapshot$k$()” refers to an operation on a simulated object $o$, while write_snapshot$_1$()” refers to an operation issued by a simulator on a simulation object $IS[o]$.

In addition to the 1-IS objects, the simulator processes $Q_0$ and $Q_1$ manage the following variables.

- $REG[k][o]$ is an array made up of two atomic read/write registers associated with each simulated $k$-IS object $o$. $REG[i][o]$ is written by $Q_i$ and read by both $Q_i$ and $Q_{i-1}$. It contains (at least) the values written in $o$ by the processes simulated by $Q_i$ (lines 8 and 11). If $Q_i$ has not already simulated write_snapshot$k$() on $o$ while $Q_{i-1}$ has, $REG[i][o]$ is initialized to the result of the write_snapshot$k$() operations on $o$ issued by the processes of $A_{i-1}$ simulated by $Q_{i-1}$ (lines 6-8).

- $prop_i[o]$ is a local variable of $Q_i$ containing the values written in the $k$-IS object $o$ by the simulated processes in $A_i$ (line 5). When the next step of all the simulated processes is write_snapshot$k$() on $o$, $Q_i$ returns the initial value of $REG[i][o]$ (line 19). In the next $t$ executions of the loop, when $Q_i$ considers the simulated process $p$, this value will be returned to $p$ (line 12) by the simulation of write_snapshot$k$() on $o$ issued by $p$.

The central point of the simulation lies in the way the $k$-IS objects are simulated. For this, only when the next step of all the simulated processes in $A_i$ are $o$.write_snapshot$k$() (write_snapshot$k$() on the same object $o$), the simulator $Q_i$ performs write_snapshot$1$() on associated 1-IS object $IS[o]$ shared by $Q_0$ and $Q_1$, where the values written by the processes in $A_i$ in this $k$-IS object $o$. The result of this invocation of write_snapshot$1$() contains either all the values from all simulated processes, or only the values of the processes in $A_i$. Moreover, all processes of $Q_i$ obtain the same result, and $Q_i$ also writes this result value into $REG[i, o]$ (line 19).
Let us now consider the case in which the next step of the processes in $A_i$ is not write_snapshot\(_k\)() on the same object. If the next step of some process $p \in A_i$ is write_snapshot\(_k\)() on object $o$ and no write_snapshot\(_k\)() on $o$ by processes in $A_i$ has already returned, we prove that there is a time $\tau$ at which all processes in $A_0$, or all processes $A_1$, are inside the $k$-IS object $o$. To this end, let us assume that there is no time at which all processes in $A_i$ are inside a $k$-IS object $o$. By Lemma 1, there is a time $\tau$ at which a set of at least $k$ processes, say $C$, are inside a $k$-IS object $o$. At this time, as –by assumption– at least one process in $A_i$ is not inside a $k$-IS object, it follows that at least one process of $A_{1-i}$ is inside a $k$-IS object. But let us then consider the run in which all processes in $A_i$ may be considered as crashed before they invoked write_snapshot\(_k\)() on $o$. Hence for this run, $C$ contains no process in $A_i$ and, as $|C| \geq k$, $C$ is equal to $A_{1-i}$.

From this observation we deduce that either there is a time for which the next operation of all $p \in A_i$ is a write_snapshot\(_k\)() on $o$, or there is a time at which the next step of all processes $p \in A_{1-i}$ is a write_snapshot\(_k\)() on $o$. Hence, $Q_i$ or $Q_{1-i}$ executes writeSnapshot\(_k\)() on $IS[o]$). If $Q_{1-i}$ performs executes write_snapshot\(_k\)() on $IS[o]$, the result for each process in $A_{1-i}$ is the set $V$ made up of the values written by the processes in $A_{1-i}$. After that, $Q_i$ can read $V$ from a shared variable, and is able to compute the result of a write_snapshot\(_k\)() on $o$ (the result is $V$ union the set of values of processes in $A_i$ for which $Q_i$ has simulated the write_snapshot\(_k\)() on $o$). Hence, if $p \in A_i$ is stuck in the simulation on an object $o$, either $Q_{1-i}$ eventually executes write_snapshot\(_k\)() on $IS[o]$, and $Q_i$ eventually simulates write_snapshot\(_k\)() on $o$ for $p$, or eventually the next operation of all processes in $A_i$ is a write_snapshot\(_k\)() on $o$, and $Q_i$ can compute the result returned by these write_snapshot\(_k\)() on $o$.

To extend the result to $2t > n$, we partition $\{p_1, \cdots, p_n\}$ in 3 sets $A_0, A_1, D$ such that $|A_0| = n-t$, $|A_1| = n-t$, $|D| = 2t - n$. Then, we run the previous simulation algorithm $A$ where all processes in $D$ are initially crashed, $Q_0$ simulates the set of processes of $A_0$, and $Q_1$ simulates the processes of $A_1$. With this simulation, $Q_0$ and $Q_1$ realizes a wait-free consensus, which is known to be impossible.

\[\Box\text{Theorem 4}\]

![Figure 1: Summarizing the results](image-url)
7 Conclusion

The aim and content of the paper  The paper has first introduced the notion of a $k$-immediate snapshot ($k$-IS) object, which generalizes the notion of immediate snapshots (IS) objects to $t$-crash $n$-process systems (the IS object corresponds to the case $k = t = n - 1$). It has then shown that $k$-IS objects cannot be implemented in asynchronous read/write systems for $k < n - 1$.

The paper considered then the respective power of $k$-IS objects and $x$-set agreement objects ($x$-SA) in $t$-crash-prone systems. As both these family of objects are impossible to implement in read/write systems for $t, k < n - 1$ or $x \leq t$, respectively, the paper strove to establish which of $k$-IS and $x$-SA objects are the most “impossible to solve”. The main results are the following where the zones A, B, C, D, refer to Figure 1.

- Even if we have consensus objects, it is not possible to implement $k$-IS objects in a $t$-crash system where $t > k$ (Zone D).
- It is possible to implement $x$-SA objects, where $x = \max(1, t + k - (n - 2))$, from $k$-IS objects in systems where $1 \leq t \leq k < n - 1$ (Zone A + B + C).
- It is possible to implement $k$-IS objects from $1$-SA objects (consensus) in read/write systems where $1 \leq t \leq k \leq n - 1$ (Zone A + B + C).
- $1$-SA objects (consensus) and $k$-IS objects are equivalent in read/write systems where $1 \leq t < n/2$ and $t \leq k \leq (n - 1) - t$ (Zone A).
- It is not possible to implement $1$-SA (consensus) from $k$-IS objects in read/write systems when $n/2 \leq t \leq k < n - 1$ (Zone C).

Stated in a more operational way, these results exhibit the price of the synchronization hidden in $k$-IS object (which requires that the view returned to a process contains at least $(n - k)$ pairs, (where a pair is made up of a value plus the id of the process that deposited it in the $k$-IS object).

More generally, the previous results establish a computability map relating important problems, which are impossible to solve in pure read/write systems.

Open problems  The following problems remain to be solved to obtain a a finer relation linking $k$-IS and $x$-SA, when $1$.

- Direction “from $k$-IS to $x$-SA”. Is it possible to implement $x$-SA objects, with $1 \leq x < t + k - (n - 2)$ in $t$-crash $n$-process systems enriched with $k$-IS objects (Zone B)? We conjecture that the answer to this question is no.
- Direction “from $x$-SA to $k$-IS”. Given an $x$-SA object, which $k$-IS objects can be implemented from it? More generally, is there a “$k$-IS-like” communication object such that $x$-SA and this “$k$-SA-like” object are computationally equivalent (by “$k$-IS-like” we mean an object possibly weaker than a $k$-IS object)?

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References

[1] Afek Y., Attiya H., Dolev D., Gafni E., Merritt M. and Shavit N., Atomic snapshots of shared memory. *Journal of the ACM*, 40(4):873-890 (1993)

[2] Anderson J., Multi-writer composite registers. *Distributed Computing*, 7(4):175-195 (1994)

[3] Borowsky E. and Gafni E., Immediate atomic snapshots and fast renaming. *Proc. 12th ACM Symposium on Principles of Distributed Computing (PODC’93)*, ACM Press, pp. 41-50 (1993)

[4] Borowsky E. and Gafni E., Generalized FLP impossibility results for resilient asynchronous computations. *Proc. 25th ACM Symposium on Theory of Computation (STOC’93)*, ACM Press, pp. 91-100 (1993)

[5] Borowsky E. and Gafni E., A simple algorithmically reasoned characterization of wait-free computations. *Proc. 16th ACM Symposium on Principles of Distributed Computing (PODC’97)*, ACM Press, pp. 189-198 (1997)

[6] Borowsky E., Gafni E., Lynch N. and Rajsbaum S., The BG distributed simulation algorithm. *Distributed Computing*, 14:127-146 (2001)

[7] Castañeda A., Rajsbaum S., and Raynal M., Specifying concurrent problems: beyond linearizability and up to tasks. *Proc. 29th Symposium on Distributed Computing (DISC’15)*, Springer LNCS 9363, pp. 420-435 (2015)

[8] Chaudhuri S., More choices allow more faults: set consensus problems in totally asynchronous systems. *Information and Computation*, 105(1):132-158 (1993)

[9] Delporte C., Fauconnier H., Rajsbaum S., and Raynal M., t-Resilient immediate snapshot is impossible. *Proc. 23nd Int’l Colloquium on Structural Information and Communication Complexity (SIROCCO’16)*, Springer LNCS 9988, pp. 177-191 (2016)

[10] Fischer M.J., Lynch N.A., and Paterson M.S., Impossibility of distributed consensus with one faulty process. *Journal of the ACM*, 32(2):374-382 (1985)

[11] Gafni E. and Rajsbaum S., Recursion in distributed computing. *Proc. 12th Int’l Conference on Stabilization, Safety, and Security of Distributed Systems (SSS’10)*, Springer LNCS 6366, pp. 362-376 (2010)

[12] Herlihy M. P., Wait-free synchronization. *ACM Transactions on Programming Languages and Systems*, 31(5):124-149 (1991)

[13] Herlihy M.P., Kozlov D., and Rajsbaum S., *Distributed computing through combinatorial topology*, Morgan Kaufmann/Elsevier, 336 pages, ISBN 9780124045781 (2014)

[14] Herlihy M.P., Luchangco V., and Moir M., Obstruction-free synchronization: double-ended queues as an example. *Proc. 23th Int’l IEEE Conference on Distributed Computing Systems (ICDCS’03)*, IEEE Press, pp. 522-529, 2003.

[15] Herlihy M., Rajsbaum S., and Raynal M., Power and limits of distributed computing shared memory models. *Theoretical Computer Science*, 509:3-24 (2013)

[16] Herlihy M. P. and Shavit, N., A simple constructive computability theorem for wait-free computation. *Proc. 26th ACM Symposium on Theory of Computing (STOC’94)*, ACM Press, pp. 243-252 (1994)

[17] Herlihy M. P. and Shavit, N., The topological structure of asynchronous computability. *Journal of the ACM*, 46(6):858-923 (1999)

[18] Herlihy M. P. and Wing J. M., Linearizability: a correctness condition for concurrent objects. *ACM Transactions on Programming Languages and Systems*, 12(3):463-492 (1990)

[19] Lamport L., On interprocess communication, Part I: basic formalism. *Distributed Computing*, 1(2):77-85 (1986)

[20] Loui M. and Abu-Amara H., Memory requirements for agreement among unreliable asynchronous processes. *Advances in Computing Research*, 4:163-183, JAI Press (1987)
[21] Neiger G., Set-linearizability. Brief announcement in Proc. 13th ACM Symposium on Principles of Distributed Computing (PODC’94), ACM Press, page 396 (1994)

[22] Rajsbaum, S. and Raynal, M., An introductory tutorial to concurrency-related distributed recursion. Bulletin of the European Association of TCS, 111:57-75 (2013)

[23] Raynal M., Concurrent programming: algorithms, principles and foundations. Springer, 515 pages, ISBN 978-3-642-32026-2 (2013)

[24] Saks M. and Zaharoglou F., Wait-free k-set agreement is impossible: the topology of public knowledge. SIAM Journal on Computing, 29(5):1449-1483 (2000)