SHIFTED DERIVED POISSON MANIFOLDS ASSOCIATED WITH LIE PAIRS

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ABSTRACT. We study the shifted analogue of the “Lie Poisson” construction for $L_{\infty}$-algebroids and we prove that any $L_{\infty}$-algebroid naturally gives rise to shifted derived Poisson manifolds. We also investigate derived Poisson structures from a purely algebraic perspective and, in particular, we establish a homotopy transfer theorem for derived Poisson algebras. As main result, we prove that, given a Lie pair $(L, A)$, the space $\Omega^*_A(\Lambda^*(L/A))$ admits a $(-1)$-shifted derived Poisson algebra structure with the wedge product as associative multiplication and the Chevalley–Eilenberg differential $d_B: \Omega^*_A(\Lambda^*(L/A)) \to \Omega^{*+1}_A(\Lambda^*(L/A))$ as first $L_{\infty}$ bracket. This $(-1)$-shifted derived Poisson algebra structure on $\Omega^*_A(\Lambda^*(L/A))$ is unique up to an isomorphism having the identity map as first Taylor coefficient. The $A$-module structure on $\Lambda^*(L/A)$ at hand is the natural extension of the Bott $A$-connection on $L/A$. Consequently, the Chevalley–Eilenberg hypercohomology $H(\Omega^*_A(\Lambda^*(L/A)), d^B)$ admits a canonical Gerstenhaber algebra structure.

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Research partially supported by NSFC grant 11471179.
Research partially supported by NSF grants DMS-1406668 and DMS-1707545.
1. Introduction

The notion of Lie pairs is a natural framework encompassing a range of diverse geometric contexts including complex manifolds, foliations, and g-manifolds (that is, manifolds endowed with an action of a Lie algebra $g$). By a Lie pair $(L, A)$, we mean an inclusion $A \hookrightarrow L$ of Lie $\mathbb{k}$-algebroids over a smooth manifold $M$. (Throughout the paper, we use the symbol $\mathbb{k}$ to denote either of the fields $\mathbb{R}$ and $\mathbb{C}$.) Recall that a Lie $\mathbb{k}$-algebroid is a $\mathbb{k}$-vector bundle $L \to M$, whose space of sections is endowed with a Lie bracket $[-,-]$, together with a bundle map $\rho : L \to TM \otimes \mathbb{k}$ called anchor such that $\rho : \Gamma(L) \to \mathcal{T}(M) \otimes \mathbb{k}$ is a morphism of Lie algebras and $[X, fY] = f[X, Y] + (\rho(X) f) Y$ for all $X, Y \in \Gamma(L)$ and $f \in C^\infty(M, \mathbb{k})$. A $\mathbb{k}$-vector bundle $L \to M$ is a Lie $\mathbb{k}$-algebroid if and only if $\Gamma(L)$ is a Lie–Rinehart $\mathbb{k}$-algebra [22] over the commutative ring $C^\infty(M, \mathbb{k})$. A Lie pair over the one-point space $M = \{\ast\}$ is simply a pair of Lie algebras $(g, h)$ with an inclusion of $h$ into $g$.

Given a Lie pair $(L, A)$, the quotient $L/A$ is naturally an $A$-module: $\nabla^{\text{Bott}}_a b = q([a, l])$, where $a \in \Gamma(A), b \in \Gamma(L/A)$, $q$ denotes the projection $L \to L/A$, and $l$ is any element of $\Gamma(L)$ such that $q(l) = b$. The flat $A$-connection $\nabla^{\text{Bott}}$ on $L/A$ is known as the Bott connection [2].

Let $\mathcal{X}_A$ and $\mathcal{X}_L$ denote the differentiable stacks determined by the local Lie groupoids integrating the Lie algebroids $A$ and $L$, respectively. The dg algebra $(\Omega^*_A(\Lambda^*(L/A)), d^{\text{Bott}}_A)$ may be regarded as the space of formal polyvector fields tangent to the fibers of the differentiable stack fibration $\mathcal{X}_L \to \mathcal{X}_A$. For instance, the dg algebra $(\Omega^*_F(\Lambda^*(T_M/T_F)), d^{\text{Bott}}_F)$ associated with the Lie pair $(T_M, T_F)$ encoding a foliation $F$ of a smooth manifold $M$ may be thought of as the space of formal polyvector fields on the differentiable stack determined by the holonomy groupoid of the foliation $F$. Therefore, it is natural to wonder whether the dg algebra $(\Omega^*_A(\Lambda^*(L/A)), d^{\text{Bott}}_A)$ admits or not a “$(-1)$-shifted Lie bracket” compatible with the wedge product — an analogue of the Schouten bracket on the polyvector fields of a differentiable manifold. Unfortunately, such a bracket does not exist for arbitrary Lie pairs. However, it turns out that, for every Lie pair, a $(-1)$-shifted $L_\infty$-algebra structure does exist, which is compatible with the wedge product in the sense that all its higher brackets satisfy the graded Leibniz rule. This is exactly what we call a $(-1)$-shifted derived Poisson algebra, i.e. a “Gerstenhaber algebra up to homotopy.”

The main theorem of the paper can be summarized as follows:

**Theorem 1.1.** Given any Lie pair $(L, A)$, the space $\Omega^*_A(\Lambda^*(L/A))$ admits a structure of $(-1)$-shifted derived Poisson algebra with the wedge product as associative multiplication and the Chevalley–Eilenberg differential $d^{\text{Bott}}_A : \Omega^*_A(\Lambda^*(L/A)) \to \Omega^{*-1}_A(\Lambda^*(L/A))$ as first $L_\infty$ bracket. This $(-1)$-shifted derived Poisson algebra structure is unique up to an isomorphism of $(-1)$-shifted derived Poisson algebras having the identity map as first Taylor coefficient. The $A$-module structure on $\Lambda^*(L/A)$ at hand is the natural extension of the Bott $A$-connection on $L/A$.

As an immediate consequence, we obtain the following
Theorem 1.2. Given any Lie pair \((L, A)\), the Chevalley–Eilenberg hypercohomology
\[ \mathbb{H}(\Omega^\bullet_A(\Lambda^\bullet(L/A)), a^\text{Bott}_A) \]
admits a canonical Gerstenhaber algebra structure. The \(A\)-module structure on \(\Lambda^\bullet(L/A)\) at hand is the natural extension of the Bott \(A\)-connection on \(L/A\).

In [25], two of the authors constructed an \(L_\infty[1]\) algebra structure on \(\Omega^\bullet_A(\Lambda^\bullet(L/A))\) via Fedosov dg Lie algebroids — see Section 4.2. Their construction relies on the choice of additional geometric data: a splitting of the short exact \(0 \to A \xrightarrow{i} L \xrightarrow{pr} L/A \to 0\) and a torsion-free \(L\)-connection \(\nabla\) on \(L/A\) extending the Bott \(A\)-connection. It is natural to wonder to what extent the resulting \(L_\infty[1]\) algebra structure on \(\Omega^\bullet_A(\Lambda^\bullet(L/A))\) depends on the geometric data chosen. This question and a similar question for polydifferential operators were investigated in [1]. In this paper, we propose a more direct approach of this problem: we describe two other methods for constructing such \((-1)\)-shifted derived Poisson algebras — one involves \(L_\infty\)-algebroids and the other involves deformations of Dirac structures — and we prove that all three approaches yield exactly the same \((-1)\)-shifted derived Poisson algebra structure on \(\Omega^\bullet_A(\Lambda^\bullet(L/A))\) — see Section 4.3. The uniqueness of the \((-1)\)-shifted derived Poisson algebra structure in Theorem 1.1 then follows from a standard result of Ševera on deformations of Dirac structures [27, 26].

Let us briefly recall the construction via deformations of Dirac structures. The deformation of Dirac structures were investigated about 15 years ago by Severa, Roytenberg, and many others [27, 23]. Given a Courant algebroid \(E\) of signature \((n, n)\), the deformations of a Dirac structure \(D\) in \(E\) are governed by an \(L_\infty\) algebra structure on \(\Gamma(\Lambda^\bullet D^\vee)\), which is in fact a \((-1)\)-shifted derived Poisson algebra unique up to isomorphism [27]. Now, given a Lie pair \((L, A)\), it is well known that \(E = L \oplus L^\vee\) is a Courant algebroid of signature \((n, n)\) and \(D = A \oplus A^\perp\) is a Dirac structure in \(E\) [14]. It is easy to see that \(\Gamma(\Lambda^\bullet D^\vee)\) and \(\Omega^\bullet_A(\Lambda^\bullet(L/A))\) are isomorphic as graded vector spaces and that the isomorphism identifies the first \(L_\infty\) bracket on \(\Gamma(\Lambda^\bullet D^\vee)\) with the Chevalley–Eilenberg differential \(d^\text{Bott}_A : \Omega^\bullet_A(\Lambda^\bullet(L/A)) \to \Omega^{\bullet+1}_A(\Lambda^\bullet(L/A))\). Thus one obtains a \((-1)\)-shifted derived Poisson algebra structure on the dga \((\Omega^\bullet_A(\Lambda^\bullet(L/A)), a^\text{Bott}_A)\).

Next, we proceed to outline the construction of the \((-1)\)-shifted derived Poisson algebra structure on \(\Omega^\bullet_A(\Lambda^\bullet(L/A))\) via \(L_\infty\) algebroids.

An \(L_\infty\) algebroid is a vector bundle object \(\mathcal{L} \to \mathcal{M}\) in the category of \(\mathbb{Z}\)-graded manifolds endowed with (1) a sequence \((\lambda_i)_{i \geq 1}\) of maps \(\lambda_i : \Lambda^i \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})[2 - i]\), called multibrackets, that determine a structure of \(L_\infty\) algebra on \(\Gamma(\mathcal{L})\) and (2) a sequence \((\rho_l)_{l \geq 0}\) of bundle maps \(\rho_l : \mathcal{L}^l \to T_M \otimes_{\mathbb{R}} k[1 - l]\), called anchor maps, that determine a morphism of \(L_\infty\) algebras from \(\Gamma(\mathcal{L})\) to \(T(\mathcal{M}) \otimes_{\mathbb{R}} k\). The \(L_\infty\)-brackets \(\lambda_i\) and the anchor maps \(\rho_l\) must satisfy the usual compatibility condition [7, 3, 29].

There exists an equivalent and more compact definition of \(L_\infty\) algebroids à la Vaǐntrob via dg manifolds [28], which we will recall briefly. Let \(V\) be a \(\mathbb{Z}\)-graded vector space. A \(\mathbb{Z}\)-graded manifold \(\mathcal{M}\) with fiber type \(V\) is a sheaf of commutative algebras \(C^\infty_{\mathcal{M}}\), called the structure sheaf of \(\mathcal{M}\), over an ordinary smooth manifold \(|\mathcal{M}|\). Every point of \(|\mathcal{M}|\) must admit an open neighborhood \(U \subset |\mathcal{M}|\) such that \(C^\infty_{\mathcal{M}}(U)\), the algebra of (local) functions on \(U\), is isomorphic to \(C^\infty(U) \otimes \hat{S}(V^\vee)\). In the sequel, we shall write \(C^\infty(\mathcal{M})\) to denote the algebra \(C^\infty_{\mathcal{M}}(|\mathcal{M}|)\) of global functions on \(\mathcal{M}\). A vector field on \(\mathcal{M}\) is a sheaf derivation \(X : C^\infty_{\mathcal{M}} \to C^\infty_{\mathcal{M}}\). Since partitions of unity exist in the smooth context, one can simply think of a vector field \(X\) as a derivation of the algebra of global functions \(C^\infty(\mathcal{M})\). The set of all vector fields on \(\mathcal{M}\), which is a graded Lie algebra, is denoted by \(\mathcal{T}(\mathcal{M})\). A dg manifold is a \(\mathbb{Z}\)-graded manifold \(\mathcal{M}\) together with homological vector field, i.e. vector field
Let $\hat{T}^\bullet_{\text{poly}}(\mathcal{M})[k]$ denote the completion of the space of $k$-shifted polyvector fields on $\mathcal{M}$ — see Section C for details. A $k$-shifted derived Poisson manifold can be thought of as a dg manifold $(\mathcal{M}, Q)$ equipped with a formal series $\pi = \sum_{l=2}^{\infty} \pi_l$ of $(k-2)$-shifted polyvector fields, with $\pi_l \in \hat{T}^\bullet_{\text{poly}}(\mathcal{M})[k-2]$ of degree $+1$ in $\hat{T}^\bullet_{\text{poly}}(\mathcal{M})[k-2][k-1]$, satisfying the Maurer–Cartan equation $[Q, \pi] + \frac{1}{2}[\pi, \pi] = 0$. The well known “Lie–Poisson” construction admits the following analogue in the “shifted derived” context.

**Theorem 1.3.** Let $\mathcal{L} \rightarrow \mathcal{M}$ be a vector bundle object in the category of $\mathbb{Z}$-graded manifolds and let $k \in \mathbb{Z}$ be a fixed integer. The following statements are equivalent.

1. The vector bundle $\mathcal{L} \rightarrow \mathcal{M}$ is an $L_\infty$ algebroid.
2. The space $\Gamma(\hat{\mathcal{S}}(\mathcal{L}[k]))$ is a $k$-shifted derived Poisson algebra with $l$-bracket of weight $(1-l)$.
3. The graded manifold $(\mathcal{L}[k])^\vee$ is a $k$-shifted derived Poisson manifold and the weight of the $l$-bracket on $C^\infty((\mathcal{L}[k])^\vee)$ is $(1-l)$.

We note that, in the literature, 0-shifted derived Poisson algebras are also called $P_\infty$ algebras [19, 4]. Now, going back to a Lie pair $(L, A)$, it is easily seen that, once a splitting of the short exact $0 \rightarrow A \xrightarrow{i} L \xrightarrow{\pi} L/A \rightarrow 0$ has been chosen, the graded vector bundle $A[1] \times L/A \rightarrow A[1]$ acquires a natural $L_\infty$-algebroid structure. This $L_\infty$-algebroid was studied by Vitagliano in the special case of a Lie pair corresponding to a foliation [29]. Applying Theorem 1.3 to this $L_\infty$ algebroid and the integer $k = -1$ yields a $(-1)$-shifted derived Poisson algebra structure on $\Omega^*_A(\Lambda^*(L/A))$.

The discussion of the constructions outlined above occupies Sections 3 and 4. Section 2 is devoted to a discussion of derived Poisson structures from a purely algebraic perspective and contains, in particular, the proof of a homotopy transfer result for derived Poisson algebras.

**Notations.** In this paper, unless specified otherwise, graded means $\mathbb{Z}$-graded. Given a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V^n$, we say that an element $v \in V^n$ has degree $n$ and we write $|v| = n$. Given a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V^n$, the symbol $V[k]$ denotes the graded vector space obtained from $V$ by shifting the grading according to the rule $(V[k])^n = V^{n+k}$. We write $|v|^k = |v| + k$ to denote the degree of $v$ when regarded as an element of $V[-k]$. The dual $V^\vee$ of a graded vector space $V$ is graded according to the rule $(V^\vee)^n = (V^{-n})^\vee$.

Likewise, if $E = \bigoplus_{n \in \mathbb{Z}} E^n$ is a graded vector bundle over a manifold $M$, $E[k]$ denotes the graded vector bundle obtained by shifting the graduation of the fibers of $E$ according to the above rule.

Given a vector space $V$, the symbol $\hat{S}(V)$ denotes the $m$-adic completion of the symmetric algebra $S(V)$, where $m$ is the ideal of $S(V)$ generated by $V$. Thus, $\hat{S}(V) = \prod_{p=0}^{\infty} S^p(V)$. The symbol $\overline{S}(V)$ denotes the reduced symmetric algebra of $V$, i.e. $\overline{S}(V) = \bigoplus_{p=1}^{\infty} S^p(V)$.

The Koszul sign $\varepsilon(\sigma; v_1, \cdots, v_p)$ (which will be abbreviated as $\varepsilon(\sigma)$) of a permutation $\sigma \in \mathfrak{S}_p$ of $p$ homogeneous vectors $v_1, v_2, \ldots, v_p$ of a graded vector space $V$ is determined by the relation

$$v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(p)} = \varepsilon(\sigma; v_1, \cdots, v_p) v_1 \odot v_2 \odot \cdots \odot v_p,$$

where $\odot$ denotes the multiplication in the symmetric algebra $S(V)$.  

$Q \in T(\mathcal{M})$ of degree $+1$ satisfying $Q^2 = 0$. An $L_\infty$-algebroid is a vector bundle object $\mathcal{L} \rightarrow \mathcal{M}$ in the category of $\mathbb{Z}$-graded manifolds together with a homological vector field $Q$ on $\mathcal{L}[1]$ tangent to the zero section $\mathcal{M} \subset \mathcal{L}[1]$ — see Proposition A.2. It turns out that $L_\infty$-algebroids are closely related to \textit{shifted derived Poisson manifolds} in the sense of Pridham [20].
An \((r, s)\)-shuffle is a permutation \(\sigma\) of the set \(\{1, 2, \ldots, r + s\}\) such that \(\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(r)\) and \(\sigma(r + 1) \leq \sigma(r + 2) \leq \cdots \leq \sigma(r + s)\). We write \(\text{Sh}(r, s)\) to denote the set of \((r, s)\)-shuffles.

2. **Shifted derived Poisson algebras**

2.1. **Shifted derived Poisson algebras.** Throughout this note, the notation \(k \in \mathbb{Z}\) refers to a fixed integer.

**Definition 2.1.** A \(k\)-shifted derived Poisson algebra is a \(\mathbb{Z}\)-graded commutative \(k\)-algebra \(A = \bigoplus_{k \in \mathbb{Z}} A^k\) together with a coderivation

\[
Q : \overline{S}(A[1 - k]) \to \overline{S}(A[1 - k])[1]
\]

of the reduced symmetric tensor coalgebra \((\overline{S}(A[1 - k]), \Delta)\) satisfying (1) the cohomological condition \(Q \circ Q = 0\); and (2) the Leibniz rule

\[
p \circ Q(a_1 \odot \cdots \odot a_{n-1} \odot a_n \cdot a'_n) = (p \circ Q(a_1 \odot \cdots \odot a_{n-1} \odot a_n)) \cdot a'_n + \epsilon a_n \cdot (p \circ Q(a_1 \odot \cdots \odot a_{n-1} \odot a'_n))
\]

with \(\epsilon = (-1)^{(2-n)+(\alpha_1+\cdots+\alpha_{n-1})+k(n-1)}\alpha_n\), for all \(a_1 \in A^{\alpha_1}, a_2 \in A^{\alpha_2}, \ldots, a_n \in A^{\alpha_n}\), and \(a'_n \in A\).

The symbol \(p\) denotes the canonical projection

\[
\overline{S}(A[1 - k]) \to A[1 - k]
\]

defined by

\[
p(a) = a, \quad \forall a \in A
\]

\[
p(a_1 \odot \cdots \odot a_n) = 0, \quad \text{if} \ n \geq 2.
\]

**Remark 2.2** (Explanation of definition above). To understand Equation (1) and in particular the sign \(\epsilon\) of its last term, observe that, if

\[
a_1 \in A^{\alpha_1}, \quad a_2 \in A^{\alpha_2}, \quad \ldots, \quad a_n \in A^{\alpha_n},
\]

or equivalently

\[
a_1 \in (A[1 - k])^{\alpha_1 - 1 + k}, \quad a_2 \in (A[1 - k])^{\alpha_2 - 1 + k}, \quad \ldots, \quad a_n \in (A[1 - k])^{\alpha_n - 1 + k},
\]

then we have

\[
p \circ Q(a_1 \odot a_2 \odot \cdots \odot a_n) \in (A[1 - k])^{1+(\alpha_1 - 1 + k) + (\alpha_2 - 1 + k) + \cdots + (\alpha_n - 1 + k)},
\]

or equivalently

\[
p \circ Q(a_1 \odot a_2 \odot \cdots \odot a_n) \in A^{1-k+1+(\alpha_1 - 1 + k) + (\alpha_2 - 1 + k) + \cdots + (\alpha_n - 1 + k)}.
\]

Therefore,

\[
A \ni x \mapsto p \circ Q(a_1 \odot \cdots \odot a_{n-1} \odot x) \in A
\]

is an operator of degree

\[
1 + (\alpha_1 - 1 + k) + \cdots + (\alpha_{n-1} - 1 + k) = (2 - n) + (\alpha_1 + \cdots + \alpha_{n-1}) + k(n-1)
\]

on \(A\).

Equation (1) simply states that this operator of degree \((2 - n) + (\alpha_1 + \cdots + \alpha_{n-1}) + k(n-1)\) is a (graded) derivation of the graded algebra \(A\).
An alternative and equivalent description of the above definition is the following.

**Definition 2.3.** A \( k \)-shifted derived Poisson algebra is a \( \mathbb{Z} \)-graded commutative algebra \( A = \bigoplus_{i \in \mathbb{Z}} A^i \) over a field \( k \) of characteristic zero and endowed with a family of multi-linear maps \( \lambda_i : A^i \to A \) \((l = 1, 2, \ldots)\), such that

1. The sequence of maps \( \{ \lambda_i \}_{i=1}^{\infty} \) define an \( L_\infty \) algebra structure on \( A[-k] \);
2. For all \( a_1, \ldots, a_{l-1} \in A \), the map
   \[
   a \mapsto \lambda_l(a_1, \ldots, a_{l-1}, a)
   \]
   is a derivation of \( A \).

**Remark 2.4.** By degree conventions, the bracket \( \lambda_i \) is of degree \( k(l-1) + 2 - l \). Therefore, the derivation \( \lambda_i(a_1, \ldots, a_{l-1}, -) \) is of degree \( k(l-1) + 2 - l + |a_1| + \cdots + |a_{l-1}| \). I.e.,

\[
\lambda_l(a_1, \ldots, a_{l-1}, aa') = \lambda_l(a_1, \ldots, a_{l-1}, a)a' + (-1)^{(k(l-1)-l+|a_1|+\cdots+|a_{l-1}|)}a\lambda_l(a_1, \ldots, a_{l-1}, a').
\]

(2)

A \( k \)-shifted Poisson algebra (see Appendix) is a \( k \)-shifted derived Poisson algebra, where the only nontrivial bracket is the binary bracket \( \lambda_2 = [\cdot, -] \).

**Remark 2.5.** Note that 0-shifted derived Poisson algebras have been studied by Oh-Park [19] and Cattaneo-Felder [4], called \( P_\infty \) algebras.

Also if one considers \( \mathbb{Z}_2 \)-graded, i.e. supermanifolds instead of \( \mathbb{Z} \)-graded manifolds, Definition 2.3 reduces to homotopy Poisson algebras and homotopy Schouten algebras introduced by Bruce [3].

Given a \( k \)-shifted derived Poisson algebra \((A, \lambda_l)\), we have a cochain complex \( \lambda_1 : A^* \to A^{*+1} \).

The binary bracket \( \lambda_2 \) is of degree \( k \), which satisfies the following Jacobi identity up to homotopy:

\[
\begin{aligned}
\lambda_2(\lambda_2(a_1, a_2), a_3) - \lambda_2(a_1, \lambda_2(a_2, a_3)) &+ (-1)^{|a_1||a_2||k|} \lambda_2(a_2, \lambda_2(a_1, a_3)) \\
&= \lambda_1(\lambda_3(a_1, a_2, a_3)) - \lambda_3(\lambda_1(a_1), a_2, a_3) \\
&\quad - (-1)^{|a_1||k|} \lambda_3(\lambda_1(a_1), a_2, a_3) - (-1)^{|a_1||k|+|a_2||k|} \lambda_3(a_1, a_2, \lambda_1(a_3)),
\end{aligned}
\]

for all \( a_1, a_2, a_3 \in A \). The following is thus immediate.

**Proposition 2.6.** If \( A \) is a \( k \)-shifted derived Poisson algebra, then the binary bracket \( \lambda_2 \) induces, on the cohomology groups \( H(A, \lambda_1) \), a \( k \)-shifted Poisson algebra structure.

### 2.2. Morphisms of derived Poisson algebras

**Definition 2.7.** Let \((A, d_A, \cdot_A, \lambda_2, \ldots, \lambda_n, \ldots)\) and \((B, d_B, \cdot_B, \ell_2, \ldots, \ell_n, \ldots)\) be \( k \)-shifted derived Poisson algebras. A morphism \( f_\infty = (f_1, \ldots, f_n, \ldots) : B \to A \) of derived Poisson algebras is a collection of maps \( f_n : B^{\otimes n} \to A \), \( n \geq 1 \), such that

1. \( f_\infty \) is an \( L_\infty \) morphism from \((B[-k], d_B[-k], \ell_2, \ldots, \ell_n, \ldots)\) to \((A[-k], d_A[-k], \lambda_2, \ldots, \lambda_n, \ldots)\);
2. The following relation is satisfied for all \( n \geq 0 \) and \( x_1, \ldots, x_n, y, z \in B \)

\[
f_{n+1}(x_1, \ldots, x_n, y \cdot_B z) = \\
= \sum_{i=0}^{n} \sum_{\sigma \in \text{Sh}(i, n-i)} (-1)^{\sigma} f_{i+1}(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y) \cdot_A f_{n-i+1}(x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}, z),
\]

(3)
where \( \phi = \epsilon(\sigma) + [y]((n-i)(k-1) + |x_{\sigma(i+1)}| + \cdots + |x_{\sigma(n)}|) \). In particular, for \( n = 0 \) this says that \( f_1 : B \to A \) is a morphism of algebras, and for \( n = 1 \) it says that \( f_2 : B^{\otimes 2} \to A \) is an \( f_1 \)-biderivation.

**Remark 2.8.** By degree conventions, each \( f_n \) is of degree \((n-1)(k-1)\).

**Remark 2.9.** Since \( S(B[1-k]) \) is a graded cocommutative coalgebra via the unshuffle coproduct \( \Delta \) and \((A, \cdot_A)\) is a graded commutative algebra, the space \( \text{Hom}(S(B[1-k]), A) \) is a graded commutative algebra via the convolution product \( f \ast g = \cdot_A \circ (f \otimes g) \circ \Delta \). We denote by \( f(\ldots, y) \) the map

\[
 f(\ldots, y) : S(B[1-k]) \to A : x_1 \otimes \cdots \otimes x_n \to (-1)^{(n-k-1)+|x_1|+\cdots+|x_n|}y f_{n+1}(x_1, \ldots, x_n, y),
\]

where \( x_i \in B^{[x_i]} = B[1-k]^{[x_i]+k-1} \), \( i = 1, \ldots, n \). Equation (3) is equivalent to

\[
 f(\ldots, y \cdot_B z) = f(\ldots, y) \ast f(\ldots, z).
\]

In other words, \( f \) is a morphism of derived Poisson algebras if and only if

\[
 (B, \cdot_B) \to (\text{Hom}(S(B[1-k]), A), \ast) : y \to f(\ldots, y)
\]

is a morphism of graded algebras.

The proof of the following proposition can be done by a tedious direct computation, which we omit.

**Proposition 2.10.** With the above definition of morphisms, \( k \)-shifted derived Poisson algebras form a category.

The following Proposition 2.11 justifies the previous definition in the framework of deformation theory. Given a graded algebra \((A, \cdot_A)\), we denote by \( L = \text{Coder}(\mathcal{S}(A[1-k])) \) the graded Lie algebra of coderivations of the reduced symmetric coalgebra \( \mathcal{S}(A[1-k]) \). From the point of view of deformation theory (cf. [17]), this graded Lie algebra controls the deformations of the trivial \( L_\infty \) algebra structure on \( A[-k] \). In other words, the set \( \text{MC}(L) \) of Maurer–Cartan elements of \( L \), that is, the set of solutions \( Q \in L^1 \) of the Maurer–Cartan equation \([Q, Q] = 0\), is in bijective correspondence with the set of \( L_\infty \) algebra structures on \( A[-k] \). Furthermore, the (formal) exponential group \( \text{exp}(L^0) = \{ e^R \}_{R \in L^0} \) acts on the set \( \text{MC}(L) \) via conjugation \( Q \mapsto e^{-R}Qe^R \) and the corresponding set of orbits parametrizes \( L_\infty \) algebra structures on \( A[-k] \) up to \( L_\infty \) isomorphism. An easy computation shows that the subspace \( M \subset L \) spanned by those coderivations \( R = (r_1, \ldots, r_n, \ldots) \) whose Taylor coefficients \( r_n : A^{\otimes n} \to A \) are multiderivations (see item (2) in Definition 2.3), is a graded Lie subalgebra. Moreover, it is clear that the Maurer–Cartan elements in \( M \) are precisely the \( k \)-shifted derived Poisson algebra structures on \((A, \cdot_A)\). In the following Proposition 2.11, we show that the (formal) exponential group \( \text{exp}(M^0) \) is precisely the group of coalgebra automorphisms \( F = (f_1, \ldots, f_n, \ldots) : \mathcal{S}(A[1-k]) \to \mathcal{S}(A[1-k]) \) satisfying the identity (3).

In order to avoid convergence issues, we proceed formally. We denote by \( \mathbb{K}[[t]] \) the algebra of formal power series, by \( A[[t]] = A \otimes_{\mathbb{K}} \mathbb{K}[[t]] \), and by \( \mathcal{S}_{\mathbb{K}[[t]]}(A[[t]][1-k]) \) the reduced symmetric \( \mathbb{K}[[t]] \)-coalgebra over the \( \mathbb{K}[[t]] \)-module \( A[[t]][1-k] \). Given a degree zero coderviation \( R = (r_1, \ldots, r_n, \ldots) \in L^0 \), we consider the associated formal flow \( e^{tR} = : F^t = (f_1^t, \ldots, f_n^t, \ldots) : \mathcal{S}_{\mathbb{K}[[t]]}(A[[t]][1-k]) \to \mathcal{S}_{\mathbb{K}[[t]]}(A[[t]][1-k]) \) : it is a well defined \( \mathbb{K}[[t]] \)-linear coalgebra automorphism.

**Proposition 2.11.** Given \( R = (r_1, \ldots, r_n, \ldots) \in L^0 \) as above, then the formal flow \( e^{tR} = : F^t = (f_1^t, \ldots, f_n^t, \ldots) \) satisfies the identity (3) if and only if all the Taylor coefficients \( r_n : A^{\otimes n} \to A, n \geq 1, \) are multiderivations, that is, if and only if \( R \in M^0 \).
Proof. For notational simplicity, in the following computations we abbreviate equations such as Equation (3) by omitting the \(x_1, \ldots, x_n\) arguments, and by denoting by \(\pm K\) the appropriate Koszul sign, which we omit to write down explicitly (signs were made precise in Definition 2.7). Moreover, we write \(\cdot\) in place of \(\cdot_A\). For instance, Equation (3) becomes

\[
f_{n+1}^t(\ldots, y \cdot z) = \sum_{i=0}^{n} \sum_{\sigma \in \text{Sh}(i,n-i)} \pm K f_{i+1}^t(\ldots, y) \cdot f_{n-i+1}^t(\ldots, z).
\]

Having set these notations, we proceed with the proof. One implication is easy: assume \(F^t\) satisfies Equation (3), then for all \(x_1, \ldots, x_n, y, z \in A\)

\[
r_{n+1}(\ldots, y \cdot z) = \frac{d}{dt} f_{n+1}^t(\ldots, y \cdot z)|_{t=0} = \]

\[
= \sum_{i=0}^{n} \sum_{\sigma \in \text{Sh}(i,n-i)} \frac{d}{dt} \left( \pm K f_{i+1}^t(\ldots, y) \cdot f_{n-i+1}^t(\ldots, z) \right)|_{t=0} = \]

\[
= \sum_{i=0}^{n} \sum_{\sigma \in \text{Sh}(i,n-i)} \pm K \frac{d}{dt} f_{i+1}^t(\ldots, y)|_{t=0} \cdot f_{n-i+1}^t(\ldots, z)|_{t=0} + \]

\[
\pm K f_{i+1}^t(\ldots, y)|_{t=0} \cdot \frac{d}{dt} f_{n-i+1}^t(\ldots, z)|_{t=0} = \]

\[
= r_{n+1}(\ldots, y) \cdot z \pm K y \cdot r_{n+1}(\ldots, z),
\]

since \(F^t_{|t=0} = (id, 0, \ldots, 0, \ldots)\). This shows that \(r_{n+1}\) is a multiderivation for all \(n \geq 0\).

We turn to the other implication. Since \(r_1\) is an algebra derivation, \(f_1^t = e^{t r_1} : A[[t]] \to A[[t]]\) is an algebra morphism.

Let \(n \geq 1\), and fix \(x_1, \ldots, x_n, y, z \in A\): we consider the formal power series

\[
A[[t]] \ni \xi(t) := f_{n+1}^t(\ldots, y \cdot z) - \sum_{i=0}^{n} \sum_{\sigma \in \text{Sh}(i,n-i)} \pm K f_{i+1}^t(\ldots, y) \cdot f_{n-i+1}^t(\ldots, z).
\]
We need to prove $\xi(t) = 0$. As before, since $F^t_{|t=0} = (\id, 0, \ldots, 0, \ldots)$ we see that $\xi(0) = 0$. We notice that $\frac{d}{dt} F^t = R F^t$. Using induction on $n$,

$$ \frac{d}{dt} (f^t_{n+1}(\ldots, y \cdot z)) = r_1 f^t_{n+1}(\ldots, y \cdot z) + $$

$$ \sum_{k \geq 2, i_1, \ldots, i_{k-1} \geq 1, i_k \geq 0} \sum_{i_1 + \cdots + i_k = n} \frac{1}{(k-1)!} r_k (f^t_{i_1}(\ldots), \ldots, f^t_{i_k+1}(\ldots, y \cdot z)) = $$

$$ r_1(\xi(t)) + r_1 \left( \sum_{i=0}^n \sum_{\sigma \in \text{Sh}(i,n-i)} \pm K f^t_{i+1}(\ldots, y) \cdot f^t_{n-i+1}(\ldots, z) \right) + $$

$$ \sum_{k, i_1, \ldots, i_{k-1} \geq 1, i_k \geq 0} \sum_{i_1 + \cdots + i_k = n} \frac{1}{(k-1)!} r_k (f^t_{i_1}(\ldots), \ldots, f^t_{i_k+1}(\ldots, y) \cdot f^t_{i_k+1}(\ldots, z)) = $$

$$ r_1(\xi(t)) + \sum_{k, i_1, \ldots, i_{k-1} \geq 1, i_k \geq 0} \sum_{i_1 + \cdots + i_k = n} \pm K f^t_{i_1}(\ldots, y) \cdot f^t_{i_k+1}(\ldots, z) + $$

$$ \pm K \frac{1}{(k-1)!} f^t_{i+1}(\ldots, y) \cdot r_k (f^t_{i_1}(\ldots), \ldots, f^t_{i_k+1}(\ldots, z)) = r_1(\xi(t)) + $$

$$ \sum_{i=0}^n \sum_{\sigma \in \text{Sh}(i,n-i)} \pm K \frac{d}{dt} f^t_{i+1}(\ldots, y) \cdot f^t_{n-i+1}(\ldots, z) \pm K f^t_{i+1}(\ldots, y) \cdot \frac{d}{dt} f^t_{n-i+1}(\ldots, z) = $$

$$ r_1(\xi(t)) + \frac{d}{dt} \left( \sum_{i=0}^n \sum_{\sigma \in \text{Sh}(i,n-i)} \pm K f^t_{i+1}(\ldots, y) \cdot f^t_{n-i+1}(\ldots, z) \right). $$

To sum up, we found that

$$ \xi'(t) = r_1(\xi(t)), $$

thus

$$ \xi^{(n)}(t) = r_1^n(\xi(t)), \quad \forall n \geq 0. $$

Expanding in formal Taylor series,

$$ \xi(t) = \sum_{n \geq 0} \frac{t^n}{n!} \xi^{(n)}(0) = \sum_{n \geq 0} \frac{t^n}{n!} r_1^n(\xi(0)) = 0. $$

\[ \square \]

2.3. Homotopy transfer for derived Poisson algebras. In this section we prove a homotopy transfer theorem for derived Poisson algebras.

We begin by recalling the standard Perturbation Lemma.

**Definition 2.12.** Let $(A, d_A)$ and $(B, d_B)$ be complexes. A contraction $(\sigma, \tau, h)$ of $(A, d_A)$ onto $(B, d_B)$ is the datum of DG maps $\sigma : (A, d_A) \to (B, d_B)$, $\tau : (B, d_B) \to (A, d_A)$ and a contracting homotopy $h : A \to A$ such that

$$ \sigma \tau = \id_B, \quad hd_A + d_A h = \tau \sigma - \id_A. $$
and furthermore

\[ \sigma h = 0, \quad h\tau = 0, \quad h^2 = 0. \]

We denote such a contraction data by:

\[ (A, d_A) \xleftarrow{\sigma} (B, d_B). \]

**Lemma 2.13.** Let \((\sigma, \tau, h)\) be a contraction of \((A, d_A)\) onto \((B, d_B)\) and let \(\delta_A : A \to A\) be a perturbation of the differential \(d_A\): that is, \(\delta_A\) is a degree one map such that \(\overline{d}_A := d_A + \delta_A\) squares to zero. Then

\[ \delta_B := \sum_{i=0}^{\infty} \sigma \delta_A(h^i) \tau \]

is a perturbation of the differential \(d_B\), and \((\bar{\sigma}, \bar{\tau}, \bar{h})\) defined by

\[ \bar{\sigma} := \sum_{i=0}^{\infty} \sigma (\delta_A^i) \], \quad \bar{\tau} := \sum_{i=0}^{\infty} (h^i \delta_A^i) \], \quad \bar{h} := \sum_{i=0}^{\infty} (h^i \delta_A^i) \]

is a contraction of \((A, \overline{d}_A)\) onto \((B, \overline{d}_B) := d_B + \delta_B\).

To be more precise, we should add some technical assumption ensuring convergence of the above infinite sums, but we will be loose in that respect and proceed formally.

**Definition 2.14.** Given DG commutative algebras \((A, d_A, \cdot_A), (B, d_B, \cdot_B)\) and a contraction \((\sigma, \tau, h)\) of \((A, d_A)\) onto \((B, d_B)\), we say that \((\sigma, \tau, h)\) is a semifull algebra contraction if the following identities are satisfied for all \(a, b \in A\) and \(x, y \in B\)

\[ h((-1)^{|a|+1} a \cdot_A b + a \cdot_A h(b)) = h(a) \cdot_A h(b), \quad (4) \]
\[ h(a \cdot_A \tau(x)) = h(a) \cdot_A \tau(x), \quad (5) \]
\[ \sigma((-1)^{|a|+1} a \cdot_A b + a \cdot_A h(b)) = 0, \quad (6) \]
\[ \sigma(a \cdot_A \tau(x)) = \sigma(a) \cdot_B x, \quad (7) \]
\[ \tau(x \cdot_B y) = \tau(x) \cdot_A \tau(y). \quad (8) \]

**Remark 2.15.** This class of contractions was introduced by Real [21]: more precisely, in [21, Definition 4.5], Equations (4)-(7) are replaced by the seemingly weaker

\[ h(h(a) \cdot_A h(b)) = h(h(a) \cdot_A \tau(x)) = 0, \]
\[ \sigma(h(a) \cdot_A h(b)) = \sigma(h(a) \cdot_A \tau(x)) = 0, \]

whereas Equation (8) is maintained. It is straightforward that, if Equations (4)-(7) are satisfied, then the above equations are satisfied as well. In fact, our definition and the one in [21] are equivalent. To illustrate this fact, we show how to deduce Equation (4) from the pair of equations above. We have

\[ h(h(a) \cdot_A b) = h(h(a) \cdot_A (\tau \sigma - h d_A - d_A h))(b)) = -h(h(a) \cdot_A d_A h(b)), \]
and similarly $h(a \cdot_A h(b)) = -h(d_A h(a) \cdot_A h(b))$, thus

$$h((-1)^{|a|+1} h(a) \cdot_A b + a \cdot_A h(b)) = -h d_A (h(a) \cdot_A h(b)) = (d_A h - \tau \sigma + \text{id})(h(a) \cdot_A h(b)) = h(a) \cdot_A h(b).$$

Equations (5)-(7) can be deduced similarly.

**Definition 2.16.** Given a DG algebra $(A, d_A, \cdot_A)$, an algebra perturbation of $d_A$ is a perturbation in the usual sense, which is furthermore an algebra derivation.

**Proposition 2.17.** Given DG algebras $(A, d_A, \cdot_A), (B, d_B, \cdot_B)$, a semifull algebra contraction $(\sigma, \tau, h)$ of $(A, d_A, \cdot_A)$ onto $(B, d_B, \cdot_B)$ and an algebra perturbation $\delta_A : A \to A$ of $d_A$, we apply the Perturbation Lemma 2.13. Then $\delta_B$ is an algebra perturbation of $d_B$, and $(\tilde{\sigma}, \tilde{\tau}, \tilde{h})$ is a semifull algebra contraction of $(A, \tilde{d}_A, \cdot_A)$ onto $(B, \tilde{d}_B, \cdot_B)$.

**Proof.** This is proved in [21], see also [6, 8] for some related results. For completeness, we sketch a proof of this fact. To show that

$$\tilde{\tau}(x \cdot_B y) = \tau(x) \cdot_A \tilde{\tau}(y),$$

we prove inductively that

$$(\h\delta_A)^{i+1} \tau(x \cdot_B y) = \sum_{j=0}^{i} (\h\delta_A)^j \tau(x) \cdot_A (\h\delta_A)^{i-j} \tau(y), \quad \forall i \geq 0,$$

The basis of the induction is Equation (8). Assume the above identity for a given $i$. Then

$$\begin{align*}
(\h\delta_A)^{i+1} \tau(x \cdot_B y) &= h\delta_A \left( \sum_{j=0}^{i} (\h\delta_A)^j \tau(x) \cdot_A (\h\delta_A)^{i-j} \tau(y) \right) \\
&= h \left( \delta_A \left( (\h\delta_A)^i \tau(x) \cdot_A \tau(y) \right) \right) + \\
&+ \sum_{j=1}^{i} h \left( (-1)^{|x|} h\delta_A (h\delta_A)^{i-j} \tau(x) \cdot_A \delta_A (h\delta_A)^{i-j} \tau(y) + \delta_A (h\delta_A)^{j-1} \tau(x) \cdot_A h\delta_A (h\delta_A)^{i-j} \tau(y) \right) + \\
&+ (-1)^{|x|} h \left( \tau(x) \cdot_A \delta_A (h\delta_A)^i \tau(y) \right) \\
&= (h\delta_A)^{i+1} \tau(x) \cdot_A \tau(y) + \sum_{j=1}^{i} (h\delta_A)^j \tau(x) \cdot_A (h\delta_A)^{i-j} \tau(y) + \tau(x) \cdot_A (h\delta_A)^{i+1} \tau(y),
\end{align*}$$

using Equations (4)-(5), which proves the inductive step.

Replacing the leftmost $h$ by $\sigma$ in the above computation, and using Equations (6)-(7) in the last passage, we see that

$$\sigma \delta_A (h\delta_A)^i \tau(x \cdot_B y) = \sigma \delta_A (h\delta_A)^i \tau(x) \cdot_B y + (-1)^{|x|} x \cdot_B \sigma \delta_A (h\delta_A)^i \tau(y),$$

which implies that $\delta_B$ is indeed an algebra perturbation.

Finally, to show that $(\tilde{\tau}, \tilde{\sigma}, \tilde{h})$ satisfies Equations (4)-(7) in Definition 2.14, it is enough to show that it satisfies the equivalent conditions in Remark 2.15, which is almost straightforward from the definitions. \hfill $\square$

The main result of this section is the following theorem, which says that we can transfer derived Poisson algebras structures along semifull algebra contractions.
Theorem 2.18. Let \((A, d_A, \cdot_A), (B, d_B, \cdot_B)\) be DG commutative algebras and let \((\sigma, \tau, h)\) be a semi-full algebra contraction of \((A, d_A, \cdot_A)\) onto \((B, d_B, \cdot_B)\). Let \(\lambda_n : A^{\otimes n} \to A, n \geq 2\), be a family of maps making \((A, d_A, \cdot_A, \lambda_2, \ldots, \lambda_n, \ldots)\) into a \(k\)-shifted derived Poisson algebra. Via homotopy transfer along the contraction \((\sigma, \tau, h)\), there is an induced \(L_\infty\) algebra structure on \(B[-k]\), whose structure maps we denote by \(\ell_n : B^{\otimes n} \to B, n \geq 2\). These maps make \((B, d_B, \cdot_B, \ell_2, \ldots, \ell_n, \ldots)\) into a \(k\)-shifted derived Poisson algebra. Moreover, the \(L_\infty\) quasi-isomorphism \(\tau_\infty = (\tau_1, \tau_2, \ldots, \tau_n, \ldots)\) from \(B[-k]\) to \(A[-k]\) is a morphism of \(k\)-shifted derived Poisson algebras.

Remark 2.19. It is a well-known fact that \(L_\infty\) algebra structures can be transferred along contractions: for a proof of this fact, we refer to [9].

Proof. In the following computations we shall treat signs loosely, denoting each time by \(\pm_K\) the appropriate Koszul sign, and leaving to the reader to work out these signs explicitly.

By homotopy transfer formulas \(\tau_1 = \tau\) and \(\ell_{n+1}, \ell_{n+1}, n \geq 1\), are defined recursively by

\[
\tau_{n+1}(x_1, \ldots, x_n, y) = \sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_p)} \pm_K \frac{1}{(p-1)!} h\lambda_p(\tau_{i_1}(x_{\sigma(1)}, \ldots, x_{\sigma(i_1)}), \ldots, \tau_{i_p+1}(x_{\sigma(n-i_p+1)}, \ldots, x_{\sigma(n)}))
\]

\[
\ell_{n+1}(x_1, \ldots, x_n, y) = \sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_p)} \pm_K \frac{1}{(p-1)!} \sigma\lambda_p(\tau_{i_1}(x_{\sigma(1)}, \ldots, x_{\sigma(i_1)}), \ldots, \tau_{i_p+1}(x_{\sigma(n-i_p+1)}, \ldots, x_{\sigma(n)}))
\]

For notational simplicity, in the following computations we will omit the \(x_1, \ldots, x_n\) variables, and we shall abbreviate the above equations as

\[
\tau_{n+1}(x_1, \ldots, x_n, y) = \sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_p)} \pm_K \frac{1}{(p-1)!} h\lambda_p(\tau_{i_1}(\ldots), \ldots, \tau_{i_p+1}(\ldots))
\]

\[
\ell_{n+1}(x_1, \ldots, x_n, y) = \sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_p)} \pm_K \frac{1}{(p-1)!} \sigma\lambda_p(\tau_{i_1}(\ldots), \ldots, \tau_{i_p+1}(\ldots))
\]
We shall prove first that $\tau_{\infty}$ satisfies the required compatibilities with the products. By assumption $\tau_1 = \tau$ is a morphism of graded algebras. Proceeding inductively,
\[
\tau_{n+1}(x_1, \ldots, x_n, y \cdot_B z) =
\sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_p)} \pm K \frac{1}{(p-1)!} h_{\lambda_p}(\tau_{i_1}(\ldots), \ldots, \tau_{i_p+1}(\ldots, y \cdot_B z)) =
\sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_p)} \pm K \frac{1}{(p-1)!} h_{\lambda_p}(\tau_{i_1}(\ldots), \ldots, \tau_{i_p+1}(\ldots, y \cdot_A \tau_{i_p+1}(\ldots, z)) =
\sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{\sigma \in \text{Sh}(i_1, \ldots, i_p)} \pm K \frac{1}{(p-1)!} h_{\lambda_p}(\tau_{i_1}(\ldots), \ldots, \tau_{i_p+1}(\ldots, y \cdot_A \tau_{i_p+1}(\ldots, z)) + \sum_{i,j \geq 1} \sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{q \geq 2} \sum_{j_1, \ldots, j_q \geq 1} \sum_{j_1 + \cdots + j_q = j} \pm K \frac{1}{(q-1)!} h_{\lambda_q}(\tau_{j_1}(\ldots), \ldots, \tau_{j_q}(\ldots, z)) =
\sum_{i,j \geq 1} \sum_{p \geq 2, i_1, \ldots, i_p \geq 1, i_p \geq 0} \sum_{q \geq 2} \sum_{j_1, \ldots, j_q \geq 1} \sum_{j_1 + \cdots + j_q = j} \pm K \frac{1}{(q-1)!} h_{\lambda_q}(\tau_{j_1}(\ldots), \ldots, \tau_{j_q}(\ldots, z)) =
\sum_{\sigma \in \text{Sh}(i_1, \ldots, i_q)} \pm K \frac{1}{(q-1)!} h_{\lambda_q}(\tau_{j_1}(\ldots), \ldots, \tau_{j_q}(\ldots, z)) =
\sum_{\sigma \in \text{Sh}(i_1, \ldots, i_q)} \pm K \frac{1}{(q-1)!} h_{\lambda_q}(\tau_{j_1}(\ldots), \ldots, \tau_{j_q}(\ldots, z)) =
\sum_{n} \sum_{i=0}^{n} \pm K \tau_{i+1}(\ldots, y \cdot_A \tau_{n-i+1}(\ldots, z),
\]
where we used the identities (4)-(5).

Comparing the formula for $\tau_{n+1}$ and the one for $\ell_{n+1}$ at the beginning of the proof, to show that the $\ell_{n+1}$ are multiderivations we can follow the above computations, replacing the leftmost $h$ by $\sigma$ and using the identities (6)-(7) (instead of (4)-(5)) when appropriate. \hfill \Box

3. Shifted derived Poisson manifolds

3.1. Shifted derived Poisson manifolds.

**Definition 3.1.** A $k$-shifted derived Poisson manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ whose sheaf of functions $C^\infty_\mathcal{M}$ is a sheaf of $k$-shifted derived Poisson algebras.

**Remark 3.2.** Equivalently, a $k$-shifted derived Poisson manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ such that $C^\infty(\mathcal{M})$, the space of global functions on $\mathcal{M}$, is endowed with a $k$-shifted derived Poisson algebra structure $\lambda_i : (C^\infty(\mathcal{M}))^\otimes l \to C^\infty(\mathcal{M}), l \geq 1$. 

Example 3.3. Let \( \mathfrak{g} \) be a finite dimensional \( L_\infty \) algebra. By extending the \( L_\infty \) structure maps \( \lambda_l \) \((l \geq 1)\) on \( \mathfrak{g} \) to the completed symmetric algebra \( \hat{S}(\mathfrak{g}[k]) \) via the Leibniz rule, one obtains a \( k \)-shifted derived Poisson algebra on \( \hat{S}(\mathfrak{g}[k]) = C^\infty(\mathfrak{g}[k])^\vee \). Thus \( (\mathfrak{g}[k])^\vee \) is a \( k \)-shifted derived Poisson manifold, called \( k \)-shifted derived Lie Poisson manifold. In particular, \( \mathfrak{g}^\vee \) is a derived Lie Poisson manifold.

If \( \mathfrak{g} \) is an ordinary Lie algebra, \( (\mathfrak{g}[0])^\vee = \mathfrak{g}^\vee \) admits a Poisson manifold structure, called Lie-Poisson structure. On the other hand, we have the standard Gerstenhaber algebra \( \Lambda^*\mathfrak{g} \), which corresponds to the \((-1)\)-shifted Poisson manifold structure on \( (\mathfrak{g}[-1])^\vee \).

The following proposition relates the notion of shifted derived Poisson manifolds introduced above with the one defined by Pridham [20].

Let \( \mathcal{M} \) be a \( \mathbb{Z} \)-graded manifold. By \( \hat{T}_{\text{poly}}^\bullet(\mathcal{M})[k] \), we denote the formally completed Schouten-Nijenhuis algebra of \( k \)-shifted polyvector fields on \( \mathcal{M} \) (see Appendix).

Proposition 3.4. A \( k \)-shifted derived Poisson manifold is equivalent to a DG manifold \( (\mathcal{M}, Q) \) equipped with a formal series of \((k-2)\)-shifted polyvector fields \( \pi = \sum_{l=2}^\infty \pi_l \) satisfying the Maurer–Cartan equation

\[
[Q, \pi] + \frac{1}{2}[\pi, \pi] = 0,
\]

(9)

where \( \pi_l \in \hat{T}_{\text{poly}}^l(\mathcal{M})[k-2] \), is of degree +1 in \( \hat{T}_{\text{poly}}^\bullet(\mathcal{M})[k-2][k-1] \).

Proof. The first bracket \( \lambda_1 : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \) is a derivation of degree +1 and squares to zero. Thus it determines a homological vector field \( Q \in \mathcal{T}(\mathcal{M}) \), which will also be denoted by \( \pi_1 \).

Since the \( l \)-brackets \( \lambda_l \) \((l = 1, 2, \cdots)\) define an \( L_\infty \) algebra structure on \( C^\infty(\mathcal{M})[-k] \), each \( \lambda_l \) \((l \geq 2)\) should be skew-symmetric w.r.t. \( C^\infty(\mathcal{M})[-k] \):

\[
\lambda_l(\cdots, f, g \cdots) = -(-1)^{|f||g|^k} \lambda_l(\cdots, g, f \cdots).
\]

Let

\[
\pi_l(f_1, f_2, \cdots, f_l) = (-1)^* \lambda_l(f_1, f_2, \cdots, f_l),
\]

( where \( * = (l-1)|f_1|^{k}+(l-2)|f_2|^{k}+\cdots+|f_{l-1}|^{k} \)

then \( \pi_l \) is symmetric w.r.t. elements in \( C^\infty(\mathcal{M})[-k+1] \), i.e.

\[
\pi_l(\cdots, f, g \cdots) = (-1)^{|f||l-1|}|g|^{k-1} \pi_l(\cdots, g, f \cdots).
\]

It is clear that \( \pi_l \) is also a derivation in each argument as \( \lambda_l \) is.

According to Appendix C, for any \( l \geq 2 \), \( \pi_l \) can be considered as a \((k-2)\)-shifted \( l \)-polyvector field, i.e. \( \pi_l \in \hat{T}_{\text{poly}}^l(\mathcal{M})[k-2] \). Moreover, its total degree is

\[
\| \pi_l \|_{k-2} = |\pi_l| - l(k-1) = |\lambda_l| - l(k-1) = 2 - k.
\]

Therefore, when being considered as an element in \( \hat{T}_{\text{poly}}^l(\mathcal{M})[k-2][k-1] \), \( \pi_l \) is of degree +1. Denote \( \Lambda = Q + \pi = \sum_{l \geq 1} \pi_l \). Since \( \pi_l \) \((l \geq 1)\) is of homogeneous total degree \((2 - k)\) in \( \hat{T}_{\text{poly}}^l(\mathcal{M})[k-2] \) and \( Q \) is homological, \( \Lambda \) is of total degree +1. Hence

\[
\Pi = \frac{1}{2}[Q + \pi, Q + \pi] = \frac{1}{2}[\Lambda, \Lambda] = \Lambda \circ \Lambda = \sum_{m,n \geq 1} \pi_m \circ \pi_n \in \bigoplus_{l=2}^{\infty} \hat{T}_{\text{poly}}^l(\mathcal{M})[k-2].
\]
The weight $p$ component of $\Pi$ is

$$\Pi_p = \sum_{m+n-1=p} \pi_m \circ \pi_n.$$  \hfill (10)

Note that, for $f_1, f_2, \ldots, f_{m+n-1} \in C^\infty(\mathcal{M})$,

$$(\pi_m \circ \pi_n)(f_1, \ldots, f_{m+n-1}) = \sum_{\sigma \in \text{Sh}(m,n-1)} (-1)^* e^{[k-1]}(\sigma) \pi_m(\pi_n(f_{\sigma(1)}, \ldots, f_{\sigma(n)}), f_{\sigma(n+1)}, \ldots, f_{\sigma(m+n-1)})$$

where $* = (n-1) [f_{\sigma(1)}] + (m-n+1) [f_{\sigma(n)}] + \cdots + [f_{m+n-2}]$. According to Equation (10), the latter is equivalent to the condition that $\Pi_p = 0$ for each $p \geq 2$. According to Equation (10), it is clear that this is exactly the generalized Jacobi identity of $\{\lambda_i\}_{i=1}^\infty$, with which $C^\infty(\mathcal{M})[-k]$ becomes an $L_\infty$ algebra. This concludes the proof.

**Definition 3.5.** Let $\mathcal{M}$ and $\mathcal{M}'$ be a $k$-shifted derived Poisson manifolds with, respectively, structure maps $\lambda_i : (C^\infty(\mathcal{M}))^{\otimes l} \to C^\infty(\mathcal{M})$ and $\lambda'_i : (C^\infty(\mathcal{M}'))^{\otimes l} \to C^\infty(\mathcal{M}')$, $l \geq 1$. A morphism of $k$-shifted derived Poisson manifolds from $\mathcal{M}$ to $\mathcal{M}'$ is a map of $\mathbb{Z}$-graded manifolds $\phi : \mathcal{M} \to \mathcal{M}'$ together with a collection of maps

$$\varphi_n : (C^\infty(\mathcal{M}'))^{\otimes n} \to C^\infty(\mathcal{M})$$

such that $\varphi_\infty = (\varphi_1 = \phi^*, \varphi_2, \varphi_3, \ldots)$ is a morphism of $k$-shifted derived Poisson algebras from $(C^\infty(\mathcal{M}'), \lambda'_i)$ to $(C^\infty(\mathcal{M}), \lambda_i)$.

In particular, $\phi : \mathcal{M} \to \mathcal{M}'$ is a map of DG manifolds.

3.2. $L_\infty$ algebroids and shifted derived Poisson manifolds. We follow the definition of Bruce [3], who considered $\mathbb{Z}_2$-case.

**Definition 3.6.** An $L_\infty$ algebroid consists of a vector bundle object $\mathcal{L} \to \mathcal{M}$ in the category of $\mathbb{Z}$-graded manifolds together with

- a sequence $\lambda_i \geq 1$ of maps $\lambda_i : \Lambda^l \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})[2-l]$, called multibrackets, that determine a structure of $L_\infty$ algebra on the space $\Gamma(\mathcal{L})$ of smooth sections of $\mathcal{L} \to \mathcal{M}$
- and a sequence $\rho_i \geq 0$ of bundle maps $\rho_i : \Lambda^l \mathcal{M} \to T\mathcal{M}[1-l]$, called anchor maps, that determine a morphism of $L_\infty$ algebras from $\Gamma(\mathcal{L})$ to $\mathcal{T}(\mathcal{M})$
satisfying the compatibility condition

\[
\lambda_l(a_1, a_2, \ldots, a_{l-1}, f a_l) = \rho_{l-1}(a_1, a_2, \ldots, a_{l-1}) (f) a_l \\
+ (-1)^{|l+|a_1|+\ldots+|a_{l-1}|+|f|} f \lambda_l(a_1, a_2, \ldots, a_{l-1}, a_l)
\]

(11)

for all \( l \geq 1, a_1, \ldots, a_l \in \Gamma(L), \) and \( f \in C^\infty(M) \).

**Remark 3.7.** Note that the image of \( \rho_1 \) may not be integrable. When \( M \) is an ordinary manifold \( M \) (being considered of degree zero) and the vector bundle \( L = \bigoplus_{i \geq 0} L^i \to M \) is a non-negative graded vector bundle, due to degree reasons, all higher anchor maps \( \rho_l \) vanish except for \( \rho_1 \), which must be a bundle map \( L^0 \to TM \). Our notion of \( L^\infty \) algebroids reduces to the one studied by Laurent-Gengoux et al. [12]. In this case, \( \rho_1(L^0) \) defines a singular foliation on \( M \). When \( L \) is concentrated in degree 0, it becomes a usual Lie algebroid over \( M \).

The following is a \( \mathbb{Z} \)-graded analogue of Theorem 1 and Corollary 2 in [3].

**Theorem 3.8.** Let \( L \to M \) be a vector bundle object in the category of \( \mathbb{Z} \)-graded manifolds and let \( k \in \mathbb{Z} \) be a fixed integer. The following statements are equivalent:

1. The vector bundle \( L \to M \) is an \( L^\infty \) algebroid.
2. The space \( \Gamma(S(L[k])) \) is a \( k \)-shifted derived Poisson algebra with \( l \)-bracket of weight \((1 - l)\).
3. The graded manifold \((L[k])^\vee \) is a \( k \)-shifted derived Poisson manifold and the weight of the \( l \)-bracket on \( C^\infty((L[k])^\vee) \) is \((1 - l)\).

Recall that \( C^\infty((L[k])^\vee) \) naturally carries the weight associated with the vector bundle structure \((L[k])^\vee \to M\). Namely, elements in \( \Gamma(S^m(L[k])) \) are of weight \( m \). The weight of the \( l \)-bracket

\[
\lambda_l : \Gamma(S^*(L[k])) \times \cdots \times \Gamma(S^*(L[k])) \to \Gamma(S^*(L[k]))
\]

means the difference of weights on both sides.

**Proof.** On the one hand, an \( L^\infty \) algebroid structure on \( L \) is extended to a \( k \)-shifted derived Poisson algebra on \( \Gamma(S(L[k])) \) via the Leibniz rule and the generation relations:

\[
\lambda_l(a_1, \ldots, a_{l-1}, f) = \rho_{l-1}(a_1, \ldots, a_{l-1}) f, \quad (l \geq 1) \\
\lambda_l(f, g, \ldots) = 0, \quad (l \geq 2)
\]

where \( a_1, \ldots, a_l \in \Gamma(L[k]), f, g \in C^\infty(M) \). As the \( l \)-bracket \( \lambda_l \) maps

\[
\Gamma(S^{n_1}(L[k])) \times \cdots \times \Gamma(S^{n_l}(L[k])) \to \Gamma(S^{n_1+\ldots+n_l-l+1}(L[k]))
\]

its weight is certainly \((1 - l)\).

On the other hand, given a \( k \)-shifted derived Poisson algebra structure on \( \Gamma(S(L[k])) \) whose \( l \)-bracket is of weight \((1 - l)\), by restricting all entries of the \( l \)-bracket to \( \Gamma(L[k]) \) (and properly shifting the degrees), one gets the \( L^\infty \) algebra structure on \( \Gamma(L) \). Similarly, by restricting the first \((l - 1)\) entries to \( \Gamma(L[k]) \) and the last to \( C^\infty(M) \), one gets the \((l - 1)\)-th anchor map. Finally, the Leibniz rule of the structure maps on \( \Gamma(S(L[k])) \) implies Identity (11).

Let \( L \to M \) be an \( L^\infty \) algebroid with structure maps \( \lambda_l \) and \( \rho_l \) as in Definition 3.6. The first bracket \( \lambda_1 : \Gamma(L) \to \Gamma(L) \) and \( \rho_0 \in \mathcal{T}(M) \) are compatible:

\[
\lambda_1(fa) = \rho_0(f)a + (-1)^{|f|} f \lambda_1(a), \quad \forall a \in \Gamma(L), f \in C^\infty(M).
\]
Introduce a dual map $\lambda^\vee_1 : \Gamma(L^\vee) \to \Gamma(L^\vee)$ by the following relation
\[
\langle \lambda^\vee_1(\xi), a \rangle = \rho \langle \xi, a \rangle - (-1)^{kl} \langle \xi, \lambda_1(a) \rangle, \quad \forall \xi \in \Gamma(L^\vee), \forall a \in \Gamma(L).
\]
Since $\lambda_1^2 = 0$, it can be verified that $(\lambda_1^\vee)^2 = 0$. Hence $(\Gamma(L^\vee), \lambda_1^\vee)$ is a cochain complex.

**Definition 3.9.** Let $\mathcal{L}_1 \to \mathcal{M}_1$ and $\mathcal{L}_2 \to \mathcal{M}_2$ be $L_\infty$ algebroids.

1. A morphism of $L_\infty$ algebroids from $\mathcal{L}_1 \to \mathcal{M}_1$ to $\mathcal{L}_2 \to \mathcal{M}_2$ is a sequence of bundle maps
   
   \[
   (\mathcal{L}_1)^p \xrightarrow{\phi_p} \mathcal{L}_2
   \]
   
   for $p = 1, 2, \cdots$, such that the induced map $\phi : \mathcal{L}_1[1] \to \mathcal{L}_2[1]$ is a map of DG-manifolds.\(^1\)

2. A quasi isomorphism from $\mathcal{L}_1 \to \mathcal{M}_1$ to $\mathcal{L}_2 \to \mathcal{M}_2$ is a morphism of $L_\infty$ algebroids $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\phi_1^\vee : \Gamma(L_2^\vee) \to \Gamma(L_1^\vee)$ is a quasi-isomorphism of cochain complexes.

The following fact can be easily verified.

**Proposition 3.10.** Let $\mathcal{L}_1 \to \mathcal{M}$ and $\mathcal{L}_2 \to \mathcal{M}$ be $L_\infty$ algebroids. Let

\[
(\mathcal{L}_1)^p \xrightarrow{\phi} \mathcal{L}_2
\]

(for $p = 1, 2, \cdots$) be a morphism of $L_\infty$ algebroids. Then there induces a morphism of $k$-shifted Poisson manifolds from $(\mathcal{L}_1[k])^\vee$ to $(\mathcal{L}_2[k])^\vee$.

4. The $(-1)$-shifted derived Poisson manifold arising from a Lie pair

4.1. **First construction: $L_\infty$ algebroid arising from a Lie pair.** By a Lie pair $(L, A)$, we mean an ordinary (non-graded) Lie algebroid $(L, [\cdot, \cdot]_L, \rho_L)$ over an ordinary (non-graded) smooth manifold $M$, together with a Lie subalgebroid $(A, [\cdot, \cdot]_A, \rho_A)$ of $L$ over the same base $M$.

For convenience, let us denote the quotient vector bundle $L/A$ by $B$. Denote $pr_B : L \to B$ the projection map. Note that $B$ is naturally an $A$-module:

\[
\nabla^A_a b = pr_B[a, l]_L,
\]

where $a \in \Gamma(A), b \in \Gamma(B)$ and $l \in \Gamma(L)$ satisfying $pr_B(l) = b$. The flat $A$-connection $\nabla^A$ on $B$ is also known as the Bott connection.

Let

\[
\Omega^\bullet_A = \Gamma(L^\bullet A^\vee), \quad \Omega^\bullet_A(L^\bullet B) = \Gamma(L^\bullet A^\vee \otimes L^\bullet B).
\]

\(^1\)This means that the induced morphism of commutative algebras over $\phi_0^\vee : C^\infty(M_2) \to C^\infty(M_1)$:

\[
\phi^\vee : C^\infty(\mathcal{L}_2[1]) = \Gamma(S(\mathcal{L}_2[1][-1])) \to C^\infty(\mathcal{L}_1[1]) = \Gamma(S(\mathcal{L}_1[1][-1]))
\]

commutes with the homological vector fields $Q_1$ and $Q_2$: $Q_2 \circ \phi^\vee = \phi^\vee \circ Q_1$. Here $Q_1$ and $Q_2$ are, respectively, homological vector fields on $\mathcal{L}_1[1]$ and $\mathcal{L}_2[1]$ corresponding to the $L_\infty$ algebroid structures as in Proposition A.2.
Note that elements in $\Omega^m_\Lambda(\Lambda^n B)$ are of degree $m + n$. Denote by
\[ d_B^{\text{Bott}} : \Omega^0_\Lambda(\Lambda^* B) \to \Omega^1_\Lambda(\Lambda^* B) \tag{12} \]
the standard Chevalley–Eilenberg differential corresponding to the Bott $A$-connection on $\Lambda^* B$.

**Proposition 4.1.** Let $(L, A)$ be a Lie pair. Any splitting $j : B \to L$ of the exact sequence
\[ 0 \to A \overset{i}{\to} L \overset{pr_B}{\to} B \to 0 \tag{13} \]
induces an $L_\infty$ algebroid structure on the $\mathbb{Z}$-graded vector bundle $A[1] \oplus B \to A[1]$.

**Proof.** A splitting of (13) is a pair of maps $j : B \to L$ and $pr_A : L \to A$ such that $pr_B \circ j = \text{id}_B$, $pr_A \circ i = \text{id}_A$ and $i \circ pr_A + j \circ pr_B = \text{id}_L$:
\[ 0 \overset{}{\leftrightarrow} A \overset{i}{\underset{pr_A}{\leftrightarrow}} L \overset{pr_B}{\underset{j}{\leftrightarrow}} B \overset{}{\leftrightarrow} 0. \]
We therefore obtain an isomorphism of vector bundles over $M$:
\[ I : A \oplus B \to L, \quad (a, b) \mapsto i(a) + j(b), \tag{14} \]
for all $a \in A$ and $b \in B$.

Consider the $\mathbb{Z}$-graded vector bundle $\mathcal{L} := A[1] \oplus B \to A[1]$. Note that the base of $\mathcal{L}$ is $\mathcal{M} = A[1]$. Then $\mathcal{L}[1] = A[1] \oplus B[1] \cong L[1]$.

Since $L$ is a Lie algebroid, the standard Chevalley–Eilenberg differential $d_L : \Gamma(\Lambda^* L^\vee) \to \Gamma(\Lambda^{*+1} L^\vee)$ defines a homological vector field $Q_L$ on $L[1]$. Via the identification (14), one obtains a homological vector field $Q$ on $\mathcal{L}[1]$. Since $A$ is a Lie subalgebroid of $L$, it is simple to see that the homological vector field $Q$ on $\mathcal{L}[1]$ is indeed tangent to the zero section $A[1] \subset \mathcal{L}[1]$, whose restriction can be identified with the Chevalley–Eilenberg differential $d_A$ on $A[1]$. Thus by Proposition A.2, $\mathcal{L} = A[1] \oplus B \to A[1]$ is indeed an $L_\infty$ algebroid. \hfill \Box

**Remark 4.2.** By Proposition 4.16 in the sequel, we will see that different choice of splittings give isomorphic $L_\infty$ algebras $\Gamma(\mathcal{L}) \cong \Omega^*_\Lambda(B)$, i.e. a collection of maps
\[ \varphi_n : (\Omega^*_\Lambda(B))^\otimes n \to \Omega^*_\Lambda(B), \quad n = 1, 2, \ldots \]
whose first map $\varphi_1$ is the identity. But $\varphi_n$ ($n \geq 2$) are not $C^\infty(A[1]) = \Omega^*_A$-multilinear. So these $\varphi_n$ do not induce bundle maps $\otimes^n \mathcal{L} \to \mathcal{L}$. Therefore, different choices of splittings do not induce isomorphic $L_\infty$ algebroid structures on the $\mathbb{Z}$-graded vector bundle $\mathcal{L} = A[1] \oplus B \to A[1]$.

As an immediate consequence of Proposition 4.1, we have the following

**Proposition 4.3.** Let $(L, A)$ be a Lie pair. Then any splitting of (13) induces a $(-1)$-shifted derived Poisson algebra structure on $\Omega^*_\Lambda(\Lambda^* B)$, where the multiplication is the wedge product and the $L_\infty$ brackets are given as follows:

1. The first bracket $l_1$ is the Chevalley–Eilenberg differential $d_B^{\text{Bott}}$ as in Equation (12);
2. The binary bracket
\[ l_2 = \{-, -\} : \Omega^1_\Lambda(\Lambda^1 B) \times \Omega^0_\Lambda(\Lambda^1 B) \to \Omega^{1+0}_\Lambda(\Lambda^{1+1-1} B) \]
is generated by the following relations:
   a) $\{u, v\} = pr_B[\{u, v\}_L], \forall u, v \in \Gamma(B);$  
   b) $\{u, \omega\} = pr_A[\{u, \omega\}_A], \forall u \in \Gamma(B), \omega \in \Gamma(A^\vee);$
c) \( \{ u, f \} = \rho_L(u)f, \forall u \in \Gamma(B), \forall f \in C^\infty(M) \);

d) \( \{ \omega_1, \omega_2 \} = 0, \forall \omega_1, \omega_2 \in \Omega^*_B \);

(3) The trinary bracket

\[
l_3 = r(-, -, -) : \Omega^i_A(\Lambda^j B) \times \Omega^p_A(\Lambda^l B) \times \Omega^s_A(\Lambda^s B) \to \Omega^{i+p+r-l}_A(\Lambda^{i+l+s-2} B)
\]

is \( C^\infty(M) \)-linear in each entry and generated by the following relations:

\begin{itemize}
  \item[a)] \( r \) vanishes if restricted to \( \Gamma(B) \times \Gamma(B) \times \Gamma(B) \), \( \Gamma(B) \times \Gamma(A^\vee) \times \Gamma(A^\vee) \), and \( \Gamma(A^\vee) \times \Gamma(A^\vee) \);
  \item[b)] \( r(u, v, \omega) = \langle \text{pr}_A[u, v]_L, \omega \rangle \), for all \( u, v \in \Gamma(B) \) and \( \omega \in \Gamma(A^\vee) \);
\end{itemize}

(4) All the higher brackets \( l_i \) (\( i \geq 4 \)) vanish.

Here we have used the identification \( L \cong A \oplus B \) and \( L^\vee \cong A^\vee \oplus B^\vee \).

In order to prove Proposition 4.3, we need to give an explicit expression for the homological vector field \( Q \) on \( L[1] \). Choosing a splitting of (13), we have an identification \( L \cong A \oplus B \). Besides the Bott \( A \)-connection \( \nabla \) on \( B \), we also have a \( B \)-"connection" on \( A \):

\[
\Gamma(B) \otimes \Gamma(A) \to \Gamma(A) : (u, a) \mapsto \Delta_u a = \text{pr}_A[u, a]_L.
\]

Introduce the following maps:

\[
\rho_B : B \to TM, \quad \rho_B = \rho_L|_B;
\]

\[
[-, -]_B : \Gamma(B) \otimes \Gamma(B) \to \Gamma(B), \quad [u_1, u_2]_B = \text{pr}_B[u_1, u_2]_L,
\]

\[
\beta(-, -) : \Gamma(B) \otimes \Gamma(B) \to \Gamma(A), \quad \beta(u_1, u_2) = \text{pr}_A[u_1, u_2]_L,
\]

where \( u_1, u_2 \in \Gamma(B) \). Then the Lie algebroid structure on \( L \) can be described as follows:

\[
\begin{cases}
  [a_1, a_2]_L = [a_1, a_2]_A; \\
  [u_1, u_2]_L = \beta(u_1, u_2) + [u_1, u_2]_B; \\
  [a, u]_L = -\Delta_u a + \nabla^A_a u; \\
  \rho_L(a + u) = \rho_A(a) + \rho_B(u),
\end{cases}
\]

(15)

where \( a, a_1, a_2 \in \Gamma(A), u, u_1, u_2 \in \Gamma(B) \).

Let \( \Omega^p_A(\Lambda^q B^\vee) = \Gamma(\Lambda^p A^\vee \otimes \Lambda^q B^\vee) \). Then

\[
C^\infty(L[1]) = \Gamma(\Lambda^* L^\vee[-1]) = \bigoplus_{p \geq 0, q \geq 0} \Omega^p_A(\Lambda^q B^\vee).
\]

Since \( B \) is an \( A \)-module, we have the standard Chevalley–Eilenberg differential:

\[
d^B_{\text{Bott}} : \Omega^p_A(\Lambda^q B^\vee) \to \Omega^{p+1}_A(\Lambda^q B^\vee).
\]

In a similar fashion, we can define a map:

\[
d^A_B : \Omega^p_A(\Lambda^q B^\vee) \to \Omega^p_A(\Lambda^{q+1} B^\vee).
\]

Note that, however, \( d^A_B \) may not square to zero.

There is also a degree \( (+1) \)-derivation:

\[
d_\beta : \Omega^p_A(\Lambda^q B^\vee) \to \Omega^{p-1}_A(\Lambda^{q+2} B^\vee),
\]

generated by the following relations:

\[
d_\beta f = 0, \quad \forall f \in C^\infty(M);
\]

\[
\langle d_\beta(\omega), u_1 \wedge u_2 \rangle = -\langle \omega, \beta(u_1, u_2) \rangle, \quad \forall \omega \in \Gamma(A^\vee), u_1, u_2 \in \Gamma(B).
\]
The following lemma is immediate:

**Lemma 4.4.** The homological vector field \( Q \) on \( \mathcal{L}[1] \) is given by

\[
d_L = d_{\text{Bott}}^A + d_{\beta} + d_{\beta}.
\]

Consider the vector bundle \( \mathcal{L} = A[1] \oplus B \to A[1] \). Then, its section space \( \Gamma(\mathcal{L}) \cong \Omega^n_A(B) \) and the space of vector fields on the base manifold \( T(A[1]) \) is exactly \( \text{Der}(\Omega^n_A) \).

**Lemma 4.5.** The \( L_{\infty} \) algebroid structure on the vector bundle \( \mathcal{L} = A[1] \oplus B \to A[1] \) as in Proposition 4.1 is determined by the following relations:

1. The 0-th anchor \( \rho_0 = d_A : \Omega^n_A \to \Omega^{n+1}_A \);
2. The first anchor \( \rho_1 : \mathcal{L} \to T(A[1]) \) is determined by
   \[
   \rho_1(u) = \Delta_u, \quad \forall u \in \Gamma(B);
   \]
3. The binary anchor \( \rho_2 : \otimes^2 \mathcal{L} \to T(A[1]) \) is determined by
   \[
   \rho_2(u_1, u_2) = i_{\beta(u_1, u_2)}, \quad \forall u_1, u_2 \in \Gamma(B);
   \]
4. The first bracket \( l_1 = d_{\text{Bott}}^A : \Omega^n_A(B) \to \Omega^{n+1}_A(B) \);
5. The binary bracket \( l_2 : \otimes^2 \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L}) \) is determined by
   \[
   l_2(u_1, u_2) = [u_1, u_2]_B, \quad \forall u_1, u_2 \in \Gamma(B);
   \]
6. The trinary bracket \( l_3 : \otimes^3 \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L}) \) is determined by
   \[
   l_3(u_1, u_2, u_3) = 0, \quad \forall u_1, u_2, u_3 \in \Gamma(B);
   \]
7. All higher brackets \( l_i \) (\( i \geq 4 \)), and anchors \( \rho_j \) (\( j \geq 3 \)) vanish.

**Proof.** We follow the construction as in the proof of Proposition A.2. The data are as follows:

1. \( \mathfrak{a} = D^{\leq 1}(\mathcal{L}[1]) \cong \text{Der}(\Omega^n_A(\Lambda^* B^\vee)) \oplus \Omega^n_A(\Lambda^* B^\vee) \), the graded Lie algebra of first order differential operators on \( \mathcal{L}[1] \).
2. \( \mathfrak{a} = \Gamma(\mathcal{L}[1]) \oplus \mathcal{C}^\infty(A[1]) \cong \Omega^*_A(B) \oplus \Omega^*_A. \) The inclusion map \( \mathfrak{a} \hookrightarrow \mathfrak{a} \) is given as follows: \( \forall \alpha \in \Omega^*_A(B), \) \( \iota_\alpha \) is the contraction operator
   \[
   \iota_\alpha : \Omega^p_A(\Lambda^q B^\vee) \to \Omega^{p+q}(\Lambda^{q-1} B^\vee).
   \]
   Also, \( \Omega^*_A \) can be naturally treated as a subalgebra in \( \Omega^*_A(\Lambda^* B^\vee) \).
3. \( P : \mathfrak{a} \to \mathfrak{a}, \) the projection map that consists of \( P_1 : \text{Der}(\Omega^*_A(\Lambda^* B^\vee)) \to \Omega^*_A(B) \) and \( P_2 : \Omega^*_A(\Lambda^* B^\vee) \to \Omega^*_A: \)
   \[
   P(v + \mu) = P_1(v) + P_2(\mu), \quad v \in \text{Der}(\Omega^*_A(\Lambda^* B^\vee)), \mu \in \Omega^*_A(\Lambda^* B^\vee).
   \]
   The \( P_2 \) map is projection to the \( \Omega^*_A \)-part. The \( P_1 \) map is defined as follows: for any derivation \( v \in \text{Der}(\Omega^*_A(\Lambda^* B^\vee)), \) \( P_1(v) \in \Omega^*_A(B) \) is determined by the composition
   \[
   \Gamma(B^\vee) \xrightarrow{\nu} \Omega^*_A(\Lambda^* B^\vee) \xrightarrow{P_2} \Omega^*_A.
   \]
4. \( Q \in \mathfrak{a}, \) the homological vector field \( d_L = d_{\text{Bott}}^A + d_{\beta} + d_{\beta} \) as in Equation (16).
We now apply Equations (35) and (34) to get the structure maps of $\mathcal{L}$. For this purpose, we observe that in the decomposition $d_L = d_{\text{Bott}}^A + d_B^A + d_\beta$, $d_{\text{Bott}}^A$ is of weight 0, $d_B^A$ is of weight 1, and $d_\beta$ is of weight 2.

Now, applying Equation (34), the 0-th anchor is simply determined by
$$\rho_0(\omega) = P_2[d_L, \omega] = [d_{\text{Bott}}^A, \omega] = d_A(\omega), \quad \forall \omega \in \Omega^*_A.$$ This proves (1). The first anchor can be obtained by the same method:
$$\rho_1(u)\omega = P_2[[d_L, \iota_u], \omega] = [[d_{\text{Bott}}^A, \iota_u], \omega] = (d_B^A \circ \iota_u + \iota_u \circ d_{\text{Bott}}^A)\omega = \iota_u d_{\text{Bott}}^A \omega = \Delta_u \omega.$$ This proves (2). One can similarly prove (3).

Now we describe the $L_\infty$ brackets. For the first bracket, let us apply Equation (35). For $u \in \Gamma(B)$, note that $l_1(u) \in \Omega^1_A(B)$. For all $\eta \in \Omega^0_A(\Lambda^1 B^\vee) = \Gamma(B^\vee)$, we have
$$\iota_{l_1(u)}(\eta) = (P_1[d_L, \iota_u])\eta = ([d_{\text{Bott}}^A, \iota_u])\eta = (d_{\text{Bott}}^A \circ \iota_u + \iota_u \circ d_{\text{Bott}}^A)\eta = d_{\text{Bott}}^A \langle u, \eta \rangle + \iota_u d_{\text{Bott}}^A \eta = \iota_{d_{\text{Bott}}^A u}(\eta).$$ This proves (4).

For the binary bracket, let $u_1, u_2 \in \Gamma(B)$, $\eta \in \Gamma(B^\vee)$. Then,
$$\iota_{l_2(u_1, u_2)}(\eta) = (P_2[[d_L, \iota_{u_1}], \iota_{u_2}])\eta = [[d_{\text{Bott}}^A, \iota_{u_1}], \iota_{u_2}]\eta = (d_{\text{Bott}}^A \circ \iota_{u_1} + \iota_{u_1} \circ d_{\text{Bott}}^A)\iota_{u_2} - \iota_{u_2} \circ d_{\text{Bott}}^A \circ \iota_{u_1} - \iota_{u_2} \circ \iota_{u_1} \circ d_{\text{Bott}}^A\eta = \iota_{u_1} d_{\text{Bott}}^A \langle u_2, \eta \rangle - \iota_{u_2} d_{\text{Bott}}^A \langle u_1, \eta \rangle - \iota_{u_1} \iota_{u_2} d_{\text{Bott}}^A \eta = \langle \eta, [u_1, u_2]_B \rangle = \iota_{[u_1, u_2]}(\eta).$$ This proves (5). The rest of claims (6) and (7) can be proved in a similar way, which we omit. □

**Proof of Proposition 4.3.** According to Theorem 3.8, the $L_\infty$ algebroid $\mathcal{L} = A[1] \oplus B$ gives rise to a $(-1)$-shifted derived Poisson algebra structure on the space $$\Gamma(\hat{S}(\mathcal{L}[-1])) = \Gamma(A[1] \times \Lambda^\bullet B)$$ (as a vector bundle over $A[1]$) = $\Omega^*_A(\Lambda^\bullet B)$.

All the structure maps $\lambda_i$ are extended from anchors and brackets of $\mathcal{L}$ by the Leibniz rule. One easily obtains the specific formulas of $\lambda_i$ from the previous lemma. □

4.2. **Second construction: Fedosov DG Lie algebroid arising from a Lie pair.** Let us first recall Fedosov DG Lie algebroids as constructed in [25]. We will use the same settings as in Section 4.1: a Lie pair $(L, A)$ and $B = L/A$. Consider the graded vector bundle $\hat{\mathfrak{g}} := L[1] \oplus B \rightarrow L[1]$. It is clear that $C^\infty(\hat{\mathfrak{g}}) = \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee)$.

Given a splitting of the short exact sequence (13), the following maps are established in [25, Section 3]:

- $\delta : \Lambda^\bullet L^\vee \otimes \hat{S}B^\vee \rightarrow \Lambda^{\bullet + 1} L^\vee \otimes \hat{S}B^\vee$, a degree +1 derivation;
- $\sigma : \Lambda^\bullet L^\vee \otimes \hat{S}B^\vee \rightarrow \Lambda^\bullet A^\vee$, the projection;
- $\tau : \Lambda^\bullet A^\vee \rightarrow \Lambda^\bullet L^\vee \otimes \hat{S}B^\vee$, the inclusion;
\begin{itemize}
    \item $h : \Lambda^i L^\vee \otimes \hat{SB}^\vee \rightarrow \Lambda^{*i} L^\vee \otimes \hat{SB}^\vee$, the homotopy map;
    \item $\sigma_z = \sigma \otimes 1 : \Lambda^i L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B \rightarrow \Lambda^i A^\vee \otimes \Lambda^\bullet B$, the projection;
    \item $\tau_\check{z} = \tau \otimes 1 : \Lambda^i A^\vee \otimes \Lambda^\bullet B \rightarrow \Lambda^i L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B$, the inclusion;
    \item $h_\check{z} = h \otimes 1 : \Lambda^i L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B \rightarrow \Lambda^{*i} L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B$, the homotopy map.
\end{itemize}

We recall a result in [25]:

\textbf{Proposition 4.6} (Theorem 3.5 in [25]). Let $(L, A)$ be a Lie pair. Given a splitting of the short exact sequence (13) and a torsion-free $L$-connection $\nabla$ on $B$ extending the Bott $A$-connection, there exists a unique 1-form valued in formal vertical vector fields of $B$:

$$X^\nabla \in \Gamma(L^\vee \otimes \hat{S}^2 B^\vee \otimes B)$$

satisfying $h_\check{z}(X^\nabla) = 0$ and such that the derivation $Q : \Gamma(\Lambda^i L^\vee \otimes \hat{SB}^\vee) \rightarrow \Gamma(\Lambda^{*i+1} L^\vee \otimes \hat{SB}^\vee)$ defined by

$$Q = -\delta + d^\nabla + X^\nabla$$

satisfies $Q^2 = 0$. Here

1. $d^\nabla_L : \Gamma(\Lambda^i L^\vee \otimes \hat{SB}^\vee) \rightarrow \Gamma(\Lambda^{*i+1} L^\vee \otimes \hat{SB}^\vee)$ is the covariant derivative associated with the $L$-connection $\nabla$ on $B$;
2. $X^\nabla$ acts on the algebra $\Gamma(\Lambda^i L^\vee \otimes \hat{SB}^\vee)$ as a derivation in a natural fashion.

As a consequence, $(\mathfrak{F} = L[1] \oplus B, Q)$ is a DG manifold, called the Fedosov DG manifold. Consider the space $\Gamma(\Lambda^i L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B)$, which is identified as the space of formal polyvector fields on $\mathfrak{F}$ that are tangent to $B$. Moreover, it is a subalgebra of the Schouten-Nijenhuis algebra $T^\bullet_{\text{poly}}(\mathfrak{F})$. It can be further proved that $\Gamma(\Lambda^i L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B)$ is stable under the Lie derivative $\mathcal{L}_Q = [Q, -]$. Hence

$$\Gamma(\Lambda^i L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B), \mathcal{L}_Q$$

is differential Gerstenhaber algebra.

We need another result:

\textbf{Proposition 4.7} (Theorem 4.19 in [25]). Under the same hypothesis as in Proposition 4.6, there induces a contraction datum:

$$\begin{align*}
\leftarrow \sigma_\check{z} \quad (\Gamma(\Lambda^i L^\vee \otimes \hat{SB}^\vee \otimes \Lambda^\bullet B), \mathcal{L}_Q) \xrightarrow{\check{h}_2} (\Omega^A_*(\Lambda^\bullet B), d^\text{Bott}_A). 
\end{align*}$$

The maps $\check{h}_2$ and $\check{\tau}_2$ are defined by (see [25, Section 4.4]):

$$\begin{align*}
\check{h}_2 &= h_\check{z} + \sum_{i=1}^{\infty} (h_\check{z} \circ \mathcal{L}_Q)^i h_\check{z}, \\
\check{\tau}_2 &= \tau_\check{z} + \sum_{i=1}^{\infty} (h_\check{z} \circ \mathcal{L}_Q)^i \tau_\check{z}.
\end{align*}$$
The differential Gerstenhaber algebra \((\Gamma(\Lambda^* L^\vee \otimes \hat{S} B^\vee \otimes \Lambda^* B), \mathcal{L}_Q)\) is, by definition, a \((-1)\)-shifted derived Poisson algebra, whose first bracket is \(\mathcal{L}_Q\), binary bracket is the Schouten bracket, and all higher brackets vanishes. By homotopy transfer Theorem 2.18, there induces on \(\Omega^*_A(\Lambda^* B)\) a \((-1)\)-shifted derived Poisson algebra, whose first bracket is \(d_A^{\text{Bott}}\).

Below is our main result in this section.

**Proposition 4.8.** Let \((L, A)\) be a Lie pair. Choose a splitting of the exact sequence (13), and a torsion free \(L\)-connection \(\nabla\) on \(B\) that extends the \(A\)-module structure of \(B\). Then the \((-1)\)-shifted derived Poisson algebra structure on \(\Omega^*_A(\Lambda^* B)\) obtained by homotopy transfer from the one on \(\Gamma(\Lambda^* L^\vee \otimes \hat{S} B^\vee \otimes \Lambda^* B)\), which is induced from Fedosov DG Lie algebroid \(L[1] \oplus B \to L[1]\), coincides with the one as in Proposition 4.3.

The rest of this section is devoted to prove this proposition. We need to recall some techniques and facts that are already shown in [25]. Let us resume the settings and notions in Section 4.1. We chose a splitting of Sequence (13) so that one treats \(L = A \oplus B\) directly.

Given a torsion free \(L\)-connection \(\nabla\) on \(B\) that extends the \(A\)-module structure of \(B\), one can write
\[
\nabla_{a+b'} = \nabla_a b' + \nabla_b b', \quad \forall a \in \Gamma(A), b, b' \in \Gamma(B).
\]

Here the \(\Delta^B\) can be thought of as a \(B\)-“connection” on \(B\). The condition that \(\nabla\) being torsion free means
\[
\Delta^B_{b_1} b_2 - \Delta^B_{b_2} b_1 = [b_1, b_2]_B, \quad \forall b_1, b_2 \in \Gamma(B). \tag{19}
\]

Here \([−, −]_B\) is the \(B\)-“bracket” introduced in Equation (15).

In what follows, let us denote
\[
C^{(p, q, r)} = \Gamma(\Lambda^p A^\vee \otimes \Lambda^q B^\vee \otimes S^r B^\vee).
\]

Recall the graded vector bundle \(\mathcal{F} := L[1] \oplus B \to L[1]\). Hence
\[
C^\infty(\mathcal{F}) = \Gamma(\Lambda^* L^\vee \otimes \hat{S} B^\vee) = \Gamma(\Lambda^* A^\vee \otimes \Lambda^* B^\vee \otimes \hat{S} B^\vee) = \prod_{p, q, r \geq 0} C^{(p, q, r)}.
\]

Recall Equation (16), where we split \(d_L\) into three components. Abusing notations, let us again write
\[
d_L^{\nabla} = d_A^{\text{Bott}} + d_B^A + d_\beta : \Gamma(\Lambda^* L^\vee \otimes \hat{S} B^\vee) \to \Gamma(\Lambda^{*+1} L^\vee \otimes \hat{S} B^\vee),
\]

where
\[
d_A^{\text{Bott}} : C^{(p, q, r)} \to C^{(p+1, q, r)},
\]
\[
d_B^A : C^{(p, q, r)} \to C^{(p, q+1, r)},
\]
\[
d_\beta : C^{(p, q, r)} \to C^{(p-1, q+2, r)}.
\]

All the following are due to the varies relevant definitions and facts in [25]:

- The kernel of \(\sigma_\natural\) is \(C^{(\bullet \geq 1, \bullet)} \oplus C^{(\bullet \bullet \geq 1)} \otimes \Gamma(\Lambda^* B)\).
- The map \(h_\natural\) sends \(C^{(p, q, r)} \otimes \Gamma(\Lambda^s B)\) to \(C^{(p, q-1, r+1)} \otimes \Gamma(\Lambda^s B)\).
The map
\[ \theta = d_L^Y + X^Y \]
is a degree +1 derivation of \( C^\infty(\mathfrak{g}) \), and in fact a perturbation of the cochain complex \( (C^\infty(\mathfrak{g}), -\delta) \). Moreover, the map \( \mathcal{L}_\theta \) satisfies

\[ \mathcal{L}_\theta(C^{(p,q,r)} \otimes \Gamma(A^s B)) \subset (C^{(p+1,q,r)} \oplus C^{(p,q+1,r)} \oplus C^{(p,q+2,r)}) \otimes \Gamma(A^s B). \]

The Schouten-Nijenhuis bracket in \( \Gamma(\Lambda^* L^Y \otimes \hat{S}B^Y \otimes \Lambda^* B) = C^{(***)} \otimes \Gamma(\Lambda^* B) \) satisfies

\[ [C^{(p,q,r)} \otimes \Gamma(A^s B), C^{(a,b,c)} \otimes \Gamma(A^d B)] \subset C^{(p+a,q+b,r+c-1)} \otimes \Gamma(A^{s+d-1} B). \]

Using these facts, the following formulas can be straightforward verified.

**Lemma 4.9.**  
1) For any \( b \in \Gamma(B) \), one has

\[ \tilde{h}_z(b) = b + \sum_{i=1}^\infty (h_i \circ \mathcal{L}_\theta)^i(b) \equiv b + h_z(d_B^\Delta b) \mod C^{(0,0,\geq 2)} \otimes \Gamma(B). \]  

Note that \( h_z(d_B^\Delta b) \in C^{(0,0,1)} \otimes \Gamma(B) \).

2) For any \( \theta \in \Gamma(A^Y) \), one has

\[ \tilde{h}_z(\theta) = \theta + \sum_{i=1}^\infty (h \circ (d_B^\Delta \theta) \equiv \theta + h(d_B^\Delta \theta) \mod (C^{(1,0,\geq 2)} \oplus C^{(0,1,\geq 2)}). \]  

Note that \( h(d_B^\Delta \theta) \in C^{(1,0,1)} \) and \( h(d_B \theta) \in C^{(0,1,1)} \).

3) For any \( \varpi \in C^{(p,q,r)} \otimes \Gamma(A^s B) \), one has

\[ \tilde{h}_z(\varpi) \equiv h_z(\varpi) \mod \bigoplus_{i+j=p+q-1} C^{(i,j,\geq r+2)} \otimes \Gamma(A^s B). \]  

Note that \( h_z(\varpi) \in C^{(p,q-1,r+1)} \otimes \Gamma(A^s B) \).

An immediate consequence is the following fact.

**Lemma 4.10.** For any \( \varpi_1, \varpi_2 \in C^{(***)} \otimes \Gamma(A^s B) \), one has

\[ \sigma_z[\tilde{h}_z(\varpi_1), \tilde{h}_z(\varpi_2)] = 0. \]  

The following identities are due to the definition of \( h \) (see [25, Section 3.2]). We omit the details of verification.

**Lemma 4.11.** For all \( b, b_1, b_2 \in \Gamma(B) \), \( \theta \in \Gamma(A^Y) \), we have

\[ t_b h_z(d_B^\Delta b_2) = \Delta_B^B b_2, \]
\[ t_b h_z(d_B^\Delta \theta) = \Delta_B \theta, \]
\[ t_b h_z d_B^\Delta b_2 h(d_B \theta) = \frac{1}{2} \beta(b_1, b_2, \theta). \]

Here \( \Delta \) is the \( B \)-“connection” on \( A \) introduced in Equation (15).

We are now able to show the following
Lemma 4.12. For \( b_1, b_2 \in \Gamma(B) \), \( \theta_1, \theta_2 \in \Gamma(A^\vee) \) and \( f \in C^\infty(M) \), one has

\[
\begin{align*}
\sigma_2[\tilde{x}_2(b_1), \tilde{x}_2(b_2)] &= [b_1, b_2]_B, \\
\sigma_2[\tilde{x}_2(b_1), \tilde{x}_2(\theta_1)] &= \Delta_{b_1}\theta_1, \\
\sigma_2[\tilde{x}_2(b_1), f] &= \rho_B(b_1)f, \\
\sigma_2[\tilde{x}_2(\theta_1), \tilde{x}_2(\theta_2)] &= 0.
\end{align*}
\]  

(24)

Proof. By Equation (20), we have

\[
\begin{align*}
[\tilde{x}_2(b_1), \tilde{x}_2(b_2)] &\equiv \nu_{b_1} h_2(d_B^2 b_2) - \nu_{b_2} h_2(d_B^2 b_1) \mod C^{(0,0,\geq 1)} \otimes \Gamma(B) \\
&\equiv \Delta_{b_1} b_2 - \Delta_{b_2} b_1 \mod C^{(0,0,\geq 1)} \otimes \Gamma(B) \\
&\equiv [b_1, b_2]_B \mod C^{(0,0,\geq 1)} \otimes \Gamma(B).
\end{align*}
\]

(25)

The last step is due to \( \nabla \) being torsion free (Equation (19)). Applying \( \sigma_2 \), the first identity in Equation (24) is immediate.

Similarly, by Equations (20) and (21),

\[
\begin{align*}
[\tilde{x}_2(b), \tilde{x}_2(\theta)] &\equiv \nu_{b_1} h_2(d_B^2 \theta) + \nu_{b_2} h_2(d_B \theta) \mod (C^{(1,0,\geq 1)} \oplus C^{(0,1,\geq 1)}) \\
&\equiv \Delta_B \theta + \nu_{b_1} h_2(d_B \theta) \mod (C^{(1,0,\geq 1)} \oplus C^{(0,1,\geq 1)}).
\end{align*}
\]

(26)

Notice that \( \nu_{b_1} h_2(d_B \theta) \in C^{(0,1,0)} \). Then applying \( \sigma_2 \), the second identity in Equation (24) is immediate. The third identity easily follows from Equation (20):

\[
[\tilde{x}_2(b_1), f] \equiv \rho_B(b_1)f \mod C^{(0,0,\geq 1)}.
\]

The last identity is obvious. \( \square \)

Lemma 4.13. For \( b, b_1, b_2, b_3 \in \Gamma(B) \) and \( \theta, \theta_1, \theta_2, \theta_3 \in \Gamma(A^\vee) \), one has

\[
\begin{align*}
\sigma_2[\tilde{x}_2(b_1), \tilde{x}_2(b_2), \tilde{x}_2(\theta)] &= \frac{1}{2} \langle \beta(b_1, b_2), \theta \rangle, \\
\sigma_2[\tilde{x}_2(\theta), \tilde{x}_2(b_1), \tilde{x}_2(b_2)] &= 0, \\
\sigma_2[\tilde{x}_2(b_1), \tilde{x}_2(b_2), \tilde{x}_2(b_3)] &= 0, \\
\sigma_2[\tilde{x}_2(b), \tilde{x}_2(b_1), \tilde{x}_2(b_2), \tilde{x}_2(b_3)] &= 0, \\
\sigma_2[\tilde{x}_2(\theta_1), \tilde{x}_2(b_1), \tilde{x}_2(\theta_2), \tilde{x}_2(b_3)] &= 0, \\
\sigma_2[\tilde{x}_2(\theta_1), \tilde{x}_2(\theta_2), \tilde{x}_2(\theta_3)] &= 0.
\end{align*}
\]  

(27)

Proof. We show the first identity in (27). By Equations (26) and (22), we have

\[
\begin{align*}
\tilde{h}_2[\tilde{x}_2(b_2), \tilde{x}_2(\theta)] &\equiv \nu_{b_1} h_2(d_B \theta) \mod C^{(0,0,\geq 2)},
\end{align*}
\]

where \( h_2 \nu_{b_2} h_2(d_B \theta) \in C^{(0,0,1)} \).

Therefore, using Equation (20), we get

\[
\begin{align*}
[\tilde{x}_2(b_1), \tilde{x}_2(b_2), \tilde{x}_2(\theta)] &\equiv \nu_{b_1} h_2(d_B \theta) \mod C^{(0,0,\geq 1)} \\
&\equiv \frac{1}{2} \langle \beta(b_1, b_2), \theta \rangle \mod C^{(0,0,\geq 1)}.
\end{align*}
\]

Applying \( \sigma_2 \), one gets the first identity. The rest identities can be worked out similarly. \( \square \)

The following lemma is proved along the same lines.
Lemma 4.14. The following equation holds true for any \( x, y, z \in \Gamma(B) \) or \( \Gamma(A^\vee) \):

\[
\vspace{1em}
\hat{h}_x \left[ \hat{\tau}_2(x), \hat{h}_y [\hat{\tau}_2(y), \hat{\tau}_2(z)] \right] = 0.
\] (28)

With these preparatory work, we finally give the proof of our main result — Proposition 4.8.

Proof of Proposition 4.8. Via the contraction data \((\hat{h}_x, \sigma, \hat{\tau}_2)\) in Equation (18), one constructs the \(L_\infty\) brackets on \(\Omega^*_B(\Lambda^* B)\) (Theorem 2.18). The first one is already shown to be \(d_{\text{Bott}}^A\).

According to the proof of Theorem 2.18, the binary bracket reads

\[
\vspace{1em}
l_2(X, Y) = \sigma_2[\hat{\tau}_2(X), \hat{\tau}_2(Y)], \quad \forall X, Y \in \Omega^*_B(\Lambda^* B).
\]

Compare the identities in Equation (24) with those in (2) of Proposition 4.3, we see the generating relations of the binary bracket are exact the same.

The trinary bracket reads

\[
\vspace{1em}
l_3(X, Y, Z) = \sigma_2[\hat{\tau}_2(X), \hat{h}_y [\hat{\tau}_2(Y), \hat{\tau}_2(Z)]] + c.p.
\]

Let us examine the trinary bracket on generating elements. For \( b_1, b_2 \in \Gamma(B) \) and \( \theta \in \Gamma(A^\vee) \), we have

\[
\vspace{1em}
l_3(b_1, b_2, \theta) = \sigma_2 \left( \left[ \hat{\tau}_2(b_1), \hat{h}_y [\hat{\tau}_2(b_2), \hat{\tau}_2(\theta)] \right] \right.
\]

\[
\vspace{1em}
- \left. \left[ \hat{\tau}_2(b_2), \hat{h}_2 [\hat{\tau}_2(b_1), \hat{\tau}_2(\theta)] \right] + \left[ \hat{\tau}_2(\theta), \hat{h}_2 [\hat{\tau}_2(b_1), \hat{\tau}_2(b_2)] \right] \right)
\]

\[
\vspace{1em}
= \frac{1}{2} \langle \beta(b_1, b_2), \theta \rangle - \frac{1}{2} \langle \beta(b_2, b_1), \theta \rangle \quad (\text{by Equation (27)})
\]

\[
\vspace{1em}
= \langle \beta(b_1, b_2), \theta \rangle.
\]

For the same reasons, one sees that \( l_3 \) vanishes if restricted to \( \Gamma(B) \times \Gamma(B) \times \Gamma(B), \Gamma(B) \times \Gamma(A^\vee) \times \Gamma(A^\vee) \) and \( \Gamma(A^\vee) \times \Gamma(A^\vee) \times \Gamma(A^\vee) \). So, the generating relations of the trinary bracket are exact the same as those in (3) of Proposition 4.3.

The 4-bracket reads

\[
\vspace{1em}
l_4(X, Y, Z, W) = \lambda \sigma_2 \left[ \hat{h}_x [\hat{\tau}_2(X), \hat{\tau}_2(Y)], \hat{h}_y [\hat{\tau}_2(Z), \hat{\tau}_2(W)] \right] + c.p.
\]

\[
\vspace{1em}
+ \mu \sigma_2 \left[ \hat{\tau}_2(X), \hat{h}_2 [\hat{\tau}_2(Y), \hat{\tau}_2(Z)], \hat{\tau}_2(W) \right] + c.p.
\]

where \( \lambda, \mu \) are two constants. By Equations (23) and (28), we see that \( l_4 \) vanishes if restricted to generating elements in \( \Gamma(B) \) or \( \Gamma(A^\vee) \). Hence \( l_4 \) is trivial. It can be similarly verified that all higher brackets \( l_j \) \((j \geq 5)\) are trivial.

This completes the proof. \( \square \)

4.3. Third construction: Dirac deformation and proof of main theorems. Deformation of Dirac structures has been studied at least 15 years ago by Severa and Roytenberg [27, 23]. It is well known that the deformation is controlled by an \(L_\infty\) algebra, which is canonical up to \(L_\infty\) isomorphisms. In fact, it is a \((-1)\)-shifted derived Poisson algebra. Let us recall the construction below.

Let \( E \) be a Courant algebroid of signature \((n, n)\) over a smooth manifold \( M \), and \( D \subset E \) a Dirac structure. Choose a transversal almost Dirac (i.e. maximal isotropic) subbundle \( C \subset E \) such that \( E \cong D \oplus C \). Identify \( C \) with \( D^\vee \). Then we have \( E \cong D \oplus D^\vee \).
The Courant bracket $[-, -]_E$ on $\Gamma(E)$ and the anchor map $\rho_E : E \to TM$ induce, by restrictions, a skew-symmetric bracket $[-, -]_{D^\vee}$ on $\Gamma(D^\vee)$ and an anchor map $\rho_\vee : D^\vee \to TM$,

$$[\xi, \eta]_{D^\vee} = \text{pr}_{D^\vee} [\xi, \eta]_E, \quad \forall \xi, \eta \in \Gamma(D^\vee),$$

$$\rho_\vee = \rho_E|_{D^\vee}.$$ 

Let $\phi \in \Gamma(\Lambda^3 D)$ be the section defined by

$$\phi(\xi, \eta, \zeta) = 2 \langle [\xi, \eta]_E, \zeta \rangle_E = 2 \langle \text{pr}_D [\xi, \eta]_E, \zeta \rangle_E, \quad \forall \xi, \eta, \zeta \in \Gamma(D^\vee).$$

Here $\langle -, - \rangle_E$ is the symmetric metric on $E$. It can be easily verified that $\phi$ is indeed skew-symmetric.

Unless $C \cong D^\vee$ is again a Dirac structure, in general $\phi$ is non-zero and $(D^\vee, [-, -]_{D^\vee}, \rho_\vee)$ is not a Lie algebroid. Instead, $(D, D^\vee)$ forms a quasi-Lie bialgebroid [24].

Let

$$\lambda_1 = d_D : \Gamma(\Lambda^i D^\vee) \to \Gamma(\Lambda^{i+1} D^\vee)$$

be the Chevalley–Eilenberg differential of the Lie algebroid $D$.

Define a binary bracket

$$\lambda_2 : \Gamma(\Lambda^i D^\vee) \otimes \Gamma(\Lambda^l D^\vee) \to \Gamma(\Lambda^{i+l-1} D^\vee)$$

by extending, by Leibniz rule (see Equation (2), for $l = 2, k = -1$), the relation

$$\lambda_2(\xi, \eta, f) = \rho_\vee(\xi)(f), \quad \forall \xi, \eta \in \Gamma(D^\vee), \ f \in C^\infty(M).$$

Similarly, let

$$\lambda_3 : \Gamma(\Lambda^i D^\vee) \otimes \Gamma(\Lambda^l D^\vee) \otimes \Gamma(\Lambda^r D^\vee) \to \Gamma(\Lambda^{i+l+r-3} D^\vee)$$

be the trinary bracket extending $\phi$, by Leibniz rule, in each argument ($\lambda_3$ vanishes if one of the argument is a function on $M$).

The following result is due to Severa [27] and Roytenberg [23]. Relevant results appear in [10] and [5].

**Proposition 4.15.** Let $E$ be a Courant algebroid of signature $(n, n)$ over a smooth manifold $M$, and $D \subset E$ a Dirac structure. Choose a transversal almost Dirac structure $C$. Then

- $\Gamma(\Lambda^i D^\vee)$, together with $\lambda_1, \lambda_2, \lambda_3$ defined above and $\lambda_l = 0, l > 3$, and the wedge product, is a $(-1)$-shifted derived Poisson algebra;
- the underlying $L_\infty$ algebra structure on $\Gamma(\Lambda^i D^\vee)[1]$ controls deformations of the Dirac structures $D \subset E$ in the following sense: the graph $\{X + \omega^b(X)| X \in D\} \subset E$ of an element $\omega \in \Gamma(\Lambda^2 D^\vee)[1]$ is a Dirac structure if and only if $\omega$ satisfies the Maurer–Cartan equation:

$$\lambda_1(\omega) + \frac{1}{2} \lambda_2(\omega, \omega) + \frac{1}{6} \lambda_3(\omega, \omega, \omega) = 0.$$ 

In fact, the $(-1)$-shifted derived Poisson algebra structure on $\Gamma(\Lambda^i D^\vee)$ is canonical, up to an isomorphism of $(-1)$-shifted derived Poisson algebras. Note that there is a one-one correspondence between almost Dirac structures transversal to $D$ and elements in $\Gamma(\Lambda^2 D)$. Their relation is established as follows:

$$\pi \in \Gamma(\Lambda^2 D) \iff C_\pi = \{\pi^a(\xi) + \xi| \xi \in C \cong D^\vee\}.$$
Proposition 4.16 ([27]). Under the same hypothesis as in Proposition 4.15, assume that $C_\pi$ is another almost Dirac structure transversal to $D$, which corresponds to an element $\pi \in \Gamma(\Lambda^2 D)$. Then there is a canonical isomorphism between the induced $(-1)$-shifted derived Poisson algebra structures on $\Gamma(\Lambda^\bullet D^\vee)$, which is given by $\exp \delta_\pi$, where $\delta_\pi$ is a coderivation on $\mathcal{S}(\Gamma(\Lambda^\bullet D^\vee))[2]$ generated by
\[
S^2(\Gamma(\Lambda^\bullet D^\vee)) \to \Gamma(\Lambda^\bullet D^\vee)
\]
\[\xi \otimes \eta \mapsto (\iota_\pi \xi) \wedge \eta + \xi \wedge \iota_\pi \eta - \iota_\pi (\xi \wedge \eta).\] (29)

Note that the bilinear map in Equation (29) is a biderivation of the associative algebra $\Gamma(\Lambda^\bullet D^\vee)$ with respect to the wedge product. Therefore $\exp \delta_\pi$ is indeed compatible with respect to the associative algebra structure, according to Proposition ??.

Now consider a Lie pair $(L, A)$. Let $E = L \oplus L^\vee$ be the standard Courant algebroid [14], where the Courant bracket and the anchor are defined, respectively, by
\[
[X + \alpha, Y + \beta]_E = [X, Y]_L + (\mathcal{L}_X \beta - \iota_Y d_L \alpha),
\]
\[\rho_E(X + \alpha) = \rho_L(X),\]
where $X + \alpha, Y + \beta \in \Gamma(L \oplus L^\vee)$. The symmetric pairing is also standard:
\[
\langle X + \alpha, Y + \beta \rangle_E = \frac{1}{2}(\langle X, \beta \rangle + \langle Y, \alpha \rangle).
\]

It is a standard result that $D = A \oplus A^\perp$ is a Dirac structure of $E$. Choose a splitting of the exact sequence (13) so that $L \cong A \oplus B$. Then $A^\perp \cong B^\vee$ and $B \oplus A^\vee$ is a transversal almost Dirac structure of $D$ in $E$. That is
\[
E \cong (A \oplus B^\vee) \oplus (B \oplus A^\vee) \cong D \oplus D^\vee
\]
with $D \cong A \oplus B^\vee$ and $D^\vee \cong B \oplus A^\vee$. Thus $(D, D^\vee)$ is a quasi-Lie bialgebroid, and the induced structure maps on $D^\vee$ are, respectively, given by the following relations:

1) The bracket $[-,-]_{D^\vee}$ reads
\[
[u + \theta, v + \omega]_{D^\vee} = \text{pr}_{D^\vee}([u, v]_L + (\mathcal{L}_u \omega - \iota_v d_L \theta))
\]
\[= \text{pr}_B[u, v]_L + \text{pr}_{A^\vee}(\mathcal{L}_u \omega - \mathcal{L}_v \theta),\]
where $u + \theta, v + \omega \in \Gamma(D^\vee) = \Gamma(B \oplus A^\vee)$.

2) The anchor $\rho_{D^\vee}$ is simply
\[\rho_{D^\vee}(u + \theta) = \rho_L(u).\]

3) The 3-form $\phi$ on $D^\vee$ is:
\[
\phi(u_1 + \theta_1, u_2 + \theta_2, u_3 + \theta_3) = 2 \langle \text{pr}_B[u_1 + \theta_1, u_2 + \theta_2]_E, u_3 + \theta_3 \rangle_E
\]
\[= 2 \langle \text{pr}_A[u_1, u_2]_L + \text{pr}_{B^\vee}(\mathcal{L}_{u_1} \theta_2 - \mathcal{L}_{u_2} \theta_1), u_3 + \theta_3 \rangle_E
\]
\[= \langle \text{pr}_A[u_1, u_2]_L, \theta_3 \rangle + \langle \mathcal{L}_{u_1} \theta_2, u_3 \rangle - \langle \mathcal{L}_{u_2} \theta_1, u_3 \rangle
\]
\[= \langle \text{pr}_A[u_1, u_2]_L, \theta_3 \rangle + \langle \text{pr}_A[u_3, u_1]_L, \theta_2 \rangle + \langle \text{pr}_A[u_2, u_3]_L, \theta_1 \rangle,
\]
for all $u_i \in B, \theta_i \in A^\vee, i = 1, 2, 3$.

It thus follows that the generating relations of the $(-1)$-shifted derived Poisson algebra structure on $\Gamma(\Lambda^\bullet D^\vee) \cong \Gamma(\Lambda^\bullet (A^\vee \oplus B)) \cong \Omega_A^*(\Lambda^\bullet B)$, are exactly those of Proposition 4.3.

In summary, we have proved the following
Proposition 4.17. Let $(L, A)$ be a Lie pair, and $E = L \oplus L^\perp$ the standard Courant algebroid associated to the Lie algebroid $L$ [14]. Consider the Dirac structure $D = A \oplus A^\perp$. Then

1. A splitting $j$ of the exact sequence (13) determines an almost Dirac structure $C_j$ transversal to $D$;
2. The $(-1)$-shifted derived Poisson algebra corresponding to $C_j$ as in Proposition 4.15 coincides with the one as in Proposition 4.3.

Proof of Theorem 1.1 and Theorem 1.2. According to Proposition 4.16 and Proposition 4.17, the $(-1)$-shifted derived Poisson algebra $\Omega^*_A(\Lambda^* B)$ as in Proposition 4.15, Proposition 4.3 and Proposition 4.8 all coincide, and is canonical up to derived Poisson algebra isomorphisms with the linear map being the identity. As a consequence, the induced $(-1)$-shifted Poisson algebra, or a Gerstenhaber algebra, on the cohomology level of $(\Omega^*_A(\Lambda^* B), d_A^{\text{Bott}})$, is indeed canonical. □

4.4. Examples.

4.4.1. Matched pairs of Lie algebroids. Let $L$ be a Lie algebroid, $A$ and $B$ two Lie subalgebroids of $L$ such that $L \cong A \oplus B$ as vector bundles. Then $L/A \cong B$ is naturally an $A$-module, while $L/B \cong A$ is naturally a $B$-module. Then $(A, B)$ is said to form a matched pair. Alternatively, one can define a matched pair as follows.

Definition 4.18 ([15, 18, 16]). Lie algebroids $A$ and $B$ over the same base manifold $M$ are said to form a matched pair if there exists an action $\nabla$ of $A$ on $B$ and an action $\Delta$ of $B$ on $A$, such that the identities

$$[ho_A(X), \rho_B(Y)] = -\rho_A(\Delta_Y X) + \rho_B(\nabla_X Y),$$

$$\nabla_X [Y_1, Y_2] = [\nabla_X Y_1, Y_2] + [Y_1, \nabla_X Y_2] + \nabla_{\Delta_Y X_2} Y_1 - \nabla_{\Delta_Y X_1} Y_2,$$

$$\Delta_Y [X_1, X_2] = [\Delta_Y X_1, X_2] + [X_1, \Delta_Y X_2] + \nabla_{X_2} Y_1 - \nabla_{X_1} Y_2$$

hold for all $X_1, X_2, X \in \Gamma(A)$ and $Y_1, Y_2, Y \in \Gamma(B)$. Here $\rho_A$ and $\rho_B$ denote the anchor maps of $A$ and $B$ respectively.

Given a matched pair $(A, B)$ of Lie algebroids, there is a Lie algebroid structure on the direct sum vector bundle $L = A \oplus B$, with anchor

$$X \oplus Y \mapsto \rho_A(X) + \rho_B(Y)$$

and the Lie bracket

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = ([X_1, X_2] + \Delta_{Y_1} X_2 - \Delta_{Y_2} X_1) \oplus ([Y_1, Y_2] + \nabla_{X_2} Y_1 - \nabla_{X_1} Y_2).$$

Clearly, the pair $(L, A)$ is a Lie pair, and the Bott $A$-connection on $L/A \cong B$ coincides with $\nabla$. Applying Proposition 4.3, we see that $h$ vanishes, and $\Omega^*_A(\Lambda^* B)$ is in fact a DGLA.

Theorem 4.19. For any given matched pair $(A, B)$ of Lie algebroids, $\Omega^*_A(\Lambda^* B)$ admits a canonical differential Gerstenhaber algebra structure\(^2\), where the multiplication is the wedge product, and the differential is the Chevalley–Eilenberg differential

$$d^\nabla : \Omega^*_A(\Lambda^* B) \to \Omega^*_{A^\perp}(\Lambda^* B).$$

\(^2\)This means that the differential $d^\nabla$ is compatible with the Gerstenhaber bracket:

$$d^\nabla [X,Y] = \{d^\nabla X, Y\} + (-1)^{|X|+1} \{X, d^\nabla Y\}, \quad \forall X, Y \in \Omega^*_A(\Lambda^* B).$$
Here the A-module structure on $\Lambda^\bullet B$ is the natural extension of the $A$-action on $B$.

**Theorem 4.20.** For any given matched pair $(A, B)$ of Lie algebroids, the Chevalley–Eilenberg total cohomology $\mathbb{H}(\Omega^\bullet_A(\Lambda^\bullet B), d_X)$ admits a canonical Gerstenhaber algebra structure.

As an application, consider a complex manifold $X$. Set $A = T_{X}^{0,1}$ and $B = T_{X}^{1,0}$. Then $(A, B)$ is a matched pair of (complex) Lie algebroids, and its direct sum $A \bowtie B$ is isomorphic, as a Lie algebroid, to $T_{X} \otimes \mathbb{C}$. It is simple to see that $\Omega^\bullet_A(\Lambda^\bullet B) = \Omega^\bullet_X(T^\bullet_{X}^{0,0})$, and the differential $d^\Lambda_X$ is the standard $\bar{\partial}$ operator, and the Lie bracket $[-, -]$ is extension of the Schouten bracket and Lie derivative. Therefore, $(\Omega^\bullet_X(T^\bullet_{X}^{0,0}), [-, -], \bar{\partial})$ is a DGLA, and $(\Omega^\bullet_X(T^\bullet_{X}^{0,0}), \wedge, [-, -], \bar{\partial})$ is a differential Gerstenhaber algebra. The corresponding cohomology, which is isomorphic to the sheaf cohomology $\mathbb{H}(X, \Lambda^\bullet \Theta)$ of holomorphic polyvector fields. The Gerstenhaber algebra structure in Theorem 1.2 becomes the standard Gerstenhaber algebra structure on $\mathbb{H}(X, \Lambda^\bullet \Theta)$.

Another example of matched pairs comes from $g$-manifolds. Let $g$ be a Lie algebra, $M$ a $g$-manifold with infinitesimal action given by a Lie algebra morphism $\phi : g \to T(M)$. Let $A = M \times g$ be the associated action Lie algebroid and $B = TM$. Then $A$ and $B$ form a Lie algebroid matched pair, with mutual actions

$$\nabla_a Y = [\phi(a), Y], \quad \Delta_Y a = 0,$$

where $a \in g$ is considered as a constant section in $A$, and $Y \in T(M)$. It is clear that $\Omega^\bullet_A(\Lambda^\bullet B) \cong \Lambda^\bullet g^\vee \otimes T^\bullet_{\text{poly}}(M)$, and $d^\Lambda_X$ is the standard Chevalley–Eilenberg differential, and the Lie bracket $[-, -]$ is extension of the Schouten bracket. Hence $\Lambda^\bullet g^\vee \otimes T^\bullet_{\text{poly}}(M)$ is a differential Gerstenhaber algebra and its cohomology $\mathbb{H}_{\text{CE}}(g, T^\bullet_{\text{poly}}(M))$ is the Gerstenhaber algebra in Theorem 1.2. More details can be found in [13, Lemma 3.1].

**4.4.2. Semisimple Lie algebras.** Let $g$ be a complex semisimple Lie algebra, and $h$ a Cartan subalgebra in $g$. One has the root decomposition of $g$:

$$g = h \bigoplus_{\alpha \in \Delta} g_{\alpha}. \quad (30)$$

Here $\Delta \subset h^\vee$ is the root system of $g$, and the weight space of $\alpha \in h^\vee$,

$$g_{\alpha} = \{ u \in g \mid [h, u] = \alpha(h)u, \forall h \in h \}$$

is 1-dimensional. It is standard in Lie theory that one can find $h_{\alpha} \in h$ and base vectors $x_{\alpha} \in g_{\alpha}$, such that

$$[h_{\alpha}, x_{\alpha}] = 2x_{\alpha}, \quad [h_{\alpha}, x_{-\alpha}] = -2x_{-\alpha}, \quad [x_{\alpha}, x_{-\alpha}] = h_{\alpha}.$$  

Applying Theorems 1.1 and 1.2, we have the following

**Theorem 4.21.** Let $g$ be a complex semisimple Lie algebra and $g = h \bigoplus_{\alpha \in \Delta} g_{\alpha}$ the root decomposition. Then $\Lambda^\bullet h^\vee \otimes \Lambda^\bullet(\bigoplus_{\alpha \in \Delta} g_{\alpha})$ admits a canonical $(-1)$-shifted derived Poisson algebra structure, where the multiplication is the wedge product and the first $L_\infty$ bracket is the Chevalley–Eilenberg differential

$$d_h : \Lambda^\bullet h^\vee \otimes \Lambda^\bullet(\bigoplus_{\alpha \in \Delta} g_{\alpha}) \to \Lambda^{\bullet+1} h^\vee \otimes \Lambda^\bullet(\bigoplus_{\alpha \in \Delta} g_{\alpha}).$$

Here the $h$-module structure on $\Lambda^\bullet(\bigoplus_{\alpha \in \Delta} g_{\alpha})$ is the natural extension of the $h$-action on $\bigoplus_{\alpha \in \Delta} g_{\alpha}$. Specifically, $d_h$ is generated by

$$d_h(x_{\alpha}) = \alpha \otimes x_{\alpha}.$$
The binary bracket \( \{-, -\} \) of \( \Lambda^* \mathfrak{h}^\vee \otimes \Lambda^*( \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha) \) is generated by
\[
\{x_\alpha, x_\beta\} = x_{\alpha + \beta}, \quad \forall \alpha, \beta \in \Delta, \alpha + \beta \in \Delta.
\]
The trinary bracket \( r(-, -, -) \) is generated by
\[
r(x_\alpha, x_{-\alpha}, \xi) = \langle h_\alpha, \xi \rangle, \quad \forall \xi \in \mathfrak{h}^\vee.
\]
All other situations of generating relations are trivial.

**Theorem 4.22.** Let \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \) be a complex semisimple Lie algebra, the Chevalley–Eilenberg hypercohomology \( H(\Lambda^* \mathfrak{h}^\vee \otimes \Lambda^*( \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha), d_\mathfrak{h}) \) admits a canonical Gerstenhaber algebra structure.

4.4.3. **Transitive Lie algebroids.** Let \((L, [-,-], \rho_L)\) be a transitive Lie algebroid over \( M \), i.e. \( \rho_L : L \to TM \) is surjective. Let \( \mathfrak{K} = \ker \rho_L \) be the adjoint bundle of \( L \), which is a Lie algebra bundle over \( M \). Then \((L, \mathfrak{K})\) is a Lie pair. The quotient \( L/\mathfrak{K} \cong TM \) has the trivial \( \mathfrak{K}\)-module structure.

A connection in \( L \) is a bundle map \( \gamma : TM \to L \) such that \( \rho_L \circ \gamma = \text{id}_{TM} \). Such a connection always exists. The curvature of \( \gamma \) is the bundle map \( R^\gamma : TM \wedge TM \to \mathfrak{K} \) defined by
\[
R(X, Y) = [\gamma(X), \gamma(Y)]_L - \gamma[X, Y], \quad \forall X, Y \in \mathcal{T}(M).
\]

There also induces a \( TM \)-connection on \( \mathfrak{K} \) defined by
\[
\nabla_X u = [\gamma(X), u]_L, \quad \forall X \in \mathcal{T}(M), u \in \Gamma(\mathfrak{K}).
\]

By choosing such a connection, \( L \) can be identified with \( \mathfrak{K} \oplus TM \) as vector bundles over \( M \), while the Lie algebroid structure on \( L \) can be described as follows: the anchor is the projection to the second component, and the Lie bracket is expressed as
\[
[u, v]_L = (R^\gamma(X, Y) + [u, v]|_{\mathfrak{K}} + \nabla_X v - \nabla_Y u, [X, Y]), \quad \forall u, v \in \Gamma(\mathfrak{K}), X, Y \in \mathcal{T}(M).
\]

Now for the Lie pair \((L, \mathfrak{K})\), by Proposition 4.3, Theorems 1.1 and 1.2, we have

**Theorem 4.23.** Let \( L \) be a transitive Lie algebroid over \( M \) with anchor \( \rho_L \), and \( \mathfrak{K} = \ker \rho_L \). Then, up to \((-1\rangle\)-shifted derived Poisson algebra isomorphisms whose first Taylor coefficient is the identity map, \( \Gamma(\Lambda^* \mathfrak{K}^\vee) \otimes \mathcal{T}_{\text{poly}}(M) \) admits a unique \((-1\rangle\)-shifted derived Poisson algebra structure, where the multiplication is the wedge product, and the first \( L_\infty \) bracket
\[
d : \Gamma(\Lambda^* \mathfrak{K}^\vee) \otimes \mathcal{T}_{\text{poly}}(M) \to \Gamma(\Lambda^* \mathfrak{K}^\vee) \otimes \mathcal{T}_{\text{poly}}(M)
\]
is extended from the Chevalley–Eilenberg differential of \( \mathfrak{K} \):
\[
d(\omega \otimes W) = d_{\mathfrak{K}}\omega \otimes W, \quad \forall \omega \in \Gamma(\Lambda^* \mathfrak{K}^\vee), W \in \mathcal{T}_{\text{poly}}(M).
\]

By choosing a Lie algebroid connection \( \gamma : TM \to L \), the binary bracket of \( \Gamma(\Lambda^* \mathfrak{K}^\vee) \otimes \mathcal{T}_{\text{poly}}(M) \) is generated by the following relations
\[
\{X, Y\} = [X, Y],
\{X, \omega\} = \nabla_X \omega,
\{\omega, \eta\} = 0,
\]
where \( X, Y \in \mathcal{T}(M), \omega, \eta \in \Gamma(\mathfrak{K}^\vee) \).

The trinary bracket is generated by the only nontrivial relation
\[
r(X, Y, \omega) = \langle R^\gamma(X, Y), \omega \rangle, \quad \forall X, Y \in \mathcal{T}(M), \omega \in \Gamma(\mathfrak{K}^\vee).
\]
**Theorem 4.24.** Under the same hypothesis as in Theorem 4.23, the space $\mathbb{H}_{CE}(\mathfrak{r}) \otimes T_{poly}^\bullet(M)$ admits a canonical Gerstenhaber algebra structure.

4.4.4. **Foliations.** Consider a regular foliation $F$ on a smooth manifold $M$. Then $(TM, TF)$ is a Lie pair, where $TF$ is the distribution that integrates to $F$. Let $N_F = TM/TF$ be the associated normal bundle. Applying Theorems 1.1 and 1.2, we have

**Theorem 4.25.** Let $F$ be a regular foliation on $M$. Then, up to $(−1)$-shifted derived Poisson algebra isomorphisms whose first Taylor coefficient is the identity map, $\Gamma(\Lambda^\bullet T_F^\vee \otimes \Lambda^\bullet N_F)$ admits a unique $(−1)$-shifted derived Poisson algebra structure, where the multiplication is the wedge product and the first $L_\infty$ bracket is the Chevalley–Eilenberg differential with respect to the Bott connection of $F$ on $N_F$:

$$d_F^{\text{Bott}} : \Gamma(\Lambda^\bullet T_F^\vee \otimes \Lambda^\bullet N_F) \to \Gamma(\Lambda^{\bullet+1} T_F^\vee \otimes \Lambda^\bullet N_F).$$

**Theorem 4.26.** Under the same hypothesis as in Theorem 4.25, the Chevalley–Eilenberg hypercohomology $\mathbb{H}_{CE}(F, \Lambda^\bullet N_F)$ admits a canonical Gerstenhaber algebra structure.

Note also that a direct corollary of our Proposition 4.1 is

**Corollary 4.27.** By choosing a complementary distribution to $F$, the direct sum of vector bundles $T_F[1] \oplus N_F$ has an $L_\infty$ algebroid structure over $T_F[1]$.

In terms of pure algebraic language, Corollary 4.27 amounts to saying that the pair

$$(\Gamma(\Lambda^\bullet T_F^\vee), \Gamma(\Lambda^\bullet T_F^\vee \otimes N_F))$$

admits an $LR_{\infty}$ algebra structure [29].

We recall some notions and facts commonly used throughout the paper. We mainly follow the conventions of Bruce [3], Voronov [30], and Lada–Markl [11].

**Appendix A.** $L_\infty$ algebroids and DG-manifolds

Recall the notation of $L_\infty$ algebroids in Definition 3.6.

**Lemma A.1.** Let $\mathcal{L} \to \mathcal{M}$ be a vector bundle object in the category of $\mathbb{Z}$-graded manifolds and let $r$ be an integer number. An $L_\infty$ algebroid structure on $\mathcal{L} \to \mathcal{M}$ is equivalent to a structure of $L_\infty$ algebra on the $\mathbb{L}$-vector space $\Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r]$ with structure maps

$$\bar{\lambda}_l : \Lambda^l(\Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r]) \to (\Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r])[2-l]$$

satisfying the Leibniz rule

$$\bar{\lambda}_l(w_1, w_2, \cdots, w_{l-1}, fw_l) = \bar{\lambda}_l(w_1, w_2, \cdots, w_{l-1}, f)w_l$$

$$+ (-1)^{(l+|w_1|+\cdots+|w_{l-1}|)|f|}f\bar{\lambda}_l(w_1, w_2, \cdots, w_{l-1}, w_l),$$

for all $w_1, \cdots, w_l \in \Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r]$ and $f \in C^\infty(\mathcal{M})$, and such that

(1) $\Gamma(\mathcal{L})$ is an $L_\infty$ subalgebra of $\Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r]$;

(2) $C^\infty(\mathcal{M})[r]$ is an $L_\infty$ ideal of $\Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r]$, i.e. $\bar{\lambda}_l(w_1, w_2, \cdots, w_l) \in C^\infty(\mathcal{M})[r]$ if at least one of the arguments $w_1, w_2, \ldots, w_l$ is in $C^\infty(\mathcal{M})[r]$;

(3) and $C^\infty(\mathcal{M})[r]$ is abelian, i.e. $\bar{\lambda}_l(w_1, w_2, \cdots, w_l) = 0$ if at least two of the arguments $w_1, w_2, \ldots, w_l$ are in $C^\infty(\mathcal{M})[r]$. 
Sketch of proof. Suppose \( \mathcal{L} \) is endowed with an \( L_\infty \) algebroid structure with structure maps \((\lambda_l)_{l \geq 1}\) and anchor maps \((\lambda_l)_{l \geq 0}\) as in Definition 3.6. Define a sequence \((\bar{\lambda}_l)_{l \geq 1}\) of \( \mathbb{k} \)-linear maps
\[
\bar{\lambda}_l : \Lambda^l(\Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r]) \to (\Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r])[2 - l]
\]
by the relations
\begin{itemize}
    \item \( \bar{\lambda}_l(x_1, \cdots, x_n) = 0 \) if at least two of the arguments \( x_1, \ldots, x_n \) are in \( C^\infty(\mathcal{M})[r] \);
    \item \( \bar{\lambda}_l(a_1, \cdots, a_l) = \lambda_l(a_1, \cdots, a_l) \) for all \( a_1, \cdots, a_l \in \Gamma(\mathcal{L}) \);
    \item and \( \bar{\lambda}_{l+1}(a_1, \cdots, a_l, f) = \rho_l(a_1, \cdots, a_l)f \) for all \( a_1, \cdots, a_l \in \Gamma(\mathcal{L}) \) and \( f \in C^\infty(\mathcal{M})[r] \).
\end{itemize}

It is straightforward to verify that the sequence of brackets \((\bar{\lambda}_l)_{l \geq 1}\) endow \( \Gamma(\mathcal{L}) \oplus C^\infty(\mathcal{M})[r] \) with an \( L_\infty \) algebra structure and that the Leibniz rule (31) is satisfied. \( \square \)

**Proposition A.2** ([3, 28]). Let \( \mathcal{L} \to \mathcal{M} \) be a vector bundle object in the category of \( \mathbb{Z} \)-graded manifolds. Then \( \mathcal{L} \) is an \( L_\infty \) algebroid if and only if \( \mathcal{L}[1] \) is a DG manifold whose homological vector field \( Q \) is tangent to the zero section \( \mathcal{M} \xrightarrow{0} \mathcal{L}[1] \).

Given a vector bundle object \( \mathcal{E} \xrightarrow{\pi} \mathcal{M} \) in the category of \( \mathbb{Z} \)-manifolds, consider the graded Lie algebra \( \mathcal{D} \leq 1(\mathcal{E}) \) of first-order differential operators on \( \mathcal{E} \). It can be identified to \( \mathcal{D}(\mathcal{E}) \oplus C^\infty(\mathcal{E}) \) in a canonical way: the sum \( X + f \) of a vector field \( X \in \mathcal{D}(\mathcal{E}) \) and a function \( f \in C^\infty(\mathcal{E}) \) corresponds to the first-order differential operator
\[
C^\infty(\mathcal{E}) \ni g \mapsto X(g) + fg \in C^\infty(\mathcal{E}).
\]
Since \( C^\infty(\mathcal{E}) \cong \Gamma(\hat{\mathcal{E}}^\vee) \), the contraction \( \iota_s \) with a section \( s \in \Gamma(\mathcal{E}) \) defines a derivation of \( C^\infty(\mathcal{E}) \), i.e. a vector field on \( \mathcal{E} \). The injection \( J : \Gamma(\mathcal{E}) \oplus C^\infty(\mathcal{M}) \hookrightarrow \mathcal{D}(\mathcal{E}) \oplus C^\infty(\mathcal{E}) \) defined by \( J(s + f) = \iota_s + \pi^*(f) \) embeds \( \Gamma(\mathcal{E}) \oplus C^\infty(\mathcal{M}) \) as an abelian Lie subalgebra of \( \mathcal{D} \leq 1(\mathcal{E}) \). We proceed to define a surjection \( P : \mathcal{D} \leq 1(\mathcal{E}) \twoheadrightarrow \Gamma(\mathcal{E}) \oplus C^\infty(\mathcal{M}) \) such that \( P \circ J = \text{id} \). Given a vector field \( X \in \mathcal{D}(\mathcal{E}) \), consider the composition
\[
\Gamma(\mathcal{E}^\vee) \hookrightarrow \Gamma(\hat{\mathcal{E}}^\vee) \cong C^\infty(\mathcal{E}) \xrightarrow{X} C^\infty(\mathcal{E}) \xrightarrow{0^*} C^\infty(\mathcal{M}),
\]
where \( 0^* \) denotes the pullback of functions through the zero section of the vector bundle \( \mathcal{E} \xrightarrow{\pi} \mathcal{M} \). There exists a unique \( X^\dagger \in \Gamma(\mathcal{E}) \) such that \( \langle \xi, X^\dagger \rangle = 0^*(X(\xi)) \), for all \( \xi \in \Gamma(\mathcal{E}^\vee) \). We define \( P : \mathcal{D} \leq 1(\mathcal{E}) \twoheadrightarrow \Gamma(\mathcal{E}) \oplus C^\infty(\mathcal{M}) \) by \( P(X + f) = X^\dagger + 0^*(f) \), for all \( X \in \mathcal{D}(\mathcal{E}) \) and \( f \in C^\infty(\mathcal{E}) \). We note that the projection operator \( J \circ P \) satisfies
\[
J \circ P([x, y]) = J \circ P([J \circ P(x), y] + [x, J \circ P(y)]) , \quad \forall x, y \in \mathcal{D} \leq 1(\mathcal{E}). \quad (32)
\]

**Lemma A.3.** Given \( Q \in \mathcal{T}(\mathcal{E}) \), the following assertions are equivalent.

1. The vector field \( Q \) is tangent to the zero section of \( \mathcal{E} \xrightarrow{\pi} \mathcal{M} \).
2. There exists a unique vector field \( \Xi \) on \( \mathcal{M} \) such that the diagram
\[
\begin{array}{ccc}
C^\infty(\mathcal{E}) & \xrightarrow{Q} & C^\infty(\mathcal{E}) \\
0^* \downarrow & & \downarrow 0^* \\
C^\infty(\mathcal{M}) & \xrightarrow{\Xi} & C^\infty(\mathcal{M})
\end{array}
\]
commutes.
3. The ideal \( \ker(0^*) \) of \( C^\infty(\mathcal{E}) \) is \( Q \)-stable.
4. \( Q \in \ker(J \circ P) \).
Proof of Proposition A.2. Consider the vector bundle object $E \xrightarrow{\pi} M$, with $E = L[1]$, in the category of $\mathbb{Z}$-graded manifold and suppose $Q$ is a homological vector field on $L[1]$ tangent to the zero section of $L[1] \xrightarrow{\pi} M$. According to Lemma A.3, we have $Q \in \ker(J \circ P)$.

Together, the graded Lie algebra $\mathcal{D}^\leq(L[1])$, its abelian Lie subalgebra $J(\Gamma(L[1]) \oplus C^\infty(M))$, the projection $J \circ P$ of $\mathcal{D}^\leq(L[1])$ onto $J(\Gamma(L[1]) \oplus C^\infty(M))$, and the vector field $Q \in \ker(J \circ P)$ constitute the Voronov data [30, Theorem 1 and Corollary 1] of an $L^\infty[1]$ algebra structure on $\Gamma(L[1]) \oplus C^\infty(M)$ encoded in the sequence of derived brackets $(\mu_l)_{l \geq 1}$ defined by

$$
\mu_l(z_1, z_2, \ldots, z_l) = P\left(\left[\left[\left[Q, z_1\right], z_2\right], \ldots, z_l\right]\right),
$$

for all $z_1, z_2, \ldots, z_l \in \Gamma(L[1]) \oplus C^\infty(M)$.

Applying the décalage isomorphism, we obtain a structure of $L^\infty$ algebra $(\check{\mu}_l)_{l \geq 1}$ on $\Gamma(L) \oplus C^\infty(M)[-1]$. The multibrackets $(\check{\mu}_l)_{l \geq 1}$ and $(\check{\lambda}_l)_{l \geq 1}$ are related as follows:

$$
\check{\lambda}_l(w_1, w_2, \ldots, w_l) = (-1)^* \mu_l(w_1, w_2, \ldots, w_l),
$$

(\text{where } * = (l-1)|w_1| + (l-2)|w_2| + \cdots + |w_{l-1}|)

for all homogeneous $w_1, w_2, \ldots, w_l \in \Gamma(L) \oplus C^\infty(M)[-1]$.

It is straightforward to verify that the $L^\infty$ algebra structure $(\check{\lambda}_l)_{l \geq 1}$ on $\Gamma(L) \oplus C^\infty(M)[-1]$ satisfies the four conditions listed in Lemma A.1. Therefore, $L \rightarrow M$ is an $L^\infty$ algebroid. Its anchor maps $\rho_l : \Lambda^l \Gamma(L) \rightarrow T(M)$ (with $l \geq 0$) and multibrackets $\lambda_l : \Lambda^l \Gamma(L) \rightarrow \Gamma(L)$ (with $l \geq 1$) are defined by the relations

$$
\rho_l(a_1, a_2, \ldots, a_l) f = (-1)^b 0^* \left( \left[\left[Q, t_{a_1}\right], t_{a_2}\right], \ldots, t_{a_l}\right) (f \circ \pi)
$$

(\text{where } b = \sum |a_1| + (l-1)|a_2| + \cdots + |a_l|)

$$
\langle \lambda_l(a_1, a_2, \ldots, a_l), \xi \rangle = (-1)^b 0^* \left( \left[\left[Q, t_{a_1}\right], t_{a_2}\right], \ldots, t_{a_l}\right) (\xi)
$$

(\text{where } b = \sum |a_1| + (l-1)|a_2| + \cdots + |a_l|)

for all homogeneous $\xi \in \Gamma(L^\vee)$, $a_1, a_2, \ldots, a_l \in \Gamma(L)$ and $f \in C^\infty(M)$.

Conversely, given the $L^\infty$ algebroid $L \rightarrow M$ with its anchors $(\rho_l)_{l \geq 0}$ and its multibrackets $(\lambda_l)_{l \geq 1}$, we claim that it is possible to recover the corresponding vector field $Q$ on the $\mathbb{Z}$-graded manifold $L[1]$ tangent to the submanifold $M \xrightarrow{\pi} L[1]$ and satisfying Equations (34) and (35).

The algebra $C^\infty(L[1])$ admits the direct product decomposition

$$
C^\infty(L[1]) = \prod_{k=0}^{\infty} \Gamma(S^k(L^\vee[-1])).
$$

We will refer to $\Gamma(S^k(L^\vee[-1]))$ as the component of weight $k$ of $C^\infty(L[1])$. Together, $\pi^* C^\infty(M)$ (the component of weight 0) and $\Gamma(L^\vee[-1])$ (the component of weight 1) generate the algebra $C^\infty(L[1])$ multiplicatively. A vector field $Q$ on $L[1]$ is necessarily the sum $Q = \sum_{l=-1}^{\infty} D_l$ of derivations $D_l$ of weight $l$ of $C^\infty(L[1])$:

$$
D_l : \Gamma(S^l(L^\vee[-1])) \rightarrow \Gamma(S^{l+1}(L^\vee[-1])).
$$

Because we want the vector field $Q$ to be tangent to $M$, i.e. we want $Q \in \ker(J \circ P)$, its component $D_{-1}$ of weight $-1$ must vanish. Given a choice of local coordinates $(x^j)_{j \in I}$ on $M$; a local frame
Since each contraction \( H \) hence for all homogenous elements \((s_k)_{k \in K}\) for \( L[1] \); and the dual local frame \((\xi^k)_{k \in K}\) for \( L^\vee[-1] \), the derivation \( D_l \) admits the local expression

\[
D_l = \frac{1}{l!} \sum_{j \in J} \xi^{i_j} \cdots \xi^{i_1} \pi^*(Q^j_{i_1, i_2, \ldots, i_l}) \frac{\partial}{\partial x^j} + \frac{1}{(l+1)!} \sum_{k \in K} \xi^{i_1} \cdots \xi^{i_l} \pi^*(Q_k^{i_0, i_1, \ldots, i_l}) \frac{\partial}{\partial \xi^k},
\]

where

\[
\hat{Q}^j_{i_1, i_2, \ldots, i_l} = (-1)^{|s_{i_1}|+|s_{i_2}|+\cdots+|s_{i_l}|} \left( \left[ \left[ D_l, t_{s_{i_1}} \right], \left[ t_{s_{i_2}}, \ldots, \right], t_{s_{i_l}} \right] (\pi^*(x^j)) \right),
\]

\[
Q^k_{i_0, i_1, \ldots, i_l} = (-1)^{|s_{i_0}|+|s_{i_1}|+\cdots+|s_{i_l}|} \left( \left[ \left[ Q, t_{s_{i_1}} \right], \left[ t_{s_{i_2}}, \ldots, \right], t_{s_{i_l}} \right] (\xi^k) \right).
\]

Since each contraction \( t_a \) is a derivation of weight \((-1)\), we may rewrite the latter equations as

\[
\hat{Q}^j_{i_1, i_2, \ldots, i_l} = (-1)^{|s_{i_1}|+|s_{i_2}|+\cdots+|s_{i_l}|} \left( \left[ \left[ Q, t_{s_{i_1}} \right], \left[ t_{s_{i_2}}, \ldots, \right], t_{s_{i_l}} \right] (\pi^*(x^j)) \right),
\]

\[
Q^k_{i_0, i_1, \ldots, i_l} = (-1)^{|s_{i_0}|+|s_{i_1}|+\cdots+|s_{i_l}|} \left( \left[ \left[ Q, t_{s_{i_1}} \right], \left[ t_{s_{i_2}}, \ldots, \right], t_{s_{i_l}} \right] (\xi^k) \right).
\]

Hence \( Q \) satisfies Equations (34) and (35) if and only if

\[
\hat{Q}^j_{i_1, i_2, \ldots, i_l} = (-1)^{\circ} \rho_l (s_{i_1}, s_{i_2}, \ldots, s_{i_l}) x^j
\]

(\text{where } \circ = (l-1)|s_{i_1}|+(l-2)|s_{i_2}|+\cdots+|s_{i_l-1}|)

\[
\text{and } Q^k_{i_0, i_1, \ldots, i_l} = (-1)^{\dagger} \langle \lambda_{l+1} (s_{i_0}, s_{i_1}, \ldots, s_{i_l}), \xi^k \rangle.
\]

(\text{where } \dagger = (l+1)|s_{i_0}|+l|s_{i_1}|+\cdots+|s_{i_l}|)

The discussion above establishes our earlier claim that the anchors and multibrackets of the \( L_\infty \) algebroid \( L \rightarrow M \) determine through Equations (34) and (35) a unique vector field \( Q \) of degree 1 on \( L[1] \) tangent to the submanifold \( M \).

The \( L_\infty[1] \) algebra structure \((\mu_l)_{l \geq 1}\) on \( \Gamma(L[1]) \oplus C^\infty(M) \) determined by the \( L_\infty \) algebroid \( L \rightarrow M \) as per Lemma A.1 is related to the vector field \( Q \) through Equation (33). It follows from the generalized Jacobi identity, Equation (33), and repetive use of Equation (32) that

\[
0 = \sum_{p+r=l+1} (\mu_p \circ \mu_r) (z_1, \cdots, z_l) = P \left( \left[ \left[ [Q, Q], z_1 \right], z_2, \cdots, z_l \right] \right),
\]

for all \( z_1, z_2, \ldots, z_l \in \Gamma(L[1]) \oplus C^\infty(M) \). Careful analysis of the weight components of \([Q, Q]\) shows that they must all vanish. Thus the constructed vector field \( Q \) is homological. \(\square\)

**Appendix B. Shifted Poisson algebras**

**Definition B.1.** A \( k \)-shifted Poisson algebra is a graded commutative and associative algebra \( \mathfrak{R} \) with a degree \( k \) Poisson bracket, denoted by \([-,-]\) (i.e. \([\mathfrak{R}^i, \mathfrak{R}^j] \subset \mathfrak{R}^{i+j+k}\)), satisfying

1. \([a, b] = (-1)^{|a||b|}b[a, b],\)
2. \([a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}b[a, c],\)
3. \([a, bc] = [a, b]c + (-1)^{|a||b|}b[a, c],\)

for all homogenous elements \( a, b, c \in \mathfrak{R} \).
Note that Conditions (1) and (2) are equivalent to that $\mathfrak{g}[-k]$ is a graded Lie algebra, while (3) means that the Lie bracket is a biderivation.

Also note that 0-shifted Poisson algebras are usual Poisson algebras, and $(-1)$-shifted Poisson algebras are Gerstenhaber algebras.

In the meantime, one can obtain a $k$-shifted Poisson algebra out of a graded Lie algebra as indicated in the following

**Proposition B.2.** Let $\mathfrak{g}$ be a graded Lie algebra. Then the symmetric product $S(\mathfrak{g}[k])$ (similarly $\hat{S}(L[k])$) admits a unique $k$-shifted Poisson algebra structure which extends the original Lie bracket on $\mathfrak{g}$.

### Appendix C. Shifted polyvector fields

Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold. A vector field $X \in \mathcal{T}(\mathcal{M})$ of degree $l$ is a derivation of degree $l$: $C^\infty(\mathcal{M}) \xrightarrow{X} C^\infty(\mathcal{M})$. The degree of $X$ is denoted by $|X| = l$.

The commutator in $\mathcal{T}(\mathcal{M})$ is standard:

$$[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X,$$

for all homogeneous $X, Y \in \mathcal{T}(\mathcal{M})$. It is obvious that $\mathcal{T}(\mathcal{M})$ is a left $C^\infty(\mathcal{M})$-module, and the pair $(\mathcal{T}(\mathcal{M}), C^\infty(\mathcal{M}))$ forms a graded Lie-Rinehart algebra.

Let $k \in \mathbb{Z}$ be a fixed integer. Let $T^0_{\text{poly}}(\mathcal{M})[k] = C^\infty(\mathcal{M})$, and for each $m \geq 1$,

$$T^m_{\text{poly}}(\mathcal{M})[k] = S^m_{C^\infty(\mathcal{M})}(\mathcal{T}(\mathcal{M})[k + 1]).$$

Elements in $T^m_{\text{poly}}(\mathcal{M})[k]$ are called $k$-shifted $m$-polyvector fields on $\mathcal{M}$. Then the space

$$T^*_{\text{poly}}(\mathcal{M})[k] = \bigoplus_{m \geq 0} T^m_{\text{poly}}(\mathcal{M})[k]$$

is called the $k$-shifted Schouten-Nijenhuis algebra of $\mathcal{M}$. Its completion is denoted by $\hat{T}^*_{\text{poly}}(\mathcal{M})[k]$.

An element in $T^m_{\text{poly}}(\mathcal{M})[k]$ is a finite sum of elements of the form:

$$\Pi = \tilde{X}_1 \circ \tilde{X}_2 \circ \cdots \circ \tilde{X}_m,$$

where $X_i \in \mathcal{T}(\mathcal{M})$, and $\tilde{X}_i \in \mathcal{T}(\mathcal{M})[k+1]$ means the corresponding element with shifted degree. The number $|\Pi| = |X_1| + \cdots + |X_m|$ is called the pure degree of $X$, whereas $m$ is called the weight. By

$$\|\Pi\|_k = |\Pi| - m(k+1),$$

we denote the $(k$-shifted) total degree of $\Pi$, which is in fact the actual degree being considered as an element in $T^m_{\text{poly}}(\mathcal{M})[k]$.

The following lemma can be easily verified.

**Lemma C.1.** The space $T^*_{\text{poly}}(\mathcal{M})[k]$ is a $(k+1)$-shifted Poisson algebra, where the bracket is a natural extension of the one on $\mathcal{T}(\mathcal{M})$, similarly $\hat{T}^*_{\text{poly}}(\mathcal{M})[k]$.

**Remark C.2.** Here we follow the notion and degree conventions of shifted polyvector fields of Pridham [20].
When \( k = -2 \), the space of \((-2)\)-shifted polyvector fields \( \mathcal{T}_{\text{poly}}^\bullet(\mathcal{M})[-2] \), coincides with the usual Schouten-Nijenhuis algebra on \( \mathcal{M} \), which is simply denoted by \( \mathcal{T}_{\text{poly}}^\bullet(\mathcal{M}) \).

The following proposition provides an alternative description of shifted polyvector fields.

**Lemma C.3.** A homogeneous \( k \)-shifted \( m \)-polyvector field \( \Pi \) on \( \mathcal{M} \) is equivalently to a \( m \)-ary operation of degree \( |\Pi| \) (the pure degree of \( \Pi \)):

\[
\Pi : (C^\infty(\mathcal{M}))^\otimes m \to C^\infty(\mathcal{M})
\]

satisfying the following properties:

1) \( \Pi \) is symmetric w.r.t. \( C^\infty(\mathcal{M})[-k - 1] \):

\[
\Pi(f_1, \cdots, f_i, f_{i+1}, f_{i+2}, \cdots, f_m) = (-1)^{|f_i||f_{i+1}|\cdots|f_{i+k+1}|} \Pi(f_1, \cdots, f_{i-1}, f_{i+1}, f_i, f_{i+1}, \cdots, f_m);
\]

2) \( \Pi \) is a derivation of degree \( |\Pi| \):

\[
\Pi(f_1, \cdots, f_{m-1}, f_m f'_m) = \Pi(f_1, \cdots, f_{m-1}, f_m) f'_m + (-1)^{|\Pi|+|f_1|+\cdots+|f_{m-1}|} f_m \Pi(f_1, \cdots, f_{m-1}, f'_m).
\]

The proof is omitted as it is completely analogous to the usual unshifted polyvector fields on ordinary smooth manifolds.

To describe explicitly the \((k+1)\)-shifted Poisson bracket in \( \mathcal{T}_{\text{poly}}^\bullet(\mathcal{M})[k] \), we need to introduce compositions of such polyvector fields. For \( \Pi \in \mathcal{T}_{\text{poly}}^m(\mathcal{M})[k] \), \( \Lambda \in \mathcal{T}_{\text{poly}}^n(\mathcal{M})[k] \), we define \( \Pi \circ \Lambda \) as the \((m + n - 1)\)-ary operation \( (C^\infty(\mathcal{M}))^\otimes m+n-1 \to C^\infty(\mathcal{M}) \) given by

\[
(\Pi \circ \Lambda)(f_1, \cdots, f_{m+n-1}) = \sum_{\sigma \in \text{Sh}(m,n-1)} \epsilon^{[k+1]}(\sigma) \Pi(\Lambda(f_{\sigma(1)}), \cdots, f_{\sigma(n)}, f_{\sigma(n+1)}, \cdots, f_{\sigma(m+n-1)}).
\]

Here \( \epsilon^{[k+1]}(\sigma) \) means the Koszul sign w.r.t. the shifted degrees \( |f_i|^{[k+1]}, \cdots, |f_{m+n}|^{[k+1]} \).

Then the commutator of \( \Pi \in \mathcal{T}_{\text{poly}}^m(\mathcal{M})[k] \) and \( \Lambda \in \mathcal{T}_{\text{poly}}^n(\mathcal{M})[k] \) defines the \((k+1)\)-shifted Poisson bracket in \( \mathcal{T}_{\text{poly}}^\bullet(\mathcal{M})[k] \):

\[
[\Pi, \Lambda] = \Pi \circ \Lambda - (-1)^{|\Pi|+|\Lambda|+k+k+1)} \Lambda \circ \Pi,
\]

which is also known as the Schouten–Nijenhuis bracket of \( \Pi \) and \( \Lambda \).

Note that \( \Pi \in \mathcal{T}_{\text{poly}}^m(\mathcal{M})[k] \) as an operator \( (C^\infty(\mathcal{M}))^\otimes m \to C^\infty(\mathcal{M}) \) can also be written as a derived bracket:

\[
\Pi(f_1, \cdots, f_m) = [[\cdots [[\Pi, f_1], f_2], \cdots, f_{m-1}], f_m],
\]

where \( f_i \in \mathcal{T}_{\text{poly}}^0(\mathcal{M})[k] = C^\infty(\mathcal{M}) \).

**Acknowledgments**

We would like to thank Camille Laurent-Gengoux, Pavol Ševera, Jim Stasheff, and Luca Vitagliano for fruitful discussions and useful comments.
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