Efron’s monotonicity property for measures on \( \mathbb{R}^2 \)

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Abstract: First we prove some kernel representations for the covariance of two functions taken on the same random variable and deduce kernel representations for some functionals of a continuous one-dimensional measure. Then we apply these formulas to extend Efron’s monotonicity property, given in Efron [1965] and valid for independent log-concave measures, to the case of general measures on \( \mathbb{R}^2 \). The new formulas are also used to derive some further quantitative estimates in Efron’s monotonicity property.

AMS 2000 subject classifications: Primary 60E15, 60F10.
Keywords and phrases: covariance formulas, log-concave, monotonicity, preservation.

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*Supported by NI-AID grant 2R01 AI29168-04, and by a PIMS postdoctoral fellowship
†Supported in part by NSF Grant DMS-1566514, NI-AID grant 2R01 AI291968-04
1. Introduction: a monotonicity property

Efron [1965] proved the following proposition:

**Proposition 1.1.** Let \((X, Y)\) be a pair of real valued random variables. Then the following two statements are equivalent:

(i) For any \(\Psi : \mathbb{R}^2 \to \mathbb{R}\), a function which is nondecreasing in each argument, the conditional expectation

\[
I(s) = \mathbb{E} [\Psi(X, Y) | X + Y = s]
\]

is nondecreasing in \(s\).

(ii) For any \((x, y) \in \mathbb{R}^2\), the conditional survival functions

\[
S_X(x; s) = \mathbb{P} [X > x | X + Y = s] \quad \text{and} \quad S_Y(y; s) = \mathbb{P} [Y > y | X + Y = s]
\]

are nondecreasing in \(s\).

**Proof.** (i) implies (ii) is given by taking \(\Psi(X, Y) = 1_{\{X > x\}}\) and then by using the symmetry in \(X\) and \(Y\). To prove that (ii) implies (i), let \(F_s^{-1}\) and \(G_s^{-1}\) be the conditional quantile functions of \(X\) and \(Y\) given \(X + Y = s\); that is, for \(0 < u < 1\)

\[
x_{u,s} \equiv F_s^{-1}(u) \equiv \inf\{x: F_s(x) \geq u\},
\]

\[
y_{u,x} \equiv G_s^{-1}(u) \equiv \inf\{y: G_s(y) \geq u\},
\]

where \(F_s(x) = \mathbb{P}(X \leq x | X + Y = s)\) and \(G_s(y) = \mathbb{P}(Y \leq y | X + Y = s)\). Then, by (ii), for \(t < s\),

\[
u \leq \mathbb{P}(X \leq F_s^{-1}(u) | X + Y = s)
\]

\[
= 1 - \mathbb{P}(X > F_s^{-1}(u) | X + Y = s)
\]

\[
\leq 1 - \mathbb{P}(X > G_t^{-1}(u) | X + Y = t)
\]

\[
= \mathbb{P}(X \leq F_s^{-1}(u) | X + Y = t),
\]

and hence \(x_{u,t} = F_s^{-1}(u) \leq F_s^{-1}(u) = x_{u,s}\). By symmetry \(y_{u,s}\) is also nondecreasing in \(s\). Thus

\[
\mathbb{E} [\Psi(X, Y) | X + Y = s] = \int_{u \in (0,1)} \Psi(x_{u,s}, y_{u,s}) \, du
\]

Then (i) follows from (1.3). \(\square\)

In this paper, condition (i) of Proposition 1.1 is referred to as Efron’s “monotonicity property”. Efron [1965] used Proposition 1.1 to prove the monotonicity property for independent log-concave variables \(X\) and \(Y\). In this paper, we extend the validity of Efron’s monotonicity property to more general pairs \((X, Y)\) on the plane, see Section 3. Our main result, Theorem 3.1, provides a condition on the joint density \(h\) of \((X, Y)\), in terms of the second derivatives of \(\phi \equiv (-\log h)\) which imply (ii) of Proposition 1.1. In particular, in Section 3.3 we exhibit examples of random pairs satisfying the monotonicity property that are neither log-concave nor mutually independent. We also recover by different techniques Efron’s monotonicity for independent log-concave variables in Section 3.2. Then we obtain quantitative lower-bounds for the derivative of Efron’s \(I\) function in Section 5.

Our proofs rely on several key covariance identities which are stated in Section 2. These identities, originating in Höfling [1940] (see also Hoeffding [1994] for a translation of the German original), build on more recent results in the log-concave case due to Menz and Otto [2013].

We conclude the paper in Section 5 by providing complete proofs of the key covariance identities stated in Section 2.
Remark 1.1. It is easily seen, through standard approximation arguments, that point (ii) of Proposition 1.1 is equivalent to nondecreasingness in $s$ of the functions
\[ E[\varphi(X)|X + Y = s] \quad \text{and} \quad E[\varphi(Y)|X + Y = s] \] (1.4)
for every nondecreasing function $\varphi$. This implies that in point (i), one can take without loss of generality functions $\Psi$ to depend only on one variable. A simple proof of the monotonicity of functionals given in (1.4) for independent log-concave variables $X$ and $Y$ is established in Saumard and Wellner [2014] using symmetrization arguments.

Efron’s monotonicity property appears naturally in the theory of log-concave measures, see Saumard and Wellner [2014]. Indeed, it has been used by Johnson [2007] and Johnson, Kontoyiannis and Madiman [2013] to prove preservation of ultra log-concavity under convolution (for discrete random variables), and by Wellner [2013] to give a proof that log-concavity and strong log-concavity are preserved by convolution in the one-dimensional continuous setting. These proofs operate at the level of scores or relative scores (first derivative of the convex potentials of the log-concave measures). Without reliance on derivatives, the classical proof of preservation of log-concavity under convolution consists of a direct application of Prékopa’s theorem, Prékopa [1971]. A proof of preservation of log-concavity under convolution can also be derived via the Brascamp-Lieb inequality (Brascamp and Lieb [1976]), that operates at the second derivative level of the convex potentials and that is the local form of the Brunn-Minkowski inequality.

Efron’s monotonicity property can also be viewed as a monotonicity property for the collection of conditional laws with respect to the stochastic order (Theorem 6.B.9. in Shaked and Shanthikumar [2007]), see also Shanthikumar [1987a], Shanthikumar [1987b], Rinott and Samuel-Cahn [1991], Dubhashi and Häggström [2008], Zhuang, Yao and Hu [2010]).

Efron’s monotonicity property has been applied in the context of negative dependence theory (Jong-Dev and Proschan [1983], Block, Savits and Shaked [1985], Boland et al. [1996], Hu and Hu [1999], Pemantle [2000]), in combinatorial probability (Fill [1988], Liggett [2000], Johnson [2007], Goldschmidt, Martin and Spanò [2008], Gross et al. [2015]), in queueing theory (Shanthikumar and Yao [1986], Shanthikumar and Yao [1987], Masuda [1995], Pestien and Ramakrishnan [2002], Daduna and Szekli [2004]), in Economic theory (Ederer [2010], Wang [2012], Denuit and Dhaene [2012]), in the theory of statistical testing (Berk [1978], Cohen and Sackrowitz [1987], Cohen and Sackrowitz [1990], Benjamin and Heller [2008], Heller et al. [2016]), as well as other statistical estimation problems (Stefanski [1992], Hwang and Stefanski [1994], Ma [1999]).

Hence any extension of Efron’s monotonicity property may have several applications in statistical theory and also beyond. The questions and issues described in Hwang and Stefanski [1994] provide an interesting example of the statistical relevance of the results that we obtain below. Let us briefly recall the setting of their paper.

Hwang and Stefanski [1994] study the preservation of monotonicity of regression functions under measurement errors. Let $(T, X, U)$ be a triple of random variables where $T$ is a response variable, $U$ is an (unobserved) covariate, $X = U + Z$ is the covariate $U$ with additive “measurement error” $Z$. Hwang and Stefanski discuss monotonicity of $E[T|X = x]$ under the assumption that $E[T|U = u]$ is monotone. Preservation of monotonicity is analyzed relative to the behavior of the measurement error $Z := X - U$. Then the relationship between the “true” regression function and the regression function with “measurement error” is important for modeling purposes (see Spiegelman [1986], Gleser [1990], Stefanski and Carroll [1991], Stefanski and Carroll [1990] and Carroll et al. [2006]).

Using Efron’s monotonicity property, Hwang and Stefanski show that monotonicity of the regression function is preserved when a log-concave error $Z$ in measurement is made independently of a log-concave covariate $U$. Preservation of monotonicity of a regression function will be further discussed below in light of our results.
2. Covariance Identities

Our goal is to prove the monotonicity property with the greatest generality in terms of the law of the pair of random variables involved. By Proposition 1.1 above, it suffices to focus on the monotonicity of the conditional survival functions in (1.2) of (ii). To do this in Section 3 we will use several helpful identities for covariances which are summarized below. Proofs of the new identities in our list, along with examples and counterexamples, will be given in Section 5.

It is worth noting that covariance identities have an interest by themselves since they provide powerful tools to derive deviation and concentration inequalities (see for instance Bobkov, Götze and Houdré [2001], Houdré and Marchal [2004], Houdré [2002] and also Ledoux [2001] Section 5.5) or functional inequalities (Saumard and Wellner [2017]). From this point of view, the use of covariance identities to prove extensions of Efron’s monotonicity property may be seen as a new connection of covariance identities with functional inequalities.

**Proposition 2.1.** Suppose that \( (X, Y) \) have joint distribution function \( H \) on \( \mathbb{R}^2 \) with marginal distribution functions \( F \) and \( G \). Suppose that \( a, b \) are non-decreasing functions from \( \mathbb{R} \) to \( \mathbb{R} \) with \( \text{Var}(a(X)) < \infty \) and \( \text{Var}(b(Y)) < \infty \). Then

\[
\text{Cov}[a(X), b(Y)] = \int_{\mathbb{R}^2} \{H(x, y) - F(x)G(y)\} \, da(x)db(y). \tag{2.1}
\]

The identity (2.1) can be found in Shorack [2000], section 7.4, formula page 117, but it has its origins in Höffding [1940] (see also Hoeffding [1994] for a translation of the German original) This identity has several useful corollaries. We begin with the original inequality due to Höffding [1940], by taking \( a \) and \( b \) to be identity functions.

**Corollary 2.1.** (Hoeffding). When \( a(x) = x \) and \( b(y) = y \) for all \( x, y \in \mathbb{R} \),

\[
\text{Cov}[X, Y] = \int_{\mathbb{R}^2} \{H(x, y) - F(x)G(y)\} \, dx \, dy.
\]

**Corollary 2.2.** (a) When \( Y = X \) almost surely so that \( G = F \) and \( H(x, y) = F(x \wedge y) \), and \( a, b \) are non-decreasing and left-continuous,

\[
\text{Cov}[a(X), b(X)] = \int_{\mathbb{R}^2} \{F(x \wedge y) - F(x)F(y)\} \, da(x)db(y)
\]

\[
= \int_{\mathbb{R}^2} K_{\mu}(x, y) \, da(x)db(y) \tag{2.2}
\]

where the non-negative and symmetric kernel \( K_{\mu} \) on \( \mathbb{R}^2 \) is defined by

\[
K_{\mu}(x, y) = F(x \wedge y) - F(x)F(y), \quad \text{for all } (x, y) \in \mathbb{R}^2. \tag{2.3}
\]

and where \( F(x) = F_{\mu}(x) = \mu((-\infty, x]) \) is the distribution function associated with the probability measure \( \mu \) on \((\mathbb{R}, \mathcal{B})\).

(b) Moreover, (2.2) continues to hold if \( a = a_1 - a_2 \), \( b = b_1 - b_2 \) where \( a_j \in L_p(F) \) and \( b_j \in L_q(F) \) for \( j = 1, 2 \) with \( p^{-1} + q^{-1} = 1 \).

Now we specialize Corollary 2.2 slightly by taking \( a \) to be an indicator function.
Corollary 2.3. Suppose that \( b = b_1 - b_2 \) where \( b_1, b_2 \) are left-continuous and non-decreasing with either \( b_j \in L_2(F) \) for \( j = 1, 2 \) or \( b_j \in L_1(F) \) for \( j = 1, 2 \), and let \( z \in \mathbb{R} \). Then, with \( a(x) = 1_{(-\infty,z)}(x) \),

\[
F(z) \int_{\mathbb{R}} b dF - \int_{(-\infty,z]} b dF = - \text{Cov}[1_{[X \leq z]}, b(X)] = \int_{\mathbb{R}} K_{\mu}(z, y) db(y), \tag{2.4}
\]

and

\[
-(1 - F(z)) \int_{\mathbb{R}} b dF + \int_{(z, \infty]} b dF = \text{Cov}[1_{[X > z]}, b(X)] = \int_{\mathbb{R}} K_{\mu}(z, y) db(y). \tag{2.5}
\]

Furthermore, if \( b \in L_1(F) \) is absolutely continuous, then

\[
F(z) \int_{\mathbb{R}} b dF - \int_{(-\infty,z]} b dF = - \text{Cov}[1_{[X \leq z]}, b(X)] = \int_{\mathbb{R}} K_{\mu}(z, y)b'(y) dy, \tag{2.6}
\]

and

\[
-(1 - F(z)) \int_{\mathbb{R}} b dF + \int_{(z, \infty)} b dF = \text{Cov}[1_{[X > z]}, b(X)] = \int_{\mathbb{R}} K_{\mu}(z, y)b'(y) dy. \tag{2.7}
\]

Remark 2.1. Note that the quantities appearing on the left sides in (2.4) and (2.5) have interpretations in terms of mean residual life or reversed mean residual life functions: in particular, the left side of (2.5) can be written as

\[
(E\{h(X)|X > z\} - E\{h(X)\})(1 - F(z)),
\]

while the left side of (2.4) can be written as

\[-(E\{h(X)|X \leq z\} - E\{h(X)\}) F(z).\]

Our next corollary, a further corollary of Corollary 2.2, allows the functions \( a \) and \( b \) to be differences of left-continuous and non-decreasing functions, or absolutely continuous.

Corollary 2.4. (Menz and Otto) If \( a = a_1 - a_2 \) and \( b = b_1 - b_2 \) where \( a_j \in L_p(F) \) and \( b_j \in L_q(F) \) for \( j = 1, 2 \) with \( p^{-1} + q^{-1} = 1 \), then (2.2) continues to hold. Moreover, if \( a \) and \( b \) are absolutely continuous with \( a \in L_p(F) \) and \( b \in L_q(F) \), then

\[
\text{Cov}[a(X), b(X)] = \iint_{\mathbb{R}^2} a'(x)K_{\mu}(x, y)b'(y)dxdy. \tag{2.8}
\]

The covariance identity (2.8) appeared in Menz and Otto [2013] (but without explicit assumptions on the functions \( a \) and \( b \)). Note that this inequality implies a version of the FKG inequality: if \( a \) and \( b \) are non-decreasing, then \( a'(x) \geq 0 \) and \( b'(y) \geq 0 \) so that the right side of (2.8) is non-negative, and hence \( E\{a(X)b(Y)\} \geq E\{a(X)\}E\{b(Y)\} \).

Our last set of covariance identities involve taking \( b = \varphi' \) in the case when \( F \) has density \( f = \exp(-\varphi) \).
Corollary 2.5. Suppose that $F$ has absolutely continuous density $f = \exp(-\varphi)$.

(a) If $\varphi$ has derivative $\varphi'$ which satisfies $\varphi' = \varphi_1' - \varphi_2'$ where $\varphi_j' \in L_1(F)$ for $j = 1, 2$ and $\varphi_j'$ are left-continuous and non-decreasing, then

$$
\int_{\mathbb{R}} K_\mu(x, y) d\varphi'(y) = f(x). 
$$

(2.9)

(b) If $\varphi$ has derivative $\varphi' \in L_1(F)$ which is absolutely continuous, then

$$
\int_{\mathbb{R}} K_\mu(x, y) \varphi''(y) dy = f(x). 
$$

(2.10)

(c) In particular, if $f$ is log-concave and absolutely continuous, $f = \exp(-\varphi)$ with $\varphi$ convex, then (2.9) holds.

(d) If $f$ is log-concave and absolutely continuous, and $\varphi'$ is absolutely continuous, then (2.10) holds.

The condition $\varphi' \in L_1(F)$ in (b) of Corollary 2.5 is not overly restrictive. Indeed, it is equivalent to $f' \in L_1(\text{Leb})$ since $f' = -\varphi'f$, and the latter condition is easily checked. Also note that $\varphi' = -f'/f$ is the “score for location” in statistics.

Remark 2.2. Corollary 2.5(d) was given by Menz and Otto [2013]. The other parts of Corollary 2.5 are apparently new.

Proofs of Proposition 2.1 and Corollaries 2.1 - 2.5 will be given in Section 5.

3. The monotonicity property for general measures on $\mathbb{R}^2$

3.1. A general result

Proposition 1.1 shows that monotonicity of $s \mapsto I(s)$ in (1.1) is implied by monotonicity of the conditional survival functions $S_X(x; s)$ and $S_Y(y; s)$ in (1.2). The following theorem provides a way of verifying the monotonicity of the conditional survival functions $s \mapsto S_X(x; s)$ and $s \mapsto S_Y(y; s)$ in terms of the elements of the Hessian matrix $\text{Hess}(\varphi)$ where $\varphi = -\log h$ is the potential (perhaps non-convex) of the joint density $h$ of $(X, Y)$. First some further notation. We write

$$
\text{Hess}(\varphi)(x, y) = \left( \begin{array}{cc}
\frac{\partial^2}{\partial x^2} \varphi(x, y) & \frac{\partial^2}{\partial y \partial x} \varphi(x, y) \\
\frac{\partial^2}{\partial y \partial x} \varphi(x, y) & \frac{\partial^2}{\partial y^2} \varphi(x, y)
\end{array} \right) = \left( \begin{array}{cc}
\frac{\partial^2 \varphi}{\partial x^2} & \frac{\partial^2 \varphi}{\partial x \partial y} \\
\frac{\partial^2 \varphi}{\partial x \partial y} & \frac{\partial^2 \varphi}{\partial y^2}
\end{array} \right)
$$

for the Hessian of $\varphi \equiv -\log h$ where we suppose that $h > 0$ on some open set $S \subset \mathbb{R}^2$. We denote the conditional densities of $X$ given $X + Y = s$ and $Y$ given $X + Y = s$ by $f_1(\cdot; s) \equiv f_1$ and $f_2(\cdot; s) \equiv f_2$ respectively, and denote the corresponding conditional measures by $\mu_1 \equiv \mu_1(\cdot; s)$ and $\mu_2 \equiv \mu_2(\cdot; s)$ respectively. Thus

$$
f_1(x; s) = \exp(-\varphi_1(x; s)) = \frac{h(x, s - x)}{\int_S h(x', s - x') dx'}, 
$$

(3.1)

$$
f_2(y; s) = \exp(-\varphi_2(y; s)) = \frac{h(s - y, y)}{\int_S h(s - y', y') dy'}. 
$$

(3.2)

Furthermore we write

$$
\partial_1 \varphi(x, y) \equiv \frac{\partial}{\partial x} \varphi(x, y) \quad \text{and} \quad \partial_2 \varphi(x, y) \equiv \frac{\partial}{\partial y} \varphi(x, y).
$$
We will also need the following domination conditions:

**D1:** Fix $s_0 \in \mathbb{R}$. Suppose that $x \mapsto \partial_2 \varphi(x, s-x)$ is absolutely continuous for $s \in [s_0 - \epsilon, s_0 + \epsilon] \equiv S_\epsilon$ for some $\epsilon > 0$, and there exists a function $\overline{f} \in L_1(\text{Leb})$ such that

$$|\partial_2 \varphi(x, s-x) \exp(-\varphi(x, s-x))| \leq \overline{f}(x)$$

for almost all $x \in \mathbb{R} \cap S$ and all $s \in S_\epsilon$.

**D2:** Fix $s_0 \in \mathbb{R}$. Suppose that $y \mapsto \partial_1 \varphi(s-y, y)$ is absolutely continuous for $s \in S_\epsilon$ for some $\epsilon > 0$, and there exists a function $\overline{h} \in L_1(\text{Leb})$ such that

$$|\partial_1 \varphi(s-y, y) \exp(-\varphi(s-y, y))| \leq \overline{h}(y)$$

for almost all $y \in \mathbb{R} \cap S$ and all $s \in S_\epsilon$.

**Theorem 3.1.** Suppose that D1 holds. Then with $K_{1,0} \equiv K_{\mu_1, s_0}$,

$$\partial_2 S_X(x, s_0) = \int_{\mathbb{R}} K_{1,0}(x, x') (\partial_{22}^2 \varphi - \partial_{21}^2 \varphi)(x', s_0 - x') dx'. \tag{3.3}$$

Suppose that D2 holds. Then with $K_{2,0} \equiv K_{\mu_2, s_0}$,

$$\partial_2 S_Y(y; s_0) = \int_{\mathbb{R}} K_{2,0}(y', y) (\partial_{11}^2 \varphi - \partial_{21}^2 \varphi)(s_0 - y', y') dy'. \tag{3.4}$$

Hence,

$$\partial_2 S_Y(y; s_0) = \int_{\mathbb{R}} K_{1,0}(s_0 - y, y') (\partial_{11}^2 \varphi - \partial_{21}^2 \varphi)(y', s_0 - y') dy'. \tag{3.5}$$

Moreover, if both D1 and D2 hold, then

$$f_1(x; s_0) = \partial_2 S_X(x; s_0) + \partial_2 S_Y(s_0 - x; s_0), \tag{3.6}$$

$$f_2(y; s_0) = \partial_2 S_Y(y; s_0) + \partial_2 S_X(s_0 - y; s_0). \tag{3.7}$$

**Corollary 3.1.** If D1 and D2 hold (so the conclusions of Theorem 3.1 hold), and

$$(\partial_{22}^2 \varphi - \partial_{21}^2 \varphi)(x', s_0 - x') \geq 0 \text{ for all } x' \text{ such that } (x', s_0 - x') \in S \text{ and }$$

$$(\partial_{11}^2 \varphi - \partial_{21}^2 \varphi)(s_0 - y', y') \geq 0 \text{ for all } y' \text{ such that } (s_0 - y', y') \in S,$$

then the conditional survival functions $S_X(\cdot|s) = S_X(\cdot; s)$ and $S_Y(\cdot|s) = S_Y(\cdot; s)$ in (ii) of Proposition 1.1 are non-decreasing in $s$, and hence (i) of Proposition 1.1 also holds.

**Proof.** Let $J_1(s) \equiv \log \int_{\mathbb{R}} h(x, s-x) dx$. By the domination assumption D1, the function $J_1$ is differentiable on $S_\epsilon$ with derivative

$$J_1'(s) = -\int_{\mathbb{R}} \partial_2 \varphi(x, s-x) h(x, s-x) dx / \int_{\mathbb{R}} h(x', s-x') dx'.$$
Note that \( \varphi_1(x; s) = \varphi(x, s - x) + J_1(s) \), and we therefore find that
\[
\partial_2 \varphi_1(x; s) = \partial_2 \varphi(x, s - x) + J'_1(s),
\]
and
\[
\partial^2_{12} \varphi_1(x; s) = (\partial^2_{12} \varphi - \partial^2_{22} \varphi)(x, s - x). \tag{3.8}
\]

Multiplying by minus one and integrating this identity with respect to \( K_{\mu_1}(x, x') \) and then applying covariance identity (2.6) yields
\[
\int_{\mathbb{R}} (\partial^2_{12} \varphi - \partial^2_{22} \varphi)(x', s - x') K_{\mu_1}(x, x') dx' \\
= - \int_{\mathbb{R}} K_{\mu_1}(x, x') \partial^2_{12} \varphi_1(x'; s) dx' \quad \text{by (3.8)} \\
= - \left( \int_{(-\infty, x]} f_1(x'; s) dx' \int \partial_2 \varphi_1(\cdot; s) f_1(\cdot; s) dx' - \int_{(-\infty, x]} \partial_2 \varphi_1(x'; s) f_1(x'; s) dx' \right) \tag{3.9}
\]
where the first term is
\[
\int_{\mathbb{R}} \partial_2 \varphi_1(x'; s_0) f_1(x'; s_0) dx' = \int_{\mathbb{R}} \partial_2 \varphi_1(x'; s_0) \exp(-\varphi_1(x'; s_0)) dx' \\
= - \frac{d}{ds} \int_{\mathbb{R}} \exp(-\varphi_1(x'; s)) dx' \bigg|_{s = s_0} = 0,
\]
and where the second term is
\[
\int_{(-\infty, x]} \partial_2 \varphi_1(x'; s_0) f_1(x'; s_0) dx' = \int_{(-\infty, x]} \partial \varphi_1(x'; s_0) \exp(-\varphi_1(x'; s_0)) dx' \bigg|_{s = s_0} \\
= - \frac{\partial}{\partial s} \left( \mu_2(-\infty, x] \right)(s_0) = \frac{\partial}{\partial s} S_X(x; s_0).
\]

Combining this with (3.9) evaluated at \( s = s_0 \) yields the claimed identity (3.3).

The identity (3.4) follows from the same argument used to prove (3.3) by symmetry. To prove (3.5), let \( F_{1,s} \) and \( F_{2,s} \) denote the conditional distribution functions corresponding to the conditional densities \( f_1(\cdot; s) \) and \( f_2(\cdot; s) \). Then \( F_{1,s}(x) = 1 - F_{2,s}(s - x) \) and hence with \( K_{j,s}(x, y) = F_{j,s}(x \wedge y) - F_{j,s}(x) F_{j,s}(y), j = 1, 2 \), it follows that
\[
K_{2,s}(y, y') = K_{1,s}(s - y, s - y') \quad \text{for all} \ y, y'. \tag{3.10}
\]

Then, evaluating at \( s = s_0 \),
\[
\int_{\mathbb{R}} K_{2,0}(y, y') (\partial^2_{11} \varphi - \partial^2_{21} \varphi)(s_0 - y, y') dy' \\
= \int_{\mathbb{R}} K_{2,0}(y, y')(\partial^2_{11} \varphi - \partial^2_{21} \varphi)(s_0 - y, y') dy' \quad \text{since} \ K_{2,0}(y, y') = K_{2,0}(y', y) \\
= \int_{\mathbb{R}} K_{1,0}(s - y, s - y')(\partial^2_{11} \varphi - \partial^2_{21} \varphi)(s_0 - y, y') dy' \quad \text{by (3.10)} \\
= \int_{\mathbb{R}} K_{1,0}(s - y, v)(\partial^2_{11} \varphi - \partial^2_{21} \varphi)(v, s_0 - v) dv \\
\]
by the change of variable \( y' = s_0 - v \).
Thus (3.5) holds.

To show that (3.6) holds, note that since $\varphi_1 (x; s) = \varphi (x, s - x) + J_1 (s)$,

\[
\partial_1 \varphi_1 (x; s) = \partial_1 \varphi (x, s - x) - \partial_2 \varphi (x, s - x), \text{ and }
\partial_{11} \varphi_1 (x; s) = \partial_{11} \varphi (x, s - x) - \partial_{21} \varphi (x, s - x) + \partial_{12} \varphi_{22} (x, s - x).
\]

But then

\[
f_1 (x; s_0) = \int_{\mathbb{R}} K_{1,0} (x, x') \partial_{11} \varphi_1 (x'; s_0) dx' \text{ by (2.10), Corollary 2.5 } \\
= \int_{\mathbb{R}} K_{1,0} (x, x') \{ \partial_{11} \varphi (x', s_0 - x') - \partial_{21} \varphi (x', s_0 - x') + \partial_{12} \varphi_{22} (x', s_0 - x') \} dx' \\
= \partial_2 S_X (x; s_0) + \partial_2 S_Y (y; s_0)
\]

by using (3.3) and (3.5) in the last equality.

Now we are ready discuss examples (and counter-examples) of joint distributions on $\mathbb{R}^2$ where (ii) of Proposition 1.1 is satisfied. Identities (3.3) - (3.5) in Theorem 3.1 are very useful in this regard.

But we first consider the log-concave case in the light of Theorem 3.1.

### 3.2. Independent log-concave variables revisited

The following theorem is due to Efron [1965]. We give a different proof than Efron’s, based on formulas (3.3) and (3.4) of Theorem 3.1 above.

**Theorem 3.2 (Efron [1965]).** The monotonicity property is satisfied for any pair of independent log-concave random variables.

**Proof.** Let $(X, Y)$ be a pair of independent log-concave random variables with density $h$ on $\mathbb{R}^2$ with respect to Lebesgue measure. Then

\[
h (x, y) = g_X (x) g_Y (y) = \exp \left( - (\varphi_X (x) + \varphi_Y (y)) \right), \quad (x, y) \in \mathbb{R}^2,
\]

where $g_X$ and $g_Y$ are the densities ($\varphi_X$ and $\varphi_Y$ are the convex potentials) of $X$ and $Y$ respectively. Indeed, a log-concave random variable on $\mathbb{R}$ automatically has a density with respect to the Lebesgue measure (see for instance Saumard and Wellner [2014]). For now, let us also assume that $h > 0$ on $\mathbb{R}^2$.

Denote also $\varphi (x, y) = \varphi_X (x) + \varphi_Y (y)$, $(x, y) \in \mathbb{R}^2$. Let us first assume that $\varphi_X$ and $\varphi_Y$ are $C^2$ and that $g'_{X}, g'_{Y} \in L_{\infty}$. Define measures $\mu_1$ by $d\mu_1 (x) = f_1 (x; s) dx$ where

\[
f_1 (x; s) = \exp ( - \varphi_1 (x; s)) = \frac{h (x, s - x)}{\int_{\mathbb{R}} h (x', s - x') dx'}.
\]

Then, it follows that

\[
\varphi_1 (x; s) = \varphi_X (x) + \varphi_Y (s - x) + \log \left( \int_{\mathbb{R}} \exp \left( - (\varphi_X (x') + \varphi_Y (s - x')) \right) dx' \right).
\]

Using the assumption that that $g'_{Y} = - \varphi'_{Y} \exp ( - \varphi_Y)$ is uniformly bounded, it is easy to see that $\partial_2 \varphi_1$ exists and that

\[
\partial_2 \varphi_1 (x; s) = \varphi'_{Y} (s - x) + \frac{\int_{\mathbb{R}} \varphi'_{Y} (s - x') \exp \left( - (\varphi_X (x') + \varphi_Y (s - x')) \right) dx'}{\int_{\mathbb{R}} h (x', s - x') dx'}.
\]
Again using \( g_Y' \in L_\infty \), simple calculations show that there exists a function \( \mathcal{G} \in L_1 (\text{Leb}) \) such that for almost all \( x \in \mathbb{R} \),
\[
|\partial_2 \varphi_1 (x; s) \exp (-\varphi_1 (x; s))| \leq \mathcal{G} (x) .
\]
Furthermore, \( \partial_2 \varphi_1 (\cdot; s) \) is absolutely continuous (even \( C^1 \)), so by formula (3.3), it follows that
\[
(\partial_2 S_X) (x; s) = \int_{\mathbb{R}} K_{\mu_1} (x, u) (\partial^2_{22} \varphi - \partial^2_{12} \varphi) (u, s - u) \, du
\]
\[
= \int_{\mathbb{R}} K_{\mu_1} (x, u) \varphi''_y (s - u) \, du .
\]
Since \( \varphi''_Y \geq 0 \) by log-concavity of \( g_Y \), it follows that \( (\partial_2 S_X) (x; s) \geq 0 \). Note that the argument shows that even if \( g_X \) is is not log-concave, log-concavity of \( g_Y \) implies monotonicity of \( s \mapsto S_X (x; s) \).

By symmetry between \( X \) and \( Y \), we also have \( (\partial_2 S_Y) (y; s) \geq 0 \) and we conclude from Proposition 1.1 that the monotonicity property is satisfied for \((X, Y)\). To conclude, we have to prove that we can reduce the situation from general convex potentials to potentials \( \varphi \) that are finite on \( \mathbb{R} \) (this implies that \( h > 0 \) on \( \mathbb{R} \)), that are \( C^2 \) and that satisfy \( \| \varphi' \exp (-\varphi) \|_\infty < +\infty \).

This is done by convolution with Gaussian random variables, whose variance tends to zero (see for instance Proposition 5.5 in Saumard and Wellner [2014]). In particular, we see that any pair of independent log-concave random variables \((X, Y)\) there exists a sequence of log-concave random variables \((X_n, Y_n)\), with \( X_n \) independent of \( Y_n \), such that the densities \( g_{X_n} \), \( g_{Y_n} \) of \( X_n \) and \( Y_n \) are \( C^\infty \) and converge respectively to \( g_X \) and \( g_Y \) in \( L_\infty \). Hence, for any \((x, y, s) \in \mathbb{R}^3\),

\[
S_{X_n}(x; s) \rightarrow S_X (x; s) \quad \text{and} \quad S_{Y_n}(y; s) \rightarrow S_Y (y; s) , \quad \text{as} \ n \rightarrow \infty
\]

which gives the monotonicity in \( s \) of \( S_X (x; s) \) and \( S_Y (y; s) \).

The monotonicity extends to more than two independent log-concave variables.

**Theorem 3.3 (Efron [1965]).** Let \( m \) be greater than one. Then the monotonicity property is satisfied for any \( m \)-tuple of independent log-concave variables.

**Proof.** We proceed as in Efron [1965] by induction on \( m \). Let \((X_1, \ldots, X_m)\) be an \( m \)-tuple of log-concave variables, let \( S = \sum_{i=1}^m X_i \) be their sum, and set

\[
\Lambda (t, u) = E \left[ \Psi (X_1, \ldots, X_m) \mid \sum_{i=1}^{m-1} X_i = t , \ X_m = u \right] .
\]

Then

\[
E [\Psi (X_1, \ldots, X_m) \mid S = s] = E [\Lambda (T, X_m) \mid T + X_m = s] ,
\]

where \( T = \sum_{i=1}^{m-1} X_i \). The variable \( T \) has a log-concave density (by preservation of log-concavity by convolution). Hence, by the induction hypothesis at rank 2, it suffices to prove that \( \Lambda \) is coordinatewise non-decreasing. \( \Lambda (t, u) \) is non-decreasing in \( t \) by the induction hypothesis at rank \( m - 1 \). Also \( \Lambda (t, u) \) is non-decreasing in \( u \) since \( \Psi \) is non-decreasing in its last argument. This concludes the proof.

### 3.3. Examples

#### 3.3.1. Bivariate Gaussian

This special case, in which the joint density is log-concave but not independent, is simple but instructive. Suppose that \((X, Y) \sim N_2 (0, \Sigma)\) where

\[
\Sigma = \begin{pmatrix}
\sigma^2 & \rho \sigma \tau \\
\rho \sigma \tau & \tau^2
\end{pmatrix} .
\]
Then
\[ \varphi(x, y) = -\log \phi_2(x, y) = \frac{1}{2(1 - \rho^2)} \left( \frac{x^2}{\sigma^2} - 2\rho \frac{x y}{\sigma \tau} + \frac{y^2}{\tau^2} \right) + \text{constant}, \]
so that
\[ \frac{\partial}{\partial x} \varphi(x, y) = \frac{1}{\sigma^2(1 - \rho^2)} \left( x - \frac{\rho \sigma y}{\tau} \right), \]
\[ \frac{\partial}{\partial y} \varphi(x, y) = \frac{1}{\tau^2(1 - \rho^2)} \left( y - \frac{\rho \tau x}{\sigma} \right), \]
and
\[ \partial_{11}^2 \varphi(x, y) = \frac{1}{\sigma^2(1 - \rho^2)}, \quad \partial_{22}^2 \varphi(x, y) = \frac{1}{\tau^2(1 - \rho^2)}, \]
\[ \partial_{12}^2 \varphi(x, y) = \partial_{21}^2 \varphi(x, y) = -\frac{\rho}{\sigma \tau (1 - \rho^2)}. \]
Thus we have
\[ \partial_{11}^2 \varphi(x, y) - \partial_{21}^2 \varphi(x, y) = \frac{1}{\sigma(1 - \rho^2)} \left( \frac{1}{\sigma} + \frac{\rho}{\tau} \right), \]
\[ \partial_{22}^2 \varphi(x, y) - \partial_{21}^2 \varphi(x, y) = \frac{1}{\tau(1 - \rho^2)} \left( \frac{1}{\tau} + \frac{\rho}{\sigma} \right), \]
where the right hand sides of the last two displays are nonnegative if and only if \(-\rho \leq (\tau / \sigma) \wedge (\sigma / \tau)\) or, equivalently, if and only if \(\rho \geq -\{((\tau / \sigma) \wedge (\sigma / \tau))\}.\) It follows from Corollary 3.1 that if \(\rho \geq -\{((\tau / \sigma) \wedge (\sigma / \tau))\},\) then \(E[\Psi(X, Y) | X + Y = z]\) is a monotone function of \(z\) for any function \(\Psi\) which is monotone in each coordinate, for example \(\Psi_1(x, y) \equiv 1\{x \geq 0, y \geq 0\}\) or \(\Psi_2(x, y) \equiv ax + by\) with \(a, b \geq 0.\)

In fact, in this example we have \((X | X + Y = z) \sim N(\mu z, A^2)\) and, by symmetry, \((Y | X + Y = z) \sim N(\nu z, A^2)\) where
\[ \frac{1}{A^2} = \frac{1}{1 - \rho^2} \left\{ \frac{1}{\sigma^2} + \frac{2\rho}{\sigma \tau} + \frac{1}{\tau^2} \right\}, \]
\[ \mu = \frac{A^2}{1 - \rho^2} \left( 1 + \frac{\rho \tau}{\sigma} \right) \frac{1}{\tau^2}, \]
\[ \nu = \frac{A^2}{1 - \rho^2} \left( 1 + \frac{\rho \sigma}{\tau} \right) \frac{1}{\sigma^2} = 1 - \mu. \]
(Note that when \(\sigma = \tau = 1\) and \(\rho = 0\) this yields \((X | X + Y = z) \sim N(z/2, 1/2).\))

Now we check the claimed monotonicity for the conditional expectations in the case of \(\Psi_1\) and \(\Psi_2.\) For \(\Psi_1,\) with \(\Phi(z) \equiv P(N(0, 1) \leq z)\) on the right side,
\[ E[\Psi_1(X, Y) | X + Y = z] = P(X \geq 0, z - X \geq 0 | X + Y = z) = P_z(0 \leq X \leq z) \]
\[ = \{ \Phi(z - \mu z / A) - \Phi(-\mu z / A) \} 1\{ z \geq 0 \} \]
\[ = \{ \Phi(\nu z / A) - \Phi(-\mu z / A) \} 1\{ z \geq 0 \}. \]
This is a monotone function of \(z\) if \(\rho \geq -\{\sigma / \tau \wedge \tau / \sigma\}.\) For \(\Psi_2\) we have
\[ E[\Psi_2(X, Y) | X + Y = z] = aE(X | X + Y = z) + bE(Y | X + Y = z) = a\mu z + b\nu z \]
\[ = (a\mu + b\nu)z = \frac{A^2}{1 - \rho^2} \left\{ \frac{a}{\tau^2} \left( 1 + \frac{\rho \tau}{\sigma} \right) + \frac{b}{\sigma^2} \left( 1 + \frac{\rho \sigma}{\tau} \right) \right\} z. \]
This is monotone increasing for any \(a, b \geq 0\) if and only if \(\rho \geq -\{\tau / \sigma \wedge (\sigma / \tau)\},\) just as we concluded above via Corollary 3.1.
3.3.2. Morgenstern copula

(Not log-concave and not independent) Suppose that \((X, Y)\) has density \(c_\theta\) on \([0, 1]^2\) where
\[
c_\theta(x, y) = 1 + \theta(1 - 2x)(1 - 2y), \quad (x, y) \in [0, 1]^2
\]
for \(|\theta| \leq 1\). Then straightforward calculation yields
\[
\frac{\partial^2 \varphi(x, y)}{\partial^2 x} - \frac{\partial^2 \varphi(x, y)}{\partial^2 y} = \frac{4\theta(1 + \theta(1 - 2y)^2)}{[1 + \theta(1 - 2x)(1 - 2y)]^2},
\]
\[
\frac{\partial^2 \varphi(x, y)}{\partial^2 x} - \frac{\partial^2 \varphi(x, y)}{\partial^2 y} = \frac{4\theta(1 + \theta(1 - 2x)^2)}{[1 + \theta(1 - 2x)(1 - 2y)]^2},
\]
and the right sides in the last display are both non-negative if and only if \(\theta \geq 0\). Hence for \(\Psi\) coordinatewise monotone, \(E[\Psi(X, Y)|X + Y = z]\) is monotone in \(z\) if and only if \(\theta \geq 0\).

3.3.3. Frank copula

(Not log-concave and not independent) Suppose that \((X, Y)\) has distribution function \(C_\theta\) on \([0, 1]^2\) where
\[
C_\theta(x, y) = \begin{cases} 
\log \left\{ 1 - \frac{(1 - \theta)^2}{(1 - \theta_y)^2} \right\} / \log \theta, & \theta \neq 1, \ (x, y) \in [0, 1]^2 \\
1, & \theta = 1, \ (x, y) \in [0, 1]^2.
\end{cases}
\]
for \(0 < \theta < \infty\). Then straightforward calculation yields
\[
\frac{\partial^2 \varphi(x, y)}{\partial^2 x} - \frac{\partial^2 \varphi(x, y)}{\partial^2 y} = -\frac{2\theta^2(\theta - 2\theta^2 + \theta^2 y^2) \log \theta^2}{(\theta - \theta^2 - \theta^2 y + \theta^2 y^2)^2},
\]
\[
\frac{\partial^2 \varphi(x, y)}{\partial^2 x} - \frac{\partial^2 \varphi(x, y)}{\partial^2 y} = -\frac{2\theta^2(\theta - 2\theta^2 + \theta^2 x^2) \log \theta^2}{(\theta - \theta^2 - \theta^2 y + \theta^2 y^2)^2},
\]
and the right sides in the last display are both non-negative if and only if \(\theta \in (0, 1]\). Hence for \(\Psi\) coordinatewise monotone, \(E[\Psi(X, Y)|X + Y = z]\) is monotone in \(z\) if and only if \(0 < \theta < 1\).

Note that \(\theta = 1\) corresponds to \((X, Y)\) being independent uniform \((0, 1)\) random variables, and we know that the conditional expectation is monotone by Efron’s theorem in this case.

3.3.4. Clayton-Oakes copula

(Not log-concave and not independent) Suppose that \((X, Y)\) has distribution function \(C_\theta\) on \([0, 1]^2\) where
\[
C_\theta(x, y) = \left\{ x^{-\theta} + y^{-\theta} - 1 \right\}^{-1/\theta}, \quad (x, y) \in (0, 1]^2
\]
for \(0 < \theta < \infty\). Then straightforward calculation yields
\[
\frac{\partial^2 \varphi(x, y)}{\partial^2 x} - \frac{\partial^2 \varphi(x, y)}{\partial^2 y} = \frac{\theta^2 x(1 + 2\theta) y^\theta + \theta y^{1+2\theta} - (1 - \theta - 2\theta^2) x^\theta y^{1+\theta} (1 - y^\theta) - (1 - \theta) x^2 y (1 - y^\theta)^2}{x^2 y (y^\theta + x^\theta (1 - y^\theta))^2},
\]
\[
\frac{\partial^2 \varphi(x, y)}{\partial^2 x} - \frac{\partial^2 \varphi(x, y)}{\partial^2 y} = \frac{\theta^2 y(1 + 2\theta) x^\theta + \theta x^{1+2\theta} - (1 - \theta - 2\theta^2) y^\theta x^{1+\theta} (1 - x^\theta) - (1 + \theta) y^2 x (1 - x^\theta)^2}{y^2 x (x^\theta + y^\theta (1 - x^\theta))^2},
\]
and the right sides in the last displays are both non-negative if \(\theta \in (1/2, 1)\). Hence for \(\Psi\) coordinatewise monotone, \(E[\Psi(X, Y)|X + Y = z]\) is monotone in \(z\) if \(1/2 \leq \theta \leq 1\).
3.4. Monotonicity preservation under measurement error

Now we discuss the statistical application described at the end of Section 1 above in light of Theorem 3.1. Indeed, we are now able to extend the results of Hwang and Stefanski [1994] related to independent log-concave errors. Briefly recall the framework: we are given a triple \((T, X, U)\) of random variables, and consider the monotonicity of \(\mathbb{E}[T|X = x]\) under the assumption that \(\mathbb{E}[T|U = u]\) is monotone. The variable \(Z := X - U\) is interpreted as a measurement error and is essential in the analysis. We assume in the sequel that the triple \((T, X, U)\) has a density \(f_{T, U, X}\) with respect to Lebesgue measure on \(\mathbb{R}^3\), and that \(T\) and \(X\) are conditionally independent given \(U\). Thus

\[
f_{T, U, X}(t, u, x) = f_{T|U}(t|u)f_{X|U}(x|u)f_U(u).
\]

This yields:

\[
\mathbb{E}[T|X = x] = \int \mathbb{E}[T|U = u]f_{U|X}(u|x)du.
\]

(3.11)

Setting \(\Psi(u) = \mathbb{E}[T|U = u]\), formula (3.11) can be rewritten as follows:

\[
\mathbb{E}[T|X = x] = \mathbb{E}[\Psi(U)|U + Z = x].
\]

(3.12)

If \(\Psi\) is nondecreasing, we see by Remark 1.1 that monotonicity of \(\mathbb{E}[T|X = x]\) is thus ensured as soon the conditional quantiles

\[
S_U(u, x) = \mathbb{P}[U > u \mid U + Z = x]
\]

are nondecreasing in \(x\) for any \(u \in \mathbb{R}\). Now, setting \(\varphi = -\log(f_{U, X})\) and using Corollary 3.1, we get that monotonicity of \(\mathbb{E}[T|X = x]\) is ensured if for every \(s_0 \in \mathbb{R}\), Assumption D1 is valid for \(\varphi\) and for all \(x' \in \mathbb{R}\),

\[
(\partial_{22}^2 \varphi - \partial_{12}^2 \varphi)(x', s_0 - x') \geq 0.
\]

(3.13)

As seen in the examples above, this condition is valid for independent log-concave variables \(U\) and \(Z\), but also for possibly dependent variables which may or may not be log-concave.

4. Quantitative estimates in the monotonicity property

In this section, we establish a quantitative strengthening of Efron’s monotonicity property. Recall that we are interested in the function \(I\) of \(s \in \mathbb{R}\), given in (1.1). We thus consider a pair \((X, Y)\) of random variables with density \(h\) on \(\mathbb{R}^2\) with respect to the Lebesgue measure. By setting

\[
S_X(x, s) = \mathbb{P}[X > x \mid X + Y = s] \quad \text{and} \quad S_Y(y, s) = \mathbb{P}[Y > y \mid X + Y = s],
\]

we have seen in Section 1 that the function \(I\) is non-decreasing if for all \((x, y) \in \mathbb{R}^2\), \(S_X(x, s)\) and \(S_Y(y, s)\) are non-decreasing in \(s \in \mathbb{R}\).

Note that if \(h\) is positive and continuous on \(\mathbb{R}^2\) then \(\int h(s - y', y') dy' > 0\) and the function \(f_2\) given by (3.2) is well-defined. In this case, we may write

\[
I(s) = \int_{\mathbb{R}} \Psi(s - y, y) f_2(y; s) dy.
\]

(4.1)

By a change of variable, we may also write

\[
I(s) = \int_{\mathbb{R}} \Psi(x, s - x) f_1(x; s) dx,
\]

where \(f_1\) is given by (3.1). We define the measure \(\mu_1\) by \(d\mu_1(x) = f_1(x; s) dx\).
Theorem 4.1. Assume that the statements of Proposition 1.1, that is Efron’s monotonicity property, hold. Let us take $s_0 \in \mathbb{R}$ and $\varepsilon > 0$, and let $V(s_0) = [s_0 - \varepsilon, s_0 + \varepsilon]$. With the notations above, assume that $h = \exp(-\varphi)$ is positive and coordinatewise differentiable on $\mathbb{R}^2$. Assume also that $\Psi$ is coordinatewise differentiable on $\mathbb{R}^2$. Furthermore, assume that for any $s \in V(s_0)$, $\Psi(\cdot, s) - \varphi$ and $(\partial_1 \varphi)(\cdot, s)$ are absolutely continuous. Assume that for all $(x, y) \in \mathbb{R}^2$, the functions $S_X(x, s)$ and $S_Y(y, s)$ are non-decreasing in $s \in V(s_0)$.

If there exist four integrable functions on $\mathbb{R}$, $A, B, C, D \in L_1(\text{Leb})$ and a positive constant $M$ such that, for all $(x, y) \in V(s_0) \times \mathbb{R}^2$,

$$|\Psi(s - y, y)| \leq M, \quad (4.2)$$

$$|\partial_1 \Psi(s - y, y) h(s - y, y)| \leq A(y), \quad (4.3)$$

$$|\partial_1 \varphi(s - y, y) h(s - y, y)| \leq B(y), \quad (4.4)$$

$$|\partial_2 \varphi(x, s - x) \exp(-\varphi(x, s - x))| \leq C(x) \wedge D(s - x), \quad (4.5)$$

then the function $I$ defined in (1.1) is differentiable at the point $s_0$ and

$$I'(s_0) \geq \left(1 - \sup_{x \in \mathbb{R}} \left\{ \frac{\partial_2 S_Y(s_0 - x, s_0)}{\partial_2 S_X(x, s_0) + \partial_2 S_Y(s_0 - x, s_0)} \right\} \right) \mathbb{E}[\partial_1 \Psi(X, Y) | X + Y = s_0]. \quad (4.6)$$

Proof of Theorem 4.1. Under the assumptions of Theorem 4.1, the function $I$ in (3.2) and (4.1) is well-defined and we have (using notations above),

$$I(s) = \int_{\mathbb{R}} \Psi(s - y, y) f_2(y, s) dy .$$

By differentiating with respect to $s$ (interchanging differentiation and integral signs is allowed by the assumptions (4.2), (4.3) and (4.4)), we get

$$I'(s_0) = \mathbb{E}[\partial_1 \Psi(X, Y) | X + Y = s_0] - \text{Cov}[\Psi(X, Y), (\partial_1 \varphi)(X, Y) | X + Y = s_0]. \quad (4.7)$$

Notice that by Assumption (4.5), kernel representations hold for $\partial_2 S_X$ and $S_Y$. Now, by Corollary 2.4, Theorem 3.1 and coordinatewise monotonicity of $\Psi$, we have

$$\text{Cov}[\Psi(X, Y), (\partial_1 \varphi)(X, Y) | X + Y = s_0] \quad (4.8)$$

$$= \int \int (\partial_1 \Psi - \partial_2 \Psi)(x, s_0 - x) K_{\mu_0}(x, \tilde{x}) (\partial^2_{11} \varphi - \partial^2_{12} \varphi)(\tilde{x}, s_0 - \tilde{x}) dxd\tilde{x} \quad (4.9)$$

$$= \int (\partial_1 \Psi - \partial_2 \Psi)(x, s_0 - x) (\partial_2 S_Y)(s_0 - x, s_0) dx \quad (4.10)$$

$$\leq \int (\partial_1 \Psi)(x, s_0 - x) (\partial_2 S_Y)(s_0 - x, s_0) dx \quad (4.11)$$

$$\leq \sup_{x \in \mathbb{R}} \left\{ \frac{(\partial_2 S_Y)(s_0 - x, s_0)}{f_1(x, s_0)} \right\} \int (\partial_1 \Psi)(u, s_0 - u) f_1(u, s_0) du$$

$$= \sup_{x \in \mathbb{R}} \left\{ \frac{(\partial_2 S_Y)(s_0 - x, s_0)}{(\partial_2 S_X)(x, s_0) + (\partial_2 S_Y)(s_0 - x, s_0)} \right\} \mathbb{E}[\partial_1 \Psi(X, Y) | X + Y = s_0]. \quad (4.12)$$

Indeed, equality (4.9) comes from identity (2.8), then we used identity (3.5) to obtain (4.10). Inequality (4.11) is derived using coordinatewise monotonicity of $\Psi$ together with monotonicity of $S_Y(y, s_0)$. Finally, equality (4.12) follows from identity (3.6). \qed
Note that by symmetry between $X$ and $Y$, if the right integrability conditions are satisfied, then we could also get

$$I'(s_0) \geq \left(1 - \sup_{x \in \mathbb{R}} \frac{\partial_2 S_X(x, s_0)}{\partial_2 S_X(x, s_0) + \partial_2 S_Y(s_0 - x, s_0)}\right) \mathbb{E}[\partial_2 \Psi(X, Y) | X + Y = s_0]$$

or, mixing the latter lower bound with the one of Theorem 4.1,

$$I'(s_0) \geq \left(1 - \sup_{x \in \mathbb{R}} \frac{\partial_2 S_Y(s_0 - x, s_0)}{\partial_2 S_X(x, s_0) + \partial_2 S_Y(s_0 - x, s_0)}\right) \mathbb{E}[\partial_1 \Psi(X, Y) | X + Y = s_0]$$

$$\vee \left(1 - \sup_{x \in \mathbb{R}} \frac{\partial_2 S_X(x, s_0)}{\partial_2 S_X(x, s_0) + \partial_2 S_Y(s_0 - x, s_0)}\right) \mathbb{E}[\partial_2 \Psi(X, Y) | X + Y = s_0].$$

Let us now return to the statistical application discussed at the end of the introduction and further investigated in Subsection 3.4 above. Using the notation of Subsection 3.4, we are now able to give a lower bound on the derivative of the regression function $\mathbb{E}[T | X = x]$, relative to the derivative of $\Psi(u) = \mathbb{E}[T | U = u]$. Indeed, by setting $h = f_U, X = \exp(-\varphi)$, we find from formula (3.12) and Theorem 4.1 that, under the required integrability conditions we have

$$\frac{d}{dx}(\mathbb{E}[T | X = x]) \geq \left(1 - \sup_{u \in \mathbb{R}} \frac{\partial_2 S_Z(x - u, x)}{\partial_2 S_U(u, x) + \partial_2 S_Z(x - u, x)}\right) \mathbb{E}[\Psi'(U) | U + Z = x].$$

5. Proofs and Examples for Section 2

5.1. Proofs of the Covariance Identities

We begin by reviewing several identities in Shorack [2000], section 7.4. Let $(X, Y)$ have a joint distribution function $H$ on $\mathbb{R}^2$ with marginals $F$ and $G$ for $X$ and $Y$ respectively. Let $F^{-1}$ denote the left-continuous inverse of $F$. Thus if $\xi \sim \text{Uniform}(0, 1)$, $X \equiv F^{-1}(\xi)$ has distribution function $F$. Then we can write

$$X = \int_{(0,1)} F^{-1}(t) d\mathbb{1}_{[\xi \leq t]} = F^{-1}(\xi), \quad \text{and} \quad X = \int_{\mathbb{R}} xd\mathbb{1}_{[X \leq x]}.$$  

Similarly if the mean $\mu$ of $X$ exists, then

$$\mu = \int_{(0,1)} F^{-1}(t) dt, \quad \text{and} \quad \nu = \int_{\mathbb{R}} x dF(x).$$

By taking the differences in these identities we find that

$$X - \mu = \int_{(0,1)} F^{-1}(t) d(\mathbb{1}_{[\xi \leq t]} - t) = -\int_{(0,1)} (1_{[\xi \leq t]} - t) dF^{-1}(t), \quad \text{and}$$

$$X - \mu = \int_{\mathbb{R}} xd(\mathbb{1}_{[X \leq x]} - F(x)) = -\int_{\mathbb{R}} (1_{[X \leq x]} - F(x)) dx,$$

where the second expressions follow from integration by parts or from Fubini’s theorem. Note that the existence of $\mu$ is used in both of these proofs. Similarly, if $a$ is nondecreasing and left-continuous, with $\mathbb{E}|a(X)| < \infty$, then

$$a(X) = \int_{\mathbb{R}} a(x) d\mathbb{1}_{[X \leq x]}, \quad \mathbb{E}a(X) = \int_{\mathbb{R}} a(x) dF(x),$$

$$a(X) = \int_{\mathbb{R}} a(x) d\mathbb{1}_{[X \leq x]}, \quad \mathbb{E}a(X) = \int_{\mathbb{R}} a(x) dF(x).$$
and hence
\[ a(X) - Ea(X) = \int_{\mathbb{R}} a(x)d(1_{X \leq x}) - F(x) = -\int_{\mathbb{R}} (1_{X \leq x} - F(x))da(x). \]

A similar development for \( b(Y) \) yields
\[ b(Y) - Eb(Y) = \int_{\mathbb{R}} b(y)d(1_{Y \leq y}) - G(y) = -\int_{\mathbb{R}} (1_{Y \leq y} - G(y))db(y). \]

where the second expressions follows from integration by parts or from Fubini’s theorem together with \( \text{Var}(b(Y)) < \infty \).

Using the identities above together with Fubini’s theorem and the assumption \( \text{Var}(a(X)) < \infty \), \( \text{Var}(b(Y)) < \infty \), we obtain the covariance identity (2.1). This is just as in Shorack [2000], page 117, formula (14).

Corollary 2.1 follows immediately from Proposition 2.1 by taking \( a \) and \( b \) to be the identity functions.

**Proof of Corollary 2.2.** Part (a) follows immediately upon noting that when \( Y = X \), \( H(x, y) = F(x \wedge y) \).

Part (b): First note that \( \text{Var}(a(X)) < \infty \) since
\[ \text{Var}(a(X)) \leq Ea^2(X) = E[(a_1(X) - a_2(X))^2] \leq 2[Ea_1^2(X) + Ea_2^2(X)] \leq \infty. \]

Similarly, \( \text{Var}(b(X)) < \infty \), and \( |\text{Cov}(a_1(X), a_2(X))| < \infty \) by the Cauchy-Schwarz inequality. Then \( \text{Cov}(a_1(X), a_2(X)) \) is given by polarization:
\[ \text{Cov}(a_1(X), a_2(X)) = \frac{1}{4} \{\text{Var}(a_1(X) + a_2(X)) - \text{Var}(a_1(X) - a_2(X))\}. \]

Thus we have, by using the variance identity resulting from (2.2) with \( a = b \), the covariance identity (2.2), and the symmetry of \( F(x \wedge y) - F(x)F(y) \) in \( x \) and \( y \),
\[
\text{Var}(a(X)) \\
= \text{Var}(a_1(X) - a_2(X)) \\
= \text{Var}(a_1(X)) - 2\text{Cov}(a_1(X), a_2(X)) + \text{Var}(a_2(X)) \\
= \int_{\mathbb{R}^2} \{F(x \wedge y) - F(x)F(y)\}da_1(x)da_1(y) - \int_{\mathbb{R}^2} \{F(x \wedge y) - F(x)F(y)\}da_1(x)da_2(y) \\
- \int_{\mathbb{R}^2} \{F(x \wedge y) - F(x)F(y)\}da_1(y)da_2(x) + \int_{\mathbb{R}^2} \{F(x \wedge y) - F(x)F(y)\}da_2(x)da_2(y) \\
= \int_{\mathbb{R}^2} \{F(x \wedge y) - F(x)F(y)\}d(a_1 - a_2)(x)d(a_1 - a_2)(y) \\
= \int_{\mathbb{R}^2} \{F(x \wedge y) - F(x)F(y)\}da(x)da(y).
\]

Then (2.2) holds for \( a = a_1 - a_2 \) and \( b = b_1 - b_2 \) by polarization:
\[ \text{Cov}(a(X), b(X)) = \frac{1}{4} \{\text{Var}(a(X) + b(X)) - \text{Var}(a(X) - b(X))\}. \]

\[ \square \]
**Remark 5.1.** If $a$ is non-decreasing and right-continuous, then the identity in Corollary 2.2 can fail: for example, if $F(x) = (1 - p)1_{[0, \infty)}(x) + p1_{[1, \infty)}(x)$ so that $X \sim \text{Bernoulli}(p)$, and $a(x) = 1_{[1, \infty)}(x)$, then $a(X) \sim \text{Bernoulli}(p)$ so $\text{Var}(a(X)) = p(1 - p)$ on the left side, but

$$\int \{F(x \land y) - F(x)F(y)\}da(x)da(y) = (F(1) - F(1)F(1)) \cdot 1 \cdot 1 = 0.$$ 

(On the other hand, if $a(x) = 1_{(0, \infty)}(x)$, then $a(X) = 1$ with probability $p$ so that it is again a Bernoulli($p$) random variable and the left side is again $p(1 - p)$, but the right side equals $F(0) - F(0)^2 = (1 - p) - (1 - p)^2 = (1 - p) \cdot p$.)

**Remark 5.2.** Note that both sides in the variance version of (2.2) are infinite if $\text{Var}(a(X)) = \infty$.

**Proof of Corollary 2.3.** Let $a(x) \equiv F(z) - 1_{[x \leq z]}$. First notice that

$$F(z)\int_R b(y)dF(y) - \int_{(\infty, z]} b(y)dF(y) = \int_R b(y)a(y)dF(y).$$

Then $a$ increases from $F(z) - 1$ to $F(z)$ with the only change being a jump upward of 1 at $x = z$. Note that the first equality in (2.4) holds since $E_F a(X) = 0$. Then the second equality in (2.4) follows from (2.2). The equalities in (2.5) follow by noting that

$$a(x) = -(1 - F(z)) - (1 - 1_{[x \leq z]}) = -(1 - F(z)) + 1_{[x > z]}.$$ 

If $b \in L_1(F)$ is absolutely continuous, then

$$\int_R K_{\mu}(z, y)db(y) = \int_R K_{\mu}(z, y)b'(y)dy.$$ 

Moreover, in this case, $b$ has bounded variation, so $b = b_1 - b_2$ with $b_j$ non-decreasing, $j = 1, 2$. Since $b$ is continuous and in $L_1(F)$, we may assume without loss of generality that $b_j$, $j = 1, 2$, are also continuous and in $L_1(F)$ (see for instance Shorack [2000], Exercise 4.1, p.75). Hence, (2.4) is valid in this case and so is (2.6). The proof of (2.7) is similar using (2.5).

**Proof of Corollary 2.4.** Since $a \in L_p(F)$ and $b \in L_q(F)$ for some $p, q \in [1, \infty]$ satisfying $p^{-1} + q^{-1} = 1$, then by Hölder’s inequality

$$|\text{Cov}(a(X), b(X))| \leq \|a(X)\|_p\|b(X)\|_q < \infty$$

where

$$\|a(X)\|_p \equiv \{E|a(X)|^p\}^{1/p} = \left\{ \int |a(x)|^p dF(x) \right\}^{1/p}$$

and similarly for $\|b(X)\|_q$. The fact that $a$ and $b$ are absolutely continuous with $a \in L_p(F)$, $b \in L_q(F)$ implies that $a = a_1 - a_2$ and $b = b_1 - b_2$ where $a_j, b_j$ are non-decreasing, left-continuous and $\|a_j(X)\|_p < \infty$, $\|b_j(X)\|_q < \infty$ for $j = 1, 2$. Hence, Corollary 2.4 follows from Corollary 2.3.

**Proof of Corollary 2.5.** (a) This follows from Corollary 2.3. Indeed, we take $b = \varphi'$ in Corollary 2.3. Then from Corollary 2.3, we get

$$\int_R K_{\mu}(x, y)d\varphi'(y) = -\int_R (1_{(-\infty, x]}(y) - F(x)) \varphi'(y)f(y)dy$$

$$= -\int_{(-\infty, x]} \varphi'(y)f(y)dy + F(x) \int_{-\infty}^{\infty} \varphi'(y)f(y)dy$$

$$= f(x) + F(x) \cdot 0 = f(x)$$
since \( \int_{-\infty}^{x} \varphi'(y)f(y)dy = -f(x) \) and \( \int_{x}^{\infty} \varphi'(y)f(y)dy = f(x) \).

(b): This follows from (a) and the hypothesized absolute continuity.

(c) and (d): It remains only to show that the hypotheses of Corollary 2.5(a) always hold in the log-concave case. But in this case \( \varphi' \) is monotone non-decreasing, so by taking the right-continuous version of \( \varphi' \) and letting \( x_0 = \inf\{y \in \mathbb{R} : \varphi'(y) \geq 0\} \), we have \( \varphi'(x) \geq 0 \) for \( x \geq x_0 \) and \( \varphi'(x) < 0 \) for \( x < x_0 \). It follows that
\[
\int_{\mathbb{R}}|\varphi'(y)|e^{-\varphi(y)}dy = \int_{x_0}^{\infty} \varphi'(y)e^{-\varphi(y)}dy + \int_{-\infty}^{x_0} (-\varphi'(y))e^{-\varphi(y)}dy
\]
\[
= 2e^{-\varphi(x_0)} < \infty.
\]

Thus \( \varphi' \in L_1(F) \) and the hypotheses of Corollary 2.5(a) hold. \( \square \)

**Remark 5.3.** Another way to check finiteness of covariances is to use the following consequence of Hölder’s inequality for the kernel \( K_\mu \) in (2.3): note that if \( 1/p + 1/q = 1 \) with \( p \geq 1 \) we have
\[
F(x \wedge y) - F(x)F(y) = \text{Cov}(1[X \leq x], 1[X \leq y])
\]
\[
= \int \{(1[1 \leq x] - F(x))(1[1 \leq y] - F(y))\}dF(z)
\]
\[
\leq \left\{ \int |1[1 \leq x] - F(x)|^p dF(z) \right\}^{1/p} \cdot \left\{ \int |1[1 \leq y] - F(y)|^q dF(z) \right\}^{1/q}
\]
\[
= \{ |1 - F(x)|^p F(x) + F(x)^p(1 - F(x)) \}^{1/p} \cdot \{ |1 - F(y)|^q F(y) + F(y)^q(1 - F(y)) \}^{1/q}
\]
\[
\leq \{ F(x)(1 - F(x)) \}^{1/p} \cdot \{ F(y)(1 - F(y)) \}^{1/q}.
\]

Thus finiteness of the integrals \( \int_\mathbb{R} \{ F(x)(1 - F(x)) \}^{1/p} da_j(x) \) and \( \int_\mathbb{R} \{ F(x)(1 - F(x)) \}^{1/q} db_j(x) \) for \( j = 1, 2 \) and any conjugate pair \( (p, q) \) implies that \( |\text{Cov}(a(X), b(X))| < \infty \) follows from Corollary 2.2.

### 5.2. Examples and Counterexamples

We give five examples in connection with the formulas (2.9) and (2.10) in Corollary 2.5. In the first three examples \( f \) is log-concave, (2.9) follows from (c) and is known from Menz and Otto [2013]. The third and fourth examples give cases in which log-concavity fails, but at least one of (2.9) and (2.10) holds.

**Example 1.** (Gamma densities). Let
\[
f(x) = f_\theta(x) = \frac{x^{\theta-1}}{\Gamma(\theta)} \exp(-x)1_{(0, \infty)}(x)
\]
for \( \theta > 0 \). It is easily seen that the densities \( f_\theta \) are log-concave for \( \theta \geq 1 \) and absolutely continuous for \( \theta > 1 \). The derivative \( \varphi' \) exists everywhere if \( \theta > 1 \). Thus (2.9) holds for \( \theta > 1 \). Furthermore, note that
\[
\varphi(x) = x - (\theta - 1) \log x + \log \Gamma(\theta)
\]
and hence \( \varphi' \) is absolutely continuous for \( \theta > 1 \) with
\[
\varphi'(x) = 1 - \frac{\theta - 1}{x}, \quad \varphi''(x) = \frac{\theta - 1}{x^2}.
\]
while
\[
E|\varphi'(X)| \leq (\theta - 1) \int_0^\infty x^{-x^{\theta-1}} \frac{\exp(-x)}{\Gamma(\theta)} dx = 1 < \infty
\]
if \( \theta > 1 \). Thus by (b) of Corollary 2.5, (2.10) holds for \( \theta > 1 \). When \( \theta = 1 \), \( f(x) = \exp(-x)1_{(0,\infty)}(x) \) is log-concave, but \( \varphi \) is not absolutely continuous, and it can easily be seen that (2.9) fails. When \( \theta \in (0,1) \), \( f_\theta \) is not log-concave and \( \varphi \) is not absolutely continuous. In this case the hypotheses (and conclusions) of Corollary 2.5 fail.

**Example 2.** (Logistic density). Now let \( f \) be the logistic density, \( f(x) = e^{-x}/(1 + e^{-x})^2 \). In this case \( f \) is absolutely continuous and strictly log-concave since \( \varphi(x) = x + 2\log(1 + e^{-x}) \) is convex with \( \varphi''(x) = 2f(x) > 0 \) for all \( x \in \mathbb{R} \) and \( \varphi' \) is bounded. Thus (2.10) holds. This can also be verified by a direct calculation:

\[
\int_{\mathbb{R}} K_\mu(x,y)\varphi''(y)dy = \int_{(\infty,\infty)} F(y)(1 - F(x))^2f(y)dy + \int_{(-\infty,\infty)} F(x)(1 - F(y))^2f(y)dy
\]
\[
= 2(1 - F(x)) \int_{(-\infty,\infty)} F(y)dF(y) + 2F(x) \int_{(-\infty,\infty)} (1 - F(y))dF(y)
\]
\[
= F(x)(1 - F(x)}(F(x) + (1 - F(x))) = F(x)(1 - F(x)) = f(x).
\]

**Example 3.** (Laplace density). For the Laplace density \( \varphi \equiv (Laplace \text{ density}) \). Suppose that \( \varphi \) is not absolutely continuous and strictly log-concave since \( \varphi(x) = x + 2\log(1 + e^{-x}) \) is convex with \( \varphi''(x) = 2f(x) > 0 \) for all \( x \in \mathbb{R} \) and \( \varphi' \) is bounded. Thus (2.10) holds. This can also be verified by a direct calculation:

\[
\int_{\mathbb{R}} K_\mu(x,y)\varphi''(y)dy = \int_{(-\infty,\infty)} F(y)(1 - F(x))^2f(y)dy + \int_{(\infty,\infty)} F(x)(1 - F(y))^2f(y)dy
\]
\[
= 2(1 - F(x)) \int_{(-\infty,\infty)} F(y)dF(y) + 2F(x) \int_{(-\infty,\infty)} (1 - F(y))dF(y)
\]
\[
= F(x)(1 - F(x)}(F(x) + (1 - F(x))) = F(x)(1 - F(x)) = f(x).
\]

**Example 4.** (Cauchy). Suppose that \( f \) is the Cauchy density given by

\[
f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}.
\]

Then

\[
F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x),
\]

\( f \) is absolutely continuous, \( \varphi(x) = \log x + \log(1 + x^2) \), and \( \varphi' \) is absolutely continuous with

\[
\varphi'(x) = \frac{2x}{1 + x^2}, \quad \varphi''(x) = \frac{2(1 - x^2)}{(1 + x^2)^2}.
\]

It follows from Corollary 2.5 (b) that the identity (2.10) holds. This can also be seen by direct calculation as follows:

\[
\int_{\mathbb{R}} K_\mu(x,y)\varphi''(y)dy = (1 - F(x)) \int_{(-\infty,\infty)} F(y)\frac{2(1 - y^2)}{(1 + y^2)^2}dy + F(x) \int_{(\infty,\infty)} (1 - F(y))\frac{2(1 - y^2)}{(1 + y^2)^2}dy
\]
\[
= \frac{2(1 - F(x))(1 + \pi x + 2x \arctan(x)) + 2F(x)(1 - \pi x + 2x \arctan(x))}{2\pi(1 + x^2)}
\]
\[
= \frac{(1 - F(x))(1 + \pi x + 2x \arctan(x)) + F(x)(1 - \pi x + 2x \arctan(x))}{\pi(1 + x^2)}
\]
\[
= \frac{1 + 2\pi x(F(x)(1 - F(x)) - F(x)(1 - F(x)))}{\pi(1 + x^2)} = \frac{1}{\pi(1 + x^2)} = f(x).
\]
Example 5. (Bridge distribution; Wang and Louis [2003]). Suppose that $X \sim f \equiv f_\theta$ where, for $\theta \in (0, 1)$,

$$f_\theta(x) = \frac{\sin(\pi \theta)}{2\pi \cosh(\theta x) + \cos(\pi \theta)}.$$ 

These densities are log-concave for $\theta \in (0, 1/2]$, but log-concavity fails for $\theta \in (1/2, 1)$. They are all absolutely continuous. Here with $\varphi_\theta(x) \equiv -\log f_\theta(x)$ we have

$$\varphi'_\theta(x) = \frac{\theta \sinh(\theta x)}{\cos(\pi \theta) + \cosh(\theta x)}$$

and

$$\varphi''_\theta(x) = \frac{\theta^2 (1 + \cos(\pi \theta) \cosh(\theta x))}{(\cos(\pi \theta) + \cosh(\theta x))^2}.$$ 

Note that $\varphi'_\theta$ is bounded, and hence $\varphi'_\theta \in L_1(F_\theta)$. It follows from Corollary 2.5 (b) that (2.10) holds.

Acknowledgements:

We owe thanks to two referees for catching a number of errors and for helpful suggestions concerning the organization and exposition.

References

Benjamini, Y. and Heller, R. (2008). Screening for partial conjunction hypotheses. Biometrics 64 1215–1222. MR2522270

Berk, R. H. (1978). Some monotonicity properties of symmetric Pólya densities and their exponential families. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 42 303–307. MR0494617

Block, H. W., Savits, T. H. and Shaked, M. (1985). A concept of negative dependence using stochastic ordering. Statist. Probab. Lett. 3 81–86. MR792794

Bobkov, S. G., Götze, F. and Houdré, C. (2001). On Gaussian and Bernoulli covariance representations. Bernoulli 7 439–451. MR1836739 (2002g:60038)

Boland, P. J., Hollander, M., Joag-Dev, K. and Kochar, S. (1996). Bivariate dependence properties of order statistics. J. Multivariate Anal. 56 75–89. MR1380182

Brascamp, H. J. and Lieb, E. H. (1976). On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Functional Analysis 22 366–389. MR0450480 (56 #58774)

Carroll, R. J., Ruppert, D., Stefanski, L. A. and Crainiceanu, C. M. (2006). Measurement error in nonlinear models, second ed. Monographs on Statistics and Applied Probability 105. Chapman & Hall/CRC, Boca Raton, FL A modern perspective. MR2243417

Cohen, A. and Sackrowitz, H. B. (1987). Unbiasedness of tests for homogeneity. Ann. Statist. 15 805–816. MR888441

Cohen, A. and Sackrowitz, H. B. (1990). Unbiased tests for some one-sided testing problems. Canad. J. Statist. 18 337–346. MR1105643

Daduna, H. and Szekli, R. (2004). On the correlation structure of closed queueing networks. Stoch. Models 20 1–29. MR2036293

Denuit, M. and Dhaene, J. (2012). Convex order and comonotonic conditional mean risk sharing. Insurance Math. Econom. 51 265–270. MR2926463
Dubhashi, D. and Häggström, O. (2008). A note on conditioning and stochastic domination for order statistics. J. Appl. Probab. 45 575–579. MR2426853

Ederer, F. (2010). Feedback and motivation in dynamic tournaments. Journal of Economics & Management Strategy 19 733–769.

Efron, B. (1965). Increasing properties of Pólya frequency functions. Ann. Math. Statist. 36 272–279. MR0171335 (30 #1566)

Fill, J. A. (1988). Bounds on the coarseness of random sums. Ann. Probab. 16 1644–1664. MR958208

Gleser, L. J. (1990). Improvements of the naive approach to estimation in nonlinear errors-in-variables regression models. In Statistical analysis of measurement error models and applications (Arcata, CA, 1989). Contemp. Math. 112 99–114. Amer. Math. Soc., Providence, RI. MR1087101

Goldschmidt, C., Martin, J. B. and Spanò, D. (2008). Fragmenting random permutations. Electron. Commun. Probab. 13 461–474. MR2430713

Gross, J. L., Mansour, T., Tucker, T. W. and Wang, D. G. L. (2015). Log-concavity of combinations of sequences and applications to genus distributions. SIAM J. Discrete Math. 29 1002–1029. MR335766

Heller, R., Chatterjee, N., Krieger, A. and Shi, J. (2016). Post-selection Inference Following Aggregate Level Hypothesis Testing in Large Scale Genomic Data. bioRxiv 058404.

Hoeffding, W. (1994). The collected works of Wassily Hoeffding. Springer Series in Statistics: Perspectives in Statistics. Springer-Verlag, New York Edited and with a preface by N. I. Fisher and P. K. Sen. MR1307621

Höfling, D. (1940). Masztabinvariente Korrelationstheorie. Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin 5 181–233. MR0004426

Houdré, C. (2002). Remarks on deviation inequalities for functions of infinitely divisible random vectors. Ann. Probab. 30 1223–1237. MR1920106

Houdré, C. and Marchal, P. (2004). On the concentration of measure phenomenon for stable and related random vectors. Ann. Probab. 32 1496–1508. MR2060306

Hu, T. and Hu, J. (1999). Sufficient conditions for negative association of random variables. Statist. Probab. Lett. 45 167–173. MR1718435

Hwang, G. T. and Stefanski, L. A. (1994). Monotonicity of regression functions in structural measurement error models. Statist. Probab. Lett. 20 113–116. MR1293287

Joag-Dev, K. and Proschan, F. (1983). Negative association of random variables, with applications. Ann. Statist. 11 286–295. MR684886

Johnson, O. (2007). Log-concavity and the maximum entropy property of the Poisson distribution. Stochastic Process. Appl. 117 791–802. MR2327839 (2008k:62012)

Johnson, O., Kontoyiannis, I. and Madiman, M. (2013). Log-concavity, ultra-log-concavity, and a maximum entropy property of discrete compound Poisson measures. Discrete Appl. Math. 161 1232–1250. MR3030616

Ledoux, M. (2001). The concentration of measure phenomenon. Mathematical Surveys and Monographs 89. American Mathematical Society, Providence, RI. MR1849347 (2003k:28019)

Liggett, T. M. (2000). Monotonicity of conditional distributions and growth models on trees. Ann. Probab. 28 1645–1665. MR1813837

Ma, C. (1999). On the posterior distribution of a location parameter from a strongly unimodal distribution. Statist. Probab. Lett. 42 33–37. MR1671757

Masuda, Y. (1995). Exploiting partial information in queueing systems. Oper. Res. 43 530–536. MR1337465

Menz, G. and Otto, F. (2013). Uniform logarithmic Sobolev inequalities for conservative spin systems with super-quadratic single-site potential. Annals of Probability 41 2182-2224.

Pemantle, R. (2000). Towards a theory of negative dependence. J. Math. Phys. 41 1371–1390. Probabilistic techniques in equilibrium and nonequilibrium statistical physics. MR1757964

Pestien, V. and Ramakrishnan, S. (2002). Monotonicity and asymptotic queue-length distribution in
discrete-time networks. *Queueing Syst.* **40** 313–331. MR1897715

Prékopa, A. (1971). Logarithmic concave measures with application to stochastic programming. *Acta Sci. Math. (Szeged)* **32** 301–316. MR0315079 (47 #3628)

Rinott, Y. and Samuel-Cahn, E. (1991). Orderings of optimal stopping values and prophet inequalities for certain multivariate distributions. *J. Multivariate Anal.* **37** 104–114. MR1097308

Saumard, A. and Wellner, J. A. (2014). Log-concavity and strong log-concavity: A review. *Statist. Surv.* **8** 45–114.

Saumard, A. and Wellner, J. A. (2017). On the Isoperimetric constant, covariance inequalities and $L_p$-Poincaré inequalities in dimension one. Technical Report, Department of Statistics, University of Washington. arXiv:1711.00668.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic orders. Springer Series in Statistics.* Springer, New York. MR2265633 (2008g:60005)

Shanthikumar, J. G. (1987a). On stochastic comparison of random vectors. *J. Appl. Probab.* **24** 123–136.

Shanthikumar, J. G. (1987b). Stochastic majorization of random variables with proportional equilibrium rates. *Adv. in Appl. Probab.* **19** 854–872. MR914596

Shanthikumar, J. G. and Yao, D. D. (1986). The effect of increasing service rates in a closed queueing network. *Journal of Applied Probability* **23** 474–483.

Shanthikumar, J. G. and Yao, D. D. (1987). Stochastic monotonicity of the queue lengths in closed queueing networks. *Oper. Res.* **35** 583–588. MR924950

Shorack, G. R. (2000). *Probability for Statisticians. Springer Series in Statistics.* Springer-Verlag, New York. MR1762415 (2001d:60002)

Spiegelman, C. H. (1986). Two pitfalls of using standard regression diagnostics when both $X$ and $Y$ have measurement error. *Amer. Statist.* **40** 245–248. MR857147

Stefanski, L. A. (1992). Monotone likelihood ratio of a faulty-inspection distribution. *The American Statistician* **46** 110–114.

Stefanski, L. A. and Carroll, R. J. (1990). Score tests in generalized linear measurement error models. *J. Roy. Statist. Soc. Ser. B* **52** 345–359. MR1064421

Stefanski, L. A. and Carroll, R. J. (1991). Deconvolution-based score tests in measurement error models. *Ann. Statist.* **19** 249–259. MR1091848

Wang, Y. (2012). Capacity investment under responsive pricing: Implications of market entry choice. *Decision Sciences* **43** 107–140.

Wang, Z. and Louis, T. A. (2003). Matching conditional and marginal shapes in binary random intercept models using a bridge distribution function. *Biometrika* **90** 765–776. MR2024756 (2004j:62135)

Wellner, J. A. (2013). Strong log-concavity is preserved by convolution. In *High dimensional probability VI: the Banff volume. Progress in Probability* **66** 95–102. Birkhauser, Basel.

Zhuang, W., Yao, J. and Hu, T. (2010). Conditional ordering of order statistics. *J. Multivariate Anal.* **101** 640–644. MR2575410