The $L^2$ geometry of spaces of harmonic maps $S^2 \to S^2$ and $\mathbb{R}P^2 \to \mathbb{R}P^2$

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Abstract

Harmonic maps from $S^2$ to $S^2$ are all weakly conformal, and so are represented by rational maps. This paper presents a study of the $L^2$ metric $\gamma$ on $M_n$, the space of degree $n$ harmonic maps $S^2 \to S^2$, or equivalently, the space of rational maps of degree $n$. It is proved that $\gamma$ is Kähler with respect to a certain natural complex structure on $M_n$. The case $n = 1$ is considered in detail: explicit formulae for $\gamma$ and its holomorphic sectional, Ricci and scalar curvatures are obtained, it is shown that the space has finite volume and diameter and codimension 2 boundary at infinity, and a certain class of Hamiltonian flows on $M_1$ is analyzed. It is proved that $\tilde{M}_n$, the space of absolute degree $n$ (an odd positive integer) harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$, is a totally geodesic Lagrangian submanifold of $M_n$, and that for all $n \geq 3$, $\tilde{M}_n$ is geodesically incomplete. Possible generalizations and the relevance of these results to theoretical physics are briefly discussed.
1 Introduction

In theoretical physics, one often regards harmonic maps \((M, g) \rightarrow (N, h)\), from a Riemannian manifold of dimension 2, as static solutions of a so-called nonlinear \(\sigma\)-model on spacetime \((M \times \mathbb{R}, \eta)\), where \(\eta = dt^2 - g\) is the Lorentzian pseudometric. Those harmonic maps which minimize energy within their homotopy class are usually called “lumps” in this context, because generically their energy density is localized in lump-like structures distributed over \(M\). In many cases of interest, the homotopy classes of maps \(\phi : M \rightarrow N\) are labelled by the topological degree of \(\phi\), and the moduli space of static degree \(n\) lumps, \(M_n\), is a smooth, finite-dimensional manifold. There is a natural Riemannian metric on \(M_n\), namely the \(L^2\) metric, which assigns to each pair of tangent vectors \(X, Y \in T_\phi M_n \subset \Gamma(\phi^*TN)\) the inner product

\[
\gamma(X, Y) = \int_M d\mu_g h_{\phi}(X, Y),
\]

where \(d\mu_g\) denotes the area measure on \((M, g)\). The physical interpretation of this metric is that it is the restriction to \(M_n\) of the symmetric bilinear form defined by the kinetic energy functional of the parent \(\sigma\)-model. Note that, unlike the harmonic map energy, the kinetic energy (and hence \(\gamma\)) depends on \(g\), not just the conformal class of \(g\).

This paper presents a study of this metric in the cases \(M = N = S^2\) and \(M = N = \mathbb{R}P^2\) with their canonical metrics. These cases are convenient because one has complete, explicit parametrizations of the harmonic maps in terms of rational functions. We will focus particularly on the simplest nontrivial case, degree 1 maps \(S^2 \rightarrow S^2\), obtaining a quite thorough understanding of its \(L^2\) geometry. The choice \(N = S^2\) or \(\mathbb{R}P^2\) is rather natural from the standpoint of physics since the order parameters of ferromagnets and nematic liquid crystals are \(S^2\)- and \(\mathbb{R}P^2\)-valued respectively \([36]\). Previously, the algebraic topology of spaces of rational maps has been studied by Segal \([31]\) and Guest, Kozlowski, Murayama and Yamaguchi \([9]\), and the algebraic topology of spaces of harmonic maps \(S^2 \rightarrow S^m\) and \(S^2 \rightarrow \mathbb{R}P^m\) by Furuta, Guest, Kotani and Ohnita \([4]\). The differential topology of spaces of harmonic maps \(S^2 \rightarrow S^{2m}\) and \(S^2 \rightarrow \mathbb{C}P^m\) has been studied by Bolton and Woodward \([3]\) and Lemaire and Wood \([19]\) respectively. The present paper may be considered complementary to this body of work.

The physical motivation behind this study is that \(\sigma\)-model lumps are in many ways analogous to topological solitons in relativistic gauge theories, such as BPS monopoles and abelian Higgs vortices. In the \(S^2\) case, for example, lumps attain a Bogomol’nyi type topological lower bound on energy within their homotopy class, and consequently satisfy a first order “self-duality” equation (namely, the Cauchy-Riemann equation). Manton conjectured \([22]\) that the slow motion of \(n\) BPS monopoles is well approximated by geodesic flow with respect to the \(L^2\) metric on the \(n\)-monopole moduli space. This conjecture was extended to lumps by Ward \([37]\), and has since been formulated and proved rigorously for monopoles and vortices by Stuart \([34, 35]\). The metric in the case \(n = 2, M = \mathbb{R}^2, N = S^2\) was investigated numerically by Leese \([18]\). So the physical motivation behind the present work is the hope that the \(L^2\) metrics will shed light on slow lump dynamics in the parent \(\sigma\)-model, as the Atiyah-Hitchin \([1]\) and Samols \([32]\) metrics have done for monopole and vortex dynamics. Of course, they remain interesting and natural geometric structures in their own right.

The rest of the paper is structured as follows. Let \(M_n, n \in \mathbb{Z}\), denote the space of
degree $n$ harmonic maps $S^2 \to S^2$. In section 2, we give a simple, concrete proof that $(M_n, \gamma)$ is Kähler with respect to the complex structure induced by a natural open inclusion $M_n \subset \mathbb{C}P^{2n+1}$. This result was previously conjectured (in a rather more general setting) by Ruback [28], who gave a very persuasive formal argument in its favour. In section 3 we show that the Kähler property, along with the isometry group, almost completely determines the $L^2$ metric on $M_1$. Specifically, we show that any Kähler metric on $M_1$ invariant under the isometry group of $\gamma$ is determined by a single function of one variable, rather than 21 functions of 6 variables, as for a generic metric in 6 dimensions. An explicit formula for $\gamma$ is given, and it is shown that $M_1$, although noncompact, has finite volume and diameter. It is shown also that the boundary of $(M_1, \gamma)$ at infinity has codimension 2.

In section 4, the curvature properties of $M_1$ are studied. Explicit formulae for the holomorphic sectional curvatures of a certain unitary frame and for the Ricci and scalar curvatures are derived. It is shown that the holomorphic sectional and scalar curvatures are unbounded above, and conjectured that the Ricci curvature is positive definite. The relevance of these results to quantum lump dynamics is discussed.

It is natural to regard the Kähler form $\Omega$ on $M_n$ as a symplectic form and study the symplectic geometry of $(M_n, \Omega)$. Such symplectic geometry has recently been used to study vortex dynamics in a non-relativistic version of the abelian Higgs model, for example [27]. In section 5, the most general physically meaningful Hamiltonian flow on $(M_1, \Omega)$ is analyzed, and the corresponding 1 lump dynamics described.

In section 6, we address the geometry of spaces of harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$. Eells and Lemaire [6] have shown that, if nonconstant, such maps are classified homotopically by a certain odd positive integer, which we shall call the absolute degree (see section 6 for a definition). In section 6 it is proved that $\tilde{M}_n$, the space of absolute degree $n$ harmonic maps, is naturally identified with a certain totally geodesic Lagrangian submanifold of $M_n$, where the symplectic form is again taken to be the Kähler form. Further, it is shown that for all $n \geq 3$, $\tilde{M}_n$ is geodesically incomplete, while $\tilde{M}_1$ is compact.

Finally, in section 7 we speculate on possible generalizations of this work. As an example, it is shown that the $L^2$ metric on the space of degree 2 elliptic functions is naturally Kähler.

2 The Kähler property of $M_n$

By the Hopf degree theorem [10], homotopy classes of continuous maps $\phi : S^2 \to S^2$ are labelled by their topological degree $n \in \mathbb{Z}$. A well known argument of Lichnerowicz [20] (rediscovered independently by physicists Belavin and Polyakov [2] and Woo [40]) shows that in the degree $n$ class the harmonic map energy satisfies $E[\phi] \geq 2\pi|n|$, with equality if and only if $\phi$ is holomorphic ($n \geq 0$) or antiholomorphic ($n < 0$). Since harmonic maps are by definition local extremals of $E$, (anti)holomorphic maps are harmonic, and furthermore, minimize energy within their class. In fact, all harmonic maps $S^2 \to S^2$ are (anti)holomorphic [41]. Since degree $n$ and $-n$ maps are trivially related by a change of orientation (on domain or codomain), we may, and henceforth will, assume $n \geq 0$ without loss of generality.

Introducing complex stereographic coordinates $z, W$ on domain and codomain, the gen-
eral degree $n$ harmonic map is
\[
W(z) = \frac{a_1 + a_2 z + \cdots + a_{n+1} z^n}{a_{n+2} + a_{n+3} z + \cdots + a_{2n+2} z^n}
\] (2.1)
where $a_i \in \mathbb{C}$ are constants, $a_{n+1}$ and $a_{2n+2}$ do not both vanish, and the numerator and denominator share no common roots. So $M_n$ is the space of degree $n$ rational maps. Clearly any point $(\xi a_1, \ldots, \xi a_{2n+2}) \in \mathbb{C}^{2n+2}, \xi \in \mathbb{C} \setminus \{0\}$, determines the same rational map as $(a_1, \ldots, a_{2n+2})$, so one may identify each rational map with a point in $\mathbb{C}P^{2n+1}$. This gives a natural open inclusion $M_n \subset \mathbb{C}P^{2n+1}$ (not an identification, since the “no common roots” condition removes a complex codimension 1 algebraic variety from $\mathbb{C}P^{2n+1}$) which we use to equip $M_n$ with a topology and complex structure. This topology is natural in that it coincides with the relative topology of $M_n$ in $\mathbb{C}0(S^2, S^2)$. The metric of interest does not derive from the inclusion $M_n \subset \mathbb{C}P^{2n+1}$, of course, but rather from definition (1.1). We now establish:

**Theorem 2.1** For all $n > 0$, $(M_n, \gamma)$ is Kähler with respect to the complex structure induced by the open inclusion $M_n \subset \mathbb{C}P^{2n+1}$.

**Proof:** On the open set where $a_{2n+2} \neq 0$, we may introduce complex local coordinates $b^\alpha = a_\alpha/a_{2n+2}, \alpha = 1, 2, \ldots, 2n + 1$. We may always arrange that $a_{2n+2} \neq 0$ by a rotation of the codomain, so it suffices to show that $\gamma$ is Kähler in this coordinate system. Explicitly,
\[
\gamma = \gamma_{\alpha\beta} db^\alpha \overline{db^\beta}
\] (2.2)
where repeated indices are summed over, and
\[
\gamma_{\alpha\beta} = \int_{\mathbb{C}} \frac{dz \overline{d\bar{z}}}{(1 + |z|^2)^2} \frac{1}{(1 + |W|^2)^2} \frac{\partial W}{\partial b^\alpha} \overline{\left( \frac{\partial W}{\partial b^\beta} \right)}
\] (2.3)
\[
W = \frac{b_1 + b_2 \overline{z} + \cdots + b_{n+1} z^n}{b_{n+2} + b_{n+3} \overline{z} + \cdots + z^n}.
\] (2.4)
Note that $\gamma$ is manifestly Hermitian, that is, $\gamma_{\beta\alpha} \equiv \overline{\gamma_{\alpha\beta}}$. Hence, we need only demonstrate that
\[
\frac{\partial \gamma_{\alpha\beta}}{\partial b^\delta} = \frac{\partial \gamma_{\delta\beta}}{\partial b^\alpha}, \quad \frac{\partial \gamma_{\alpha\beta}}{\partial b^\delta} = \frac{\partial \gamma_{\alpha\delta}}{\partial b^\beta}
\] (2.5)
for all $\alpha, \beta, \delta$. In fact (2.3) follow immediately from (2.3) and (2.4) provided one may interchange the order of partial derivative and integral in $\partial \gamma_{\alpha\beta}/\partial b^\delta$. But this is an immediate consequence of the following lemma, whose proof is presented in the appendix:

**Lemma 2.2** Let $X$ be a compact Riemannian manifold, $F : X \times (-\epsilon, \epsilon) \to \mathbb{R}$ be smooth and $f : (-\epsilon, \epsilon) \to \mathbb{R}$ such that
\[
f(x) = \int_X F(\cdot, x).
\]
Then
\[
f'(0) = \int_X F_2(\cdot, 0)
\]
where $F_2 : X \times (-\epsilon, \epsilon) \to \mathbb{R}$ is the partial derivative of $F$ with respect to the second entry.
Applying this to the integrand of \((2.3)\), with \(x\) representing the (real or imaginary part of) any of the coordinates \(b^\alpha\), the result is proved. \(\square\)

Before specializing to the case \(n = 1\), we note two facts about \(M_n\). First, \((M_n,\gamma)\) is geodesically incomplete. This is a special case of a more general result \([29]\). Second, both domain and codomain spheres are isometric under the group of rotations and reflections of \(\mathbb{R}^3\), \(O(3) \cong SO(3) \cup SO(3)\). Here \(SO(3)\) denotes the orientation reversing component. The induced action of \(O(3) \times O(3)\) on the set of continuous maps \(S^2 \to S^2\) decomposes \(O(3) \times O(3)\) into degree preserving and degree reversing components:

\[
O(3) \times O(3) \cong [(SO(3) \times SO(3)) \cup (SO(3) \times SO(3))] \cup [(SO(3) \times SO(3)) \cup (SO(3) \times SO(3))].
\]

(2.6)

The degree preserving subgroup, call it \(G\), acts isometrically on \((M_n,\gamma)\). It is convenient to define \(P : M_n \to M_n\) such that \(P : W(z) \mapsto \overline{W(z)}\). Then \(G \cong SO(3) \times SO(3) \times \mathbb{Z}_2\), where \(\mathbb{Z}_2 = \{\text{Id}, P\}\). We shall denote the identity component of \(G\) by \(G_0\).

\section{The metric on \(M_1\)}

In the case \(n = 1\), the isometric action of \(G_0 \cong SO(3) \times SO(3)\) described above has cohomogeneity 1, that is, generic \(G_0\) orbits have codimension 1. This is most easily seen by identifying \(M_1\) with \(PL(2,\mathbb{C})\). Note that the case \(n = 1\) is special in that degree 1 rational maps are closed under composition, so \(M_1\) has a natural Lie group structure, namely that of the M"{o}bius group \(PL(2,\mathbb{C}) \cong SL(2,\mathbb{C})/\mathbb{Z}_2\). Explicitly, one identifies a rational map

\[
W : z \mapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}
\]

with a projective equivalence class of \(GL(2,\mathbb{C})\) matrices,

\[
[M] = \left\{ \xi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \xi \in \mathbb{C}^\times \right\},
\]

(3.2)

noting that map composition and matrix multiplication correspond under the identification.

Then the \(PU(2) \cong SU(2)/\mathbb{Z}_2 \cong SO(3)\) subgroup of \(PL(2,\mathbb{C})\) consists of rotations of \(S^2\), so in matrix language \(G_0\) acts on \(PL(2,\mathbb{C})\) by left and right \(PU(2)\) matrix multiplication.

A particularly convenient moving coframe for \(PL(2,\mathbb{C})\) is defined as follows. Let \(\tau_a\), \(a = 1, 2, 3\) be the standard Pauli matrices

\[
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(3.3)

Then any \([M] \in PL(2,\mathbb{C})\) has a unique polar decomposition

\[
[M] = [U](\Lambda I_2 + \lambda \cdot \tau)
\]

(3.4)

where \([U] = \{\pm U\} \in PU(2), \lambda \in \mathbb{R}^3, \lambda = |\lambda|, \Lambda = \sqrt{1 + \lambda^2}\) and \(\cdot\) denotes the \(\mathbb{R}^3\) scalar product \([27]\). The moving coframe is \(\{d\lambda_a, \sigma_a : a = 1, 2, 3\}\), where \(\sigma_a\) are the left-invariant one-forms on \(PU(2)\) associated with the basis \(\{i/2 \tau_a : a = 1, 2, 3\}\) for \(su(2) \cong T_{[12]} PU(2)\). So
M₁ ≃ PU(2) × ℝ³ as a manifold (though not as a group). Physically, the lump parametrized by ([U], ☢) should be thought of as located at −ξ ∈ S² (where ξ = λ/λ), with “sharpness” λ and internal orientation [U]. The action of ([L], [R]) ∈ PU(2) × PU(2) ≃ G₀ on M₁ in terms of the polar decomposition is

\[(L, R) : ([U], \lambda) \mapsto ([LR], R\lambda)\]

(3.5)

where \(R \in SO(3)\) is the rotation corresponding to \([R] \in PU(2)\) (explicitly, it has matrix components \(R_{ab} = \frac{1}{2}\text{tr}(\tau_0 R^T \tau_0 R)\)). From this one sees that the G₀ action indeed has cohomogeneity 1, the orbits being level sets of λ. The orbit space M₁/G₀ may be identified with the radial curve \(Γ = \{([I₂], (0, 0, λ)) : λ ≥ 0\}\) of rational maps \(W_λ : z \mapsto μ(λ)z\), where \(μ(λ) = (Λ + λ)^2\). There is one exceptional orbit, namely \(λ = 0\), which has codimension 3.

The main aim of this section is to obtain an explicit formula for \(γ\), by applying the following:

**Proposition 3.1** Let \(τ\) be a G invariant symmetric (0, 2) tensor on M₁ which is Hermitian \((τ(JX, JY) ≡ τ(X, Y))\), and whose J-associated 2-form \(τ\) \((τ(X, Y) := τ(JX, Y))\) is closed. Then there exists a smooth function \(A : [0, \infty) → ℝ\) such that

\[τ = A₁dλ⋅dλ + A₂(λ⋅dλ)^2 + A₃σ⋅σ + A₄(λ⋅σ)^2 + A₅λ⋅(σ×dλ),\]

(3.6)

where

\[A₁ = A(λ), \quad A₂ = \frac{A(λ)}{1 + λ²} + \frac{A'(λ)}{λ}, \quad A₃ = \left(\frac{1 + 2λ²}{4}\right)A(λ),\]

\[A₄ = \left(\frac{1 + λ²}{4λ}\right)A'(λ), \quad A₅ = A(λ),\]

(3.7)

A' denotes the derivative of A, × denotes the ℝ³ vector product and juxtaposition of covectors denotes symmetrized tensor product.

**Proof:** We first show that the most general G₀ invariant symmetric (0, 2) tensor on M₁ is

\[τ = A₁dλ⋅dλ + A₂(λ⋅dλ)^2 + A₃σ⋅σ + A₄(λ⋅σ)^2 + A₅λ⋅(σ×dλ) + A₆σ⋅dλ + A₇(λ⋅dλ)(λ⋅σ),\]

(3.8)

where \(A₁, \ldots, A₇\) are functions of \(λ\) only.

That such a \(τ\) is G₀ invariant follows from the pulled back action of G₀ on our moving coframe:

\[(L, R) : (dλ, σ) \mapsto (Rdλ, Rσ).\]

(3.9)

We may prove that (3.8) is the most general G₀ invariant symmetric (0, 2) tensor possible by means of the representation theory of SO(N). Any such tensor is uniquely determined by the 1-parameter family of symmetric bilinear forms \(τ_λ : V_λ ⊕ V_λ → ℝ\), where \(V_λ = T_{W_λ}M₁\), and each \(τ_λ\) must be invariant under the isotropy group \(H_λ < G₀\) of \(W_λ\). Explicitly,

\[H_λ = \begin{cases}
  \{([\exp(-\frac{λ}{2}ψτₐ]), [\exp(\frac{λ}{2}ψτₐ)]) : ψ ∈ ℝ\} ≅ SO(2) & λ > 0 \\
  \{([U^†], [U]) : [U] ∈ PSU(2)\} ≅ SO(3) & λ = 0.
\]

(3.10)
The induced action of $H_\lambda$ on $V_\lambda^* \otimes V_\lambda^*$ leaves the subspaces of symmetric and antisymmetric bilinear forms invariant, that is, preserves the splitting

$$V_\lambda^* \otimes V_\lambda^* = [V_\lambda^* \otimes V_\lambda^*] + [V_\lambda^* \wedge V_\lambda^*] =: V_\lambda^+ \oplus V_\lambda^-.$$  \hfill (3.11)

One may compute the dimension of the subspace of $V_\lambda^+$ on which $H_\lambda$ acts trivially (i.e. the subspace of $H_\lambda$ invariant symmetric bilinear forms) by counting the number of copies of the trivial representation in the decomposition of $(H_\lambda, V_\lambda^+)$ into irreducible representations, using character orthogonality. Equation (3.8) captures all possibilities if and only if this dimension is 7 for $\lambda > 0$ and 3 for $\lambda = 0$.

Consider first the generic case, $\lambda > 0$, $H_\lambda \cong SO(2)$. The $H_\lambda$ action on $V_\lambda^+$ has matrix representation,

$$R(\psi) = \begin{pmatrix} \cos \psi & \sin \psi & 0 & 0 & 0 & 0 \\ -\sin \psi & \cos \psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \psi & \sin \psi & 0 \\ 0 & 0 & 0 & -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$  \hfill (3.12)

relative to the ordered basis $(\partial/\partial \lambda_1, \ldots, \theta_3)$, where $\{\theta_a\}$ are the left-invariant vector fields dual to $\{\sigma_a\}$. Hence the character $\chi : H_\lambda \to \mathbb{R}$ of this representation is

$$\chi(\psi) = \text{tr} R(\psi) = 2 + 4 \cos \psi.$$  \hfill (3.13)

The character of the induced representation of $SO(2)$ on $V_\lambda^\pm$ is

$$\tilde{\chi}_\pm(\psi) = \frac{1}{2} \{ [\text{tr} R(\psi)]^2 \pm 2 \text{tr}[R(\psi)^2] \} = \begin{cases} 7 + 8 \cos \psi + 6 \cos 2\psi & \text{symmetric} \\ 5 + 8 \cos \psi + 2 \cos 2\psi & \text{antisymmetric.} \end{cases}$$  \hfill (3.14)

We shall make use of the result for $V_\lambda^-$ when analyzing the $J$-associated 2-form $\tilde{\tau}$. Since $SO(N)$ is a compact Lie group, the characters of inequivalent irreducible representations are orthogonal functions on $SO(N)$ with respect to the Haar measure. One may therefore extract the coefficient $a_0^\pm$ of the trivial character ($\chi_0(\psi) = 1$) from the decomposition

$$\overline{\chi}_\pm = \sum_n a_n^\pm \chi_n$$  \hfill (3.15)

of $\overline{\chi}_\pm$ into irreducible representations by taking the character inner product of both sides of (3.15) with $\chi_0$,

$$a_0^\pm \int_{SO(N)} d\mu \chi_0^2 = \int_{SO(N)} d\mu \chi_0 \overline{\chi}_\pm$$  \hfill (3.16)

where $d\mu$ is the Haar measure. For $SO(2)$, $d\mu = d\psi/2\pi$, so

$$a_0^\pm = \int_0^{2\pi} \frac{d\psi}{2\pi} \overline{\chi}_\pm(\psi) = \begin{cases} 7 & \text{symmetric} \\ 5 & \text{antisymmetric,} \end{cases}$$  \hfill (3.17)

in agreement with (3.8).
In the special case $\lambda = 0$, the isotropy group is $H_0 \cong SO(3)$ whose action on $V_\lambda$ has matrix representation

$$ R(\psi, \hat{n}) = \begin{pmatrix} O(\psi, \hat{n}) & 0 \\ 0 & O(\psi, \hat{n}) \end{pmatrix} \tag{3.18} $$

where $(\psi, \hat{n})$ parametrizes the rotation through angle $\psi$ about axis $\hat{n} \in S^2$ and $O(\psi, \hat{n})$ is the associated $SO(3)$ matrix. The character of this representation is

$$ \chi(\psi, \hat{n}) = 2 \text{tr} O(\psi, \hat{n}) = 2(1 + e^{i\psi} + e^{-i\psi}) = 2 + 4 \cos \psi. \tag{3.19} $$

It follows from (3.19), (3.13) and (3.14) that the characters of the induced representations on $V^\pm_\lambda$ are the same trigonometric functions $\tilde{\chi}_\pm(\psi)$ above, independent of $\hat{n}$. Once again, we may extract $a_0^\pm$ using character orthogonality, but now we must integrate over $SO(3)$ using the Haar measure, which is

$$ d\mu = \frac{1}{\pi} \sin^2 \psi \frac{1}{2} d\psi \tag{3.20} $$

after integrating over $\hat{n} \ [12]$. The result is

$$ a_0^\pm = \frac{1}{\pi} \int_0^{2\pi} d\psi \sin^2 \psi \frac{1}{2} \tilde{\chi}_\pm(\psi) = \begin{cases} 
3 \text{ symmetric} \\
1 \text{ antisymmetric}, 
\end{cases} \tag{3.21} $$

which proves the initial claim.

Since $\tau$ is $G$ invariant (not merely $G_0$ invariant), it must also be invariant under the discrete isometry $P$, which in matrix terms is $P: [M] \mapsto [\bar{M}]$ (entrywise complex conjugation). The pulled-back action on the moving coframe is

$$ P^*: (d\lambda, \sigma) \mapsto (d\lambda_1, -d\lambda_2, d\lambda_3, -\sigma_1, \sigma_2, -\sigma_3), \tag{3.22} $$

implying that $A_6 \equiv A_7 \equiv 0$.

It remains to show that the coefficient functions $A_1, \ldots, A_5$ are determined by the single function $A$ as in (3.7). This follows from Hermiticity of $\tau$ and closure of $\hat{\tau}$. Recall that the complex structure on $M_1$ is inherited from the open inclusion $M_1 \subset CP^3$. For example, on the open set where $a_{11} \neq 0$ (equation (3.1)), we may use the inhomogeneous coordinates

$$ b_1 = \frac{a_{12}}{a_{11}}, \quad b_2 = \frac{a_{21}}{a_{11}}, \quad b_3 = \frac{a_{22}}{a_{11}}, \tag{3.23} $$

to define a complex coordinate chart. This chart contains the curve $\Gamma$ we are using to parametrize the orbit space $M_1/G_0$. It is a simple matter to write down the almost complex structure $J$ associated with this complex structure, in terms of the basis $\{\partial/\partial \lambda_a, \theta_a : a = 1, 2, 3\}$ for $V_\lambda$, namely

$$ J: \frac{\partial}{\partial \lambda_1} \mapsto \frac{2}{\Lambda} \left( \theta_1 - \frac{\lambda}{2} \frac{\partial}{\partial \lambda_2} \right), \quad J: \frac{\partial}{\partial \lambda_2} \mapsto \frac{2}{\Lambda} \left( \theta_2 + \frac{\lambda}{2} \frac{\partial}{\partial \lambda_1} \right), \quad J: \frac{\partial}{\partial \lambda_3} \mapsto \frac{2}{\Lambda} \theta_3, $$

$$ J: \theta_1 \mapsto -\frac{1}{2\Lambda} \left( \frac{\partial}{\partial \lambda_1} - 2\lambda \theta_2 \right), \quad J: \theta_2 \mapsto -\frac{1}{2\Lambda} \left( \frac{\partial}{\partial \lambda_2} + 2\lambda \theta_1 \right), \quad J: \theta_3 \mapsto -\frac{\Lambda}{2} \frac{\partial}{\partial \lambda_3}. \tag{3.24} $$

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We emphasize that (3.24) is valid only on tangent spaces based at points on the curve Γ. By $G$ invariance of $\tau$, this will be all the information we need.

Hermiticity of $\tau$, $\tau_J(JX,JY) \equiv \tau_J(X,Y)$ for all $X,Y \in V_\lambda$, produces two nontrivial constraints on the coefficients $A_1, \ldots, A_5$, namely,

$$A_3 \approx \frac{A_1}{4} + \frac{\lambda^2}{2} A_5, \quad A_1 + \lambda^2 A_2 \equiv \frac{4}{1 + \lambda^2} (A_3 + \lambda^2 A_4). \quad (3.25)$$

Let $f \in G$, and denote by the same symbol its action on $M_1$, $f : M_1 \rightarrow M_1$. The 2-form $\hat{\tau}(\cdot,\cdot) = \tau(J\cdot,\cdot)$ is invariant, $f^*\hat{\tau} = \hat{\tau}$, under any holomorphic $f \in G$ since $f^*\tau = \tau$ ($G$-invariance of $\tau$) and $df_W \circ J_W = J_{f(W)} \circ df_W$ (holomorphicity). Similarly, $f^*\hat{\tau} = -\hat{\tau}$ for antiholomorphic $f \in G$. Now each $f \in G_0$ is holomorphic, so $\hat{\tau}$ is $G_0$ invariant. We claim that the most general $G_0$ invariant 2-form on $M_1$ is

$$\hat{\tau} = \hat{A}_1(d\lambda \cdot \sigma - \sigma \cdot d\lambda) + \hat{A}_2((\lambda \cdot d\lambda)(\lambda \cdot \sigma) - (\lambda \cdot \sigma)(\lambda \cdot d\lambda)) + \hat{A}_3\lambda \cdot (\sigma \times \sigma) + \hat{A}_4\lambda \cdot (d\lambda \times d\lambda) + \hat{A}_5(d\lambda \cdot (\lambda \times \sigma) - (\lambda \times \sigma) \cdot d\lambda), \quad (3.26)$$

where $\hat{A}_1, \ldots, \hat{A}_5$ are functions of $\lambda$ only, and juxtaposition of 1-forms indicates *unsymmetrized* tensor product. Clearly, such a 2-form is $G_0$ invariant by (3.23), and is the most general such form possible by (3.17) and (3.21). In fact, since $P : [M] \rightarrow [\hat{M}]$ is antiholomorphic, $P^*\hat{\tau} = -\hat{\tau}$, and we may immediately conclude that $\hat{A}_5 \equiv 0$.

It is a simple matter to match $\hat{\tau}(\cdot,\cdot)$ with $\gamma_\lambda(J\cdot,\cdot)$ on $V_\lambda$ using (3.24), and hence determine $\hat{A}_1, \ldots, \hat{A}_4$ in terms of $A_1, \ldots, A_5$. The result is

$$\hat{A}_1 = \frac{A}{2} A_1, \quad \hat{A}_2 = \frac{A}{2} A_2, \quad \hat{A}_3 = \frac{1}{4\lambda}(A_1 + 4A_3), \quad \hat{A}_4 = \frac{\lambda}{A}(A_5 - A_1). \quad (3.27)$$

Closure of $\hat{\tau}$ then gives extra constraints on the metric coefficients $A_1, \ldots, A_5$. Using the standard exterior differential algebra for the left-invariant 1-forms of $SO(3)$,

$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = \sigma_3 \wedge \sigma_1, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2, \quad (3.28)$$

one finds that at any $W_\lambda \in \Gamma$,

$$d\hat{\tau} = (\hat{A}'_1 - \lambda \hat{A}_2)d\lambda_3 \wedge (d\lambda_1 \wedge \sigma_1 + d\lambda_2 \wedge \sigma_2) + (\hat{A}_3 - \hat{A}_1)(d\lambda_1 \wedge \sigma_2 \wedge \sigma_3 + d\lambda_2 \wedge \sigma_3 \wedge \sigma_1 + d\lambda_3 \wedge \sigma_1 \wedge \sigma_2) + \lambda(\hat{A}'_4 - \lambda \hat{A}_2)d\lambda_3 \wedge \sigma_1 \wedge \sigma_2 + (\lambda \hat{A}'_4 + 3\hat{A}_4)d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3. \quad (3.29)$$

Hence, $d\hat{\tau} = 0$ if and only if

$$\hat{A}_1 = \hat{A}_3, \quad \hat{A}'_1 = \lambda \hat{A}_2, \quad \hat{A}_4 = 0, \quad (3.30)$$

the last of these following from nonsingularity of $\hat{\tau}$ at $\lambda = 0$. Rearranging these using (3.27) and the Hermiticity constraints (3.25), one finds that all the metric coefficients are determined by the single smooth function $A_1 = A(\lambda)$ as in (3.7). \hfill \Box

**Corollary 3.2** The $L^2$ metric on $M_1$ is

$$\gamma = A_1 d\lambda \cdot d\lambda + A_2 (\lambda \cdot d\lambda)^2 + A_3 \sigma \cdot \sigma + A_4 (\lambda \cdot \sigma)^2 + A_5 \lambda \cdot (\sigma \times d\lambda),$$

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where $A_1, \ldots, A_5$ are functions of $\lambda$ only, determined as in (3.3) by the single function

$$A = \frac{4\pi \mu^4 - 4\mu^2 \log \mu - 1}{(\mu^2 - 1)^3},$$

where $\mu = (\sqrt{1 + \lambda^2} + \lambda)^2$.

**Proof:** By theorem 2.1, $\gamma$ is Hermitian and its $J$-associated 2-form (the Kähler form, henceforth denoted $\Omega$, rather than $\hat{\gamma}$) is closed. Furthermore, $\gamma$ is $G$ invariant. Hence proposition 3.1 applies. The formula for $A$ is obtained by computing $\gamma_\lambda(\partial/\partial \lambda_1, \partial/\partial \lambda_1)$ using (1.1).

Given a tensor $\tau$ satisfying the hypotheses of proposition 3.1, it is convenient to define a second coefficient function, $B(\lambda) := \tau_\lambda(\theta_3, \theta_3)$. Of course, $B$ is determined by $A$, according to (3.7):

$$B(\lambda) = A_3 + \lambda^2 A_4 \equiv \frac{1 + 2\lambda^2}{4} A(\lambda) + \frac{\lambda + \lambda^3}{4} A'(\lambda).$$

One finds for $\tau = \gamma$, the $L^2$ metric, that

$$B = \frac{4\pi \mu^2[(\mu^2 + 1) \log \mu - \mu^2 + 1]}{(\mu^2 - 1)^3}.$$

An explicit formula for $\gamma$ has previously appeared in the physics literature [32], although its Kähler property and the resulting interdependence of the coefficient functions was not understood, nor was a rigorous classification of $G$ invariant tensors on $M_1$ performed. The geodesic flow on $(M_1, \gamma)$ has been extensively studied, also in [32], revealing quite complicated lump dynamics. We finish this section by examining the large $\lambda$ behaviour of $\gamma$. Specifically, we will prove that $(M_1, \gamma)$ has finite volume and diameter, and describe its boundary at infinity.

**Theorem 3.3** $(M_1, \gamma)$ has finite volume and diameter.

**Proof:** The volume form is

$$\text{vol} = \frac{1}{2} BA^2 d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3$$

(3.34)

Hence,

$$\text{Vol}(M_1, \gamma) = 4\pi \text{Vol}(SO(3)) \int_0^\infty d\lambda \lambda^2 \frac{\sqrt{1 + \lambda^2}}{2} BA^2$$

$$= \frac{\pi}{16} \text{Vol}(SO(3)) \int_1^\infty \frac{d\mu}{\mu} \left(\mu - \frac{1}{\mu}\right)^2 BA^2$$

$$< c + \pi^3 \text{Vol}(SO(3)) \int_2^\infty d\mu \mu \left(2^4 \log \frac{\mu}{\mu^2}\right) \left(\frac{2^3}{\mu}\right)^2$$

(3.35)

where $c$ is a constant (the volume from $\mu = 1$ to $\mu = 2$). Hence $(M_1, \gamma)$ has finite volume.
One may similarly bound the diameter of \((M_1, \gamma)\),
\[
\text{diam}(M_1, \gamma) := \sup_{W_1, W_2 \in M_1} d(W_1, W_2). \tag{3.36}
\]
By the triangle inequality,
\[
\text{diam}(M_1, \gamma) \leq 2 \sup_{W \in M_1} d(W, \text{Id}). \tag{3.37}
\]
The distance of any map \(W\) from \(\text{Id}\) is bounded above by the sum of the length of the radial curve from \(([U], \lambda)\) to \(([U], 0)\) and the distance in \(SO(3)\) from \([U]\) to \([I]\) with respect to the bi-invariant metric \(A_3(0)\sigma \cdot \sigma\). The latter contribution is bounded independent of \([U]\) by compactness of \(SO(3)\), and the former is, by \(G_0\) invariance, bounded above by the length of the curve \(\Gamma\). But
\[
\text{length}(\Gamma) = \int_0^\infty d\lambda \sqrt{A_1 + \lambda^2 A_2} = \int_1^\infty \frac{d\mu}{\mu} \sqrt{B}
\]
\[
< c + 8\sqrt{\pi} \int_2^\infty \frac{d\mu}{\mu^2} \sqrt{\log \mu} \mu^2 < \infty. \tag{3.38}
\]
Hence \((M_1, \gamma)\) has finite diameter. \(\square\)
For both estimates, the key point is that \(A(\lambda)\) and \(A'(\lambda)\) decay sufficiently rapidly as \(\lambda \to \infty\) to guarantee convergence of the integrals. Note that while every \(G\) invariant Kähler metric on \(M_1\) is determined by a single function \(A(\lambda)\), the converse is false: not every \(A(\lambda)\) defines such a metric since one must also demand that \(\gamma\) be positive definite. This places one nontrivial constraint on \(A\),
\[
\gamma(\frac{\partial}{\partial \lambda_3}, \frac{\partial}{\partial \lambda_3}) > 0 \quad \Rightarrow \quad \frac{A'}{A} > -\frac{1 + 2\lambda^2}{\lambda + \lambda^3}, \tag{3.39}
\]
and one trivial constraint \((A > 0)\), which together bound the decay rate of \(A(\lambda)\) as \(\lambda \to \infty\). Integrating inequality (3.39) yields, for example,
\[
A(\lambda) > \frac{\sqrt{2}A(1)}{\lambda \sqrt{1 + \lambda^2}} \quad \forall \lambda > 1, \tag{3.40}
\]
so the decay of \(A\) cannot be faster than \(O(1/\lambda^2\)). It is interesting to note that the asymptotic behaviour of the \(L^2\) metric saturates this bound, namely \(\lim_{\lambda \to \infty} \lambda \Lambda A = \pi\).

As shown above, the boundary of \((M_1, \gamma)\) at infinity lies at finite proper distance, so the space is geodesically incomplete. One expects, however, that generic geodesics do not escape to infinity, since the boundary has codimension 2. In fact, this boundary may be identified with the base space \(B^4\) of the fibration of generic (that is \(\lambda > 0\)) orbits by circles
\[
SO(3) \times S^2
\]
\[
\pi
\]
\[
B^4
\]
defined as follows: for each \(\tilde{\lambda} \in S^2\), the fibre containing \(\mathcal{R} \in SO(3) \times \{\tilde{\lambda}\}\) is the orbit of \(\mathcal{R}\) under the isotropy group (with respect to the standard \(SO(3)\) action on \(S^2\)) of \(\tilde{\lambda}\). To see
this, identify $G_0/H_\lambda (\lambda > 0)$ with the $G_0$ orbit of $W_\lambda \in \Gamma$, and note that the image of the left invariant frame

$$\{(\theta_a,0),(-\theta_a,\theta_a) : a = 1,2,3\}$$

of $G_0$ maps under the linearized coset projection at the identity to

$$\{\theta_1,\theta_2,\theta_3,-\lambda\frac{\partial}{\partial \lambda_1},\lambda\frac{\partial}{\partial \lambda_2},0\} \subset V_\lambda.$$ (3.43)

All but one of the non-zero image vectors have length bounded away from 0 for $\lambda > 0$. However, $||\theta_3||^2 = B(\lambda) \sim \pi \log \lambda/2\lambda^4 \to 0$ as $\lambda \to \infty$. This is the tangent vector along the fibres of $\pi : SO(3) \times S^2 \to B^4$ defined above, so as $\lambda \to \infty$, the fibres collapse leaving a boundary diffeomorphic to $B^4$.

## 4 Curvature properties

### 4.1 Holomorphic sectional curvature

Recall that the sectional curvature of a plane $P \in \text{Gr}_2(T\mathbb{W}_1)$ is

$$\sigma(X,Y) := \langle R(X,Y)Y,X \rangle,$$ (4.1)

where $X,Y$ are orthonormal and span $P$, $(\cdot,\cdot) = \gamma(\cdot,\cdot)$ and $R$ is the Riemann curvature tensor $[38]$. Recall also that, since $\gamma$ is Hermitian, $\gamma(X,JX) \equiv 0$ and $||JX|| = ||X||$, so one may assign to a line $L \in \text{Gr}_1(T\mathbb{W}_1)$ containing $X$, $||X|| = 1$, the holomorphic sectional curvature

$$\text{Hol}(X) := \sigma(X,JX).$$ (4.2)

In fact, given that $\gamma$ is Kähler, Hol uniquely determines $\sigma$ and hence $R [13]$.

We shall compute the holomorphic sectional curvature of the unitary frame $\{e_a,Je_a : a = 1,2,3\}$ for $V_\lambda$, where

$$e_1 = \frac{1}{\sqrt{A_1}} \frac{\partial}{\partial \lambda_1}, \quad e_2 = \frac{1}{\sqrt{A_1}} \frac{\partial}{\partial \lambda_2}, \quad e_3 = \frac{1}{\sqrt{A_1 + \lambda^2 A_2}} \frac{\partial}{\partial \lambda_3}.$$ (4.3)

Hermiticity implies that $\text{Hol}(X) \equiv \text{Hol}(JX)$, and $G$ invariance implies that $\text{Hol}(e_1) \equiv \text{Hol}(e_2)$, so we shall calculate only $\text{Hol}(e_1)$ and $\text{Hol}(e_3)$. These will vary with basepoint $W_\lambda \in \Gamma$, and hence be functions of $\lambda$.

The simpler of the two is $\text{Hol}(e_3)$:

$$\text{Hol}(e_3) = \frac{4}{(1+\lambda^2)(A_1 + \lambda^2 A_2)^2} \left( \nabla_{\frac{\partial}{\partial \lambda_3}} \nabla_{\theta_3} \theta_3 - \nabla_{\theta_3} \nabla_{\frac{\partial}{\partial \lambda_3}} \theta_3 - \nabla_{\frac{\partial}{\partial \lambda_3}} \theta_3, \frac{\partial}{\partial \lambda_3} \right)$$

$$= \frac{4}{(1+\lambda^2)(A_1 + \lambda^2 A_2)^2} \left\{ \frac{\partial}{\partial \lambda_3} \langle \nabla_{\theta_3} \theta_3, \frac{\partial}{\partial \lambda_3} \rangle - \langle \nabla_{\theta_3} \theta_3, \nabla_{\frac{\partial}{\partial \lambda_3}} \frac{\partial}{\partial \lambda_3} \rangle + ||\nabla_{\frac{\partial}{\partial \lambda_3}} \theta_3||^2 \right\}$$

$$= \frac{1 + \lambda^2}{8B^2} \left\{ \left( \frac{B'}{B} - \frac{\lambda}{1 + \lambda^2} \right) B' - B'' \right\}.$$ (4.4)
To obtain (4.4), we have used metric compatibility and torsionlessness of $\nabla$, left $SO(3)$ invariance of $\gamma$ and the Lie algebra $su(2) \oplus \mathbb{R}^3$, namely,

$$\left[ \frac{\partial}{\partial \lambda_a}, \frac{\partial}{\partial \lambda_b} \right] = \left[ \frac{\partial}{\partial \lambda_a}, \theta_b \right] = 0, \quad [\theta_a, \theta_b] = -\epsilon_{abc} \theta_c.$$  \hspace{1cm} (4.5)

Formula (4.4) may be written in terms of $A$ alone using (3.32), but the result is rather messy.

Due to the more complicated expression for $J_{e_1}$, in comparison with $J_{e_3}$ (see (3.24)), the calculation of $\text{Hol}(e_1)$ is considerably lengthier, though no more technically difficult. We merely record the result, which, unlike $\text{Hol}(e_3)$, simplifies somewhat when expressed purely in terms of $A$:

$$\text{Hol}(e_1) = \frac{1}{A^2 \Lambda^2} \left\{ \frac{\lambda A + \frac{1}{2} \Lambda^2 A'}{(\Lambda^2 + \lambda^2)A + \lambda \Lambda^2 A'} \left( \frac{\lambda A}{\Lambda^2} + A' \right) - \frac{2 + \lambda^2}{1 + \lambda^2} A - \frac{3 + 2 \lambda^2}{2 \lambda} A' \right\}. \hspace{1cm} (4.6)$$

Substituting the formulae (3.31) and (3.33) for $A(\lambda)$ and $B(\lambda)$ into (4.4) and (4.6), one obtains (very complicated) explicit expressions for $\text{Hol}(e_3)$ and $\text{Hol}(e_1)$. Plots of these are presented in figure 1. Note that, although $\text{Hol}(e_1)$ is bounded, $\text{Hol}(e_3)$ is unbounded above. In fact, one finds (using Maple, for example) that

$$\lim_{\lambda \to \infty} \text{Hol}(e_1) = \frac{1}{\pi}, \quad \lim_{\lambda \to \infty} \frac{(\log \lambda)^3}{\lambda^4} \text{Hol}(e_3) = \frac{1}{4\pi}. \hspace{1cm} (4.7)$$

which proves:

**Theorem 4.1** The holomorphic sectional curvature of $(M_1, \gamma)$ is unbounded above. Hence, no isometric compactification of $(M_1, \gamma)$ exists, despite its finite volume and diameter.
4.2 Ricci curvature

Recall that the Ricci curvature \( \rho \) of a Riemannian manifold is the symmetric \((0,2)\) tensor

\[
\rho(X,Y) := \text{tr}(V \mapsto R(V,X)Y)
\]

where \( R \) is the Riemann curvature tensor, as before.

**Proposition 4.2** Let \( \gamma \) be a G invariant Kähler metric on \( M_1 \), determined as in proposition 3.1 by the function \( \tilde{A} \). Then the Ricci curvature of \( (M_1,\gamma) \) is

\[
\rho = \tilde{A}_1 d\lambda \cdot d\lambda + \tilde{A}_2 (\lambda \cdot d\lambda)^2 + \tilde{A}_3 \sigma \cdot \sigma + \tilde{A}_4 (\lambda \cdot \sigma)^2 + \tilde{A}_5 \lambda \cdot (\sigma \times d\lambda).
\]

(4.9)

where \( \tilde{A}_1, \ldots, \tilde{A}_5 \) are functions of \( \lambda \) only, determined as in (3.7) by the single function \( \tilde{A}(\lambda) = -2\lambda(1 + \lambda^2)(A')^2 + (9\lambda^2 + 4)AA' + \lambda(1 + \lambda^2)AA'' + 4A^2\lambda + 4\lambda^2\lambda A + \lambda A' + \lambda^3A' \).

(4.10)

**Proof:** Since the \( G \) action is isometric, \( \rho \) is \( G \) invariant. Furthermore, since \( \gamma \) is Kähler, \( \rho(JX,JY) \equiv \rho(X,Y) \) \[15\], and the associated Ricci form, \( \tilde{\rho} \) is closed \[16\]. Hence, proposition 3.1 applies to \( \rho \) just as it applies to \( \gamma \), and all the coefficient functions are determined by \( \rho \lambda \) according to equation (4.8), which yields formula (4.10). \( \square \)

As with the metric, it is convenient to define the associated coefficient function

\[
\tilde{B}(\lambda) := \rho_\lambda(\theta_3,\theta_3) = \tilde{A}_3 + \lambda^2\tilde{A}_4 = \frac{1 + 2\lambda^2}{4}\tilde{A}(\lambda) + \frac{\lambda + \lambda^3}{4}\tilde{A}'(\lambda).
\]

(4.11)

An explicit formula for the Ricci curvature of the \( L^2 \) metric is obtained by substituting (3.31) into (4.10). Unfortunately, this formula is far too complicated to be instructive. However, it leads us to:

**Conjecture 4.3** The Ricci curvature of the \( L^2 \) metric on \( M_1 \) is positive definite.

In support of this, note that, relative to the ordered basis \( (\partial/\partial\lambda_1,\theta_2,\partial/\partial\lambda_2,\theta_1,\partial/\partial\lambda_3,\theta_3) \), the coefficient matrix of \( \rho_\lambda \) is block diagonal with blocks

\[
\tilde{A} \left( \begin{array}{cc}
\frac{1}{2} & -\frac{1}{4} \\
\frac{1}{2} & 1 + 2\lambda^2
\end{array} \right), \quad \tilde{A} \left( \begin{array}{cc}
\frac{1}{2} & \frac{1}{4} \\
\frac{1}{2} & 1 + 2\lambda^2
\end{array} \right), \quad \tilde{B} \left( \begin{array}{cc}
\frac{4}{1 + \lambda^2} & 0 \\
0 & 1
\end{array} \right),
\]

(4.12)

whence it follows that \( \rho_\lambda \) is positive definite if and only if \( \tilde{A}(\lambda) > 0 \) and \( \tilde{B}(\lambda) > 0 \). Now \( \tilde{A}(0) = 4 \) and \( \tilde{B}(0) = 1 \), so \( \rho \) is certainly positive definite in a neighbourhood of \( \text{Id} \), and

\[
\lim_{\lambda \to \infty} \lambda^2\tilde{A}(\lambda) = 4, \quad \lim_{\lambda \to \infty} (\log \lambda)^2\tilde{B}(\lambda) = \frac{1}{8},
\]

(4.13)

so \( \rho \) is asymptotically positive definite also. Convincing graphical evidence in favour of the conjecture is presented in figure 2, which contains plots of \( \tilde{A} \) and \( \tilde{B} \).
We note in passing that the Einstein field equations for $G$ invariant Kähler metrics,

$$\rho = \frac{\kappa}{6}\gamma$$  \hspace{1cm} (4.14)

reduce to a single second order nonlinear ODE, explicit solutions to which may be constructed in the Ricci flat case. The results will be described in detail elsewhere.

### 4.3 Scalar curvature

While Hol and $\rho$ are not directly relevant to soliton dynamics, the scalar curvature $\kappa$ certainly is, at least in the quantum regime. The standard approach to low energy quantum $n$-soliton dynamics \cite{8} is to assume that the quantum state is well described by a wavefunction on the $n$-soliton moduli space $\psi : M_n \rightarrow \mathbb{C}$ (which receives the usual probabilistic interpretation) subject to a Schrödinger equation of the form

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta_\gamma \psi + V\psi$$ \hspace{1cm} (4.15)

where $\Delta_\gamma$ is the covariant Laplacian on $(M_n, \gamma)$ and $V : M_n \rightarrow \mathbb{R}$ is a potential function. The question of precisely what terms should be included in $V$ is somewhat controversial, and the answer likely varies according to exact context. However, there seems to be general agreement that, following De Witt \cite{5}, one should include (a positive multiple of) $\kappa$ in $V$. For a recent discussion of this subject, specifically in the context of $\sigma$-models, see \cite{23}.

So the relevance of $\kappa$ to quantum lump dynamics, as well as simple geometric curiosity, motivate us to calculate it.

**Proposition 4.4 Let $\gamma$ be a $G$ invariant Kähler metric on $M_1$, determined as in proposition 3.1 by the function $A(\lambda)$. Then the scalar curvature of $(M_1, \gamma)$ is**

$$\kappa = 4\frac{\bar{A}}{A} + 2\frac{\bar{B}}{B},$$ \hspace{1cm} (4.16)
where $\bar{A}$ and $B$ are determined by $A$ as in equations (4.10) and (3.32), and $\bar{B}$ is determined by $\bar{A}$ as in equation (4.14).

Proof: By $G$ invariance, $\kappa$ is a function of $\lambda$ only, so it suffices to compute it at $W_\lambda \in \Gamma$. Making use of the unitary frame $\{e_a, Je_a : a = 1, 2, 3\}$ and recalling that $\rho(JX, JY) \equiv \rho(X, Y)$, one finds

$$\kappa = 2 \sum_{a=1}^{3} \rho(e_a, e_a) = 2 \left[ 2 \frac{\bar{A}_1}{A} + \frac{\bar{A}_1 + \lambda^2 \bar{A}_2}{A_1 + \lambda_2 A_2} \right],$$

in the notation of proposition 4.2. Formula (4.16) follows from applying the relations (3.25), (3.32) and (4.11) to (4.17). $\square$

Corollary 4.5 The scalar curvature of the $L^2$ metric on $M_1$ is unbounded above.

Proof: From equations (3.31) and (3.33) one has the limits

$$\lim_{\lambda \to \infty} \lambda^2 A(\lambda) = \pi, \quad \lim_{\lambda \to \infty} \frac{\lambda^4}{\log \lambda} B(\lambda) = \frac{\pi}{2},$$

which together with (4.13) and proposition 4.4 imply that

$$\lim_{\lambda \to \infty} \frac{(\log \lambda)^3}{\lambda^4} \kappa(\lambda) = \frac{1}{2\pi}. \quad \square$$

Remark 4.6 Numerical evidence suggests that the $L^2$ metric on $M_1$ has strictly positive scalar curvature (see figure 1), as one would expect, given conjecture 4.3.

Since $(M_1, \gamma)$ is noncompact, but of finite volume, the question of what boundary conditions to impose on the quantum wavefunction $\psi$ at $\lambda = \infty$ when seeking bound states is non-trivial. The fact that $\kappa \to \infty$ as $\lambda \to \infty$ supports the imposition of vanishing boundary conditions for all quantum states of finite energy. One would expect the quantum 1-lump energy spectrum to be discrete, therefore.

4.4 The Fubini-Study metric

There is another natural Kähler metric on $M_1$ given by the open inclusion $M_1 \subset \mathbb{C}P^3$, namely the Fubini-Study metric on $\mathbb{C}P^3$. In terms of the local inhomogeneous coordinates $b_1, b_2, b_3$ (3.23) this takes the form

$$\gamma_{FS} = \frac{(1 + \sum |b_a|^2)(1 + \sum d\bar{b_a} \bar{db}_b) - (\sum \bar{b_a} db_a)(\sum b_b \bar{db}_b)}{(1 + \sum |b_a|^2)^2}. \quad (4.20)$$

Proposition 4.7 The Fubini-Study metric on $M_1$ is

$$\gamma_{FS} = A_1 d\lambda \cdot d\lambda + A_2 (\lambda \cdot d\lambda)^2 + A_3 (\sigma \cdot d\sigma)^2 + A_4 (\lambda \cdot \sigma)^2 + A_5 \lambda \cdot (\sigma \times d\lambda),$$

$A_1, \ldots, A_5$ being determined as in (3.3) by the single function

$$A_{FS}(\lambda) = \frac{2\mu(\lambda)}{1 + \mu(\lambda)^2}, \quad (4.21)$$

where $\mu(\lambda) = (\sqrt{1 + \lambda^2} + \lambda)^2$. 

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**Proof:** The isometric action of $PU(4)$ on $(\mathbb{CP}^3, \gamma_{FS})$ obtained by projecting the standard $U(4)$ action on $\mathbb{C}^4$ contains the $G_0$ action on $M_1$ we have been considering. Furthermore, $\gamma_{FS}$ is manifestly invariant under $M \mapsto \overline{M}$ (i.e. $b_a \mapsto \overline{b}_a$) from (4.20). Hence proposition 3.1 applies. It remains to compute $A_{FS}(\lambda) = \gamma_{FS}(\partial/\partial \lambda_1, \partial/\partial \lambda_1)$ at $W_\lambda \in \Gamma$, using (4.20), which is straightforward algebra.

Proposition 4.7 gives us several checks on our curvature calculations. It is known that $(\mathbb{CP}^3, \gamma_{FS})$ has constant holomorphic sectional curvature (i.e. $\text{Hol}(\lambda) = 2\rho_H$) and base point $p$, determined as in proposition 3.1 by $A_{FS}(\lambda)$. Thinking of $M_1$ as the 1-lump moduli space, only $SO(3) \times SO(3)$ invariant Hamiltonians make physical sense, so $H$ must be a function of $\lambda$ only.

**Proposition 5.1** Let $\Omega$ be the Kähler form associated with a $G$ invariant Kähler metric on $M_1$, determined as in proposition 3.1 by $A(\lambda)$, and $H(\lambda)$ be a smooth, $G$ invariant function on $M_1$. The Hamiltonian vector field corresponding to $(\Omega, H)$ is

$$X_H = \frac{2\sqrt{1 + \lambda^2}H'(\lambda)}{(1 + 2\lambda^2)A(\lambda) + (\lambda + \lambda^3)A'(\lambda)} \hat{\lambda} \cdot \theta. \quad (5.2)$$

**Proof:** It is convenient to decompose vector fields relative to the moving frame $\{\partial/\partial \lambda_1, \ldots, \partial/\partial \lambda_3\}$ using the notation

$$Y = Y \cdot \frac{\partial}{\partial \lambda} + \bar{Y} \cdot \theta, \quad (5.3)$$

that is, collecting the coefficients into a pair of $\mathbb{R}^3$-vector valued functions. Recall from the proof of proposition 3.1 that the Kähler form is

$$\Omega = \hat{A}_1(\lambda \cdot \sigma - \sigma \cdot d\lambda) + \hat{A}_2(\lambda \cdot d\lambda) \wedge (\lambda \cdot \sigma) + \hat{A}_1 \lambda \cdot (\sigma \times \sigma), \quad (5.4)$$

so the defining equation for the Hamiltonian vector field $X_H = X \cdot \partial/\partial \lambda + \bar{X} \cdot \theta$ reads

$$\hat{A}_1(\lambda \cdot \bar{X} - \bar{Y} \cdot X) + \hat{A}_2[(\lambda \cdot Y)(\lambda \cdot \bar{X}) - (\lambda \cdot \bar{Y})(\lambda \cdot X)]$$

$$+ \hat{A}_1 \lambda \cdot (\bar{Y} \times \bar{X}) = \frac{H'}{\lambda} \lambda \cdot Y \quad \forall Y$$

$$\Rightarrow \hat{A}_1 X + \hat{A}_2(\lambda \cdot X)\lambda + \hat{A}_1 \lambda \times \bar{X} = 0 \quad (5.5)$$

$$\Rightarrow \hat{A}_1 \bar{X} + \hat{A}_2(\lambda \cdot \bar{X})\lambda - \frac{H'}{\lambda} \lambda = 0. \quad (5.6)$$

5 Hamiltonian flows

The Kähler form $\Omega$ is a closed 2-form, nondegenerate by nondegeneracy of $\gamma$, and hence a natural symplectic form on $M_1$. Associated with any smooth function $H : M_1 \to \mathbb{R}$ there is a Hamiltonian flow, defined as the flow along the smooth vector field $X_H$ defined such that

$$\Omega(Y, X_H) = dH(Y) \quad (5.1)$$

for all vector fields $Y$. Thinking of $M_1$ as the 1-lump moduli space, only $SO(3) \times SO(3)$ invariant Hamiltonians make physical sense, so $H$ must be a function of $\lambda$ only.

**5.1** Let $\Omega$ be the Kähler form associated with a $G$ invariant Kähler metric on $M_1$, determined as in proposition 3.1 by $A(\lambda)$, and $H(\lambda)$ be a smooth, $G$ invariant function on $M_1$. The Hamiltonian vector field corresponding to $(\Omega, H)$ is

$$X_H = \frac{2\sqrt{1 + \lambda^2}H'(\lambda)}{(1 + 2\lambda^2)A(\lambda) + (\lambda + \lambda^3)A'(\lambda)} \hat{\lambda} \cdot \theta. \quad (5.2)$$

**Proof:** It is convenient to decompose vector fields relative to the moving frame $\{\partial/\partial \lambda_1, \ldots, \partial/\partial \lambda_3\}$ using the notation

$$Y = Y \cdot \frac{\partial}{\partial \lambda} + \bar{Y} \cdot \theta, \quad (5.3)$$

that is, collecting the coefficients into a pair of $\mathbb{R}^3$-vector valued functions. Recall from the proof of proposition 3.1 that the Kähler form is

$$\Omega = \hat{A}_1(\lambda \cdot \sigma - \sigma \cdot d\lambda) + \hat{A}_2(\lambda \cdot d\lambda) \wedge (\lambda \cdot \sigma) + \hat{A}_1 \lambda \cdot (\sigma \times \sigma), \quad (5.4)$$

so the defining equation for the Hamiltonian vector field $X_H = X \cdot \partial/\partial \lambda + \bar{X} \cdot \theta$ reads

$$\hat{A}_1(\lambda \cdot \bar{X} - \bar{Y} \cdot X) + \hat{A}_2[(\lambda \cdot Y)(\lambda \cdot \bar{X}) - (\lambda \cdot \bar{Y})(\lambda \cdot X)]$$

$$+ \hat{A}_1 \lambda \cdot (\bar{Y} \times \bar{X}) = \frac{H'}{\lambda} \lambda \cdot Y \quad \forall Y$$

$$\Rightarrow \hat{A}_1 X + \hat{A}_2(\lambda \cdot X)\lambda + \hat{A}_1 \lambda \times \bar{X} = 0 \quad (5.5)$$

$$\Rightarrow \hat{A}_1 \bar{X} + \hat{A}_2(\lambda \cdot \bar{X})\lambda - \frac{H'}{\lambda} \lambda = 0. \quad (5.6)$$
The pair (5.5), (5.6) is easily solved for $X, \tilde{X}$, yielding
\[ X_H = \frac{H'}{A_1 + \lambda^2 A_2} \cdot \theta. \] (5.7)

One now uses (3.7) and (3.27) to rewrite $\hat{A}_1$ and $\hat{A}_2$ in terms of $A$. ✷

Flow along $X_H$ corresponds physically to a lump which maintains constant shape $\lambda$ and position $-\tilde{\lambda}$, while spinning internally at constant speed about its axis. The variation of spin speed and sense with $\lambda$ depends on the specifics of $H(\lambda)$.

6 The space of harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$

We begin by recalling some relevant results of Eells and Lemaire [6]. The homotopy classes of continuous maps $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ fall into distinct families labelled by the induced endomorphism of the fundamental group, $\phi_\ast : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$. Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, there are two families, one for which $\phi_\ast$ is the zero morphism ($\phi$ maps all loops to contractible loops), the other where $\phi_\ast$ is the identity morphism ($\phi$ maps noncontractible loops to noncontractible loops). The zero morphism family contains two classes, one of which is the trivial class. The identity morphism family contains infinitely many classes. Any map in this family lifts to $\tilde{\phi} : S^2 \to S^2$,

\[ S^2 \quad \tilde{\phi} \quad S^2 \]
\[ \pi \quad \quad \quad \pi \]
\[ \mathbb{R}P^2 \quad \phi \quad \mathbb{R}P^2 \] (6.1)

where $\pi$ denotes the covering projection, and the different classes are distinguished by the absolute value of the degree of $\tilde{\phi}$, which may take any odd value. We shall refer to this homotopy invariant as the absolute degree $|\text{deg}|$ of $\phi$.

Turning to harmonic maps, all but one of the homotopy classes described above contain harmonic representatives. Again following [6], if $\phi$ belongs to the zero morphism family, it lifts to a map $\tilde{\phi} : \mathbb{R}P^2 \to S^2$ which is also harmonic since the covering projection $\pi : S^2 \to \mathbb{R}P^2$ is a local isometry. All harmonic maps from $\mathbb{R}P^2$ to $S^2$ are constant, so the nontrivial class has no harmonic representative. The moduli space of harmonic maps in the trivial class is thus $\mathbb{R}P^2$, and the $L^2$ metric on this space is a constant multiple of the canonical metric. If $\tilde{\phi}$ is harmonic and belongs to the identity morphism family, it lifts to a harmonic map $\hat{\phi} : S^2 \to S^2$ (again, because $\pi$ is a local isometry), and the space of these is well understood in terms of rational maps. So the task is to identify those harmonic maps $\hat{\phi} : S^2 \to S^2$ which factor through the quotient in (6.1). Let $p : S^2 \to S^2$ be the antipodal map ($p : z \mapsto -1/z$ in stereographic coordinates). Then $\hat{\phi}$ projects to a well defined map $\phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ if and only if $\tilde{\phi} \circ p = p \circ \hat{\phi}$, or in terms of the associated rational map $W(z)$,

\[ [W(z)]^{-1} = W(\bar{z}^{-1}). \] (6.2)

We now note that given such a rational map, of degree $n > 0$ say, no other degree $n$ map projects to the same $\phi$, although $W(-1/\bar{z})$, which has degree $-n$, does. So we may identify $M_n$, the moduli space of $|\text{deg}| n$ harmonic maps $\mathbb{R}P^2 \to \mathbb{R}P^2$, with the subset of $M_n$ on which (6.2) holds.
Theorem 6.1 $\tilde{M}_n$, where $n \geq 1$ is odd, is a totally geodesic Lagrangian submanifold of $(M_n, \gamma, \Omega)$.

**Proof:** Let $P : M_n \to M_n$ such that

$$P : \tilde{\phi} \mapsto p \circ \tilde{\phi} \circ p. \quad (6.3)$$

Then $\tilde{M}_n \subset M_n$ is precisely the fixed point set of $P$. Since $P$ is an isometry of $(M_n, \gamma)$, in the component $\tilde{SO}(3) \times \tilde{SO}(3)$, $\tilde{M}_n$ is totally geodesic if it is a submanifold (i.e. nonsingular). Extending $P$ naturally to $\mathbb{C}P^{2n+1}$, one finds that

$$P : [a_1, \ldots, a_{n+1}, a_{n+2}, \ldots, a_{2n+2}] \mapsto \left[-(1)^n\sigma_{2n+2}, -(1)^{n-1}\sigma_{2n+1}, \ldots, \sigma_{n+3}, \sigma_{n+2}, -(1)^n\sigma_{n+1}, -(1)^{n-1}\sigma_n, \ldots, \sigma_2, -\sigma_1\right], \quad (6.4)$$

which is manifestly antiholomorphic. Hence $P^*\Omega = -\Omega$, and the Kähler (symplectic) form restricts to 0 on the fixed point set. So $\tilde{M}_n$ is a Lagrangian submanifold if it is nonsingular and has (real) dimension $2n + 1$.

It remains to check that $\tilde{M}_n$ is indeed nonsingular and has half the dimension of $M_n$. A short calculation in inhomogeneous coordinates demonstrates that the fixed point set of $P$ in $\mathbb{C}P^{2n+1}$ is smooth with real dimension $2n + 1$ if $n$ is odd, and is empty if $n$ is even (the latter being a special case of the topological fact that no even degree map $S^2 \to S^2$ projects to a map $\mathbb{R}P^2 \to \mathbb{R}P^2$ in (6.3)). This does not suffice for our purposes, however, since a real codimension 2 algebraic variety must be removed from $\mathbb{C}P^{2n+1}$ to yield $M_n$.

We must verify, therefore, that the intersection of $M_n$ with this singular set has dimension lower than $2n + 1$.

Since the question is local, we may work in a neighbourhood of any fixed map $\tilde{\phi}$, and choose stereographic coordinates on the codomain which are projected from neither $\tilde{\phi}((0, 0, 1))$ nor $\tilde{\phi}((0, 0, -1))$. Then, in a sufficiently small neighbourhood, all harmonic maps have rational form

$$W(z) = \mu \frac{(z - z_1) \cdots (z - z_n)}{(z - w_1) \cdots (z - w_n)} \quad (6.5)$$

where $\mu \in \mathbb{C}^\times$. These should be thought of as parametrized by $\mu$ and a pair of unordered $n$-tuples of complex numbers $\{w_i\}, \{z_i\} \in \mathbb{C}^n/P_n$, $P_n$ being the permutation group on $n$ objects. Of course, in this context $\mathbb{C}^n/P_n \cong \mathbb{C}^n$ diffeomorphically through the global coordinates $\{a_i\}$ where $(z - z_1) \cdots (z - z_n) = z^n + a_nz^{n-1} + \ldots + a_1$. The singular set, on which $\deg W < n$, is that piece where $\{w_i\} \cap \{z_i\} \neq \emptyset$. The fixed point set of $P$ in this neighbourhood consists of maps for which

$$\{z_1, \ldots, z_n\} = \left\{-\frac{1}{w_1}, \ldots, -\frac{1}{w_n}\right\} \quad (6.6)$$

and

$$|\mu| = |w_1w_2 \cdots w_n|. \quad (6.7)$$

Equations (6.6) and (6.7) determine a $2n + 1$ dimensional submanifold of $\mathbb{C}^\times \times \mathbb{C}^n/P_n \times \mathbb{C}^n/P_n$, parametrized by $\{w_i\} \in [\mathbb{C}^\times]^n/P_n$ and $\arg \mu \in S^1$. From this must be excluded, if
n ≥ 3, the 2n − 3 dimensional variety on which \( w_i = -1/\pi_j \) for some \( i, j \). This still leaves a nonsingular 2n + 1 dimensional fixed point set, as was to be proved. □

Note that \( \mathcal{P} \) is also an antiholomorphic isometry of \( \gamma_{FS} \), the Fubini-Study metric inherited from the open inclusion \( \tilde{M}_n \subset \mathbb{C}P^{2n+1} \). So \( \tilde{M}_n \) is a totally geodesic Lagrangian submanifold of \( (\tilde{M}_n, \gamma_{FS}, \Omega_{FS}) \) also, by identical reasoning. The metric induced on \( \tilde{M}_n \) by \( \gamma \) is more interesting than that induced by \( \gamma_{FS} \), however, since it coincides with the \( L^2 \) metric on \( \tilde{M}_n \). The geodesic approximation to \( \mathbb{R}P^2 \) lump dynamics on \( \mathbb{R}P^2 \) is thus a special case of \( S^2 \) lump dynamics on \( S^2 \).

It is clear from the proof above that \( \tilde{M}_n \) is generically noncompact. The case \( n = 1 \) is exceptional, however. Here, as described in section 3, one may identify a rational map with a projective equivalence class \([M]\) of \( GL(2, \mathbb{C}) \) matrices. Let \([M]\) be a fixed point of \( \mathcal{P} : PL(2, \mathbb{C}) \to PL(2, \mathbb{C}) \). Then

\[
\mathcal{P} : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} \overline{a}_{22} & \overline{a}_{21} \\ \overline{a}_{12} & \overline{a}_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]

(6.8)

So there exists \( \xi \in \mathbb{C}^\times \) such that

\[
a_{11} = -\xi \overline{a}_{22}, \quad a_{12} = \xi \overline{a}_{21}, \quad a_{21} = \xi \overline{a}_{12}, \quad a_{22} = -\xi \overline{a}_{11},
\]

(6.9)

whence it follows that \( |\xi| = 1 \). But then

\[
MM^\dagger = \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 & a_{11}\overline{a}_{21} + a_{12}\overline{a}_{22} \\ a_{21}\overline{a}_{11} + a_{22}\overline{a}_{12} & |a_{21}|^2 + |a_{22}|^2 \end{pmatrix} = (|a_{11}|^2 + |a_{12}|^2)\mathbb{I}_2,
\]

(6.10)

so \([M] \in PU(2) \cong SO(3)\). Hence \( \tilde{M}_1 \) consists of the rotation orbit of \( \text{Id} : z \mapsto z \), and the induced metric \( \tilde{\gamma} \) on \( \tilde{M}_1 \) is

\[
\tilde{\gamma} = A_3(0)\sigma \cdot \sigma,
\]

(6.11)

the standard bi-invariant metric on \( SO(3) \), up to a constant factor. Each \( \phi \in \tilde{M}_1 \) has completely uniform energy density, so it is rather misleading to call these solutions "\( \mathbb{R}P^2 \) lumps."

For higher \( n \) the possibilities are more varied. For example, the energy density of \([z \mapsto z^n] \in \tilde{M}_n, n ≥ 3\), is concentrated in a symmetric band centred on a (projected) great circle on \( \mathbb{R}P^2 \), the band being narrower for larger \( n \). Considering rational maps of the form (6.5), with parameters satisfying (6.6) and (6.7), a sharp lump-like structure may be induced by arranging that one of the poles of \( W \) be close to one of the zeroes, for example by choosing \( w_2 \) close to \(-1/\pi_1 \), while keeping the other poles and zeroes well separated. Since lumps are associated with close pole-zero pairs, and the poles determine the zeroes (they must be antipodal), for \( \phi \in \tilde{M}_n \) at most \((n - 1)/2 \) distinct lumps in the energy distribution are possible.

The origin of the noncompactness of \( \tilde{M}_n, n ≥ 3 \), is that when \( w_2 \to -1/\pi_1 \), say, the degree of \( \tilde{\phi} \) drops by 2, that is a lump (or, in the lifted picture, an antipodal pair of lumps) forms, collapses to an infinitely sharp spike and disappears. In fact, there are geodesics with respect to \( \tilde{\gamma} \) which reach such singularities in finite time. We conclude this section by establishing:

**Theorem 6.2** For all \( n ≥ 3 \), \((\tilde{M}_n, \tilde{\gamma})\) is geodesically incomplete.
Proof: It suffices to exhibit a curve of finite length which converges to infinity, that is, escapes every compact subset of $\tilde{M}_n$. Consider the curve $\Gamma : \left(\frac{1}{2}, 1\right) \ni \rho \mapsto W_\rho \in \tilde{M}_n$ where

$$W_\rho(z) = \rho z^{n-2} \frac{(z + 1)(z - \rho^{-1})}{(z - 1)(z + \rho)}, \quad (6.12)$$

which certainly converges to infinity (as $\rho \to 1$). The induced metric on $\Gamma$ is $\tilde{\gamma}_\Gamma = f(\rho)d\rho^2$, where

$$f(\rho) = \int_\mathbb{C} \frac{dz d\overline{z}}{(1 + |z|^2)^2} \frac{1}{(1 + |W_\rho|^2)^2} \left| \frac{\partial W_\rho}{\partial \rho} \right|^2. \quad (6.13)$$

We now appeal to a technical lemma whose proof is postponed to the appendix:

**Lemma 6.3** There exist $C > 0$ and $\rho_* \in (0, 1)$ such that for all $\rho \in (\rho_*, 1)$,

$$f(\rho) < C \left[ 1 + \log \left( \frac{1}{\rho} \left( \frac{1}{1 - \rho} \right) \right) \right].$$

Hence, the length of $\Gamma$,

$$\int_{\frac{1}{2}}^1 d\rho \sqrt{f(\rho)} < C \left[ 1 + \int_{\rho_*}^1 d\rho \sqrt{1 + \log \left( \frac{1}{\rho} \right)} \right]. \quad (6.14)$$

is finite. $\square$

Note that this result does not follow directly from the results of [29] previously mentioned (incompleteness of $M_n$), although the method of proof is similar. Recall that geodesic flow on $(\tilde{M}_n, \tilde{\gamma})$ is conjectured to approximate closely the low energy dynamics of the $\mathbb{R}P^2$ model on spacetime $\mathbb{R}P^2 \times \mathbb{R}$. So the geodesic approximation predicts that $\mathbb{R}P^2$ lumps on $\mathbb{R}P^2$ may collapse and form singularities in finite time, just as it does for $S^2$ lumps on any compact Riemann surface. In fact, little is known about singularity formation in the full $(2 + 1)$ dimensional system, although there is some numerical evidence in favour of lump collapse [21, 26].

### 7 Concluding remarks

One could hope to generalize the results of this paper in at least two directions. Replacing the domain 2-sphere by an arbitrary compact Riemann surface $\Sigma$, one could study the $L^2$ metric on the space $\text{Hol}_n(\Sigma)$ of degree $n$ holomorphic maps $\Sigma \to S^2$. If nonempty, $\text{Hol}_n(\Sigma)$ is the space of minimal energy degree $n$ harmonic maps (if empty, for example $\text{Hol}_{\pm 1}(T^2) = \emptyset$, there exists no minimal energy degree $n$ harmonic map), which is the space of most direct interest to physicists, rather than the space of all harmonic maps. $\text{Hol}_n(\Sigma)$ has the structure of a complex algebraic variety, so one would expect theorem 2.1, the Kähler property of the $L^2$ metric, to generalize to this situation. (In fact, $\text{Hol}_n(\Sigma)$ may not be smooth if $|n| \leq 2\text{genus}(\Sigma) - 2$, but the Kähler property should still hold in the complement of the singular set.)

As an example, consider $\text{Hol}_2(T^2)$. It was proved in [33] that $\text{Hol}_2(T^2)$ is homeomorphic (in $C^0$ topology) to the complex homogeneous space $[\mathbb{P}L(2, \mathbb{C}) \times T^2]/V_4$, where $V_4$ is a
certain Viergruppe (finite group of order 4, each element being its own inverse). So $\text{Hol}_2(T^2)$ inherits a natural complex structure from the covering space $PL(2, \mathbb{C}) \times T^2$, and it suffices to show that the lift of the $L^2$ metric is Kähler. Explicitly, a point

$$\left( \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, s \right) \in PL(2, \mathbb{C}) \times T^2$$

is identified with the degree 2 holomorphic map

$$W(z) = \frac{a_1 \wp(z-s) + a_2}{a_3 \wp(z-s) + a_4}$$

where $\wp$ is the Weierstrass $\wp$-function. Introducing inhomogeneous coordinates on $PL(2, \mathbb{C})$, an essentially identical argument to that of the proof of theorem 2.1 establishes:

**Theorem 7.1** The $L^2$ metric $\gamma$ on $\text{Hol}_2(T^2)$ is Kähler with respect to the complex structure induced by the identification with $[PL(2, \mathbb{C}) \times T^2]/V_4$.

It is interesting to note that $(\text{Hol}_2(T^2), \gamma)$, like $(M_1, \gamma)$ has finite diameter, leading one to expect that theorem 3.3 should generalize to $(\text{Hol}_n(\Sigma), \gamma)$ also.

The second natural generalization would be to replace the codomain $S^2 \cong \mathbb{CP}^1$ by a general projective space, $\mathbb{CP}^N$. Lemaire and Wood [19] have shown that the space of degree $n$, energy $4\pi E$ harmonic maps $S^2 \to \mathbb{CP}^2$, $\text{Harm}_{n,E}(\mathbb{CP}^2)$ is, in $C^j$ topology ($j \geq 2$), a disjoint union of smooth manifolds indexed by total ramification index. Further, there is an explicit identification between each smooth component of $\text{Harm}_{n,E}(\mathbb{CP}^2)$ and an appropriate space of linearly full holomorphic maps $S^2 \to \mathbb{CP}^2$ of fixed degree and ramification index. So again one has a natural complex structure on the moduli space, and again one would expect the $L^2$ metric to be Kähler with respect to this structure. It is even possible that the Kähler property of the $L^2$ metric may persist when the codomain itself is not Kähler. Bolton and Woodward [3] have conjectured that $\text{Harm}_E(S^{2m})$, the space of energy $4\pi E$ harmonic maps $S^2 \to S^{2m}$, is a complex algebraic variety (of dimension $2E + m^2$). If true, it would be natural to ask again whether $\gamma$ is Kähler, at least on the smooth part of $\text{Harm}_E(S^{2m})$. Note that both these generalizations lie beyond the scope of Ruback’s formal argument [28].

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**Appendix: Proofs of Lemmas 2.2 and 6.3**

**Lemma 2.2**

Since $F : X \times (-\epsilon, \epsilon) \to \mathbb{R}$ is smooth, its partial derivative with respect to the second entry, $F_2$ is continuous. Hence the restriction $F_2 : X \times [0, x] \to \mathbb{R}, 0 < x < \epsilon$, is integrable (its domain is compact). Thus, by the Fubini theorem [4],

$$\int_X \left\{ \int_{[0, x]} F_2 \right\} = \int_{[0, x]} \left\{ \int_X F_2 \right\}.$$
But \( \int_{[0, x]} \bar{F}_2 \equiv F(\cdot, x) - F(\cdot, 0) \), so the left hand side is \( f(x) - f(0) \). Hence, by the fundamental theorem of calculus,

\[
f'(0) = \left\{ \int_X \bar{F}_2 \right\}_{x=0} = \int_X F_2(\cdot, 0).
\]

**Lemma 6.3**

From (6.12) and (6.13) one finds that

\[
\rho > \alpha
\]

where \( \alpha \) is such that

\[
\int \bar{F} \leq (1 - \epsilon) \int F \leq \int \tilde{F}.
\]

Fix \( \epsilon \in (0, \frac{1}{2}) \), and assume that \( \rho \) is close to \( 1 \), that is \( 0 < \rho - 1 < \epsilon \). Then \( F(\cdot, \rho) \) may be bounded independent of \( \rho \) except on the union of disks \( D_\epsilon(-1) \cup D_\epsilon(1) \), where one or other of the terms in the denominator may vanish (here \( D_\epsilon(z_0) = \{ z \in C : |z - z_0| < \rho \} \)). We shall denote positive constants (independent of \( z \) and \( \rho \)) by \( C_1, C_2 \) etc.. On \( D_\epsilon(-1) \) there exists \( C_1 \) such that

\[
F(z, \rho) < \frac{C_1|z + 1|^2}{(|z + \rho|^2 + \alpha \rho^2 |z + 1|^2)^2}
\]

where \( \alpha = (1 - \epsilon)^2 < 1 \). Hence, defining \( re^{i\theta} := z + 1 \),

\[
\int_{D_\epsilon(-1)} dz d\bar{z} F(z, \rho) < C_1 \int_0^{2\pi} d\theta \int_0^\epsilon dr \frac{r^3}{(1 + \alpha \rho^2)r^2 - 2(1 - \rho) \cos \theta r + (1 - \rho)^2]^2}
\]

provided \( \rho > (2\alpha)^{-1/2} \in (0, 1) \). Then, rescaling \( r \rightarrow r/(1 - \rho) \), one finds that

\[
\int_{D_\epsilon(-1)} dz d\bar{z} F(z, \rho) < C_2 \int_0^{(1 - \rho)^{-1}} dr \frac{r^3}{(1 + \alpha \rho^2)r^2 - 2(1 - \rho) \cos \theta r + (1 - \rho)^2]^2}
\]

\[
< C_3 + C_4 \int_1^{(1 - \rho)^{-1}} dr \frac{r^3}{(1 + \alpha \rho^2)r^2 - 2(1 - \rho) \cos \theta r + (1 - \rho)^2]^2}
\]

\[
< C_5[1 - \log(1 - \rho)]
\]

Noting that \( \rho > (2\alpha)^{-1/2} \) implies \( 1 + \alpha \rho^2 > 3\alpha/2 \), one finds a similar estimate for the contribution from \( D_\epsilon(1) \):

\[
\int_{D_\epsilon(1)} dz d\bar{z} F(z, \rho) < C_6 \int_{D_\epsilon(1)} dz d\bar{z} \frac{|z - 1|^2}{(|z - \rho|^2 + \alpha \rho^2 |z - 1|^2)^2}
\]

\[
< C_7 \int_0^1 dr \frac{r^3}{(1 + \alpha \rho^2)r^2 - 2(1 - \rho) \cos \theta r + (1 - \rho)^2]^2}
\]

\[
< C_8 \int_0^{(1 - \rho)^{-1}} dr \frac{r^3}{(1 + \alpha \rho^2)r^2 - 2(1 - \rho) \cos \theta r + (1 - \rho)^2]^2}
\]

\[
< C_9[1 - \log(1 - \rho)]
\]
Since $F$ is bounded independent of $\rho$ on $U = \mathbb{C}\backslash[D_\epsilon(-1) \cup D_\epsilon(1)]$,

$$
\int_U d\z d\bar{\z} F(z, \rho) < C_{10} + C_{11} \int_1^\infty \frac{dr}{r^{2n-1}} < C_{12}.
$$

Defining $C = C_5 + C_9 + C_{12} > 0$ and $\rho_* = (2\alpha)^{-1/2} \in (0, 1)$, the lemma is proved.
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