Abstract

We develop a notion of wavefront set aimed at characterizing in Fourier space the directions along which a distribution behaves or not as an element of a specific Besov space. Subsequently we prove an alternative, albeit equivalent characterization of such wavefront set using the language of pseudodifferential operators. Both formulations are used to prove the main underlying structural properties. Among these we highlight the individuation of a sufficient criterion to multiply distributions with a prescribed Besov wavefront set which encompasses and generalizes the classical Young’s theorem. At last, as an application of this new framework we prove a theorem of propagation of singularities for a large class of hyperbolic operators.

Keywords: Besov spaces, wavefront set, propagation of singularities

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1 Introduction

Microlocal analysis and the associated Hörmander’s wavefront set [Hor94] have been an unmitigated success in analysis which has found in addition manifold applications ranging from engineering to mathematical physics. One of the most recent interplay with modern theoretical physics is related to the rôle played by microlocal techniques in the construction of a full-fledged theory of quantum fields on generic Lorentzian and Riemannian backgrounds as well as in the development of a mathematical formulation of renormalization with the language of distributions, see e.g. [BF00, Rej16, DDR20, CDDR20].

In the early developments of the interplay between microlocal analysis and renormalization, it has become clear that the original framework developed by Hörmander aimed at disentangling the directions of rapid decrease in Fourier space of a given distribution from the singular ones suffered from a substantial limitation. As a matter of fact, in many concrete scenarios one is interested in having a more refined estimate of the singular behavior of a distribution, for instance comparing it with that of an element lying in a suitable Sobolev space. This has lead to considering more specific forms of wavefront set, among which a notable rôle in application has been played by the so-called Sobolev wavefront set, see [Hor97].
Still having in mind the realm of quantum field theory, one of the first remarkable uses has been discussed in \cite{JS02}, while nowadays it has become an essential ingredient in many modern results among which noteworthy are those concerning the analysis of the wave equations on manifolds with boundaries or with corners, see e.g. \cite{Vas08,Vas12}.

An apparently completely detached branch of analysis in which distributions and their specific singular behavior plays a distinguished rôle is that of stochastic partial differential equations. Without entering in too many technical details, far from the scope of this work, remarkable leaps forward have been obtained in the past few years both within the framework of the theory of regularity structures \cite{Hai14,Hai15} and in that of paracontrolled distributions \cite{GIP12}. In both approaches, despite the necessity of dealing with specific problems, such as renormalization, calling for the analysis of products or of extensions of a priori ill-defined distributions, microlocal techniques never enter the game.

The reasons are manifold but the main one lies in the fact that, in the realm of stochastic partial differential equations, often one considers Hölder distributions, i.e. elements of $C^\alpha(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$. The latter can be read as a specific instance of the so-called Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$, $p,q \in [1, \infty]$, \cite{Tri06}. When working in this framework, one relies often on Bony paradifferential calculus \cite{Bo81} as it is devised to better catch the specific features of elements lying in a Besov space. To this end microlocal techniques and the wavefront set in particular appear at first glance to be far from the optimal tool to be used, since it appears to be unable to grasp the peculiar singular behaviour of a distribution in comparison to an element of $B^\alpha_{p,q}(\mathbb{R}^d)$.

Nonetheless it has recently emerged that, in the analysis of a large class of nonlinear stochastic partial differential equations, microlocal analysis can be used efficiently to devise a recursive scheme to construct both solutions and correlation functions, while taking into account intrinsically the underlying renormalization freedoms. \cite{DDRZ20,BDR21}. One of the weak point of this novel approach lies in the lack of any control of the convergence of the underlying recursive scheme. This can be ascribed mainly to the fact that employing microlocal techniques appears to wash out all information concerning the behaviour of the underlying distributions as elements of a Besov space. Observe that each $B^\alpha_{p,q}(\mathbb{R}^d)$ is endowed with the structure of a Banach space which is pivotal in setting up a fixed point argument to prove the existence of solutions for the considered class of nonlinear stochastic partial differential equations.

Hence, it appears natural to seek a way to combine the best of both worlds, trying to use the language of microlocal analysis on the one hand, while keeping track of the underlying Besov space structure on the other hand. In this paper we plan to make the first step in this direction, developing a modified notion of wavefront set, specifically devised to keep track of the behaviour of a distribution in comparison to that of an element of a Besov space. For definiteness and in order to avoid unnecessary technical difficulties, focusing instead on the main ideas and constructions, we shall focus on the Besov spaces $B^\alpha_{\infty,\infty}(\mathbb{R}^d) \equiv C^\alpha(\mathbb{R}^d)$, which are, moreover, the most relevant ones in concrete applications. We highlight that an investigation in this direction, complementing our own, has appeared in \cite{GM15}.

Specifically our proposal hinges on the following starting point, a definition of Besov wavefront set which focuses on the behaviour of a distribution in Fourier space.

**Definition 1:** Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$. We say that $(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ does not lie in the $B^\alpha_{\infty,\infty}$-wavefront set, denoting $(x,\xi) \not\in WF^\alpha(u)$, if there exist $\phi \in \mathcal{D}(\mathbb{R}^d)$ with $\phi(x) \neq 0$ as well as an open conic neighborhood $\Gamma$ of $\xi$ in $\mathbb{R}^d \setminus \{0\}$ such that

$$\left| \int_{\Gamma} \hat{\phi}(\eta) \hat{\omega}(\eta) e^{iy \cdot \eta} d\eta \right| \lesssim 1,$$

(1.1)
for any \( \kappa \in \mathcal{D}(B(0,1)) \) with \( \tilde{\kappa}(0) \neq 0 \), \( \lambda \in (0,1) \), \( y \in \text{supp}(\phi) \) and \( \kappa \in \mathcal{R}_{[\alpha]} \), see Definition 5.

While conceptually the above definition enjoys all desired structural properties, from an operational viewpoint, it is rather difficult to use it concretely both in examples and in the proof of various results. For this reason we give an alternative, albeit equivalent, characterization of WF\( ^\alpha (u) \), \( u \in \mathcal{D}'(\mathbb{R}^d) \), in terms of the intersection of the characteristic set of a suitable class of order zero, properly supported pseudodifferential operators, see Proposition 33. Using this tool we are able to prove a large set of structural properties of the Besov wavefront set. In particular we discuss its interplay with pullbacks, we devise a supported pseudodifferential operators. Section 4 contains the main results concerning the structural properties and discussing a few notable examples. In Section 3.1 we prove that the Besov wavefront set can be equivalently characterized in terms of the characteristic set of a suitable class of properly supported pseudodifferential operators, see Proposition 53. Using this tool we are able to prove a large set of structural properties of the Besov wavefront set. The three main results that we obtain are the following:

- We prove that, given an embedding \( f \in C^\infty(\Omega, \Omega') \) between two open subsets \( \Omega \subseteq \mathbb{R}^d \) and \( \Omega' \subseteq \mathbb{R}^m \), one can establish a criterion, see Theorem 33 for the existence of the pull-back \( f^* u \), \( u \in \mathcal{D}'(\Omega') \) which generalizes the one devised by Hörmander in the smooth setting, [Hör94, Thm. 8.2.4]. A noteworthy byproduct of this analysis is that, whenever \( f \) is a diffeomorphism, then, for any \( \alpha \in \mathbb{R} \), \( f^* \text{WF}^\alpha (u) = \text{WF}^\alpha (f^* u) \), see Theorem 15. This result is noteworthy since it entails that the notion of Besov wavefront set can be applied also to distributions supported on an arbitrary smooth manifold [RS21].

- We establish a sufficient criterion for the existence of the product of two distributions with prescribed Besov wavefront set and we provide an estimate for the wavefront set of the product, see Theorem 38. This result contains and actually extends the renown Young’s theorem on the product of two Hölder distributions, which is often used in the applications to stochastic partial differential equations.

- We apply the whole construction of the Besov wavefront set to prove a propagation of singularities theorem for a large class of hyperbolic partial differential equations, see Theorem 35. This result is strongly tied to a preliminary analysis on the wavefront set \( \text{WF}^\alpha (\mathcal{K}(u)) \) where \( \mathcal{K} \) is a linear map from \( C^\infty_0 (\Omega') \to \mathcal{D}'(\Omega) \) where \( \Omega \subseteq \mathbb{R}^d \) while \( \Omega' \subseteq \mathbb{R}^m \).

The paper is organized as follows: In Section 2 we present the definition of Besov spaces outlining some of its main properties and alternative, equivalent characterizations. Subsequently we review succinctly the basic notions of pseudodifferential operators and of the associated operator wavefront set. In Section 3 we present the main object of our investigation, giving the definition of Besov wavefront set in terms of the behaviour of a distribution in Fourier space, outlining subsequently some of the basic structural properties and discussing a few notable examples. In Section 4 we prove that the Besov wavefront set can be equivalently characterized in terms of the characteristic set of a suitable class of properly supported pseudodifferential operators. Section 4 contains the main results concerning the structural properties of the Besov wavefront set. In particular we discuss its interplay with pullbacks, we devise a sufficient criterion for the product of two distributions with prescribed Besov wavefront set and we prove a theorem of propagation of singularities for a class of hyperbolic partial differential operators.

**Notations** In this short paragraph we fix a few recurring notations used in this manuscript. With \( \mathcal{E}(\mathbb{R}^d) \) (resp. \( \mathcal{D}(\mathbb{R}^d) \)), we denote the space of smooth (resp. smooth and compactly supported) functions on \( \mathbb{R}^d \), \( d \geq 1 \), while \( \mathcal{S}(\mathbb{R}^d) \) stands for the space of rapidly decreasing smooth functions. Their topological dual spaces are denoted respectively \( \mathcal{E}'(\mathbb{R}^d) \), \( \mathcal{D}'(\mathbb{R}^d) \) and \( \mathcal{S}'(\mathbb{R}^d) \). In addition, given \( u \in \mathcal{S}(\mathbb{R}^d) \), we adopt the following convention to define its Fourier transform

\[
\mathcal{F}(u)(k) = \hat{u}(k) := \int_{\mathbb{R}^d} e^{-ik \cdot x} u(x) \, dx.
\]
At the same time, we indicate with the symbol \( \hat{\cdot} \) the inverse Fourier transform \( \mathcal{F}^{-1} \), namely, for any \( f \in \mathcal{S}(\mathbb{R}^d) \), \( f = \hat{f} = \hat{f} \). Similarly, for any \( v \in \mathcal{S}'(\mathbb{R}^d) \), we indicate with \( \hat{v} \in \mathcal{S}'(\mathbb{R}^d) \) its Fourier transform, defining it per duality as \( \hat{v}(u) = \langle \hat{u}, v \rangle \) for all \( u \in \mathcal{S}(\mathbb{R}^d) \). In general, given a function \( f \in \mathcal{E}(\mathbb{R}^d) \), \( x \in \mathbb{R}^d \) and \( \lambda \in (0, 1] \), we shall denote \( f_\lambda(x) := \lambda^{-d} f(\lambda^{-1}(y - x)) \). At last with \( (x) := (1 + |x|^2)^{\frac{1}{2}} \) we denote the Japanese bracket, while the symbol \( \lesssim \) refers to an inequality holding true up to a multiplicative finite constant. Observe that, depending on the case in hand, such constant might depend on other data, such as for example the choice of an underlying compact set. For the ease of notation we shall omit making such dependencies explicit, since they shall become clear from the context.

## 2 Preliminaries

The aim of this section is to introduce the key function spaces and some of their notable properties. The content of this specific subsection is mainly inspired by [BCD11, Tri06]. The starting point lies in the notion of a Littlewood-Paley partition of unity.

### Definition 2:
Let \( N \in \mathbb{N} \) and let \( \psi \in \mathcal{D}(\mathbb{R}^d) \) be a positive function supported in \( \{2^{-N} \leq |\xi| \leq 2^{N}\} \). We call Littlewood-Paley partition of unity a sequence \( \{\psi_j\}_{j \in \mathbb{N}_0} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) such that

- \( \psi_0 \in \mathcal{D}(\mathbb{R}^d) \) and \( \text{supp}(\psi_0) \subseteq \{ |\xi| \leq 2^{N}\} \);
- \( \psi_j(x) := \psi(2^{-j}x) \) for \( j \geq 1 \);
- \( \sum_{j \in \mathbb{N}_0} \psi_j(\xi) = 1 \) for all \( \xi \in \mathbb{R}^d \);
- for any multi-index \( \alpha \), \( \exists C_{\alpha} > 0 \) such that
  \[ |D^\alpha \psi_j(\xi)| \leq C_{\alpha} |\xi|^{-|\alpha|}, \quad j \geq 1; \]
- \( \psi_j(-\xi) = \psi_j(\xi) \) for all \( j \geq 0 \).

In the following we shall always assume for definiteness \( N = 1 \).

### Definition 3:
Let \( \alpha \in \mathbb{R} \). We call **Besov space** \( B_{p,q}^\alpha(\mathbb{R}^d) \), \( p, q \in [1, \infty] \), the Banach space whose elements \( u \) are such that

\[
\|u\|_{B_{p,q}^\alpha(\mathbb{R}^d)} := \sum_{j \geq 0} 2^{j\alpha q} \|\psi_j(D)u\|_{L^p(\mathbb{R}^d)} < \infty, \tag{2.1}
\]

At the same time if \( q = \infty \), while \( p \in [1, \infty] \), we set

\[
\|u\|_{B_{p,\infty}^\alpha(\mathbb{R}^d)} := \sup_{j \geq 0} 2^{j\alpha} \|\psi_j(D)u\|_{L^p(\mathbb{R}^d)} < \infty, \tag{2.2}
\]

where we used the Fourier multiplier notation \( \psi_j(D)u(x) := \mathcal{F}^{-1}\{\psi_j(\xi)\hat{u}(\xi)\}(x) \). At the same time, we say that \( u \in B_{p,\infty}^\alpha(\mathbb{R}^d) \) if \( \varphi u \in B_{p,\infty}^\alpha(\mathbb{R}^d) \) for any \( \varphi \in \mathcal{D}(\mathbb{R}^d) \).

### Remark 4:
By definition of Fourier multiplier, it descends that

\[
\psi_j(D)u(x) = \mathcal{F}^{-1}\{\psi_j(\xi)\hat{u}(\xi)\}(x) = (\bar{\psi}_j * u)(x) = u(2^{jd}\bar{\psi}(2^j(-x))),
\]

where we exploited \( \mathcal{F}^{-1}\{uv\} = \bar{u} * \bar{v} \) and \( \bar{\psi}_j(x) = 2^{jd}\bar{\psi}(2^jx) \). As a consequence, if \( u \in B_{\infty,\infty}^\alpha(\mathbb{R}^d) \),

\[
|u(\bar{\psi}_j^2x)| \lesssim 2^{-j\alpha}, \quad \forall j \geq 0, \quad \forall x \in \mathbb{R}^d. \tag{2.3}
\]
In our analysis it will be often convenient not to consider directly Definition 3 rather to work with an equivalent characterization, dubbed the local means formulation – see [Tri06] Sec. 1.4 & Thm.1.10. This is based on the following tool.

**Definition 5:** Let \( B(0,1) = \{ y \in \mathbb{R}^d : |y| < 1 \} \). For \( s \in \mathbb{N}_0 \), we call \( \mathcal{B}_s \) the subset of \( \mathcal{D}(B(0,1)) \) whose elements \( \kappa \) are such that there exists \( \epsilon > 0 \)

\[
\kappa(\xi) \neq 0 \quad \text{if} \quad \frac{\epsilon}{2} < |\xi| \leq 2\epsilon, \quad \text{and} \quad (\partial^\beta \kappa)(0) = 0 \quad \text{if} \quad |\beta| \leq s.
\]  

(2.4)

Observe that the second condition in Equation (2.4) is empty if \( s < 0 \).

**Definition 6:** Let \( \alpha \in \mathbb{R}, \kappa \in \mathcal{B}_{[\alpha]} \), with \( [\alpha] \) the biggest integer \( N \) such that \( N \leq \alpha \). Let \( \kappa \in \mathcal{D}(B(0,1)) \) be such that \( \kappa(0) \neq 0 \). We call \( B_{p,\infty}^\alpha(\mathbb{R}^d), p \in [1, \infty] \), the space of distributions \( u \in \mathcal{S}'(\mathbb{R}^d) \) such that

\[
\|u\|_{B_{p,\infty}^\alpha(\mathbb{R}^d)} := \|u(\kappa,.)\|_{L^p(\mathbb{R}^d)} + \sup_{\lambda \in (0,1)} \frac{\|u(\kappa,.)\|_{L^\infty(\mathbb{R}^d)}}{\lambda^\alpha} < \infty,
\]

(2.5)

where the \( L^\infty \)-norm is taken with respect to the variable \( x \).

**Remark 7:** We observe that different choices for \( \kappa \) and \( \kappa' \) yield in Equation (2.5) equivalent norms. Therefore, henceforth we shall omit to indicate the superscripts \( \kappa \) and \( \kappa' \).

If \( \alpha < 0 \), there exists a further equivalent characterization for Besov spaces – see [BL21] Prop. A.5, [Tri06] Cor. 1.12. We focus on the case \( p = \infty \).

**Proposition 8:** Let \( \alpha < 0 \) and \( \kappa \in \mathcal{D}(B(0,1)) \) be such that \( \kappa(0) \neq 0 \). Then \( u \in B_{\infty,\infty}^\alpha(\mathbb{R}^d) \) if and only if

\[
\sup_{\lambda \in (0,1)} \frac{\|u(\kappa,.)\|_{L^\infty(\mathbb{R}^d)}}{\lambda^\alpha} < \infty,
\]

(2.6)

where the \( L^\infty \)-norm is taken with respect to the variable \( x \).

We conclude this subsection proving a last, useful characterization of the element lying in \( B_{\infty,\infty}^\alpha \).

**Proposition 9:** Let \( u \in \mathcal{S}'(\mathbb{R}^d) \) and \( \alpha \in \mathbb{R} \). Then \( u \in B_{\infty,\infty}^\alpha(\mathbb{R}^d) \) if and only if, given \( \kappa \in \mathcal{B}_{[\alpha]} \) and \( \kappa \in \mathcal{D}(B(0,1)) \) such that \( \kappa(0) \neq 0 \), it holds that

\[
|\langle \hat{u}(\xi), e^{ix \cdot \xi} \hat{\kappa}(\lambda \xi) \rangle| \leq 1, \quad |\langle \hat{u}(\xi), e^{ix \cdot \xi} \hat{\kappa}(\lambda \xi) \rangle| \leq \lambda^\alpha,
\]

(2.7)

for any \( \lambda \in (0,1) \) and \( x \in \mathbb{R}^d \).

**Proof.** The statement is a direct consequence of Definition 6 combined with the following identities

\[
u(\varphi) = \langle \hat{u}(\xi), e^{ix \cdot \xi} \varphi(\xi) \rangle, \quad u(\varphi^\lambda) = \langle \hat{u}(\xi), e^{ix \cdot \xi} \hat{\varphi}(\lambda \xi) \rangle,
\]

(2.8)

where \( \varphi \in \mathcal{S}(\mathbb{R}^d), u \in \mathcal{S}'(\mathbb{R}^d), x \in \mathbb{R}^d, \lambda \in (0,1) \). In turn these are a by-product of the identities \( u(\varphi) = \hat{u}(\varphi) \) and \( \langle \hat{\varphi}(\lambda \xi) \rangle = e^{ix \cdot \xi} \hat{\varphi}(\lambda \xi) \).

**Remark 10:** Observe that, if \( \alpha < 0 \), then it is sufficient to verify the second of the two conditions in Equation (2.7).
2.1 Pseudodifferential Operators

In this section we shall focus on the second functional tool which plays a distinguished rôle in our analysis. Hence we recall succinctly the definition and some notable properties of pseudodifferential operators. For later convenience, this section is mainly inspired by [Hin21], though further details can be found in [GS94] [Hor94]. We start by recalling the definition both of a symbol and of its quantization.

**Definition 11:** Let \( m \in \mathbb{R} \) and \( n \in \mathbb{N} \). A function \( a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) is called a symbol of order \( m \) if, for all \( \alpha \in \mathbb{N}_0^n \), \( \beta \in \mathbb{N}_0^n \), it satisfies

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} |\xi|^{m-|\beta|}
\]

for some constant \( C_{\alpha\beta} > 0 \) and for any \( x \) in a compact set of \( \mathbb{R}^d \). We denote the space of symbols of order \( m \) with \( S^m(\mathbb{R}^d; \mathbb{R}^N) \). In addition, we define the space of residual symbols by

\[
S^{-\infty}(\mathbb{R}^d; \mathbb{R}^N) := \bigcap_{m \in \mathbb{R}} S^m(\mathbb{R}^d; \mathbb{R}^N).
\]

At last we call \( S^m_{\text{hom}}(\mathbb{R}^d; \mathbb{R}^N) \subset S^m(\mathbb{R}^d; \mathbb{R}^N) \) the collection of homogeneous symbols of order \( m \), namely, when \( |\xi| > 1 \), \( a(x, \lambda \xi) = \lambda^m a(x, \xi) \) for all \( \lambda > 0 \) and, for all \( \alpha \in \mathbb{N}_0^n \), \( \beta \in \mathbb{N}_0^n \)

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} |\xi|^{m-|\beta|}.
\]

**Definition 12:** Let \( m \in \mathbb{R} \), \( n \in \mathbb{N} \) and let \( a \in S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^n) \). We define its quantization \( \text{Op}(a) : S(\mathbb{R}^d) \to S(\mathbb{R}^d) \) as

\[
(\text{Op}(a)u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) dy d\xi, \quad u \in S(\mathbb{R}^d).
\]

\( \text{Op}(a) \) is called a pseudodifferential operator \( \Psi DO \) of order \( m \) and the whole set of these operators is denoted by \( \Psi^m(\mathbb{R}^d) \). Moreover, we set

\[
\Psi^{-\infty}(\mathbb{R}^d) := \bigcap_{m \in \mathbb{R}} \Psi^m(\mathbb{R}^d).
\]

Since it plays a rôle in our analysis, we remark that Equation (2.11) can be replaced either by the right quantization \( \text{Op}_R(a) \) or by the left quantization \( \text{Op}_L(a) \)

\[
(\text{Op}_R(a')u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} a'(y, \xi) u(y) dy d\xi, \quad \forall a' \in S^m(\mathbb{R}^d; \mathbb{R}^d) \quad (2.12a)
\]

\[
(\text{Op}_L(a)u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) dy d\xi, \quad \forall a \in S^m(\mathbb{R}^d; \mathbb{R}^d) \quad (2.12b)
\]

It is important to stress that, at the level of pseudodifferential operators, the choices of quantization procedure is to a certain extent immaterial, since, for any \( a \in S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \), there always exist \( a_L, a_R \in S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \) such that – see [Hin21] Thm. 4.8

\[
\text{Op}(a) = \text{Op}_L(a_L) = \text{Op}_R(a_R).
\]

**Remark 13:** By means of a standard duality argument one can extend continuously the action of a pseudodifferential operator of order \( m \), \( m \in \mathbb{R} \), to tempered distributions. In order not to burdening the reader with an unnecessarily baroque notation, we still indicate any such extension as \( \text{Op}(a) : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) for all \( a \in S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \).
As a last step we give a characterization of a notable subclass of pseudodifferential operators, based on their support properties.

**Definition 14:** Let $A \in \Psi^m(\mathbb{R}^d)$ and let $K_A \in S'((\mathbb{R}^d \times \mathbb{R}^d)^*)$ be the associated Schwartz kernel. We say that $A$ is properly supported if the canonical projections $\pi_1 : \text{supp}(K) \subseteq \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\pi_2 : \text{supp}(K) \subseteq \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are proper maps.

Associated to a pseudodifferential operator, one can introduce the notion of *operator wavefront set*, which is a key ingredient in our construction outlined in Section 3.

**Definition 15:** Let $a \in S^m(\mathbb{R}^d, \mathbb{R}^N)$. We say that a point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^N \setminus \{0\})$ does not lie in the *essential support* of $a$,

$$\text{ess supp}(a) \subseteq \mathbb{R}^d \times (\mathbb{R}^N \setminus \{0\}),$$

if there exists $\varepsilon > 0$ such that for all $\ell \in \mathbb{N}^d_0$, $\beta \in \mathbb{N}^N_0$, $k \in \mathbb{R}$, it holds

$$|\partial_\ell^\beta \partial_\xi^k a(x, \xi)| \leq C(\xi)^{-k}, \quad \forall (x, \xi), \text{ such that } |\xi| \geq 1, \text{ and } |x - x_0| + \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \varepsilon. \quad (2.13)$$

Observe that ess supp$(a)$ is a closed subset of $\mathbb{R}^d \times (\mathbb{R}^N \setminus \{0\})$ whereas, for each $x \in \mathbb{R}^d$, $\pi_\xi[\text{ess supp}(a)] \subseteq \mathbb{R}^N \setminus \{0\}$ is a conical subset. At last we can state the main definition of this whole section:

**Definition 16:** Let $A = \text{Op}_L(a) \in \Psi^m(\mathbb{R}^d)$. The *operator wavefront set* of $A$ is

$$WF^*(A) := \text{ess supp}(a) \subseteq \mathbb{R}^d \times (\mathbb{R}^N \setminus \{0\}). \quad (2.14)$$

In the following proposition we summarize a few notable properties of the operator wave set. Since the proof is a direct application of Definitions 14 and 15 we omit it.

**Proposition 17:** Let $A, B \in \Psi^m(\mathbb{R}^d)$. The following properties hold:

1. If $A$ has compactly supported Schwartz kernel, then $WF^*(A) = \emptyset$ if and only if $A \in \Psi^{-\infty}(\mathbb{R}^d)$.
2. $WF^*(A + B) \subseteq WF^*(A) \cup WF^*(B)$.
3. $WF^*(AB) \subseteq WF^*(A) \cap WF^*(B)$.
4. $WF^*(A^*) = WF^*(A)$, where $A^*$ is the adjoint of $A$ defined so that for all $u, v \in S(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} dx \,(A^* u)(x)\overline{v(x)} = \int_{\mathbb{R}^d} dx \,u(x)(Av)(x).$$

A further concept, related to \(\Psi\)DOs and of great relevance in the following sections is that of microlocal parametrix. Here we recall its construction. Without entering into many details, for which we refer in particular to [GS94, Chap. 3], we underline that, given any $A \in \Psi^m(\mathbb{R}^d)$, $m \in \mathbb{R}$, one can always associate to it a principal symbol $|\sigma_m(A)| \in S^m(\mathbb{R}^d; \mathbb{R}^d) / S^{m-1}(\mathbb{R}^d; \mathbb{R}^d)$. In the following, when we do not write explicitly the square brackets, we are considering a representative within the equivalence class identifying the principal symbol.

**Definition 18:** Given $A \in \Psi^m(\mathbb{R}^d)$, a point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ does not lie in the *elliptic set* of $A$, $\text{Ell}(A)$, if there exists $\varepsilon > 0$ and a constant $C > 0$ such that

$$|\sigma_m(A)(x, \xi)| \geq C|\xi|^m, \quad \forall (x, \xi) \text{ such that } |\xi| \geq 1, \text{ and } |x - x_0| + \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \varepsilon, \quad (2.15)$$

where $|\sigma_m(A)|$ is the principal symbol of $A$. We call *characteristic set* of $A$, $\text{Char}(A)$, the complement of $\text{Ell}(A)$.
Remark 19: Definition [18] can be reformulated as follows: a point \((x_0, \xi_0) \in \text{Ell}(A)\) if there exist 
\(b \in S^{-m}(\mathbb{R}^d; \mathbb{R}^d)\) and a conic neighbourhood of \((x_0, \xi_0)\) such that therein \(P_m(A)b - 1 \in S^{-1}(\mathbb{R}^d; \mathbb{R}^d)\).

**Proposition 20:** Let \(A \in \Psi^m(\mathbb{R}^d)\) and let \(\mathcal{C} \subset \text{Ell}(A)\) be a closed subset. Then there exists \(B \in \Psi^{-m}(\mathbb{R}^d)\) such that
\[
\mathcal{C} \cap WF'(AB - I) = \emptyset, \quad \mathcal{C} \cap WF'(BA - I) = \emptyset. \tag{2.16}
\]

\(B\) is called microlocal parametrix for \(A\) on \(\mathcal{C}\).

The proof of this proposition can be found in [Hin21] Prop. 6.15. For later convenience we conclude the section stating a result on the properties of pseudodifferential operators acting on Besov spaces, see [Abe12] Sect 6.6.

**Theorem 21:** Let \(m \in \mathbb{R}, \alpha \in \mathbb{R}\) and let \(a \in S^m(\mathbb{R}^d; \mathbb{R}^d)\). Let \(A : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)\) be the associated element of \(\Psi^m(\mathbb{R}^d)\) as per Definition [12] Equation (2.12b) and Remark [13]. Then the restriction of \(A\) to a Besov space as per Definition [5] setting \(p = q = \infty\) is a bounded linear operator \(A : B^\infty_{\infty, \infty}(\mathbb{R}^d) \to B^\infty_{\infty, \infty}(\mathbb{R}^d)\).

### 2.1.1 Localization of a \(\Psi\)DO

In the next sections, we will be interested in the behaviour of \(\Psi\)DOs under the action of a local diffeomorphism. To this end we adapt to our framework and to our notations the analysis in [Hor94] Chap. 18.1.

Hence, let \(\Omega \subset \mathbb{R}^d\) be an open subset, we say that a function \(a \in C^\infty(\Omega \times \mathbb{R}^d)\) identifies a local symbol on \(\Omega \times \mathbb{R}^d\), i.e. \(a \in S^m(\Omega; \mathbb{R}^d)\) if \(\phi a \in S^m(\mathbb{R}^d; \mathbb{R}^d)\) for all \(\phi \in C^\infty_0(\Omega)\). Using Equation (2.12b) one identifies an operator
\[
\text{Op}_L(a) : S'(\mathbb{R}^d) \to \mathcal{D}'(\Omega). \tag{2.17}
\]

Observing that \(C^\infty_0(\Omega) \hookrightarrow \mathcal{E}'(\Omega) \hookrightarrow S'(\mathbb{R}^d)\), one can restrict the domain in Equation (2.17) to an operator \(\text{Op}_L(a) : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)\) or \(\text{Op}_L(a) : C^\infty_0(\Omega) \to C^\infty(\Omega)\), where with a slight abuse of notation we keep on using the same symbol \(\text{Op}_L(a)\). In full analogy with Definition [12] we indicate the ensuing collection of pseudodifferential operators by \(\Psi^m(\Omega)\). The following theorem is the direct adaptation to our setting and notations of [Hor94] Thm. 18.1.17.

**Theorem 22:** Let \(\Omega, \Omega' \subset \mathbb{R}^d\) be open subsets, \(f \in \text{Diff}(\Omega; \Omega')\) and let \(A \in \Psi^m(\Omega')\). Then
\[
A_f : C^\infty_0(\Omega) \to C^\infty(\Omega), \quad u \mapsto A_fu := A((f^{-1})^*u) \circ f \tag{2.18}
\]
is a pseudodifferential operator of order \(m\). Moreover,
\[
\sigma_m(A_f)(x, \xi) = \sigma_m(A)(f(x), (\text{det} f(x))^{-1}\xi), \tag{2.19}
\]
where \(\sigma_m(A_f)\) and \(\sigma_m(A)\) are the principal symbols of \(A_f\) and \(A\) respectively while \(\text{det} f\) stands for the differential map associated with \(f\).

## 3 Besov Wavefront Set

The aim of this section is to introduce our main object of investigation. We shall therefore give a definition of Besov wavefront set, discussing subsequently its main structural properties. We proceed in two different, albeit ultimately equivalent ways. The first is based on the prototypical notion of wavefront set based on Fourier transforms – [Her03] Ch. 8], while the second, outlined in Section 3.1 relies on pseudodifferential operators as introduced in Section 3. Observe that, in the following, we rely heavily on Proposition [9] as well as on Definition [5].
Definition 23: Let \( u \in \mathcal{D}'(\mathbb{R}^d) \) and \( \alpha \in \mathbb{R} \). We say that \( (x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \) does not lie in the \( B^\alpha_{\infty, \infty} \)-wavefront set, denoting \( (x_0, \xi_0) \not\in \text{WF}^\alpha(u) \), if there exist \( \phi \in \mathcal{D}(\mathbb{R}^d) \) with \( \phi(x) \neq 0 \) as well as an open conic neighborhood \( \Gamma \) of \( \xi \) in \( \mathbb{R}^d \setminus \{0\} \) such that for any compact set \( K \subset \mathbb{R}^d \)

\[
\left| \int_{\Gamma} \hat{\phi}(\xi) \check{\xi}(\xi) e^{ix \cdot \xi} d\xi \right| \lesssim 1, \tag{3.1}
\]

\[
\left| \int_{\Gamma} \hat{\phi}(\xi) \check{\xi}(\lambda \xi) e^{ix \cdot \xi} d\xi \right| \lesssim \lambda^\alpha, \tag{3.2}
\]

for any \( \kappa \in \mathfrak{N}(\alpha), K \subset \mathcal{D}(B(0,1)) \) with \( \check{\kappa}(0) \neq 0, \lambda \in (0,1) \) and \( x \in K \).

Remark 24: Observe that, on account of Proposition 9 and of Remark 10, whenever \( \alpha < 0 \) in Definition 23 it suffices to check that Equation (3.2) holds true.

We are now in a position to prove some basic properties of the Besov wavefront set which are a direct consequence of its definition.

Proposition 25: Let \( u \in \mathcal{D}'(\mathbb{R}^d) \). Then

\[ u \in B^\alpha_{\infty, \infty}(\mathbb{R}^d) \iff \text{WF}^\alpha(u) = \emptyset. \]

Proof. The implication

\[ u \in B^\alpha_{\infty, \infty}(\mathbb{R}^d) \implies \text{WF}^\alpha(u) = \emptyset, \]

follows immediately combining Definition 3 and Proposition 9 with Definition 23. Conversely, if \( \text{WF}^\alpha(u) = \emptyset \), then once more Definition 23 entails that, for any \( \phi \in \mathcal{D}(\mathbb{R}^d) \), it holds

\[
\left| \int_{\mathbb{R}^d} \hat{\phi}(\eta)e^{iy \cdot \eta} \check{\kappa}(\lambda \eta) d\eta \right| \lesssim \lambda^\alpha, \quad \left| \int_{\mathbb{R}^d} \hat{\phi}(\eta)e^{iy \cdot \eta} \check{\kappa}(\eta) d\eta \right| \lesssim 1.
\]

From Proposition 9 it descends that \( \hat{\phi}u \in B^\alpha_{\infty, \infty}(\mathbb{R}^d) \) for any \( \phi \in \mathcal{D}(\mathbb{R}^d) \). This proves the sought statement.

Proposition 26: Let \( u, v \in \mathcal{D}'(\mathbb{R}^d) \). Then

\[ \text{WF}^\alpha(u + v) \subset \text{WF}^\alpha(u) \cup \text{WF}^\alpha(v). \]

Proof. Assume \( (x_0, \xi_0) \in \text{WF}^\alpha(u + v) \). Then, for any test function \( \phi \in \mathcal{D}(\mathbb{R}^d) \), open conic neighborhood \( \Gamma \) of \( \xi_0 \), there exists a compact set \( K \subset \mathbb{R}^d \) such that, for any \( N \in \mathbb{N} \), it holds true

\[
\left| \int_{\Gamma} \phi(u + v)(\xi) \check{\kappa}(\lambda \xi) e^{ix \cdot \xi} d\xi \right| > Nx, \quad \text{for some } x \in K \text{ and } \lambda \in (0,1].
\]

Applying the triangle inequality, it descends

\[
Nx \lesssim \left| \int_{\Gamma} \hat{\phi}(\xi) \check{\kappa}(\lambda \xi) e^{ix \cdot \xi} d\xi \right| + \left| \int_{\Gamma} \hat{\phi}(\xi) \check{\kappa}(\lambda \xi) e^{ix \cdot \xi} d\xi \right|,
\]

which entails that \( (x_0, \xi_0) \in \text{WF}^\alpha(u) \cup \text{WF}^\alpha(v) \).
Corollary 27: Let \( u \in \mathcal{D}'(\mathbb{R}^d) \). If \( \alpha_1 \leq \alpha_2 \), then
\[
\text{WF}^{\alpha_1}(u) \subseteq \text{WF}^{\alpha_2}(u).
\] (3.3)

Proof. The inclusion in Equation (3.3) follows immediately from Definition 25 particularly Equation 3.2.

Remark 28: Observe that, on account of the inclusion \( C^\infty(\mathbb{R}^d) \subset B_{\infty,\infty}^\alpha(\mathbb{R}^d) \) for all \( \alpha \in \mathbb{R} \), Proposition 25 entail that, for every \( f \in C^\infty(\mathbb{R}^d) \)
\[
\text{WF}^{\alpha}(f) = \emptyset \quad \forall \alpha \in \mathbb{R}
\]

In particular, this result entails that, given any \( u \in \mathcal{D}'(\mathbb{R}^d) \), if \( x \notin \text{singsupp}(u) \), then \( (x, \xi) \notin \text{WF}^{\alpha}(u) \) for all \( \alpha \in \mathbb{R} \). Here \( \text{singsupp}(u) \) refers to the singular support of \( u \), see [Hör03, Def. 2.2.3] for the definition. In the following, we give some explicit examples of Besov wavefront sets. Observe that the results of Remark 28 are always implicitly taken into account.

Example 29: Let \( u = \delta \in \mathcal{D}'(\mathbb{R}^d) \) be the Dirac delta centered at the origin. Recalling that for any \( \phi \in \mathcal{D}(\mathbb{R}^d) \) \( \phi \delta = \phi(0) \delta \), Equation (3.1) translates to
\[
\left| \int_{\Gamma} \tilde{\kappa}(\eta)e^{iy \cdot \eta}d\eta \right| \leq \int_{\Gamma} |\tilde{\kappa}(\eta)| \, d\eta \lesssim 1,
\]
since \( \tilde{\kappa} \in \mathcal{S}(\mathbb{R}^d) \). Here we have neglected \( \phi(0) \) since it plays no rôle. Focusing instead on Equation (3.2), for any choice of \( \phi \in \mathcal{D}(\mathbb{R}^d) \) with \( \phi(0) \neq 0 \), it descends, neglecting once more \( \phi(0) \), that
\[
\left| \int_{\Gamma} \tilde{\kappa}(\eta)e^{iy \cdot \eta}d\eta \right| \leq \int_{\Gamma} |\tilde{\kappa}(\lambda \eta)| \, d\eta \lesssim \lambda^{-d},
\]
where the last inequality descends from the change of variable \( \eta \mapsto \eta' := \lambda \eta \). While this estimate entails that \( \text{WF}^{\alpha}(\delta) = \emptyset \) if \( \alpha \leq -d \), in order to obtain a sharp estimate observe that we can set \( y = 0 \) in Equation (3.2) since it lies in \( \text{supp}(\phi) \) for any admissible \( \phi \), being \( \phi(0) \neq 0 \). Hence it descends
\[
\left| \int_{\Gamma} \tilde{\kappa}(\lambda \eta)d\eta \right| = \lambda^{-d} \left| \int_{\Gamma} \tilde{\kappa}(\eta')d\eta' \right| = C_\lambda \lambda^{-d},
\]
where \( \eta' := \lambda \eta \) and where we used implicitly both that \( \Gamma \) is a cone and that \( \tilde{\kappa} \in \mathcal{S}(\mathbb{R}^d) \). At this stage, comparing with Definition 25 we can conclude that
\[
\text{WF}^{\alpha}(\delta) = \begin{cases} 
\emptyset & \alpha \leq -d, \\
(0, \xi) : \xi \in \mathbb{R}^d \setminus \{0\} & \alpha > -d.
\end{cases}
\]

Example 30: Let \( u = \partial_j \delta \in \mathcal{D}'(\mathbb{R}^d) \) be a derivative of the Dirac delta centered at the origin, i.e. \( \partial_j = \frac{\partial}{\partial x_j}, \ x_j \) being an Euclidean coordinate on \( \mathbb{R}^d \). Following Definition 25 and using the identity \( \phi \partial_j \delta = \phi(0) \partial_j \delta - (\partial_j \phi)(0) \delta \) for any \( \phi \in \mathcal{D}(\mathbb{R}^d) \), Equation (3.1) translates to
\[
\left| (\partial_j \phi)(0) \int_{\Gamma} \eta_j \tilde{\kappa}(\eta)e^{iy \cdot \eta}d\eta - \phi(0) \int_{\Gamma} \tilde{\kappa}(\eta)e^{iy \cdot \eta}d\eta \right| \leq \int_{\Gamma} |(\partial_j \phi)(0)\eta_j - \phi(0))\tilde{\kappa}(\eta)| \, d\eta \lesssim 1,
\]
where, similarly to Example 29, we exploited that \( \tilde{\kappa} \in \mathcal{S}(\mathbb{R}^d) \). Focusing on Equation (3.2), we can repeat the same procedure as in Example 29. For the sake of conciseness we focus directly only on \( y = 0 \) since
Example 32: Let

\[
\int_{\Gamma} \eta_j \tilde{k}(\lambda \eta) d\eta = \lambda^{-d-1} \left| \int_{\Gamma} \eta_j' \tilde{k}(\eta') d\eta' \right| = \tilde{C}_k \lambda^{-d-1},
\]

where \( \eta' := \lambda \eta \) and where we used implicitly both that \( \Gamma \) is a cone and that \( \tilde{k} \in S(\mathbb{R}^d) \). Adding to this equality the outcome of Example 29, it descends

\[
\text{WF}^\alpha(\partial_x \delta) = \begin{cases}
\emptyset & \alpha \leq -d - 1, \\
(0, \xi) : \xi \in \mathbb{R}^d \setminus \{0\} & \alpha > -d - 1.
\end{cases}
\]

Example 31: Let \( u \in E'(\mathbb{R}^d) \). Observe that there exists \( C > 0 \) such that

\[
|\hat{u}(\xi)| \leq C |\xi|^M
\]

where \( M \) is the order of \( u \) and \( (\xi) := (1 + |\xi|^2)^{\frac{7}{4}} \), see [TM99]. Fix \( \Gamma \) an open conic neighborhood of \( \xi \in \mathbb{R}^d \setminus \{0\} \). Given \( \kappa \) as per Definition 5, \( \lambda \in (0,1) \) and \( y \in \text{supp}(u) \), it holds

\[
\left| \int_{\Gamma} \hat{u}(\eta)e^{i\eta \cdot \kappa(\lambda \eta)} d\eta \right| \leq \int_{\Gamma} |\hat{u}(\eta)||\tilde{k}(\lambda \eta)| d\eta \leq C \int_{\Gamma} |\eta|^M |\hat{k}(\lambda \eta)| d\eta \approx \lambda^{-M-d} \int_{\Gamma} |\eta|^M |\hat{k}(\eta)| d\eta \lesssim \lambda^{-M-d},
\]

where, with reference to Equation (3.11) and (3.2), we have implicitly chosen \( \phi \in \mathcal{D}(\mathbb{R}^d) \) such that \( \phi = 1 \) on \( \text{supp}(u) \). As a result, we get \( \text{WF}^\alpha(u) = \emptyset \) if \( \alpha \leq -d-M \).

Example 32: Let \( u: \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( u(x_1, x_2) = (x_1^2 + x_2^2)^{\frac{7}{4}} \). We recall that \( \hat{u}(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{-\frac{3}{2}} \), which should be interpreted as the integral kernel of an element lying in \( S'(\mathbb{R}^2) \). Since \( \text{singsupp}(u) = \{(0,0)\} \), we consider \( (0,0, \xi_1, \xi_2) \) such that \( (\xi_1, \xi_2) \neq (0,0) \). Given \( \phi \in \mathcal{D}(\mathbb{R}^2) \) with \( \phi(0,0) = 1 \) and an open conic neighborhood \( \Gamma \) of \( (\xi_1, \xi_2) \), we can still use the rationale followed in Example 29 studying Equation (3.2) with \( y = (0,0) \). It reads

\[
\left| \int_{\Gamma} \hat{\tilde{u}}(\eta_1, \eta_2) \hat{\kappa}(\lambda \eta_1, \lambda \eta_2) d\eta_1 d\eta_2 \right| = \left| \int_{\Gamma} (\eta_1^2 + \eta_2^2)^{-\frac{3}{2}} \hat{\kappa}(\lambda \eta_1, \lambda \eta_2) d\eta_1 d\eta_2 \right| = \\
\left( \lambda \eta_1^2 + \lambda \eta_2^2 \right)^{-\frac{3}{2}} |\hat{\kappa}(\eta_1, \eta_2)| d\eta_1 d\eta_2 = C_k \lambda^\frac{1}{2},
\]

where no singularity at the origin occurs since \( \kappa \) is chosen in agreement with Definition 9. This entails that

\[
\begin{align*}
\text{WF}^\alpha(u) &= \emptyset & & \alpha \leq -\frac{1}{2} \\
\text{WF}^\alpha(u) &= \{(0,0, \xi_1, \xi_2) : (\xi_1, \xi_2) \neq (0,0)\} & & \alpha > -\frac{1}{2}
\end{align*}
\]

3.1 Pseudodifferential Characterization

The aim of this section is to give a second, albeit equivalent, characterization of the Besov wavefront set of a distribution by means of pseudodifferential operators. This is in spirit very much akin to the one outlined in [CS94] for the smooth wavefront set and it is especially useful in discussing operations between distributions with a prescribed Besov wavefront set, see Section 4. In the following, we shall make use of the notions introduced in Definition 12 and 14.
Proposition 33: Let \( \alpha \in \mathbb{R} \) and \( u \in \mathcal{D}'(\mathbb{R}^d) \). Then

\[
\text{WF}^\alpha(u) = \bigcap_{A \in \Psi^0(\mathbb{R}^d), A \in B^\alpha_{\infty,\infty}(\mathbb{R}^d)} \text{Char}(A),
\]

(3.6)

where the intersection is taken only over properly supported pseudodifferential operators.

Proof. Suppose that \((x_0, \xi_0) \notin \text{WF}^\alpha(u)\). By Definition 20, there exist \( \phi \in \mathcal{D}(\mathbb{R}^d) \) with \( \phi(x_0) \neq 0 \) and \( \Gamma \), a conic neighbourhood of \( \xi_0 \), such that for any compact set \( K \subset \mathbb{R}^d \)

\[
\left| \int_{\Gamma} \hat{\phi}(\xi) \kappa(\lambda \xi) e^{ix \cdot \xi} d\xi \right| \lesssim \lambda^\alpha \quad \forall x \in K, \forall \lambda \in (0, 1],
\]

where \( \kappa \in \mathcal{R}_\alpha \). Calling \( \Gamma(\xi) \) the characteristic function on \( \Gamma \), it descends that

\[
\mathcal{F}^{-1} \left[ \Gamma(\xi) \hat{\phi}(\xi) \right] \in B^\alpha_{\infty,\infty}(\mathbb{R}^d).
\]

(3.7)

Set \( \chi \in C^\infty(\mathbb{R}^d) \) to be such that \( \chi(\xi) = 0 \) if \( |\xi| \leq a \) and \( \chi(\xi) = 1 \) if \( |\xi| \geq 2a \) where \( a \) is a non vanishing constant chosen so that \( \chi(\xi_0) \neq 0 \). In addition choose \( \psi \in C^\infty(S^{\alpha-1}) \) such that \( \text{supp}(\psi) \subset B_{\varepsilon}(\xi_0/|\xi_0|) \subset \Gamma \), \( \varepsilon > 0 \) and \( \psi(\xi_0/|\xi_0|) \neq 0 \). Consequently we can introduce \( A := \text{Op}(a) \in \Psi^0(\mathbb{R}^d) \), where

\[
a(x, y, \xi) = \phi(x) \psi \left( \frac{\xi}{|\xi|} \right) \chi(\xi) \phi(y) \in S^0(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d).
\]

(3.8)

Observe that, following standard arguments, \( A \) is by construction properly supported and elliptic at \((x_0, \xi_0)\). To conclude it suffices to notice that, combining Equation 3.7 and Theorem 21, we can conclude that \( Au \in B^\alpha_{\infty,\infty}(\mathbb{R}^d) \).

Conversely, let \((x_0, \xi_0) \notin \bigcap_{A \in \Psi^0(\mathbb{R}^d)} \text{Char}(A) \). Hence, taking into account Definition 15 there exists \( B \in \Psi^0 \), elliptic at \((x_0, \xi_0)\), such that \( Bu \in B^\alpha_{\infty,\infty}(\mathbb{R}^d) \). Consider once more \( \phi, \psi \) and \( \chi \) as in the previous part of the proof, so that

\[
\text{WF}'(A) \subset \text{Ell}(B).
\]

where \( A := \text{Op}_R(\psi(\xi/|\xi|) \chi(\xi) \phi(y)) \) and where \( \text{WF}' \) is as per Definition 16. We claim that \( Au \in B^\alpha_{\infty,\infty}(\mathbb{R}^d) \). In view of Proposition 20 there exists a microlocal parametrix \( Q \in \Psi^0(\mathbb{R}^d) \) of \( B \) such that \( QB = I - R \) with \( R \in \Psi^{-1}(\mathbb{R}^d) \) and \( \text{WF}'(R) \cap \text{WF}'(A) = \emptyset \). Thus, \( Au = A(QB + R)u = (AQ)(Bu) + ARu \),

where \( ARu \in C^\infty(\mathbb{R}^d) \). Given \( \rho \in \mathcal{D}(\mathbb{R}^d) \) such that \( \rho = 1 \) on \( \text{supp}(\phi) \), it descends

\[
(ARu)(Bu) = (AQ)(\rho Bu) + (AQ)((1 - \rho) Bu).
\]

Since \( 1 - \rho = 0 \) on \( \text{supp}(\phi) \), then \( (AQ)((1 - \rho) Bu) = 0 \). At the same time \( (AQ)(\rho Bu) \in B^\alpha_{\infty,\infty}(\mathbb{R}^d) \) on account of Theorem 21. This entails that \( Au \in B^\alpha_{\infty,\infty}(\mathbb{R}^d) \). Hence, given \( \kappa \in \mathcal{R}_\alpha \), see Definition 5 it holds

\[
\left| \int_{\text{Ell}(\psi(D/(D)) \chi(D))} \psi \left( \frac{\xi}{|\xi|} \right) \chi(\xi) \hat{\phi}(\xi) \kappa(\lambda \xi) e^{ix \cdot \xi} d\xi \right| \lesssim \lambda^\alpha, \quad \forall \lambda \in (0, 1], \quad \forall x \in K.
\]

(3.9)
On account of Remark 19 there exists a symbol $p \in S^0(\mathbb{R}^n; \mathbb{R}^n)$ such that
\[ r(\xi) := 1 - \psi \left( \frac{\xi}{|\xi|} \right) \chi(\xi)p(\xi) \in S^{-1} \]
for any $\xi \in \text{Ell}(\psi(D/D)|\chi(D))$. It descends
\[ \left| \int_{\text{Ell}(\psi(D/D)|\chi(D))} \widehat{\phi u}(\xi) \hat{\kappa}(\lambda \xi)e^{ix\cdot \xi}d\xi \right| = \left| \int_{\text{Ell}(\psi(D/D)|\chi(D))} \left( \psi \left( \frac{\xi}{|\xi|} \right) \chi(\xi)p(\xi) + r(\xi) \right) \widehat{\phi u}(\xi) \hat{\kappa}(\lambda \xi)e^{ix\cdot \xi}d\xi \right| \]
\[ \leq \left| \int_{\text{Ell}(\psi(D/D)|\chi(D))} \psi \left( \frac{\xi}{|\xi|} \right) \chi(\xi)p(\xi) \widehat{\phi u}(\xi) \hat{\kappa}(\lambda \xi)e^{ix\cdot \xi}d\xi \right| + \left| \int_{\text{Ell}(\psi(D/D)|\chi(D))} r(\xi) \widehat{\phi u}(\xi) \hat{\kappa}(\lambda \xi)e^{ix\cdot \xi}d\xi \right| \]
\[ = \left| \left\langle p(D)\psi \left( \frac{D}{|D|} \right) \chi(D)(\phi u), \kappa^2 \right\rangle \right| + \left| \int_{\text{Ell}(\psi(D/D)|\chi(D))} r(\xi) \widehat{\phi u}(\xi) \hat{\kappa}(\lambda \xi)e^{ix\cdot \xi}d\xi \right|, \]
for any $x \in K$ and $\lambda \in (0, 1]$. On the one hand, as a result of Theorem 21 and Equation (3.9), it holds that
\[ |A| \lesssim \lambda^\alpha. \]
On the other hand,
\[ |B| \leq \left| \left\langle r(D)p(D)\psi \left( \frac{D}{|D|} \right) \chi(D)(\phi u), \kappa^2 \right\rangle \right| + \left| \int_{\text{Ell}(\psi(D/D)|\chi(D))} r^2(\xi) \widehat{\phi u}(\xi) \hat{\kappa}(\lambda \xi)e^{ix\cdot \xi}d\xi \right| \lesssim \lambda^{\alpha+1} + \left| \int_{\text{Ell}(\psi(D/D)|\chi(D))} r^2(\xi) \widehat{\phi u}(\xi) \hat{\kappa}(\lambda \xi)e^{ix\cdot \xi}d\xi \right|, \]
where we applied once more Theorem 21 with $r(D) \in \Psi^{-1}(\mathbb{R}^d)$ and $p(D)\psi \left( \frac{D}{|D|} \right) \chi(D)(\phi u) \in B_{\infty,\infty}^\alpha(\mathbb{R}^d)$. This concludes the proof. \(\square\)

**Remark 34:** The content of Proposition 33 is an adaptation to the case in hand of the characterization of the smooth wavefront set of a distribution in terms of pseudodifferential operators, see \cite{Hin21} Cor. 6.18. For later convenience and to fix the notation, we recall it. Let $v \in \mathcal{D}'(\mathbb{R}^d)$. It holds
\[ WF(v) = \bigcap_{A \in \Psi^0(\mathbb{R}^d), A \in C^\infty(\mathbb{R}^d)} \text{Char}(A), \]
where Char(A) is the characteristic set of A introduced in Definition 18.

We prove a proposition aimed at stating another useful characterization of the Besov wavefront set of a distribution.

**Proposition 35:** Let $u \in \mathcal{D}'(\mathbb{R}^d)$. It holds that
\[ (x, \xi) \in WF^\alpha(u) \iff (x, \xi) \in WF(u - v) \quad \forall v \in B_{\infty,\infty}^\alpha(\mathbb{R}^d), \]
where WF stands for the (smooth) wavefront set.
Proof. Suppose \((x, \xi) \in \WF^\alpha(u)\). On account of Remark 34, given \(v \in B^\infty_{\infty, \infty}(\R^d)\) we consider \(A \in \Psi^0(\R^d)\) such that \(A(u-v) \in C^\infty(\R^d)\). This entails that \(Au \in B^\infty_{\infty, \infty}(\R^d)\). Yet, since \((x, \xi) \in \WF^\alpha(u)\), Proposition 35 entails that \((x, \xi) \in \Char(A)\).

Conversely, let \((x, \xi) \notin \WF^\alpha(u)\). By Definition 23, there exist \(\phi \in \mathcal{D}(\R^d)\) normalized so that \(\phi(x) = 1\) and an open conic neighborhood \(\Gamma\) of \(\xi\) such that Equation (3.1) is satisfied. Let \(v \in B^\infty_{\infty, \infty}(\R^d)\) be such that

\[
\hat{v}(\eta) = \begin{cases} 
\hat{\phi}(\eta) & \text{if } \eta \in \Gamma, \\
0 & \text{otherwise} 
\end{cases}
\]  

(3.13)

Then \(\hat{\theta} = \hat{\phi}u - \hat{v}\) vanishes on \(\Gamma\) and therefore \((x, \xi) \notin \WF(\theta)\). Consider \(\chi \in \mathcal{D}(\R^d)\) such that \(\chi \phi = 1\) in a neighbourhood of \(x \in \R^d\). Then \(\chi v \in B^\infty_{\infty, \infty}(\R^d)\) and \((x, \xi) \notin \WF(\chi \theta)\). After observing that \(u - \chi v = (1 - \chi \phi)u + \chi \theta\), we conclude \((x, \xi) \notin \WF(u - \chi v)\) exploiting that \((1 - \chi \phi)u\) vanishes in a neighbourhood of \(x\). Since \(\chi = 1\) at \(x\), we can conclude that \((x, \xi) \notin \WF(u - v)\).

We can now establish a relation between the Besov wavefront set and the smooth counterpart, see Remark 34. The second part of the proof of the following corollary is inspired by a similar one, valid in the context of the Sobolev wavefront set Hin21 Prop. 6.32.

Corollary 36: Let \(u \in \mathcal{D}'(\R^d)\). It holds that

\[
\WF(u) = \bigcup_{\alpha \in \mathbb{R}} \WF^\alpha(u).
\]  

(3.14)

Proof. Assume \((x, \xi) \in \WF^\alpha(u)\) for any \(\alpha \in \mathbb{R}\). Using Proposition 35, we can choose \(v \in C^\infty_0(\R^d) \subset B^\infty_{\infty, \infty}(\R^d)\) concluding that \((x, \xi) \in \WF(u - v) = \WF(u)\). Hence \(\bigcup_{\alpha \in \mathbb{R}} \WF^\alpha(u) \subseteq \WF(u)\). Taking the closure and recalling that \(WF(u)\) is per construction a closed set, it descends \(\bigcup_{\alpha \in \mathbb{R}} \WF^\alpha(u) \subseteq \WF(u)\).

To prove the other inclusion, assume \((x, \xi) \notin \WF^\alpha(u)\) for all \(\alpha \in \mathbb{R}\). Hence there must exist a conic, open set \(\Gamma \subseteq \R^d \times \R^d \setminus \{0\}\) such that \((x, \xi) \in \Gamma\) and \(\Gamma \cap \WF^\alpha(u) = \emptyset\) for all \(\alpha \in \mathbb{R}\). We can thus choose \(A \in \Psi^0(\R^d)\) to be properly supported, elliptic at \((x, \xi)\) and such that \(WF'(A) \subset \Gamma\) and \(Au \in B^\infty_{\infty, \infty}(\R^d)\) for all \(\alpha \in \mathbb{R}\). This entails that \(Au \in C^\infty(\R^d)\). It descends that, since \((x, \xi) \notin \Char(A)\), then \((x, \xi) \notin \WF(u)\).

4 Structural Properties

In this section we discuss the main structural properties of distributions with a prescribed Besov wavefront set as per Definition 23 and Proposition 35 including notable operations.

Transformation Properties under Pullback – We start by investigating the interplay between Definition 23 and the pull-back of a distribution. In the following we enjoy the analysis outlined in Section 2.1.1.

Remark 37: In Definition 23 as well as in Proposition 35 we have always assumed implicitly that the underlying distribution is globally defined, i.e. \(u \in \mathcal{D}'(\R^d)\). Yet, mutatis mutandis, the whole construction and the results obtained so far can be slavishly adapted to distributions \(v \in \mathcal{D}'(\Omega), \Omega \subseteq \R^d\).

Theorem 38 (Pull-back - I): Let \(\Omega \subseteq \R^d, \Omega' \subseteq \R^m\) be open sets and let \(f \in C^\infty(\Omega; \Omega')\) be an embedding. Moreover let

\[
N_f := \{(f(x), \xi) \in \Omega' \times \R^m : \, ^tdf(x)\xi = 0\},
\]  

(4.1)
be the set of normals of \( f \). For any \( u \in \mathcal{D}'(\Omega') \) such that there exists \( \alpha > 0 \) so that

\[
N_f \cap \text{WF}^\alpha(u) = \emptyset,
\]

there exists \( f^* u \in \mathcal{D}'(\Omega) \). In addition

\[
\text{WF}^\alpha(f^* u) \subseteq f^* \text{WF}^\alpha(u),
\]

for every \( u \in \mathcal{D}'(\Omega') \) abiding to Equation (4.2), where

\[
f^* \text{WF}^\alpha(u) := \{(x, \partial_x f(x) \eta) : (f(x), \eta) \in \text{WF}^\alpha(u)\}.
\]

**Proof.** As a consequence of Proposition 33, Equation (1.2) is equivalent to

\[
N_f \cap \text{WF}(u - v) = \emptyset, \quad \forall v \in B^{\alpha,\infty}_{\infty,\infty}(\Omega)
\]

Then, there exists the pullback \( f^*(u - v) \in \mathcal{D}'(\Omega) \). Taking into account that \( B^{\alpha,\infty}_{\infty,\infty}(\Omega') \subset C^0(\Omega') \) for \( \alpha > 0 \), we have that \( f^* v = v \circ f \). Thus,

\[
f^* u = f^*(u - v) + f^* v
\]

identifies an element lying in \( \mathcal{D}'(\Omega) \). Focusing on Equation (4.3), let \( (x, \partial_x f(x) \eta) \notin f^* \text{WF}^\alpha(u) \). It implies \( (f(x), \eta) \notin \text{WF}^\alpha(u) \). By Proposition 33, there exists \( A \in \Psi^0(\Omega') \), elliptic in \( (f(x), \eta) \), such that \( A u \in B^{\alpha,\infty}_{\infty,\infty}(\Omega') \). Bearing in mind that \( f \) is a diffeomorphism on \( f[\Omega] \),

\[
A_f(f^* u) = (Au) \circ f,
\]

identifies a pseudodifferential operator of order 0 per Theorem 22. Since \( (Au) \circ f \in B^{\alpha,\infty}_{\infty,\infty}(\mathbb{R}^d) \), Theorem 22 entails

\[
\sigma_0(A_f)(x, \partial_x f(x) \eta) = \sigma_0(A)(f(x), \eta) \neq 0.
\]

This proves \( (x, \partial_x f(x) \eta) \notin \text{WF}^\alpha(f^* u) \).

To conclude this first part of the section, we shall prove that Besov wavefront set is invariant under the diffeomorphisms.

**Theorem 39** (Pull-back - II): Let \( \Omega, \Omega' \subseteq \mathbb{R}^d \) be two open subsets and let \( f : \Omega \to \Omega' \) be a diffeomorphism. Then, given \( u \in \mathcal{D}'(\Omega') \), for any \( \alpha \in \mathbb{R} \), it holds

\[
\text{WF}^\alpha(f^* u) = f^* \text{WF}^\alpha(u).
\]

**Proof.** We prove the inclusion \( \text{WF}^\alpha(f^* u) \subseteq f^* \text{WF}^\alpha(u) \). Let \( (x, \xi) \notin f^* \text{WF}^\alpha(u) \), i.e., \( (f(x), (\partial_x f(x))^{-1} \xi) \notin \text{WF}^\alpha(u) \). Thus, there exists \( A \in \Psi^0(\Omega') \), elliptic at \( (f(x), (\partial_x f(x))^{-1} \xi) \), such that \( A u \in B^{\alpha,\infty}_{\infty,\infty}(\Omega') \). If one introduces \( A_f \in \Psi^0(\Omega) \) such that

\[
A_f(f^* u) = f^*(Au),
\]

Equation (2.19) entails that

\[
\sigma_0(A_f)(x, \xi) = \sigma_0(A)(f(x), (\partial_x f(x))^{-1} \xi) \neq 0,
\]

that is \( A_f \) is elliptic at \( (f(x), (\partial_x f(x))^{-1} \xi) \). To conclude, we need to prove that \( A_f(f^* u) \in B^{\alpha,\infty}_{\infty,\infty}(\Omega) \). For any but fixed \( \phi \in C_0^\infty(\Omega) \), it holds

\[
|\langle \phi A_f(f^* u), \kappa \rangle| < |\langle f^*(Au), \phi \kappa \rangle| = |\{Au, (f_\eta)(f_* \kappa)^{-1}(E)\}| \lesssim |\langle Au, (f_* \phi)(f_* \kappa)^{-1}(E)\}| \lesssim \lambda^\alpha,
\]

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for any $z \in \Omega$, $\lambda \in (0, 1]$ and $\kappa \in \mathcal{B}_\alpha$. An analogous estimate yields

$$||\phi_A(f^*u), z|| \leq 1,$$

for any $z \in \Omega$ and $x \in \mathcal{D}(B(0, 1))$ such that $\bar{\phi}(0) \neq 0$. This proves that $(x, \xi) \not\in WF^\alpha(f^*u)$. Conversely, let $(x, \xi) \not\in WF^\alpha(f^*u)$. Then, there exists $\bar{A} \in \Psi^0(\Omega)$, elliptic in $(x, \xi)$, such that $\bar{A}(f^*u) \in B^\alpha_{\infty, \infty}(\Omega)$. Let $A \in \Psi^0(\Omega')$ be such that

$$Au = (f^{-1})^*(\bar{A}(f^*u)).$$

Still on account of Theorem 22 it holds that

$$\sigma_0(A)(f(x), (\xi d(f(x))^{-1}) = \sigma_0(\bar{A})(x, \xi) \neq 0.$$

As a consequence, $A$ is elliptic at $(f(x), (\xi d(f(x))^{-1})$. Reasoning as in the first part of the proof, it turns out that $Au \in B^\alpha_{\infty, \infty}(\Omega')$. This entails that $(x, \xi) \not\in f^*WF^\alpha(u)$. \qed

**Remark 40:** Theorem [33] is especially noteworthy since it is the building block to extend the notion of Besov wavefront set to distributions supported on any arbitrary smooth manifold $M$, following the same rationale used when working with the smooth counterpart. On a similar note, we observe that for the sake of simplicity of the presentation, we decided to stick to individuating a point of $WF^\alpha(u)$ belonging to the cotangent bundle $T^*\mathbb{R}^d \setminus \{0\}$. Yet, from a geometrical viewpoint each element of $WF^\alpha(u)$ should be better read as lying in the cotangent bundle $T^*\mathbb{R}^d \setminus \{0\}$. For the sake of conciseness, we shall not dwell into further details which are left to the reader.

**Microlocal Properties of $\Psi$DOs** — Our next task is the study of the interplay between pseudodifferential operators and distributions at the level of wavefront set. To this end we recall a notable result, valid in the smooth setting, see [Hin21] Prop. 6.27, namely, if $A \in \Psi^m(\mathbb{R}^d)$ and $u \in \mathcal{D}'(\mathbb{R}^d)$, then $A$ is microlocal:

$$WF(Au) \subseteq WF^r(A) \cap WF(u),$$

where $WF^r$ stands for the operator wavefront set as per Definition 16. At the level of Besov wavefront set the counterpart of this statement is the following proposition.

**Proposition 41:** Let $A \in \Psi^m(\mathbb{R}^d)$, $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$. Then

$$WF^{\alpha-m}(Au) \subseteq WF^r(A) \cap WF^\alpha(u).$$

**Proof.** Suppose that $(x_0, \xi_0) \not\in WF^r(A)$. As a consequence of Proposition 20 there exists $B \in \Psi^0(\mathbb{R}^d)$, elliptic at $(x_0, \xi_0)$. In addition, we find $B$ such that $WF^r(A) \cap WF^r(B) = \emptyset$. Proposition 17 entails that $BA \in \Psi^{-\infty}(\mathbb{R}^d)$, which implies in turn that $B(Au) \in C^\infty(\mathbb{R}^d) \subseteq B^{m, \infty}_{\infty, \infty}(\mathbb{R}^d)$. Proposition 33 yields that $(x_0, \xi_0) \not\in WF^{\alpha-m}(Au)$. Conversely, suppose $(x_0, \xi_0) \not\in WF^\alpha(u)$. Then, still in view of Proposition 33 there exists $\tilde{A} \in \Psi^0(\mathbb{R}^d)$, elliptic at $(x_0, \xi_0)$, such that $\tilde{A}u \in B^m_{\infty, \infty}(\mathbb{R}^d)$. Take $B \in \Psi^0(\mathbb{R}^d)$ elliptic at $(x_0, \xi_0)$ with $WF^r(B) \subseteq Ell(\tilde{A})$. On account of Proposition 20 there exists a parametrix $Q \in \Psi^0(\mathbb{R}^d)$ of $\tilde{A}$, that is, $Q\tilde{A} = I - R$ with $R \in \Psi^0(\mathbb{R}^d)$ and $WF^r(R) \cap WF^r(B) = \emptyset$. Therefore,

$$B(Au) = BA(Q\tilde{A} + R)u = BAQ(\tilde{A}u) + (BAR)u.$$

Since $BAR \in \Psi^{-\infty}(\mathbb{R}^d)$, then $(BAR)u \in C^\infty(\mathbb{R}^d)$. At the same time $BAQ(\tilde{A}u) \in B^{m, \infty}_{\infty, \infty}(\mathbb{R}^d)$ because $\tilde{A}u \in B^m_{\infty, \infty}(\mathbb{R}^d)$ and $BAR \in \Psi^m(\mathbb{R}^d)$. Yet, since $(x_0, \xi_0) \not\in Char(B)$, it descends $(x_0, \xi_0) \not\in WF^{\alpha-m}(Au)$. This concludes the proof. \Box
The second result we present in this section provides a sort of inverse result, with respect to the previous one, which is more relevant from a PDE viewpoint.

**Proposition 42:** Let \( u \in \mathcal{D}'(\mathbb{R}^d) \), \( A \in \Psi^m(\mathbb{R}^d) \) and \( m, \alpha \in \mathbb{R} \). Then

\[
\WF^\alpha(u) \subseteq \Char(A) \cup \WF^\alpha_m(Au).
\] (4.7)

*Proof.* Let \((x_0, \xi_0) \notin \Char(A) \cup \WF^\alpha_m(Au)\). Thus there exists \( B \in \Psi^0(\mathbb{R}^d) \), elliptic at \((x_0, \xi_0)\), such that \( B(Au) \in B^\alpha_{\infty, \infty, \text{loc}}(\mathbb{R}^d) \). Let \( K \) be any properly supported pseudodifferential operator lying in \( \Psi^{-m}(\mathbb{R}^n) \), which can be chosen without loss of generality to be elliptic at \((x_0, \xi_0)\). It descends \((BKA)u \in B^\alpha_{\infty, \infty, \text{loc}}(\mathbb{R}^d)\). Since \((x_0, \xi_0) \notin \Char(BKA)\), it descends that \((x_0, \xi_0) \notin \WF^\alpha(u)\). \(\square\)

**Corollary 43** (Elliptic Regularity): Let \( u \in \mathcal{D}'(\mathbb{R}^d) \), \( m, \alpha \in \mathbb{R} \) and let \( A \in \Psi^m(\mathbb{R}^d) \) be elliptic. Then

\[
\WF^\alpha(u) = \WF^\alpha_m(Au).
\]

*Proof.* Since \( A \) is an elliptic pseudodifferential operator \( \Char(A) = \emptyset \) and, in view of Definition \([16]\), \( \WF^\alpha(A) = \emptyset \). The statement is thus a direct consequence of Propositions \([11]\) and \([12]\). \(\square\)

**Product of Distributions** — In the following we investigate the formulation of a version of Hörmander’s criterion for the product of distributions, tied to the Besov wavefront set. In the spirit of \([\text{Hor03}]\), we rely on two ingredients. The first has already been discussed in Theorem \([33]\) while the second one concerns the tensor product of two distributions. In particular we wish to establish an estimate on the singular behaviour of \( u \otimes v \) for given \( u, v \in \mathcal{D}'(\mathbb{R}^d) \). This can be read as a direct adaptation to this context of \([\text{JS02}, \text{Prop. B.5}]\) which is based in turn on \([\text{Hor97}, \text{Lemma 11.6.3}]\). For this reason we shall omit the proof.

**Proposition 44** (Tensor product): Let \( \Omega \subseteq \mathbb{R}^d \) and \( \Omega' \subseteq \mathbb{R}^m \) be two open sets. If \( u \in \mathcal{D}'(\Omega) \) and \( v \in \mathcal{D}'(\Omega') \), then the following two inclusions hold true:

\[
\WF^{\alpha + \beta}(u \otimes v) \subseteq \WF^\alpha_0(u) \times WF(v) \cup WF(u) \times WF^\beta_0(v),
\] (4.8)

and, calling \( \gamma := \min\{\alpha, \beta, \alpha + \beta\} \),

\[
\WF^{\gamma}(u \otimes v) \subseteq \WF^\alpha(u) \times WF_0(v) \cup WF_0(u) \times WF^\beta(v),
\] (4.9)

where we adopted the notation \( WF_0(u) := WF(u) \cup (\supp(u) \times \{0\}) \) and similarly \( WF^\alpha_0(u) := WF^\alpha(u) \cup (\supp(u) \times \{0\}) \).

At last, we are in a position to prove a counterpart of Hörmander’s criterion for the product of two distributions within the framework of the Besov wavefront set.

**Theorem 45:** Let \( u, v \in \mathcal{D}'(\Omega) \) where \( \Omega \subseteq \mathbb{R}^d \) is any open set. If \( \forall (x, \xi) \in \Omega \times \mathbb{R}^d \setminus \{0\} \) there exist \( \alpha, \beta \in \mathbb{R} \) with \( \alpha + \beta > 0 \) such that

\[
(x, \xi) \notin \WF^{\alpha}(u) \cup (-WF^\beta(v))
\]

then the product \( uv \in \mathcal{D}'(\Omega) \) can be defined by

\[
uv = \Delta^\gamma(u \otimes v),
\]

where \( \Delta : \Omega \to \Omega \times \Omega \) is the diagonal map. Moreover, calling \( \gamma := \min\{\alpha, \beta\} \),

\[
\WF^{\gamma}(uv) \subset \{(x, \xi + \eta) : (x, \xi) \in \WF^{\alpha}(u), (x, \eta) \in WF_0(v) \text{ or } (x, \xi) \in WF_0(u), (x, \eta) \in WF^\beta(v)\}. \] (4.10)
Proof. Observe that, per hypothesis, there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$ such that

$$WF^{\alpha + \beta}(u \otimes v) \cap N_{\Delta} = \emptyset,$$

where $N_{\Delta} = \{(x, x, \xi, -\xi)\}$ is the set of normal directions of the diagonal map as defined in Equation (4.1). Hence, on account of Theorem 38 combined with Proposition 44, Equation (4.9) in particular, there exists $\Delta^*(u \otimes v) \in D'(\mathbb{R}^d)$ and

$$WF^\gamma(\Delta^*(u \otimes v)) \subset \Delta^*WF^\gamma(u \otimes v) = \{(x, \xi + \eta) : (x, \xi) \in WF^\alpha(u), (x, \eta) \in WF^\beta(v) \text{ or } (x, \xi) \in WF^\alpha(u), (x, \eta) \in WF^\beta(v)\}.$$

This concludes the proof. \qed

**Remark 46:** Observe that, if we consider $u \in B^\alpha_{\infty, \infty}(\mathbb{R}^d)$ and $v \in B^\beta_{\infty, \infty}(\mathbb{R}^d)$ with $\alpha + \beta > 0$, it descends that $WF^\alpha(u) = WF^\beta(v) = \emptyset$. Hence, on account of Theorem 45, there exists $uv \in D'(\mathbb{R}^d)$ and $WF^\gamma(uv) = \emptyset$ with $\gamma = \min(\alpha, \beta)$. This is nothing but the statement of the renown Young’s theorem on the product of two Hölder distributions, see [BCD11, DRS21].

To conclude this section, we discuss an application of Theorem 45 which is of relevance in many concrete scenarios. More precisely, we consider a continuous $\mathcal{K} : C^\alpha_0(\Omega') \to D'(\Omega)$ with kernel $K \in D'((\Omega \times \Omega')$. Given $u \in E'(\Omega')$, we investigate the existence of $\mathcal{K}u$ and we seek to establish a bound on the associated Besov wavefront set. As a preliminary step, we need to prove two ancillary results.

**Corollary 47:** Let $\Omega \times \Omega' \subseteq \mathbb{R}^d \times \mathbb{R}^m$ and let $v \in B^\alpha_{\infty, \infty}(\Omega \times \Omega')$, $\alpha \in \mathbb{R}$. Calling $\pi : \Omega \times \Omega' \to \Omega$ the projection map on the first factor, it holds that $\pi_* v \in B^\alpha_{\infty, \infty}(\Omega)$, $\pi_*$ being the push-forward map.

**Proof.** Without loss of generality, let us consider $v \in B^\alpha_{\infty, \infty}(\Omega \times \Omega') \cap E'(\Omega \times \Omega')$. We recall that

$$(\pi_* v)(\phi) := v(\phi \circ 1),$$

where $\phi \in E(\Omega)$. Then, for any $\kappa \in \mathcal{B}(\alpha)(\Omega)$ as per Definition 5, $x' \in \Omega$, $\lambda \in (0, 1]$,

$$|((\pi_* v)[\kappa^\lambda_2])| = |v((\kappa \circ 1)[\lambda_{x', y'}])| \lesssim \lambda^\alpha.$$

Observe that $\kappa \circ 1 \in \mathcal{B}(\alpha)(\Omega \times \Omega')$. At the same time, for any $\xi \in \mathcal{D}(B(0, 1))$ with $\xi(0) \neq 0$, $x' \in \Omega$, $\lambda \in (0, 1]$, it holds true

$$|((\pi_* v)[\kappa_{x'}])| = |v((\xi \circ 1)[x', y'])| \lesssim \lambda^\alpha,$$

which concludes the proof. \qed

**Proposition 48:** Let $v \in E'(\Omega \times \Omega')$, where $\Omega \times \Omega' \subseteq \mathbb{R}^d \times \mathbb{R}^m$ is an open subset. Assume that the projection map on the first factor $\pi : \Omega \times \Omega' \to \Omega$ is proper on $\text{supp}(v)$. Then it holds that, for all $\alpha \in \mathbb{R}$,

$$WF^\alpha(\pi_* v) \subset \{(x, \xi) \in (\Omega \times \mathbb{R}^d \setminus \{0\}) \mid \exists y \in \text{supp}(v) \text{ for which } (x, y, \xi, 0) \in WF^\alpha(v)\},$$

where $\pi_*$ is the push-forward map.

**Proof.** Since $v$ is compactly supported and since the action of $\pi_*$ is tantamount to a partial evaluation against the constant function $1 \in C^\infty(\Omega')$, i.e., $\pi_*(v)(\phi) = v(\phi \circ 1)$ for all $\phi \in E(\Omega)$, then $\pi_*(v) \in E'(\Omega)$. On account of Proposition 55 a pair $(x, \xi) \in WF^\alpha(\pi_* v)$ if and only if $(x, \xi) \in WF(\pi_* v - u)$ for all $u \in B^\alpha_{\infty, \infty}(\Omega) \cap E'(\Omega)$. Here we can restrict the attention to compactly supported elements lying in $B^\alpha_{\infty, \infty}(\Omega)$ since $\pi_* v \in E'(\Omega)$. 

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In turn, on account of Corollary 47 we can replace \( u \) by \( \pi_∗(\tilde{u}) \), where \( \tilde{u} \in B^\omega_{∞,∞}(Ω × Ω′) \cap E′(Ω × Ω′) \). In other words it turns out that

\[(x, ξ) ∈ WF^α(\pi_∗v) ⇐⇒ (x, ξ) ∈ WF(\pi_∗(v − \tilde{u})) \quad ∀ \tilde{u} ∈ B^\omega_{∞,∞}(Ω × Ω′) ∩ E′(Ω × Ω′). \]

Applying [Hör03, Thm. 8.2.12], it descends that

\[WF(\pi_∗(v − \tilde{u})) ⊆ \{(x, ξ) \mid ∃ y ∈ supp(v − \tilde{u}) \text{ for which } (x, y, ξ, 0) ∈ WF(v − \tilde{u})\}. \]

Yet, on account of the arbitrariness of \( \tilde{u} \) and using Proposition 55 it descends that \( y ∈ supp(v) \) and \( (x, y, ξ, 0) ∈ WF^α(v) \), which is nothing but the sought statement.

We can prove the main result of this part of our work and we divide it in two statements.

**Theorem 49:** Let \( Ω ⊆ \mathbb{R}^n, Ω′ ⊆ \mathbb{R}^m \) be open subsets, \( K ∈ D′(Ω × Ω′) \) be the kernel of \( K : C^∞_0(Ω′) → D′(Ω) \). Then, for all \( α ∈ \mathbb{R} \) and for all \( u ∈ C^∞_0(Ω′) \),

\[WF^α(K(u)) ⊆ \{(x, ξ) \mid ∃ y ∈ supp(u) \text{ for which } (x, y, ξ, 0) ∈ WF^α(K)\}. \]

**Proof.** Let \( π : Ω × Ω′ → Ω \) be the projection map on the second factor and assume for the time being that \( K ∈ E′(Ω × Ω′) \). It descends that \( \pi(\alpha) = \pi_∗(K ∗ (1 ⊙ u)) \), where \( \pi_∗ \) is the push-forward along \( π \) while \( ∗ \) stands for the product of distributions. Observe that, since \( WF^α(1 ⊙ u) = ∅ \) for all \( α ∈ \mathbb{R} \) then, the pointwise product is well-defined on account of Theorem 50. The latter also entails that, for all \( α ∈ \mathbb{R} \),

\[WF^α(K ∗ (1 ⊙ u)) ⊆ \{(x, y, ξ, η) ∈ WF^α(K) \mid y ∈ supp(u)\}. \]

At this stage, observing that by localizing the underlying distribution around each point of the wavefront set, we can apply Proposition 50. It descends

\[WF^α(\pi_∗(K ∗ (1 ⊙ u))) ⊆ \{(x, ξ) \mid ∃ y ∈ supp(u) \text{ for which } (x, y, ξ, 0) ∈ WF^α(K)\}, \]

which concludes the proof.

At last we generalize the preceding theorem so to investigate under which circumstances \( u \) can be taken to be an element lying \( E′(Ω′) \) and with a non empty wavefront set.

**Theorem 50:** Let \( Ω ⊆ \mathbb{R}^n, Ω′ ⊆ \mathbb{R}^m \) be open subsets, \( K ∈ D′(Ω × Ω′) \) be the kernel of \( K : C^∞_0(Ω′) → D′(Ω) \) and \( u ∈ E′(Ω′) \). In addition, for any \( α ∈ \mathbb{R} \), we call

\[-WF^α_Ω(K) := \{(y, η) ∈ Ω′ × \mathbb{R}^m \setminus \{0\} : ∃ x ∈ Ω \mid (x, y, 0, −η) ∈ WF^α(K)\}. \tag{4.11} \]

If for any \( (y, η) ∈ Ω′ × (\mathbb{R}^m \setminus \{0\}) \) there exists \( α_1, α_2 ∈ \mathbb{R} \) with \( α_1 + α_2 > 0 \) such that

\[(y, η) ∉ -WF^α_Ω(K) ∪ WF^α_Ω(u), \tag{4.12} \]

then there exists \( K u ∈ D′(Ω) \). Furthermore, if \( α ≤ α_1 + α_2 \), then

\[WF^α(K(u)) ⊆ \{(x, ξ) ∈ Ω × (\mathbb{R}^n \setminus \{0\}) : ∃ (y, η) ∈ Ω′ × (\mathbb{R}^m \setminus \{0\}) \mid (x, y, ξ, η) ∈ X ∪ Y\}, \]

where

\[X := \{(x, y, ξ, η) ∈ WF^α_1(K) \mid (y, −η) ∈ WF_0(u)\}, \quad Y = \{(x, y, ξ, η) ∈ WF(K) \mid (y, −η) ∈ WF^α_2(u)\}. \]
Proof. Following the same strategy as in the proof of Theorem 10, we aim at writing $X(u) := \pi_x(K \cdot (\cdot \cup u))$ where $\pi_x$ is the push-forward built out of the projection map $\pi : \Omega \times \Omega' \to \Omega$. Given $\alpha_2 \in \mathbb{R}$, Equation \[ (4.9) \] entails that $WF^{\alpha_2}(1 \otimes \eta) \subseteq (supp(u) \times 0) \times WF^{\alpha_2}(u)$, which combined with Theorem 15 and Equation \[ (4.12) \], entails that there exists $K \in D'(\Omega \times \Omega')$. Yet, being $u$ compactly supported we can act with the push-forward along the map $\pi : \Omega \times \Omega' \to \Omega$, hence obtaining that $\pi_x(K \cdot (\cdot \cup u)) \subseteq D'(\Omega)$. A straightforward adaptation to the case in hand of Proposition 18 entails that, for every $\alpha \in \mathbb{R}$, $WF^{\alpha}(\pi_x(K \cdot (\cdot \cup u)))$ is contained within the collection of points $(x, \xi) \in \Omega \times \mathbb{R}^d \setminus (\{0\})$ for which there exists $y \in \Omega'$ such that $(x, y, \xi, 0) \in WF^{\alpha}(K \cdot (\cdot \cup u))$.

Suppose now that $\alpha = \alpha_1 + \alpha_2$. Theorem 15, Equation \[ (4.10) \] in particular entails that the collection of points $(x, y, \xi, 0) \in WF^\alpha(K \cdot (\cdot \cup u))$ is contained in those of the form $(x, y, \xi, 0)$ such that one of the two following conditions is met:

1. there exists $\eta \in \mathbb{R}^m$ such that $(x, y, \xi, \eta) \in WF^{\alpha_1}(K)$ and $(y, -\eta) \in WF_0(u)$,

2. there exists $\eta \in \mathbb{R}^m$ such that $(x, y, \xi, \eta) \in WF(K)$ and $(y, -\eta) \in WF^{\alpha_2}(u)$.

To conclude it suffices to recall that, on account of Equation \[ (4.3) \] $WF^\alpha(K \cdot (\cdot \cup u)) \subseteq WF^{\alpha_1 + \alpha_2}(K \cdot (\cdot \cup u))$ whenever $\alpha \leq \alpha_1 + \alpha_2$. \qed

To conclude, we prove a statement which adapts to the current scenario an important result for the Sobolev wavefront set, see \[ [JS02, Prop. B.9]. \]

**Corollary 51**: Let $\Omega \subseteq \mathbb{R}^m$ be open subsets, $K \in D'(\Omega \times \Omega')$ be the kernel of $X : C^\infty_0(\Omega') \to D'(\Omega)$ and $u \in E'(\Omega')$. Assume in addition that for any $(y, \eta) \in \Omega' \times (\mathbb{R}^m \setminus \{0\})$ there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 > 0$ such that

$$(y, \eta) \notin WF^{\alpha_1}(K) \cup WF^{\alpha_2}(u).$$

If $WF_X(K) = \emptyset$ and if there exists $\gamma \in \mathbb{R}$ such that $X(B^{\alpha_2 + \gamma}_{\infty, \infty}(\Omega') \cap \Omega') \subset B^{\alpha_2 + \gamma}_{\infty, \infty}(X)$, then

$WF^{\alpha_2 + \gamma}(Xu) \subseteq WF'(K) \cup WF^\alpha(u) \cup WF_K.$

where $WF'(K) \cup WF^\alpha(u) := \{ (x, \xi) \mid \exists (y, \eta) \in WF^\alpha(u) \text{ for which } (x, y, \xi, -\eta) \in WF(K) \}$ while $WF_K := \{ (x, \xi) \in \Omega \times \mathbb{R}^m : \exists y \in \Omega' \mid (x, y, \xi, 0) \in WF(K) \}$.

**Proof.** On account of Theorem 54, Equation \[ (4.13) \] entails that $X(u) \in D'(\Omega')$. Bearing in mind Proposition 15, we can find an open conic neighborhood $\Gamma \subset WF^\alpha(u)$ such that $WF(u - v) \subset \Gamma$ for all $v \in B^{\alpha_2 + \gamma}_{\infty, \infty}(\Omega')$. Per assumption $X(v) \in B^{\alpha_2 + \gamma}_{\infty, \infty}(\Omega)$, which entails in turn on account of \[ [Hor03, Theorem 8.2.13] \]

$WF^{\alpha_2 + \gamma}(Xu) \subseteq WF'(K) \cup WF^\alpha(u) \cup WF_X(K).$

To conclude, in view of the arbitrariness of $\Gamma$, we infer

$WF^{\alpha_2 + \gamma}(Xu) \subseteq WF'(K) \cup WF^\alpha(u) \cup WF_K.$ \qed

**Example 52**: Let us consider the heat kernel operator, namely the fundamental solution of the heat equation $G \in D'(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, whose integral kernel reads in standard Cartesian coordinates

$G(t, x, t', x') = \frac{\Theta(t - t')}{(4\pi(t - t'))^{d/2}} e^{-\frac{|x - x'|^2}{4(t - t')}}.$
where $\Theta$ is the Heaviside function. By Schauder estimates, c.f. [Sim97], $G$ can also be read as the kernel of an operator $\mathcal{G}: B^\infty_{\infty,\infty}(\mathbb{R}^{1+d}) \to B_{\infty,\infty}^\alpha(\mathbb{R}^{1+d})$. Furthermore it holds that

$$WF(G) = \{(t,x,t,x,\tau,\xi, -\tau,-\xi) \mid (t,x) \in \mathbb{R}^{d+1} \text{ and } (\tau,\xi) \in \mathbb{R}^{d+1} \setminus \{0\}\}. \quad (4.15)$$

Therefore, we are in position to apply [13]. Considering any $u \in \mathcal{E}'(\mathbb{R})$, we can infer that the hypotheses of Corollary [7] are met since $WF^\alpha_{\mathbb{R}^{d+1}}(G) = \emptyset$ for all $\alpha \in \mathbb{R}$, where the subscript $\mathbb{R}^{d+1}$ should be read in the sense of Equation (4.11). At the same time, on account of Remark [7], there must exist $\alpha < 0$ such that $WF^\alpha(u) = \emptyset$. This entails that

$$WF^\alpha+2(\mathcal{G}(u)) \subseteq WF'(G) \circ WF^\alpha(u),$$

which, combined with Equation (4.15), yields $WF'(G) \circ WF^\alpha(u) = WF^\alpha(u)$. This leads to the inclusion

$$WF^\alpha+2(\mathcal{G}(u)) \subseteq WF^\alpha(u).$$

### 4.1 Besov Wavefront Set and Hyperbolic Partial Differential Equations

As an application of the results of the previous sections, we study the interplay between the Besov wavefront set and a large class of hyperbolic partial differential equations of the form

$$\partial_t u = ia(D_x)u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad (4.16)$$

where we assume $a = a_1 + a_0$ where $a_1 \in S^1_{\text{hom}}(\mathbb{R}^d)$, while $a_0 \in S^0(\mathbb{R}^d)$ see Definition [11]. Using standard Fourier analysis, we can infer that the fundamental solution associated to the operator $\partial_t - ia(D_x)$ is the distribution $G \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$, whose integral kernel reads

$$G(t,x) = \Theta(t)[e^{ita(D)}\delta](x),$$

where $\Theta$ is once more the Heaviside function.

**Proposition 53:** Let $\alpha \in \mathbb{R}$. Then $B^\infty_{\infty,\infty}(\mathbb{R}^d) \cap \mathcal{E}'(\mathbb{R}^d) \subset B^\alpha_{2,\infty}(\mathbb{R}^d)$.

**Proof.** Let $v \in B^\alpha_{2,\infty}(\mathbb{R}^d) \cap \mathcal{E}'(\mathbb{R}^d)$. For any $\kappa \in \mathcal{B}_{[\alpha]}$ as per Definition [5] it

$$\|v(\kappa_2^\alpha)\|_{L^2(\mathbb{R}^d)} \lesssim \|v(\kappa_1^\alpha)\|_{L^\infty(\mathbb{R}^d)} \lesssim \lambda^\alpha,$$

where the first estimate is a byproduct of $v$ being compactly supported. A similar reasoning applies when considering any $\underline{\kappa} \in \mathcal{D}(B(0,1))$ such that $\underline{\kappa}(0) \neq 0$. As a consequence of Definition [6] we infer that $v \in B^\alpha_{2,\infty}(\mathbb{R}^d)$. \hfill $\blacksquare$

**Proposition 54:** Let $G \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ be the fundamental solution of the hyperbolic operator $\partial_t - ia(D_x)$. Then, $G(t,\cdot) \in B^{\frac{d}{2}}_{2,\infty}(\mathbb{R}^d)$ for any $t \in \mathbb{R}$. Moreover, given $v \in B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha \in \mathbb{R}$,

$$G(t,\cdot) \ast v \in B^\alpha_{\infty,\infty}(\mathbb{R}^d),$$

where $\ast$ stands for the convolution.
Proof. Let \( \{\psi_j\}_{j \geq 0} \) be a Littlewood-Paley partition of unity as per Definition. For any \( j \geq 1 \), it descends
\[
\|\psi_j(D_x)e^{it\Phi(D_x)}\delta\|_{L^2(\mathbb{R}^d)} = \|\psi_j\|_{L^2(\mathbb{R}^d)} = 2^j \|\psi\|_{L^2(\mathbb{R}^d)},
\]
where we applied Fourier-Plancherel theorem in the first equality. Hence we can conclude that
\[
\sup_{j \geq 0} 2^{-j \frac{d}{2}} \|\psi_j(D_x)e^{it\Phi(D_x)}\delta\|_{L^2(\mathbb{R}^d)} < \infty,
\]
which entails that \( G(t, \cdot) \in B^{\frac{d}{2}}_{2, \infty}(\mathbb{R}^d) \). Observe that, for every \( \phi \in \mathcal{D}(\mathbb{R}^d) \), \( \phi v \in B^{\frac{d}{2}}_{2, \infty}(\mathbb{R}^d) \) on account of Proposition. Then, as a consequence of [KS21, Thm 2.2], we can infer that \( G(t, \cdot) * (\phi v) \in B^{\frac{d}{2}}_{\infty, \infty}(\mathbb{R}^d) \) for any \( t \in \mathbb{R} \).

Proposition can be read as a statement that the solution map associated to Equation (4.16)
\[
S(t, 0) : u(0) \mapsto u(t)
\]
is continuous from \( B^{\alpha, \text{loc}}_{\infty, \infty}(\mathbb{R}^d) \) to \( B^{\alpha - \frac{d}{2}, \text{loc}}_{\infty, \infty}(\mathbb{R}^d) \). Moreover, \( S(t, 0) \) can be inverted and \( S(t, 0)^{-1} = S(0, t) \).

Theorem 55: Let \( a \) be as per Equation (4.16) and let \( u_0 \in S(\mathbb{R}^d) \). Suppose that \( u \) is the solution of the initial value problem
\[
\begin{aligned}
\partial_t u &= i a(D_x) u, \\
u(0) &= u_0.
\end{aligned}
\]
Then, for every \( \alpha \in \mathbb{R} \),
\[
WF^{\alpha - \frac{d}{2}}(u(t)) = \mathcal{C}(t)WF^\alpha(u_0),
\]
where \( \mathcal{C}(t) \) is the flow from \( t \) to \( 0 \) associated the Hamiltonian vector field \( H_{a(t)} \).

Proof. We just prove the inclusion \( \subset \), the other following suite. Let us consider \( (x, \xi) \notin WF^\alpha(u_0) \). Then there exists \( A \in \Psi^0(\mathbb{R}^d) \), elliptic at \( (x, \xi) \), such that \( Au_0 \in B^{\alpha, \text{loc}}_{\infty, \infty}(\mathbb{R}^d) \). Let us define \( A(t) := S(t, 0) \circ A \circ S(0, t) \) so that \( A(t)u(t) = S(t, 0)Au_0 \in B^{\alpha - \frac{d}{2}, \text{loc}}_{\infty, \infty}(\mathbb{R}^d) \). On account of Egorov’s theorem, see e.g. [Hin21], we can conclude that \( A(t) \) still lies in \( \Psi^0(\mathbb{R}^d) \) and it is elliptic at \( \mathcal{C}(t)^{-1}(x, \xi) \). This implies \( \mathcal{C}(t)^{-1}(x, \xi) \notin WF^{\alpha - \frac{d}{2}}(u(t)) \). \( \square \)

Remark 56: It is worth mentioning that the estimate on the Besov wavefront set as per Theorem might be improved if working with a generic Besov space \( B^\alpha_{p, q}(\mathbb{R}^d) \) rather than with \( B^\alpha_{\infty, \infty}(\mathbb{R}^d) \). Yet this step requires first of all to establish an improved version of Proposition which appears to be elusive at this stage.

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