$E(k, L)$ level statistics of classically integrable quantum systems based on the Berry-Robnik approach

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Theory of the quantal level statistics of classically integrable system, developed by Makino et al. in order to investigate the non-Poissonian behaviors of level-spacing distribution (LSD) and level-number variance (LNV)[1, 2], is successfully extended to the study of $E(K, L)$ function which constitutes a fundamental measure to determine most statistical observables of quantal levels in addition to LSD and LNV. In the theory of Makino et al., the eigenenergy level is regarded as a superposition of infinitely many components whose formation is supported by the Berry-Robnik approach in the far semiclassical limit[4]. We derive the limiting $E(K, L)$ function in the limit of infinitely many components and elucidates its properties when energy levels show deviations from the Poisson statistics.

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1 Introduction

One of the main objectives in the research field of quantum chaology is to elucidate the quantum manifestation of regular and chaotic features of classical dynamical systems. Energy-level statistics, which were initially developed in nuclear physics, played an important role in elucidating the universal properties of these manifestations. In 1977, Berry and Tabor conjectured that, for a quantum system whose classical dynamical system is integrable, the energy eigenvalues in an unfolded scale, behave like uncorrelated random numbers from the Poisson process in the semiclassical limit, and that the fluctuation properties of these eigenvalues obey the Poisson statistics. This conjecture is in contrast with the conjecture of Bohigas, Giannoni, and Schmit of 1984, which states that the unfolded energy eigenvalues of a quantum system, whose classical dynamical system is fully chaotic, are well characterized by the GOE or GUE statistics of Random Matrix Theory (RMT) in the semiclassical limit. These two conjectures have been examined using various statistical observables, e.g., level spacing distribution (LSD), level number variance (LNV), spectral rigidity, mode fluctuation distribution, skewness and excess kurtosis, and these observables can be calculated from knowledge of the $E(K, L)$ function.

The $E(K, L)$ function is defined as a distribution function that stands for the probability of finding $K$ levels in a randomly chosen energy-interval of length $L$. For a given arbitrary value of non-negative integer $K = 0, 1, 2 \cdots$, $E(K, L)$ characterizes the fluctuation property of energy levels at the particular scale of $L = K$. Once $E(K, L)$ is determined, the LSD is calculated as

$$P(K, L) = \frac{\partial^2}{\partial L^2} \sum_{j=0}^{K} (K - j + 1)E(j, L).$$

The LSD $P(K, L)$ is introduced as a distribution function that denotes a probability density to find two adjacent levels of spacing $L$ containing $K$ levels in between. The nearest-neighbor LSD (NNLS), $P(0, L)$, which is frequently used to analyze the short-range spectral fluctuation, is a special case of $K = 0$. In a similar way, the LNV $\Sigma^2(L)$, skewness $\gamma_1(L)$ and excess kurtosis $\gamma_2(L)$, which are respectively, the two, three and four-point correlation functions, are calculated as $\Sigma^2(L) = C_2(L)$, $\gamma_1(L) = C_3(L)/C_2(L)^{3/2}$ and $\gamma_2(L) = C_4(L)/C_2(L)^2 - 3$, respectively, where $C_n(L)$ is $n$th moment of the level number fluctuation around its average value $L$, obtained from $E(K, L)$ as

$$C_n(L) = \sum_{K=0}^{+\infty} (K - L)^n E(K, L).$$
Moreover, the spectral rigidity $\Delta_3(L)$, which is conventionally used to analyze the two-point correlation instead of $\Sigma^2(L)$, is also calculated from $E(K, L)$ as\[13\]

$$\Delta_3(L) = \frac{2}{L^4} \int_0^L dS (L^3 - 2L^2S + S^3)C_2(S). \tag{3}$$

In this way, it is quite important to determine the $E(K, L)$ function, which provides a basis for the energy level statistics. For the Poissonian level sequence of the unfolded scale, $E(K, L)$ can be characterized by the Poisson distribution:

$$E_{\text{Poisson}}(K, L) = \frac{L^K}{K!} \exp (-L), \tag{4}$$

which obviously leads to the results from the Poisson statistics: $P(K, L) = L^K e^{-L}/K!$, $\Sigma^2(L) = L$, $\gamma_1(L) = L^{-1/2}$, $\gamma_2(L) = 1/L$, and $\Delta_3(L) = L/15$.

Many works have examined the Berry-Tabor conjecture\[1, 2, 8, 14–27\], and the statistical property of eigenenergy levels that the Poisson statistics can characterize is now widely accepted as a universal property of generic integrable quantum systems. However, the mechanism supporting this conjecture is still unclear, and deviation from the Poisson statistics is observed in some classically integrable systems that have a spatial or time-reversal symmetry.

One possible mechanism underlying the deviation from Poisson statistics has been proposed by Makino et al.\[1\], on the basis of the Berry-Robnik approach\[3, 4, 11\]. We briefly review the outline as follows. For an integrable system, individual orbits are confined in each inherent torus whose surface is defined by holding its action variable constant, and the whole region of the phase space is densely covered with infinitely many invariant tori, which have infinitesimal volumes in the Liouville measure. Because of the suppression of quantum tunneling in the semiclassical limit $\hbar \to 0$, the Wigner function of each quantal eigenstate is expected to be localized in the phase space region explored by a typical trajectory, and to form independent components\[4, 33\]. For a classically integrable quantum system, the Wigner function localizes on the infinitesimal region in $\hbar \to 0$ and tends to a $\delta$ function on a torus\[34\]. Then, the eigenenergy levels can be represented as a statistically independent superposition of infinitely many components, each of which contributes infinitesimally to the level statistics. Therefore, if the individual spectral components are sparse enough, one would expect Poisson statistics to be observed as a result of the law of small numbers\[35\].

The statistical independence of spectral components is assumed to be justified by the principle of uniform semiclassical condensation of eigenstates in the phase space and by the lack of their mutual overlap, and thus can be expected only in the semiclassical limit\[4\]. This mechanism was initially introduced as a basis for the Berry-Robnik approach to investigate the energy level statistics of the generic mixed quantum system, and its validity is confirmed.
by numerical computations in the extremely deep semiclassical region which is called the Berry-Robnik regime\[32\].

On the basis of this view, Makino and Tasaki investigated the NNLSD of systems with infinitely many components\[1\]. They derived the cumulative function of NNLSD, $M(L) = \int_0^L P(0, S) dS$, which is characterized by a single monotonically increasing function $\bar{\mu}(0, S) \in [0, 1]$ of the nearest level spacing $S$ as

$$M(L) = 1 - [1 - \bar{\mu}(0, L)] \exp \left( - \int_0^L [1 - \bar{\mu}(0, S)] dS \right). \tag{5}$$

The function $\bar{\mu}(0, S)$ classifies $M(L)$ into three cases: Case 1, Poisson distribution $M(L) = 1 - e^{-L}$ for all $L \geq 0$ if $\bar{\mu}(0, +\infty) = 0$; case 2, asymptotic Poisson distribution, which converges to the Poisson distribution for $L \to +\infty$, but possibly not for small spacing $L$ if $0 < \bar{\mu}(0, +\infty) < 1$; case 3, sub-Poisson distribution, which deviates from the Poisson distribution for $\forall L$ in such a way that $M(L)$ converges to 1 for $L \to +\infty$ more slowly than does the Poisson distribution if $\bar{\mu}(0, +\infty) = 1$. This argument is extended later to the study of LNV\[2\], whose properties are evaluated for cases 1-3 as follows: Case 1, the LNV is the Poissonian $\Sigma^2(L) = L$; case 2, the LNV deviates from the Poissonian in such a way that the slope is greater than 1 for $L > 0$ and approaches a number $\geq 1 + 2\bar{\mu}(0, +\infty)$ as $L \to +\infty$; case 3, the LNV deviates from the Poissonian in such a way that the slope is greater than 1 for $L > 0$ and approaches a number $\geq 3$ as $L \to +\infty$. Therefore, the Berry-Robnik approach, when applied to classically integrable systems, allows the NNLSD and LND to deviate from the Poisson statistics.

In this paper, extending the above arguments of Makino et al.\[1\][2], we investigate the $E(K, L)$ function of a quantum system whose energy level consists of infinitely many independent components, and elucidate its property when the NNLSD of eigenenergy levels shows cases 2 and 3. This paper suggests the possibility of a new statistical law to be observed in the $E(K, L)$ level statistics of classically integrable quantum systems.

The limiting $E(K, L)$ function is derived as follows: We consider a system whose phase space is decomposed into $N$ disjoint regions that give distinct spectral components. The Liouville measures of these regions are denoted by $\rho_n (n = 1, 2, 3, \cdots, N)$, which satisfy the normalization $\sum_{n=1}^N \rho_n = 1$. In the Berry-Robnik approach, these quantities are equivalent to the statistical weights of individual spectral components.

When the entire sequence of energy levels is a product of statistically independent superpositions of $N$ subsequences, $E(K, L)$ is decomposed into the $E(K, L)$ function of
 subsequences, $e_n(k, L)$, as

$$E_N(K, L) = \sum_{n=1}^{N} \prod_{k_n=K}^{N} e_n(k_n, L),$$

(6)

where $e_n$ satisfies the normalizations $\sum_{k=0}^{+\infty} e_n(k, L) = 1$ and $\sum_{k=0}^{+\infty} ke_n(k, L) = L$. In terms of the normalized level-spacing distribution $p_n(k, S)$ of the subsequence, $e_n(k, L)$ is described as

$$e_n(k, L) = \rho_n \int_{L}^{+\infty} dx \int_{x}^{+\infty} [p_n(k, S) - 2p_n(k - 1, S) + p_n(k - 2, S)] dS,$$

(7)

where $p_n(j < 0, S) = 0$, and $p_n$ satisfies the normalization conditions $\int_{0}^{+\infty} p_n(k, S) dS = 1$ and $\int_{0}^{+\infty} Sp_n(k, S) dS = (k + 1)/\rho_n$. Eq. (7) is known as the formula in the theory of point process, which is derived as corollaries of the Palm-Khintchine theorem[36].

In addition to Eq. (6), we introduce two assumptions that were introduced in Refs.[1, 2]:

**Assumption (i).** The statistical weights of individual components vanish uniformly in the limit of infinitely many components: $\max_n \rho_n \to 0$ as $N \to +\infty$.  

**Assumption (ii).** The weighted mean of the cumulative level-spacing distribution of spectral components, $\mu(k, S) \equiv \sum_{n=1}^{N} \rho_n \mu_n(k, S)$ with $\mu_n(k, L) = \int_{0}^{L} p_n(k, S) dS$, converges as $N \to +\infty$ to $\bar{\mu}(k, S)$, where the convergence is uniform on each closed interval: $S \in [0, L]$. It is noted that $\mu(k, S)$ is monotonically decreasing for increasing $k$.

In the Berry-Robnik approach, Eq. (6) relates the level statistics in the semiclassical limit with the phase space geometry.

Under assumptions (i) and (ii), Eq. (6) leads to the following new expression in the limit of $N \to +\infty$:

$$\bar{E}(K, L) = \alpha_K(L) e^{\beta(L)L} E_{\text{Poisson}}(K, L),$$

(8)

where the factor $\alpha_K(L)$ and exponent $\beta(L)$ of the distribution function are described by the parameter function $\bar{\mu}(k, L)$. When the lowest-order moment of this function shows $\bar{\mu}(0, L) = 0$ for all $L$, one has $\alpha_K(L) = 1$ for all $K$ and $\beta(L) = 0$, and the limiting function $\bar{E}(K, L)$ of the whole energy sequence reduces to the Poisson distribution[4]. As shown in Ref.[1], this condition is expected to arise when the individual spectral components are sparse enough. In general, one may expect $\bar{\mu}(0, L) > 0$, which corresponds to a certain accumulation of levels of individual components. In this case, the limiting function $\bar{E}(K, L)$ deviates from the Poisson distribution.

The organization of this paper is as follows. In Section 2, the limiting function $\bar{E}(K, L)$ is derived from Eq. (6) and assumptions (i) and (ii). In Section 3, the property of the limiting function $\bar{E}(K, L)$ is analyzed for cases 1–3, where the possibilities of deviation from the
Poisson statistics are discussed. In Section 4, the numerical investigation of the $E(K, L)$ function is carried out for the rectangular billiard, whose numerical results for the NNLSD have been shown to deviate from the Poisson distribution\[1, 2\]. In Section 5, we discuss some relations between our results and those of related works.

2 Limiting $\bar{E}(K, L)$ function

Starting from Eq.(6) and assumptions (i) and (ii), we derive the limiting function of $E_N(K, L)$ for a system with infinitely many components($N \to +\infty$). First we transfer a sequence of nonnegative integers $\{k_n\}_{n=1,\ldots,N}$ of Eq.(6) into a sequence of non-duplicate natural numbers $\{\kappa_m\}_{m=1,2,3,\ldots}$ with $\{d_m\}_{m=1,2,3,\ldots}$ being their individual duplications. Then, the polynomial of Eq.(6) is factorized using ratios $e'_n(\kappa_m, \rho_n L) = e_n(\kappa_m, \rho_n L)/e_n(0, \rho_n L)$ as

$$E_N(K, L) = E_N(0, L) \left[ \sum_{M=1}^{K} \sum_{\sum_{m=1}^{M} d_m \kappa_m = K} \prod_{m=1}^{M} d_m! \left( \sum_{n=1}^{N} e'_n(\kappa_m, \rho_n L) \right)^{d_m} + \sum_{n=1}^{N} O(\rho_n^2) \right],$$

with

$$E_N(0, L) = \exp \left( \sum_{n=1}^{N} \ln e_n(0, \rho_n L) \right),$$

where we have used properties $e_n(0, \rho_n L)^{-1} = 1 + O(\rho_n)$ and $e_n(k > 0, \rho_n L) = O(\rho_n)$ (see also Eq.(7)). Since $e'_n(k > 0, \rho_n L) = e_n(k, \rho_n L) + O(\rho_n^2)$ and Eq.(7), $\sum_{n=1}^{N} e'_n(\kappa_m, \rho_n L)$ and $\sum_{n=1}^{N} \ln e_n(0, \rho_n L)$ are described by the weighted mean $\mu(k, S) = \sum_{n=1}^{N} \rho_n \int_{0}^{S} p_n(k, x) dx$ as

$$\sum_{n=1}^{N} e'_n(\kappa_m, \rho_n L) = \int_{0}^{L} dS \left[ \mu(\kappa_m, S) - 2\mu(\kappa_m - 1, S) + \mu(\kappa_m - 2, S) \right] + \sum_{n=1}^{N} O(\rho_n^2),$$

and

$$\sum_{n=1}^{N} \ln e_n(0, \rho_n L) = -L + \int_{0}^{L} \mu(0, S) dS + \sum_{n=1}^{N} O(\rho_n^2).$$

Here, $\sum_{n=1}^{N} O(\rho_n^2)$ in the equations (9), (11) and (12) shows the convergence,

$$\left| \sum_{n=1}^{N} O(\rho_n^2) \right| \leq C \max_{n} \rho_n \sum_{n=1}^{N} \rho_n = C \max_{n} \rho_n \to 0 \quad \text{as} \quad N \to +\infty,$$

which results from assumption (i). Therefore, by applying assumption (ii), we have the limiting formula in the limit of $N \to +\infty$,

$$\bar{E}(K, L) = \alpha_K(L)e^{\beta(L)L}E_{\text{Poisson}}(K, L),$$

(14)
where \( \alpha_0(L) = 1 \),

\[
\alpha_{K>0}(L) = \frac{K!}{L^K} \sum_{M=1}^{K} \sum_{\sum_{m=1}^{M} d_m = K} \prod_{m=1}^{M} 1 \left( \int_0^L dS \begin{bmatrix} \bar{\mu}(\kappa_m, S) \\ -2\bar{\mu}(\kappa_m - 1, S) \\ +\bar{\mu}(\kappa_m - 2, S) \end{bmatrix} \right)^{d_m},
\]

(15)

and

\[
\beta(L) = \frac{1}{L} \int_0^L \bar{\mu}(0, S) dS.
\]

(16)

For \( K = 1 - 3 \), the factors \( \alpha_K(L) \) are specified as

\[
\alpha_1(L) = \frac{1}{L} \int_0^L \left[ 1 + \bar{\mu}(1, S) - 2\bar{\mu}(0, S) \right] dS,
\]

(17)

\[
\alpha_2(L) = \frac{2}{L^2} \int_0^L \left[ \bar{\mu}(2, S) - 2\bar{\mu}(1, S) + \bar{\mu}(0, S) \right] dS \\
\quad + \frac{1}{L^2} \left( \int_0^L \left[ 1 + \bar{\mu}(1, S) - 2\bar{\mu}(0, S) \right] dS \right)^2,
\]

(18)

and

\[
\alpha_3(L) = \frac{6}{L^3} \int_0^L dS \left[ \bar{\mu}(3, S) - 2\bar{\mu}(2, S) + \bar{\mu}(1, S) \right] \\
\quad + \frac{6}{L^3} \int_0^L dS \left[ \bar{\mu}(2, S) - 2\bar{\mu}(1, S) + \bar{\mu}(0, S) \right] \times \int_0^L dS \left[ 1 + \bar{\mu}(1, S) - 2\bar{\mu}(0, S) \right] \\
\quad + \frac{1}{L^3} \left( \int_0^L dS \left[ 1 + \bar{\mu}(1, S) - 2\bar{\mu}(0, S) \right] \right)^3.
\]

(19)
3 Properties of limiting $\bar{E}(K, L)$ function

Since $\bar{\mu}(k, S)$ monotonically increases for $S \geq 0$ and $0 \leq \bar{\mu}(k, S) \leq 1$, $\alpha_K(L)$ and $\beta(L)$ in the limit $L \to +\infty$ show

$$
\alpha_K(L) = K! \sum_{M=1}^{K} \frac{1}{L^{K-M} \sum_{m=1}^{M} d_m \kappa_m = K} \sum_{m=1}^{M} \frac{1}{d_m!} \left( \frac{1}{L} \int_0^L \left[ \begin{array}{c}
-2\mu(k_m - 1, S) \\
+\mu(k_m - 2, S)
\end{array} \right] dS \right)^{d_m}
$$

$$
\times \prod_{m=1}^{M} \frac{1}{d_m!} \left( \frac{1}{L} \int_0^L \left[ \begin{array}{c}
\mu(k_m, S) \\
-2\mu(k_m - 1, S) \\
+\mu(k_m - 2, S)
\end{array} \right] dS \right)^{d_m}
$$

$$
\longrightarrow \lim_{L \to +\infty} \left[ \frac{1}{L} \int_0^L (1 + \bar{\mu}(1, S) - 2\bar{\mu}(0, S)) dS \right]^{K}
$$

$$
= \left[ 1 + \bar{\mu}(1, +\infty) - 2\bar{\mu}(0, +\infty) \right]^K \equiv \alpha_K(+\infty),
$$

(20)

and

$$
\beta(L) \longrightarrow \bar{\mu}(0, +\infty).
$$

(23)

Note that $K - \sum_{m=1}^{M} d_m = 0$ only when $M = 1$ and $d_1 = K(k_1 = 1)$. From these convergences and the limiting value of the lowest-order function $\bar{\mu}(0, +\infty)$, all $E(K, L)$ of $K = 0, 1, 2 \cdots$ are classified into the following three cases.

Case-1 [$\bar{\mu}(0, +\infty) = 0$]: $E(K, L)$ is the Poisson distribution. This condition is equivalent to $\bar{\mu}(K, L) = 0$ for all $L$ and $K$ since $\bar{\mu}(K, L)$ is monotonically increasing for $L$ and decreasing for $K$. Thus, one has

$$
\alpha_K(L) = 1 \text{ and } \beta(L) = 0 \text{ for all } L \text{ and } K.
$$

(24)

Case-2 [$0 < \bar{\mu}(0, +\infty) < 1$]: $E(K, L)$ deviates from the Poisson distribution, i.e., $E(K, L)$ for a large value of $L$ is well approximated by the Poisson distribution, $\alpha_K(+\infty)L^K e^{-[1-\bar{\mu}(0, +\infty)]L/K!}$, where $\alpha_K(+\infty)$ is a value bounded by the inequality

$$
0 \leq \alpha_K(+\infty) \leq [1 - \bar{\mu}(0, +\infty)]^K.
$$

On the other hand for small value of $L$, $E(K, L)$ may deviate from the Poisson distribution. Since $\alpha_{K+1}(+\infty) = [1 + \bar{\mu}(1, +\infty) - 2\bar{\mu}(0, +\infty)]\alpha_K(+\infty)$ and $\bar{\mu}(0, +\infty) \geq \bar{\mu}(1, +\infty)$, the factor $\alpha_K(+\infty)$ monotonically decreases for $K$, i.e.,

$$
\alpha_K(+\infty) > \alpha_{K+1}(+\infty) \text{ for all } K,
$$

(25)

where $\alpha_0(+\infty) = 1$ and $\lim_{K \to +\infty} \alpha_K(+\infty) = 0$. 

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Case-3[\bar{\mu}(0, +\infty) = 1]: \bar{E}(K, L) in \ L \rightarrow +\infty \ approaches \ 0 \ more \ slowly \ than \ does \ the \ Poisson \ distribution, \ where \ the \ factor \ corresponds \ to \ \alpha_0(+\infty) = 1 \ and

\[
\alpha_K(+\infty) = 0 \ \forall \ K > 0, \quad (26)
\]

and deviates from the Poisson distribution for all L.

It should be noted that Case 1 and Case 3 are extreme cases where all factors \ \alpha_K(L) of \ K > 0 \ in the limit \ L \rightarrow +\infty \ converge \ to \ 1 \ in \ Case 1 \ and \ to \ 0 \ in \ Case 3.

As is also shown in Ref.[1], one observes Case 1 if the scaled NNLSD of individual components \ f_n(0, \rho_n S) = p_n(0, S)/\rho_n, \ which \ satisfy \ \int_0^{+\infty} f_n(0, x)dx = 1 \ and \ \int_0^{+\infty} x f_n(0, x)dx = 1 \ are \ uniformly \ bounded \ by \ a \ positive \ constant \ D : |f_n(0, S)| \leq D (1 \leq n \leq N). \ Indeed, \ in \ N \rightarrow +\infty, \ the \ following \ holds:

\[
|\mu(0, S)| \leq \sum_{n=1}^{N} \rho_n^2 \int_{0}^{S} |f_n(0, \rho_n x)| \ dx \leq DS \sum_{n=1}^{N} \rho_n^2 \leq DS \max_n \rho_n \sum_{n=1}^{N} \rho_n \rightarrow 0 \equiv \bar{\mu}(0, S).
\]

(27)

Such a bounded condition is possible when the individual spectral components are sparse enough.

In general, one may observe Case 2 or Case 3, each of which corresponds to strong accumulation of energy levels, leading to a singular NNLSD of the individual components. Such an accumulation can arise when the physical system has a symmetry[1, 2, 8, 20–27]. In the next section, we numerically analyze the \ E(K, L) \ function \ for \ the \ rectangular \ billiard \ system \ whose \ NNLSD \ of \ the \ eigenenergy \ levels \ has \ been \ shown \ to \ obey \ Cases \ 2 \ and \ 3[2].

4 Numerical studies of rectangular billiard

We analyze the property of \ E(K, L) \ for \ a \ rectangular \ quantal \ billiard \ whose \ eigen-energy \ levels \ are \ given \ by \ \epsilon_{n,m} = n^2 + \gamma m^2, \ where \ n \ and \ m \ are \ positive \ integers \ and \ \gamma \ is \ the \ square \ ratio \ of \ two \ sides, \ a \ and \ b, \ denoted \ as \ \gamma = a^2/b^2. \ The \ unfolding \ transformation \ \{\epsilon_{m,n}\} \rightarrow \{\bar{\epsilon}_{m,n}\} \ is \ carried \ out \ by \ using \ the \ leading \ Weyl \ term \ of \ the \ integrated \ density \ of \ states, \ \mathcal{N}(\epsilon), \ as \ \bar{\epsilon}_{m,n} = \mathcal{N}(\epsilon_{m,n}) = \pi \epsilon_{m,n}/4\sqrt{\gamma}. \ Berry \ and \ Tabor \ observed \ that \ the \ NNLSD \ of \ this \ system \ agrees \ with \ the \ Poisson \ distribution \ (Case 1) \ when \ \gamma \ is \ far \ from \ rational, \ while \ it \ deviates \ from \ the \ Poisson \ distribution \ when \ \gamma \ is \ rational[8]. \ The \ deviation \ from \ the \ Poisson \ distribution \ was \ precisely \ analyzed \ for \ \gamma = 1 \ by \ Connors \ and \ Keating[20]. \ Working \ on \ the \ basis \ of \ Landau’s \ number-theoretical \ result[31, \ they \ proved \ that \ the \ mean \ degeneracy \ of \ the \ eigen-energy \ levels \ increases \ logarithmically \ as \ the \ energy \ becomes \ higher. \ This \ property \ has \ been \ confirmed \ numerically \ by \ Robnik \ and \ Veble \ in \ Ref.[21], \ where \ the \ NNLSD
$P(0, L)$ converges to the delta function in the high-energy (semiclassical) limit $\epsilon \to +\infty$. In Ref.\,[2], Makino et al. have shown that the logarithmic degeneracy of levels at $\gamma = 1$ leads to $\bar{\mu}(0, +0) = 1$ in the high-energy limit. Since $\bar{\mu}(0, L)$ is monotonically increasing and $0 \leq \bar{\mu}(0, L) \leq 1$, the square billiard with $\gamma = 1$ obviously shows $\bar{\mu}(0, +\infty) = 1$, which corresponds to Case 3.

In this section, we evaluate numerically the behaviors of quantities that converge to $\alpha_K(L)$ and $\beta(L)$ in the semiclassical limit. We carry out a numerical study for an irrational case in addition to a rational case ($\gamma = 1$), which is described by a finite continued fraction of the golden mean number $(\sqrt{5} + 1)/2$,

$$\gamma = 1 + \frac{1}{1 + 1 + \cdots + 1 + \frac{1}{1 + \delta}} = [1; 1, 1, \cdots, 1, 1 + \delta], \quad (28)$$

with an irrational truncation parameter $\delta \in [0, 1)$.

Figure 1 shows semi-logarithmic plots of $E(K, L)$ for $K = 0 - 4$ and $K = 10$. In each figure, we show results for three values of $\gamma$ corresponding to the (a) 41st and (b) fourth approximations of the golden mean, and (c) $\gamma = 1$. The solid curve in each figure represents the Poisson distribution $[4]$. Our analysis is valid for $K < L_{\text{max}}$ as shown in Refs.\,[11, 21], where $L_{\text{max}}$ is determined by the shortest period of classical periodic orbit\,[29], and it is calculated for the rectangular billiard as $L_{\text{max}} = \sqrt{\pi \epsilon_{n, m} \gamma}^{-1/4}$. We used eigen-energy levels $\tilde{\epsilon}_{n, m} \in [100 \times 10^{10}, 101 \times 10^{10}]$ corresponding to $L_{\text{max}} \sim 1.8 \times 10^6$, which is sufficiently large for our numerical study. The numerical computation in this paper was carried out using the double-precision real number operation. When the continued fraction is close to the golden mean number, $E(K, L)$ is well approximated by the Poisson distribution[plot (a)], and this result corresponds to Case 1 given in section 3. In cases in which the continued fractions are far from the golden mean number, $E(K, L)$ clearly deviates from the Poisson distribution [plots (b) and (c)]. Since all levels at $\gamma = 1$ are degenerate except those with $n = m$, $E(K, L)$ with odd $K$ is very small.

Figure 2 shows numerical plots of $\tilde{\alpha}_K(L)$ for $K = 1 - 5$ and $K = 10$, which are obtained by $E(K, L)$ as

$$\tilde{\alpha}_K(L) = \frac{K!}{L^K} \left( \frac{E(K, L)}{E(0, L)} \right). \quad (29)$$

In each figure, we show three results for $\gamma$ corresponding to plots (a)–(c) in figure 1. Note that function $[29]$ is equivalent to $\alpha_K(L)$ in the semiclassical limit $\epsilon \to +\infty$. In case $\gamma$ is close to the golden mean and $E(K, L)$ is well approximated by the Poisson distribution, $\tilde{\alpha}_K(L)$ agrees with 1 very well [plot (a)]. On the other hand, in case $\gamma$ is far from the golden mean and $E(K, L)$ deviates from the Poisson distribution, $\tilde{\alpha}_K(L)$ approaches a number $\tilde{\alpha}_K(+\infty)$ such that $0 < \tilde{\alpha}_K(+\infty) < 1$[plot (b)], and this result corresponds to Case 2. In case $\gamma = 1$,
\( \tilde{\alpha}_K(L) \) quickly converges to 0 as \( L \to +\infty \), and this result corresponds to Case 3. It is quite interesting that \( \tilde{\alpha}_K(L) \) of the 4th approximation, whose result corresponds to Case 2, obviously shows relation (25) of monotonically decreasing as \( K \) increases as shown in Figure 3.

Figure 4 shows numerical plots of \( \tilde{\beta}(L) \equiv 1 + \frac{1}{\sqrt{2 \ln E(0, L)}} \) for the three values of \( \gamma \) corresponding to plots (a)–(c) in Figure 1. This function is equivalent to \( \beta(L) \) in the semiclassical limit \( \epsilon \to +\infty \), which satisfies \( \beta(0) = \bar{\mu}(0, 0) \) and \( \lim_{L \to +\infty} \beta(L) = \bar{\mu}(0, +\infty) \). When \( \gamma \) is close to the golden mean, \( \tilde{\beta}(L) \) agrees with 0 very well [plot (a)], and this result obeys the property of Case 1. On the other hand, in case \( \gamma \) is far from the golden mean, \( \tilde{\beta}(L) \) approaches a number \( \tilde{\beta}(+\infty) \) such that \( 0 < \tilde{\beta}(+\infty) < 1 \) [plots (b)], and this result obeys the property of Case 2. However, in case \( \gamma = 1 \) where Case 3 is expected to arise in the semiclassical limit, \( \tilde{\beta}(L) \) does not reproduce \( \beta(L) = 1 \) even in the region \( L >> 1 \). This is because we are not yet far enough in the high-energy region where \( \tilde{\beta}(L) \) agrees well with \( \beta(L) \). In order to estimate the convergence of \( \tilde{\beta}(L) \) to 1, we analyze \( \tilde{\beta}(+0) \), which can be described by the cumulative NNLSD \( M(L) \) as \( \tilde{\beta}(+0) = M(+0) \) (see Appendix A). According to the theoretical prediction of Connors and Keating\[20\], and additional argument of Makino et. al.\[2\], \( M(+0) \) of the square billiard is described as \( M(+0) \approx 1 - \frac{4c}{\pi \sqrt{\ln \epsilon}} \), and thus we also have

\[
\tilde{\beta}(+0) \approx 1 - \frac{4}{\pi} \frac{c}{\sqrt{\ln \epsilon}}, \tag{30}
\]

with \( c \approx 0.764 \). This indicates an extremely slow convergence of \( \tilde{\beta}(+0) \) to 1 as the energy \( \epsilon \) becomes higher and \( \beta(+0) \leq \beta(+\infty) = 1 \) (Case 3) observed in the semiclassical limit (\( \epsilon \to +\infty \)). We finally confirm the approximate expression (30).

Figure 5 shows \( 1 - \tilde{\beta}(0) \) vs \( 4c/\pi \sqrt{\ln \epsilon} \) for various energy ranges. The solid line represents the theoretical curve (30), which is valid in the semiclassical (high energy) region. Although we are not yet far enough in the high-energy region where \( 1 - \tilde{\beta}(0) << 1 \), the agreement between them is very good.

5 Summary and Conclusion

The basic ideas of our study were to apply the Berry-Robnik approach to a classically integrable quantum system, whose phase space consists of infinitely many fine regions, and to discuss the possibility of deviations from the Poisson statistics. In this paper, we successfully applied these ideas to the study of the \( E(K, L) \) function which is one of the most fundamental observables in the research field of energy-level statistics.

In the Berry-Robnik approach, the quantal eigenfunctions, localizing on different phase-space regions in the neighborhood of the semiclassical (high-energy) limit, form mutually
independent spectral components, where the statistical weight of each component corresponds to the volume ratio (Liouville measure) of the phase-space region. Therefore, we considered a situation where the system consists of infinitely many components and each of them contributes infinitesimally to the spectral statistics. Then, starting from the superposition formula \( \bar{E}(K, L) \) and assumptions (i) and (ii), the limiting distribution function \( \bar{E}(K, L) \) is derived, which is described by the monotonically increasing functions \( \bar{\mu}(K, L) \), \( K = 0, 1, 2, \cdots \) of the level-spacing \( L \). The limiting distribution function \( \bar{E}(K, L) \) is distinguished into three cases: Case 1, Poissonian if \( \bar{\mu}(0, +\infty) = 0 \); Case 2, Poissonian for large \( L \), but possibly to be non-Poissonian for small \( L \) if \( 0 < \bar{\mu}(0, +\infty) < 1 \); and Case 3, non-Poissonian for all \( L \) if \( \bar{\mu}(0, +\infty) = 1 \). Thus, we showed that deviations from Poisson statistics can be observed, not only in the properties of NNLSD and LNV as shown in the previous works of Refs.[1, 2], but also in the properties of individual \( E(K, L) \) functions. Note that cases 2 and 3 are possible when there is a strong accumulation of levels, which leads to a singular level spacing distribution of individual components, and such accumulation is expected to arise for a system that has a symmetry, e.g. spatial symmetry or time-reversal symmetry.

As was shown in the numerical studies of Refs.[1, 2], the rectangular billiard is one possible example by which to show case 2 in addition to case 1 when the aspect parameter of the system is irrational, and case 3 when the aspect parameter is rational. In this paper, the numerical study of the rectangular billiard is extended to the analysis of the \( E(K, L) \) function, where the theoretical arguments of \( \bar{E}(K, L) \) for cases 1–3 are well reproduced. Similar results are expected also in the torus billiard[21, 25], equilateral-triangular billiard[23, 24], and integrable Morse oscillator[26], where the deviations from Poisson statistics are reported to be associated with a spatial symmetry.

The limiting function \( \bar{E}(K, L) \) obtained in the present paper, gives a basis on which to investigate the non-Poissonian behaviors of the other statistical observables. For example, the \( n \)-point correlation function for a system with infinitely many components is calculated from the moment function,

\[
\bar{C}_n = \sum_{K=0}^{+\infty} (K - L)^n \bar{E}(K, L),
\]

and this function for \( n = 2 \) is described using \( \bar{\mu}(K, L) \) in the following simple form[2]:

\[
\bar{C}_2(L) = L + 2 \int_0^L \sum_{K=0}^{+\infty} \bar{\mu}(K, S)dS.
\]

For \( 0 < \bar{\mu}(0, +\infty) \leq 1 \), \( C_2(L) \) provides the non-poissonian limiting LNV, \( \Sigma^2(L) = \bar{C}_2(L) \), whose slope is larger than that of the Poissonian LNV \( \Sigma^2(L) = L \). In a similar way, it is also possible to calculate the skewness \( \bar{\gamma}_1(L) \) and excess \( \bar{\gamma}_2(L) \) from \( \bar{C}_3(L) \) and \( \bar{C}_4(L) \) respectively,
the LSD $P(K, L)$ of $K > 0$ from Eq. (1), and $\Delta_3(L)$ from Eq. (3), which, for $\bar{\mu}(0, L) \neq 0$, show the non-Poissonian behaviors.

This paper reveals the three different classes of $E(K, L)$ statistics possibly observed for the eigenenergy levels consisting of infinitely many components, and also suggests a possibility of new statistical laws (cases 2 and 3) to be observed in the classically integrable quantum systems that have spatial or time-reversal symmetry. Further case study for individual physical systems will be shown elsewhere.

![Figure 1](image)

**Fig. 1** Semilog plots of $E(K, L)$, $K = 0 - 4$ and 10 for the rectangular billiard systems[(a) 41th and (b) fourth approximations of $\gamma = (\sqrt{5} + 1)/2$] and for the square billiard system[(c) $\gamma = 1$]. The truncation parameter is provided as $\delta = \pi \times 10^{-9}$. The solid curve corresponds to the Poisson distribution.

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Fig. 2 Numerical plots of $\tilde{\alpha}_K(L)$, $K = 1 - 4$ and 10 for the rectangular billiard systems [(a) 41th and (b) fourth approximations of $\gamma = (\sqrt{5} + 1)/2$ and for the square billiard system [(c) $\gamma = 1$]. The solid line, $\tilde{\alpha}_K(L) = 1$, corresponds to the Poisson distribution.

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Fig. 3  Numerical plots of $\tilde{\alpha}_K(L), K = 1 - 5$ and 10 for the rectangular billiard system with the 4th approximation of $\gamma = (\sqrt{5} + 1)/2$. The solid line, $\tilde{\alpha}_K(L) = 1$, corresponds to the Poisson distribution.

Fig. 4  Numerical plots of $\tilde{\beta}(L)$ for the rectangular billiard systems [(a) 41th and (b) fourth approximations of $\gamma = (\sqrt{5} + 1)/2$] and for the square billiard system [(c) $\gamma = 1$]. The solid line, $\tilde{\beta}(L) = 0$, corresponds to the Poisson distribution.

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Fig. 5  Numerical test of approximate expression (30) for the square billiard system\((\gamma = 1)\). The solid line represents the theoretical prediction, which is valid in the semiclassical limit \(\epsilon \to +\infty\). For each plots, we used \(4 \times 10^7\) unfolded energy levels.

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A  Appendix A: Derivation of equation \(\tilde{\beta}(0) = M(0)\)

As corollaries of the Palm-Khintchine formula[36], the cumulative NNLSID \(M(L) = \int_0^L P(S) dS\) is rewritten in terms of \(E(0, L)\) as

\[
M(S) = 1 + \frac{d}{dL} E(0, L).
\]  \hspace{1cm} (A1)

Since \(E(0, 0) = 1\) and the equation (A1), \(\ln E(0, L)\) for \(L << 1\) is expanded as

\[
\ln E(0, L) = L \frac{dE(0, L)}{dL} + O(L^2)
\]  \hspace{1cm} (A2)

\[
= L [1 - M(L)] + O(L^2),
\]  \hspace{1cm} (A3)

and we have the relation

\[
\tilde{\beta}(0) = 1 + \frac{1}{L} \ln E(0, L)|_{L=0} = M(0).
\]  \hspace{1cm} (A4)