COMPATIBLE ALMOST COMPLEX STRUCTURES ON
TWISTOR SPACES AND THEIR GRAY-HERVELLA CLASSES

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Abstract. In this paper we determine the Gray-Hervella classes of the compatible almost complex structures on the twistor spaces of oriented Riemannian four-manifolds considered by G. Deschamps in [6].

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1. Introduction

In the 80’s of the last century A. Gray and L.M. Hervella [9] have proposed a natural classification of almost Hermitian manifolds by studying a representation of the unitary group on the space of tensors satisfying the same identities as the covariant derivative of the Kähler form of an almost Hermitian manifold. This representation has four irreducible components, which determine sixteen classes of almost Hermitian manifolds playing an important role in Hermitian geometry.

The main purpose of the present paper is to determine the Gray-Hervella classes of the so-called compatible almost Hermitian structures on the twistor space of an oriented four-dimensional Riemannian manifold \((M, g)\) considered by G. Deschamps in [6]. The (positive) twistor space of \((M, g)\) is the total space of the 2-sphere bundle \(\pi: Z \rightarrow M\) consisting of all unit \((+1)\)-eigenvectors of the Hodge star operator acting on \(\Lambda^2 TM\). The fibre of \(Z\) at a point \(p \in M\) can be identified with space of all complex structures on the tangent space \(T_p M\) compatible with the metric and orientation of \(M\). The Levi-Civita connection of \((M, g)\) gives rise to a splitting \(TZ = H \oplus V\) of the tangent bundle of \(Z\) into horizontal and vertical subbundles. Then, following Atiyah-Hitchin-Singer [3] and Eells-Salamon [7], one can define two almost complex structures \(J_1\) and \(J_2\) on the six-manifold \(Z\) as follows. On every horizontal space \(H_\sigma, \sigma \in Z\), \(J_1\) and \(J_2\) are both the horizontal lift of the complex structure \(\sigma: T_{\pi(\sigma)} M \rightarrow T_{\pi(\sigma)} M\). The vertical space \(V_\sigma\) is the tangent space at the point \(\sigma\) of the fibre of \(Z\) through \(\sigma\). This fibre is a unit 2-sphere and \(J_1\) is defined on \(V_\sigma\) as the standard complex structure of the sphere, while \(J_2\) is the conjugate complex structure \(-J_1\). By a famous theorem of Atiyah-Hitchin-Singer [3] the almost complex structure \(J_1\) is integrable (i.e. comes from a complex structure) if and only if the base manifold \((M, g)\) is anti-self-dual. In contrast, the almost complex structure \(J_2\) is never integrable by a result of Eells-Salamon [7] but it is very useful in harmonic maps theory. The almost complex structures \(J_1\) and \(J_2\) are compatible with the 1-parameter family of Riemannian metrics \(h_t = \pi^*g + tg^v\), \(t > 0\), where \(g^v\) is the restriction to the fibres of \(Z\) of the metric on \(\Lambda^2 TM\) induced...
by \( g \). The Gray-Hervella classes of the almost Hermitian structures \((h_t, J_1)\) and \((h_t, J_2)\) have been determined in [10].

It has been observed by G. Deschamps in [6] that one can obtain other almost complex structures on \(Z\) compatible with the metrics \(h_t\) by means of a fibre-preserving map \(f : Z \to Z\). Given such a map, the corresponding almost complex structure \(J_f\) on \(H_{\sigma}\) is defined as the horizontal lift of \(f(\sigma)\); on \(V_{\sigma}\) it is set to be equal to \(J_1\). Thus, if \(f = id\) we obtain the almost complex structure \(J_1\) and if \(f\) is the antipodal map \(\sigma \to -\sigma\) we get \(-J_2\). In Section 3 of the present paper we derive coordinate-free formulas for the covariant derivative of the Kähler 2-form of the almost Hermitian structure \((h_t, J_f)\) corresponding to an arbitrary \(f\). We use these formulas to determine the Gray-Hervella classes of \((h_t, J_f)\) for some particular fibre-preserving maps \(f : Z \to Z\). More precisely, let \((M, g, J)\) be an almost Hermitian manifold of real dimension four. Consider \(M\) with the orientation induced by the almost complex structure \(J\). Then \(J\) is a section of the (positive) twistor space \(Z\) of \((M, g)\). At any point \(p \in M\), the complex structure \(J_p : T_p M \to T_p M\) is a point of the fibre \(Z_p\) at \(p\) of \(Z\). Take a complex number \(\lambda\). Since \(Z_p\) is the unit 2-sphere, we can compose the stereographic projection of \((h_t, J_f)\) from the point \(J_p\) with the linear map \(z \to \lambda z\) of the complex plane, then go back to the sphere by the inverse stereographic projection. In this way we obtain a fibre-preserving map \(f^\pm_\lambda : Z \to Z\) whose restriction to every fibre is a holomorphic map. If we use in a similar way the stereographic projection from the point \(-J_p\), we get a map \(f^-_\lambda\) whose restrictions to the fibres of \(Z\) are anti-holomorphic maps. In particular, \(f^1(t) = \sigma\), so \(J_{f^+} = J_1\), the Atiyah-Hitchin-Singer almost complex structure, whereas \(f^-_1(\sigma) = -\sigma\) and \(J_{f^-} = -J_2\), the conjugate structure of the Eells-Salamon almost complex structure. For \(\lambda = 0\), we have \(f^\pm_0 \equiv \mp J\); note that the structures \(J\) and \(-J\) induce the same orientation since \(\dim M = 4\) and belong to the same Gray-Hervella classes. In the case \(\lambda = 0\), the integrability condition for the corresponding almost complex structure \(J_{f^\pm_0}\) has been given in [6] (where this structure is denoted by \(J_\infty\)). In Section 4, Theorem 1 we establish all possible Gray-Hervella classes of the almost Hermitian structure \((h_t, J_{f^\pm_0})\) on the twistor space \(Z\) and found the geometric conditions on the base manifold \(M\) under which this structure belongs to each of these classes. In the case when \(\lambda \neq 0,1\) and the base manifold \(M\) is Kähler, the integrability condition for the almost complex structure \(J_{f^\pm_1}\) has been obtained in [6] (the structure being denoted there by \(J_{\lambda, id}\)). Under the assumptions that \(\lambda \neq 0\) and \(M\) is Kähler, in Section 5, Theorem 2 we determine the Gray-Hervella classes of the almost Hermitian structure \((h_t, J_{f^\pm_1})\).

At the end of this section, we discuss also the case when \(|\lambda| = 1\) without the Kähler assumption on the base manifold \(M\). In order to keep the length of the paper reasonable, we discuss only some of the basic Gray-Hervella classes.

2. Preliminaries

Let \((M, g)\) be an oriented Riemannian manifold of dimension four. The metric \(g\) induces a metric on the bundle of two-vectors \(\pi : \Lambda^2 TM \to M\) by the formula

\[
g(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{2} \det[g(v_i, v_j)].
\]
The Levi-Civita connection of \((M, g)\) determines a connection on the bundle \(\Lambda^2 T M\), both denoted by \(\nabla\), and the corresponding curvatures are related by

\[
R(X \wedge Y)(Z \wedge T) = R(X, Y)Z \wedge T + Z \wedge R(X, Y)T
\]

for \(X, Y, Z, T \in T M\). Let us note that we adopt the following definition for the curvature tensor \(R : \{X, Y\} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]\). Then the curvature operator \(\mathcal{R}\) is the self-adjoint endomorphism of \(\Lambda^2 T M\) defined by

\[
g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T).
\]

The Hodge star operator defines an endomorphism * of \(\Lambda^2 T M\) with \(s^2 = \text{Id}\). Hence we have the decomposition

\[
\Lambda^2 T M = \Lambda^2_+ T M \oplus \Lambda^2_- T M
\]

where \(\Lambda^2_\pm T M\) are the subbundles of \(\Lambda^2 T M\) corresponding to the \((\pm 1)\)-eigenvalues of the operator *.

For every \(p \in M\), the group \(SO(4)\) acts in a natural way on the space of 4-tensors on \(T_p M\) having the same symmetries as the Riemannian curvature tensor. The irreducible decomposition of this space under the action of \(SO(4)\), found by Singer and Thorpe [11], gives the following decomposition of the curvature operator

\[
\mathcal{R} = \frac{s}{6} \text{Id} + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_- \quad (1)
\]

where \(s\) is the scalar curvature, the operator \(\mathcal{B}\) represents the traceless Ricci tensor and \(\mathcal{W}_\pm\) are the restrictions on \(\Lambda^4_\pm T M\) of the operator \(\mathcal{W}\) corresponding the the Weyl conformal tensor. These operators are symmetric and \(\mathcal{B}\) sends \(\Lambda^2_+ T M\) into \(\Lambda^2_- T M\), while \(\mathcal{W}_\pm|\Lambda^4_\pm T M = 0\).

A manifold \(M\) is Einstein if and only if \(\mathcal{B} = 0\). It is conformally flat if and only if \(\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-\) vanishes. Recall also that \(M\) is called self-dual (anti-self-dual) if \(\mathcal{W}_- = 0\) (\(\mathcal{W}_+ = 0\)).

For every \(a \in \Lambda^2 T M\), define a skew-symmetric endomorphism of \(T_{\pi(a)} M\) by

\[
g(K_\sigma X, Y) = 2g(a, X \wedge Y), \quad X, Y \in T_p M. \quad (2)
\]

If \(\sigma \in \Lambda^2_+ T M\) is a unit vector, then \(K_\sigma\) is a complex structure on the vector space \(T_{\pi(\sigma)}\) compatible with the metric and the orientation of \(M\). Conversely, the 2-vector \(\sigma\) dual to one half of the Kähler 2-form of such a complex structure is a unit vector in \(\Lambda^2_+ T M\). Thus the unit sphere subbundle \(Z\) of \(\Lambda^2_+ T M\) parametrizes the complex structures on the tangent space of \(M\) compatible with its metric and the orientation. This subbundle is called the twistor space of \(M\).

**Remark.** If we endow \(\Lambda^2 T M\) with the metric \(2g\), as many authors do, then the curvature operator acting on \(\Lambda^2 T M\) is one half of the operator used here and the twistor space of \(M\) is the sphere subbundle of \(\Lambda^2_+ T M\) of radius \(\sqrt{2}\).

Let \((E_1, E_2, E_3, E_4)\) be a local oriented orthonormal frame of \(T M\). Set

\[
\begin{align*}
    s_1 &= E_1 \wedge E_2 + E_3 \wedge E_4, & \bar{s}_1 &= E_1 \wedge E_2 - E_3 \wedge E_4 \\
    s_2 &= E_1 \wedge E_3 + E_4 \wedge E_2, & \bar{s}_2 &= E_1 \wedge E_3 - E_4 \wedge E_2 \\
    s_3 &= E_1 \wedge E_4 + E_2 \wedge E_3, & \bar{s}_3 &= E_1 \wedge E_4 - E_2 \wedge E_3.
\end{align*}
\]

for the curvature tensor. These operators are symmetric and \(\mathcal{B}\) sends \(\Lambda^2_+ T M\) into \(\Lambda^2_- T M\), while \(\mathcal{W}_\pm|\Lambda^4_\pm T M = 0\).
Then \((s_1, s_2, s_3)\), resp. \((\bar{s}_1, \bar{s}_2, \bar{s}_3)\), is a local oriented orthonormal frame of \(\Lambda^2_sTM\), resp. \(\Lambda^2TM\).

For every \(\sigma \in Z\), the tangent space \(T_{\pi(\sigma)}M\) has an orthonormal basis of the form \(E'_1, K_\sigma E'_1, E''_1, K_\sigma E''_1\). This basis yields the orientation of \(T_{\pi(\sigma)}\), so setting\(E_1 = E'_1, E_2 = K_\sigma E'_1, E_3 = E''_1, E_4 = K_\sigma E''_1\) we obtain an oriented orthonormal basis for which \(\sigma = s_1\); for \(E_1 = E'_1, E_2 = E''_1, E_3 = K_\sigma E'_1, E_4 = -K_\sigma E''_1\), we have \(\sigma = s_2\) and if \(E_1 = E'_1, E_2 = E''_1, E_3 = K_\sigma E''_1, E_4 = K_\sigma E'_1\), we have \(\sigma = s_3\).

The Levi-Civita connection \(\nabla\) of \(M\) induces a metric connection on the bundle \(\Lambda^2_sTM\) whose horizontal distribution is tangent to the twistor space \(Z\). Thus we have the decomposition \(TZ = H \oplus V\) of the tangent bundle of \(Z\) into horizontal and vertical components. The vertical space at a point \(\sigma \in Z\) is the space \(V_\sigma = \{V \in T_\sigma Z : \pi_*V = 0\}\).

This is the tangent space to the fibre of \(Z\) through \(\sigma\), thus, considering \(T_\sigma Z\) as a subspace of \(T_\sigma(\Lambda^2^sTM)\) (as we shall always do), \(V_\sigma\) is the orthogonal complement of \(\mathbb{R}\sigma\) in \(\Lambda^2_sT_{\pi(\sigma)}M\). The map \(V \ni V_\sigma \rightarrow KV\) gives an identification of the vertical space with the space of skew-symmetric endomorphisms of \(T_{\pi(\sigma)}M\) that anti-commute with \(K_\sigma\). Let \(s\) be a local section of \(Z\) such that \(s(p) = \sigma\) where \(p = \pi(\sigma)\). Considering \(s\) as a section of \(\Lambda^2_sTM\), we have \(\nabla_Xs \in V_\sigma\) for every \(X \in T_pM\) since \(s\) has a constant length. Moreover, \(X^h_\sigma = s_\ast X - \nabla_Xs\) is the horizontal lift of \(X\) at \(\sigma\).

Denote by \(\times\) the usual vector cross product on the oriented 3-dimensional vector space \(\Lambda^2_sT_pM, p \in M\), endowed with the metric \(g\). Then it easy to check that

\[
g(R(a)b, c) = g(R(b \times c), a)
\]

for \(a \in \Lambda^2T_pM, b, c \in \Lambda^2_sT_pM\), and

\[
g(\sigma \times V, X \wedge K_\sigma Y) = g(\sigma \times V, K_\sigma X \wedge Y) = g(V, X \wedge Y)
\]

for \(V \in V_\sigma, X, Y \in T_pM\).

It is also easy to show that for every \(a, b \in \Lambda^2_sT_pM\)

\[
K_a \circ K_b = -g(a, b)Id + K_{a \times b}.
\]

The action of \(SO(4)\) on \(\Lambda^2\mathbb{R}^4\) preserves the decomposition \(\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4\). Thus, considering \(S^2\) as the unit sphere in \(\Lambda^2_+\mathbb{R}^4\), we have an action of the group \(SO(4)\) on \(S^2\). Then, if \(SO(M)\) denotes the principal bundle of the oriented orthonormal frames on \(M\), the twistor space \(Z\) is the associated bundle \(SO(M) \times_{SO(4)} S^2\). It follows from the Vilms theorem (see, for example, [4, Theorem 9.59]) that the projection map \(\pi : (Z, h_t) \rightarrow (M, g)\) is a Riemannian submersion with totally geodesic fibres (this can also be proved by a direct computation).

Let \((G, x_1, \ldots, x_4)\) be a local coordinate system of \(M\) and let \((E_1, \ldots, E_4)\) be an oriented orthonormal frame of \(TM\) on \(G\). If \((s_1, s_2, s_3)\) is the local frame of \(\Lambda^2_sTM\) define by (3), then \(\overline{\alpha} = x_0 \circ \pi, y_j(\sigma) = g(\sigma, (s_j \circ \pi)(\sigma))\), \(1 \leq \alpha \leq 4, 1 \leq j \leq 3\), are local coordinates of \(\Lambda^2_sTM\) on \(\pi^{-1}(G)\).

The horizontal lift \(X^h\) on \(\pi^{-1}(G)\) of a vector field

\[
X = \sum_{\alpha=1}^4 X^\alpha \frac{\partial}{\partial x_\alpha}
\]
is given by
\[ X^h = \sum_{\alpha=1}^{4} (X^\alpha \circ \pi) \frac{\partial}{\partial x^{\alpha}} - \sum_{j,k=1}^{3} y_j (g(\nabla_X s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}. \]  
(7)

Hence
\[ [X^h, Y^h] = [X,Y]^h + \sum_{j,k=1}^{3} y_j (g(R(X \wedge Y)s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}. \]  
(8)

for every vector fields \( X, Y \) on \( G \). Let \( \sigma \in Z \). Using the standard identification \( T_\sigma (\Lambda^2 T_p M) \cong \Lambda^2 T_p M \) formula (8) can be rewritten as
\[ [X^h, Y^h]_{\sigma} = [X,Y]^h_{\sigma} + R_p (X \wedge Y)_{\sigma}, \quad p = \pi(\sigma). \]  
(9)

Denote by \( D \) the Levi-Civita connection of \((Z, h_t)\). Then we have the following.

**Lemma 1.** ([5]) If \( X, Y \) are vector fields on \( M \) and \( V \) is a vertical vector field on \( Z \), then
\[ (D_X h^h)_{\sigma} = (\nabla_X Y)^h_{\sigma} + \frac{1}{2} R_p (X \wedge Y)_{\sigma}, \]  
(10)
\[ (D_V X^h)_{\sigma} = \mathcal{H}(D_X^h V)_{\sigma} = -\frac{t}{2} (R_p (\sigma \times V) X)^h_{\sigma} \]  
(11)

where \( \sigma \in Z, p = \pi(\sigma) \) and \( \mathcal{H} \) means "the horizontal component".

**Proof.** Identity (10) follows from the Koszul formula for the Levi-Civita connection and (9).

Let \( W \) be a vertical vector field on \( Z \). Then
\[ h_t(D_V X^h, W) = -h_t(X^h, D_V W) = 0 \]
since the fibres are totally geodesic submanifolds, so \( D_V W \) is a vertical vector field. Therefore \( D_V X^h \) is a horizontal vector field. Moreover, \([V, X^h]\) is a vertical vector field, hence \( D_V X^h = \mathcal{H} D_X^h V \). Then
\[ h_t(D_V X^h, Y^h) = h_t(D_X^h V, Y^h) = -h_t(V, D_X^h Y^h). \]

Now (11) follows from (10) and (4). Q.E.D.

3. Compatible almost complex structures on twistor spaces

Let \( f : Z \to Z \) be a morphism of the bundle \( Z \), i.e. a smooth map with \( \pi \circ f = \pi \). Following [6] we define an almost complex structure \( J_f \) on the 6-manifold \( Z \) setting
\[ J_f V = \sigma \times V \quad \text{for } V \in V_{\sigma}, \]
\[ J_f X^h = (K_f(\sigma) X)^h_{\sigma} \quad \text{for } X \in T_{\pi(\sigma)} M. \]

Note that, since the fibres of \( Z \) are spheres, the restriction of \( J_f \) to any fibre is the standard complex structure of the unite sphere.

In the case when \( f = Id \), the almost complex structure \( J_f \) coincides with that defined by Atiyah-Hitchin-Singer [3]. In this case the almost complex structure \( J_f \) is integrable if and only if the base manifold \( M \) is anti-self-dual [3]. If \( f \) is the antipodal map \( f(\sigma) = -\sigma, J_f \) is the conjugate structure of the almost complex structure defined by Eells-Salamon [7]. This structure is never integrable [7].

**Remark.** One can also consider the sphere bundle \( Z^- \) in \( \Lambda^2 TM \) as the twistor space of \( M \). Then the Atiyah-Hitchin-Singer almost complex structure is defined...
as $\mathcal{J}V = -\sigma \times V$ for $V \in \mathcal{V}_\sigma$ and $\mathcal{J}X^h = (K_\sigma X)^h$ for $X \in T_{\pi(\sigma)}M$. It is integrable if and only if $M$ is self-dual. The complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric is self-dual but not anti-self-dual. Thus the Atiyah-Hitchin-Singer almost complex structure on $Z$ is not integrable while it is integrable on $Z^-$. This is one of the reasons some authors to consider the sphere bundle $Z^-$ in $\Lambda^2 T^*M$ instead of that in $\Lambda^2_+ TM$.

The almost complex structure $\mathcal{J}_f$ is compatible with the Riemannian metrics $h_t, t > 0$ defined above and let $\Omega(A, B) = h_t(\mathcal{J}_f A, B)$ be the Kähler 2-form of the almost Hermitian manifold $(Z, h_t, \mathcal{J}_f)$. We now compute the covariant derivative of $\Omega$.

**Proposition 1.** Let $\sigma \in Z, X, Y, Z \in T_p M, p = \pi(\sigma), U, V, W \in \mathcal{V}_\sigma$. Then

$$
(D_{X^h\Omega})(Y^h, Z^h) = 2g(V f_s(X^h), Y \wedge Z);
$$

(12)

$$
(D_{X^h\Omega})(Y^h, U) = -\frac{t}{2}g(R(U), X \wedge Y) + \frac{t}{2}g(R(\sigma \times U), X \wedge K_{f(\sigma)}Y);
$$

(13)

$$
(D_{U\Omega})(Y^h, Z^h) = -\frac{t}{2}g(R(\sigma \times U), Y \wedge K_{f(\sigma)}Z) + K_{f(\sigma)}Y \wedge Z + 2g(f_s U, Y \wedge Z);
$$

(14)

$$
(D_{X^h\Omega})(U, V) = 0, \quad (D_{U\Omega})(Y^h, V) = 0, \quad (D_{U\Omega})(V, W) = 0. \quad (15)
$$

**Proof.** Extend the tangent vector $Y, Z$ to vector fields in a neighbourhood of $p$ such that $\nabla Y |_p = \nabla Z |_p = 0$.

To prove the first formula, we note that

$$(D_{X^h\Omega})(Y^h, Z^h) = X^h(h_t(\mathcal{J}_f Y^h, Z^h)) - h_t(\mathcal{J}_f D_{X^h\Omega} Y^h, Z^h) + h_t(Y^h, \mathcal{J}_f D_{X^h\Omega} Z^h).$$

The vectors $\mathcal{J}_f D_{X^h\Omega} Y^h$ and $\mathcal{J}_f D_{X^h\Omega} Z^h$ are vertical in view of (10). Hence

$$(D_{X^h\Omega})(Y^h, Z^h) = X^h(h_t(\mathcal{J}_f Y^h, Z^h)).$$

Let $s$ be a section of the bundle $Z$ around $p$ such that $s(p) = \sigma$ and $\nabla s |_p = 0$. Then

$$X^h(h_t(\mathcal{J}_f Y^h, Z^h)) = X(h_t(\mathcal{J}_f Y^h, Z^h) \circ s) = X(g(K_{f(s)} Y, Z)) = 2X(g(f \circ s, Y \wedge Z)).$$

The map $\tilde{s} = f \circ s$ is a section of $Z$ with $\tilde{s}(p) = f(\sigma)$. Then

$$X^h_{f(s)} + \nabla_X (f \circ s) = \tilde{s}(X) = f(\sigma)(s_{\ast p}(X)) = f_s(X^h).$$

Therefore $\nabla_X (f \circ s) = \mathcal{V} f_s(X^h)$. Thus

$$(D_{X^h\Omega})(Y^h, Z^h) = 2g(\mathcal{V} f_s(X^h), Y \wedge Z).$$

Extend $U$ to a vertical vector field in a neighbourhood of $\sigma$. Identities (10) and (11) imply that

$$
(D_{X^h\Omega})(Y^h, U) = -h_t(\mathcal{J}_f D_{X^h\Omega} Y^h, U) - h_t(\mathcal{J}_f Y^h, D_{X^h\Omega} U) = -\frac{t}{2}g(\sigma \times R(X, Y) \sigma U) + \frac{t}{2}g(K_{f(\sigma)} Y, R(\sigma \times U) X)
$$

This gives the second formula of the lemma since, in view of (4),

$$
g(\sigma \times R(X, Y) \sigma U) = -g(R(X, Y) \sigma U) = g(R(U), X \wedge Y).$$
Next, we have
\[(D_U \Omega)(Y^h, Z^h) = U(h_t(J_f Y^h, Z^h)) + h_t(D_U Y^h, J_f Z^h) - h_t(J_f Y^h, D_U Z^h).\]  

(16)

Moreover, \(f = \sum_{i=1}^{3} (y_i \circ f)(s_i \circ \pi),\) therefore
\[U(h_t(J_f Y^h, Z^h)) = 2 \sum_{i=1}^{3} U((y_i \circ f)(s_i, Y \wedge Z) \circ \pi) = 2 \sum_{i=1}^{3} U(y_i \circ f)(s_i, Y \wedge Z)_p.\]

The map \(f\) sends fibres to fibres, hence \(f_*\) sends vertical vectors to vertical vectors.

In particular, \(f_*U = \sum_{i=1}^{3} U(y_i \circ f)(\frac{\partial}{\partial y_i})_{f(\sigma)}\). It follows that \(U(h_t(J_f Y^h, Z^h)) = 2g(f_*U, Y \wedge Z)\) and the third formula of the lemma follows from (16) and (11).

To prove the remaining formulas fix a point \(\sigma \in Z\) and set \(p = \pi(\sigma)\). Take an oriented orthonormal frame \((E_1, ..., E_4)\) of \(M\) around the point \(p\) such that \(\nabla E_\alpha|_p = 0, \alpha = 1, ..., 4\), and define an oriented orthonormal frame \((s_1, s_2, s_3)\) of \(\Lambda^2 V, T M\) by means of (3). We have \(\nabla s_i|_p = 0, i = 1, 2, 3\), for the latter frame. Choose also a local coordinate system \((x_1, ..., x_4)\) of \(M\) near \(p\), then define local coordinates \((x_\alpha, y_i)\) \(\alpha = 1, ..., 4, i = 1, 2, 3\), on \(Z\) as above.

Every section \(a\) of \(\Lambda^2_v T M\) on an open set \(G\) gives rise to a vertical vector field \(\bar{a}\) on \(\pi^{-1}(G)\) defined by
\[\bar{a}_\tau = a \circ \pi(\tau) - g(a \circ \pi(\tau), \tau) \tau, \quad \tau \in \pi^{-1}(G).\]

Note that, around every point of \(Z\), there exists a frame of vertical vector fields of this type.

Further on, we shall use this notation without explicitly saying so.

Now take sections \(a\) and \(b\) of \(Z\) defined in a neighbourhood of \(p = \pi(\sigma)\) and such that \(a(p) = U, b(p) = V, \nabla a|_p = \nabla b|_p = 0\). Let \(\bar{a}\) and \(\bar{b}\) be the vertical vector fields associated to \(a\) and \(b\). Then \(\bar{a}_\sigma = U, \bar{b}_\sigma = V\) and
\[\frac{D_X a}{\partial_y} = X^h_t(h_t(J_f \bar{a}, \bar{b}) - h_t(J_f D_X \bar{a}, V) + h_t(U, J_f D_X \bar{b})].\]  

(17)

Set
\[\bar{a} = \sum_{i=1}^{3} \bar{a}_i \frac{\partial}{\partial y_i}, \quad \bar{b} = \sum_{i=1}^{3} \bar{b}_i \frac{\partial}{\partial y_i}.\]

Then
\[\bar{a}_i = \sum_{j=1}^{3} (\delta_{ij} - y_i y_j) (g(a, s_j) \circ \pi),\]  

(18)

and similar for \(\bar{b}_i\). Moreover,
\[\frac{J_f \bar{a} = (y_2 \bar{a}_3 - y_3 \bar{a}_2) \frac{\partial}{\partial y_1} + (y_3 \bar{a}_1 - y_1 \bar{a}_3) \frac{\partial}{\partial y_2} + (y_1 \bar{a}_2 - y_2 \bar{a}_1) \frac{\partial}{\partial y_3}.\]  

(19)

Hence
\[h_t(J_f \bar{a}, \bar{b}) = (y_2 \bar{a}_3 - y_3 \bar{a}_2) \bar{b}_1 + (y_3 \bar{a}_1 - y_1 \bar{a}_3) \bar{b}_2 + (y_1 \bar{a}_2 - y_2 \bar{a}_1) \bar{b}_3.\]
If \( X = \sum_{\alpha=1}^{4} X^\alpha (\frac{\partial}{\partial x_\alpha})_p \), we have \( X^h_\sigma = \sum_{\alpha=1}^{4} X^\alpha (\frac{\partial}{\partial x_\alpha})_\sigma \), hence
\[
X^h_\sigma (\tilde{a}_i) = \sum_{j=1}^{3} (\delta_{ij} - y_i y_j) X(g(a, s_j)) = 0
\]
since \( \nabla_X a = \nabla_X s_j = 0 \). Similarly, \( X^h_\sigma (\tilde{b}_i) = 0 \), \( i = 1, 2, 3 \). It follows that \( X^h_\sigma (h_t(J_f \tilde{a}, \tilde{b})) = 0 \). Using (7) and (18), one obtains by a straightforward computation that \( [X^h, \tilde{a}]_\sigma = (\nabla_X a)_\sigma = 0 \). Hence \( D_{X^h} \tilde{a} = -D_{a^h} X^h \in \mathcal{H}_a \) in view of (11). Then \( h_t(J_f D_{X^h} \tilde{a}, V) = 0 \). Similarly \( h_t(U, J_f D_{X^h} \tilde{b}) = 0 \). Thus, \( (D_{X^h})_\Omega(U, V) = 0 \) by (17). Also
\[
(D_{U})_\Omega(Y^h, V) = U(h_t(J_f Y^h, \tilde{b})) - h_t(J_f D_{U} Y^h, V) - h_t(J_f Y^h, D_{U} \tilde{b}) = 0
\]
since \( J_f Y^h \), \( J_f D_{U} Y^h \) are horizontal vectors and \( D_{U} \tilde{b} \) is vertical.

Finally, the identity \( (D_{U})_\Omega(V, W) = 0 \) is a consequence of the fact that the fibres of \( Z \) are totally geodesic submanifolds and \( J_f \) preserves the vertical distribution.

Proposition 1 and the formula
\[
d\Omega(A, B, C) = \sum_{\textrm{cyc}} (D_A \Omega)(B, C)
\]
give the following

**Corollary 1.** Let \( \sigma \in Z \), \( X, Y, Z \in T_p M \), \( p = \pi(\sigma) \), \( U, V, W \in \mathcal{V}_\sigma \). Then
\[
\begin{align*}
&d\Omega(X^h, Y^h, Z^h) = 2g(\nabla f, (X^h, Y \wedge Z) + 2g(\nabla f, (Y^h, Z \wedge X) + 2g(\nabla f, (Z^h, X \wedge Y)) \\
&d\Omega(X^h, Y^h, U) = g(2fU - t\mathcal{R}(U), X \wedge Y) \\
&d\Omega(X^h, U, V) = 0, \quad d\Omega(U, V, W) = 0.
\end{align*}
\]

**Corollary 2.** Let \( \sigma \in Z \), \( X \in T_p M \), \( p = \pi(\sigma) \), \( U \in \mathcal{V}_\sigma \). Then
\[
(\delta \Omega)(X^h_\sigma) = \text{Trace}\{T_p M \ni A \to 2g(\nabla f, (A^h_\sigma), X \wedge A)\} = \\
\text{Trace}\{\mathcal{V}_\sigma \ni \tau \to g(\nabla f, ((K_\tau X)^h_\sigma), \tau)\}.
\]
\[
\delta \Omega(U) = -tg(\mathcal{R}(\sigma \times U), f(\sigma)).
\]

**Proof.** Let \( E_1, ..., E_4 \) be an orthonormal basis of \( T_p M \), \( p = \pi(\sigma) \) and \( \tau_1, \tau_2 \) a \( g \)-orthogonal basis of \( \mathcal{V}_\sigma \). Then, by (12) and (15),
\[
\begin{align*}
\delta \Omega(X^h_\sigma) &= -\sum_{i=1}^{4} (D_{(E_i)_\sigma})_\Omega((E_i)^h_\sigma, X^h_\sigma) - \sum_{m=1}^{2} (D_{\tau_m})_\Omega(\tau_m, X^h_\sigma) = \\
&-2\sum_{i=1}^{4} g(\nabla f, ((E_i)^h_\sigma), E_i \wedge X) = -2\sum_{i=1}^{4} \sum_{m=1}^{2} g((f_*(r(E_i)^h_\sigma)), \tau_m)g(\tau_m, E_i \wedge X) = \\
&\sum_{i=1}^{4} \sum_{m=1}^{2} g(f_*(r(E_i)^h_\sigma), \tau_m)g(K_{\tau_m} X, E_i) = \sum_{m=1}^{2} g(f_*(r(K_{\tau_m} X)^h_\sigma), \tau_m).
\end{align*}
\]
This proves (20).

In view of (13) and (15), we have
\[
\delta \Omega(U) = -\frac{t}{2} g(\mathcal{R}(\sigma \times U), \sum_{i=1}^{4} E_i \wedge K f(\sigma) E_i)
\]
Moreover, for $Y, Z \in T_pM$,
\[
\sum_{i=1}^{4} g(E_i \wedge K_{f(\sigma)}E_i, Y \wedge Z) = \\
\frac{1}{2} \sum_{i=1}^{4} [-g(Y, E_i)g(K_{f(\sigma)}Z, E_i) + g(Z, E_i)g(K_{f(\sigma)}Y, E_i)] = \\
g(K_{f(\sigma)}Y, Z) = 2g(f(\sigma), Y \wedge Z).
\]
Thus $\sum_{i=1}^{4} E_i \wedge K_{f(\sigma)}E_i = 2f(\sigma)$ and the second formula of the corollary is proved.

Denote the Nijenhuis tensor of $J_f$ by $N$. The next statement follows from Proposition 1, identity (5) and the well-known formula
\[
h_t(N(A, B), C) = (D_A \Omega)(J_f B, C) - (D_Jf, B)(A, C) - (D_B \Omega)(J_f A, C) + (D_Jf, A)(B, C).
\]

**Corollary 3.** Let $\sigma \in Z$, $X, Y, Z \in T_pM$, $p = \pi(\sigma)$, $U, V \in V_\sigma$. Then
\[
h_t(N(X^h_\sigma, Y^h_\sigma), Z^h_\sigma) = 2g(Vf_\sigma(X^h_\sigma), K_{f(\sigma)}Y \wedge Z) - 2g(Vf_\sigma(Y^h_\sigma), K_{f(\sigma)}X \wedge Z) + 2g(Vf_\sigma((K_{f(\sigma)}Y)_\sigma), X \wedge Z) \]
\[
- 2g(Vf_\sigma((K_{f(\sigma)}X)_\sigma), Y \wedge Z) \quad (22)
\]

Since $h_t(N(X^h, U), Z^h) = -2g(f(\sigma) \times f_\sigma(U), X \wedge Z) + 2g(f_\sigma(\sigma \times U), X \wedge Z)$, we have the following.

**Corollary 4.** ([6]) $\mathcal{H}(N(X^h, U)) = 0$ if and only if the restriction of $f$ to every fibre is a holomorphic map.

4. **Gray-Hervella classes of the almost complex structures $J_\omega$**

In what follows we use the same notation for the Gray-Hervella classes as in [9]. For example, $K$ is the class of Kähler manifolds, $W_1$ is the class of nearly Kähler manifolds, $W_2$ is the class of almost Kähler manifolds, $W_3 \oplus W_4$ is the class of Hermitian manifolds, $W_1 \oplus W_2 \oplus W_3$ is the class of semi-Kähler or balanced manifolds, etc.

Let $(g, J)$ be an almost Hermitian structure on a four-manifold $M$. Define a section $\omega$ of $\Lambda^2TM$ by
\[
g(\omega, X \wedge Y) = \frac{1}{2} g(JX, Y), \quad X, Y \in TM.
\]
Clearly, at any point, $\omega$ is the dual 2-vector of one half of the Kähler 2-form $F$ of the almost Hermitian manifold $(M, g, J)$. Consider $M$ with the orientation yielded by the almost complex structure $J$. Then $\omega$ is a section of the twistor bundle $Z$. As in [6], define a bundle map $f: Z \to Z$ setting $f = \omega \circ \pi$. Since the restriction of $f$ to any fibre is a constant map, $f_*|\mathcal{V} = 0$. We also have
\[
f_*(X^h_\sigma) = X^h_{\omega(p)} + \nabla_X \omega,
\]
where \( p = \pi(\sigma) \) and \( X \in T_p M \). Note that
\[
2g(\nabla_X \omega, Y \wedge Z) = (\nabla_X F)(Y, Z).
\]

Denote by \( J_\omega \) the almost complex structure on \( Z \) determined by the map \( f \) defined by \( \omega \). In the next theorem we determine the Gray-Hervella classes of the almost Hermitian manifolds \((Z, h_t, J_\omega)\).

**Theorem 1.** Let \((M, g, J)\) be an almost Hermitian 4-manifold with Kähler 2-vector \( \omega \), self-dual Weyl tensor \( W_+ \), and scalar curvature \( s \). The possible Gray-Hervella classes of its twistor space \((Z, h_t, J_\omega)\) are \( W, K, W_3, H = W_3 \oplus W_4, SK = W_1 \oplus W_2 \oplus W_3, G_1 = W_1 \oplus W_3 \oplus W_4 \) and \( G_2 = W_2 \oplus W_3 \oplus W_4 \). Moreover
(i) \((Z, h_t, J_\omega) \in K\) if and only if \((M, g, J)\) is Kähler and Ricci flat.
(ii) \((Z, h_t, J_\omega) \in SK \cap H = W_3\) if and only if \((M, g, J)\) is Kähler and scalar flat.
(iii) \((Z, h_t, J_\omega) \in H = W_3 \oplus W_4\) if and only if \((M, g, J)\) is Hermitian and
\[
W_+(\sigma) = \frac{s}{2}g(\sigma, \omega) - \frac{s}{6} \sigma
\]
for all \( \sigma \in \Lambda^2 T^* M \).
(iv) \((Z, h_t, J_\omega) \in SK = W_1 \oplus W_2 \oplus W_3\) if and only if \((M, g, J)\) is almost Kähler and
\[
W_+(\omega) = -\frac{s}{6} \omega.
\]
(v) \((Z, h_t, J_\omega) \in G_1 = W_1 \oplus W_3 \oplus W_4\) if and only if \((M, g, J)\) is Hermitian.
(vi) \((Z, h_t, J_\omega) \in G_2 = W_2 \oplus W_3 \oplus W_4\) if and only if
\[
W_+(\sigma) = \frac{s}{2}g(\sigma, \omega) - \frac{s}{6} \sigma
\]
for all \( \sigma \in \Lambda^2 T^* M \).

**Proof.** To determine the possible Gray-Hervella classes of the twistor space \((Z, h_t, J_\omega)\) we shall need several technical lemmas.

Given a point \( Z \), we take a basis \( E_1, E_2 = JE_1, E_3, E_4 = JE_3 \) of \( T_{\pi(\sigma)} M \). Such a basis induces the orientation of \( M \) we have chosen and we define \( s_1, s_2, s_3 \) and \( \bar{s}_1, \bar{s}_2, \bar{s}_3 \) via (3) (so \( \omega = s_1 \)). This notation will be used in the proofs of the next statements.

**Lemma 2.** \((Z, h_t, J_\omega) \in K\) if and only if \((M, g, J)\) is Kähler and Ricci flat.

**Proof.** It follows from Proposition 1 and (22) that \((Z, h_t, J_\omega)\) is Kähler if and only if \((M, g, J)\) is Kähler and for every \( \sigma \in Z, U \in V_\sigma \) and \( X, Y \in T_{\pi(\sigma)} M \)
(i) \(-g(\mathcal{R}(U), X \wedge Y) + g(\mathcal{R}(\sigma \times U), X \wedge JY) = 0,\)
(ii) \( g(\mathcal{R}(U), X \wedge JY + JX \wedge Y) = 0.\)

The latter identity implies
\[
g(\mathcal{R}(U), s_2) = g(\mathcal{R}(U), s_3) = 0.
\]

It follows from identity (i) that
\[
g(\mathcal{R}(U), E_1 \wedge E_2) = g(\mathcal{R}(U), E_3 \wedge E_4) = 0,
\]

(23)
\[ g(\mathcal{R}(U), E_1 \wedge E_3) = g(\mathcal{R}(\sigma \times U), E_1 \wedge E_4), \]
\[ g(\mathcal{R}(U), E_3 \wedge E_1) = g(\mathcal{R}(\sigma \times U), E_3 \wedge E_2). \]  
(25)

We obtain from (24) that \( g(\mathcal{R}(U), s_i) = g(\mathcal{R}(U), s_1) = 0 \). Thus \( g(\mathcal{R}(U), s_i) = 0 \) for \( i = 1, 2, 3 \) and every \( U \in \Lambda^2_+ T_p M \). It follows from (25) that 
\[ g(\mathcal{R}(\sigma \times U), s_3) = 0. \]

Moreover, identities (23) and (25) imply 
\[ g(\mathcal{R}(U), s_2) = 2g(\mathcal{R}(U), E_1 \wedge E_3) = 2g(\mathcal{R}(\sigma \times U), E_1 \wedge E_4) = g(\mathcal{R}(\sigma \times U), s_3). \]

Therefore \( g(\mathcal{R}(U), s_2) = 0 \), thus \( g(\mathcal{R}(U), s_i) = 0 \), \( i = 1, 2, 3 \). Hence \( \mathcal{R}(U) = 0 \) for every \( U \in \Lambda^2_+ T_p M \). This shows that if \( (Z, h_t, \mathcal{J}_w) \) is Kähler, then \((M, g, J)\) is a Kähler and Ricci flat manifold.

Conversely, suppose that \((M, g, J)\) is Kähler and Ricci flat. Using the curvature decomposition (1), the Kähler curvature identities and the first Bianchi identity, one can see that 
\[ g(\mathcal{R}(s_1), s_1) = \frac{s}{3} s_1, \quad \mathcal{R}(s_2) = \mathcal{R}(s_3) = 0. \]

This implies the well-known fact (which can be traced back to [8]) that the eigenvalues of the operator \( W_+ \) on a Kähler surface are \( \frac{1}{3}, -\frac{1}{6}, -\frac{1}{6} \). It follows that \( \mathcal{R}(U) = 0 \) for every \( U \in \Lambda^2_+ T_p M \), thus identities (i) and (ii) obviously are satisfied.

**Lemma 3.** \((Z, h_t, \mathcal{J}_w) \in W_1 \oplus W_2 \oplus W_4\) if and only if \((Z, h_t, \mathcal{J}_w) \in \mathcal{K}\).

**Proof.** The condition for \((Z, h_t, \mathcal{J}_w)\) to be in the class \( W_1 \oplus W_2 \oplus W_4\) is
\[
(DA \Omega)(B, C) + (D_{\mathcal{J}_w} A \Omega)(\mathcal{J}_w B, C) = 
-\frac{1}{4} \{ h_t(A, B) \delta \Omega(C) - h_t(A, C) \delta \Omega(B) - h_t(A, \mathcal{J}_w B) \delta \Omega(\mathcal{J}_w C) 
+ h_t(A, \mathcal{J}_w C) \delta \Omega(\mathcal{J}_w B) \} 
\]
(26)

for every \( A, B, C \in T Z \). Proposition 1 and (22) imply that this condition is satisfied if and only if for every \( \sigma \in Z, X, Y, Z \in T \pi(\sigma) M \) and \( U, V, W \in \mathcal{V}_\sigma \) we have
\[
(i) \quad \nabla_X F(Y, Z) + \nabla_X F(JY, Z) = 
-\frac{1}{4} \{ g(X, Y) \delta F(Z) - g(X, Z) \delta F(Y) - g(X, JY) \delta F(JZ) + g(X, JZ) \delta F(JY) \} + 
- g(\mathcal{R}(U), X \wedge Y + JX \wedge JY) + g(\mathcal{R}(\sigma \times U), X \wedge JY - JX \wedge Y) = 
- g(\mathcal{R}(U), X \wedge Y + JX \wedge JY) + g(\mathcal{R}(\sigma \times U), X \wedge JY - JX \wedge Y) = 
0, 
\]
\[
(ii) \quad g(U, V) \delta F(X) - g(U, \sigma \times V) \delta F(JX) = 0, 
\]
\[
(iv) \quad g(U, V)g(\mathcal{R}(\sigma \times W), \omega) - g(U, W)g(\mathcal{R}(\sigma \times V), \omega) + 
\]
\[
(v) \quad g(U, \sigma \times V)g(\mathcal{R}(W), \omega) - g(U, \sigma \times W)g(\mathcal{R}(V), \omega) = 0. 
\]

Clearly, identity (v), obtained from (26) for vertical vectors \( A = U, B = V, C = W \), holds when \( U = 0 \). If \( U \neq 0 \), then \( U, \sigma \times U \) is a basis of \( \mathcal{V}_\sigma \) and it is easy to check that this identity is also satisfied. Thus identity (v) does not impose any restriction on the base manifold \( M \). Identity (iv) implies that \( \delta F = 0 \). Then it follows from (i) that \( (\nabla_X F)(Y, Z) + (\nabla_X F)(JY, Z) = 0 \). It is well-known (and easy to see) that, in dimension 4, the latter identity is equivalent to \( dF = 0 \). Take a point \( p \in M \) and let \( X \in T_p M \) be a unit vector. For every
Since $s \in Z$ with $\pi(s) = p$ and every $U \in V_\sigma$, identity (ii) with $Y = JX$ gives 
$$2g(\mathcal{R}(U), X \wedge JX) = g(\mathcal{R}(U), s_1).$$ 
Thus we have $2g(\mathcal{R}(U), E_1 \wedge E_2) = g(\mathcal{R}(U), s_1)$ and 
$$2g(\mathcal{R}(U), E_3 \wedge E_4) = g(\mathcal{R}(U), s_1).$$ 
This implies
$$g(\mathcal{R}(U), E_1 \wedge E_2) = g(\mathcal{R}(U), E_3 \wedge E_4) = g(\mathcal{R}(U), s_1) = 0$$

since $s_1 = E_1 \wedge E_2 + E_3 \wedge E_4$. It follows that
$$g(\mathcal{R}(U), s_1^\perp) = 0, \quad g(\mathcal{R}(s_1), s_1) = g(\mathcal{R}(s_2), s_1) = 0.$$ 

(27)

Identity (iii) with $X = E_1, Y = E_3$ becomes
$$g(\mathcal{R}(U), s_2) + g(\mathcal{R}(\sigma \times U), s_3) = 0, \quad U \in V_\sigma.$$ 

Applying the latter identity for $\sigma = s_2$ and $\sigma = s_3$ and taking into account (27) we see that
$$g(\mathcal{R}(s_2), s_2) = g(\mathcal{R}(s_2), s_3) = g(\mathcal{R}(s_3), s_3) = 0.$$ 

It follows that $g(\mathcal{R}(s_1), s_j) = 0, i, j = 1, 2, 3$. This means that $(M, g)$ is anti-self-dual with zero scalar curvature.

Since $g(\mathcal{R}(U), \omega) = 0$ for every vertical vector $U$, identity (ii) takes the form
$$g(\mathcal{R}(U), X \wedge Y + JX \wedge JY) = g(\mathcal{R}(\sigma \times U), X \wedge JY - JX \wedge Y) = 0.$$ 

Setting in this identity $(X, Y) = (E_1, E_3)$ and $(X, Y) = (E_3, E_1)$ we obtain
$$g(\mathcal{R}(U), \tilde{s}_2) - g(\mathcal{R}(\sigma \times U), \tilde{s}_3) = 0, \quad g(\mathcal{R}(U), \tilde{s}_2) + g(\mathcal{R}(\sigma \times U), \tilde{s}_3) = 0.$$ 

This, together with (27), implies $g(\mathcal{R}(U), s_j^\perp) = 0, j = 1, 2, 3$. Thus
$$g(\mathcal{R}(s_i), s_j^\perp) = 0, \quad i, j = 1, 2, 3$$

which means that $\mathcal{B} = 0$. We note also that, since $\dim M = 4$, $dF = \theta \wedge F$, where $\theta = \delta F \circ J$, so the identity $\delta F = 0$ is equivalent to $dF = 0$, i.e. to $(M, g, J)$ being almost Kähler. It follows that if $(Z, h_t, \mathcal{J}_\omega) \in W_1 \oplus W_2 \oplus W_4$, then $(M, g, J)$ is almost Kähler, anti-self-dual and Ricci flat manifold. According to [2, Proposition 1] these conditions are equivalent to the base manifold being Kähler and Ricci flat. For such a manifold we have $\nabla F = \delta F = 0$ and $R(U) = 0$ for every vertical vector $U$, thus conditions (i) - (iv) are clearly satisfied. Now the lemma follows from Lemma 2.

**Lemma 4.** $(Z, h_t, \mathcal{J}_\omega) \in \mathcal{SK} = W_1 \oplus W_2 \oplus W_3$ if and only if $(M, g, J)$ is almost Kähler and $\mathcal{W}_+(\omega) = -\frac{8}{6} \omega$.

**Proof.** The defining condition for the class of semi-Kähler manifolds is $\delta \Omega = 0$. According to Corollary 2 and (22), the twistor space is semi-Kähler if and only if $g(\delta \omega, X) = 0$, i.e. $\delta F(X) = 0$ and $g(\mathcal{R}(U), \omega \circ \pi(s)) = 0$ for every $\sigma \in Z, U \in V_\sigma, X \in T_{\pi(\sigma)}M$. As we have mentioned the identity $\delta F = 0$ is equivalent to $dF = 0$ since $\dim M = 4$. The identity $g(\mathcal{R}(U), \omega \circ \pi(s)) = 0$ for all $U \in V_\sigma$ holds if and only if $g(\mathcal{R}(\omega), s_i) = 0, i = 1, 2, 3$. This is equivalent to $\frac{8}{6} \omega + \mathcal{W}_+(\omega) = 0$.

**Lemma 5.** $(Z, h_t, \mathcal{J}_\omega) \in \mathcal{G}_1 = W_1 \oplus W_3 \oplus W_4$ if and only if the almost complex structure $J$ is integrable.
Lemma 6. \((Z, h_t, \mathcal{J}_\omega)\) belongs to the class \(\mathcal{G}_1\) when 
\[ (D_A\Omega)(A, B) - (D_{\mathcal{J}_\omega} A)(\mathcal{J}_\omega A, B) = 0, \quad A, B \in TZ. \] 
(28)

It follows from Proposition 1 and (22) that this condition holds if and only if for every \(X, Y \in TM\) 
\[ (\nabla_X F)(X, Y) - (\nabla_{JX} F)(JX, Y) = 0. \]
In dimension 4, the latter identity is equivalent to \(J\) being integrable.

Lemma 6. \((Z, h_t, \mathcal{J}_\omega)\) \(\in \mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4\) if and only if 
\[ W_+(\sigma) = \frac{s}{2}g(\sigma, \omega) - \frac{s}{6}\sigma \]
for all \(\sigma \in \Lambda^3 TM\).

Proof. The condition for \((Z, h_t, \mathcal{J}_\omega)\) to be in the class \(\mathcal{G}_2\) is 
\[ \otimes_{A, B, C} \{ (D_A\Omega)(B, C) - (D_{\mathcal{J}_\omega} A)(\mathcal{J}_\omega B, C) \} = 0. \] 
(29)

By Proposition 1 and (22) this is equivalent to the following identities 
(i) \(\nabla_X F)(Y, Z) - (\nabla_{JX} F)(JY, Z) = 0, \quad X, Y, Z \in TM.\)
(ii) \(g(\mathcal{R}(U), X \wedge Y - JX \wedge JY) - g(\mathcal{R}(\sigma \times U), X \wedge JY + JX \wedge Y) = 0,\)
for every \(\sigma \in Z, U \in \mathcal{V}_\sigma, X, Y \in T_{\pi(\sigma)}M.\) Identity (i) is always satisfied in dimension 4. Identity (ii) gives 
\[ g(\mathcal{R}(U), s_2) - g(\mathcal{R}(\sigma \times U), s_3) = 0. \]
Applying the latter identity for \(\sigma = s_1, s_2, s_3\), it is easy to see that 
\[ g(\mathcal{R}(s_i), s_j) = 0 \text{ for } (i, j) \neq (1, 1). \]
The curvature decomposition and the fact that \(\text{Trace} \mathcal{W}_+ = 0\) then imply 
\[ \frac{s}{6} + g(\mathcal{W}_+(\omega), \omega) = g(\mathcal{R}(\omega), \omega) = \frac{s}{2}. \]
Thus the matrix of \(\mathcal{W}_+\) with respect to the basis \(s_1 = \omega, s_2, s_3\) is diagonal with 
\[ \text{diagonal entries } \frac{s}{3}, -\frac{s}{6}, -\frac{s}{6}. \]
Therefore \(\mathcal{W}_+(\sigma) = \frac{s}{2}g(\sigma, \omega) - \frac{s}{6}\sigma.\)

Conversely, suppose that this identity is fulfilled. Then \(\mathcal{R}(\sigma) = \frac{s}{2}g(\sigma, \omega) + \mathcal{B}(\sigma).\)
It is easy to check that if \(\sigma \in \Lambda^2 T_p M\) and \(\tau \in \Lambda^2 T_p M\), the endomorphisms \(K_\sigma\) and \(K_\tau\) of \(T_p M\) commute, \(K_\sigma \circ K_\tau = K_\tau \circ K_\sigma.\) This implies that, for every \(X, Y \in T_p M,\)
the 2-vector \(X \wedge Y - K_\sigma X \wedge K_\tau Y\) is orthogonal to \(\Lambda^2 T_p M\), so it lies in \(\Lambda^2 T_p M.\) In particular, \(g(\mathcal{B}(\sigma), X \wedge Y - JX \wedge JY) = 0.\) We also have \(g(\omega, X \wedge Y - JX \wedge JY) = 0.\)
Thus \(g(\mathcal{R}(\sigma), X \wedge Y - JX \wedge JY) = 0\) for every \(\sigma \in Z, X, Y \in T_{\pi(\sigma)}M.\) It follows that condition (ii) is satisfied, hence \((Z, h_t, \mathcal{J}_\omega) \in \mathcal{G}_2.\)

Lemma 7. \((Z, h_t, \mathcal{J}_\omega) \in \mathcal{W}_1 \oplus \mathcal{W}_3\) if and only if \((M, g, J)\) is Kähler and scalar flat.

Proof. Note that 
\[ \mathcal{W}_1 \oplus \mathcal{W}_3 = (\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3) \cap (\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4).\]
Hence it follows from Lemmas 4 and 5 that \((Z, h_t, \mathcal{J}_\omega) \in \mathcal{W}_1 \oplus \mathcal{W}_3\) if and only if \((M, g, J)\) is Kähler and \(\mathcal{W}_+(\omega) = -\frac{s}{6}\omega.\) But, as we have already mentioned, it is
well-known that for Kähler manifolds $W_+^{(\omega)} = \frac{s}{3} \omega$ and the above identity implies that $s = 0$. The converse statement follows from the fact that a Kähler manifold is scalar flat if and only if it is anti-self-dual.

**Lemma 8.** $(Z, h_t, J_\omega) \in W_2 \oplus W_3$ if and only if $(M, g, J)$ is Kähler and scalar flat.

**Proof.** It follows from Lemmas 4 and 6 that $(Z, h_t, J_\omega) \in W_2 \oplus W_3$ if and only if $(M, g, J)$ is almost Kähler, anti-self-dual and scalar flat. Now the lemma follows from Proposition 1 in [2] according to which these conditions are equivalent to the base manifold being Kähler and scalar flat.

**Lemma 9.** $(Z, h_t, J_\omega) \in W_3$ if and only if $(M, g, J)$ is a Kähler and scalar flat.

**Proof.** The lemma follows from Lemmas 7 and 8.

**Lemma 10.** $(Z, h_t, J_\omega) \in H = W_3 \oplus W_4$ if and only if the almost complex structure $J$ is integrable and

$$W_+^{(\sigma)} = \frac{s}{3} g(\sigma, \omega) \omega - \frac{s}{6} \sigma$$

for all $\sigma \in \Lambda^2 TM$.

**Proof.** The proof follows from Lemmas 5 and 6.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.**

It follows from Lemmas 2 and 3 that

$$K = W_1 = W_2 = W_4 = W_1 \oplus W_2 = W_1 \oplus W_4 = W_2 \oplus W_4 = W_1 \oplus W_2 \oplus W_4.$$

Lemmas 7, 8 and 9 imply that

$$W_3 = W_1 \oplus W_3 = W_2 \oplus W_3.$$

Hence the first part of the theorem follows from Lemmas 4, 5, 6 and 10.

The statements (i)–(vi) follow respectively from Lemmas 2, 9, 10, 4, 5 and 6.

**Remark.** Concerning the geometric conditions in Theorem 1 we note that:

(i). Any compact Kähler and Ricci flat surface is either a complex torus, a hyper-elliptic surface with the flat metric, a $K3$-surface with a Calabi-Yau metric or its $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ quotient.

(ii). The spectrum of the anti-self-dual Weyl tensor $W_+$ of a Hermitian surface $M$ is equal to $(\frac{k}{3}, -\frac{k}{6}, -\frac{k}{6})$, where $k$ is the conformal scalar curvature [1]. Hence the curvature condition in Theorem 1, (iii) implies that $k = s$, i.e. $\delta \theta = \|\theta\|^2$, where $\theta$ is the Lee form of $M$ [1]. If $M$ is compact, then integrating this identity and using Stock's formula, we see that $\theta = 0$, i.e. the surface $M$ is locally conformally Kähler.

(iii). We do not know non-Kähler examples of compact almost Kähler 4-manifolds whose anti-self-dual Weyl tensor $W_+$ satisfies the condition of Theorem 1, (iv).
5. Gray-Hervella classes of the almost complex structures $J^\pm_\lambda$

As in [6], in order to define a fibre-preserving map $f : Z \to Z$ in an explicit way, we shall use the stereographic projection of every fibre $Z_{\pi(\sigma)}$ from the point $\omega_{\pi(\sigma)}$ onto the plane $({\mathbb R}\omega_{\pi(\sigma)})^1$, the orthogonal complement being taken in $\Lambda^2_+T_{\pi(\sigma)}M$. This stereographic projection $\Phi_\sigma$ and its inverse $\Phi_\sigma^{-1}$ are given by

$$\Phi_\sigma(\tau) = \frac{\tau - g(\tau, \omega_{\pi(\sigma)})\omega_{\pi(\sigma)}}{1 - g(\tau, \omega_{\pi(\sigma)})}, \quad \tau \in Z_{\pi(\sigma)} \setminus \{\omega_{\pi(\sigma)}\},$$

$$\Phi_\sigma^{-1}(\zeta) = \frac{2\zeta + ||\zeta||^2 - 1}{||\zeta||^2 + 1}, \quad \zeta \in ({\mathbb R}\omega_{\pi(\sigma)})^1.$$  

The map $\Phi_\sigma$ is holomorphic with respect to the standard complex structure of $Z_{\pi(\sigma)}$ and the complex structure of $({\mathbb R}\omega_{\pi(\sigma)})^1$ given by $\zeta \to \omega_{\pi(\sigma)} \times \zeta$ (the latter structure is compatible with the metric $g$ of $\Lambda^2_+T_{\pi(\sigma)}M$). As usual, we also set $\Psi_\sigma(\sigma) = \infty$, the "ideal" element of the plane $({\mathbb R}\omega_{\pi(\sigma)})^1$.

Let $\lambda = a + ib \in \mathbb{C}$ and set $F_\lambda(\zeta) = \lambda \zeta$ for $\zeta \in ({\mathbb R}\omega_{\pi(\sigma)})^1$. Then

$$f^+_\lambda(\sigma) = \Phi_\sigma^{-1} \circ F_\lambda \circ \Phi_\sigma(\sigma)$$

is a self-map of $Z$ whose restriction to any fibre is holomorphic. Similarly, denote by $\Psi_\sigma$ the stereographic projection of $Z_{\pi(\sigma)}$ from the point $-\omega_{\pi(\sigma)}$ onto the plane $({\mathbb R}\omega_{\pi(\sigma)})^1$. Set $f^-_\lambda(\sigma) = \Psi_\sigma^{-1} \circ F_\lambda \circ \Psi_\sigma(\sigma)$. In this way we obtain another self-map of $Z$ whose restriction to any fibre is anti-holomorphic. Clearly, the points $f^+_\lambda(\sigma)$ and $f^-_\lambda(\sigma)$ are symmetric with respect to the plane $({\mathbb R}\omega_{\pi(\sigma)})^1$.

The maps $f^\pm_\lambda : Z \to Z$ are given by the following explicit formula:

$$f^+_\lambda(\sigma) = [(a^2 + b^2 + 1) + (a^2 + b^2 - 1)g(\sigma, \omega_{\pi(\sigma)})]^{-1} \times$$

$$\{2\sigma - 2a\sigma \times \omega_{\pi(\sigma)} - 2ag(\sigma, \omega_{\pi(\sigma)})\omega_{\pi(\sigma)}$$

$$\pm[(a^2 + b^2 - 1) + (a^2 + b^2 + 1)g(\sigma, \omega_{\pi(\sigma)})][\omega_{\pi(\sigma)}].$$

Denote by $J^\pm_\lambda$ the almost complex structure on $Z$ defined by means of the map $f^\pm_\lambda(\sigma)$. Note that $f^\pm_0 = \mp \omega$ and $J^\pm_0$ is the almost complex structure on $Z$ yielded by the almost complex structure $\mp J$ on $M$ and discussed in the preceding section. The structure $J^\pm_\lambda$ is denoted by $J_{\lambda,t}$ in [6] where the integrability condition for this structure is found when the base manifold $(M, g, J)$ is Kähler. Note also that $f^+_\lambda(\sigma) = \sigma$ and $J^+_\lambda$ is the Atiyah-Hitchin-Singer almost complex structure, whereas $f^-_\lambda(\sigma) = -\sigma$ and $J^-_\lambda$ is the Eells-Salamon almost complex structure. The Gray-Hervella classes of these structures have been determined in [10].

Since the restrictions to the fibres of the map $f^-_\lambda(\sigma)$ are not holomorphic, Corollary 4 implies the following.

**Corollary 5.** The almost complex structure $J^-_\lambda$ is never integrable.
Theorem 2. Let $(M, g, J)$ be a Kähler manifold and $\lambda \neq 0$ be a complex number. 
(i) The possible Gray-Hervella classes of the twistor space $(Z, h_t, J^\pm)$ are $\mathcal{W}$, $\mathcal{Q}_K = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $\mathcal{S}_K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. Moreover
\[(i_1) (Z, h_t, J^-) \in \mathcal{S}_K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \text{ if and only if } (M, g, J) \text{ is scalar flat.} \]
\[(i_2) (Z, h_t, J^+) \in \mathcal{Q}_K = \mathcal{W}_1 \oplus \mathcal{W}_2 \text{ if and only if } (M, g, J) \text{ is Ricci flat.} \]
(ii) The possible Gray-Hervella classes of the twistor space $(Z, h_t, J^\pm)$ are $\mathcal{W}$ and $\mathcal{W}_3 = \mathcal{S}_K \cap \mathcal{H}$. The latter case occurs if and only if $(M, g, J)$ is scalar flat.

Proof. Given a point $p \in M$, we choose an orthonormal frame of vector fields $A_1, \ldots, A_4$ around $p$ such that $A_3 = JA_2$, $A_4 = JA_1$ and use this frame to define sections $s_1, s_2, s_3$ of $\Lambda^2_* TM$ via (3). Then $\omega = s_3$ and

\[s_1 = A_1 \wedge A_2 - JA_1 \wedge JA_2, s_2 = A_1 \wedge JA_2 + JA_1 \wedge A_2, s_3 = A_1 \wedge JA_1 + A_2 \wedge JA_2.\]

Suppose that $(M, g, J)$ is a Kähler 4-manifold. Then, as we have mentioned,

\[\mathcal{R}(s_1) = \mathcal{R}(s_2) = 0, \quad g(\mathcal{R}(s_3), s_3) = \frac{8}{3}s_3. \quad (31)\]

In particular the Kähler metric $g$ is anti-self-dual if and only if it is scalar flat.

To determine the possible Gray-Hervella classes of the twistor space $(Z, h_t, J^\pm)$ of an almost Hermitian manifold $(M, g, J)$ we shall need several technical lemmas. Next we shall always assume that $(M, g, J)$ is a Kähler 4-manifold with Kähler 2-vector $\omega$ and scalar curvature $s$ and that $\lambda \neq 0$ is an arbitrary complex number.

Lemma 11. $(Z, h_t, J^\pm) \in \mathcal{S}_K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ if and only if $s = 0$.

Proof. We know that $\mathcal{V}(f_\lambda^\pm)_*\omega(X^h_p) = 0$ for all $X \in T_{\pi(\lambda)}M$, hence, by Corollary 2, the fundamental 2-form of $(h_t, J^\pm)$ is co-closed if and only if

\[g(\mathcal{R}(U), f^\pm(\sigma)) = 0. \quad (32)\]

for every $\sigma \in Z$ and $U \in \mathcal{V}_\sigma$. Setting in this identity $\sigma = s_1(p)$, $U = s_3(p)$ for $p \in M$ and taking into account (31), we obtain $(a^2 + b^2 - 1)g(\mathcal{R}(s_3), s_3) = 0$. Hence $g(\mathcal{R}(s_3), s_3) = 0$ if $a^2 + b^2 \neq 1$. If $a^2 + b^2 = 1$ we set $\sigma = \frac{1}{\sqrt{2}}(s_1 + s_3)$ and $U = s_1 - s_3$.

Then $\sqrt{2}f(\sigma)$ is $s_1 + bs_2 \pm s_3$ and identity (32) gives again $g(\mathcal{R}(s_3), s_3) = 0$. It follows that $s = 0$.

Conversely, if $s = 0$, we have $g(\mathcal{R}(s_i), s_j) = 0$, $i, j = 1, 2, 3$, so $g(\mathcal{R}(\sigma), \tau) = 0$ for every $\sigma, \tau \in \Lambda^2_* TM$. In particular, identity (32) is fulfilled, hence $(h_t, J^\pm)$ is semi-Kähler.

Lemma 12. (i) $(Z, h_t, J^-) \in \mathcal{Q}_K = \mathcal{W}_1 \oplus \mathcal{W}_2$ if and only if $(M, g, J)$ is Ricci flat. 
(ii) $(Z, h_t, J^+) \text{ never belongs to the class } \mathcal{W}_1 \oplus \mathcal{W}_2$. 

Proof.
Proof. Suppose that \((Z, h, J^\pm) \in \mathcal{W}_1 \oplus \mathcal{W}_2\). By the defining condition for the class of quasi-Kähler manifolds
\[
(D_A \Omega)(B, C) + (D_B \Omega)(J^\pm \circ B, C) = 0, \quad A, B \in T \mathcal{Z}.
\] (33)
Hence, according to the second formula of Proposition 1,
\[
g(\mathcal{R}(U), X \wedge Y + K_{f^\pm \circ (\sigma)} X \wedge K_{f^\pm \circ (\sigma)} Y)
- g(\mathcal{R}(\sigma \times U), X \wedge K_{f^\pm \circ (\sigma)} Y - K_{f^\pm \circ (\sigma)} X \wedge Y) = 0.
\] (34)
for every \(\sigma \in Z\), \(X, Y \in T_{\pi(\sigma)} M\), \(U \in \mathcal{V}_\sigma\). Setting \(Y = K_{f^\pm \circ (\sigma)} X\) we get
\[
g(\mathcal{R}(U), X \wedge K_{f^\pm \circ (\sigma)} X) = 0,
\] (35)
or, equivalently,
\[
g(\mathcal{R}(U), X \wedge K_{f^\pm \circ (\sigma)} Y - K_{f^\pm \circ (\sigma)} X \wedge Y) = 0.
\] (36)
It is easy to check by means of (6) that for any \(\tau \in \Lambda^2 T_p M\) and \(X, Y \in T_p M\) with \(X \perp Y\), the 2-vector \(X \wedge K_\tau Y - K_\tau X \wedge Y\) is orthogonal to \(\Lambda^2 T_p M\), hence it lies in \(\Lambda^2 T_p M\). Moreover, for every \(\tau \in \Lambda^2 T_p M\), every vector of \(\Lambda^2 T_p M\) is a linear combination of vectors of the form \(X \wedge K_\tau Y - K_\tau X \wedge Y\) with \(X \perp Y\) and vectors of the form \(Z \wedge K_\tau Z\), \(X, Y, Z \in T_p M\). Indeed, if \(a_1, \ldots, a_4\) is an orthonormal basis of \(T_p M\) such that \(a_3 = K_\tau a_2\), \(a_4 = K_\tau a_1\), then it is positively oriented and \(a_1 \wedge a_2 - a_3 \wedge a_4 = -(a_1 \wedge K_\tau a_3 - K_\tau a_1 \wedge a_3), a_1 \wedge a_3 - a_4 \wedge a_2 = a_1 \wedge K_\tau a_2 - K_\tau a_1 \wedge a_2, a_1 \wedge a_4 - a_2 \wedge a_3 = a_1 \wedge K_\tau a_4 - a_2 \wedge K_\tau a_2\). Thus it follows from (35) and (36) that
\[
g(\mathcal{R}(U), s^-) = 0
\]
for every \(\sigma \in Z\), \(U \in \mathcal{V}_\sigma\), \(s^- \in \Lambda^2 T_{\pi(\sigma)} M\). In particular, \(g(\mathcal{R}(s_3), s^-) = 0\), hence in view of (31), \(\mathcal{R}(s_3) = \frac{s}{3} s_3\). Now, setting \(\sigma = s_4(p)\) and \(U = s_3(p), p \in M\), in (36), we obtain
\[
sg(s_3, X \wedge Y + K_{f^\pm \circ (s_1)} X \wedge K_{f^\pm \circ (s_1)} Y) = 0.
\] (37)
This identity for \((X, Y) = (A_1, A_2)\) and \((X, Y) = (A_1, A_3)\) gives
\[
sa(a^2 + b^2 - 1) = 0, \quad sb(a^2 + b^2 - 1) = 0.
\]
Hence \(s = 0\) if \(a^2 + b^2 \neq 1\). If \(a^2 + b^2 = 1\), we set \(\sigma = \frac{1}{\sqrt{2}}(s_1 + s_3)\) and \(U = s_1 - s_3\). We have \(\sqrt{2} f(\sigma) = as_1 + bs_2 \pm s_3\) and identity (37) with \((X, Y) = (A_1, A_2)\) and \((A_1, A_3)\) gives \(sa = 0\) and \(sb = 0\). Therefore \(s = 0\). It follows that \(\mathcal{R}(\tau) = 0\) for every \(\tau \in \Lambda^2 TM\). Since \((M, g, J)\) is a Kähler manifold this is equivalent to the metric \(g\) being Ricci flat. Moreover, in view of the third formula of Proposition 1, we get from (33) that
\[
g((f^\pm)_*(U), Y \wedge Z) + g((f^\pm)_*(J^\pm U), K_{f^\pm \circ (\sigma)} Y \wedge Z) = 0
\] (38)
for \(Y, Z \in T_{\pi(\sigma)} M\) and \(U \in \mathcal{V}_\sigma\). The restriction of \(f^\pm\) to any fibre of \(Z\) is holomorphic, hence
\[
(f^\pm)_*(J^\pm U) = J^\pm (f^\pm)_*(U) = f^\pm (\sigma) \times (f^\pm)_*(U).
\]
Then, by (5)
\[
g((f^\pm)_*(J^\pm U), K_{f^\pm \circ (\sigma)} Y \wedge Z) = g((f^\pm)_*(U), Y \wedge Z).
\]
This and (38) imply \((f_\lambda^\pm)_*(U) = 0\). Therefore the restriction of \(f_\lambda^\pm\) to every fibre of \(Z\) is a constant map which is a contradiction.

Now suppose that the Kähler manifold \(M\) is Ricci flat. In this case \(\mathcal{R}(\tau) = 0\) for every \(\tau \in \Lambda^2_+ TM\). Then, in view of Proposition 1, in order to prove that \((Z, h_t, \mathcal{J}_-) \in W_1 + W_2\) it is enough to show that, for every \(U \in \mathcal{V}_\sigma\) and \(Y, Z \in T_{\pi(\sigma)} M\), we have

\[
(D_U \Omega)(Y_h^\pm, Z_h^\pm) + (D_{\mathcal{J}_- U} \Omega)(K_{f_\lambda^-(\sigma)} Y_h^\pm, Z_h^\pm) = 0
\]

This is equivalent to identity (38) for \(f_\lambda^-\). The restriction of the map \(f_\lambda^-\) to any fibre of \(Z\) is anti-holomorphic, hence

\[
(f_\lambda^-)_*(\mathcal{J}_- U) = -\mathcal{J}_-(f_\lambda^-)_*(U) = -(f_\lambda^-)(\sigma) \times (f_\lambda^-)_*(U).
\]

This, in view of (5), implies that identity (38) for \(f_\lambda^-\) is fulfilled. Therefore \((Z, h_t, \mathcal{J}_-) \in W_1 + W_2\).

**Lemma 13.** (i) \((Z, h_t, \mathcal{J}_+) \in W_1 + W_2 + W_4\) if and only if \((M, g, J)\) is Ricci flat. (ii) \((Z, h_t, \mathcal{J}_+)\) never belongs to the class \(W_1 + W_2 + W_4\).

**Proof.** Suppose that \((Z, h_t, \mathcal{J}_+)\) is of class \(W_1 + W_2 + W_4\). Then

\[
(D_X \Omega)(X_h^\pm, U) + (D_{\mathcal{J}_+ X} \Omega)(X_h^\pm, U) = -\frac{1}{2} ||X||^2 \delta \Omega(U)
\]

for \(X \in T_{\pi(\sigma)} M, U \in \mathcal{V}_\sigma\). By Proposition 1 and Corollary 2, this is equivalent to

\[
2g(\mathcal{R}(U), X \wedge K_{f_\lambda^+(\sigma)} X) = ||X||^2 g(\mathcal{R}(U), f_\lambda^+(\sigma))
\]

(39)

or

\[
g(\mathcal{R}(U), X \wedge K_{f_\lambda^+(\sigma)} Y - K_{f_\lambda^+(\sigma)} X \wedge Y) = g(X, Y)g(\mathcal{R}(U), f_\lambda^+(\sigma)).
\]

(40)

for \(X, Y \in T_{\pi(\sigma)} M, U \in \mathcal{V}_\sigma\). Take an orthonormal basis \(E_1, ..., E_4\) of \(T_{\pi(\sigma)} M\) such \(E_3 = K_{f_\lambda^+(\sigma)} E_2, E_4 = K_{f_\lambda^+(\sigma)} E_1\). Then identity (39) for \(X = E_1\) and \(X = E_2\) gives

\[
2g(\mathcal{R}(U), E_1 \wedge E_4) = g(\mathcal{R}(U), f_\lambda^+(\sigma)), \quad 2g(\mathcal{R}(U), E_2 \wedge E_3) = g(\mathcal{R}(U), f_\lambda^+(\sigma)).
\]

These identities imply \(g(\mathcal{R}(U), E_1 \wedge E_4) = g(\mathcal{R}(U), f_\lambda^+(\sigma))\). Moreover, setting in (40) \((X, Y) = (E_1, E_3)\) and \((X, Y) = (E_1, E_2)\) we get

\[
g(\mathcal{R}(U), E_1 \wedge E_2 - E_3 \wedge E_4) = 0, \quad g(\mathcal{R}(U), E_1 \wedge E_3 - E_4 \wedge E_2) = 0.
\]

It follows that \(g(\mathcal{R}(U), s^-) = 0\) for every \(s^- \in \Lambda^2_+ T_{\pi(\sigma)} M\), hence \(g(\mathcal{R}(\sigma), s^-) = 0\) for \(\sigma \in Z\) and \(s^- \in \Lambda^2_+ T_{\pi(\sigma)} M\). This and (31) imply

\[
\mathcal{R}(\sigma) = \frac{s}{3} g(\sigma, \omega), \quad \sigma \in \Lambda^2_+ TM.
\]

Then, by Corollary 2,

\[
\delta \Omega(s_3) = -tg(\mathcal{R}(s_1 \times s_3), f_\lambda^+(s_1)) = t \frac{s}{3} g(s_2, s_3) g(s_3, f_\lambda^+(s_1)) = 0.
\]

Moreover, \(\delta \Omega(s_1) = tg(\mathcal{R}(s_2), f_\lambda^+(s_3)) = 0\) and \(\delta \Omega(s_2) = -tg(\mathcal{R}(s_1), f_\lambda^+(s_3)) = 0\). It follows that \(\delta \Omega = 0\), hence \(s = 0\) by Lemma 11. Finally note that an almost Hermitian manifold with \(\delta \Omega = 0\) belongs to the class \(W_1 + W_2 + W_4\) if and only if it belongs to the class \(W_1 + W_2\). Hence the lemma follows from Lemmas 12 and 11.
Lemma 14. ([6]) \((Z, h_t, J^\perp_\lambda) \in \mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4\) if and only if \((M, g, J)\) is scalar flat.

**Proof.** By Corollaries 4 and 3, the almost complex structure \(J^\perp_\lambda\) is integrable if and only if

\[
\begin{align*}
g(\mathcal{R}(X \wedge K_{f^\perp_\lambda} Y + K_{f^\perp_\lambda} X \wedge Y), U) \\
+ g(\mathcal{R}(X \wedge Y - K_{f^\perp_\lambda} X \wedge K_{f^\perp_\lambda} Y), \sigma \times U) = 0
\end{align*}
\]

(41)

It is easy to check that for every \(\tau \in \Lambda^2_T M\) and \(X, Y \in T_p M\), the 2-vector \(X \wedge K_{\tau} Y + K_{\tau} X \wedge Y \in \Lambda^2_T M\) (and is orthogonal to \(\tau\)). Therefore, in view of (31), \(J^\perp_\lambda\) is integrable if and only if

\[
\begin{align*}
g(X \wedge K_{f^\perp_\lambda} Y + K_{f^\perp_\lambda} X \wedge Y, s_3)g(\mathcal{R}(s_3), U) \\
+ g(X \wedge Y - K_{f^\perp_\lambda} X \wedge K_{f^\perp_\lambda} Y, s_3)g(\mathcal{R}(s_3), \sigma \times U) = 0
\end{align*}
\]

(42)

for \(X, Y \in T_{\pi(\sigma)} M\) and \(U \in \mathcal{V}_\sigma\). Set \(\sigma = s_1\) and \(U = s_3\). Then, since \(\mathcal{R}(s_2) = 0\), identity (42) becomes

\[
g(X \wedge K_{f^\perp_\lambda} Y + K_{f^\perp_\lambda} X \wedge Y, s_3)g(\mathcal{R}(s_3), s_3).
\]

(43)

For \((X, Y) = (A_1, A_2)\) and \((X, Y) = (A_1, A_3)\), the vector \(X \wedge K_{f^\perp_\lambda} Y + K_{f^\perp_\lambda} X \wedge Y\) is collinear to \(-2\sigma s_3 + (a^2 + b^2 - 1)s_2\) and \(2\sigma s_3 - (a^2 + b^2 - 1)s_1\), respectively. Then identity (43) gives

\[
bg(\mathcal{R}(s_3), s_3) = 0, \quad ag(\mathcal{R}(s_3), s_3) = 0.
\]

Therefore \(g(\mathcal{R}(s_3), s_3) = 0\), thus \(s = 0\). This shows that if \(J^\perp_\lambda\) is integrable, then \((M, g, J)\) is scalar flat.

Conversely, suppose that \((M, g, J)\) is Kähler and scalar flat. Then \(\mathcal{V}(f^\perp_\lambda)_* (X^h_{\sigma}) = 0\) for every \(\sigma \in \mathcal{Z}\) and \(X \in T_{\pi(\sigma)} M\). Hence, by Corollary 3, \(\mathcal{H}(X^h, Y^h) = 0\) for every \(X, Y\). We also have \(g(\mathcal{R}(s_i), s_j) = 0\), \(i, j = 1, 2, 3\) since \(s = 0\). Thus \(g(\mathcal{R}(\sigma), \tau) = 0\) for every \(\sigma, \tau \in \Lambda^2_T M\). Recall that for every \(\tau \in \Lambda^2_T M\) and \(X, Y \in T_{\pi(\sigma)} M\), the 2-vector \(X \wedge K_{\tau} Y + K_{\tau} X \wedge Y\) lies in \(\Lambda^2_T M\). Then by Corollary 3 we get \(\mathcal{V}(N(X^h, Y^h)) = 0\). Finally, the map \(f^\perp_\lambda\) is holomorphic, hence, by Corollary 4 we have \(\mathcal{H}(N(X^h_{\sigma}, U)) = 0\) for every \(U \in \mathcal{V}_\sigma\). Now Corollary 3 implies that \(N = 0\).

Lemma 15. (i) \((Z, h_t, J^\perp_\lambda)\) never belongs to the class \(\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4\).

(ii) \((Z, h_t, J^\perp_\lambda) \in \mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4\) if and only if \((M, g, J)\) is scalar flat.

**Proof.** By the definition of the class \(\mathcal{G}_1\) ([9]), \((Z, h_t, J^\perp_\lambda) \in \mathcal{G}_1\) if and only if

\[
h_t(N(A, B), C) + h_t(N(C, B), A) = 0
\]

for all \(A, B, C \in T^\perp Z\). By Corollary 3 this is equivalent to the identity

\[
\begin{align*}
tg(\mathcal{R}(U), X \wedge K_{f^\perp_\lambda} Y + K_{f^\perp_\lambda} X \wedge Y) \\
+ tg(\mathcal{R}(\sigma \times U), X \wedge Y - K_{f^\perp_\lambda} X \wedge K_{f^\perp_\lambda} Y) \\
+ 2g((f^\perp_\lambda)_* (\sigma \times U) - f^\perp_\lambda (\sigma) \times (f^\perp_\lambda)_* (U), X \wedge Y) = 0
\end{align*}
\]

(44)

To prove (i) note that the restriction of \(f^\perp_\lambda\) on the fibre is anti-holomorphic, thus \(f^\perp_\lambda (\sigma) \times (f^\perp_\lambda)_* (U) = -(f^\perp_\lambda)_* (\sigma \times U)\). Hence, if \((Z, h_t, J^\perp_\lambda) \in \mathcal{G}_1\), then, setting \(\sigma = s_3(p), U = s_1(p)\), and taking into account that \(\mathcal{R}(s_1) = \mathcal{R}(s_2) = 0\), we
obtain from (44) $(f^-_\lambda)_{s, s_1}(s_1) = 0$. But a straightforward computation shows that $(f^-_\lambda)_{s, s_1}(s_1) \neq 0$, a contradiction.

To prove (ii) notice that the restriction of $f^+_\lambda$ on the fibre is holomorphic and identity (44) takes the form (41). Hence $(Z, h_1, J^+\sigma)$ is of class $G_1$ if and only if its of class $H$, the later condition being equivalent to $s = 0$ by Lemma 14.

**Lemma 16.** (i) $(Z, h_1, J^-\lambda)$ never belongs to the class $G_2 = W_2 \oplus W_3 \oplus W_4$.

(ii) $(Z, h_1, J^+\lambda) \in G_2 = W_2 \oplus W_3 \oplus W_4$ if an only if $(M, g, J)$ is scalar flat.

**Proof.** The structure $(h_1, J^\pm\lambda)$ is of class $G_2$ if and only if $[9]$

$$\otimes_{A, B, C} h_t(N(A, B), J_{A, C}) = 0, \quad A, B, C \in TZ.$$

For $A = X_h^h, B = U \in \mathcal{V}_\sigma, C = Y_h^h$ this identity and Corollary 3 imply

$$2g(J^-\lambda(f^-_\lambda)_{s, U}(U) - (f^+_\lambda)_{s, U}(J^+\lambda U), X \wedge K_{f^+_\lambda}(\sigma) Y + K_{f^+_\lambda}(\sigma) X \wedge Y)$$

$$-tg(R(X \wedge K_{f^+_\lambda}(\sigma) Y + K_{f^+_\lambda}(\sigma) X \wedge Y), \sigma \times U)$$

$$+tg(R(X \wedge Y - K_{f^+_\lambda}(\sigma) X \wedge K_{f^+_\lambda}(\sigma) Y, U) = 0. \quad (45)$$

Set $\sigma = \omega(p), p \in M$. We have $f^+_\lambda(\omega) = \pm \omega$, so $K_{f^+_\lambda}(\omega) = \pm J$. Then, setting $(X, Y) = (A_1, A_3), (X, Y) = (A_1, A_2)$ and taking into account (31), we obtain form (45) that

$$g(J^-\lambda(f^-_\lambda)_{s, U}(U) - (f^+_\lambda)_{s, U}(J^+\lambda U), s_1) = 0, \quad i = 1, 2.$$

The map $f^-_\lambda$ is anti-holomorphic on the fibres of $Z$, so the latter identity gives

$$g((f^-_\lambda)_{s, s_1}(J^-\lambda U), s_1) = 0, \quad i = 1, 2, \quad U \in \mathcal{V}_{s_2}.$$

We set $U = s_2(p)$ and $U = s_1(p)$ and compute

$$f^-_{s, s_1}(s_1) = \frac{2as_1 + 2bs_2 - (a^2 + b^2 - 1)s_3}{2(a^2 + b^2)}, f^-_{s, s_3}(s_2) = \frac{2as_2 - 2bs_1 - (a^2 + b^2 - 1)s_3}{2(a^2 + b^2)}.$$

It follows that $a = 0, b = 0$, which contradicts to the assumption $\lambda \neq 0$. This proves statement (i).

Now suppose that $(Z, h_1, J^+\lambda)$ is of class $G_2$. Then identity (45) becomes

$$g(R(X \wedge K_{f^+_\lambda}(\sigma) Y + K_{f^+_\lambda}(\sigma) X \wedge Y), \sigma \times U)$$

$$-g(R(X \wedge Y - K_{f^+_\lambda}(\sigma) X \wedge K_{f^+_\lambda}(\sigma) Y, U) = 0. \quad (46)$$

We have $f^+_\lambda(s_1) = (a^2 + b^2 + 1)^{-1}(2as_1 + 2bs_2 + cs_3)$ where $c = a^2 + b^2 - 1$. Then

$$A_1 \wedge K_{f^+_\lambda(s_1)}A_2 + K_{f^+_\lambda(s_1)}A_1 \wedge A_2 = (a^2 + b^2 + 1)^{-1}(-2bs_1 + cs_2).$$

Thus, setting $\sigma = s_1, (X, Y) = (A_1, A_2), U = s_2$ in (46) and taking into account that $R(s_1) = R(s_3) = 0$, we obtain $bq(R(s_3), s_3) = 0$. Similarly, since $f^+_\lambda(s_2) = (a^2 + b^2 + 1)^{-1}(-2bs_1 + 2as_2 + cs_3)$, setting $\sigma = s_2, (X, Y) = (A_1, A_2), U = s_1$ we get $ag(R(s_3), s_3) = 0$. It follows $g(R(s_3), s_3) = 0$, hence $s = 0$. By Lemma 14 $s = 0$ if and only if the almost complex structure $J^\lambda$ is integrable. In particular, if $s = 0, (Z, h_1, J^+\lambda)$ is of class $G_2$ and (ii) is proved.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.**

(i). It follows from statements (i) of Lemmas 15 and 16, and [9, Table I] that the possible nontrivial Gray-Hervella classes of $(Z, h_1, J^-\lambda)$ are subclasses of $W_1 \oplus$
\( \mathcal{W}_2 \oplus \mathcal{W}_3 \) or \( \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4 \). Moreover statements (i) of Lemmas 12 and 13 imply that
\[
\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3.
\]

Hence the first part of the theorem follows from statements (i) of Lemmas 11, 15 and 16.

(ii). Using statements (ii) of Lemmas 11-16 we prove the second part of the theorem in a similar way.

Now we shall discuss the case when \( |\lambda| = 1 \). In this case we have the simple formula (30) for \( V(f^\lambda_n)_\sigma(X^{\lambda}_h), \sigma \in \mathcal{Z}, X \in T_{\pi(\sigma)}M \). This simplifies the computations that should be done in order to determine the possible Gray-Hervella types of the almost Hermitian manifold \((h_t, \mathcal{J}^\lambda_t)\). Here we shall address only a few of the basic classes.

**Proposition 2.** Suppose that \( |\lambda| = 1 \) and \( \text{Re}(\lambda) \neq 0, \pm 1 \) for \( \mathcal{J}^\pm_h \). Then:

(i) The almost Hermitian structure \((h_t, \mathcal{J}^\lambda_t)\) on the twistor space \( \mathcal{Z} \) is (non-integrable) quasi-Kähler if and only if \( M \) is Ricci flat.

(ii) The structure \((h_t, \mathcal{J}^\pm_h)\) is never quasi Kähler.

(iii) The structures \((h_t, \mathcal{J}^\pm_h)\) are not nearly Kähler or almost Kähler.

**Proof.** It is convenient to prove first the following.

**Lemma 17.** If \((h_t, \mathcal{J}^\pm_h)\) is quasi Kähler, then \((M, g, J)\) is Kähler and Ricci flat.

**Proof of the lemma.** Let \( p \in M, X, Y, Z \in T_pM \). We have \( f^\pm_h(\omega(p)) = \pm(\omega(p)) \), hence by Proposition 1 and (30)
\[
0 = \frac{1}{2}[(D_{X^h_{\omega(p)}}^h \Omega)(Y^h_{\omega(p)}, Z^h_{\omega(p)}) + (D_{(JX)^h_{\omega(p)}}^h \Omega)(JY^h_{\omega(p)}, Z^h_{\omega(p)})] = \\
- bg(\omega \times \nabla_X \omega, Y \wedge Z) + (\pm 1 - a)g(\nabla_X \omega, Y \wedge Z)
\]
This and identity (6) give
\[
- bg(\nabla_X J(JY), Z) + (\pm 1 - a)g(\nabla_X J)(Y), Z)
+ bg(\nabla_{JX J}(Y), Z) + (\pm 1 - a)g(\nabla_{JX J})(JY), Z).
\]

Thus
\[
bJ[\nabla_X J](Y) - J(\nabla_{JX J}(Y)) + (\pm 1 - a)[\nabla_X J](Y) - J(\nabla_{JX J})(Y) = 0.
\]
By assumption \( a \neq 1 \) when considering \( J_+ \) and \( a \neq -1 \) for \( J_- \). Thus \( b^2 + (\pm 1 - a)^2 \neq 0 \) and the latter equation implies
\[
(\nabla_X J)(Y) - J(\nabla_{JX J})(Y) = 0.
\]
This means that the almost Hermitian structure \((g, J)\) on \( M \) is quasi Kähler. It follows that it is Kähler since \( \dim M = 4 \). The assumption that \( \mathcal{J}^\pm \) is quasi Kähler implies identity (34) and, as in the proof of Lemma 12, we see that \( s = 0 \). Thus \( M \) is Kähler and Ricci flat.

Now we are ready to prove Proposition 2.

(i). If \((h_t, \mathcal{J}^\lambda_t)\) is quasi Kähler, \( M \) is Kähler and Ricci flat by the lemma. Conversely, if \( M \) is such a manifold, \((h_t, \mathcal{J}^\lambda_t)\) is quasi Kähler by Lemma 12.

(ii). This statement follows from the lemma and Lemma 12.
(iii). If \((h_t, \mathcal{J}_\kappa^+)\) is nearly Kähler or almost Kähler, it is quasi Kähler, hence \(M\) is Kähler by the lemma. Then, according to Lemma 13 (ii), \((h_t, \mathcal{J}_\kappa^+)\) does not belong to the class \(\mathcal{N} \mathcal{K} = \mathcal{W}_1\) or to the class \(\mathcal{A} \mathcal{K} = \mathcal{W}_2\). Also \((h_t, \mathcal{J}_\kappa^+)\) is not of class \(\mathcal{N} \mathcal{K}\) or \(\mathcal{A} \mathcal{K}\) by Lemmas 15 (i) and 16 (i).

**Proposition 3.** Let \(|\lambda| = 1\) and \(\text{Re}(\lambda) \neq 0, 1\). Then the almost complex structure \(\mathcal{J}_\kappa^+\) is integrable if and only if \((M, g, J)\) is Kähler and scalar flat. In this case \((h_t, \mathcal{J}_\kappa^+) \in \mathcal{W}_3 = \mathcal{SK} \cap \mathbf{H}^2\).

**Proof.** Suppose that the almost complex structure \(\mathcal{J}_\kappa^+\) is integrable. Let \(p \in M\). For \(\sigma = s_1(p)\) and \(X \in T_p M\), we have
\[
\sigma \times \nabla_X \omega = g(s_2, s_1) \nabla_X s_3) + g(s_3, s_1) \nabla_X s_3) = -g(s_3, \nabla_X s_3) = g(s_2, \nabla_X s_3) = g(s_2, \nabla_X s_3).
\]
Thus by (30)
\[
\mathcal{V}(f_s)^+ = \left[ -bg(\nabla_X s_3, s_2) + (1 - a)g(\nabla_X s_3, s_1) \right] s_3.
\]
It is convenient to set
\[
\phi_1 = -bg(\nabla_A, s_3, s_2) + (1 - a)g(\nabla_A, s_3, s_1),
\]
We have \(K_{(s_1)} = aK_{s_1} + bK_{s_2}\), if, using Corollary 3, it is easy to see that
\[
h_t(N(A^h_1, A^h_2), A^h_3)_{s_1(p)} = 2b\phi_1, \quad h_t(N(A^h_1, A^h_2), A^h_3)_{s_2(p)} = -2a\phi_1.
\]
It follows that \(\phi_1 = 0\) since \(N = 0\) and \(a^2 + b^2 \neq 0\). We also have
\[
h_t(N(A^h_1, A^h_2), A^h_3)_{s_1(p)} = 2b\phi_2, \quad h_t(N(A^h_1, A^h_2), A^h_3)_{s_2(p)} = 2a\phi_2,
\]
\[
h_t(N(A^h_1, A^h_2), E_{h}s_1(p) = 2a\phi_3, \quad h_t(N(A^h_1, A^h_2), A^h_3)_{s_2(p)} = 2b\phi_3.
\]
It follows that \(\phi_2 = \phi_3 = \phi_4 = 0\). Thus
\[
- bg(\nabla_A, s_3, s_2) + (1 - a)g(\nabla_A, s_3, s_1) = 0, \quad i = 1, ..., 4.
\]
Now set \(\sigma = s_2(p)\). We have
\[
\mathcal{V}(f_s)^+ = bg(\nabla_X s_3, s_1) + (1 - a)g(\nabla_X s_3, s_2)
\]
and \(K_{(s_2)} = -bK_{s_1} + aK_{s_2}\). Then a similar computation as above gives
\[
bg(\nabla_A, s_3, s_1) + (1 - a)g(\nabla_A, s_3, s_2) = 0, \quad i = 1, ..., 4.
\]
It follows from (47) and (48) that
\[
bg(\nabla_A, s_3, s_1) = 0, \quad i = 1, ..., 4.
\]
Also we have \(g(\nabla_A, s_3, s_3) = 0\) since \(s_3\) is of constant length, hence, by (49), the almost complex structure \(J\) is Kählerian. By Corollary 3, the identity \(\mathcal{V}N(X^h_p, Y^h_p) = 0\) is equivalent to
\[
g(\mathcal{R}(X \wedge K_{(s_1)} Y + K_{(s_1)} X \wedge Y, U)
+ g(\mathcal{R}(X \wedge Y - K_{(s_1)} X \wedge K_{(s_1)} Y), \sigma \times U) = 0
\]
for \(X, Y \in T_{\sigma} M\) and \(U \in \mathcal{V}_{\sigma}\). Setting in the latter identity \(\sigma = s_1(p)\), \(U = s_3(p)\), \((X, Y) = (A_1, A_2)\) and \((X, Y) = (A_1, A_3)\), and taking into account (31), we get
\[
bg(\mathcal{R}(s_3), s_3) = 0, \quad ag(\mathcal{R}(s_3), s_3) = 0.
\]
Therefore $g(R(s_3), s_3) = 0$, thus $s = 0$. This proves that if $\mathcal{J}^+_{\lambda}$ is integrable, then $(M, g, J)$ is Kähler and scalar flat. The converse follows from Theorem 2 (ii).

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