THE MAXIMUM-LIKELIHOOD DECODING
THRESHOLD FOR GRAPHIC CODES

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Abstract. For a class \( \mathcal{C} \) of binary linear codes, we write \( \theta_{\mathcal{C}} : (0, 1) \rightarrow [0, \frac{1}{2}] \) for the maximum-likelihood decoding threshold function of \( \mathcal{C} \), the function whose value at \( R \in (0, 1) \) is the largest bit-error rate \( p \) that codes in \( \mathcal{C} \) can tolerate with a negligible probability of maximum-likelihood decoding error across a binary symmetric channel. We show that, if \( \mathcal{C} \) is the class of cycle codes of graphs, then \( \theta_{\mathcal{C}}(R) \leq \frac{(1-\sqrt{R})^2}{2(1+R)} \) for each \( R \), and show that equality holds only when \( R \) is asymptotically achieved by the cycle codes of regular graphs.

1. Introduction

For a class \( \mathcal{C} \) of binary linear codes and for some rate \( R \in (0, 1) \), we consider the maximum-likelihood decoding threshold \( \theta_{\mathcal{C}}(R) \) for \( \mathcal{C} \) at \( R \). This is the unique \( \theta \in [0, \frac{1}{2}] \) such that

- for each \( p \in [0, \theta) \) and all \( \epsilon > 0 \), given a binary symmetric channel of bit-error rate \( p \), there exists a code \( C \in \mathcal{C} \) of rate at least \( R \) such that the probability of an error in maximum-likelihood decoding on \( C \) is less then \( \epsilon \), and
- for each \( p \in (\theta, \frac{1}{2}] \) there exists \( \epsilon > 0 \) such that, given a binary symmetric channel of bit-error rate \( p \), for each code \( C \in \mathcal{C} \) of rate at least \( R \) the probability of an error in maximum-likelihood decoding on \( C \) is at least \( \epsilon \).

The function \( \theta_{\mathcal{C}}(R) \) is the threshold function for \( \mathcal{C} \); it essentially measures the maximum bit-error rate that can be ‘tolerated’ by rate-\( R \) codes in \( \mathcal{C} \) with vanishing probability of a decoding error. Our main result proves an upper bound on this function for the class of graphic codes (that is, cycle codes of graphs):

**Theorem 1.1.** If \( \mathcal{G} \) is the class of graphic codes and \( R \in (0, 1) \), then \( \theta_{\mathcal{G}}(R) \leq \frac{(1-\sqrt{R})^2}{2(1+R)} \). If equality holds, then \( R = 1 - \frac{3}{2d} \) for some \( d \in \mathbb{Z} \).

This research was partially supported by a grant from the Office of Naval Research [N00014-12-1-0031] and by a grant from the National Science Foundation Division of Mathematical Sciences [1501985].
This generalises a result of Decreusefond and Zémor [4], who proved the same upper bound for the class of cycle codes of regular graphs. Our proof follows theirs conceptually, although our exposition and notation are somewhat different. The proof in [4] implicitly involves a problem of enumerating ‘non-backtracking’ walks that is trivial for regular graphs but not in general; much of the original material in our proof is related to this difficulty.

When \( R = 1 − \frac{2}{d} \) for some \( d \in \mathbb{Z} \) (that is, when the cycle codes of large \( d \)-regular graphs have rate close to \( R \)) our theorem does not improve the bound \( \theta_G(R) \leq \frac{(1−\sqrt{R})^2}{2(1+R)} \), but constructions due to Alon and Bachmat [1] show that for all \( d \geq 3 \) there exist large \( d \)-regular graphs whose cycle codes attain this threshold (ie. can tolerate a bit-error rate of \( p \) for any \( p < \frac{(1−\sqrt{R})^2}{2(1+R)} \)). Combining this with Theorem 1.1, we have the following:

**Theorem 1.2.** If \( G \) is the class of graphic codes, and \( R = 1 − \frac{2}{d} \) for some integer \( d \geq 3 \), then \( \theta_G(R) = \frac{(1−\sqrt{R})^2}{2(1+R)} \).

Theorem 1.1 implies that this equality holds for no other \( R \in (0, 1) \); this can be interpreted as a statement that the cycle codes of regular graphs are ‘best’ among all graphic codes.

**Minor-Closed Classes.** The main result of [12] shows that the failure of the graphic codes to be ‘asymptotically good’ extends to every proper minor-closed subclass of binary codes; that is, every proper subclass that is closed under puncturing and shortening. The proof uses a deep result in matroid structure theory due to Geelen, Gerards and Whittle [6] that states, roughly, that the ‘highly connected’ members of any such class of codes are close to being either graphic or cographic (that is, the dual of a cycle code of a graph).

We believe that this paradigm that the members of any minor-closed subclass of binary codes are ‘nearly’ graphic or cographic will also apply to the threshold function. We predict that the threshold function \( \theta_G(R) \) for any minor-closed class agrees with that of either the class \( G \) of graphic codes or the class \( G^* \) of cographic codes. It is easily shown (see [6]) that \( \theta_{G^*}(R) = 0 \) for all \( R \in (0, 1) \). Geelen, Gerards and Whittle [6] made the following striking conjecture:

**Conjecture 1.3.** Let \( C \) be a proper subclass of the binary linear codes that is closed under puncturing and shortening. Either

- \( G \subseteq C \) and \( \theta_C = \theta_G \), or
- \( \theta_C = 0 \).
In other words, the presence or absence of the class of graphic codes should be all that determines the threshold function for any minor-closed class. Proving this conjecture would likely require a combination of the matroidal techniques in [12] and the algebraic and probabilistic ideas in this paper.

2. Preliminaries

We give some basic definitions in coding theory that, together with the definition of threshold function in the introduction, are all that are required for this paper; a more comprehensive reference is found in [11]. We also use some standard graph theory terminology from [5] and [7].

For integers \( n \geq k \geq 0 \), a binary linear \([n,k]\)-code is a \( k \)-dimensional subspace \( C \) of some \( n \)-dimensional vector space \( V \) over GF(2). We call the elements of \( C \) codewords. The rate of \( C \) is the ratio \( R = \frac{k}{n} \).

2.1. Graphic Codes. This paper is concerned solely with the cycle codes of graphs. For a finite simple graph \( G = (V,E) \), the cycle code of \( G \) is the subspace of \( GF(2)^E \) whose elements are exactly the characteristic vectors of cycles of \( G \) (that is, edge-disjoint unions of circuits of \( G \), or equivalently edge-sets of even subgraphs of \( G \)). We call such a code graphic, writing \( \mathcal{G} \) for the class of all graphic codes; it is well-known that every graphic code is the cycle code of a connected graph.

If \( G \) is connected, then its cycle code \( C \) is a binary linear \([n,k]\)-code, where \( n = |E| \) and \( k = |E| - |V| + 1 \), giving \( R = 1 - \frac{|V|}{|E|} + \frac{1}{|E|} \).

The ratio \( \frac{|V|}{|E|} \) is exactly \( 2\mu(G) \), where \( \mu(G) \) denotes the average degree of \( G \); we adopt this notation \( \mu(G) \) throughout the paper. The above formula implies that a large connected graph \( G \) has a cycle code of rate \( R \approx 1 - \frac{2}{\mu(G)} \). A simple ‘error-tolerance’ parameter of \( C \) is the minimum Hamming distance \( d \) between two codewords of \( C \); this is equal to the girth of \( G \) (the length of a shortest circuit of \( G \)) – we will write \( d(G) \) for the girth of a graph \( G \).

2.2. Maximum-likelihood decoding. Suppose that some codeword \( c \) of a linear \([n,k]\)-code \( C \subseteq V \) is transmitted across a binary symmetric channel with bit-error rate \( 0 < p < \frac{1}{2} \), giving some \( x \in V \) obtained by switching the value of each entry of \( c \) independently with probability \( p \). Maximum-likelihood decoding (abbreviated ML-decoding) is the process where, given \( x \), we attempt to recover \( c \) by choosing the codeword \( c' \in C \) with the highest probability to have been sent, given that \( x \) has been received. If this choice is ambiguous (that is, if this maximum is not unique) or gives an incorrect answer (that is, if \( c' \neq c \)), then we say a decoding error has been made; this occurs with some probability
depending on \( p \) and \( C \) but, by linearity, not on the particular codeword \( c \). In this particular setting of a constant bit-error probability \( p < \frac{1}{2} \) that behaves independently on each bit, ML-decoding is equivalent to nearest-neighbour decoding, where \( c' \) is simply chosen to be the closest codeword to \( x \) in Hamming distance.

ML-decoding is hard for general binary codes \([3]\), but an attractive property of graphic codes (and an important motivating factor for this paper) is that ML-decoding can be implemented efficiently for graphic codes using standard techniques in combinatorial optimization \([9]\). This is the case because the probability of a decoding error can be understood purely graphically: if \( C \) is the cycle code of a graph \( G = (V,E) \) and codewords of \( C \) are transmitted across a channel of bit-error rate \( p \in (0,\frac{1}{2}) \), then the probability of an ML-decoding error is exactly the probability, given a set \( X \subseteq E \) formed by choosing each edge uniformly at random with probability \( p \), that \( X \) contains at least half of the edges of some circuit of \( G \). Thus, to prove our main theorem, we study random subsets of edges of a graph. From this point on, given a set \( E \) and some \( p \in [0,1] \), we refer to a random set \( X \subseteq E \) formed by including each element of \( E \) independently at random with probability \( p \) as a \( p \)-random subset of \( E \).

### 3. Non-backtracking walks

A non-backtracking walk of length \( \ell \) in a graph \( G \) is a walk \((v_0,v_1,\ldots,v_\ell)\) of \( G \) so that \( v_{i+1} \neq v_{i-1} \) for all \( i \in \{1,\ldots,\ell-1\} \). In all nontrivial cases, the number of such walks grows roughly exponentially in \( \ell \); in this section we estimate the base of this exponent, mostly following \([2]\), Theorem 1).

Let \( G = (V,E) \) be a simple connected graph of minimum degree at least 2. Let \( \bar{E} = \{ (u,v) \in V^2 : u \sim_G v \} \) be the \( 2|E| \)-element set of arcs of \( G \). Let \( B = B(G) \in \{0,1\}^{E \times \bar{E}} \) be the matrix so that \( B_{(u,v),(u',v')} = 1 \) if and only if \( u' = v \) and \( u \neq v' \). It is easy to see that

1. \( B \) is the adjacency matrix of a strongly connected digraph (essentially the ‘line digraph’ of \( G \)), and
2. For each integer \( \ell \geq 1 \), the entry \( (B^\ell)_{e,f} \) is the number of non-backtracking walks of length \( \ell + 1 \) in \( G \) with first arc \((v_0,v_1) = e\) and last arc \((v_\ell,v_{\ell+1}) = f\).

By (1) and the Perron-Frobenius theorem (see \([7]\), section 8.8), there is a positive real eigenvalue \( \lambda_* \) of \( B \) and an associated positive real eigenvector \( w_* \), so that \( |\lambda_*| \geq |\lambda| \) for every eigenvalue \( \lambda \) of \( B \). Furthermore, by Gelfand’s formula \([8]\) we have \( \lambda_* = \lim_{n \to \infty} \|B^n\|^{1/n} \), where \( \|B^n\| \) denotes the sum of the absolute values of the entries of \( B^n \). By
(2), the parameter \( \lambda_* = \lambda_*(B(G)) \) thus governs the growth of non-backtracking walks in \( G \).

Note that \( B^\ell \) has only nonnegative entries, so \( \|B^\ell\| = \bar{1}^T B^\ell \bar{1} \). Let \( \mu = \mu(G) = \frac{1}{n} \mid E \mid \) denote the average degree of \( G \). The proof of Theorem 1 of [2] contains the following:

**Lemma 3.1.** Let \( G \) be a connected graph of minimum degree at least 2 and let \( B = B(G) \). Then \( \bar{1}^T B^\ell \bar{1} \geq (n\mu) \Lambda^\ell \), where

\[
\Lambda = \Lambda(G) = \prod_{v \in V} (d_G(v) - 1)^{d_G(v)/(n\mu)}.
\]

It follows in turn from this lemma that \( \|B^\ell\|^{1/\ell} \geq \Lambda(G) \), so \( \lambda_*(B(G)) \geq \Lambda(G) \). As observed in [2], the log-convexity of the function \((x - 1)^x \) (for \( x > 1 \)) implies that \( \Lambda(G) \geq \mu(G) - 1 \). For each real number \( x \), let \( \eta(x) = \min(x - \lfloor x \rfloor, \lfloor x \rfloor - x) \) denote the distance from \( x \) to the nearest integer. The proof of the following lemma, which slightly improves the bound \( \Lambda(G) \geq \mu(G) - 1 \) when \( \mu(G) \) is not an integer, is an unilluminating exercise in calculus.

**Lemma 3.2.** Let \( \mu_0 \in \mathbb{R} \) satisfy \( \mu_0 \geq 2 \) and let \( G \) be a connected graph with minimum degree at least 2 and average degree at least \( \mu_0 \). Then \( \lambda_*(B(G)) \geq \mu_0 - 1 + \frac{\eta(\mu_0)^3}{8\mu_0^3} \).

**Proof.** Let \( n = |V(G)| \), let \( d_1, \ldots, d_n \) be the degrees of the vertices of \( G \), and let \( \mu = \frac{1}{n} \sum_{i=1}^n d_i \geq \mu_0 \) be the average degree of \( G \). Let \( \eta = \eta(\mu) \); note that \( \mu \geq 2 + \eta \). Define \( g : (1, \infty) \rightarrow \mathbb{R} \) by \( g(x) = x \log(x - 1) \); observe that \( g'(x) = \frac{x}{x - 1} + \log(x - 1) \) and \( g''(x) = \frac{x - 2}{(x - 1)^2} \). We have \( \log(\Lambda(G)) = \frac{1}{n\mu} \sum_{i=1}^n g(d_i) \); for each \( i \), Taylor’s theorem gives

\[
g(d_i) = g(\mu) + g'(\mu)(d_i - \mu) + \frac{1}{2} g''(\xi_i)(d_i - \mu)^2
\]

for some \( \xi_i \) between \( d_i \) and \( \mu_0 \). We now estimate the ‘error’ terms.

**Claim 3.2.1.** \( \frac{1}{2} g''(\xi_i)(d_i - \mu)^2 \geq \frac{\eta^3}{8\mu^3} \) for each \( i \).

**Proof of claim:** First suppose that \( d_i = 2 \). Then \( g(d_i) = 0 \), so

\[
\frac{1}{2} g''(\xi_i)(2 - \mu)^2 = -g(\mu) - g'(\mu)(2 - \mu)
   = (\mu - 2) \left( \frac{\mu}{\mu - 1} + \log(\mu - 1) \right) - \mu \log(\mu - 1)
   = \frac{\mu(\mu - 2)}{\mu - 1} - 2 \log(\mu - 1).
\]

Note that the above expression is equal to \( 1.174 \ldots > 1 \) for \( \mu = \frac{7}{3} \), and is increasing in \( \mu \) for \( \mu \in (2, \infty) \). If \( \mu \geq \frac{7}{3} \) then we therefore have
\(\frac{1}{2}g(\xi_i)(2 - \mu)^2 > 1.\) If \(\mu < \frac{7}{3}\) then \(\mu = 2 + \eta\) and \(\eta < \frac{1}{3}\), so

\[
\frac{\mu(\mu - 2)}{\mu - 1} - 2 \log(\mu - 1) = \frac{\eta(2 + \eta)}{1 + \eta} - 2 \log(1 + \eta)
\geq \frac{\eta(2 + \eta)}{1 + \eta} - 2(\eta - \frac{1}{2}\eta^2 + \frac{1}{3}\eta^3)
= \frac{\eta^3}{3(1 + \eta)}(1 - 2\eta)
> \frac{1}{12}\eta^3,
\]

where the last inequality uses \(\eta < \frac{1}{3}\). Therefore if \(d_i = 2\) we have

\[
\frac{\frac{1}{2}g''(\xi_i)(d_i - \mu)^2}{2\eta^2 \xi_i^2} \geq \frac{\eta(d_i - \mu)^2}{2\max(\mu, d_i)^2}.
\]

It is easy to show, since \(d_i \in \mathbb{Z}\), that \(\left|\frac{d_i - \mu}{\max(\mu, d_i)}\right| \geq \frac{\eta}{\mu + \eta} \geq \frac{\eta}{2\mu}\), so

\[
\frac{1}{2}g''(\xi_i)(d_i - \mu)^2 \geq \frac{\eta^3}{8\mu^3} \text{ and the claim follows.}
\]

Using the claim, we have

\[
\log(\Lambda(G)) = \frac{1}{n\mu} \sum_{i=1}^{n} g(d_i)
= \frac{1}{n\mu} \sum_{i=1}^{n} \left( g(\mu) + g'(\mu)(d_i - \mu) + \frac{1}{2}g''(\xi_i)(d_i - \mu)^2 \right)
= \frac{1}{n\mu} \left( ng(\mu) + \sum_{i=1}^{n} \frac{1}{2}g''(\xi_i)(d_i - \mu)^2 \right)
\geq \log(\mu - 1) + \frac{1}{n\mu} \left( \frac{\eta^3}{8\mu^3} \right)
= \log(\mu - 1) + \frac{\eta^3}{8\mu^3}.
\]

So \(\Lambda(G) \geq (\mu - 1)\exp\left(\frac{\eta^3}{8\mu^3}\right) \geq \mu - 1 + (\mu - 1)\left(\frac{\eta^3}{8\mu^3}\right) \geq \mu - 1 + \frac{\eta^3}{8\mu^3}.\)

One easily checks that the function \(h(y) = y - 1 + \frac{\eta y}{8\mu^3}\) is strictly increasing on \((2, \infty)\); since \(\mu \geq \mu_0\) and \(\lambda_*(B(G)) \geq \Lambda(G)\), it follows that \(\lambda_*(B(G)) \geq \mu_0 - 1 + \frac{\eta(\mu_0)^3}{8\mu_0^3}\), as required. \(\square\)
4. Covering trees

A locally finite, infinite rooted tree (hereafter just a tree) is a connected acyclic infinite graph $\Gamma$ of finite maximum degree together with a particular vertex $r$ called the root. Adopting some notation of [4] and [10], for $x \in V(\Gamma)$ we write $|x|$ for the distance of $x$ from $r$, and we write $x \leq y$ if $x$ is on the path from $r$ to $y$. We write $x \wedge y$ for the join of $x$ and $y$, the vertex of largest distance from $r$ that is on both the path from $r$ to $x$ and the path from $r$ to $y$.

The trees we are interested in are 'covering trees' for finite graphs. Let $G = (V,E)$ be a finite graph of minimum degree at least 2 and let $e = (u,v)$ be an arc of $G$. The covering tree of $G$ rooted at $e$ is the tree $\Gamma = \Gamma_e(G)$ where the root is the length-zero walk $(u)$ of $G$, the other vertices are the non-backtracking walks of $G$ with first arc $e$ and the children of each walk $(u,v,v_2,\ldots,v_\ell)$ of length $\ell$ are its extensions $(u,v,v_2,\ldots,v_\ell,v_{\ell+1})$ to nonbacktracking walks of length $\ell + 1$ (ie. where $v_{\ell+1}$ is adjacent to $v_\ell$ in $G$ and is not equal to $v_{\ell-1}$). Note that the number of vertices of $\Gamma_e(G)$ at distance $\ell$ from the root is the total number of length-\ell non-backtracking walks of $G$ with first arc $e$, which is exactly the sum of the entries of the $e$-column of $B(G)^{\ell-1}$.

There is a natural homomorphism that associates each walk with its final vertex; if $G$ has large girth, this map preserves much of the local structure of $G$. To analyse the ubiquity of cycles in a random sample of edges of $G$, we follow [4] and study a problem of ‘fractional percolation’ on covering trees, bounding the probability that, given a $p$-random subset of $E(\Gamma_e(G))$, there is a long path starting at $r$ that is, in a certain sense, dense with edges in the subset.

Let $\Gamma$ be such a tree, and let $\alpha \in (0,1)$. Given $X \subseteq E(\Gamma)$, we say that a finite path $(v_0,v_1,\ldots,v_n)$ of $\Gamma$ is $\alpha$-adapted with respect to $X$ if, for each $i \in \{1,\ldots,n\}$, the subpath $(v_0,\ldots,v_i)$ contains at least $\alpha i$ edges of $X$. If $t_1,t_2,\ldots,$ is a sequence of positive integers and $T_n = \sum_{i=1}^n t_i$ is its sequence of partial sums (with $T_0 = 0$), then we say that a path $(x_0,x_1,\ldots,x_n)$ of $\Gamma$ is $(\alpha,t)$-adapted with respect to $X$ if for each $i \in \mathbb{Z}_{>0}$ for which $T_{i+1} < n$, the path $(x_{T_i},x_{T_i+1},\ldots,x_{T_{i+1}-1})$ is $\alpha$-adapted, and also the path $(x_{T_j},x_{T_j+1},\ldots,x_n)$ is $\alpha$-adapted, where $j$ is minimal so that $T_{j+1} > n$. Note that any initial subpath of an $(\alpha,t)$-adapted path is $(\alpha,t)$-adapted.

We will be considering $p$-random subsets $X$ of $E(\Gamma)$. We first estimate, with an argument used in ([4], Proposition 2), the probability that a given path is $\alpha$-adapted with respect to $X$. Henceforth, we
denote the ‘relative entropy’ between \( \alpha \) and \( p \) by
\[
D(\alpha \| p) = \alpha \ln \left( \frac{\alpha}{p} \right) + (1 - \alpha) \ln \left( \frac{1 - \alpha}{1 - p} \right).
\]
We remark that \([4] \) defines \( D(\alpha \| p) \) as the negative of this formula.

**Lemma 4.1.** Let \( 0 < p < \alpha < 1 \). There exists \( c > 0 \) so that, if \([x_0, x_1, \ldots, x_n] \) is a finite path, and \( X \) is a \( p \)-random subset of the edges of the path, then
\[
P \left( [x_0, \ldots, x_n] \text{ is } \alpha \text{-adapted w.r.t. } X \right) \geq \frac{c}{n} - \frac{5}{2} \exp(-nD(\alpha \| p)).
\]

**Proof.** Suppose that \(|X| \geq \alpha n\), and consider the cycle \([x_0, x_1, \ldots, x_{n-1}, x_n = x_0]\) formed by identifying \( x_0 \) and \( x_n \). Observe that there is some \( i \in \{0, \ldots, n-1\} \) such that the path corresponding to the cyclic ordering \([x_i, x_{i+1}, \ldots, x_n = x_0, x_1, \ldots, x_i]\) is \( \alpha \)-adapted with respect to \( X \) (choose \( i \) so that \(|X \cap E([x_0, \ldots, x_i])| - i \) is as small as possible). Thus, by symmetry, the required probability is at least \( \frac{1}{n} P(|X| \geq \alpha n) \).

It is straightforward to show using \( 0 < \alpha < 1 \) and Stirling’s approximation that all sufficiently large \( n \) satisfy
\[
[\alpha n]! \leq (\alpha n + 1)[\alpha n]! \leq \sqrt{2\pi n^3} \left( \frac{\alpha n}{e} \right)^{\alpha n}
\]
\[
(n - [\alpha n])! \leq \sqrt{2\pi n} \left( \frac{(1-\alpha)n}{e} \right)^{(1-\alpha)n},
\]
so Stirling’s approximation gives \( \left( \frac{n}{[\alpha n]} \right) \geq \frac{1}{\sqrt{2\pi n}^{3/2}} (\alpha^\alpha (1-\alpha)^{1-\alpha})^{-n} \) for all large \( n \). All large enough \( n \) thus satisfy
\[
\frac{1}{n} P(|X| \geq \alpha n) \geq \frac{1}{n} P(|X| = [\alpha n])
\]
\[
= \frac{1}{n} \left( \frac{n}{[\alpha n]} \right) p^{[\alpha n]} (1 - p)^{n-[\alpha n]}
\]
\[
\geq \frac{1}{\sqrt{2\pi n}^{5/2}} \left( \frac{p^\alpha (1-p)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \right)^n p^{[\alpha n]-[\alpha n]} (1 - p)^{\alpha n-[\alpha n]}
\]
\[
\geq \frac{p}{\sqrt{2\pi n}^{5/2}} \exp(-nD(\alpha \| p));
\]
since the probability of a path being \( \alpha \)-adapted is clearly positive for all \( n \), some \( c \in (0, \frac{p}{\sqrt{2\pi}}] \) now satisfies the lemma. \( \square \)

We say a positive integer sequence \( t = (t_i : i \geq 1) \) is **slow** if it is nondecreasing and satisfies \( \lim_{n \to \infty} t_n = \infty \) and \( \lim_{n \to \infty} \frac{t_{n+1}}{\sum_{i=1}^n \frac{1}{t_i}} = 0 \).

The next lemma is the main technical result of this section. It shows that, if \( t \) is a slow sequence, \( G \) is a graph, and \( \alpha \) and \( p \) are chosen so that \( \exp(D(\alpha \| p)) \) is less than the graph invariant \( \lambda_*(B(G)) \) of the
previous section, then there is some arc $e_0$ of $G$ for which a $p$-random subset of $E(\Gamma_{e_0}(G))$ will give an arbitrarily long $(\alpha, t)$-adapted path with probability bounded away from zero. The independence of $\delta$ on $n$ and $G$ in this lemma is crucial.

**Lemma 4.2.** For all $0 < p < \alpha < 1$, every slow sequence $t$, and all $\lambda > \exp(D(\alpha\|p))$, there is some $\delta = \delta(t, \lambda, \alpha, p) > 0$ such that, if $n \geq 1$ is an integer and $G$ is a connected graph of minimum degree at least 2 with $\lambda_*(B(G)) \geq \lambda$, then there is an arc $e_0$ of $G$ so that, given a $p$-random subset $X \subseteq E(\Gamma_{e_0}(G))$, we have

$$P(\Gamma_{e_0}(G) \text{ contains an } (\alpha, t)\text{-adapted path of length } n \text{ w.r.t. } X) > \delta.$$ 

**Proof.** Let $\lambda_1 = \lambda_*(B(G))$. Let $t = (t_i : i \geq 1)$ and $T_\ell = \sum_{i=1}^\ell t_i$ for each $\ell \geq 0$. Let $\lambda_0 = \exp(D(\alpha\|p))$ and $\lambda_1, \lambda_2$ be real numbers so that $\lambda_0 < \lambda_1 < \lambda_2 < \lambda$. Note that $\lambda_0 > 1$ and $\lambda_* \geq \lambda$.

Let $\Pi(m)$ denote the probability that a path of length $m$ is $\alpha$-adapted with respect to a $p$-random subset of its edges, and for each $\ell \geq 0$ let $f(\ell) = \prod_{i=1}^\ell \Pi(t_i)^{-1}$ be the reciprocal of the probability that a path of length $T_\ell$ is $(\alpha, t)$-adapted. To determine $\delta$, we first estimate $f$:

**Claim 4.2.1.** There exists $M > 0$ such that $f(\ell + 1) \leq M\lambda_2^{T_{\ell+1}}$ for all $\ell$.

**Proof of claim:** Let $c > 0$ be given by Lemma 4.1 for $p$ and $\alpha$. We have

$$f(\ell + 1) = \prod_{i=1}^{\ell+1} \Pi(t_i)^{-1} \leq \prod_{i=1}^{\ell+1} \frac{t_i^{5/2}}{c} \exp\left(D(\alpha\|p) \sum_{j=1}^{\ell+1} t_j \right)$$

$$= \lambda_0^{T_{\ell+1}} \prod_{i=1}^{\ell+1} \frac{t_i^{5/2}}{c}$$

$$= \lambda_1^{T_{\ell+1}} \prod_{i=1}^{\ell+1} \frac{t_i^{5/2}}{c} \left( \frac{\lambda_0}{\lambda_1} \right)^{t_i}$$

$$= \lambda_1^{(1+t_{\ell+1}/T_\ell) T_\ell} \prod_{i=1}^{\ell+1} \frac{t_i^{5/2}}{c} \left( \frac{\lambda_0}{\lambda_1} \right)^{t_i}.$$ 

Since $\lambda_0 < \lambda_1 < \lambda_2$ and $t_{\ell+1}/T_\ell \to 0$ and $t_\ell \to \infty$, this expression is at most $\lambda_2^{T_{\ell+1}}$ for large enough $\ell$. The claim follows by taking a maximum over all small $\ell$. \qed

Set $\delta = M^{-1}(\frac{1}{\lambda_2} - \frac{1}{\lambda})$. Let $\bar{E}$ be the set of arcs of $G$, let $B = B(G)$ and let $w_*$ be the (strictly positive) eigenvector of $B$ for $\lambda_*$, normalised to have largest entry 1. Choose $e_0 \in \bar{E}$ such that $w_*(e_0) = 1$. We show that $\delta$ and $e_0$ satisfy the lemma.
For each \( e \in \overline{E} \), let \( b_e \) be the standard basis vector in \( \mathbb{R}^{\overline{E}} \) corresponding to \( e \), and let \( N_h(e_0, e) = h^T b_e b_e^T \) be the number of non-backtracking walks of length \( h \) in \( G \) with first arc \( e_0 \) and last arc \( e \).

Let \( \Gamma = \Gamma_{e_0}(G) \) and \( r \) be the root of \( \Gamma \). Let \( \rho \colon V(\Gamma) \setminus \{r\} \to \overline{E} \) be the map assigning each walk to its last arc. Set \( \phi \) corresponding to \( e \) for each vertex \( x \).

For each \( z \) of \( \phi \) backtracking walks of length \( T \). (In other words, \( E \) and all \( x \) with \( \rho(x) = e \), the sum of \( \phi(y) \) over the children \( y \) of \( x \) is
\[
\sum_{e' \in \overline{E}, B_{e,e'} = 1} (w_*(e')) = \phi(x).
\]
(In other words, \( \phi \) is a unit flow on \( \Gamma \).) It follows that for every \( h \geq 0 \) and all \( x \) with \( |x| \leq h \), we have \( \sum (\phi(y) : y \geq x, |y| = h) = \phi(x) \).

For \( X \subseteq E(\Gamma) \), we say that a vertex \( v \) of \( \Gamma \) is \((\alpha, t)\)-reachable with respect to \( X \) if the path of \( \Gamma \) from \( r \) to \( x \) is \((\alpha, t)\)-adapted with respect to \( X \); let \( R(X) \) denote the set of \((\alpha, t)\)-reachable vertices. Fix \( \ell \) so that \( T_\ell \geq n \), and define a random variable \( Q = Q(X) \) by
\[
Q = f(\ell) \sum_{|x| = T_\ell} \phi(x) 1_{R(X)}(x).
\]
The \( \phi(x) \) sum to 1 over all \( x \) with \( |x| = T_\ell \), so \( \mathbb{E}(Q) = 1 \). We now bound the variance of \( Q \).

**Claim 4.2.2.** \( \mathbb{E}(Q^2) < \delta^{-1} \).

**Proof of claim:** We have
\[
\mathbb{E}(Q^2) = f(\ell)^2 \sum_{|x| = |y| = T_\ell} \phi(x) \phi(y) \mathbb{P}(x, y \in R(X)).
\]
For each \( z \in V(\Gamma) \), let \( k(z) \) be the maximum integer \( k \geq 0 \) so that \( T_k \leq |z| \). There are edge-disjoint paths of lengths \( t_1, t_2, \ldots, t_\ell \) and \( t_k(x \land y) + 2, t_k(x \land y) + 3, \ldots, t_\ell \) that all must be \( \alpha \)-adapted for both \( x \) and \( y \) to be in \( R(X) \) (the first set of paths make up the path from \( r \) to \( x \) and the second set are contained in the path from \( x \land y \) to \( y \)), so
\[
\mathbb{P}(x, y \in R(X)) \leq \prod_{i=1}^\ell \Pi(t_i) \prod_{i=k(x \land y) + 2}^\ell \Pi(t_i)
\]
\[
= f(k(x \land y) + 1) f(\ell)^{-2}
\]
\[
= M \lambda_2^{\overline{T}(x \land y)} f(\ell)^{-2}
\]
If

\[ \text{EN} \]

Thus \( E \)

For each \( e \) over a channel of bit-error rate \( G \) precisely the probability of an ML-decoding error in the cycle code of \( \beta \)

\( a \)

probability, given a

contains an (\( T \)

\( P \))

so

\( \ell \)

with respect to

\( x \)

\( \sum \)

\( z \)

\( i \)

\( E \)

\( \delta \)

\( \bar{\lambda} \)

\( \lambda \)

\( \phi \)

\( \lambda_{2}^{2}w_{\ast}(\rho(z)) \)

For each \( e \in \tilde{E} \), the number of \( z \in V(\Gamma) \) with \( |z| = i \) and \( \rho(z) = e \) is \( N_{i}(e_{0}, e) = b_{e_{0}}^{T}B_{i}^{-1}b_{e} \), so since \( Bw_{\ast} = \lambda_{\ast}w_{\ast} \) and \( w_{\ast}(e_{0}) = 1 \), we have

\[ \sum_{|z|=i} \phi(z)^{2} \leq \lambda_{\ast}^{2-2i}b_{e_{0}}^{T}B_{i}^{-1}\sum_{e \in E} b_{e}w_{\ast}(e) = \lambda_{\ast}^{2-2i}b_{e_{0}}^{T}B_{i}^{-1}w_{\ast} = \lambda_{\ast}^{1-i} \leq \lambda^{1-i}. \]

Thus \( \mathbf{E}(Q^{2}) < M \sum_{i=1}^{\infty} \lambda_{2}^{i}\lambda^{1-i} = M(\frac{1}{\lambda_{2}} - \frac{1}{\lambda})^{-1} = \delta^{-1}. \]

Now by the Cauchy-Schwartz inequality we have

\[ 1 = \mathbf{E}(Q)^{2} = \mathbf{E}(Q \cdot 1_{Q>0})^{2} \leq \mathbf{E}(Q^{2})\mathbf{E}(1_{Q>0}^{2}) < \delta^{-1}\mathbf{P}(Q > 0), \]

so \( \mathbf{P}(Q > 0) > \delta \). Therefore \( \Gamma \) has an \((\alpha, t)\)-adapted path of length \( T_{\ell} \) with respect to \( X \) with probability greater than \( \delta \). Such a path contains an \((\alpha, t)\)-adapted path of length \( n \), giving the result.

\[ \square \]

5. Graphs

For a graph \( G = (V, E) \) and for \( p, \beta \in [0, 1] \), let \( f_{p}^{\beta}(G) \) denote the probability, given a \( p \)-random subset \( X \subseteq E \), that \( X \) contains at least a \( \beta \)-fraction of the edges of some circuit of \( G \). Recall that \( f_{p}^{1/2}(G) \) is precisely the probability of an ML-decoding error in the cycle code of \( G \) over a channel of bit-error rate \( p < \frac{1}{2} \).
For each $x > 0$, let $\epsilon(x) = \frac{\eta(x)^3}{8x^2}$, where $\eta(x)$ as before denotes the distance from $x$ to the nearest integer. This is the term found in Lemma 3.2; Note that $\epsilon(x) \geq 0$, with equality if and only if $x \in \mathbb{Z}$.

The next theorem, whose proof closely follows that of ([4], Theorem 4), would be slightly weakened if $\epsilon(\mu)$ were dropped from the statement; we prove the stronger result below (in which $\epsilon$ can be viewed as a small ‘penalty’ term for nonintegral $\mu$) in order to obtain the equality characterisation in our main theorem.

**Theorem 5.1.** For all $\mu_0 \geq 2$ and $0 < p < \beta < 1$ satisfying $\exp(D(\beta||p)) < \mu_0 - 1 + \epsilon(\mu_0)$, there exists $\delta = \delta(\mu_0, p, \beta) > 0$ such that, if $G$ is a connected graph with $\mu(G) \geq \mu_0$, then $f_p^\beta(G) \geq \delta$.

**Proof.** It suffices to show this just for graphs of minimum degree at least 2, since deleting a degree-1 vertex from a graph $G$ with $\mu(G) \geq 2$ does not change $f_p^\beta$ or connectedness, and does not decrease $\mu(G)$. Suppose the result fails. Then there exists a sequence $G_1, G_2, \ldots$, of graphs of average degree at least $\mu_0$ and minimum degree at least 2, such that $\lim_{n \to \infty} (f_p^\beta(G_i)) = 0$. We clearly have $f_p^\beta(G) \geq p^{d(G)}$ for every graph (this is the probability of a $p$-random subset containing *every* edge in a given shortest cycle), so we may assume by considering a subsequence that $d(G_i) \geq i$ for each $i$. It is easy to see that there exists a slow integer sequence $t = (t_k : k \geq 1)$ so that $t_{|V(G_k)|} \leq \sqrt{k}$ for each $k$.

Note that $D(x||p)$ is increasing in $x$ for $x > p$. Set $\alpha \in (\beta, 1)$ so that $\exp(D(\beta||p)) < \exp(D(\alpha||p)) < \mu_0 - 1 + \epsilon(\mu_0)$, and let $\delta = \delta(t, \mu_0 - 1 + \epsilon(\mu_0), \alpha, p) > 0$ be given by Lemma 4.2. We argue that if $k$ is sufficiently large so that $\frac{2\sqrt{k+1}}{k} \leq \alpha - \beta$, then the graph $G = G_k$ satisfies $f_p^\beta(G) \geq \delta$. This contradicts $\lim_{n \to \infty} f_p^\beta(G_n) = 0$.

Let $G = G_k$ for such a $k$, and let $\Gamma = \Gamma_e(G)$ be the covering tree of $G$ with respect to the arc $e = (r, s)$ given by Lemma 4.2. Let $\pi : V(\Gamma) \to V(G)$ assign each path to its final vertex. By Corollary 3.2 we have $\lambda_e(B(G)) \geq \mu_0 - 1 + \epsilon(\mu_0)$. We now relate $f_p^\beta(G)$ to the probability that a $p$-random subset of $E(\Gamma)$ gives a long $(\alpha, t)$-adapted path. For each set $Z \subseteq V(G)$, let $G(Z)$ denote the subgraph of $G$ induced by $Z$.

Recalling notation from the proof of Lemma 4.2, for $X \subseteq E(G)$ we say a vertex $v$ of $G$ is *reachable* with respect to $X$ if $v = r$, or there is an $(\alpha, t)$-adapted path of $G$ (with respect to $X$) having first arc $e$ and last vertex $v$. We write $R(X)$ for the set of all such vertices. Similarly, for $Y \subseteq E(\Gamma)$, we say a vertex $v$ of $\Gamma$ is *reachable* with respect to $Y$ if there is an $(\alpha, t)$-adapted path of $\Gamma$ (with respect to $Y$) from the root.
to $v$. Let $R(Y)$ denote the set of all such vertices. Note, for any $X$ and $Y$, that each of the sets $R(X)$ and $\pi(R(Y))$ either is equal to \{r\}, or induces a connected subgraph of $G$ containing $r$ and $s$.

Suppose that $X$ is a $p$-random subset of $E(G)$ and $Y$ is a $p$-random subset of $E(\Gamma)$. Let $C_G$ denote the event that $G(R(X))$ contains a circuit, and $C_T$ denote the event that $G(\pi(R(Y)))$ contains a circuit.

**Claim 5.1.1.** $P(C_G) = P(C_T)$.

*Proof of claim:* Let $Z'$ denote the family of subsets of $V(G)$ that induce an acyclic connected subgraph of $G$ containing $r$ and $s$, and let $Z = Z' \cup \{r\}$. The event $C_G$ fails to hold exactly when $R(X) \in Z$, so

$$1 - P(C_G) = \sum_{Z \in Z} P(R(X) = Z).$$

Similarly, we have

$$1 - P(C_T) = \sum_{Z \in Z} P(\pi(R(Y)) = Z).$$

If $Z = \{r\}$, then clearly $P(R(X) = Z) = P(\pi(R(Y)) = Z) = 1 - p$. Suppose that $Z \in Z'$. By acyclicity of $G(Z)$, there is a unique subtree $\Gamma_Z$ of $\Gamma$ that contains the root of $\Gamma$ and satisfies $\pi(V(\Gamma_Z)) = Z$, and moreover $G(Z)$ and $\Gamma_Z$ are isomorphic finite trees. Now $G(Z)$ and $\Gamma_Z$ have the same number of edges, and the number of edges of $G$ with exactly one end in $Z \setminus \{r\}$ is equal to the number of edges of $\Gamma$ with exactly one end in $V(\Gamma_Z)$, so

$$P(R(X) = Z) = P(R(Y) = V(\Gamma_Z)) = P(\pi(R(Y)) = Z).$$

The claim now follows from the two summations above. \hfill \Box

**Claim 5.1.2.** $P(C_T) \geq \delta$.

*Proof of claim:* By Lemma 4.2, the tree $\Gamma$ contains, with probability at least $\delta$, a length-$|V(G)|$ path $[v_1, v_2, \ldots]$ that is $(\alpha, t)$-adapted with respect to $Y$. For any such path, there must be some $i < j$ so that $\pi(v_i) = \pi(v_j)$; now $\{\pi(v_i), \pi(v_{i+1}), \ldots, \pi(v_j)\}$ is the vertex set of a closed non-backtracking walk of $G(\pi(R(Y)))$, which must contain a circuit. This implies the claim. \hfill \Box

**Claim 5.1.3.** $f_\beta(G) \geq P(C_G)$.

*Proof of claim:* Suppose that $X \subseteq E$ satisfies $C_G$; i.e. $G(R(X))$ contains a circuit $C$. It suffices to show that $X$ contains a $\beta$-fraction of the edges of some circuit of $G$. Let $V(C) = [x_0, x_1, \ldots, x_m]$, where $x_0$ is the end of a shortest $(\alpha, t)$-adapted path $P_0$ from $r$ to $V(C)$. If there is some $i \in \{1, \ldots, m\}$ such that there exists in $G$ an $(\alpha, t)$-adapted path
$P_i$ from $r$ to $x_i$ not containing $x_{i-1}$ and an $(\alpha, t)$-adapted path $P_{i-1}$ from $r$ to $x_{i-1}$ not containing $x_i$, then $E(P_i) \cup E(P_{i-1}) \cup \{x_{i-1}x_i\}$ contains a circuit $C'$ of $G$. Moreover, this circuit is the disjoint union of the edge $x_{i-1}x_i$, a set of subpaths that are $\alpha$-adapted with respect to $X$, and at most two extra subpaths each of length at most $t_{\|V(G)\|}$, so $|X \cap E(C')| \geq \alpha|E(C')| - 2t_{\|V(G)\|} - 1$. Now $G = G_k$, so $|E(C')| \geq d(G) \geq k$ and $t_{\|V(G)\|} \leq \sqrt{k}$, giving

$$\frac{|X \cap E(C')|}{|E(C')|} \geq \alpha - \frac{2t_{\|V(G)\|} + 1}{|E(C')|} \geq \alpha - \frac{2\sqrt{k} + 1}{k} \geq \beta,$$

so $X$ contains a $\beta$-fraction of the edges of $C'$.

If no such $i$ exists, then an easy inductive argument implies for each $j \geq 1$ that every $(\alpha, t)$-adapted path from $r$ to $x_j$ passes through $x_{j-1}$, so $E(P_0) \cup E(C) - \{x_0x_m\}$ is the edge set of an $(\alpha, t)$-adapted path from $r$ to $x_m$. By a similar argument to the above, we have $|E(C) \cap X| \geq \alpha|E(C)| - 2t_{\|V(G)\|} - 1$, and thus $X$ contains a $\beta$-fraction of the edges of $C$, giving the claim. \hfill \Box

By the three claims, we have $f_{p^\beta}(G) \geq \delta$, implying the theorem. \hfill \Box

6. Thresholds

In this section, we prove Theorem 1.1. Recall that, if $C$ is the cycle code of a graph $G$, then the probability of a maximum-likelihood decoding error in $C$ over a channel of bit-error rate $p \in (0, \frac{1}{2})$ is exactly the parameter $f_{p^{1/2}}(G)$ of the previous section. We use this fact to derive Theorem 1.1 (which we now restate) from Theorem 5.1.

**Theorem 6.1.** If $\mathcal{G}$ is the class of graphic codes and $R \in (0, 1)$, then $\theta_{\mathcal{G}}(R) \leq \frac{(1-\sqrt{R})^2}{2(1+R)}$. If equality holds then $R = 1 - \frac{3}{d}$ for some integer $d$.

**Proof.** Let $\mu = \frac{2}{1-R}$. Let $\theta = \frac{(1-\sqrt{R})^2}{2(1+R)}$; note that $\theta \leq \frac{1}{2}$ and $\exp(D(\frac{1}{2}||\theta)) = \mu - 1$. There is therefore some $\theta' \leq \theta$ such that $\exp(D(\frac{1}{2}||\theta')) = \mu - 1 + \varepsilon(\mu)$, with $\theta' = \theta$ if and only if $\mu \in \mathbb{Z}$ (that is, if $R = 1 - \frac{3}{d}$ for some $d \in \mathbb{Z}$).

It is enough to show that $\theta_{\mathcal{G}}(R) \leq \theta'$, that is, to show that for all $p \in (\theta', \frac{1}{2})$ there is some $\epsilon > 0$ such that the probability of an error in maximum-likelihood decoding of a graphic code of rate at least $R$, over a channel with bit-error rate $p$, is at least $\epsilon$.

Let $p \in (\theta', \frac{1}{2})$. Since $p > \theta'$ we have $\exp(D(\frac{1}{2}||p)) < \mu - 1 + \varepsilon(\mu); \text{let} \mu_0 < \mu$ be such that $\exp(D(\frac{1}{2}||p)) < \mu_0 - 1 + \varepsilon(\mu_0)$. Let $\delta = \delta(\mu_0, p, \frac{1}{2})$ be given by Theorem 5.1 and set $\epsilon = \min(\delta, p^b)$, where $b = \frac{2\mu_0}{\mu - \mu_0}$.
Let $C$ be a graphic code of rate $R(C) \geq R$, and let $G$ be a connected graph whose cycle code is $C$. Note, since $R > 0$, that $G$ contains a circuit, so $f_p^{1/2}(G) \geq p|E(G)|$. If $\mu(G) \geq \mu_0$ then $f_p^{1/2}(G) \geq \delta \geq \epsilon$ by Theorem 5.1. Otherwise

$$1 - \frac{2}{\mu} = R \leq R(C) = 1 - \frac{2}{\mu(C)} + \frac{1}{|E(G)|} < 1 - \frac{2}{\mu_0} + \frac{1}{|E(G)|},$$

so $|E(G)| < \frac{2(\mu - \mu_0)}{\mu \mu_0} = b$ and thus $f_p^{1/2}(G) \geq p^b \geq \epsilon$, as required. □

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