STRENGTHENED GRUNSKY AND MILIN INEQUALITIES

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Abstract. The method of Grunsky inequalities has many applications and has been extended in many directions, even to bordered Riemann surfaces. However, unlike the case of functions univalent in the disk, a quasiconformal variant of this theory has not been developed so far. In this paper, we essentially improve the basic facts concerning the classical Grunsky inequalities for univalent functions on the disk and extend these results to arbitrary quasiconformal disks. Several applications are given.

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1. The Grunsky and Grunsky-Milin coefficients

1.1. The Grunsky operator. In 1939, H. Grunsky discovered the necessary and sufficient conditions for univalence of a holomorphic function in a finitely connected domain on the extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) in terms of an infinite system of the coefficient inequalities. In particular, his theorem for the canonical disk \( \Delta^* = \{ z \in \hat{\mathbb{C}} : |z| > 1 \} \) yields that a holomorphic function \( f(z) = z + \text{const} + O(z^{-1}) \) in a neighborhood of \( z = \infty \) can be extended to a univalent holomorphic function on the \( \Delta^* \) if and only if its Grunsky coefficients \( \alpha_{mn} \) satisfy

\[
\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq 1, \tag{1.1}
\]

where \( \alpha_{mn} \) are defined by

\[
\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2, \tag{1.2}
\]

the sequence \( x = (x_n) \) runs over the unit sphere \( S(l^2) \) of the Hilbert space \( l^2 \) with norm \( \|x\|^2 = \sum |x_n|^2 \), and the principal branch of the logarithmic function is chosen (cf. [Gr]). The quantity

\[
\kappa(f) = \sup \left\{ \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| : x = (x_n) \in S(l^2) \right\} \leq 1 \tag{1.3}
\]

is called the Grunsky norm of \( f \).

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For the functions with $k$-quasiconformal extensions ($k < 1$), we have instead of (1.3) a stronger bound

$$
\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq k \text{ for any } x = (x_n) \in S(l^2),
$$

(1.4)

established first in [Ku] (see also [Ku1]). Then $\varpi(f) \leq k(f)$, where $k(f)$ denotes the Teichmüller norm of $f$ which is equal to the infimum of dilatations $k(w^\prime) = ||\mu||_\infty$ of quasiconformal extensions of $f$ to $\hat{\mathbb{C}}$. Here $w^\mu$ denotes a homeomorphic solution to the Beltrami equation $\partial w = \mu \partial_{\bar{z}} w$ on $\mathbb{C}$ extending $f$; accordingly, $\mu$ is called the Beltrami coefficient (or complex dilatation) of $w$.

Note that the Grunsky (matrix) operator $G(f) = (\sum_{m,n=1}^{\infty} \alpha_{mn}(f))_{m,n=1}^{\infty}$ acts as a linear operator $l^2 \to l^2$ contracting the norms of elements $x \in l^2$; the norm of this operator equals $\varpi(f)$.

For most functions $f$, we have the strong inequality $\varpi(f) < k(f)$, while the functions with the equal norms play a crucial role in many applications.

1.2. Generalization. The method of Grunsky inequalities was generalized in several directions, even to bordered Riemann surfaces $X$ with a finite number of boundary components (cf. [Gr], [Le], [Mi], [Po], [SS]). In the general case, the generating function (1.2) must be replaced by a bilinear differential

$$
- \log \frac{f(z) - f(\zeta)}{z - \zeta} - R_X(z, \zeta) = \sum_{m,n=1}^{\infty} \beta_{mn} \varphi_m(z) \varphi_n(\zeta) : X \times X \to \mathbb{C},
$$

(1.5)

where the surface kernel $R_X(z, \zeta)$ relates to the conformal map $j_\theta(z, \zeta)$ of $X$ onto the sphere $\hat{\mathbb{C}}$ slit along arcs of logarithmic spirals inclined at the angle $\theta \in [0, \pi)$ to a ray issuing from the origin so that $j_\theta(z, \zeta) = 0$ and

$$
j_\theta(z) = (z - z_\theta)^{-1} + \text{const} + O(1/(z - z_\theta)) \quad \text{as } z \to z_\theta = j_\theta^{-1}(\infty)
$$

(in fact, only the maps $j_\theta$ and $j_{\pi/2}$ are applied). Here $\{\varphi_n\}_\infty^1$ is a canonical system of holomorphic functions on $X$ such that (in a local parameter)

$$
\varphi_n(z) = \frac{a_{n,n}}{z^n} + \frac{a_{n+1,n}}{z^{n+1}} + \ldots \quad \text{with } a_{n,n} > 0, \quad n = 1, 2, \ldots,
$$

and the derivatives (linear holomorphic differentials) $\varphi_n^\prime$ form a complete orthonormal system in $B^2(X)$.

We shall deal only with simply connected domains $X = D^* \supset \infty$ with quasiconformal boundaries (quasidisks). For any such domain, the kernel $R_D$ vanishes identically on $D^* \times D^*$, and the expansion (1.5) assumes the form

$$
- \log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \frac{\beta_{mn}}{\sqrt{mn} \chi(z)^m \chi(\zeta)^n},
$$

(1.6)

where $\chi$ denotes a conformal map of $D^*$ onto the disk $\Delta^*$ so that $\chi(\infty) = \infty$, $\chi'(\infty) > 0$.

Each coefficient $\beta_{mn}(f)$ in (1.6) is represented as a polynomial of a finite number of the initial coefficients $b_1, b_2, \ldots, b_s$ of $f$; hence it depends holomorphically on Beltrami coefficients of quasiconformal extensions of $f$ as well as on the Schwarzian derivatives

$$
S_f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2, \quad z \in D^*.
$$

(1.7)

These derivatives range over a bounded domain in the complex Banach space $B(D^*)$ of hyperbolically bounded holomorphic functions $\varphi \in \Delta^*$ with norm

$$
\|\varphi\|_B = \sup_{D^*} \lambda_{D^*}(z)|\varphi(z)|,
$$
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where \( \lambda_{D^*}(z)|dz| \) denotes the hyperbolic metric of \( D^* \) of Gaussian curvature \(-4\). This domain models the \textbf{universal Teichmüller space} \( T \) with the base point \( \chi'(\infty)D^* \) (in holomorphic Bers’ embedding of \( T \)).

A theorem of Milin extending the Grunsky univalence criterion for the disk \( \Delta^* \) to multiply connected domains \( D^* \) states that a holomorphic function \( f(z) = z + \text{const} + O(z^{-1}) \) in a neighborhood of \( z = \infty \) can be continued to a univalent function in the whole domain \( D^* \) if and only if the coefficients \( \alpha_{mn} \) in (1.6) satisfy, similar to the classical case of the disk \( D^* \), the inequality

\[
\left| \sum_{m,n=1}^{\infty} \beta_{mn} x_m x_n \right| \leq 1 \quad (1.8)
\]

for any point \( x = (x_n) \in S(l^2) \). We call the quantity

\[
\kappa_{D^*}(f) = \sup \left\{ \left| \sum_{m,n=1}^{\infty} \beta_{mn} x_m x_n \right| : x = (x_n) \in S(l^2) \right\}, \quad (1.9)
\]

the \textbf{generalized Grunsky norm} of \( f \).

Note that in the case \( D^* = \Delta^* \), \( \beta_{mn} = \sqrt{mn} \alpha_{mn} \); for this disk, we shall use the notations \( \Sigma \) and \( \kappa(f) \).

By (1.8), \( \kappa_{D^*}(f) \leq 1 \) for any \( f \) from the class \( \Sigma(D^*) \) of univalent functions in \( D^* \) with hydrodynamical normalization

\[
f(z) = z + b_0 + b_1 z^{-1} + \ldots \quad \text{near} \quad z = \infty.
\]

However, unlike the case of functions univalent in the disk, a quasiconformal variant of this theory has not been developed so far.

1.3. The technique of the Grunsky inequalities is a powerful tool in geometric complex analysis having fundamental applications in the Teichmüller space theory and other fields and concerns mainly the classical case of univalent functions on the disk \( \Delta^* \) with hydrodynamical normalization, which has been investigated by many authors from different points of view.

In this paper, we create the quasiconformal theory of generic Grunsky coefficients and essentially improve the basic facts and estimates concerning the classical Grunsky inequalities. These results are extended to univalent functions on arbitrary quasiconformal disks.

2. \textbf{Main results}

2.1. First recall the fundamental property of extremal Beltrami coefficients which plays a crucial role in applications of univalent functions with quasiconformal extensions. Consider the unit ball of Beltrami coefficients

\[
\text{Belt}(D)_1 = \{ \mu \in L_{\infty}(C) : \mu(z)|D^* = 0, \| \mu \|_{\infty} < 1 \}
\]

and their pairing with \( \psi \in L_1(D) \) by

\[
\langle \mu, \psi \rangle_D = \iint_D \mu(z)\psi(z)dxdy \quad (z = x + iy).
\]

The following two sets of holomorphic functions \( \psi \) (equivalently, of holomorphic quadratic differentials \( \psi dz^2 \))

\[
A_1(D) = \{ \psi \in L_1(D) : \psi \text{ holomorphic in } D \},
\]

\[
A_1^2(D) = \{ \psi = \omega^2 \in A_1(D) : \omega \text{ holomorphic in } D \}
\]
are intrinsically connected with the extremal Beltrami coefficients (hence, with the \( \text{Teichmüller} \) norm) and Grunsky inequalities. The well-known criterion for extremality (the Hamilton-Krushkal-Reich-Strebel theorem) implies that a Beltrami coefficient \( \mu_0 \in \text{Belt}(D) \) is extremal if an only if
\[
\|\mu_0\|_\infty = \sup_{\|\psi\|_{A_1(D)}=1} |\langle \mu_0, \psi \rangle_D|.
\]
(2.1)
The same condition is necessary and sufficient for the infinitesimal extremality of \( \mu_0 \) (i.e., at the origin of \( \mathbf{T} \) in the direction to \( \phi_T(\mu_0) \), where \( \phi_T \) is the defining (factorizing) holomorphic projection \( \text{Belt}(D) \to \mathbf{T} \)); see, e.g., [EKK], [GL]. In contrast, the Grunsky norm relates to the functions from \( A_1^2(D) \), i.e. to abelian differentials.

For an element \( \mu \in \text{Belt}(D) \) we define
\[
\mu^*(z) = \mu(z)/\|\mu\|_\infty,
\]
so that \( \|\mu^*\|_\infty = 1 \), and associate with the corresponding map \( f^\mu \) the quantity
\[
\alpha_D(f^\mu) = \sup \left\{ \left| \int_D \mu^*(z) \varphi(z) dxdy \right| : \varphi \in A_1^2(D), \|\varphi\|_{A_1} = 1 \right\} \leq 1.
\]
(2.2)
For the disk \( D = \Delta \), we shall use the notation \( \alpha(f^\mu) \).

### 2.2. Strengthened bounds for Grunsky norm

Now we can formulate our results. The following theorem essentially improves the basic estimate (1.4).

**Theorem 2.1.** For any quasidisk \( D^* \), the generalized Grunsky norm \( \kappa_{D^*}(f) \) of every function \( f \in \Sigma^0(D^*) \) is estimated by its \( \text{Teichmüller} \) norm \( k = k(f) \) by
\[
\kappa_{D^*}(f) \leq \frac{k + \alpha_D(f)}{1 + \alpha_D(f)} k,
\]
and \( \kappa_{D^*}(f) < k \) unless \( \alpha_D(f) = 1 \). The last equality occurs if and only if \( \kappa_{D^*}(f) = k(f) \).

**Theorem 2.2.** The equality \( \kappa_{D^*}f = k(f) \) holds if and only if the function \( f \) is the restriction to \( \overline{D^*} \) of a quasiconformal self-map \( w^{\mu_0} \) of \( \hat{\mathbb{C}} \) with Beltrami coefficient \( \mu_0 \) satisfying the condition
\[
\sup |\langle \mu_0, \varphi \rangle_D| = \|\mu_0\|_\infty,
\]
(2.4)
where the supremum is taken over holomorphic functions \( \varphi \in A_1^2(D) \) with \( \|\varphi\|_{A_1(D)} = 1 \).

If, in addition, the equivalence class of \( f \) (the collection of maps equal \( f \) on \( \partial D^* \)) is a Strebel point, then \( \mu_0 \) is necessarily of the form
\[
\mu_0(z) = \|\mu_0\|_\infty|\psi_0(z)|/\psi_0(z) \quad \text{with} \quad \psi_0 \in A_1^2(D).
\]
(2.5)
The condition (2.4) has a geometric nature based on the properties of the invariant Carathéodory and Kobayashi distances of the universal \( \text{Teichmüller} \) space \( \mathbf{T} \).

The assertion of Theorem 2.2 was earlier established in [Ku2] only for the functions univalent in the canonical disk \( \Delta^* \), i.e., for \( f \in \Sigma \). This special result answered a question posed by several mathematicians and has many applications.

Shiga and Tanigawa gave an essential extension of this phenomena to \( \text{Teichmüller} \) spaces of elementary groups (see [ShT]). In particular, it holds for covers of conformal maps of the punctured disk \( \{ 1 < |z| < \infty \} \).

For \( f \in \Sigma \), mapping the unit circle onto an analytic curve, the equality (2.5) was obtained by a different method in [Ku2].
2.3. **Two corollaries.** Both Theorems 2.1 and 2.2 have many interesting consequences. In this paper we present the consequences of Theorem 2.1. We start with corollaries concerning the maps with small dilatations. From (2.3), for all \( f \in \Sigma^0(D^*) \) with small dilatation \( k(f) \),

\[
\kappa_{D^*}(f) \leq \alpha_D(f)k + O(k^2),
\]

where the bound for the remainder is uniform when \( k \leq k_0 \) and \( k_0 < 1 \) is fixed. On the other hand, as was established in [Kr4], if a function \( f \in \Sigma^0(D^*) \) admits quasiconformal extension \( \psi \) of Teichmüller type, i.e. with \( \mu = k|\psi|/\psi, \psi \in A_1(D) \), then its Grunsky norm is estimated from below by

\[
\kappa_{D^*}(f) \geq \alpha_D(f)k(f), \tag{2.6}
\]

with \( \alpha_D(f) \) given by (2.2). Hence, the inequalities (2.3) and (2.6) imply

**Corollary 2.3.** The generalized Grunsky norm of any \( f \in \Sigma^0(D^*) \) with Teichmüller quasiconformal extension satisfies the asymptotic equality

\[
\kappa_{D^*}(f) = \alpha_D(f)k + O(k^2), \quad k = k(f) \to 0. \tag{2.7}
\]

In the case of the canonical disk \( \Delta^* \), one obtains from the last equality a quantitative relation between the Grunsky norm and the Schwarzian derivative of \( f \). Namely, using the Ahlfors-Weill quasiconformal extension of univalent functions and letting

\[
\nu(\varphi(z)) = \frac{1}{2}(1 - |z|^2)\varphi(1/\bar{z})1/z^2, \quad \varphi \in B = B(\Delta^*),
\]

one derives

**Corollary 2.4.** For \( f \in \Sigma(\Delta^*) \) with sufficiently small norm \( \|S_f\|_B \) of its Schwarzian,

\[
\kappa(f) = \sup\{|\nu(\varphi)| : \varphi \in A_1, \|\varphi\|_{A_1(\Delta)} = 1\} + O(\|S_f\|^2_B), \tag{2.9}
\]

where the ratio \( O(\|S_f\|^2_B)/\|S_f\|^2_B \) remains bounded as \( \|S_f\|_B \to 0 \).

The Beltrami coefficients of the form (2.8) are called harmonic, in view of their connection with the deformation theory for conformal structures.

2.4. **Continuity.** It is well known that the classical Grunsky norm \( \kappa(f) \) regarded as a curve functional is lower semicontinuous in the weak topology on the space \( \Sigma^0 \) (i.e., with respect to locally uniform convergence of sequences \( \{f_n\} \subset \Sigma^0 \) on the disk \( \Delta^* \)) and continuous with respect to convergence of \( f_n \) in Teichmüller metric (see [Sc, Sh]). The arguments exploited in the proofs essentially use the univalence on the canonical disk \( \Delta^* \). The continuity of \( \kappa(f) \) plays a crucial role in some applications of the Grunsky inequalities technique to Teichmüller spaces.

We consider univalent functions on generic quasidisks \( D^* \) and show that in either case the Grunsky norm is lower semicontinuous in the weak topology on \( \Sigma^0(D^*) \) and locally Lipschitz continuous with respect to Teichmüller metric.

**Theorem 2.5.** (i) If a sequence \( \{f_n\} \subset \Sigma^0(D^*) \) is convergent locally uniformly on \( D^* \) to \( f_0 \), then

\[
\kappa_{D^*}(f_0) \leq \liminf_{n \to \infty} \kappa_{D^*}(f_n). \tag{2.10}
\]

(ii) The functional \( \kappa_{D^*}(\varphi) \) regarded as a function of points \( \varphi = S_f \) from the universal Teichmüller space \( T \) (with base point \( D^* \)) is locally Lipschitz continuous and logarithmically plurisubharmonic on \( T \).

This key theorem is essential in the proof of other theorems.
2.5. Generalization of Moser’s conjecture. In 1985, J. Moser conjectured that the set of functions $f \in \Sigma^0$ with $\varphi(f) = k(f)$ is rather sparse in $\Sigma^0$ so that any function $f \in \Sigma^0$ is approximated by functions $f_n$ satisfying $\varphi(f_n) < k(f_n)$ uniformly on compact sets in $\Delta^*$. This conjecture was proved in [KK1] and in a strengthened form in [Kr]. The constructions applied in the proofs essentially used the univalence in the canonical disk $\Delta^*$. Theorem 2.1 allows us to solve a similar question for the generalized Grunsky norm $\varphi_{D^*}$ of the functions univalent in an arbitrary quasidisk $D^*$.

**Theorem 2.6.** For any function $f \in \Sigma^0(D^*)$, there exists a sequence $\{f_n\} \subset \Sigma^0(D^*)$ with $\varphi_{D^*}(f_n) < k(f_n)$ convergent to $f$ locally uniformly in $D^*$.

2.6. There is a related conjecture posed in [KK1] that $\varphi(f) = k(f)$ cannot be the limit functions of locally uniformly convergent sequences $\{f_n\} \subset \Sigma^0$ with $\varphi(f_n) = k(f_n)$.

Its proof is given in [Kr]. The main arguments involve a special holomorphic motion of the disk and can be appropriately extended to generic quasidisks, i.e., to the generalized Grunsky norm (cf. Section 7).

2.7. Connection with Fredholm eigenvalues. The Fredholm eigenvalues $\rho_n$ of a smooth closed Jordan curve $L \subset \hat{C}$ are the eigenvalues of its double-layer potential, i.e., of the integral equation

$$u(z) + \frac{\rho}{\pi} \int_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} d\zeta_s = h(z),$$

which has many applications. The least positive eigenvalue $\rho_1 = \rho_L$ plays a crucial role, since by the Kühnau-Schiffer theorem it is reciprocal to the Grunsky norm of the Riemann mapping function of the exterior domain of $L$. This value is defined for any oriented closed Jordan curve $L \subset \hat{C}$ by

$$\frac{1}{\rho_L} = \sup \frac{|D_G(u) - D_{G^*}(u)|}{D_G(u) + D_{G^*}(u)},$$

where $G$ and $G^*$ are, respectively, the interior and exterior of $L$; $D$ denotes the Dirichlet integral, and the supremum is taken over all functions $u$ continuous on $\hat{C}$ and harmonic on $G \cup G^*$.

Until now, no general algorithms exist for finding these values for the given quasiconformal curves. The problem was solved only for some specific classes of curves, so in general one can use only a rough estimate for $\rho_L$ by Ahlfors’ inequality

$$\frac{1}{\rho_L} \leq q_L,$$  \hspace{1cm} (2.11)

where $q_L$ is the minimal dilatation of quasiconformal reflections across the given curve $L$, (that is, of the orientation reversing quasiconformal homeomorphisms of $\hat{C}$ preserving $L$ point-wise); see, e.g., [Ah2], [Kr], [Ku].

Corollary 2.3 provides the following improvement of Ahlfors’ inequality.

**Theorem 2.7.** For any quasicircle $L = f(S^1)$, $f \in \Sigma^0$,

$$\frac{1}{\rho_L} = \sup \{|(\mu_S, \psi)_\Delta| : \psi \in A^2_1, \|\psi\|_{A_1(\Delta) = 1} + O(\|S_f\|_B^2)\}$$

$$= \sup \{|(\mu_0(1/s) \zeta/\zeta, \psi)_\Delta| : \psi \in A^2_1, \|\psi\|_{A_1(\Delta) = 1} + O(\|\mu_0\|_{A_2}^2)\},$$  \hspace{1cm} (2.12)

where $\mu_0(z) = g_z/g^*$ is the complex dilatation of extremal quasireflection over the curve $L$ and both remainders are estimated uniformly for $\|\mu_0\| \leq k_0 < 1$. 

3. Proof of Theorem 2.5

(i) First observe that the generalized Grunsky coefficients \( \beta_{mn}(f^\mu) \) of the functions \( f^\mu \in \Sigma(D^*) \) generate for each \( x = (x_n) \in l^2 \) with \( \|x\| = 1 \) the holomorphic maps

\[
h_x(\mu) = \sum_{m,n=1}^{\infty} \beta_{mn}(f^\mu)x_n x_m : \text{Belt}(D)_1 \to \Delta,
\]

and \( \sup_x |h_x(f^\mu)| = \kappa_{D^*}(f^\mu) \).

The holomorphy of these functions follows from the holomorphy of coefficients \( \beta_{mn} \) with respect to Beltrami coefficients \( \mu \in \text{Belt}(D)_1 \) mentioned above using the estimate

\[
\left| \sum_{j=2}^{M} \sum_{n=1}^{N} \beta_{mn}(f^\mu)x_n x_m \right|^2 \leq \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2
\]

which holds for any finite \( M, N \) and \( 1 \leq j \leq M, \ 1 \leq l \leq N \). This estimate is a simple corollary of the Milin univalence theorem (cf. [Mi, p. 193], [Po, p. 61]).

Similar arguments imply that the maps (3.1) regarded as functions of points \( \varphi^\mu = Sf^\mu \) in the universal Teichmüller space \( T \) (with the basepoint \( D^* \)) are holomorphic on \( T \).

Now, let a sequence \( \{f_p\} \subset \Sigma^0(D^*) \) be convergent to \( f_0 \) uniformly on compact subsets of \( \Delta^* \). Denote their generalized Grunsky coefficients by \( \beta_{mn}(f_p) \). Then, for any \( M, N < \infty \) and any fixed \( x = (x_n) \in l^2 \),

\[
\sum_{1 \leq j \leq M} \sum_{1 \leq n \leq N} \beta_{mn}(f^\mu)x_n x_m = \lim_{p \to \infty} \sum_{1 \leq j \leq M} \sum_{1 \leq n \leq N} \beta_{mn}(f_p)x_n x_m \leq \liminf_{p \to \infty} \kappa_{D^*}(f^\mu) - \kappa_{D^*}(f_0).
\]

Taking the supremum over \( x \) in the left-hand side yields the desired inequality

\[
\kappa_{D^*}(f_0) \leq \liminf_{p \to \infty} \kappa_{D^*}(f_p).
\]

(ii) Since for any \( \mu \in \text{Belt}(D)_1 \),

\[
\kappa_{D^*}(\varphi^\mu) = \sup_{\varphi \in l^2} \|h_x(\varphi^\mu)\|, \quad \varphi^\mu = Sf^\mu,
\]

the function \( \kappa(\varphi) \) possesses, together with \( h_x(\varphi) \), the mean value inequality property. To get the plurisubharmonicity of \( \kappa(\varphi) \), one needs to establish its upper semicontinuity. Using the holomorphy of functions (3.1), one can derive much more.

For any fixed \( x \in l^2 \), the function \( h_x(\varphi) - h_x(\varphi_0) \) is a holomorphic map of the ball

\[
\{ \varphi \in T : \|\varphi - \varphi_0\|_B < d \}, \quad d = \text{dist}(\varphi_0, \partial T)
\]

into the disk \( \{|w| < 2\} \). Hence, by Schwarz’s lemma,

\[
|h_x(\varphi) - h_x(\varphi_0)| \leq \frac{2}{d} \|\varphi - \varphi_0\|,
\]

and

\[
|h_x(\varphi)| \leq |h_x(\varphi_0)| \leq |h_x(\varphi) - h_x(\varphi_0)| \leq \frac{2}{d} \|\varphi - \varphi_0\|.
\]

Now assume that \( \kappa_{D^*}(\varphi) \geq \kappa_{D^*}(\varphi_0) \) and pick a maximizing sequence \( h_{x_m}(\varphi) \) so that

\[
\lim_{m \to \infty} |h_{x_m}(\varphi)| = \kappa_{D^*}(\varphi).
\]

Then, since the estimate holds for any \( x \in l^2 \), one gets

\[
0 < \kappa_{D^*}(\varphi) - \kappa_{D^*}(\varphi_0) \leq \kappa_{D^*}(\varphi) - \limsup_{m \to \infty} |h_{x_m}(\varphi_0)| \leq \frac{2}{d} \|\varphi - \varphi_0\|.
\]
In the same way, if $\kappa_{D^*}(\varphi_0) \geq \kappa_{D^*}(\varphi)$,

$$0 < \kappa_{D^*}(\varphi_0) - \kappa_{D^*}(\varphi) \leq \frac{2}{d}\|\varphi - \varphi_0\|,$$

which implies the Lipschitz continuity of $\kappa_{D^*}$ in a neighborhood of $\varphi_0$, completing the proof of the theorem.

4. Proofs of Theorem 2.1

Note that if $\kappa_{D^*}(f^\mu) = k(f^\mu) = \|\mu\|_\infty$, then

$$\kappa_{D^*}(f^\mu) = k(f^{t\mu}) \quad \text{for all} \quad |t| < 1.$$ 

This follows, for example, from subharmonicity of the function $\kappa_{D^*}(f^{t\mu})$ in $t$ on the unit disk giving subharmonicity of the ratio

$$g(t) = \frac{\kappa_{D^*}(f^{t\mu})}{k(f^{t\mu})} = \frac{\kappa_{D^*}(f^{t\mu})}{|t|} \quad \text{for all} \quad |t| < 1.$$ 

We first consider the case $D^* = \Delta^*$ which sheds light to key features. For $f \in \Sigma^0$, the functions (3.1) are of the form

$$h_\mu(x) = \sum_{m=1}^{\infty} \sqrt{m} \alpha_{mn}(f^\mu) x_m x_n : \text{Belt}(\Delta) \to \Delta. \quad (4.1)$$

Take, for a given function $f$, an extremal coefficient $\mu$ (i.e., such that $k(f) = \|\mu\|_\infty$) and consider its extremal disk

$$\Delta(\mu) = \{t\mu/\|\mu\|_\infty : |t| < 1\} \subset \text{Belt}(\Delta).$$

Put $\mu^* = \mu/\|\mu\|_\infty$. We apply to $h_\mu(f^{t\mu^*})$ the well-known improvement of the classical Schwarz lemma (see [BM, Go]) which asserts that a holomorphic function

$$g(t) = c_m t^m + c_{m+1} t^{m+1} + \cdots : \Delta \to \Delta \quad (c_m \neq 0, \ m \geq 1),$$

in $\Delta$ is estimated by

$$|g(t)| \leq |t|^m \frac{|t| + |c_m|}{1 + |c_m||t|}, \quad (4.2)$$

and the equality occurs only for

$$g_0(t) = t^m (t + c_m)/(1 + \overline{c_m} t).$$

To calculate the corresponding constant $\alpha(f)$ in (2.3), one can use the variational formula for $f^\mu(z) = z + b_0 + b_1 z^{-1} + \cdots \in \Sigma^0$ with extensions satisfying $f^\mu(0) = 0$. Namely, for small $\|\mu\|_\infty$,

$$f^\mu(z) = z - \frac{1}{\pi} \int_{\Delta} \mu(w) \left( \frac{1}{w - z} - \frac{1}{\overline{w}} \right) dudv + O(\|\mu^2\|_\infty), \quad w = u + iv, \quad (4.3)$$

where the ratio $O(\|\mu^2\|_\infty^2)/\|\mu^2\|_\infty^2$ is uniformly bounded on compact sets of $\mathbb{C}$. Then

$$b_n = \frac{1}{\pi} \int_{\Delta} \mu(w) w^{n-1} dudv + O(\|\mu^2\|_\infty), \quad n = 1, 2, \ldots,$$

and from (1.2),

$$\alpha_{mn}(\mu) = -\pi^{-1} \int_{\Delta} \mu(z) z^{m+n-2} dxdy + O(\|\mu^2\|_\infty), \quad \|\mu\|_\infty \to 0. \quad (4.4)$$
Hence, the differential at zero of the corresponding map \( h_\alpha(t\mu^*) \) with \( x = (x_n) \in S(l^2) \) is given by
\[
dh_\alpha(0)\mu^* = -\frac{1}{\pi} \int_\Delta \mu^*(z) \sum_{m+n=2}^\infty \sqrt{mn} x_m x_n z^{m+n-2} dxdy. \tag{4.5}
\]
On the other hand, as was established in [Kr2], the elements of \( A_1(\Delta)^2 \) are represented in the form
\[
\psi(z) = \omega(z)^2 = \frac{1}{\pi} \sum_{m+n=2}^\infty \sqrt{mn} x_m x_n z^{m+n-2},
\]
with \( ||\xi||_2^2 = ||\omega||_{L^2} \). Thus, by (4.2), for any \( \mu = t\mu^* \),
\[
|h_\alpha(\mu)| \leq |t| + \left| \frac{\langle \mu^*, \psi \rangle}{1 + \langle \mu^*, \psi \rangle} \right| t
\]
and \( k(f^\mu) = |t| \). Taking the supremum over \( x \in S(l^2) \), one derives the estimate (2.3).

To analyze the case of equality, observe that if \( \alpha(f^\mu) = 1 \), the second factor in the right-hand side of (2.3) equals 1, and this inequality is reduced to \( \omega(f^\mu) \leq |t| = k(f^\mu) \). But it was shown in [Kr2] that the equality \( \alpha(f^\mu) = 1 \) is the necessary and sufficient condition to have \( \omega(f) = k(f) \). This completes the proof of the theorem for the canonical disk \( \Delta^* \).

The case of a generic quasidisk \( D \) is investigated along the same lines using the results established by Milin [Mi] for the kernels and orthonormal systems in multiply connected domains. We apply these results to simply connected quasiconformal domains \( D^* \). Similar to (4.3),
\[
f^\mu(z) = z - \frac{1}{\pi} \int_D \mu(w) \left( \frac{1}{w-z} - \frac{1}{w} \right) dudv + O(||\mu^2||_\infty), \tag{4.6}
\]
but now the kernel of this variational formula is represented for \( z \) running over a subdomain of \( D^* \) bounded by the level line \( G(z, \zeta) = \rho(w) \) of the Green function of \( D^* \) in the form
\[
\frac{1}{w-z} = \sum_1^\infty P'_n(w) \varphi_n(z), \tag{4.7}
\]
where \( \varphi_n = \chi^n \) are given in (1.6) and \( P_n \) are well-defined polynomials; the degree of \( P_n \) equals \( n \). These polynomials satisfy
\[
\frac{1}{\pi} \int_D P'_m(z) P'_n(z) dx dy + \frac{1}{\pi} \int_{D^*} r'_m(z) r'_n(z) dxdy = \delta_{mn}, \tag{4.8}
\]
where the functions \( r_n \) are generated by
\[
R_{D^*}(z, \zeta) = \sum_1^\infty r_n(z) \varphi_n(\zeta)
\]
(see (1.4)) and in our case, due to what was mentioned in Section 1.2, vanish identically on \( D^* \).
Hence, (4.8) assumes the form
\[
\langle P'_m, P'_n \rangle_D = \pi \delta_{mn},
\]
which means that the polynomials \( P'_n(z)/\sqrt{\pi} \) form an orthonormal system in \( A^2_1(D) \). It is proved in [Mi] that this system is complete.

Noting that for any fixed \( z \) the equality (4.7) is extended holomorphically to all \( w \in D \), one derives from (4.6) and (4.8) the following generalization of (4.4). From (1.5),
\[
\frac{f^\mu(z) - f^\mu(\zeta)}{z - \zeta} = 1 - \frac{1}{\pi} \int_D \frac{\mu(w)dudv}{(w-z)(w-\zeta)} + O(||\mu^2||_\infty)
\]
and

\[-\log \frac{f^\mu(z) - f^\mu(\zeta)}{z - \zeta} = -\log \left[ 1 - \frac{1}{\pi} \iint_D \frac{\mu(w)dudv}{(w - z)(w - \zeta)} \right] + O(\|\mu^2\|_\infty) \]

\[= \frac{1}{\pi} \iint_D \frac{\mu(w)dudv}{(w - z)(w - \zeta)} + O(\|\mu^2\|_\infty) \]

\[= \frac{1}{\pi} \iint_D \mu(w) \sum_{1}^{\infty} P'_m(w) \varphi_m(z) \sum_{1}^{\infty} P'_n(w) \varphi(\zeta)dudv + O(\|\mu^2\|_\infty), \]

where the ratio \(O(\|\mu^2\|_\infty)/\|\mu^2\|_\infty\) is uniformly bounded on compact sets of \(\mathbb{C}\). Comparison with the representation

\[-\log \frac{f^\mu(z) - f^\mu(\zeta)}{z - \zeta} = \sum_{1}^{\infty} \beta_{mn} \varphi_m(z) \varphi_n(\zeta) \quad (\varphi_n = \chi^n) \]

yields

\[\hat{\beta}_{mn}(\mu) = -\frac{1}{\pi} \iint_D \mu(z) P'_m(w) P'_n(w)dudv + O(\|\mu^2\|_\infty), \quad (4.9)\]

which provides the representation of differentials of holomorphic functions \(\mu \mapsto \hat{\beta}_{mn}(\mu)\) on \(\text{Belt}(D)_1\) at the origin. Using the estimate \((3.2)\) ensuring the holomorphy of the corresponding functions \((3.1)\) on this ball, we get instead of \((4.5)\) that the differential of \(h_x(\mu)\) at zero is represented in the form

\[dh_x(0)\mu^* := \beta_{mn}(f^\mu) = -\frac{1}{\pi} \iint_\Delta \mu^*(z) \sum_{m,n=1}^{\infty} x_m x_n P'_m(z) P'_n(z)dxdy, \quad x = (x_n) \in S(\ell^2). \quad (4.10)\]

Now one can apply the same arguments as in the concluding part of the proof in the previous special case and get straightforwardly the estimate \((2.3)\) for the general case.

**Remark.** The equality \((4.5)\) yields that in the case \(D^* = \Delta^*\) the constant \((2.2)\) for every \(f \in \Sigma^0(D^*)\) is represented in the form

\[\alpha_D(f) = \sup_{x = (x_n) \in S(\ell^2)} \frac{1}{\pi \|\mu\|_\infty} \left| \iint_{|z| < 1} \mu(z) \sum_{m,n \geq 2} \sqrt{mn} x_m x_n z^{m+n-2}dxdy \right|, \quad (11.1)\]

where \(\mu\) is any extremal Beltrami coefficient in the equivalence class \([f]\).

### 5. Proof of Theorem 2.2

It follows from the proof of Theorem 2.1 that \(\kappa_D(f) = k(f)\) if and only if \(\alpha_D(f) = 1\). So it remains to establish the equality \((2.5)\), provided that the extremal extension of \(f\) to \(D\) is of Teichmüller type, with Beltrami coefficient \(\mu_0 = k|\psi_0|/\psi_0\), \(k = k(f)\).

Pick a sequence \(\{\psi_p = \omega_p^2\} \subset A^2(D)\) with \(\|\omega_p\|_{L^2(D)} = 1\) for which

\[\lim_{p \to \infty} \|(|\psi_0|/\psi_0, \psi_p)_{L^2(D)}\| = 1. \]

This sequence is convergent uniformly on compact sets in \(D\) to a holomorphic function \(\varphi \in A^2_2\).

If \(\varphi(z) \equiv 0\), the sequence \(\{\psi_p\}\) should be degenerate for the coefficient \(|\psi_0|/\psi_0\), which is impossible for Teichmüller extremal coefficients. Thus \(\varphi \neq 0\), and

\[|(|\psi_0|/\psi_0, \varphi)_{L^2(D)}| \leq \lim_{p \to \infty} |(|\psi_0|/\psi_0, \psi_p)_{L^2(D)}| = 1. \quad (5.1)\]

It remains to show that, under assumptions of the theorem, the left inequality in \((5.1)\) must be an equality (hence \(\psi_0 = \varphi\)). We may assume that \(f(0) = 0\) (passing if needed to \(f_1(z) = f(z) - f(0)\)).
Noting that in view of (2.2) and (4.10) each \( \varphi_p \) is represented in the form

\[
\varphi_p(z) = \frac{1}{\pi} \sum_{m,n=1}^{\infty} x_m^{(p)} x_n^{(p)} P_m'(z)P_n'(z) \quad \text{with} \quad \mathbf{x}^{(p)} = (x_n^{(p)}) \in S(l^2)
\]

and selecting if needed a subsequence from \( \mathbf{x}^{(p)} \) convergent in \( l^2 \) to \( \mathbf{x}^{(0)} = (x_n^{(0)}) \), one gets \( \lim_{p \to \infty} x_n^{(p)} = x_n^{(0)} \) for each \( n \geq 1 \), and by the above remark \( \mathbf{x}^{(0)} \neq 0 \). This implies that \( \varphi \) as the weak limit of \( \varphi_p \) is of the form

\[
\varphi(z) = \pi^{-1} \sum_{m,n=1}^{\infty} x_m^{(0)} x_n^{(0)} P_m'(z)P_n'(z),
\]

and \( \|x_0\|_\varphi = 1 \) (in view of maximality of \( \varphi \)). The variation (4.9) yields that the Grunsky coefficients of \( f^{\mu_0} = f \) and \( f^{t|\psi|/\varphi} \) are related by

\[
\beta_{mn}(f^{\mu_0}) = \beta_{mn}(f^{t|\psi|/\varphi}) + O(t^2), \quad t \to 0,
\]

and, letting \( t \to 0 \),

\[
\langle |\psi_0|/\psi_0, P_m'P_n' \rangle_\Delta = \langle |\varphi|/\varphi, P_m'P_n' \rangle_\Delta \quad \text{for all} \quad m,n \geq 1.
\]

Extension of these functionals to \( A_1(D) \) by Hahn-Banach yields

\[
\langle |\psi_0|/\psi_0 - |\varphi_1|/\varphi_1, \psi \rangle_\Delta = 0 \quad \text{for any} \quad \psi \in A_1(\Delta).
\]

As is well known (see, e.g., [GL], [Kr1]), such equality is impossible for the Teichmüller extremal coefficients unless \( \psi_0 = \varphi \). This completes the proof of the theorem.

6. Proof of Corollary 2.4 and of Theorem 2.7

**Proof of Corollary 2.4.** Theorem 2.1 and the inequality (2.6) (for \( D^* = \Delta^* \)) yield that the equality (2.1) holds for all \( f \in \Sigma^0 \) admitting the Teichmüller extremal extensions \( f^{\mu_0} \) across \( S^1 \).

The Schwarzians derivatives \( S_{f^{\mu_0}} \) of such \( f \) are Strebel’s points of the space \( T \) (cf., e.g., [GL], [Sl]).

For sufficiently small \( |t| \), the Schwarzians \( \varphi_t = S_{f^{t\mu_0}} \) determine by (2.8) the harmonic Beltrami coefficients of the Ahlfors-Weill extension of the maps \( f^{t\mu_0} \) across the unit circle \( S^1 = \partial \Delta^* \).

In view of the characteristic property of extremal Beltrami differentials, we have for any such \( \mu_0 \) the equality

\[
\nu_{\mu_0} = t \mu_0^* + \sigma_0, \quad \sigma \in A_1(\Delta)^-\perp,
\]

where

\[
A_1(\Delta)^-\perp = \{ \nu \in \text{Bel}(\Delta)_1 : \langle \nu, \psi \rangle_\Delta = 0 \quad \text{for all} \quad \psi \in A_1(\Delta) \}
\]

is the set of infinitesimally trivial Beltrami coefficients (see e.g., [GL], [Kr1]).

Since, due to [GL], the set of Strebel’s points are open and dense in Teichmüller spaces, the equality (2.9) (and its equivalent (2.7)) must hold for all points \( \varphi = S_f \) (with sufficiently small norms), which completes the proof of the corollary.

Note that by the same reasons the inequality (2.4) holds for all \( f \in \Sigma^0 \).

**Proof of Theorem 2.7.** Since all quantities in (2.11) are invariant under the action of the Möbius group \( \text{PSL}(2, \mathbb{C})/\pm 1 \), it suffices to use quasiconformal homeomorphisms \( f \) of the sphere \( \hat{\mathbb{C}} \) carrying the unit circle \( S^1 \) onto \( L \) whose Beltrami coefficients \( \mu_f(z) = \partial f/\partial \bar{z} \) are supported in the unit disk \( \Delta \) and which are hydrodynamically normalized near the infinite point, i.e., with restrictions \( f|\Delta^* \in \Sigma^0 \). Then the reflection coefficient \( q_L \) equals the minimal dilatation \( k(w^\mu) = \|\mu\|_\infty \) of quasiconformal extensions \( w^\mu \) of \( f|\Delta^* \) to \( \hat{\mathbb{C}} \), and Theorem 2.7 immediately follows from Corollaries 2.3 and 2.4.
8.2. For example, if $D$ is a very difficult problem. Their representation is known only for some special domains. The unit disk with $g(0) = 0$.

Hence, by (2.3), for any $n$,

$$
\kappa_d(f_n) \leq k \frac{k + \alpha(f)}{1 + \alpha(f)k} < k,
$$

completing the proof.

8. Examples

8.1. It follows from Theorem 2.2 (equality (2.5)) that $\kappa_d(f) < k(f)$ for any quasidisk $D \ni \infty$ and any $f \in \Sigma^0(D^*)$ having the Teichmüller extension $f^\mu$ to $\mathbb{C}$ with $\mu = k|\psi_0|/\psi_0$, where $\psi_0$ is holomorphic and has zeros of odd order in $D$. The simplest example of such $\psi_0$ is given by $\psi_0(z) = z^p$ with an odd integer $p \geq 1$.

To get other examples, one can pick $\psi_0 = g(z)^p$, where $g(z)$ is a conformal map of $D$ onto the unit disk with $g(0) = 0$, $g'(0) > 0$.

An explicit construction of the Riemann mapping functions of simply connected domains is a very difficult problem. Their representation is known only for some special domains.

8.2. For example, if $D_\mathcal{E}$ is the exterior of the ellipse $\mathcal{E}$ with the foci at $1$ and semiaxes $a, b$ ($a > b$), then the branch of the function

$$
\chi(z) = (z + \sqrt{z^2 - 1})/(a + b)
$$

positive for real $z > 1$ maps this exterior onto $\Delta^*$. A conformal map of the interior of this ellipse $D_\mathcal{E}$ onto the disk involves an elliptic function.

As is well known (see [Ne]), an orthonormal basis in the space

$$
A_2(D_\mathcal{E}) = \{\omega \in L_2(D_\mathcal{E}) : \omega \text{ holomorphic in } D_\mathcal{E}\}
$$

is formed by the polynomials

$$
P_n(z) = 2 \sqrt{\frac{n + 1}{\pi}} (r^{n+1} - r^{-n-1}) U_n(z),
$$

where $r = (a + b)^2$ and $U_n(z)$ are the Chebyshev polynomials of the second kind,

$$
U_n(z) = \frac{1}{\sqrt{1 - z^2}} \sin[(n + 1) \arccos z], \quad n = 0, 1, \ldots
$$

Using the Riesz-Fisher theorem, one obtains that each function $\psi \in A_2(D_\mathcal{E})$ is of the form (cf. [Kr2])

$$
\psi(z) = \sum_{n=0}^{\infty} x_n P_n(z), \quad x = (x_n) \in l^2,
$$
with \( \| \psi \|_{A_2} = \| x \|_{L^2} \).

By Theorem 2.2, a function \( f \in \Sigma^0(D^*_E) \) with Teichmüller extension \( f^\mu \) to \( D_E \) satisfies
\[
\varphi_{D^*_E}(f) = k(f)
\]
if and only if
\[
\mu(z) = k \sum_{0}^{\infty} x_n^0 P_n(z) / \sum_{0}^{\infty} x_n P_n(z)
\]
with some \( x_0^0 = (x_n^0) \in S(l^2) \). More generally, a function \( f \in \Sigma^0(D^*_E) \) obeys (8.1) if and only if any its extremal Beltrami coefficient \( \mu \in \text{Belt}(D_E) \) satisfies
\[
\sup_{x=(x_n) \in S(l^2)} \left| \left( \mu, \sum_{m,n \geq 0} x_m x_n P_m P_n \right)_{D_E} \right| = \| \mu \|_{\infty},
\]
taking the supremum over all \( x = (x_n) \in l^2 \) with \( \| x \| = 1 \). Note also that for every \( f \in \Sigma^0(D^*_E) \), its constant \( \alpha_{D^*_E}(f) \) is given explicitly by
\[
\alpha_{D^*_E}(f) = \sup_{x=(x_n) \in S(l^2)} \left| \left( \mu, \sum_{m,n \geq 0} x_m x_n P_m P_n \right)_{D_E} \right| = \| \mu \|_{\infty},
\]
taking any extremal \( \mu \) in the equivalence class \([f]\).

8.3. The expansion (1.6) contains a conformal map \( \chi : D^* \rightarrow \Delta^* \), while the basic quantity \( \alpha_D(f) \) is connected with conformal maps of the complementary quasidisk \( D \). The only known non-trivial example with a simple connection between these maps is the Cassini curve \( L = \{ z: |z^2 - 1| = c \} \) with \( c > 1 \). It is given in [HK]. Here \( \chi^{-1}(z) = \sqrt{1 + cz^2} \), and the branch of
\[
g(z) = z\sqrt{(c^2 - 1)/(c - z^2)}
\]
maps conformally the unit disk onto the interior of \( L \) with \( g(0) = 0 \), \( g'(0) > 0 \).

Using the function (8.2), one gets that for every univalent function \( f(z) \) in the domain \( D^* = \{ |z^2 - 1| > c \} \) with hydrodynamical normalization, its constant \( \alpha_D(f) \) is given, due to (4.11), by
\[
\alpha_D(f) = \sup_{x \in S(l^2)} \frac{1}{\pi \| \mu \|_{\infty}} \left| \int_{|z| < 1} \mu \circ g(z) \frac{g'(z)}{g(z)} \sum_{m+n \geq 2} \sqrt{mn} x_m x_n z^{m+n-2} dx dy \right|,
\]
taking again an extremal Beltrami coefficient \( \mu \) in the class \([f]\).

9. GRUNSKY NORM AND COMPLEX HOMOTOPY

Every function \( f \in \Sigma(D^*) \) generates a holomorphic homotopy by
\[
f(z,t) = f_t(z) := tf \circ g^{-1}[tg(z)] : D^* \times \Delta \rightarrow \hat{C},
\]
where \( g \) maps conformally \( D^* \) onto \( D^* \) with \( g(\infty) = \infty \), \( g'(\infty) > 0 \). This homotopy satisfies \( f(z,0) = z \), \( f(z,1) = f(z) \) and \( f(z,t) = z + b_0 t + b_1 t^2 z^{-1} + \ldots \) near \( z = \infty \). The curves \( \{ z = -\log |t| \} \) are the level lines of Green’s function \( g_D(z,\infty) = -\log |g(z)| \) of \( D^* \).

Consider the Schwarzians \( S_{f_t}(z) = S_f(z,t) \). Then the map \( t \mapsto S_f(\cdot,t) \) is holomorphic in \( t \) for any \( z \in D^* \) and, due to the well-known properties of the functions with sup norm depending holomorphically on complex parameters, this pointwise map induces a holomorphic map
\[
\chi_f : t \mapsto S_f(\cdot,t), \quad \chi_f(t) = \chi_f(0) + t \chi'_f(0) + \ldots, \quad \chi_f(0) = S_g^{-1},
\]
of the disk \( \{|t| < 1\} \) into the space \( T \). We call a level \( r = |t| > 0 \) \textbf{noncritical} if \( \chi'_f(re^{i\eta}) \neq 0 \) for any \( \eta \in [0,2\pi] \). If \( \chi'_f(t_0) = 0 \) then \( \chi'_{f_\eta}(r) = 0 \) for \( \eta = -t_0/|t_0| \). In the simplest case of the disk \( \Delta^* \),
\[
f_t(z) = tf(z/t) = z + b_0 t + b_1 t^2 z^{-1} + \ldots
\]
and $S_f(t) = t^{-2} S_f(z/t)$ for all $|t| < 1$; then the map (9.1) takes the form

$$
\chi_f(t) = \frac{\chi''(0)}{2!} t^2 + \frac{\chi'''(0)}{3!} t^3 + \ldots ,
$$

and the Grunsky coefficients of $f$ are homotopically homogeneous:

$$
\alpha_{mn}(f_t) = \alpha_{mn}(f) t^{m+n} \quad \text{for all} \quad m, n \geq 1. \quad (9.3)
$$

The homotopy disk

$$
\Delta(S_f) := \chi_f(\Delta) = \{ S_f : |t| < 1 \}
$$

has cuspidal singularities in the critical points of $\chi_f$.

For any quasidisk $D^*$ containing the infinite point, we have

**Theorem 9.1.** Let the homotopy function $f_r(z)$ of $f \in \Sigma(D^*)$ given by (9.1) satisfy

$$
\kappa_{D^*}(f_r) = k(f_r) \quad (9.4)
$$

for a noncritical level $\rho \in (0, 1)$. Then

$$
\kappa_{D^*}(f_r) = k(f_r) \quad \text{for all} \quad r < \rho. \quad (9.5)
$$

This theorem answers some questions stated by R. Kühnau in [KK2]. It also has some other interesting applications. Apart from some special cases, there is no connection between the defining constant holomorphic sectional curvature $\kappa$ for a noncritical level $\rho$ and the Grunsky coefficients of $f$, of a map $f$ and its homotopies $f_r$. Theorems 2.2 and 9.1 give the conditions ensuring the evenness of zeroes of $\psi$ and $\psi_r$ (cf. [Kr7]).

The proof of Theorem 9.1 essentially involves the curvature properties of the Kobayashi metric of universal Teichmüller space $T$. We first recall some background facts underlying the proof.

We shall use the following strengthening of the fundamental Royden-Gardiner theorem given in [Ki3].

**Proposition 9.2.** The differential (infinitesimal) Kobayashi metric $\kappa_T(\varphi, v)$ on the tangent bundle $T(T)$ of the universal Teichmüller space $T$ is logarithmically plurisubharmonic in $\varphi \in T$, equals the canonical Finsler structure $F_T(\varphi, v)$ on $T(T)$ generating the Teichmüller metric of $T$ and has constant holomorphic sectional curvature $\kappa_C(\varphi, v) = -4$ on $T(T)$.

The *generalized Gaussian curvature* $\kappa_\lambda$ of an upper semicontinuous Finsler metric $ds = \lambda(t)|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by

$$
\kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2}, \quad (9.6)
$$

where $\Delta$ is the generalized Laplacian

$$
\Delta \lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta}) d\theta - \lambda(t) \right\}
$$

(provided that $-\infty \leq \lambda(t) < \infty$). Similar to $C^2$ functions, for which $\Delta$ coincides with the usual Laplacian, one obtains that $\lambda$ is subharmonic on $\Omega$ if and only if $\Delta \lambda(t) \geq 0$; hence, at the points $t_0$ of local maxima of $\lambda$ with $\lambda(t_0) > -\infty$, we have $\Delta \lambda(t_0) \leq 0$.

The sectional *holomorphic curvature* of a Finsler metric on a complex Banach manifold $X$ is defined in a similar way as the supremum of the curvatures (9.6) over appropriate collections of holomorphic maps from the disk into $X$ for a given tangent direction in the image. The holomorphic curvature of the Kobayashi metric $\kappa_X(x, v)$ of any complete hyperbolic manifold $X$ satisfies $\kappa_X \geq -4$ at all points $(x, v)$ of the tangent bundle $T(X)$ of $X$, and for the Carathéodory metric $C_X$ we have $\kappa_C(x, v) \leq -4$ (cf., e.g., [AP], [Di], [Ko]).

It was established in [EE] that the metric $\kappa_T(\varphi, v) = F_T(\varphi, v)$ is Lipschitz continuous on $T$ (in its Bers’ embedding).
We shall deal with subharmonic circularly symmetric (radial) metrics $\lambda(t)|dt|$ on a disk \( \{|t| < a\} \), i.e., such that $\lambda(t) = \lambda(|t|)$. Any such function $\lambda(t)$ is monotone increasing in $r = |t|$ on $[0,a]$ and convex with respect to $\log r$, has one-sided derivatives for each $r < a$ (in particular $u'(0) \geq 0$), and $ru'(r)$ is monotone increasing (see, e.g., [Ro]).

**Proof of Theorem 9.1.** First consider the more simple case of the circular disk $D^* = \Delta^*$ which we use to illustrate the main ideas.

The relations (1.4), (9.3), (9.4) imply that the stretching $f_\rho$ possesses a Teichmüller extension to $\Delta$ defined by a quadratic differential $\psi_\rho \in A^2_1(\Delta)$ so that $\mu_{f_\rho}(z) = k(f_\rho)|\psi_\rho(z)|/\psi_\rho(z)$ for $|z| < 1$ and

$$\kappa(f_\rho) = k(f_\rho) = \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(f) \rho^{m+n} x_m^0 x_n^0 \right|$$

(this common value is attained on some point $x^0 = (x^0_n) \in \Omega(t^2)$). Indeed, the corresponding function (3.1) for this $x^0$ (with $\beta_{mn} = \sqrt{mn} \alpha_{mn}$) being restricted to the disk $\Delta(S_f)$ assumes the form

$$\widetilde{h}_{x^0}(t) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(f) x_m^0 x_n^0 \rho^{m+n}, \quad (9.7)$$

and by (9.4),

$$|\widetilde{h}_{x^0}(\rho)| = \kappa(f_\rho) = k(f_\rho). \quad (9.8)$$

The series (9.7) defines a holomorphic selfmap of the disk \( \{|t| < 1\} \).

Noting that the homotopy $f(z,t)$ is a holomorphic motion of the disk $\Delta^*$ parametrized by $t \in \Delta$ and applying to it the basic lambda-lemma for these motions, one obtains that each fiber map $f_t(z) = f(z,t)$ extends to a quasiconformal automorphism of the whole sphere $\hat{\Sigma}$ so that the Beltrami coefficient $\mu(z,t) = \overline{\partial}f_t/\partial f_t \in \text{Belt}(\Delta)_1$ is a $L_{\infty}$-holomorphic function of $t \in \Delta$ (and generically not extremal). If the derivative of the map (9.2) vanishes at some point $t_0$, $\chi_f(t_0) = 0$, then also $\frac{d}{dt}\mu(z,t)|_{t=t_0} = 0$, and the holomorphic dependence of the function (9.7) on $S_f$ and on $\mu(\cdot,t)$ implies

$$\widetilde{h}_{x^0}(t_0) = 0.$$ 

Hence, all critical points of the map (9.2) are simultaneously critical for the function (9.7) (though $\widetilde{h}_{x^0}$ can have extra critical points which are regular for $\chi_f$).

We apply the functions (3.1) to the explicit construction of some subharmonic Finsler metrics on holomorphic disks $\Omega = g(\Delta) \subset \text{T}$, pulling back the hyperbolic metric $\lambda_\Delta(\Delta)|dt| = |dt|/(1-|t|^2)$ of $\Delta$ (assuming that the Grunsky coefficients $\alpha_{mn}$ are given). In fact, we shall use these metrics only on the homotopy disk $\Delta(S_f)$ and on geodesic Teichmüller disks passing through the origin and points of $\Delta(S_f)$. These metrics are dominated by the Kobayashi-Teichmüller metric of the space $\text{T}$. The functions

$$h_{x,g}(t) := h_{x}(S_f \circ g(t)) = \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn}(S_f \circ g(t)) x_m x_n, \quad x \in S(t^2),$$

define holomorphic maps $\Delta \to \Omega \to \Delta$ and conformal metrics $\lambda_{h_{x,g}}(t)|dt|$ with

$$\lambda_{h_{x,g}}(t) = |h_{x,g}(t)|/(1-|h_{x,g}(t)|^2), \quad t \in \Delta$$

of Gaussian curvature $-4$ at noncritical points. We take the upper envelope of these metrics

$$\tilde{\lambda}_{x}(t) = \sup\{\lambda_{h_{x,g}}(t) : x \in S(t^2)\} \quad (9.9)$$

and its upper semicontinuous regularization

$$\lambda_{x}(t) = \lim_{t'\to t} \tilde{\lambda}_{x}(t'),$$
getting a logarithmically subharmonic metric on $\Omega$. In fact, one can show, similarly to Theorem 2.5, that this regularization does not change (increase) $\lambda_\infty$, i.e. $\lambda_\infty = \tilde{\lambda}_\infty$, in view of continuity.

Now recall that a conformal metric $\lambda(t)|dt|$ is called supporting for $\lambda(t)|dt|$ at a point $t_0$ if $\lambda_\infty(t_0) = \lambda_0(t_0)$ and $\lambda_0(t) < \lambda_\infty(t)$ for all $t \not\in \{t_0\}$ from a neighborhood of $t_0$.

**Lemma 9.3.** If a conformal metric $\lambda$ in a domain $\Omega$ has at any its noncritical point $t_0$ a supporting subharmonic metric $\lambda_0$ of Gaussian curvature at most $-4$, then $\lambda$ is subharmonic on $\Omega$ and its generalized Gaussian curvature also is at most $-4$ in all noncritical points.

**Proof.** Since the space $\mathcal{B}(D^*)$ is dual to $A_1(D^*)$, the sequences $\{h_\infty(\varphi)\}$ are convergent, by the Alaoglu-Bourbaki theorem, in weak* topology to holomorphic functions $T \to \Delta$. This yields that the metric (9.9) has a supporting metric $\lambda_0(t)$ in a neighborhood $U_0$ of any noncritical point $t_0 \in \Delta$, which means that $\lambda_\infty(t_0) = \lambda_0(t_0)$ and $\lambda_0(t) < \lambda_\infty(t)$ for all $t \not\in \{t_0\}$. Hence, for sufficiently small $r > 0$,

$$\frac{1}{r^2} \left( \int_0^{2\pi} \log \lambda_\infty(t_0 + re^{i\theta}) \, d\theta - \lambda_\infty(t_0) \right) \geq \frac{1}{r^2} \left( \int_0^{2\pi} \log \lambda_0(t_0 + re^{i\theta}) \, d\theta - \lambda_0(t_0) \right);$$

and $\Delta \log \lambda_\infty(t_0) \geq \Delta \log \lambda_0(t_0)$. Since $\lambda_\infty(t_0) = \lambda_0(t_0)$, one gets

$$- \frac{\Delta \log \lambda_\infty(t_0)}{\lambda_\infty(t_0)^2} \leq - \frac{\Delta \log \lambda_0(t_0)}{\lambda_0(t_0)^2} \leq -4,$

which completes the proof of the lemma.

Note that the inequality $\kappa_\lambda \leq -4$ is equivalent to

$$\Delta \log \lambda \geq 4\lambda^2,$$

where $\Delta$ again means the generalized Laplacian. Letting $u = \log \lambda$, one gets $\Delta u \geq 4e^{2u}.

In particular, all this holds for the metrics $\lambda_\infty$ (cf. [Kru]). Indeed, the space $\mathcal{B}(D^*)$ is dual to $A_1(D^*)$, thus by the Alaoglu-Bourbaki theorem the family $\{h_\infty(\varphi)\}$, $\varphi = S_f \in T$ is compact in weak* topology. The limit functions of its subsequences are holomorphic maps of $T$ and into the unit disk. This yields that each of the metrics (8.9) has a supporting metric in a neighborhood $U_0$ of any noncritical point $t_0 \in \Delta$.

We proceed to the proof of the theorem and note that in the case $\Omega = \Delta(S_f)$ the enveloping metric (9.9) and both norms $\kappa(f_t)$ and $\kappa'(f_t)$ are circularly symmetric in $t$. We determine on this disk also another circularly symmetric subharmonic conformal metric majorated by $\lambda_\infty$.

Namely, the map (9.7) generates the metric

$$\lambda_{\tilde{h}_{\kappa_0}}(t) = |\tilde{h}'_{\kappa_0}(t)|/(1 - |\tilde{h}_{\kappa_0}(t)|^2)$$

of Gaussian curvature $-4$ on $\Delta$ (again at noncritical points), which is supporting for $\lambda_\infty(t)$ at $t = \rho$.

Replacing $x^0$ by the points $x^0 = (\epsilon x^0) \in S(t^2)$ with $|\epsilon| = 1$, one gets the corresponding subharmonic metrics $\lambda_{\tilde{h}_{\kappa_0}}(t) = |\tilde{h}'_{\kappa_0}(t)|/(1 - |\tilde{h}_{\kappa_0}(t)|^2)$. Take their envelope

$$\lambda_0(t) := \sup_{\epsilon} \lambda_{\tilde{h}_{\kappa_0}}(t);$$

its curvature also is at most $-4$ in both supporting and holomorphic senses.

Our goal now is to prove the equility

$$\lambda_\infty(t) = \lambda_d(S_f, v),$$

where $\lambda_d$ is the restriction to $\Delta(S_f)$ of the infinitesimal Kobayashi-Teichmüller metric on the space $T$ and $v$ is a tangent vector to the Teichmüller disk touching $\Delta(S_f)$ at the point $t$. We apply Minda’s maximum principle given by
Lemma 9.4. [Min] If a function \( u : D \to [-\infty, +\infty) \) is upper semicontinuous in a domain \( \Omega \subset \mathbb{C} \) and its generalized Laplacian satisfies the inequality \( \Delta u(z) \geq Ku(z) \) with some positive constant \( K \) at any point \( z \in D \), where \( u(z) > -\infty \), and if

\[
\limsup_{z \to \zeta} u(z) \leq 0 \quad \text{for all } \zeta \in \partial D,
\]

then either \( u(z) < 0 \) for all \( z \in D \) or else \( u(z) = 0 \) for all \( z \in \Omega \).

First observe that

\[
\lambda_\omega(\rho) = \lambda_d(S_{f_\rho}, v),
\]

(9.13)

which follows from the reconstruction lemma for Grunsky norm.

Lemma 9.5. [Kr4] On any extremal Teichmüller disk \( \Delta(\mu_0) = \{ \phi_T(t\mu_0) : t \in \Delta \} \) (and its isometric images in \( T \)), we have the equality

\[
\tanh^{-1}[\varphi(f^{t\mu_0})] = \int_0^r \lambda_\omega(t)dt.
\]

(9.14)

Indeed, assuming \( \lambda_\omega(\rho) < \lambda_d(S_{f_\rho}, v) \), one would have from semicontinuity of both sides that such strong inequality must hold in a neighborhood of \( S_{f_\rho} \) in \( T \), but this violates the equalities (9.4) and (9.14) for \( r = \rho \) (along the corresponding Teichmüller disk). This proves (9.13).

The equality (9.13) yields that each of the metrics (9.11), (9.12) and \( \lambda_d(S_{f_\rho}, v) \) is supported at \( t = \rho \) by the same metric (9.10). Take the annulus \( \mathcal{A}_{r_1,r_2} = \{ r_1 < \rho < r_2 \} \) with \( r_1 < \rho < r_2 \), which does not contain the critical points of function (9.7), and put

\[
M = \{ \sup \lambda_d(t) : t \in \mathcal{A}_{r_1,r_2} \};
\]

then \( \lambda_d(t) + \lambda_0(t) \leq 2M \). Consider on this annulus the function

\[
u(r) = \log \frac{\lambda_0(r)}{\lambda_d(r)}.
\]

Then (cf. [Min], [Kr4]),

\[
\Delta u(r) = \log \lambda_0(r) - \lambda_d(r) = 4(\lambda_0^2(r) - \lambda_d^2(r)) \geq 8M(\lambda_0(r) - \lambda_d(r)),
\]

and the elementary estimate \( M \log(t/s) \geq t - s \) for \( 0 < s \leq t < M \) (with equality only for \( t = s \)) implies

\[
M \log \frac{\lambda_0(r)}{\lambda_d(r)} \geq \lambda_0(r) - \lambda_d(r),
\]

and hence, \( \Delta u(t) \geq 4M^2u(t) \).

One can apply Lemma 9.4 which implies, in view of the equality (9.13), that \( \lambda_0(r) = \lambda_\omega(r) = \lambda_d(r) \) for all \( r \in [r_1, r_2] \) (equivalently, \( \varphi_f(r) = k_f(r) \)).

Now one can fix \( \rho < r' < r_2 \) and compare the metrics \( \lambda_\omega \) and \( \lambda_d \) on the disk \( \{|t| < r'\} \) in a similar way, which yields the desired equalities \( \lambda_\omega(r) = \lambda_d(r) \) and \( \varphi_f(r) = k_f(r) \) for all \( r \leq \rho \), completing the proof for the disk \( \Delta^* \).

The proof for the functions on generic quasidisks \( D^* \) follows the same lines using the homotopy (9.2).
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