OPERATOR-VALUED DYADIC HARMONIC ANALYSIS
BEYOND DOUBLING MEASURES

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Abstract. We obtain a complete characterization of the weak-type $(1, 1)$ for Haar shift operators in terms of generalized Haar systems adapted to a Borel measure $\mu$ in the operator-valued setting. The main technical tool in our method is a noncommutative Calderón-Zygmund decomposition valid for arbitrary Borel measures.

1. Introduction

We say that $\Phi = \{\phi_Q\}_{Q \in \mathcal{D}}$ is a generalized Haar system in $\mathbb{R}^d$ adapted to a locally finite Borel measure $\mu$ and a dyadic lattice $\mathcal{D}$ if the following conditions hold:

(a) For every $Q \in \mathcal{D}$, $\text{supp}(\phi_Q) \subset Q$.

(b) If $Q', Q \in \mathcal{D}$ and $Q' \subset Q$, then $\phi_Q$ is constant on $Q'$.

(c) For every $Q \in \mathcal{D}$, $\int_{\mathbb{R}^d} \phi_Q \, d\mu = 0$.

(d) For every $Q \in \mathcal{D}$, either $\|\phi_Q\|_{L^2(\mu)} = 1$ or $\phi_Q \equiv 0$ and $\mu(Q) = 0$.

If the vanishing integral condition (c) is not imposed, the Haar system is said to be noncancellative. Let $\Phi = \{\phi_Q\}_{Q \in \mathcal{D}}$ and $\Psi = \{\psi_Q\}_{Q \in \mathcal{D}}$ be two nonnecessarily cancellative generalized Haar systems in $\mathbb{R}^d$. A Haar shift operator of complexity $(r, s) \in \mathbb{N} \times \mathbb{N}$ is an operator of the form

$$(1.1) \quad III_{r,s}f(x) = \sum_{Q \in \mathcal{D}} \sum_{R \in \mathcal{D}_k(Q)} \alpha_{Q,R,S}^r(f, \phi_R)\psi_S(x), \quad \text{with} \quad \sup_{Q,R,S} |\alpha_{Q,R,S}^r| < \infty,$$

where $\langle f, g \rangle = \int_{\mathbb{R}^d} fg \, d\mu$ and $\mathcal{D}_k(Q)$, for $k \in \mathbb{N}$, denotes the family of $k$-dyadic descendants of $Q$: the partition of $Q$ into subcubes $R \in \mathcal{D}$ of side-length $\ell(R) = 2^{-k}\ell(Q)$. Several objects in dyadic harmonic analysis have the general form (1.1), including Haar multipliers, dyadic paraproducts, the dyadic model of the Hilbert transform and their adjoints. Haar shift operators have served as important tools in the study of many different problems in harmonic analysis since the form (1.1) is a fruitful source of models of Calderón-Zygmund operators. In particular, in

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the case where \( \mu \) is the Lebesgue measure, Calderón-Zygmund operators can be expressed as weak limits of certain averages of cancellative Haar shift operators and paraproducts [4] and are pointwise dominated by positive dyadic operators, which are Haar shift operators relative to noncancellative Haar systems [1].

The boundedness behavior of Haar shift operators with respect to arbitrary locally finite Borel measures in the commutative setting was studied in [5]. There the authors characterize the weak-type \((1, 1)\) of such operators. In this paper we extend the scope of this result to the setting of semicommutative \(L_p\) spaces. The main technique that we will use in our approach is a generalization of the Calderón-Zygmund decomposition constructed in [7] which is valid for operator-valued functions, in the spirit of the Calderón-Zygmund decomposition constructed in [6].

We will work in the following framework: consider a pair \((\mathcal{M}, \nu)\) where \(\mathcal{M}\) is a von Neumann algebra and \(\nu\) is a normal semifinite faithful trace on \(\mathcal{M}\), and let \(\mu\) be a locally finite Borel measure on \(\mathbb{R}^d\). Let \(\mathcal{A}_B\) be the algebra of essentially bounded \(\mathcal{M}\)-valued functions

\[
\mathcal{A}_B = \left\{ f : \mathbb{R}^d \to \mathcal{M} \mid f \text{ strongly measurable s.t. } \operatorname{ess sup}_{x \in \mathbb{R}^d} \|f(x)\|_{\mathcal{M}} < \infty \right\}
\]

equipped with the normal, semifinite, faithful \((n.s.f.)\) trace \(\tau(f) = \int_{\mathbb{R}^d} \nu(f) \, d\mu\). The weak-operator closure \(\mathcal{A}\) of \(\mathcal{A}_B\) is a von Neumann algebra isomorphic to \(L_{\infty}(\mathbb{R}^d, \mu) \overline{\otimes} \mathcal{M}\). Given a rearrangement invariant quasi-Banach function space \(X\), let us write \(X(\mathcal{M}, \nu)\) and \(X(\mathcal{A})\) for their associated noncommutative symmetric spaces. In particular \(L_p(\mathcal{M})\) and \(L_p(\mathcal{A})\) denote the noncommutative \(L_p\) spaces associated to the pairs \((\mathcal{M}, \nu)\) and \((\mathcal{A}, \tau)\). It can be readily seen that for \(1 \leq p < \infty\) the noncommutative \(L_p\) space \(L_p(\mathcal{A})\) is isometric to the Bochner \(L_p\) space \(L_p(\mathbb{R}^d, \mu; L_p(\mathcal{M}))\). The lattices of projections are denoted by \(\mathcal{P}(\mathcal{M})\) and \(\mathcal{P}(\mathcal{A})\), while \(1_{\mathcal{M}}\) and \(1_{\mathcal{A}}\) stand for the unit elements and \(\mathcal{M}'\) and \(\mathcal{A}'\) stand for their respective commutants. For a more detailed discussion on noncommutative \(L_p\) spaces we refer to [6] and the references therein. The reader unfamiliar with the theory of noncommutative \(L_p\) spaces may think of \(\mathcal{M}\) as the algebra \(\mathcal{B}(\ell_2^n)\) of \(n \times n\) matrices equipped with the standard trace \(T\), thereby recovering the classical Schatten \(p\)-classes. The reader should take into account that, with this setting in mind, we provide estimates uniform on \(n\).

Before stating our results let us introduce some notation first. By \((E_k)_{k \in \mathbb{Z}}\) we will denote the family of conditional expectations associated to \(\mathcal{D}_k\)—the dyadic cubes \(Q\) of side-length \(\ell(Q) = 2^{-k}\)—and write \(D_k\) for the corresponding martingale difference operators. The tensor product \(E_k \otimes id_{\mathcal{M}}\) acting on \(\mathcal{A}\) will also be denoted by \(E_k\), which yields a filtration \((\mathcal{A}_k)_{k \in \mathbb{Z}}\) on \(\mathcal{A}\). We thus have that

\[
E_k(f) = f_k = \sum_{Q \in \mathcal{D}_k} \langle f \rangle_Q 1_Q,
\]

\[
D_k(f) = df_k = \sum_{Q \in \mathcal{D}_k} \left( \langle f \rangle_Q - \langle f \rangle_{\hat{Q}} \right) 1_Q,
\]

which correspond to projections to the class of operators constant at scale \(\mathcal{D}_k\). Here \(1_Q\) denotes the characteristic function of \(Q\), \(\langle f \rangle_Q = \mu(Q)^{-1} \int_Q f \, d\mu\) and \(\hat{Q}\) is the dyadic parent of \(Q\): the only dyadic cube that contains \(Q\) with twice its side-length.

We will construct the Calderón-Zygmund decomposition for functions in the class

\[
\mathcal{A}_{+, K} = \{ f : \mathbb{R}^d \to \mathcal{M} \mid f \geq 0, \supp_{\mathbb{R}^d}(f) \text{ is compact} \},
\]
whose span is dense in $L_1(\mathcal{A})$. Here, $\text{supp}_{\mathbb{R}^d}(f)$ stands for the support of $f$ as an operator-valued function, as opposed to its support projection as an element of a von Neumann algebra. As the Calderón-Zygmund decomposition introduced in [7]—which is suitable for the Lebesgue measure and doubling measures—the Calderón-Zygmund decomposition presented here is comprised of diagonal and off-diagonal terms, reflecting the lack of commutativity in the operator-valued framework. Taking $i \lor j = \max\{i, j\}$ and $i \land j = \min\{i, j\}$ for $i, j \in \mathbb{Z}$ we have:

**Theorem A.** Let $f \in \mathcal{A}_{+,K}$ and let $\lambda > 0$. Then, there exist a family of pairwise disjoint projections $(p_k)_{k \in \mathbb{Z}}$ adapted to $(A_k)_{k \in \mathbb{Z}}$ and a projection $q := 1_{\mathcal{A}} - \sum_k p_k \in \mathcal{P}(\mathcal{A})$ such that $f$ can be decomposed as $f = g + b + \beta$, where each term has a diagonal and an off-diagonal part given by

- $g = g_\Delta + g_{\text{off}}$, where
  \[ g_\Delta = qfq + \sum_{k \in \mathbb{Z}} E_{k-1}(p_kfp_k), \]
  \[ g_{\text{off}} = (1_{\mathcal{A}} - q)f + q(1_{\mathcal{A}} - q) + \sum_{i \neq j} E_{i \lor j-1}(p_if_{i \lor j}p_j); \]

- $b = b_\Delta + b_{\text{off}}$, where
  \[ b_\Delta = \sum_{k \in \mathbb{Z}} p_k(f - f_k)p_k, \quad b_{\text{off}} = \sum_{i \neq j} p_i(f - f_{i \lor j})p_j; \]

- $\beta = \beta_\Delta + \beta_{\text{off}}$, where
  \[ \beta_\Delta = \sum_{k \in \mathbb{Z}} D_k(p_kfp_k), \quad \beta_{\text{off}} = \sum_{i \neq j} D_{i \lor j}(p_if_{i \lor j}p_j). \]

The diagonal terms satisfy the classical properties
(a) $g_\Delta \in L_1(\mathcal{A}) \cap L_2(\mathcal{A})$, with
\[ \|g_\Delta\|_{L_1(\mathcal{A})} = \|f\|_{L_1(\mathcal{A})}, \quad \|g_\Delta\|^2_{L_2(\mathcal{A})} \leq 39\lambda \|f\|_{L_1(\mathcal{A})}; \]

(b) $b_\Delta = \sum_{k \in \mathbb{Z}} b_k$, with $\int_{\mathbb{R}^d} b_k \, d\mu = 0$, and satisfies the estimate
\[ \|b_\Delta\|_{L_1(\mathcal{A})} = \sum_{k \in \mathbb{Z}} \|b_k\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}; \]

(c) $\beta_\Delta = \sum_{k \in \mathbb{Z}} \beta_k$, with each $\beta_k$ a $k$-th martingale difference, and is such that
\[ \|\beta_\Delta\|_{L_1(\mathcal{A})} \leq \sum_{k \in \mathbb{Z}} \|\beta_k\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}. \]

The off-diagonal terms are such that
(d) $g_{\text{off}}$ decomposes as $g_{\text{off}} = \sum_{k \in \mathbb{Z}, h \geq 1} g_{k,h}$, where $g_{k,h}$ is the $(k+h)$-th martingale difference $g_{k,h} = D_{k+h}(p_kf_kp_k + q_{k+h}f_{k+h}p_k)$, and satisfies the estimate
\[ \sup_{h \geq 1} \sum_{k \in \mathbb{Z}} \|g_{k,h}\|^2_{L_2(\mathcal{A})} \leq 16\lambda \|f\|_{L_1(\mathcal{A})}; \]

(e) $b_{\text{off}} = \sum_{k \in \mathbb{Z}, h \geq 1} b_{k,h}$, where $b_{k,h} = p_k(f - f_{k+h})p_k + p_{k+h}(f - f_{k+h})p_k$, $\int_{\mathbb{R}^d} b_{k,h} \, d\mu = 0$ and
\[ \sum_{k \in \mathbb{Z}} \|b_{k,h}\|_{L_1(\mathcal{A})} \leq 8(h + 1)\|f\|_{L_1(\mathcal{A})}; \]
(f) $\beta_{\text{off}} = \sum_{k \in \mathbb{Z}, h \geq 1} \beta_{k,h}$, where $\beta_{k,h} = D_{k+h}(p_k f_{k+h} p_{k+h} + p_{k+h} f_{k+h} p_k)$ and
\[ \sum_{k \in \mathbb{Z}} \| \beta_{k,h} \|_{L_1(A)} \leq 8(h + 1) \| f \|_{L_1(A)}. \]

Observe that the diagonal terms satisfy estimates similar to those of their commutative counterparts found in [5]. However, in contrast to the classical setting, there are additional difficulties in proving the estimates even for diagonal terms due to the noncommutativity of $A$. In particular, the estimates of $g_{\Lambda}$ are proved in a different way and only hold for $p \leq 2$. In addition, the fact that $\mu$ is allowed to be nondoubling brings other difficulties not present in [7]. On the other hand, at first glance the off-diagonal estimates in (d), (e) and (f) seem to fail as hinted in [7]. However, the estimates at hand will prove to be sufficient for our purposes, as the operators under consideration are localized in a sense stronger than in [6,7]. In that respect, one can think of our result as a partial answer to the question posed in [6] about the existence of a Littlewood-Paley theory for nondoubling measures in the semicommutative context.

Let $\Phi = \{ \phi_Q \}_{Q \in \mathcal{D}}$ and $\Psi = \{ \psi_Q \}_{Q \in \mathcal{D}}$ be two not necessarily cancellative generalized Haar systems. A commuting Haar shift operator is an $L_2(A)$ bounded operator of the form
\[
\mathbb{I}_{r,s} f(x) = \sum_{Q \in \mathcal{D}} \sum_{R \in \mathcal{D}_r(Q)} \alpha^Q_{R,S}(f, \phi_R) \psi_S(x), \quad \sup_{Q,R,S} \| \alpha^Q_{R,S} \|_M < \infty.
\]

Here, the symbols $\alpha^Q_{R,S}$ lie in $M \cap M'$, the center of $M$. Notice that in this definition the pairing $\langle f, g \rangle = \int_{\mathbb{R}} f g d\mu$ is in fact a partial trace and hence operator-valued. Our second result determines conditions for which the weak-type $(1,1)$ for these operators hold.

**Theorem B.** Let $\mathbb{I}_{r,s}$ be given as in (1.2). Assume that $\mathbb{I}_{r,s}$ satisfies the restricted local vector-valued $L_2$ estimate
\[
\int_{\mathbb{R}} \| \mathbb{I}_{r,s}^Q(1_Q)(x) \|_M^2 d\mu(x) \leq C\mu(Q_0),
\]
uniformly over $Q_0 \in \mathcal{D}$. Here
\[
\mathbb{I}_{r,s}^Q f(x) := \sum_{Q \in \mathcal{D}(Q_0)} \sum_{R \in \mathcal{D}_r(Q)} \alpha^Q_{R,S}(f, \phi_R) \psi_S(x),
\]
where $\mathcal{D}(Q)$ denotes the family of all dyadic subcubes of $Q$ including $Q$ itself. If
\[
\| \Phi, \Psi, r, s \| := \sup_{Q \in \mathcal{D}} \{ \| \phi_R \|_{L_\infty(\mu)} \| \psi_S \|_{L_1(\mu)} : R \in \mathcal{D}_r(Q), S \in \mathcal{D}_s(Q) \} < \infty,
\]
then $\mathbb{I}_{r,s}$ maps $L_1(A)$ continuously into $L_{1,\infty}(A)$ (and consequently $L_p(A)$ into $L_p(A)$ for $1 < p < \infty$).

**Remark 1.5.** A testing argument with simple functions is used in [5] to show that the condition (1.4) is also necessary when the symbols are all nonzero. One can show that this is also the case in the present setting by following similar ideas, and hence they will not be repeated here.
Remark 1.6. As in the commutative case, if the Haar systems \( \Phi = \{ \phi_Q \}_{Q \in \mathcal{D}} \) and \( \Psi = \{ \psi_Q \}_{Q \in \mathcal{D}} \) are cancellative, orthogonality arguments may be used to verify that the condition (1.3) and the \( L^2 \) boundedness of \( \mathbb{M}_{r,s} \) are satisfied.

The condition (1.4) may be interpreted as a certain restriction on the measure \( \mu \) in terms of its degeneracy over generations of dyadic cubes. The resulting class of measures depends strongly on the Haar shift operator in question. For some operators the associated class of measures is shown to be strictly bigger than the doubling class, but nevertheless different from the class of measures of polynomial growth, for which nonstandard Calderón-Zygmund theories are available; see [5] and the references therein.

2. The Calderón-Zygmund decomposition

This section is devoted to the proof of Theorem A. First, some reductions are in order. For simplicity we will assume that \( \mu(\mathbb{R}^d) = \infty \) and that the dyadic lattice \( \mathcal{D} \) has no quadrants. Namely, that \( \mathcal{D} \) is such that for every compact \( K \) there exists \( Q \in \mathcal{D} \) with \( K \subset Q \). These assumptions can be removed arguing as in [5]. However, we find the second assumption very natural since—in a probabilistic sense—almost all dyadic lattices satisfy it. Also, as argued in [5], we are confident that our results also hold in the context of geometrically doubling metric spaces.

From the previous assumptions, it can be seen that for a fixed \( f \in A_+, K \) and \( \lambda > 0 \) there exists \( m_\lambda(f) \in \mathbb{Z} \) such that \( f_k \leq \lambda 1_A \) for all \( k \leq m_\lambda(f) \) (see [7]). Without loss of generality, we may also assume that \( f \) has only finite nonvanishing martingale differences.

Remark 2.1. To ease notation, we will use the normalization \( m_\lambda(f) = 0 \). It is safe to assume since in the proofs of Theorems A and B both \( f \in A_+, K \) and \( \lambda > 0 \) will remain fixed, but otherwise arbitrary.

We start with the construction of the projections \( (p_k)_{k \in \mathbb{Z}} \) and \( q \) of Theorem A. To that end we will use the so-called Cuculescu construction. Here we state it in the precise form that we will use, although the construction can be done in any semifinite von Neumann algebra.

Cuculescu’s construction [2]. Let \( f \in A_+, K \) and consider the associated positive martingale \( (f_k)_{k \in \mathbb{Z}} \) relative to the dyadic filtration \( (A_k)_{k \in \mathbb{Z}} \). Given \( \lambda > 0 \), the decreasing sequence of projections \( (q_k)_{k \in \mathbb{Z}} \) defined recursively by \( q_k = 1_A \) for \( k \leq 0 \) and

\[
q_k = q_k(f, \lambda) := 1_{(0, \lambda]}(q_{k-1} f_k q_{k-1})
\]

is such that
(a) \( q_k \) is a projection in \( A_k \).
(b) \( q_k \) commutes with \( q_{k-1} f_k q_{k-1} \).
(c) \( q_k f_k q_k \leq \lambda q_k \).
(d) \( q = \bigwedge_k q_k \) satisfies

\[
\|q f q\|_{A} \leq \lambda \text{ for all } k \geq 1 \quad \text{and} \quad \tau(1_A - q) \leq \frac{1}{\lambda} \|f\|_{L^1(A)}.
\]

Define the sequence \( (p_k)_{k \geq 1} \) of pairwise disjoint projections by

\[
p_k = q_{k-1} - q_k.
\]
In particular
\[ \sum_{k \geq 1} p_k = 1_A - q \]
and also \( p_k f_k p_k \geq \lambda p_k \).

**Remark 2.2.** Since the projection \( q_k \) is \( \mathcal{D}_k \)-measurable, we have the useful expression
\[ q_k = \sum_{Q \in \mathcal{D}_k} q_Q \otimes 1_Q, \]
where \( q_Q = q_Q(f, Q) \) are projections in \( \mathcal{M} \) defined by
\[ q_Q = \begin{cases} 1_M & \text{if } k < 0, \\ 1_{(0, \lambda]}(q_Q^\ast f_q q_{\tilde{Q}}) & \text{if } k \geq 0. \end{cases} \]

As in Cuculescu’s construction, these projections satisfy
(a) \( q_Q \leq q_{\tilde{Q}} \).
(b) \( q_Q \) commutes with \( q_{\tilde{Q}}^\ast f_q q_{\tilde{Q}} \).
(c) \( q_Q^\ast f_q q_Q \leq \lambda q_Q \).

One can then express the projections \( p_k \) as
\[ p_k = \sum_{Q \in \mathcal{D}_k} (q_Q - q_{\tilde{Q}}) 1_Q =: \sum_{Q \in \mathcal{D}_k} p_Q \otimes 1_Q, \]
and we analogously have that \( p_Q \in \mathcal{P}(\mathcal{M}) \) is such that \( p_Q^\ast f_q p_Q \geq \lambda p_Q \). As detailed in [7], one could interpret the projections \( p_k \) as the union of dyadic cubes of side-length \( 2^{-k} \) into which the classical level set \( \Omega_\lambda = \{ \sup_k f_k > \lambda \} \) is decomposed.

One can thus view \( q \) as the complementary set of \( \Omega_\lambda \).

**Proof of Theorem A.** By construction \( f = g + b + \beta \). We now turn to the estimates of the diagonal part. For the \( L_1 \) estimate of \( g_\Delta \) observe that by the tracial property
\[ \| g_\Delta \|_{L_1(A)} = \tau(f_q) + \sum_{k \geq 1} \tau(E_{k-1}(p_k f_k p_k)) = \tau(f_q) + \tau(f(1_A - q)) = \| f \|_{L_1(A)}, \]
since \( E_k \) preserves the trace. The proof of the \( L_2 \) estimate of \( g_\Delta \) is a bit more involved since \( \mu \) is not necessarily doubling. Also, the lack of commutativity of \( \mathcal{M} \) prevents us from following the argument used in [5]. However, standard arguments in noncommutative martingale theory apply. First notice that since \( q_k \) commutes with \( q_{k-1} f_k q_{k-1} \),
\[ E_{k-1}(p_k f_k p_k) = q_{k-1} f_{k-1} q_{k-1} - E_{k-1}(q_k f_k q_k). \]
Thus,
\[ \left\| \sum_{k \geq 1} E_{k-1}(p_k f_k p_k) \right\|_{L_2(A)}^2 \leq 2 \left( \left\| \sum_{k \geq 1} q_k f_k q_k - E_{k-1}(q_k f_k q_k) \right\|_{L_2(A)}^2 + \left\| \sum_{k \geq 1} q_k f_k q_k - q_{k-1} f_{k-1} q_{k-1} \right\|_{L_2(A)}^2 \right) = 2(I + II). \]
As it is proved in [9, Lemma 3.4], we have that
\[
\|q_k f_k q_k - E_{k-1}(q_k f_k q_k)\|_{L^2(A)}^2 \leq 2\left(\|q_k f_k q_k\|_{L^2(A)}^2 - \|q_{k-1} f_{k-1} q_{k-1}\|_{L^2(A)}^2\right) + 6\lambda^2(q_{k-1} f_{k-1} q_{k-1} - q_k f_k q_k).
\]
Therefore, by orthogonality of martingale differences and the previous estimate, summation over \(k\) gives
\[
I = \sum_{k \geq 1} \|q_k f_k q_k - E_{k-1}(p_k f_k p_k)\|_{L^2(A)}^2 \\
\leq \lim_{k \to \infty} \left(2\left(\|q_k f_k q_k\|_{L^2(A)}^2 - \|q_0 f_0 q_0\|_{L^2(A)}^2\right) + 6\lambda^2(q_0 f_0 q_0 - q_k f_k q_k)\right) \\
\leq \lim_{k \to \infty} \left(2\|q_k f_k q_k\|_{L^2(A)}^2 + 6\lambda^2(q_0 f_0)\right) \leq 8\lambda\|f\|_{L^1(A)},
\]
where Hölder’s inequality and (c) of Cuculescu’s construction were used. To estimate \(II\) we perform the telescopic sum in order to get
\[
II \leq 2\|q f q\|_{L^2(A)}^2 + 2\|q_0 f_0 q_0\|_{L^2(A)}^2 \leq 4\lambda\|f\|_{L^1(A)},
\]
which follows from the estimate \(q f q \leq \lambda q\), which in turn can be deduced from Cuculescu’s construction (see [7, Section 4.1]). By this last estimate and using that \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2\) for positive numbers \(a, b, c\), we finally obtain
\[
\|g_\Delta\|_{L^2(A)} \leq 39\lambda\|f\|_{L^1(A)}.
\]

The bad terms are easier to handle. Clearly the bad term \(b_\Delta\) is comprised of the self-adjoint terms \(b_k = p_k(f - f_k)p_k\) with the mean zero property \(E_k(b_k) = 0\), so that \(\int_{\mathbb{R}^d} b\,d\mu = 0\). Moreover, by the orthogonality of the projections \(p_k\), the tracial property of \(\tau\), and since conditional expectations are bimodular and trace preserving, we have that
\[
\|b_\Delta\|_{L^1(A)} = \sum_{k \geq 1} \|b_k\|_{L^1(A)} \leq \sum_{k \geq 1} \tau(p_k(f + f_k)p_k) \\
= 2\tau(f(1_A - q)) \leq 2\|f\|_{L^1(A)}.
\]
Similarly, \(\beta_\Delta = \sum b_k\), where \(b_k = D_k(p_k f_k p_k) = D_k\beta_\Delta\) is a \(k\)-th martingale difference—and hence of mean zero. Moreover, as conditional expectations are contractive on \(L^1(A)\),
\[
\|\beta_\Delta\|_{L^1(A)} \leq \sum_{k \geq 1} \|\beta_k\|_{L^1(A)} \leq 2\sum_{k \geq 1} \tau(p_k f_k p_k) = 2\tau(f(1_A - q)) \leq 2\|f\|_{L^1(A)}.
\]

We now turn to the off-diagonal terms, which require some more work. To get the appropriate estimate for \(g_{off}\), first we need to obtain a manageable expression for its \(k\)-th martingale difference. Rewrite \(g_{off}\) as
\[
g_{off} = (1_A - q)f q + qf(1_A - q) + \sum_{k \geq 1} \sum_{h \geq 1} E_{k+h-1}(p_k f_k p_{k+h} + p_k f_k p_{k+h} p_k).
\]
Since \(p_i \wedge j, p_i \vee j \leq q_i \wedge j - 1\) and by the commutation property (d) of Cuculescu’s construction we have that
\[
(2.4) \quad p_i f_i \wedge j p_j = p_i q_i \wedge j - 1 f_i \wedge j q_i \wedge j - 1 p_j = 0, \quad i \neq j, \quad i, j \in \mathbb{N} \cup \{\infty\}.
\]
Thus,
\[
\sum_{k \geq 1} \sum_{h \geq 1} E_{k+h-1}(p_k f_{k+h} p_{k+h} + p_{k+h} f_{k+h} p_k) \\
= \sum_{k \geq 1} \sum_{h \geq 1} E_{k+h-1}(p_k (f_{k+h} - f_k) p_{k+h} + p_{k+h} (f_{k+h} - f_k) p_k) \\
= \sum_{k \geq 1} \sum_{h \geq 1} \sum_{i=1}^h E_{k+h-1}(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k).
\]

We may now proceed to calculate \( D_j(g_{off}) \) for \( j \geq 1 \). Taking into account that, for \( h \geq 1 \), \( D_j E_{k+h-1} = D_j \) if \( j < k + h \) and zero otherwise, we get that
\[
D_j(g_{off}) = D_j((1_A - q)f q + qf(1_A - q)) \\
+ \sum_{k < j} \sum_{h > j-k} \sum_{i=1}^h D_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k) \\
+ \sum_{k \geq j} \sum_{h \geq 1} \sum_{i=1}^h D_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k) = I + II + III.
\]

We deal first with \( II \). By Fubini’s theorem we obtain that
\[
II = \sum_{k < j} \left( \sum_{i=1}^{j-k} \sum_{h > j-k} D_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k) \\
+ \sum_{i > j-k} \sum_{h \geq i} D_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k) \right)
= \sum_{k < j} \left( \sum_{i=1}^{j-k} D_j(p_k df_{k+i} q_j + q_j df_{k+i} p_k) \\
+ \sum_{i > j-k} D_j(p_k df_{k+i} q_{k+i-1} + q_{k+i-1} df_{k+i} p_k) \\
- \sum_{i \geq 1} D_j(p_k df_{k+i} q + q df_{k+i} p_k) \right) = II_1 + II_2 + II_3.
\]

After summing over \( i \) in \( II_1 \) and noticing that by \([1]\) of Cuculescu’s construction (recall that \( k < j \))
\[
p_k f_k q_j = p_k q_{k-1} f_k q_{k-1} j = 0 = q_j f_k p_k,
\]
we find that
\[
II_1 = \sum_{k < j} D_j(p_k f_j q_j + q_j f_j p_k) = D_j((1_A - q_j-1)f_j q_j + q_j f_j(1_A - q_j-1)).
\]

The term \( II_2 \) vanishes since
\[
(2.5) \quad p_k df_{k+i} q_{k+i-1} + q_{k+i-1} df_{k+i} p_k = D_{k+i}(p_k f q_{k+i-1} + q_{k+i-1} f p_k)
\]
and \( D_j D_{k+i} = 0 \), as \( k + i > j \). Performing the summation over \( i \) in \( II_3 \) and using
with \( i \land j = k \) and \( i \lor j = \infty \), we get that
\[
II = D_j((1_A - q_{j-1})f_j q_j + q_j f_j (1_A - q_{j-1}))
- D_j((1_A - q_{j-1})f q + q f (1_A - q_{j-1})) \]
Changing the order of summation we obtain
\[
III = \sum_{k \geq j} \sum_{i \geq 1} \sum_{h \geq i} D_j(p_k d f_{k+i} p_{k+h} + p_{k+h} d f_{k+i} p_k)
= \sum_{k \geq j} \left( \sum_{i \geq 1} D_j(p_k d f_{k+i} q_{k+i-1} + q_{k+i-1} d f_{k+i} p_k) - \sum_{i \geq 1} D_j(p_k d f_{k+i} q + q d f_{k+i} p_k) \right)
= -D_j((q_{j-1} - q) f q + q f (q_{j-1} - q))
\]
Here, we have also used (2.5), as \( k + i > j \), and (2.4) with \( i \lor j = \infty \). Finally, summing everything we get that for \( j \geq 1 \),
\[
D_j(g_{off}) = D_j((1_A - q_{j-1})f_j q_j) + D_j(q_j f_j (1_A - q_{j-1}))
\]
On the other hand, \( D_j(g_{off}) = 0 \) for \( j \leq 0 \). Indeed,
\[
D_j(g_{off}) = D_j((1_A - q) f q + q f (1_A - q))
+ \sum_{k \geq 1} \sum_{h \geq 1} \sum_{i = 1}^h D_j(p_k d f_{k+i} p_{k+h} + p_{k+h} d f_{k+i} p_k)
\]
and, arguing as with \( III \) above and since \( q_0 = 1_A \), we have that
\[
\sum_{k \geq 1} \sum_{h \geq 1} \sum_{i = 1}^h D_j(p_k d f_{k+i} p_{k+h} + p_{k+h} d f_{k+i} p_k) = -D_j((1_A - q) f q + q f (1_A - q))
\]
Thus, in the \( L_2 \) sense
\[
g_{off} = \sum_{j \geq 1} D_j(g_{off}) = \sum_{j \geq 1} \sum_{k < j} D_j(p_k f_j q_j + q_j f_j p_k)
= \sum_{k \geq 1} \sum_{h \geq 1} D_{k+h}(p_k f_{k+h} q_{k+h} + q_{k+h} f_{k+h} p_k) =: \sum_{k \geq 1} \sum_{h \geq 1} g_{k,h}
\]
We are now in a position to prove the estimate in (c) of Theorem A. Notice first that by Hölder’s inequality, the \( C^* \)-algebra property and (c) of Cuculescu’s construction,
\[
\|g_{k,h}\|_{L^2(A)}^2 \leq 16 \|q_{k+h} f_{k+h} p_k\|_{L^2(A)}^2
= 16 \tau(p_k f_{k+h} q_{k+h} f_{k+h} p_k)
\leq 16 \|f_{1/2}^{1/2} q_{k+h} f_{1/2}^{1/2} f_{1/2}^{1/2} f_{k+h}\|_{L^2} \tau(f_k p_k f_{k+h}^{1/2})
= 16 \|q_{k+h} f_{k+h} q_{k+h} f_{k+h}\|_{A} \tau(p_k f_{k+h} p_k) \leq 16 \lambda \tau(f p_k).
\]
This proves that for all \( h \geq 1 \)
\[
\sum_{k \geq 1} \|g_{k,h}\|_{L^2(A)}^2 \leq 16 \lambda \tau(f (1_A - q)) \leq 16 \lambda \|f\|_{L^1(A)}.
\]
For the bad terms we follow [7]. First, rewrite $b_{\text{off}}$ as

$$b_{\text{off}} = \sum_{h \geq 1} \sum_{k \geq 1} p_k (f - f_{k+h}) p_{k+h} + p_{k+h} (f - f_{k+h}) p_k =: \sum_{h \geq 1} \sum_{k \geq 1} b_{k,h}.$$  

Clearly, the terms $b_{k,h}$ have mean zero and satisfy the estimate

$$\|b_{k,h}\|_{L_1(A)} \leq 2\|p_k f p_{k+h} + p_{k+h} f p_k\|_{L_1(A)}.$$  

Next, observe that we can decompose the off-diagonal terms $p_k f p_{k+h} + p_{k+h} f p_k$ into a sum of four positive overlapping box-diagonal terms

$$p_k f p_{k+h} + p_{k+h} f p_k = \left( \sum_{j=0}^{h} p_{k+j} \right) f \left( \sum_{j=0}^{h} p_{k+j} \right) - \left( \sum_{j=0}^{h-1} p_{k+j} \right) f \left( \sum_{j=0}^{h-1} p_{k+j} \right) - \left( \sum_{j=1}^{h} p_{k+j+1} \right) f \left( \sum_{j=1}^{h} p_{k+j} \right) + \left( \sum_{j=0}^{h-1} p_{k+j+1} \right) f \left( \sum_{j=1}^{h} p_{k+j+1} \right).$$

The previous expression implies that

$$\sum_{k \geq 1} \|p_k f p_{k+h} + p_{k+h} f p_k\|_{L_1(A)} \leq 4 \sum_{k \geq 1} \sum_{j=0}^{h} \|f p_{k+j}\| \leq 4 \sum_{j=0}^{h} \| f (q_j - q) \| \leq 4(h + 1) \|f\|_{L_1(A)},$$

and hence the estimate in (3) holds. On the other hand, we have

$$\beta_{\text{off}} = \sum_{k \geq 1} \sum_{h \geq 1} D_{k,h} (p_k f_{k+h} p_{k+h} + p_{k+h} f_{k+h} p_k) =: \sum_{k \geq 1} \sum_{h \geq 1} \beta_{k,h}.$$  

Each term in the previous sum satisfies the same estimate

$$\|\beta_{k,h}\|_{L_1(A)} \leq 2\|p_k f p_{k+h} + p_{k+h} f p_k\|_{L_1(A)},$$

which yields the corresponding estimate for $\beta_{\text{off}}$. □  

3. COMMUTING HAAR SHIFT OPERATORS  

We now turn to the proof of Theorem [13] Namely that

$$\lambda \tau \left( \{ |f| > \lambda \} \right) \leq \|f\|_{L_1(A)}$$

for all $\lambda > 0$. Here $\tau(\{|f| > \lambda\})$ denotes the trace of the spectral projection of $|f|$ associated to the interval $(\lambda, \infty)$, which defines a noncommutative distribution function. We find this terminology more intuitive, since it is reminiscent of the classical one. Following the construction of noncommutative symmetric spaces (see [6] and the references therein), the resulting $L_{1,\infty}(A)$ space is a quasi-Banach space with quasi-norm $\|f\|_{L_{1,\infty}(A)} = \sup_{\lambda > 0} \lambda \tau(\{|f| > \lambda\})$ which interpolates with $L_2(A)$. It should be mentioned that the weak Bochner space $L_{1,\infty}(\mathbb{R}^d, \mu; L_1(M))$ is of no use for our purposes since $L_1(M)$ is not a UMD space and thus even Haar multipliers may not be bounded, which rules out the use of this space as an appropriate setting for providing weak-type $(1,1)$ estimates for the operators in question. The same applies if one considers $M$ instead of $L_1(M)$ as a target space.
Proof of Theorem 4.1. Let $f \in A_{+,K}$. The general case follows by the density of the span of $A_{+,K}$ in $L_1(A)$. Consider the Calderón-Zygmund decomposition $f = g_\Delta + b_\Delta + \beta_\Delta + g_{\text{off}} + b_{\text{off}} + \beta_{\text{off}}$ associated to $(f,\lambda)$ for a given $\lambda > 0$. By the quasi-triangle inequality in $L_{1,\infty}(A)$ it suffices to show that

$$\lambda \tau(\{|\texttt{III}_{r,s}(\gamma)| > \lambda\}) \lesssim \|f\|_{L_1(A)}$$

for all $\gamma \in \{g_\Delta, b_\Delta, \beta_\Delta, g_{\text{off}}, b_{\text{off}}, \beta_{\text{off}}\}$. We start with the diagonal terms, for which the estimates are very similar to the classical ones. For $g_\Delta$ we use Chebyshev’s inequality, the $L_2$ boundedness of $\texttt{III}_{r,s}$ and the $L_2$ estimate in (2) of Theorem A to get

$$\lambda \tau(\{|\texttt{III}_{r,s}(\gamma)| > \lambda\}) \leq 39 \|\texttt{III}_{r,s}\|_{L_2(A)}^2 \|f\|_{L_1(A)},$$

where $\|\texttt{III}_{r,s}\|_{L_2(A)}$ denotes the operator norm of $\texttt{III}_{r,s}$ on $L_2(A)$. For the remaining $\gamma$, we decompose $\texttt{III}_{r,s}(\gamma)$ as

$$\texttt{III}_{r,s}(\gamma) = (1_A - q) \texttt{III}_{r,s}(\gamma)(1_A - q) + q \texttt{III}_{r,s}(\gamma)q + q \texttt{III}_{r,s}(\gamma)(1_A - q) + (1_A - q) \texttt{III}_{r,s}(\gamma)q.$$

Since the distribution function is adjoint-invariant and by the second estimate in (2) of Cuculescu’s construction, we get that

$$\lambda \tau(\{|\texttt{III}_{r,s}(\gamma)| > \lambda\}) \leq 12 \|f\|_{L_1(A)} + \lambda \tau(\{|q \texttt{III}_{r,s}(\gamma)q| > \lambda/4\}).$$

To prove the estimate for $\gamma = b_\Delta$, observe that we may further decompose each term $b_k$ in (6) of Theorem A as

$$b_k = \sum_{L \in \mathcal{D}_k} p_L(f(\langle f \rangle_L)p_L1_L =: \sum_{L \in \mathcal{D}_k} b_L,$$

where the projections $p_L$ are defined as in (2.3). Since the Haar function $\phi_R$ is constant on dyadic subcubes of $R$ and $b_L$ has zero integral, $\langle b_L, \phi_R \rangle$ is nonzero only for $R \subset L$, i.e., $R^{(r)} \subset L^{(r)}$ for their respective $r$-dyadic ancestors. On the other hand, if $x \in L$ we have that $q(x) \leq q_L(x) = q_L$ in the order of the lattice $\mathcal{P}(\mathcal{M})$. This together with (2.3) gives that for $x \in L$,

$$(3.1) \quad q(x)\langle b_L, \phi_R \rangle q(x) = q(x)q_L p_L \langle b_L, \phi_R \rangle p_L q_L q(x) = 0.$$

Using that $\alpha_{R,S}^Q \in \mathcal{M} \cap \mathcal{M}'$ we find the estimate

$$\|q \texttt{III}_{r,s}(b_L)q\|_{L_1(A)} \leq \sum_{Q \in \mathcal{D}} \sum_{L \subset Q \subset L^{(r)}} \|\alpha_{R,S}^Q\|_{\mathcal{M}} \|\langle b_L, \phi_R \rangle\|_{L_1(A)} \|q\|_{L_1(\mu)} \lesssim \sup_{Q,R,S} \|\alpha_{R,S}^Q\|_{\mathcal{M}} \sum_{L \subset Q \subset L^{(r)}} \|\phi_R\|_{L_1(\mu)} \|q\|_{L_1(\mu)} \|b_L\|_{L_1(A)} \leq \frac{r2^{(r+s)d}}{} \sup_{Q,R,S} \|\alpha_{R,S}^Q\|_{\mathcal{M}} \|\zeta(\Phi, \Psi; r, s)\|b_L\|_{L_1(A)}.$$

This estimate, Chebyshev’s inequality, the fact that dyadic cubes in $\mathcal{D}_k$ are disjoint and (6) of Theorem A give the estimate

$$\lambda \tau(\{|q \texttt{III}_{r,s}b_Lq| > \lambda\}) \leq r2^{1+(r+s)d} \sup_{Q,R,S} \|\alpha_{R,S}^Q\|_{\mathcal{M}} \|\zeta(\Phi, \Psi; r, s)\|f\|_{L_1(A)}.$$
For $\gamma = \beta_\Delta$ we proceed likewise by writing

$$\beta_k = D_k(\beta_\Delta) = \sum_{L \in D_k} \sum_{J \in D_1(L)} p_J(f) p_J \left( 1_J - \frac{\mu(J)}{\mu(L)} 1_L \right)$$

$$=: \sum_{L \in D_k} \sum_{J \in D_1(L)} \beta_{L,J} =: \sum \beta_L,$$

where each term $\beta_L$ is supported (as an operator-valued function) on $L$, is constant on the dyadic descendants of $L$ and has mean zero. By Chebyshev’s inequality we have

$$\lambda_\tau(\{ |q \Pi_{r,s}(\beta_\Delta)q| > \lambda \}) \leq \sum_{k \geq 1} \sum_{L \in D_k} \left( \int_{\mathbb{R}^d \setminus L} \nu(|q(x) \Pi_{r,s} \beta_L(x)q(x)|) \, d\mu(x) \right)$$

$$+ \int_L \nu(|q(x) \Pi_{r,s} \beta_L(x)q(x)|) \, d\mu(x).$$

Since $\langle \beta_L, \phi_R \rangle$ is nonzero only for dyadic cubes $R \subset L$, proceeding as in (3.2) we obtain

$$\int_{\mathbb{R}^d \setminus L} \nu(|q(x) \Pi_{r,s} \beta_L(x)q(x)|) \, d\mu(x)$$

$$\leq r^{2(r+s)d} \sup_{Q,R,S} \| \alpha^Q_{R,S} \|_{\mathcal{M}} \Xi(\Phi, \Psi; r, s) \| \beta_L \|_{L_1(A)}.$$ 

Arguing as above and recalling that $\alpha^Q_{R,S} \in \mathcal{M} \cap \mathcal{M}'$, for $x \in L$ we obtain

$$q(x) \Pi_{r,s}(\beta_L)(x)q(x) = \sum_{Q \in D} \sum_{L \subset Q \subset L^{(r)}} \sum_{R \in D_1(Q), R \subset L} \sum_{S \in D_1(Q)} \alpha^Q_{R,S} q(x) \langle \beta_L, \phi_R \rangle q(x) \psi_S(x)$$

$$+ \sum_{Q \in D} \sum_{R \in D_1(Q), Q \subset L} \sum_{S \in D_1(Q)} \alpha^Q_{R,S} q(x) \langle \beta_L, \phi_R \rangle q(x) \psi_S(x)$$

$$= F_L(x) + G_L(x).$$

As in (3.2) we get the estimate

$$\int_L \nu(|F_L(x)|) \, d\mu(x) \leq (r + 1)2^{(r+s)d} \sup_{Q,R,S} \| \alpha^Q_{R,S} \|_{\mathcal{M}} \Xi(\Phi, \Psi; r, s) \| \beta_L \|_{L_1(A)}.$$ 

To estimate $G_L(x)$ we further decompose $\beta_L$ and get

$$G_L(x) = \sum_{J \in D_1(L)} \sum_{Q \in D_1(Q), Q \subset L} \sum_{R \in D_1(Q), S \subset D_1(Q)} \alpha^Q_{R,S} q(x) \langle \beta_{L,J}, \phi_R \rangle q(x) \psi_S(x) = \sum_{J \in D_1(L)} G_{L,J}(x).$$

Given $J \in D_1(L)$ and a dyadic cube $Q \subset L$ we either have $Q \subset J$ or $Q \subset L \setminus J$. However, the former case leads to zero terms since, as in (3.1), for $x \in Q \subset J$ we
have $q(x) \leq q_J$ and thus $q(x)(\beta_{L,J}, \phi_R)q(x) = 0$. Hence,

$$G_{L,J}(x) = -p_J(f)_J p_J \frac{\mu(J)}{\mu(L)} \sum_{Q \in \mathcal{Q}} \sum_{R \in \mathcal{Q}(Q)} \alpha_{R,S}^Q q(x)(1_{L \setminus J}, \phi_R)q(x)\psi_S(x)$$

$$= -p_J(f)_J p_J \frac{\mu(J)}{\mu(L)} \sum_{Q' \in \mathcal{Q}(Q')} \sum_{R \in \mathcal{Q}(Q)} \alpha_{R,S}^Q q(x)(1_{Q'}, \phi_R)q(x)\psi_S(x)$$

$$= -p_J(f)_J p_J \frac{\mu(J)}{\mu(L)} \sum_{Q' \in \mathcal{Q}(Q')} q(x) \mathcal{I} Q' \mathbb{I}_{r,s}(1_{Q'}) q(x).$$

Then, by Hölder’s inequality and the fact that $\text{supp}_{R,d}(\mathcal{I} Q' \mathbb{I}_{r,s}(1_{Q'})) \subset Q'$,

$$\int_L \nu(|G_L(x)|) d\mu(x)$$

$$= \int_L \nu \left( \left| \sum_{J \in \mathcal{Q}(L)} p_J(f)_J p_J \frac{\mu(J)}{\mu(L)} \sum_{Q' \in \mathcal{Q}(Q')} q(x) \mathcal{I} Q' \mathbb{I}_{r,s}(1_{Q'}) q(x) \right| \right) d\mu(x)$$

$$\leq \sum_{J \in \mathcal{Q}(L)} \|p_J(f)_J p_J\|_{L_1(M)} \frac{\mu(J)}{\mu(L)} \sum_{Q' \in \mathcal{Q}(Q')} \int_L \|\mathcal{I} Q' \mathbb{I}_{r,s}(1_{Q'}) q(x)\|_M d\mu(x)$$

$$\leq \sum_{J \in \mathcal{Q}(L)} \|p_J(f)_J p_J\|_{L_1(M)} \frac{\mu(J)}{\mu(L)} \times \left( \sum_{Q' \in \mathcal{Q}(Q')} \left( \int_{\mathbb{R}^d} \|\mathcal{I} Q' \mathbb{I}_{r,s}(1_{Q'}) q(x)\|_M^2 d\mu(x) \right)^{\frac{1}{2}} \mu(Q')^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\leq \sup_{Q \in \mathcal{Q}, \mu(Q) \neq 0} \frac{1}{\mu(Q)^{\frac{1}{2}}} \left( \int_{\mathbb{R}^d} \|\mathcal{I} Q \mathbb{I}_{r,s}(1_{Q}) q(x)\|_M^2 d\mu(x) \right)^{\frac{1}{2}} \left\| \sum_{J \in \mathcal{Q}(L)} p_J(f)_J p_J 1_J \right\|_{L_1(A)}^{\frac{1}{2}},$$

which is finite by the local vector-valued $L_2$ estimate \([1,3]\). By the estimate in \((c)\) of the Calderón-Zygmund decomposition,

$$\sum_{k \geq 1} \sum_{L \in \mathcal{G}_k} \left( \|\beta_L\|_{L_1(A)} + \left\| \sum_{J \in \mathcal{Q}(L)} p_J(f)_J p_J 1_J \right\|_{L_1(A)} \right) \leq \sum_k (\|\beta_k\|_1 + \|p_k f_k p_k\|_1) \leq 3\|f\|_1.$$
Thus, by gathering the previous estimates
\[ \lambda \tau \left( \{ |q \, \text{III}_{r,s}(\beta_\Delta)| > \lambda \} \right) \]
\[ \leq \left( (r+2)^{2+|s|} \sup_{Q,R,S} \|o^Q_{R,S}\|_{L^2} |J_{s}(\Phi, \Psi; r, s) \right) \]
\[ + \sup_{Q \in \mathcal{G}, \mu(Q) \neq 0} \frac{1}{\mu(Q)^{2}} \left( \int |q \, \text{III}_{r,s}(1_Q)(x)|^2 \, d\mu(x) \right)^{1/2} \|f\|_{L^1(\mathcal{A})}. \]

We now turn to the weak-type estimates for the off-diagonal terms, starting with \( g_{\text{off}} \). By Chebyshev’s inequality
\[ \lambda \tau \left( \{ |q \, \text{III}_{r,s}(g_{\text{off}})| > \lambda \} \right) \leq \frac{1}{\lambda} \left( \sum_{h \geq 1} \sum_{k \geq 1} q \, \text{III}_{r,s}(g_{k,h})q \right)^2_{L^2(\mathcal{A})}. \]

We further decompose the terms \( g_{k,h} \) as
\[ g_{k,h} = \sum_{L \in \mathcal{Y}_k} \sum_{J \in \mathcal{Y}_h(L)} (p_L(f) J q_J + q_J(f) J p_L) \left( 1_J - \frac{\mu(J)}{\mu(\hat{J})} \right) =: \sum_{L \in \mathcal{Y}_k} g_{L,h}. \]

Clearly, each term \( g_{L,h} \) is such that \( \text{supp}_{\mathbb{R}^d}(g_{L,h}) \subset L \) and has mean zero on the \( (h-1) \)-descendants of \( L \). Thus, \( \langle g_{L,h}, \phi_R \rangle \) is nonzero only for \( R \subset \hat{J} \) for some \( J \in \mathcal{D}_h(L) \), which amounts to saying that \( R \in \mathcal{D}_{h+j-1}(L) \) for some \( j \geq 0 \). Furthermore
\[ g_{L,h} = p_L \, A_{L,h} + A_{L,h}^* p_L, \]
where
\[ A_{L,h} = p_L \langle f, J \rangle q_J \left( 1_J - \frac{\mu(J)}{\mu(\hat{J})} \right). \]

Proceeding as in (3.1) we get that \( q(x) \langle g_{L,h}, \phi_R \rangle q(x) = 0 \) if \( x \in L \). In other words, only the cubes \( R \) such that \( R^{(r)} \supset L \) lead to nonzero terms. These two observations in terms of side-lengths provide that \( h \) must be such that \( \ell(L) = 2^{-k} < \ell(R^{(r)}) = 2^{-h+h+j-1-r} \), namely \( h \leq r \). This and the assumption \( o^Q_{R,S} \) are in \( \mathcal{M} \cap \mathcal{M}' \) allow us to deduce that \( q(x) \, \text{III}_{r,s} \, g_{k,h}(x)q(x) = 0 \) whenever \( h > r \). This localization property and the orthogonality of martingale differences in \( L^2(\mathcal{A}) \) enable us to obtain that
\[ \sum_{h \geq 1} \sum_{k \geq 1} q \, \text{III}_{r,s}(g_{k,h})q \leq \| \text{III}_{r,s} \|_{L^2(\mathcal{A})} \sum_{h=1}^r \| \sum_{k \geq 1} g_{k,h} \|_{L^2(\mathcal{A})} \]
\[ = \| \text{III}_{r,s} \|_{L^2(\mathcal{A})} \sum_{h=1}^r \left( \sum_{k \geq 1} \| g_{k,h} \|^2_{L^2(\mathcal{A})} \right)^{1/2}. \]

Therefore, by the estimate in (3) of Theorem A we arrive at
\[ \lambda \tau \left( \{ |q \, \text{III}_{r,s}(g_{\text{off}})| > \lambda \} \right) \leq 16r^2 \| \text{III}_{r,s} \|_{L^2(\mathcal{A})}^2 \|f\|_{L^1(\mathcal{A})}. \]

To get the estimate for \( b_{\text{off}} \) we proceed in an entirely similar way by decomposing the terms \( b_{k,h} \) in (3) of Theorem A as
\[ b_{k,h} = \sum_{L \in \mathcal{Y}_k} \sum_{J \in \mathcal{Y}_h(L)} (p_L(f - \langle f, J \rangle)Jp_J + p_J(f - \langle f, J \rangle)p_L) 1_J =: \sum_{L \in \mathcal{Y}_k} b_{L,h}. \]
It is clear that \(\text{supp}_{\mathbb{R}^d}(b_{L,h}) \subset L\), that \(b_{L,h}\) has mean zero over the \(h\)-dyadic descendants of \(L\) and that \(b_{L,h} = p_LB_{L,h} + B_{L,h}^1p_L\), with \(B_{L,h} = p_L(f - \langle f \rangle_J)p_J1_J\). Arguing as above, \(q(x)(b_{L,h},\phi_R)q(x)\) is nonzero only for \(R \subset L \subset R^{(r)} \subset L^{(r)}\), and hence \(q(x)\) \(\Pi_{r,s}(b_{L,h})(x)q(x)\) vanishes if \(h > r\). Thus, for \(h \leq r\) we follow the steps in (3.2) to get the estimate
\[
\sum_{L \in \mathcal{D}_k} \|q \Pi_{r,s}(b_{L,h})q\|_{L_1(A)} \leq (r-1)2^{(r+s)d} \sup_{Q,R,S} \|\alpha^{Q}_{R,S}\|_{\mathcal{M}} \Xi(\Phi;\Psi;r,s)\|b_{k,h}\|_{L_1(A)}.
\]
By Chebyshev’s inequality and the estimate in (c) of the Calderón-Zygmund decomposition we obtain
\[
\lambda\tau(\{|q \Pi_{r,s}(b_{off})q| > \lambda\}) 
\leq (r-1)2^{(r+s)d} \sup_{Q,R,S} \|\alpha^{Q}_{R,S}\|_{\mathcal{M}} \Xi(\Phi;\Psi;r,s) \sum_{h=1}^{r} (h+1)\|f\|_{L_1(A)}
= r(r-1)(r+3)2^{(r+s)d} \sup_{Q,R,S} \|\alpha^{Q}_{R,S}\|_{\mathcal{M}} \Xi(\Phi;\Psi;r,s)\|f\|_{L_1(A)}.
\]
Finally, for \(\gamma = \beta_{off}\) observe that
\[
\beta_{k,h} = \sum_{L \in \mathcal{D}_h} \sum_{J \in \mathcal{D}_h(L)} \left(\mu_{L,J}(f)Jp_J + \mu_{J}(f)p_JL\right) \left(1 - \frac{\mu(J)}{\mu(J)} 1_J\right)
=:\sum_{L \in \mathcal{D}_h} \beta_{L,h} = \sum_{L \in \mathcal{D}_h} \left(p_LC_{L,h} + C_{L,h}^*p_L\right).
\]
Here we may repeat the analysis made for \(b_{L,h}\), as each \(\beta_{L,h}\) is a \((k+h)\)-martingale difference operator with \(\text{supp}_{\mathbb{R}^d}(\beta_{L,h}) \subset L\). This and (f) of Theorem A render the desired estimate
\[
\lambda\tau(\{|q \Pi_{r,s}(\beta_{off})q| > \lambda\}) 
\leq r(r-1)(r+3)2^{(r+s)d} \sup_{Q,R,S} \|\alpha^{Q}_{R,S}\|_{\mathcal{M}} \Xi(\Phi;\Psi;r,s)\|f\|_{L_1(A)},
\]
with which we complete the proof of Theorem B.

Remark 3.3. It is worth mentioning that we have not truly needed the assumption that the symbols are commuting to obtain the estimates for the diagonal terms. Indeed, all the calculations for the diagonal terms in the proof of Theorem B can be done without this assumption simply by rearranging multiplications. Unlike (3.1), in the case when \(\gamma \in \{g_{off}, b_{off}, \beta_{off}\}\) and \(x \in L\), \(q(x)\) is required to be multiplied on both sides of \(\langle \gamma_{L,h}, \phi_R \rangle\) in order to annihilate it.

Remark 3.4. The consideration of noncommuting symbols in (1.2) introduces considerable additional difficulties when trying to provide a priori estimates. Firstly, different operators arise depending on whether the symbols act by right or left multiplication on each coefficient \(\langle f, \phi_R \rangle\). More specifically, in the case of Haar multipliers, a pair of column/row operators are introduced by
\[
M_c(f) = \sum_{Q \in \mathcal{Q}} \alpha_Q \langle f, \phi_Q \rangle \phi_Q, \quad M_r(f) = \sum_{Q \in \mathcal{Q}} \langle f, \phi_Q \rangle \alpha_Q \phi_Q,
\]
with uniformly bounded \(\alpha_Q \in \mathcal{M}\). Even in the Lebesgue setting, Haar multipliers with noncommuting symbols may lack weak-type \((1,1)\) and strong \((p,p)\) estimates for \(p \neq 2\). This problem was solved in [3] where weak-type \((1,1)\) estimates for
Haar shift operators relative to the Lebesgue measure were obtained in terms of a column/row decomposition of the input function. To be more precise, given \( f \in A_{+,K} \) and \( a, k \in \mathbb{Z} \), consider Cuculescu’s projections \( q_k(2^c) = q_k(f, 2^c) \) and

\[
\pi_{a,k} = \bigwedge_{c \geq a} q_k(2^c) - \bigwedge_{c \geq a-1} q_k(2^c).
\]

For fixed \( k \) the projections \( \pi_{a,k} \) are pairwise disjoint. Thus, \( f \) decomposes in column/row components as \( f = f_c + f_r \) in terms of the multiscale triangle truncations

\[
f_c = \sum_{k \geq 1} \sum_{a \leq b} \pi_{a,k-1} f_k \pi_{b,k-1}, \quad f_r = \sum_{k \geq 1} \sum_{a > b} \pi_{a,k-1} f_k \pi_{b,k-1}.
\]

This decomposition is used in [3] in conjunction with the Calderón-Zygmund decomposition found in [7] to obtain that

\[
\|M_r f_r\|_{\ell_1, \infty} + \|M_c f_c\|_{\ell_1, \infty} \lesssim \|f\|_1,
\]

among analogous estimates for other Haar shift operators. The key to this argument is that the terms \( \gamma \) in the Calderón-Zygmund decomposition not having a proper \( L_2 \) estimate are such that \( D_k(\gamma) = (1_A - q_{k-1})A_k + A_k^*(1_A - q_{k-1}) \), which leads to vanishing triangular truncations. A major setback for extending this argument to the nondoubling setting is that \( D_k(\beta \Delta) = \beta_k = q_{k-1} \beta_k q_{k-1} \), reflecting that its classical counterpart decomposes into terms supported in the dyadic parents of the maximal dyadic cubes of \( \Omega \). This forces one to estimate \( L_1 \) norms of triangular truncations of \( \beta_k \), which in the \( B(\ell^\infty_2) \)-valued setting brings constants at best of order \( \log(n+1) \). Furthermore, higher integrability of \( \beta_k \)—such as \( L \log L \) (see [8])—might be hindered since \( \mu \) is permitted to be nondoubling.

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