A GLOBAL PINCHING THEOREM OF COMPLETE λ-HYPERSURFACES

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ABSTRACT. In this paper, the pinching problems of complete λ-hypersurfaces in a Euclidean space \( \mathbb{R}^{n+1} \) are studied. By making use of the Sobolev inequality, we prove a global pinching theorem of complete \( \lambda \)-hypersurfaces in a Euclidean space \( \mathbb{R}^{n+1} \).

1. Introduction

Let \( M^n \) be an \( n \)-dimensional manifold, and \( X : M^n \to \mathbb{R}^{n+1} \) an immersed hypersurface in a Euclidean space \( \mathbb{R}^{n+1} \). If \( X : M^n \to \mathbb{R}^{n+1} \) satisfies
\[
H + \langle X, N \rangle = 0,
\]
one calls that \( X : M^n \to \mathbb{R}^{n+1} \) is a self-shrinker of mean curvature flow, where \( H \) and \( N \) are the mean curvature and the unit normal vector of \( X : M^n \to \mathbb{R}^{n+1} \), respectively, and \( \langle \cdot, \cdot \rangle \) is the standard inner product of \( \mathbb{R}^{n+1} \).

Remark 1.1. If \( X : M^n \to \mathbb{R}^{n+1} \) is a self-shrinker of mean curvature flow, then \( X(t) = \sqrt{1 - 2t}X \) is a self-similar solution of mean curvature flow.

It is well-known that the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), the \( n \)-dimensional sphere \( S^n(\sqrt{n}) \) and the \( n \)-dimensional cylinder \( S^k(\sqrt{k}) \times \mathbb{R}^{n-k} \), for \( 1 \leq k \leq n - 1 \), are the standard self-shrinkers of mean curvature flow. For the other examples of self-shrinkers of mean curvature flow, see [1], [6], [7], [8] and [10].

\( X(t) : M^n \to \mathbb{R}^{n+1} \) is called a variation of \( X : M^n \to \mathbb{R}^{n+1} \) if \( X(t) : M^n \to \mathbb{R}^{n+1} \), \( t \in (-\varepsilon, \varepsilon) \), are a family of immersions with \( X(0) = X \). We define a weighted area functional \( A : (-\varepsilon, \varepsilon) \to \mathbb{R} \) as follows:
\[
A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu,
\]
where \( d\mu_t \) is the area element of \( X(t) : M^n \to \mathbb{R}^{n+1} \). In [4], Colding and Minicozzi have proved that \( X : M^n \to \mathbb{R}^{n+1} \) is a self-shrinker of mean curvature flow if and only if \( X(t) : M^n \to \mathbb{R}^{n+1} \) is a critical point of the weighted area functional \( A(t) \). By using the following Sobolev inequality for \( n \)-dimensional complete hypersurfaces:
\[
\kappa^{-1} \left( \int_M g^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \leq \int_M |\nabla g|^2 d\mu + \frac{1}{2} \int_M H^2 g^2 d\mu, \quad \forall g \in C_c^\infty(M),
\]
where \( \kappa > 0 \) is a constant, Ding and Xin [5] have proved a rigidity theorem of complete self-shrinkers of mean curvature flow as follows:
Theorem 1.1. Let $X: M^n \to \mathbb{R}^{n+1}$ be a complete immersed self-shrinker of mean curvature flow in $\mathbb{R}^{n+1}$.

If $X: M^n \to \mathbb{R}^{n+1}$ satisfies

$$\left( \int_M |S|^n d\mu \right)^{\frac{2}{n}} < \frac{4}{3nK},$$

then $X: M^n \to \mathbb{R}^{n+1}$ is isometric to the Euclidean space $\mathbb{R}^n$, where $S$ denotes the squared norm of the second fundamental form of $X: M^n \to \mathbb{R}^{n+1}$.

In [3], Cheng and Wei have introduced a notation of so-called $\lambda$-hypersurfaces of the weighted volume-preserving mean curvature as follows:

Definition 1.1. Let $X: M^n \to \mathbb{R}^{n+1}$ be an immersed hypersurface in $\mathbb{R}^{n+1}$. If

$$H + \langle X, N \rangle = \lambda$$

is satisfied, where $\lambda$ is constant, then $X: M^n \to \mathbb{R}^{n+1}$ is called a $\lambda$-hypersurface of the weighted volume-preserving mean curvature. For simple, we call it a $\lambda$-hypersurface.

Remark 1.2. From definition, we know that if $\lambda = 0$, $X: M^n \to \mathbb{R}^{n+1}$ is a self-shrinker of mean curvature flow.

Example 1.1. All of self-shrinkers of mean curvature flow is $\lambda$-hypersurfaces with $\lambda = 0$.

Example 1.2. The $n$-dimensional sphere $S^n(r)$ with $r > 0$ is a compact $\lambda$-hypersurface with $\lambda = \frac{n}{r} - r$. We should notice that only the $n$-dimensional sphere $S^n(\sqrt{n})$ is the self-shrinker of mean curvature flow.

Example 1.3. The $n$-dimensional cylinder $S^k(r) \times \mathbb{R}^{n-k}$ with $r > 0$ for $1 \leq k \leq n-1$ is a complete and non-compact $\lambda$-hypersurface with $\lambda = \frac{k}{r} - r$. We remark that only the $n$-dimensional cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, for $1 \leq k \leq n-1$, is the self-shrinker of mean curvature flow.

Let $X(t): M^n \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ be a variation of $X: M^n \to \mathbb{R}^{n+1}$. The weighted volume $V: (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined in [3] as follows:

$$V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu.$$
then $X : M \to \mathbb{R}^{n+1}$ is isometric to one of the following:

1. the sphere $S^n(r)$ with radius $r \leq \sqrt{n}$,
2. the Euclidean space $\mathbb{R}^n$,
3. the cylinder $S^1(r) \times \mathbb{R}^{n-1}$ with radius $r > 0$ and $n = 2$ or with radius $r \geq 1$ and $n > 2$,
4. the cylinder $S^{n-1}(r) \times \mathbb{R}$ with radius $r > 0$ and $n = 2$ or with radius $r \leq \sqrt{n-1}$ and $n > 2$,
5. the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ for $2 \leq k \leq n-2$.

In this paper, we study a global pinching theorem of complete $\lambda$-hypersurfaces in $\mathbb{R}^{n+1}$. We prove the following:

**Theorem 1.3.** Let $X : M^n \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete proper $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$. If $X : M^n \to \mathbb{R}^{n+1}$ satisfies

$$\int_M \left\| \frac{n\lambda((n-2)H^2 - n(2n+2)}{2\sqrt{n+1}} B + \frac{n^2 - 2n + 2}{2n(n-1)} H \right\| d\mu < \frac{n-2}{n} k(n)^{-1},$$

then $X : M^n \to \mathbb{R}^{n+1}$ is isometric to the Euclidean space $\mathbb{R}^n$ or the sphere $S^n(r)$ with

$$\left| \frac{3n}{n-1}(n+2) \right| < \frac{(n-2)^3}{4^{2(n+1)}n^2(n-1)(3n-4)(n+2)} \left( \frac{\omega_{n-1}}{\omega_n} \right)^\frac{2}{n},$$

where $B = S - \frac{H^2}{n}$, $k(n) = 2 \frac{4^{2(n+1)}(n-1)(3n-4)}{(n-2)^2} \left( \frac{n}{\omega_{n-1}} \right)^\frac{2}{n}$, and $\omega_k$ denotes the area of the $k$-dimensional unit sphere $S^k(1)$.

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### 2. The Sobolev Inequality

In order to prove our theorem, the following Sobolev inequality in [9] plays a very important role.

**Theorem 2.1.** Let $X : M^n \to \mathbb{R}^{n+1}$ be a hypersurface in $\mathbb{R}^{n+1}$. For any Lipschitz function $f \geq 0$ with compact support on $M$,

$$\left( \int_M f^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}} \leq C_n \int_M \{ |\nabla f| + |H| f \} d\mu$$

holds, where

$$C_n = 4^{n+1} \left( \frac{n}{\omega_{n-1}} \right)^\frac{1}{n}.$$
From the above theorem, we have

**Corollary 2.1.** Let \( X : M^n \to \mathbb{R}^{n+1} \) be a hypersurface in \( \mathbb{R}^{n+1} \). For any Lipschitz function \( f \geq 0 \) with compact support on \( M \),

\[
(2.2) \quad k(n)^{-1} \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 d\mu + \frac{n-2}{2(n-1)} \int_M H^2 f^2 d\mu
\]

holds, where

\[
k(n) = 2 \cdot 4^{2(n+1)} (n-1)(3n-4) \left( \frac{n}{\omega_{n-1}} \right)^{\frac{2}{n}}.
\]

**Proof.** Replacing \( f \) in the theorem 2.1 with \( \frac{f^{2(n-1)}}{f^{2(n-2)}} \), we get

\[
\left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-1}{n}} \leq C_n \int_M \left\{ |\nabla f^{\frac{2(n-1)}{n-2}}| + |H| f^{\frac{2(n-1)}{n-2}} \right\} d\mu
\]

\[
= C_n \int_M \left\{ \frac{2(n-1)}{n-2} f^{\frac{n}{n-2}} |\nabla f| + |H| f^{\frac{2(n-1)}{n-2}} \right\} d\mu.
\]

By Hölder’s inequality, we have

\[
\int_M f^{\frac{n}{n-2}} |\nabla f| d\mu \leq \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{1}{2}} \left( \int_M |\nabla f|^2 d\mu \right)^{\frac{1}{2}},
\]

\[
\int_M |H| f^{\frac{2(n-1)}{n-2}} d\mu \leq \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{1}{2}} \left( \int_M H^2 f^2 d\mu \right)^{\frac{1}{2}}.
\]

Hence,

\[
\left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-1}{n}} \leq C_n \frac{2(n-1)}{n-2} \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{1}{2}} \left( \int_M |\nabla f|^2 d\mu \right)^{\frac{1}{2}}
\]

\[+ C_n \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{1}{2}} \left( \int_M H^2 f^2 d\mu \right)^{\frac{1}{2}}.
\]

Therefore

\[
\left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{2n}} \leq C_n \left\{ \frac{2(n-1)}{n-2} \left( \int_M |\nabla f|^2 d\mu \right)^{\frac{1}{2}} + \left( \int_M H^2 f^2 d\mu \right)^{\frac{1}{2}} \right\}.
\]

According to \( \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{2n}} \geq 0 \), we have

\[
\left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{2n}} \leq C_n^2 \left\{ \frac{4(n-1)^2}{(n-2)^2} \int_M |\nabla f|^2 d\mu + \int_M H^2 f^2 d\mu \right. \]

\[+ \frac{4(n-1)}{n-2} \left( \int_M |\nabla f|^2 d\mu \right)^{\frac{1}{2}} \left( \int_M H^2 f^2 d\mu \right)^{\frac{1}{2}} \right\}.
\]
Because of $\int_M |\nabla f|^2 d\mu \geq 0$ and $\int_M H^2 f^2 d\mu \geq 0$, we get

$$\left( \int_M f^2 \right)^{\frac{n-2}{2}} \leq C_n^2 \left\{ \frac{4(n-1)^2}{(n-2)^2} \int_M |\nabla f|^2 d\mu + \int_M H^2 f^2 d\mu + \frac{2(n-1)}{n-2} \left( \int_M |\nabla f|^2 d\mu + \int_M H^2 f^2 d\mu \right) \right\}$$

$$= C_n^2 \frac{2(n-1)}{n-2} \left( \frac{2(n-1)}{n-2} + 1 \right) \left\{ \int_M |\nabla f|^2 d\mu + \frac{n-2}{2(n-1)} \int_M H^2 f^2 d\mu \right\}.$$

Let $k(n) = C_n^2 \frac{2(n-1)}{n-2} \left( \frac{2(n-1)}{n-2} + 1 \right)$, then we get

$$k(n)^{-1} \left( \int_M f^2 \right)^{\frac{n-2}{2}} \leq \int_M |\nabla f|^2 d\mu + \frac{n-2}{2(n-1)} \int_M H^2 f^2 d\mu.$$

\[\square\]

3. Proof of our global pinching theorem

In order to prove the theorem \[1.3\] we prepare several lemmas. For the differential operator $\mathcal{L}$ defended by

$$\mathcal{L}f = \Delta f - \langle \nabla f, X \rangle,$$

where $\Delta$ and $\nabla$ denote the Laplace operator and the gradient operator, we have proved the following lemma in \[2\].

**Lemma 3.1.** For $B = S - \frac{H^2}{n}$, we have

$$\frac{1}{2} \mathcal{L}B \geq - \frac{|\lambda|(n-2)}{\sqrt{n(n-2)}} B^{\frac{3}{2}} + B - B^2 - \frac{1}{n} H^2 B + \frac{2\lambda}{n} H B.$$ (3.1)

Define a function $\rho$ by

$$\rho = e^{-\frac{|X|^2}{4}}.$$

**Lemma 3.2.** For any smooth function $\eta$ with compact support on $M$ and an arbitrary positive constant $\varepsilon$, we have

$$\int_M \left\{ \frac{|\lambda|(n-2)}{\sqrt{n(n-1)}} B^{n+\frac{1}{2}} \eta^2 \rho - B^n \eta^2 \rho + B^{n+1} \eta^2 \rho + \frac{1}{n} H^2 B^n \eta^2 \rho + \frac{1}{2\varepsilon} B^n |\nabla \eta|^2 \rho \right\} d\mu$$

$$\geq \frac{n-1 - \varepsilon}{2} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \ d\mu.$$ (3.2)
Proof. Multiplying $B^{n-1} \eta^2 \rho$ on both sides of (3.1) and taking integral, we obtain

$$0 \geq \int_M \left\{ - \frac{\lambda |(n-2)|}{\sqrt{n(n-1)}} B^{n+\frac{1}{2}} \eta^2 \rho + B^n \eta^2 \rho - B^{n+1} \eta^2 \rho \
- \frac{1}{n} H^2 B^n \eta^2 \rho + \frac{2\lambda}{n} HB^n \eta^2 \rho - \frac{1}{2} \mathcal{L} B \cdot B^{n-1} \eta^2 \rho \right\} d\mu.$$ 

Since $\eta$ has compact support on $M$, according to Stokes theorem, we get

$$\frac{1}{2} \int_M \mathcal{L} B \cdot B^{n-1} \eta^2 \rho \, d\mu = \frac{1}{2} \int_M \text{div}(\rho \nabla B) \cdot B^{n-1} \eta^2 \, d\mu = \frac{1}{2} \int_M \langle \rho \nabla B, \nabla (B^{n-1} \eta^2) \rangle \, d\mu = \frac{n-1}{2} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, d\mu + \int_M B^{n-1} \eta \langle \nabla B, \nabla \eta \rangle \rho \, d\mu.$$ 

Moreover, for an arbitrary constant $\varepsilon > 0$, we have

$$\int_M B^{n-1} \eta \langle \nabla B, \nabla \eta \rangle \rho \, d\mu \geq -\frac{\varepsilon}{2} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, d\mu - \frac{1}{2\varepsilon} \int_M B^n |\nabla \eta|^2 \rho \, d\mu.$$ 

Hence, we obtain

$$\int_M \left\{ - \frac{\lambda |(n-2)|}{\sqrt{n(n-1)}} B^{n+\frac{1}{2}} \eta^2 \rho + B^n \eta^2 \rho - B^{n+1} \eta^2 \rho \
- \frac{1}{n} H^2 B^n \eta^2 \rho + \frac{2\lambda}{n} HB^n \eta^2 \rho - \frac{1}{2} \mathcal{L} B \cdot B^{n-1} \eta^2 \rho \right\} d\mu \geq \frac{n-1}{2} - \frac{\varepsilon}{2} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, d\mu.$$ 

Lemma 3.3. Putting $f := B^{\frac{n}{2}} \eta^\frac{1}{2}$, we know that

$$(3.3) \quad \int_M |\nabla f|^2 \, d\mu = \frac{n^2}{4} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, d\mu + \int_M B^n |\nabla \eta|^2 \rho \, d\mu + n \int_M B^{n-1} \eta \langle \nabla B, \nabla \eta \rangle \rho \, d\mu$$

$$- \frac{1}{4} \int_M |X^\top|^2 B^n \eta^2 \rho \, d\mu + \frac{\lambda}{2} \int_M \langle X, N \rangle B^n \eta^2 \rho \, d\mu - \frac{1}{2} |X^\perp|^2 B^n \eta^2 \rho \, d\mu$$

$$+ \frac{n}{2} \int_M B^n \eta^2 \rho \, d\mu$$
and

\[
\frac{1}{2} \int_M H^2 f^2 d\mu = \frac{\lambda^2}{2} \int_M B^n \eta^2 \rho \ d\mu - \lambda \int_M \langle X, N \rangle B^n \eta^2 \rho \ d\mu + \frac{1}{2} \int_M |X^\perp|^2 B^n \eta^2 \rho \ d\mu.
\]

hold.

Proof. Calculating the left hand side of (3.3), we know

\[
\int_M |\nabla f|^2 d\mu = \int_M |\nabla (B^n \eta^2)|^2 \rho \ d\mu + \int_M B^n \eta^2 |\nabla \rho^\perp|^2 d\mu + 2 \int_M B^n \eta^2 \rho^\perp \langle \nabla (B^n \eta), \nabla \rho^\perp \rangle d\mu.
\]

Putting

\[
T_1 := \int_M |\nabla (B^n \eta^2)|^2 \rho \ d\mu,
T_2 := \int_M B^n \eta^2 |\nabla \rho^\perp|^2 d\mu,
T_3 := 2 \int_M B^n \eta^2 \rho^\perp \langle \nabla (B^n \eta), \nabla \rho^\perp \rangle d\mu.
\]

Because of \(|\nabla \rho^\perp|^2 = \frac{1}{4} |X^\top|^2 \rho\) and \(\Delta X = HN\), we have

\[
T_2 = \frac{1}{4} \int_M |X^\top|^2 B^n \eta^2 \rho \ d\mu
\]

and

\[
\Delta \rho = |X^\top|^2 \rho - \langle \Delta X, X \rangle \rho - n\rho
\]

\[
= |X^\top|^2 \rho - \lambda \langle X, N \rangle \rho + |X^\perp|^2 \rho - n\rho.
\]

Hence,

\[
T_3 = \frac{1}{2} \int_M \langle \nabla (B^n \eta^2), \nabla \rho \rangle d\mu
\]

\[
= -\frac{1}{2} \int_M B^n \eta^2 \cdot \Delta \rho \ d\mu
\]

\[
= -\frac{1}{2} \int_M |X^\top|^2 B^n \eta^2 \rho \ d\mu + \frac{\lambda}{2} \int_M \langle X, N \rangle B^n \eta^2 \rho \ d\mu
\]

\[
- \frac{1}{2} \int_M |X^\perp|^2 B^n \eta^2 \rho \ d\mu + \frac{n}{2} \int_M B^n \eta^2 \rho \ d\mu.
\]
Therefore, we get

\[
\int_M |\nabla f|^2 d\mu = \frac{n^2}{4} \int_M B^{n-2}\eta^2 |\nabla B|^2 \rho \, d\mu + \int_M B^n |\nabla \eta|^2 \rho \, d\mu + n \int_M B^{n-1}\eta \langle \nabla B, \nabla \eta \rangle \, \rho \, d\mu - \frac{1}{4} \int_M |X|^2 B^n \eta^2 \rho \, d\mu + \frac{n}{2} \int_M B^n \eta^2 \rho \, d\mu.
\]

From \( H = \lambda - \langle X, N \rangle \), we get

\[
\frac{1}{2} \int_M H^2 f^2 \, d\mu = \frac{1}{2} \left( \lambda - \langle X, N \rangle \right)^2 B^n \eta^2 \rho \, d\mu = \frac{\lambda^2}{2} \int_M B^n \eta^2 \rho \, d\mu - \lambda \int_M \langle X, N \rangle B^n \eta^2 \rho \, d\mu + \frac{1}{2} \int_M |X|^2 B^n \eta^2 \rho \, d\mu.
\]

\[\Box\]

**Lemma 3.4.** For an arbitrary constant \( \delta > 0 \), we have

\[
k(n)^{-1} \left( \int_M f^{\frac{2n}{n-2}} \, d\mu \right)^{\frac{n-2}{n}} \leq \frac{(1 + \delta)n^2}{4} \int_M B^{n-2}\eta^2 |\nabla B|^2 \rho \, d\mu + \left( 1 + \frac{1}{\delta} \right) \int_M B^n |\nabla \eta|^2 \rho \, d\mu - \frac{1}{2(n-1)} \int_M H^2 B^n \eta^2 \rho \, d\mu + \frac{\lambda}{2} \int_M H B^n \eta^2 \rho \, d\mu + \frac{n}{2} \int_M B^n \eta^2 \rho \, d\mu,
\]

where \( k(n) \) is the assertion of the Corollary 2.1.

Proof. From Corollary 2.1, we have, for any function \( f \) with compact support on \( M \),

\[
k(n)^{-1} \left( \int_M f^{\frac{2n}{n-2}} \, d\mu \right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 \, d\mu + \frac{n-2}{2(n-1)} \int_M H^2 f^2 \, d\mu \leq \int_M |\nabla f|^2 \, d\mu + \frac{1}{2} \int_M H^2 f^2 \, d\mu - \frac{1}{2(n-1)} \int_M H^2 f^2 \, d\mu.
\]
Taking $f = B^{\frac{n}{2}} \eta \rho^{\frac{1}{2}}$, from Lemma 3.3 we infer
\[
k(n)^{-1} \left( \int_M f^{\frac{2n}{n-2}} \mathrm{d}\mu \right)^{\frac{n-2}{n}} \leq \frac{n^2}{4} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, \mathrm{d}\mu + \int_M B^n |\nabla \eta|^2 \rho \, \mathrm{d}\mu + n \int_M B^{n-1} \eta \langle \nabla B, \nabla \eta \rangle \rho \, \mathrm{d}\mu - \frac{1}{4} \int_M |X|^2 B^n \eta^2 \rho \, \mathrm{d}\mu + \frac{\lambda^2}{2} \int_M B^n \eta^2 \rho \, \mathrm{d}\mu - \frac{\lambda}{2} \int_M \langle X, N \rangle B^n \eta^2 \rho \, \mathrm{d}\mu + \frac{\lambda^2}{4} \int_M B^{n-1} \eta \langle \nabla B, \nabla \eta \rangle \rho \, \mathrm{d}\mu \leq \frac{n^2}{4} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, \mathrm{d}\mu + \frac{1}{2(n-1)} \int_M H^2 f^2 \, \mathrm{d}\mu - \frac{\lambda}{2} \int_M \langle X, N \rangle B^n \eta^2 \rho \, \mathrm{d}\mu + \frac{\lambda^2}{4} \int_M B^{n-1} \eta \langle \nabla B, \nabla \eta \rangle \rho \, \mathrm{d}\mu + \frac{\lambda^2}{4} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, \mathrm{d}\mu + \frac{n^2}{4} \int_M B^n |\nabla \eta|^2 \rho \, \mathrm{d}\mu.
\]

For an arbitrary constant $\delta > 0$, we have
\[
n \int_M B^{n-1} \eta \langle \nabla B, \nabla \eta \rangle \rho \, \mathrm{d}\mu \leq \frac{\delta n^2}{4} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, \mathrm{d}\mu + \frac{\lambda^2}{4} \int_M B^n |\nabla \eta|^2 \rho \, \mathrm{d}\mu.
\]

Hence, we get
\[
k(n)^{-1} \left( \int_M f^{\frac{2n}{n-2}} \mathrm{d}\mu \right)^{\frac{n-2}{n}} \leq \frac{(1 + \delta)n^2}{4} \int_M B^{n-2} \eta^2 |\nabla B|^2 \rho \, \mathrm{d}\mu + \frac{\lambda^2}{4} \int_M B^n |\nabla \eta|^2 \rho \, \mathrm{d}\mu - \frac{1}{2(n-1)} \int_M H^2 B^n \eta^2 \rho \, \mathrm{d}\mu + \frac{n^2}{4} \int_M B^n |\nabla \eta|^2 \rho \, \mathrm{d}\mu.
\]

\[\Box\]

**Proof of Theorem 1.3** If $B \neq 0$ holds, we can choose $\eta$ such that, for $f = B^{\frac{n}{2}} \eta \rho^{\frac{1}{2}}$,
\[
\left( \int_M f^{\frac{2n}{n-2}} \mathrm{d}\mu \right)^{\frac{n-2}{n}} \neq 0.
\]
From Lemma 3.2 and Lemma 3.4, then for arbitrary constants $\varepsilon > 0$ and $\delta > 0$,

\[
k(n)^{-1} \left( \int_M f^{2n} \, d\mu \right)^{\frac{n-2}{n}} \leq \frac{(1 + \delta)n^2}{2} \frac{1}{n - 1 - \varepsilon} \int_M \left\{ \frac{|\lambda|(n-2)}{\sqrt{n(n-1)}} B^{n+\frac{1}{2}} \eta^2 \rho - B^{n} \eta^2 \rho + B^{n+1} \eta^2 \rho \right. \\
+ \frac{1}{n} H^2 B^{n} \eta^2 \rho - \frac{2\lambda}{n} H B^{n} \eta^2 \rho + \frac{1}{2\varepsilon} B^{n} |\nabla \eta|^2 \rho \right\} d\mu \\
+ \left(1 + \frac{1}{\delta}\right) \int_M B^n |\nabla \eta|^2 \rho \, d\mu - \frac{1}{2(n-1)} \int_M H^2 B^n \eta^2 \rho \, d\mu \\
+ \frac{\lambda}{2} \int_M H B^n \eta^2 \rho \, d\mu + \frac{n}{2} \int_M B^n \eta^2 \rho \, d\mu.
\]

Letting $1 + \delta = \frac{n - 1 + \varepsilon}{n}$, then, we derive

\[
k(n)^{-1} \left( \int_M f^{2n} \, d\mu \right)^{\frac{n-2}{n}} \leq \frac{n - 1 + \varepsilon}{n - 1 - \varepsilon} \int_M \left\{ \frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}} B^{n+\frac{1}{2}} \eta^2 \rho + \frac{n}{2} B^{n+1} \eta^2 \rho \right. \\
+ \frac{1}{2} \left(1 - \frac{1}{n - 1 - \varepsilon}\right) H^2 B^n \eta^2 \rho + \left(-1 + \frac{1}{2 \varepsilon} \right) \lambda H B^n \eta^2 \rho \right\} d\mu \\
+ \frac{n}{2} \left(-\frac{n - 1 + \varepsilon}{n - 1 - \varepsilon} + 1\right) \int_M B^n \eta^2 \rho \, d\mu + C(n, \varepsilon) \int_M B^n |\nabla \eta|^2 \rho \, d\mu \\
+ \leq \frac{n - 1 + \varepsilon}{n - 1 - \varepsilon} \int_M \left\{ \frac{n|\lambda|(n-2)}{2\sqrt{n(n-1)}} B^{n+\frac{1}{2}} \eta^2 \rho + \frac{n}{2} B^{n+1} \eta^2 \rho \right. \\
+ \frac{1}{2} \left(1 - \frac{1}{n - 1 - \varepsilon}\right) H^2 B^n \eta^2 \rho + \left(-1 + \frac{1}{2 \varepsilon} \right) \lambda H B^n \eta^2 \rho \right\} d\mu \\
+ C(n, \varepsilon) \int_M B^n |\nabla \eta|^2 \rho \, d\mu,
\]
where $C(n, \varepsilon)$ is a positive constant only depending on $n$ and $\varepsilon$. From $f^2 = B^n \eta^2 \rho$ and using Hölder’s inequality, we obtain

$$k(n)^{-1} \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \leq \frac{n-1 + \varepsilon}{n-1 - \varepsilon} \left( \int_M \frac{|n|\lambda|(n-2)|}{2\sqrt{n(n-1)}} B^\frac{\lambda}{2} + \frac{n}{2} B + \frac{1}{2} \left( 1 - \frac{1}{(n-1) n-1 + \varepsilon} \right) H^2 \right) \left( -1 + \frac{1}{2} \frac{n-1 - \varepsilon}{2n-1 + \varepsilon} \right) \lambda H \left( \int_{\mu} \eta \right)^\frac{n-2}{n} \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} + C(n, \varepsilon) \int_M B^n |\nabla \eta|^2 \rho d\mu.$$

Therefore, we have

$$k(n)^{-1} \leq \frac{n-1 + \varepsilon}{n-1 - \varepsilon} \left( \int_M \frac{|n|\lambda|(n-2)|}{2\sqrt{n(n-1)}} B^\frac{\lambda}{2} + \frac{n}{2} B + \frac{1}{2} \left( 1 - \frac{1}{(n-1) n-1 + \varepsilon} \right) H^2 \right) \left( -1 + \frac{1}{2} \frac{n-1 - \varepsilon}{2n-1 + \varepsilon} \right) \lambda H \left( \int_{\mu} \eta \right)^\frac{n-2}{n} \left( \int_M f^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} + C(n, \varepsilon) \int_M B^n |\nabla \eta|^2 \rho d\mu.$$

Since $X : M^n \to \mathbb{R}^{n+1}$ is proper, it is proved by Cheng and Wei in [3] that $X : M^n \to \mathbb{R}^{n+1}$ has at most polynomial area growth. Hence, we know that

$$\int_M B^n \rho d\mu < \infty.$$

Taking $\eta = \phi(\frac{|X|}{r})$ for any $r > 0$, where $\phi$ is a nonnegative function on $[0, \infty)$ such that

$$\phi(t) = \begin{cases} 1, & \text{if } t \in [0, 1] \\ 0, & \text{if } t \in [2, \infty) \end{cases}$$

and $|\phi'| \leq c$ for some absolute constant. Taking $r \to \infty$, we have

$$\int_M B^n |\nabla \eta|^2 \rho d\mu \to 0.$$
Therefore, we get
\[
k(n)^{-1} \leq \frac{n-1+\varepsilon}{n-1-\varepsilon} \left( \int_{\mathcal{M}} \frac{n!|\lambda|(n-2)}{2\sqrt{n(n-1)}} B^ \frac{3}{2} + \frac{n}{2} B + \frac{1}{2} \left( 1 - \frac{1}{(n-1)n-1+\varepsilon} \right) H^2 \right.
\]
\[
+ \left. \left( -1 + \frac{1}{2} \frac{n-1-\varepsilon}{n-1+\varepsilon} \right) \frac{\lambda H}{\sqrt{\mu}} \right) \frac{n}{n} \right)^{\frac{1}{2}}.
\]
Letting \( \varepsilon \to 1 \), we obtain
\[
k(n)^{-1} \leq \frac{n}{n-2} \left( \int_{\mathcal{M}} \frac{n!|\lambda|(n-2)}{2\sqrt{n(n-1)}} B^ \frac{3}{2} + \frac{n}{2} B + \frac{n^2-2n+2}{2n(n-1)} H^2 \right.
\]
\[
- \frac{n+2}{2n} \frac{\lambda H}{\sqrt{\mu}} \right) \frac{n}{n} \right)^{\frac{1}{2}} < \frac{n-2}{nk(n)} = k(n)^{-1}.
\]
It is a contradiction. Thus, we have \( B = S - \frac{H^2}{n} \equiv 0 \), that is, \( X : M^n \to \mathbb{R}^{n+1} \) is totally umbilical. Hence, we know that \( X : M^n \to \mathbb{R}^{n+1} \) is isomorphic to \( \mathbb{R}^n \) or a sphere \( S^n(r) \) with radius \( r \), which satisfies (1.3) from (1.2).

\[\square\]

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