Quasi-local holographic dualities in non-perturbative 3D quantum gravity

Bianca Dittrich, Christophe Goeller, Etera R Livine and Aldo Riello

1 Perimeter Institute, 31 Caroline St North, N2L 2Y5, Waterloo ON, Canada
2 Laboratoire de Physique, ENS Lyon, CNRS-UMR 5672, 46 allée d’Italie, Lyon 69007, France

E-mail: bdittrich@perimeterinstitute.ca, christophe.goeller@ens-lyon.fr, etera.livine@ens-lyon.fr and ariello@perimeterinstitute.ca

Received 15 March 2018, revised 10 May 2018
Accepted for publication 18 May 2018
Published 11 June 2018

Abstract

We present a line of research aimed at investigating holographic dualities in the context of three dimensional quantum gravity within finite bounded regions. The bulk quantum geometrodynamics is provided by the Ponzano–Regge state-sum model, which defines 3D quantum gravity as a discrete topological quantum field theory (TQFT). This formulation provides an explicit and detailed definition of the quantum boundary states, which allows a rich correspondence between quantum boundary conditions and boundary theories, thereby leading to holographic dualities between 3D quantum gravity and 2D statistical models as used in condensed matter. After presenting the general framework, we focus on the concrete example of the coherent twisted torus boundary, which allows for a direct comparison with other approaches to 3D/2D holography at asymptotic infinity. We conclude with the most interesting questions to pursue in this framework.

Keywords: spin foams, holography, 3D gravity quantum gravity, integrable models, dual theories

(Some figures may appear in colour only in the online journal)

1. Overview

In this short paper, we present a research program aimed at understanding the bulk-boundary relationship from the bulk non-perturbative quantum gravity perspective. Our goal is to clarify the role of boundaries and the physics of edge modes in quantum gravity. In non-perturbative approaches to quantum gravity, it is natural to consider finite, or ‘quasi-local’...
boundaries. On these, we can derive the boundary theory induced by the choice of classical boundary conditions or class of quantum boundary states. In a holographic approach, these boundary theories define holographic duals, still encoding the same physical content as the bulk theory, providing a non-trivial representation of bulk observables and allowing to reconstruct the bulk quantum geometry from boundary correlations. Our objective is to test this non-perturbative holographic approach, which provides a quasi-local version of the AdS/CFT correspondence [1].

Three-dimensional gravity is an ideal testbed for this program. Indeed, it can be formulated as a topological field theory, which allows for an exact non-perturbative quantization [2]. In this context, we propose to study the Ponzano–Regge state-sum model [3–6]. This is a three-dimensional discrete topological quantum field theory (TQFT), whose relation to the Turaev–Viro [7] and Reshetikhin–Turaev topological invariants is explicitly understood [8–10].

Concretely, the Ponzano–Regge model provides a quantized path integral for 3D Regge calculus, which is a well-defined discretization of general relativity [11, 12]. Specifically, it corresponds to a theory of quantum gravity with vanishing cosmological constant in Euclidean signature. Generalizations of the model to non-vanishing cosmological constant exists [5, 7, 13, 14], but we postpone their investigation to future work.

From the Ponzano–Regge state-sum, transition amplitudes for physical states of quantum geometry can be explicitly computed. However, the question of holographic dualities has not yet been systematically explored in this framework, despite being a very natural one. This is because, the Ponzano–Regge model provides a bulk-local description of the 3D quantum geometry. It is thus natural to focus on finite bounded regions of space-time, explore the various possible quantum boundary conditions and investigate the resulting quantum boundary theories.

Since the Ponzano–Regge model is intrinsically discrete, we expect the Ponzano–Regge amplitude with boundaries to induce discrete statistical physics models on the 2D boundary. Such models typically lead to 2D conformal field theories (CFTs) in their continuum limit in their critical regime. This scenario would lead to a quasi-local version of the AdS3/CFT2 correspondence [15–19].

An important question is whether the continuum limit can be reached for finite boundaries at all. Indeed, quantum gravity naturally sets a fundamental minimal length scale and for this reason it is likely that the infinite refinement limit needed for reaching the continuum cannot be achieved for finite boundaries. In turn, this would mean that the dual statistical physics boundary theories can only reach their critical regime for asymptotic boundaries. In other words, they would flow towards their limit CFT as the typical length scale of the boundary geometry is much larger than the fundamental quantum gravity length scale, and to the extent that the latter scale can be neglected. For finite boundaries, we would then be left with the non-critical discrete statistical models as holographic duals. Their correlations should still allow to probe the bulk 3D quantum geometry and faithfully represent the 3D quantum gravity amplitudes.

At this point a remark is in order. Three-dimensional gravity is topological, i.e. it does not possess any local physical degrees of freedom. For this reason, discrete models such as the Ponzano–Regge topological state-sum can capture all the relevant degrees of freedom of the continuum theory, irrespectively of the coarseness of the employed discretization. However, the introduction of a boundary with metric boundary conditions does reveal the discretization, which explains our remarks in the previous paragraph.

In this paper, we present and streamline our recent results derived in details [20, 21] and push their analysis further. In an effort to clarify the context of that work, we explain how to derive the holographic duals on the boundary of the Ponzano–Regge model (see also [22]),
and discuss the critical problem of identifying a quantum notion of asymptotic infinity. In particular, we discuss how we are able to reproduce results found in perturbative contexts [23, 24] while at the same time extending these results by non-perturbative corrections.

Since in the Ponzano–Regge model there are two ways of increasing the size of the boundary and the number of degrees of freedom it carries, two naive notions of large scale limit exist: one consists in taking a large number of Planck-sized building blocks, the other in taking a large value for the spins $j$ at fixed number of building blocks. In the former approach, a renormalization flow can be defined that describes effective actions taking the effects of a finer and finer lattice into account (see e.g. [25]). On the top of this refinement limit, a semi-classical limit can also be taken. The latter approach, instead, turns out to be best understood not in in terms of a large-scale limit, but rather in terms of a semiclassical (e.g. 0 or 1-loop) limit on a fixed discretization (e.g. [3, 26–29]). On the top of this semiclassical limit, a continuum limit can also be taken. Therefore, what seemed to be two different ways of taking a large scale limit, are rather two different limits altogether3. When both are taken one after the other, the result will not a priori be independent of the order in which the limits are taken. Nonetheless, we showed in [20, 21] that some important features agree in the two approaches. Indeed, both these choices remarkably lead to the same divergence behavior of the associated amplitudes [20, 21]. Crucially, the same behavior that characterizes the perturbative results as well. In the case of the semi-classical calculation, the one-loop result not only reproduces the perturbative ones obtained both in the continuum [23] and in the discrete [24], but also includes contributions of saddle points associated to non-classical ‘quantum’ backgrounds.

We hope that the present program will also help elucidate renormalization in discrete quantum gravity models [25, 32–34] and in particular provide first steps to implement holographic renormalization [35, 36] in non-perturbative quantum gravity models.

2. Statistical duals

The Ponzano–Regge model [3–6] proposes a topological path integral for discretized 3D geometries. Initially defined for triangulations, it can readily be extended to arbitrary 3D cellular decomposition. The amplitudes are well-defined through suitable gauge-fixing [4, 37, 38] and define a topological invariant [6, 10]. They have two equivalent definitions, either in terms of products of spin recoupling symbols (such as the $\{6j\}$-symbol representing a quantized tetrahedron) or as a discretized path integral over holonomies encoding the parallel transport along the triangulated manifold. Here, we will not review these definitions of the Ponzano–Regge bulk amplitude, but we will focus on the boundary states and induced boundary theory.

The general relation between boundary conditions, or boundary states, and (dual) boundary theories, is to be looked for in the correspondence between spin-network states and statistical models [22, 39–45].

Spin-network states are gauge-invariant Wilson graph observables naturally associated to the connection representation of gravity. Denoting by $\omega$ the relevant connection 1-form and

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3 For completeness, let us mention an extra possibility with regards to the continuum limit, which we will not pursue here. It is possible to consider the discretization itself as a quantum degree of freedom to be ‘summed over’. This is actually the main insight for the dynamical triangulation approach to quantum gravity. In the context of spinfoam models, this point of view leads to group field theory and tensor models, for which one define and study a renormalization flow [30, 31]. It is an open question how these various limits and renormalization flows connect to each other.
by \( g_l = P \exp \int_\omega \omega \) the parallel transport along the \( i \)th link \( l_i \) of the Wilson graph \( \Gamma \), such states have the form

\[
\Psi[\omega] = \Psi^\Gamma(\{g_{l_i}\}) = \Psi^\Gamma(\{h_{h(l_i)}g_{l_i}h_{h(l_i)}^{-1}\}),
\]

where the last equality is a representation of gauge invariance, with \( s(l) \) and \( t(l) \) denoting the source and target vertices of \( l \), respectively. Using standard loop quantum gravity techniques, when the supporting graph is dual to a surface, spin-network states—or some specific superpositions thereof—can be interpreted as quantum boundary metrics.

Being supported on a graph, these are intrinsically discrete objects. It is important to notice, however, that the Hilbert space of spin-network states in loop quantum gravity can be understood as a space of continuum quantum connections\(^4\), and in this sense all spin-network states are continuum states. Nonetheless, their operational geometrical interpretation still reposes on discrete structures, and their discreteness can be interpreted as the result of a ‘physical’ finite-resolution pre- or post-selecting measurement. In this case, the graph itself can be interpreted quite literally as a network of physical beacons and space-measuring devices.

On top of the graph-induced discreteness, when dealing with Euclidean theories, another type of discreteness comes into play. This is the Planck-scale discreteness associated to the spectrum of metric operators. The mathematical origin of this discreteness lies in the compactness of the parallel transport variables \( g_l \) along the edges of the graph, which replaced their infinitesimal counterparts—the connection variable \( \omega \)—as the fundamental variable.

In three Euclidean dimensions, \( g_l \in SU(2) \) and the specifics of the quantum metric described by a spin-network state are encoded in the choice of (i) \( SU(2) \) spins \( j_l \) attached to the edges of the spin-network graph, and of (ii) gauge-invariant tensors, aka intertwiners \( \iota_v \), attached to its vertices. Spin-network states with fixed spins and intertwiners will be denoted by

\[
\Psi^\Gamma_{(j,\iota)}(g_l).
\]

More specifically, intertwiners are \( SU(2) \)-invariant states in the tensor product of several \( SU(2) \) representations, say labeled by \( N \) spins \( j_1, \ldots, j_N \),

\[
\iota_v = \bigotimes_{i=1}^N D^j(h) \triangleright \iota_v, \quad \forall h \in SU(2),
\]

that is, explicitly showing the sum over magnetic indices,

\[
(t_v)_{n_1 \ldots n_N} = \sum_m \left( \prod_{i=1}^N D^j(h)_{n_i m_i} \right) (t_v)_{m_1 \ldots m_N}.
\]

With this notation, \( \Psi^\Gamma_{(j,\iota)}(g_l) \) is defined as the contraction of the vertex intertwiners with the \( j_l \)-representation Wigner matrices of the parallel transports \( g_l \), according to the combinatorics imposed by the underlying graph \( \Gamma \):

\[
\Psi^\Gamma_{(j,\iota)}(g_l) = \text{Tr}_\Gamma \left( \bigotimes_v \iota_v \otimes \bigotimes_l D^j(g_l) \right).
\]

\(^4\)This is guaranteed by cylindrical consistency of the wave functions together with the existence of an inductive limit on the wave functions: spin networks are actually defined on infinite sequences of refining graphs. The inductive continuum limit can moreover be defined both for the Ashtekar–Lewandowski and BF representations of LQG [46–49].
(here the trace stands for the sum over the magnetic indices at both ends of each link of the graph $\Gamma$.)

The amplitude $Z_M$ of a quantum gravitational process in a finite region $M$ subjected to the boundary conditions imposed by a given spin-network $\Psi^\Gamma$, is of course a function of the spin-network state itself. When dealing with the dynamics of flat space, that is of three-dimensional (quantum) gravity with vanishing cosmological constant, this function is extremely simple. If $M = B_3$ with the 2-sphere boundary, it essentially reduces to what is known as a spin-network evaluation, i.e.

$$Z_{B_3}(\Psi^\Gamma) = \Psi^\Gamma(g_t = 1).$$

Crucially, this evaluation gets twisted in non-trivial ways by the presence of topological features such as (bulk) non-contractible cycles [5]—see section 3.

Spin-network evaluations are complete contractions of intertwiners, associated to the vertices of the spin-network graph. Such an evaluation can be read as the sum over all magnetic-index configurations one can attach to the edges of the graph, with specific Boltzmann weights given by the entries of the intertwiner tensors themselves. Symbolically,

$$Z_{B_3}(\Psi^\Gamma_{(j, \iota)}) = \text{Tr}\Gamma\left( \bigotimes_{v \in \Gamma} \iota_v \right) = \sum_{\{m_l \in \Gamma\}} \prod_{v \in \Gamma} (\iota_v)_{m...}$$

where in the last term the $m$-indices of $\iota_v$ label states in those representations $V_{k_l}$ of $SU(2)$ which are attached to the links of $\Gamma$ starting at or leaving from the vertex $v$. From this perspective, $SU(2)$ spin-network evaluations is essentially the computation of a partition function of a statistical model, whose rotational invariance is automatically guaranteed by the $SU(2)$-invariance of the intertwiners themselves.

In particular, for homogeneous spins $j = 1/2$ for all edges on the boundary spin network, when the graph is a regular square lattice, which we denote $\Gamma = \square$, we can show the correspondence of the Ponzano–Regge amplitude with boundary with the partition function of the ‘isotropic’ 6-vertex model [20] (see also [39, 40] for a broader perspective and equivalence of 3D quantum gravity with statistical models), as illustrated on figure 1:

$$Z(\Psi_{(1/2, \iota)}) = \sum_{\text{arrows}} a^{\#i+\#r} b^{\#m+\#t} c^{\#v+\#s}$$

with

$$\Delta := \frac{a^2 + b^2 - c^2}{2ab} = 1.$$ 

In the case of non-trivial topologies, the above equality still holds locally, but further non-local operator insertions are needed to account for the existence of non-trivial holonomies along the non-contractible cycles. We explore this case in the next section (see also [22]).

This is an integrable statistical model, whose transfer matrix coincides with that of the XXX Heisenberg spin-chain with spectral parameter $\lambda = i(a/c - 1/2)$, e.g. [50]. The degrees of freedom of the spin-chain are indeed the same magnetic indices appearing in equation (7), however the system is not quite in a pure Gibbs ensemble of the form $\text{Tr}(e^{-\beta H_{XXX}})$, but rather in some involved generalized ensemble, whose ‘Hamiltonian’ is given by a combination of conserved quantities weighted by $\lambda$, e.g. [51]. Notice that the XXX spin chain is isotropic and hence $SO(3)$ invariant. See [22] for more on these correspondences.

For generic spins, a special class of spin-network graphs is constituted by Wilson line weaves. In a weave, a set of Wilson lines pass above and beneath each other, forming links of knots, without ever intersecting. This situation corresponds to intertwiners whose recoupling
The channel is in a state of vanishing spin (notice that these form a basis of intertwiners among four spins $j = 1/2$). Such peculiar states were observed to be related to integrable models of various kinds almost thirty years ago [39–42]. Nevertheless, at the time, the perspective was different, and the correspondence was made with knot expectation values in Chern–Simons theory, rather than with boundary spin-network evaluations.

The appearance of magnetic indices as degrees of freedom of the boundary theory is an aspect that deserves attention. Magnetic indices are quantum states in a co-adjoint orbit of $su^*(2)$. Therefore, following the LQG quantum geometrical interpretation, they represent the possible (quantum) orientations of a vector of fixed length in $\mathbb{R}^3$. Thus, the present correspondence explicitly fulfills the expectation of much of the contemporary work in the context of gravity and gauge theories in presence of boundaries, that the dual boundary degrees of freedom are constituted by reference frame orientation of a ‘would-be-gauge’ symmetry [16, 52–57].

For increasing values of the spins, the number of allowed magnetic indices grows: the number of Planck-sized cells present in the larger co-adjoint orbits grows, and the latter can be better and better approximated by their classical counterparts. This explains why, in the large spin regime, semiclassical methods exist to analyze the relevant spin-network evaluations. These methods are indeed well developed, both in the mathematical and physical literature, and the emergence of geometrical objects from the large spin asymptotics of such evaluations is well studied [3, 14, 29, 58–62].

So far, we considered only spin-network states at fixed spins. This, however, need not be a necessary restriction, and considering superpositions of all spins has its own interest. For example, this fact can be used to peak the state not only on an intrinsic geometry,

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5 The degrees of freedom in those statistical models are more naturally expressed in terms of the spins $j$, rather than the magnetic indices $m$. This ‘spin representation’ also has a holographic/gravitational interpretation, as explained in [22].
corresponding to metric boundary conditions, but also on its extrinsic geometry, leading to more general boundary conditions [26, 63].

Another interesting example is given by the construction of so-called spin-network generating functions. These are boundary states which depend analytically on one parameter per edge, in such a way that their power expansion in these parameters gives all possible (fixed-spin) spin-network evaluations. Pictorially, these parameters can be interpreted as chemical potentials—or, more precisely, fugacities—for the edge spins. Crucially, generating-function boundary states on specific graph types encode the (bosonized dual of the) Ising model [44, 45].

3. Handlebodies

As it was mentioned earlier, these correspondences are enriched by the presence of boundary non-contractible cycles, which stay non-contractible in the bulk. At this point a clarification is necessary.

It is often assumed that in a theory of quantum gravity, one has to sum over all manifolds compatible with the boundary data, including all compatible bulk topologies. However natural and appealing, this idea is plagued with difficulties ranging from defining evolution on non-globally-hyperbolic manifolds in a canonical setting, to the very classification of topologies in dimensions 4 and higher in a covariant setting. For these reasons, in a large part of the quantum gravitational literature (see however [30, 31, 64]), the issue of summing over topologies has been put aside, and the focus restricted to the ‘integration’ over metrics.

Here, we will adopt this conservative viewpoint, and hence postpone all investigations of topology change, and large diffeomorphisms alike.

This being clarified, we can address the question of how the boundary theory ‘knows’ about its bulk-contractible and bulk-non-contractible cycles. From the bulk perspective, non-contractible cycles correspond to non-trivial holonomies that need to be integrated over. Using gauge invariance, these holonomies can be made to have support on a single thin cylinder $D_2 \times [0, \epsilon]$, whose boundary counterpart is a ring $R = S^1 \times [0, \epsilon]$ winding around the conjugate boundary cycle. Specializing, for notational simplicity, to a solid torus $M_3 \cong D_2 \times S^1$, this means that the spin-network amplitude ceases to be a mere spin-network evaluation, as in (6), and rather takes the form

$$Z_{M_3}(\Psi^\Gamma) = \int_{SU(2)} dh \Psi^\Gamma(g_{l \in R} = \mathbb{1}, g_{l \in R} = h),$$

where $h$ is the non-trivial holonomy around the cycle, and $R$—with a slight abuse of notation—denotes the set of edges $l$ that (transversally) cross the thin ring $R$.

Therefore, $R$ is the support of a non-local ‘topological’ operator (which is actually a projector) acting on the boundary spin-network state. The topological nature of this operator, i.e. the unimportance of its precise location, is a consequence of the flatness of the fundamental connection.

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6 In four dimensions, the problem is even subtler, because of the non-uniqueness and proliferation of differential structures.

7 A Haar intertwiner is an intertwiner of the form

$$\langle \text{Haar} \rangle_{m, \tilde{m}} = \int_{SU(2)} dh \prod_{l=1}^L D^l(h)_{m_l}^{m'_{l}},$$

where $D^l(h)$ is a Wigner matrix in the representation of spin $j$; $m, m' \in \{-j, \ldots, j\}$ are magnetic indices; $dg$ is the Haar measure on the group. Finally, in this formula, $l$ labels the edges crossing the above-mentioned ring.
At the spin-network level, in the analogue of equation (7), the operator arising from the integration over all possible holonomies—when unconstrained—can be easily recognized to correspond to the insertion along the ring of a Haar intertwiner:

$$Z_{\mathcal{M}}(\Psi_{\Gamma}, \psi_{\frac{\beta}{\tau}}, \mathcal{H}) = \text{Tr}_{\Gamma \setminus R} \left( \bigotimes_{v \in \Gamma} \psi_{\frac{\beta}{\tau}} \right) \bullet_{R} \psi_{\mathcal{H}}$$

where the trace on $\Gamma \setminus R$ and the symbol $\bullet_{R}$ mean that the spin-network has been ‘cut open’ along $R$, and its open ends along the cut are contracted to those of the Haar intertwiner $\psi_{\mathcal{H}}$—see [20–22].

The modification of the boundary theory via this operator insertions explicitly breaks the symmetry between the various non-contractible cycles of the surface, thus imprinting on the boundary theory some knowledge of the bulk topology. In practical computations, these topological operators play a crucial role. We will present a concrete example of this in the next section.

4. Example: coherent torus

The simplest example that can provide insights on the above program is given by a twisted solid torus spacetime in three-dimensional Euclidean quantum gravity with vanishing cosmological constant. Twisted means that the solid torus $M_{3} \cong \mathbb{D}_{2} \times S_{1}$ is obtained by identifying the bottom and the top of the cylinder $\mathbb{D}_{2}(a) \times [0, \beta]$ up to a rotation of $\gamma$ radians. Here, $a$ stands for the radius of the two-disk, and $\beta$ for the Euclidean time extension of the cylinder.

This example has already been studied with other techniques—most notably via the perturbative quantum Einstein–Hilbert theory at 1-loop over a flat background, both in the continuum [23] and in the ‘discretum’ [24], and as a limit of the corresponding AdS$_{3}$/CFT$_{2}$ computation [64–67]—and can therefore serve as a benchmark for the present methods.

The most prominent feature of all these approaches, is a partition function which depends on $\gamma$ in a way that admits a (formal) expansion over boundary momentum eigenmodes $p \in \mathbb{N}$—i.e. Fourier modes in the bulk-contractible, ‘spacelike’, direction—as

$$Z(\beta, \gamma) \sim e^{-\frac{16 \pi a_{s}}{\ell_{p}}} \prod_{p \neq 2} \frac{1}{1 - 2 \cos(\gamma p)},$$

where $\ell_{p} = 8 \pi G_{N} \hbar$, and the factor of 2 is in parenthesis because its presence depends on the chosen boundary conditions (standard Gibbons–Hawking–York boundary conditions require it; for details, see the first section of [20]).

Notice the peculiar fact that for $\gamma \in 2 \pi \mathbb{Q}$, there are $p$’s whose contribution explodes. In AdS space, this issue—as well as the convergence of $Z$ at $p \to \infty$—is cured by the fact that the role of $\gamma$ is played by the torus’ modulus, $2\pi \tau = \gamma + i \sqrt{\Lambda} \beta$. The resulting “thermal AdS” partition function is then closely related to Dedekind’s $\eta(\tau)$ function, a modular form strictly speaking defined on the upper half complex plane, $\text{Im}(\tau) > 0$ [64]. In this respect the flat limit is expected to be quite singular, and deserves to be studied independently.

In any case, quite remarkably, even in the flat case multiple different approaches [23, 24, 67] agree with the formal result of equation (12), although the mechanisms leading to this formula and the regularizations that make sense of it are technically very different. In all cases, a crucial fact is that the modes $p = 0$ and $p = 1$ do not appear in the product as a consequence of diffeomorphism symmetry (see [20] for a detailed comparison).

8 The identification of $p$ as spacelike eigenmodes is clearest in some formulations [24, 64, 67], but can remain rather obscure in others [23, 65].
Maybe one of the most interesting derivations of the formula (12) is as a character of BMS3 in the 'vacuum' representation ('massive' ones have a different prefactor, and a product over $p$ that starts at $p = 1$) [67]. This is taken as a hint that a dual boundary theory indeed exists whose symmetry group is (a central extension of) BMS3, in analogy to the Virasoro symmetry of the AdS3/CFT2 case. This idea is further supported by the fact that the BMS3 characters can be obtained through a zero-cosmological-constant limit of the Virasoro ones.

Here, we present results on the analysis of precisely this situation within the bulk local non-perturbative approach provided by the Ponzano–Regge (PR) model [20, 21].

The first question one faces in setting up the computation within the PR model is what boundary state one is going to use. A survey of the previous method suggests that the boundary state should encode the geometry of a rectangular cylinder. Remarkably, this fact is made explicit in the only other quasi-local computation [24], while it is implicitly used in the other computations intrinsic to the boundary—since they assume translational symmetry along the two boundary directions. It is, however, completely hidden in the perturbative Einstein–Hilbert approach.

In order to impose the desired boundary conditions, coherent spin network techniques, developed in the context of loop quantum gravity were used to design a state corresponding to an $N_y \times N_x$ regular rectangular lattice. The twisted toroidal topology is implemented by an identification of the lattice appropriately shifted by $N_y$ units in the ‘spatial’ direction (see figure 2), so that

$$\gamma = 2\pi \frac{N_y}{N_x},$$

while the spin-network intertwiner encoding the rectangular plaquette geometry dual to the spin-network vertex $v$ is

$$\beta^{a(b)}(a)T_L \ldots 1 \ldots N_x = 6.$$ 

Figure 2. Discretization of the twisted solid torus for parameters $N_y = 3, N_x = 6$ and a shift $N_y = 1$. (a): on the left hand side, we draw the cylinder with base the 2-disk of radius $a$ and with Euclidean time extension $\beta$. To get the twisted solid torus, we identify the top and the bottom of the cylinder up to the discrete shift $i \rightarrow i + N_y$ for all lattice sites $i$, which causes the twist $\gamma = 2\pi \frac{N_y}{N_x}$ in the gluing. (b): on the right hand side, we draw the boundary lattice associated to the discretization of the solid torus. We attach a spin $T$ (resp. $L$) to each horizontal (resp. vertical) edges on the boundary, and we attach to each boundary vertex $v$ an intertwiner as defined by formula (14). Arrows indicate the shift of $N_y$ units upon gluing in the vertical direction.

See footnote 7 for details on the $D$-matrix notation. Notice that the second magnetic index, $m'_L$ in the notation of footnote 7, is here fixed to its maximal value $j_L$. This choice ultimately ensures that these states are coherent states.
where the bar stands for complex conjugation.

Equation (14) defines a coherent intertwiner, designed to encode the geometry of a flat rectangle. The four spinors $\xi^l$ represent in a precise sense the sides of the rectangular plaquette along the $\bar{z}$, $-\bar{z}$, $\bar{z}$ and $\bar{z}$ directions respectively, while the spins $L, T$ correspond to their lengths, and the lift of the group elements $G_v \in \text{SU}(2)$ to $\text{SO}(3)$ corresponds to the orientation of the plaquette’s reference frame.\footnote{The complex conjugation accounts for the matching of link orientations.} Gauge invariance implies that all orientations are weighed equally. Being coherent means that in the large spin regime in which $T, L \to \infty$ uniformly, the above state optimally minimizes the extrinsic curvature along the rectangle diagonals compatibly with Heisenberg uncertainty principle—see e.g. [68, 69].

The PR amplitude for such a boundary state $\Phi_{\text{coh}}$ can be written as a purely boundary discrete sigma-model (the label ’coh’ stands for ’coherent’). This follows essentially from equations (10) and (11) evaluated for the specific coherent state described above. In particular, on the square lattice of figure 2, $R$ can be taken to correspond to the last row of vertical links, while $h$ can be partially gauge fixed to be of the form $h = e^{-i \varphi} \sigma_z$ and then evenly redistributed across all the rows of the lattice. Also, the coherent intertwiners (14) can be explicitly contracted using\footnote{Using the fact that at each vertex $\sum_{i=1}^{4} j_i$ is an integer, one can show that $G_v \mapsto (-1)^{j_v}$ is a symmetry of the integrand. Using this fact, one can consistently consider the $G_v$ to be elements of $\text{SU}(2)/\mathbb{Z}_2 \cong \text{SO}(3)$, hence truly representing frame orientations. This identification will be implicitly assumed in the following treatment.} $\sum_m D^m(g_1)w^mD^m(g_2)w^m_j = D^j(g_1^{-1} g_2)\langle j \rangle$ and the equality $\langle g \rangle_j = (\langle g \rangle_{1/2})^2$, where $\langle g \rangle = D^j(g)$. The combination of these steps finally gives,

$$Z_{\text{PR}}(\Phi_{\text{coh}}) = \int d\varphi \prod_v \int dG_v \sin^2 \left( \frac{\varphi}{2} \right) e^{i \sum_l S_l}. \quad (17)$$

The angle $\varphi \in [0, 2\pi]$ corresponds to the conjugacy class of the only non-trivial holonomy $h$, while $S_l$ denotes the contribution to the ‘action’ of the spin-network edge $l$:

$$S_l = \begin{cases} \frac{2}{\hbar} \ln \langle \xi^{(l)}\rangle G^{-1}_v G_\ell \langle \xi^{(l)} \rangle & l \text{ horiz.} \\ \frac{2}{\hbar} \ln \langle \xi^{(l)}\rangle e^{-i \hat{\sigma}_z} G_\ell \langle \xi^{(l)} \rangle & l \text{ vert.} \end{cases} \quad (18)$$

Here, the edges $l$ are oriented to point either to the right or upwards, and $v = s(l)$, $t(l)$ are therefore its source and target vertices, respectively. Notice that the branch-cut of the logarithm plays no role.
The $G_v$ are SU($2$) elements associated to the vertices $v$ of the boundary coming from (14); in this formulation, they constitute the degrees of freedom of the dual field theory, replacing the magnetic indices of the statistical model formulation (the two formulations are in the end equivalent, see [22] for a more thorough discussion of this point). Geometrically, the $G_v$ can be understood as providing a ‘potential’ for a flat boundary connection and thus describe the embedding of the boundary into flat 3D space.

Remark now that the action $S_{coh} = \sum_l S_l$ is complex and such that $\text{Re}(S_{coh}) \leq 0$. One can think of its imaginary part as providing the actual action for the boundary theory, and its real part as providing the quantum measure. For notational simplicity, the two contributions will not be explicitly distinguished.

Now, using that $j_l$ is the eigenvalue in Planck units of the length operator along (the dual of) $l$:

$$\ell_l = \ell_P j_l,$$

one sees that in the semiclassical limit $\hbar \to 0$, at fixed boundary geometry—that is at fixed $\ell_l$—the coherent intertwiners get peaked on the classical geometry, while the amplitude $Z_{PR}$ can be evaluated at 1-loop via a stationary phase approximation.

The stationary phase equations for the boundary action are $\text{Re}(\sum_l S_l) = \max_{\{G_v\}} (\text{Re}(\sum_l S_l)) = 0$, and $\delta_{G_v} (\sum_l S_l) = 0$ which is satisfied if and only if

$$\begin{cases} G_{\ell(l)} G_{\ell(l)} = e^{\pm \psi^v_l} G_{\ell(l)} G_{\ell(l)} & l \text{ horizontal} \\ G_{\ell(l)} G_{\ell(l)} = e^{\pm \psi^h_l} e^{- \frac{\pi}{2} \sigma} G_{\ell(l)} G_{\ell(l)} & l \text{ vertical}. \end{cases}$$

Using standard spinfoam techniques [29, 62, 70–72], these equations can be geometrically interpreted as describing an immersion (i.e. a local embedding) of the tiled toroidal surface in the ambient flat space $\mathbb{R}^3$ [73, 74]. According to this interpretation, $\hat{G}_v = A_{\ell_P}$ represents the SO(3) frame of the rectangular plaquette dual to $v$—which is defined up to some global

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rotation along the $\hat{z}$ axis—while $\psi_l$ represents the dihedral angle (extrinsic curvature) between two neighboring plaquettes connected by $l$ (see figure 3).

The equation resulting from the stationarity of the bulk holonomy’s conjugacy class $\varphi$, $\delta_\varphi (\sum_l S_l) = 0$ i.e.

$$\hat{z} \cdot \sum_v \hat{G}_v \cdot \hat{x} = 0,$$

breaks the symmetry between the two cycles of the torus (here implicit in the appearance of $\hat{x}$ rather than $\hat{y}$ in the equation), and implies that

$$\psi_l^y = 0 \text{ if } l \text{ vertical.} \quad \text{(22)}$$

(An alternative solution is $\psi_l = \pi$, always for $l$ vertical; solutions where this happens are called ‘folded’ solutions [21] and will be ignored in this article—however, see the comment at the end of this section on Planck-scale values of the extrinsic curvature.)

The above equation means that the equation of motion for $\varphi$ dictates along which cycle the torus boundary can be curved extrinsically, and this happens precisely in the spatial direction, as it was intuitively expected.

Thus, the solutions to the equations of motion are given by rectangular cylinders whose ‘spatial’ sections are (not-necessarily convex) $N_x$-sided polygons and whose ends are identified modulo a shift of $N_x$ units.

Boundary conditions further require that along each ‘time slice’

$$\sum_{x=1}^{N_x} \psi_{lx}^h = 0 \text{ mod } 2\pi \quad \text{(23)}$$

and

$$\varphi = \sum_{x=1}^{N_y} \psi_{lx}^h \text{ mod } 2\pi \quad \text{(24)}$$

where the latter equation is understood to hold for any choice of $N_y$ consecutive horizontal edges $\{l_x\}$.

Fixing $N_x$ and $N_y$ to be coprime,

$$K \equiv \text{GCD}(N_x, N_y) = 1, \quad \text{(25)}$$

the analysis of the stationary phase equations can be pushed further. In particular, $K = 1$ implies that the polygonal sections must be regular, since it implies that

$$\psi_{lx}^h = \frac{2\pi}{N_x} n \text{ for all horizontal } l_x \text{’s} \quad \text{(26)}$$

for some $n \in \{0, \pm 1, \ldots, \pm \frac{N_x-1}{2}\}$, and hence

$$\varphi = 2\pi \frac{N_y}{N_x} n \equiv \gamma n. \quad \text{(27)}$$

For $K \neq 1$, all solutions are part of a continuous family, and the 1-loop determinant (the Hessian determinant of $\sum_l S_l$ at the stationary point) contains null directions. Geometrically, these degeneracies correspond to the fact that one can deform the polygonal section of the cylinder while staying on-shell.

Back to the case $K = 1$, the integer $n$ labels the different stationary points. We interpret it geometrically as a winding number which counts how many times the toroidal surface winds around the cylinder’s axis before closing. The existence of these solutions is due to the
fact that the models relies on (compactified) holonomy rather than connection variables, and consequently—loosely speaking—the flatness condition just states that the deficit angle must be an integer multiple of $2\pi$, rather than strictly zero. The degenerate case $n = 0$ is suppressed because of the integration measure, while the case $n = 1$ is of course the ‘geometrical’ one.

At these stationary points, the spin-network action takes the on-shell value (recall that for vertical links, $\psi^v_l = 0$)

$$
\sum_l S_l = iN_t \sum_{x=1}^{N_x} T \psi_h^{lx} = 2\pi i N_t \sum_{x=1}^{N_x} \psi_h^{lx}.
$$

(28)

Formally, this action has the structure of a discretized Gibbons–Hawking–York term on the rectangular cylinder, as desired from what should correspond to the on-shell value of the Einstein–Hilbert action on a flat spacetime and in agreement with equation (12). However, once exponentiated, $e^{\sum_{l}^b}$ reduces to a sign. This is a consequence of the discreteness of the spin $T \in \frac{1}{2} \mathbb{N}$.

If $K = 1$, moreover, the 1-loop determinant is non-degenerate and can be readily analyzed. Using a Fourier transform adapted to the presence of the twist, one can diagonalize the Hessian, while an exact resummation formula gets rid of the explicit energy dependence. The ensuing result for the determinant, combined with the on-shell value of the $\varphi$ measure, is

$$
(1\text{-loop det}) \times \sin^2 \frac{\varphi}{2} = A(n) \times D(\gamma,n). 
$$

(29)

where the $\gamma$ dependence is fully contained in

$$
D(\gamma,n) = (2 - 2 \cos \gamma n) \prod_{p=1}^{N_x - 1} \frac{1}{2 - 2 \cos \gamma p}.
$$

(30)

When focusing on the geometrical background geometry corresponding to $n = 1$, this reproduces precisely the sought result of equation (12). For this to work, it is crucial to notice the fundamental role played by the integration over the non-trivial holonomy wrapping around the non-trivial cycle of the torus, which contributes precisely the factor $2 - 2 \cos \gamma n$ above.

Interestingly, the product over $p$ in the equation above can be explicitly computed once we remember that the angle $\gamma$ comes from the shift $N_\gamma$ in the gluing of the lattice:

$$
\prod_{p=1}^{N_x - 1} \frac{1}{2 - 2 \cos \frac{2\pi N_\gamma p}{N_x}} = N_x
$$

(31)

(this formula holds as long as $N_\gamma$ and $N_x$ are coprime, i.e. $K = 1$). This gives a surprisingly simple result for the amplitude for a given winding number $n$,

$$
D(\gamma,n) = 2 N_x \left(1 - \cos \gamma n\right).
$$

(32)

Notice that, despite this simplification, the dependence on the twist $\gamma$ does not completely disappear and we keep a non-trivial result.

A first remark is that it is very interesting that our lattice computation leads to such a straightforward finite truncation of the BMS$_3$ character formula for the partition function (12) to the product over modes $p$ bounded by the lattice size $N_x$. Moreover the simplification of this product is a great coincidence, which likely points out towards a underlying powerful symmetry. This point needs to be investigated further.

A second remark is that the equation (30) is actually more useful than the simplified expression (32) in order to understand the physical content of the theory. Indeed, it provides
The mode decomposition of the theory, in terms of the Fourier modes $p$. Although the partition function $Z$ might simplify, what matters is that the various Fourier modes have different weights with a specific $\gamma$-dependence, i.e. $\left(2 - 2 \cos \gamma p\right)^{-1}$, which is probed e.g. by the correlations of the boundary theory.

In this sense and to this extent, this calculation perfectly agrees with the $\Lambda \to 0$ limit of the AdS case given by equation (12) when focusing on the geometrical background corresponding to the classical solution with winding number $n = 1$.

Finally, the amplitude factor $A(n)$ carries the dependence on the winding number $n$. It is a rather intricate function of the spins $L$ and $T$, the lattice sizes $N_x$ and $N_t$ and of course of the label $n$. It is nevertheless possible to considerably simplify the formula derived in [21] (by explicitly performing the product over Fourier modes $p$). The result is

$$A(n) = \left[ \frac{iL - (L + T) \tan \frac{\psi n}{2}}{iL - (L + 4TN_x) \tan \frac{\psi n}{2}} \left[ \frac{2T_N}{L(T + L)} \right]^{\frac{n}{2}} \left[ 2T_N(a_n) - 2 \right]^{-\frac{n}{2}} \right],$$

where $\psi$ is the dihedral angle unit for the lattice and $a_n$ is a simple complex trigonometric function,

$$\psi = \frac{2\pi}{N_x}, \quad a_n = \cos n\psi + \frac{iL}{T + L} \sin n\psi.$$ \hspace{1cm} (34)

Also, the notation $T_N(a)$ stands for the Chebyshev polynomial (of the first kind) of order $N$\(^{15}\).

From these expressions one can first check the reality property

$$A(-n) = \bar{A}(n).$$ \hspace{1cm} (35)

\(^{15}\text{Recall, } T_N(a) = \text{ch}(n\text{arccos}a)\text{ and } (2T_N(a) - 2)^{\frac{1}{2}} = \left(a + \sqrt{a^2 - 1}\right)^{\frac{1}{2}} - \left(a + \sqrt{a^2 - 1}\right)^{-\frac{1}{2}}.$$
This is consistent with a Hamilton–Jacobi functional, which cannot distinguish a momentum from its opposite, and more specifically with a first-order Einstein–Cartan formulation of general relativity, of which the Ponzano–Regge model is a quantization.

Moreover, $A(n)$ is, in the large $N_x, N_t$ limit (but already true when they are larger than 5), overwhelmingly peaked in modulus at the minimal and maximal values of $n$, that is at $n = 1$ and $n = \lceil N_x/2 \rceil$ as illustrated by the plots on figure 4. This behavior is entirely due to the factor with the Chebyshev polynomial, $\left| (2\sin(a_n) - 2) \right|^{-\frac{T}{L}}$. We can actually plot this at fixed $N_x, N_t$ as a function of the continuous variable $x = n\psi \in [0, \pi]$, as in figure 5. In order to understand better the asymptotic behavior at large $N_x$, let us focus on this factor. The function $\left| (2\sin(a_n) - 2) \right|$ is well-behaved, both in modulus and phase, as one can see on figure 6. The moot point is that the first winding number $n = 1$ corresponds to the angle $x = \psi = \frac{\pi}{N_x}$ which goes to $x \to 0$ as $N_t$ grows large but not fast enough so as the asymptotics of $\left| (2\sin(a_n) - 2) \right|$ be simply $\left| (2\sin(0) - 2) \right| = 0$. Indeed the two appearances of the lattice size $N_x$ conspires to give a non-trivial asymptotics:

The symmetry of $\left| (2\sin(a(x)) - 2) \right|$ depends on the sign of $(-1)^N$. Under the reflection $x \to \pi - x$, it changes to its complex conjugate for even $N$, while it further gets an extra minus sign for odd $N$. This means that the modulus $\left| (2\sin(a_n) - 2) \right|$ is exactly symmetric under $n \leftrightarrow \frac{N_t}{2} - n$ for even $N_t$, while it is slightly skewed under the exchange $n \leftrightarrow \frac{N_t}{2} - n$ for odd $N_t$, due to the $\frac{1}{2}$ shift.
\[\left[ (2T_N(a_n) - 2) \right]^\dagger \sim e^{(1+i)\sqrt{\frac{2\pi N_n}{\lambda x}}}, \quad \forall n \ll N_c. \]  

(36)

This also gives the behavior for large winding numbers \( n \lesssim \lfloor \frac{N_x}{2} \rfloor \) since the function \((2T_N(x) - 2)\) is (almost) symmetric\(^{16}\) under reflections \( x \leftrightarrow \pi - x \). This explains the peakedness of the amplitude pre-factor \( A(n) \) on the two limiting winding numbers \( n = 1 \) and \( n = \lfloor \frac{N_x}{2} \rfloor \).

The intuition behind the peakedness at the minimal and maximal winding numbers, \( n = 1 \) and \( n = \lfloor \frac{N_x}{2} \rfloor \), is that these solutions reconstruct locally almost-flat geometries (although one can be visualized as being folded onto itself), and at flat geometries the Hessian degenerates. This mechanism is analogous to ‘Ditt-invariance’\(^{[75, 76]}\). This peakedness can be used to argue that the slightest (semiclassical) knowledge of the extrinsic curvature, such as the fact that it is non-Planckian as in the maximal \( n \) case, collapses the result onto the desired classical solution at \( n = 1 \)\(^{17}\).

To summarize the role of the amplitude pre-factor \( A(n) \) is that it selects the first winding number \( n = 1 \), which corresponds to the semi-classical embedding to our toric surface in flat \( \mathbb{R}^3 \) space and allows to reproduce the expected semi-classical partition function (12) for 3D quantum gravity as a function of the twist angle \( \gamma \).

5. Outlook

In this note, we have proposed a concrete framework to analyze quasi-local holographic dualities in three dimensional quantum gravity. In such a framework, both the bulk quantum geometry—described by the Ponzano–Regge model—and the boundary theories are readily accessible, and the bulk-boundary correspondence can be readily read from the choice of boundary conditions. After highlighting the general character of the correspondence and the general questions one hopes to address in its context, we have reviewed the main results of\(^{[20, 21]}\).

In this outlook, we wish to discuss more specific questions which arise when studying the holographic setup that is here advocated for.

5.1. Boundary phase transitions

The boundary statistical theory might undergo phase transitions for specific choices of the parameters. The question is what this means from the geometrical perspective of the bulk theory. The question is particularly cogent in the case of second order phase transitions. First steps to answer this question have been taken in\(^{[44, 45]}\). There it was shown that—for an infinite superposition of trivalent spin-networks, which turns out to be dual to the Ising model—criticality corresponds to peakedness around geometrical boundary conditions (modulo a global scale).

5.2. Asymptotic infinity

The framework presented here is well-adapted to the study of quasi-local boundaries. Which theory arises when pushing these boundaries to infinity is an independent question. In order to reach infinity, the first guess is that one needs to consider boundaries of infinite circumference.

\(^{17}\) The very same physical argument allows to discard the ‘folded’ solutions mentioned above.
This can \emph{a priori} be done in two ways: by considering an infinite number of building blocks, possibly Planck-sized (e.g. with $j = 1/2$), or by rescaling a given boundary configuration by scaling the relative spins to infinity. The first procedure resembles a continuum limit of the statistical model, provided one at the same time scales down the lattice separation (and rescales the relevant boundary fields). The continuum limit, however, is usually taken by keeping the total physical size of the system constant. The second procedure, on the other hand, is related the usual semi-classical limit of spinfoams, which leads to a classical, albeit discrete, theory of gravity. It is possible that the scaling to asymptotic infinity is a mixture of the two procedures above. Given the relationship between continuum limits and second order phase transitions, identifying the correct notion of asymptotic limit might well shed new light on phase transitions in spinfoam models.

5.3. Lorentzian signature

The Ponzano–Regge model, as used here and usually studied, defines a discrete topological path integral for quantized three-metrics with Euclidean signature. More precisely, the Ponzano–Regge amplitudes are a quantized version of a path integral for three dimensional discretized Euclidean general relativity as defined by the classical Regge action [11], thus yielding amplitudes whose leading order at large scales reproduces the complex exponential of this action [3]. In the optics of exploring holographic dualities in three space-time dimensions as a stepping stone towards better understanding the bulk-boundary relation for 4d quantum general relativity, it is crucial to study path integrals for Lorentzian metrics. This would clarify the distinctions among time-like, space-like, and null boundaries, and thus among time-like, space-like, and null infinities in the asymptotic limit. An important step in this direction is to use the Lorenztian version of the Ponzano–Regge model [5, 77–81] based on the gauge group $SU(1, 1)$ instead of $SU(2)$ and involving the 6j-symbols of the recoupling theory of irreducible unitary representations of $SU(1, 1)$. This is a necessary step towards a true description of a quantum space-time.

5.4. Cosmological constant

Introducing a cosmological constant $\Lambda$ seems a natural goal, especially in relation to the (A)dS/CFT correspondence. Euclidean three-dimensional models with a cosmological constant are known and some of their properties have been widely studied [7, 14]. They correspond to real ($\Lambda < 0$) and root-of-unity ($\Lambda > 0$) $q$-deformations of the Ponzano–Regge model used in this note and in [20, 21]. In particular, the model with $\Lambda > 0$ coincides with the original Turaev–Viro topological field theory for $U_q(\mathfrak{su}(2))$, $q$ root of unity. This model is finite and hence mathematically well defined from the outset, i.e. without the need of gauge fixing (which was implicit in the present treatment, see [20]). There is therefore little difficulty in generalizing the present treatment to those cases. Difficulties, however, arise when one tries to compare the ensuing geometries to the (A)dS/CFT or even AdS/MERA setups (see also [82]). geometry encoded in the Turaev–Viro weights is that of homogeneously curved building blocks\footnote{Hence zero deficit angles in the corresponding Ponzano–Regge-like model correspond to absence of ‘extra’ curvature defects, and thus to a homogeneously curved manifold. See [13, 14, 83–85] for a treatment in three dimensions. Generalization to four dimensions also exist [86, 87].}. While this fact is well adapted to a MERA-like triangulation of the bulk manifold (in the AdS case), it is much less adapted to describe the common conformally-flat boundary at AdS time-like infinity. Fortunately, this seems to be more a nuisance than a fundamental
problem. Matching the boundary theories obtained in AdS/CFT and in the present setup seems to be a more cogent problem, which might however be inextricable from the previous problem on how to encode asymptotic infinity in the present setup.

5.5. \textit{BMS}_3

To conclude, let us mention one more issue. Although the calculation of the partition function nicely matches previous calculations and in particular the formal expression of BMS\textsubscript{3} characters of Oblak \cite{67}, it is still unclear whether and in which sense BMS\textsubscript{3} emerges (in the continuum limit) as a symmetry of the boundary theory discussed above. Of course, this is a crucial question to answer in order to claim full understanding of the setup reviewed in the previous section. Summarizing, we have described a program pushing the study of the Ponzano–Regge non-perturbative path integral for three dimensional quantum gravity towards a deeper and more explicit understanding of the bulk-boundary relation. This study comes in two stages: in the first one we ask how much of the bulk quantum geometry is determined by the choice of an extended boundary spin-network state, and which such boundary states encode (semi)classically meaningful geometries; in the second stage, on the other hand, we reinterpret the bulk quantum gravitational amplitude for such boundary states in terms of a condensed matter model intrinsic to the boundary. This reinterpretation induced us to propose to understand the refinement, or asymptotic, limit in the critical regime as conformal field theories (CFTs). In the present note, and in the more detailed papers \cite{20, 21}, we focused on the twisted torus in Euclidean signature, because of its relevance to the AdS\textsubscript{3}/CFT\textsubscript{2} correspondence and the simple interpretation of the results in terms of the asymptotic BMS symmetry. Nevertheless, it would be enlightening to consider general handlebody topologies and derive a decomposition of the Ponzano–Regge amplitudes in terms of a non-critical version of the conformal blocks for the boundary statistical models, which would lead back to the standard conformal block description of 3D quantum gravity in a continuum limit (e.g. \cite{88}).

\textbf{Acknowledgments}

We would like to thank an anonymous referee for his interesting comments and promising suggestions.

This work is supported by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

Plots were performed using Mathematica 10.

\textbf{ORCID iDs}

Bianca Dittrich \href{https://orcid.org/0000-0002-0072-5242}{https://orcid.org/0000-0002-0072-5242}
Christophe Goeller \href{https://orcid.org/0000-0003-4379-750X}{https://orcid.org/0000-0003-4379-750X}
Etera R Livine \href{https://orcid.org/0000-0001-7757-4684}{https://orcid.org/0000-0001-7757-4684}
Aldo Riello \href{https://orcid.org/0000-0003-4448-1435}{https://orcid.org/0000-0003-4448-1435}
References

[1] Witten E 1998 Anti-de Sitter space and holography Adv. Theor. Math. Phys. 2 253
[2] Witten E 1988 (2+1)-dimensional gravity as an exactly soluble system Nucl. Phys. B 311 46
[3] Ponzano G and Regge T 1968 Semiclassical limit of racah coefficients Spectroscopic and Group
Theoretical Methods in Physics ed F Bloch (Amsterdam: North-Holland) pp 1–58
[4] Freidel L and Louapre D 2004 Ponzano–Regge model revisited I: Gauge fixing, observables and
interacting spinning particles Class. Quantum Grav. 21 5685
[5] Freidel L and Livine E R 2006 Ponzano–Regge model revisited III: Feynman diagrams and
effective field theory Class. Quantum Grav. 23 2021
[6] Barrett W and Naish-Guzman J I 2009 The Ponzano–Regge model Class. Quantum Grav.
26 155014
[7] Turaev V G and Viro O Y 1992 State sum invariants of 3 manifolds and quantum 6j symbols
Topology 31 865
[8] Reshetikhin N Y and Turaev V G 1990 Ribbon graphs and their invariants derived from quantum
groups Commun. Math. Phys. 127 1
[9] Reshetikhin N and Turaev V G 1991 Invariants of 3-manifolds via link polynomials and quantum
groups Inventory Math. 103 547
[10] Freidel L and Louapre D 2004 Ponzano–Regge model revisited II: equivalence with Chern–Simons
(arXiv:gr-qc/0410141)
[11] Regge T 1961 General relativity without coordinates Il Nuovo Cimento 19 558
[12] Regge T and Williams R M 2000 Discrete structures in gravity J. Math. Phys. 41 3964
[13] Mizoguchi S and Tada T 1992 Three-dimensional gravity from the Turaev–Viro invariant Phys.
Rev. Lett. 68 1795
[14] Turaev V G and Viro O Y 1992 State sum invariants of 3 manifolds and quantum 6j symbols
Topology 31 865
[15] Reshetikhin N Y and Turaev V G 1990 Ribbon graphs and their invariants derived from quantum
groups Commun. Math. Phys. 127 1
[16] Reshetikhin N and Turaev V G 1991 Invariants of 3-manifolds via link polynomials and quantum
groups Inventory Math. 103 547
[17] Freidel L and Louapre D 2004 Ponzano–Regge model revisited II: equivalence with Chern–Simons
(arXiv:gr-qc/0410141)
[18] Regge T 1961 General relativity without coordinates Il Nuovo Cimento 19 558
[19] Regge T and Williams R M 2000 Discrete structures in gravity J. Math. Phys. 41 3964
[20] Freidel L and Louapre D 2004 Ponzano–Regge model revisited II: equivalence with Chern–Simons
(arXiv:gr-qc/0410141)
[21] Regge T and Williams R M 2000 Discrete structures in gravity J. Math. Phys. 41 3964
[22] Mizoguchi S and Tada T 1992 Three-dimensional gravity from the Turaev–Viro invariant Phys.
Rev. Lett. 68 1795
[23] Taylor Y U and Woodward C T 2005 6j symbols for U_q(sl2) and non-Euclidean tetrahedra Sel.
Math. 11 539
[24] Brown D and Henneaux J M 1986 Central charges in the canonical realization of asymptotic
symmetries: an example from three dimensional gravity Commun. Math. Phys. 104 207
[25] Carlip S 2005 Conformal field theory, (2+1)-dimensional gravity and the BTZ black hole Class.
Quantum Grav. 22 R85
[26] Heydeman M, Marcoll M, Saberi I and Stoica B 2016 Tensor networks, p-adic fields, and algebraic
curves: arithmetic and the AdS3/CFT2 correspondence (arXiv:1605.07639)
[27] Sfondrini A 2015 Towards integrability for AdS3/CFT2 J. Phys. A: Math. Theor. 48 023001
[28] Kraus P 2008 Lectures on black holes and the AdS(3)/CFT(2) correspondence Lecture Notes
Phys. 755 193
[29] Dittrich B, Goeller C, Livine E and Riello A 2017 Quasi-local holographic dualities in
non-perturbative 3d quantum gravity I—convergence of multiple approaches and examples of
Ponzano–Regge statistical duals (arXiv:1710.04202)
[30] Dittrich B, Goeller C, Livine E R and Riello A 2017 Quasi-local holographic dualities in
non-perturbative 3d quantum gravity II—from coherent quantum boundaries to BMS3 characters
(arXiv:1710.04237)
[31] Riello A 2018 Quantum edge modes in 3d gravity and 2+1d topological phases of matter
(arXiv:1802.02588)
[32] Barnich G, Gonzalez H A, Maloney A and Oblak B 2015 One-loop partition function of three-
dimensional flat gravity J. High Energy Phys. JHEP04(2015)178
[33] Bonzom V and Dittrich B 2016 3D holography: from discrete to continuum J. High Energy Phys.
JHEP03(2016)208
[34] Dittrich B 2017 The continuum limit of loop quantum gravity—a framework for solving the theory
Loop Quantum Gravity: the First 30 Years ed A Ashtekar and J Pullin (Singapore: World
Scientific) pp 153–79
[35] Bianchi E, Modesto L, Rovelli C and Speziale S 2006 Graviton propagator in loop quantum gravity
Class. Quantum Grav. 23 6989
[36] Livine E R, Speziale S and Willis J L 2007 Towards the graviton from spinfoams: higher order
corrections in the 3D toy model Phys. Rev. D 75 024038
[37] Han M 2014 On spinfoam models in large spin regime Class. Quantum Grav. 31 015004
[38] Barrett J W, Dowdall R J, Fairbairn W J, Hellmann F and Pereira R 2010 Lorentzian spin foil
amplitudes: graphical calculus and asymptotics Class. Quantum Grav. 27 165009
[30] Carrozza S 2013 Tensorial methods and renormalization in group field theories PhD Thesis LPT, Orsay
[31] Rivasseau V 2011 Quantum gravity and renormalization: the tensor track AIP Conf. Proc. 1444 18
[32] Dittrich B, Mizera S and Steinhaus S 2016 Decorated tensor network renormalization for lattice gauge theories and spin foam models New J. Phys. 18 053009
[33] Riello A 2013 Self-energy of the Lorentzian Engle–Pereira–Rovelli–Livine and Freidel–Krasnov model of quantum gravity Phys. Rev. D 88 024011
[34] Banburski A, Chen L-Q, Freidel L and Hnybida J 2015 Pachner moves in a 4d Riemannian holomorphic spin foam model Phys. Rev. D 92 124014
[35] de Boer J, Verlinde E P and Verlinde H L 2000 On the holographic renormalization group J. High Energy Phys. JHEP08(2000)003
[36] de Haro S, Solodukhin S N and Skenderis K 2001 Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence Commun. Math. Phys. 217 595
[37] Bonzom V and Smerlak M 2012 Bubble divergences from twisted cohomology Commun. Math. Phys. 312 399
[38] Bonzom V and Smerlak M 2012 Gauge symmetries in spinfoam gravity: the case for ‘cellular quantization’ Phys. Rev. Lett. 108 241303
[39] Witten E 1989 Gauge theories and integrable lattice models Nucl. Phys. B 322 629
[40] Witten E 1990 Gauge theories, vertex models, and quantum groups Nucl. Phys. B 330 285
[41] Turaev V G 1992 Shadow links and face models of statistical mechanics J. Differ. Geom. 36 35
[42] Kirillov A N and Reshetikhin N Y 1989 Representations of the algebra $U_q(sl(2))$ $q$-orthogonal polynomials and invariants of links Infinite Dimensional Lie Algebras and Groups vol 7, ed V G Kac (Singapore: World Scientific) p 285
[43] Westbury B 1998 A generating function for spin network evaluations Knot Theory (Warsaw, 1995) (Banach Center Publications vol 42) (Warsaw: Polish Academy of Sciences) pp 447–56
[44] Dittrich B and Hnybida J 2016 Ising model from intertwiners Ann. Inst. Henri Poincaré Comb. Phys. Interact. 3 363
[45] Bonzom V, Costantino F and Livine E R 2016 Duality between spin networks and the 2D Ising model Commun. Math. Phys. 344 531
[46] Ashtekar A and Lewandowski J 1995 Projective techniques and functional integration for gauge theories J. Math. Phys. 36 2170
[47] Baez J C 1996 Spin network states in gauge theory Adv. Math. 117 253
[48] Freidel L, Geiller M and Ziprick J 2013 Continuous formulation of the loop quantum gravity phase space Class. Quantum Grav. 30 085013
[49] Bahr B, Dittrich B and Geiller M 2015 A new realization of quantum geometry (arXiv:1506.08571)
[50] Gomez C, Ruiz-Altaba M and Sierra G 1996 Quantum Groups in Two-Dimensional Physics (Cambridge: Cambridge University Press)
[51] Faddeev L D 1996 How algebraic Bethe Ansatz works for integrable model Les Houches Lectures
[52] Donnelly W and Freidel L 2016 Local subsystems in gauge theory and gravity J. High Energy Phys. JHEP9(2016)102
[53] Gomez H and Riello A 2017 The observer’s ghost: notes on a field space connection J. High Energy Phys. JHEP05(2017)017
[54] Geiller M 2017 Lorentz-diffeomorphism edge modes in 3d gravity J. High Energy Phys. JHEP02(2018)029
[55] Carlip S 2005 Dynamics of asymptotic diffeomorphisms in (2+1)-dimensional gravity Class. Quantum Grav. 22 3055
[56] Balachandran A P, Bimonte G, Gupta K S and Stern A 1992 Conformal edge currents in Chern–Simons theories Int. J. Mod. Phys. A 07 4655
[57] Balachandran A P, Momen A and Chandar L 1996 Edge states in gravity and black hole physics Nucl. Phys. B 461 581
[58] Roberts J 1999 Classical 6j-symbols and the tetrahedron Geom. Topol. 3 21
[59] Gukov S 2005 Three-dimensional quantum gravity, Chern–Simons theory, and the a polynomial Commun. Math. Phys. 255 577
[60] Barrett J W, Dowdall R J, Fairbairn W J, Gomes H and Hellmann F 2009 Asymptotic analysis of the EPRL four-simplex amplitude J. Math. Phys. 50 112504
[61] Haggard H M, Han M, Kaminski W and Riello A 2015 SL(2,C) Chern–Simons theory, a non-planar graph operator, and 4D quantum gravity with a cosmological constant: semiclassical geometry Nucl. Phys. B 900 1
[62] Barrett J W and Williams R M 1999 The asymptotics of an amplitude for the four simplex Adv. Theor. Math. Phys. 3 209
[63] Bahr B and Thiemann T 2009 Gauge-invariant coherent states for loop quantum gravity. II. Non-Abelian gauge groups Class. Quantum Grav. 26 045012
[64] Maloney A and Witten E 2007 Quantum gravity partition functions in three dimensions J. High Energy Phys. JHEP10(2007)29
[65] Giombi S, Maloney A and Yin X 2008 One-loop partition functions of 3D gravity J. High Energy Phys. JHEP08(2008)007
[66] Barnich G and Oblak B 2014 Notes on the BMS group in three dimensions: I. Induced representations J. High Energy Phys. JHEP6(2014)129
[67] Oblak B 2015 Characters of the BMS group in three dimensions Commun. Math. Phys. 340 413
[68] Kapovich M and Millson J J 1996 The symplectic geometry of polygons in Euclidean space J. Differ. Geom. 44 479
[69] Barbieri A 1998 Quantum tetrahedra and simplicial spin networks Nucl. Phys. B 518 714
[70] Freidel L and Louapre D 2003 Asymptotics of 6j and 10j symbols Class. Quantum Grav. 20 1267
[71] Livine E R and Speziale S 2006 Group integral techniques for the spinfoam graviton propagator J. High Energy Phys. JHEP11(2006)092
[72] Livine E R and Speziale S 2007 A new spinfoam vertex for quantum gravity Phys. Rev. D 76 084028
[73] Dowdall R I, Gomes H and Hellmann F 2010 Asymptotic analysis of the Ponzano–Regge model for handlebodies J. Phys. A: Math. Theor. 43 115203
[74] Costantino F and Marche J 2011 Generating series and asymptotics of classical spin networks (arXiv:1103.5644)
[75] Dittrich B 2012 From the discrete to the continuous: towards a cylindrically consistent dynamics New J. Phys. 14 123004
[76] Rovelli C 2011 Discretizing parametrized systems: the magic of ditt-invariance (arXiv:1107.2310)
[77] Freidel L 2000 A Ponzano–Regge model of Lorentzian 3-dimensional gravity Nucl. Phys. Proc. Suppl. 88 237
[78] Davids S 2000 Semiclassical limits of extended Racah coefficients J. Math. Phys. 41 924
[79] Davids S 2000 A State sum model for (2+1) Lorentzian quantum gravity PhD Thesis University of Nottingham
[80] Freidel L, Livine E R and Rovelli C 2003 Spectra of length and area in (2+1)-Lorentzian loop quantum gravity Class. Quantum Grav. 20 1463
[81] Garcia-Islas J M 2004 (2+1)-dimensional quantum gravity, spin networks and asymptotics Class. Quantum Grav. 21 445
[82] Dittrich B, Donnelly W and Riello A 2018 Tensor network states and dual boundary theories from quantum gravity (in preparation)
[83] Bahr B and Dittrich B 2009 Improved and perfect actions in discrete gravity Phys. Rev. D 80 124030
[84] Bonzom V, Dupuis M and Girelli F 2014 Towards the Turaev–Viro amplitudes from a Hamiltonian constraint Phys. Rev. D 90 104038
[85] Livine E R 2017 3d quantum gravity: coarse-graining and q-deformation Ann. Henri Poincaré A 18 1465
[86] Haggard H M, Han M and Riello A 2016 Encoding curved tetrahedra in face holonomies: phase space of shapes from group-valued moment maps Ann. Henri Poincaré A 17 2001
[87] Dittrich B 2017 (3+1)-dimensional topological phases and self-dual quantum geometries encoded on Heegaard surfaces J. High Energy Phys. JHEP5(2017)123
[88] Kraus P, Maloney A, Maxfield H, Ng G S and Wu J-Q 2017 Witten diagrams for torus conformal blocks J. High Energy Phys. JHEP09(2017)149