CHOW GROUPS OF GUSHEL-MUKAI FIVEFOLDS

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ABSTRACT. We compute the Chow groups of smooth Gushel-Mukai varieties of dimension 5.

1. INTRODUCTION

Let \( n \in \{3, 4, 5, 6\} \). A Gushel-Mukai \( n \)-fold is a smooth Fano variety of dimension \( n \), with Picard number 1, index \( n - 2 \) and degree 10. By Mukai [15] (together with the later result of Mella [14]), a smooth Gushel-Mukai (GM for short) \( n \)-fold \( X \) can be obtained as a smooth dimensionally transverse intersection [7, (1)]

\[
X = \text{CGr}(2, V_5) \cap \mathbb{P}(W) \cap Q \subset \mathbb{P}(\mathbb{C} \oplus \Lambda^2 V_5),
\]

where \( V_5 \) is a 5-dimensional complex vector space, \( \text{CGr}(2, V_5) \subset \mathbb{P}(\mathbb{C} \oplus \Lambda^2 V_5) \) is the projective cone over the image of the Grassmannian variety \( \text{Gr}(2, V_5) \) under the Plücker embedding, \( W \) is a vector subspace of \( \Lambda^2 V_5 \oplus \mathbb{C} \) of dimension \( n + 5 \), and \( Q \) is a quadric hypersurface.

In this article, we are interested in the study of Chow groups of smooth GM fivefolds defined over the field of complex numbers \( \mathbb{C} \).

The main result of this article is the following.

**Theorem 1.** Let \( X \) be a smooth GM fivefold defined over \( \mathbb{C} \). Then:

(i) The cycle class maps induce isomorphisms \( \text{CH}_i(X) \cong H_{2i}(X, \mathbb{Z}) \cong \mathbb{Z} \), for \( i = 0, 1, 4 \), and \( \text{CH}_3(X) \cong H_6(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \).

(ii) The \( i \)-th Griffiths group of \( X \) vanishes for any \( i \in \mathbb{N} \).

(iii) The Abel-Jacobi map induces an isomorphism \( \Phi : \text{CH}_2(X)_{\text{alg}} \to J^5(X) \), and we have the following splitting short exact sequence

\[
0 \to J^5(X) \xrightarrow{\Phi^{-1}} \text{CH}_2(X) \xrightarrow{\text{cl}} \mathbb{Z} \oplus \mathbb{Z} \to 0,
\]

where \( J^5(X) \) is the intermediate Jacobian of \( X \), and \( \text{cl} : \text{CH}_2(X) \to \mathbb{Z} \oplus \mathbb{Z} \) is the cycle class map to \( H^6(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \) [7, Proposition 2.6].
Remark 2. The facts $\text{CH}_0(X) \cong \mathbb{Z}$ and $\text{CH}_4(X) \cong \mathbb{Z}$ follow from the rationality of $X$ [4, Proposition 4.2] and the fact that $\text{Pic}(X) = \mathbb{Z}H$ [4, Lemma 2.29], where $H$ is the hyperplane section class of $X \subset \mathbb{P}(W)$. Moreover, Laterveer [13, Proposition 3.1] proved that $\text{CH}^j(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2j}(X, \mathbb{Q})$ is an injection for $j \neq 3$, which led to the description of the $i$-th ($i \neq 2$) Chow group with rational coefficient.

This article is inspired by the work of Fu and Tian [9] on cubic fivefolds. The main tools are decomposition of the diagonal which is initiated by Bloch-Srinivas [1], and the relation between unramified cohomology and Chow groups (see [2]).

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2. Preliminaries

2.1. Gushel-Mukai fivefolds. A GM fivefold $X$ is a smooth GM variety defined as in the introduction (1) with $n = 5$. There is a natural polarization on $X$, given by the restriction of the hyperplane class on $\mathbb{P}(W)$.

Thanks to the series of work of Debarre and Kuznetsov on GM varieties ([4], [7], [8], [6]), we know some of their properties, like their moduli spaces and period map (cf. [8], [6]) and their Fano varieties of linear spaces (cf. [6]). Here we list some facts about GM fivefolds that we are going to use.

Proposition 3. Let $X$ be a smooth GM fivefold and $(V_6, V_5, A)$ the Lagrangian data associated with $X$ (cf. [4, Section 3]). We set $Y^a_1 := \{ [v] \in \mathbb{P}(V_6) \mid \dim(A \cap (v \wedge \wedge^2 V_6)) \geq 1 \}$ and endow it with a natural scheme structure (cf. [16]).

(i) [4, Proposition 4.2] $X$ is rational;

(ii) [6, Propositions 3.1 and 3.4] The group $H_*(X, \mathbb{Z})$ is torsion-free. The Hodge diamond of $X$ is as follows

\[
\begin{array}{cccccc}
1 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
(iii) [7, Theorems 1.1 and 5.3] Let $f_A : \tilde{Y}_A^{\geq 2} \to Y_A^{\geq 2}$ be the double cover of $Y_A^{\geq 2}$ branched along $Y_A^{\geq 3}$ (we call $\tilde{Y}_A^{\geq 2}$ the double EPW surface with respect to $A$ (cf. [5])). There is an isomorphism of integral Hodge structures
\[ H_1(\tilde{Y}_A^{\geq 2}, \mathbb{Z}) \cong H_5(X, \mathbb{Z}) \]
which induces an isomorphism
\[ \text{Alb}(\tilde{Y}_A^{\geq 2}) \cong J(X) \]
of principally polarized abelian varieties. In particular, if $Y_A^{\geq 3} = \emptyset$, this isomorphism can be induced by a subscheme $Z \subset X \times \tilde{Y}_A^{\geq 2}$.

(3) $\text{AJ}_Z : H_1(\tilde{Y}_A^{\geq 2}, \mathbb{Z}) \sim H_5(X, \mathbb{Z})$.

Corollary 4. Let $X$ be a smooth GM fivefold, then the cycle class map induces an isomorphism $\text{CH}_3(X) \cong H_6(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. By [20, Theorem 2.21], the Abel-Jacobi map $\text{CH}_3(X)_{\text{hom}} \to J^3(X)(\mathbb{C})$ is an isomorphism because of the rationality of $X$. But $H^3(X, \mathbb{Z}) = 0$ by Proposition 3 (ii) and $X$ satisfies the integral Hodge conjecture of degree 4 by [20, Lemma 1.14], which implies that $\text{CH}_3(X) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. \hfill \Box

We will use the followings results concerning the varieties of linear spaces contained in smooth GM fivefolds.

Proposition 5. Let $X$ be a smooth GM fivefold and let $(V_6, V_5, A)$ be the Lagrangian data associated with $X$. We set $Y_{A,V_5}^{\geq l} := \{ [v] \in P(V_5) \mid \dim(A \cap (v \wedge \wedge^2 V_6)) \geq l \}$ and endow it with a natural scheme structure (cf. [16]). Let $Y_{A,V_5} := Y_{A,V_5}^{\geq 1}$.

(i) [6, Theorem 4.7] Let $F_1(X)$ be the Fano variety of lines of $X$. The map $\sigma : F_1(X) \to P(V_5)$, which sends $[v \wedge V_3]$ to $[v]$ for $0 \neq v \in V_3$, factors as
\[ F_1(X) \xrightarrow{\tilde{\sigma}} \overline{P(V_5)} \to P(V_5), \]
where $\overline{P(V_5)} \to P(V_5)$ is the double cover branched along the sextic hypersurface $Y_{A,V_5} \subset P(V_5)$ and $\tilde{\sigma}$ is a $P^1$-bundle over the complement of the preimage of $Y_{A,V_5}^{\geq 2} \cup \Sigma_1(X)$, where $\Sigma_1(X)$ is the kernel locus of $X$ defined as [4, Lemma 4.4].
(ii) [6, Lemma 4.3(b)] Let $F_2(X)$ be the Fano variety of planes of $X$. Then $F_2(X)$ has two connected components, $F^\sigma_2(X)$ and $F^\tau_2(X)$, parametrizing the so-called $\sigma$-planes and $\tau$-planes. For $\sigma$-planes:

- If $X$ is ordinary, or special with $p_X \notin \text{pr}_{Y,2}(E)$, the map $\sigma$ factors as

$$F^\sigma_2(X) \xrightarrow{\sim} \tilde{Y}^\geq 2_{A,V_5} \rightarrow Y^\geq 2_{A,V_5} \hookrightarrow \mathbf{P}(V_5),$$

where $\tilde{Y}^\geq 2_{A,V_5} \rightarrow Y^\geq 2_{A,V_5}$ is a double covering of the curve $Y^\geq 2_{A,V_5}$ branched along $Y^\geq 3_{A,V_5}$.

- If $X$ is special with $p_X \in \text{pr}_{Y,2}(E)$, the scheme $F^\sigma_2(X)$ is the union of a double cover $\tilde{Y}^\geq 2_{A,V_5}$ and one double or two reduced components isomorphic to $\mathbf{P}^1$ and contracted by the map $\sigma$ onto $\Sigma_1(X)$.

(iii) [7, Lemma 5.8] The Abel-Jacobi map

$$H_1(F^2_\sigma(X), Z) \rightarrow H_5(X, Z)$$

is a surjective morphism of Hodge structures for general $X$.

**Corollary 6.** Let $X$ be a smooth GM fivefold. Then $F_1(X)$ is rationally chain connected. In particular, $\text{CH}_0(F_1(X)) = \mathbb{Z}$.

**Proof.** By Proposition 5 (i), and Graber-Harris-Starr [10, Corollary 1.3], it is enough to show that $\mathbf{P}(V_5)$ is rationally chain connected. Suppose that $Y_{A,V_5} \subset \mathbf{P}(V_5)$ is defined by $f_0 = 0$, note that $Y_{A,V_5}$ is not a smooth sextic hypersurface, since $Y^\geq 2_{A,V_5} \subset \text{Sing}(Y_{A,V_5})$ and $Y^\geq 2_{A}$ is an integral normal surface [4, Theorem B.2]. Consider the flat family

$$\tilde{\mathbf{P}} \rightarrow \mathbf{P}(1,1,1,1,3)$$

where $\tilde{\mathbf{P}} := \mathbf{P}(H^0(\mathbf{P}(V_5), \mathcal{O}(6)))$. The morphism $p$ is flat and for any $[f] \in \mathbb{H}$, $p^{-1}([f])$ is the hypersurface in $\mathbf{P}(1,1,1,1,3)$ defined by $x^2 - f = 0$ (here $\deg(x) = 3$). Therefore, the smoothness of $p^{-1}([f])$ is equivalent to the smoothness of the hypersurface $Z(f) \subset \mathbf{P}(V_5)$, in which case $p^{-1}([f])$ is a Fano manifold of index $2$. By Kollár-Miyaoka-Mori [12, Theorem 0.1], it is rationally chain connected (indeed is rationally connected) (cf. [11, Theorem 2.13, P.254]). By specialization, $p^{-1}([f_0]) = \tilde{\mathbf{P}}(V_5)$ is also rationally chain connected. \qed
2.2. Unramified Cohomology. Let \( X \) be a smooth projective complex variety. Consider the identity continuous morphism
\[
\pi : X_{\text{cl}} \longrightarrow X_{\text{Zar}},
\]
where \( X_{\text{cl}} \) is with the classical topology and \( X_{\text{Zar}} \) is with the Zariski topology. For any abelian group \( A \) (e.g. \( \mathbb{Z}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z} \)) and any positive integer \( i \), we introduce a sheaf on \( X_{\text{Zar}} \) defined by the derived direct image of the constant sheaf \( A \), that is, \( \mathcal{H}^i(A) := R^i\pi_*A \). By definition, the \( i \)-th unramified cohomology (cf. [1], [2]) of \( X \) with coefficients in \( A \) is
\[
H_{\text{nr}}^i(X, A) := H^0(X_{\text{Zar}}, \mathcal{H}^i(A)).
\]
It is clear that \( H_{\text{nr}}^i(X, A) = E_2^{0,i} \) in the Leray spectral sequence of \( (\pi, A) \).

On the other hand, the cohomology \( H^k(X_{\text{cl}}, A) \) has a filtration by coniveau: for any \( c \in \mathbb{N} \),
\[
N^c H^k(X, A) = \sum Z \ker(H^k(X, A) \to H^k(X \setminus Z, A))
\]
where \( Z \) runs through all the closed algebraic subsets of \( X \) of codimension \( \geq c \). From this filtration, we can get the coniveau spectral sequence (cf. [1, Section 3]):
\[
\begin{align*}
'E_1^{p,q} & = \bigoplus_{Y \in X^{(p)}} H^{q-p}(\mathbb{C}(Y), A(-p)) \Rightarrow N^\bullet H^{p+q}(X, A) \\
\end{align*}
\]
where \( X^{(p)} \) is the set of integral subschemes of \( X \) of codimension \( p \).

Thanks to the work of Bloch–Ogus on the Gersten conjecture [1, Proposition 6.4], we have the following proposition due to Deligne:

**Proposition 7.** Let \( X \) be a smooth projective complex variety. Starting from \('E_2\), the coniveau spectral sequence
\[
'E_1^{p,q} = \bigoplus_{Y \in X^{(p)}} H^{q-p}(\mathbb{C}(Y), A(-p)) \Rightarrow N^\bullet H^{p+q}(X, A)
\]
coincides with the Leray spectral sequence associated with \((\pi, A)\)
\[
E_2^{p,q} = H^p(X, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(X, A).
\]

Here we list some facts we will need about unramified cohomology and the Leray spectral spectral of \((\pi, A)\).
Proposition 8. (i) [2, Proposition 3.3(iii)] If \(X = \mathbb{P}^d_C\), then
\[
H^p(X, \mathcal{H}^q(A)) = \begin{cases} 
0 & \text{if } p \neq q, \\
A & \text{if } p = q \leq d.
\end{cases}
\]

(ii) [1, Corollary 7.4] There is a natural isomorphism
\[
\text{CH}^p(X)/\text{alg} \cong H^p(X, \mathcal{H}^p(\mathbb{Z}))
\]
for any smooth projective complex variety \(X\), where \(\text{CH}^p(X)/\text{alg}\) is the group of algebraic cycles of codimension \(p\) modulo algebraic equivalence.

(iii) [2, Theorem 3.1] For \(i, j \in \mathbb{N}\) and \(n \in \mathbb{N}^*\), we have a short exact sequence
\[
0 \to H^j(X, \mathcal{H}^i(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z} \to H^j(X, \mathcal{H}^i(\mathbb{Q}/\mathbb{Z})) \to H^{j+1}(X, \mathcal{H}^i(\mathbb{Z}))_{\text{tor}} \to 0.
\]

(iv) [1, Corollary 6.2] \(H^p(X, \mathcal{H}^q(A)) = 0\) if \(p > q\), for any smooth projective complex variety \(X\).

As unramified cohomology is a birational invariant for smooth projective varieties [20, Theorem 1.21]. One deduce the following from Proposition 3(i).

Corollary 9. For any smooth GM fivefold \(X\) and any abelian group \(A\), \(H^0_{\text{nr}}(X, A) \cong A\), and \(H^i_{\text{nr}}(X, A) = 0\) for \(i \neq 0\).

3. Chow group of 1-cycles

This section is devoted to the proof of \(\text{CH}_1(X) \cong \mathbb{Z}\). First, we are going to show that \(\text{CH}_1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}\), see Laterveer [13] for an alternative proof using the so-called Franchetta property.

Lemma 10. Let \(X\) be a smooth GM fivefold, for any two points \(x, y \in X\), \(x\) and \(y\) can be connected by at most 4 lines.

Proof. Let \(\gamma_X(x) = [U_2]\) and \(\gamma_X(y) = [V_2]\) be two projective lines in \(\mathbb{P}(V_5)\) associated with \(x, y\), where \(\gamma_X : X \to \text{Gr}(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)\) is the Gushel map of \(X\) defined as [4, Section 2.1].

Claim: \(x\) and \(y\) can be connected by lines if the lines \([U_2]\) and \([V_2]\) can be connected by lines in \(\gamma_X(X) \subset \text{Gr}(2, V_5)\).

Proof of the claim: Let \(F_1(X, x) \subset F_1(X)\) be the subscheme parametrizing lines passing through \(x\), then \(\sigma(F_1(X, x)) = [U_2]\) by [6, Proposition 4.1]. By the same reason, \(\sigma(F_1(X, y)) = [V_2]\).
Note that \(x_1\) and \(x_2\) can be connected by at most two lines if \(\gamma_X(x_1) \cap \gamma_X(x_2) \neq \emptyset\) (indeed if these two lines intersect at \([v] \in P(V_5)\), \(x_1\) and \(x_2\) will be contained in a quadratic hypersurface \(\rho_1^{-1}(v)\) in \(P^3\) or \(P^4\) by [4, Proposition 4.5]). Therefore, if there are lines \([U_2] = L_0, L_1, \ldots, L_n = [V_2]\) in \(P(V_5)\) such that \(L_i \in \gamma_X(X)\) and \(L_i \cap L_{i+1} \neq \emptyset\) for \(i = 0, \ldots, n - 1\), \(x\) and \(y\) can be connected by \(2n\) lines.

Therefore, it is enough to show that \([U_2]\) and \([V_2]\) can be connected by lines in \(\gamma_X(X) \subset Gr(2, V_5)\). Here \(\dim \gamma_X(X) = 5\) as \(\gamma_X\) is finite\(^1\). Note that fixed a point \([v] \in [U_2]\), there exist at least 2-dimensional lines in \(\gamma_X(X)\) passing \([v]\) because \(\dim \rho_1^{-1}(v) \geq 2\) and \(\gamma_X\) is finite. Thus the universal family of lines in \(\gamma_X(X)\) intersecting \([U_2]\) is at least 4-dimensional, which must intersect with the projection line \([V_2]\) in \(P(V_5)\). In other words, there exist lines \([U_2] = L_0, L_1, L_2 = [V_2]\) in \(\gamma_X(X)\) such that \(L_i \cap L_{i+1} \neq \emptyset\) for \(i = 0, 1\).

**Proposition 11.** Let \(X\) be a smooth GM fivefold, then \(\text{CH}_1(X) \otimes \mathbb{Z} \mathbb{Q} \cong \mathbb{Q}\). Specifically, there exist an integer \(N \neq 0\), such that for any 1-cycle \(C\), \(NC = m[l]\) in \(\text{CH}_1(X)\) for some \(m\), where \([l]\) is the class of a line in \(X\).

**Proof.** By Corollary 6, all lines in \(X\) have the same class in \(\text{CH}_1(X)\). Therefore, it suffices to show that for any curve \(C\) in \(X\), there is a positive integer \(N\), such that \(NC\) is rationally equivalent to sum of lines.

Let \(g : F \to B\) parametrize the family of effective 1-cycles with at most four lines components, and \(u : F \to X\) be the evaluation map such that the map

\[m : F \times_B F \to X \times X\]

is generically finite of degree \(N\) and surjective (this is possible, thanks to Lemma 10). By the argument in [17, Proposition 3.1], there are integers \(m_i\)'s and lines \(F_i\)'s in the fibers of \(g\), such that \(NC\) is rationally equivalent to \(\sum_i m_i u_*(F_i)\).

\[\square\]

**Corollary 12.** Let \((X, H)\) be a polarized smooth GM fivefold. There exists a nonzero integer \(m \in \mathbb{N}^\ast\) and a decomposition in \(\text{CH}_5(X \times X)\)

\[(6)\]

\(m \Delta_X = mx \times X + ml \times h + Z',\)

\(^1\text{deg} \gamma_X = 1\) if \(X\) is ordinary, \(\text{deg} \gamma_X = 2\) if \(X\) is special in the sense of [4, Section 2.5].
where $x$ is an arbitrary point of $X$, $l$ is an arbitrary projective line contained in $X$, $h := c_1(H)$ is the hyperplane section class, and $Z'$ is supported on $X \times T$ with $T$ a closed algebraic subset of pure dimension 3 in $X$.

**Proof.** By the argument of decomposition of the diagonal [19, Theorem 10.29], there is a nonzero integer $m$ and a decomposition in $\text{CH}_5(X \times X)$

\[(7) \quad m \triangle_X = Z_0 + Z_1 + Z',\]

where $Z_i$ is supported on $W_i \times W_i'$ with $\dim W_i = i$ and $\dim W_i' = n - i$ for $i = 0, 1$, and $Z'$ is supported on $X \times T$ with $T$ a closed algebraic subset of codimension $\geq 2$ in $X$. We suppose that $T$ is of pure dimension 3. In particular, since $\text{CH}_0(X)_\mathbb{Q}$ and $\text{CH}_1(X)_\mathbb{Q}$ are generated by a point and a line in $X$ respectively, and $\text{Pic}(X) = \mathbb{Z}H$ by [4, Lemma 2.29], we can write $Z_0 = m_1 x \times X$ and $Z_1 = m_2 l \times h$ for some integers $m_1$ and $m_2$.

It remains to show $m = m_1 = m_2$. Let (7) act by correspondences on the class of a point $x$ and the class of a line $l$. Note that $(x \times X)^*(x) = x$, $(x \times X)^*(l) = 0$, $(l \times h)^*(x) = 0$, $(l \times h)^*(l) = l$ and $Z'^*(x) = Z'^*(l) = 0$ by reason of dimension and $\triangle_X$ always acts as the identity, we get $mx = m_1 x$ and $ml = m_2 l$. Hence, $m = m_1 = m_2$. \hfill \square

Before proving $\text{CH}_1(X) \cong \mathbb{Z}$, we recall the following well-known result.

**Lemma 13.** [1, Lemma 7.10] The Chow group of algebraically trivial cycles is divisible.

**Corollary 14.** For a smooth GM fivefold $X$, let $\text{CH}_1(X)_\text{alg}$ and $\text{CH}_1(X)_\text{hom}$ be the subgroups of $\text{CH}_1(X)$ of algebraically trivial cycles, resp. homologically trivial cycles. Then $\text{Griff}_1(X) = 0$, where $\text{Griff}_i(X) := \text{CH}_i(X)_\text{hom}/\text{CH}_i(X)_\text{alg}$ is the $i$-th Griffiths group of $X$. In other words, $\text{CH}_1(X)_\text{alg} = \text{CH}_1(X)_\text{hom}$.

**Proof.** By [20, Proposition 1.30], the group $\text{Griff}_1(X)$ is a birational invariant for smooth projective varieties. Note that $X$ is birationally isomorphic to $\mathbb{P}^5$ [4, Proposition 4.2] and $\text{Griff}_1(\mathbb{P}^5) = 0$. \hfill \square

**Corollary 15.** For any smooth GM fivefold $X$, $\text{CH}_1(X)/\text{alg} \cong \mathbb{Z}$, i.e. any curve $C$ in $X$ can algebraically equivalent to sum of lines. Here $\text{CH}_1(X)/\text{alg}$ is the group of 1-cycles modulo algebraic equivalence.
Proof. Indeed, \( H^8(X, \mathbb{Z}) = \mathbb{Z}l \), where \( l \) is an arbitrary projective line in \( X \). For any curve \( C \subset X \), there is an integer \( d \) such that \( C - dl = 0 \) in \( H^8(X, \mathbb{Z}) \) (where \( d \) is the degree of \( C \)). Therefore \( (C - dl) \in CH_1(X)_{\text{alg}} \). That means \( [C] = d[l] \) in \( CH_1(X)_{/\text{alg}} \). □

**Proposition 16.** For any smooth GM fivefold \( X \), \( CH_1(X) \cong \mathbb{Z} \).

Proof. By Corollary 15, for any 1-cycle \( C \) of \( X \), there is an integer \( d \) such that \( [C] - d[l] \) is an element in \( CH_1(X)_{\text{alg}} \), where \( l \) is an arbitrary line in \( X \). But \( CH_1(X)_{\text{alg}} \) is divisible, \( [C] - d[l] = NC' \) in \( CH_1(X) \) for some 1-cycle \( C' \), where \( N \) is the integer in Proposition 11. Note that \( NC' \) is rationally equivalent to, by Proposition 11, a sum of lines \( m[l] \). Thus \( [C] = d[l] + NC' = d[l] + m[l] = (d + m)[l] \) in \( CH_1(X) \). □

4. The vanishing of the Griffiths groups

For a smooth GM fivefold \( X \), we have shown that \( \text{Griff}_0(X) = \text{Griff}_1(X) = \text{Griff}_4(X) = 0 \) by Corollary 14 and the definition of Griffiths groups. We prove in this section that \( \text{Griff}_2(X) = \text{Griff}_3(X) = 0 \). The main tool is unramified cohomology, which is a birational invariant for smooth projective varieties.

We need the following lemma:

**Lemma 17.** [5, Theorem 5.2 (2)] Suppose that \( X \) is a smooth GM fivefold, and \( A = A(X) \) is the Lagrangian subspace associated with \( X \). There is a unique double cover \( f_2 : \tilde{Y}^{\geq 2} \rightarrow Y^{\geq 2} \) branched along the finite set \( Y_A^3 \) (note that \( Y_A^3 = Y_A^{\geq 3} \) as \( A \) contains no decomposable vectors by [4, Theorem 3.16] and [4, Theorem B.2]), which is empty when \( X \) is general. Moreover, the scheme \( \tilde{Y}^{\geq 2}_A \) is integral and normal, and it is smooth away from \( f_2^{-1}(Y_A^3) \) and has ordinary double points along \( f_2^{-1}(Y_A^3) \).

**Proposition 18.** Let \( X \) be a smooth GM fivefold.

(i) There exist a smooth projective curve \( C \), and an algebraic 3-cycle \( W \subset C \times X \), such that the induced map \( H^1(C, \mathbb{Z}) \rightarrow H^5(X, \mathbb{Z}) \) is a surjective morphism of Hodge structures.

(ii) \( H^5(X, \mathbb{Z}) \) is of algebraic coniveau 2, i.e. \( N^2 H^5(X, \mathbb{Z}) = H^5(X, \mathbb{Z}) \).
Proof. (1) Consider the Hilbert closure $Z \subset X \times \widetilde{Y}_{\Delta}^{\geq 2}$ defined in [7, Lemma 5.2]. By [7, Theorem 5.3], if $Y_A^{\geq 3} = \emptyset$ (this condition holds for general $X$) so that $\widetilde{Y}_{\Delta}^{\geq 2}$ is a smooth surface, the Abel-Jacobi map

$$AJ_Z : H^3(\widetilde{Y}_{\Delta}^{\geq 2}, Z) \to H^5(X, Z)$$

is an isomorphism of integral Hodge structures. We can choose a very ample smooth divisor $i : C \to \widetilde{Y}_{\Delta}^{\geq 2}$, then the Lefschetz hyperplane theorem ([19, Theorem 1.23]) says that the composition

$$H^1(C, Z) \to H^3(\widetilde{Y}_{\Delta}^{\geq 2}, Z) \to H^5(X, Z)$$

is a surjection of Hodge structures. Indeed, we can just set $C := \widetilde{Y}_{\Delta V_5}^{\geq 2} \cong F_2^2(X)$ [7, (26) and Lemma 5.8].

As for any $X$, consider the universal family $\widetilde{Y}_{\Delta}^{\geq 2} \to \mathbb{D}$, $\pi : \mathcal{X} \to \mathbb{D}$ and $Z \to \mathbb{D}$ such that $Z_t \subset \mathcal{X}_t \times \widetilde{Y}_{\Delta t}^{\geq 2}$ is defined as [7, Lemma 5.2]. Here $\mathbb{D}$ is an analytical disk of small radius in $\mathbb{C}$ of center 0 such that for all $t \in \mathbb{D} \setminus \{0\}$ the fiber $\mathcal{X}_t$ is general in the sense above and $\mathcal{X}_0 \cong X$. In other words, we have the universal family:

$$\begin{array}{ccc}
Z & \xrightarrow{q} & \mathcal{X} \\
\downarrow p & & \downarrow \pi \\
\widetilde{Y}_{\Delta}^{\geq 2} & \xrightarrow{i} & \mathbb{D}
\end{array}$$

Choose a very ample smooth divisor $C \subset \widetilde{Y}_{\Delta}^{\geq 2}$ away from the union of the singular points of $\widetilde{Y}_{\Delta}^{\geq 2}$ and the finite subset of $\widetilde{Y}_{\Delta}^{\geq 2}$ such that $Z \to \widetilde{Y}_{\Delta}^{\geq 2}$ is not flat [7, Lemma 5.2], and deform this smooth curve along $\mathbb{D}$, we get a flat family of smooth curves $\mathcal{C} \to \mathbb{D}$ (if has singular fiber, shrinking $\mathbb{D}$), the diagram (8) becomes the following when restricting to $\mathcal{C}$:

$$\begin{array}{ccc}
\mathcal{W} & \xrightarrow{q} & \mathcal{X} \\
\downarrow p & & \downarrow \pi \\
\mathcal{C} & \xrightarrow{\pi} & \mathbb{D}
\end{array}$$

Here $p$ and $\pi$ are flat. Note that $H^1(C_t, Z) \cong H^1(C_t, Z)$ for any $t \neq 0$, since $C \to \mathbb{D}$ is an Ehresmann’s fibration [18, Theorem 9.3]. And for any $t \neq 0$, the Abel-Jacobi map induced by $\mathcal{W}_t$, i.e. $H^1(C_t, Z) \to H^5(\mathcal{X}_t, Z)$, is surjective. By continuity of local systems, the Abel-Jacobi map $AJ_{\mathcal{W}} : H^1(C, Z) \to H^5(X, Z)$ is a surjection, where $W := \mathcal{W}_0$. 
(2) Consider the algebraic cycle \( W \subset C \times X \) constructed in (1). The Abel-Jacobi map \( H_1(C, \mathbb{Z}) \to H_5(X, \mathbb{Z}) \) can factor as following:

\[
H_1(C, \mathbb{Z}) \overset{g_\ast p_\ast}{\to} H_5(q(W), \mathbb{Z}) \to H_5(X, \mathbb{Z})
\]

Thus \( H_5(q(W), \mathbb{Z}) \to H_5(X, \mathbb{Z}) \) is surjective. Equivalently, \( H^5(X, \mathbb{Z}) \) is supported on \( q(W) \), which is an algebraic subset of codimension \( k \geq 2 \). Therefore, \( H^5(X, \mathbb{Z}) = N^k H^5(X, \mathbb{Z}) \subset N^2 H^5(X, \mathbb{Z}) \).

\[\square\]

**Proposition 19.** Let \( X \) be a smooth projective complex GM fivefold.

(i) \( \text{Griff}^2(X) = 0 \).

(ii) \( \text{coker}\{H^5(X, \mathbb{Z}) \to H^1(X, H^4(\mathbb{Z}))\} \cong \text{Griff}^3(X) \).

**Proof.** This proof is similar to the proof of [9, Proposition 13]. First, we have an exact sequence from the spectral sequence [1, (8.2)]

\[
H^3(X, \mathbb{Z}) \to H^0(X, H^3(\mathbb{Z})) \overset{d_2}{\to} H^2(X, H^2(\mathbb{Z})) \overset{cl}{\to} H^4(X, \mathbb{Z})
\]

where the first arrow is the composition \( H^3(X, \mathbb{Z}) \to E^0_{3,0} \hookrightarrow E^0_{2,3} = H^0(X, H^3(\mathbb{Z})) \), the last arrow is the cycle class map. Therefore, by Corollary 9, \( \text{ker}\{H^2(X, H^2(\mathbb{Z})) \overset{cl}{\to} H^4(X, \mathbb{Z})\} = 0 \). But this is exactly \( \text{Griff}^2(X) \) by Proposition 8 (ii). (Indeed, the vanishing of \( \text{Griff}^2(X) \) can be obtained directly from [20, Theorem 2.21] as \( CH_0(X) \cong \mathbb{Z} \).)

Similarly, from the spectral sequence, we get an exact sequence [9, (5)] (we use \( H^4_{\text{nr}}(X, \mathbb{Z}) = 0 \) by Corollary 9, and \( H^3(X, H^2(\mathbb{Z})) = 0 \) by Proposition 8(iv).)

\[
H^2(X, H^3(\mathbb{Z})) \to H^5(X, \mathbb{Z}) \to H^1(X, H^4(\mathbb{Z})) \overset{d_2}{\to} H^3(X, H^3(\mathbb{Z})) \to H^6(X, \mathbb{Z}).
\]

Hence \( \text{coker}\{H^5(X, \mathbb{Z}) \to H^1(X, H^4(\mathbb{Z}))\} \cong \text{ker}\{H^3(X, H^3(\mathbb{Z})) \to H^6(X, \mathbb{Z})\} \). But the later is exactly \( \text{Griff}^3(X) \) by Proposition 8 (ii).  

\[\square\]

**Corollary 20.** Let \( X \) be a smooth GM fivefold, \( \text{Griff}_2(X) \) is torsion free.

**Proof.** By Proposition 8 (iv) the first arrow of (11) is the composition of

\[
H^2(X, H^3(\mathbb{Z})) \to N^2 H^5(X, \mathbb{Z}) \to H^5(X, \mathbb{Z}).
\]
It is surjective, since $X$ is of algebraic coniveau 2 by Proposition 18. Consequently, the second arrow in Equation (11) is zero, and

$$H^1(X, \mathcal{H}^4(Z)) \cong \text{Griff}^3(X)$$

by Proposition 19 (ii).

Let $i = 4$ and $j = 1$, the exact sequence (5) implies $H^1(X, \mathcal{H}^4(Z)) \cong H^0(X, \mathcal{H}^4(Q/Z))$ as $H^0(X, \mathcal{H}^4(Z))$ vanishes by Corollary 9. But $H^0(X, \mathcal{H}^4(Q/Z)) = 0$ again by Corollary 9. Therefore, $\text{Griff}_2(X) \cong H^1(X, \mathcal{H}^4(Z))$.

\begin{prop}
Let $X$ be a smooth GM fivefold, $\text{Griff}_2(X) = 0$.
\end{prop}

\begin{proof}
The proof is the same as [9, Proposition 18]. By Corollary 20, it is enough to show that $\text{Griff}_2(X)$ is of torsion.

We need the decomposition of the diagonal (6) in the proof of Corollary 12. Choose a desingularization $\tilde{T} \to T$ equipped with a cycle $\tilde{Z}' \in \text{CH}^3(X \times \tilde{T})$ whose image in $\text{CH}^3(X \times T)$ is a multiple $m'Z'$ of $Z'$. We denote $\tilde{i}: \tilde{T} \to X$ the composition of desingularization and inclusion. Note that $\tilde{T}$ is not necessarily connected and we can write $\text{Pic}^0(\tilde{T}) = \prod \text{Pic}^0(\tilde{T}_i)$ and $\text{Alb}(\tilde{T}) = \prod \text{Alb}(\tilde{T}_i)$, where $\tilde{T} = \bigsqcup_i \tilde{T}_i$ is the decomposition into connected components.

For any 2-cycle $\alpha$, let the correspondences in (6) act on $\alpha$, then $\Delta_*(\alpha) = \alpha$, $(x \times X)_*(\alpha) = (l \times h)_*(\alpha) = 0$ by the reason of dimension and $m'Z'_*(\alpha) = \tilde{i}_* \circ \tilde{Z}'_*(\alpha)$. Therefore, $mm'\alpha = \tilde{i}_* \circ \tilde{Z}'_*(\alpha)$, i.e. $\tilde{i}_* \circ \tilde{Z}'_* = mm'$ on $Z_2(X)$.

If $\alpha$ is homologically trivial, $\tilde{Z}'_*(\alpha)$ is a 1-cocycle in $\tilde{Z}'$ which is homologically trivial. So $\tilde{i}_* (\tilde{Z}'_*(\alpha))$ is algebraically trivial as $\text{Griff}^1(\tilde{T}) = 0$, so as $mm'\alpha$. Hence $\alpha$ is a torsion element in $\text{Griff}_2(X)$.

\end{proof}

\section{Proof of Theorem 1 (iii)}

In the rest of this article, $X$ is a smooth GM fivefold defined on $\mathbb{C}$. In this section, we consider the Abel–Jacobi map

$$\Phi : \text{CH}_2(X)_{\text{alg}} \to J^5(X)(\mathbb{C})$$

and show that $\Phi$ is an isomorphism. Note that $\text{CH}_2(X)_{\text{alg}}$ is a divisible group by Lemma 13.
**Proposition 22.** The Abel-Jacobi map \( \Phi : CH_2(X)_{\text{alg}} \to J^5(X) \) is surjective with a finite kernel, i.e. is an isogeny.

**Proof.** Let us show the surjectivity of \( \Phi \) first. By Proposition 18 (i), we can consider the following commutative diagram by the compatibility between the Abel-Jacobi maps and the correspondence actions [18, Theorem 12.17]:

\[
\begin{array}{ccc}
CH_0(C)_{\text{alg}} & \xrightarrow{\text{alb}} & J(C) \\
\downarrow W^* & & \downarrow \\
CH_2(X)_{\text{alg}} & \xrightarrow{\Phi} & J^5(X) \\
\end{array}
\]

the top horizontal arrow is surjective since \( CH_0(C)_{\text{alg}} = CH_0(C)_{\text{hom}} \) and [18, Lemma 12.11], and the right arrow is a surjection with connected fibers, because \( H^1(C, \mathbb{Z}) \to H^5(X, \mathbb{Z}) \) is surjective. Therefore, the bottom arrow \( \Phi \) is a surjection.

It remains to show the finiteness of the kernel of \( \Phi \). This part is the same as the proof of [9, Proposition 18]. Let \( \tilde{T} \) and \( \tilde{Z} \) be as in the proof of Proposition 21. We have a commutative diagram

\[
\begin{array}{ccc}
CH^3(X)_{\text{alg}} & \xrightarrow{\Phi} & J^5(X) \\
\downarrow \tilde{Z}_* & & \downarrow \bar{Z}_* \\
CH^1(\tilde{T})_{\text{alg}} & \xrightarrow{\cong} & \text{Pic}^0(\tilde{T}) \\
\downarrow \tilde{i}_* & & \downarrow \bar{i}_* \\
CH^3(X)_{\text{alg}} & \xrightarrow{\Phi} & J^5(X) \\
\end{array}
\]

Similarly, (7) implies that the composites of the vertical arrows in (13) are both the multiplication by \( mm' \). Now for all \( \alpha \in \ker(\Phi) \), by the divisibility of \( CH^3(X)_{\text{alg}} \), there exists a \( \beta \in CH^3(X)_{\text{alg}} \), such that \( \alpha = mm' \beta \). So \( \Phi(\beta) \in J^5(X)[mm'] \). Since the arrow in the middle of (7) is an isomorphism, we find that \( \tilde{Z}_*(\beta) \in CH^1(\tilde{T})_{\text{alg}}[mm'] \). From where,

\[
\alpha = mm' \beta = \tilde{z}_* \circ \tilde{Z}_*(\beta) \in \tilde{z}_*(CH^1(\tilde{T})_{\text{alg}}[mm'])
\]

which is a finite set. \( \Box \)

*Proof of Theorem 1(iii)*
This proof is similar to [9, Proposition 19]. For any smooth \( X \), there exists a commutative diagram for groups

\[
\begin{array}{ccc}
\text{CH}_0(C)_{\text{alg}} & \xrightarrow{\sim} & J(C) \\
\downarrow^{W^*} & & \downarrow \\
\text{CH}_2(X)_{\text{alg}} & \xrightarrow{\Phi} & J^5(X)
\end{array}
\]

Let \( A \) be the kernel of the right arrow, which is an abelian subvariety of \( J(C) \). By Proposition 22, as a morphism between abelian groups, \( \Phi \) is surjective with a finite kernel. Since any morphism from a divisible group to a finite group is trivial, the image of \( A \) in \( \ker(\Phi) \) is trivial so that the image of the group \( \text{CH}_0(C)_{\text{alg}} \) in \( \text{CH}_2(X)_{\text{alg}} \) is isomorphic to \( J^5(X) \). So we have a split of \( \Phi \) and \( \text{CH}_2(X)_{\text{alg}} \cong J^5(X) \oplus \ker(\Phi) \). However the group \( \text{CH}_2(X)_{\text{alg}} \) is divisible (Lemma 13), and therefore \( \ker(\Phi) = 0 \), i.e. \( \Phi \) is an isomorphism. Finally, the class map \( \text{CH}_2(X) \to H^6(X, \mathbb{Z}) \) is surjective since \( X \) satisfies the integral Hodge conjecture of degree 6 by [3, Remark 4.2]. Moreover, we have the split exact sequence (2). \( \square \)

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