EXISTENCE AND STABILITY OF GENERALIZED TRANSITION WAVES FOR TIME-DEPENDENT REACTION-DIFFUSION SYSTEMS

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Abstract. The current paper is devoted to the study of the existence and stability of generalized transition waves of the following time-dependent reaction-diffusion cooperative system

\[ \frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + F(t, u(t, x)), \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \quad u \in \mathbb{R}^K, \quad K > 1. \]

Here \( F(t, u(t, x)) \) depends on \( t \in \mathbb{R} \) in a general way. Recently, the spreading speeds and linear determinacy of the above time-dependent system have been studied by Bao et al. [J. Differential Equations 265 (2018) 3048-3091]. In this paper, using the principal Lyapunov exponent and principal Floquet bundle theory of linear cooperative systems, we prove the existence of generalized transition waves in any given direction with speed greater than the spreading speed by constructing appropriate subsolutions and supersolutions. When the initial value is uniformly bounded with respect to a weighted maximum norm, we further show that all solutions converge to the generalized transition wave solutions exponentially in time.

1. Introduction. Since the pioneering works by Fisher [13] and Kolomogorov et al. [19], traveling wave solutions of reaction-diffusion systems have widely been studied due to its important applications in biology, chemistry, epidemiology and physics, see Conley and Gardner [10], Mischaikow and Huston [25], Volpert et al. [45], Weinberger [47] and references therein. Due to the influence of the environmental heterogeneity, the mathematical study on the space and/or time periodic traveling wave solutions for various variants of reaction-diffusion equations/systems in periodic habitats have been studied by many people. See, for example, Alikakos et al. [1], Bao and Wang [3], Bao et al. [4], Berestycki and Hamel [6], Fang et al. [12], Han et al. [16], Kong et al. [20], Liang and Zhao [23], Liang et al. [21], Nadin [26], Pan [33], Weinberger [48], Yang et al. [50], Yu and Zhao [51], Zhao and Ruan [52], and so on. However, in nature, the environments or media of biological
models are subject to various seasonal variations, which are roughly and generally but not exactly periodic. It is then very important and interesting to investiga-
tive traveling wave solution and spreading speed of reaction-diffusion systems with more general time dependence. In this paper, we are interested in studying the ex-
istence, uniqueness and stability of generalized transition waves for time-dependent reaction-diffusion cooperative systems of the form
\[
\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + F(t, u(t, x)), \quad (x, t) \in \mathbb{R}^N \times \mathbb{R},
\]
where the vector-valued function \( u(t, x) = (u_1(t, x), \ldots, u_K(t, x)) \) \((K \geq 1)\) represents the densities of the population of \( K \)-species at time \( t \in \mathbb{R} \) and point \( x \in \mathbb{R}^N \), and the reaction term
\[
F(t, u(t, x)) = (F^1(t, u), \ldots, F^K(t, u)) \text{ satisfies the following standard assumptions:}
\]
\begin{enumerate}[(H1)]
\item \( F(t, u) \) is \( C^2 \) in \( u; F^i(t, u), \frac{\partial F^i}{\partial u_j}(t, u) \text{ and } \frac{\partial^2 F^i}{\partial u_i \partial u_j}(t, u) \) are globally Hölder
\item \( F(t, u) \) is cooperative in \( u \) in the sense that for \( 1 \leq i \neq j \leq K \),
\end{enumerate}
Together with comparison principle for cooperative system, for any \( u_0 \in [0, u^*(s)] \),
\( u(t; s, u_0) \) exists and \( u(t; s, u_0) \in [0, u^*(t)] \) for all \( t \geq s \), where \( u(t; s, u_0) \) is the solution of (1.2) with \( u(s; s, u_0) = u_0 \).
Assumptions (H1) and (H2) imply that \( u^*(t) \) is a globally stable solution of (1.2) in the sense that for any \( u_0 \in (0, \inf_{t \in \mathbb{R}} u_1^*(t)) \times \ldots \times (0, \inf_{t \in \mathbb{R}} u_K^*(t)) \),
\[
\|u(t + s; s, u_0) - u^*(t + s)\|_{\mathbb{R}^K} \to 0 \quad \text{as } t \to \infty \text{ uniformly for } s \in \mathbb{R}.
\]
Moreover, if \( u^{**}(t) \in [0, u^*(t)] \) is an entire solution of (1.2) and \( u^{**}(\cdot) \neq u^*(\cdot) \), then there is \( 1 \leq i \leq K \) such that \( u_i^{**}(t) \equiv 0 \), which means at least one of the species is extinct.

When \( K = 1 \), system (1.1) becomes the following reaction-diffusion equation
\[
\frac{\partial u}{\partial t} = \Delta u(t, x) + f(t, u(t, x)), \quad x \in \mathbb{R}^N,
\]
where \( f(t, u) \) depends on \( t \in \mathbb{R} \) and is of KPP type in \( u \). The study of spatial spreading speeds and transition waves of reaction-diffusion equation (1.4) in general time and/or space dependence environments have attracted much more attention, see Berestycki and Hamel \[7, 8\] and Shen \[35\] for the notion of generalized transition waves. Following from the work of \[17, 37, 38\], for any \( \xi \in S^{N-1} = \{ \xi \in \mathbb{R}^N \mid \| \xi \| = 1 \} \), there is a so-called spreading speed interval \( [c_{\text{inf}}^*(\xi), c_{\text{sup}}^*(\xi)] \) of (1.4) and generalized propagating speed interval \( [c_{\text{inf}}(u_0, \xi), c_{\text{sup}}(u_0, \xi)] \) of a front \( u_0 \) or a front solution \( u(t, x; s, u_0) \) in the direction of \( \xi \). It is proved by Shen \[37, 38\] that for any \( \xi \in S^{N-1}, \) (1.4) has a spreading speed \( c^*(\xi) := c_{\text{inf}}^*(\xi) = c_{\text{sup}}^*(\xi) \) in the direction of \( \xi \) if \( f(t, u) \leq f_u(t, 0)u \) for \( t \in \mathbb{R} \) and \( u \geq 0 \). Shen \[40\] proved the existence, uniqueness and stability of generalized transition waves for time recurrent monostable equations (1.4), see Shen \[36\] for time heterogeneous bistable case. Nadin and Rossi \[28\] also proved the existence of generalized transition waves of reaction-diffusion equation
with general time dependent KPP nonlinearity by a different method. The set of admissible asymptotic past and future speeds of transition fronts for KPP equation depending on $t \in \mathbb{R}$ has been studied by Hamel and Rossi [15]. Shen and Shen [41] proved the existence, uniqueness and stability of generalized transition waves of nonlocal Fisher-KPP equation with time dependent, see Cao and Shen [9] for the spreading speeds and transition fronts of discrete fisher KPP equation in time heterogeneous. We further refer to Hamel [14], Lim and Zlatos [24], Nadin [29], Nadin and Rossi [27, 30], Nolen et al. [31], Rossi and Ryzhik [34], Shen [35, 38, 39], Tao et al. [44], Zlatos [53], and so on for the general transition wave of various reaction-diffusion equations.

For $K > 1$, system (1.1) has been widely used to describe the evolution of population density of $K$-species with the internal interaction and spreading speeds and traveling waves are two important dynamical issues of system (1.1). In 2017, Bao et al.[5] have introduced the notion of spreading speeds for general time dependents cooperative system (1.1) with different dispersal types. Roughly, $c^*(\xi)$ is called the spread speeds of (1.1) in the direction of $\xi \in \mathbb{S}^{N-1}$ if for any continuous function $u_0 : \mathbb{R}^N \to \mathbb{R}^K$ with

$$\lim_{x \cdot \xi \to -\infty} \inf u_0(\cdot) > 0 \quad \text{and} \quad u_0(\cdot) = 0 \quad \text{for} \quad x \cdot \xi \gg 1,$$

then

$$\limsup_{x \cdot \xi \leq c' t, t \to \infty} \| u(t + s, x; s, u_0) - u^*(t + s) \|_{X} = 0 \quad \text{for} \quad c' < c^*(\xi)$$

uniformly in $s \in \mathbb{R}$ and

$$\limsup_{x \cdot \xi \leq c' t, t \to \infty} \| u(t + s, x; s, u_0) \|_{X} = 0 \quad \text{for} \quad c' > c^*(\xi)$$

uniformly in $s \in \mathbb{R}$, where $u(t, x; s, u_0)$ is the solution of (1.1) with $u(s, x; s, u_0) = u_0$. Recently, Bao [2] also studied the spatial spreading speeds and linear determinacy for general time-dependent cooperative system with random dispersal, discrete dispersal and nonlocal dispersal, including (1.1). In this paper, we continue to explore the existence, uniqueness and stability of generalized transition waves for time-dependent cooperative system (1.1). We mainly use principal Lyapunov theory of linear cooperative system and the method of sub- and supersolution to prove the existence of generalized transition waves. Since system (1.1) mainly considers the interaction between different components in a higher dimensional system, many method used in the study of stability of traveling waves for scalar equation can not be used to system (1.1). Motivated by the stability results in [32, 46], we show that the transition waves attract all solution with initial function in its bounded neighborhood with respect to a weighted maximum norm exponentially in time.
The rest of the paper is organized as follows. In Section 2, we collect some fundamental properties of principal Lyapunov exponent and principal Floquent bundle theory for linear cooperative system and give our main results. In Section 3, by constructing some important super- and subsolutions of (1.1), we study the existence of generalized wave solution of (1.1). The stability and uniqueness of generalized wave solution of (1.1) is proved in Section 4.

2. Principal Lyapunov theory and main results. In this section, we first present some results on the principal Lyapunov exponent and principal Floquent bundle theory for linear cooperative system of ordinary differential equation. We then introduce the definition of generalized transition waves and state the main results of the current paper.

2.1. Principal Lyapunov theory for linear cooperative system. Consider the following ordinary differential system

$$\frac{d}{dt} u(t) = A(t)u(t),$$

where $A(t) = (a_{ij}(t))_{K \times K}$ for $t \in \mathbb{R}$ and $u(\cdot) \in \mathbb{R}^K$. Define the hull of $A(t)$ as $Y := H(A) = cl\{A \cdot \tau(\cdot) = A(\cdot + \tau) | \tau \in \mathbb{R}\}$, where the closure is taken under the compact open topology. For any $B \in Y$ and $t \in \mathbb{R}$, $\sigma_tB := B \cdot t := (b_{ij}(\cdot + t))_{K \times K} \in Y$. Then $(Y, \{\sigma_t\}_{t \in \mathbb{R}})$ is a compact flow. Recall that $(Y, \{\sigma_t\}_{t \in \mathbb{R}})$ is called a compact flow if $\sigma_t : Y \rightarrow Y (t \in \mathbb{R})$ satisfies: $[(t, z) \rightarrow \sigma_tz]$ is jointly continuous in $(t, z) \in \mathbb{R} \times Y$, $\sigma_0 = id$, and $\sigma_s \circ \sigma_t = \sigma_{s+t}$ for any $s, t \in \mathbb{R}$. We may write $z \cdot t$ or $(z, t)$ for $\sigma_tz$.

For any $B \in H(A)$, consider also

$$\frac{d}{dt} u(t) = B(t)u(t).$$

The classical theory for ordinary differential systems implies that, for each $u_0 \in \mathbb{R}^K$ and $s \in \mathbb{R}$, (2.2) has a unique solution $u(t; s, u_0, B)$ with $u(s; s, u_0, B) = u_0$. Put $U(t, B)u_0 = u(t; 0, u_0, B)$ for any $u_0 \in \mathbb{R}^K$. Let $(Y, \{\sigma_t\}_{t \in \mathbb{R}})$ is a compact flow and $B(Y)$ is the Borel $\sigma$–algebra of $Y$. Recall that a probability measure $\mu$ on $(Y, B(Y))$ is called an invariant measure for $(Y, \{\sigma_t\}_{t \in \mathbb{R}})$ if for any $E \in B(Y)$ and any $t \in \mathbb{R}$,

$$\mu(\sigma_t(E)) = \mu(E).$$

We say that $(Y, \{\sigma_t\}_{t \in \mathbb{R}})$ is said to be uniquely ergodic if it has a unique invariant measure (in such case, the unique invariant measure is necessarily ergodic). Then $A(t)$ is called uniquely ergodic if the compact flow $(H(A), \{\sigma_t\}_{t \in \mathbb{R}})$ is unique ergodic.

Assume that the square matrix $A(t)$ is quasi-positive and strongly irreducible. For convenience, we also remark that a square matrix $A(t) = (a_{ij}(t))_{K \times K}$ is quasi-positive if its off-diagonal entries are nonnegative for any $t \in \mathbb{R}$ and a square matrix $A(t) = (a_{ij}(t))_{K \times K}$ is called strongly irreducible if there is a $\delta > 0$ such that for any two nonempty subsets $S, S'$ of $\{1, 2, ..., K\}$ form a partition of $\{1, 2, ..., K\}$ and any $t \in \mathbb{R}$, there exist $i \in S$, $j \in S'$ satisfying that

$$|a_{ij}(t)| \geq \delta.$$

Then by [42, Definition 4.6 and Theorem 4.4], there are subspaces $X_1(B), X_2(B) \subset \mathbb{R}^K$ satisfying that $X_1(B) \oplus X_2(B) = \mathbb{R}^K$; $X_1(B), X_2(B)$ are continuous in $B \in \mathbb{R}$; $X_1(B) = \text{span}\{\phi(B)\}$ for some $\phi(B) \gg 0$ and $\|\phi(B)\| = 1$, $X_2(B) \cap (\mathbb{R}^K)^+ = \{0\}$; and $U(t, B)X_i(B) = X_i(\sigma_tB)$ for any $t \in \mathbb{R}$ and $B \in H(A)$. Moreover, there are
\[ \gamma > 0 \text{ and } M > 0 \text{ such that} \]
\[ \frac{\|U(t,B)\psi\|}{\|U(t,B)\phi(B)\|} \leq Me^{-\gamma t} \|\psi\| \]
for any \( \psi \in X_2(B) \) and \( B \in H(A) \). Moreover, if \( A(t) \) or \( (Y, \{\sigma_t\}_{t \in \mathbb{R}}) \) is unique ergodic, then the limit \( \lim_{t \to \infty} \frac{\ln \|U(t,B)\|}{t} \) exists and is independent of \( B \in Y \), also see \([18, 38]\).

Thus we give the definition of principal Lyapunov exponent of (2.1) as follows.

**Definition 2.1.** Assume that \( A(t) \) is quasi-positive, strongly irreducible, and unique ergodic. \( \lambda(A) \) is called the principal Lyapunov exponent of (2.1) or \( A(\cdot) \), where
\[
\lambda(A) = \lim_{t \to \infty} \frac{1}{t} \ln \|U(t,B)\| \]
for any \( B \in Y = H(A) \), and \( \{\text{span}(\phi(\sigma_t A))\}_{t \in \mathbb{R}} \) is called the principal Floquet bundle of (2.1) or \( A(\cdot) \) associated to \( \lambda(A) \).

Let
\[
\kappa(B) = \langle B(0)\phi(B), \phi(B) \rangle, \tag{2.3}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product of \( \mathbb{R}^K \). It is easy to see that \( \kappa(B) \)

for all \( B \in Y \), and the limit is independent of \( B \).

Let
\[
A(t) = \left( \frac{\partial F^t}{\partial u_{ij}}(t,0) \right)_{K \times K}. \tag{2.4}
\]

Note that \( F(t,0) = 0 \). Consider the following linearized system of (1.1) at \( u = 0 \)
\[
\frac{\partial u}{\partial t} = \Delta u(t,x) + A(t)u. \tag{2.5}
\]

We point out that, if \( u(t,x) = e^{-\mu x - \xi t} \eta(t) \) with the initial value \( u(0,x) = e^{-\mu x - \xi} \alpha \)
and \( \alpha \in \mathbb{R}^K \) is a solution of (2.5), then the vector-valued function \( \eta(t) \)
satisfies the following ordinary differential system
\[
\frac{d\eta(t)}{dt} = A^\mu(t)\eta(t), \tag{2.6}
\]
where \( A^\mu(t) = (a^\mu_{ij}(t))_{K \times K} \) with \( a^\mu_{ij}(t) = a_{ij}(t) \) for \( i \neq j \), \( i, j = 1, \ldots, K \), and \( a^\mu_{ii}(t) = \mu^2 + a_{ii}(t) \), for \( i = 1, \ldots, K \).

Let \( H(A^\mu) \) be the hull of \( A^\mu(t) \). By (H2), we know that \( A^\mu(t) \) is quasi-positive for any \( \mu \in \mathbb{R} \) and \( t \in \mathbb{R} \). If \( A(t) \) is strongly irreducible and unique ergodic for any \( t \in \mathbb{R} \), then \( A^\mu(t) \) is also unique ergodic and strongly irreducible for any \( t \in \mathbb{R} \) and \( \mu \in \mathbb{R} \). Then \( (H(A^\mu), \{\sigma_t\}_{t \in \mathbb{R}}) \) is unique ergodic for any \( \mu \in \mathbb{R} \). By Definition 2.1, we have the following lemma for (2.6).

**Lemma 2.1.** Assume that \( A(t) \) is unique ergodic and strongly irreducible for any \( t \in \mathbb{R} \). Then \( \lambda(A^\mu) \) is the principal Lyapunov exponent of (2.6) and \( \{\text{span}(\phi(\mu, \sigma_t A))\}_{t \in \mathbb{R}} \) be the principal Floquet bundle associate to \( \lambda(A^\mu) \) for any \( \mu \in \mathbb{R} \).
Let 
\[ \lambda(\mu, A) := \lambda(A^\mu) \quad \text{and} \quad \lambda(0) = \lambda(0, A). \] 
(2.7)

By (2.4),
\[ \lambda(\mu, A) = \lim_{t-s \to \infty} \frac{1}{t-s} \int_s^t \kappa(\sigma^\tau A^\mu) d\tau \]
Hence \( \lambda(\mu, A) \) is continuous and smooth in \( \mu \in \mathbb{R} \) and is also continuous in \( A(\cdot) \) with respect to uniform convergence topology. Moreover, we have the following lemma for \( \lambda(\mu, A) \) and \( \{\text{span}(\phi(\mu, \sigma_t A))\}_{t \in \mathbb{R}} \).

**Lemma 2.2.**

(i) For any given \( \xi \in S^{N-1} \) and \( \mu > 0 \),
\[ u(t, x) = e^{-\mu x \cdot \xi - \frac{1}{\mu} \int_0^t \kappa(\sigma_\tau A^\mu) d\tau} \phi(\mu, \sigma_t A) \]
is a solution of (2.5).

(ii) If \( \lambda(0) > 0 \), then there exists a \( \mu^*(A) > 0 \) such that
\[ \inf_{\mu > 0} \frac{\lambda(\mu, A)}{\mu} = \frac{\lambda(\mu^*(A), A)}{\mu^*(A)} \] 
(2.8)
and
\[ \frac{\lambda(\mu, A)}{\mu} \geq \frac{\lambda(\mu^*(A), A)}{\mu^*(A)} \quad \text{for} \ 0 < \mu < \mu^*(A). \] 
(2.9)

**Proof.** (i) directly follows from the definition of principal Lyapunov exponent \( \lambda(\mu, A) \) and the principal Floquet bundle \( \{\text{span}(\phi(\mu, \sigma_t A))\}_{t \in \mathbb{R}} \). (ii) follows from Lemma 3.4 in [5]. \( \square \)

### 2.2. Definition and main results

In this subsection, we extend the concept of generalized transition waves to generalized reaction-diffusion system (1.1) and state the main results on the existence, uniqueness and stability of generalized transition waves for (1.1).

As point in the Introduction, we have studied the spreading speeds and its properties of time dependent cooperative system including (1.1) in [5]. To state the results on spreading speeds and generalized transition waves for (1.1), we further introduce the following assumptions. Recall that \( A(\cdot) = \left( \frac{\partial F}{\partial u_j}(t, 0) \right)_{K \times K} \).

Assume that

(H3) \( A(t) \) is unique ergodic and strongly irreducible for any \( t \in \mathbb{R} \) and \( \lambda(\mu, A) > 0 \) for any \( \mu \in \mathbb{R} \).

Then following from Lemma 2.1, \( \lambda(\mu, A) \) is the principal Lyapunov exponent of (2.6) for any \( \mu \in \mathbb{R} \) and \( \lambda(\mu, A) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \kappa(\sigma_\tau A^\mu) d\tau \). Moreover, there is \( \{\text{span}(\phi(\mu, \sigma_t A))\}_{t \in \mathbb{R}} \) be the principal Floquet bundle associate to \( \lambda(\mu, A) \). Generally, denote \( \phi(t, \mu) := \phi(\mu, \sigma_t A) \).

We also assume that

(H4) \( F(t, \rho \phi(t, \mu)) \leq \rho A(t) \phi(t, \mu) \) for any \( \rho > 0, \mu \in \mathbb{R} \) and \( t \in \mathbb{R} \).

In fact, assumption (H4) plays an important role in the study of spreading speed and will also be used to establish the supersolution and subsolution of system (1.1). In the following, we first list the result on the spreading speeds of (1.1), which has been proved in [5].
**Theorem 2.1.** (See [5, Theorem 1.3]) Assume that (H1)-(H4) hold true. Then (1.1) has a spreading speed $c^*$ and

$$c^* := \inf_{\mu > 0} \frac{\lambda(\mu, A)}{\mu},$$

which is linearly determinant.

For any given $\mu > 0$, let

$$c(t, \mu) = \frac{k(\sigma_t A^\mu)}{\mu}.$$  

Note that

$$c^* = \inf_{\mu > 0} \frac{\lambda(\mu, A)}{\mu}$$

and

$$\lambda(\mu, A) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \kappa(\sigma_\tau A^\mu) d\tau.$$  

Then we have

$$c^* = \inf_{\mu > 0} \frac{1}{\mu} \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \kappa(\sigma_\tau A^\mu) d\tau.$$  

By Lemma 2.2 (ii), there is $\mu^* > 0$ such that

$$c^* = \frac{\lambda(\mu^*, A)}{\mu^*}.$$  

The objective of this paper is to study the existence, uniqueness and stability of generalized transition waves connecting $0$ and $u^*(t)$ for time dependent system (1.1). Next, we give the definition of generalized transition wave for (1.1).

**Definition 2.2.** (Generalized transition waves) An (almost planar) generalized transition wave in the direction $\xi \in S^{N-1}$ of (1.1) is a time-globally solution

$$u(t, x) = \Phi \left( t, x \cdot \xi - \int_0^t c(s) ds \right),$$

where $c(s) : \mathbb{R} \to \mathbb{R}$ is bounded and $\Phi : \mathbb{R} \times \mathbb{R} \to [0, u^*(t)]$ satisfying that

$$\lim_{z \to -\infty} \Phi(t, z) = u^*(t) \quad \text{and} \quad \lim_{z \to +\infty} \Phi(t, z) = 0$$

uniformly in $t \in \mathbb{R}$. Here $c$ is called the speed of the generalized transition waves $u(t, x)$ and $\Phi$ is the profile of $u(t, x)$.

The main results of current paper can be then stated as follows.

**Theorem 2.2.** Assume that (H1)-(H4) hold true.

(i) For all $\gamma > c^*$, let $0 < \mu < \mu^*$ and $c(t) = \frac{k(\sigma_t A^\mu)}{\mu}$ be such that $\gamma = \frac{\lambda(\mu, A)}{\mu}$. Then there exists a generalized transition wave

$$U(t, x) = \Phi \left( x \cdot \xi - \int_0^t c(\tau) d\tau, t \right)$$

in the direction $\xi \in S^{N-1}$ with $\Phi(t, z)$ being nonincreasing in $z$ and

$$\lim_{z \to -\infty} (\Phi(t, z) - u^*(t)) = 0 \quad \text{and} \quad \lim_{z \to +\infty} \frac{\Phi(t, z)}{e^{-\mu z} \phi_i(t, \mu)} = 1 \quad \forall i = 1, \ldots, K$$

uniformly in $t \in \mathbb{R}$.

(ii) For any $\gamma < c^*$, there is no such generalized transition wave $U(t, x)$ in the direction $\xi$ with a speed $c(t)$.
Recall that $\mathbf{u}^*(t)$ is the unique globally stable of

$$\frac{du}{dt} = \mathbf{F}(t, \mathbf{u}(t)).$$

(2.10)

Consider the linearized system of (2.10) at point $\mathbf{u}^*(t)$:

$$\frac{du}{dt} = A^*(t)\mathbf{u}(t),$$

(2.11)

where

$$A^*(t) = \left( \frac{\partial F^i}{\partial u_j}(t, \mathbf{u}^*) \right)_{K \times K}.$$  

Note that $\mathbf{F}(t, \mathbf{u})$ is cooperative and then $A^*(t)$ is quasi-positive. In order to get the stability of generalized transition wave $\mathbf{U}(t, x)$ established in Theorem 2.2, we further assume that

(H5) $A^*(t)$ is strongly irreducible and unique ergodic for any $t \in \mathbb{R}$ and $\lambda(A^*) < 0$.

(H6) $\mathbf{F}(t, \mathbf{u})$ is subhomogeneous on $\mathbf{u} \in [0, \mathbf{u}^*(t)]$ in the sense that $\mathbf{F}(t, \nu \mathbf{u}) \geq \nu \mathbf{F}(t, \mathbf{u})$ for all $t \in \mathbb{R}$, $\nu \in (0, 1)$ and $\mathbf{u} \in [0, \mathbf{u}^*(t)]$.

Under the assumption (H5), ordinary differential system (2.11) has the principal Lyapunov exponent $\lambda(A^*)$ and $\{\text{span}(\varphi_\sigma(A^*))\}_{\sigma \in \mathbb{R}}$ is the principal Floquent bundle of (2.11) associated to $\lambda(A^*)$, see Definition 2.1 and Lemma 2.1. By (2.4), we also have

$$\lambda(A^*) = \lim_{t \to \infty} \frac{1}{t-s} \int_s^t \kappa(\sigma \tau, A^*) d\tau.$$

Assumption (H6) usually plays an important role in the study of coupled cooperative system. In (H5), $\lambda(A^*) < 0$ means that $\mathbf{u}^*(t)$ is locally stable. Note that $\mathbf{U}(t, x) = (U_1(t, x), \ldots, U_K(t, x))$. Choose $\eta_0 \in \mathbb{R}$ such that $|U_i(t, x) - u_i^*(t)| < \delta$ for $x \cdot \xi - \int_0^t c(\tau) d\tau < \eta_0$ and $i = 1, \ldots, K$. For a sufficiently small $\varepsilon > 0$, let $\mu^\varepsilon := \mu + \varepsilon$. We then define a weight function $\omega^\varepsilon(\eta)$ as

$$\omega^\varepsilon(\eta) := \begin{cases} e^{\mu^\varepsilon(\eta - \eta_0)}, & \eta \geq \eta_0, \\ 1, & \eta < \eta_0. \end{cases}$$

Then we have the following results on the stability and uniqueness of generalized transition waves for system (1.1).

**Theorem 2.3.** Assume that (H1)-(H5) hold true. Let $\mathbf{U}(t, x) = \Phi \left( x \cdot \xi - \int_0^t c(\tau) d\tau, t \right)$ be a generalized transition wave of (1.1) in the direction of $\xi$ connecting $\mathbf{u}^*(t)$ and $0$ with speed $c(t)$ for $\gamma > c^*$. Then there exist a real number $\kappa > 0$ such that for any given initial value $\mathbf{u}_0(x)$ with $0 \leq \mathbf{u}_0(x) \leq \mathbf{u}^*(t)$ and

$$[\mathbf{u}_0(x) - \mathbf{U}(0, x)] \omega^\varepsilon(x \cdot \xi) \in L^\infty(\mathbb{R}^N, \mathbb{R}^K),$$

there is

$$\sup_{x \in \mathbb{R}^N} |u_i(t, x; \mathbf{u}_0) - U_i(t, x)| \leq C e^{-\kappa t}, \ \forall t > 0$$

for some constant $C > 0$ and $i = 1, \ldots, K$.

**Theorem 2.4.** Assume that (H1)-(H5) hold true. Let $\mathbf{U}(t, x) = \Phi \left( x \cdot \xi - \int_0^t c(\tau) d\tau \right)$ and $\bar{\mathbf{U}}(t, x) = \Phi \left( x \cdot \xi - \int_0^t c(\tau) d\tau \right)$ are two generalized transition waves of (1.1)
connecting \( \mathbf{u}^*(t) \) and \( \mathbf{0} \) in the direction of \( \xi \) with speed \( c(t) \) for \( \gamma > c^* \), and

\[
\lim_{z \to +\infty} \frac{\Phi_i(t, z)}{e^{-\mu z} \Phi_i(t, \mu)} = 1 \quad \text{and} \quad \lim_{z \to -\infty} \frac{\Phi_i(t, z)}{e^{-\mu z} \Phi_i(t, \mu)} = 1
\]

uniformly in \( t \in \mathbb{R} \) for all \( i = 1, \ldots, K \). Then \( U_i(t, x) = \bar{U}_i(t, x) \) for \( (t, x) \in \mathbb{R} \times \mathbb{R}^N \) and \( i = 1, \ldots, K \).

3. Existence of generalized transition waves. In this section, we investigate the existence of generalized transition waves for reaction-diffusion system (1.1) and prove Theorem 1.1 by using the suitable sub-and supersolutions and comparison principle.

In the following, we will construct some important super- and subsolutions for the following system

\[
\frac{\partial \mathbf{u}}{\partial t} = \partial_{zz} \mathbf{u}(t, z) + c(t) \partial_z \mathbf{u}(t, z) + \mathbf{F}(t, \mathbf{u}(t, z)). \tag{3.1}
\]

We can define the supersolutions (subsolution) of (3.1) if the equalities are replaced by inequalities \( \leq (\geq) \). Recall that \( \lambda(\mu, A) = \lim_{T \to -\infty} \frac{1}{T} \int_t^{t+T} \kappa(\sigma A^\mu) \mathbf{d}\tau \).

By Lemma 2.2 (i),

\[
\exp \left( -\mu \left( x \cdot \xi - \int_0^t \kappa(\sigma A^\mu) \mathbf{d}\tau \right) \right) \phi(t, \mu)
\]

is a solution of \( \frac{\partial \mu}{\partial t} = \Delta \mathbf{u}(t, x) + A(t) \mathbf{u} \), that is,

\[
\frac{d\phi}{dt} = \mu^2 \phi - \kappa(\sigma A^\mu) \phi + A(t) \phi. \tag{3.2}
\]

Recall that \( c(t, \mu) := \frac{\kappa(\sigma A^\mu)}{\mu} \) for any given \( \mu > 0 \). Then for any \( \gamma > c^* \), there must be a \( \mu \in (0, \mu^*) \) such that \( \gamma = \frac{\lambda(\mu, A)}{\mu} \). By the definition of \( c(t, \mu) \) and \( \lambda(\mu, A) \), there is

\[
\gamma = \frac{\lambda(\mu, A)}{\mu} = \lim_{T \to -\infty} \frac{1}{T} \int_t^{t+T} c(\tau, \mu) \mathbf{d}\tau.
\]

**Proposition 3.1.** For any \( \gamma > c^* \), let \( 0 < \mu < \mu^* \) be such that \( \frac{\lambda(\mu, A)}{\mu} = \gamma \) and \( c(t) = c(t, \mu) \). There exists a continuous supersolution \( \mathbf{u}^*(t, z) = (u^i_1(t, z), \ldots, u^i_K(t, z)) \) for system (3.1) with \( 0 \leq u^i_1(t, z) \leq u^i_2(t, z) \leq \ldots \leq u^i_K(t, z) \) is nonincreasing in \( z \in \mathbb{R} \) and \( \lim_{z \to \pm\infty} \frac{u^i_j(t, z)}{e^{-\mu z} \Phi_i(t, \mu)} = 1 \) uniformly in \( t \in \mathbb{R} \) for any \( j = 1, \ldots, K \).

**Proof.** Let \( \mathbf{v}(t, z) = e^{-\mu z} \phi(t, \mu) \). Since \( \kappa(\sigma A^\mu) = c(t) \mu \), then we know that \( \mathbf{v}(t, z) \) is a solution of the system

\[
\partial_t \mathbf{v}(t, z) - \partial_{zz} \mathbf{v}(t, z) - c(t) \partial_z \mathbf{v}(t, z) - A(t) \mathbf{v}(t, z) = 0.
\]

Note that \( \mathbf{F}(t, \rho \phi(t, \mu)) \leq \rho A(t) \phi(t, \mu) \) for any \( \rho > 0 \) and \( t \in \mathbb{R} \). Then for any \( i = 1, \ldots, K \), we have

\[
\mathcal{N}_i[\mathbf{v}(t, z)] = \frac{\partial \Sigma_i}{\partial t}(t, z) \cdot \mathbf{v}_i(t, z) - \partial_{zz} \mathbf{v}_i(t, z) - c(t) \partial_z \mathbf{v}_i(t, z) - F^i(t, \mathbf{v}(t, z))
\]

\[
= e^{-\mu z} \frac{d\phi_i}{dt}(t, \mu) - \mu^2 e^{-\mu z} \phi_i(t, \mu) + c(t) \mu e^{-\mu z} \phi_i(t, \mu) - F^i(t, e^{-\mu z} \phi(t, \mu))
\]

\[
= \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t, 0) e^{-\mu z} \phi_j(t, \mu) - F^i(t, e^{-\mu z} \phi(t, \mu)) \geq 0.
\]
Note that $u^*(t)$ is a solution of (3.1). Define $u^+(t, z) = (u^+_1(t, z), ..., u^+_K(t, z))$ by

$$u^+_i(t, z) = \min\{\bar{u}_i(t, z), u^+_i(t)\} \quad \text{for } i = 1, ..., K.$$  

Then $u^+(t, z)$ is a continuous supersolution of (3.1). Note that $e^{-\mu z} \phi_i(t, \mu) \to 0$ as $z \to +\infty$, then there is a constant $L > 0$ such that $e^{-\mu z} \phi_i(t, \mu) < u^+_i(t)$ for $z > L$ and $i = 1, ..., K$. Then for $z > L$, we have $u^+_i(t, z) = e^{-\mu z} \phi_i(t, \mu)$ and then $u^+_i(t, z)$ is nonincreasing in $z \in \mathbb{R}$ for any $i = 1, ..., K$. It is clear to see that $u^+_i(t, +\infty) = 0$, $u^+_i(t, -\infty) = u^*_i(t)$, $0 \leq u^+_i(t, z) \leq u^*_i(t)$ for any $i = 1, ..., K$. Moreover, we have

$$\lim_{z \to +\infty} e^{-\mu z} \phi_i(t, \mu) = 1 \quad \text{uniformly in } t \in \mathbb{R} \quad \text{for any } i = 1, ..., K.$$

By (H1), for any $i = 1, ..., K$, there are positive constants $\varpi$ and $\gamma$ such that

$$F^i(t, u) \geq \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t, 0) u_j - \varpi \left( \sum_{j=1}^K |u_j| \right)^{1+\gamma} \quad \text{for all } t \in \mathbb{R}. \quad (3.3)$$

We then present a technical lemma as follows, which has been proved in [28, Lemma 3.2].

**Lemma 3.1.** Let $B \in L^\infty(\mathbb{R})$. Define $B = \lim_{T \to \infty} \inf_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} B(s) ds$. Then

$$\bar{B} = \sup_{A \in W^{1,\infty}(\mathbb{R})} \text{essinf}_{t \in \mathbb{R}} (A(t) + B(t)).$$

Let $\langle B \rangle := \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} B(s) ds$ be the mean value of $B$. If $B$ admits a mean value $\langle B \rangle$, then $\langle B \rangle = \bar{B}$.

Fix $\xi \in S^{N-1}$ and $\gamma > c^*$. Take $\bar{\mu}$ with $0 < \mu < \bar{\mu} < \min\{\mu^*, 2\mu\}$. Then we have

$$\gamma = \frac{\lambda(\mu, A)}{\mu} \quad \text{and} \quad \frac{\lambda(\mu, A)}{\mu} > \frac{\lambda(\bar{\mu}, A)}{\bar{\mu}}.$$  

Note that

$$\lambda(\bar{\mu}, A) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \kappa(\sigma_A \tilde{A}) d\tau.$$  

Then we have

$$\lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \left( c(\tau) - \frac{\kappa(\sigma_A \tilde{A})}{\mu} \right) d\tau > 0.$$  

By Lemma 3.1, there is $M(t) \in W^{1,\infty}(\mathbb{R})$ such that

$$\text{essinf}_{t \in \mathbb{R}} \left( c(t) - \frac{\kappa(\sigma_A \tilde{A})}{\mu} + M(t) \right) > 0.$$  

Let $\psi(t, \bar{\mu}) = (\psi_1(t, \bar{\mu}), ..., \psi_K(t, \bar{\mu}))$ is the principal Floquent bundle associate to $\lambda(\bar{\mu}, A)$. Thus we have

$$\frac{d\psi_i}{dt} = \mu^2 \psi_i - \kappa(\sigma_A \tilde{A}) \psi_i + \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t, 0) \psi_j, \quad \forall 1 \leq i \leq K. \quad (3.4)$$

For a given positive constant $d$ and function $M(t)$, let $u(t, z) = (u_1(t, z), ..., u_K(t, z))$ is defined as

$$u_i(t, z) = e^{-\mu z} \phi_i(t, \mu) - d e^{M(t) - \bar{\mu} z} \psi_i(t, \bar{\mu}) \quad \forall i = 1, ..., K.$$
Let 
\[ d_0 = \max_{1 \leq i \leq K} \left\{ 1/ \min_{t \in \mathbb{R}} \psi_i(t, \mu) \right\}. \]

**Proposition 3.2.** Let \( d \geq d_0 \) be a given constant. Then \( \mathbf{u}(t, z) = (\mathbf{u}_1(t, z), \ldots, \mathbf{u}_K(t, z)) \) satisfies
\[ \frac{\partial \mathbf{u}_i}{\partial t} \leq \partial_z \mathbf{u}_i + c(t) \partial_{zz} \mathbf{u}_i + F^i(t, \mathbf{u}(t, z)), \quad \forall 1 \leq i \leq K, \quad (3.5) \]
for any \( z \geq z_0 \), where
\[ z_0 = \max \left\{ \frac{1}{\mu} \max_{1 \leq i \leq K} \ln \left\{ \max_{t \in \mathbb{R}} \frac{\phi_i(t, \mu)}{u^*_i(t)} \right\}, \right. \]
\[ \frac{1}{\mu - \mu} \left( \max_{t \in \mathbb{R}} M(t) + \max_{1 \leq i \leq K} \ln \left\{ \max_{t \in \mathbb{R}} \frac{d\psi_i(t, \mu)}{\phi_i(t, \mu)} \right\} \right) \}

**Proof.** Note that \( \phi(\mu, \sigma, A) \) satisfies system (3.2) for any \( \mu \in (0, \mu^*) \). Then, for \( c(t) = c(t, \mu) = \frac{\kappa(\sigma, A^\mu)}{\mu} \), we have
\[ \frac{d\phi_i}{dt} = \mu^2 \phi_i - c(t) \mu \phi_i + \sum_{j=1}^K \frac{\partial F_i}{\partial u_j}(t, 0) \phi_j(t, \mu), \quad \forall i = 1, \ldots, K. \]

By definition of \( \kappa(\sigma, A^\mu) \) and (3.4), we have
\[ N(t) := \frac{\partial \psi_i}{\partial t} - \mu^2 \psi_i(t, \mu) + c(t) \mu \psi_i(t, \mu) + \sum_{j=1}^K \frac{\partial F_i}{\partial u_j}(t, 0) \psi_j(t, \mu) \]
\[ = c(t) \mu \psi_i(t, \mu) - \kappa(\sigma, A^\bar{\mu}) \psi_i(t, \mu) \]
\[ = \mu \psi_i(t, \mu) \left[ c(t) - \frac{\kappa(\sigma, A^\bar{\mu})}{\mu} \right], \quad \forall i = 1, \ldots, K. \]

It is easy to see that \( \mathbf{u}_i(t, x; d) \geq 0 \) for
\[ z \geq \frac{1}{\mu - \mu} \left( \max_{t \in \mathbb{R}} M(t) + \max_{1 \leq i \leq K} \ln \left\{ \max_{t \in \mathbb{R}} \frac{d\psi_i(t, \mu)}{\phi_i(t, \mu)} \right\} \right) \]
and \( \mathbf{u}_i(t, x; d) \leq u^*_i(t) \) for \( z \geq \frac{1}{\mu} \ln \left\{ \max_{t \in \mathbb{R}} \frac{d\psi_i(t, \mu)}{\phi_i(t, \mu)} \right\} \). By adding a large constant \( \alpha \) to \( M(t) \), we have that
\[ \frac{1}{\mu - \mu} \left( \max_{t \in \mathbb{R}} M(t) + \max_{1 \leq i \leq K} \ln \left\{ \max_{t \in \mathbb{R}} \frac{d\psi_i(t, \mu)}{\phi_i(t, \mu)} \right\} \right) > 0 \]
and
\[ M'(t) + N(t) \geq \omega \left( \sum_{j=1}^K |u_j| \right)^{1+\gamma} e^{\mu z - M(t)} \text{ for any } t \in \mathbb{R}. \quad (3.6) \]

For any \( i = 1, \ldots, K \), by (3.3)-(3.6), we have
\[ N_i[\mathbf{u}] = \frac{\partial \mathbf{u}_i}{\partial t} - \partial_{zz} \mathbf{u}_i - c(t) \partial_{zz} \mathbf{u}_i - F^i(t, \mathbf{u}) \]
\[ = e^{-\mu z} \left[ \frac{\partial \phi_i}{\partial t} - \mu^2 \phi_i(t, \mu) + c(t) \mu \phi_i(t, \mu) \right] - F^i(t, \mathbf{u}) \]
Thus (3.5) holds true.

**Proposition 3.3.** For any $\gamma > c^*$, let $0 < \mu < \mu^*$ be such that $\gamma = \frac{\lambda(\mu, A)}{\mu}$ and $c(t) = c(t, \mu)$. There is a continuous subsolution $u^-(t, z) = (u^1_1(t, z), ..., u^K(t, z))$ of system (3.1) satisfying for any $i = 1, ..., K$, $0 < u^i_1(t, z) < u^*_i(t, z)$, $\inf_{z \leq z_0, t \in \mathbb{R}} u^i_1(t, z) > 0$ for any $z_0 \in \mathbb{R}$ and $\lim_{t \to +\infty} \frac{u^i(t, \mu)}{e^\nu(t, \mu)} = 1$ uniformly in $t \in \mathbb{R}$.

**Proof.** Let $\lambda_0 = \lambda(0, A)$ and $\phi(t)$ be the principal Floquet bundle of $\frac{\partial u}{\partial t} = A(t)u$ corresponding to $\lambda_0$. Let $\rho = (\rho_1, ..., \rho_K)$ be a constant vector with $\rho_i > 0$ for any $i = 1, ..., K$. Note that $u(t, x; \rho) = (\rho_1 \phi_1(t), ..., \rho_K \phi_K(t))$ satisfies $\frac{\partial u}{\partial t} \leq \partial_{zz} u' + c(t) \partial_z u' + F(t, u')$ for some positive constant $\rho_i$ (i = 1, ..., K) small. In fact, for any $i = 1, ..., K$, we have

$$
\mathcal{N}_i[u'] = \frac{\partial u'_i}{\partial t} - \partial_{zz} u'_i - c(t) \partial_z u'_i + F^i(t, u')
$$

$$
= \rho_i \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t, 0) \phi_j(t) - F^i(t, \rho \phi) - \kappa(\sigma_t A^0) \rho_i \phi_i(t)
$$

$$
\leq \omega \left( \sum_{j=1}^K (\rho_j \phi_j) \right)^{1+\gamma} - \kappa(\sigma_t A^0) \rho_i \phi_i
$$

$$
\leq \rho_i \left[ -\kappa(\sigma_t A^0) \phi_i(t) + \omega \max_{1 \leq i \leq K} \rho_i \gamma C_\gamma \sum_{j=1}^K (\phi_j)^{1+\gamma} \right] \leq 0
$$
Thus, for any $i = 1, ..., K$, which implies that $u^i(t; z; \rho)$ is a subsolution of (3.1) for any $t \in \mathbb{R}$ and $z \in \mathbb{R}$. By Proposition 3.2, we know that $u_i(t; z; d) > 0$ for $z \geq z_0$ and $i = 1, ..., K$. Then there is $z_i \geq z_0$ and $\rho_i$ such that $u_i(t; z; d) = \rho_i \phi_i(t)$ for some $z = z_i$ and $i = 1, ..., K$. Define $u^-(t; z; d; \rho) = (u_1^-(t; z; d; \rho), ..., u_K^-(t; z; d; \rho))$ by $u_i^-(t; z; d; \rho) = \begin{cases} u_i(t; z; d) & \text{for } z \geq z_i, \\ \rho_i \phi_i(t) & \text{for } z < z_i \end{cases}$ for all $i = 1, ..., K$. (3.7)

Then $u_i^-(t; z; d; \rho)$ is a continuous function in $t \in \mathbb{R}$ and $z \in \mathbb{R}$ for any $i = 1, ..., K$ and $u^-(t; z; d; \rho)$ is a subsolution of (3.1). It is clear that $u_i^-(t; z; d) > 0$ for any $z \in \mathbb{R}$ and $\lim_{z \to +\infty} u_i^-(t; z) = 1$ uniformly in $t \in \mathbb{R}$.

Let $X := \{u = (u_1, ..., u_K) \in C(\mathbb{R}^N, \mathbb{R}^K) | u_i(x) \text{ is uniformly continuous and } \sup_{x \in \mathbb{R}^N} |u_i(x)| < \infty\}$ with the supremum norm $\|u\|_X$. For a given $\xi \in S^{N-1}$, let $X^+(\xi) = \{u_0 \in X | 0 \leq u_0 \ll u_{\text{inf}}, \liminf_{x \xi \to -\infty} u_0(x) \gg 0, u_0(x) = 0, \forall x : \xi \gg 1\}$. In order to show the non-existence of generalized transition waves for speed $c(t)$ with $\gamma < c^*$, we introduce the following Lemma which has been proved in [5, Lemma 2.7].

**Proposition 3.4.** For given $c \in \mathbb{R}$ and $u_0(\cdot) \in X^+(\xi)$, if there are constants $\delta_0 > 0$ such that $\min_{i=1, ..., K} \left\{ \liminf_{x \xi \to -\infty} u_i(t+s,x; s, u_0) \right\} \geq \delta_0 \text{ uniformly in } s \in \mathbb{R}$. Then for any $c' < c$, $\limsup_{x \xi \to -\infty} \|u(t+s,x; s, u_0) - u^*(t+s)\|_X = 0 \text{ uniformly in } s \in \mathbb{R}$.

Now, we are in the position to prove the existence and nonexistence of generalized transition waves for system (1.1).

**Proof of Theorem 2.2.** (i) For all $\gamma > c^*$, let $0 < \mu < \mu^*$ and $c(t) = \frac{k(\gamma \mu)}{\mu}$ be such that $\gamma = \frac{\lambda(\mu, A)}{\mu}$. Let $u^+(t,z)$ and $u^-(t,z)$ be as in Proposition 3.1 and 3.3, respectively. For any $n \in \mathbb{N}$, let $\Phi_n(t,z)$ is the solution of the following problem $\begin{cases} \partial_t \Phi - \partial_z \Phi = c(t) \phi, & z \in \mathbb{R}, t > -n, \\ \Phi(-n,z) = u^*(z), & z \in \mathbb{R}. \end{cases}$ By the comparison principle of parabolic system, we have that $u^-(t,z) \leq \Phi_n(t,z) \leq u^*(t,z)$. Since $u^*(t,z)$ is nonincreasing in $z \in \mathbb{R}$, then $\Phi_n(t,z)$ is also nonincreasing in $z \in \mathbb{R}$. Due to the parabolic estimates and the embedding theorem, there is a
subsequence of \( \{ \Phi_n \}_{n \in \mathbb{N}} \) converging weakly to \( \Phi(t, z) \) in \( W^{2,1}_p(K) \) and strongly in \( L^\infty(K) \) for any compact \( K \subset \mathbb{R} \times \mathbb{R} \) and \( p < \infty \), which is a solution of the following system

\[
\partial_t \Phi - \partial_z \Phi_{z z} - c(t) \partial_z \Phi = F(t, \Phi), \quad z \in \mathbb{R}, \ t \in \mathbb{R}.
\]

Then the function \( \Phi(t, z) \) is nonincreasing in \( z \) and satisfies \( u^-(t, z) \leq \Phi(t, z) \leq u^+(t, z) \) for any \( z \in \mathbb{R} \) and \( t \in \mathbb{R} \). For every \( z_0 > 0 \), applying the parabolic strong maximum principle to \( \Phi(z - z_0, t) - \Phi(z, t) \), we find that \( \Phi(z, t) \) is decreasing in \( z \).

By the definition of the supersolution \( u^+(t, z) \) and subsolution \( u^-(t, z) \), we have

\[
\lim_{z \to +\infty} \frac{\Phi_i(t, z)}{e^{-\mu z} \phi_i(t, \mu)} = 1 \quad \text{uniformly in } t \in \mathbb{R} \text{ for any } 1 \leq i \leq K.
\]

As shown in Proposition 3.3, \( \inf_{z \leq 0, t \in \mathbb{R}} u_i(t, z) > 0 \) for \( i = 1, \ldots, K \). Then we must have \( \inf_{z \leq 0, t \in \mathbb{R}} \Phi_i(t, z) > 0 \) for any \( i = 1, \ldots, K \). By (H1), \( u^*(t) \) is globally stable for (1.1) and then

\[
\lim_{z \to -\infty} (\Phi(t, z) - u^*(t)) = 0 \quad \text{uniformly in } t \in \mathbb{R}.
\]

(ii) Assume that \( U(t, x) = \Phi(t, x \cdot \xi - \int_0^t c(\tau) d\tau) \) is a generalized transition wave solution of (1.1) in the direction of \( \xi \in S^{N-1} \) with speed \( c(t) \) for \( \gamma < c^* \). Then for any \( t \in \mathbb{R}, \inf_{z \leq 0, t \in \mathbb{R}} \Phi_i(t, z) > 0 \) for \( i = 1, \ldots, K \).

Fix \( \gamma, \gamma' \) and \( \gamma'' \) such that \( \gamma < \gamma' < \gamma'' < c^* \). For any \( u_0(\cdot) \in X^+(\xi) \), by Proposition 3.4, we know that

\[
\lim_{s \to -\infty} \| u(t + s, x; s, u_0) - u^*(t + s) \|_X = 0 \quad \text{uniformly in } s \in \mathbb{R}, \quad (3.8)
\]

where \( u(t + s, x; s, u_0) \) is the solution of (1.1) with the initial value \( u(s, \cdot; s, u_0) = u_0(\cdot) \). Let

\[
U^s(x) = \Phi \left( s, x \cdot \xi + \int_0^s c(\tau) d\tau \right), \quad \forall s \in \mathbb{R}.
\]

By (i), we can choose \( u_0(\cdot) \in X^+(\xi) \) such that

\[
u_0(x) \leq U^s(x) \quad \text{for any } x \in \mathbb{R}^N, \ s \in \mathbb{R}.
\]

Hence,

\[
u(t + s, x; s, u_0) \leq u(t + s, x; s, U^s) \quad \text{for } x \in \mathbb{R}^N, t \geq 0.
\]

This together with (3.8) implies that

\[
\lim_{x, t \to -\infty} \sup_{s \leq \gamma'', t \to -\infty} \| u(t + s, x; s, U^s) - u^*(t + s) \| = 0.
\]

By (i), we have \( u(t + s, x; s, U^s) = \Phi \left( t + s, x \cdot \xi - \int_0^{t + s} c(\tau) d\tau + \int_0^s c(\tau) d\tau \right) \). It then must be

\[
\lim_{x, t \to -\infty} \| u(t + s, x; s, U^s) \|_X = 0
\]

uniformly in \( s \in \mathbb{R} \). This is a contradiction. Hence, there is no such generalized transition wave \( U(t, x) \) of (1.1) connecting \( 0 \) and \( u^*(t) \) in the direction \( \xi \) with a speed \( c(t) \) for \( \gamma < c^* \). This completes the proof. \( \square \)
4. Stability of generalized traveling wave solutions. In this section, we investigate the uniqueness and stability of traveling wave solution of (1.1) with respect to the perturbation along the direction of the wave propagation. Throughout this section, we fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$.

Recall that $u^*(t)$ is the unique globally stable of $\frac{du}{dt} = F(t, u(t))$. Let

$$A^*(t) = \left( \frac{\partial F_i}{\partial u_j}(t, u^*) \right)_{K \times K}.$$ 

Under the assumption (H5), (2.11) has the principal Lyapunov exponent $\lambda(A^*)$ with $\lambda(A^*) < 0$ and $\{\text{span}(\varphi(\sigma_i A^*))\}_{i \in \mathbb{R}}$ is the principal Floquet bundle associated to $\lambda(A^*)$. Denote $\varphi(t) := \varphi(\sigma_i A^*)$. Note that

$$\lambda(A^*) = \lim_{t-s \to \infty} \frac{1}{t-s} \int_s^t \kappa(\sigma_t A^*) d\tau.$$ 

Thus, $\kappa(\sigma_i A^*)$ and $\varphi(\sigma_i A^*)$ satisfy

$$\varphi_t = A^*(t) \varphi(t) - \kappa(\sigma_i A^*) \varphi(t). \quad (4.1)$$

Note that $\lambda(A^*) < 0$. Choose $0 < \varepsilon < -\lambda(A^*)$ and $\varepsilon_1 > 0$ small enough such that

$$\varepsilon_1 \sum_{i=1}^K \varphi_i(t) < \varepsilon \min_{1 \leq i \leq K} \varphi_i(t), \quad \forall t \in \mathbb{R}, \ i = 1, ..., K. \quad (4.2)$$

Choose $\delta > 0$ small such that for any function $v^\pm = (v^+_1, ..., v^+_K)$ satisfying $|v^+_i(t,x) - u^*_i(t)| < \delta$ and $v^-_i(t,x) \leq v^+_i(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $i = 1, ..., K$, we have

$$|F^i(t,v^+) - F^i(t,v^-)| < \sum_{j=1}^K \left( \frac{\partial F^i}{\partial u_j}(t, u^*(t)) + \varepsilon_1 \right) (v^+_j - v^-_j) \quad (4.3)$$

for all $t \in \mathbb{R}$ and $i = 1, ..., K$. For any sufficiently small positive constant $\varepsilon$, let $\mu^\varepsilon = \mu + \varepsilon$. Recall that the weight function is defined as

$$\omega^\varepsilon(\eta) := \begin{cases} e^{\mu^\varepsilon(\eta-\eta_0)}, & \eta \geq \eta_0, \\
1, & \eta < \eta_0, \end{cases}$$

where $\eta_0 \in \mathbb{R}$ is chosen such that $|U_i(t,x) - u^*_i(t)| < \delta$ for $x \cdot \xi - \int_0^t c(\tau) d\tau < \eta_0$ and $i = 1, ..., K$.

In the following, we prove the stability of generalized transition wave of (1.1) and prove Theorem 2.3.

Proof of Theorem 2.3. Let $u(t,x; u_0) = (u_1(t,x; u_0), ..., u_K(t,x; u_0))$ is the solution of (1.1) with the initial value $u_0(\cdot) \in X$ with $0 \leq u_0(x) \leq u_{\text{inf}}(x) := \inf_{t \in \mathbb{R}} u^*(t)$ for any $x \in \mathbb{R}^N$. For any $\gamma > c^*$, let $U(t,x) = \Phi \left( x \cdot \xi - \int_0^t c(\tau) d\tau \right)$ be the generalized transition wave of (1.1) in the direction of $\xi$ connecting $u^*(t)$ and $0$ with speed $c(t)$. Thus $U(t,x)$ satisfies (1.1), that is,

$$\frac{\partial U}{\partial t} (t,x) = \Delta U(t,x) + F(t, U(t,x)). \quad (4.4)$$

Define

$$U^+(0,x) := \max \{ u_0(x), U(0,x) \} \quad \text{and} \quad U^-(0,x) := \min \{ u_0(x), U(0,x) \}.$$
Let $U^\pm(t, x)$ be the solution of (1.1) with initial values $U^\pm(0, x)$. Then

$$\frac{\partial U^\pm}{\partial t}(t, x) = \Delta U^\pm(t, x) + F(t, U^\pm(t, x)).$$

By the comparison principle for parabolic system, we obtain

$$0 \leq U^-(t, x) \leq u(t, x; u_0), U(t, x) \leq U^+(t, x) \leq u^*(t)$$

for any $t > 0$ and $x \in \mathbb{R}^N$, which implies that

$$\|u(t, x; u_0) - U(t, x)\| \leq \max\{\|U^+(t, x) - U(t, x)\|, \|U^-(t, x) - U(t, x)\|\}$$

for $t > 0$ and $x \in \mathbb{R}^N$. In the following, we aim to show that $U^\pm(t, x)$ converges to $U(t, x)$ exponentially in time. Thus the desired stability result follows from (4.7) directly.

Define

$$V(t, x) := U^+(t, x) - U(t, x).$$

Then the function $V(t, x)$ satisfies the following initial value condition:

$$0 \leq V(0, x) := U^+(0, x) - U(0, x) \leq \|u_0(x) - U(0, x)\|$$

for any $x \in \mathbb{R}^N$. We then obtain that $V(0, x)\omega^\varepsilon(x \cdot \xi)$ is uniformly bounded on $\mathbb{R}^N$ with respect to the weight function $\omega^\varepsilon(\cdot)$. To show the convergence of $U^+(t, x)$ to $U(t, x)$, we consider the following two cases: $x \cdot \xi - \int_0^t c(\tau)d\tau \geq \eta_0$ and $x \cdot \xi - \int_0^t c(\tau)d\tau < \eta_0$, respectively.

**Case (I).** $x \cdot \xi - \int_0^t c(\tau)d\tau \geq \eta_0$.

By (4.4), (4.5) and (4.8), we have

$$\frac{\partial V_i}{\partial t}(t, x) = \Delta V_i(t, x) + F^i(t, U(t, x) + V(t, x)) - F^i(t, U(t, x))$$

for any $i = 1, ..., K$. By the subhomogeneous of $F^i(t, u)$ in $u = (u_1, ..., u_K)$ (see (H6)), we obtain

$$F^i(t, U_1 + V_1, ..., U_K + V_K) - F^i(t, U_1, ..., U_K) \leq \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t, 0)V_j(t, x)$$

for any $i = 1, ..., K$. Then from (4.9), we have

$$\frac{\partial V_i}{\partial t}(t, x) \leq \Delta V_i(t, x) + \sum_{j=1}^K \frac{\partial F^i}{\partial u_j}(t, 0)V_j(t, x), \quad \forall i = 1, ..., K.$$  

Next, we define

$$\overline{V}_i(t, x) := C_1\phi^\varepsilon_i(t, \mu^\varepsilon)\exp(-\mu^\varepsilon(x \cdot \xi - \eta_0) + \kappa(\sigma, A^\varepsilon)t), \quad \forall i = 1, ..., K,$$

where $\mu^\varepsilon = \mu + \varepsilon$, $\phi^\varepsilon(t, \mu^\varepsilon) = (\phi^\varepsilon_1, ..., \phi^\varepsilon_K)$ is the principal Floquent bundle of (2.6) associated with the principal Lyapunov exponent $\lambda(\mu^\varepsilon, A)$ and

$$\lambda(\mu^\varepsilon, A) = \lim_{t-s \to \infty} \frac{1}{t-s} \int_s^t \kappa(\sigma_t A^\varepsilon)d\tau,$$

Since that $V(0, x)\omega^\varepsilon(x \cdot \xi)$ is uniformly bounded on $\mathbb{R}^N$, there is a positive constant $C_1 > 0$ sufficiently large such that $V(0, x) \leq \overline{V}(0, x)$ for any $x \in \mathbb{R}^N$. A
straightforward calculation yields
\[ \frac{\partial \bar{V}_i}{\partial t}(t, x) = \Delta \bar{V}_i(t, x) + \sum_{j=1}^{K} \frac{\partial F_i}{\partial u_j}(t, 0) \bar{V}_j(t, x) \]
for \( i = 1, ..., K \). Comparison to (4.10), using the comparison principle, we have
\[ \bar{V}(t, x) \leq \bar{V}(t, x) \quad \text{for any } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \]
Note that \( \int_0^t c(\tau)d\tau - \kappa(\sigma_t A^{\nu^*}) > 0 \). This implies that
\[ V_i(t, x) \leq \bar{V}_i(t, x) = C_1 \phi^*_i(t, \mu^*) e^{-\mu^*(x - \int_0^t c(\tau)d\tau - m_0) e^{-\mu^*(\int_0^t c(\tau)d\tau - \kappa(\sigma_t A^{\nu^*})) t}} \leq \tilde{C}_1 e^{-\mu^*(\int_0^t c(\tau)d\tau - \kappa(\sigma_t A^{\nu^*})) t}, \quad \forall i = 1, ..., K. \] (4.11)
where we have used the fact that \( x \cdot \dot{\xi} - \int_0^t c(\tau)d\tau \geq \eta_0 \).

**Case (II).** \( x \cdot \dot{\xi} - \int_0^t c(\tau)d\tau < \eta_0 \).

From (4.6) and (4.8), we have
\[ \bar{U}(t, x) \leq U(t, x) = U(t, x) + \bar{V}(t, x) \leq u^*(t). \]
Note that the point \( \eta_0 \in \mathbb{R} \) is chosen such that \( U(t, x) \) is very close to \( u^*(t) \) for all \( x \cdot \dot{\xi} - \int_0^t c(\tau)d\tau < \eta_0 \), that is, \( |U(t, x) - u^*(t)| < \delta \) for \( x \cdot \dot{\xi} - \int_0^t c(\tau)d\tau < \eta_0 \) and \( i = 1, ..., K \). We then obtain from (4.3) that for any \( x \cdot \dot{\xi} - \int_0^t c(\tau)d\tau < \eta_0 \),
\[ F'(t, U^+(t, x)) - F(t, U(t, x)) \leq \sum_{i=1}^{K} \left( \frac{\partial F_i}{\partial u_j}(t, u^*) + \varepsilon_1 \right) V_j(t, x), \quad i = 1, ..., K. \] (4.12)
Hence, from (4.9) and (4.12), we have that
\[ \frac{\partial \bar{V}_i}{\partial t}(t, x) \leq \Delta \bar{V}_i(t, x) + \sum_{j=1}^{K} \left( \frac{\partial F_i}{\partial u_j}(t, u^*) + \varepsilon_1 \right) \bar{V}_j(t, x), \quad i = 1, ..., K \]
for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N\) with \( x \cdot \dot{\xi} - \int_0^t c(\tau)d\tau < \eta_0 \).

Recall that \( \varphi(t) \) the principal Floquent bundle associated to the principal Lyapunov exponent \( \lambda(A^*) \) and
\[ \lambda(A^*) = \lim_{t \to \infty} \frac{1}{t} \int_{s}^{t} \kappa(\sigma_t A^{*})d\tau. \]
Since \( \lambda(A^*) < 0 \), we must have \( \kappa(\sigma_t A^*) < 0 \). Choose a positive number \( \tilde{\lambda} \) with \( \tilde{\lambda} < \lambda < -\kappa(\sigma_t A^*) \). Define
\[ \bar{V}_i(t, x) := C_2 \varphi_i(t)e^{-(\tilde{\lambda} - \tau)t}, \quad i = 1, ..., K, \]
where \( C_2 > 0 \) is sufficiently large such that \( V(0, x) \leq \bar{V}(0, x) \) for \( x \in \mathbb{R}^N \). Now, we verify that the function \( \bar{V}(t, x) \) is a supersolution of the following linear system
\[ \frac{\partial \bar{V}_i}{\partial t}(t, x) = \Delta \bar{V}_i(t, x) + \sum_{j=1}^{K} \left( \frac{\partial F_i}{\partial u_j}(t, u^*) + \varepsilon_1 \right) \bar{V}_j(t, x), \quad i = 1, ..., K, \]
for any \((t, x) \in \mathbb{R} \times \mathbb{R}^N\) with \( x \cdot \dot{\xi} - \int_0^t c(\tau)d\tau < \eta_0 \).
By (4.1), direct calculation yields that
\[
\frac{\partial \tilde{V}_i(t, x)}{\partial t} = C_2 e^{-(\tilde{\lambda} - \tau)t} \left[ \frac{d \varphi_i(t)}{dt} + (-\tilde{\lambda} + \tau) \varphi_i(t) \right] = C_2 e^{-(\tilde{\lambda} - \tau)t} \left[ \sum_{j=1}^{K} \frac{\partial F_i}{\partial u_j}(t, u^*) \varphi_j(t) + \kappa(\sigma_1 A^*) \varphi_i(t) \right] \geq C_2 e^{-(\tilde{\lambda} - \tau)t} \left[ \sum_{j=1}^{K} \frac{\partial F_i}{\partial u_j}(t, u^*) \varphi_j(t) + \tau \varphi_i(t) \right] = \Delta \tilde{V}_i(t, x) + \sum_{j=1}^{K} \frac{\partial F_i}{\partial u_j}(t, u^*) \tilde{V}_i(t, x) + \varepsilon_1 \tilde{V}_i(t, x).
\]

By (4.2), we have \( \varepsilon_1 \sum_{i=1}^{K} \tilde{V}_i(t, x) \leq \tau \tilde{V}_i(t, x) \) for any \( i = 1, \ldots, K \), hence,
\[
\frac{\partial \tilde{V}_i(t, x)}{\partial t} \geq \Delta \tilde{V}_i(t, x) + \sum_{j=1}^{K} \left( \frac{\partial F_i}{\partial u_j}(t, u^*) + \varepsilon_1 \right) \tilde{V}_j(t, x), \quad \forall i = 1, \ldots, K.
\]

By the comparison principle, we have
\[
V(t, x) \leq \bar{V}(t, x)
\]
for any \( (t, x) \in \mathbb{R} \times \mathbb{R}^N \) with \( x \cdot \xi - \int_0^t c(\tau)d\tau < \eta_0 \) and hence
\[
V_i(t, x) \leq \bar{V}_i(t, x) = C_2 \varphi_i(t) e^{-(\tilde{\lambda} - \tau)t} \leq C_2 e^{-(\tilde{\lambda} - \tau)t}
\]  
(4.13)
for all \( t > 0, x \cdot \xi - \int_0^t c(\tau)d\tau < \eta_0 \) and some positive constant \( C_2 \).

Choose
\[
\kappa := \min \left\{ \int_0^t c(\tau)d\tau - \kappa(\sigma_1 A^*) \tilde{\lambda} - \tau \right\}.
\]

Combining with (4.11) and (4.13), we have
\[
V_i(t, x) \leq C e^{-\kappa t}, \quad i = 1, \ldots, K, \quad t \geq 0, \quad x \in \mathbb{R}^N,
\]
where \( C > 0 \) denotes a large fixed constant. By the similar way, we can prove that
\[
\bar{V}_i(t, x) = |U_i^-(t, x) - U_i(t, x)| \leq C e^{-\kappa t}, \quad i = 1, \ldots, K, \quad t \geq 0, \quad x \in \mathbb{R}^N.
\]
Hence, from (4.7), we have
\[
|u_i(t, x; u_0) - U_i(t, x)| \leq C e^{-\kappa t}, \quad i = 1, \ldots, K, \quad t \geq 0, \quad x \in \mathbb{R}^N.
\]
This completes the proof. \( \square \)

As a consequence of Theorem 2.3, we can prove the uniqueness result of generalized transition waves of (1.1), that is Theorem 2.4.

**Proof of Theorem 2.4.** Since \( U(t, x) \) and \( \bar{U}(t, x) \) are two solution of (1.1) and
\[
\lim_{z \to +\infty} \frac{\tilde{U}_i(t, z)}{\tilde{U}_i(t, x)} = 1 \quad \text{for any } i = 1, \ldots, K.
\]
Hence, \([U(t, x) - \bar{U}(t, x)]e^{\mu^*t}z^{-\int_0^t c(\tau)d\tau} \) is uniformly bounded for some small \( \varepsilon = \mu^* - \mu > 0 \). Especially, \([U(0, x) - \bar{U}(0, x)]e^{\varepsilon(x \cdot \xi)} \in L^\infty(\mathbb{R}^N, \mathbb{R}^K) \). It then follows from Theorem 2.3 that
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}^N} |U_i(t, x) - \bar{U}_i(t, x)| = 0, \quad i = 1, \ldots, K.
\]
Hence we can conclude that \( U(t, x) = \tilde{U}(t, x) \) for all \( t > 0 \) and \( x \in \mathbb{R}^N \).

\[ \square \]

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