Exploring helical dynamos with machine learning: Regularized linear regression outperforms ensemble methods

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ABSTRACT

We use ensemble machine learning algorithms to study the evolution of magnetic fields in magnetohydrodynamic (MHD) turbulence that is helically forced. We perform direct numerical simulations of helically forced turbulence using mean field formalism, with electromotive force (EMF) modeled both as a linear and non-linear function of the mean magnetic field and current density. The form of the EMF is determined using regularized linear regression and random forests. We also compare various analytical models to the data using Bayesian inference with Markov chain Monte Carlo (MCMC) sampling. Our results demonstrate that linear regression is largely successful at predicting the EMF and the use of more sophisticated algorithms (random forests, MCMC) do not lead to significant improvement in the fits. We conclude that the data we are looking at is effectively low dimensional and essentially linear.

Finally, to encourage further exploration by the community, we provide all of our simulation data and analysis scripts as open source IPython notebooks.

Key words. dynamo – magnetohydrodynamics (MHD) – turbulence

1. Introduction

Large scale magnetic fields are observed on the Earth, the Sun, galaxies. Understanding the origin and sustenance of these magnetic fields is the subject of dynamo theory. Magnetic fields observed in the galaxies are thought to have their origin in the early phases of cosmological evolution (Subramanian 2019). On relatively smaller scales, magnetic fields likely play a significant role in star (Wurster & Li 2018) and planet formation (Ercolano & Pasucci 2017). For the Sun, a long standing question has been to understand the origin of the cyclical behavior of the magnetic North and South poles (Brun & Browning 2017).

A number of outstanding questions remain about dynamo theory. Classical models of dynamo theory (Krause & Steenbeck 1967; Krause & Rädler 1980) assume that the background flow is stationary and the magnetic field is too weak to feedback on the flow. This reduces the problem of magnetic field generation to a closure problem: how does the fluctuating electromotive force (EMF) depend on the mean field? Previous work on dynamo theory has often modeled the EMF as a linear function of the mean field (see Brandenburg & Subramanian 2005, for an extensive review). This approximation is reasonable in the growth (kinematic) phase in a flow-driven magnetohydrodynamic (MHD) system where the initial magnetic energy is orders of magnitude smaller than the kinetic energy of the forced fluid. But the limitations become apparent when the magnetic fields have gained enough energy to modify the background flow through the Lorentz force. To model this backreaction of the mean magnetic field on the flow, various proposals have been put forward including models where EMF is a non-linear function of the mean field (Jepps 1975; Pouquet et al. 1976; Vainshtein & Cattaneo 1992).

Two methods have been widely used in numerical simulations of dynamos to get an explicit form for the electromotive force and both are essentially linear: the test field method (Schirmer et al. 2007) and linear regression using the least squares method (Brandenburg & Sokoloff 2002). The kinematic test field method uses a set of orthogonal “test” fields that are pre-determined and can be used to compute the various coefficients in the EMF expansion corresponding to terms linear in the large scale field and its gradients. As input, the test field requires both the small and large scale velocity field and dynamically solves (on the fly) the test field equations to compute the test field coefficients. This has the advantage of retaining dynamical information (Hubbard & Brandenburg 2009; Rheinhardt & Brandenburg 2012). Linear regression, in contrast, is mostly used in post-processing to model the “inverse” problem of determining the coefficients corresponding to different terms in the linear expansion of the EMF.

Machine learning and deep learning have revolutionized image processing, object detection, natural language processing (LeCun et al. 2015). Recently there has been a surge of interest in using machine and deep learning tools in dynamical systems and computational fluid dynamics (Hennigh 2017; Kutz 2017; Brunton & Kutz 2019). Data-driven modeling is not new to fluid dynamics; in meteorology, observational data has been used for decades to guide weather predictions (see Navon 2009, for a historical overview). Considerable increase in computational power with advances in hardware (GPUs) has allowed modern optimization methods to become increasingly tractable. The end goal is to not only build predictive data-driven models but also to understand underlying complex physics that cannot be captured by simple linear regression.

The motivation to study reduced descriptions of the full 3D MHD simulations comes primarily from the extremely large, and currently numerically intractable, magnetic Reynolds number of most astrophysical systems ($Rm \sim 10^{6}-10^{12}$).
Brandenburg & Subramanian 2005). Reduced-order modeling of
dynamical systems is currently a highly active field of research
that aims to provide description of physical systems in terms of
the minimal possible degrees of freedom (Kutz 2013, 2017). In
this paper, we attempt to build reduced-order models for MHD
dynamos by applying non-linear regression tools to the widely
studied $\alpha^2$ dynamo (Brandenburg & Subramanian 2005). In
this context, we follow the best practices in machine learning that
emphasize generalization by using train-validation-test splits in
data. If one uses all the data to compute correlations (or use any
other algorithm for that matter), it amounts to “descriptive” mod-
eling. By separating the data into distinct training, validation
and test sets, one enters into the realm of “predictive” modeling
where the generalization power of different algorithms can be
tested by first fitting data on the training set, and then test-
ing it on the validation before making predictions on the test set.
A high score on the training set but a poor score on validation
set indicates poor generalization. This is in contrast to standard
linear regression techniques employed in the dynamo literature
where the entire data set is taken to be the training set with no
validation set to test overfitting (Brandenburg & Sokoloff 2002).

We use data from Direct Numerical Simulations (DNS) of
forced helical turbulence to construct a reduced model where
the EMF is an unknown (linear and non-linear) function of the
large scale field. This allows us to quantitatively assess the stan-
dard assumption of modeling EMF as linearly proportional to the
mean magnetic field and current density. We employ three classes
of modern statistical and machine learning models: (i) Linear
regression, (ii) Random forests, (iii) Bayesian inference (with
Markov chain Monte Carlo model minimization). Our selection of
a well-studied systems like forced helical MHD turbulence
supports our second main motivation: we can compare our results
from machine learning tools against relatively well-known results
in the literature. The insights gained from such a comparison
could then help guide future work on more complex MHD tur-
bulent systems such as MHD turbulence with differential rotation
(Blackman & Brandenburg 2002; Charbonneau 2014).

We describe our data and numerical simulations in the next
section. This is followed by a section that begins by explain-
ing the fitting methods considered in this paper. We then apply
these machine learning and statistical methods on the data. We
discuss our results in Sect. 4 putting out work in the context of
previous work and some caveats. Section 5 is conclusions. All
of our analysis scripts and data are freely available as interactive
IPYTHON notebooks.1

## 2. Data

We describe the equations, setup and code that was used to
conduct DNS of forced turbulence in magnetized flows.

### 2.1. Description of DNS

We use the publicly available 6th order finite difference PENCIL-
CODE (Brandenburg & Dobler 2002)2. The MHD equations are
solved in a triply periodic cubic domain of size $(2\pi)^3$:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho V)$$

(1)

$$\frac{\partial V}{\partial t} = -\frac{1}{\rho} \left[ V \cdot \nabla \cdot V + \nabla p - 2\nabla \cdot (\eta \nabla S) - J \times B \right] + f$$

(2)

$$\frac{\partial A}{\partial t} = V \times B - \eta \nabla J.$$  

(3)

1. https://github.com/fnauman/ML_alpha2
2. http://github.com/pencil-code

### Table 1. Summary of the simulations used in this paper.

| Name  | $c_s/(k_1\eta)$ | $Rm$ | $t_{res}$ | $v_{rms}$ |
|-------|-----------------|------|-----------|----------|
| R5e2  | 500             | 1.68 | 4.97      | 0.0337   |
| R1e3  | 1000            | 4.44 | 9.94      | 0.0446   |
| R2e3  | 2000            | 10.31| 19.88     | 0.052    |
| R3e3  | 3000            | 16.71| 30.12     | 0.055    |
| R4e3  | 4000            | 22.64| 39.76     | 0.057    |
| R5e3  | 5000            | 28.72| 49.70     | 0.058    |
| R6e3  | 6000            | 34.61| 59.63     | 0.058    |
| R7e3  | 7000            | 40.41| 69.58     | 0.058    |
| R8e3  | 8000            | 46.22| 79.52     | 0.058    |
| R9e3  | 9000            | 50.16| 89.55     | 0.056    |
| R1e4* | 10000           | 55.79| 99.40     | 0.056    |
| R15e4*| 15000           | 82.12| 149.1     | 0.055    |

**Notes.** We used a resolution of 256$^3$ and a box size of $(2\pi)^3$ for all of our simulations. The forcing scale $k_f/k_1 = 10$ is used for all runs. Shock viscosity was used in starred (*) simulations.

where $v$ is the viscosity, $S = \frac{1}{2}(V_{ij} + V_{ji}) - \frac{1}{2}\delta_{ij}\nabla \cdot V$, $f$ is the forcing function, $\eta$ is the resistivity, and $J = \mu_0^{-1} \nabla \times B$ is the current density.

The forcing function is taken to be homogeneous and isotropic with explicit control over the fractional helicity determined by the $\sigma$ parameter. It is defined as:

$$f(t, r) = Re \left\{ N f(k_t) \exp[i k(t) \cdot x + i \phi(t)] \right\},$$

(4)

where the wave vectors $k(t)$ and the phase $|\phi(t)| < \pi$ are random at each time step. The wavevectors are chosen within a band to specify a range of forcing centered around $k_f$. The normalization factor is defined such that the forcing term matches the physical dimensions of the other terms in the Navier Stokes equation, $N = f_0\sigma_{ij}(|k_{ij}|/\delta t)^{1/2}$, where $f_0$ is the forcing amplitude, $c_s$ is the sound speed, $\delta t$ is the time step. The Fourier space forcing has the form (Haugen et al. 2004):

$$f_k = R \cdot f_{k_{ehol}}$$

with $R_{ij} = \delta_{ij} - \frac{i \sigma_{ij} \hat{k}_k}{\sqrt{1 + \sigma^2}},$

(5)

where $\sigma$ is a measure of the helicity of the forcing; for positive maximum helicity, $\sigma = 1$, and $f_{k_{ehol}}^2 = k \times \hat{e} / \sqrt{k^2 - (\hat{k} \cdot \hat{e})^2}$.

The initial velocity and density are set to zero while the magnetic field is initialized through a vector potential with a small amplitude ($10^{-3}$) Gaussian noise. We define the units (Brandenburg 2001):

$$c_s = \rho_0 = \mu_0 = 1,$$

(6)

such that the magnetic field, $B$, is in the units of Alfvén speed ($c_s \sqrt{\mu_0 \rho_0}$). The domain size is $L^3 = (2\pi)^3$, which leads to $k_1 = 2\pi/L = 1$. Moreover, we use the following dimensionless numbers:

$$Rm = v_{rms}/k_1\eta, \quad t_{res} = 1/k_1^2 \eta,$$

(7)

where $k_f/k_1 = 10$ is the forcing wavemode. We provide a summary of the simulations in Table 1.

### 2.2. Reduced model

Numerical simulations of large scale astrophysical flows with
magnetic Reynolds number well in the excess of $10^6$ are pro-
hibitive. Moreover, state-of-the-art simulations requires $O(10^6)$
CPU hours to compute for moderate Reynolds numbers. To make modeling such systems more tractable, it is therefore necessary to look for simpler surrogate models that can capture the most significant properties of the high fidelity simulations (DNS).

Large scale dynamo theory concerns itself with the evolution of the filtered induction equation,

$$\frac{\partial}{\partial t} \overline{b} = \nabla \times (\overline{v} \times \overline{b}) + \nabla \times e + \eta \nabla^2 \overline{b},$$

(8)

where EMF is given as $e = \nabla \times b$ and $\eta$ is the turbulent resistivity, and the overbar represents a filtered quantity. The filter could be ensemble, temporal or spatial depending on the problem. The first term on the right-hand side is typically ignored unless there is no strong non-constant mean velocity field such as in the case of galaxies (the mean differentially rotating flow for the galactic dynamo is considered in Shukurov et al. 2006). The dynamics of the velocity field (Eq. (2)) are ignored based on the assumption that the forcing term is the dominant term and it causes the velocity field to be in a steady state. This assumption is only reasonable in the "kinematic" regime where the magnetic fields are too weak to modify the velocity field dynamically.

In keeping with previous work that has used horizontal averaging as the filter in Eq. (8) (Brandenburg 2001; Brandenburg & Subramanian 2005), our data has been averaged over the whole $xy$ plane:

$$\overline{b} = \frac{1}{L_xL_y} \int_0^{L_x} \int_0^{L_y} B \, dx \, dy.$$

This averaging scheme has the interesting property that $\nabla \times (\overline{v} \times \overline{b}) \sim 0$ since $\overline{B} = 0$ (due to incompressibility and periodic boundary conditions). We only consider the fields (let $Q$ be an arbitrary field) reduced to 1D by furthermore averaging them in two different ways:

1. $z$ averaged: (\overline{Q} (t)) = 1/(z_2 - z_1) \int_{z_1}^{z_2} Q \, dz$ where $z_1$, $z_2$ are chosen such that they do not cover the entire $z$-domain since that will correspond to volume averaged quantities that are zero.

2. $t$ averaged: (\overline{Q} (z)) = 1/(t_2 - t_1) \int_{t_1}^{t_2} Q \, dt$ where $t_1$, $t_2$ are chosen to correspond to either the kinematic or saturation regime for different runs.

This leaves two independent magnetic fields and corresponding EMF components. Analytical models that use $e = \alpha B - \eta J$ are termed $a^2$ dynamos and have been extensively studied in the literature. Here $\eta$ represents the turbulent resistivity.

In Fig. 1, we show a three-dimensional phase-space visualization of $E_x$ vs. $B_x$ vs. $B_y$ that shows the relation between the EMF and the mean magnetic fields (situation is the same when considering $E_y$, too). The circle in $B_x - B_y$ plane represents the fact that the total magnetic energy $B_x^2 + B_y^2$ is constant. The slight tilt with respect to the EMF (i.e., towards $z$ direction) implies a linear correlation with the EMF. The increasing disorder with increasing $Rm$ is characteristic of highly turbulent systems.

3. Model fits

We describe the 1D linear and non-linear regression for temporal and vertical profiles in the following subsections. In the following, we will refer to the magnetic fields as both “fields” and “features”, and EMF as “target” interchangeably. For most of the following, we will focus on $E_x$, but similar results hold for $E_y$, since we are looking at isotropically forced MHD turbulence with no background field.

3.1. Description of algorithms considered in this paper

Ordinary Least Squares (OLS) assumes that the data is linear, features/terems are independent, variance for each point is roughly constant (homoscedasticity), features/terms do not interact with one another. Moreover, causal inference based on linear regression can be misleading (Bollen & Pearl 2013), a problem shared by all curve fitting methods. We should highlight the distinction between linearity of the regression as opposed to linearity of the features. For example, one can construct a non-linear basis (example: $x, x^2, x^3, \ldots$ or $\sin x, \cos x, \ldots$) and do linear regression with this basis. Linear regression on non-linear data can potentially represent non-linear data well assuming that the non-linear basis terms are independent and have low variance. Constructing the correct interacting/non-linear terms requires domain expertise. For this reason, it is important to also consider non-linear regression methods such as random forests that are robust to noise and are capable of modeling interactions implicitly.
3.1.1. LASSO

Most real data have considerable variance that can lead to misleading fits from linear regression. Regularized linear regression aims to construct models that are more robust to outliers. Two of the most common regularization schemes for linear regression are (Tibshirani 1996; Hastie et al. 2015): (i) LASSO (Least Absolute Shrinkage and Selection Operator; also known simply as L1 norm), (ii) Ridge (also known as L2 norm). As opposed to ridge, LASSO has the advantage of doing feature selection by reducing coefficients of insignificant features. We will use LASSO regression in this work.

Intuitively LASSO reduces variance at the cost of introducing more bias. LASSO works better than ridge regression if the data is low dimensional (or the coefficient matrix is sparse). For high dimensional complex data, neither ridge nor LASSO do well. Because of this, the “Bet on sparsity” principle (Hastie et al. 2009) suggests trying LASSO for all datasets. Formally, LASSO optimizes for:

\[
\text{argmin}_w \left[ \|Y - Xw\|^2_2 + \alpha \|w\|_1 \right],
\]

where \( y \) represents the target vector (in our case, \( \mathcal{E} \)), \( X \) is the feature matrix (\( B_1, B_2, \ldots \) are the columns), \( w \) is coefficient vector and \( \alpha \) is a parameter with a range between 0 and 1. The subscripts “1” and “2” represent L1 (\( \sum_n \|w\|_1 = |w_1| + |w_2| + \ldots + |w_n| \)) and L2 (\( \sum_n \|w\|_2 = \sqrt{w_1^2 + w_2^2 + \ldots + w_n^2} \)) norms, respectively. A higher \( \alpha \) penalizes outliers more strongly and shrinks coefficients of unimportant features to zero while a lower \( \alpha \) is similar to linear regression with no regularization (OLS).

3.1.2. Random forests

Random forests belong to the class of ensemble machine learning algorithms (Breiman 2001a; Hastie et al. 2009). A random forest consists of a large number of decision trees that each take a random subset of features with a bootstrapped sample of the data (Raschka 2018). This helps in getting rid of strong correlations among different trees but some correlations might still remain. Random forests are one of the most popular and widely used machine learning algorithms because they require little hyperparameter tuning and can handle large noisy data sets. From a computational point of view, each decision tree in the ensemble is independent and can thus be easily processed in parallel. Moreover, because of the weighted average over the ensemble, random forests are robust to strong variance in the data but a few deep decision trees can lead to strong bias.

Ensembles of decision trees (random forests and gradient boosting) are non-parametric models meaning they do not require information about the underlying data distribution. In other words, unlike in linear regression where the shape of the function that connects the independent variables to the dependent variable is fixed, the functional form is not known in random forests and is instead determined through training. This offers a different perspective on modeling, termed “algorithmic” modeling as opposed to “data” modeling (Breiman 2001b). The non-parametric nature of random forests and gradient boosting make them applicable to a wide class of problems since in most real-world situations, data distribution is not known a-priori. However, this comes at the cost of interpretability making ensembles of decision trees harder to understand than linear regression (Strobl et al. 2008).

Feature importance and selection: As opposed to linear regression, random forests do not directly measure the coefficient of the features. A single decision tree is highly interpretable but a whole ensemble of decision trees is harder to interpret (Breiman 2001b). Random forests offer some insight through feature importances (Hastie et al. 2009). We use the “mean decrease impurity” method implemented in PYTHON library SCIKIT-LEARN (Pedregosa et al. 2011; Breiman et al. 1984; Louppe 2014). With this method, the feature importance corresponds to the frequency of a particular feature appearing across different trees combined with the reduction of error corresponding to that particular feature, weighted by the fraction of samples in that particular node. One popular alternative is to randomly permute the data for a particular feature and see whether it has an effect on the mean squared error (or information gain/gini importance for classification tasks). If the mean squared error increases or remains the same, it implies that the particular feature is not important. This is termed as “permutation” importance (Breiman 2001a; Strobl et al. 2008).

3.1.3. Bayesian inference

Bayesian inference is a powerful statistical framework that presents an alternative way to perform the model parameter estimation. The main strengths of Bayesian-based approaches are their ability to quantify the uncertainty of the model and the possibility to incorporate previous a-priori knowledge into the estimation. The downside, when compared to many conventional machine learning methods, is that the model needs to be known before the fit.

Let us quickly summarize the idea behind Bayesian inference. In general, let us present our model with \( M \) and our empirical (simulation) data with \( D \). We are then interested in how well can our model describe the data, or quantitatively what is the probability that our model is valid given the data, \( \text{Pr}(M|D) \). By directly comparing our model and the data, what we actually get is, however, the probability of our model given the data, known as the likelihood \( \mathcal{L} = \text{Pr}(D|M) \). Our original question of the validity of our model, can be answered by applying the Bayes’ theorem presented as (see e.g., Grinstead & Snell 1997)

\[
\text{Pr}(M|D) = \frac{\text{Pr}(D|M) \text{Pr}(M)}{\text{Pr}(D)},
\]

where \( \text{Pr}(M) \) is our prior probability of the model and \( \text{Pr}(D) = \int \text{Pr}(D|M) \text{d}M \) is the prior probability. Therefore, because \( \mathcal{L} \) can be computed and \( \text{Pr}(M) \) is known a-priori, we can use the Bayes’ theorem to quantitatively assess the validity of, for example, our \( \alpha^2 \) dynamo models given the simulation data results.

In practice, not only are we interested in the validity of the model, but also what are the parameters \( \Theta \) of the model, \( M(\Theta) \), that best describe the data. In Bayesian inference, these model parameters are determined using a marginal estimation where the resulting posterior parameter for model parameter \( \Theta_j \) are obtained by integrating over the probability of all the other model parameters, \( \Theta \) (i \( \neq \) j). Then, this (one-dimensional) distribution represents the probability that the \( j \)th model parameter, \( \Theta_j \), will take a particular value given our data \( D \).

Here we solve the Eq. (11) (and therefore obtain also posterior distributions for our model parameters) using Monte Carlo Markov chain (MCMC) integration techniques. To perform the MCMC fit we use the publicly available EMCEE library (Foreman-Mackey et al. 2013). We employ the affine-invariant stretch-move ensemble sampler with typically \( 3N_{\text{param}} \) walkers (members of ensemble), where \( N_{\text{param}} \) is the number of fit parameters. No prior knowledge is incorporated into the fits and so we use uniform priors for all of our model parameters.
3.2. Vertical profiles

In this subsection, we apply various methods on time averaged data that only has vertical dependence (256 grid points).

In Fig. 2, we show the linear (Pearson) correlation coefficients among the various fields. The correlation between the current densities and the magnetic fields is nearly 1 indicating that from a curve fitting point of view, it has redundant information. Since we are using helical forcing leading to a helical state, \( \mathbf{J} = \nabla \times \mathbf{B} \sim \mathbf{B} \). At high \( R_m \), one might expect that the power in the velocity and magnetic field spectrum will be distributed across multiple modes implying weaker correlations. However, Fig. 2 (right panel) indicates the correlation between \( \mathbf{B} \) and \( \mathbf{J} \) is still perfect. This means that we can eliminate current density as an independent variable. See, however, the work Tilgner & Brandenburg (2008) that suggests that the tensorial form of \( \alpha_i, \eta_{ij} \) in the non-linear regime depends on \( \mathbf{B} \). In keeping with previous work, for linear basis we do consider the current density as an independent variable but eliminate it for the non-linear (polynomial) basis where its inclusion will lead to several redundant terms.

We consider two sets of basis in this subsection:

1. Linear: \( \mathbf{B}_x, \mathbf{B}_y, \mathbf{J}_x, \mathbf{J}_y \).
2. Polynomial of \( O(3) \): \( \mathbf{B}, \mathbf{B}_x, \mathbf{B}_y, \mathbf{B}_x^2, \mathbf{B}_y^2, \mathbf{B}_x \mathbf{B}_y, \mathbf{B}_x^2 \mathbf{B}_y, \mathbf{B}_x \mathbf{B}_y^2 \).

We note that for kinetically forced simulations, \( \nu^2 \sim \text{constant} \), which means that the total energy conservation equation yields, \( \mathbf{B}_x^2 + \mathbf{B}_y^2 \sim \mathbf{B}^2 \sim \text{constant} \) in the saturated state. This makes \( \mathbf{B}_x \) and \( \mathbf{B}_y \) linearly dependent terms and thus we do not consider them separately. Another motivation to use \( \mathbf{B}^2 \) is because work on dynamo quenching often considers \( \alpha \sim 1/(1 + \mathbf{B}^2 / \mathbf{B}_eq^2) \), which when expanded for the \( \mathbf{E} \) leads to cubic terms of the form \( \mathbf{B} \mathbf{B}_i \). We will be using the publicly available PYTHON library scikit-learn (see Guido & Müller 2017; Géron 2017, for introductions) for modeling the data.

Previous work on dynamo theory has often relied on linear regression methods (Brandenburg & Sokoloff 2002; Brandenburg 2018) with some effort to incorporate non-linear effects through magnetic helicity conservation (Pouquet et al. 1976; Blackman 2015; see also non-linear test field method Rheinhardt & Brandenburg 2010). In this section, we use linear regression with a regularization term for both the linear and polynomial basis discussed previously.

In Fig. 3, we show the fits using random forests regression and LASSO. Because our data is small (only 256 points), we use two-fold cross validation to determine various hyperparameters of LASSO and random forests. We found that bootstrapping leads to spurious results so we turned it off. A similar problem was found with random train/test splits: the spatial structure of the fields is like one full sinusoidal wave and randomly picking a fraction of the points leads to strange patterns. Moreover, while deep trees can lead to overfitting, in our case we found that a maximum depth of 8 was optimal through 2-fold cross validation. The end results are still not as good as LASSO perhaps owing to the fact that the training set is only 204 entries (80% of the data) – this is too low for an algorithm like random forest to really excel. That is the key lesson we have learnt while modeling this dataset throughout: “small” datasets mean simpler algorithms will outperform more sophisticated algorithms like random forests.

In Table 2, we compare the coefficients from OLS, LASSO, randomized LASSO (stability), Recursive Feature Elimination (RFE) from OLS and RFE from random forests, Mutual Information Criterion (MIC), Pearson correlation (Corr) and the average of all these predictions in the last column. The meaning of these numbers is different. For OLS and LASSO, the numbers corresponding to the fields \( \mathbf{B}_i \) represent coefficients from linear regression and can be negative or positive. For random forests, the numbers represent the relative feature importance based on “mean decrease impurity” that is only positive. These feature importance numbers are scaled to sum to unity. Stability selection is a general term that refers to checking robustness of model predictions by testing it on different random (bootstrapped) subsets of data with different hyperparameter values. In our
3.3. Time series analysis: Vertically averaged data

In contrast to the previous section, the magnetic fields and the EMF show exponential growth in time. To model this behavior,
we found it convenient to do pre-processing where we took the logarithm of squared vertically averaged fields. Because of this choice, we do not consider polynomial expansions like we did before and instead focus on either the linear $E \propto B$ form, or the quenched form $E \propto B/(1 + B^2/B_{eq}^2)$.

3.3.1. Preprocessing

For the vertically averaged data set, we find that the fluctuations in the EMF are quite significant (see Fig. 4). Such large fluctuations make any kind of statistical analysis quite prohibitive. In this section, we therefore decided to use the log of $E^2$ averaged over the $z$ direction. Moreover, to reduce the fluctuations we used a moving average of size 20 (right panel of Fig. 4). This allows for cleaner fits.

3.3.2. LASSO and random forest

In Fig. 5, we show fits for the $E^2$ time series using LASSO and random forests for $Rm = 1.5 \times 10^4$. Just like the fits for spatial data in Fig. 3, we find that LASSO does considerably better than random forests, the latter of which again has a step like shape. Left: we show the time series only up to the kinematic phase ($\sim 0.01$ resistive times) with the inset describing the train/test split. Right: time series is between kinematic and saturation phases. LASSO seems to capture the shape of the curve but is offset. In both cases, random forest prediction returns a nearly horizontal line, characteristic of decision tree regression. The black vertical dashed line presents the train-test split (80% training data).

Fig. 4. EMF evolution for $Rm = 1.5 \times 10^4$. Left: $E_z$ component is shown without any pre-processing ($z = \pi$). Right: here we show $\log(E^2)$, that is the squared EMF averaged over $z$. Moreover, we use a moving average with window size 20 for the orange line to smooth out the fluctuations.

Fig. 5. The fits for $E^2$ time series using LASSO and random forests for $Rm = 1.5 \times 10^4$. Just like the fits for spatial data in Fig. 3, we find that LASSO does considerably better than random forests, the latter of which again has a step like shape. Left: we show the time series only up to the kinematic phase ($\sim 0.01$ resistive times) with the inset describing the train/test split. Right: time series is between kinematic and saturation phases. LASSO seems to capture the shape of the curve but is offset. In both cases, random forest prediction returns a nearly horizontal line, characteristic of decision tree regression. The black vertical dashed line presents the train-test split (80% training data).

3.3.3. Bayesian inference

In this section we focus on reconstructing square of the “true” EMF field $E^2$ as a function of time $t$. As one of the simplest forms possible, we construct the reconstructed EMF, $E_{R}(z; t)^2 = E_{x,R}(z; t)^2 + E_{y,R}(z; t)^2$, from the corresponding magnetic field components as

$$E_{x,R}(z; t) = \frac{\alpha}{1 + B^2(z; t)/B_{eq}^2} B_x(z; t), \quad E_{y,R}(z; t) = \frac{\alpha}{1 + B^2(z; t)/B_{eq}^2} B_y(z; t),$$

where $E_{x,R}$ and $E_{y,R}$ are the reconstructed $x$ and $y$ components of the EMF field. Here $\alpha$ and $B_{eq}$ are our model parameters to optimize. This EMF form has the nice property that in the limit $B_{eq} \to \infty$ it reduces to the linear model.

After this, we are left with the freedom to define our likelihood or score-function that describes the quality of the reconstruction. Here we define it via a simple residual function $\mathcal{R}$,

$$\mathcal{R} = \frac{E_{R} - E}{E_{R}} - 2,$$
that penalizes the fit from over- and underestimating the reconstructed EMF. Note that other forms of the residual functions are also possible. For discrete simulation data results (defined on an array) the likelihood then simplifies to $L_q = \sum_{k} |R_{q}|$, where the summation index $k$ is taken to be over the time dimension or the $z$ direction, depending on the type of fit we are considering. Finally, when employing the MCMC fit note that we are actually minimizing log $\sum_{q} L_{q}$, as is typically done, to get a more shallow and slowly varying score-function.

Examples of the fit results are shown in Fig. 6 for three different $Rm$ values of $500$, $5 \times 10^3$, and $1.5 \times 10^4$ depicting the maximum a posteriori model. The (quenched) EMF model does reproduce the qualitative behavior of the saturating $B$ field but especially for higher $Rm$ the fits become more worse. Most importantly, we have found that constraints on the saturation field strength $B_{eq}$ are always very loose. Results without using the quenching form also lead to similar fits. This gives an indirect support for the previous machine learning results: The data strongly favors isotropic $\alpha$ with a simplified model of $E = \alpha B$. The meaning of this “$\alpha$” could be confusing since we have already eliminated the mean current density $J$ as an independent variable. The $\alpha$ coefficient represents the saturated value where the growth term roughly balances the dissipation term, leading to the effective expression: $\alpha \sim k_{1}(\eta + \eta_i)$.

### 3.4. Results summary

For both the vertical and temporal profiles, various models suggest that $E \propto B$ is a good description of the data. But if $dB/dt \propto aB$, this will lead to exponential growth without any saturation? Is there an inconsistency here? In the induction equation, the fields at time “$t$” are related to the fields at earlier times.

The machine learning models we use here are not capable of storing long term memory. The fits shown in this paper are comparing $E$ and $\overline{B}$ in some nearest neighbor sense and because both $E$ and $\overline{B}$ are growing exponentially before saturating at about the same time, they appear to be linearly correlated. Capturing the dynamical information requires more sophisticated algorithms that we leave for future work.

### 4. Discussion

#### 4.1. Caveats

We summarize the properties of the different machine learning algorithms considered in this work in Table 3. Linear regression (both OLS and LASSO) are by definition capable of dealing with linear data only. However, they can handle some non-linear properties using a non-linear basis (for example, polynomial regression). The biggest advantage of linear models is the interpretability: the sensitivity of the target variable (EMF) with respect to the features $(\overline{B}, \overline{E})$ is as easy as determining the coefficients of these terms. Linear models are also good at dealing with data of all sizes but matrix inverses are expensive for large data. Linear models might also be able to model trends in data if the features they consider have the same trend as the target variable.

Random forests are based on ensembles of decision trees. While random forests are quite robust to outliers and missing data, the feature importances computed from random forests can be misleading (Strobl et al. 2008). Moreover, random forests are not as interpretable as linear regression or single decision trees. A typical random forest fit can involve 10–100 trees implying that interpretation is difficult. But computation of feature importances and partial dependence plots help in making sense of random forest results (Breiman 2001b; Hastie et al. 2009). Random

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**Table 3.** Summary of the advantages and disadvantage of different models considered in this work.

| Property                  | OLS | LASSO | RF | Bayesian |
|---------------------------|-----|-------|----|----------|
| Non-linear                | No  | No    | Yes| Yes      |
| Interpretable             | Yes | Yes   | Yes| Yes      |
| Suitable for all data sizes? | Yes | Yes   | Mid-sized to big data | Small to mid-sized data |
| Interactions              | No  | No ($^b$) | Yes| Yes      |
| Prediction of trends      | No  | No    | No | Yes      |
| Poor signal to noise ratio| No  | No    | Yes| Yes      |

Notes. ($^a$) Feature importances and partial dependence plots can help interpret random forests. ($^b$) Including cross terms like $\overline{B}, \overline{E}$, can help model interactions but it still assumes each feature is linearly independent.
folds typically bootstrap data and use a random subset of features for each decision tree to determine which one leads to the lowest error. These two sources of randomness imply that random forests are best suited for intermediate to big data. Indeed in our “small data” case, we find that bootstrapping had to be turned off and the best fit models used less than 10 decision trees.

Bayesian methods offer an interesting complementary toolkit to various machine learning techniques. They also enable us to properly take the noisy nature of DNS into account. Here the big caveat is, that the model(s) needs to be known beforehand. This, however, further arguments in favor of our hybrid approach applied in this paper. We have used machine learning techniques to detect important features and then applied our physical knowledge to build valid models out from these. Biggest caveat here is that Bayesian model optimization becomes more and more computationally demanding with multidimensional (and often multi-modal) data. This puts a limitation on the real-life usability of the method when dealing with increasingly complex data.

The data considered in this paper is either spatially or temporally averaged reducing to $O(100-1000)$ entries. Big data typically refers to the case where the observations/rows are $\geq 10^9$. “Small” data faces problems of overfitting and is more likely to be sensitive to outliers. Because of these issues, simpler models like linear regression with regularization tend to do well. It is, therefore, no surprise that LASSO has outperformed random forests. For larger datasets with irregular non-linear patterns, random forests and gradient boosting tend to do much better (Olson et al. 2017).

4.2. Comparison with previous work

We used the simplest model: isotropic with no non-local effects in either space and/or time. While earlier work has focused primarily on linear models of dynamos where $E \sim B$ using linear regression and the test field method (Brandenburg & Subramanian 2005), non-linear models have also been considered (Rheinhardt & Brandenburg 2010). We assumed isotropy, which could be a misleading assumption in the presence of shear (see Brandenburg & Sokoloff 2002; Karak et al. 2014). Furthermore, in our model, we did not incorporate non-local effects that were considered by Brandenburg & Sokoloff (2002), Hubbard & Brandenburg (2009), Rheinhardt & Brandenburg (2012). Combining machine learning algorithms with anisotropic, local, non-linear models might lead to new insights, and we leave this for future work.

The $a^2$ dynamo considered in this work is an idealized system but has the advantage that due to maximally helical forcing and periodic boundary conditions, magnetic helicity is both conserved and is gauge invariant (see Blackman 2015). This simplistic setup allows developing analytical theory to explain the origin and saturation of magnetic fields (Pouquet et al. 1976; Blackman & Field 2002). Numerical results seem to be consistent with theory (Brandenburg 2001; Brandenburg et al. 2012). In this work, we considered existing analytical formulations and used machine learning tools to test whether the EMF is linear or non-linear in the mean magnetic fields. Machine learning models can complement analytical theory in looking for reduced descriptions of high dimensional systems (Brunton & Kutz 2019).

5. Conclusions

Our main result can be summarized as follows: because we are in the “small” data regime and we are dealing with ordered data (high signal to noise ratio due to helically forced turbulence), regularized linear regression (LASSO) provides the best fit. For small organized data, sources of randomness such as random train/test splits, bootstrapping, cross validation with random subsets do not help. For this particular dataset, many of the features are strongly correlated with one another complicating the model fits even further. Strong correlations also imply that feature engineering to construct polynomial basis is not particularly useful.

Random forests and MCMC do better with mid-sized to big data. Moreover, these ensemble methods tend to outperform linear and regularized linear regression when the signal to noise ratio is poor (see Table 3). For the data we consider in this paper, it is therefore no surprise that LASSO provides the best fit.

What we intended to demonstrate using the example of $a^2$ is that sophisticated machine learning algorithms can be used to analyze data from DNS of MHD turbulence thanks to the publicly available machine learning libraries. This provides an exciting opportunity to re-visit some long-standing problems in astro-physical flows where linear regression has dominated modeling efforts. For example, one can look at the full data cube without planar averaging and study whether 3D localized structures play a dominant role in the sustenance of the turbulence. Here, deep learning (Lusch et al. 2018) will be a useful alternative as it is known to outperform classical machine learning methods in image processing, voice recognition, natural language processing. One particular problem of interest is shear-driven dynamos (Tobias & Cattaneo 2013; Nauman & Blackman 2014).

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