Abstract

From the subsubleading chiral three-nucleon force [intermediate-range contributions, published in Phys. Rev. C 87, 054007 (2013)] a density-dependent NN-interaction \( V_{\text{med}} \) is derived in isospin-symmetric nuclear matter. Following the division of the pertinent 3N-diagrams into two-pion-one-pion exchange topology and ring topology, one evaluates for these all selfclosings and concatenations of nucleon-lines to an in-medium loop. In the case of the \( 2\pi 1\pi \)-exchange topology, the momentum- and \( k_f \)-dependent potentials associated with the isospin-operators (1 and \( \vec{\tau}_1 \cdot \vec{\tau}_2 \)) and five independent spin-structures require at most one numerical integration. For the more challenging (concatenations of the) ring diagrams proportional to \( c_{1,2,3,4} \), one ends up with regularized double-integrals \( \int_0^\lambda dr \int_0^{\pi/2} d\psi \) from which the \( \lambda^2 \)-divergence has been subtracted and the logarithmic piece \( \sim \ln(m_\pi/\lambda) \) is isolated. The derived semi-analytical results are most helpful to implement the subsubleading chiral 3N-forces into nuclear many-body calculations.

1 Introduction and summary

Three-nucleon forces are an indispensable ingredient in accurate few-nucleon and nuclear structure calculations. Nowadays, chiral effective field theory is the appropriate tool to construct systematically the nuclear interactions in harmony with the symmetries of QCD. Three-nucleon forces appear first at \( N^2\text{LO} \), where they consist of a zero-range contact-term (\( \sim c_E \)), a mid-range \( 1\pi \)-exchange component (\( \sim c_D \)) and a long-range \( 2\pi \)-exchange component (\( \sim c_{1,3,4} \)). The complete calculation of the chiral 3N-forces to subleading order \( N^3\text{LO} \) [1, 2] and even to subsubleading order \( N^4\text{LO} \) [3, 4] has been achieved during the past decade by the Bochum-Bonn group. At present the focus lies on constructing 3N-forces in chiral effective field theory with explicit \( \Delta(1232) \)-isobars, for which the long-range \( 2\pi \)-exchange component has been derived recently in ref. [5] at order \( N^3\text{LO} \).

However, for the variety of existing many-body methods, that are commonly employed in calculations of nuclear matter or medium mass and heavy nuclei, it is technically very challenging to include the chiral three-nucleon forces directly. An alternative and approximate approach is to use instead a density-dependent two-nucleon interaction \( V_{\text{med}} \) that originates from the underlying 3N-force. When restricting to on-shell scattering of two nucleons in isospin-symmetric spin-saturated nuclear matter, the resulting in-medium NN-potential \( V_{\text{med}} \) has the same isospin- and spin-structure as the free NN-potential. The analytical expressions for \( V_{\text{med}} \) from the leading chiral 3N-force at \( N^2\text{LO} \) (involving the parameters \( c_{1,3,4}, c_D \) and \( c_E \)) have been presented in ref. [6] and these have found many applications (e.g. to thermodynamics of nuclear matter) in recent years [7, 8, 10, 11, 12, 13, 14, 15, 16]. But in order to perform nuclear many-body calculations that are consistent with their input at the two-body level, one needs also \( V_{\text{med}} \) derived from the subleading chiral 3N-forces at order \( N^3\text{LO} \). In two recent works this task has been completed for the short-range terms and relativistic \( 1/M \)-corrections in ref. [17], and for the long-range terms in ref. [18]. In the latter case one is dealing with 3N-diagrams which were divided in ref. [11] into classes of \( 2\pi \)-exchange topology, \( 2\pi 1\pi \)-exchange topology, and ring topology. For these topologies the selfclosings of a nucleon-line and the concatenations of any two nucleon-lines to an in-medium loop had to be worked out to together with the summation/integration

\[ \int_0^\lambda dr \int_0^{\pi/2} d\psi \]
over the filled Fermi-sea of density \( \rho = 2k_f^3/3\pi^2 \). The momentum- and \( k_f \)-dependent potentials associated with the isospin operators (1 and \( \vec{\tau}_1 \cdot \vec{\tau}_2 \)) and five independent spin-structures (1, \( \vec{\sigma}_1 \cdot \vec{\sigma}_2, \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \vec{q}, i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{p}), \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p}') \) could all be expressed in terms of functions, which were either given in closed analytical form or required at most one numerical integration. In order to obtain for the (non-factorizable) 3N-diagrams such an expedient form it was crucial to invert the order the original loop-integration and the added Fermi-sphere integral. Moreover, the method of dimensional regularization, as it was implicitly used in ref. [1], could be recovered by subtracting asymptotic constants from the integrands of \( \int_0^\infty dl \).

![Figure 1: 2π-exchange topology, 2π1π-exchange topology and ring topology which comprise the long- and intermediate-range chiral 3N-forces at subsleading order N^4LO.](image)

The purpose of the present paper is to extend the calculation of the in-medium NN-potential \( V_{\text{med}} \) to the subsleading chiral 3N-forces at order N^4LO. The long-range 2π-exchange component, symbolized by the left diagram in Fig. 1, has already been treated in section 4 of ref. [18] through appropriate contributions to the two structure functions \( \tilde{g}_{\pi}(q_2) \) and \( \tilde{h}_{\pi}(q_2) \). As indicated by the notation, these structure functions are equal to \( f_{\pi}^2 \) times the isoscalar non-spin-flip and isovector spin-flip πN-scattering amplitudes at zero pion-energy \( \omega = 0 \) and squared momentum-transfer \( t = -q_2^2 \). The present paper is organized as follows. We start in section 2 with the computation of \( V_{\text{med}} \) from the intermediate-range 2π1π-exchange component, symbolized by the middle diagram in Fig. 1. In comparison to section 3 of ref. [18] one encounters at N^4LO a richer spin- and momentum-dependence for this part of the chiral 3N-force, and 12 instead of 8 functions \( f_j(q_1) \) are needed to represent all diagrams belonging to this topology. The contributions to \( V_{\text{med}} \) as they arise from selfclosures, vertex-correction by 1π-exchange, vertex-correction by 2π-exchange, and double-exchange are given by semi-analytical expressions that comply with this extended structure. Note that ref. [1] has concluded from a study of the 3N-potential in coordinate space at the equilateral triangle configuration, that the N^4LO corrections to the intermediate-range topologies are numerically large and dominate in most cases over the nominally leading N^3LO terms. This feature could be traced back to the large coefficients \( c_{2,3,4} \), which reflect the importance of the \( \Delta(1232) \)-isobar coupled to the \( \pi N \)-system. At N^4LO the 3N-diagrams belonging to the ring topology, symbolized by the right diagram in Fig. 1, fall into three classes according to their scaling with \( g_A^3 \). Section 3 is devoted to the simplest ring interaction proportional to \( g_A^3 c_{1,2,3,4} \) and the contributions to \( V_{\text{med}} \) from selfclosures and concatenations are given in three subsections. After angular integration the remaining double-integral \( \int dl_0 dl \) is treated in polar coordinates and regularized by a (euclidean) cutoff \( \lambda \). In this form the \( \lambda^2 \)-divergence can be easily subtracted and the subsequent logarithmic piece \( \sim \ln(m_\pi/\lambda) \) is isolated. A good check is provided by the fact that the total \( \lambda^2 k_f^2 \)-divergence is of isoscalar central type and thus can be absorbed on the 3N short-distance parameter \( c_E \). In section 4 the analogous calculations are carried out for the more involved ring interaction proportional to \( g_A^3 c_{1,2,3,4} \). Finally, one considers in section 5 the ring interaction proportional to \( g_A^3 c_{1,2,3,4} \), which consists of a large number of terms with different isospin-, spin- and momentum-dependence. At that point we elaborate also a bit on euclidean
loop-integrals over four or three pion-propagators. The self-closing contributions to \( V_{\text{med}} \) are given in closed analytical form in subsection 5.1 and one observes that these central, spin-spin and tensor potentials linear in density \( \rho \) depend either on \( c_2 + c_3 \) or on \( c_1 \) and \( c_3 \). Concerning the contributions to \( V_{\text{med}} \) from concatenations, we present in subsection 5.2 the pertinent expressions only for two selected (yet simple) terms from the ring interaction \( \sim g_A^4 c_{1,2,3,4} \). These give rise either to an isovector spin-orbit potential, or to isoscalar and isovector central and spin-spin potentials. A complete list of the lengthy formulas for the remaining contributions to \( V_{\text{med}} \) from the concatenations of the 3N-ring interaction \( \sim g_A^4 c_{1,2,3,4} \) can be obtained from the author upon request.

In summary, after eventual partial-wave projection the presented results for \( V^\text{med}_N \) are suitable for an approximate implementation of the subleading chiral 3N-forces of intermediate range into nuclear many-body calculations.

2 Two-pion-one-pion exchange topology

The \( 2\pi 1\pi \)-exchange 3N-interaction arises from a large set of loop-diagrams, and according to eq. (3.1) in ref. [4] it can be written in the general form:

\[
V_{3N} = \frac{g_A^4}{256\pi f_\pi^6 m_\pi^2 + \frac{g_3^2}{q_3^2}} \left\{ \bar{q}_1 \cdot \bar{q}_3 \left[ \bar{q}_2 \cdot q_1 \cdot q_3 f_1(q_1) + \bar{q}_2 \cdot q_1 f_2(q_1) + \bar{q}_2 \cdot q_3 f_3(q_1) \right] \right. \\
+ \bar{q}_2 \cdot \bar{q}_3 \left[ \bar{q}_1 \cdot q_1 \cdot q_3 f_4(q_1) + \bar{q}_1 \cdot q_3 f_5(q_1) + \bar{q}_1 \cdot q_1 \cdot q_3 f_6(q_1) \right] \\
+ \bar{q}_2 \cdot q_1 f_7(q_1) + \bar{q}_2 \cdot q_3 f_8(q_1) + \bar{q}_2 \cdot q_3 f_9(q_1) \\
+ \left( \bar{q}_1 \times \bar{q}_2 \right) \cdot \bar{q}_3 \left[ (\bar{q}_1 \times \bar{q}_2) \cdot q_1 f_{10}(q_1) + f_{11}(q_1) \right] + \bar{q}_1 \cdot q_1 \cdot q_3 \bar{q}_2 \cdot q_1 f_{12}(q_1) \right\}, \tag{1}
\]

where \( \bar{q}_j \) denotes the momentum-transfer at nucleon \( j \in \{1,2,3\} \), and \( q_1 + q_2 + q_3 = 0 \) holds due to momentum-conservation. Since a common prefactor \( g_A^4/(256\pi f_\pi^6) \) has been pulled out in eq. (1), the contributions to the reduced functions \( f_j(s) \) at \( N^3\text{LO} \) read according to eq. (3.2) in ref. [4]:

\[
f_1(s) = \frac{m_\pi}{s^2} (1 - 2 g_A^2) - \frac{g_A^2 m_\pi}{4 m_\pi^2 + s^2} + \left[ 1 + g_A^2 + \frac{4 m_\pi^2}{s^2} (2 g_A^2 - 1) \right] A(s), \tag{2}
\]

\[
f_2(s) = f_7(s) = (4 m_\pi^2 + 2 s^2) A(s), \tag{3}
\]

\[
f_3(s) = \left[ 4 (1 - 2 g_A^2) m_\pi^2 + (1 - 3 g_A^2) s^2 \right] A(s), \tag{4}
\]

\[
f_5(s) = - s^2 f_4(s) = 2 g_A^2 s^2 A(s), \tag{5}
\]

\[
f_{11}(s) = - \left( 2 m_\pi^2 + \frac{s^2}{2} \right) A(s), \tag{6}
\]

\[
f_{6,8,9,10,12}(s) = 0, \tag{7}
\]

with the heavy-baryon loop-function

\[
A(s) = \frac{1}{2 s} \arctan \frac{s}{2 m_\pi}. \tag{8}
\]

Likewise, one extracts from eq.(3.3) in ref. [4] the following contributions to the reduced functions \( f_j(s) \) at \( N^4\text{LO} \):

\[
f_1(s) = \frac{16 c_4}{3 \pi} \left\{ (4 - g_A^2) m_\pi^2 \frac{m_\pi}{s^2} + \left[ (g_A^2 - 4) \frac{m_\pi^2}{s^2} + \frac{1 - g_A^2}{2} - \frac{3 m_\pi^2}{4 m_\pi^2 + s^2} \right] L(s) \right\}, \tag{9}
\]

\[
f_3(s) = \frac{16 c_4}{3 \pi} \left[ (g_A^2 - 1) m_\pi^2 + (g_A^2 - 4) s^2 - \frac{12 m_\pi^4}{4 m_\pi^2 + s^2} \right] L(s), \tag{10}
\]

\[
f_5(s) = - s^2 f_4(s) = \frac{16 c_4}{\pi} s^2 L(s), \tag{11}
\]

3
\[ f_6(s) = \frac{8}{3\pi} \left( (6c_1 + c_2 - 3c_3) \frac{m^2}{s^2} + \left( (3c_3 - 6c_1 - c_2) \frac{m^2}{s^2} + \frac{c_2}{2} + \frac{3(2c_1 + c_3) m^2}{4m^2 + s^2} \right) L(s) \right), \]

\[ f_7(s) = \frac{8}{\pi \cdot g^2_A} \left[ \left( 2\left( c_3 + \frac{c_2}{3} - 2c_1 \right) m^2 + \left( 2c_2 + c_3 \right) s^3 \right) L(s) + \left( (8\pi f^2 \bar{c}_{14} - \frac{5c_2}{18} - c_3) \frac{s^4}{2} \right) \right], \]

\[ f_9(s) = \frac{8}{\pi} \left[ (8c_1 - c_2 - 4c_3) m^2 - (3c_2 + 13c_3) \frac{s^2}{4} - \frac{4(2c_1 + c_3) m^2}{4m^2 + s^2} \right] L(s), \]

\[ f_{10}(s) = f_{12}(s) = \frac{4c_4}{\pi} L(s), \]

\[ f_{2,8,11}(s) = 0. \]

with the frequently occurring logarithmic loop-function

\[ L(s) = \sqrt{\frac{4m^2 + s^2}{s}} \ln \frac{s + \sqrt{4m^2 + s^2}}{2m}, \]

Note that we have supplied in eq.(13) through the last term proportional to \( s^2/2 \) that particular polynomial piece\(^2\) which cannot be absorbed on the short-distance parameters \( c_D \) and \( c_E \). One notices from eqs.(7,16) that there is yet no contribution to \( f_8(s) \), but the corresponding structure \( \vec{\sigma}_2 \cdot \vec{q}_3 \vec{q}_1 \cdot \vec{q}_3 \) will arise once explicit \( \Delta(1232) \)-isobars are considered in the derivation of the chiral \( 2\pi\) interaction.

### 2.1 Contributions to in-medium NN-potential

Now we turn to the contributions of the \( 2\pi\) interaction NN-potential \( V_{NN} \) written in eq.(1) to the in-medium NN-potential \( V_{med} \). Only the selfclosings of nucleon line 1 gives a non-vanishing spin-isospin trace, and after relabeling 3 \( \rightarrow \) 1 one obtains the contribution

\[ V_{med}^{(0)} = \frac{g_A^{4}k^{3}f^{0}}{3(4\pi f_{\pi}^{2})^{3}} \frac{\vec{\sigma}_1 \cdot \vec{q}_2}{m^2 + q^2} \frac{\vec{q}_1 \cdot \vec{q}}{24\pi^4 f_{\pi}^{4}} \frac{\vec{q}_1 \cdot \vec{q}_2}{2 \vec{q}_2 \cdot \vec{q}_3 (6c_1 - c_2 - 5c_3)}, \]

which is of the form: \( 1\pi\) interaction NN-interaction times a factor linear in density \( \rho = 2k_f^3/3\pi^2 \).

The last expression in eq.(18) comes from evaluating \( f_9(s) \) in eq.(14) at \( s = 0 \). In all forthcoming formulas for \( V_{med} \) we denote by \( \vec{q} = \vec{p}' - \vec{p} \) the momentum-transfer for the on-shell scattering process \( N_1(\vec{p}) + N_2(-\vec{p}) \rightarrow N_1(\vec{p}') + N_2(-\vec{p}') \) in the nuclear matter rest-frame. On the other hand the vertex corrections by \( 1\pi\)-exchange, apparent in eq.(1) through the factor \( \vec{\sigma}_3 \cdot \vec{q}_3/(m^2 + q^2) \), produce the contribution

\[ V_{med}^{(1)} = \frac{g^{4}}{8\pi f_{\pi}^{2}} \left\{ \left( 2m^2 \Gamma_0 - \frac{4k_f^3}{3} \right) \frac{\vec{\sigma}_1 \cdot \vec{q}_2}{m^2 + q^2} \frac{\vec{q}}{2 \vec{q}_2 \cdot \vec{q}_3 (6c_1 - c_2 - 5c_3)} \right\}, \]

\(^2\)This important additional information was provided by H. Krebs. The value of the low-energy constant \( \bar{c}_{14} \) as extracted from \( \pi N \)-scattering is \( 1.52 \text{ GeV}^{-3} \) or \( \bar{c}_{14} = 1.18 \text{ GeV}^{-3} \).
with the \((p, k_f)\)-dependent functions \(\Gamma_0, \tilde{\Gamma}_1 = \Gamma_0 + \Gamma_1, \Gamma_2\) and \(\tilde{\Gamma}_3 = \Gamma_0 + 2\Gamma_1 + \Gamma_3\) defined in the appendix of ref. [17]. Moreover, the vertex corrections by \(2\pi\)-exchange, represented by the expression in curly brackets of eq.\,(1), can be summarized as the \(1\pi\)-exchange NN-interaction times a \((p, q, k_f)\)-dependent factor

\[
V^{(2)}_{\text{med}} = \frac{g_A^4}{(8\pi f^2)^3} \frac{\vec{r}_1 \cdot \vec{r}_2}{m^2 + q^2} \sigma_1 \cdot q \vec{q} \cdot \vec{q} \left[ S_1(p, k_f) + q^2 S_2(p, k_f) \right].
\]

(20)

The two auxiliary functions \(S_{1,2}(p, k_f)\) are computed as integrals over \(f_2(s)\) in the following way:

\[
S_1(p, k_f) = \int_{p-k_f}^{p+k_f} ds \frac{s}{p} \left[ k_f^2 - (p - s)^2 \right] \left\{ 2s^2 f_{12}(s) - f_3(s) - f_5(s) - f_0(s) \\
+ \frac{1}{8p^2} [(p + s)^2 - k_f^2] (f_2(s) + f_7(s) - 4f_{11}(s)) + \frac{1}{24p^2} [k_f^2 - (p - s)^2] \\
\times (s^2 + 4sp + p^2 - k_f^2) [4f_{10}(s) - 2f_{12}(s) - f_1(s) - f_4(s) - f_6(s)] \right\},
\]

(21)

\[
S_2(p, k_f) = \int_{p-k_f}^{p+k_f} ds \frac{s}{8p^3} \left[ k_f^2 - (p - s)^2 \right] [(p + s)^2 - k_f^2] \left\{ f_8(s) \\
+ \frac{1}{4p^2} (s^2 + p^2 - k_f^2) [4f_{10}(s) - 2f_{12}(s) - f_1(s) - f_4(s) - f_6(s)] \right\}.
\]

(22)

Finally, there is the contribution \(V^{(3)}_{\text{med}}\) from the double-exchange. We separate it into an isoscalar part:

\[
V^{(3)}_{\text{med}} = \frac{3g_A^4}{(8\pi f^2)^3} \left[ \vec{\sigma}_1 \cdot \vec{\sigma}_2 \left( 2I_{2,2} - 2I_{3,2} - H_{1,2} - \tilde{I}_{1,2} \right) + \vec{\sigma}_1 \cdot \vec{q} \vec{q} \cdot \vec{q} \left( \frac{H_{1,1} + \tilde{I}_{1,4}}{2} - I_{2,4} - I_{3,5} \right) \right. \\
\left. + (\vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p}' \vec{\sigma}_2 \cdot \vec{p}') \left( I_{2,3} - I_{3,3} - \frac{H_{1,3} + \tilde{I}_{1,3}}{2} \right) \right],
\]

(23)

and an isovector part:

\[
V^{(3)}_{\text{med}} = \frac{g_A^4 \tilde{r}_1 \cdot \tilde{r}_2}{(8\pi f^2)^3} \left\{ 2m^2 I_{5,0} - 2H_{5,0} + \frac{q^2}{2} (H_{4,1} + \tilde{I}_{4,1}) - p^2 (H_{4,3} + \tilde{I}_{4,3}) - 3H_{4,2} - 3\tilde{I}_{4,2} \right. \\
+ i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p} \times \vec{q}) \left[ H_{10,1} + \tilde{I}_{10,1} - \frac{1}{2} (H_{4,1} + \tilde{I}_{4,1}) - 2I_{11,1} \right] \\
+ \vec{\sigma}_1 \cdot \vec{\sigma}_2 \left[ 2I_{7,2} - H_{6,2} + \tilde{I}_{7,2} + H_{8,2} + \tilde{I}_{8,2} - 2I_{9,2} + 4H_{10,2} + 2\tilde{I}_{10,2} - 2I_{12,2} - 4m^2 I_{12,2} + 2p^2 (H_{10,3} + \tilde{I}_{10,3} - 2I_{11,3}) - q^2 (H_{10,1} + \tilde{I}_{10,1} - 2I_{11,4}) \right] \\
+ \vec{\sigma}_1 \cdot \vec{q} \vec{q} \cdot \vec{q} \left[ \frac{H_{6,1} + \tilde{I}_{6,4} + \tilde{I}_{8,5}}{2} - I_{7,4} + H_{8,0} - H_{8,1} - I_{9,5} + H_{10,1} + \tilde{I}_{10,4} \\
- 2I_{11,4} + H_{12,1} + \tilde{I}_{12,4} + 2m^2 (2I_{12,4} + I_{12,5} - 2I_{12,0}) \right] + (\vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 \cdot \vec{p} + \vec{\sigma}_1 \cdot \vec{p}' \vec{\sigma}_2 \cdot \vec{p}') \\
\times \left[ \frac{H_{8,3} - H_{6,3}}{2} + I_{7,3} - I_{9,3} - H_{10,3} - \tilde{I}_{10,3} + 2I_{11,3} + H_{12,3} - \tilde{I}_{12,3} - 2m^2 I_{12,3} \right].
\]

(24)

The double-indexed functions \(H_{j,\nu}(p)\) are defined by:

\[
H_{j,0}(p) = \frac{1}{2p} \int_{p-k_f}^{p+k_f} ds s f_2(s) [k_f^2 - (p - s)^2],
\]

(25)
\[ H_{j,1}(p) = \frac{1}{8p^3} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) [k_f^2 - (p - s)^2] [(p + s)^2 - k_f^2], \]  
\[ H_{j,2}(p) = \frac{1}{48p^3} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) [k_f^2 - (p - s)^2]^2 (s^2 + 4sp + p^2 - k_f^2), \]  
\[ H_{j,3}(p) = \frac{1}{16p^5} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) [k_f^2 - (p - s)^2] [(p + s)^2 - k_f^2] (p^2 + s^2 - k_f^2). \]

The other double-indexed functions \( I_{j,\nu}(p, q) \) are defined by:

\[ I_{j,0}(p, q) = \frac{1}{2q} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) \ln \frac{qX + 2\sqrt{W}}{(2p + q)[m_\pi^2 + (s - q)^2]}, \]  
\[ I_{j,1}(p, q) = \frac{1}{4p^2 - q^2} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) \left[ \frac{p(s^2 + m_\pi^2) - \sqrt{W}}{q^2} + \frac{p^2 + k_f^2 - s^2}{2p} \right], \]  
\[ I_{j,2}(p, q) = \frac{1}{8q^2} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) \left\{ s(m_\pi^2 + s^2 + q^2) - p \left( m_\pi^2 + s^2 + \frac{3q^2}{4} \right) \right. \right. \]  
\[ - \frac{1}{2q} \left[ m_\pi^2 + (s + q)^2 \right] \left[ m_\pi^2 + (s - q)^2 \right] \ln \frac{qX + 2\sqrt{W}}{(2p + q)[m_\pi^2 + (s - q)^2]} \]  
\[ - \frac{X \sqrt{W}}{4p^2 - q^2} - \frac{(k_f^2 - s^2)^2}{p} + \frac{p}{4p^2 - q^2} \left( m_\pi^2 + 2k_f^2 - s^2 + \frac{q^2}{2} \right)^2 \}, \]  
\[ I_{j,3}(p, q) = \frac{1}{(4p^2 - q^2)^2} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) \left\{ \frac{X \sqrt{W}}{q^2} + \frac{q^2}{8p^3} (k_f^2 - s^2)^2 - \frac{3pq^2}{8} \right. \right. \]  
\[ + \frac{p}{q^2} (2s^2 + m_\pi^2)(2p^2 + s^2 - 2k_f^2 - m_\pi^2) + \frac{p}{2} (2k_f^2 - 3m_\pi^2 + p^2 - s^2) \]  
\[ + \frac{1}{4p} \left[ s^2 (2m_\pi^2 + q^2 - 4s^2) + k_f^2 (10s^2 - 2m_\pi^2 - 3q^2) - 6k_f^4 \right] \}, \]  
\[ I_{j,4}(p, q) = \frac{1}{4q^4} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) \left\{ \frac{X}{(4p^2 - q^2)^2} \left[ \sqrt{W} (3q^2 - 4p^2) - 8pq^2 X \right] \right. \]  
\[ + \left[ \frac{q^3}{2} + q(s^2 - m_\pi^2) - \frac{3}{2q} (s^2 + m_\pi^2)^2 \right] \ln \frac{qX + 2\sqrt{W}}{(2p + q)[m_\pi^2 + (s - q)^2]} \]  
\[ - \frac{p}{4p^2 - q^2} \left[ 16k_f^4 + 8k_f^2 (2m_\pi^2 + q^2 - 2s^2) + 3m_\pi^4 + 3s^4 + 2q^2 (m_\pi^2 - s^2) + 10s^2 m_\pi^2 \right] \]  
\[ + s(3s^2 - q^2 + 3m_\pi^2) + 2p^2 - p (4k_f^2 + 4m_\pi^2 + q^2) + \frac{2}{p} (k_f^2 + q^2 - s^2)(s^2 - k_f^2) \}, \]  
\[ I_{j,5}(p, q) = -I_{j,4}(p, q) + \frac{1}{2q^2} \int_{p-k_f}^{p+k_f} ds \, s f_j(s) \left[ \frac{q^2 - s^2 - m_\pi^2}{q} \ln \frac{qX + 2\sqrt{W}}{(2p + q)[m_\pi^2 + (s - q)^2]} + \frac{k_f^2 - (p - s)^2}{p} \right]. \]

with the auxiliary polynomials

\[ X = m_\pi^2 + 2(k_f^2 - p^2) + q^2 - s^2, \]
\[ W = k_f^2 q^4 + p^2 (m_\pi^2 + s^2)^2 + q^2 [(k_f^2 - p^2)^2 + m_\pi^2 (k_f^2 + p^2) - s^2 (k_f^2 + p^2 + m_\pi^2)]. \]
Furthermore, the functions \( \tilde{I}_{j,\nu}(p,q) \) with \( j = 1, 4, 6, 8, 10, 12 \) appearing in eqs.(23,24) are computed analogously by replacing in the integrand \( f_j(s) \) by \( \tilde{f}_j(s) = (s^2 - m^2 - q^2) f_j(s) \). The decomposition into \( H_{j,\nu} \) and \( I_{j,\nu} \) is obtained by canceling momentum-factors against a pion-propagator, while \( \tilde{I}_{j,\nu} \) takes care of \( s^2 \)-dependent remainder terms.

3 Ring interaction proportional to \( g_A^0 \)

Next, we turn to the 3N-ring interaction at \( N^4 \)LO, which consists of three pieces with different dependence on the axial-vector coupling constant: \( g_A^{2n}, n = 0,1,2 \). The \( g_A^0 \)-part can be obtained directly from the well-known Feynman rules for the \( \pi\pi NN \) Tomozawa-Weinberg vertex and the second-order \( \pi\pi NN \)-contact vertex proportional to \( c_{1,2,3,4} \). Altogether, the 3N-ring interaction proportional to \( g_A^0 c_{1,2,3,4} \) is given by a euclidean loop-integral of the form

\[
V_{3N} = -\frac{1}{f^2_{\pi}} \int_0^\infty dl_0 \int \frac{d^3l_2}{(2\pi)^3} \frac{l_2^2}{(l_0^2 + l_2^2)(m^2 + l_2^2)(m^2 + l_3^2)} \times \left\{ \tilde{\tau}_2 \cdot \tilde{\tau}_3 \left[ c_2 c_3 + (c_2 + c_3) l_0^2 + c_3 l_2 \cdot \tilde{l}_3 \right] + \frac{c_4}{4} \tilde{\tau}_1 \cdot (\tilde{\tau}_2 \times \tilde{\tau}_3) \tilde{\tau}_1 \cdot (\tilde{l}_3 \times \tilde{l}_2) \right\}, \tag{36}
\]

with \( m = \sqrt{m^2 + l_0^2} \) and one has to set \( \tilde{l}_1 = \tilde{l}_2 - \tilde{q}_3 \) and \( \tilde{l}_3 = \tilde{l}_2 + \tilde{q}_1 \). Alternatively, one can take the (Fourier-transformed) coordinate-space potential in eq.(4.8) of ref. [1] and translate spatial gradients back to momentum factors \( \tilde{l}_{1,2,3} \). Note that the 4-dimensional loop integral in eq.(36) is quadratically divergent and therefore the 3N-ring interaction \( V_{3N} \) requires a regularization (e.g. by an ultraviolet cutoff) and a renormalization (by absorbing cutoff-dependent pieces on the 3N short-distance parameters \( c_E \) and \( E_{1,...,10} \) [20]).

3.1 In-medium NN-potential from selfclosing of nucleon-lines

Only the selfclosing of nucleon line 1 gives a non-vanishing spin-isospin trace, and after relabeling \( 3 \to 1 \) one recognizes an isovector central potential \( \sim k^3 \tilde{\tau}_1 \cdot \tilde{\tau}_2 \). Evaluating the pertinent loop-integral in spherical coordinates \( l_0 = r \cos \psi, l_2 = r \sin \psi, \hat{l}_2 \cdot \hat{q} = \cos \theta \) and introducing a cutoff \( \lambda \) for the radial integral \( \int \rho^\lambda dr \), one gets the following contribution to the in-medium NN-potential:

\[
V_{\text{med}}^{(0)} = k^2 \frac{\tilde{\tau}_1 \cdot \tilde{\tau}_2}{48\pi^4 f^6_\pi} \left\{ \left[ (2c_1 - 3c_2) m^2 + c_2 c_3 q^2 \right] \ln \frac{m_\pi}{\lambda} + \left( \frac{3c_2}{2} - 2c_1 + \frac{13c_3}{3} \right) \frac{m^2_\pi}{4} \right. \\
+ \left. \left( \frac{13c_2}{8} + \frac{11c_3}{3} \right) \frac{q^2}{12} \right\} L(q), \tag{37}
\]

with the function \( L(q) \) defined in eq.(17). Note that the power divergence proportional to \( \lambda^2 k^2 \) has been dropped in eq.(37), but it will be considered in the total balance at the end of this section. Note also that the coefficient \( c_2/4 + c_3/3 \) appears twice, such that the chiral limit \( m_\pi \to 0 \) of \( V_{\text{med}}^{(0)} \) exists.

3.2 In-medium potential from concatenations \( N_3 \) on \( N_2 \) and \( N_2 \) on \( N_3 \)

Next, one has to work out for \( V_{3N} \) in eq.(36) the six possible concatenations of two nucleon-lines and their mirror graphs. The proper assignments of \( \tilde{l}_1, \tilde{l}_2, \tilde{l}_3, \) with \( \tilde{l} \) the unconstrained loop-momentum and \( \tilde{l}_4 \) from the interior of a Fermi-sphere \( |l_4| < k_f \), are given for each concatenation in Table 1.
Tab. 1: Assignment of pion momenta, where \( \vec{l} \) is unconstrained and \( |\vec{l}| < k_f \) from a Fermi sphere.

The integral over the Fermi-sphere and the angular part of the loop-integral can always be solved analytically in terms of the following functions:

\[
\bar{\Gamma}_0(l) = k_f - \bar{m} \left[ \arctan \left( \frac{k_f + l}{\bar{m}} \right) + \arctan \left( \frac{k_f - l}{\bar{m}} \right) \right] + \frac{\bar{m}^2 + k_f^2 - l^2}{4l} \ln \frac{\bar{m}^2 + (k_f + l)^2}{\bar{m}^2 + (k_f - l)^2},
\]

\[
\bar{\Gamma}_1(l) = \frac{k_f^3}{6l^2} (\bar{m}^2 + k_f^2 + l^2) - \frac{1}{16\bar{m}^3} \left[ \bar{m}^2 + (k_f + l)^2 \right] \left[ \bar{m}^2 + (k_f - l)^2 \right] \ln \frac{\bar{m}^2 + (k_f + l)^2}{\bar{m}^2 + (k_f - l)^2},
\]

\[
\bar{\Gamma}_2(l) = \frac{k_f^3}{9} - \frac{\bar{m}^2}{3} \bar{\Gamma}_0(l) + \frac{1}{6} (k_f^2 + \bar{m}^2 - l^2) \bar{\Gamma}_1(l),
\]

\[
\bar{\Gamma}_3(l) = \frac{k_f^3}{3l^2} + \frac{l^2 - \bar{m}^2 - k_f^2}{2l^2} \bar{\Gamma}_1(l),
\]

\[
\Lambda(l) = \frac{1}{4p} \ln \frac{\bar{m}^2 + (l + p)^2}{\bar{m}^2 + (l - p)^2},
\]

\[
\Omega(l) = \frac{1}{q\sqrt{B + q^2l^2}} \ln q + \sqrt{B + q^2l^2},
\]

with the abbreviation \( B = [\bar{m}^2 + (l + p)^2][\bar{m}^2 + (l - p)^2] \). For remaining integration over \( dl_0dl \) one chooses polar coordinates \( l_0 = r \cos \psi, l = r \sin \psi \) and sets a radial cutoff \( \lambda \). By performing these calculational steps one obtains from the concatenations \( N_3 \) on \( N_2 \) and \( N_2 \) on \( N_3 \) an isoscalar central potential of the form

\[
V^{(1)}_{\text{med}} = \frac{3}{4\pi^3 f_\pi^6} \int_0^\lambda dr \int_0^{\pi/2} d\psi \left\{ \frac{1}{4p} \left[ \bar{m}^2 + (l + p)^2 \right] \ln \frac{\bar{m}^2 + (l + p)^2}{\bar{m}^2 + (l - p)^2} \right\}.
\]

The purpose of the subtraction term in the second line is to remove a power divergence proportional to \( \lambda^2 k_f^3 \). After that the double-integral in eq.(44) has only a logarithmic dependence on the cutoff:

\[
\frac{\pi k_f^3}{48} \left[ c_2 \left( 3m^2_\pi + \frac{3k_f^2}{10} + \frac{p^2}{2} + \frac{q^2}{4} \right) - 4c_1 m^2_\pi + c_3 \left( 6m^2_\pi + \frac{2k_f^2}{5} + \frac{2p^2}{3} + q^2 \right) \right] \ln \frac{m_\pi}{\lambda},
\]

and this detailed knowledge may be useful for numerical checks.

The last \( c_4 \)-term in eq.(36) produces in the same way a contribution to the isovector spin-orbit potential

\[
V^{(1)}_{\text{med}} = \frac{c_4 \vec{r}_1 \cdot \vec{r}_2}{8\pi^5 f_\pi^6} i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{p}) \int_0^\lambda dr \int_0^{\pi/2} d\psi \left[ \frac{1}{4p^2 - q^2} \left[ \Lambda(l) + \left( \vec{p}^2 - l^2 - \bar{m}^2 - \frac{q^2}{2} \right) \Omega(l) \right] \right],
\]

with a large-\( \lambda \) behavior of the double-integral: \(- (\pi k_f^3/144) \ln(m_\pi/\lambda)\).
3.3 In-medium NN-potential from remaining four concatenations

The other four concatenations, \( N_3 \) on \( N_1 \), \( N_1 \) on \( N_3 \), \( N_1 \) on \( N_2 \), and \( N_2 \) on \( N_1 \), applied to the \( c_{1,2,3} \)-term in eq.(36) give rise to an isovector central potential of the form

\[
V_{\text{med}}^{(cc)} = \frac{\vec{t}_1 \cdot \vec{t}_2}{4\pi^3 f_\pi^2} \int_0^\infty \! dr \int_0^{\pi/2} \! d\psi \left\{ l_0^2 l_0 \left[ 4c_1 m_{\pi}^2 + 2(c_2 + c_3)l_0^2 \right] \vec{\Gamma}_0(l) \Omega(l) \right. \\
+ c_3 \vec{\Gamma}_1(l) \left[ \Lambda(l) + (l^2 - \bar{m}^2 - \bar{p}^2)\Omega(l) \right] \} - \frac{k_3^2}{3} (c_3 + c_2 \cos^2 \psi) \sin^2 2\psi \right\},
\]

with a suitable subtraction term to have only a logarithmic \( \lambda \)-dependence of the double-integral:

\[
\frac{\pi k_3^2 \lambda^2}{24} \left[ c_2 \left( 3m_{\pi}^2 + \frac{3k_3^2}{10} + \frac{p^2}{2} + \frac{q^2}{4} \right) + \frac{2}{2} \right] \frac{c_3}{4c_1 m_{\pi}^2 + 2(c_2 + c_3)l_0^2} \ln \frac{m_{\pi}}{\lambda}.
\]

Under the same calculational treatment, the \( c_{4} \)-term in eq.(36) produces a further contribution to the isovector spin-orbit potential

\[
V_{\text{med}}^{(cc)} = \frac{c_4 \vec{t}_1 \cdot \vec{t}_2}{8\pi^3 f_\pi^2} \int_0^\infty \! dr \int_0^{\pi/2} \! d\psi \left\{ l_0^2 l_0 \left[ \frac{c_2}{3} \left( c_2 + c_3 \right) + \frac{c_3}{3} \right] \vec{\Gamma}_1(l) \right. \\
- \frac{k_3^2 \lambda^2}{192\pi^4 f_\pi^6} \left[ - \vec{t}_1 \cdot \vec{t}_2 (c_2 + 2c_3) + 3 \left( \frac{c_2}{2} + c_3 \right) + \vec{t}_1 \cdot \vec{t}_2 (c_2 + 2c_3) \right] = \frac{k_3^2 \lambda^2}{128\pi^4 f_\pi^6} (c_2 + 2c_3),
\]

such that the remaining isoscalar piece can be absorbed on the 3N short-distance parameter \( c_E \). This perfect matching gives an a posteriori justification to drop or subtract the \( \lambda^2 \)-divergences at any place. In the case of the pieces proportional to \( \ln(m_{\pi}/\lambda) \) one can verify that these can be absorbed on the parameters \( E_{1,...,10} \) of the subleading 3N-contact interaction \([20]\) (see also eq.(49) in ref. \([17]\)).

4 Ring interaction proportional to \( g_{\Lambda}^2 \)

The 3N-ring interaction proportional to \( g_{\Lambda}^2 c_{1,2,3,4} \) can be inferred from the coordinate-space potential written in eq.(4.7) of ref. \([4]\). By exploiting the permutational symmetry (and parity-invariance) one can obtain the following somewhat simpler form

\[
V_{3N} = -\frac{g_{\Lambda}^2}{f_\pi^6} \int_0^\infty \! dl_0 \int d^3l_2 \frac{1}{(2\pi)^3 (\bar{m}^2 + l_0^2)(\bar{m}^2 + l_0^2)} \left\{ \vec{\sigma}_2 \cdot \vec{\sigma}_3 \vec{t}_2 \cdot \vec{t}_3 + \frac{c_4}{2} \left[ \vec{t}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3) \vec{l}_1 \cdot \vec{l}_2 \vec{\sigma}_3 \cdot \vec{l}_3 + \vec{t}_1 \cdot (\vec{\sigma}_2 + \vec{\sigma}_3) \right] \right. \\
\times \left[ 2c_1 m_{\pi}^2 (c_2 + c_3)l_0^2 \bar{c}_3 \vec{l}_2 \cdot \vec{l}_3 + \frac{c_4}{2} \left[ \vec{t}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3) \vec{l}_1 \cdot \vec{l}_2 \vec{\sigma}_3 \cdot \vec{l}_3 + \vec{t}_1 \cdot (\vec{\sigma}_2 + \vec{\sigma}_3) \right] \right] \right\} \right\},
\]

which involves only three different isospin-operators: \( \vec{t}_2 \cdot \vec{t}_3, \vec{t}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3) \) and \( \vec{t}_1 \cdot (\vec{\sigma}_2 + \vec{\sigma}_3) \). Again, \( \bar{m} \) stands for \( \sqrt{m_{\pi}^2 + l_0^2} \) and one has to set \( \vec{l}_1 = \vec{l}_2 - \vec{q}_3 \) and \( \vec{l}_3 = \vec{l}_2 + \vec{q}_3 \).
4.1 In-medium NN-potential from selfclosing of nucleon-lines

Following the same procedure as in subsection 3.1, one obtains from the selfclosing of nucleon-line 1 (providing a non-vanishing spin-isospin trace) a further contribution to the isovector central potential:

\[
V_{\text{med}}^{(0)} = \frac{g_A^2 k_f^3}{48 \pi^4 f^6_\pi} \left\{ \left[ 3 m^2_\pi (4 c_1 - c_2 - 6 c_3) - \frac{5 q^2}{6} (c_2 + 4 c_3) \right] \ln \frac{m_\pi}{\lambda} + \left( c_1 + \frac{c_2}{12} + \frac{11 c_3}{6} \right) m^2_\pi \right. \\
+ \left( \frac{41 c_2}{8} + 43 c_3 \right) q^2 + \left[ 4 m^2_\pi (6 c_1 - c_2 - 7 c_3) m^2_\pi - \frac{5 q^2}{6} (c_2 + 4 c_3) + \frac{2 m^2_\pi q^2}{4 m^2_\pi + q^2} (2 c_1 - c_3) \right] L(q) \right\}
\]  
(52)

where the \( \lambda^2 \)-divergence has been dropped. Again, the appearance of the same coefficient \( c_2 + 4 c_3 \) in the first and second line of eq.(52) guarantees the chiral limit \( m_\pi \to 0 \) of \( V_{\text{med}}^{(0)} \).

4.2 In-medium potential from concatenations \( N_3 \) on \( N_2 \) and \( N_2 \) on \( N_3 \)

The concatenations \( N_3 \) on \( N_2 \) and \( N_2 \) on \( N_3 \) give for the first term with isospin-factor \( \bar{\tau}_2 \bar{\tau}_3 \) in eq.(51) an isoscalar central potential of the form

\[
V_{\text{med}}^{(1)} = \frac{c_1}{12} \left( 6 m^2_\pi + k^2_f + \frac{5 p^2}{3} + \frac{q^2}{6} \right) - 6 c_1 m^2_\pi + c_3 \left( 9 m^2_\pi + k^2_f + \frac{5 p^2}{3} + q^2 \right) \ln \frac{m_\pi}{\lambda}.
\]

(54)

The second term \( \sim c_3 \bar{\tau}_1 \cdot (\bar{\tau}_2 \times \bar{\tau}_3) \) in eq.(51) produces an isovector spin-orbit potential of the form

\[
V_{\text{med}}^{(1)} = \frac{c_3 g_A^2 k_f^3}{16 \pi^5 f^6_\pi} \int_0^\lambda \! dr \! \int_0^{\pi/2} \! d\psi \left\{ i \bar{\tau}_1 \cdot (\bar{\tau}_2 \times \bar{\tau}_3) \cdot (\bar{\bar{q}} \times \bar{p}) - \frac{l}{4 p^2 - q^2} \left[ 2 \bar{\Gamma}_2 (l) \right] - 2 \Lambda (l) \right. \\
+ (2 l^2 + 2 m^2 - 2 p^2 + q^2) \Omega (l) \left. \right] + \Gamma_3 (l) (m^2 + p^2 - l^2) \left[ \Lambda (l) - (m^2 + l^2 + p^2) \Omega (l) \right] \right\}
\]

(55)

with a large-\( \lambda \) behavior of the double-integral: \( -(\pi k^3_f /18) \ln (m_\pi /\lambda) \). The third term proportional to \( c_1 \bar{\tau}_1 \cdot (\bar{\tau}_2 \times \bar{\tau}_3) \) in eq.(51) gives on the one hand rise to an isovector central potential of the form

\[
V_{\text{med}}^{(1)} = \frac{c_3 g_A^2 k_f^3}{8 \pi^5 f^6_\pi} \int_0^\lambda \! dr \! \int_0^{\pi/2} \! d\psi \left\{ 2 l \left[ m^2 \bar{\Gamma}_0 (l) + 2 \bar{\Gamma}_2 (l) \right] \left[ 2 \Lambda (l) - (2 m^2 + q^2) \Omega (l) \right] \\
+ l \bar{\Gamma}_3 (l) \left[ \frac{l}{2} - (2 l^2 - m^2 - p^2) \right] \Lambda (l) + \left( \frac{l}{2} - l^2 (4 m^2 + q^2) \right) \Omega (l) \right\}
\]

(56)

where the double-integral has the large-\( \lambda \) behavior: \( \pi k^3_f \left[ m^2_\pi + k^2_f /10 + p^2 /6 + 7 q^2 /36 \right] \ln (m_\pi /\lambda) \). On the other hand one gets a contribution to the isovector spin-orbit potential of the form:

\[
V_{\text{med}}^{(1)} = \frac{c_3 g_A^2 k_f^3}{8 \pi^5 f^6_\pi} \left\{ i \bar{\tau}_1 \cdot (\bar{\tau}_2 \times \bar{\tau}_3) \cdot (\bar{\bar{q}} \times \bar{p}) \int_0^\lambda \! dr \! \int_0^{\pi/2} \! d\psi \left\{ \left[ (m^2 \bar{\Gamma}_0 (l) + 2 \bar{\Gamma}_2 (l) \right] \left[ 2 \Lambda (l) \\
+ (2 p^2 - 2 l^2 - 2 m^2 - q^2) \Omega (l) \right] + \Gamma_3 (l) (m^2 + p^2 - l^2) \left[ (m^2 + l^2 + p^2) \Omega (l) - \Lambda (l) \right] \right\} \right\}
\]

(57)

with a large-\( \lambda \) behavior of the double-integral: \( (\pi k^3_f /24) \ln (m_\pi /\lambda) \).
4.3 In-medium NN-potential from remaining four concatenations

The other four concatenations, \(N_3\) on \(N_1\), \(N_1\) on \(N_3\), \(N_1\) on \(N_2\), and \(N_2\) on \(N_1\), applied the first term \(\tilde{\tau}_2 \cdot \tilde{\tau}_2\) in eq.(51) produce an isovector central potential of the form

\[
V_{\text{med}}^{(cc)} = \frac{g_A^2 \tilde{\tau}_1 \cdot \tilde{\tau}_2}{4\pi^5 f_\pi^6} \int_0^\Lambda dr \int_0^{\pi/2} d\psi \left\{ l [2c_1 m_\pi^2 + (c_2 + c_3) k_5^2] \left\{ \bar{\Gamma}_0(l) [2\Lambda(l) - (2\bar{m}^2 + q^2)\Omega(l)]
+ \bar{\Gamma}_1(l) [(l^2 - p^2 - \bar{m}^2)\Omega(l)] \right\}
+ c_3 \left\{ \bar{\Gamma}_2(l) [2\Lambda(l) - (2\bar{m}^2 + q^2)\Omega(l)]
+ \bar{\Gamma}_3(l) \left( \frac{l}{2} + (l^2 - p^2 - \bar{m}^2)\Lambda(l) + \frac{1}{2} (l^2 - p^2 - \bar{m}^2)^2 \Omega(l) \right) \right\}
+ \left( l^2 - p^2 - 2\bar{m}^2 - \frac{q^2}{4} + \frac{q^2}{4p^2} (l^2 + \bar{m}^2) \right) \Lambda(l) + \left( \bar{m}^2 + \frac{q^2}{2} \right) (\bar{m}^2 + p^2 - l^2) \Omega(l) \right\}
- \frac{8k_3^3}{3} (c_3 + c_2 \cos^2 \psi) \sin^4 \psi \right\},
\]

(58)

where the double-integral depends (after subtraction in the last line) logarithmically on the cutoff:

\[
\frac{\pi k_f^3}{4} \left[ c_2 \left( m_\pi^2 + \frac{k_5^2}{10} + \frac{p^2}{6} + \frac{5q^2}{36} \right) - 4c_1 m_\pi^2 + c_3 \left( 6m_\pi^2 + \frac{11k_5^2}{15} + \frac{11p^2 + 5q^2}{9} \right) \right] \ln \frac{m_\pi}{\lambda},
\]

(59)

In the same way one obtains from the second term \(c_4 \tilde{\tau}_1 \cdot \tilde{\tau}_2 \times \tilde{\tau}_3\) in eq.(51) a contribution to the isovector spin-orbit potential of the form

\[
V_{\text{med}}^{(cc)} = \frac{c_4 g_A^2 \tilde{\tau}_1 \cdot \tilde{\tau}_2}{8\pi^5 f_\pi^6} i(\tilde{\sigma}_1 + \tilde{\sigma}_2) \cdot (\vec{q} \times \vec{p}) \int_0^\Lambda dr \int_0^{\pi/2} d\psi \frac{l \bar{\Gamma}_1(l)}{4p^2 - q^2} \left\{ \left( l^2 - p^2 \right) \Lambda(l) - \left[ \left( l^2 - p^2 \right)^2 + q^2 l^2 + m^2 (l^2 + p^2) \right] \Omega(l) \right\}
\]

(60)

with a large-\(\lambda\) behavior of the double-integral: \(- (\pi k_f^3/18) \ln(m_\pi/\lambda)\). Under the same calculational treatment the third term \(c_4 \tilde{\tau}_1 \cdot (\tilde{\tau}_2 + \tilde{\tau}_3)\) in eq.(51) gives rise to three spin-dependent potentials with the common isospin-factor \(3 + \tilde{\tau}_1 \cdot \tilde{\tau}_2\). The pertinent spin-spin potential has the form

\[
V_{\text{med}}^{(cc)} = \frac{c_4 g_A^2 \tilde{\tau}_1 \cdot \tilde{\tau}_2}{16\pi^5 f_\pi^6} \int_0^\Lambda dr \int_0^{\pi/2} d\psi \left\{ \frac{l \bar{\Gamma}_1(l)}{2p^2} \left( 3p^2 - l^2 - \bar{m}^2 - q^2 \right)
+ \left( l^2 - \bar{m}^2 - \frac{3p^2}{2} + \left( \bar{m}^2 + l^2 + q^2 \right) \frac{\bar{m}^2 + l^2}{2p^2} + \frac{q^2 (4\bar{m}^2 + 4l^2 + q^2)}{8p^2 - 2q^2} \right) \Lambda(l)
+ \frac{q^2 (\bar{m}^2 + l^2 + p^2)}{4p^2 - q^2} \left( 2p^2 - 2l^2 - 2\bar{m}^2 - q^2 \right) \Omega(l) \right\} - \frac{16k_3^3}{9} \sin^4 \psi \right\},
\]

(61)

with a large-\(\lambda\) behavior of the (subtracted) double-integral: \((\pi k_f^3/9) \left[ 6m_\pi^2 + k_5^2 + 2p^2 + 3q^2/4 \right] \ln(m_\pi/\lambda)\). Note that the subtraction term acts only in the \(^1S_0\) state with total isospin 1, therefore one can replace (of course only for this \(\lambda^2 k_f^3\)-term) the operator \(\tilde{\sigma}_1 \cdot \tilde{\sigma}_2 (3 + \tilde{\tau}_1 \cdot \tilde{\tau}_2)\) by \(-3(3 + \tilde{\tau}_1 \cdot \tilde{\tau}_2)\). Next, there is a tensor-type potential of the form

\[
V_{\text{med}}^{(cc)} = \frac{c_4 g_A^2}{8\pi^5 f_\pi^6} \int_0^\Lambda dr \int_0^{\pi/2} d\psi \frac{l \bar{\Gamma}_1(l)}{4p^2 - q^2} \left\{ \frac{l}{2p^2} \left[ 3\bar{m}^2 + 3l^2 - p^2 + \frac{5q^2}{4} - \frac{3q^2}{4p^2} (\bar{m}^2 + l^2) \right] + \left[ \frac{p^2}{2} + l^2 - \bar{m}^2 - \frac{5q^2}{8} \right]
- \frac{6(\bar{m}^2 + l^2)^2 + q^2 (\bar{m}^2 + 3l^2)}{4p^2} + \frac{3q^2}{8p^4} (\bar{m}^2 + l^2)^2 - \frac{q^2 (4\bar{m}^2 + 4l^2 + q^2)}{4p^2 - q^2} \right] \Lambda(l)
+ \frac{q^2}{4p^2 - q^2} \left[ 4(\bar{m}^2 + l^2)^2 + 4p^2 (\bar{m}^2 - l^2) + q^2 (\bar{m}^2 + 3l^2 + p^2) \right] \Omega(l) \right\},
\]

(62)
with a large-\(\lambda\) behavior of the double-integral: \(-(\pi k_f^3/36)\ln(m_\pi/\lambda)\). Finally, one gets an ordinary tensor potential which has the form

\[
V^{(cc)}_{\text{med}} = \frac{c_4 g_A^2}{8\pi^5 f_\pi^6} (3 + \vec{\tau}_1 \cdot \vec{\tau}_2) \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} \int_0^\lambda \int_0^{\pi/2} d\psi \int_0^{l_1(l)} \frac{\Gamma_1(l)}{4p^2 - q^2} \left\{ \frac{l}{2p^2} (2p^2 - q^2) \right\}
\]

\[
+ \left[ q^2 (4\tilde{m}^2 + 4l^2 + q^2) \right] + \frac{\tilde{m}^2 + l^2}{2p^2} (q^2 - 4p^2) \right\} \Lambda(l) + \left[ 3(\tilde{m}^2 + l^2)^2 \right.
\]

\[
+ p^2 (4\tilde{m}^2 + 4l^2 + p^2) + q^2 (\tilde{m}^2 + 2l^2 + p^2) - \frac{8p^2}{4p^2 - q^2} (\tilde{m}^2 + l^2 + p^2)^2 \right\} \Omega(l) \right\}, \tag{63}
\]

with a large-\(\lambda\) behavior of the double-integral: \(-(\pi k_f^3/24)\ln(m_\pi/\lambda)\). Let us again consider the balance of \(\lambda^2\)-divergences in the total sum \(V^{(0)}_{\text{med}} + V^{(1)}_{\text{med}} + V^{(cc)}_{\text{med}}\). With the equivalent form of the piece from eq.(61), the balance reads

\[
\frac{g_A^4 \tilde{\lambda}^2}{96\pi^4 f_\pi^6} \left\{ - \vec{\tau}_1 \cdot \vec{\tau}_2 (c_2 + 6c_3) + \left[ 3 \left( \frac{c_2}{2} + 3c_3 \right) + 3c_4 \vec{\tau}_1 \cdot \vec{\tau}_2 \right] \right\}
\]

\[
= \frac{g_A^4 \tilde{\lambda}^2}{64\pi^4 f_\pi^6} (c_2 + 6c_3 - 6c_4), \tag{64}
\]

and one observes that the remaining isoscalar piece can again be absorbed on the 3N short-distance parameter \(c_E\).

### 5 Ring interaction proportional to \(g_A^4\)

The 3N-ring interaction proportional to \(g_A^4c_{1,2,3,4}\) can be inferred from the coordinate-space potential written in eq.(4.6) of ref. [4]. In momentum-space this part of \(V_{3N}\) at N^4LO is given by a euclidean loop-integral over three pion-propagators (one of them squared) times a long series of terms with different spin-, isospin- and momentum-dependence, which reads

\[
V_{3N} = \frac{g_A^4}{f_\pi^6} \int_0^\infty dl_0 \int_0^{d_3} \frac{d^3l_2}{(2\pi)^4 (\tilde{m}^2 + l_1^2)(\tilde{m}^2 + l_2^2)(\tilde{m}^2 + l_3^2)}
\]

\[
\times \left\{ \tilde{m}^2 \left[ (\vec{\sigma}_1 \cdot \vec{l}_3) \cdot (\vec{\sigma}_3 \cdot \vec{l}_1) \left[ 2l_2^6 (c_2 + c_3) \vec{\tau}_1 \cdot \vec{\tau}_3 - 6c_1 m_\pi^2 + \vec{l}_1 \cdot \vec{l}_3 (c_4 (\vec{\tau}_1 + \vec{\tau}_3) \cdot \vec{\tau}_2 - 3c_3) \right] \right] + 2(\vec{\sigma}_1 \cdot \vec{l}_2) \cdot (\vec{\sigma}_2 \cdot \vec{l}_1) \left[ 2l_2^6 (c_2 + c_3) \vec{\tau}_1 \cdot \vec{\tau}_2 - 6c_1 m_\pi^2 + \vec{l}_1 \cdot \vec{l}_2 (c_4 (\vec{\tau}_1 + \vec{\tau}_2) \cdot \vec{\tau}_3 - 3c_3) \right] \right\}
\]

\[
\times \left\{ 2c_3 \vec{\tau}_1 \cdot (2\vec{\tau}_2 + \vec{\tau}_3) + 2\vec{\sigma}_2 \cdot \vec{\sigma}_3 (c_4 \vec{\tau}_1 \cdot (\vec{\tau}_2 + \vec{\tau}_3) - 3c_3) + \vec{\sigma}_1 \cdot \vec{\sigma}_3 (c_4 (\vec{\tau}_1 + \vec{\tau}_3) \cdot \vec{\tau}_2 - 3c_3) \right\}
\]

\[
+ 2\vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 \left[ 4c_1 m_\pi^2 \vec{\tau}_1 \cdot \vec{\tau}_3 - 3l_0^2 (c_2 + c_3) + 2\vec{\sigma}_1 \cdot \vec{\sigma}_3 (l_0^2 (c_2 + c_3) \vec{\tau}_1 \cdot \vec{\tau}_3 - 3c_1 m_\pi^2) + 2\vec{\sigma}_1 \cdot \vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 \right.
\]

\[
\times \left( 3c_3 - c_4 (\vec{\tau}_1 + \vec{\tau}_2) \cdot \vec{\tau}_3 \right) \right\} + 2\vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 \left[ 4c_1 m_\pi^2 \vec{\tau}_1 \cdot \vec{\tau}_3 + \vec{\sigma}_1 \cdot \vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 (c_4 \vec{\tau}_1 \cdot \vec{\tau}_2 - 3c_3) \right.
\]

\[
+ \vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 \left[ 
\left. 4 \vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 (c_4 \vec{\tau}_1 \cdot \vec{\tau}_2 - 3c_3) \right] + 2(\vec{l}_1 \cdot \vec{l}_2) \cdot \vec{l}_3 (c_4 \vec{\tau}_1 \cdot \vec{\tau}_2 - 3c_3)
\right]
\]

\[
+ 4(\vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{\sigma}_3 \vec{\sigma}_4 \vec{\sigma}_5 \vec{\sigma}_6 \vec{\sigma}_7 \vec{\sigma}_8) \left( 3c_1 m_\pi^2 - l_0^2 (c_2 + c_3) \vec{\tau}_1 \cdot \vec{\tau}_3 \right)
\]

\[
+ 2(\vec{l}_1 \cdot \vec{l}_2 \cdot \vec{l}_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{\sigma}_3 \vec{\sigma}_4 \vec{\sigma}_5 \vec{\sigma}_6 \vec{\sigma}_7 \vec{\sigma}_8) \left( 3c_1 m_\pi^2 - l_0^2 (c_2 + c_3) \vec{\tau}_1 \cdot \vec{\tau}_3 \right) \right\}, \tag{65}
\]

with \(\tilde{m} = \sqrt{m_\pi^2 + l_0^2}\) and one has to set \(\vec{l}_1 = \vec{l}_2 - \vec{q}_3\) and \(\vec{l}_3 = \vec{l}_2 + \vec{q}_1\). Without the prefactor \(g_A^4 f_\pi^6\) the first line in eq.(65) defines a euclidean three-point function \(\vec{J}(q_1, q_2, q_3)\) that is symmetric.
under \( q_1 \leftrightarrow q_3 \). By applying the Cutkosky cutting rule to the first and third pion-propagator, one can easily compute its imaginary part \( \text{Im}\tilde{J}(q_1, \mu, q_3) \) as a 2\( \pi \)-phase space integral over a squared pion-propagator, and obtains the following spectral-representation

\[
\tilde{J}(q_1, q_2, q_3) = \frac{1}{16\pi^2} \int_{2m_\pi}^\infty d\mu \frac{\sqrt{\mu^2 - 4m^2_{\pi}}}{\mu^2 + q_2^2} \left[ (\mu q_1 q_3)^2 + m^2_{\pi} G \right]^{-1}, \tag{66}
\]

with the abbreviation \( G = [\mu^2 + (q_1 + q_3)^2][\mu^2 + (q_1 - q_3)^2] \). By a partial-fraction decomposition of the two denominators in eq.(66) one is able to find an analytical solution of the spectral-integral in terms of the even loop-function \( L(s) = L(-s) \), defined in eq.(17). The final result for \( \tilde{J}(q_1, q_2, q_3) \) reads:

\[
\tilde{J}(q_1, q_2, q_3) = \frac{1}{16\pi^2} \left\{ b^2_+ \left[ L(q_2) - L(b_-) \right] + b^2_- \left[ L(b_-) - L(q_2) \right] \right\}, \tag{67}
\]

with the auxiliary variables \( b_\pm = (q_1 \sqrt{4m^2_{\pi} + q_3^2} \pm q_3 \sqrt{4m^2_{\pi} + q_1^2})/(2m_\pi) \) and the symmetric combination \( C = q_1 q_3 \sqrt{(4m^2_{\pi} + q_1^2)(4m^2_{\pi} + q_3^2)} \). Likewise, the (bare) euclidean loop-integral over three pion-propagators in the first line of eq.(51) defines a totally symmetric three-point function \( J(q_1, q_2, q_3) \). It possesses a more involved spectral-representation

\[
J(q_1, q_2, q_3) = \frac{1}{16\pi^2} \int_{2m_\pi}^\infty d\mu \frac{\mu}{(\mu^2 + q_1^2) \sqrt{\mu^2 - 4m^2_{\pi}}} \ln \frac{\mu (\mu^2 + q_1^2 + q_3^2) + \sqrt{\mu^2 - 4m^2_{\pi}} G}{\mu (\mu^2 + q_1^2 + q_3^2) - \sqrt{\mu^2 - 4m^2_{\pi}} G}, \tag{68}
\]

which does not allow for a solution in terms of elementary functions.

5.1 In-medium NN-potential from selfclosing of nucleon-lines

In this subsection the in-medium NN-potential \( V_{\text{med}}^{(0)} \) is computed as it arises from the selfclosing of nucleon-lines for the (extremely lengthy) 3N-ring interaction written in eq.(65). One gets a non-vanishing spin-isospin trace from closing \( N_1, N_2 \) and \( N_3 \), respectively. After performing the angular and radial integrals the summed contributions are sorted according to their two-body spin- and isospin-operators. The complete list of contributions to \( V_{\text{med}}^{(0)} \) consists of an isoscalar central potential:

\[
V_{\text{med}}^{(0)} = \frac{g_A^4 (c_2 + c_3)^3}{96\pi^4 f^6_{\pi}} \left\{ \ln \frac{m_\pi}{\lambda} - \frac{135m^2_{\pi}}{16} + \frac{53q^2_{\pi}}{4} \right\}, \tag{69}
\]

an isovector central potential:

\[
V_{\text{med}}^{(0)} = \frac{g_A^4 k^3 f^1_{\pi} f^2_{\pi}}{96\pi^4 f^6_{\pi}} \left\{ 15 \left( \frac{7c_3 - 4c_1}{2} \right) m^2_{\pi} + 233c_2^2\ln \frac{m_\pi}{\lambda} + \frac{389c_3}{24} - 27c_1 \right\} m^2_{\pi} + \frac{8m^4_{\pi}(2c_1 - c_3)}{4m^2_{\pi} + q^2}, \tag{70}
\]

an isoscalar spin-spin potential:

\[
V_{\text{med}}^{(0)} = \frac{g_A^4 k^3 f^1_{\pi} f^2_{\pi}}{480\pi^4 f^6_{\pi}} \left\{ 80(7c_1 - 10c_3) m^2_{\pi} - \frac{83c_3 q^2_{\pi}}{2} \right\} \ln \frac{m_\pi}{\lambda} + \frac{1220c_1 - 3407c_3}{4} \frac{m^2_{\pi}}{3}, \tag{71}
\]

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an isoscalar tensor potential:

\[
V^{(0)}_{\text{med}} = \frac{g_{A}^{4}(c_{2} + c_{3})k_{f}^{3}}{720\pi^{4}f_{\pi}^{6}}\hat{\sigma}_{1}\cdot\hat{\sigma}_{2}\tilde{\tau}_{1}\cdot\tilde{\tau}_{2}\left\{ \left[ 140m_{\pi}^{2} + \frac{17q^{2}}{2} \right] \ln \frac{m_{\pi}}{\Lambda} + \frac{653m_{\pi}^{2}}{12} + \frac{731q^{2}}{240} + \frac{4m_{\pi}^{4}}{q^{2}} + \left[ \frac{30m_{\pi}^{4}}{4m_{\pi}^{2} + q^{2}} + \frac{17q^{2}}{2} - \frac{4m_{\pi}^{2}}{q^{2}} \right] L(q) \right\} ,
\]

(72)

an isovector spin-spin potential:

\[
V^{(0)}_{\text{med}} = \frac{g_{A}^{4}k_{f}^{3}}{48\pi^{4}f_{\pi}^{6}}\hat{\sigma}_{1}\cdot\hat{q}\hat{\sigma}_{2}\cdot\hat{q}\tilde{\tau}_{1}\cdot\tilde{\tau}_{2}\left\{ 4c_{3}\ln \frac{m_{\pi}}{\Lambda} + \frac{61c_{3}}{48} + \frac{2m_{\pi}^{2}}{q^{2}}(3c_{1} + c_{3}) + \left[ 4c_{3} - \frac{2m_{\pi}^{2}}{q^{2}}(3c_{1} + c_{3}) + \frac{3m_{\pi}^{2}(c_{3} - 2c_{1})}{4m_{\pi}^{2} + q^{2}} \right] L(q) \right\} ,
\]

(73)

and an isovector tensor potential:

\[
V^{(0)}_{\text{med}} = \frac{g_{A}^{4}k_{f}^{3}\lambda^{2}}{72\pi^{4}f_{\pi}^{6}}\hat{\sigma}_{1}\cdot\hat{q}\hat{\sigma}_{2}\cdot\hat{q}\tilde{\tau}_{1}\cdot\tilde{\tau}_{2}\left\{ -\ln \frac{m_{\pi}}{\Lambda} - \frac{7}{48} + \frac{m_{\pi}^{2}}{q^{2}} - \left( 1 + \frac{m_{\pi}^{2}}{q^{2}} \right) L(q) \right\} .
\]

(74)

Note that \( c_{4} \) has dropped out and the dependence on the other three low-energy constants \( c_{1,2,3} \) is well structured. The isoscalar central and isovector spin-dependent potentials are solely proportional to the sum \( c_{2} + c_{3} \), whereas the other potentials depend separately on \( c_{1} \) and \( c_{3} \). The total \( \lambda^{2} \)-divergence behind the central and spin-spin potentials written in eqs.(69-72) is (for S-waves) equivalent to

\[
\frac{g_{A}^{4}k_{f}^{3}\lambda^{2}}{4\pi^{4}f_{\pi}^{6}}\left\{ \frac{163c_{3}}{2} - \frac{271c_{2}}{6} + \frac{295c_{3}}{3} \right\} .
\]

(75)

In combination with the \( \lambda^{2} \)-divergences from concatenations \( V^{(2)}_{\text{med}} + V^{(cc)}_{\text{med}} \) for all interaction terms in eq.(65) it will reduce to an isoscalar component only, that can be absorbed on the 3N short-distance parameter \( c_{E} \). This property serves as an excellent check on our extensive calculations.

### 5.2 In-medium NN-potential from concatenations for two selected terms

The 3N-ring interaction \( V_{3N} \) written in eq.(65) consists of a very large number of terms. In this paper we consider for the contributions to \( V_{\text{med}} \) from concatenations of two nucleon-lines only two selected terms. The analogous formulas for all the other terms can be obtained from the author upon request.

a) The term proportional to \( c_{4}\tilde{\tau}_{1}\cdot(\tilde{\tau}_{2}\times\tilde{\tau}_{3}) \) in the fourth line of eq.(65). It gives for the concatenations \( N_{3} \) on \( N_{1} \) and \( N_{1} \) on \( N_{3} \) an isovector spin-orbit potential of the form

\[
V^{(2)}_{\text{med}} = \frac{c_{4}g_{A}^{4}\tilde{\tau}_{1}\cdot\tilde{\tau}_{2}}{16\pi^{3}f_{\pi}^{6}}i(\tilde{\sigma}_{1}\cdot\tilde{\sigma}_{2})(\hat{q}\times\hat{p})\int^{\lambda}_{0}dr\int^{\pi/2}_{0}d\psi\int^{l}_{0}\frac{l}{4p^{2} - q^{2}}\left\{ \gamma_{2}(l)\left[ \frac{l}{4p^{2} - q^{2}}(4p^{2} - q^{2})\Omega(l) \right] + \frac{q^{2}}{p^{2}}(\bar{m}^{2} + l^{2}) \right. - 4l^{2} - 8\bar{m}^{2} - 3q^{2} \right\} \Lambda(l) + (2\bar{m}^{2} + q^{2})(2\bar{m}^{2} + l^{2} - 2p^{2} + q^{2})\Omega(l) \left. \right\} \gamma_{3}(l)\left[ \frac{l}{4p^{2} - q^{2}}(4p^{2} - q^{2})(\bar{m}^{2} + l^{2} - p^{2}) \right] + \frac{q^{2}}{4p^{2}}(\bar{m}^{2} + l^{2} - p^{2})^{2} + 2(p^{2} - l^{2})(\bar{m}^{2} + l^{2} + p^{2}) \right.
\]

\[
+ \frac{q^{2}}{4}(2\bar{m}^{2} - 2l^{2} + p^{2})\Lambda(l) + (l^{2} - \bar{m}^{2} - p^{2})(B + \frac{q^{2}}{2}(\bar{m}^{2} + 3l^{2} + p^{2}))\Omega(l) \right\} ,
\]

(76)
with a large-$\lambda$ behavior of the double-integral: $(5 \pi k_f^3 / 72) \ln (m_\pi / \lambda)$. The new functions $\tilde{\gamma}_{2,3}(l)$ appearing in eq.(76) are $\tilde{\gamma}_{2,3}(l) = -\partial \tilde{\Gamma}_{2,3}(l) / \partial m^2$ with $\tilde{\Gamma}_{2,3}(l)$ given in eqs.(40,41). The other four concatenations produce also an isovector spin-orbit potential of the form:

$$V^{(cc)}_{\text{med}} = \frac{c_4 g_A^4}{16 \pi^5 f_\pi^6} \left( i (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{p}) \right) \int_0^\lambda \int_0^{\pi / 2} d\vec{r} \, d\psi \left\{ \tilde{\Gamma}_{2}(l) \left\{ \frac{l}{p^2} + \frac{4l}{4p^2 - q^2} \right\} \Lambda(l) + \frac{m^2}{B} \left[ \frac{1}{p^2} + \frac{2q^2}{q^2 (l^2 - m^2 - p^2)} \right] - K^2 (0, 0) \right\} \Lambda(l) \right\} \right\} , (77)$$

with a large-$\lambda$ behavior of the double-integral: $(5 \pi k_f^3 / 36) \ln (m_\pi / \lambda)$.

b) The term proportional to $\tilde{l}_1 \cdot \vec{t}_1 \vec{t}_1 \cdot \tilde{l}_3 \cdot \vec{t}_3$ multiplied by the fifth line in eq.(65). It gives for the concatenations $N_3$ on $N_1$ and $N_3$ on $N_1$ a combination of central and spin-spin potentials of the form

$$V^{(2)}_{\text{med}} = \frac{g_A^4}{32 \pi^3 f_\pi^2} \left[ \frac{3c_3 - 3c_4}{2} \right] + 3 (c_3 - c_2) \vec{\sigma}_1 \cdot \vec{\sigma}_2 - (2c_3 + 3c_4) \vec{t}_1 \cdot \vec{t}_2 - c_4 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{t}_1 \cdot \vec{t}_2 \right] \int_0^\lambda \int_0^{\pi / 2} d\vec{r} \, d\psi \left\{ \begin{array}{l}
\frac{1}{p^2} (4p^2 - q^2) + \frac{q^2}{p^2} (m^2 + l^2) - 8m^2 - 3q^2 \Lambda(l) + (2m^2 + q^2)^2 \Omega(l) \\
+ K^2 (0, 0) \right\} \Lambda(l) \right\} \right\} , (78)$$

with a large-$\lambda$ behavior of the double-integral: $(\pi k_f^3 / 24) [35m_\pi^2 + 18k_f^2 / 5 + 6p^2 + 43q^2 / 12] \ln (m_\pi / \lambda)$. The other four concatenations produce the same combination of central and spin-spin potentials, which takes the form:

$$V^{(cc)}_{\text{med}} = \frac{g_A^4}{8 \pi^5 f_\pi^6} \left[ \frac{3c_3 - 3c_4}{2} \right] + 3 (c_3 - c_2) \vec{\sigma}_1 \cdot \vec{\sigma}_2 - (2c_3 + 3c_4) \vec{t}_1 \cdot \vec{t}_2 - c_4 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{t}_1 \cdot \vec{t}_2 \right] \int_0^\lambda \int_0^{\pi / 2} d\vec{r} \, d\psi \left\{ \begin{array}{l}
8 - \frac{q^2}{p^2} \right\} \Lambda(l) + \left( \frac{2m^2 + q^2}{B + q^2 l^2} (m^2 + l^2 + p^2) - 4 \right) (2m^2 + q^2) \Omega(l) \\
+ K^2 (0, 0) \right\} \Lambda(l) \right\} \right\} , (79)$$

with a large-$\lambda$ behavior of the double-integral: $(\pi k_f^3 / 12) [35m_\pi^2 + 79k_f^2 / 20 + 79p^2 / 12 + 3q^2] \ln (m_\pi / \lambda)$. 

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6 Summary and outlook

In this work the density-dependent in-medium NN-interaction $V_{\text{med}}$ has been derived from the subleading chiral 3N-forces. This is necessary since for the intermediate-range topologies ($2\pi 1\pi$-exchange and ring-diagrams) the $N^4\text{LO}$ corrections of ref. [4] dominate in most cases over the nominally leading $N^3\text{LO}$ terms. The loop-integrals representing the 3N-ring interaction proportional to $c_{1,2,3,4}$ have been regularized by a (euclidean) cutoff $\lambda$ and each contribution to $V_{\text{med}}$ has been presented such that the absorption of $(\lambda^2$ and $\ln(m_\pi/\lambda)$) divergences on the 3N short-distance parameters becomes obvious. In the next step, partial-wave matrix elements of $V_{\text{med}}$ will be calculated numerically [21] in order to study quantitatively the effects of the subleading [17, 18] as well as subsubleading chiral 3N-forces. At the same time the construction of 3N-forces in chiral effective field theory with explicit $\Delta(1232)$-isobars by the Bochum group should be accompanied by a calculation of the corresponding density-dependent NN-potential $V_{\text{med}}$. On the other hand, neutron matter calculations with (sub)-subleading chiral 3n-forces require the $\rho_n$-dependent nn-interaction in pure neutron matter.

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