Resolution of dark matter problem in $f(T)$ gravity

Mubasher Jamil$^{1,a}$, D. Momeni$^{2,b}$, Rathbay Myrzakulov$^{2,c}$

$^1$Center for Advanced Mathematics and Physics (CAMP), National University of Sciences and Technology (NUST), H-12, Islamabad, Pakistan
$^2$Eurasian International Center for Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan

Abstract In this paper, we attempt to resolve the dark matter problem in $f(T)$ gravity. Specifically, from our model we successfully obtain the flat rotation curves of galaxies containing dark matter. Further, we obtain the density profile of dark matter in galaxies. Comparison of our analytical results shows that our torsion-based toy model for dark matter is in good agreement with empirical data-based models. It shows that we can address the dark matter as an effect of torsion of the space.

1 Introduction

As we know Dark (non-luminous and non-absorbing) Matter (DM) is an old idea even stemming from before the dark energy problem, which is causing the accelerating expansion of the Universe on the large scale [1, 2]. The most accepted observational evidence for the existence of such component comes from the astrophysical measurements of several galactic rotation curves. From the point of view of classical mechanics, we expect that the rotational velocity $v_\phi$ of any astrophysical object moving in a stable (quasi stable) Newtonian circular orbit with radius $r$ must be of the form

$$v_\phi(r) \propto \sqrt{M(r)/r},$$

where $M(r)$ is identified as the mass (effective mass) profile thoroughly inside the orbit. For many spiral and elliptical galaxies this velocity $v_\phi$ remains approximately constant for large galactic radii, for instance in the Milky way galaxy, $v \approx 240 \text{ km/s}$. This estimation is valid only near the position of our solar system. There is a lower bound on the DM mass density, $\Omega_{DM} \approx 0.1$ from phenomenological particle physics. There are different kinds of dark matter [3–15]. To solve the DM problem several proposals were introduced. The problem can be interpreted as an effect of the extra dimensions in a cosmological special relativity (CSR) model, proposed by Carmelli [16–21]. From the particle physics point of view we have LSP in supersymmetric theories or LKP in higher dimensional theories in which the SM (standard model) predicts some extra dimensions. Stability condition of any candidate for dark matter is a very important problem which must be checked. For example, for stabilization checking in SUSY (super symmetry) we must check the validity of R-parity and in supergravity alternatives we must follow the KK parity. In brief, “Any candidate for dark matter need not be stable if its abundance at the time of its decay is sufficiently small”. There are several classical candidates for dark matter, as perfect fluid models [22, 23] and as the geometrical modifications of the Einstein–Hilbert action, for example $R^2$ modification of the usual Einstein gravity [24] or the anisotropic and diffeomorphism invariance model of Horava–Lifshitz as an integration constant [25].

In this Letter we focus on the mechanism of $f(T)$ gravity and show that in the context of this new proposed non-Riemannian extension of general relativity (GR), it is possible to explain the rotation curves of the galaxies without introducing dark matter. Our plan in this letter is as follows. In Sect. 2 we propose the basis of the $f(T)$ gravity. In Sect. 3, we investigate the spherically symmetric solutions of the model. In Sect. 4 we solve the equations and show that the rotation curve of the galaxies in this toy model of the spherically symmetric-static model can be recovered by the effects of torsion alone. In Sect. 5 we obtain the halo density profile and compare it with two well-known astrophysical models. We conclude in the final section.

2 Formalism of $f(T)$ gravity

A gauge theory of gravity is based on the equivalence principle. For example, $SL(2, C)$ gauge theory on the gravitational field can be used for quantization of this fundamental
force [26]. We are working with a curved manifold for the construction of a gauge theory for gravitational field. It is not necessary to use only Riemannian manifolds. The general form of a gauge theory for gravity, with metric, non-metricity, and torsion can be constructed easily [27]. If we relax the non-metricity, our theory is defined on Weitzenböck spacetime, with torsion and with zero local Riemann tensor \( R_{\alpha\beta\rho\nu} = 0 \). In this theory, which is called teleparallel gravity, we use a non-Riemannian spacetime manifold.

The basic quantities in teleparallel or the natural extension of the teleparallel theory and that of “ether” in physics before the creation of special relativity theory by Einstein [46]. In this Letter we focus only on \( f(T) \) models, without curvature and with non-zero torsion. We should remark that an attempt to explain the flat rotation curve of galaxies has been made earlier in the framework of ECSK theory [47].

### 3 Spherically symmetric geometry

As is well known, a typical spiral galaxy contains two forms of matter: luminous matter in the form of stars and stellar clusters which are found in the galactic disk, while another form is dark matter, which is generally found in the galactic halo and encapsulates the galaxy disk. In the early Universe, the DM played a crucial role in the formation of galaxies when the dense concentration of DM was in the galactic centers which helped in the accumulation of more dust and gas to form proto-galaxies. In the later stages of galactic evolution, the DM slowly drifted towards the outer regions of the galaxies forming huge (but less concentrated/dense) DM halos. Although the precise form of the distribution of dark matter in the halos is not known, we assume that the spatial geometry of galactic halo is spherically symmetric. Moreover, the dark matter halo is isotropic: the spherical DM expansion (hypothetically) only radially while having no tangential or orthogonal motions relative to the radial one. From the point of view of Grand Unified Theories (GUT), the most likely candidate of DM is the neutralino, which is a weakly interacting massive particle [48] with additional minor contribution form primordial black holes, \( \Omega_{\text{PBH}} = 10^{-8} \), which were formed in the early Universe and are also candidate for other violent cosmic events like Gamma Ray Bursts [49–51]. Note that we are not interested in any particular form of DM and deal only with its characteristic role in the rotation of galactic disks. Our theoretical model suggests that the flat galactic rotation curves can be explained in terms of torsion of space without invoking dark matter. In other words, the huge DM halo is nothing but mysterious and elusive torsion of space. Now we construct a model for the galaxy, based on the above assumptions. The metric of static spherically symmetric (SSS) spacetimes can be described, without loss of generality, by

\[
ds^2 = e^{\alpha(r)} dt^2 - e^{\beta(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]
This form of metric is updated by the Schwarzschild gauge, and is useful for the construction of a toy model for galaxy. In order to re-write the metric (3) into the invariant form under the Lorentz transformations, we use the tetrad matrix [52]

\[ \{ e_\mu \} = \text{diag} \{ e^{a(r)/2}, e^{b(r)/2}, r, r \sin \theta \}. \] (4)

Although \( f(T) \) gravity is not local Lorentz invariant [53], we can impose local invariance symmetry on the metric components. Here we make some remarks regarding the relation between local Lorentz invariance of the \( f(T) \) as a scalar gravitational theory and choice of the tetrads, especially in the case of spherically symmetric metric, given by (4). As we know, teleparallel gravity with \( f(T) = T \) has local Lorentz invariance, for any set of the constants \( c_{1,2} \), since it is equivalent to the Einstein theory. This means the total Lagrangian of the linear \( T \) theory is equivalent to the Einstein plus a surface boundary term which can be canceled in the derivation of the equations of motion [41, 42]. Boundarity terms like \( n^\nu T_\mu^\nu \) (\( T \) is torsion) have thermodynamical meaning but are free of the dynamics. Torsion-based \( f(T) \) theory, constructed from the tetrad basis \( e_\mu \) must be local invariant under proper Lorentz transformations. But in this form and with usual tetrad basis, it has been shown that this invariance breaks [53]. Recently (without any direct proof), the authors of [54] have shown that with another choice of the tetrads (they called them “good tetrads”), which leads to the non-diagonal metric components, the restriction on the form of the \( f(T) \) as a linear \( T \) theory, coming from the equation

\[ f_{TT} = 0, \]

relaxes. This relaxation can be interpreted as a rotation of the tetrad basis. In this new tetrad basis, there are free Euler angles (all local functions of the coordinates \( r, t \)), and the expression of the scalar torsion has some extra terms. In this case the system of equations even for vacuum case is very complicated. Further, they found it is possible to recover the Schwarzschild–de Sitter case as an exact black hole solution in this theory, and test the parameters of the model, using the usual tests as PPN, etc. There are some points that must be clarified by those authors [54] to support their conclusions, for example does this local invariance help the power-counting renormalizability of the \( f(T) \) model as an alternative theory? There are many kinds of such rotational transformations. How may we prefer one kind of the rotations over others? We will come back to these problems in a forthcoming paper on this topic [55]. Any way in this letter we want to explore the effect of the torsion for construction of a geometrical model for DM. For this purpose, the usual tetrads are enough.

Using (4), one obtains \( e = \det \{ e_\mu \} = e^{(a+b)/2} r^2 \sin(\theta) \), and the torsion scalar in terms of \( r \) is given by

\[ T(r) = \frac{2e^{-b}}{r} \left( a' + \frac{1}{r} \right), \] (5)

where the prime (‘) denotes the derivative with respect to the radial coordinate \( r \). The equations of motion for an anisotropic fluid are [52]

\[ 4\pi \rho = \frac{f_T}{4} - \left( T - \frac{1}{r^2} - \frac{e^{-b}}{r} (a' + b') \right) \frac{f_T}{2}, \] (6)

\[ 4\pi p_r = \left( T - \frac{1}{r^2} \right) \frac{f_T}{2} - \frac{f}{4}, \] (7)

\[ 4\pi p_t = \frac{f_T}{2} \left[ T + e^{-b} \left( \frac{a''}{2} + \left( \frac{a'}{2} + \frac{1}{2r} \right) (a' - b') \right) \right] - \frac{f}{4}, \] (8)

\[ \cot \theta \frac{2r^2}{f_T^2} f_{TT} = 0, \] (9)

where \( p_r \) and \( p_t \) are the radial and tangential pressures, respectively; \( \rho \) is density profile. This last quantity is very important in our astrophysical predictions. Here if we use the ‘good’ tetrads [54], the outcome system becomes very complicated; it is the following:

\[ f_{TT} T' \cot \theta = 0, \] (10)

\[ f_{TT} [2 - 2e^{b} + 2r^{2} e^{b} T - 2r b'] \] (11)

\[ f_{TT} [2 - 2e^{b} + 2r^{2} e^{b} T - 2r a'], \] (12)

\[ f_{TT} b = 0, \] (13)

\[ \left( 2e^{b} f_{TT} + 2f_{TT} \right) \left( 1 + \frac{e^{b}}{2} \sin \gamma \right) \] (14)

\[ \left( 2e^{b} + r^{2} e^{b} T - 2r a' \right) \] (15)

\[ 4e dT'+ b^{2} - 2e^{b} + r^{2} e^{b} T \] (16)

\[ + 2r a', \] (17)

\[ + 4r \left( 1 + \frac{e^{b}}{2} \sin \gamma \right) \] (18)
which always relapses into the particular case of Teleparallel Theory, with \( f(T) \) a constant or a linear function. We adopt this linear teleparallel choice for our physical discussions of the possible explanation of the DM in the context of the torsion-based gravity, \( f(T) \). In the next section, we will solve the above equations (6), (7), and (8) for the metric function \( a(r) \). In the language of the 3 + 1 decomposition of the metrics, determining \( a(r) \) is equivalent to finding the lapse (or redshift) function \( N(r) = e^{\alpha(r)} \).

4 Dark matter problem in \( f(T) \) gravity

The quasi global solution for (3) with the assumption \( a(r) = -b(r) \) and by imposing the isotropicity in the pressure components \( p_r = p_t \) is the Schwarzschild–(A)dS case represented by [52]

\[
e^{a(r)} = e^{-b(r)} = 1 - \frac{c_0}{r} + \frac{c_1}{3} r^2.
\]

Obviously such trial metric cannot be successful in generating the rotation curve of the spiral galaxies. Indeed this classical solution leads to a zero torsion, \( T = 0 \). From physical intuition we know the DM problem must come from a non-zero torsion, and specially from a variable one, \( T \equiv T(r) \). Now we introduce an ansatz for the solution and choose

\[
b(r) = c,
\]

where \( c \) is an arbitrary constant.\(^1\) Another choice is the polynomial form for the lapse function \( e^a \) [57], but with such choices, independent from the origin, the rotation curve is fixed by a desirable linear form, and it seems that such choices are ad hoc and not physically acceptable. Further we choose another ansatz, \( c_1 = 1 \) and \( c_2 = 0 \). It means \( f(T) = T \). The model and the field equations still remain scale invariant. The main reason for the choice of the metric function \( b \) as a constant goes back to the scale invariance of the system, and further we are interested in a lapse function \( N = e^a \), which may explain the flat rotation curve. We discuss more our choice of such a restricted gauge. Let us consider the following static form of the metric:

\[
g_{\mu\nu} dx^\mu dx^\nu = h_{AB} dx^A dx^B - \Phi^2 d\Sigma_2^2
\]

instead of (3), which is a four dimensional dual of the following renormalizable effective action, defined on the two dimensional induced metric \( h_{AB}(x, t) \), \( A, B = 0, 1 \):

\[
S = -\frac{1}{16\pi} \int d^2x \sqrt{-h} \left[ f(\Phi)T + \epsilon \Phi,_{\Phi} \Phi^A - U(\Phi) \right],
\]

\[
\epsilon = \text{constant} \equiv O(1).
\]

This action is a generalization of the action which is proposed in the Einstein gravity for some dilaton fields [56]. We assume that the free functions \( f, U \) are analytic in \( \Phi \) in the limit of large \( \Phi \). Indeed this action is power-counting renormalizable.\(^2\) This power-counting renormalizability is valid for any polynomial form of the interaction coupling \( f, U \). Comparing (3), (20) we find there exists a gauge freedom for the choice of the field \( \Phi = b(r) \). One trivial gauge is the constant gauge given by (19). Thus our ansatz can be interpreted as a gauge free term in the effective action. Now we come back to the analytic investigation of the solutions. Without loss of any generalization, we assume the isotropic ansatz for the matter distribution

\[
p_t = p_r.
\]

The expression for rotation curves of galaxies is [59, 60]

\[
v_\psi = \left( \frac{r (e^a)'}{2 e^a} \right)^{1/2},
\]

where a prime denotes differentiation with respect to the radial coordinate \( r \). This formula is the same as in Einstein gravity. We must clarify this point here. The path of the free particle can be obtained using the usual minimization method of the action for a free particle,

\[
b \int ds = 0,
\]

\[
d s = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\eta.
\]

Here \( \eta \) is the affine parameter. Using this equation we obtain the following geodesic equation:

\[
\ddot{x}^\mu + \Gamma^\mu_{\nu\alpha} \dot{x}^\nu \dot{x}^\alpha = 0.
\]

Here \( \Gamma^\mu_{\nu\alpha} \) is the Levi-Civita connection, defined by the symmetric part of the general connection from the metricity equation. The asymmetric part of the connection \( \gamma^\mu_{\nu\alpha} \) has no part in this geometrical equation. Thus if we mean by the geodesic equation the non-auto-parallel motion, the same expression can be used. But using the auto-parallel formalism is another story and we will not enter in it. It is easy to show that this geodesic equation is equivalent by the Euler–Lagrange equations, derived from the following point like Lagrangian for the test particle:

\[
2L = e^{a(t)} - e^{b(r)} - r^2 (b^2 + \sin^2 \theta \dot{\phi}^2).
\]

The purely radial equation for test particle reads

\[
\dot{r}^2 + U(r) = 0,
\]

\[
U(r) = e^{-b(r)} \left( e^{-a(r)} e^2 - \frac{b^2}{r^2} - 1 \right),
\]

\[
h^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}.
\]

Here \( p_\theta, p_\phi \) are the conjugate momenta of the corresponding coordinates, \( \epsilon = \ell e^a \) is the energy (local). Here we are

\(^1\)Here \( c \) does not represent the speed of light.

\(^2\)This calculation is done most conveniently in the gauge theoretic formulation analogous to [58]. We will present the details in a forthcoming paper on renormalizability of the \( f(T) \) models.
assuming that there exists a ZAMO (zero angular momentum) observer, located at the spatial infinity \( r \to \infty \). We note that the tangential (rotation) velocity reads

\[
v_\psi = \sqrt{\frac{\dot{a}(r)}{r^2 e^2}} = \frac{r a'(r)}{2},
\]

which coincides with (23). Making use of isotropic ansatz (22) in (7) and (8), we obtain

\[
e^{a(r)} = \frac{r^3 - \sqrt{13 - 4e^2}}{16(-13 + 4e^2)} \left(C_1 r \frac{\sqrt{13 - 4e^2}}{r} - C_2\right)^4,
\]

where \( C_1 \) and \( C_2 \) are new constants of integration. A simple direct calculation of torsion using (5) and with metric function given by (25) shows that in this case \( T(r) \neq 0 \) (see Fig. 1). By substituting (25) in (23) we can plot the rotation (tangential) velocity as in Fig. 2. We used a large set of data coming from the local tests of the \( f(T) \) models based on the cosmographic description [61]. Figure 1 resembles the rotation curves for large (spiral) galaxies [62–64]. The scale for the velocity profile \( v_\psi \) is \( v_\psi \approx 10^{-3} \), which roughly corresponds to \( r \approx 300 \text{ km/s} \), in reasonable agreement with the data. It is important to mention that the velocity profile shown in Fig. 2 is constructed basically on a phenomenological toy model. It is gratifying that our \( f(T) \) theory predicts a good velocity profile that was argued to be a good phenomenological fit to the data.

5 On central density profile \( \rho(r) \) in \( f(T) \) model

From observational data we know that there exists a core with roughly constant density (mass density) in the galaxy. Many models have been proposed for this mass profile density. In brief these models are used in the numerical simulations, specially for \( \Lambda \text{CDM} \):

\[
\rho(r) = \frac{\rho_i}{(\frac{r}{r_s})(1 + \frac{r}{r_s})^2}.
\]

(26)

\[
\rho(r) = \frac{\rho_0}{1 + (\frac{r}{r_c})^2}.
\]

(27)

Here \( \rho_i \) represents the density of Universe at the collapse time, \( \rho_0 \) is central density of the halo, \( r_s \) is a characteristic radius for the halo and \( r_c \) is the radius of the core [65–67]. Further, very close to the center, this density profile is characterized by \( r^n \) where \( n \approx -1 \), which means we have a density cusp. The observations often favor \( n \approx 0 \), i.e. a constant-density core. Now we want to compare our estimation on the form of \( \rho(r) \) from model of \( f(T) \) by metric function (25). Substituting (25) in (6) we obtain

\[
\rho(r) = \frac{1}{16\pi} \left[c_2 + \frac{2c_1(1 - e^{-c})}{r^2}\right].
\]

(28)

This is the exact mass profile density of our model of DM, which is comparable with the two recently proposed models of the density (26, 27) (see Fig. 3 for a comparison of the estimated model of us given in (28) and two models (26, 27) from astrophysical data given from [65–67]).
The lapse function \(N\) remains spherically symmetric and static. Then by solving \(f(T)\) in the torsion field around a uniformly rotating massive galaxy, we obtained the rotation curve of the galaxy. In this letter, we obtained the rotation curve of the galaxy in the \(f(T)\) gravity. Solid is our prediction given by (28), dotted is the density of NFW model (26) and dashed is the estimation of pseudo-isothermal sphere approximation introduced by (27). The behavior of our model is very close to the estimations of the NFW model [65–67].

## 6 Conclusion

In this letter, we obtained the rotation curve of galaxies in the \(f(T)\) gravity. We proposed that the galaxy metric remains spherically symmetric and static. Then by solving the general equations of the metric components, we obtained the lapse function \(N = e^\theta\) as a function of the radial coordinate \(r\). Our discussions show a very good agreement between the rotation curves in this model and other curves obtained from the data. The scale for the velocity profile \(v_\phi\) is \(v_\phi \approx 10^{-3}\), which roughly corresponds to \(r \approx 300\) km/s, in reasonable agreement with the data. Further, the exact mass profile density of our model of DM is in good agreement with the two models of the density. It proves that the dark matter problem can easily be resolved as the effect of torsion of the space time.

It is well known from ECSK theory that torsion couples to the spin of matter, hence one can envisage to measure the torsion produced by any massive spinning object (including massive spiral and elliptical galaxies). However, there have been assumptions about testing these theories, such as “all torsion gravity theories predict observationally negligible torsion in the solar system, since torsion (if it exists) couples only to the intrinsic spin of elementary particles, not to rotational angular momentum” [68]. Mao et al. have shown that Gravity Probe B is an ideal experiment for constraining several torsion-based theories [68]. Although their analysis is based on the torsion field around a uniformly rotating spherical mass such as Earth, the task of constraining torsion around massive galaxies is still open for exploration.

## References

1. G. Bertone, *Particle Dark Matter* (Cambridge University Press, Cambridge, 2010)
2. G. Bertone, D. Hooper, J. Silk, Phys. Rep. 405, 279 (2005)
3. J. Angle et al., Phys. Rev. Lett. 107, 051301 (2011)
4. C. Kouvaris, P. Tinyakov, Phys. Rev. Lett. 107, 091301 (2011)
5. A. Loeb, N. Weiner, Phys. Rev. Lett. 106, 171302 (2011)
6. M. Jamil, D. Momeni, Chin. Phys. Lett. 28, 099801 (2011)
7. H. Davoudiasl, D.E. Morrissey, K. Sigurdson, S. Tulin, Phys. Rev. Lett. 105, 211304 (2010)
8. S. Chang, R.F. Lang, N. Weiner, Phys. Rev. Lett. 106, 011301 (2011)
9. A.D. Simone, V. Sanz, H.P. Sato, Phys. Rev. Lett. 105, 121802 (2010)
10. M.T. Frandsen, S. Sarkar, Phys. Rev. Lett. 105, 011301 (2010)
11. D. Momeni, A. Azadi, Astrophys. Space Sci. 317, 231 (2008)
12. T. Cohen, K.M. Zurek, Phys. Rev. Lett. 104, 101301 (2010)
13. Y. Bai, M. Carena, J. Lykken, Phys. Rev. Lett. 103, 261803 (2009)
14. J. McDonald, Phys. Rev. Lett. 103, 151301 (2009)
15. P. Sikivie, Q. Yang, Phys. Rev. Lett. 103, 111301 (2009)
16. M. Carmeli, Int. J. Theor. Phys. 38, 1993 (1999)
17. M. Carmeli, Int. J. Theor. Phys. 37, 2621 (1998)
18. F.J. Oliveira, Int. J. Mod. Phys. D 15, 1963 (2006)
19. F.J. Oliveira, arXiv:gr-qc/0508094
20. S. Behar, M. Carmeli, Int. J. Theor. Phys. 39, 1375 (2000)
21. S. Behar, M. Carmeli, Int. J. Theor. Phys. 39, 1397 (2000)
22. T. Harko, G. Mocanu, Phys. Rev. D 85, 084012 (2012)
23. J. Magana, T. Matus, V. Robles, A. Suarez, arXiv:1201.6107
24. J.A.R. Cembranos, Phys. Rev. Lett. 102, 141301 (2009)
25. S. Mukohyama, Phys. Rev. D 80, 064005 (2009)
26. M. Carmeli, S. Malin, Int. J. Theor. Phys. 37, 2615 (1998)
27. L. Smalley, Phys. Lett. A 61, 436 (1977)
28. E.V. Linder, Phys. Rev. D 81, 127301 (2010). [Erratum-ibid. D 82, 109902 (2010)]
29. R. Ferraro, F. Fiorini, Phys. Rev. D 75, 084031 (2007)
30. R. Ferraro, F. Fiorini, Phys. Rev. D 78, 124019 (2008)
31. M. Jamil, D. Momeni, R. Myrzakulov, Eur. Phys. J. C 72, 1959 (2012)
32. R. Myrzakulov, Eur. Phys. J. C 71, 1752 (2011)
33. K.K. Yerzhanov, Sh.R. Myrzakul, I.I. Kulnazarov, R. Myrzakulov, arXiv:1006.3879
34. K. Bamba, M. Jamil, D. Momeni, R. Myrzakulov, arXiv: 1202.6114
35. M. Jamil, D. Momeni, R. Myrzakulov, Cent. Eur. J. Phys. (2012). doi:10.2478/s11534-012-0103-2
36. P.Y. Tsyba, I.I. Kulnazarov, K.K. Yerzhanov, R. Myrzakulov, Int. J. Theor. Phys. 50, 1876 (2011)
37. K. Bamba, R. Myrzakulov, S. Nojiri, S.D. Odintsov, Phys. Rev. D 85, 104036 (2012)
38. R. Myrzakulov, arXiv:1008.4486
39. R. Myrzakulov, arXiv:1205.5266
40. P.A. Gonzalez, E.N. Saridakis, Y. Vasquez, arXiv:1110.4024
41. K. Hayashi, T. Shirafuji, Phys. Rev. D 19, 3524 (1979)
42. K. Hayashi, T. Shirafuji, Phys. Rev. D 24, 3312 (1981)
43. Y.D. Obukhov, J.G. Pereira, Phys. Rev. D 67, 044016 (2003)
44. V.C. de Andrade, J.G. Pereira, Phys. Rev. D 56, 4689 (1997)
45. F.W. Hehl, P.v.d. Heyde, G.D. Kerlick, J.M. Nester, Rev. Mod. Phys. 48, 393 (1976)
46. A.V. Minkevich, Phys. Lett. B 678, 423 (2009)
47. M.L. Fil'chenkova, Astron. Astrophys. Trans. 19, 115 (2000)
48. C. Grupen, *Astroparticle Physics* (Springer, Berlin, 2005)
49. B. Nayak, M. Jamil, Phys. Lett. B 709, 118 (2012)
50. M. Jamil, Int. J. Theor. Phys. 49, 1706 (2010)
51. M. Jamil, A. Qadir, Gen. Relativ. Gravit. 43, 1069 (2011)
52. M.H. Daouda, M.E. Rodrigues, M.J.S. Houndjo, Eur. Phys. J. C 72, 1890 (2012)
53. B. Li, T.P. Sotiriou, J.D. Barrow, Phys. Rev. D 83, 064035 (2011)
54. N. Tamanini, C.G. Bohmer, arXiv:1204.4593
55. D. Momeni et al., work in progress
56. J.G. Russo, A.A. Tseytlin, Nucl. Phys. B 382, 259 (1992)
57. F. Rahaman, M. Kalam, A. DeBenedictis, A.A. Usmani, S. Ray, Mon. Not. R. Astron. Soc. 389, 27 (2008)
58. M. Visser, Phys. Rev. D 80, 025011 (2009)
59. A. Boyarsky, A. Neronov, O. Ruchayskiy, I. Tkachev, Phys. Rev. Lett. 104, 191301 (2010)
60. N. Fornengo, R. Lineros, M. Regis, M. Taoso, Phys. Rev. Lett. 107, 271302 (2011)
61. S. Capozziello, V.F. Cardone, H. Farajollahi, A. Ravanpak, Phys. Rev. D 84, 043527 (2011)
62. M.S. Seigar, Y. Sofue, V. Rubin, Annu. Rev. Astron. Astrophys. 39, 137 (2001)
63. E. Corbelli, P. Salucci, Mon. Not. R. Astron. Soc. 311, 441 (2000)
64. G. Angloher et al., Eur. Phys. J. C 72, 1971 (2012)
65. J.F. Navarro, C.S. Frenk, S.D.M. White, Astrophys. J. 463, 563 (1996)
66. K. Spekkens, R. Giovanelli, M.P. Haynes, Astron. J. 129, 2119 (2005)
67. W.J.G. de Blok, A. Bosma, S. McGaugh, Mon. Not. R. Astron. Soc. 340, 657 (2003)
68. Y. Mao, M. Tegmark, A. Guth, S. Cabi, Phys. Rev. D 76, 104029 (2007)