Gorenstein deformations of nonnormal surfaces

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Abstract. Let $\Delta \subset H$ be the germ of a nonnormal surface along a proper curve with smooth components such that the high index points of $H$ are semi-log-terminal and the Gorenstein singular points are semi-log-canonical of embedding dimension at most 4. We describe the sheaf $T^{1}_{qG}(H)$ of $\mathbb{Q}$-Gorenstein deformations of $H$ and we obtain criteria for $\Delta \subset H$ to have $\mathbb{Q}$-Gorenstein terminal smoothings.

1. Introduction. The purpose of this paper is to describe the $\mathbb{Q}$-Gorenstein smoothings of a germ $\Delta \subset H$ of a nonnormal surface with semi-log-canonical (slc) singularities along a proper curve with smooth irreducible components. The motivation for doing so comes from many different problems.

A stable surface is a proper two-dimensional reduced scheme $H$ such that $H$ has only semi-log-canonical singularities and $\omega^{[k]}_H$ is locally free and ample for some $k > 0$. The boundary of the compactification of surfaces of general type consists of smoothable stable surfaces. Therefore it is interesting to know which stable surfaces are smoothable.

Let $0 \in X$ be the germ of a rational surface singularity. Kollár [Ko91] has made a series of conjectures concerning the components of the versal deformation space Def($X$). A $P$-resolution of $X$ [KSB88] is a proper birational map $g: H \rightarrow X$ such that:

1. $H$ is Cohen Macaulay and Gorenstein outside finitely many points.
2. $K_H$ is $g$-ample.
3. $H$ has a $\mathbb{Q}$-Gorenstein smoothing.

Kollár conjectures that there is a one-to-one correspondence between the components of the versal deformation space Def($X$) of $X$, and $qG$-components of deformations of $P$-resolutions [Ko91]. This conjecture is known to be true for Gorenstein surface singularities, quotient singularities [KSB88] and rational quadruple points [Ste91].

If a $P$-resolution $H$ of a rational surface singularity is normal, then its deformation theory can be described by the local deformations around the singular points. However, there are examples [Ko91] that show that a $P$-resolution may not be normal. In such a case it is possible that locally $H$ is smoothable but...
It is not. Therefore it is important to give a more detailed study of the deformation theory of nonnormal surface germs along smooth proper curves.

There is also a close relation between ∆ ⊂ H as above and 3-fold terminal extremal neighborhoods. A 3-fold terminal extremal neighborhood [Ko-Mo92] is a proper birational map ∆ ⊂ Y → X ∋ P such that Y is the germ of a 3-fold along a proper curve ∆, ∆ = f⁻¹(P)_red, X and Y are terminal, and −KY is f-ample. Then Y is a one-parameter Q-Gorenstein smoothing of the general member H ∈ |O_Y|. In general H may or may not be normal and its singularities may be hard to describe. Kollár and Mori [Ko-Mo92] showed that given a terminal extremal neighborhood as above, the general members of |−KY| and |−KX| have DuVal singularities. If the neighborhood is isolated and the general member of |−KX| is of type An, then the general member H ∈ |O_Y| has semi-log-canonical (slc) singularities, and in fact the points of index bigger than 1 are semi-log-terminal (slt) [Tzi05]. The same is true if the neighborhood is divisorial and the general member S ∈ |−KX| that contains Γ = f(E), where E is the f-exceptional divisor, is of type Am. Hence a detailed description of the Q-Gorenstein deformations of nonnormal surface germs ∆ ⊂ H with slc singularities will provide information about the structure of terminal extremal neighborhoods.

Let ∆ ⊂ H be the germ of a surface with slc singularities along a smooth proper curve such that the points of index bigger than 1 are slt. By the classification of slc singularities [KSB88], if such a germ has a terminal smoothing then its singularities are either normal crossing points, pinch points, degenerate cusps of embedding dimension at most 4 or slt points analytically isomorphic to (xy = 0)/Z_n(a, −a, 1), (a, n) = 1. Such singularities we call singularities of class qG. A surface germ ∆ ⊂ H with singularities of class qG has local Q-Gorenstein smoothings but there is not necessarily a global one extending the local.

The main purpose of this paper is to describe the tangent space of the Q-Gorenstein deformation functor Def^{qG}(H) and in particular the sheaf of first order Q-Gorenstein deformations T^1_{qG}(H).

In Theorem 3.1 it is shown that the 1-dimensional part p(T^1_{qG}(H)) of T^1_{qG}(H) is a rank 1 sheaf on ∆ that is locally free at all points except degenerate cusps of embedding dimension 4 that correspond to singular points of ∆. Moreover, it is shown that there exists a nontrivial exact sequence

\[ 0 \rightarrow L \rightarrow p(T^1_{qG}(H)) \rightarrow \oplus P_k(P) \rightarrow 0 \]

where L is a line bundle on ∆ and P goes over all degenerate cusps of embedding dimension 4 that correspond to singular points of ∆. A formula for the degree of L with data on the minimal log-resolution of ∆ ⊂ H is also given. Theorem 3.2 gives a formula for L with data in the normalization ̃H of H. In the case that ∆ is Gorenstein and rational, an explicit formula that describes p(T^1_{qG}(H)) in terms of L is also given in Corollary 3.3.
Proposition 3.4 shows that if $H^2(T_H) = 0$, $\Delta$ is rational and $\deg (L \otimes O_C) \geq 0$, for any irreducible component $C$ of $\Delta$, then $H$ has global $\mathbb{Q}$-Gorenstein smoothings. In particular this applies to the case when $H$ is a modification of an isolated rational singularity.

As applications, Corollary 3.5 gives a necessary and sufficient condition for a germ $\Delta \subset H$ to have a one parameter terminal $\mathbb{Q}$-Gorenstein smoothing in the case that $\Delta$ is a rational cycle of curves, and Corollary 3.6 shows that under some conditions there exists a terminal extremal neighborhood $\Delta \subset Y \to X \ni P$, such that $H \in |O_Y|$.

Acknowledgments. At this point I must mention that a special case of Theorem 3.1 appeared without proof in [Ko91], and in fact this was the original motivation for this work.

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2. Preliminaries. Let $\mathcal{F}$ be a coherent sheaf of dimension $d$ on a Noetherian scheme $X$. We denote $\mathcal{F}^{[n]} = (\mathcal{F}^{\otimes n})^{**}$ and $p(F) = F/F_d - 1$, where $F_d - 1$ is the maximal subsheaf of $F$ of dimension less or equal to $d - 1$. We call $p(F)$ the pure part of $F$ (if $X$ is integral then this is just the quotient of $F$ by its torsion part).

Definition 2.1. Let $X$ be either a variety, an analytic space or the germ of a singularity, such that it is Cohen Macaulay and Gorenstein in codimension 1. Let $Y$ be the total space of a one parameter deformation of $X$. Then we say that $Y$ is a $\mathbb{Q}$-Gorenstein deformation if $\omega^{[n]}_Y$ is locally free for some $n$. The assumptions on $X$ assure that this makes sense even for nonnormal varieties.

Next we define the $\mathbb{Q}$-Gorenstein deformation functor. Let $\text{Art}(\mathbb{C})$ be the category of finite local Artin $\mathbb{C}$-algebras.

Definition 2.2. (Definition 3.17 [KSB88]) The functor of $\mathbb{Q}$-Gorenstein deformations is the functor $\text{Def}^{qG}(X): \text{Art}(\mathbb{C}) \to \text{Sets}$ such that for any finite local $\mathbb{C}$-algebra $A$, $\text{Def}^{qG}(X)(A)$ is the set of isomorphism classes of flat morphisms $f: Y \to S = \text{Spec}(A)$, such that $Y \otimes k(A) \cong X$, the sheaf $\omega^{[n]}_{Y/S}$ is invertible for some $n$.

It is not immediately clear that $\text{Def}^{qG}(X)$ as defined above is indeed a functor. For this to be true, the property that the relative dualizing sheaf $\omega_{Y/S}$ is $\mathbb{Q}$-Gorenstein should be stable under base extension. However, this is true [Has-Kov04, Lemma 2.6] and therefore $\text{Def}^{qG}(X)$ is a functor and moreover a subfunctor of the versal deformation functor $\text{Def}(X)$.
Let $k$ be a field and $F: \text{Art}(k) \rightarrow \mathcal{C}$ be any functor from the category of finite local $k$-algebras to a category $\mathcal{C}$. Then $F(k[t]/(t^2))$ is called the tangent space of $F$ [Sch68].

**Definition 2.3.** We denote by $T^1_{qG}(X)$ the tangent space of the functor $\text{Def}^{qG}(X)$, i.e., the space of first order $\mathbb{Q}$-Gorenstein deformations of $X$. Moreover, we denote by $T^1_{qG}(X)$ the sheaf of $\mathcal{O}_X$-modules such that for any affine open subset $U \subset X$, $T^1_{qG}(X)(U)$ is the $\mathcal{O}_X(U)$-module of isomorphism classes of first order $\mathbb{Q}$-Gorenstein deformations of $U$. If $U$ is Gorenstein, then $T^1_{qG}(X)(U) = T^1(U)$, the space of first order deformations of $U$.

Standard results from deformation theory show that there is an exact sequence

$$0 \rightarrow H^1(T_X) \rightarrow T^1_{qG}(X) \rightarrow H^0(T^1_{qG}(X)) \rightarrow H^2(T_X).$$

Moreover, it is well known that the obstructions to globalize local deformations are in $H^1(T^1_{qG}(X))$ and $H^2(T_X)$. $T^1_{qG}(X)$ is a sheaf supported on the singular locus of $X$ and hence in the isolated singularity case the first obstruction space is trivial.

The purpose of this paper is to describe the sheaf $T^1_{qG}(X)$ (and hence the tangent space of the functor $\text{Def}^{qG}(X)$), in the case when $X$ has singularities of class $qG$ [Definition 2.4] and to obtain criteria for the existence of $\mathbb{Q}$-Gorenstein smoothings of $X$.

For the definitions and basic properties of minimal semi-resolution, semi-log-canonical and semi-log-terminal singularities, we refer the reader to [KSB88].

**Definition 2.4.** A nonnormal surface singularity $P \in X$ is called a singularity of class $qG$, if it is analytically isomorphic to one of the following:

1. Normal crossing point: $(xy = 0) \subset \mathbb{C}^3$.
2. Pinch point: $(x^2 - y^2z = 0) \subset \mathbb{C}^3$.
3. Degenerate cusp of embedding dimension at most 4.
4. Semi-log-terminal ($slt$): $(xy = 0)/\mathbb{Z}_n(a, -a, 1), (a, n) = 1$.

**Remark 2.5.** The normal crossing points, pinch points and degenerate cusps of embedding dimension at most 4 are exactly the semi-log-canonical singularities of embedding dimension at most 4 [KSB88]. Moreover, the last singularity is the only nonnormal semi-log-terminal singularity that admits a terminal $\mathbb{Q}$-Gorenstein smoothing. The main motivation of this work was to investigate which surfaces with semi-log-canonical singularities admit a terminal $\mathbb{Q}$-Gorenstein smoothing. Such a surface must have singularities of class $qG$ [Tzi05]. This is the motivation behind the previous definition.

The next lemma gives the classification of degenerate cusps of embedding dimension at most 4.
LEMMA 2.6. (Karras [Ka77], Shepherd-Barron [Sh-B83], Stevens [Ste98])

Let $P \in X$ be a degenerate cusp and let $f : Y \to X$ be the minimal semi-resolution. Let $\Gamma = f^{-1}(P)_{\text{red}}$. Then

1. $\text{mult}_P(X) = \max\{2, -\Gamma^2\}$.

2. $\text{embdim}_P(X) = \max\{3, -\Gamma^2\}$.

3. If $\Gamma^2 = -1$, then in local analytic coordinates

   $$(P \in X) \cong (x^2 = y^3 + y^2z^2) \subset \mathbb{C}^3.$$ 

4. If $\Gamma^2 = -2$ then in suitable local analytic coordinates

   $$(P \in X) \cong (x^2 + z^2(z^{n+1} - y^2) = 0) \subset \mathbb{C}^3$$

with $2 \leq n \leq \infty$. In the case that $n = \infty$ we set $z^n = 0$.

   If $n < \infty$, then the singular locus of $X$ is a smooth irreducible curve. If $n = \infty$, then the singular locus of $X$ is a reducible curve with exactly two smooth irreducible components. We will call such a singularity a degenerate cusp of type $T^2_n$.

5. If $\Gamma^2 = -3$ then in local analytic coordinates

   $$(P \in X) \cong (x^{p+2} + y^{q+2} - xyz = 0) \subset \mathbb{C}^3$$

where $1 \leq p, q \leq \infty$ and if $u$ is any of $x, y, z$, we set $u^\infty = 0$. Moreover, the singular locus of $X$ has exactly $1 + k$ smooth irreducible components, where $k$ is the number of exponents that are $\infty$. We will call such a singularity a degenerate cusp of type $T^3_{p,q}$. 

6. If $\Gamma^2 = -4$ then in local analytic coordinates

   $$(P \in X) \cong (xy - z^p - t^q = 0, zt - x^r = 0) \subset \mathbb{C}^4$$

where $2 \leq p, q, r \leq \infty$ and if $u$ is any of $x, y, z, t$, we set $u^\infty = 0$. Moreover, the singular locus of $X$ has exactly $1 + k$ smooth irreducible components, where $k$ is the number of exponents that are $\infty$. We will call such a singularity a degenerate cusp of type $T^4_{p,q,r}$.

In particular we see that any degenerate cusp of embedding dimension at most 4 is a complete intersection and in fact a degenerate cusp is a complete intersection if and only if it has embedding dimension at most 4 [Ste98]. Hence they all have smoothings and in fact $T^1_qG(P \in X) = T^1(P \in X)$. 
3. Description of $T^1_{qG}(H)$. The next theorem describes the pure part of $T^1_{qG}(H)$. Its embedded part is described in Lemma 3.9

**THEOREM 3.1.** Let $\Delta \subset H$ be the germ of a surface along a reduced proper curve. Assume that every irreducible component of $\Delta$ is smooth and that locally along $\Delta$ the surface $H$ has singularities of class $qG$. Let $\Delta' \subset H \xrightarrow{f} H \supset \Delta$ be the minimal log resolution of $\Delta \subset H$. Then

1. $p(T^1_{qG}(H))$ is a rank 1 pure sheaf on $\Delta$ that fits in an exact sequence

$$0 \longrightarrow L \longrightarrow p(T^1_{qG}(H)) \longrightarrow \bigoplus p k(P) \longrightarrow 0$$

where $L$ is a line bundle on $\Delta$ and $P \in H$ runs over all degenerate cusps of embedding dimension 4 of type other than $T^4_{p,q,\infty}$ such that $P$ is a singular point of $\Delta$ too. Moreover, the restriction of the above extension on any irreducible component of $\Delta$ is split.

2. Let $C$ be an irreducible component of $\Delta$ and $C'$ be the divisorial part of $f^{-1}(C)$ that dominates $C$.

Then

$$\deg (L \otimes \mathcal{O}_C) = C'^2 + q + p + 6c_1 + 4c_2 + 3c_3 + 2c_4$$

where $q, p$ are the numbers of slt and pinch points that lie on $C$ and $c_i$ are the numbers of degenerate cusps with $\Gamma^2 = -i$, $1 \leq i \leq 4$, that lie on $C$.

The proof of Theorem 3.1 is given in section 4.

It is relatively easy to rewrite the formula 3.1.2 in terms of the normalization $\tilde{H}$ of $H$ as follows.

**THEOREM 3.2.** With assumptions as in Theorem 3.1, let $\pi: \tilde{H} \longrightarrow H$ be the normalization of $H$ and $\tilde{C}$ the divisorial part of $\pi^{-1}(C)$.

Let $U_3, U_4$ be the sets of points $P \in H$ that lie on $C$ such that $P \in H$ is a degenerate cusp with $\Gamma^2 = -3$ and $-4$ respectively. Then

$$\deg (L \otimes \mathcal{O}_C) = C'^2 + p + 2c_1 + 2c_2 + \sum_{P \in U_3} \alpha_3(P) + \sum_{Q \in U_4} \alpha_4(Q)$$

where

$$\alpha_3(P) = 1 + \frac{p + q}{pq + p + q}$$

if $P \in U_3$ is of type $T^3_{p,q'}$ and

$$\alpha_4(Q) = \frac{r(p + q) - 4}{rpq - p - q}$$
if $Q \in U_4$ is of type $T_{p,q,r}^4$. If any of $p$, $q$, $r$ is infinity, then we understand $\alpha_3(P)$ and $\alpha_4(Q)$ to mean the corresponding limits at infinity.

Theorem 3.2 can be derived by rather long but straightforward calculations from 3.1.2 and its proof is omitted.

The next corollary shows that if the singularities of $\Delta$ are at most ordinary nodes, then $p(T^1_{qG}(H))$ is completely determined by $L$. Moreover, if $\Delta$ does not contain any cycles, then it is also possible to write down an explicit formula for $p(T^1_{qG}(H))$ in terms of the restriction of $L$ on the irreducible components of $\Delta$. This happens in many cases of geometric interest, like construction of flips and divisorial contractions.

**Corollary 3.3.** With assumptions as in Theorem 3.1, suppose in addition that the singularities of $\Delta$ are at worst ordinary nodes. Then $p(T^1_{qG}(H))$ is completely determined by $L$. Moreover, suppose that $\Delta$ does not contain any cycles and let $P_1, \ldots, P_k$ be the degenerate cusps of $H$ of embedding dimension 4 and of type other than $T_{p,q,\infty}^4$ that correspond to singular points of $\Delta$. Then there is a decomposition $\Delta = \Delta_1 + \cdots + \Delta_{k+1}$ such that for each $i$, $\Delta_i$ is connected, $\bigcup_{i\neq j}(\Delta_i \cap \Delta_j) = \{P_1, \ldots, P_k\}$, and

$$p(T^1_{qG}(H)) = L|_{\Delta_1} \oplus \cdots \oplus L|_{\Delta_{k+1}}.$$ 

**Proof.** By Theorem 3.1, there is an exact sequence

$$0 \longrightarrow L \longrightarrow p(T^1_{qG}(H)) \longrightarrow \bigoplus_{i=1}^k k(P_i) \longrightarrow 0,$$

where $P_1, \ldots, P_k$ are the degenerate cusps of $H$ of embedding dimension 4 and of type other than $T_{p,q,\infty}^4$ that correspond to singular points of $\Delta$.

Moreover, since $p(T^1_{qG}(H))$ is pure, the sequence is not split at any of the $P_i$, for all $i$. But there is only one extension of $\bigoplus_{i=1}^k k(P_i)$ by $O_\Delta$ with these properties. Indeed, since $P_i \in \Delta$ is an ordinary node, it follows that $\text{Ext}^1_\Delta(k(P_i), O_\Delta) = k(P_i)$.

Hence locally around $P_i$ there is exactly one nonsplit extension of $k(P_i)$ by $O_\Delta$, for any $P_i$, and therefore there is a unique extension of $O_\Delta$ by $\bigoplus_{i=1}^k k(P_i)$ that is not split at any of the $P_i$. Therefore $p(T^1_{qG}(H))$ is completely determined by $L$.

Suppose now in addition that $\Delta$ does not contain any cycles. In this case it is possible to write down explicitly the unique nonsplit extension of $\bigoplus_{i=1}^k k(P_i)$ by $L$. Since $\Delta$ contains no cycles and there are at most two irreducible components passing through any point, there is a decomposition $\Delta = \Delta_1 + \cdots + \Delta_{k+1}$ such that $\Delta_i$ is connected for each $i$ and

$$\bigcup_{i\neq j}(\Delta_i \cap \Delta_j) = \{P_1, \ldots, P_k\}.$$
Then there is an extension
\[ 0 \to \mathcal{O}_\Delta \xrightarrow{f} \bigoplus_{j=1}^{k+1} \mathcal{O}_{\Delta} \xrightarrow{g} \bigoplus_{s<r} \mathcal{O}_{\Delta} I_s + I_r \to 0, \]
or equivalently
\[ 0 \to \mathcal{O}_\Delta \to \bigoplus_{i=1}^{k+1} \mathcal{O}_{\Delta_i} \to \bigoplus_{i=1}^{k} k(p_i) \to 0, \]
where \( I_j \) is the ideal sheaf of \( \Delta_j \) in \( \Delta \), \( f(1) = (1, 1, \ldots, 1) \) and \( g((a_j)) = (b_s,r) \), with \( b_{s,r} = a_s - a_r \). Tensoring with \( L \) we get the nonsplit extension
\[ 0 \to L \to \bigoplus_{i=1}^{k+1} L|_{\Delta_i} \to \bigoplus_{i=1}^{k} k(p_i) \to 0. \]

Therefore, from the uniqueness of such extension, \( p(T^1_{\mathbb{Q}G}(H)) \cong \bigoplus_{i=1}^{k+1} L|_{\Delta_i}. \)

**Proposition 3.4.** Let \( \Delta \subset H \) be a surface germ as in Theorem 3.1 such that \( H^2(T_H) = 0 \). Assume that \( \Delta \) is rational and let \( C_i, i = 1, \ldots, k \) be its irreducible components. Let \( L \in \text{Pic}(\Delta) \) as in Theorem 3.1 and suppose that \( d_i = \deg (L \otimes \mathcal{O}_{C_i}) \geq 0 \), for all \( i \). Then there is a one parameter \( \mathbb{Q} \)-Gorenstein deformation \( X \to T \) of \( H \) such that the general fiber \( X_g \) of \( X \) is smooth and the singular locus of \( X \) consists only of isolated cyclic quotient singularities of type \( 1/n(a,-a,1) \) corresponding to the slt points of \( H \) and ordinary double points.

**Remark.** In general it is rather difficult to calculate \( H^2(T_H) \). However if \( H \) is a modification of an isolated surface singularity \( 0 \in S \), i.e., there is a birational morphism \( f: \Delta \subset H \to S \) \( \ni 0 \), then by the formal functions theorem it follows that \( H^2(T_H) = 0 \). Moreover, if \( R^{i_f}f_*\mathcal{O}_H = 0 \) (for example if the singularity \( 0 \in S \) is rational), then \( \Delta \) is rational and hence the assumptions of Proposition 3.4 are satisfied.

**Proof.** Since \( \Delta \) is rational and \( \deg (L \otimes \mathcal{O}_{C_i}) \geq 0 \), it follows that \( H^1(L) = 0 \). Hence from the exact sequence
\[ 0 \to L \to p(T^1_{\mathbb{Q}G}(H)) \to \bigoplus p k(P) \to 0 \]
it follows that \( H^1(p(T^1_{\mathbb{Q}G}(H))) = 0 \). From our assumption, \( H^2(T_H) = 0 \) as well, and hence there are no obstructions to lift local deformations.

Since \( d_i = \deg (L \otimes \mathcal{O}_{C_i}) \geq 0 \), there is a section \( s \) of \( L \) that does not vanish at slt points or degenerate cusps, and that it vanishes with order 1 at \( d_i \) normal crossing points of \( C_i \), for all \( i \) [Art66]. Then locally \( s \) gives one-parameter smoothings with either cyclic quotient singularities or ordinary double points as claimed in the statement which globalize since all obstructions vanish. \( \square \)
Corollary 3.5. Let $\Delta \subset H$ be the germ of a surface along a proper curve $\Delta$ as in Theorem 3.1. In addition assume that $H^1(O_\Delta) = 0$, $H^2(T_H) = 0$ and

$$\tilde{C}^2 + p + 2c_1 + 2c_2 + \sum_{P \in U_3} \alpha_3(P) + \sum_{Q \in U_4} \alpha_4(Q) \geq 0$$

for every irreducible component $C$ of $\Delta$. Then there exists a one-parameter $\mathbb{Q}$-Gorenstein smoothing $X$ of $H$ whose singularities are isolated cyclic quotient singularities of type $1/n(a, -a, 1)$, or ordinary double points $(xy - zt = 0) \subset \mathbb{C}^4$.

The following corollary shows that under certain conditions, it is possible to construct three dimensional terminal extremal neighborhoods $Y \rightarrow X$ such that $H \in |O_Y|$.

Corollary 3.6. Let $g: \Delta \subset H \rightarrow S \ni 0$ be a modification of an isolated surface singularity $0 \in S$. Assume that $H$ has singularities of class $qG$ along $\Delta$ and that $R^1 f_* O_H = 0$. Then a one parameter smoothing of $g, f: X \rightarrow Y$, with $X$ terminal, exists, if and only if

$$\tilde{C}^2 + p + 2c_1 + 2c_2 + \sum_{P \in U_3} \alpha_3(P) + \sum_{Q \in U_4} \alpha_4(Q) \geq 0$$

for every irreducible component $C$ of $\Delta$. In particular, a terminal extremal neighborhood $f: \Delta \subset X \rightarrow Y \ni P$ exists such that $H \in |O_X|$, if and only if the above inequality holds and $K_H \cdot C < 0$, for every irreducible component of $\Delta$.

Proof. Since $R^1 f_* O_H = 0$ and the fibers of $f$ are at most 1-dimensional, it follows that $\Delta$ is a rational cycle of curves and $H^2(T_H) = 0$. Hence by Corollary 3.5 there exists a terminal smoothing $X$ of $H$. Then by [Wa76], $g$ extends to $X$, and hence there exists a 3-fold contraction $g: X \rightarrow Y$, as claimed. If it happens that $K_H \cdot C < 0$, then $K_X \cdot C = K_H \cdot C < 0$, and hence $Y$ is terminal as well and the contraction is an extremal neighborhood.

Example 3.7. Let $U$ be the germ of a smooth surface around a chain of smooth rational curves with the following dual graph

$$\begin{align*}
-2 & \quad -2 \quad -3 \quad -k \quad -3 \quad -1 \quad -2 \quad -4 \\
\circ & \quad \circ & \quad \circ & \quad \bullet & \quad \circ & \quad \bullet & \quad \circ & \quad \circ \\
C_2 & \quad C_1 & \quad \circ & \quad \circ & \quad \circ & \quad \circ
\end{align*}$$

Contracting all curves except $C_1$ and $C_2$ we get a map $\tilde{H} \rightarrow S$, such that $0 \in S$ is smooth if $k = 2$ and a cyclic quotient singularity of type $1/15(1, 11)$ if $k = 3$. Moreover, $\tilde{H}$ has exactly three singular points $P_1 \in C_1, P_2 \in C_2$ and $Q \in C_1 \cap C_2$, analytically isomorphic to $1/7(1, 3), 1/7(1, -3)$ and $1/3(1, 1)$, respectively. We can now identify $C_1$ and $C_2$ to a smooth rational curve $C$ and get a nonnormal surface $C \subset H$. $H$ has an slt singular point analytically isomorphic to $(xy = 0)/\mathbb{Z}_7(3, -3, 1)$, a degenerate cusp of type $T^3_{1,1}$, and is normal crossing everywhere.
else. Moreover, $\tilde{H}$ is the normalization of $H$ and there is a morphism $g: H \rightarrow S$. From Theorem 3.1.2 it follows that $d = \deg L = -k - 1 + 3 = -k + 3$. Hence $L \cong O_C$, if $k = 3$, and $L \cong \mathcal{O}_C(-1)$ if $k = 2$. Hence, if $k = 2$ then $H$ is not smoothable and if $k = 3$ then there exists a one parameter terminal smoothing $Y$ of $H$ extending the local ones. More precisely, $Y$ has exactly 1 singular point and it is of type $1/7(3, -3, 1)$. The morphism $g$ extends to $Y$ [Wa76], and hence we get an extremal neighborhood $f: C \subset Y \rightarrow X \ni 0$, such that $H \in |O_Y|$.

The next lemma describes the local deformations of singularities of class $qG$.

**Lemma 3.8.** Let $(P \in H)$ be the germ of a singularity as in Theorem 3.1. Then $(P \in H)$ has a smoothing $(P \in X)$ such that:

1. $(P \in X)$ is smooth if $(P \in H)$ is not a degenerate cusp of embedding dimension 4 or an slt point.

2. $(P \in X) \cong (xy - zt = 0) \subset \mathbb{C}^4$ if $(P \in H)$ is a degenerate cusp of embedding dimension 4.

3. $(P \in X) \cong \mathbb{C}^3/\mathbb{Z}_a(a, -a, 1)$ if $(P \in H)$ is an slt point.

**Proof.** The first part is obvious. Now suppose that $(P \in H)$ is a degenerate cusp of embedding dimension 4. Then in suitable local analytic coordinates it is given by equations $xy - z^p - r^4 = 0, zt - x^r = 0$. Then take $X$ to be $xy + zt - z^p - r^4 - x^r = 0$ which is analytically equivalent to $xy - zt = 0$.

Let $(P \in H)$ be an slt point. Then $(P \in H) \cong (xy = 0)/\mathbb{Z}_a(a, -a, 1)$. A $\mathbb{Q}$-Gorenstein smoothing is $(xy + t = 0)/\mathbb{Z}_a(a, -a, 1, 0) \cong \mathbb{C}^3/\mathbb{Z}_a(a, -a, 1)$. \qed

The next lemma describes the support and the torsion part of $T^{1}_{qG}(H)$.

**Lemma 3.9.** Let $\Delta \subset H$ be the germ of a surface along a proper curve as in Theorem 3.1. Then:

1. The divisorial part of the support of $T^{1}_{qG}(H)$ is $\Delta$ and it has an embedded point over any pinch point or degenerate cusp of type other than $T_{3, \infty, \infty}^4$ and $T_{4, \infty, \infty}^4$.

2. The pure part $p(T^{1}_{qG}(H))$ of $T^{1}_{qG}(H)$ is a rank 1 sheaf on $\Delta$ and it is free at all points except at the singular points of $\Delta$ that correspond to degenerate cusps of embedding dimension 4 of $H$ of type other than $T_{p, q, \infty}^4$ ($p, q < \infty$).

3. Let $C$ be any component of $\Delta$. Then the torsion parts of $T^{1}_{qG}(H) \otimes O_C$ and $p(T^{1}_{qG}(H)) \otimes O_C$ are $\oplus p P$ and $\oplus q Q$, respectively, where $P \in H$ is a degenerate cusp of embedding dimension 4 that lies on $C$ and $Q \in H$ is a degenerate cusp of embedding dimension 4 of type other than $T_{p, q, \infty}^4$ that lies on $C$ and is a singular point of $\Delta$. 

Proof. It is well known that for any hypersurface singularity \( X \) given by an equation \( f = 0 \) in \( \mathbb{C}^n \),

\[
T^1(X) = T^1_{qG}(X) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(f, J(f))},
\]

where \( J(f) \) is the Jacobian ideal of \( f \). At any generic point of \( \Delta \), \( H \) is normal crossing, i.e., \( H \) is given by \( xy = 0 \) in \( \mathbb{C}^3 \) and therefore

\[
T^1_{qG}(H) = \mathbb{C}[x, y, z]/(x, y) = \mathbb{C}[z].
\]

Hence \( T^1_{qG}(H) \) is free of rank 1 at any generic point of \( \Delta \). We proceed to describe the embedded and pure part of \( T^1_{qG}(H) \).

Let \( P \in H \) be an slt point. Let \( P \in X \) be a \( qG \)-smoothing and let \( \pi: \hat{X} \rightarrow X \) be the index 1 cover of \( X \). Then \( \pi^{-1}(H) = \hat{H} \rightarrow H \) is the index 1 cover of \( H \) [KSB88] and hence \( \hat{H} = (xy = 0) \subset \mathbb{C}^3 \). Therefore \( qG \)-smoothing of \( H \) are quotients of \( qG \)-smoothing of \( \hat{H} \) and therefore, \( T^1_{qG}(P \in H) = T^1(\hat{H})^{2n} \). Hence \( T^1_{qG}(P \in H) \) is a line bundle at the slt points.

Let \( P \in H \) be an index 1 singular point of \( H \). Then this is either a pinch point or a degenerate cusp of embedding dimension at most 4. Pinch points and cusps of embedding dimension 3 are hypersurface singularities and hence \( T^1_{qG}(H) = T^1(H) \) can be calculated using (3.1). Cusps of embedding dimension 4 are codimension 2 complete intersection singularities given by two equations \( f = g = 0 \) in \( \mathbb{C}^4 \). According to [Kas-Sch72]

\[
T^1_{qG}(P \in H) = \frac{\mathbb{C}[x, y, z, t] \oplus \mathbb{C}[x, y, z, t]}{M},
\]

where \( M \) is the \( \mathbb{C}[x, y, z, t] \)-submodule of \( \mathbb{C}[x, y, z, t] \oplus \mathbb{C}[x, y, z, t] \) that is generated by \( (f, 0), (0, f), (g, 0), (0, g), (\partial f/\partial x, \partial g/\partial x), (\partial f/\partial y, \partial g/\partial y), (\partial f/\partial z, \partial g/\partial z) \) and \( (\partial f/\partial t, \partial g/\partial t) \). Now explicit calculations using the equations of the singularities given in Lemma 2.6 show part (1) of the lemma and that if \( C \) is any irreducible component of \( \Delta \), then the torsion part of \( T^1_{qG}(H) \otimes \mathcal{O}_C \) is \( \oplus p_k(P) \), where \( P \in H \) is a degenerate cusp of embedding dimension 4 that lies on \( C \).

It remains to show the claims about the pure part \( p(T^1_{qG}(H)) \) of \( T^1_{qG}(H) \). Let \( P \) be a point of \( \Delta \). If \( \Delta \) is smooth at \( P \) then \( p(T^1_{qG}(H)) \) is a line bundle at \( P \) since it is pure. Suppose that \( P \) is a singular point of \( \Delta \). Then \( P \in H \) is a degenerate cusp. Suppose that it has embedding dimension at most 3. Then locally there is an embedding \( H \subset \mathbb{C}^3 \). Dualizing the exact sequence

\[
0 \rightarrow I_{H, \mathbb{C}^3}/I_{H, \mathbb{C}^3}^2 \rightarrow \Omega_{\mathbb{C}^3} \otimes \mathcal{O}_H \rightarrow \Omega_H \rightarrow 0
\]
we get the exact sequence

\[
\cdots \longrightarrow \mathcal{N}_{H/\mathbb{C}^3} \overset{\phi}{\longrightarrow} \text{Ext}^1_{\mathcal{O}_H}(\Omega_H, \mathcal{O}_H) = T^1(H) \longrightarrow 0.
\]

Restricting on \( \Delta \) it follows that \( p(T^1_{qG}(H)) = \mathcal{N}_{H/\mathbb{C}^3} \otimes \mathcal{O}_\Delta \) and hence it is a line bundle. It remains to consider the case of a cusp \( P \in H \) of embedding dimension 4. In this case we will show the claims by doing explicit calculations using the equations of the cusps. From the previous discussion, \( p(T^1_{qG}(H)) \) is a line bundle at \( P \) unless \( P \) corresponds to a singular point of \( \Delta \). Hence it must be of type \( T^4_{4, q, r} \) where at least one of \( p, q \) and \( r \) is infinite. We will only do the cases \( T^4_{4, q, \infty} \) and \( T^4_{p, q, \infty} \). The rest are done similarly.

So let \( P \in H \) be of type \( T^4_{4, q, \infty} \). Then according to Lemma 2.6, \( P \in H \) is given by the equations \( xy + z^p = zt + x^t = 0 \) in \( \mathbb{C}^4 \). Using (3.2) it follows that

\[
T^1_{qG}(H) = \frac{\mathbb{C}[x, y, z, t] \oplus \mathbb{C}[x, y, z, t]}{M},
\]

where \( M \) is generated by \((y, rx^{t-1}), (x, 0), (pz^{p-1}, t), (0, z), (z^p, 0), (0, xy), (zt, 0), (0, x^t)\). The ideal sheaf of \( \Delta \) in \( H \) is \( I_\Delta = (x, z, yt) \) and hence restricting on \( \Delta \) we find that

\[
T^1_{qG}(H) \otimes \mathcal{O}_\Delta = \frac{\mathbb{C}[y, t] \oplus \mathbb{C}[y, t]}{(y, 0, (0, t))},
\]

which is clearly a pure \( \mathcal{O}_\Delta \)-module. Hence this is \( p(T^1_{qG}(H)) \). Moreover straightforward calculations show that its restriction on any of the two irreducible components of \( \Delta \) has one dimensional torsion and hence \( p(T^1_{qG}(H)) \) is not a line bundle on \( \Delta \).

Let \( P \in H \) be of type \( T^4_{p, q, \infty} \). Then again according to Lemma 2.6, \( P \in H \) is given by the equations \( xy + z^p + t^q = zt = 0 \) in \( \mathbb{C}^4 \). Similar calculations as in the previous case show that

\[
T^1_{qG}(H) = \frac{\mathbb{C}[x, y, z, t] \oplus \mathbb{C}[x, y, z, t]}{M},
\]

where \( M \) is generated by \((zt, 0), (0, zt), (y, 0), (x, 0), (pz^{p-1}, t), (qt^{t-1}, z), (0, xy)\). The ideal sheaf of \( \Delta \) in \( H \) is \( I_\Delta = (z, t, xy) \) and hence restricting on \( \Delta \) we find that

\[
T^1_{qG}(H) \otimes \mathcal{O}_\Delta = \frac{\mathbb{C}[x, y] \oplus \mathbb{C}[x, y]}{(y, 0, (0, xy)).}
\]

This is still not pure since \((1, 0)\) is annihilated by the maximal ideal \((x, y)\) of \( \mathcal{O}_\Delta \). However, there is a surjection \( T^1_{qG}(H) \otimes \mathcal{O}_\Delta \longrightarrow \mathcal{O}_\Delta = \mathbb{C}[x, y]/(xy) \) given by projection to the second variable. Hence \( p(T^1_{qG}(H)) = \mathcal{O}_\Delta \) and hence in this case it is a line bundle on \( \Delta \). This concludes the proof of the lemma. \( \Box \)
The next lemma shows how to calculate the pure part of $T^1_{qG}(H)$ from a suitable embedding.

**Lemma 3.10.** Let $\Delta \subset H$ be the germ of a surface along a proper curve as in the statement of Theorem 3.1. Let $H \subset X$ be an embedding such that $X$ is a $\mathbb{Q}$-Gorenstein 3-fold, $H$ is Cartier in $X$, and for all $P \in \Delta \subset H$, $(P \in X)$ is a general $\mathbb{Q}$-Gorenstein smoothing of $(P \in H)$. Then there is a short exact sequence

$$0 \to L \to p(T^1_{qG}(H)) \to \oplus_P k(P) \to 0,$$

where $L = \mathcal{N}_{H/X} \otimes \mathcal{O}_\Delta$ is a line bundle on $\Delta$ and $P \in H$ runs over all degenerate cusps of embedding dimension 4 of type other than $T^4_{p,q,\infty}$ such that $P$ is a singular point of $\Delta$ too. Moreover, the restriction of the above extension on any irreducible component of $\Delta$ is trivial.

**Proof.** Dualizing the exact sequence

$$0 \to \mathcal{I}_{H,X}/ \mathcal{I}_{H,X}^2 \to \Omega_X \otimes \mathcal{O}_H \to \Omega_H \to 0$$

we get the exact sequence

$$0 \to \mathcal{I}_{H,X}/ \mathcal{I}_{H,X}^2 \to \mathcal{I}_{H,X} \to \mathcal{O}_H \to 0$$

(3.4) \[ \cdots \to \mathcal{N}_{H/X} \stackrel{\phi}{\to} \text{Ext}^1_{H}(\Omega_H, \mathcal{O}_H) = T^1(H) \to \text{Ext}^1_{H}(\Omega_X \otimes \mathcal{O}_H, \mathcal{O}_H) \to 0, \]

where $\text{Ext}^1_{H}(\Omega_X \otimes \mathcal{O}_H, \mathcal{O}_H)$ is torsion and it is supported over the slt points of $H$ and the degenerate cusps of embedding dimension 4. We claim that the image of $\phi$ is $T^1_{qG}(H)$. This is local around the slt singular points of $H$. So let $P \in U \subset H$ be a small neighborhood of a slt singular point $P$ of $H$. Let $V = U \cap H$. Then $\mathcal{N}_{H/X}|_V = \mathcal{N}_{V/U} = \mathcal{O}_V$ and $\phi(1) \in T^1(V)$ is the class of the first order deformation of $V$ induced by $U$. But by construction this is $\mathbb{Q}$-Gorenstein. Therefore $\phi(\mathcal{N}_{V/U}) \subset T^1_{qG}(V)$. In fact we will show that $\phi(\mathcal{N}_{V/U}) = T^1_{qG}(V)$. Let $\pi: \hat{U} \to U$ be the index 1 cover of $U$. Then it is well known that $\hat{V} = \pi^{-1}(V) \to V$ is the index 1 cover of $V$ and in fact any $\mathbb{Q}$-Gorenstein deformation of $V$ comes from a deformation of $\hat{V}$ [KSB88]. In fact, if $n$ is the index of $H$ at $P$, then the action of $\mathbb{Z}_n$ on $\hat{V}$ induces an action on $T^1(\hat{V})$ and $T^1_{qG}(V) = T^1(\hat{V})^{\mathbb{Z}_n}$. Now since $\hat{V}$ is a normal crossing singularity, it follows that there is a surjection

$$\hat{\phi}: \mathcal{N}_{\hat{V}/\hat{U}} \to T^1(\hat{V}),$$

and therefore by taking $\mathbb{Z}_n$-invariants, there is a surjection

$$\phi: \mathcal{N}_{\hat{V}/\hat{U}}^{\mathbb{Z}_n} = \mathcal{N}_{V/U} \to T^1_{qG}(V) = T^1(\hat{V})^{\mathbb{Z}_n}.$$
as claimed. Hence from (3.4) there is an exact sequence

$$\mathcal{N}_{H/X} \xrightarrow{\phi} T^1_{qG}(H) \longrightarrow \oplus \text{Ext}_{\mathcal{O}_{H,P}}^1(\Omega_{X,P}, \mathcal{O}_{H,P}) \longrightarrow 0 \tag{3.5}$$

where $P$ runs through all degenerate cusps of $H$ of embedding dimension 4.

Next we calculate the torsion part that appears in the above sequence. This is local at degenerate cusps $P \in H$ of embedding dimension 4. Locally around $P$, $X = (xy - zw = 0) \subset \mathbb{C}^4$. Hence there is an exact sequence

$$0 \longrightarrow I_{X,C^5}/I_{X,C^5}^2 \longrightarrow \Omega_{C^5} \otimes \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow 0.$$

Considering that $I_{X,C^5}/I_{X,C^5}^2 \cong \mathcal{O}_X$, this gives that

$$\text{Hom}_H(\Omega_{C^5} \otimes \mathcal{O}_H, \mathcal{O}_H) \longrightarrow \text{Hom}_H(\mathcal{O}_H, \mathcal{O}_H) \longrightarrow \text{Ext}_H^1(\Omega_X \otimes \mathcal{O}_H, \mathcal{O}_H) \longrightarrow 0.$$

Now writing explicitly all the maps involved in the above sequence and the isomorphisms $\text{Hom}_H(\Omega_{C^5} \otimes \mathcal{O}_X, \mathcal{O}_H) \cong \mathcal{O}^{\otimes 4}_H$, $\text{Hom}_H(\mathcal{O}_H, \mathcal{O}_H) \cong \mathcal{O}_H$, it follows that the last exact sequence is equivalent to

$$\mathcal{O}^{\otimes 4}_H \xrightarrow{\phi} \mathcal{O}_H \longrightarrow \text{Ext}_H^1(\Omega_X \otimes \mathcal{O}_H, \mathcal{O}_H) \longrightarrow 0,$$

where the map $\phi$ is given by sending the standard basis elements $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ and $(0, 0, 0, 1)$ to $x$, $y$, $z$, $t$, respectively. Hence it is now clear that $\text{Ext}_H^1(\Omega_X \otimes \mathcal{O}_H, \mathcal{O}_H) \cong \mathbb{C} = k(P)$.

Passing to the pure part of $T^1_{qG}(H)$ and using (3.5) we get a commutative diagram

$$0 \longrightarrow \mathcal{N}_{H/X} \otimes \mathcal{O}_\Delta \longrightarrow T^1_{qG}(H) \otimes \mathcal{O}_\Delta \longrightarrow \oplus p(k(P)) \longrightarrow 0$$

where $P \in H$ is a degenerate cusp of embedding dimension 4. But at any $P$ that is not a singular point of $\Delta$, $p(T^1_{qG}(H))$ is free of rank one and in fact it is the free part of $T^1_{qG}(H) \otimes \mathcal{O}_\Delta$. Also, from Lemma 3.9, $T^1_{qG}(H) \otimes \mathcal{O}_\Delta$ has one dimensional torsion at any degenerate cusp of embedding dimension 4. Therefore, $Q$ is supported on the singular points of $\Delta$ of the type claimed. Moreover, again from Lemma 3.9, $p(T^1_{qG}(H))$ is not a line bundle at any degenerate cusp of embedding dimension 4 of type other than $T^4_{p,q,\infty}$ that is a singular point of $\Delta$. Hence $Q = \oplus p(k(P))$, where $P$ runs over all degenerate cusps of embedding dimension 4 and type other than $T^4_{p,q,\infty}$ that are also singular points of $\Delta$, as claimed. This concludes the proof of the lemma. \qed
The next lemma was communicated to the author by János Kollár and it shows that it is always possible to find a 3-fold $X$ with the properties of Lemma 3.10.

**Lemma 3.11.** Let $\Delta \subset H$ be the germ of a surface along a proper curve $\Delta$. Assume that $H$ is smooth away from $\Delta$ and that generically along every component of $\Delta$ it has at worst singularities of type $(xy = 0) \subset \mathbb{C}^3$. Assume that for each $c \in \Delta$ we specify a neighborhood $U_c \subset X$ and a closed embedding $U_c \subset V_c$. Assume that $V_c$ is smooth away from $c$ and singular for finitely many only $c$. Then there is a global embedding $\Delta \subset H \subset X$ that patches the local embeddings.

**Proof.** Let $W = H - \{\text{a few transverse cuts}\}$. Then $W$ is Stein and it embedds in a $\mathbb{C}^n$, for some $n$. We want to get an embedding $W \subset V_W$ into something three dimensional that has at most finitely many singularities along $\Delta$. Let $f: Y \rightarrow \mathbb{C}^n$ be the blow up of $\mathbb{C}^n$ along $W$. Let $E$ be the $f$--exceptional set. Then by the assumptions on the singularities of $H$, a local calculation shows that away from finitely many singularities of $H$ that correspond to nonnormal crossing points, $E = E_1 + E_2$, where $E_1$ and $E_2$ are both smooth intersecting transversally. Moreover, $E_1 + E_2$ is Cartier and the singular locus of $Y$ has codimension 3 in $Y$. More precisely, locally the singularities of $Y$ are of the type $(xy - zt = 0) \subset \mathbb{C}^{n+1}$. Moreover, $\dim f^{-1}(c) = n - 3$, for all $c \in \Delta$. Then by Bertini’s theorem, the general $V' = Z_1 \cap Z_2 \cap \cdots \cap Z_{n-3}$, with $Z_i \in \{ - E_1 - E_2 \}$, has finitely many singularities and $V' \rightarrow V_W = f_c(V')$ is birational. $V_W$ has the required properties.

Therefore, by removing a few more transverse cuts, $W = H - \{\text{a few transverse cuts}\}$ has a smooth embedding into a three dimensional space $V_W$. Let $(H - W) \cap \Delta = \{c_1, \ldots, c_k\}$. Let $U_{c_i}$ be a Stein neighborhood of $c_i \in H$ and let $U_{c_i} \subset V_{c_i}$ be an embedding such that $V_{c_i}$ is smooth away from $c_i$. We want to glue $V_W$ and $V_{c_i}$ into a 3-fold $V$. This will follow from the following.

**Claim.** Let $c \in \Delta \subset U$ be the germ of a surface singularity along a curve $\Delta$. Assume that $U$ is singular along $\Delta$. Let $R \subset U$ be a small ring around $c \in U$. Let $R \subset X^1_R, R \subset X^2_R$ be two embeddings, where $X^1_R$ and $X^2_R$ are smooth, which we may also assume that they are Stein. Then, after shrinking $R$ if necessary, $X^1_R \cong X^2_R$. The map $R \subset X^2_R$ extends to a map $X^1_R \rightarrow X^2_R$. Since $R$ is singular along $\Delta$, then for all $c \in \Delta$, $T_c(R) = T_c(X^1_R) = T_c(X^2_R)$, and hence by shrinking $R$ if necessary, it follows from the inverse function theorem that $X^1_R \cong X^2_R$.

Now gluing $V_W$ and $V_{c_i}$, $i = 1, \ldots, k$, we obtain the required embedding. $\square$

An immediate consequence of Lemmas 3.11 and 3.6 is the following:

**Corollary 3.12.** Let $\Delta \subset H$ be a surface germ as in Theorem 3.1. Then there exists an embedding $H \subset X$ such that $X$ is a terminal 3-fold and $H$ is Cartier in $X$. Moreover, let $P \in X$ be a singular point of $X$. Then either $P \in H$ is a degenerate cusp of embedding dimension 4 and $(P \in X) \cong (xy - zt = 0) \subset \mathbb{C}^4$ or $P \in H$ is an slt point, and $(P \in X) \cong 1/n(a, -a, 1)$.

The next proposition is needed in the proof of Theorem 3.1.
PROPOSITION 3.13. Let $\Delta \subset X$ be the germ of a 3-fold along an irreducible smooth curve $\Delta$ (proper or not). Suppose that $(P \in \Delta \subset X) \cong (x = y = 0) \subset \mathbb{C}^3_{x,y,z}/\mathbb{Z}_n(a,-a,1)$. Then there exists a divisorial contraction $E \subset Y \rightarrow X \supset \Delta$, contracting an irreducible divisor $E$ onto $\Delta$ such that

1. $(f^{-1}(P))_{\text{red}} = F = \mathbb{P}^1$.
2. $E^{\text{sing}} \cap F = \{Q_1, Q_2\}$ such that $(Q_1 \in E) \cong \mathbb{C}^2/\mathbb{Z}_n(1,2a)$ and $(Q_2 \in E) \cong \mathbb{C}^2/\mathbb{Z}_n(1,-2a)$.
3. $(Q_1 \in Y) \cong \mathbb{C}^3/\mathbb{Z}_n(a,-2a,1)$, $(Q_2 \in Y) \cong \mathbb{C}^3/\mathbb{Z}_n(-a,2a,1)$, $K_Y \cdot F = -1/n$, if $n$ is odd, and $K_Y \cdot F = -2/n$, if $n$ is even.

Proof. If a divisorial contraction $f: E \subset Y \rightarrow X \supset \Delta$ that satisfies the conditions of the proposition exists, then it is unique and in fact

$$Y \cong \text{Proj} \oplus_d I_{\Delta,X}^{(d)}.$$

Therefore its existence is equivalent to the finite generation of the sheaf of algebras $\oplus_d I_{\Delta,X}^{(d)}$ [Tzi03]. This can be checked locally and hence the existence of local contractions imply the existence of a global one. We proceed to construct the local contraction explicitly.

Let $P \in X$ be a singular point such that $(P \in X) \cong (x = y = 0) \subset \mathbb{C}^3/\mathbb{Z}_n(a,-a,1)$, where $x, y, z$ are the coordinates for $\mathbb{C}^3$. Hence $\Delta = \pi(L)$, where $L$ is the line $x = y = 0$. Then the divisorial contraction can be constructed from the following diagram:

$$\begin{array}{ccc}
B_1\hat{X} = \hat{Y} & \stackrel{\#}{\longrightarrow} & Y = \hat{X}/\mathbb{Z}_n \\
\downarrow j & & \downarrow f \\
\mathbb{C}^3 = \hat{X} & \stackrel{\pi}{\longrightarrow} & X,
\end{array}$$

where $\hat{Y}$ is the blow up of $\mathbb{C}^3$ along the $z$-axis, $x = y = 0$. A straightforward calculation now of the lifting of the action of $\mathbb{Z}_n$ on $\hat{Y}$ shows that the singular locus of $E$ is two points $Q_1$ and $Q_2$, and $(Q_1 \in E) \cong 1/n(1,2a)$, $(Q_2 \in E) \cong 1/n(1,-2a)$. Moreover, $(Q_1 \in Y) \cong 1/n(a,-2a,1)$ and $(Q_2 \in Y) \cong 1/n(-a,2a,1)$. If $n$ is odd then the singular locus of $Y$ consists of the points $Q_1$ and $Q_2$. If on the other hand $n$ is even, then $Y$ may be singular along $f^{-1}(P)$.

It remains to find $K_Y \cdot F$, where $F = f^{-1}(P)_{\text{red}}$. Let $\hat{F} = \hat{f}^{-1}(0)$. Then $F = \hat{F}/\mathbb{Z}_n$. Let $u, \bar{u}$ be homogeneous coordinates for $\mathbb{P}^1 = \hat{F}$. Let $\zeta$ be a primitive $n$-th root of unity. Then $\mathbb{Z}_n$ acts on $\hat{F}$ by $\zeta \cdot [u,\bar{u}] = [\zeta^a u, \zeta^{-a} \bar{u}]$. Hence $\hat{F} \rightarrow \mathbb{P}^1$ is $n$-to-one if $n$ is odd and $n/2$-to-one of $n$ is even. Now $\hat{f}$ is étale in codimension 1 and hence $K_{\hat{F}} = \pi^*K_Y$. Therefore

$$-1 = K_{\hat{F}} \cdot \hat{F} = \pi^*K_Y \cdot \hat{F} = K_Y \cdot \pi_*(\hat{F})$$
and therefore \( K_Y \cdot F = -1/n \), if \( n \) is odd, and \( K_Y \cdot F = -2/n \), if \( n \) is even, as claimed. \( \square \)

4. Proof of Theorem 3.1. By Corollary 3.12, there exists an embedding \( H \subset X \) such that \( X \) is a terminal 3-fold and \( H \) is Cartier in \( X \).

Moreover, let \( P \in X \) be a singular point of \( X \). Then either \( P \in H \) is a degenerate cusp of embedding dimension 4 and \((P \in X) \cong (xy - zt = 0) \subset \mathbb{C}^4\) or \( P \in H \) is an slt point, and \((P \in X) \cong 1/n(a, -a, 1)\). Then from Lemma 3.10 it follows that there is an exact sequence

\[
0 \to L \to p(T_{qG}^1(H)) \to \oplus p k(P) \to 0, \tag{4.1}
\]

where \( L = N_{H/X} \otimes O_\Delta \) is a line bundle on \( \Delta \) and \( P \in H \) runs over all degenerate cusps of embedding dimension 4 of type other than \( T_{p,q,\infty}^4 \) such that \( P \) is a singular point of \( \Delta \) too. This shows Theorem 3.1.

Let \( C \) be an irreducible component of \( \Delta \). Then from (4.1) it follows that \( \deg (L \otimes O_C) = C \cdot H \). The proof of Theorem 3.1.2 consists of two steps.

1. After suitable weighted blow ups over the degenerate cusps we get a partial resolution \( f: \hat{X} \to X \) such that \( \hat{\Delta} = f^{-1}_* \Delta \) is smooth and the singularities of \( \hat{H} \) around \( \hat{\Delta} \) are either normal crossings, pinch points and slt points. Moreover we show that

\[
H \cdot C = \hat{H} \cdot \hat{C} + 6c_1 + 4c_2 + 3c_3 + 2c_4
\]

where \( \hat{H} \) and \( \hat{C} \) are the birational transforms of \( H \) and \( C \) in \( \hat{X} \).

2. Resolve the remaining singularities of \( \hat{H} \) by doing a divisorial extraction over \( \hat{\Delta} \) to get the minimal log-resolution of \( \Delta \subset H \).

Step 1. Partial resolution of the cusps. Let \( P \in H \) be a degenerate cusp. Fix notation as in Lemma 2.6. We will consider four cases according to \( \Gamma^2 \).

Case 1. Suppose \( \Gamma^2 = -1 \). Then in suitable local analytic coordinates, \( P \in H \) is given by \( x^2 + y^3 + y^2z^2 = 0 \) in \( X = \mathbb{C}^3 \). Assign weights 3, 2, 1 on \( x, y, z \) and let \( f: \hat{X} \to X \) be the \((3, 2, 1)\)-weighted blow up. Let \( E = \mathbb{P}(3, 2, 1) \) be the exceptional divisor and \( \hat{H} = f^{-1}_* H \). Then by direct calculation we see that the singularities of \( \hat{H} \) are normal crossings only. Moreover, in the affine chart \( z \neq 0 \) we have coordinates \( u, v, w \) with \( x = uw^3 \), \( y = vw^2 \) and \( y = w \). Then \( f^* H \) is given by \( u^2w^6 + v^3w^6 + u^3w^6 = 0 \) and \( E \) by \( w = 0 \). Therefore

\[
f^*(H) = \hat{H} + 6E. \tag{4.2}
\]

Case 2. Suppose \( \Gamma^2 = -2 \). Then in suitable local analytic coordinates, \( P \in H \) is given by \( x^2 + y^2z^2 - z^{p+3} = 0 \) in \( X = \mathbb{C}^3 \), where as usual we set \( z^\infty = 0 \) if
\( n = \infty \). Assign weights 2, 1, 1 on \( x, y, z \) and let \( f: \hat{X} \to X \) be the (2, 1, 1)-weighted blow up. Let \( E = \mathbb{P}(2, 1, 1) \) be the exceptional divisor and \( \hat{H} = f_1^{-1}H \). Explicit calculations again show that the singularities of \( \hat{H} \) are normal crossings and one isolated DuVal singularity of type \( A_{n-2} \). Moreover, \( \hat{\Delta} = f_1^{-1}\Delta \) is smooth. In the affine chart \( y \neq 0 \) we have coordinates \( u, v, w \) with \( x = uv^2, y = v \) and \( z = wv \). In this chart \( f^*H \) is given by \( u^2v^4 + w^2v^4 + wn + 3v^n + 3 = 0 \) and \( E \) by \( v = 0 \). Therefore

\[
(4.3) \quad f^*(H) = \hat{H} + 4E.
\]

**Case 3.** Suppose \( \Gamma^2 = -3 \). Then let \( f: \hat{X} \to X = \mathbb{C}^3 \) be the blow up of \( P \in X \). Explicit calculations show that \( \hat{\Delta} \) is smooth and that the singularities of \( \hat{H} \) are normal crossings and two isolated DuVal singularities of type \( A_{p-2}, A_{q-2} \). Moreover

\[
(4.4) \quad f^*(H) = \hat{H} + 3E.
\]

**Case 4.** Suppose that \( \Gamma^2 = -4 \). Then let \( f: \hat{X} \to X \) be the blow up of \( P \in X \). Then \( X \) is smooth and \( E = \mathbb{P}^1 \times \mathbb{P}^1 \). Explicit calculations show that \( \hat{\Delta} \) is smooth and that the singularities of \( \hat{H} \) are normal crossings and three isolated DuVal singularities of type \( A_{p-2}, A_{q-2}, A_{r-2} \). Moreover

\[
(4.5) \quad f^*(H) = \hat{H} + 2E.
\]

In all of the above cases, \( A_\infty \) denotes a smooth point.

Resolving all degenerate cusps of \( H \) with the previous method, we get a birational morphism \( f: \hat{X} \to X \) such that \( \hat{\Delta} \) is smooth and that around it the singularities of \( \hat{H} = f_1^{-1}H \) are normal crossings, pinch points and \( slt \) points. The exceptional set of \( f \) are divisors \( E_{1,p}, E_{2,Q}, E_{3,R} \) and \( E_{4,S} \), where \( P, Q, R, S \) run over all degenerate cusps with \( \Gamma^2 = -1, -2, -3, -4 \), respectively. Then (4.2), (4.3) (4.4) (4.5) show that

\[
f^*(H) = \hat{H} + 6 \sum_p E_{1,p} + 4 \sum_p E_{2,Q} + 3 \sum_p E_{3,R} + 2 \sum_s E_{4,S}.
\]

Therefore if \( C \) is an irreducible component of \( \Delta \) and \( c_i \) the numbers of degenerate cusps of \( H \) with \( \Gamma^2 = -i \) that lie on \( C \), then

\[
(4.6) \quad H \cdot C = \hat{H} \cdot \hat{C} + 6c_1 + 4c_2 + 3c_3 + 2c_4,
\]

where \( \hat{C} = f_1^{-1}C \). Now \( \hat{H} \cdot \hat{C} \) depends only on a neighborhood of \( \hat{\Delta} \) in \( \hat{H} \). But by construction, in a neighborhood of \( \hat{\Delta} \), \( \hat{H} \) has only normal crossings, pinch points and \( slt \) points. So taking a sufficiently small neighborhood of \( \hat{\Delta} \) we may assume that the singularities of \( \hat{H} \) are only normal crossings, pinch points and \( slt \) points.
Step 2. We now proceed to resolve all the singularities of $\hat{H}$ and get the claimed formula in the minimal log-resolution of $\Delta \subset H$. Set $\hat{d} = \hat{H} \cdot \hat{C}$.

Let $g: E \subset \tilde{X} \rightarrow \hat{X} \supset \hat{C}$ be the divisorial contraction constructed in Proposition 3.13. The explicit construction of the contraction shows that $\hat{H} = g^{-1}\hat{H}$ is the normalization of $\hat{H}$. Therefore it is smooth over the normal crossing and pinch points of $\hat{H}$ and over any slt point of type $(xy = 0)/\mathbb{Z}_n(a, -a, 1)$, it has two quotient singularities of type $1/n(1, a)$ and $1/n(1, -a)$. Moreover, $E \cdot \hat{H} = \tilde{C} = g^{-1}\tilde{C}$.

Since $\hat{H}$ is normal crossing at the generic point of $\tilde{C}$, it follows that

\[ g^* \hat{H} = \hat{H} + 2E. \tag{4.7} \]

Moreover, $E^2 = -\delta + F$, where $\delta$ is a section over $\tilde{C}$ and $F$ is supported on finitely many fibers. Intersecting (4.7) with $E$ we get

\[ f^*(D) = \hat{H} \cdot E + 2E^2 = \tilde{C} - 2\delta + 2F, \tag{4.8} \]

where $D \in \text{Pic}(\tilde{C})$ and $\deg D = \hat{H} \cdot \hat{C}$. Now it is possible to write $K_E = a \tilde{C} + g^*(D_1)$ for some $a \in \mathbb{Q}$ and $D_1 \in \text{Pic}(\tilde{C}) \otimes \mathbb{Q}$. Intersecting with a general fiber $l$ we find that $a = -1$ and hence

\[ K_E = -\tilde{C} + g^*(D_1). \tag{4.9} \]

Intersecting now with $\tilde{C}$ in $E$, we find that

\[ K_E \cdot \tilde{C} + \tilde{C}^2 = 2\deg(D_1). \tag{4.10} \]

Now by adjunction

\[ K_E \cdot \tilde{C} + \tilde{C}^2 = 2p_a(\tilde{C}) - 2 + \text{Diff}(\tilde{C}, E), \]

where $\text{Diff}(\tilde{C}, E)$ is the different of $\tilde{C}$ in $E$ [Ko93, Proposition-Definition 16.5]. Hence (4.10) becomes

\[ \deg(D_1) = p_a(\tilde{C}) - 1 + (1/2)\text{Diff}(\tilde{C}, E). \tag{4.11} \]

Intersecting (4.9) with $\delta$ in $E$ we get

\[ K_E \cdot \delta = -\tilde{C} \cdot \delta + g^*(D_1) \cdot \delta = -\tilde{C} \cdot \delta + \deg(D_1). \tag{4.12} \]

Intersecting (4.8) with $\delta$ we get

\[ \hat{d} = \tilde{C} \cdot \delta - 2\delta^2 + 2F \cdot \delta. \tag{4.13} \]
Now (4.11), (4.12), and (4.13) give that

\[ KE \cdot \delta + \delta^2 = -\hat{d} - \delta^2 + 2F \cdot \delta + p_a(\tilde{C}) - 1 + (1/2)\text{Diff}(\tilde{C}, E). \]

**Claim.**

\[ -\delta^2 + 2F \cdot \delta = \frac{1}{2} \tilde{C}^2 + \frac{1}{2} \hat{d}, \]

where \( \tilde{C}^2 \) is taken in \( \tilde{H} \). Intersecting (4.8) with \( E \) and taking into account that \( E \cdot \delta = -\delta^2 + \delta \cdot F, \tilde{C}^2 = E \cdot \tilde{C} \) we find that

\[ -\hat{d} = E \cdot \tilde{C} - 2(E \cdot \delta - E \cdot F) = \tilde{C}^2 - 2(E \cdot \delta - E \cdot F). \]

But \( E \cdot F = (-\delta + F) \cdot F = -\delta \cdot F \). Hence \( E \cdot \delta - E \cdot F = -\delta^2 + 2\delta \cdot F \). Therefore

\[ -\hat{d} = \tilde{C}^2 - 2(-\delta^2 + 2\delta \cdot F) \]

and hence

\[ -\delta^2 + 2F \cdot \delta = \frac{1}{2} \tilde{C}^2 + \frac{1}{2} \hat{d} \]

as claimed. Taking this into account, (4.14) becomes

\[ KE \cdot \delta + \delta^2 = -\hat{d} + \frac{1}{2} \tilde{C}^2 + \frac{1}{2} \hat{d} + p_a(\tilde{C}) - 1 + \frac{1}{2} \text{Diff}(\tilde{C}, E). \]

Let \( \text{Diff}(\delta, E) \) be the different of \( \delta \) in \( E \). Then by adjunction

\[ KE \cdot \delta + \delta^2 = 2p_a(\tilde{C}) - 2 + \text{Diff}(\delta, E) \]

and hence (4.15) becomes

\[ \hat{d} = \tilde{C}^2 + 2(p_a(\tilde{C}) - 1) - 2(2p_a(\tilde{C}) - 2) + \text{Diff}(\tilde{C}, E) - 2\text{Diff}(\delta, E). \]

We now want to calculate the different \( \text{Diff}(\tilde{C}, E) \) and \( \text{Diff}(\delta, E) \). The next lemma allows us to do so.

**Lemma 4.1.** (Corollary 16.7, [Ko93]) Let \( 0 \in C \subset H \) be the germ of a cyclic quotient singularity of type \( \frac{1}{n}(1, a) \) around a smooth curve \( C \). Let \( f: U \rightarrow H \) be the minimal resolution. Assume that the extended dual graph is

\[ \bullet \rightarrow \circ \rightarrow \cdots \rightarrow \circ. \]
Then
\[ \text{Diff}(C, 0 \in H) = 1 - 1/n. \]

From Proposition 3.13 it follows that the singular points of \( E \) lie over \( slt \) points of \( H \). More precisely, let \( Q_1, \ldots, Q_s \) be the \( slt \) points of \( H \) and \( P_i, P_{-i} \) the corresponding singular points of \( E \) over \( Q_i \). Then if \( Q_i \in H \cong (xy = 0)/\mathbb{Z}_{n_i}(a_i, -a_i, 1) \), then \( P_i \in \tilde{H}, P_{-i} \in \tilde{H} \) are quotient singularities of type \( 1/n_i(1, a_i), 1/n_i(1, -a_i) \), respectively. In the minimal resolution \( H' \), the \( h \)-exceptional set are two disjoing chains of rational curves, \( B_1, \ldots, B_s, D_1, \ldots, D_k \),

\[ \hat{d} = \hat{c}^2 + p. \]
over $1/n(1,a)$ and $1/n(1,-a)$ respectively. Their self intersection numbers $-b_i$, $-d_j$ are computed from the continued fraction expansions for $\frac{n}{a}$, $\frac{n}{n-a}$ and $C'$ meets $B_1$ and $D_1$. Let

$$h^*\tilde{C} = C' + \sum_{i=1}^{s} a_iB_i + \sum_{j=1}^{k} c_iD_i.$$ 

Then

$$\tilde{C}^2 = C'^2 + a_1 + c_1.$$ 

We claim that $a_1 + c_1 = 1$. Indeed, the $a_i$ satisfy the equations

$$a_{i-1} - b_ia_i + a_{i+1} = 0.$$ 

This can be written as

$$\frac{a_{i-1}}{a_i} = b_i - \frac{1}{a_i/a_{i+1}}.$$ 

This is the same formula that computes the $b_i$ and hence $a_1 = b_1 = a/n$. Similarly we find that $c_1 = d_1 = (n-a)/n$. Hence $a_1 + c_1 = 1$, as claimed. Hence we have shown that

(4.20) $$\tilde{C}^2 = C'^2 + q,$$

where $q$ is the number of slt points of $H$ that lie on $C$. Now from (4.6), (4.19) and (4.20) it follows that

$$\deg(L \otimes O_C) = C'^2 + q + p + 6c_1 + 4c_2 + 3c_3 + 2c_4$$

as claimed. This concludes the proof of Theorem 3.1.

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