Operadic curvature as a tool for gravity

Eugen Paal

Department of Mathematics, Tallinn Technical University
Ehitajate tee 5, 19086 Tallinn, Estonia
e-mail: eugen@edu.ttu.ee

Abstract

The deformation equation and its integrability condition (Bianchi identity) of a non-associative deformation in operad algebra are found. Their relation to the theory of gravity is discussed.

1 Introduction

Non-associativity is sometimes said to be an algebraic equivalent of the differential geometrical concept of curvature (e. g. [1, 2]). By adjusting this for physics, one may surmise that gravity and gauge fields geometry can be described in algebraic terms. In particular, instead of the curvature of the space-time, associator rises to the fore [3, 4]. In this sense, gravity can be seen to have an algebraic representation. When non-associativity of space-time becomes large, operadic structure will become important and one must use operad algebra to understand the algebraic underground of the gravity and how the gravity could be quantized. Instead of the quantum gravity, the operadic gravity rises to the fore.

In this paper, the equivalence is clarified from the linear deformation theoretical point of view. By using the Gerstenhaber brackets and a coboundary operator in a pre-operad, the (formal) associator can be represented as a curvature form in differential geometry. This (structure) equation is called a deformation equation. Its integrability condition is the Bianchi identity. Their relation to the theory of gravity is discussed.

2 Operad algebra

Let $K$ be a unital associative commutative ring, char $K \neq 2, 3$, and let $C^n$ ($n \in \mathbb{N}$) be unital $K$-modules. For homogeneous $f \in C^n$, $n$ is called the degree of $f$ and (when it does not cause confusion) $f$ is written instead of deg $f$. For example, $(-1)^f := (-1)^n$, $C^f := C^n$ and $\circ f := \circ_n$. Also, it is convenient to use the reduced degree $|f| := n - 1$. Throughout this paper, it is assumed that $\otimes := \otimes_K$. 
Definition 1. A linear pre-operad (composition system) with coefficients in $K$ is a sequence $C := \{C^n\}_{n \in \mathbb{N}}$ of unital $K$-modules (an $\mathbb{N}$-graded $K$-module), such that the following conditions hold.

1. For $0 \leq i \leq m - 1$ there exist (partial) compositions
   \[\circ_i \in \text{Hom}(C^m \otimes C^n, C^{m+n-1}), \quad |\circ_i| = 0.\]

2. For all $h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g$, the composition (associativity) relations hold,
   \[(h \circ_i f) \circ_j g = \begin{cases} \begin{align*} (-1)^{|f||g|} (h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i - 1, \\ h \circ_i (f \circ_{j-i} g) & \text{if } i \leq j \leq i + |f|, \\ (-1)^{|f||g|} (h \circ_{j-|f|} g) \circ_i f & \text{if } i + f \leq j \leq |h| + |f|. \end{align*} \end{cases}\]

3. There exists a unit $I \in C^1$ such that
   \[I \circ_0 f = f = f \circ_i I, \quad 0 \leq i \leq |f|.\]

In the 2nd item, the first and third parts of the defining relations turn out to be equivalent.

Elements of an operad may be called operations. Operad can be seen as a system of operations closed with respect to compositions.

Example 2 (composition pre-operad). Let $L$ be a unital $K$-module and $C^n_L := \text{Hom}(L^\otimes n, L)$. Define the partial compositions for $f \otimes g \in C^f \otimes C^g$ as
   \[f \circ_i g := (-1)^{|g|} f \circ (\text{id}_L^\otimes i \otimes g \otimes \text{id}_L^\otimes (|f|-i)), \quad 0 \leq i \leq |f|.\]

Then $C_L := \{C^n_L\}_{n \in \mathbb{N}}$ is a pre-operad (with the unit $\text{id}_L \in C^1_L$) called the composition pre-operad of $L$.

3 Gerstenhaber brackets

The total composition $\bullet : C^f \otimes C^g \to C^{f+|g|}$ is defined by
   \[f \bullet g := \sum_{i=0}^{|f|} f \circ_i g \in C^{f+|g|}, \quad |\bullet| = 0.\]

The pair $\text{Com} C := \{C, \bullet\}$ is called the composition algebra of $C$.

The Gerstenhaber brackets $[\cdot, \cdot]$ are defined in $\text{Com} C$ by
   \[[f, g] := f \bullet g - (-1)^{|f||g|} g \bullet f = (-1)^{|f||g|}[g, f], \quad ||[\cdot, \cdot]| = 0.\]

The commutator algebra of $\text{Com} C$ is denoted as $\overline{\text{Com} C} := \{C, [\cdot, \cdot]\}$. It turns out that $\overline{\text{Com} C}$ is a graded Lie algebra. The Jacobi identity reads
   \[(-1)^{|f||h|}[f, [g, h], f] + (-1)^{|g||f|}[g, [h, f], f] + (-1)^{|h||g|}[[h, f], g] = 0.\]
In a pre-operad $C$, define a pre-coboundary operator $\partial_\Delta$ by

$$\partial_\Delta f := \text{ad}^{\text{right}}_{\Delta} f := [f, \Delta], \quad |\partial_\Delta| = |\Delta|.$$  

It follows from the Jacobi identity the (right) derivation property

$$\partial_\Delta[f, g] = (-1)^{|\Delta||g|} |\partial_\Delta f, g| + [f, \partial_\Delta g]$$

and the commutation relation

$$[\partial f, \partial g] := \partial_f \partial_g - (-1)^{|f||g|} \partial_g \partial_f = \partial_{[f, g]}.$$  

Thus, if $|\Delta|$ is odd, then

$$\partial_\Delta^2 = \frac{1}{2} [\partial_\Delta, \partial_\Delta] = \frac{1}{2} \partial_{\Delta \cdot \Delta} = \partial_{\Delta^2}.$$  

### 4 Deformation equation

For an operad $C$, let $\Delta, \Delta_0 \in C^2$. The difference $\omega := \Delta - \Delta_0$ is called a deformation, and $\Delta$ is said to be a deformation of $\Delta_0$. Let $\partial := \partial_\Delta$, and denote the (formal) associators of $\Delta$ and $\Delta_0$ as follows:

$$A := \Delta \cdot \Delta = \frac{1}{2} [\Delta, \Delta], \quad A_0 := \Delta_0 \cdot \Delta_0 = \frac{1}{2} [\Delta_0, \Delta_0].$$

The deformation is called associative if $A = 0 = A_0$.

To find the deformation equation, calculate

$$A = \frac{1}{2} [\Delta_0 + \omega, \Delta_0 + \omega]$$

$$= \frac{1}{2} [\Delta_0, \Delta_0] + \frac{1}{2} [\Delta_0, \omega] + \frac{1}{2} [\omega, \Delta_0] + \frac{1}{2} [\omega, \omega]$$

$$= A_0 - \frac{1}{2} (-1)^{|\Delta_0||\omega|} [\omega, \Delta_0] + \frac{1}{2} [\omega, \Delta_0] + \frac{1}{2} [\omega, \omega]$$

$$= A_0 + [\omega, \Delta_0] + \frac{1}{2} [\omega, \omega].$$

So we get the deformation equation

$$A - A_0 = \partial \omega + \frac{1}{2} [\omega, \omega].$$

The deformation equation can be seen as a differential equation for $\omega$ with given associators $A_0, A$. Note that if the associator is fixed, i.e. $A = A_0$, one obtains the Maurer-Cartan (master) equations, well-known from the theory of associative deformations.
5 Prolongation

Now differentiate the deformation equation,
\[
\partial(A - A_0) = \partial^2 \omega + \frac{1}{2} \partial \lbrack \omega, \omega \rbrack \\
= \partial^2 \omega + \frac{1}{2} (-1)^{\lbrack \partial \omega, \omega \rbrack} \lbrack \partial \omega, \omega \rbrack + \frac{1}{2} \lbrack \omega, \partial \omega \rbrack \\
= \partial^2 \omega - \frac{1}{2} \partial \omega, \omega \rbrack + \frac{1}{2} \lbrack \omega, \partial \omega \rbrack \\
= \partial^2 \omega - \partial \omega, \omega \rbrack.
\]

Again using the deformation equation, we obtain
\[
\partial(A - A_0) = \partial^2 \omega - [\partial \omega, \omega] \\
= \partial^2 \omega - \lbrack A - A_0, \partial \omega \rbrack - \frac{1}{2} \lbrack \omega, \omega \rbrack] \\
= \partial^2 \omega - [A - A_0, \omega] + \frac{1}{2} \lbrack \omega, \omega \rbrack] \\
= \partial^2 \omega - [\partial \omega, \omega].
\]

Finally use \([\omega, \omega], \omega] = 0\) to obtain the condition
\[
\partial(A - A_0) = \partial^2 \omega - [A - A_0, \omega] \]

6 Associativity constraint and Bianchi identity

We know that \(\partial^2 = \partial A_0\). Hence, if associativity constraint \(A_0 = 0\) holds, then
\(\partial^2 = 0\)

The deformation equation for such a non-associative deformation reads
\[
A = \partial \omega + \frac{1}{2} \lbrack \omega, \omega \rbrack
\]

One can see that associator is a formal curvature while the deformation is working as a connection. One can say that associator is an operadic equivalent of the curvature. The integrability condition of the deformation equation reads as the Bianchi identity
\[
\partial A + [A, \omega] = 0
\]

One can easily check that further differentiation does not add new conditions.
7 Covariant derivation

Note that
\[ \partial_\Delta f = [f, \Delta] = [f, \Delta_0 + \omega] = [f, \Delta_0] + [f, \omega] = \partial f + [f, \omega]. \]

One can say that \( \nabla := \partial_\Delta \) is a covariant derivation. The Bianci identity reads
\[ \nabla A = \partial A + [A, \omega] = 0 \]

Also note that
\[ \nabla^2 f = [f, A] \]

So the condition \( \nabla^2 = 0 \) does not imply \( A = 0 \). Instead, \( \nabla^2 = 0 \) implies that \( A \) lies in the center of \( \text{Com}^- C \). In particular,
\[ \nabla^2 = 0 \implies \partial A = 0 \implies A \in \text{Ker } \partial. \]

Note that \( \Delta \) may nevertheless remain non-associative.

8 Discussion: operadic gravity

Thus the differential geometrical notion of curvature can be easily adjusted for deformations in a pre-operad. Rather than to speak about algebraic deformation theory, one may speak about the geometrical one. Geometry performs the pioneering role in creating of the exact scientific world picture. One may ask that how far one can proceed with geometrical notions in operad theoretical deformation theory. In particular, this question may be adjusted for physics as well.

In General Relativity, gravity is a fundamental interaction associated with the space-time curvature (associator [3, 4]). Operadic curvature may be used for representing gravity in a form suitable for deformation quantization. Geodesic multiplication [3, 4] can here been seen as a prospective model.

One may also follow the Maxwell (gauge field) equations. It is well-known that the first pair of the Maxwell equations can be represented as the Bianchi identity. To introduce the second pair, one must define a dualization \( ^\dagger \) and an operad current \( J \). Then the (gauge field) Maxwell-like equations read
\[ \nabla A = \partial A + [A, \omega] = 0, \quad \nabla A^\dagger = \partial A^\dagger + [A^\dagger, \omega] = J^\dagger \]

In this approach, one must study the physical equations in non-associative deformation complexes.

Acknowledgements

I would like to thank P. Kuusk, J. Lõhmus and J. Stasheff for helpful comments. Research was supported in part by the ESF grant 3654.
References

[1] A. I. Nesterov and L. V. Sabinin, Nonassociative geometry: Towards discrete structure of spacetime. Phys. Rev. D62 (2000), 081501; hep-th/0010159.

[2] A. I. Nesterov and L. V. Sabinin, Non-associative geometry and discrete structure of spacetime. Comment. Math. Univ. Carolinae, 41 (2000), 347-357; hep-th/0003238.

[3] M. Kikkawa, On local loops in affine manifolds. J. Hiroshima Univ. Ser. A-1. Math. 28 (1964), 199-207.

[4] M. Akivis, Geodesic loops and local triple systems in a space with an affine connection. Sibirski Math. J. 19 (1978), 243-253 (in Russian).

[5] M. Gerstenhaber, On cohomology structure of an associative ring. Ann. of Math. 78 (1963), 267-288.