Dilation Theory and Functional Models for Tetrablock Contractions

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Abstract
A classical result of Sz.-Nagy asserts that a Hilbert space contraction operator \( T \) can be dilated to a unitary \( U \), i.e., \( T^n = P_H U^n | H \) for all \( n = 0, 1, 2, \ldots \). A more general multivariable setting for these ideas is the setup where (i) the unit disk is replaced by a domain \( \Omega \) contained in \( \mathbb{C}^d \), (ii) the contraction operator \( T \) is replaced by an \( \Omega \)-contraction, i.e., a commutative operator \( d \)-tuple \( T = (T_1, \ldots, T_d) \) on a Hilbert space \( \mathcal{H} \) such that \( \| r(T_1, \ldots, T_d) \|_{\mathcal{L}(\mathcal{H})} \leq \sup_{\lambda \in \Omega} | r(\lambda) | \) for all rational functions with no singularities in \( \overline{\Omega} \) and the unitary operator \( U \) is replaced by an \( \Omega \)-unitary operator tuple, i.e., a commutative operator \( d \)-tuple \( U = (U_1, \ldots, U_d) \) of commuting normal operators with joint spectrum contained in the distinguished boundary \( b\Omega \) of \( \Omega \). For a given domain \( \Omega \subset \mathbb{C}^d \), the rational dilation question asks: given an \( \Omega \)-contraction \( T \) on \( \mathcal{H} \), is it always possible to find an \( \Omega \)-unitary \( U \) on a larger Hilbert space \( \mathcal{K} \supset \mathcal{H} \) so that, for any \( d \)-variable rational function without singularities in \( \overline{\Omega} \), one can recover \( r(T) \) as \( r(T) = P_H r(U) | \mathcal{H} \). We focus here on the case where \( \Omega = \mathbb{E} \), a domain in \( \mathbb{C}^3 \) called the tetrablock. (i) We identify a complete set of unitary invariants for a...
\( E \)-contraction \((A, B, T)\) which can then be used to write down a functional model for \((A, B, T)\), thereby extending earlier results only done for a special case, (ii) we identify the class of pseudo-commutative \(E\)-isometries (a priori slightly larger than the class of \(E\)-isometries) to which any \(E\)-contraction can be lifted, and (iii) we use our functional model to recover an earlier result on the existence and uniqueness of a \(E\)-isometric lift \((V_1, V_2, V_3)\) of a special type for a \(E\)-contraction \((A, B, T)\).

**Keywords** Commutative contractive operator-tuples · Functional model · Unitary dilation · Isometric lift · Spectral set · Pseudo-commutative contractive lift

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### 1 Introduction

Suppose that we are given a commutative tuple \( T = (T_1, \ldots, T_d) \) of operators on a Hilbert space \( \mathcal{H} \) together with a bounded domain \( \Omega \) contained in \( d \)-dimensional Euclidean space \( \mathbb{C}^d \). We now recall the notion of \( \Omega \) being a *spectral set* and \( \Omega \) being a *complete spectral set* for the commutative \( d \)-tuple \( T \) and refer to the original paper of Arveson [7] for additional details on this and the related matters which follow. We say that \( \Omega \) is a *spectral set* for \( T \) if it is the case that

\[
\| r(T) \|_{B(\mathcal{H})} \leq \sup_{\lambda \in \Omega} |r(\lambda)| \quad \text{for } r \in \text{Rat}(\Omega) \tag{1.1}
\]

where we set \( \text{Rat}(\Omega) \) equal to the space of all \( d \)-variable scalar-valued rational functions \( r \) having no singularities in \( \overline{\Omega} \) and \( B(\mathcal{H}) \) equal to the Banach algebra of all bounded linear operators on \( \mathcal{H} \). Here \( r(T) \) can be defined via the functional calculus given by

\[
r(T) = p(T_1, \ldots, T_d)q(T_1, \ldots, T_d)^{-1}
\]

where \((p, q)\) is a coprime pair of \( d \)-variable polynomials such that \( r = p/q \). In analogy with what happens for the case \( \Omega \) equal to the unit disk \( \mathbb{D} \) (see the discussion below), we say simply that \( T \) is an \( \Omega \)-contraction if it is the case that \( \Omega \) is a spectral set for \( T \).

We say that \( \Omega \) is a *complete spectral set* for \( T \) if (1.1) continues to hold when one substitutes matrix rational functions \( R(\lambda) = [r_{ij}(\lambda)]_{i,j=1,\ldots,n} = [p_{ij}(\lambda)q_{ij}(\lambda)^{-1}] \) having no singularities in \( \overline{\Omega} \):

\[
\| R(T) \|_{B(\mathcal{H}^n)} \leq \sup_{\lambda \in \Omega} \| R(\lambda) \|_{\mathbb{C}^n \times n}.
\]

The seminal result of Arveson (see [7]) is that \( \Omega \) is a complete spectral set for \( T \) if and only if there is a commutative \( d \)-tuple of normal operators \( N = (N_1, \ldots, N_d) \) on a larger Hilbert space \( \tilde{\mathcal{K}} \supset \mathcal{H} \) with joint spectrum contained in the distinguished boundary \( b\Omega \) of \( \Omega \) (in which case we say that \( N \) is a \( \Omega \)-unitary for short) so that, for
any rational function \( r \) with no singularities in \( \overline{\Omega} \) as above, it is the case that \( r(T) \) on \( \mathcal{H} \) can be represented as the compression of \( r(N) \) to \( \mathcal{H} \), i.e.,

\[
r(T) = P_\mathcal{H}r(N)|_{\mathcal{H}}
\]

where \( P_\mathcal{H} \) is the orthogonal projection of \( \mathcal{K} \) onto \( \mathcal{H} \). It is easy to see that a necessary condition for \( \Omega \) to be a complete spectral set for a given operator \( d \)-tuple \( T \) is that \( \Omega \) be a spectral set for \( \mathcal{T} \). The rational dilation problem for a given domain \( \Omega \) is to determine if the converse holds: given \( \Omega \), is it always the case that an operator tuple \( T \) having \( \Omega \) as a spectral set in fact has \( \Omega \) as a complete spectral set (and hence then any \( T \) having \( \Omega \) as a spectral set has a \( d\Omega \)-normal dilation \( N \))? 

Let us mention that it is often convenient to reformulate the problem of existence of an \( \Omega \)-unitary dilation instead as the problem of existence of a \( \Omega \)-isometric lift (see e.g. the introduction of [9]). Here we say that the operator tuple \( V = (V_1, \ldots, V_d) \) on a Hilbert space \( \mathcal{K} \) is a \( \Omega \)-isometry if \( V \) extends to a \( \Omega \)-unitary operator tuple \( U = (U_1, \ldots, U_d) \) on a Hilbert spaces \( \tilde{\mathcal{K}} \supset \mathcal{K} \). We say that \( V = (V_1, \ldots, V_d) \) on \( \mathcal{K} \) is a lift of \( T = (T_1, \ldots, T_d) \) on \( \mathcal{H} \) if \( \mathcal{H} \subset \tilde{\mathcal{K}} \) and \( r(V)^*|_{\mathcal{H}} = r(T)^* \) for \( r \in \text{Rat}(\Omega) \), or equivalently, \( V \) is a coextension of \( T \) in the sense that \( P_\mathcal{H}r(V)|_{\mathcal{H}} = r(T) \) and \( (r(V)|_{\mathcal{H}})^{\perp} \subset (r(T)|_{\mathcal{H}})^{\perp} \) for \( r \in \text{Rat}(\Omega) \).

It suffices to consider only minimal \( \Omega \)-unitary dilations and minimal \( \Omega \)-isometric lifts. It is always the case that the restriction of a \( \Omega \)-unitary dilation to the subspace \( \bigvee_{r \in \text{Rat}(\Omega)} r(U)\mathcal{H} \) gives rise to a minimal \( \Omega \)-isometric lift, and conversely, the minimal \( \Omega \)-unitary extension of a minimal \( \Omega \)-isometric lift gives rise to a minimal \( \Omega \)-unitary dilation for \( \mathcal{T} \). Finally we point out that it is often convenient to be more flexible in the definition of an \( \Omega \)-isometric lift and of an \( \Omega \)-unitary dilation by not insisting that \( \mathcal{H} \) is a subspace of \( \mathcal{K} \) or \( \tilde{\mathcal{K}} \) but rather allow an isometric identification map \( \Pi : \mathcal{H} \to \mathcal{K} \) and \( \tilde{\Pi} : \mathcal{H} \to \tilde{\mathcal{K}} \). Thus we say that the pair (\( \Pi, V \)) is an \( \Omega \)-isometric lift for \( T \) on \( \mathcal{H} \) if \( \Pi : \mathcal{H} \to \mathcal{K} \) is an isometric embedding, \( V \) is \( \Omega \)-isometric on \( \mathcal{K} \) and \( r(V)^*\Pi = \Pi r(T)^* \) for \( r \in \text{Rat}(\Omega) \), while (\( \tilde{\Pi}, U \)) is a \( \Omega \)-unitary dilation of \( T \) if \( \tilde{\Pi} : \mathcal{H} \to \tilde{\mathcal{K}} \) is an isometric embedding, \( U \) is \( \Omega \)-unitary on \( \tilde{\mathcal{K}} \), and \( \Pi^*r(U)\Pi = r(T) \) for \( r \in \text{Rat}(\Omega) \).

The motivating classical example for this setup is the case where \( \Omega \) is the unit disk \( \mathbb{D} \subset \mathbb{C} \). In this case, the distinguished boundary \( \partial \mathbb{D} \) of \( \mathbb{D} \) is the same as the boundary \( \partial \mathbb{D} \) which is the unit circle \( \mathbb{T} \) and a \( \partial \mathbb{D} \)-normal operator is just a unitary operator. Since \( \mathbb{D} \) is polynomially convex, it suffices to work with polynomials rather than rational functions with no poles in \( \overline{\mathbb{D}} \). By choosing the polynomial \( p \) to be \( p = \chi \) and \( \chi(\lambda) = \lambda \), we see that \( \|T\| \leq 1 \) (i.e., that \( T \) be a contraction) is necessary for \( \mathbb{D} \) to be a spectral set for \( T \). The fact that this condition is also sufficient, i.e., that the inequality

\[
\|p(T)\| \leq \sup_{\lambda \in \mathbb{D}} |p(\lambda)|
\]

holds for any contraction operator \( T \) and polynomial \( p \), is a classical inequality known as von Neumann’s inequality going back to [34]. to show that \( \mathbb{D} \) is a complete spectral
set for any contraction operator $T$, we may use the easier side of Arveson’s theorem and show instead that any contraction operator $T$ has a $𝒟$-unitary dilation. But for the case $Ω = 𝒟$, according to our conventions, a $𝒟$-unitary operator is just a unitary operator $U$ (i.e., $U^*U = UU^* = I_𝒟$). But any contraction operator $T$ on $ℋ$ dilating to a unitary operator $U$ on $ onHide$ $ℋ$ is exactly the content of the Sz.-Nagy dilation theorem (see [32, Chapter II]).

Over the ensuing decades there have been sporadic attempts to find other domains (both contained in $ℂ$ or more generally contained in $ℂ^d$) for which one can settle the rational dilation question one way or the other (i.e., positively or negatively). Among single-variable domains (as observed in the introduction of [9] where precise references are given), it is known that rational dilation holds if $Ω \subset ℂ$ is a simply connected domain (simply use a conformal map to reduce to the disk case) or is doubly-connected, but fails if $Ω$ has two or more holes (see [2, 21]). As for multivariable domains, perhaps the first class to be understood are the polydisks $𝒟^d$ with $d \geq 2$: for $d = 2$ rational dilation holds due to the Andô dilation theorem [5] while for $d \geq 3$ rational dilation fails (see [29, 35]).

More recently the rational dilation problem has been investigated for other concrete multivariable domains originally discovered due to connections with the $₅$-synthesis problem in Robust Control Theory (see the original Doyle-Packard paper [20] as well as the book [24] for a more expository treatment). We mention in particular the symmetrized bidisk

$$Γ = \{(s, p) ∈ ℂ^2 : s = (λ₁ + λ₂), p = λ₁λ₂ \text{ for some } (λ₁, λ₂) ∈ ℂ^2\} \quad (1.2)$$

and a domain in $ℂ^3$ called the tetrablock and denoted by $Ẽ$:

$$Ẽ := \{(a, b, \det X) : X = \begin{bmatrix} a & a' \\ b' & b \end{bmatrix} \text{ with } \|X\| < 1\} \quad (1.3)$$

As might be expected, the domain $Γ$ behaves like $𝒟^2$ with respect to the rational dilation problem as both domains are contained in $ℂ^2$: specifically, rational dilation holds for the domain $Γ$ (see [3, 4, 13]) and there is a functional model analogous to the Sz.-Nagy–Foias model for the disk case (see [4, 14]), at least for the pure case. The situation of the rational dilation problem for the tetrablock $Ẽ$ is less clear: there is a sufficient and a necessary condition for the existence of a $Ẽ$-isometric lift of a certain form [9, 12] but a definitive resolution of the problem in full generality remains elusive (see [9, 26]). However it is shown in [30] that, at least in the pure case, it is still possible to construct a functional representation of a pure $Γ$-contraction as the compression to $ℋ$ of a certain lift triple $(Aℓ, Bℓ, Tℓ)$ which formally looks like an tetrablock isometry but is not guaranteed to satisfy all of the required commutativity conditions. A similar phenomenon holds for the case where $Ω = ℂ^d$ with $d \geq 2$ (see [11]): for this case, as pointed out above, there are indeed counterexamples to show that rational dilation fails, but there is nevertheless a weaker type of lift (called pseudo-commutative $𝒟^d$-isometric lift)) which generates a functional model for the given $𝒟^d$-contractive $d$-tuple $T = (T₁, \ldots, T_d)$ even when rational dilation fails.
In this paper we focus on the case $\Omega = \mathbb{E}$. As was the case in [12], the most definitive results are for the case of what we shall call a \textbf{special tetrablock contraction}, i.e., a tetrablock contraction $(A, B, T)$ which has a tetrablock isometric lift $(V_1, V_2, V_3)$ such that $V_3 = V$ is a Sz.-Nagy–Foias minimal isometric lift for the single contraction operator $T$. As in [12], we identify the additional commutativity conditions (2.8) which must be imposed on the Fundamental Operator pair $(G_1, G_2)$ of $(A^*, B^*, T^*)$ which characterizes when $(A, B, T)$ is special. There results a Douglas-type functional model (as in [19] for the single contraction operator setting) for the tetrablock contraction which also exhibits the tetrablock isometric lift $(V_1, V_2, V_3)$, all in a functional-model form rather than via block-matrix constructions as in [12]. This Douglas-type model can in turn be converted to a Sz.-Nagy–Foias-type model; the Sz.-Nagy–Foias characteristic function $\Theta_T$ for the contraction operator $T$, together with the the fundamental operators $(G_1, G_2)$ for the adjoint tetrablock contraction $(A^*, B^*, T^*)$, along with some additional information needed to handle the case where $T$ is not a pure contraction, form what we call a characteristic \textbf{tetrablock data set} for $(A, B, T)$ in terms of which one can write down the functional model. Conversely, we identify a collection of objects which we call a \textbf{special tetrablock data set}: specifically, (i) a pure contractive operator function $(D, D_\omega, \Theta)$, (ii) a pair of operators $(G_1, G_2)$ on the coefficient space $D_\omega$, (iii) a tetrablock unitary $(R, S, W)$ acting on $D_\Theta \cdot L^2(D)$, such that (iv) all these together satisfy a natural invariant-subspace compatibility condition. From such a characteristic tetrablock data set we construct a functional model such that the embedded functional-model operator triple is the most general special tetrablock contraction up to unitary equivalence, with its special tetrablock isometric lift also embedded in the functional model. We also are careful to push the theory as far as we can without the assumption that the original tetrablock contraction is special. In this case we identify a class of operator triples $(V_1, V_2, V_3)$ with $V_3$ equal to a minimal isometric lift for $T$ to which $(A, B, T)$ can be lifted: here $V_1$ and $V_2$ commute with $V_3$ but not necessarily with each other and it appears that $V_1, V_2$ need not be contractions. In this case there is no converse direction: there is no guarantee that the compression of a general pseudo-commutative tetrablock isometry $(V_1, V_2, V_3)$ on $\mathcal{K}$ back to $\mathcal{H}$ will yield a tetrablock contraction.

Let us note that the recent paper of Bisai and Pal [16] contains closely related results. These authors basically compute the $Z$-transform of the Schäffer-type construction of the unique special tetrablock isometric lift $(V_1, V_2, V_3)$ (where $V_3$ is equal to the minimal Sz.-Nagy isometric lift of $T$) to arrive at a functional model for this lift. Our approach on the other hand uses the Douglas lifting approach to construct the functional model directly with the existence and uniqueness of the special tetrablock isometric lift falling out as part of the construction. When the tetrablock contraction is not special and no such lift is possible, the same construction still leads to a functional model but $(V_1, V_2, V_3)$ is only a pseudo-commutative tetrablock isometry and there is no tetrablock isometric lift constructed in this way. The results for the special case arise as a special case (the case where the Fundamental Operator pair $(G_1, G_2)$ for the tetrablock contraction $(A^*, B^*, T^*)$ satisfy the additional commutativity conditions (2.8)) of the general functional-model construction. The paper [16] also obtains a noncommutative functional model for a non-special case, based on the work of Durszt...
[25] (a variation of the approach of Douglas for the construction of the minimal isometric lift for the case of a single contraction operator \( T \)), but with the additional hypothesis that \( A \) and \( B \) commute not only with \( T \) but also with \( T^* \). It is clear that the complete unitary invariant for a pure tetrablock contraction \( A, B, T \) consists of the characteristic function \( \Theta_T \) of \( T \) together with the Fundamental Operator pair \( \{ G_1, G_2 \} \) of \( \{ A^*, B^*, T^* \} \); for the non-pure case (where \( \Theta_T \) is no longer inner) we add a certain tetrablock unitary \( (R, S, W) \) acting on \( \Delta^2 H^2(D_T) \) which is part of our model (see Theorem 4.5 below), while Bisai-Pal add the Fundamental Operator pair \( (F_1, F_2) \) for \( A, B, T \) and argue that \( (\Theta_T, (F_1, F_2), G_1, G_2) \) is a complete unitary invariant. It remains to be seen which is the more relevant and useful in the future.

It is now becoming clear that the domains \( \mathbb{D}^d \) (polydisk), \( \Gamma \) (symmetrized bidisk), \( \mathbb{E} \) (tetrablock) as well as \( \mathbb{D}^d_s \) (symmetrized polydisk) all have common features with respect to the associated operator theory and applications to the rational dilation problem for each of these domains. The paper [11] shows how a program completely parallel to that done here for the tetrablock case can be worked out equally well for the polydisk case \( \Omega = \mathbb{D}^d \) (where rational dilation is known to fail when \( d \geq 3 \)). In all these settings, there appear a pair of unitary invariants called Fundamental Operators which play a key role as part of a set (including the Sz.-Nagy–Foias characteristic function of an appropriate contraction operator determined by the operator tuple) of unitary invariants for the operator tuple of whatever class. The notion of Fundamental Operators as a fundamental object of interest seems to have appeared first in connection with the symmetrized bidisk \( \Gamma \) [14], then in connection with the un-symmetrized polydisk [11, 31], and now also in connection with the symmetrized polydisk (see [28]). Often the proper notion of Fundamental Operators for one setting is found by making a correspondence of the less understood setting with some other better understood setting, and then adapting definitions for the first to become definitions for the second. In particular, many of the results for the tetrablock case were originally found by adapting from results for the symmetrized bidisk case (see e.g. [12]), and it has been shown how one can deduce the bidisk functional model from the tetrablock functional model (see [31]). In this spirit in a future publication we plan to show how the results from [11] for the polydisk case (most of which are just statements parallel to what is done here for the tetrablock case) can alternatively be derived as a corollary of the corresponding results for the tetrablock case via the simple observation: if \( T = (T_1, \ldots, T_d) \) is a commutative, contractive operator \( d \)-tuple, then for \( 1 \leq i \leq d \), if we set \( T_{\{i\}} = \Pi_{1 \leq j \leq d: j \neq i} T_j \), then for each \( i = 1, \ldots, d \) the \( d \)-tuple \( (A_i, B_i, P) = (T_i, T_{\{i\}}, \Pi_{1 \leq j \leq d} T_j) \) is a tetrablock contraction; the \( d = 2 \) case can be found in [31, Section 3, Version 3].

Finally, let us point out that it is possible to reformulate the rational dilation problem for a given domain \( \Omega \) as a problem about unital representations of a unital function algebra: given a contractive representation \( \pi: f \in \mathcal{A} \mapsto \pi(f) \in \mathcal{B}(\mathcal{H}) \) which is contractive \( \|\pi(f)\|_{\mathcal{B}(\mathcal{H})} \leq \|f\|_{\mathcal{B}(\mathcal{H})} \) where the unital representation property is that \( \pi(1_{\mathcal{A}}) = 1_{\mathcal{H}} \) and \( \pi(f_1 \cdot f_2) = \pi(f_1)\pi(f_2) \), it is automatically the case that the representation is completely contractive, i.e., still contractive after tensoring with \( \mathbb{C}^{n \times n} \) for any \( n \in \mathbb{N} \)? To recover the original formulation as a special case, one can take \( \mathcal{A} = \text{Rat}(\overline{\Omega}) \) where the closure is in the \( C^* \)-algebra \( C(b\Omega) \) (continuous functions on
the distinguished boundary $\partial \Omega$). However with this more general formulation one can consider function algebras which go beyond $\text{Rat}(\Omega)$, e.g., the constrained subalgebra $\mathbb{C} \cdot 1 + z^2 \mathbb{A}(\mathbb{D})$ of the disk algebra $\mathbb{A}(\mathbb{D}) = \text{Rat}(\mathbb{D})$. Alternatively, it is often possible to represent the algebra $\mathbb{A}$ as conformally equivalent to the algebra of all functions analytic on some algebraic curve $\Omega = \mathbb{C}$ embedded in some higher-dimensional closed complex manifold (the Neil parabola intersected with the bidisk for the case of $\mathbb{C} \cdot 1 + z^2 \mathbb{A}(\mathbb{D})$). For the state of knowledge (up to 2018) on this direction of dilation theory including much discussion and references on earlier work, we refer to the paper of Dritschel and Undrakh [23]. We shall not pursue this direction here.

The paper is organized as follows. After the present Introduction, in Sect. 2 we collect assorted definitions and illustrative results concerning tetrablock contractions, tetrablock isometries, and tetrablock unitaries, including a direct proof of the existence of the Fundamental Operator pair for a given tetrablock contraction, which will be needed in the sequel. Here we also show how to associate a tetrablock unitary $(R, S, W)$ with a tetrablock contraction $(A, B, T)$ in a canonical way; this is the key ingredient needed to eliminate the purity assumption on the contraction operator $T$ required in earlier work on this problem (see [30]). Section 3 shows how a lifting framework for the tetrablock-contraction setting can be constructed as an embellishment of the Douglas-model lifting framework [19] originally formulated as an approach to the Sz.-Nagy dilation theorem for a single contraction operator $T$, with the pseudo-commutative tetrablock-isometric lift $(V_1, V_2, V_3)$ having $V_3 = T$ and $V_1$ and $V_2$ constructed by making use of the Fundamental Operator pair for the tetrablock contraction $(A^*, B^*, T^*)$. The final Sect. 4 identifies the invariants required to write down a functional model equipped with a model operator triple $(A, B, T)$ which is concrete functional-model version of a general tetrablock contraction.

2 The Fundamentals of Tetrablock Contractions

This section gives a brief introduction to the operator theory associated with the tetrablock.

2.1 Tetrablock Contractions

The tetrablock, denoted by $\mathbb{E}$, is the non-convex but polynomially convex domain in $\mathbb{C}^3$ given by (1.3). From this formula for $\mathbb{E}$ is easy to read off the following symmetry properties.

**Proposition 2.1** The tetrablock $\mathbb{E}$ has the following symmetry properties:

1. $\mathbb{E}$ is invariant under complex conjugation:
   
   $$(a, b, t) \in \mathbb{E} \iff (\bar{a}, \bar{b}, \bar{t}) \in \mathbb{E}.$$  

2. $\mathbb{E}$ is invariant under interchange of the first two coordinates:
   
   $$(a, b, t) \in \mathbb{E} \iff (b, a, t) \in \mathbb{E}.$$
The distinguished boundary of $\mathbb{E}$, i.e., the Šilov boundary with respect to the algebra of functions that are analytic in $\mathbb{E}$ and continuous on $\overline{\mathbb{E}}$, is given by

$$b\mathbb{E} := \left\{ (a, b, \text{det}X) : X = \begin{bmatrix} a & a' \\ b' & b \end{bmatrix} \text{ is a unitary} \right\}$$

(see [1, Theorem 7.1]). From this characterization it is easy to see that $b\mathbb{E}$ is also invariant under the two involutions $(a, b, t) \mapsto (\overline{a}, \overline{b}, \overline{t})$ and $(a, b, t) \mapsto (b, a, t)$.

Several tractable characterizations of the tetrablock can be found in [1, Theorem 2.2]; we pick two of these that will be used in what follows.

**Theorem 2.2** For a point $(a, b, t) \in \mathbb{C}^3$, the following are equivalent:

(i) $(a, b, t) \in \mathbb{E}$;

(ii) with the rational function $\Psi : \overline{\mathbb{D}} \times \mathbb{C}^3 \to \mathbb{C}$ defined as

$$\Psi(z, (a, b, t)) = \frac{a - zt}{1 - zb}, \quad (2.1)$$

$$\sup_{z \in \overline{\mathbb{D}}} |\Psi(z, (a, b, t))| < 1; \text{ and if } ab = t \text{ then, in addition, } |b| < 1;$$

(iii) with $\Psi$ as in (2.1), $\sup_{z \in \overline{\mathbb{D}}} |\Psi(z, (b, a, t))| < 1; \text{ and if } ab = t \text{ then, in addition, } |a| < 1$.

Moreover, when item (i) is replaced by $(a, b, t) \in \mathbb{E}$, then all the strict inequalities in items (ii) and (iii) are replaced by non-strict inequalities.

**Remark 2.3** Note that in Theorem 2.2, the equivalence of (i) ⇔ (iii) is an immediate consequence of the equivalence of (i) ⇔ (ii) in view of the invariance of $\mathbb{E}$ under the involution $(a, b, t) \mapsto (b, a, t)$.

Recall the notions of tetrablock unitary, tetrablock isometry and tetrablock contraction given in the Introduction. Several algebraic characterizations of tetrablock isometries and tetrablock unitaries are known; see Theorems 5.4 and 5.7 in [12]. We recall the ones that are useful for our purposes here. Here we use the notation $r(X)$ for the spectral radius of a Hilbert-space operator $X$. It is then not difficult to see that the $\mathbb{E}$-symmetries noted in Proposition 2.1 imply the same symmetries on the respective operator classes (with respect to the class of $\mathbb{E}$-isometries which requires a little extra care), as noted in the next result. We leave the easy verification as an exercise for the reader.

**Proposition 2.4** Suppose that $(A, B, T)$ is a triple of bounded operators on a Hilbert space $\mathcal{H}$. Then:

1. $(A, B, T)$ is a $\mathbb{E}$-contraction $\iff (A^*, B^*, T^*)$ is a $\mathbb{E}$-contraction $\iff (B, A, T)$ is a $\mathbb{E}$-contraction.
2. $(A, B, T)$ is a $\mathbb{E}$-isometry $\iff (B, A, T)$ is a $\mathbb{E}$-isometry.
3. $(A, B, T)$ is a $\mathbb{E}$-unitary $\iff (A^*, B^*, T^*)$ is a $\mathbb{E}$-unitary $\iff (B, A, T)$ is a $\mathbb{E}$-unitary.
Theorem 2.5 Let \((A, B, T)\) be a commutative triple of bounded Hilbert space operators. Then the following are equivalent:

(i) \((A, B, T)\) is a tetrablock isometry (respectively unitary);
(ii) \((A, B, T)\) is a tetrablock contraction and \(T\) is an isometry (respectively unitary);
(iii) \(A = B^*T\), \(B\) is a contraction and \(T\) is an isometry (respectively unitary); and
(iv) \(B = A^*T\), \(A\) is a contraction and \(T\) is an isometry (respectively unitary).
(v) \(B = A^*T\), \(r(A) \leq 1\) and \(r(B) \leq 1\), and \(T\) is an isometry (respectively unitary).

2.2 Pseudo-commutative Tetrablock Isometries and Unitaries

We propose to introduce the notions of pseudo-commutative tetrablock unitary and pseudo-commutative tetrablock isometry for an operator triple \((A, B, T)\) by using criterion (iii) or equivalently (iv) in Theorem 2.5 but with the weakening the commutativity hypothesis imposed on the whole triple \((A, B, T)\) to just the condition that \(A\) and \(B\) commute with \(T\) (but not necessarily with each other). As we are also dropping the condition that \(A\) or \(B\) be a contraction, a more proper term would be noncontractive pseudo-commutative tetrablock isometry, but, as this term will be consistent throughout, we settle on the shorter term for brevity. The resulting definition is as follows. We leave it to the reader to verify that the two formulations are equivalent.

Definition 2.6 Let \((A, B, T)\) be a triple of bounded Hilbert-space operators. We say that the triple \((A, B, T)\) is a pseudo-commutative tetrablock isometry (respectively, unitary) if any of the following equivalent conditions holds:

\begin{align}
(1) & \quad T \text{ is an isometry (respectively, unitary) and} \\
& \quad AT = TA, \quad BT = TB, \quad A = B^*T. \\
(2.2) & \\
(2) & \quad T \text{ is an isometry (respectively, unitary), and} \\
& \quad AT = TA, \quad BT = TB, \quad B = A^*T. \\
(2.3) &
\end{align}

Remark 2.7 From Definition 2.6 and Theorem 2.5, we see that any tetrablock isometry/unitary is also a pseudo-commutative tetrablock isometry/unitary but not conversely. If we wish to emphasize that we are referring to the logically more special tetrablock isometry/unitary rather than the more general pseudo-commutative tetrablock isometry/unitary, we often will say strict tetrablock isometry/unitary for emphasis.

Remark 2.8 We now present a couple of elementary observations on pseudo-commutative versus strict tetrablock unitaries which we hope give the reader some additional insight.

\begin{enumerate}
\item We remark that if \((A, B, T)\) is a pseudo-commutative tetrablock unitary, then \(A\) and \(B\) are not necessarily normal operators as would happen in the strict case. For example, pick a non-normal contraction \(G_1\) acting on a Hilbert space \(\mathcal{E}\) and consider the triple \((M_{G_1^*}, M_{\xi G_1}, M_{\xi})\) on \(L^2(\mathcal{E})\). It is easy to see that this triple
is a pseudo-commutative tetrablock unitary. However, neither $A$ nor $B$ is normal unless $G_1$ is so. Also note that if $(A, B, T)$ is a pseudo-commutative tetrablock unitary, then so is the adjoint triple $(A^*, B^*, T^*)$. This can be seen by observing that the adjoint of the identities in (2.2) with $T$ unitary can be converted to the identities (2.3) for $(A^*, B^*, T^*)$ with $T^*$ still unitary. Note next that if $(A, B, T)$ is a pseudo-commutative tetrablock unitary, then

$$A^*A = T^*BB^*T = BT^*TB^* = BB^*, \quad B^*B = T^*AA^*T = AT^*TA^* = AA^*.$$ 

Thus we always have

$$A^*A = BB^*, \quad AA^* = B^*B \quad (2.4)$$

for a pseudo-commutative tetrablock unitary $(A, B, T)$. As a first consequence of (2.4), we see that if $A$ is normal, then

$$B^*B = AA^* = A^*A = BB^*$$

and $B$ is also normal. Similarly if $B$ is normal, then $A$ is also normal. In conclusion, if $(A, B, T)$ is a pseudo-commutative unitary such that one of $A$ or $B$ is normal, then so is the other.

(2) We note as a consequence of (iii) $\Rightarrow$ (i) in Theorem 2.5 that in particular if $(A, B, T)$ is a strict tetrablock unitary (so we also have $AB = BA$), then the operators $A$ and $B$ are normal. One can see this directly from the considerations here as follows. As a strict tetrablock unitary in particular meets all the requirements for membership in the pseudo-commutative tetrablock unitary class, we know that (2.4) holds. Combining this with the commutativity relation $AB = BA$ then gives us

$$A^*A = A^*B^*T = B^*A^*T = B^*B = AA^*$$

showing that $A$ is normal. The same computation with the roles of $A$ and $B$ interchanged then shows that $B$ is also normal. The full strength of (iii) $\Rightarrow$ (i) in Theorem 2.5 is that in addition the commutative normal triple $(A, B, T)$ has joint spectrum in the boundary of the tetrablock $\mathcal{E}$; for this somewhat deeper fact we refer to [12].

The next result gives a feel for how close pseudo-commutative tetrablock isometries come to being strict tetrablock isometries.

**Theorem 2.9** Let $(A, B, T)$ be a pseudo-commutative tetrablock isometry on a Hilbert space $\mathcal{H}$.

(1) Then the spectral radius $r(AB)$ of the product operator $AB$ is given by

$$r(AB) = \max\{\|A\|^2, \|B\|^2\}. \quad (2.5)$$
(2) Suppose in addition that $AB = BA$ and $r(A) \leq 1$, $r(B) \leq 1$. Then both $A$ and $B$ are contraction operators ($\max\{\|A\|, \|B\|\} \leq 1$) and $(A, B, T)$ is a strict tetrablock isometry.

**Proof** The proof follows the ideas of Bhattacharyya [12, pp.1619-1620]. We first consider statement (1). Form two operators $X_1 = [0 \ A \ B \ 0]$ and $X_2 = [T \ T \ 0 \ 0]$ on $\mathbb{H}^2$. From the two relations $B^*T = A$ and $A^*T = B$ we deduce that $X_1 = X_1^*X_2$ where $X_2X_2^* = [T^*T \ 0 \ 0 \ I]$ since $T$ is an isometry. Hence $X_1X_1^* = X_1^*X_2X_2^*X_1 \preceq X_1^*X_1$, i.e., $X_1$ is a hyponormal operator. By a theorem of Stampfli (see [17, Proposition 4.6], it follows that $r(X_1) = \|X_1\|$. We compute the operator norm of $X_1$ as follows:

$$\|X_1\|^2 = \left\|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} 0 & B^* \\ A^* & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} AA^* & 0 \\ 0 & BB^* \end{bmatrix}\right\| = \max\{\|A\|^2, \|B\|^2\}.$$  

and hence $\|X_1\| = \max\{\|A\|, \|B\|\}$. To compute $r(X_1)$, note first that $X_1^*X_1 = \begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}$ and hence

$$X_1^{2n} = \begin{bmatrix} (AB)^n & 0 \\ 0 & (BA)^n \end{bmatrix}.$$  

Consequently,

$$r(X_1) = \lim_{n \to \infty} \max \frac{1}{2^n} \|\|AB\|^n\|^{\frac{1}{2^n}}, \|BA\|^n\|^{\frac{1}{2^n}} = \max\{r(AB)^{\frac{1}{2}}, r(BA)^{\frac{1}{2}}\}.$$  

However a general fact is that the nonzero spectrum of $AB$ is the same as the nonzero spectrum of $BA$, and hence $r(AB) = r(BA)$. Thus $r(X_1) = \|X_1\|$ gives us (2.5) and the proof of statement (1) is complete.

As for statement (2), a known fact is that if $A$ and $B$ commute, then the spectrum of the product operator $AB$ is given by

$$\sigma(AB) = \{\lambda \cdot \mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}.$$  

Hence the hypothesis that $r(A) \leq 1$ and $r(B) \leq 1$ implies that $r(AB) \leq 1$ as well. But then from the conclusion of statement (1) already proved, we conclude that both $A$ and $B$ are contraction operators, and the proof of statement (2) is now complete. (Note that this also proves (v) $\Rightarrow$ (iii) or (iv) in Theorem 2.5.) \qed

**Example 2.10** (1) A pseudo-commutative/strict tetrablock isometry. Let $\mathcal{E}$ be a coefficient Hilbert space and $H^2(\mathcal{E}) = H^2 \otimes \mathcal{E}$ be the associated Hardy space of $\mathcal{E}$-valued functions. Let $G_1$ and $G_2$ be operators on $\mathcal{E}$ and set

$$A = MG_1^* + zG_2, \quad B = MG_2^* + zG_1, \quad T = M_z \text{ on } H^2(\mathcal{E}).$$  

(2.6)
Then it is immediate that $T$ is an isometry and that $A$ and $B$ commute with $T$. The special coupled form of the pencils defining $A$ and $B$ enables us to show that $A = B^*T$:

$$B^*T = (I_{H^2} \otimes G_2^* + M_z \otimes G_1)^* \cdot (M_z \otimes I_G)$$

$$= (I_{H^2} \otimes G_2 + M_z^* \otimes G_1^*) \cdot (M_z \otimes I_G)$$

$$= (I_{H^2} \otimes G_1^*) + (M_z \otimes G_2) = MG_1^* + zG_2 = A$$

and similarly $B = A^*T$.

For $(A, B, T)$ to be a strict tetrablock isometry, we need in addition that $AB = BA$ and that $\|A\| \leq 1$ (in which case also $\|B\| = \|A^*T\| \leq 1$ as well). To ensure that $\|A\| \leq 1$ requires that $G_1$ and $G_2$ are not too large in the precise sense that

$$\sup_{z \in \mathbb{T}} \|G_1^* + zG_2\| \leq 1. \quad (2.7)$$

To check the condition $AB = BA$, we compute

$$AB = MG_1^* + zG_2MG_2^* + zG_1$$

$$= (I_{H^2} \otimes G_1^* + M_z \otimes G_2) \cdot (I_{H^2} \otimes G_2^* + M_z \otimes G_1)$$

$$= I_{H^2} \otimes G_1^*G_2^* + M_z \otimes (G_1^*G_1 + G_2G_2^*) + M_z^2 \otimes G_1G_2$$

while a similar computation gives us

$$BA = I_{H^2} \otimes G_2G_1^* + M_z \otimes (G_1G_1^* + G_2^*G_2) + M_z^2 \otimes G_2G_1.$$

We conclude that in this example, $(A, B, T)$ is a strict tetrablock isometry exactly when $(2.7)$ together with the following commutativity conditions hold:

$$G_1G_2 = G_2G_1, \quad G_1^*G_1 + G_2^*G_2 = G_1G_1^* + G_2G_2^*, \quad (2.8)$$

sometimes also written more compactly in terms of commutators as

$$[G_1, G_2] = 0, \quad [G_1^*, G_1] = [G_2^*, G_2]$$

where in general $[X, Y]$ is the commutator:

$$[X, Y] = XY - YX.$$

(2) A pseudo-commutative/strict tetrablock unitary. It is easy to use the spectral theory for unitary operators (a particular case of the spectral theory of normal operators) to write down a model for the general pseudo-commutative/strict tetrablock unitary $(R, S, W)$, as follows. By Definition 2.6 we see in particular that $W$ is unitary. By the spectral theory for general normal operators (see e.g., any
of [6, 18] or [8, Chapter 2]), after a unitary change of coordinates, we can represent $W$ as the operator $M_\xi$ of multiplication by the coordinate function $(M_\xi : h(\xi) \mapsto \xi h(\xi))$ on a direct-integral space $\bigoplus \int_\mathbb{T} \mathcal{H}_\xi \nu(d\xi)$ determined by a scalar spectral measure $\nu$ supported on $\mathbb{T}$ and a measurable multiplicity function $\xi \mapsto \dim \mathcal{H}_\xi$. Since the operators $R$ and $S$ commute with $W = M_\xi$, it follows that $R$ and $S$ are represented as decomposable operators on $\bigoplus \int_\mathbb{T} \mathcal{H}_\xi \delta^*(1)$, i.e.,

$$R = M_\phi : h(\xi) \mapsto \phi(\xi)h(\xi)$$

and

$$S = M_\psi : h(\xi) \mapsto \psi(\xi)h(\xi)$$

for measurable functions such that $\phi(\xi) \in \mathcal{B}(\mathcal{H}_\xi)$, $\psi(\xi) \in \mathcal{B}(\mathcal{H}_\xi)$ for a.e. $\xi$. The fact that in addition $R = S^*W$ then forces $\phi(\xi) = \psi(\xi)^* \cdot \xi$ for a.e. $\xi$. Thus any pseudo-commutative tetrablock unitary has the form

$$(R, S, W) = (M_{\psi^*\xi}, M_\psi, M_\xi)$$

acting on $\bigoplus \int_\mathbb{T} \mathcal{H}_\xi \nu(d\xi)$ (2.9)

If $(R, S, W)$ is a strict tetrablock unitary, then in addition we must have that $\psi(\xi)$ is a contractive normal operator on $\mathcal{H}_\xi$ for a.e. $\xi$ in order to guarantee in addition that $RS = SR$ and that $\|R\| \leq 1$, $\|S\| \leq 1$. By this analysis we conclude that (2.9) (with $\psi(\xi)$ constrained to be contractive normal for a.e. $\xi$ for the strict case) is the general form for a pseudo-commutative/strict tetrablock unitary. In a less-functional form, to write a pseudo-contractive tetrablock triple $(R, S, W)$, the free parameters are: (i) a unitary operator $W$, and (ii) an operator $S$ commuting with $W$; then the associated pseudo-commutative tetrablock contraction is $(W^*S, S, W)$; for this to be strict, one must require in addition that the operator $S$ in the commutant of $W$ be a normal contraction.

To deduce the von Neumann-Wold decomposition for a tetrablock isometry, the next lemma is useful. Only the special case where the operator $S$ in the statement is a shift will be needed for our application, in which case the result is well-known (see e.g. [33, page 22]). For completeness we present here a proof of the general result.

**Lemma 2.11** Let $W$ be a unitary operator on $\mathcal{H}_2$, $S$ an operator on $\mathcal{H}_1$ such that $S^{*n} \to 0$ in the strong operator topology as $n \to \infty$. If $X$ is a bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ such that $WX = SX$, then $X = 0$.

**Proof** From $WX = SX$ we get by iteration that $XW^n = S^nX$ for $n = 1, 2, \ldots$. Taking adjoints gives then $W^{*n}X^* = X^*S^{*n}$. Apply this identity to an arbitrary fixed vector $x \in \mathcal{H}_2$ to get $W^{*n}X^*x = X^*S^{*n}x$ for all $n \geq 1$. Apply $W^n$ to both sides of this equation to get $W^nW^{*n}X^*x = W^nX^*S^{*n}x$ for $n = 1, 2, \ldots$. As $W$ is unitary, this becomes $X^*x = W^nX^*S^{*n}x$. Taking norms then gives

$$\|X^*x\| = \|W^nX^*S^{*n}x\| = \|X^*S^{*n}x\| \leq \|X^*\| \|S^{*n}x\| \to 0 \text{ as } n \to \infty$$

by the assumed strong convergence of powers of $S^*$ to zero, forcing $X^*$ (and hence also $X$) to be the zero operator.  

The von Neumann-Wold decomposition (see [32, 34, 36]) ensures that if $T$ is an isometry acting on a Hilbert space $\mathcal{H}$, then $T$ can be represented as an operator as the
external direct sum $M_z \oplus U$ of a shift operator $M_z$ acting on a Hardy space $H^2(\mathcal{E})$ and a unitary operator $W$ on $\mathcal{F}$ for some coefficient Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$. The following result not only gives a model for an arbitrary pseudo-commutative/strict tetrablock isometry $(A, B, T)$, but also can be seen as a pseudo-commutative/strict tetrablock-isometry analogue of the classical von Neumann–Wold decomposition for a single isometric Hilbert space operator $T$.

**Theorem 2.12** Let $(A, B, T)$ be an operator-triple on the Hilbert space $\mathcal{H}$.

(1) Then $(A, B, T)$ is a pseudo-commutative tetrablock isometry on $\mathcal{H}$ if and only if there exist Hilbert spaces $\mathcal{E}, \mathcal{F}$, operators $G_1, G_2$ acting on $\mathcal{E}$ subject to (2.7), along with a pseudo-commutative tetrablock unitary $(R, S, W)$ acting on $\mathcal{F}$, such that $\mathcal{H}$ is isomorphic to $\begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix}$ and under the same isomorphism $(A, B, T)$ is unitarily equivalent to

$$
\begin{bmatrix}
M_{G_1^*+zG_2} & 0 & 0 \\
0 & M_{G_2^*+zG_1} & 0 \\
0 & 0 & M_z
\end{bmatrix}.
$$

(2.10)

(2) Then $(A, B, T)$ is a strict tetrablock isometry on $\mathcal{H}$ if and only if $(A, B, T)$ is unitarily equivalent to the operator triple as in (2.10) (with $G_1, G_2$ subject to (2.7)) acting on a space $\begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix}$, where in addition the operator-pencil coefficients $(G_1, G_2)$ satisfy the system of operator identities (2.8), and the triple $(R, S, W)$ is a strict tetrablock unitary (i.e., we also have the relation $RS = SR$ with $R$ and $S$ contraction operators).

**Remark 2.13** We shall think of a triple of operators on $\begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix}$ as in (2.10) as a functional model for a pseudo-commutative/strict tetrablock isometry/unitary. The $H^2(\mathcal{E})$-component clearly has a functional form while the second component can be brought to a measure-theoretic functional form as in item (2) in Example 2.10.

**Proof** The sufficiency (for both the pseudo-commutative and the strict case) follows from Example 2.10.

We now suppose that $(A, B, T)$ is a strict tetrablock isometry. Let us apply the Wold decomposition to the isometry $T$: there exist Hilbert spaces $\mathcal{E}, \mathcal{F}$, and a unitary $\tau : \mathcal{H} \rightarrow \begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix}$ such that

$$
\tau T \tau^* = \begin{bmatrix} M_z & 0 \\
0 & W \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix} \rightarrow \begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix}
$$

for some unitary $W$ on $\mathcal{F}$. Next assume that

$$
\tau(A, B) \tau^* = \begin{bmatrix} A_{11} & A_{12} \\
A_{21} & R \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\
B_{21} & S \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix} \rightarrow \begin{bmatrix} H^2(\mathcal{E}) \\ \mathcal{F} \end{bmatrix}.
$$

Now use these matrix representations and equate the (12)-entries of the relation $AT = TA$ to get $A_{12}W = M_zA_{12}$. Therefore $A_{12} = 0$ by Lemma 2.11. Similarly, from
the relation $BT = TB$ we have $B_{12} = 0$. Compare the (21)-entries of the relation $A = B^*T$ to get $A_{21} = 0$. The same treatment for the relation $B = A^*T$ gives $B_{21} = 0$. Therefore we are left with the following relations

$$A_{11}M_z = M_z A_{11}, \quad B_{11}M_z = M_z B_{11}, \quad A_{11} = B^*_{11}M_z \text{ (and } B_{11} = A^*_{11}M_z);$$

$$RW = WR, \quad SW = WS, \quad R = S^*W \text{ (and } S = WR^*).$$

The second set of the above relations together with the fact that $W$ is a unitary implies that $(R, S, W)$ is a pseudo-commutative tetrablock unitary. The first two intertwining relations in the first set implies that there exist bounded analytic functions $\Phi, \Psi : \mathbb{D} \to \mathcal{B}(\mathcal{E})$ such that $A_{11} = M_\Phi, \quad B_{11} = M_\Psi$. The remaining relations in the first set then give us

$$M_\Phi = M^*_\Psi M_z, \quad M_\Psi = M^*_\Phi M_z.$$ 

There now only remains a tedious computation with the power series expansions of $\Phi$ and $\Psi$ to see that the remaining relations in the second set forces $\Phi$ and $\Psi$ to have the coupled linear forms

$$\Phi(z) = G^*_1 + zG_2 \quad \text{and} \quad \Psi(z) = G^*_2 + zG_1$$

for some operators $G_1, \quad G_2 \in \mathcal{B}(\mathcal{E})$. Again the relation (2.7) is equivalent to $M_\Phi$ being a contraction operator. From Definition 2.6 it follows that $(M_\Phi, M_\Psi, M_z)$ is a pseudo-commutative tetrablock isometry. The completes the proof for the pseudo-commutative setting.

Suppose now that $(A, B, T)$ is a tetrablock isometry. Then in particular $(A, B, T)$ satisfies all the requirements to be a pseudo-commutative tetrablock isometry, so all the preceding analysis applies. We then see that $(A, B, T)$ is unitarily equivalent to the triple in (2.10). As $(A, B, T)$ now is actually a tetrablock isometry, we have that $AB = BA$. The unitary equivalence then forces

$$\begin{bmatrix} M_{G_1^*+zG_2} & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} M_{G_2^*+zG_1} & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} M_{G_2^*+zG_1} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} M_{G_1^*+zG_2} & 0 \\ 0 & R \end{bmatrix}$$

which can be split up as two commutativity conditions

$$M_{G_1^*+zG_2}M_{G_2^*+zG_1} = M_{G_2^*+zG_1}M_{G_1^*+zG_2} \quad (2.11)$$

$$RS = SR. \quad (2.12)$$

By reversing the computations done in item (1) of Example 2.10, we see that the intertwining (2.11) forces the set of conditions (2.8) Moreover, the condition (2.12) is exactly the missing ingredient needed to promote $(R, S, W)$ from a pseudo-commutative tetrablock unitary to a strict tetrablock unitary. This completes the proof.
Remark 2.14 It will be useful to have a terminology for an intermediate class of operator triples \((A, B, T)\) which sits between strict tetrablock isometries and general pseudo-commutative tetrablock isometries. Let us say that the triple \((A, B, T)\) is a **semi-strict tetrablock isometry** if \((A, B, T)\) is a pseudo-commutative isometry with Wold decomposition as in (2.10) is such that the pseudo-commutative tetrablock unitary component \((R, S, W)\) is actually a strict tetrablock unitary, i.e., \(R\) and \(S\) are contractions which commute with each other as well as with \(W\) which is unitary.

We next present an analogue of the single-variable operator theory fact that any isometry can always be extended to a unitary.

**Corollary 2.15** Pseudo-commutative/strict tetrablock isometries can be extended to pseudo-commutative/strict tetrablock unitaries. More precisely:

1. A triple \((A, B, T)\) is a pseudo-commutative tetrablock isometry if and only if it extends to a pseudo-commutative tetrablock unitary. Moreover there exists an extension that acts on the space of minimal unitary extension of the isometry \(T\).
2. A triple \((A, B, T)\) is a strict tetrablock isometry if and only if it extends to a strict tetrablock unitary acting on the space of the minimal unitary extension of the isometry \(T\).

**Proof** If a triple extends to a pseudo-commutative tetrablock unitary, then from Definition 2.6 we can read off that it is also a pseudo-commutative tetrablock isometry. Similarly, if a triple extends to a strict tetrablock unitary, we can read off from criterion (iii) or (iv) in Theorem 2.5 that the triple itself must be a strict tetrablock isometry.

We now address the converse. In view of Theorem 2.12, we can assume without loss of generality that a pseudo-commutative tetrablock isometry \((A, B, T)\) is given in the form:

\[
\begin{pmatrix}
M_{G_1^{*}+\zeta G_2} & 0 \\
0 & R
\end{pmatrix}
\begin{pmatrix}
M_{G_2^{*}+\zeta G_1} & 0 \\
0 & S
\end{pmatrix}
\begin{pmatrix}
M_{\zeta} & 0 \\
0 & W
\end{pmatrix}
\begin{pmatrix}
H^2(\mathcal{E}) \\
\mathcal{F}
\end{pmatrix}
\]

for some operators \(G_1, G_2\) on \(\mathcal{E}\) and for some pseudo-commutative tetrablock unitary \((R, S, W)\) acting on \(\mathcal{F}\). Consider \(H^2(\mathcal{E}) \oplus \mathcal{F}\) as a subspace of \(L^2(\mathcal{E}) \oplus \mathcal{F}\) in the natural way. Then the triple

\[
\begin{pmatrix}
M_{G_1^{*}+\zeta G_2} & 0 \\
0 & R
\end{pmatrix}
\begin{pmatrix}
M_{G_2^{*}+\zeta G_1} & 0 \\
0 & S
\end{pmatrix}
\begin{pmatrix}
M_{\zeta} & 0 \\
0 & W
\end{pmatrix}
\begin{pmatrix}
L^2(\mathcal{E}) \\
\mathcal{F}
\end{pmatrix}
\]

is an extension of \((A, B, T)\). The unitary \(M_\zeta \oplus W\) is clearly a minimal unitary extension of the isometry \(M_\zeta \oplus W\). And since \(M_{G_1^{*}+\zeta G_2} = M_{G_2^{*}+\zeta G_1} M_\zeta\) and \((R, S, W)\) is a pseudo-commutative tetrablock unitary, the above triple is a pseudo-commutative tetrablock unitary by Definition 2.6.

If we start with a strict tetrablock isometry, then we shall also have that

\[
\begin{pmatrix}
M_{G_1^{*}+\zeta G_2} & 0 \\
0 & R
\end{pmatrix}
\]

commutes with

\[
\begin{pmatrix}
M_{G_2^{*}+\zeta G_1} & 0 \\
0 & S
\end{pmatrix}
\]

on \(H^2(\mathcal{E}) \oplus \mathcal{F}\), or equivalently, \(RS = SR\) and the Toeplitz operator symbols equal to the pencils \(G_1^{*} + \zeta G_2\) and \(G_2^{*} + \zeta G_1\) commute:

\[(G_1^{*} + \zeta G_2)(G_2^{*} + \zeta G_1) = (G_2^{*} + \zeta G_1)(G_1^{*} + \zeta G_2).\]
But then it is straightforward to see that this implies the commutativity of the associated Laurent operators acting on $L^2(\mathcal{E})$:

$$M_{G_1^1+\xi G_2}M_{G_2^2+\xi G_1} = M_{G_2^2+\xi G_1}M_{G_1^1+\xi G_2}.$$  

and hence also

$$\begin{bmatrix} M_{G_1^1+\xi G_2} & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} M_{G_2^2+\xi G_1} & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} M_{G_2^2+\xi G_1} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} M_{G_1^1+\xi G_2} & 0 \\ 0 & R \end{bmatrix}.$$  

on $L^2(\mathcal{E})$. Moreover the extension of $M_{G_1^1+\xi G_2}$ on $H^2(\mathcal{E})$ to $M_{G_1^1+\xi G_2}$ on $L^2(\mathcal{E})$ is norm-preserving, and hence the latter is contractive whenever the former is contractive, and similarly for $M_{G_2^2+\xi G_1}$ and $M_{G_2^2+\xi G_1}$. We now have enough observations to conclude by criterion (iii) or (iv) in Theorem 2.5 that

$$\left( \begin{bmatrix} M_{G_1^1+\xi G_2} & 0 \\ 0 & R \end{bmatrix}, \begin{bmatrix} M_{G_2^2+\xi G_1} & 0 \\ 0 & S \end{bmatrix}, \begin{bmatrix} M_c & 0 \\ 0 & W \end{bmatrix} \right) \text{ on } L^2(\mathcal{E})$$

is a tetrablock unitary as required. 

\[\Box\]

Another one-variable fact is the result due to Sz.-Nagy–Foias (see [32, Theorem I.3.2]): any contraction operator $T$ on a Hilbert space $\mathcal{H}$ can be decomposed as $T = \begin{bmatrix} T_{cnu} & 0 \\ 0 & U \end{bmatrix}$ where $T_{cnu}$ is a completely nonunitary (c.n.u.) contraction operator (meaning there is no reducing subspace of $\mathcal{H}$ such that $T|_{\mathcal{H}}$ is unitary) and where $U$ is unitary. There is a analogous result for the setting of tetrablock unitaries and tetrablock contractions. We say that the tetrablock contraction $(A, B, T)$ on $\mathcal{H}$ is a c.n.u. tetrablock contraction if there is no nontrivial jointly reducing subspace $\mathcal{H}_u \subset \mathcal{H}$ for $(A, B, T)$ such that $(A, B, T)|_{\mathcal{H}_u}$ is a tetrablock unitary. The following result appears in [27]

**Theorem 2.16** Let $(A, B, T)$ be a tetrablock contraction on $\mathcal{H}$. Then $\mathcal{H}$ has an internal orthogonal direct-sum decomposition $\mathcal{H} = \mathcal{H}_{c.n.u.} \oplus \mathcal{H}_u$ with $\mathcal{H}_{c.n.u.}$ and $\mathcal{H}_u$ jointly reducing for $(A, B, T)|_{\mathcal{H}_{c.n.u.}}$ equal to a c.n.u. tetrablock contraction and $(A, B, T)|_{\mathcal{H}_u}$ equal to a tetrablock unitary.

The at first surprising fact is that the same decomposition $\mathcal{H} = \mathcal{H}_{c.n.u.} \oplus \mathcal{H}_u$ inducing the canonical decomposition of the contraction operator $T$ into is c.n.u. part $T_{c.n.u.}$ and its unitary part $T_u$ turns out to also be jointly reducing for the whole operator triple $(A, B, T)$ and induces the canonical tetrablock decomposition for the tetrablock contraction $(A, B, T)$, i.e., $(A, B, T)|_{\mathcal{H}_{c.n.u.}}$ is a c.n.u. tetrablock contraction, and $(A, B, T)|_{\mathcal{H}_u}$ is a tetrablock unitary.

**Remark 2.17** The import of Theorem 2.16 for model theory is the same as is the case for the classical case: since the model theory for a (strict) tetrablock unitary is already well understood (see item (2) in Remark 2.8, it follows that it is perfectly satisfactory to focus on the case where $(A, B, T)$ is a c.n.u. tetrablock contraction for the purposes of model theory.
2.3 A Canonical Construction of a Tetrablock Unitary from a Tetrablock Contraction

In this section we start with a tetrablock contraction \((A, B, T)\) and construct a tetrablock unitary which is uniquely associated with \((A, B, T)\) in a sense that will be made precise later in this subsection. This will be used later to construct a concrete functional model for a pseudo-commutative tetrablock-isometric lift for \((A, B, T)\) which can be viewed as a functional model for \((A, B, T)\) itself.

We start by using the fact that the last entry \(T\) in our tetrablock contraction \((A, B, T)\) is a contraction operator. Hence there exist a positive semidefinite operator \(QT^*\) such that

\[
Q^2_T := \text{SOT- lim } T^n T^{*n}.
\]  

(2.13)

Define the operator \(X^*_T : \text{Ran } QT^* \rightarrow \text{Ran } QT^*\) densely by

\[
X^*_T QT^* h = QT^* T^* h.
\]

(2.14)

This is an isometry because for all \(h \in \mathcal{H}\),

\[
\|X^*_T QT^* h\|^2 = \langle Q^2_T T^* h, T^* h \rangle = \lim_{n \to \infty} \langle T^n T^{*n} T^* h, T^* h \rangle
\]

\[
= \lim_{n \to \infty} \langle T^{n+1} T^{*(n+1)} h, h \rangle = \langle Q^2_T h, h \rangle = \|Q_T h\|^2.
\]

(2.15)

Since \(A\) is a contraction, we have for all \(h \in \mathcal{H}\)

\[(AQ^2_T, A^* h, h) = \lim_n \langle T^n AA^* T^{*n} h, h \rangle \leq \lim_n \langle T^n T^{*n} h, h \rangle = \langle Q^2_T h, h \rangle.\]

The same computation for the contraction \(B\) will yield the same inequality involving \(B\) in place of \(A\). Consequently, the operators \(AT^*, BT^* : \text{Ran } QT^* \rightarrow \text{Ran } QT^*\) defined densely by

\[
A^*_T QT^* h = QT^* A^* h \text{ and } B^*_T QT^* h = QT^* B^* h
\]

(2.16)

are contractions, and extend contractively to all of \(\text{Ran } QT^*\) by a limiting process. Furthermore, from the definitions it is easy to see that \((AT^*, BT^*, XT^*)\) is a commutative triple since by assumption we know that \((A^*, B^*, T^*)\) is commutative. This and (2.14) imply that if \(f\) is a three-variable polynomial, then for all \(h \in \mathcal{H}\),

\[
\|f(A^*_T, B^*_T, X^*_T) QT^* h\| = \|Q_T f(A^*, B^*, T^*) h\|
\]

\[
\leq \|f(A^*, B^*, T^*) ||h|| \leq \|f(A^*, B^*, T^*) ||h||
\]

\[
\leq (\sup |f|) ||h||
\]

where in the last inequality we used the fact that \((A, B, T)\) is a tetrablock contraction. This inequality together with the fact that \(X^*_T\) is an isometry implies that
(\(A_T^{**}, B_T^{**}, X_T^{**}\)) is a tetrablock isometry. By Corollary 2.15, \((A_T^{**}, B_T^{**}, X_T^{**})\) has a tetrablock unitary extension \((R_D^*, S_D^*, W_D^*)\) acting on a space which we shall call \(Q_T^* \supseteq \text{Ran} Q_T^*\), where \(W_D^*\) acting on \(Q_T^*\) is the minimal unitary extension of \(X_T^{**}\).

**Definition 2.18** Let \((A, B, T)\) be a tetrablock contraction and let \((R_D, S_D, W_D)\) be the tetrablock unitary constructed from \((A, B, T)\) as above. We say that \((R_D, S_D, W_D)\) is the **canonical tetrablock unitary** associated with the tetrablock contraction \((A, B, T)\).

The next result assures us that canonical tetrablock unitaries associated with the same tetrablock contraction \((A, B, T)\) are the same up to unitary equivalence.

**Theorem 2.19** Let \((A, B, T)\) on \(\mathcal{H}\) and \((A', B', T')\) on \(\mathcal{H}'\) be two tetrablock contractions with \((R_D, S_D, W_D)\) and \((R_D', S_D', W_D')\) equal to the respective canonical tetrablock unitaries. If \((A, B, T)\) and \((A', B', T')\) are unitarily equivalent via \(\tau\), then \((R_D, S_D, W_D)\) and \((R_D', S_D', W_D')\) are unitarily equivalent via \(\omega_\tau : Q_T^* \rightarrow Q_{T'}^*\).

\[
\omega_\tau : W_D^n Q_T^* h \mapsto W_D'^n Q_{T'^*} \tau h \tag{2.17}
\]

for all \(n \geq 0\) and \(h \in \mathcal{H}\).

**Proof** Let the spaces \(Q_T^*, Q_{T'^*}\) and the operators \(\{A_T^{**}, B_T^{**}, Q_T^*\}, \{A_{T'^*}, B_{T'^*}, Q_{T'^*}\}\) be obtained as above from \((A, B, T)\) and \((A', B', T')\), respectively. Since \(\tau\) is a unitary intertwining \(T\) and \(T'\), it intertwines \(T^*\) and \(T'^*\) and thus \(\tau Q_T^* = Q_{T'^*} \tau\). Therefore by definition (2.16) it follows that \(\tau (A_T^{**}, B_T^{**}, Q_T^*) = (A_{T'^*}, B_{T'^*}, Q_{T'^*}) \tau\). By definition of \(\omega_\tau\) it is clear that \(\omega_\tau W_D = W_D' \omega_\tau\). Therefore for every \(h \in \mathcal{H}\) and \(n \geq 0\),

\[
\omega_\tau R_D W_D^n Q_T^* h = \omega_\tau W_D^{n+1} W_D^* R_D Q_T^* h
\]

\[
= \omega_\tau W_D^{n+1} S_D^* Q_T^* h \quad \text{[using Theorem 2.5, part (iv)]}
\]

\[
= W_D^{n+1} \tau B_T^{**} Q_T^* h = W_D^{n+1} B_T^{**} Q_{T'^*} \tau h
\]

\[
= W_D^{n+1} S_D^{**} Q_{T'^*} \tau h
\]

\[
= S_D^{**} W_D^{n+1} Q_T^* \quad \text{[since \(S_D^{**} W_D = W_D' S_D^*\)]}
\]

\[
= R_D' W_D^n Q_{T'^*} \tau h \quad \text{[using Theorem 2.5, part (iii)]}
\]

\[
= R_D' \omega_\tau W_D^n Q_T^* h.
\]

A similar computation shows that \(\omega_\tau S_D = S_D' \omega_\tau\). \(\square\)

**2.4 The Fundamental Operators**

Much of the theory of tetrablock contractions is heavily based on a pair of operators that is uniquely associated with a tetrablock contraction. These are called the fundamental operators, the existence of which was proved in [12] with appeal to connections between tetrablock contractions and symmetrized-bidisk contractions. We state the result and sketch a more self-contained proof with the appeal to symmetrized-bidisk...
theory eliminated. In the sequel we shall use the notation \( \nu(X) \) to denote the numerical radius of the operator \( X \) on the Hilbert space \( \mathcal{H} \):

\[
\nu(X) := \sup_{x \in \mathcal{H}: \|x\| = 1} |\langle Xx, x \rangle_{\mathcal{H}}|.
\]

**Theorem 2.20** Let \( (A, B, T) \) be a tetrablock contraction on a Hilbert space \( \mathcal{H} \).

(i) (See [12, Theorem 3.4]) There exist two unique operators \( F_1 \) and \( F_2 \) acting on \( D_T \) with the numerical radii at most one such that

\[
A - B^*T = D_T F_1 D_T \quad \text{and} \quad B - A^*T = D_T F_2 D_T. \tag{2.18}
\]

Moreover, the operators \( F_1, F_2 \) are such that \( \nu(F_1 + zF_2) \leq 1 \) for all \( z \in \overline{D} \).

(ii) (See [12, Corollary 4.2]) The operators \( F_1, F_2 \) are alternatively characterized as the unique bounded operators on \( D_T \) such that \( (X_1, X_2) = (F_1, F_2) \) satisfies the system of operator equations

\[
D_T A = X_1 D_T + X_2^* D_T T \quad \text{and} \quad D_T B = X_2 D_T + X_1^* D_T T. \tag{2.19}
\]

**Proof** Let \( (A, B, T) \) be a tetrablock contraction on \( \mathcal{H} \). Since for every \( z \in \overline{D} \), \( \Psi(z, \cdot) \) as in item (ii) of Theorem 2.2 is analytic in an open set containing \( \overline{E} \), and \( \overline{E} \) is polynomially convex, a limiting argument implies that \( \Psi(\zeta, (A, B, T)) \) is a contraction for every \( \zeta \in \mathbb{T} \), or equivalently, on simplifying \( I - \Psi(\zeta, (A, B, T))^* \Psi(\zeta, (A, B, T)) \geq 0 \) we get

\[
(I - T^*T) + (B^*B - A^*A) - \zeta(B - A^*T) - \overline{\zeta}(B - A^*T)^* \geq 0.
\]

Similarly, applying item (iii) of Theorem 2.2 we have for every \( \alpha \in \mathbb{T} \),

\[
(I - T^*T) + (A^*A - B^*B) - \alpha(A - B^*T) - \overline{\alpha}(A - B^*T)^* \geq 0.
\]

On adding the above two positive operators and then simplifying we get

\[
D_T^2 \geq \Re \alpha \left[ (A - B^*T) + \beta(B - A^*T) \right] \tag{2.20}
\]

for every \( \alpha, \beta \in \mathbb{T} \). We now make use the following lemma of independent interest:

**Lemma 2.21** (See [13, Lemma 4.1]) Let \( \Sigma \) and \( D \) be two operators such that

\[
DD^* \geq \Re \alpha \Sigma \quad \text{for all} \quad \alpha \in \mathbb{T}.
\]

Then there exists an operator \( F \) acting on \( \overline{\text{Ran} \; D^*} \) with numerical radius at most one such that \( \Sigma = DFD^* \).
**Sketch of proof** Apply the Fejer-Riesz factorization theorem of Dritschel-Rovnyak [22, Theorem 2.1] to the Laurent operator-valued polynomial $P(e^{i\theta}) = 2DD^* - e^{i\theta} \Sigma - e^{-i\theta} \Sigma^*$. Along the way one makes use of the standard Douglas lemma ($\exists X \in B(H)$ with $AX = B \Leftrightarrow BB^* \preceq AA^*$) and a criterion for a Hilbert space operator to have numerical radius at most 1: $X \in B(H)$ has $\nu(X) \leq 1 \Leftrightarrow \Re(\beta X) \preceq I_H$ for all $\beta \in \mathbb{T}$. Note that Lemma 2.21 can itself be viewed as a quadratic, numerical-radius version of the Douglas lemma.

We apply Lemma 2.21 to the case (2.20) for each $\beta$ to get a numerical contraction $F(\beta)$ such that

$$(A - B^*T) + \beta(B - A^*T) = DT F(\beta)DT. \quad (2.21)$$

On adding equations (2.21) for the cases $\beta = 1$ and $-1$, we get

$$A - B^*T = DT F_1 DT \quad \text{where} \quad F_1 := \frac{F(1) + F(-1)}{2}. \quad (2.22)$$

Thus putting $\beta = 1$ in (2.21) and combining with (2.21) gives us

$$B - A^*T = DT(F(1) - F_1)DT = DT F_2 DT \quad \text{where} \quad F_2 := \frac{F(1) - F(-1)}{2}. \quad (2.23)$$

We conclude that $F_1$ and $F_2$ so constructed satisfy equations (2.18). It is easy to see that in general

$$X \in B(D_T) \text{ with } DTXD_T = 0 \Rightarrow X = 0. \quad (2.24)$$

Applying this observation to the homogeneous version of equations (2.18) implies that the solutions $(F_1, F_2)$ of (2.18) must be unique whenever they exist.

On the other hand, if we combine (2.22) with (2.23) we see that $\tilde{F}(\beta) := F_1 + \beta F_2$ gives us a second solution of (2.21). Hence

$$DT(\tilde{F}(\beta) - F(\beta))DT = 0 \text{ where } \tilde{F}(\beta) - F(\beta) \in B(D_T) \text{ for all } \beta \in \mathbb{T}.$$

Again by (2.24), we see that $\tilde{F}(\beta) = F(\beta)$ for all $\beta \in \mathbb{T}$. But we saw above (as a consequence of Lemma 2.21) that $F(\beta)$ is a numerical contraction for all $\beta \in \mathbb{T}$. As we now know that $\tilde{F}(\beta) = F(\beta)$, we conclude that the pencil $\tilde{F}(\beta) = F_1 + \beta F_2$ is a numerical contraction for all $\beta \in \mathbb{T}$. By applying the Maximum Modulus Theorem to the holomorphic function $\langle (F_1 + \beta F_2)h, h \rangle$ for each fixed $h \in H$, we see that $F_1 + zF_2$ is a numerical contraction for all $z \in \overline{D}$. This completes the proof of item (i) in Theorem 2.20.

To see that $F_1$ and $F_2$ satisfy equations (2.19), simply multiply $D_T$ on the left of each equation and use the identities (2.18) to simplify. To show that $F_1, F_2$ are the
unique operators on $D_T$ satisfying these two equations, it is enough to show that $X = 0$ and $Y = 0$ are the only operators in $\mathcal{B}(D_T)$ satisfying

$$XD_T + Y^* D_T T = 0, \quad YD_T + X^* D_T T = 0.$$ 

To show that $X = 0$, compute

$$D_T XD_T = -D_T Y^* D_T = T^* D_T XD_T = -T^* D_T Y^* D_T T^2 = T^*^2 D_T XD_T T^2.$$ 

Thus by iteration of the above process $D_T XD_T = T^*^n D_T XD_T T^n$. This shows that $X = 0$ because for every $h \in H$

$$\lim_n \|D_T T^n h\|^2 = \lim_n \|T^n h\|^2 - \lim_n \|T^{n+1} h\|^2 = 0.$$ 

A similar argument gives $Y = 0$. This is the idea of the proof due by Bhattacharyya [12].

The unique operators $F_1, F_2$ in item (i) of Theorem 2.20 will be referred to as the fundamental operators for the tetrablock contraction $(A, B, T)$, as in [12].

As we have seen in Proposition 2.1, if $(A, B, T)$ is a $\mathbb{E}$-contraction, so also is $(A^*, B^*, T^*)$. For the construction of the functional model for a tetrablock contraction $(A, B, T)$, it turns out to be more convenient to work with the fundamental operators for the adjoint tetrablock contraction $(A^*, B^*, T^*)$ which we denote as $(G_1, G_2)$. Thus there exists exactly one solution $(X_1, X_2) = (G_1, G_2)$ of the system of equations

$$A^* - BT^* = D_T X_1 D_T^*, \quad B^* - AT^* = D_T X_2 D_T^* \tag{2.25}$$

with equivalent characterization as the unique solution $(X_1, X_2) = (G_1, G_2)$ of the second system of equations

$$D_T A^* = X_1 D_T^* + X_2^* D_T T^*, \quad D_T B^* = X_2 D_T^* + X_1^* D_T T^* \tag{2.26}$$

Then from Example 2.10 we see immediately that the operator pair $(M_{G_1^* + zG_2}, M_{G_2^* + zG_1}, M_z)$ acting on the Hardy space $H^2(D_T^*)$ is of the correct form to be a pseudo-commutative tetrablock isometry. We would like to establish conditions under which this a priori only pseudo-commutative $\mathbb{E}$-isometry is actually a strict $\mathbb{E}$-isometry. This follows from the following result.

**Theorem 2.22** (See [12]) Suppose that $(G_1, G_2)$ is the Fundamental Operator pair for the $\mathbb{E}$-contraction $(A^*, B^*, T^*)$. Let $(V_1, V_2, V_3)$ be the operator triple

$$(V_1, V_2, V_3) = (M_{G_1^* + zG_2}, M_{G_2^* + zG_1}, M_z)$$

on $H^2(D_T^*)$. Then:
(1) \((V_1, V_2, V_3)\) is a pseudo-commutative tetrablock isometry having the additional property that
\[
r(V_1) \leq 1, \quad r(V_2) \leq 1.
\]

(2) Suppose in addition that the Fundamental Operator pair \((G_1, G_2)\) satisfy the commutativity conditions. Then \((V_1, V_2, V_3)\) is a strict tetrablock isometry.

**Corollary 2.23** Let \((G_1, G_2)\) be the Fundamental Operator pair for the tetrablock contraction \((A^*, B^*, T^*)\) and set \((V_1, V_2, V_3)\) as in (2.27). Then \((V_1, V_2, V_3)\) is a (strict) tetrablock isometry if and only if the commutativity conditions (2.8) hold.

**Proof of Corollary 2.23** If the commutativity conditions (2.8) are satisfied, then statement (2) of Theorem 2.22 says that \((V_1, V_2, V_3)\) is a tetrablock isometry. Conversely, if \((V_1, V_2, V_3)\) is a tetrablock isometry, in particular \((V_1, V_2, V_3)\) is commutative so the commutativity conditions (2.8) are satisfied. \(\square\)

**Proof of Theorem 2.22** We first consider statement (1). That \((V_1, V_2, V_3)\) is a pseudo-commutative tetrablock isometry follows from the fact that it has the required form (2.6) as presented in Example 2.10. It remains to use the fact that \((G_1, G_2)\) is a Fundamental Operator pair for the tetrablock contraction \((A^*, B^*, T^*)\) (to see that we also have \(r(V_1) \leq 1\) and \(r(V_2) \leq 1\).

By Theorem 2.20 (applied to \((A^*, B^*, T^*)\) in place of \((A, B, T)\)), we know that \(\nu(G_1 + zG_2) \leq 1\) for all \(z \in \mathbb{D}\). By the Andô criterion for the numerical radius of an operator to be no more than 1, this means that
\[
\beta(G_1 + \alpha G_2) + \overline{\beta}(G_1^* + \overline{\alpha}G_2^*) \leq 2I_{D^*_T} \forall \alpha, \beta \in \mathbb{D}.
\]
Rearrange this inequality as
\[
(\overline{\beta}G_1^* + \beta \alpha G_2) + (\beta G_1 + \overline{\beta} \overline{\alpha}G_2^*)
\]
\[
= \overline{\beta}(G_1^* + \beta^2 \alpha G_2) + \beta(G_1 + \overline{\beta}^2 \overline{\alpha}G_2^*) \leq 2I_{D^*_T}.
\]
By the Andô criterion applied in the reverse direction, this tells us that
\[
\nu(G_1^* + zG_2) \leq 1\ for all \ z \in \overline{\mathbb{D}}.
\]
But in general the numerical radius dominates the spectral radius; thus
\[
r(G_1^* + zG_2) \leq v(G_1^* + zG_2) \leq 1\ for all \ z \in \overline{\mathbb{D}}.
\]
If we choose \(\lambda \in \mathbb{C}\) with \(|\lambda| > 1\), then \(\lambda I_{D^*_T} - (G_2^* + zG_1)\) is invertible, and in fact the operator-valued function \(z \mapsto (\lambda I_{D^*_T} - (G_2^* + zG_1))^{-1}\) is in \(H^\infty(\mathcal{B}(D^*_T))\). We thus conclude that in fact
\[
r(M_{G_2^* + zG_1}) \leq 1.
\]
All the above analysis applies to the pair \((G_2, G_1)\) in place of \((G_1, G_2)\), as \((G_2, G_1)\) is the Fundamental Operator pair for the \(\mathcal{E}\)-contraction \((B^*, A^*, T^*)\); hence we also have also

\[
r(M_{G_2^*+zG_1}^*) \leq 1.
\]

This completes the proof of statement (1).

We now consider statement (2). As we are now assuming that \((G_1, G_2)\) satisfy the commutativity conditions (2.8), it follows that \(M_{G_1^*+zG_2}^*\) commutes with \(M_{G_2^*+zG_1}^*\). We are now in a position to apply statement (2) in Theorem 2.9 to conclude that the a priori only pseudo-commutative \(\mathcal{E}\)-contraction \((M_{G_1^*+zG_2}^*, M_{G_2^*+zG_1}^*, M_z^*)\) is in fact a strict \(\mathcal{E}\)-contraction. \(\square\)

3 Functional Models for Tetrablock Contractions

In this section we produce two functional models for tetrablock contractions, the first inspired by model theory of Douglas [19], and the second by the model theory of Sz.-Nagy and Foias [32]. We note that so far only a functional model is known for the special case when the last entry is a pure contraction; see [30, Theorem 4.2].

3.1 A Douglas-Type Functional Model

Let \(T\) be any contraction on \(\mathcal{H}\). Define the operators \(\mathcal{O}_{DT^*,T^*} : \mathcal{H} \to H^2(D_{T^*})\) as

\[
\mathcal{O}_{DT^*,T^*}(z)h = \sum_{n=0}^{\infty} z^n D_{T^*}T^{*n}h, \quad \text{for every } h \in \mathcal{H},
\]

and \(\Pi_D : \mathcal{H} \to \left[ H^2(D_{T^*}) \right.\] by

\[
\Pi_Dh = \begin{bmatrix} \mathcal{O}_{DT^*,T^*}(z)h \\ Q_{T^*}h \end{bmatrix} \quad \text{for all } h \in \mathcal{H},
\]

where \(Q_{T^*}\) is as in (2.13). Then the computation

\[
\|\Pi_Dh\|^2 = \|\mathcal{O}_{DT^*,T^*}(z)h\|_{H^2(D_{T^*})}^2 + \|Q_{T^*}h\|^2
\]

\[
= \sum_{n=0}^{\infty} \|D_{T^*}T^{*n}h\|^2 + \lim_{n \to \infty} \|T^{*n}h\|^2
\]

\[
= (\|h\|^2 - \|T^*h\|^2) + (\|T^*h\|^2 - \|T^{*2}h\|^2 + \cdots) + \lim_{n \to \infty} \|T^{*n}h\|^2
\]

\[
= \|h\|^2
\]
shows that $\Pi_D$ is an isometry. Note the following intertwining property of $O_{D_T^*, T^*}$:

$$O_{D_T^*, T^*}(z)T^* h = \sum_{n=0}^{\infty} z^n D_{T^*} T^{*n+1} h = M_z^n \sum_{n=0}^{\infty} z^n D_{T^*} T^{*n} h = M_z^n O_{D_T^*, T^*}(z) h. \quad (3.3)$$

This together with intertwining (2.14) of $Q_{T^*}$ implies

$$\Pi_D T^* = \begin{bmatrix} M_z & 0 & 0 \\ 0 & 0 & W_D \end{bmatrix} \Pi_D. \quad (3.4)$$

This shows that the the pair

$$V_D := \begin{bmatrix} M_z & 0 & 0 \\ 0 & 0 & W_D \end{bmatrix} : \begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix} \rightarrow \begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}$$

is an isometric lift of $T$. This construction is by Douglas, where he also showed that this lift is minimal (see [19]).

Now let $(A^*, B^*, T^*)$ be a tetrablock contraction acting on $\mathcal{H}$ and $G_1, G_2$ be the fundamental operators of $(A^*, B^*, T^*)$. Let $(R_D, S_D, W_D)$ acting on $Q_{T^*}$ be the canonical tetrablock unitary associated with $(A, B, T)$. Consider the operators

$$\begin{pmatrix} M_{G_1^*+zG_2} & 0 \\ 0 & R_D \end{pmatrix}, \begin{pmatrix} M_{G_2^*+zG_1} & 0 \\ 0 & S_D \end{pmatrix}, \begin{pmatrix} M_z & 0 \\ 0 & W_D \end{pmatrix}$$

on $\begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}$. \quad (3.5)

We claim that

$$\Pi_D(A^*, B^*, T^*) = \begin{pmatrix} M_{G_1^*+zG_2} & 0 \\ 0 & R_D \end{pmatrix}^*, \begin{pmatrix} M_{G_2^*+zG_1} & 0 \\ 0 & S_D \end{pmatrix}^*, \begin{pmatrix} M_z & 0 \\ 0 & W_D \end{pmatrix}^* \Pi_D, \quad (3.6)$$

where $\Pi_D : \mathcal{H} \rightarrow \begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}$ is the isometry as in (3.2).

Recalling the definition $\Pi_D = \begin{bmatrix} O_{D_T^*, T^*} \\ Q_{T^*} \end{bmatrix} : \mathcal{H} \rightarrow \begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}$, we see that the three-fold intertwining condition (3.6) splits into two three-fold intertwining conditions

$$O_{D_T^*, T^*}(A^*, B^*, T^*) = (M_{G_1^*+zG_2}^*, M_{G_2^*+zG_1}^*, M_z^*) O_{D_T^*, T^*}, \quad (3.7)$$

$$Q_{T^*}(A^*, B^*, T^*) = (R_D^*, S_D^*, W_D^*) Q_{T^*}. \quad (3.8)$$

The last equation in (3.7) combined with the last equation in (3.8) we have already seen as the condition that $\Pi_D$ is the isometric identification map implementing $\begin{pmatrix} M_z & 0 \\ 0 & W_D \end{pmatrix}$ as the Douglas minimal isometric lift of $T$ (see (3.4)). We shall next check only the
first equation in (3.7) and the first equation in (3.8) as the verification of the respective second equations is completely analogous. Thus it remains to check

\[ O_{D_{T^*}, T^*}^{-1} A^* = M_{G_1^* + zG_2} O_{D_{T^*}, T^*}, \]  
\[ Q_{T^*} A^* = R_D^* Q_{T^*}. \]  

(3.9)  

(3.10)

Note that (3.10) is part of the construction of the canonical tetrablock unitary associated with the original tetrablock contraction \((A, B, T)\) (see (2.16)). To check (3.9), let us rewrite the condition in function form:

\[ D_{T^*}(I - zT^*)^{-1} A^* = G_1 D_{T^*}(I - zT^*)^{-1} + G_2^* D_{T^*}(I - zT^*)^{-1} T^* \]

As \(A\) commutes with \(T\), we can rewrite this as

\[ D_{T^*} A^* (I - zT^*)^{-1} = (G_1 D_{T^*} + G_2^* D_{T^*} T^*) (I - zT^*)^{-1}. \]

We may now cancel off the resolvent term \((I - zT^*)^{-1}\) to get a pure operator equation

\[ D_{T^*} A^* = G_1 D_{T^*} + G_2^* D_{T^*} T^*. \]

(3.11)

Let us now recall that the operators \((G_1, G_2)\) on \(D_{T^*}\) were chosen to be the Fundamental Operators for the tetrablock contraction \((A^*, B^*, T^*)\). Thus by our earlier discussion of Fundamental Operators for tetrablock contractions (see Theorem 2.20), we know that \((X_1, X_2) = (G_1, G_2)\) satisfies the identities (2.26), the first of which gives the same condition on \((G_1, G_2)\) as (3.11). Thus this choice of \((G_1, G_2)\) indeed leads to a solution of (3.9), and the proof of (3.6) is complete. (Of course the second equation in (2.26) amounts to the verification of the second equation in (3.7).)

This then means that the operator triple with isometric embedding operator \(\Pi_D\)

\[(V_1, V_2, V_3) := \left( \Pi_D, \left[ \begin{array}{ccc} M_{G_1^* + zG_2} & 0 & 0 \\ 0 & R_D & 0 \\ 0 & 0 & S_D \end{array} \right], \left[ \begin{array}{cc} M_{G_2^* + zG_1} & 0 \\ 0 & 0 & W_D \end{array} \right] \right) \]

(3.12)

is a lift of the tetrablock contraction \((A, B, T)\). As \((R_D, S_D, W_D)\) is \(E\)-unitary as part of the canonical construction in Sect. 2.3, we see from the form of the top components of \((V_1, V_2, V_3)\) in (3.12) that \((V_1, V_2, V_3)\) is a semi-strict \(E\)-isometry and is a strict \(E\)-isometry exactly when the top-component triple \((M_{G_1^* + zG_2}, M_{G_2^* + zG_1}, M_z)\) is a strict \(E\)-isometry. By Theorem 2.22, this in turn happens exactly when the Fundamental Operator pair \((G_1, G_2)\) satisfies the commutativity conditions (2.8). In summary we have verified most of the following result. We note that item (2) recovers a result of Bhattacharyya-Sau [15] via functional-model methods rather than by block-matrix-construction methods.

**Theorem 3.1** (1) Let \((A, B, T)\) be a tetrablock contraction on \(\mathcal{H}\) and let \(V_3\) on \(\mathcal{K} \supset \mathcal{H}\) be the (essentially unique) minimal isometric lift for the contraction operator \(T\). Then there is a unique choice of operators \((V_1, V_2)\) on \(\mathcal{K}\) so that the triple \(V = (V_1, V_2, V_3)\) is a semi-strict tetrablock isometric lift for \((A, B, T)\).
(2) A necessary and sufficient condition that there be a strict tetrablock isometric lift $V = (V_1, V_2, V_3)$ for $(A, B, T)$ with the isometry $V_3$ equal to a minimal isometric lift for $T$ is that the fundamental operators $(G_1, G_2)$ for the adjoint tetrablock contraction $(A^*, B^*, T^*)$ satisfy the system of operator equations (2.8). In this case the operator pair $(V_1, V_2)$ on $K$ is uniquely determined once one fixes a choice (essentially unique) for a minimal isometric lift $V_3$ for $T$.

(3) The lift $(V_1, V_2, V_3)$ can be given in functional form in the coordinates of the Douglas model as follows. For $(A, B, T)$ equal to a tetrablock contraction on a Hilbert space $H$ let $(G_1, G_2)$ be the fundamental operators of $(A^*, B^*, T^*)$, let $(R_D, S_D, W_D)$ be the tetrablock unitary canonically associated with $(A, B, T)$ as in Definition 2.18, and let $\Pi_D = \begin{bmatrix} O_{D^{T+}, T^*} \\ Q_{T^*} \end{bmatrix}$ be the Douglas isometric embedding map from $H$ into $\begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}$. Then

$$\left( \Pi_D, \left( \begin{bmatrix} M_{G_1^* + G_2} & 0 \\ 0 & R_D \end{bmatrix}, \begin{bmatrix} M_{G_2^* + G_1^*} & 0 \\ 0 & S_D \end{bmatrix}, \begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix} \right) \right)$$

(3.13)

is a semi-strict (strict exactly when $(G_1, G_2)$ satisfies (2.8)) tetrablock isometric lift for $(A, B, T)$. In particular $(A, B, T)$ is jointly unitarily equivalent to

$$P_H \left( \begin{bmatrix} M_{G_1^* + G_2} & 0 \\ 0 & R_D \end{bmatrix}, \begin{bmatrix} M_{G_2^* + G_1^*} & 0 \\ 0 & S_D \end{bmatrix}, \begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix} \right) \big|_{H_D},$$

(3.14)

where $H_D$ is the functional model space given by

$$H_D := \text{Ran} \: \Pi_D \subset \begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}.$$  

(3.15)

and any other semi-strict $E$-isometric lift $(V_1', V_2', V_3')$ with $V_3' = \begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix}$ on $\begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}$ is equal to (3.13).

**Proof** The discussion preceding the statement of the theorem amounts to a proof of statement (3) in Theorem 3.1. Statements (1) and (2) apart from the uniqueness assertion amounts to a coordinate-free (abstract, model-free) interpretation of the results of statement (3). It remains only to discuss the uniqueness assertion in statements (1) and (2). This can also be formulated in terms of the model as follows: Given a tetrablock contraction $(A, B, T)$, let $\Pi_D = \begin{bmatrix} O_{D^{T+}, T^*} \\ Q_{T^*} \end{bmatrix}$ be the Douglas embedding map and let $\mathcal{K}_D = \begin{bmatrix} H^2(D_{T^*}) \\ Q_{T^*} \end{bmatrix}$ be the Douglas minimal isometric lift space for $T$ with $V_D = \begin{bmatrix} M_z & 0 \\ 0 & W_D \end{bmatrix}$ on $\mathcal{K}_D$ equal to the Douglas minimal isometric lift for $T$. Suppose that

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}$$
are two operators on $\mathcal{K}_D = \left[ H^{(D_T^*)}_{Q_T^*} \right]$ such that $(\tilde{A}, \tilde{B}, \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right])$ is a pseudo-commutative tetrablock isometric lift for $T$. Then necessarily
\[
\tilde{A} = \begin{bmatrix} M_{G_1^*+zG_2} & 0 \\ 0 & R_D \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} M_{G_2^*+zG_1} & 0 \\ 0 & S_D \end{bmatrix}.
\]

where the pair $(G_1, G_2)$ is equal to the pair of Fundamental Operators for the tetrablock contraction $(A^*, B^*, T^*)$, and where $(R_D, S_D, W_D)$ is the tetrablock unitary canonically associated with the tetrablock contraction $(A, B, T)$ as in Definition 2.18.

To prove this model-theoretic reformulation of the uniqueness problem, we proceed as follows. We are given first of all that the triple
\[
(\tilde{A}, \tilde{B}) = \left( \begin{array}{cc} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{array} \right), \quad \tilde{B} = \left( \begin{array}{cc} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{array} \right), \quad \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right] (3.16)
\]
is a pseudo-commutative tetrablock isometry. According to Definition 2.6 we have the following:

- (i) $\tilde{A} \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right] = \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right] \tilde{A}$ and $\tilde{B} \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right] = \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right] \tilde{B}$;
- (ii) $\tilde{B} = \tilde{A}^* \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right]$ and $\tilde{B} = \tilde{A}^* \left[ M_{\xi} \begin{array}{cc} 0 \\ 0 \end{array} W_D \right]$; and
- (iii) $\|\tilde{A}\| \leq 1$.

As in the proof of Theorem 2.12, conditions (i) and (ii) force $\tilde{A}$ and $\tilde{B}$ to have the block-diagonal form
\[
(\tilde{A}, \tilde{B}) = \left( \begin{array}{cc} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{array} \right), \quad \left( \begin{array}{cc} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{array} \right).
\]

for some pseudo-commutative tetrablock unitary $(\tilde{A}_{22}, \tilde{B}_{22}, W_D)$, and operators $\tilde{G}_1, \tilde{G}_2 \in \mathcal{B}(D_T^*)$ so that the linear pencils $\tilde{G}_1^* + z\tilde{G}_2$ and $\tilde{G}_1^* + z\tilde{G}_2$ are contraction-valued for all $z \in \mathbb{D}$. We now use the fact that the triple $(3.16)$ is a pseudo-commutative lift of $(A, B, T)$, i.e., the operators $\tilde{A}$, $\tilde{B}$ satisfy the conditions
\[
\left[ M_{\tilde{G}_1^*+z\tilde{G}_2} \begin{array}{cc} 0 \\ 0 \end{array} \tilde{A}_{22}^* \right] \left[ O_{D_T^*, T^*} Q_{T^*} \right] = \left[ O_{D_T^*, T^*} Q_{T^*} \right] A^*
\]
and
\[
\left[ M_{\tilde{G}_2^*+z\tilde{G}_1} \begin{array}{cc} 0 \\ 0 \end{array} \tilde{B}_{22}^* \right] \left[ O_{D_T^*, T^*} Q_{T^*} \right] = \left[ O_{D_T^*, T^*} Q_{T^*} \right] B^*.
\]

Equivalently,
\[
M_{\tilde{G}_1^*+z\tilde{G}_2} O_{D_T^*, T^*} = O_{D_T^*, T^*} A^*, \quad M_{\tilde{G}_2^*+z\tilde{G}_1} O_{D_T^*, T^*} = O_{D_T^*, T^*} A^* \quad (3.17)
\]
and
\[
(\tilde{A}_{22}^*, \tilde{B}_{22}^*, W_D^*) Q_{T^*} = Q_{T^*} (A^*, B^*, T^*). \quad (3.18)
\]
We first show that $(\tilde{A}_{22}, \tilde{B}_{22}) = (R_D, S_D)$. Note that since $\tilde{A}_{22}$ commutes with $W_D$ and $W_D$ is a unitary, $\tilde{A}_{22}$ commutes with $W_D^*$ as well, and so we use (3.18) to compute

$$\tilde{A}_{22}(W^n_D Q_T^*h) = W^n_D \tilde{A}_{22}^* Q_T^*h = W^n_D Q_T^* A^*h$$

$$= W^n_D R_D^* Q_T^*h = R_D^*(W^n_D Q_T^*h).$$

Since $\{W^n_D Q_T^*h : n \geq 0 \text{ and } h \in \mathcal{H}\}$ is dense in $\mathcal{Q}_{T^*}$, we have $\tilde{A}_{22} = R_D$. Similarly, $\tilde{B}_{22} = S_D$. Next we show that $(\tilde{G}_1, \tilde{G}_2)$ are the fundamental operators of $(A^*, B^*, T^*)$. To this end, we can use (3.17) and the power series expansion of $O_{D^*, T^*}(z) = \sum_{n \geq 0} D_{T^*} T^* n h$ to arrive at the equations

$$\tilde{G}_1 D_{T^*} + \tilde{G}_2^* D_{T^*} T^* = D_{T^*} A^* \quad \text{and} \quad \tilde{G}_2 D_{T^*} + \tilde{G}_1^* D_{T^*} T^* = D_{T^*} B^*.$$ 

By part (ii) of Theorem 2.20 applied to the tetrablock contraction $(A^*, B^*, T^*)$, $(\tilde{G}_1, \tilde{G}_2)$ must be equal to the Fundamental Operator pair for $(A^*, B^*, T^*)$. \hfill $\square$

### 3.2 A Sz.-Nagy–Foias Type Functional Model

Sz.-Nagy and Foias gave a function-space realization of $\mathcal{Q}_{T^*}$ and thereby produced a concrete functional model for a contraction $T$. In their analysis a crucial role is played by what they called the characteristic function associated with $T$:

$$\Theta_T(z) := -T + z \mathcal{O}_{D^*, T^*}|_{\mathcal{D}_T} : \mathcal{D}_T \mapsto \mathcal{D}_T^*.$$ \hfill (3.19)

The name suggests the fact the characteristic function $\Theta_T$ enables one to write down an explicit functional model on which there is a compressed multiplication operator $T$ which recovers the original operator $T$ up to unitary equivalence in case $T$ is a c.n.u. contraction (see Chapter VI of [32]). Let $\Theta_T(\xi)$ be the radial limit of the characteristic function defined almost everywhere on $T$. Consider

$$\Delta_T(\xi) := (I - \Theta_T(\xi)^* \Theta_T(\xi))^{1/2}.$$ \hfill (3.20)

Sz.-Nagy and Foias showed in [32] that

$$V_{NF} := \begin{bmatrix} M_z & 0 \\ 0 & M_\xi \end{bmatrix} \begin{bmatrix} H^2(\mathcal{D}_T^*) \\ \Delta_T L^2(\mathcal{D}_T) \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{D}_T^*) \\ \Delta_T L^2(\mathcal{D}_T) \end{bmatrix} \rightarrow \begin{bmatrix} H^2(\mathcal{D}_T^*) \\ \Delta_T L^2(\mathcal{D}_T) \end{bmatrix}$$

is a minimal isometric lift of $T$ via some isometric embedding

$$\Pi_{NF} : \mathcal{H} \rightarrow \begin{bmatrix} H^2(\mathcal{D}_T^*) \\ \Delta_T L^2(\mathcal{D}_T) \end{bmatrix} =: K_{NF}$$
such that
\[
\mathcal{H}_{\text{NF}} := \text{Ran} \Pi_{\text{NF}} = \left[ \begin{array}{c} H^2(D_{T^*}) \\ \Delta T L^2(D_T) \end{array} \right] \O \left[ \begin{array}{c} \Theta_T \\ \Delta T \end{array} \right] \cdot H^2(D_T). \tag{3.21}
\]

Any two minimal isometric lifts of a given contraction $T$ are unitarily equivalent; see Chapter I of [32]. In [10] an explicit unitary $u_{\text{min}} : Q_{T^*} \to \Delta T L^2(D_T)$ is found that intertwines $W_D$ and $M_\zeta|_{\Delta T L^2(D_T)}$ and
\[
\Pi_{\text{NF}} = \left[ \begin{array}{c} I_{H^2(D_{T^*})} \\ 0 \\ u_{\text{min}} \end{array} \right] \Pi_D. \tag{3.22}
\]

It is possible to give a concrete Sz.-Nagy–Foias type functional model using the transition map $u_{\text{min}} : Q_{T^*} \to \Delta T L^2(D_T)$ as appeared (see (3.22)) in the case of a single contractive operator above. We must replace the canonical tetrablock unitary $(R_D, S_D, W_D)$ by its avatar on the function space $\Delta T L^2(D_T)$:
\[
(R_{\text{NF}}, S_{\text{NF}}, W_{\text{NF}}) = u_{\text{min}}^*(R_D, S_D, W_D)u_{\text{min}}. \tag{3.23}
\]

Then the following functional model is a straightforward consequence of Theorem 3.1 and (3.22).

**Theorem 3.2** Let $(A, B, T)$ be a tetrablock contraction on a Hilbert space $\mathcal{H}$ such that $T$ is c.n.u., and $(G_1, G_2)$ be the fundamental operators of $(A^*, B^*, T^*)$. Then $(A, B, T)$ is jointly unitarily equivalent to
\[
P_{\mathcal{H}_{\text{NF}}} \left( \left[ \begin{array}{c} M_{G_1^* + zG_2} \\ 0 \\ R_{\text{NF}} \end{array} \right], \left[ \begin{array}{c} M_{G_2^* + zG_1} \\ 0 \\ S_{\text{NF}} \end{array} \right], \left[ \begin{array}{c} M_\zeta \\ 0 \\ W_{\text{NF}} \end{array} \right] \right) \mid_{\mathcal{H}_{\text{NF}}}, \tag{3.24}
\]
where $\mathcal{H}_{\text{NF}}$ is the functional model space given by (see (3.21))
\[
\mathcal{H}_{\text{NF}} = \text{Ran} \Pi_{\text{NF}} = \left[ \begin{array}{c} H^2(D_{T^*}) \\ \Delta T L^2(D_T) \end{array} \right] \O \left[ \begin{array}{c} \Theta_T \\ \Delta T \end{array} \right] \cdot H^2(D_T).
\]

Note that in the special case when $T^{*n} \to 0$ strongly as $n \to \infty$, the space $Q_{T^*} = 0$ and hence also $\Delta T L^2(D_T) = 0$, i.e., $\Theta_T$ is inner. Therefore in this special case the models above simply boil down to the following which was obtained in [30].

**Theorem 3.3** (See [30, Theorem 4.2]) Let $(A, B, T)$ be a pure tetrablock contraction on a Hilbert space $\mathcal{H}$ and $(G_1, G_2)$ be the fundamental operators of $(A^*, B^*, T^*)$. Then $(A, B, T)$ is jointly unitarily equivalent to
\[
P_{\text{Ran} \mathcal{O}_{D_{T^*}, T^*}}(M_{G_1^* + zG_2}, M_{G_2^* + zG_1}, M_\zeta) \mid_{\text{Ran} \mathcal{O}_{D_{T^*}, T^*}}.
\]
4 Tetrablock Data Sets: Characteristic and Special

In this section we provide some preliminary results towards a Sz.-Nagy–Foias-type model theory for tetrablock-contraction operator tuples \((A, B, T)\). Note that if \(T\) is unitary, then the characteristic function \(\Theta_T\), as in (3.19), is trivial (i.e., equal to the zero operator between the zero spaces). As the most general contraction operator \(T\) is the direct sum of a unitary \(T_u\) with a completely nonunitary (c.n.u.) part \(T_{\text{c.n.u}}\) and the model theory for unitary operators is easily handled by spectral theory, it is natural for model theory purposes to restrict to the case where \(T\) is c.n.u. One then associates a functional model spaces

\[
\mathcal{K}_T = \left[ \frac{H^2(D_T^*)}{\Delta_T L^2(D_T)} \right], \quad \mathcal{H}_T = \mathcal{K}_T \ominus \left[ \frac{\Theta_T}{\Delta_T} \right] H^2(D_T) \subset \mathcal{K}_T
\]

together with functional-model operators \(T_T\) and \(V_T\) by

\[
T_T = P_{\mathcal{H}_T} \begin{bmatrix} M_z & 0 \\ 0 & M_\xi \end{bmatrix} \quad \text{on } \mathcal{H}_T, \quad V_T = \begin{bmatrix} M_z & 0 \\ 0 & M_\xi \end{bmatrix} \quad \text{on } \mathcal{K}_T.
\]

Then it is immediate that:

(NF1) \(V_T\) is an isometry on \(\mathcal{K}_T\).

(NF2) \(\mathcal{H}_T^+ = \left[ \frac{\Theta_T}{\Delta_T} \right] H^2(D_T)\) is invariant for \(V_T\), and hence \(V_T\) is an isometric lift for \(T_T\).

Less immediately obvious are other features of the model:

(NF3) (See [32, Theorem VI.2.3]) If \(T\) is c.n.u., then \(T_T\) is unitarily equivalent to \(T\).

(NF4) (See [32, Theorem VI.3.4]) If \(T, T'\) are two c.n.u. contraction operators, then \(T\) is unitarily equivalent to \(T'\) if and only if \(\Theta_T\) coincides with \(\Theta_{T'}\) in the following sense: there exist unitary change-of-coordinate maps \(\phi: D_T \rightarrow D_{T'}\) and \(\phi_*: D_T^* \rightarrow D_{T'}^*\) so that \(\phi_* \Theta_T(\lambda) = \Theta_{T'}(\lambda) \phi\) for all \(\lambda \in \mathbb{D}\). Often in literature this property is described simply as: the characteristic function \(\Theta_T\) is a complete unitary invariant for c.n.u. contraction operator \(T\).

The Sz.-Nagy–Foias theory goes still further by identifying the coincidente-envelop of the characteristic functions (3.19), i.e., the set of all contractive operator functions \((D, D_\ast, \Theta)\) coinciding with the characteristic function \(\Theta_T\) for some c.n.u. contraction operator \(T\), as simply any contractive operator function \((D, D_\ast, \Theta)\) which is pure in the sense that

\[
\|\Theta(0)u\| < \|u\| \quad \text{for any } u \in D \text{ such that } u \neq 0.
\]

Then we can start with any pure COF \((D, D_\ast, \Theta)\), form \(\mathcal{K}(\Theta)\) and \(\mathcal{H}(\Theta)\) according to

\[
\mathcal{K}(\Theta) = \left[ \frac{H^2(D_\ast)}{D_\Theta \cdot L^2(D)} \right], \quad \mathcal{H}(\Theta) = \mathcal{K}(\Theta) \ominus \left[ \frac{\Theta}{D_\Theta} \right] H^2(D) \subset \mathcal{K}(\Theta)
\]
where $D_{\Theta}$ is the $\Theta$-defect operator function $D_{\Theta}(\zeta) = (I_D - \Theta(\zeta)^*\Theta(\zeta))^{\frac{1}{2}}$. Then we can form the model operators

$$V(\Theta) = \begin{bmatrix} M_\zeta & 0 \\ 0 & M_\zeta \end{bmatrix} \text{ on } K(\Theta), \quad T(\Theta) = \left. P_{\mathcal{H}(\Theta)} V(\Theta) \right|_{\mathcal{H}(\Theta)}.$$

Then we have the additional results:

(NF5) (See [32, Theorem VI.3.1].) Given any pure COF $\Theta$, $T(\Theta)$ is a c.n.u. contraction operator on $\mathcal{H}(\Theta)$ with characteristic operator function $\Theta_{T(\Theta)}$ coinciding with $\Theta$.

In this way the loop is closed: the study of c.n.u. contraction operators is the same as the study of pure COFs.

To explain generalizations to the setting of tetrablock contractions, we first introduce some useful terminology. For this discussion, just as in the classical Sz.-Nagy–Foias settings, it makes sense to restrict to c.n.u. tetrablock contractions (see Theorem 2.16 and Remark 2.17).

**Definition 4.1** Let us say that any collection of objects

$$\Xi = (\Theta, (G_1, G_2), \psi)$$

consisting of

(i) a pure COF function $(D, D_*, \Theta)$,
(ii) a pair of operators $(G_1, G_2)$ on the coefficient space $D_*$, and
(iii) a measurable function $\psi$ on $\mathbb{T}$ such that, for a.e. $\zeta \in \mathbb{T}$, $\psi(\zeta)$ is a contractive normal operator on $D_{\Theta}(\zeta) = \text{Ran } D_{\Theta}(\zeta)$.

is a tetrablock data set.

**Remark 4.2** Canonically associated with any such $\psi$ as in item (iii) in Definition 4.1 is the tetrablock unitary triple $(R, S, W)$ on the direct integral space $\bigoplus \int_{\mathbb{T}} D_{\Theta}(\zeta) \frac{|d\zeta|}{2\pi}$ given by

$$R = M_{\psi^*\zeta}, \quad S = M_\psi, \quad W = M_\zeta$$

and (as one sweeps over all possible such $\psi$), this is the general tetrablock unitary operator triple $(R, S, W)$ on the space $\bigoplus \int_{\mathbb{T}} D_{\Theta}(\zeta) \frac{|d\zeta|}{2\pi}$ with the unitary operator $W$ equal to $W = M_\zeta$ (multiplication by the coordinate function) (see Example 2.10 (2)). Thus item (iii) in the definition of tetrablock data set can be equivalently rephrased as:

(iii’) a tetrablock unitary operator-triple $(R, S, W)$ on the direct-integral space $\int_{\mathbb{T}} D_{\Theta}(\zeta) \frac{|d\zeta|}{2\pi}$ such that the last unitary component $W$ is equal to multiplication by the coordinate function $W = M_\zeta$.

However, for convenience of notation, we shall continue to use the notation $(R, S, W)$ for the third component of a tetrablock data set $\Xi = (\Theta, (G_1, G_2), (R, S, W))$ with the convention (iii’) also part of the definition.
Definition 4.3 Given a c.n.u. tetrablock contraction \((A, B, T)\) we say that \(\Xi_{A,B,T} = (\Theta, (G_1, G_2), \psi)\) is the \textbf{characteristic tetrablock data set} for \((A, B, T)\) if

(i) \((D, D_s, \Theta)\) is equal to the Sz.-Nagy–Foias characteristic function \((D_T, D_T^*, \Theta_T)\) for the c.n.u. contraction operator \(T\),

(ii) \((G_1, G_2)\) is equal to the Fundamental Operator pair for the adjoint tetrablock contraction \((A^*, B^*, T^*)\), and

(iii) \((R, S, D)\) is given by

\[
((R, S, W)) = (R_{\text{NF}}, S_{\text{NF}}, W_{\text{NF}}) := u_{\min}^*(R_D, S_D, W_D)u_{\min}
\]

where \((R_D, S_D, W_D)\) is the tetrablock unitary on \(Q_T^\ast\) determined by tetrablock contraction \((A, B, T)\) according to Definition 2.18, and where \(u_{\min}: \Delta_TL^2(D_T) \to Q_T^\ast\) is the unitary identification map identifying the Sz.-Nagy–Foias lifting residual space \(\Delta_TL^2(D_T)\) with the Douglas lifting residual space \(Q_T^\ast\).

Then it is clear that the characteristic tetrablock data set \(\Xi_{A,B,T}\) for a c.n.u. tetrablock contraction \((A, B, T)\) is a tetrablock data set. The natural notion of equivalence for tetrablock-data sets is the following.

Definition 4.4 Let \((D, D_s, \Theta), (D', D'_s, \Theta')\) be two purely contractive analytic functions. Let \(G_1, G_2 \in B(D_s), G'_1, G'_2 \in B(D'_s)\), and \((R, S, W)\) on \(\Delta_\Theta L^2(D)\) and \((R', S', W')\) on \(\Delta_{\Theta'} L^2(D')\) be two tetrablock unitaries (with \(W\) and \(W'\) equal to \(M_\zeta\) on their respective spaces). We say that the two triples \((\Theta, (G_1, G_2), (R, S, W))\) and \((\Theta', (G'_1, G'_2), (R', S', W'))\) \textbf{coincide} if:

(i) \((D, D_s, \Theta)\) and \((D', D'_s, \Theta')\) coincide,

(ii) the unitary operators \(\phi, \phi_*\) involved in the coincidence of \((D, D_s, \Theta)\) and \((D', D'_s, \Theta')\) satisfy the additional intertwining conditions:

\[
\phi_*(G_1, G_2) = (G'_1, G'_2)\phi_* \quad \text{and} \quad \omega_\phi(R, S, W) = (R', S', W')\omega_\phi,
\]

where \(\omega_\phi: \Delta_\Theta L^2(D) \to \Delta_{\Theta'} L^2(D')\) is the unitary map induced by \(\phi\) according to the formula

\[
\omega_\phi := (I_{L^2} \otimes \phi)|_{\Delta_\Theta L^2(D)}.
\]

Given a characteristic tetrablock data set \(\Xi_{(A,B,T)} = (\Theta, (G_1, G_2), (R, S, W))\) for a tetrablock contraction \((A, B, T)\) we can write down a functional model:

\[
\mathcal{K}(\Xi) = \left[ \frac{H^2(D_s)}{\Delta_\Theta L^2(D)} \right], \quad \mathcal{H}(\Xi) = \mathcal{K} \ominus \left[ \frac{\Theta}{\Delta_\Theta} \right] H^2(D)
\]

with functional-model operators

\[
\mathbf{V}(\Xi) = \begin{bmatrix} M_{G_1} + G_2 & 0 \\ 0 & M_{G_2} + zG_1 \\ R & 0 \end{bmatrix}, \quad \mathbf{T}(\Xi) = \mathcal{P}_{\mathcal{H}(\Xi)}\mathbf{V}(\Xi)|_{\mathcal{H}(\Xi)}.
\]
The tetrablock analogue of items (NF1)-(NF2) in our discussion of the Sz.-Nagy–Foias model above fails without the additional assumptions: namely, it is not the case that \( V \) is a tetrablock isometry as well as that \( V \) is a lift for \( T \) unless we also impose the condition (2.8) on \((G_1, G_2)\). Nevertheless, the analogue of (NF3) does hold: given that \( \Xi \) is the characteristic tetrablock triple for \((A, B, T)\), it is the case that \( (A, B, T) \) is unitarily equivalent to \( T(\Xi) \): this is the content of the last part of item (3) in Theorem 3.1 (after a conversion to Sz.-Nagy–Foias rather than Douglas coordinates): see (3.14) and (3.15). The next theorem amounts to the analogue of (NF4) in our list of features for the Sz.-Nagy–Foias model above.

**Theorem 4.5** Let \((A, B, T)\) and \((A', B', T')\) be two tetrablock contractions acting on \( \mathcal{H} \) and \( \mathcal{H}' \), respectively. Let

\[
(\Theta_T, (G_1, G_2), (R_{NF}, S_{NF}, W_{NF})), \quad (\Theta_{T'}, (G'_1, G'_2), (R'_{NF}, S'_{NF}, W'_{NF})),
\]

be the characteristic tetrablock data sets for \((A, B, T)\) and \((A', B', T')\), respectively.

1. If \((A, B, T)\) and \((A', B', T')\) are unitarily equivalent, then their characteristic tetrablock data sets coincide.
2. Conversely, if \( T \) and \( T' \) are c.n.u. contractions and the characteristic tetrablock data sets of \((A, B, T)\) and \((A', B', T')\) coincide, then \((A, B, T)\) and \((A', B', T')\) are unitarily equivalent.

**Proof** First suppose that \((A, B, T)\) and \((A', B', T')\) are unitarily equivalent via a unitary \( \tau : \mathcal{H} \to \mathcal{H}' \). The fact that \( \Theta_T \) and \( \Theta_{T'} \) coincide is a part of the Sz.-Nagy–Foias theory [32]. Indeed, note that

\[
\tau(I - T^* T) = (I - T'^* T') \tau \quad \text{and} \quad \tau(I - TT^*) = (I - T'T'^*) \tau
\]

and therefore by the functional calculus for positive operators,

\[
\tau D_T = D_{T'} \tau \quad \text{and} \quad \tau D_{T^*} = D_{T'^*} \tau \quad (4.2)
\]

and thereby inducing two unitary operators

\[
\phi := \tau|_{D_T} : D_T \to D_{T'} \quad \text{and} \quad \phi_* := \tau|_{D_{T^*}} : D_{T^*} \to D_{T'^*}. \quad (4.3)
\]

Consequently \( \phi_* \Theta_T = \Theta_{T'} \phi \). Next, since the fundamental operators are the unique operators that satisfy the equations for \((X_1, X_2) = (G_1, G_2)\):

\[
A^* - BT^* = D_{T^*} X_1 D_T \quad \text{and} \quad B^* - AT^* = D_{T^*} X_2 D_T,
\]

one can easily obtain using (4.2) that

\[
\phi_* (G_1, G_2) = (G'_1, G'_2) \phi_*. \quad (4.4)
\]
Finally the proof of the forward direction will be complete if we establish that

\[(R_{NF}, S_{NF}, W_{NF}) = \omega^*_\phi(R'_{NF}, S'_{NF}, W'_{NF}) \omega_{\phi} \] (4.5)

where \(\omega_{\phi} = (I_L \otimes \phi)|_{\Delta T L^2(D_T)} : \Delta T L^2(D_T) \rightarrow \Delta T' L^2(D_{T'}).\) For this we first note that

\[
\begin{bmatrix}
I & 0 \\
0 & u_{\min}
\end{bmatrix}
\begin{bmatrix}
O_{D_{T^*}, T^*} \\
Q_T^*
\end{bmatrix} = \Pi_{NF} =
\begin{bmatrix}
I_{H^2} \otimes \phi^* & 0 \\
0 & \omega^*_{\phi}
\end{bmatrix} \Pi_{NF} \tau
\]

\[
= \begin{bmatrix}
I_{H^2} \otimes \phi^* & 0 \\
0 & \omega^*_{\phi'}
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & u_{\min}'
\end{bmatrix}
\begin{bmatrix}
O_{D_{T'^*}, T'^*} \\
Q_{T'^*}
\end{bmatrix} \tau,
\]

from which we read off that

\[u_{\min} Q_{T^*} = \omega^*_\phi u_{\min}' Q_{T'^*} \tau.\] (4.6)

Now since \(Q_{T'^*} = \text{span}(W_{D}^n Q_{T'^*} \tau h : h \in \mathcal{H}, n \geq 0)\) and \(u_{\min}\) has the intertwining property \(u_{\min} W_{D} = M_{\xi} u_{\min},\) we use (4.6) to compute

\[
\omega_{\phi} \cdot u_{\min} \cdot \omega^*_\tau (W_{D}^n Q_{T'^*} \tau h) = \omega_{\phi} \cdot u_{\min} (W_{D}^n Q_{T^*} h) = \omega_{\phi} M_{\xi} u_{\min} Q_{T^*} h = M_{\xi}^n \omega_{\phi} \cdot u_{\min} Q_{T'^*} h
\]

\[= M_{\xi}^n u_{\min}' Q_{T'^*} \tau h = u_{\min}' (W_{D}^n Q_{T'^*} \tau h).\]

Consequently

\[\omega_{\phi} \cdot u_{\min} \cdot \omega^*_\tau = u_{\min}'.\] (4.7)

Using this identity and the intertwining properties of the unitaries involved, it is now easy to establish (4.5).

Conversely, suppose \(T\) and \(T'\) are c.n.u. contractions, \(\phi : D_T \rightarrow D_{T^*}\) and \(\phi^* : D_{T^*} \rightarrow D_{T'^*}\) be the unitary operators involved in the coincidence of the characteristic tetrablock data sets

\[((G_1, G_2), (R_{NF}, S_{NF}, W_{NF}), \Theta_T), ((G'_1, G'_2), (R'_{NF}, S'_{NF}, W'_{NF}), \Theta_{T'}).\]

By Definition 4.4, it follows that the unitary

\[U = \begin{bmatrix}
I_{H^2} \otimes \phi^* & 0 \\
0 & \omega_{\phi'}
\end{bmatrix} : \begin{bmatrix}
H^2(D_{T^*}) \\
\Delta T L^2(D_T)
\end{bmatrix} \rightarrow \begin{bmatrix}
H^2(D_{T'^*}) \\
\Delta T' L^2(D_{T'})
\end{bmatrix}\]

identifies the model spaces \(\mathcal{H}_{NF}\) and \(\mathcal{H}_{NF}'\) and intertwines the model operators as in (3.24) associated with \((A, B, T)\) and \((A', B', T')\), respectively. This completes the proof of Theorem 4.5. \(\square\)
The question remains as to what additional coupling conditions must be imposed on a tetrablock data set $\Xi$ to assure that $\Xi$ coincides with the characteristic tetrablock data set for a c.n.u. tetrablock contraction $(A, B, T)$. In the Sz.-Nagy–Foias theory, the data set (or invariant) consists of a single COF, and the only additional requirement is that it must be pure.

From the results of Sect. 2.4 we see that any characteristic tetrablock data set $\Xi$ for a c.n.u. tetrablock contraction $(A, B, T) = ((\Theta, (G_1, G_2), (R, S, W))$

for a c.n.u. tetrablock contraction operator-triple $(A, B, T)$ satisfies the additional conditions (expressed directly in terms of the components of $\Xi$ rather than in terms of $(A, B, T)$):

(i) $\Theta$ is a pure COF (see [32, Theorem VI.3.1]),
(ii) the numerical radius conditions

$$v(G_1^* + zG_2) \leq 1, \quad v(G_2^* + zG_1) \leq 1 \text{ for all } z \in \mathbb{D}$$

hold, implying that also the spectral radius conditions

$$r(M_{G_1^*+zG_2}) \leq 1, \quad r(M_{G_2^*+zG_1}) \leq 1$$

(see Theorems 2.22 (1)).

(iii) It is almost the case that the spectral radius and norm agree for $M_{G_1^*+zG_2}$ and $M_{G_2^*+zG_1}$ in the following sense (see Theorem 2.9 (1)):

$$r(M_{G_1^*+zG_2} \cdot M_{G_2^*+zG_1}) = \max\{\|M_{G_1^*+zG_2}\|, \|M_{G_2^*+zG_1}\|^2\}.$$ 

However we do not expect that just imposing these conditions is sufficient to guarantee that such a tetrablock data set $\Xi$ will coincide with the characteristic tetrablock data set for some tetrablock contraction, so we do not expect to have an analogue of (NF5) at this level of generality.

Let us now specialize our class of tetrablock contractions to what we call special tetrablock contractions, i.e., any tetrablock contraction $(A, B, T)$ with the special property that the Fundamental Operator pair $(G_1, G_2)$ for $(A^*, B^*, T^*)$ satisfies the additional pair of operator equations (2.8). By Theorem 3.1, this is equivalent to $(A, B, T)$ having a minimal tetrablock isometric lift $(V_1, V_2, V_3)$ acting on a minimal Sz.-Nagy isometric-lift space for $T$ with $V_3$ equal to a minimal isometric lift for the single contraction operator $T$.

This suggests that we define a special tetrablock data set as follows. For convenience in later discussion we shall now write the last component $(R, S, W)$ simply as $\psi$ for a measurable contractive-normal operator-valued function $\zeta \mapsto \psi(\zeta) \in \mathcal{B}(\mathcal{D}_{\Theta(\zeta)})$ according to the convention explained in Remark 4.2.

**Definition 4.6** We say that the tetrablock data set

$$\Xi = ((\mathcal{D}, \mathcal{D}_*, \Theta), (G_1, G_2), \psi)$$

(4.8)
is a **special tetrablock data set** if the following conditions hold:

(i) The operators $G_1, G_2 \in \mathcal{B}(\mathcal{D})$ satisfy the commutativity conditions (2.8), i.e.,

$$[G_1, G_2] = 0, \quad [G_1^*, G_1] = [G_2^*, G_2]$$

as well as the pencil-contractivity condition

$$\|G_1^* + zG_2\| \leq 1 \text{ for all } \zeta \in \mathbb{D}$$

and then also

$$\|G_2^* + zG_1\| \leq 1 \text{ for all } \zeta \in \mathbb{D}.$$ 

(ii) the space $\left\{ \left[ \begin{array}{cc} G_1 & f \\
G_2 & \end{array} \right] : f \in H^2(\mathcal{D}) \right\}$ is jointly invariant under the operator triple

$$\left( \begin{array}{ccc}
M_{G_1} & 0 & 0 \\
0 & M_{\psi} & 0 \\
0 & 0 & M_\zeta
\end{array} \right). \quad (4.9)$$

Given a special tetrablock data set $(\Theta, (G_1, G_2), \psi)$, we say that the space

$$\mathcal{H} = \left[ \begin{array}{c} H^2(\mathcal{D}) \\
D_\Theta L^2(\mathcal{D}) \end{array} \right] \Theta \left[ \begin{array}{c} H^2(\mathcal{D}) \\
D_\Theta \end{array} \right] \quad (4.10)$$

is the **functional model space** and the (commutative) operator triple $(A, B, T)$ given by

$$P_{\mathcal{H}}(\left[ \begin{array}{ccc}
M_{G_1} & 0 & 0 \\
0 & M_{\psi} & 0 \\
0 & 0 & M_\zeta
\end{array} \right]) \quad (4.11)$$

the **functional-model operator triple** associated with the data set. The following theorem is our one analogue of item (NF5) in our list of features of the Sz.-Nagy–Foias model.

**Theorem 4.7** If $\Xi = (\Theta, (G_1, G_2), \psi)$ is a special tetrablock data set, then the associated model operator triple $(A, B, T)$ as in (4.11), is a tetrablock contraction that lifts to the tetrablock isometry as in (4.9). Moreover, the tetrablock data set $\Xi$ coincides with the characteristic triple of $(A, B, T)$.

**Proof** By part (i) of Definition 4.6, the triple as in (4.9) is a strict tetrablock isometry, and by part (ii), it lifts the model triple $(A, B, T)$ as in (4.11). Thus in particular, $(A, B, T)$ is a tetrablock contraction and the first part of the theorem follows. For the second part, we use the Sz.-Nagy–Foias model theory for single contractions and Theorem 3.1 as follows. Apply Theorem VI.3.1 in [32] to the purely contractive analytic function $\Theta$ to conclude that the characteristic function $\Theta_T$ of $T$ coincides with $\Theta$, i.e., there exists unitary operators $u : \mathcal{D} \to \mathcal{D}_T$ and $u_* : \mathcal{D}_* \to \mathcal{D}_{T*}$ such that $u_* \cdot \Theta(z) = \Theta_T(z) \cdot u$ for all $z \in \mathbb{D}$. Let us set $(G_1', G_2') := u_*(G_1, G_2)u_*$ and
\( (R', S', W') := \omega_u (M_{\psi(\zeta) + \zeta}, M_{\psi(\zeta)}, M_\zeta) \omega_u^* \). Since \( G_1, G_2 \) satisfy the commutativity conditions, \( G_1', G_2' \) also satisfy the same conditions, and consequently the triple

\[
\left( \begin{bmatrix} M_{G_1' + zG_2'} & 0 \\ 0 & \omega_u M_{\psi(\zeta) + \zeta} \omega_u^* \end{bmatrix}, \begin{bmatrix} M_{G_2' + zG_1'} & 0 \\ 0 & \omega_u M_{\psi(\zeta)} \omega_u^* \end{bmatrix}, \begin{bmatrix} M_\zeta & 0 \\ 0 & M_\zeta \end{bmatrix} \right)
\]

(4.12)
is a strict tetrablock isometry. Note that since

\[
\begin{bmatrix} u^* & 0 \\ 0 & \omega_u \end{bmatrix} \begin{bmatrix} \Theta & 0 \\ 0 & \Delta \Theta \end{bmatrix} = \begin{bmatrix} \Theta T & 0 \\ 0 & \Delta \Theta T \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \omega_u \end{bmatrix},
\]

the unitary operator

\[
\tau := \begin{bmatrix} u^* & 0 \\ 0 & \omega_u \end{bmatrix} : \begin{bmatrix} H^2(D_a) \\ \Delta \Theta L^2(D) \end{bmatrix} \to \begin{bmatrix} H^2(D_{T'}) \\ \Delta \Theta T L^2(D_T) \end{bmatrix}
\]

takes the functional model space \( \mathcal{H} \) as in (4.10) onto

\[
\begin{bmatrix} H^2(D_{T'}) \\ \Delta \Theta T L^2(D_T) \end{bmatrix} \ominus \begin{bmatrix} \Theta T & 0 \\ 0 & \Delta \Theta T \end{bmatrix} H^2(D_T).
\]

Therefore by part (ii) of Definition 4.6, the tetrablock isometry as in (4.12) is a lift of \((\mathbf{A}, \mathbf{B}, \mathbf{T})\) via the embedding \( \iota : \tau |_{\mathcal{H}} \), where

\[
\iota : \begin{bmatrix} H^2(D_{T'}) & \Theta T \\ \Delta \Theta T L^2(D_T) & \Delta \Theta T \end{bmatrix} \to \begin{bmatrix} H^2(D_T) \\ \Delta \Theta T L^2(D_T) \end{bmatrix}
\]

is the inclusion map. Since

\[
\begin{bmatrix} M_\zeta & 0 \\ 0 & M_\zeta \end{bmatrix} : \begin{bmatrix} H^2(D_T) \\ \Delta \Theta T L^2(D_T) \end{bmatrix} \to \begin{bmatrix} H^2(D_T) \\ \Delta \Theta T L^2(D_T) \end{bmatrix}
\]
is a minimal isometric lift of \( \mathbf{T} \), by part (2) of Theorem 3.1, there is a unique such tetrablock isometric lift with the last entry of the lift fixed. If \( \mathbf{G}_1, \mathbf{G}_2 \) are the Fundamental Operators of \((\mathbf{A}^*, \mathbf{B}^*, \mathbf{T}^*)\), then by Theorem 3.2,

\[
\left( \begin{bmatrix} M_{G_1' + zG_2} & 0 \\ 0 & R_{NF} \end{bmatrix}, \begin{bmatrix} M_{G_2' + zG_1} & 0 \\ 0 & S_{NF} \end{bmatrix}, \begin{bmatrix} M_\zeta & 0 \\ 0 & W_{NF} \end{bmatrix} \right)
\]
is another tetrablock isometric lift of \((\mathbf{A}, \mathbf{B}, \mathbf{T})\), where \((R_{NF}, S_{NF}, W_{NF})\) is the canonical tetrablock unitary associated with \((\mathbf{A}, \mathbf{B}, \mathbf{T})\). Therefore we must have \((G_1, G_2) = (G_1', G_2') = u_*(G_1, G_2)u_*)^*\) and

\[
(R_{NF}, S_{NF}, W_{NF}) = \omega_u (M_{\psi(\zeta) + \zeta}, M_{\psi(\zeta)}, M_\zeta) \omega_u^*.
\]

This is what was needed to be shown. \( \square \)
**Epilogue.** It is easy to write down tetrablock data sets as in (4.8). Given such a tetrablock data set, it may not be very tractable to determine if in addition it satisfies conditions (i) and (ii) in Definition 4.6.

However it is not so difficult to cook up viable examples. For example, we note that the commutativity conditions in (i) are automatic if we choose $G_1$ and $G_2$ to be scalar operators on $D_*$. We can arrange the pencil contractivity condition to hold just by choosing $G_1$ and $G_2$ to be sufficiently small. If we choose the operator $\psi(\zeta)$ to be a scalar for each $\zeta$, then we are forced to choose $\psi(\zeta) = G_2^*\zeta + G_1$. Since we have already chosen $G_2$ and $G_1$ so that the pencil contractivity condition holds, then $\psi(\zeta)$ is contractive and of course a scalar operator is normal. Then all conditions are satisfied. In this way we get a whole class of tractable examples of special tetrablock contractions for any pure COF $\Theta$. If $D_*$ and $D$ are both at most one-dimensional, all the examples are of this form.

**Data availability** Data sharing is not applicable to this article as no datasets were generated or analysed during this current study.

**Declarations**

**Conflict of interest** The authors state that there are no conflicts of interest.

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