Atiyah and Todd classes of regular Lie algebroids

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Abstract For any regular Lie algebroid \( A \), the kernel \( K \) and the image \( F \) of its anchor map \( \rho_A \), together with \( A \) itself fit into a short exact sequence, called the Atiyah sequence, of Lie algebroids. We prove that the Atiyah and Todd classes of dg manifolds arising from a regular Lie algebroid respect the Atiyah sequence, i.e., the Atiyah and Todd classes of \( A \) restrict to the Atiyah and Todd classes of the bundle \( K \) of Lie algebras on the one hand, and project onto the Atiyah and Todd classes of the integrable distribution \( F \subseteq T_M \) on the other hand.

Keywords Atiyah classes, Todd classes, regular Lie algebroids, dg manifolds

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1 Introduction

The Atiyah class was introduced by Atiyah [2] to characterize the obstruction to the existence of a holomorphic connection on a holomorphic vector bundle. Kapranov [8] showed that the Atiyah class of a Kähler manifold \( X \) induces an \( L_\infty \) algebra structure on the shifted tangent complex \( \Omega_{T_X}^{\bullet-1}(T_X) \), which plays an important role in his reformulation of Rozansky-Witten theory. In his celebrated work [9], Kontsevich found a deep link between the Todd genus of complex manifolds and the Duflo element of Lie algebras. Liao et al. [12] explained this link via the formality theorem for smooth differential graded (dg for short) manifolds, where the Atiyah class of a dg vector bundle introduced by Mehta et al. [17] is an essential ingredient. See the survey article [20] on Atiyah classes of dg manifolds and how they work in the Duflo-Kontsevich type theorems for dg manifolds.

A dg manifold (or a \( Q \)-manifold) is a \( \mathbb{Z} \)-graded smooth manifold equipped with a homological vector field \( Q \), i.e., a degree +1 derivation of square zero on the algebra of smooth functions. Dg manifolds arise naturally in many situations in geometry, Lie theory, and mathematical physics. For example, according to Vaïntrob [21], to any Lie algebroid \( A \) is associated a dg manifold \((A[1], d_A)\), where the homological vector field \( d_A \) is identified with the Chevalley-Eilenberg differential of the Lie algebroid \( A \). In fact, any homological vector field on the graded manifold \( A[1] \) arises from a Lie algebroid structure on the vector bundle \( A \) in this manner. Since graded manifolds enjoy many similar properties as smooth manifolds, such as the existence of connections, curvatures, etc., one is able to generalize the same constructions in...
topology and differential geometry, for example, characteristic classes by the Chern-Weil theory, to dg manifolds (see [5,10]).

In this paper, we study the Atiyah and Todd classes of dg manifolds arising from regular Lie algebroids. A Lie algebroid \((A, \rho_A, [-, -]_A)\) over a smooth manifold \(M\) is said to be regular if its anchor \(\rho_A\) is of constant rank. The kernel \(K = \ker(\rho_A)\) together with the restriction \([-]_K\) of the Lie bracket \([-]_A\) onto \(\Gamma(K)\) is a bundle of Lie algebras; the image \(F = \text{Im}(\rho_A) \subseteq T_M\), being as the tangent bundle of the regular characteristic foliation, is a Lie subalgebroid of the tangent Lie algebroid \(T_M\). In other words, there is a short exact sequence of Lie algebroids over \(M\), i.e.,

\[
0 \to K \xrightarrow{i} A \xrightarrow{\rho_A} F \to 0,
\]

known as the Atiyah sequence of \(A\) (see [11]). The main purpose of this paper is to investigate whether Atiyah classes of dg manifolds arising from a regular Lie algebroid \(A\) respect the above Atiyah sequence. Our main theorem states that the Atiyah class of \(A\) restricts to the Atiyah class of the bundle \(K\) of Lie algebras along the inclusion \(i\) on the one hand, and projects onto the Molino class [18] of the integrable distribution \(F \subseteq T_M\) on the other hand (see Theorem 4.6).

For this purpose, observe that both \(K\) and the quotient bundle \(B := T_M/F\) carry canonical \(A\)-module structures. Thus, the pullback \(\pi^*(E)\) of the Whitney sum \(E := K[1] \oplus B\) along the bundle projection \(\pi: [A[1] \to M\), equipped with the Chevalley-Eilenberg differential \(d_{CE}\) of the \(A\)-module \(E\), is a dg vector bundle over \((A[1], d_A)\). Combining the results in [1,6], we see that there exists a contraction of dg vector bundles for the tangent dg vector bundle \((T_{A[1]}, L_{d_A})\) of \((A[1], d_A)\):

\[
\eta \hookrightarrow (T_{A[1]}, L_{d_A}) \xrightarrow{\phi} (E := \pi^*(E), Q_E := d_{CE} - \Omega),
\]

where

\[
-\Omega \in \Omega^2_A(\text{Hom}(B, K[1])) := \Gamma(\wedge^2 A^* \otimes \text{Hom}(B, K[1]))
\]

is a perturbation of the differential \(d_{CE}\). Via this contraction, we give an explicit description of the Atiyah class of the dg manifold \((A[1], d_A)\) (see Theorem 4.1). In particular, we obtain the Atiyah class of a bundle of Lie algebras (see Proposition 4.3), and rediscover the fact in [4] that the Atiyah class of the dg manifold arising from an integrable distribution \(F \subseteq T_M\) is related via a canonical isomorphism to the Atiyah class of the Lie pair \((T_M, F)\) introduced in [3] (see Proposition 4.5). Since scalar Atiyah classes and Todd classes are generated by Atiyah classes, we also prove that the scalar Atiyah and Todd classes arising from a regular Lie algebroid \(A\) respect the Atiyah sequence of \(A\) (see Propositions 5.3 and 5.6).

Note that the perturbation term \(-\Omega \in \Omega^2_A(\text{Hom}(B, K[1]))\) is indeed \(d_{CE}\)-closed. It was proved in [6] that the Chevalley-Eilenberg cohomology class

\[
\omega = [\Omega] \in H^1_{CE}(A; \text{Hom}(B, K[1])) \cong H^2_{CE}(A; \text{Hom}(B, K))
\]

is a characteristic class of \(A\). In fact, the cohomology class \(\omega\) measures whether the Lie algebroid structure locally splits around leaves of the submanifold foliated from \(F\). More precisely, according to [6, Theorem 7.3], \(\omega\) vanishes if and only if around each leaf \(L\) foliated from \(F\), there exist a tubular neighborhood \(U \subseteq M\) of \(L\) and an identification \(U = L \times N\) such that the restriction \(\pi|_U\) is isomorphic to the cross product of \(A|_L\) and the trivial Lie algebroid over \(N\). For this reason, we say that a regular Lie algebroid \(\text{locally splits}\), if the cohomology class \(\omega\) vanishes. For a locally splittable regular Lie algebroid \(A\), we prove that the kernel \(K\) of the anchor map is a Lie algebra bundle, and the Atiyah class of \(A\) only consists of two components: one is the Atiyah class of \(K\) which is represented by the Lie bracket on \(\Gamma(K)\); the other is the Atiyah class of its characteristic distribution \(F\), which is related via a canonical isomorphism to the Molino class of \(F\). The Todd class of \(A\) is given by the product of the Todd class of this Lie algebra bundle \(K\) that is represented by the Duflo element, and the Todd class of the characteristic distribution \(F\) (see Theorem 5.8).
2 Atiyah and Todd classes of dg vector bundles

2.1 Dg manifolds and dg vector bundles

In this subsection, we briefly recall the definitions of dg manifolds and dg vector bundles (see, for details, [13, 19]).

Let $M$ be a smooth manifold, and $\mathcal{O}_M$ be the sheaf of smooth functions over $M$. By a $(\mathbb{Z}_2)$-graded manifold $\mathcal{M}$ with the support $M$, we mean a pair $(\mathcal{M}, \mathcal{O}_\mathcal{M})$, where $\mathcal{O}_\mathcal{M}$ is a sheaf of $\mathbb{Z}$-graded commutative $\mathcal{O}_M$-algebras over $M$ such that for every contractible open subset $U \subseteq M$, $\mathcal{O}_\mathcal{M}(U)$ is locally isomorphic to $C^\infty(U) \otimes SV$ for a fixed finite-dimensional $\mathbb{Z}$-graded vector space $V$. The space $C^\infty(\mathcal{M}) = \Gamma(\mathcal{O}_\mathcal{M})$ of global sections of the structure sheaf $\mathcal{O}_\mathcal{M}$ will be referred to as the algebra of smooth functions on $\mathcal{M}$. A morphism $\phi$ of graded manifolds from $\mathcal{M}$ to $\mathcal{N}$ is a pair $(\phi_0, \phi^*)$, where $\phi_0 : M \to N$ is a smooth map of base manifolds, and $\phi^* : \mathcal{O}_\mathcal{N} \to \mathcal{O}_\mathcal{M}$ is a morphism of structure sheaves covering $\phi_0$. In particular, $\phi^*$ induces a morphism of algebras of smooth functions $\phi^* : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M})$. The collection of graded manifolds and their morphisms constitutes a category called the category of graded manifolds.

A dg manifold is a graded manifold $\mathcal{M}$ together with a homological vector field, i.e., a vector field $Q_{\mathcal{M}}$ of degree +1 satisfying the integrable condition $[Q_{\mathcal{M}}, Q_{\mathcal{M}}] = 0$. In other words, the vector field $Q_{\mathcal{M}}$ arises from an infinitesimal action of the super Lie group $\mathbb{R}^{0|1}$ on the graded manifold $\mathcal{M}$. The associated cohomology $H^*(\mathcal{M}) := H^*(C^\infty(\mathcal{M}), Q_{\mathcal{M}})$ is called the cohomology of the dg manifold $(\mathcal{M}, Q_{\mathcal{M}})$. A morphism $\phi$ of dg manifolds from $(\mathcal{M}, Q_{\mathcal{M}})$ to $(\mathcal{N}, Q_{\mathcal{N}})$ is a morphism of the underlying graded manifolds such that the morphism $\phi^*$ of algebras of smooth functions is a cochain map, i.e.,

$$\phi^* \circ Q_{\mathcal{N}} = Q_{\mathcal{M}} \circ \phi^* : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M}).$$

The collection of dg manifolds together with their morphisms forms a category called the category of dg manifolds.

A dg vector bundle is an $\mathbb{R}^{0|1}$-equivariant vector bundle in the category of graded manifolds, or equivalently, a vector bundle in the category of dg manifolds, i.e., a graded vector bundle $\pi : E \to M$, equipped with two homological vector fields $Q_E$ and $Q_M$ coming from infinitesimal actions of $\mathbb{R}^{0|1}$ on $E$ and $M$, respectively, such that the bundle projection $\pi$ is a morphism of dg manifolds. Here, the $\mathbb{R}^{0|1}$-action on $E$ is required to be fibre-preserving and fiberwise linear. In other words, the homological vector field $Q_E$ is linear with $Q_M$ being its restriction onto the base manifold $M$.

**Example 2.1.** Let $(\mathcal{M}, Q_{\mathcal{M}})$ be a dg manifold. Then the vector bundle $(T^*_E)^0 \otimes (T_M)^0$ together with the Lie derivative $L_{Q_{\mathcal{M}}}$ along the homological vector field $Q_{\mathcal{M}}$ is a dg vector bundle over $(\mathcal{M}, Q_{\mathcal{M}})$ for all $p, q \geq 0$.

2.2 Atiyah and Todd classes

Let $(E, Q_E)$ be a dg vector bundle over a dg manifold $(\mathcal{M}, Q_{\mathcal{M}})$. Mehta et al. [17] introduced a cohomology class $At_{(E, Q_E)}$, called the Atiyah class of $(E, Q_E)$, to measure the obstruction to the existence of a linear connection on the graded vector bundle $E$ that is compatible with the homological vector fields $Q_E$ and $Q_{\mathcal{M}}$ (see also [13]). More precisely, to any linear connection $\nabla$ on the graded vector bundle $E$ is associated a degree +1 element $At_{(E, Q_E)} \in \Gamma(T^*_E \otimes \text{End}(E))$ defined by

$$At_{(E, Q_E)}(X, e) = Q_E(\nabla_X e) - \nabla_{Q_{\mathcal{M}}(X)} e - (-1)^{|X|} \nabla_X Q_E(e)$$

(2.1)

for all the homogeneous $X \in \Gamma(T^*_M)$ and $e \in \Gamma(E)$. This element is closed under the homological vector field of the dg vector bundle $T^*_M \otimes \text{End}(E)$. Its cohomology class

$$At_{(E, Q_E)} := [\nabla_{(E, Q_E)}] \in H^1(\Gamma(T^*_E \otimes \text{End}(E)))$$

is independent of the choice of the linear connection $\nabla$, and is called the Atiyah class of the dg vector bundle $(E, Q_E)$. 
For any positive integer \( k \), one can form \( \text{At}^k_{(E,Q_E)} \), the image of \( \text{At}^k_{(E,Q_E)} \) under the natural map
\[
\otimes^k H^1(\Omega^1(M) \otimes_{C^\infty(M)} \Gamma(\text{End}(E))) \to H^k(\Omega^k(M) \otimes_{C^\infty(M)} \Gamma(\text{End}(E)))
\]
induced by the wedge product in the space \( \Omega^p(M) \) of differential forms on \( M \) and the composition in \( \text{End}(E) \). The \( k \)-th scalar Atiyah class [17] of the dg vector bundle \((E,Q_E)\) is defined by
\[
\text{ch}_k(E,Q_E) := \frac{1}{k!} \left( \frac{i}{2\pi} \right)^k \text{str}(\text{At}^k_{(E,Q_E)}) \in H^k(\Omega^k(M), L_{Q,M}),
\]
where \( \text{str} : \Gamma(\text{End}(E)) \to C^\infty(M) \) is the supertrace map (see [15]).

The Todd class of the dg vector bundle \((E,Q_E)\) is defined by
\[
\text{Td}(E,Q_E) := \text{Ber} \left( \frac{\text{At}(E,Q_E)}{1 - e^{-\text{At}(E,Q_E)}} \right) \in \prod_{k \geq 0} H^k(\Omega^k(M), L_{Q,M}),
\]
where \( \text{Ber} : \Gamma(\text{End}(E)) \to C^\infty(M) \) is the Berezinian (or the superdeterminant) map (see [15]).

### 2.3 Invariance under contractions

A contraction of dg vector bundles over a dg manifold \((M,Q_M)\) from \((E,Q_E)\) to \((F,Q_F)\) is given by three bundle maps \((\phi, \psi, h)\) fitting in the following diagram:

\[
\begin{array}{ccc}
\begin{array}{c}
h \subset (E,Q_E) \\
\end{array} & \xrightarrow{\phi} & (F,Q_F) \\
\begin{array}{c}
\psi \\
\end{array} & \downarrow & \downarrow
\end{array}
\]

where both \( \phi : (E,Q_E) \to (F,Q_F) \) and \( \psi : (F,Q_F) \to (E,Q_E) \) are morphisms of dg vector bundles, and \( h : E \to E[-1] \) is a bundle map satisfying
\[
\phi \circ \psi = \text{id}_E, \quad \psi \circ \phi = \text{id}_E + [Q_E,h] = \text{id}_E + Q_E \circ h + h \circ Q_E
\]
and the side conditions \( h^2 = 0, \phi \circ h = 0 \) and \( h \circ \psi = 0 \). Given a contraction \((\phi,\psi,h)\) of bounded dg vector bundles over \((M,Q_M)\) from \((E,Q_E)\) to \((F,Q_F)\), by taking the dual, we obtain a contraction \((\phi^*,\psi^*,h^*)\) of bounded dg vector bundles from \((E^*,Q_{E^*})\) to \((F^*,Q_{F^*})\). Then applying the tensor product of these two contractions, we obtain the following contraction of bounded dg vector bundles over \((M,Q_M)\):

\[
\begin{array}{ccc}
\begin{array}{c}
h \subset (E^* \otimes E \cong \text{End}(E),[Q_E,-]) \\
\end{array} & \xrightarrow{\Phi} & (F^* \otimes F \cong \text{End}(F),[Q_F,-]) \\
\begin{array}{c}
\Phi := \text{id}_{\text{T}_M^*} \otimes \Phi : H^* (\text{T}_M^* \otimes \text{End}(E)) \xrightarrow{\cong} H^* (\text{T}_M^* \otimes \text{End}(F))
\end{array}
\end{array}
\]

where the bracket \([-,-]\) means the graded commutator, and the three bundle maps are defined by

\[
\Phi := \psi^* \circ \phi, \quad \Psi := \phi^* \circ \psi, \quad H := (\psi\phi)^* \circ h + h^* \circ \text{id}_E.
\]

Passing to the cohomology of section spaces, we obtain an isomorphism
\[
\Phi_{E,F} := \text{id}_{\text{T}_M^*} \otimes \Phi : H^* (\text{T}_M^* \otimes \text{End}(E)) \xrightarrow{\cong} H^* (\text{T}_M^* \otimes \text{End}(F)).
\] (2.2)

Atiyah classes of bounded graded vector bundles are invariant under contractions in the following sense.

**Proposition 2.2.** Suppose that there exists a contraction \((\phi,\psi,h)\) of bounded dg vector bundles over \((M,Q_M)\) from \((E,Q_E)\) to \((F,Q_F)\). Then the isomorphism \(\Phi_{E,F}\) in (2.2) sends the Atiyah class \(\text{At}_{(E,Q_E)}\) of \((E,Q_E)\) to the Atiyah class \(\text{At}_{(F,Q_F)}\) of \((F,Q_F)\).

**Proof.** Let us choose a \(T_M\)-connection \(\nabla^F\) on the graded vector bundle \(F\). Via the inclusion \(\psi : F \to E\), we may identify \(F\) as a subbundle of \(E\). For each direct sum decomposition \(E = F \oplus F^c\) of graded vector bundles, we choose a \(T_M\)-connection \(\nabla^c\) on the complement bundle \(F^c\) of \(F\) such that \(\nabla^F\) and \(\nabla^c\) define a \(T_M\)-connection \(\nabla^E\) on \(E\). This connection satisfies
\[
\nabla^E_X \psi(v) = \psi(\nabla^F_X v)
\] (2.3)
for all $X \in \Gamma(T_M)$ and all $v \in \Gamma(F)$. Then for all the homogeneous $X \in \Gamma(T_M)$ and $v \in \Gamma(F)$, we have

\[
\Phi_{F,F}(\mathcal{A}_{\mathcal{E}_F})(X,v) = \phi(\mathcal{A}_{\mathcal{E}_F})(X,\psi(v))
\]
\[
= \phi(Q_{\mathcal{E}}(\nabla^F_X \psi(v)) - \nabla^F_{[Q_{\mathcal{M}},X]} \psi(v) - (-1)^{|X|} \nabla^F_{\mathcal{E}_F} \psi(v)) \quad \text{(since $\psi$ is a cochain map)}
\]
\[
= \phi(Q_{\mathcal{E}}(\nabla^F_X \psi(v)) - \nabla^F_{[Q_{\mathcal{M}},X]} \psi(v) - (1)^{|X|} \nabla^F_{\mathcal{E}_F} \psi(Q_{\mathcal{F}}(v))) \quad \text{(by (2.3))}
\]
\[
= \phi(\nabla^F_{[F,Q_{\mathcal{M}}]}(\nabla^F_X v) - \nabla^F_{[Q_{\mathcal{M}},X]} v - (1)^{|X|} \nabla^F_{\mathcal{E}_F} Q_{\mathcal{F}}(v))
\]
\[
= Q_{\mathcal{F}}(\nabla^F_X v) - \nabla^F_{[Q_{\mathcal{M}},X]} v - (1)^{|X|} \nabla^F_{\mathcal{E}_F} Q_{\mathcal{F}}(v) = \mathcal{A}_{\mathcal{F},\mathcal{F}}(X,v).
\]

Passing to the cohomology level, we conclude the proof. \hfill \Box

As an immediate consequence, we have the following invariance on scalar Atiyah and Todd classes.

**Corollary 2.3.** Assume that there exists a contraction of bounded dg vector bundles from $(\mathcal{E}, Q_{\mathcal{E}})$ to $(\mathcal{F}, Q_{\mathcal{F}})$. Then we have that for all positive integers $k$, $\text{ch}_k(\mathcal{E}, Q_{\mathcal{E}}) = \text{ch}_k(\mathcal{F}, Q_{\mathcal{F}})$ and $\text{Td}(\mathcal{E}, Q_{\mathcal{E}}) = \text{Td}(\mathcal{F}, Q_{\mathcal{F}})$.

### 3 Cohomology of regular Lie algebroids

In this section, we study the cohomology of tensor fields on the dg manifold $(A[1], d_A)$ arising from a regular Lie algebroid $A$.

#### 3.1 The graded geometry of regular Lie algebroids

**3.1.1 The graded geometry of Lie algebroids**

We start with a brief discussion on the graded geometry of general Lie algebroids. According to Vaintrob [21], a Lie algebroid structure on a smooth vector bundle $A$ is one-to-one correspondent to a homological vector field $d_A$ on the graded manifold $A[1]$. More precisely, given a Lie algebroid $A$ over $M$, we obtain a dg manifold $(A[1], d_A)$ whose space of functions is the Chevalley-Eilenberg dg algebra $(\Omega_A = \Gamma(\wedge A^+), d_A)$ of $A$. Furthermore, each $A$-module $(E, \nabla^E)$ consisting of an ungraded vector bundle $E \to M$ and a flat Lie algebroid $A$-connection $\nabla^E$ gives rise to a dg vector bundle over the dg manifold $(A[1], d_A)$ via the following pullback diagram:

\[
\begin{array}{ccc}
(\mathcal{E} := \pi^* E, d^\mathcal{E}) & \longrightarrow & E \\
\downarrow & & \downarrow \\
(A[1], d_A) & \longrightarrow & M.
\end{array}
\]

However, the converse is not true in general, since any graded vector bundle over the graded manifold $A[1]$ does not necessarily arise from the pullback of an ungraded vector bundle over $M$ along $\pi$, but rather from the pullback of a graded vector bundle over $M$ (see [16]). Indeed, dg vector bundles over the dg manifold $(A[1], d_A)$ are one-to-one correspondent to representations up to homotopy (or $\infty$-representations) of $A$ consisting of graded vector bundles over $M$ equipped with flat $A$-superconnections (see [1,6]).

**Definition 3.1 (See [1,6]).** A representation up to homotopy, or an $\infty$-representation, of $A$ on a graded vector bundle $E = \bigoplus_{i \in \mathbb{Z}} E^i$ over $M$ is a square zero operator $D$ of degree 1 on

\[
\Omega_A(E) := \Omega_A \otimes_{C^\infty(M)} \Gamma(E) = \Gamma(\wedge A^+ \otimes E)
\]

satisfying the Leibniz rule

\[
D(\alpha \omega) = (d_A \alpha) \omega + (-1)^{|\alpha|} \alpha (D \omega)
\]
for all the homogeneous $\alpha \in \Omega_A$ and $\omega \in \Omega_A(E)$. The cohomology of the resulting complex is denoted by

$$H^\bullet(A; E) = \bigoplus_k H^k(A; E) := \bigoplus_{p+q=k} H^k(\Omega^q_A(E^p), D).$$

It was proved in [1, 6] that there exists a correspondence between the tangent dg vector bundle $(T_{A[1]}, L_{d_A})$ of $(A[1], d_A)$ and a representation up to homotopy of the Lie algebroid $A$ on the graded vector bundle $A[1] \oplus T_M$ over $M$, called the adjoint representation in [1], which we now recall.

Observe that there is a short exact sequence of vector bundles over the graded manifold $A[1]$:

$$0 \to \pi^\ast(A[1]) \xrightarrow{\rho} T_{A[1]} \xrightarrow{\pi} \pi^\ast(T_M) \to 0,$$

where $\pi_\ast: T_{A[1]} \to \pi^\ast(T_M)$ is the tangent map of the bundle projection $\pi: A[1] \to M$, and $I$ is the canonical vertical lifting. Taking global sections gives rise to a short exact sequence of left graded $\Omega_A$-modules, i.e.,

$$0 \to \Omega_A \otimes_{C^\infty(M)} \Gamma(A[1]) \xrightarrow{\pi} \Gamma(T_{A[1]}) \xrightarrow{\pi} \Omega_A \otimes_{C^\infty(M)} \Gamma(T_M) \to 0.$$

The canonical vertical lifting $I$ is given by the $\Omega_A$-linear contraction defined as follows: for all $\omega \in \Omega_A$ and $a[1] \in \Gamma(A[1])$,

$$I(\omega \otimes a[1]) = \omega \otimes_{\mathfrak{a}_U} \in \mathrm{Der}(\Omega_A) \cong \Gamma(T_{A[1]}).$$

Let us choose a linear connection $\nabla^A$ on the vector bundle $A$ over $M$. This connection $\nabla^A$ induces a splitting of the short exact sequence (3.1) such that $T_{A[1]} \cong A[1] \times_M (A[1] \oplus T_M)$. Thus, one has an isomorphism of $\Omega_A$-modules

$$\Gamma(T_{A[1]}) \xrightarrow{\cong} \Omega_A(A[1] \oplus T_M) = \Omega_A \otimes_{C^\infty(M)} \Gamma(A[1] \oplus T_M).$$

The isomorphism (3.3) transfers the Lie derivative $L_{d_A}$ on $\Gamma(T_{A[1]})$ to a square zero derivation

$$D_{\nabla^A} = \begin{pmatrix} d_{\nabla^\ast_A} & R_{\nabla^A} \\ \rho_A & d_{\nabla^\ast_A} \end{pmatrix} : \begin{pmatrix} \Omega^\bullet_A(A[1]) \\ \Omega^\bullet_A(T_M) \end{pmatrix} \to \begin{pmatrix} \Omega^{\bullet+1}_A(A[1]) \\ \Omega^{\bullet+1}_A(T_M) \end{pmatrix},$$

where

- $\rho_A$ is the $\Omega_A$-linear extension of the anchor map $\rho_A: A[1] \to T_M$;
- $d_{\nabla^\ast_A}: \Omega^\bullet_A(A[1] \oplus T_M) \to \Omega^{\bullet+1}_A(A[1] \oplus T_M)$ is the covariant derivative of the basic $A$-connection $\nabla^\ast_A$ on $A[1] \oplus T_M$ defined by $\nabla^\ast_A(u) := \rho_A(\nabla^A_u a) + [\rho_A(a), u]$, and $\nabla^\ast_A(a[1]) := (\nabla^A_{\rho_A(a')} a + [a, a']_A)[1]$ for all $a', a'' \in \Gamma(A)$ and $u \in \Gamma(T_M)$;
- $R_{\nabla^A} \in \Omega^2_A(\mathrm{Hom}(T_M, A[1]))$, called the basic curvature of $\nabla^A$, defines an $\Omega_A$-linear map $\Omega^{\bullet+1}_A(T_M) \to \Omega^{\bullet+1}_A(A[1])$ by

$$R_{\nabla^A}(u)(a', a'') := ([\nabla^A_u a', a'']_A - [\nabla^A_u a', a''_A] - [a', \nabla^A_u a''_A] - \nabla^A_{\nabla^\ast_A u} a' + \nabla^A_{\nabla^\ast_A u} a'')[1],$$

for all $a', a'' \in \Gamma(A)$, and $u \in \Gamma(T_M)$.

In particular, we obtain an isomorphism of cochain complexes

$$\Gamma(T_{A[1]}, L_{d_A}) \xrightarrow{\cong} (\Omega^1_A(\mathcal{A}[1] \oplus T_M), D_{\nabla^A} = \rho_A + d_{\nabla^\ast_A} + R_{\nabla^A}).$$

By taking dual operations, we obtain another isomorphism of cochain complexes

$$(\Omega^1(A[1]) = \Gamma(T^\ast_{A[1]}, L_{d_A}) \xrightarrow{\cong} (\Omega^A(T^\ast M \oplus A^\ast[-1]), D_{\nabla^A} = \rho_A + d_{\nabla^\ast_A} + R_{\nabla^A}),$$

where $\rho_A: \Omega_A(T^\ast M) \to \Omega_A(A^\ast[-1])$ and $(R_{\nabla^A})^* \in \Omega^2_A(\mathrm{Hom}(A^\ast[-1], T^\ast M))$ are the $\Omega_A$-linear duals of $\rho_A$ and $R_{\nabla^A}$, respectively, while $d_{\nabla^\ast_A}$ is the covariant derivative of the dual $A$-connection on $T^\ast M \oplus A^\ast[-1]$. 
3.1.2 Applications to regular Lie algebroids

Let \((A, \rho_A, [-, -]_A)\) be a regular Lie algebroid over \(M\) with the characteristic distribution

\[ F := \text{Im} \rho_A \subseteq T_M \]

of constant rank. Let \(K \subseteq A\) be the kernel of \(\rho_A\), which is a bundle of Lie algebras over \(M\) such that the inclusion \((\Gamma(K), [-, -]_K) \hookrightarrow (\Gamma(A), [-, -]_A)\) is a morphism of Lie algebras. We thus obtain a short exact sequence of Lie algebroids over \(M\)

\[ 0 \to K \overset{j}{\to} A \overset{\rho_A}{\to} F \to 0, \tag{3.6} \]

known as the Atiyah sequence of the regular Lie algebroid \(A\) (see [11]). According to [6], the differential \(D_{\nabla A}(3.4)\) can be simplified if we choose a special linear connection on the regular Lie algebroid \(A\). Note that there is another short exact sequence of vector bundles over \(M\)

\[ 0 \to F \to T_M \overset{\text{pr}_B}{\to} B \to 0, \tag{3.7} \]

where we denote by \(B\) the quotient bundle \(T_M/F\), which can be thought of as the normal bundle of the characteristic foliation \(F \subseteq M\). We fix a quadruple

\[ (\tau, j, \nabla^K, \nabla^F), \tag{3.8} \]

where

- \(\tau: F \to A\) is a splitting of the short exact sequence (3.6) of vector bundles, and \(j: B \to T_M\) is a splitting of the short exact sequence (3.7) of vector bundles;
- \(\nabla^K\) is a linear connection on \(K\) extending the \(F\)-connection on \(K\) defined by \(\nabla^K_A a_K = [\tau(u_F), a_K]_A\) for all \(u_F \in \Gamma(F)\) and \(a_K \in \Gamma(K)\);
- \(\nabla^F\) a linear connection on \(F\) extending a torsion-free \(F\)-connection on \(F\) satisfying

\[ \nabla^F_b u_F = \text{pr}_F[j(b), u_F] \]

for all \(b \in \Gamma(B)\) and \(u_F \in \Gamma(F)\).

Let

\[ \nabla^A_a = \nabla^K_a (a_K) + \tau(\nabla^F_a u_F) + R^F(u_F, a_F) \tag{3.9} \]

for all \(u \in \Gamma(T_M)\) and \(a \in \Gamma(A)\), where \(u_F\) is the component of \(u\) in \(\Gamma(F)\), \(a_K\) and \(a_F\) are the components of \(a\) in \(\Gamma(K)\) and \(\Gamma(F)\), respectively, and \(R^F \in \Omega^2_F(K)\), called the curvature of the splitting \(\tau\), is defined as follows: for all \(u_F, v_F \in \Gamma(F)\),

\[ R^F(u_F, v_F) = [\tau(u_F), \tau(v_F)]_A - \tau([u_F, v_F]). \]

It is easy to see that \(\nabla^A\) defined above is indeed a linear connection on \(A\).

**Lemma 3.2** (See [6]). The basic connection \(\nabla^\text{bas}_A\) and the basic curvature \(R^\text{bas}_A \in \Omega^2_A(\text{Hom}(T_M, A))\) of the linear connection \(\nabla^A\) defined in (3.9) satisfy

\[ \nabla^\text{bas}_F u = \nabla^\text{bas}_F u + \nabla^\text{Bat}_F u_F = \nabla_F u_F + \text{pr}_B(\rho_A(a), j(u_B)), \]

\[ \nabla^\text{bas}_a a' = [a, a_k^A]_A + \tau(\nabla^F_{\rho_A(a)}(a')), \]

\[ R^\text{bas}_A(a, a')(u_F) = -\tau(R^F(\rho_A(a), \rho_A(a')) u_F) \]

and

\[ R^\text{bas}(a, a')(j(b)) \]

\[ = \nabla^F_{j(b)} [a, a_k^A]_A - \nabla^F_{j(b)} [a, a_k^A]_A - [a, a_k, \nabla^F_{j(b)}(a')]_A \]

\[ - R^F(\rho_A(a), j(b))a_k^A + R^F(\rho_A(a'), j(b))a_k^A \]
Proposition 3.3 (See [1]). For all $a, a', a'' \in \Gamma(A)$, $u \in \Gamma(T_M)$ and $b \in \Gamma(B)$, where $u_F$ and $u_B$ are the components of $u$ in $\Gamma(F)$ and $\Gamma(B)$, respectively, $a_K = \rho_K(a)$ is the component of $a$ in $\Gamma(K)$, and $R^{\nabla^F}$ and $R^{\nabla^K}$ are the curvatures of the chosen linear connections $\nabla^F$ and $\nabla^K$, respectively.

As a consequence, the basic curvature defines an $\Omega_A$-linear map of degree 1, i.e., $\Omega: \Omega^*_A(B) \to \Omega^*_A(K[1])$ by

$$\Omega(b)(a, a') := -R^{\nabla^A}_B([\rho(a), j(b)])[1]$$

for all $a, a' \in \Gamma(A)$ and $b \in \Gamma(B)$.

### 3.2 Contractions of tangent dg vector bundles

Given a regular Lie algebroid $(A, \rho_A, [-, -]_A)$, there exist ordinary representations of $A$ on the vector bundles $K$ and $B$, i.e., flat Lie algebroid $A$-connections on $K$ and $B$, defined by

$$\nabla^K: \Gamma(A) \times \Gamma(K) \to \Gamma(K), \quad \nabla^K_a^{\nabla^K} = [a, \rho_K]_A$$

and

$$\nabla^B: \Gamma(A) \times \Gamma(B) \to \Gamma(B), \quad \nabla^B_a^{\nabla^B} = \rho_B([\rho(a), j(b)),$$

respectively, for all $a \in \Gamma(A)$, $a_K \in \Gamma(K)$ and $b \in \Gamma(B)$. Here, $j: B \to T_M$ is any splitting of the short exact sequence (3.7) of vector bundles.

According to [1, Example 4.17], the adjoint representation of the regular Lie algebroid $A$ on the 2-term complex $A[1] \xrightarrow{\rho_A} T_M$ is quasi-isomorphic to a representation up to homotopy on its cohomology $K[1] \to B$. This representation up to homotopy on $K[1] \oplus B$ is indeed a perturbation of the ordinary representation of $A$ defined in (3.11) and (3.12). However, the explicit construction was skipped. For completeness, we give a thorough description in the following proposition.

**Proposition 3.3 (See [1]).** Let $A$ be a regular Lie algebroid over $M$. For any quadruple $(\tau, j, \nabla^K, \nabla^F)$ as in (3.8), there is a contraction for the adjoint representation of $A$ on the graded vector bundle $A[1] \oplus T_M$.

$$\kappa \subset (\Omega_A(A[1] \oplus T_M), D_{\nabla^A}) \xrightarrow{\psi} (\Omega_A(K[1] \oplus B), d_{\text{CE}} - \Omega),$$

where $\nabla^A$ is the linear connection on $A$ defined in (3.9), $D_{\nabla^A}$ is the differential defined in (3.4), $d_{\text{CE}}$ is the Chevalley-Eilenberg differential of the $A$-module $K[1] \oplus B$ defined in (3.11) and (3.12), and $\Omega$ is the degree 1 map defined in (3.10).

To prove this proposition, we need the following perturbation lemma (see [14] and the references therein).

**Lemma 3.4 (Perturbation lemma).** Given a contraction of cochain complexes

$$\kappa \subset (P, \delta) \xrightarrow{\varphi \psi} (T, d)$$

and a perturbation $\varphi$ of the differential $\delta$, i.e., a linear map $\varphi: P \to P[1]$ satisfying $\delta + \varphi$ is a new differential on $P$ and the following constraints:

$$\bigcup_n \ker((h\varphi)^n) = T, \quad \bigcup_n \ker(\varphi(h)^n) = P, \quad \bigcup_n \ker(h(\varphi)^n) = P,$$

the series

$$\varphi := \sum_{k=0}^{\infty} \varphi(h^k) \psi, \quad \phi := \sum_{k=0}^{\infty} \varphi(h^k),$$

$$\psi := \sum_{k=0}^{\infty} (h^k) \psi, \quad h := \sum_{k=0}^{\infty} h(h)^k$$

all converge, and the datum
Combining the side conditions is defined by perturbation Lemma 3.4 to the contraction (3.17) and the perturbation of sequences (3.6) and (3.7) of vector bundles, respectively, there is a contraction of cochain complexes for all $\bar{h} \in (\Gamma(A[1] \oplus T_M), \rho_A) \xrightarrow{\phi} \Gamma(K[1] \oplus B), 0)$, where the three maps $\phi, \psi, h$ are defined by

$$\phi(a[1]+u) = pr_K(a) + pr_B(u), \quad \psi(a_K[1]+b) = a_K[1] + j(b), \quad h(a[1]+u) = -\tau(pr_F(u))[1]$$

for all $a \in \Gamma(A), a_K \in \Gamma(K)$ and $u \in \Gamma(T_M)$. The $\Omega_A$-linear extension of this contraction gives rise to a contraction of $\Omega_A$-modules

$$\bar{h} \in (\Omega_A(A[1] \oplus T_M), \rho_A) \xrightarrow{\phi} \Omega_A(K[1] \oplus B), 0).$$

Observe that $d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}$ defines a perturbation of the differential $\rho_A$. Since

$$\bar{h}((d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})(u[1]+u)) = h(d_{\nabla_{bas}}(u)) \in \Omega^1_A(A[1])$$

is defined by

$$\bar{h} : (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})(h(u[1]+u)) = -d_{\nabla_{bas}}(\tau(u_F)) \in \Omega^1_A(A[1])$$

is defined by

$$\bar{h} : (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})(h(u[1]+u)) = -d_{\nabla_{bas}}(\tau(u_F)) \in \Omega^1_A(A[1])$$

it follows that

$$h \circ (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) = -(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) \circ h : \Omega_A(A[1] \oplus T_M) \to \Omega_A(A[1] \oplus T_M).$$

Combining the side conditions $\phi \circ h = 0, h \circ \psi = 0$ and $h^2 = 0$, we have

$$\phi \circ (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) \circ h = -\phi \circ h \circ (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) = 0,$n

$$h \circ (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) \circ \psi = -(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) \circ h \circ \psi = 0,$n

$$h \circ (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) \circ h = h^2 \circ (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) = 0.$$

Thus, the maps $\phi, \psi, h$ and the perturbation $d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}$ satisfy the constraints in (3.14). Applying the perturbation Lemma 3.4 to the contraction (3.17) and the perturbation $d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}$, we obtain a new contraction

$$h' \in (\Omega_A(A[1] \oplus T_M), D_{\nabla_A} = \rho_A + d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) \xrightarrow{\phi'} (\Omega_A(K[1] \oplus B), D'),$$

where

$$\phi' = \sum_{l=0}^{\infty} \phi((d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})h)^l = \phi + \sum_{l=1}^{\infty} (\phi((d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})h)((d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})h)^{l-1}) = \phi,$$
\[ \psi' = \sum_{l=0}^{\infty} (h(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}))^l \psi = \psi + \sum_{l=1}^{\infty} (h(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}))^{l-1} (h(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})) \psi = \psi, \]
\[ h' = \sum_{l=0}^{\infty} (h(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}))^l h = h + \sum_{l=1}^{\infty} (h(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}))^{l-1} (h(d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A})) h = h, \]

and the new differential \( D' \) on \( \Omega_A(K[1] \oplus B) \) is \( D' = \phi' \circ (d_{\nabla_{bas}} + R^\text{bas}_{\nabla_A}) \circ \psi = d_{CE} - \Omega, \) according to Lemma 3.2.

Combining (3.5) with Proposition 3.3, we obtain a contraction for the tangent dg vector bundle of the dg manifold \( (A[1], d_A) \).

**Corollary 3.5.** For each choice of quadruple \((j, \tau, \nabla^K, \nabla^F)\), there exists a contraction of dg vector bundles over the dg manifold \( (A[1], d_A) \):

\[
H \subset (T_{A[1]}, L_{d_A}) \xrightarrow{\phi} (E, Q_E),
\]

where \( E = \pi^*(E) \) is the pullback bundle of the Whitney sum \( E := K[1] \oplus B \) over \( M \) along the projection \( \pi: A[1] \to M, Q_E = d_{CE} - \Omega \) is the differential on \( \Gamma(E) \) induced by the pair of homological vector fields carried by the vector bundle \( E \) and its base manifold \( A[1], \) and the inclusion \( \Psi \) is defined by

\[
\Psi(\alpha \otimes a_K[1]) = I(\psi(\alpha \otimes a_K[1])) = \alpha \otimes \iota_{a_K}, \quad \Psi(\alpha \otimes b) = \alpha \otimes \nabla^A_j(b)
\]

(3.18)

for all \( \alpha \in \Omega_A, a_K \in \Gamma(K) \) and \( b \in \Gamma(B) \). Here, \( \nabla^A \) is the linear connection on \( A \) defined in (3.9) and \( I: \Omega_A(A[1]) \to \Gamma(T_{A[1]}) \) is the canonical vertical lifting in (3.2).

**Remark 3.6.** When the anchor \( \rho_A \) is injective, the regular Lie algebroid \( A \) is identified with its characteristic distribution \( F = \text{Im} \rho_A \). The above contraction was explicitly constructed in \([4]\) to compute Atiyah and Todd classes of integrable distributions.

As a consequence, the map \( \Psi \) induces an isomorphism from the cohomology of the representation up to homotopy \((\Omega_A(K[1] \oplus B), Q_E)\) of \( A \) to the cohomology of the tangent dg vector bundle \( (T_{A[1]}, L_{d_A}) \), i.e.,

\[
\Psi: H^*(A; K[1] \oplus B) := H^*(\Omega_A(K[1] \oplus B), Q_E) \xrightarrow{\cong} H^*(\Gamma(T_{A[1]}), L_{d_A}).
\]

By taking dual operations, we obtain a contraction for the cotangent dg vector bundle of the dg manifold \((A[1], d_A)\).

**Corollary 3.7.** For each choice of quadruple \((j, \tau, \nabla^K, \nabla^F)\), there exists a contraction for the cotangent dg vector bundle of the dg manifold \((A[1], d_A)\):

\[
H^\ast \subset (T^*_{A[1]}), L_{d_A}) \xrightarrow{\Phi^\ast} (E^\ast = \pi^*(B^\ast \oplus K^*[1]), Q_{E^\ast} := d_{CE} + \Omega^\ast),
\]

where \( d_{CE} \) is the Chevalley-Eilenberg differential of the dual \( A \)-module \( B^\ast \oplus K^*[1] \), and \( \Omega^\ast \) is the \( \Omega_A \)-linear dual of \( \Omega \).

By taking tensor products of the two contractions in Corollaries 3.5 and 3.7, we obtain contractions from the space \( \Gamma(T_{A[1]}^{\otimes m} \otimes T_{A[1]}^{\otimes n}) \) of \((m, n)\) tensor fields on the dg manifold \((A[1], d_A)\) to the section space \( \Gamma(E^{\otimes m} \otimes E^{\otimes n}) \) of the dg vector bundle \( E^{\otimes m} \otimes E^{\otimes n} \) over \((A[1], d_A)\). For simplicity, we also denote by \( Q_E \) the induced differential on the section space \( \Gamma(E^{\otimes m} \otimes E^{\otimes n}) \) by abuse of notation. When passing to cohomology, for all the pairs \((m, n)\) of non-negative integers, we obtain isomorphisms

\[
\Phi_{m,n}: H^\ast(\Gamma(T_{A[1]}^{\otimes m} \otimes T_{A[1]}^{\otimes n}), L_{d_A}) \xrightarrow{\cong} H^\ast(\Gamma(E^{\otimes m} \otimes E^{\otimes n}), Q_E) = H^\ast(A; E^{\otimes m} \otimes E^{\otimes n}).
\]

(3.19)

Here, \( H^\ast(A; E^{\otimes m} \otimes E^{\otimes n}) \) is the cohomology of the representation up to homotopy of \( A \) on the tensor products \( E^{\otimes m} \otimes E^{\otimes n} \) of graded vector bundles over \( M \).
4 Atiyah classes of regular Lie algebroids

In this section, we study the Atiyah class of a regular Lie algebroid $A$ over $M$, i.e., the Atiyah class of the tangent dg vector bundle $(T_{A[1]}, L_{dA})$ of the dg manifold $(A[1], dA)$ arising from $A$.

4.1 Atiyah classes

Let us fix a quadruple $(\tau, j, \nabla^K, \nabla^F)$ as in (3.8) and denote by $\nabla^A$ the associated linear connection on $A$ defined in (3.9). By Proposition 2.2 and Corollary 3.5, the Atiyah class of the tangent dg vector bundle $(T_{A[1]}, L_{dA})$ is related via an isomorphism to the Atiyah class of the dg vector bundle

$$(\mathcal{E} := \pi^*(E) = \pi^*(K[1] \oplus B), Q_\mathcal{E} := d_{CE} - \Omega).$$

We summarize these into our main theorem.

**Theorem 4.1.** For each choice of quadruple $(j, \tau, \nabla^K, \nabla^F)$, the Atiyah class $At_{(A[1], dA)}$ of the dg manifold $(A[1], dA)$ arising from a regular Lie algebroid $A$ is related to the Atiyah class $At_{(\mathcal{E}, Q_\mathcal{E})}$ of the dg vector bundle $(\mathcal{E}, Q_\mathcal{E})$ via the isomorphism

$$\text{id}_{T_{A[1]}^*} \otimes \Phi_{1,1} : H^1(\Gamma(T_{A[1]}^* \otimes \text{End}(T_{A[1]}))) \xrightarrow{\sim} H^1(\Gamma(T_{A[1]}^* \otimes \text{End}(\mathcal{E}))).$$

Here, $\Phi_{1,1}$ is the isomorphism for the integer pair $(1, 1)$ in (3.19). Moreover, the cohomology class

$$(\Phi_{1,1} \circ \text{id}_{\text{End}(\mathcal{E})})(At_{(\mathcal{E}, Q_\mathcal{E})}) \in H^1(\Gamma(\mathcal{E} \otimes \text{End}(\mathcal{E}))) = H^1(A; E^* \otimes \text{End}(E))$$

is represented by a (formal) sum $\alpha_A + \beta_A + \alpha_B + \beta_B \in \Omega_A(E^* \otimes \text{End}(E))$ of cocycles, where

1. $\alpha_A \in \Gamma(K^*[-1] \otimes \text{End}(K[1]))$ is given by the Lie bracket $[-, -]_K$ on $\Gamma(K)$;
2. $\beta_A \in \Omega_A^1(B^* \otimes \text{End}(K[1]))$ is defined by

$$\beta_A(b, a_K[1])(a_0) = \Omega(b)(a_0, a_K)$$

$$= (R^K_r(\rho_A(a_0), j(b))a_K)[1] + (\nabla^K_r[a_K, pr_K(a_0)]_K - [\nabla^K_r(a_K), \text{pr}_K(a_0)]_K$$

$$- [a_K, \nabla^K_r(\text{pr}_K(a_0))][1]$$

for all $a_0 \in \Gamma(A)$ and $b \in \Gamma(B)$; here, $R^K_r$ is the curvature of the linear connection $\nabla^K$ on $K$;
3. $\alpha_B \in \Omega_A^1(B^* \otimes \text{End}(B))$ is given by the $(1, 1)$-component of the curvature $R^{\nabla_B}_{1,1}$ of a linear connection $\nabla_B$ on $B$, extending the Bott $F$-connection, i.e.,

$$\alpha_B(b, b')(a) = R^{\nabla_B}(\rho_A(a), b)b' = [\nabla^{\rho_A}(a), \nabla^{\rho_A}(b)]b' - \nabla^{\rho_A}(a, b)$$

for all $a \in \Gamma(A)$ and $b, b' \in \Gamma(B)$;
4. $\beta_B \in \Omega_A^2(B^* \otimes B^* \otimes K[1])$ is defined by

$$\beta_B(b_1, b_2)(a_0, a_1) = \nabla_{j(b_2)}(\Omega)(b_1; a_0, a_1)$$

$$:= \nabla^{\rho_A}_{j(b_2)}(\Omega)(b_1; a_0, a_1) - \Omega(\nabla^{\rho_A}_{j(b_1)}b_2)(a_0, a_1) - \Omega(b_2)(a_0, \nabla^{\rho_A}_{j(b_1)}a_1)$$

$$- \Omega(b_2)(\nabla^{\rho_A}_{j(b_1)}a_0, a_1)$$

for all $a_0, a_1 \in \Gamma(A)$, $b_1, b_2 \in \Gamma(B)$.

These four components satisfy the following conditions:

$$d_{CE}(\alpha_A) = 0, \quad d_{CE}(\alpha_B) = 0, \quad \Omega(\beta_A) = 0,$$

$$\Omega(\beta_B) = 0, \quad \Omega(\alpha_A) = d_{CE}(\beta_A), \quad \Omega(\alpha_B) = d_{CE}(\beta_B).$$
To compute this Atiyah class, we need a $T_{A[1]}$-connection on $\mathcal{E}$. For this purpose, we first choose a linear connection $\nabla K[1] \oplus B$ on the Whitney sum $K[1] \oplus B$ defined by $\nabla_{aK[1]}^K(bK[1] + b) := (\nabla^K_a K[1]) + \nabla^B b$ for all $a \in \Gamma(TM)$, $aK \in \Gamma(K)$ and $b \in \Gamma(B)$, where $\nabla^K$ is the chosen linear connection on $K$, and $\nabla^B$ is a linear connection $B$ extending the Bott $F$-connection. We denote by $\nabla^\epsilon$ the pullback connection of $\nabla K[1] \oplus B$ on $\mathcal{E}$ along the projection $\pi: A[1] \to M$ defined by

$$
\nabla^\epsilon_X (a \otimes (aK[1] + b)) = \mathcal{X}(a) \otimes (aK[1] + b) + (-1)^{|\mathcal{X}|}a \otimes \nabla_{\mathcal{X}(a)}^K (aK[1] + b)
$$

for all the homogeneous $\mathcal{X} \in \Gamma(T_{A[1]})$, $a \in \Omega_A$ and all $aK \in \Gamma(K), b \in \Gamma(B)$.

**Lemma 4.2.** Under the quasi-isomorphism $\Psi: (\Omega_A(K[1] \oplus B), Q_{\mathcal{E}}) \to (\Gamma(T_{A[1]}), L_{d_A})$ defined in Corollary 3.5, the Atiyah cocycle $A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon$ of $(\mathcal{E}, Q_{\mathcal{E}})$ with respect to the pullback connection $\nabla^\epsilon$ in (4.1) is given by

$$
\begin{align*}
A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(aK[1]), aK[1]) &= [aK, aK_T][1], \\
A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(aK[1]), b_1) &= A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(b_1), aK[1]) = \Omega(b_1)(-, aK) = -(R^\text{bas}_{\mathcal{X}}(-, aK)j(b_1))[1], \\
A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(b_1), b_2) &= \alpha_B(b_1, b_2) + \beta_B(b_1, b_2)
\end{align*}
$$

for all $aK, aK_T \in \Gamma(K)$ and $b_1, b_2 \in \Gamma(B)$, where $\alpha_B(b_1, b_2) \in \Omega^1_A(B)$ and $\beta_B(b_1, b_2) \in \Omega^2_A(K[1])$ are given, respectively, by

$$
\begin{align*}
\alpha_B(b_1, b_2)(a_0) &= R_{\mathcal{X}}(\rho_A(a_0), j(b_1))b_2 = \nabla_{\rho_A(a_0)}^B \nabla_{j(b_1)}^B b_2 - \nabla_{j(b_1)}^B \nabla_{\rho_A(a_0)}^B b_2 - \nabla_{\rho_A(a_0), j(b_1)}^B b_2 \\
\beta_B(b_1, b_2)(a_0, a_1) &= \nabla_{j(b_1)}^B (\Omega(b_2)(a_0, a_1)) - \Omega(\nabla_{j(b_1)}^B b_2)(a_0, a_1) - \Omega(b_2)(a_0, \nabla_{j(b_1)}^A a_1) \\
&= \Omega(b_2)(\nabla_{j(b_1)}^A a_0, a_1)
\end{align*}
$$

for all $a_0, a_1 \in \Gamma(A)$ and $b_1, b_2 \in \Gamma(B)$.

**Proof.** Using the definition of the Atiyah cocycle, we compute case by case: for all $aK, aK_T \in \Gamma(K)$, we have

$$
\begin{align*}
A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(aK[1]), aK_T[1]) &= Q_{\mathcal{E}}(\nabla^\epsilon_{\Psi(aK[1])} aK_T[1]) - \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(aK[1]))} aK_T[1] + \nabla^\epsilon_{\Psi(aK[1])} Q_{\mathcal{E}}(aK_T[1]) \quad \text{(since $\Psi$ is a cochain map)} \\
&= Q_{\mathcal{E}}(\nabla^\epsilon_{\Psi(aK[1])} aK_T[1]) - \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(aK[1]))} aK_T[1] + \nabla^\epsilon_{\Psi(aK[1])} Q_{\mathcal{E}}(aK_T[1]) \quad \text{(by (3.18) and (4.1))} \\
&= Q_{\mathcal{E}}((\nabla^\epsilon_{\Psi(aK[1])} aK_T[1]) - \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(aK[1]))} aK_T[1] + \nabla^\epsilon_{\Psi(aK[1])} Q_{\mathcal{E}}(aK_T[1]) \quad \text{(since $\Psi \circ \Omega = 0$)} \\
&= \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(aK[1])} aK_T[1]) = [aK, aK_T][1].
\end{align*}
$$

Similarly, for all $aK \in \Gamma(K)$ and $b_1 \in \Gamma(B)$, we have

$$
\begin{align*}
A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(aK[1]), b_1) &= Q_{\mathcal{E}}(\nabla^\epsilon_{\Psi(aK[1])} b_1) - \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(aK[1]))} b_1 + \nabla^\epsilon_{\Psi(aK[1])} Q_{\mathcal{E}}(b_1) \\
&= \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(aK[1])} (\Omega(b_1) - \Omega(b_1)) = -\epsilon_{aK} \Omega(b_1) = \Omega(b_1)(-, aK)
\end{align*}
$$

and

$$
\begin{align*}
A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(b_1), aK[1]) &= Q_{\mathcal{E}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) - \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(b_1))} aK[1] + \nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= Q_{\mathcal{E}}((\nabla^\epsilon_{\Psi(b_1)} aK[1]) - \nabla^\epsilon_{d_{\mathcal{X}}(\Psi(b_1))} aK[1] + \nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = (\nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = (\nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = (\nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = (\nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
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&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = (\nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = (\nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = [aK, aK_T][1],
\end{align*}
$$

and thus we have that for all $a_0 \in \Gamma(A)$,

$$
A_{(\mathcal{E}, Q_{\mathcal{E}})}^\epsilon (\Psi(b_1), aK[1]) = (\nabla^\epsilon_{\Psi(b_1)} Q_{\mathcal{E}}(aK[1]) \\
&= d_{\mathcal{X}}(\nabla^\epsilon_{\Psi(b_1)} aK[1]) = [aK, aK_T][1].
$$
For each linear connection $s_1$ satisfying for all $a_1$, we have

$$ \Gamma_{\epsilon,\Omega_{\epsilon}}(\Psi(b_1), a_1[1]) = \Omega(b_1)(-a_1, a_K) = \Gamma_{\epsilon,\Omega_{\epsilon}}(\Psi(a_K[1]), b_1). $$

Finally, for all $b_1, b_2 \in \Gamma(B)$, we have

$$ \Gamma_{\epsilon,\Omega_{\epsilon}}(\Psi(b_1), b_2) = Q_{\epsilon}(\nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{\Psi(b_1)} Q_{\epsilon}(b_2) (\text{since } \Psi \text{ is a cochain map}) $$

$$ = Q_{\epsilon}(\nabla_{\epsilon,\Psi(b_1)} b_2) - \nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{\Psi(b_1)} Q_{\epsilon}(b_2) (\text{by (3.18) and (4.1)}) $$

$$ = d_{CE}(\nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{d_{CE}(b_1)} b_2 - \nabla_{\epsilon,\Psi(b_1)} d_{CE}(b_2) + \nabla_{\epsilon,\Psi(b_1)} \Omega(b_2) - \Omega(\nabla_{\epsilon,\Psi(b_1)} b_2). $$

These five terms are separated into the following two parts:

$$ \alpha_B(b_1, b_2) := d_{CE}(\nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{d_{CE}(b_1)} b_2 - \nabla_{\epsilon,\Psi(b_1)} d_{CE}(b_2) \in \Omega_A(B), $$

$$ \beta_B(b_1, b_2) := \nabla_{\epsilon,\Psi(b_1)} \Omega(b_2) - \Omega(\nabla_{\epsilon,\Psi(b_1)} b_2) \in \Omega^2(K[1]) $$

satisfying for all $a_0, a_1 \in \Gamma(A)$,

$$ \alpha_B(b_1, b_2)(a_0) $$

$$ = \nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{d_{CE}(b_1)} b_2 - \nabla_{\epsilon,\Psi(b_1)} d_{CE}(b_2) \in \Omega_A(B), $$

$$ = \nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{d_{CE}(b_1)} b_2 - \nabla_{\epsilon,\Psi(b_1)} d_{CE}(b_2) \in \Omega_A(B), $$

$$ = \nabla_{\epsilon,\Psi(b_1)} b_2 - \nabla_{d_{CE}(b_1)} b_2 - \nabla_{\epsilon,\Psi(b_1)} d_{CE}(b_2) \in \Omega^2(K[1]) $$

and

$$ \beta_B(b_1, b_2)(a_0, a_1) = (\nabla_{\epsilon,\Psi(b_1)} \Omega(b_2))(a_0, a_1) - \Omega(\nabla_{\epsilon,\Psi(b_1)} b_2)(a_0, a_1) $$

$$ = \nabla_{\epsilon,\Psi(b_1)} \Omega(b_2)(a_0, a_1) - \Omega(\nabla_{\epsilon,\Psi(b_1)} b_2)(a_0, a_1) $$

$$ = \nabla_{\epsilon,\Psi(b_1)} \Omega(b_2)(a_0, a_1) - \Omega(\nabla_{\epsilon,\Psi(b_1)} b_2)(a_0, a_1). $$

This completes the proof. 

Theorem 4.1 is an immediate consequence of Proposition 2.2 and Lemma 4.2.

4.2 Examples

4.2.1 Bundles of Lie algebras

Recall that a bundle $K$ of Lie algebras over $M$ is a regular Lie algebroid $(K, [-,-]_K)$ with zero anchor. For each linear connection $\nabla^K$ on $K$, the isomorphism in (3.5) becomes

$$ (\Gamma(T_{K[1]}), L_{\nabla^K}) \overset{\cong}{\rightarrow} \left( \Omega_K(K[1] \oplus T_M), D_{\nabla^K} = \begin{pmatrix} d_{CE} & \Omega \\ 0 & 0 \end{pmatrix} \right), $$

(4.2)
where \( d^K_{\text{CE}} : \Omega^*_K(K[1]) \to \Omega^{*+1}_K(K[1]) \) is the Chevalley-Eilenberg differential of the \( K \)-module \( K[1] \), and \( \Omega \) is given by the basic curvature \( R^{\text{bas}}_{\Omega} \in \Omega^*_K(\text{Hom}(T_M, K[1])) \) defined by
\[
\Omega(u)(x, y) = -(R^{\text{bas}}_{\Omega}(x, y)u)[1] := -([\nabla^K u]_x y)_{K} - [\nabla^K y]_x y - [x, \nabla^K y]_K[1]
\]
for all \( x, y \in \Gamma(K) \) and \( u \in \Gamma(T_M) \). It is clear that the basic curvature, in this case, measures the compatibility between \( \nabla^K \) and the Lie bracket \([-,-]_K\).

As an immediate consequence of Theorem 4.1, we obtain the following description on the Atiyah class of \( \Omega \):

**Proposition 4.3.** Let \((K, [-,-]_K)\) be a bundle of Lie algebras over \( M \). Given a linear connection \( \nabla^K \) on \( K \), under the isomorphism of cohomologies induced from (4.2),
\[
H^1(\Gamma(T^*_K[1] \otimes \text{End}(T_K[1])), L_{d_K}) \xrightarrow{\cong} H^1(K, (T^*_M \otimes K^*[1]) \otimes \text{End}(K[1] \oplus T_M)),
\]
where the right-hand side denotes the cohomology of the representation up to homotopy of \( K \) on the tensor product \((T^*_M \otimes K^*[1]) \otimes \text{End}(K[1] \oplus T_M)\) of graded vector bundles over \( M \), and the Atiyah class \( \text{At}(K[1], d_K) \) of the associated dg manifold \((K[1], d_K)\) is represented by the sum of three terms \( \alpha_K + \beta_K + \beta_M \). Here,

1. \( \alpha_K \in H^1(K, (T^*_M \otimes K)[1]) \) is given by the Lie bracket \([-,-]_K\);
2. \( \beta_K \in H^1(K, (T^*_M \otimes K)[1]) \) is defined by
\[
\beta_K(u, x[1])(y) := (R^{\text{bas}}_{\Omega}(x, y)u)[1] = ([\nabla^K u]_x y)_{K} - [\nabla^K y]_x y - [x, \nabla^K y]_K[1]
\]
for all \( u \in \Gamma(T_M) \) and \( x, y \in \Gamma(K) \);
3. \( \beta_M \in H^1(K, (T^*_M \otimes T_M \otimes K)[1]) \) is defined by
\[
\beta_M(u, v)(x, y) = -[\nabla^K u]_x v(x, y) + \beta_K([\nabla^K v]_x y, x[1])(y) + \beta_K(v, [\nabla^K y]_x x[1])(y) + [\nabla^K y]_x v(x, y)
\]
for all \( u, v \in \Gamma(T_M) \) and \( x, y \in \Gamma(K) \). Here, \( \nabla^K \) is an affine connection on \( M \).

In particular, if \((K, [-,-]_K)\) is a Lie algebra bundle, i.e., the fiber Lie algebra is fixed in local trivialization, then there exists a linear connection \( \nabla^K \) on \( K \) such that its basic curvature \( R^{\text{bas}}_{\Omega} \) vanishes (see [1, Proposition 2.13]). Thus, we obtain the following corollary.

**Corollary 4.4.** Let \((K, [-,-]_K)\) be a Lie algebra bundle. Then the Atiyah class \( \text{At}(K[1], d_K) \) of the associated dg manifold \((K[1], d_K)\) is represented by the Lie bracket \([-,-]_K\), viewed as a degree 1 element in \( H^1(K^*[1] \otimes \text{End}(K[1])) \).

### 4.2.2 Integrable distributions

Consider a regular Lie algebroid \( \mathcal{A} \) whose anchor map \( \rho \) is injective. In this case, \( \mathcal{A} \) is identified with its characteristic distribution \( F := \rho(A) \subseteq T_M \), the tangent bundle of a regular foliation \( \mathcal{F} \subseteq M \). Note that \((T_M, F)\) is a Lie algebroid pair (or Lie pair for short) over \( M \). We briefly recall from [3] the regular class of the Lie pair \((T_M, F)\). For each splitting \( j : B = T_M/F \to T_M \) of the short exact sequence (3.7) and a linear connection \( \nabla^B \) on \( B \) extending the Bott \( F \)-connection, there is a Chevalley-Eilenberg 1-cocycle of \( F \):
\[
\text{At}^B \in C^1(F; B^* \otimes \text{End}(B)) = \Gamma(F^* \otimes B^* \otimes \text{End}(B)),
\]
defined by
\[
\text{At}^B(u_F, b_1)b_2 := R_F^{B}(u_F, j(b_1))b_2 = \nabla^B_{u_F} \nabla^B_{j(b_1)}b_2 - \nabla^B_{j(b_1)} \nabla^B_{u_F}b_2 - \nabla^B_{u_F \cdot j(b_1)}b_2
\]
for all \( u_F \in \Gamma(F) \) and \( b_1, b_2 \in \Gamma(B) \). The cohomology class
\[
\text{At}_B = [\text{At}^B] \in H^1_{\text{CE}}(F; B^* \otimes \text{End}(B))
\]
does not depend on the choice of \( j \) and \( \nabla^B \), and is called the Atiyah class of the Lie pair \((T_M, F)\), which is also known as the Molino class [18] of the foliation \( \mathcal{F} \) induced from \( F \). Applying Theorem 4.1 to this case, we rediscover the following result on Atiyah classes of integrable distributions.
Proposition 4.5 (See [4]). Let $F \subseteq T_M$ be an integrable distribution, i.e., the tangent bundle of a regular foliation in $M$. Then the isomorphism $\Phi_{2,1}$ for the integer pair $(2,1)$ in (3.19), which now becomes

$$\Phi_{2,1}: H^1(\Gamma(T_F^*[1] \otimes \text{End}(T_{[1]}[1]))) \xrightarrow{\sim} H^1(\Omega_F(B^* \otimes \text{End}(B)), d_{CE}) = H^1_{CE}(F; B^* \otimes \text{End}(B)),$$

sends the Atiyah class $\text{At}_{[1]}[1]$ of the dg manifold $(F[1], d_F)$ to the Atiyah class $\text{At}_B$ of the Lie pair $(T_M, F)$.

4.3 Functoriality with respect to the Atiyah sequence

Given three Atiyah classes $\text{At}_{(K[1],d_K)}$, $\text{At}_{(A[1],d_A)}$, and $\text{At}_{(F[1],d_F)}$. We now study their relationship.

By Theorem 4.1, the Atiyah class $\text{At}_{(A[1],d_A)}$ is related to $\text{At}_{(\mathcal{E},Q_K)}$ via the isomorphism

$$\text{At}_{(A[1],d_A)} \in H^1(\Gamma(T_A[1] \otimes \text{End}(T_A[1]))) \cong H^1(A; E^* \otimes \text{End}(E)) \cong H^1(\Gamma(T_A[1] \otimes \text{End}(E))) \cong \text{At}_{(\mathcal{E},Q_K)}.$$

Note that the inclusion $i$ in the sequence (4.3) induces an inclusion $i: (K[1], d_K) \hookrightarrow (A[1], d_A)$ of dg manifolds. The restriction of the homological vector field $Q_K$ onto

$$\mathcal{E} |_{K[1]} = i^* \mathcal{E}, \quad Q_K |_{K[1]} = \begin{pmatrix} d_{CE} & -i^* \Omega \\ 0 & 0 \end{pmatrix} : \Gamma(\mathcal{E} |_{K[1]}) = \Omega_K(K[1] \otimes B) \rightarrow \Omega_K(K[1] \otimes B)[1]$$

makes $(\mathcal{E} |_{K[1]}, Q_K |_{K[1]})$ into a dg vector bundle over the dg manifold $(K[1], d_K)$. Here, $d_{CE}$ is the Chevalley-Eilenberg differential of the adjoint $K$-module $K[1]$, and $i^* \Omega \in \Omega^2_K(Hom(B,K[1]))$ is the pullback of $\Omega$ along the inclusion $i$.

Theorem 4.6. Let $A$ be a regular Lie algebroid with the Atiyah sequence (4.3).

1. The standard projection map

$$\text{Pr}: H^1(A; E^* \otimes \text{End}(E)) \rightarrow H^1_{CE}(A; B^* \otimes \text{End}(B))$$

sends the Atiyah class $\text{At}_{(\mathcal{E},Q_K)} \in H^1(\Gamma(T_A[1] \otimes \text{End}(E))) \cong H^1(A; E^* \otimes \text{End}(E))$ of the dg vector bundle $(\mathcal{E},Q_K)$, which is related to the Atiyah class $\text{At}_{(A[1],d_A)}$ of $A$ via an isomorphism according to Theorem 4.1, to the pullback class $\rho_A^*(\text{At}_B)$ of the Atiyah class $\text{At}_B \in H^1_{CE}(F; B^* \otimes \text{End}(B))$ of the Lie pair $(T_M, F)$, the latter of which is also known as the Molino class of $F$. It is related via an isomorphism to the Atiyah class $\text{At}_{(\mathcal{F}[1],d_F)}$ of the dg manifold $(F[1], d_F)$ according to Proposition 4.5.

2. For each splitting $j: B \rightarrow T_M$ of the short exact sequence (3.7), there exists an inclusion

$$H^1(K; E^* \otimes \text{End}(E)) \hookrightarrow H^1(K; T_M^* \otimes \text{End}(K[1] \otimes T_M)) \cong H^1(\Gamma(T_K^*[1] \otimes \text{End}(T_K[1])))$$

The Atiyah class $\text{At}_{(K[1],d_K)}$ of the bundle $K$ of Lie algebras, which lives in $H^1(K; E^* \otimes \text{End}(E))$, is equal to the Atiyah class of the dg vector bundle $(\mathcal{E} |_{K[1]}, Q_K |_{K[1]})$ over $(K[1], d_K)$:

$$\text{At}_{(\mathcal{E} |_{K[1]}, Q_K |_{K[1]})} \in H^1(K; E^* \otimes \text{End}(E)) \subset H^1(K; T_M^* \otimes K^*[1] \otimes \text{End}(E)) \xrightarrow{\sim} H^1(\Gamma(T_K^*[1] \otimes \text{End}(\mathcal{E} |_{K[1]}))).$$

Therefore, the restriction map

$$i^*: H^1(A; E^* \otimes \text{End}(E)) \rightarrow H^1(K; E^* \otimes \text{End}(E)) \hookrightarrow H^1(\Gamma(T_K^*[1] \otimes \text{End}(T_K[1])))$$

sends the Atiyah class $\text{At}_{(\mathcal{E},Q_K)}$ to the Atiyah class $\text{At}_{(K[1],d_K)}$. 


Hence, applying Theorem 4.1, we obtain
\[ \nabla^M_u j(b) = \text{pr}_B[u_F, j(b)], \quad \nabla^M_u F = \nabla^F_u F \]
for all \( u_F \in \Gamma(F), \ u \in \Gamma(T_M) \) and \( b \in \Gamma(B) \). The dual map \( \text{pr}^*_B : B^* \to T^*_M \) of the projection \( \text{pr}_B : T_M \hookrightarrow B \) and the chosen splitting \( j : B \to T_M \) induce an inclusion of the cohomology spaces
\[ H^1(K; E^* \otimes \text{End}(E)) \hookrightarrow H^1(K; (T^*_M \otimes K^*[-1]) \otimes \text{End}(K[1] \otimes T_M)). \]

By Proposition 4.3, the Atiyah class \( \text{At}(K[1], d_K) \) is represented by a formal sum \( \alpha_K + \beta_K + \beta_M \). Here, \( \alpha_K \) is the Lie bracket \([\cdot, \cdot]_K\) on \( \Gamma(K) \), and the element \( \beta_K \in \Omega^1_K(T_M^* \otimes \text{End}(K[1])) \) satisfies that for all \( x, y \in \Gamma(K) \) and \( u_F \in \Gamma(F) \),
\[ \beta_K(u_F, x[1])(y) = (\nabla^K_u[x, y]_K - \nabla^K_{u_F}x, y)_K - [x, \nabla^K_{u_F}y]_K)[1] \]
\[ = ([\tau(u_F), x, y]_A - [[\tau(u_F), x], y]_A - [x, [\tau(u_F), y]]_A) \]
(4.4)

Thus, we have
\[ \beta_K = i^*\beta_A \in \Omega^1(K; B^* \otimes \text{End}(K[1])) \subset \Omega^1(K; T^*_M \otimes \text{End}(K[1])). \]

Meanwhile, by (4.4), the element \( \beta_M \in \Omega^1_K(T_M^* \otimes T_M^* \otimes K[1]) \) satisfies
\[ \beta_M(u, v_F)(x, y) \]
\[ = \nabla^K_u[1](\beta_K(v_F; x[1])(y) - \beta_K(\nabla^K_u v_F, x[1])(y) - \beta_K(v_F, (\nabla^K_u x)[1])(y) - \beta_K(v_F, x[1])(\nabla^K_u y) \]
\[ = 0 \]
for all \( u \in \Gamma(T_M), \ v_F \in \Gamma(F) \) and \( x, y \in \Gamma(K) \), and
\[ \beta_M(u_F, j(b))(x, y) = \nabla^K_{u_F}[1](\beta_K(j(b); x[1])(y)) \]
\[ - \beta_K(\nabla^K_{u_F} j(b), x[1])(y) - \beta_K(\nabla^K_{u_F} j(b), (\nabla^K_{u_F} x)[1])(y) - \beta_K(j(b), x[1])(\nabla^K_{u_F} y) \]
\[ = (R^K_K(\nabla^K_{u_F} j(b), x, y)_K - R^K_K(u_F, j(b)) x, y)_K - [x, R^K_K(u_F, j(b)) y]_K) \]
\[ = (u^K_C R^K_K(u_F, j(b))(x, y)) \]
for all \( u_F \in \Gamma(F), \ b \in \Gamma(B) \) and \( x, y \in \Gamma(K) \). Thus, we also have
\[ \beta_M = i^*\beta_B \in H^1(K; B^* \otimes B^* \otimes K[1]) \subset H^1(K; T^*_M \otimes T^*_M \otimes K[1]). \]

Hence, applying Theorem 4.1, we obtain
\[ \text{At}(K[1], d_K) = \alpha_K + \beta_K + \beta_M = i^*(\alpha_A + \beta_A + \beta_B) \]
\[ = i^*(\text{At}(\xi, Q_\ell)) = \text{At}(\xi|_{K[1]}, Q_\ell|_{K[1]}). \]

This completes the proof. \( \square \)

As an immediate consequence, we obtain the following vanishing result.

**Corollary 4.7.** If the Atiyah class \( \text{At}(A[1], d_A) \) of a regular Lie algebroid \( A \) vanishes, then both the Atiyah class \( \text{At}(K[1], d_K) \) of the bundle \( K = \ker \rho_A \) of Lie algebras and the Atiyah class \( \text{At}(F[1], d_F) \) of the characteristic distribution \( F = \text{Im} \rho_A \) vanish.

5 Scalar Atiyah and Todd classes

5.1 Scalar Atiyah classes

Now we study the scalar Atiyah classes of the dg manifold \( (A[1], d_A) \). As the first step, by taking the tensor product on the isomorphism in Corollary 3.7, we obtain an isomorphism between the cohomology
of differential $k$-forms on the dg manifold $(A[1], d_A)$ of total degree $k$ and the cohomology of the representation up to homotopy of $A$ on the graded vector bundle $\wedge^k E^* \cong \wedge^k (K^*[-1] \oplus B^*)$ of total degree $k$, i.e.,

$$H^k(\Omega(1), L_{d_A}) \cong H^k(A; \wedge^k E^*) = \bigoplus_{q=0}^k H^k(A; \wedge^{k-q} B^* \otimes (S^q K^*)[-q]).$$

(5.1)

Recall that the differential on $\Omega_A(\wedge^k E^*)$ is induced by the Leibniz rule from the differential $Q_{CE} = d_{CE} + \Omega^*$ on $E^*$, where $\Omega^* \in \Omega^2_A(\text{Hom}(K^*[-1], B^*))$ is the $\Omega_A$-linear dual of $\Omega \in \Omega_A^2(\text{Hom}(B, K[1]))$. The projection onto the $(q = 0)$-component defines a cochain map

$$\text{Pr}: \bigoplus_{q=0}^k \Omega_A(\wedge^{k-q} B^* \otimes (S^q K^*)[-q], Q_{CE}) \to (\Omega_A(\wedge^k B^*), d_{CE}),$$

and thus induces a projection from the cohomology $H^k(A; \wedge^k E^*)$ of the representation up to homotopy of $A$ on the graded vector bundle $\wedge^k E^*$ of total degree $k$ to the $k$-th Chevalley-Eilenberg cohomology $H^k_{CE}(A; \wedge^k B^*)$ of the $A$-module $\wedge^k B^*$:

$$\text{Pr}: H^k(A; \wedge^k E^*) \to H^k_{CE}(A; \wedge^k B^*).$$

(5.2)

Note that each homogeneous element $T \in \Omega_A(E^* \otimes \text{End}(E))$ has the matrix form

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where

$$T_1 \in \Omega_A(E^* \otimes \text{End}(K[1])), \quad T_2 \in \Omega_A(E^* \otimes \text{Hom}(B, K[1])), \quad T_3 \in \Omega_A(E^* \otimes \text{Hom}(K[1], B)), \quad T_4 \in \Omega_A(E^* \otimes \text{End}(B)).$$

The supertrace $\text{str}(T)$ of $T$ is defined by $\text{str}(T) := \text{tr}(T_4) - \text{tr}(T_1) \in \Omega_A(E^*)$. With the help of Theorem 4.1, we obtain the following description on the scalar Atiyah classes of $(A[1], d_A)$.

**Proposition 5.1.** Let $A$ be a regular Lie algebroid. For each choice of quadruple $(j, \tau, \nabla^K, \nabla^F)$, under the isomorphism in (5.1), the $k$-th scalar Atiyah class $\text{ch}_A(A[1], d_A)$ of the associated dg manifold $(A[1], d_A)$ is represented by

$$\frac{1}{k!} \left( \frac{i}{2\pi} \right)^k \left( \text{tr}(\alpha_B^k) - \sum_{q=0}^{k-1} \frac{k!}{q!(k-q)!} \text{tr}(\alpha_A^q \beta_A^{k-q}) \right),$$

where $\alpha_A \in \Gamma(K^*[-1] \otimes \text{End}(K[1]))$, $\beta_A \in \Omega_A^1(B^* \otimes \text{End}(K[1]))$ and $\alpha_B \in \Omega_A^1(B^* \otimes \text{End}(B))$ are elements defined in Theorem 4.1 satisfying the conditions

$$\text{tr}(\alpha_B^k) \in \Omega_A^k(\wedge^k B^*) \subset \Omega_A^k(\wedge^k E^*),$$

$$\text{tr}(\alpha_A^q \beta_A^{k-q}) \in \Omega_A^{k-q}(\wedge^{k-q} B^* \otimes (S^q K^*)[-q])$$

and

$$\frac{k!}{q!(k-q)!} d_{CE}(\text{tr}(\alpha_A^q \beta_A^{k-q})) = \frac{k!}{(q+1)!(k-q-1)!} \Omega(\text{tr}(\alpha_A^{q+1} \beta_A^{k-q-1})).$$

for all $0 \leq q \leq k$.

**Proof.** By Theorem 4.1, the $k$-th power of the Atiyah class is represented by

$$\begin{pmatrix} (\alpha_A + \beta_A)^k \\ 0 \end{pmatrix},$$

where
where \( \alpha_A \in \Gamma(K^*[-1] \otimes \mathrm{End}(K[1])) \) is given by the Lie bracket \([-,-]_K \) on \( \Gamma(K) \), \( \beta_A \in \Omega^1_A(B^* \otimes \mathrm{End}(K[1])) \) and \( \alpha_B \in \Omega^1_A(B^* \otimes \mathrm{End}(B)) \). Thus, we have

\[
\mathrm{str}(\mathrm{At}^k_{(A[1],d_A)}) = \frac{1}{k!} \left( \frac{i}{2\pi} \right)^k \mathrm{tr}(\mathrm{At}^k_B),
\]

Substituting into the definition of scalar Atiyah classes, we complete the proof. \( \square \)

As an immediate application, we recall that the scalar Atiyah classes of the Lie pair \((T_M, F)\) (see [3]) is defined by

\[
\chi_k(B) := \frac{1}{k!} \left( \frac{i}{2\pi} \right)^k \mathrm{tr}(\mathrm{At}^k_B),
\]

where \( \mathrm{At}^k_B \) denotes the image of \( \pi^* \mathrm{At}^k \) under the natural map

\[
H^1_{CE}(F; B^* \otimes \mathrm{End}(B)) \times \cdots \times H^1_{CE}(F; B^* \otimes \mathrm{End}(B)) \to H^k_{CE}(F; \wedge^k B^*)
\]

induced by the composition in \( \mathrm{End}(B) \) and the wedge product in \( \wedge B^* \). Hence, applying Propositions 4.5 and 5.1 to this particular regular Lie algebroid \( F \), we rediscover the following result on scalar Atiyah classes of integrable distributions.

**Proposition 5.2** (See [4]). Let \( F \subseteq T_M \) be an integrable distribution, i.e., the tangent bundle of some regular foliation in \( M \). Then the isomorphism (5.1) becomes \( H^k(\Omega^k(F[1]), L_{d_F}) \cong H^k_{CE}(F; \wedge^k B^*) \), which sends the scalar Atiyah classes \( \chi_k(F[1], d_F) \) of the dg manifold \((F[1], d_F)\) to the scalar Atiyah classes \( \chi_k(B) \) of the Lie pair \((T_M, F)\).

Finally, using Theorem 4.6 and Proposition 5.1, we see that scalar Atiyah classes of dg manifolds arising from a regular Lie algebroid respect the associated Atiyah sequence.

**Proposition 5.3.** Let \( A \) be a regular Lie algebroid with the Atiyah sequence (4.3). For each positive integer \( k \),

1. the projection map \( \Pr \) in (5.2) sends the \( k \)-th scalar Atiyah class of \( A \):

\[
\chi_k(A[1], d_A) \in H^k(\Omega^k(A[1]), L_{d_A}) \cong H^k(A; \wedge^k E^*)
\]

to the pullback class \( \rho^* \chi_k(B) \) of the \( k \)-th scalar Atiyah class \( \chi_k(B) \) of the Lie pair \((T_M, F)\), which is related to the \( k \)-th scalar Atiyah class \( \chi_k(F[1], d_F) \) of the integrable distribution \( F \) by an isomorphism according to Proposition 5.2;

2. the restriction map

\[
i^* : H^k(A; \wedge^k E^*) \to H^k(K; \wedge^k E^*) \subseteq H^k(\Omega^k(K[1]), L_{d_K})
\]

induced from the inclusion \( i : K \hookrightarrow A \) sends the \( k \)-th scalar Atiyah class \( \chi_k(A[1], d_A) \) of \( A \) to the \( k \)-th scalar Atiyah class \( \chi_k(K[1], d_K) \) of \( K \).

### 5.2 Todd classes

We now study the Todd class of the dg manifold \((A[1], d_A)\). Let

\[
P(x) = \frac{x}{1 - e^{-x}} = \sum_{k \geq 0} \frac{(-1)^k}{k!} B_k x^k,
\]

where \( B_k \) is the \( k \)-th Bernoulli number.

**Proposition 5.4.** Let \( A \) be a regular Lie algebroid. For each choice of quadruple \((j, \tau, \nabla^K, \nabla^F)\), under the isomorphism

\[
\prod_{k \geq 0} H^k(\Omega^k(A[1]), L_{d_A}) \cong \prod_{k \geq 0} H^k(A; \wedge^{k-q} B^* \otimes (S^q K^*)[-q]),
\]
the Todd class $\text{Td}_{(A[1], d_A)}$ of the dg manifold $(A[1], d_A)$ arising from a regular Lie algebroid $A$ is represented by 

$$\det(P(\alpha_B)) \det(P^{-1}(\alpha_A + \beta_A)) = \det(P(\alpha_B)) \exp \left( \sum_{k \geq 1} \frac{B_k}{k!} \text{tr}(\alpha_A^q \beta_A^{k-q}) \right),$$

where $\alpha_A \in \Gamma(K^- [-1] \otimes \text{End}(K[1]))$, $\beta_A \in \Omega_A^1(B^* \otimes \text{End}(K[1]))$ and $\alpha_B \in \Omega_A^1(B^* \otimes \text{End}(B))$ are elements defined in Theorem 4.1.

**Proof.** Substituting the Atiyah cocycle $\text{At}^\nabla := \alpha_A + \beta_A + \alpha_B + \beta_B$ representing the Atiyah class $\text{At}(\xi, Q) \cong \text{At}_{(A[1], d_A)}$ as in Theorem 4.1 into the definition of Todd classes, we see that the Todd class $\text{Td}_{(A[1], d_A)}$ is indeed represented by

$$\text{Ber}(P(\text{At}^\nabla)) = \text{Ber} \left( P \left( \begin{array}{c} \alpha_A + \beta_A & \alpha_A + \beta_B \\ 0 & \alpha_B \end{array} \right) \right)$$

$$= \text{Ber} \left( P(\alpha_A + \beta_A) \ast P(\alpha_B) \right)$$

$$= \det(P(\alpha_B)) \det(P^{-1}(\alpha_A + \beta_A)).$$

Note further that we can rewrite the polynomial $P(x)$ in the following way (see [7]):

$$P(x) = \exp \left( - \sum_{k \geq 1} \frac{B_k}{k!} x^k \right).$$

It follows that

$$\text{Ber}(P(\text{At}^\nabla)) = \text{Ber} \left( \exp \left( - \sum_{k \geq 1} \frac{B_k}{k!} \text{At}_{(A[1], d_A)}^k \right) \right)$$

$$= \exp \left( - \sum_{k \geq 1} \frac{B_k}{k!} \text{str}(\text{At}_{(A[1], d_A)}^k) \right) \quad \text{(by (5.3))}$$

$$= \exp \left( - \sum_{k \geq 1} \frac{B_k}{k!} \text{tr}(\alpha_A^k) \right) \exp \left( \sum_{k \geq 1} \sum_{q=0}^k \frac{B_k}{k!} \text{tr}(\alpha_A^q \beta_A^{k-q}) \right)$$

$$= \det(P(\alpha_B)) \exp \left( \sum_{k \geq 1} \sum_{q=0}^k \frac{B_k}{k!} \text{tr}(\alpha_A^q \beta_A^{k-q}) \right).$$

This completes the proof. \hfill \Box

Recall that the Todd class [3] of the Lie pair $(T_M, F)$ is the cohomology class

$$\text{Td}_B = \det(P(\text{At}_B)) = \det \left( \frac{\text{At}_B}{1 - e^{-\text{At}_B}} \right) \in \bigoplus_{k \geq 0} H^k_{CE}(F; \wedge^k B^*).$$

Applying Theorem 4.1 and Proposition 5.4, we rediscover the following result on Todd classes of integrable distributions.

**Corollary 5.5** (See [4]). *Let $F \subseteq T_M$ be an integrable distribution, i.e., the tangent bundle of some regular foliation in $M$. Then the isomorphism

$$\prod_{k \geq 0} H^k(\Omega^k(F[1]), L_{d_F}) \cong \prod_{k \geq 0} H^k_{CE}(F; \wedge^k B^*)$$

sends the Todd class $\text{Td}_{(F[1], d_F)}$ of the dg manifold $(F[1], d_F)$ to the Todd class $\text{Td}_B$ of the Lie pair $(T_M, F)$.***
Finally, applying Theorem 4.6 and Proposition 5.4, we see that Todd classes of dg manifolds arising from a regular Lie algebroid respect the associated Atiyah sequence as well.

**Proposition 5.6.** Let $A$ be a regular Lie algebroid with the Atiyah sequence (4.3).

1. The projection map

$$\text{Pr}: \prod_{k \geq 0} \bigoplus_{q=0}^{k} H^k(A; \wedge^{k-q} B^* \otimes (S^q K^*)[-q]) \to \prod_{k \geq 0} H^k_{CE}(A; \wedge^K B^*)$$

induced from (5.2) sends the Todd class $\text{Td}_{\text{A}[\text{I}],d_A}$ of $A$ to the pullback class $\rho^*_{\text{A}} \text{Td}_B$ of the Todd class $\text{Td}_B$ of the Lie pair $(T_M,F)$, the latter of which is related to the Todd class $\text{Td}_{\text{I}[\text{I}],d_F}$ of the integrable distribution $F$ via an isomorphism by Corollary 5.5.

2. Under the isomorphisms

$$\prod_{k \geq 0} H^k(\Omega^k(A[1]), L_{d_A}) \cong \prod_{k \geq 0} \bigoplus_{q=0}^{k} H^k(A; \wedge^{k-q} B^* \otimes (S^q K^*)[-q])$$

and

$$\prod_{k \geq 0} H^k(\Omega^k(K[1]), L_{d_K}) \cong \prod_{k \geq 0} \bigoplus_{q=0}^{k} H^k(A; \wedge^{k-q} T^*_M \otimes (S^q K^*)[-q]),$$

the restriction map

$$\prod_{k \geq 0} \bigoplus_{q=0}^{k} H^k(A; \wedge^{k-q} B^* \otimes (S^q K^*)[-q]) \to \prod_{k \geq 0} \bigoplus_{q=0}^{k} H^k(K; \wedge^{k-q} T^*_M \otimes (S^q K^*)[-q])$$

induced from the inclusion $i: K \hookrightarrow A$ and the projection $\text{pr}_B: T_M \to B$ sends the Todd class $\text{Td}_{\text{A}[\text{I}],d_A}$ of $A$ to the Todd class $\text{Td}_{\text{I}[\text{I}],d_K}$ of $K$ represented by

$$\det(P^{-1}(\alpha_K + \beta_K)) = \det(P^{-1}(\alpha_K)) \exp \left( \sum_{k \geq 1} \sum_{q=0}^{k-1} B_k \frac{\text{tr}(\alpha_K \beta_K^{k-q})}{q!(k-q)!} \right),$$

where

$$\det(P^{-1}(\alpha_K)) = \frac{1 - e^{-\alpha_K}}{\alpha_K} \in \prod_{k \geq 0} \Gamma((S^k K^*)[-k])$$

consists of Duflo elements of this bundle of Lie algebras.

### 5.3 The application to locally splittable cases

Assume that $A$ is a locally splittable regular Lie algebroid, i.e., the characteristic class of $A$

$$[\omega] \in H^2_{CE}(A; \text{Hom}(B, K))$$

vanishes. According to [6, Proposition 7.2], one can choose a quadruple $(\tau, j, \nabla^K, \nabla^F)$ such that $\Omega = 0$. By Corollary 3.5, we obtain a contraction

$$H^* \subset \bigwedge^+ (T_{A[1]}, L_{d_A}) \xrightarrow{\psi} (E = \pi^*(K[1] \oplus B), Q_E = d_{CE}),$$

and thus an isomorphism on the cohomology level

$$H^*(\Gamma(T_{A[1]}, L_{d_A})) \cong H^*_{CE}(A; E) = H^*_{CE}(A; K[1] \oplus B) = \bigoplus_n H^n_{CE}(A; K[1] \oplus B),$$

where $H^n_{CE}(A; K[1] \oplus B)$ is the Chevalley-Eilenberg cohomology of the graded $A$-module $K[1] \oplus B$ of total degree $n$. 
By taking the dual and the tensor product, we obtain an isomorphism

\[ H^*(\Gamma(T^*_A \otimes \text{End}(T_A)), L_{d_A}) \cong H^*_\text{CE}(A; E^* \otimes \text{End}(E)) = H^*_\text{CE}(A; (B^* \oplus K^*[−1]) \otimes \text{End}(K[1] \oplus B)). \]

In particular, we have

\[ H^1(\Gamma(T^*_A \otimes \text{End}(T_A)), L_{d_A}) \cong H^1_{\text{CE}}(A; (B^* \oplus K^*[−1]) \otimes \text{End}(K[1] \oplus B)) \]

\[ = H^1_{\text{CE}}(A; K^*[−1] \otimes \text{End}(K[1])) \oplus H^1_{\text{CE}}(A; (B^* \oplus K^*[−1]) \otimes \text{Hom}(B, K[1])) \]

\[ \oplus H^1_{\text{CE}}(A; B^* \otimes \text{Hom}(K[1], B)) \oplus H^1_{\text{CE}}(A; B^* \otimes \text{End}(B)). \]

Under this isomorphism, we write each class \( w \in H^1(\Gamma(T^*_A \otimes \text{End}(T_A)), L_{d_A}) \) in the following matrix form:

\[
\begin{pmatrix}
    w_1 & w_2 \\
    w_3 & w_4
\end{pmatrix},
\]

where

\[ w_1 \in H^1_{\text{CE}}(A; K^*[−1] \otimes \text{End}(K[1])), \quad w_2 \in H^1_{\text{CE}}(A; (B^* \oplus K^*[−1]) \otimes \text{Hom}(B, K[1])), \]

\[ w_3 \in H^1_{\text{CE}}(A; B^* \otimes \text{Hom}(K[1], B)), \quad w_4 \in H^1_{\text{CE}}(A; B^* \otimes \text{End}(B)). \]

Meanwhile, in this case, the isomorphism (5.1) of cohomology of \( k \)-forms on the dg manifold \((A[1], d_A)\) becomes

\[ H^k(\Omega^k(A[1]), L_{d_A}) \cong \bigoplus_{q=0}^k H^*_\text{CE}(A; \wedge^{k−q} B^* \otimes (S^q K^*)[−q]). \]

**Lemma 5.7.** Let \( A \) be a locally splittable regular Lie algebroid over \( M \). Then the kernel \( K \) of its anchor \( \rho_A \) is a Lie algebra bundle, i.e., the fiber Lie algebra is fixed in local trivialization of \( K \).

**Proof.** According to [1, Proposition 2.13], it suffices to show that there exists a linear connection \( \nabla^K \) on \( K \) such that its basic curvature \( R^K \) vanishes. By the assumption, one can choose a quadruple \((\tau, j, \nabla^K, \nabla^F)\) as in (3.8) such that \( \Omega = 0 \). For all \( a_K, a'_K \in \Gamma(K), b \in \Gamma(B) \) and \( u_F \in \Gamma(F) \), we have

\[ R^K_{\text{bas}}(a_K, a'_K)(u_F) = \nabla^K_{u_F}[a_K, a'_K]^K = [\nabla^K_{u_F} a_K, a'_K]^K - [a_K, \nabla^K_{u_F} a'_K]^K \]

\[ = [\tau(u_F), [a_K, a'_K]]_A - [[\tau(u_F), a_K]_A, a'_K]_A - [a_K, [\tau(u_F), a'_K]]_A = 0 \]

and

\[ R^K_{\text{bas}}(a_K, a'_K)(j(b)) = -\Omega(b)(a_K, a'_K) = 0. \]

Thus, the basic curvature \( R^K_{\text{bas}} \) of the linear connection \( \nabla^K \) on \( K \) vanishes.

As a consequence, the isomorphism (4.2) induced from the chosen linear connection \( \nabla^K \) on \( K \) becomes

\[ (\Gamma(T_K^*[1]), L_{d_K}) \rightarrow (\Omega_K(K[1] \oplus T_M), d_K^*). \]

By taking the dual and tensor products on cohomology spaces, we obtain isomorphisms

\[ H^1(\Gamma(T^*_K \otimes \text{End}(T_K))), L_{d_K}) \cong H^1_{\text{CE}}(K; (T^*_M \oplus K^*[−1]) \otimes \text{End}(K[1] \oplus T_M)) \]

and

\[ H^k(\Omega^k(K[1]), L_{d_K}) \cong \bigoplus_{q=0}^k H^k_{\text{CE}}(K; \wedge^{k−q} T^* M \otimes (S^q K^*)[−q]) \]

\[ = \bigoplus_{q=0}^k \Omega^k−q(M) \otimes_{\text{CE}(M)} H^0_{\text{CE}}(K; S^q K^*)[−q]. \]
Theorem 5.8. Let $A$ be a locally splittable regular Lie algebroid, i.e., the characteristic class

$$[\omega] \in H^2_{CE}(A; \Hom(B, K))$$

vanishes.

(1) The Atiyah class

$${\text{At}}_{(A[1], d_A)} \in H^1(\Gamma(T^*_A[1] \otimes \End(T_A[1])) \cong H^1_{CE}(A; E^* \otimes \End(E))$$

of the dg manifold $(A[1], d_A)$, when written in the matrix form as in (5.4), is of the block-diagonal type

$$\begin{pmatrix} [\alpha_A] & 0 \\ 0 & \rho^*_A \text{At}_B \end{pmatrix},$$

where

- $[\alpha_A] \in H^1_{CE}(A, K^*[−1] \otimes \End(K[1]))$ is represented by the Lie bracket $[−, −]_K$ on $\Gamma(K)$ satisfying

$$i^*[\alpha_A] = \text{At}_{(K, d_K)} \in H^1_{CE}(K, K^*[−1] \otimes \End(K[1]))$$

$$\subset H^1_{CE}(K; (T_M \otimes K^*[−1]) \otimes \End(K[1] \otimes T_M))$$

under the isomorphism (5.6);

- $\rho^*_A \text{At}_B \in H^1_{CE}(A, B^* \otimes \End(B))$ is the pullback of the Atiyah class $\text{At}_B \in H^1_{CE}(F, B^* \otimes \End(B))$ of the Lie pair $(T_M, F)$ via the anchor map $\rho_A$.

(2) Under the isomorphisms (5.5) and (5.7), the $k$-th scalar Atiyah class $\text{ch}_k(A[1], d_A)$ is given by the sum

$$\text{ch}_k(A[1], d_A) = \text{ch}_k(K[1], d_K) + \rho^*_A(\text{ch}_k(B)),$$

where

- $\text{ch}_k(K[1], d_K) \in H^0_{CE}(K, S^k K^*)[−k]$ is the $k$-th scalar Atiyah class of $(K[1], d_K)$, which is also an element in $H^0_{CE}(A, S^k K^*)[−k]$;

- $\rho^*_A(\text{ch}_k(B)) \in H^0_{CE}(A; \wedge^k B^*)$ is the pullback (by the anchor map $\rho_A$) of the $k$-th scalar Atiyah class $\text{ch}_k(B) \in H^0_{CE}(F; \wedge^k B^*)$ of the Lie pair $(T_M, F)$.

(3) Under the isomorphism

$$\prod_{k \geq 0} H^k(\Omega^k(A[1]), L_{d_A}) \cong \bigoplus_{k \geq 0, q \geq 0} H^k_{CE}(A; \wedge^k \Omega^q + (S^q K^*)[−q])$$

induced by (5.5), the Todd class $\text{Td}(A[1], d_A) \in \prod_{k \geq 0} H^k(\Omega^k(A[1]), L_{d_A})$ is given by

$$\text{Td}(A[1], d_A) = \text{Td}_{(K[1], d_K)} \cdot \rho^*_A \text{Td}_B,$$

where

- $\text{Td}_{(K[1], d_K)} \in \prod_{k \geq 0} H^0_{CE}(A; S^k K^*)[−k] \subset \prod_{k \geq 0} H^0_{CE}(K; S^k K^*)[−k]$ is the Todd class of the dg manifold $(K[1], d_K)$, which is represented by the Duflo element of the Lie algebra bundle $K$;

- $\rho^*_A(\text{Td}_B) \in \bigoplus_{k \geq 0} H^k_{CE}(A; \wedge^k \Omega^q)$ is the pullback of the Todd class $\text{Td}_B \in \bigoplus_{k \geq 0} H^k_{CE}(F; \wedge^k \Omega^q)$ of the Lie pair $(T_M, F)$.

Proof. By Lemma 5.7 and Corollary 4.4, under the isomorphism (5.6), the Atiyah class $\text{At}_{(K[1], d_K)}$ of the bundle $K = \ker \rho_A$ of Lie algebras arising from a locally splittable regular Lie algebroid $(A, \rho_A, [−, −], A)$ lives in

$$H^1_{CE}(K; K^*[−1] \otimes \End(K[1])) \subset H^1_{CE}(K; (T_M \otimes K^*[−1]) \otimes \End(K[1] \otimes T_M)),$$

and is represented by the Lie bracket $[−, −]_K$ on $\Gamma(K)$.

Applying Theorem 4.1 to the locally splittable regular Lie algebroid $A$, we see that the Atiyah class $\text{At}_{(A[1], d_A)}$ of $(A[1], d_A)$ is related via a canonical isomorphism to

$$\text{At}_{(\xi, Q\xi)} \in H^1_{CE}(A; (K^*[−1] \otimes B^*) \otimes \End(K[1] \otimes B)),$$
which, when written in the matrix form as in (5.4), is represented by
\[
\begin{pmatrix}
\alpha_A & 0 \\
0 & \alpha_B
\end{pmatrix}.
\]

Here,
- \(\alpha_A \in \Gamma(K[-1] \otimes \text{End}(K[1]))\) is given by the Lie bracket \([-,-]_K\) on \(\Gamma(K)\) satisfying \(i^*\alpha_A = \text{At}_{(K[1],dK)}\);
- \(\alpha_B \in \Omega^1_A(B^* \otimes \text{End}(B))\) coincides with the pullback \(\rho_*\text{At}_B\) of the Atiyah cocycle \(\text{At}_B\) of the Lie pair \((T_M,F)\).

This proves the first statement.

For the statement (2), by Proposition 5.1, we have
\[
\text{ch}_k(A[1],dA) = \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \left(\text{tr}([\alpha_B]^k) - \text{tr}([\alpha_A]^k)\right)
= \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \text{tr}([\alpha_B]^k) + \text{ch}_k(K[1],dK) \text{ (by the definition of } \alpha_B \text{ and } \text{At}_B)\n= \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \text{tr}(\rho_*\text{At}_B) + \text{ch}_k(K[1],dK)
= \rho_*\text{ch}_k(B) + \text{ch}_k(K[1],dK).
\]

For the statement (3), we apply Proposition 5.4 to obtain
\[
\text{Td}_{(A[1],dA)} = \det(P^{-1}([\alpha_A])) \cdot \det(P([\alpha_B]))
= \det(P^{-1}([\alpha_A])) \cdot \det(P(\rho_*\text{At}_B))
= \det(P^{-1}([\alpha_A])) \cdot \rho_*\text{det}(P(\text{At}_B))
= \text{Td}_{(K[1],dK)} \cdot \rho_*\text{Td}_B.
\]

This completes the proof.

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