Transformations of multilevel coherent states under coherence-preserving operations

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Received January 13, 2021; accepted March 23, 2021; published online April 28, 2021

Quantum coherence, emerging from the “superposition” of quantum states, is widely used in various information processing tasks. Recently, the resource theory of multilevel quantum coherence is attracting substantial attention. In this paper, we mainly study the transformations of resource pure states via free operations in the theoretical framework for multilevel coherence. We prove that any two multilevel coherent resource pure states can be interconverted with a nonzero probability via a completely positive and trace non-increasing $k$-coherence-preserving map. Meanwhile, we present the condition of the interconversions of two multilevel coherent resource pure states under $k$-coherence-preserving operations. In addition, we obtain that in the resource-theoretic framework of multilevel coherence, no resource state is isolated, that is, given a multilevel coherent pure state $|\psi\rangle$, there exists another multilevel coherent pure state $|\phi\rangle$ and a $k$-coherence-preserving operation $\Lambda_k$, such that $\Lambda_k(|\phi\rangle) = |\psi\rangle$.

multilevel coherent state, $k$-coherence-preserving operation, interconversion of two multilevel coherent resource pure states

PACS number(s): 03.65.Ta, 03.65.Ud, 03.67.-a

1 Introduction

Quantum coherence [1, 2], one embodiment of the superposition principle of quantum states, plays a central role in fundamental physics. It is widely used in various information processing tasks, such as quantum cryptography [3], metrology [4, 5], thermodynamics [6, 7], and even quantum biology [8]. Recently, the resource theory of quantum coherence is flourishing [2, 9-15]. Within the standard framework of the resource theory of quantum coherence [1], the diagonal states in the prefixed reference basis are incoherent, and incoherent operations are the free operations. While in the resource theory of quantum coherence, incoherent operation is not the single and physically motivated choice of free operations that best describe the allowed state manipulations, unlike the standard choice of local operations and classical communication (LOCC) for the theory of entanglement [16-18]. Hence there has been extensive work on the operational properties and applications of quantum coherence under different sets of free operations [1, 9, 19-22].

Quantum coherence is now recognized as a relatively fledged resource [2, 10], and great progress has been made

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on it. However, the coarse grained description in the most current literature is ultimately insufficient to reach fully understanding the essential role of the quantum superposition in the aforementioned tasks. Thus, one needs to further consider the number of classical states in the coherent superposition, which leads to the concept of multilevel coherence [23-25]. The study of the rich structure of multilevel coherence has a tangible impact on many fields of physics [6, 23, 26]. Ringbauer et al. [27] developed the theoretical and experimental groundwork for characterizing and quantifying multilevel coherence, and present the robustness of multilevel coherence as a bona fide measure. Their results contribute to a better understanding of multilevel coherence. The theoretical framework [27] consists of two basic ingredients: the set of multilevel coherence-free states which are defined by the coherence rank of the states, and the set of multilevel coherence-free operations. The \( k \)-coherence-preserving operations \( (k \in \{1, 2, \ldots, d\} \) in a \( d \)-dimensional quantum system), which parallel the nonentangling operations [28] in entanglement theory, belong to a class of multilevel coherence-free operations [27].

In quantum resource theories, one of the most significant aspects is the rules governing state transformations via the free operations [2, 12, 17]. For the state-to-state transformations, quantum coherence has similar features with quantum entanglement [17]. The celebrated Nielsen theorem gave the necessary and sufficient conditions for a class of entanglement transformations by LOCC [29]. This connects quantum entanglement to the linear-algebraic theory of majorization. Vidal [30] generalized Nielsen theorem and presented an optimal local conversion strategy of bipartite entangled pure states. Du et al. [31] built the counterpart of Nielsen theorem for quantum coherence and also used the majorization as a key ingredient in the context of the interconvertibility of coherent states. The strategy [30] was adapted to the optimal conversion of coherent states under incoherent operations [32]. However, the situations change drastically for multipartite entanglement; almost all pure multipartite entangled states are isolated under LOCC [33, 34]. To solve this problem, the researchers relaxed the class of LOCC to wider free operations [35]. Accordingly, the interconversions of pure coherent states under different free operations were intensively investigated [9, 19, 32, 36-38]. Although many valuable results have been obtained, the transformations of multilevel coherent states have remained relatively unexplored. We will focus on the transformations of multilevel coherent states via \( k \)-coherence-preserving operations.

The paper is organized as follows. We first briefly review the frameworks of coherence and multilevel coherence, and introduce two multilevel coherent measures: the robustness of multilevel coherence and the geometry measure of multilevel coherence in sect. 2. Then in sect. 3, we present our main result: that any two multilevel coherent resource pure states can be interconverted with a nonzero probability via a completely positive and trace non-increasing \( k \)-coherence-preserving map, where the nonzero probability can be validly evaluated. Additionally, we answer the question given two resource pure states \(|\psi_1\rangle\) and \(|\psi_2\rangle\), whether \(|\psi_1\rangle\) can be transformed into \(|\psi_2\rangle\) under \( k \)-coherence-preserving operations, and also we prove no resource state is isolated. Finally, we end with conclusions in sect. 4.

## 2 Preliminaries

Before discussing the transformation of multilevel coherent states, it is instructive to review the general framework of the resource theory of quantum coherence introduced in ref. [1], and the framework of multilevel coherence [27]. Throughout the paper, we consider the general \( d \)-dimensional Hilbert space \( \mathcal{H} \). Let \( \mathcal{D}(\mathcal{H}) \) be the set of all density matrices on \( \mathcal{H} \). Since coherence is a basis dependent concept, we hereafter fix a particular basis, \(|i\rangle\in1,...,d\), of \( \mathcal{H} \).

Let \( I \subset \mathcal{D}(\mathcal{H}) \) be the set of incoherent states. All incoherent density matrices \( \delta \in I \) are of the form:

\[
\delta = \sum_{i=1}^{d} p_i |i\rangle\langle i|,
\]

where \( p_i \in [0, 1] \) and \( \sum_i p_i = 1 \).

In the resource theory of quantum coherence, the definition of free operations is not unique. Within the framework of Baumgatz et al. [1], free operations are those of the incoherent operations, which act as a Kraus decomposition, i.e., \( \Lambda(\cdot) = \sum n K_n \cdot \) and the Kraus operators \( \{K_n\} \) satisfy \( \sum_n K_n K_n^\dagger = I_d \) and \( K_n I K_n^\dagger \subset I \) for all \( n \).

Another free operation in coherence theory is the incoherence-preserving operation [39]. A quantum operation \( \Lambda \) (a completely positive and trace-nonincreasing map) which maps incoherent quantum states on the input space \( I_{in} \) to incoherent quantum states on the output space \( I_{out} \), is defined as incoherence-preserving operation. More succinctly, \( \rho \in I_{in} \Rightarrow \Lambda(\rho) \in I_{out} \).

Incoherence-preserving operations are the largest class of free operations for the resource theory of quantum coherence and are parallel to the non-entangling operations [28] in entanglement theory.

Equipped with the incoherent states and incoherent operations, Baumgatz et al. [1] introduced the coherence measurement \( C \) mapping a quantum state \( \rho \) to a nonnegative real number and satisfying

1. Nonnegativity, \( C(\rho) \geq 0 \), and \( C(\delta) = 0 \) if \( \delta \in I \).
(2a) Monotonicity under incoherent operations, i.e., $C(\Lambda(\rho)) \leq C(\rho)$ for any incoherent operation $\Lambda$.

(2b) Monotonicity under selective incoherent operations on average, i.e., $\sum_i p_i C(\rho_i) \leq C(\rho)$ with probabilities $p_i = \text{Tr}[K_i \rho K_i^\dagger]$, postmeasurement states $\rho_i = K_i \rho K_i^\dagger / p_i$, and Kraus operators $\{K_i\}$.

(3) Convexity, i.e., $C(\sum_i p_i \rho_i) \leq \sum_i p_i C(\rho_i)$ for any set of states $\{\rho_i\}$ and probability distribution $\{p_i\}$.

Remarkably, conditions (2b) and (3) automatically imply condition (2a) [1].

Similar to the resource-theoretic framework of coherence, Ringbauer et al. [27] developed the resource theory of multilevel coherence. In particular, they provided a new characterization of multilevel coherence-free states, free operations, rigorously unfolding the hierarchy of multilevel coherence. They also formalized the robustness of multilevel coherence (an efficiently computable measure of multilevel coherence).

**Multilevel coherence-free quantum states** For a $d$-dimensional Hilbert space $\mathcal{H}$ and the aforementioned particular basis, $|i\rangle_{1,\ldots,d}$, any pure state $|\psi\rangle \in \mathcal{H}$ can be written as $|\psi\rangle = \sum_{i=1}^d c_i |i\rangle$, with $\sum_{i=1}^d |c_i|^2 = 1$. If there exist exactly $k$ nonzero coefficients $c_i$, then the coherence rank $r_c$ of $|\psi\rangle$ is $k$, i.e., $r_c(|\psi\rangle) = k$. The sets $I_k \subseteq \mathcal{D}(\mathcal{H})$ with $k \in \{1,2,\ldots,d\}$ are defined as all probabilistic mixtures of pure density operators $|\psi_i\rangle \langle \psi_i|$ with a coherence rank of at most $k$ [27,41]:

$$I_k = \left\{ \sum_i p_i |\psi_i\rangle \langle \psi_i| : p_i \geq 0, \sum_i p_i = 1, r_c(|\psi_i\rangle) \leq k \right\}, \quad (3)$$

where $I_k$ is the set of $(k+1)$-level coherence-free states. Note that the intermediate sets obey a strict hierarchy $I_1 \subset I_2 \subset \cdots \subset I_d$ and are the free states in the resource theory of multilevel coherence, where $I_1$ is the set of (fully) incoherent states and $I_d = \mathcal{D}(\mathcal{H})$ is the set of all states. The coherence rank $r_c$, the number of nonzero coefficients $c_i$, reveals the multilevel nature of coherence.

**Multilevel coherence-free operations** In the resource theory of multilevel coherence, the second ingredient is the set of free operations which are quantum operations that cannot create multilevel coherence. Generalizing the formalism introduced for standard coherence [1,2], one can refer to a linear completely positive and trace-preserving (CPTP) map $\Lambda$ as a $k$-coherence-preserving operation [27] if it cannot increase the coherence level, i.e., $\Lambda(I_k) \subseteq I_k$.

**$k$-coherence-preserving map** A linear map $\Lambda$ is defined as a $k$-coherence-preserving map, if $\Lambda(\rho) / \text{Tr}[\Lambda(\rho)] \in I_k$, for any $\rho \in I_k$.

**Measure of multilevel coherence** In the resource theory of multilevel coherence, quantifying multilevel coherence is a crucial task. Here, we mainly introduce two well-defined measures of multilevel coherence: the robustness, and the geometric measure.

For a quantum state $\rho \in \mathcal{D}(\mathcal{H})$, the robustness of $k$-coherence is defined as [41]:

$$R_k(\rho) = \min_{\delta \in I_k} \left\{ s \geq 0 \mid \rho + s \delta, \delta \in I_k \right\}, \quad (4)$$

for $k \in \{2,3,\ldots,d\}$.

For a pure state $|\psi\rangle$, the geometric measure of $k$-coherence is [42]

$$G_k(|\psi\rangle) = 1 - \max_{|\phi\rangle \in I_{k-1}} |\langle \phi|\psi\rangle|^2. \quad (5)$$

Geometric measure of $k$-coherence is a computable quantifier of multilevel coherence, extending previous work [42].

### 3 Main results

Multilevel quantum coherence is a powerful, yet experimentally accessible quantum resource [27], and it is a key ingredient for practical applications from the transfer phenomena in many-body and complex systems to quantum technologies [27]. In the resource theory of multilevel quantum coherence, the study of transformations of resource states via free operations is an important task. In this section, we mainly consider the transformation of multilevel coherent pure states. Given two multilevel coherent pure states, we study whether they can be interconverted via coherence-preserving maps/operations.

Let us start the main text with two very useful lemmas.

**Lemma 1** [43] Suppose $\rho_1, \rho_2$ are density matrices, then there exists an operator $\Lambda$.

$$\Lambda(\sigma) = \text{Tr}(A\sigma) \rho_1 + \text{Tr}[(1-A)\sigma] \rho_2, \quad (6)$$

which is a CPTP map if $0 \leq A \leq 1$. Here $\sigma \in \mathcal{D}(\mathcal{H})$.

**Lemma 2** For any quantum state $\sigma \in I_k$ and multilevel coherent pure states $\psi_1, \psi_2 \not\in I_k (\psi_1$ and $\psi_2$ are the corresponding density matrices of $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively), suppose $0 < p \leq 1$ and $\frac{1}{p} \left( \frac{1}{\text{Tr}[\psi_1\sigma]} - 1 \right) \geq R_k(\psi_2)$, then the map $\Lambda_k$:

$$\Lambda_k(\sigma) = p \text{Tr}(\psi_1\sigma) \psi_2 + \text{Tr}[(1-\psi_1)\sigma] \delta \quad (7)$$

is a trace non-increasing $k$-coherence-preserving map. Here, $\delta \in I_k$ is the optimal state achieving the robustness of $k$-coherence of $\psi_2$. Specially, if $p = 1$, then $\Lambda_k$ in eq. (7) is a $k$-coherence-preserving operation.

**Proof** For any quantum state $\sigma \in I_k$, there is

$$\text{Tr}[\Lambda_k(\sigma)] = p \text{Tr}(\psi_1\sigma)\text{Tr}(\psi_2) + \text{Tr}[(1-\psi_1)\sigma]\text{Tr}(\delta) = \text{Tr}(\sigma) - (1-p)\text{Tr}(\psi_1\sigma) \leq \text{Tr}(\sigma), \quad (8)$$

as we require.
with equality for $p = 1$. Hence $\Lambda_4$ is a trace non-increasing map, while by Lemma 1 it is a CPTP map for $p = 1$.

Due to eq. (7), we have

$$\Lambda_k(\sigma) \leq \psi_2 + \frac{1}{p} \left( \frac{1}{\text{Tr}(\psi_1\sigma)} - 1 \right) \delta. \quad (9)$$

If $\frac{1}{p} \left( \frac{1}{\text{Tr}(\psi_1\sigma)} - 1 \right) \geq R_k(\psi_2)$, then

$$\Lambda_k(\sigma) \leq \psi_2 + R_k(\psi_2)\delta + \frac{1}{p} \left( \frac{1}{\text{Tr}(\psi_1\sigma)} - 1 \right) - R_k(\psi_2) \delta. \quad (10)$$

where $R_k(\psi_2)$ denotes the robustness of $k$-coherence of $\psi_2$. Then according to the definition, one has

$$\frac{\psi_2 + R_k(\psi_2)\delta}{1 + R_k(\psi_2)} \in I_k, \quad (11)$$

with $\delta \in I_k$, thus $\Lambda_k(\sigma)/\text{Tr}[\Lambda_k(\sigma)] \in I_k$.

So $\Lambda_k$ is a trace non-increasing and $k$-coherence-preserving map, while $\Lambda_4$ is a $k$-coherence-preserving operation for $p = 1$.

**Theorem 1** In the resource theory of multilevel quantum coherence, suppose free operations are $k$-coherence-preserving operations, and the resource states are quantum states not belonging to $I_k$. All resource pure states are interconvertible with a non-zero probability. That is, for multilevel coherent pure states $\psi_1, \psi_2 \notin I_k$, there exists a completely positive and trace non-increasing $k$-coherence-preserving map $\Lambda_k$ such that $\Lambda_k(\psi_1) = p\psi_2$ with $p \leq \frac{G_{\infty}(\psi_1)}{R_k(\psi_2)[1 - G_{\infty}(\psi_1)]}, 0 < p \leq 1$.

**Proof** For any pure states $\psi_1, \psi_2 \notin I_k$, let

$$\Lambda_k(\cdot) = p \text{Tr}(\psi_1\cdot)\psi_2 + \text{Tr}[\mathbb{I} - \psi_1\cdot] \delta, \quad (12)$$

where $\delta$ is the optimal state achieving the robustness of $\psi_2$. By Lemma 1 and the proof of Lemma 2, $\Lambda_k$ is a completely positive and trace non-increasing map.

Next we prove that $\Lambda_k$ is a $k$-coherence-preserving map. From the definition of geometric measure, for a multilevel coherent pure state $\psi_1 \notin I_k$ and any $\sigma \in I_k$, one has

$$1 - G_{k+1}(\psi_1) = \max_{|\phi| \in I_k} |\langle \phi | \psi_1 \rangle|^2$$

$$= \max_{|\phi| \in I_k} \text{Tr}(\psi_1|\phi\rangle\langle \phi |) \geq \text{Tr}(\psi_1\sigma). \quad (13)$$

Note that $G_{k+1}(\psi_1) < 1$, if we can choose $p \leq \frac{G_{k+1}(\psi_1)}{R_k(\psi_2)[1 - G_{\infty}(\psi_1)]}$, and $0 < p \leq 1$, then

$$R_k(\psi_2) \leq \frac{G_{k+1}(\psi_1)}{p[1 - G_{k+1}(\psi_1)]}$$

$$= \frac{1}{p} \left( \frac{1}{1 - G_{k+1}(\psi_1)} - 1 \right). \quad (14)$$

where the last inequality is due to inequality eq. (13). Thus, it follows from Lemma 2 that $\Lambda_k$ is a $k$-coherence-preserving map, and

$$\Lambda_k(\psi_1) = p \text{Tr}(\psi_1\psi_2) + \text{Tr}[\mathbb{I} - \psi_1\psi_2] \delta = p\psi_2. \quad (15)$$

This concludes the proof.

For a pure state $|\psi\rangle = (v_1, v_2, \ldots, v_d)^T$ with real entries satisfying $v_1 \geq v_2 \geq \cdots \geq v_d \geq 0$, an analytical formula for the robustness of $k$-coherence $(k \in \{2, 3, \ldots, d\})$

$$R_k(|\psi\rangle) = \frac{\sum_{i=1}^d v_i^2}{k - l + 1 - \sum_{i=1}^d v_i^2}, \quad (16)$$

was derived in ref. [41], where $l \in \{2, 3, \ldots, k\}$ is the largest integer such that $v_{l-1} \geq \frac{s_l}{\sum_{i=1}^l v_i}$ with $s_l \equiv \sum_{i=1}^l v_i$ (if no such integer exists, then set $l = 1$). If $|\psi\rangle$ is not of this form then it can be converted to this form via a diagonal unitary and/or a permutation matrix, and these operations do not affect the value of $R_k$.

Regula et al. [42] obtained a closed formula for the geometric measure of $k$-coherence of arbitrary pure states $|\psi\rangle$ as:

$$G_k(|\psi\rangle) = 1 - \sum_{i=1}^{k-1} |\mu_i|^2, \quad (17)$$

where $\mu_i$ is the $i$th largest coefficient (by absolute value) of $|\psi\rangle$.

Using eqs. (16) and (17), the bound in Theorem 1 can be easily evaluated. Suppose $|\psi_1\rangle = (\mu_1, \mu_2, \ldots, \mu_d)^T$ with $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_d|$ and $|\psi_2\rangle = (v_1, v_2, \ldots, v_d)^T$ with real entries satisfying $v_1 \geq v_2 \geq \cdots \geq v_d \geq 0$ in Theorem 1, then one can validly evaluate $p$. That is, for multilevel coherent pure states $\psi_1, \psi_2 \notin I_k$, there exists a completely positive and trace non-increasing $k$-coherence-preserving map $\Lambda_k$ such that $\Lambda_k(\psi_1) = p\psi_2$ with $p \leq \frac{\sum_{i=1}^l v_i^2}{k - l + 1 - \sum_{i=1}^l v_i^2}$, where $l \in \{2, 3, \ldots, k\}$ is the largest integer such that $v_{l-1} \geq \frac{s_l}{\sum_{i=1}^l v_i}$ with $s_l \equiv \sum_{i=1}^l v_i$, and $0 < p \leq 1$. This is useful as these bounds are often not easy to compute for other resources, e.g., in multipartite entanglement theory.

**Theorem 2** For the resource theory of multilevel coherence, $k$-coherence-preserving operations are free operations. Then no resource state is isolated. For any resource pure state $\psi \notin I_k$, there always exists another resource pure state $\phi \notin I_k$, and a $k$-coherence-preserving operation $\Lambda_k$ such that $\Lambda_k(\phi) = \psi$. 
Proof For the resource pure state $ψ ∉ I_k$, $R_k(ψ)$ is the robustness of $k$-coherence of $ψ$. If the geometric measure of $(k + 1)$-coherence of the resource pure state $φ$ satisfies

$$\frac{G_{k+1}(φ)}{R_k(φ)[1 - G_{k+1}(φ)]} ≥ 1,$$  \hspace{1cm} (18)

that is, $G_{k+1}(φ) ≥ 1 - \frac{1}{R_k(φ)+1}$, then from Lemma 2, one can construct a $k$-coherence-preserving operation:

$$Λ_k(·) = \text{Tr}(φ·ψ_1 + \text{Tr}[(I - φ)·] δ),$$  \hspace{1cm} (19)

where $δ$ is the optimal state achieving the robustness of $k$-coherence of $ψ$. It implies that

$$Λ_k(φ) = \text{Tr}(φφ·ψ_1 + \text{Tr}[(I - φ)φ] δ = ψ,$$  \hspace{1cm} (20)

which completes the proof.

Theorem 3 In the resource theory of multilevel quantum coherence, $k$-coherence-preserving operations are the free operations. For any resource pure state $ψ_1, ψ_2 ∉ I_k$, $ψ_1$ can be transformed to $ψ_2$ via a $k$-coherence-preserving operation $Λ_k$ if the geometric measure of $(k + 1)$-coherence of $ψ_1$ and the robustness of $k$-coherence of $ψ_2$ satisfy the condition $\frac{G_{k+1}(ψ_1)}{R_k(ψ_2)[1 - G_{k+1}(ψ_2)]} ≥ 1$.

Proof Let

$$Λ_k(·) = \text{Tr}(ψ_1·ψ_2 + \text{Tr}[(I - ψ_1)·] δ),$$  \hspace{1cm} (21)

where $δ$ is the optimal state achieving the robustness of $k$-coherence of $ψ_2$.

If the geometric measure of $(k + 1)$-coherence of $ψ_1$ and the robustness of $k$-coherence of $ψ_2$ satisfy

$$\frac{G_{k+1}(ψ_1)}{R_k(ψ_2)[1 - G_{k+1}(ψ_2)]} ≥ 1,$$  \hspace{1cm} (22)

then by Lemma 2, $Λ_k$ is a $k$-coherence-preserving operation and

$$Λ_k(ψ_1) = \text{Tr}(ψ_1ψ_1ψ_2 + \text{Tr}[(I - ψ_1)ψ_1] δ = ψ_2.$$  \hspace{1cm} (23)

This proof is complete.

Corollary In the theoretical framework for multilevel coherence, $k$-coherence-preserving operations are the free operations. The coherence state $ψ = |ψ⟩⟨ψ|$ where $|ψ⟩ = \frac{1}{√d} \sum_{i=1}^{d} |i⟩$ can be transformed to any other resource state $φ ∉ I_k$ via $k$-coherence-preserving operations, for any $k$.

Remarkably, the proof of Theorem 1 actually gives the explicit construction of a completely positive and trace non-increasing $k$-coherence-preserving map $Λ_k$ such that $Λ_k(ψ_1) = pψ_2$ when the nonzero probability meets $p ≤ \frac{G_{k+1}(ψ_1)}{R_k(ψ_2)[1 - G_{k+1}(ψ_2)]}$. If pure states $ψ_1, ψ_2 ∉ I_k$, we have $R_k(ψ_2) > 0$ and $0 < G_{k+1}(ψ_1) < 1$, hence there always exists $p ∈ (0, 1]$ such that probability condition holds. The Theorems 2 and 3, to some extent, are corollaries. In Theorems 2, for a pure state $ψ ∉ I_k$, one can always find a pure state $φ ∉ I_k$ with $G_{k+1}(φ)$ large enough to satisfy the required condition. Given pure states $ψ_1, ψ_2 ∉ I_k$, if $R_k(ψ_2)$ is sufficiently smaller than $G_{k+1}(ψ_1)$, then one can pick $p = 1$ in Theorem 1, such that $ψ_1$ is transformed to $ψ_2$ via a $k$-coherence-preserving operation.

4 Conclusion

The study of multilevel quantum coherence reveals further parallels between the resource theories of coherence and entanglement. The characterization and quantification of multilevel coherence can apply semidefinite programming rather than general convex optimization. Hence it verifies that multilevel quantum coherence is a powerful, yet experimentally accessible quantum resource. In this paper, we focus on the transformation of multilevel coherent resource pure states which has remained relatively unexplored. In the theoretical framework for multilevel coherence, the free operations are $k$-coherence-preserving operations. We show that any two multilevel coherent resource pure states can be interconverted with a nonzero probability via a completely positive and trace non-increasing $k$-coherence-preserving map. Here the nonzero probability is related with the robustness and the geometric measure of multilevel coherence. We present the condition of the interconversions of any two multilevel coherent resource pure states under $k$-coherence-preserving operations. Moreover, we obtain that in the resource theory of multilevel quantum coherence, no resource state is isolated. Finally, we show that in the resource theory of multilevel quantum coherence, the coherence state $|ψ⟩ = \frac{1}{√d} \sum_{i=1}^{d} |i⟩$ can be transformed to any other multilevel coherent pure state under $k$-coherence-preserving operations. We expect that our approach can be used to future study the framework of multilevel coherence and interconversion of resource states.

This work was supported by the National Natural Science Foundation of China (Grant No. 12071110), the Hebei Natural Science Foundation of China (Grant Nos. A2020205014, and A2018205125), and the Science and Technology Project of Hebei Education Department (Grant Nos. ZD2020167, and ZD2021066).

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