Mixmaster universe in the $z = 3$ deformed Hořava-Lifshitz gravity

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The $z = 3$ deformed Hořava-Lifshitz gravity with coupling constants $\omega$ and $\epsilon$ leads to a non-relativistic “mixmaster” cosmological model. The potential of theory is given by the sum of IR and UV potentials in the ADM Hamiltonian formalism. It turns out that the presence of the UV-potential cannot suppress chaotic behaviors existing in the IR-potential, which comes from curvature anisotropy.

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I. INTRODUCTION

Recently, quantum gravity at a Lifshitz point, which is power-counting renormalizable and hence potentially UV complete, was proposed by Hořava [1–3]. This theory for quantum gravity is not intended to be a unified theory like string theory.

A hot issue of the Hořava-Lifshitz gravity is to answer to the question of whether it can accommodate the Hořava scalar $\psi$, in addition to two degrees of freedom (DOF) for a massless graviton. This additional scalar degree of freedom inevitably appears as a result of the reduced symmetry of diffeomorphism known as “foliation diffeomorphism” [4–9]. The authors [4] have shown that without the projectability condition, a perturbative general relativity cannot be reproduced in the IR-limit of the $z = 3$ deformed Hořava-Lifshitz gravity because of the strong coupling problem. With the projectability condition, the authors [6] have argued that $\psi$ is propagating around the Minkowski space but it has a negative kinetic term, showing a ghost mode. Moreover, it was found that the Hořava scalar is a ghost if the sound speed squared is positive (strong coupling problem) [7]. Even the Lorentz-violating mass term was included, the mass term did not cure the ghost problem of the Hořava scalar [10]. In order to resolve the strong coupling problem, Blas, Pujolas, and Sibiryakov have proposed an extended version of the Hořava-Lifshitz gravity where the lapse function $N$ may depend on the spatial coordinate $r$ (the theory is not projectable) and thus, terms of $\partial_i \ln N$ are included in the action [11]. It was argued that this extended version are free from the strong coupling pathology. However, the extended theory could still suffer from the strong coupling at low energies in the kinetic term [12], but it can be evaded by including higher spatial derivative terms [13]. Hence, up to now, the strong coupling issue is not completely resolved even though the extended theory was seriously considered.

Specific cosmological implications of the $z = 3$ Hořava-Lifshitz gravity with the Friedmann-Robertson-Walker (FRW) metric based on isotropy and homogeneity have recently been shown in [14–16], including homogeneous vacuum solution with chiral primordial gravitational waves [17] and nonsingular cosmological evolution with the big bang of standard and inflationary universe replaced by a matter bounce [18–20]. As far as the isotropic solutions are concerned, there is no difference between $z = 2$ [1] and $z = 3$ [2] Hořava-Lifshitz gravities because the Cotton tensor vanishes when using the isotropic FRW metric. Furthermore, one has introduced the $z = 3$ deformed Hořava-Lifshitz gravity to find asymptotically flat background [21, 22].

On the other hand, the equations of general relativity lead to singularities when we look at the equations backwards the origin of time. Especially, we concentrate on a temporal singularity of the solutions to the Einstein equations for the mixmaster model (Bianchi IX Universe) describing an anisotropic and homogeneous cosmology. It was well known that the approach to singularity shows a chaotic behavior. The mixmaster universe [23–30] could be described by a Hamiltonian dynamical system in a 6D phase space. Belinsky, Khalatnikov, and Lifshitz (BKL) had conjectured that this 6D phase system could be well approximated by a 1D discrete Gauss map that is known to be chaotic as one approaches the singularity [31]. Chernoff and Barrow have suggested that the mixmaster 6D phase space could be split into the product of a 4D phase space and a 2D phase space having regular variables [25]. Following Cornish and
Levin, Lehner and Di Menza have found that the chaos in the mixmaster universe is obtained for the Hamiltonian system with potential having fixed walls, which describes the curvature anisotropy.

However, it turned out that the mixmaster chaos could be suppressed by (loop) quantum effects. In the loop quantum cosmology, the effective potential at decreasing volume labeled by “discreteness $j$” are significantly changed in the vicinity of $(0,0)$-isotropy point in the anisotropy plane ($\beta_+, p_+$.). The potential at larger volumes exhibits a potential wall of finite height and finite extension. As the volume is decreased, the wall moves inward and its height decreases. Progressively, the wall disappears completely making the potential negative everywhere at a dimensionless volume of $(2.172)^{3/2}$ in the Planck units. Eventually, the potential approaches zero from below. This shows that classical reflections will stop after a finite amount of time, implying that classical arguments about chaos are inapplicable. Once quantum effects are taken into account, the reflections stop just when the volume of a given patch is about the size of Planck volume.

We point out that loop quantum gravity is a non-perturbative and background independent canonical quantization of general relativity, while loop quantum cosmology is a cosmological mini-superspace model quantized with methods of loop quantum gravity. Hence the discreteness of spatial geometry and the simplicity of setting allow for complete study of cosmological evolution. The difference between loop quantum cosmology and other approaches of quantum cosmology is that the input is plugged by a full quantum gravity theory, which introduces a discreteness to space-time. That is, in order to quantize general relativity, this discreteness manifests itself as quanta of space.

Recently, we have investigated the $z = 2$ deformed Hořava-Lifshitz gravity with coupling constant $\omega$ which leads to a nonrelativistic “mixmaster” cosmological model. We have obtained that for $\omega > 0$, there always exists chaotic behavior. This contrasts to the case of the loop mixmaster dynamics based on loop quantum cosmology, where the mixmaster chaos is suppressed by loop quantum effects. We recognize that the role of UV coupling parameter $\omega$ is intrinsically different from the area quantum number $j$ of the loop quantum cosmology which controls the volume of the universe. In our case, time variable (related to the volume of $V = e^{3\omega}$) as well as two physical degrees of anisotropy $\beta_{\pm}$ are treated in the standard way without quantization. However, in the loop quantum framework, all three scale factors were quantized using the loop techniques. Hence two are quite different: the potential wells at the origin never disappear for any $\omega > 0$ in the $z = 2$ Hořava-Lifshitz gravity, while in the loop quantum gravity the height of potential wall rapidly decreases until they disappears completely as the Planck scale is reached.

On the other hand, it was interestingly shown that adding 4D curvature squared term $(^4 R)^2$ (and possibly other) curvature squared terms to the Einstein gravity leads to an interesting result that the chaotic behavior is absent. Hence it is very curious to see why $(^4 R)^2$ does suppress chaotic behavior but 3D curvature squared terms of $\frac{3}{4\omega} R^2 - \frac{2}{\omega} R_{ij} R^{ij}$ does not suppress chaotic behavior. It was argued that the absence of chaos in covariant higher curvature generalization of Einstein gravity $f( ^4 R)$ is due to the presence of a scalar $\varphi = \log f'( ^4 R)$. This scalar slows down the velocity of the point particle (the universe) relative to the moving walls and thus, the universe will bounce back only if it moves not too oblique relative to the walls. A few of collisions are sufficient to make it so oblique that it will not bounce off another wall. The universe will enter quickly in a definite Kasner trajectory and stay there all the time in its approach to the singularity. Hence, the evolution of the universe is not chaotic.

Hence it is very interesting to investigate cosmological application of the $z = 3$ Hořava-Lifshitz gravity in conjunction with the mixmaster universe based on the anisotropy and homogeneity because this Hořava-Lifshitz gravity may be regarded as a strong candidate for quantum gravity. The mixmaster universe in the $z = 3$ Hořava-Lifshitz gravity was discussed in Ref. The authors have focused on the Cotton bilinear term $C_{ij} C^{ij}$ only and thus, have briefly sketched possible dynamical behaviors of the universe when approaching the initial singularity.

For the isotropic case of the $z = 3$ Hořava-Lifshitz gravity, the $k = 1$ FRW universe with dark radiation and dust matter ($w = 0$) has led to a matter bounce. If the Hořava-Lifshitz gravity is true, the universe did not bang-it bounced. That is, a universe filled with matter will contract down to a small but finite size and then bounce again, giving us the expanding universe that we see today. This bounce scenario indicates a key feature of the Hořava-Lifshitz gravity, showing an essential difference from the big bang scenario. On the other hand, for the anisotropic case of Einstein gravity, the mixmaster universe filled with stiff matter ($w = 1$) has led to a non-chaotic universe because there is a slowing down of particle velocity, which is unable to reach any more the walls after some time in the moving wall picture. Hence, an urgent issue for an anisotropic mixmaster universe is to see whether there exists a mechanism to slow down the particle velocity in the $z = 3$ Hořava-Lifshitz gravity.

In this work, we wish to find whether a mechanism to stop chaotic behaviors exists in the Hořava-Lifshitz gravity. We will analyse the $z = 3$ deformed Hořava-Lifshitz gravity without cosmological constant to make the situation simple.
II. $z = 3$ DEFORMED HOŘAVA-LIFSHITZ GRAVITY

In order to get an associated Hamiltonian within the ADM formalism \cite{36}, we have to find three potentials in 6D phase space: IR-potential $V_{IR}$ from 3D curvature $R$ and two UV-potentials: $V_{UV}$ from curvature squared terms of $R^2$ and $R_{ij}R^{ij}$ with UV coupling parameter $\omega$ and $V_{IR}^{(II)}$ from $C_{ij}R^{ij}$ and $C_{ij}C^{ij}$ with additional coupling constant $\epsilon$.

We start with the action of the $z = 3$ deformed Hořava-Lifshitz gravity \cite{1, 21}

$$S_\lambda = \int dt d^3x \sqrt{|g|} N \left[ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \mu^4 R + \frac{\kappa^2 \mu^2 (1 - 4 \lambda)}{32 (1 - 3 \lambda)} R^2 - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu}{2 \eta^2} C_{ij} R^{ij} - \frac{\kappa^2}{2 \eta^2} C_{ij} C^{ij} \right]$$

(1)

with four parameters $\kappa$, $\mu$, $\lambda$, and $\eta$. In the case of $\lambda = 1$, the above action leads to

$$S_{\lambda=1} = \int dt d^3x \sqrt{|g|} N \mu^4 \left[ \frac{1}{\epsilon^2} (K_{ij} K^{ij} - K^2) + R + \frac{3}{4 \omega} R^2 - \frac{2}{\omega} R_{ij} R^{ij} + \frac{8 \sqrt{2}}{\omega^{3/2} \epsilon} C_{ij} R^{ij} - \frac{16}{\omega^{4/3} \epsilon^2} C_{ij} C^{ij} \right]$$

(2)

where the two UV coupling parameters $\omega = 16 \mu^2 / \kappa^2$ and $\epsilon = \eta^2 \epsilon^{1/3}$ are introduced to control curvature and Cotton squared terms \cite{37}. In the limit of $\omega \to \infty$ ($\kappa^2 \to 0$), $S_{\lambda=1}$ reduces to Einstein gravity (GR) with the speed of light $c^2 = \kappa^2 \mu^2 / 2$ and Newton’s constant $G = \kappa^2 / (32 \pi)$.

Now, let us introduce the metric for the mixmaster universe to distinguish between expansion (volume change: $\alpha$) and anisotropy (shape change: $\beta_{ij}$) as follows

$$ds^2 = -dt^2 + e^{2\alpha} e^{2\beta_{ij}} \sigma^i \otimes \sigma^j,$$

(3)

where $\sigma^i$ are the 1 forms given by

$$\sigma^1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi,$$

$$\sigma^2 = \sin \psi d\theta - \cos \psi \sin \theta d\phi,$$

$$\sigma^3 = d\psi + \cos \theta d\phi$$

(4)

on the three-sphere parameterized by Euler angles ($\psi, \theta, \phi$) with $0 \leq \psi < 4\pi$, $0 \leq \theta < \pi$, and $0 \leq \phi < 2\pi$. The shape change $\beta_{ij}$ is a $3 \times 3$ traceless symmetric tensor with det[$e^{2\beta_{ij}}$] = 1 expressed in terms of two independent shape parameters $\beta_{\pm}$ as

$$\beta_{11} = \beta_+ + \sqrt{3} \beta_- , \quad \beta_{22} = \beta_+ - \sqrt{3} \beta_- , \quad \beta_{33} = -2 \beta_+.$$  

(5)

Then, the evolution of the universe can be described by giving $\beta_{\pm}$ as function of $\alpha$. Note that the $k = 1$ FRW universe is the special case of $\beta_{\pm} = 0$.

Now we concentrate on the behavior near singularity. Then, the empty space without matter is sufficient to display the generic local evolution close to singularity because the terms due to dust matter or radiation are negligible near singularity.

Before we proceed, let us consider the Einstein gravity. Using Eq. (3), the 3D curvature takes the form

$$R = -12e^{-2\alpha} V_{IR}(\beta_+, \beta_-),$$

(6)

where the IR-potential of curvature anisotropy is given by

$$V_{IR}(\beta_+, \beta_-) = \frac{1}{24} \left[ 2e^{4\beta_+} \cosh(4\sqrt{3}\beta_-) + e^{-8\beta_+} \right] - \frac{1}{12} \left[ 2e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + e^{4\beta_+} \right].$$

(7)

Figure \[\text{I}\] depicts a typical IR-potential where three canyon lines located at $\beta_- = 0$ and $\beta_- = \pm \sqrt{3} \beta_+$, showing an axial symmetry. It has the shape of an equilateral triangle in the space labeled by $(\beta_+, \beta_-)$ and exponentially steep walls far away from the origin. As is shown in Fig. \[\text{I}\] the potential is the well close to the origin $(0, 0)$: left panel shows equipotential curves viewed from the top, while right panel is the shape of potential. The origin $(0, 0)$, which corresponds to the isotropic case, is the global minimum with negative value. Near the origin, the IR-potential takes concentric forms of equipotential curves as

$$V_{IR}(0, 0) \approx -\frac{1}{8} \left( \beta_+^2 + \beta_-^2 \right).$$

(8)
On the other hand, the asymptotic form of the IR-potential for the case of \( \beta_- \ll 1 \) is either \( V_{IR} \approx 2e^{4\beta_+} \beta_+^2 \) if \( \beta_+ \to \infty \) or \( V_{IR} \approx \frac{1}{2e^{8\beta_+}} \) if \( \beta_+ \to -\infty \). That is, the both walls grow exponentially. \( V_{IR} \) has no local maxima along \( \beta_- = 0 \), compared to \( V_{UV}^{(I)} \) and \( V_{UV}^{(II)} \). For \( \beta_- = 0 \), the IR-potential approaches zero from below if \( \beta_+ \to \infty \). Hence the point particle with positive energy \( E > 0 \) can escape to infinity along the canyon lines. The smallest deviation from axial symmetry will turn the particle against the infinitely steep walls.

The evolution of the universe is described by the motion of a point \( \beta = (\beta_+, \beta_-) \) as a function of \( \alpha \) using the time-dependent Lagrangian. The exponential wall picture of the IR-potential implies that a particle (the universe) runs through almost free (Kasner) epochs where the potential could be neglected, and it is reflected at the walls, resulting infinite number of oscillations. This implies that Einstein gravity with the IR-potential \( V_{IR} \) shows chaotic behaviors when the singularity is approached [38].

The action (2) provides the time-dependent Lagrangian

\[
\mathcal{L}_{\lambda=1} = (4\pi)^2 \mu^4 e^{3\alpha} \left[ -6(\dot{\alpha}^2 - \dot{\beta}_+^2 - \dot{\beta}_-^2) - 12e^{-2\alpha} V_{IR}(\beta_+, \beta_-) - \frac{e^{-4\alpha}}{16\omega} V_{UV}^{(I)}(\beta_+, \beta_-) - e^{-6\alpha} V_{UV}^{(II)}(\beta_+, \beta_-) \right],
\]

where the dot denotes \( \frac{d}{dt} \). One needs to introduce an emergent speed of light \( c \) in order to see the UV behaviors, while for the IR behaviors, one chooses \( c = 1 \) simply. Here, the UV-potential \( V_{UV}^{(I)} \) is defined from the curvature squared terms as

\[
\frac{3}{4\omega} R^2 - \frac{2}{\omega} R^{ij} R_{ij} \equiv \frac{e^{-4\alpha}}{16\omega} V_{UV}^{(I)},
\]

where \( R_{ij} \) are the Riemannian curvature tensors.
where the UV-potential $V_{UV}^{(I)}$ takes the form of

$$V_{UV}^{(I)}(\beta_+, \beta_-) \equiv -\left[40\left(e^{8\beta} \cosh(4\sqrt{3}\beta_-) + e^{2\beta} \cosh(6\sqrt{3}\beta_-) + e^{-10\beta} \cosh(2\sqrt{3}\beta_-) - 40e^{2\beta} \cosh(2\sqrt{3}\beta_-) + 4e^{-4\beta} \cosh(4\sqrt{3}\beta_-) + 2e^{8\beta} - 20e^{-4\beta} - 42e^{8\beta} \cosh(8\sqrt{3}\beta_-) - 21e^{-16\beta}\right]\right].$$

(11)

On the other hand, the other UV-potential $V_{UV}^{(II)}$ is found from the Cotton terms as

$$V_{UV}^{(II)}(\beta_+, \beta_-) \equiv -e^{-6\alpha}V_{UV}^{(II)}(\beta_+, \beta_-)$$

with

$$V_{UV}^{(II)}(\beta_+, \beta_-) \equiv \frac{8\sqrt{2}}{\omega^{3/2} \epsilon} C_{ij} R^{ij} - \frac{16}{\omega^{4/3} \epsilon^2} C_{ij} C^{ij} \equiv -e^{-6\alpha}V_{UV}^{(II)}(\beta_+, \beta_-)$$

(12)

We have thoroughly studied the $V_{UV}^{(II)} = 0$ case of the $z = 2$ deformed Hořava-Lifshitz gravity in [34], indicating that chaotic behavior persists, as the Einstein gravity did show. Thus, we point out that a key feature of the $z = 3$ deformed Hořava-Lifshitz gravity is the presence of the UV-potential $V_{UV}^{(I)}$. As was mentioned in [33], the Cotton bilinear term $V_{UV}^{CC}$ contributes to $V_{UV}^{(II)}$ without $\alpha$. Near the origin $(\beta_+, \beta_-) = (0, 0)$, the UV-potential of $V_{UV}^{(II)}$ is approximated by

$$V_{UV}^{(II)}(\beta_+, \beta_-) \approx \left(\frac{288\sqrt{2} \epsilon^3}{\omega^{7/6} \epsilon} + \frac{864}{\omega^{4/3} \epsilon^2}\right)(\beta_+^2 + \beta_-^2).$$

(14)

This means that $V_{UV}^{(II)}(0, 0) = 0$ has no contribution to the isotropic point $(0, 0)$, in contrast to the IR-potential $V_{IR}(0, 0) = -1/8$ and UV-potential $V_{UV}^{(I)}(0, 0) = -3$. Fig. 3 indicates shape changes of the UV-potential $V_{UV}^{(II)}$ for different values $\epsilon$, showing different local maxima $V_{uv}(\beta_+, 0, 1, \epsilon)$ for different $\epsilon$. The asymptotic form for $\beta_- \ll 1$ is either $V_{UV}^{(II)} \approx \frac{6444\beta_+^2}{\omega^{11/6} \epsilon^2}$ if $\beta_+ \to \infty$, or $V_{UV}^{(II)} \approx \frac{242\beta_+^2}{\omega^{11/6} \epsilon^2}$ if $\beta_+ \to -\infty$. An important point is that unlike $V_{IR}$ and $V_{UV}^{(I)}$, the asymptotic form of $V_{UV}^{(II)} \to V_{UV}^{CC}$ is independent of volume change $\alpha$. As before, for $E > V_{uv}$, the point particle can escape to infinity along the canyon lines $\beta_+ = 0$ and $\beta_- = \pm \sqrt{3} \beta_+$. In order to appreciate implications of chaotic approach to the $z = 3$ deformed Hořava-Lifshitz gravity, we have to calculate the Hamiltonian density by introducing three canonical momenta as

$$p_+ = \frac{\partial L^{(III)}_{L, R}}{\partial \dot{\beta}_+} = 12(4\pi)^2 \mu^4 e^{3\alpha} \beta_+^3, \quad p_\alpha = \frac{\partial L^{(III)}_{L, R}}{\partial \dot{\alpha}} = -12(4\pi)^2 \mu^4 e^{3\alpha} \dot{\alpha}.$$  

(15)

The normalized canonical Hamiltonian in 6D phase space is given by

$$H_{6D} = \frac{1}{2} (p_+^2 + p_-^2 - p_\alpha^2) + e^{4\alpha} \left(V_{IR} + \frac{e^{-2\alpha}}{192 \omega} V_{UV}^{(I)} + \frac{e^{-4\alpha}}{12} V_{UV}^{(II)} \right)$$

$$\equiv \frac{1}{2} (p_+^2 + p_-^2 - p_\alpha^2) + V_{c}(\beta_+, \beta_-, \omega, \epsilon),$$

(16)

where we have redefined $H_{6D} = 12(4\pi)^2 \mu^4 e^{3\alpha} H_c$ using the canonical Hamiltonian $H_c$, and chosen the parameter $12(4\pi)^2 \mu^4 = 1$ for simplicity. Then, the Hamiltonian equations of motion are

$$\dot{\beta}_\pm = p_\pm, \quad \dot{p}_\pm = -e^{4\alpha} \frac{\partial V_{IR}}{\partial \beta_\pm} - e^{2\alpha} \frac{\partial V_{UV}^{(I)}}{192 \omega} \frac{\partial V_{UV}^{(II)}}{\partial \beta_\pm} - \frac{1}{12} \partial \beta_\pm$$

(17)

$$\dot{\alpha} = -p_\alpha, \quad \dot{p}_\alpha = -4e^{4\alpha} V_{IR} - \frac{e^{2\alpha}}{96 \omega} V_{UV}^{(I)} - \frac{1}{12} V_{UV}^{CR}.$$  

(18)
FIG. 3: The UV-potentials of $V^{(II)}_{UV}$ for $\alpha = 0$, $\beta_{-} = 0$, and $\omega = 1$ with $\epsilon = 1$ (solid), 10 (dotted), and 100 (dashed), respectively. There exist local maxima $V_{lm}$, compared to the IR-potential $V_{IR}$.

in 6D phase space.

III. ISOTROPIC EVOLUTION

Now, let us see what happens in the isotropic point of $(0,0)$ where the $k = 1$ FRW universe pops up. Since the isotropic potential does not receive any contribution from the Cotton tensor, it is given by

$$V_{\alpha}(0,0,\omega) = -\left(\frac{e^{4\alpha}}{8} + \frac{e^{2\alpha}}{64\omega}\right).$$  \hfill (19)

From the Hamiltonian constraint of $\mathcal{H}_{6D} \approx 0$, we have the first Friedmann equation

$$\dot{\alpha}^2 = -\frac{1}{4} \left(\frac{1}{e^{2\alpha}} + \frac{1}{8\omega} \frac{1}{e^{4\alpha}}\right).$$  \hfill (20)

Introducing the scaling factor $a = 2e^{\alpha}$ with $H = \frac{\dot{a}}{a} = \dot{\alpha}$, the above equation leads to

$$H^2 = -\left(\frac{1}{a^2} + \frac{1}{2\omega} \frac{1}{a^4}\right),$$  \hfill (21)

which is the same equation appeared for the Hořava-Lifshitz cosmology \cite{18,20,37}. The second term of right handed side represents the dark radiation with negative energy density. This means that the universe cannot evolve isotropically in vacuum without turning on some shearing components. Adding a matter density of $\rho = \rho_0 a^3(1+w)$ to the above equation leads to

$$H^2 = -\left(\frac{1}{a^2} + \frac{1}{2\omega} \frac{1}{a^4}\right) + \frac{\rho_0}{a^{3(1+w)}}.$$  \hfill (22)

The solution to this equation can be obtained for $-1/3 < \omega < 1/3$. Neglecting the first term of curvature, there can be a bounce in $a$ that replaces the initial singularity of the universe. This is the only case that dark radiation with negative energy density can grow with respect to a regular matter energy density.

However, small derivations from isotropy will be dominant in the small volume limit of $a \to 0 (\alpha \to -\infty)$ because the Cotton bilinear term, which is independent of $\alpha(a)$, kicks in $V_{\alpha}(\beta_{+},\beta_{-},\omega,\epsilon)$ and washes away the effects of dark radiation term. The kinetic energy of anisotropy parameter $\beta_{\pm}$ also contributes to the universe evolution. This implies that the cosmological bounce is unstable against anisotropy and the universe can be in singular state of Kasner universe. In the next section, we wish to study the mixmaster universe of the $z = 3$ Hořava-Lifshitz gravity explicitly.

IV. CHAOTIC BEHAVIORS IN REDUCED 4D PHASE SPACE

Chernoff and Barrow have showed that the mixmaster 6D phase space could be split into the product of a 4D phase space showing chaotic behavior and a 2D phase space showing regular behavior \cite{25}. Hence, we confine the dynamical
system to a 4D phase space describing the 4D static billiard in this section. Setting $\alpha = 1$, let us consider the motion of a particle (the universe) of coordinates $(\beta_+, \beta_-)$ under the potential of

$$V(\beta_+, \beta_-, \omega, \epsilon) = e^4 \left[ V_{IR} + \frac{V_{UV}^{(I)}}{192e^2\omega} + \frac{V_{UV}^{(II)}}{12e^2\omega} \right].$$

(23)

This potential has the symmetry of an equilateral triangle reflecting the equivalence of three axes in the metric. Explicitly, a particle is moving in the potential with exponential walls bounding a triangle. We mention again that near the origin $(0,0)$, the potential $V(\beta_+, \beta_-, \omega, \epsilon)$ takes approximately the form of

$$V(0, 0, \omega, \epsilon) \approx -\left( \frac{e^4}{8} + \frac{e^2}{64\omega} \right) + \left( e^4 + \frac{17e^2}{4\omega} + \frac{24\sqrt{2}e}{\omega^3/\epsilon^2} + \frac{72}{\omega^{3/2}\epsilon^2} \right) \left( \beta_+^2 + \beta_-^2 \right).$$

(24)

Comparing (24) with (8), the former reduces to the latter up to $e^4$ in the IR-limit of $\omega \to \infty$. It turned out that adding the UV-potential $V_{UV}^{(I)}$ makes the potential well deeper, compared to $V_{UV}^{(II)}$.

An important thing is to check whether the inflection point at the origin $(\beta_+, \beta_-) = (0,0)$ appears as $\omega$ varies, which might show a signal of changing from chaotic to non-chaotic behavior. This inflection point is determined by the condition of

$$V''(\beta_+, 0, \omega, \epsilon)|_{\beta_+ = 0} = V''(0, \beta_-, \omega, \epsilon)|_{\beta_- = 0} = 0,$$

(25)

which leads to an algebraic equation

$$2e^4 + \frac{17e^2}{2\omega} + \frac{48\sqrt{2}e}{\omega^{3/2}\epsilon^2} + \frac{144}{\omega^{3/2}\epsilon^2} = 0.$$  

(26)

However, we find that there is no such positive solution $\omega$ and $\epsilon$ to Eq. (26). This shows clearly that an inflection point could not be developed by adjusting $\omega$ and $\epsilon$. We have the same result even for negative $\epsilon$ because the Cotton potential $V_{IR}^{(II)}$ is always zero at the origin. This means that we could not make a transition from chaotic to non-chaotic behavior in the 4D phase space. Explicitly, as is shown in Fig. 4, the shapes of potential near the origin $(0,0)$ is not changed significantly as the parameter $\omega$ is changed from 0.01 to 1. For $\omega = 100, 1$ cases, there are no essential differences when comparing with the IR case of $\omega = \infty$ (Einstein gravity: EG). It is found that the origin of $(0,0)$ always remains global minimum, regardless of any value of $\omega$ which regulates the UV effects. The only difference is the appearance of local maxima $V_{lm}$ as $\omega$ decreases.

The chaos could be defined as being such that (i) the periodic points of the flow associated to the Hamiltonian are dense, (ii) there is a transitive orbit in the dynamical system, and (iii) there is sensitive dependence on the initial conditions. Our reduced system is described by the 4D Hamiltonian

$$\mathcal{H}_{4D} = \frac{1}{2}(p_+^2 + p_-^2) + V(\beta_+, \beta_-, \omega, \epsilon).$$

(27)

FIG. 4: Three types of potential graphs $V(\beta_+, 0, \omega, \epsilon)$ for (a) the $\omega = 100$ and $\epsilon = 100$ case (EG) without local maximum; (b) the $\omega = 0.01$ and $\epsilon = 100$ case ($z = 2$ HL) with local maximum $V_{lm} = 7.116$; (c) the $\omega = 0.01$ and $\epsilon = 1$ case ($z = 3$ HL) with local maximum $V_{lm} = 113.744$. 
Now, let us perform simulations of the dynamics and represent Poincaré sections, which describe the trajectories in phase space \((p_+, \beta_+)\) by varying the total energy \(E\) or \(H_{4D}\) of the system. We perform the analysis for three cases: \(\omega = 100, \epsilon = 100\) (EG), \(\omega = 0.01, \epsilon = 100\) (\(z = 2\) HL gravity), and \(\omega = 0.01, \epsilon = 1\) (\(z = 3\) HL gravity). We have found that the chaotic behavior persists for all \(\omega > 0\). Figs. 5, 6, and 7 represent as Einstein gravity (EG), \(z = 2\) HL gravity, and \(z = 3\) HL gravity, respectively. These show that the intersections of several computed trajectories are displaced in \((p_+, \beta_+)\) with \(\beta_− = 0\) for different values of energies. In each plot, we choose an initial point which corresponds to a prescribed kinetic energy. The results of Poincaré sections show that for lower energy within the potential well, the integrable behavior dominates and the intersections of trajectories represent closed curves. On the other hand, for higher energy within the potential well, the closed curves are broken up gradually and the bounded phase space fills with a chaotic sea. The same kinds of plots have been obtained for the other phase space \((p_−, \beta_−)\) with \(\beta_+ = 0\).

As a result, we have obtained that for \(\omega > 0\), there always exists chaotic behavior. This may contrast to the case of the loop mixmaster dynamics based on loop quantum cosmology [33], where the mixmaster chaos is suppressed by loop quantum effects [32]. However, we have to distinguish the parameter “\(\omega\)” with the quantum number “\(j\)”. The former describes the regulation of UV effects without spacetime quantization, while the latter depicts the spacetime quantization and, handles the size of the universe. Hence, the role of UV coupling parameter \(\omega\) is different from the quantum number \(j\) of the loop quantum cosmology. In our case, time variable (related to the volume of \(V = e^{3\alpha}\)) as well as two physical degrees of anisotropy \(\beta_{\pm}\) are treated in the standard way without quantization. However, in the loop quantum framework, all three scale factors were quantized using the loop techniques. Hence two are quite different: the potential wells at the origin never disappear for any \(\omega > 0\) in the \(z = 3\) Hořava-Lifshitz gravity, while in the loop quantum cosmology the height of potential wall rapidly decreases until they disappear completely as the Planck scale is reached. In order to see the similar effect like decreasing \(j\), we consider the volume change \(\alpha\) as the dynamical variable seriously and thus, need a further work in the 6D phase space.

V. CHAOTIC BEHAVIOR IN 6D PHASE SPACE

We remind the reader that the true phase space is 6D for the vacuum universe, and thus, we have a movable billiard with the potential \(V_{\alpha}(\beta_+, \beta_-, \omega, \epsilon)\) in Eq. (16) because the walls are moving with time since the logarithm of the
FIG. 6: Poincaré sections for the $\omega = 0.01$ and $\epsilon = 100$ case ($z = 2$ HL gravity) with (a) $E = -15.0$ (b) $E = -13.0$ (c) $E = -7.0$ (d) $E = -4.0$.

FIG. 7: Poincaré sections for the $\omega = 0.01$ and $\epsilon = 1$ case ($z = 3$ HL gravity) (a) $E = 3.0$ (b) $E = 20.0$ (c) $E = 50.0$ (d) $E = 60.0$. 
volume change $\alpha = \frac{1}{4}\ln V$ and its derivative are entering in the system. In this case, $\alpha$ and $p_\alpha$ are regular variables as functions of time. In this section, we investigate a possibility of finding non-chaotic behaviors by considering the small volume limit of $\alpha \to -\infty$. To this end, it would be better to introduce a new time $\tau$ defined by

$$\tau = \int \frac{dt}{V}, \quad V = e^{3\alpha},$$

(28)

which makes decoupling of the volume $\alpha$ from the shape $\beta_\pm$ explicitly. Starting from the action (2) and integrating out the space variables, we have

$$\bar{S}_{\lambda=1} = (4\pi)^2 \mu^4 \int d^7 \tau \frac{e^{3\alpha} N}{V} \left[ 6(-\alpha^2 + \beta_+^2 + \beta_-^2) - V^2 \left( 12 e^{-2\alpha} V_{IR}(\beta_+, \beta_-) + \frac{e^{-4\alpha}}{16\omega} V_{UV}(\beta_+, \beta_-) + e^{-6\alpha} V_{UV}(\beta_+, \beta_-) \right) \right],$$

(29)

where the prime ('') denotes the derivatives with respect to $\tau$. Plugging $N = 1$ into (29), we have the Lagrangian as

$$\bar{\mathcal{L}}_{\lambda=1} = (4\pi)^2 \mu^4 \left[ 6(-\alpha^2 + \beta_+^2 + \beta_-^2) - 12 e^{4\alpha} \left( V_{IR}(\beta_+, \beta_-) + \frac{e^{-2\alpha}}{192\omega} V_{UV}(\beta_+, \beta_-) + \frac{e^{-4\alpha}}{12} V_{UV}(\beta_+, \beta_-) \right) \right].$$

(30)

The canonical momenta are given by

$$\bar{p}_\pm = \frac{\partial \bar{\mathcal{L}}_{\lambda=1}}{\partial \beta'_\pm} = 12(4\pi)^2 \mu^4 \beta'_\pm, \quad \bar{p}_\alpha = \frac{\partial \bar{\mathcal{L}}_{\lambda=1}}{\partial \alpha'} = -12(4\pi)^2 \mu^4 \alpha'.$$

(31)

Then, the canonical Hamiltonian in 6D phase space is obtained to be

$$\bar{\mathcal{H}}_{6D} = \bar{p}_\alpha \alpha' + \bar{p}_+ \beta'_+ + \bar{p}_- \beta'_- - \bar{\mathcal{L}}_{\lambda=1}$$

$$= \frac{1}{2}(\bar{p}_+^2 + \bar{p}_-^2 - \bar{p}_\alpha^2) + e^{4\alpha} \left( V_{IR} + \frac{e^{-2\alpha}}{192\omega} V_{UV}(\beta_+, \beta_-) + \frac{e^{-4\alpha}}{12} V_{UV}(\beta_+, \beta_-) \right),$$

(32)

where we have chosen the parameter $12(4\pi)^2 \mu^4 = 1$ for simplicity. Then, the Hamiltonian equations of motion are obtained as

$$\beta'_\pm = \bar{p}_\pm, \quad \bar{p}_\pm = -e^{4\alpha} \frac{\partial V_{IR}}{\partial \beta_\pm} - \frac{e^{2\alpha}}{192\omega} \frac{\partial V_{UV}(\beta_+, \beta_-)}{\partial \beta_\pm} - \frac{1}{12} \frac{\partial V_{UV}(\beta_+, \beta_-)}{\partial \beta_\pm},$$

(33)

$$\alpha' = -\bar{p}_\alpha, \quad \bar{p}_\alpha = -4e^{4\alpha} V_{IR} - \frac{e^{2\alpha}}{96\omega} V_{UV}(\beta_+, \beta_-) - \frac{1}{12} V_{UV}^{CR}.$$  

(34)

We note that comparing Eqs. (33) and (34) with Eqs. (17) and (18), there is no change in the Hamiltonian and its equations of motion except replacing $t$ by $\tau$. From Eq. (34), the evolution of $\alpha$ is determined by

$$\alpha'' = 4e^{4\alpha} V_{IR} + \frac{e^{2\alpha}}{96\omega} V_{UV}(\beta_+, \beta_-) + \frac{1}{12} V_{UV}^{CR}.$$  

(35)

Then, we obtain a 6D phase space consisting in the product of a 4D chaotic one times a 2D regular phase space for the $\alpha$ and $p_\alpha$ variables. As the volume goes to zero near singularity ($e^{4\alpha} \to 0$, $p_\alpha \to 0$), one finds the limit

$$\bar{\mathcal{H}}_{6D} \to \frac{1}{2}(\bar{p}_+^2 + \bar{p}_-^2) + K \neq \bar{\mathcal{H}}_{4D}.$$  

(36)

Hence, we note that the 6D system is not asymptotic in $\tau$ to the previous 4D system.

Now, we are in a position to show whether the presence of the UV-potential can suppress chaotic behaviors existing in the IR-potential. In order to carry out it, we have to introduce two velocities: particle velocity $v_p$ and wall velocity $v_w$ defined by

$$v_p = \sqrt{\bar{p}_+^2 + \bar{p}_-^2}, \quad v_w = \frac{d\beta^w}{d\tau},$$

(37)

where the wall location $\beta^w_+$ is determined by the fact that the asymptotic potential $K$ is significantly felt by the particle as

$$\bar{p}_\alpha^2 \approx 2K = \left[ \frac{e^{4\alpha - 8\beta_+}}{12} + \frac{7e^{2\alpha - 16\beta_+}}{32\omega} + \frac{4\sqrt{2}e^{\alpha - 20\beta_+}}{3\omega^{7/6}} + \frac{4e^{-24\beta_+}}{\omega^{1/3}\epsilon^2} \right].$$

(38)
in the limit of $\beta_+ \to -\infty$. On the other hand, the particle velocity is given by

$$v_p = \sqrt{2\mathcal{H}_{6D} + \bar{p}_\alpha^2 - 2K}. \tag{39}$$

We would like to mention three limiting cases: IR-limit dominated by $V_{IR}$ and two UV-limits dominated by $V_{UV}^{CR}$ and $V_{UV}^{CC}$, respectively. In the IR-limit ($\omega \to \infty$) of Einstein gravity, the wall location is determined by

$$\beta_+^w \approx \frac{\alpha}{2} - \frac{1}{8} \ln \left[12\bar{p}_\alpha^2\right]. \tag{40}$$

Then, the wall velocity is given by

$$v_{IR}^w = -\frac{d\beta_+^w}{d\tau} \approx \frac{\bar{p}_\alpha}{2} + \frac{e^{4\alpha - 8\beta_+}}{24\bar{p}_\alpha}, \tag{41}$$

which leads to

$$|v_{IR}^w| \approx \frac{\bar{p}_\alpha}{2}. \tag{42}$$

As a result, we find that the particle velocity is always greater than the wall velocity as

$$v_p^{IR} = \sqrt{2\mathcal{H}_{6D} + \bar{p}_\alpha^2 - 2e^{4\alpha}V_{IR}} \approx |\bar{p}_\alpha| > v_{IR}^w. \tag{43}$$

Thus, there will be an infinite number of collisions of the particle against the wall since it will always catch a wall [29, 30].

Next, let us investigate what happens in the UV-limit. We mention that the Cotton bilinear term $C_{ij}C^{ij}$ is marginal in the $z = 3$ Hořava-Lifshitz action and it is expected to dominate in the UV regime. As its potential $V_{UV}^{CC}$ is shown, it is independent of volume change $\alpha$. Hence, in this UV regime, one may approximate Eq. (41) to be

$$\alpha' = -\bar{p}_\alpha, \quad \bar{p}_\alpha' \approx 0 \quad (\alpha'' \approx 0), \tag{44}$$

which imply that the scale factor ($V = \alpha_1 \alpha_2 \alpha_3 = e^{3\alpha}$) of the universe will evolve as a free particle with the fixed momentum $\bar{p}_\alpha$ and thus, the volume of space diminishes linearly at early time. Concerning the shape $\beta_\pm$ of the universe, however, the potential $V_{UV}^{CC}$ plays no role in determining the wall and particle velocities definitely. The wall velocity is zero as

$$v_{w}^{CC} = \frac{d\beta_+^w}{d\tau} = -\frac{1}{12} \bar{p}_\alpha \approx 0, \tag{45}$$

while the particle velocity is determined to be imaginary

$$v_p^{CC} \approx \sqrt{\bar{p}_\alpha^2 - \frac{4e^{-24\beta_+}}{\omega^2\epsilon^2}}, \tag{46}$$

for $\bar{p}_\alpha^2 < \frac{4e^{-24\beta_+}}{\omega^2\epsilon^2}$ in the limit of $\beta_+ \to -\infty$. In this case, the role of Cotton bilinear term is trivial in the 6D phase space.

Finally, we consider the $V_{UV}^{CR}$ term. The wall velocity takes the form

$$|v_{w}^{CR}| = \frac{|\bar{p}_\alpha|}{20}, \tag{47}$$

and the particle velocity leads to

$$v_p^{CR} \approx \sqrt{\bar{p}_\alpha^2 - \frac{4\sqrt{2}e^{\alpha - 20\beta_+}}{3\omega^2\epsilon}} \approx \frac{|\bar{p}_\alpha|}{|v_{w}^{CR}|} \tag{48}$$

in the limit of $\alpha \to -\infty$. This case is similar to the IR-limit of Einstein gravity.

In summary, we could not observe a slowing down of the particle velocity due to the UV effects. However, similar to the Einstein gravity, the mixmaster universe of the $z = 3$ deformed Hořava-Lifshitz gravity filled with stiff matter ($w = 1$) has led to a non-chaotic universe because there is a slowing down of particle velocity which is unable to reach any more the walls after some time in the moving wall picture [29].
VI. DISCUSSIONS

First of all, we wish to mention that the mixmaster universe has provided another example that the Hořava-Lifshitz gravity has shown chaotic behavior, as other chaotic dynamics of string or M-theory cosmology models [39]. This may be because we did not quantize the Hořava-Lifshitz gravity and we have studied its classical aspects only.

The two relevant parameters, which characterize the \( z = 3 \) Hořava-Lifshitz gravity, are \( \omega \) and \( \epsilon \). In the reduced 4D phase space (static billiard), there is no essential difference in the potentials between \( z = 1 \) (Einstein gravity) and \( z = 3 \) Hořava-Lifshitz gravity except the appearance of local maxima. Unfortunately, the local maxima does not change the chaotic motion significantly and thus, the chaotic behaviors persist in the \( z = 3 \) Hořava-Lifshitz gravity without a matter. In the 6D phase space (movable billiard), the important issue was to see whether the potential \( V_{UV}^{CC} \) from the Cotton bilinear term could slow down the particle velocity \( v_p \) relative to the wall velocity \( v_w \). However, we could not observe a slowing down of the particle velocity.

At this stage, we compare our results with the mixmaster universe in the generalized uncertainty principle (GUP) [40]. Considering a close connection between the \( z = 2 \) Hořava-Lifshitz gravity and GUP [37], there may exist a cosmological relation between them. The chaotic behavior of the Bianchi IX model, which was not tamed by GUP effects, means that the deformed mixmaster universe is still a chaotic system. This is mainly because two physical degrees of anisotropy \( \beta_A \) are considered as deformed, while the time variable is treated in the standard way. This supports that our approach (without quantization) is correct.

Furthermore, it was shown that adding \( (\mathbf{R})^2 \) (and possibly other) curvature terms to the general relativity leads to the interesting result that the chaotic behavior is absent [41,42]. Hence it is very curious to see why \( (\mathbf{R})^2 \) does suppress chaotic behavior but \( \frac{3}{2}R^2 - \frac{2}{3}R_{ij}R^{ij} \) does not suppress chaotic behavior. In the latter case, \( f(R) \)-action may be appropriate for this purpose [43].

Consequently, the presence of the UV-potentials from the \( z = 3 \) deformed Hořava-Lifshitz gravity cannot suppress chaotic behaviors existing in the IR-potential, which comes from the Einstein gravity.

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