A Classification of Weak Asynchronous Models of Distributed Computing

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Abstract
We conduct a systematic study of asynchronous models of distributed computing consisting of identical finite-state devices that cooperate in a network to decide if the network satisfies a given graph-theoretical property. Models discussed in the literature differ in the detection capabilities of the agents residing at the nodes of the network (detecting the set of states of their neighbors, or counting the number of neighbors in each state), the notion of acceptance (acceptance by halting in a particular configuration, or by stable consensus), the notion of step (synchronous move, interleaving, or arbitrary timing), and the fairness assumptions (non-starving, or stochastic-like). We study the expressive power of the combinations of these features, and show that the initially twenty possible combinations fit into seven equivalence classes. The classification is the consequence of several equi-expressivity results with a clear interpretation. In particular, we show that acceptance by halting configuration only has non-trivial expressive power if it is combined with counting, and that synchronous and interleaving models have the same power as those in which an arbitrary set of nodes can move at the same time. We also identify simple graph properties that distinguish the expressive power of the seven classes.

1 Introduction
Distributed computing is increasingly interested in the study of networks of natural or artificial devices, like molecules, cells, microorganisms, or nano-robots. These devices have very limited computational and communication capabilities, and are indistinguishable. In particular, a device cannot recognize whether its current communication partner is the same as a past one. This stands in stark contrast to the devices of standard computer networks, which has motivated researchers to question the suitability of traditional distributed computing models for the study of these networks, and to propose new ones. Examples include population protocols [3, 1], chemical reaction networks [14], networked finite state machines [7], the weak models of distributed computing of [9], and the beeping model [5]. A survey discussing many of them, and more, can be found in [12].

All these models share several common features, introduced to capture the limitations of the devices [7]: the network can have an arbitrary topology; all nodes of the network have a finite number of states, independent of the size of the network or its topology; all nodes...
run the same protocol; and state changes only depend on the states of a bounded number of neighbors, again independent of the size of the network.

Unfortunately, despite this very substantial common ground, the models still differ in many aspects, which makes it hard to compare results across them, or decide which features are essential for a particular result. A study of the models allows one to identify four specific junctions at which they choose different paths:

- **Detection.** In some models, agents can only detect the existence of neighbors in a certain state [9]. In others, they can count their number, up to a fixed threshold [7, 9]. For example, in biological models, cells communicate by emitting special kinds of proteins, and detecting them; in some models the cells may detect the presence of the protein when its concentration exceeds a given threshold, while in others they are able to detect different concentration levels.

- **Acceptance.** Some models compute by stable consensus, which requires all nodes to eventually agree on the outcome of the computation (but the nodes do not need to know that consensus has been reached) [3, 1, 14], while others require the nodes to reach a consensus in a halting configuration [9]. Acceptance by stable consensus is computationally powerful, since it permits the algorithm designer to concentrate on ensuring that every bad input is eventually rejected; declaring all non-rejecting states accepting ensures that every good input is eventually accepted.

- **Selection.** In some models, at each step a scheduler chooses an arbitrary set of nodes to make a step [7, 13], while in others it is exactly one, or exactly one pair of neighboring nodes [3, 1, 14]. We call the latter exclusive or interleaving models. Intuitively, interleaving models are useful when it can be assumed that process steps are much faster than the time interval between them, while the former policy does not need this assumption. In addition, they help the algorithm designer, who can assume that agents act in mutual exclusion. (Examples where this is useful can be found in the proofs of Propositions 16 and 20.) Another common option for selection is the synchronous execution model [9], where all nodes are selected in each step. Again this can be helpful for designing algorithms, but it is incompatible with exclusive selection.

- **Fairness.** Some models use fairness assumptions designed to model or approximate stochastic behavior [3, 1, 14], while others choose minimal notions, like “all nodes make a step infinitely often”, which only assume the absence of crash faults (see, e.g., [8, 10]). Stochastic-like assumptions are reasonable for biological or chemical models, but can be too strong for networks of artificial nodes, which may follow non-random execution policies. Stochastic models may be able to solve problems that cannot be solved with weaker fairness assumptions.

The goal of this paper is to explore the space of models spanned by the above parameters, and compare their computational power within a specific framework. For this we use distributed automata, a generic formalism for the description of finite-state distributed algorithms. Such an automaton consists of a set of rules that tell the nodes of a graph how to change their state depending on the states of their neighbors. Intuitively, the automaton describes an algorithm that allows the nodes of an input graph to decide, in a distributed way, whether the graph satisfies a given property. The computational power of a class of distributed automata is then given by the class of graph languages recognized by the automata in the class, or, in other words, by the graph properties that the class of automata can decide.

We start with twenty classes of distributed automata, and show that with respect to their computational power, they fall into seven different classes. This reduction is a consequence of two results presented in this paper: (1) acceptance by halting configuration only has non-
trivial expressive power if it is combined with counting; (2) both interleaving and synchronous selection have the same power as liberal selection where arbitrarily many nodes can move at the same time (and therefore, one can design an automaton in an interleaving or synchronous model, which is less error prone, and then translate it to a liberal model). Some of the simulations we design to prove the results are of independent interest. In particular, we give explicit constructions showing how to simulate interleaving models by non-interleaving ones.

The paper is organized as follows. Section 2 introduces distributed automata and their variants. Sections 3 to 5 show that the variants collapse to at most the seven equivalence classes mentioned above. Section 6 contains separation results showing that the seven classes are different. Finally, Section 7 presents further results on their expressive power. Proofs missing or only sketched in the main text can be found in the Appendix.

2 A taxonomy of distributed automata

Given sets \( X, Y \), we denote by \( 2^X \) the power set of \( X \), and by \( X^Y \) the set of functions \( Y \to X \). We define \([m : n] := \{ i \in \mathbb{Z} \mid m \leq i \leq n \}\) and \([n] := [0 : n]\), for any \( m, n \in \mathbb{Z} \) such that \( m \leq n \).

Let \( A \) be a finite set. A (\( A \)-labeled, undirected) graph is a triple \( G = (V, E, \lambda) \), where \( V \) is a finite nonempty set of nodes, \( E \) is a set of undirected edges of the form \( e = \{u, v\} \subseteq V \) such that \( u \neq v \), and \( \lambda : V \to A \) is a labeling. Isomorphic graphs are considered to be equal.

Convention: Throughout the paper, all graphs have at least two nodes and are connected.

2.1 Distributed automata

Distributed automata take a graph as input, and either accept or reject it. To define them we first introduce distributed machines.

Distributed machines. Let \( A \) be a finite set of symbols and let \( \beta \in \mathbb{N}_+ \). A (distributed) machine with input alphabet \( A \) and counting bound \( \beta \) is a tuple \( M = (Q, \delta_0, \delta, Y, N) \), where \( Q \) is a finite set of states, \( \delta_0 : A \to Q \) is an initialization function, \( \delta : Q \times [\beta]^Q \to Q \) is a transition function, and \( Y, N \subseteq Q \) are two sets of accepting and rejecting states, respectively. The function \( \delta \) updates the state of a node \( v \) based on the number of neighbors \( v \) has in each state, but it can only detect if \( v \) has 0, 1, \ldots, \( (\beta - 1) \), or at least \( \beta \) neighbors in a given state.

Selections, schedules, configurations, runs, and acceptance. A selection of a \( A \)-labeled graph \( G = (V, E, \lambda) \) is a set \( S \subseteq V \), and a schedule of \( G \) is an infinite sequence of selections \( \sigma = (S_0, S_1, S_2, \ldots) \in (2^V)^\omega \). Intuitively, the selection \( S_t \) is the set of nodes activated by the scheduler at time \( t \).

Let \( M = (Q, \delta_0, \delta, Y, N) \) be a distributed machine with input alphabet \( A \). A configuration of \( M \) on \( G \) is a mapping \( C : V \to Q \). Given a configuration \( C \) and a node \( v \in V \), we let \( N_v^C : Q \to [\beta] \) denote the function that assigns to each state \( q \) the number of neighbors of \( v \) that are in state \( q \) up to threshold \( \beta \), i.e., \( \min\{\beta, \text{card}(\{u \mid \{u, v\} \in E \land C(u) = q\})\} \). We call \( N_v^C \) the \( \beta \)-bounded multiset of states of \( v \)'s neighbors.

For any selection \( S \), we define the successor configuration of \( C \) via \( S \) to be the configuration \( \text{succ}_\delta(C, S) \) that one obtains from \( C \) if all nodes in \( S \) evaluate the transition function \( \delta \) simultaneously while the remaining nodes keep their current state. Formally, for all \( v \in V \),

\[
\text{succ}_\delta(C, S)(v) = \begin{cases} 
C(v) & \text{if } v \notin S \\
\delta(C(v), N_v^C) & \text{if } v \in S.
\end{cases}
\]
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This brings us directly to the notion of a run. Given a schedule \( \sigma = (S_0, S_1, S_2, \ldots) \), the run of \( M \) on \( G \) scheduled by \( \sigma \) is the infinite sequence \( \rho = (C_0, C_1, C_2, \ldots) \) of configurations that are defined inductively as follows, where \( \circ \) denotes function composition, and \( t \in \mathbb{N} \):

\[
C_0 = \delta_0 \circ \lambda \quad \text{and} \quad C_{t+1} = \text{succ}_s(C_t, S_t).
\]

A configuration \( C \) is accepting if \( C(v) \in Y \) for every \( v \in V \), and rejecting if \( C(v) \in N \) for every \( v \in V \). A run \( \rho = (C_0, C_1, C_2, \ldots) \) of \( M \) on \( G \) is accepting if there is a time \( t \in \mathbb{N} \) such that \( C_t \) is accepting for every \( t' \geq t \). In other words, a run is accepting if from some time on it only visits accepting configurations. Similarly, \( \rho \) is rejecting if eventually all visited configurations are rejecting. Following [3], we call this acceptance by stable consensus.

Distributed automata. Not every schedule of a distributed machine models an execution; for example, schedules in which a node is never activated are usually considered illegal. We assume that distributed machines are controlled by a scheduler that ensures that the machine executes a legal run. Formally, a scheduler is a pair \( \Sigma = (s, f) \), where \( s \) is a selection constraint that assigns to every graph \( G = (V, E, \lambda) \) a set \( s(G) \subseteq 2^V \) of permitted selections such that every node \( v \in V \) occurs in at least one selection \( S \in s(G) \), and \( f \) is a fairness constraint that assigns to every graph \( G \) a set \( f(G) \subseteq s(G) \) of fair schedules of \( G \). We call the runs with schedules in \( f(G) \) fair runs (with respect to \( \Sigma \)).

A distributed automaton is a pair \( A = (M, \Sigma) \), where \( M \) is a machine and \( \Sigma \) is a scheduler satisfying the consistency condition: for every graph \( G \), either all fair runs of \( M \) on \( G \) are accepting, or all fair runs of \( M \) on \( G \) are rejecting. Intuitively, the machine is “immune” to the scheduler because its answer is independent of the scheduler’s choices. This formalizes the standard notion of “asynchronous distributed algorithm”. Notice that the consistency condition is a very strong semantic requirement. Although we will not do so in this paper, one can prove that it is undecidable whether a given pair \( (M, \Sigma) \) satisfies it.

A accepts \( G \) if every fair run of \( A \) on \( G \) is accepting, and rejects \( G \) otherwise. The language \( L(A) \) recognized by \( A \) is the set of graphs it accepts. Two automata are equivalent if they recognize the same language.

2.2 Classifying distributed automata.

We classify automata according to four criteria: detection capabilities, acceptance condition, selection constraint, and fairness constraint. The first two criteria concern the distributed machine, and the other two the scheduler. For each criterion, we investigate some of the major options that have been considered in the literature.

Detection. In some models, agents can only detect the existence of neighbors in a certain state. This corresponds to non-counting machines, i.e., machines with counting bound \( \beta = 1 \). Other models can detect the number of neighbors up to a higher bound [9].

Acceptance. As mentioned above, distributed machines accept by stable consensus. This is the acceptance condition of population protocols and chemical reaction networks [3, 1, 14]. Other models consider a notion of acceptance where each node explicitly decides to accept or reject [9]. This notion is captured by halting automata. A machine \( M \) is halting if its transition function does not allow the nodes to leave accepting or rejecting states, i.e., if \( \delta(q, P) = q \) for every \( q \in V \cup N \) and every \( \beta \)-bounded multiset \( P \in [\beta]^Q \). In halting machines, each node knows whether the input graph will be accepted the moment it enters an accepting or rejecting state. Indeed, by the consistency condition, in every fair run, eventually either all nodes occupy accepting states, or all nodes occupy rejecting states. Since nodes can never
leave an accepting state once they enter it, each node that enters such a state knows that all other nodes will eventually do likewise. The same applies to rejecting states.

Selection. A scheduler \( \Sigma = (s, f) \) is synchronous on \( G = (V, E, \lambda) \) if \( s(G) = \{ V \} \). Intuitively, at every step all nodes make a move. \( \Sigma \) is exclusive or interleaving-based on \( G \) if \( s(G) = \{ \{ v \} \mid v \in V \} \). Intuitively, at every step exactly one node makes a move, i.e., nodes execute steps in mutual exclusion. Finally, \( \Sigma \) is liberal on \( G \) if \( s(G) = 2^V \). Intuitively, at every step an arbitrary subset of nodes makes a move. A scheduler is called synchronous if it is synchronous on every graph. Exclusive and liberal schedulers are defined analogously.

Fairness. A schedule \( \sigma = (S_0, S_1, \ldots) \) of a graph \( G \) is weakly fair if for every node \( v \) of \( G \), there exist infinitely many indices \( t \) such that \( v \in S_t \). In other words, a schedule is weakly fair if every node is active infinitely often. A scheduler \( \Sigma = (s, f) \) is weakly fair if \( f(G) \) contains precisely the weakly-fair schedules of \( s(G)^\omega \) for every graph \( G \). This is the weakest fairness constraint one can impose on distributed automata; it only excludes runs in which a node crashes, and does not participate in the computation anymore.

With respect to a given selection constraint \( s \), a schedule \( \sigma = (S_0, S_1, \ldots) \in s(G)^\omega \) of a graph \( G \) is strongly fair if for every finite sequence \( (T_0, \ldots, T_n) \in s(G)^* \) there exist infinitely many indices \( t \) such that \( (S_t, S_{t+1}, \ldots, S_{t+n}) = (T_0, T_1, \ldots, T_n) \). Intuitively, strong fairness requires that every possible finite sequence of selections is scheduled infinitely often. If every node is selected independently with positive probability, stochastic schedules are almost surely strongly fair. A scheduler \( \Sigma = (s, f) \) is strongly fair if for every graph \( G \), the set \( f(G) \) contains precisely the strongly-fair schedules of \( s(G)^\omega \).

\begin{itemize}
\item \textbf{Remark 1.} Whether a schedule \( \sigma \) of a graph \( G = (V, E, \lambda) \) is strongly fair or not depends on \( s(G) \). For example, if \( s(G) = \{ V \} \), then the synchronous schedule \( V^\omega \) is strongly fair, but if \( s(G) = 2^V \), then it is not.
\end{itemize}

Our notion of strong fairness implies an apparently stronger one, used frequently in the literature, stating that in a strongly fair run, a sequence of configurations that is enabled infinitely often must occur infinitely often:

\begin{itemize}
\item \textbf{Lemma 2.} Let \( A \) be a strongly fair automaton and \( (D_0, \ldots, D_n) \) be a sequence of configurations of \( A \) such that \( D_{i+1} \) is the successor configuration of \( D_i \) via some selection \( S_i \) permitted by \( A \), for \( i \in [0: n] \). For any fair run \( \rho = (C_0, C_1, \ldots) \) of \( A \), if \( C_i = D_0 \) for infinitely many indices \( i \in \mathbb{N} \), then \( (C_j, \ldots, C_{j+n}) = (D_0, \ldots, D_n) \) for infinitely many indices \( j \in \mathbb{N} \).
\end{itemize}

The classification above yields 24 classes of automata (four classes of machines and six classes of schedulers). To assign mnemonics to them, we use lowercase letters for the most restrictive machine variants (i.e., non-counting and halting), and the same letters in uppercase for the other variants. With schedulers we proceed the other way round, assigning lowercase letters to the most liberal variants (i.e., liberal selection and weak fairness). Intuitively, due to the consistency condition, the more liberal a scheduler, the harder it is for an automaton to recognize a graph language, because more runs have to yield the same result. So, loosely speaking, we expect the expressive power to increase with the number of uppercase letters.

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Detection} & \textbf{Acceptance} & \textbf{Selection} & \textbf{Fairness} \\
\hline
d: non-counting & a: halting & a: liberal & f: weak \\
D: counting & A: stable consensus & S: exclusive & F: strong \\
\hline
\end{tabular}
\end{center}

We denote each class of automata by a string \( wxyz \in \{d, D\} \times \{a, A\} \times \{s, S, \$\} \times \{f, F\} \). The class of languages recognized by \( \mathcal{G}(wxyz) \) is denoted \( \mathcal{G}(wxyz) \). The following
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Lemma states all relations between language classes that follow directly from the definitions. Statement 1 abbreviates “$G(dxyz) \subseteq G(Dxyz)$ for all $x \in \{a,A\}, y \in \{a,S,\$\}, z \in \{f,F\}$”. We use the same convention in Statements 2 to 5, and throughout the paper. That is, any statement with four-letter strings containing the wildcard symbol * must be expanded into the list of all statements that can be obtained by replacing identically positioned occurrences of * with the same letter.

Lemma 3. 1. $G(d***) \subseteq G(D***)$, 2. $G(*a**) \subseteq G(*A**)$, 3. $G(*f**) \subseteq G(*F**)$, 4. $G(**sf) \subseteq G(**Sf)$, 5. $G(**sf) \subseteq G(**$f)$, 6. $G(**$F$) \subseteq G(**f)$.

Lemma 3 leads to the diagram in Figure 1, showing 20 automata classes (we have $G(**$f$) = G(**$F$) by Statements 3 and 6). An arrow between two classes means that every graph language recognized by the source class is also recognized by the target class.

The reader probably finds Figure 1 very complicated. We also do, and this was the motivation for the present paper. How many of these classes are really different? In the next sections we show that classes with the same color have the same expressivity, and thus that the diagram of Figure 1 collapses to the one of Figure 4, which contains only seven classes.

![Figure 1](image-url)

**Figure 1** Initial classification of the models according to the class of graph languages they recognize. Arrows indicate inclusion between classes of languages. The diagram can be thought of as lying in four-dimensional space, where each dimension represents one of our four parameters. The vectors of the “coordinate system” are labeled with the statement number of Lemma 3 that proves the inclusions in the corresponding direction. In the coming sections, classes are shown to be equal if and only if they have the same color, reducing the 20 classes to 7, as shown in Figure 4. This means in particular that we completely eliminate the dimension of selection (shown in dotted lines), leaving us with only three dimensions.

### 3 The weakest classes have no expressiveness

We prove that das*-automata have no expressive power, and the results in Sections 4 and 5 will generalize this to da**-automata. Intuitively, if agents cannot count their neighbors, and must reach a halting configuration, then they cannot distinguish any two graphs. Formally, a graph property is **trivial** if either every graph satisfies it, or no graph satisfies it. We have:
Theorem 4. Every das*-automaton recognizes a trivial graph property.

Proof sketch. By Statement 3 of Lemma 3, it suffices to prove the claim for dasF-automata. So let $A$ be a dasF-automaton, and let $G$ and $H$ be two graphs (connected and with at least two nodes by convention). Assume that $A$ accepts $G$ but rejects $H$. By the consistency condition, all fair runs of $A$ on $G$ accept, and all fair runs on $H$ reject. Now let $\rho^G$ and $\rho^H$ be any such runs, and let $t \in \mathbb{N}$ be a time at which all nodes in $\rho^G$ and $\rho^H$ have halted. We define a new graph $K$ that consists of $t$ copies $\{G_i\}_{i \in [1:t]}$ and $\{H_i\}_{i \in [1:t]}$ of $G$ and $H$, with additional edges defined as follows. For each node $w^X$ of the original graph $X \in \{G, H\}$, we denote its copy in $X_i$ by $w^X_i$, where $i \in [1:t]$. Let $u^G$ and $v^G$ be two adjacent nodes of $G$, and $u^H$ and $v^H$ be two adjacent nodes of $H$. We add the connecting edges $\{w^X_i, v^X_{i+1}\}$ for all $i \in [1:t]$ and $X \in \{G, H\}$, as well as the edge $\{u^G_i, u^H_i\}$. This is illustrated in Figure 2.

Figure 2 Graph $K$ used in the proof of Theorem 4.

We show that there is a fair run $\rho$ of $A$ on $K$ that neither accepts nor rejects. It follows that $A$ does not satisfy the consistency condition, contradicting the hypothesis. Since $A$ is a non-counting automaton, initially every node $w^X$ except for $u^G_i$ and $u^H_i$ “sees” the same neighborhood as the corresponding node $w^X$ in the original graph $X$. Only the two nodes $u^G_i$ and $u^H_i$ may have a different neighborhoods than $u^G$ and $u^H$, and this might affect their behavior starting at time 1. Their different behavior can be propagated to other nodes in subsequent rounds, but it takes time before it reaches every node. We exploit this to construct $\rho$ in such a way that some nodes of $K$ (those of $G_1$) reach an accepting state, while others (those of $H_1$) reach a rejecting state. Since $A$ is a halting automaton, these nodes will never change their state again, and so the run is neither accepting nor rejecting. □

4 Synchronicity can always be simulated

We show that every class with synchronous selection is equivalent to the corresponding class with liberal selection. Albeit non-trivial, this is easy to prove by a standard technique of distributed computing known as alpha synchronizer. (The term was introduced in [4], but a similar idea appeared earlier in cellular automata theory [11].) Given a machine $M = (Q, \delta_0, \delta, Y, N)$, we define a machine $\tilde{M} = (\tilde{Q}, \tilde{\delta}_0, \tilde{\delta}, \tilde{Y}, \tilde{N})$ such that for every graph $G$, the unique synchronous run of $M$ on $G$ accepts (rejects) iff every weakly fair run $\rho$ of $\tilde{M}$ on $G$ accepts (rejects). The gadget achieving this is called a “synchronizer”, because it ensures that the nodes of $G$ behave “as in the synchronous case”, even when selection is liberal.

The set of states of $\tilde{M}$ is $\tilde{Q} := Q \times Q \times \{0, 1, 2\}$. Given $(q, q', i) \in \tilde{Q}$, we call $q$ the past $M$-state, $q'$ the current $M$-state, and $i$ the phase. The initialization function is given by $\tilde{\delta}_0(a) := (\delta_0(a), \delta_0(a), 0)$. In order to define the transition function $\tilde{\delta}$, let $v$ be a node in state $(q, q', i)$. If $v$ is selected by the scheduler, its next state is determined as follows:

- If at least one neighbor of $v$ is in phase $(i - 1) \mod 3$, then $v$ does not change state.
- Intuitively, if some neighbor is still one phase behind, then $v$ waits for it to “catch up”.
- If every neighbor of $v$ is in phase $i$ or $(i + 1) \mod 3$, then $v$ moves to $(q', q''(i + 1) \mod 3)$, where $q''$ is defined as follows. Let $N_v$ be the set of neighbors of $v$, and for each
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We show that this is possible, exploiting the fact that strongly fair models, i.e., we compare a class of the form $\mathbf{SF}$ with the corresponding class $\mathbf{sF}$. On an intuitive level, their equivalence might be less surprising than the subsequent result presented in Section 5.2 because strong fairness provides a way to break symmetry, which can be exploited to simulate exclusivity. Nevertheless, neither class trivially subsumes the other, so we have to prove inclusions in both directions.

### 5.1 Exclusivity under strong fairness

We start by considering strongly fair models, i.e., we compare a class of the form $\mathbf{SF}$ with the corresponding class $\mathbf{sF}$. On an intuitive level, their equivalence might be less surprising than the subsequent result presented in Section 5.2 because strong fairness provides a way to break symmetry, which can be exploited to simulate exclusivity. Nevertheless, neither class trivially subsumes the other, so we have to prove inclusions in both directions.

**Theorem 6.** For every $\mathbf{SF}$-automaton there is an equivalent $\mathbf{sF}$-automaton.

**Proof sketch.** Given a $\mathbf{SF}$-automaton $A$, we construct a $\mathbf{sF}$-automaton $B$ such that for all input graphs $G$, every strongly fair run of $B$ on $G$ simulates a strongly fair run of $A$ on $G$. The difficulty lies in the fact that $A$ and $B$ do not share the same notion of strong fairness because they have different selection constraints. While $A$’s liberal scheduler guarantees that arbitrary sequences of selections will occur infinitely often, $B$’s exclusive scheduler can select only one node at a time. To simulate $A$’s behavior with $B$, we adapt the synchronizer from Section 4. Just like there, nodes keep track of their previous and current state in $A$, as well as the current phase number modulo 3. However, instead of updating their state in every phase, they only do so if an additional activity flag is set. Thus, we can simulate an arbitrary selection $S$ by raising the flags of exactly those nodes that lie in $S$. The outcome of a phase simulated in this way will be the same as if all the nodes in $S$ made a transition simultaneously. The main issue is how to set the activity flags in each phase in such a way that every finite sequence $(S_1, \ldots, S_n)$ of selections is guaranteed to occur infinitely often. We show that this is possible, exploiting the fact that $B$’s scheduler is strongly fair.

**Theorem 7.** For every $\mathbf{SF}$-automaton there is an equivalent $\mathbf{sF}$-automaton.
Proof sketch. First, we note that the only way exclusivity could possibly be useful is to break symmetry between adjacent nodes. This is because for an independent set (i.e., a set of pairwise non-adjacent nodes), the order of activation is irrelevant: whether the scheduler activates them all at once or one by one in some arbitrary order, the outcome will always be the same. Consequently, to simulate a run with exclusivity, it suffices to simulate a run where no two adjacent nodes are active at the same time. We provide a simple protocol that makes use of the strong fairness constraint (in an environment with liberal selection) to ensure that if a node wants to execute a transition, then it will eventually be able to do so while all its neighbors remain passive.

5.2 Exclusivity under weak fairness

We now show that even in the absence of strong fairness, the restriction to interleaving schedulers does not increase expressive power. At first sight, this may be quite surprising because exclusivity inherently breaks symmetry, whereas an automaton with liberal selection and weak fairness can always be assumed to run synchronously and thus be incapable of breaking symmetry. In fact, it is easy to come up with examples of automata that exploit exclusivity to ensure termination.

Proposition 8. For every **sf-automaton, there exists a **Sf-automaton that recognizes the same graph language but makes use of exclusive selection to ensure termination. If run synchronously, it never terminates (and hence it is not a valid **sf-automaton).

However, although the automata described in Proposition 8 make use of exclusivity, they do not really benefit from it; they only recognize languages that can also be recognized by liberal automata. As we will see in Theorem 11, this observation can be generalized to arbitrary **Sf-automata. Intuitively, since exclusivity does not add any expressive power, it can in a certain sense be simulated without needing to break symmetry.

The proof of Theorem 11 is based on the notion of Kronecker cover. The Kronecker cover (also known as bipartite double cover) of a graph \( G = (V, E, \lambda) \) is the bipartite graph \( G' = (V', E', \lambda') \) where \( V' = V \times \{0, 1\} \), \( E' = \bigcup_{\{u, v\} \in E} \{(u, 0), (v, 1)\}, \{(u, 1), (v, 0)\}\), and \( \lambda'((v, i)) = \lambda(v) \) for all \( (v, i) \in V' \). An example is provided in Figure 3.

![Figure 3](image)

Figure 3 A graph (on the left) and its Kronecker cover (on the right).

The Kronecker cover in Figure 3 is connected because the nodes in \( \{u, v, w\} \times \{0, 1\} \) form a cycle. The following lemma generalizes this observation.

Lemma 9. The Kronecker cover of a connected graph \( G \) is connected if and only if \( G \) contains a cycle of odd length, (i.e., if and only if \( G \) is non-bipartite).

If a Kronecker cover is connected, then it constitutes a legal input for a distributed automaton. The next key lemma shows that, in this case, a weakly fair automaton cannot even distinguish between a graph and its Kronecker cover.
Lemma 10. For every $***f$-automaton $A$ with input alphabet $\Lambda$ and every non-bipartite $\Lambda$-labeled graph $G$, $A$ accepts $G$ if and only if it accepts the Kronecker cover of $G$.

We can now prove the main technical result of this section:

Theorem 11. For every $***f$-automaton there is an equivalent $**sf$-automaton.

Proof sketch. Given a $**sf$-automaton $A$, we construct an equivalent $**$f-automaton $B$ (i.e., a synchronous automaton). This is sufficient to prove the claim, because we know from Theorem 5 that $B$ can always be simulated by a $**sf$-automaton using a synchronizer.

Let $G$ be an input graph for $A$. If we were guaranteed that the labels of $G$ define a proper vertex coloring (i.e., edges connect nodes of different colors), then the task would be straightforward. Indeed, since each color of a proper coloring represents an independent set, $B$ could simply operate in cyclically repeating phases, each one activating precisely the nodes of one of the colors. As explained in the proof of Theorem 7, such a run is equivalent to a run of an exclusive scheduler that activates the nodes of each independent set one by one (in some arbitrary order).

This approach can be adapted to bipartite graphs because a bipartite graph has exactly two possible 2-colorings. However, computing one of the two 2-colorings would require to break symmetry, which a $**$f-automaton cannot do. So instead, the states of automaton $B$ have two components, one corresponding to each coloring, and nodes update both components when they are activated.

Using these ideas, we construct $B$ in such a way that it recognizes the same bipartite graphs as $A$. Then we use Lemmas 9 and 10 to prove that $L(A) = L(B)$. Indeed, if $G$ is not bipartite, then by Lemma 9, its Kronecker cover $G'$ is connected and therefore constitutes a legal input for a distributed automaton. By Lemma 10, $B$ accepts $G$ if and only if it accepts $G'$. Since Kronecker covers are bipartite by definition, we know from the above discussion that $B$ accepts $G'$ if and only if $A$ accepts $G'$. Finally, again by Lemma 10, $A$ accepts $G'$ if and only if it accepts $G$. From this chain of equivalences, we can conclude that $G$ is accepted by $B$ if and only if it is accepted by $A$. ◀

6 Separations

In Sections 3, 4 and 5 we have shown that the classes of graph languages in Figure 1 collapse to at most the seven classes shown on the left of Figure 4. In this section we show that the seven classes are all different. For this we examine four graph languages, and determine which classes are expressive enough to recognize them:

- $\mathcal{B}$: The language of graphs with set of labels \{black, white\} having at least one black node.
- $\mathcal{S}$: The language of star graphs, i.e., the set of all connected, unlabeled graphs in which one node (the center) has degree at least 2, and all others (the leaves) have degree 1.
- $\mathcal{C}_3$: The language containing one single graph, namely the cycle $C_3$ with three nodes labeled by 0, 1, and 2, respectively.
- $\mathcal{S}_{even}$: The language of even stars, i.e., the graphs of $\mathcal{S}$ with an even number of leaves.

The results are summarized on the right of Figure 4.

Recognizing properties of labeled graphs: the language $\mathcal{B}$

The main difference between the two types of acceptance is that halting automata cannot recognize properties that require nodes to wait an unlimited amount of time for some information that may never arrive, while even the simplest class of automata accepting by stable consensus can recognize some of those properties, such as $\mathcal{B}$. 
Proposition 12. $B$ is recognizable by a $dAsF$-automaton, but not by any $*a**$-automaton.

Proof sketch. The $dAsf$-automaton has two states, called black and white. The initial state of a node is given by its label. Black nodes remain always black, and white nodes with a black neighbor become black. Since graphs are connected by assumption, if a graph contains some black node then eventually all nodes are black, otherwise all nodes stay white.

For the second part, one can show that $DasF$-automata cannot distinguish between an entirely white cycle and a sufficiently long path graph whose nodes are all white except for two black nodes at the endpoints. (The argument is similar to the proof of Theorem 4.)

Recognizing properties of unlabeled graphs: the language $S$

We show in Proposition 13 that $dAsf$-automata cannot recognize any non-trivial property of unlabeled graphs (which we identify with the labeled graphs whose nodes all carry the same label). That is, while $dAsf$-automata can recognize properties of the labeling of a graph, they cannot recognize any non-trivial property of its structure. Then we show in Proposition 14 that the strong fairness of $dAsF$-automata allows them to recognize $S$.

Proposition 13. $dAsf$-automata can only recognize trivial properties of unlabeled graphs. In particular, $S$ is not recognizable by a $dAsf$-automaton.

Proof. Let $A$ be a $dAsf$-automaton, and let $\rho = (C_0, C_1, \ldots)$ be the synchronous run of $A$ on an unlabeled graph $G = (V, E)$, i.e., the run scheduled by $V^\omega$. We show that $A$ either accepts all unlabeled graphs, or rejects all unlabeled graphs. Since $V^\omega$ is a weakly fair schedule, $\rho$ is a fair run, and so by the consistency condition $A$ accepts $G$ iff $\rho$ is accepting. Since $G$ is unlabeled, in $C_0$ every node of $G$ is in the same state $q_0$, which is independent of $G$. Moreover, since $\rho$ is synchronous and $A$ is non-counting, in each configuration $C_i$ every node of $G$ is in the same state $q$, which is also independent of $G$. So the states visited by $\rho$ are independent of $G$, and so $A$ either accepts all unlabeled graphs, or rejects all unlabeled graphs.

Proposition 14. $S$ is recognizable by a $dAsF$-automaton and by a $DasF$-automaton.

Proof sketch. We give a $dAsF$-automaton that recognizes $S$. The states of the automaton are pairs $(d, c)$, where $d \in \{\text{leaf, center, unknown, neither}\}$ is the estimate of $v$, and $c \in \{0, 1\}$ is its color. Every time a node is selected it flips its color. When a node with estimate unknown sees two neighbors with different colors, it switches to center, and if from then on it sees a neighbor with estimate center, it moves to neither. Strong fairness is crucial for...
correctness: by Lemma 2, it ensures that a node that is not a leaf will eventually be selected in a configuration in which at least two of its neighbors have different colors.

Now we give a DAsf-automaton with $\beta = 2$ that recognizes $S$. Since $\beta = 2$, a node can determine for each state $q$ if it has 0, 1, or at least 2 neighbors in $q$. The automaton’s states are $\{\text{init}, \text{leaf}, \text{non-leaf}, \text{accept}, \text{reject}\}$. Initially all nodes are in state $\text{init}$. The nodes update their estimates depending on the number of neighbors (0, 1, or at least 2) in each state.

Symmetry breaking: the language $C_3$

We show that the language $C_3$ requires both acceptance by stable consensus and strong fairness to be recognizable. Intuitively, both of them are required to distinguish $C_3$ from arbitrarily long cycles that repeat the labeling of $C_3$ cyclically.

 Proposition 15. $C_3$ is recognizable by a DAsF-automaton, but neither by DA*F-automata nor by Da*F-automata.

Proof sketch. Our DAsF-automaton for $C_3$ checks two conditions: first, that the input graph is a cycle with cyclic labeling $0 \rightarrow 1 \rightarrow 2$, and second, that it contains exactly one node labeled by 2 (which implies that the cycle has length 3). For both conditions, we use a similar trick as in Proposition 14, relying on acceptance by stable consensus and strong fairness to eventually break symmetry between otherwise indistinguishable nodes. To verify the second condition, each node labeled by 2 successively sends signals in both directions through the cycle, and checks that those signals always come back from the expected direction.

For the second part of the claim, we show that DA*F- and Da*F-automata cannot distinguish $C_3$ from $C_6$, the hexagon whose nodes are labeled by $0 \rightarrow 1 \rightarrow 2 \rightarrow 0 \rightarrow 1 \rightarrow 2$ (and back to 0). To do so, given a fair run $\rho_3$ of such an automaton on $C_3$, we construct a fair run $\rho_6$ on $C_6$ that “duplicates” the behavior of $\rho_3$. In the case of Da*F-automata, this duplication is performed only until $\rho_3$ has reached a halting configuration (because otherwise $\rho_6$ would violate the strong fairness constraint).

Counting neighbors modulo a number: the language $S_{\text{even}}$

Since counting automata can only count up to a threshold $\beta$, no node can directly observe that it has an even number of neighbors. This makes the language $S_{\text{even}}$ rather difficult to recognize. We now show that the combination of counting and strong fairness can do the job. The proof also provides a good example where exclusivity helps to design an algorithm.

 Proposition 16. $S_{\text{even}}$ is recognizable by a DasF-automaton.

Proof sketch. In Proposition 14 we have exhibited a Dasf-automaton $A$ recognizing $S$. We now give a DasF-automaton $B$ that uses counting, exclusivity, and strong fairness to further decide if the number of leaves is even. Loosely speaking, $B$ first executes $A$; if $A$ rejects, then $B$ rejects, because the graph is not even a star. If $A$ accepts, then $B$ enters a new phase during which it counts the number of leaves modulo 2. By Theorem 7, $B$ is equivalent to a DasF-automaton.

We can assume that when $A$ accepts, all nodes are labeled with either leaf or center (the unique non-leaf). We give an informal description of $B$. Leaves can be in states visible, invisible, dead, even, or odd. While leaves have not been counted by the center, they alternate between the states visible and invisible. The center only increments its modulo-2 counter if exactly one leaf is visible. After a leaf is counted, it moves to dead. When all leaves become dead, i.e., when they have all been counted, the center decides whether to accept or reject; the leaves read the decision from the counter, and move to even or odd accordingly.
The next two results show that recognizing $S_{\text{even}}$ needs both counting and strong fairness.

1. **Proposition 17.** $S_{\text{even}}$ is not recognizable by $\text{dA*f}$-automata.

   **Proof.** We show that for every $\text{dA*f}$-automaton $A$ there exist stars $G$ and $G'$ such that exactly one of $G$ and $G'$ belongs to $S_{\text{even}}$, but $A$ either accepts both of them or rejects both of them. Let $\beta \geq 1$ be $A$'s counting bound, and let $G$ and $G'$ be the stars with $\beta + 1$ and $\beta + 2$ leaves, respectively. Now consider the synchronous runs $\rho$ and $\rho'$ of $A$ on $G$ and $G'$. By symmetry, and since the number of leaves exceeds $\beta$ in both $G$ and $G'$, at every time $t \in \mathbb{N}$, the center is in the same state in $\rho$ and $\rho'$, and likewise all leaves are in the same state. So the sequences of states visited by the center and the leaves are the same in both $\rho$ and $\rho'$, and therefore $\rho$ is accepting iff $\rho'$ is accepting.

2. **Proposition 18.** $S_{\text{even}}$ is not recognizable by $\text{dA*f}$-automata.

   **Proof sketch.** Given a $\text{dA*f}$-automaton $A$, the proof identifies an even number $n$, depending on $A$, such that if $A$ accepts the star with $n$ leaves, then it cannot reject the star with $n + 1$ leaves. The proof is involved, and can be found in the Appendix.

### 7 Expressive power

As a first application of our results, we investigate the expressivity of our models for graph languages that depend only on the labeling function of a graph, and not on its topology.

Given a $\mathcal{L}$-labeled graph $G = (V, E, \lambda)$, where $\mathcal{L} = \{\ell_1, \ldots, \ell_k\}$, let $\#_G: \mathcal{L} \to \mathbb{N}$ be the mapping that assigns to each label $\ell$ the number $\#_G(\ell)$ of nodes of $V$ such that $\lambda(v) = \ell$. A language is **Presburger-definable** if there is a formula $\varphi(x_1, \ldots, x_k)$ of Presburger arithmetic such that a $\mathcal{L}$-labeled graph $G$ belongs to the language if and only if $\varphi(\#_G(\ell_1), \ldots, \#_G(\ell_k))$ holds. An example of such a language is $\mathcal{B}$, the set of graphs that contain a black node.

We show that $\text{DAsF}$-automata recognize all Presburger languages, but none of the other six classes do. The negative part of the result follows easily from the table in Figure 4.

1. **Proposition 19.** There exist Presburger-definable languages that are not recognizable by $\text{d**f*}^*$, $\text{sa**}^*$, or $\text{**f*}$-automata.

   **Proof.** By Proposition 12, $\text{sa**}$-automata cannot recognize the language $\mathcal{B}$, which is Presburger-definable. Furthermore, by Propositions 14, 17 and 18, $\text{dA*f}$- and $\text{DA*f}$-automata can recognize the language $S$ of star graphs but not the language $S_{\text{even}}$ of stars with an even number of leaves. This implies that $\text{dA*f}$- and $\text{DA*f}$-automata cannot recognize the Presburger-definable language of graphs with an odd number of nodes, because the intersection of this language with $S$ is equal to $S_{\text{even}}$, and languages recognizable by distributed automata are closed under intersection (by a standard product construction).

For the positive part, we proceed in three steps: First, following [1] and Section 5 of [3], we introduce **graph population protocols**, a graph variant of the well-known population protocol model introduced in [2, 3]. Then we recall a result of [3] showing that graph population protocols recognize all Presburger-definable languages. Finally, we show that every graph population protocol can be simulated by a $\text{DAsF}$-automaton.

Our definition of graph population protocols is equivalent to that of [1, 3], but reuses the notation of Section 2 as far as possible. A **graph population protocol** $\Pi = (Q, \delta_0, \delta, Y, N)$ is defined like a $\text{DAsF}$-automaton with machine $M = \Pi$, except for the following differences:

- The transition function is of the form $\delta: Q^2 \to Q^2$. 

A selection of a graph $G = (V, E, \lambda)$ is an ordered pair $S = (u, v) \in V^2$ of adjacent nodes (instead of a singleton $\{u\} \subseteq V$), and the selection constraint on $G$ is $\{(u, v) \mid \{u, v\} \in E\}$. $C_t(v)$ is defined inductively as follows, for $t \in \mathbb{N}$ and $v \in V$:

$$C_0(v) = \delta_0(\lambda(v)) \quad \text{and} \quad C_{t+1}(v) = \begin{cases} 
\delta(C_t(v), C_t(u))_{\text{fst}} & \text{if } S_t = (v, u) \text{ for some } u, \\
\delta(C_t(u), C_t(v))_{\text{snd}} & \text{if } S_t = (u, v) \text{ for some } u, \\
C_t(v) & \text{otherwise,}
\end{cases}$$

where $P_{\text{fst}}$ and $P_{\text{snd}}$ denote the first and second component of a pair $P$. So, intuitively, the scheduler selects two adjacent nodes, which update their states according to $\delta$. The definitions of all other relevant notions remain the same. This holds in particular for acceptance by stable consensus and strong fairness (which are baked into the model), and the consistency condition. Standard population protocols correspond to graph population protocols on complete graphs, where every pair of distinct nodes is connected by an edge.

It is shown in [3] that standard population protocols recognize all Presburger-definable languages. Further, Theorem 7 of [3] shows that every language recognized by population protocols is also recognized by graph population protocols. Loosely speaking, given a population protocol, one constructs the protocol on graphs in which, when an edge of the graph is selected, either the two nodes connected by it interact as in the population protocol, or they swap their states. By strong fairness, the states of the nodes can “move around the graph”, and any pair of states eventually interacts infinitely often. The choice between interacting or swapping is nondeterministic, but it can be simulated by deterministic transitions (see [3]). Therefore, in order to show that $\mathcal{DA}^+\mathcal{F}$-automata recognize all Presburger-definable languages, it suffices to simulate graph population protocols with distributed automata. As in the proof of Proposition 16, we make use of exclusivity to simplify the construction.

▶ **Proposition 20.** For every graph population protocol there is an equivalent $\mathcal{DA}^+\mathcal{F}$-automaton.

**Proof sketch.** We present a simulation that runs a population protocol on a distributed automaton. To this end, the automaton has to simulate a scheduler that selects ordered pairs of adjacent nodes instead of arbitrary sets of nodes. For any pair $(u, v)$ that is selected to perform a transition, let us call $u$ the initiator and $v$ the responder of the transition. By Theorem 7, we may assume that the automaton’s scheduler selects a single node in each step.

The main idea is as follows: When a node $u$ is selected and sees that it can become the initiator of a transition, it declares its intention to do so by raising the flag “?”. Then $u$ waits until some neighbor $v$ is selected and raises the flag “!”, which signals that $v$ wants to become the responder of a transition. If this happens, the next time $u$ is selected, it computes its new state according to the state of $v$ and the transition function of the population protocol, but also keeps its old state in memory so that $v$ can still see it. After that, $v$ also updates its state, and finally $u$ deletes its old state, which completes the transition. Throughout this protocol, the nodes verify that they have exactly one partner during each transition. If this condition is violated, they raise the error flag “⊥” and abort their current transition. ◀

▶ **Corollary 21.** $\mathcal{DA}^+\mathcal{F}$-automata recognize all Presburger-definable languages.

8 Conclusions

We have conducted an extensive comparative analysis of the expressive power of weak asynchronous models of distributed computing. Our analysis has reduced the initial “jungle”
of twenty different models to only seven. This reduction in complexity is achieved by Theorems 4, 5, 6, 7, and 11, all of which have a clear and intuitive interpretation.

We have also shown that the seven classes are distinct, and have identified inclusions and non-inclusions between them. However, two inclusions remain open: Are \( \text{Dasf} \) or \( \text{DAsf} \) included in \( \text{dAsF} \)? Intuitively, this asks if strong fairness and acceptance by stable consensus can be used to simulate counting. We can provide a positive answer for graphs of bounded degree (a limitation common in practice), because in this case even \( \text{dA*F} \) and \( \text{DAsF} \) coincide.

▶ **Proposition 22.** For every \( \text{DA*F} \)-automaton \( A \) and every \( k \in \mathbb{N} \) there is a \( \text{dA*F} \)-automaton \( B \) equivalent to \( A \) on graphs of maximum degree \( k \).

However, for arbitrary graphs we conjecture that neither \( \text{Dasf} \) nor \( \text{DAsf} \) are included in \( \text{dAsF} \).

Finally, we have made a first step towards characterizing the graph languages recognizable by the different classes, by transferring a characterization for population protocols.

As a last note, observe that our results hold for decision problems on undirected graphs that can be solved by consensus in the framework of distributed automata. Several of our constructions (e.g., those in Theorems 5 and 7) rely on bidirectional communication, which is not guaranteed on directed graphs. Furthermore, exclusive selection leads to higher computational power for non-decision problems. For instance, it can be used to solve the vertex coloring problem on graphs of bounded degree (by a standard greedy algorithm), which, for symmetry reasons, is impossible in a model with synchronous selection.

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A Appendix

A.1 Proofs of Section 2

\textbf{Lemma 2.} Let $A$ be a strongly fair automaton and $(D_0, \ldots, D_n)$ be a sequence of configurations of $A$ such that $D_{i+1}$ is the successor configuration of $D_i$ via some selection $S_i$ permitted by $A$, for $i \in [0:n]$. For any fair run $\rho = (C_0, C_1, \ldots)$ of $A$, if $C_i = D_0$ for infinitely many indices $i \in \mathbb{N}$, then $(C_j, \ldots, C_{j+n}) = (D_0, \ldots, D_n)$ for infinitely many indices $j \in \mathbb{N}$.

\textbf{Proof.} Let $\mathcal{E} = \{E_1, \ldots, E_k\}$ be the set of configurations that occur infinitely often in $\rho$. Notice that these configurations can all reach each other because otherwise they could not occur infinitely often. The assumption is that $D_0 \in \mathcal{E}$. We construct a finite sequence $\sigma$ of selections permitted by $A$ such that for every $i \in [1:k]$, the sequence of configurations visiting starting from $E_i$ and applying $\sigma$ contains either the subsequence $(D_0, \ldots, D_n)$, or some configuration $E'_i \notin \mathcal{E}$. This suffices to prove the claim because from a certain point on, \rho visits only configurations in $\mathcal{E}$, and by strong fairness the schedule fragment $\sigma$ is guaranteed to be chosen infinitely often by the scheduler. Since any configuration $E'_i \notin \mathcal{E}$ may only occur finitely often, the only possibility is that the subsequence $(D_0, \ldots, D_n)$ occurs infinitely often.

It remains to construct a suitable sequence $\sigma$. We proceed by induction, constructing a series of sequences $\sigma_0, \sigma_1, \ldots, \sigma_k$ such that for $j \in [0:k]$, the sequence $\sigma_j$ satisfies the desired property for every $i \in [1:j]$. It then suffices to choose $\sigma = \sigma_k$. As the base case, we set $\sigma_0 = \varepsilon$ (the empty sequence). Now, given $\sigma_j$, we distinguish two cases in order to construct $\sigma_{j+1}$. If starting from $E_{j+1}$ and applying $\sigma_j$, the automaton visits some configuration $E'_{j+1} \notin \mathcal{E}$, then we simply set $\sigma_{j+1} = \sigma_j$. Otherwise, let $E'_{j+1}$ be the final configuration reached from $E_{j+1}$ by applying $\sigma_j$. Since $E'_{j+1} \in \mathcal{E}$ and $D_0 \in \mathcal{E}$, there exists a sequence of selections $\sigma'$ that leads the automaton from $E'_{j+1}$ to $D_0$. Therefore, if starting from $E_{j+1}$, the automaton applies the schedule fragment $\sigma_{j+1} = \sigma_j \cdot \sigma' \cdot S_0 \cdots S_{n-1}$, then it traverses a sequence of configurations ending with $(D_0, \ldots, D_n)$. Moreover, since $\sigma_j$ is a prefix of $\sigma_{j+1}$, the property already established for $\sigma_j$ with respect to $E_1, \ldots, E_j$ also holds for $\sigma_{j+1}$.

\textbf{Lemma 3.} 1. $G(\text{d***}) \subseteq G(\text{D***})$. 2. $G(\text{*A**}) \subseteq G(\text{A**})$. 3. $G(\text{**f}) \subseteq G(\text{***F})$. 4. $G(\text{**sf}) \subseteq G(\text{***F})$. 5. $G(\text{**sf}) \subseteq G(\text{**f})$. 6. $G(\text{**F}) \subseteq G(\text{**f})$.

\textbf{Proof.} 1. Non-counting automata are a subclass of counting automata.

2. Halting automata are a subclass of automata accepting by stable consensus.

3. Let $A = (M, s, f)$ be a $\text{**sf}$-automaton, and let $G = (V, E, \lambda)$ be a graph. The set $f(G)$ contains the weakly-fair runs of $s(G)^\omega$. Now consider $A' = (M, s, f')$, where $f'(G)$ contains the strongly-fair runs of $s(G)^\omega$. Since the set of permitted selections is the same for $A$ and $A'$, we have $f'(G) \subseteq f(G)$. Therefore, since $A$ satisfies the consistency condition, so does $A'$, and thus $A'$ is a $\text{***F}$-automaton with $L(A') = L(A)$.

4. Let $A = (M, s, f)$ be a $\text{**sf}$-automaton, and let $G = (V, E, \lambda)$ be a graph. We have $s(G) = 2^V$, and $f(G)$ contains the weakly-fair runs of $s(G)^\omega$. Let $s'(G) = \{\{v\} \mid v \in V\}$, and let $f'(G)$ be the weakly-fair runs of $s'(G)^\omega$. We have $f'(G) \subseteq f(G)$. Proceed now as in 3.

5. The argument is fully analogous to that of 4., the only difference being that $s'(G) = \{V\}$.

6. Let $A = (M, s, f)$ be a $\text{**FSF}$-automaton. We have $s(G) = \{V\}$. Further, the run $\rho$ scheduled by $V^\omega$ is strongly fair (because $V$ is the only possible selection). So $f(G) = \{\rho\}$. Let $A' = (M, s, f')$ be the unique $\text{***F}$-automaton with machine $M$. Since the run $\rho$ scheduled by $V^\omega$ is also weakly fair, we have $f'(G) = \{\rho\} = f(G)$. It follows that $L(A) = L(A')$. \hfill $\blacksquare$
A.2 Proofs of Section 3

Theorem 4. Every das\textsuperscript{F}-automaton recognizes a trivial graph property.

Proof. By Statement 3 of Lemma 3, it suffices to prove the claim for das\textsuperscript{F}-automata. So let us consider a das\textsuperscript{F}-automaton \( A \), and assume for the sake of contradiction that there exist two graphs \( G \) and \( H \) such that \( A \) accepts \( G \) and rejects \( H \). Let \( \rho^G = (C_0^G, C_1^G, \ldots) \) and \( \rho^H = (C_0^H, C_1^H, \ldots) \) be strongly fair runs of \( A \) on \( G \) and \( H \), respectively. By the consistency condition, \( \rho^G \) is accepting and \( \rho^H \) is rejecting. Based on that, we will construct a new graph \( K \) and a strongly fair run \( \rho \) of \( A \) on \( K \) that is neither accepting nor rejecting. This means that \( A \) does not satisfy the consistency condition, and therefore does not qualify as a distributed automaton, a contradiction.

We start by constructing \( K \). Let \( t \in \mathbb{N} \) be a time at which all nodes in \( \rho^G \) and \( \rho^H \) have halted (i.e., all nodes in \( C_t^G \) and \( C_t^H \) have reached an accepting or rejecting state). Our new graph \( K \) consists of \( t \) copies \( \{G_i\}_{i \in [1:t]} \) of \( G \) and \( t \) copies \( \{H_i\}_{i \in [1:t]} \) of \( H \), which are connected as follows. For each node \( w^X \) of the original graph \( X \in \{G, H\} \), we denote its copy in \( X_i \) by \( w_i^X \), where \( i \in [1:t] \). Let \( u^G \) and \( \rho^G \) be two adjacent nodes of \( G \), and \( u^H \) and \( \rho^H \) be two adjacent nodes of \( H \). (Recall that all graphs are assumed to be connected and have at least two nodes.) In addition to the edges in each copy \( X_i \), graph \( K \) also contains the connecting edges \( \{u_i^X, v_i^X\} \) for all \( i \in [1:t] \) and \( X \in \{G, H\} \), as well as the edge \( \{u_i^G, u_i^H\} \). An illustration of this construction is provided in Figure 5.

![Figure 5 Graph K used in the proof of Theorem 4.](image)

The important feature of \( K \) is that every node \( w_i^X \) except for \( u_i^G \) and \( u_i^H \) has a neighborhood equivalent to the neighborhood of the corresponding node \( w_i^X \) in the original graph \( X \). This is because \( A \) is a non-counting automaton, where each node can only see the set of states of its neighbors, without being able to count them. So initially, the additional edges between different copies of the same graph \( X \) do not change the “perception” of the nodes they connect. However, the two nodes \( u_i^G \) and \( u_i^H \) may have a different neighborhoods than \( u_i^G \) and \( u_i^H \), and this might affect their behavior starting at time 1. Their different behavior can be propagated to other nodes in subsequent rounds, but this proposition takes time before it can reach nodes in the extreme parts of the graph.

We now construct a suitable run \( \rho = (C_0, C_1, \ldots) \) of \( A \) on \( K \). During the first \( t \) steps, \( \rho \) tries to copy the behavior of \( \rho^G \) and \( \rho^H \). More precisely, let \( \sigma^G \) and \( \sigma^H \) be schedules that schedule \( \rho^G \) and \( \rho^H \), respectively. We use them to define a schedule \( \sigma \) of \( K \) that schedules \( \rho \): at every time \( r \in [0:t] \), each copied node \( w_i^X \) is selected by \( \sigma \) if and only if the original node \( w_i^X \) is selected by \( \sigma^X \), where \( i \in [1:t] \) and \( X \in \{G, H\} \). Note that this does not violate the strong fairness constraint because we have only fixed a finite prefix of \( \sigma \). We can therefore extend \( \sigma \) in such a way that it satisfies the strong fairness constraint.

It remains to show that \( \rho \) is either accepting or rejecting. For this, we prove by induction over \( r \) that for all \( r \in [0:t] \) and \( i \in [1:t-r] \), \( X \in \{G, H\} \), every copied node \( w_i^X \) in \( \rho \) at time \( r \) is in the same state as the original node \( w_i^X \) in \( \rho^X \) at time \( r \), i.e., \( C_r(w_i^X) = C_r^X(w_i^X) \). This obviously holds for \( r = 0 \), since every copy \( w_i^X \) has the same label as \( w_i^X \). For \( r \in [1:t] \),
the induction hypothesis tells us that at time \( r - 1 \), each copy \( w_i^X \) with \( i \in [1 : t - r + 1] \) is in the same state as \( w^X \), and if \( i \leq t - r \), then \( w_i^X \) also sees the same set of states as \( w^X \) in its neighborhood. Moreover, by the definition of \( \sigma \), node \( w_i^X \) is selected if and only if \( w^X \) is selected. Hence, provided \( i \in [1 : t - r] \), the two nodes are also in the same state at time \( r \).

Since at time \( t \) all nodes of \( G \) are in an accepting state in \( \rho^G \), and all nodes of \( H \) are in a rejecting state in \( \rho^H \), the same holds in \( \rho \) for the copies of those nodes in \( G_1 \) (the “left-most” copy of \( G \)) and \( H_1 \) (the “right-most” copy of \( H \)). And since \( A \) is a halting automaton, these nodes will never change their state again. But this means that \( \rho \) never reaches a stable consensus, and therefore that it is neither accepting nor rejecting.

\[ \square \]

A.3 Proofs of Section 4

\[ \textbf{Theorem 5.} \text{ For every } **s\$**-automaton there is an equivalent } **s\$**-automaton. \]

\[ \textbf{Proof.} \text{ Let } A = (M, s, f) \text{ be a } **s\$**-automaton, and let } G = (V, E, \lambda) \text{ be a graph. Let } \hat{A} = (\hat{M}, \hat{s}, \hat{f}), \text{ where } \hat{M} \text{ is as described in Section 4, } \hat{s} \text{ is liberal, and } \hat{f} \text{ is weakly (strongly) fair if } f \text{ is so. By the consistency condition, the unique run } \rho \text{ of } A \text{ on } G \text{ is either accepting or rejecting. By the definition of } \hat{M}, \text{ and since all runs of } \hat{f}(G) \text{ are at least weakly fair, if } \rho \text{ is accepting then every fair run of } \hat{A} \text{ is accepting, and if } \rho \text{ is rejecting then every fair run of } \hat{A} \text{ is rejecting. So } \hat{A} \text{ also satisfies the consistency condition, and } L(A) = L(\hat{A}). \] \[ \square \]

A.4 Proofs of Section 5.1

\[ \textbf{Theorem 6.} \text{ For every } **sF**-automaton there is an equivalent } **SF**-automaton. \]

\[ \textbf{Proof.} \text{ Given a } **sF**-automaton } A, \text{ we construct a } **SF**-automaton } B \text{ such that for all input graphs } G, \text{ every strongly fair run of } B \text{ on } G \text{ simulates a strongly fair run of } A \text{ on } G. \text{ Since } A \text{ satisfies the consistency condition by hypothesis, this property implies that } B \text{ does too, and moreover that } B \text{ accepts a graph if and only if } A \text{ accepts it. The difficulty lies in the fact that } A \text{ and } B \text{ do not share the same notion of strong fairness because they have different selection constraints. While } A \text{'s liberal scheduler guarantees that arbitrary sequences of selections will occur infinitely often, } B \text{'s exclusive scheduler can select only one node at a time.}

To simulate } A \text{'s behavior with } B, \text{ we slightly adapt the synchronizer construction from Section 4. Just like there, nodes keep track of their previous and current state in } A, \text{ as well as the current round number modulo 3. However, instead of updating their state in every round, they only do so if an additional } \textit{activity flag} \text{ is set. Thus, we can simulate an arbitrary selection } S \text{ by raising the flags of exactly those nodes that lie in } S. \text{ The outcome of a round simulated in this way will be the same as if all the nodes in } X \text{ made a transition simultaneously.}

Now, the main issue is how to set the flags in each round in such a way that every finite sequence \( (S_1, \ldots, S_n) \) of selections is guaranteed to occur infinitely often. To achieve this, we take advantage of the fact that } B \text{'s scheduler is strongly fair with respect to exclusive selection. We use the following (deterministic) rules: If node } v \text{ is selected while it is in round } i \text{ mod 3 and none of its neighbors are yet in round } (i + 1) \text{ mod 3, then } v \text{ raises its flag; the next time } v \text{ is selected and allowed to move, it will simulate a transition of } A, \text{ lower its flag, and move to round } (i + 1) \text{ mod 3. Otherwise, if } v \text{ is selected when its flag is down and some of its neighbors have already reached the next round, it simply moves to round } (i + 1) \text{ mod 3 without simulating a transition.
Formally, if the machine of $A$ is $M = (Q, \delta_0, \delta, Y, N)$ with input alphabet $A$ and counting bound $\beta$, we define the machine of $B$ as $M' = (Q', \delta'_0, \delta', Y', N')$, where

$$Q' = \frac{Q_{\text{previous}} \times Q_{\text{current}} \times \{0, 1, 2\} \times \{\bot, \top\}}{\text{round} \text{flag}},$$

$Y'$ and $N'$ are defined analogously, and $\delta'_0(a) = (\delta(a), \delta(a), 0, \bot)$ for all $a \in A$. The transition function $\delta'$ is described as follows. Let $v$ be a node, and assume it is selected by the scheduler.

- In case $v$ is in state $(q, q', i, \bot)$:
  - if none of $v$'s neighbors are yet in round $(i + 1)$ mod 3, then $v$ moves to $(q, q', i, \top)$;
  - else, if some neighbor of $v$ is still in round $(i - 1)$ mod 3, then $v$ stays in $(q, q', i, \bot)$;
  - else, $v$ moves to state $(q'', q', (i + 1) \text{ mod } 3, \bot)$.

- In case $v$ is in state $(q, q', i, \top)$:
  - if some neighbor of $v$ is still in round $(i - 1)$ mod 3, then $v$ stays in $(q, q', i, \top)$;
  - else, $v$ moves to $(q'', q', (i + 1) \text{ mod } 3, \bot)$, where $q'' = \delta(q', P)$ and $P$ is the $\beta$-bounded multiset consisting of the current states of the neighbors who are in round $i$, and the previous states of the neighbors who are in round $(i + 1)$ mod 3.

Notice that the above construction allows the scheduler of $B$ to choose an arbitrary selection $S$ in each round. For instance, the scheduler can first bring all nodes to round $i$ mod 3, next select all nodes in $S$ (one by one) to raise their flags, then select those same nodes again so that they can perform their transitions and move to round $(i + 1)$ mod 3, and finally select all the remaining nodes to bring them to the next round as well. To prevent nodes outside of $S$ from being activated, the scheduler has to select them in some order that ensures that at least one of their neighbors is already in the next round (for example, a breadth-first or depth-first order starting from the nodes in $S$). Since the scheduler is strongly fair, by Lemma 2, every finite sequence of selections appears infinitely often.

**Theorem 7.** For every **SF-automaton there is an equivalent **SF-automaton.

**Proof.** First, we note that the only way exclusivity could possibly be useful is to break symmetry between adjacent nodes. This is because for an independent set (i.e., a set of pairwise non-adjacent nodes), the order of activation is irrelevant: whether the scheduler activates them all at once or one by one in some arbitrary order, the outcome will always be the same. More precisely, if we consider a graph $G = (V, E, \lambda)$, a configuration $C$ on $G$, and some independent set of nodes $U \subseteq V$, then the scheduler can choose any sequence of selections $(S_1, \ldots, S_n)$ such that $\bigcup_{i \in [1:n]} S_i = U$ and $\text{card}([i \in [1:n] | v \in S_i]) = 1$ for all $v \in U$. Regardless of the scheduler’s choice, the configuration $C'$ reached from $C$ via the schedule fragment $(S_1, \ldots, S_n)$ will always be the same. Consequently, to simulate a run with exclusivity, it suffices to simulate a run where no two adjacent nodes are active at the same time.

We now describe a simple protocol that makes use of the strong fairness constraint (in an environment with liberal selection) to ensure that if a node wants to execute a transition, then it will eventually be able to do so while all of its neighbors remain passive. Suppose that an active node $v$ wants to transition from state $q$ to state $q'$. To this end, it first goes into an intermediate state $(q, q')$ that declares this intention. Then, the next time $v$ is activated by the scheduler, it checks that none of its neighbors are in an intermediate state of the form $(p, p')$. If the check passes, $v$ switches to state $q'$. Otherwise, it goes back to state $q$ and tries again the next time it is activated. By Lemma 2, the strong fairness constraint guarantees that $v$ will infinitely often be able to execute a transition.
More formally, given a **SF-automaton with machine \( M = (Q, \delta_0, \delta, Y, N) \) and counting bound \( \beta \), we can simulate it by a ***sF-automaton with machine \( M' = (Q', \delta'_0, \delta', Y', N') \), where

\[
Q' = Q \cup (Q \times Q), \quad Y' = Y \cup (Y \times Y), \quad N' = N \cup (N \times N),
\]

\( \delta'_0 \) is the extension of \( \delta_0 \) to the codomain \( Q' \), and \( \delta' \) is defined as follows: For \( q, q' \in Q \) and \( P \in [\beta]^{Q'} \) such that \( P \) contains no state \( (p, p') \in Q' \), we have

\[
\delta'(q, P) = (q, \delta(q, P)) \quad \text{and} \quad \delta'((q, q'), P) = q',
\]

and for \( q, q' \in Q \) and \( P \in [\beta]^{Q'} \) such that \( P \) contains at least one state \( (p, p') \in Q' \), we have

\[
\delta'(q, P) = q \quad \text{and} \quad \delta'((q, q'), P) = q.
\]

The first case corresponds to the situation where a node can make progress because none of its neighbors are in an intermediate state, whereas the second case corresponds to the situation where a node must wait for some neighbors to either complete or abort their current transition attempt.

### A.5 Proofs of Section 5.2

**Proposition 8.** For every **sf-automaton, there exists a ***sf-automaton that recognizes the same graph language but makes use of exclusive selection to ensure termination. If run synchronously, it never terminates (and hence it is not a valid ***sf-automaton).**

**Proof.** We first describe the machine of a very simple \( \text{daSF} \)-automaton \( A \) that recognizes the trivial language of all unlabeled graphs but relies on exclusive selection to terminate. It has the state set \( Q = \{p, q, h\} \), where \( p \) is initial, and \( h \) is halting and accepting. The transition function \( \delta \) is defined as follows: if \( v \) moves to \( q \); if \( v \) and all its neighbors are in state \( q \), then \( v \) moves to \( p \); otherwise, \( v \) moves to \( h \). For every unlabeled graph \( G \), in the synchronous run of \( A \) on \( G \) all nodes keep alternating forever between states \( p \) and \( q \) (recall that graphs are connected and have at least two nodes), whereas in a run with exclusive selection, all nodes eventually end up in the accepting state \( h \).

Now, using a standard product construction, we can easily transform any **sf-automaton \( B \) into an equivalent ***sf-automaton \( C \) whose machine never halts under synchronous execution: \( C \) simply simulates \( A \) and \( B \) in parallel and accepts precisely when both accept.

**Lemma 9.** The Kronecker cover of a connected graph \( G \) is connected if and only if \( G \) contains a cycle of odd length, (i.e., if and only if \( G \) is non-bipartite).

**Proof.** If \( G = (V, E, \lambda) \) does not contain any cycle of odd length, it is easy to see that its Kronecker cover consists of two disjoint copies of \( G \). Indeed, since containing no odd cycle is equivalent to being bipartite (see, e.g., [6, Prp. 1.6.1]), we know that \( V \) can be partitioned into two sets \( V_0 \) and \( V_1 \) such that every edge of \( G \) connects a node in \( V_0 \) to one in \( V_1 \). Hence, in the Kronecker cover \( G' \), we obtain one copy of \( G \) over the set of nodes \( (V_0 \times \{0\}) \cup (V_1 \times \{1\}) \) and another (disjoint one) over the set \( (V_0 \times \{1\}) \cup (V_1 \times \{0\}) \).

It remains to show that if \( G \) contains an odd cycle, then \( G' \) is connected. We proceed in two steps. First, consider some cycle \( v_1v_2\ldots v_nv_1 \) of odd length \( n \) in the original graph \( G \). Since \( n \) is odd, this cycle is replicated in \( G' \) by the cycle

\[(v_1, 0)(v_2, 1)\ldots(v_n, 0)(v_1, 1)(v_2, 0)\ldots(v_n, 1)(v_1, 0)\]
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of length $2n$. (If $n$ were even, we would get two disjoint cycles of length $n$ instead.) Second, since $G$ is connected, for any node $u \in V$ there exists a path $w_1w_2 \ldots w_m$ in $G$ such that $w_1 = u$ and $w_m = v_1$. This path is replicated in $G'$ by the two paths

$$(w_1, 0)(w_2, 1) \ldots (w_m, i) \quad \text{and} \quad (w_1, 1)(w_2, 0) \ldots (w_m, j),$$

where $(i, j) = (1, 0)$ if $m$ is even, and $(i, j) = (0, 1)$ if $m$ is odd. This means that both $(u, 0)$ and $(u, 1)$ are connected to the aforementioned cycle of length $2n$, and since $u$ was chosen arbitrarily, it follows that $G'$ is connected.

Lemma 10. For every **$f$-automaton $A$ with input alphabet $A$ and every non-bipartite $A$-labeled graph $G$, $A$ accepts $G$ if and only if it accepts the Kronecker cover of $G$.\)

Proof. It suffices to prove the claim for **$f$- and **$f$-automata, since **$f$-automata can be regarded as a special case of both. In the following, let $A = (M, \Sigma)$ and $G = (V, E, \lambda)$. Since $G$ is non-bipartite (i.e., it contains a cycle of odd length), we know by Lemma 9 that its Kronecker cover $G' = (V', E', \lambda')$ is connected and therefore qualifies as valid input for $A$.

Let us begin with the case where $A$ is synchronous, i.e., a **$f$-automaton, and consider the (unique) runs $\rho$ and $\rho'$ of $A$ on $G$ and $G'$, respectively. Recall that $V' = V \times \{0, 1\}$. For every node $v$ of $G$, its copies $(v, 0)$ and $(v, 1)$ in $G'$ have the same label as $v$ and an equivalent multiset of neighbors (i.e., all their neighbors are copies of $v$’s neighbors). It is thus easy to see by induction that in every round $i \in \mathbb{N}$, $(v, 0)$ and $(v, 1)$ are in the same state in $\rho'$ as $v$ is in $\rho$. Therefore, the $i$-th configuration of $\rho'$ is accepting if and only if the $i$-th configuration of $\rho$ is accepting, and hence $A$ accepts $G'$ precisely if it accepts $G$.

We now turn to the case where $A$ is a **$f$-automaton. Consider any schedule $\sigma = (S_0, S_1, \ldots) \in (2^V)^\omega$ that satisfies the constraints of the scheduler $\Sigma$. To prove the claim, it suffices to show that there exists a schedule $\sigma'$ of $G'$ that also satisfies the constraints of $\Sigma$ such that the run $\rho'$ of $A$ on $G'$ scheduled by $\sigma'$ is accepting if and only if the run $\rho$ of $A$ on $G$ scheduled by $\sigma$ is accepting. Indeed, by the consistency condition, this implies that $A$ accepts $G$ if and only if it accepts $G'$.

We choose $\sigma' = (S'_0, S'_1, \ldots) \in (2^{V'})^\omega$ such that

$$S'_{2t} = S_t \times \{0\} \quad \text{and} \quad S'_{2t+1} = S_t \times \{1\}$$

for all $t \in \mathbb{N}$. That is, for every node $v$ of $G$, if $v$ is active at time $t$, then its copy $(v, 0)$ in $G'$ is active at time $2t$, and its copy $(v, 1)$ is active at time $2t + 1$. Note that since $\sigma$ is weakly fair, so is $\sigma'$. Furthermore, the exclusivity of $\sigma$ also carries over to $\sigma'$ (this is why we do not schedule $(v, 0)$ and $(v, 1)$ simultaneously). However, $\sigma'$ is not strongly fair in general, and therefore the assumption that $A$ is a **$f$-automaton is essential.

Now, since $(v, 0)$ and $(v, 1)$ are not connected, and since both have the same label as $v$ and an equivalent multiset of neighbors, it is again easy to see by induction that the following holds: at every even time $2i$, both copies are in the same state as $v$ is at time $i$, while at every odd time $2i + 1$, copy $(v, 0)$ is already in the same state as $v$ at time $i + 1$, but copy $(v, 1)$ is still in the state $v$ had at time $i$. Here we rely on the fact that each selection $S_i$ is a singleton, which ensures that if $v$ is active in $G$ at time $i$, then no other node is active at the same time. This means that $(v, 0)$ and $(v, 1)$ receive the same multiset of states from their neighbors in $G'$ at times $2i$ and $2i + 1$, respectively. Consequently, the $(2i)$-th configuration of $\rho'$ is accepting if and only if the $i$-th configuration of $\rho$ is accepting, and the $(2i + 1)$-th configuration of $\rho'$ is accepting if and only if both the $i$-th and the $(i + 1)$-th configurations of $\rho$ are accepting. Given that legal runs must eventually reach a stable consensus (i.e., only accepting or only rejecting configurations after a certain time), this means that $\rho'$ is accepting if and only if $\rho$ is accepting.

$\blacksquare$
Theorem 11. For every $\ast \ast f$-automaton there is an equivalent $\ast \ast sf$-automaton.

Proof. In the following, we show how, for a given $\ast \ast sf$-automaton $A$, we can construct an equivalent $\ast \ast sf$-automaton $B$ (i.e., a synchronous automaton). This is sufficient to prove the claim because we know from Theorem 5 that $B$ can always be simulated by a $\ast \ast sf$-automaton using a synchronizer.

First of all, let us observe that the task would be straightforward if we were guaranteed that the labels of the input graph define a proper vertex coloring. Indeed, since each color of a proper coloring represents an independent set, $B$ could simply operate in cyclically repeating phases that correspond to the different colors. More precisely, if the given colors were 0, ..., $k - 1$, then in the $i$-th round (i.e., the $i$-th time all nodes change state synchronously), only the ($i \mod k$)-colored nodes would evaluate the transition function of the simulated automaton $A$. As explained in the first paragraph of the proof of Theorem 7, such a run is equivalent to a run of an exclusive scheduler that activates the nodes in each independent set one by one (in some arbitrary order).

Obviously the above approach only works if we are given a proper coloring. Nevertheless, it can be adapted to a special case of uncolored graphs: if the input graph happens to be bipartite, then there exist exactly two possible 2-colorings. This is because as soon as we fix the color of a single node, there is only one possible choice of color for all the remaining nodes. However, choosing one of the two 2-colorings would require to break symmetry, which a $\ast \ast sf$-automaton cannot do. So instead, we simply work with both colorings in parallel.

We now go into more details on how to simulate a $\ast \ast sf$-automaton $B$ on bipartite graphs. Let $M = (Q, \delta_0, \delta, Y, N)$ be the machine of $A$ with input alphabet $A$ and counting bound $\beta$, and let $\{0, 1\}$ be a set of colors that we will use to color the graph. At any point in time in an execution of $B$, each node $v$ stores a pair of states $(q_0, q_1) \in Q \times Q$, where $q_0$ represents $v$'s current state in case its color is 0, and similarly $q_1$ represents $v$'s current state in case its color is 1. This way, $B$ can run the aforementioned round-based simulation of $A$ for both possible 2-colorings in parallel. To simulate the case where $v$ is 0-colored, $v$ looks at the state in its own 0-component but at the states in its neighbors' 1-component (since the neighbors must be 1-colored if $v$ is 0-colored). To simulate the case where $v$ is 1-colored, the procedure is the other way around.

More formally, the machine of $B$ can be defined as $M' = (Q', \delta'_0, \delta', Y', N')$, where

$$Q' = Q \times Q \times \{0, 1\}, \quad Y' = Y \times Y \times \{0, 1\}, \quad N' = N \times N \times \{0, 1\},$$

$\delta'_0(a) = (\delta(a), \delta(a), 0)$ for all $a \in A$, and the transition function $\delta'$ is defined as follows, for $q_0, q_1 \in Q$ and $P \in [\beta]^{Q'}$:

$$\delta'((q_0, q_1, 0), P) = (\delta(q_0, P_1), q_1, 1),$$

$$\delta'((q_0, q_1, 1), P) = (q_0, \delta(q_1, P_0), 0),$$

where $P_0$ and $P_1$ are the $\beta$-bounded projections of $P$ to the two first state components, i.e.,

$$P_0 : p \mapsto \min\{\beta, \sum_{p_1 \in Q, i \in \{0, 1\}} P(p, p_1, i)\},$$

$$P_1 : p \mapsto \min\{\beta, \sum_{p_0 \in Q, i \in \{0, 1\}} P(p_0, p, i)\},$$

for all $p \in Q$. The third state component counts the number of synchronous rounds modulo 2. If the round number is even, each node behaves as if it were 0-colored and its neighbors were 1-colored. Thus, each node updates its 0-component according to its neighbors' 1-components. Meanwhile, the 1-component remains unchanged because 1-colored nodes are supposed to
remain passive in even rounds. If the round number is odd, everything is the other way around.

The above construction of $B$ is based on the assumption that the input graph is bipartite. However, we now argue that in fact this assumption is not necessary. To do so, we have to distinguish two cases:

- If the input graph $G$ is bipartite, then by construction, the synchronous run of $B$ on $G$ simulates in parallel two runs of $A$ on $G$ with exclusive selection. By the consistency condition, this implies that $G$ is accepted by $B$ if and only if it is accepted by $A$.
- If $G$ is not bipartite, then by Lemma 9, its Kronecker cover $G'$ is connected and therefore constitutes a legal input for a distributed automaton. Now, by Lemma 10, $B$ accepts $G$ if and only if it accepts $G'$. Since $G'$ is bipartite (by the definition of a Kronecker cover), we know from the above discussion that $B$ accepts $G'$ if and only if $A$ accepts $G'$.

Finally, again by Lemma 10, $A$ accepts $G'$ if and only if it accepts $G$. From this chain of equivalences, we can conclude that $G$ is accepted by $B$ if and only if it is accepted by $A$. Notice that in the case where the input graph is not bipartite, $B$ simulates $A$ on the Kronecker cover $G'$ instead of the actual graph $G$. So in some sense, our construction only performs a “pseudo simulation”, where the simulated run may not correspond to any possible run on $G$. Nevertheless, this is sufficient because $\mathsf{dAsF}$-automata cannot distinguish between $G$ and $G'$.

A.6 Proofs of Section 6

Proposition 14. $S$ is recognizable by a $\mathsf{dAsF}$-automaton and by a $\mathsf{DAsf}$-automaton.

Proof. We first present a $\mathsf{dAsF}$-automaton that recognizes $S$. The states of the automaton are pairs $(d, c)$, where $d \in \{\text{leaf, center, unknown, neither}\}$ is the estimate of $v$, and $c \in \{0, 1\}$ is its color. The accepting states are those with estimate leaf or center, and the rejecting states are those with estimate unknown or neither. Initially all nodes are in state (unknown, 0). Let $(d, c)$ be the current state of a node $v$, and let $NE(v)$ denote the current set of estimates of the neighbors of $v$. If $v$ is selected by the scheduler, then it moves to the state $(d', c')$, where $c' = 1 - c$, and $d'$ is given by:

(a) If neither $\in NE(v)$, then $d' = \text{neither}$.
(b) If neither $\notin NE(v)$, $d = \text{unknown}$, center $\notin NE(v)$, and at least two neighbors of $v$ have different colors, then $d' = \text{center}$.
(c) If neither $\notin NE(v)$, $d = \text{unknown}$, center $\in NE(v)$, and at least two neighbors of $v$ have different colors, then $d' = \text{neither}$.
(d) If neither $\notin NE(v)$, $d = \text{unknown}$, $NE(v) = \{\text{center}\}$, and all neighbors of $v$ have the same color, then $d' = \text{leaf}$.
(e) If neither $\notin NE(v)$, $d = \text{center}$, and center $\in NE(v)$, then $d' = \text{neither}$.
(f) If neither $\notin NE(v)$, $d = \text{leaf}$, and at least two neighbors of $v$ have different colors, then $d' = \text{neither}$.
(g) Otherwise $d' = d$.

Assume that $G$ is not a star. If it consists of exactly two nodes connected by an edge, then it is easy to see that the estimate of both nodes remains forever unknown, so $G$ is rejected. Otherwise, $G$ contains at least one edge $\{u, v\}$ such that both $u$ and $v$ have degree at least 2. We show that eventually at least one of $u$ and $v$ reaches estimate neither. By (a), every node eventually reaches estimate neither, and so $G$ is rejected.
First we claim that both $u$ and $v$ eventually reach states with estimate \textit{center} or \textit{neither}. This is the point at which we make crucial use of strong fairness: by Lemma 2, it ensures that $v$ is eventually selected in a configuration in which at least two neighbors of $v$ have different colors. If in this configuration $v$ has estimate \textit{unknown}, then $v$ moves either to \textit{neither} (cases (a) and (c)) or \textit{center} (case (b)), and if it has estimate \textit{leaf}, then $v$ moves to \textit{neither} (cases (a) and (f)). The same holds for $u$, and so the claim is proved.

By the claim, at least one of $u$ and $v$ eventually reaches estimate \textit{neither}, in which case we are done, or both eventually reach \textit{center}; in this case, the next time one of the two is selected it moves to \textit{neither} (case (e)), and we are also done.

Assume now that $G$ is a star. We show that every node ends up with estimate \textit{leaf} or \textit{center}. Since leaves have only one neighbor, cases (b), (c), and (f) never apply, and so they can never reach estimate \textit{center}. This implies that case (e) also never applies for leaves. Further, as long as the center has estimate \textit{unknown}, all leaves remain in \textit{unknown}, because (a) and (d) do not apply. It follows that the center also remains in \textit{unknown} until it is selected in a configuration in which at least two neighbors have different colors, which eventually happens by strong fairness; at that moment it moves to \textit{center} (case (b)). Since (e) never applies, the center maintains the estimate \textit{center} forever. Once the center has reached estimate \textit{center}, whenever a leaf is selected it changes its estimate to \textit{leaf} (case (d)). After that, no other rule than (g) ever applies, and so the leaf maintains estimate \textit{leaf} forever. This concludes the proof of the first part of the proposition.

For the second part we present a \texttt{Dzasf}-automaton with counting bound $\beta = 2$ that recognizes $S$. We only sketch the automaton, since the ability to count makes the task of recognizing $S$ easy. Recall that $\beta = 2$ means that for each state $q$ a node can detect if it has zero, exactly one, or at least two neighbors in $q$.

The states of the automaton are $\{\text{init}, \text{leaf}, \text{non-leaf}, \text{accept}, \text{reject}\}$. The yes and no states are $\text{accept}$ and $\text{reject}$, respectively.

Initially all nodes are in state $\text{init}$. Let $v$ be a node. Observe that, since the automaton can count, a selected node can directly observe if it is a leaf or not. When $v$ is selected:

(a) If $v$ has only one neighbor, then
   (a.1) if the neighbor is in state $\text{init}$ or $\text{non-leaf}$, $v$ moves to $\text{leaf}$;  
   (a.2) if the neighbor is in state $\text{leaf}$ or $\text{reject}$, $v$ moves to $\text{reject}$; and  
   (a.3) if the neighbor is in state $\text{accept}$, $v$ moves to $\text{accept}$.

(b) If $v$ has more than one neighbor, then
   (b.1) if at least one neighbor is in state $\text{reject}$ or $\text{non-leaf}$, $v$ moves to $\text{reject}$;  
   (b.2) else if at least one neighbor is in state $\text{init}$, $v$ moves to $\text{non-leaf}$;  
   (b.3) else (all neighbors in states $\text{leaf}$ or $\text{accept}$), $v$ moves to $\text{accept}$.

Assume $G$ is a star. By (a.1) and (b.2), a node can only reach state $\text{leaf}$ (non-leaf) if it really is a leaf (non-leaf) of $G$. This fact, together with an inspection of (a.2) and (b.1), shows that a node can only reach $\text{reject}$ if $G$ is not a star. Further inspection of (a.3) and (b.3) shows that it can only reach state $\text{accept}$ if $G$ is a star. So it only remains to prove that every node eventually reaches $\text{accept}$ or $\text{reject}$. By (a.2) and (a.3) it suffices to show that eventually some node reaches $\text{accept}$ or $\text{reject}$. If all nodes are leaves, then there are at most two nodes, and by (a.2) they eventually move to $\text{reject}$. Assume now that there is at least one non-leaf. By (a), (b), and weak fairness, eventually all nodes leave state $\text{init}$, and so all non-leaves are in one of $\text{non-leaf}$, $\text{accept}$, or $\text{reject}$. If at least one non-leaf is in $\text{accept}$ or $\text{reject}$, we are done. Otherwise, if $G$ is a star, then by (b.3) the (unique) non-leaf eventually moves to $\text{accept}$; if $G$ is not a star, then two neighbors are in state $\text{non-leaf}$, and by (b.1) the next time any of them is selected it moves to $\text{reject}$.
A Classification of Weak Asynchronous Models of Distributed Computing

Proposition 15. $C_3$ is recognizable by a $d\bar{A}sF$-automaton, but neither by $DA*f$-automata nor by $Da*F$-automata.

Proof. (a) $C_3$ is recognizable by a $d\bar{A}sF$-automaton.

We sketch the behavior of a $d\bar{A}sF$-automaton for $C_3$. Recall that the nodes of the cycle $C_3$ are labeled by 0, 1, and 2. First, if a node with label $i$ detects that it has more than two neighbors, or that the set of labels of its neighbors is different from $\{(i-1) \mod 3, (i+1) \mod 3\}$, then the node moves to a rejecting state. Nodes with a neighbor in a rejecting state also move to a rejecting state. To detect that a node has more than two neighbors, the automaton uses the same trick as in Proposition 14: the state of each node has a color component with three possible values, which changes whenever the node is active. By strong fairness and Lemma 2, if the node has more than two neighbors, then it will eventually see that its neighbors have three different colors, and reject.

As we consider only connected graphs, the preceding tests ensure that graphs which are not cycles with cyclic labeling $0 \rightarrow 1 \rightarrow 2$ are eventually rejected. It remains to ensure that a cycle of length other than 3 is eventually rejected too. For this, the automaton checks an equivalent condition: the cycle contains exactly one node labeled by 2. Nodes labeled by 2 alternate between two phases, 0 and 1. In phase $b \in \{0, 1\}$, the node asks its neighbor labeled by $b$ to propagate a signal through the cycle, and then waits until a signal arrives. (For this, the node moves to a state indicating that it wants the signal to be propagated, and waits for the neighbor to reach a state indicating it has received the message.) If the next signal arrives through the $(1 - b)$ neighbor, the node moves to phase $(1 - b)$; if it arrives through the $b$ neighbor, the node moves to a rejecting state. If the cycle contains only one node labeled by 2, then every signal sent through one neighbor arrives through the other. However, if the cycle contains at least two nodes labeled by 2, then by strong fairness, eventually two consecutive 2-nodes send a clockwise and a counterclockwise signal, and so eventually a 2-node sends a signal through a node, receives the next signal through the same node, and moves to the rejecting state.

(b) $C_3$ is not recognizable by $DA*f$-automata. Let $C_6$ be the hexagon whose nodes are labeled by 0–1–2–0–1–2 (and back to 0). We show that every $DA*f$-automaton $A$ that accepts $C_3$ also accepts $C_6$. For this, consider the synchronous schedules $\sigma_3$ and $\sigma_6$ of $A$ on $C_3$ and $C_6$. Observe that $\sigma_3$ and $\sigma_6$ are weakly fair, and so the runs $\rho_3 = (C_3, 0, C_3, 1, \cdots)$ and $\rho_6 = (C_6, 0, C_6, 1, \cdots)$ scheduled by them are fair too. By the consistency condition, $\rho_3$ is accepting. Let $v, v'$ be nodes of $C_3$ and $C_6$, respectively, carrying the same label. It is easy to see that $C_3, (v) = C_6, (v')$ for every time $t \geq 0$. So $\rho_6$ is also accepting, and thus, by the consistency condition, $A$ accepts $C_6$.

(c) $C_3$ is not recognizable by $Da*F$-automata. We proceed as in part (b): we show that every $Da*F$-automaton $A$ that accepts $C_3$ also accepts $C_6$. Let $\sigma_3 = (S_{3,0}, S_{3,1}, \cdots)$ be a strongly fair schedule of $A$ on $C_3$, and let $\rho_3 = (C_3, 0, C_3, 1, \cdots)$ be the run scheduled by it. Since $C_3$ is accepted, $\rho_3$ is accepting, and so there is a configuration $C_{3,t_0}$ in which every agent is in an accepting state.

For every $1 \leq t \leq t_0$, let $S_{6,t}$ be the selection that for every label $\ell = 0, 1, 2$ contains the two nodes of $C_6$ labeled by $\ell$ iff $S_{3,t}$ contains the node of $C_3$ labeled by $\ell$ (loosely speaking, $S_{6,t}$ "duplicates" $S_{3,t}$). Let $\sigma_6$ be the result of choosing an arbitrary strongly fair schedule $(S'_{6,0}, S'_{6,1}, \cdots)$ of $A$ on $C_6$, and replacing $S'_{6,0}, \cdots, S'_{6,t_0}$ by $S_{6,0}, \cdots, S_{6,t_0}$. Since $\sigma_6$ satisfies the definition of strong fairness, the run $\rho_6 = (C_6, 0, C_6, 1, \cdots)$ scheduled by it is also strongly fair.

Let $v, v'$ be nodes of $C_3$ and $C_6$, respectively, carrying the same label. By the definition
of the selection \( S_{0,t} \) for \( 1 \leq t \leq t_0 \), we have \( C_{3,t_0}(v) = C_{6,t_0}(v') \). So, in particular, every node of \( C_{6,t_0}(v') \) is in an accepting state. Since \( A \) is a halting automaton, nodes that have accepted can no longer change their state, so \( \rho_6 \) is accepting, and therefore \( A \) accepts \( C_6 \). ▶

**Proposition 16.** \( S_{even} \) is recognizable by a \( \text{DasF} \)-automaton.

**Proof.** In Proposition 14 we have exhibited a \( \text{Dasf} \)-automaton \( A \) recognizing \( S \). We now give a \( \text{DasF} \)-automaton \( B \) with \( \beta = 2 \) that uses counting, exclusivity, and strong fairness to further decide if the number of leaves is even. Loosely speaking, \( B \) first executes \( A \); if \( A \) rejects, then \( B \) rejects, because the graph is not even a star. If \( A \) accepts, then \( B \) enters a new phase during which it counts the number of leaves modulo 2. By Theorem 7, \( B \) is equivalent to a \( \text{DasF} \)-automaton.

We can assume that when \( A \) accepts, all nodes are labeled with either leaf or center (the unique non-leaf). We first give an informal description of \( B \). Leaves can be in states visible, invisible, dead, even, or odd. Intuitively, while leaves have not been counted by the center, they alternate between the states visible and invisible. The center only increments its modulo-2 counter if exactly one leaf is visible. After a leaf is counted, it moves to dead. When all leaves become dead, i.e., when they have all been counted, the center decides whether to accept or reject; the leaves read the decision from the counter, and move to even or odd accordingly.

Formally, the state of a leaf is one out of \{visible, invisible, dead, even, odd\}, where even is accepting, and odd is rejecting. Initially all leaves are invisible. The states of the center are of the form

\[
(ph, p, d) \in \{0, 1, 2\} \times \{0, 1\} \times \{\text{none}, 0, 1\},
\]

where \( ph \) is the phase, \( p \) the parity, and \( d \) the decision, respectively. The initial state is \((0, 0, \text{none})\), and the accepting and rejecting states are those with decision 0 and 1, respectively. The transition function is as follows. Let \( v \) be a node selected by the scheduler.

- If \( v \) is a leaf, and its current state is \( s \), then:
  - If \( s = \text{invisible} \) (visible) and the center is in phase 0, then \( v \) moves to visible (invisible).
    Intuitively, while the center is in phase 0, \( v \) keeps making itself visible and invisible to the center. By Lemma 2, strong fairness guarantees that eventually exactly one leaf will be visible to the center.
  - If \( s = \text{visible} \) and the center is in phase 1, then \( v \) moves to dead.
    Intuitively, \( v \) knows that it has been counted by the center, and dies.
  - If \( s = \text{dead} \) and the center is in phase 2, then \( v \) moves to even or odd, depending on the decision made by the center.
  - Otherwise \( v \) remains in state \( s \).
- If \( v \) is the center, and its current state is \( \alpha = (ph, p, d) \), then \( v \) changes its state as follows:
  - If exactly one leaf is visible and \( ph = 0 \), then the center moves to \( \alpha[ph \rightarrow 1, p \rightarrow 1 \rightarrow p] \).
    (Where \( \alpha[ph \rightarrow 1, p \rightarrow 1 \rightarrow p] \) denotes the result of substituting 1 for \( ph \) and \( 1 \rightarrow p \) for \( p \) in \( \alpha \).) Intuitively, the center counts the visible leaf. Since the scheduler is exclusive, no other leaf can change its visibility status at the same time as the center performs this operation. This guarantees that multiple leaves are not counted as one, and that the unique counted leaf remains visible.
  - If all leaves are invisible or dead, at least one leaf is invisible, and \( ph = 1 \), then the center moves to \( \alpha[ph \rightarrow 0] \).
    Intuitively, after counting a leaf the center sees that the leaf knows it has been counted and died.
If all leaves are dead and \( ph = 1 \), then the center moves to \( \alpha[ph \to 2, d \to p] \).

Intuitively, the counting is done, and the center takes the current parity as the decision.

Otherwise the center remains in state \( \alpha \).

In every strongly fair run, eventually the center is selected in a configuration in which exactly one leaf, say \( v \) is visible. This is detected by the center, which updates its counter and moves to phase 1. The center stays in phase 1 until it sees that all leaves are invisible or dead, which guarantees that \( v \) knows it has been counted and died. The center then moves to phase 0 again, to count the next leaf. When all leaves have been counted (which the center can detect by observing that they are all dead), the center knows that its parity bit is the correct one, and moves to phase 2. By fairness, all leaves eventually read the result from the center, and move to even or odd.

Notice how the use of an exclusive scheduler simplifies our design. Indeed, the distributed machine described above would not be correct under a liberal scheduler, because the center, and moves to phase 2. By fairness, all leaves eventually read the result from the center, and move to even or odd.

\begin{proposition}
\( S_{\text{even}} \) is not recognizable by \( \text{dA}^*\text{F}-\text{automata} \).
\end{proposition}

\textbf{Proof.} For the sake of obtaining a contradiction, let us assume that there exists a \( \text{dA}^*\text{F}-\text{automaton} \) \( A \) with machine \( M = (Q, \delta_0, \delta, Y, N) \) that recognizes \( S_{\text{even}} \). We must first introduce several concepts related to \( M \) before we can get to the actual contradiction argument.

Without loss of generality, we assume that the language of star graphs is \( S = \{ ST_i \mid i \geq 2 \} \), where \( ST_i \) is the unlabeled graph with nodes \( \{r, l_1, \ldots, l_i\} \) and edges \( \{r, l_1\}, \ldots, \{r, l_i\} \). We call \( r \) the \textit{root} and \( l_1, \ldots, l_i \) the \textit{leaves} of the star. Throughout this proof, we consider only configurations of \( M \) whose underlying graph is \( ST_i \) for some \( i \geq 2 \), and call them \textit{star configurations}. For notational simplicity, we sometimes identify a star configuration with a tuple \( C = (q, f) \), where \( q \in Q \) is the state of \( r \) and \( f : Q \to \mathbb{N} \) is a function that assigns to each state \( p \) the number of leaves of \( ST_i \) that are in state \( p \). We denote the total number of nodes of \( C \) by \( \text{card}(C) \), i.e., \( \text{card}(C) = 1 + \sum_{p \in Q} f(p) \). Clearly, a configuration \( C \) of \( ST_i \) satisfies \( \text{card}(C) = i + 1 \).

A \textit{base configuration} is a star configuration in which every state \( p \in Q \) occurs at most once on a leaf node. We write \textit{Base} for the set of all base configurations, i.e., \( \text{Base} = Q \times \{0, 1\} \). The base configuration associated with \( C = (q, f) \) is the configuration \( \text{base}(C) = (q, f') \) such that \( f'(p) = \min\{ f(p), 1 \} \) for all \( p \in Q \). Intuitively, \( \text{base}(C) \) is the smallest star configuration in which the root sees the same set of states as \( C \).

Given two configurations \( C = (q, f) \) and \( C' = (q', f') \), we let \( C \preceq C' \) denote that \( q = q' \), \( f(p) \leq f'(p) \) for all \( p \in Q \), and \( f(p) = 0 \) if and only if \( f'(p) = 0 \). Observe that \( \preceq \) is a partial order. The \textit{upward closure} of \( C \) is the set \( \uparrow C := \{ C' \mid C' \succeq C \} \). In other words, \( \uparrow C \) is the set of configurations that one can obtain by duplicating some leaves of \( C \). Notice that the root of such a configuration also sees the same set of states as in \( C \).
The successor relation on configurations of \( M \) will be denoted by \( \rightarrow \). That is, for two configurations \( C \) and \( D \), we write \( C \rightarrow D \) if and only if \( C \) can reach \( D \) in a single execution step of \( M \). (This means that there exists a selection \( S \) of \( C \)'s underlying graph such that one obtains \( D \) by evaluating \( M \)'s transition function \( \delta \) at the nodes of \( C \) selected by \( S \).) We lift this relation to sets of configurations \( C \) and \( D \) in a rather natural way, writing \( C \rightarrow D \) if and only if for every \( C \in C \) there exists some \( D \in D \) such that \( C \rightarrow D \). Furthermore, we use the standard notation \( \rightarrow^* \) for the reflexive-transitive closure of \( \rightarrow \), and \( \rightarrow^i \) for the \( i \)-fold composition of \( \rightarrow \) with itself, where \( i \in \mathbb{N} \).

**Claim 1.** If \( C \rightarrow^* D \), then \([C] \rightarrow^* [D]\).

Proceeding by induction over \( i \in \mathbb{N} \), we show that \( C \rightarrow^i D \) implies \([C] \rightarrow^i [D]\). The case \( i = 0 \) is trivial, since \( C \rightarrow^0 D \) means that \( C = D \).

For \( i = 1 \), we observe that for every configuration \( C' \in [C] \), the roots of \( C \) and \( C' \) can behave identically (as they see the same set of states), and if \( C' \) has more leaves than \( C \), then the additional leaves can copy the behavior of their indistinguishable siblings. So \( C \rightarrow^1 D \) implies that there is some \( D' \in [D] \) such that \( C' \rightarrow^1 D' \). More precisely, let \( S, ST \in S \) be the underlying graphs of \( C \) and \( C' \), respectively. Since \( C' \succeq C \), we know that the set of leaves of \( ST' \) is a superset of the set of leaves of \( ST \). Let \( S \) be the selection of \( ST \) underlying the step \( C \rightarrow^1 D \). We now define the selection \( S' \) of \( ST' \) as follows:

1. The root \( r \) belongs to \( S' \) if and only if it belongs to \( S \).
2. For every state \( q \): if \( S \) does not select any leaves in state \( q \), then neither does \( S' \); otherwise, \( S' \) selects all leaves in state \( q \) selected by \( S \), plus all other leaves in state \( q \) that do not belong to \( ST \).

It follows that \( S' \succeq S \), and moreover a leaf of \( ST' \) is selected in \( S' \) only if some leaf of \( ST \) in the same state is selected in \( S \). So a node of \( S' \) can only move to a state, say \( q \), if some node of \( S \) also moves to \( q \). Letting \( D' \) be the configuration reached by selecting \( S' \), this implies \( D' \in [D] \), and thus \([C] \rightarrow^1 [D]\).

For \( i \geq 2 \), the premise \( C \rightarrow^i D \) tells us that there exists a configuration \( E \) such that \( C \rightarrow^1 E \rightarrow^{i-1} D \). By the induction hypothesis, this implies \([C] \rightarrow^1 [E] \rightarrow^{i-1} [D]\), and therefore \([C] \rightarrow^i [D]\) \(\Box\).

As a direct consequence of Claim 1 we obtain:

**Claim 2.** If \( \{C\} \rightarrow^* [D] \), then \([C] \rightarrow^* [D]\).

Indeed, \( \{C\} \rightarrow^* [D] \) means that there is some \( D' \succeq D \) such that \( C \rightarrow^* D' \). By Claim 1, it follows that \([C] \rightarrow^* [D'] \). Moreover, \( D' \succeq D \) implies \([D'] \subseteq [D]\). Therefore we get \([C] \rightarrow^* [D]\) \(\Box\).

Claim 2 provides the motivation for the last notion we need to introduce: if we want to represent the set \( \text{Pre}^*([D]) \) of predecessors of \([D]\) (i.e., the configurations from which one can reach a configuration of \([D]\) in zero or more steps), and if \( C, C' \in \text{Pre}^*([D]) \) such that \( C \prec C' \), then the representation of \( \text{Pre}^*([D]) \) does not need to mention \( C' \) explicitly, since \( C \in \text{Pre}^*([D]) \) already implies \( C' \in \text{Pre}^*([D]) \). This leads us to represent \( \text{Pre}^*([D]) \) by its set of minimal elements with respect to \( \preceq \). Formally, we define \( \text{MinPre}^*([D]) \) to be the set of all configurations \( C \) such that \( \{C\} \rightarrow^* \{D\} \) and there exists no configuration \( C' \prec C \) such that \( \{C'\} \rightarrow^* \{D\} \).

**Claim 3.** For every star configuration \( D \), the set \( \text{MinPre}^*([D]) \) is finite.

Since there are only finitely many base configurations, and every star configuration lies in the upward closure of its base configuration, it suffices to show that \( \text{MinPre}^*([D]) \cap \{C\} \) is finite for all \( C \in \text{Base} \). This follows easily from Dickson’s Lemma, which states that for every infinite sequence \( \vec{v}_1, \vec{v}_2, \ldots \) of vectors of \( \mathbb{N}^k \), there exist two indices \( i < j \) such that \( \vec{v}_i \leq \vec{v}_j \).
with respect to the pointwise partial order on vectors. Indeed, assume \( \text{MinPre}^*(\lceil D \rceil) \cap C \) is infinite, and let \( C_1, C_2, \ldots \) be an enumeration of its elements, where \( C_i = (q, f_i) \). By Dickson’s Lemma, there are \( i < j \) such that \( f_i(p) \leq f_j(p) \) for all \( p \in Q \). This implies \( C_i \preceq C_j \), and thus contradicts the minimality of \( C_j \). \( \square \)

With all these notions in place, we can finally come back to the contradiction argument that proves Proposition 18. Let \( m \) be the maximum cardinality of any configuration that lies in the set \( \text{MinPre}^*(\lceil D \rceil) \) of some base configuration \( D \), i.e.,

\[
m := \max \{ \text{card}(C) \mid \text{there exists } D \in \text{Base} \text{ such that } C \in \text{MinPre}^*(\lceil D \rceil) \}.
\]

Observe that \( m \) is well-defined because \( \text{Base} \) is finite by definition, and \( \text{MinPre}^*(\lceil D \rceil) \) is finite by Claim 3.

Now consider a star \( ST_n \) whose number of leaves is chosen such that \( n \) is even and \( n \geq (m \cdot |Q|) \), where \( |Q| \) is the number of states of \( A \). Let \( \rho = (C_0, C_1, \ldots) \) be a fair run of \( A \) on \( ST_n \). Since \( n \) is even, \( \rho \) is accepting, which means that there is a time \( r \in \mathbb{N} \) such that for every \( r' \geq r \), the configuration \( C_{r'} \) is accepting. Moreover, since the total number of configurations of \( A \) on \( G \) is finite, there is \( s \geq r \) such that the (accepting) configuration \( C_s \) is visited infinitely often in \( \rho \). Since \( A \) is strongly fair, no rejecting configuration is reachable from \( C_s \), because otherwise, by Lemma 2, \( \rho \) must visit that configuration. Let \( C_s = (q, f) \), and let \( p_{\max} \) be a state that occurs maximally often at a leaf node of \( C_s \), i.e., \( f(p_{\max}) \geq f(p) \) for all \( p \in Q \).

Based on \( \rho \), we construct a fair run \( \rho' = (C_0', C_1', \ldots) \) of \( A \) on the star \( ST_{n+1} \) such that the first \( s + 1 \) configurations \( (C_0', \ldots, C_s') \) copy the behavior of \( \rho \). More precisely, the leaves \( l_1, \ldots, l_n \) behave exactly as in \( \rho \). For the leaf \( l_{n+1} \), let \( l_i \) be any of the leaves of \( ST_n \) such that \( C_s(l_i) = p_{\max} \). During the first \( s \) steps, the schedule of \( \rho' \) selects \( l_{n+1} \) if and only if the schedule of \( \rho \) selects \( l_i \). It follows that \( l_{n+1} \) visits the same sequence of states as \( l_i \), and so \( C_s'(l_{n+1}) = p_{\max} \). Note that this construction does not contradict the strong fairness constraint because we only fix a finite prefix of \( \rho' \). We now extend \( \rho' \) in such a way that it satisfies the strong fairness constraint.

Since \( n + 1 \) is odd, the run \( \rho' \) must eventually visit only rejecting configurations. In particular, some rejecting configuration \( C'_i \) is reachable from \( C_s' \), and so \( C'_i \succeq D \) for some \( D \in \text{MinPre}^*(\lceil \text{base}(C'_i) \rceil) \).

Claim 4. \( C_s \succeq D \).

Recall that \( C_s = (q, f) \), and let \( C'_s = (q, f') \) and \( D = (q, g) \). We have to show that \( f(p) \geq g(p) \) for every state \( p \in Q \). To do so, we distinguish two cases:

- If \( p \neq p_{\max} \), then by the definition of \( C'_s \), we have \( f(p) = f'(p) \), and since \( C'_s \succeq D \), it follows immediately that \( f(p) \geq g(p) \).

- If \( p = p_{\max} \), then by the pigeonhole principle and the definitions of \( n \) and \( p_{\max} \), we have \( f(p) \geq n/|Q| \geq m \). Moreover, we have \( g(p) \leq m \) because the definition of \( m \) ensures that \( \text{card}(D) \leq m \). Hence, \( f(p) \geq g(p) \). \( \square \)

Since \( D \in \text{MinPre}^*(\lceil \text{base}(C'_i) \rceil) \), Claim 4 tells us that \( C_s \) can also reach some rejecting configuration in \( \lceil \text{base}(C'_i) \rceil \). This contradicts what we have established above. We therefore conclude that \( d\text{AsF-automata cannot recognize} \ S_{\text{even}}, \) and by Theorem 7, the same holds for \( d\text{AsF-automata} \).

A.7 Proofs of Section 7

\( \blacktriangleright \) Proposition 20. For every graph population protocol there is an equivalent \( d\text{AsF-automaton} \).
Proof. We present a simulation that runs a graph population protocol on a distributed automaton. To this end, the automaton has to simulate a scheduler that selects ordered pairs of adjacent nodes instead of arbitrary sets of nodes. For any pair \((u, v)\) that is selected to perform a transition, let us call \(u\) the initiator and \(v\) the responder of the transition. By Theorem 7, we may assume that the automaton’s scheduler selects a single node in each step.

The main idea of the construction is as follows: When a node \(u\) is selected and sees that it can become the initiator of a transition, it declares its intention to do so by raising the flag “?”. Then \(u\) waits until some neighbor \(v\) is selected and raises the flag “!”; which signals that \(v\) wants to become the responder of a transition. If this happens, the next time \(u\) is selected, it computes its new state according to the state of \(v\) and the transition function of the population protocol, but also keeps its old state in memory so that \(v\) can still see it. After that, \(v\) also updates its state, and finally \(u\) deletes its old state, which completes the transition. Throughout this protocol, the nodes verify that they have exactly one partner during each transition. If this condition is violated, they raise the error flag “⊥” and abort their current transition.

Formally, let \(I = (Q, \delta_0, \delta, Y, N)\) be a population protocol on \(A\)-labeled graphs. We construct the DASF-automaton \(A\) with machine \(M = (Q', \delta'_0, \delta', Y', N')\), where

\[
Q' = Q \cup (Q \times \{?, !, \perp\}) \cup Q^2,
\]

the sets \(Y'\) and \(N'\) are defined analogously, \(\delta'_0(a) = \delta_0(a)\) for all \(a \in A\), and \(\delta'\) is defined as follows. Let \(v\) be the node currently selected by the scheduler.

1. In case \(v\) is in state \(q \in Q\):
   a. if all of \(v\)'s neighbors are in states of \(Q\), then \(v\) moves to \((q, ?)\);
   b. if exactly one of \(v\)'s neighbors is in some state of \(Q \times \{?\}\) and all others are in states of \(Q\), then \(v\) moves to \((q, !)\);
   c. if several of \(v\)'s neighbors are in states of \(Q \times \{?\}\), then \(v\) moves to \((q, \perp)\);
   d. otherwise, \(v\) remains in state \(q\).

   Intuitively, in rule 1a, \(v\) makes a request for a transition partner, in rule 1b, \(v\) accepts the request of some other node, and in rule 1c, \(v\) signals an error because it has received multiple requests. Signaling the error is necessary to guarantee that two requesting nodes with a common neighbor do not end up in a deadlock. In rule 1d, \(v\) simply waits for ongoing transitions in its neighborhood to be completed.

2. In case \(v\) is in state \((q, ?)\):
   a. if all of \(v\)'s neighbors are in states of \(Q\), then \(v\) remains in \((q, ?)\);
   b. if exactly one of \(v\)'s neighbors is in a state of the form \((p, !)\) and all others are in states of \(Q\), then \(v\) moves to \((q, \delta(q, p))\);
   c. otherwise, \(v\) moves to \((q, \perp)\).

   Intuitively, in rule 2a, \(v\) waits for some node to accept its request, in rule 2b, \(v\) initiates a transition of \(I\) with the unique responder that has accepted its request, and in rule 2c, \(v\) aborts its attempt to make a transition. The latter happens either if some neighbor of \(v\) has received multiple requests, or if several nodes have accepted \(v\)'s request (in which case \(v\)'s new state informs those nodes of the error).

3. In case \(v\) is in state \((q, !)\):
   a. if exactly one of \(v\)'s neighbors is in some state of \(Q \times \{?\}\) and all others are in states of \(Q\), then \(v\) remains in \((q, !)\);
   b. if exactly one of \(v\)'s neighbors is in a state of the form \((p, p')\) and all others are in states of \(Q\), then \(v\) moves to \(\delta(p, q)\).
c. otherwise, \( v \) moves to state \( q \).

Intuitively, in rule 3a, \( v \) waits for its potential transition partner to initiate the transition, in rule 3b, \( v \) performs its own part of the transition, and in rule 3c, \( v \) aborts the transition attempt. The latter happens if the initiator of the transition signals an error.

4. In case \( v \) is in state \((q, \perp)\):
   a. if some neighbor of \( v \) is in a state of \( Q \times \{?, !\} \), then \( v \) remains in \((q, \perp)\);
   b. otherwise, \( v \) moves to state \( q \).

Intuitively, in rule 4a, \( v \) waits for its affected neighbors to see that an error has occurred, and in rule 4b, \( v \) returns to the state it had before the last failed transition attempt.

5. In case \( v \) is in state \((q, q') \in Q^2\):
   a. if some neighbor of \( v \) is in a state of \( Q \times \{!\} \), then \( v \) remains in \((q, q')\);
   b. otherwise, \( v \) moves to state \( q' \).

Intuitively, in rule 5a, \( v \) waits for its transition responder to perform its part of the transition; to make this possible, \( v \) must still keep its old state \( q \) in memory. In rule 5b, the transition has been completed, so \( v \) can remove its old state.

By Lemma 2, strong fairness guarantees that every ordered pair of nodes will be able to perform a transition infinitely often, and more generally, every finite sequence of pairs will be selected infinitely often by the simulated scheduler. Moreover, if several pairs make transitions simultaneously, the construction ensures that none of these pairs have a node in common. This means that the outcome of the transitions would not change if they were rescheduled sequentially. Hence, every fair run of automaton \( A \) simulates a fair run of population protocol \( \Pi \), and since \( \Pi \) satisfies the consistency condition, so does \( A \). Therefore the two devices are equivalent.

Notice that the above construction relies on the fact that \( A \) is a DASF-automaton: nodes must be able to count to verify that they have exactly one partner during each transition; acceptance by stable consensus and strong fairness are required to match the way population protocols are executed; and just as in the proof of Proposition 16, exclusive selection is used to simplify the design of the automaton. In particular, when a responder accepts the request of an initiator (rule 1b), it is guaranteed that none of its other neighbors make a new request at the same time. Similarly, when a node initiates a transition with a responder (rule 2b), it can be sure that its request is not simultaneously accepted by another node.

\[\text{\bf A.8 Proofs of Section 8}\]

\textbf{Proposition 22.} For every D\(\text{AsF}\)-automaton \( A \) and every \( k \in \mathbb{N} \) there is a d\(\text{AsF}\)-automaton \( B \) equivalent to \( A \) on graphs of maximum degree \( k \).

\textbf{Proof.} Given a D\(\text{AsF}\)-automaton \( A \), we have to describe a d\(\text{AsF}\)-automaton \( B \) such that for every graph \( G \) of maximum degree \( k \), every fair run of \( B \) on \( G \) simulates some fair run of \( A \) on \( G \). Observe that this is enough to prove that \( A \) and \( B \) are equivalent on graphs of maximum degree \( k \). Indeed, since by assumption \( A \) satisfies the consistency condition, either all fair runs of \( A \) on \( G \) are accepting, or all are rejecting. If every fair run of \( B \) on \( G \) simulates some fair run of \( A \) on \( G \), then \( B \) also satisfies the consistency condition and accepts \( G \) iff \( A \) accepts \( G \).

In the following, we construct a d\(\text{AsF}\)-automaton \( B \) that simulates a D\(\text{AsF}\)-automaton \( A \) on any graph of maximum degree \( k \). (The same construction can also be used to go from D\(\text{ASF}\)-automata to d\(\text{ASF}\)-automata.)
Let $Q$ be the set of states of $A$. A state of $B$ is a fivetuple $\alpha = (q_0, q, p, fc, sc)$, where $q_0, q \in Q$ are the initial and current state, respectively, $p \in \{0, 1, 2\}$ is the phase, and $fc \in [k^2]$ is the first color, and $sc \in \{0, 1\}$ is the second color, respectively.

Let $G = (V, E, \lambda)$ be a graph of maximum degree $k$. The initial state of a node $v$ of $G$ in $B$ is $(q_0, q_0, 0, 0, 0)$, where $q_0 = \delta_0(\lambda(v))$ and $\delta_0$ is the initialization function of $A$. Let us now give a more precise but still intuitive description of the intended meaning of “a node $v$ of a graph $G$ is currently in state $\alpha = (q_0, q, p, fc, sc)$”. The first two components are straightforward:

- $q_0$ is always $\delta_0(\lambda(v))$. (That is, the transition function of $B$, introduced below, never changes the first component of a state.) Sometimes the node needs to go back to its initial state, and this component just tells the node where to go.
- $q$ is the current state of $v$ in the run of $A$ being simulated.

The other three components require some further explanation. Given a node $v$, let $NE(v)$ be the set containing $v$ and its neighbors. We say that a configuration $C$ is well colored if for every node $v$ the first colors of $v$ and all its neighbors are pairwise distinct in $C$ (i.e., each first color occurs at most once in $v$’s neighborhood). A goal of the protocol is to eventually reach a well-colored configuration $C_{wc}$, such that from then on no node ever changes its first color. Intuitively, the first color of a node at $C_{wc}$ becomes its locally unique identifier: an identifier that never changes, different from the identities of all its neighbors and neighbors’ neighbors. With locally unique identifiers the nodes can then easily simulate the moves of $A$: Indeed, in order to know how many neighbors they have in a state of $A$, say $q_1$, they just count the number of different states they see of the form $(q_0, q_1, p, fc, sc)$.

To achieve this goal, the protocol uses the second colors. In phase 0 the nodes restart their states (initially this is superfluous because they are already there), and move to phase 1. In phase 1, the nodes select an arbitrary distribution of first colors. Since the nodes are deterministic, they rely on strong fairness to ensure that eventually a well-colored distribution is chosen. The nodes then move to phase 2, where they start simulating $A$ under the assumption that the current configuration is well colored. However, at the same time they keep changing their second colors, and start to watch out for neighbors with the same first color as themselves, and for pairs of neighbors with the same first color but distinct second colors. Whenever they detect one of these two situations, they know that their assumption was incorrect, which implies that the simulation they have carried out so far is useless. So they move back to phase 0. We recall that, as in some other proofs, the nodes do not move synchronously from phase to phase; instead, a node moves to a new phase, and waits for its neighbors to follow.

Let us now describe the transition function of $B$. Let $C$ denote the current configuration of $B$. Fix a node $v$ of $G$, and let $\alpha = (q_0, q, p, fc, sc)$ be the current state of $v$ in $C$. Further, let $q'$ be the state $v$ would move to in machine $A$ from the configuration of $A$ corresponding to $C$. Finally, let $(fc + 1)$ denote $(fc + 1) \mod (k^2 + 1)$, and $(p + 1)$ and $(p - 1)$ denote $(p + 1) \mod 3$ and $(p - 1) \mod 3$, respectively. If $v$ is selected by the scheduler at $C$, then its next state is determined as follows:

(0) If $v$ is in phase 0 then:

- (0.a) If some neighbor of $v$ is in phase 2, then $v$ stays in $\alpha$.
- (0.b) If all neighbors of $v$ are in phase 0 or 1, then $v$ moves to $\alpha[q \rightarrow q_0, p \rightarrow 1]$.

(1) If $v$ is in phase 1 then:

- (1.a) If at least one neighbor of $v$ is in phase 0, then $v$ moves to $\alpha[fc \rightarrow fc + 1]$; (Intuitively, $v$ waits for its neighbors in phase 0 to catch up.)
(1.b) If all neighbors of \( v \) are in phase 1, then \( v \) moves to \( \alpha[p \to 2, fc \to fc + 1] \);
(The node initiates a new phase.)

(1.c) If at least one neighbor of \( v \) is in phase 2, then \( v \) moves to \( \alpha[p \to 2] \).

(2) If \( v \) is in phase 2 then:

(2.a) If some neighbor of \( v \) is in phase 1, then \( v \) moves to \( \alpha[fc \to fc + 1] \).

(2.b) If all neighbors of \( v \) are in phase 2, and any two nodes of \( NE(v) \) with the same first color also have the same second color, then \( v \) moves to \( \alpha[q \to q', sc \to 1 - sc] \).
(In this case \( v \) sees no local violation of the well-coloring condition, and so it simulates a move of \( A \), and changes its second color.)

(2.c) If all neighbors of \( v \) are in phase 2, and \( NE(v) \) contains two nodes with the same first color but distinct second colors, then \( v \) moves to \( \alpha[p \to 0] \);

(2.d) If some neighbor of \( v \) is in phase 0, then \( v \) moves to \( \alpha[p \to 0] \).

This concludes the description of \( B \). In the rest of the proof we show that \( B \) is a distributed automaton, i.e., that it satisfies the consistency condition, and that every fair run of \( B \) on \( G \) simulates some fair run of \( A \) on \( G \). The proof is in four steps.

**Claim 1.** Every run of \( B \) eventually reaches a well-colored configuration with all nodes in phase 2.

By strong fairness and Lemma 2, it suffices to show that for every configuration there exists a finite sequence of selections such that the configuration reached after executing them is well colored with all nodes in phase 2. First we show that it is possible to color the nodes of \( G \) with at most \( k^2 + 1 \) different colors so that the colors of every set of nodes \( NE(v) \) are pairwise distinct. Let \( G' \) be the result of triangulating \( G \), i.e., adding an edge \( \{v_1, v_3\} \) for every pair of edges \( \{v_1, v_2\}, \{v_2, v_3\} \in G \) such that \( v_1 \neq v_3 \). Since \( G \) has maximum degree \( k \), the graph \( G' \) has maximum degree at most \( k^2 \). Clearly, a coloring of \( G' \) in the usual graph-theoretical sense (i.e., for every edge \( \{v_1, v_2\} \) of \( G' \) the nodes \( v_1 \) and \( v_2 \) have different colors) satisfies that the colors of every set \( NE(v) \) in \( G \) are pairwise distinct. So it suffices to exhibit a coloring of \( G' \) with \( k^2 + 1 \) colors. Such a coloring can be obtained by applying the standard greedy algorithm that produces a coloring of a graph with maximum degree \( m \) using \( m + 1 \) colors (in our case \( m = k^2 \)).

We prove the existence of a reachable well-colored configuration with all nodes in phase 2 in two steps:

1. Every reachable configuration can reach either a well-colored configuration with all nodes in phase 2, or a configuration with all nodes in phase 0.

Let \( C \) be a reachable configuration. Inspection of (0)-(2) shows that from \( C \) we can reach \( C' \) with all nodes in phase 2. If \( C' \) is well colored we are done. Otherwise, there is a node \( v \) such that two nodes of \( NE(v) \) have the same first color in \( C' \). If these nodes have distinct second colors, we can select \( v \) and bring it to phase 0 with (2.c), and then (2.d) yields the result. If the nodes have the same second colors, we select one of them. If (2.b) applies, then its second color changes, and we can select \( v \) as before. If (2.c) applies, then this node moves to phase 0, and then (2.d) yields the result.

2. Every configuration with all nodes in phase 0 can reach a well-colored configuration with all nodes in phase 2.

Take a spanning tree \( T \) of \( G \). Starting with \( T' := T \), repeatedly select a leaf \( v \) of \( T' \) as many times as necessary to give it any first color we wish (this is possible by (0.b) and (1.a)); we then remove \( v \) from \( T' \) and iterate. When \( T' \) consists of just one node, we proceed similarly, but using (1.b) and (2.a). This yields a well-colored configuration with one node in phase 2 and all others in phase 1. We repeatedly select nodes in phase 1 with a neighbor in phase 2 and apply (1.c). \( \square \)
Claim 2. The set of well-colored configurations with all nodes in phase 2 is closed under the transition relation. In such configurations only (2.b) is enabled, which changes neither the phase nor the first color of a node. So after any transition the new configuration is also well-colored, and all nodes stay in phase 2.

Let us now prove that \( B \) satisfies the consistency condition, and that it is equivalent to \( A \) on graphs of maximum degree \( k \). Let \( \rho^B = (C^B_0, C^B_1, C^B_2, \ldots) \) be an arbitrary strongly fair run of \( B \) on \( G \). It suffices to show that there exists a strongly fair run \( \rho^A \) of \( A \) on \( G \) such that \( \rho^B \) is accepting iff \( \rho^A \) is accepting. Indeed, since \( A \) satisfies the consistency condition by hypothesis, it follows that \( B \) is also consistent, and that \( B \) accepts \( G \) iff \( A \) does, which implies the equivalence of \( A \) and \( B \) on \( k \)-bounded graphs.

Let \( \sigma^B = (S^B_0, S^B_1, S^B_2, \ldots) \in (2^V)^\omega \) be a schedule that schedules \( \rho^B \). We now define a schedule \( \sigma^A = (S^A_0, S^A_1, S^A_2, \ldots) \), and then choose \( \rho^A \) as the run scheduled by \( \sigma^A \). For every node \( v \), let \( t_v \) be the smallest time after which \( v \) and its neighbors reach phase 2 and stay in it forever (in run \( \rho^B \)), which exists by Claims 1 and 2. For every \( t \in \mathbb{N} \), we decide whether \( v \in S^A_t \) or not as follows:

If \( t \leq t_v \), then \( v \notin S^A_t \); if \( t > t_v \), then \( v \in S^A_t \) iff \( v \in S^B_t \).

So, intuitively, in \( \sigma^A \) a node \( v \) is never selected before \( NE(v) \) has “stabilized”, and after that it is selected whenever \( \sigma^B \) selects it. It remains to show that \( \rho^A \) is strongly fair, and that \( \rho^A \) is accepting iff \( \rho^B \) is accepting.

Claim 3. \( \rho^A \) is strongly fair.

By Claims 1 and 2 and the definition of \( \sigma^A \), there is a time \( t \) such that \( S^A_t = S^B_t \) for every \( t' \geq t \) (intuitively, \( t \) is the time at which all nodes have stabilized in phase 2). Since \( \sigma^B \) is strongly fair by hypothesis, and strong fairness is independent of the properties of any finite prefix, \( \sigma^A \) is also strongly fair. So \( \rho^A \) is strongly fair.

Claim 4. \( \rho^A \) is accepting iff \( \rho^B \) is accepting.

Let \( \rho^A = (C^A_0, C^A_1, C^A_2, \ldots) \), and let \( v \) be an arbitrary node of \( G \). It suffices to prove that \( C^A_t(v) = C^B_t(v) \) holds for every \( t \geq t_v \). (Indeed, by definition a run is accepting iff every node eventually visits accepting states only, and so, since \( C^A_t(v) = C^B_t(v) \) for every \( t \geq t_v \), this holds for \( \rho^A \) iff it holds for \( \rho^B \).) We proceed by induction on \( t \).

Base: \( t = t_v \). Let \( q_{0v} \) be the initial state of \( v \). We have \( C^A_t(v) = q_{0v} = C^B_t(v) \). We have \( C^A_t(v) = q_{0v} \) for every \( t \leq t_v \) because \( v \notin S^A_t \) for any \( t \leq t_v \). Moreover, we have \( C^B_t(v) = q_{0v} \) because \( v \) moves to \( q_{0v} \) the last time it moves to phase 1 (case (2.b)), and stays in \( q_{0v} \) until \( t \) and all its neighbors reach phase 2 (case (2.b)). But this is precisely the time \( t_v \): Since \( v \) never leaves phase 2 again, neither do its neighbors (otherwise they would “drag” \( v \) to phase 0 with them).

Step: \( t > t_v \). By induction hypothesis we have \( C^A_{t-1}(v) = C^B_{t-1}(v) \), and by the definition of \( \sigma^A \) we have \( v \in S^A_{t-1} \) iff \( v \in S^B_{t-1} \). So it suffices to show \( C^A_{t-1}(u) = C^B_{t-1}(u) \) for every neighbor \( u \) of \( v \). Fix a neighbor \( u \). Consider two cases:

= \( t \geq t_u \). Then \( C^A_{t-1}(u) = C^B_{t-1}(u) \) follows from the induction hypothesis applied to the node \( u \).

= \( t < t_u \). Let \( q_{0u} \) be the initial state of \( u \). Since, by definition, \( \sigma^A \) never selects \( u \) before time \( t_u \), we have \( C^A_{t-1}(u) = q_{0u} \). We show \( C^B_{t-1}(u) = q_{0u} \). Since \( t < t_u \) holds but \( u \) will never leave phase 2 after \( t \) by hypothesis, some neighbor of \( u \) will still change its phase after \( t \). So its neighbor is in phase 1. But all nodes in phase 1 are in their initial state. ▷