Novel Non-iterative Least Square Method with Adjustable Weighting Vector for Filter Design

Jie Chen\textsuperscript{1, a} and Yingzeng Yin\textsuperscript{2}

\textsuperscript{1}School of Electronic Engineering, Xi’an Aeronautical University, No.259 Xi’an West Second Ring Road, Xi’an 710077, China.
\textsuperscript{2}National Key Laboratory of Antenna and Microwave Technology, School of Electronic Engineering, Xidian University, No.2 Taibai South Road, Xi’an 710071, China.
\textsuperscript{a}Email: chenbinglin88888@163.com

Abstract. A novel non-iterative weighted least square method is presented. The new method advances the traditional non-iterative least square formula into a new non-iterative weighted least square formula with a controllable weighting vector. The weighting vector can be adjusted to control the stop-band attenuation of the filters designed by the new formula. In addition, how to choose the reasonable weighting values is discussed through examples. The simulations of the new method are provided to demonstrate its performance. The simulations results show, when compared with the traditional least square method, the stop-band attenuation of the filters designed by the new formula can easily attain the higher grade by adjusting the weighting vector.

1. Introduction

The least square method and its variants have being widely investigated in various areas [1]-[6]. In the literature [1], [2], the iterative least square method and a constrained least square method were applied to synthesize conformal array antenna. A high resolution velocity analysis method using the $\ell_1$-norm regularized least square method for pavement inspection was presented in the article [3]. A meshless method based on the iterative weighted least square method for electro hydrodynamic problems was presented in the article [4]. The two-stage orthogonal least square methods for neural network construction were proposed in the literature [5]. The literature [6] used a hierarchical partial least square method to assess gait characteristics in total knee arthroplasty patients. The most researched or used least square method in these literatures was the iterative least square method.

Similarly, the digital filters were widely studied [7]. In the literature [8], the digital filter implementation and performance analysis were presented. In general, the digital filters can be divided into two types of filters—a finite impulse response filter (FIR) and an infinite impulse response filter (IIR). Either of FIR and IIR has many kinds, such as the high-pass filters, the band-pass filters, and the band-stop filters. In addition, numerous filter design methods have been exploited.

The digital filters can be designed by the least square related methods. However, when the non-iterative least square method is used to design filters, the stop-band attenuation of the designed filters is uncontrollable and less than that of the filters designed by some window functions. Therefore, its application is limited. To overcome these drawbacks, this letter presents a novel filter design method based on a non-iterative weighted least square formula with a controllable weighting vector that can be used to control the stop-band attenuation of the digital filters. Simulation examples are provided to verify the validity of the new method. Compared with the traditional non-iterative least
square method, the new method can flexibly improve and control the stop-band attenuation of the
designed filter through an adjustment to the weighting vector.

The remainder of this letter is organized as follows: Section 2 presents the general non-iterative
least square method for filter design; Section 3 introduces the new non-iterative weighted least square
for filter design; Section 4 verifies the validity of the new method by several examples; and Section 5
draws conclusions.

2. General Non-Iterative Least Square Method for Filter Design
Let an $N$-order linear FIR filter’s amplitude-frequency response be $H_g(\omega)$. When $N$ is an odd integer,$H_g(\omega)$ can be written as

$$H_g(\omega) = \sum_{n=0}^{M} f(n) \cos(n\omega), \tag{1}$$

where $f(n)$ is the time-domain sequences of the designed FIR filter, $M=\frac{N-1}{2}$. Let the targeted
filter be $H_d(\omega)$.

Set the digital frequency $\omega$ in a cycle interval $[0, \pi]$; let its discretized values be $\omega_0, \omega_1, \omega_2, ..., \omega_{179}, \omega_{180}$ sequentially. Let

$$B = \begin{bmatrix}
1 & \cos(\omega_0) & \cos(2\omega_0) & \cdots & \cos(M\omega_0) \\
1 & \cos(\omega_1) & \cos(2\omega_1) & \cdots & \cos(M\omega_1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \cos(\omega_{180}) & \cos(2\omega_{180}) & \cdots & \cos(M\omega_{180})
\end{bmatrix}, \tag{2}$$

$$F = [f(0) \ f(1) \ \cdots \ f(M)]^T, \tag{3}$$

$$A = [H_d(\omega_0) \ H_d(\omega_1) \ \cdots \ H_d(\omega_{180})]^T, \tag{4}$$

wherein superscript $T$ denotes the transpose operation.

Hence, equation (1) can be expressed as

$$BF = A. \tag{5}$$

The traditional least square solution of equation (5) can be obtained as

$$F_{LS} = (B^H B)^{-1} B^H A, \tag{6}$$

where the superscript $H$ denotes the conjugate transpose operation.

3. New Non-Iterative Least Square Method for Filter Design
To distinguish from the traditional least square method, let $C=F$, therefor, the error vector of equation
(5) is

$$E_{LS} = BC - A. \tag{7}$$

Therefore, the mean squared error is

$$E_{LS}^2 = (BC - A)^H (BC - A). \tag{8}$$

The weighted mean squared error is

$$E^2 = (BC - A)^H D_1 (BC - A), \tag{9}$$

where $D_1$ is defined as
\[
D_i = \begin{bmatrix}
H_i & 0 & \cdots & 0 \\
0 & H_i & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_K
\end{bmatrix},
\]

(10)

where \(H_i (i=1, 2, ..., K)\) is the weighting value for the error corresponding to \(\omega_i\).

For the convenience of expression, let
\[
H = [H_1, H_2, ..., H_i, ..., H_K]^T.
\]

(11)

So the following equation can be obtained
\[
D_i = \text{diag}(H),
\]

(12)

where \(\text{diag}\) denotes the operation to form a diagonal matrix using elements of \(H\).

To make \(E^2\) reach the minimum, equation (9) should satisfy
\[
\frac{\partial E^2}{\partial C} = 0.
\]

(13)

From equation (9), we can obtain
\[
\frac{\partial E^2}{\partial C} = (\partial(BC-A)^* / \partial C)D_i(BC-A)^* + (\partial(D_i(BC-A))^T / \partial C)((BC-A)^*)^T = B^H D_i^H (BC-A)^* = 0.
\]

(14)

wherein superscript * denotes conjugate operation. Hence,
\[
B^H D_i^H BC = B^H D_i^H A = 0.
\]

(15)

Therefore, we can obtain
\[
B^H D_i^H BC = B^H D_i^H A.
\]

(16)

From equation (16) we can obtain the weighted least square solution of equation (5) as
\[
C_{\text{WLS}} = (B^H D_i^H B)^{-1} B^H D_i^H A.
\]

(17)

This equation is called the weighted least square formula (WLSF). While \(H=1 \ (i=1, 2, ..., K)\), equation (17) degrades into the traditional standard least square solution of equation (6).

4. Simulations of the New Method

Matrix laboratory (MATLAB) is used to simulate the method of equation (6) and (17), and their performances are compared.

Let \(\omega\)'s discretized values \(\omega_0, \omega_1, \omega_2, ..., \omega_{179}, \omega_{180}\) be equal to 0, \(\pi/180, 2\pi/180, ..., 179\pi/180, \pi\) sequentially. Three examples are provided to demonstrate the merit of the new method.

In the first example, a band-stop filter is designed using equation (6) and equation (17). Let the targeted filter be
\[
H_d(\omega) = \begin{cases}
1 & 0 < \omega < \omega_p \\
10^{-5} & \omega_p \leq \omega < \omega_s \\
1 & \omega_s \leq \omega \leq \pi
\end{cases}
\]

(18)

Let \(M=25\), \(\omega_p=1/3\pi\), and \(\omega_s=2/3\pi\). According to equation (6) and equation (17), the solutions \(F_{\text{LS}}\) and \(C_{\text{WLS}}\) can be obtained, respectively. Then, the amplitude-frequency responses of the two filters can be derived using equation (1). The elements of the weighting vector \(H\) is set as
\[ H_l = \begin{cases} 
1 & 0 < \omega < \omega_p \\
10^l & \omega_p \leq \omega < \omega_s \\
1 & \omega_s \leq \omega \leq \pi 
\end{cases} \quad (19) \]

where \( l \) is an integer variable. The simulation results are shown in figure 1.

In figure 1, the \( x \)-axis denotes digital frequency variable \( \omega \), and the \( y \)-axis is the amplitude of the designed filter’s frequency response. In the legend for figure 1, the “target” denotes the amplitude-frequency response of the targeted filter; the “ls” indicates the curve of the filter designed by equation (6); the “wls” marks the curve of the filter designed by equation (17); the “\( l \)” is the parameter in equation (19).

From figure 1, it can be learned that the least attenuation of the filter designed by the traditional least square method is -10.9 dB, which is far less than the -50 dB attenuation of the targeted filter. However, the least attenuation of the filter designed by equation (17), while \( l = 1 \), reaches -22.9 dB; while \( l = 2 \), reaches -37.9 dB; while \( l = 3 \), reaches -48.5 dB. It can be seen from figure 1 that in stop-band, while \( l = 3 \), the attenuation fluctuates little and nearly lock at the targeted filter’s curve. Moreover, the transition band width gradually increases along with the rising attenuation.

In the second example, a low-pass filter is designed using equation (6) and equation (17). Let the targeted filter be

\[ H_d(\omega) = \begin{cases} 
1 & 0 < \omega < \omega_p \\
10^{-5} & \omega_p \leq \omega \leq \pi \n\end{cases} \quad (20) \]

Let \( M=25 \) and \( \omega_p=2/3\pi \). According to equation (6) and equation (17), the solutions \( F_{\text{LS}} \) and \( C_{\text{WLS}} \) can be obtained, respectively. Then, the amplitude-frequency responses of the two filters can be derived using equation (1).

The elements of the weighting vector \( H \) is set as

\[ H_l = \begin{cases} 
1 & 0 < \omega < \omega_p \\
10^l & \omega_p \leq \omega \leq \pi 
\end{cases} \quad (21) \]

wherein \( l \) is an integer variable. The simulation results are shown in figure 2.

In figure 2, the \( x \)-axis denotes digital frequency variable \( \omega \), and the \( y \)-axis is the amplitude of the designed filter’s frequency response. In the legend for figure 2, the “target” denotes the amplitude-frequency response of the targeted filter; the “ls” indicates the curve of the filter designed by equation (6); the “wls” marks the curve of the filter designed by equation (17); the “\( l \)” is the parameter in equation (21).

From figure 2, it can be learned that the least attenuation of the filter designed by the traditional least square method is -11.2 dB, which is far less than the -50 dB attenuation of the targeted filter.
However, the least attenuation of the filter designed by equation (17), while \( l = 1 \), reaches -21.2 dB; while \( l = 2 \), reaches -38 dB; while \( l = 3 \), reaches -48.3 dB. It can be seen from figure 2 that in stop-band, while \( l = 3 \), the attenuation fluctuates little and nearly lock at the targeted filter’s curve. Moreover, the transition band width gradually increases along with the rising attenuation.

![Figure 2](image-url)

**Figure 2.** Amplitude-frequency response curves of the designed filters of the second example.

In the third example, a high-pass filter is designed using equation (6) and equation (17). Let the targeted filter be

\[
H_d(\omega) = \begin{cases} 
10^{-5} & 0 < \omega < \omega_p \\
1 & \omega_p \leq \omega \leq \pi 
\end{cases}
\] (22)

Let \( M = 25 \) and \( \omega_p = 2/3\pi \). According to equation (6) and equation (17), the solutions \( F_{\text{LS}} \) and \( C_{\text{WLS}} \) can be obtained, respectively. Then, the amplitude-frequency responses of the two filters can be derived using equation (1).

The elements of the weighting vector \( H \) is set as

\[
H_i = \begin{cases} 
10^l & 0 < \omega < \omega_p \\
1 & \omega_p \leq \omega \leq \pi 
\end{cases}
\] (23)

Where in \( l \) is an integer variable. The simulation results are shown in figure 3.

![Figure 3](image-url)

**Figure 3.** Amplitude-frequency response curves of the designed filters of the third example.

In figure 3, the \( x \)-axis denotes digital frequency variable \( \omega \), and the \( y \)-axis is the amplitude of the designed filter’s frequency response. In the legend for figure 3, the “target” denotes the amplitude-frequency response of the targeted filter; the “ls” indicates the curve of the filter designed
by equation (6); the “wls” marks the curve of the filter designed by equation (17); the “l” is the parameter in equation (23).

From figure 3, it can be learned that the least attenuation of the filter designed by the traditional least square method is -10.3 dB, which is far less than the -50 dB attenuation of the targeted filter. However, the least attenuation of the filter designed by equation (17), while l=1, reaches -21.9 dB; while l=2, reaches -36.3 dB; while l=3, reaches -48 dB. It can be seen from figure 3 that in stop-band, while l=3, the attenuation fluctuates little and nearly lock at the targeted filter’s curve. Moreover, the transition band width gradually increases along with the rising attenuation.

The results of all three examples prove that the filters designed by the traditional non-iterative least square method of equation (6) performed worse than filters designed by the non-iterative weighted least square method equation (17) in terms of filter attenuation. The filter attenuation designed by equation (17) can be controlled by an arbitrarily determined weighting vector, and the filter designed by equation (17) can attain greater attenuation at minimal cost to transition bandwidth expansion.

5. Conclusions
This letter presents a novel filter design method based on a non-iterative weighted least square formula with a weighting vector that can be arbitrarily determined. The weighting vector can be used to control the stop-band attenuation of filters. Compared with the traditional non-iterative least square method, the new method can easily control and raise the stop-band attenuation of filters at minimal cost to transition bandwidth expansion.

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