ANALYSIS OF UNBALANCED BLACK RING SOLUTIONS
WITHIN THE QUASILOCAL FORMALISM

Liu Zhen-Xing
Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences,
West No.30 Xiao Hong Shan, Wuhan 430071, China
liunenu@tom.com

Chen Ze-Qian
Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences,
West No.30 Xiao Hong Shan, Wuhan 430071, China
zqchen@wipm.ac.cn

We investigate the properties of rotating asymptotically flat black ring solutions in five-dimensional Einstein-Maxwell-dilaton gravity with the Kaluza-Klein coupling. Within the quasilocal formalism, the balance condition for these solutions is derived by using the conservation of the renormalized boundary stress-energy tensor, which is a new method proposed by Dumitru Astefanesei and his collaborators. We also study the thermodynamics of unbalanced black rings. The conserved charges and the thermodynamical quantities are computed. Due to the existence of a conical singularity in the boundary, these quantities differ from the original regular ones. It is shown that the Smarr relation and the quantum statistical relation are still satisfied. However, we get an extra term in the first law of thermodynamics. As the balance condition is imposed this extra term vanishes.

Keywords: quasilocal formalism; black ring; conical singularity.

1. Introduction

The uniqueness theorems valid in four dimensions tell us the simplicity of black holes. That is, an asymptotically flat, stationary black hole solution of Einstein-Maxwell theory is completely characterized by its mass, angular momentum and charge, and the spherical topology is the only allowed horizon topology. However, all these conventional notions do not apply in the framework of higher-dimensional physics. Since the discovery of a rotating black ring solution in five dimensions by Emparan and Reall, many new properties not shared by four-dimensional black holes were found and studied. A black ring is an object equipped with an event horizon of topology $S^1 \times S^2$ rather than the much more familiar $S^3$ topology. It was also explicitly proved in Ref. 1 the breakdown of the uniqueness in five dimensions: for vacuum solutions, there exist one black hole and two black rings all with the same values of the mass and the angular momentum. The spectrum of black objects contains both black holes and black rings in five dimensions, which is far richer than
in four dimensions. This is why the uniqueness is so weak in higher-dimensional gravity.

Many examples of black ring solutions have been constructed in various gravity theories by now. An important question raised in studying these solutions refers to the dynamic balance condition. The balance condition is a constraint on the parameters of a black ring which can be understood on physical grounds: the angular momentum of the ring must be tuned so that the centrifugal force balances the tension and gravitational self-attraction. In some previous literature this condition for the thin rings was obtained by demanding the absence of all conical singularities. Recently, Astefanesei, Rodriguez and Theisen came up with a new method for dealing with the general (thin or fat) rings. They obtained the balance condition for a vacuum solution by considering the conservation of the renormalized boundary stress-energy tensor. This closes a gap left unanswered in black ring physics. The main goal of the present paper is to apply the new method proposed in Ref. 5 to analyzing the charged dilatonic black ring solutions. We will give a systematical computation showing how to derive the dynamic balance condition.

Employing the quasilocal formalism supplemented with boundary counterterms, we will also carry out a preliminary study of the thermodynamics of the unbalanced rings. The Smarr relation and the quantum statistical relation will be checked. In particular, we are able to prove that the balance condition is equivalent with the satisfaction of the first law of thermodynamics.

The remainder of this paper is organized as follows. In the next section we introduce the rotating black ring solutions of the Einstein-Maxwell-dilaton (EMd) theory with a coupling constant $\alpha = \sqrt{8/3}$. In section 3 we give a brief review of the quasilocal formalism supplemented with the counterterms. In section 4 we compute the divergence-free boundary stress tensor in detail and derive the balance condition. Section 5 investigates the thermodynamics preliminarily. Finally, we conclude in section 6 with a discussion of our results.

2. Charged Dilatonic Black Rings

We consider the five-dimensional EMd system. The field equations are given by

$$ R_{\mu\nu} = 2\partial_\mu \Phi \partial_\nu \Phi + 2e^{-2\alpha \Phi} \left( F_{\mu\rho} F_{\nu}{}^\rho - \frac{1}{6} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), $$ (1)

$$ \nabla_\mu \left( e^{-2\alpha \Phi} F^{\mu\nu} \right) = 0, $$ (2)

$$ \nabla_\mu \nabla^\mu \Phi + \frac{\alpha}{2} e^{-2\alpha \Phi} F_{\rho\sigma} F^{\rho\sigma} = 0, $$ (3)

where $\alpha = \sqrt{8/3}$ is the Kaluza-Klein (KK) coupling constant. A family of rotating solutions to this theory were presented in Ref. 6:

$$ ds^2 = \frac{F(x)}{F(y)} \left( dt + \sqrt{R} \cosh(\beta)(1 + y) d\psi \right)^2 + \frac{R^2}{(x - y)^2} V_\beta(x, y)^{4/3} $nabla_\beta \left( G_\Phi(x, y)y^2 \right) F(y)^2 \left( \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\Phi^2 \right), $$ (4)
\[ A = \frac{1}{2} \frac{\sinh \beta}{\cosh^2 \beta F(y) - \sinh^2 \beta F(x)} \left[ \cosh \beta (F(y) - F(x)) dt - R \sqrt{\lambda \nu (1 + y) F(x)} d\psi \right], \]  
\[ e^{-\Phi} = V_\beta(x, y)^\sqrt{\epsilon}, \]  
where \[ F(\xi) = 1 - \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 - \nu \xi), \]  
and \[ V_\beta(x, y) = \cosh^2 \beta - \sinh^2 \beta \frac{F(x)}{F(y)}. \]  

\( R, \lambda, \nu \) and \( \beta \) are parameters whose appropriate combinations give various physical quantities. The parameters \( \lambda \) and \( \nu \) lie in the range \( 0 \leq \nu < \lambda \leq 1 \). To obtain a Lorentzian signature, the coordinates \( x \) and \( y \) must take values in \[ -1 \leq x \leq 1, \quad -\infty < y \leq -1, \quad \lambda^{-1} < y < \infty. \]  
Asymptotic spatial infinity is reached as \( x \to y \to -1 \).

We should emphasize that these solutions are not physically regular. In order to eliminate the conical singularities at \( x = -1 \) and \( y = -1 \) we identify the angles \( \phi \) and \( \psi \) with equal periods
\[ \Delta \phi = \Delta \psi = \frac{4\pi \sqrt{F(-1)}}{|G'(1)|} = 2\pi \frac{\sqrt{1 + \lambda}}{1 + \nu}. \]  
Then the orbits of \( \partial_\phi \) and \( \partial_\psi \) close off smoothly at \( x = -1 \) and \( y = -1 \), respectively.

There are now two cases depending on the value of \( \lambda \). One of them corresponds to the black rings and the other to the black holes.

Case one is defined by \( \lambda < 1 \). In this case, there will also be a conical singularity at \( x = +1 \) unless \( \phi \) is identified with period
\[ \Delta \phi' = \frac{4\pi \sqrt{F(+1)}}{|G'(1)|} = 2\pi \frac{\sqrt{1 - \lambda}}{1 - \nu}. \]  
Demanding \( \Delta \phi = \Delta \phi' \) yields
\[ \lambda = \frac{2\nu}{1 + \nu^2} \quad \text{(black ring)}, \]  
which makes the circular orbits of \( \partial_\phi \) close off smoothly also at \( x = +1 \). Then \( (x, \phi) \) parametrize a two-sphere \( S^2 \), \( \psi \) parametrizes a circle \( S^1 \), and the sections at constant \( t \), \( y \) have the topology of a ring \( S^1 \times S^2 \). Physically, (12) is interpreted as the balance condition for a black ring. We will rederive this condition from a new perspective in section 4.

Case two is defined by
\[ \lambda = 1 \quad \text{(black hole)}. \]  

In this case, the orbits of $\partial_\phi$ do not close at $x = +1$. Then $(x, \phi, \psi)$ parametrize a three-sphere $S^3$ at constant $t, y$. The line element (4) describes the charged dilatonic black holes. To see this, it is convenient to introduce the transformation of coordinates

$$
x = -1 + \frac{2m \cos^2 \theta}{r^2 + a^2 \cos^2 \theta},
$$

$$
y = -1 - \frac{2m \sin^2 \theta}{r^2 - m + a^2 \cos^2 \theta},
$$

$$
\psi = \sqrt{2} \tilde{\psi},
$$

$$
\phi = \sqrt{2} \tilde{\phi},
$$

with

$$
m = \frac{4R^2}{1 + \nu}, \quad a = \frac{R\sqrt{8\nu}}{1 + \nu}.
$$

In these new coordinates, the solutions take the form

$$
\begin{align*}
A & = \frac{1}{2} \frac{m \sinh \beta}{\Sigma_1 + m \sinh^2 \beta} \left( \cosh \beta dt + a \sin^2 \theta d\tilde{\psi} \right), \\
e^{-\Phi} & = \left( 1 + \frac{m \sinh^2 \beta}{\Sigma_1} \right)^{\frac{1}{\Sigma_1}},
\end{align*}
$$

where

$$
\Sigma_1 \equiv r^2 + a^2 \cos^2 \theta, \quad \Sigma_2 \equiv r^2 + a^2 - m.
$$

If we set $\beta = 0$, the neutral Myers-Perry black hole solutions with only one non-vanishing angular momentum (see Ref. 7) will be recovered.

Both for black rings and black holes, $|y| = \infty$ is an ergosurface, $y = 1/\nu$ is the event horizon, and the inner, spacelike singularity is reached as $y \to \lambda^{-1}$ from above.

We will focus on the black rings ($0 < \lambda < 1$) in this paper.

### 3. Quasilocal Formalism

It is well known that the concept of local energy (defining the energy at a spacetime point) in general relativity is meaningless. However, one can define a quasilocal energy in a spatially bounded region by employing the quasilocal formalism of Brown.
and York.\cite{York} For asymptotically flat spacetimes, the quasilocal energy agrees with the ADM energy\cite{ADM} in the limit that the boundary tends to spatial infinity.

The quasilocal formalism provides a powerful method to compute the conserved charges and to study the thermodynamics of black objects. Within this formalism, the conserved charges are related to the divergence-free boundary stress tensor: 10

$$
\tau_{ij} \equiv -\frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{ij}} = \frac{1}{8\pi} \left( K_{ij} - h_{ij} K - \Psi (R_{ij} - R h_{ij}) - h_{ij} \Box \Psi + \Psi \eta_{ij} \right),
$$

where $R_{ij}$ is the Ricci tensor of the induced boundary metric $h_{ij}$, $\Psi \equiv \sqrt{3/2} R$, $K_{ij}$ is the extrinsic curvature of the boundary, and $I = I_B + I_{\partial B} = -\frac{1}{16\pi} \int_B \left( R - 2g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - e^{-2\alpha} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} d^5 x

- \frac{1}{8\pi} \int_{\partial B} \left( K - \sqrt{3/2} R \right) \sqrt{-h} d^4 x
$$

is the total action which has been renormalized by the counterterm. 11

$$\text{I}_{ct} = \frac{1}{8\pi} \int_{\partial B} \sqrt{3/2} R \sqrt{-h} d^4 x.\quad (25)$$

We use Greek indices and Latin indices to denote the bulk coordinates and the boundary coordinates, respectively.

The boundary metric can be written as

$$h_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab} (dy^a + N^a dt)(dy^b + N^b dt),\quad (26)$$

where $N$ is the lapse function, $N^a$ is the shift vector, and $\{y^a\}$ are the intrinsic coordinates on a closed hypersurface $\Sigma$ (with normal $\hat{n}^i$). Then the mass and the angular momentum are defined by

$$M = -\int_{\Sigma} d^3 y \sqrt{\sigma} \hat{n}^i \tau_{ij} \xi^j_t,\quad (27)$$

$$J_\psi = -\int_{\Sigma} d^3 y \sqrt{\sigma} \hat{n}^i \tau_{ij} \xi^j_\psi,\quad (28)$$

where $\xi^i_t = \partial_t$ and $\xi^i_\psi = \partial_\psi$ are normalized Killing vectors.

Note that the boundary stress tensor satisfies an approximate local conservation law\cite{York}

$$D^i \tau_{ij} = -n^\mu T_{\mu ij} \equiv -T_{ij},\quad (29)$$

where $D^i$ is the covariant derivative of $h_{ij}$, $n^\mu$ is the normal on the boundary, and $T_{\mu\nu}$ is the energy-momentum tensor. We will use this conservation law to derive the balance condition for a charged black ring in the next section.
4. Balance Condition

We consider the black rings, i.e., \( \lambda \neq 1 \). The aim of this section is to rederive the balance condition (12) from the conservation law (29). As done in Ref. 5, we just remove the conical singularity in the bulk by identifying \( \phi \) and \( \psi \) with an equal period

\[
\Delta \phi = \Delta \psi = \frac{4\pi \sqrt{F(+1)}}{|G'(+1)|} = \frac{2\pi \sqrt{1-\lambda}}{1-\nu}.
\]

The conical singularity in the boundary \( x = y = -1 \) still exists. We introduce the transformation of coordinates

\[
x = -1 + \frac{2m \cos^2 \theta}{r^2 + a^2 \cos^2 \theta},
\]

\[
y = -1 - \frac{2m \sin^2 \theta}{r^2 - m + a^2 \cos^2 \theta},
\]

\[
\psi = \sqrt{1-\lambda} \frac{\psi}{1-\nu},
\]

\[
\phi = \sqrt{1-\lambda} \frac{\phi}{1-\nu},
\]

with

\[
m = \frac{(1+\lambda)^2 R^2}{1+\nu}, \quad a = \frac{R \sqrt{(1+\lambda)(\lambda-1+\nu+3\lambda\nu)}}{1+\nu}.
\]

In these new coordinates, the asymptotic form of the metric is

\[
g_{tt} = -1 + \frac{2}{3} \frac{(1 + 2 \cosh^2 \beta) R^2 \lambda(1 + \lambda)}{1 + \nu} \frac{1}{r^2} + O(1/r^4),
\]

\[
g_{t\psi} = \frac{2 R^2 (1 + \lambda)^2 \sqrt{\nu(1 - \lambda)} \sin^2 \theta \cosh \beta}{1 - \nu^2} \frac{1}{r^2} + O(1/r^4),
\]

\[
g_{rr} = 1 + \frac{2}{3} R^2 \frac{(1 + \lambda)}{(1 + \nu)^2} \left[ (2\nu - 1) \cos^2 \theta + (1 + \nu) \lambda \cosh^2 \beta + (2\lambda - \lambda\nu - 3\nu) \right] \frac{1}{r^2} + O(1/r^4),
\]

\[
g_{\theta\theta} = r^2 + \frac{2}{3} R^2 \frac{(1 + \lambda)}{(1 + \nu)^2} \left[ (2\nu - 1) \cos^2 \theta + (1 + \nu) \lambda \cosh^2 \beta + (2\lambda - 3) \right] + O(1/r^2),
\]

\[
g_{r\theta} = -R^4 (1 - \nu)(1 - \lambda)(1 + \lambda)^3 \sin \theta \frac{1}{4(1 + \nu)^3} \frac{1}{r^2} + O(1/r^5),
\]

\[
g_{\psi\psi} = \frac{(1 - \lambda)(1 + \nu)^2}{(1 - \nu)^2 (1 + \lambda)} r^2 \sin^2 \theta + \frac{2}{3} R^2 \frac{(1 - \lambda)}{(1 - \nu)(1 + \lambda)} \left[ (1 + \nu) \lambda \cosh^2 \beta + (2\nu - 1) \lambda + 3\nu \right] \sin^2 \theta + O(1/r^2),
\]

\[
g_{\phi\phi} = \frac{(1 - \lambda)(1 + \nu)^2}{(1 - \nu)^2 (1 + \lambda)} r^2 \cos^2 \theta + \frac{2}{3} R^2 \frac{(1 - \lambda)}{(1 - \nu)(1 + \lambda)} \left( 1 + \nu \right) \lambda \cosh^2 \beta + 2\lambda - 3 \cos^2 \theta + O(1/r^2).
\]
Note that we still use $\psi$ and $\phi$ to denote the angles after performing the transformation. The non-vanishing stress tensor components are

$$\tau_{tt} = \frac{1}{8\pi} \left( - R^2 \frac{(1 + \lambda)}{(1 + \nu)^2} \left[ \lambda(1 + \nu)(1 + 2 \cosh^2 \beta) ight. \right.$$

$$\left. + \frac{10}{3}(2\nu - \lambda + 1) \cos 2\theta \frac{1}{r^3} + O(1/r^5) \right)$$

$$\tau_{\theta\theta} = \frac{1}{8\pi} \left( - \frac{4R^2(1 + \lambda)^2}{(1 + \nu)^4} \frac{\lambda(1 - \lambda)}{1 - \nu^2} \sin^2 \beta \cosh \beta \frac{1}{r^3} + O(1/r^5) \right)$$

$$\tau_{\phi\phi} = \frac{1}{8\pi} \left( - \frac{4R^2(1 - \lambda)^2}{(1 - \nu)^4} (2\nu - \lambda + 1)(1 + 2 \cos 2\theta) \cos^2 \theta \frac{1}{r^3} + O(1/r^5) \right).$$

We also compute $T_{nj}$ and find that

$$T_{nt} = T_{n\psi} = T_{n\phi} = 0,$$

$$T_{n\theta} = \frac{1}{8\pi} \left[ 4R^6 \frac{\lambda(1 + \lambda)^2}{(1 + \nu)^4} \left( \frac{2}{3} \frac{\lambda}{3} \lambda \cosh^2 \beta + \frac{\lambda}{3} - 1 \right) \sin^2 \beta \sin 2\theta \frac{1}{r^3} + O(1/r^5) \right].$$

In order to obtain the balance condition, we use the conservation law

$$D^i \tau_{it} = 0,$$

$$D^i \tau_{i\theta} = -T_{n\theta},$$

$$D^i \tau_{i\psi} = 0,$$

$$D^i \tau_{i\phi} = 0,$$

where $D^i$ corresponds to the regular boundary metric (induced from the five-dimensional flat metric)

$$ds^2 = -dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2 + \cos^2 \theta d\phi^2).$$

The coordinate $r$ in (54) is a constant, which determines the location of the boundary. One can check that (50), (52) and (53) are exactly satisfied. The only non-trivial equation is (51), which has an explicit expression as below:

$$\frac{8}{3} R^2 \left[ \frac{(1 + \lambda)}{(1 + \nu)^2} - \frac{(1 - \lambda)}{(1 - \nu)^2} \right] (2\nu - \lambda + 1) \frac{(1 - 2 \sin^2 \theta)}{\sin \theta} \frac{1}{r^3} + O(1/r^5)$$

$$= -4R^6 \frac{\lambda(1 + \lambda)^4}{(1 + \nu)^4} \left( \frac{2}{3} \lambda \cosh^2 \beta + \frac{\lambda}{3} - 1 \right) \sin^2 \beta \sin 2\theta \frac{1}{r^3} + O(1/r^5).$$
It turns out that the source term on the right-hand side of (55) falls off sufficiently fast at infinity. This strongly supports the assumption made in Ref. 5. Comparing both sides of (55), we find that the coefficient of $r^{-3}$ must be identically equal to zero, namely,

$$\left[\frac{(1 + \lambda)}{(1 + \nu)^2} - \frac{(1 - \lambda)}{(1 - \nu)^2}\right](2\nu - \lambda + 1) = 0. \quad (56)$$

Note that $2\nu - \lambda + 1 \neq 0$. Therefore, we obtain

$$\lambda = \frac{2\nu}{1 + \nu^2},$$

which is exactly the balance condition (12) for a charged dilatonic black ring.

5. Thermodynamics

We will discuss the thermodynamics of unbalanced rings in this section. The quasilocal formalism defines a renormalized action (24), which is related to the following grand-canonical potential:

$$G = IT, \quad (57)$$

where $T$ is the temperature. One finds that

$$\lim_{r \to \infty} \left(K - \sqrt{\frac{2}{3} R} \right) \sqrt{-h} = -\frac{1}{6} R^2 \frac{\lambda(1 - \lambda)(1 + \nu)}{(1 - \nu)^2} (1 + 2 \cosh^2 \beta) \sin 2\theta + O(1/r^2). \quad (58)$$

The expression for the boundary action is

$$I_{\partial B} = \frac{\pi}{12} R^2 \frac{\lambda(1 - \lambda)(1 + \nu)}{(1 - \nu)^2} (1 + 2 \cosh^2 \beta) \frac{1}{T}. \quad (59)$$

The bulk action is computed by adopting the “quasi-Euclidean” method of Ref. 12. We obtain

$$I_B = -\frac{\pi}{6} R^2 \frac{\lambda(1 - \lambda)(1 + \nu)}{(1 - \nu)^2} \sinh^2 \beta \frac{1}{T}. \quad (60)$$

Then

$$G = (I_B + I_{\partial B})T = \frac{\pi}{4} R^2 \frac{\lambda(1 - \lambda)(1 + \nu)}{(1 - \nu)^2}. \quad (61)$$

We use (27) and (28) to compute the mass and the angular momentum

$$M = \frac{\pi}{4} R^2 \frac{\lambda(1 - \lambda)(1 + \nu)}{(1 - \nu)^2} (1 + 2 \cosh^2 \beta), \quad (62)$$

$$J_\psi = \frac{\pi}{2} R^3 \frac{(1 + \nu)(1 + \lambda)(1 - \lambda)^{3/2} \sqrt{\lambda \nu}}{(1 - \nu)^3} \cosh \beta. \quad (63)$$
The temperature, entropy, angular velocity, electric charge and electric potential (evaluated on the horizon) are given by

\[ T = \frac{1}{4\pi R} \frac{1 - \nu}{\sqrt{\lambda(\lambda - \nu)}} \frac{1}{\cosh \beta}, \]  

(64)

\[ S = 2\pi^2 R^3 \frac{(1 - \lambda)(\lambda - \nu)^{3/2}\sqrt{\lambda}}{(1 - \nu)^3} \cosh \beta, \]  

(65)

\[ \Omega = \frac{1}{R} \frac{(1 - \nu)}{(1 + \nu)} \frac{\sqrt{\nu}}{\sqrt{\lambda(1 - \lambda)}} \frac{1}{\cosh \beta}, \]  

(66)

\[ Q = \pi R^2 \frac{\lambda(1 - \lambda)(1 + \nu)}{(1 - \nu)^2} \sinh \beta \cosh \beta, \]  

(67)

\[ \phi_H = \frac{1}{2} \tanh \beta. \]  

(68)

Note that the electric charge is defined by

\[ Q = \frac{1}{8\pi} \oint_{\infty} e^{-2\alpha \Phi} F_{\mu\nu} d\Sigma^{\mu\nu}. \]  

(69)

One can easily verify that the Smarr relation and the quantum statistical relation for unbalanced rings are still satisfied, i.e.,

\[ M = \frac{3}{2}(T S + \Omega J_\psi) + \phi_H Q, \]  

(70)

\[ G = M - T S - \Omega J_\psi - \phi_H Q. \]  

(71)

However, we get an extra term in the first law of thermodynamics, i.e.,

\[ dM - T dS - \Omega dJ_\psi - \phi_H dQ = -\frac{1}{4} \pi R^2 \frac{1}{1 - \nu} d\lambda + \frac{1}{2} \pi R^2 \frac{(1 + \lambda)(1 - \lambda)}{(1 + \nu)(1 - \nu)^2} d\nu. \]  

(72)

The extra term on the right-hand side of (72) stems from the stresses due to the conical singularity in the boundary. In other words, the fact that the first law of thermodynamics is not satisfied is reflected in the existence of additional stresses that deform the boundary metric. As the balance condition is imposed this extra term vanishes. Conversely, solving the differential equation

\[-\frac{1}{4} \pi R^2 \frac{1}{1 - \nu} d\lambda + \frac{1}{2} \pi R^2 \frac{(1 + \lambda)(1 - \lambda)}{(1 + \nu)(1 - \nu)^2} d\nu = 0 \]  

(73)

can also yield the balance condition (12). Therefore, we have proved that the dynamic balance condition is equivalent with the satisfaction of the first law of thermodynamics.

Supplemented with counterterms, the quasilocal formalism is a very powerful tool to study the thermodynamics of various black objects. See Refs. 10, 5, 13 and 14 for further details.
6. Conclusions

In this work, we derived the dynamic balance condition for a charged dilatonic black ring in a new way: considering the conservation of the renormalized boundary stress-energy tensor. We have also investigated the properties of unbalanced rings within the quasilocal formalism. By computing the conserved charges and the thermodynamical quantities, we found that although the black ring is in a state of non-equilibrium, the Smarr relation and the quantum statistical relation are still satisfied. The conical singularity is reflected in an extra term occurring in the first law of thermodynamics. Particularly, we proved that the dynamic balance condition is equivalent with the satisfaction of the first law of thermodynamics.

The quasilocal formalism supplemented with counterterms is very robust and it provides a complete way to study the thermodynamics of black objects. An appealing feature of the counterterm method is that the difficulties associated with the choice of a reference background can be avoided. Generally speaking, an action (containing both bulk term and Gibbons-Hawking term) can be regularized by adding counterterms (boundary terms) that depend on the intrinsic geometry of the regularizing surface. We emphasize that the counterterm (25) used in this paper applies only to asymptotically flat black ring solutions. It is expected that the counterterm method will also be developed in studying the properties of nonasymptotically flat black rings (e.g., Ref. 15).

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References

1. R. Emparan and H. S. Reall, *Phys. Rev. Lett.* **88** (2002) 101101.
2. H. Elvang and R. Emparan, *JHEP* **11** (2003) 035.
3. R. Emparan, *JHEP* **03** (2004) 064.
4. R. Emparan, T. Harmark, V. Niarchos, N. A. Obers and M. J. Rodriguez, *JHEP* **10** (2007) 110.
5. D. Astefanesei, M. J. Rodriguez and S. Theisen, *JHEP* **12** (2009) 040.
6. H. K. Kunduri and J. Lucietti, *Phys. Lett. B* **609** (2005) 483.
7. R. C. Myers and M. J. Perry, *Ann. Phys.* **172** (1986) 304.
8. J. D. Brown and J. W. York, Jr., *Phys. Rev. D* **47** (1993) 1407.
9. R. Arnowitt, S. Deser and C. W. Misner, arXiv:gr-qc/0405109.
10. D. Astefanesei and E. Radu, *Phys. Rev. D* **73** (2006) 044014.
11. P. Kraus, F. Larsen and R. Siebelink, *Nucl. Phys. B* **563** (1999) 259.
12. J. D. Brown, E. A. Martinez and J. W. York, Jr., *Phys. Rev. Lett.* **66** (1991) 2281.
13. D. Astefanesei, R. B. Mann, M. J. Rodriguez and C. Stelea, *Class. Quant. Grav* **27** (2010) 165004.
14. D. Astefanesei, M. J. Rodriguez and S. Theisen, *JHEP* **08** (2010) 046.
15. S. S. Yazadjiev, *Phys. Rev. D* **72** (2005) 104014.