2+1 Dimensional Quantum Gravity as a Gaussian Fermionic System and the 3D-Ising Model

Giuseppe Bonacina

Dipartimento di Fisica, Università di Milano, I-20133 Milano, Italy

Maurizio Martellini*

Dipartimento di Fisica, Università di Roma “La Sapienza”, I-00185 Roma, Italy
I.N.F.N., sezione di Pavia, Pavia, Italy

and

Mario Rasetti

Dipartimento di Fisica, Politecnico di Torino, I-10129 Torino, Italy
I.N.F.M., Unità Torino Politecnico, Torino, Italy

ABSTRACT

We show that 2+1-dimensional Euclidean quantum gravity is equivalent, under some mild topological assumptions, to a Gaussian fermionic system. In particular, for manifolds topologically equivalent to $\Sigma_g \times \mathbb{R}$ with $\Sigma_g$ a closed and oriented Riemann surface of genus $g$, the corresponding 2+1-dimensional Euclidean quantum gravity may be related to the 3D-lattice Ising model before its thermodynamic limit.

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* Permanent address: Dipartimento di Fisica, Università di Milano, I-20133 Milano, Italy
1. Introduction

A few years ago, Witten\cite{1} showed that 2+1-dimensional quantum gravity in a first order dreibein formalism is exactly soluble at the classical and quantum levels. The key point in Ref. 1 is the observation that the dreibein $e^a_\mu$ and the spin connection $\omega_\mu^a \equiv \varepsilon^{abc}\omega^b_\mu$ form a gauge field of the group $ISO(2, 1)$ ($ISO(3)$) in Lorentzian (Euclidean) signature. Thus, the Einstein-Hilbert action

$$
\begin{aligned}
\left\{ I = \frac{k}{2} \int_{M^3} \varepsilon^{\mu\nu\rho} e^a_\mu R^a_{\nu\rho}(\omega) \\
R^a_{\nu\rho}(\omega) \equiv \partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu + [\omega_\nu, \omega_\rho]^a
\end{aligned}
$$

becomes the non-Abelian Chern-Simons action on $M^3$ with gauge group $G = ISO(2, 1)$ or $ISO(3)$ depending on the signature of the 3D-manifold $M^3$. Here, and in the following, we shall assume $M^3$ closed and oriented than otherwise stated. However, in this context the meaning of solvability is quite obscure, since in Witten’s approach solvability is ascribed to the fact that the Hilbert space is essentially the space of half-densities on the moduli space of flat $SO(2, 1)$ ($SO(3)$) connections on $\Sigma_g$, where $\Sigma_g$ is a spacelike surface of $M^3$, which is a closed Riemann surface of genus $g$. Witten resorts to a canonical quantization scheme, which requires that $M^3$ is topologically $\sim \Sigma_g \times \mathbb{R}$. This result clearly doesn’t tell us anything on the full quantum dynamics (i.e. the inclusion in the scheme of correlation functions), it is restricted to three manifolds topologically equivalent to $\Sigma_g \times \mathbb{R}$ and it essentially prescribes solving the Hamiltonian constraints of 3D-QG before quantizing, which is a procedure not necessarily equivalent to the standard (covariant) BRST-quantization.

In this note, we shall show how for each fixed generic (closed) three manifold $M^3$, the partition function $Z_{\text{EQG}}(M^3)$ of the Euclidean continued 3D-QG is equivalent (up to a normalization factor) to the partition function of a Gaussian discrete fermionic system whose action encodes the topological nature of $M^3$. Namely, we shall represent $M^3$ as the manifold obtained by Dehn surgery on $S^3$ along a link $L \subset S^3$ and show that (under some suitable conditions) $Z_{\text{EQG}}(M^3) = Z_{\text{EQG}}(L; S^3)$ is the partition function of the free fermions propagating on the link diagram $D_L$ and on its r-parallel versions. One may then study
the unknown correlation functions of 3D-Euclidean quantum gravity, in a way similar to the 3D-Ising model,\textsuperscript{3} directly in the fermionic formulation, which is Gaussian, rather than in the (non-linear) Chern-Simons gauge description\textsuperscript{.} In particular, in the case $M^3$ is a hyperbolic three-manifold $N^3$, $\partial N^3 \neq \emptyset$, we shall show that $Z_{EQG}(N^3)$ is related to the reduced partition function of the 3D-Ising model on the lattice $\Lambda$ which is embeddable in $\partial N^3^*$.

2. The Euclidean 3D-QG Partition Function and the Alexander-Conway Polynomial

Our starting point is Witten’s result about the partition function of Euclidean 3D-QG. He shows\textsuperscript{[6]} that if one selects a non-degenerate metric $\bar{g}_{\alpha\beta}$ on $M^3$ and a (background) flat $SO(3)$ spin connection $\bar{\omega}^a_{\mu(\alpha)}$, where $\alpha$ is a labelling index, and uses the Landau background gauge condition

$$\bar{D}_\mu(\alpha) e^a_\mu = \bar{D}_\nu(\alpha) \nu^a = 0$$  \hspace{1cm} (2.1)

where $\bar{D}^a_\mu(\alpha) \equiv \bar{g}^{\mu\nu} \bar{D}_\nu(\alpha)$ is the covariant derivative with respect to the Levi-Civita connection $\bar{\nabla}_\nu$ associated to $\bar{g}_{\mu\nu}$ plus the flat connection $\bar{\omega}_{\mu(\alpha)}$ of interest, i.e. $\bar{D}_\nu(\alpha) = \bar{\nabla}_\nu + [\bar{\omega}_{\nu(\alpha)}, \cdot]$; then the Euclidean partition function for the 3D-QG including the Fadeev-Popov ghosts reads

$$Z_{EQG}(M^3) = \sum_{(\alpha)} Z_{EQG(\alpha)}(M^3)$$

$$Z_{EQG(\alpha)}(M^3) = \frac{[\text{Det}'(\bar{\Delta}_0(\alpha))]^2}{[\text{Det}'(\bar{D}(\alpha))]} = \frac{[\text{Det}'(\bar{\Delta}_1(\alpha))]^{\frac{1}{2}}}{[\text{Det}'(\bar{\Delta}_2(\alpha))]} [\text{Det}'(\bar{\Delta}_3(\alpha))]^{\frac{1}{2}}$$  \hspace{1cm} (2.2)

(since: $\text{Det}'(\bar{\Delta}_k(\alpha)) = \text{Det}'(\bar{\Delta}_{3-k}(\alpha))$) where $\bar{D}(\alpha) = \ast \bar{D}(\alpha) + \bar{D}(\alpha) \ast$ (here $\ast$ is the Hodge duality operator) and $\bar{\Delta}_{\mu(\alpha)} \equiv (D_{\mu(\alpha)} D^\mu_{(\alpha)})$ is the Laplacian operator acting on twisted i-forms. Furthermore $\text{Det}'(\ast)$ in (2.2) is a functional determinant,\textsuperscript{⋄}

\textsuperscript{adox} Till now it is not known which kind of generalized Jones polynomials give the non-Abelian Chern-Simons theory with a non-compact gauge group $Iso(3)$ and on a generic three manifold $M^3$ not homeomorphic to $S^3$.

\textsuperscript{*} Clearly, the 3D-Ising model is understood before the thermodynamic limit $N \to \infty$ is taken. In this limit, one should also perform $g \to \infty$. In our picture, this amounts to considering\textsuperscript{[4]} a sort of double scaling limit (DSL)\textsuperscript{[5]} at the level of the reduced EQG-partition function on the Riemann surface $\Sigma_g \sim \partial N^3$. Indeed, the DSL is the usual way of formulating the genus expansion, i.e. the sum over all genera.
regularized, for instance, by zeta-function technique \[7\] and omitting zero-modes. Equation (2.2) is derived under the following assumptions:

i) that the moduli space \( \mathcal{N} \) of flat SO(3) connections modulo SO(3)-gauge transformations consists of finitely many points, and \( \tilde{\omega}_{\mu(\alpha)} \) is an arbitrary representative of \( \mathcal{N} \). If \( M^3 \sim \Sigma_g \times \mathbb{R} \), is an orientable closed Riemann surface of genus \( g \), \( \mathcal{N} \) has connected components corresponding to Euler classes \( 2g - 2, 2g - 3, \ldots, -(2g - 2) \). Here, the relevant component is that, say \( \tilde{\mathcal{N}} \in \mathcal{N} \), of maximal Euler class \( 2g - 2 \) (Ref. 6);

ii) that all \( \mathcal{N} \) connections are irreducible. Of course, these conditions on \( \mathcal{N} \) drastically restrict the allowed topologies of \( M^3 \), however, as this set of “good topologies” is not empty, it is reasonable to work out a quantization scheme for Euclidean 3D-gravity only for this particular set of topological three manifolds. On the basis of such assumption, we notice first that the ratio of determinants in (2.2) is in fact the Ray-Singer analytic torsion \[8\] (R.S.-torsion), \( T_{\rho(\alpha)}(M^3) \), relative to the orthogonal representation \( \rho(\alpha) : \pi_1(M^3) \to O(3) \) (i.e. the i-forms on the universal cover of \( M^3 \) transform according to \( \rho(\alpha) \)). This is a topological invariant of \( M^3^* \), which labels homotopy equivalent spaces.

Let us assume that \( \rho(\alpha) \) is acyclic, i.e. that \( H^*(M^3; \rho(\alpha)) \) is zero. Cheeger and Müller have shown \[9\] that in this case the R.S.-torsion \( T_{\rho(\alpha)}(M^3) \) is equivalent to the so-called Reidemeister torsion \[10\] (R.-torsion), \( \tau_{\rho(\alpha)}(M^3) \in \mathbb{R}^+ \), which is a non-homotopy topological invariant that may be computed from the twisted (by \( \rho(\alpha) \)) cochain complex associated to \( M^3 \) by the suitable alternating product of determinants. In this case, therefore, one may set \[o\] :

\[
Z_{EQG(\alpha)}(M^3) = T_{\rho(\alpha)}(M^3) = \tau_{\rho(\alpha)}(M^3). \tag{2.3}
\]

A few comments are in order:

i) In the definition of the Reidemeister torsion \( \tau_{\rho(\alpha)}(M^n) \) one must start with a PL-manifold; but every 3-manifold may be triangulated and hence the PL-assumption is unnecessary.

* Therefore, it is independent of the metric used in the gauge fixing and Fadeev-Popov terms.

\[\diamond\] Fried\[11\] has shown that this identification survives also in the non acyclic case if \( \rho(\alpha) \) is orthogonal.
ii) The representation \( \rho(\alpha) : \pi_1(M^3) \to O(m) \) extends to a unique ring homomorphism from the integral group ring \( \mathbb{Z}(\pi_1(M^3)) \) to the ring of all real \( m \times m \)-matrices. Now the Reidemeister torsion \( \tau_{\rho(\alpha)}(M^3) \), as defined in Ref. 10, is an element of the so-called Whitehead group \( \tilde{K}_1M_m(\mathbb{R}) \). It is known (Ref. 10) that \( \tilde{K}_1M_m(\mathbb{R}) \cong \tilde{K}_1\mathbb{R} \), which, in terms of the Reidemeister torsion, is equivalent to saying that the representation of \( \pi_1(M^3) \) is given by the ring homomorphism \( \varphi(\alpha) : \pi_1(M^3) \to F_0 \), where \( F_0 \) is the commutative multiplicative group of a field, e.g. the field of real numbers \( \mathbb{R} \). Thus, we have:

\[
\begin{cases}
\tau_{\rho(\alpha)}(M^3) = \tau_{\varphi(\alpha)}(M^3) \\
\rho(\alpha) : \pi_1(M^3) \to O(m) \\
\varphi(\alpha) : \pi_1(M^3) \to F_0(\mathbb{R})
\end{cases}
\] (2.4)

Our next step will be connecting Eq. (2.4) to an appropriate Alexander polynomial\(^{[12]}\) \( \triangle_L \). For this purpose, we need the general definition of Dehn surgery on a 3-variety. Following Lickorish,\(^{[13]}\) we may always construct \( M^3 \) by Dehn surgery along a link \( L = K_1 \cup \ldots \cup K_n \) in \( S^3 \) in the following way:

\[ M^3 = [S^3 - (K^\circ_{f_1} \cup \ldots \cup K^\circ_{f_n})] \cup_h (K_{f_1} \cup \ldots \cup K_{f_n}) \equiv (S^3 - L_f) \cup_h L_f \] (2.5)

where \( K^\circ_{f_i} \) is the interior of \( K_{f_i} \) and \( K_{f_i} \) is the preferred framing \( f_i \) of each component \( K_i \) of \( L \subset S^3 \), i.e. the map \( K_i \to K_{f_i} \sim S^1 \times D^2 \) in which the longitude \( \lambda_i \) is oriented in the same way as \( K_i \) and the meridian \( \mu_i \) has linking number +1 with \( L_i \). In (2.5), \( h \) is the union of homomorphisms \( h_i : \partial K_{f_i} \to \partial K_{f_i} \subset M^3 \) defined by \( h_i(\mu_i) = [J_i] = a_i\lambda_i + b_i\mu_i \), where \( b_i \) is the linking number between \( L_i \) and \( J_i \), whereas \( J_i \) is a specified fixed simple closed curve in each \( \partial K_{f_i} \) (clearly \( a_i, b_i \in \mathbb{Z} \)). Notice that the homeomorphism type of \( M^3 \) does not depend on the choice of \( h \). Then we have the

**Fundamental Theorem** (Ref. 13): every closed oriented three-manifold may be obtained by Dehn surgery on a link \( L \) in \( S^3 \) with surgery coefficients \( r_i = \frac{b_i}{a_i} \).

The key point in the proof of the fundamental theorem and in what we shall show later about the 3D-Ising model (Sec. 3), is the noticing that \( h_i \), defined
equivalently as \( h : S^3 - \{ K_{f_i} \} \to M^3 - \{ K_{f_i} \} \), may be characterized by an element \( \tau \) of the genus-\( g \) mapping class group (extensively defined in Sec. 3). We do it in the following way. Every closed, orientable three-manifold \( M^3 \) admits, apart from the Dehn surgery representation quoted previously, a Heegaard decomposition: \( M^3 = H_1 \cup_\tau H_2, \partial H_1 \sim -\partial H_2 \sim \Sigma_g, \tau : \partial H_2 \to \partial H_1 \), where \( H_1 \) and \( H_2 \) are handlebodies of genus \( g \), \( \tau \) is the homeomorphism \( \partial H_2 \to \partial H_1 \) (Ref. 13). Thus, the above homeomorphism \( h \) may be defined as \( I_d \times l \), where \( l \) is the homeomorphism \( l : H_2 - \{ K_{f_i}^\circ \} \to H_2' - \{ K_{f_i}^\circ \} \). Now, one could show (Ref. 13) that \( l \) can always be obtained as an extension of the mapping class group element \( \tau = (f')^{-1} f : \partial H_2 \to \partial H_2' \sim \partial H_2 \), i.e. \( \tau \) belongs to the group of isotopy classes of orientation preserving self-diffeomorphisms of the orientable, closed Riemann surface \( \Sigma_g \sim \partial H_2 \). This latter statement follows from the fact that the handlebodies \( H_2' \) and \( H_2 \) are homeomorphic since they have the same genus \( g \) (Ref. 2).

As Milnor (Ref. 10) first noticed, there is a close connection between the Alexander polynomial \( \Delta_L(t_1, \ldots, t_n) \) and the Reidemeister torsion \( \tau_\varphi(S^3 - L_f) \). Let us remind briefly the definition of Alexander polynomial for a link \( L = K_1 \cup \ldots \cup K_n \) in \( S^3 \). If \( V_L \) is the exterior of \( L \), i.e. \( V_L = S^3 - L_f \), then the homology group \( H_1(V_L) \) is canonically isomorphic to a free Abelian multiplicative group with \( n \) free generators \( (t_1, \ldots, t_n) \). The generator \( t_i \) corresponds to the homology class of a meridian \( \mu_i \) of the preferred framing \( f_i \) of \( K_i \), \( f_i : K_i \to K_{f_i} \). Clearly, if \( L \) is a knot, that is if \( n = 1 \), we simply write \( t_1 \) instead of \( t \). The integral group ring \( \mathbb{Z}[H_1(V)] \) is identified via this correspondence with the Laurent polynomial ring \( \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \). The Alexander polynomial \( \Delta_L(t_1, \ldots, t_n) \) of the link \( L \subset S^3 \) is this Laurent polynomial in the variables \( (t_1, \ldots, t_n) \) determined up to multiplication by polynomials of the form \( \pm t_1^{r_1} \cdots t_n^{r_n} \) with integral \( r_1, \ldots, r_n \). To summarize, the Alexander polynomial is a homology invariant computable from the one-dimensional homology group of the exterior of the link with appropriate twisted coefficients. Then the Milnor-Turaev theorem\textsuperscript{[14]} states that:
Milnor-Turaev Theorem (Ref. 14): the Alexander polynomial $\Delta_L$ of a link in $S^3$ is equal (up to a standard factor) to the Reidemeister torsion $\tau_\varphi$ of the exterior of the link, i.e.

$$\tau_\varphi(S^3 - L_f) \simeq \Delta_L(t_1, \ldots, t_n) \quad (2.6)$$

where $\varphi$ is the ring homomorphism: $\pi_1(S^3 - L_f) \to \Z[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. Indeed, $S^3 - K_f$ is a space with the homology of a solid torus, i.e. $H_{i=0,1}(S^3 - K_f) = \Z$ (otherwise zero), and hence in the case when $M^3 = S^3 - L_f$ we may identify the commutative multiplicative group $F_0$ defined in (2.4) with the Laurent polynomial ring $\Z(t_i, t_i^{-1})$. In general, if we have a set of homomorphisms $\rho_{(\alpha)} : \pi_1(S^3 - L_f) \to F_{0(\alpha)}(\Z) \simeq \Z(t_{i(\alpha)}, t_{i(\alpha)}^{-1})$, the correspondence (2.6) will take the form $\tau_{\varphi(\alpha)}(S^3 - L_f) \simeq \Delta_{L(\alpha)}(t_i)$, where $\Delta_{L(\alpha)}(t_i) \equiv \Delta_L(t_i)$. However, the sums (in $(\alpha)$) over $\tau_{\varphi(\alpha)}$ will become products (in $(\alpha)$) over $\Delta_{L(\alpha)}$ since $\Delta_{L(\alpha)}$ is by definition an element of the multiplicative group, namely of $\Z[H_{1(\alpha)}(S^3 - L_f)]$.

To summarize, we have shown that if $M^3$ is obtained by Dehn surgery along a certain link $L$ (with $n$ components) in $S^3$ with the preferred framing $f$, then the 3D-Euclidean quantum gravity partition function in the background Landau gauge is given by (up to some irrelevant normalization factors):

$$Z_{\text{EQG}}[M^3 = (S^3 - L_f) \cup_h L_f] = \sum_{(\alpha)} \tau_{\varphi(\alpha)}[(S^3 - L_f) \cup_h L_f], \quad (2.7)$$

where in particular

$$\sum_{(\alpha)} \tau_{\varphi(\alpha)}(S^3 - L_f) \simeq \prod_{(\alpha)} \Delta_{L(\alpha)}(t_i) i = 1, \ldots, n. \quad (2.8)$$

In the next section we shall show that the argument of the sum in (2.7) may be rewritten as the vacuum average of the link operator $L$ in terms of the partition function provided by the Alexander polynomial $\Delta_L$. This shall be a natural consequence of the fact that $\Delta_L$ can be represented by a free fermionic Berezin-type path integral and $L$ by non-local composite free fermion operators.

It is also worth noticing in particular that if $M^3$ is a-priori a fixed hyperbolic three-manifold $N^3$, then it is homeomorphic, by the ring homomorphism $\varphi$, to
the exterior of a knot $K$, e.g. to $S^3 - K_f$ if and only if $K$ is not a satellite knot and a torus knot.\[^{[15]}\] Thus in that case, equation (2.7) becomes formally

$$Z_{EQG}(N^3) = \sum_{(\alpha)}\tau_{\varphi(\alpha)}[\delta(S^3 - K_f)] \simeq \prod_{(\alpha)}\delta^* \triangle K(\alpha), \quad (2.9)$$

where $\delta$ is the homomorphism $S^3 - K_f \rightarrow N^3$, and $\delta^*$ denotes the lift to $\tau_{\varphi(\alpha)}$ (and hence $\triangle K$) of $\delta$, i.e. formally: $\tau_{\varphi(\alpha)}[\delta(\cdot)] = \delta^* \tau_{\varphi(\alpha)}(\cdot)$. $\delta^*$ may be obtained from a matrix representation of the mapping class group $\mathcal{M}_g$ canonically associated with the Heegaard decompositions of genus $g$ for $S^3$ and $N^3$ (Ref. 13)\(^*\).

A recent result by Kohno\[^{[17]}\] allows to define topological invariants $K(M^3)$ of closed orientable three-manifolds $M^3$ using the representations of $\mathcal{M}_g$ in such a way that $K(M^3)$ is an invariant under the Heegaard decomposition. As noticed discussing the fundamental theorem, any closed oriented 3-d manifold admits a Heegaard decomposition which, via Lickorish theorem\[^{[13]}\], is naturally in one-to-one correspondence with an element of the mapping class group of genus equal to the Heegaard genus. Kohno’s construction provides a projective linear representation of $\mathcal{M}_g$

$$\Phi_k : \mathcal{M}_g \rightarrow GL(Z_k(\Gamma))/\sim, \quad (2.10)$$

where $k$ is a positive integer labelling representations, and $Z_k$ is a finite dimensional complex vector space, each element of which is in one-to-one correspondence (via Kohno’s $k + 1$ admissible weights) with the edges of the dual graph $\Gamma$, which is a trivalent graph associated with the pants decomposition of the

\[^{[15]}\] Let us recall that under (any) framing $K_i \in L$ becomes a solid torus $T_i \equiv K_f, \partial T_i \neq \emptyset$, and that the set of all homomorphisms, up to isotopies, of a surface is defined as the mapping class group of that surface.\[^{[16]}\] Thus, the proof that $\delta \in \mathcal{M}_g$ follows directly from Thurston construction (Ref. 15) of hyperbolic three-manifolds $N^3$ as $N^3 = N^3 - (T'_1, \ldots, T'_r)$, where $N^3$ is hyperbolic and the $T'_i$’s are disjoint solid tori obtained by framing a suitable link $L'$ with $r$ components $K_i$ and by the so called Lickorish twist theorem (Ref. 13). In fact one may choose Heegaard decompositions of the same genus for $S^3$ and $N^3$, i.e. $S^3 = H_1 \cup g H_2, N^3 = H'_1 \cup g' H'_2$, where $g : \partial H_2 \rightarrow \partial H_1$ and $g' : \partial H'_2 \rightarrow \partial H'_1$. Here we assume that $\partial N^3 = \emptyset$. Since all handlebodies of a given genus are homeomorphic, choose any homeomorphism $h : H_1 \rightarrow H'_1$. It follows, as a consequence of Lickorish twist theorem, that the homeomorphism $f \equiv (g')^{-1} g : \partial H_2 \rightarrow \partial H'_2$ belonging to the genus-$g$ mapping class group $\mathcal{M}_g$ extends to a homeomorphism $f : H_2 - (T_1, \ldots, T_r) \rightarrow H'_2 - (T'_1, \ldots, T'_r)$. This extends the chosen $h : H_1 \rightarrow H'_1$ to a homeomorphism $\delta \equiv (h, f) : S^3 - (T_1, \ldots, T_r) = S^3 - L_f \rightarrow N^3 - (T'_1, \ldots, T'_r) = N^3 - L'_f = N^3$. So, $\delta : S^3 - L_f \rightarrow N$ carries an action of the mapping class group $\mathcal{M}_g$.\[^{[17]}\]
Heegaard surface $\Sigma_g$. $\sim$ denotes equivalence with respect to a cyclic group implying only phase factors. $K(M^3)$ is, up to a normalization factor, the trace of $\Phi$ meant as the 00 entry of the matrix $\Phi_k(h)$, $h \in \mathcal{M}_g$ being the Heegaard glueing homeomorphism, with respect to the basis of $Z_k$.

The problem we are faced with here, on the other hand, is the construction of the representations of the genus $g$ mapping class group $\mathcal{M}_g$ starting from a Dehn surgery presentation for $M^3$, for the following two reasons:

i) the euclidean 3-d quantum gravity partition function (2.7) is a topological invariant by way of the Reidemeister torsion of the 3-manifold $M^3$, given by a Dehn surgery presentation, and it is therefore interesting to investigate the relation between such an invariant and Kohno’s $K(M^3)$;

ii) in view of the features of the 3-d Ising model, whose partition function is based on the whole set of irreducible representations of the mapping class group of $\Sigma_g$ (in turn presented in terms of Dehn twists), and of the expected equivalence between the two partition functions, we have to express Dehn’s surgery invariants in terms of Heegaard invariants.

In other words, the problem is understanding the connection existing between $K(M^3)$ and the topological invariants $I(M^3)$ of the three-manifolds $M^3$ obtained\textsuperscript{[18]} by performing Dehn surgery on a framed link.

We sketch here a procedure to obtain representations of the mapping class group from the Dehn surgery prescription, due to Kohno.\textsuperscript{[19]} The data is a set of trivalent graphs $\gamma_i$ (3-holed spheres) and a link $L_0$ such that $L_0 \cup \gamma_i$ is a link $L \in S^3$. At this point one has two options: either performing surgery on the link $L = (L_0, \gamma_i)$ in $S^3$ with a choice of framing (e.g. the preferred framing discussed below Eq. (2.5)) thus obtaining a three-manifold $M^3$ as shown above, or, equivalently, regarding the trinions $\gamma_i$ as the complementary space of the so-called pants decomposition of a Riemann surface $\Sigma_g$. In other words, the $\gamma_i$’s with, say, $i = 1, \ldots, n$, characterize a Riemann surface of genus $g = \frac{1}{2}(n + 2)$. In this second case, the three-manifold $M^3$ is obtained by glueing, with a homeomorphism $f$, the cylinder $\Sigma_g \times I$ with another copy of $\Sigma_g$ (the link $L_0$ is inside the cylinder). Then it turns out that $f$, called the cylinder map, belongs to the mapping class group $\mathcal{M}_g$ of $\Sigma_g$.\textsuperscript{○} Two comments are now in order:

\textsuperscript{○} This construction of $\mathcal{M}_g$ representations may be understood also in terms of the so-called
a) $f$ provides a representation of the Heegaard decomposition;

b) whereas the process leading from the Heegaard decomposition to the Dehn surgery (and to the mapping class group representations) is one-to-one (naturally up to Heegaard equivalence), the inverse construction leading from Dehn surgery to Heegaard decomposition (and once more to a representation of $M_g$) is not necessarily one-to-one.

In other words, the surgery link $L$ depends on $f$, whereas $f$ in general does not depend on $L$ alone.

We argue that the two invariants derived one within the Dehn surgery scheme, the other in the Heegaard decomposition, should be related. To begin with, we recall that Cappell-Lee-Miller\[21\] have recently shown, in the frame of a conformal field theory approach to problem of topological invariants of a 3-manifold, that the above invariants $K(M^3)$ and $I(M^3)$ are the same up to a phase factor. In our specific case, if we take $I(M^3) \equiv Z_{EQG}(M^3)$, where $Z_{EQG}(M^3)$ is given by Eq. (2.7), and recall that $Z_{EQG}(M^3)$ is also equal to the Reidemeister torsion $\tau(M^3)$, Eq. (2.3), the equivalence with Kohno’s invariant $K(M^3)$ follows immediately from the fact that the Reidemeister torsion can distinguish homotopy equivalent spaces\[22\] just like Kohno’s invariant $K(M^3)$. For example, the invariant $K(M^3)$ can distinguish the Lens spaces $L(7, 1)$ and $L(7, 2)$, which are not homeomorphic three-manifolds with the same homotopy, like $\tau(M^3)$\[23\]. Of course, the equivalence between $K(M^3)$ and $Z_{EQG}(M^3) \equiv \tau(M^3)$, is up to a suitable irrelevant phase factor which, in the physical picture of $Z_{EQG}(M^3)$ as a path integral (see

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plat representation of a link.\[20\] Namely, if $L$ denotes the link carrying the Dehn surgery before the framing, we may represent it by a $(2g + 2)$-plat. Recall that a $(2g + 2)$-plat representation of the above link $L$ in $S^3$ is a triad $(S^3, \Sigma_0, L)$ where $(S^3, \Sigma_0)$ is a Heegaard splitting of genus zero of $S^3$ which separates $S^3$ into 3-balls $B^{(1)}$ and $B^{(2)}$ so that $B^{(i)} \cap L$ is a collection of $g + 1$ unknotted and unlinked arcs with $\partial B^{(i)} \cap L$ a set of $2g + 2$ points on $\Sigma_0 \equiv \partial B^{(i)}$ for $i = 1, 2$. The topological type of the triad $(S^3, \Sigma_0, L)$ is fully described by a Heegaard sewing map $\varphi$ which is required to preserve the $2g + 2$ points in $\partial (B^{(i)} \cap L)$; hence, up to isotopy, it is an element of the mapping class group $\mathcal{M}_{0,2g+2} \sim B_{2g+2}$ ($B_n$ is Artin’s $n$-strings braid group) of the $(2g + 2)$-punctured sphere. Here, $\mathcal{M}_{g,n}$ stands for the mapping class group for an $n$-punctured genus $g$ Riemann surface $\Sigma_{g,n}$. Then, as it is well known (Ref. 20), if $\sigma_i$ is the standard braid generator of $B_{2g+2}$ which interchanges the $i$-th and the $(i + 1)$-th points of $\partial (B^{(1)} \cap L)$, $\sigma_i$ lifts to the Dehn twist $\tau_{C_i}$ $(i = 1, \ldots, 2g + 1)$ along the non-contractible circle $C_i$ decomposing the Heegaard surface $\Sigma_g$ in a standard way. We may visualize $\Sigma_g$ as the 2-fold covering of the sphere $\Sigma_0$ branched over $\partial (B^{(i)} \cap L)$, $i = 1, 2$. Thus, $\mathcal{M}_g$ is minimally generated by a homomorphic image of $B_{2g+2}$ and one further element (Ref. 16).
next section), can be always reabsorbed into the functional measure.

Furthermore, the above problem can be seen in the ampler framework of distinguishing between homology and homotopy equivalence of manifolds in 3-d. Two 3-manifolds, say \( M \) and \( N \), are said to be simple homotopy equivalent if their CW complexes can be obtained one from the other by a CW deformation (namely adding a finite sequence of cells). The question whether homotopy equivalence implies simple homotopy equivalence was answered by Whitehead\cite{[24]} by a theorem stating that the obstruction to such implication is just the non-vanishing of the Whitehead torsion, of which the Reidemeister torsion is a representation.

As both the Kohno connection and our invariant (2.7) are simple homotopy invariants, yet not homotopy invariants, the above arguments, together with the results of Turaev and Viro\cite{[25]} who succeed in connecting the invariant constructed from the \( q \)-6 \( j \) symbols of the quantum group \( U_q(s\ell 2) \) (which has the same semiclassical limit as our \( EQG \) partition function) to the Kohno invariant, we conjecture, and shall henceforth assume, the equivalence of \( I(M^3) \) and \( K(M^3) \).

3. Free Fermions and the 3D-Euclidean Quantum Gravity Partition Function

The mapping class group \( \mathcal{M}_g \) of an orientable 2-manifold \( \Sigma_g \) of genus \( g \) is defined as the group of path components (i.e. modulo isotopy) of the group of all orientation preserving homeomorphisms of \( \Sigma_g \). Baer-Nielsen’s theorems gives us the equivalent definition: the mapping class group of a surface is isomorphic to the outer automorphism group of its fundamental group.

It is interesting to recall here a few basic facts about the representations of \( \mathcal{M}_g \). First, one fixes the cut system \( \mathcal{C}_0 \equiv \{ \alpha_1, \ldots, \alpha_g \} \), namely a collection of disjoint circles on \( \Sigma_g \) such that \( \Sigma_g \setminus \bigcup_{i=1}^{g} \alpha_i \) is a connected manifold, isomorphic with a 2\( g \)-punctured sphere. The simplest choice is \( \alpha_1 \) goes once around the first handle, \( \alpha_i, 2 = 1, \ldots, g \) goes once around the \( g \)-th handle separating the \( (i - 1) \)-th from the \( i \)-th hole. One defines then the new family of closed simple curves on \( \Sigma_g \), \( \{ \omega_{i,j} ; 1 \leq i < j \leq 2g \} : \omega_{i,j} \) interlaces handles \( i \) and \( j \) [more precisely, \( \omega_{i,j} \) enters hole \( i \), goes around handle \( i \), comes out of hole \( (i - 1) \), enters...
hole $j$, goes around handle $j$, comes out of hole $(j - 1)$ and closes]. Denote by $W_{i,j}$ the Dehn’s twist with respect to $\omega_{i,j}$. One defines moreover the following new homeomorphisms of $\Sigma_g$ : $P := A_g B_g A_g$, which is a simple move permuting $\alpha_g$ and $\beta_g$; $L := B_g A_g A_g B_g$, which reverses the orientation of $\alpha_g$, and $T_i := B_i A_i A_{i+1} B_{i+1}$, $i = 1, \ldots, g - 1$ which permutes the circles $\alpha_i$ and $\alpha_{i+1}$.

The mapping class group $M_g$ is generated by $\{ L; P; A_i, i = 1, \ldots, g; T_j, j = 1, \ldots, g - 1; W_{i,j}, 1 \leq i < j \leq g \}$.

Let now $H_0$ be the stabilizer subgroup, generated by $\{ A_i; W_{i,j} \}$, of elements of $M_g$ which leave the circles $\{ \alpha_i \}$ fixed; and $H$ the subgroup, generated by $\{ H_0; L; T_i \}$, of elements which leave the cut system $C_0$ invariant. $H$ is defined by the exact sequences:

$$1 \longrightarrow H_0 \longrightarrow H \xrightarrow{\vartheta} \pm S_g \longrightarrow 1 ;$$

$$1 \longrightarrow [\mathbb{Z}/2\mathbb{Z}]^g \longrightarrow \pm S_g \longrightarrow S_g \longrightarrow 1 ;$$

where $\vartheta(L) \in [\mathbb{Z}/2\mathbb{Z}]^g$ and $\vartheta(T_i)$ is the transposition $(i, i+1)$ in the symmetric group $S_g$.

All the relations of $M_g$ are generated by $\{ H, P \}$:

(I) $P$ commutes with $H_g$ (the subgroup of elements of $H$ which leave $\alpha_g$ and $\beta_g$ invariant);

(II) $P^2 \equiv A_g L A_g \in H$ ;

(III) $PFPFPFP \in H$ whenever $\exists$:

(1). a circle $\gamma$ on $\Sigma_g$ which intersects once transversally both $\alpha_g$ and $\beta_g$ and does not intersect any other $\alpha_i, i \neq g$, and

(2). a map $F \in H$ such that $[P F]^{-1} \gamma P F = \beta_g$ ; $[P F]^{-1} \beta_g P F = \alpha_g$ ; $[P F]^{-1} \gamma P F = \gamma$ .

(IV) $P$ commutes with $\tilde{F} P \tilde{F}^{-1}$ where $\tilde{F} \in H$ maps the simple closed curve $\tilde{\beta}$ encircling holes $(g - 1)$ and $g$ onto $\beta_g$ .

(V) $P F_1 P F_2 P F_3 P F_4 P \in H$ whenever $\exists$:

(1). a circle $\delta$ on $\Sigma_g$ which intersects once transversally both $\alpha_{g-1}$ and $\beta_g$ and does not intersect $\beta_{g-1}$ nor any other $\alpha_i, i \neq g - 1$, and
the maps $F_j \in H; j = 1, \ldots, 4$ satisfy – upon defining $E(0) := I; E(n) := E(n-1)P F_n; n = 1, \ldots, 4$ (in terms of which the element of $H$ we are considering reads $E(4)P$) – the four relations: $E(n)B_{g-1}E^{-1}(n) = B_g; n = 1, \ldots, 4$. When $\Sigma_g$ has no punctures the isotropy subgroup $H$ is included in the exact sequence

$$
\mathbb{Z} \to \mathbb{Z}^g \oplus B_{2g-1} \to H \to \pm S_g \to 1;
$$

where $B_{2g-1}$ is the Artin coloured braid group over $(2g-1)$ strings, whereas $\pm S_g$, the group of signed permutations of $g$ objects, isomorphic with the group of $g \times g$ matrices having just one non-zero entry, equal to $\pm 1$, in each row and column.

The above presentation allows deriving information about faithful representations of $M_g$ \cite{26} \cite{27}. For $g = 1$, $M_g \sim SL(2, \mathbb{Z})$, the classical (as opposed to the Teichmüller or many-handled) Modular Group. The related moduli space is a space whose points correspond to conformal isomorphism classes of tori. For arbitrary $g > 1$, upon denoting by $I(\Sigma_g)$ the set of isotopy classes of all the closed (non oriented) curves embedded in $\Sigma_g$, and by $\Phi_g$ any foliation whose leaves are geodesics for some metric on $\Sigma_g$ (since $\Sigma_g$ has negative Euler characteristics, the metric is hyperbolic), with transverse measure $\mu$, we have the following results. $\mu(\bullet)$, which is a positive real function assigning to each arc $\sigma \in \Sigma_g$ transverse to the leaves of $\Phi_g$ and with extremal points in $\Sigma_g \setminus \Phi_g$ an invariant weight, is determined by the conditions:

(a). $\mu(\sigma) = \mu(\sigma')$ if $\sigma$ is homotopic to $\sigma'$ through arcs transverse to $\Phi_g$ and with endpoints in $\Sigma_g \setminus \Phi_g$;  

(b). if $\sigma = \bigcup_i \sigma_i$; with $\sigma_i \cap \sigma_j \subset \partial \sigma_i \cap \partial \sigma_j$; then $\mu(\sigma) = \sum_i \mu(\sigma_i)$; 

(c). $\mu(\sigma) \neq 0$ if $\sigma \cap \Phi_g \neq \emptyset$.

The collection of all these measured geodesic foliations constitutes a space $\Xi_g$ on which $M_g$ acts in a natural way. In particular, in this (faithful) representation, the elements $m \in M_g$ are classified according to the following scheme: $m$ is said to be

periodic, if it is of finite order in $M_g$; 

reducible, if there is a point in $I(\Sigma_g)$ which is invariant with respect to the element $m$ itself.

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reducible, if there is a point in $I(\Sigma_g)$ which is invariant with respect to the element $m$ itself;
pseudo-Anosov, if \( \exists \) mutually transverse geodesic foliations \( \Phi_g^{(s)}, \Phi_g^{(u)} \in \Xi_g \) (\( s \) stands for \textit{stable}, \( u \) for \textit{unstable}), such that \( m(\Phi_g^{(s)}) = \frac{1}{\varepsilon} \Phi_g^{(s)} \) and \( m(\Phi_g^{(u)}) = \varepsilon \Phi_g^{(u)} \) for some real \( \varepsilon > 1 \).

In order to derive a faithful representation from our finite presentation, one should first prove that no normal subgroup \( N_{M_g} \) of \( M_g \) can have all of its elements \( \neq I \) which are pseudo-Anosov, because only in this case one can identify an homeomorphism \( m_o \in N_{M_g} \) fixing some \( \iota \in \mathcal{I}(\Sigma_g) \) and then proceed in the construction of an \textit{induced} faithful representation of \( M_g \) as a group of matrices (possibly with entries in a field of characteristics \( \neq 0 \) or of anticommuting variables).

For example, let \( \pi \) be a path on \( \Sigma_g \) which crosses the curve \( \alpha_i \) at a finite number \( \ell \) of points \( \{p_1^{(i)}, \ldots, p_{\ell}^{(i)}\} \). When we act on \( \Sigma_g \) with \( A_i \), the effect on \( \pi \) is that it is broken at each point \( p_k^{(i)} \) and a copy of \( \alpha_i \) is inserted at the discontinuity in such a way as to coalesce (also in orientation) with the adjacent fragments of \( \pi \). Resorting to the property that on any compact surface such as \( \Sigma_g \) there exists at least a pair of essential simple closed curves, say \( \gamma, \gamma' \), which \textit{fill} the surface but such that one can find another essential closed curve \( \tilde{\gamma}' \), disjoint from \( \gamma' \), such that \( \gamma \cup \tilde{\gamma}' \) does \textit{not} fill the surface, one can show (Ref. 26) that \( \mathcal{D}_\gamma \mathcal{D}_\gamma^{-1} \) is isotopic to a pseudo-Anosov map [28]. Then \( \gamma'' \equiv \mathcal{D}_\gamma \mathcal{D}_\gamma^{-1} \circ \tilde{\gamma}' \) is a curve disjoint from any essential simple curve \( \tilde{\gamma} \) having no intersections with \( \gamma \cup \tilde{\gamma}' \). Thus there exists a map

\[
\mathcal{D}_\gamma^{-1} \mathcal{D}_\gamma \mathcal{D}_\gamma^{-1} \mathcal{D}_\gamma \mathcal{D}_\gamma \mathcal{D}_\gamma^{-1} \equiv \mathcal{D}_\gamma^{-1} \mathcal{D}_\gamma''
\]

which fixes \( \tilde{\gamma} \) and hence is \textit{not} pseudo-Anosov.

Considering the action of \( M_g \) on the projective space \( \Xi_g \) of measured geodesic foliations, Dehn’s twists should be treated as maps with parabolic action, since they are locally conjugate to the element \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in PSL(2, \mathbb{Z}) \). Moreover, recalling the presentation of the fundamental group,

\[
\pi_1(\Sigma_g) \sim \langle A_1, B_1, \ldots, A_g, B_g \mid \prod_{i=1}^g [A_i, B_i] \rangle
\]

where \( \{A_i, B_i \mid 1 = 1, \ldots, g\} \) are assumed as a canonical basis for the first homology group \( H_1(\Sigma_g) \), and noticing that its elements which act parabolically on
the hyperbolic projective space are only those which may be freely homotoped into cusps and that just these elements are non-Anosov, all that remains to be done is to check – by using the presentation – whether \( \mathcal{M}_g \) has a geometrically finite subgroup \( \mathcal{S}_{\mathcal{M}_g} \) on which it acts by conjugation. Then, unless the normal closure in \( \pi_1(\Sigma_g) \) of the elements of the action of \( \mathcal{M}_g \) on \( \mathcal{S}_{\mathcal{M}_g} \) excludes all the cusp generators, not all of its elements \( \neq \mathbb{I} \) are pseudo-Anosov.

It is worth pointing out that this conclusion holds for \( g \geq 2 \), when \( \mathcal{M}_g \) there is a set of elementary homeomorphisms equivalent to global braids. The corresponding matrix representation, when it exists, is that induced from the monodromy representation associated with the Lefschetz fibration\(^{[29]} \) of \( \Sigma_g \).

The approach to the Ising model which so far appeared to be the most promising for extension to the \( d = 3 \) case is that referred to as the Pfaffian (or dimer) method, whose formulation holds – to a certain extent – for any number \( d \) of dimensions. We briefly review here the formulation of such a method that was proposed in Ref. 21\(^{[30]} \) as a possible candidate to attack some three-dimensional cases. It holds when \( \Lambda \) is homogeneous under some finitely presented (not necessarily finite) group \( G \), and consists of a number of steps:

[a] the decorated lattice \( \Lambda_\delta \) is derived from \( \Lambda \) following Fisher’s scheme\(^{[31]} \);

[b] the positional degrees of freedom in \( \Lambda_\delta \) are relabelled in terms of a set of anticommuting Grassmann variables \( \eta_{g_\ell} \), in one-to-one correspondence with the group elements \( g_\ell \) of \( G \);

[c] the group \( G \) is extended to the group \( \hat{G} \) in such a way that all the bond orientations of \( \Lambda_\delta \) compatible with the combinatorial constraints imposed by the global generalization of the Kasteleyn’s theorem* to a non-planar

* See Ref. 30 for a complete discussion of this delicate issue. Intuitively, what is done in this approach is that a generalized Kasteleyn’s theorem is obtained by first extending the planar case to the \( 2^{2g} \) covering of one of the surfaces, say \( \Sigma_g \), in which the lattice \( \Lambda_\delta \) is embedded, and by recovering then the non planar case by summing over all possible boundary conditions (spinwise) for the polygon whereby by suitable sides identifications \( \Sigma_g \) – which can of course be thought of as a Riemann surface – is obtained. Such sum over all possible choices of the boundary conditions is included in natural way in the configuration sum giving the model partition function. Successively, that sum is shown to be equivalent to summing over all possible ways of embedding \( \Lambda_\delta \) in a surface of genus \( g \), and the latter in turn to be identical to a sum over all the images of \( \Sigma_g \) with respect to the mapping class group, i.e. essentially with respect to all PL diffeomorphisms modulo isotopy of \( \sigma_g \) itself. This identifies the partition function with the zeta function for the (infinite dimensional) set
case (i.e. to one in which the lattice \( \Lambda \) cannot be embedded into a surface of genus zero, but can – yet preserving the lattice coordination – be embedded into one, say \( \Sigma_g \) of genus \( g \geq 0 \) and only those can be obtained as the invariant (under \( \tilde{G} \)) set of configurations of the graph \( \Gamma \) covering \( \Lambda_\delta \) \( 2^g \) times.

\[ [d] \text{ The partition function of the model on } \Lambda \text{ is then given by } \]

\[
Z(\Lambda) = \prod_{\alpha=1}^{d} \left\{ \cosh(K_\alpha) \right\}^{N_\alpha} \text{Pf} \tilde{\mathcal{S}} \quad ; \quad \prod_{\alpha=1}^{d} N_\alpha = N \quad ; \quad (3.1)
\]

where \( K_\alpha = \frac{J_\alpha}{k_B T} \) is the coupling constant of the model in direction \( \alpha \), and \( N \) the total number of sites of \( \Lambda \). \( \tilde{\mathcal{S}} \) is the incidence matrix of \( \Lambda_\delta \), extended with respect to \( \tilde{G} \) and \( \text{Pf} \) denotes the Pfaffian (for a skew-symmetric matrix such as \( \tilde{\mathcal{S}} \), \( \text{Pf} \equiv \sqrt{\det} \)).

\[ [e] \text{ If both } G \text{ and } g \text{ are finite, then } \tilde{G} \text{ is finite, and recalling that the regular representation } \mathcal{R} \text{ of a finite group } \tilde{G} \text{ is the direct sum of its irreducible representations } \text{, labelled by an index } j \text{ } , \text{ each contained as many times as its dimension } \dim j \text{, (3.1) reduces in a natural way to : } \]

\[
Z(\Lambda) = \prod_{\alpha=1}^{d} \left\{ \cosh(K_\alpha) \right\}^{N_\alpha} \prod_{j(F)} \left( \det \mathcal{R} \left[ \tilde{\mathcal{S}}(j(F)) \right] \right)^{\frac{1}{2} \dim j(F)} ; \quad (3.2)
\]

where the extra-index \( F \) refers to Fermionic representations , as required by the generalized Kasteleyn’s theorem, and \( \mathcal{S}(j) \) is a matrix of rank \( j \).

of flows induced by the diffeomorphisms. Finally, the same partition function is expressed as a product of (theta regularized) determinants, closely reminiscent of Dirichlet-type zeta functions. These, resorting to Fried’s definition \[^{[32]}\] D. Fried, Counting Circles, in Dynamical Systems, Springer-Verlag Lect. Notes Math. 1342, 196 (1988) can then be treated as the formal dynamical zeta function associated with a flow at zero value of its indeterminate. It has been recently proven by Moscovici and Stanton \[^{[33]}\] that this zeta function coincides with the \( R \)-torsion, with coefficients in any flat acyclic bundle, for \( \Sigma_g \). There is a complete consistency between such result, and the similarity pointed out by Milnor \[^{[34]}\] between the algebraic formalism of \( R \)-torsion in topology and zeta functions in the sense of Weil in dynamical systems. Also, the result perfectly bridges the approach to the Ising model of 30and the present approach to quantum gravity, with the identity exhibited by Ray and Singer in the second of refs. Sexpressing the holomorphic analog of \( R \)-torsion for surfaces of genus \( g \) in terms of classical Selberg zeta functions, whence the conjecture of equality between \( R \)-torsion and analytic torsion, subsequently proven by Cheeger and Müller 9was derived.
Let us recall first that the group $G$ is called the extension of the group $G$ by the group $\Pi$ if: having $G$ presentation $G \approx < \Xi | \Omega >$, where $\Xi$ denotes the set of generators and $\Omega$ the set of relations, and similarly having $\Pi$ presentation $\Pi \approx < \Upsilon | \Theta >$; we have the exact sequence

$$1 \rightarrow G \xrightarrow{\iota} G \xrightarrow{\pi} \Pi \rightarrow 1 ; \quad (3.3)$$

and – upon denoting by $\varphi$ a mapping which is the inverse of the inclusion $\iota$; $\varphi : \Pi \rightarrow G$ with $\pi \circ \varphi = 1_{\Pi}$; and by $\Upsilon(\varphi) \sim \varphi(\Upsilon)$ the restriction of the relations of $\Pi$ to $G - G$ has presentation:

$$G \approx < \Xi \cup \Upsilon(\varphi) | \Omega \cup \{ \varphi^{-1}(v) \xi \varphi(v) \lambda^{-1}_v(\xi) : \xi \in \Xi; v \in \Upsilon \} \cup$$

$$\cup \{ W_\vartheta(\xi) \vartheta(\varphi(v)) : \vartheta \in \Theta \} > . \quad (3.4)$$

where $\lambda_{\varphi} : G \rightarrow G$ is the automorphism of $G$ induced by the action of the element $\varphi \in \Pi$ on $G : g_\ell \mapsto \varphi^{-1}(\varphi(\varphi(g_\ell)))$; and $W_\vartheta$ is some suitable word (one is to be selected for each $\vartheta \in \Theta$) bringing each element of $G$ into the form $W(\xi) \cdot \gamma(\varphi(v))$ for some $\gamma \in \iota(G)$.

Of course, each automorphism $\lambda_{\varphi}$ can be altered by an inner automorphism of $G$ with no essential effect. If we factor out the group of inner automorphisms we obtain a new mapping $\kappa : \Pi \rightarrow Out G = Aut G / Inn G$ which is a homomorphism and is basic for the extension, in that equivalent extensions define the same homomorphism. The triple $\{ \Pi, G, \kappa \}$ is called an abstract kernel, and a group $G$ together with the exact sequence (3.3) is called an extension with respect to the abstract kernel if for $\gamma \in \pi^{-1}(\varphi), \varphi \in \Pi$, the automorphism of $G$ defined by $g_\ell \mapsto \iota^{-1}[\gamma^{-1}\iota(g_\ell)\gamma]$ belongs to the equivalence class of $\kappa(\varphi)$.

The cases of physical interest are those in which $G$ is a Fuchsian group and $\Sigma_g$ is a factor surface of $G$. The center $\mathcal{A}$ of $G$ can therefore be considered as a $\Pi$-module with an operation in the equivalence class of $\kappa(\varphi), \varphi \in \Pi$, if $\Pi$ is identified with the fundamental group $\pi_1(\Sigma_g)$. Considering now the family of cohomology groups $H^n(\Pi, G), n \geq 1$ of $\Pi$ with coefficients in $G$ (i.e. the cohomology groups of the cochain complexes defined by $\{ C^n(\Pi, G), \partial^n \}_{n \in \mathbb{Z}}$, \nl
where $\partial^n : C^n(\Pi, G) \rightarrow C^{n+1}(\Pi, G)$ is the boundary operator and $C^n$ is an $n$-dimensional cochain\(^*\), one notices that $C^3(\Pi, \mathbb{R})$ – upon regarding $C^n(\Pi, \mathbb{R})$ as an abelian group whose operation we write multiplicatively – is zero (one says that there is a trivial obstruction). The theorem of Zieschang (Ref. 35) states then the extension $G$ of the abstract kernel $\{\Pi, G, \kappa\}$ exists, and that $G$ is a proper subgroup of the mapping class group $\mathcal{M}_g$.

Thus the homeomorphism $\text{Ext} : G \rightarrow \tilde{G}$ required in $[c]^{\circ}$ acts locally by attaching a Kasteleyn’s phase to the circuits on $\Sigma_g$ homotopic to zero, and globally by an extension by the fundamental group, \textit{i.e.} mapping $\pi_1(\Sigma_g)$ to $\mathbb{Z}_2$. On the other hand, all possible surfaces in which $\Lambda_\delta$ can be embedded are equivalent from the combinatorial point of view, and we can restrict to one \textit{e.g.} by fixing a cut system on $\Sigma_g$. Moreover, as stated above, the relations of the mapping class group all follow from relations supported in certain subsurfaces of $\Sigma_g$ finite in number and of genus at most 2. There follows (Ref. 30) that the most general choice for $\tilde{G}$ is:

$$\tilde{G} = \mathcal{R} \bigotimes_{wr} S_{2g}; \quad (3.5)$$

where $\otimes_{wr}$ denotes the wreath-product\([37]\), whose elements can be taken to be all $2g \times 2g$ permutation matrices in which the non-zero elements have been replaced by elements of $\mathcal{R}$; whereas $\mathcal{R} = \mathcal{M}_g / \mathcal{H}$, namely the subgroup of elements of $\Sigma_g$ which preserve the isotopy class of a maximal, unordered, non separating system of $g$ disjoint, smoothly embedded cycles (non contractible and non isotopic), \textit{e.g.} just the cut system $\{\alpha_i; i = 1, \ldots, g\}$. $\mathcal{R}$ is then essentially generated by the elements representing homology exchange between any pair of circles $(\alpha_i, \alpha_j); i,j = 1, \ldots, g$.

Eq’s (3.2) and (3.5) allow us now to write the free energy $F \equiv -\kappa_B T \ln \mathcal{Z}$ as

$$-\beta F = \sum_{\alpha=1}^{d} N_{\alpha} \ln \cosh(\beta J_{\alpha}) + \frac{1}{2} \sum_{j^{(F)}} \text{dim} j^{(F)} \text{Tr} \left( \ln \mathcal{R} \left[ \tilde{\mathcal{S}}^{(j^{(F))}} \right] \right); \quad (3.6)$$

\(^*\) Recall that $C^n(\Pi, G), n \geq 1$ is the group of all functions $f : \Pi^n \equiv \Pi \times \cdots \times \Pi \rightarrow G$ such that $f(\varpi_1, \ldots, \varpi_n) = 0$ if some $\varpi_i, 1 \leq i \leq n$ equals $1$.

\(^\diamond\) It should be kept in mind that maps and spaces are to be thought of in the $PL$ (piecewise-linear) category, namely all morphisms referred to in present discussion should be meant in the corresponding definition as given in Ref. 24\([36]\).
from which it appears clearly that while $\mathcal{Z}$ can be expanded in terms of characters of $\mathcal{R}$, $\mathcal{F}$, as given by the latter equation, could be rewritten in terms of invariant symmetric functions for $\mathcal{R}$. The coefficients of such an invariant expansion retain some of the original combinatorial flavour of the problem: they count the numbers of words in $\mathcal{R}$ equivalent to the identity, i.e. provide a solution for the Dehn’s word problem for the subgroup $\mathcal{H}$ of $\mathcal{M}_g$.

An unexpected bridge between $\mathcal{Z}_{\text{EQG}}(N^3)$ as given in (2.9), and $\mathcal{Z}(\Lambda)$ as given in (3.1), can be cast by the following argument. Recalling the representation of $\mathcal{Z}(\Lambda)$ as grassmannian path integral, as given by Itzykson$^{[38]}$, we consider for simplicity the particular case in which $M^3$ is obtained by Dehn surgery along a knot $K$ in $S^3$. It follows, that the associated Eq. (2.7) is a special form of the generalized surgery formula for a non-Abelian 3D-Euclidean Chern-Simons gauge theory defined over a generic three-manifold $\tilde{M}^3$ suggested by Witten.$^{[39]}$ Witten, in Ref. 39, argues that:

$$
\begin{cases}
Z[\tilde{M}^3] = \sum_j h_j^0 Z[M^3; R_j] \\
Z[M^3; R_j] \equiv \langle W_{R_j}(K) \rangle_{M^3},
\end{cases}
$$

(3.7)

where $\tilde{M}^3 = M^3 \cup_h K_f$, $h$ is the glueing homeomorphism on the solid torus $K_f$ and $Z[M^3; R_j]$ is the CS-partition function of $M^3$ with an extra Wilson line $W_{R_j}(K)$ in the $R_j$ representation (of the CS-gauge group $\mathcal{G}$) included on the knot $K$. When the CS-coupling $k$ is an integer, using the techniques of rational conformal field theories (see e.g. Ref. 39), one could show that $R_j$ is a finite-dimensional modulus of the representation ring of $\mathcal{G}$ with $j < \infty$. Then it turns out that the knot diagram $D_K$ parametrized by $R_j$ has a nice (equivalent) interpretation$^{[40]}$ in terms of the so-called “r-parallel version” $C \ast D_K$ of $D_K$. That is, for any $j \in \{1, \ldots, N\}$ we associate a non-negative integer $C(j)$, called the “colouring” of $D_K$, from the set $\{1, 2, \ldots, n\} \in \mathbb{Z}_+$. Let $C(j) \ast D_K$ be the diagram which can be formed by taking $C(j)$-copies all parallel, in the plane, to $D_K$. In this picture Eq. (3.7) becomes:

$$
Z[M^3; R_j] = \langle W_{R_j}(K) \rangle_{M^3} = \langle W_{R}[C(j) \ast D_K] \rangle_{M^3} \equiv \langle C(j) \ast D_K \rangle_{M^3},
$$

(3.8)

the symbol $W_{R}$ denoting the Wilson line in the fundamental representation $\mathcal{R}$ of $\mathcal{G}$. Similarly, one finds that the coefficients $h_j \equiv h_j^0$ (in general complex
numbers) can be written as $h_j = h_{C^{-1}(\mathbb{Z}_+)} \equiv \lambda_c$ by definition of the colouring map $C$. Therefore, one can also write Eq. (3.7) as (remind that $K$ denotes a knot):

$$Z[\tilde{M}^3] = \sum_{c \in C} \lambda_c \langle c \ast D_K \rangle_{M^3}. \quad (3.9)$$

Eq. (3.9) has recently been rigorously stated by Lickorish (Ref. 18) in the case of the one-variable Jones polynomial for $\langle c \ast D_K \rangle_{M^3}$ if $M^3 = S^3$ and $G = su(2)$.

It is now immediate to notice that the partition function (2.7) of the 3D-Euclidean quantum gravity has the form (3.9) with $\tilde{M}^3 = (S^3 - K_f) \cup_h K_f$ and $M^3 = S^3 - K_f$, if one sets

$$\langle \ldots \rangle_{(S^3 - K_f)} \simeq \sum_{(\alpha)} \tau_{\varphi(\alpha)}(S^3 - K_f) \simeq \prod_{(\alpha)} \Delta_K(t_{(\alpha)}) \equiv \prod_{(\alpha)} \Delta_{K(\alpha)}(t), \quad (3.10)$$

where we have used Eq. (2.8) and $t_{(\alpha)} \in H_1(S^3 - K_f) \equiv \rho_{(\alpha)}[\pi_1(S^3 - K_f)]_{[\ast, \ast]}$, and one regards $C \ast D_K$ as an extra “field” on which to compute the vacuum-to-vacuum expectation value given by the “partition function” $\prod_{(\alpha)} \Delta_{K(\alpha)}$. Now, such an identification of $\Delta_{K(\alpha)}$ with a certain path integral for each $(\alpha)$ is just what one in fact has!

Indeed, Kauffman and Saleur$^{[41]}$ have recently shown that the Alexander-Conway polynomial of a knot $K$ is the fermionic path integral over free fermions propagating on the knot diagram $D_K$. Their basic idea is to describe the tangle diagram $D_K$ as a planar Feynman graph $\Gamma_K$ for a Gaussian fermionic theory. The Feynman graph is obtained by projection of the tangle diagram on a two-dimensional planar four-valent graph. To each crossing $i$ of an oriented tangle diagram one associates four complex Grassmannian variables $\psi^\alpha_i, \psi^\beta_j$ where the labels $\alpha = \beta = \text{up (u)}, \text{down (d)}$ refer to edges going up and down with respect to the direction of the crossing at the point $i$.

All $\psi$’s anticommute

$$[\psi^\alpha_i, \psi^\beta_j]_+ = 0; \ \alpha, \beta = n \text{ or } d; \ i \neq j$$

and in particular $(\psi^\alpha_i)^2 = 0$. The Berezin path integral is defined as usual by the
\[ \int \Pi_i \psi_i^u \psi_i^{u\dagger} d\psi_i^d d\psi_i^{d\dagger} \Pi_i \psi_i^u \psi_i^{u\dagger} \psi_i^d d\psi_i^{d\dagger} = 1. \]

At Lagrangian level, if along the link \((i,j)\) the edge is oriented from vertex \(i\) to vertex \(j\), the propagator is \(\psi_i^{\alpha\dagger} \psi_j^\beta\) with labels \(\alpha, \beta = u \text{ or } d\) depending on the particular configuration. For instance, the tangle \(D_K\) or equivalently the associated Feynman graph \(\Gamma_K\), both shown in Fig. 1, correspond to the kinetic term \(\psi_i^u \psi_j^d\). Thus, the Kauffman-Saleur’s result is

**KS-Theorem** (Ref. 41): the Alexander-Conway polynomial \(\nabla_K(q)\) for a fixed knot \(K\) has the fermionic path integral representation

\[
\left\{ \begin{array}{l}
\nabla_K(q) = \langle \psi_+ | \psi_- \rangle \equiv N(K; q) \int d\psi^\dagger d\psi \exp \left\{ \sum_{i,j} \sum_{\alpha, \beta = u, d} \psi_i^{\alpha\dagger} M_{\alpha, \beta}^{i,j}(K; q) \psi_j^\beta \right\}, \\
N(K; q) \equiv q^{-L(K) - I(K)}
\end{array} \right. \tag{3.11}
\]

where \(M = [M_{\alpha, \beta}^{i,j}(K; q)]\), which depends on the type of knot \(K\) selected, is a \([2 \times \# (crossings)][\otimes^2]\)-matrix whose entries are \(\pm 1\) or certain rational functions of \(q\). Furthermore, \(N\) is a normalization factor specified by the number of internal edges \(I(K)\) (loops \(L(K)\)) of the Feynman graph \(\Gamma_K\) associated to \(D_K\).

Since the usual Alexander polynomial \(\triangle_K(t)\) is given by \(\nabla_K(q)\) in terms of the formula (Ref. 42)

\[ \triangle_K(t) = \nabla_K(q \equiv \sqrt{t} - \frac{1}{\sqrt{t}}), \tag{3.12} \]

it follows that the Gaussian Berezin path integral (3.11) extends also to \(\triangle_K(t)\) and hence to \(\tau_\psi(S^3 - K_f)\) via Eq. (2.6). In our case we have a family of Alexander-Conway polynomial for a knot is defined by the skein relation

\[ \nabla_{K+}(q) - \nabla_{K-}(q) = q \nabla_{K_0}(q) \]

and by the normalization: \(\nabla_K = 0\) for \(K = \text{(unknot)}\) and \(\nabla_K = 1\) for \(K = \text{(unknotted strand)}\).

\[ \diamond \text{ As it is well known, the Berezin path integral in (3.11) gives the square of the Pfaffian } Pf(M) \equiv \sqrt{\det[M(K; q)]}. \]
der polynomials \( \Delta_{K(\alpha)} \), thus we shall have \( \Delta_{K(\alpha)} \equiv \Delta_K(t(\alpha)) = \langle \psi_+|\psi_- \rangle_{(\alpha)} \propto \sqrt{Pf(M_{K(\alpha)})} \), where \( M_{K(\alpha)} \) inherits the dependence on the labelling \((\alpha)\) by \( t(\alpha) \).

Collecting all together, when \( M^3 \) is obtained by Dehn surgery along a knot \( K \) in \( S^3 \) we have the formula:

\[
Z_{EQG}[M^3 = (S^3 - K_f) \cup_h K_f] = \sum_{c \in C} \lambda_c \prod_{(\alpha)} \langle \psi_+|[c * D_K](\psi^\alpha \psi^\beta)]|\psi_- \rangle_{(\alpha)} \tag{3.13}
\]

where \( h \ni h_j = h_{C^{-1}(\mathbb{Z}_+)} \equiv \lambda_c, c \in \mathbb{Z}_+, \) and \([c * D_K](\psi \psi^\dagger)\) denotes the operator associated to the c-parallel version of \( D_K \) in the Kauffman-Saleur fermionic representation. Clearly, following Ref. 41, we may identify \( 1 * D_K = D_K \) with the action. For instance, to the trefoil diagram \( D_T \) and to the associated Feynman graph \( \Gamma_T \), shown in Fig. 2 corresponds the matrix element \( \langle \psi_+|(\psi^\dagger M(T; t)|\psi)\rangle \) where (Ref. 18)

\[
\psi^\dagger M(T; t)\psi = \psi^d_1 \psi^u_2 + \psi^d_1 \psi^d_2 + \psi^u_2 \psi^u_3 + \psi^d_2 \psi^u_3 + \psi^u_3 \psi^d_1 + \\
(\sqrt{t} - \frac{1}{\sqrt{t}}) \sum_{i=1}^3 (\psi^u_i \psi^u_i + \psi^d_i \psi^d_i) + (t + \frac{1}{t} - 3) \sum_{i=1}^3 \psi^u_i \psi^d_i.
\]

As we already observed, the c-parallel version of \( D_K \) is a link analogue of the technique used to characterize the representation ring of a Lie algebra by tensor products of the fundamental representation, rather than by the irreducibles. In our fermionic picture, this corresponds to using integer powers \((\psi^\dagger M_K \psi)^{l(c)}\), \( l(c) \in \mathbb{Z}_+ \), of \( \psi^\dagger M_K \psi \) to describe \( c * D_K \). As a consequence of this description, the quantity under the product symbol in the r.h.s. of Eq. (3.13) becomes the Berezin path integral of a polynomial in \( \psi^\alpha_i \psi^\beta_j \) and it may be interpreted as the Green’s function obtained from \( \int d\psi^\dagger d\psi \exp(\psi^\dagger M_K \psi) \).

It is not surprising at this point that there exist an intimate relationship between torsion invariants and partition function which can be formally interpreted as dynamical zeta functions and, on the one side, Gaussian fermionic (i.e. grassmannian) stochastic systems, on the other topological invariants. On one front

\[
\langle \psi_+|(\psi^\dagger M_K \psi)^{l(c)}|\psi_- \rangle_{(\alpha)} \sim \left( \frac{\partial}{\partial \beta} \right)^{l(c)} \int d\psi^\dagger d\psi \exp(\beta \psi^\dagger M_K(\alpha)\psi)|_{\beta=1}.
\]

* Clearly, one has that:
we have representatives of anosov flows entering the partition function, whose relative weight heavily exceeds that of closed orbits of the discrete periodic set and hence emphasizes the stochastic features of the model. On the other side the same partition function, which describes the global dynamical (and/or thermodynamical) behaviour of the system can be viewed as the generating function of all closed loops (i.e. links with possibly knotted components) embedded in a Riemann surface $\Sigma_g$. The problem of characterizing the asymptotic images of the manifold $M_{X \cap Y}$ intersection between two sub-varieties $X$ and $Y$ of a given manifold $\mathcal{W}$ under iteration of the group of diffeomorphisms of $\mathcal{W}$ as, say, $X$ is kept fixed ($\dim \mathcal{W} = \dim X + \dim Y$, $\dim M_{X \cap Y} = |\dim X - \dim Y|$), has been recently studied by V.I. Arno’ld\[43\]. Under an ergodic hypothesis, well motivated physically, one expects that the equilibrium features of the model we are considering are indeed controlled just by these images (where one identifies obviously $X$ with $\Sigma_g$, $Y$ with the isotopy-equivalents of $\Sigma_g$ itself, and $M_{X \cap Y}$ with the set of loops generated by intersection). The result of Arno’ld implies that it is just the set of topological invariants of $M_{X \cap Y}$ and it alone which completely characterizes the asymptotic action of the group of diffeomorphisms of $\mathcal{W}$ (that in our correspondence can be thought of as the manifold of dimension $\dim \Sigma_g + 1$ (in general non Euclidean also in the Ising case, due to the choice of boundary conditions).

A final comment is in order. The procedure described in Sect. 3, whereby we have essentially mapped the 3D Euclidean gravity to a free-fermion system over a lattice is an homeomorphism between the two theories. This is due to the conceptual passage through the Ising model, which allows us in principle to reconstruct from the lattice model the whole group of diffeomorphisms in 3D, $Diff^3$. The profound meaning of such a reconstruction can be understood in the following way: the 3D Euclidean quantum gravity partition function is clearly invariant with respect to $Diff^3$, as it essentially coincides with the Reidemeister torsion which is diffeomorphically invariant by its very construction. On the other hand, at the fermionic lattice model level one has (ref. 41) a hidden quantum symmetry $U_q[s\ell(1,1)]$, in other words, the link fermions $\psi^\alpha_i$, $\psi^\beta_j$ have a non-trivial statistics whose dual is the quantum group $U_q[s\ell(1,1)]$. The key notion here is then the following: the quantum group symmetry $U_q[g]$ of a lattice model appears on very short distances of the order of the lattice step; in the continuum
limit, it appears at a single point, valued in the group of the Kac-Moody algebra \( \hat{g} \) associated with the Lie algebra \( g \) of some finite-dimensional Lie group \( G \).†

Thus we can equivalently affirm that our Gaussian fermionic system has a gauge symmetry of type \( G \approx SU(1, 1) \), since the corresponding Kac-Moody group acts just as a local gauge symmetry. It is known (ref. 1) that the action in the continuum theory of this local gauge \( A_1 \) symmetry on the first order fields \((e_\mu, \omega_\mu) \equiv A_\mu \) describing the 3D gravitational field \( g_{\mu,\nu} \) is, on shell, equivalent to the to the action of \( Diff^3 \) on \( A_\mu \). This leads us to interpreting the presence of the group of diffeomorphisms in 3 dimensions in the continuum Einstein gravity theory as the manifestation of the *quantum* internal symmetry of the underlying lattice model (3.11).

4. Conclusions

Summarizing, we have shown that 3D-Euclidean quantum gravity in first order dreibein formalism and in the Landau gauge, when quantized on a generic three-manifold obtained by Dehn surgery along a knot \( K \) (link \( L \)) in \( S^3 \), is equivalent to a Gaussian fermionic theory propagating on the c-parallel versions of the knot (link) diagram \( D_K \) \( (D_L) \). In particular we have shown in the Berezin path integral picture that the 3D-EQG partition function \( Z(N^3) \) for a 3D-hyperbolic manifold \( N^3 \) is equivalent (up to some irrelevant normalization factor) to that one \( Z(\Lambda) \) of a 3D-Ising model on a lattice \( \Lambda \) embedded in \( \mathbb{R}^3 \).

Let us conclude with two remarks:

i) In the previous section we have shown that 3D-Euclidean quantum gravity admits a free fermion representation as well as the 3D-Ising model (Ref. 26) and that these two models seem in mutual relation. Furthermore, we have proved that for fixed tridimensional topologies the 3D-EQG partition function is given by a suitable Alexander-Conway polynomial, which can explicitly be computed by combinatorial or (Gaussian) path integral techniques. So, thanks to the aforementioned equivalence, we also have a computable algorithm for solving the quantum 3D-Ising model before performing the thermodynamic limit.

† This observation first due to Alekseev, Faddeev and Volkov\[^{44}\] in their study of the WZNW-model, applies here as well, with \( \hat{g} = sl(1, 1) \).
Now, a related question is whether these are the only 3D-models which allow a free fermion description, or rather it is a general property of (integrable) models in 3D.

ii) We would like to notice that an indirect hint of the possible quantum connection between the 3D-Euclidean quantum gravity partition function and the semiclassical limit of a polynomial link invariant may be found – in view mainly of the recent work by Turaev – in the analysis performed by Ponzano and Regge. In Ref. 45 it was argued that when $M^3$ is “close” to $S^3$, the path integral of the 3D-Euclidean quantum gravity in the simplicial approximation known as the Regge calculus is actually proportional to the semiclassical (large angular momentum) limit of the standard $su(2)$ 6j-symbol. We conjecture that the realization of the Regge Ponzano program of understanding the Feynman summation of histories for the lattice 3D euclidean Einstein-Hilbert action as a sort of state model associated with the Racah coefficients, can be fully completed at quantum level by our eq. (2.9).‡

Such a conjecture is based on the following facts:

a) it is known that the standard $su(2)$ 6j-symbol is the “semiclassical” limit ($q \to 1$) of the quantum 6j symbol

$$\{a \ b \ c\ \ d \ e \ f\}_q \equiv [D]_q$$

of the quantum group $U_q(sl(2; \mathbb{R}))$;

b) an intrinsic combinatorial approach is known which allows to associate with the quantum 6j symbols of $U_q(sl(2; \mathbb{R}))$ the two-variable HOMFLY-polynomial $P_K(q, z)^{[49]}$;

c) the Alexander polynomial $\triangle_K(t)$ entering eq. (2.9), is a particular case of $P_K(q, z)$ when $q = 1^{[50]}$ and $z = \sqrt{t} - \frac{1}{\sqrt{t}}$.

‡ Very recently, Ooguri et al. and Mizoguchi et al. argued that the Turaev-Viro model provides indeed a $q$-analogue lattice regularization of the Ponzano-Regge model, where the cut-off is given by $\frac{1}{\ln q}$. In their approach the equivalence with the 3D Euclidean quantum gravity in the form (1.1) follows as continuum limit of the Turaev-Viro piecewise-linear model, equivalent to the limit $q \to 1$. 
On the other hand, the possibility of constructing directly “quantum” invariants for a closed 3-manifold $M$ from the $q$-$6j$ symbols has been recently stressed by Turaev and Viro in Ref. 25, supporting once more our conjecture that the Regge Ponzano idea might be extended to full quantum level. Indeed we may recall that the Turaev-Viro 3-manifold invariant of $M$ is given by

$$|M|_q \equiv C^{\mathcal{V}} \sum_{\{\text{Col}\}} \prod_{i \in \mathcal{E}} (-1)^i [2i + 1]_q \prod_T [D]_q$$

where $q$ is a complex root of unit of a certain degree $k \in \mathbb{Z}_+$, $k \geq 0$, $C$ is a constant, $\mathcal{V}$, $\mathcal{E}$, $\mathcal{T}$ denote respectively the numbers of vertices, edges, and the set of tetrahedra of the simplicial complex $\mathcal{X}$, $[n]_q$ is the quantum dimension, whereas $\text{Col}$ is the map which associates with the edges of $\mathcal{X}$ elements of the set $\{\text{Col}\}$ of colours $\left\{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\right\}$ related in standard way to the framing map introduced in (2.5).

Notice that the construction of the invariant is associated with a specific triangulation of $M$; however the main theorem of Ref. 25 show just that $|M|_q \in \Phi$ does not depend in fact on the choice of the triangulation $\mathcal{X}$, namely it is a bona fide topological invariant.

Moreover, Turaev and Viro show that

$$|M|_q = |I_k(M)|^2,$$

where $I_k(M)$ is the Dehn surgery invariant for $M$ discussed in Sect. 2, and $q = \exp \left(\frac{2\pi i}{k + 2}\right)$.

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FIGURE CAPTIONS

1. An example of the correspondence between a knot diagram $D_K$ and its relative planar Feynman graph $\Gamma_K$ for a Gaussian fermionic theory.

2. The trefoil, its planar representation and relative correspondences.