Action-angle map and duality for the open Toda lattice in the perspective of Hamiltonian reduction

L. Fehér

Department of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, H-6720 Szeged, Hungary, and
Department of Theoretical Physics, WIGNER RCP, RMKI
H-1525 Budapest, P.O.B. 49, Hungary
e-mail: lfeher@physx.u-szeged.hu

Abstract

An alternative derivation of the known action-angle map of the standard open Toda lattice is presented based on its identification as the natural map between two gauge slices in the relevant symplectic reduction of the cotangent bundle of $GL(n, \mathbb{R})$. This then permits to interpret Ruijsenaars’ action-angle duality for the Toda system in the same group-theoretic framework which was established previously for Calogero type systems.

Keywords: open Toda lattice, action-angle duality, Hamiltonian reduction
1 Introduction

It is common knowledge that Liouville integrable systems admit action-angle variables which trivialize the flows of the commuting Hamiltonians. There exist powerful methods (see e.g. [1]) to construct action-angle variables from the original variables, but then one still has to face the difficult and relevant problem of reconstructing the original variables from the action-angle variables. For purely scattering systems, one can in principle use the asymptotic momenta and their conjugates to obtain globally well-defined action-angle variables. However, even among such topologically trivial systems it is very exceptional that one can describe the map from the action-angle variables to the original variables explicitly.

A beautiful example is provided by the standard open Toda lattice encoded as a Hamiltonian system by $(M, \omega, H)$, where $M := \mathbb{R}^n \times \mathbb{R}^n$ with the Darboux form

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i$$

and

$$H(q, p) = \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i=1}^{n-1} e^{q_i-q_{i+1}}. \quad (1.1)$$

The phase space of the corresponding action-angle variables, here denoted by $\hat{p}$ and $\hat{q}$, is

$$\hat{M} := \{(\hat{p}, \hat{q}) \in \mathbb{R}^n \times \mathbb{R}^n | \hat{p}_1 > \hat{p}_2 > \cdots > \hat{p}_n\} \quad (1.2)$$

equipped with the symplectic form $\hat{\omega} = \sum_{i=1}^n d\hat{q}_i \wedge d\hat{p}_i$. One can find the explicit action-angle map, $R: \hat{M} \rightarrow M$, in the paper [2] by Ruijsenaars\(^1\). The map $R$ operates according to the following formula:

$$q_j = \ln(\sigma_{n+1-j}/\sigma_{n-j}), \quad p_j = \dot{\sigma}_{n+1-j}/\sigma_{n+1-j} - \dot{\sigma}_{n-j}/\sigma_{n-j}, \quad \forall j = 1, \ldots, n, \quad (1.3)$$

where $\sigma_0 := 1$ and

$$\sigma_k := \sum_{|I|=k} e^{\sum_{i\in I} \hat{q}_i} \prod_{i\in I, j \notin I} |\hat{p}_i - \hat{p}_j|^{-1}, \quad \forall k = 1, \ldots, n. \quad (1.4)$$

The sum is taken over the subsets $I \subset \{1, 2, \ldots, n\}$ of cardinality $|I| = k$ and $\dot{\sigma}_k$ is defined by $\dot{\sigma}_k := \{\sigma_k, \frac{1}{2} \sum_{i=1}^n \hat{p}_i^2\}_M$. This map $R$ converts $H$ into the free form $H \circ R = \frac{1}{2} \sum_{i=1}^n \dot{p}_i^2$. Ruijsenaars’ derivation [2] relied on scattering theory and careful analysis of the Toda dynamics both for obtaining the map $R$ and for proving that it is a symplectomorphism. His construction was inspired by the pioneering work of Moser [3] and formulas from the papers of Kostant [4] and Olshanetsky-Perelomov [5] dealing with the explicit solution of the open Toda systems attached to all simple root systems [6]. For reviews of Toda systems, see e.g. [1, 7, 8, 9].

The first goal of this Letter is to present an alternative derivation of the Toda action-angle map. We recall [5] (or [1, 7, 9]) that the system (1.1) results from Hamiltonian reduction of the free geodesic motion on the symmetric space $GL(n, \mathbb{R})/O(n)$, and thus it can be also viewed as a reduction of free motion on the larger configuration space $GL(n, \mathbb{R})$. We shall explain how this basic fact leads readily to the map $R$ and all its pertinent properties.

The second (and main) goal is to demonstrate that the action-angle dual of the Toda system, introduced in [2], can be interpreted in the same manner as is now well-known (see e.g. [10, 11]) regarding Ruijsenaars’ duality relations [12, 13] between Calogero type systems. Recall\(^1\)

---

\(^1\)Our variables $(\hat{p}, \hat{q})$ correspond to $(-\hat{\theta}, \hat{q})$ as used in [2].
that two integrable many-body systems form a dual pair if their phase spaces are related by a symplectomorphism that converts the particle-positions and the action-variables of one system, respectively, into the action-variables and the particle-positions of the other system. The usual group-theoretic picture behind this kind of duality (called action-angle duality or Ruijsenaars duality) is as follows. There exists a “big phase space” equipped with two distinguished Abelian Poisson algebras which descend to particle-positions and action-variables of integrable many-body systems upon a suitable reduction. The key point is that the single reduced phase space admits two alternative models, typically given by two gauge slices, which are identified with the phase spaces of the many-body systems in duality. In particular, the roles of the two reduced Abelian Poisson algebras as positions and actions are interchanged in the two models.

In fact, we shall identify the phase spaces \((M, \omega)\) and \((\hat{M}, \hat{\omega})\) with two gauge slices in the relevant symplectic reduction of \(T^*GL(n, \mathbb{R})\). The Toda system lives on \(M\), while \(\hat{M}\) supports the dual many-body Hamiltonian defined by

\[
\hat{H}(\hat{p}, \hat{q}) := \sigma_1(\hat{p}, \hat{q}) = \sum_{i=1}^{n} e^{\hat{q}_i} \prod_{j \neq i} \frac{1}{|\hat{p}_i - \hat{p}_j|}.
\]  

We shall then explain that the geometrically engendered symplectomorphism between the two slices yields the action-angle map \(R\), and confirm that the above sketched interpretation of the duality relation holds in the Toda case. Conceptually, the whole picture is the same as for Calogero type systems. The special feature is that now the map from \(\hat{M}\) to \(M\) can be described fully explicitly.

The description of the reduction picture of the Toda duality is the principal achievement of this Letter. This represents a step forward in the research program aimed at understanding all action-angle dualities in group-theoretic terms based on Hamiltonian reduction. As for the action-angle map itself, it must be stressed that several other alternative derivations are possible. For example, derivations of the Toda action-angle variables can be found in [14, 15, 16, 17], although the author is not aware of any reference other than [2] where the reconstruction of the original variables is given in as explicit form as displayed in equations (1.3), (1.4) above.

The rest of this text is organized as follows. In subsections 2.1 and 2.2, we present two models of the reduced phase space: one is the standard Toda slice [5, 7] and the other is a new one, which we shall term “Moser gauge”. In subsection 2.3, we derive the explicit action-angle symplectomorphism \(R\) as the map from the Moser gauge to the Toda gauge. In section 3, we explain the duality issues and conclude with comments on open problems.

## 2 Two descriptions of the reduced phase space

In what follows we denote \(G := GL(n, \mathbb{R})\) and let \(\mathfrak{g}\) stand for the corresponding Lie algebra \(gl(n, \mathbb{R})\). We consider the maximal compact subgroup \(K := O(n, \mathbb{R})\), the group \(A\) of positive diagonal matrices and the group \(N_+\) of upper triangular matrices having 1 along the diagonal.

We are going to reduce the cotangent bundle of \(G\), which we realize as

\[
T^*G \simeq G \times \mathfrak{g} = \{(g, \mathcal{J})\},
\]  

where left-translations are used for trivializing \(T^*G\) and \(\mathfrak{g}^*\) is identified with \(\mathfrak{g}\) by means of the invariant bilinear form of \(\mathfrak{g}\) provided by the matrix trace. The Hamiltonian of the free particle
moving on $G$ is the $k = 2$ member of the Poisson commuting family

$$H_k(g, J) := \frac{1}{k} \text{tr}(J^k), \quad k = 1, \ldots, n. \quad (2.2)$$

The symmetry group whereby we reduce is the direct product $N_+ \times K$. An arbitrary element $(\eta_+, \eta_K)$ from this group acts on $T^*G$ by the map $\Psi_{(\eta_+, \eta_K)}$ defined by

$$\Psi_{(\eta_+, \eta_K)}(g, J) := (\eta_+ g \eta_K^{-1}, \eta_K J \eta_K^{-1}). \quad (2.3)$$

We equip $T^*G$ with the symplectic form

$$\Omega := 2d\text{tr}(J g^{-1}dg), \quad (2.4)$$

which is invariant under $\Psi_{(\eta_+, \eta_K)}$. The action of $N_+ \times K$ is generated by a moment map, $\Phi$. To describe the moment map, we use the vector space decompositions

$$g = g_+ + g_0 + g_- \quad \text{and} \quad g = \mathfrak{R} + \mathfrak{R}^\perp, \quad (2.5)$$

where the first one refers to strictly upper-triangular, diagonal and lower-triangular matrices, while $\mathfrak{R}$ and $\mathfrak{R}^\perp$ consist of antisymmetric and symmetric matrices, respectively. Accordingly, we can decompose any $X \in \mathfrak{g}$ as

$$X = X_+ + X_0 + X_- \quad \text{and} \quad X = X_\mathfrak{R} + X_\mathfrak{R}^\perp. \quad (2.6)$$

By using the trace scalar product, the duals of the Lie algebras of $N_+$ and $K$ can be identified with $\mathfrak{g}_-$ and with $\mathfrak{R}$ itself. Applying these conventions, the moment map

$$\Phi: T^*G \to (\mathfrak{g}_+)^* \times \mathfrak{R}^* \simeq \mathfrak{g}_- \times \mathfrak{R} \quad (2.7)$$

operates as

$$\Phi(g, J) = ((gJg^{-1}g)^{-1}, -J_\mathfrak{R}). \quad (2.8)$$

The relevant moment map constraint then reads

$$\Phi(g, J) = \mu_0 := (I_-, 0), \quad (2.9)$$

where the matrix $I_- := \sum_{i=1}^{n-1} E_{i+1,i}$ contains 1 in its entries just below the diagonal. It is easily seen that the constraint surface $\Phi^{-1}(\mu_0)$ is preserved by the full symmetry group $N_+ \times K$. In other words, the constraints

$$(gJg^{-1})_- - I_- = 0 \quad \text{and} \quad J_\mathfrak{R} = 0 \quad (2.10)$$

are of first class in Dirac’s sense. The corresponding reduced phase space,

$$(T^*G)_{\text{red}} := \Phi^{-1}(\mu_0)/(N_+ \times K), \quad (2.11)$$

is guaranteed to be a smooth manifold since the action of $N_+ \times K$ is free and proper. This follows from the Iwasawa decomposition whereby any $g \in G$ can be uniquely represented in the form

$$g = g_+ g_A g_K, \quad (g_+, g_A, g_K) \in N_+ \times A \times K. \quad (2.12)$$
2.1 Toda gauge

Taking arbitrary \((q, p) \in \mathbb{R}^n \times \mathbb{R}^n\), we now parametrize \(g_A\) in (2.12) as

\[
g_A = e^{Q(q)/2} \quad \text{with} \quad Q(q) := -\sum_{i=1}^{n} q_{n+1-i}E_{i,i},
\]

and introduce the Jacobi matrix \(L(q, p)\) by

\[
L(q, p) := P(p) + e^{-Q(q)/2}I - e^{Q(q)/2} + e^{Q(q)/2}I + e^{-Q(q)/2}
\]

with \(P(p) := -\sum_{i=1}^{n} p_{n+1-i}E_{i,i}\) and \(I_+ := (I_-)^t\). (The usage of \(Q(q)\) and \(P(p)\) is justified by later convenience.) The Iwasawa decomposition implies that every orbit of \(N_+ \times K\) in \(T^*G\) contains a unique representative of the form \((e^{Q(q)/2}, J)\). If \(g = e^{Q(q)/2}\), then the constraint (2.10) is solved by \(J = L(q, p)\). This means that the manifold \(S := \{(e^{Q(q)/2}, L(q, p)) \mid (q, p) \in M\}, \quad M = \mathbb{R}^n \times \mathbb{R}^n\) (2.15)

is a global cross-section (“gauge slice”) of the orbits of the symmetry group in the constraint surface \(\Phi^{-1}(\mu_0)\). If we identify the reduced phase space \((T^*G)_{\text{red}}\) with the gauge slice \(S\), then the reduced symplectic form, \(\Omega_{\text{red}}\), turns into the pull-back \(\iota_S^*(\Omega)\), where \(\iota_S : S \rightarrow T^*G\) is the tautological inclusion. One immediately finds that

\[
\iota_S^*(\Omega) = \sum_{i=1}^{n} dp_i \wedge dq_i \equiv \omega,
\]

which is the Darboux form on \(M\). These observations are summarized by the identifications

\[
((T^*G)_{\text{red}}, \Omega_{\text{red}}) \simeq (S, \iota_S^*(\Omega)) \simeq (M, \omega).
\]

The equality

\[
H(q, p) = \frac{1}{2} \text{tr}(L(q, p)^2)
\]

shows that the Toda Hamiltonian (1.1) is the reduction of the free one, \(\mathcal{H}_2\) in (2.2). The matrix \(L(q, p)\) appearing in the slice \(S\) (2.15) is the standard Toda Lax matrix, and for this reason \(S\) can be called the “Toda gauge”. Of course, all the above are very standard results [7, 9].

2.2 Moser gauge

Let \(\mathbb{R}^n_{++}\) denote the set of vectors \(\hat{p} \in \mathbb{R}^n\) subject to the condition \(\hat{p}_1 > \hat{p}_2 > \cdots > \hat{p}_n\), and \(\mathbb{R}^n_+\) the set of vectors with positive components. For any \((\hat{p}, w) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_+\) define the matrices

\[
\Lambda(\hat{p}) := \text{diag}(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n),
\]

\[
\Gamma(\hat{p}, w) := [w, \Lambda(\hat{p})w, \Lambda(\hat{p})^2w, \ldots, \Lambda(\hat{p})^{n-1}w] = WV(\hat{p}).
\]

Here \(w\) is represented as a column vector, \(W\) and \(V\) are invertible diagonal and Vandermonde matrices

\[
W := \text{diag}(w_1, \ldots, w_n) \quad \text{and} \quad V(\hat{p})_{i,j} := (\hat{p}_i)^{j-1}.
\]
The Toda gauge was defined by bringing $g \in G$ to diagonal form by the action of $N_+ \times K$. Now we introduce another gauge, in which the component $\mathcal{J}$ of the pair $(g, \mathcal{J}) \in \Phi^{-1}(\mu_0)$ is diagonalized. In fact, we claim that the manifold

$$\hat{S} := \{ (\Gamma(\hat{p}, w)^{-1}, \Lambda(\hat{p})) \mid (\hat{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \}$$

(2.22)

is a global cross-section of the orbits of $N_+ \times K$ in the constraint surface $\Phi^{-1}(\mu_0)$.

Note first that $\hat{S}$ indeed lies in the constraint surface $\Phi^{-1}(\mu_0)$. This can be seen, for example, from the identity $\Lambda \Gamma - \Gamma I_- = [0, \ldots, 0, \Lambda^u w]$, which implies that $(\Gamma^{-1} \Lambda \Gamma)_- = I_-$ holds. To proceed, consider the Iwasawa decomposition

$$\Gamma(\hat{p}, w)^{-1} = \eta_+(\hat{p}, w) \rho(\hat{p}, w) \eta_K(\hat{p}, w)$$

(2.23)

with unique matrices $\eta_+(\hat{p}, w) \in N_+$, $\eta_K(\hat{p}, w) \in K$ and diagonal positive matrix

$$\rho(\hat{p}, w) = \text{diag}(\rho_1(\hat{p}, w), \ldots, \rho_n(\hat{p}, w)).$$

(2.24)

The factors of $\Gamma^{-1}$ enjoy the scaling properties

$$\rho(\hat{p}, \lambda w) = \lambda^{-1} \rho(\hat{p}, w), \quad \eta_+(\hat{p}, \lambda w) = \eta_+(\hat{p}, w), \quad \eta_K(\hat{p}, \lambda w) = \eta_K(\hat{p}, w), \quad \forall \lambda \in \mathbb{R}_+.$$  

(2.25)

Acting by $(\eta_+(\hat{p}, w)^{-1}, \eta_K(\hat{p}, w)) \in N_+ \times K$, we obtain

$$\Psi_{(\eta_+(\hat{p}, w)^{-1}, \eta_K(\hat{p}, w))}(\Gamma(\hat{p}, w)^{-1}, \Lambda(\hat{p})) = (\rho(\hat{p}, w), \mathcal{J}(\hat{p}, w)),$$

(2.26)

$$\mathcal{J}(\hat{p}, w) = \eta_K(\hat{p}, w) \Lambda(\hat{p}) \eta_K(\hat{p}, w)^{-1}.$$  

(2.27)

Then $(\rho(\hat{p}, w), \mathcal{J}(\hat{p}, w))$ belongs to the Toda gauge slice $S$ (2.15), because $\rho(\hat{p}, w) \in A$ and the moment map constraint (2.10) holds. Therefore we have the equalities

$$\mathcal{J}(\hat{p}, w) = L(q, p) \quad \text{and} \quad \rho(\hat{p}, w) = e^{Q(q)/2}$$

(2.28)

with uniquely determined $(q, p) \in M$. Now we quote (see e.g. [3, 18]) the following well-known result: every Jacobi matrix $L(q, p)$ can be written in the form (2.27) and this yields a one-to-one parametrization of the set of Jacobi matrices after one fixes the norm of the vector $w$. Combining this result with the scaling properties (2.25), we see that every element of the Toda gauge $S$ is obtained as a gauge transform (2.28) of a unique element of $\hat{S}$. Since we know that $S$ is a global cross-section, the claim that $\hat{S}$ is global cross-section follows.

We call $\hat{S}$ “Moser gauge” since the variables $(\hat{p}, w)$ were first introduced in Moser’s seminal work [3] on the Toda system. The reduced symplectic form is easily calculated in Moser’s variables. One finds that

$$\iota_{\hat{S}}^*(\Omega) = 2 \sum_{i=1}^{n} d \ln w_i \wedge d \hat{p}_i + \sum_{j,k=1}^{n} \frac{d \hat{p}_j \wedge d \hat{p}_k}{\hat{p}_j - \hat{p}_k}.$$  

(2.29)

By viewing $\hat{S}$ as a coordinate system on $S$, which is allowed since both are models of $(T^*G)_{\text{red}}$, the Toda Hamiltonian becomes

$$(\iota_{\hat{S}}^* \mathcal{H}_2)(\hat{p}, w) = \frac{1}{2} \sum_{i=1}^{n} \hat{p}_i^2.$$  

(2.30)

\footnote{The calculation, which the author learned from C. Klimčík, is described in Appendix A for convenience.}
Moser’s variables linearize the Toda dynamics, but are not quite action-angle variables since their Poisson brackets read
\[
\{\hat{p}_i, \hat{p}_j\} = 0, \quad \{\hat{p}_i, w_j\} = \frac{w_j}{2} \delta_{ij}, \quad \{w_j, w_k\} = \frac{1}{2} \frac{w_j w_k}{\hat{p}_j - \hat{p}_k} \quad (j \neq k). \tag{2.31}
\]
One can construct true action-angle variables \(\hat{p}, \hat{q}\) by using a primitive of the de Rham exact second term that appears in \((2.29)\). This is of course not unique, and the choice which will be useful for our purpose corresponds to the following parametrization:
\[
w_i(\hat{p}, \hat{q}) := e^{\frac{i}{2} \hat{q}_i} \prod_{j \neq i}^{n} |\hat{p}_j - \hat{p}_i|^{-\frac{1}{2}}, \quad (\hat{p}, \hat{q}) \in \mathbb{R}^n_+ \times \mathbb{R}^n \equiv \hat{M}. \tag{2.32}
\]
Direct substitution shows that
\[
\iota_S^*(\Omega) = \frac{2}{\lambda} \sum_{i=1}^{n} d \ln w_i(\hat{p}, \hat{q}) \wedge d\hat{p}_i + \sum_{j,k=1}^{n} \frac{d\hat{p}_j \wedge d\hat{p}_k}{P_j - P_k} = \sum_{i=1}^{n} d\hat{q}_i \wedge d\hat{p}_i \equiv \hat{\omega}. \tag{2.33}
\]
The identification \((\hat{S}, \iota_S^*(\Omega)) \equiv (\hat{M}, \hat{\omega})\) given by the above equations anticipates that \((\hat{p}, \hat{q})\) as defined here coincide with the action-angle variables of \([2]\) described in the Introduction.

For completeness, we note that Moser’s variables are usually presented by means of the resolvent function
\[
f(z) := e^{qn}((z - L(q, p))^{-1})_{1,1} = \sum_{i=1}^{n} \frac{w_i^2}{z - \hat{p}_i}, \tag{2.34}
\]
where \(z\) is an auxiliary complex variable. The second equality, which originally served as the definition of \((\hat{p}, w)\), follows from \((2.28)\). To see how this comes about, notice that equations \((2.27)\) and \((2.28)\) imply the relation \(((z - L(q, p))^{-1})_{1,1} = \sum_{i=1}^{n} \frac{\eta_K, i}{z - \hat{p}_i}\). Then substitute the relation \((\eta_K)_{1,i} = w_i \rho_1 = w_i e^{-q_1/2}\), which is obtained from the Iwasawa decomposition of \(\Gamma\) (cf. \((2.23)\)) and the second equality in \((2.28)\). By using the identity \((2.33)\), one can verify that the variables \((\hat{p}, \hat{q})\) coincide also with the action-angle variables exhibited in \([15, 16]\) relying on an interesting recasting of the Toda Poisson brackets in terms of the resolvent function.

### 2.3 Explicit action-angle map from gauge transformation
The results described so far give rise to a symplectomorphism between the two models,
\[
(S, \iota_S^*(\Omega)) \equiv (M, \omega) \quad \text{and} \quad (\hat{S}, \iota_S^*(\Omega)) \equiv (\hat{M}, \hat{\omega}), \tag{2.35}
\]
of the reduced phase space \((T^*G)_{\text{red}}, \Omega_{\text{red}}\). Under this symplectomorphism, any point of \(\hat{S}\) corresponds to the unique gauge equivalent point of \(S\). It turns out that, when operating in the direction \(\hat{S} \to \hat{S}\), the explicit formula of this map is easily computed. We below denote the map in question as \(R: \hat{S} \to S\), and will in the end identify it with the action-angle map \(R: \hat{M} \to M\) presented in the Introduction.

Let \(m_k(X) := \det(X_k)\) denote the \(k\)-th leading principal minor of any \(n \times n\) matrix \(X\), i.e., the determinant of the matrix \(X_k\) obtained by deleting the last \((n - k)\) rows and columns of \(X\). Introduce the following Poisson commuting Hamiltonians on \(T^*G\):
\[
\mathcal{H}_k(g, \mathcal{J}) := m_k((gg^t)^{-1}), \quad k = 1, 2, \ldots, n. \tag{2.36}
\]
These Hamiltonians are invariant with respect to the action of the reduction group \( N_+ \times K \).

If \( \mathcal{R} \) sends a point \((\Gamma^{-1}, \Lambda) \in \hat{S}\) to \((e^{Q(q)/2}, L(q, p)) \in S\), then the equality

\[
m_k(\Gamma^t \Gamma) = \prod_{j=1}^{k} e^{n+1-j}
\]

(2.37)

holds, since on the two sides we have the values of the invariant function \( \hat{H}_k \) at gauge equivalent points. By (2.35), the components of \( q \) and \( p \), and that of \( \hat{p} \) and \( \hat{q} \), give coordinates on \( S \) and on \( \hat{S} \), respectively. We then see from (2.37) that the sought-after formula of \( \mathcal{R} \) is provided by

\[
e^{\eta_k} \circ \mathcal{R} = \frac{m_{n+1-k}(X(\hat{p}, \hat{q}))}{m_{n-k}(X(\hat{p}, \hat{q}))}
\]

with \( X(\hat{p}, \hat{q}) := \Gamma(\hat{p}, w(\hat{p}, \hat{q}))^t \Gamma(\hat{p}, w(\hat{p}, \hat{q})) \),

(2.38)

\[
p_k \circ \mathcal{R} = \{q_k, H\}_M \circ \mathcal{R} = \{q_k \circ \mathcal{R}, H \circ \mathcal{R}\}_M = \{q_k \circ \mathcal{R}, \frac{1}{2} \sum_{i=1}^{n} \hat{p}_i^2\}_M.
\]

(2.39)

We here used that \( \mathcal{R} \) is a symplectomorphism and that the Toda Hamiltonian \( H \) (1.1) satisfies \( H \circ \mathcal{R} = \tau_S^\ast (H_2) \) with (2.30), which are immediate consequences of the reduction.

To make the above formula explicit, we need to calculate the minors \( m_k(\Gamma^t \Gamma) \). For this, it is convenient to write

\[
X(\hat{p}, \hat{q}) = Y(\hat{p}, w(\hat{p}, \hat{q})) \quad \text{with} \quad Y(\hat{p}, w) = \Gamma(\hat{p}, w)^t \Gamma(\hat{p}, w) = V(\hat{p})^t W V(\hat{p}),
\]

(2.40)

using the matrices \( W \) and \( V \) defined in (2.21). Since \( Y_k \) is the product of the \( k \times n \) matrix obtained by deleting the last \((n-k)\) rows of \( V' W \) and its transpose, we can effortlessly calculate \( m_k(Y) = \det(Y_k) \) by applying the standard Cauchy-Binet formula (e.g. [19]). Letting \( I = \{i_1, \ldots, i_k\} \), \( 1 \leq i_1 < \cdots < i_k \leq n \), run over the subsets of \( \{1, \ldots, n\} \), the Cauchy-Binet formula now reads

\[
\det(Y_k) = \sum_{|I|=k} [\det(W_I) \det(V_I)]^2,
\]

(2.41)

where \( W_I \) is the \( k \times k \) diagonal matrix with entries \( w_{i_a} \) and \( V_I \) is the \( k \times k \) Vandermonde matrix with entries \( (V_I)_{a,b} = \hat{p}_a^{i_b-1} \). Thus we find the following result:

\[
m_k(Y(\hat{p}, w)) = \sum_{|I|=k} \left( \prod_{l \in I} w_l^2 \prod_{\substack{i,j \in I \\ i \neq j}} |\hat{p}_i - \hat{p}_j| \right).
\]

(2.42)

Substitution of \( w_I(\hat{p}, \hat{q}) \) from (2.32), then directly leads to the formula

\[
m_k(X(\hat{p}, \hat{q})) = \sum_{|I|=k} e^{\sum_{i \in I} \hat{q}_i} \prod_{i \in I, j \notin I} |\hat{p}_i - \hat{p}_j|^{-1},
\]

(2.43)

which is just the expression \( \sigma_k \) in (1.4).

Finally, the comparison of equations (1.3), (1.4) and (2.38), (2.39) shows that the gauge transformation \( \mathcal{R} : \hat{S} \to S \) is nothing but the action-angle map \( R : \hat{M} \to M \) derived by Ruijsenaars in [2] with the aid of a different method.
3 Group-theoretic interpretation of Toda duality

We have seen that the Hamiltonian reduction approach to the open Toda lattice permits an alternative derivation of the explicit action-angle map (1.3). As a useful spin-off, we can now interpret the Toda duality in the same manner as was done previously for Calogero type systems.

Remember that we have two natural Abelian Poisson algebras, \( \{ H_k \} \) in (2.2) and \( \{ \hat{H}_k \} \) in (2.36), before reduction. Their generators are \( N_+ \times K \) invariant functions possessing complete Hamiltonian flows. These Abelian algebras survive the reduction and any generator of them descends to a Liouville integrable reduced Hamiltonian. The only statement to check is that the generators of the reduced Abelian Poisson algebras consist of \( n \) independent functions, which is readily verified using the two alternative models of \((T^*G)_{\text{red}}\) given by the Toda gauge and the Moser gauge (2.35). As detailed below, these two Abelian algebras on \((T^*G)_{\text{red}}\) engender a dual pair of integrable systems.

In terms of the model \((S, \iota_2^*(\Omega)) \equiv (M, \omega)\) of \(((T^*G)_{\text{red}}, \Omega_{\text{red}})\), the reduction of the Abelian Poisson algebra \(\{ H_k \}\) yields the commuting Toda Hamiltonians and the reduction of \(\{ \hat{H}_k \}\) becomes equivalent to the algebra of the position-variables of the Toda system (1.1). In terms of the alternative model \((\tilde{S}, \iota'_2(\Omega)) \equiv (\tilde{M}, \tilde{\omega})\) of \(((T^*G)_{\text{red}}, \Omega_{\text{red}})\), the reduction of \(\{ \hat{H}_k \}\) gives the commuting Hamiltonians of the dual many-body system (1.5), whose position-variables are equivalent to the reduction of the algebra \(\{ H_k \}\). This means that the components of \(q\) can be viewed both as position-variables of the Toda system and as action-variables of the dual system, whose main Hamiltonian \(\sigma_1(1.5)\) is the reduction of \(\hat{H}_1\). Similarly, the components of \(\hat{p}\) encode action-variables for the Toda system and position-variables for the dual system. Thus one has two integrable many-body systems living on each other’s action-angle phase spaces. This is the essence of the duality relation discovered originally by Ruijsenaars [2, 12].

The commuting Hamiltonians of the systems in duality are generated by the Jacobi matrix \(L(q, p)\) (2.14) and the Hankel matrix \(X(\hat{p}, \hat{q}) = Y(\hat{p}, w(\hat{p}, \hat{q}))\) (2.40). Here, it should be noted that the matrix \(Y\), which has the entries \(Y_{i,j} = \sum_{k=1}^{n} (\hat{p}_k)^i (\hat{q}_k)^j w_k^2\), appears in many papers dealing with Toda systems and related questions. Indeed, equation (2.37) represents the key ingredient of the reconstruction of the Jacobi matrix from its spectral data, which goes back to classical work by Stieltjes (see e.g. [20] and references therein). One could construct the Toda action-angle map by relying directly on the relations (2.31) and (2.37). In our work we obtained these relations by following the standard procedure of Hamiltonian reduction.

We conclude with a list of open problems. First, it could be interesting to apply quantum Hamiltonian reduction [21] to gain a better understanding of the quantum mechanical version of Toda duality. For the state of art of this subject, see the papers [22, 23, 17] and references therein. Second, it would be important to generalize the group-theoretic framework presented in this paper so as to accommodate the open relativistic Toda lattice, whose dual was also derived in [2]. Finally, we remark that action-angle duals of closed Toda lattices are not yet known, and this issue, as well as the cases of other root systems, should be investigated in the future. For constructions of action-angle variables of closed Toda lattices, the reader may consult, e.g., [1, 24].

Acknowledgements. I am indebted to C. Klimčík for important technical clarifications. I thank the referees for useful comments. I also wish to thank I. Tsutsui for discussions and for suggesting the term “Moser gauge”. This work was supported in part by the Hungarian Scientific Research Fund (OTKA) under the grant K 77400 and by the project TAMOP-4.2.2.A-11/1/KONYV-2012-0060 financed by the EU and co-financed by the European Social Fund.
A Calculation of the symplectic form in the Moser gauge

In this appendix we present the derivation of the formula \((2.29)\). We start by noting that direct substitution of \(J = \Lambda(\hat{p})\) and \(g^{-1} = \Gamma(w, \hat{p}) = WV(\hat{p})\) into the symplectic form \(\Omega \ (2.4)\) yields

\[
\iota_s^*(\Omega) = -2dtr(\Lambda dW W^{-1}) - 2dtr(\Lambda dV V^{-1}).
\]

The first term is trivial to evaluate. The second term can be spelled out as

\[
tr(\Lambda dV V^{-1}) = \sum_{i,j=1}^n \hat{p}_i (dV_{ij}) \tilde{V}_{ij} (\det V)^{-1}
\]

where \(\tilde{V}_{ij}\) the co-factor entering the inverse matrix, \((V^{-1})_{ji} = \frac{\tilde{V}_{ij}}{\det V}\). Since (on account of \((2.21)\)) \(V_{ij}\) depends only on \(\hat{p}_i\) and thus \(\tilde{V}_{ij}\) does not depend on \(\hat{p}_i\), the expression \((A.2)\) can be recast as

\[
tr(\Lambda dV V^{-1}) = \sum_{i=1}^n \hat{p}_i (d\hat{p}_i) \tilde{V}_{ij} (\det V)^{-1} = \sum_{i=1}^n \hat{p}_i (d\hat{p}_i) \partial_{\hat{p}_i} \ln |\det V|.
\]

Therefore

\[
dtr(\Lambda dV V^{-1}) = \sum_{i,j=1}^n \frac{\hat{p}_i (d\hat{p}_j) \partial_{\hat{p}_i} \ln |\det V|) d\hat{p}_j \wedge d\hat{p}_i = \frac{1}{2} \sum_{i,j=1}^n \frac{d\hat{p}_j \wedge d\hat{p}_i}{\hat{p}_i - \hat{p}_j}.
\]

The last equality was obtained using the Vandermonde determinant \(\det V = \prod_{a<b}(\hat{p}_b - \hat{p}_a)\).

References

[1] O. Babelon, D. Bernard and M. Talon, Introduction to Classical Integrable Systems, Cambridge University Press, 2003
[2] S.N.M. Ruijsenaars, Commun. Math. Phys. 133 (1990) 217
[3] J. Moser, Lect. Notes in Phys. 38 (1975) 467
[4] B. Kostant, Adv. Math. 34 (1979) 195
[5] M.A. Olshanetsky and A.M. Perelomov, Invent. math. 54 (1979) 261
[6] O.I. Bogoyavlensky, Commun. Math. Phys. 51 (1976) 201
[7] A.M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkhäuser, 1990
[8] P. Deift, L.-C. Li and C. Tomei, pp. 511-536 in: Important Developments in Soliton Theory, A.S. Fokas and V.E. Zakharov (Eds.), Springer, 1993
[9] A.G. Reyman and M.A. Semenov-Tian-Shansky, pp. 116-259 in: Encyclopedia of Mathematical Sciences, Vol. 16, V.I. Arnold and S.P. Novikov (Eds.), Springer, 1994
[10] V. Fock, A. Gorsky, N. Nekrasov and V. Rubtsov, JHEP **0007** (2000) 028
[11] L. Fehér and C. Klimčík, Nucl. Phys. B **860** (2012) 464
[12] S.N.M. Ruijsenaars, Commun. Math. Phys. **115** (1988) 127
[13] S.N.M. Ruijsenaars, Publ. RIMS **31** (1995) 247
[14] P. Deift, L.-C. Li, T. Nanda and C. Tomei, Comm. Pure Appl. Math. **XXXIX** (1986) 183
[15] L. Faybusovich and M. Gekhtman, Phys. Lett. A **272** (2000) 236
[16] Y.A. Grigoryev and A.V. Tsiganov, SIGMA **2** (2006) 097
[17] E. Sklyanin, J. Phys. A: Math. Theor. **46** (2013) 382001
[18] T. Nanda, Siam. J. Numer. Anal. **22** (1985) 310
[19] F.R. Gantmacher, The Theory of Matrices, Vol 1, Chelsea Publ. Co., 1959
[20] R. Beals, D.H. Sattinger and J. Szmigielski, Comm. Pure Appl. Math. **LIV** (2001) 0091
[21] M.A. Semenov-Tian-Shansky, pp. 226-259 in: Encyclopedia of Mathematical Sciences, Vol. 16, V.I. Arnold and S.P. Novikov (Eds.), Springer, 1994
[22] O. Babelon, Lett. Math. Phys. **65** (2003) 229
[23] M. Hallnäs and S. Ruijsenaars, Journ. Math. Phys. **53** (2012) 123512
[24] A. Henrici and T. Kappeler, Int. Math. Res. Not. **Vol. 2008** (2008) article ID rnn031