Abstract

Greenberger, Horne, Shimony and Zeilinger gave a new version of the Bell theorem without using inequalities (probabilities). Mermin summarized it concisely; but Bohm and Hiley criticized Mermin’s proof from contextualists’ point of view. Using the Branching Space-time language, in this paper a proof will be given that is free of these difficulties. At the same time we will also clarify the limits of the validity of the theorem when it is taken as a proof that quantum mechanics is not compatible with a deterministic world nor with a world that permits correlated space-related events without a common cause.

1 Greenberger, Horne, Shimony, Zeilinger and Mermin

Greenberger, Horne, Shimony and Zeilinger (1990) developed a proof of the Bell theorem without using inequalities. Following Mermin, one can
extract from their ideas not only a simple Kochen-Specker-like illustration that there are quantities that have no values independently of the measurements, but also some simple Bell-EPR like results emphasizing locality (spatio-temporal) principles.

In the GHSZ example we consider three spin-half particles originated in a gedanken spin-conserving decay flying apart along three different straight lines in the horizontal plane. Let us denote the spin of particle $i$ along its direction of motion by $\frac{1}{2} \hbar \sigma_i^z$, the spin along the vertical direction by $\frac{1}{2} \hbar \sigma_i^x$ and the spin along the horizontal direction orthogonal to the trajectory by $\frac{1}{2} \hbar \sigma_i^y$. Assume that the quantum state of the three-particle system is

$$\Psi = \frac{1}{\sqrt{2}} \left( |1^z_1 \rangle \otimes |1^z_2 \rangle \otimes |1^z_3 \rangle - |1^z_1 \rangle \otimes |1^z_2 \rangle \otimes |1^z_3 \rangle \right)$$

(1)

where $\sigma_i^z |1^z_i \rangle = |1^z_i \rangle$ and $\sigma_i^z |1^z_i \rangle = |1^z_i \rangle$. In this state, the possible measurement results can be $R(\sigma_i^x) = \pm 1$ and $R(\sigma_i^y) = \pm 1$ for $a = 1, 2, 3$. Consider now the following operators:

$$\Omega_1 = \sigma_1^x \sigma_2^y \sigma_3^z$$
$$\Omega_2 = \sigma_1^y \sigma_2^x \sigma_3^z$$
$$\Omega_3 = \sigma_1^y \sigma_2^y \sigma_3^x$$
$$\Omega_4 = \sigma_1^x \sigma_2^x \sigma_3^z$$

(2)

The only known way to measure these quantities is to perform the measurements $\sigma_1^x, \sigma_2^x, \sigma_3^x$, $\sigma_1^y, \sigma_2^y, \sigma_3^y$, etc., and to take the product of the corresponding results. Of course, the quantum state $\Psi$ does not determine the measurement results $R(\sigma_1^x), R(\sigma_2^y), R(\sigma_3^z), R(\sigma_1^x), R(\sigma_2^z), R(\sigma_3^z), \ldots$. But we can know in advance each $R(\Omega_i)$! The state $\Psi$ is an eigenstate of $\Omega_1, \Omega_2$ and $\Omega_3$ with eigenvalue 1. $\Psi$ is an eigenstate of $\Omega_4$, too, with eigenvalue $-1$. Consequently, in any measurement of these quantities the results are

$$R(\Omega_1) = R(\Omega_2) = R(\Omega_3) = -R(\Omega_4) = 1$$

(3)

which implies a constraint on the measurement results for the spin-components of the three separated particles. This constraint provides, what we shall call an *inconsistency-type correlation* among the outcomes at the three stations; that is, there are combinations of outcomes that are not possible, even though each *individual* outcome is possible.

The measurement results $R(\Omega_i)$ are fixed in advance. That is why, according to the minimal understanding of a “value” assigned to a quantum
observable, one may say that the values of $\Omega_1, \ldots, \Omega_4$ are

$$V(\Omega_1) = V(\Omega_2) = V(\Omega_3) = -V(\Omega_4) = 1 \quad (4)$$

Consider now the product of these operators

$$\Omega = \Omega_1\Omega_2\Omega_3\Omega_4. \quad (5)$$

$\Psi$ is an eigenstate of $\Omega$ too with eigenvalue $-1$. This fact is consistent with

$$V(\Omega_1)V(\Omega_2)V(\Omega)V(\Omega_4) = -1, \quad (6)$$

that is,

$$V(\sigma_x^1\sigma_y^1\sigma_y^1)V(\sigma_y^1\sigma_x^1\sigma_y^1)V(\sigma_y^1\sigma_y^1\sigma_x^1) = -1. \quad (7)$$

And the same holds for the measurement results

$$R(\sigma_x^1\sigma_y^1\sigma_y^1)R(\sigma_y^1\sigma_x^1\sigma_y^1)R(\sigma_y^1\sigma_x^1\sigma_x^1) = -1. \quad (8)$$

Until now we did not do anything beyond standard quantum mechanics. Here we come, however, to a Kochen-Specker-type argument. Let us make an additional assumption and see whether it goes through or not: Assume that we can assign values not only to the products $\sigma_x^1\sigma_y^1\sigma_y^1, \sigma_y^1\sigma_x^1\sigma_y^1, \ldots$, but also to the spin-operators $\sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2, \sigma_x^3, \sigma_y^3$ themselves, such that (7) can be written as

$$V(\sigma_x^1)V(\sigma_y^1)V(\sigma_y^2)V(\sigma_y^3)V(\sigma_y^2)V(\sigma_y^3)$$

$$V(\sigma_x^1)V(\sigma_y^2)V(\sigma_y^3)V(\sigma_y^2)V(\sigma_y^3) = -1. \quad (9)$$

This is, however, impossible, because each $V(\sigma_y^3)$ appears twice, so whatever the values $V(\sigma_y^3)$ are, the left hand side is a positive number, instead of $-1$. In this way, one finds evidence against the idea of pre-existing values for the quantities $\sigma_y^3$. 

From the Kochen-Specker point of view locality considerations are absent or not so important. From the EPR point of view, one takes into account that the various measurements are distant, and finds evidence that in spite of the correlations among them, there are no “elements of reality” corresponding to the values $V(\sigma_y^3)$. One uses the term “element of reality” partly because one thinks of a pre-settled value of a physical quantity as corresponding to an objective feature of our world, a feature that exists
Independently of the measurement that reveals it. Another reason is that
\( V(\sigma_i^a) \) seems to satisfy the often-quoted EPR reality criterion: "If, without
in any way disturbing a system, we can predict with certainty the value of
a physical quantity, then there exists an element of physical reality corre-
sponding to this physical quantity." By virtue of the spatial separation of
the three measurements, one can predict any of \( V(\sigma_i^a) \), without apparent
disturbance of the system, by carrying out two suitable measurements on
the other two particles. (See Mermin (1990a)).

Since the values of the spin-components cannot be prearranged in ad-
\textit{vance before the measurements, the individual experiments cannot merely
reveal values settled in advance. In this sense one can use the GHSZ exam-
ple in the Bell-EPR-common cause context, too: The non-existence of pre-
arranged values/measurement outcomes seems to imply the non-existence
of a common cause that can account for all the correlations provided by
the constraint (3). In the remainder of this paper we will be discussing
the GHSZ example in this context. In order to establish the correctness of
the implication to no-common-cause, we use language that is more rigorous
than customary. In this way our discussion will not only provide a more
solid nonlocality theorem, but will avoid the vulnerable points about which
the contextualists’ complaints have been made.

2 Contextualists’ criticism

From the contextualists’ point of view the above proof is not completely
acceptable. Their critique is focused in the following remarks:

- From a contextualist point of view any Kochen-Specker-type argu-
  ment is vacuous. The disagreement (\ref{disagreement}), for example, proves the non-
  existence of values, the existence of which has never been assumed by
  a contextual theory, such as the Bohm mechanics. The "beables", if
  they exist, determine the result of each individual measurement \textit{op-
  eration}. But these results are not present before the measurement
  operations have been completed.

- The Bell-EPR or the Bell-EPR-common cause context concerns the
  question of what we can tell about the measurement results

\[ R(\sigma_i^1), R(\sigma_i^2), \ldots R(\sigma_i^1). \]
To avoid a contradiction analogous to (9),

\[
\begin{align*}
R(\sigma_1^x)R(\sigma_2^y)R(\sigma_3^y) & \quad R(\sigma_1^y)R(\sigma_2^x)R(\sigma_3^x) \\
1 & \quad 1 \\
R(\sigma_1^y)R(\sigma_2^y)R(\sigma_3^x) & \quad R(\sigma_1^x)R(\sigma_2^y)R(\sigma_3^y) \\
1 & \quad -1
\end{align*}
\]

\[\begin{align*}
R(\sigma_1^y)R(\sigma_2^y)R(\sigma_3^x) & \quad R(\sigma_1^x)R(\sigma_2^y)R(\sigma_3^y) \\
1 & \quad -1
\end{align*}\]

\[\begin{align*}
1 = -1,
\end{align*}\]

(10)

the measurement results must be context dependent in the sense that, for example, \(R(\sigma_1^x)\) cannot be the same in the first place, when \(\sigma_1^x\) is assumed to be measured along with \(\sigma_2^y\) and \(\sigma_y^3\), as it is in the second place when it is measured with \(\sigma_2^y\) and \(\sigma_3^y\) (see Bohm and Hiley 1993). So, it would be better to use a special index, to express the context in which the measurement is completed:

\[
\begin{align*}
R_{xyy}(\sigma_1^x) & \quad R_{xyy}(\sigma_1^y) \quad R_{xyy}(\sigma_3^y) \\
1 & \quad 1 \\
R_{xyy}(\sigma_1^y) & \quad R_{xyy}(\sigma_2^x) \quad R_{xyy}(\sigma_3^y) \\
1 & \quad -1
\end{align*}
\]

\[\begin{align*}
R_{xyy}(\sigma_1^y) & \quad R_{xyy}(\sigma_2^x) \quad R_{xyy}(\sigma_3^y) \\
1 & \quad -1
\end{align*}\]

\[\begin{align*}
1 = -1,
\end{align*}\]

(11)

Such an expression is not contradictory at all.

One might argue that \(R_{xyy}(\sigma_1^x)\), for example, should be the same as \(R_{xxx}(\sigma_1^x)\), because of the spatial separation of the three stations. There is no influence on the measurement at station 1 of the choices among the possible measurements at the other two stations. But this is irrelevant. It is a matter of fact that \(R_{xyy}(\sigma_1^x)\) differs from \(R_{xxx}(\sigma_1^x)\). These two measurements belong to different runs of the experiment.

We agree that both of the above remarks are relevant, but we think that this is not the last word. In the following sections we will present a reformulation of Mermin version of the GHSZ thought-experiment. It will avoid vague notions such as “value of a quantity”, concentrating on the Bell-EPR-common cause context only. Instead,

1. we will reformulate the problem using only some elementary objects, such as events and their causal relations,

2. in this language we will describe exactly what it could mean to say that there is no common-cause explanation of inconsistency-type correlations among separated spin measurements,
3. and we will refer to the observed correlations among the separated spin measurements results only in such a way that everything will have a correct empirical meaning.

On this purified basis, we will give an airtight proof that in a certain limited sense, the GHSZ thought-experiment does in fact imply a violation of the common-cause principle.

3 Basics of the branching space-time theory

Our formulation will use some of the very basic elements of the Branching Space-time Theory (BST). The aim of this theory was to solve the problem: How can we combine relativity and indeterminism in a rigorous theory? The underlying idea is that a true description of our world may require fusing Einstein spacetime with Prior/Thomason branching time. For further motivation and details of BST we refer to Belnap 1992.

From BST we rely on the following postulates and definitions on the primitives \( \langle \text{Our World}, < \rangle \), where \( \text{Our World} \) is the totality of (possible) point events connected to us by a succession of causal paths, and \(<\) is an adaptation of the causal ordering relation among space-time points (the existence of a time-like or light-like path) to the domain of possible point events. We stress that no other primitives whatsoever figure in our discussion.

**Postulate**  Our World is postulated to be a nonempty set; its members are called point events. \(<\) is postulated a dense partial order on Our World such that every nonempty lower bounded chain has an infimum. A history is defined as a subset \( h \) of Our World maximal with respect to the property that if \( h \) contains two point events, then it contains an upper bound for them. It is postulated that each nonempty upper bounded chain has a supremum in each history of which it is a subset. A choice point for two histories is defined as a point event maximal in their intersection. Given any nonempty lower bounded chain \( E \) of point events in \( h_1 - h_2 \), it is postulated that there is at least one choice point \( e_1 \) for \( h_1 \) and \( h_2 \) such that \( e_1 < e_2 \) for all \( e_2 \in E \) (this is the “prior choice principle”). An event is defined as a nonempty set of point events.

A good picture to have in mind is that each history is a space-time of General Relativity that is free from causal degeneracy (for example the
Minkowski space-time). We do not use anything of the more detailed geometry of a space-time beyond its causal structure. So, if we wish, we can regard a history at a more abstract level as a causal space of point events satisfying the Kronheimer-Penrose (1967) axioms. The postulates above, especially the prior choice principle, constrain how these histories branch one from the other. The details are not, however, pertinent. We add the following definitions for current purposes.

**Definition 3.1** An initial event $I$ is a nonempty upper bounded chain. An outcome event $O$ is a nonempty lower bounded chain. A stable event is both initial and outcome, and is contained in every history it overlaps.

Initial events represent situations that can have any one of a variety of outcomes, which are represented by outcome events. Both initial and outcome events have definite loci in *Our World*; but the two sorts of events have different structures.

Initial events and outcome events, because of their respective structures, can fit together into “spreads.” Roughly speaking, a spread – which we are going to define just below – consists in a single initial event, and a variety of individually possible but mutually inconsistent outcomes of that initial. A spread with its outcomes is one way of representing an “experiment” and its possible outcomes. It also sometimes makes sense to think of a spread as a choice-situation for an experimenter, with the outcomes representing the available choices. The meaning of a spread is, however, more general; it can describe any kind of indeterministic situation without implying either the presence or the absence of any human activity at the “preparation” of the initial event or “observation” of the outcomes. Thus the following formulation draws no distinction between the “a measurement process” and any other process going on in *Our World*:

**Definition 3.2** A spread $\sigma$ is an ordered pair the first member of which is an initial event $I$ and the second of which is a nonempty set $\Omega$ of outcome events; we write $\sigma = I \rightarrow \Omega$. Each spread must satisfy the following three conditions:

1. $I$ causally precedes each $O \in \Omega$.
2. Every history containing $I$ overlaps some $O \in \Omega$.
3. No history overlaps two distinct members of $\Omega$. 

7
Thus given any historical course in which the initial event \( I \) comes to a close, exactly one of the outcome events in \( \Omega \) commences. In all cases spreads have two features: (i) they have a definite locus in Our World; and (ii) their internal structure has a definite causal structure, since their mutually inconsistent “outcomes” are located after their initials in the causal ordering.

We next define an “n-spread” with its “outcome vectors” in order to give us a convenient way of simultaneously considering a number of experiments and their possible results.

**Definition 3.3** An n-spread is a finite sequence of spreads:

\[
\Sigma = \{ \sigma_i = I_i \rightarrow \Omega_i \}_{i=1,2,...,n}
\]

The collection of the initial events \( \{I_1, I_2, ..., I_n\} \) is called the set of initials of \( \Sigma \). An outcome vector \( O \) is a vector whose \( i \)th term is one of the outcomes of the \( i \)th spread of \( \Sigma \), \( O \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \).

The next definition uses the expressive power of BST to give an exact account of what it means to say that a set of events is “consistent.”

**Definition 3.4** A set of initial and outcome events is said to be consistent if there is a history that contains each of its initial events and overlaps each of its outcome events. An outcome vector is said to be consistent if its terms form a consistent set of outcome events.

Observe that there is an important difference in how initial events and outcome events enter into consistency-relations: In the history in question, initial events must finish, while outcome events must begin. Given the difference in their structures, that is to be expected.

We also distinguish several senses of “consistency” of an n-spread \( \Sigma \).

**Definition 3.5** \( \Sigma \) is minimally consistent if its set of initials is consistent, and maximally consistent if all of its outcome vectors are consistent. And \( \Sigma \) is 1-consistent if its set of initials is consistent with each single outcome of each of its spreads in the following sense: for each outcome event \( O \in \Omega_i \), where \( 1 \leq i \leq n \), the set of events \( \{I_1, I_2, ..., I_n, O\} \) is consistent.

Lastly, in contrast to the language available in most discussions of Bell-like phenomena, the language of BST permits us to say with absolute rigor what it is for a set of experimental situations to be spatially separated.
Definition 3.6 \( \Sigma \) is space-like iff it is minimally consistent, and no initial of one of its spreads causally precedes any outcome of a distinct spread.

The following sums up what will be used explicitly.

**Fact 1** Let \( \sigma \) be a spread, let \( \Sigma \) be an \( n \)-spread, and let \( E \) be an initial or outcome event of \( \Sigma \).

- \( E \) is consistent with the initial of \( \sigma \) if and only if \( E \) is consistent with some outcome of \( \sigma \).
- \( E \) is consistent with the set of initials of an \( n \)-spread if and only if \( E \) is consistent with some outcome vector of \( \Sigma \).
- If \( E \) is consistent with a set of initials and outcomes, then \( E \) is consistent with each element of the set.
- If \( \Sigma \) is maximally consistent then it is 1-consistent. If it is 1-consistent then it is minimally consistent.

Using the definitions just given, BST suggests the following partial (and pre-probabilistic) analysis of the concept of the “common cause.” The idea is that the question of a common cause does not arise unless you are given a spacelike \( n \)-spread \( \Sigma \) that is 1-consistent, so that the joint coming to a close of its initials is consistent with each outcome of each of its spreads. (One does not even look for a common cause if there is a violation of 1-consistency; if, that is, an otherwise possible outcome of one the experiments is prevented merely by initializing the other experiments.) But you are also given an outcome vector \( C \) that is inconsistent – in spite of the fact that each term of \( C \) is consistent with the set of initials of \( \Sigma \). It is this combination that raises a question. And when the question does arise, one looks for a causal locus in the common past of the outcomes of spreads in \( \Sigma \). What one looks for can be understood in terms of Reichenbach’s (1956) concept of “screening off,” here adapted to the language of consistency: The inconsistency of \( C \) represents a “correlation” among the terms of the \( n \)-spread, in spite of their space-like separation, and screening off makes the “correlation” disappear in the context of each outcome of the common cause \( \sigma \). As a partial analysis, we collect the necessary conditions for a spread to be a common cause for an inconsistent outcome vector. If our analysis weren’t restricted to inconsistency-type correlations, we would perhaps need to enrich or refine these conditions, but for present purposes the following definition works out:
Definition 3.7 Let $\sigma$ be a spread, $\Sigma$ a space-like $n$-spread that is 1-consistent, and $C$ an inconsistent outcome vector of $\Sigma$. $\sigma$ is a common cause for $C$ in $\Sigma$ only if

CC1. (Causal priority) The initial of $\sigma$ is causally prior to every outcome of every spread in $\Sigma$.

CC2. (Consistency) Each outcome of $\sigma$ is consistent with the set of initials of $\Sigma$.

CC3. (Screening off of inconsistency) $\sigma$ “screens off” $C$. That is, each outcome $O$ of $\sigma$ is inconsistent with some one term of $C$.

The expression “common cause for the inconsistent outcome vector $C$” sounds a little awkward in ordinary language; we use it as short for something like “common causal locus for $C$” or “common-cause explanation for $C$”. Whichever, it clearly and rigorously refers to an actual place in Our World such that what happens there “explains away” the space-like correlation represented by $C$.

4 BST formulation of the GHSZ-Mermin theorem

We now turn to redescribing Mermin’s version of the GHSZ theorem in the language of BST, dropping all language of particles, state, and systems. Nor do we even use much of the language of BST itself, confining ourselves to the notions introduced in the previous section. We shall, however, help ourselves to heuristic interpretations, with the caution that the hard information is contained in the explicit numbered stipulations below.

At each of the three “stations” 1, 2, or 3 there can be one of the two “measurements” of $\sigma_x$ or $\sigma_y$ each of which can have one of two “outcomes” $-$ or $+$. We interpret this situation as follows. There are to begin with three widely separated “pre-preparation” initial events $I_1, I_2$ and $I_3$, with the interpretations e.g.

$I_1$: Station 1 has been pre-prepared (so to speak) so that it will be set to measure either on the $x$ axis or on the $y$ axis (but not both).

Next, the pre-preparation initial $I_1$ (for example) has two possible outcomes $x_1$ and $y_1$, with the interpretation e.g.
$x_1$: Station 1 is prepared to measure on the $x$ axis.

Now $x_1$, for example, is not only an outcome event, but also an initial event, with its own two outcomes, $x_1^-$ and $x_1^+$, with the interpretation e.g. $x_1^-$: At station 1 the outcome was spin ‘$-$’ on the $x$ axis.

The picture to have is that the three stations are widely separated, but that at each station the events come in rapid succession. Keep in mind that our theoretical language implies little or nothing in the system/state vocabulary, much less in the language of classical physics, much less in the language of quantum mechanics. What happened happened; that’s it. We record in the following “stipulations” the minimal story that quantum mechanics, no doubt, tells.

**Stipulation 1**

- We use labels $\{1, 2, 3\}$ for the three “stations,” labels $\{x, y\}$ for the two “measurement-types,” and labels $\{-, +\}$ for the two “outcome-types.” (The three quoted phrases are to be taken only as heuristic. Theoretically the labels we use are just labels; the suggested concepts do not figure as part of theory.) We use $i$ and $j$ as ranging over $\{1, 2, 3\}$.

- We shall consider three initial events $I_i$, six stable events $x_i, y_i$, and twelve outcome events $x_i^-, x_i^+, y_i^-, y_i^+$.

- These events fit together into spreads as follows. For $i = 1, 2, 3$, $\sigma_i, \sigma_i^+$, and $\sigma_i^y$ are spreads, where $\sigma_i = I_i \rightarrow \{x_i, y_i\}$ and $\sigma_i^+ = x_i \rightarrow \{x_i^-, x_i^+\}$ and $\sigma_i^y = y_i \rightarrow \{y_i^-, y_i^+\}$. It follows from the stability of $x_i$ and $y_i$ that for $i = 1, 2, 3$, $\sigma_i^+ = I_i \rightarrow \{x_i^-, x_i^+ y_i^-, y_i^+\}$ is a spread.

- These spreads fit together into 3-spreads as follows: $\Sigma_{123} = \{\sigma_1 \sigma_2 \sigma_3\}$, $\Sigma_{123} = \{\sigma_1^+ \sigma_2^+ \sigma_3^+\}$. We will also consider the following 3-spreads:

\[
\begin{align*}
\Sigma_{xxx} &= \{\sigma_1^x \sigma_2^x \sigma_3^x\} \\
\Sigma_{xxy} &= \{\sigma_1^x \sigma_2^y \sigma_3^y\} \\
\Sigma_{xyx} &= \{\sigma_1^x \sigma_2^y \sigma_3^x\} \\
\Sigma_{xyy} &= \{\sigma_1^x \sigma_2^y \sigma_3^y\}
\end{align*}
\]  

(12)
That $\sigma_i$ and the $\sigma_i^x, \sigma_i^y$ are spreads already gives us the following spatio-
temporal information: for $i = 1, 2, 3$, we have $I_i < x_i, y_i < x_i^-, x_i^+, y_i^-, y_i^+$. The following stipulation is additional.

**Stipulation 2** The 3-spread $\Sigma_{123}$ is space-like. This means that the sta-
tions are sufficiently far apart in a space-like sense, and the events at each station are sufficiently close in a time-like sense, so as to guarantee that no measurement outcome at one station has the initial of a distinct station in its causal past. A fortiori, $\Sigma_{123}$ and all the 3-spreads in (12) are spacelike.

**Stipulation 3** The following stipulations are based on quantum mechanics
(see Section 1).

1. An outcome vector of the 3-spread $\Sigma_{123}$ (and hence an outcome vector
of a 3-spread in (12)) is consistent if and only if (i) there is a mixture
of $x$ and $y$ and an even number of minuses, or (ii) there is no mixture
of $x$ and $y$ and an odd number of minuses.

2. Here are four examples, all used below. $(x_1^+ x_2^- x_3^-)$, which is an outcome
vector of $\Sigma_{xxx}$, is consistent (no mixture of $x$ and $y$, odd $-$).
Each of $(x_1^+ x_2^- y_3^+), (x_1^+ y_2^+ y_3^+)$ and $(x_1^+ y_2^- x_3^+)$ (an outcome vector of
respectively $\Sigma_{xxy}, \Sigma_{xyy}$ and $\Sigma_{xyx}$), is inconsistent (mixed $x$ and $y$, odd
number of minuses).

3. Furthermore, $\Sigma_{123}$ is maximally consistent: no setting of measurement
type at one station can be prohibited by any selection of measurement
types at the other stations. The same fact can be restated by saying
that all of the 3-spreads in question are minimally consistent. The 3-spreads in (12) are none of them, however, maximally consistent.
Instead these 3-spreads each has a property intermediate between min-
imal consistency and maximal consistency: Each is 1-consistent, which
is to say of each that the joint realization of its measurement settings
(initials) permits the later occurrence of any single outcome of any of
its spreads (though of course not jointly!).

An inconsistent outcome vector of a 1-consistent space-like $n$-spread is
what sends us in search of a common cause. And maybe there is one. We
do not prove that there isn’t. Instead, we prove something weaker.
Theorem 4.1 There is no single spread \( \sigma \) that is a common cause for every inconsistent outcome vector of each of the four 3-spreads \( \Sigma_{xxx}, \Sigma_{xxy}, \Sigma_{xyy} \) and \( \Sigma_{xyx} \).

This theorem rules out, or seems to rule out, that the “gedanken spin-conserving decay,” if analyzed as an initial having a variety of possible outcomes, could serve as the single common-cause explanation of all the measurement correlations.

Proof. Suppose for reductio that \( \sigma \) is a common cause of each inconsistent outcome vector of each of the 3-spreads \( \Sigma_{xxx}, \Sigma_{xxy}, \Sigma_{xyy} \) and \( \Sigma_{xyx} \). Therefore, even without a full analysis of the concept of a common cause, it is a part of our supposal that CC2 and CC3 hold for \( \sigma \) and each of \( \Sigma_{xxx}, \Sigma_{xxy}, \Sigma_{xyy} \) and \( \Sigma_{xyx} \).

We begin by choosing an outcome \( O \) of \( \sigma \). CC2 for \( \Sigma_{xxx} \) implies that \( O \) is consistent with the set of initials \( \{x_1x_2x_3\} \) of \( \Sigma_{xxx} \). This implies by Fact 1 that \( O \) is consistent with at least one consistent outcome vector of \( \Sigma_{xxx} \). At this point we ought to pick an arbitrary such consistent outcome vector, but instead we ask you to settle for a persuasive example: \( (x_1^+x_2^-x_3^+) \), known to be consistent by Stipulation 3. So since \( O \) is consistent with the outcome vector \( (x_1^+x_2^-x_3^+) \), by Fact 1

\[ O \text{ is consistent individually with each of } x_1^+, x_2^- \text{ and } x_3^+. \]

Now consider that \( (x_1^+x_2^-y_3^+) \) must by Stipulation 3 be inconsistent. So by CC3 for \( \Sigma_{xxy} \), \( O \) must be inconsistent with one of those outcomes individually. But \( O \) is consistent with each of the first two, so it must be that

\[ O \text{ is inconsistent with } y_3^+. \]

CC2 for \( \Sigma_{xxy} \) implies that \( O \) is consistent with the initials \( \{x_1x_2y_3\} \) of \( \Sigma_{xxy} \). By Fact 1 \( O \) is therefore consistent with the initial \( y_3 \) of \( \sigma_y^3 \). The outcomes of \( \sigma_y^3 \) are \( y_3^+ \) and \( y_3^- \). By Fact 1 \( O \) must be consistent with one of these two outcomes. Since \( O \) is inconsistent with the former, it must be that

\[ O \text{ is consistent with } y_3^- \].

Next consider the outcome vector \( (x_1^+y_2^-y_3^-) \) of \( \Sigma_{xyy} \). By Stipulation 3 it is inconsistent, and so by CC3 for \( \Sigma_{xyy} \), together with previously established consistencies, it follows that
Lastly, consider \( (x_1^+ y_2^- x_3^+) \), which by Stipulation \( \text{III} \) is inconsistent. By CC3 for \( \Sigma_{xyx} \), together with previously established consistencies, we have that

\[
O \text{ is inconsistent with } y_2^+.
\]

But the inconsistency of \( O \) with both \( y_2^+ \) and \( y_2^- \) implies, by Stipulation \( \text{III} \) and Fact \( \Box \), the inconsistency of \( O \) with \( y_2 \). This contradicts the implication of CC2 for \( \Sigma_{xyz} \), by way of Fact \( \Box \) that \( O \) is consistent with \( y_2 \), and concludes the \textit{reductio}.

That of course was a proof by example. To generalize, recast the argument by replacing the definite labels with variables.

\[\Box\]

5 Limitations

There seem to be three ways Our World might be:

Level I. Deterministic, that is, there is only one history.

Level II. Indeterministic, but without any “strange” correlations between spatially separated happenings. In other words, each inconsistent outcome vector of each space-like n-spread has a common-cause explanation.

Level III. Indeterministic, and with “strange” correlations between spatially separated happenings.

The strength of our result is that it establishes the relation between a Bell-like phenomenon and a no-common-cause-like phenomenon with absolute rigor, relying on the causal ordering in Our World as sole primitive. As we announced at the outset, however, there are limitations.

- A frequent interpretation of the GHSZ story as well as the other Bell-like theorems is that they show that certain phenomena predicted by quantum mechanics (but describable pre-theoretically, without quantum mechanics) are \textit{incompatible with determinism}. One of us has elsewhere questioned the legitimacy of this interpretation as arising from insufficient care in applying the usual formalism (Szabó 1995a,b). Certainly our proof cannot bear such an interpretation, because it begins
by stipulating the existence of certain spreads, which are explicitly indeterministic phenomena. We may in fact be permitted to doubt that other proofs fare better in this respect. Although these proofs are not sufficiently rigorous to be sure, they seem to share with our proof the hypothesis that indeterministic phenomena occur.

- Our proof of Theorem 4.1 makes no use of the causal-priority condition CC1 of Definition 3.7. More work is needed in understanding the interplay of the idea of causal priority with other concepts. We need especially to advance our understanding of the proposition that the values of certain measurements cannot be arranged in advance.

- Bell-like theorems are often interpreted as showing that certain phenomena predicted by quantum mechanics are inconsistent with the principle of the common cause; or, in other words, that these phenomena involve space-like correlations without a common cause. We are in this paper explicit that Theorem 4.1 does not say, simply, that there are correlations without a common cause. Instead, it says what it can prove: There are certain sets of correlations such that one cannot find a single common-cause locus for all of them. In other words, although Theorem 4.1 does not strictly imply that Our World belongs to Level III, it moves us sharply in that direction. So even though our result succeeds in ruling out that a single “gedanken decay” can account for all the correlations, its limited nature suggests the interest in finding a better theorem.

References

Belnap, N. (1992): "Branching space-time," Synthese, 92, 385-434.

Bohm, D. and Hiley, B. J. (1993): The Undivided Universe, Routledge, London and NY.

Brown, H. R. and Svetlichny, G. (1990): Found. Phys. 20, 1379-1387.

Greenberger, D. M., Horn, M. A., Shimony, A. and Zeilinger, A. (1990): “Bell’s theorem without inequalities." Am. J. Phys. 58, pp. 1131-1143.

Kronheimer, E. H. and Penrose, R. (1967): “On the structure of causal spaces,” Proceedings of the Cambridge Philosophical Society, 63, 418.
Mermin, N. D. (1990a): “What’s wrong with these elements of reality?”,
*Physics Today*, June.

Mermin, N. D. (1990b): “Simple Unified Form for the Major No-Hidden-
Variables Theorem”, *Phys. Rev. Lett.*, **65**, 3373-3376.

Reichenbach, H. (1956): *Direction of Time*, University of California Press,
Berkeley.

Szabó, L. E. (1995a): “Is quantum mechanics compatible with a determinis-
tic universe? Two interpretations of quantum probabilities” *Foundations
of Physics Letters*, **8**, 421-440.

Szabó, L. E. (1995b): “Quantum mechanics in an entirely deterministic
universe” *Int. J. Theor. Phys.*, **34**, 1751-1766.