Quasi-morphisms on Free Groups

Pascal Rolli

Abstract

Let $F$ be the free group over a set of two or more generators. In [2] R. Brooks constructed an infinite family of quasi-morphisms $F \to \mathbb{R}$ such that an infinite subfamily gives rise to independent classes in the second bounded cohomology $H^2_b(F, \mathbb{R})$, which proves that this space is infinite dimensional, cf. [7]. We give a simpler proof of this fact using a different type of quasi-morphisms. After computing the Gromov norm of the corresponding bounded classes, we generalize our example to obtain quasi-morphisms on free products, as well as quasi-morphisms into groups without small subgroups, also known as $\varepsilon$-representations.

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1 Introduction

Let $\Gamma$ be a group. A map $f : \Gamma \to \mathbb{R}$ is called a quasi-morphism if

$$\sup_{x,y \in \Gamma} |f(x) + f(y) - f(xy)| < \infty$$

If $\varphi : \Gamma \to \mathbb{R}$ is a homomorphism and $b : \Gamma \to \mathbb{R}$ is a bounded map then $f = \varphi + b$ is a quasi-morphism. Any quasi-morphism of this form is said to be trivial. Given a group, how many (in a suitable sense) non-trivial quasi-morphisms does it admit? An answer to this question may be found in the second bounded cohomology.

Recall that the ordinary group cohomology of $\Gamma$ with (trivial) real coefficients, denoted $H^\ast(\Gamma, \mathbb{R})$, can be defined as the homology of the complex $(C^\ast(\Gamma), \partial)$ where $C^n(\Gamma)$ is the space of maps $\Gamma^n \to \mathbb{R}$ and $\partial : C^n(\Gamma) \to C^{n+1}(\Gamma)$ is given by

$$\partial f(x_0, \ldots, x_n) = f(x_1, \ldots, x_n) + \sum_{i=0}^{n-1} (-1)^i f(x_0, \ldots, x_ix_{i+1}, \ldots, x_n)$$

$$+ (-1)^n f(x_0, \ldots, x_{n-1}).$$

Restricting the boundary maps $\partial$ to the subspaces $C^n_b(\Gamma) := \{ f : \Gamma \to \mathbb{R} ; f \text{ is bounded} \} \subset C^n(\Gamma)$ gives a subcomplex $(C^\ast_b(\Gamma), \partial)$ whose homology $H^\ast_b(\Gamma, \mathbb{R})$ is called the bounded cohomology of $\Gamma$. More precisely, let

$$Z^n_b(\Gamma) := \ker (\partial : C^n_b(\Gamma) \to C^{n+1}_b(\Gamma))$$

$$B^n_b(\Gamma) := \operatorname{im} (\partial : C^{n-1}_b(\Gamma) \to C^n_b(\Gamma))$$

be the spaces of bounded $n$-cocycles and bounded $n$-coboundaries respectively. The $n$-th bounded cohomology of $\Gamma$ is then the quotient

$$H^n_b(\Gamma, \mathbb{R}) := \frac{Z^n_b(\Gamma)}{B^n_b(\Gamma)}.$$

The inclusion $C^\ast_b(\Gamma) \hookrightarrow C^\ast(\Gamma)$ induces the comparison map

$$c : H^\ast_b(\Gamma, \mathbb{R}) \to H^\ast(\Gamma, \mathbb{R})$$

Its kernel in degree two, denoted $\operatorname{EH}_b^2(\Gamma, \mathbb{R})$, turns out to be relevant to the study of quasi-morphisms:
Let $QM(\Gamma) \subset C^1(\Gamma)$ denote the space of quasi-morphisms on $\Gamma$. For $f \in QM(\Gamma)$, the quantity $\sup_{x,y \in \Gamma} |f(x) + f(y) - f(xy)| =: \text{def} \ f$ is called the defect of $f$. Since $\partial f(x,y) = f(y) - f(xy) + f(x)$ we get

$$\|\partial f\|_{\infty} = \text{def} \ f < \infty,$$

so $\partial f$ is a bounded 2-cocycle and we have a linear map

$$\Phi : QM(\Gamma) \rightarrow H^2_b(\Gamma, \mathbb{R}), \ f \mapsto [\partial f]_b,$$

where $[\cdot]_b$ denotes the bounded cohomology class.

One has $[\partial f]_b = 0$ if and only if there is some $b \in C^1_b(\Gamma)$ such that $\partial(f - b) = 0$, or equivalently, such that $f - b$ is a homomorphism. So the kernel of the above map is the subspace of trivial quasi-morphisms:

$$\ker \Phi = \text{Hom}(\Gamma, \mathbb{R}) \oplus C^1_b(\Gamma) \subset QM(\Gamma).$$

Note that the class $[\partial f]_b$ lies in the kernel of the comparison map, since the corresponding unbounded class $[\partial f] \in H^2(\Gamma, \mathbb{R})$ is zero. On the other hand, if $\alpha \in H^2_b(\Gamma, \mathbb{R})$ satisfies $c(\alpha) = 0$ then $\alpha = [\partial f]_b$ for some $f \in C^1(\Gamma)$ which must be a quasi-morphism. That is, $\text{im} \ \Phi = EH^2_b(\Gamma, \mathbb{R})$, so we have the following

**Proposition 1.1.** The map $\Phi$ induces an isomorphism

$$\frac{QM(\Gamma)}{\text{Hom}(\Gamma, \mathbb{R}) \oplus C^1_b(\Gamma)} \cong EH^2_b(\Gamma, \mathbb{R}).$$

**Corollary 1.2.** If $H^2(\Gamma, \mathbb{R}) = 0$ then $\Phi$ induces an isomorphism

$$\frac{QM(\Gamma)}{\text{Hom}(\Gamma, \mathbb{R}) \oplus C^1_b(\Gamma)} \cong H^2_b(\Gamma, \mathbb{R}).$$

2 The second bounded cohomology of a free group

Let $F = F(S)$ be the free group over a set $S$, $|S| \geq 2$. Since $H^2(F, \mathbb{R}) = 0$ (cf. [3]), the calculation of $H^2_b(F, \mathbb{R})$ amounts to finding non-trivial quasi-morphisms on $F$. In [2], R. Brooks constructed an infinite family of quasi-morphisms on $F$ such that an infinite subfamily is mapped to independent
classes under $\Phi$ (cf. [7]), which proves that space $H^2_b(F, \mathbb{R})$ is infinite dimensional. In the following we describe another type of quasi-morphisms on $F$ which allows us to give a simpler proof of this fact.

We say $x \in F$ is a power if $x = s^k$ for some $s \in S$ and some $k \in \mathbb{Z}$. Each non-trivial $x \in F$ has a unique shortest factorization into powers, which we simply call factorization of $x$. Let $\ell^\infty$ be the space of bounded real sequences. For $\sigma \in \ell^\infty$ we define a map $g_\sigma : F \to \mathbb{R}$ as follows: For a power $x = s^k$ set $g_\sigma(x) = \sigma(k)$, where $\sigma$ is extended to an odd function on $\mathbb{Z}$. In general, for $x \in F$ with factorization $x = x_0 \cdots x_n$, set

$$g_\sigma(x) = \sum_{i=0}^n g_\sigma(x_i).$$

**Proposition 2.1.** The map $g_\sigma$ is a quasi-morphism.

**Proof.** Let $x, y \in F$ have factorizations $x = x_0 \cdots x_n$ and $y = y_0 \cdots y_m$. The factorization of $xy$ has the form

$$x_0 \cdots x_{n-r} \cdot z \cdot y_{r} \cdots y_m \quad \text{or} \quad x_0 \cdots x_{n-r} \cdot y_{r} \cdots y_m$$

for some $r \geq 0$. The first case occurs if $x_{n-i} = y_{i-1}$ for $0 \leq i \leq r - 2$. Since $\sigma$ is odd, this implies $g_\sigma(x_{n-i}) + g_\sigma(y_i) = 0$, and hence,

$$|g_\sigma(x) + g_\sigma(y) - g_\sigma(xy)| = \left| \sum_{i=0}^{n-r+1} g_\sigma(x_i) + \sum_{i=r-1}^m g_\sigma(y_i) - g_\sigma(xy) \right|$$

$$= |g_\sigma(x_{n-r+1}) + g_\sigma(y_{r-1}) - g_\sigma(z)|$$

$$\leq 3 \|\sigma\|_\infty.$$

In the second case $x_{n-i} = y_{i-1}$ holds as well for $i = r - 1$, so $\partial g_\sigma(x, y) = 0$. \square

**Proposition 2.2.** The linear map $\ell^\infty \to H^2_b(F, \mathbb{R})$, $\sigma \mapsto [\partial g_\sigma]_b$ is injective.

**Proof.** Assume that $[\partial g_\sigma]_b = 0$. This means $g_\sigma \in \ker \Phi$, i.e. $g_\sigma - b = \varphi$ for some $b \in C^0_b(F)$ and some $\varphi \in \text{Hom}(F, \mathbb{R})$. For $s \in S$, evaluating this equation at $s^k$ yields $\sigma(k) - b(s^k) = k \varphi(s)$. The left-hand side is bounded as a function of $k$, so $\varphi(s) = 0$. Hence, $\varphi = 0$ and $g_\sigma$ is bounded.

Let $s, t \in S$ be two distinct generators. For $k, l \in \mathbb{Z}$ the equation $g_\sigma((s^k t^l)^k) = 2k \sigma(l)$ holds. Since $g_\sigma$ is bounded, this implies $\sigma(l) = 0$, and so $\sigma = 0$. \square
Corollary 2.3. The space $H^2_b(F, \mathbb{R})$ has infinite dimension.

Remarks. (i) Note that the argument holds with slight modification if we define $g_\sigma(s^k) = \sigma_s(k)$ where $\sigma = (\sigma_s)_{s \in S} \in (\ell^\infty)^S$ is a uniformly bounded family of sequences, cf. Section 4.

(ii) Free groups $F(S)$ as above belong to the class of non-elementary hyperbolic groups. D.B.A. Epstein and K. Fujiwara proved that $H^2_b(\Gamma, \mathbb{R})$ is infinite dimensional for any such group $\Gamma$. (\[4\]).

(iii) For a group $\Gamma$, a complex Hilbert space $\mathcal{H}$ and a unitary representation $\pi : \Gamma \to U(\mathcal{H})$, the bounded cohomology of $\Gamma$ with coefficients in $\mathcal{H}$, denoted $H^*_b(\Gamma, \mathcal{H})$, is defined as the homology of the complex $(C^n_\pi(\Gamma, \mathcal{H}), \partial_\pi)$. Here $C^n_\pi(\Gamma, \mathcal{H})$ is the space of bounded maps $\Gamma^n \to \mathcal{H}$ and $\partial_\pi : C^n_\pi(\Gamma, \mathcal{H}) \to C^{n+1}_\pi(\Gamma, \mathcal{H})$ is given by

$$\partial_\pi f(x_0, \ldots, x_n) = \pi(x_0)f(x_1, \ldots, x_n) + \sum_{i=0}^{n-1} (-1)^i f(x_0, \ldots, x_i x_{i+1}, \ldots, x_n) + (-1)^n f(x_0, \ldots, x_{n-1}).$$

It seems to be unknown whether $H^2_b(F, \mathcal{H}) \neq 0$ for non-trivial representations $\pi : F \to U(\mathcal{H})$.

Consider the following generalization of the above construction: Let $\sigma = (\sigma_s)_{s \in S} \in (\ell^\infty(\mathcal{H}))^S$ be a uniformly bounded family of sequences in $\mathcal{H}$, where each $\sigma_s$ is extended to a map on $\mathbb{Z}$ such that $\sigma_s(k) + \pi(s^k)\sigma_s(-k) = 0$ for $k \in \mathbb{Z}$. Define $g_\sigma : F \to \mathcal{H}$ on powers by $g_\sigma(s^k) = \sigma_s(k)$, and for $x \in F$ with factorization $x = x_0 \cdots x_n$ set

$$g_\sigma(x) := g_\sigma(x_0) + \pi(x_0)g_\sigma(x_1) + \pi(x_0x_1)g_\sigma(x_2) + \cdots + \pi(x_0x_1 \cdots x_{n-1})g_\sigma(x_n).$$

As above $\|\partial_\pi g_\sigma\|_\infty < \infty$, so there is a linear map $(\ell^\infty(\mathcal{H}))^S \to \text{EH}^2_b(F, \mathcal{H})$, $\sigma \mapsto [\partial_\pi g_\sigma]$. The space $H^2_b(F, \mathcal{H})$ could be shown to be non-zero by proving that the image of some $\sigma \in (\ell^\infty(\mathcal{H}))^S$ is a non-trivial bounded class, which means that $g_\sigma$ is not the sum of a crossed homomorphism and a bounded map. This is most likely to work if $\pi$ has a trivial subrepresentation.

We conclude this section with the following observation (which concerns the case of trivial real coefficients):

Proposition 2.4. $\|\partial g_\sigma\|_\infty = \text{def } g_\sigma = \text{def } \sigma$. 

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Proof. If $x, y \in F$ are such that the first case occurs in the proof of Proposition 2.1, then in fact the three terms of $\partial g_\sigma(x, y)$ that remain after cancellation are powers of the same generator. So $|\partial g_\sigma(x, y)| \leq \text{def } \sigma$. On the other hand,

$$\sup_{k, l \in \mathbb{Z}} |\partial g_\sigma(s^k, s^l)| = \sup_{k, l \in \mathbb{Z}} |\partial \sigma(k, l)| = \text{def } \sigma.$$ 

3 Homogenization and the Gromov norm

Let $\Gamma$ be a group. There is a canonical semi-norm on bounded cohomology, the quotient semi-norm, that for a class $\alpha \in H^n_b(\Gamma, \mathbb{R})$ is given by

$$\|\alpha\| = \inf_{f \in \alpha} \|f\|_\infty$$

In dimension two this semi-norm is a proper norm ([6]), called the Gromov norm.

For a cocycle $f \in Z^n_b(\Gamma)$ one obviously has $\|[f]_b\| \leq \|f\|_\infty$. Using an estimate by C. Bavard we show that this is an equality in case $f$ is one of the cocycles of the previous section. For this purpose we consider the notion of a homogenous quasi-morphism: $\varphi \in \text{QM}(\Gamma)$ is called homogenous if $\varphi(x^n) = n\varphi(x)$ for $x \in \Gamma, n \in \mathbb{Z}$. Let $\text{QM}^h(\Gamma) \subset \text{QM}(\Gamma)$ be the subspace of homogenous quasi-morphisms. The process of homogenizing quasi-morphisms relies on the following result:

Lemma 3.1. If $\{a_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers satisfying

$$|a_n + a_m - a_{n+m}| < C$$

for $m, n \in \mathbb{N}$ and a constant $C > 0$ then $|ma_n - a_{mn}| < (m-1)C$ and the sequence $\{\frac{a_n}{m}\}_{n \in \mathbb{N}}$ converges.

Proof. For $m, n \in \mathbb{N}$ we have

$$|ma_n - a_{mn}| = \sum_{k=1}^{m-1} a_{kn} + a_n - a_{(k+1)n} < (m-1)C$$

and analogously $|na_m - a_{mn}| < (n-1)C$. I.e. $|ma_n - na_m| < (m + n - 2)C$, so

$$\left|\frac{a_n}{n} - \frac{a_m}{m}\right| < \left(\frac{1}{m} + \frac{1}{n} - \frac{2}{mn}\right)C,$$
which shows that \( \{ a_n \} \) is a Cauchy sequence.

**Proposition 3.2.** There is a direct sum decomposition
\[
\text{QM}(\Gamma) = \text{QM}^h(\Gamma) \oplus C_b^1(\Gamma)
\]
where the homogenous part \( f^h \) of \( f \in \text{QM}(\Gamma) \) is given by
\[
f^h(x) = \lim_{n \to \infty} \frac{f(x^n)}{n}.
\]

**Proof.** Let \( f \in \text{QM}(\Gamma) \) and \( x \in \Gamma \). The previous Lemma applies to the sequence \( \{ f(x^n) \}_{n \in \mathbb{N}} \), so we can define \( f^h \) as in the proposition. The first part of the lemma implies \( |nf(x) - f(x^n)| < (n - 1)C \), where \( C = \text{def} f \), so
\[
|f(x) - f^h(x)| = \lim_{n \to \infty} \frac{1}{n}|nf(x) - f(x^n)| \leq C,
\]
which means that \( f = f^h + f^b \) for some \( f^b \in C_b^1(\Gamma) \). In particular, \( f^h \) is a quasi-morphism, and it is easily seen to be homogenous. This is the unique such decomposition, since any bounded homogenous quasi-morphism is zero.

**Corollary 3.3.** For \( f \in \text{QM}(\Gamma) \) the cocycles \( \partial f \) and \( \partial f^h \) represent the same class in \( H^2_b(\Gamma, \mathbb{R}) \) and there is an isomorphism
\[
\frac{\text{QM}^h(\Gamma)}{\text{Hom}(\Gamma, \mathbb{R})} \cong \text{EH}^2_b(\Gamma, \mathbb{R}).
\]

**Corollary 3.4.** On abelian groups all quasi-morphisms are trivial.

**Proof.** If \( \Gamma \) is abelian and \( \varphi \in \text{QM}^h(\Gamma) \) then
\[
|\varphi(x) + \varphi(y) - \varphi(xy)| = \frac{1}{n} |\varphi(x^n) + \varphi(y^n) - \varphi((xy)^n)|
= \frac{1}{n} |\varphi(x^n) + \varphi(y^n) - \varphi(x^n y^n)| \to 0 \quad (n \to \infty).
\]
That is, \( \text{QM}^h(\Gamma) = \text{Hom}(\Gamma, \mathbb{R}) \).

**Remark.** This is a special case of a more general result: It has been shown that \( H^n_b(\Gamma, \mathbb{R}) = 0, n \geq 1, \) for any amenable group \( \Gamma \), see [5]. By Proposition 1.1 this implies that any such group admits only trivial quasi-morphisms. Abelian groups, as well as finite groups and solvable groups are amenable.

We now determine the homogenization of the quasi-morphisms \( g_\sigma \in \text{QM}(F) \) defined in the previous section.

**Lemma 3.5.** Let \( x, y \in F \).
(i) If \( x \) is a power then \( g^h_\sigma(x) = 0 \), otherwise \( g^h_\sigma(x) = g_\sigma(x) - \partial g_\sigma(x,x) \).

(ii) If none of \( x, y \) is a power then

\[
\partial g^h_\sigma(x,y) = \partial g_\sigma(x,y) + \partial g_\sigma(xy,xy) - \partial g_\sigma(x,x) - \partial g_\sigma(y,y)
\]

**Proof.** (i) \( g^h_\sigma(s^k) = \lim_{n \to \infty} \frac{1}{n} \sigma(kn) = 0 \) since the sequence \( \sigma \) is bounded. If \( x \in F \) is not a power then \( \partial g_\sigma(x^k,x) = \partial g_\sigma(x,x) \) for \( k \geq 1 \) and the claim follows from

\[
(n-1)\partial g_\sigma(x,x) = \sum_{k=1}^{n-1} \partial g_\sigma(x^k,x) = ng_\sigma(x) - g_\sigma(x^n).
\]

(ii) Apply the first part. \( \square \)

**Proposition 3.6.** \( \|\partial g^h_\sigma\|_\infty \geq 2\|\partial g_\sigma\|_\infty \).

**Proof.** Let \( \varepsilon > 0 \), let \( k, l \in \mathbb{Z} \) such that \( d := \partial \sigma(k,l) > \text{def} \sigma - \varepsilon \), and let \( s, t \in S \) be distinct generators. Consider \( x, y \in F \) given by

\[
x = s^{-k}l^{-k}st^{-l}s^k, \\
y = s^lt^{-l}st^{-k}s^l.
\]

These words satisfy

\[
\partial g_\sigma(x,y) = d, \quad \partial g_\sigma(x,x) = \partial g_\sigma(y,y) = \partial g_\sigma(xy,xy) = -d.
\]

Hence, by Lemma 3.5.(ii) and Proposition 1.4,

\[
\partial g^h_\sigma(x,y) = 2d > 2 \text{def} \sigma - 2\varepsilon = 2\|\partial g_\sigma\|_\infty - 2\varepsilon.
\]

The following estimate by C. Bavard holds for any group \( \Gamma \):

**Proposition 3.7.** (i) Each \( \varphi \in \text{QM}^h(\Gamma) \) satisfies

\[
\|[\partial \varphi]_b\| \geq \frac{1}{2}\|\partial \varphi\|_\infty.
\]

**Proposition 3.8.** \( \|[\partial g_\sigma]_b\| = \text{def} \sigma \).

**Proof.** Combining the estimates of the preceeding propositions we get

\[
\|[\partial g_\sigma]_b\| \leq \|\partial g_\sigma\|_\infty \leq \frac{1}{2}\|\partial g^h_\sigma\|_\infty \leq \|[\partial g^h_\sigma]_b\|.
\]
Since $\partial g_\sigma$ and $\partial g_\sigma^h$ represent the same class these are all equalities, so $\|\partial g_\sigma\|_b = \|\partial g_\sigma\|_\infty = \text{def } \sigma$ by Proposition 2.4. 

**Remark.** In particular, we have shown that the estimate in Proposition 3.7 is an equality in case of the quasi-morphisms $g_\sigma^h$. It has been conjectured that equality holds for any homogenous quasi-morphism on any group.

### 4 Free products

We show how the results of Section 2 can be adapted to obtain non-trivial quasi-morphisms on free products of groups.

Let $\Gamma, \Gamma'$ be groups. A map $f : \Gamma \to \Gamma'$ is called *odd* if $f(x^{-1}) = f(x)^{-1}$ for all $x \in \Gamma$. We denote $\hat{C}_b^1(\Gamma) \subset C_b^1(\Gamma)$ the space of bounded odd maps $\Gamma \to \mathbb{R}$.

Let $\{\Gamma_s\}_{s \in S}, |S| \geq 2$, be a family of non-trivial groups and let $\Gamma = \ast_{s \in S} \Gamma_s$ be the associated free product. We consider the space

$$V(\Gamma) := \prod_{s \in S} \hat{C}_b^1(\Gamma_s)$$

and its subspace of uniformly bounded families:

$$V_0(\Gamma) := \left\{ (\sigma_s)_{s \in S} \in V(\Gamma) : \sup_{s \in S} \|\sigma_s\|_\infty < \infty \right\} \subset V(\Gamma).$$

We identify each $\Gamma_s$ with its image under the natural map $\Gamma_s \hookrightarrow \Gamma$. The *factorization* of an element $x \in \Gamma$ is the unique way of writing $x$ as a product $x = x_0 \cdots x_n$ such that $x_i \in \Gamma_{s_i}$ is nontrivial and $s_i \neq s_{i+1}$ for $0 \leq i < n$. For $\sigma = (\sigma_s)_{s \in S} \in V_0(\Gamma)$ and $x \in \Gamma$ with factorization as above, define $g_\sigma : \Gamma \to \mathbb{R}$ by

$$g_\sigma(x) = \sum_{i=0}^n \sigma_{s_i}(x_i).$$

**Proposition 4.1.** The map $g_\sigma$ is a quasi-morphism.

**Proof.** The argument from Proposition 2.1 holds as well in this context, so for $x, y \in \Gamma$ we have

$$|\partial g_\sigma(x, y)| \leq 3 \sup_{s \in S} \|\sigma_s\| < \infty.$$
Proposition. 4.2. The map \( V_0(\Gamma) \to EH^2_b(\Gamma, \mathbb{R}) \) given by \( \sigma \mapsto [\partial g_\sigma]_b \) is a linear injection.

Proof. Assume that \( [\partial g_\sigma]_b = 0 \), i.e. \( g_\sigma - b = \varphi \) for some \( b \in C^1_b(\Gamma) \) and some \( \varphi \in \text{Hom}(\Gamma, \mathbb{R}) \). For \( x \in \Gamma_s \), evaluating this equation at \( x^k \) yields \( \sigma_s(x^k) - b(x^k) = k \varphi(x) \). The left-hand side is bounded as a function of \( k \), so \( \varphi(x) = 0 \). Since \( \Gamma \) is generated by the subset \( \bigcup_{s \in S} \Gamma_s \), we get \( \varphi = 0 \) and \( g_\sigma \) is bounded.

Let \( s, t \in S \) be distinct indices and let \( x \in \Gamma_s, y \in \Gamma_t \). For \( k \in \mathbb{Z} \) the equation \( g_\sigma((xy^{\pm 1})^k) = k(\sigma_s(x) \pm \sigma_t(y)) \) holds. Since \( g_\sigma \) is bounded, this implies \( \sigma_s(x) \pm \sigma_t(y) = 0 \), so \( \sigma_s(x) = \sigma_t(y) = 0 \) and therefore, \( \sigma = 0 \).

Remarks. (i) The free group \( F(S) \) over a set \( S \) is naturally isomorphic to the free product of a set of copies of \( \mathbb{Z} \) indexed by \( S \). Letting \( \sigma_s = \sigma_s \in \ell^{\infty} \) for all \( s \in S \) gives exactly the quasi-morphisms of Section 2.

(ii) The Gromov norm of the classes \( [\partial g_\sigma]_b \) can be calculated by modifying the arguments of the previous section.

As an example we state the following

Corollary 4.3. For \( \Gamma := \text{PSL}_2(\mathbb{Z}) \) there exists a non-trivial quasi-morphism \( \Gamma \to \mathbb{R} \) and hence, \( \dim H^2_b(\Gamma, \mathbb{R}) \geq 1 \).

Proof. Since \( \text{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3 \) there is a linear map

\[
V_0(\Gamma) \cong \widehat{C}^1_b(\mathbb{Z}_2) \times \widehat{C}^1_b(\mathbb{Z}_3) \hookrightarrow EH^2_b(\Gamma, \mathbb{R}).
\]

Any odd map on \( \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \) is characterized by its values on the elements \( 1, 2, \ldots, [\frac{n-1}{2}] \), so we have

\[
\dim \widehat{C}^1_b(\mathbb{Z}_n) = \left\lfloor \frac{n-1}{2} \right\rfloor,
\]

and hence,

\[
\dim V_0(\Gamma) = 0 + 1 = 1.
\]
5 Groups without small subgroups

We show how the target of the quasi-morphisms of Section 2 can be replaced by a group without small subgroups.

Let $G$ be a group with neutral element $e$, and let $d$ be a metric on $G$. For a set $X$ the distance of two maps $f, g : X \to G$ is given by

$$d(f, g) := \sup_{x \in X} d(f(x), g(x)).$$

If $d(f, g) < \infty$ we say $f$ and $g$ are at bounded distance. If $f$ is at bounded distance from the trivial map $x \mapsto e$ then we say $f$ is bounded and we write $\|f\|_\infty$ for this distance.

Let $\Gamma$ be another group. A map $f : \Gamma \to G$ is called a quasi-morphism or an $\varepsilon$-representation if the maps $\Gamma^2 \to G$, $(x, x') \mapsto f(xx')$ and $(x, x') \mapsto f(x)f(x')$

are at bounded distance.

A subgroup $H \leq G$ is called $\varepsilon$-small if $H \subset B_\varepsilon(e)$. $G$ is said to be a group without small subgroups if there exists $\varepsilon > 0$ such that every $\varepsilon$-small subgroup is trivial. For example, $\mathbb{R}$ equipped with the usual metric belongs to this class of groups, and the two definitions of a quasi-morphism into $\mathbb{R}$ agree.

A metric on a group is called bi-invariant if it turns left and right translation on into isometries. Any compact Lie group carries a bi-invariant metric such that there are no small subgroups.

From now on, let $G$ be equipped with a bi-invariant metric $d$. We construct quasi-morphisms $F \to G$, where $F = F(S)$ as in Section 2.

Let $\sigma \in \ell^\infty(G)$. We define $g_\sigma : F \to G$ by $g_\sigma(x) = \sigma(s)$ if $x = s^k$ for some $s \in S$, where $\sigma$ is again extended to an odd map $\mathbb{Z} \to G$. In general, for $x \in F$ with factorization $x = x_0 \cdots x_n$, let

$$g_\sigma(x) = \prod_{i=0}^n g_\sigma(x_i).$$

**Proposition 5.1.** The map $g_\sigma$ is a quasi-morphism.

**Proof.** Let $x, y \in F$ with factorizations $x = x_0 \cdots x_n$ and $y = y_0 \cdots y_m$. 

As in Section 2 the factorization of $xy$ takes one of two possible forms. Consider the case $xy = x_0 \cdots x_{n-r} \cdot z \cdot y_r \cdots y_m$, i.e. $x_{n-i} = y_i^{-1}$ for $0 \leq i \leq r-2$. Due to the bi-invariance of $d$, powers cancel as in the proof of Proposition 2.1, so

$$d(g_\sigma(xy), g_\sigma(x)g_\sigma(y))$$
$$= d\left(\prod_{i=1}^{n-r} g_\sigma(x_i) \cdot g_\sigma(z) \cdot \prod_{i=r}^{m} g_\sigma(y_i), \prod_{i=1}^{n} g_\sigma(x_i) \prod_{i=1}^{m} g_\sigma(y_i)\right)$$
$$= d(g_\sigma(z), g_\sigma(x_{n-r+1})g_\sigma(y_{r-1}))$$
$$\leq d(g_\sigma(z), e) + d(g_\sigma(x_{n-r+1}), e) + d(g_\sigma(y_{r-1}^{-1}), e) \leq 3\|\sigma\|_\infty.$$

In the second case we have complete cancellation and the distance is zero. \qed

Let $G$ be a group with a bi-invariant metric $d$ and without $\varepsilon$-small subgroups, and let $\sigma$ and $g_\sigma$ be as above.

**Proposition 5.2.** Let $\varepsilon_0 := \|\sigma\|_\infty$. If $0 < \varepsilon_0 < \frac{\varepsilon}{2}$ then the quasi-morphism $g_\sigma$ is non-trivial, in the sense that there is no $\varphi \in \text{Hom}(F, G)$ such that $d(g_\sigma, \varphi) \leq \varepsilon_0$.

**Proof.** Assume there is such a $\varphi$. For $s \in S$ and $k \in \mathbb{Z}$ we have

$$d(\varphi(s^k), e) \leq d(\varphi(s^k), g_\sigma(s^k)) + d(g_\sigma(s^k), e) \leq \varepsilon_0 + \varepsilon_0 < \varepsilon.$$

That is, the cyclic group $\langle \varphi(s) \rangle \leq G$ is small and therefore $\varphi(s) = e$. It follows that $\varphi$ is trivial, so $g_\sigma$ is bounded by $\frac{\varepsilon}{2}$.

Now pick $t \in S$ distinct from $s$ and $k, l \in \mathbb{Z}$. Since $(g_\sigma(s^l t^\pm 1))^k = g_\sigma((s^l t^\pm 1)^k)$, the cyclic group $\langle g_\sigma(s^l t^\pm 1) \rangle \leq G$ is small. So $g_\sigma(s^l t^\pm 1) = \sigma(s^l)\sigma_\sigma(1)^\pm 1 = e$. This implies $\sigma(s)^{-1} = \sigma(s)$. Therefore the group $\langle \sigma(s) \rangle = \{\sigma(s), \sigma(s)^{-1}\} \leq G$ is small, so $\sigma(s) = e$. Since this holds for any $s \in S$ and any $l \in \mathbb{Z}$, we have $\|\sigma\|_\infty = 0$, a contradiction. \qed

**Remark.** In the case $G = \mathbb{R}$ we get once more the non-triviality of the quasi-morphisms of Section 2, since $\varepsilon$-small subgroups of $\mathbb{R}$ are trivial for any $\varepsilon > 0$. 

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Department of Mathematics
ETH Zürich, CH-8092 Zürich, Switzerland

prolli@student.ethz.ch