ROOT CLOSED FUNCTION ALGEBRAS
ON COMPACTA OF LARGE DIMENSION

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Abstract. Let $X$ be a Hausdorff compact space and let $C(X)$ be the algebra of all continuous complex-valued functions on $X$, endowed with the supremum norm. We say that $C(X)$ is (approximately) $n$-th root closed if any function from $C(X)$ is (approximately) equal to the $n$-th power of another function.

We characterize the approximate $n$-th root closedness of $C(X)$ in terms of $n$-divisibility of the first Čech cohomology groups of closed subsets of $X$. Next, for each positive integer $m$ we construct an $m$-dimensional metrizable compactum $X$ such that $C(X)$ is approximately $n$-th root closed for any $n$. Also, for each positive integer $m$ we construct an $m$-dimensional compact Hausdorff space $X$ such that $C(X)$ is $n$-th root closed for any $n$. 

1. Introduction

Relations between algebraic closedness of the algebra of continuous bounded complex-valued functions $C(X)$ on a space $X$ and topological properties of $X$ have been studied since the 1960s [5]. Recall that the algebra $C(X)$ is called algebraically closed if each monic polynomial with coefficients in $C(X)$ has a root in $C(X)$. For a locally connected compact Hausdorff space, the algebra $C(X)$ is algebraically closed if and only if $\dim X \leq 1$ and $H^1(X; \mathbb{Z}) = 0$ [8], [13], where $H^1(X; \mathbb{Z})$ denotes the first Čech cohomology group of $X$ with the integer coefficient (see section 2). It is proved in [13] that for a first-countable compact Hausdorff space $X$, algebraic closedness of $C(X)$ is equivalent to a weaker property of square root closedness. The latter means that every function from $C(X)$ is a square of another function. It should be noted that this property appears in the study of subalgebras of $C(X)$ [4].

An even weaker property of approximate square root closedness was introduced by Miura [12] and was proved to be equivalent to the square root closedness when the underlying compact Hausdorff space $X$ is locally connected.

There is a nice characterization of algebraic closedness of $C(X)$ when $X$ is a metrizable continuum. Namely, in this case $C(X)$ is algebraically closed if and only if $X$ is a dendrite (i.e. a Peano continuum containing no simple closed curves) [10], [13].
The approximate $n$-th root closedness of $C(X)$ was studied by Kawamura and Miura and was proved to be equivalent to $n$-divisibility of $H^1(X;\mathbb{Z})$ under the additional assumption $\dim X \leq 1$. The universal space for metrizable compacta with the approximately $n$-th root closed $C(X)$ is constructed in [3].

In this paper we characterize the approximate $n$-th root closedness of $C(X)$ for any Hausdorff paracompact space $X$. Namely, $C(X)$ is approximately $n$-th root closed if and only if the group $H^1(A;\mathbb{Z})$ is $n$-divisible for every closed subset $A$ of $X$. If $\dim X \leq 1$, then the $n$-divisibility of $H^1(X;\mathbb{Z})$ implies the $n$-divisibility of $H^1(A;\mathbb{Z})$, so this generalizes Theorem 1.3 of [10]. Further, for each positive integer $m$ we construct an $m$-dimensional metrizable compactum $X$ such that $C(X)$ is approximately $n$-th root closed for any $n$. Note that such examples were known in dimension 1 only. Also, for each positive integer $m$ we construct an $m$-dimensional compact Hausdorff space $X$ such that $C(X)$ is $n$-th root closed for any $n$. This example solves the problem posed in [10]: for a compact Hausdorff space $X$, does square root closedness of $C(X)$ imply $\dim X \leq 1$?

2. Notations, definitions, and ideas of constructions

All maps considered in this paper are continuous. For spaces $X$ and $Y$, we denote the set of all maps from $X$ to $Y$ by $C(X,Y)$. As usual, by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ we denote the integers, the rational numbers, and the complex numbers, respectively. We let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the multiplicative subgroup of $\mathbb{C}$. An inverse spectrum over a directed partially ordered set $(\mathcal{A},\prec)$ consisting of spaces $X_\alpha$, $\alpha \in \mathcal{A}$, and projections $p_\alpha^\beta : X_\beta \to X_\alpha$, $\alpha, \beta \in \mathcal{A}$, $\beta \succ \alpha$, is denoted by $\{X_\alpha, p_\alpha^\beta, \mathcal{A}\}$. Throughout this section $n > 1$ denotes an integer.

By $H^k(X;G)$ we denote the $k$-th Čech cohomology group of the space $X$ with an abelian coefficient group $G$. Note that in the case when $X$ is a Hausdorff paracompact space the Čech cohomologies are naturally isomorphic to the Alexander-Spanier cohomologies [14, p. 334]. Note also that due to the Huber’s theorem [9] for a Hausdorff paracompact space $X$ there exists a natural isomorphism between the group of all homotopy classes of maps from $X$ to $K(G,k)$ and the group $H^k(X;G)$ if $G$ is countable. Here $K(G,k)$ denotes the Eilenberg-MacLane complex.

For a space $X$, by $C(X)$ we denote the algebra of all bounded complex-valued functions on $X$, endowed with the supremum norm. We say that $C(X)$ is approximately $n$-th root closed if for every $f \in C(X)$ and every $\varepsilon > 0$ there exists $g \in C(X)$ such that $||f - g^n|| < \varepsilon$. The algebra $C(X)$ is said to be $n$-th root closed if any $f \in C(X)$ has an $n$-th root, which means that there exists $g \in C(X)$ such that $f = g^n$. Note that if $C(X)$ is (approximately) $n$-th root closed, then $C(A)$ is also (approximately) $n$-th root closed for any closed subset $A$ of $X$.

We consider $C(X,\mathbb{C}^*)$ as a multiplicative subgroup of $C(X)$ with a metric inherited from $C(X)$. We say that $C(X,\mathbb{C}^*)$ is (approximately) $n$-th root closed if any $f \in C(X,\mathbb{C}^*)$ has an (approximate) $n$-th root in $C(X,\mathbb{C}^*)$.

The basic idea explored in this paper — the construction of a projective $n$-th root resolution — is outlined as follows. The simplest case has been known in the theory of uniform algebra, and it is called the Cole construction (cf. [15], Chapter 3, §19, pp. 194-197).

Given a space $X$ and a function $f : X \to \mathbb{C}$ it is not always possible to solve, even approximately, the problem of finding an $n$-th root of $f$ (consider for instance any homotopically non-trivial map from a circle $S^1$ to $\mathbb{C}^*$). Nevertheless, it is
always possible to solve the $n$-th root problem \textit{projectively} in the following sense.

There exists a space denoted $R_n(X,f)$ and a map $\pi^f: R_n(X,f) \to X$ such that the composition $f \circ \pi^f$ has an $n$-th root. The space $R_n(X,f)$ is simply the graph of the (multivalued) $n$-th root of $f$,

$$R_n(X,f) = \{ (x,z) \mid f(x) = z^n \} \subset X \times \mathbb{C},$$

and the map $\pi^f$ is the natural projection on $X$. Obviously, the projection of $R_n(X,f)$ to $\mathbb{C}$ is an $n$-th root of the composition $f \circ \pi^f$. We say that the space $R_n(X,f)$ together with the map $\pi^f$ resolve the $n$-th root problem for $f$ projectively.

Given any family of maps $\mathcal{M} \subset C(X)$ we can projectively resolve the $n$-th root problem for all maps from $\mathcal{M}$ simultaneously by using the space

$$R_n(X,\mathcal{M}) = \{ (x,(z_f)_{f \in \mathcal{M}}) \mid f(x) = z_f^n \ \forall f \in \mathcal{M} \} \subset X \times C^\mathcal{M}$$

and defining $\pi^\mathcal{M}: R_n(X,\mathcal{M}) \to X$ to be the natural projection. Let $A$ and $B$ be two subsets of $C(X)$ such that $A \subset B$. There is a natural projection $\pi^A_B: R_n(X,B) \to R_n(X,A)$ defined by $\pi^A_B[(x,(z_f)_{f \in B})] = (x,(z_f)_{f \in A})$. We let $R_n(X,\emptyset) = X$ and $\pi^\emptyset_B = \pi^B$.

We outline the ideas of our constructions in Sections \[\] and \[\]. Suppose that we want to construct a space $X$ with $n$-th root closed $C(X)$. Take any space $X_1$ and resolve the $n$-th root problems for $X_1$ projectively using the space $X_2 = R_n(X_1, C(X_1))$. Then resolve all $n$-th root problems for $X_2$ projectively using $X_3$, and so on. This way we obtain an inverse spectrum of spaces $X_\lambda$ and define $X$ to be the inverse limit of this spectrum. To guarantee that the $n$-th root problems for $X$ can be solved, we need this spectrum to be \textit{factorizing} in the following sense: for any map $f: X \to \mathbb{C}$ there exist a space $X_\lambda$ in the spectrum and a map $f_\lambda: X_\lambda \to \mathbb{C}$ such that $f = f_\lambda \circ p_\lambda$, where $p_\lambda: X_\lambda \to X$ is the limit projection. Then the projective resolution of the $n$-th root problem for $f_\lambda$ gives us a solution of the $n$-th root problem for $f$. In order to obtain a factorizing spectrum we make its length uncountable. Namely, we construct the spectrum over $\omega_1$, the first uncountable ordinal.

The space described above is not metrizable for two reasons. First, the length of the spectrum used is not countable. Second, for a metric compactum $X_\lambda$ and a subset $\mathcal{M} \subset C(X_\lambda)$, the space $R_n(X_\lambda, \mathcal{M})$ is metrizable if and only if the set $\mathcal{M}$ is countable. If we want $C(X)$ to have just the \textit{approximate} $n$-th root property, it is enough to construct a countable spectrum and for each space $X_\lambda$ to resolve the projective $n$-th root problem for a countable dense set of maps from $C(X_\lambda)$. Then the limit space $X$ is a metrizable compactum, if we start with a metrizable compactum $X_1$.

To guarantee that for the limit space $X = \lim \{ X_\lambda, p^\mu_\lambda, \Lambda \}$ we can (approximately) solve the $n$-th root problem for any function from $C(X)$ and for any $n > 1$, we represent the index set $\Lambda$ as the union of disjoint cofinal subsets $\{ \Lambda_n \}_{n=2}^\infty$. Then we construct the spectrum by transfinite induction so that the space $X_{\lambda+1}$ and the projection $p^\lambda_{\lambda+1}$ resolve projectively (almost) all $n$-th root problems on $X_\lambda$, where $\lambda \in \Lambda_n$. Since every set $\{ \Lambda_n \}$ is cofinal, for any $n$ and any $\alpha$, (almost) every $n$-th root problem on $X_\alpha$ will be projectively resolved at some level $\lambda > \alpha$ where $\lambda \in \Lambda_n$.

To guarantee that the limit space $X$ has dimension $\dim X \geq m$, we start the construction with the space $X_1$ homeomorphic to the $m$-dimensional sphere $S^m$. Then we show that the homomorphism $(p_1)^*: H^m(S^m; \mathbb{Q}) \to H^m(X; \mathbb{Q})$ induced
by the limit projection is a monomorphism. Therefore the mapping \( p_1 : X \rightarrow S^m \) is essential and hence \( \dim X \geq m \). To prove that the homomorphism above is a monomorphism, we use a construction called transfer, that briefly can be described as follows. Suppose \( G \) is a finite group acting on a compact Hausdorff space \( Y \). Let \( Y/G \) be the quotient space and let \( \pi : Y \rightarrow Y/G \) be the natural projection. Then there exists a homomorphism \( \mu^* : H^*(Y; \mathbb{Q}) \rightarrow H^*(Y/G; \mathbb{Q}) \) such that the composition \( \mu^* \pi^* \) is the multiplication by the order of \( G \) in the group \( H^*(Y/G; \mathbb{Q}) \). Therefore \( \pi^* \) is a monomorphism. See Chapter II, §19 of [1] for more information on transfers.

3. Projective resolutions

In this section we establish some properties of projective resolutions needed for our constructions in Sections 5 and 6. We begin with a summary of basic properties of the space \( R_n(X, M) \).

**Proposition 3.1.** Let \( X \) be a space, let \( M \) be a subset of \( C(X) \), and let \( n > 1 \) be an integer.

(a) \( R_n(X, M) \) is the pull-back in the following diagram:

\[
\begin{array}{ccc}
R_n(X, M) & \longrightarrow & \mathbb{C}^M \\
\downarrow \pi^M & & \downarrow N \\
X & \longrightarrow & \mathbb{C}^M \\
\end{array}
\]

where \( F : X \rightarrow \mathbb{C} \) is defined by \( F(x) = (f(x))_{f \in M} \) and \( N : \mathbb{C}^M \rightarrow \mathbb{C}^M \) is defined by \( N((z_f)_{f \in M}) = (z_f^n)_{f \in M} \).

(b) For any \( f \in M \) there exists \( g \in C(R_n(X, M)) \) such that \( f \circ \pi^M = g^n \).

(c) If \( X \) is a compact Hausdorff space, then \( R_n(X, M) \) is also a compact Hausdorff space and \( \dim R_n(X, M) \leq \dim X \).

**Proof.** The statement (a) is obvious. To prove (b) we just let \( g([x, (z_h)_{h \in A}]) = z_f \). To verify (c) we note first of all that \( R_n(X, M) \) is a subset of the product \( X \times \prod_{f \in M} \{z \mid z^n \in f(X)\} \) of compact Hausdorff spaces. Moreover, \( R_n(X, M) \) is closed in this product due to (a), and the compactness follows. For the dimension part, observe that \( \pi^M \) has zero-dimensional fibers and apply [7, Theorem 3.3.10].

In what follows we shall omit the index \( n \) when this does not cause ambiguities.

**Proposition 3.2.** For any space \( X \) and any two subsets \( A \) and \( B \) of \( C(X) \) there exists a natural homeomorphism \( h : R(R(X, A), B \circ \pi^A) \rightarrow R(X, A \cup B) \), where \( B \circ \pi^A = \{f \circ \pi^A \mid f \in B\} \). This homeomorphism makes the following diagram commutative:

\[
\begin{array}{ccc}
R(R(X, A), B \circ \pi^A) & \xrightarrow{h} & R(X, A \cup B) \\
\downarrow \pi_{B \circ \pi^A} & & \downarrow \pi^{A \cup B} \\
R(X, A) & \xrightarrow{\pi^A} & X \\
\end{array}
\]
Proof. Note that both $R(X, A \cup B)$ and $R(R(X, A), B \circ \pi^A)$ can be viewed as subsets of $X \times \mathbb{C}^A \times \mathbb{C}^B$. Namely,

$$R(X, A \cup B) = \{(x, (z_f)_{f \in A}, (z_g)_{g \in B}) \mid z_f^n = f(x), z_g^n = g(x)\},$$

$$R(R(X, A), B \circ \pi^A) = \{(x, (z_f)_{f \in A}, (z_{g \circ \pi^A})_{g \in B}) \mid z_f^n = f(x), z_{g \circ \pi^A}^n = (g \circ \pi^A)([(x, (z_f)_{f \in A})])\}.$$ 

It remains to note that these subsets coincide since

$$(g \circ \pi^A)([(x, (z_f)_{f \in A})]) = g(x)$$

by the definition of $\pi^A$. \hfill \Box

**Proposition 3.3.** Let $X$ be a compact Hausdorff space and let $S$ be a subset of $C(X)$. Let $A$ be a family of subsets of $S$, partially ordered by inclusion. Assume that $A$ is a directed set with respect to this order and that $\bigcup A = S$. Then $R(X, S)$ is naturally homeomorphic to the limit of the inverse spectrum $\{R(X, A), \pi_S^A, \{\}\}$.

**Proof.** Put $\mathcal{R} = \operatorname{lim}(R(X, A), \pi_S^A, \{\})$. Define $h_A: R(X, S) \rightarrow R(X, A)$ for each $A \in \mathcal{A}$ letting $h_A = \pi_S^A$. The family of maps $\{h_A \mid A \in \mathcal{A}\}$ induces the limit map $h: R(X, S) \rightarrow \mathcal{R}$. We claim that $h$ is a homeomorphism. Since both $R(X, S)$ and $\mathcal{R}$ are Hausdorff compacta, it is enough to check that $h$ is surjective. Since all maps $\pi_S^A$ are surjective, $h$ is surjective by Theorem 3.2.14 in [6].

To verify the injectivity, it is enough, for any two distinct points from $R(X, S)$, to find $A \in \mathcal{A}$ such that the images of these two points under $h_A$ are distinct. Let $y = (x, (z_f)_{f \in S})$ and $y' = (x', (z'_f)_{f \in S})$ be two distinct points from $R(X, S)$. If $x \neq x'$, then any $A \in \mathcal{A}$ will do. Otherwise there exists $f \in S$ such that $z_f \neq z'_f$. Since $\bigcup \mathcal{A} = S$ there exists $A \in \mathcal{A}$ such that $f \in A$, and one can easily see that $h_A(y) \neq h_A(y')$. \hfill \Box

Later we use the following special case of Corollary 14.6 from [1].

**Proposition 3.4.** Let $S = \{X_\alpha, p^3_\alpha, A\}$ be an inverse spectrum consisting of Hausdorff compact spaces. Then there exists a natural isomorphism

$$\operatorname{lim} H^*(X_\alpha; \mathbb{Q}) \cong H^*(\operatorname{lim} S; \mathbb{Q}).$$

**Proposition 3.5.** Let $X$ be a compact Hausdorff space and let $S$ be any subset of $C(X)$. Then for any integer $n > 1$

$$(\pi^S)^*: H^*(X; \mathbb{Q}) \rightarrow H^*(R_n(X, S); \mathbb{Q})$$

is a monomorphism.

**Proof.** (i) First, we prove the proposition for any space $X$ and a set $S$ consisting of a single function $f$. There is an action of $\mathbb{Z}_n$ on $R_n(X, f)$ whose orbit space is $X$, with $\pi^f$ being the quotient map. Namely, represent $\mathbb{Z}_n$ as the group of $n$-th roots of $1$ and put $g \cdot (x, z_f) = (x, g \cdot z_f)$. The proposition now follows from Theorem 19.1 in [1]. Repeating the argument and applying Proposition 3.2 finitely many times, we see that the proposition holds for every finite set $S$.

(ii) Finally, let $S$ be any subset of $C(X)$. Let $S_{\text{fin}}$ denote the set of all finite subsets of $S$, partially ordered by inclusion. Proposition 3.3 implies that $R_n(X, S)$ is the limit of the inverse spectrum $\{R_n(X, A), \pi_{S_{\text{fin}}}^A, \{\}\}$. We apply step (i) of this proof to conclude that $(\pi_{S_{\text{fin}}}^A)^*: H^*(R_n(X, A); \mathbb{Q}) \rightarrow H^*(R_n(X, B); \mathbb{Q})$ is a monomorphism for all $A \subset B$ in $S_{\text{fin}}$. An application of Proposition 3.4 completes the proof. \hfill \Box
4. Characterizations

Lemma 4.1. If a map \( f : X \to \mathbb{C}^* \) has an \( n \)-th root, then any map \( g : X \to \mathbb{C}^* \) which is homotopic to \( f \) also has an \( n \)-th root.

Proof. Apply the homotopy lifting property to the \( n \)-th degree covering map \( \mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^n \).

\[ \square \]

Lemma 4.2. Let \( X \) be a normal space. The following conditions are equivalent:

(a) \( C(X) \) is approximately \( n \)-th root closed.

(b) \( C(A, \mathbb{C}^*) \) is approximately \( n \)-th root closed for any closed subset \( A \) of \( X \).

(c) \( C(A, \mathbb{C}^*) \) is \( n \)-th root closed for any closed subset \( A \) of \( X \).

Proof. For a positive number \( r \), let \( A(0, r) = \{ z \in \mathbb{C} : |z| \geq r \} \) and \( B(0, r) = \{ z \in \mathbb{C} : |z| \leq r \} \). Let \( \rho_z : \mathbb{C}^* \to A(0, \varepsilon) \) be the radial retraction. Note that \( \rho_z \) is homotopic to the identity map of \( \mathbb{C}^* \).

(a) \( \Rightarrow \) (b) Take \( \varepsilon > 0 \) and consider a closed subset \( A \) of \( X \). Pick \( f \in C(A, \mathbb{C}^*) \) and put \( h = \rho_z \circ f \). Extend \( h \) to a function \( F \) on \( X \), applying the hypothesis (a) to find an \( n \)-th root \( g \), and restricting \( g \) to \( A \), we obtain a function \( g : A \to \mathbb{C}^* \) such that \( ||h - g^n|| < \varepsilon/2 \). This condition guarantees that \( g \in C(A, \mathbb{C}^*) \). It is easy to see that \( ||f - g''|| < \varepsilon + \varepsilon/2 < 2\varepsilon \).

(b) \( \Rightarrow \) (c) Again, consider \( f \in C(A, \mathbb{C}^*) \), where \( A \) is a closed subset of \( X \), and put \( h = \rho_z \circ f \). Note that \( h \) is homotopic to \( f \). Find \( g : A \to \mathbb{C}^* \) such that \( ||h - g^n|| < \varepsilon/2 \). This condition guarantees that \( g^n \) is homotopic to \( h \) and hence to \( f \). An application of Lemma 4.1 completes the proof.

(c) \( \Rightarrow \) (a) Take \( f \in C(X) \) and fix \( \varepsilon > 0 \). Consider \( A = f^{-1}(A(0, \varepsilon)) \) and \( B = f^{-1}(B(0, \varepsilon)) \). Find \( g \in C(A, \mathbb{C}^*) \) such that \( f|_A = g^n \). Note that \( g(A \cap B) \subset B(0, \sqrt[2n]{\varepsilon}) \), and we can extend \( g \) over \( X \) to \( \overline{g} \) such that \( \overline{g}(B) \subset B(0, \sqrt[n]{\varepsilon}) \). It is easy to check that \( \|f - \overline{g}''\| < 2\varepsilon \).

We let \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Suppose \( Y \) is a Hausdorff paracompact space. Huber’s Theorem \[9\] implies the existence of a canonical isomorphism \( H^1(Y; \mathbb{Z}) \cong [Y, S^1] \). Here \( [Y, S^1] \) denotes the group of all homotopy classes of maps from \( Y \) to \( S^1 \) with the group operation induced by the multiplication of maps in \( C(Y, S^1) \). We denote the homotopy class of a map \( f \in C(Y, S^1) \) by \( [f] \).

Theorem 4.3. Let \( X \) be a Hausdorff paracompact space. Then \( C(X) \) is approximately \( n \)-th root closed iff \( H^1(A; \mathbb{Z}) \) is \( n \)-divisible for every closed subset \( A \) of \( X \).

Proof. Consider a closed subset \( A \) of \( X \). First, suppose that \( C(X) \) is approximately \( n \)-th root closed. Let \( f : A \to S^1 \) be a representative of an arbitrary element of \( H^1(A; \mathbb{Z}) \). By condition (c) of Lemma 4.2 there exist \( g : A \to S^1 \) such that \( g^n = f \) and hence \( n[g] = [f] \) in \( H^1(A; \mathbb{Z}) \).

In order to prove the converse part, we verify condition (c) of Lemma 4.2. Pick \( f \in C(A, \mathbb{C}^*) \). Then \( f \) is homotopic to a map \( \tilde{f} : A \to S^1 \). Since \( [\tilde{f}] \in H^1(A; \mathbb{Z}) \) is divisible by \( n \) there exists \( h : A \to S^1 \) such that \( h^n \) is homotopic to \( f \) and hence to \( f \). Lemma 4.1 implies that \( f \) has an \( n \)-th root.

\[ \square \]

5. Compacta with approximately root closed \( C(X) \)

Lemma 5.1. Let \( S = \{ X_i, p_i^{i+1} \} \) be an inverse sequence of compact metrizable spaces and let \( X = \lim S \). Consider the following two conditions.

---
(a) $C(X)$ is approximately $n$-th root closed.

(b) For any $i$, any closed subset $A_i$ of $X_i$ and any map $h: A_i \to \mathbb{C}^*$, there exists $j > i$ such that the map $h \circ p^j_i: A_j \to \mathbb{C}^*$ has an $n$-th root, where $A_j = (p^j_i)^{-1}(A_i)$.

Condition (b) implies condition (a). Moreover if all projections of $S$ are surjective, then the converse implication (a)→(b) also holds.

**Proof.** Put $X = \lim S$. First, we show that $C(X)$ is approximately $n$-th root closed by checking condition (b) of Lemma 6.4. Let $A$ be a closed subset of $X$ and let $f \in C(A, \mathbb{C}^*)$ be a function. Take any $\varepsilon > 0$. There exist $i$ and a mapping $f_i: p_i(A) \to \mathbb{C}^*$ such that $f_i \circ p_i|_A$ is $\varepsilon$-close to $f$. Let $A_i = p_i(A)$ and find $j > i$ such that the map $f_i \circ p^j_i: A_j \to \mathbb{C}^*$ has an $n$-th root. Then $g = h \circ f_j$ is an $n$-th root of $f_i \circ p_i|_A$. Obviously, $\|f - g^n\| < \varepsilon$.

Conversely, suppose $C(X)$ is approximately $n$-th root closed and all projections of $S$ are surjective. Pick $i$ and consider a closed subset $A_i$ of $X_i$ and a map $h: A_i \to \mathbb{C}^*$. Let $\varepsilon = \min\{|h(x)|: x \in A_i\}$. Put $A = (p_i)^{-1}(A_i)$. There exists $g: A \to \mathbb{C}^*$ such that $g^n$ is $\varepsilon$-close to $h \circ p_i|_A$. We can find $j > i$ and a map $g_j: p_j(A) \to \mathbb{C}^*$ such that $(g_j \circ p_j)^n$ is $\varepsilon$-close to $g^n$. Let $A_j = (p^j_i)^{-1}(A_i)$. Since all projections of $S$ are surjective, $p_j(A) = A_j$. Using this, it is not hard to verify that $(g_j)^n$ is $\varepsilon/2$-close, and hence homotopic to $h \circ p^j_i$. Lemma 6.4 implies that $h \circ p^j_i$ has an $n$-th root.

**Theorem 5.2.** For every positive integer $m$ there exists an $m$-dimensional compact metrizable space $X$ such that $C(X)$ is approximately $n$-th root closed for all positive integers $n$.

**Proof.** We obtain $X$ as the inverse limit of a sequence $S = \{X_i, p^{i+1}_i\}$, consisting of $m$-dimensional metrizable compacta. The sequence is constructed by induction as follows. Represent the set of all positive integers as a union of disjoint infinite subsets $\{\Lambda_n\}_{n=2}^{\infty}$. Put $X_1 = S^{m_1}$, the $m$-dimensional sphere. Suppose the space $X_k$ has already been constructed. Fix a countable collection $B_k$ of closed subsets of $X_k$ such that for each closed subset $A$ of $X_k$ and for any open neighborhood $U$ of $A$, there exists $B \in B_k$ such that $A \subset B \subset U$. For each $B \in B_k$ fix a family $F_B$ of maps from $B$ to $\mathbb{C}^*$ which is dense in the space $C(B, \mathbb{C}^*)$. For every map from the family $F_B$, we fix its extension to a map from $X_k$ to $\mathbb{C}$ and denote the family of these extensions by $F_B$. Let $\Phi_k = \bigcup\{F_B \mid B \in B_k\}$. Define $X_{k+1} = R_n(X_k, \Phi_k)$ where $n$ is such that $k \in \Lambda_n$, and let $p^{k+1}_k = \pi_{\Phi_k}$.

Put $X = \lim S$. To verify that $C(X)$ is approximately $n$-th root closed for each $n > 1$, it is enough to show that condition (b) of Lemma 6.4 is satisfied for the inverse sequence $S$. Fix $n > 1$. Pick $i$ and consider a closed subset $A_i$ of $X_i$ and a function $h: A_i \to \mathbb{C}^*$. Take a number $j > i$ such that $j - 1 \in \Lambda_k$. Let $A_j = (p^j_i)^{-1}(A_i)$. We show that the map $h \circ p^j_i: A_j \to \mathbb{C}^*$ has an $n$-th root. Put $A_{j-1} = (p_i^{j-1})^{-1}(A_i)$. Let $g$ be an extension of the map $h \circ p_i^{j-1}: A_{j-1} \to \mathbb{C}^*$ to some neighborhood $U$ of $A_{j-1}$. There exists $B \in B_k$ and a function $f: B \to \mathbb{C}^*$ such that $A_{j-1} \subset B \subset U$ and the restriction $g|B$ is homotopic to $f$. Let $\tilde{f}: B \to \mathbb{C}$ be the extension of $f$ that belongs to the family $\Phi_k$. Since the map $p_i^{j-1}$ resolves the projective $n$-th root problem for $\tilde{f}$, the map $f \circ p_i^{j-1}|_{A_j}$ has an $n$-th root.
By Lemma 4.1 the map \( g \circ p_j^j|_{A_j} \) has an \( n \)-th root. It remains to note that 
\[ h \circ p_j^j|_{A_j} = g \circ p_j^j|_{A_j}. \]

Note that \( \dim X \leq m \) since all \( X_k \) are at most \( m \)-dimensional. Proposition 3.5 implies that 
\( (p_{k+1}^k)^*: H^m(X_k; \mathbb{Q}) \rightarrow H^m(X_{k+1}; \mathbb{Q}) \) is a monomorphism. Applying 
Proposition 3.3 we conclude that \( (p_1)^*: H^m(S^m; \mathbb{Q}) \rightarrow H^m(X; \mathbb{Q}) \) is a monomorphism. Thus the limit projection \( p_1: X \rightarrow S^m \) is essential and therefore \( \dim X \geq m \).

Let \( \mathcal{K} \) be a class of spaces. A space \( Z \in \mathcal{K} \) is called a universal space for the class 
\( \mathcal{K} \), if every space in \( \mathcal{K} \) is topologically embedded in \( Z \). For a positive integer \( n \) and 
\( \tau \geq \omega \), let \( \mathcal{A}_\tau(n) \) (\( \mathcal{A}_\tau \) resp.) be the class of all compact Hausdorff spaces \( X \) such 
that \( w(X) \leq \tau \) and \( C(X) \) is \( n \)-th root closed (\( C(X) \) is \( n \)-th root closed for each \( n > 1 \) resp.). It was shown in [3, Corollary 1.3], that 
\( \mathcal{A}_\tau(n) \) contains a universal space for any \( \tau \geq \omega \) and any \( n > 1 \). Using the idea of 
the proof of Theorem 1.2 from [3] one can show that \( \mathcal{A}_\tau \) also contains a universal space.

**Corollary 5.3.** Let \( Y \) be a universal space with respect to the class \( \mathcal{A}_\omega \) or \( \mathcal{A}_\omega(n) \). 
Then \( Y \) is infinite dimensional.

Hence, any universal space for the class \( \mathcal{A}_\tau(n) \) (\( \mathcal{A}_\tau \) resp.) must be infinite 
dimensional for any \( \tau \geq \omega \).

Also we may consider the subclass \( \mathcal{A}_{m,\tau}(n) \) (\( \mathcal{A}_{m,\tau} \) resp.) consisting of all spaces 
in \( \mathcal{A}_\tau(n) \) (\( \mathcal{A}_\tau \) resp.) of dimension at most \( m \). Theorem 1.2 of [3] also proves 
that the class \( \mathcal{A}_{1,\tau}(n) \) contains a universal space. A similar proof, based on the 
Marden factorization theorem [11], works to prove that the class \( \mathcal{A}_{m,\tau}(n) \) (\( \mathcal{A}_{m,\tau} \) 
resp.) contains a universal space.

### 6. Compacta with root closed \( C(X) \)

In this section, for any positive integer \( m \) we construct a compact Hausdorff 
space \( X \) with \( \dim X = m \) such that \( C(X) \) is \( n \)-th root closed for all \( n \). Note that 
for a metrizable continuum \( Y \) the algebra \( C(Y) \) is square root closed if and only if 
\( Y \) is a dendrite, and therefore \( \dim Y \leq 1 \) [10], [13]. This forces the space \( X \) above 
to be non-metrizable.

**Lemma 6.1.** Let \( S = \{ X_\alpha, p_\alpha^\beta, \mathcal{A} \} \) be a factorizing spectrum. In order for \( C(\lim S) \) 
to be \( n \)-th root closed it is sufficient that for any \( \alpha \in \mathcal{A} \) and any function \( h \in C(X_\alpha) \) 
there exists \( \beta > \alpha \) such that \( h \circ p_\alpha^\beta \) has an \( n \)-th root. If all limit projections of \( S \) 
are surjective, the above condition is also necessary.

**Proof.** Put \( X = \lim S \). Consider \( f \in C(X) \). Since \( S \) is factorizing there exists \( \alpha \) 
and \( f_\alpha \in C(X_\alpha) \) such that \( f = f_\alpha \circ p_\alpha \). By the condition of the lemma we can 
find \( \beta > \alpha \) and \( g_\beta: X_\beta \rightarrow \mathbb{C} \) such that \((g_\beta)^n = f_\alpha \circ p_\beta^\alpha \). It is easy to verify that 
\( g = g_\beta \circ p_\beta \) is an \( n \)-th root of \( f \).

Now suppose that all limit projections of \( S \) are surjective and \( C(X) \) is \( n \)-th root closed. Consider \( \alpha \in \mathcal{A} \) and \( h \in C(X_\alpha) \). There exists \( g \in C(X_\alpha) \) such that 
\( g^n = h \circ p_\alpha \). Since \( S \) is factorizing, there exists \( \beta > \alpha \) and \( g_\beta: X_\beta \rightarrow \mathbb{C} \) such that 
\( g = g_\beta \circ p_\beta \). Since the projection \( p_\beta \) is surjective, \((g_\beta)^n = h \circ p_\beta^\alpha \).

**Theorem 6.2.** For each positive integer \( m \), there exists a compact Hausdorff space 
\( X \) with \( \dim X = m \) and such that \( C(X) \) is \( n \)-th root closed for any \( n \).
Proof. Represent the ordinal $\omega_1$ as the union of countably many disjoint uncountable subsets $\{\Lambda_n\}_{n=2}^\infty$. Starting with $X_0 = S^m$, where $S^m$ denotes an $m$-dimensional sphere, by transfinite induction we define an inverse spectrum $S = \{X_\alpha, p_\alpha, \omega_1\}$ as follows. If $\beta = \alpha + 1$, then define $X_\beta = R_n(X_\alpha, C(X_\alpha))$, where $n$ is such that $\alpha \in \Lambda_n$, and let $p_\alpha^\beta = \pi^{C(X_\alpha)}$. If $\beta$ is a limit ordinal, then define $X_\beta = \lim\{X_\alpha, p_\alpha^\beta, \alpha < \beta\}$ and, for $\alpha < \beta$, let $p_\alpha^\beta$ be the limit projection.

Put $X = \lim S$. To verify that $C(X)$ is $n$-th root closed for each $n > 1$, it is enough to check the condition of Lemma 6.1 for the spectrum $S$. Consider $n > 1$. Since the spectrum $S$ has length $\omega_1$, it is factorizing [2 Corollary 1.3.2]. Consider a function $h: X_\alpha \to \mathbb{C}$ and take an ordinal $\gamma > \alpha$ such that $\gamma \in \Lambda_n$. Since the map $p_\gamma^{\gamma+1}$ resolves the projective $n$-th root problem for $h \circ p_\alpha^\gamma$, the map $h \circ p_\gamma^{\gamma+1}$ has an $n$-th root.

Note that $\dim X_\alpha \leq m$ for each $\alpha$ and hence $\dim X \leq m$. We claim that $(p_\beta^\alpha)^* : H^*(X_\alpha; \mathbb{Q}) \to H^*(X_\beta; \mathbb{Q})$ is a monomorphism for all $\alpha < \beta < \omega_1$. Indeed, in the case $\beta = \alpha + 1$ it follows from Proposition 3.4, and then in a general case it is due to Proposition 3.4. Finally, again with the help of Proposition 3.4 we conclude that $p_0^*: H^m(S^m; \mathbb{Q}) \to H^m(X; \mathbb{Q})$ is a monomorphism and hence the map $p_0 : X \to S^m$ is essential. This implies $\dim X \geq m$. \hfill \Box

It is not hard to verify that if $C(Y)$ is $n$-th root closed for some (completely regular) space $Y$, then $C(\beta Y)$ is also $n$-th root closed. Here by $\beta Y$ we denote the Stone-Čech compactification of $Y$.

**Corollary 6.3.** There exists an infinite-dimensional compact Hausdorff space $X$ such that $C(X)$ is $n$-th root closed for all $n$.

**Proof.** For each $m$, let $X_m$ denote compactum provided by Theorem 6.2. We put $X = \beta(\bigoplus\{X_m \mid m \in \omega\})$. \hfill \Box

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